NEW REAL VARIABLE METHODS IN $H$ SUMMABILITY OF FOURIER SERIES.

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Abstract. In this paper we shall be concerned with $H_\alpha$ summability, for $0 < \alpha \leq 2$ of the Fourier series of arbitrary $L^1([-\pi, \pi])$ functions. The method to be employed is a refinement of the real variable method introduced by Marcinkiewicz in [8].

Dedicated to the memory of A. Eduardo Gatto

1. Introduction

Let $f$ be a function in $L^1([-\pi, \pi])$, denote by $S_n(f, \cdot)$ the partial sum of order $n$ of the Fourier series of $f$,

\begin{equation}
S_n(f, x) = \sum_{|k| \leq n} c_k e^{-inx}, \quad x \in [-\pi, \pi].
\end{equation}

We say that $f$ is $H_2$ summable at $x$ if there exists a number $s$ such that,

\[ \frac{1}{n} \sum_{k=1}^{n} |S_k(f, x) - f(x)|^2 \longrightarrow 0 \quad \text{a.e.} \]

This can be extended easily to $\alpha > 0$; i.e. we say that its Fourier series is $H_\alpha$ summable to some $f(x)$ or that it is a strongly $\alpha$-summable to sum $f(x)$, if

\begin{equation}
\frac{1}{n} \sum_{k=1}^{n} |S_k(f, x) - f(x)|^\alpha \longrightarrow 0 \quad \text{a.e.}
\end{equation}

Historically the problem goes back to H. Hardy and J. H. Littlewood in [6]. There the problem is restricted to $H_2$ summability of $L^2([-\pi, \pi])$ functions (i.e. $\alpha = 2$ and $f \in L^2([-\pi, \pi])$, see also T. Carleman [5].

In 1935 Hardy and Littlewood proved the case $\alpha > 0$ and $f \in L^p$, with $1 < p < \infty$ and posed the problem of whether “any arbitrary periodic function in $L^1([-\pi, \pi])$ is $H_2$ summable a. e. in $[-\pi, \pi]$. The answer to this question came only on January of 1939, when J. Marcinkiewicz presented his remarkable result [8], developing a real variable method to establish it.

Finally the case of $H_\alpha$ summability a. e. for $\alpha > 2$ and $f \in L^1([-\pi, \pi])$ was proved by A. Zygmund in 1941, [12] using complex methods. In view of the negative results concerning convergence a.e. of the Fourier series of functions in $L^1([-\pi, \pi])$ the $H_\alpha$ summability
acquires a special meaning.

In this paper we shall be concerned with $H_\alpha$ summability, for $0 < \alpha \leq 2$ of Fourier series for arbitrary $L^1([−\pi, \pi])$ functions. The methods used here are a refinement of the real variable method by Marcinkieicz in [8], and could be applied also to the case $\alpha > 2$. Nevertheless this requires a modification of the Marcinkiewicz function and a change of kernel function (to be defined later).

2. Preliminaries

Consider the following maximal operator

\[(\sigma_\alpha f)(x) = \sup_{n>0} \left[ \frac{|S_1(f, x)|^\alpha + |S_2(f, x)|^\alpha + \ldots + |S_n(f, x)|^\alpha}{n} \right]^{1/\alpha} \]

where, as before, $S_k(f, \cdot)$ stands for the $k$-th partial sum of the Fourier series of $f \in L^1([−\pi, \pi])$ and $0 < \alpha \leq 2$.

Also, let us consider for $f \in L^1([−\pi, \pi])$ the non-centered Hardy-Littlewood function, namely,

\[f^*(x) = \sup_{I \supset \{x\}} \frac{1}{|I|} \int_I |f(y)| \, dy\]

where $I$ is taken to be an open interval, containing $x$. Observe that the set

\[F = \left\{ x : f^*(x) \leq \lambda \right\}\]

is a closed set, and the set

\[G = \left\{ x : f^*(x) > \lambda \right\},\]

is an open set.

The class $A_1$ of weights is defined using the non-centered Hardy-Littlewood function, $f^*$, we say $\omega \in A_1$ if the inequality

\[\omega(\left\{ x : f^*(x) > \lambda \right\}) \leq \frac{C}{\lambda} \int_{-\pi}^{\pi} |f(x)| \omega(x) \, dx\]

holds true for any $f \in L^1([−\pi, \pi]), f \geq 0 C$ depends only on $\omega$.

A well known result gives a characterization of the weights in the case $(-\infty, \infty)$, see Stein [11]; a positive weight $w \geq 0$ belongs to the class $A_1$ if and only if

\[w^*(x) \leq C\omega(x)\].
In order to prove the problem of $H_2$ summability for $L^1$ functions Marcinkiewicz proved that $\sigma_2^* f$ is finite a.e. and he refined that to $H_2$ summability. Moreover, it can be proved, see Stein [9],

\begin{equation}
|\{x : (\sigma_2^* f)(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1,
\end{equation}

$C$ depends only on $\omega$.

In order to tackle that problem Marcinkiewicz introduced the so called Marcinkiewicz function. If $F$ is a perfect set and $G = F^c$ its complement, if $d(x, F) = \inf_{z \in F} |x - z|$, denotes the distance from $x$ to $F$; then he defined,

\begin{equation}
F(x) = \int_G \frac{1}{(x - y)^2} d(x, F) dy
\end{equation}

which is finite a.e. for $x \in F$.

The function $F(x)$ has important implications in the $L^1$ theory of singular integrals. In particular, in 1966, L. Carleson in his famous $L^2$ theorem uses a variation of this function, Carleson function is denoted as $\Delta$ in his article. Also Zygmund shows the realtion between $\Delta(x)$ and $F(x)$, see [13].

By Kolmogorov’s counterexample in 1926, we know that there exists a function $f \in L^1([-\pi, \pi])$ such that $S_n(f, \cdot)$ diverges a.e., thus the maximal function,

$$f^{**}(x) = \sup_n |S_n(f, x)|$$

can not be weak type $(1, 1)$, see Zygmund [14], Vol II.

Now, consider firstly functions supported on an interval of length $\pi/8$, centered at the origin. For those functions the maximal function $\sigma_\alpha^* f$ satisfies the inequality

$$(\sigma_\alpha^* f)(x) \leq C f^*(x),$$

for $x$ such that $d(0, x) \geq \pi/4$, where $C$ is a constant depending on $\pi$ only. The above inequality is consequence of

$$\int_{-\pi}^\pi |f(t)| dt \leq 2\pi f^*(x),$$

and the estimate

$$\left| \frac{\sin(n + 1/2)u}{\sin u/2} \right| \leq \frac{1}{|u|} + \frac{1}{2},$$

see A. Zygmund [14], Vol I pages 50–51. In what follows we shall introduce a majorization of the kernel used by Marcinkiewicz in [8].

Returning to the case of a general function $f$, such such a function can be decomposed into a sum of pieces, each supported on an interval of length $\pi/8$. By a shift each one of the pieces is moved to the origin, thus each piece can be studied as if it were supported on an interval of length $\pi/8$ centered at the origin.

The method to be employed in this paper is a refinement of the real variable method introduced by Marcinkiewicz in [8].
3. Main results

The main results obtained in this paper are the following,

**Theorem 3.1.** Given \(0 < \alpha \leq 2\), \(f \in L^1([-\pi, \pi])\) and \(\sigma_\alpha^* f\) as above, then

\[
|\{x : (\sigma_\alpha^* f)(x) > \lambda\}| \leq \frac{C_\alpha}{\lambda} \|f\|_1,
\]

where \(C_\alpha\) is a constant that depends only on the \(\alpha\) (but not on \(f\)).

Moreover, if \(\omega\) is an \(A_1\)-weight, then

\[
\omega \left( \left|\{x : (\sigma_\alpha^* f)(x) > \lambda\}\right| \right) \leq \frac{C_\omega}{\lambda} \int_{-\pi}^{\pi} |f(x)| \omega(x) \, dx,
\]

\(C_\omega\) is a constant that depends only on the weight (but not on \(f\)).

As a particular case, if \(\omega(x) \equiv 1\) we have the Lebesgue’s measure case.

The following result was given by J. Marcinkiewicz in [8] (see Lemma 3),

**Lemma 3.1.** Let \(P\) be a perfect set (i.e., a set without isolated points), \(\Delta_v\) a sequence of contiguous segments, \(\varphi(x)\) the function of period \(2\pi\) equal to zero in \(P\) and to \(|\Delta_v|\) for \(x \in \Delta_v\). We have almost everywhere in \(P\)

\[
\int_{-\pi}^{\pi} \varphi(t + x) \, dt < \infty.
\]

For the proof see [8]. We will give an alternative proof of this result.

3.1. Whitney type decomposition. We start considering an special type of covering for open sets in \(\mathbb{R}\) which has the same type of building principle than the Whitney decomposition in \(\mathbb{R}^n\), see Stein [10].

**Lemma 3.2.** Let \(G\) be an open set, and consider its decomposition into open disjoint intervals \(\{I_k\}\) (i.e. \(G\) can be written as \(G = \bigcup_k I_k\), and \(I_k \cap I_l = \emptyset\) for \(k \neq l\)) such that \(I_k\) are its connected components and the end points of \(I_k\) are in \(F = G^c\). Then it is possible to find a countable refinement \(\{I'_j\}\) such that,

i) whenever \(I_j\) and \(I'_j\) are not adjacent, i.e. \(\bar{I}_j \cap \bar{I}_j' = \emptyset\) then, for a suitable \(C\)

\[
d(I_j, I'_j) \geq C |I_j| \text{ and } d(I_j, I'_j) \geq C |I'_j|.
\]

\(C\) can be chosen to be bigger or equal to \(1/2\).

ii) For any \(j\), \(d(I_j, F) = |I_j|\).

**Proof.** We have \(G = \bigcup_k I_k = \bigcup_k (a_k, b_k)\) with \(a_k, b_k \in F = G^c\). For each fixed \(k\) we do the following:

- We decompose \(I_k\) into three subintervals \(I_{k,1}, I_{k,2}, I_{k,3}\) having equal length, thus

\[
I_k = \bigcup_{i=1}^3 I_{k,i} \quad \text{and} \quad d(I_{k,i}, F) = |I_{k,i}|
\]
$J_{k,2}$ is selected to be a closed interval and $J_{k,1}$, $J_{k,3}$ are open intervals.

- $J_{k,2}$, the central subinterval, will be an element of the new refinement $\{I_j\}$.
- Each of the side open intervals, $J_{k,1}$ and $J_{k,3}$ are broken up into two subintervals of the same length such that, the ones that are adjacent to the central interval $J_{k,2}$, are taken such that the left one $J_{k,1,2}$ is closed on the left and open on the right and the right one $J_{k,3,1}$ is open on the left and closed to the right and therefore

$$d(J_{k,1,2}, F) = |J_{k,1,i}| \quad i = 1, 2$$
$$d(J_{k,3,1}, F) = |J_{k,3,i}|. \quad i = 1, 2$$

The intervals, $J_{k,1,2}$ and $J_{k,3,1}$ will be part of the new refinement $\{I_j\}$.

- As before, the remaining side open intervals $J_{k,1,1}$ and $J_{k,3,2}$ are broken up into two subintervals of the same length, the one that is adjacent to the interval $J_{k,1,2}$, is taken such that is closed on the left and open on the right and the one that is adjacent to the interval $J_{k,3,1}$ is open on the left and closed to the right and they will be part of the new refinement $\{I_j\}$.

- Iterating this argument over and over again and doing the same process for each $J_k$ of the original decomposition of $G$ we obtain a sequence of intervals $\{I_j\}$, such that $d(I_j, F) = |I_j|$.

It is important to note that if $I_j$ and $I_{j'}$ are not adjacent, i.e. $\overline{I_j} \cap \overline{I_{j'}} = \emptyset$, then there will be among them at least one subinterval satisfying the construction conditions, and therefore they satisfy

$$d(I_j, I_{j'}) \geq C |I_j|$$
$$\geq C |I_{j'}|$$

$□$

3.2. Consequences of this Whitney type decomposition. Now, given $f \in L^1[-\pi, \pi])$, $f \geq 0$ and $\lambda > 0$ consider the set

$$G = \{ x : f^*(x) = \sup \frac{1}{|I|} \int_I f(t) \, dt > \lambda \},$$

then $G$ is an open set, and consider the Whitney type decomposition for $G$, $\{I_j\}$, as above, i.e. $G = \cup_{j=1}^{\infty} I_j$. We take its average

$$\frac{1}{|I_j|} \int_{I_j} f \, dx \leq \frac{1}{|I_j|} \int_{I_j} f \, dt$$

where $\overline{I_j}$ has been obtained from $I_j$ by expanding it 2 times, i.e. $|\overline{I_j}| \geq 2|I_j|$. If $I_j$ is one of the central subintervals of the original decomposition $J_{k,2}$, one of the adjacent subintervals is
also included. If \( I_j \) is not the central subintervals, we choose another subinterval, adjacent to the central one. In this way we have \( \tilde{I}_j = I_j \cup I \) and therefore \( |\tilde{I}_j| = 2|I_j| \)

\[
\frac{1}{|I_j|} \int_{I_j} f \, dx \leq \frac{1}{|I_j|} \int_{I_j} f \, dt = \frac{2}{2|I_j|} \int_{\tilde{I}_j} f \, dt \\
\leq \frac{1}{|I_j|} \int_{\tilde{I}_j} f \, dt \leq 2\lambda.
\]

Regardless of \( I_j \), the \( \tilde{I}_j \) we have defined has points from the complement of \( G \), and therefore its integral is less or equal than \( 2\lambda \). In other words,

\[
\int_{I_j} f \, dx \leq 2\lambda|I_j| \quad \text{if and only if} \quad \int_{\tilde{I}_j} f \, dx \leq 2\lambda|\tilde{I}_j|
\]

given that \( \tilde{I}_j \) contains at least one point from \( F = G^c = \{ f^* \leq \lambda \} \).

Suppose now that we have a Poisson kernel and a function \( f \) such that \( f \geq 0, \text{supp}(f) \subset J_k \) and \( f \) is \textit{bad}, by which we mean that \( f \) is infinite in a dense subset (i.e., for all \( n \) there exists \( E_n \subset J_k \) such that \( |f| > n \)). Even though \( f \) is bad, we know that for some \( k_0 \)

\[
\frac{1}{|J_{k_0}|} \int_{J_{k_0}} f \, dx \leq \lambda \quad \text{and} \quad d(u, J_{k_0}) \geq c|J_{k_0}|
\]

(3.4)

\[
\int_{J_{k_0}} \frac{\epsilon}{\epsilon^2 + (u - v)^2} f(v) \, dv \leq C(c)\lambda
\]

(3.5)

\[
\int_{J_{k_0}} \frac{\epsilon}{\epsilon^2 + (u - v)^2} f(v) \, dv \leq C \int_{J_{k_0}} \frac{\epsilon}{\epsilon^2 + (u - v)^2} \phi(v) \, dv
\]

where \( \phi(v) \leq C(c)\lambda \) and \( |u - v| > c|J_{k_0}| \).

**Proof.** We will prove (3.2). Take \( v \in J_{k_0} \) such that \( d(u, J_{k_0}) > C|J_{k_0}| \).

\[
\int_{J_{k_0}} \frac{\epsilon}{\epsilon^2 + (u - v)^2} f(v) \, dv \leq \int_{J_{k_0}} \frac{\epsilon}{\epsilon^2 + c^2|J_{k_0}|^2} f(v) \, dv \quad \text{if} \quad |u - v| < C|J_{k_0}|
\]

\[
\leq \frac{\epsilon}{\epsilon^2 + c^2|J_{k_0}|^2} \int_{J_{k_0}} f(v) \, dv = \frac{\epsilon|J_{k_0}|}{\epsilon^2 + c^2|J_{k_0}|^2} \frac{1}{|J_{k_0}|} \int_{J_{k_0}} f(v) \, dv
\]

\[
\leq \frac{\epsilon|J_{k_0}|}{\epsilon^2 + c^2|J_{k_0}|^2} \lambda = \frac{\epsilon\lambda}{\epsilon^2 + \frac{c^2}{4} (2|J_{k_0}|^2)^2 |J_{k_0}|}
\]

\[
\leq \frac{\epsilon\lambda}{\epsilon^2 + \frac{c^2}{4} (|J_{k_0}|^2 + u)^2}
\]

\[
\frac{2 \cdot |J_{k_0}|}{|J_{k_0}| - u}
\]
Therefore this must hold for the average on $v$

$$\int_{J_k} \frac{\varepsilon}{\varepsilon^2 + (u-v)^2} f(v) \, dv \leq \varepsilon \lambda \left| J_k \right| \frac{1}{\left| J_k \right|} \int_{J_k} \frac{dv}{\varepsilon^2 + \frac{\varepsilon^2}{4} \left( \left| J_k \right| + v \right)^2}$$

$$= \int_{J_k} \frac{\varepsilon}{\varepsilon^2 + \frac{\varepsilon^2}{4} \left( \left| J_k \right| + v \right)^2} \lambda \, dv = C \lambda$$

(3.4) was the key point in Marcinkiewicz’s proof.

3.3. On the Marcinkiewicz function. Let $F(x)$ be the Marcinkiewicz function, defined in (2.6) and $\{ I_k \}$ the covering of $G$ satisfying the properties of Lemma 3.2 and consider

$$(3.6) \quad \sum_{k=1}^{\infty} \left( \int_{I_k} \frac{|I_k|}{(x-y)^2} f_k \, dy \right)$$

where $f_k = \frac{1}{|I_k|} \int_{I_k} f(t) \, dt$. Then it is not difficult to see that if $x \in F = G^c$, then (3.6) is finite and

$$\int_F \left[ \sum_{k=1}^{\infty} \left( \int_{I_k} \frac{|I_k|}{(x-y)^2} f_k \, dy \right) \right] = \sum_{k=1}^{\infty} \int_{I_k} \left[ f_k \left( \int \frac{1}{(x-y)^2} \, dx \right) \right] \, dy$$

$$\leq C \sum_{k=1}^{\infty} \int_{I_k} f_k \, dy$$

$$\leq \lambda C |G|$$

where the last inequality follows from the construction of the covering and the first reflects the fact that $|I_k| \int \frac{1}{(x-y)^2} \, dx \leq C$ because $|I_k| \int \frac{1}{x^2} \, dx \leq C$ for $|x| \geq |I_k|$. Furthermore

$$\left| \{ F(x) > \lambda \} \right| \leq \frac{C}{\lambda} \sum_k \left( \int f_k(y) \, dy \right)$$

$$\leq \frac{C}{\lambda} \lambda |G|$$

$$\leq C \frac{1}{\lambda} \int f(x) \, dx$$

In other words, the Marcinkiewicz function $F$ is $(1,1)$-weak.

**Proposition 3.1.** Let $\mu$ be a measure such that $\mu \in A_1$ and $d\mu = g \, dx$, where $g$ is the density of the measure $\mu$. Then

$$\int_G g \, dx \leq \frac{C}{\lambda} \int f \, g \, dx$$
Proof. Let $F(x)$ be the Marcinkiewicz function
\[
\int F(x)\,d\mu(x) \leq \int_F \frac{1}{(x-y)^2}|I_k|f_k(y)\,dy
\leq \int_{I_k} f(y)\frac{d\mu(x)}{|y-x| + |I_k|}\,dy
\leq Fg(y) \leq Cg(y) \leq C \int f(y)g(y)\,dy
\]
since $(x-y)^2 \sim [(y-x) + |I_k|]^2$. Therefore using Chebyshev’s inequality
\[
\left(\mu\{x: F(x) > \lambda\}\right) \leq \frac{C}{\lambda} \int_F F(x)\,d\mu(x)
\]
over $F$. Then by Marcinkiewicz’s theorem
\[
\mu(G) \leq \frac{C}{\lambda} \int f(y)g(y)\,dy.
\]
\[\Box\]

**Theorem 3.2** (surprising result).

(3.7) \[
\int \frac{|f(x+y)|}{y^2}\,dy < \infty \quad \text{a.e. in } F
\]

Proof. Let $f_G = f|_G$ and $G = \bigcup_k I_k$, where $I_k$ denotes maximal intervals, and $f_k \cap F \neq \emptyset$. We shall see that

(3.8) \[
\int \frac{1}{(x-y)^2} f_G(y)\,dy = \int \frac{f_G(x+y)}{y^2}\,dy < \infty \quad \text{a.e.}
\]

We shall show next that

\[
\sum_k \int_{I_k} f_G(y)\,dy \int \frac{1}{(x-y)^2}\,dx
\]

Consider $\tilde{I}_k = (1+\varepsilon)I_k$, a dilation by a factor of $1 + \varepsilon$ from the center of $I_k$, therefore

$|\tilde{I}_k| = |(1+\varepsilon)I_k| = (1 + \varepsilon)|I_k|
$

If $x \in \tilde{F} = \left(\bigcup (1+\varepsilon)I_k\right)^c \subset F$ then

\[
\int_{\tilde{F}} \frac{1}{(x-y)^2}\,dx \leq \int_{|x-y| > \varepsilon|I_k|} \frac{1}{(x-y)^2}\,dy \leq \frac{1}{\varepsilon|I_k|}
\]

Then

\[
\int_{I_k} f_G(y)\left(\int_{\tilde{F}} \frac{dx}{(x-y)^2}\right)\,dy \leq \frac{1}{\varepsilon|I_k|} \int_{I_k} f_G(y)\,dy \leq \frac{1}{\varepsilon} \lambda
\]
We observe that when \( x \) is outside the intervals \( I_k \), that is whenever \( x \notin I_k \), \( y_k \in I_k \) and if we note that \( d(x, I_k) \geq \epsilon|I_k| \)

\[
\int_{I_k} \frac{1}{(x-y)^2} f_c(y) \, dy \sim \frac{1}{(x-y)^2} \int_{I_k} f_c(y) \, dy \\
= \frac{|I_k|}{(x-y)^2} \int_{I_k} f_c(y) \, dy \\
\leq \frac{1}{|I_k|} \int_{I_k} f_c(y) \, dy \int_{I_k} \frac{1}{(x-y)^2} \, dy \\
= \lambda \int_{I_k} \frac{1}{(x-y)^2} \, dy
\]

Summing up over \( k \) we obtain

\[
\sum_k \int_{I_k} \left( \int_{F} \frac{dx}{(x-y)^2} \right) \, dy \leq C \lambda \int_{\bar{G}} \frac{1}{(x-y)^2} \, dy.
\]

Now we shift our attention on a more pure version. If \( G = \bigcup I_k, F = G^c, \delta_k = c|I_k| \) and defining \( \phi \) by

\[
\phi(x) = \begin{cases} 
  c|I_k| & \text{if } x \in I_k \\
  0 & \text{if } x \in F
\end{cases}
\]

we have

\[
\phi(x) = \sum_k c|I_k| \chi_k(x)
\]

**Lemma 3.3.**

\[
\int_{F'} \frac{1}{(x-y)^2} \phi(y) \, dy < \infty \quad \text{a.e. on } F
\]

**Proof.** Let \( F' = (G')^c \), where \( G' = \bigcup (1 + \epsilon) I_k \). Denoting by \( I'_k = (1 + \epsilon) I_k \)

\[
\int_{F'} \int \frac{1}{(x-y)^2} \phi(y) \, dy \, dx = \sum_k \int_{I_k} \sum_k \int_{I'_k} \frac{1}{(x-y)^2} \, dx \, dy \\
= \sum_k \int_{I_k} c \, dy \\
= C''|G|
\]
where we have used that $(x - y)^2 \geq (1 + \varepsilon)|I_k|$ if $x \in F'$. Therefore

$$
\int_{F'} \left( \int \frac{1}{(x - y)^2} \phi(y) \, dy \right) \, dx < \infty
$$

(3.13)

$$
\int \frac{1}{(x - y)^2} \phi(y) \, dy < \infty \quad \text{a.e. on } F
$$

We plan to discuss the following chain of inequalities

(3.14)

$$
\lambda \leq \frac{1}{|I_k|} \int_{I_k} f(u) \, du \leq 2\lambda
$$

For $f \geq 0$, $u \geq 0$ and $v \geq 0$

$$
(1 - r^2) \int_{0}^{\pi/2} \frac{f(u + x)}{(1 - r^2)(u + v)^2} \int_{0}^{\pi/2} \frac{f(v + x)}{(1 - r^2)(u - v)^2} \, dv \, du
$$

$$
\leq (1 - r^2) \int_{0}^{\pi/2} \frac{f(u + x)}{(1 - r^2)u^2} \int_{0}^{\pi/2} \frac{f(v + x)}{(1 - r^2)(u - v)^2} \, dv \, du \quad \text{since } v \geq 0
$$

$$
\leq (1 - r^2) \int_{x}^{x + \pi/2} \frac{f(u)}{(1 - r^2)(u - x)^2} \int_{0}^{\pi/2} \frac{f(v)}{(1 - r^2)(u - v)^2} \, dv \, du
$$

$$
\leq (1 - r^2) \int_{-\pi}^{\pi} \frac{f(u)}{(1 - r^2)(u - x)^2} \int_{-\pi}^{\pi} \frac{f(v)}{(1 - r^2)(u - v)^2} \, dv \, du
$$

Then

$$
(1 - r^2) \int_{-\pi}^{\pi} \frac{f_k(u)}{(1 - r)(u - x)^2} \sum_{j=k'} \int_{-\pi}^{\pi} \frac{f_j(\cdot)}{(1 - r)(u - v)^2} \, dv \, du
$$

$$
\leq (1 - r^2) \int_{-\pi}^{\pi} \frac{f_k(u)}{(1 - r)(u - x)^2} \sum_{j=k} \int_{-\pi}^{\pi} \frac{f_j(\cdot)}{(1 - r)^2} \, dv \, du
$$

$$
\leq \int_{-\pi}^{\pi} \frac{f_k(u)}{(u - x)^2} C\lambda |I_k| \, du
$$

Let

$$
f_k = \begin{cases} 
    f & \text{on } I_k \\
    0 & \text{elsewhere.}
\end{cases}
$$
If $x \in I_j$, then
\[
q_k(x) = \int_{\Lambda_k} \frac{1 - r}{(x - y)^2 + (1 - r)^2} f_k(y) \, dy \\
\leq \int_{\Lambda_k} \frac{1 + r}{(1 - r)^2 + c(x - y)^2} \mu_k \, dy
\]
\[
\int \frac{1 - r}{(1 - r)^2 + (x - y)^2} f_k(y) \, dy \leq \left( \frac{1 - r}{(1 - r)^2 + (x - y)^2} \right) \mu_k \\
\leq \left( \frac{1 - r}{(1 - r)^2 + C(x - y)^2} \right) \mu_k
\]
\[
\int \frac{\epsilon^{p-1}}{\epsilon^p + (x - y)^p} \, dy = \int \frac{\epsilon^{p-1}}{\epsilon^p + |y|^p} \, dy = \int \frac{\epsilon^{p-1}}{1 + |y|^p} \, dy \\
= \int \frac{\epsilon^{-1}}{1 + |s|^p} \, dy = \int \frac{ds}{1 + |s|^p}, \quad y = s.
\]

### 3.4. Power Series

Now, consider the power series, $\sum_{k=0}^{\infty} a_k x^k$, for $x$ a complex variable, which is convergent for $|x| < 1$ and let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $f$ is analytic.

Denoting the partial sum of $\sum_{k=0}^{\infty} a_k$ by $S_n = \sum_{k=0}^{n} a_k$, then, as $S_k - S_{k-1} = a_k$, then
\[
f(x) = \sum_{k=0}^{\infty} (S_k - S_{k-1}) x^k = \sum_{k=0}^{\infty} S_k x^k - \sum_{k=1}^{\infty} S_{k-1} x^k
\]
\[
= \sum_{k=0}^{\infty} S_k x^k - \sum_{k=0}^{\infty} S_k x^{k+1} = \sum_{k=0}^{\infty} S_k (x^{k+1} - x^k) = (1 - x) \sum_{k=0}^{\infty} S_k x^k
\]

Suppose now $x \in B_1 = \{ x : |x| < 1 \}$, then
\[
f(x) = \sum_{k=0}^{\infty} S_k x^k
\]
Writing $x^k = r^k e^{i k \theta}$,
\[
f(x) = \sum_{k=0}^{\infty} (S_k r^k) e^{i k \theta}
\]

If $1 < p < 2$, then using the Hausdorff-Young inequality we have
\[
\left[ \sum_{k=0}^{\infty} |S_k r^k|^q \right]^{1/q} \leq C_p \left( \int_0^{2\pi} \left| \frac{f(re^{i \theta})}{1 - re^{i \theta}} \right|^p d\theta \right)^{1/p} \\
\times [ \int_0^{2\pi} \frac{1}{1 - re^{i \theta}} \left( \int_0^{2\pi} |P(r, \theta - \psi) f(\psi) \, d\psi \right)^{p} ]^{1/p}
\]
We start with case when the $1 < \theta < 2$ and $2 < q < \infty$

Now, since we have the equivalence

\[ (\sum_{k=0}^{\infty} r^k q |S_k|^q)^{1/q} \leq C_p \left( \int_0^{2\pi} \frac{1}{1 - re^{is}} |f(r, \theta + s)|^p \right)^{1/p}. \]

By the Hausdorff-Young inequality which bounds the $L^p$ norm the Fourier coefficients for $1 < p < 2$ and $2 < q < \infty$

Now, since we have the equivalence $|1 - re^{is}| \sim |(1 - r)^2 + s^2|^{1/2}$ in the sense it is bounded by (3.19) multiplied by a constant

\[
\leq \left( \int_0^{2\pi} \frac{1}{[(1 - r)^2 + s^2]^{p/2}} \left( \int P(r, \theta + s - u) p(u) \, du \right)^p \right)^{1/p} \\
= \left( \int_0^{2\pi} \frac{1}{[(1 - r)^2 + (s - \theta)^2]^{p/2}} \left( \int P(r, \theta + s - u) p(u) \, du \right)^p \right)^{1/p}
\]

Using a Whitney type decomposition with $f = \sum_{k=0}^{\infty} f_k$ we can rewrite the previous equation as

\[
\left( \int_0^{2\pi} \frac{1}{[(1 - r)^2 + (s - \theta)^2]^{p/2}} \left( P(r, s - u) f_k(u) \sum_j \left( \int P(r, s - v) f_j(v) \, dv \right) \right) \right)^{p-1}
\]

where $1 - r = \epsilon$ and where for now we shall work with a single $k$ to bound $f_k$ and then we shall consider all $k$

\[
\left( \int_0^{2\pi} \frac{1}{[\epsilon^2 + (s - \theta)^2]^{p/2}} \left( P(r, s - u) f_k(u) \sum_j \left( \int P(r, s - v) f_j(v) \, dv \right) \right) \right)^{p-1}
\]

We consider the different possible cases separately:

**Case a:** We start with case when the $j$ are not adjacent to $I_k$.

\[ 2 \leq \left( \int_0^{2\pi} \frac{1}{[\epsilon^2 + (s - \theta)^2]^{p/2}} \left( \int P(r, s - v) \lambda \chi_k C \lambda^{p-1} \right) \right) \]

considering $s$ not in $I_k$

\[ \leq \left( \int_0^{2\pi} \frac{1}{[\epsilon^2 + (s - \theta)^2]^{p/2}} \left( \int P(r, s - v) C \lambda \lambda^{p-1} \right) \right) \]
Case b: In the case that \( j \) touch the adjacent, the measures are comparable and we have

\[
3.4 \leq \left( \int_0^{2\pi} \frac{1}{[\epsilon^2 + (s - \theta)^2]^{p/2}} \left( \int P(r, s - u)C(\lambda|I_k|^{p-1}) \right) \right)
\]

In the case that \( s \) not in \( I_k \)

\[
\leq \left( \int_0^{2\pi} \frac{1}{[\epsilon^2 + (s - \theta)^2]^{p/2}} \left( \int_{I_k} P(r, s - u)\lambda\chi_k(u)C(\lambda|I_k|^{p-1}) \right) \right)
\]

3.4.1. Abel Sums analog. Consider the series \( (1 - r) \sum_{v=0}^{\infty} r^v (S_v(f, x))^2 \), then taking \( r = 1 - \frac{1}{n} \) then

\[
(1 - r) \sum_{v=0}^{\infty} r^v (S_v(f, x))^2 = (1 - (1 - \frac{1}{n})) \sum_{v=0}^{\infty} (1 - \frac{1}{v})^v (S_v(f, x))^2
\]

\[
= \frac{1}{n} \sum_{v=0}^{\infty} (1 - \frac{1}{v})^v (S_v(f, x))^2
\]

\[
\geq e^{-1} \frac{1}{n} \sum_{v=0}^{\infty} (S_v(f, x))^2.
\]

Therefore,

\[
(S^*f)(x) \leq e^{1/2} \sup_{0 < r < 1} [(1 - r)(1 - r) \sum_{v=0}^{\infty} r^v (S_v(f, x))^2]^{1/2}.
\]

The key will be to study \( (1 - r) \sum_{v=0}^{\infty} r^v (S_v(f, x))^2 \). We will construct a kernel

\[
(3.20) \quad D(r, x, y) = \sum_{v=0}^{\infty} r^v D_v(x)D_v(y),
\]

where \( D_v \) is the Dirichlet kernel

\[
D_v(x) = \frac{\sin(v + 1/2)x}{2\sin(x/2)}.
\]

Thus,

\[
D(r, x, y) = \frac{(1 - r)[(1 - r)^2 + 2r(2 + \cos x + \cos y)]}{4(1 - 2r\cos(x - y) + r^2)(1 - 2r\cos(x + y) + r^2)}.
\]

For \( -\pi + \epsilon < (x + y) < \pi - \epsilon \) and \( -\pi + \epsilon < (x - y) < \pi - \epsilon \) we get,

\[
D(r, x, y) \leq \frac{9(1 - r)}{[(1 - r)^2 + rC_3(x - y)^2][(1 - r)^2 + rC_3(x + y)^2]}.
\]

We will assume \( f = 0 \) if \( |x| > \pi/2 - \epsilon/2 \), and we will estimate

\[
(1 - r) \sum_{v=0}^{\infty} r^v (S_v(f, x))^2 = \frac{1 - r}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + u)g(x + v)D(x, r, u, v)du dv,
\]
We shall consider the integral

\[(1 - r)^2 \int_0^\infty \int_0^\infty \frac{9f(x + u)g(x + v)}{[(1 - r)^2 + rc(u - v)^2][(1 - r)^2 + rc + v^2]} \, du \, dv.\]

We move from the periodic case to the continuous case. The above integral is dominated by the integral (3.21)

\[(1 - r)^2 \int_0^\infty \int_0^\infty \frac{f(x + u)g(x + v)}{[(1 - r)^2 + rc(u - v)^2][(1 - r)^2 + rc(v^2)]} \, du \, dv,\]

and the analogous integrals on the regions \((-\infty, 0) \times (-\infty, 0), (-\infty, 0) \times (0, \infty)\) and \((0, \infty) \times (-\infty, 0)\), that can be argue as the preceding one using symmetry arguments.

Let us study the integral (3.21), which is typical

\[
\begin{align*}
(1 - r)^2 \int_0^\infty \int_0^\infty & \frac{f(x + u)g(x + v)}{[(1 - r)^2 + rc(u - v)^2][(1 - r)^2 + rc(v^2)]} \, du \, dv \\
& \leq (1 - r)^2 \int_0^\infty \int_0^\infty \frac{f(x + u)f(x + v)}{[(1 - r)^2 + rc(u - v)^2][(1 - r)^2 + rc(v^2)]} \, du \, dv
\end{align*}
\]

\(f\) has already been decomposed as \(f = \sum_{k=0}^\infty f_k\) where

\[
f_k = \begin{cases} f & \text{on } I_k \\ 0 & \text{otherwise.} \end{cases}
\]

\(\{I_k\}\) are such that

\[
\frac{1}{|I_k|} \int_{I_k} f(t) \, dt \leq C \lambda.
\]

The above integral (3.21) is dominated by

\[
C(1 - r)^2 \int_\infty \int_\infty \frac{f(x + u)}{[(1 - r)^2 + rc(u - v)^2]} \frac{f(x + v)}{[rc^2/v^2]} \, du \, dv,
\]

and after a change of variables, we have,

\[
\frac{c}{r} \int_\infty \int_\infty \frac{f(v)}{(v - x)^2} \frac{(1 - r)^2 f(u)}{(1 - r)} \, du \, dv.
\]

The intervals \(I_k\) have been constructed so that

i) If \(I_j\) and \(I_k\) are adjacent then

\[
\frac{1}{2} |I_j| \leq |I_k| \leq 2 |I_j|.
\]

ii) \(d(I_j, I_k) \geq \frac{1}{2} |I_j|\), and \(d(I_j, I_k) \geq \frac{1}{2} |I_k|\)

For \(x \in F = \bigcup_v I_v\) decompose the double integral as the sum

\[
\sum_{i,j} \int_{I_i} \int_{I_j} (1 - r)^2 \int_{I_i} \int_{I_j} \frac{f(v)}{(1 - r)^2 + c(v - x)^2} \, dv \int_{I_i} \int_{I_j} \frac{f(u)}{(1 - r)^2 + c(u - v)^2} \, du
\]

i) adjacent \(I_j\): there are at most 2 of those intervals,

\[
(1 - r)^2 \int_{I_i} \int_{I_j} \frac{f(v)}{(1 - r)^2 + c(v - x)^2} \, dv \int_{I_i} \int_{I_j} \frac{f(u)}{(1 - r)^2 + c(u - v)^2} \, du \leq \int_{I_i} \int_{I_j} \frac{f(v)}{(v - x)^2} \, dv \times C \lambda |I_i| \leq C \lambda F(x).
\]
ii) non adjacent $I_j$:
\[
(1 - r)^2 \int_{I_j} \frac{f(v)}{(1 - r)^2 + c(v - x)^2} dv \times \sum_j \frac{C}{\lambda} \int_{I_j} \frac{\lambda}{(1 - r)^2 + (v - x)^2} \leq C\lambda^2.
\]

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