Uniqueness and hyperconic geometry of positioning with biased distance measurements

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Abstract
Positioning an object with biased distance measurements is exactly solvable if exact knowledge of the reference locations and noise-free range measurements are assumed. By examining the positioning algebra, this paper obtains explicit necessary and sufficient conditions for the positioning problem to have a unique or twin solutions. The intersection of negative hyperconic halves is shown to be an exact geometric interpretation of the positioning solutions. The placement of non-coplanar references largely ensures but does not guarantee unique positioning. Given a set of references, object region for non-unique positioning is identifiable by using the conditions derived in this work. The placement of five references to form an asymmetric hexahedron is postulated to be sufficient for unique positioning in a three-dimensional environment. Illustrative examples explain these findings.

Keywords Source localization · GPS solutions · Explicit solutions · Hyperconic nappes

Introduction
Locating energy-emitting/reflecting sources is an interesting and important problem found in many applications (Dogandzic et al. 2005). Theoretically locating a source with known receivers is equivalent to locating a receiver with known sources. The most important application of the latter is in the global positioning system (GPS), where the receiver location and clock bias are determined by processing data transmitted from satellites. There are mainly two types of methods for solving the positioning problem, one involves iteratively solving the linearized measurement equations as often adopted in applications (Tsui 2000; Doong 2009; Larsson and Danev 2010; Seco-Granados et al. 2012), or solving a nonlinear least-squares problem which may need partial iterations only (Stoica and Li 2006; Beck et al. 2008), and the other, as pursued in this study, finds closed-form solutions. Both iterative and non-iterative methods for range-based positioning are reviewed and compared by Yan et al. (2013).

The first closed-form solution by Bancroft (1985) is simple and captures the essence of 3D positioning with a biased time of arrivals (TOAs) as pseudorange measurements. In many investigations, biased TOAs are converted to unbiased time differences of arrivals (TDOAs). With different treatments, closed-form solutions have been derived using, for instance, the Lorentz inner product (Bancroft 1985; Chaffee and Abel 1994), spherical interpolation (Smith and Abel 1987; Gillette and Silverman 2008), hyperbolic intersection (Schau and Robinson 1987; Chan and Ho 1994; Leva 1996), the Gröbner basis method (Grafarend and Shan 2002; Awange and Grafarend 2002, 2003), and generalized Cramer identities (Caravantes et al. 2017) among others (Awange et al. 2018). Most of these closed-form solutions have been derived for nD positioning with n + 1 biased TOAs. Recently, a closed-form solution has been obtained for dual-system positioning, which is also an over-determined problem (Zhao et al. 2021).

The positioning geometry with unbiased TOAs and that with TDOAs are well understood, as the former corresponds to a spherical intersection (multilateration) (Fang 1986), and the latter hyperbolic intersection (Leva 1996). Differences in positioning geometries using, respectively, biased and unbiased TOAs were analyzed (Abel and Chaffee 1991; Chaffee and Abel 1994), see also a comprehensive study of local GPS geometry (Santerre et al. 2017;
Santerre and Geiger 2018), and even more in-depth exploration of 2D TDOA geometric maps (Compagnoni et al. 2014). Most studies correctly considered 3D positioning geometry hyperbolic due to use of TDOAs converted from biased TOAs, while the conic interpretation (Abel and Chaffee 1991; Chaffee and Abel 1994) was not really applicable after the conversion.

Algebra and geometry of positioning with biased TOAs have not been fully explored in the literature due to a lack of a complete set of explicit necessary and sufficient conditions for the uniqueness and exact geometric interpretations of positioning. For example, following an analogy of positioning with unbiased TOAs, it might lead to believe, as postulated in the case of 3D positioning by Abel and Chaffee (1991), that nD positioning is unique if the number of non-coplanar references is greater than n + 2. This hypothesis has not been confirmed or falsified since then, although some sufficient conditions were derived by Chaffee and Abel (1994) for unique positioning. An application of the implicit function theorem by Grafarend and Shan (2002) led to a local sufficient condition for unique positioning.

The current study examines the general case of positioning an nD object using n + 2 biased TOAs and derives explicit necessary and sufficient conditions for the positioning problem to have a single or twin solutions. Moreover, it shows that the intersection of negative hyperconic halves in an (n + 1)D space is an exact geometric interpretation of nD positioning with biased TOAs. Given a configuration of references, the conditions derived in this work can be used to identify object regions where positioning has single and twin solutions, respectively. Analysis of positioning algebra and geometry as well as simulation studies in this work lead to the postulation that asymmetric configurations of n + 2 non-coplanar references suffice for unique nD positioning for n > 1.

### Preliminaries

This section explains general notations and concepts of hyperplanes and hypercones.

### General notations

Real numbers are considered throughout. Vectors and matrices are customarily boldfaced capital and low case letters, respectively. Set \( \{ x_i \} \) collects a known number of elements, and \( \min \{ x_i \} \) is the minimum of elements in \( \{ x_i \} \). Elements in a set can also be vectors of the same dimension. \( A^T \), \( A^{-1} \) and rank \( A \) are the transpose, inverse and rank of \( A \), respectively. The norm of \( a \) is \( |a| = \sqrt{a^T a} \).

### Hyperplane

In an \((n + 1)\)D space, a hyperplane parameterized by \((n, d)\) is an \(n\)D linear subspace, where the normal \( n \) is a unit vector perpendicular to all vectors in the subspace, and scalar \( d \) is the signed perpendicular distance to the origin of the reference frame. Algebraic equation \( n^T x = d \) represents such a hyperplane with \( x \) being an arbitrary point in it. A hyperplane is a plane, a straight line, or just a point in a 3D, 2D or 1D space. Points in a space are non-coplanar if they are not in a hyperplane in that space. The intersection of two hyperplanes with non-parallel normals forms another hyperplane with the dimension reduced by one. In an \((n + 1)\)D space, the intersection of \( n \) hyperplanes with linearly independent normals is a straight line, and that of \( n + 1 \) such hyperplanes is reduced to a point.

### Hypercone

A hypercone is a surface of revolution formed by rotating a straight line around an axis to which it is not parallel and intersects. The intersection is the apex. The two identical unbounded hyperconic halves are called the nappes positioned apex to apex. The one extending indefinitely in the direction of the rotation axis is the positive nappe, and the other negative nappe. In an \((n + 1)\)D space, a hypercone parameterized by \((\overline{x}, \theta)\) is an \(n\)D quadratic subspace, where \((n + 1)\)D point \( \overline{x} \) is the apex, and \( \theta \in (0, \pi/2) \) the half apex angle. To separate its first \( n \) elements and the last, denote the apex by \( \overline{x} = (\overline{x}_1, n, \overline{x}_{n+1}) \), and similarly an arbitrary point of the hypercone by \( x = (x_1, n, x_{n+1}) \). Algebraic equation \( |\overline{x}_1, n - x_1, n| = |\overline{x}_{n+1} - x_{n+1}|\tan \theta \) represents hypercone \((\overline{x}, \theta)\). With a fixed \( x_{n+1} \), the hypercone becomes a hypersphere. The negative and positive nappes correspond to \( \overline{x}_{n+1} > x_{n+1} \) and \( \overline{x}_{n+1} < x_{n+1} \) respectively, meeting at \( \overline{x} \), namely \( x_{n+1} \) reaching \( \overline{x}_{n+1} \).

### Problem formulation

The biased distance measurements of the TOA type are described by

\[
 r_i = |p_i - p| + r, \quad i = 1, 2, \ldots, n + 2 \tag{1}
\]

where \( r \) is an unknown bias, and \( |p_i - p| \) the distance between known reference \( p_i \) and unknown object \( p \) in an \(n\)D Euclidean space.

In GPS applications, \( p_i \) and \( p \) represent emitter \( i \) and the receiver, respectively. Usually, \( r_i = v(t_i - t) \) and \( r = v t_d \) with \( v \) being the signal transmission speed, \( t \) the arrival time of signals, \( t_i \) the emission time of signal \( i \), and \( t_d \) the clock bias.
between the receiver and the synchronized emitters. Equiva-
ently, in a different application, \( r_i = v (t_i - t) \) with \( t \) being
the arrival time of signal \( i \) and \( t \) the synchronized time of emitters.

The objective of positioning is to find an \((n+1)\)D point
\((p, r)\) satisfying (1) for given \( \{p_i\} \) and \( \{r_i\} \). It is of theoretical
and practical interests to examine the uniqueness of the posi-
tioning. Two further related problems will also be explored:
a) determination of the region of \( p \) where positioning is non-
unique with a given configuration of \( \{p_i\} \), and b) specification
of a configuration of \( \{p_i\} \) which guarantees unique positioning
everywhere.

To focus on essential positioning algebra and geometry,
a few assumptions implied in the problem formulation are
explained below. Stating these assumptions clarifies the scope
of the investigation undertaken without loss of generality.

**Assumption 1** The reference locations are exactly known
and distance measurements are free of noise. This ensures
solvability of (1) and that of any further equations deduced
from it.

**Assumption 2** The distance measurements are greater than
the bias, i.e. \( \min \{r_i\} > r \). This comes from \( |p_i - p| > 0 \)
because \( r_i < r \) is impossible and \( r_i = r \) implies \( p = p_i \), and
hence \( p \) becomes known.

**Assumption 3** The references are not coplanar. If they are,
let \( \bar{p} \) be the mirroring point of \( p \) across the hyperplane where
\( \{p_i\} \) are in, then \( \{\bar{p}, r\} \) is also a solution of (1) if \( \{p, r\} \) is.

**Assumption 4** The dimension of the object is greater than
one, i.e., \( n > 1 \). A general example will show that simple 1D
positioning may have infinite solutions.

**Assumption 5** The number of references is two plus the
dimension of the object. Examples will show non-unique
positioning in the case of having \( n+1 \) references regardless
of their placements. Theoretically, there is no need to con-
sider more than \( n+2 \) references because of Assumption 1.

### Equivalent equations

This section derives several equations from (1) for finding
closed-form solutions and necessary and sufficient conditions
on unique positioning.

To derive linear equations for positioning, squaring both
sides of (1) yields

\[
(r_i - r)^2 = |p_i - p|^2, \quad i = 1, 2, \ldots, n + 2
\]  

(2)

\( r < \min \{r_i\} \) is sought in solving (2). Subtracting the pre-
ceding equation from the next in (2) with \( i \) up to \( n+1 \) and
stacking the outcomes lead to

\[
A p = br + c
\]  

(3)

with

\[
A = \left[ p_2 - p_1, \ldots, p_{n+1} - p_n \right],
\]

(4)

\[
b = \left[ r_2 - r_1, \ldots, r_{n+1} - r_n \right],
\]

(5)

\[
c = \left[ c_1, \ldots, c_n \right], \quad c_i = \left( |p_{i+1} - p_i|^2 - r_{i+1}^2 + r_i^2 \right)/2.
\]  

(6)

Assumption 3 implies, without loss of generality, rank \( A = n \)
and hence from (3), \( p \) is uniquely determined in terms of \( r \) as

\[
p = A^{-1} (br + c).
\]  

(7)

Substituting equation (7) into equation (2) yields quadratic
equations

\[
a = |A^{-1} b|^2 - 1, \quad a_i = e_i^t A^{-1} b, \quad a_{i0} = |e_i|^2, \quad e_i = p_i - A^{-1} (br + c).
\]  

(9)

Note \( a_{i0} \neq 0 \) because \( a_{i0} = 0 \) implies \( a_i = 0 \), and hence (8)
would have solution \( r = r_i \) contradicting Assumption 2. Con-
sequently, no equation in (8) vanishes. Moreover, if \( a \leq 0 \),
each equation in (8) has a single solution which must be
identical to all single solutions from other equations in (8)
due to Assumption 1. When \( a > 0 \), each equation in (8) has
twin solutions that are not degraded to one if and only if
\( a_i^2 \neq a a_{i0} \) for any \( i \). In the case of \( a > 0 \), all equations in (8)
share the same twin solutions if and only if \( a_1 = 0 \) with \( a_1 \)
consisting of

\[
a_i = a_i (r_i - r_1) + a_{i1} - a_{11}, \quad i = 2, \ldots, n + 2.
\]  

(10)

This is because subtracting the \( i \)th equation from the first
equation in (8) gives

\[
a_i r = b_i, \quad i = 2, \ldots, n + 2
\]  

(11)

with

\[
b_i = a (r_i^2 - r_1^2)/2 + a_{i1} r_i - a_{11} r_1 + (a_{i1} - a_{11})/2.
\]  

(12)
and due to Assumption 1, Eq. (11) vanishes if and only if \( a_i = 0 \). In view of \( a > 0 \), by writing the \( i \)th equation in (8) as \(( r - \hat{r}_{ai}) ( r - r_{\beta,j}) = 0 \) with \( r_{ai} \) and \( r_{\beta,j} \) being its twin solutions, it is ready to verify the argument leading to (10).

A special way of determining \( r \) is to substitute (7) into

\[
(p_{n+2} - p_{n+1})' p = (r_{n+2} - r_{n+1}) r + c_{n+1}
\]  

which succeeds (3) with \( c_{n+1} \) being defined in (6) for \( i = n + 1 \), and assume

\[
(p_{n+2} - p_{n+1})' A^{-1} b \neq (r_{n+2} - r_{n+1}).
\]  

The following theorem summarizes the analysis in this section.

**Theorem**  The positioning problem has twin solutions if and only if \( a > 0, a_{1i}^2 \neq a a_{10} \) and \( a_1 = 0 \), and a single solution otherwise. The twin solutions are obtained by solving any quadratic equation in (8), and the single solution is obtained from (11) as \( r = b_i/a_i \) for any \( i \) so that \( a_i \neq 0 \).

### Positioning geometry

This section explains the essential positioning geometry leading to placements of the object and references for unique positioning.

#### Essential geometry

For any \( i \), \(|p_i - p| = |r_i - r|\) is the hypercone with apex \( X = (p_i, r_i) \) and fixed half angle \( \theta = \pi/4 \). Due to Assumption 2, (1) represents \( n + 2 \) identical negative hyperconic nappes but with different apexes. A solution \((p, r)\) to (1) is an \((n + 1)\) D point belonging to the intersection of the \( n + 2 \) negative nappes.

Being equivalent to \(|p_i - p| = |r_i - r|\), (2) corresponds to whole hypercones with both positive and negative nappes. A solution \((p, r)\) to (1) must also be in the linear subspace defined by (3). Understanding and particularly visualization of the hyperconic geometry are probably in vain for \( n \geq 3 \). The difficulties could be circumvented if intersection of \( n \) hyperplanes defined in (3) and the individual nappes defined in (1) is considered instead. In particular, the intersection of the \( n \) hyperplanes in (3) is a straight line described by the parametric equation

\[
\begin{align*}
\begin{bmatrix}
 p \\
 r
\end{bmatrix} &= \begin{bmatrix}
 A' \\
 b'
\end{bmatrix} ( AA' + b b' )^{-1} c + qu, \\
 u &= \begin{bmatrix}
 A^{-1} b \\
 1
\end{bmatrix} / \sqrt{a + 2},
\end{align*}
\]  

with free parameter \( q \) and direction vector \( u \). Importantly, \( 1/\sqrt{a + 2} = \cos \theta \), is the \( r \)-directional cosine of the straight line, and \( \theta \) is the associated angle. Note the correspondences between \( \theta <, =, > \pi/4 \) and \( a <, =, > 0 \).

The negative nappes in (1) intersect due to solvability of (1), and the intersection must be identical to that of the straight line and each negative nappe. If \( \theta < \pi/4 \), the straight line goes into but never comes out of each negative nappe, and hence (1) has a unique solution. When \( \theta = \pi/4 \), it may lie on the surface or goes into but never comes out of a negative nappe, and hence (1) can have only one solution because the straight line cannot be simultaneously on the surfaces of all nappes, as that would imply the references being coplanar. When \( \theta > \pi/4 \), there are two possibilities: (a) the straight line is tangent to a negative nappe, which means (1) having a single solution; (b) the straight line goes in and comes out of a negative nappe, which means (1) having twin solutions if the pair of intersections are identical for all nappes, or a single solution otherwise.

On the straight line, \( p \) uniquely relates to \( r \) as given in (7). The straight line intersects the negative nappes at a single point if and only if the parabolas \( f_i(r) = a_i (r_i - r)^2 + 2a_{1i}(r_i - r) + a_{10} \) for all \( i \) defined in (8) have a single \( r \)-intercept with \( r < \min \{ r_i \} \). Finally, the straight line intersects the negative nappes at two common points if and only if the parabolas have the same twin \( r \)-intercepts with \( r < \min \{ r_i \} \), which means that these parabolas must be identical and not tangent to the \( r \)-axis. When the \( r \)-directional angle of the straight line is \( \pi/4 \), these parabolas are degraded to a single straight line defined in (8) for \( a = 0 \), and it has a single \( r \)-intercept with \( r < \min \{ r_i \} \).

### Object regions for positioning with single and twin solutions

If exists, any of the regions of \( p \) for (1) having a single and twin solutions can normally be determined by numerical computations only. These two regions are complement to each other, and each of them could be formed by disjoint sub-regions. The satisfaction of the necessary and sufficient conditions in the theorem determines the region of \( p \) for (1) having twin solutions. In 2D positioning, from 3D surface plots of \( a, |a_1| \) and \( a_{11}^2 - a a_{10} \) with given \( \{ p_i \} \) against variations of \( p \), the regions of \( p \) for (1) having twin solutions can be inspected, while in general for 3D positioning, only the regions of \( p \) can be shown in a 3D graph.

### Reference configurations for unique positioning

The conditions derived for single and twin positioning solutions with a given set of references do not directly indicate but help through simulation studies to postulate reference configurations that ensure unique positioning wherever the object is.
In the nD case, regardless of their placement, n + 1 references cannot ensure unique positioning. This is rooted in that nD positioning with n unbiased distance measurements always has twin solutions if the object is not located in the hyperplane defined by the n references. Its mirroring point across the hyperplane is the other solution.

It is postulated that if n + 2 references are in a configuration where neither are they symmetric about a hyperplane, nor are every n + 1 of them in a hyperplane, nD positioning for n > 1 is unique everywhere. Not included is the special case n = 1, where positioning has infinite solutions if the object is not between any two of references lying on a straight line.

In the 2D case, the positioning will be unique by placing four references asymmetrically and ensuring no three of them being collinear. This is to require four references to form an asymmetric quadrilateral. In the 3D case, the positioning will be unique by placing five references asymmetrically and ensuring no four of them are coplanar. This is to require five references to form an asymmetric hexahedron. Using analytic geometry, simple algebraic conditions can be derived for characterization of these general 2D and 3D asymmetric configurations, but omitted for brevity.

**Illustrative examples**

To illustrate the closed-form solutions and their algebraic and geometric properties, examples of nD positioning with n up to 3 are considered. Although being straightforward, the construction of examples with n > 3 is not of interest due to a lack of obvious applications.

**1D positioning**

Stemming from a particular case study by Bancroft (1985), this general example has two references and three cases:

(a) \( p_1 < p < p_2 : \) Since \( a < 0, \) directly solving (1) gives the unique

\[
p = (p_1 + p_2 + r_1 - r_2)/2, r = (r_1 + r_2 + p_1 - p_2)/2.
\]

(b) \( p < p_1 < p_2 : \) Infinite solutions

\[
p = (p_1 + p_2 - r_1 - r_2)/2 + r.
\]

(c) \( p_1 < p < p_2 : \) Infinite solutions

\[
p = (p_1 + p_2 + r_1 + r_2)/2 - r.
\]

In cases (b) and (c), \( a = a_1 = a_2 = 0, \) hence (8) vanishes, and \( p \) is obtained from (7) with arbitrary \( r < \min \{r_1, r_2\}. \) Figure 1 illustrates the geometries of cases a) and c). Negative hyperconic nappes become downward open triangles in 1D positioning, and (3) is the straight line \( (p_2 + p_1)p = (r_2 - r_1)r + c_1 \) which intersects the open triangles in case (a), and overlaps one side of them in cases (b) and (c). As expected, there is nowhere for object \( p \) to make 1D positioning have twin solutions regardless of configurations of \( \{p_i\} \) because of \( a \leq 0 \) in all cases.

**2D positioning**

Let three non-collinear references be \( p_1 = [1, 0]' , p_2 = [2, 0]', p_3 = [0, 1]' \). Various \( \{r_i\} \) generate different cases:

(a) With \( r_1 = r_3 = 2 \) and \( r_2 = 1 + \sqrt{2}, \) since \( a < 0, \) (1) has a unique solution \( \{1, 0\}', 1 \).

(b) With \( r_1 = r_3 = 1 + \sqrt{5} \) and \( r_2 = 1 + \sqrt{10}, \) since \( a > 0, \) \( a_{11} = 0, \) and \( a_{11}^2 \neq aa_{10}, \) (1) has twin solutions: \( \{0.4042, 0.4042\}', 0.5161 \) and \( \{-1, -1\}', -1 \).

(c) With \( r_1 = r_2 = 3, \) since \( a > 0 \) and \( a_1 = 0, \) but \( a_{11}^2 = aa_{10}, \) (1) has a unique solution \( \{0, 0\}', 1 \).
Three conic nappes in case (a) and one nappe in cases (b) and (c) along with the straight line as the intersection of two planes are shown in the top, middle and bottom panels of Fig. 2, respectively. As expected, it is difficult to visualize the intersection of three nappes, while it is easy to see the intersection of one nappe and one straight line. In the middle panel of Fig. 2, the straight line goes in and comes out of the nappe, while in the bottom panel of the figure, it is tangent to the nappe.

With unchanged $r_1$, $r_2$ and $r_3$ as in case b), but added $p_4 = [0, 2]$ and $r_4 = r_2$, (1) still has the twin solutions, although no three of the four references are collinear. The
top panel of Fig. 3 shows plots of \(a_1 = 0\) and \(a_2^2 - aa_{10}\) against variations of \(p = [x, y]\) with the four references. The intersection of regions of \(x\) and \(y\) ensuring \(a > 0\), \(a_1 = 0\) and \(a_2^2 \neq aa_{10}\) determines the locations where (1) has twin solutions. These locations are on part of a straight line described by \(0 \neq x = y < 0.75\) as shown in the bottom panel of Fig. 3.

In a further case, references are 

\[ p_1 = [0, 0, 0], p_2 = [1, 0, 0], p_3 = [0, 1, 0], p_4 = [1, 1, 0], \]

no three of which are collinear, and measurements are 

\[ r_1 = 1 + 2\sqrt{2}, r_2 = 1 + \sqrt{5}, r_3 = r_2, r_4 = 1 + \sqrt{2}. \]

Since \(a = 0\), (1) has a unique solution \([2, 2], 1\). Algebraically, this is the case where the quadratic equations in (8) are degraded to linear equations. Geometrically, the straight line in (15) is with \(r\)-directional angle \(\theta = \pi/4\). It goes into all four negative nappes at the same place and never comes out of them.

### 3D positioning

Let four non-coplanar references be

\[ p_1 = [0, 0, 0], p_2 = [1, 0, 0], \]

\[ p_3 = [0, 1, 0], p_4 = [0, 0, 1], \]

produce the first two of the following cases:

**Case 1** With \(r_1 = 1 + \sqrt{3}\) and \(r_2 = r_3 = r_4 = 1 + \sqrt{2}\), due to \(a < 0\), (1) has a unique solution \([1, 1, 1], 1\).

**Case 2** With \(r_1 = \sqrt{3} - 1\) and \(r_2 = r_3 = r_4 = \sqrt{6} - 1\), due to \(a > 0\), \(a_1 = 0\), and \(a_2^2 \neq aa_{10}\). (1) has twin solutions: 

\[ \left(0.1082, 0.1082, 0.1082\right), 0.5447 \]

and 

\[ 
\left([-1, -1, -1], -1\right). 
\]

**Case 3** With five non-coplanar references

\[ p_1 = [1, 0, 0], p_2 = [0, 1, 0], \]

\[ p_3 = [0, 0, 1], p_4 = 2p_1, p_5 = 2p_3, \]

\[ r_1 = r_3 = 2, r_2 = r_4 = 1 + \sqrt{2} \text{ and } \sqrt{3}, \text{ due to } a > 0, \]

\(a_1 = 0\), and \(a_2^2 \neq aa_{10}\). twin solutions are: 

\[ \left([1, 1, 0], 1\right) \text{ and } \left([-0.7830, -0.7830, -4.9342], -3.3046\right), \]

**Case 4** With \(r_1\) to \(r_4\) unchanged in case 2, but additional \(r_5 = 2\sqrt{3} - 1\) and

\[ p_5 = [1, 1, 1], \]

due to \(a_1 \neq 0\), (1) has a unique solution \((-1, -1, -1), -1\) as one of the twin solutions obtained in case 2.

With the four non-coplanar references in (16) and (17) used in cases 1) and 2), the top panel of Fig. 4 shows the region of \(p\) where positioning has twin solutions. With the five non-coplanar references in (18) and (19) considered in case 3, when the object is located in the partial plane \(x = y\) or at either of the isolated points [0.4, 2, −0.8] and [2, 0.4, −0.8] as shown in the bottom panel of Fig. 4, the positioning will have twin solutions. However, the references in (16) and (17) together with \(p_5 = 2p_1\) or that in (20) are also non-coplanar, but positioning with \(\{p_i\}\) in either of the configurations is unique everywhere.
Concluding remarks

Given a set of references, the object region for positioning with non-unique solutions is identifiable by using the conditions derived. Although such a region may exist even for over-determined problems, in general, the asymptotic placement of a sufficient number of references can reduce or even remove it. This number is the dimension of the object plus two. The exploration of positioning algebra and geometry contributes toward an improved understanding of the nature of positioning with biased TOAs.

The explicit necessary and sufficient conditions derived for unique and non-unique positioning are new and easy to use and cover all degraded cases with a clear indication for obtaining closed-form positioning solutions. With some adjustments, these conditions can be extended to solvability and uniqueness of positioning under the influence of uncertainties in biased pseudorange measurements and locations of references.

Positioning with unbiased TDOAs converted from biased TOAs cannot avoid the problems of solvability and uniqueness. This is because each TDOA equation represents a particular sheet of a hyperboloid of two sheets. While positioning with biased TOAs finds the intersection of negative hyperconic nappes, TDOA-based positioning seeks the intersection of one-side hyperbolic sheets. Because of one-to-one correspondence between the two positioning schemes in the absence of measurement noise, the results of positioning with biased TOAs are indirectly applicable to the positioning problem using unbiased TDOAs.

This study has postulated asymmetric configuration of a minimal number of non-coplanar references for unique positioning. Supported by algebraic and geometrical analysis as well as simulation studies, this postulation has, however, not been verified directly. Furthermore, in the case of non-existence of exact solutions in most practical applications due to measurement noise, the closed-form solutions cannot theoretically achieve the least squares of the original nonlinear navigation equations. Finally, practical applications of positioning problems in the case of the dimension greater than three remain to be found despite a belief in their existence.

Data availability All data generated or analyzed during this study are included in this manuscript.

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