Charged coherent states related to \( su_q(2) \) covariance*

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Abstract

A new kind of \( q \)-deformed charged coherent states is constructed in Fock space of two-mode \( q \)-boson system with \( su_q(2) \) covariance and a resolution of unity for these states is derived. We also present a simple way to obtain these coherent states using state projection method.

With the study of solutions of the Yang-Baxter equation, quantum groups and algebras have been extensively developed by Jimbo[1] and Drinfeld[2]. To realize the Jordan-Schwinger mapping of the quantum algebra \( su_q(2) \), Biedenharn and Macfarlane[3][4] introduced a kind of \( q \)-deformed harmonic oscillators. They also discussed some properties of these oscillators, for example, the Fock space structure and the \( q \)-deformed coherent states for such oscillator. And Sun and Fu[5] gave the \( q \)-deformed boson realization of the quantum algebra \( su_q(n) \) constructed in terms of this kind of deformed oscillators. But the oscillators in their discussions are all mutually commuting. To our knowledge, in 1989, Pusz and Woronowicz first introduced multimode \( q \)-deformed oscillators[6] which are not mutually commuting but satisfy the following relations

\[
\begin{align*}
    a_i^\dagger a_j^\dagger &= \sqrt{q} a_j^\dagger a_i^\dagger \quad (i < j) \quad (i, j = 1, \cdots, n) \\
    a_i^\dagger a_j &= \frac{1}{\sqrt{q}} a_j^\dagger a_i \quad (i < j) \\
    a_i^\dagger a_j &= \sqrt{q} a_j^\dagger a_i \\
    a_j^\dagger &= \sqrt{q} a_i^\dagger a_j \\
    a_i a_i^\dagger &= 1 + qa_i a_i + (q - 1) \sum_{k>i} a_k^\dagger a_k .
\end{align*}
\]

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Kulish and Damaskinsky pointed that these coupled multimode q-deformed oscillators can be expressed in terms of independent q-deformed harmonic oscillators. Wess and Zumino developed a differential calculus on the quantum hyperplane covariant with respect to the action of the quantum group GL_q(n). Recently Chung studied multiboson realization of the two-mode q-boson algebra relations with su_q(2) covariance. The algebra relations are given by

\begin{align*}
a^\dagger b^\dagger &= \sqrt{q} b^\dagger a^\dagger \\
ab &= \frac{1}{\sqrt{q}} ba \\
a^\dagger b &= \sqrt{q} b^\dagger a \\
b^\dagger a^\dagger &= 1 + qa^\dagger a + (q-1)b^\dagger b \\
ba^\dagger &= \sqrt{q} a^\dagger b \\
b^\dagger a &= 1 + qb^\dagger b.
\end{align*}

In fact, these two-mode q-boson algebra relations are exactly the eq(1) for n = 2 case. Comparing (2) with Wess-Zumino’s differential calculus relations on the covariant quantum plane

\begin{align*}
xy &= \sqrt{q}yx \\
\partial_x \partial_y &= \frac{1}{\sqrt{q}} \partial_y \partial_x \\
\partial_x y &= \sqrt{q} y \partial_x \\
\partial_y x &= \sqrt{q} x \partial_y \\
\partial_x x &= 1 + qx \partial_x + (q-1)y \partial_y \\
\partial_y y &= 1 + qy \partial_y.
\end{align*}

we can easily find that they are the same thing in sense of Bargmann representation according to the corresponding relations \((a^\dagger \leftrightarrow x, b^\dagger \leftrightarrow y\) and \(a \leftrightarrow \partial_x, b \leftrightarrow \partial_y\)). We also gave the the structure of Fock space for the coupled two-mode q-boson system and discussed its simply application to Jordan-Schwinger realization of quantum algebra su_q(2) and su_q(1,1).

In this letter, we construct a new kind of q-deformed charged coherent states in the Fock space of the coupled two-mode q-boson oscillators (we emphasize that these coherent states are not so-called su_q(2) coherent states but they are related to q-boson algebra with su_q(2) covariance) and show a resolution of unity for these states. We also discuss a simple way to obtain these states using projection method. Bhaumik et al discussed the undeformed and also independent case. We use the method developed by them.

To begin with, we recall the Fock space representation of (2). We take \(a^\dagger, a\) and \(b^\dagger, b\) as q-deformed creation and destruction boson operators, and define the number operators \(N_a, N_b\) following by
\[ b^\dagger b = [N_b] \]
\[ q^{-N_b} a^\dagger a = [N_a] \] (4)

where
\[ [N] = \frac{q^N - 1}{q - 1}. \]

Using the algebraic relation (2), it is easy to verify
\[ [N_a, a] = -a, \quad [N_b, b] = -b, \]
\[ [N_a, b] = 0, \quad [N_b, a] = 0, \] (5)

and \([N_a, N_b] = 0\). Let \(|0, 0\rangle\) be the ground state satisfying
\[ a |0, 0\rangle = 0, \quad b |0, 0\rangle = 0 \]
\[ N_i |0, 0\rangle = 0, \quad (i = a, b) \] (6)

and \(\{ |n, m\rangle \mid n, m = 0, 1, 2, \ldots \}\) be the set of the orthogonal number eigenstates
\[ N_a |n, m\rangle = n |n, m\rangle \]
\[ N_b |n, m\rangle = m |n, m\rangle \]
\[ \langle n, m \mid n', m' \rangle = \delta_{nn'} \delta_{mm'} . \] (7)

From the algebra (2), the Fock space representation for the \(q\)-bosons \(a\) and \(b\) is given by
\[ a |n, m\rangle = \sqrt{q^m [n]} |n - 1, m\rangle , \quad b |n, m\rangle = \sqrt{[m]} |n, m - 1\rangle \]
\[ a^\dagger |n, m\rangle = \sqrt{q^m [n + 1]} |n + 1, m\rangle , \quad b^\dagger |n, m\rangle = \sqrt{[m + 1]} |n, m + 1\rangle . \] (8)

The general number eigenstate \(|n, m\rangle\) is obtained by applying \(b^\dagger\) \(m\) times after applying \(a^\dagger\) \(n\) times on the ground state \(|0, 0\rangle\)
\[ |n, m\rangle = \frac{(b^\dagger)^m (a^\dagger)^n}{\sqrt{[n]! [m]!}} |0, 0\rangle \] (9)

where
\[ [n]! = [n] [n - 1] \cdots [2] [1] , \quad [0]! = 1 . \]

Because of the noncommutative character of \(a, b\) showing in (2), they have not common eigenstate-coherent states. But using the above number operators \(N_a\) and \(N_b\), we may define the charge operators given by
\[ Q = N_a - N_b \] (10)
which means that each of the $a$ quanta possesses a charge $'+1'$, and each of the $b$ quanta a charge $'−1'$. From (2), we can easily find

$$[Q, ab] = [Q, ba] = [ab, ba] = 0 .$$  \hspace{1cm} (11)

Hence, the operator $Q$, $ab$, and $ba$ may have common eigenstates, which is called $q$-deformed charged coherent state[12]. We denote this coherent state as $|z, e\rangle$ ($z$ is a complex number and $e$ an integer) with the requirements that

$$Q |z, e\rangle = e |z, e\rangle \hspace{1cm} ab |z, e\rangle = z |z, e\rangle .$$  \hspace{1cm} (12)

To obtain an explicit expression for this coherent state, we consider the following expansion

$$|z, e\rangle = \sum_{n,m=0}^\infty C_{nm}(z) |n, m\rangle .$$  \hspace{1cm} (13)

Since $|z, e\rangle$ is an eigenstate of $Q$, for $e \geq 0$, we obtain

$$|z, e\rangle = \sum_{m=0}^\infty C_{m+e,m}(z) |m + e, m\rangle .$$  \hspace{1cm} (14)

Substituting this expression into the second equation in (12), using (8), we obtain

$$C_{m+e,m}(z) = C_{e,0} \frac{\sqrt{[e]!} z^n}{\sqrt{q^{m(m-1)/2}} [m]! [m + e]!} .$$  \hspace{1cm} (15)

so the charged coherent state is given by

$$|z, e\rangle = N_{e} \sum_{m=0}^\infty \frac{z^n}{\sqrt{q^{m(m-1)/2}} [m]! [m + e]!} |m + e, m\rangle$$  \hspace{1cm} (16)

with the normalization constant

$$N_{e}^{-2} = \sum_{m=0}^\infty \frac{|z|^{2n}}{q^{m(m-1)/2} [m]! [m + e]!} .$$  \hspace{1cm} (17)

To resolve the unity for these coherent states, we introduce the $q$-analogue of integral defined by Jackson[13][14][15]. We hereby give a few words about this integral. The $q$-derivative is defined as

$$\frac{df}{dq} = f(qx) - f(x) \frac{1}{(q - 1)x} \hspace{1cm} (q-1)x$$  \hspace{1cm} (18)

and the corresponding integral is given by

$$\int f(x) dq = (1-q) \sum_{l=0}^\infty q^lx f(q^lx) + const .$$  \hspace{1cm} (19)
If $a$ is finite, we have

$$\int_0^a f(x) \, dq_x = (1 - q) \sum_{l=0}^{\infty} q^l x f(q^l a)$$

(20)

and if $a$ is infinitive, we have

$$\int_0^\infty f(x) \, dq_x = (1 - q) \sum_{l=-\infty}^{\infty} q^l x f(q^l) .$$

(21)

The $q$-integration by parts formula is

$$\frac{d}{dq} (f(x) g(x)) = \left( \frac{d}{dq} f(x) \right) g(qx) + f(x) \left( \frac{d}{dq} g(x) \right)$$

(22)

and

$$\int_0^a f(x) \left( \frac{d}{dq} g(x) \right) \, dq_x = f(x) g(x) \bigg|_{x=a}^{x=0} - \int_0^a \left( \frac{d}{dq} f(x) \right) g(qx) \, dq_x .$$

(23)

It is well known that Exton\[16\] defines a family of $q$-exponential functions by

$$E(q, \lambda; x) = \sum_{n=0}^{\infty} x^n q^{\lambda n(n-1)} [n]! .$$

(24)

Using the definition of $q$-derivative (18), one can derive the following expression

$$\frac{dE(q, \lambda; x)}{dq_x} = E(q, \lambda; q^2 \lambda x)$$

(25)

so with the parts formula(22) and (23), the following result can be carefully calculated.

$$\int_0^\infty E(q, \lambda; -x) x^n \, dq_x = q^{h(n, \lambda)} [n]! ,$$

(26)

where

$$h(x, \alpha) = \alpha (x + 1) (x + 2) - x (x + 1) / 2 .$$

(27)

Next step we define $q(\lambda, \rho)$-modified Bessel function which has the same form as the common modified Bessel function when $q, \lambda, \rho$ approach 1 simultaneously

$$K_n(\lambda, \rho; x) = \frac{1}{[2]^2_q} \left( \frac{x}{[2]_{q^{1/2}}} \right)^n \int_0^\infty t^{-n-1} E(q, \lambda; -t) E\left(q, \rho; -\frac{x^2}{[2]_{q^{1/2}}^2 t} \right) \, dq t$$

(28)

where $n$ is integer. With the above definition and (26), we have
\[ \int_{0}^{\infty} d_{q^{1/2}} r \cdot r^\mu K_\nu \left( \lambda, \rho; \left[ a \right]_{q^{1/2}} r \right) = q^{h \left( \frac{\mu+\nu+1}{2}, \rho \right) + h \left( \frac{\mu+\nu-1}{2}, \lambda \right)} \left[ a \right]_{q^{1/2}}^{-\mu-1} \Gamma_q \left( \frac{\mu+\nu+1}{2} \right) \Gamma_q \left( \frac{\mu-\nu+1}{2} \right) \]  

(29)

where \( \Gamma_q (x) \) is \( q \)-deformed Gamma Function. When \( q \) approaches 1, it reduces into the common Gamma Function. If \( x \) is integer, it equals \( [x!] \). By setting \( \Phi_e (z) = q^{\epsilon+3} [2]_{q^{1/2}} |z|^\epsilon N_e^{-2} K_e \left( \lambda, \rho; \left[ 2 \right]_{q^{1/2}} |z| \right) \)  

(30)

where

\[ \lambda = 1 + 2e^{-1}, \quad \rho = 1/2 - 2e^{-1} \]

these charged coherent states for the two-mode coupled \( q \)-bosons satisfy the completeness relation

\[ \sum_{e=-\infty}^{\infty} \int d_{q^{1/2}}^{2} z \Phi_e (z) |z, e\rangle \langle z, e| = I \]  

(31)

where

\[ d_{q^{1/2}}^{2} z = \frac{1}{2} d_{q^{1/2}} \left( |z|^2 \right) d\theta = \frac{[2]_{q^{1/2}}}{2} |z| d_{q^{1/2}} (|z|) d\theta. \]

The coherent states for this system may also be obtained by projecting out a state of a definite charge from the following state

\[ |\alpha \beta\rangle = \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{q^{m(n-1)/2} |n|! |m|!} \left( b^{\dagger} \right)^m \left( a^{\dagger} \right)^n |0, 0\rangle \]

\[ = \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{q^{m(n-1)/2} |n|! |m|!} |n, m\rangle \]  

(32)

which does not have a definite charge. It is easily found that this state is the associated coherent state of \( ab \) and \( ba \) satisfying

\[ ab |\alpha \beta\rangle = \alpha \beta |\alpha \beta\rangle, \quad ba |\alpha \beta\rangle = \sqrt{q} \alpha |\alpha \beta\rangle \]  

(33)

If we set

\[ \alpha = \xi \exp (-i (\theta + \varphi)) , \quad \beta = \eta \exp (-i (\theta - \varphi)) \]  

(34)

we can find that the state

\[ |\xi \eta \theta; e\rangle = \frac{1}{2\pi} \int_{0}^{2\pi} d\varphi \exp (ie \varphi) |\alpha \beta\rangle \]  

(35)
yields
\[ Q |\xi\eta; e\rangle = e |\xi\eta; e\rangle, \quad ab |\xi\eta; e\rangle = \alpha\beta |\xi\eta; e\rangle, \quad (36) \]
so this state is identical to the charged coherent state \(|z, e\rangle\) for this coupled \(q\)-boson system if we only set \(\xi\eta\exp(-i\theta) = z\). Here \(\exp(x)\) is the ordinary exponential function of \(x\).

In summary, we explicitly construct the \(q\)-deformed charged coherent states for the two-mode coupled \(q\)-bosons with \(su_q(2)\) covariance and give a resolution of unity for these states. We also find a simple way to obtain these coherent states using state projection. Similarly, one can construct deformation of the charged coherent states for multimode coupled \(q\)-bosons(1). Work on this direction is in progress.

References

[1] M. Jimbo, *Lett. Math. Phys.* **10**, 63 (1985); **11**, 247 (1985)
[2] V. Drinfeld, *In Proceedings of the International Congress of Mathematicians* (Berkely) p. 78 (1986)
[3] L. C. Biedenharn, *J. Phys. A: Math. Gen.* **22** L873 (1989)
[4] A. J. Macfarlane, *J. Phys. A: Math. Gen.* **22** 4581 (1989)
[5] Chang-Pu Sun and Hong-Chen Fu, *J. Phys. A: Math. Gen.* **22** L983 (1989)
[6] W. Pusz and S. L. Woronowisz, *Rep. Math. Phys.* **27** 231 (1989)
[7] P. P. Kulish and E. V. Damaskinsky, *J. Phys. A: Math. Gen.* **23** L415 (1990)
[8] J. Wess and B. Zumino, CERN preprint CERN-TH-5697/90 (1990)
[9] W. S. Chung, *Inter. J. Theor. Phys* Vol. 39, **10** 2407 (2000)
[10] V. Bargmann, *Pure Appl. Math.* **14** 87 (1967)
[11] T. Deng and S. Jing, *Journal of Anhui Institute of Architecture*, Vol 5, **2** 30 (1997)
[12] D. Bhaumik, K. Bhaumik and B. Dutta-Roy, *J. Phy. A: Math. Gen.* **9** 1507 (1976)
[13] F. H. Jackson, *Trans. R. Soc.* **46** 253 (1908)
[14] F. H. Jackson, *Quart. J. Math.* **41** 193 (1910)
[15] F. H. Jackson, *Quart. J. Math.* **2** 1 (1951)
[16] H. Exton, *q-Hypergeometric Functions and Applications* (Chichester: Ellis Horwood) (1983)