On the BRST structure of $W_3$ gravity coupled to $c = 2$ matter

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Abstract

We present some explicit results on the structure of singular vectors in $c = 2$ Verma modules of the $W_3$ algebra. Using the embedding patterns of those vectors we construct resolutions for the $c = 2$ irreducible modules, and thus are able to compute some of the BRST cohomology of $W_3$ gravity coupled to $c = 2$ matter. In particular, we determine the states in the ground ring of the theory.
On the BRST structure of $\mathcal{W}_3$ gravity coupled to $c = 2$ matter

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ABSTRACT

We present some explicit results on the structure of singular vectors in $c = 2$ Verma modules of the $\mathcal{W}_3$ algebra. Using the embedding patterns of those vectors we construct resolutions for the $c = 2$ irreducible modules, and thus are able to compute some of the BRST cohomology of $\mathcal{W}_3$ gravity coupled to $c = 2$ matter. In particular, we determine the states in the ground ring of the theory.

1. Introduction

The BRST quantization of two dimensional gravity coupled to conformal matter has posed the interesting mathematical problem of computing the semi-infinite cohomology of the Virasoro algebra with values in a tensor product of two highest weight modules [1,2]. In this note we report on some explicit results in a partial solution to the similar problem of quantizing $\mathcal{W}_3$ gravity coupled to $\mathcal{W}_3$ matter.

More precisely, we study the cohomology of the BRST operator recently constructed in [3], on the product of two two-scalar Fock space modules of the $\mathcal{W}_3$ algebra. This so called ‘$d = (2,2)$ $\mathcal{W}_3$ string’ is a natural generalization of the usual $d = 2$ string.

The main source of differences between the present problem and its analogue for the Virasoro algebra is the nonlinear nature of the $\mathcal{W}_3$ algebra. As a result, many of the computational techniques of Lie algebra cohomology either cannot be used, or are too difficult to implement due to the much greater algebraic complexity of the problem. In addition, the structure of positive energy modules of the $\mathcal{W}_3$ algebra, most notably of $c = 2$ Verma modules, has only partially been understood.

This note is organized as follows. In Section 2, apart from basic definitions, we introduce positive energy modules of the $\mathcal{W}_3$ algebra, and summarize some of the known results. In Section 3 we discuss in some detail the structure of Verma modules of the $\mathcal{W}_3$ algebra at $c = 2$. In particular, we summarize our explicit studies of singular vectors at low-lying weights in those modules, and give their embedding patterns. From those results we conjecture resolutions for a class of irreducible modules in terms of (generalized) Verma modules. Finally, in Section 4 we introduce the BRST complex of the $\mathcal{W}_3$ algebra and discuss its cohomology on the tensor product of two two-scalar Fock spaces. For other and/or related results we refer the reader to e.g. [4-7] and references therein.
2. The $W_3$ algebra and its modules

The $W_3$ algebra is generated by the operators $L_m, W_m, m \in \mathbb{Z}$, which satisfy the following commutation relations [8] (for a review on $W$-algebras see [9] and references therein).

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \quad (2.1)
\]
\[
[L_m, W_n] = (2m - n)W_{m+n}, \quad (2.2)
\]
\[
[W_m, W_n] = (m - n)\left(\frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2)\right)L_{m+n}
+ \beta(m - n)\Lambda_{m+n} + \frac{c}{360}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}, \quad (2.3)
\]

where $\beta = 16/(22 + 5c)$ and

\[
\Lambda_m = \sum_n (L_{m-n}L_n) - \frac{3}{10}(m + 3)(m + 2)L_m, \quad (2.4)
\]

with the normal ordered product given by

\[
(L_m L_n) = \begin{cases} 
L_m L_n & \text{for } m \leq -2, \\
L_n L_m & \text{for } m > -2.
\end{cases} \quad (2.5)
\]

As for Lie algebras, the commutators satisfy the Jacobi identities, but the normal ordered terms introduce a nonlinear structure into the algebra. The $W_3$ algebra clearly contains the Virasoro algebra, generated by the $L_n, n \in \mathbb{Z}$, as a Lie subalgebra. The maximal abelian subalgebra (‘Cartan subalgebra’) $W_{3,0}$ of $W_3$ is spanned by $L_0$ and $W_0$, but, because $(\text{ad } W_0)$ is not diagonalizable, $W_3$ does not admit a root space decomposition. Nevertheless, it is still convenient to decompose the generators of $W_3$ according to the $(-\text{ad } L_0)$ eigenvalue, and, in particular, define $W_3^{(\pm)} = \{L_n, W_n \mid \pm n > 0\}$.

In the following we will consider only the so-called positive energy modules of $W_3$ [10], which are defined by the condition that (the energy operator) $-L_0$ is diagonalizable with finite dimensional eigenspaces, and with the spectrum bounded from below. If the lowest energy eigenspace is one dimensional we denote the eigenvalue of $L_0$ and $W_0$ on this highest weight state by $h$ and $w$, respectively.

In particular, the generalized Verma module $M(h, w, c)_N$ is defined as the positive energy module induced from an $N$-dimensional indecomposable representation of $W_{3,0}$. More explicitly, let $v_0, \ldots, v_{N-1}$ be a canonical basis such that

\[
L_0v_i = hv_i, \quad i = 0, \ldots, N - 1, \quad (2.6)
\]
\[
W_0v_0 = wv_0, \quad W_0v_i = wv_i + v_{i-1}, \quad i = 1, \ldots, N - 1. \quad (2.7)
\]

\[1\] We follow here the physicists convention in which the central element, $c$, of the algebra is set to a constant.
Then \( M(h, w, c)_N \) is spanned by the monomials of the form

\[
L_{-n_1} \ldots L_{-n_i} W_{-m_1} \ldots W_{-m_j} v_k, \quad i, j \geq 0, \quad k = 0, \ldots, N - 1, \quad (2.8)
\]
on which the generators in \( \mathcal{W}_3^{(-)} \) act freely, while the action of those in \( \mathcal{W}_3^{(+)} \) and \( \mathcal{W}_{3,0} \) is determined using (2.1)-(2.3), (2.6)-(2.7) and

\[
L_n v_i = W_n v_i = 0, \quad n > 0, \quad i = 0, \ldots, N - 1. \quad (2.9)
\]

Clearly, for \( N = 1 \) we recover the usual definition of the Verma module \( M(h, w, c) \). By the standard argument, \( M(h, w, c) \) contains a maximal submodule. We denote the corresponding irreducible quotient module by \( L(h, w, c) \). The module contragradient to \( M(h, w, c) \) will be denoted by \( M(h, w, c)^* \).

From the physics viewpoint the most natural class of positive energy modules of \( \mathcal{W}_3 \) are the Fock space modules \( F(\Lambda, \alpha_0) \), which arise in the free field realization of the \( \mathcal{W}_3 \) algebra in terms of two scalar fields, and thus generalize the Feigin-Fuchs modules of the Virasoro algebra. Here \( \Lambda \) is an \( \mathfrak{sl}(3) \) weight. [In the following we will use the notation \( P_+ \) for the set of dominant integral weights of \( \mathfrak{sl}(3) \), \( \Delta_+ \) for its positive roots, and \( Q_+ = \mathbb{Z}_+ \Delta_+ \) for the positive root lattice.] The background charge, \( \alpha_0 \), is determined by \( c = 2 - 24\alpha_0^2 \).

The eigenvalues of \( L_0 \) and \( W_0 \) on the highest weight state \( |\Lambda, \alpha_0\rangle \) of \( F(\Lambda, \alpha_0) \) are

\[
h(\Lambda) = -(\theta_1 \theta_2 + \theta_1 \theta_3 + \theta_2 \theta_3) - \alpha_0^2 = \frac{1}{2} (\Lambda, \Lambda + 2\alpha_0 \rho),
\]

\[
w(\Lambda) = \sqrt{3/2} \theta_1 \theta_2 \theta_3, \quad (2.10)
\]

where

\[
\theta_1 = (\Lambda + \alpha_0 \rho, \Lambda_1), \quad \theta_2 = (\Lambda + \alpha_0 \rho, \Lambda_2 - \Lambda_1), \quad \theta_3 = (\Lambda + \alpha_0 \rho, -\Lambda_2). \quad (2.11)
\]

The weights \( \Lambda_1 \) and \( \Lambda_2 \) are the fundamental weights of \( \mathfrak{sl}(3) \), and \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \) is the Weyl vector. Note that \( h(\Lambda) \) and \( w(\Lambda) \) as in (2.10) determine \( \Lambda \) only up to a Weyl rotation \( \Lambda \to w(\Lambda + \alpha_0 \rho) - \alpha_0 \rho, \ w \in W \). In the following we will also write \( M(\Lambda, c) \), or simply \( M(\Lambda) \) for the (generalized) Verma module \( M(h(\Lambda), w(\Lambda), c) \).

The following two theorems summarize some of the known results on the structure of Fock space modules \( F(\Lambda, \alpha_0) \). We have written them in a form which allows an immediate generalization to higher rank \( \mathcal{W} \) algebras. The first theorem can be proven by examining the Shapovalov form on the Verma module [5], while the second one essentially follows from the unitarizability of \( F(\Lambda, 0) \), an explicit construction of singular vectors, and direct comparison of the character for each side [5].

\[\text{2} \] For a detailed discussion see [9] and references therein.
Theorem 2.1.
(a) \( F(\Lambda, \alpha_0) \cong \begin{cases} M(\Lambda, c) & \text{if } (\Lambda + \alpha_0 \rho, \alpha) \notin (IN_+ + IN_-) \text{ for all } \alpha \in \Delta_+, \\ M(\Lambda, c) & \text{if } (\Lambda + \alpha_0 \rho, \alpha) \notin -(IN_+ + IN_-) \text{ for all } \alpha \in \Delta_+. \end{cases} \)

In particular, if \( (\Lambda + \alpha_0 \rho, \alpha) \notin (IN_+ + IN_-) \text{ for all } \alpha \in \Delta, \) then \( M(\Lambda, c) \) (and thus also \( F(\Lambda, \alpha_0) \)) is irreducible.
(b) For \( \alpha_0^2 \leq -4 \) or, equivalently, \( c \geq c_{crit} - \ell = \ell + 48|\rho|^2 \) we have
\[
F(\Lambda, \alpha_0) \cong \begin{cases} M(\Lambda, c) & \text{for } i(\Lambda + \alpha_0 \rho) \in \eta D_+, \\ M(\Lambda, c) & \text{for } -i(\Lambda + \alpha_0 \rho) \in \eta D_+, \end{cases}
\]
where \( \ell \) is the rank of the \( W \) algebra (\( \ell = 2 \) for \( W_3 \)), \( D_+ \) denotes the fundamental Weyl chamber, and \( \eta = \text{sign}(-i\alpha_0) \).

Theorem 2.2. If \( w \in W \) such that \( w\Lambda \in P_+ \) then
\[
F(\Lambda, 0) = \bigoplus_{\beta \in Q_+} m(w\Lambda; \beta) L(w\Lambda + \beta),
\]
(2.12)
where, for \( \Lambda \in P_+ \) and \( \beta \in Q_+ \), the multiplicity \( m(\Lambda; \beta) \) (with which the \( c = \ell = 2 \) irreducible \( W_3 \)-module \( L(\Lambda + \beta) \) occurs in the direct sum decomposition of \( F(\Lambda, 0) \)) is equal to the multiplicity of the weight \( \Lambda \) in the irreducible finite dimensional representation of \( sl(3) \) with highest weight \( \Lambda + \beta \).

3. The structure of \( W_3 \) Verma modules for \( c = 2 \)

It is an interesting mathematical problem, as well as being crucial for our results in the next section, to construct resolutions of the irreducible modules \( L(\Lambda, c) \) in terms of Verma modules. For that reason we have studied explicitly the singular vectors and their embedding structure in \( c = 2 \) (generalized) Verma modules \( M(\Lambda)_N \) for low lying weights \( \Lambda \). Except for the simplest cases, all the computations were performed using algebraic manipulation programs [11]. We discuss below our results, which are summarized in Tables I, Ia, II and IIa in Appendix A.

For convenience we denote weights by their Dynkin labels \((a_1, a_2)\), instead of \( \Lambda \), where \( \Lambda = a_1 \Lambda_1 + a_2 \Lambda_2 \). We recall that, as usual, singular vectors are defined to be those vectors in \( M(\Lambda) \) that are annihilated by all the generators in \( W_3^{(+)} \). The vectors in the tables are such that \( L_0 \) is diagonal and \( W_0 \) is in a canonical Jordan form. Then \( u_{a_1a_2} \) denotes a singular vector at energy level \( h(a_1, a_2) \), which is an eigenstate of \( W_0 \) with the eigenvalue \( w(a_1, a_2) \).

Similarly, \( v_{a_1a_2} \) is a singular vector for which \( W_0 v_{a_1a_2} = w(a_1, a_2)v_{a_1a_2} + u_{a_1a_2} \). The \( v \)-type singular vectors are determined only up to addition of an \( u \)-type vector. [Although one might expect to also find higher dimensional indecomposable representations of \( W_{3,0} \) on
the subspaces of singular vectors, it is a curious fact that they do not arise in the examples below. However, in many cases a singular vector \( u \) or a doublet \((v, u)\) is found inside a higher dimensional indecomposable representation of \( \mathcal{W}_{3,0} \).

A singular vector \( u_{a_1a_2} \) in \( M(b_1, b_2)_N \) determines a canonical embedding of \( M(a_1, a_2) \) into \( M(b_1, b_2)_N \). Similarly, any \( v_{a_1a_2} \) in \( M(b_1, b_2)_N \) defines a homomorphism of \( M(a_1, a_2)_2 \) into \( M(b_1, b_2)_N \) which, in general, has a nontrivial kernel. Singular vectors which vanish under such homomorphism are denoted by primes; e.g. in Table I, \( u'_30 \) and \( u'_03 \) in \( M(1, 1)_2 \) vanish in the submodule generated by \((v_{11}, u_{11})\) in \( M(0, 0) \). By examining various columns for the same entries one can also determine singular vectors that lie in intersections of submodules, e.g. from Tables II and IIa we find that the intersection of \( M(2, 0) \) and \( M(1, 2)_2 \) in \( M(0, 1) \) contains \( u_{12} \) and \((v_{31}, u_{31})\).

The singular vectors and their embedding patterns lead to the following conjecture for resolutions of irreducible modules \( L(a_1, a_2) \) at \( c = 2 \).

**Conjecture 3.1.** The resolution of an irreducible module \( L(\Lambda, 2) \) in terms of generalized Verma modules is one of three types as given in Appendix B. The type of the resolution depends on whether \( \Lambda \) lies in the interior of the fundamental Weyl chamber, on the boundary, or, for \( \Lambda = 0 \), on the intersection of two boundaries.

Note that these resolutions are finite, as is also the case for resolutions of \( c = 2 \) irreducible modules in terms of Fock space modules. It is a nontrivial check that the resolutions in Appendix B give rise to characters which agree with those obtained from the Fock space resolutions. Our explicit results confirm those resolutions for the weights \((0, 0), (0, 1), (1, 0), (1, 1), (0, 2) \) and \((2, 0)\), and partially for some of the other weights. However, it is possible, though unlikely, that the resolutions may get modified because of unaccounted singular vector structure beyond the levels which we have studied explicitly.

**4. The BRST cohomology**

The basic idea of the explicit construction of the semi-infinite cohomology complex of the \( \mathcal{W}_3 \)-algebra [12,3] (in the physics literature called the BRST complex) is essentially the same as for the Virasoro or affine Lie algebras (see e.g. [13-15]). First one introduces an infinite dimensional Clifford algebra of ghost operators, \((b^{[1]}_m, c^{[1]}_m)\) and \((b^{[2]}_m, c^{[2]}_m)\), corresponding to the generators \( L_m \) and \( W_m, \ m \in \mathbb{Z} \), respectively. These ghost operators satisfy the anti-commutation relations \( \{c^{[1]}_m, b^{[1]}_n\} = \delta_{m+n,0} \) and \( \{c^{[2]}_m, b^{[2]}_n\} = \delta_{m+n,0} \). The corresponding standard positive energy module, \( F^{gh} \), of the ghost algebra is freely generated by the negative mode operators acting on the highest weight state \(|0\rangle_{gh} \) satisfying

\[
\begin{align*}
    b^{[1]}_n|0\rangle_{gh} &= b^{[2]}_n|0\rangle_{gh} = 0, \quad n \geq 0, \\
    c^{[1]}_n|0\rangle_{gh} &= c^{[2]}_n|0\rangle_{gh} = 0, \quad n > 0.
\end{align*}
\]

(4.1)
This ghost Fock space has a canonical grading by the ghost number $\text{gh}(\cdot)$, where $\text{gh}(c_n^{[1]}) = \text{gh}(b_n^{[1]}) = 1$, while the ghost number of the vacuum state $|0\rangle_{gh}$ is zero. Given a vector space $V$ on which the $\mathcal{W}_3$ algebra acts, the space of semi-infinite forms is simply the tensor product $V \otimes F^{gh}$, with the degree given by the ghost number. The problem is then to construct a nilpotent operator of degree one, the BRST operator, which becomes the differential in the complex. In addition, the structure of the leading terms should be the same as for the ordinary differential in the semi-infinite complex of Lie algebras. The two cases in which this construction has been carried out, and which appear to be relevant in $\mathcal{W}_3$ gravity theories, are:

i) $V$ is a $\mathcal{W}_3$ algebra module with the central charge $c = 100$ [12].

ii) $V$ is a tensor product of two $\mathcal{W}_3$-modules, $V = V^M \otimes V^L$, with $c^M + c^L = 100$ [3].

In the second case, the differential is given by the following normal ordered operator

$$
\begin{align*}
   d &= \sum_m (c_{-m}^{[2]}(\tilde{W}_m^M - i\tilde{W}_m^L) + c_{-m}^{[1]}(L_m^M + L_m^L)) \\
   &\quad + \sum_{m,n} \left( -\frac{1}{2}(m-n)c_{-m}^{[1]}c_{-n}^{[1]}b_{m+n}^{[1]} - (2m-n)c_{-m}^{[1]}c_{-n}^{[2]}b_{m+n}^{[2]} \\
   &\quad - \frac{1}{6}\mu(m-n)(2m^2 - mn + 2n^2 - 8)c_{-m}^{[2]}c_{-n}^{[2]}b_{m+n}^{[1]} \\
   &\quad + \sum_{m,n,p} \left( -\frac{1}{2}(m-n)c_{-m}^{[2]}c_{-n}^{[2]}b_{-p}^{[1]}(L_{m+n+p}^M - L_{m+n+p}^L) \right),
\end{align*}
$$

(4.2)

where $\tilde{W}_{m,l} = W_{m,l}/\sqrt{\beta_{m,l}}$ and $\mu = (1 - 17\beta^M)/(10\beta^M)$. We will denote the cohomology of $d$ at degree $n$ by $H^{(n)}(d, V^M \otimes V^L)$, and call it the semi-infinite cohomology, or the BRST cohomology, of the $\mathcal{W}_3$ algebra on $V^M \otimes V^L$. When either $V^M$ or $V^L$ is a trivial module, the BRST operator (4.2) reduces to the one of [12].

In the context of $\mathcal{W}_3$ gravity one is interested in computing the BRST cohomologies when $V^M$ is either an irreducible module $L(h^M, c^M)$ or a Fock space module $F(\Lambda^M, \alpha_0^M)$. In both cases $V^L$ is a Fock space module $F(\Lambda^L, \alpha_0^L)$. The cohomology of $L(h^M, c^M) \otimes F(\Lambda^L, \alpha_0^L)$, when $c^M < 2$, for the so-called $\mathcal{W}_3$ minimal models coupled to $\mathcal{W}_3$ gravity, has been discussed in [5]. In the following we will consider only the second case when $c^M = 2$, i.e. the cohomology of $F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i)$. 

There are several reasons why understanding this cohomology is important. Firstly, it gives the spectrum of physical states of the $d = (2,2)$ $\mathcal{W}_3$ string, which one expects to have a similar structure to that of the $d = 2$ string. In particular, the nontrivial cohomology should arise only in degrees between $-3$ and $5$, and the states with the lowest, i.e. $-3$, ghost number form a “ground ring” which to a large extent reflects the properties of the theory [16]. The other reason is that, provided our intuition from the case of the Virasoro algebra is correct, one may be able to continuously deform the cohomology
of two Fock space modules to other values of $c^M$. [In fact the explicit construction of some cohomology states in [4] strongly supports such a possibility.] If this is indeed true, then all the interesting cohomologies could easily be computed because the resolutions of irreducible modules in terms of Fock space modules are known [17,10].

A direct computation of the cohomology of two Fock space modules appears to be quite difficult (unlike the Virasoro algebra [2]) because of the algebraic complexity of the problem. However, given the results of Sections 2 and 3, we may obtain partial results by reducing the computation to the one of Verma modules. In particular, a direct modification of standard arguments yields the following ‘reduction theorem’ [5]

**Theorem 4.1.** The cohomology $H(d, M(\Lambda^M)_N \otimes \overline{M}(\Lambda^L))$ is nonvanishing if and only if $w(\Lambda^M + \alpha^M_0 \rho) = -i(\Lambda^L + \alpha^L_0 \rho)$ for some $w \in W$, in which case it is spanned by the states $v_0, c^1_0 v_0, c^2_0 v_{N-1}$ and $c^{[1]}_0 c^{[2]}_0 v_{N-1}$, where $v_i = v^M_i \otimes \overline{v}^L \otimes |0\rangle_{gh}$ (cf (2.6) – (2.9)).

Let us define $\alpha_\pm = \frac{1}{2}(\alpha^M_0 \mp i\alpha^L_0)$ so that $\alpha_+ \alpha_- = -1$. An immediate consequence of Theorems 2.1 and 4.1 is that the cohomology is essentially trivial for a generic class of Fock spaces. More precisely, we parametrize the momenta $\Lambda^M$ and $\Lambda^L$ by

$$
\Lambda^M + \alpha^M_0 \rho = \alpha_+ \Lambda^M(+) + \alpha_- \Lambda^M(-),
$$

$$
-i(\Lambda^L + \alpha^L_0 \rho) = \alpha_+ \Lambda^L(+) - \alpha_- \Lambda^L(-).
$$

We call them generic iff there is no positive root $\alpha \in \Delta_+$ such that

$$(\Lambda^M(+), \alpha) \in \mathbb{Z}, \quad (\Lambda^L(-), \alpha) \in \mathbb{Z}, \quad (\Lambda^M(+), \alpha)(\Lambda^L(-), \alpha) > 0.$$  \hspace{1cm} (4.4)

Then we have

**Theorem 4.2.** For generic momenta $\Lambda^M$ and $\Lambda^L$ as defined above, $H(d, F(\Lambda^M, \alpha^M_0) \otimes F(\Lambda^L, \alpha^L_0)) \neq 0$ iff there exists $w \in W$ such that $w(\Lambda^M + \alpha^M_0 \rho) = -i(\Lambda^L + \alpha^L_0 \rho)$, in which case it is spanned by the states $v, c^1_0 v, c^2_0 v$ and $c^{[1]}_0 c^{[2]}_0 v$, where $v = |\Lambda^M\rangle \otimes |\Lambda^L\rangle \otimes |0\rangle_{gh}$ is the physical vacuum.

Returning to the $c^M = 2$ case (i.e. $\alpha_\pm = \pm 1$) we first note that the structure of the cohomology simplifies considerably because of the following observation

**Theorem 4.3.** The cohomology spaces $H^{(n)}(d, F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i))$ carry a fully reducible representation of $sl(3)$. The $sl(3)$ generators are explicitly given by the zero modes of a level-1 Frenkel-Kac-Segal vertex operator construction in terms of matter fields only.

We now consider a non-generic case, when both $\Lambda^M$ and $-i\Lambda^L$ are integral weights.\(^3\) Moreover, we further restrict the Liouville momenta to the region $-i\Lambda^L + 2\rho \in D_+$. Then

\(^3\) The remaining cases, when $\Lambda^M, \Lambda^L$ are integral along certain roots, can be analyzed similarly.
we can combine Theorems 2.1 and 2.2 and reduce the computation of \(H(d, F(\Lambda^M, 0) \otimes F(\Lambda^L, \alpha_0^L))\) to that of \(H(d, L(\Lambda) \otimes M(\Lambda^L))\) for the \(c = 2\) irreducible modules, \(L(\Lambda)\), which appear in the decomposition of \(F(\Lambda^M, 0)\). In turn, the latter cohomology can be determined by standard arguments using the resolutions of Section 3 and Theorem 4.1 – in particular one finds that indeed the lowest ghost number states have ghost number \(-3\). From this follows directly the enumeration of all the states in the corresponding (chiral) ground ring \(\mathcal{R}\), which we summarize as

**Conjecture 4.4.** Given a pair of integral weights \((\Lambda^M, -i\Lambda^L)\), let \(w \in W\) be such that \(w\Lambda^M \in P_+\). Then there are ground ring elements \(\phi_{(\Lambda^M, -i\Lambda^L)}\) provided \(w\Lambda^M + \beta = -i\Lambda^L\) for some \(\beta \in Q_+\). The multiplicity depends on whether \(w\Lambda^M + \beta\) is in the interior or on the boundary of the fundamental Weyl chamber, and is given by \(2m(w\Lambda^M; \beta)\) or \(m(w\Lambda^M; \beta)\), respectively. The (relative) energy level at which \(\phi_{(\Lambda^M, -i\Lambda^L)}\) occurs is given by \(\frac{1}{2} - i\Lambda^L + 2\rho|2 - \frac{1}{2}|\Lambda^M|^2\).

It is not difficult to see that under the \(sl(3)\) symmetry in Theorem 4.3 the ground ring \(\mathcal{R}\) decomposes into the following direct sum of irreducible finite-dimensional \(sl(3)\) modules

\[
\mathcal{R} = \bigoplus_{\Lambda \in P_+} \mathcal{R}_\Lambda^{(1)} \oplus \bigoplus_{\Lambda \in P_{++}} \mathcal{R}_\Lambda^{(2)},
\]

\(i.e.\) each representation corresponding to an integral weight in the interior of the fundamental Weyl chamber \((P_{++})\) occurs exactly twice, while those on the boundary \((P_+ \setminus P_{++})\) exactly once. Since the \(sl(3)\) symmetry acts by automorphisms of the ground ring, a simple counting of the states indicates that the generators of the ring will have to satisfy vanishing relations.

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## Appendix A

### Table I

| $h$ | $M(0,0)$ | $M(1,1)$ | $M(3,0)$ | $M(0,3)$ | $M(2,2)$ | $M(4,1)$ | $M(1,4)$ | $M(3,3)$ | \ldots |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|--------|
| 0   | $u_{00}$| $u_{11}$|         |         |         |         |         |         |        |
| 1   | $(v_{11}, u_{11})$| $u_{11}$|         |         |         |         |         |         |        |
| 3   | $u_{30}$| $u_{30}$| $u_{30}$|         |         |         |         |         |        |
| 4   | $(v_{22}, u_{22})$| $(v_{22}, u_{22})$| $u_{22}$| $u_{22}$| $u_{22}$|         |         |         |        |
| \vdots | $(v_{41}, u_{41})$| $(v_{41}, u_{41})$| $u_{41}$| $u_{41}$| $u_{41}$| $u_{41}$|         |         |        |
| 7   | $(v_{41}, u_{41})$| $(v_{41}, u_{41})$| $u_{41}$| $u_{41}$| $u_{41}$| $u_{41}$| $u_{41}$|         |        |
| \vdots | $(v_{14}, u_{14})$| $(v_{14}, u_{14})$| $u_{14}$| $u_{14}$| $u_{14}$| $u_{14}$| $u_{14}$|         |        |
| 9   | $(v_{33}, u_{33})$| $(v_{33}, u_{33})$| $(v_{33}, u_{33})$| $(v_{33}, u_{33})$| $u_{33}$| $u_{33}$| $u_{33}$|         |        |
| \vdots | \vdots| \vdots| \vdots| \vdots| \vdots| \vdots| \vdots| \vdots| \vdots|

### Table Ia

| $h$ | $M(1,1)_2$ | $M(2,2)_2$ | $M(4,1)_2$ | $M(1,4)_2$ | $M(3,3)_2$ | \ldots |
|-----|---------|---------|---------|---------|---------|--------|
| 1   | $(v_{11}, u_{11})$|         |         |         |         |        |
| 3   | $u_{30}, u'_{30}$| $u_{03}, u'_{03}$|         |         |         |        |
| 4   | $(v_{22}, u'_{22})$, $u'_{22}$| $(v_{22}, u_{22})$|         |         |         |        |
| \vdots | \vdots| $(v_{41}, u_{41})$| $(v_{41}, u_{41})$| $(v_{14}, u_{14})$| $(v_{14}, u_{14})$|        |
| 7   | \vdots| $(v_{41}, u_{41})$| $(v_{41}, u_{41})$| $(v_{14}, u_{14})$| $(v_{14}, u_{14})$|        |
| \vdots | \vdots| $(v_{33}, u_{33})$, $u'_{33}$| $(v_{33}, u_{33})$, $u'_{33}$| $(v_{33}, u_{33})$, $u'_{33}$| $(v_{33}, u_{33})$, $u'_{33}$|        |
| \vdots | \vdots| \vdots| \vdots| \vdots| \vdots| \vdots|
**Table II**

| $h$ | $M(0,1)$ | $M(2,0)$ | $M(1,2)$ | $M(3,1)$ | $M(0,4)$ | $M(2,3)$ | $M(5,0)$ | $M(4,2)$ | ... |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|-----|
| $\frac{1}{3}$ | $u_{01}$ |          |          |          |          |          |          |          |     |
| $\frac{4}{3}$ | $u_{20}$ | $u_{20}$ |          |          |          |          |          |          |     |
| $\frac{7}{3}$ | $(v_{12}, u_{12})$ | $u_{12}$ | $u_{12}$ |          |          |          |          |          |     |
| $\frac{13}{3}$ | $(v_{31}, u_{31})$ | $(v_{31}, u_{31})$ | $u_{31}$ | $u_{31}$ |          |          |          |          |     |
| $\frac{16}{3}$ | $u_{04}$ | $u_{04}$ | $u_{04}$ |          |          |          |          |          |     |
| $\frac{19}{3}$ | $(v_{23}, u_{23})$ | $(v_{23}, u_{23})$ | $(v_{23}, u_{23})$ | $u_{23}$ | $u_{23}$ | $u_{23}$ |          |          |     |
| $\frac{25}{3}$ | : | $u_{50}$ | $u_{50}$ | $u_{50}$ |          |          |          |          |     |
| $\frac{28}{3}$ | : | $(v_{42}, u_{42})$ | $(v_{42}, u_{42})$ | $(v_{42}, u_{42})$ | $u_{42}$ | $u_{42}$ | $u_{42}$ | $u_{42}$ |     |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

**Table IIa**

| $h$ | $M(1,2)_2$ | $M(3,1)_2$ | $M(2,3)_2$ | $M(4,2)_2$ | ... |
|-----|-------------|-------------|-------------|-------------|-----|
| $\frac{7}{3}$ | $(v_{12}, u_{12})$ | $(v_{31}, u_{31})$ | $(v_{31}, u_{31})$ |          |     |
| $\frac{13}{3}$ | $(v_{31}, u_{31})$ | $(v_{31}, u_{31})$ |          |          |     |
| $\frac{16}{3}$ | $u_{04}, u_{04}'$ | $(v_{23}, u_{23})$ | $(v_{23}, u_{23})$ |          |     |
| $\frac{19}{3}$ | $(v_{23}, u_{23})$ | $(v_{23}, u_{23})$ | $(v_{23}, u_{23})$ |          |     |
| $\frac{25}{3}$ | : | $u_{50}, u_{50}'$ |          |          |     |
| $\frac{28}{3}$ | : | $(v_{42}, u_{42})$, $u_{42}'$ | $(v_{42}, u_{42})$, $u_{42}'$ | $(v_{42}, u_{42})$, $u_{42}'$ |     |
| ... | ... | ... | ... | ... | ... |