An unexpected Minkowskian Solution of Einstein’s Equation of General Relativity with Cosmological Constant.

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Abstract

We suggest the following solution of Friedman’s equations: parameter of curvature $K = 0$, scale factor $R(t) = 1$ and Cosmological Constant ($CC$) $\Lambda \neq 0$. In this case Robertson-Walker’s metric becomes Minkowskian. This special solution of Einstein’s equation of General Relativity ($GR$) forces therefore us into renormalizing Einstein’s Special Relativity ($SR$) with $\Lambda \neq 0$.

By introducing a maximal interval (Hyperbolic Horizon), we deduce the law of Hubble and transform in this way $SR$ into $HCR$ (Hyperbolic Cosmological Relativity). Euclidean Einstein’s rigid ruler is replaced with Lobatchevskian LIGHT-distance. Both basic parameters of Cosmology, $H$ (Hubble) and $q$ (acceleration) are deduced on the only basis of Lorentz Transformation ($LT$). Usual ad hoc scale factor of Lemaître $R(t)$ is replaced with Bondi’s Doppler scale $k(\tau)$-factor. We induce finally a global principle of equivalence between centrifugal (hyperbolic) acceleration and repulsive gravitation. Hidden density of energy (minimal acceleration) is a relativistic effect of globally curved Minkowski’s space-time ($\Lambda \neq 0$). We obtain in this way an unexpected relationship between Einstein’s $CC$ (1917) and Poincaré’s gravitational pressure (1905) of vacuum.

1 Special unexplored case of Friedman-Lemaître’s equations

Let us consider standard Robertson-Walker’s ($RW$) metric (1-a) and Einstein’s basic equation (1-b) of General Relativity ($GR$) with Cosmological Constant ($CC$) at one space dimension (radial) $x (\mu, \nu = 0, 1)$ and light velocity ($c = 1$)

$$ds^2 = dt^2 - \frac{R^2(t)}{1-Kx^2}dx^2 \quad (a) \quad G^\mu_\nu - \Lambda g^\mu_\nu = 8\pi G T^\mu_\nu \quad (b)$$

(1)

where $G^\mu_\nu$ is Einstein’s standard curvature tensor, $g^\mu_\nu$ metric tensor and $T^\mu_\nu$ energy tensor of perfect relativistic fluid. Let us examine special case with parameter of Gaussian curvature $K = 0$, scale factor $R(t) = 1$ ($\dot{R} = \ddot{R} = 0$) but $\Lambda \neq 0$. From 2 non-static Friedman-Lemaître’s standard equations

$$\frac{K}{R^2} + \frac{\dot{R}^2}{R^2} - \Lambda = -8\pi Gp \quad (a) \quad \frac{K}{R^2} + \frac{\dot{R}^2}{R^2} - \frac{\Lambda}{3} = \frac{8\pi}{3} G \rho \quad (b)$$

(2)

with $p + \rho = 0 \quad (c)$ or $p + \rho c^2 = 0$ null density of enthalpy ($V = 1$)

we deduce a special solution (that is not de Sitter’s solution: $K = 0, \Lambda \neq 0$ but $R(t) \neq 1, \rho = 0$) (3)

$$ds^2 = dt^2 - dx^2 \quad (a) \quad \Lambda = 8\pi G \rho = -8\pi G \rho \quad (b) \quad \text{with} \quad p + \rho = 0 \quad (c)$$

(3)

This is a special Minkowskian (3a) solution because Einstein’s $CC$ (1917) seems to correspond 3b, 3c), in the framework of $SR$ (Special Relativity), to Poincaré’s negative ”gravitational” (non-electromagnetic) pressure of vacuum (1905). This Minkowskian (3-a) solution of $GR$ ($Ag^\mu_\nu = pg^\mu_\nu$) is unexplored because it is generally

1On the only basis of $LT$, Poincaré introduced in 1905 a non-electromagnetic pressure of ether (vacuum) [H. Poincaré 1905]. He wrote at the end of his paragraph 8: Comme l’attraction newtonienne est proportionnelle à cette masse expérimentale, on est tenté de conclure qu’il y a quelque relation entre la cause qui engendre la gravitation et celle qui engendre ce potentiel supplémentaire. He did not succeed to find (in his last paragraph 9) the exact relation between his ”gravitational” scalar field $p$ (”supplementary potential”) and the gravitational constant $G$ (see 27 & 28).
admitted in SR that we have \( p = \rho = \Lambda = 0 \). Cosmological Constant introduced in static Einstein’s model in 1917 ([A. Einstein] \( p = 0 \)) has, in usual point of view, nothing to do with Einstein’s 1905 relativistic kinematics ([A. Einstein (1905)]). However we underline that "vacuum without matter" is represented by CC and Minkowski’s space-time as well and that CC is a purely kinematics (standard value \( \Lambda = 10^{-57} \text{cm}^{-2} \) quantity.

2 From Finite Interval to Renormalized Hyperbolic Special Relativity \((c, \Lambda)\)

How can we introduce CC in SR? Let us consider basic invariant, space-time Interval (4b), by Lorentz Transformation \((LT, 4a)\) between two systems \( K' \) and \( K'' \) with \( \beta \) is velocity and \( \gamma = (1 - \beta^2)^{-\frac{1}{2}} \):

\[
x = \gamma(x' + \beta t') \quad t = \gamma(t' + \beta x') \quad (a) \quad x^2 - t^2 = x'^2 - t'^2 \quad (b)
\]

Let us determine respectively spacelike unit and timelike unit in such a way that \( c = 1 \) (light "cone" \( x = t \)). In primed system \( K' \), we have therefore, \( x' = 1 \) with \( t' = 0 \) \((5a)\), and \( t' = 1 \) with \( x' = 0 \) \((5b)\) (standard Minkowskian diagram of scale hyperbolas respectively along \( Ox \) and along \( Ot \)):

\[
x^2 - t^2 = x'^2 = 1 \quad (t' = 0) \quad (a) \quad \text{and} \quad t^2 - x^2 = t'^2 = 1 \quad (x' = 0) \quad (b) \quad \Rightarrow \quad x^2 - t^2 = 0 \quad (c)
\]

We use Varicak’s hyperbolic approach (without imaginary number \( i = \sqrt{-1} \)) for spacelike interval (Hyperbola along \( Ox \)) and timelike interval (Hyperbola along \( Ot \)). DurationUnit \( t' = 1 \) between two successive events in \( x' = 0 \) is irreducible to DistanceUnit \( x' = 1 \) between two simultaneous events in \( t' = 0 \).

Let us now demonstrate that we have \((5a)\) and \((5b)\) \( \Leftrightarrow 5c \). It seems that it is not true because from hyperbolic null norm \( x'^2 - t'^2 = 0 \) we obtain pseudoEuclidean \( x^2 + t^2 = 0 \) (and inversely from \( x^2 - t^2 = 0 \) we obtain \( x'^2 + t'^2 = 0 \)). This previous reasoning is however wrong because the subtraction of intervals \( (x^2 - t^2 = (x'^2))_{t' = 0} - (t^2 - x^2 = x'^2)_{x' = 0} \) is undefined. Let us therefore consider geometrically the coordinates \((x' = 1 \text{ and } t' = 1)\) of LightPoint \((1, 1)\) in \( K' \) \((FIGA, Y. Pierseaux (2009))\). These coordinates are also components of lightlike 4-vector \((1, 1)\) here \( 2 \)-vector in \( K' \). These components are \( L T e d \) into \( 2 \)-vector in \( K \) \((\gamma(1 + \beta), \gamma(1 + \beta))\) with the Bondi scale factor \( k = \sqrt{\frac{1+\beta}{1-\beta}} (5)\):

\[
\begin{array}{c}
(1, 1) \quad LT \quad (k, k) \\
\frac{x}{x} = k = \frac{t}{t}
\end{array}
\]

\[
k^2(x^2 - t^2) = 0 = x^2 - t^2
\]

This scale factor is a ratio between two LightDistances between simultaneous events \( k \) respectively \( Ox \) in \( K \) and \( O'x' \) in \( K' \). From one LightPoint \((x', t') \to (kx', kt')\) we deduce two hyperbolas \((x^2 - t^2 = x'^2)_{t' = 0} \) and \((t^2 - x^2 = t'^2)_{x' = 0} \) with \( x^2 - t^2 = 0 = x^2 - t^2 \) and therefore \((5a)\) and \((5b)\) \( \Leftrightarrow 5c \). Any distance is therefore given by space component of a light \( 2 \)-vector. Our new hyperbolic scale factor of \( k \)–dilation seems incompatible with Einstein’s interpretation of contraction of rigid rod (see §3, 11). In vacuum, we have only light-units and we didn’t need “Rigid Ruler”. If \( \Lambda = 0 \) these light-units are arbitrary.

If \( \Lambda \neq 0 \) Basic LightUnits are determined by space and time components of a lightlike \( 2 \)-vector \( \Lambda(R_H, T_H) \) where Radius of Hubble \( R_H (\Lambda = \frac{1}{T_H}) \) is a LightDistance travelled in a maximal Hubble Time \( T_H \).

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Attention, Warning: our two simultaneous events, \((0, 0)\) and \((kr, 0)\), in \( K \), are not the image by \( LT, ((0, 0), (\gamma r, \gamma \beta r)) \), of two simultaneous events in \( K' (0, 0) \) and \((r, 0)\) (see §5, Bondi). Given that any relativistic distance must be defined between simultaneous events (proper \( K' \)-distance and moving \( K \)-distance as well) we must take space components of \((1, 1) \quad LT \quad (k, k) \) in \( K' \) \((x' = 1) \) and in \( K (x = k) \) as well.

This LightPoint \((k, k)\) is longitudinal point of Poincaré’s elongated LightEllipse (LightEllipsoid) [H. Poincaré 1908]. According to Poincaré, any spherical wavefront is \( L T e d \) into an elongated ellipsoidal wavefront. We showed that Poincaré’s ellipse involves, in Minkowski’s metric, a new definition of lightdistance \((Y. Pierseaux 2006, Y. Pierseaux 2007)\). LightDistance in \( K \) between two simultaneous events, \((0, 0)\) and \((0, k)\), is by definition LightDuration in \( K \) between two successive events, \((0, 0)\) and \((0, k)\) see 6 quater).

Bondi had already a problem with Einstein’s rigid rods: “The next thing that has bedevilled special relativity is a somewhat awkward attitude towards the measurement of distance. It has always bothered me a great deal, i don’t hink all my colleagues are quite so bothered, that it is frequently stated in the books that for relativity you require a rigid ruler. However since it emerges from relativity that sound must not be faster than light, and since in perfectly rigid rule the velocity of sound is infinite, whereas the velocity of light is finite, you have an inconsistency which has always caused me no end of bother. I now think that we need to consider this particularly difficult instrument, the rigid ruler”. I have discovered that what I call the \( k \)-calculus is still unknown in educated mathematical circles.” Bondi deduced from Milne’s radar method the relativistic addition of velocities (\( \beta_H \), the hyperbolic velocity), Einstein’s time dilation, but not Einstein’s contraction. Bondi didn’t know Poincaré’s interpretation of Lorentz contraction (11) and did not succeed to replace Euclidean ruler with Lobatchevskian ruler.
We have therefore a Finite (Maximal) spacelike interval (5-H, a) and Finite (Maximal) timelike interval (5-H, c). Hyperbolas are inseparable from their asymptotes. If hyperbolas disappears $\Lambda \to 0$ (units $\to \infty$), light cone (Finite velocity) also disappears $c \to \infty$. Finite including units of space and time $\Lambda (R_H, T_H)$ determine a null 4-vector ($c = 1$). By introducing an Hyperbolic (Lobatchevskian) Horizon $R_H$ (inaccessible Max $x^2 - t^2 < 1$), we transform SR into HC(S)R, (Hyperbolic Cosmological Relativity). $HCR$ is a $D$(onably)$SR(c, \Lambda)$ where $R_H$ is a classical invariant length. Unlike Friedman case $K = -1$ with variable Gaussian curvature ($K/R^2(t)$) we can deduce an hyperbolic distance from a Circle Limit with a fixed Horizon. In Beltrami’s model, Cayley-Klein’s definition of hyperbolic radial distance $s_H$ (from origin $O$ of the circle to a point $P$) with cross-ratio formula and Cartesian radial distance $x' = r < R_H$ is:

$$s_H = \frac{R_H}{2} \ln \frac{R_H(R_H + r)}{R_H(R_H - r)} = R_H \tanh^{-1} \frac{r}{R_H} \quad \Rightarrow \quad s_H = R_H \ln \sqrt{\frac{1 + \frac{\varepsilon}{R_H}}{1 - \frac{\varepsilon}{R_H}}} = \ln \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} (R_H = 1)$$

$HCR$ consists of a $\Lambda$–renormalization of $SR$ ($r = \varepsilon R_H$) ($0 \leq \varepsilon < 1$) where $r$ is comobile ($K'$) distance of material point (galaxy). This new definition of distance has to be consistent with Varicak’s model of velocity space ($v < c$, horizon of limit circle with hyperbolic velocity $\beta_H = \ln \sqrt{1 + R^2}$). Milne’s model was only based on a three dimensional version of Beltrami’s model of velocity space. Unfortunately in $HCR$, a constant scale factor $R(t) = 1$ seems incompatible with an expanding Universe because Hubble ”constant” $H(t) = \frac{R(t)}{R(t)} = 0$ seems to be cancelled (see 21).

We suggest a development in four stages: deductions of Hubble law (§3), non static Minkowskian Friedman’s metric (4a, §4), natural scale factor (§5) and finally (§7) principle of global equivalence because $HCR$ must be a coherent singular solution of GR (2c or 3c).

### 3 From comobile proper time to Hubble’s law

In Varicak’s representation $LT$ is an Hyperbolic Rotation ($HR$) of axis $O'x'$ and $Ot^1$ ”(in scissors towards $x' = t'$ \(^5\)) ([V. Varicak \(\text{[J. F. Barrett]}\) see annex §8). In basic Minkowski’s diagram (§2, FIGA), Hyperbola along $Ot$ represents all possible (hyperbolic) velocities $\beta_H = \ln k$ (angles) of system $K'$ whilst hyperbola along $Ox$ (a worldline, §4) represents all successive angles $\beta_H$ (velocities) of one system $K'$ (Rindler §4). Suppose an $HR$ ($LT$) from 0 to $\beta_H$ along arc of $basic$ hyperbola $Ox(R_H)$ in polar hyperbolic coordinates $(R_H, \beta_H)$. Einstein’s element of proper time is by definition a duration $dt' = dt$ between two successive events at the same place in proper system $K'$: $dx' = 0$ ($dt^2 = dt^2(1 - \frac{dx'^2}{dt^2})$. We can easily calculate Hyperbolic Proper Time (length of hyperbolic arc) $\tau_H$ by integration (given that $x = R_H \cosh \beta_H$ and $t = R_H \sinh \beta_H$)

$$\left(\frac{dx}{dt}\right)_{dx'=0} = \beta \quad \tau_H = \int \frac{\beta_H}{R_H} \, dt = \int_{Ox} \sqrt{1 - \beta^2} \, dt = R_H \sinh^{-1} \frac{t}{R_H} = R_H \beta_H$$

This result ”$\tau_H = R_H \beta_H"$ is obvious from analogy (§8) with ”$s = R\theta"$ in circular ($R, \theta$) polar coordinates ($s_H$ and $\tau_H$ are components of lightlike 4-vector, see 11). Both hyperbolic distance $s_H$ and hyperbolic velocity $\beta_H$ are connected to scale factor $k$ (6 with7):

$$s_H = \tau_H = \beta_H = \ln \sqrt{\frac{1 + \beta}{1 - \beta}} (R_H = 1) = \ln k \quad \Rightarrow \quad \varepsilon = \beta$$

Let us examine now the serious consequences of renormalization for comobile $K$-distance of galaxy $r < 1$. We first deduce a basic proportionality ”distance-velocity”

$$r = \beta R_H \quad \beta = Hr \quad (10)$$

\(^5\)”Finite interval” is a natural relativistic idea because Infinite interval would mean the Return of Minkowski’s Shadow (Absolute Space).

\(^6\)Milne admitted $\Lambda = 0$ and variable $R(t)$ and his relation distance-velocity was only postulated and not deduced, §3.
i.e. the law of Hubble with Hubble true constant $H = \sqrt{\Lambda}$, which is a basic constant angular velocity ($\S 8$; \textit{"$v = \omega r$"}). So if we have large velocity we apply Einstein's kinematics (without $CC$) but if we have large velocity and distance (galaxy) we apply $HCR$ (Renormalized $SR$ with $CC$).

With only light signals in vacuum (without material rigid rods), we secondly obtain, with a source at $O'$, (see 6) a $k$--dilation of LightDistance. Let us indeed introduce $r_O$ a "moving LightDistance" for observer $O$ in $K$, defined by spacelike interval between two simultaneous events $(0,0)$ and $(kr,0)$ in $K$ (6) (space component of light 2-vector $(kr, kr)$):

$$\beta_H = \ln k \quad (a) \implies \quad r_O = kr \quad (b) \implies \quad (r, r) \to (kr, kr) \quad (c) \quad (6\text{bis})$$

\text{Lobatchevskian LightDistance is not compatible with Einstein's 1905 Euclidean Rigid Ruler (7 from 5-H) (with $\gamma^{-1}$--contraction). If any distance is a light-distance (light travel duration) and if light is a monochromatic electromagnetic wave, any length ($r$) must be $LT$ed as a wavelength ($\lambda_{source}$) which is by definition a light-distance (travel period). Like Einstein's $\gamma^{-1}$--contracted rigid ruler, $k$ -- dilated LightDistance is completely reciprocal: lightwave of a moving source is always redshifted. So we underline that factor $k$ is a natural scale factor (see $\S 5$). Nothing is changed with Einstein's element of proper time but by symmetry we have with integration of the element of proper length $dt' = 0 \ (dx'^2 = dx^2(1 - \frac{dt'^2}{dx^2})$ on the other $Ot$ Hyperbola: :

$$\frac{dt}{dx}_{dt=0} = \beta \quad s_H = \int_{Ot} dx' = \int_{Ot} \sqrt{1 - \beta^2} dx = R_H \beta_H = \ln k \ (R_H = 1) \quad (11)$$

We have therefore also an element of proper length which is, like proper time, contracted in $K'$ (Poincaré's interpretation of Lorentz contraction). Until now these integrations (8 and 11) seem purely geometrical. But the integration on angles suppose an hidden acceleration because angles are velocities $\beta_H$ ($\S 8$). Finally we remark that basic hyperbola embedded in Minkowski’s space-time

$$x^2 - t^2 = \frac{1}{\alpha_M^2} = R_H^2 = \varrho_H^{-2} \quad (12)$$

has a constant radius of curvature $\varrho_H$ exactly as circle in Euclidean space ($\S 8$). There is no contradiction with $GR$ (whose $HCR$ is a special case, $\S 1$) because there is no local curvature (Christoffel) but a global constant curvature of space-time.

4 From Rindler’s Metric to Minkowski’s globally curved Space-time with minimal acceleration

Let us consider now proper hyperbolic acceleration of a test ("material", see $\S 6$) point $P'$ on our basic worldline $Ox$ hyperbola (5-H). To basic angular velocity (constant of Hubble) corresponds a minimal Milgrom proper acceleration $\alpha_M = \frac{H}{c}$ also directly connected with $CC$ ($\Lambda = \frac{1}{R_H^2}$) in vacuum space-time.

$$x^2 - t^2 = \frac{1}{\alpha_M^2} = R_H^2 = \varrho_H^{-2} \quad \alpha_M = \frac{d\beta_H}{dt} = H \quad \alpha_M \tau_H = \beta_H \quad (13)$$

If we have now a proper minimal acceleration in vacuum space-time, a basic objection could be: "therefore inertial motion becomes impossible". However introduction of a minimal centrifugal acceleration involves immediately global Lobatchevskian negative ($\varrho_H = -R_H^{-1}$) curvature of space-time (12) and therefore

$$K = 0 \ \text{but} \ \varrho_H = -R_H^{-1} \quad \Rightarrow \quad \varrho_H c^2 + \alpha_M = 0 \quad (14)$$

(norm is square root of interval, see $\S 6$). So the contradiction with inertial motion disappears because a free "material" point $P'$ follows a geodesic of curved $\varrho_H$ space-time (like any geodesics in $GR$). This negative inverse curvature $\varrho_H$ has to be distinguish from hyperbolic Gaussian variable curvature $k/R^2(t)$. $\varrho_H \neq 0$ (from geometrical $CC$ in $GR$) and $K = 0$ (in metrics) are compatible with constant curvature (remind that our special solution of Friedman’s equation $HCR$ never has to be in contradiction with general consequences of $RW$’s metrics). Rindler’s metric issued from "Born’s Rigid Motion" of accelerated system $K'$, is written (with Rindler’s notations) for polar coordinates $X = \alpha^{-1}$ and $T = \beta_H$

$$ds^2 = X^2dT^2 - dX^2 \quad \rightarrow \quad ds^2 = dT^2 - dX^2 \quad (15)$$
Rindler’s metric is a non-Minkowskian Metric ([W. Rindler]). With a minimal acceleration $\alpha = \alpha_M$, Rindler’s metric becomes, by renormalization (5-H), Minkowski’s metric ($X \rightarrow R_H = 1$) In our notations, we have with cosmic time $\tau_H$ (8) for a free point
\[ ds^2 = R_H^2 d\beta_H^2 - dr^2 = d\tau^2 - dr^2 \quad (= dt^2 - dx^2) \] (16)
It is no longer an accelerated motion but an inertial geodesic motion... in a curved Lobatchevskian vacuum:
\[ dt^2 - dx^2 = ds^2 = d\tau^2 - dr^2 \] (17)
So Minkowskian metric becomes a singular case of non-static Friedman’s metric (4-a) and therefore compatible with expanding space: In standard pseudo-Euclidean SR (with imaginary number), Minkowski space-time is flat but in Hyperbolic Renormalized SR it is globally curved by $\Lambda \neq 0$ ($§1, 12$).

Let us examine now the serious consequences of renormalization for acceleration $\alpha > 1$ (comobile distance $r < 1$). Unlike $GT$ (Galilean Transformation), $LT$ is an hyperbolic rotation. Any rotation involves an acceleration. Einstein’s boost (active $LT$) defines an hidden phase of acceleration of $P'(K')$ along arc of hyperbola ($FIGA, 0 \rightarrow \beta_H$) before the phase of cruise velocity $\beta$ (uniform translation) along tangent to $P'(LT - HR - Boost, §8)$.

In any rotation we have a direction of acceleration towards a fixed center of acceleration $O = O'$, §8. In Euclidean Rotation Motion we have a centripetal acceleration ($\alpha = -\frac{r^2}{r^2}$) whilst in Hyperbolic Rotation Motion ($LT$) we have a centrifugal ($\alpha = \frac{r^2}{r^2}$) acceleration (see 14) that corresponds to expanding LightDistance (6bis, factor $k = 1 + z$). In Born-Rindler’s model of Rigid Motion (According to Rindler, Einstein’s rigid rod “ends in a photon”) there is no LightDistance because a photon dispatched to chase the particle $P'$ at $t = 0$ from a source $S$ in $O$ never catch up (Rindler’s event horizon) with a mirror $M'$ fixed in $P'$. Rindler’s boost never stops and basic connection between Einstein’s boost and $LT$ disappears (Einstein’s boost stops, see §8).

Suppose a mirror $M'(K')$, at distance $r < R_H$, reaching its “cruise velocity” at time $t = \beta \gamma$ (by $LT$) ($FIGA$, geometrically tangent to hyperbola at $M$) and therefore the photon, emitted at the origin of time $t = 0$, catch up with mirror $M'$. The time of catching up is given by intersection between tangent to $M(\gamma r, \gamma b\beta)$ and light cone $c = 1$ ($FIGA, \ Y. \ Pierseaux(2009)$).
\[ \text{tangent } t - \gamma \beta r = \frac{1}{\beta}(x - \gamma r) \quad \text{and cone } x = t \implies x = \gamma(1 + \beta) = kr = r_O \] (6ter)
So the distance Source-Mirror $SM = r_O$ in $K$ will be not contracted but elongated (6, 6bis) by $k - s$cake factor (see $quarter$ when respective roles Source-Mirror are inverted). We underline that redshift cannot be changed into blueshift because in basic boost origin $O'$ (with $S$) moves away from origin $O$ ($\beta \geq 0, k \geq 1$). We have to change Einstein’s interpretation of Lorentz contraction that remains unobservable until today. Einstein’s contraction is impossible to demonstrate by using only light signals in vacuum ($§5$). Moreover we must find a principle of equivalence between $K$ (gravitation) and $K'$ (acceleration).

5 From Bondi’s Scale factor $k$ to accelerated expanding Universe

In standard Cosmology ($K = 0$) with Lemaitre’s scale factor $R(t)$ the definitions of first $H(t)$ and second parameter of Cosmology $q$ are:
\[ H(t) = \frac{\dot{R}(t)}{R(t)} \quad (1) \quad q = -\frac{R(t)\ddot{R}(t)}{R(t)^2} \quad (2) \quad \text{with } 1 + z = \frac{R(t_e)}{R(t)} = 1 + \beta + (1 + \frac{1}{2}q)\beta^2 + ... (3). \] (18)
and for radial trajectory of photons
\[ dt^2 - R^2(t)dx^2 = 0 \quad (4) \]
Let us begin with (18, 3 and 4). In $HCR$ radial trajectory of photons on null geodesics (17) is
\[ dt^2 - dx^2 = 0 = d\tau^2 - dr^2 \quad \text{with } dx = kdr \quad \text{or } \quad dt = kdr \] (19)
Doppler measurable spectral redshift is by definition $z = \frac{\Delta \lambda}{\lambda} = k - 1$ with an emission by a remote moving source and reception by $O$. Let us therefore inverse in details respective roles of source and mirror (6ter). Consider a light source $S$ at rest in $K'$ emitting towards $O'$ and $O$ ($M$), when origins coincide in $t = t' = 0$. Time $t'$ of
\[ \text{Poincaré disagrees (visibility of Lorentz contraction by elongation distance) with Terrel-Penrose (invisibility of Lorentz contraction) on this point.} \]
arrival in $O'$ determines distance $r(t_e = 0) = O'S$ in $K'$. By $LT$ $(0, r) \rightarrow (\gamma \beta r, \gamma r)$ light signal reaches $O'$ in time $\gamma r$. Light signal not yet reaches $O$. It has still to travel distance $O'O = \gamma \beta r$ in $K$. So observer $O$ receives on his telescope (Mirror) light signal at the time $t_r = \gamma \tau + \gamma \beta r = kr$. So the distance at time $t = 0$ for the observer and source was precisely $kr$. So we have a relativistic effect of motion on measurements of large distances (note 3). With monochromatic $S(\lambda)$ source we have (6, 6bis, 6ter):

$$k = 1 + z = \frac{r(t_r)}{r(t_e)} = \frac{\lambda(t_r)}{\lambda(t_e)} = 1 + \beta - \frac{1}{2} \beta^2 + q = -1 \quad (6\text{quater})$$

LightDistance $r$ is directly connected with LuminosityDistance $d$ on mirror of telescope of Tolman-Robertson formula in cosmology (20, [M.P. Robertson, H.C. Tolman [Y. Pierseaux (2009)].] In the case of stationary source we have luminosity at distance $d$ with $l_e = \frac{L_S}{4\pi(1+z)^2}$ ($L_S$ absolute luminosity). When the source is receded according to Tolman we have a double relativistic reduction of luminosity $l_r$ on a small area of mirror $l_r = \frac{L_S}{4\pi(1+z)^2} l_e$ and apparent luminosity is then $l_r = \frac{L_S}{4\pi(1+z)^2} r^2$ and therefore

$$k = 1 + z = \frac{d(t_r)}{d(t_e)} \quad (20)$$

Like radial LuminosityDistance, our new LightDistance is also deduced from $LT$ ([Y. Pierseaux (2009)]). According to Robertson, sphere of emission is $LT ed$ into another sphere of reception (Einstein-1905, see note 3). With Poincaré’s light ellipsoid (note 3) the element of solid angle of aperture in radial direction is $LT ed$ into $d\Omega_e = k^{-2} d\Omega_e$ and therefore invariant element of surface (small area of mirror) $dS = r(t_e)^2 d\Omega_e = dS' = r(t_e)^2 d\Omega_e$ involves immediately an expanding radial space (6quater). Einstein’s Doppler formula (invariance of phase of monochromatic plane wave) and Poincaré’s “Doppler” formula (invariance of phase of monochromatic spherical wave) only coincide (6-quater) in quasi-radial configuration ([Y. Pierseaux 2007]).

How is it now possible to deduce a Hubble constant without introducing Lemaitre scale factor $R(t)$ in Friedman’s metric (18, 1 & 2)? With basic minimal acceleration in vacuum, Bondi’s scale factor in Renormalized $HCR$ depends of Cosmic time $k(\tau)$ (13, "point" means derivation with respect to proper cosmic time $\dot{H}\tau$) only in Hoyle-Gold-Bondi’s model relativistic concept of interval is taken into account ([H. Bondi]). But even in Hoyle’s Steady State theory, based on perfect space-time Cosmological Principle ([Y. Pierseaux (2010)], a time dependent scale factor $R(t) = R_0 e^{Ht}$ ($K = 0$, $\Lambda = 0$) with exponential expanding ($K = 0$) is introduced. In $HCR$ we have $\ddot{k} = H \dot{k}$ and $k = k$ given that Hubble constant is a true constant:

$$\dot{k} = \frac{d\beta_H}{d\tau_H} \quad H \quad k(\tau) = e^{H\tau} \quad \text{versus} \quad (18-1) \quad (21)$$

Only in Hoyle-Gold-Bondi’s model relativistic concept of interval is taken into account ([H. Bondi]). But even in Hoyle’s Steady State theory, based on perfect space-time Cosmological Principle ([Y. Pierseaux (2010]), a time dependent scale factor $R(t) = R_0 e^{Ht}$ ($K = 0$, $\Lambda = 0$) with exponential expanding ($K = 0$) is introduced. In $HCR$ we have $\ddot{k} = H \dot{k}$ and $k = k$ given that Hubble constant is a true constant:

$$q = \frac{-\ddot{k}}{\dot{k}} = -1 \quad \dot{k} - H^2 k = 0 \quad \text{versus} \quad (18-2) \quad (22)$$

that is entirely induced from $LT$, involves then a basically accelerated expansion. Let us note that we have basic analogy (see §8) of this harmonic hyperbolic expansion $e^{H\tau} H$ with harmonic $x + \omega^2 x = 0$ elongated oscillation $e^{int}$ (with expected hyperbolic change of sign): elongation $k$ is proportional not only with velocity but also with "acceleration" $\ddot{k}$. Our scale factor $k(\tau)$ predicts directly a redshift effect

$$z(\tau) = e^{H\tau} - 1 \quad \delta z(\tau) = H e^{H\tau} \quad \delta^2 z(\tau) = H^2 e^{H\tau} = H \delta z(\tau) \quad (23)$$

where the "acceleration" of elongating $\delta^2 z(\tau)$ is proportional to the velocity of expanding $\delta z(\tau)$. In $HCR$ we don’t need artificial scale factor $R(t) \geq 0$ because we have a natural scale factor $k(\tau) \geq 1$ Standard singularity $R(t) = 0$ ("Big Bang") is changed into "Big Boost".

6 From global hyperbolic principle of equivalence to "dilated non-diluted” medium

Until now vacuum in $HCR$ seems without matter (3a). How is it possible to introduce a pressure or a density of energy (3c)? Classical vacuum in $HCR$ is however not completely empty because minimal scalar field of

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8This is a new result from relativity of simultaneity.” Coincidence $O - O'$ in $t' = t = 0'$ and "emission in $t_e = 0'$ are simultaneous events only in $K'$ but not in $K$ ($t_e = \beta \gamma r$). So the time of reception at chronometer of $O$ is $t_r = \gamma \tau + \beta \gamma r = kr$ (6, 6bis, 6ter). In prerelativistic kinematics distance does not depend on state of motion of source. With Lorentz boost (LT) we see that distance of moving source is always $k - \text{diluted}$.. This relativistic effect on large distance is perfectly compatible with proper Lorentz contraction in (11).
acceleration $\alpha_M$ corresponds to density of an hidden energy $E_H$ with pseudo-mass $M_H = \frac{E_H}{c^2}$. Given that $HCR$ is a special solution of $GR$ ($\S 1$), this theory has to be based on a principle of equivalence. Unlike local (parabolic) principle of equivalence (between centripetal acceleration $\alpha_p = \frac{GM}{c^2}$ and gravitation) we have now in $HCR$ a global (hyperbolic) principle of equivalence between centrifugal acceleration and gravitation with

$$ \frac{GM_H}{R_H^2} = -\alpha_M(a), \quad \phi = c^2 = -\frac{GM_H}{R_H} \quad (b), \quad \alpha_M = \frac{\phi}{R_H}(c) \quad (24) $$

Centrifugal acceleration supposes therefore a repulsive gravitation ($24(b)$ means that Universe is an anti black hole). Global field of acceleration in $K'$ is identical to global scalar field of gravitation in $K$. In the same way, proper constant acceleration in $K'(\tau)$ corresponds to a global curvature in $K(t)$. Until today the deep connection between $SR$ invariance and $GR$ equivalence was not completely clarified. We obtain now a new synthesis between $GR - logic$ and $SR - logic$ because in $HCR$, $dt^2 - dx^2$ and $dr^2 - dv^2$ ($17$) are "Lorentz-invariant" and "Einstein-equivalent" as well. Moreover by multiplying with proper pseudo-mass $M_H$ 4-vector acceleration ($\S 3$) $\vec{A}'(\alpha, 0) \xrightarrow{LT} \vec{A}(\gamma \alpha, \gamma \beta \alpha)$ with ($||A'|| = ||A|| = \alpha_M$), we obtain dynamics spacelike 4-vector force,$\vec{F}'(M_H \alpha_M, 0) \xrightarrow{LT} \vec{F}(\gamma M_H \alpha_M, \gamma \beta M_H \alpha_M)$ whose norm (square root of interval) is invariant

$$ \left| \vec{F}' \right| = F_H = \pm M_H \alpha_M = \pm \frac{M_H}{R_H} c^2 \quad \left| \vec{F} \right| = \pm \frac{GM_H^2}{R_H^2} = \pm \frac{c^4}{G} \quad (25) $$

Everything happens as if a repulsive force acts between two ends of our non-rigid rod (elastic segment or string). Unlike non-invariant vectorial attractive gravitation law for baryonic matter ($M \rightarrow \gamma M$), scalar auto-repulsive gravitational law is "Lorentz-invariant" (remind that $R_H$ is an interval, $5-H$). Change of sign, that is allowed by relativistic algebra, does not modify the type of 4-vector or interval (square of norm). So we have Friedman’s formula of sphere with kinetics energy and potential energy (without prerelativistic factor $\frac{1}{2}(v^2)$, see $8$)

$$ \alpha_M + \frac{GM_H}{R_H^2} = 0 \quad (a) \quad M_H c^2 + \frac{GM_H^2}{R_H^2} = 0 \quad (b) \quad (26) $$

By multiplying ($26$) with $\frac{M_H}{R_H}$ we obtain basic relationship between Poincaré’s pressure $p_H$ and density of energy $\rho_E$ of hidden relativistic fluid ($3c$)

$$ \frac{F_H}{R_H^2} + \frac{GM_H^2}{R_H} \frac{1}{R_H} = 0 \quad \Rightarrow \quad p_H + \rho_E = 0 \quad p_H + \rho_M c^2 = 0 \quad (27) $$

we obtain basic relationship with negative pressure connected to "fictive" mass $M_H$ or "fictive" energy $E_H$. Fictive fluid is a $GR$ effect of negative curvature and by multiplying ($14$) with $M_H^2 R_H^4$ we obtain the same result ($27$) $M_H^2 \theta_H^3 c^2 + \frac{M_H}{R_H} \alpha_M = 0 \quad \Rightarrow \quad \rho_M c^2 + p_H = 0$ (Y. Pierseaux (2009)): that characterizes a vacuum medium "in dilution without dilution" (null density of enthalpy, $3-c$). This is an extraordinary result: from renormalization of $SR$ ($5-H$) that involves isenthalpic motion ($27$) (Y. Pierseaux 2001). $HCR$ is based on a perfect cosmological principle: it is a Steady State theory without problem of dilution (creation of matter) because Poincaré’s pressure (Einstein’s $CC$) is a Lorentz invariant. We can define negative density-curvature (and then positive pressure) or negative pressure-acceleration (and then positive density) as well. Let us choose negative pressure. We have then with measurable Hubble constant a density

$$ p_H = -\frac{c^2 H^2}{G} \quad \rho_M = \frac{\Lambda}{G} \quad (28) $$

So density of energy of gravitational Poincaré’s ether is very small. This is the reason why Einstein’s 1905 kinematics remains valid, except in Cosmology. Einstein’s $GR$ remains valid, even in Cosmology, except maybe for a factor of normalization. Indeed $GR$ coefficient of normalization $8\pi$ ($3b$, without $HR$ factor $\frac{1}{2}$, $8$) we have $\rho_M = \frac{\Lambda}{8\pi G} = \frac{H^2}{8\pi G}$ that is the third of critical density $cd = \frac{3H^2}{8\pi G}$. With relativistic correction ($8$) we deduce $\rho_M = \frac{\Lambda}{4\pi G}$ observed value with two thirds of $cd$.

7 Conclusion: from the ”Big Bang”to the ”big boost”

We prove that there is already a serious problem with only one space dimension between $SR$ and $GR$. The generalization at two (three) space dimensions have to be based on Poincaré’s ellipse (ellipsoids, note3) and Penrose’s hyperboloids ([Y. Pierseaux (2010)]). Under strong ”pressure” of Einstein’s $GR$, we was forced into
modifying Einstein’s SR. HCR is a new solution of standard Friedman’s equations, completely compatible with recent observations without introducing standard ad hoc scale factor $R(t)$. The observed accelerated expansion is a relativistic Lorentzian effect on measurements of large distances. In standard literature CC is generally associated to quantum vacuum. In HCR, CC is associated to classical vacuum but we note that we have an undulatory space-time because any lightlength is $LTed$ like a wavelength. Our special solution HCR is a purely anti-Machian solution of $GR$ because the pseudo-mass of remote object is, dialectically, a physical effect of inertial curved geodesic. With only $LT$ but the whole $LT$ ($§8$), we replaced the Big Bang (Absolute Origin of time $t = 0$) with a Big (Lorentz) Boost where the choice of origin $O–O'$ in $t = \tau = 0$, is possible in any point of space-time (Hubble’s Hyperbolic Horizon remains identical). Finally, we remark that the "Big Lorentz Boost" with minimal acceleration involves a basic emission of radiation by electron⁹ (see next paper about hyperbolic electrodynamics).

8 Annex: Euclidean Rotation versus Hyperbolic Rotation

Standard analogy between Hyperbolic Rotation (Lorentz Transformation, LT) and Euclidean Rotation is always limited, in standard literature, to mathematical concept (without motion). This limitation is a nonsense because, between "circular" $s = R\theta$ versus "hyperbolic" $\tau_H = R_H\beta_H$, the angle $\beta_H$ is already a velocity and the length $s_H$ is already a time $\tau_H(c = 1)$. This $LT$-analogy therefore is also valid between physical Uniform Circular Motion (UCM) and Uniform Hyperbolic Motion (UHM). On hyperbolic point of view, Minkowski’s scale hyperbola (Lobatchevskian Curvature) and Born’s hyperbola of acceleration (Milgrom acceleration) are exactly identical: it is impossible to separate $LT$ from $LB$ (Lorentz Boost). Given that any Rotation involves an acceleration, we have therefore an acceleration in any LT. Einstein showed in 1905 that any (active) details three successive phases for ($P$ is already a time)

| Standard analogy between Hyperbolic Rotation (Lorentz Transformation, LT) and Euclidean Rotation | Uniform Circular Motion (UCM) | Uniform Hyperbolic Motion (UHM, LT) |
|---|---|---|
| Velocity $v$ | $\alpha = -\frac{v^2}{R}$ | $\alpha_M = \frac{v^2}{R_H}$ |
| Euclidean Radius (Rigid Ruler) $R$ | $\omega = \frac{v}{R}$ | Hubble Constant $H = \frac{R_H}{\tau_H}$ |
| Period of Rotation $T$ | $\omega = \frac{v}{R}$ | Law of Hubble within segment $\beta = H\tau$ |
| Centripetal Acceleration $\alpha = -\frac{v^2}{R}$ | constant angular velocity within disc $v = \omega r$ | negative global curvature $\rho_H = -R_H^{-1}$ |
| $\alpha = -\frac{v^2}{R}$ | positive curvature of trajectory $R^{-1}$ | Cosmic proper time $\tau_H = R_H\beta_H$, $d\tau = R_Hd\beta_H$ |
| $\omega = \frac{v}{R}$ | Length of arc $s = R\theta$, $ds = Rd\theta$ | Hyperbolic elongation elastic factor $k = e^{H\tau_H}$ |
| Harmonic elongation $R + \omega^2 R = 0$ | Harmonic elongation $\dot{R} + \omega^2 R = 0$ | Hyperbolic Elongation $\dot{k} = H^2 k = 0$ ($q = -1$) |
| standard rotation:: fixed $R$ and variable $\theta(t)$ | fixed $R$ and variable $\theta(t)$ | fixed $R_H$ and variable $\beta_H(\tau)$ |
| Scale factor $R(t)$ in Euclidean Cosmology ($K = 0$) | Scale factor $k = 1 + z$ in HCR ($K = 0$, $\Lambda \neq 0$) | scale factor $k = 1 + z$ in HCR ($K = 0$, $\Lambda \neq 0$) |
| non-minkovskian metric $ds^2 = dt^2 - R^2(t)dx^2$ | minkovskian metric $ds^2 = dt^2 - dx^2$ | minkovskian metric $ds^2 = dt^2 - dx^2$ |
| variable $R(t)$ and fixed $\theta (\theta = 0, \text{radial})$ | Fixed Horizon $R_H = 1$ and variable $\beta_H = \ln k(\tau)$ | Fixed Horizon $R_H = 1$ and variable $\beta_H = \ln k(\tau)$ |
| artificial logarithmic derivative $\frac{d}{dt}(\ln R(t)) = \frac{\dot{R}(t)}{R(t)} = H$ | Natural logarithmic derivative $H = \frac{d}{dt}(\ln k) = \frac{k_0}{k(\tau)\beta_H}$ | Natural logarithmic derivative $H = \frac{d}{dt}(\ln k) = \frac{k_0}{k(\tau)\beta_H}$ |
| parameter of deceleration $q = -\frac{R(t)}{R(t)}\dot{R}(t)$ | parameter of acceleration $q = -\frac{k}{k_0} = -1$ | parameter of acceleration $q = -\frac{k}{k_0} = -1$ |
| radial photons $dt^2 - R^2(t)dx^2 = 0$ | radial photons $dt^2 - dx^2 = 0$ | radial photons $dt^2 - dx^2 = 0$ |
| Centripetal non invariant acceleration $\alpha_{cp} = \frac{GM}{R^2}$ | Centrifugal invariant acceleration $\alpha_M = \frac{GM}{R_H^2}$ | Centrifugal invariant acceleration $\alpha_M = \frac{GM}{R_H^2}$ |

⁹Historically Poincaré introduced his gravitational constraint on electromagnetic field (e-m) in such a way that impulsion and energy of an electron, entirely induced from e-m field, forms a 4-vector. Without parameters of electron (classical radius and charge) we cannot deduce the value of CMB.
because we have an centrifugal acceleration (Hubble constant) connected with scale Bondi’s factor.

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