Fixed points of analytic actions of supersoluble Lie groups on compact surfaces

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Abstract

We show that every real analytic action of a connected supersoluble Lie group on a compact surface with nonzero Euler characteristic has a fixed point. This implies that E. Lima’s fixed point free $C^\infty$ action on $S^2$ of the affine group of the line cannot be approximated by analytic actions. An example is given of an analytic, fixed point free action on $S^2$ of a solvable group that is not supersoluble.

Introduction

Let $M$ denote a compact connected surface, with possibly empty boundary $\partial M$, endowed with a (real) analytic structure. $T_p M$ is the tangent space to $M$ at $p \in M$. The Euler characteristic of $M$ is denoted by $\chi(M)$.

Let $G$ be a Lie group with Lie algebra $\mathcal{L}(G) = \mathcal{G}$; all groups are assumed connected unless the contrary is indicated. An action of $G$ on $M$ is a homomorphism $\alpha$ from $G$ to the group $\mathcal{H}(M)$ of homeomorphisms of $M$ such that the evaluation map

$$\operatorname{ev}^\alpha = \operatorname{ev}: G \times M \to M, \ (g, x) \to \alpha(g)(x)$$

is continuous. We usually suppress notation for $\alpha$, denoting $\alpha(g)(x)$ by $g(x)$. The action is called $C^r$, $r \in \{1, 2, \ldots; \omega\}$ if $\operatorname{ev}$ is a $C^r$ map, where $C^\omega$ means analytic.

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The set $\mathcal{A}(G, M)$ of actions of $G$ on $M$ is embedded in the space of continuous maps $G \times M \to M$ by the correspondence $\alpha \mapsto \text{ev}^\alpha$. We endow $\mathcal{A}(G, M)$ with the topology of uniform convergence on compact sets.

A point $p \in M$ is a fixed point for an action $\alpha$ of $G$ if $\alpha(g)(p) = p$ for all $g \in G$. The set of fixed points is denoted by \text{Fix}$(G)$ or \text{Fix}$(\alpha(G))$.

In this paper we consider the problem of finding conditions on solvable group actions that guarantee existence of a fixed point.

When $\chi(M) \neq 0$, every flow (action of the real line $\mathbb{R}$) on $M$ has a fixed point; this was known to Poincaré for flows generated by vector fields, and for continuous actions it is a well known consequence of Lefschetz’s fixed point theorem. E. Lima [4] showed that every abelian group action on $M$ has a fixed point, and J. Plante [6] extended this to nilpotent groups.

These results do not extend to solvable groups: Lima [4] constructed a fixed point free action on the 2-sphere of the solvable group $A$ of homeomorphisms of $\mathbb{R}$ having the form $x \mapsto ax + b$, $a > 0, b \in \mathbb{R}$; and Plante [4] constructs fixed point free action of $A$ on all compact surfaces. These actions are not known to be analytic; but Example 3 below describes a fixed point free, analytic action of a 3-dimensional solvable group on $S^2$.

Recall that $G$ is supersoluble if every element of $G$ belongs to a codimension one subalgebra (see Barnes [1]). Our main result is the following theorem:

\textbf{Theorem 1} Let $G$ be a connected supersoluble Lie group and $M$ a compact surface $M$ such that $\chi(M) \neq 0$. Then every analytic action of $G$ on $M$ has a fixed point.

Since the group $A$ described above is supersoluble, Lima’s $C^\infty$ action cannot be improved to a fixed point free analytic action. The following result shows it cannot be approximated by analytic actions:

\textbf{Corollary 2} Let $G$ and $M$ be as in Theorem 1. If $\alpha \in \mathcal{A}(G, M)$ has no fixed point, then $\alpha$ has a neighborhood in $\mathcal{A}(G, M)$ containing no analytic action.

\textbf{Proof} By Theorem 1 and compactness of $M$, it suffices to prove the following: For all convergent sequences $\beta_n \to \beta$ in $\mathcal{A}(G, M)$ and $p_n \to p$ in $M$, with $p_n \in \text{Fix}(\beta_n(G))$, we have $p \in \text{Fix}(\beta(G))$. Being a connected locally compact group, $G$ is generated by a compact neighborhood $K$ of the identity. Then $\beta_n(g) \to \beta(g)$ uniformly for $g \in K$, so $\beta(g)(p) = p$ for all $g \in K$. Since $K$ generates $G$, this implies that $p \in \text{Fix}(\beta(G))$.

In Theorem 1, the hypothesis that $G$ is connected is essential: the abelian group of rotations of $S^2$ generated by reflections in the three coordinate axes is a well known counterexample. And every Lie group with a nontrivial homomorphism to the group of integers acts analytically without fixed point on every compact surface admitting
a fixed point free homeomorphism, thus on every surface except the disk and the projective plane.

The following example shows that supersolubility is essential:

**Example 3**

Let $Q$ be the 3-dimensional Lie group obtained as the semidirect product of the real numbers $\mathbb{R}$ acting on the complex numbers $\mathbb{C}$ by $t \cdot z = e^{it}z$; this group is solvable but not supersoluble. Identify $Q$ with the space $\mathbb{R} \times \mathbb{C} \approx \mathbb{R}^3$ and note that left multiplication defines a linear action of $Q$ on $\mathbb{R}^3$. The induced action on the 2-sphere $S$ of oriented lines in $\mathbb{R}^3$ through the origin has no fixed point, and $\chi(S) = 2$. Geometrically, one can see this as the universal cover of the proper euclidean motions of the plane, acting on two copies of the plane joined along a circle at infinity.

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**Proof of Theorem 1**

We assume given an action $\alpha: G \to H(M)$. The orbit of $p \in M$ is $G(p) = \{g(x): g \in G\}$. The isotropy group of $p \in M$ is the closed subgroup $I_p = \{g \in G: \alpha(g)(p) = p\}$. The evaluation map $ev_p: G \to M$ at $p \in M$ is defined by $g \mapsto g(p)$.

Suppose that the action is $C^r$, $r \geq 1$. Then $ev_p$ induces a bijective $C^r$ immersion $i_p: G/I(p) \to G(p)$. The tangent space $E(p) \subset T_pM$ to this immersed manifold at $p$ is the image of $T_eG$ under the differential of $ev_p$ at the identity $e \in G$.

For $j = 0, 1, 2$, let $V_j = V_j(G) \subset M$ denote the union of the $j$-dimensional orbits. Then $M = V_2 \cup V_1 \cup V_0$. Each $V_j$ is invariant, $V_2$ is open, $V_1 \cup V_0$ is compact, and $V_0 = \text{Fix}(G)$.

**Lemma 4 (Plante)** Assume that $G$ is solvable and that $G(p)$ is a compact 1-dimensional orbit. Then there is a closed normal subgroup $H \subset G$ of codimension 1 such that every point of $G(p)$ has isotropy group $H$.

**Proof** Choose a homeomorphism $f: G(p) \approx S^1$ (the circle). Let $\beta: G \to H(S^1)$ be the action defined by $\beta(g) = f \circ \alpha(g) \circ f^{-1}$. Because $G$ is solvable, by a result of Plante ([7], Theorem 1.2) there exists a homeomorphism $h$ of $S^1$ conjugating $\beta(G)$ to the rotation group $SO(2)$. Since $\beta(G)$ is abelian and acts transitively on $S^1$, all points of $S^1$ have the same isotropy group for $\beta$; this isotropy group is the required $H$.

Analyticity is used to establish the following useful property:
Lemma 5 Assume that $G$ acts analytically and that $\text{Fix}(G) = \emptyset$. Then either $V_1 = M$ and $\chi(M) = 0$, or else $V_1$ is the (possibly empty) union of a finite family of orbits, each of which is a smooth Jordan curve contained in $\partial M$ or in $M \setminus \partial M$.

Proof Since there are no orbits of dimension 0, $V_1$ is a compact set comprising the points $p$ such that $\dim E_p \leq 1$. It is easy to see that $V_1$ is a local analytic variety.

If $V_1 = M$ then the map $p \mapsto E_p$ is a continuous field of tangent lines to $M$, tangent to $\partial M$ at boundary points. The existence of such a field implies that $\chi(M) = 0$.

Assume that $V_1 \neq M$. Note that $\dim_p V_1 \geq 1$ at each $p \in V_1$. Since $M$ is connected and $V_1$ is a variety, $V_1$ must have dimension 1 at each point. The set of points where $V_1$ is not smooth is a compact, invariant 0-dimensional subvariety, i.e., a finite set of fixed points, hence empty. Since $V_1$ consists of 1-dimensional orbits, $V_1$ must be a compact, smooth invariant 1-manifold without boundary, i.e. each component of $V_1$ is a Jordan curve. Since $\partial M$ is the union of invariant Jordan curves, any component of $V_1$ that meets $\partial M$ is a component of $\partial M$.

In view of Lemma 5, it suffices to prove the following more general result:

Proposition 6 Let $G$ be a connected supersoluble Lie group acting continuously on the compact connected surface $M$. Assume that

(a) there are no fixed points

(b) for each closed subgroup $H$, $V_1(H)$ is the union (perhaps empty) of finitely many disjoint Jordan curves.

Then $\chi(M) = 0$.

By passing to a universal covering group we assume that $G$ is simply connected. This implies that every closed subgroup is simply connected (see Hochschild, Theorem XII.2.2.)

We proceed by induction on $\dim G$, the case $G = \mathbb{R}$ having been covered in the introduction. Henceforth assume inductively that $\dim G = n \geq 2$ and that the proposition holds for all supersoluble groups of lower dimension. With this hypothesis in force, we first rule out the case that $M$ is a disk:

Proposition 7 If $M$ is as in Proposition 6, then $\chi(M) \neq 1$

Proof Suppose not; then $M$ is a closed 2-cell. Since there are no fixed points, $\partial M$ is an orbit, hence a component of $V_1$. Every component of $V_1$ bounds a unique 2-cell in $M$, and there are only finitely many such 2-cells. Let $D$ be one that contains no other. Then $D$ is invariant under $G$, and the action of $G$ on $D$ is fixed point free. Therefore we may assume that $M = D$, so that $V_1 = \partial M$. 

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By Lemma 4 there exists a closed normal subgroup $H$ of codimension one with $\partial M \subset \text{Fix}(H)$. Let $R \subset G$ be a 1-parameter subgroup transverse to $H$ at the identity; then $RH = G$.

Because $G$ is supersoluble, there is a codimension one subalgebra $K \subset G$ containing the Lie algebra $R$ of $R$. Because $G$ is simply connected and solvable $K$ is the Lie algebra of a closed subgroup $K \subset G$ of dimension $n - 1$, and $KH = G$. By the induction hypothesis there exists $p \in \text{Fix}(K)$. Then $\dim G(p) \leq \dim G - \dim K = 1$. Therefore $p \in V_1 = \partial D$. We now have $p \in \text{Fix}(K) \cap \text{Fix}(H) = \text{Fix}(G)$, a contradiction.

We return now to the case of general $M$.

Denote the connected components of $M \setminus V_1$ by $U_i, \ldots, U_r$, $r \geq 1$. Each $U_i$ is an open orbit, whose set theoretic boundary $\text{bd } U_i$ is a (possibly empty) union of components of $V_1$. The closure $\overline{U_i}$ is a compact surface invariant under $G$, whose boundary as a surface is $\partial U_i = \text{bd } U_i$.

We show that $U_i$ is an open annulus. Let $H \subset G$ be the isotropy subgroup of $p \in U_i$. Evaluation at $p$ is a surjective fibre bundle projection $G \rightarrow U_i$ with standard fibre $H$. Therefore there is an exact sequence of homotopy groups

$$
\cdots \rightarrow \pi_j(G) \rightarrow \pi_j(U_i) \rightarrow \pi_{j-1}(H) \rightarrow \pi_{j-1}(G) \rightarrow \cdots \rightarrow \pi_0(G) = \{0\}
$$

ending with the trivial group $\pi_0(G)$ of components of $G$. The component group $\pi_0(H)$ is solvable (see Raghunathan [7], Proposition III.3.10), so taking $j = 1$ shows that $\pi_1(U_i)$ is solvable. Therefore $U_i$ is a sphere, torus, open 2-cell, or open annulus. If $U_i$ is a torus then $U_i = M$, contradicting $\chi(M) \neq 0$. The sphere is ruled out by the exact sequence $\pi_2(G) \rightarrow \pi_2(U_i) \rightarrow \pi_1(H)$, because $\pi_2(G) = 0$ for every Lie group and $\pi_1(H) = 0$. Proposition 7 rules out the 2-cell.

It follows that $\overline{U_i}$ is a closed annulus, so $\chi(\overline{U_i}) = 0$. By the additivity property $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ of the Euler characteristic, any space $M$ built by gluing annuli along their boundary circles must have $\chi(M) = 0$.

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