Hilbert series of generic ideals in products of projective spaces

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Abstract

If \( k[x_1, \ldots, x_n]/I = R = \sum_{i \geq 0} R_i \), \( k \) a field, is a standard graded algebra, the Hilbert series of \( R \) is the formal power series \( \sum_{i \geq 0} \dim_k R_i t^i \). It is known already since Macaulay which power series are Hilbert series of graded algebras \([12]\). A much harder question is which series are Hilbert series if we fix the number of generators of \( I \) and their degrees, say for ideals \( I = (f_1, \ldots, f_r) \), \( \deg f_i = d_i, \ i = 1, \ldots, r \). In some sense "most" ideals with fixed degrees of their generators have the same Hilbert series. There is a conjecture for the Hilbert series of those "generic" ideals, see below. In this paper we make a conjecture, and prove it in some cases, in the case of generic ideals of fixed degrees in the coordinate ring of \( \mathbb{P}^1 \times \mathbb{P}^1 \), which might be easier to prove.

KEYWORDS: \( \mathbb{P}^1 \times \mathbb{P}^1 \), Hilbert series, generic ideals
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1 Background

The conjecture and the results in section 2 below are inspired by the corresponding conjecture and results in the singly graded case.

Conjecture 1 \([5]\) Let \( I = (f_1, \ldots, f_r) \subset k[x_1, \ldots, x_n] \), \( k \) an infinite field, be an ideal generated by generic forms \( f_i \) with \( \deg(f_i) = d_i \), and let \( R = k[x_1, \ldots, x_n]/I \). Then

\[
R(u) = \left[ \prod_{i=1}^r (1 - u^{d_i})/(1 - u)^n \right]_+ + \sum a_i u^i = \sum b_i u^i, \text{ where } b_i = a_i \text{ if for all } j \leq i \text{ we have } a_j > 0, \text{ and } b_i = 0 \text{ otherwise.}
\]

We first comment on the use of the word "generic". A polynomial of degree \( d \) in \( k[x_1, \ldots, x_n] \) is a linear combination of \( \binom{n+d-1}{d} \) monomials. Thus an ideal \( (f_1, \ldots, f_r) \), \( \deg(f_i) = d_i \), can be considered as a point in \( A = \mathbb{A}^N, N = \sum_{i=1}^r \binom{n+d_i-1}{d_i} \). There is a Zariski-open subset of \( A \), for which the Hilbert series is constant. Ideals corresponding to points in that Zariski-open set are what we call generic, see \([8]\).

The conjecture is proved for \( r \leq n \) (trivial), for \( n \leq 2 \) \([3]\), for \( n = 3 \) \([1]\), for \( r = n + 1 \) \([16]\). There are partial results in \([10], [7], [2], [13], [3], [15], [14]\).
If $J = (l_1^{d_1}, \ldots, l_r^{d_r})$, where $l_i$ are generic linear forms. Sometimes, but not always, the Hilbert series of $k[x_1, \ldots, x_n]/J$ equals the one in the conjecture. There is a conjecture on when it does. [11] [4].

2 $\mathbb{P}^1 \times \mathbb{P}^1$

We are considering homogeneous ideals in the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, let $k$ be an infinite field, $S = k[x_0, x_1, y_0, y_1]$ be bigraded, $\deg(x_i) = (1, 0)$, $\deg(y_j) = (0, 1)$, and let $I$ be a bihomogeneous ideal, so generated by bihomogeneous elements. Hence $R = S/I$ is bigraded, $R = \oplus_{i,j \geq 0} R_{i,j}$. The Hilbert series of $R$ is defined as $R(u, v) = \sum \dim_k R_{i,j} u^i v^j$.

We are interested in the case when the ideal is generated by "generic" elements. Given an ideal $I$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we denote the space of ideals $I = (f_1, \ldots, f_r)$ where $\deg f_i = d_i$ by $I_{d_1, \ldots, d_r}$. An element of degree $(d, e)$ is a linear combination of $(d+1)(e+1)$ monomials. Thus an ideal in $I_{d_1, \ldots, d_r}$ can be considered as a point in $\mathbb{A}^N_k$, $N = \sum_{i=1}^r (d_i+1)(e_i+1)$.

We partially order Hilbert series termwise, so that $\sum a_{i,j} u^i v^j \geq \sum b_{i,j} u^i v^j$ if $a_{i,j} \geq b_{i,j}$ for all $i, j$.

**Theorem 2** There are only a finite number of possibilities for Hilbert series of ideals in $I_{d_1, \ldots, d_r}$. There is a nonempty Zariski open part of of $I_{d_1, \ldots, d_r}$ where the Hilbert series is constant. This constant Hilbert series is the smallest possible for ideals in $I_{d_1, \ldots, d_r}$.

**Proof** The corresponding theorems in the singly graded case, [8] Theorem 1] , [4] Theorem p.120, and [6] Proposition 1 are easily adapted. We call points in this nonempty Zariski open set generic.

We define $(d, e) \leq (f, g)$ if $d \leq f$ and $e \leq g$, and $(d, e) \geq (f, g)$ if $d \geq f$ and $e \geq g$. Furthermore $|\sum a_{i,j} u^i v^j|_+ = \sum b_{i,j} u^i v^j$, where $b_{i,j} = a_{i,j}$ if $a_{k,l} > 0$ for all $(k, l) \leq (i, j)$ and $b_{i,j} = 0$ otherwise.

**Lemma 3** Let $R = S/I$ be bigraded and $f \in R_{i,j}$. Then $(R/f)(u, v) \geq [(1-u^j v^j)(R(u, v))]_+$.

**Proof** Consider the map $f: R_{d-i,e-j} \rightarrow R_{d,e}$. The image is largest if the map is of maximal rank, i.e., either injective or surjective, so $\dim_k (R/f)_{d,e} \geq \max\{0, \dim R_{d,e} - \dim R_{i,j}\}$. If $\dim(R/f)_{d,e} = 0$, then $\dim(R/f)_{d+f,e+g} = 0$ for all $(f, g) \geq 0$.

**Lemma 4** $|(1-u^j v^j)(\sum a_{i,j} u^i v^j) |_+ = [(1-u^j v^j) \sum a_{i,j} u^i v^j]_+$.

**Proof** Easy calculation.

These two lemmas give the following.

**Theorem 5** Let $I = (f_1, \ldots, f_r) \subset k[x_0, x_1, y_0, y_1] = S$, $\deg f_i = (d_i, e_i)$. Then $S/I(u, v) \geq \prod_{i=1}^r (1-u^{d_i} v^{e_i})/(1-u)^2 (1-v)^2$.
We now give a conjecture in the case when the $f_i$’s are generic, c.f. [9]. To prove the conjecture for some fixed $(d_1, \ldots, d_r)$ it suffices to give one example with the conjectured series. If the conjecture is true for these parameters, then almost all ideals have the conjectured series, so we must be very unlucky if we miss the series with a random choice of coefficients.

**Conjecture 6** Let $I = (f_1, \ldots, f_r) \subset k[x_0, x_1, y_0, y_1] = S$, deg $f_i = (d_i, e_i)$ generic. Then $(S/I)(u, v) = \left[ \prod_{i=1}^r (1 - u^{d_i}v^{e_i}) / ((1 - u)^2(1 - v)^2) \right]_+.$

We have checked that the conjecture is true in the following cases. Some of these were checked by Alessandro Oneto. Except for the first class, we have used computer calculations.

1. For small $r$ the concepts of ideal generated by generic forms and complete intersection agrees. It is well known that the conjecture is true for complete intersections.
2. deg $f_i = (1, 1)$ for all $i$, any $r$.
3. Some $f_i$ of degree (1,1), some of degree (1,2), any $r$.
4. deg $f_i = (1, 2)$ for all $i$, any $r$.
5. deg $f_i = (2, 2)$ for all $i$, any $r$.

We also checked that the corresponding conjecture is true for deg $f_i = (1,1,1)$ for all $i$, any $r$, in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

On the other hand, the corresponding conjecture in $\mathbb{P}^2 \times \mathbb{P}^2$ cannot be true. For four generic forms of degree (1,1) the conjecture would give that $R_{d,d} = 0$ if $d >> 0$. The correct statement is that $\dim_k R_{d,d} = 6$ if $d >> 0$.

We think that the conjecture is challenging enough, but we also give some questions.

**Question** What is the Hilbert series for generic ideals in $(\mathbb{P}^1)^k = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, $k > 2$ times?

**Question** What is the Hilbert series for generic ideals in $\mathbb{P}^m \times \mathbb{P}^n$?

**Question** Let $f_i$ be generic linear forms in $k[x_1, x_2]$ and $g_i$ generic linear forms in $k[y_1, y_2]$, and let $I = (f_1^{e_1}g_1^{e_1}, \ldots, f_r^{e_r}g_r^{e_r})$. What is the Hilbert series of $k[x_1, x_2, y_1, y_2]/I$?

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