On the gauge dependence of the singlet and adjoint potentials

V.A. Belavin\textsuperscript{a}, V.G. Bornyakov\textsuperscript{a,b}, V.K. Mitrjushkin\textsuperscript{a,c}

\textsuperscript{a} Institute of Theoretical and Experimental Physics, B.Cheremushkinskaya 25, Moscow, 117259, Russia
\textsuperscript{b} Institute for High Energy Physics, Protvino 142284, Russia
\textsuperscript{c} Joint Institute for Nuclear Research, 141980 Dubna, Russia

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Abstract

We study gauge dependence of the recently suggested definition of the singlet and adjoint potentials in SU(2) lattice gauge theory. We find that in the (time local) maximal tree axial gauge the singlet potential obtained from the gauge dependent correlator Tr$L(x)L^\dagger(y)$ differs from that computed in the Coulomb gauge. In the generalized Coulomb gauge we find the range of the parameter values in which the singlet potential differs from that in the Coulomb gauge.

1 Introduction

The free energy of a static quark–antiquark pair – static quark potential – is of great importance for the understanding of the confinement–deconfinement transition, long distance properties of chromoplasma and heavy quark phenomenology at finite temperatures. One expects that the static quark potential depends on the color channel one chooses, e.g., singlet or adjoint, and that these potentials are gauge invariant \textsuperscript{1,2}. It is well known that the correlator of the Polyakov loop gives rise to the gauge invariant definition of the color averaged potential $V_\text{av}(\vec{R})$ \textsuperscript{1}. However, there is a problem with the gauge invariant definition of the singlet $V_\text{sing}(\vec{R})$ and adjoint $V_\text{adj}(\vec{R})$ potentials. Recently, it has been argued \textsuperscript{3} that it is possible to overcome this problem and to arrive at a gauge invariant definition of the singlet and adjoint potentials using the so called dressed Polyakov lines. The dressing of the source may be viewed as a gauge transformation, and this approach is equivalent to the definition of the singlet and adjoint potentials in a certain gauge. It has been claimed \textsuperscript{3} that one can use any unique gauge that is local in time, i.e.
the gauge that does not change the spectrum of the transfer matrix (contrary to the case of the Lorentz/Landau gauge used in \([2, 4]\)). Therefore, to find \(V_{\text{sing}}(\vec{R})\) one must calculate the average \(\langle \text{Tr} \left( L(\vec{R}) L^\dagger(\vec{0}) \right) \rangle\) in the chosen (time local) gauge, \(L(\vec{x})\) being the Polyakov line defined in the next section. This approach has been used to calculate \(V_{\text{sing}}(\vec{R})\) and \(V_{\text{adj}}(\vec{R})\) in 3D \([3]\) and 4d \([5]\) SU(2), and in 4d SU(3) \([6]\) lattice gauge theories.

The main goal of this note is to study more closely the dependence of \(V_{\text{sing}}(\vec{R})\) and \(V_{\text{adj}}(\vec{R})\) on the choice of the gauge. To this purpose we calculated the gauge invariant correlator \(\langle \text{Tr} \left( L(\vec{R}) \cdot L^\dagger(\vec{0}) \right) \rangle\) as well as the averages \(\langle \text{Tr} \left( L(\vec{R}) L^\dagger(\vec{0}) \right) \rangle\) in various local in time gauges: maximal tree axial gauge (AG), Coulomb gauge (CG) and generalized Coulomb gauge (GCG).

All our calculations were made in 4d SU(2) lattice gauge theory in the finite volume with periodic boundary conditions. Main definitions and computational details are given in the next section. In section 3 we present our results and make conclusions.

## 2 Potentials, gauges and details of simulations

Let \(N_\tau\) and \(N_\sigma\) be the number of sites along the timelike and spacelike directions, respectively, and \(a\) the lattice spacing. We choose \(0 \leq x_i \leq N_\sigma a - a\) \((i = 1, 2, 3)\) and \(0 \leq x_4 \leq N_\tau a - a\).

In the case of SU(2) lattice gauge theory the color averaged potential \(V_{\text{av}}(\vec{R})\) is given by \([1]\)

\[
e^{-V_{\text{av}}(\vec{R})/T} = \frac{1}{4} \langle \text{Tr} \left( L(\vec{R}) \cdot L^\dagger(\vec{0}) \right) \rangle .
\]

(2.1)

where \(T = 1/aN_\tau\) is the temperature, \(L(\vec{x})\) is the Polyakov line defined as \(L(\vec{x}) = \frac{i}{2} \text{Tr} \prod_{x_0=0}^{N_\sigma a-a} U_{x_0,\vec{x},0} \), \(U_{x\mu}\) is the lattice field. Following \([2]\) one can introduce the color singlet and adjoint potentials as follows

\[
e^{-V_{\text{sing}}(\vec{R})/T} = \frac{1}{2} \langle \text{Tr} \left( L(\vec{R}) L^\dagger(\vec{0}) \right) \rangle ;
\]

(2.2)

\[
\frac{3}{4} e^{-V_{\text{adj}}(\vec{R})/T} = e^{-V_{\text{av}}(\vec{R})/T} - \frac{1}{4} e^{-V_{\text{sing}}(\vec{R})/T} .
\]

(2.3)

Eq.'s (2.2), (2.3) have meaning only when some gauge fixing condition is imposed. Without gauge fixing \(V_{\text{sing}}(\vec{R}) = V_{\text{adj}}(\vec{R}) = V_{\text{av}}(\vec{R})\). In this work we employ the following local in time gauges.

- 3D maximal tree axial gauge.

In this case the gauge is fixed at every time slice independently as follows:
– first we fix $U_{x1} = 1$ for all $x_1, x_2, x_3$ apart from $x_1 = N_\sigma a - a$;
– then we fix $U_{x2} = 1$ for $x_1 = N_\sigma a - a$ and all $x_2, x_3$ apart from $x_2 = N_\sigma a - a$;
– finally we fix $U_{x3} = 1$ for $x_1 = x_2 = N_\sigma a - a$ and all $x_3$ apart from $x_3 = N_\sigma a - a$.

• Generalized Coulomb gauge.
  
  To fix this gauge one should find the maximum of the functional
  \[ F^\lambda_\Omega(U) = \sum_x \frac{1}{2} \text{Tr} U^\Omega_{x1} + \lambda \sum_x \left[ \frac{1}{2} \text{Tr} U^\Omega_{x2} + \frac{1}{2} \text{Tr} U^\Omega_{x3} \right], \]
  with respect to gauge transformations $\Omega$, where $U^\Omega_{x\mu} = \Omega_x U_{x\mu} \Omega^\dagger_{x+\mu}$ and $\lambda$ is some parameter taking values between zero and unity. Evidently, at $\lambda = 1$ one arrives at the standard Coulomb gauge.

Our main simulations were made on $6 \times 18^3$ lattice at $\beta = 2.35$. At this $\beta$ $T/T_c = \sqrt{\sigma(\beta_c)/\sigma(2.35)} = 0.76(1)$, where $\sigma$ is string tension, and $\beta_c = 2.43$ \[1\]. 5,000 configurations, separated by 100 combined sweeps consisting of one heat bath and two overrelaxation sweeps, were collected. To compare our results for the Coulomb gauge with those of ref. \[5\] we also made simulations on $4 \times 16^3$ lattice at $\beta = 2.234$ ($T/T_c = 0.80$ \[5\]). To fix the Coulomb and the generalized Coulomb gauges we used the ordinary procedure alternating the relaxation and overrelaxation sweeps.

Statistical errors are calculated by the blocked jackknife method. To see the dependence of the potentials on the choice of the Gribov copy in the Coulomb gauge we generated 5 random gauge copies for every equilibrium configuration at $\beta = 2.234$. No sizable effect of the Gribov copies has been found.

3 Results in various gauges

In the Coulomb gauge our results are in agreement with earlier results \[5\] for the $4d$ case. The singlet potential $V^{CG}_{\text{sing}}(\vec{R})$ is below the color averaged $V^{\text{av}}_{\text{adj}}(\vec{R})$ at small distances and approaches it from below with increasing $|\vec{R}|$, while the adjoint potential $V^{CG}_{\text{adj}}(\vec{R})$ is above $V^{\text{av}}_{\text{adj}}(\vec{R})$ at small distances and approaches it from above at large distances (see Figure \[1\]).

Of special interest for us is the comparison of the singlet and adjoint potentials in various time local gauges. Figure \[1\] shows the behavior of the singlet potential $V_{\text{sing}}^{AG}(\vec{R})$ in the maximal tree axial gauge (filled circles). The important observation

\[1\]We took $\sigma(2.35) = 0.311(2)$ from \[8\] and $\sigma(2.43)$ was estimated by interpolation of the data for string tension taken from the literature.
is that this potential is far from being equal to the singlet potential \( V^{CG}_{\text{sing}}(\vec{R}) \) defined in the Coulomb gauge (triangles up Figure [1]). In fact, this observation represents one of the main results of our work.

Numerically, the potential \( V^{AG}_{\text{sing}}(\vec{R}) \) is very close to the color averaged potential \( V^{CG}_{\text{sing}}(\vec{R}) \) for most values of the distance \( R \). At the same time at distances along the lattice axes one can see clear deviations from \( V_{av}(\vec{R}) \) which are due to the lack of the rotational invariance of maximal tree axial gauge. The effects of this lack of the rotational invariance look even more impressive for the potentials measured along the axes in the 1st, 2nd and 3rd directions separately (we do not show it in the Figure).

In this gauge the correlator eq.(2.2) can be explicitly expressed as a linear combination of the periodic Wilson loops (PWL), introduced in [2]. The path connecting points \( \vec{0} \) and \( \vec{R} \) (space-like leg of PWL) depends on direction, e.g. the path is a straight line for \( \vec{R} = \{R,0,0\} \), while it deviates from a straight line for \( \vec{R} = \{0,R,0\} \) and \( \vec{R} = \{0,0,R\} \). Thus the lack of the rotational invariance can be ascribed to the strong dependence of the PWL on this path. Let us also mention that the correlator eq.(2.2) in AG is exactly the same in the maximal tree gauge, which includes all four directions.

In the case of another gauge we used – GCG – the gauge fixing functional defined in eq.(2.4) depends on the parameter \( \lambda \) chosen to be \( 0 < \lambda \leq 1 \). Evidently, the choice \( \lambda = 1 \) corresponds to the standard Coulomb gauge. We found that in this gauge both potentials, i.e the singlet potential \( V^{\lambda}_{\text{sing}}(\vec{R}) \) and adjoint potential \( V^{\lambda}_{\text{adj}}(\vec{R}) \), demonstrate a nontrivial dependence on \( \lambda \).

In Figure [2] we show the \( R \)-dependence of the singlet potential \( V^{\lambda}_{\text{sing}}(\vec{R}) \) in GCG at some comparatively small values of \( \lambda \) as well as \( R \)-dependence of \( V^{CG}_{\text{sing}}(\vec{R}) \) \((\lambda = 1)\) and \( V_{av}(\vec{R}) \). One can see that with decreasing \( \lambda \) the singlet potential \( V^{\lambda}_{\text{sing}}(\vec{R}) \) moves towards the color averaged potential \( V_{av}(\vec{R}) \).

Figure [3] shows the dependence on \( \lambda \) of the singlet potential \( V^{\lambda}_{\text{sing}}(\vec{R}) \) at some particular values of \( R \) and reveals a nontrivial character of this dependence. Indeed, the data suggest that for \( \lambda \geq \lambda_c \) \((\lambda_c \simeq 0.6)\) there is no any sizable dependence on \( \lambda \) and, therefore, \( V^{\lambda}_{\text{sing}}(\vec{R}) = V^{CG}_{\text{sing}}(\vec{R}) \) within our errorbars. At \( \lambda < \lambda_c \) this is not true anymore and the deviation of \( V^{\lambda}_{\text{sing}}(\vec{R}) \) from \( V^{CG}_{\text{sing}}(\vec{R}) \) increases with decreasing \( \lambda \). Thus we find the class of the gauges, namely the GCG for \( \lambda \geq \lambda_c \), which give rise to the same singlet potential as the Coulomb gauge does. On the other hand, we find further evidence for the gauge dependence of the correlator eq.(2.2).

It is worthwhile to note that the ”transition” at \( \lambda = \lambda_c \) has nothing to do with the restoration of the rotational invariance. Indeed, we found that at \( \lambda > \lambda_c \) the singlet potential measured along the 1st direction is different from the potentials measured along the 2nd or 3rd directions.

At the end we want to make a remark about the calculation of the singlet
potential at \( T = 0 \) using gauge noninvariant correlators of the Wilson lines. It was demonstrated in [3] that the singlet potential calculated from such correlators in the local in time Laplacian gauge was in full agreement with the gauge independent calculation using Wilson loops \(^2\). We have calculated the correlator of the Wilson lines in GCG at \( \lambda = 0.2 \) and in CG on \( 12^4 \) lattice at \( \beta = 2.3 \). We found that the singlet potentials in both gauges agreed nicely with the potential determined from the Wilson loops. Thus at \( T = 0 \) the gauge dependent correlators of the Wilson lines give rise to the same singlet potentials in GCG as in the Laplacian gauge or in CG. In AG the noise was too strong to make any definite conclusion.

We conclude that the statement made in [3] about uniqueness of the singlet and adjoint potentials defined by eq.’s (2.2), (2.3) in any unique and local in time gauge is possibly too strong. We believe that further study of the definition of the singlet and adjoint potentials at finite temperature will be useful. Our study can be extended to other, rotationally invariant gauges, different from the Laplacian gauge and the Coulomb gauge. An example of such gauge is the 3D Maximal Abelian gauge [10] with additional fixing of the abelian degrees of freedom or its analog using corresponding Laplacian operators.

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\(^2\)The correlator of the Wilson lines summed over spatial 2-plane was used to calculate the string tension in SU(3) theory in [9].
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Figure 1: Dependence on $R = |\vec{R}|$ of color averaged potential $V_{av}(\vec{R})$, singlet and adjoint potentials $V_{sing}^{CG}(\vec{R})$, $V_{adj}^{CG}(\vec{R})$ in the Coulomb gauge and singlet potential $V_{sing}^{AG}(\vec{R})$ in the maximal tree axial gauge. Errorbars are smaller than symbol sizes.
Figure 2: $R$–dependence of $V_{\text{sing}}^\lambda(\vec{R})$, $V_{\text{sing}}^{\text{CG}}(\vec{R})$ and $V_{\text{av}}(\vec{R})$. Errorbars are smaller than symbol sizes.

$\beta=2.35$

lattice $18^3 \times 6$
Figure 3: The dependence of the singlet potential $V_{\text{sing}}^\lambda (\vec{R})$ in GCG on $\lambda$ at various values of the distance $R$. Errorbars are smaller than symbol sizes. Lines correspond to the standard Coulomb gauge ($\lambda = 1$).