Investigations of the $\pi N$ total cross sections at high energies using new FESR: log $\nu$ or $(\log \nu)^2$

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We propose to use rich informations on $\pi p$ total cross sections below $N(\sim 10 \text{ GeV})$ in addition to high-energy data in order to discriminate whether these cross sections increase like log $\nu$ or $(\log \nu)^2$ at high energies, since it is difficult to discriminate between asymptotic log $\nu$ and $(\log \nu)^2$ fits from high-energy data alone. A finite-energy sum rule (FESR) which is derived in the spirit of the $P'$ sum rule as well as the $n = 1$ moment FESR have been required to constrain the high-energy parameters. We then searched for the best fit of $\sigma_{tot}^{(+)}$ above 70 GeV in terms of high-energy parameters constrained by these two FESR. We can show from this analysis that the $(\log \nu)^2$ behaviours is preferred to the log $\nu$ behaviours.

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The high energy behavior of $\pi N$ total cross sections has been one of the longstanding problems in particle physics. The sum of $\pi^+ p$ and $\pi^- p$ total cross sections has a tendency to increase above 70 GeV experimentally\[3]. It is well-known as the Froissart unitarity bound\[2] that the increase of total cross sections is at most log $\nu$. It has not been possible\[4], however, to discriminate between asymptotic log $\nu$ and log$^2\nu$ fits if one uses high-energy data alone above $\sim 70$ GeV.

The purpose of this paper is to propose to use rich informations of $\pi p$ total cross sections at low and intermediate energy regions through new finite-energy sum rules (FESR) as constraints in addition to high-energy data, in order to discriminate the high energy behaviours of $\pi p$ total cross sections above 70 GeV.

Such a kind of attempt has been initiated in Ref. \[4]. The s-wave $\pi N$ scattering length $a^{(+)}$ of the crossing-even amplitude had been expressed as

\begin{equation}
(1 + \frac{\mu}{M}) a^{(+)} = -\frac{g_F^2}{3\pi} \left( \frac{\mu}{2M} \right)^2 \frac{1}{M} \left[ 1 - \left( \frac{\mu}{M} \right)^2 \right]^2 \int_0^\infty dk \left( \sigma_{tot}^{(+)}(k) - \sigma_{tot}^{(+)}(\infty) \right)
\end{equation}

with pion mass $\mu$ under the assumption that there are no singularities with the vacuum quantum numbers in the $J$ plane except for the Pomeron ($P$). The evidence that this sum rule had not been satisfied led us to the prediction of the $P'$ trajectory with $\alpha_{P'}(0) \approx 0.5$, and the $f$ meson ($f_2(1275)$) has been uncovered on this $P'$ trajectory.

(FESR(1)): Taking into account the present situation of increasing total cross section data, we derive FESR in the spirit of the $P'$ sum rule\[4]. We consider the crossing-even (spin-averaged) forward scattering amplitude for $\pi p$ scattering\[4]

\begin{equation}
f^{(+)}(\nu) = \frac{1}{4\pi} [A^{(+)}(\nu) + \nu B^{(+)}(\nu)].
\end{equation}

We assume

\begin{equation}
\text{Im } f^{(+)}(\nu) \simeq \text{Im } R(\nu) + \text{Im } f_{P'}(\nu)
\end{equation}

\begin{equation}
= \frac{\nu}{\mu} (c_0 + c_1 \log \frac{\nu}{\mu} + c_2 \log^2 \frac{\nu}{\mu}) + \frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{P'}(0)}
\end{equation}

at high energies ($\nu \geq N$). Since this amplitude is crossing-even, we have

\begin{equation}
R(\nu) = \frac{i\nu}{2\mu^2} (2c_0 + c_2 \pi^2 + c_1 \left( \log^2 \frac{\nu}{\mu} \right) + \frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{P'}(0)}),
\end{equation}

\begin{equation}
f_{P'}(\nu) = -\frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{P'}(0)} \frac{\sin \pi \alpha_{P'}(0)}{\sin \pi \alpha_{P'}(0)}
\end{equation}

and subsequently we obtain

\begin{equation}
\text{Re } R(\nu) = \frac{\pi\nu}{2\mu^2} \left( c_1 + 2c_2 \log \frac{\nu}{\mu} \right),
\end{equation}

\begin{equation}
\text{Re } f_{P'}(\nu) = -\frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{\alpha_{P'}(0)} \frac{\cot \pi \alpha_{P'}(0)}{2}
\end{equation}

\begin{equation}
= -\frac{\beta_{P'}}{\mu} \left( \frac{\nu}{\mu} \right)^{0.5},
\end{equation}

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substituting $\alpha_{P'}(0) = \frac{1}{2}$ in Eq. (3). Let us define

$$\tilde{f}^{(+)}(\nu) = f^{(+)}(\nu) - R(\nu) - f_{P'}(\nu)$$

and write dispersion relation for $\frac{\tilde{f}^{(+)}(\nu)}{\nu - \mu}$. Since this amplitude is superconvergent, we obtain

$$\text{Re} \tilde{f}^{(+)}(\mu) = \frac{P}{\pi} \int_{-\infty}^{\infty} d\nu' \frac{\text{Im} \tilde{f}^{(+)}(\nu')}{\nu' - \mu}$$

$$= \frac{2P}{\pi} \int_{0}^{\infty} \frac{\nu' \text{Im} \tilde{f}^{(+)}(\nu')}{k^2} d\nu' .$$

Using Eqs. (8) and (9), we have

$$\text{Re} f^{(+)}(\mu) = \text{Re} R(\mu) + \text{Re} f_{P'}(\mu) - \frac{g_r^2}{4\pi} \left( m \right)^2 \frac{1}{M} \frac{1}{1 - (\frac{\mu}{2M})^2}$$

$$+ \frac{1}{2\pi^2} \int_{0}^{N} \sigma_{\text{tot}}^{(+)}(k) dk$$

$$- \frac{2P}{\pi} \int_{0}^{N} \frac{\nu}{k^2} \left( \text{Im} R(\nu) + \beta \nu \left( \frac{\nu}{\mu} \right)^{0.5} \right) d\nu ,$$

where $N \equiv \sqrt{N^2 - \mu^2} \simeq N$. Let us call Eq. (10) as the FESR(1) which we use as the first constraint. It is important to notice that Eq. (10) reduces to the $P'$ sum rule in ref. [4] if $c_1, c_2 \rightarrow 0$.

The FESR (3), (4), (5)

$$\int_{0}^{N} \nu \sigma_{\text{tot}}^{(+)}(k) dk$$

holds for even positive integer $n$ when $f^{(+)}(\nu)$ is crossing even. We can also derive negative-integer moment FESR. The only significant FESR is a one for $f^{(+)}(\nu)/\nu$ corresponding to $n = -1$. FESR(1) belongs to this case.

It is important to emphasize that the FESR should not depend so much on the value of $N$.

$$(\text{FESR(2))}: \text{The second FESR corresponding to } n = 1 \text{ is:}$$

$$\frac{\pi \mu}{4\pi} \left( \frac{g_r^2}{4\pi} \right) \left( \frac{m}{2M} \right)^3 + \frac{1}{4\pi} \int_{0}^{N} dk k^2 \sigma_{\text{tot}}^{(+)}(k) = \int_{0}^{N} \nu \text{Im} R(\nu) d\nu + \int_{0}^{N} \nu \text{Im} f_{P'}(\nu) d\nu .$$

We call Eq. (12) as the FESR(2). It is to be noticed that the contribution from higher energy regions is enhanced.

(Data) The numerical values,

$$-\frac{g_r^2}{4\pi} \left( \frac{m}{2M} \right)^2 \frac{1}{M} \frac{1}{1 - (\frac{\mu}{2M})^2} = -0.0854 \text{GeV}^{-1} ,$$

$$\frac{\pi \mu}{4\pi} \left( \frac{g_r^2}{4\pi} \right) \left( \frac{m}{2M} \right)^3 = 0.0026 \text{GeV} ,$$

have been evaluated using $\frac{g_r^2}{4\pi} = 14.4$.

$$\text{Re} f^{(+)}(\mu) = \frac{1 + \frac{\mu}{M}}{\frac{\mu}{M}} a^{(+)} = \frac{1 + \frac{\mu}{M}}{\frac{\mu}{M}} \left( \sum_{n} a_n \right) = - (0.014 \pm 0.026) \text{GeV}^{-1} \quad (14)$$

was obtained from [3] $a_{\frac{1}{2}} = (0.171 \pm 0.005) \mu^{-1}$ and $a_{\frac{3}{2}} = -(0.088 \pm 0.004) \mu^{-1}$.

We have used rich data [3] of $\sigma_{\pi^{+}p}$ and $\sigma_{\pi^{-}p}$ to evaluate the relevant integrals of cross sections appearing in FESR(1) and (2). We connect each data point [4] of $\sigma_{\pi^{+}p}(k)$ or $k^2 \sigma_{\pi^{+}p}(k)$ with the next point by straight line in order, from $k = 0$ to $k = N$, and regard the area of this polygonal line graph as the relevant integral in the region $0 \leq k \leq N$. The integrals of $\sigma_{\text{tot}}^{(+)}(k)$ ($k^2 \sigma_{\text{tot}}^{(+)}(k)$) are given by averaging these of $\sigma_{\pi^{+}p}(k)$ and
\[ \sigma^{-}\pi^{-}(k) (k^2 \sigma^{-}\pi^{-}(k) \text{ and } k^2 \sigma^{-}\pi^{-}(k)). \] We have obtained

\[ \frac{1}{2\pi^2} \int_0^\infty dk \sigma_{\text{tot}}^{(+)}(k) = 38.75 \pm 0.25 \text{ GeV}^{-1}, \]
\[ \frac{1}{4\pi} \int_0^\infty dk k^2 \sigma_{\text{tot}}^{(+)}(k) = 1817 \pm 31 \text{ GeV} \]

for \( \mathcal{N} = 10 \text{ GeV} \). The errors of relevant integrals, which are from the error of each data point, are very small (\( \sim 1 \) percent), and thus, we regard the central values are exact ones in the following analysis.

When \( \sigma^{-}\pi^{-} \) and \( \sigma^{-}\pi^{-} \) data points are listed at the same value of \( k \), we make \( \sigma_{\text{tot}}^{(+)}(k) \) data point by averaging these values. Totally 183 points are obtained in the region \( 0.16 \leq k \leq 340 \text{ GeV} \) as \( \sigma_{\text{tot}}^{(+)}(k) \) data. There are 12 points in \( k \geq 70 \text{ GeV} \) region, which will be used in the following analysis.

(Analysis) The FESR(1) and (2) are our starting points. Armed with these two, we expressed high-energy parameters \( c_0, c_1, c_2, \beta_P \) in terms of the Born term and the \( N \) scattering length \( a^{(+)} \), as well as the total cross sections up to \( N \). We then attempt to fit the \( \sigma_{\text{tot}}^{(+)} \) above 70GeV. We set \( N = 10 \text{ GeV} \) (corresponding to \( \sqrt{s_{\text{tot}}} = 4.43 \text{GeV} \)) since there are no resonances above this energy. The FESR(2) has also contributions from the lower trajectory \( P^\prime \) which may pass through \( f_2(1810) \). Since \( \alpha_P(0) \) is expected to be around -1, we can assume \( P^\prime \) contribution to be suppressed compared with that from \( P \).

Let us first define the log\( \nu \) model and the log \( \nu \) model. The log\( \nu \) model is a model for which the imaginary part of \( f^{(+)}(\nu) \) behaves as \( a + b \log \nu + c(\log \nu)^2 \) as \( \nu \) becomes large. The log \( \nu \) model is a model for which the imaginary part of \( f^{(+)}(\nu) \) behaves as \( a' + b' \log \nu \) for large \( \nu \). So we generally assume that the \( \text{Im } f^{(+)}(\nu) \) behaves as Eq. (3) at high energies (\( \nu \gg N \)).

(1) log \( \nu \) model: This model has three parameters \( c_0, c_1 \) and \( \beta_P \) with two constraints FESR (1), (2). (Note that the number of independent parameters is one.) We set \( N = 10 \text{ GeV} \) and expressed both \( c_0, \beta_P \) as a function of \( c_1 \) using the FESR(1) and (2). We obtained

\[ c_0(c_1) = 0.0879 - 4.94c_1, \]
\[ \beta_P(c_1) = 0.1290 + 7.06c_1. \]

We then tried to fit 12 data points of \( \sigma_{\text{tot}}^{(+)}(k) \) between 70GeV and 340GeV. The result is shown by thick solid line in FIG. 1. The best fit we obtained is \( c_1 = 0.00185 \) which gives \( c_0 = 0.0787 \) and \( \beta_P = 0.142 \) with the bad “reduced \( \chi^2 \),” \( \chi^2/(N_{\text{data}} - N_{\text{param}}) = 29.04/(12 - 1) \approx 2.6 \). Therefore it turned out that this model has difficulties to reproduce the experimental increase of \( \pi \pi \) total cross sections above 70GeV (see, thick solid line in FIG. 1 (b)). In this log \( \nu \) fit, the results also depend on the value of \( N \), which is not so good.

(2) log\( \nu \) model: This model has four parameters \( c_0, c_1, c_2 \) and \( \beta_P \) with two constraints FESR(1), (2). (So the number of independent parameters is two.) We again set \( N = 10 \text{ GeV} \) and required both FESR(1) and (2) as constraints. Then \( c_0, \beta_P \) are expressed as functions of \( c_1 \) and \( c_2 \) as

\[ c_0(c_1, c_2) = 0.0879 - 4.94c_1 - 21.50c_2, \]
\[ \beta_P(c_1, c_2) = 0.1290 + 7.06c_1 + 41.46c_2. \]

We then searched for the fit to 12 data points of \( \sigma_{\text{tot}}^{(+)}(k) \) above 70GeV. The result is shown by thick solid line in FIG. 2. The best fit in terms of two parameters \( c_1 \) and \( c_2 \) led us to greatly improved value of “reduced \( \chi^2 \),” \( \chi^2/(N_{\text{data}} - N_{\text{param}}) = 0.746/(12 - 2) \approx 0.075 \) for \( c_1 = -0.0215 < 0 \) and \( c_2 = 0.00182 > 0 \), which give \( c_0 = 0.155 \) and \( \beta_P = 0.0524 \). This is an excellent fit to the data (see, thick solid line in FIG. 2 (b)).

(\( \alpha_P \) dependence) So far, we have assumed the intercept of the \( P^\prime \) trajectory \( \alpha_P(0) \) to be 0.5. The value \( \alpha_P(0) \) is estimated to be 0.586 according to the Chew-Frautschi plot, using the universal slope \( \alpha' = 1/1.15 \text{GeV}^{-2} \) and the mass of \( f_2 \) to be 1275MeV. Let us check if the results change for this value of \( \alpha_P(0) \).

Suppose we take \( \alpha_P(0) = 0.586 \) and discuss the two cases, \( \log \nu \) and log\( \nu^2 \).

(1) log \( \nu \) model: We again set \( N = 10 \text{ GeV} \) and expressed both \( c_0, \beta_P \) as functions of \( c_1 \) using FESR (1), (2). We then obtained

\[ c_0(c_1) = 0.0817 - 5.28c_1, \]
\[ \beta_P(c_1) = 0.1238 + 6.77c_1. \]

We then searched for the fit to 12 data points of \( \sigma_{\text{tot}}^{(+)}(k) \) above 70GeV. The result is shown by thin solid line in FIG. 1. The best fit we obtained is \( c_1 = 0.00353 \) which gives \( c_0 = 0.0630 \) and \( \beta_P = 0.148 \) with “reduced \( \chi^2 \),” \( \chi^2/(N_{\text{data}} - N_{\text{param}}) = 22.30/(12 - 1) = 2.03 \). So, this model has difficulties again to reproduce the experimental increase of \( \pi \pi \) total cross sections above 70 GeV (see, thin solid line in FIG. 1 (b)).

(2) log\( \nu^2 \) model: We also set \( N = 10 \text{ GeV} \) and required both FESR (1) and (2) as constraints. Then we obtained

\[ c_0(c_1, c_2) = 0.0817 - 5.28c_1 - 23.50c_2, \]
\[ \beta_P(c_1, c_2) = 0.1238 + 6.77c_1 + 39.80c_2. \]

We again searched for the fit to 12 points of \( \sigma_{\text{tot}}^{(+)}(k) \) above 70 GeV. The best fit in terms of two parameters \( c_1 \) and \( c_2 \) again led us to greatly improved value of “reduced \( \chi^2 \),” \( \chi^2/(N_{\text{data}} - N_{\text{param}}) = 0.750/(12 - 2) = 0.075 \) for \( c_1 = -0.0197 < 0 \) and \( c_2 = 0.00173 > 0 \), which give \( c_0 = 0.145 \) and \( \beta_P = 0.0593 \). This is again an excellent fit to the data (see, the caption of FIG. 2).

We have also searched for \( \alpha_P(0) = 0.543 \) (average of 0.5 and 0.586) and for \( \alpha_P(0) = 0.642 \) due to the Particle Data Group [4]. We found that the results do not change so much.

It is remarkable to notice that the wide range of data \( (k \geq 5 \text{GeV}) \) have been reproduced within the error even
FIG. 1: Fit to the $\sigma_{\text{tot}}^{(+)}$ data above 70GeV by the log $\nu$ model. Thick(Thin) solid line shows the result in the case of $\alpha_{P'} = 0.5(0.586)$. Correspondingly, the contribution from $\text{Im} \ R(\nu)$ (with $c_2 = 0$) is shown by thick(thin) dashed line. Recently a datum\[12\] for $\pi^-N$ total cross section at very high energy ($k=610\text{GeV}$) was reported by the SELEX collaboration. This point is included in (b). The log $\nu$ model with $\alpha_{P'}(0) = 0.5(0.586)$ predicts 24.2(24.4)mb for $\sigma_{\text{tot}}^{(+)}$ at 610GeV which is inconsistent with their value on $\pi^-N$, (26.6 $\pm$ 0.9)mb.

FIG. 2: Fit to the $\sigma_{\text{tot}}^{(+)}$ data above 70GeV by the log $^2\nu$ model. The result in the case of $\alpha_{P'} = 0.5$ is shown by thick solid line, which overlaps in all energy region with the result of $\alpha_{P'} = 0.586$ shown by thin solid line, and both results cannot be distinguished from each other. The contribution from $\text{Im} \ R(\nu)$ with $c_2 > 0$ is shown by thick(thin) dashed line. A datum at 610 GeV obtained by SELEX collaboration is included in (b). Our log $^2\nu$ model predicts 25.9mb for $\sigma_{\text{tot}}^{(+)}$ at 610GeV which is consistent with their value on $\pi^-N$, (26.6 $\pm$ 0.9)mb.

Recently a datum\[13\] for $\pi^-N$ total cross section at very high energy ($k=610\text{GeV}$)\[15\] was reported by the SELEX collaboration. Our log $^2\nu$ model(log $\nu$ model) with $\alpha_{P'}(0) = 0.5$ predicts 25.9mb(24.2mb) for $\sigma_{\text{tot}}^{(+)}$ at 610GeV which is consistent(inconsistent) with their value on $\pi^-N$, (26.6 $\pm$ 0.9)mb. This fact also suggests the validity of the log $^2\nu$ model.

Therefore, we can conclude that our analysis in terms of high-energy parameters constrained by the FESR
(1),(2) prefers the log:2 ν/µ behaviours satisfying the Froissart unitarity bound. Finally we should add a note that the origin of the log:2 ν behaviour of the amplitude at high energy is argued to be explained from the effect of gluon saturation. 

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[14] We take the error Δy for each data point y as Δy = \sqrt{\sum y_i/(\Delta y_i)^2}. When several data points, denoted as yi, with error Δyi (i = 1, · · · , n), are listed at the same value of k, these points are replaced by \overline{y} with \Delta\overline{y}, given by \overline{y} = (\sum y_i/(\Delta y_i)^2)/(\sum 1/(\Delta y_i)^2) and \Delta\overline{y} = \sqrt{1/(\sum 1/(\Delta y_i)^2)}. At this energy the difference between σπ−p and σπ−N (= σπ−N) being estimated as ∼0.2mb is negligible, and the σπ−N can be regarded as σπ−N.