Bosonization and density-matrix renormalization group studies of the Fulde-Ferrell-Larkin-Ovchinnikov phase and irrational magnetization plateaus in coupled chains

G. Roux,1, E. Orignac,2, P. Pujol,1 and D. Poilblanc1

1Laboratoire de Physique Théorique, Université Paul Sabatier, CNRS, 118 Route de Narbonne, 31400 Toulouse, France.
2Laboratoire de Physique de l’Ecole Normale Supérieure de Lyon, CNRS, 46 Allée d’Italie, 69007 Lyon, France.

(Dated: December 29, 2021)

We review the properties of two coupled fermionic chains, or ladders, under a magnetic field parallel to the lattice plane. Results are computed by complementary analytical (bosonization) and numerical (density-matrix renormalization group) methods which allows a systematic comparison. Limiting cases such as coupled bands and coupled chains regimes are discussed. We particularly focus on the evolution of the superconducting correlations under increasing field and on the presence of irrational magnetization plateaus. We found the existence of large doping-dependent magnetization plateaus in the weakly-interacting and strong-coupling limits and in the non-trivial case of isotropic couplings. We report on the existence of extended Fulde-Ferrell-Larkin-Ovchinnikov phases within the isotropic t-J and Hubbard models, deduced from the evolution of different observables under magnetic field. Emphasis is put on the variety of superconducting order parameters present at high magnetic field. We have also computed the evolution of the Luttinger exponent corresponding to the ungapped spin mode appearing at finite magnetization. In the coupled chain regime, the possibility of having polarized triplet pairing under high field is predicted by bosonization.

PACS numbers: 71.10.Pm, 74.20.Mn, 74.81.-g, 75.40.Mg

I. INTRODUCTION

Low-dimensional strongly correlated systems have attracted strong interest in the last years because fluctuations and energetic competition drive these systems into exotic phases. Quasi-one-dimensional or strongly anisotropic two-dimensional superconductors are also known to be good candidates for the realization of Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phases1,2,3,4,5 because the orbital effects of the magnetic field, which induce the Hc2 critical field in type-II superconductors, can be strongly suppressed. In a singlet superconductor without orbital effect, there is a competition between polarizing the spins of the electrons and binding them into Cooper pairs leading to a theoretical critical field called the Pauli limit or the paramagnetic limit.6,7,8,9,10 However, this limit can be exceeded with an inhomogeneous order parameter (FFLO) which can be energetically favorable, allowing pockets of polarized electrons and paired electrons. Qualitatively, the FFLO mechanism consists in giving singlet Cooper pairs a finite momentum which leads to this inhomogeneous superconducting order parameter. Among these intriguing low-dimensional systems, ladders, which consist of a few coupled chains, proved to display deep new physical behaviors and sustained considerable experimental and theoretical work.6,9

At half-filling, ladders are Mott insulators and have a spin gap if the number of chains is even (this gap goes to zero in the limit of an infinite number of coupled chains) and no spin gap if it is odd. The two-leg ladder has thus a spin-liquid ground state with exponentially decaying magnetic correlations. Spin gaps can also open under magnetic field for rational values of the magnetization per site leading to plateaus in the magnetization curve.11 Experimental evidences of zero magnetization plateaus have been reported on ladder and coupled-dimer compounds, such as Cu2 (C7H12N2)2Cl4, (C5H12N)2CuBr4 and (5IAP)2CuBr4·2H2O. Away from half-filling, a few systems are known to develop irrational magnetization plateaus controlled by hole concentration.12,13,14 Furthermore, when holes are introduced into the spin-liquid two-leg ladder, these charge carriers generically bring the system into either a metallic or a superconducting state. The isotropic two-leg ladder is known to have a wide superconducting phase14,15 and also a metallic phase with dominant charge-density wave (CDW) fluctuations. Another interesting feature is the appearance of commensurate CDW for a commensurate hole concentration.16 The theoretically proposed framework to account for superconductivity in doped ladders relies on magnetic fluctuations and is based on the resonating-valence-bond (RVB) mechanism for superconductivity proposed by Anderson in the context of high-Tc superconductors.17 Within the isotropic Hubbard and t-J models, singlet pairing with an unconventional modified d-wave structure is found. The competition between superconductivity and CDW has indeed been observed in the copper oxide ladder compound Sr14−xCa2xCu24O41+δ (SCCO) for which the superconducting state only appears under high pressure.18 However, the mechanism responsible for superconductivity in SCCO has not reached a full agreement yet. In this context, the upper critical magnetic fields determined from transport measurements19,20 suggest that the Pauli limit is exceeded in SCCO which reassesses the issue of the nature of the pairing. Note that superconductivity has been discovered in the zig-zag ladder subsystem of the copper oxide compound Pr2Ba4Cu7O15−δ at ambi-
ent pressure.\textsuperscript{22}

Recently, a superconducting two-leg t-J ladder under a strong magnetic field in the plane of the ladder was studied numerically\textsuperscript{23} using the density-matrix renormalization group\textsuperscript{24,25} (DMRG) method. The magnetic curve displays a doping-dependent magnetization plateau, as predicted\textsuperscript{26} by Cabra \textit{et al.} for a Hubbard ladder. In addition to this non trivial magnetic behavior, an exceeding of the Pauli limit was found. Within the t-J model, this exceeding of the Pauli limit was explained by a one-dimensional analogue of the FFLO phase, hence reconciling the expectation of singlet pairing and the exceeding of the Pauli limit. Lastly, the behavior of the superconducting correlations studied in Ref. \textsuperscript{23} showed a surprising behavior in and outside the plateau with a notable emergence of $S^z = 0$ triplet superconducting correlations.

In this paper, we propose to greatly complete Ref. \textsuperscript{23} by an extensive comparison of bosonization and numerical calculations and by extending the results to discuss the case of weakly-interacting or strongly anisotropic ladders and also to the case of weakly-coupled chains. In particular, we show that the picture developed in Ref. \textsuperscript{23} is consistent with bosonization and extends to both Hubbard and t-J models, supporting its generality. While most studies on the FFLO phase resort to both Hubbard and t-J models, supporting its generalization. In particular, we show that the picture developed in Ref. \textsuperscript{23} is consistent with bosonization and extends to both Hubbard and t-J models, supporting its generality.

The article is organized as follow: first, after introducing microscopic models (Sec. \textbf{II}), we briefly examine the non-interacting and strong-coupling limits in Sec. \textbf{III} where large doping-dependent magnetization plateaus can occur. The coupled bands regime is discussed in detail in Sec. \textbf{IV} and corresponds to the case of doped isotropic ladders which is the most studied at zero magnetic field. Lastly, the coupled chains regime is studied under magnetic field (Sec. \textbf{V}).

\section{II. MICROSCOPIC MODELS AND CONVENTION}

We describe the ladder system with a standard one-orbital Hubbard model which can have different hopping terms along the chains (\|) and between the chains (\perp). We consider a situation where no magnetic orbital effect is present and thus only keep a Zeeman coupling to the spin degree of freedom. This would experimentally correspond to the situation where the magnetic field $H$ is in the plane of the ladder (and even along the direction of the ladder to minimize all orbital effects). Then, one can write the Hubbard Hamiltonian

\begin{equation}
\mathcal{H} = -t_{\parallel} \sum_{i,p=1,2,\sigma} [c^\dagger_{i+1,p,\sigma} c_{i,p,\sigma} + h.c.] -t_{\perp} \sum_{i,\sigma} [c^\dagger_{i,\sigma} c_{i,1,\sigma} + h.c.] +U \sum_{i,p=1,2} n_{i,p,\uparrow} n_{i,p,\downarrow} - \sum_{i,p=1,2} H \cdot \mathbf{S}_{i,p}
\end{equation}

where $c^\dagger_{i,p,\sigma}$, $\mathbf{S}_{i,p}$ and $n_{i,p,\sigma}$ are respectively electron creation, spin and density operators at site $i$ on chain $p$ and $\sigma$ is the spin index. The $g\mu_B$ prefactor has been absorbed in the definition of $H$ for convenience. Free bands dispersion is given by

\begin{equation}
\varepsilon_{k_y,\sigma}(k) = -2t_{\parallel} \cos(k) - t_{\perp} \cos(k_y) - H \frac{\sigma}{2}
\end{equation}

(with $k = k_x$ for simplicity and $k_y = 0, \pi$) and are sketched in FIG. \textbf{II}. If we denote by $k^\sigma_{F,k_y}$ the Fermi wavevectors, we have the Luttinger sum rule

\begin{equation}
n = \frac{1}{2\pi} \sum_{(k_y,\sigma)\text{occ.}} k^\sigma_{F,k_y},
\end{equation}

where $n = N^e/(2L) = 1 - \delta$ is the electron density and $\delta$ the hole density and $L$ the length of the ladder. Experimentally, copper-oxide systems have a fixed hole doping $\delta$ rather than a fixed chemical potential $\mu$ so $\delta$ will be kept fixed in this paper. Similarly, the magnetization per site $m = (N^+ - N^-)/(2L)$ satisfies

\begin{equation}
m = \frac{1}{2\pi} \sum_{(k_y,\sigma)\text{occ.}} \sigma k^\sigma_{F,k_y}.
\end{equation}

Relations \textsuperscript{29} and \textsuperscript{30} are also valid in the presence of interactions. For quasi one-dimensional interacting systems, a useful theorem is the generalization of the Lieb-Schultz-Mattis theorem to doped and magnetized states.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Splitting of the bands dispersion in a non-interacting doped ladder ($\mu$ is the chemical potential) due to Zeeman effect at low magnetic field $H$.}
\end{figure}
done by Yamanaka-Oshikawa-Affleck (YOA). This theorem relates gap openings to commensurability conditions. When dealing with spinful fermions, the demonstration is based on the definition of 2 twist operators for spins $\uparrow, \downarrow$ which gives two commensurability conditions for parameters in each sector. In the case of doped two-leg ladders, YOA parameters take the simple form $1 - \delta + \sigma m$. If a parameter is non-integer, the spectrum has gapless excitations in this sector. If a parameter is non-integer but rational, a gap can open in the sector ($\sigma m = 29$) so that the usual spin mode-charge mode separation present when $\sigma m = 26$ is no longer valid. In the strong-coupling limit of the Hubbard model $U \gg t$ and for small hole doping, Hamiltonian (1) reduces to the t-J model:

$$\mathcal{H} = -t_{\parallel} \sum_{i,p=1,2,\sigma} \mathcal{P} [c_{i+1,p,\sigma}^\dagger c_{i,p,\sigma} + h.c.] \mathcal{P} - t_{\perp} \sum_{i,\sigma} \mathcal{P} [c_{i,\sigma}^\dagger c_{i,\sigma} + h.c.] \mathcal{P} + J_{\parallel} \sum_{i,p=1,2} [S_{i,\sigma} \cdot S_{i+1,\sigma} - \frac{1}{4} n_{i,p} n_{i+1,p}]$$

$$+ J_{\perp} \sum_{i=1} S_{i,1} \cdot S_{i,2} - \frac{1}{4} n_{i,1} n_{i,2} - \sum_{i,p=1,2} H \cdot S_{i,p},$$

in which $\mathcal{P}$ is the Gutzwiller projector, $n_{i,p} = \sum_{\sigma} n_{i,p,\sigma}$ and the same convention is used to label the antiferromagnetic couplings $J_{\parallel,\perp} = 4t_{\parallel,\perp}^2/U$. In what follows, these two microscopic models will be studied numerically and the isotropic model will assume $t = t_{\parallel} = t_{\perp}$ and $J = J_{\parallel} = J_{\perp}$.

Bosonization allows to study the low-energy properties and correlation functions of 1D-like systems using field theoretical methods. The bosonization Hamiltonians used to describe ladders$^{39,31,32,33,34}$ fall into two classes, coupled bands models to be reviewed in Sec. IV and coupled chains models to be reviewed in Sec. V. The first approach is more appropriate to study the system with isotropic parameters. There is however the possibility, by strongly reducing the interchain hopping amplitude $t_{\perp}$, to obtain a cross-over to a regime where the coupled chain model is best suited to describe the behavior of the system (Sec. V).

III. NON INTERACTING AND STRONG-COUPLED LIMITS

Non interacting system. It is interesting to discuss first the non-interacting system using the relations (2) and keeping $\delta$ fixed instead of the chemical potential. Two main cases are possible: either a strong interband coupling with $t_{\perp} > 2t_{\parallel}$ or a small interband coupling for $t_{\perp} < 2t_{\parallel}$.

In the first case, the only bands which are partially filled at low magnetic field are the up and down spins bonding bands (0, $\uparrow$) and (0, $\downarrow$) (see FIG. 2 (a)). A first critical field corresponds to the complete filling of (0, $\uparrow$) (FIG. 2 (b)). This induces a plateau with $m = \delta$ in the magnetization curve. This plateau is doping-dependent and similar to what has been predicted for other doped systems. The width of the plateau can be deduced from energetic considerations to be $2(t_{\perp} - 2t_{\parallel})$. When the plateau ends (FIG. 2 (c)), the band ($\pi, \uparrow$) starts to be partially filled. When the band (0, $\downarrow$) gets empty, all electrons are polarized and the magnetization curve is constant with value $m = 1 - \delta$. For open boundary conditions (OBC), a $m = \delta$ plateau is also found as displayed in FIG. 3 (see Sec. V.E.1 for computational details). The width is in good agreement with $2(t_{\perp} - 2t_{\parallel})$. Note that this plateau does not originate from interactions contrary to what will be discussed in the strong-coupling approach below and in Sec. V.E.1 but is simply due to band-filling effects.

In the second case, all four bands are partially filled at low magnetic field. No plateaus are found but filling or emptying successively the bands will induce cusps in the magnetization curve and a decrease of the slope of the magnetization because fewer electrons contribute to the magnetization. First, the band ($\pi, \downarrow$) gets empty.
We chose $J_\perp = 2.5$, $J_{\parallel} = 0.3$ and $t_{\parallel} = t_{\perp} = 1.0$. The $m = \delta$ plateau is an effect of interactions and the ground-state in this plateau has unpaired holes. Still, holes are paired for magnetizations with $m \leq m_c < \delta$.

![FIG. 4: (Color online) Plateaus in the strong-coupling limit.](image)

Adding interactions in the system offers the possibility to study magnetization plateaus due to interactions and also pairing, which does not necessarily mean superconductivity. In this limit we have $J_\parallel, t_{\parallel}, t_{\perp} \ll J_\perp$ and the pairing energy defined as in Eq. (33) is estimated to be $J_\perp - 2t_{\perp} - 2t_{\parallel}$ and remains approximately constant under magnetic field because the average magnetization in Eq. (33) is zero. Two ground states are possible at finite magnetization: either holes are paired up and the polarization is due to triplets, or holes pairs are split apart and the partner electron on the rung holds the polarization. The difference between the energies per rung of these two states is:

$$m < \delta: \quad \Delta e \sim (J_\perp - 2t_{\perp} - 2t_{\parallel})\delta - J_\perp m$$

$$m > \delta: \quad \Delta e \sim (2J_\perp - 2t_{\perp} - 2t_{\parallel})\delta - 2J_\perp m$$

If $J_\perp/(t_{\perp} + t_{\parallel}) > 2$, a transition is then possible from the state with paired holes to the state with unpaired holes. The corresponding critical magnetization $m_c = [1 - 2(t_{\perp} + t_{\parallel})/J_\perp]\delta$ is always smaller than $\delta$. This also gives a possible scenario for a wide $m = \delta$ plateau due to interactions. For large $J_\perp$ and $m = 0$, we expect the ground state to have pairs of holes and singlets mostly on rungs so that the system has a large spin gap, which gives a $m = 0$ plateau. However, the $m = 0$ plateau is much smaller than the half-filling spin gap because polarized spins in fact gain kinetic energy if they are localized next to holes so that it is easier to create them. As the magnetic field is increased, hole pairs are split as suggested by the above argument and the partner electron of the hole on the rung becomes polarized. When all of them are fully polarized, we enter the $m = \delta$ plateau because the next spin excitation one can do is to flip a singlet on a rung, which cost is approximately $J_\perp$. The $m = \delta$ plateau can thus be wider than the $m = 0$ one. This scenario corresponds to FIG. 4, where no pairing is found in the plateau from the computation of local densities of spins and holes with DMRG. Such a scenario could be relevant for compounds with lightly coupled dimers which could be doped with holes.

When one starts with unpaired holes at zero magnetic field, for instance with $t_{\parallel} = J_{\parallel} \ll t_{\perp} = J_\perp$, the large half-filled spin gap of order $J_\perp - J_{\parallel}$ is immediately destroyed by doping (see FIG. 5). Increasing the magnetic field further brings quickly the system into a plateau phase with unpaired holes. While the magnetization curve is similar to the non-interacting one below the $m = \delta$ plateau, its width is clearly controlled by $J_\perp - J_{\parallel}$ rather than by $2(t_{\perp} - 2t_{\parallel})$. 

![FIG. 5: (Color online) Plateaus phase diagram of a 2-leg ladder for $t_{\perp} = J_\perp = 1.0$ and $t_{\parallel} = J_{\parallel} = 0.2$. Dopings immediately destroys the $m = 0$ plateau present at half-filling while a large doping-dependent $m = \delta$ plateau appears for a small critical field.](image)

![FIG. 6: (Color online) Local densities of holes and spins in a system with spins in the ground state of the $m = \delta$ phase computed on a ladder with $L = 48$ and 6 holes, $t_{\perp} = t_{\parallel} = 1.0$ and $J_\perp = 0.5$. As $J_\perp$ increases from 0 to $J_{\parallel}$, hole pairs form in the isotropic limit. Pairing in and above $m = \delta$ is not expected in the strong-coupling limit.](image)
An interesting feature on the pairing of holes in the 
\( m = \delta \) plateau is the occurrence of a transition when 
\( J_{||} \) is increased from 0 to \( J_{\perp} \) from a state with unpaired 
holes to paired holes (see FIG. 7). Having pairing in this 
plateau is a situation that is not expected in the strong-
coupling limit as we have seen, but for intermediate \( J_{\perp} \), 
we expect that hopping between hole pairs and magnons 
can stabilize hole pairs\(^{25}\) (see Sec. IV). FIG. 7 suggests 
that a \( J_{||} \) comparable to \( J_{\perp} \) is also needed to have pairing 
in this phase.

IV. COUPLED BANDS REGIME

A. Bosonized Hamiltonian

When the interactions are not too strong compared to 
the interchain hopping, it is reasonable to begin to solve 
the non-interacting band structure and then add interac-
tions. This is the approach followed in Refs. [32,33,34]. 
The non-interacting band structure is simply formed of 
the bonding band of energy \( \varepsilon_{0,\sigma}(k) \) and an antibonding 
band of energy \( \varepsilon_{\pi,\sigma}(k) \). The annihilation operators of the 
fermions in these bands are respectively given by:

\[
\psi_{0,\sigma} = \frac{1}{\sqrt{2}}(\psi_{1,\sigma} + \psi_{2,\sigma})
\]

\[
\psi_{\pi,\sigma} = \frac{1}{\sqrt{2}}(\psi_{1,\sigma} - \psi_{2,\sigma})
\]

with 1 and 2 the labels of the chains. In the continuum 
limit, the fermions 0 and \( \pi \) are bosonized in terms of the 
boseon fields \( \phi_{k_y,\sigma}(k_y = 0, \pi) \) so that we write the fermion 
operator:

\[
c_{n,k_y,\sigma} \rightarrow \sqrt{\alpha} \psi_{k_y,\sigma}(x)
\]

\[
= \sqrt{\alpha} [e^{i k_y x} \psi_{R,k_y,\sigma}(x) + e^{-i k_y x} \psi_{L,k_y,\sigma}(x)]
\]

in which \( x = na \) (\( a \) is the lattice spacing) and

\[
\psi_{R/L,k_y,\sigma} = \frac{\eta_{R/L,k_y,\sigma}}{\sqrt{2\pi}} e^{i \epsilon R/L \phi_{R/L,k_y,\sigma}}
\]

where \( R, L \) are the labels for the right and left movers (see 
FIG. 7). \( \alpha \) is a cutoff (typically \( \alpha \)), \( \epsilon_{R/L} = \mp 1 \) and the \( \eta \) 
are Klein factors needed to make the annihilation operators 
of the different fermions species anticommute. We also have the definitions \( \phi_{k_y,\sigma} = \frac{1}{\sqrt{2}}(\phi_{L,k_y,\sigma} + \phi_{R,k_y,\sigma}) \) while 
dual fields are \( \theta_{k_y,\sigma} = \frac{1}{\sqrt{2}}(\phi_{L,k_y,\sigma} - \phi_{R,k_y,\sigma}) = \pi \int \Pi_{k_y,\sigma} \).

We then introduce the useful bosons \( \phi_{\nu,k_y} \) with \( \nu = c, s \) 
for charge (resp. spin) corresponding to the symmetric (resp. 
antisymmetric) combination of \( \phi_{k_y,\uparrow} \) and \( \phi_{k_y,\downarrow} \).
The same transformation is performed on the dual fields, 
leading to the Hamiltonian:

\[
\mathcal{H} = \sum_{k_y \in c,s} \int dx \left[ \frac{1}{2\pi} [u_{k_y} (\pi \Pi_{k_y,\theta})^2 + u_{k_y} (\partial_x \phi_{\nu,k_y})^2] \right]
\]

where \( [\phi_{\nu,k_y}(x), \Pi_{\nu,k_y}(x')] = i \delta_{\nu,\nu'} \delta_{k_y,k_y'} (x-x') \). In 
the following, we will make the usual approximation\(^{25}\) of neglecting the difference between the velocities \( u_{k_y} \) 
of the 0 and \( \pi \) bands. This allows us to introduce the linear 
combinations:

\[
\phi_{c,\pm} = \frac{1}{\sqrt{2}} (\phi_{c,0} \pm \phi_{c,\pi})
\]

\[
\phi_{s,\pm} = \frac{1}{\sqrt{2}} (\phi_{s,0} \pm \phi_{s,\pi})
\]

with similar definitions for the dual fields \( \theta_{\nu,\pm} = \mp \int \Pi_{\nu,\pm} \). In the most general case, we have to use a 
\( Z \) matrix to describe the evolution of the system under 
magnetic field\(^{37}\). When comparing the results with 
the chain models of Sec. IV it is useful to note that while 
\( \phi_{c+} = \phi_{p+} \) and \( \phi_{s+} = \phi_{\sigma+} \) (we use Greek letters for 
the fields defined in the chain models and Latin letters for 
the fields defined in the band models), while there is no 
simple relation between \( \phi_{p-} \) and \( \phi_{c-} \) and between \( \phi_{s-} \) 
and \( \phi_{\sigma-} \). The magnetic field couples to the system by a term:

\[
\mathcal{H} = \frac{H}{\pi} \int dx \partial_x \phi_{s+}
\]

Once interactions are turned on, two types of terms appear 
in the Hamiltonian. The terms of the first type are forward scattering interaction terms that are 
quadratic in the fields \( \phi_{\nu,\pm} \). The terms of the second type 
are backscattering interaction terms, the expressions of which were derived in Ref. [32]. Since terms containing 
\( \cos 2\phi_{s+} \) cannot appear in the Hamiltonian when the 
magnetization in nonzero, the expression of the backscattering terms in a magnetized ladder reads:

\[
\mathcal{H}_{\text{back}} = \int dx \cos 2\theta_{c-} \left[ \frac{2g_c}{(2\pi \alpha)^2} \cos 2\phi_{s-} + \frac{2g_s}{(2\pi \alpha)^2} \cos 2\theta_{s-} \right]
\]

From the Hamiltonian\(^{39}\), one sees that in the ground 
state the field \( \theta_{c-} \) is pinned to \( \langle \theta_{c-} \rangle = 0 \). By using the results of Ref. [32] [Sec. 3.1 and Eqs. (20) and (56)], 
one can argue that in the presence of repulsive interactions, 
one must have \( K_{s-} < 1 \). Thus, one obtains a 
freezing of the field related to the most relevant operator 
\( \phi_{s-} = \frac{\pi}{\sqrt{2}} \). Briefly, when \( m = \delta = 0 \), all 
fields are massive and we have the famous spin-liquid phase, 
often denoted by \( \text{COS0} \). When \( m = 0 \) but \( \delta \neq 0 \), the system 
with repulsive interactions is in a C1S0 phase, or 
Luttinger-Emery phase (LE), with sectors \( c^- \) and \( s^- \) being 
massive while the sector \( c^+ \) is massless, corresponding to the 
charge mode. When \( m \neq 0 \) and \( \delta \neq 0 \), the sector \( s^+ \) 
becomes massless giving rise to a C1S1 phase. Luttinger 
exponents associated with these sectors will be denoted by \( K_{c/s} \). 
Till now, we have discussed the case of generic nonzero magnetization. For the specific case \( m = \delta \), however, 
a magnetization plateau can be expected. Following 
Ref. [26], we introduce the fields

\[
\phi_{\sigma} = \frac{1}{\sqrt{2}} (\phi_{c,\pm} + \sigma \phi_{s,\pm})
\]
which appear in the term that induces the opening of the \( m = \delta \) plateaus

\[
\int dx \cos \left[ 2(k_{F,0}^\sigma + k_{F,\pi}^\sigma)x - 2\sqrt{2}\phi_\sigma^+ \right]
\]

(10)

with, from \( [8] \) and \( [11] \), the relation \( k_{F,0}^\sigma + k_{F,\pi}^\sigma = \pi(n + \sigma m) \). This term leads to the opening of plateaus when \( n \pm m \in \mathbb{Z} \) as expected in YOA theorem. This condition is a commensurability condition which combines both spin and charge degrees of freedom. The \( m = \delta \) plateau corresponds to the locking of the \( \phi_\sigma^+ \) mode. The origin of the term (10) giving rise to the plateau is the Hubbard model becomes necessary.

\[
\int dx \cos \left[ 2(k_{F,0}^\sigma + k_{F,\pi}^\sigma)x - 2(\phi_0, + \phi_\pi, \uparrow) \right].
\]

(12)

\[A. \text{ Superconducting order parameters and most divergent fluctuations} \]

In this section, we define the order parameters for superconductivity at nonzero magnetization (high field), derive their bosonized expressions and deduce their long range correlations. Since the SU(2) symmetry is broken by the magnetic field, we have to compute the superconducting correlation functions \( \langle \Delta_\sigma^x(x)\Delta_\sigma^y(0) \rangle \) in various channels \( \lambda \). We use the following microscopic definitions for the pairing operators \( \Delta_\sigma^x(n) \) at rung \( n \):

\[
\text{singlet: } \Delta_{\uparrow\downarrow}^x(n) = \sum \sigma c_{\uparrow\sigma} c_{\downarrow\sigma}^\dagger
\]

\[
\text{triplet: } \begin{cases}
\Delta_{\uparrow\downarrow}^x(n) = \sum \sigma c_{\uparrow\sigma} c_{\downarrow\sigma}^\dagger \\
\Delta_{\uparrow\uparrow}^x(n) = \sum \sigma c_{\uparrow\sigma} c_{\uparrow\sigma}^\dagger \\
\Delta_{\downarrow\downarrow}^x(n) = \sum \sigma c_{\downarrow\sigma} c_{\downarrow\sigma}^\dagger
\end{cases}
\]

(13)

(14)

with \( r = (n,1) \) and \( r' = (n,2) \) for next-nearest neighbor pairs created on a rung and \( r = (n,p) \) and \( r' = (n+1,p) \) if created on the leg \( p \). Contrarily to the case of the coupled chain regime (Sec. [X]) where the Fermi wave-vectors are the same in both chains, in the case of the band regime the Fermi wave-vectors of the two bands are different (\( k_{F,0}^\sigma \neq k_{F,\pi}^\sigma \)) as can be seen on FIG. [I]. As a result, a more detailed derivation of the bosonized expressions starting from lattice expressions in a two-chain Hubbard model becomes necessary.

1. Bosonized form

Using the bosonized form \( [8] \) of the fermion operators, we can express these order parameters as a function of products of the \( \psi_{R/L,k}\sigma \). In this section, only components with the dominant contribution will be kept, i.e. we will neglect terms of the form \( \psi_R \psi_R \) and \( \psi_L \psi_L \). These dominant contributions correspond to pairing with the lowest total momentum \( q \). The case of \( 2k_F \) triplet pairing will be discussed in Sec. [IV B 2]. At finite magnetization, the equality of the velocities and equation \( [8] \) ensures that the lowest momentum is \( q = k_{F,0}/\pi - k_{F,\pi}/\pi = \pi m \).

To simplify the expression of the operators we will use extensively the following results on a pinned field \( \varphi \) (for which \( \langle \varphi \rangle = \text{cste} \)): we can replace \( \langle f(\varphi) \rangle \) by \( f(\langle \varphi \rangle) \) and the dual field has exponentially decaying correlation functions \( [8] \).

Starting with the interband order parameters, which read:

\[
\psi_{R,0,\sigma} \psi_{L,\pi,-\sigma} \sim e^{i[\theta_+ - \phi_{-\sigma} - \sigma(\phi_+ + \phi_-)]},
\]

\[
\psi_{R,0,\sigma} \psi_{L,\pi,\sigma} \sim e^{i[\theta_- - \phi_{-\sigma} - \sigma(\phi_+ + \phi_-)]},
\]

we note that they are all proportional to \( e^{i\phi_-} \) and thus, since \( \theta_- \) is pinned, their correlations decay exponentially. In other words, power law decay is possible only for intraband superconducting correlations.

The intraband superconducting order parameters in the three spin channels read respectively:

- Intraband singlet:

\[
\sum_{\sigma} \psi_{R,\sigma} \psi_{L,\sigma,-\sigma} \sim \sum_{\sigma} e^{i[\theta_+ + \theta_- - \sigma(\phi_+ + \phi_-)]},
\]

\[
\sum_{\sigma} \psi_{R,\sigma,\pi} \psi_{L,\pi,\sigma} \sim \sum_{\sigma} e^{i[\theta_- + \theta_- - \sigma(\phi_+ + \phi_-)]}.
\]

- Intraband triplet \( S^z = 0 \):

\[
\sum_{\sigma} \psi_{R,\sigma} \psi_{L,\sigma,-\sigma} \sim \sum_{\sigma} e^{i[\theta_+ + \theta_- - \sigma(\phi_+ + \phi_-)]},
\]

\[
\sum_{\sigma} \psi_{R,\pi,\sigma} \psi_{L,\pi,\sigma} \sim \sum_{\sigma} e^{i[\theta_- + \theta_- - \sigma(\phi_+ + \phi_-)]}.
\]

- Intraband triplet \( S^z = 1 \):

\[
\psi_{R,0,\sigma} \psi_{L,0,-\sigma} \sim e^{i[\theta_+ + \theta_- - \sigma(\phi_+ + \phi_-)]},
\]

\[
\psi_{R,\pi,\sigma} \psi_{L,\pi,\sigma} \sim e^{i[\theta_- + \theta_- - \sigma(\phi_+ + \phi_-)]}.
\]

To determine which forms of superconductivity will be dominant, we need to express the leg and rung order parameters in terms of the intraband order parameters.

We have for the rung singlet order parameter,

\[
\Delta_{\uparrow\downarrow}^x(x) \sim \sum_{\sigma} e^{i\sqrt{2}q \sigma} \psi_{R,0,\sigma} \psi_{L,0,-\sigma} - \psi_{R,\pi,\sigma} \psi_{L,\pi,\sigma}.
\]

(15)
Whereas the rung triplet $S^z = 0$ is given by:

$$\Delta_{t\downarrow\downarrow}(x) \sim \sum_{\sigma} \frac{1}{2} e^{i\sigma q x} (\psi_{R,\pi,\sigma} \psi_{L,0,-\sigma} - \psi_{R,0,-\sigma} \psi_{L,\pi,\sigma}),$$

therefore, since it is composed only of interband terms, we expect that its correlation will present exponential decay. Lastly, for the rung triplet $S^z = 1$ order parameter, we find:

$$\Delta_{t\uparrow\uparrow}(x) \sim \frac{1}{2} (\psi_{R,0,\uparrow} \psi_{L,0,\uparrow} - \psi_{R,\pi,\uparrow} \psi_{L,\pi,\uparrow}).$$

Note that because we can neglect interband coupling, the expression $K$ is the same as found numerically in FIG. [3].

Turning to the leg singlet order parameter, we find that it reads:

$$\Delta_{s\downarrow\downarrow}(x) \sim \sum_{\sigma} \frac{1}{2} e^{i\sigma q x} \left[ e^{i2K_{F,\sigma}} \psi_{R,0,\sigma} \psi_{L,0,-\sigma} + (0 \rightarrow \pi) \right].$$

Note that if we take the limit $a \rightarrow 0$ this term disappears as happened for the rung triplet. Finally, the leg triplet $S^z = 0$ or parameter reads:

$$\Delta_{t\downarrow\downarrow}(x) \sim \sum_{\sigma} \frac{1}{2} e^{i\sigma q x} \left[ \sin(k_{F,\sigma} \theta) \psi_{R,0,\sigma} \psi_{L,0,-\sigma} + (0 \rightarrow \pi) \right].$$

Since $K_{s-} < 1$, we have $\langle \phi_{s-} \rangle = \frac{\pi}{2}$. Then, all the intraband triplet $S^z = 1$ have exponentially decaying correlations. As a consequence, both the leg and the rung triplets with $S^z = 1$ have exponentially decaying correlations. The behavior of these correlations with higher $2k_F$ momentum will be discussed in Sec. [2B].

A consequence of the ordering of the field $\phi_{s-}$ is that:

$$\psi_{R,0,\sigma} \psi_{L,0,-\sigma} \sim e^{-i\frac{\pi}{2} \sigma} e^{i(\theta_{+} - \sigma \phi_{+})},$$

$$\psi_{R,\pi,\sigma} \psi_{L,\pi,-\sigma} \sim e^{i\frac{\pi}{2} \sigma} e^{i(\theta_{-} - \sigma \phi_{-}).}$$

Provided that $k_{F,0} \neq k_{F,\pi}$, the leg triplet $S^z = 0$ order parameter does not vanish. We find that it has the same critical exponent as the rung singlet order parameter, namely $1/4(K_{s+}^{-1} + K_{s+})$. It is larger than the $m = 0$ critical exponent $1/(2K_{c+})$ because of the appearance of the massless boson $\theta_{+}$. The fact that we have a finite momentum pairing for $q = \pi m$ which is very likely to occur in a one-dimensional like system and is the signature of the FFLO mechanism.

2. $S^z = 1$ Triplet with a $2k_F$ momentum

Another notable result is the presence of rung-rung triplet $S^z = 1$ correlations with $2k_F$ oscillations. Within the band representation, it is indeed possible to find a component of the rung-rung triplet correlations that has power law decay. Let us consider the next terms in the expansion of the rung triplet operator:

$$\psi_{1,\sigma} \psi_{2,\sigma} \sim -2e^{i(k_{F,\sigma} + k_{F,\sigma}^2) x} \psi_{R,0,\sigma} \psi_{L,\pi,\sigma} + \ldots$$

In bosonized form, this operator reads:

$$\Delta_{s\sigma,2k_F}(x) \sim e^{i(k_{F,\sigma} + k_{F,\sigma}^2) x} e^{i(\theta_{+} - \phi_{+} + \sigma(\theta_{+} - \phi_{+}))}$$

It has power law correlations with a critical exponent

$$\frac{1}{2} (K_{c+} + K_{c+}^{-1} + K_{s+} + K_{s+}^{-1}),$$

and, from equations (3) and (4), the associated wave-vector is $k_{F,0}^2 + k_{F,\pi}^2 = \pi(n + \sigma m)$. Note that its critical exponent is always larger than the one of the rung singlet order parameter.

3. Charge-density waves order parameters

Here, we address the question of the CDW exponents under magnetic field. First, the $2k_F$ CDW order parameter $n_s$, which vanishes exponentially at zero magnetization, contains terms such as:

$$\psi_{r,0,\sigma} \psi_{r,\pi,\sigma} \sim e^{i(\theta_{+} - \phi_{+} + \sigma_0 \pi_{\sigma})}.\) Note that its critical exponent is $n_s^2$ and has an exponent $2K_{c+}$ at zero magnetization. This order parameter contains terms such as

$$\psi_{R,0,\sigma} \psi_{L,\pi,\sigma} \sim e^{i(\theta_{+} - \phi_{+} + \sigma_0 \pi_{\sigma})},$$

with a “$2k_F$” wave vector $\pi(n + \sigma m)$. It decays exponentially because of $e^{i\theta_{+}}$. Finally, no $2k_F$ CDW are expected under magnetic field.

Secondly, the $4k_F$ CDW order parameter is $n_s^2$ and has an exponent $2K_{c+}$ at zero magnetization. This order parameter contains terms such as

$$\psi_{R,0,\sigma} \psi_{L,\pi,\sigma} \sim e^{i(\theta_{+} - \phi_{+} + \sigma_0 \pi_{\sigma})},$$

which have a “$4k_F$” wave vector $2\pi(n + m)$ and, for nonzero magnetization, an exponent $2K_{c+}$. We also have terms like

$$\psi_{R,0,\sigma} \psi_{L,\sigma} \sim e^{i(\theta_{+} - \phi_{+})},$$

with a wave-vector $2(k_{F,\sigma}^2 + k_{F,\pi}^2)$ and an exponent $2K_{c+}$ which is not affected by the magnetic field. This last term can compete with the superconducting order parameter to be the most diverging fluctuations depending on $K_{c+}$ and $K_{s+}$. 
TABLE I: The superconducting operators for the Hubbard model with an empty band corresponding to the coupled band regime just above $H_c$. 

| Type of operator | fermion expression | dimension | wave-vector |
|------------------|-------------------|-----------|-------------|
| $S^z = 1$ triplet | $\psi_{R0}^\dagger \psi_{L0}$ | $\frac{1}{4k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | 0 |
| $\psi_{R1} \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | 0 |
| $\psi_{R0}^\dagger \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | 0 |
| $\psi_{R+} \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | 0 |

$\psi_{R0}^\dagger \psi_{L0}$, $\psi_{R+} \psi_{L0}$

$S^z = 0$ triplet or singlet

| Type of operator | fermion expression | dimension | wave-vector |
|------------------|-------------------|-----------|-------------|
| SDW$^z$/CDW | $\psi_{R0}^\dagger \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | $k_{F \pi}^\dagger - k_{F \sigma}$ |
| $\psi_{R+}^\dagger \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | $k_{F \pi}^\dagger - k_{F \sigma}$ |

$\psi_{R+}^\dagger \psi_{L0}$, $\psi_{R0}^\dagger \psi_{L0}$

| Type of operator | fermion expression | dimension | wave-vector |
|------------------|-------------------|-----------|-------------|
| SDW$^{\pi,\nu}$ | $\psi_{R0}^\dagger \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | $k_{F \pi}^\dagger + k_{F \sigma}$ |
| $\psi_{R+}^\dagger \psi_{L0}$ | $\frac{1}{2k_F^2} + \frac{1}{2k_F} + \frac{1}{2k_F}$ | $k_{F \pi}^\dagger + k_{F \sigma}$ |

$\psi_{R+}^\dagger \psi_{L0}$, $\psi_{R0}^\dagger \psi_{L0}$

C. Interpreting the superconducting critical field $H_c$ as a band-filling transition

In this section, we identify the observed superconducting upper critical field $H_c$ with a band-filling transition. Such a transition will ungap all 3 remaining sectors, leading to an enhancement of the correlations exponents. We describe the system just above the transition and compute the various possible exponents using bosonization. We consider a case where the Fermi energy is positioned in such a way that the band ($\pi, \downarrow$) is empty while the three other bands remain partially filled (see FIG. 1 for illustration). The band ($\pi, \downarrow$) being empty has important consequences. Projecting out the high energy subspace where the band ($0, \downarrow$) is occupied by a single electron, one gets in lowest order:

$$c_{n,0,\downarrow}^\dagger c_{n,0,\downarrow} \rightarrow \frac{1}{2} c_{n,0,\downarrow}^\dagger c_{n,0,\downarrow},$$

and as a result the on-site Hubbard interaction reduces to:

$$\frac{U}{2} \sum_n n_{i,0,\downarrow} (n_{i,0,\uparrow} + n_{i,\pi,\uparrow}).$$

In the absence of commensuration between the different bands, the resulting bosonized Hamiltonian has only forward scattering interactions. Thus, its excitations are completely gapless. The bosonized Hamiltonian reads:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_U$$

$$\mathcal{H}_0 = \sum_{\nu \in \{0, \uparrow\}, \{\pi, \uparrow\}} \int \frac{dx}{2\pi} \nu F,\nu \left[ (\pi \nu)^2 + (\partial_x \phi_{\nu})^2 \right]$$

$$\mathcal{H}_U = \frac{U\sigma}{2\pi^2} \int dx \partial_x \phi_{\sigma,\downarrow} (\partial_x \phi_{\nu,\uparrow} + \partial_x \phi_{\nu,\uparrow})$$

This Hamiltonian can be fully diagonalized. In the general case, where the Fermi velocities are all different, one needs first to perform a rescaling: $\Pi_{\nu} \rightarrow \sqrt{u/v_{F,\nu}} \Pi_{\nu}$ and $\phi_{\nu} \rightarrow \phi_{\nu}/\sqrt{u/v_{F,\nu}}$, where $u$ is an arbitrary quantity with the dimension of a velocity and then diagonalize the matrix:

$$\begin{pmatrix}
\frac{U}{2\pi^2} \nu F,0,\nu & \frac{U}{2\pi^2} \nu F,0,\nu & \frac{U}{2\pi^2} \nu F,\pi,\nu \\
\frac{U}{2\pi^2} \nu F,0,\nu & \frac{U}{2\pi^2} \nu F,0,\nu & \frac{U}{2\pi^2} \nu F,\pi,\nu \\
0 & \frac{U}{2\pi^2} \nu F,0,\nu & \frac{U}{2\pi^2} \nu F,\pi,\nu
\end{pmatrix}$$

Afterwards, one performs the inverse rescaling on the diagonal matrix to obtain the velocities of the modes and the associated Luttinger exponents. To simplify the algebra, we will assume that all the Fermi velocities are equal. Then, we find that the following combination of fields

$$\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{pmatrix} \begin{pmatrix}
\phi_{0,\downarrow} \\
\phi_{0,\uparrow} \\
\phi_{\pi,\uparrow}
\end{pmatrix},$$

(26)
are smoothed out by taking the absolute value. States were kept with discarded weight of order $10^{-8}$ for the wave-vectors. Note that the $S^z = 1$ "2$k_F$" oscillations are smoothed out by taking the absolute value. $M = 1000$ states were kept with discarded weight of order $10^{-6}$.

diagonalizes the interaction, and:

$$u_1 K_1 = u_2 K_2 = u_3 K_3 = v_F$$

$$\frac{u_1}{K_1} = v_F + \frac{U}{2\pi \alpha \sqrt{2}}$$

$$\frac{u_2}{K_2} = v_F - \frac{U}{2\pi \alpha \sqrt{2}}$$

$$\frac{u_3}{K_3} = v_F$$

In terms of these new fields, the fermion operators read:

$$\psi_{r,0,\uparrow} = \frac{\eta_{r,\alpha,\uparrow}}{\sqrt{2\pi \alpha}} \left[ \psi_1(\theta_1 - r\phi_1) - \frac{1}{\sqrt{2}} (\theta_2 - r\phi_2) + \frac{1}{\sqrt{2}} (\theta_3 - r\phi_3) \right]$$

$$\psi_{r,\pi,\uparrow} = \frac{\eta_{r,\pi,\uparrow}}{\sqrt{2\pi \alpha}} \left[ \psi_1(\theta_1 - r\phi_1) - \frac{1}{\sqrt{2}} (\theta_2 - r\phi_2) - \frac{1}{\sqrt{2}} (\theta_3 - r\phi_3) \right]$$

$$\psi_{r,0,\uparrow} = \frac{\eta_{r,\alpha,\downarrow}}{\sqrt{2\pi \alpha}} \left[ \psi_1(\theta_1 - r\phi_1) + \psi_2(\theta_2 - r\phi_2) \right]$$

From these expressions, it is possible to obtain the various superconducting order parameters and their critical exponents. The results are gathered in table I and table II on which one must note that the correlation exponent will be twice the dimension of the operator.

D. Numerical results on superconducting correlation functions

We have computed superconducting correlation functions with DMRG using the definitions of Eqs. (13) and

As predicted by bosonization results, only modified $d$-wave singlet superconductivity is found when $m = 0$ (rung-leg correlations have an opposite sign to rung-rung and leg-leg correlations, see Ref. [10] for a bosonization discussion), while $S^z = 0$ leg-leg triplet and $S^z = 1$ rung-rung triplet correlations emerge under high magnetic field (see FIG. 8 for the t-J model and FIG. 9 for the repulsive Hubbard model). Furthermore, these correlations oscillate with wave-vectors $q = \pi m$ for the $S^z = 0$ channel and $q = 2k_F$ for the $S^z = 1$ channel. The $q = \pi m$ relation has been checked in all the FFLO phase (see FIG. 17 and discussion in Sec. IV C), by fitting the oscillating correlations with $A \cos(qr + \phi)/r^\alpha$ (see FIG. 10), confirming the FFLO-like mechanism. We observe that $S^z = 0$ rung-rung triplet correlations have an exponential behavior while rung-singlet correlations are algebraic, in agreement with the prediction from bosonization. Since from the point of view of the rotation symmetry these two order parameter transform in the same way, the difference must be caused by their different transformation properties under interchange of the legs. Note that because of the $\sin(k_{F,\pi,0} a)$ factors in the leg singlet and $S^z = 0$ leg triplet, the cancellation of the intraband terms obtained in the case of rung order parameters is absent. This explains the observation of power-law correlations. Concerning the orbital part of pairs, singlet Cooper pairs have a mixed $s$-wave and $d$-wave structure and $S^z = 0$ triplet can be considered to first approximation as $p$-wave pairs along the legs (with a symmetric superposition of the two legs).

The good agreement with bosonization predictions relies on the fact that in the doped isotropic t-J ladder, 4 Fermi points exist with approximately $3\pi/5$ and $k_{F,\pi} \sim 2\pi/5$ when $m = 0$ and that the assumption of equal Fermi velocities is numerically reasonable at

FIG. 8: (Color online) Absolute value of superconducting correlations for zero and a finite magnetization in an isotropic doped ladder with $\delta = 0.063$. Notation are "Rg" for rung, "Lg" for leg, "0" for numerically irrelevant signal, "exp" for exponentially decaying correlations and "$\pi m$" and "$2k_F$" for the wave-vectors. For $\delta = 0.063$ notations are "Rg" for rung, "Lg" for leg, "0" for numerically irrelevant signal, "exp" for exponentially decaying correlations and "$\pi m$" and "$2k_F$" for the wave-vectors. Note that the $S^z = 1$ "2$k_F$" oscillations are smoothed out by taking the absolute value. $M = 1000$ states were kept with discarded weight of order $10^{-6}$.

FIG. 9: (Color online) Same correlations as FIG. 8 but using the Hubbard model with $\delta = 0.063$ and $U/t = 8.0$. $M = 1600$ states were kept with discarded weight of order $10^{-6}$.
low hole doping. The existence of a small leg-leg triplet component (about 100 times smaller than the rung rung triplet) in the two leg t-J ladder considered may therefore be explained by the persistence of a small band splitting in this strongly coupled ladder system.

We now would like to compare precisely the algebraic decay exponents $\alpha$ and $\alpha'$ of the singlet and $S^z = 0$ leg-leg triplet correlations. DMRG is known to often underestimate correlation functions for a fixed number of state kept $M$. In order to capture the behavior in the thermodynamic limit, we computed the correlation functions for fixed $M$ ranging from 800 to 2000 and lengths $L = 32, 64, 96$ and 128 (we worked at fixed $\delta = 1/16$ and $m = 3/32$ so that these are the only accessible cluster sizes). We extracted $\alpha(M, L)$ by fitting the data. Then, we can extrapolate $\alpha$ in the $M \to \infty$ limit to get a correct exponent $\alpha(L)$ at size $L$ from which we can do a finite size scaling. On FIG. 11 we see that for a given size, $\alpha(M)$ decreases as $M$ is enlarged, roughly like $\alpha(M) \sim 1/M$ (see FIG. 13(b)) as was previously found. We note from FIG. 12 that the larger the system, the larger $M$ is needed to reach a good convergence. Convergence as a function of the discarded weight is also given on FIG. 13 and has qualitatively the same behavior. For $L = 128$, the convergence is slower with $M$ than for $L = 96$, probably because we would need larger $M$ to have a correct accuracy (the $1/M$ might be realized for large enough $M$) so that we believe the results are not as reliable as for $L = 64$ and 96 (larger $M$ would be very expensive numerically). On one side of FIG 13 (c), $M = 2000$ is too small (for $L = 128$) while on the other side, it is difficult to extract $\alpha$ because the system is too small to resolve enough oscillations (when $L = 32$). Extrapolations can be tentatively done with some uncertainties which can be roughly estimated. Therefore, we can infer from FIG. 13 that $\alpha' = 1.54 \pm 0.15$ is greater than $\alpha^s = 1.17 \pm 0.15$ which pleads in favor of dominat-
and because $M(c)$ given in the caption of FIG.11.

FIG. 14: (Color online) Normalized 4-points spin correlations $M$ number of kept states in the isotropic t-J ladder with $J/t = 1200$ states kept. Their decay exponent is $2K_{s+}^{-1}$ which gives access to $K_{s+}(H)$. Correlations were computed on a system with $L = 64$ and $M = 1200$ states kept.

**Evolution of the Luttinger parameter $K_{s+}$**

We now want to have access to the evolution of the Luttinger exponent $K_{s+}$ of the gapless mode which appears at finite magnetization. We remark that the bosonized form of the triplet creation operator on a rung $S_{dL}^+(x)S_{dL}^+(x)$ contains dominant terms such as

$$
\psi_{L,0\uparrow}^\dagger \psi_{L,0\downarrow}^\dagger \psi_{R,\pi\uparrow}^\dagger \psi_{R,\pi\downarrow} \sim e^{-i2[\phi_{s+}+\theta_{s-}]}.
$$

(giving an exponent $2K_{s+}^{-1}$ since $\langle \phi_{s-} \rangle = \pi/2$. The wave-vector associated with this operator is $(k_F^+, k_F^-) - (k_{F,x}^+, k_{F,y}^-) \sim 0$ since, if the difference between the Fermi velocities of the 0 and $\pi$ bands is negligible, we have $k_{F,x}^+ - k_{F,y}^- = \pi m$. Note that $2K_{s+}^{-1}$ is the smaller exponent for terms with this wave-vector. Because $K_{s+} < 1$, we could expect smaller possible exponents such as $2K_{s+}^2$ or $2K_{s+} + K_{s+}^2$. For the first, one can show that fields $2[\phi_{s+} \pm \phi_{s-}]$ or $2[\phi_{s+} + \sigma \theta_{s-}]$ cannot appear in the decomposition of $S_{dL}^+ S_{dL}^+$

The general behavior is that $K_{s+} < 1$ and increases with magnetic field towards the limit $K_{s+} = 1$ at high fields. Similarly, the Luttinger parameter $K_{s+}$ has been characterized using Bethe ansatz in the context of the SO(8) description of two-leg half-filled Hubbard ladder which gave $K_{s+} < 1$. When doped, the SO(6) description gives the same constraint (results not shown) as well as an increase of $K_{s+}$ at high magnetic field. However, a detailed comparison of SO(6) predictions and numerics would require to determine first the conditions and parameters at which both approaches match and this is beyond the scope of this article. Lastly, the values of $K_{s+}$ obtained numerically should be slightly larger in the thermodynamic limit than what is found because of finite size and finite $M$ effects as explained above.
FIG. 16: (Color online) Magnetization curves for the isotropic t-J, Hubbard and non-interacting systems. Irrational plateaus are clearly visible at $m = \delta$. $H_c$ represents the superconducting critical field. The energy scale $J$ is defined as $4t^2/U$ in the Hubbard model. The non-interacting system curve has been rescaled ($H \rightarrow H/4$) for clarity.

E. Generic phase diagram of the t-J model

In this section, we discuss the generic phase diagram in the $(H, \delta)$ plane for the isotropic t-J model on the basis of DMRG and bosonization results.

1. Doping-dependent magnetization plateaus

The magnetization curves $m(H)$ of FIG. 16 obtained within the Hubbard and t-J models, display plateaus for $m = 0$ and $m = \delta$. Energies $E(n_h, S^z)$ were computed keeping $M = 1600$ states with the single-site method proposed by White (we used a noise level of $10^{-6}$) at fixed hole number $n_h$ and total magnetization $S^z$. Magnetic fields are deduced using $H(S^z) = E(n_h, S^z + 1) - E(n_h, S^z)$, and interpolated with $[H(S^z) + H(S^z - 1)]/2$ if they do not belong to a plateau. The $m = 0$ plateau simply corresponds to the well-known spin-gap of the doped ladder. The $m = \delta$ plateau, exists at small doping for continuous values of $\delta$ (see FIG. 17 and Ref. 23) and thus falls into the classes of doping-dependent irrational magnetization plateaus predicted by Cabra et al. It can be understood as a Commensurate-Incommensurate (C-IC) transition so that we expect the critical exponent of the magnetization as a function of the magnetic field to be $1/2$. In the plateau phase, the mode $\phi^\uparrow$ is locked but the sector $(\downarrow, +)$ remains gapless, leading to a metallic phase. Note that $\phi^\uparrow$ is a superposition of both spin and charge modes so that charge and spin are no more independent modes in the plateau phase.

The Hubbard and t-J models give qualitatively the same behavior at low magnetization. The spin gap ($m = 0$ plateau) in the Hubbard model with $U/t = 8$ is about half of the spin gap of the t-J model with $J/t = 0.5$ as it was previously found, but the irrational plateau has roughly the same width. A larger $U$ gives a slightly larger plateau. Both models display a singularity of the magnetization curve, more or less pronounced, near the superconducting critical field $H_c$. The location of $H_c$ in FIG. 16 have been roughly determined by looking at local densities (see Ref. 23 for method) since Cooper pairs break down above this field. For larger $U$, the superconducting critical field is smaller. These results are also observed in the t-J model having in mind that $J/t \sim t/U$ (see Sec. IV E 2 and FIG. 19). For higher magnetization states, Gutzwiller projection induces a rather different behavior between strongly interacting systems (Hubbard $U = 16$ and t-J) the Hubbard model with $U = 8$ which displays the expected square-root-like behaviors near critical fields in good agreement with the band-emptying scenario proposed in Sec. IV C. At last, a quick comparison with the isotropic non-interacting system proves the non-trivial role of interactions in the apparition of these plateaus.

The magnetic part of the phase diagram of the t-J model with $J/t = 0.35$ has been computed (see FIG. 17) and is very similar to the one obtained with $J/t = 0.5$ (see Ref. 23) but have larger $m = \delta$ plateaus. We propose that such a phase diagram is generic for the isotropic t-J and Hubbard models under Zeeman effect at low doping and for parameters $0.25 \leq (J/t \sim 4t/U) \lesssim 1.0$ corresponding to the strong-coupling regime. Varying $J/t$ modifies the width of the different phases as proposed in FIG. 19. Lastly, we note that $\delta = 1/4$ corresponds to the end of the plateau both for $J/t = 0.35$ and $J/t = 0.5$. Furthermore, the magnetization curves are strongly mod-
fields are gapful. This suggests that the system is in a...

$\phi_m$ from $t_\text{f}$.

The evolution of these gaps under increasing magneti-

tations are also algebraic when the pairing energy is finite

Thus, one has to pay the energy gap of the $m' = \delta'$ plateau for that, and this gap equals the $m = \delta$ gap in the thermodynamical limit.

FIG. 18: (Color online) Two-particle gap (a) and pairing energy (b) for isotropic ladders as a function of magnetiza-

An anomaly in the two-particle gap at $m = \delta$ is clearly visible while the pairing energy remains finite up to

The finite value of $\Delta_{2p}$ in the thermodynamical limit within the plateau phase can be inferred by us-

$\Delta_p(S^z = 0)$. Paired electrons in a singlet state do not take advantage of Zeeman effect contrary to unpaired electrons which can be polarized along the magnetic field. The total energy of these two electrons is $2E(n_h - 1, +1/2) - 2 \times \frac{J}{t} \times H$, to be compared with the total energy of the same system with paired electrons, $E(n_h, 0) + E(n_h - 2, 0)$. By looking at Eq. (32) with $S^z = 0$, we find that the Pauli field simply reads $H_p = \Delta_p(S^z = 0)$. What actually occurs in the system is a transition to a FFLO state which changes the nature of the ground-state so that the Pauli limit can be exceeded. The superconducting upper critical field is deduced from $H_p = \Delta_p(S^z = 0)$. It simply means that Pauli limit is exceeded at low doping and for a wide region of parameters. This exceeding can be associated with the FFLO mechanism discussed in Sec. [IV.D.

The evolution of these gaps under increasing magneti-

$\Delta_{2p}(S^z) = E(n_h + 2, S^z) + E(n_h - 2, S^z)$

and

$$\Delta_p(S^z) = E(n_h - 1, S^z + 1/2) + E(n_h - 1, S^z - 1/2) - E(n_h, S^z) - E(n_h - 2, S^z).$$  

The finite value of $\Delta_{2p}$ in the thermodynamical limit within the plateau phase can be inferred by us-

$\Delta_p(S^z = 0)$. Paired electrons in a singlet state do not take advantage of Zeeman effect contrary to unpaired electrons which can be polarized along the magnetic field. The total energy of these two electrons is $2E(n_h - 1, +1/2) - 2 \times \frac{J}{t} \times H$, to be compared with the total energy of the same system with paired electrons, $E(n_h, 0) + E(n_h - 2, 0)$. By looking at Eq. (32) with $S^z = 0$, we find that the Pauli field simply reads $H_p = \Delta_p(S^z = 0)$. What actually occurs in the system is a transition to a FFLO state which changes the nature of the ground-state so that the Pauli limit can be exceeded. The superconducting upper critical field is deduced from $H_p = \Delta_p(S^z = 0)$. It simply means that Pauli limit is exceeded at low doping and for a wide region of parameters. This exceeding can be associated with the FFLO mechanism discussed in Sec. [IV.D.

At last, we remark from FIG. 18 that the pairing energy has a discontinuity upon magnetizing the ladder (in the $m \to 0$ limit) very similar to the known discontinuity of the spin gap upon doping the ladder (in the $\delta \to 0$ limit). Note also that the strong reduction of the pairing energy and of the spin gap for $\delta = 1/4$ observed on FIG. 17 is related to the proximity of the CDW phase which occurs for $J/t \sim 0.2$. 

![Figure 18: (Color online) Two-particle gap (a) and pairing energy (b) for isotropic ladders as a function of magnetization. An anomaly in the two-particle gap at $m = \delta$ is clearly visible while the pairing energy remains finite up to the superconducting-metallic state transition. $m_z$ represents the magnetization corresponding to the superconducting critical field $H_c$.](image-url)
The phase diagram of FIG. 17 is proposed to be generic in the coupled band regime for strong repulsive interactions. In addition to the magnetic properties described above, the system is in a LE superconducting state in the $m = 0$ plateau, in a Luttinger Liquid superconducting phase below $H_c$ and in a metallic phase in the $m = \delta$ plateau phase. Above $H_c$, the system switches to a metallic phase with three ungapped sectors which couple both spin and charge degrees of freedom. The central charge $c$ is expected to take the following values in each phase encountered as the magnetic field is increased (taking for example a vertical cut on FIG. 17 with $\delta = 0.063$): $c = 1$ ($m = 0$ plateau), 2 (FFLO below the $m = \delta$ plateau), 1 ($m = \delta$ plateau), 2 (FFLO above the $m = \delta$ plateau), 3 (above $H_c$), 2 (saturation phase). Note that one could expect a last transition $c = 2 \rightarrow 1$ for certain parameters, corresponding to a situation where the $(0, \uparrow)$ is completely filled. Since such a transition would induce a cusp in the magnetization curve and there is no sign of it on FIG. 16 we conclude that $c = 2$ is the value for the saturation in this regime.

To evaluate the role of $J/t$ on this generic phase diagram, we computed various energy gaps and critical fields in the $\delta \rightarrow 0$ limit, i.e. for two holes on a ladder with $L \rightarrow \infty$. Results are extrapolated from systems of length $L = 32, 48, 64$ and are displayed on FIG. 19. First, we note that for $J/t$ very small, a Nagaoka phase competes the LE phase and induces strong finite size effects in the ground states, thus the displayed boundary between these phases does not correspond to the thermodynamical transition. Hence, we only focus on points for which the LE phase is stable (on finite size systems) which are found to be qualitatively for $0.25 \leq J/t \leq 1.0$. For large $J/t$ the model undergoes a phase separation between holes and spins and we are not interested in the behavior near this instability. In this region of parameters, we have the following behavior: the width of the plateau increases for decreasing $J/t$ (for $J/t = 1.0$, the $m = \delta$ plateau is hardly visible at finite density, data not shown), while the exceeding of the Pauli limit increases with $J/t$ (which simply follows the increase of the pairing energy).

These elementary excitation gaps at $\delta \rightarrow 0^+$ can be related to the dynamics of the system when $m = 0$. The difference between $\min(\Delta_p, \Delta_M)$ and the spin gap corresponds to the binding energy of the resonant magnetic mode. It is interesting to note that the upper limit of the $m = \delta$ plateau is approximatively independent of $J/t$ and corresponds to a free magnon (which is nothing but the spin gap of the undoped system $\Delta_M \sim J/2$). Indeed, once the hole pair is bound to a magnon, the next magnetic excitation is to create a magnon in the remaining undoped background, which cost is $\Delta_M$, slightly renormalized by the scattering with the holes-magnon bound-state. This supports the phenomenological mechanism proposed in Ref. 22 for the opening of the irrational plateau. The binding of the hole pair to the magnon comes from the gain in kinetic energy holes have in a locally ferromagnetic environment. It is thus expected that the binding energy increases with $t/J$. It is also likely to find such a bound state in the vicinity of a Nagaoka phase which here competes the LE phase. It also explains the decreasing of the pairing energies as $J/t$ is reduced. Note that previous studies of this resonant magnetic mode with exact diagonalization47 and periodic boundary conditions agree with these results, and hence suggest that open boundary conditions do not affect this generic phase diagram.

V. COUPLED CHAIN REGIME

A. Dominant exchange model

By strongly reducing the interchain hopping amplitude, one can reach the coupled chain regime in which interchain hopping is an irrelevant perturbation48,49,50,51 and the relevant interchain couplings are either the Josephson coupling (for attractive intrachain interactions) or the exchange coupling (for repulsive interchain interaction). An important point is that even if the bare model has only interchain hopping, Josephson and exchange couplings are generated by the Renormalization Group (RG) flow49,50,51 and, as a result, the effective model always contain this type of interactions. In the
rest of the paper, we will call this strong coupling limit the “chain representation”. The bosonized Hamiltonian describing the two uncoupled chains reads

$$\mathcal{H} = \sum_{\nu = 1, 2} \int \frac{dx}{2\pi} \left[ u_\nu K_\nu (\pi \Pi_{\nu,i})^2 + \frac{u_\nu}{K_\nu} (\partial_x \phi_{\nu,i})^2 \right],$$

(34)

where we have dropped terms $\propto g_{\perp} \cos \sqrt{8} \phi_{\sigma,i}$ since these terms are marginally irrelevant in the case of repulsive interactions. In the case of attractive interactions, they also become irrelevant upon the application of a magnetic field strong enough to induce a C-IC transition.\(^{43}\)

1. Dominant exchange

For small $J_\parallel/t_\parallel$ ratios, the analysis of the scaling dimensions in the t-J model on a single chain\(^{32}\) shows that the dominant interchain coupling is the exchange one. This term reads

$$\frac{2J_\perp}{(2\pi\alpha)^2} \cos \sqrt{2}(\phi_{\rho,1} - \phi_{\rho,2}) \left[ \cos \sqrt{2}(\phi_{\sigma,1} - \phi_{\sigma,2}) + \frac{1}{2} \cos \sqrt{2}(\phi_{\sigma,1} + \phi_{\sigma,2} + 2\pi m) \right],$$

(35)

where we have the usual definition $\phi_{\nu,i} = \int \pi \Pi_{\nu,i}$. For nonzero magnetization, the last term is oscillating and drops from the Hamiltonian. It is convenient to introduce the new fields

$$\phi_{\nu,\pm} = \frac{1}{\sqrt{2}}(\phi_{\nu,1} \pm \phi_{\nu,2}),$$

$$\theta_{\nu,\pm} = \frac{1}{\sqrt{2}}(\phi_{\nu,1} \pm \phi_{\nu,2}).$$

With this transformation, the exchange term is rewritten as

$$\frac{2J_\perp}{(2\pi\alpha)^2} \int dx \cos 2\phi_{\rho,-} \left[ \cos 2\phi_{-} + \frac{1}{2} \cos 2\phi_{-} \right].$$

(36)

Moreover, the two chains being equivalent, the chain Hamiltonian is rewritten as

$$\mathcal{H} = \sum_{\nu = 1, 2} \int \frac{dx}{2\pi} \left[ u_\nu K_\nu (\pi \Pi_{\nu,r})^2 + \frac{u_\nu}{K_\nu} (\partial_x \phi_{\nu,r})^2 \right].$$

(37)

Obviously, the Hamiltonians describing the fields $\phi_{\sigma,+}$ and $\phi_{\rho,+}$ are purely quadratic indicating that the total charge and the total spin excitations are gapless. The fields $\phi_{\sigma,-}$ and $\phi_{\rho,-}$ are described by a generalized sine Gordon model. The sine Gordon interaction term \(^{36}\) can be treated within a RG analysis. The scaling dimensions of the term $\cos 2\phi_{\rho,-} \cos 2\phi_{\sigma,-}$ is $K_{\rho,-} + K_{\sigma,-}$ and scaling dimensions of the term $\cos 2\phi_{\rho,-} \cos 2\phi_{\sigma,-}$ is $K_{\rho,+} + K_{\sigma,+}$. From the analysis of Ref. \(^{30}\) we can conclude that in the case of interest we have $K_{\rho,+} > 1$ and $\cos 2\phi_{\rho,-} \cos 2\phi_{\sigma,-}$ is the most relevant term and for antiferromagnetic $J_\parallel$, the ground state has $\langle \phi_{\rho,-} \rangle = 0$ (mod $\pi$) and $\langle \theta_{\sigma,-} \rangle = \frac{\pi}{2}$ (mod $\pi$).

2. Most divergent superconducting fluctuations

The expression of the intrachain order parameters in terms of the fields in \(^{47}\) can be found in Ref. \(^{39}\). When re-expressed in terms of the $\pm$ fields and denoting $q = \pi m$, the singlet operator reads:

$$O_{SS,i} = \sum_{\sigma} \sigma e^{i\sigma x} \psi_{R,1,\sigma} \psi_{L,1,-\sigma}$$

(38)

$$\sim \sum_{\sigma} \sigma e^{i\sigma x} e^{i(\theta_{\rho,+} - \langle \theta_{\rho,+} \rangle) e^{-i(\theta_{\sigma,+} - \langle \theta_{\sigma,+} \rangle)}}$$

and, for triplets operators, we have:

$$O_{TS,i,S^z=0} = \sum_{\sigma} \sigma e^{i\sigma x} \psi_{R,1,\sigma} \psi_{L,1,-\sigma}$$

(39)

$$\sim \sum_{\sigma} \sigma e^{i\sigma x} e^{i(\theta_{\rho,+} - \langle \theta_{\rho,+} \rangle) e^{-i(\theta_{\sigma,+} - \langle \theta_{\sigma,+} \rangle)}}$$

(40)

For the interchain operators, the singlet operator reads:

$$O_{SS}' = \sum_{\sigma} \sigma e^{i\sigma x} \psi_{R,1,\sigma} \psi_{L,2,-\sigma}$$

(41)

$$\sim \sum_{\sigma} \sigma e^{i\sigma x} e^{i[\theta_{\rho,+} - \phi_{\rho,+} + \langle \theta_{\sigma,-} - \phi_{\sigma,+} \rangle]}$$

while the triplet reads:

$$O_{TS,S^z=0} = \sum_{\sigma} \sigma e^{i\sigma x} \psi_{R,1,\sigma} \psi_{L,2,-\sigma}$$

(42)

$$\sim \sum_{\sigma} \sigma e^{i\sigma x} e^{i[\theta_{\rho,+} - \phi_{\rho,+} + \langle \theta_{\sigma,-} - \phi_{\sigma,+} \rangle]}$$

(43)

$$O_{TS,S^z=1} = \psi_{R,1,\uparrow} \psi_{L,2,\uparrow}$$

$$\sim e^{i(\theta_{\rho,+} - \phi_{\rho,+} + \theta_{\sigma,+} - \phi_{\sigma,-})}.$$
Since $\langle \phi_{\rho,-} \rangle = 0$, we have $\langle e^{i\phi_{\rho,-}} \rangle \neq 0$ and by duality $\langle e^{i\phi_{\rho,-}(x)}e^{i\phi_{\rho,-}(0)} \rangle \sim e^{-x/\xi}$. This property implies that all the intrachain superconducting correlations should decay exponentially along the chains. On the other hand, the interchain correlations are reinforced. The physical picture is that in this situation fermions of opposite spin on each chain are bound together by the exchange interaction. Whether the dominant superconductivity is the interchain one or the intrachain one depends on the value of $K_{\sigma,-}$. Since we have $K_{\sigma,-} > 1$ in our case, the dominant superconducting correlations are the interchain singlet and the interchain triplet $S^z = 0$. Note that these two operators have exactly the same critical exponents. In the case of dominant Josephson coupling, the situation is reversed.

3. $S^z = 1$ triplet with a $2k_F$ momentum

It is also straightforward to derive an expression of the operator $e^{i2k_F,\sigma \cdot \psi} \psi_{R,1,\sigma} \psi_{R,2,\sigma}$ associated to the $2k_F$ triplet correlations. This expression reads:

$$e^{2ik_F,\sigma \cdot \psi}(\theta_{\rho,+} - \phi_{\rho,+}) e^{i\sigma(\theta_{\sigma,+} - \phi_{\sigma,+})}$$

This expression again shows power law correlations with the critical exponent:

$$\frac{1}{2} (K_{\rho,+} + K_{\sigma,+}^{-1} + K_{\sigma,-} + K_{\sigma,-}^{-1})$$

very similar to what has been obtained in Sec. [V B 2].

B. favoring the Josephson coupling in the coupled chain regime

Let us consider the regime of small interchain hopping and assume that now the intra-chain terms are such that $J_{\parallel} \sim 4t_{\parallel}$. In this case, the dominant term becomes the Josephson coupling and the perturbation term Eq. (36) is replaced by:

$$\frac{2\lambda}{(2\pi\alpha)^2} \cos \sqrt{2}(\theta_{\rho,1} - \theta_{\rho,2}) \left[ \cos \sqrt{2}(\theta_{\sigma,1} - \theta_{\sigma,2}) + \frac{1}{2} \cos \sqrt{2}(\phi_{\sigma,1} - \phi_{\sigma,2}) + \frac{1}{2} \cos \sqrt{2}(\phi_{\sigma,1} + \phi_{\sigma,2} + 2\pi m x) \right].$$

The treatment parallels the one of the exchange coupling in Sec. [V A 1]. The Josephson term is rewritten in the form:

$$\frac{2\lambda}{(2\pi\alpha)^2} \int dx \cos 2\theta_{\rho,-} \left[ \cos 2\theta_{\sigma,-} + \frac{1}{2} \cos 2\phi_{\sigma,-} \right].$$

The scaling dimensions of the term $\cos 2\theta_{\rho,-} \cos 2\theta_{\sigma,-}$ is $K_{\rho,-}^{-1} + K_{\sigma,-}^{-1}$ and scaling dimensions of the term $\cos 2\phi_{\rho,-} \cos 2\phi_{\sigma,-}$ is $K_{\rho,-}^{-1} + K_{\sigma,-}$. A similar RG analysis to the one of Sec. [V A 1] yields for $K_{\sigma,-} > 1$ $\langle \theta_{\rho,-} \rangle = 0$ (mod $\pi$) and $\langle \theta_{\sigma,-} \rangle = \frac{\pi}{2}$ (mod $\pi$) and for $K_{\sigma,-} < 1$ $\langle \theta_{\rho,-} \rangle = 0$ (mod $\pi$) and $\langle \phi_{\sigma,-} \rangle = \frac{\pi}{2}$ (mod $\pi$). This time, interchain correlations are decaying exponentially, and the dominant superconducting correlations are the interchain ones. Since $K_{\sigma,-} > 1$, $\langle \theta_{\sigma,-} \rangle = \frac{\pi}{2}$ and the interchain $S^z = 1$ triplet superconductivity is dominant. For $K_{\sigma,-} < 1$ the dominant superconducting correlations are the intrachain singlet and interchain triplet $S^z = 0$.

The results are summarized in the table [III] where the critical exponents $\eta_{\sigma}$ are defined by $\langle O_{\sigma}(x)O_{\alpha}(0) \rangle \sim |x|^{-\eta_{\sigma}}$. We note that triplet $S^z = 1$ superconducting correlations can never coexist with singlet superconducting correlations.

We have tried to observe these predictions numerically but using both the t-J model with small $t_{\perp}, J_{\perp}$ and Hubbard model with small $t_{\perp}$. Unfortunately, for large $J_{\parallel}/t_{\parallel}$, the system with open boundary conditions has edge effects due to the proximity of the phase separation which occurs generically in the t-J model for large $J/t$. When edge effects are absent and in the Hubbard model, we did not find evidences of the proposed predictions, i.e. we mostly found cases related to the coupled bands regime fixed point. Still, contrary to the isotropic case which has been widely studied numerically at zero magnetization, a systematic study of the phase diagram at zero magnetization would be necessary before tackling the system under magnetic field. This systematic study is out of the scope of the present article.

VI. CONCLUSION

In conclusion, we studied the case of two fermionic coupled chains, or ladders, under a magnetic field inducing a Zeeman effect in the system. The first situation we addressed was the free and strong-coupling limits. We found that large doping-dependent magnetization plateaus occur for “coupled-dimers” systems (with large interactions on the rungs) and that pairing is not expected in the $m = \delta$ magnetization plateau phase. For a system with isotropic couplings, we showed that $m = \delta$ plateaus also exist and but pairing survives to much higher magnetizations. Furthermore, the computation of the one and two-particles gap and bosonization interpretation proved that the plateau phase is metallic while the system is in a superconducting state below and
The different dominant superconducting fluctuations with the associated critical exponents. Josephson coupling dominates for $K_{\sigma_-} > 1$ and exchange coupling dominates for $K_{\rho_-} < 1$.

| Dominant interaction | Dominant superconducting correlations | Critical exponent |
|----------------------|--------------------------------------|------------------|
| Exchange coupling ($K_{\sigma_-} > 1$) | rung singlet and rung triplet $S^z = 0$ | $\frac{1}{2K_{\rho+}} + \frac{K_{\sigma_-}}{2}$ |
| Josephson coupling ($K_{\sigma_-} > 1$) | leg triplet $S^z = 1$ | $\frac{1}{2K_{\rho+}} + \frac{1}{2K_{\sigma_-}}$ |

TABLE III: The different dominant superconducting fluctuations with the associated critical exponents. Josephson coupling dominates for $K_{\sigma_-} > 1$ and exchange coupling dominates for $K_{\rho_-} < 1$.

above this plateau. In addition, we computed the superconducting upper critical field which is much larger than the Pauli limit for a wide range of the parameters of the t-J model. Superconducting correlation functions precisely agree with the bosonization predictions up to the accuracy of our methods. Turning to the coupled-chain regime, other interesting unconventional behaviors are predicted for the superconducting correlations with, for instance, the possibility of having polarized triplet correlations under high magnetic fields. However, quick numerical investigations study were not able to find good parameters providing evidences of such predictions.

Discussion on experiments – We now briefly discuss consequences for experiments. First of all, the possibility of measuring irrational magnetization plateau would give a direct access to the hole doping $\delta$ which has recently been estimated to be $\delta \sim 0.1$ in the superconducting phase of doped ladders. However, the magnetization is a global measurement including the contribution of all subsystems. For instance, the magnetization curve of SCCO has very recently been measured showing that the main contribution comes from the chains subsystem. On the other hand, measurements such as NMR under high magnetic field can provide local information on each subsystem.

The superconducting critical field $H_c(T)$ of SCCO has been measured under high pressure and displays a strong anisotropy and an anomalous $H_c(T)$ curvature. Furthermore, $H_c(T)$ much larger than standard Pauli limit are found, suggesting an exceeding of the latest. The nature of the superconductivity in SCCO is a long-standing debate because of its complex structure and the requirement of high pressure experiments. Questions such as the nature of the pairing or the dimensionality of superconductivity have not reached full agreement yet. One possibility is that superconducting fluctuations of the RVB type develops in the ladder subsystem leading to true superconducting order once these ladders are coupled through Josephson coupling ($T_c$ being controlled by this coupling rather than by the magnetic energy scale $J$). In this case our study suggests that the FFLO mechanism is relevant at the ladder subsystem level and thus could be stabilized when coupling the ladders. The other possibility would be that the superconducting phase is really two-dimensional allowing other explanations of the exceeding of Pauli limit such as triplet pairing when $H=0$. NMR measurements also showed that superconductivity survives under rather high field and $p$-wave superconductivity was suggested. Note that our study also proposes the possibility for the emergence at high fields of pairing channels not present at $H=0$ since magnetic fluctuations from which pairing originates are strongly affected by the magnetic field. Lastly, we mention that anomalous curvature of $H_c(T)$ as found in experiments were predicted in the mean-field approach of FFLO states in quasi-one and two-dimensional $d$-wave superconductors.

Acknowledgments

G.R. would like to thank IDRIS (Orsay, France) and CALMIP (Toulouse, France) for use of supercomputer facilities. G. R., P. P. and D. P. thank Agence Nationale de la Recherche (France) for support.

---

1. P. Fulde and R. A. Ferrell, Phys. Rev. 135, A550 (1964); A. I. Larkin and Y. N. Ovchinnikov, Sov. Phys. JETP 20, 762 (1965); R. Casalbuoni and G. Nardulli, Rev. Mod. Phys. 76, 263 (2004).
2. T. Ishiguro in High Magnetic Fields, vol. 595 of Lecture notes in physics (Springer, 2002).
3. N. Dupuis, Phys. Rev. B 51, 9074 (1995).
4. H. Shimahara and D. Rainer, J. Phys. Soc. Jpn. 66, 3591 (1997).
5. M. Sigrist, AIP Conf. Proc. 789, 165 (2005).
6. E. Dagotto and T. M. Rice, Science 271, 618 (1996); T. M. Rice, Z. Phys. B 103, 165 (1997); E. Dagotto, Rep. Prog. Phys. 62, 1525 (1999).
7. D. C. Cabra, A. Honecker, and P. Pujol, Phys. Rev. Lett. 79, 5126 (1997); Phys. Rev. B 58, 6241 (1998).
8. G. Chaboussant, M.-H. Julien, Y. Fagot-Revurat, L. P. Lévy, C. Berthier, M. Horvatić, and O. Piovesana, Phys. Rev. Lett. 79, 925 (1997); G. Chaboussant, Y. Fagot-Revurat, M.-H. Julien, M. E. Hanson, C. Berthier, M. Horvatić, L. P. Lévy, and O. Piovesana, Phys. Rev. Lett. 80, 2713 (1998).
9. B. C. Watson, V. N. Kotov, M. W. Meisel, D. W. Hall, G. E. Grauroth, W. T. Montfrooij, S. E. Nagler, D. A. Jensen, R. Backov, M. A. Petruska, G. E. Fanucci, and D. R. Talham, Phys. Rev. Lett. 86, 5168 (2001).
10. C. P. Landee, M. M. Turnbull, C. Galeriu, J. Giantsidis,
S. R. White, I. Affleck, and D. J. Scalapino, Phys. Rev. B 18, 5579 (1999).
D. C. Cabra, A. De Martino, A. Honecker, P. Pujol, and P. Simon, Phys. Lett. A 268, 418 (2000).
D. C. Cabra, A. De Martino, A. Honecker, P. Pujol, and P. Simon, Phys. Rev. B 63, 094406 (2001).
C. A. Hayward, D. Poilblanc, R. M. Noack, D. J. Scalapino, and W. Hanke, Phys. Rev. Lett. 75, 926 (1995).
E. Orignac and D. Poilblanc, Phys. Rev. B 68, 052504 (2003); D. Poilblanc, D. J. Scalapino and S. Capponi, Phys. Rev. Lett. 91, 137203 (2003); D. Poilblanc and D. J. Scalapino, Phys. Rev. B 71, 174403 (2005).
S. R. White, I. Affleck, and D. J. Scalapino, Phys. Rev. B 65, 165122 (2002).
P. W. Anderson, Science 235, 1196 (1987).
A. Gozar and G. Blumberg, *Frontiers in Magnetic Materials* (Springer-Verlag, 2005), pp. 653-695.
M. Uehara, T. Nagata, J. Akimitsu, H. Takahashi, N. M°ri, and K. Kinoshita, J. Phys. Soc. Jpn. 65, 2764 (1996); D. Jerome, P. Auba n-Senzer, and Y. Piskunov in *High Magnetic Fields*, vol. 595 of Lecture notes in physics (Springer, 2002).
D. Braithwaite, T. Nagata, I. Sheikin, H. Fujino, J. Akimitsu, and J. Flouquet, Solid State Com. 114, 533 (2000).
T. Nakainishi, N. Motoyama, H. Mitamura, N. Takeshita, H. Takahashi, H. Eisaki, S. Uchida, and N. Mori, Phys. Rev. B 72, 054520 (2005).
M. Matsukawa, Y. Yamada, M. Chiba, H. Ogasawara, T. Shibata, A. Matsushita, and Y. Takano, Physica C 411, 101 (2004).
G. Roux, S. R. White, S. Capponi, and D. Poilblanc, Phys. Rev. Lett. 97, 087207 (2006).
S. R. White, Phys. Rev. Lett. 69, 2863 (1992); Phys. Rev. B 48, 10345 (1993).
U. Schollwöck, Rev. Mod. Phys. 77, 259 (2005).
D. C. Cabra, A. De Martino, P. Pujol, and P. Simon, Europhys. Lett. 57, 402 (2002).
T. Pruschke and H. Shiba, Phys. Rev. B 46, 356 (1992).
K. Yang, Phys. Rev. B 63, 140511(R) (2001).
M. Yamanaka, M. Oshikawa, and I. Affleck, Phys. Rev. Lett. 79, 1110 (1997); M. Oshikawa, M. Yamanaka, and I. Affleck, Phys. Rev. Lett. 78, 1984 (1997); P. Gagliardini, S. Haas, and T. M. Rice, Phys. Rev. B 58, 9603 (1998).
N. Nagaosa, Sol. State Comm. 94, 495 (1995); N. Nagaosa and M. Oshikawa, J. Phys. Soc. Jpn. 65, 2241 (1996).
D. V. Khveshchenko and T. M. Rice, Phys. Rev. B 50, 252 (1994).
A. M. Finkelstein and A. I. Larkin, Phys. Rev. B 47, 10461 (1993).
H. J. Schulz, Phys. Rev. B 53, R2959 (1996).
L. Balents and M. P. A. Fisher, Phys. Rev. B 53, 12133 (1996).
M. Troyer, H. Tsumetsugu, and T. M. Rice, Phys. Rev. B 53, 251 (1996).
M. Fabrizio, Phys. Rev. B 48, 15838 (1993).
H. Frahm and V. E. Korepin, Phys. Rev. B 42, 10553 (1990); Phys. Rev. B 43, 5653 (1991).
E. Orignac and Y. Suzumura, Eur. Phys. J. B 23, 57 (2001).
T. Giamarchi, *Quantum Physics in one Dimension*, vol. 121 of *International series of monographs on physics* (Oxford University Press, Oxford, UK, 2004).
R. Konik, F. Lesage, A. W. W. Ludwig and H. Saleur, Phys. Rev. B 61, 4983 (2000).
H. Schulz (1998), cond-mat/9808167.
S. R. White, Phys. Rev. B 72, 180403(R) (2005).
G. I. Dzhaparidze and A. A. Nersesyan, JETP Lett. 27, 334 (1978); V. L. Pokrovsky and A. L. Talapov, Phys. Rev. Lett. 42, 65 (1979); H. J. Schulz, Phys. Rev. B 22, 5274 (1980).
E. Jeckelmann, D. J. Scalapino, and S. R. White, Phys. Rev. B 58, 9492 (1998).
A. Buzdin, S. Tollis, and J. Cayssol, Phys. Rev. Lett. 95, 167003 (2005).
H. H. Lin, L. Balents, and M. P. A. Fisher, Phys. Rev. B 58, 1794 (1998).
D. Poilblanc, O. Chiappa, J. Riera, S. R. White, and D. J. Scalapino, Phys. Rev. B 62, 14633 (2000); D. Poilblanc, E. Orignac, S. R. White, and S. Capponi, Phys. Rev. B 69, 220406(R) (2004).
X. G. Wen, Phys. Rev. B 42, 6623 (1990).
D. Boies, C. Bourbonnais, and A.-M. S. Tremblay, in *Proceedings of the XXXIst Rencontres de Moriond, edited by T. Martin, G. Montambaux, and J. Tran Thanh Van* (Editions Frontières, Gif sur Yvette, France, 1996).
V. M. Yakovenko, JETP Lett. 56, 510 (1992).
S. Brazovskii and V. Yakovenko, J. de Phys. (Paris) Lett. 46, L111 (1985).
M. Ogata, M. U. Luchini, S. Sorella, and F. F. Assaad, Phys. Rev. Lett. 66, 2388 (1991).
Y. Piskunov, D. Jérôme, P. Auban-Senzer, P. Wietek, and A. Yakubovsky, Phys. Rev. B 72, 064512 (2005).
R. Klingeler et al., Phys. Rev. B 72, 184406 (2005); Phys. Rev. B 73, 014426 (2006).
T. Nagata, M. Uehara, J. Goto, J. Akimitsu, N. Motoyama, H. Eisaki, S. Uchida, H. Takahashi, T. Nakainishi, and N. M°ri, Phys. Rev. Lett. 81, 1090 (1998).
Y. Piskunov, D. Jérôme, P. Auban-Senzer, P. Wietek, and A. Yakubovsky, Phys. Rev. B 69, 014510 (2004); N. Motoyama, H. Eisaki, S. Uchida, T. Nakahata, N. Morii, T. Nakainishi and H. Takahashi, Europhys. Lett. 58, 758 (2002).
N. Fujiwara, N. M°ri, Y. Uwatoko, T. Matsumoto, N. Motoyama, and S. Uchida, Phys. Rev. Lett. 90, 137001 (2003); J. of Phys.: Condens. Matter 17, S929 (2005).