Derived simple algebras and restrictions of recollements of derived module categories

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Abstract. Recollements of derived module categories are investigated. First, some known results on homological dimensions of algebras appearing in a recollement are complemented and extended and new results on K-theoretic invariants are established. Secondly, it is clarified when recollements can be lifted or, in particular, restricted between different levels of derived categories. Using these characterisations, examples and criteria are given to show that the notion of derived simplicity depends on the choice of derived categories. Thirdly, new classes of derived simple algebras are given; in particular, it is shown that indecomposable commutative rings are derived simple. Finally, a finite-dimensional counterexample to the derived Jordan–Hölder theorem is given.

Keywords: derived module categories, recollements, lifting and restricting recollements, derived simple algebras, homological dimensions, commutative noetherian rings.

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1. INTRODUCTION

Derived categories, introduced by Grothendieck and Verdier, have been playing an increasingly important role in various areas of mathematics, including representation theory, algebraic geometry, microlocal analysis and mathematical physics. Major topics of current interest are substructures of derived categories, such as bounded $t$-structures, which form the ‘skeleton’ of Bridgeland’s stability manifold, as well as comparisons of derived categories. One way to compare derived categories and their invariants, such as Grothendieck groups and Hochschild cohomologies, is by derived equivalences, for instance by tilting. Another way has been introduced by Beilinson, Bernstein and Deligne, who defined the concept of a recollement [9] of a triangulated category. A recollement (‘gluing’) of a derived category by another two derived categories is a diagram of six functors

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between these categories, generalising Grothendieck’s six functors. Beilinson, Bernstein and Deligne
used recollements to define the category of perverse sheaves over a stratified topological space as
the heart of a t-structure that is obtained by ‘gluing’ standard t-structures on the strata.

Viewing a recollement as a short exact sequence of categories, deconstructing the middle term
into smaller and possibly less complicated outer terms, suggests to use recollements for computing
homological invariants inductively along sequences of recollements. The induction start then has
to be based on investigating ‘derived simple’ algebras that cannot be deconstructed further. This
raises a number of questions to which we are going to provide (positive or negative) answers.

(1) Which invariants can be computed inductively along recollements?

(2) Does the concept of derived simplicity depend on the choice of (unbounded, bounded, ...) deferred
category?

(3) When do recollements lift or restrict between different levels of derived categories?

(4) Which algebras are derived simple?

(5) Which algebras satisfy a derived version of the Jordan–Hölder theorem?

With respect to inductively computing invariants along recollements, Happel showed in [26]
that conjectures on homological dimensions, such as the finitistic dimension conjecture, can be
transfered to smaller algebras. It is known that a recollement of derived module categories induces
long exact sequences on algebraic K-theory [19, 51, 39, 46], on Hochschild homologies and cyclic
homologies [30], and on Hochschild cohomologies [32] and [23]. In [34], the third named author and
Vitória showed that for a finite-dimensional piecewise hereditary algebra over a field any bounded
t-structure can be obtained by ‘gluing’ via recollements from standard t-structures on the derived
category of vector spaces.

Complementing and extending results in the literature, we show the following results in Proposition
2.14, Lemma 2.10 (b) and Proposition 6.5.

**Theorem I.** Let \(A, B, C\) be algebras. Suppose that \(D(\text{Mod}A)\) admits a recollement by \(D(\text{Mod}B)\)
and \(D(\text{Mod}C)\).

(a) The global dimension of \(A\) is finite if and only if those of \(B\) and \(C\) are finite.

(b) If \(A\) is finite-dimensional over a field, then so are \(B\) and \(C\). Moreover, the ranks of the
Grothendieck groups of \(\text{mod}B\) and \(\text{mod}C\) sum up to that of \(\text{mod}A\).

The following result on higher K-groups, Theorem 6.7, is motivated by yet unpublished work of
Chen and Xi [15], which also has been extended by these authors in parallel and independent work
[17].

**Theorem II.** Let \(A, B, C\) be finite-dimensional algebras over a field. Suppose that \(D^{-}(\text{Mod}A)\)
admits a recollement by \(D^{-}(\text{Mod}B)\) and \(D^{-}(\text{Mod}C)\). Then we have isomorphism of K-groups
\(K_{*}(A) \cong K_{*}(B) \oplus K_{*}(C)\).

Concerning the question (2), Example 5.8 provides a finite-dimensional algebra that is \(D^{-}(\text{Mod})\)-
simple but not \(D(\text{Mod})\)-simple, that is, it has non-trivial recollements at \(D(\text{Mod})\)-level, but not at
\(D^{-}(\text{Mod})\)-level, and Example 5.10 provides a finite-dimensional algebra that is \(K^{b}(\text{proj})\)-simple
but not \(D^{-}(\text{Mod})\)-simple. So, the concept of being derived simple strongly depends on the choice
of derived categories. We will clarify the connection between the different choices by characterising
derived simplicity on each level in terms of recollements on the unbounded level. Our main tool
here is the concept of ladders, which are collections of ‘adjacent’ recollements, whose number is called ‘height’ of the ladder (see Section 3). The following result combines Theorems 5.5, 5.9 and 5.12.

**Theorem III.** Let $A$ be a finite-dimensional algebra over a field. Then

(a) $A$ is $\mathcal{D}(\text{Mod})$-simple if and only if all non-trivial ladders of $\mathcal{D}(\text{Mod}A)$ have height 0.

(b) $A$ is $\mathcal{D}^-(\text{Mod})$-simple if and only if all non-trivial ladders of $\mathcal{D}(\text{Mod}A)$ have height $\leq 1$.

(c) $A$ is $\mathcal{D}^b(\text{Mod})$-simple if and only if it is $\mathcal{K}^b(\text{proj})$-simple if and only if it is $\mathcal{D}^b(\text{mod})$-simple if and only if all non-trivial ladders of $\mathcal{D}(\text{Mod}A)$ have height $\leq 2$.

The proof of this theorem relies on our answer to the question (3) above. While lifting is easily seen not to be problematic, restricting recollements is in general not possible, see Example 4.13. Therefore, we provide several criteria for restrictions, see Section 4. These criteria are in terms of particular objects and in terms of two of the six functors occurring in the recollement.

A stratification is a sequence of recollements deconstructing a derived module category into ‘simple factors’. These simple factors can be viewed as ‘composition factors’ of the given derived module category. The derived Jordan–Hölder theorem in Question (5) asks whether a finite stratification exists and whether any two stratifications have the same set of simple factors (up to equivalence). The validity of the derived Jordan–Hölder theorem has been disproved by Chen and Xi [12, 13] for a certain family of infinite-dimensional algebras, proved by the first three authors [5, 6] for hereditary artin algebras and finite-dimensional piecewise hereditary algebras, proved by the third and fourth authors [35] for finite group algebras, and it was proposed as an open question by Chen and Xi in [14] for finite-dimensional algebras over a field. In this paper, we give a finite-dimensional counterexample. Our results also imply that the example in [14, Section 5 (4)] is indeed a counterexample too.

Concerning the problem of classifying derived simple algebras, a large class of such algebras was recently discovered in [35]: blocks of group algebras of finite groups in any characteristic. Here we add another large class of examples by proving that indecomposable commutative rings always are derived simple. In particular, this result implies the validity of the Jordan–Hölder Theorem for derived module categories over commutative noetherian rings.

Finally, we point out that Section 4.4 corrects a mistake in [6].

The paper is structured as follows. In Section 2 we collect some preliminaries on recollements of triangulated and derived categories, derived functors, derived simple algebras and stratifications. Moreover, we prove Theorem I(a) and the first statement of Theorem I(b). In Section 3 we recall the definition of a ladder and discuss when a ladder can be extended upwards or downwards. In Section 4 we study the lifting and restricting problem between different levels of derived categories; we give criteria when a recollement on $\mathcal{D}(\text{Mod})$ can be restricted to a recollement on $\mathcal{D}^-(\text{Mod})$, $\mathcal{D}^b(\text{Mod})$, $\mathcal{K}^b(\text{proj})$ and $\mathcal{D}^b(\text{mod})$ respectively. Section 5 is devoted to studying the dependence of derived simplicity on the choice of (unbounded, bounded, ...) derived categories, in particular, proving Theorem III. Further, we show that indecomposable commutative algebras are derived simple for any choice of derived categories. As a consequence, we recover some results on silting and tilting over commutative algebras in the literature [20, 42, 48]. In Section 6 we study algebraic
K-theory of finite-dimensional algebras and prove Theorem II. In the final Section we discuss the derived Jordan–Hölder theorem.

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2. Derived categories and recollements

In this section, the basic definitions are given and various technical results are established that will be crucial in the later sections. Subsection contains some preliminaries on triangulated categories, while Subsection focusses on derived module categories. In Subsection we study the problem when triangle functors of derived module categories restrict to subcategories. In we collect techniques to investigate functors and objects involved in recollements of derived module categories, and in we recall some important constructions of such recollements. The notions of derived simple ring and stratification are reviewed in and we show that given a recollement of derived module categories the middle algebra has finite global dimension if and only if so do the two outer algebras.

Let be a commutative ring. When is a field, let be the k-dual.

2.1. Triangulated categories. Let be a triangulated -category with shift functor [1]. An object of is exceptional if for all unless . Let be a set of objects of . We denote by the smallest triangulated subcategory of containing and closed under taking direct summands, and by the right perpendicular category of , i.e.

where the maps are given by adjunctions.

Assume further that has all (set-indexed) infinite direct sums. An object of is compact if the functor commutes with taking direct sums. For a set of objects of , we denote by the smallest triangulated subcategory of containing and closed under taking direct summands. is called a set of compact generators of if all objects in are compact and .

In this case is said to be compactly generated by .

2.1.1. Recollements. A recollement of triangulated -categories is a diagram of triangulated categories and triangle functors such that

(1) are adjoint pairs;
(2) are full embeddings;
(3) (and thus also ) and ;
(4) for each there are triangles

where the maps are given by adjunctions.
Thanks to (1) and (3), the two triangles (often called the canonical triangles) in (4) are unique up to unique isomorphisms.

We say that this is a recollement of \( C \) by \( C' \) and \( C'' \). We say the recollement is trivial if one of the triangulated categories \( C' \) and \( C'' \) is trivial, or equivalently, one of the full embeddings \( i_* \), \( j! \), and \( j_* \) is a triangle equivalence.

2.1.2. TTF triples. An equivalent language for studying recollements is that of TTF triples. We are going to extend \([40, 4.2]\), which covers the case of compactly generated triangulated categories.

Let \( C \) be a triangulated \( k \)-category with shift functor \([1]\). A t-structure on \( C \) is a pair of full subcategories \((C_{\leq 0}, C_{\geq 0})\) which are closed under isomorphisms and which satisfy the following conditions

- \( C_{\leq 0}[1] \subseteq C_{\leq 0} \) and \( C_{\geq 0}[-1] \subseteq C_{\geq 0} \),
- \( \text{Hom}(X, Y) = 0 \) for \( X \in C_{\leq 0} \) and \( Y \in C_{\geq 0}[-1] \),
- for any object \( X \) of \( C \) there is a triangle

\[
\begin{array}{c}
X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]
\end{array}
\]

with \( X' \in C_{\leq 0} \) and \( X'' \in C_{\geq 0}[-1] \).

A TTF triple of \( C \) is a triple \((X, Y, Z)\) of full subcategories of \( C \) such that \((X, Y)\) and \((Y, Z)\) are t-structures on \( C \). It follows that \( X, Y \) and \( Z \) are actually triangulated subcategories of \( C \). Given a recollement of the form (2.1), one easily checks that

\((\text{Im}(j!), \text{Im}(i_*), \text{Im}(j_*))\)

is a TTF triple of \( C \), which we shall call the associated TTF triple. We say that two recollements are equivalent if the TTF triples associated to them coincide. The following result is essentially included in \([9, \text{Section 1.4.4}]\), see also \([38, \text{Section 9.2}]\), \([40, \text{Section 4.2}]\).

**Proposition 2.1.** Let \( C \) be a triangulated \( k \)-category. There is a one-to-one correspondence between equivalence classes of recollements of \( C \) and TTF triples of \( C \).

The key point of the proof is the following lemma.

**Lemma 2.2.** Let \( 0 \rightarrow C' \xrightarrow{F} C \xrightarrow{G} C'' \rightarrow 0 \) be a short exact sequence of triangulated \( k \)-categories, i.e. \( F \) is fully faithful and \( G \) induces a triangle equivalence \( C/\text{Im} F \cong C'' \) (possibly up to direct summands). Then \( F \) has a left (respectively, right) adjoint if and only if only if so does \( G \). In this case, if \( G' \) denotes the left (respectively, right) adjoint of \( G \), then \((\text{Im} G', \text{Im} F)\) (respectively, \((\text{Im} F, \text{Im} G')\)) is a t-structure on \( C \).

When \( C \) is compactly generated by one object - for example, \( C \) is the derived module category of some \((\text{dg})-k\)-algebra - a TTF triple is uniquely determined by its first term, see \([40, 4.4.14]\). In other words, the associated recollement is uniquely determined by \( \text{Im}(j!) \). Following \([40]\), if \( C'' \) is compactly generated by one object \( T \), we say the recollement is generated by \( j!(T) \).

The following result shows that recollements preserve direct sum decompositions, which generalises a result implicitly appearing in the proof of \([35, \text{Corollary 3.4}]\).
Lemma 2.3. Assume that a recollement of the form (2.1) is given. Suppose that \( C = C_1 \oplus \ldots \oplus C_s \) is a direct sum decomposition of triangulated categories. Then there are direct sum decompositions \( C' = C'_1 \oplus \ldots \oplus C'_s \) and \( C'' = C''_1 \oplus \ldots \oplus C''_s \) such that the given recollement restricts to recollements

\[
\begin{array}{c}
C_i' \xrightarrow{s=i} C_i \xrightarrow{j} C''_i
\end{array}
\]

and the direct sum of these restricted recollements is the given recollement.

Proof. Let \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) be the TTF triple corresponding to the given recollement. For \( i = 1, \ldots, s \), let \( \mathcal{X}_i = \mathcal{X} \cap C_i \), \( \mathcal{Y}_i = \mathcal{Y} \cap C_i \) and \( \mathcal{Z}_i = \mathcal{Z} \cap C_i \). Then \( \mathcal{X} = \bigoplus_i \mathcal{X}_i \), \( \mathcal{Y} = \bigoplus_i \mathcal{Y}_i \) and \( \mathcal{Z} = \bigoplus_i \mathcal{Z}_i \), because each object \( C \) of \( C \) is the direct sum \( C = \bigoplus_i C_i \) of objects \( C_i \in C_i \) and \( \mathcal{X} \), \( \mathcal{Y} \) and \( \mathcal{Z} \) are closed under taking direct summands. Moreover, each triangle in \( C \) is the direct sum of \( s \) triangles, respectively lying in \( C_1, \ldots, C_s \). It follows that for any \( i = 1, \ldots, s \), the triple \((\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)\) is a TTF triple for \( C_i \). \( \square \)

2.2. Derived categories. Let \( A \) be a \( k \)-algebra. We denote by \( \text{mod}A \) the category of (right) \( A \)-modules, by \( \text{mod}A \) its subcategory consisting of modules with projective resolution by finitely generated projectives, and by \( \text{proj}A \) its subcategory of finitely generated projective modules. The analogous categories of left modules will be denoted by \( \text{A-Mod} \), \( \text{A-mod} \) and \( \text{A-proj} \) respectively. For \( * \in \{ b, -, +, 0 \} \), let \( \mathcal{D}^*(\text{mod}A) \) denote the derived category of complexes of objects in \( \text{mod}A \) satisfying the corresponding boundedness condition. When \( k \) is a field and \( A \) is a finite-dimensional \( k \)-algebra, we also consider the corresponding derived categories \( \mathcal{D}^*(\text{mod}A) \) of objects in \( \text{mod}A \). Notice that \( \mathcal{D}^b(\text{mod}A) \) then coincides with the full subcategory \( \mathcal{D}_{fi}(A) \) of \( \mathcal{D}(\text{mod}A) \) consisting of complexes of \( A \)-modules whose total cohomology has finite length over \( k \). The shift functor will be denoted by \([1]\).

Let \( K^b(\text{proj}A) \) denote the homotopy category of bounded complexes of finitely generated projective \( A \)-modules. We often view it as a full subcategory of the categories \( \mathcal{D}^*(\text{mod}A) \) and \( \mathcal{D}^*(\text{Mod}A) \) and identify it with its essential image. The objects in \( K^b(\text{proj}A) \) are, up to isomorphism, precisely the compact objects in \( \mathcal{D}(\text{Mod}A) \). The free module \( A_A \) of rank 1 is a compact generator of \( \mathcal{D}(\text{Mod}A) \). For a complex \( X \in \mathcal{D}(\text{Mod}A) \) we write \( X^{tr,A} \) for \( R\text{Hom}_A(X, A) \). When it does not cause confusion, we will drop the subscript \( A \) and simply write \( X^{tr} \).

A complex \( P \) in \( K^b(\text{proj}A) \) is said to have length \( n \) if \( n \) is the minimal integer such that there is \( Q \in K^b(\text{proj}A) \) such that \( Q \cong P \) and \( n \) equals the number of non-zero components of \( Q \).

If \( k \) is a field and \( A \) is finite-dimensional over \( k \), then any object in \( D^- (\text{mod}A) \) admits a minimal representative, that is, a complex of finitely generated projective \( A \)-modules such that the images of the differentials lie in the radicals.

Lemma 2.4. (a) The category \( \mathcal{D}_{fi}(A) \) is the full subcategory of \( \mathcal{D}(\text{mod}A) \) consisting of those objects \( X \) such that the total cohomology of the complex \( R\text{Hom}_A(P, X) \) has finite length over \( k \), i.e. \( \bigoplus_{n \in \mathbb{Z}} \text{Hom}(P, X[n]) \) has finite length, for any \( P \in K^b(\text{proj}A) \).
(b) The category \( \mathcal{D}^b(\text{mod}A) \) is the full subcategory of \( \mathcal{D}(\text{Mod}A) \) consisting of those objects \( X \) such that the the complex \( R\text{Hom}_A(P, X) \) has bounded total cohomology, i.e. \( \text{Hom}(P, X[n]) \neq 0 \) for only finitely many \( n \in \mathbb{Z} \), for any \( P \in K^b(\text{proj}A) \).
Assume that \( k \) is a field and that \( A \) is a finite-dimensional \( k \)-algebra.
(c) The category $K^b(\text{proj}A)$ is the full subcategory of $\mathcal{D}(\text{Mod}A)$ consisting of those objects $P$ such that the complex $R\text{Hom}_A(P,X)$ has finite-dimensional total cohomology for any $X \in \mathcal{D}^b(\text{mod}A)$. 

(d) The category $K^b(A\text{-proj})$ is the full subcategory of $\mathcal{D}(A\text{-Mod})$ consisting of those objects $P$ such that the complex $X \otimes_A \mathbf{L}P$ has finite-dimensional total cohomology for any $X \in \mathcal{D}^b(\text{mod}A)$.

Proof. (a) It follows by dévissage that objects in $\mathcal{D}_{fl}(A)$ satisfy the condition. Since dévissage will be often used later, we give the details here. Let $N$ be an object of $\mathcal{D}_{fl}(A)$ and let $\mathcal{A}$ be the full subcategory of $\mathcal{D}(\text{Mod}A)$ consisting of objects $M$ such that the total cohomology of $R\text{Hom}_A(M,N)$ has finite length over $k$. Then $\mathcal{A}$ contains $A$ and is closed under direct summands, shifts and extensions. This shows that $\mathcal{A}$ contains $\text{tria}(A) = K^b(\text{proj}A)$ and we are done. Conversely taking $P = A$ we see that the total cohomology space of $X = R\text{Hom}_A(A,X)$ in the latter category has finite length, i.e. $X \in \mathcal{D}_{fl}(A)$.

(b) Similar to (a).

(c) It follows by dévissage that objects of $K^b(\text{proj}A)$ belong to the latter category. Conversely, let $P$ be any object of the latter category. Then $DP = R\text{Hom}_A(P,DA)$, and hence, $P$, have finite-dimensional total cohomology, so $P \in \mathcal{D}^b(\text{mod}A)$. Thus we may assume that $P$ is a minimal right bounded complex of finitely generated projective $A$-modules. If such a complex is not bounded, then some indecomposable projective $A$-module with simple top $S$ occurs infinitely many times. It follows that there are nonzero morphisms from $P$ to infinitely many shifts of $S$, that is, the total cohomology of $R\text{Hom}_A(P,S)$ is infinite dimensional, a contradiction.

(d) Similar to (c). \qed

2.2.1. Restricting triangle functors. In this subsection we discuss restriction of triangle functors between derived modules categories to subcategories.

Let $A$ and $B$ be $k$-algebras and $F : \mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B)$ a $k$-linear triangle functor. Let $\mathcal{D} = K^b(\text{proj}), D_{fl}, D^b(\text{mod}), D^b(\text{Mod})$ or $D^{-}(\text{Mod})$. Let $\mathcal{D}$ be the essential image of $\mathcal{D}$ under the canonical embedding into $\mathcal{D}(\text{Mod})$. We say that $F$ restricts to $\mathcal{D}$ if $F$ restricts to a triangle functor $\mathcal{D}_A \to \mathcal{D}_B$. The following two well-known lemmas follow by dévissage.

Lemma 2.5. The following two conditions are equivalent:

(i) $F$ restricts to $K^b(\text{proj})$, 
(ii) $F(A) \in K^b(\text{proj}B)$.

Lemma 2.6. Assume that $k$ is a field and that $A$ and $B$ be finite-dimensional $k$-algebras. The following two conditions are equivalent:

(i) $F$ restricts to $D^b(\text{mod})$, 
(ii) $F(S) \in D^b(\text{mod}B)$ for any simple $A$-module $S$.

Lemma 2.7. Assume that $F$ admits a right adjoint $G$. Consider the following conditions

(i) $F$ restricts to $K^b(\text{proj})$, 
(ii) $G$ restricts to $D^b(\text{Mod})$, 
(iii) $G$ restricts to $D_{fl}$. 

Then (i) implies (ii) and (iii). If \( k \) is a field and \( A \) and \( B \) are finite-dimensional \( k \)-algebras, then (iii) implies (i).

**Proof.** (i)⇒(ii) Let \( M \) be in \( \mathcal{D}^b(\text{Mod}B) \) and \( P \) be any object in \( K^b(\text{proj}A) \). Then
\[
\text{Hom}_A(P, G(M[n])) = \text{Hom}_B(F(P), M[n]).
\]
The condition (i) implies that \( F(P) \in K^b(\text{proj}B) \). It follows from Lemma 2.4 (b) applied to the algebra \( B \) that the above space does not vanish for only finitely many \( n \in \mathbb{Z} \). Applying Lemma 2.4 (b) to the algebra \( A \) shows that \( G(M) \in \mathcal{D}^b(\text{Mod}A) \).

(i)⇒(iii) This follows from Lemma 2.4 (a) by a similar argument as in (i)⇒(ii).

(iii)⇒(i) This follows from Lemma 2.4 (c) by a similar argument as in (i)⇒(ii).

Let \( X \) be a complex of \( A\text{-}B \)-bimodules. Then there is a pair of adjoint triangle functors
\[
? \otimes_A X : \mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B), \quad \text{RHom}_B(X, ?) : \mathcal{D}(\text{Mod}B) \to \mathcal{D}(\text{Mod}A).
\]
Assume that \( A \) is projective as a \( k \)-module. Let \( pX \) be a bimodule \( K \)-projective resolution of \( X \) (see [17]). Then \( (pX)_B \) is a \( K \)-projective resolution of \( X \) as a complex of right \( B \)-modules, and hence \( \text{RHom}_B(X, ?) \) is naturally isomorphic to \( \text{RHom}_B(pX, ?) \), which is computed as the total complex of the Hom bicomplex. In particular, for a complex \( Y \) of \( C \)-\( B \)-bimodules, \( \text{RHom}_B(pX, Y) \) has the structure of a complex of \( C \)-\( A \)-bimodules. By abuse of notation, we will identify \( \text{RHom}_B(X, Y) \) and \( \text{RHom}_B(pX, Y) \).

**Lemma 2.8.** Let \( X \) be a complex of \( A\text{-}B \)-bimodules and assume that \( A \) is projective as a \( k \)-module. Consider the following conditions

(i) \( \mathcal{D}(\mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B) \)

(ii) \( \mathcal{D}(\mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B) \)

(iii) \( \mathcal{D}(\mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B) \)

Then (i)⇒(ii)⇒(iii). If \( k \) is a field and \( A \) and \( B \) are finite-dimensional \( k \)-algebras, then all three conditions are equivalent.

**Proof.** (i)⇒(ii): It follows by dévissage that \( (X^{tr_A})_A = \text{RHom}_{A^{op}}(X, A^{op})_A \in K^b(\text{proj}A) \). Therefore, by [29] Lemma 6.2 (a)], the derived functor
\[
\text{RHom}_A((X^{tr_A})^{op}, ?) : \mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B)
\]
is isomorphic to \( \mathcal{D}(\mathcal{D}(\text{Mod}A^{op}) \otimes B) \)

\[
\text{RHom}_A((X^{tr_A})^{op}, ?) : \mathcal{D}(\text{Mod}A) \to \mathcal{D}(\text{Mod}B)
\]
is isomorphic to \( \mathcal{D}(\mathcal{D}(\text{Mod}A^{op}) \otimes B) \) and thus has a left adjoint \( \otimes_B X^{tr_A^{op}} \), which restricts to \( \otimes_B X^{tr_A^{op}} \) by Lemma 2.5

(iii)⇒(i): This follows from Lemma 2.7.

Finally, when \( k \) is a field and \( A \) and \( B \) are finite-dimensional \( k \)-algebras, the implication (iii)⇒(i) follows from Lemma 2.4 (d). \( \square \)
2.2.2. Recollements of derived categories. Suppose there is a recollement

$$\begin{align*}
\mathcal{D} & \overset{j^*}{\leftarrow} \mathcal{D}(\text{Mod}B) \overset{i_*}{\to} \mathcal{D}(\text{Mod}A) \overset{j^!}{\to} \mathcal{D}(\text{Mod}C) \overset{j^!}{\to} \mathcal{D}(\text{Mod}B) \overset{i_*}{\to} \mathcal{D}(\text{Mod}A) \overset{j^*}{\to} \mathcal{D}(\text{Mod}B)
\end{align*}$$

(R)

where $A$, $B$ and $C$ are $k$-algebras. By [10, 5.2.9], [11], or [12, Theorem 2.2], the recollement is generated by the compact exceptional object $T = j_!(C) \in \mathcal{D}(\text{Mod}A)$. Moreover, the object $i^*(A)$ is a compact generator of $\mathcal{D}(\text{Mod}B)$ by [10, 4.3.6, 4.4.8]. The following lemma is well-known.

**Lemma 2.9.** With the above notation, the four objects $T = j_!(C)$, $j_!j^!(A)$, $i_*i^*(A)$ and $T' = i_*(B)$ of $\mathcal{D}(\text{Mod}A)$ satisfy

(a) $j_!(C)$ and $j_!j^!(A)$ are left orthogonal to both $i_*i^*(A)$ and $i_*(B)$, and $j_!j^!(A) \in \text{Tria}(j_!(C))$;
(b) $j_!(C)$ and $i_*(B)$ are exceptional objects with $C \cong \text{End}_A(T)$ and $B \cong \text{End}_A(T')$;
(c) $\text{Tria} i_*i^*(A) = \text{Tria} i_*(B)$;
(d) $i_*i^*(A) \cong \text{RHom}_A(i_*i^*(A), i_*i^*(A))$ as complexes of $k$-modules;
(e) $i^*$ and $j_!$ restrict to $K^b(\text{proj})$, and $i_*$ and $j^*$ restrict to $\mathcal{D}^b(\text{Mod})$ as well as to $\mathcal{D}_{fl}$;
(f) $j_!j^!(A)$, $i_*i^*(A)$ and $i_*(B)$ belong to $\mathcal{D}^b(\text{Mod}A)$.

**Proof.** (a) and (b) follows immediately from the definition of recollement.

(c) Since $i^*(A)$ is a compact generator of $\mathcal{D}(\text{Mod}B)$, it follows that $\text{Tria} i^*(A) = \text{Tria} B$. Thus $\text{Tria} i_*i^*(A) = \text{Tria} i_*(B)$ for $i_*$ is a full embedding, cf. [11, Lemma 2.2].

(d) This is obtained by applying $\text{RHom}_A(\cdot, i_*i^*(A))$ to the canonical triangle

$$j_!j^!(A) \to A \to i_*i^*(A) \to j_!j^!(A)[1].$$

(e) The statement on $i^*$ and $j_!$ follows from Lemma 2.5 since $T$ and $T'$ are compact. For the second statement apply Lemma 2.7 on the right adjoints $i_*$ and $j^!$.

(f) It follows from the above canonical triangle and (c) that $j_!j^!(A)$, $i_*i^*(A)$ and $i_*(B)$ all belong to $\mathcal{D}^b(\text{Mod}A)$ if and only if so does one of them. Since $i_*(B)$ does by (e), the claim is proven. \[\square\]

**Lemma 2.10.** Assume that a recollement of the form (R) is given. The following hold true.

(a) ([23, Proposition 3]) Assume that $A$, $B$ and $C$ are projective over $k$. Then there exists a (unique) right bounded complex $X$ of projective $C$-$A$-bimodules which as a complex of $A$-modules is quasi-isomorphic to $T$, and up to equivalence, we can assume

$$j_! = \mathbb{L} \otimes_C X, \quad j^! = \text{RHom}_A(X, ?) = \mathbb{L} \otimes_A X^{\text{tr}_A} \quad \text{and} \quad j_* = \text{RHom}_C(X^{\text{tr}_A}, ?).$$

In particular, $j_!j^!(A) = X^{\text{tr}_A} \otimes_C X$, and there is a canonical triangle

$$X^{\text{tr}_A} \otimes_C X = j_!j^!(A) \to A \to i_*i^*(A) \to j_!j^!(A)[1].$$

Similarly, there exists a (unique) right bounded complex $Y$ of projective $A$-$B$-bimodules which as a complex of $B$-modules is quasi-isomorphic to $i^*(A)$, and up to equivalence, we can assume

$$i^* = \mathbb{L} \otimes_A Y, \quad i_* = \text{RHom}_B(Y, ?) = \mathbb{L} \otimes_B Y^{\text{tr}_B} \quad \text{and} \quad i^! = \text{RHom}_A(Y^{\text{tr}_B}, ?).$$
(b) Suppose now that $k$ is a field and $A$ is a finite-dimensional $k$-algebra. Then $B$ and $C$ are also finite-dimensional $k$-algebras. Moreover, the functor $j_i$ restricts to a triangle functor $\mathcal{D}^b(\text{mod} C) \rightarrow \mathcal{D}^- (\text{mod} A)$ and the objects $j_* j^!(A)$, $i_* i^*(A)$ and $i_*(B)$ belong to $\mathcal{D}^b(\text{mod} A)$. Further, the object $X$ (respectively, $Y$) in (a) can be taken as a right bounded complex of finitely generated projective $C$-$A$-bimodules (respectively, $A$-$B$-bimodules).

**Proof.** (a) This was proved in [23] for the case when $k$ is a field. But the proof there works in our more general setting.

(b) Recall that $T = j_i(C) \in \mathcal{D}(\text{Mod} A)$ is compact and exceptional and has endomorphism algebra $C$. It follows that $C$ is a finite-dimensional $k$-algebra.

Next we show that $j_i$ restricts to a triangle functor $\mathcal{D}^b(\text{mod} C) \rightarrow \mathcal{D}^- (\text{mod})$. Let $M \in \mathcal{D}^b(\text{mod} C)$. For $n \in \mathbb{Z}$, we have

$$DH^n(j_i(M)) = D\text{Hom}_{\mathcal{D}(\text{Mod} A)}(A, j_i(M)[n])$$
$$\cong \text{Hom}_{\mathcal{D}(\text{Mod} A)}(j_i(M)[n], D(A))$$
$$\cong \text{Hom}_{\mathcal{D}(\text{Mod} C)}(M, j_i^! D(A)[-n]).$$

Here the first isomorphism follows from the Auslander–Reiten formula and the second one follows by adjunction. Since $j_i(C) = T \in K^b(\text{proj} A)$, Lemma 2.4 and Lemma 2.7 imply that both $j_i^!(A)$ and $j_i^!(D(A))$ belong to $\mathcal{D}^b(\text{mod} C)$. Consequently, the space $\text{Hom}_{\mathcal{D}(\text{Mod} A)}(M, j_i^! D(A)[-n])$ is finite-dimensional for each $n$ and vanishes for sufficiently large $n$. Therefore $j_i(M)$ has right bounded cohomologies, i.e. $j_i(M) \in \mathcal{D}^-(\text{mod} A)$.

In particular $j_i j^!(A) \in \mathcal{D}^-(\text{mod} A)$. In view of (a), we have $j_i j^!(A) \in \mathcal{D}^b(\text{mod} A)$. But then the canonical triangle $X^{tr} \otimes_X Y \cong j_i j^!(A) \rightarrow A \rightarrow i_* i^*(A) \rightarrow j_i^!(A)[1]$ and Lemma 2.9(c) yield that $i_* i^*(A)$ and $i_*(B)$ belong to $\mathcal{D}^b(\text{mod} A)$ as well. This also shows that $B = \text{End}_A(i_*(B))$ is finite-dimensional.

The last statement holds because the complexes $X$ and $Y$ have finite-dimensional total cohomology and the algebras $C^{op} \otimes A$ and $A^{op} \otimes B$ are finite-dimensional.

**Remark 2.11.** The converse of the first statement of Lemma 2.10(b) is not true: $B$ and $C$ being finite-dimensional algebras does not imply that $A$ is finite-dimensional. For an example, take $A$ to be the path algebra of the infinite Kronecker quiver, i.e. the quiver which has two vertices 1 and 2 and which has infinitely many arrows from 1 to 2 and no arrows from 2 to 1. Then the projective module $e_1 A$ generates a recollement

$$\begin{align*}
\mathcal{D}(\text{Mod} A/A e_1 A) & \quad \longrightarrow \quad \mathcal{D}(\text{Mod} A) \\
\mathcal{D}(\text{Mod} A) & \quad \longrightarrow \quad \mathcal{D}(\text{Mod} e_1 A e_1)
\end{align*}$$

where both $A/A e_1 A$ and $e_1 A e_1$ are isomorphic to $k$, while $A$ is infinite-dimensional. This example is taken from [31]: Example 9].

**2.2.3. Construction of recollements.** For finite-dimensional algebras $A$, $B$ and $C$ over a field $k$, every recollement of the form

$$\begin{align*}
\mathcal{D}^b(\text{mod} B) & \quad \quad \leftarrow \quad i_*=i \longrightarrow \quad \mathcal{D}^b(\text{mod} A) \\
\mathcal{D}^b(\text{mod} A) & \quad \quad \leftarrow \quad j_!=j \longrightarrow \quad \mathcal{D}^b(\text{mod} C)
\end{align*}$$

(R)
is given by a pair of compact exceptional objects \( T = j_!(C), T' = i_*(B) \in \mathcal{D}(\text{Mod}A) \) with \( T' \) being right orthogonal to \( T \), see \([6, 2.2 \text{ and } 2.5]\). A similar result holds true for recollements of the form \((R)\), as recalled in \([22]\). In particular, in both cases the recollement is generated by the compact exceptional object \( T = j_!(C) \).

Conversely, every compact exceptional object \( T \in \mathcal{D}(\text{Mod}A) \) with \( C = \text{End}_A(T) \) generates a recollement

\[
\mathcal{D}(\text{Mod}B) \xrightarrow{i_* = i} \mathcal{D}(\text{Mod}A) \xrightarrow{j_! = j'} \mathcal{D}(\text{Mod}C)
\]

where \( B = \mathcal{R}\text{Hom}_A(T', T') \) for a certain object \( T' \cong i_* i^*(A) \in \mathcal{D}(\text{Mod}A) \) which occurs, up to isomorphism, in the canonical triangle \( j_! j^!(A) \to A \to i_* i^*(A) \to j_! j^!(A)[1] \).

We stress, however, that \( B \) is just a \( \text{dg algebra} \), unless the object \( T' \) is exceptional, in which case we obtain a recollement of the form \((R)\) with \( B \cong \text{End}_A(T') \).

In the latter case, the recollement is also induced by a homological ring epimorphism \( \lambda : A \to B \), that is, a ring epimorphism such that \( \text{Tor}^A_i(B, B) = 0 \) for all \( i > 0 \). Then, up to isomorphism,

\[
i_* = \lambda^* = \text{L}\otimes_B B_A = \text{R}\text{Hom}_B(A_B, ?), \quad i^* = \text{L}\otimes_A B, \quad i^! = \text{R}\text{Hom}_A(B_A, ?), \quad \text{and } j^* = \text{L}\otimes_A X
\]

with \( X \) given by the triangle \( X \to A \xrightarrow{\lambda} B \to X[1] \). For details we refer to \([4, 1.6 \text{ and } 1.7]\). A method for determining \( T' \) is given in \([4, \text{Appendix}]\).

In the examples in Sections \([4, 5, \text{and } 7]\) we will often consider the following special case of the construction above. Let \( T = eA \) where \( e = e^2 \in A \) is an idempotent such that \( AeA \) is a \textit{stratifying ideal}, that is, \( Ae \otimes_{eAe} eA \cong AeA \), or equivalently, \( \lambda : A \to A/eAeA \) is a homological ring epimorphism. Then \( T \) generates a recollement of the form \((R)\) where \( C = eAe \) and \( B = A/eAeA \), and \( T' \cong B_A \) occurs in the canonical triangle \( AeA \to A \xrightarrow{\lambda} B \to AeA[1] \). Here the functors \( i_*, i^*, i^! \) are as above, and

\[
j_! = \text{L}\otimes_{eAe} eA, \quad j^* = \text{R}\text{Hom}_A(eA, ?) = \text{L}\otimes_A eA, \quad j_* = \text{R}\text{Hom}_A(eA, ?).
\]

\textbf{Lemma 2.12.} Let \( A \) be a \( k \)-algebra and \( e \in A \) be an idempotent. Assume that \( A/eAeA \), as a (right) \( A \)-module, admits a projective resolution with components in \( \text{add}(eA) \) except in degree 0. Then \( AeA \) is a stratifying ideal of \( A \) and the projection \( A \to A/eAeA \) is a homological epimorphism.

\textbf{Proof.} The surjection \( A \to A/eAeA \) is a ring epimorphism, and under the above assumptions, \( \text{Tor}^A_i(A/eAeA, A/eAeA) = 0 \) for \( i > 0 \). The desired result follows. \( \square \)

2.2.4. \textbf{Derived simple algebras.} Derived simplicity (sometimes also called derived simplicity) of an algebra was introduced by Wiedemann \([50]\). Let \( \mathcal{D} = K^b(\text{proj}), \mathcal{D}_f, \mathcal{D}^b(\text{mod}), \mathcal{D}^b(\text{Mod}), \mathcal{D}^-(\text{Mod}) \) or \( \mathcal{D}(\text{Mod}) \). By definition, a \( k \)-algebra \( A \) is said to be \textit{derived simple} with respect to \( \mathcal{D} \) (or \( \mathcal{D} \)-\textit{simple} for short) if there is no nontrivial \( \mathcal{D} \)-recollement, namely, a recollement of the form

\[
\mathcal{D}_B \xrightarrow{i_* = i} \mathcal{D}_A \xrightarrow{j_!} \mathcal{D}_C
\]

where \( B \) and \( C \) are also \( k \)-algebras. We reserve the short term \textit{derived simple} for \( \mathcal{D}(\text{Mod}) \)-simple.

Fields, and more generally, local algebras are derived simple. First examples of derived simple algebras over a field with more than one simple module have been constructed by Wiedemann \([50]\) and by Happel \([25]\). In \([35]\) derived simplicity has been established for blocks of group algebras.
of finite groups, in any characteristic, and also for indecomposable symmetric algebras of finite representation type, provided that the base ring \( k \) is a field.

A \( k \)-algebra is said to be \textit{indecomposable} if it is not isomorphic to a direct product of two nonzero rings. A non-trivial decomposition of a ring yields a non-trivial recollement. Hence a decomposable ring never is derived simple in any sense.

2.2.5. \textbf{Stratifications.} Having defined derived simplicity, we can study stratifications. Roughly speaking, a \textit{stratification} is a way of breaking up a given derived category into simple pieces using recollements. More rigorously, let \( \mathcal{D} = K^b(\text{proj}), \mathcal{D}_{\text{fl}}, \mathcal{D}^b(\text{mod}), \mathcal{D}^b(\text{Mod}), \mathcal{D}^-(\text{Mod}) \) or \( \mathcal{D}(\text{Mod}) \) and let \( A \) be an algebra; a stratification of \( \mathcal{D}_A \) (or a \( \mathcal{D} \)-stratification of \( A \)) is a full rooted binary tree whose root is the given derived category \( \mathcal{D}_A \), whose nodes are derived categories of type \( \mathcal{D} \) of algebras and whose leaves are derived categories of type \( \mathcal{D} \)-simple algebras such that a node is a recollement of its two child nodes unless it is a leaf. The leaves are called the simple factors of the stratification. By abuse of language, we will also call the algebras whose derived categories are the leaves the simple factors of the stratification.

2.2.6. \textbf{Finiteness of Global Dimension.} A major reason for the interest in recollements and stratifications of derived module categories is that the algebras \( B \) and \( C \) in the two outer terms frequently are less complicated than \( A \). One can then study \( A \) by investigating the two outer algebras. This reduction works well with respect to homological dimensions. We will consider the global dimension.

Let \( A, B \) and \( C \) be finite-dimensional algebras over a field forming a \( D^b(\text{mod}) \)-recollement of the form \( (R^2) \). Wiedemann \cite{50} Lemma 2.1] showed that \( A \) has finite global dimension if and only if so do \( B \) and \( C \). This was generalised in \cite{31} Corollary 5] to \( D^-(\text{Mod}) \)-recollements of algebras over general commutative rings. For completeness we include a detailed proof here.

\textbf{Proposition 2.13.} (\cite{31} Corollary 5) Let \( A, B \) and \( C \) be \( k \)-algebras and assume there is a recollement of the following type

\[
\begin{array}{c}
\mathcal{D}^-(\text{Mod}B) \xleftarrow{} \mathcal{D}^-(\text{Mod}A) \xrightarrow{} \mathcal{D}^-(\text{Mod}C).
\end{array}
\]

Then \( A \) is of finite global dimension if and only if so are \( B \) and \( C \).

\textbf{Proof.} We first observe the following \textbf{Fact:} An algebra \( A \) has finite global dimension if and only if \( \mathcal{D}^b(\text{Mod}A) = K^b(\text{Proj}A) \). If \( A \) has finite global dimension, the equality holds by definition. If \( A \) has infinite global dimension, there is a module of infinite projective dimension (if there are infinitely many \( n \in \mathbb{N} \) and modules \( M_n \) with \( \text{proj. dim}(M_n) = n \), take the direct sum of those \( M_n \)), which is in the bounded derived category, but not in \( K^b(\text{Proj}A) \). This proves the fact.

Now suppose a recollement on \( D^-(\text{Mod}) \) level is given. If \( A \) has finite global dimension, by \cite{31} Proposition 4] the given recollement can be restricted to recollements on \( D^b(\text{Mod}) \) level and on \( K^b(\text{Proj}) \) level. We compare the two recollements: the middle terms coincide, the left hand terms satisfy \( K^b(\text{Proj}B) \subset D^b(\text{Mod}B) \) and the right hand terms also satisfy \( K^b(\text{Proj}C) \subset D^b(\text{Mod}C) \). But the inclusion \( K^b(\text{Proj}B) \subset D^b(\text{Mod}B) \) in the left hand side of the recollement implies the converse inclusion \( K^b(\text{Proj}C) \supset D^b(\text{Mod}C) \) on the right hand side of the recollement. Therefore equality holds and \( C \) and also \( B \) have finite global dimension. Conversely if \( B \) and \( C \) have finite global dimension, the same argument of restricting recollements works. \( \square \)
Based on this result, we prove the analogue for $\mathcal{D}(\text{Mod})$-recollements.

**Proposition 2.14.** In a recollement of the form $(\mathcal{R})$ the algebra $A$ is of finite global dimension if and only if so are $B$ and $C$.

**Proof.** We will show below that if $\text{gl.dim}(A) < \infty$ or $\text{gl.dim}(C) < \infty$ then $i_*(B) \in K^b(\text{Proj}A)$. As $j_!(C) \in K^b(\text{proj}A)$, [31, Theorem 1] implies the existence of a recollement

$\mathcal{D}^-\text{(Mod}B) \longleftarrow \mathcal{D}^-\text{(Mod}A) \longleftarrow \mathcal{D}^-\text{(Mod}C) \ .$

Applying the previous proposition then concludes the proof of the assertion.

What is left to show is that $i_*(B) \in K^b(\text{Proj}A)$. If $\text{gl.dim}(A) < \infty$, then by Lemma 2.9 (e) the object $i_*(B) \in D^b(\text{Mod}A) \cong K^b(\text{Proj}A)$. Suppose $\text{gl.dim}(C) < \infty$. Again using Lemma 2.9 (e) yields $j^*(A) \in D^b(\text{Mod}C) = K^b(\text{Proj}C)$. On the other hand, since $j_!$ commutes with direct sums and $j_!(C) \in K^b(\text{proj}A)$, it follows that $j_!(\text{Proj}C) \subseteq K^b(\text{Proj}A)$. Hence $j_!(K^b(\text{Proj}C)) \subseteq K^b(\text{Proj}A)$. Therefore, $j_!j^*(A) \in K^b(\text{Proj}A)$, and hence $i_*i^*(A) \in K^b(\text{Proj}A)$ due to the triangle

$$j_!j^*(A) \to A \to i_*i^*(A) \to j_!j^*(A)[1].$$

By Lemma 2.9 (c), $i_*(B) \in K^b(\text{Proj}A)$. \hfill \Box

3. Ladders

When considering all possible recollements for a fixed triangulated or derived category, it is convenient to group them, for example, into ladders. A **ladder** $\mathcal{L}$ is a finite or infinite diagram of triangulated categories and triangle functors

\[
\begin{array}{cccccc}
\vdots & \vdots & & & & \\
& j_{n-2} & \leftarrow & j_{n-2} & \rightarrow & \\
& j_{n-1} & & j_{n-1} & & \\
\mathcal{C}' & j_{n} & \rightarrow & \mathcal{C} & \leftarrow & \mathcal{C}'' \\
& j_{n+1} & \rightarrow & j_{n+1} & \leftarrow & \\
& j_{n+2} & & j_{n+2} & & \\
& & \vdots & & \vdots & \\
\end{array}
\]

such that any three consecutive rows form a recollement. The rows are labelled by a subset of $\mathbb{Z}$ and multiple occurrence of the same recollement is allowed. This definition is taken from [10, Section 1.5] with a minor modification. The **height** of a ladder is the number of recollements contained in it (counted with multiplicities). It is an element of $\mathbb{N} \cup \{0, \infty\}$. A recollement is considered to be a ladder of height 1.

The ladder $\mathcal{L}$ induces a **TTF tuple** $(\text{Im}i_n)_n$, and is uniquely determined, up to equivalence, by any entry of the tuple. A ladder can be **unbounded**, **bounded above**, **bounded below** or **bounded** in the obvious sense, and the corresponding TTF tuple is **unbounded**, **left bounded**, **right bounded** or **bounded**. The ladder $\mathcal{L}$ is a **ladder of derived categories** if $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ are derived categories of algebras. A ladder is **complete** if it is not a proper subladder of any other ladder. Then the induced TTF tuple also is said to be complete.
**Lemma 3.1.** Let $C$ be a compactly generated triangulated category. Let $(\ldots, C_{-2}, C_{-1}, C_0)$ be a TTF tuple of $C$, and assume that $C_{-1}$ is compactly generated by one object. Then this TTF tuple is complete if and only if some (or any) compact generator of $C_{-1}$ is not compact in $C$.

**Proof.** The pair $(C_{-1}, C_0)$ can be completed to a TTF triple $(C_{-1}, C_0, C_1)$ if and only if a (or any) compact generator of $C_{-1}$ is compact in $C$, see [40] Lemma 4.4.8 and Remark 5.2.6]. □

**Proposition 3.2.** Let $A$, $B$ and $C$ be $k$-algebras forming a recollement

\[
\begin{array}{ccc}
\text{D(ModB)} & \xrightarrow{i_{=1}} & \text{D(ModA)} \\
\text{D(ModC)} & \xrightarrow{j_{=1}} & \text{D(ModC)}
\end{array}
\]

\[(R)\]

(a) The recollement can be extended one step downwards (in the obvious sense) if and only if $j_*$ (equivalently $i_!$) has a right adjoint. This occurs precisely when $j_!$ (equivalently $i_*$) restricts to $K^b(\text{proj})$.

(b) The recollement can be extended one step upwards if and only if $j^!$ (equivalently $i^*$) has a left adjoint. If $A$ is a finite-dimensional algebra over a field, this occurs precisely when $j^!$ (equivalently $i^*$) restricts to $D^b(\text{mod})$.

**Proof.** The first statements of both (a) and (b) follow from Lemma 2.2. Further, it follows from Lemma 3.1 that the recollement can be extended downwards if and only if $i_!(B)$ is compact, which means by Lemma 2.5 that $i_*$ restricts to $K^b(\text{proj})$. This proves the second statement of (a) concerning $i_*$; the equivalence with the statement in parentheses is part of Lemma 4.3 below, which will be proven without using any result of the present section. The second statement of (b) is just the equivalence of (ii) and (iii) in Lemma 2.8. Indeed, both $i^*$ and $j_!$ are derived tensor products, by Lemma 2.10 (a). Moreover, the left adjoints of $i^*$ and $j_!$ always restrict to $K^b(\text{proj})$ by Lemma 2.9 (e), since they form the top row of a recollement. □

Proposition 3.2 can be extended by induction as follows. Let $X$ and $Y$ be as in Lemma 2.10 (a) and define $X_n, Y_n$ recursively by setting $X_0 = X$, $Y_0 = Y$ and

\[
X_n = \begin{cases} 
(X_{n-1})^A & \text{if } n \text{ is odd and } n > 0 \\
(X_{n-1})^C & \text{if } n \text{ is even and } n > 0 \\
(X_{n+1})^C_{\text{op}} & \text{if } n \text{ is odd and } n < 0 \\
(X_{n+1})^A_{\text{op}} & \text{if } n \text{ is even and } n < 0 
\end{cases}
\]

\[
Y_n = \begin{cases} 
(Y_{n-1})^B & \text{if } n \text{ is odd and } n > 0 \\
(Y_{n-1})^A & \text{if } n \text{ is even and } n > 0 \\
(Y_{n+1})^A_{\text{op}} & \text{if } n \text{ is odd and } n < 0 \\
(Y_{n+1})^B_{\text{op}} & \text{if } n \text{ is even and } n < 0 
\end{cases}
\]

All of these are complexes of bimodules.

**Proposition 3.3.** Suppose that $k$ is a field and $A$, $B$, $C$ are finite-dimensional $k$-algebras forming a recollement of the form (R). Let $m \in \mathbb{N}$.

(a) The following statements are equivalent:

(i) (R) can be extended $m$ steps downwards.

(ii) $(X_n)_A$ is compact in $D(\text{ModA})$ for $n$ even with $1 \leq n \leq m$ and $(X_n)_C$ is compact in $D(\text{ModC})$ for $n$ odd with $1 \leq n \leq m$.

(iii) $(Y_n)_B$ is compact in $D(\text{ModB})$ for $n$ even with $1 \leq n \leq m$ and $(Y_n)_A$ is compact in $D(\text{ModA})$ for $n$ odd with $1 \leq n \leq m$. 

If this is the case, we obtain a ladder of height \( m + 1 \). We label the rows of this ladder by \( 0, \ldots, m + 2 \) from the top. For \( n \) even (respectively, odd) with \( 0 \leq n \leq m + 1 \), the two functors of the \( n \)-th row are \( \mathbb{L} \otimes_A Y_n \) and \( \mathbb{L} \otimes_C X_n \) (respectively, \( \mathbb{L} \otimes_B Y_n \) and \( \mathbb{L} \otimes_A X_n \)). The two functors of the \((m + 2)\)-nd row are \( \mathsf{RHom}_B(Y_{m+1},?) \) and \( \mathsf{RHom}_A(X_{m+1},?) \) if \( m \) is odd and \( \mathsf{RHom}_A(Y_{m+1},?) \) and \( \mathsf{RHom}_C(X_{m+1},?) \) if \( m \) is even.

**(b)** The following statements are equivalent:

(i) \( \{\mathbb{L}\} \) can be extended \( m \) steps upwards,

(ii) \( C(X_n) \) is compact in \( \mathcal{D}(\mathcal{C}) \) for \( n \) even with \( 1 - m \leq n \leq 0 \) and \( A(X_n) \) is compact in \( \mathcal{D}(\mathcal{A}) \) for \( n \) odd with \( 1 - m \leq n \leq 0 \),

(iii) \( A(Y_n) \) is compact in \( \mathcal{D}(\mathcal{A}) \) for \( n \) even with \( 1 - m \leq n \leq 0 \) and \( B(Y_n) \) is compact in \( \mathcal{D}(\mathcal{B}) \) for \( n \) odd with \( 1 - m \leq n \leq 0 \).

If this is the case, we obtain a ladder of height \( m + 1 \). We label the rows of this ladder by \( -m, \ldots, 0, 1, 2 \) from the top. For \( n \) even (respectively, odd) with \( -m \leq n \leq 1 \), the two functors of the \( n \)-th row are \( \mathbb{L} \otimes_A Y_n \) and \( \mathbb{L} \otimes_C X_n \) (respectively, \( \mathbb{L} \otimes_B Y_n \) and \( \mathbb{L} \otimes_A X_n \)). The two functors of the second row are \( \mathsf{RHom}_A(Y_1,?) \) and \( \mathsf{RHom}_C(X_1,?) \).

**Proof.** This follows by induction on \( m \) from Proposition 3.2, Lemma 2.10 and Lemma 2.8. We leave the details to the reader. \( \square \)

Next we give some examples of ladders and TTF tuples.

**Example 3.4.** Let \( k \) be a field, \( B \) and \( C \) be finite-dimensional \( k \)-algebras and \( M \) a finitely generated \( C \)-\( B \)-bimodule. Consider the matrix algebra \( A = \left( \begin{array}{cc} B & 0 \\ C & M_B \\ \end{array} \right) \). Put \( e_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \end{array} \right) \) and \( e_2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \end{array} \right) \). Using Lemma 2.12, one checks that both \( Ae_1A \) and \( Ae_2A \) are stratifying ideals of \( A \). Moreover, they produce a ladder of height 2

\[ \mathcal{D} (\mathcal{M}obB) \xrightarrow{\mathbb{L} \otimes_{\mathbb{A}} e_1A} \mathcal{D} (\mathcal{M}odA) \xrightarrow{\mathbb{L} \otimes_{\mathbb{A}} e_2A} \mathcal{D} (\mathcal{M}odC). \]

By Proposition 3.2(b) and Lemma 2.8, this ladder can be extended one step upwards if and only if \( C(e_2A) \in K^b(\mathcal{C} \text{-proj}) \) if and only if \( C M \) has finite projective dimension over \( C \). By Proposition 3.2(a) and Lemma 2.5, it can be extended one step downwards if and only if \( (Ae_1)_B \in K^b(\text{proj}B) \) if and only if \( M_B \) has finite projective dimension over \( B \).

**Example 3.5.** Let \( k \) be a field and let \( A \) be the path algebra of the quiver \( 1 \rightarrow \rightarrow 2 \). Up to isomorphism and shift there are three indecomposable exceptional objects in \( K^b(\text{proj}A) \), given by the following representations

\[ P_1 = ( k \leftarrow \leftarrow 0 ), \quad P_2 = ( k \leftarrow \text{id} \leftarrow k ), \quad S_2 = ( 0 \leftarrow \leftarrow k ). \]

They generate an unbounded TTF tuple which is periodic of period 3:

\[ \ldots, \text{Tria} (P_2), \text{Tria} (P_1), \text{Tria} (S_2), \text{Tria} (P_2), \text{Tria} (P_1), \ldots. \]

Note that the TTF triple \( (\text{Tria} (P_2), \text{Tria} (P_1), \text{Tria} (S_2)) \) corresponds to the recollement generated by \( P_2 = e_2A \), while the TTF triple \( (\text{Tria} (P_1), \text{Tria} (S_2), \text{Tria} (P_2)) \) corresponds to the recollement generated by \( P_1 = e_1A \), where \( e_1 \) and \( e_2 \) are trivial paths at 1 and 2, respectively.
Example 3.6. Let $k$ be a field and let $A$ be the algebra given by the quiver with relations

$$1 \xrightarrow{\alpha} 2, \quad \alpha \beta.$$  

This is a quasi-hereditary algebra (with respect to the ordering $1 < 2$); its global dimension is two.

An object of $K^b(\text{proj} A)$ is indecomposable if and only if it is isomorphic to one of the following complexes

- $P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1$,  
- $P_2 \xrightarrow{\beta} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1$,  
- $P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\alpha} P_2$,  
- $P_2 \xrightarrow{\beta} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\beta \alpha} P_1 \xrightarrow{\alpha} P_2$,

see for example [11]. The objects in the second and third families are exceptional with endomorphism algebra $k$, the projective $P_1$ is exceptional with endomorphism algebra $k[x]/x^2$, while the other objects in the first and fourth families are not exceptional.

The projective module $P_2$ generates a $\mathcal{D}(\text{Mod})$-recollement (this is the one from the quasi-hereditary structure of $A$), which sits in the infinite ladder corresponding to the TTF tuple $(C_n)_{n \in \mathbb{Z}}$, where

$$C_n = \begin{cases} 
\text{Tria} (\nu^2 P_2) & \text{if } n \text{ is even}, \\
\text{Tria} (\nu S_1) & \text{if } n \text{ is odd}.
\end{cases}$$

Here $\nu$ is the derived Nakayama functor for $A$. One directly checks that up to shift all exceptional indecomposable objects of $K^b(\text{proj} A)$ other than $P_1$ already occur in the tuple

$$(\ldots, \nu^{-1} P_2, \nu^{-1} S_1, P_2, S_1, \nu P_2, \nu S_1, \ldots).$$

To construct this ladder explicitly, one may start with the recollement given by the quasi-hereditary structure and first compute the outer terms: The algebra on the left hand side is $B := A/Ae_2 A \simeq k$. The algebra on the right hand side is $C := e_2 Ae_2 \simeq k$. Next, the images of $B$ and $C$, respectively, can be determined: $i_* (B) \simeq B \otimes_B B A \simeq S_1$ is simple, $j_*(C) \simeq e_2 Ae_2 \otimes_{e_2 Ae_2} e_2 A \simeq e_2 A$ is projective and $j_*(C) \simeq \mathbf{R}\text{Hom}_{e_2 Ae_2} (Ae_2, e_2 Ae_2) \simeq \text{Hom}_k (Ae_2, k)$ is the injective $A$-module $I_2$. Observing that $I_2 \simeq \nu P_3$ shows that the exceptional objects $P_2, S_1, \nu P_2$ correspond to three subsequent recollements in the ladder, which therefore is determined by iteratively applying $\nu$ and its inverse to these objects.

Observe that the endomorphism algebra of $P_1$ has infinite global dimension, while $A$ has finite global dimension. It follows from Proposition 2.14 that $P_1$ cannot generate a recollement of derived module categories. Therefore up to equivalence the above ladder is the unique non-trivial ladder of derived categories for $\mathcal{D}(\text{Mod}A)$. Note that the above tuple of objects, read from the right to the left, is a helix in the sense of the Rudakov school [15].

In Example 3.3 the ladder has a repeating pattern given by the Nakayama functor. This is a general phenomenon, compare [28, 52].

Proposition 3.7. Keep the notation and assumptions as in Proposition 3.3. Assume further that $A$ has finite global dimension and let $\nu : \mathcal{D}(\text{Mod} A) \to \mathcal{D}(\text{Mod} A)$ be the derived Nakayama functor
for A. Set \( T = j_i(C) \) and \( T' = i_*(B) \). Then \([R]\) sits in an unbounded ladder which corresponds to the TTF tuple \((C_n)_{n \in \mathbb{Z}}\), where

\[
C_n = \begin{cases} 
\text{Tria}(\nu^n T) & \text{if } n \text{ is even}, \\
\text{Tria}(\nu^{n-1} T') & \text{if } n \text{ is odd}.
\end{cases}
\]

\textbf{Proof.} By Proposition\[2.13\] both \( B \) and \( C \) have finite global dimension. Then it follows by induction on \( n \) that for all \( n \in \mathbb{Z} \) the complex \( X_n \) is compact in \( \mathcal{D}(\text{Mod} A) \), \( \mathcal{D}(\text{Mod} C) \), \( \mathcal{D}(C \text{-Mod}) \) or \( \mathcal{D}(A \text{-Mod}) \), depending on the sign and parity of \( n \). Thus by Proposition\[3.3\] the given recollement \([R]\) sits in an unbounded ladder which corresponds to the TTF tuple \((C_n)_{n \in \mathbb{Z}}\), where

\[
C_n = \begin{cases} 
\text{Tria}((X_n)_A) & \text{if } n \text{ is even}, \\
\text{Tria}((Y_n)_A) & \text{if } n \text{ is odd}.
\end{cases}
\]

It remains to show \( \text{Tria}((X_n)_A) = \text{Tria}(\nu^n T) \) if \( n \) is even and \( \text{Tria}((Y_n)_A) = \text{Tria}(\nu^{n-1} T') \) if \( n \) is odd. Up to shift of the TTF tuple, it suffices to show \( \text{Tria}((X_2)_A) = \text{Tria}(\nu T) \). By definition, \( X_2 = (X^{t_A})^{t_C} = R\text{Hom}_C(R\text{Hom}_A(X, A), C) \). Since \( C \) has finite global dimension, the projective generator \( C \) and injective cogenerator \( DC \) each generate each other in \( \mathcal{D}(\text{Mod} C) \) in finitely many steps. Therefore, \( (X_2)_A \) and \( \nu(T) \cong \nu(X) = DR\text{Hom}_A(X, A) \cong R\text{Hom}_C(R\text{Hom}_A(X, A), DC) \) generate each other in \( \mathcal{D}(\text{Mod} A) \) in finitely many steps. In particular, \( \text{Tria}((X_2)_A) = \text{Tria}(\nu(T)) \), as desired. \( \square \)

\section*{4. Lifting and Restricting Recollements}

Recollements of derived module categories can be defined on all levels — bounded or unbounded derived categories and finitely generated or general module categories. While lifting of recollements from bounded to left or right bounded level and from left or right bounded to unbounded level is not problematic (see \cite{31} and \cite{5} Section 4), going in the opposite direction is not always possible. In general, recollements on unbounded level need not restrict to recollements on bounded or left or right bounded level. In the first and second subsections, characterisations are given, when lifting and restricting is possible. In the third subsection, an example is provided to illustrate the conditions in these characterisations.

\subsection*{4.1. Lifting recollements to \( \mathcal{D}(\text{Mod}) \)}

Let \( A \) be a \( k \)-algebra. Let \( \mathcal{D}_A \) be one of the following derived categories associated to \( A \): \( \mathcal{D}^-(\text{Mod} A) \), \( \mathcal{D}^b(\text{Mod} A) \), \( \mathcal{D}^b(\text{mod} A) \) (for \( k \) being a field and \( A \) being finite-dimensional over \( k \)) and \( K^b(\text{proj} A) \).

\textbf{Proposition 4.1.} Let \( A, B \) and \( C \) be \( k \)-algebras.

(a) ([\cite{5} Lemma 4.1, 4.2, 4.3] and \cite{5} Corollary 2.7]) \textit{Any recollement}

\[
\begin{array}{c}
\mathcal{D}_B \xleftarrow{i^*} \mathcal{D}_A \xrightarrow{j_*} \mathcal{D}_C \\
\mathcal{D}(\text{Mod} B) \xleftarrow{i_*} \mathcal{D}(\text{Mod} A) \xrightarrow{j_*} \mathcal{D}(\text{Mod} C)
\end{array}
\]

\textit{can be lifted to a recollement}

\[
\begin{array}{c}
\mathcal{D}(\text{Mod} B) \dashv j_* \mathcal{D}(\text{Mod} A) \dashv ij_*
\end{array}
\]

\textit{such that } \( j_!(C) \cong j_!(C) \), \( i_*(B) \cong i'_*(B) \) \textit{and } \( j_*(C) \cong j'_*(C) \).
Lemma 4.3. (a) Then

(b) The lifted recollement \([4.2]\) in (a) restricts, up to equivalence, to the recollement \([4.1]\).

Proof. (b) We will show that the TTF triple \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) associated with the lifted recollement \([4.2]\) restricts to the TTF triple \((\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1)\) associated with the given recollement \([4.1]\). The desired result then follows from Proposition \([2.1]\). Recall that \(\mathcal{X} = \text{Tria}(j_!(C))\), \(\mathcal{Y} = \text{Tria}(i_*(B)) = (\text{Tria}(j_!(C)))^\perp\) and \(\mathcal{Z} = (\text{Tria}(i_*(B)))^\perp\). It is clear that \(\mathcal{X}_1 \subseteq \mathcal{X}\). Moreover, \(j_!(C) \in \mathcal{X}_1\) being left orthogonal to \(\mathcal{Y}_1\) implies \(\mathcal{Y}_1 \subseteq \mathcal{Y}\). Similarly, \(\mathcal{Z}_1 \subseteq \mathcal{Z}\). Consequently, \(\mathcal{X}_1 \subseteq \mathcal{X} \cap \mathcal{D}_A\), \(\mathcal{Y}_1 \subseteq \mathcal{Y} \cap \mathcal{D}_A\) and \(\mathcal{Z}_1 \subseteq \mathcal{Z} \cap \mathcal{D}_A\). In particular, \(\mathcal{X}_1 \perp (\mathcal{Y} \cap \mathcal{D}_A)\), which implies that \(\mathcal{Y} \cap \mathcal{D}_A \subseteq \mathcal{Y}_1\) because \((\mathcal{X}_1, \mathcal{Y}_1)\) is a \(t\)-structure of \(\mathcal{D}_A\). Therefore the equality \(\mathcal{Y}_1 = \mathcal{Y} \cap \mathcal{D}_A\) holds. Similarly \(\mathcal{X}_1 = \mathcal{X} \cap \mathcal{D}_A\) and \(\mathcal{Z}_1 = \mathcal{Z} \cap \mathcal{D}_A\). □

Remark. It is not clear if the restriction of the induced recollement \([4.2]\) coincides with (instead of being equivalent to) the given recollement \([4.1]\). This would positively answer Rickard’s question asking whether any derived equivalence is a derived tensor functor given by a bimodule complex.

4.2. Restricting recollements from \(\mathcal{D}^{(\text{Mod})}\). Suppose we are given three \(k\)-algebras \(A, B\) and \(C\) together with a recollement of the form

\[
\begin{array}{ccc}
\mathcal{D}(\text{Mod}B) & \xrightarrow{i_* = i} & \mathcal{D}(\text{Mod}A) \\
\llcorner & & \llcorner \\
\mathcal{D}(\text{Mod}C) & \xrightarrow{j^* = j} & \mathcal{D}(\text{Mod}C) \\
\rlcorner & & \rlcorner \\
\end{array}
\]

(R)

We are interested in conditions under which this recollement can be restricted to a recollement on \(K^b(\text{proj})\), \(\mathcal{D}^b(\text{mod})\) (when \(k\) is a field and the algebras are finite-dimensional \(k\)-algebras), \(\mathcal{D}^b(\text{Mod})\) and \(\mathcal{D}^{-}(\text{Mod})\).

4.2.1. Restricting recollements to \(K^b(\text{proj})\). We start with an auxiliary result.

Lemma 4.2. Let \(F : \mathcal{C} \to \mathcal{C}'\) be a fully faithful triangle functor commuting with direct sums. If \(X \in \mathcal{C}\) is not compact, then \(F(X)\) is not compact.

Proof. Let \(\{Y_i \mid i \in I\}\) be a set of objects of \(\mathcal{C}\). There is a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(X, \bigoplus_{i \in I} Y_i) & \xrightarrow{\approx} & \text{Hom}(F(X), F(\bigoplus_{i \in I} Y_i)) \\
\mid & & \mid \\
\bigoplus_{i \in I} \text{Hom}(X, Y_i) & \xrightarrow{\approx} & \bigoplus_{i \in I} \text{Hom}(F(X), F(Y_i)).
\end{array}
\]

If \(F(X)\) is compact in \(\mathcal{C}'\), then the right lower map is bijective, implying that the left map is bijective, i.e. \(X\) is compact in \(\mathcal{C}\). □

Next we show that \(i_*\) restricts to \(K^b(\text{proj})\) if and only if so does \(j^*\). Moreover, if this happens, then \(j^*\) restricts to \(K^b(\text{proj})\) if and only if so does \(j_*\).

Lemma 4.3. (a) \(i_*(B) \in K^b(\text{proj}A)\) if and only if \(j^*(A) \in K^b(\text{proj}C)\), which happens if and only if \(j_!j^*(A) \in K^b(\text{proj}A)\).

(b) Assume that \(i_*(B) \in K^b(\text{proj}A)\). Then \(i'(A) \in K^b(\text{proj}B)\) if and only if \(j_*(C) \in K^b(\text{proj}A)\).
Proof. (a) Consider the canonical triangle

\[ \xymatrix{ j_! j^*(A) \ar[r] & A \ar[r] & i_* i^*(A) \ar[r] & j_! j^*(A)[1]. } \]

If \( i_*(B) \in K^b(\text{proj} A) \), then \( i_* i^*(A) \in K^b(\text{proj} A) \) by Lemma 2.20(c). Therefore \( j_! j^*(A) \in K^b(\text{proj} A) \), and hence \( j^*(A) \in K^b(\text{proj} A) \) by Lemma 2.22. Conversely, if \( j^*(A) \in K^b(\text{proj} C) \), then \( j_! j^*(A) \in K^b(\text{proj} A) \) by Lemma 2.20(e). So the above triangle implies that \( i_* i^*(A) \in K^b(\text{proj} A) \). Now by Lemma 2.20(c), \( i_*(B) \in K^b(\text{proj} A) \).

(b) Assume that \( i_*(B) \in K^b(\text{proj} A) \) and \( i^!(A) \in K^b(\text{proj} B) \) hold. Consider the following canonical triangle associated to \( j_!(C) \)

\[ \xymatrix{ i_* i^! j_!(C) \ar[r] & j_!(C) \ar[r] & j_* j^! j_!(C) \ar[r] & i_* i^! j_!(C)[1]. } \]

It follows from the assumption that \( i_* i^! j_!(C) \in K^b(\text{proj} A) \) since \( j_!(C) \in K^b(\text{proj} A) \). Thus \( j_!(C) \in K^b(\text{proj} A) \).

Assume that \( i_*(B) \in K^b(\text{proj} A) \) and \( j_!(C) \in K^b(\text{proj} A) \) hold. Consider the following canonical triangle associated to \( A \):

\[ \xymatrix{ i_* i^! (A) \ar[r] & A \ar[r] & j_* j^! (A) \ar[r] & i_* i^! (A)[1]. } \]

Applying \( i^* \) to this triangle, we obtain

\[ \xymatrix{ i^* i_* i^! (A) \ar[r] & i^* (A) \ar[r] & i^* j_* j^! (A) \ar[r] & i^* i_* i^! (A)[1]. } \]

We know from (a) and the assumption that \( j^*(A) \in K^b(\text{proj} C) \). Since \( j_!(C) \in K^b(\text{proj} A) \) and \( i^* \) restricts to \( K^b(\text{proj}) \), it follows that \( i^* j_* j^! (A) \in K^b(\text{proj} B) \), and hence \( i^! (A) \in K^b(\text{proj} B) \).

**Theorem 4.4.** The following are equivalent:

(i) the recollement \( [R] \) restricts to a recollement

\[ \xymatrix{ K^b(\text{proj} B) \ar[r]_{i_*} & K^b(\text{proj} A) \ar[r]_{j_*} & K^b(\text{proj} C) \ar[r]_{j_*} & K^b(\text{proj} B), } \]

(ii) \( i_*(B) \in K^b(\text{proj} A) \) and \( i^!(A) \in K^b(\text{proj} B) \),

(iii) \( j^*(A) \in K^b(\text{proj} C) \) and \( j_!(C) \in K^b(\text{proj} A) \).

**Proof.** By Lemma 2.20(e) both functors \( j_! \) and \( i^* \) can be restricted to \( K^b(\text{proj}) \). Therefore a recollement of the form \( [R] \) restricts to \( K^b(\text{proj}) \) if and only if \( i_*(B) \in K^b(\text{proj} A) \), \( i^!(A) \in K^b(\text{proj} B) \), \( j^*(A) \in K^b(\text{proj} C) \) and \( j_!(C) \in K^b(\text{proj} A) \). The desired result then follows from Lemma 4.3.

Example 4.13(c) will show that requiring only the condition \( i^!(A) \in K^b(\text{proj} A) \) in (ii) (or \( B \) has finite global dimension, cf. Corollary 4.7 below) is not enough for the recollement \( [R] \) to restrict to \( K^b(\text{proj}) \). The next example shows that requiring only the condition \( j_!(C) \in K^b(\text{proj} A) \) in (iii) also is not sufficient.
Lemma 2.8 to

Proof.

The following are equivalent for a finite-dimensional algebra $A$:

(i) the recollement (R) restricts to a recollement

$\mathcal{D}(\text{Mod}B) \xleftarrow{i_*} \mathcal{D}(\text{Mod}A) \xrightarrow{j_*} \mathcal{D}(\text{Mod}C)$,

(ii) $C X \in K^b(C\text{-proj})$ and $(X^\text{tr} \otimes_C X)_A \in K^b(\text{proj}A)$;

(iii) $C X \in K^b(C\text{-proj})$ and $X^\text{tr}_C \in K^b(\text{proj}C)$.

(iv) $j_!$ restricts to $\mathcal{D}^b(\text{mod})$ and $i_*(B) \in K^b(\text{proj}A)$.

Theorem 4.6. The following are equivalent for a finite-dimensional algebra $A$:

Example 4.5. Let $k$ be a field, $T = k^{|\mathbb{N}|}$ and $A = \text{End}_k(T)$. Then elements in $A$ are identified with $\mathbb{N} \times \mathbb{N}$-matrices such that there are only finitely many non-zero entries in each column. Let $e$ be the elementary matrix with entry 1 in position $(1,1)$ and with entry 0 in any other position. Then $AeA$ is a stratifying ideal and hence yields a recollement

$$
\mathcal{D}(\text{Mod}A/AeA) \xleftarrow{i_*} \mathcal{D}(\text{Mod}A) \xrightarrow{j_*} \mathcal{D}(\text{Mod}k).
$$

which is even a stratification. Indeed, $AeA$ consists of the endomorphisms of $T$ with finite-dimensional image, so $A/AeA$ is a simple von Neumann regular ring by [33, 4.27] and [19, the second Example on p. 16] and therefore derived simple by [5, 4.11]. One checks that $j_*(k) = eA$ belongs to $K^b(\text{proj}A)$, while $j^*(A) = T$ does not belong to $K^b(\text{proj}k)$. So this recollement does not restrict to $K^b(\text{proj})$, although $j_*$ restricts to $K^b(\text{proj})$.

More generally, this phenomenon occurs in every recollement induced by a good tilting module. Recall that a module $T_R$ over a ring $R$ is said to be a **good tilting module** if it has finite projective dimension, $\text{Ext}_R^i(T, T^{(\alpha)}) = 0$ for any cardinal $\alpha$, and there is an exact sequence of right $R$-modules

$$0 \rightarrow R \rightarrow T_0 \rightarrow \ldots \rightarrow T_n \rightarrow 0$$

with $T_0, T_1, \ldots, T_n \in \text{add} T$. It was shown by Chen and Xi in [12, 10] that $T_R$ then induces a recollement

$$
\mathcal{D}(\text{Mod}B) \xleftarrow{i_*} \mathcal{D}(\text{Mod}A) \xrightarrow{j_*} \mathcal{D}(\text{Mod}R),
$$

where $A = \text{End}_R(T)$, $j_* = \mathbb{R}\text{Hom}_R(T, ?)$, and $j^* = L \otimes_A T$.

Since $R$ is quasi-isomorphic to the complex $T_0 \rightarrow \ldots \rightarrow T_n$, we see that $j_*(R) \in K^b(\text{proj}A)$, so $j_*$ restricts to $K^b(\text{proj})$ by Lemma 2.3. On the other hand, $j^*$ restricts to $K^b(\text{proj})$ if and only if $j^*(A) \cong T_R \in K^b(\text{proj}R)$, that is, if and only if $T_R$ is a classical tilting module.

4.2.2. Restricting recollements to $\mathcal{D}^b(\text{mod})$. Let $k$ be a field. Suppose we are given a recollement of the form (R) where $A$, and thus also $B$ and $C$, are finite-dimensional $k$-algebras. Recall that in this case $\mathcal{D}_{jh}$ coincides with $\mathcal{D}^b(\text{mod})$.

Let $X$ and $X^\text{tr} = \mathbb{R}\text{Hom}_A(X, A)$ be chosen as in Lemma 2.10(a).

Theorem 4.6. The following are equivalent for a finite-dimensional algebra $A$:

(i) the recollement (R) restricts to a recollement

$\mathcal{D}^b(\text{mod}B) \xleftarrow{i_*} \mathcal{D}^b(\text{mod}A) \xrightarrow{j_*} \mathcal{D}^b(\text{mod}C)$,

(ii) $C X \in K^b(C\text{-proj})$ and $(X^\text{tr} \otimes_C X)_A \in K^b(\text{proj}A)$;

(iii) $C X \in K^b(C\text{-proj})$ and $X^\text{tr}_C \in K^b(\text{proj}C)$.

(iv) $j_!$ restricts to $\mathcal{D}^b(\text{mod})$ and $i_*(B) \in K^b(\text{proj}A)$.

Proof. (i) $\Rightarrow$ (ii): Suppose the recollement can be restricted to the level of $\mathcal{D}^b(\text{mod})$. Applying Lemma 2.8 to $j_! = ? \otimes_C X$ yields that $C X \in K^b(C - \text{proj})$. Furthermore by [6, Corollary 2.4], the functors $j_!$ and $j_*$ send compact objects to compact objects. In particular, $j_!j^!(A) \cong X^\text{tr} \otimes_C X$, as a complex of right $A$-modules, is compact.
The equivalence of (ii), (iii), and (iv) holds true by Lemma 2.8 and by Lemma 4.3 (a) since $j^*(A) = X^i_L$ and $jj^*(A) = (X^u \otimes C X)_A$.

(iv) $\Rightarrow$ (i): $i_*$ and $j^*$ always restrict to $\mathcal{D}^b(\text{mod})$ by Lemma 2.9 (c), and $j_!$ restricts to $\mathcal{D}^b(\text{mod})$ by assumption. Moreover, since $i_*(B) \in K^b(\text{proj}A)$, it follows from Lemma 2.5 and Lemma 2.7 that $i^!$ restricts to $\mathcal{D}^b(\text{mod})$. Similarly, one shows that $j_*$ restricts to $\mathcal{D}^b(\text{mod})$ since $j^*(A) = X^r_C \in K^b(\text{proj}C)$. It remains to check $i^*$. For $M \in \mathcal{D}^b(\text{mod}A)$, there is a triangle

$$j_!j^*(M) \rightarrow M \rightarrow i_*i^*(M) \rightarrow j_!j^*(M)\{1\}.$$ 

Since both $j_!$ and $j^*$ restrict to $\mathcal{D}^b(\text{mod})$, the object $jj^*(M)$, and hence $i_*i^*(M)$, are in $\mathcal{D}^b(\text{mod}A)$. Because $i_*$ is a full embedding, there are isomorphisms

$$H^n(i^*(M)) = \text{Hom}_B(B, i^*(M)[n]) = \text{Hom}_A(i_*(B), i_*i^*(M)[n]).$$

Since $i_*(B) \in K^b(\text{proj}A)$ and $i_*i^*(M) \in \mathcal{D}^b(\text{mod}A)$, it follows that $i^*(M)$ has bounded finite-dimensional cohomologies, that is, $i^*(M) \in \mathcal{D}^b(\text{mod}B)$. \hfill \Box

A special case in which condition (iii) is satisfied is when the algebra $C$ has finite global dimension. Indeed, since $X^i_A$ is compact in $\mathcal{D}(\text{Mod}A)$, as a complex $X$ has finite-dimensional total cohomological space. Hence $CX \in \mathcal{D}^b(C\text{-mod})$. When $C$ has finite global dimension, $CX$ is compact in $\mathcal{D}(C\text{-Mod})$. A similar argument applies to $X^u$.

**Corollary 4.7.** Given a finite-dimensional algebra $A$, a recollement of the form $(\mathcal{R})$ can be restricted to $\mathcal{D}^b(\text{mod})$ provided the algebra $C$ has finite global dimension.

However, this corollary fails when replacing $C$ by $B$. Counterexamples will be given below in Example 4.13.

**4.2.3. Restricting Recollements to $\mathcal{D}^b(\text{Mod})$.** The following result is included in the proof of [31, Proposition 4] and in [5, Lemma 4.1, Example 4.6].

**Proposition 4.8.** The following are equivalent for a $k$-algebra $A$:

(i) the recollement $(\mathcal{R})$ restricts to a recollement

$$\mathcal{D}^b(\text{Mod}B) \overset{i_*}{\leftarrow} \mathcal{D}^b(\text{Mod}A) \overset{j_!}{\rightarrow} \mathcal{D}^b(\text{Mod}C),$$

(ii) $i_*(B) \in K^b(\text{Proj}A)$ and $j_!$ restricts to $\mathcal{D}^b(\text{Mod})$.

**Corollary 4.9.** Let $k$ be a field and let $A$ be a finite-dimensional $k$-algebra. Then the recollement $(\mathcal{R})$ restricts to $\mathcal{D}^b(\text{Mod})$ if and only if it restricts to $\mathcal{D}^b(\text{mod})$.

**Proof.** By Theorem 4.3 and Proposition 4.6, we have to show that the following are equivalent:

(a) $i_*(B) \in K^b(\text{Proj}A)$ and $j_!$ restricts to $\mathcal{D}^b(\text{Mod})$.
(b) $i_*(B) \in K^b(\text{proj}A)$ and $j_!$ restricts to $\mathcal{D}^b(\text{mod})$.

Condition (a) implies (b) by Lemma 2.10 (b), which shows that $i_*(B) \in \mathcal{D}^b(\text{mod}A)$ and $j_!$ restricts to $\mathcal{D}^b(\text{mod}C) \rightarrow \mathcal{D}^b(\text{mod}A)$. Conversely, let $X$ be as in Lemma 2.10 (a) such that $j_! = L \otimes_C X$. Under condition (b), the equivalence (ii)$\Leftrightarrow$(iii) of Lemma 2.8 and the implication (i)$\Rightarrow$(ii) of Lemma 2.7 imply that $j_!$ restricts to $\mathcal{D}^b(\text{Mod})$. This completes the proof. \hfill \Box
There is a $\mathcal{D}^b(\text{Mod})$-counterpart of Corollary 4.7.

**Corollary 4.10.** If $C$ has finite global dimension, then the recollement $(\mathcal{R})$ restricts to $\mathcal{D}^b(\text{Mod})$.

**Proof.** Assume that $C$ has finite global dimension. In the proof of Proposition 2.14 it has been shown that $i_*(B) \in K^b(\text{Proj}A)$ and $j_!$ restricts to $K^b(\text{Proj})$. Since $\mathcal{D}^b(\text{Mod}C) \cong K^b(\text{Proj}C)$ and $K^b(\text{Proj}A) \subseteq \mathcal{D}^b(\text{Mod}A)$, it follows that $j_!$ restricts to $\mathcal{D}^b(\text{Mod})$. The desired result is obtained by applying Proposition 4.8. □

4.2.4. Restricting recollements to $\mathcal{D}^-\text{(Mod)}$.

**Proposition 4.11.** The following are equivalent:

(i) the recollement $(\mathcal{R})$ restricts to a recollement

\[ \mathcal{D}^-\text{(Mod}B) \mathcal{R} \mathcal{D}^-\text{(Mod}A) \mathcal{R} \mathcal{D}^-\text{(Mod}C), \]

(ii) $i_*(B) \in K^b(\text{Proj}A)$.

If $k$ is a field and $A$ is finite-dimensional over $k$, then both conditions are equivalent to

(iii) $i_*(B) \in K^b(\text{proj}A)$.

**Proof.** The equivalence between (i) and (ii) is [5, Lemma 4.4]. The rest is as in the proof of Corollary 4.9. □

4.2.5. Special case: $A$ has finite global dimension.

**Proposition 4.12.** If $A$ has finite global dimension, then the recollement $(\mathcal{R})$ restricts to $\mathcal{D}^-\text{(Mod)}$ and $\mathcal{D}^b(\text{Mod})$. If in addition $k$ is a field and $A$ is a finite-dimensional $k$-algebra, the recollement $(\mathcal{R})$ restricts to $\mathcal{D}^b(\text{mod})$ as well.

**Proof.** The first statement follows from the proof of Proposition 2.14 and Corollary 4.10. The second statement follows from Corollary 4.9. □

4.3. An example. The following example [4.13] illustrates, in particular, the lack of symmetry in Corollary 4.7. There are three $\mathcal{D}(\text{Mod})$-recollements. One of them cannot be restricted to any of $\mathcal{D}^-\text{(Mod)}$, $\mathcal{D}^b(\text{Mod})$, $\mathcal{D}^b(\text{mod})$ or $K^b(\text{proj})$. The other two restrict to $\mathcal{D}^-\text{(Mod)}$, with one further restricting to $\mathcal{D}^b(\text{Mod})$ and $\mathcal{D}^b(\text{mod})$ and the other one further restricting to $K^b(\text{proj})$.

**Example 4.13.** Let $k$ be a field and $A$ be the $k$-algebra given by quiver and relations

\[ 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3, \quad \beta^2, \alpha \beta. \]

Denote by $P_i = e_iA$ and $S_i$, respectively, the indecomposable projective and simple module at the vertex $i = 1, 2$. So $P_1$ has composition series $\frac{1}{2}$ and $P_2$ has composition series $\frac{1}{3}$.

As explained in Section 2.2.3 any recollement is generated by a compact exceptional object. So we start out by showing that $P_1$, $P_2$ and $\text{Cone}(P_2 \xrightarrow{\alpha} P_1)$ are the only indecomposable exceptional objects in $K^b(\text{proj}A)$ up to shift.

Let $X$ be an indecomposable exceptional object in $K^b(\text{proj}A)$. Its terms are direct sums of copies of $P_1$ and of $P_2$. If the first and the last non-zero term have a common summand, the identity map on this summand gives a morphism, which is not homotopic to zero, from $X$ to a shifted
copy of $X$. There are no non-zero maps from $P_1$ to $P_2$, and non-zero endomorphisms of $P_1$ are isomorphisms. Thus, if $X$ is chosen minimal and has length at least two, it cannot start with copies of $P_1$. Therefore, we may assume that $X$ is minimal of the form

$$P_2^{n_a} \longrightarrow P_1^{m_{a+1}} \oplus P_2^{n_{a+1}} \longrightarrow \ldots \longrightarrow P_1^{m_b} \oplus P_2^{n_b} \longrightarrow P_1^{m_{b+1}},$$

where $a$ and $b$ are integers indicating the degrees, and $n_a$ and $n_b$ are positive. Let $f : P_2^{n_{b-1}} \to P_2^{n_a}$ be a nonzero morphism which, in matrix form, has entries from $e_2 J e_2$. Then

$$P_2^{n_a} \longrightarrow P_1^{m_{a+1}} \oplus P_2^{n_{a+1}} \longrightarrow \ldots \longrightarrow P_1^{m_{b-1}} \oplus P_2^{n_{b-1}} \longrightarrow P_1^{m_b}$$

is a chain map which is not homotopic to zero. Therefore $\text{Hom}(X, X[a - b]) \neq 0$. Since $X$ is exceptional, this forces $a = b$. It follows that up to shift $X$ is isomorphic to one of the stated objects.

(a) Consider the recollement generated by $P_1 = e_1 A$, which has endomorphism algebra $\text{End}_A(P_1) = e_1 A e_1 \cong k$. As a right $A$-module $A/A e_1 A$ is isomorphic to $P_2$. So by Lemma 2.12, the canonical map $A \to A/A e_1 A \cong k[x]/x^2$ is a homological epimorphism, and the recollement has the form

$$\mathcal{D}(\text{Mod}A/A e_1 A) \leftarrow_{i_* = i} \mathcal{D}(\text{Mod}A) \leftarrow_{i_! = j} \mathcal{D}(\text{Mod} e_1 A e_1).$$

Then

- $i_*(A/A e_1 A) = P_2 \in K^b(\text{proj}A)$,
- the algebra on the right hand side $e_1 A e_1 \cong k$ has global dimension 0,
- $j_*(e_1 A e_1) = S_1 \not\in K^b(\text{proj}A)$.

It follows that the recollement restricts to $D^-(\text{Mod})$ (Proposition 4.11), $D^b(\text{Mod})$ (Corollary 4.10) and $D^b(\text{mod})$ (Corollary 4.7), but not to $K^b(\text{proj})$ (Theorem 4.4).

(b) Consider the recollement generated by $M = \text{Cone}(P_2 \xrightarrow{\alpha} P_1)$. The endomorphism ring $\text{End}(M)$ of $M$ is $k[x]/x^2$, where $x$ is represented by the endomorphism

$$P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} 0.$$ 

The object $M$ admits a real action (not up to homotopy, actually there are no homotopies) by $\text{End}(M)$ on the left.

In order to determine the recollement, we consider the following triangle:

$$M[-1] \longrightarrow A = P_1 \oplus P_2 \longrightarrow P_1 \oplus P_1 \longrightarrow M.$$

Since $M[-1] \in \text{Tria} M$ and $P_1 \oplus P_1 \in (\text{Tria} M)^\perp$, and canonical triangles are unique, this must be the canonical triangle of $A$. It also can be found by the method explained in [4, Appendix].

Clearly $P_1 \oplus P_1 = i^* i_*(A)$ is compact and exceptional with endomorphism algebra $M_2(k)$, the algebra of $2 \times 2$-matrices with entries in $k$. It follows from [4, Proposition 1.7] that the recollement
generated by $M$ is induced by a homological epimorphism $\lambda : A \to M_2(k)$ and has the form

$$
\begin{array}{c}
\xymatrix{
\mathcal{D}(\text{Mod}M_2(k)) & \mathcal{D}(\text{Mod}A) & \mathcal{D}(\text{ModEnd}(M)) \\
\langle i_* \rangle & \langle j_0 \rangle & \langle j_\lambda \rangle
}
\end{array}
$$

where the left column is induced from $\lambda$, cf. Section 2.2.3. Moreover, since $M$ is a complex of $\text{End}(M)$-$A$-bimodules, $j_\lambda \cong \otimes_k \mathcal{L} M$. Then

- it follows from the canonical triangle that $M[-1] = j_\lambda i^! (A) \in K^b(\text{proj}A)$;
- $i_* (M_2(k)) = P_1 \oplus P_1 \in K^b(\text{proj}A)$, by Lemma 2.9(c), using that $M_2(K)$ is semisimple;
- $\text{End}(M)M \notin K^b(\text{End}(M)\text{-proj})$, because $M \cong k \oplus k[1]$ over $\text{End}(M)$, where $k$ is the unique simple $\text{End}(M)$-module;
- $i^! (A) = \mathcal{R}\text{Hom}_A(M_2(k), A) \cong \mathcal{R}\text{Hom}_A(P_1 \oplus P_1, A) \in K^b(\text{proj}M_2(k))$.

It follows that the recollement restricts to $\mathcal{D}^-(\text{Mod})$ (Proposition 4.11) and $K^b(\text{proj})$ (Theorem 4.4), but not to $\mathcal{D}^b(\text{Mod})$, nor to $\mathcal{D}^b(\text{mod})$ (Theorem 4.6 and Corollary 4.9). This is an example where $B$ has finite global dimension but the recollement cannot be restricted to $\mathcal{D}^b(\text{mod})$.

(c) Consider the recollement generated by $P_2 = e_2 A$, whose endomorphism algebra is $\text{End}_A(P_2) = e_2 A e_2 \cong k[x]/x^2$. As a right $A$-module $A/A e_2 A$ is isomorphic to $S_1$ and it has a projective resolution

$$
\ldots \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow A/A e_2 A \longrightarrow 0.
$$

Therefore by Lemma 2.12 the canonical projection $\mu : A \to A/A e_2 A \cong k$ is a homological epimorphism. The recollement has the form

$$
\begin{array}{c}
\xymatrix{
\mathcal{D}(\text{Mod}k) & \mathcal{D}(\text{Mod}A) & \mathcal{D}(\text{Mod}[x]/x^2) \\
\langle i_* \rangle & \langle j_0 \rangle & \langle j_\lambda \rangle
}
\end{array}
$$

where the left column is induced by the projection $\mu$ and $j_\lambda \cong \otimes_k \mathcal{L} P_2$. Then

- $e_2 A e_2 P_2 \cong e_2 A e_2 \in K^b(e_2 A e_2\text{-proj})$;
- $i_* (k) = S_1 \notin K^b(\text{proj}A)$;
- $i^! (A) = k \in K^b(\text{proj}k)$.

It follows that this recollement cannot be restricted to $\mathcal{D}^-(\text{Mod})$ (Proposition 4.11), $\mathcal{D}^b(\text{Mod})$ (Proposition 4.8), $\mathcal{D}^b(\text{mod})$ (Theorem 4.6), nor to $K^b(\text{proj})$ (Theorem 4.3). This shows that $i^! (A)$ being compact or even the algebra on the left side having finite global dimension is not enough for the recollement to restrict to $K^b(\text{proj})$.

To summarise, up to equivalence there are only three non-trivial recollements of $\mathcal{D}(\text{Mod}A)$ by derived module categories. In fact, the three recollements can be put into one (complete) ladder of height 3

$$
\begin{array}{c}
\xymatrix{
\mathcal{D}(\text{Mod}k) & \mathcal{D}(\text{Mod}A) & \mathcal{D}(\text{Mod}[x]/x^2)
}
\end{array}
$$

which corresponds to the TTF quintuple

$$(\text{Tria}(M), \text{Tria}(P_1), \text{Tria}(P_2), \text{Tria}(S_1), \text{Tria}(S_1)^\perp).$$
4.4. **A correction.** We take the opportunity to correct a mistake from [6]. In [6] Section 1.3, it is stated that a homological ring epimorphism \( \varphi : A \to B \) between two finite-dimensional algebras always induces a recollement of \( \mathcal{D}^b(\text{mod}A) \) by \( \mathcal{D}^b(\text{mod}B) \) and a triangulated category \( \mathcal{X} \). This is then used in [6] Theorem 3.3 for showing that a finitely generated tilting module over a finite-dimensional algebra \( A \) induces a recollement of \( \mathcal{D}^b(\text{mod}A) \) by the bounded derived categories \( \mathcal{D}^b(\text{mod}B) \) and \( \mathcal{D}^b(\text{mod}C) \) of two algebras \( B, C \) constructed from \( T \).

Unfortunately, the assumption that \( B \) has finite projective dimension both as a right \( A \)-module and as a left \( A \)-module is missing in both statements. In fact, without this assumption the statements fail: a counterexample is provided by the recollement (b) in Example 4.13 (take the ring epimorphism \( \lambda : A \to M_2(k) \) and the tilting module \( T = A \) with the \( T \)-resolution \( 0 \to A \to A \oplus e_2A \to e_2A \to 0 \)).

Let us now prove the statements under the additional assumption. We remark that the statements were later employed in the context when \( A \) has finite global dimension. Then it holds naturally that \( \text{proj. dim}(B_A) < \infty \) and \( \text{proj. dim}(A_B) < \infty \), thus the remaining results of [6] are not affected by this mistake.

Take a homological ring epimorphism \( \varphi : A \to B \) between two finite-dimensional algebras. It induces a recollement at \( D(\text{Mod}) \)-level

\[
\begin{array}{c}
\mathcal{D}(\text{Mod}B) \\
\uparrow \quad i_* = i^! \quad D(\text{Mod}A) \\
\downarrow \quad i^! \quad \mathcal{X}
\end{array}
\]

where \( i_* \) restricts to \( \mathcal{D}^b(\text{mod}) \). Combining Lemma 2.7 and 2.5 we see that \( i^! \) restricts to \( \mathcal{D}^b(\text{mod}) \) if and only if \( i_* \) restricts to \( K^b(\text{proj}) \), which in turn means that \( B_A \) is a compact object. Moreover, we infer from Lemma 2.8 that \( i^* = ? \otimes_A B \) restricts to \( \mathcal{D}^b(\text{mod}) \) if and only if \( A_B \) is compact.

For the proof of [6] Theorem 3.3, we proceed as in [6] to obtain a recollement at \( D(\text{Mod}) \)-level, and then we apply Theorem 4.6 and the second part of Proposition 3.2 (b), cf. [6] Corollary 2.7.

5. Derived simplicity

In this section, the term ladder refers to a ladder of unbounded derived module categories. A ladder is **trivial** if one of its three terms is trivial. We characterise derived simplicity with respect to different choices of derived categories in terms of heights of ladders (Theorems 5.5, 5.9 and 5.12). Using this characterisation, it is shown by examples that the concept of derived simplicity depends on the choice of derived categories.

Finally, in Section 5.6 we exhibit a large family of derived simple algebras: all indecomposable commutative rings.

5.1. **Restricting recollements along ladders.** Recall that in Example 4.13 there is a ladder of height 3. The upper recollement of the ladder restricts to \( K^b(\text{proj}) \), the middle recollement restricts to \( \mathcal{D}^b(\text{Mod}) \) and \( \mathcal{D}^b(\text{mod}) \), and both the upper and the middle recollements restrict to \( \mathcal{D}^-(\text{Mod}) \). This is a general phenomenon.
Let \( k \) be a commutative ring and suppose \( A, B \) and \( C \) are \( k \)-algebras forming a recollement

\[
\begin{array}{ccc}
\mathcal{D}(\text{Mod}B) & \xleftarrow{i_*} & \mathcal{D}(\text{Mod}A) \\
\xrightarrow{i^*} & & \xrightarrow{j^*} \mathcal{D}(\text{Mod}C).
\end{array}
\] (R)

Recall that the two functors in the upper row always restrict to \( K^b(\text{proj}) \). Following [23], we say that this recollement is **perfect** if \( i_*(B) \in K^b(\text{proj}A) \) holds. By Proposition 4.11, a perfect recollement restricts to \( \mathcal{D}^-(\text{Mod}) \).

A ladder of height 2

\[
\begin{array}{ccc}
\mathcal{D}(\text{Mod}B) & \xleftarrow{i_*} & \mathcal{D}(\text{Mod}A) \\
\xrightarrow{i^*} & & \xrightarrow{j^*} \mathcal{D}(\text{Mod}C),
\end{array}
\]
contains two recollements, which we call the upper and the lower recollement in the obvious sense.

**Lemma 5.1.** Let \( \mathcal{L} \) be a ladder of height 2. Then the upper recollement is a perfect recollement. In particular, it restricts to \( \mathcal{D}^-(\text{Mod}) \).

**Proof.** The middle row of the upper recollement is the upper row of the lower recollement, and hence both functors in this row restrict to \( K^b(\text{proj}) \).

\( \square \)

A ladder of height 3

\[
\begin{array}{ccc}
\mathcal{D}(\text{Mod}B) & \xleftarrow{i_*} & \mathcal{D}(\text{Mod}A) \\
\xrightarrow{i^*} & & \xrightarrow{j^*} \mathcal{D}(\text{Mod}C),
\end{array}
\]
contains three recollements, which we call the upper, the middle and the lower recollement in the obvious sense.

**Proposition 5.2.** Let \( \mathcal{L} \) be a ladder of height 3. Then the upper recollement of \( \mathcal{L} \) restricts to \( K^b(\text{proj}) \) and the middle recollement of \( \mathcal{L} \) restricts to \( \mathcal{D}^b(\text{Mod}) \) and \( \mathcal{D}_{fl} \).

**Proof.** The three rows of the upper recollement are respectively the upper row of the upper, the middle and the lower recollement of \( \mathcal{L} \), and hence all six functors in these three rows restrict to \( K^b(\text{proj}) \). So, the upper recollement restricts to \( K^b(\text{proj}) \). Being the right adjoints of these six functors, the functors in the middle recollement restrict to \( \mathcal{D}^b(\text{Mod}) \) and \( \mathcal{D}_{fl} \), by Lemma 2.7. This shows that the middle recollement of \( \mathcal{L} \) restricts to \( \mathcal{D}^b(\text{Mod}) \) and \( \mathcal{D}_{fl} \).

\( \square \)

5.2. \( \mathcal{D}(\text{Mod}) \)-simplicity. Being \( \mathcal{D}(\text{Mod}) \)-simple is the strongest property among all versions of derived simplicity.

**Proposition 5.3.** Let \( A \) be \( \mathcal{D}(\text{Mod}) \)-simple. Then \( A \) is derived simple with respect to any of \( \mathcal{D}^-(\text{Mod}), \mathcal{D}^b(\text{Mod}), \mathcal{D}_{fl} \) or \( K^b(\text{proj}) \).

**Proof.** Let \( \mathcal{D} = \mathcal{D}^-(\text{Mod}), \mathcal{D}^b(\text{Mod}), \mathcal{D}_{fl} \) or \( K^b(\text{proj}) \). Suppose that \( A \) is not \( \mathcal{D} \)-simple. Then there is a non-trivial recollement of \( A \) on the level \( \mathcal{D} \). By Proposition 4.11 there is a non-trivial recollement of \( A \) on the level of \( \mathcal{D}(\text{Mod}) \), contradicting the assumption.

\( \square \)

In general the concepts of derived simplicity with respect to \( \mathcal{D}(\text{Mod}), \mathcal{D}^-(\text{Mod}), \mathcal{D}^b(\text{Mod}), \mathcal{D}_{fl} \) and \( K^b(\text{proj}) \) are different. However, for algebras of finite global dimension, some of them coincide, as a consequence of Proposition 4.12.
Corollary 5.4. Let $A$ be a $k$-algebra of finite global dimension. Then $A$ is $\mathcal{D}(\text{Mod})$-simple if and only if it is $\mathcal{D}^{-}(\text{Mod})$-simple if and only if it is $\mathcal{D}^{b}(\text{Mod})$-simple. If in addition $k$ is a field and $A$ is finite-dimensional over $k$, then the derived simplicity of $A$ does not depend on the choice of the derived category.

In Section 5.6 we will provide new examples of $\mathcal{D}(\text{Mod})$-simple algebras.

5.3. $\mathcal{D}^{-}(\text{Mod})$-simplicity. This means that all ladders have height at most one:

Theorem 5.5. Let $A$ be a $k$-algebra. Consider the following conditions:

(i) $A$ is $\mathcal{D}^{-}(\text{Mod})$-simple,
(ii) there are no non-trivial perfect recollements of the form $(\mathcal{R})$,
(iii) every non-trivial ladder of $\mathcal{D}(\text{Mod}A)$ has height $\leq 1$.

Then $(i) \Rightarrow (ii) \Leftrightarrow (iii)$. If $k$ is a field and $A$ is finite-dimensional over $k$, then all three conditions are equivalent.

Proof. $(i) \Rightarrow (ii)$ Suppose that $A$ has a recollement of the form $(\mathcal{R})$ with $i_{*}(B) \in K^{b}(\text{proj}A)$. By Proposition 4.11 this recollement restricts to $\mathcal{D}^{-}(\text{Mod})$, yielding a non-trivial recollement on the $\mathcal{D}^{-}(\text{Mod})$-level and implying that $A$ is not $\mathcal{D}^{-}(\text{Mod})$-simple.

$(ii) \Rightarrow (iii)$ Suppose that $L$ is a non-trivial ladder of $\mathcal{D}(\text{Mod}A)$ of height 2. By Lemma 5.1 the upper recollement of $L$ is a non-trivial perfect recollement.

$(iii) \Rightarrow (ii)$ Suppose there is a non-trivial recollement of the form $(\mathcal{R})$ with $i_{*}(B) \in K^{b}(\text{proj}A)$. By Proposition 3.2(a), this recollement, viewed as a ladder of height 1, can be extended downwards by one step, yielding a non-trivial ladder of height 2.

$(ii) \Rightarrow (i)$ Let $k$ be a field and $A$ be a finite-dimensional $k$-algebra. Suppose that $A$ is not $\mathcal{D}^{-}(\text{Mod})$-simple, i.e., there is a non-trivial recollement of $A$ on the $\mathcal{D}^{-}(\text{Mod})$-level. By Proposition 4.1 there is a non-trivial recollement of the form $(\mathcal{R})$ which restricts to $\mathcal{D}^{-}(\text{Mod})$. By Proposition 4.11 $i_{*}(B) \in K^{b}(\text{proj}A)$, i.e. this recollement is a non-trivial perfect recollement.

For general algebras, it is not the case that any recollement of the form $(\mathcal{R})$ which can be restricted to $\mathcal{D}^{-}(\text{Mod})$ is part of a ladder of height 2, as shown in the next example.

Example 5.6. (Example 4.5 continued.) Since $j_{!}(k) = eA \not\in \mathcal{D}_{fl}(A)$, it follows from Lemma 2.7 and Lemma 2.9(e) that $j_{!}$ does not admit a left adjoint, and hence the recollement cannot be extended upwards (Proposition 3.2(b)). Moreover, $j^{*}(A) \not\in K^{b}(\text{proj}k)$, equivalently, $i_{*}(A/AeA) \not\in K^{b}(\text{proj}A)$, implying that the recollement cannot be extended downwards (Proposition 3.2(a)). So the recollement is a complete ladder of height 1. However, $AeA$ is projective, so it follows by the short exact sequence

$$0 \longrightarrow AeA \longrightarrow A \longrightarrow A/AeA \longrightarrow 0$$

that $i_{*}(A/AeA) \in K^{b}(\text{Proj}A)$, which implies that the recollement restricts to $\mathcal{D}^{-}(\text{Mod})$ (Proposition 4.11). In fact, the recollement restricts further to $\mathcal{D}^{b}(\text{Mod})$ (Corollary 4.10).

It is proved in [35] that finite-dimensional symmetric algebras over a field do not admit any non-trivial perfect recollement, see [35, Remark 4.3]. Hence:

Theorem 5.7. Let $k$ be a field and $A$ be a connected (i.e. indecomposable as an algebra) finite-dimensional symmetric $k$-algebra. Then $A$ is $\mathcal{D}^{-}(\text{Mod})$-simple.
Below we construct a $\mathcal{D}^-(\text{Mod})$-simple finite-dimensional algebra which is not $\mathcal{D}(\text{Mod})$-simple.

**Example 5.8.** Let $k$ be a field and let $A$ be the $k$-algebra given by the quiver with relations

$$\alpha \begin{array}{ccc} 1 & \xrightarrow{\gamma} & 2 \\ \downarrow & \cdots & \downarrow \\ \delta, & \beta \gamma \beta, & \alpha^2, \gamma \alpha, \delta^2, \delta \gamma. \end{array}$$

This algebra is 14-dimensional and the composition series of the two indecomposable projectives $P_1$ and $P_2$ are depicted as follows

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 1 & 2 & 2 \\
\end{array}
\]

We claim that, up to shift and isomorphism, $P_1$ and $P_2$ are the only indecomposable compact exceptional objects in $\mathcal{D}(\text{Mod}A)$. Indeed, let $X$ be an indecomposable exceptional object of $K^b(\text{proj}A)$, minimal of the form

$$P_1^m \oplus P_2^m \rightarrow P_1^m \oplus P_2^m \rightarrow \cdots \rightarrow P_1^m \oplus P_2^m,$$

where one of $m$ and $n$ is nonzero and one of $m$ and $n$ is nonzero. Assume that $m \neq 0$ and $n \neq 0$: the other cases can be treated similarly. Consider the morphism $f : P_2 \rightarrow P_1$, which maps the top of $P_2$ to the last radical layer of $P_1$. The map $g : P_2 \rightarrow P_1$ which, in matrix form, has all entries $f$, induces a self-extension of $X$ in degree $b - a$:

$$\begin{array}{ccc}
\cdots & \rightarrow & P_1^{mb-1} \oplus P_2^{mb-1} \rightarrow P_1^{mb} \oplus P_2^{mb} \\
& | & \\
& \left( \begin{array}{cc} 0 & g \\ 0 & 0 \end{array} \right) & \\
& & \\
& & P_1^{ma} \oplus P_2^{ma} \rightarrow P_1^{ma+1} \oplus P_2^{ma+1} \rightarrow \cdots
\end{array}$$

The object $X$ being exceptional implies that $a = b$, and further, either $m = 0$ and $n = 1$, or vice versa.

Next we show that each of $P_1$ and $P_2$ generates a recollement. Consider the case for $P_1 = e_1 A$. As a right $A$-module the quotient $A/Ae_1 A$ admits the following projective resolution

$$\cdots \rightarrow P_1 \rightarrow P_1 \rightarrow P_2 \rightarrow A/Ae_1 A \rightarrow 0.$$ 

Thus by Lemma 2.12, $Ae_1 A$ is a stratifying ideal, and hence $P_1$ generates a recollement of $\mathcal{D}(\text{Mod}A)$ by $\mathcal{D}(\text{Mod}A)/Ae_1 A \cong \mathcal{D}(\text{Mod}k[x]/(x^2))$ and $\mathcal{D}(\text{Mod}Ae_1) \cong \mathcal{D}(\text{Mod}k(x,y)/(x^2,y^2,x,y)) (e_1 A = k \{ e_1, \beta \gamma, \alpha, \alpha \beta \gamma \}).$

The case for $P_2$ is similar: it generates a recollement of $\mathcal{D}(\text{Mod}A)$ by $\mathcal{D}(\text{Mod}Ae_2 A) \cong \mathcal{D}(\text{Mod}k[x]/(x^2))$ and $\mathcal{D}(\text{Mod}e_2 A \cong \mathcal{D}(\text{Mod}k(x,y)/(x^2,y^2,x,y)) (e_2 A = k \{ e_2, \gamma, \delta \gamma \}).$

In particular, $A$ is not $\mathcal{D}(\text{Mod})$-simple. However, the two recollements are not in the same ladder, and hence both recollements are already complete ladders. So all non-trivial ladders of $\mathcal{D}(\text{Mod}A)$ have height 1. By Theorem 5.9, $A$ is $\mathcal{D}^-(\text{Mod})$-simple.

### 5.4. $K^b(\text{proj})$-simplicity.

This means that all ladders have height at most two:

**Theorem 5.9.** Let $A$ be a $k$-algebra. The following are equivalent:

(i) $A$ is $K^b(\text{proj})$-simple,

(ii) all non-trivial ladders of $\mathcal{D}(\text{Mod}A)$ have height $\leq 2$. 

Example 5.10. Let $k$ be a field and let $A$ be the radical square zero $k$-algebra whose quiver is

$$
\begin{array}{ccc}
\gamma & 1 & \alpha \\
\alpha & 2 & \beta \\
\beta & 0 & 0
\end{array}
$$

with indecomposable projective modules $P_1 = 1 \begin{array}{c} 1 \\
\alpha \\
\beta \\
0
\end{array} 2$ and $P_2 = \begin{array}{c} 2 \\
\alpha \\
\beta \\
0
\end{array} 0$. As in Example 4.13, it can be checked that $P_1, P_2,$ and $X := \text{Cone}(P_2 \xrightarrow{\alpha} P_1)$ are the only indecomposable exceptional compact objects, up to shift and up to isomorphism. We will show that $X$ does not generate a recollement of derived module categories, while $P_1$ and $P_2$ are in the same ladder.

Consider the recollement of $D(\text{Mod}A)$ generated by $X = \text{Cone}(P_2 \rightarrow P_1)$. The endomorphism ring of $X$ is $\text{End}_A(X) = k[x,y]/(x^2, y^2, xy) =: C$, where

$$
\begin{array}{ccc}
x_1 & P_2 & P_1 \\
\alpha & \beta \\
0 & 0 \\
0 & 0
\end{array}
$$

As a complex of left $C$-modules, $X$ is isomorphic to $k[x]/x^2 \xrightarrow{\alpha} k \oplus k[y]/y^2$, where the underlined term is in degree 0, $k[x]/x^2$ and $k[y]/y^2$ are identified as quotients of $C$ by factoring out the ideal generated by $y$ and $x$, respectively, and $\alpha$ is the projection onto the trivial $C$-module $k$. This complex splits into the direct sum of $k[1]$ and $k[y]/y^2$. It is straightforward to show that as a complex of right $C$-modules $X^{tr} := \text{Hom}_A(X, A)$ is isomorphic to $k[y]/y^2 \rightarrow k \oplus k[x]/x^2$, which splits into the direct sum of $k$ and $(k[x]/x^2)[1]$. Hence the total cohomology of $X^{tr} \otimes_C X$ is infinite-dimensional. In particular it does not belong to $D^b(\text{mod}A)$. By Lemma 2.10 the recollement

$$
\begin{array}{ccc}
D' & \xrightarrow{\iota} & D(\text{Mod}A) \\
\iota \xrightarrow{\gamma} & j_1 & \iota \xrightarrow{\gamma} D(\text{Mod}C)
\end{array}
$$

generated by $X$ cannot be a recollement of ordinary algebras. More precisely the right perpendicular category $D'$ of $X$ is not a derived category of any ordinary algebra.

Consider the recollement generated by $P_1 = e_1 A$, whose endomorphism ring $\text{End}_A(P_1) = e_1 A e_1$ is isomorphic to $k[x]/x^2$. As a right $A$-module $A/e_1 A$ is isomorphic to $P_2$. Thus it follows from Lemma 2.12 that $P_1$ generates a recollement of $D(\text{Mod}A)$ by $D(\text{Mod}A/e_1 A) \cong D(k[x]/x^2)$ and $D(\text{Mod}e_1 A) \cong D(k[x]/x^2)$. The corresponding TTF triple is $(\text{Tria}(P_1), \text{Tria}(P_2), \text{Tria}(P_2)^\perp)$.
Consider the recollement generated by $P_2 = e_2A$. As a right $A$-module $A/Ae_2A$ admits the following projective resolution

$$\ldots \rightarrow P_2 \xrightarrow{\beta} P_2 \xrightarrow{\beta} P_2 \xrightarrow{\alpha} P_1 \rightarrow A/Ae_2A \rightarrow 0.$$ 

Thus it follows from Lemma 2.12 that $P_2$ generates a recollement of $\mathcal{D}(\text{Mod}A)$ by $\mathcal{D}(\text{Mod}A/Ae_2A) \cong \mathcal{D}(k[x]/x^2)$ and $\mathcal{D}(\text{Mod}_A e_2A) \cong \mathcal{D}(k[x]/x^2)$. The TTF triple corresponding to this recollement is $(\text{Tria}(P_2), \text{Tria}(A/Ae_2A), \text{Tria}(A/Ae_2A)^{\perp})$.

Clearly the above two TTF triples together form one TTF quadruple $(\text{Tria}(P_1), \text{Tria}(P_2), \text{Tria}(A/Ae_2A), \text{Tria}(A/Ae_2A)^{\perp})$, which is complete. Since this is the unique non-trivial TTF tuple of $\mathcal{D}(\text{Mod}A)$, it follows that $A$ is $K^b(\text{proj})$-simple but not $\mathcal{D}^-(\text{Mod})$-simple.

5.5. $\mathcal{D}^b(\text{Mod})$-simplicity and $\mathcal{D}^{fl}$-simplicity. The following proposition follows immediately from Proposition 5.11.

**Proposition 5.11.** Let $A$ be a $k$-algebra. If $A$ is $\mathcal{D}^b(\text{Mod})$-simple or $\mathcal{D}^{fl}$-simple, then all non-trivial ladders of $\mathcal{D}^-(\text{Mod}A)$ have height $\leq 2$.

**Theorem 5.12.** Let $k$ be a field and $A$ be finite-dimensional over $k$. The following are equivalent:

(i) $A$ is $\mathcal{D}^b(\text{Mod})$-simple,

(ii) $A$ is $\mathcal{D}^b(\text{mod})$-simple,

(iii) $A$ is $K^b(\text{proj})$-simple,

(iv) all non-trivial ladders of $\mathcal{D}(\text{Mod}A)$ have height $\leq 2$.

**Proof.** Recall that in this case $\mathcal{D}^{fl}(A) = \mathcal{D}^b(\text{mod}A)$.

(i)⇔(ii) This follows from Proposition 4.11 and Corollary 4.9.

(iii)⇔(iv) This is Theorem 5.9.

(i)⇒(iv) This follows from Proposition 5.11.

(iv)⇒(ii) Suppose that $A$ is not $\mathcal{D}^b(\text{mod})$-simple. Then there is a non-trivial recollement of $A$ on $\mathcal{D}^b(\text{mod})$-level. By Proposition 4.11 this lifts to a recollement of the form $(R)$ where $j_i$ restricts to $\mathcal{D}^b(\text{mod})$ and $i_1(B) \in K^b(\text{proj}A)$, see Theorem 4.6. But then this recollement of $\mathcal{D}(\text{Mod}A)$ extends to a non-trivial ladder of height 3 by Proposition 3.2(a) and (b). □

5.6. Indecomposable commutative rings are derived simple. Let $A$ be a commutative ring. Given $p \in \text{Spec}A$ and a complex of $A$-modules $X : \ldots \rightarrow X^i \xrightarrow{d^i} X^{i+1} \rightarrow \ldots$ we consider the complex $X_p : \ldots \rightarrow X^i \otimes_A A_p \xrightarrow{d^i \otimes_A A_p} X^{i+1} \otimes_A A_p \rightarrow \ldots$

Since $? \otimes_A A_p$ is an exact functor, we have $H^i(X_p) = H^i(X) \otimes_A A_p$.

**Lemma 5.13.** For $X \in K^b(\text{proj}A)$, $Y \in \mathcal{D}(\text{Mod}A)$, $p \in \text{Spec}A$ and any integer $n$, there is an isomorphism $\text{Hom}_{\mathcal{D}(\text{Mod}A)}(X, Y[p]) \otimes_A A_p \cong \text{Hom}_{\mathcal{D}(\text{Mod}A_p)}(X_p, Y[p])$. 

Proof. $\text{Hom}_{D(\text{Mod}A)}(X, Y[n]) = \text{Hom}_{K(\text{Mod}A)}(X, Y[n])$ is the $n$-th cohomology of the total complex $\text{Hom}_A(X, Y)$. The well-known formula [21, 3.2.4]

$$\text{Hom}_A(X^i, Y^j) \otimes A_p \simeq \text{Hom}_{A_p}(X^i_p, Y^j_p)$$

implies that $\text{Hom}_A(X, Y)_p \simeq \text{Hom}_{A_p}(X_p, Y_p)$. The claim now follows from the fact that localisation preserves cohomologies and $X_p \in K^b(P_A)$. □

**Proposition 5.14.** If $X$ is a compact object in $D(\text{Mod}A)$ of length $n$, then

$$\text{Hom}_{D(\text{Mod}A)}(X, X[n]) \neq 0$$

Proof. We can assume that $X$ has a $K^b(\text{proj}A)$-representative of the form $\ldots \rightarrow P^{-n} \rightarrow \cdots \rightarrow P^0 \rightarrow 0 \ldots$. By assumption any split embedding of $P^{-n}$ into a finitely generated free module $F$ gives rise to a chain map $X \rightarrow F[n]$ which is not homotopic to zero, so $\text{Hom}_{D(\text{Mod}A)}(X, A[n]) \neq 0$. By Lemma 5.13 there is $p \in \text{Spec} - A$ such that $\text{Hom}_{D(\text{Mod}A)}(X_p, A_p[n]) \neq 0$. Then also $X_p$ is a compact complex of length $n$ over the local ring $A_p$, and as in [5, 4.9] we obtain $\text{Hom}_{D(\text{Mod}A_p)}(X_p, X_p[n]) \neq 0$. The claim now follows by applying Lemma 5.13 again. □

The Proposition shows that every compact exceptional object in $D(\text{Mod}A)$ is projective up to shift. In particular:

**Theorem 5.15.** Every indecomposable commutative ring $A$ is derived simple with respect to $D(\text{Mod})$.

Proof. As recalled in 2.2.2, every recollement of the form (R) is generated by a compact exceptional object, hence by an object of the form $P[n]$ where $P$ is a finitely generated projective module and $n \in \mathbb{Z}$. This means that the right hand side in (R) equals $\text{Tria} P[n] = \text{Tria} P$. By [33, 2.44], the trace $\tau_P(A)$ of $P$ in the commutative ring $A$ is a direct summand of $A$, thus either zero or equal to $A$. But then $P$ is zero or a finitely generated projective generator of $\text{Mod}A$, in particular, $\text{Tria} P$ equals zero or $D(\text{Mod}A)$, and the claim is proven. □

Moreover, we recover a result from [20, 42] as a special case.

**Corollary 5.16.** Every finitely generated tilting module over a commutative ring is projective.

Proof. It is well known that every finitely generated tilting module $T$ has a projective resolution with finitely generated projective modules (e.g. by combining the fact that every tilting class is definable [8] with [3, 9.13 (5)]). Then $T$ is a compact exceptional object, so it is projective. □

Here is another consequence of the Proposition above.

**Corollary 5.17.** A commutative ring $A$ is derived equivalent to a ring $B$ if and only if $A$ and $B$ are Morita equivalent.

Proof. By a well known result due to Rickard [43], $A$ and $B$ are derived equivalent if and only if $B$ is the endomorphism ring of a tilting complex $T$ over $A$. But, as shown above, $T$ is of the form $P[n]$ where $P$ is a finitely generated projective generator of $\text{Mod}A$ and $n \in \mathbb{Z}$. Hence $B \simeq \text{End}_A(P)$ is Morita equivalent to $A$. □
As a consequence, the derived Picard group is the direct product of the infinite cyclic group generated by the shift in the derived category and the classical Picard group, which describes the Morita equivalences.

Finally we briefly mention a result on co-t-structures and leave the details to the interested readers. Let $A$ be an indecomposable commutative ring. As in the proof of Theorem 5.1 of the associated triangulated category $D = 0$, the group $i$ (full subcategory of $D_{-1}$) of Frobenius pairs and K-groups. A algebra for any integer $i$ Chen and Xi to show that the $K$-groups of the two outer algebras. Moreover, since any object of the co-heart of a co-t-structure on $K^b(\proj A)$ is presilting, it follows that if the co-heart is non-trivial, then it is $(\proj A)/n$ for some integer $n$, in particular, the co-t-structure is bounded. Consequently, by [35, 5.9], any co-t-structure on $K^b(\proj A)$ with non-trivial co-heart is a shift of the standard one, compare [35, 5.6].

6. Algebraic $K$-theory

This section is devoted to the study of $K$-groups. We restrict to finite-dimensional algebras. In this case Schlichting’s ($-1$)-st $K$-group vanishes. Using this result and a result on silting objects we show that if a $D(\Mod)$-recollement is given, the Grothendieck group of the middle algebra is the direct sum of those of the two outer algebras. For a $D^-(\Mod)$-recollement, we use a method of Chen and Xi to show that the $i$-th $K$-group is the direct sum of the $i$-th $K$-groups of the two outer algebras for any integer $i$.

6.1. Frobenius pairs and $K$-groups. We follow [46]. A Frobenius pair is a pair $(\mathcal{A}, \mathcal{A}_0)$, where $\mathcal{A}$ is a small Frobenius category and $\mathcal{A}_0$ is a full Frobenius subcategory, i.e. an extension closed full subcategory of $\mathcal{A}$ which inherits the structure of a Frobenius category. To a Frobenius pair $(\mathcal{A}, \mathcal{A}_0)$ and $i \in \mathbb{Z}$ we associate the $i$-th $K$-group $K_i(\mathcal{A}, \mathcal{A}_0)$ of $(\mathcal{A}, \mathcal{A}_0)$, see [46, Section 12]. For $i = 0$, the group $K_0(\mathcal{A}, \mathcal{A}_0)$ is the Grothendieck group of the idempotent completion ([7, Definition 1.2]) of the associated triangulated category $D(\mathcal{A}, \mathcal{A}_0) = \mathcal{A}/\mathcal{A}_0$.

For example, for a finite-dimensional algebra $A$ over a field $k$, let $C^b(\mod A)$ be the category of bounded complexes of $A$-modules from $\mod A$ and $\acyc^b(\mod A)$ be its full subcategory of acyclic complexes. Then $K_0(C^b(\mod A), \acyc^b(\mod A)) = K_0(D^b(\mod A))$, which is isomorphic to the usual Grothendieck group $K_0(A)$ of the algebra $A$, cf. [24, III.1].

A functor of Frobenius pairs $(\mathcal{A}, \mathcal{A}_0) \to (B, B_0)$ is a functor $\mathcal{A} \to B$ of Frobenius categories which restricts to a functor $\mathcal{A}_0 \to B_0$. A sequence $(\mathcal{A}, \mathcal{A}_0) \to (B, B_0) \to (C, C_0)$ of Frobenius pairs is a short exact sequence if the induced sequence of triangulated categories $D(\mathcal{A}, \mathcal{A}_0) \xrightarrow{i} D(B, B_0) \xrightarrow{p} D(C, C_0)$ is short exact, i.e. $i$ is fully faithful and $p$ induces an equivalence $D(B, B_0)/\text{im}(i) \xrightarrow{\sim} D(C, C_0)$ up to direct summands.

Theorem 6.1. ([46, Theorem 9]) Let $(\mathcal{A}, \mathcal{A}_0) \to (B, B_0) \to (C, C_0)$ be a short exact sequence of Frobenius pairs. Then there is a long exact sequence of $K$-groups

$$\ldots \to K_i(\mathcal{A}, \mathcal{A}_0) \to K_i(B, B_0) \to K_i(C, C_0) \to K_{i-1}(\mathcal{A}, \mathcal{A}_0) \to K_{i-1}(B, B_0) \to K_{i-1}(C, C_0) \to \ldots$$

From now on, let $k$ be a field and let $A$ be a finite-dimensional $k$-algebra. Recall that $\proj A$ denotes the category of finitely generated projective $A$-modules. By abuse of notation, we will also denote by $\proj A$ its skeleton, which by definition consists of one representative from each isomorphism class of objects. Let $C^b(\proj A)$ be the category of bounded complexes of finitely generated
projective $A$-modules. It has a natural structure of a Frobenius category: the conflations are the componentwise split short exact sequence of complexes. Let $C^b_0(proj A)$ be the full subcategory of $C^b(proj A)$ consisting of null-homotopic complexes. Then $(C^b(proj A), C^b_0(proj A))$ is a Frobenius pair and we call it a Frobenius model of $K^b(proj A)$ since the associated triangulated category $C^b(proj A)/C^b_0(proj A) = K^b(proj A)$. We denote $K_i(A) = K_i(C^b(proj A), C^b_0(proj A))$.

6.2. Vanishing of $K_{-1}$. Recall that mod$A$ denotes the category of finitely generated $A$-modules. By abuse of notation, we will also denote by mod$A$ its skeleton. Define the singularity category $D_{sg} (A)$ as the triangle quotient $D^b(mod A)/K^b(proj A)$.

Let $C^b_0(mod A)$ be the Frobenius subcategory of $C^b(mod A)$ corresponding to the essential image of the embedding $K^b(proj A) \hookrightarrow D^b(mod A)$. Then $(C^b(mod A), acyc^b(mod A))$ and $(C^b(mod A), C^b_0(mod A))$ are respectively Frobenius models of $D^b(mod A)$ and $D_{sg}(A)$. Therefore we have a short exact sequence of Frobenius pairs

$$(C^b(proj A), C^b_0(proj A)) \longrightarrow (C^b(mod A), acyc^b(mod A)) \longrightarrow (C^b(mod A), C^b_0(mod A)),$$

which induces a long exact sequence

$$\ldots \longrightarrow K_0(A) \longrightarrow K_0(C^b(mod A), acyc^b(mod A)) \longrightarrow K_0(C^b(mod A), C^b_0(mod A)) \longrightarrow K_{-1}(A) \longrightarrow \ldots$$

By [46] Theorem 6], $K_{-1}(C^b(mod A), acyc^b(mod A))$ vanishes. As a consequence, $K_{-1}(A)$ is exactly the obstruction of the idempotent completeness of $D_{sg}(A)$.

Proposition 6.2. $K_{-1}(A)$ vanishes if and only if $D_{sg}(A)$ is idempotent complete.

Proof. We obtain from above an exact sequence

$$K_0(C^b(mod A), acyc^b(mod A)) \xrightarrow{p} K_0(C^b(mod A), C^b_0(mod A)) \longrightarrow K_{-1}(A) \longrightarrow 0.$$ 

Therefore $K_{-1}(A) = 0$ holds if and only if $p$ is surjective. The latter condition is satisfied if and only if $D_{sg}(A)$ is idempotent complete, see for example [46] Remark 1].

Corollary 6.3. $K_{-1}(A) = 0$.

Proof. By [18] Corollary 2.4, $D_{sg}(A)$ is idempotent complete. The desired result follows immediately from Proposition 6.2.□

6.3. The long exact sequence. For our purpose it will be useful to employ another Frobenius model for $K^b(proj A)$. Let $C^{-\epsilon}(proj A)$ denote the category of right bounded complexes which are homotopy equivalent to complexes in $C^b(proj A)$ and let $C^{-\epsilon}_0(proj A)$ denote its full subcategory consisting of null-homotopic complexes. Let $K^{-\epsilon}(proj A)$ denote the stable category of $C^{-\epsilon}(proj A)$. Then the canonical embedding $(C^b(proj A), C^b_0(proj A)) \rightarrow (C^{-\epsilon}(proj A), C^{-\epsilon}_0(proj A))$ induces a triangle equivalence $K^b(proj A) \rightarrow K^{-\epsilon}(proj A)$, and by [46] Theorem 9], we have a canonical isomorphism for $i \in \mathbb{Z}$

$$K_i(A) = K_i(C^b(proj A), C^b_0(proj A)) \cong K_i(C^{-\epsilon}(proj A), C^{-\epsilon}_0(proj A)).$$

Assume that there is a recollement of the form [8]. Recall from Lemma 2.10 and Lemma 2.9 that $B$ and $C$ are necessarily finite-dimensional over $k$, the functors $i^*$ and $j_!$ restrict to $K^b(proj)$, and
Moreover, there is a right bounded complex of finitely generated projective $C$-$A$-bimodules $X$ such that $j_1 = L \otimes_C X = ? \otimes_C X$. As a complex of $A$-modules, $X$ belongs to $\mathcal{C}^{-c}(\text{proj} A)$ and it follows that $? \otimes_C X : \mathcal{C}^{-c}(\text{proj} C) \rightarrow \mathcal{C}^{-c}(\text{proj} A)$ is a well-defined functor of Frobenius categories, which induces a functor $(\mathcal{C}^{-c}(\text{proj} C), \mathcal{C}_0^{-c}(\text{proj} C)) \rightarrow (\mathcal{C}^{-c}(\text{proj} A), \mathcal{C}_0^{-c}(\text{proj} A))$ of Frobenius pairs. Similarly, there is a right bounded complex of finitely generated $A$-$B$-bimodules $Y$ such that $i^* = ? \otimes_A Y$ induces a functor $(\mathcal{C}^{-c}(\text{proj} A), \mathcal{C}_0^{-c}(\text{proj} A)) \rightarrow (\mathcal{C}^{-c}(\text{proj} B), \mathcal{C}_0^{-c}(\text{proj} B))$ of Frobenius pairs. So, we obtain a sequence of Frobenius pairs

$$(\mathcal{C}^{-c}(\text{proj} C), \mathcal{C}_0^{-c}(\text{proj} C)) \rightarrow (\mathcal{C}^{-c}(\text{proj} A), \mathcal{C}_0^{-c}(\text{proj} A)) \rightarrow (\mathcal{C}^{-c}(\text{proj} B), \mathcal{C}_0^{-c}(\text{proj} B)).$$

We claim that this is a short exact sequence. In fact, by [37] Theorem 2.1, there is an equivalence of triangulated categories up to direct summands

$$K^b(\text{proj} A)/\text{tria}(j_1(C)) \xrightarrow{\sim} K^b(\text{proj} B) \quad (6.1)$$

which is even an equivalence, because $\mathbb{K}_{-1}(C) = 0$ and $K^b(\text{proj} A)/\text{tria}(j_1(C))$ is thus idempotent complete by [46] Remark 1.

It follows from [46] Theorem 9 that there is a long exact sequence for $i \in \mathbb{Z}$

$$\ldots \rightarrow \mathbb{K}_i(C) \rightarrow \mathbb{K}_i(A) \rightarrow \mathbb{K}_i(B) \rightarrow \mathbb{K}_{i-1}(C) \rightarrow \mathbb{K}_{i-1}(A) \rightarrow \mathbb{K}_{i-1}(B) \rightarrow \ldots$$

By Corollary [6.3] the groups $\mathbb{K}_1(A)$, $\mathbb{K}_1(B)$ and $\mathbb{K}_1(C)$ are trivial. Thus we have the following corollary.

**Corollary 6.4.** Let $A$, $B$ and $C$ be finite-dimensional $k$-algebras admitting a recollement of the form [37]. Then there are long exact sequences of $K$-groups

$$\ldots \rightarrow \mathbb{K}_i(C) \rightarrow \mathbb{K}_i(A) \rightarrow \mathbb{K}_i(B) \rightarrow \ldots \rightarrow \mathbb{K}_0(C) \rightarrow \mathbb{K}_0(A) \rightarrow \mathbb{K}_0(B) \rightarrow 0, \quad i \geq 0,$n

$$0 \rightarrow \mathbb{K}_{-2}(C) \rightarrow \mathbb{K}_{-2}(A) \rightarrow \mathbb{K}_{-2}(B) \rightarrow \ldots \rightarrow \mathbb{K}_i(C) \rightarrow \mathbb{K}_i(A) \rightarrow \mathbb{K}_i(B) \rightarrow \ldots, \quad i \leq -2.$n

6.4. The Grothendieck group. Let $k$ be a field and let $A$, $B$ and $C$ be finite-dimensional $k$-algebras admitting a recollement of the form [37]. We can assume that $A$, $B$ and $C$ are basic.

Denote by $r(A)$ the number of isomorphism classes of simple $A$-modules, which equals the rank of the Grothendieck group $\mathbb{K}_0(A)$.

**Proposition 6.5.** Let $A$, $B$ and $C$ be basic finite-dimensional $k$-algebras admitting a recollement of the form [37]. Then $r(A) = r(B) + r(C)$. In particular, there is a short exact sequence

$$0 \rightarrow \mathbb{K}_0(C) \rightarrow \mathbb{K}_0(A) \rightarrow \mathbb{K}_0(B) \rightarrow 0.$$

We need some preparation. An object $X$ of $K^b(\text{proj} A)$ is called a presilting object if $\text{Hom}(X, \Sigma^i X) = 0$ for any $i > 0$ and a silting object if in addition $K^b(\text{proj} A) = \text{tria}(X)$. The following is a special case of a result of [27].

**Proposition 6.6.** Let $X$ be a basic presilting object of $K^b(\text{proj} A)$. Then the quotient functor $\pi : K^b(\text{proj} A) \rightarrow K^b(\text{proj} A)/\text{tria}(X)$ induces a one-to-one correspondence between the set of isomorphism classes of basic silting objects in $K^b(\text{proj} A)$ containing $X$ as a direct summand and the set of isomorphism classes of basic silting objects in $K^b(\text{proj} A)/\text{tria}(X)$. Let $T$ be an element of the former set; then $\text{End}_{K^b(\text{proj} A)/\text{tria}(X)}(\pi(T))$ is isomorphic to the quotient of $\text{End}_{K^b(\text{proj} A)}(T)$ by the ideal generated by the idempotent corresponding to $X$. 
Corollary 6.8. Let $A$ be a stratification with simple factors $A_1, \ldots, A_s$. Then there are isomorphisms of $K$-groups $K_i(A) \cong \bigoplus_{j=1}^s K_i(A_j)$ for $i \in \mathbb{Z}$. 

Proof. Observe that $j_i(C)$ is a basic presilting object in $K^b(projA)$ (it is actually a partial tilting object). Let $\pi$ denote the composition of the quotient functor $\pi : K^b(projA) \to K^b(projA)/\text{tria}(j_i(C))$ and the triangle equivalence $\delta_i$. By Proposition 6.6, there is a silting object $T$ in $K^b(projA)$ which contains $j_i(C)$ as a direct summand such that $\pi(T) = B$. Then $B = \text{End}_{K^b(projA)}(B) = \text{End}_{K^b(projA)}(T)/(e)$, where $e$ is the idempotent corresponding to the direct summand $X$ of $T$, and $(e)$ is the ideal generated by $e$. Hence

$$r(B) = r(\text{End}_{K^b(projA)}(T)) - r(e \text{End}_{K^b(projA)}(T)e) = r(\text{End}_{K^b(projA)}(T)) - r(C).$$

Since $r(\text{End}_{K^b(projA)}(T))$ equals the number of indecomposable direct summands of $T$, which equals $r(A)$ by [2, Theorem 2.26], it follows that $r(A) = r(B) + r(C)$, as desired. \hfill \Box

6.5. Decomposing $K$-groups along recollements. The following theorem has been motivated and inspired by the results in [15], which it complements and strengthens in the case of finite-dimensional algebras. We are indebted to Changchang Xi for informing us about these results.

Theorem 6.7. Let $A$, $B$ and $C$ be finite-dimensional $k$-algebras admitting a $D^-(\text{Mod})$-recollement of the form $[R]$. Then there are isomorphisms $K_i(A) \cong K_i(B) \oplus K_i(C)$ for $i \in \mathbb{Z}$.

Proof. By Proposition 4.11, we may view the given recollement as a recollement restricted from a $D(\text{Mod})$-recollement. As in Section 6.3, consider the short exact sequence of Frobenius pairs

$$((C^{-c}(projC), C_0^{-c}(projC))) \xrightarrow{\iota} (C^{-c}(projA), C_0^{-c}(projA)) \xrightarrow{\pi} (C^{-c}(projB), C_0^{-c}(projB)), $$

which induces a short exact sequence of triangulated categories (recall that the canonical embedding $K^b(proj) \to K^{-c}(proj)$ is a triangle equivalence)

$$K^{-c}(projC) \xrightarrow{j_i} K^{-c}(projA) \xrightarrow{i_*} K^{-c}(projB)$$

and a long exact sequence of $K$-groups

$$\cdots \to K_{i+1}(B) \xrightarrow{\delta_{i+1}} K_i(C) \xrightarrow{K_i(\iota)} K_i(A) \xrightarrow{K_i(\pi)} K_i(B) \xrightarrow{\delta_i} K_{i-1}(C) \xrightarrow{K_{i-1}(\iota)} K_{i-1}(A) \to \cdots \quad (6.2)$$

Recall from Section 6.3 that there is a right bounded complex of finitely generated $C$-$A$-bimodules $X$ such that $j_i = ? \otimes_A X$. It follows that $j_* = ? \otimes_A X'^* \cong [L]$. By Proposition 4.11, $i_* (B) \in K^b(projA)$ holds. It follows from Lemma 4.3 that $j_* (A) = X'^* \in K^b(projC)$. So there is a right bounded complex of finitely generated projective $A$-$C$-bimodules $X'$ which is quasi-isomorphic to $X'^*$ as a complex of bimodules; so $j_* \cong ? \otimes_A X' = ? \otimes_A X'$. Moreover, as a complex of $C$-modules $X'$ belongs to $C^{-c}(projC)$. Therefore $? \otimes_A X'$ defines a functor of Frobenius pairs $\kappa : (C^{-c}(projA), C_0^{-c}(projA)) \to (C^{-c}(projC), C_0^{-c}(projC))$. The composition $\kappa \iota : (C^{-c}(projC), C_0^{-c}(projC)) \to (C^{-c}(projC), C_0^{-c}(projC))$ induces the triangle functor $j_i \circ j : K^{-c}(projC) \to K^{-c}(projC)$, which is equivalent to the identity. Thus by [16, Theorem 9], $K_i(\kappa) \circ K_i(\iota) = K_i(\kappa \iota) : K_i(C) \to K_i(C)$ is an isomorphism. In particular, $K_i(\iota)$ is a split monomorphism for any $i \in \mathbb{Z}$. We are done. \hfill \Box

The following corollary is a direct consequence of Theorem 6.7.

Corollary 6.8. Let $k$ be a field and $A$ be a finite-dimensional $k$-algebra admitting a $D^-(\text{Mod})$-stratification with simple factors $A_1, \ldots, A_s$. Then there are isomorphisms of $K$-groups $K_i(A) \cong \bigoplus_{j=1}^s K_i(A_j)$ for $i \in \mathbb{Z}$. 

Prominent finite-dimensional algebras whose derived categories admit stratifications are quasi-hereditary algebras. Here, the factors in the stratification are derived categories of vector spaces over the endomorphism rings of the simple modules, i.e., over division rings. In particular, the K-theory of Schur algebras of algebraic groups and of blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of a semisimple complex Lie algebra decomposes into a direct sum of as many copies of the K-theory of the ground field as there are simple modules. In a similar way, the K-theory of hereditary algebras and of algebras of global dimension two can be decomposed.

7. The derived Jordan–Hölder theorem

When recollements of derived module categories were studied first, around 1990, the question came up whether a derived Jordan–Hölder theorem holds true, that is, whether finite stratifications exist and are unique in the sense that the simple factors of any two stratifications (multiplicities counted) are the same.

In this section, we study only $\mathcal{D}(\text{Mod})$-stratifications. Recently it has been shown that such a Jordan–Hölder theorem holds true for hereditary artin algebras [6], and for symmetric algebras satisfying certain homological condition [35]. It turns out, however, to be false in general, see [12] and [13] for infinite-dimensional counterexamples.

We first show that, for algebras with a block decomposition, the validity of the Jordan–Hölder theorem reduces to the blocks, that is, to the indecomposable ring direct summands. The number of blocks is a derived invariant. For a stratification $\mathcal{S}$, we denote by $\text{SF}(\mathcal{S})$ the sequence of simple factors of $\mathcal{S}$. For two stratifications $\mathcal{S}$ and $\mathcal{S}'$, we say that $\text{SF}(\mathcal{S})$ and $\text{SF}(\mathcal{S}')$ are equivalent and denote by $\text{SF}(\mathcal{S}) \sim \text{SF}(\mathcal{S}')$ if the two sequences $\text{SF}(\mathcal{S})$ and $\text{SF}(\mathcal{S}')$ are the same up to triangle equivalence and reordering of their elements.

Lemma 7.1. Let $A$ be a $k$-algebra with a block decomposition $A = A_1 \oplus \ldots \oplus A_s$. Then the Jordan–Hölder theorem holds true $A$ if and only if it holds true for each $A_i$ (for any choice of derived category). Moreover, if $\mathcal{S}, \mathcal{S}_1, \ldots, \mathcal{S}_s$ are stratifications of $\mathcal{D}(\text{Mod}A), \mathcal{D}(\text{Mod}A_1), \ldots, \mathcal{D}(\text{Mod}A_s)$, respectively, then $\text{SF}(\mathcal{S})$ is equivalent to the sequence $(\text{SF}(\mathcal{S}_1), \ldots, \text{SF}(\mathcal{S}_s))$.

Proof. We only prove the lemma for $\mathcal{D}(\text{Mod})$; similar arguments work for other derived categories.

For each $i = 1, \ldots, s$, let $\mathcal{S}_i$ be a stratification of $\mathcal{D}(\text{Mod}A_i)$. Then we can ‘glue’ these $\mathcal{S}_i$’s to obtain a stratification $\mathcal{S}$ of $\mathcal{D}(\text{Mod}A)$, which satisfies $\text{SF}(\mathcal{S}) = (\text{SF}(\mathcal{S}_1), \ldots, \text{SF}(\mathcal{S}_s))$. Now fix any $i$ and let $\mathcal{S}_i'$ be another stratification of $\mathcal{D}(\text{Mod}A_i)$, then glueing $\mathcal{S}_1, \ldots, \mathcal{S}_i', \ldots, \mathcal{S}_s$ we obtain a stratification $\mathcal{S}'$ of $\mathcal{D}(\text{Mod}A)$ with $\text{SF}(\mathcal{S}') = (\text{SF}(\mathcal{S}_1), \ldots, \text{SF}(\mathcal{S}_{i-1}), \text{SF}(\mathcal{S}_i'), \text{SF}(\mathcal{S}_{i+1}), \ldots, \text{SF}(\mathcal{S}_s))$. If the Jordan–Hölder theorem holds for $A$, then $\text{SF}(\mathcal{S}) \sim \text{SF}(\mathcal{S}')$. It follows that $\text{SF}(\mathcal{S}_i) \sim \text{SF}(\mathcal{S}_i')$, namely, the Jordan–Hölder theorem holds for $A_i$.

Conversely, by Lemma 2.3, a recollement

\[
\begin{array}{ccc}
\mathcal{D}(\text{Mod}B) & \xrightarrow{i} & \mathcal{D}(\text{Mod}A) \\
\xleftarrow{j} & & \xrightarrow{j'} \\
\mathcal{D}(\text{Mod}C) & \xleftarrow{j''} & \\
\end{array}
\]

restricts to recollements

\[
\begin{array}{ccc}
\mathcal{D}(\text{Mod}B_i) & \xrightarrow{i} & \mathcal{D}(\text{Mod}A_i) \\
\xleftarrow{j} & & \xrightarrow{j'} \\
\mathcal{D}(\text{Mod}C_i) & \xleftarrow{j''} & \\
\end{array}
\]
such that \( B = \bigoplus_i B_i \) and \( C = \bigoplus_i C_i \). Thus it follows that a stratification \( \mathcal{S} \) of \( \mathcal{D}(\text{Mod}A) \) can be glued from stratifications \( \mathcal{S}_i \) of \( \mathcal{D}(\text{Mod}A_i) \). In particular, \( \text{SF}(\mathcal{S}) = (\text{SF}(\mathcal{S}_1), \ldots, \text{SF}(\mathcal{S}_s)) \). Let \( \mathcal{S}' \) be another stratification for \( \mathcal{D}(\text{Mod}A) \), then there are stratifications \( \mathcal{S}'_i \) of \( \mathcal{D}(\text{Mod}A_i) \) such that \( \text{SF}(\mathcal{S}') = (\text{SF}(\mathcal{S}'_1), \ldots, \text{SF}(\mathcal{S}'_s)) \). If the Jordan–Hölder theorem holds true for all \( A_i \), then \( \text{SF}(\mathcal{S}_i) \sim \text{SF}(\mathcal{S}'_i) \) holds for all \( i \). It follows that \( \text{SF}(\mathcal{S}) \sim \text{SF}(\mathcal{S}') \), namely, the Jordan–Hölder theorem holds true for \( A \).

Since the derived Jordan–Hölder theorem holds true for derived simple algebras, we have

**Corollary 7.2.** Let \( A \) be a \( k \)-algebra with a block decomposition \( A = A_1 \oplus \cdots \oplus A_s \). If \( A_1, \ldots, A_s \) are derived simple, then the derived Jordan–Hölder theorem holds true for \( A \).

Since noetherian rings always admit a block decomposition, Theorem 5.15 yields the following.

**Corollary 7.3.** The derived Jordan–Hölder theorem holds true for commutative noetherian rings.

Let \( k \) be a field. Next we restrict our attention to finite-dimensional algebras over \( k \). For these algebras the finiteness of any stratification is an easy corollary of Proposition 6.5. However, as we will show later, the uniqueness property in general fails for finite-dimensional algebras.

**Corollary 7.4.** Let \( k \) be a field and let \( A \) be a finite-dimensional \( k \)-algebra. Then any stratification of \( \mathcal{D}(\text{Mod}A) \) is finite.

We give some more examples for which the derived Jordan–Hölder theorem is valid. All these algebras have two non-isomorphic simple modules, so it follows from Proposition 6.5 that the outer algebras of any non-trivial recollement are local and hence are derived simple. Therefore any non-trivial recollement is already a stratification.

**Example 7.5.** For the algebra in Example 4.13 or Example 5.10, there is a unique ladder. This in particular verifies the derived Jordan–Hölder theorem for these two algebras.

For the algebra in Example 5.8, up to equivalence there are precisely two non-trivial recollements, which have the same factors. In particular, the derived Jordan–Hölder theorem holds true for this algebra.

The validity of the derived Jordan–Hölder theorem was proposed as an open question in the end of the article [14]. At the same time, the authors of [14] suggested a candidate ([14, Section 5 (4)]) for a counterexample. Using our techniques, it is seen indeed to be a counterexample: The algebra \( R \ltimes S \) in [14, Section 5 (4)] has only two isomorphism classes of compact indecomposable exceptional objects: the two indecomposable projective modules. Moreover, using Lemma 2.10 (b), one shows that the ideal generated by the two primitive idempotents are not stratifying. Hence \( R \ltimes S \) is derived simple. Consequently, the algebra \( B \) is stratified on the one hand by \( S \) and \( R \ltimes S \) and on the other hand by \( R \) and \( S \ltimes S \).

Finally we present an example of different flavour.

**Example 7.6.** Let \( A \) be the \( k \)-algebra given by the following quiver with relations

\[
\begin{align*}
\alpha & \bigcirc 1 & \gamma \\
\downarrow & & \downarrow \\
\beta & \bigleftarrow 2 & \\
\end{align*}
\]

\( \beta \gamma \beta, \alpha^2, \gamma \alpha. \)
We are going to show that the derived Jordan–Hölder theorem does not hold for $A$.

The composition series of the two indecomposable right projective $A$-modules $P_1 = e_1A$ and $P_2 = e_2A$ are depicted as follows

$$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

We list, without proof, some properties of the algebra $A$. It has infinite global dimension. Its finitistic dimension is finite; this follows from the recollements below, since local algebras have finite (in fact, zero) finitistic dimension, and therefore the middle term has so, too, thanks to [26 Theorem 2]. $A$ is a monomial algebra of wild representation type (see table W in [22]). Its opposite algebra is a standardly stratified algebra (in the sense of [1]; that is, its regular representation has a filtration by standard modules, using the order $1 < 2$).

Consider $e = e_1$. We have isomorphisms $A/e_1A = k\{e_2\} \cong k$ and $e_1Ae_1 = k\{e_1, \alpha, \beta \gamma, \alpha \beta \gamma\} \cong k(x, y)/(x^2, y^2, xy)$ of algebras. The algebra $e_1Ae_1$ is local and its regular module has a simple socle; thus the algebra is self-injective. As a right $A$-module $A/e_1A$ is isomorphic to the simple module at the vertex 2 and admits the following projective resolution

$$\ldots \longrightarrow P_1 \overset{\alpha}{\longrightarrow} P_1 \overset{\alpha}{\longrightarrow} P_1 \overset{\gamma}{\longrightarrow} P_2 \longrightarrow A/e_1A \longrightarrow 0.$$

It follows from Lemma 2.12 that $Ae_1A$ is a stratifying ideal. Consequently, $\mathcal{D}(\text{Mod}A)$ admits a recollement by $\mathcal{D}(\text{Mod}A/Ae_1A)$ and $\mathcal{D}(\text{Mode}_1Ae_1)$, i.e. by $\mathcal{D}(\text{Mod}k)$ and $\mathcal{D}(\text{Mod}k(x, y)/(x^2, y^2, xy))$.

The case for $e = e_2$ is similar. We have algebra isomorphisms $A/e_2A = k\{e_1, \alpha\} \cong k[x]/(x^2)$ and $e_2Ae_2 = k\{x, \gamma, \beta\} \cong k[x]/(x^2)$. As a right $A$-module $A/e_2A$ has composition series \(1\) and admits the following projective resolution

$$\ldots \longrightarrow P_2 \oplus P_2 \overset{(\beta, \alpha \beta)}{\longrightarrow} P_2 \oplus P_2 \overset{\gamma \beta}{\longrightarrow} P_2 \oplus P_2 \longrightarrow A/e_2A \longrightarrow 0.$$

It follows from Lemma 2.12 that $Ae_2A$ is a stratifying ideal. Consequently, $\mathcal{D}(\text{Mod}A)$ admits a recollement by $\mathcal{D}(\text{Mod}A/Ae_2A)$ and $\mathcal{D}(\text{Mode}_2Ae_2)$, i.e. by $\mathcal{D}(\text{Mod}k[x]/(x^2))$ and $\mathcal{D}(\text{Mod}k[x]/(x^2))$.

To summarise:
The category $\mathcal{D}(\text{Mod}A)$ admits two recollements: one has factors $k$ and $k(x, y)/(x^2, y^2, xy)$, while the other one has factors $k[x]/(x^2)$ and $k[x]/(x^2)$. All these factors are local algebras, hence derived simple, and clearly pairwise not Morita (and thus also not derived) equivalent. This shows that the derived Jordan–Hölder theorem fails for $A$.

In both recollements, the functor $i_*$ sends the respective algebra $B$ on the left hand side to an $A$-module of infinite global dimension. Hence all criteria given in section four fail and neither recollement restricts to $K^b(\text{proj})$, $D^b(\text{mod})$ or $D^-(\text{Mod})$. In the second recollement the functor $j_1 = \Omega e_2A$ does restrict to $D^b(\text{mod})$, since $e_2A$ is free of rank two over its endomorphism ring $e_2Ae_2$. Therefore, by Proposition 3.2(b), the second recollement can be extended upwards and thus it is part of a non-trivial ladder of height $\geq 2$. The first recollement, associated with $e_1$, cannot be extended upwards. Neither the first nor the second recollement can be extended downwards, since the criterion in Proposition 3.2(a) fails in both cases.

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