Berline–Vergne Valuation and Generalized Permutohedra

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Abstract Generalizing a conjecture by De Loera et al., we conjecture that integral generalized permutohedra all have positive Ehrhart coefficients. Berline and Vergne construct a valuation that assigns values to faces of polytopes, which provides a way to write Ehrhart coefficients of a polytope as positive sums of these values. Based on available results, we pose a stronger conjecture: Berline–Vergne’s valuation is always positive on permutohedra, which implies our first conjecture. This article proves that our strong conjecture on Berline–Vergne’s valuation is true for dimension up to 6, and is true if we restrict to faces of codimension up to 3. In addition to investigating the positivity conjectures, we study the Berline–Vergne’s valuation, and show that it is the unique construction for McMullen’s formula used to describe number of lattice points in permutohedra under certain symmetry constraints. We also give an equivalent statement to the strong conjecture in terms of mixed valuations.

Keywords Ehrhart polynomials · Generalized permutohedra · Mixed valuations · Berline–Vergne construction

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1 Introduction

A lattice point is a point in $\mathbb{Z}^D$. Given any bounded set $S \subseteq \mathbb{R}^D$, we let $\text{Lat}(S) := |S \cap \mathbb{Z}^D|$ be the number of lattice points in $S$. Given a polytope $P$ in $\mathbb{R}^D$, a natural enumerative problem is to compute $\text{Lat}(P)$. In this paper, we will focus on integral polytopes, i.e., polytopes whose vertices are all lattice points, and generalized permutohedra – a special family of polytopes.

1.1 Motivation: Ehrhart Positivity for Generalized Permutohedra

One approach to study the question of computing $\text{Lat}(P)$ for an integral polytope $P$ is to consider a more general counting problem: For any nonnegative integer $t$, let $tP := \{tx : x \in P\}$ be the $t$-th dilation of $P$, and then consider the function

$$i(P, t) := \text{Lat}(tP)$$

that counts the number of lattice points in $tP$. It is a classic result that $i(P, t)$ is a polynomial in $t$. More precisely:

**Theorem 1.1** (Ehrhart [11]) There exists a polynomial $f(x)$ such that $f(t) = i(P, t)$ for any $t \in \mathbb{Z}_{\geq 0}$. Moreover, the degree of $f(x)$ is equal to the dimension of $P$.

We call the function $i(P, t)$ the Ehrhart polynomial of $P$. A few coefficients of $i(P, t)$ are well understood: the leading coefficient is equal to the normalized volume of $P$, the second coefficient is one half of the sum of the normalized volumes of facets, and the constant term is always 1. Although formulas are derived for the other coefficients, they are quite complicated. One notices that the leading, second and last coefficients of the Ehrhart polynomial of any integral polytope are always positive; but it is not true for the rest of the coefficients for general polytopes. We say a polytope has Ehrhart positivity or is Ehrhart positive if it has positive Ehrhart coefficients.

A few families of polytopes are known to be Ehrhart positive. Zonotopes, in particular regular permutohedra, are Ehrhart positive [20, Theorem 2.2]. Cyclic polytopes also have this property. Their Ehrhart coefficients are given by the volumes of certain projections of the original polytope [13]. Stanley–Pitman polytopes are defined in [22] where a formula for their Ehrhart polynomials is given and from which Ehrhart positivity follows. Recently in [10] De Loera, Haws, and Koepp study the case of matroid base polytopes and conjecture that they are Ehrhart positive. Both Stanley–Pitman polytopes and matroid base polytopes fit into a bigger family: generalized permutohedra.

In [18] Postnikov defines generalized permutohedra as polytopes obtained by moving the vertices of a usual permutohedron while keeping the same edge directions. He also considers a strictly smaller family, type $\gamma$, consisting of sums of dilated simplices. He describes the Ehrhart polynomial for the type $\gamma$ family in [18, Theorem 11.3], from which Ehrhart positivity follows. The type $\gamma$ family includes the Stanley–Pitman polytopes, associahedra, cyclohedra, and more (see [18, Sect. 8]), but fails to contain matroid base polytopes [1, Proposition 2.4]. We consider the following con-
jecture on generalized permutohedra, generalizing the conjecture on Ehrhart positivity of matroid base polytopes given in [10] by De Loera et al.

**Conjecture 1.2** Integral generalized permutohedra are Ehrhart positive.

The above conjecture is the original motivation of this paper. However, instead of studying it directly, we reduce it to another conjecture which only concerns regular permutohedra, a smaller family of polytopes.

### 1.2 McMullen’s Formula, α-Positivity and Uniqueness

In 1975 Danilov [9] asked, in the context of toric varieties, whether the following formula holds for any integral polytope $P$:

$$\text{Lat}(P) = \sum_{F: \text{a face of } P} \alpha(F, P) \text{nvol}(F),$$

where the value of $\alpha(F, P)$ depends only on the normal cone of $P$ at $F$.

McMullen [15] was the first to confirm the existence of formula (1.1) in a non-constructive way. Therefore, we call (1.1) **McMullen’s formula**. Morelli [16] was the first to supply a deterministic algorithm to choose $\alpha(F, P)$. In [17], Pommersheim and Thomas gave a canonical construction of $\alpha(F, P)$ from the Todd class of a toric variety. In 2007, Berline and Vergne gave the first construction of $\alpha(F, P)$ in [5] in the primary space without using toric varieties.

One immediate consequence of the existence of formula (1.1) is that if $\alpha(F, P)$ is positive for each face $F$ of $P$, then Ehrhart positivity follows. (See Theorem 3.1 and Lemma 3.2.) Hence, it is natural to say that a polytope $P$ has **α-positivity** or is **α-positive** if all $\alpha$’s associated to $P$ are positive.

Although there are different constructions for $\alpha(F, P)$, Berline–Vergne’s construction has certain nice properties that are good for our purpose, and thus we will use their construction in our paper. We refer to their construction for $\alpha(F, P)$ as the **BV-α-valuation**. To be more precise, we will use the terminologies **BV-α-positivity** and **BV-α-positive** to indicate that the $\alpha$’s we use are from the BV-α-valuation.

At present, the explicit computation of the BV-α-valuation is a recursive, complicated process, but we carry it out in the special example of regular permutohedra of small dimensions, whose symmetry simplifies the computations. Based on our empirical results, we conjecture the following:

**Conjecture 1.3** Every regular permutohedron is BV-α-positive.

One important property of the BV-α-valuation enables us to reduce the problem of proving the Ehrhart positivity of all generalized permutohedra to proving the positivity of all the $\alpha$’s arising from the regular permutohedra.

**Theorem 1.4** Conjecture 1.3 implies Conjecture 1.2.

Therefore, we focus on proving Conjecture 1.3 instead. In this paper, we provide partial progress on proving Conjecture 1.3 (and thus Conjecture 1.2):
Theorem 1.5  (1) For all $n \leq 6$, the regular permutohedron $\Pi_n$ is BV-$\alpha$-positive. Therefore, all the integral generalized permutohedra (including matroid base polytopes) of dimension at most 6 are Ehrhart positive.  

(2) For any $n$, and any face $F$ of $\Pi_n$ of codimension 2 or 3, we have $\alpha(F, \Pi_n)$ is positive, where $\alpha$ is the BV-$\alpha$-valuation. Hence, the third and fourth coefficients of the Ehrhart polynomial of any integral generalized permutohedron (including matroid base polytopes) are positive.

During the process of investigating positivity of the $\alpha$’s, another natural question arises: Although there are different constructions for $\alpha$, is it possible that under certain constraints, the construction is unique? Our second main result of this paper is an affirmative answer to this question when we focus on regular permutohedra.

Theorem 1.6  Suppose $\alpha$ is a construction such that McMullen’s formula holds and it is symmetric about the coordinates (see Definition 3.13). Then all the $\alpha$’s arising from the regular permutohedra $\Pi_n$ are uniquely determined.

Even though the above theorem does not seem to be related to attempts of proving Conjecture 1.3, as a consequence of techniques used in proving the theorem, we give in Corollary 5.6 an equivalent statement to Conjecture 1.3 in terms of mixed valuations.

Organization of the Paper

In Sect. 2, we give basic definitions and review background results that are relevant to our paper. In Sect. 3, we give details of the BV-$\alpha$-valuation, discuss consequences of the properties of this construction. In particular, we complete the proof of Theorem 1.4.  

In Sect. 4, assuming the $\alpha$ function in McMullen’s formula (1.1) is symmetric about the coordinates (a property of the BV-$\alpha$-valuation), we derive a combinatorial formula for $\text{Lat}(\text{Perm}(v))$ indexed by subsets of $[n]$, where $\text{Perm}(v)$ is a “generic permutohedron”, which belongs to a family of generalized permutohedra containing the regular permutohedra. We then finish with a proof for Theorem 1.5.

In Sect. 5, using the combinatorial formula we derived in Sect. 4 and theories of mixed valuations, we show that the $\alpha$-values arising from the regular permutohedron are unique as long as we assume $\alpha$ is symmetric about the coordinates, as well as present equivalent statements to Conjecture 1.3 in terms of mixed valuations.  

We finish with some natural questions arising from this paper in Sect. 6.

2 Background

In this section and the next section, we assume the ambient space is $\mathbb{R}^D$, and $\mathbb{Z}^D$ is the lattice in $\mathbb{R}^D$. For any $D$-vector $\beta$, $\beta_i(\cdot)$ is the linear function that maps $x \in \mathbb{R}^D$ to the scalar product of $\beta$ and $x$. Since we can consider $\beta(\cdot)$ as a point in the dual space $(\mathbb{R}^D)^*$ of $\mathbb{R}^D$, we will use the notation $\beta$ (or any bold letter) to denote both the $D$-vector and the linear function.

We assume familiarity with basic definitions of polyhedra and polytopes as presented in [3,23], and only review terminologies and setups that are relevant to us.
polyhedron is the set of points defined by a linear system of inequalities. A polytope is a bounded polyhedron. (A polytope can also be defined as the convex hull of a finite set of points.) An integral polyhedron is a polyhedron whose vertices are all lattice points, i.e., points with integer coordinates.

Notation 2.1 For any set of points \( A \subseteq \mathbb{R}^D \), we denote by \( \text{aff}(A) \) the affine span of \( A \), and by \( \text{lin}(A) \) the linear space obtained by shifting \( \text{aff}(A) \) to the origin.

Let \( V \) be a subspace of \( \mathbb{R}^D \), and \( \Lambda := V \cap \mathbb{Z}^D \) the lattice in \( V \). For any polytope \( P \) such that \( \text{lin}(P) \subseteq V \), we define the volume of \( P \) normalized to the lattice \( \Lambda \) to be the integral \( \text{vol}_\Lambda(P) := \int_P 1 \, d\Lambda \), where \( d\Lambda \) is the canonical Lebesgue measure defined by the lattice \( \Lambda \). In the case where \( \dim P = \dim \Lambda \), we get the normalized volume of \( P \), denoted by \( \text{nvol}(P) \).

2.1 Cones and Fans

A (polyhedral) cone is the set of all nonnegative linear combinations of a finite set of vectors. A cone is pointed if it does not contain a line. A cone \( C \) is rational if the cone \( C \) is generated by vectors with rational coordinates.

Definition 2.2 Suppose \( V \) is a subspace of \( \mathbb{R}^D \) and \( P \) is a polytope satisfying \( \text{lin}(P) \subseteq V \). Given any face \( F \) of \( P \), the normal cone of \( P \) at \( F \) with respect to \( V \) is

\[
\text{ncone}_V(F, P) := \{ u \in V^* : u(p_1) \geq u(p_2), \forall p_1 \in F, \forall p_2 \in P \}.
\]

Therefore, \( \text{ncone}_V(F, P) \) is the collection of linear functions \( u \) in \( V^* \) such that \( u \) attains maximum value at \( F \) over all points in \( P \). The normal fan \( \Sigma_V(P) \) of \( P \) with respect to \( V \) is the collection of all normal cones of \( P \).

Definition 2.3 Suppose \( W \) is a subspace of \( \mathbb{R}^D \). Let \( K \subseteq W \) be a cone. The polar cone of \( K \) with respect to \( W \) is the cone

\[
K_W^\circ = \{ y \in W^* : y(x) \leq 0, \forall x \in K \}.
\]

In the situation where \( K \) is full-dimensional in \( W \), we will omit the subscript \( W \) and the words “with respect to \( W \”).

One can check that \( K^\circ(= K^\circ_{\text{span} K}) \) is always a pointed cone. We state without proofs the following straightforward results for normal cones, which will be useful for our paper.

Lemma 2.4 Let \( F \) be a face of a polytope \( P \subseteq \mathbb{R}^D \), and suppose \( L = \text{lin}(F) \) and \( \text{lin}(P) \subseteq V \). (Recall Notation 2.1.) Then the followings are true:

(a) \( \text{ncone}_V(F, P) \) spans the orthogonal complement of \( L^* \) with respect to \( V^* \). Hence, \( \dim \text{ncone}_V(F, P) = \dim V - \dim F \).
The pointed cone \( \text{ncone}_V(F, P)^\circ \) is invariant under the choice of \( V \) (as long as \( \text{lin}(P) \subseteq V \)). So we may omit the subscript \( V \) and just write \( \text{ncone}(F, P)^\circ \). Furthermore, \( \text{ncone}(F, P)^\circ \) is full-dimensional in the orthogonal complement of \( L \) with respect to \( \text{lin}(P) \), and is of dimension \( \dim P - \dim F \).

We remark that this unique pointed cone asserted in (b) is known as the pointed feasible cone of \( P \) at \( F \), which is important in Berline–Vergne’s construction. (See Sect. 3.1.)

### 2.2 Generalized Permutohedra

We introduce generalized permutohedra, the main family of polytopes we study in this paper. In this part and any later part that is related to generalized permutohedra, we assume \( D = n + 1 \), i.e., the ambient is \( \mathbb{R}^{n+1} \). First, we present the usual permutohedron as the convex hull of a finite number of points.

**Definition 2.5** Given a point \( v = (v_1, v_2, \ldots, v_{n+1}) \in \mathbb{R}^{n+1} \), we define the usual permutohedron

\[
\text{Perm}(v) = \text{Perm}(v_1, v_2, \ldots, v_{n+1}) := \text{conv}\left( (v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n+1)}) : \sigma \in \mathfrak{S}_{n+1} \right).
\]

In particular, if \( v = (1, 2, \ldots, n + 1) \), we obtain the regular permutohedron, denoted by \( \Pi_n \),

\[
\Pi_n := \text{Perm}(1, 2, \ldots, n + 1).
\]

As long as there are two different entries in \( v \) we have \( \dim \text{Perm}(v) = n \).

The generalized permutohedra is originally introduced by Postnikov [18, Definition 6.1] as polytopes obtained from usual permutohedra by moving vertices while preserving all edge directions. In [19], Postnikov, Reiner, and Williams give several equivalent definitions, one of which uses concepts of normal fans.

**Definition 2.6** The Braid arrangement fan, denoted by \( \mathfrak{B}_n \), is the complete fan in \( \mathbb{R}^{n+1} \) given by the hyperplanes

\[
x_i - x_j = 0 \quad \text{for all } i \neq j.
\]

**Proposition 2.7** (Proposition 3.2 of [19]) A polytope \( P \) in \( V := \mathbb{R}^{n+1} \) is a generalized permutohedron if and only if its normal fan \( \Sigma_V(P) \) with respect to \( V \) is refined by the Braid arrangement fan \( \mathfrak{B}_n \).

It follows from [18, Proposition 2.6] that as long as \( v = (v_1, v_2, \ldots, v_{n+1}) \) has distinct coordinates, the associated usual permutohedron \( \text{Perm}(v) \) has the Braid arrangement \( \mathfrak{B}_n \) as its normal fan. We call \( \text{Perm}(v) \) with \( v \) of distinct coordinates a generic permutohedron. In particular, the regular permutohedron \( \Pi_n \) is a generic permutohedron.
2.3 Algebra of Polyhedra and Mixed Valuations

For a set $S \subseteq \mathbb{R}^D$, the indicator function $[S] : \mathbb{R}^D \to \mathbb{R}$ of $S$ is defined as $[S](x) = 1$ if $x \in S$, and $[S](x) = 0$ if $x \notin S$. Let $V$ be a subspace of $\mathbb{R}^D$. The algebra of polyhedra, denoted by $\mathcal{P}(V)$, is the vector space defined as the span of the indicator functions of all polyhedra in $V$. We similarly define $\mathcal{P}_b(V)$ as the algebra of polytopes, and $\mathcal{C}(V)$ as the algebra of cones.

A linear transformation $\phi : \mathcal{P}(V), \mathcal{P}_b(V), \mathcal{C}(V) \to W$, where $W$ is a vector space, is a valuation. Both volume $\text{Vol}_\Lambda(\cdot)$ and number of lattice points $\text{Lat}(\cdot)$ are valuations.

Let $\Lambda$ be a sublattice of $\mathbb{Z}^D$ and $V$ is the span of $\Lambda$. A valuation is a $\Lambda$-valuation if it is invariant under $\Lambda$-translation. We say a valuation $\phi$ is homogeneous of degree $d$ if $\phi([tP]) = t^d \phi([P])$ for any integral polytope $P$ and $t \in \mathbb{Z}_{\geq 0}$. It is clear that $\text{Vol}_\Lambda$ is homogeneous of degree $\dim \Lambda$, but $\text{Lat}$ is not homogeneous.

The following important theorem by McMullen is a special case of [15, Theorem 6].

**Theorem 2.8** Suppose $\phi$ is a homogeneous $\Lambda$-valuation on $\mathcal{P}_b(V)$ of degree $d$. Then there exists a function $\mathcal{M}$ which takes $d$ integral polytopes as inputs such that

$$
\phi(t_1 P_1 + t_2 P_2 + \cdots + t_k P_k) = \sum_{j_1, \ldots, j_d \in [k]} \mathcal{M}(P_1, P_2, \ldots, P_d) t_{j_1} \cdots t_{j_d},
$$

(2.1)

for any $k \in \mathbb{Z}_{\geq 0}$, any integral polytopes $P_1, \ldots, P_k \subset V$ and $t_1, \ldots, t_k \in \mathbb{Z}_{\geq 0}$.

The following definition and lemma are stated in [12, Sect. 3 of Chapter IV] for the volume valuation (which is a homogeneous valuation). We give the general forms here.

**Definition 2.9** Let $\phi$ and $\mathcal{M}$ be as in Theorem 2.8. We define another function $\mathcal{M}_\phi$ that takes $d$ integral polytopes as inputs as an average of the function $\mathcal{M}$:

$$
\mathcal{M}_\phi(P_1, \ldots, P_d) := \frac{1}{d!} \sum_{\sigma \in S_d} \mathcal{M}(P_{\sigma(1)}, \ldots, P_{\sigma(d)}).
$$

It is easy to see that $\mathcal{M}_\phi$ is uniquely chosen for each $\phi$, and (2.1) still holds for $\mathcal{M}_\phi$:

$$
\phi(t_1 P_1 + t_2 P_2 + \cdots + t_k P_k) = \sum_{j_1, \ldots, j_d \in [k]} \mathcal{M}_\phi(P_1, P_2, \ldots, P_d) t_{j_1} \cdots t_{j_d},
$$

(2.2)

We call $\mathcal{M}_\phi$ the mixed valuation of $\phi$.

The lemma below gives two properties of the mixed valuation $\mathcal{M}_\phi$. We omit the proof which is very similar to that is given in [12, Sect. 3 of Chapter IV] for mixed volume valuation.

**Lemma 2.10** (i) For any integral polytopes $P_1, \ldots, P_d$, and any permutation $\sigma \in S_d$, we have $\mathcal{M}_\phi(P_1, \ldots, P_d) = \mathcal{M}_\phi(P_{\sigma(1)}, \ldots, P_{\sigma(d)})$. 

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(ii) The function $M\phi$ is a multi-linear function, that is, it is linear in each component.

Apply the above results to volume valuation, a homogeneous valuation, we obtain the following:

**Theorem 2.11** (Theorem 3.2 of [12]) Suppose $P_1, \ldots, P_k$ are integral polytopes with $\dim(P_1 + \cdots + P_k) = d$. Let $\Lambda$ be the $d$-dimensional lattice span($P_1 + \cdots + P_k$) $\cap \mathbb{Z}^D$. Then

$$\text{Vol}_\Lambda(t_1 P_1 + t_2 P_2 + \cdots + t_k P_k) = \sum_{j_1, \ldots, j_d \in [k]} M\text{Vol}_\Lambda(P_{j_1}, P_{j_2}, \ldots, P_{j_d}) t_{j_1} \cdots t_{j_d},$$

where the sum is carried out independently over the $j_i$. The function $M\text{Vol}_\Lambda(P_{j_1}, P_{j_2}, \ldots, P_{j_d})$ is called the mixed volume of $P_{j_1}, P_{j_2}, \ldots, P_{j_d}$.

Furthermore we have the following properties:

**Theorem 2.12** (Theorem 4.13 of [12]) Let $P_1, \ldots, P_d$ be integral polytopes. Then,

1. $M\text{Vol}_\Lambda(P_1, \ldots, P_d) \geq 0$.
2. $M\text{Vol}_\Lambda(P_1, \ldots, P_d) > 0$ if and only if each $P_i$ contains a line segment $I_i = [a_i, b_i]$ such that $b_1 - a_1, \ldots, b_d - a_d$ are linearly independent.

The lattice point, or counting, valuation $\text{Lat}$ is not homogeneous. However it can be decomposed into homogeneous parts.

**Theorem 2.13** (Theorem 5 of [15]) Suppose $V$ is $d$-dimensional. Then we can decompose the valuation $\text{Lat}$ as

$$\text{Lat} = \text{Lat}^d + \cdots + \text{Lat}^1 + \text{Lat}^0,$$

where $\text{Lat}^r$ is homogeneous of degree $r$.

This decomposition corresponds to the coefficients of the Ehrhart polynomial, in particular $\text{Lat}^d$ corresponds to the volume valuation $\text{Vol}_\Lambda$, where $\Lambda = V \cap \mathbb{Z}^D$. Applying Theorem 2.8 and Lemma 2.10 to each homogeneous function $\text{Lat}^r$ gives us the following result:

**Theorem 2.14** Suppose $P_1, \ldots, P_k$ are integral polytopes with $\dim(P_1 + \cdots + P_k) = d$. Then

$$\text{Lat}(t_1 P_1 + t_2 P_2 + \cdots + t_k P_k) = \sum_{e=0}^d \sum_{j_1, \ldots, j_e=1}^k M\text{Lat}^e(P_{j_1}, P_{j_2}, \ldots, P_{j_e}) t_{j_1} \cdots t_{j_e}.$$

We cannot expect $M\text{Lat}^r$ or any other mixed valuation $M\phi$ to be nonnegative in general. However we have a way to compute them.

**Theorem 2.15** Suppose $\phi$ is a homogeneous $\Lambda$-valuation on $\mathcal{P}_b(V)$ of degree $d$. For any integral polytopes $P_1, P_2, \ldots, P_d \subset V$, we have

$$d! M\phi(P_1, P_2, \ldots, P_d) = \sum_{J \subseteq [d]} (-1)^{d-|J|} \phi \left( \sum_{j \in J} P_j \right).$$
We define two functions $f$ and $g$ on the Boolean algebra of order $d$. (See [21, Chapter 3].) For any subset $T \subseteq [d]$, let

$$f(T) := \phi \left( \sum_{i \in T} P_i \right) \quad \text{and} \quad g(T) := \sum_{j_1, \ldots, j_d \in [d]} M \phi(P_{j_1}, P_{j_2}, \ldots, P_{j_d}).$$

Apply (2.2) with $t_i = 1$ to $f(T)$, one sees that $f(T) = \sum_{S \subseteq T} g(S)$. Therefore, by Mobius inversion, we get

$$g(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} f(S).$$

Then the theorem follows from evaluating the above equality at $T = [d]$.

\section{McMullen’s Formula and the BV-$\alpha$-Valuation}

Recall that in the introduction we discuss the existence of the following McMullen’s formula:

$$\text{Lat}(P) = \sum_{F: \text{a face of } P} \alpha(F, P) \text{ nvol}(F), \quad (3.1)$$

where $\alpha(F, P)$ depends only on the normal cone of $P$ at $F$.

One immediate consequence of the existence of McMullen’s formula (3.1) is that it provides another way to prove Ehrhart’s theorem. Moreover, it gives a description of each Ehrhart coefficient. We state the following modified version of Theorem 1.1.

**Theorem 3.1** For an integral polytope $P \subset \mathbb{Z}^D$ and any $t \in \mathbb{Z}_{\geq 0}$, the function $i(P, t) = \text{Lat}(tP) = |tP \cap \mathbb{Z}^n|$ is a polynomial in $t$ of degree $\dim P$. Furthermore, the coefficient of $t^k$ in $i(P, t)$ is given by

$$\sum_{F: \text{a } k\text{-dim face of } P} \alpha(F, P) \text{ nvol}(F). \quad (3.2)$$

**Proof** The desired formula follows from applying McMullen’s formula to $tP$ and observing that $\alpha(tF, tP) = \alpha(F, P)$.

Formula (3.2) for the coefficients of the Ehrhart polynomial $i(P, t)$ gives a sufficient condition for Ehrhart positivity.

**Lemma 3.2** Let $P$ be an integral polytope. For a fixed $k$, if $\alpha(F, P)$ is positive for any $k$-dimensional face of $P$, then the coefficient of $t^k$ in the Ehrhart polynomial $i(P, t)$ of $P$ is positive.

Hence, if $\alpha(F, P) > 0$ for every face $F$ of $P$, then $P$ is Ehrhart positive.

As discussed in the introduction, different constructions of $\alpha(F, P)$ were given in the literature. In our paper, we will use Berline–Vergne’s construction, which we refer to as the BV-$\alpha$-valuation.
3.1 Berline–Vergne’s Construction

For any rational quotient subspace \( W \) of \( \mathbb{R}^D \), Berline and Vergne construct in [5] a function \( \Psi_W([C]) \) on indicator functions of rational cones \( C \) in \( W \) with the following properties:

(P1) \( \Psi_W(\cdot) \) is a valuation on the algebra of rational cones in \( W \).
(P2) McMullen’s formula (3.1) holds for integral polytopes in \( \mathbb{R}^D \) if we set

\[
\alpha(F, P) := \Psi_{\mathbb{R}^D/\text{lin}(F)}([\text{ncone}(F, P)^C]).
\]

(3.3)

(P3) If a cone \( C \) contains a line, then \( \Psi_W([C]) = 0 \).
(P4) Its value on the zero-dimensional cone \( \emptyset \) is 1, i.e. \( \Psi_W([\emptyset]) = 1 \).
(P5) \( \Psi_W \) is invariant under the action of \( O_D(\mathbb{Z}) \), the group of orthogonal unimodular transformations. More precisely, if \( T \) is an orthogonal unimodular transformation, for any cone \( C \), we have \( \Psi_W([C]) = \Psi_W([T(C)]) \).
(P6) Let \( L \) be the orthogonal complement of \( \text{lin}(C) \). Then \( \Psi_W([C]) = \Psi_{\mathbb{R}^D/L}([C]) \).

This indicates that we may omit the subscript \( W \) in \( \Psi_W \).

It is important to remark that their main construction is actually a valuation with values on certain classes of holomorphic functions, and the function \( \Psi_W \) we described above comes from the (non-trivial) evaluation at zero. For more details we refer the reader to the original paper [6] and to the exposition in [3, Chapters 19–20].

3.2 Reduction Theorem

We have already discussed a consequence of the existence of McMullen’s formula, which reduces the problem of proving Ehrhart positivity to proving \( \alpha \)-positivity. Now we will discuss a very important consequence of the BV-\( \alpha \)-valuation – the reduction theorem – applying which we complete the proof of Theorem 1.4.

For the rest of the section, we assume \( \alpha(F, P) \) comes from the BV-\( \alpha \)-valuation, and take (3.3) as the definition of \( \alpha(F, P) \). We start with the following preliminary lemma.

**Lemma 3.3** Suppose \( V \) is a subspace of \( \mathbb{R}^D \), and \( P \) and \( Q \) are two integral polytopes in \( V \). Let \( F \) be a face of \( P \). Suppose there exist faces \( G_1, G_2, \ldots, G_r \) of \( Q \) of the same dimension such that

\[
\text{ncone}_V(F, P) = \bigcup_{i=1}^r \text{ncone}_V(G_i, Q).
\]

(3.4)

Then \( F \) is of the same dimension as \( G_i \)'s, and \( \alpha(F, P) = \sum_{i=1}^r \alpha(G_i, Q) \).
Proof The first consequence of (3.4) is that ncone\(_{V}(F, P)\) and ncone\(_{V}(G_i, Q)\)’s all span the same subspace, say \(W^*\). Let \(L\) be the orthogonal complement of \(W\) with respect to \(V\). Then by Lemma 2.4 (a),

\[
\text{lin}(F) = L = \text{lin}(G_i) \quad \text{for each } i.
\]

Hence, \(F\) has the same dimension as \(G_i\)’s. Next, since ncone\(_{V}(G_i, Q)\) \(\cap\) ncone\(_{V}(G_j, Q)\) is a lower dimensional cone for any \(i \neq j\), we have

\[
[ncone\(_{V}(F, P)] = \sum_{i=1}^{r} [ncone\(_{V}(G_i, Q)]
\]

modulo polyhedra contained in proper subspaces of \(W\).

Taking the polar of the above identity with respect to \(W\) and applying Lemma 2.4 (b) yields

\[
[ncone(F, P)]^o = \sum_{i=1}^{r} [ncone(G_i, Q)]^o \quad \text{modulo polyhedra with lines.}
\]

Then our desired identity follows from Properties (P1) and (P3) of the Berline–Vergne construction. \(\Box\)

Our reduction theorem is a clear consequence of the above lemma.

**Theorem 3.4** (Reduction Theorem) Suppose \(V\) is a subspace of \(\mathbb{R}^D\), and lin\((P)\) and lin\((Q)\) are both subspaces of \(V\). Assume further the normal fan \(\Sigma_V(P)\) of \(P\) with respect to \(V\) is a refinement of the normal fan \(\Sigma_V(Q)\) of \(Q\) with respect to \(V\). Then for any fixed \(k\), if \(\alpha(F, P) > 0\) for every \(k\)-dimensional face \(F\) of \(P\), then \(\alpha(G, Q) > 0\) for every \(k\)-dimensional face \(G\) of \(Q\). Therefore, BV-\(\alpha\)-positivity of \(P\) implies BV-\(\alpha\)-positivity of \(Q\).

Theorem 3.4 and Proposition 2.7 immediately give the following result and complete the proof for Theorem 1.4

**Theorem 3.5** (Reduction Theorem, special form) Let \(Q \subset \mathbb{R}^{n+1}\) be a generalized permutohedron. Then for any fixed \(k\), if \(\alpha(F, \Pi_n) > 0\) for every \(k\)-dimensional face \(F\) of \(\Pi_n\), then \(\alpha(G, Q) > 0\) for every \(k\)-dimensional face \(G\) of \(Q\). Therefore, BV-\(\alpha\)-positivity of \(\Pi_n\) implies BV-\(\alpha\)-positivity of \(Q\).

**Proof of Theorem 1.4** The theorem follows from Theorem 3.5 and Lemma 3.2. \(\Box\)

**Remark 3.6** All permutohedra of dimension at most \(n\) can be realized in \(\mathbb{R}^{n+1}\), and Proposition 2.7 applies to all of these permutohedra. Therefore, the polytope \(Q\) in Theorem 3.5 could be any permutohedron of dimension up to \(n\).

**Remark 3.7** Theorem 3.5 still holds if we replace \(\Pi_n\) with any generic permutohedron, that is, any Perm\((v)\) where \(v \in \mathbb{R}^{n+1}\) is a vector with distinct coordinates.
3.3 Examples of Computing $\Psi_W$

Let $W = \mathbb{R}^D/L$ be a rational quotient space of $\mathbb{R}^D$. We might identify $W$ with $L^\perp$, the orthogonal complement of $L$ with respect to $\mathbb{R}^D$. We consider

$$\Lambda(W) := \text{the orthogonal projection of } \mathbb{Z}^D \text{ to } L^\perp$$

to be the lattice of $W$, which is important in computing $\Psi_W$ associated to Berline–Vergne’s construction.

The computation of the function $\Psi_W$ is carried out recursively. Hence, it is quicker to compute $\Psi_W$ for lower dimensional cones. Since the dimension of $(\text{cone}(F, P))^\circ$ is equal to the codimension of $F$ with respect to $P$, the value of $\alpha(F, P)$ is easier to compute if $F$ is a higher dimensional face. The following easy results follow from comments in [3, Example 20.2].

**Lemma 3.8** Suppose $\alpha$ is the BV-$\alpha$-valuation. Then for any integral polytope $P$ and any facet $F$ of $P$, we have $\alpha(P, P) = 1$ and $\alpha(F, P) = 1/2$.

In general, the computation of $\Psi_W([C])$ is quite complicated. However, when $C$ is a unimodular cone with respect to the lattice $\Lambda(W)$, that is, $C$ is generated by a set of rays that can be extended to a basis of $\Lambda(W)$, computations are greatly simplified. In small dimensions we can even give a simple closed expression for $\Psi_W$ of unimodular cones. The following result is given in [3, Example 19.3].

**Lemma 3.9** Suppose $C = \text{Cone}(u_1, u_2) \subset W$, where $\{u_1, u_2\}$ can be extended to a basis of the lattice $\Lambda(W)$. Then

$$\Psi_W([C]) = \frac{1}{4} + \frac{1}{12} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} \right).$$

With the help of Maple code provided by Berline and Vergne, one can obtain a formula for computing $\Psi_W$ of a 3-dimensional unimodular cone.

**Lemma 3.10** Suppose $C = \text{Cone}(u_1, u_2, u_3) \subset W$, where $\{u_1, u_2, u_3\}$ can be extended to a basis of the lattice $\Lambda(W)$. Then

$$\Psi_W([C]) = \frac{1}{8} + \frac{1}{24} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_3, u_3 \rangle} + \frac{\langle u_3, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_3, u_2 \rangle}{\langle u_3, u_3 \rangle} \right).$$

**Remark 3.11** The formulas for 2-dimensional and 3-dimensional unimodular cones appear to be simple. However, the apparent simplicity breaks down for dimension 4. The formula for 4-dimensional unimodular cones include (way) more than 1000 terms.
### 3.4 Symmetry Property

Theorem 3.4 mainly follows from Properties (P1) and (P3) of the BV-\(\alpha\)-valuation. In this subsection, we will focus on Property (P5) (as well as Property (P6)), showing reasons why this particular construction of \(\alpha\) is convenient.

**Lemma 3.12** The valuation \(\Psi_W\) is symmetric about the coordinates, i.e., for any cone \(C \in W\) and any permutation \(\sigma \in \mathfrak{S}_D\), we have

\[
\Psi_W([C]) = \Psi_{\sigma(W)}([\sigma(C)]),
\]

where \(\sigma(T) = \{(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(D)}) : (x_1, \ldots, x_D) \in T\}\) for any set \(T \subseteq \mathbb{R}^D\).

**Proof** Let \(M_\sigma\) be the permutation matrix corresponding to \(\sigma\). Then the lemma follows from the observation that \(T\) is mapped to \(\sigma(T)\) under the linear transformation \(M_\sigma\) and any permutation matrix is orthonormal and unimodular.

The above result motivates the following definition.

**Definition 3.13** Suppose \(\alpha\) is a construction such that McMullen’s formula (3.1) holds. We say it is symmetric about the coordinates if \(\alpha(F, P) = \alpha(G, Q)\) whenever \((\text{ncone}(F, P))^\circ = \sigma((\text{ncone}(G, Q))^\circ)\) for some \(\sigma \in \mathfrak{S}_D\).

Therefore, we have the following:

**Lemma 3.14** The BV-\(\alpha\)-valuation is symmetric about the coordinates.

**Proof** Suppose \(\Psi_W\) is the construction given by Berline–Vergne described in Sect. 3.1. Let \(C = (\text{ncone}(F, P))^\circ\) and \(C' = (\text{ncone}(G, Q))^\circ\). Then by Property (P5), one sees that it is enough to show that

\[
\Psi_{\mathbb{R}^D/\text{lin}(F)}([C]) = \Psi_{\mathbb{R}^D/\sigma(\text{lin}(G))}([C]).
\]

which follows from Property (P6).

Below we give an example of how to use Lemma 3.14 to compute BV-\(\alpha\)-values directly without using Berline–Vergne’s recursive computation for \(\Psi_W\) (which could be quite complicated as we have seen in Sect. 3.3).

**Example 3.15** Consider the regular permutohedron \(\Pi_3\). (See the polytope on the left below.) The group \(O_4(\mathbb{Z})\) acts on faces of it. In particular, its edges split into two orbits:

1. One orbit consists of edges between a square and an hexagon. There are 24 of these, and we denote their common \(\alpha\)-value by \(\alpha_1\).
2. The other orbit consists of edges between the hexagons. There are 12 of these, and we denote their common \(\alpha\)-value by \(\alpha_2\).
Note that all edges have normalized volume 1. By equation (3.2), in addition to the knowledge that the linear coefficient of $i(\Pi_3, t)$ is 6, we get an equation on $\alpha_1$ and $\alpha_2$:

$$24\alpha_1 + 12\alpha_2 = 6.$$ 

We need one more equation in order to find values of $\alpha_i$’s, so we look at a deformation of $\Pi_3$. We can push away the square faces to get the polytope $\text{Perm}(1, 1, 4, 4)$, which after translating by $-(1, 1, 1, 1)$ and shriek by a factor of 3 becomes the hypersimplex $\Delta_{2,4}$ or $\text{Perm}(0, 0, 1, 1)$. This is a regular octahedron shown on the right side of the figure above. Notice that all the edges in this new polytope have the same normal cones as the edges in the second orbit above. Hence, they must have the same $\alpha$-value, which we have denoted $\alpha_2$. By (3.2), in addition to the knowledge that the linear coefficient of $i(\Delta_{2,4}, t)$ is $7/3$, we get our second equation

$$12\alpha_2 = \frac{7}{3}.$$ 

Solving the two linear equations above, we obtain $\alpha_1 = 11/72$ and $\alpha_2 = 14/72$.

Property (P5), in particular Lemma 3.14, does not hold for all the solutions for McMullen’s formula. For example, the construction given by Pommersheim and Thomas in [17] depends on an ordering of a basis for the vector space, which means their construction is not symmetric about coordinates. This is the main reason why we work with the BV-$\alpha$-valuations for this paper.

### 4 Generic Permutohedron

Since the proof of Theorem 1.4 was completed in the last section, we will focus on the BV-$\alpha$-valuation arising from the regular permutohedron $\Pi_n$, or any generic permutohedron. (See Remark 3.7.) We use the following setup.

**Setup 4.1**

(i) Let $v = (v_1, v_2, \ldots, v_{n+1})$ be a vector with strictly increasing entries, and consider the generic permutohedron $\text{Perm}(v) = \text{Perm}(v_1, v_2, \ldots, v_{n+1})$.

(ii) Suppose $\alpha$ is a construction such that McMullen’s formula (3.1) holds and it is symmetric about the coordinates (see Definition 3.13).

It is clear that (i) covers all generic permutohedron, and the BV-$\alpha$-valuation is a special case of (ii). Under this setup, we will analyze (4.1) below for computing $\text{Lat}(\text{Perm}(v))$ further, and derive a more combinatorial formula for computing $\alpha$-values arising from
Perm(\(v\)). Applying McMullen’s formula to \(P = \text{Perm}(v)\), we get

\[
\text{Lat}(\text{Perm}(v)) = \sum_{F: \text{a face of } \text{Perm}(v)} \alpha(F, \text{Perm}(v)) \cdot n\text{vol}(F).
\] (4.1)

Because of the symmetric properties of \(\text{Perm}(v)\) and \(\alpha\), a lot of terms in the above summand coincide. Hence, it is natural to group them together as in Example 3.15. In order to do this, we need the following definition and proposition.

**Definition 4.2** The symmetric group \(\mathbb{S}_{n+1}\) acts linearly on \(\mathbb{R}^{n+1}\) by permuting the coordinates. Two subsets \(A_1, A_2 \subset \mathbb{R}^{n+1}\) are said to be symmetric if they lie in the same orbit, i.e. if there exist \(\sigma \in \mathbb{S}_{n+1}\) such that \(\sigma(A_1) = A_2\). Since the action is orthogonal, two symmetric sets are congruent, in particular, they have the same volume (if measurable).

The following results are well-known. (See for example [2, Proposition 2.2 of Chapter VI].)

**Proposition 4.3** There is a one-to-one correspondence between ordered set partitions of \([n+1]\) and faces of \(\text{Perm}(v)\) defined as follows:

For any ordered set partition \(\mathcal{P} = (P_1, P_2, \ldots, P_l)\) of \([n+1]\), the corresponding face is obtained by maximizing any linear functional given by a vector \(c \in \mathbb{R}^{n+1}\) with the property that

(a) \(c_i = c_j\) if \(i\) and \(j\) are both in \(P_k\) for some \(k\), and

(b) \(c_i < c_j\) if \(i \in P_{k_1}\) and \(j \in P_{k_2}\) with \(k_1 < k_2\).

Let \(m_i = |P_i|\). Then the corresponding face has dimension \(n+1-l\) and it is congruent to

\[
\text{Perm}(v_{M_1}) \times \text{Perm}(v_{M_2}) \times \cdots \times \text{Perm}(v_{M_l}),
\]

where \(v_{M_i} = (v_j : \sum_{k=1}^{i-1} m_k < j \leq \sum_{k=1}^{i} m_k)\). In other words, \(v_{M_i}\) consists of the first \(m_i\) entries of \(v = (v_1, \ldots, v_{n+1})\), \(v_{M_2}\) consists of the next \(m_2\) entries, and so on.

We call the ordered tuple \(\mathbf{m} := (m_1, m_2, \ldots, m_l)\) the composition of \(\mathcal{P}\).

**Example 4.4** Let \(n = 5\) and consider the ordered set partition \(\mathcal{P} = ((1, 4, 6), (2, 5), (3))\). Then the composition of \(\mathcal{P}\) is \((3, 2, 1)\). The face of \(\text{Perm}(v)\) corresponding to \(\mathcal{P}\) is the face which optimizes any linear functional \(c = (c_1, c_2, c_3, c_4, c_5, c_6)\) with \(c_1 = c_4 = c_6 < c_2 = c_5 < c_3\). In order to figure out this corresponding face, we look for vertices of \(\text{Perm}(v)\) optimizing such a functional \(c\). One sees that \(v_1, v_2\) and \(v_3\) should be in positions 1, 4 and 6 of these vertices, \(v_4\) and \(v_5\) in positions 2 and 5, and \(v_6\) in position 3. Therefore, the desired vertices are

\[
\left\{ (v_{\mu(1)}, v_{\tau(1)}, v_6, v_{\mu(2)}, v_{\tau(2)}, v_{\mu(3)}) : \mu \in \mathbb{S}_{\{1,2,3\}}, \tau \in \mathbb{S}_{\{4,5\}} \right\}.
\]

Hence, we conclude that the face that is corresponding to the ordered set partition \(\mathcal{P} = ((1, 4, 6), (2, 5), (3))\) is congruent to \(\text{Perm}(v_1, v_2, v_3) \times \text{Perm}(v_4, v_5) \times \text{Perm}(v_6)\).
By Proposition 4.3, two faces of Perm($v$) are in the same orbit, i.e. they are symmetric, if and only if their corresponding ordered set partitions have the same composition. Therefore, the orbits of the $S_{n+1}$-action on the faces of Perm($v$) are indexed by compositions $m$ of $n + 1$. We denote the orbit corresponding to the composition $m$ by $O_n(m)$.

Furthermore, under Setup 4.1, the $\alpha$-construction is symmetric about the coordinates. Hence, for any fixed $m$, the value $\alpha(F, \text{Perm}(v))$ is a constant on $O_n(m)$, and thus we can define $\alpha_n(m)$ to be this constant.

Finally, a canonical representative of $O_n(m)$ is chosen as below.

**Definition 4.5** Let $m = (m_1, m_2, \ldots, m_l)$ be a composition of $n + 1$. Define an ordered set partition $P(m) = (P(m)_i)$ where

$$P(m)_i = \left[ \sum_{k=1}^{i-1} m_k + 1, \sum_{k=1}^{i} m_k \right].$$

In other words, the first subset $P(m)_1$ consists of the first $m_1$ positive integers, the second subset $P(m)_2$ consists of the next $m_2$ positive integers, and so on.

Then we define $F_m$ to be the face corresponding to the ordered set partition $P(m)$ under the bijection given in Proposition 4.3.

**Example 4.6** Let $n = 5$ and $m = (3, 2, 1)$. Then $P(m) = ([1, 2, 3], [4, 5], [6])$, and

$$F_m = \text{conv}\{ (v_{\mu(1)}, v_{\mu(2)}, v_{\mu(3)}, v_{\tau(1)}, v_{\tau(2)}, v_6) : \mu \in S_{\{1, 2, 3\}}, \tau \in S_{\{4, 5\}} \}. \quad (4.2)$$

Applying the above discussions to (4.1), we get

$$\text{Lat}(\text{Perm}(v)) = \sum_{m: \text{composition of } n+1} |O_n(m)| \cdot \alpha_n(m) \cdot n\text{vol}(F_m). \quad (4.3)$$

Note that one of the terms in the above formula can be explicitly described: For a fixed $m = (m_1, \ldots, m_l)$, the number of faces in $O_n(m)$ is equal to the number of ordered set partitions whose compositions are $m$. Thus,

$$|O_n(m)| = \binom{n + 1}{m_1, m_2, \ldots, m_l}. \quad (4.4)$$

It is easy to see that $F_m$ is always adjacent to the vertex $v = (v_1, \ldots, v_n, v_{n+1})$. In fact, we show below that every face adjacent to $v$ arises as $F_m$ for a unique $m$. Note that the vertex cone of $\text{Perm}(v)$ at $v$ is spanned by the following $n$ vectors:

$$e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}.$$

Hence, subsets of these $n$ vectors are in one-to-one correspondence to faces of $\text{Perm}(v)$ that are adjacent to $v$. 

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Lemma 4.7 Let \( m \) be a composition of \( n + 1 \), and \( F_m \) the face of \( \text{Perm}(v) \) that is associated to \( m \). Define \( S = S(m) \) to be the subset of \([n]\) such that

\[
\text{aff}(F_m) = v + \text{span}\{e_i - e_{i+1} : i \in S\}.
\]

The map \( S \) is a bijection from compositions of \( n + 1 \) to subsets of \([n]\). Hence, \( \{F_m : m \text{ is a composition of } n + 1\} \) consists of all faces of \( \text{Perm}(v) \) that are adjacent to \( v \).

Example 4.8 Let \( n = 5 \) and \( m = (3, 2, 1) \). Then \( F_m \) is given by (4.2). One checks that

\[
\text{aff}(F_m) = v + \text{span}\{e_1 - e_2, e_2 - e_3, e_4 - e_5\}.
\]

Hence, \( S = S(m) = \{1, 2, 4\} \).

Proof of Lemma 4.7 We can define an inverse to \( S \) in the following way: Suppose \( S \) is a subset of \([n]\). Let \( T = [n + 1] \setminus S \). Suppose \( T = \{t_1 < t_2 < \cdots < t_l\} \). Then one verifies that

\[
S \mapsto m := (t_1, t_2 - t_1, \ldots, t_l - t_{l-1})
\]
is an inverse to \( S \), completing the proof.

By the above lemma, we may abuse notation and use subsets of \([n]\) to index formula (4.3):

\[
\text{Lat}(\text{Perm}(v)) = \sum_{S \subseteq [n]} |\mathcal{O}_n(S)| \cdot \alpha_n(S) \cdot n\text{vol}(F_S).
\]

(4.5)

Note that \( \dim(F_S) = |S| \).

Partial Results on Conjectures

We will explore consequences of formula (4.5) further in Sect. 5, and will devote the rest of this section to proving Theorem 1.5, providing partial results on our main conjectures. Clearly, in order to prove Theorem 1.5, we just need to verify the following two statements respectively, assuming \( \alpha \) is the BV-\( \alpha \)-valuation:

\[
\alpha_n(S) > 0, \quad \forall S \subseteq [n], \quad \forall n \leq 6,
\]

(4.6)

\[
\alpha_n(S) > 0, \quad \forall S \subset [n], \quad |S| = n - 2, n - 3.
\]

(4.7)

Applying formula (3.3) to our situation, we get

\[
\alpha_n(S) = \Psi_{\mathbb{R}^{n+1}/\text{lin}(F_S)}((\text{ncone}(F_S, \Pi_n)))\).
\]

(4.8)

Recall we discussed briefly how to compute Berline–Vergne’s \( \Psi_W \) in Sect. 3.3. In particular, we discussed that \( \Psi_W([C]) \) is relatively easier to compute if \( C \) is a unimodular cone with respect to the lattice \( \Lambda(W) \). The following result on \( \text{ncone}(F_S, \Pi_n)\) shows that the computation of \( \alpha_n(S) \) fits into this situation.
Lemma 4.9 Let $S \subseteq [n]$ and $V = \mathbb{R}^{n+1}$. Then
\[ \text{ncone}_V(F_S, \Pi_n) = \{ c = (c_1, \ldots, c_{n+1}) \in V^* : c_i = c_{i+1} \forall i \in S \text{ and } c_i \leq c_{i+1} \forall i \notin S \}. \]

Furthermore, if $[n] \setminus S = \{ i_1 < i_2 < \cdots < i_\ell \}$ and for each $1 \leq j \leq \ell$, we define a vector
\[
R_{ij} = \left( \frac{1}{i_1 - i_{j+1}}, \ldots, \frac{1}{i_j - i_{j+1}}, \frac{1}{i_1 - i_{j-1}}, \ldots, \frac{1}{i_j - i_{j-1}}, 0, \ldots, 0 \right),
\]
where by convention $i_0 = 0$ and $i_{\ell+1} = n + 1$, then $\text{ncone}(F_S, \Pi_n)^{\circ}$ is spanned by $R_{ij}$. Moreover, $\{ R_{ij} \}$ can be extended to a basis of the lattice $\Lambda(\mathbb{R}^{n+1}/\text{lin}(F_S))$.

Proof of Lemma 4.7 The formula for $\text{ncone}_V(F_S, \Pi_n)$ follows from Proposition 4.3 and Lemma 4.7. The second conclusion then follows from a direct calculation.

By Lemma 4.7, $\text{lin}(F_S) = \text{span}(e_i - e_{i+1} : i \in S)$. Note that $\{ e_1 \} \cup \{ e_i - e_{i+1} : i \in [n] \}$ is a basis for $\mathbb{Z}^{n+1}$. Hence, the orthogonal projections of $\{ e_1 \} \cup \{ e_j - e_{j+1} : 1 \leq j \leq \ell \}$ onto $\text{lin}(F_S)^{\perp}$ is a basis for $\Lambda(\mathbb{R}^{n+1}/\text{lin}(F_S))$. One checks for each $j$, the orthogonal projection of $e_{i_j} - e_{i_j+1}$ is $R_{ij}$. Thus, the last assertion follows.

Applying Lemmas 3.9 and 3.10 to the above lemma with $\ell = 2$ and 3, we obtain precise formulas for $\alpha_n(S)$ when $|S| = n - 2$ or $n - 3$.

Corollary 4.10 Suppose $S \subseteq [n]$.

(i) If $[n] \setminus S = \{ i, j \}$ with $i < j$, then $\alpha_n(S) = \frac{1}{4} - \frac{1}{12} \left( i + \frac{n + 1 - j}{n + 1 - i} \right)$.

(ii) If $[n] \setminus S = \{ i, j, k \}$ with $i < j < k$, then $\alpha_n(S) = \frac{1}{8} - \frac{1}{24} \left( i + \frac{n + 1 - k}{n + 1 - j} \right)$.

Proof of Theorem 1.5 We are able to verify that (4.6) is true by directly applying Berline–Vergne’s construction $\Psi_W$ and using (4.8) and Lemma 4.9. (We omit all the data to save space. For interested readers, please see examples in [8, Sect. 5.1].) So part (1) of the theorem follows.

Next, it is easy to check that $\alpha_n(S)$ are positive in both formulas provided in Corollary 4.10. Hence, (4.7) and part (2) of the theorem follow.

Remark 4.11 By Remark 3.6, if $\alpha_6(S)$ is positive for all $S \subseteq [6]$, then we immediately have $\alpha_n(S) > 0$ for all $n < 6$ and any $S \subseteq [n]$. Hence, the proof of Theorem 1.5 (1) can be reduced to proving $\alpha_6(S) > 0$ only.

Similarly, it is not necessary to show $\alpha_n(S)$ for both $|S| = n - 2$ and $|S| = n - 3$ to complete a proof of Theorem 1.5 (2). In fact, it follows from Remark 3.6 that if $\alpha_n(S) > 0$ for all $n$ and $S$ of size $n - 3$, then $\alpha_n(S) > 0$ for all $n$ and $S$ of size $n - 2$.

5 Uniqueness

In this section, we take a different point of view and investigate the uniqueness of the $\Psi_W/\alpha$ constructions for McMullen’s formula. We will apply the mixed valuation theories introduced in Sect. 2.3 to Minkowski sums of hypersimplices.
**Definition 5.1** The **hypersimplex** $\Delta_{k,n+1}$ is defined as

$$
\Delta_{k,n+1} = \text{Perm}(0, \ldots, 0, 1, \ldots, 1).
$$

The main goal of this part is to prove Theorem 1.6, which will be rephrased below as Theorem 5.5 stating that the $\alpha$-values of faces of $\text{Perm}(v)$ are uniquely determined as a scalar of mixed valuation of hypersimplices if we require $\alpha$ and $v$ to be given under Setup 4.1. Furthermore, as a consequence of this result, we give an equivalent statement of Conjecture 1.3 in Corollary 5.6.

As in Setup 4.1, we consider the generalized permutohedron $\text{Perm}(v) = \text{Perm}(v_1, v_2, \ldots, v_n, v_{n+1})$ with $v_1 < v_2 < \cdots < v_n < v_{n+1}$. We have the following expression for $\text{Perm}(v)$ as Minkowski sum [18, Sect. 16]:

$$
\text{Perm}(v) = w_1 \Delta_{1,n+1} + w_2 \Delta_{2,n+1} + \cdots + w_n \Delta_{n,n+1},
$$

where

$$
w_i := v_{i+1} - v_i \quad \text{for} \quad i = 1, 2, \ldots, n.
$$

(5.1)

(The $w_i$’s are actually lengths of edges of $\text{Perm}(v)$. But this is not relevant to our discussion.)

Using the results on mixed volumes – Theorems 2.11 and 2.12 – we have the following:

**Lemma 5.2** The normalized volume of $\text{Perm}(v)$ is a homogeneous polynomial in $w_i$’s with strictly positive coefficients.

In [18], the coefficients of the above homogeneous polynomial are called **mixed Eulerian numbers**, and some basic properties are established. One of the properties is the following:

**Lemma 5.3** The coefficient of $w_1 w_2 \cdots w_n$, the unique squarefree monomial, in the homogeneous polynomial assumed in Lemma 5.2 is $n!$.

Note that Lemma 5.3 says that $\sum_{\sigma \in \mathfrak{S}_n} \mathcal{M}\text{Lat}^n(\Delta_{\sigma(1),n+1}, \Delta_{\sigma(2),n+1}, \ldots, \Delta_{\sigma(n),n+1}) = n!$, which by Lemma 2.10(i) is equivalent to

$$
\mathcal{M}\text{Lat}^n(\Delta_{1,n+1}, \Delta_{2,n+1}, \ldots, \Delta_{n,n+1}) = 1.
$$

(5.2)

Recall in Sect. 4, we associate a face $F_m$ of $\text{Perm}(v)$ to any composition $m$ of $n + 1$, establish a bijection $S$ from $m$ to subsets $S$ of $[n]$, and rewrite $F_m$ as $F_S$. We have the following result on the normalized volume of $F_S$.

**Proposition 5.4** Suppose $P = \text{Perm}(v)$ and $S \subseteq [n]$. Let $F_S$ be the corresponding face of $P$ as defined in Sect. 4, and $m = (m_1, \ldots, m_l) := S^{-1}(S)$ is the composition in bijection to $S$. Then $\text{nvol}(F_S)$ is a homogeneous polynomial in $\{w_i : i \in S\}$, whose coefficient of $\prod_{i \in S} w_i$ – the unique squarefree monomial – is

$$
C_n(S) := (m_1 - 1)! (m_2 - 1)! \cdots (m_l - 1)!. \quad \text{(5.3)}
$$
Proof of Lemma 4.7 Suppose \( [n+1] \setminus S = \{ t_1 < t_2 < \cdots < t_{i-1} < t_i = n+1 \} \), and by convention let \( t_0 = 0 \). Note that by the proof of Lemma 4.7, we have \( \sum_{k=1}^i m_k = t_i \) for each \( i \). By Proposition 4.3, the face \( F_S \) is congruent to \( \text{Perm}(\nu_{M_1}) \times \cdots \times \text{Perm}(\nu_{M_l}) \), where \( \nu_{M_i} = (v_j : t_{i-1} < j \leq t_i) \). Hence, \( \text{nvol}(F_S) = \prod_{i=1}^l \text{nvol}(\text{Perm}(\nu_{M_i})) \). Let

\[
T_i := \{ j : t_{i-1} < j < t_i \}.
\]

Then by Lemmas 5.2 and 5.3, the normalized volume of \( \text{Perm}(\nu_{M_i}) \) is a homogeneous polynomial in \( \{ t_j : j \in T_i \} \), and the coefficient of \( \prod_{j \in T_i} w_j \) – the unique squarefree monomial – in this homogeneous polynomial is \( (m_i - 1)! \). Therefore, the conclusion follows from the observation that \( S = \bigcup_{i=1}^l T_i \). \( \square \)

The following is the main result of this section, which is an expanded version of Theorem 1.6.

**Theorem 5.5** Suppose \( \alpha \) and \( \nu \) are as in Setup 4.1 and \( C_n(S) \) is defined as in (5.3). Then the \( \alpha \) values of faces of \( \text{Perm}(\nu) \) are uniquely determined. More specifically, if \( S = \{ s_1, s_2, \ldots, s_k \} \), we have

\[
\alpha_n(S) = \frac{1}{C_n(S)|\mathcal{O}_n(S)|} k! \mathcal{M}\text{Lat}^k(\Delta_{s_1,n+1}, \Delta_{s_2,n+1}, \ldots, \Delta_{s_k,n+1})
\]

(5.4)

(5.5)

\[
= \frac{m_1 \cdot m_2 \cdots m_l}{(n+1)!} k! \mathcal{M}\text{Lat}^k(\Delta_{s_1,n+1}, \Delta_{s_2,n+1}, \ldots, \Delta_{s_k,n+1}).
\]

In particular the above formula applies to the BV-\( \alpha \)-valuation.

One sees that the above theorem gives a connection between the \( \alpha \) arising from the regular permutohedron and the mixed lattice point valuation \( \mathcal{M}\text{Lat}^k \) on hypersimplices. Therefore, we have the following:

**Corollary 5.6** The following two statements are equivalent:

1. For any \( S = \{ s_1, \ldots, s_k \} \subseteq [n] \), we have \( \mathcal{M}\text{Lat}^k(\Delta_{s_1,n+1}, \Delta_{s_2,n+1}, \ldots, \Delta_{s_k,n+1}) > 0 \).
2. The regular permutohedron \( \Pi_n \) is BV-\( \alpha \)-positive.

**Proof of Theorem 5.5** Let \( w_i \) be defined as in (5.1). Theorem 2.14 or Theorems 2.8 and 2.13 tell us that the number of lattice points in

\[
\text{Perm}(\nu) = w_1 \Delta_{1,n+1} + w_2 \Delta_{2,n+1} + \cdots + w_n \Delta_{n,n+1}
\]

is a polynomial in the \( w_i \) variables. We denote this polynomial by \( E = E(w_1, w_2, \ldots, w_n) \). We focus on the coefficient of squarefree monomials \( w_S := \prod_{i \in S} w_i \) in \( E \). On the one hand, by (2.2) and Lemma 2.10 (i), this coefficient is equal to

\[
k! \mathcal{M}\text{Lat}^k(\Delta_{s_1,n+1}, \Delta_{s_2,n+1}, \ldots, \Delta_{s_k,n+1}).
\]

(5.6)
Next by (4.5), we have

\[ E(w_1, \ldots, w_n) = \sum_{\mathcal{O}_n(S)} |\mathcal{O}_n(S)| \cdot C_n(S) \cdot \text{nvol}(F_S). \]

(Proposition 5.4 guarantees that the right hand side of the above expression is indeed polynomial on the \(w_i\) variables.) Note that according to Proposition 5.4, the only contribution to the monomial \(w_S = \prod_{i \in S} w_i\) in the summand above is the term corresponding to \(S\), and it is given by \(C_n(S)\). Using these, we conclude that the coefficient of \(w_S\) in \(E(w_1, \ldots, w_n)\) is

\[ \alpha_n(S) \cdot |\mathcal{O}_n(S)| \cdot C_n(S). \]  

(5.7)

Our two expressions, (5.6) and (5.7), for the coefficient of \(w_S\) in \(E\) have to agree. Hence, (5.4) follows.

Finally, (5.5) follows from (5.3), (4.4), and the bijection defined in Lemma 4.7. \(\square\)

Formula (5.5) allows us to obtain values of \(\alpha_n(S)\) by computing mixed lattice point valuations of hypersimplices.

**Example 5.7** Consider \(n = 3\) and \(S = \{1, 3\} \subseteq [3]\). The corresponding composition is \(m = (2, 2)\). Applying (5.5), we get

\[ \alpha_3([1, 3]) = \frac{2 \cdot 2}{24} 2! \mathcal{M}\text{Lat}^2(\Delta_{14}, \Delta_{34}). \]

Furthermore, Theorem 2.15 provides a way to compute mixed valuations:

\[ 2! \mathcal{M}\text{Lat}^2(\Delta_{14}, \Delta_{34}) = \text{Lat}^2(\Delta_{14} + \Delta_{34}) - \text{Lat}^2(\Delta_{14}) - \text{Lat}^2(\Delta_{14}). \]

By the comment after Theorem 2.13, for any polytope \(\text{Lat}'(P)\) is the coefficient of \(t^r\) in the Ehrhart polynomial \(i(P, t)\). Hence, we can figure out the terms in the right hand side of the above equality by computing corresponding Ehrhart polynomials:

\[ i(\Delta_{14} + \Delta_{34}, t) = \frac{10}{3} t^3 + 5t^2 + \frac{11}{3} t + 1, \]
\[ i(\Delta_{14}, t) = \frac{1}{6} t^3 + t^2 + \frac{11}{6} t + 1, \]
\[ i(\Delta_{34}, t) = \frac{1}{6} t^3 + t^2 + \frac{11}{6} t + 1. \]

Therefore, \(2! \mathcal{M}\text{Lat}^2(\Delta_{14}, \Delta_{34}) = 5 - 1 - 1 = 3\), and \(\alpha_3([1, 3]) = \frac{2 \cdot 2}{24} \cdot 3 = \frac{1}{2}\), which agrees with Lemma 3.8 since \(F_{[1,3]}\) is a facet.
6 Further Questions and Remarks

We finish the article with a brief description of other progress we have made on proving our conjectures and a discussion on questions/problems arising from this paper.

Other Results

In addition to the results presented in this paper, we have two other related results, which are omitted because they are less important than those appeared in the paper, and we want to keep the paper within a reasonable length.

The first one was another partial result on our strong conjecture (Conjecture 1.3). Recall in Example 3.15 we found the \(\alpha\)-values of two kinds of edges of \(\Pi_3\) by setting up a triangular linear system. Using similar strategy, we can set up an explicit linear system for \(\alpha\)-values of edges of \(\Pi_n\) for any \(n\). Since solving linear systems is very fast, we can find \(\alpha\)-values of edges of \(\Pi_n\) quickly for \(n\) that is not too large. For example, we computed \(\alpha\)-values of edges of \(\Pi_{500}\) and verified that they were all positive. By Remark 3.6, this implies that our strong conjecture is true for edges of generalized permutohedra of dimension up to 500. (See [7, Sect. A.2.4] for details.) Even though it will be easy for us to push the number 500 to a much larger number by solving linear systems explicitly, a systematic way to show all solutions are positive for all \(n\) will be more desirable.

The second result is another equivalent statement to Conjecture 1.3 in addition to the equivalent statement in terms of mixed lattice point valuations provided in Corollary 5.6. The Berline–Vergne’s construction gives one way to write the Todd class of the permutohedral variety in terms of the toric invariant cycles. We can show that if there is any way of writing such class as a positive combination of such cycles, then the BV-\(\alpha\)-valuation is one of them. (See [8, Proposition 7.2] or [7].) This is important since there are other constructions that may work. For instance, if there is an appropriate choice of flags in Pommersheim–Thomas method that yields positive values, then it will prove Conjecture 1.3.

Questions

Naturally, the main question left is still whether Conjecture 1.3, or any of its equivalences, is true. Other than that, the following questions may be of further interest.

1. **Uniqueness of the BV-\(\alpha\)-valuation.**
   - Can we generalize Theorem 5.5 to other families of polytopes that come from a certain normal fan? In other words, is there any other normal fan \(\Sigma\), such that the BV-\(\alpha\)-values arising from polytopes whose normal fan is \(\Sigma\) are uniquely determined?
   - More importantly, is the BV-\(\alpha\)-valuation the unique solution to McMullen’s formula that is a valuation and is invariant under permutations of coordinates?

2. **Compute the BV-\(\alpha\)-valuation for some specific polytopes.**
• For the case of generalized permutohedra, we believe that knowledges of α-values on hypersimplices would be a very useful step in understanding the general case. We note that it seems that there are very few examples explicitly computed in the literature.

It is worth mentioning that in a recent paper [14], G. Liu describes a recursive way to compute mixed Eulerian numbers, which are mixed volumes of hypersimplices of the same dimension. The method extends to a recursive way of computing the mixed lattice point valuations of hypersimplices, but so far we cannot prove positivity that way.

Lastly, we would like to mention a related idea. The exterior angle γ(\(F, P\)) of \(P\) at \(F\) is the “intrinsic measure” of \(\text{ncone}_V(F, P)\). It is clear from the definition of exterior angles, \(γ\) is symmetric about the coordinates. Moreover, the following result on \(γ(F, P)\) indicates that \(γ\) is a partial solution to McMullen’s formula.

**Theorem 6.1** (Corollary 7.8 of [4]) Let \(P \subset \mathbb{R}^D\) be an integral zonotope, i.e. a Minkowski sum of elements in \(\mathbb{Z}^D\), then we have

\[
\text{Lat}(P) = \sum_{F: \text{a face of } P} γ(F, P) \cdot \text{nvol}(F)
\]

The above theorem applies to the regular permutohedron; however, unfortunately it does not apply to its deformations, not even for generic permutohedra. So this construction \(γ\) does not satisfy Setup 4.1. Our arguments in Sect. 5 depend on the fact that McMullen’s formula holds for all generic permutohedra, and thus will not hold for the exterior angle construction. Indeed, the exterior angles of the edges of \(\Pi_3\) are not even rational numbers, and hence are different from the formulas we derived in Theorem 5.5.

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