Characterization of completely $k$-magic regular graphs

A A Eniego$^1$ and I J L Garces$^2$

$^1$ Science and Mathematics Department, National University, Manila, The Philippines
$^2$ Department of Mathematics, Ateneo de Manila University, Quezon City, The Philippines

E-mail: aaeniego@national-u.edu.ph, ijlgarces@ateneo.edu

Abstract. Let $k \in \mathbb{N}$ and $c \in \mathbb{Z}_k$. A graph $G$ is said to be $c$-sum $k$-magic if there is a labeling $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$ such that $\sum_{uv \in E(G)} \ell(uv) \equiv c \pmod{k}$ for every vertex $v$ of $G$, where $N(v)$ is the neighborhood of $v$ in $G$. We say that $G$ is completely $k$-magic whenever it is $c$-sum $k$-magic for every $c \in \mathbb{Z}_k$. In this paper, we characterize all completely $k$-magic regular graphs.

1. Introduction
Let $G = (V(G), E(G))$ be a finite, simple (unless otherwise stated) graph with vertex set $V(G)$ and edge set $E(G)$. A factor of $G$ is a subgraph $H$ with $V(H) = V(G)$. In particular, if a factor $H$ of $G$ is $h$-regular, then we say that $H$ is an $h$-factor of $G$. An $h$-factorization of $G$ is a partition of $E(G)$ into disjoint $h$-factors. If such factorization of $G$ exists, then we say that $G$ is $h$-factorable.

The following theorem is attributed to Petersen [7], which we state using the versions of Akiyama and Kano [2] and Wang and Hu [10].

Theorem 1.1 ([2, Theorem 3.1], [7], [10, Theorem 10]). Let $G$ be a $2r$-regular connected general graph (not necessarily simple), where $r \geq 1$. Then $G$ is 2-factorable, and it has a $2k$-factor for every $k$, $1 \leq k \leq r$. Moreover, if $G$ is of even order, then it is $r$-factorable.

A graph $G$ is $\lambda$-edge connected if it remains connected whenever fewer than $\lambda$ edges are removed.

Theorem 1.2. [6] Let $r$ and $k$ be integers such that $1 \leq k < r$, and $G$ be a $\lambda$-edge connected $r$-regular general graph, where $\lambda \geq 1$. If one of the following conditions holds:

1. $r$ is even, $k$ is odd, $|G|$ is even, and $\frac{r}{2} \leq k \leq r(1 - \frac{1}{k})$,
2. $r$ is odd, $k$ is even, and $2 \leq k \leq r(1 - \frac{1}{k})$, or
3. $r$ and $k$ are both odd and $\frac{r}{2} \leq k$,

then $G$ has a $k$-regular factor.

Let $k$ be a positive integer. A finite simple graph $G = (V(G), E(G))$ is said to be $k$-magic if there exists an edge labeling $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$, where $\mathbb{Z}_1 = \mathbb{Z}$ the group of integers, and $\mathbb{Z}_k = \{0, 1, 2, \ldots, k - 1\}$ the group of integers modulo $k \geq 2$, such that the induced vertex labeling $\ell^+ : V(G) \to \mathbb{Z}_k$, defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$, is a constant map. If $c \in \mathbb{Z}_k$
and $\ell^+(v) = c$ for all $v \in V(G)$, then we call $c$ a magic sum of $G$. In particular, if $G$ is $k$-magic with magic sum $c$, then we say that $G$ is $c$-sum $k$-magic. If $G$ is $c$-sum $k$-magic for all $c \in \mathbb{Z}_k$, then it is said to be completely $k$-magic. The set of all magic sums $c \in \mathbb{Z}_k$ of $G$ is the sum spectrum of $G$ with respect to $k$ and is denoted by $\Sigma_k(G)$. If $c = 0$, then we say that $G$ is zero-sum $k$-magic. The null set of $G$, denoted by $N(G)$, is the set of all positive integers $k$ such that $G$ is a zero-sum $k$-magic graph.

Remark 1.3. If $c \in \mathbb{Z}_k$ and $\ell$ is a $c$-sum $k$-magic labeling of $G$, then the labeling $\ell'$, defined by $\ell'(e) = k - \ell(e)$, is a $(k-c)$-sum $k$-magic labeling of $G$.

Remark 1.4. Any 2-magic graph is not completely 2-magic.

The concept of $A$-magic graphs is due to Sedlacek [9]. Over the years, many papers have been published in connection with magic graphs. Akbari, Rahmati, and Zare [1] investigated the zero-sum $k$-magic labelings and null sets of regular graphs. Dong and Wang [4] solved affirmatively a conjecture posed in [1] on the existence of a zero-sum 3-magic labeling of 5-regular graphs. Salehi [8] determined the integer-magic spectra of certain classes of cycle-related graphs. Using the term “index set,” Wang and Hu [10] initially studied the concept of completely $k$-magic graphs. They gave a partial list of completely 1-magic regular graphs. Eniego and Garces [5] completely added the remaining cases in this list. They also presented the sum spectra of some regular graphs that are not completely $k$-magic.

Theorem 1.5 ([1, Theorem 13]). Let $G$ be an $r$-regular graph, where $r \geq 3$ and $r \neq 5$. If $r$ is even, then $N(G) = \mathbb{N}$ (the set of positive integers); otherwise, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.

Theorem 1.6 ([4, Theorem 2.1]). Every 5-regular graph admits a zero-sum 3-magic labeling.

Theorem 1.7 ([5, Theorem 3.3]). Let $n \geq 3$ and $k \geq 3$ be integers, and $C_n$ the cycle with $n$ vertices.

1. If $n$ is even, then $C_n$ is completely $k$-magic for all $k$.
2. If $n$ is odd, then $C_n$ is not completely $k$-magic for any $k$. Moreover, we have

$$\Sigma_k(C_n) = \begin{cases} \mathbb{Z}_k \setminus \{0\} & \text{if } k \text{ is odd}, \\ \{0, 2, \ldots, k-2\} & \text{if } k \text{ is even}. \end{cases}$$

Theorem 1.8 ([5, Lemma 3.4]). Let $k \geq 4$ be an even integer. Then there exists no $k$-magic graph of odd order that is completely $k$-magic. In particular, if $c$ is a magic sum of a $k$-magic graph of odd order, then $c$ must be even.

Theorem 1.9 ([5, Theorem 3.6]). Let $k, r \geq 3$ be integers, and $G$ an $r$-regular graph. If $\gcd(r, k) = 1$, then $\{1, 2, \ldots, k-1\} \subseteq \Sigma_k(G)$.

Theorem 1.10 ([5, Theorem 3.7]). Let $G$ be a zero-sum $k$-magic $r$-regular graph, where $k \geq 3$ and $r \geq 3$. If $G$ has a 1-factor, then $G$ is completely $k$-magic.

Theorem 1.11 ([10, Theorem 13], [5, Theorem 2.1]). Let $G$ be an $r$-regular graph of order $n$. Then

$$\Sigma_1(G) = \begin{cases} \mathbb{Z} \setminus \{0\} & \text{if } r = 1, \\ \mathbb{Z} & \text{if } r = 2 \text{ and } G \text{ contains even cycles only}, \\ 2\mathbb{Z} \setminus \{0\} & \text{if } r = 2 \text{ and } G \text{ contains an odd cycle}, \\ 2\mathbb{Z} & \text{if } r \geq 3, r \text{ even, and } n \text{ odd}, \\ \mathbb{Z} & \text{if } r \geq 3 \text{ and } n \text{ even}, \end{cases}$$

where $2\mathbb{Z}$ is the set of all even integers.
With Remark 1.4 and Theorem 1.11, it remains to characterize all completely $k$-magic regular graphs for $k \geq 3$. This characterization is the main theorem of this paper, which we state as follows.

**Theorem 1.12 (Main Theorem).** Let $r \geq 2$ and $k \geq 3$ be integers, and $G$ an $r$-regular graph of order $n \geq 3$. Then $G$ is completely $k$-magic if and only if one of the following properties holds:

1. $k \geq 3$, $r = 2$, and $G$ contains even cycles only,
2. $k \geq 5$ and $r \geq 3$ odd,
3. $k \geq 5$, $r \geq 4$ even, and $n$ even,
4. $k \geq 5$ odd, $r \geq 4$ even, and $n$ odd,
5. $k = 4$, $r \geq 3$, $n$ even, and $G$ zero-sum 4-magic, or
6. $k = 3$ and any one of the following conditions holds:
   i. $r \equiv 0 \pmod{3}$, 
   ii. $r \equiv 0 \pmod{6}$, or 
   iii. $r \equiv 0 \pmod{3}$, $r$ odd, and $G$ has a factor $H$ such that $d_H(v) \equiv 1 \pmod{3}$ for all $v \in V(H)$.

For convenience, we only consider graphs that are finite and simple (unless otherwise stated). We also write $\mathbb{Z}_k^*$ to mean $\mathbb{Z}_k \setminus \{0\}$. For graph-theoretic terms that are not explicitly defined in this paper, see [3].

2. **Proof of the Main Theorem**

We divide the proof into several results.

It is not difficult to see that if $G$ is 1-regular, then $\Sigma_k(G) = \mathbb{Z}_k^*$. For 2-regular graphs, the following remark is a consequence of Theorem 1.7.

**Remark 2.1.** Let $k \geq 3$ and $G$ a 2-regular graph. If $G$ has an odd cycle, then

$$\Sigma_k(G) = \begin{cases} 
\mathbb{Z}_k^* & \text{if } k \text{ is odd} \\
\{0, 2, \ldots, k-2\} & \text{if } k \text{ is even}.
\end{cases}$$

Otherwise, we have $\Sigma_k(G) = \mathbb{Z}_k$.

Clearly, if $G$ is 1-factorable, then $G$ is completely $k$-magic. The following theorem considers regular graphs that has a factor that is completely $k$-magic.

**Theorem 2.2.** Let $r \geq 2$, $2 \leq h \leq r$, $k \neq 2$, and $G$ an $r$-regular graph. If $G$ has an $h$-factor that is completely $k$-magic, then $G$ is completely $k$-magic.

**Proof.** The case when $h = r$ is trivial, so we assume $h < r$. Let $H$ be an $h$-factor of $G$ that is completely $k$-magic. Let $\alpha = c - (r-h) \pmod{k}$ and $f_\alpha$ be an $\alpha$-sum $k$-magic labeling of $H$ for each $c \in \mathbb{Z}_k$.

Define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} 
f_\alpha(e) & \text{if } e \in E(H) \\
1 & \text{if } e \in E(G \setminus H).
\end{cases}$$

Observe that $\ell_c$ is a $c$-sum $k$-magic labeling of $G$ for each $c \in \mathbb{Z}_k$. Hence, $G$ is completely $k$-magic.
The following construction will be useful.

**Remark 2.3.** Let $G$ be an $r$-regular graph with $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$, where $r \geq 1$. Then we can construct a graph $G'$ (with parallel edges) such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{e'_1, e'_2, e'_3, \ldots, e'_m\}$, where $e'_i$ is a duplicate edge of $e_i$ in $G$ for each $i$ (that is, edges $e_i$ and $e'_i$ have the same end vertices). By Theorem 1.1, $G'$ has a $2h$-factor $H'$ for each $h$, $1 \leq h \leq r$. Also, $G' \setminus H'$ is a $(2r-2h)$-factor of $G'$ obtained by removing the edges of $H'$ from $G'$.

**Theorem 2.4.** Let $G$ be a $r$-regular graph. Then $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.

*Proof.* We know from Theorem 1.11 and Theorem 1.6 that $1, 3 \in N(G)$. For $k \geq 5$, we consider two cases.

**Case 1.** Suppose $k \geq 5$ and $k \neq 8$. Using the construction described in Remark 2.3, let $H'$ and $G' \setminus H'$ be a 2-factor and 8-factor of $G'$, respectively.

Define a zero-sum $k$-magic labeling $\ell'$ on $G'$ by

$$
\ell'(e) = \begin{cases} 
  k - 4 & \text{if } e \in E(H') \\
  1 & \text{if } e \in E(G' \setminus H').
\end{cases}
$$

Note that the labeling $\ell$ on $G$ defined by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $e_i \in E(G)$ is a zero-sum $k$-magic labeling on $G$.

**Case 2.** Suppose $k = 8$. Using again the construction in Remark 2.3, let $H'$ and $G' \setminus H'$ be a 4-factor and 6-factor of $G'$, respectively.

Define a zero-sum labeling $\ell'$ on $G'$ by

$$
\ell'(e) = \begin{cases} 
  2 & \text{if } e \in E(H') \\
  4 & \text{if } e \in E(G' \setminus H').
\end{cases}
$$

Observe that the labeling $\ell$ on $G$ defined by $\ell(e_i) = \frac{1}{2}[\ell'(e_i) + \ell'(e'_i)]$ for $e_i \in E(G)$ is a zero-sum 8-magic labeling on $G$.

Therefore, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$. 

Note that an odd-regular graph may not be zero-sum 4-magic. It was remarked in [1, Remark 10] that an odd-regular graph $G$ is not zero-sum 4-magic if $G$ has a vertex such that every edge incident to it is a cut-edge.

**Theorem 2.5.** Let $G$ be an $r$-regular graph, where $r \geq 3$ is odd and $k \geq 5$. Then $G$ is completely $k$-magic.

*Proof.* We know from Theorems 1.5 and 2.4 that $0 \in \Sigma_k(G)$. Let $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$. As constructed in Remark 2.3, let $H'$ and $G' \setminus H'$ be a 2-factor and $(2r-2)$-factor of $G'$, respectively. We consider two cases.

**Case 1.** Suppose $r \equiv 1 \pmod{k}$. Then $\gcd(r, k) = 1$. By Theorem 1.9, $G$ is completely $k$-magic.

**Case 2.** Suppose $r \not\equiv 1 \pmod{k}$. Assume $\gcd(r, k) = d$ so that $r = ad$ and $k = bd$ for some positive integers $a$ and $b$. Note that, since $r$ is odd, $d$ is also odd. We consider two sub-cases.

**Sub-Case 2.1.** Suppose $k \geq 5$ is odd. Then $b$ is odd.

For each $c \in \mathbb{Z}_k^* \setminus \{k - b, k - 2b\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^*$ by

$$
\ell'_c(e) = \begin{cases} 
  x & \text{if } e \in E(H') \\
  \frac{1}{2}(k + b) & \text{if } e \in E(G' \setminus H').
\end{cases}
$$

where $x \in \mathbb{Z}_k$ is chosen such that $\ell'_c(e) \equiv x \pmod{k}$. Then $\ell'_c$ is a zero-sum $k$-magic labeling on $G'$.

By Remark 2.3, we have $\ell'_c(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $e_i \in E(G)$. Hence, $G$ is completely $k$-magic.
where \( x = \frac{1}{2}(b + c) \) if \( c \) is odd, and \( x = \frac{1}{2}(b + c + k) \) if \( c \) is even. Observe that \( \ell'_c \) is a c-sum \( k \)-magic labeling of \( G' \) for each \( c \neq 0 \).

For each \( c \notin \{0, k - h, k - 2b\} \), define \( \ell_c : E(G) \to \mathbb{Z}_k^* \) by \( \ell_c(e_i) = \ell'_c(e_i) + \ell'_c(e'_i) \) for \( 1 \leq i \leq m \). Since \( \ell'_c \) is a c-sum \( k \)-magic labeling of \( G' \), \( \ell_c \) is a c-sum \( k \)-magic labeling of \( G \) for each \( c \in \mathbb{Z}_k^* \setminus \{k-b, k-2b\} \).

If \( k \neq 3b \), then, by Remark 1.3, \( k - b, k - 2b \notin \Sigma_k(G) \). If \( k = 3b \), it is enough to show that \( k - 2b \notin \Sigma_k(G) \). To do that, we provide a different labeling using a different set of factors of \( G' \). Let \( J' \) and \( G' \setminus J' \) be a 4-factor and \((2r - 4)\)-factor of \( G' \) respectively. In addition, we let \( J' = J'_1 \cup J'_2 \), where \( J'_1 \) and \( J'_2 \) are 2-factors of \( J' \).

Define \( \ell' : E(G') \to \mathbb{Z}_k^* \) by

\[
\ell'(e) = \begin{cases} 
\frac{1}{2}(b + 1) & \text{if } e \in E(J'_1) \\
\frac{1}{2}(b - 1) & \text{if } e \in E(J'_2) \\
b & \text{if } e \in E(G' \setminus J').
\end{cases}
\]

Since \( k = 3b \), \( d = 3 \) and \( r = 3a \). Thus, the magic sum in \( G' \) is given by \( 2\left[\frac{1}{2}(b + 1)\right] + 2\left[\frac{1}{2}(b - 1)\right] + b(2r - 4) \equiv 0 \) (mod \( k \)). Define \( \ell : E(G) \to \mathbb{Z}_k^* \) by \( \ell(e_i) = \ell'(e_i) + \ell'(e'_i) \) for \( 1 \leq i \leq m \). Note that \( \ell \) is also a \((k - 2b)\)-sum \( k \)-magic labeling of \( G \).

**Sub-Case 2.2.** Suppose \( k \geq 6 \) is even. Then \( b \) is even.

By labeling all the edges of \( G \) with \( \frac{1}{2}k \), we see that \( \frac{1}{2}k \in \Sigma_k(G) \).

Suppose \( r - 1 \equiv \frac{1}{2}k \) (mod \( k \)). For each \( c \in \mathbb{Z}_k^* \setminus \{k - 1, \frac{1}{2}k\} \), define \( \ell'_c : E(G') \to \mathbb{Z}_k^* \) by

\[
\ell'_c(e) = \begin{cases} 
c & \text{if } e \in E(H') \\
1 & \text{if } e \in E(G' \setminus H').
\end{cases}
\]

Observe that the sum of the labels of the edges incident to each vertex in \( G' \) is \( 2(r - 1) + 2c \equiv 2c \) (mod \( k \)). Using a similar argument as in Sub-Case 2.1, it can be shown that \( G \) is also c-sum \( k \)-magic for all even \( c \neq 0 \). Thus, we are left to show that \( G \) is c-sum \( k \)-magic as well for all odd \( c \).

For each odd \( c \neq k - 1 \), define \( \ell_c : E(G) \to \mathbb{Z}_k^* \) by \( \ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)] \) for each \( i, 1 \leq i \leq m \). Note that, since \( \ell'_c \) is a \( 2c \)-sum \( k \)-magic labeling of \( G' \), \( \ell_c \) is a c-sum \( k \)-magic labeling of \( G \) for each odd \( c \neq k - 1 \). Again, by Remark 1.3, we see that \( k - 1 \notin \Sigma_k(G) \).

Suppose \( r - 1 \equiv r_0 \) (mod \( k \)), where \( r_0 \neq \frac{1}{2}k \). For each \( c \in \mathbb{Z}_k^* \setminus \{r_0, r_0 + \frac{1}{2}k, r_0 - 1\} \), define \( \ell'_c : E(G') \to \mathbb{Z}_k^* \) by

\[
\ell'_c(e) = \begin{cases} 
c - r_0 & \text{if } e \in E(H') \\
1 & \text{if } e \in E(G' \setminus H').
\end{cases}
\]

Observe that the sum of the labels of the edges incident to each vertex in \( G' \) is \( 2r_0 + 2c - 2r_0 \equiv 2c \) (mod \( k \)). As in Sub-Case 2.1, it can be shown that \( G \) is also even-sum \( k \)-magic. So again, we are left to show that \( G \) is odd-sum \( k \)-magic.

As what we did earlier, for each odd \( c \neq r_0 - 1 \) (and, possibly, \( r_0 + \frac{1}{2}k \)), define \( \ell_c : E(G) \to \mathbb{Z}_k^* \) by \( \ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)] \) for all \( i, 1 \leq i \leq m \). Since \( \ell'_c \) is a \( 2c \)-sum \( k \)-magic labeling of \( G' \), \( \ell_c \) is a c-sum \( k \)-magic labeling of \( G \) for each odd \( c \neq r_0 - 1 \) (and, possibly, \( r_0 + \frac{1}{2}k \)). If \( r_0 - 1 \) and \( r_0 + \frac{1}{2}k \) are not inverses, then, by Remark 1.3, \( \mathbb{Z}_k^* \subset \Sigma_k(G) \).

If \( r_0 - 1 \) and \( r_0 + \frac{1}{2}k \) are inverses, then it is enough to show that \( r_0 - 1 \in \Sigma_k(G) \). Define \( \ell' \) on \( G' \) by

\[
\ell'(e) = \begin{cases} 
k - 1 & \text{if } e \in E(H') \\
1 & \text{if } e \in E(G' \setminus H').
\end{cases}
\]
Note that the magic sum using \( \ell' \) is \( 2r_0 - 2 \). Define \( \ell \) on \( G \) by \( \ell(e_i) = \frac{1}{3}[\ell'(e_i) + \ell'(e'_i)] \) for \( e_i \in E(G) \). Clearly, \( \ell \) is an \((r_0 - 1)\)-sum \( k \)-magic labeling on \( G \). Thus, by Remark 1.3, \( r_0 + \frac{1}{2}k \in \Sigma_k(G) \), and so \( \mathbb{Z}_k^* \subseteq \Sigma_k(G) \).

In any case, \( G \) is completely \( k \)-magic. \( \square \)

**Theorem 2.6.** Let \( k \geq 5 \) and \( G \) a \( 2r \)-regular graph of order \( n \geq 3 \), where \( r \geq 2 \).

1. If \( n \) is even, then \( G \) is completely \( k \)-magic.
2. If \( n \) is odd, then
   - \( G \) is completely \( k \)-magic if \( k \) is odd, and
   - \( \Sigma_k(G) = \{0, 2, 4, \ldots, k - 2\} \) if \( k \) is even.

**Proof.** Let \( E(G) = \{e_1, e_2, e_3, \ldots, e_m\} \). By Theorem 1.5, \( G \) is zero-sum \( k \)-magic.

1. Suppose \( r = 2 \). To prove the theorem, we only show that \( \mathbb{Z}_k^* \subseteq \Sigma_k(G) \). We consider two cases.

   **Case 1.** Suppose \( k \) is odd. Then \( \gcd(4, k) = 1 \). By Theorem 1.9, \( \mathbb{Z}_k^* \subseteq \Sigma_k(G) \).

   **Case 2.** Suppose \( k \) is even. It is not difficult to see that, being 4-regular, \( G \) is 2-edge connected. By Remark 2.3, we can construct \( G' \) so that \( G' \) is a 4-connected 8-regular graph. By Theorem 1.2, \( G' \) has a 3-factor, say \( H' \). Let \( G' \setminus H' \) be the 5-factor of \( G' \) obtained by removing the edges of \( H' \) from \( G' \).

   **Sub-Case 2.1.** Let \( k = 2d, d \) even. For each \( c \in \mathbb{Z}_k^* \setminus \left\{\frac{1}{2}k, \frac{1}{3}k\right\} \), define \( f_c : E(G') \to \mathbb{Z}_k^* \) by
   
   \[
   f_c(e) = \begin{cases} 
   2c & \text{if } e \in E(H') \\
   k - c & \text{if } e \in E(G' \setminus H'). 
   \end{cases}
   \]

   Observe that the sum of the labels of the edges incident to each of the vertices in \( G' \) is equal to \( 5(k - c) + 3(2c) \equiv c \mod k \). This shows that \( f_c \) is a \( c \)-sum \( k \)-magic labeling of \( G' \) for all \( c \neq 0, \frac{1}{2}k, \frac{1}{3}k \). By Remark 1.3, \( \frac{1}{2}k \in \Sigma_k(G') \).

   For each \( c \in \mathbb{Z}_k^* \setminus \left\{\frac{1}{2}k, \frac{1}{3}k\right\} \), define \( \ell_c : E(G) \to \mathbb{Z}_k^* \) by \( \ell_c(e_i) = f_c(e_i) + f_c(e'_i) \) for all \( i, 1 \leq i \leq m \). Clearly, \( \ell_c \) is a \( c \)-sum \( k \)-magic labeling of \( G \) for each \( c \in \mathbb{Z}_k^* \setminus \left\{\frac{1}{2}k, \frac{1}{3}k\right\} \). By Remark 1.3, we see that \( \mathbb{Z}_k^* \setminus \left\{\frac{1}{2}k\right\} \subseteq \Sigma_k(G) \).

   By Theorem 1.1, \( G \) is 2-factorable. Let \( G_1 \) and \( G_2 \) be the two 2-factors of \( G \). Label the edges in \( G_1 \) with \( d \) and the edges in \( G_2 \) with \( \frac{1}{2}(k - d) \). This shows that \( d = \frac{1}{2}k \in \Sigma_k(G) \).

   **Sub-Case 2.2.** Let \( k = 2d, d \geq 3 \) odd. Observe that, for \( c \neq 0, \frac{1}{2}k \), the labeling \( \ell_c \) in Sub-Case 2.1 is a \( c \)-sum \( k \)-magic labeling of \( G \). We are left to show that \( \frac{1}{2}k \in \Sigma_k(G) \).

   Let \( d \neq 3 \) and 9. We give a labeling for the factors of \( G' \) defined above (namely, \( H' \) and \( G' \setminus H' \)) and the 2-factors of \( G \) (namely, \( G_1 \) and \( G_2 \)) to show that \( G \) is \( d \)-sum \( k \)-magic.

   Let \( f : E(G) \to \mathbb{Z}_k^* \) be defined by
   
   \[
   f(e) = \begin{cases} 
   d + 1 & \text{if } e \in E(G_1) \\
   \frac{1}{2}(k - d - 1) & \text{if } e \in E(G_2). 
   \end{cases}
   \]

   Clearly, \( f \) is \((d + 1)\)-sum \( k \)-magic labeling of \( G \).

   Let \( g' : E(G') \to \mathbb{Z}_k^* \) be defined by
   
   \[
   g'(e) = \begin{cases} 
   k - 2 & \text{if } e \in E(H') \\
   1 & \text{if } e \in E(G' \setminus H'). 
   \end{cases}
   \]

   Define also \( g : E(G) \to \mathbb{Z}_k^* \) by \( g(e_i) = g'(e_i) + g'(e'_i) \) for all \( i, 1 \leq i \leq m \). Note that \( g' \) is a \((k - 1)\)-sum \( k \)-magic labeling of \( G' \), so \( g \) is a \((k - 1)\)-sum \( k \)-magic labeling of \( G \).
Finally, define \( \ell : E(G) \to \mathbb{Z}_k^* \) by \( \ell(e) = f(e) + g(e) \) for all \( e \in E(G) \). Since \( f \) and \( g \) are \((d + 1)\)-sum and \((k - 1)\)-sum \( k \)-magic labeling of \( G \), respectively, \( \ell \) is a \( d \)-sum \( k \)-magic labeling of \( G \).

Suppose \( d = 3 \) or \( 9 \). Define \( g' : E(G') \to \mathbb{Z}_k^* \) be defined by

\[
g'(e) = \begin{cases} 2x & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H') \end{cases},
\]

where \( x = 1 \) if \( d = 3 \), and \( x = 3 \) if \( d = 9 \). Note that \( g' \) is a \( 5 \)-sum \( k \)-magic labeling of \( G' \). Define a labeling \( g \) on \( G \) by \( g(e_i) = g'(e_i) + g'(e_i') + 1 \) for all \( i, 1 \leq i \leq m \). Note that \( g \) is a \( d \)-sum \( k \)-magic labeling on \( G \). Thus, \( d = \frac{1}{2} k \in \Sigma_k(G) \), and so \( G \) is completely \( k \)-magic.

Suppose \( r \geq 3 \) is odd. By Theorem 1.1, \( G \) is \( r \)-factorable. By Theorem 2.5, the \( r \)-factors of \( G \) are completely \( k \)-magic for all \( k \geq 5 \). Thus, by Theorem 2.2, \( G \) is also completely \( k \)-magic.

If \( r \geq 4 \) is even, then, by Theorem 1.1, \( G \) has a \( 6 \)-factor, say \( H \). Using the case for \( r \) is odd, \( H \) is completely \( k \)-magic. Thus, by Theorem 2.2, \( G \) is also completely \( k \)-magic.

(2(i)) By Theorem 1.1, \( G \) is 2-factorable. Let \( G_1, G_2, \ldots, G_r \) be the 2-factors of \( G \). If \( k \) is odd, then, by Remark 2.1, \( \mathbb{Z}_k^* \subseteq \Sigma_k(G_i) \) for all \( i, 1 \leq i \leq r \). For each \( i \) and \( c \in \mathbb{Z}_k^* \), let \( \ell_c \) be a \( c \)-sum \( k \)-magic labeling of \( G_i \). We consider two cases.

Case 1. Suppose \( r \equiv 1 \pmod{3} \). For each \( c \in \mathbb{Z}_k^* \), define \( \ell_c : E(G) \to \mathbb{Z}_k^* \) by

\[
\ell_c(e) = \begin{cases} \ell_c(e) & \text{if } e \in E(G_1) \\ \ell_c(e) & \text{if } e \in E(G_i) \text{ for some } i = 2, 3, \ldots, r. \end{cases}
\]

Note that \( \ell_c \) is a \( c \)-sum \( k \)-magic labeling of \( G \) for all \( c \neq 0 \).

Case 2. Suppose \( r \not\equiv 1 \pmod{3} \). For each \( c \in \mathbb{Z}_k^* \setminus \{r - 1 \pmod{3}\} \), define \( \ell_c : E(G) \to \mathbb{Z}_k^* \) by

\[
\ell_c(e) = \begin{cases} \ell_{c-x}(e) & \text{if } e \in E(G_1) \\ \ell_c(e) & \text{if } e \in E(G_i) \text{ for some } i = 2, 3, \ldots, r, \end{cases}
\]

where \( x \equiv r - 1 \pmod{3} \). The sum of the labels of the edges incident to each vertex is \( c \pmod{3} \). Thus, \( G \) is \( c \)-sum \( k \)-magic for each \( c \neq x \). By Remark 1.3, \( G \) is \( x \)-sum \( k \)-magic since \( G \) is \((k - x)\)-sum \( k \)-magic. In this case, \( G \) is completely \( k \)-magic.

(2(ii)) This follows from Remark 2.1, Lemma 1.8, and Theorem 2.2.

The proof of the following theorems are similar to Theorem 2.5 and Theorem 2.6.

**Theorem 2.7.** Let \( r \geq 3 \), and \( G \) a zero-sum 4-magic \( r \)-regular graph. Then

1. If the order of \( G \) is even, then \( G \) is completely 4-magic.
2. If the order of \( G \) is odd, then \( \Sigma_4(G) = \{0, 2\} \).

**Theorem 2.8.** Let \( G \) be an \( r \)-regular graph, where \( r \geq 3 \).

1. If \( r \equiv 0 \pmod{3} \) or \( r \equiv 0 \pmod{6} \), then \( G \) is completely 3-magic.
2. If \( r \equiv 0 \pmod{3} \) and \( r \) odd, then \( G \) is completely 3-magic if and only if \( G \) has a factor \( H \) such that \( d_H(v) \equiv 1 \pmod{3} \) for all \( v \in V(H) \).
References

[1] Akbari S, Rahmati F, and Zare S 2014 Zero-sum magic labelings and null sets of regular graphs *Electron. J. Combin.* **21**(2) #P2.17

[2] Akiyama J and Kano M 2011 *Factors and Factorizations of Graphs* (Springer-Verlag)

[3] Bondy J A and Murty U S R 2008 *Graph Theory* (Springer)

[4] Dong G and Wang N 2014 A conjecture on zero-sum 3-magic labeling of 5-regular graphs *arXiv* 1406.6870v1

[5] Eniego A A and Garces I J L 2015 Completely k-magic regular graphs *Appl. Math. Sci. (Ruse)* **103** pp 5139–5148

[6] Gallai T 1950 On factorisation of graphs *Acta Math. Hungar.* **1**(1) pp 133–53

[7] Petersen J 1891 Die theorie der regulären graphs *Acta Math.* **15** pp 193–220

[8] Salehi E 2006 Integer-magic spectra of cycle-related graphs *Iran. J. Math. Sci. Inform.* **2** pp 53–63

[9] Sedlacek J 1976 On magic graphs *Math. Slovaca* **26** pp 329–35

[10] Wang T M and Hu S W 2011 Constant sum flows in regular graphs *Frontiers in Algorithmics and Algorithmic Aspects in Information and Management*, ed M Attalah, X Y Li and B Zhu (Berlin Heidelberg: Springer) pp 168–175