LEARNING, COMPLEXITY AND INFORMATION DENSITY

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Abstract. What is the relationship between the complexity of a learner and the randomness of his mistakes? This question was posed in [7] who showed that the more complex the learner the higher the possibility that his mistakes deviate from a true random sequence. In the current paper we report on an empirical investigation of this problem. We investigate two characteristics of randomness, the stochastic and algorithmic complexity of the binary sequence of mistakes. A learner with a Markov model of order $k$ is trained on a finite binary sequence produced by a Markov source of order $k^*$ and is tested on a different random sequence. As a measure of learner’s complexity we define a quantity called the sysRatio, denoted by $\rho$, which is the ratio between the compressed and uncompressed lengths of the binary string whose $i^{th}$ bit represents the maximum a posteriori decision made at state $i$ of the learner’s model. The quantity $\rho$ is a measure of information density. The main result of the paper shows that this ratio is crucial in answering the above posed question. The result indicates that there is a critical threshold $\rho^*$ such that when $\rho \leq \rho^*$ the sequence of mistakes possesses the following features: (1) low divergence $\Delta$ from a random sequence, (2) low variance in algorithmic complexity. When $\rho > \rho^*$, the characteristics of the mistake sequence changes sharply towards a high $\Delta$ and high variance in algorithmic complexity.

1. Overview

In computer science, the notion of computational complexity serves as a measure of how difficult it is to compute a solution for a given problem. Computations take time and complexity here means the time rate of growth to solve the problem. Another related kind of complexity measure (studied in theoretical computer science) is the so-called algorithmic (or Kolmogorov) complexity which measures how long a computer program (on some generic computational machine) needs to be in order that it produces a complete description of an object. Interestingly, the theory says that if we consider as an object a system that can process input information (available as a binary sequence of high entropy) and which produces another sequence as an output then the amount of randomness in the output sequence is inversely proportional to the algorithmic complexity of the system.

This has been traditionally studied in the context of algorithmic randomness (see [1] and references within) and it has been only until recently unknown whether such a relationship between complexity and randomness exists for more general systems, for instance, those governed by physical laws. In [3] the complexity of a general static system (for instance, a physical solid) is modeled algorithmically, i.e., by its description length. Using the model it is proposed that the stability of a static system (from the physical perspective) is related to its level of algorithmic

\textit{Key words and phrases.} Structural complexity, algorithmic information theory, binary sequence prediction.
complexity. This is explained by the relationship between the complexity of a system and its ability to 'distort' the randomness in its environment. The first proof of this concept appeared in a recent paper [8] where it was shown that this inverse relationship between system complexity and randomness exists also in a physical system. The particular system investigated consisted of a one-dimensional vibrating solid-beam to which a random sequence of external input forces is applied.

The current paper is yet another proof of concept of the model of [5]. We proceed along the line of [8] but instead of considering a physical system (the static solid with input force sequence) we consider a decision system and study its influence on a random binary data sequence on which prediction decisions are made. The decision system is based on the maximum a posteriori probability decision where probabilities are defined by a statistical parametric model which is estimated from data. The learner of this model is a computer program that trains from a given random data sequence and then produces a decision rule by which it is able to predict (or decide) the value of the next bit in future (yet unseen) random binary sequences.

While this paper is in the realm of machine-learning we are not proposing a new algorithm nor are we interested in the performance of the learner. But rather, our interest is in displaying a learning (and decision) system from the perspective of static system complexity and its influence on random inputs [5].

2. Introduction

Let \( X^{(n)} = X_1, \ldots, X_n \) be a sequence of binary random variables drawn according to some unknown joint probability distribution \( P(X^{(n)}) \). Consider the problem of learning to predict the next bit in a binary sequence drawn according to \( P \). For training, the learner is given a finite sequence \( x^{(m)} \) of bits \( x_t \in \{0, 1\}, 1 \leq t \leq m \), drawn according to \( P \) and estimates a model \( M \) that can be used to predict the next bit of a partially observed sequence. After training, the learner is tested on another sequence \( x^{(n)} \) drawn according to the same unknown distribution \( P \). Using \( M \) he produces the bit \( y_t \) as a prediction for \( x_t \), \( 1 \leq t \leq n \). Denote by \( \xi^{(n)} \) the corresponding binary sequence of mistakes where \( \xi_t = 1 \) if \( y_t \neq x_t \) and is 0 otherwise. In [7] the following question was posed: how random is \( \xi^{(n)} \) ?

It is clear that the sequence of mistakes should be random since the test sequence \( x^{(n)} \) is random. It may also be that because the learner is using a model of a finite structure (or a finite description-length) that it may somehow introduce dependencies and cause \( \xi^{(n)} \) to be less random than \( x^{(n)} \). And yet by another intuition, perhaps the fact that the learner is of a finite complexity limits its ability to 'deform' (or distort) randomness of \( x^{(n)} \)? These are all valid initial guesses that relate to this main question. We note that our basis for saying that \( M \) has a finite structure stems from it being an element of some regular hypothesis class, for instance, having a finite VC-dimension as is often the case in a learning setting (see for instance structural risk minimization of [10]). In the current paper, we are not interested in the learner’s performance (as modeled for instance by Valiant’s PAC framework [4, 5]) but instead we take a black-box view of a learner and ask how much influence does he has on the stochastic properties of the errors. We view the learner as an entity that ‘interferes’ with the randomness that is inherent in the sequence to be predicted and through his predictions creates a sequence of mistakes that has a different stochastic character. This view in a broader sense is taken in [3] and is shown (empirically) in [8] to explain how static structures may ‘deform’ random external forces.

The question raised above was answered in [7] for a particular learning setting where the teacher uses a probability distribution \( P \) based on a Markov model with a
certain complexity. The learner has access to a hypothesis class of Boolean decision rules that are based on Markov models. Hence, learning amounts to the estimation of parameters of a finite-order Markov model (see for instance [3, 4]). The answer shows theoretically that the random characteristics of the subsequence of mistakes corresponding to the 0-predictions of a learner changes in accordance with the complexity of the learner’s decision rule’s complexity. The more complex the rule the higher the possibility of ‘distortion’ of randomness, i.e., the farther away it is from being truly-random.

In the current paper we take an experimental approach to answering the above question. As in [7] we focus on Markov source and a Markov learner whose orders may differ. In the next section we describe the setup.

3. Experimental setup

The learning problem consists of predicting the next bit in a given sequence generated by a Markov chain (model) $M^*$ of order $k^*$. There are $2^k$ states in the model each represented by a word of $k^*$ bits. During a learning problem, the source’s model is fixed. A learner, unaware of the source’s model, has a Markov model of order $k$. We denote by $p(1|i)$ the probability of transitioning from state $i$ whose binary $k$-word is $b_i = [b_i(1), \ldots, b_i(k)]$ to the state whose word is $[b_i(2), \ldots, b_i(k), 1]$. Given a random sequence of length $m$ generated by the source the learner estimates its own model’s parameters $p(1|i)$ by $\hat{p}(1|i)$, $1 \leq i \leq 2^k$, which is the frequency of the event “$b_i$ is followed by a 1” in the training sequence. We denote by $M$ the learnt model with parameters $\hat{p}(1|i)$, $1 \leq i \leq 2^k$. We denote by $p^*(1|i)$ the transition probability from state $i$ of the source model, $1 \leq i \leq 2^k$.

A simulation run is characterized by the parameters, $k$ and $m$. It consists of a training and testing phases. In the training phase we show the learner a binary sequence of length $m$ and he estimates the transition probabilities. In the testing phase we show the learner another random sequence (generated by the same source) of length $n$ and test the learner’s predictions on it. For each bit in the test sequence we record whether the learner has made a mistake. When a mistake occurs we indicate this by a 1 and when there is no mistake we write a 0. The resulting sequence of length $n$ is the generalization mistake sequence $\xi^{(n)}$. We denote by $\xi^{(n)}_0$ the binary subsequence of $\xi^{(n)}$ that corresponds to the mistakes that occurred only when the learner predicted a 0.

For a fixed $k$ denote by $N_{k,m}$ the number of runs with a learner of order $k$ and training sample of size $m$. The experimental setup consists of $N_{k,m} = 10$ runs with $1 \leq k \leq 10$, $m \in \{100, 200, \ldots, 10000\}$ with a total of $100 \cdot 10 \cdot N_{k,m} = 10000$ runs. The testing sequence is of length $n = 1000$. Each run results in a file called system which contains a binary vector $d$ whose $i^{th}$ bit represents the maximum a posteriori decision made at state $i$ of the learner’s model, i.e.,

$$d_i = \begin{cases} 1 & \text{if } \hat{p}(1|i) > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq 2^k$. Let us denote by $\alpha_i = P(\hat{p}(1|i) > \frac{1}{2})$, thus $d_i$ are Bernoulli random variables with parameters $\alpha_i$, $1 \leq i \leq 2^k$. The learner’s system is its decision rule at every possible state.

Another file generated is the errorT0 which contains the mistake subsequence $\xi^{(n)}_0$. At the end of each run we measure the lengths of the system file and its compressed length where compression is obtained via the Gzip algorithm (a variant of [11]) and compute the sysRatio (denoted as $\rho$) which is the ratio of the compressed to uncompressed length of the system file. Note that $\rho$ is a measure of information density since it captures the number of bits of useful information (useful for
describing the system) there are per bit of representation (in the uncompressed file).

We do similarly for the mistake-subsequence $\xi_0(n)$ obtaining the length $\ell_0$ of the compressed file that contains $\xi_0(n)$ (henceforth referred to as the estimated algorithmic complexity of $\xi_0(n)$ since it is an approximation of the Kolmogorov complexity of $\xi_0^{(n)}$, see [5]). We measure the KL-divergence $\Delta_0$ between the probability distribution $P(w|\bar{p})$ of binary words $w$ of length 4 and the empirical probability distribution $\hat{P}_m(w)$ as measured from the mistake subsequence $\xi_0(n)$. Note, $P(w|\bar{p})$ is defined according to the Bernoulli model with parameter $\bar{p}$, that is, $P(w|\bar{p}) = \bar{p}^i(1-\bar{p})^{4-i}$ for a word $w$ with $i$ ones, where $\bar{p}$ is the frequency of ones in the subsequence $\xi_0^{(n)}$. The distribution $\hat{P}_m(w)$ equals the frequency of a word $w$ in $\xi_0^{(n)}$. Hence $\Delta_0$ reflects by how much $\xi_0^{(n)}$ deviates from being random according to a Bernoulli sequence.

4. Results

We are interested in the determining the following relationships: (1) the system ratio $\rho$ versus the learner’s model order $k$, (2) the estimated algorithmic complexity $\ell_0$ of the subsequence $\xi_0^{(n)}$ versus the $\rho$, and (3) the deviation $\Delta_0$ versus $\rho$.

We choose four different levels of learning problems, controlled by the order of the source model $k^* = 3, 4, 5, 6$. For each problem we choose for the source model a transition matrix of probabilities $p^*(1|i) = 1 - p$, $p^*(0|i) = p$, where for some of the states $i$ we set $p = 0.3$ and for others $p = 0.7$, $1 \leq i \leq 2^{k^*}$. Thus the Bayes optimal error is 0.3. To ensure that the problem is sufficiently challenging we set the first half of the states (those ranging from the $k^*$-dimensional vector 00...0 to 011...1) to have $p = 0.3$ and the second half (10...0 to 11...1) to have $p = 0.7$. This ensures that a Markov model of order $k < k^*$ cannot approximate the true transition probabilities well, i.e., the infinite-sample limit estimate based on a Markov model of order $k$ which is smaller than $k^*$ will still be $\tilde{p}(1|i) = 0.5$, $1 \leq i \leq 2^k$. But for a Markov model of order $k \geq k^*$ the infinite-sample size estimates will converge to the true values of $p$ or $1 - p$.

Before we start to investigate the three relationships stated above we perform a sanity check to see how the prediction generalization error (for any of the two prediction types, not just when predicting a zero) varies with respect to the model complexity $k$. Figure 4.1 displays this relationship for a learning problem with $k^* = 3$. The curve (with x) is the mean error over all learning runs of a fixed $k$ value, the upper and lower curves are the standard deviation above and below the mean, respectively. As seen, when the learner’s model order $k$ is smaller than $k^*$ his generalization error stays at the maximum level of 0.5. At $k = k^*$ there is a drop to an error close to the Bayes error of 0.3 Then as $k$ increases beyond $k^*$ the mean (as well as the standard deviation) of the generalization error start to increase. This is due to overfitting of the model to the training data and also because the variance of the error estimate increases with $k$ due to the fact that the maximum sample size of any run is fixed at $m = 10000$ and is not increasing with respect to $k$.

We now proceed to describe the first result which concerns the relationship between the sysRatio $\rho$ and $k$. Figure 4.2 shows the mean and standard deviation of the SysRatio $\rho$ as a function of $k$. The mean decreases as the learner’s model order $k$ increases. To explain this, first note that the uncompressed length of the system is always $c \cdot 2^k$ for some constant $c > 0$ since the vector $d$ is of length $2^k$ (see section 3). The length of the compressed system file also grows, but at a slower rate with respect to $k$ and this gives rise to the decrease in $\rho$ with respect to $k$. Why is the rate of the compressed system file growing more slowly?
The reason is that for values of $k < k^*$ the learner’s model is incapable (by design of the learning problem) of estimating the Bayes optimal prediction and the probability of the events “$b_i$ is followed by a 1” is $p(1|i) = 1/2$, $1 \leq i \leq 2^k$. Thus the average value $\hat{p}(1|i)$ of the indicators of such events is a Binomial random variable with a distribution symmetric at $1/2$ and hence from (3.1) the probability $\alpha_i$ that $\hat{p}(1|i) > 1/2$ equals $1/2$. The components of the random vector $d$ are independent Bernouli random variables with parameter $\alpha_i$ when conditioned on the sample size vector $v$ (this is the vector whose components $v_i$ are the number of times that $b_i$ appeared in the training sequence, see [7] for details). Since in this case $\alpha_i = 1/2$ then each component has a maximum entropy $H(d_i) = -\alpha_i \log \alpha_i - (1 - \alpha_i) \log (1 - \alpha_i) = \log 2 = 1$ and hence the expected value of the entropy of the vector $d$ (with respect to the random sample size vector $v$) is maximal and equals $E_v H(d|v) = E_v \sum_{i=1}^{2^k} H(d_i|v_i) = E_v 2^k = 2^k$. Hence the expected compressed length of the system file (which contains the vector $d$) is large as the expected description length of any random variable is at least as large as its entropy.

As $k$ increases beyond $k^*$ the model becomes more capable of estimating the true transition probabilities (recall, these are either 0.3 or 0.7) and the probability $p(1|i)$ of the events “$b_i$ is followed by a 1” get farther away from $1/2$ in the direction of 0.3 or 0.7, depending on the particular state $i$, $1 \leq i \leq 2^k$. Thus the average value $\hat{p}(1|i)$ of the indicators of such events is a Binomial random variable with an asymmetric distribution with a mean $p(1|i)$. Hence from (3.1) the probability $\alpha_i$ that $\hat{p}(1|i) > 1/2$ gets either very close to 0 or 1 as the training size $m$ increases. Thus the components of the random vector $d$ tend to be closer to deterministic. They are still random since the training sequence length is not increasing with $k$ and the variance of the estimates $\hat{p}(1|i)$ does not converge to zero. Therefore for each of the $2^k$ components of the vector $d$ the entropy is smaller than when $k < k^*$. However as there are exponentially many components $d_i$, on the whole, the entropy of $d$ (and hence the expected compressed length of the system file) still increase but at a lower rate than when $k < k^*$.

Next, we discuss the characteristics of the mistake subsequence $\xi_0^{(n)}$. Figure 4.3 shows the graph (with $x$) of the mean of the estimated algorithmic complexity $\ell_0$ of $\xi_0^{(n)}$ versus the mean of the system ratio $\rho$ on the horizontal axis. The dashed lines are the upper and lower envelopes of the standard deviation from the mean. The arrow points at the value of $\rho^*$ that corresponds to $k^*$ (the source model order).
As can be seen, for low values of sysRatio the spread $\ell_0$ is low. There is a sharp threshold at $\rho^*$ where the spread around the mean value of $\ell_0$ increases significantly.

Next, Figure 4.4 displays the graph (with x) of the mean of the divergence $\Delta_0$ of the mistake subsequence $\xi_0^{(n)}$ versus the mean of the system ratio $\rho$ on the horizontal axis. The dashed lines are the upper and lower envelopes of the standard deviation from the mean. The arrow points at the value of $\rho^*$ that corresponds to $k^*$ (the source model order). As can be seen, for low values of sysRatio the spread of $\Delta_0$ is low. As the result above for $\ell_0$, we see a threshold at $\rho^*$ where the standard deviation around the mean value of $\Delta_0$ increases significantly.
The paper introduces the notion of sysRatio $\rho$ which is a measure of information density of the learner's model. It is similar to the notion of rate of information transmission [2] as it measures the ratio of the number of useful information bits contained in a file that describes the learner decision rule per bit of representation (in the file). The results of this paper depict that this information density influences the level of randomness of the mistakes made by a learner. The sysRatio $\rho$ is a proper measure of complexity of a learner decision rule. It is with respect to $\rho$ that the characteristics of the random mistake subsequence $\xi^{(a)}_0$ follow what the theory [2] predicts. The higher the sysRatio the more significant the deviation $\Delta_0$ of $\xi^{(a)}_0$ compared to a pure Bernoulli random sequence. In addition, we have shown that the higher the sysRatio the larger the possible fluctuations in the algorithmic complexity $\ell_0$ of $\xi^{(a)}_0$. The interesting point is the sharp non-linearity in this relationship. We showed that there is a threshold $\rho^*$ at which the spread in values of $\ell_0$ and $\Delta_0$ increases and it corresponds to the point where the learner's model becomes too simple and is incapable of predicting well.

5. Conclusions

Figure 4.4. Divergence $\Delta_0$ of the mistake subsequence $\xi^{(a)}_0$ versus the SysRatio $\rho$

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