Boundedness and decay for the Teukolsky equation of spin $\pm 1$ on Reissner–Nordström spacetime: the $\ell = 1$ spherical mode

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Abstract
We prove boundedness and polynomial decay statements for solutions to the spin $\pm 1$ Teukolsky-type equation projected to the $\ell = 1$ spherical harmonic on Reissner–Nordström spacetime. The equation is verified by a gauge-invariant quantity which we identify and which involves the electromagnetic and curvature tensor. This gives a first description in physical space of gauge-invariant quantities transporting the electromagnetic radiation in perturbations of a charged black hole.

The proof is based on the use of derived quantities, introduced in previous works on linear stability of Schwarzschild (Dafermos et al 2019 Acta Math. 222 1–214). The derived quantity verifies a Fackerell–Ipser-type equation, with right hand side vanishing at the $\ell = 1$ spherical harmonics. The boundedness and decay for the projection to the $\ell \geq 2$ spherical harmonics are implied by the boundedness and decay for the Teukolsky system of spin $\pm 2$ obtained in Giorgi (2018 (arXiv:1811.03526)).

The spin $\pm 1$ Teukolsky-type equation is verified by the curvature and electromagnetic components of a gravitational and electromagnetic perturbation of the Reissner–Nordström spacetime. Consequently, together with the estimates obtained in Giorgi (2018 (arXiv:1811.03526)), these bounds allow to prove the full linear stability of Reissner–Nordström metric for small charge to coupled gravitational and electromagnetic perturbations.

Keywords: Teukolsky equation, Reissner–Nordström spacetime, Electromagnetic radiation
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1. Introduction

The problem of stability of the Kerr family as solution to the Einstein vacuum equation is one of the main open problems in general relativity. The equivalent problem in the setting of electrovacuum spacetimes is the stability of the Kerr–Newman family as solution to the Einstein–Maxwell equations.

The Reissner–Nordström family of spacetimes \((M, g_{M,Q})\) is the simplest non-trivial solution of the Einstein–Maxwell equation. It can be expressed in local coordinates as

\[
g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \(M\) and \(Q\) are two real parameters verifying \(|Q| < M\).

The ultimate goal of the resolution to the problem of non-linear stability of a solution of the Einstein equation consists in showing that it is stable to small perturbations of initial data as a solution to the fully non-linear Einstein equation. In particular, a complete resolution would need to prove that small perturbations of a solution remain bounded for all times, and converge to another member of the family of solutions. Intermediate steps are the formal mode analysis of the linearized Einstein equation and the proof of decay for the physical space linearized equation.

The linearized gravity around a solution can be formally decomposed into modes, and such decomposition in modes allows one to prove what is known as mode stability, i.e the lack of exponentially growing modes for all metric or curvature components. Extensive literature by the physics community covers the formal study of fixed modes from the point of view of metric perturbations and of Newman–Penrose formalism.

The original approach to mode stability of the Reissner–Nordström spacetime are the metric perturbations, leading to a generalization of the Regge–Wheeler and Zerilli equations. In his studies of the metric perturbations of Reissner–Nordström spacetime in [19–21], Moncrief reduced the governing equations to two pairs of one-dimensional wave equations which govern the odd and the even-parity perturbations. In [5, 6], Chandrasekhar showed that the two pairs are related to each other: the solutions to one parity can be deduced from the solutions to the opposite parity. See also [26, 27].

Analysis of the fixed mode perturbations of the Reissner–Nordström black hole via the Newman–Penrose formalism were treated in [1, 4], where the corresponding Teukolsky equations are derived (see [25]). In [5], Chandrasekhar showed that the Newman–Penrose equations can be transformed to one dimensional wave equation appropriate for the odd and even parity perturbations, now called the (fixed-frequency) Chandrasekhar transformation.

Numerical works have supported the study of mode stability in Kerr–Newman and Reissner–Nordström spacetime, see for example [14, 23]. Mode stability for charged spacetimes in higher dimensions has been treated in [18].

All the above results rely on the derivation of the equations in separated forms and are sufficient to prove that there are no exponentially growing modes, and to obtain control on the quasi-normal modes of the perturbations. However, this weak version of stability is far from
sufficient to prove boundedness and decay of the solution even to the linearized Einstein equation. In particular, the behavior of the quasi-normal modes in the asymptotically flat regime is not sufficient to deduce any information on the solution in physical space. One needs instead to derive sufficiently strong decay estimates through a physical space analysis in order to be able to apply them in the nonlinear framework.

In [15], we started a program towards the proof of (physical space, non-modal) linear stability of the Reissner–Nordström spacetime to coupled gravitational and electromagnetic perturbations. See the introduction of [15] for an introduction on the problem of stability of charged black holes. Since we perturb the curvature tensor using null frames, our analysis is related to the one in Newman–Penrose formalism, but will not make any use of decomposition in modes. The analysis is entirely in physical space, and no choice of gauge are performed in this paper. We will instead treat gauge-invariant quantities, as defined in section 4.

The main result in [15] is the proof of boundedness and decay for two symmetric traceless two tensors $\alpha$ and $f$, governing the gravitational radiation of the Reissner–Nordström spacetime. The decay for $\alpha$ and $f$, verifying Teukolsky-type equations of spin 2, was obtained by applying a physical space version of the Chandrasekhar transformation to both $\alpha$ and $f$. This transformation consists in taking two null derivatives of $\alpha$ and one derivative of $f$ to obtain a system of generalized Regge–Wheeler equations for which combined estimates were derived.

In the linear stability of Schwarzschild spacetime to gravitational perturbations in [9], the decay for $\alpha$ implies the decay of all the other curvature components and Ricci coefficients supported in $\ell \geq 2$ spherical harmonics. In addition, an intermediate step of the proof is the following theorem: Solutions of the linearized gravity around Schwarzschild supported only on $\ell = 0, 1$ spherical harmonics are a linearized Kerr plus a pure gauge solution.

In the setting of linear stability of Reissner–Nordström to coupled gravitational and electromagnetic perturbations, we expect to have electromagnetic radiation supported in $\ell \geq 1$ spherical mode, as for solutions to the Maxwell equations in Schwarzschild (see [2, 3, 24]).

In particular, the decay for the two tensors $\alpha$ and $f$ obtained in [15] will not give any decay information about the $\ell = 1$ spherical mode of the perturbations. It turns out that, in the case of solutions to the linearized gravitational and electromagnetic perturbations around Reissner–Nordström spacetime, the projection to the $\ell = 0, 1$ spherical harmonics is not exhausted by the linearized Kerr–Newman and the pure gauge solutions. Indeed, the presence of the Maxwell equations involving the extreme curvature component of the electromagnetic tensor, which is a one-form, transports electromagnetic radiation supported in $\ell \geq 1$ spherical harmonics.

In [21], Moncrief first analyzes the case of mode perturbations supported in $\ell = 1$ spherical harmonics, in particular the ones which correspond to the electromagnetic radiation. He derives lapse and shift functions in the metric perturbation formalism which govern the perturbation in $\ell = 1$.

In [7], a complex scalar quantity, written as $2\Psi_1\phi_1 - 3\phi_0\Psi_2$ in Newman–Penrose formalism, was identified to be invariant to the first order for the infinitesimal rotations but was not used in the subsequent analysis. Indeed, it was used to show that a gauge where $\Psi_1$ and $\phi_1$ vanish identically cannot be chosen, while a gauge where $\phi_0 = \phi_2 = 0$, the so called phantom gauge, can be chosen. In [7], the equations governing the perturbations in the Newman–Penrose formalism were written in the phantom gauge, and all the analysis was performed in such a gauge. In particular, by choosing the phantom gauge, the above quantity was being reduced to essentially a rescaled version of the curvature component $\Psi_1$.

Fixing a gauge in the derivation of the equations essentially prevents one to identify the gauge-invariant quantities, which instead ought to be interpreted as radiation and carry physical significance. For this reason, the gauge-independent quantities involved in the electromagnetic radiation in Reissner–Nordström spacetime were not clearly identified up to this point.
In this paper we introduce a gauge-independent one-form \( \tilde{\beta} \), which is a mixed curvature and electromagnetic component (see (71) for the definition). This is a tensorial version of the scalar combination identified in [7] and has the additional interesting property of vanishing for linearized Kerr–Newman solutions in the setting of linear stability of Reissner–Nordström. In particular, we identify this quantity to be carrying the electromagnetic radiation in the coupled gravitational and electromagnetic perturbations of Reissner–Nordström spacetime.

It is remarkable that such \( \tilde{\beta} \) verifies a spin \( \pm 1 \) Teukolsky-type equation (derived in appendix A), with non-trivial right hand side, which can be schematically written as

\[
\Box_{\text{grav}} \tilde{\beta} + c_1 L(\tilde{\beta}) + c_2 L(\tilde{\beta}) + V_1 \tilde{\beta} = \text{R.H.S.}
\]

(2)

where \( L \) and \( L \) are outgoing and ingoing null vectors and the right hand side involves curvature components, electromagnetic components and Ricci coefficients.

By applying a physical version of the Chandrasekhar transformation, we obtain a derived quantity \( p \) at the level of one derivative of \( \tilde{\beta} \). We define \( p \) as (see (74))

\[
p = \frac{1}{\kappa} \nabla^3 (r^5 \kappa \tilde{\beta})
\]

where \( \kappa = \text{tr} \chi \) is the trace of the second null fundamental form. Similar physical space versions of the Chandrasekhar transformations were introduced in [9, 24] (see section 6.1 for a comparison of the derived quantities) and also used in Kerr in [10]. This transformation has the remarkable property of turning the Teukolsky-type equation of spin 1 (2) into a Fackerell–Ipser-type1 equation, with right hand side which vanishes in \( \ell = 1 \) spherical harmonics.

A computation, carried out in appendix B, reveals that \( p \) verifies an equation of the schematic form:

\[
\Box_{\text{grav}} p + V p = J
\]

(3)

where \( J \) is supported in \( \ell \geq 2 \) spherical harmonics.

Projecting equation (3) in \( \ell = 1 \) spherical harmonics, we obtain a scalar wave equation with vanishing right hand side, for which techniques developed in [12, 13, 22] can be straightforwardly applied. This proves boundedness and decay for the projection of \( p \), and therefore \( \tilde{\beta} \), to the \( \ell = 1 \) spherical mode, in Reissner–Nordström spacetimes with small charge. The boundedness and decay for its projection into \( \ell \geq 2 \) is implied by using the result for the spin \( \pm 2 \) Teukolsky equation in [15], for small charge.

We emphasize that the analysis of the Teukolsky-type and the Fackerell–Ipser-type equation is obtained in physical space, and not in frequency fixed modes.

A rough version of the main result is the following. The precise statement will be given as main theorem in section 7.2.

Main theorem. (Rough version) Let \( |Q| \ll M \). Solutions \( \tilde{\beta} \) and \( \tilde{\beta} \) to the generalized Teukolsky equation of spin \( \pm 1 \) on Reissner–Nordström exterior spacetimes arising from regular localized initial data remain uniformly bounded and satisfy an \( r^\ell \)-weighted energy hierarchy and polynomial decay.

The main theorem above gives motivation for identifying the gauge-invariant quantities \( \tilde{\beta} \) and \( \tilde{\beta} \) to the electromagnetic radiation in the emission of gravitational and electromagnetic waves in a perturbation of a charged black hole. They are the electromagnetic analogue of the quantities \( \alpha \) and \( \alpha \) (corresponding to \( \Phi_0 \) and \( \Phi_4 \) in Newman–Penrose formalism) which

1 The Fackerell–Ipser equation was encountered in the study of Maxwell equations in Schwarzschild spacetime, see [24].
transport gravitational radiation for vacuum black holes. The description of electromagnetic radiation supported in \( \ell \geq 1 \) for charged black holes has been explicitly obtained in physical space for the first time here, since it was eluded in the choice of phantom gauge chosen in [7].

The main theorem in the present paper and the main theorem in [15] provide decay for the three quantities \( \alpha, f, \tilde{\beta} \), and their corresponding negative spin versions. These decays imply boundedness and decay of all the remaining quantities in the linear stability for coupled gravitational and electromagnetic perturbations of Reissner–Nordström spacetime for small charge, as has been proved in [16].

The outline of the paper is as follows.

In section 2 we recall the null frame decomposition of the spacetime. In section 3 we present the relevant structure of the Reissner–Nordström spacetime and the linearized Einstein–Maxwell equations around it.

In section 4 we define the main quantities \( \tilde{\beta} \) and \( \tilde{\beta} \) verifying the Teukolsky-type equation of spin 1, presented in section 5. In section 6, we present the Chandrasekhar transformation relating the Teukolsky-type equation to the Fackerell–Ipser-type equation.

In section 7 we define the main weighted energies and state the theorem. In section 8 we derive the estimates for the projection of \( p \) into the \( \ell = 1 \) spherical mode. In section 9, we derive the estimates for the solutions to the Teukolsky equation of spin \( \pm 1 \), therefore proving the main theorem.

In appendix A we collect the computations in the derivation of the generalized Teukolsky equation of spin \( \pm 1 \) and in appendix B we show the derivation of the generalized Fackerell–Ipser equation through the Chandrasekhar transformation.

2. Decomposition in null frames

In this section, we review the formalism of local null frames of a Lorentzian manifold.

Let \((\mathcal{M}, g)\) be a \(3 + 1\)-dimensional Lorentzian manifold, and let \(\mathbf{D}\) be the covariant derivative associated to \(g\).

Suppose that the the Lorentzian manifold \((\mathcal{M}, g)\) can be foliated by spacelike two-surfaces \((S, g)\), where \(g\) is the pullback of the metric \(g\) to \(S\). To each point of \(\mathcal{M}\), we can associate a null frame \(N = \{e_A, e_3, e_4\}\), with \(\{e_A\}_{A=1,2}\) being tangent vectors to \((S, g)\), such that the following relations hold

\[
\begin{align*}
g(e_3, e_3) &= 0, & g(e_4, e_4) &= 0, & g(e_3, e_4) &= -2 \\
g(e_3, e_A) &= 0, & g(e_4, e_A) &= 0, & g(e_A, e_B) &= \hat{g}_{AB}.
\end{align*}
\]  

We define the Ricci coefficients associated to the metric \(g\) with respect to the null frame \(N\) in the following way (see [8]):

\[
\begin{align*}
\chi_{AB} &:= g(\mathbf{D}_A e_3, e_B), & \chi_{AB} &:= g(\mathbf{D}_A e_3, e_B) \\
\eta_A &:= \frac{1}{2} g(\mathbf{D}_A e_4, e_A), & \eta_A &:= \frac{1}{2} g(\mathbf{D}_A e_4, e_A), \\
\xi_A &:= \frac{1}{2} g(\mathbf{D}_A e_4, e_A), & \xi_A &:= \frac{1}{2} g(\mathbf{D}_A e_4, e_A) \\
\omega &:= \frac{1}{4} g(\mathbf{D}_4 e_3, e_3), & \omega &:= \frac{1}{4} g(\mathbf{D}_4 e_3, e_3) \\
\zeta_A &:= \frac{1}{2} g(\mathbf{D}_A e_4, e_3).
\end{align*}
\]
We decompose the two-tensor $\chi_{AB}$ into its tracefree part $\hat{\chi}_{AB}$, a symmetric traceless two-tensor on $S$, and its trace $\kappa := \text{tr} \chi$. Similarly for $\chi_{AB}$.

Let $W$ denote the Weyl curvature of $g$ and let $\ast W$ denote the Hodge dual on $(\mathcal{M}, g)$ of $W$. We define the null curvature components in the following way (see [8]):

\[
\alpha_{AB} := W(e_1, e_4, e_B, e_4) \quad \alpha_{AB} := W(e_A, e_3, e_B, e_3)
\]
\[
\beta_A := \frac{1}{2} W(e_A, e_4, e_3, e_4) \quad \beta_A := \frac{1}{2} W(e_A, e_3, e_3, e_4)
\]
\[
\rho := \frac{1}{4} W(e_1, e_4, e_3, e_4) \quad \sigma := \frac{1}{4} \ast W(e_1, e_3, e_3, e_4).
\]

Let $F$ be a two-form in $(\mathcal{M}, g)$, and let $\ast F$ denote the Hodge dual on $(\mathcal{M}, g)$ of $F$. We define the null electromagnetic components in the following way:

\[
(F) \beta_A := F(e_A, e_4), \quad (F) \beta_A := F(e_A, e_3)
\]
\[
(F) \rho := \frac{1}{2} F(e_3, e_4), \quad (F) \sigma := \frac{1}{2} \ast F(e_3, e_4).
\]

If $(\mathcal{M}, g)$ satisfies the Einstein–Maxwell equations

\[
R_{\mu \nu} = 2F_{\mu \lambda} F_{\nu}^{\lambda} - \frac{1}{2} g_{\mu \nu} F_{\alpha \beta} F_{\alpha \beta},
\]

\[
D_{[\alpha} F_{\beta]} = 0, \quad D^\beta F_{\alpha \beta} = 0
\]

the Ricci coefficients, curvature and electromagnetic components defined in (5)–(7) satisfy a system of equations, presented in section 2.3 of [15].

3. The Reissner–Nordström spacetime

In this section, we introduce the Reissner–Nordström exterior metric, as well as relevant background structure. We remind of section 3 of [15] for a more complete description.

Define the manifold with boundary

\[
\mathcal{M} := \mathcal{D} \times S^2 := (-\infty, 0] \times (0, \infty) \times S^2
\]

with Kruskal coordinates $(U, V, \theta^1, \theta^2)$, as defined in section 3 of [15]. The boundary $\mathcal{H}^+$ will be referred to as the horizon. We denote by $\mathcal{S}^2_{U,V}$ the 2-sphere $\{U, V\} \times S^2 \subset \mathcal{M}$ in $\mathcal{M}$.

Fix two parameters $M > 0$ and $Q$, verifying $|Q| < M$. Then the Reissner–Nordström metric $g_{M,Q}$ with parameters $M$ and $Q$ is defined to be the metric:

\[
g_{M,Q} = -4 \Omega^2_k (U, V) dU dV + r^2 (U, V) \gamma_{AB} d\theta^A d\theta^B
\]

where

\[
\Omega^2_k (U, V) = \frac{r_+ r_-}{4r(U, V)^2} \left( \frac{r(U, V) - r_-}{r_+} \right)^{1+\left(\frac{n}{r} \right)^2} \exp \left( -\frac{r_+ - r_-}{r_+} r(U, V) \right)
\]

and

\[
\gamma_{AB} = \text{standard metric on } S^2
\]
\( r_{\pm} = M \pm \sqrt{M^2 - Q^2} \) \tag{12}

and \( r \) is an implicit function of the coordinates \( U \) and \( V \).

### 3.1. Double null coordinates \( u, v \)

We define another double null coordinate system that covers the interior of \( \mathcal{M} \), modulo the degeneration of the angular coordinates. This coordinate system, \( (u, v, \theta^1, \theta^2) \), is called **double null coordinates** and are defined via the relations

\[
U = -\frac{2r_+^2}{r_+ - r_-} \exp \left( -\frac{r_+ - r_-}{4r_+^2} u \right) \quad \text{and} \quad V = \frac{2r_+^2}{r_+ - r_-} \exp \left( \frac{r_+ - r_-}{4r_+^2} v \right) . \tag{13}
\]

Using (13), we obtain the Reissner–Nordström metric on the interior of \( \mathcal{M} \) in \( (u, v, \theta^1, \theta^2) \)-coordinates:

\[
g_{M,Q} = -4\Omega^2 (u, v) \, du \, dv + r^2 (u, v) \gamma_{AB} d\theta^A d\theta^B \tag{14}
\]

with

\[
\Omega^2 := 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{15}
\]

In \( (u, v, \theta^1, \theta^2) \)-coordinates, the horizon \( \mathcal{H}^+ \) can still be formally parametrised by \( (\infty, v, \theta^2, \theta^2) \) with \( v \in \mathbb{R}, (\theta^1, \theta^2) \in S^2 \). We denote \( r_\mathcal{H} = r_+ = M + \sqrt{M^2 - Q^2} \). We denote by \( S_{u,v} \) the sphere \( S^2_{u,v} \) where \( U \) and \( V \) are given by (13).

The photon sphere of Reissner–Nordström corresponds to the hypersurface in which null geodesics are trapped. It is the hypersurface given by \( \{r = r_\mathcal{P}\} \) where \( r_\mathcal{P} \) is given by

\[
r_\mathcal{P} = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2} . \tag{16}
\]

### 3.2. Null frames: Ricci coefficients and curvature components

We define in this section two normalized null frames associated to Reissner–Nordström.

1. The symmetric null frame \( (e_3, e_4) \) is given by

\[
e_4 = \frac{1}{\Omega} \partial_v , \quad e_3 = \frac{1}{\Omega} \partial_u . \tag{17}
\]

In this null frame,

\[
\kappa = -\kappa = \frac{2\Omega}{r}, \quad \omega = -\omega = -\frac{M}{2\Omega r^2} + \frac{Q^2}{2\Omega r^3} .
\]

We also have

\[
\nabla_3 \Omega = -2\omega \Omega, \quad \nabla_4 \Omega = -2\omega \Omega. \tag{18}
\]

2. The regular\(^3\) null frame \( (e_3^*, e_4^*) \) is given by

\[
e_3^* = \Omega^{-1} e_3, \quad e_4^* = \Omega e_4 . \tag{19}
\]

\(^2\)See equation (38) in [15] for the implicit definition.

\(^3\)This frame extends regularly to a non-vanishing null frame on \( \mathcal{H}^+ \).
The curvature and electromagnetic components which are non-vanishing do not depend on the particular null frame. They are given by

\[(r)\rho = \frac{Q}{r^2}, \quad \rho = -\frac{2M}{r^2} + \frac{2Q^2}{r^6}.\]

We also have that

\[K = \frac{1}{r^2}\]

for the Gauss curvature of the round \(S^2\)-spheres.

### 3.3. Killing fields of the Reissner–Nordström metric

We discuss now the Killing fields associated to the metric \(g_{M,Q}\).

We define the vectorfield \(T\) to be the timelike Killing vector field \(\partial_t\) of the \((t, r)\) coordinates in (1), which in double null coordinates is given by

\[T = \frac{1}{2}(\partial_u + \partial_v).\]

The vector field extends to a smooth Killing field on the horizon \(\mathcal{H}^+\), which is moreover null and tangential to the null generator of \(\mathcal{H}^+\).

We can also define a basis of angular momentum operator \(\Omega_i, i = 1, 2, 3\) (see for example section 3.3 of [15]). The Lie algebra of Killing vector fields of \(g_{M,Q}\) is then generated by \(T\) and \(\Omega_i\), for \(i = 1, 2, 3\).

### 3.4. The spherical harmonics

We collect some known definitions and properties of the Hodge decomposition of scalars, one forms and symmetric traceless two tensors in spherical harmonics. We also recall some known elliptic estimates. See section 4.4 of [9] for more details.

#### 3.4.1. The \(\ell = 0, 1\) spherical harmonics and tensors supported on \(\ell \geq 2\)

We denote by \(\hat{Y}_{m}^{\ell}\) with \(|m| \leq \ell\), the well-known spherical harmonics on the unit sphere, i.e.

\[\Delta_0^n Y_{m}^{\ell} = -\ell(\ell + 1)Y_{m}^{\ell}\]

where \(\Delta_0^n\) denotes the laplacian on the unit sphere \(S^2\). The \(\ell = 0, 1\) spherical harmonics are given explicitly by

\[\hat{Y}_{m=0}^{\ell=0} = \frac{1}{\sqrt{4\pi}},\]

\[\hat{Y}_{m=1}^{\ell=0} = \sqrt{\frac{3}{8\pi}} \cos \theta, \quad \hat{Y}_{m=-1}^{\ell=0} = \sqrt{\frac{3}{8\pi}} \sin \theta \cos \phi, \quad \hat{Y}_{m=1}^{\ell=1} = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi.\]

This family is orthogonal with respect to the standard inner product on the sphere, and any arbitrary function \(f \in L^2(S^2)\) can be expanded uniquely with respect to such a basis.

In the foliation of Reissner–Nordström spacetime, we are interested in using the spherical harmonics with respect to the sphere of radius \(r\). For this reason, we normalize the definition of the spherical harmonics on the unit sphere above to the following.
We denote by $Y_{m}^{\ell}$, with $|m| \leq \ell$, the spherical harmonics on the sphere of radius $r$, i.e.
\[
\triangle Y_{m}^{\ell} = -\frac{1}{r^2} \ell(\ell+1) Y_{m}^{\ell}
\]
where $\triangle$ denotes the laplacian on the sphere $S_{u,v}$ of radius $r = r(u,v)$. Such spherical harmonics are normalized to have $L^2$ norm in $S_{u,v}$ equal to 1, so they will in particular be given by $Y_{m}^{\ell} = \frac{1}{\ell} Y_{m}^{\ell}$. We use these basis to project functions on Reissner–Nordström manifold in the following way.

**Definition 3.1.** We say that a function $f$ on $\mathcal{M}$ is supported on $\ell \geq 2$ if the projections
\[
\int_{S_{u,v}} f \cdot Y_{m}^{\ell} = 0
\]
vanish for $Y_{m}^{\ell} = 1$ for $m = -1, 0, 1$. Any function $f$ can be uniquely decomposed orthogonally as
\[
f = c(u,v) Y_{m=0}^{\ell=1} + \sum_{i=-1}^{1} c_i(u,v) Y_{m=i}^{\ell=1} (\theta, \varphi) + f_{\ell \geq 2}
\]
where $f_{\ell \geq 2}$ is supported in $\ell \geq 2$.

In particular, we can write the orthogonal decomposition
\[
f = f_{\ell=0} + f_{\ell=1} + f_{\ell \geq 2}
\]
where
\[
f_{\ell=0} = \frac{1}{4\pi r^2} \int_{S_{u,v}} f \quad (24)
\]
\[
f_{\ell=1} = \sum_{i=-1}^{1} \left( \int_{S_{u,v}} f \cdot Y_{m=i}^{\ell=1} \right) Y_{m=i}^{\ell=1} \quad (25)
\]

Recall that an arbitrary one-form $\xi$ on $S_{u,v}$ has a unique representation $\xi = r D^{i}_{\epsilon} (f, g)$, for two uniquely defined functions $f$ and $g$ on the unit sphere, both with vanishing mean. In particular, the scalars $\text{div} \xi$ and $\text{curl} \xi$ are supported in $\ell \geq 1$. As in [9], we define

**Definition 3.2.** We say that a smooth $S_{u,v}$ one form $\xi$ is supported on $\ell \geq 2$ if the functions $f$ and $g$ in the unique representation
\[
\xi = r D^{i}_{\epsilon} (f, g)
\]
are supported on $\ell \geq 2$. Any smooth one form $\xi$ can be uniquely decomposed orthogonally as
\[
\xi = \xi_{\ell=1} + \xi_{\ell \geq 2}
\]
where the two scalar functions $r D^{i}_{\epsilon} \xi = (r \text{div} \xi_{\ell=1}, r \text{curl} \xi_{\ell=1})$ are in the span of (21) and $\xi_{\ell \geq 2}$ is supported on $\ell \geq 2$.

Recall that an arbitrary symmetric traceless two-tensors $\theta$ on $S_{u,v}$ has a unique representation
\[
\theta = r^2 D^{i}_{\epsilon} D^{j}_{\epsilon} (f, g)
\]

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for two uniquely defined functions \( f \) and \( g \) on the unit sphere, both supported in \( \ell \geq 2 \). In particular, the scalars \( \text{div} \, \theta \) and \( \text{curl} \, \text{div} \, \theta \) are supported in \( \ell \geq 2 \).

For future reference, we recall the following lemma.

**Lemma 3.1 (Lemma 4.4.1 in [9])**. The kernel of the operator \( T = r^2 \mathcal{D} \frac{1}{r} r \mathcal{D} \frac{1}{r} \) is finite dimensional. More precisely, if the pair of functions \((f_1, f_2)\) is in the kernel, then
\[
f_1 = c Y_{m=0}^{\ell=0} + \sum_{i=-1}^{1} c_i Y_{m=1}^{\ell=1}, \quad f_2 = \tilde{c} Y_{m=0}^{\ell=0} + \sum_{i=-1}^{1} \tilde{c}_i Y_{m=1}^{\ell=1}
\]
for constants \( c, c_i, \tilde{c}, \tilde{c}_i \).

**3.4.2. Elliptic estimates.** We recall the relations between the angular operators and the Laplacian \( \Delta \) on \( S \):
\[
\begin{align*}
\mathcal{D}_1 \mathcal{D}_1^{\frac{1}{r}} &= -\Delta_0, \quad \mathcal{D}_1^{\frac{1}{r}} \mathcal{D}_1 = -\Delta_1 + K, \\
\mathcal{D}_2 \mathcal{D}_2^{\frac{1}{r}} &= -\frac{1}{2} \Delta_1 - \frac{1}{2} K, \quad \mathcal{D}_2^{\frac{1}{r}} \mathcal{D}_2 = -\frac{1}{2} \Delta_2 + K
\end{align*}
\]
(26)
where \( \Delta_0 \) and \( \Delta_1 \) are the Laplacian on scalars and on one-form respectively, and \( K \) is the Gauss curvature of the surface \( S \).

We recall the following \( L^2 \) elliptic estimates. See for example [8].

**Proposition 3.1 (Proposition 2.1.3 in [17])**. Let \((S, \gamma)\) be a compact surface with Gauss curvature \( K \). Then the following identities hold for vectors \( \xi \) on \( S \):
\[
\begin{align*}
\int_S (|\nabla \xi|^2 + K |\xi|^2) &= \int_S (|\text{div} \, \xi|^2 + |\text{curl} \, \xi|^2) = \int_S |D_1 \xi|^2 \\
\int_S (|\nabla \xi|^2 - K |\xi|^2) &= 2 \int_S |D_2 \xi|^2.
\end{align*}
\]
(27) (28)
Moreover, suppose that the Gauss curvature is bounded. Then there exists a constant \( C > 0 \) such that the following estimate holds for all vectors \( \xi \) on \( S \) orthogonal to the kernel of \( D_2^{\frac{1}{r}} \):
\[
\int_S \frac{1}{r^2} |\xi|^2 \leq C \int_S |D_2 \xi|^2.
\]
(29)
Consider a one-form \( \xi \) on \( M \) and its decomposition \( \xi = \xi_{\ell=1} + \xi_{\ell \geq 2} \) as in definition 3.2. Then we have the following elliptic estimate.

**Corollary 3.1.** Let \( \xi \) be a one-form on \( M \). Then there exists a constant \( C > 0 \) such that the following estimate holds:
\[
\int_S |\xi|^2 \leq C \left( \int_S |r \text{div} \, \xi_{\ell=1}|^2 + |r \text{curl} \, \xi_{\ell=1}|^2 + |r \mathcal{D}_2 \xi|^2 \right).
\]

**Proof.** Using the orthogonal decomposition of \( \xi \), we have
\[
\int_S |\xi|^2 = \int_S |\xi_{\ell=1}|^2 + \int_S |\xi_{\ell \geq 2}|^2.
\]
Observe that, according to lemma 3.1, $\xi_{l=2}$ is in the kernel of $\mathcal{D}_f^\sharp$. Applying (27) to $\xi_{l=1}$ and (29) to $\xi_{l=2}$ we obtain the desired estimate.

We derive the transport equation for the projection to the $l = 1$ spherical harmonics of a function $f$ on $\mathcal{M}$.

**Lemma 3.2.** Let $f$ be a scalar function on $\mathcal{M}$. Then

$$\nabla_4(f_{l=1}) = (\nabla f)_{l=1} \quad \nabla_3(f_{l=1}) = (\nabla f)_{l=1}.$$ 

Consequently we have

$$\Box_{\mu\nu}(f_{l=1}) = (\Box_{\mu\nu}f)_{l=1}. \quad (30)$$

**Proof.** Applying $\nabla_4$ to the expression for the projection to the $l = 1$ spherical harmonics given by (24), we obtain

$$\nabla_4(f_{l=1}) = \sum_{i=1}^{1} \nabla_4\left(\left(\int_S f \cdot Y_{m=1}^{l=1}\right) Y_{m=1}^{l=1}\right).$$

Recall that the normalized spherical harmonics are defined as $Y_{m}^{l=1} = \frac{1}{r} Y_{m}^{l=1}$, where $Y_{m}^{l=1}$ are given by (22), and therefore $\nabla_4(Y_{m}^{l=1}) = 0$. This implies

$$\nabla_4(Y_{m}^{l=1}) = \nabla_4\left(\frac{1}{r} Y_{m}^{l=1}\right) = \nabla_4\left(\frac{1}{r}\right) Y_{m}^{l=1} = -\frac{1}{2r} \nabla Y_{m}^{l=1} = -\frac{1}{2} r Y_{m}^{l=1}.$$

We therefore obtain

$$\nabla_4(f_{l=1}) = \sum_{i=1}^{1} \nabla_4\left(\left(\int_S f \cdot Y_{m=1}^{l=1}\right) Y_{m=1}^{l=1}\right) + \sum_{i=1}^{1} \left(\int_S f \cdot Y_{m=1}^{l=1}\right) \nabla_4 Y_{m=1}^{l=1}$$

$$= \sum_{i=1}^{1} \left(\int_S \nabla_4(f \cdot Y_{m=1}^{l=1}) + sf \cdot Y_{m=1}^{l=1}\right) Y_{m=1}^{l=1} + \sum_{i=1}^{1} \left(\int_S f \cdot Y_{m=1}^{l=1}\right) (-\frac{1}{2} r Y_{m}^{l=1})$$

$$= \sum_{i=1}^{1} \left(\int_S \nabla_4(f) \cdot Y_{m=1}^{l=1} + f \cdot \nabla_4(Y_{m=1}^{l=1}) + \frac{1}{2} sf \cdot Y_{m=1}^{l=1}\right) Y_{m=1}^{l=1}$$

$$= \sum_{i=1}^{1} \left(\int_S \nabla_4(f) \cdot Y_{m=1}^{l=1}\right) Y_{m=1}^{l=1} = (\nabla f)_{l=1},$$

as desired. Similarly for $\nabla_3 f$. \hfill $\Box$

### 3.5. Linearized Einstein–Maxwell equations

We collect here the equations for linearized gravitational and electromagnetic perturbation of Reissner–Nordström metric. Recall that in Reissner–Nordström metric the following Ricci coefficients, curvature and electromagnetic components vanish:

$$\tilde{\chi}, \tilde{\eta}, \eta, \zeta, \xi, \xi, \alpha, \beta, \sigma, \tilde{\beta}, \tilde{\alpha}, (F)\beta, (F)\sigma, (F)\beta.$$
In particular, in writing the linearization of the equations of section 2, we will neglect the quadratic terms, i.e. product of terms above which vanish in Reissner–Nordström background. See section 4 of [15].

3.5.1. Linearised null structure equations.

\[ \nabla_3 \hat{\chi} + (\kappa + 2\omega) \hat{\chi} = -2 \mathcal{D} \xi - \alpha, \]  
\[ \nabla_4 \hat{\chi} + (\kappa + 2\omega) \hat{\chi} = -2 \mathcal{D} \xi - \alpha, \]  
\[ \nabla_3 \hat{\chi} + \left( \frac{1}{2} \kappa - 2\omega \right) \hat{\chi} = -2 \mathcal{P}_\eta - \frac{1}{2} \kappa \hat{\chi}, \]  
\[ \nabla_4 \hat{\chi} + \left( \frac{1}{2} \kappa - 2\omega \right) \hat{\chi} = -2 \mathcal{P}_\eta - \frac{1}{2} \kappa \hat{\chi}. \]  
\[ \nabla_3 \zeta + \left( \frac{1}{2} \kappa - 2\omega \right) \zeta = 2 \mathcal{D} (\omega, 0) - \left( \frac{1}{2} \kappa + 2\omega \right) \eta + \left( \frac{1}{2} \kappa + 2\omega \right) \xi - \beta - (F) \rho (F) \beta, \]  
\[ \nabla_4 \zeta + \left( \frac{1}{2} \kappa - 2\omega \right) \zeta = -2 \mathcal{D} (\omega, 0) + \left( \frac{1}{2} \kappa + 2\omega \right) \eta - \left( \frac{1}{2} \kappa + 2\omega \right) \xi - \beta - (F) \rho (F) \beta, \]  
\[ \nabla_4 \xi - \nabla_3 \eta = -\frac{1}{2} \kappa (\eta - \eta) + 4 \omega \xi - \beta - (F) \rho (F) \beta, \]  
\[ \nabla_3 \zeta - \nabla_4 \eta = \frac{1}{2} \kappa (\eta - \eta) + 4 \omega \xi + \beta + (F) \rho (F) \beta, \]  
\[ \nabla_3 \kappa + \frac{1}{2} \kappa^2 + 2\omega \kappa = 2 \text{div} \xi, \]  
\[ \nabla_4 \kappa + \frac{1}{2} \kappa^2 + 2\omega \kappa = 2 \text{div} \xi, \]  
\[ \nabla_3 \kappa + \frac{1}{2} \kappa^2 - 2\omega \kappa = 2 \text{div} \eta + 2\rho, \]  
\[ \nabla_4 \kappa + \frac{1}{2} \kappa^2 - 2\omega \kappa = 2 \text{div} \eta + 2\rho, \]  
\[ \text{div} \hat{\chi} = -\frac{1}{2} \kappa \zeta - \frac{1}{2} \mathcal{D} (\kappa, 0) + \beta - (F) \rho (F) \beta, \]  
\[ \text{div} \hat{\chi} = \frac{1}{2} \kappa \zeta - \frac{1}{2} \mathcal{D} (\kappa, 0) - \beta + (F) \rho (F) \beta \]  
\[ \nabla_4 \omega + \nabla_3 \omega = 4 \omega \omega + \rho + (F) \rho^2, \]  
\[ \text{curl} \eta = \sigma, \]
\[ \text{curl} \eta = -\sigma \] (47)
\[ K = -\frac{1}{4} \kappa \xi - \rho + (F) \rho^2. \] (48)

### 3.5.2. Linearised Maxwell equations.
\[ \nabla_3 (F) \beta + \left( \frac{1}{2} \kappa - 2 \omega \right) (F) \beta = -\nabla_1^* (F) \rho, (F) \sigma + 2 (F) \rho \eta, \] (49)
\[ \nabla_4 (F) \beta + \left( \frac{1}{2} \kappa - 2 \omega \right) (F) \beta = \nabla_1^* (F) \rho, -(F) \sigma - 2 (F) \rho \eta \] (50)
\[ \nabla_3 (F) \rho + \kappa (F) \rho = -\text{div} (F) \beta \] (51)
\[ \nabla_4 (F) \rho + \kappa (F) \rho = \text{div} (F) \beta \] (52)
\[ \nabla_3 (F) \sigma + \kappa (F) \sigma = \text{curl} (F) \beta \] (53)
\[ \nabla_4 (F) \sigma + \kappa (F) \sigma = \text{curl} (F) \beta. \] (54)

### 3.5.3. Linearised Bianchi identities.
\[ \nabla_3 \alpha + \left( \frac{1}{2} \kappa - 4 \omega \right) \alpha = -2 \nabla_1^* \beta - 3 \rho \hat{\chi} - 2 (F) \rho \left( \nabla_1^* (F) \beta + (F) \rho \hat{\chi} \right), \] (55)
\[ \nabla_4 \alpha + \left( \frac{1}{2} \kappa - 4 \omega \right) \alpha = 2 \nabla_1^* \beta - 3 \rho \hat{\chi} + 2 (F) \rho \left( \nabla_1^* (F) \beta - (F) \rho \hat{\chi} \right), \] (56)
\[ \nabla_3 \beta + (\kappa - 2 \omega) \beta = \nabla_1^* (-\rho, \sigma) + 3 \rho \eta + (F) \rho \left( \nabla_1^* (F) \rho, -(F) \sigma - \kappa (F) \beta - \frac{1}{2} \kappa (F) \beta \right), \] (57)
\[ \nabla_4 \beta + (\kappa - 2 \omega) \beta = \nabla_1^* (\rho, \sigma) - 3 \rho \eta + (F) \rho \left( \nabla_1^* (F) \rho, -(F) \sigma - \kappa (F) \beta - \frac{1}{2} \kappa (F) \beta \right), \] (58)
\[ \nabla_3 (F) \beta + (2 \kappa - 2 \omega) \beta = -\text{div} \alpha - 3 \rho \hat{\xi} + (F) \rho \left( \nabla_3 (F) \beta + 2 \omega (F) \beta + 2 (F) \rho \xi \right), \] (59)
\[ \nabla_4 (F) \beta + (2 \kappa + 2 \omega) \beta = \text{div} \alpha + 3 \rho \hat{\xi} + (F) \rho \left( \nabla_4 (F) \beta + 2 \omega (F) \beta - 2 (F) \rho \xi \right) \] (60)
\[ \nabla_3 \rho + \frac{3}{2} \kappa \rho = -\kappa (F) \rho^2 - \text{div} \beta - (F) \rho \text{ div} (F) \beta, \] (61)
\[ \nabla_4 \rho + \frac{3}{2} \kappa \rho = -\kappa (F) \rho^2 + \text{div} \beta + (F) \rho \text{ div} (F) \beta, \] (62)
\[ \nabla_3 \sigma + \frac{3}{2} \kappa \sigma = -\text{curl} \beta - (F) \rho \text{ curl} (F) \beta. \quad (63) \]
\[ \nabla_4 \sigma + \frac{3}{2} \kappa \sigma = -\text{curl} \beta - (F) \rho \text{ curl} (F) \beta. \quad (64) \]

### 4. Gauge-invariant quantities \( \tilde{\beta} \) and \( \bar{\beta} \)

In this section, we define the tensors for which we prove decay in the main theorem of this paper.

In order to identify the gauge-invariant quantities we consider null frame transformations, i.e. linear transformations which take null frames into null frames. They are given by

\[
e' = e^a (e^4 + f_A e_A),
\]
\[
e' = e^{-a} (e^3 + f^A e_A),
\]
\[
e' = O_A B e^B + \frac{1}{2} f^A e^4 + \frac{1}{2} f^A e^3,
\]

where \( a \) is a scalar function, \( f \) and \( f^a \) are \( S_{\mu,\nu} \)-tensors and \( O_A B \) is an orthogonal transformation of \( (S_{\mu,\nu}, g) \), i.e. \( O_A C O_B D g_{CD} = g_{AB} \). See lemma 2.3.1 of [17] for a classification of such transformations. When \( a, f \) and \( f^a \) are small perturbations of 0 and \( O_{AB} \) is a small perturbation of the identity, we check that the above transformation transform a null frame into another null frame. Indeed, for example

\[
g(e'_3, e'_4) = g(e^{-a} (e^3 + f^B e_B), e^a (e^4 + f^B e_B)) = g(e_3, e_4) = -2
\]
\[
g(e'_3, e'_A) = g(e^{-a} (e^3 + f^B e_B), O_A C e^C + \frac{1}{2} f^A e_4 + \frac{1}{2} f^B e_3)
\]
\[
= g(e^{-a} e_3, \frac{1}{2} f^A e_4) + g(e^{-a} f^B e_B, O_A C e^C) + \text{quadratic terms in } f, f^a
\]

These transformations modify the Ricci coefficients and the curvature components of the spacetime at the linear level. We summarize here some transformation rules for Ricci coefficients and curvature components under a general null transformation. See also proposition 2.3.4 of [17].

**Proposition 4.1.** Under a general null transformation, the linear transformations of curvature and electromagnetic components are the following:

\[
\beta' = \beta + \frac{3}{2} \rho f
\]
\[
(\rho)' = \rho,
\]
\[
\bar{\beta}' = \bar{\beta} - \frac{3}{2} \rho f
\]
\[
(F) \beta' = (F) \beta + (F) \rho,
\]
\[
(F) (\rho)' = (F) \rho,
\]
\[
(F) \bar{\beta}' = (F) \bar{\beta} - \frac{1}{2} (F) \rho.
\]
Proof. We have
\[
\beta' = \frac{1}{2} W(e', e_4, e_3, e_1)
\]
\[
= \frac{1}{2} W(O_A b e B + \frac{1}{2} f_A e_4 + \frac{1}{2} f_A e_3, e_4 + f^c e_c + e^e (e_3 + f^e e_e), e^a (e_4 + f^d e_d))
\]
\[
= \beta + \frac{1}{2} f^D W(e_3, e_4) + \frac{1}{4} f_A W(e_3, e_4, e_4) + \text{quadratic terms}
\]
\[
= \beta + \frac{3}{2} f_A \rho + \text{quadratic terms}.
\]

We have
\[
(F) \beta' = F(e'_4) = F(O_A b e B + \frac{1}{2} f_A e_4 + \frac{1}{2} f_A e_3, e_4 + f^c e_c + e^e (e_3 + f^e e_e))
\]
\[
= (F) \beta + \frac{1}{2} f_A F(e_4) + \text{quadratic terms}
\]
\[
= (F) \beta + (F) \rho.
\]

Similarly we see that \(\rho' = \rho + \text{quadratic terms}\) and \((F) \rho' = (F) \rho + \text{quadratic terms}\). \(\Box\)

We define a gauge-invariant quantity in the context of linear stability in the following way.

Definition 4.1. We say that \(\Psi\) is a (linear) gauge-invariant quantity if under a linear null frame transformation it is modified only quadratically, i.e. if \(\Psi' = \Psi + \text{quadratic terms}\).

The Teukolsky equations we shall consider are wave equations for gauge-invariant quantities for linear gravitational and electromagnetic perturbations of Reissner–Nordström spacetime.

From the study of Maxwell equations in Schwarzschild spacetime (see [3, 24]), it is known that the extreme null components \((F)\beta\) and \((F)\beta\) verify a decoupled wave equation, called the Teukolsky equation of spin \(\pm 1\). However, it is immediate from (68) and (70) in proposition 4.1, that the extreme electromagnetic components \((F)\beta\) and \((F)\beta\) are not gauge-invariant if \((F)\rho\) is not zero in the background. In particular, in the case of coupled gravitational and electromagnetic perturbations of Reissner–Nordström spacetime, for which \((F)\rho = \frac{3}{2}\), the spin \(\pm 1\) Teukolsky equations verified by \((F)\beta\) and \((F)\beta\) (derived in proposition A.1) cannot be used to obtain decay, since they are not gauge-invariant.

A new gauge-invariant quantity to be used in this context has to be found. This quantity has to be a one-tensor, in order to transport electromagnetic radiation, supported in \(\ell \geq 1\).

We define the following one-tensors
\[
\tilde{\beta} := 2 (F) \rho \beta - 3 (F) \beta, \quad \tilde{\beta} := 2 (F) \rho \beta - 3 (F) \beta.
\]

They are gauge invariant quantities. Indeed, using proposition 4.1, we obtain
\[
\tilde{\beta}' = 2 (F) \rho' \beta' - 3 (F) \beta' = 2 (F) \rho (\beta + \frac{3}{2} \rho f) - 3 (F) \beta + (F) \rho f = 2 (F) \rho \beta - 3 (F) \beta = \tilde{\beta}
\]
and similarly for \((F) \beta\).

Remark 4.1. Observe that in Schwarzschild \(\tilde{\beta}\) reduces, at the linear level, to
\[
\tilde{\beta} := -3 (F) \beta = \frac{6M}{r^3} (F) \beta
\]
and the Teukolsky equation in proposition A.2 reduces to the Teukolsky equation in proposition A.1 verified by \((p^i)\beta\).

The gauge-invariant quantities \(\tilde{\beta}\) and \(\tilde{\bar{\beta}}\) have the additional remarkable property that they vanish for Kerr–Newman solutions, as it becomes clear in linearizing the Reissner–Nordström spacetime for small variation of angular momentum (see remark 5.2.1 in [16]).

This implies that, in order to prove linear stability of Reissner–Nordström spacetime, since \(\tilde{\beta}\) and \(\tilde{\bar{\beta}}\) vanish for any pure gauge solutions and for any linearized Kerr–Newman solution, these quantities have to decay.

Indeed, \(\tilde{\beta}\) and \(\tilde{\bar{\beta}}\) verify a Teukolsky-type equation which can be treated through a Chandrasekhar transformation, and for which decay for the projection to the \(\ell = 1\) mode can be proved independently from the remaining projection. The boundedness and decay of the \(\ell \geq 2\) modes are implied by the results for \(\alpha\) and \(\tilde{\bar{f}}\) in [15].

5. Generalized spin ±1-Teukolsky and Fackerell–Ipser equation in \(\ell = 1\) mode

In this section, we introduce a generalization of the celebrated spin ±1 Teukolsky equations and the Fackerell–Ipser equation, and explain the connection between them and their relation to the linear stability of Reissner–Nordström spacetime to gravitational and electromagnetic perturbations.

5.1. Generalized spin ±1 Teukolsky equation

The generalized spin ±1 Teukolsky equation concerns 1-tensors which we denote \(\tilde{\beta}\) and \(\tilde{\bar{\beta}}\) respectively.

**Definition 5.1.** Let \(\tilde{\beta}\) be a \(S_1^2\) one-tensor defined on a subset \(D \subset M\). We say that \(\tilde{\beta}\) satisfy the generalized Teukolsky equation of spin +1 if it satisfies the following PDE:

\[
\Box (r^3 \tilde{\beta}) = -2\omega \nabla_4 (r^3 \tilde{\beta}) + (\kappa + 2\omega) \nabla_3 (r^3 \tilde{\beta}) + \left( \frac{1}{4} \kappa r - 3\omega r + \omega r - 2\rho + 3 (p^i)\rho^2 - 8\omega \rho + 2\nabla_\rho \right) r^3 \tilde{\beta} - 2r^3 (p^i)\rho^2 \left( \nabla_4 (r^3) \beta + \left( \frac{3}{2} \kappa r + 2\omega \right) (p^i)\beta - 2 (p^i)\rho^2 \right) + I
\]

where \(\Box = g^{\mu\nu} D_\mu D_\nu\) denotes the wave operator in Reissner–Nordström spacetime, and \(I\) is a one-tensor with vanishing projection to the \(\ell = 1\) spherical harmonics, i.e. \(\text{div} I_{\ell=1} = \text{curl} I_{\ell=1} = 0\).

Let \(\tilde{\bar{\beta}}\) be a \(S_1^2\) one-tensor defined on a subset \(D \subset M\). We say that \(\tilde{\bar{\beta}}\) satisfy the generalized Teukolsky equation of spin –1 if it satisfies the following PDE:

\[
\Box (r^3 \tilde{\bar{\beta}}) = -2\omega \nabla_4 (r^3 \tilde{\bar{\beta}}) + (\kappa + 2\omega) \nabla_3 (r^3 \tilde{\bar{\beta}}) + \left( \frac{1}{4} \kappa r - 3\omega r + \omega r - 2\rho + 3 (p^i)\rho^2 - 8\omega \rho + 2\nabla_\rho \right) r^3 \tilde{\bar{\beta}} - 2r^3 (p^i)\rho^2 \left( \nabla_4 (r^3) \beta + \left( \frac{3}{2} \kappa r + 2\omega \right) (p^i)\beta - 2 (p^i)\rho^2 \right) + I
\]

where \(I\) is a one-tensor with vanishing projection to the \(\ell = 1\) spherical harmonics.

5.2. Generalized Fackerell–Ipser equation in \(\ell = 1\) mode

The other generalized equation in \(\ell = 1\) to be defined here is the generalized Fackerell–Ipser equation, to be satisfied by a one tensor \(p\).
Definition 5.2. Let $p$ be a one-tensor on $D \subset M$. We say that $p$ satisfies the generalized Fackerell–Ipser equation in $\ell = 1$ if it satisfies the following PDE:

$$\Box_g p + \left( \frac{1}{4} \kappa^2 \xi - 5(F)^2 \right) p = J$$

where $J$ is a one-tensor with vanishing projection to the $\ell = 1$ spherical harmonics, i.e. $\text{div} J_{\ell=1} = 0$ and $\text{curl} J_{\ell=1} = 0$.

In section 6, we will show that given a solution $\tilde{\beta}$ and $\tilde{\beta}$ of the generalized spin $\pm 1$ Teukolsky equations in $\ell = 1$, respectively, we can derive two solutions $p$ and $\tilde{p}$, respectively, of the generalized Fackerell–Ipser equation in $\ell = 1$.

5.3. The characteristic initial value formulation

For completeness, we state here a standard well-posedness theorem for both the generalized Teukolsky equation and the generalized Fackerell–Ipser equation. We formulate it in the context of a characteristic initial value problem. We fix a sphere $S_{u_0, v_0}$ in $M$ and consider the outgoing Reissner–Nordström light cone $C_{u_0} = \{ u = u_0 \} \times \{ v \geq v_0 \} \times S^2$ and the ingoing Reissner–Nordström light cone $C_{v_0} = \{ u \geq u_0 \} \times \{ v = v_0 \} \times S^2$ on which data are being prescribed.

Proposition 5.1 (Well-posedness for generalized Teukolsky equation of spin $+1$). Given a sphere with corresponding null cones $C_{u_0}$ and $C_{v_0}$ prescribe

- along $C_{v_0}$ a one-$S_{u,v}$-tensor $\tilde{\beta}_{0,\text{in}}$, such that $\Omega \tilde{\beta}$ is smooth
- along $C_{u_0}$ a smooth one-$S_{u,v}$-tensor $\tilde{\beta}_{0,\text{out}}$, satisfying $\tilde{\beta}_{0,\text{out}} = \tilde{\beta}_{0,\text{in}}$ on $S_{u_0,v_0}$.

Then there exists a unique smooth one-$S_{u,v}$-tensor $\Omega \tilde{\beta}$ defined on $M \cap \{ u \geq u_0 \} \cap \{ v \geq v_0 \}$ such that

- $\tilde{\beta}$ satisfies the generalized Teukolsky equation of spin $+1$ in $M \cap \{ u \geq u_0 \} \cap \{ v \geq v_0 \}$
- $\Omega \tilde{\beta}|_{C_{u_0}} = \tilde{\beta}_{0,\text{in}}$ and $\tilde{\beta}|_{C_{v_0}} = \tilde{\beta}_{0,\text{out}}$

A similar result holds for the generalized Teukolsky equation of spin $-1$.

The well-posedness statement for the Fackerell–Ipser equation is entirely analogous.

6. Generalized Chandrasekhar transformation into Fackerell–Ipser

We now describe a transformation theory relating solutions of the generalized Teukolsky equation to solutions of the generalized Fackerell–Ipser equation in $\ell = 1$.

We introduce the following operators for a $n$-rank $S$-tensor $\Psi$, previously defined in [15]:

$$P(\Psi) = \frac{1}{\kappa} \nabla_3 (r \Psi), \quad P(\Psi) = \frac{1}{\kappa} \nabla_4 (r \Psi).$$

Observe that the operators which allow to define the Chandrasekhar transformation into the Regge–Wheeler equations in [15] are the same as the operators for the Chandrasekhar transformation into Fackerell–Ipser equation used here.
Given a solution $\tilde{\beta}$ of the generalized Teukolsky equation of spin $+1$, we can define the following derived quantities for $\tilde{\beta}$:

$$\psi_5 = r^4 \kappa \tilde{\beta},$$
$$\psi_6 = P(\psi_5) := p.$$  \hfill (74)

Similarly, given a solution $\tilde{\beta}$ of the generalized Teukolsky equation of spin $-1$, we can define the following derived quantities for $\tilde{\beta}$:

$$\psi_5 = r^4 \kappa \tilde{\beta},$$
$$\psi_6 = P(\psi_5) := p.$$  \hfill (75)

The following proposition is proven in appendix B.

**Proposition 6.1.** Let $\tilde{\beta}$ be a solution of the generalized Teukolsky equation of spin $+1$ on $\mathcal{M} \cap \{u \geq u_0\} \cap \{v \geq v_0\}$. Then the one-tensor $p$ as defined through (74) satisfies the generalized Fackerell–Ipser in $l = 1$ system on $\mathcal{M} \cap \{u \geq u_0\} \cap \{v \geq v_0\}$.

Similarly, let $\tilde{\beta}$ be a solution of the generalized Teukolsky equation of spin $-1$ on $\mathcal{M} \cap \{u \geq u_0\} \cap \{v \geq v_0\}$. Then the one-tensor $p$ as defined through (75) satisfies the generalized Fackerell–Ipser equation in $l = 1$ on $\mathcal{M} \cap \{u \geq u_0\} \cap \{v \geq v_0\}$.

**Proof.** In Lemma A.3, we compute the wave equation verified by a derived quantity of the form $P(\Psi)$. We use this lemma to derive the wave equation for $p$ in proposition B.1, from the Teukolsky equation for $\beta$. See appendix B.

The fact that the derived quantity $p$ satisfies the generalized Fackerell–Ipser equation in $\ell = 1$, together with the transport relations (74), will be the key to estimating the projection to $\ell = 1$ spherical harmonics of $\tilde{\beta}$ and control such projection of the electromagnetic contribution of the radiation.

### 6.1. Relation with higher order quantities defined in Schwarzschild

As observed in remark 4.1, the one-form $\tilde{\beta}$ defined above for Reissner–Nordström spacetime reduces to $\tilde{\beta} = \frac{\mathcal{M}}{\mathcal{F}} (F) \beta$ in the particular case of Schwarzschild. Indeed, in the case of Maxwell equations in Schwarzschild spacetime, the one-form $(F) \beta$ is gauge-invariant and verifies a Teukolsky equation of spin $\pm 1$ given by

$$\Box_g (F) \beta = -2 \omega \Box_4 (F) \beta + (\kappa + 2 \omega) \nabla_4 (F) \beta + \left( \frac{1}{4} (\kappa E - 3 \omega E + \omega \kappa - 2 \nabla_4 \omega) \right) (F) \beta + O(\epsilon^2)$$

as derived in proposition A.1. This is consistent with the fact that in Schwarzschild, with $(F) \rho = O(\epsilon)$, the wave equation verified by $\tilde{\beta}$ in proposition A.2 reduces to

5 Recall the other derived quantities defined in [15]:

$$\psi_0 = r^2 \kappa \tilde{\beta},$$
$$\psi_1 = P(\psi_0),$$
$$\psi_2 = P(\psi_1) = P(P(\psi_0)) := q,$$
$$\psi_3 = r^2 \kappa l,$$
$$\psi_4 = P(\psi_3) := q^\prime.$$
\[ \Box (r^3 \tilde{\beta}) = -2\omega \nabla_4 (r^3 \tilde{\beta}) + (\kappa + 2\omega) \nabla_3 (r^3 \tilde{\beta}) + \left( \frac{1}{4} \kappa \kappa - 3\omega \omega + \omega \kappa - 2\nabla_4 \omega \right) r^3 \tilde{\beta} + O(\epsilon^2). \]

Observe that it coincides with the Teukolsky equation previously known for the Maxwell equations in Schwarzschild.

In the analysis of Maxwell equations in Schwarzschild in [24], the transformation theory is defined by the author in the following way \(6\) (see section 3.2 in [24]):

\[ \phi := \frac{r^2}{1 - \mu} \nabla_4 (r \Omega (F) \beta) \]

(76)

where \(l = \mu = \Omega^2 = 1 - \frac{2\mu}{r} \) and \(L = \Omega \omega\). Recall that in double null coordinates used in [24], \(\tau = \text{tr} \nabla = -\frac{2}{r} \Omega \) in (76), we obtain

\[ \phi = \frac{r^2}{\Omega} \nabla_3 (r \Omega (F) \beta) = \frac{r}{\Omega} \nabla_3 (r^2 \kappa (F) \beta) = \frac{1}{6M} \nabla_3 \left( r^2 \kappa \beta \right) = \frac{r}{6M} \beta \]

which relate the \(\phi\) in [24] in Schwarzschild to the \(\beta\) defined in this paper.

### 6.2. Relation with the linear stability of Reissner–Nordström spacetime

We will now finally relate the equations presented above to the full system of linearized gravitational and electromagnetic perturbations of Reissner–Nordström spacetime in the context of linear stability of Reissner–Nordström, as proved in [16].

Consider a solution to the linearized Einstein–Maxwell equations around Reissner–Nordström spacetime, as presented in section 3.5. Then, the quantities \(\tilde{\beta} \lambda = 2 (F) \rho \beta \lambda - 3 \rho (F) \beta \lambda\) and \(\tilde{\beta} \lambda = 2 (F) \rho \beta \lambda - 3 \rho (F) \beta \lambda\) verify the generalized Teukolsky equation of spin \(\pm 1\) respectively. We obtain the following theorem.

**Theorem 6.1.** Let \(\tilde{\beta}, \tilde{\beta}\) be the curvature components of a solution to the linearized Einstein–Maxwell equations around Reissner–Nordström spacetime as in section 3.5. Then \(\tilde{\beta}\) satisfies the generalized Teukolsky equation of spin \(+1\), and \(\tilde{\beta}\) satisfies the generalized Teukolsky equation of spin \(-1\). Moreover, the derived quantities \(\bar{p}\) and \(\bar{p}\) defined in (74) and (75) verify the generalized Fackerell–Ipser equation in \(\ell = 1\).

**Proof.** See proposition A.2 in appendix A for the derivation of the generalized Teukolsky equation and proposition B.1 in appendix B for the derivation of the generalized Fackerell–Ipser equation in \(\ell = 1\).

The quantity \(\tilde{\beta}\) verifies the following wave equation:

\[ \Box (r^3 \tilde{\beta}) = -2\omega \nabla_4 (r^3 \tilde{\beta}) + (\kappa + 2\omega) \nabla_3 (r^3 \tilde{\beta}) + \left( \frac{1}{4} \kappa \kappa - 3\omega \omega + \omega \kappa - 2\nabla_4 \omega \right) r^3 \tilde{\beta} \]

\[ - 2r^3 \kappa (F) \rho \beta + \left( \nabla_3 (r^2 \kappa) + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \beta \right) + 8r^3 (F) \rho ^2 \nabla \mathfrak{f} \]

which is of the form given in definition 5.1, with \(\mathcal{L} = 8r^3 (F) \rho ^2 \nabla \mathfrak{f}\), which indeed has vanishing projection to the \(\ell = 1\) mode. Indeed, recall that the tensor \(\mathfrak{f}\) defined in [15] is a symmetric

\( \text{Observe that in [24] the extreme null component is defined as } \alpha(V) = F(V, L) = F(V, \Omega \epsilon) = \Omega (F) \beta. \)
traceless two tensor, and therefore by results about spherical harmonics recalled in section 3.4, its divergence has vanishing projection to the \( \ell = 1 \) spherical harmonics. By proposition 6.1, the derived quantity \( p \) verifies the following generalized Fackerell–Ipser equation in \( \ell = 1 \):

\[
\square g p + \left( \frac{1}{4} \kappa \xi - 5 \rho^2 \right) p = 8 r^2 \rho^2 \text{div} (q F)
\]

which is of the form given in definition 5.2, with \( J = 8 r^2 \rho^2 \text{div} (q F) \), which has vanishing projection to the \( \ell = 1 \) mode. □

Using proposition 6.1, we can associate to any solution to the linearized Einstein–Maxwell equations around Reissner–Nordström spacetime a one form which verifies the generalized Fackerell–Ipser equation in \( \ell = 1 \).

6.3. The \( \ell \geq 2 \) modes of the electromagnetic radiation

Recall the definition of the gauge-invariant quantities defined in [15]. From the Weyl curvature component \( \alpha \) defined in (6), we defined the derived quantity \( \psi_1 \) as follows:

\[
\psi_1 = P (r^2 \kappa^2 \alpha) = \frac{1}{2} r (r^2 \kappa^2 \alpha) + \frac{1}{2} \kappa \nabla_3 (r^2 \kappa^2 \alpha).
\]

We also defined the gauge-invariant quantity \( f \) as

\[
f = D_\beta \beta + F \rho \hat{\chi}
\]

and their equivalent spin \(-2\) versions. These quantities defined in [15] are related to the one tensor \( \tilde{\beta} \) defined in (71).

Lemma 6.1. The following relations hold true:

\[
\begin{align*}
\kappa r^3 D_\beta \beta & = -(F) \rho \psi_1 - \left( 2 \varphi r^2 + 3 \rho \right) r^3 \kappa f, \\
\kappa r^3 D_\beta \tilde{\beta} & = (F) \rho \psi_1 - \left( 2 \varphi r^2 + 3 \rho \right) r^3 \kappa f.
\end{align*}
\]

Proof. By definition of \( f \) we have

\[
r^2 \kappa f = r^2 \kappa \left( D_\beta \beta + \rho \hat{\chi} \right) - (77)
\]

and by definition of \( \psi_1 \), we have

\[
\psi_1 = \frac{1}{2} r (r^2 \kappa^2 \alpha) + \frac{1}{2} \kappa \nabla_3 (r^2 \kappa^2 \alpha) - 4 \omega r^2 \kappa^2 \alpha = r^3 \kappa \left( \nabla_3 (\alpha) + \frac{1}{2} \kappa \alpha + 4 \omega \alpha \right).
\]

Using Bianchi identity (55), we obtain

\[
\psi_1 = r^3 \kappa \left( -2 D_\beta \beta - 3 \rho \hat{\chi} - 2 \varphi r^2 \rho \right) \quad (78)
\]

Multiplying (77) by \( 3 \rho \) and summing it to the (78) multiplied by \( \rho \), we obtain

\[
(F) \rho \psi_1 + (2 \rho^2 + 3 \rho) r^3 \kappa f = r^3 \kappa \left( 3 \rho D_\beta \beta - 2 \rho D_\beta \tilde{\beta} \right) = -r^3 \kappa D_\beta \tilde{\beta}
\]

as desired. Similarly for \( \tilde{\beta} \). □
By these relations, it is clear that the bounds and decay obtained for \( \psi_1 \) and \( f \) in the main theorem in [15] imply bounds and decay for \( D^{\frac{1}{2}} \beta \), therefore on the projection to the \( \ell \geq 2 \) spherical harmonics of \( \beta \). Using the control for the \( \ell = 1 \) mode of \( \beta \) and \( \tilde{\beta} \) obtained through the generalized Fackerell–Ipser equation in \( \ell = 1 \) and elliptic estimates, we can derive control for the one-tensors \( \tilde{\beta} \) and \( \widetilde{\beta} \).

7. Energy quantities and statements of the main theorem

We give the definitions of weighted energy quantities, and we provide the precise statement of the main theorem of this paper.

7.1. Definition of weighted energies

We define in this section a number of weighted energies.

We define the following vectorfields:

- \( T = \frac{1}{2} [\Omega \nabla_3 + \Omega \nabla_4] \)
- \( R^* = \frac{1}{2} [-\Omega \nabla_3 + \Omega \nabla_4] \)

Notice that \( T \) coincides with the Lie-differentiation \( L_T \) with respect to the Killing field \( T \) as defined in section 3.3.

7.1.1. Weighted energies for \( \phi \).

The energies in this section will be applied to \( \phi = (r^2 \text{div} p)_{\ell=1} \), \( \phi = (r^2 \text{curl} p)_{\ell=1} \), or \( \phi = (r^2 \text{div} p)_{\ell=1} \), \( \phi = (r^2 \text{curl} p)_{\ell=1} \).

We introduce the following weighted energies for \( \phi \).

1. Flux energy quantities:

   - \( T \)-energy null fluxes
     \[
     F_T^u[\phi](v_1, v_2) = \int_{\partial \Sigma} d\sigma \sin \theta d\sigma d\phi \left\{ |\Omega \nabla_4 \phi|^2 + \Omega^2 |\nabla \phi|^2 + V |\phi|^2 \right\},
     \]
     \[
     F_T^v[\phi](u_1, u_2) = \int_{\Sigma} d\sigma \sin \theta d\sigma d\phi \left\{ |\Omega \nabla_3 \phi|^2 + \Omega^2 |\nabla \phi|^2 + V |\phi|^2 \right\}
     \]
     where \( V = \frac{\alpha}{r} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^7 \).

   - Non-degenerate energy null fluxes
     \[
     F_u[\phi](v_1, v_2) = \int_{\partial \Sigma} d\sigma \sin \theta d\sigma d\phi \left\{ |\Omega \nabla_4 \phi|^2 + |\nabla \phi|^2 + V |\phi|^2 \right\},
     \]
     \[
     F_v[\phi](u_1, u_2) = \int_{\Sigma} d\sigma \sin \theta d\sigma d\phi \left\{ |\Omega^{-1} \nabla_3 \phi|^2 + |\nabla \phi|^2 + V |\phi|^2 \right\}
     \]

   - Weighted energy quantity in null-infinity

\( ^7 \) Hence these fluxes are manifestly coercive in the exterior region.
\[
F^T_w[\phi](v_1, v_2) = \int_{v_1}^{v_2} dv \sin \theta d\theta d\phi [r^2|\Omega \nabla_4 \phi|^2 + |\nabla \phi|^2 + V|\phi|^2].
\]
\[
F^T_v[\phi](u_1, u_2) = \int_{u_1}^{u_2} du \sin \theta d\theta d\phi \Omega^2 \left[|\Omega^{-1} \nabla_3 \phi|^2 + r^2|\nabla \phi|^2 + |\phi|^2\right].
\] (79)

- Total energy flux:
  \[
  F[\phi] = \sup_v F^T_w[\phi](v_0, \infty) + \sup_v F^T_v[\phi](u_0, \infty)
  \]
- Initial energy at \((u_0, v_0)\):
  \[
  F_0[\phi] = F^T_w[\phi](v_0, \infty) + F^T_v[\phi](u_0, \infty).
  \]

2. Weighted spacetime bulk energies:
   - Degenerate Morawetz bulk
     \[
     I_{\text{deg}}[\phi] = \int_{u_0}^{\infty} \int_{v_0}^{\infty} du dv \sin \theta d\theta d\phi \Omega^2 \left[\frac{1}{r^2}|R^* \phi|^2 + \frac{1}{r^3} |\phi|^2\right] + \left(r^2 - 3Mr + 2Q^2\right)^2 \left[|\nabla \phi|^2 + \frac{1}{r^2} |\Omega \nabla_4 \phi|^2 + \frac{1}{r^2} |\Omega^{-1} \nabla_3 \phi|^2\right]
     \]
   - Weighted bulk norm in the far-away region
     \[
     I_2[\phi] = \int_{u_0}^{\infty} \int_{v_0}^{\infty} du dv \sin \theta d\theta d\phi \cdot i_{r \geq R} \left[r|\Omega \nabla_4 \phi|^2 + r^{-1-\delta}|\nabla_3 \phi|^2 + r^{1+\delta}|\nabla \phi|^2 + r^{-1-\delta}|\phi|^2\right]
     \]
     where \(i_{r \geq R}\) is the indicator function which equals 1 for \(r \geq R\) and is zero otherwise.

7.1.2. Weighted energies for \(\tilde{\beta}_{l=1}\) and \(\tilde{\beta}_{l=1}'\)

We define the following weighted energies for \(\tilde{\beta}_{l=1}\) and \(\tilde{\beta}_{l=1}'\).

1. Total energy fluxes:
   \[
   F[\phi, \tilde{\beta}_{l=1}] = F[\phi] + \sup_v \int_{v_0}^{\infty} dv \sin \theta d\theta d\phi r^{10-\delta} \left(|(r \div \tilde{\beta})_{l=1}|^2 + |(r \curl \tilde{\beta})_{l=1}|^2\right)
   \]
   \[
   F[\phi, \tilde{\beta}_{l=1}'] = F[\phi] + \sup_v \int_{v_0}^{\infty} dv \sin \theta d\theta d\phi r^{8} \left(|(r \div \tilde{\beta}')_{l=1}|^2 + |(r \curl \tilde{\beta}')_{l=1}|^2\right). \tag{80}
   \]

2. Total energy fluxes with first derivative:
   \[
   F[\phi, D_{\tilde{\beta}_{l=1}}] = F[\phi, \tilde{\beta}_{l=1}] + \sup_v \int_{v_0}^{\infty} dv \sin \theta d\theta d\phi r^{10-\delta} \left(|D(\Omega (r \div \tilde{\beta})_{l=1})|^2 + |D(\Omega (r \curl \tilde{\beta})_{l=1})|^2\right)
   \]
   \[
   F[\phi, D_{\tilde{\beta}_{l=1}'}] = F[\phi, \tilde{\beta}_{l=1}'] + \sup_v \int_{v_0}^{\infty} dv \sin \theta d\theta d\phi r^{8} \left(|D(\Omega^{-1} (r \div \tilde{\beta}')_{l=1})|^2 + |D(\Omega^{-1} (r \curl \tilde{\beta}')_{l=1})|^2\right)
   \] (81)
   \]
   where the notation \(D\) is defined\(^8\) by \(|D\xi|^2 = |r \Omega^{-1} \nabla_x \xi|^2 + |r \Omega \nabla_4 \xi|^2\) and \(D \Omega \) is \(|D\xi|^2 = |\Omega^{-1} \nabla_3 \xi|^2 + |r \Omega \nabla_4 \xi|^2|.

\(^8\)Observe that since those are energies for scalar supported in \(l = 1\) spherical harmonics, the control on the angular derivative is not needed, but is given by the zero-th order term.
3. Initial energy at \((u_0, v_0)\):
\[
F_0[\phi, \beta_{\ell=1}] = F_0[\phi] + \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \Omega^2 r^{10-\delta} \left( |(r \text{div } \beta)_{\ell=1}|^2 + |(r \text{curl } \beta)_{\ell=1}|^2 \right) (u, v)
\]
and the equivalent definition for \(F_0[\phi, D\beta_{\ell=1}]\) and \(F_0[\phi, D\beta_{\ell=1}]\).

4. Bulk norms:
\[
\mathbb{I}[\phi, \beta_{\ell=1}] = \mathbb{I}[\phi] + \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \Omega^4 r^{9-\delta} \left( |(r \text{div } \beta)_{\ell=1}|^2 + |(r \text{curl } \beta)_{\ell=1}|^2 \right)
\]
\[
\mathbb{I}[\phi, D\beta_{\ell=1}] = \mathbb{I}[\phi] + \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial r^3 \left( |(r \text{div } \beta)_{\ell=1}|^2 + |(r \text{curl } \beta)_{\ell=1}|^2 \right).
\]

5. Bulk norms with first derivatives:
\[
\mathbb{I}[\phi, \partial \beta_{\ell=1}] = \mathbb{I}[\phi] + \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \Omega^3 r^{10-\delta} \left( |(r \text{div } \beta)_{\ell=1}|^2 + |(r \text{curl } \beta)_{\ell=1}|^2 \right)
\]
\[
\mathbb{I}[\phi, \partial \beta_{\ell=1}] = \mathbb{I}[\phi] + \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \beta \left( |\text{curl } \partial \beta_{\ell=1}|^2 + |\text{curl } \partial \beta_{\ell=1}|^2 \right).
\]

7.13. Weighted energies for \(\mathcal{D}_{r} \beta\) and \(\mathcal{D}_{r} \beta\). We define the following weighted energies for \(\mathcal{D}_{r} \beta\) and \(\mathcal{D}_{r} \beta\).

1. Total energy fluxes:
\[
F[\mathcal{D}_{r} \beta] = \sup_u \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \Omega^2 r^{10-\delta} |r \mathcal{D}_{r} \beta|^2
\]
\[
F[\mathcal{D}_{r} \beta] = \sup_v \int_{v_0}^{\infty} dv \sin \theta \partial \partial_{\phi} \partial r^3 |r \mathcal{D}_{r} \beta|^2.
\]

2. Total energy fluxes with first derivative:
\[
F[\mathcal{D}_{r} \beta] = F[\mathcal{D}_{r} \beta] + \sup_u \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \Omega^4 r^{10-\delta} |\mathcal{D}(r \mathcal{D}_{r} \beta)|^2
\]
\[
F[\mathcal{D}_{r} \beta] = F[\mathcal{D}_{r} \beta] + \sup_v \int_{v_0}^{\infty} dv \sin \theta \partial \partial_{\phi} \partial r^3 |\mathcal{D}(r \mathcal{D}_{r} \beta)|^2
\]

where the notation \(\mathcal{D}\) is defined by \(|\mathcal{D} \xi|^2 = |r \Omega^{-1} \nabla_3 \xi|^2 + |r \nabla \nabla_4 \xi|^2 + |r \nabla \xi|^2\) and \(\nabla\) is \(|\nabla \xi|^2 = |\Omega^{-1} \nabla_3 \xi|^2 + |r \Omega^{-1} \nabla_4 \xi|^2 + |r \nabla \xi|^2|\).

We define in the obvious way the initial energy \(F_0[\mathcal{D}_{r} \beta], F_0[\mathcal{D}_{r} \beta], F_0[\mathcal{D}_{r} \beta]\) and \(F_0[\mathcal{D}_{r} \beta]\).

3. Bulk norms:
\[
\mathbb{I}[\mathcal{D}_{r} \beta] = \int_{u_0}^{\infty} du \sin \theta \partial \partial_{\phi} \partial \Omega^4 r^{10-\delta} |r \mathcal{D}_{r} \beta|^2
\]
\[
\mathbb{I}[\mathcal{D}_{r} \beta] = \int_{v_0}^{\infty} dv \sin \theta \partial \partial_{\phi} \partial r^3 |r \mathcal{D}_{r} \beta|^2.
\]
4. Bulk norms with first derivatives:
\[
I[\mathcal{D} \mathcal{F}^{1/2}] = \int_{\Omega_0} \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \, \sin \theta d\theta d\phi \, \Omega^1 r^{4-\delta} |\mathcal{D}(\Omega r \mathcal{D}^{1/2})|^2 \\
I[\mathcal{DF}^{1/2}] = \int_{\Omega_0} \int_{\Omega_0} \mathcal{DF}^{1/2} \, \sin \theta d\theta d\phi \, r^{7-\delta} |\mathcal{D}(\Omega^{1/2} r \mathcal{D}^{1/2})|^2.
\]

7.1.4. **Weighted energies for \( \tilde{\beta} \) and \( \tilde{\beta}' \).** We define the following weighted energies for \( \tilde{\beta} \) and \( \tilde{\beta}' \).

1. Total energy fluxes:
\[
\mathcal{F}[\tilde{\beta}] = \sup_{u} \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \, \sin \theta d\theta d\phi \, \Omega^1 r^{10-\delta} |\tilde{\beta}|^2 + \Omega^2 r^{10-\delta} |\mathcal{D}(\Omega \tilde{\beta})|^2 \\
\mathcal{F}[\tilde{\beta}'] = \sup_{v} \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \, \sin \theta d\theta d\phi \, \Omega^1 r^{8} |\tilde{\beta}'|^2 + \Omega^2 r^{8} |\mathcal{D}(\Omega \tilde{\beta}')|^2
\]

and the initial energies \( \mathcal{F}_0[\tilde{\beta}] \) and \( \mathcal{F}_0[\tilde{\beta}'] \).

2. Bulk norms:
\[
I[\tilde{\beta}] = \int_{\Omega_0} \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \, \sin \theta d\theta d\phi \, \Omega^1 r^{9-\delta} |\tilde{\beta}|^2 + \Omega^2 r^{9-\delta} |\mathcal{D}(\Omega \tilde{\beta})|^2 \\
I[\tilde{\beta}'] = \int_{\Omega_0} \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \, \sin \theta d\theta d\phi \, r^{7-\delta} |\mathcal{D}^{1/2} \tilde{\beta}|^2 + r^{7-\delta} |\mathcal{D}(\Omega^{1/2} \tilde{\beta})|^2.
\]

7.1.5. **Weighted energies defined in [15].** We recall here the initial energy defined in [15]. This energy represents the initial data in the main theorem.

The initial energy for \( q, q^F, q \) and \( q^F \) are given by
\[
\mathcal{F}_0[q] = \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \left[ r^2 |\nabla q|^2 + |\nabla q|^2 + r^{-2} |q|^2 \right] (u_0, v) \\
+ \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \left[ |\Omega^{-1} \nabla q|^2 + r^2 |\nabla q|^2 + |q|^2 \right] (u, v_0),
\]
\[
\mathcal{F}_0[q^F] = \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \left[ r^2 |\nabla q^F|^2 + |\nabla q^F|^2 + r^{-2} |q^F|^2 \right] (u_0, v) \\
+ \int_{\Omega_0} \mathcal{D} \mathcal{F}^{1/2} \left[ |\Omega^{-1} \nabla q^F|^2 + r^2 |\nabla q^F|^2 + |q^F|^2 \right] (u, v_0)
\]

and identical energies for \( q \) and \( q^F \).
The initial energy for $f$, $\psi_1$ and $\alpha$ are given by

$$F_0[f] = \int_{0}^{\infty} dv r^2 \left[ r^2 |\nabla \psi_1|^2 + |\psi_1|^2 + r^{-2} |f|^2 \right] (u_0, v)
+ \int_{0}^{\infty} du \Omega^2 r^2 \left[ r^2 |\nabla \psi_1|^2 + r^2 |\nabla |^2 + r^2 |\psi_1|^2 \right] (u, u_0)$$

$$F_0[\psi_1] = \int_{0}^{\infty} dv r^2 \left[ r^2 |\nabla \psi_1|^2 + |\psi_1|^2 + r^{-2} |\psi_1|^2 \right] (u_0, v)
+ \int_{0}^{\infty} du \Omega^2 r^2 \left[ r^2 |\nabla \psi_1|^2 + r^2 |\psi_1|^2 + r^2 |\psi_1|^2 \right] (u, u_0)$$

$$F_0[\alpha] = \int_{0}^{\infty} dv r^2 \left[ r^2 |\nabla \alpha|^2 + |\alpha|^2 + r^{-2} |\alpha|^2 \right] (u_0, v)
+ \int_{0}^{\infty} du \Omega^2 r^2 \left[ r^2 |\nabla \alpha|^2 + r^2 |\alpha|^2 + r^2 |\alpha|^2 \right] (u, u_0).$$

We define

$$F_0[q, q^F, f, \psi_1, \alpha] = F_0[q] + F_0^{1,T,Y}[q^F] + F_0^{1,T,Y}[f] + F_0[\psi_1] + F_0[\alpha]$$

$$F_0[q, q^F, f, \psi_1, \alpha] = F_0[q] + F_0^{1,T,Y}[q^F] + F_0^{1,T,Y}[f] + F_0[\psi_1] + F_0[\alpha].$$

The total initial energies are defined as

$$F_0[q, q^F, f, \psi_1, \alpha, \phi, \beta] = F_0[q, q^F, f, \psi_1, \alpha] + F_0[\phi, \beta_{0-1}] + F_0[D \beta] + F_0[D \alpha] + F_0[D \beta_0^2]$$

$$F_0[q, q^F, f, \psi_1, \alpha, \phi, \beta] = F_0[q, q^F, f, \psi_1, \alpha] + F_0[\phi, \beta_{0-1}] + F_0[D \beta] + F_0[D \alpha] + F_0[D \beta_0^2].$$

**7.1.6. Higher order energies.** To estimate higher order energies we also introduce the following notation, motivated by the fact that the Fackerell–Ipser equation commutes with $T$ and the angular momentum operators $\Omega$. We define
1. Higher derivative energies for $n \geq 1$:
\[
\mathcal{P}^n_{\ell, \mathcal{V}}[\phi] = \sum_{i+j \leq n} \sup_{u} F^n_T [T'(r \mathcal{V}_A) / \phi](u_0, \infty) + \sum_{i+j \leq n} \sup_{v} F^n_T [T'(r \mathcal{V}_A) / \phi](v_0, \infty)
\]
\[
\mathcal{P}^n_0[\phi] = \sum_{i+j \leq n} F^n_T [T'(r \mathcal{V}_A) / \phi](v_0, \infty) + \sum_{i+j \leq n} F^n_T [T'(r \mathcal{V}_A) / \phi](u_0, \infty)
\]

and similarly for $\mathcal{P}^n_{\ell, \mathcal{V}}[\phi, D_{\ell=1}], \mathcal{P}^n_{\ell, \mathcal{V}}[\phi, D_{\ell=1}]$, and $\mathcal{P}^n_{\ell, \mathcal{V}}[\beta]$. We also similarly define $\mathcal{P}^n_{\ell, \mathcal{V}}[\beta], \mathcal{P}^n_0[\phi, q^\ell, f, \psi_1, \alpha, \phi, \beta]$.

2. Higher derivative spacetime bulks:
\[
\mathcal{P}^n_{\ell, \mathcal{V}}[\phi] = \sum_{i+j \leq n} \| \mathcal{I}[T'(r \mathcal{V}_A) / \phi] \|_{L^2}, \quad \mathcal{P}^n_0[\phi] = \sum_{i+j \leq n} \| \mathcal{I}[T'(r \mathcal{V}_A) / \phi] \|_{L^2} (90)
\]

and similarly for $\mathcal{P}^n_{\ell, \mathcal{V}}[\phi, D_{\ell=1}], \mathcal{P}^n_{\ell, \mathcal{V}}[\phi, D_{\ell=1}]$, and $\mathcal{P}^n_{\ell, \mathcal{V}}[\beta], \mathcal{P}^n_{\ell, \mathcal{V}}[\beta]$.

72. Precise statement of the main theorem

We are now ready to state the boundedness and decay theorem for solutions $\tilde{\beta}$ and $\tilde{\beta}$ of the generalized Teukolsky equation of spin $\pm 1$.

In the estimates below we denote $A \lesssim B$ if there exists an universal constant $C$ such that $A \leq CB$.

**Main theorem.** Let $\tilde{\beta}$ and $\tilde{\beta}$ be solutions to the generalized Teukolsky equation of spin $\pm 1$ respectively in Reissner–Nordström spacetime for $|Q| \ll M$. Then the following estimates hold:

1. weighted boundedness and integrated decay estimate for $\tilde{\beta}$ and $\tilde{\beta}$:
\[
\mathcal{P}[\tilde{\beta}] + \mathcal{I}[\tilde{\beta}] \lesssim F_0[q, q^\ell, f, \psi_1, \alpha, \phi, \beta] (91)
\]

2. higher order energy and integrated decay estimates for any integer $n \geq 1$:
\[
\mathcal{P}^n_{\ell, \mathcal{V}}[\tilde{\beta}] + \mathcal{P}^n_{\ell, \mathcal{V}}[\tilde{\beta}] \lesssim \mathcal{P}^n_0[q, q^\ell, f, \psi_1, \alpha, \phi, \beta] (92)
\]

3. pointwise decay estimates:
\[
|r^{j+1} \tilde{\beta}| \lesssim C_0^{-\frac{j+1}{2}}, \quad |r^j \tilde{\beta}| \lesssim C_0^{-\frac{j+1}{2}} (93)
\]

where $C$ depends on appropriate higher Sobolev norms.

73. The logic of the proof

The remainder of the paper concerns the proof of the main theorem. We outline here the main steps of the proof.

1. Given a solution $\tilde{\beta}$ and $\tilde{\beta}$ of the generalized Teukolsky equation of spin $\pm 1$ we can associate a solution $\mathcal{P}$ to the generalized Fackerell–Ipser equation in $\ell = 1$ (72) through
proposition 6.1. We commute equation (72) with \( r \mathcal{D}_1 \) and project it into the \( \ell = 1 \) mode. By corollary B.1 and (30), we have that the scalar quantities \((r \text{div} \mathbf{p})_{\ell=1}\) and \((r \text{curl} \mathbf{p})_{\ell=1}\) verify the wave equation

\[
\Box_g (r \text{div} \mathbf{p})_{\ell=1} - \left( \rho + 4 (F) \right) (r \text{div} \mathbf{p})_{\ell=1} = 0
\]

\[
\Box_g (r \text{curl} \mathbf{p})_{\ell=1} - \left( \rho + 4 (F) \right) (r \text{curl} \mathbf{p})_{\ell=1} = 0
\]

(94)

and similarly for \((r \text{div} \mathbf{p})_{\ell=1}\) and \((r \text{curl} \mathbf{p})_{\ell=1}\). Equation (94) are scalar wave equations for which integrated local energy estimates can be obtained in a standard way. We obtain energy conservation, degenerate Morawetz estimates, redshift estimates and \(\mathcal{F}^n\) hierarchy estimates for the scalar quantity \((r \mathcal{D}_1 \mathbf{p})_{\ell=1}\). This is done in section 8.

2. Using the relations (74), and applying enhanced transport estimates, from the control obtained for \((r \mathcal{D}_1 \mathbf{p})_{\ell=1}\) we will get estimates for the scalar quantities \((r \mathcal{D}_1 \tilde{\mathcal{B}})_{\ell=1}\) and \((r \mathcal{D}_1 \tilde{\mathcal{B}})_{\ell=1}\), which give the \( \ell = 1 \) spherical mode of \( \tilde{\mathcal{B}} \) and \( \tilde{\mathcal{B}} \). This is done in section 9.1.

3. To control the projection to the \( \ell \geq 2 \) spherical modes of \( \tilde{\mathcal{B}} \) and \( \tilde{\mathcal{B}} \) we will make use of the relations between \( \tilde{\mathcal{B}} \) and \( \tilde{\mathcal{B}} \) and \( f \), and similarly for the negative spin quantities, shown in lemma 6.1. We will then make use of the estimates obtained in [15] for \( \tilde{\mathcal{B}} \) and \( f \). This is done in section 9.2.

The two points above then imply the main theorem, i.e. the integrated decay statements about the solutions \( \tilde{\mathcal{B}} \) and \( \tilde{\mathcal{B}} \) of the generalized Teukolsky equation of spin \( \pm 1 \), through standard elliptic estimates. This is finally done in section 9.3.

8. Estimates for the \( \ell = 1 \) spherical mode of \( \mathbf{p} \)

The present section contains the proof of the estimates for the projection to the \( \ell = 1 \) spherical mode of the solution to the generalized Fackerell–Ipser equation \( \mathbf{p} \). This proof follows closely previous works for the scalar wave equation (see [11–13]), for the Regge–Wheeler equation (see [9]) and for the Fackerell–Ipser equation (see [24]).

We summarize the main result in the following proposition.

**Proposition 8.1.** Let \( \mathbf{p} \) and \( \mathbf{p} \) be solutions to the generalized Fackerell–Ipser equation in \( \ell = 1 \) (72) in Reissner–Nordström spacetime for \( |Q| < M \). Let \( \phi = (r^2 \text{div} \mathbf{p})_{\ell=1} \) or \( \phi = (r^2 \text{curl} \mathbf{p})_{\ell=1} \) or \( \phi = (r^2 \text{div} \mathbf{p})_{\ell=1} \) or \( \phi = (r^2 \text{curl} \mathbf{p})_{\ell=1} \). Then the following estimates hold for all \( \delta \leq p \leq 2 \):

1. energy boundedness, degenerate integrated local energy decay and \( \mathcal{F}^n \) hierarchy estimates for \( \phi \):

\[
\mathcal{F}[\phi] + \mathcal{I}_{\text{deg}}[\phi] + \mathcal{I}_\mathcal{L}[\phi] \lesssim \mathcal{F}_0[\phi]
\]

(95)

2. higher order energy and integrated decay estimates for any integer \( n \geq 1 \):

\[
\mathcal{F}_n[T,\mathcal{F}][\phi] + \mathcal{I}_{\text{deg}}[\phi] + \mathcal{I}_{\mathcal{F}_p}[\phi] \lesssim \mathcal{F}_n[T,\mathcal{F}][\phi].
\]

(96)

We begin in section 8.1 to write the projection to the \( \ell = 1 \) mode of (72) in double null coordinates. We derive the energy conservation for the obtained scalar wave equation in
section 8.2. We then show a version of integrated decay which degenerates at the photon sphere, at the horizon and at null infinity in section 8.3. We remove the degeneration at the horizon making use of the redshift vectorfield in section 8.4, and we refine the degeneration at null infinity through the $j^p$ hierarchy estimates in section 8.5. Finally, we derive higher order energy estimates in section 8.6.

8.1. The projection of the Fackerell–Ipser equation to the $\ell = 1$ mode

Let $p$ and $\mathbf{p}$ be a solution of the generalized Fackerell–Ipser equation in $\ell = 1$ (72).

By (94), defining $\bar{\phi}$ as $\bar{\phi} = (r\text{div}\mathbf{p})_\ell = 1$ or $\bar{\phi} = (r\text{curl}\mathbf{p})_\ell = 1$, or $\bar{\phi} = (r\text{div}\mathbf{p})_\ell = 1$, then $\bar{\phi}$ verifies

$$\Box g \bar{\phi} - (\rho + 4(F) \rho^2) \bar{\phi} = 0.$$  

(97)

By corollary A.1, equation (97) can be written in double null coordinates as

$$\Omega \nabla_3(\Omega \nabla_4(r \bar{\phi})) - \Omega^2 \Delta(r \bar{\phi}) + 4(F) \rho^2 \Omega^2 r \bar{\phi} = 0.$$  

Define $\phi = r \bar{\phi}$ the equation becomes

$$\Omega \nabla_3(\Omega \nabla_4(\phi)) - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \Delta \phi + V \phi = 0$$  

(98)

where $V = 4(F) \rho^2 \Omega^2 = \frac{4Q^2}{r^2} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$.

Remark 8.1. Equation (98) reduces to the scalar form of the Fackerell–Ipser equation in Schwarzschild $(1 - \mu)^{-1} L u - \Delta u = 0$ when $Q = 0$ (see equation (2.14) in [24]).

8.2. Energy conservation

Let $\phi$ be a smooth solution of (98) on $\{u \geq u_0\} \cap \{v \geq v_0\}$. By standard computations, the equation (98) implies

$$[\Omega \nabla_3 + \Omega \nabla_4] \int \sin \theta d\theta d\phi [\{\Omega \nabla_4 \phi\}^2 + |\Omega \nabla_3 \phi|^2 + 2 \frac{2M}{r} + \frac{Q^2}{r^2} |r \nabla \phi|^2 + 2V|\phi|^2]$$

$$+ [\Omega \nabla_3 - \Omega \nabla_4] \int \sin \theta d\theta d\phi [|\Omega \nabla_4 \phi|^2 - |\Omega \nabla_3 \phi|^2] = 0.$$  

Integrating the above with respect to $du dv$ yields a conservation law: for any $u \geq u_0$ and $v \geq v_0$ the $\phi$ of proposition 8.1 satisfies

$$F^T_u[\psi](v_0, v) + F^T_v[\psi](u_0, u) = F^T_{u_0}[\psi](v_0, v) + F^T_{v_0}[\psi](u_0, u).$$  

(99)

8.3. Morawetz estimates

Let $f$ be a function on $\mathcal{M}$ of $r^* = v - u$ only, and denote $f' := R^*(f)$. Following [9], equation (98) implies the identity
where the symbol $=S$ means that the above identity holds after integration over $\int \sin \theta d\theta d\phi$.

After integrating the above identity in the spacetime with respect to the measure $\int dudv \sin \theta d\theta d\phi$, the last line gives a spacetime energy, while the other lines give boundary terms.

We want to choose $f(r)$ such that the last line in (100) gives a coercive spacetime energy. The choice below is a generalization of the choice appeared in [9, 24] to the case of the subextremal Reissner–Nordström spacetime for which $Q \ll M$.

Recall that $\phi$ is a scalar function supported in $\ell = 1$ spherical harmonics. Therefore we have the Poincaré inequality

$$\int_{S^2} |\nabla \phi|^2 \geq \frac{2}{r^2} \int_{S^2} |\phi|^2.$$  

Therefore, the last line of (100) integrated on the sphere can be bounded from below by

$$\int \sin \theta d\theta d\phi \left\{ f'(r)|R^*\psi|^2 + |r \nabla \phi|^2 \left( -4f \left( \frac{\Omega^2}{r^2} \right) \right) + |\phi|^2 (-4f'V' - 2f'') \right\} \geq \int \sin \theta d\theta d\phi \left\{ f'(r)|R^*\psi|^2 + \left( -4f \left( V + \frac{2\Omega^2}{r^2} \right)' - 2f'' \right) |\phi|^2 \right\}.  \tag{101}$$

We want to choose $f(r)$ such that

1. $\frac{f'}{r^2} \geq \frac{c}{r}$,
2. $-4f'\left( V + \frac{2\Omega^2}{r^2} \right)' - 2f'' \geq \frac{c}{r}$

for some positive constants $c$. Indeed, the above conditions will give a coercive estimate for the bulk term in the Morawetz estimates.

We define

$$f = \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \left( 1 + \frac{M}{r} + \frac{Q^2}{2r^2} \right).$$

We denote with the subscript $r$ the derivative with respect to $r$. We compute

$$f_r = \frac{1}{2r^2} \left( 4Mr^3 + (12M^2 - 10Q^2)r^2 - 3MQ^2r - 8Q^4 \right),$$
$$f_{rr} = \frac{1}{r^2} \left( -4Mr^3 + (-18M^2 + 15Q^2)r^2 + 6MQ^2r + 20Q^4 \right),$$
$$f_{rrr} = \frac{1}{r^3} \left( 12Mr^3 + (72M^2 - 60Q^2)r^2 - 30MQ^2r - 120Q^4 \right).$$

In Reissner–Nordström metric, define $0 < \gamma < 1$ such that $Q^2 = \gamma M^2$. Then we obtain

$$f_r = \frac{1}{2r^2} \left( 4Mr^3 + (12 - 10\gamma)M^2r^2 - 3\gamma M^4r - 8\gamma^2 M^4 \right).$$
The horizon is then given by \( r_H = M + \sqrt{M^2 - Q^2} = M(1 + \sqrt{1 - \gamma}) \). Define \( r = M(1 + \sqrt{1 - \gamma})x \) we have that

\[
f_h = \frac{M^4}{2\gamma^3} \left( 4(1 + \sqrt{1 - \gamma})^3x^3 + (12 - 10\gamma)(1 + \sqrt{1 - \gamma})^2x^2 - 3\gamma(1 + \sqrt{1 - \gamma})x - 8\gamma^2 \right).
\]

The polynomial on the right hand side is clearly positive in \( x \in [1, \infty] \) for \( \gamma = 0 \). Therefore, it is also positive for \( \gamma \) small enough.

Now we analyze the expression \(-\frac{4f}{\Omega^2}(V + \frac{2\Omega^2}{r^2}) - \frac{2f'''}{\Omega^2} \). Recall that \( V = \frac{4Q^2}{\gamma}\Omega^2 \), therefore

\[
-\frac{4f}{\Omega^2}(V + \frac{2\Omega^2}{r^2}) - \frac{2f'''}{\Omega^2} = -\frac{4f}{\Omega^2} \left( \frac{2Q^2}{r^2} + 1 \right) \left( \frac{2\Omega^2}{r^2} \right)' - \frac{2f'''}{\Omega^2}.
\]

(102)

We compute the derivative of \( f \) with respect to \( R \), where recall that \( R^*(r) = \Omega^2 \). We therefore have

\[
f''(r) = \Omega^2 f''(r) + (\Omega^2)^2 \Omega f''(r),
\]

\[
f'''(r) = 3(\Omega^2)^2 \Omega f'' + (\Omega^2)^2 (\Omega^2)'^2 + \Omega^4 (\Omega^2)'^2, f(r).
\]

Therefore the expression (102) becomes

\[
\frac{1}{r^4} \left( 656Q^8 - 1822MQ^6r^3 + 1604Q^6r^3 + 2412M^3Q^2r^3 - 2986MQ^4r^3 - 1152M^4r^3 + 404M^2Q^2r^4 + 1072Q^2r^4 + 744M^4r^3 - 1224MQ^2r^3 + 64M^6r^5 + 256Q^4r^5 - 104Mr^7 + 16x^3 \right).
\]

Plugging in \( Q^2 = \gamma M^2 \) and \( r = M(1 + \sqrt{1 - \gamma})x \), we obtain the polynomial

\[
656\gamma^4 - 1822\gamma^3(1 + \sqrt{1 - \gamma})x + 1604\gamma^3(1 + \sqrt{1 - \gamma})x^2 + 2412\gamma^2(1 + \sqrt{1 - \gamma})x^3 - 2986\gamma\Omega^2x^4 - 1152\gamma\Omega^2x^4 + 404\gamma^2x^4 - 1072\gamma(1 + \sqrt{1 - \gamma})x^5 + 744\gamma(1 + \sqrt{1 - \gamma})x^6 - 1224\gamma(1 + \sqrt{1 - \gamma})x^7 + 64\gamma(1 + \sqrt{1 - \gamma})x^8 + 256\gamma(1 + \sqrt{1 - \gamma})x^9 - 104(1 + \sqrt{1 - \gamma})x^10 + 16(1 + \sqrt{1 - \gamma})x^11 - 72.
\]

Consider the case of \( \gamma = 0 \). Then the above polynomial, up to dividing by \( 2^4x^4 \), reduces to

\[
16x^3 - 52x^3 + 16x^2 + 93x - 72.
\]

The above polynomial evaluated at \( x = 1 \) is equal to 1, and therefore positive. We now prove that its derivative, \( 64x^3 - 156x^2 + 32x + 93 \) is positive for \( x \geq 1 \). The derivative has its minimum for \( x = 1 \) at \( x = \frac{31 + \sqrt{1465}}{4} \), where its value can be checked being positive. This implies positivity of the polynomial in \( x \in [1, \infty] \) for \( \gamma = 0 \). For \( \gamma \) small enough, the above expression is therefore also positive.

Upon integrating (100) with respect to the measure \( du dv \sin \theta dv d\phi \) over any spacetime region of the form \([u_0, u] \times [v_0, v] \times S_{\alpha \beta} \) with \( f \) chosen as above, we see that we can estimate all boundary terms by the \( T \) fluxes at \( u, v, u_0 \) and \( v_0 \). By the conservation of energy (99), they can therefore be bounded by \( F_{\mu\nu}^U[u_0, u] + F_{\mu\nu}^U[v_0, v] \). We estimate the total term by (101) and making use of conditions 1. and 2. and of the positivity of the angular term, we can obtain

\[
\int_{u_0}^u \int_{v_0}^v du dv \sin \theta dv d\phi \Omega^2 \left\{ \frac{1}{r^2} |R^*(r)|^2 + \frac{1}{r^2} |\phi|^2 + \frac{(r^2 - 3Mr + 2Q^2)^2}{r^5} |\nabla \phi|^2 \right\}
\]

\[
\lesssim F_{\mu\nu}^U[u_0, u] + F_{\mu\nu}^U[v_0, v] \]
Finally, a standard argument allows to recover the missing derivative in the above bulk estimate, integrating the relation above using an increasing function $f$, vanishing at third order at $r = r_p$. We finally get the following Morawetz estimate:

$$
\int_{\mathbb{M}} \int_{\mathbb{S}} d\nu d\omega \sin \theta d\theta d\phi d\Omega \left\{ \frac{1}{r^2} |R^r \phi|^2 + \frac{1}{r^2} |\phi|^2 + \frac{(r^2 - 3Mr + 2G)^2}{r^5} \left( |\nabla \phi|^2 + \frac{1}{r^2} |T \phi|^2 \right) \right\} 
\lesssim F_{\text{deg}}^n[\phi](v_0, v) + F_{\text{deg}}^n[\phi](u_0, u).
$$

Observe that the above bulk has a degeneration at the photon sphere caused by the trapping phenomenon at $r = r_p$.

8.4. Redshift estimates

Observe that the above bulk has a degeneration at the horizon, since it does not control all derivatives at $r = r_H$. We can eliminate this degeneracy by making use of the redshift vectorfield. See [24] for a derivation of the redshift estimate.

Through the standard techniques of the redshift vectorfield, we can improve the energy conservation (99) to the following non-degenerate version:

$$
F_u[\phi](v_0, v) + F_v[\phi](u_0, u) \lesssim F_{v_0}[\phi](u_0, u) + F_{u_0}[\phi](v_0, v)
$$

and the following improved version of the Morawetz estimate (103):

$$
\int_{\mathbb{M}} \int_{\mathbb{S}} d\nu d\omega \sin \theta d\theta d\phi d\Omega \left\{ \frac{1}{r^2} |R^r \phi|^2 + \frac{1}{r^2} |\phi|^2 + \frac{(r^2 - 3Mr + 2G)^2}{r^5} \left( |\nabla \phi|^2 + \frac{1}{r^2} |\Omega \nabla \phi|^2 + \frac{1}{r^2} |\Omega^{-1} \nabla \phi|^2 \right) \right\} 
\lesssim F_{u_0}[\phi](v_0, v) + F_{u_0}[\phi](u_0, u).
$$

Taking the limit $u, v \to \infty$, the left hand side is given by the degenerate Morawetz bulk $\mathbb{I}^\text{deg}[\phi]$. The two estimates above therefore give, for a $\phi$ as in proposition 8.1 the following boundedness estimate:

$$
\sup_u F_u[\phi](v_0, \infty) + \sup_v F_v[\phi](u_0, \infty) \lesssim F_{u_0}[\phi](v_0, \infty) + F_{u_0}[\phi](u_0, \infty)
$$

and the integrated decay estimate

$$
\mathbb{I}^\text{deg}[\phi] \lesssim F_{u_0}[\phi](v_0, \infty) + F_{v_0}[\phi](u_0, \infty).
$$

8.5. The $r^p$ hierarchy estimates

We will adapt the $r^p$ hierarchy estimates [12] to the Fackerell–Ipser equation, as done in [9, 24].

Equation (98) implies the following identity\(^9\), for $p$ and $k$ real numbers:

$$
0 = \partial_u \left( \frac{r^p}{\Omega^{2k-2}} |\nabla \phi|^2 \right) + \partial_v \left( \frac{r^p}{\Omega^{2k-2}} |\nabla \phi|^2 \right) + \partial_u \left( \frac{r^p V}{\Omega^{2k}} |\phi|^2 \right) - \partial_v \left( \frac{r^p V}{\Omega^{2k}} |\phi|^2 \right) + \left( (2 - p) r^{p-1} \Omega^{2-k} + r^p (k - 1) \Omega^{-2k+2} (\Omega^2)^\prime \right) |\nabla \psi|^2.
$$

We integrate (107) for $1 \leq p \leq 2$ with respect to the measure $d\nu d\omega \sin \theta d\theta d\phi$ in a region of the form

$$
\mathcal{R} = \{(u, v) \in \mathcal{M} : r \geq R, u_0 \leq u \leq u_{\text{fin}}, v_0 \leq v \leq v_{\text{fin}}\}.
$$

\(^9\) See equation (296) in [9].
for sufficiently large $R$, $u_{\text{fin}}$ and $v_{\text{fin}}$. We fix $R$ big enough so that the following holds in the region $\{ r \geq R \}$:

$$-\partial_u \left( \frac{r^p}{\Omega^{2k}} \right) \geq \frac{1}{2} r^p$$

for all $1 \leq p \leq 2$ and $k \leq 5$.

We also calculate

$$-\partial_r \left( \frac{r^p V}{\Omega^{2k}} \right) = -4Q^2 \partial_r \left( \frac{r^{p-4}}{\Omega^{2k-2}} \right) = -4Q^2 (p-4) r^{p-5} \Omega^{-2k+2} \partial_r r - 4Q^2 (1-k) r^{p-4} \Omega^{-2k} (\Omega^2)' \partial_r r$$

$$= 4Q^2 \frac{\partial_r r}{\Omega^{2k}} ((4-p) r^{p-5} \Omega^2 + (k-1) r^{p-4} (\Omega^2)').$$

Recall that

$$\Omega^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (\Omega^2)' = \frac{2M}{r^2} - \frac{2Q^2}{r^3}.$$

The above computation becomes

$$-\partial_v \left( \frac{r^p V}{\Omega^{2k}} \right) = 4Q^2 \partial_v \left( \frac{r^{p-4}}{\Omega^{2k-2}} \right) = ((4-p) r^{p-5} + 2M r^{p-6} (k + p - 5) + Q^2 r^{p-7} (-2k - p + 6)).$$

This implies that given any $1 \leq p \leq 2$, the choice $k = 5$ ensures that

$$-\partial_v \left( \frac{r^p V}{\Omega^{2k}} \right) \geq 2r^{p-6}$$

holds in $\mathcal{R}$ for sufficiently large $R$. Integrating (107) in the spacetime region $\mathcal{R}$ for $p = 2 - \delta$, and using the Morawetz estimate (106) to bound the terms on the timelike hypersurface $r = R$, we obtain

$$\int_\mathcal{R} dudv \sin \theta d\theta d\phi \left[ r |\nabla_4 \psi|^2 + r^{1-\delta} |\nabla \phi|^2 + r^{-1-\delta} |\phi|^2 \right]$$

$$\leq C \int_\mathcal{R} dudv \sin \theta d\theta d\phi \left[ r^2 |\nabla_4 \phi|^2 \right] (u_0, v) + C(F_u[u](v_0, \infty) + F_v[u](u_0, \infty)) \quad (108)$$

where we applied the Poincaré inequality.

Taking the limit as $u_{\text{fin}}, v_{\text{fin}} \to \infty$ and summing (108) to (106) and to (105), we obtain estimate (95) of proposition 8.1.

8.6. Higher order estimates

We note that the Fackerell–Ipser equation (98) trivially commutes with Lie differentiation with respect to the Killing fields on Reissner–Nordström spacetime. Recalling the definition of the higher order energies (90), we immediately obtain estimate (96) of proposition 8.1.

Exploiting the $r^p$-hierarchy in the standard way, polynomial decay estimates can be obtained for $\phi$ (see for example [9, 24]).

9. Estimates for $\tilde{\beta}$ and $\tilde{\beta}$

We prove here the main theorem. The proof is divided in two parts. First we estimate the projection to $\ell = 1$ spherical mode of $\tilde{\beta}$ and $\tilde{\beta}$, and then we estimate the projection to the $\ell \geq 2$ modes. Finally, we combine the two to prove the theorem.
9.1. Estimates for the $\ell=1$ spherical mode of $\tilde{\beta}$ and $\tilde{\beta}$

In this section we make use of the differential relation between $\tilde{\beta}$ and $p$ (74) together with the estimates obtained for the projection to the $\ell=1$ mode of $p$ in proposition 8.1.

We summarize the estimates in the following proposition.

Proposition 9.1. Let $\tilde{\beta}$ and $\tilde{\beta}$ be solutions to the generalized Teukolsky equation of spin $\pm 1$ respectively in Reissner–Nordström spacetime for $|Q| \ll M$. Then the following estimates hold:

1. weighted boundedness and integrated decay estimate for $\tilde{\beta}_{\ell=1}$ and $\tilde{\beta}_{\ell=1}$:

$$F[\phi, \tilde{\beta}_{\ell=1}] + I[\phi, \tilde{\beta}_{\ell=1}] \lesssim F_0[\phi, \tilde{\beta}_{\ell=1}]$$

$$F[\phi, \tilde{\beta}_{\ell=1}] + I[\phi, \tilde{\beta}_{\ell=1}] \lesssim F_0[\phi, \tilde{\beta}_{\ell=1}]$$ (109)

2. higher order energy and integrated decay estimates for any integer $n \geq 0$:

$$\mathbb{P}^{n,T}[\phi, D\tilde{\beta}_{\ell=1}] + \mathbb{P}^{n,T}[\phi, D\tilde{\beta}_{\ell=1}] \lesssim \mathbb{P}^{n,T}[\phi, D\tilde{\beta}_{\ell=1}]$$

$$\mathbb{P}^{n,T}[\phi, D\tilde{\beta}_{\ell=1}] + \mathbb{P}^{n,T}[\phi, D\tilde{\beta}_{\ell=1}] \lesssim \mathbb{P}^{n,T}[\phi, D\tilde{\beta}_{\ell=1}]$$ (110)

We prove the proposition in the following two subsections, using transport estimates for $\tilde{\beta}$ and $\tilde{\beta}$.

9.1.1. Transport estimates for the projection to $\ell=1$ mode of $\tilde{\beta}$ and $\tilde{\beta}$. We estimate the projection to $\ell=1$ mode of $\tilde{\beta}$ and $\tilde{\beta}$ through some basic transport estimates.

Lemma 9.1. Let $\tilde{\beta}$ be a solution to the generalized Teukolsky equation of spin $+1$ in Reissner–Nordström spacetime for $|Q| \ll M$. Then along any null hypersurface of constant $u$

$$\int_{r_0}^\infty dv \sin \theta d\theta d\phi \Omega^2 r^{10-\delta} \left( |(r\text{div} \tilde{\beta})_{\ell=1}|^2 + |(r\text{curl} \tilde{\beta})_{\ell=1}|^2 \right) \lesssim F_0[\phi, \tilde{\beta}_{\ell=1}].$$

In addition,

$$\int_{\phi_0}^{\infty} \int_{r_0}^{\infty} dv \sin \theta d\theta d\phi \Omega^2 r^{9-\delta} \left( |(r\text{div} \tilde{\beta})_{\ell=1}|^2 + |(r\text{curl} \tilde{\beta})_{\ell=1}|^2 \right) \lesssim F_0[\phi, \tilde{\beta}_{\ell=1}].$$

Let $\tilde{\beta}$ be a solution to the generalized Teukolsky equation of spin $-1$ in Reissner–Nordström spacetime for $|Q| \ll M$. Then along any null hypersurface of constant $\nu$

$$\int_{\phi_0}^{\infty} dv \sin \theta d\theta d\phi \Omega^2 r^{8-\delta} \left( |(r\text{div} \tilde{\beta})_{\ell=1}|^2 + |(r\text{curl} \tilde{\beta})_{\ell=1}|^2 \right) \lesssim F_0[\phi, \tilde{\beta}_{\ell=1}].$$

In addition,

$$\int_{\phi_0}^{\infty} \int_{r_0}^{\infty} dv \sin \theta d\theta d\phi \Omega^2 r^{7-\delta} \left( |(r\text{div} \tilde{\beta})_{\ell=1}|^2 + |(r\text{curl} \tilde{\beta})_{\ell=1}|^2 \right) \lesssim F_0[\phi, \tilde{\beta}_{\ell=1}].$$

Proof. By commuting the differential relation (74) with $r\partial_1$ and projecting into $\ell = 1$ spherical mode, we obtain
\begin{equation}
\text{rdiv } p_{\ell=1} = \frac{1}{\kappa} \nabla_3(r^5 \kappa (\text{rdiv } \tilde{\beta}_{\ell=1})), \quad \text{rcurl } p_{\ell=1} = \frac{1}{\kappa} \nabla_3(r^5 \kappa (\text{rcurl } \tilde{\beta}_{\ell=1}))
\end{equation}

and similarly commuting (75) for spin \(-1\) quantities:

\begin{equation}
\text{rdiv } p_{\ell=1} = \frac{1}{\kappa} \nabla_4(r^5 \kappa (\text{rdiv } \tilde{\beta}_{\ell=1})), \quad \text{rcurl } p_{\ell=1} = \frac{1}{\kappa} \nabla_4(r^5 \kappa (\text{rcurl } \tilde{\beta}_{\ell=1})).
\end{equation}

From (111), we have

\begin{equation}
\nabla_3(r^{10+\kappa^2}|\text{rdiv } \tilde{\beta}_{\ell=1}|^2) = 2r^5 \kappa^2 (\text{rdiv } \tilde{\beta}_{\ell=1})(\text{rdiv } p_{\ell=1}).
\end{equation}

Multiplying by \(r^\delta\) and using that \(\nabla_3 r = \frac{1}{r} \times r\), we have

\begin{equation}
\nabla_3(r^{10+\kappa^2}|\text{rdiv } \tilde{\beta}_{\ell=1}|^2) + \frac{n}{2} r^{10+\kappa^2} (\text{rdiv } \tilde{\beta}_{\ell=1})^2 = 2r^5 \kappa^2 (\text{rdiv } \tilde{\beta}_{\ell=1})(\text{rdiv } p_{\ell=1})
\end{equation}

by Cauchy–Schwarz. This therefore simplifies to

\begin{equation}
\nabla_3(r^{10+\kappa^2}|\text{rdiv } \tilde{\beta}_{\ell=1}|^2) + \frac{n}{4} r^{10+\kappa^2} (\text{rdiv } \tilde{\beta}_{\ell=1})^2 \leq \frac{4}{n} r^n (\text{rdiv } p_{\ell=1})^2.
\end{equation}

Recalling that \(\nabla_3 = \Omega^{-1} \partial_\kappa\) and \(\kappa = -\frac{2\Omega}{r}\) and \(\phi = r^2 \text{div } p_{\ell=1}\) or \(\phi = r^2 \text{curl } p_{\ell=1}\), we obtain

\begin{equation}
\partial_\kappa(r^{8+n}\Omega^2 |\text{rdiv } \tilde{\beta}_{\ell=1}|^2) + \frac{n}{2} r^{7+n}\Omega^4 |\text{rdiv } \tilde{\beta}_{\ell=1}|^2 \leq \frac{2}{n} r^{n-3}\Omega^2 |\phi|^2
\end{equation}

\begin{equation}
\partial_\kappa(r^{8+n}\Omega^2 |\text{rcurl } \tilde{\beta}_{\ell=1}|^2) + \frac{n}{2} r^{7+n}\Omega^4 |\text{rcurl } \tilde{\beta}_{\ell=1}|^2 \leq \frac{2}{n} r^{n-3}\Omega^2 |\phi|^2.
\end{equation}

From (112), we have

\begin{equation}
\nabla_4(r^{10+\kappa^2}|\text{rdiv } \tilde{\beta}_{\ell=1}|^2) = 2r^5 \kappa^2 (\text{rdiv } \tilde{\beta}_{\ell=1})(\text{rdiv } p_{\ell=1}).
\end{equation}

Multiplying by \(r^{-\delta}\) and using that \(\nabla_4 r = \frac{1}{r} \times r\), we have

\begin{equation}
\nabla_4(r^{10-\delta-\kappa^2}|\text{rdiv } \tilde{\beta}_{\ell=1}|^2) + \frac{\delta}{2} r^{10-\delta-\kappa^2} |\text{rdiv } \tilde{\beta}_{\ell=1}|^2 \leq \frac{4}{\delta} r^{-\delta-\kappa} (\text{rdiv } p_{\ell=1})^2.
\end{equation}

which simplifies to

\begin{equation}
\nabla_4(r^{10-\delta-\kappa^2}|\text{rdiv } \tilde{\beta}_{\ell=1}|^2) + \frac{\delta}{4} r^{10-\delta-\kappa^2} |\text{rdiv } \tilde{\beta}_{\ell=1}|^2 \leq \frac{4}{\delta} r^{-\delta-\kappa} (\text{rdiv } p_{\ell=1})^2.
\end{equation}

Recall that \(\nabla_4 = \Omega^{-1} \partial_\kappa\) and \(\kappa = \frac{2\Omega}{r}\) and \(\phi = r^2 \text{div } p_{\ell=1}\) or \(\phi = r^2 \text{curl } p_{\ell=1}\), we obtain

\begin{equation}
\partial_\kappa(r^{8-\delta}\Omega^2 |\text{rdiv } \tilde{\beta}_{\ell=1}|^2) + \frac{\delta}{2} r^{7-\delta}\Omega^4 |\text{rdiv } \tilde{\beta}_{\ell=1}|^2 \leq \frac{2}{\delta} r^{n-3-\delta}\Omega^2 |\phi|^2
\end{equation}

\begin{equation}
\partial_\kappa(r^{8-\delta}\Omega^2 |\text{rcurl } \tilde{\beta}_{\ell=1}|^2) + \frac{\delta}{2} r^{7-\delta}\Omega^4 |\text{rcurl } \tilde{\beta}_{\ell=1}|^2 \leq \frac{2}{\delta} r^{n-3-\delta}\Omega^2 |\phi|^2.
\end{equation}

We can also multiply (114) by \(\frac{1}{r^2}\) for which \(\partial_\kappa\Omega^{-2} = -\frac{1}{r^2} \left( \frac{2M}{r^2} - \frac{2G}{r^4} \right)\) and obtain
\( \partial_r (r^8 |\text{div} \tilde{\beta}_{\ell=1}|^2) + \left( \frac{2M}{r^2} + \frac{2Q^2}{r^3} \right) r^8 |\text{div} \tilde{\beta}_{\ell=1}|^2 = 2r^3 \Omega (\text{div} \tilde{\beta}_{\ell=1}) (\text{div} p_{\ell=1}) \)

and hence

\[ \partial_r (r^8 |\text{div} \tilde{\beta}_{\ell=1}|^2) + \left( M - \frac{Q^2}{r^2} \right) r^8 |\text{div} \tilde{\beta}_{\ell=1}|^2 \leq \left( \frac{r}{Mr - Q^2} \right) r^{-2} \Omega^2 |\phi|^2. \]

(116)

The right hand side in (113), applied with \( n = 2 - \delta \), as well as the right hand side of (115) and (116) are estimated from initial data upon integration over a spacetime region \([u_0, u] \times [v_0, v] \times S^2\) according to proposition 8.1, giving the desired bounds.

Recalling the definitions of the energies and the bulks (88) and (89), the above lemma implies estimate (109).

9.1.2. Higher derivative estimates. We estimate higher derivative of the projection to the \( \ell = 1 \) mode of \( \tilde{\beta} \) and \( \tilde{\beta} \).

Note that we already control the \( \nabla_3 (\text{div} \tilde{\beta}_{\ell=1}) \) and \( \nabla_4 (\text{div} \tilde{\beta}_{\ell=1}) \) by the differential relations (74) and (75) they satisfy.

To estimate the remaining derivative, we commute the differential relations by \( 2R_s = -\Omega \nabla_3 + \Omega \nabla_4 \). We compute

\[ \Omega \nabla_3 (R_s (r^4 \Omega (\text{div} \tilde{\beta}_{\ell=1}))) = R_s (\Omega^3 \text{div} p_{\ell=1}). \]

Since the right hand side satisfies a non-degenerate estimate, we can use proposition 8.1 to bound such term from initial data.

In order to obtain the optimal weights in \( r \), we commute the relation (111) with \( \Omega \nabla_4 \). Indeed commuting

\[ \Omega \nabla_3 (r^4 \Omega (\text{div} \tilde{\beta}_{\ell=1})) = \Omega^3 r^{-2} \phi \]

by \( \Omega \nabla_4 \), we obtain

\[ \Omega \nabla_3 [\Omega \nabla_4 (r^4 \Omega (\text{div} \tilde{\beta}_{\ell=1}))] = \Omega \nabla_4 (\Omega^3 r^{-2} \phi). \]

Using a cut-off function which vanishes for \( r \leq R \), we obtain the optima weights for the \( \Omega \nabla_4 \) derivative. This proves (110) for \( n = 0 \).

The result for \( \tilde{\beta}_{\ell=1} \) is analogous, with the difference that there is no improvement in powers of \( r \) when taking the \( \nabla_3 \) derivative.

Since the Fackerell–Ipser equation commutes with the Killing fields \( T \) and angular momentum operators, the higher order estimates in (110) are implied.

9.2. Estimate for the \( \ell \geq 2 \) spherical modes of \( \hat{\beta} \) and \( \hat{\beta} \)

In this section, we obtain estimates for \( r \nabla^2 \hat{\beta} \) and \( r \nabla^2 \hat{\beta} \). We make use of the relations obtained in lemma 6.1:

\[ r^3 \kappa \nabla^2 \hat{\beta} = - (\kappa^{(F)} \rho \psi_1 - \left( 2 (\kappa^{(F)} \rho^2 + 3 \rho) \right) r^3 \kappa \mathbf{f}, \]

\[ r^3 \kappa \nabla^2 \hat{\beta} = - (\kappa^{(F)} \rho \psi_1 - \left( 2 (\kappa^{(F)} \rho^2 + 3 \rho) \right) r^3 \kappa \mathbf{f}. \]
From the first relation, we can write
\[ r \mathcal{D}_\gamma^\beta = \frac{Q}{2\Omega r^3} \psi_1 + \left( \frac{6M}{r^2} - \frac{8Q^2}{r^3} \right) f. \] (117)

This implies that
\[ \int_{\mathbb{R}_+} \int_{S^2} \sin \theta d\theta d\phi \int_{\mathbb{R}^+} r^4 |r \mathcal{D}_\gamma^\beta|^2 r^4 \lesssim \int_{\mathbb{R}_+} \int_{S^2} \sin \theta d\theta d\phi \int_{\mathbb{R}^+} r^4 \left( r^{-6+n} |\psi_1|^2 + r^{-4+n} |f|^2 \right). \]

The right hand side is controlled by the initial data of \( q, q^F, f, \psi_1, \alpha \) by main theorem of [15], for \( n = 9 - \delta \). Recall that the theorem in [15] gives bound for the non-degenerate spacetime energies for the quantities \( \psi_1 \) and \( f \), in Reissner–Nordström spacetime with \( |Q| \ll M \). Therefore, we can integrate (117) in the whole spacetime region, and still control the right hand side by initial data. This proves
\[ \mathcal{F}[\mathcal{D}_\gamma^\beta] + \mathcal{I}[\mathcal{D}_\gamma^\beta] \lesssim F_0[q, q^F, f, \psi_1, \alpha] + F_0[\mathcal{D}_\gamma^\beta]. \]

For the spin \(-2\) quantities, we have the relation
\[ r \mathcal{D}_\gamma^\beta = \frac{Q}{2\Omega r^3} \psi_1 + \left( \frac{6M}{r^2} - \frac{8Q^2}{r^3} \right) f, \]
from which we can similarly obtain
\[ \int_{\mathbb{R}_+} \int_{S^2} \sin \theta d\theta d\phi \int_{\mathbb{R}^+} r^4 |r \mathcal{D}_\gamma^\beta|^2 r^4 \lesssim \int_{\mathbb{R}_+} \int_{S^2} \sin \theta d\theta d\phi \int_{\mathbb{R}^+} r^4 \left( r^{-6+n} |\psi_1|^2 + r^{-4+n} |f|^2 \right). \]

The right hand side is controlled by the initial data of \( q, q^F, f, \psi_1, \alpha \) by main theorem for spin \(-2\) of [15], for \( n = 7 - \delta \). As above, this gives
\[ \mathcal{F}[\mathcal{D}_\gamma^\beta] + \mathcal{I}[\mathcal{D}_\gamma^\beta] \lesssim \mathcal{F}_0[q, q^F, f, \psi_1, \alpha] + \mathcal{F}_0[\mathcal{D}_\gamma^\beta]. \]

Upon taking derivative with respect to \( \nabla_n, \nabla_4 \) and \( \nabla_\gamma \) of (117) and integrating in spacetime, we can bound the left hand side by the initial data using main theorem in [15]. Indeed, both \( \psi_1 \) and \( f \) are quantities obtained through transport estimates, therefore their spacetime norms are not degenerate. Observe all those derivatives improve of a power of \( r \) in the estimates. This therefore implies
\[ \mathcal{F}[\Box \mathcal{D}_\gamma^\beta] + \mathcal{I}[\Box \mathcal{D}_\gamma^\beta] \lesssim \mathcal{F}_0[q, q^F, f, \psi_1, \alpha] + \mathcal{F}_0[\Box \mathcal{D}_\gamma^\beta]. \] (119)

In commuting (118) we obtain a similar estimate, observing that the energies for the spin \(-2\) quantities \( \psi_1 \) and \( f \) do not improve in powers of \( r \). This implies as above
\[ \mathcal{F}[\Box \mathcal{D}_\gamma^\beta] + \mathcal{I}[\Box \mathcal{D}_\gamma^\beta] \lesssim \mathcal{F}_0[q, q^F, f, \psi_1, \alpha] + \mathcal{F}_0[\Box \mathcal{D}_\gamma^\beta]. \] (120)

By commuting the relations (117) and (118) by the Killing fields \( T \) and angular momentum operator, we trivially obtain the equivalent estimates for higher order derivatives.

9.3. Proof of the main theorem

We are now ready to prove the main theorem. The proof is a straightforward application of the elliptic estimates in section 3.4.2, and the estimates obtained in section 9.1 for the projection to the \( \ell = 1 \) spherical mode and in section 9.2 for the projection to the \( \ell \geq 2 \) spherical harmonics.
By corollary 3.1 applied to $\tilde{\beta}$ and $\tilde{\beta}_e$, we have
\[ \int_s |\tilde{\beta}|^2 \leq C \left( \int_s |r \div \tilde{\beta}_{e=1}|^2 + |r \curl \tilde{\beta}_{e=1}|^2 + |r \bar{\mathcal{D}}_2 \tilde{\beta}|^2 \right). \]
Recalling that $r \div$, $r \curl$ and $r \bar{\mathcal{D}}_2$ commute with $\bar{\mathcal{V}}_3$ and $\bar{\mathcal{V}}_4$, from the above elliptic estimate we can immediately infer that
\[
\begin{align*}
F \tilde{\beta} &\leq F[\phi, \tilde{\beta}_{e=1}] + F[\phi, D \tilde{\beta}_{e=1}] + F[\mathcal{D}_2 \tilde{\beta}] + F[\mathcal{D}_2 \tilde{\beta}] \\
F \tilde{\beta} &\leq F[\phi, \tilde{\beta}_{e=1}] + F[\phi, D \tilde{\beta}_{e=1}] + F[\mathcal{D}_2 \tilde{\beta}] + F[\mathcal{D}_2 \tilde{\beta}]
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{I} \tilde{\beta} &\leq \mathcal{I}[\phi, \tilde{\beta}_{e=1}] + \mathcal{I}[\phi, D \tilde{\beta}_{e=1}] + \mathcal{I}[\mathcal{D}_2 \tilde{\beta}] + \mathcal{I}[\mathcal{D}_2 \tilde{\beta}] \\
\mathcal{I} \tilde{\beta} &\leq \mathcal{I}[\phi, \tilde{\beta}_{e=1}] + \mathcal{I}[\phi, D \tilde{\beta}_{e=1}] + \mathcal{I}[\mathcal{D}_2 \tilde{\beta}] + \mathcal{I}[\mathcal{D}_2 \tilde{\beta}]
\end{align*}
\]
Combining (109), (110), (119) and (120), we prove estimate (91) of main theorem.

By applying the equivalent of the above estimates for higher derivative, we prove estimate (92) of main theorem.

As a standard application of the $r^p$ hierarchy estimates, we can prove the pointwise decay estimates (93). The constant depends on two derivatives in $T$ of the initial data.

### Appendix A. Derivation of the Teukolsky equations

The computations in this appendix are done in full generality, in physical space, with no gauge assumption. To perform the computations, we use the equations in section 3.5.

#### A.1. Preliminaries

We collect here few lemmas of [15].

**Lemma A.1 (Lemma A.1.1. of [15]).** The wave operator of a $p$-rank $S$-tensor $\Psi$ is given by
\[
\square \Psi = -\frac{1}{2} \bar{\mathcal{V}}_3 \bar{\mathcal{V}}_4 \Psi - \frac{1}{2} \bar{\mathcal{V}}_4 \bar{\mathcal{V}}_3 \Psi + \phi \Psi + \left( -\frac{1}{2} \kappa + \omega \right) \bar{\mathcal{V}}_4 \Psi + \left( \eta^C + \eta^F \right) \bar{\mathcal{V}}_3 \Psi
\]
\[ = -\bar{\mathcal{V}}_3 \bar{\mathcal{V}}_4 \Psi + \phi \Psi + \left( -\frac{1}{2} \kappa + 2 \omega \right) \bar{\mathcal{V}}_4 \Psi - \frac{1}{2} \kappa \bar{\mathcal{V}}_3 \Psi + 2 \eta^C \bar{\mathcal{V}}_3 \Psi. \tag{A.1} \]
\[ = -\bar{\mathcal{V}}_4 \bar{\mathcal{V}}_3 \Psi + \Delta \Psi + \left( -\frac{1}{2} \kappa + 2 \omega \right) \bar{\mathcal{V}}_3 \Psi - \frac{1}{2} \kappa \bar{\mathcal{V}}_4 \Psi + 2 \eta^C \bar{\mathcal{V}}_3 \Psi \tag{A.2} \]
where $\Delta \Psi = g^{CD} \bar{\mathcal{V}}_C \bar{\mathcal{V}}_D \Psi$ is the Laplacian for $p$-tensors.

**Corollary A.1.** In double null gauge, we have for a $p$-tensor $\Psi$
\[ r \Omega^2 \square \Psi = -\Omega \bar{\mathcal{V}}_3 (\Omega \bar{\mathcal{V}}_4 (r \Psi)) + \Omega^2 \Delta (r \Psi) + \Omega^2 \rho (r \Psi). \]
\textbf{Proof.} We compute
\[
\Omega \nabla_3 (\Omega \nabla_4 (r \Psi)) = \Omega \nabla_3 \left( \frac{1}{2} \Omega \nabla r \Psi + \Omega \nabla_4 \Psi \right)
\]
\[
= \frac{1}{2} \Omega (\nabla_3 \Omega) r \nabla \Psi + \frac{1}{2} \Omega^2 (\nabla_3 r) \nabla \Psi + \frac{1}{2} \Omega^2 \nabla_3 \nabla_4 \Psi + \frac{1}{2} \Omega^2 \nabla_3 \nabla_4 \Psi
\]
Using (18) and (41), we obtain
\[
\Omega \nabla_3 (\Omega \nabla_4 (r \Psi)) = \frac{1}{2} \Omega (-2 \omega \Omega) r \nabla \Psi + \frac{1}{2} \Omega^2 (\frac{1}{2} r \nabla \Omega) \nabla \Psi + \frac{1}{2} \Omega^2 (-\frac{1}{2} m \nabla n \nabla \Psi - \frac{1}{2} \Omega - 2 \omega \Omega + 2 \rho) \Psi + \frac{1}{2} \Omega^2 \nabla_3 \nabla_4 \Psi
\]
Using (A.1), we easily obtained the desired relation. \hfill \Box

\textbf{Lemma A.2 (Lemma A.1.3. of [15]).} Consider a rescaled tensor \( \rho^m r^m \Psi \), for \( n \) and \( m \) two numbers. Then
\[
r^m \rho^m \nabla_3 (r^m \rho^m \Psi) = \nabla_3 (r^m \rho^m \Psi) + \left( \frac{m-n}{2} \kappa + 2 m \omega \right) r^m \rho^m \Psi
\]  \quad (A.3)
\[
r^m \rho^m \nabla_4 (r^m \rho^m \Psi) = \nabla_4 (r^m \rho^m \Psi) + \left( \frac{m-n}{2} \kappa - 2 m \omega - 2 m \rho \right) r^m \rho^m \Psi
\]  \quad (A.4)
Moreover, it verifies the following wave equation:
\[
\square (r^m \rho^m \Psi) = \left( - m (n+1) + m(m-1) - 2 m \kappa \right) r^m \rho^m \Psi + m \left( n \omega + \frac{1}{2} m \omega + 2 n \omega + 4 m^2 \rho \right) r^m \rho^m \Psi + m \left( n \omega + \frac{1}{2} m \omega + 2 n \omega + 4 m^2 \rho \right) r^m \rho^m \Psi
\]
We recall
\[
\left( - r D_1 \square + \square_0 r D_1 \right) \Phi = - K r D_1 \Phi
\]  \quad (A.5)
\[
\left( - r D_1 \square + \square_0 r D_1 \right) \phi = - K r D_1 \phi
\]  \quad (A.6)
We recall here the spin +1 Teukolsky equation for the electromagnetic component \((F)\beta\).

\textbf{Proposition A.1 (Spin +1 Teukolsky equation for \((F)\beta—proposition A.2.1 in [15]).} Let \( S \) be a linear gravitational and electromagnetic perturbation around Reissner–Nordström. Consider the (gauge-dependent) curvature component \((F)\beta\), which is part of the solution \( S \). Then \((F)\beta\) satisfies the following equation:
\[
\square (F) \beta = - 2 \omega \nabla_4 (F) \beta + (\kappa + 2 \omega) \nabla_3 (F) \beta + \left( \frac{1}{4} m \kappa + \omega \kappa - 3 \omega \kappa - (F) \rho^2 - 2 \nabla_4 \omega \right) (F) \beta
\]  \quad (F) \rho \left( 2 \div \kappa + 4 \beta - 2 \nabla_3 \xi + (\kappa + 8 \omega) \xi \right).
We recall here the fundamental lemma in [15] to derive the wave equation for a quantity \( \mathcal{P}(\Psi) \).

**Lemma A.3 (Lemma B.0.2 in [15]).** Let \( \Psi \) be a \( p \)-tensor which verifies a wave equation of the form:

\[
\square_p \Psi = A \mathcal{P}^{-1}(\Psi) + B \Psi + C \mathcal{P}(\Psi) + M, \tag{A.7}
\]

where \( \mathcal{P}^{-1}(\Psi) \) is a \( p \)-tensor such that \( \mathcal{P} \mathcal{P}^{-1}(\Psi) = \Psi \), and \( A, B, C \) are coefficients of the wave equation. Then the \( p \)-tensor \( \mathcal{P}(\Psi) \) verifies the wave equation:

\[
\square_p(\mathcal{P}(\Psi)) = (\xi^{-1}\nabla_i(A) + r\mathcal{A}) \mathcal{P}^{-1}(\Psi) + \left(\xi^{-1}r\nabla_i(B) + rB + A + \frac{1}{2}(\rho + r^{(F)}\rho^2)\right) \Psi \\
+ \left(B + \xi^{-1}r\nabla_i(C) + rC + \frac{1}{2}(\kappa - 2\rho - 2^{(F)}\rho^2)\right) \mathcal{P}(\Psi) + \left(C + \frac{1}{r}(-\kappa + 2\rho)\right) \mathcal{P}(\mathcal{P}(\Psi)) \\
+ \xi^{-1}\nabla_i M + \frac{3}{2}rM.
\]

**A.2. Spin +1 Teukolsky-type equation for \( \tilde{\beta} \)**

We derive here the spin +1 Teukolsky equation for the gauge invariant quantity \( \tilde{\beta} \).

**Proposition A.2 (Generalized spin +1 Teukolsky equation in \( l = 1 \) for \( \tilde{\beta} \)).** Let \( \mathcal{S} \) be a linear gravitational and electromagnetic perturbation around Reissner–Nordström. Consider the gauge-invariant curvature component \( \tilde{\beta} = 2^{(F)}\rho \beta - 3 \rho^{(F)} \beta \). Then \( \tilde{\beta} \) satisfies the following equation:

\[
\square_p(r^3 \tilde{\beta}) = -2 \xi \nabla_4 (r^3 \tilde{\beta}) + (\kappa + 2\omega) \nabla_i (r^3 \tilde{\beta}) + \left(\frac{1}{4}\kappa - 3\omega + \omega + 2\rho - 3^{(F)}\rho^2 - 8\omega + 2\nabla_i \omega\right) r^3 \tilde{\beta} \\
- 2 \xi \nabla_4 \omega \left(\nabla_4 \beta + \left(\frac{3}{2}\kappa + 2\omega\right)^{(F)} \beta - 2^{(F)} \rho \kappa\right) + 8 \rho^{(F)}\rho^2 \text{div} \mathcal{J}.
\]

**Proof.** We compute the following, using (52), (60) and (62):

\[
\nabla_4 \tilde{\beta} + (3\kappa + 2\omega) \tilde{\beta} = \nabla_4 \left(2^{(F)} \rho^2 \tilde{\beta} - 3 \rho^{(F)} \beta\right) + (3\kappa + 2\omega)(2^{(F)} \rho \beta - 3 \rho^{(F)} \beta) \\
= 2 \nabla_4 \rho \beta + 2^{(F)} \rho \nabla_4 \beta - 3 \nabla_4 \rho \beta - 3 \rho \nabla_4 \beta + (3\kappa + 2\omega)(2^{(F)} \rho \beta - 3 \rho^{(F)} \beta) \\
= 2(\kappa - 2 \rho \beta) \nabla_4 \beta + (3\kappa - 2 \rho \beta) \nabla_4 \beta + 2^{(F)} \rho \beta + (3\kappa + 2\omega)(2^{(F)} \rho \beta - 3 \rho^{(F)} \beta) \\
- 3(\frac{3}{2} \kappa \rho - \rho^{(F)} \beta) \nabla_4 \beta + 3 \rho \nabla_4 \beta + (3\kappa + 2\omega)(2^{(F)} \rho \beta - 3 \rho^{(F)} \beta) \\
= 2^{(F)} \rho \text{div} \mathcal{J} + \left(2^{(F)} \rho^2 - 3 \rho\right) \left(\nabla_4 \beta + \left(\frac{3}{2} \kappa + 2\omega\right)^{(F)} \beta - 2^{(F)} \rho \kappa\right).
\]

We therefore obtain

\[
\nabla_4 \tilde{\beta} = -(3\kappa + 2\omega) \tilde{\beta} + 2^{(F)} \rho \text{div} \mathcal{J} + \left(2^{(F)} \rho^2 - 3 \rho\right) \left(\nabla_4 \beta + \left(\frac{3}{2} \kappa + 2\omega\right)^{(F)} \beta - 2^{(F)} \rho \kappa\right).
\]

(A.8)

We apply \( \nabla_3 \) to (A.8) to derive \( \nabla_3 \nabla_4 \tilde{\beta} \). We consider the following pieces:
\[ \nabla \delta \left( -3\kappa + 2\omega \right) \beta = -3\nabla \rho \beta - 2\nabla \omega \beta - (3\kappa + 2\omega) \nabla \beta = \left( \frac{3}{2} \kappa - 6\omega - 6\rho - 2\nabla \omega \right) \beta - (3\kappa + 2\omega) \nabla \beta. \]

\[ \nabla \delta \left( 2 (F) \rho \nabla \alpha \right) = 2 \nabla \delta \left( F \right) \rho \nabla \alpha + 2 (F) \rho \nabla \delta \left( \nabla \alpha \right) \]

\[ = -2 \left( \frac{3}{2} \kappa - 4\omega \right) \rho \nabla \alpha - 2 (F) \rho \nabla \delta \left( \nabla \alpha \right) - 2 (F) \rho \left( \nabla \delta \left( \nabla \alpha \right) \right). \]

\[ = 2 (F) \rho \left( -2 \left( \frac{3}{2} \kappa - 4\omega \right) \rho \nabla \alpha + (\delta \nabla + K) \beta + (\delta \alpha + K) (F) \beta \right). \]

In the latter expression, we can eliminate \( \beta \) by writing \( 2 (F) \rho \beta = \beta + 3 \rho (F) \beta \), which gives

\[ \nabla \delta \left( 2 (F) \rho \nabla \alpha \right) = 2 (F) \rho \left( -2 \left( \frac{3}{2} \kappa - 4\omega \right) \rho \nabla \alpha + (\delta \nabla + K) \beta + (\delta \alpha + K) (F) \beta \right) + (\delta \alpha + K) (F) \beta \]

\[ = 2 (F) \rho \left( -2 \left( \frac{3}{2} \kappa - 4\omega \right) \rho \nabla \alpha + (\delta \nabla + K) \beta + (\delta \alpha + K) (F) \beta \right) \]

\[ + (\delta \alpha + K) (F) \beta. \]

The derivative \( \nabla \delta \) applied to the last term of (A.8) gives two terms. The first one is:

\[ \nabla \delta \left( 2 (F) \rho \nabla \alpha \right) \left( \nabla \delta \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \]

\[ = \left( - \frac{3}{2} \kappa + 2\omega \right) \nabla \delta \left( \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \]

\[ = - \frac{3}{2} \kappa \rho \left( \nabla \delta \left( \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \right) + \frac{3}{2} \kappa \rho \left( \nabla \delta \left( \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \right). \]

We simplify the term multiplied by \( \rho \), writing \( (3 \rho) \left( \nabla \delta \left( \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \right) = - \nabla \delta \left( -3 \kappa + 2\omega \right) \beta + 2 (F) \rho \nabla \alpha + (2 (F) \rho \nabla \alpha \right) \left( \nabla \delta \left( \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \right), \)

so it gives:

\[ \nabla \delta \left( 2 (F) \rho \beta - 3 \rho \right) \left( \nabla \delta \left( \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \right) = 2 \left( \frac{3}{2} \kappa + 2\omega \right) \beta - 2 (F) \rho \xi \]

\[ - \frac{3}{2} \kappa \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) \beta + (F) \rho \left( \frac{9}{2} \kappa - 3 \omega \right) \beta - 2 (F) \rho \xi \].

The second term is given by

\[ \left( 2 (F) \rho \beta - 3 \rho \right) \nabla \delta \left( \nabla \delta \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \]

\[ = \left( 2 (F) \rho \beta - 3 \rho \right) \nabla \delta \left( \nabla \delta \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \]

\[ = \left( 2 (F) \rho \beta - 3 \rho \right) \left( \nabla \delta \nabla \delta \rho \beta + \left( \frac{3}{2} \kappa + 2\omega \right) (F) \beta - 2 (F) \rho \xi \right) \]

\[ - \left( \frac{3}{2} \kappa + 2\omega \right) \beta + \left( \frac{3}{2} \kappa + 2\omega \right) \beta + (F) \rho \left( \frac{9}{2} \kappa - 3 \omega \right) \beta - 2 (F) \rho \xi \].

By proposition A.1, we have that

\[ \nabla \delta \nabla \nabla \beta = \left( \frac{1}{4} \kappa - \omega \right) \nabla \beta - 3 \omega \kappa - (2 (F) \rho \beta + 3 \omega \nabla \xi) \]

\[ + \left( 2 (F) \rho \beta - 3 \rho \right) \beta - \frac{3}{2} \kappa \nabla \beta + (F) \rho \beta + (\delta \nabla + K) (F) \beta \]

\[ + (F) \rho (-2 \nabla \delta \beta - 4 \beta + 2 \nabla \delta \xi - \frac{3}{2} \kappa - 8 \omega \xi). \]
and writing $2^{(F)}\rho\beta = \tilde{\beta} + 3\rho^{(F)}\beta$ as above, we have

$$\nabla_4 \nabla_3 \beta^{(F)} = \left( -\frac{1}{4} \nabla_4 \tilde{\rho} - \omega_4 \kappa + 3\omega_4 \kappa - 6\rho - (F)\rho^2 + 2\nabla_4 \omega \right)^{(F)}\beta$$

$$+ \left( -\frac{1}{2} \tilde{\kappa} + \omega_4 \right) \nabla_4 \beta^{(F)} - \frac{3}{2} \nabla_3 \beta^{(F)} + \tilde{\alpha}_1^{(F)}\beta$$

$$+ (F)\rho \left( -2\text{div} \tilde{\kappa} + 2\nabla_3 \xi - 8\omega_4 \xi \right) - 2\tilde{\beta}.$$ 

We therefore obtain:

$$(2^{(F)}\rho^2 - 3\rho) \nabla_3 \left( \nabla_4 \beta^{(F)} + \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)}\beta - 2^{(F)}\rho\xi \right)$$

$$= (2^{(F)}\rho^2 - 3\rho) \left( -\frac{1}{2} \kappa - \omega_4 + 6\omega_4 - 3\rho - (F)\rho^2 + 2\nabla_4 \omega + 2\nabla_3 \omega \right)^{(F)}\beta + \left( -\frac{1}{2} \kappa + \omega_4 \right) \nabla_4 \beta^{(F)} + \tilde{\alpha}_1^{(F)}\beta$$

$$+ (F)\rho \left( -2\text{div} \tilde{\kappa} + 2\omega_4 \xi - 8\omega_4 \xi \right) - 2\tilde{\beta}.$$ 

The term $2^{(F)}\rho^2 - 3\rho \left( -\frac{1}{2} \kappa + 4\omega_4 \right) \nabla_4 \beta^{(F)} = (2^{(F)}\rho^2 - 3\rho) \nabla_4 \beta^{(F)}$ can be simplified by using $2^{(F)}\rho^2 - 3\rho \nabla_4 \beta^{(F)} = \nabla_4 \tilde{\beta}^{(F)} + (3\kappa + 2\omega)_\beta^{(F)} - 2(F)\rho\text{div} \alpha - (2^{(F)}\rho^2 - 3\rho) \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)}\beta - 2^{(F)}\rho\xi$, which finally gives, using (45)

$$(2^{(F)}\rho^2 - 3\rho) \nabla_3 \left( \nabla_4 \beta^{(F)} + \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)}\beta - 2^{(F)}\rho\xi \right)$$

$$= (2^{(F)}\rho^2 - 3\rho) \left( -\frac{1}{2} \kappa - \omega_4 + 6\omega_4 - 3\rho - (F)\rho^2 + 2\nabla_4 \omega + 2\nabla_3 \omega \right)^{(F)}\beta + \tilde{\alpha}_1^{(F)}\beta$$

$$+ (F)\rho \left( -2\text{div} \tilde{\kappa} + 2\omega_4 \xi - 8\omega_4 \xi \right) - 2\tilde{\beta}$$

$$+ \left( -\frac{1}{2} \kappa + 4\omega_4 \right) \nabla_4 \tilde{\beta}^{(F)} + (3\kappa + 2\omega)_\beta^{(F)} - 2(F)\rho\text{div} \alpha - (2^{(F)}\rho^2 - 3\rho) \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)}\beta - 2^{(F)}\rho\xi.$$

We can finally put all this together and compute $\nabla_3 \nabla_4 \beta$, using (48):

$$\nabla_3 \nabla_4 \beta = \left( \frac{3}{2} \kappa - \omega_4 - 6\omega_4 - 6\rho - 2\nabla_4 \omega \right) \beta - (3\kappa + 2\omega) \nabla_3 \beta$$

$$+ 2^{(F)}\rho \left( -2\kappa - 4\omega_4 \right) \text{div} \alpha + (3\kappa + 2\omega) \nabla_4 \beta$$

$$+ \left( 3\kappa + 2\omega \right)^{(F)}\beta + \left( \tilde{\alpha}_1^{(F)} + \kappa \right) \beta + \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)}\beta - 2^{(F)}\rho\text{div} \alpha$$

$$+ 2^{(F)}\rho^2 \left( \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)} \beta - 2^{(F)}\rho\xi \right) - \frac{3}{2} \nabla_4 \beta^{(F)} + \left( \frac{9}{2} \kappa - 3\omega_4 \right) \beta + (F)\rho \left( 3\omega_4 \text{div} \alpha \right)$$

$$+ (F)\rho \left( \frac{3}{2} \kappa - 4\omega_4 \right) \text{div} \alpha$$

$$+ \left( -\frac{9}{2} \kappa + 4\omega_4 \right) \nabla_4 \beta$$

$$+ \left( -\frac{3}{2} \kappa - \omega_4 + 12\omega_4 - 8\omega_4 \right) \beta + (F)\rho \left( 2\omega_4 - 8\omega_4 \text{div} \alpha \right)$$

$$= \left( -\frac{19}{4} \kappa + 6\omega_4 - 4\omega_4 - 3\kappa \rho - 8\omega_4 - 2\nabla_4 \omega \right) \beta - (3\kappa + 2\omega) \nabla_4 \beta$$

$$+ \left( -\frac{3}{2} \kappa + 4\omega_4 \right) \nabla_4 \beta + \tilde{\alpha}_1^{(F)} \beta$$

$$+ 2^{(F)}\rho \left[ 2\tilde{\alpha}_1^{(F)} + \kappa \right] \beta - 4^{(F)}\rho\text{div} \tilde{\kappa} + \tilde{\omega} \left( \nabla_4 \beta^{(F)} + \left( \frac{3}{2} \kappa + 2\omega \right)^{(F)}\beta - 2^{(F)}\rho\xi \right).$$
We can therefore compute the wave equation using (A.1):

\[
\Box g = -\nabla_\beta \nabla_\beta + \nabla_\beta \beta + \left( -\frac{1}{2} \kappa + 2\omega \right) \nabla_\beta \beta - \frac{1}{2} \kappa \nabla_\beta \beta
\]

\[
= \left( \frac{19}{4} \kappa \omega - 6\omega + 4\omega \right) - 8\omega + 2\nabla_\omega \omega \right) \nabla_\beta \beta + \left( -\frac{5}{2} \kappa + 2\omega \right) \nabla_\beta \beta + \left( -\frac{3}{2} \kappa + 2\omega \right) \nabla_\beta \beta
\]

\[
+ 2^{(F)} \rho^2 \left[ -2(\beta + K)^{(F)} \beta + 4^{(F)} \rho \text{div} \chi - \kappa \left( \nabla_\beta \beta + (\frac{3}{2} \kappa + 2\omega) (\beta - 2^{(F)} \rho \xi) \right) \right].
\]

Using lemma A.2 for \( n = 3, m = 0 \), we compute \( \Box g (r^3 \beta) \):

\[
\Box g (r^3 \beta) = ( -3 \kappa \omega - 3\rho ) r^3 \beta + r^3 \Box g (\beta) - \frac{3}{2} r^2 \nabla_\beta (\beta) - \frac{3}{2} \kappa \rho^2 \nabla_\beta (\beta)
\]

\[
= \left( \frac{7}{4} \kappa \omega - 6\omega + 4\omega \right) - 2\rho + 3^{(F)} \rho^2 - 8\omega + 2\nabla_\omega \omega \right) r^3 \beta + \left( \kappa + 2\omega \right) r^3 \nabla_\beta \beta - 2\omega r^3 \nabla_\beta \beta
\]

\[
+ 2^{(F)} \rho^2 \left[ -2(\beta + K)^{(F)} \beta + 4^{(F)} \rho \text{div} \chi - \kappa \left( \nabla_\beta \beta + (\frac{3}{2} \kappa + 2\omega) (\beta - 2^{(F)} \rho \xi) \right) \right].
\]

and using (A.3), we obtain

\[
\Box g (r^3 \beta) = -2\omega \nabla_\beta (r^3 \beta) + (\kappa + 2\omega) \nabla_\beta (\beta) + \left( \frac{1}{4} \kappa \omega - 3\omega + 2\omega \right) - 2\rho + 3^{(F)} \rho^2 - 8\omega + 2\nabla_\omega \omega \right) r^3 \beta
\]

\[
+ 2^{(F)} \rho^2 \left[ -2(\beta + K)^{(F)} \beta + 4^{(F)} \rho \text{div} \chi - \kappa \left( \nabla_\beta \beta + (\frac{3}{2} \kappa + 2\omega) (\beta - 2^{(F)} \rho \xi) \right) \right].
\]

Consider the term \(-2(\Delta_1 + K)^{(F)} \beta + 4^{(F)} \rho \text{div} \chi \). Using (26) and recalling the definition of the symmetric traceless two-tensor \( f = \text{D} \xi (\beta) + (\beta) \rho \chi \) in [15], we have

\[
-2(\Delta_1 + K)^{(F)} \beta + 4^{(F)} \rho \text{div} \chi = -2(-2 \text{D}_2 \text{D}_2 \beta + K)^{(F)} \beta + 4^{(F)} \rho \text{D}_2 \chi
\]

\[
= 4 \text{D}_2 \text{D}_2 (\beta) + 4^{(F)} \rho \text{D}_2 \chi = 4 \text{div} f
\]

(A.9)

which gives the desired expression.

\[ \square_g \psi_S = \left[ -\frac{1}{4} \kappa \omega + 5^{(F)} \rho^2 \right] \psi_S + \frac{1}{r} \left( \kappa \omega - 2\rho \right) \psi_S
\]

\[
+ 2^{(F)} \rho^2 \left[ 4 \text{div} f - \kappa \left( \nabla_\beta \beta + (\frac{3}{2} \kappa + 2\omega) (\beta - 2^{(F)} \rho \xi) \right) \right].
\]

**Proof.** Using lemma A.2 for \( \psi_S = r^\kappa (r^3 \beta) \) with \( n = m = 1 \), we obtain

Corollary A.2. The derived quantity \( \psi_S = r^\kappa (r^3 \beta) \) verifies the following wave equation:
\[ \Box_g (\psi_s) = \left( -\omega_K + \rho + 2 (^F) \rho^2 + \omega_K + 8 \omega \omega + 4 \rho \omega \omega^{-1} - 2 \nabla_3 \omega \right) \psi_s + r \Box_g \Box_g (r^3 \beta) \\
+ \left( 2 \omega \right) r \Box_g \nabla_4 (r^3 \beta) + \left( -2 \omega - 2 \rho \omega^{-1} \right) r \Box_g \nabla_3 (r^3 \beta) \\
= \left( -\omega_K + \rho + 2 (^F) \rho^2 + \omega_K + 8 \omega \omega + 4 \rho \omega \omega^{-1} - 2 \nabla_3 \omega \right) \psi_s \\
+ r \Box_g \left[ -2 \omega \nabla_4 (r^3 \beta) + (\kappa + 2 \omega) \nabla_3 (r^3 \beta) \\
+ \left( \frac{1}{4} \kappa \kappa - 3 \omega \kappa + \omega \kappa - 2 \rho + 3 (^F) \rho^2 - 8 \omega \omega + 2 \nabla_3 \omega \right) r^3 \beta \\
+ 2 r^F (^F) \rho^2 \left[ 4 \text{div} \tilde{f} - \kappa \left( \nabla_4 (^F) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) (^F) \beta - 2 (^F) \rho \xi \right) \right] \right] \\
+ \left( 2 \omega \right) r \Box_g \nabla_4 (r^3 \beta) + \left( -2 \omega - 2 \rho \omega^{-1} \right) r \Box_g \nabla_3 (r^3 \beta) \\
= \left( \frac{1}{4} \kappa \kappa - \rho + 5 (^F) \rho^2 - 2 \omega \kappa + 4 \rho \omega \omega^{-1} \right) \psi_s + r (\kappa \kappa - 2 \rho) \nabla_3 (r^3 \beta) \\
+ 2 r^F (^F) \rho^2 \left[ 4 \text{div} \tilde{f} - \kappa \left( \nabla_4 (^F) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) (^F) \beta - 2 (^F) \rho \xi \right) \right] \right] \\
\text{and writing } r \Box_g \nabla_3 (r^3 \beta) = \nabla_3 (\psi_s) + 2 \omega \psi_s = \frac{1}{\kappa} \psi_b + \left( -\frac{1}{4} \kappa + 2 \omega \right) \psi_s, \text{ we obtain} \\
\Box_g (\psi_s) = \left( \frac{1}{4} \kappa \kappa - \rho + 5 (^F) \rho^2 - 2 \omega \kappa + 4 \rho \omega \omega^{-1} \right) \psi_s + (\kappa - 2 \rho \omega^{-1}) \left( \frac{1}{4} \kappa \kappa - 2 \rho \right) \psi_b \\
+ 2 r^F (^F) \rho^2 \left[ 4 \text{div} \tilde{f} - \kappa \left( \nabla_4 (^F) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) (^F) \beta - 2 (^F) \rho \xi \right) \right] \right] \\
= \left( -\frac{1}{4} \kappa \kappa + 5 (^F) \rho^2 \right) \psi_s + \psi_b + \left( \frac{1}{4} \kappa \kappa - 2 \rho \right) \psi_b \\
+ 2 r^F (^F) \rho^2 \left[ 4 \text{div} \tilde{f} - \kappa \left( \nabla_4 (^F) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) (^F) \beta - 2 (^F) \rho \xi \right) \right] \right] \\
as desired. \]
By lemma A.3, we compute
\[
\Box_q(p) = \left( \kappa^{-1} r \nabla_3(-\frac{1}{4} \kappa \omega + 5 \langle F \rangle \rho^2) + r \left( -\frac{1}{4} \kappa \omega + 5 \langle F \rangle \rho^2 \right) + \frac{1}{2} r \rho + r \langle F \rangle \rho^2 \right) \psi_3 \\
+ \left( -\frac{1}{4} \kappa \omega + 5 \langle F \rangle \rho^2 + \kappa^{-1} r \nabla_3 \left( \frac{1}{r} (\kappa \omega - 2 \rho) \right) + r \frac{1}{r} (\kappa \omega - 2 \rho) + \frac{1}{2} \kappa \omega - 4 \rho - 2 \langle F \rangle \rho^2 \right) p \\
+ \left( \frac{1}{r} (\kappa \omega - 2 \rho) + \frac{1}{r} (-\kappa \omega + 2 \rho) \right) \Box p + \kappa^{-1} r \nabla_3 M + \frac{3}{2} r M \\
= \left( \kappa^{-1} r \left( \frac{1}{4} \kappa \omega^2 - \frac{1}{2} \kappa \omega - 10 \kappa \langle F \rangle \rho^2 \right) + r \left( -\frac{1}{4} \kappa \omega + 5 \langle F \rangle \rho^2 \right) + \frac{1}{2} r \rho + r \langle F \rangle \rho^2 \right) \psi_3 \\
+ \left( -\frac{1}{4} \kappa \omega + 5 \langle F \rangle \rho^2 - \frac{1}{2} (\kappa \omega - 2 \rho) + \kappa^{-1} (-\kappa \omega^2 + 5 \kappa \rho + 2 \kappa \langle F \rangle \rho^2) \\
+ r \frac{1}{r} (\kappa \omega - 2 \rho) + \frac{1}{2} \kappa \omega - 4 \rho - 2 \langle F \rangle \rho^2 \right) p \\
+ \kappa^{-1} r \nabla_3 M + \frac{3}{2} r M \\
\]
which gives
\[
\Box_q p = \left( \frac{1}{4} \kappa \omega + 5 \langle F \rangle \rho^2 \right) p + \left( -4 \langle F \rangle \rho^2 \right) p + \kappa^{-1} \nabla_3 M + \frac{3}{2} r M.
\]
We compute now the right hand side $\kappa^{-1} r \nabla_3 (M) + \frac{3}{2} r M$. Recalling that $\langle F \rangle \rho = \frac{\rho}{\kappa}$, we write $M$ as
\[
M = Q^2 \kappa \left( 2 \delta \psi + 2 \kappa \nabla_4 \langle F \rangle \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) \langle F \rangle \beta - 2 \langle F \rangle \rho \chi \right) \\
= Q^2 (M_1 + M_2)
\]
with
\[
M_1 = 8 \kappa \delta \psi, \\
M_2 = -2 \kappa \nabla_4 \langle F \rangle \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) \langle F \rangle \beta - 2 \langle F \rangle \rho \chi
\]
We compute separately:
\[
\kappa^{-1} r \nabla_3 (M_1) + \frac{3}{2} r M_1 = \kappa^{-1} r \nabla_3 (8 \kappa \delta \psi) + 12 r \delta \psi \nabla f = 8 \kappa^{-1} r \nabla_3 (8 \kappa \delta \psi) + 8 \kappa^{-1} r \nabla_3 (8 \delta \psi) + 12 r \delta \psi f + \nabla_3 (8 \kappa^{-1} r \nabla \beta - \frac{1}{2} 2 \kappa \delta \psi f) + 12 r \delta \psi f.
\]
Recall that\(^{10}\)
\[
\nabla_3 (f) + (\kappa - 2 \omega) f = -D_2 D_2 \langle F \rangle \rho \langle F \rangle \sigma - \frac{1}{2} \langle F \rangle \rho \left( \kappa \chi + \kappa \chi \right)
\]
therefore we obtain
\(^{10}\) Equation (238) in [15].
\[ \kappa^{-1} \nabla_3 ( M_1 ) + \frac{3}{2} r M_1 = r ( 4 \kappa - 16 \omega ) \nabla f + 8 r \nabla ( - ( \kappa - 2 \omega ) f - D \nabla \nabla ( F \rho, ( F ) \sigma ) - \frac{1}{2} ( F ) ( \eta \nabla + \kappa \nabla ) ) \]

\[ = - 4 r \nabla f - 8 r D \nabla \nabla ( F \rho, ( F ) \sigma ) - 4 r ( F ) ( 2 \nabla f + \kappa \nabla ) . \]

We now compute the second term:

\[ \kappa^{-1} r \nabla_3 ( M_2 ) + \frac{3}{2} r M_2 = \kappa^{-1} r \nabla_3 \left( - 2 \kappa^2 \left( \nabla_4 ( F ) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) ( ( F ) \beta - 2 ( F ) \rho \xi ) \right) + \frac{3}{2} r ( - 2 \kappa^2 \left( \nabla_4 ( F ) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) ( ( F ) \beta - 2 ( F ) \rho \xi ) \right) \right) \]

which gives, using computations in proposition 2.1,\[ \kappa^{-1} r \nabla_3 ( M_2 ) + \frac{3}{2} r M_2 = - 4 r \left( - \frac{1}{2} \kappa^2 - 2 \omega \xi \right) \left( \nabla_4 ( F ) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) ( ( F ) \beta - 2 ( F ) \rho \xi ) \right) + 2 \kappa r \left( \left( - \kappa \xi - \omega \kappa - 6 \omega \rho - ( F ) \rho^2 + 2 \nabla_4 ( F ) \beta + ( F ) \rho ( - 2 \nabla f + \kappa \xi - 8 \omega \xi ) - 2 \beta \right) \right) \]

\[ + \left( \frac{1}{2} \kappa + 4 \xi \right) \left( \nabla_4 ( F ) \beta + \left( \frac{3}{2} \kappa + 2 \omega \right) ( ( F ) \beta - 2 ( F ) \rho \xi ) \right) \]

\[ = - 2 \kappa r \left( ( \phi_4 + \kappa ) ( ( F ) \beta + ( F ) \rho ( - 2 \nabla f - 2 \beta ) \right) . \]

Using again (2.9), we obtain

\[ \kappa^{-1} r \nabla_3 ( M_2 ) + \frac{3}{2} r M_2 = 4 \kappa r ( \nabla f + \beta ) . \]

Putting the two pieces together, we obtain

\[ \kappa^{-1} r \nabla_3 ( M ) + \frac{3}{2} r M = Q^2 \left( - 4 r \kappa \nabla f - 8 r D \nabla \nabla ( F \rho, ( F ) \sigma ) - 4 r ( F ) ( 2 \nabla f + \kappa \nabla ) \right) \]

\[ + Q^2 ( 4 \kappa r ( \nabla f + \beta ) ) \]

\[ = Q^2 ( 8 r \nabla f - D \nabla \nabla ( F \rho, ( F ) \sigma ) - \frac{1}{2} ( F ) ( \eta \nabla + \kappa \nabla ) ) + Q^2 ( 4 \kappa r \beta ) . \]

Using that

\[ q^F = - r^3 D \nabla \nabla ( F \rho, ( F ) \sigma ) - \frac{1}{2} ( F ) ( \eta \nabla + \kappa \nabla ) \]

we can finally write

\[ \kappa^{-1} r \nabla_3 ( M ) + \frac{3}{2} r M = \frac{Q^2}{r^2} ( 8 \nabla ( q^F ) ) + Q^2 ( 4 \kappa r \beta ) = 8 r^2 ( F ) ( \kappa \nabla + \kappa \nabla ) + 4 r ( F ) \rho^2 ( \psi_3 . ) \]
Finally the equation simplifies to
\[ \Box_g p = \left( -\frac{1}{4} \kappa \kappa + 5 \ell (F) \rho^2 \right) p + \left( -4r(F) \rho^2 \right) \psi_5 + 8r^2(F) \rho^2 \text{div} (q^F) + 4r(F) \rho^2 \psi_5 \]
\[ = \left( -\frac{1}{4} \kappa \kappa + 5 \ell (F) \rho^2 \right) p + 8r^2(F) \rho^2 \text{div} (q^F) \]
as desired.

**Corollary B.1.** The scalar quantity supported in \( \ell = 1 \) verifies the wave equation
\[ \Box_g (r \text{div} p)_{\ell=1} = (\rho + 4(F) \rho^2)(r \text{div} p)_{\ell=1} = 0. \]

**Proof.** Using (A.5) and (30), we obtain
\[ \Box_g (r \text{div} p)_{\ell=1} + (K + \frac{1}{4} \kappa \kappa - 5(F) \rho^2)(r \text{div} p)_{\ell=1} = 0 \]
and using Gauss equation, we finally have the desired formula.
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