Accounting for anisotropy of properties of substructural elements in study of strength of macroinhomogeneous media in the case of plane stressed state

B S Reznikov and K A Kuzmin
Novosibirsk State Technical University, Novosibirsk 630073, Russia
E-mail: kuzmin.2011@stud.nstu.ru

Abstract. In this paper on the mathematical model of a multiphase medium (with longitudinally transverse arrangement of phases) and the algorithm for calculating strength under combined thermomechanical loading the method is proposed for analyzing the limiting state of a composite, which allows one to take into account the internal structure of the material and the anisotropy of the elastic and thermal properties of substructural elements. For each element of the composition the strength condition of P. P. Balandin is used. It takes into account various limits of tensile strength and compression of phase materials. A numerical study of condition of initial destruction of composites and its various sections in space of parameters of the external action under isothermal loading in the case of plane stressed state is made.

1. Introduction
In modern composites, where essentially anisotropic crystalline inclusions, "whiskers", carbon and boron fibers, etc., are used as reinforcing elements, analysis of effect of anisotropy of properties of phase materials on initial destruction of structurally inhomogeneous medium is urgent task.

To study limiting state of a composite with longitudinally transverse arrangement of "M+N" phases (M is the number of longitudinal phases, N is the number of transverse phases, Fig. 1), we will use the defining relations from [1], obtained on the correctly formulated conjugation conditions for deformations, stresses and temperature at the interface boundaries [1-3].

![Figure 1. Characteristic structural element of composite with longitudinally transverse arrangement of "M + N" phases in the case of plane stressed state.](image-url)
2. The basic relations and algorithm for studying initial destruction of media with internal structure

For composite consisting of "M+N" phases (Figure 1), in case of action of homogeneous temperature field in case of plane stressed state (i.e., by loading in the plane 103 by forces \( \sigma_1, \sigma_3, \sigma_5 \)), taking into account the defining relations from [1-3], we have the following expressions for stresses in elements of composition:

in longitudinal phases

\[
\sigma_j^{(s)} = \sum_{i=1,3} \psi_{ji}^{(s)} \cdot \sigma_i + \theta \cdot \psi_{ji}^{(s)},
\]

\[
\sigma_5^{(s)} = \sigma_5 = \psi_5^{(s)} \cdot \sigma_5,
\]

\( j = 1,2,3; \ s = 1,2,\ldots,M; \)

in transverse phases

\[
\sigma_j^{(l)} = \sum_{i=1,3} \psi_{ji}^{(l)} \cdot \sigma_i + \theta \cdot \psi_{ji}^{(l)},
\]

\[
\sigma_5^{(l)} = \sigma_5 = \psi_5^{(l)} \cdot \sigma_5,
\]

\( j = 2,3; \ l = M + 1, M + 2,\ldots,M + N. \)

The values \( \psi_{ji}^{(s)}, \psi_{ji}^{(l)}, \psi_5^{(s)}, \psi_5^{(l)}, \psi_5^{(l)} \) are determined by effective compliance and linear thermal expansion coefficients of structurally inhomogeneous material, which were obtained in [1], \( \theta = T - T_0 \) is temperature increment, \( T_0 \) is temperature in initial state; we will use the matrix designations for stresses from [1-4] and notation from [1-3].

Studying initial destruction of multiphase composites, we will use the structural approach [5] and the strength condition P.P. Balandin [6] for each phase, which under the considered form of loading, taking into account (1), (2), has the form:

\[
\sum_{j=1}^{3} \left( \sigma_j^{(m)} \right)^2 + \left( \sigma_5^{(m)} \right)^2 - \sigma_1^{(m)} \cdot \sigma_2^{(m)} - \sigma_1^{(m)} \cdot \sigma_3^{(m)} - \sigma_2^{(m)} \cdot \sigma_3^{(m)} + \left( \sigma_5^{(m)} - \sigma_5^{(m)} \right) \sum_{j=1}^{3} \sigma_j^{(m)} = \sigma_4^{(m)} \sigma_5^{(m)},
\]

where \( \sigma_1^{(m)} \) and \( \sigma_3^{(m)} \) are the tensile and compressive strength limits of the material of the m-th phase, \( m = 1,2,\ldots,M; M+1,\ldots,M+N \).

To construct in space of external actions \( \sigma_10\sigma_3\sigma_5\theta \) hypersurface of initial destruction of macroinhomogeneous material and its various sections, we will use the algorithm from [7], that is, to take

\[
\sigma_i = L_i \cdot p \ (i = 1,3,5); \ \theta = L_i \cdot p; \ \sum_{i=1,3,5} |L| + |L| = 1,
\]

where \( p > 0 \) is load parameter, \( L_1, L_3, L_5, L_5 \) - intensity of corresponding type of external influence. Using the expressions for stresses in substructural elements (1), (2), the strength condition (3), for fixed \( L_i (i = 1,3,5), L_i \), we determine the value of \( P_a^{(m)} \) parameter, corresponding to beginning of destruction of the m phase:

\[
P_a^{(m)} = \frac{-B^{(m)} + \sqrt{(B^{(m)})^2 + 4\sigma_1^{(m)} \sigma_3^{(m)} A^{(m)}}}{2A^{(m)}},
\]

where \( m = s, s = 1,2,\ldots,M; m = l, l = M+1,M+2,\ldots,M+N \);

for quantity with index of «s» (s = 1,2,\ldots,M) we have
\[
A^{(i)} = \sum_{j=1}^{3} \left( \varphi_j^{(i)} \right)^2 + 3 \left( L_j \psi_j^{(3)} \right)^2 - \varphi_1^{(i)} \varphi_2^{(i)} - \varphi_1^{(i)} \varphi_3^{(i)} - \varphi_2^{(i)} \varphi_3^{(i)}, \tag{6}
\]

\[
B^{(i)} = \left( \sigma_{(3)}^{(i)} - \sigma_{(1)}^{(i)} \right) \sum_{j=1}^{3} \varphi_j^{(i)},
\]

\[
\varphi_j^{(i)} = L_j \psi_j^{(1)} + L_j \psi_j^{(3)} + L_j \psi_j^{(4)} \quad (j = 1, 2, 3);
\]

for quantity with index of «l» (l = M+1, M+2, ..., M+N) –

\[
A^{(l)} = L_l^2 + \left( \varphi_2^{(l)} \right)^2 + \left( \varphi_3^{(l)} \right)^2 + 3 L_3^2 - L_1 \varphi_2^{(l)} - L_1 \varphi_3^{(l)} - \varphi_2^{(l)} \varphi_3^{(l)}, \tag{7}
\]

\[
B^{(l)} = \left( \sigma_{(1)}^{(l)} - \sigma_{(3)}^{(l)} \right) \left( L_1 + \varphi_2^{(l)} + \varphi_3^{(l)} \right),
\]

\[
\varphi_j^{(l)} = L_j \psi_j^{(1)} + L_j \psi_j^{(3)} + L_j \psi_j^{(4)} \quad (j = 1, 2, 3).
\]

The magnitude of loading parameter \( P_{v_i} \), corresponding to beginning of destruction of multiphase composite, is determined from solution of minimization problem:

\[
P_{v_i} = \min \{ P^{(m)}_{v_i} \}, \quad m = 1, 2, ..., M, M+1, M+2, ..., M+N. \tag{8}
\]

Then, taking into account the first two relations from (4), we find external forces and the magnitude of temperature effect at which structurally inhomogeneous material begins to break down:

\[
\sigma_{v_i} = L_i p_{v_i} (i = 1, 3, 5), \quad \theta_{v_i} = L_i \cdot p_{v_i}. \tag{9}
\]

Considering all possible values of \( L_1 (1, 3, 5) \) and \( L_i \) from (4), we obtain hypersurface of strength of composite in space of external forces \( \sigma_{v}, \theta_{v}, \sigma_{v_i}, \theta_{v_i} \).

It should be noted that the proposed approach makes it possible to determine the type of initial destruction (i.e., due to destruction of which substructural element destruction of composite material begins) for each type of external action.

3. Analysis of influence of anisotropy of properties of substructure elements on strength of composite

As the first example to consider structurally inhomogeneous material consisting of two longitudinal phases: \( M = 2, N = 0, s = 1, 2 \). The first phase is transversally isotropic with the following parameters:

\[
\bar{E}_1^{(i)} = 5, \quad \bar{E}_2^{(i)} = 50, \quad \nu_1^{(i)} = \nu_2^{(i)} = 0.2; \tag{10}
\]

\[
\bar{G}_1^{(i)} = \frac{\bar{E}_1^{(i)}}{2(1 + \nu_1^{(i)})}, \quad \bar{G}_2^{(i)} = 25; \quad \bar{\alpha}_1^{(i)} = 0.3, \quad \bar{\alpha}_2^{(i)} = 0.4; \quad \bar{\sigma}_{(+)}^{(i)} = 20, \quad \bar{\sigma}_{(-)}^{(i)} = 2,
\]

and the second phase is isotropic with parameters:

\[
\bar{E}(2) = 1; \quad \bar{\sigma}_{(+)}^{(2)} = 1, \quad \bar{\sigma}_{(-)}^{(2)} = 0.5; \quad \nu^{(2)} = 0.2; \quad \bar{\alpha}(2) = 1. \tag{11}
\]

Here and below the values with the bar are dimensionless:
\begin{equation}
\bar{E}_j^{(1)} = \frac{E_j^{(1)}}{E_0} (j = 1,3), \quad \bar{E}_j^{(2)} = \frac{E_j^{(2)}}{E_0};
\end{equation}

\begin{equation}
\bar{\alpha}_{j}^{(1)} = \frac{\alpha_j^{(1)}}{\alpha_0} (j = 1,3); \quad \bar{\sigma}_{i}^{(s)} = \frac{\sigma_{i}^{(s)}}{\sigma_0} (s = 1,2),
\end{equation}

\begin{equation}
\bar{\sigma}_{j} = \frac{\sigma_j}{\sigma_0}, \quad \bar{\theta}_s = \frac{\theta E_0 \alpha_0}{\sigma_0},
\end{equation}

where \( E_j^{(1)}, E_j^{(2)}, \nu_j^{(1)}, \nu_j^{(2)}, \alpha_j^{(1)}, \alpha_j^{(2)} \) - Young's moduli, Poisson's, linear thermal expansion coefficients of materials of the 1st and 2nd phases; \( \sigma_0, E_0 \) – constant dimensions of stress, \( \alpha_0 \) – constant [1/deg].

Figures 2 - 4 show different sections of hypersurface of initial destruction of composite with the specific volume content of first and second phases \( \omega_1 = 0.2 \) and \( \omega_2 = 0.8 \). For comparison, the dashed lines indicate the sections of hypersurface of initial destruction when both phases are different isotropic materials: for the second phase we have the parameters (11), and for the first phase the parameters from (10) with the "1" subscript. Numerical calculations have shown that the greatest qualitative and quantitative influence on strength of composite is anisotropy of Young's modulus of the first phase.

On the Fig. 2 - 4 parts of the curves marked with digits "1" and "2" correspond to destruction of the first and second phases, at the corner points we have simultaneous destruction of both phases.

Figure 2. Section \( \bar{\sigma}_{1e} \bar{\sigma}_{3e} : \omega_1 = 0.2, \omega_2 = 0.8 \). Figure 3. Section \( \bar{\sigma}_{1e} \bar{\theta}_s : \omega_1 = 0.2, \omega_2 = 0.8 \).

Figure 4. Section \( \bar{\sigma}_{1e} \bar{\sigma}_{3e} : \omega_1 = 0.2, \omega_2 = 0.8 \).

To analyze the effect on the initial destruction condition of composite of the spatial arrangement of anisotropic phase and its specific volume content, calculations were carried out for three-phase composite materials with longitudinal arrangement of substructure elements: \( M = 2 \) (\( s = 1,2 \)) and \( N = 1 \) (\( l = 3 \)). In this case, the following parameters were adopted for transversely isotropic phase:
\[ E^{(m)}_1 = 5, \quad E^{(m)}_3 = 100; \quad \nu_1^{(m)} = 0.2, \quad \nu_3^{(m)} = 0.4; \]

\[ \overline{G}_i^{(m)} = \frac{E^{(m)}}{2(1 + \nu_1^{(m)})}, \quad \overline{G}_3^{(m)} = 50; \quad \overline{\alpha}_{11}^{(m)} = 0.3, \quad \overline{\alpha}_{33}^{(m)} = 0.7; \]

\[ \overline{\sigma}^{(m)}_1 = 20, \quad \overline{\sigma}^{(m)}_3 = 50. \]

For the isotropic phase for \( s = 2 \) we have the parameters (11), and for the other isotropic phase we have the following parameters:

\[ E^{(n)} = 5; \quad \overline{\sigma}^{(n)}_1 = 20; \quad \overline{\sigma}^{(n)}_3 = 5; \quad \nu^{(n)} = 0.2; \quad \overline{\sigma}^{(n)}_1 = 0.3. \]  

(14)

Values of the indices "\( m \)" and "\( n \)" in (13) and (14) depend on location of these phases in space, which will be indicated below.

To consider the two cases location of anisotropic phase in space:
1) the longitudinal arrangement of transversally isotropic phase \( m = s = 1 \) and \( n = l = 3 \);
2) the transverse location of transversally isotropic phase \( m = l = 3 \) and \( n = s = 1 \).

Figures 5 - 7 show different sections of hypersurface of initial destruction of three-phase composites. The solid lines correspond to the case (15) for the following values of specific volume content of the phases: \( \overline{\omega}_1 = 0.3; \quad \overline{\omega}_2 = \overline{\omega}_3 = 0.35 \); and the dashed curves correspond to the case (16) with \( \overline{\omega}_1 = \overline{\omega}_2 = 0.35; \quad \overline{\omega}_3 = 0.3 \). Digits 1, 2, 3, as before, indicate the type of initial destruction.
The results of calculations of different sections of the strength condition of three-phase composites as function of the specific volume content of anisotropic phase are shown on Fig. 8 - 13. For the case (15), we have Fig. 8 - 11, the solid lines correspond to $\tilde{\omega}_1 = 0.3; \tilde{\omega}_2 = \tilde{\omega}_3 = 0.35$; dashed - $\tilde{\omega}_1 = 0.5; \tilde{\omega}_2 = \tilde{\omega}_3 = 0.25$; dot-dash - $\tilde{\omega}_1 = 0.7; \tilde{\omega}_2 = \tilde{\omega}_3 = 0.15$;

Under the conditions (16), we have Fig. 12, 13, where solid lines correspond $\tilde{\omega}_1 = 0.3; \tilde{\omega}_2 = \tilde{\omega}_3 = 0.35$; dashed - $\tilde{\omega}_1 = 0.5; \tilde{\omega}_2 = \tilde{\omega}_3 = 0.25$; dot-dash - $\tilde{\omega}_1 = 0.7; \tilde{\omega}_2 = \tilde{\omega}_3 = 0.15$.

The results on Fig. 5-13 show that strength of macroinhomogeneous media essentially depends both on location in space of anisotropic phase and on its specific volume content.
4. Conclusion

In this way, the proposed approach allows us to predict limiting state of composites, depending on the nature of anisotropy of substructure elements, their specific volume content and mutual arrangement. It allows us to purposefully design multiphase materials with the required strength characteristics in terms of operating conditions for constructions.

References

[1] Reznikov B S and Gobysh A V 2014 Prediction structure of multiphase size-stable composites under temperature influence Reports of the 3rd All-Russian Conference Problems of Optimal Design of Constructions Novosibirsk State University of architecture and Civil Engineering (Sibstrin) (Novosibirsk: Sibstrin) pp 345-52 (in Russian)

[2] Reznikov B S, Nikitenko A F and Kucherenko I V 2008 Prediction macroscopic properties of structurally inhomogeneous media Report 1 Izvestia vuzov, Stroitelstvo (Novosibirsk: Sibstrin) 2 pp 10-17 (in Russian)

[3] Reznikov B S and Gobysh A V 2013 Calculation effective coefficients of thermal expansion of macroinhomogeneous composites Reports of Academy of Sciences of Higher School 2(21), pp 139-49 (in Russian)

[4] Shermergor T D 1977 The Theory of Elasticity of Microinhomogeneous Media (Moscow: Nauka) (in Russian)

[5] Nemirovskiy Y V and Reznikov B S 1986 Strength of Structural Elements of Composite Materials (Novosibirsk: Nauka) (in Russian)

[6] Balandin P P 1937 On question of strength hypotheses Vestnik inzhenerov i tekhnikov 1 19-24 (in Russian)

[7] Reznikov B S 1989 Predicting destruction of annular plates taking into account actual structure and stochastic nature of reinforced material Krayevyye zadachi i ikh prilozheniya (Cheboksary: Chuvash state university) pp 89–99 (in Russian)