Semichiral Sigma Models with 4D Hyperkähler Geometry

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Abstract

Semichiral sigma models with a four-dimensional target space do not support extended \( N = (4, 4) \) supersymmetries off-shell \cite{1}, \cite{2}. We contribute towards the understanding of the non-manifest on-shell transformations in (2,2) superspace by analyzing the extended on-shell supersymmetry of such models and find that a rather general ansatz for the additional supersymmetry (not involving central charge transformations) leads to hyperkähler geometry. We give non-trivial examples of these models.
1 Introduction

In a previous paper [2], we presented the general structure of semichiral sigma models with (4, 4) supersymmetry. We found conditions for invariance of the action and interesting geometric structures related to simultaneous integrability (Magri-Morosi concomitants) and a weaker conditions than (almost) complex structures for the transformation matrices (Yano f-structures). This rich mathematical context prompt us to take a closer look at specific models.

In [2] we treated both off-shell and on-shell supersymmetric (4, 4) models with manifest (2, 2) supersymmetry. One particular model where the non-manifest supersymmetry can only close on-shell, is the case of one left and one right semichiral field corresponding to a four-dimensional target space. These models are simple enough that a lot of the calculations can be carried out explicitly. They also enjoy a number of special properties

\[ \text{[1]} \] The off-shell \( N = (4, 4) \) pseudo-supersymmetry for a semichiral sigma model with four-dimensional target space was discussed in detail in [1].
such as carrying an almost (pseudo-) hyperkähler structure and having a $B$-field that is
governed by a single function.

In the present paper we start from the same general ansatz for the extra supersym-
metries as in [2], we then solve the conditions for invariance of the action and discover
that on-shell closure of the algebra follows from these, with one additional input from
the algebra. The solution leads to a geometry which is necessarily hyperkähler. Note
that we are not proving that $(4, 4)$ supersymmetry in four dimensions restricts the target
space geometry to be hyperkähler, since there may be more general ansätze combining
supersymmetry with central charge transformations. We briefly discuss this option in our
conclusions.

We relate our solution to geometric conditions from [2] and illustrate our findings in
an explicit (non-trivial) example.

The paper is organized as follows. Section 2 contains background material, definitions
and sets the notation for the paper. In section 3 we give the derivation of our conditions
for invariance of the action and on-shell closure while section 4 examplifies them. Section
5 contains our conclusions and in the appendix we have collected the special case of linear
transformations, the explicit form of the complex structures and the metric for the example
in section 4.

\section{Semichiral sigma models}

Consider a generalized Kähler potential with one left- and one right semichiral field and
their complex conjugates, $K(X^L, X^R)$, where $L = (\ell, \bar{\ell})$ and $R = (r, \bar{r})$. The action,
\begin{equation}
S = \int d^2x d^2\theta d^2\bar{\theta} K(X^L, X^R) \tag{2.1}
\end{equation}
has manifest $N = (2, 2)$ supersymmetry. The supersymmetry algebra is defined in terms
of the anti-commutator of the covariant supersymmetry derivatives as
\begin{equation}
\{ \mathcal{D}_\pm, \mathcal{D}_\pm \} = i \partial_\pm \tag{2.2}
\end{equation}
and the semichiral fields are defined by their chirality constraints as
\begin{equation}
\mathcal{D}_± X^\ell = 0, \quad \mathcal{D}_± X^r = 0. \tag{2.3}
\end{equation}

The geometry of the model is governed by two complex structures $J^{(+)}$ and $J^{(-)}$ that
both preserve the metric $g$
\begin{equation}
J^{(±) t} g J^{(±)} = g \tag{2.4}
\end{equation}
as well as by an anti-symmetric $B$-field whose field strength $H$ enters in the form of torsion in the integrability conditions

$$0 = \nabla^{(\pm)} J^{(\pm)} = \left(\partial + \Gamma^{(0)} \pm \frac{1}{2} H g^{-1}\right) J^{(\pm)},$$

(2.5)

where $\Gamma^{(0)}$ is the Levi-Civita connection. These conditions identify the geometry as bi-hermitean [3], or generalized Kähler geometry (GKG) [4].

The fact that our superfields are semichiral specifies the GKG as being of symplectic type where the metric $g$ and the $B$-field take the form

$$g = \Omega[J^{(+)} , J^{(-)}],$$
$$B = \Omega\{J^{(+)} , J^{(-)}\} .$$

(2.6)

The matrix $\Omega$ is defined as

$$\Omega = \frac{1}{2} \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}$$

(2.7)

and the submatrix $K_{LR}$ is the Hessian

$$K_{LR} = \begin{pmatrix} K_{\ell r} & K_{\ell \bar{r}} \\ \bar{K}_{\ell r} & K_{\bar{\ell} \bar{r}} \end{pmatrix} .$$

(2.8)

An additional condition results from the target space being four-dimensional and reads [5]

$$\{J^{(+)} , J^{(-)}\} = 2c \mathbb{I}, \quad \Rightarrow B = 2c \Omega ,$$

(2.9)

where $c = c(X^L, X^R)$. For reference, we rewrite this relation as

$$(1 - c)|K_{\ell \bar{r}}|^2 + (1 + c)|K_{\ell r}|^2 = 2K_{\ell \bar{r}}K_{\bar{\ell} \bar{r}} .$$

(2.10)

The condition (2.9) allows us to construct an $SU(2)$ worth of almost (pseudo-) complex structures [1], $J^{(1)} , J^{(2)} , J^{(3)} , [1], [3], [6],$

$$J^{(1)} := \frac{1}{\sqrt{1 - c^2}} [J^{(-)} + cJ^{(+)}] ,$$
$$J^{(2)} := \frac{1}{2\sqrt{1 - c^2}} [J^{(+)} , J^{(-)}] ,$$
$$J^{(3)} := J^{(+)} .$$

(2.11)

For $|c| < 1$ the geometry is almost hyperkähler, while for $|c| > 1$ the geometry is almost pseudo-hyperkähler [1].

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2This gives the $B$ field in a particular global gauge as $B = B^{(2,0)} + B^{(0,2)}$ with respect to both complex structures.

3In higher dimensions than four, almost hyperkähler implies hyperkähler [7].
3 On-shell \( N = (4, 4) \) supersymmetry

3.1 The ansatz

In this first subsection we recapitulate some definitions from \[2\]. Additional supersymmetry transformations on the semichiral field \( s \) must preserve the chirality constraints (2.3). We make the following ansatz for the additional supersymmetry,

\[
\begin{align*}
\delta X^\ell &= \bar{\epsilon}^\ell + \bar{D}^\ell + f(X^L, X^R) + g(X^r)\epsilon^\ell - \bar{D}\epsilon^\ell, \\
\delta X^\bar{\ell} &= \epsilon^\bar{\ell} + D^\bar{\ell} + \bar{f}(X^L, X^R) + \bar{g}(X^\ell)\epsilon^\bar{\ell} - D\epsilon^\bar{\ell}, \\
\delta X^r &= \epsilon^r - \bar{D} + \tilde{f}(X^L, X^R) + \tilde{g}(X^r)\epsilon^r + \bar{D}\epsilon^r, \\
\delta X^\bar{r} &= \bar{\epsilon}^\bar{r} - D + \tilde{f}(X^L, X^R) + \tilde{g}(X^\bar{r})\epsilon^r + \bar{D}\epsilon^r.
\end{align*}
\] (3.1)

For later convenience, a compact form of these transformations will be useful. We thus introduce transformation matrices

\[
U^{(+)} = \begin{pmatrix}
* & f_\ell & f_r & f_r \\
0 & 0 & 0 & 0 \\
0 & \bar{g} & 0 & 0 \\
0 & 0 & \bar{h} & 0
\end{pmatrix}, \quad V^{(+)} = \begin{pmatrix}
0 & * & 0 & 0 \\
\bar{f}_\ell & \bar{f}_r & \bar{f}_r & 0 \\
0 & \bar{h} & 0 & 0 \\
0 & 0 & \bar{g} & 0
\end{pmatrix}, \quad (3.2)
\]

\[
U^{(-)} = \begin{pmatrix}
g & 0 & * & 0 \\
0 & \bar{h} & * & 0 \\
\bar{f}_\ell & \bar{f}_r & \bar{f}_r & 0 \\
0 & 0 & * & 0
\end{pmatrix}, \quad V^{(-)} = \begin{pmatrix}
h & 0 & 0 & * \\
0 & \bar{g} & 0 & 0 \\
0 & 0 & 0 & * \\
\bar{f}_\ell & \bar{f}_r & \bar{f}_r & 0
\end{pmatrix}. \quad (3.2)
\]

One column in each of the matrices is arbitrary. Further writing the semichiral fields as a vector \( X^i \) where \( i = (\ell, \bar{\ell}, r, \bar{r}) \), the transformations read

\[
\delta X^i = \epsilon^a U^{(a)i}_j \bar{\partial}_a X^j + \epsilon^a V^{(a)i}_j \partial_a X^j, \quad (3.3)
\]

where the spinor index \( \alpha \) takes the values + and −.

3.2 No off-shell supersymmetry

Consider supersymmetry closure of the transformations in (3.1). The supersymmetry algebra defined in (2.2) requires that two subsequent transformations commute to a translation, e.g., that \([\delta_1^{(+)}, \delta_2^{(+)}]X^\ell = i\epsilon_2^{(+)} \partial X^\ell \). But the transformations in (3.1) commute
to
\[
[\delta_1, \delta_2]X^\ell = -\epsilon^+_2 \epsilon^-_1 \left( |f_\ell|^2 i \partial_+ X^\ell + (f_\ell \tilde{f}_{r\ell} + f_{r\ell} \tilde{h}) \bar{D}_+ D_+ X^\ell + (f_\ell \tilde{f}_{r\ell} + f_{r\ell} \tilde{g}) \bar{D}_+ D_+ X^\ell \right) \\
+ \epsilon^-_2 \epsilon^-_1 (-gh) i \partial_- X^\ell + \ldots ,
\]
where the dots represent the mixed $\epsilon^+ \epsilon^-$-terms in the algebra. Since $|f_\ell|^2 \neq -1$, the $\epsilon^+ \bar{\epsilon}^+$-part of the algebra (3.1) can never close off-shell\(^4\). In section 3.5, we will see that the algebra closes on-shell.

We do not get a contradiction from the $\epsilon^- \bar{\epsilon}^-$-part of the algebra, however; it closes if and only if
\[
gh = -1 .
\]

### 3.3 Invariance of action

The action in (2.1) is invariant under the $\bar{\delta}^{(+)}$-transformations if and only if the Lagrangian satisfies the following partial differential equations [2]
\[
(K_i U^{(+)}_{ij}) \bar{D}_+ X^j \bar{D}_+ X^k = 0 ,
\]
(3.6)
together with the corresponding equations for the other transformation matrices. Explicitly
\[
f_r K_{\ell \ell} - f_\ell K_{r \ell} + \tilde{g} K_{r \ell} = 0 ,
\]
\[
f_r K_{\ell \ell} - f_\ell K_{r \ell} + \tilde{h} K_{r \ell} = 0 ,
\]
\[
(\tilde{g} - \tilde{h}) K_{r \ell} - f_{r \ell} K_{r \ell} + f_\ell K_{r \ell} = 0 ,
\]
(3.7)
and
\[
\tilde{f}_\ell K_{r \bar{r}} - \tilde{f}_{r \ell} K_{r \ell} + g K_{r \ell} = 0 ,
\]
\[
\tilde{f}_\ell K_{r \bar{r}} + \tilde{h} K_{r \ell} - \tilde{f}_{r \ell} K_{r \ell} = 0 ,
\]
\[
(g - \tilde{h}) K_{\ell \ell} - \tilde{f}_\ell K_{r \ell} + \tilde{f}_{r \ell} K_{r \ell} = 0 .
\]
(3.8)

In addition we have the relations complex conjugate to those in (3.7) and (3.8) and a derived useful relation between $\tilde{h}$ and $\tilde{g}$:
\[
\frac{\tilde{h}}{g} = \frac{\tilde{g}}{\bar{g}} = \frac{K_{\ell \ell} K_{r \bar{r}} - |K_{r \ell}|^2}{K_{\ell \ell} K_{r \bar{r}} - |K_{r \ell}|^2} .
\]
(3.9)

\(^4\)From the derivation we see that the full statement is that we cannot have a left (or right) supersymmetry off-shell.
3.4 Integrability

In [2] we discuss on-shell closure of the $(4,4)$ algebra in terms of the $SU(2)$ set of complex structures $J_{(\pm)}^{(A)}$ known to exist from the $(1,1)$ reduction. We show that all the $(2,2)$ closure conditions are satisfied on-shell by relating them to expressions involving the $J_{(\pm)}^{(A)}$s. In other words, we prove that the $(2,2)$ conditions needed for additional supersymmetry of the $(2,2)$ model are equivalent to the $(1,1)$ conditions needed for extra (on-shell) supersymmetry of the corresponding $(1,1)$ model.

Here we follow a different route. We assume that the systems of linear partial differential equations (3.7) and (3.8) are integrable and show that this, together with $g, \tilde{g}, h$ and $\tilde{h}$ all being constant, is sufficient to prove that all the $(2,2)$ closure conditions are satisfied on-shell.

Integrability of (3.7) and (3.8) in turn, may be discussed in terms of the usual machinery for analyzing systems of linear partial differential equations. We do not include such an analysis, but solve the equations in examples below.

A final comment on the relation to the analysis in [2] is relegated to Appendix B. There we show the $(2,2)$ relation

$$\frac{1}{2} \alpha \left( J_{(\pm)}^{(1)} - i J_{(\pm)}^{(2)} \right) = U^{(\pm)} \pi^{(\pm)}, \quad (3.10)$$

found in [2] and relating the abstract $J^{(A)}$s to the transformation matrices (3.2), indeed holds for the two additional complex structures $J_{(+)}^{(1)}$ and $J_{(+)}^{(2)}$ explicitly constructed in (2.11).

3.5 On-shell algebra closure

Before discussing how to close the algebra on-shell, we point to some consequences of imposing the full algebra.

In (3.5), we have seen that the $(−)$-part of the algebra for $\mathbb{X}^\ell$ closes if and only if $gh = -1$. The same is true for the $(+)$-part of the algebra for $\mathbb{X}^r$; it closes if and only if $\tilde{g} \tilde{h} = -1$. From (3.9) we then deduce that

$$\det(K_{LR}) \neq 0. \quad (3.11)$$

This is a familiar condition which, amongst other things, ensures that a non-degenerate geometry can be extracted from the action [6]. A further consequence of $\tilde{g} \tilde{h} = -1 = gh$ in conjunction with (3.7) and (3.8) is that (2.10) is satisfied with

$$c = \frac{1 - |g|^2}{1 + |g|^2} = \frac{1 - |\tilde{g}|^2}{1 + |\tilde{g}|^2}. \quad (3.12)$$
Since we have assumed that $g = g(X^\ell)$ and $\bar{g} = \bar{g}(\bar{X}^r)$, the relations (3.12) tell us that $g, \bar{g}, h$ and $\bar{h}$ are all constant.

We now turn to the on-shell closure. Two subsequent transformations defined in (3.1) acting on a left-semichiral field commute to

$$[\delta_1, \delta_2]X^\ell = \bar{c}^{+\epsilon^+}_{[2 \epsilon^+]_1} [\mathcal{M}(U^{(\pm)}, \bar{V}^{(\pm)})_{jk} \bar{D}_+ X^j \bar{D}_+ X^k - (U^{(\pm)} \bar{V}^{(\pm)})_j^{\ell} \bar{D}_+ \bar{D}_+ X^j - (V^{(\pm)} U^{(\pm)})_j^{\ell} \bar{D}_+ \bar{D}_+ X^j]$$

$$+ \bar{c}^{-\epsilon^-}_{[2 \epsilon^-]_1} [\mathcal{M}(V^{(\pm)}, \bar{U}^{(\pm)})_{jk} \bar{D}_+ X^j \bar{D}_+ X^k - (U^{(\pm)} \bar{V}^{(\pm)})_j^{\ell} \bar{D}_+ \bar{D}_+ X^j - (V^{(\pm)} U^{(\pm)})_j^{\ell} \bar{D}_+ \bar{D}_+ X^j]$$

where the transformation matrices $U^{(\pm)}$ and $V^{(\pm)}$ are defined in (3.2). The Nijenhuis tensor $\mathcal{N}$ and the Magri-Morosi concomitant $\mathcal{M}$ are defined as

$$\mathcal{N}(I)^{i}_{jk} = I^{I}_{[j}^i [k], I^{I}_{j} [k], I^{I}_{[j}^i [k]},$$

$$\mathcal{M}(I, J)^{i}_{jk} = I^{I}_{[j} J^{J}_{k}] - J^{J}_{[j} I^{I}_{k}] + J^{J}_{i} J^{I}_{j} - I^{I}_{i} J^{J}_{j},$$

(3.14)

We will now show that each of the terms in (3.13) close to a supersymmetry algebra on-shell and discuss the geometric interpretation. In section 3.2 we have seen that the transformations in (3.1) cannot close to a supersymmetry off-shell. Hence we have to go on-shell. The field equations that follow from the action in (2.1) are

$$\bar{D}_+ K_\ell = 0,$$

$$\bar{D}_- K_\ell = 0,$$

(3.15)

and their complex conjugates. To investigate on-shell closure, we use the the first equation to solve for, e.g., $\bar{D}_+ X^\ell$:

$$\bar{D}_+ X^\ell = -\frac{1}{K_{\ell \ell}} (K_\ell X^\ell + K_{\ell r} \bar{D}_+ X^r).$$

(3.16)

Using the expressions for the transformation functions in (3.14)-(3.9) and the on-shell relation (3.10), the last term in the algebra for $X^\ell$, (3.13), becomes

$$[\delta_1^{(+)}, \delta_2^{(+)}]X^\ell = \bar{c}^{+\epsilon^+}_{[2 \epsilon^+]_1} [\mathcal{N}(U^{(\pm)})_{jk} \bar{D}_+ X^j \bar{D}_+ X^k]$$

$$\bar{c}^{-\epsilon^-}_{[2 \epsilon^-]_1} [(f_{r,\ell} f_{r,\ell} - f_{r,\ell} f_{r,\ell} + f_{r,\ell} \bar{g}) \bar{D}_+ X^r \bar{D}_+ X^r]$$

$$+(f_{r,\ell} f_{r,\ell} - f_{r,\ell} f_{r,\ell} + f_{r,\ell} \bar{g}) \bar{D}_+ X^r \bar{D}_+ X^r + (f_{r,\ell} f_{r,\ell} - f_{r,\ell} f_{r,\ell} + f_{r,\ell} \bar{g}) \bar{D}_+ X^r \bar{D}_+ X^r]$$

$$= \bar{c}^{+\epsilon^+}_{[2 \epsilon^+]_1} \frac{K_{\ell r}}{K_{\ell \ell}} \left( \frac{2 \bar{g}}{c-1} \right) \bar{D}_+ X^r \bar{D}_+ X^r$$

$$= 0,$$

(3.17)
The vanishing of the Nijenhuis tensor for an almost complex structure means that the structure is integrable, hence a complex structure. Here we see that the relevant parts of the Nijenhuis tensor for the transformation matrix $U^{(+)}$ vanish. The same is true for the relevant parts of the Nijenhuis tensor for $U^{(-)}$ and $V^{(±)}$.

We now move over to the terms in the algebra (3.13) involving the Magri-Morosi concomitant. For clarity, we define the following combinations of the parameter functions,

$$\mu = f_\ell f_r + f_r h, \quad \nu = f_\ell f_r + f_r \tilde{g}, \quad \tau = f_\ell (g - \tilde{h}) - f_r \tilde{f}_r, \quad \omega = f_r g - \tilde{f}_r f_r. \quad (3.18)$$

To investigate on-shell closure of the $^{(+)}$-supersymmetry for $X^\ell$, we use the conjugate version of the field equation (3.15) to solve for $D^+ X^r$. The algebra becomes

$$[\delta_1^{(+)} + \delta_2^{(+)}, X^\ell] = \bar{\epsilon}^+ \epsilon_1^+ \left[ \mathcal{M}(U^{(+)}, V^{(+)}) f_{\ell k} \bar{D}^+_+ X^j \bar{D}^+_+ X^k + (U^{(+)} V^{(+)}) f_{\ell r} \bar{D}^+_+ X^j \right]$$

$$= \bar{\epsilon}^+ \epsilon_1^+ \left[ -|f_\ell|^2 \bar{D}^+_+ X^\ell - \mu \bar{D}^+_+ X^r - \nu \bar{D}^+_+ X^r \right]$$

$$= \bar{\epsilon}^+ \epsilon_1^+ \left[ \left( \mu \frac{K_\ell f_\ell}{K_{\ell r}} - |f_\ell|^2 \right) \bar{D}^+_+ X^\ell + \left( \mu \frac{K_{\ell r}}{K_\ell f_\ell} - \nu \right) \bar{D}^+_+ X^r \right]$$

$$= \bar{\epsilon}^+ \epsilon_1^+ \left[ \left( |f_\ell|^2 - \tilde{g} h - |f_r|^2 \right) \bar{D}^+_+ X^\ell + \frac{K_{\ell r}}{K_\ell f_\ell} (\tilde{g} \tilde{h} - \tilde{g} h) \bar{D}^+_+ X^r \right]$$

$$= \bar{\epsilon}^+ \epsilon_1^+ \left( i \partial^+_+ X^\ell \right), \quad (3.19)$$

where in the last line we used that $\tilde{g} h = -1$. We already know that the $^{(-)}$-part of the algebra for $X^\ell$ closes to a supersymmetry if and only if $\tilde{g}$ and $\tilde{h}$ satisfies this constraint. The vanishing of the Magri-Morosi concomitant for two commuting complex structures is equivalent to the statement that the structures are simultaneous integrable [8]. Here we see that the relevant parts of the Magri-Morosi concomitant of $U^{(+)}$, $V^{(+)}$ combines with products of the transformation matrices to vanish on-shell, such that the algebra closes to a supersymmetry.

Now we focus on the mixed $\bar{\epsilon}^+ \epsilon^-$-terms in the algebra. Again using the field equations (3.15) to write $\bar{D}_- X^r$ in terms of $\bar{D}_- X^\ell$ and $\bar{D}_- X^\ell$, together with the expressions for the
transformation functions in (3.7)-(3.8), the algebra closes to
\[ [\bar{\delta}_1^{(+)}, \bar{\delta}_2^{(-)}] X^\ell = \bar{\epsilon}_1^{+} \epsilon_1^{+} [M(U^{(+)}, U^{(-)})^j_k \bar{D}_+ X^j \bar{D}_- X^k - [U^{(+)}, U^{(-)}]^j_k \bar{D}_+ \bar{D}_- X^j] \]
\[ = \bar{\epsilon}_1^{+} \epsilon_1^{+} [\tau \bar{D}_- X^\ell + \omega \bar{D}_- X^\ell + (-f_r \bar{f}_r) \bar{D}_- X^\ell] \]
\[ = \bar{\epsilon}_1^{+} \epsilon_1^{+} [\left( \tau - \frac{K_{lr}}{K_{rr}} \omega \right) \bar{D}_- X^\ell + \left( -f_r \bar{f}_r - \frac{K_{lr}}{K_{rr}} \omega \right) \bar{D}_- X^\ell] \]
\[ = \bar{\epsilon}_1^{+} \epsilon_1^{+} (g \bar{g} - \bar{g} \bar{g}) \bar{D}_- X^\ell \]
\[ = 0 , \quad (3.20) \]
where in the last line we use that \( g \) and \( \bar{g} \) are constants. A similar derivation can be done for the \( \bar{\epsilon}^+ \epsilon^- \)-term in the algebra, which also vanishes when using the field equations. The closure of the algebra on the right semichiral field follows in exactly the same way.

As a summary, we see that the transformations defined in (3.1) close to a supersymmetry algebra on the semichiral fields on-shell,
\[ [\delta_1, \delta_2] X^i = \bar{\epsilon}_1^{+} \epsilon_1^{+} i \partial_+ X^i + \bar{\epsilon}_1^{-} \epsilon_1^{-} i \partial_- X^i , \quad (3.21) \]
and that the action is invariant under the same transformations, if and only if the transformation functions take the expressions in (3.7)-(3.9) and
\[ |g|^2 = |\bar{g}|^2 = -\frac{K_{lr} K_{rr} - |K_{tr}|^2}{K_{lr} K_{rr} - |K_{tr}|^2} \quad (3.22) \]
is a constant.

4 Hyperkähler solutions

The four-dimensional target space geometry is hyperkähler if \( c \) in (2.10) is a constant with absolute value less than one. We see from (3.12) that this is the case at hand. In this section we explore some additional properties of this hyperkähler geometry. We notice that the structures in (2.11) are now integrable and give us an \( SU(2) \) of complex structures,
\[ [J^{(A)}, J^{(B)}] = \delta^{AB} + \epsilon^{ABC} J^{(C)} . \quad (4.1) \]

To describe the hyperkähler (HK) geometry, a generalized potential \( K \) must satisfy (2.10) with constant \( |c| < 1 \). An additional requirement is that the determinant of the matrix \( K_{LR} \) is nonvanishing (3.11). The transformations functions are then found from (derivatives of) this \( K \) as solutions of (3.7)-(3.9).
As discussed in the appendix, there are many quadratic actions which satisfy (2.10) and (3.11), e.g.,

\[ K = (X^\ell - X^\bar{\ell})(X^r - X^\bar{r}) + (X^r + X^\ell + X^\bar{r} + X^\bar{\ell})(X^r + X^\ell + X^\bar{r} + X^\bar{\ell}) \].

Solutions to (3.7)-(3.9) are easily found for this \( K \) since all the coefficients are constants. The supersymmetry transformations are linear. More on this in the appendix.

There are a number of nontrivial examples of HK geometries written in semichiral coordinates. In particular in \[9\], the relation between the Kähler potential and the generalized Kähler potential is discussed, and HK geometries in semichiral coordinates are generated from Kähler potentials with certain isometries amongst their chiral and twisted chiral coordinates (see also \[7\] and \[10\] for further discussions of semichiral formulations of hyperkahler geometries). The method is an adaption of the Legendre transform construction of \[11\], \[12\]. In brief, the construction yields a semichiral description of a four-dimensional HK manifold from any function \( F(x, v, \bar{v}) \) which satisfies Laplace’s equation

\[ F_{xx} + F_{v\bar{v}} = 0 \].

Here \( x \) and \( v \) correspond to certain expressions in the chiral and twisted coordinates that need not concern us here. The T-duality between a chiral and twisted chiral model with \( N = (4, 4) \) supersymmetry and its semichiral dual counterpart is investigated in detail in \[13\]. The generalized Kähler potential is obtained via the Legendre transform

\[ K(X^\ell - X^\bar{\ell}, X^\ell + X^\bar{\ell} + 2X^r, X^\ell + X^\bar{\ell} + 2X^\bar{r}) = F(x, v, \bar{v}) - \frac{1}{2} v(X^\ell + X^\bar{\ell} + 2X^r) - \frac{1}{2} \bar{v}(X^\ell + X^\bar{\ell} + 2X^\bar{r}) - \frac{i}{2} x(X^\ell - X^\bar{\ell}) \].

with

\[
\begin{align*}
F_x & = i \frac{1}{2} (X^\ell - X^\bar{\ell}) =: i \frac{1}{2} z , \\
F_v & = \frac{1}{2} (X^\ell + X^\bar{\ell} + 2X^r) =: \frac{1}{2} y , \\
F_{\bar{v}} & = \frac{1}{2} (X^\ell + X^\bar{\ell} + 2X^\bar{r}) =: \frac{1}{2} \bar{y} .
\end{align*}
\]  

The resulting \( K \) satisfies (2.10) with \( c = 0 \) and has \( \text{det}(K_{LR}) \neq 0 \).

We now use this construction to generate a nontrivial example illustrating the discussion in the previous sections. Since we do not need the connection to a Kähler potential, we can start from any \( F \) satisfying (4.3). A convenient example is

\[ F(x, v, \bar{v}) = r - x \ln(x + r) , \quad r^2 := x^2 + 4v\bar{v} . \]  

\[ \text{The restriction to } c = 0 \text{ is not necessary for this kind of construction--see } \[7\]. \]

\[ \text{When } v \text{ and } x \text{ are chiral and real linear superfields, respectively, } F \text{ is the superspace Lagrangian of the improved tensor multiplet } \[13\]. \]
Solving the relations (4.5) and plugging into (4.4) results in the generalized Kähler potential

\[ K = \frac{1}{2} e^{-\frac{1}{2}iz} \left( 1 - \frac{1}{4}y\bar{y} \right), \]  

(4.7)

which indeed satisfies all the relevant requirements. Note that it is written in coordinates invariant under the Abelian symmetry

\[ \delta X^\ell = \varepsilon, \quad \delta X^r = -\varepsilon, \quad \varepsilon \in \mathbb{R}. \]  

(4.8)

To find the transformation functions, we calculate the various second derivatives of \( K \) and insert into (3.7)-(3.9). The resulting partial differential equations may then be solved to yield

\[ f = 2i(\lambda - \bar{g})\ln(2 - iy) + 2i(\lambda + \bar{g})\ln(2 - i\bar{y}) + \lambda z, \]

\[ \tilde{f} = -i\kappa \ln(\bar{y}) + ig\frac{1}{8}y^2 - \frac{1}{2}(\kappa + g)z, \]  

(4.9)

where \( \lambda \) and \( \kappa \) are integration constants. The appearance of these constants may seem surprising, since we expect the transformations to be unique. Below we shall see how they are determined.

As in the derivation of the supersymmetry transformations in [15], we identify part of (4.9) as field equation symmetries, that is symmetries of a Lagrangian \( L(\varphi) \) of the form

\[ \delta \varphi^i = A^{ij} \frac{\partial L}{\partial \varphi^j}, \]  

(4.10)

with \( A^{ij} \) anti-symmetric (or symmetric for spinorial indices). These transformations will leave the action invariant and vanish on-shell. When inserted into (3.1) the \( \lambda \) part of \( f \) gives an expression that vanishes due to the \( X^\ell \) field equation,

\[ (2 - iy)(2 - i\bar{y})\bar{D}_+ z + 2(2 - iy)\bar{D}_+ y + 2(2 - i\bar{y})\bar{D}_+ \bar{y} = 0, \]  

(4.11)

and the \( \kappa \) part of \( \tilde{f} \) gives an expression that vanishes due to the \( X^r \) field equation,

\[ \bar{y}\bar{D}_- z + 2i\bar{D}_- \bar{y} = 0. \]  

(4.12)

This means that the transformations (4.9) reduce to

\[ f = 2i\bar{g}(\ln(2 - iy) + \ln(2 - i\bar{y})), \]

\[ \tilde{f} = g(i\frac{1}{8}y^2 - \frac{1}{2}z). \]  

(4.13)
According to (3.22), \( \tilde{g} \) and \( g \) are phases when \( c = 0 \), and from (3.1) we see that they may be absorbed by an R-transformation of the charges \( Q_\pm \) for the extra supersymmetries:

\[
\begin{align*}
\tilde{g} &= e^{i\varphi}, & g &= e^{i\psi}, \\
\bar{\epsilon}^+ e^{i\varphi} &\rightarrow \bar{\epsilon}^+, & \bar{\epsilon}^- e^{i\psi} &\rightarrow \bar{\epsilon}^-.
\end{align*}
\] (4.14)

The final form of the functions \( f \) and \( \tilde{f} \) thus becomes

\[
\begin{align*}
f &= 2i \ln \left( \frac{2 - iy}{2 - i\bar{y}} \right), \\
\tilde{f} &= i\frac{1}{2} \left( i\bar{z} + \frac{1}{2}y^2 \right).
\end{align*}
\] (4.15)

Inserting this in (3.1) yields the transformations

\[
\begin{align*}
\delta X^\ell &= -\frac{\bar{\epsilon}^+}{(2 - iy)(2 - i\bar{y})} 2\left( i(y - \bar{y})\bar{\mathcal{D}}_+ X^\ell + 2(2 - iy)\bar{\mathcal{D}}_+ X^r - 2(2 - iy)\bar{\mathcal{D}}_+ X^r \right) \\
&\quad + \bar{\epsilon}^- \bar{\mathcal{D}}_- X^\ell - \epsilon^- \mathcal{D}_- X^\ell, \\
\delta X^r &= -\frac{\epsilon^-}{4} \left( (2 - iy)\bar{\mathcal{D}}_- X^\ell - (2 + iy)\bar{\mathcal{D}}_- X^\ell \right) + \bar{\epsilon}^+ \bar{\mathcal{D}}_+ X^r - \epsilon^+ \mathcal{D}_+ X^r,
\end{align*}
\] (4.16)

and their complex conjugates.

5 Summary and conclusions

We have found that our ansatz (3.1) for additional supersymmetries of a semichiral sigma model with four-dimensional target space corresponds to hyperkähler geometry on the target space. For this case we have provided the form of the transformation functions, related them to previous general discussions in the literature and described generalized Kähler potentials satisfying the invariance conditions.

The existence of four-dimensional examples of semichiral sigma models with non-trivial \( B \)-field [16] indicates that the ansatz (3.1) has to be modified for on-shell algebras. It is well known, e.g., from four-dimensional sigma models with chiral fields \((\phi, \bar{\phi})\) as coordinates, that extra supersymmetries may come together with central charge transformations [11, [17] in the form

\[
\delta \phi^i = \tilde{D}^2(\epsilon \Omega^i),
\] (5.1)

where the scalar transformation superfield \( \epsilon \) contains the supersymmetry at the \( \theta \) level. Central charge transformations vanish on-shell, but will have effect, e.g., on the conditions
that follow from invariance of the action. Such a generalization of the transformations (3.1) will presumably cover the \( dB \neq 0 \) case.

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A Linear transformations

In [1] we were able to extract interesting information from a semichiral sigma model with four-dimensional target space and additional non-manifest linear pseudo-supersymmetry. In this section we show that linear supersymmetry transformations are only possible for quadratic potentials (flat geometry) or when the metric is degenerate.

There is a very direct argument why linear transformations lead to quadratic actions. If the transformation on a field \( X \) is linear and the algebra closes on-shell, we have, schematically,

\[
\delta X = QX, \quad \Rightarrow Q^2 X = [\delta, \delta]X = \partial X + F
\]

where \( F \) is (derivatives of) a field equation. The latter then reads

\[
F = OX = (Q^2 - \partial)X = 0 , \quad (A.2)
\]

which means that the Lagrangian must be quadratic. In the present case, it still turns out to be instructive to explicitly consider the linear case.

The transformations are the same as in the general case (3.1), with the difference that the transformation functions are now all constant. Again, off-shell closure of the algebra cannot occur, since \( |f_\ell|^2 \neq -1 \). On-shell closure, however, can be obtained just as in the general case.

Invariance of the action (2.1) under the linear transformations implies the same system of partial differential equations (3.7)-(3.8), but with constant parameters. The equations imply that the Lagrangian \( K \) must satisfy

\[
K_{\ell r} = \frac{\bar{\nu}}{\mu} K_{\ell r} , \quad (A.3)
\]

among other relations. The parameters \( \mu \) and \( \nu \) are defined in (3.18) and are here constant. Taking derivative with respect to \( X^r \) on both sides implies that \( (|\mu|^2 - |\nu|^2)K_{\ell r \bar{r} \bar{\nu}} = 0 \). This
implies that either $|\mu|^2 = |\nu|^2$, leading to degenerate metric, as we have seen in (3.11), or that $K_{\ell\ell\bar{\ell}} = 0$. Similar results can be derived from the other PDEs and as a result we draw the conclusions that linear supersymmetry transformations imply either degenerate metric, or a generalized Kähler potential with vanishing third derivatives, i.e. a quadratic potential.

For a quadratic potential $K$, everything works as in the general case. From the equation (2.10) we again find that

$$c = \frac{1 - |g|^2}{1 + |g|^2} \iff |g|^2 = \frac{1 - c}{1 + c}. \quad (A.4)$$

This again implies that $|c| < 1$ and thus the geometry is necessarily hyperkähler.

As a final comment we note that there are a large set of non-trivial (non-quadratic) generalized potentials invariant under the linear transformations with $\det(K_{LR}) = 0$. As mentioned, this makes it impossible to extract a metric, so they do not correspond to sigma models. One may speculate that these have an application in models where the background is in some sense topological.

**B  Complex structures**

In [2] it is concluded on general grounds that the relation (3.10) holds on-shell

$$\frac{1}{2} \alpha \left( J_{(\pm)}^{(1)} - i J_{(\pm)}^{(2)} \right) = U^{(\pm)} \pi^{(\pm)}, \quad (B.1)$$

where

$$\pi^{(\pm)} := \frac{1}{2} (I + i J^{(\pm)}) \quad (B.2)$$

$J^{(A)}_{(\pm)}$ are the complex structures obeying an $SU(2)$ algebra with $J^{(3)}_{(\pm)} := J^{(\pm)}$ and $\alpha$ is a phase.\footnote{This phase is not included in [2], but represents an ambiguity in the choice of $\epsilon$ related to $R$-symmetry.} Here we verify this relation by explicitly constructing $J^{(1)}$ and $J^{(2)}$ as in (2.11).
From [6], $J^{(±)}$ takes the form

\[
J^{(±)} = \frac{i}{D} \begin{pmatrix}
D & 0 & 0 & 0 \\
0 & -D & 0 & 0 \\
2K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & 2K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & S & 2K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
-2K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -2K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -2K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -S \\
0 & 0 & D & 0 \\
0 & 0 & 0 & -D
\end{pmatrix},
\]

where we have defined the sum and difference

\[
S := |K_{\bar{\ell}\ell}|^2 + |K_{\bar{\ell}\bar{\ell}}|^2, \quad D := |K_{\bar{\ell}\ell}|^2 - |K_{\bar{\ell}\bar{\ell}}|^2.
\]

Inserting the expressions for $J^{(±)}$ in (2.11) yields

\[
J^{(1)} = \frac{2i}{D\sqrt{1-c^2}} \begin{pmatrix}
K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
-K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
cK_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & cK_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & \frac{1}{2}(D + cS) & cK_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
-cK_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -cK_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -cK_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -\frac{1}{2}(D + cS)
\end{pmatrix},
\]

and

\[
J^{(2)} = -\frac{2}{D\sqrt{1-c^2}} \begin{pmatrix}
K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
-K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} \\
-K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}} & -K_{\bar{\ell}\ell}K_{\bar{\ell}\bar{\ell}}
\end{pmatrix}.
\]

Finally, we have

\[
D\sqrt{1-c^2} U^{(+)\pi^{(+)}} =
\begin{pmatrix}
(f_{\bar{\ell}}K_{\bar{\ell}\ell} - f_rK_{\bar{\ell}\ell})K_{\bar{\ell}\ell} & f_{\bar{\ell}}D - (f_rK_{\bar{\ell}\ell} - f_{\bar{\ell}}K_{\bar{\ell}\ell})K_{\bar{\ell}\ell} & (f_{\bar{\ell}}K_{\bar{\ell}\ell} - f_rK_{\bar{\ell}\ell})K_{\bar{\ell}\ell} & (f_{\bar{\ell}}K_{\bar{\ell}\ell} - f_rK_{\bar{\ell}\ell})K_{\bar{\ell}\ell} \\
0 & 0 & 0 & 0 \\
-gK_{\bar{\ell}\ell}K_{\bar{\ell}\ell} & -gK_{\bar{\ell}\ell}K_{\bar{\ell}\ell} & -g|K_{\bar{\ell}\ell}|^2 & -gK_{\bar{\ell}\ell}K_{\bar{\ell}\ell} \\
\bar{\ell}K_{\bar{\ell}\ell}K_{\bar{\ell}\ell} & \bar{\ell}K_{\bar{\ell}\ell}K_{\bar{\ell}\ell} & \bar{\ell}K_{\bar{\ell}\ell}K_{\bar{\ell}\ell} & \bar{\ell}K_{\bar{\ell}\ell}K_{\bar{\ell}\ell}
\end{pmatrix}.
\]

Using these expressions for the HK models at hand, we verify explicitly that (3.10) is satisfied with $i\alpha = \bar{g} = -\bar{\ell}$ due to the invariance conditions (3.7)-(3.9).
C Example metric

The metric that follows from the potential (4.7) may be found from the formulae in [6]. It is

\[ g = \begin{pmatrix} g_{LL} & g_{LR} \\ g_{RL} & g_{RR} \end{pmatrix} = \frac{i e^{-i \frac{1}{2} z}}{2(y - \bar{y})} \begin{pmatrix} A & \mathcal{B} \\ \mathcal{B}^t & C \end{pmatrix} \]  

(C.1)

with

\[ A = \frac{1}{16} (4 + y\bar{y}) \begin{pmatrix} 4 - y\bar{y} - 2i(y + \bar{y}) & 4 + y\bar{y} \\ 4 + y\bar{y} & 4 - y\bar{y} + 2i(y + \bar{y}) \end{pmatrix} \]

\[ B = \frac{1}{4} \begin{pmatrix} (2 - i\bar{y})[4 + y\bar{y} - i(y + \bar{y})] & (2 - iy)[4 + y\bar{y} + i(y + \bar{y})] \\ (2 + i\bar{y})[4 + y\bar{y} - i(y + \bar{y})] & (2 + iy)[4 + y\bar{y} + i(y + \bar{y})] \end{pmatrix} \]

\[ C = \begin{pmatrix} 4 + \bar{y}^2 & 4 + y\bar{y} \\ 4 + y\bar{y} & 4 + y^2 \end{pmatrix} \]  

(C.2)

---

8In a semichiral model with four-dimensional target space, the (trivial) \(B\)-field, is proportional to \(c\) for a HK metric. Since \(c = 0\) in the example, it should vanish. The formulae in [6] confirm this.
References

[1] M. Göteman and U. Lindström, “Pseudo-hyperkähler Geometry and Generalized Kähler Geometry,” Lett. Math. Phys. 95, 211 (2011) [arXiv:0903.2376 [hep-th]].

[2] M. Göteman, U. Lindström, M. Roček and I. Ryb, “Sigma models with off-shell N=(4,4) supersymmetry and noncommuting complex structures,” JHEP 1009, 055 (2010) [arXiv:0912.4724 [hep-th]].

[3] S. J. Gates, Jr., C. M. Hull and M. Roček, “Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models,” Nucl. Phys. B 248, 157 (1984).

[4] M. Gualtieri, “Generalized complex geometry,” Oxford University DPhil thesis, [arXiv:math/0401221].

[5] T. Buscher, U. Lindström and M. Roček, “New supersymmetric sigma models with Wess Zumino terms,” Phys. Lett. B 202, 94 (1988).

[6] U. Lindström, M. Roček, R. von Unge and M. Zabzine, “Generalized Kähler manifolds and off-shell supersymmetry,” Commun. Math. Phys. 269, 833 (2007) [hep-th/0512164].

[7] P. M. Crichigno, “The Semi-Chiral Quotient, Hyperkahler Manifolds and T-Duality,” [arXiv:1112.1952 [hep-th]].

[8] P. S. Howe and G. Papadopoulos, “Further Remarks On The Geometry Of Two-dimensional Nonlinear Sigma Models,” Class. Quant. Grav. 5 (1988) 1647.

[9] J. Bogaerts, A. Sevrin, S. van der Loo and S. Van Gils, “Properties of semichiral superfields,” Nucl. Phys. B 562, 277 (1999) [hep-th/9905141].

[10] M. Dyckmanns, “A twistor sphere of generalized Kahler potentials on hyperkahler manifolds,” [arXiv:1111.3893 [hep-th]].

[11] U. Lindström, M. Roček, “Scalar Tensor Duality and N=1, N=2 Nonlinear Sigma Models,” Nucl. Phys. B 222, 285 (1983).

[12] N. J. Hitchin, A. Karlhede, U. Lindström, M. Roček, “Hyperkähler Metrics and Supersymmetry,” Commun. Math. Phys. 108, 535 (1987).

[13] M. Göteman, “N = (4, 4) supersymmetry and T-duality”, to appear soon.
[14] A. Karlhede, U. Lindström, M. Roček, “Selfinteracting Tensor Multiplets In N=2 Superspace,” Phys. Lett. B 147, 297 (1984).

[15] U. Lindström, “Generalized N = (2,2) supersymmetric nonlinear sigma models,” Phys. Lett. B 587, 216 (2004) [hep-th/0401100].

[16] U. Lindström, R. von Unge, M. Roček, I. Ryb and M. Zabzine, “T-duality for the S1 piece in the S3 × S1 model”. Work in preparation.

[17] C. M. Hull, A. Karlhede, U. Lindström and M. Roček, “Nonlinear sigma models and their gauging in and out of superspace,” Nucl. Phys. B 266, 1 (1986).