Rolling friction for hard cylinder and sphere on viscoelastic solid

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Received 6 August 2010
Published online: 25 November 2010 – © EDP Sciences / Società Italiana di Fisica / Springer-Verlag 2010

Abstract. We calculate the friction force acting on a hard cylinder or spherical ball rolling on a flat surface of a viscoelastic solid. The rolling-friction coefficient depends non-linearly on the normal load and the rolling velocity. For a cylinder rolling on a viscoelastic solid characterized by a single relaxation time Hunter has obtained an exact result for the rolling friction, and our result is in very good agreement with his result for this limiting case. The theoretical results are also in good agreement with experiments of Greenwood and Tabor. We suggest that measurements of rolling friction over a wide range of rolling velocities and temperatures may constitute a useful way to determine the viscoelastic modulus of rubber-like materials.

1 Introduction

Rubber friction is a topic of huge practical importance, e.g., for tires, rubber seals, wiper blades, conveyor belts and syringes [1–16]. Many experiments have been performed with a hard spherical ball rolling on a flat rubber substrate [17–20]. Nearly the same friction force is observed during sliding as during rolling, assuming that the interface is lubricated and that the sliding velocity and fluid viscosity are such that a thin lubrication film is formed with a thickness much smaller than the indentation depth of the ball, but larger than the amplitude of the roughness on the surfaces [17]. The results of rolling friction experiments have often been analyzed using a very simple model of Greenwood and Tabor [17], which however contains an (unknown) factor $\alpha$, which represents the fraction of the input elastic energy lost as a result of the internal friction of the rubber. In this paper we present a very simple theory for the friction force acting on a hard cylinder or spherical ball rolling on a flat rubber surface. For a cylinder rolling on a viscoelastic solid characterized by a single relaxation time Hunter [21] has obtained an exact result for the rolling friction, and our result is in very good agreement with his result for this limiting case.

2 Theory

Using the theory of elasticity (assuming an isotropic viscoelastic medium), one can calculate the displacement field $u_i$ on the surface $z = 0$ in response to the surface stress distributions $\sigma_i = \sigma_3i$. Let us define the Fourier transform

$$u_i(q, \omega) = \frac{1}{(2\pi)^3} \int d^2x \, dt \, u_i(x, t) e^{-i(q \cdot x - \omega t)}$$

and similar for $\sigma_i(q, \omega)$. Here $x = (x, y)$ and $q = (q_x, q_y)$ are two-dimensional vectors. In ref. [3] we have shown that

$$u_i(q, \omega) = M_{ij}(q, \omega)\sigma_j(q, \omega),$$

or, in matrix form,

$$u(q, \omega) = M(q, \omega)\sigma(q, \omega),$$

where the matrix $M$ is given in ref. [3].

We now assume that $|\nabla u_z(x)| \ll 1$ and that the surface stress only acts in the $z$-direction, so that

$$u_z(q, \omega) = M_{zz}(q, \omega)\sigma_z(q, \omega).$$

Since in the present case $\omega$ is of order $vq$ (where $v$ is the sliding or rolling velocity) we get $\omega/c_T q = v/c_T \ll 1$ in most cases of practical interest, where $c_T$ is the transverse sound velocity in the rubber. In this case (see ref. [3])

$$(M_{zz})^{-1} = \frac{E q}{2(1 - \nu^2)},$$

It is interesting to note that if, instead of assuming that the displacement $u$ points along the $z$-direction, we assume that the displacement $u$ points along the $z$-direction, then

$$\sigma_z(q, \omega) = (M^{-1})_{zz}(q, \omega) u_z(q, \omega).$$
where in the limit \( \omega/c_Tq \ll 1 \),
\[
(M^{-1})_{zz} = -\frac{2Eq(1-\nu)}{(1+\nu)(3-4\nu)},
\]
which differs from (2) only with respect to a factor \( 4(1-\nu)^2/(3-4\nu) \). For rubber-like materials (\( \nu \approx 0.5 \)) this factor is of order unity. Hence, practically identical results are obtained independently of whether one assumes that the interfacial stress or displacement vector is perpendicular to the nominal contact surface. In reality, neither of these two assumptions holds strictly, but the result above indicates that the theory is not sensitive to this approximation.

Now, assume that
\[
\sigma_z(x,t) = \sigma_z(x-vt),
\]
then
\[
\sigma_z(q,\omega) = \frac{1}{(2\pi)^3} \int d^2q dt \ \sigma_z(x-vt) \ e^{-iq\cdot(x-vt)}
= \delta(-q \cdot v) \ \sigma_z(q),
\]
where
\[
\sigma_z(q) = \frac{1}{(2\pi)^2} \int d^2x \ \sigma_z(x)e^{-iq\cdot x}.
\]
If \( F_t \) denotes the friction force, then the energy dissipated during the time period \( t_0 \) equals
\[
\Delta E = F_t v t_0. \tag{4}
\]

But this energy can also be written as
\[
\Delta E = \int d^2q dt \ u \cdot \sigma
= (2\pi)^3 \int d^2q d\omega (\omega u(q,\omega) \cdot \sigma(-q,-\omega), \tag{5}
\]
where \( \omega = v \cdot q \). Substituting (1) in (5) and using (3) and that (for large \( t_0 \))
\[
[\delta(-q \cdot v)]^2 = \delta(-q \cdot v) \frac{1}{2\pi} \int dt \ \epsilon(\omega-q \cdot v)t
= \delta(-q \cdot v) \frac{1}{2\pi} \int_{-t_0/2}^{t_0/2} dt
= \delta(-q \cdot v)(t_0/2\pi),
\]
gives
\[
\Delta E = (2\pi)^2 t_0 \int d^2q (-i\omega)
\times M_{zz}(-q,-\omega)\sigma_z(q)\sigma_z(-q).
\]
Comparing this expression with (4) gives the friction force
\[
F_t = \frac{(2\pi)^2}{v} \int d^2q (-i\omega) \ M_{zz}(-q,-\omega)\sigma_z(q)\sigma_z(-q).
\]

Since \( F_t \) is real and since \( \sigma_z(-q) = \sigma_z^*(q) \), we get
\[
F_t = \frac{(2\pi)^2}{v} \int d^2q \ \omega \ \text{Im} \ M_{zz}(-q,-\omega)|\sigma_z(q)|^2. \tag{6a}
\]

Using (2) we can also write
\[
F_t = \frac{2(2\pi)^2}{v} \int d^2q \ \omega \ \text{Im} \ \frac{1}{E_{\text{eff}}(\omega)}|\sigma_z(q)|^2, \tag{6b}
\]
where \( E_{\text{eff}} = E/(1-\nu^2) \). In principle, \( \nu \) depends on the frequency but the factor \( 1/(1-\nu^2) \) varies from \( 4/3 \approx 1.33 \) for \( \nu = 0.5 \) (rubbery region) to \( \approx 1.19 \) for \( \nu = 0.4 \) (glassy region) and we can neglect the weak dependence on frequency. Within the assumptions given above, eq. (6) is exact. Note that even if we use \( \sigma_z(q) \) calculated to zero order in \( \tan \delta \), the friction force (6) will be correct to linear order in \( \tan \delta \). We also note that the present approach is very general and flexible. For example, instead of a semi-infinite solid as assumed above one may be interested in a thin viscoelastic film on a hard flat substrate. In this case, (assuming slip-boundary conditions) the \( M(q,\omega) \)-function which enters in (6a) is determined by [8,22]
\[
M^{-1} = -\frac{E_T}{2(1-\nu^2)}S,
\]
\[
S = \frac{(3-4\nu)\cosh(2qd) + 2(qd)^2 - 4\nu(3-2\nu) + 5}{(3-4\nu)\sinh(2qd) - 2qd}.
\]

Note that as \( d \to \infty \), \( S \to 1 \) and the present result reduces to (2). Substituting this in (6a) gives the rolling friction for a sphere (or cylinder) on a thin rubber film adsorbed to a hard flat substrate. We now apply the theory to a) a rigid cylinder and b) a rigid sphere rolling on a semi-infinite viscoelastic solid.

### 2.1 Cylinder

Consider a hard cylinder (radius \( R \) and length \( L_y \gg R \)) rolling on a viscoelastic solid. The same result is obtained during sliding if one assumes lubricated contact and if one can neglect the viscous energy dissipation in the lubrication film. As discussed above, when calculating the friction force to linear order in \( \tan \delta \) we can neglect dissipation when calculating the contact pressure \( \sigma_z(x) \) and assume that the stress is of the Hertz form. Thus if we introduce a coordinate system with the \( y \)-axis parallel to the cylinder axis and with the origin of the \( x \)-axis in the middle of the contact area (of width \( 2a \)), then the contact stress for \(-a < x < a \) is
\[
\sigma_z(x) = \frac{2f_N}{\pi a} \left[ 1 - \left( \frac{x}{a} \right)^2 \right]^{1/2}, \tag{7}
\]