Abstract. The purpose of this paper is to establish the almost sure weak ergodic convergence of a sequence of iterates \((x_n)\) given by
\[ x_{n+1} = (I + \lambda_n A(\xi_{n+1}, \cdot))^{-1}(x_n) \]
where \((A(s, \cdot) : s \in E)\) is a collection of maximal monotone operators on a separable Hilbert space, \((\xi_n)\) is an independent identically distributed sequence of random variables on \(E\) and \((\lambda_n)\) is a positive sequence in \(\ell^2 \setminus \ell^1\). The weighted averaged sequence of iterates is shown to converge weakly to a zero (assumed to exist) of the Aumann expectation \(\mathbb{E}(A(\xi_1, \cdot))\) under the assumption that the latter is maximal. We consider applications to stochastic optimization problems of the form
\[ \min \mathbb{E}(f(\xi_1, x)) \quad \text{w.r.t.} \quad x \in \bigcap_{i=1}^m X_i \]
where \(f\) is a normal convex integrand and \((X_i)\) is a collection of closed convex sets. In this case, the iterations are closely related to a stochastic proximal algorithm recently proposed by Wang and Bertsekas.

Key words. Proximal point algorithm, Stochastic approximation, Convex optimization.

AMS subject classifications. 90C25, 65K05

1. Introduction. The proximal point algorithm is a method for finding a zero of a maximal monotone operator \(A : \mathcal{H} \to 2^{\mathcal{H}}\) on some Hilbert space \(\mathcal{H}\) i.e., a point \(x \in \mathcal{H}\) such that \(0 \in A(x)\). The approach dates back to [15] [28] [10] and has aroused a vast literature. The algorithm consists in the iterations
\[ y_{n+1} = (I + \lambda_n A)^{-1}y_n \]
for \(n \in \mathbb{N}\) where \(\lambda_n > 0\) is a positive step size. When the sequence \((\lambda_n)\) is bounded away from zero, it was shown in [28] that \((y_n)\) converges weakly to some zero of \(A\) (assumed to exist). The case of vanishing step size was investigated by several authors including [10], [20], see also [1]. The condition \(\sum_n \lambda_n = +\infty\) is generally unsufficient to ensure the weak convergence of the iterates \((y_n)\) unless additional assumptions on \(A\) are made (typically, \(A\) must be demi-positive). A counterexample is obtained when \(A\) is a \(\pi/2\)-rotation in the 2D-plane and \(\sum_n \lambda_n^2 < \infty\). However, the condition \(\sum_n \lambda_n = +\infty\) is sufficient to ensure that \(y_n\) converges weakly in average to a zero of \(A\). Here, by convergence in average we mean that the weighted averaged sequence
\[ \bar{y}_n = \frac{\sum_{k=1}^n \lambda_k y_k}{\sum_{k=1}^n \lambda_k} \]
converges weakly to a zero of \(A\).

This paper extends the above result to the case where the operator \(A\) is no longer fixed but is replaced at each iteration \(n\) by one operator randomly chosen amongst a collection \((A(s, \cdot) : s \in E)\) of maximal monotone operators. We study the random sequence \((x_n)\) given by
\[ x_{n+1} = (I + \lambda_n A(\xi_{n+1}, \cdot))^{-1}x_n \]

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where \((\xi_n)\) is an independent identically distributed sequence with probability distribution \(\mu\) on some probability space \((\Omega, \mathcal{F}, P)\). We refer to the above iterations as the stochastic proximal point algorithm. Under mild assumptions on the collection of operators, the random sequence \((x_n)\) generated by the algorithm is shown to be bounded with probability one. The main result is that almost surely, \((x_n)\) converges weakly in average to some random point within the set of zeroes (assumed non-empty) of the mean operator \(A\) defined by

\[
A: x \mapsto \int A(s, x) \, d\mu(s)
\]

where \(\int\) represents the Aumann integral [3, Chapter 8]. While the operator \(A\) is always monotone, our key assumption is that it is also maximal. This condition is satisfied in a number of particular cases. For instance when the random variable \(\xi_1\) belongs almost surely to a finite set, say \(\{1, \ldots, m\}\), \(A(x)\) coincides with the Minkowski sum

\[
A(x) = \sum_{i=1}^{m} P(\xi_1 = i) A(i, x)
\]

for every \(x \in \mathcal{H}\), and \(A\) is maximal under the sufficient condition that the interiors of the domains of all operators \(A(i, .)\) \((i = 1, \ldots, m)\) have a non-empty intersection [23].

### Related works

In the literature, numerous works have been devoted to iterative algorithms searching for zeroes of a sum of maximal operators. One of the most celebrated approach is the Douglas-Rachford algorithm analyzed by [14]. Though suited to a sum of two operators, the Douglas-Rachford algorithm can be adapted to an arbitrary finite sum using the so-called product space trick. The authors of [10] and [20] consider applying product of resolvents in a cyclic manner. Numerically, the above deterministic approaches become difficult to implement when the number of operators in the sum is large, or a fortiori infinite (i.e. the mean operator is an integral). In parallel, stochastic approximation techniques have been developed in the statistical literature to find a root of an integral functional \(h: \mathcal{H} \to \mathcal{H}\) of the form

\[
h(x) = \int H(s, x) \, d\mu(s).
\]

The archetypal algorithm writes \(x_{n+1} = x_n - \lambda_n H(\xi_{n+1}, x_n)\) as proposed in the seminal work of Robbins and Monro [21]. It turns out that the iterates (1.1) have a similar form

\[
x_{n+1} = x_n - \lambda_n A_{\lambda_n}(\xi_{n+1}, x_n)
\]

where \(A_{\lambda}(s, .)\) is the so-called Yosida approximation of the monotone operator \(A(s, .)\). As a matter of fact, our analysis borrows some proof ideas from the stochastic approximation literature [2].

Applications of stochastic approximation include the minimization of integral functionals of the form \(x \mapsto E(f(\xi_1, x))\) where \((f(s, .) : s \in E)\) is a collection of proper lower-semicontinuous convex functions on \(\mathcal{H} \to (-\infty, +\infty]\). We refer to [8] for a survey. For instance, the stochastic subgradient algorithm writes \(x_{n+1} = x_n - \lambda_n \nabla f(\xi_{n+1}, x_n)\) where \(\nabla f(\xi_{n+1}, x_n)\) represents a subgradient of \(f(\xi_{n+1}, .)\) at point \(x_n\) (assumed in this case to be everywhere well defined). The algorithm is often analyzed under a somehow restrictive uniform boundedness assumption of the subgradients [8]. In practice, an artificial reprojection step is often introduced to enforce boundedness of the iterates. In contrast, the stochastic proximal point algorithm given by (1.1) is more inclined to avoid numerical unstabilities. Denoting by \(A(s, .)\) the subdifferential of
\( f(s, .) \), the resolvent \((I + \lambda A(s, .))^{-1}\) coincides with the proximity operator associated with \( f(s, .) \) given by

\[
\text{prox}_{\lambda f(s, .)}(x) = \arg \min_{t \in \mathcal{H}} \lambda f(s, t) + \frac{\|t - x\|^2}{2}
\]

for any \( x \in \mathcal{H} \). The iterations (1.1) can be equivalently written as

\[
x_{n+1} = \text{prox}_{\lambda_n f(\xi_{n+1}, .)}(x_n).
\]

A related algorithm is studied (among others) by Bertsekas in [9] under the assumption that \( \xi_1 \) has a finite range and \( f(s, .) \) is defined on \( \mathbb{R}^d \rightarrow \mathbb{R} \). As functions are supposed to have full domain, [9] introduces a projection step onto a closed convex set in order to cover the case of constrained minimization. When there exists a constant \( c \) such that the functions \( f(s, .) \) are \( c \)-Lipschitz continuous for all \( s \), and under other technical assumptions, the algorithm of [9] is proved to converge to a sought minimizer. In [30], the finite range assumption is dropped and random projections are introduced. Extension to Hadamard spaces is considered in [4]. Extension to variational inequalities is considered in [31].

**Organization and contributions.** The paper is organized as follows. After some preliminaries in Section 2, the main algorithm is introduced in Section 3. The aim of Section 4 is to establish that the algorithm is stable in the sense that the sequence \((x_n)\) is bounded almost surely. We actually prove a stronger result: for any zero \( x^* \) of \( A \), the sequence \( \|x_n - x^*\| \) converges almost surely. This point is the first key element to prove the weak convergence in average of the algorithm. The second element is provided in Section 5 where it is shown that any weak cluster point of the weighted averaged sequence \((\tau_n)\) is a zero of \( A \). Putting together these two arguments and using Opial’s lemma [20], we conclude that, almost surely, \((\tau_n)\) converges weakly to a zero of \( A \). The proofs of Section 5 rely on two major assumptions. First, the operator \( A \) is assumed maximal, as discussed above. Second, the averaged sequence of (random) Yosida approximations evaluated at the iterates is supposed to be uniformly integrable with probability one. The latter assumption is easily verifiable when all operators are supposed to have the same domain. The case where operators have different domains is more involved. We introduce a linear regularity assumption of the set of domains of the operators inspired by [6] (a similar assumption is also used in [30]). We provide estimates of the distance between the iterate \( x_n \) and the essential intersection of the domains. The latter estimates allow to verify the uniform integrability condition, and yield the almost sure weak convergence in average of the algorithm in the general case.

In Section 6, we study applications to convex optimization. We use our results to prove weak convergence in average of \((x_n)\) given by (1.3) to a minimizer of \( x \mapsto \mathbb{E}(f(\xi_1, x)) \). As an illustration, we address the problem

\[
\min \mathbb{E}(f(\xi_1, x)) \text{ w.r.t. } x \in \bigcap_{i=1}^{m} X_i
\]

where \( X_1, \ldots, X_m \) are closed convex sets of \( \mathbb{R}^d \) and \( f(s, .) \) is a convex function on \( \mathcal{H} \rightarrow \mathbb{R} \) for each \( s \in E \). We propose a random algorithm quite similar to [30] and whose convergence in average can be established under verifiable conditions.
2. Preliminaries.

2.1. Random closed sets. Let $\mathcal{H}$ be a separable Hilbert space (identified with its dual) equipped with its Borel $\sigma$-algebra $\mathcal{B}(\mathcal{H})$. We denote by $\|x\|$ the Euclidean norm of any $x \in \mathcal{H}$ and by $d(x, Q) = \inf\{\|y - x\| : y \in Q\}$ the distance between a point $x \in \mathcal{H}$ and a set $Q \subset \mathcal{H}$ (equal to $+\infty$ when $Q = \emptyset$). We denote by $\text{cl}(Q)$ the closure of $Q$. We note $|Q| = \sup\{\|x\| : x \in Q\}$.

Let $(T, T)$ be a measurable space. Let $\Gamma : T \to 2^\mathcal{H}$ be a multifunction such that $\Gamma(t)$ is a closed set for all $t \in T$. The domain of $\Gamma$ is denoted by $\text{dom}(\Gamma) = \{t \in T : \Gamma(t) \neq \emptyset\}$. The graph of $\Gamma$ is denoted by $\text{gr}(\Gamma) = \{(t, x) : x \in \Gamma(t)\}$.

We say that $\Gamma$ is $T$-Effros-measurable if $\{t \in T : \Gamma(t) \cap U \neq \emptyset\} \in \mathcal{T}$ for each open set $U \subset \mathcal{H}$. This is equivalent to say that for any $x \in \mathcal{H}$, the mapping $t \mapsto d(x, \Gamma(t))$ is a random variable [11], [17]. We say that $\Gamma$ is graph-measurable if $\text{gr}(\Gamma) \in \mathcal{T} \otimes \mathcal{B}(\mathcal{H})$. Effros-measurability implies graph measurability and the converse is true if $(T, T)$ is complete for some $\sigma$-finite measure [11, Chapter III], [17, Theorem 2.3, pp.28].

Given a probability measure $\nu$ on $(T, T)$, a function $\phi : T \to \mathcal{H}$ is called a measurable selection of $\Gamma$ if $\phi(t) \in \Gamma(t)$ for all $t$ $\nu$-a.e. We denote by $S(\Gamma)$ the set of measurable selections of $\Gamma$. If $\Gamma$ is Effros-measurable, the measurable selection theorem states that $S(\Gamma) \neq \emptyset$ if and only if $\Gamma(t) \neq \emptyset$ for all $t$ $\nu$-a.e. [17, Theorem 2.13, pp.32], [3, Theorem 8.1.3]. For any $p \geq 1$, we denote by $L^p(T, \mathcal{H}, \nu)$ the set of measurable functions $\phi : T \to \mathcal{H}$ such that $\int \|\phi\|^p d\nu < \infty$. We set $S^p(\Gamma) = S(\Gamma) \cap L^p(T, \mathcal{H}, \nu)$. The Aumann integral of the Effros-measurable map $\Gamma$ is the set

$$\int \Gamma d\nu = \left\{ \int \phi d\nu : \phi \in S^1(\Gamma) \right\}$$

where $\int \phi d\nu$ is the Bochner integral of $\phi$.

2.2. Monotone operators. An operator $A : \mathcal{H} \to 2^\mathcal{H}$ is said monotone if $\forall (x, y) \in \text{gr}(A), \forall (x', y') \in \text{gr}(A), \langle y - y', x - x' \rangle \geq 0$. The operator $A$ is maximal monotone if it is monotone and if for any other monotone operator $A' : \mathcal{H} \to 2^\mathcal{H}$, $\text{gr}(A) \subset \text{gr}(A')$ implies $A = A'$. A maximal monotone operator $A$ has closed convex images and $\text{gr}(A)$ is closed [7, pp. 300]. We denote the identity by $I : x \mapsto x$. For some $\lambda > 0$, the resolvent of $A$ is the operator $J_{\lambda} = (I + \lambda A)^{-1}$ or equivalently: $y \in J_{\lambda}(x)$ if and only if $(x - y)/\lambda \in A(y)$. The Yosida approximation of $A$ is the operator $A_{\lambda} = (I - J_{\lambda})/\lambda$. Assume from now on that $A$ is a maximal monotone operator. Then $J_\lambda$ is a single valued map on $\mathcal{H}$ and is firmly non-expansive in the sense that $\langle J_\lambda(x) - J_\lambda(y), x - y \rangle \geq \|J_\lambda(x) - J_\lambda(y)\|^2$ for every $(x, y) \in \mathcal{H}^2$. The Yosida approximation $A_{\lambda}$ is $1/\lambda$-Lipschitz continuous and satisfies $A_{\lambda}(x) \in A(J_{\lambda}(x))$ for every $x \in \mathcal{H}$ [16], [7, Corollary 23.10]. For any $x \in \text{dom}(A)$, we denote by $A_0(x)$ the element of least norm in $A(x)$ i.e., $A_0(x) = \text{proj}_{A(x)}(0)$ where $\text{proj}_C$ represents the projection operator onto a closed convex set $C$. When $A$ is maximal monotone and $x \in \text{dom}(A)$, then $\|A_{\lambda}(x)\| \leq \|A_0(x)\|$. In that case, $A_{\lambda}(x)$ and $J_{\lambda}(x)$ respectively converge to $A_0(x)$ and $x$ as $\lambda \downarrow 0$ [7, Section 23.5].

3. Algorithm.

3.1. Description. Let $(E, \mathcal{E}, \mu)$ be a complete probability space and let $\mathcal{H}$ be a separable Hilbert space equipped with its Borel $\sigma$-algebra $\mathcal{B}(\mathcal{H})$. Consider a mapping $A : E \times \mathcal{H} \to 2^\mathcal{H}$ and define for any $\lambda > 0$, the resolvent and the Yosida approximation
of $A$ as the mappings $J_\lambda$ and $A_\lambda$ respectively defined on $E \times H \to 2^H$ by

$$
J_\lambda(s, x) = (I + \lambda A(s, .))^{-1}(x)
A_\lambda(s, x) = (x - J_\lambda(s, x))/\lambda
$$

for all $(s, x) \in E \times H$.

**Assumption 1.**

(i) For every $s \in E$ $\mu$-a.e., $A(s, .)$ is maximal monotone.

(ii) For every $x \in H$, $A(., x)$ is $\mathcal{E}$-Effros measurable.

(iii) For any $\lambda > 0$ and $x \in H$, $J_\lambda(., x)$ is $\mathcal{E}/\mathcal{B}(H)$-measurable.

As $A(., .)$ is maximal monotone, $J_\lambda(., .)$ is a single-valued continuous map for each $s \in E$. Together with Assumption 1(iii), this implies that $J_\lambda$ is a Carathéodory map. As such, $J_\lambda$ is $\mathcal{E} \otimes \mathcal{B}(H)/\mathcal{B}(H)$-measurable by [3, Lemma 8.2.6].

**Remark 1.** Assumptions 1(ii) and 1(iii) can be together replaced with the single stronger statement that $A$ is $\mathcal{E} \otimes \mathcal{B}(H)$-Effros measurable. In that case, $J_\lambda$ is $\mathcal{E} \otimes \mathcal{B}(H)/\mathcal{B}(H)$-measurable as stated by [29, Prop. 3.25] generalizing the arguments of [13, Theorem 3.2] to the multivalued case.

Consider another probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\xi_n : n \in \mathbb{N}^*)$ be a sequence of random variables on $\Omega \to E$. For an arbitrary initial point $x_0 \in H$ (assumed fixed throughout the paper), we consider the following iterations

$$
x_{n+1} = J_{\lambda_n}(\xi_{n+1}, x_n).
$$

**Assumption 2.**

(i) The sequence $(\lambda_n : n \in \mathbb{N})$ is positive and belongs to $\ell^2 \setminus \ell^1$.

(ii) The random sequence $(\xi_n : n \in \mathbb{N}^*)$ is independent and identically distributed with probability distribution $\mu$.

Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by the r.v. $\xi_1, \ldots, \xi_n$. We denote by $\mathbb{E}$ the expectation on $(\Omega, \mathcal{F}, \mathbb{P})$ and by $E_n = \mathbb{E}(\cdot | \mathcal{F}_n)$ the conditional expectation w.r.t. $\mathcal{F}_n$.

**3.2. Mean operator.** For any $x \in H$, we define $S_A(x) = S(A(., x))$ as the set of measurable selections of $A(., x)$. We define similarly $S^p_A(x) = S^p(A(., x))$. For each $s \in E$, we set $D_s = \text{dom}(A(s, .))$. Following [12], we define the essential intersection (or continuous intersection) of the domains $D_s$ as

$$
\mathcal{D} = \bigcup_{N \in \mathcal{N}} \bigcap_{s \in E \setminus N} D_s
$$

where $\mathcal{N}$ is the set of $\mu$-negligible subsets of $E$. Otherwise stated, a point $x$ belongs to $\mathcal{D}$ if $x \in D_s$ for every $s$ outside a negligible set. We define

$$
\mathbf{A}(x) = \int A(s, x)d\mu(s).
$$

For any $s \in E$ and any $x \in D_s$, we define $A_0(s, x) = \text{proj}_{A(s, x)}(0)$ as the element of least norm in $A(s, x)$.

**Lemma 3.1.** Under Assumption 1, $\mathbf{A}$ is monotone and has convex values. Moreover, if $\int \|A_0(s, x)\|d\mu(s) < \infty$ for all $x \in \mathcal{D}$, then

$$
\text{dom}(\mathbf{A}) = \mathcal{D}.
$$
Proof. The first point is clear. For any \( x \in \mathcal{D} \), \( A_0(\ldots, x) \) is well defined \( \mu \)-a.e. and is measurable as the pointwise limit of measurable functions \( A_{\lambda}(\ldots, x) \) for \( \lambda \downarrow 0 \). By the measurable selection theorem, \( \mathcal{D} = \text{dom}(S_A) \). On the other hand, \( \text{dom}(A) = \text{dom}(S_A^1) \subset \mathcal{D} \). For any \( x \in \mathcal{D} \), \( A_0(\ldots, x) \) is an integrable selection of \( A(\ldots, x) \) by the standing hypothesis. Thus, \( x \in \text{dom}(\overline{A}) \). As a consequence, \( \mathcal{D} \subset \text{dom}(\overline{A}) \). \( \square \)

**Example 1.** Consider the case where \( \mu \) is a finitely supported measure, say \( \text{supp}(\mu) = \{1, \ldots, m\} \) for some integer \( m \geq 1 \). Set \( w_i = \mu(\{i\}) \) for each \( i \). Then \( \overline{A} = \sum_{i=1}^{m} w_i A(i, \cdot) \) and its domain is equal to

\[
\mathcal{D} = \bigcap_{i=1}^{m} D_i.
\]

Moreover, if the interiors of the respective sets \( D_1, \ldots, D_m \) have a non-empty intersection, then \( \overline{A} \) is maximal by [23].

**Example 2.** Set \( \mathcal{H} = \mathbb{R}^d \). Assume \( A \) is non-empty valued and for all \( x \in \mathcal{H}, |A(x)| \leq g(x) \) for some \( g \in L^1(E, \mathcal{R}, \mu) \). Then \( \overline{A} \) is non-empty (convex) valued and has a closed graph by [32]. Thus \( \overline{A} \) is maximal monotone by [5, pp. 45].

We denote by \( \text{zer}(\overline{A}) = \{x \in \mathcal{H} : 0 \in \overline{A}(x)\} \) the set of zeroes of \( \overline{A} \). We define for each \( p \geq 1 \)

\[
\mathcal{Z}_A(p) = \{x \in \mathcal{H} : \exists \phi \in S_A^p(x) : \int \phi \, d\mu = 0\}.
\]

For any \( p \geq 1 \), \( \mathcal{Z}_A(p) \subset \mathcal{Z}_A(1) \) and \( \mathcal{Z}_A(1) = \text{zer}(\overline{A}) \).

**4. Stability and cluster points.** The following simple Lemma will be used twice.

**Lemma 4.1.** Let Assumption 1 hold true. Consider \( u \in \mathcal{H}, \phi \in S_A^1(u), x \in \mathcal{H}, \lambda > 0, \beta > 0 \). Then, for every \( s \mu \)-a.e.,

\[
\langle A_{\lambda}(s, x) - \phi(s), x - u \rangle \geq \lambda(1 - \beta)\|A_{\lambda}(s, x)\|^2 - \frac{\lambda}{4\beta}\|\phi(s)\|^2.
\]

**Proof.** As \( \langle A_{\lambda}(s, x) - \phi(s), J_{\lambda}(s, x) - u \rangle \geq 0 \) for all \( s \mu \)-a.e., we obtain

\[
\langle A_{\lambda}(s, x) - \phi(s), x - u \rangle \geq \langle A_{\lambda}(s, x) - \phi(s), x - J_{\lambda}(s, x) \rangle
\]

\[
= \lambda\langle A_{\lambda}(s, x) - \phi(s), A_{\lambda}(s, x) \rangle
\]

\[
= \lambda\|A_{\lambda}(s, x)\|^2 - \lambda\langle \phi(s), A_{\lambda}(s, x) \rangle.
\]

Use \( \langle a, b \rangle \leq \beta\|a\|^2 + \frac{1}{2\beta}\|b\|^2 \) with \( a = A_{\lambda}(s, x) \) and \( b = \phi(s) \), the result is proved. \( \square \)

**4.1. Boundedness.** The following proposition establishes that the stochastic proximal point algorithm is stable whenever \( \mathcal{Z}_A(2) \) is non-empty.

**Proposition 1.** Let Assumptions 1, 2 hold true. Suppose \( \mathcal{Z}_A(2) \neq \emptyset \) and let \( (x_n) \) be defined by (3.1). Then,

(i) There exists an event \( B \in \mathcal{F} \) such that \( \mathbb{P}(B) = 1 \) and for every \( \omega \in B \) and every \( x^* \in \mathcal{Z}_A(2) \), the sequence \( (\|x_n(\omega) - x^*\|) \) converges as \( n \to \infty \).

(ii) \( \mathbb{E}\left(\sum_{n=1}^{\infty} \lambda_n^2 \int \|A_{\lambda_n}(s, x_n(\omega))\|^2 d\mu(s)\right) < \infty \).

(iii) For any \( p \in \mathbb{N}^+ \) such that \( \mathcal{Z}_A(2p) \neq \emptyset \), \( \sup_n \mathbb{E}(\|x_n\|^{2p}) < \infty \).

**Proof.** Consider \( u \in \mathcal{Z}_A(2), \phi \in S_A^2(u) \) such that \( \int \phi \, d\mu = 0 \). Choose \( 0 < \beta \leq \frac{1}{2} \).

Note that \( x_{n+1} = x_n - \lambda_n A_{\lambda_n}(\xi_{n+1}, x_n) \). We expand

\[
\|x_{n+1} - u\|^2 = \|x_n - u\|^2 + 2\lambda_n \langle x_{n+1} - x_n, x_n - u \rangle + \lambda_n^2 \|x_{n+1} - x_n\|^2
\]

\[
= \|x_n - u\|^2 - 2\lambda_n \langle A_{\lambda_n}(\xi_{n+1}, x_n), x_n - u \rangle + \lambda_n^2 \|A_{\lambda_n}(\xi_{n+1}, x_n)\|^2.
\]
Using Lemma 4.1, for all $s$ $\mu$-a.e.,

\[ (A_{\lambda_n}(s, x_n), x_n - u) \geq \lambda_n (1 - \beta) \|A_{\lambda_n}(s, x)\|^2 - \frac{\lambda_n}{4\beta} \|\phi(s)\|^2 + \langle \phi(s), x_n - u \rangle. \]

Therefore,

\[
(4.1) \quad \|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - \lambda_n^2 (1 - 2\beta) \|A_{\lambda_n}(\xi_{n+1}, x)\|^2 \\
+ \frac{\lambda_n^2}{2\beta} \|\phi(\xi_{n+1})\|^2 - 2\lambda_n \langle \phi(\xi_{n+1}), x_n - u \rangle.
\]

Take the conditional expectation of both sides of the inequality:

\[
\mathbb{E}_n \|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - \lambda_n^2 (1 - 2\beta) \mathbb{E} \|A_{\lambda_n}(s, x)\|^2 + \frac{\lambda_n^2 c}{2\beta}
\]

where we set $c = \int \|\phi\|^2 d\mu$ and used $\int \phi d\mu = 0$. By the Robbins-Siegmund theorem (see [22, Theorem 1]) and choosing $0 < \beta < \frac{1}{2}$, we deduce that:

\[
\sum \lambda_n^2 \int \|A_{\lambda_n}(s, x_n)\|^2 d\mu < \infty
\]

Thus, point (ii) is proved, $\sup_n \mathbb{E}(\|x_n\|^2) < \infty$ and finally, the sequence $\{\|x_n - u\|^2\}$ converges almost surely as $n \to \infty$. Let $Q$ be a dense countable subset of $Z_A(2)$. There exists $B \in \mathcal{F}$ such that $\mathbb{P}(B) = 1$ and for all $\omega \in B$, all $u \in Q$, $\{\|x_n(\omega) - u\|\}$ converges. Consider $\omega \in B$ and $x^* \in Z_A(2)$. For any $\epsilon > 0$, choose $u \in Q$ such that $\|x^* - u\| \leq \epsilon$ and define $\ell_u = \lim_{n \to \infty} \|x_n(\omega) - u\|$. Note that $\|x_n(\omega) - u\| \leq \|x_n(\omega) - x^*\| + \epsilon$ thus $\ell_u \leq \lim \inf \|x_n(\omega) - x^*\| + \epsilon$. Similarly, $\|x_n(\omega) - x^*\| \leq \|x_n(\omega) - u\| + \epsilon$ thus $\lim \sup \|x_n(\omega) - x^*\| \leq \ell_u + \epsilon$. Finally, $\lim \sup \|x_n(\omega) - x^*\| \leq \lim \inf \|x_n(\omega) - x^*\| + 2\epsilon$. As $\epsilon$ is arbitrary, we conclude that $\{\|x_n(\omega) - x^*\|\}$ converges. Point (i) is proved.

We prove point (iii) by induction. Set $u \in Z_A(2p)$. We have shown above that $\sup_n \mathbb{E}(\|x_n - u\|^2) < \infty$. Consider an integer $q \leq p$ such that $\sup_n \mathbb{E}(\|x_n - u\|^{2q-2}) < \infty$. We will show that $\sup_n \mathbb{E}(\|x_n - u\|^{2q}) < \infty$ and the proof will be complete. Use equation (4.1) with $\beta = \frac{1}{2}$,

\[
\mathbb{E}(\|x_{n+1} - u\|^{2q}) \leq \mathbb{E}(\|x_n - u\|^2 + \lambda_n^2 \|\phi(\xi_{n+1})\|^2 - 2\lambda_n \langle \phi(\xi_{n+1}), x_n - u \rangle)^q
\]

where for any $\vec{k} = (k_1, k_2, k_3)$ such that $k_1 + k_2 + k_3 = q$, we define

\[
T_n^{\vec{k}} = (-2)^{k_3} \lambda_n^{2k_2+k_3} \mathbb{E}(\|x_n - u\|^{2k_1} \|\phi(\xi_{n+1})\|^{2k_2} \langle \phi(\xi_{n+1}), x_n - u \rangle)^{k_3}.
\]

Note that $T_n^{(q, 0, 0)} = \mathbb{E}(\|x_n - u\|^{2q})$. We now prove that there exists a constant $c''$ such that for any $\vec{k} \neq (q, 0, 0)$, $T_n^{\vec{k}} \leq c'' \lambda_n^2$. Consider a fixed value of $\vec{k} \neq (q, 0, 0)$ such that $k_1 + k_2 + k_3 = q$ and consider the following cases.

- If $k_3 = 0$, then $k_1 \leq q - 1$ and $k_2 \geq 1$. In that case,

\[
|T_n^{\vec{k}}| \leq \lambda_n^{2k_2} \mathbb{E}(\|x_n - u\|^{2k_1}) \int \|\phi\|^{2k_2} d\mu \\
\leq \alpha \lambda_n^2 \mathbb{E}(1 + \|x_n - u\|^{2q-2}) \int \|\phi\|^{2p} d\mu
\]
where $\alpha$ is a constant chosen in such a way that $\lambda_n^{2k_2} \leq \alpha \lambda_n^2$ for any $1 \leq k_2 \leq q$ and where we used the inequality $a^k \leq 1 + a^q - 1$ for any $k_1 \leq q - 1$. The constant $c' = \alpha \sup_n \mathbb{E}(1 + \|x_n - u\|^{2q-2}) \int \|x\|^{2p} d\mu$ is finite and we have $|T_n^k| \leq c' \lambda_n^2$.

- If $k_3 = 1$ and $k_2 = 0$, then $T_n^k = 0$ using that $\int \phi d\mu = 0$.
- In all remaining cases, $k_1 \leq 2$ and $k_2 + k_3 \geq 2$. By the Cauchy-Schwarz inequality,

$$|T_n^k| \leq 2^{k_2} \lambda_n^{2k_2 + k_3} \mathbb{E}[\|x_n - u\|^{2k_2 + k_3} \|\phi(x_{n+1})\|^{2k_2 + k_3}]$$

$$= 2^{k_2} \lambda_n^{2k_2 + k_3} \mathbb{E}[\|x_n - u\|^{2k_2 + k_3}] \int \|x\|^{2k_2 + k_3} d\mu.$$

Now $2k_2 + k_3 = k_2 + k_1 \leq 2p$ and $2k_1 + k_3 = k_1 + q - k_2 \leq k_1 + p \leq 2q - 2$.

Using again that $\sup \mathbb{E}(1 + \|x_n - u\|^{2q-2}) < \infty$ and $\int \|x\|^{2p} d\mu < \infty$, we conclude that there exists an other constant $c'' \geq c'$ such that $|T_n^k| \leq c'' \lambda_n^2$.

We have shown that $|T_n^{(k_1,k_2,k_3)}| \leq c'' \lambda_n^2$ whenever $k_1 + k_2 + k_3 = q$ and $(k_1, k_2, k_3) \neq (q, 0, 0)$. Bounding the rhs of (4.2), we obtain

$$\mathbb{E}[\|x_{n+1} - u\|^{2q}] \leq \mathbb{E}[\|x_n - u\|^{2q}] + c'' \lambda_n^2$$

which in turn implies that $\sup_n \mathbb{E}[\|x_n - u\|^{2q}] < \infty$. \(\square\)

4.2. Weak cluster points. For an arbitrary sequence $(a_n : n \in \mathbb{N})$, we use the notation $\overline{a}$ to represent the weighted averaged sequence $\overline{a} = \sum_{k=1}^{n} \lambda_k a_k / \sum_{k=1}^{n} \lambda_k$.

Assumption 3. The monotone operator $A$ is maximal.

Special cases where Assumption 3 is satisfied are discussed in Examples 1 and 2 above. An application will also be provided in Section 6.4.

Recall that a family $(f_i : i \in I)$ of measurable functions on $E \to \mathbb{R}_+$ is uniformly integrable if

$$\lim_{\alpha \to +\infty} \sup_{i} \int_{\{f_i > \alpha\}} f_i d\mu = 0.$$

Definition 4.2. We say that a sequence $(u_n) \in \mathcal{H}^{\infty}$ has the property $\overline{U}$ if the sequence

$$\frac{\sum_{k=1}^{n} \lambda_k \|A_{\lambda_k} (\ldots, u_k)\|}{\sum_{k=1}^{n} \lambda_k}$$

is uniformly integrable.

Proposition 2. Let Assumptions 1–3 hold true and suppose that $Z_A(2) \neq \emptyset$. Consider the random sequence $(x_n)$ given by (3.1) with weighted averaged sequence $(\overline{x_n})$. Let $G \in \mathcal{F}$ be an event such that for almost every $\omega \in G$, $(x_n(\omega))$ has the property $\overline{U}$. Then, there exists $B \in \mathcal{F}$ such that $\mathbb{P}(B) = 1$ and such that for every $\omega \in B \cap G$, all weak cluster points of the sequence $(\overline{x_n(\omega)})$ belong to $\text{zer}(A)$.

Proof. Denote $h_\lambda(x) = \int A_\lambda(s, x) d\mu(s)$ for any $\lambda > 0$, $x \in \mathcal{H}$. We justify the fact that $h_\lambda(x)$ is well defined. As $A$ is maximal, its domain contains at least one point $u \in \mathcal{H}$. For such a point $u$, there exists $\phi \in S_1(u)$. As $A_\lambda(\cdot, u)$ is $1/\lambda$-Lipschitz continuous, $\|A_\lambda(s, x)\| \leq \|A_\lambda(s, u)\| + \|x - u\|$. Moreover $\|A_\lambda(s, u)\| \leq \|A_0(s, u)\| \leq \|\phi(s)\|$ and since $\phi \in L^1(E, \mathcal{H}, \mu)$, we obtain that $A_\lambda(\cdot, x) \in L^1(E, \mathcal{H}, \mu)$. This implies that $h_\lambda(x)$ is well defined for all $x \in \mathcal{H}$, $\lambda > 0$. We write

$$x_{n+1} = x_n - \lambda_n h_{\lambda_n}(x_n) + \lambda_n \eta_{n+1}.$$
where $\eta_{n+1} = -A_{\lambda_n}(\xi_{n+1}, x_n) + h_{\lambda_n}(x_n)$ is a $\mathcal{F}_n$-adapted martingale increment sequence i.e., $\mathbb{E}_n(\eta_{n+1}) = 0$. Note that

$$
\mathbb{E}_n\|\eta_{n+1}\|^2 \leq \int \|A_{\lambda_n}(s, x_n)\|^2 \, d\mu(s)
$$

and by Proposition 1(ii), it holds that $\sum_k \lambda_k^2 \mathbb{E}_n\|\eta_{k+1}\|^2 < \infty$ almost surely. As a consequence, the $\mathcal{F}_n$-adapted martingale $\sum_{k \leq n} \lambda_k \eta_{k+1}$ converges almost surely to a random variable which is finite $\mathbb{P}$-a.e. Along with Proposition 1, this implies that there exists an event $B \in \mathcal{E}$ of probability one such that for any $\omega \in B \cap G$,

(i) $(\sum_{k \leq n} \lambda_k \eta_{k+1}(\omega))$ converges,
(ii) $(x_n(\omega))$ is bounded,
(iii) $\sum_n \lambda_n^2 \int \|A_{\lambda_n}(\cdot, x_n(\omega))\|^2 \, d\mu$ is finite,
(iv) $(x_n(\omega))$ has the property $\mathbb{E}_n(\eta_{n+1})$.

From now on to the end of this proof, we fix such an $\omega$. As it is fixed, we omit the dependency w.r.t. $\omega$ to keep notations simple. We write for instance $x_n$ instead of $x_n(\omega)$ and what we refer to as constants can depend on $\omega$.

Let $(u, v) \in \text{gr}(A)$ and consider $\phi \in S^1_A(u)$ such that $v = \int \phi d\mu$. Denote by $\epsilon > 0$ an arbitrary positive constant.

We need some preliminaries. By (i), there exists an integer $N = N(\epsilon)$ such that for all $n \geq N$, $\|\sum_{k=N}^n \lambda_k \eta_{k+1}\| \leq \epsilon$. Define $Y_n(s) = \|A_{\lambda_n}(s, x_n)\|$ and let $(\overline{Y}_n)$ represent the corresponding weighted averaged sequence. As $(\overline{Y}_n)$ is uniformly integrable, the same holds for the sequence $(\overline{Y}_n^{(N)})$ defined by

$$
\overline{Y}_n^{(N)} = \frac{\sum_{k=N}^n \lambda_k Y_k}{\sum_{k=N}^n \lambda_k}.
$$

In particular, there exists a constant $c$ such that

$$
\sup_n \int \overline{Y}_n^{(N)} \, d\mu < c.
$$

Moreover, by [18, Proposition II-5-2], there exists $\kappa_\epsilon > 0$ such that

$$
\forall H \in \mathcal{E}, \quad \mu(H) < \kappa_\epsilon \Rightarrow \int_H \overline{Y}_n^{(N)} \, d\mu < \epsilon.
$$

Since $\mu(\{|\phi| > K\}) \to 0$ as $K \to +\infty$, there exists $K_1$ (depending on $\epsilon$) such that for all $K \geq K_1$, $\mu(\{|\phi| > K\}) < \kappa_\epsilon$. For any such $K$,

$$
\int_{\{|\phi| > K\}} \overline{Y}_n^{(N)} \, d\mu < \epsilon.
$$

Denote $v_K = \int_{\{|\phi| > K\}} \phi \, d\mu$. Note that $v_K \to v$ by the dominated convergence theorem. Thus, there exists $K_2$ such that for all $K \geq K_2$, $\|v_K - v\| \to 0$. From now on, we set $K \geq \max(K_1, K_2)$.

Using an idea from [2], we define a sequence $(y_n : n \geq N)$ such that $y_N = x_N$ and $y_{n+1} = y_n - \lambda_n h_{\lambda_n}(x_n)$ for all $n \geq N$. By induction, $y_n = x_n - \sum_{k=N}^{n-1} \lambda_k \eta_{k+1}$. In particular, $\|y_n - x_n\| \leq \epsilon$. We expand

$$
\|y_{n+1} - u\|^2 = \|y_n - u\|^2 - 2 \lambda_n(h_{\lambda_n}(x_n), y_n - u) + \|y_{n+1} - y_n\|^2
\leq \|y_n - u\|^2 - 2 \lambda_n(h_{\lambda_n}(x_n), x_n - u) + 2\epsilon \lambda_n \|h_{\lambda_n}(x_n)\| + \lambda_n^2 \|h_{\lambda_n}(x_n)\|^2.
$$
Define \( \delta_{K, \lambda}(x) = \int_{\{\|\phi\| > K\}} A_{\lambda}(s, x) d\mu(s) \) and use Lemma 4.1 with \( \beta = 1 \):

\[
\langle h_{\lambda_n}(x_n) - v_K, x_n - u \rangle \geq -\|\delta_{K, \lambda_n}(x_n)\| \|x_n - u\| - \frac{\lambda_n K^2}{4}
\]

\[
\geq -c \int_{\{\|\phi\| > K\}} Y_n d\mu - \frac{\lambda_n K^2}{4}
\]

where the constant \( c \) is selected in such a way that \( c > \sup_n \|x_n - u\| \). Using that \( \|v_K - v\| < \epsilon \),

\[
\langle h_{\lambda_n}(x_n) - v, x_n - u \rangle \geq -c \epsilon - c \int_{\{\|\phi\| > K\}} Y_n d\mu - \frac{\lambda_n K^2}{4}.
\]

As a consequence,

\[
\|y_{n+1} - u\|^2 \leq \|y_n - u\|^2 - 2 \lambda_n \langle v, x_n - u \rangle + r_n
\]

where we define

\[
r_n = 2c \epsilon \lambda_n + \lambda_n^2 s_n + 2 \lambda_n c t_{n,K} + 2 \epsilon \lambda_n t_{n,0}
\]

\[
s_n = \|h_{\lambda_n}(x_n)\|^2 + K^2/2
\]

\[
t_{n,a} = \int_{\{\|\phi\| \geq a\}} Y_n d\mu \quad (\forall a \in \{0, K\}).
\]

For any \( a \in \{0, K\} \), denote

\[
\tau_{n,a}^{(N)} = \frac{\sum_{k=N+1}^{n} \lambda_k t_{k,a}}{\sum_{k=N}^{n} \lambda_k}.
\]

By inequality (4.3), \( \tau_{n,0}^{(N)} < c \). By inequality (4.4), \( \tau_{n,0}^{(N)} < c \). By point (iii), \( \sum \lambda_n^2 \|h_{\lambda_n}(x_n)\|^2 < \infty \). Using Assumption 2(i), it follows that

\[
0 \leq \frac{\sum_{k=N+1}^{n} \lambda_k t_{k,a}}{\sum_{k=N}^{n} \lambda_k} < 6 \epsilon + o_n(1)
\]

where \( o_n(1) \) stands for a sequence which converges to zero as \( n \to \infty \). Summing the inequalities (4.5) down to rank \( N \), and dividing by \( 2 \sum_{k=N}^{n} \lambda_k \), we obtain

\[
0 \leq -\frac{\sum_{k=N}^{n} \lambda_k \langle v, x_k - u \rangle}{\sum_{k=N}^{n} \lambda_k} + 3 \epsilon + o_n(1).
\]

Let \( \hat{x} \) be a weak cluster point of the weighted averaged sequence \( x_n \). Then, \( \hat{x} \) is also a weak cluster point of the sequence

\[
\frac{\sum_{k=N}^{n} \lambda_k x_k}{\sum_{k=N}^{n} \lambda_k}.
\]

We obtain \( 0 \leq -\langle v, \hat{x} - u \rangle + 3 \epsilon \). The inequality holds for any \( \epsilon > 0 \), thus \( 0 \leq -\langle v, \hat{x} - u \rangle \). As the inequality holds for any \( (u, v) \in \text{gr}(\mathcal{A}) \) and \( \mathcal{A} \) is maximal monotone, this means that \( (\hat{x}, 0) \in \text{gr}(\mathcal{A}) \) [7, Theorem 20.21]. \( \square \)
4.3. Weak ergodic convergence. The aim of Corollary 1 below is to merge Propositions 1 and 2 into a weak ergodic convergence result. We need the following condition to hold.

**Assumption 4.** \( \mathrm{zer}(A) \neq \emptyset \) and \( \mathrm{zer}(A) \subset Z_A(2) \).

The condition \( \mathrm{zer}(A) \neq \emptyset \) means that there exists \( x^* \in \mathcal{H} \) for which one can find a selection \( \phi \) of \( A(\cdot, x^*) \) such that \( \int \phi d\mu = 0 \). The condition \( \mathrm{zer}(A) \subset Z_A(2) \) means that moreover, such a \( \phi \) can be chosen to be square integrable. For instance, this holds under the stronger condition that for any zero \( x^* \) of \( A \), \( |A(\cdot, x^*)| \) is square integrable.

**Lemma 4.3** (Passty). Let \( (\lambda_n) \) be a non-summable sequence of positive reals, and \( (a_n) \) any sequence in \( \mathcal{H} \) with weighted averaged sequence \( (\bar{a}_n) \). Assume there exists a non-empty closed convex subset \( Q \) of \( \mathcal{H} \) such that (i) weak subsequential limits of \( \bar{a}_n \) lie in \( Q \); and (ii) \( \lim_n \|a_n - b\| \) exists for all \( b \in Q \). Then \( (\bar{a}_n) \) converges weakly to an element of \( Q \).

**Proof.** See [20]. \( \square \)

**Corollary 1.** Let Assumptions 1–4 hold true. Consider the random sequence \( (x_n) \) given by (3.1) with weighted averaged sequence \( (\bar{x}_n) \). Let \( G \in \mathcal{F} \) be an event such that for almost every \( \omega \in G \), \( (x_n(\omega)) \) has the property \( UT \). Then, almost surely on \( G \), \( (\bar{x}_n) \) converges to a point in \( \mathrm{zer}(A) \).

**Proof.** The corollary is a consequence of Proposition 1(i), Proposition 2 and Lemma 4.3. \( \square \)

Corollary 1 establishes the almost sure weak ergodic convergence of the stochastic proximal point algorithm under the abstract condition that w.p.1, \( (x_n) \) has the property \( UT \). We must now provide verifiable conditions under this property indeed holds w.p.1. This is the purpose of the next section.

5. Main results.

5.1. Case of a common domain. We first address the case where the domains \( D_s \) of the operators \( A(s, \cdot) \) (\( s \in E \)) are equal (at least for all \( s \) outside a negligible set).

We also need an additional assumption.

**Assumption 5.** For any compact set \( K \subset \mathcal{H} \), the family \( (\|A_0(\cdot, x)\| : x \in K \cap D) \) is uniformly integrable.

Assumption 5 is satisfied if the following stronger condition holds for any compact set \( K \subset \mathcal{H} \):

\[
\exists r_K > 0, \quad \sup_{x \in K \cap D} \int \|A_0(s, x)\|^{1+r_K} \, d\mu(s) < \infty.
\]

**Theorem 1.** Let Assumptions 1–5 hold true. Assume that the domains \( D_s \) coincide for all \( s \) outside a \( \mu \)-negligible set. Consider the random sequence \( (x_n) \) given by (3.1) with weighted averaged sequence \( (\bar{x}_n) \). Then, almost surely, \( (\bar{x}_n) \) converges weakly to a zero of \( A \).

**Proof.** By Proposition 1 and the fact that \( D_s = D \) for all \( s \) \( \mu \)-a.e., there is a set of probability one such that for any \( \omega \) in that set, there is compact set \( K = K_\omega \) such that \( x_n(\omega) \in K \cap D \) for all \( n \in \mathbb{N}^* \). By Assumption 5, the sequence \( (\|A_0(\cdot, x_n(\omega))\|) : n \in \mathbb{N}^* \) is uniformly integrable. As \( \|A_\lambda_n(\cdot, x_n(\omega))\| \leq \|A_0(\cdot, x_n(\omega))\| \), the same holds for the sequence \( (\|A_\lambda_n(\cdot, x_n(\omega))\|) : n \in \mathbb{N}^* \) and holds as well for the corresponding weighted averaged sequence. The conclusion follows from Corollary 1. \( \square \)

5.2. Case of distinct domains. Theorem 1 does not immediately extends to the case when the respective domains of the operators are distinct. The reason is that the inequality \( \|A_\lambda_n(s, x_n)\| \leq \|A_0(s, x_n)\| \) used to prove Theorem 1 does no longer...
hold when \( x_n \notin D_s \). Nonetheless, using that \( A_\lambda(s, \cdot) \) is \( \frac{1}{\lambda} \)-Lipschitz continuous, the argument can be adapted provided that \( (x_n) \) stays “close enough” to the essential domain \( D \). This statement will be made precise in Theorem 2.

We define the mapping \( \Pi : E \times \mathcal{H} \to \mathcal{H} \) by

\[
\Pi(s, x) = \text{proj}_{\text{cl}(D_s)}(x).
\]

Note that \( \Pi(s, x) = \lim_{\lambda \to 0} J_\lambda(s, x) \) by [7, Theorem 23.47]. By Assumption 1, \( \Pi \) is \( \mathcal{E} \otimes \mathcal{B}(\mathcal{H})/\mathcal{B}(\mathcal{H}) \)-measurable as a pointwise limit of measurable maps. The distance between a point \( x \in \mathcal{H} \) and \( D_s \) coincides with \( d(x, D_s) = \|x - \Pi(s, x)\| \).

**Assumption 6.** There exists \( \kappa > 0 \) such that for all \( x \in \mathcal{H} \),

\[
\int d(x, D_s)^2 d\mu(s) \geq \kappa d(x, D)^2.
\]

Following [6, Definition 5.6] (see also [30]), we say that a finite collection of subsets \( (X_1, \ldots, X_m) \) of \( \mathcal{H} \) is **linearly regular** if

\[
\exists \kappa' > 0, \forall x \in \mathcal{H}, \max_{i=1,\ldots,m} d(x, X_i) \geq \kappa' d(x, \bigcap_{i=1}^m X_i).
\]

In the special case of Example 1 (\( \mu \) is finitely supported), it is routine to check that Assumption 6 holds if and only if the domains \( D_1, \ldots, D_m \) of the operators \( A(1, \cdot), \ldots, A(m, \cdot) \) are linearly regular.

**Lemma 5.1.** Let Assumptions 1, 2 and 6 hold true. Assume that \( \lambda_n/\lambda_{n+1} \to 1 \) as \( n \to +\infty \) and \( D \neq \emptyset \). For each \( n \), consider a \( \mathcal{F}_n \)-measurable random variable \( \delta_n \) on \( \mathcal{H} \). Assume that the sequence \( (E_n\|\delta_{n+1}\|^2) \) is bounded almost surely and in \( L^1(\Omega, \mathcal{H}, \mathbb{P}) \). Then, the sequence \( (x_n) \) given by

\[
x_{n+1} = \Pi(\xi_{n+1}, x_n) + \lambda_n \delta_{n+1}
\]

satisfies

\[
\sup_n \mathbb{E} \left[ \frac{d(x_n, D)^2}{\lambda_n^2} \right] < \infty
\]

\[
\sup_n \frac{\sum_{k \leq n} d(x_k, D)}{\sum_{k \leq n} \lambda_k} < \infty \text{ a.s.}
\]

**Proof.** Consider an arbitrary point \( u \in D \). By definition of \( D \), \( u \in D_s \) for all \( s \) \( \mu \)-a.e. For any \( \beta > 0 \),

\[
\|x_{n+1} - u\|^2 \leq (1 + \beta) \left\| \Pi(\xi_{n+1}, x_n) - u \right\|^2 + \lambda_n^2 (1 + \frac{1}{\beta}) \|\delta_{n+1}\|^2.
\]

As \( \Pi(\xi_{n+1}, \cdot) \) is firmly non-expansive,

\[
\|x_{n+1} - u\|^2 \leq (1 + \beta) \left( \|x_n - u\|^2 - \|x_n - \Pi(\xi_{n+1}, x_n)\|^2 \right) + \lambda_n^2 (1 + \frac{1}{\beta}) \|\delta_{n+1}\|^2.
\]

The above inequality holds for any \( u \in D \) and thus for any \( u \in \text{cl}(D) \). It holds in particular when substituting \( u \) with \( \text{proj}_{\text{cl}(D)}(x_n) \). Remarking that \( d(x_{n+1}, D) \leq \|x_{n+1} - \text{proj}_{\text{cl}(D)}(x_n)\| \), it follows that

\[
d(x_{n+1}, D)^2 \leq (1 + \beta) \left( d(x_n, D)^2 - \|x_n - \Pi(\xi_{n+1}, x_n)\|^2 \right) + \lambda_n^2 (1 + \frac{1}{\beta}) \|\delta_{n+1}\|^2.
\]
By Assumption 6,
\[ \mathbb{E}_{n} ||x_n - \Pi(x_{n+1}, x_n)||^2 = \int ||x_n - \Pi(s, x_n)||^2 d\mu(s) \geq \kappa d(x_n, D)^2. \]

Therefore,
\[ \mathbb{E}_{n} d(x_{n+1}, D)^2 \leq (1 + \beta)(1 - \kappa) d(x_n, D)^2 + \lambda_n^2 (1 + \frac{1}{\beta}) \mathbb{E}_{n} ||\delta_{n+1}||^2. \]

Define \( \Delta_n = d(x_n, D)/\lambda_n \). Using that \( \lambda_n/\lambda_{n+1} \to 1 \) and choosing \( \beta \) small enough, there exists constants \( 0 < \rho < 1, c > 0 \) and a deterministic integer \( n_0 \) depending on the sequence \( (\lambda_n) \) and the constants \( \beta, \kappa \) such that for all \( n \geq n_0 \),

\( \mathbb{E}_{n} \Delta_{n+1}^2 \leq \rho \Delta_n^2 + c \mathbb{E}_{n} ||\delta_{n+1}||^2. \)

Taking expectation of both sides and using that \( (\mathbb{E}||\delta_{n+1}||^2) \) is bounded, we obtain that the sequence \( (\Delta_n) \) is uniformly bounded in \( L^2(\Omega, \mathbb{R}_+, \mathbb{P}) \). Now consider the sums
\[ T_n = \sum_{k=n_0+1}^{n} d(x_k, D) \quad \text{and} \quad \varphi_n = \sum_{k=n_0+1}^{n} \lambda_k. \]

Decompose \( T_n = \sum_{k=n_0+1}^{n} \mathbb{E}_{k-1} d(x_k, D) + R_n \) where
\[ R_n = \sum_{k=n_0+1}^{n} (d(x_k, D) - \mathbb{E}_{k-1} d(x_k, D)). \]

Note that \( R_n \) is an \( \mathcal{F}_n \)-adapted martingale and
\[ \mathbb{E}(d(x_k, D) - \mathbb{E}_{k-1} d(x_k, D))^2 \leq \mathbb{E}(d(x_k, D)^2) \leq C \lambda_k^2 \]
for some finite constant \( C = \sup_n \mathbb{E}(\Delta_n^2) \). As \( \sum_k \lambda_k^2 < \infty \), we deduce that \( R_n \) converges a.s. to some r.v. \( R_\infty \) which is finite \( \mathbb{P} \)-a.e. As a consequence, \( R_n/\varphi_n \) tends a.s. to zero. On the otherhand, by Jensen’s inequality,
\[ T_n \leq \sum_{k=n_0+1}^{n} (\mathbb{E}_{k-1} d(x_k, D)^2)^{1/2} + ||R_n||. \]

By (5.4) again and the assumption that \( \mathbb{E}_{n} ||\delta_{n+1}||^2 \) is bounded a.s., there exists a finite r.v. \( Z > 0 \) such that, almost surely, \( \mathbb{E}_{n}(\Delta_{n+1}^2) \leq \rho \Delta_n^2 + c Z \). Thus, there exists other constants \( \rho < \rho_1 < 1 \) and \( c_1 \) such that \( \mathbb{E}_{n}(\Delta_{n+1}^2)^{1/2} \leq \rho_1 \Delta_n + c_1 Z \). Using that \( \lambda_n/\lambda_{n+1} \to 1 \), we obtain
\[ \mathbb{E}_{n}(d(x_{n+1}, D)^2)^{1/2} \leq \rho_2 d(x_n, D) + c_1 \lambda_{n+1} Z \]
for some constants \( \rho_1 < \rho_2 < 1 \). As a consequence,
\[ \frac{T_n}{\varphi_n} \leq \frac{c_2 Z}{1 - \rho_2} + \frac{\|R_n\|}{(1 - \rho_2) \varphi_n}. \]

Therefore, \( T_n/\varphi_n \) is bounded a.s. and the Lemma is proved. \( \square \)
As λ ↓ 0, we have already noted that the resolvent \( J_\lambda(s,x) \) converges to the best approximation \( \Pi(s,x) \) of \( x \) in \( D_s \). We need a finer assumption on the rate.

**Assumption 7.** There exist \( p \in \mathbb{N}^* \) and \( C \in L^2(\mathbb{E},\mathbb{R}_+,\mu) \) such that for any \( x \in \mathcal{H}, \lambda > 0, \)

\[
\| J_\lambda(s,x) - \Pi(s,x) \| \leq \lambda C(s)(1 + \|x\|^p)
\]

and \( Z_A(2p) \neq \emptyset. \)

The second condition \( Z_A(2p) \neq \emptyset \) means that there exists a zero of \( A \), say \( x^* \), for which one can find a \((2p)\)-integrable selection \( \phi \in A(.,x^*) \) such that \( \int \phi d\mu = 0 \). This is for instance the case if \( |A(.,x^*)|^{2p} \) is integrable.

**Proposition 3.** Let Assumptions 1, 2, 6 and 7 hold true. Suppose that \( \lambda_n/\lambda_{n+1} \to 1 \) as \( n \to \infty \). Then, the sequence \((x_n)\) given by (3.1) satisfies almost surely

\[
\sup_n \frac{\sum_{k \leq n} d(x_k,D)}{\sum_{k \leq n} \lambda_k} < \infty.
\]

**Proof.** The sequence \((x_n)\) satisfies (5.3) if we set

\[
\delta_{n+1} = (J_{\lambda_n}(\xi_{n+1},x_n) - \Pi(\xi_{n+1},x_n))/\lambda_n.
\]

By Assumption 7, \( E_n\|\delta_{n+1}\|^2 \leq c(1 + \|x_n\|^{2p}) \) for some constant \( c > 0 \). Therefore, by Proposition 1(iii), \( E_n\|\delta_{n+1}\|^2 \) is uniformly bounded almost surely and in \( L^1(\Omega,\mathcal{H},\mathbb{P}) \).

The conclusion of Lemma 5.1 applies. \( \square \)

**Theorem 2.** Let Assumptions 1–7 hold true and let \( \lambda_n/\lambda_{n+1} \to 1 \) as \( n \to \infty \). Consider the random sequence \((x_n)\) given by (3.1) with weighted averaged sequence \((\overline{x_n})\). Then, almost surely, \((\overline{x_n})\) converges weakly to a zero of \( A \).

**Proof.** For every \( n \), choose any point \( z_n \in \mathcal{D} \) such that \( \|z_n - x_n\| \leq 2d(x_n,D) \).

As \( A_\lambda(s,. \) is \( \frac{1}{\lambda} \)-Lipschitz continuous,

\[
\| A_{\lambda_n}(s,z_n) \| \leq \frac{2d(x_n,D)}{\lambda_n}.
\]

Using moreover that \( \| A_{\lambda_n}(s,z_n) \| \leq \| A_0(s,z_n) \|, \)

\[
\frac{\sum_{k=1}^n \lambda_k \| A_{\lambda_k}(s,x_k) \|}{\sum_{k=1}^n \lambda_k} \leq \frac{\sum_{k=1}^n \lambda_k \| A_0(s,z_k) \|}{\sum_{k=1}^n \lambda_k} + \frac{2 \sum_{k=1}^n d(x_k,D)}{\sum_{k=1}^n \lambda_k}.
\]

By Proposition 3,

\[
\sum_{k=1}^n \lambda_k \| A_{\lambda_k}(s,x_k) \| \leq \sum_{k=1}^n \lambda_k \| A_0(s,z_k) \| + C'
\]

where \( C' \) is a r.v. independent of \( n \) and \( s \) and which is finite \( \mathbb{P} \)-a.e. By Assumption 5, the family \( \| A_0(.,z_k(\omega)) \| \) is uniformly integrable for almost every \( \omega \). Thus, the same holds for the corresponding averaged sequence, which in turn implies that the functions of \( s \) given by the lhs of (5.5) are uniformly integrable. The conclusion follows from Proposition 2. \( \square \)
6. Application to convex optimization.

6.1. Problem and Algorithm. A function $f : E \times \mathcal{H} \to (-\infty, +\infty]$ is called a normal convex integrand if it is $\mathcal{E} \otimes \mathcal{B}(\mathcal{H})$-measurable and if $f(s, \cdot)$ is lower semicontinuous proper and convex for each $s \in E$ [27]. For such a function $f$, we search for minimizers of the integral functional $F$ defined for all $x \in \mathcal{H}$ by

$$F(x) = \int f(s, x)d\mu(s)$$

if $f(\cdot, x)^+ \mu$-integrable and $F(x) = +\infty$ otherwise (we use the notation $a^+ = \max(\pm a, 0)$). We introduce the subdifferential operator $\partial f : E \times \mathcal{H} \to \mathcal{H}$ defined for all $(s, x) \in E \times \mathcal{H}$ by

$$\partial f(s, x) = \{u \in \mathcal{H} : \forall y \in \mathcal{H}, f(s, y) \geq f(s, x) + \langle u, y - x \rangle \}.$$

Identifying $\partial f$ with the operator $A$ of Section 3, the resolvent $J_\lambda$ coincides with the proximity operator $(s, x) \mapsto \text{prox}_{\lambda f(\cdot, \cdot)}(x)$ defined in (1.2). The iterations (3.1) write

$$(6.2) \quad x_{n+1} = \text{prox}_{\lambda_n f(\xi_{n+1}, \cdot)}(x_n).$$

The aim is to prove the almost sure weak convergence in average of $(x_n)$ to a minimizer of $F$ (assumed to exist). We denote by $\partial f_0(s, x)$ the element of $\partial f(s, x)$ with smallest norm. We denote by $\mathcal{D}$ the essential intersection of the sets $D_s = \text{dom}(\partial f(s, \cdot))$ for $s \in E$.

Assumption 8.

(i) $f : E \times \mathcal{H} \to (-\infty, +\infty]$ is a normal convex integrand.

(ii) $F$ is proper and lower semicontinuous.

(iii) For all $x \in \mathcal{H}$, $\partial F(x) = \int \partial f(s, x)d\mu(s)$.

(iv) The set of minimizers of $F$ is non-empty and included in $\mathcal{Z}_{\partial f}(2)$.

We discuss Assumption 8(iii). The inclusion $\int \partial f(s, x)d\mu(s) \subset \partial F(x)$ always hold. Conditions under which it holds with equality are provided in [25], [19]. As stated in [25], a sufficient condition for the equality to hold is that the functions $F, f(s, \cdot)$ $(s \in E)$ have the same domain and $F$ is continuous at some point. An other practical example will be discussed in Section 6.4.

6.2. Case of a common domain.

Theorem 3. Let Assumptions 2 and 8 hold true. Assume that the domains $D_s$ coincide for all $s$ outside a $\mu$-negligible set. Assume that for any compact set $K \subset \mathcal{H}$, the family $(\|\partial f_0(\cdot, x)\| : x \in K \cap \mathcal{D})$ is uniformly integrable. Consider the random sequence $(x_n)$ given by (6.2) with weighted averaged sequence $(\mathfrak{T}_n)$. Then, almost surely, $(\mathfrak{T}_n)$ converges weakly to a minimizer of $F$.

Proof. We prove that $A = \partial f$ satisfies the conditions of Assumptions 1 and 3 and the conclusion follows from Theorem 1. Operator $\partial f(s, \cdot)$ is maximal monotone for any given $s \in E$, see e.g. [7, Theorem 21.2]. For a fixed $x \in \mathcal{H}$, $\partial f(\cdot, x)$ is Effros-measurable, see [26, Corollary 4.6] and [19, Theorem 3] in the infinite dimensional case. The proximity operator $J_\lambda(\cdot, x)$ is $\mathcal{E}/\mathcal{B}(\mathcal{H})$ measurable, see [24, Lemma 4] (combined with [27, Proposition 2] in the infinite dimensional case). Therefore, $A = \partial f$ satisfies the conditions in Assumption 1.

Note that $F$ is a convex function. By Assumption 8(ii) and [7, Theorem 21.2], $\partial F$ is maximal monotone. Using moreover Assumption 8(iii), the condition in Assumption 3 is satisfied. Finally, Assumptions 1–5 are fulfilled and the conclusion follows from Theorem 1. □
6.3. Case of distinct domains. We now address the case where the domains $D_s$ are possibly distinct. We need the following lemma.

**Lemma 6.1.** Let $g : \mathcal{H} \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. Consider $x \in \mathcal{H}$ and $\lambda > 0$. Let $\pi$ be the projection of $x$ onto $\text{dom}(g)$. Assume that $\partial g(\pi) \neq \emptyset$. Then, $\|\text{prox}_{\lambda g}(x) - \pi\| \leq 2\lambda\|\partial g(\pi)\|$.

**Proof.** When $x = \pi$, the result is standard [7, Corollary 23.10] (and the factor 2 in the inequality can even be omitted). We assume in the sequel that $x \neq \pi$. Define

$$j = \text{prox}_{\lambda g}(x), \, \varphi = \partial g(\pi)$$

and

$$q = \arg\min_{y \in H} g(\pi) + \langle \varphi, y - \pi \rangle + \frac{\|y - x\|^2}{2\lambda}$$

where $H$ is the half-space $\{y \in \mathcal{H} : \langle y - \pi, x - \pi \rangle \leq 0\}$. It holds that

$$0 \in \lambda\varphi + q - x + N_H(q)$$

where $N_H(q)$ is the normal cone to $H$ at $q$. Upon noting that $N_H(q) \subset \mathbb{R}_+(x - \pi)$, there exists $\alpha \geq 0$ such that $\lambda\varphi = -q + x - \alpha(x - \pi)$. Now as $\varphi \in \partial g(\pi)$ and $(x - j)/\lambda \in \partial g(j)$, it follows by monotonicity of $\partial g$ that

$$0 \leq \langle \lambda\varphi - x + j, \pi - j \rangle = \langle j - q, \pi - j \rangle + \alpha(x - \pi, j - \pi).$$

As $\langle x - \pi, j - \pi \rangle \leq 0$, we have $0 \leq \langle j - q, \pi - j \rangle$ which in turn implies that $\|j - \pi\| \leq \|q - \pi\|$. The minimizer $q$ in (6.3) is determined by solving a linear programming problem. It is easy to show $\|q - \pi\| \leq 2\lambda\|\varphi\|$. The lemma is proved. \qed

**Assumption 9.** There exists $p \in \mathbb{N}^*$ and $C \in L^2(E, \mathbb{R}_+, \mu)$ such that for all $s \in E \mu$-a.e. and all $x \in \text{dom}(\partial f(s, .))$,

$$\|\partial f_0(s, x)\| \leq C(s)(1 + \|x\|^p)$$

and $Z_A(2p) \neq \emptyset$. Moreover, $\text{dom}(\partial f(s, .))$ is closed $\mu$-a.e.

**Theorem 4.** Let Assumptions 2, 8, 6 and 9 hold true. Suppose that $\lambda_n/\lambda_{n+1} \to 1$ as $n \to \infty$. Consider the random sequence $(x_n)$ given by (6.2) with weighted averaged sequence $(\overline{x}_n)$. Then, almost surely, $(\overline{x}_n)$ converges weakly to a minimizer of $F$.

**Proof.** When letting $A = \partial f$, the conditions in Assumptions 1–4 are fulfilled by using the same arguments as in the proof of Theorem 3. Moreover, Assumption 9 implies that the uniform integrability condition of Assumption 5 holds. To apply Theorem 2, it is sufficient to verify the condition of Assumption 7 replacing $J_\lambda(s, .)$ with $\text{prox}_{\lambda f(s, .)}$. By Lemma 6.1 and using $\Pi(s, x) \in D_s$, the following holds $\mu$-a.e.

$$\|\text{prox}_{\lambda f(s, .)}(x) - \Pi(s, x)\| \leq 2\lambda\|\partial f_0(s, \Pi(s, x))\|$$

$$\leq 2\lambda C(s)(1 + \|\Pi(s, x)\|^p).$$

Let $x^*$ be an arbitrary point in $D$. One has $\|\Pi(s, x)\| \leq \|x^*\| + \|\Pi(s, x) - \Pi(s, x^*)\|$ where we used the fact that $x^* = \Pi(s, x^*)$ for all $s \mu$-a.e. By non-expansiveness of $\Pi(s, .)$, $\|\Pi(s, x)\| \leq \|x^*\| + \|x - x^*\|$. Finally, there exists a constant $\alpha$ depending only on $p$ and $x^*$ such that $\|\text{prox}_{\lambda f(s, .)}(x) - \Pi(s, x)\| \leq \lambda\alpha C(s)(1 + \|x\|^p)$. The conclusion follows from Theorem 2. \qed
6.4. A constrained optimization problem. Let \((X_1, \ldots, X_m)\) be a collection of non-empty closed convex subsets of \(\mathcal{H} = \mathbb{R}^d\) where \(d \in \mathbb{N}^*\). We consider the problem

\[
(6.4) \quad \min F(x) \text{ w.r.t. } x \in \bigcap_{i=1}^{m} X_i
\]

where \(F(x) = \int f(s, x) d\mu(s)\) for all \(x \in \mathcal{H}\). Consider a random sequence \((I_n)\) on \(\{0, 1, \ldots, m\}\) independent of \((\xi_n)\), with distribution \(p_i = \mathbb{P}(I_n = i)\) for every \(i \in \{0, 1, \ldots, m\}\). Consider the iterations

\[
(6.5) \quad x_{n+1} = \begin{cases} 
\text{prox}_{\lambda_n f(\xi_{n+1}, \ldots)}(x_n) & \text{if } I_{n+1} = 0 \\
\text{proj}_{X_{n+1}}(x_n) & \text{otherwise.}
\end{cases}
\]

Assumption 10.

(i) The set \(\bigcap_{i=1}^{m} X_i\) has non-empty interior.

(ii) The sets \(X_1, \ldots, X_m\) are linearly regular in the sense of (5.2).

(iii) \(f : E \times \mathcal{H} \to \mathbb{R}\) is a normal convex integrand and \(f(., x)\) is integrable for each \(x \in \mathcal{H}\).

(iv) A solution to (6.4) exists and any solution \(x^*\) satisfies \(|\partial f(., x^*)| \in L^2(E, \mathbb{R}, \mu)\).

(v) There exists \(p \in \mathbb{N}^*\) and a solution \(x_n^*\) such that \(|\partial f(., x_n^*)| \in L^{2p}(E, \mathbb{R}, \mu)\).

(vi) There exists \(C \in L^2(E, \mathbb{R}^+, \mu)\) such that for any \(x \in \mathcal{H}\), \(||\partial f_0(s, x)|| \leq C(s)(1 + ||x||^p)\) \(\mu\text{-a.e.}\).

Theorem 5. Let Assumptions 2 and 10 hold. Consider the iterates \((x_n)\) given by (6.5) with weighted averaged sequence \((\tau_n)\) where the random sequence \((I_n)\) is defined above. Assume that \(p_i > 0\) for all \(i \in \{0, 1, \ldots, m\}\) and let \(\lambda_n/\lambda_{n+1} \to 1\) as \(n \to \infty\). Then, almost surely, \((\tau_n)\) converges in average to a solution to (6.4).

Proof. We introduce the random sequence \(\xi_n = (\xi_n, I_n)\) on the set \(\bar{E} = E \times \{0, 1, \ldots, m\}\) equipped with the corresponding product \(\sigma\)-algebra. We denote by \(\nu = \mu \otimes (\sum_{i=0}^{m} p_i \delta_i)\) the probability distribution of \(\xi_n\) where \(\delta_i\) stands for the Dirac measure at \(i\). For all \(s = (s, i)\) in \(\bar{E}\) and \(x \in \mathcal{H}\), define

\[
\tilde{f}(s, x) = f(s, x) \chi_{\{0\}}(i) + \sum_{j=1}^{m} \epsilon_{X_j}(x) \chi_{\{j\}}(i)
\]

where \(\chi_C\) is the characteristic function of a set \(C\) (equal to 1 on that set and zero outside) and \(\epsilon_C\) is the indicator function of a set \(C\) (equal to 0 on that set and \(+\infty\) outside). We use the convention \(0 \times (+\infty) = 0\). The iterations (6.5) also write

\[
x_{n+1} = \text{prox}_{\lambda_n f(\xi_{n+1}, \ldots)}(x_n).
\]

We show that \(\tilde{f}\) satisfies the conditions in Assumption 8 where \(f\) is replaced by \(\tilde{f}\).

(i) \(\tilde{f}\) is a normal convex integrand on \(\bar{E} \times \mathcal{H} \to \mathbb{R}\).

(ii) As \(f(., x)\) is integrable for any \(x\), it follows that \(F = \int f(., x)\) is proper, convex and continuous. Since \(p_i > 0\) for all \(i\), the integral functional \(\bar{F}(x) = \int \tilde{f}(., x) d\nu\) is equal to

\[
\bar{F}(x) = p_0 F(x) + \epsilon_X(x)
\]

where \(X = \bigcap_{i=1}^{m} X_i\). By Assumption 10(i), \(\bar{F}\) is proper and lower semicontinuous.

(iii) Using again Assumption 10(i) and the fact that \(\text{dom}(F) = \mathcal{H}\),

\[
\partial \bar{F}(x) = p_0 \partial F(x) + N_X(x)
\]
where $N_X(x)$ is the normal cone to $X$ at $x$. Moreover, for any $\tilde{s} = (s, i)$,

$$\partial \tilde{f}(\tilde{s}, x) = \partial f(s, x)\chi_{\{0\}}(i) + \sum_{j=1}^{m} N_{X_j}(x) \chi_{\{j\}}(i)$$  \tag{6.6}$$

and it follows that

$$\int \partial \tilde{f}(\cdot, x) d\nu = p_0 \int \partial f(\cdot, x) d\mu + \sum_{i=1}^{m} N_{X_i}(x).$$

Assumption 10(i) implies that $\sum_{i=1}^{m} N_{X_i}(x) = N_X(x)$ by [7, Corollary 16.39]. Moreover, as $F$ is everywhere finite, $\int \partial f(\cdot, x) d\mu = \partial F(x)$ by [25]. We conclude that for every $x \in \mathbb{H}$,

$$\int \partial \tilde{f}(\cdot, x) d\nu = \partial \tilde{F}(x).$$  \tag{6.7}$$

(iv) The minimizers of $\tilde{F}$ are the solutions to (6.4) and vice-versa. In particular, $\tilde{F}$ admits minimizers. Let us prove that each minimizer $x^*$ belongs to $Z_{\partial f}(2)$. By Fermat’s rule, $0 \in \partial \tilde{F}(x^*)$. Using successively (6.7) and (6.6), there exists $\phi \in S_{\partial f}(x^*)$ and $(u_1, \ldots, u_m) \in N_{X_1}(x^*) \times \cdots \times N_{X_m}(x^*)$ such that $0 = p_0 \int \phi d\mu + \sum_{i=1}^{m} p_i u_i$. Define for any $(s, i) \in \tilde{E}$, $\phi(s, i) = \phi(s)\chi_{\{0\}}(i) + \sum_{j=1}^{m} u_j \chi_{\{j\}}(i)$. Clearly, $\phi(s, i) \in \partial \tilde{f}((s, i), x^*)$ and $\int \phi d\nu = 0$. By Assumption 10(iv), $\int \|\phi\|^2 d\nu < +\infty$. Therefore, $x^* \in Z_{\partial f}(2)$.

We have checked that the four conditions in Assumption 8 are fulfilled when $f$ and $F$ are respectively replaced by $\tilde{f}$ and $\tilde{F}$. Now set $A = \partial \tilde{f}$. Using the same arguments as in the proof of Theorem 3, the operator $A$ fulfills the conditions in Assumptions 1, 3 and 4. Assumption 2 being granted, it remains to check that $A = \partial \tilde{f}$ fulfills Assumptions 5, 6 and 7.

The uniform integrability condition in Assumption 5 is an immediate consequence of Assumption 10(vi) along with (6.6). Using the linear regularity of the sets $X_1, \ldots, X_m$, Assumption 6 is satisfied when substituting $D_s$ with $\text{dom}(\partial \tilde{f}(s, \cdot))$.

We finally check that $A = \partial \tilde{f}$ fulfills Assumption 7. Let $p \in \mathbb{N}^*$ and $x_p^*$ be defined as in Assumption 10(v). Following the exact same line as above, one can construct $\tilde{\phi}$ such that $\tilde{\phi}(s, i) \in \partial \tilde{f}((s, i), x_p^*)$, $\int \tilde{\phi} d\nu = 0$ and $\int \|\tilde{\phi}\|^2 d\nu < +\infty$. Therefore, $Z_{\partial f}(2p) \neq \emptyset$. Denote by $\tilde{J}_f(\tilde{s}, x) = \text{prox}_{\tilde{f}(\cdot, x)}(x)$ and $\tilde{\Pi}(s, x)$ the projection of $x$ onto the domain of $\partial \tilde{f}(\cdot, \cdot)$. For any $\tilde{s} = (s, i)$, one has $\tilde{J}_f(\tilde{s}, x) - \tilde{\Pi}(\tilde{s}, x) = 0$ if $i \geq 1$. When $i = 0$, $\tilde{J}_f(\tilde{s}, x) = \text{prox}_{\tilde{f}(s, \cdot)}(x)$ and $\tilde{\Pi}(\tilde{s}, x) = x$. Thus, $\frac{1}{\lambda}\|\tilde{J}_f(\tilde{s}, x) - \tilde{\Pi}(\tilde{s}, x)\| \leq \|\partial f_0(s, x)\|$ which is no larger than $C(s)(1 + \|x\|)$. As $C$ is square-integrable, we conclude that the operator $A = \partial \tilde{f}$ fulfills Assumption 7.

By Theorem 2, the iterates (6.5) almost surely converge weakly in average to a zero of $\partial \tilde{F}$. As zeroes of $\partial \tilde{F}$ coincide with solutions to (6.4), the proof is complete. $\blacksquare$

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