Killing forms and toric Sasaki-Einstein spaces

Vladimir Slesar¹, Mihai Visinescu² and Gabriel Eduard Vîlcu³,⁴

¹ Department of Mathematics, University of Craiova, Str. Al.I. Cuza, Nr. 13, Craiova 200585, Romania
² Department of Theoretical Physics, National Institute for Physics and Nuclear Engineering, Magurele, P.O.Box M.G.-6, Romania
³ Department of Mathematical Modelling, Economic Analysis and Statistics, Petroleum-Gas University of Ploiești, Bulevardul București, Nr. 39, Ploiești, 100680, Romania
⁴ Faculty of Mathematics and Computer Science, Research Center in Geometry, Topology and Algebra, University of Bucharest, Str. Academiei, Nr. 14, Sector 1, Bucharest, 060042, Romania

E-mail: ¹vlslesar@central.ucv.ro
E-mail: ²mvisin@theory.nipne.ro
E-mail: ³gvilcu@upg-ploiesti.ro
E-mail: ⁴gvilcu@gta.math.unibuc.ro

Abstract. The construction of the special Killing forms on toric Sasaki-Einstein manifolds is presented. This goal is achieved using the interplay between complex coordinates of the Calabi-Yau metric cone and the special Killing forms on the toric Sasaki-Einstein space. As a concrete example, we present the complete set of special Killing forms on the five-dimensional Einstein-Sasaki $Y^{p,q}$ spaces. It is pointed out the existence of two additional special Killing forms associated with the complex holomorphic volume form of Calabi-Yau cone manifold.

1. Introduction

In the last time Sasaki-Einstein geometry has become of high interest in connection with some recent developments in mathematics and theoretical physics [1, 2]. Among them, we mention the construction of explicit inhomogeneous Sasaki-Einstein metrics [3, 4]. In particular, in 5-dimensional case an interesting class of toric contact structures on $S^2 \times S^3$ denoted $Y^{p,q}$ (where $q < p$ are positive integers) have been studied by physicists [4]. On the other hand, the Sasaki-Einstein spaces provide supersymmetric backgrounds relevant to the AdS/CFT correspondence [5]. For example Sasaki-Einstein spaces proved to be associated with Euclideanised Kerr-NUT-(A)dS spaces in a certain scaling limit [6]. Moreover, the Kerr-AdS black holes with equal angular momenta in arbitrary odd spacetime dimensions equal to or greater than 5 were considered in [7], the authors showing that Sasaki-Einstein metrics on compact manifolds can be obtained by twisting the Killing vector fields of the black holes. They also discussed the implication of this correspondence to string and M-theory as well as to supersymmetric Yang-Mills theory on Kerr-AdS black hole metrics, focusing on both AdS$_5 \times S^5$ and AdS$_4 \times S^7$ solutions. We recall that AdS$_5 \times S^5$ is a supersymmetric solution of type IIB supergravity that is expected to be dual to a four-dimensional superconformal field theory arising from a stack of D3-branes placed at the tip.
of the corresponding Calabi-Yau cone, while AdS$_4 \times X_7$ is a supersymmetric solution of eleven-dimensional supergravity that is expected to be dual to a three-dimensional superconformal field theory arising on a stack of M2-branes sitting at the tip of the corresponding metric cone.

In this paper we want to take a closer look at the special Killing forms on toric Sasaki-Einstein spaces. For this purpose we use foliated coordinates on the metric cone. The interest in the Killing forms is motivated by the significance of symmetries in physics. Killing forms play a fundamental role being related to hidden symmetries, separability of field equations, existence of quantum symmetry operators, supersymmetries, etc.

We shall exemplify the general approach giving an explicit construction of the special Killing forms on 5-dimensional $Y^{p,q}$ spaces.

The paper is organized as follows. In the next Section we describe the Killing forms and emphasize the importance of the special Killing forms in the case of Sasaki spaces. In Section 3 we present the Sasakian geometry and its interrelation with Kähler geometry; also, some particular features of special Killing forms in this framework are emphasized. In Section 4 we exemplify the construction of the special Killing forms on $Y^{p,q}$ spaces. The paper ends with conclusions in Section 5.

2. Special Killing forms

Killing vector fields represent a basic object of differential geometry connected with the infinitesimal isometries. The flow of a Killing vector field preserves a given metric, and there exists a conserved quantity for the geodesic motions. A natural generalization of Killing vector fields is represented by the conformal Killing vector fields. In this case the flows preserve a given conformal class of metrics. More generally, one can consider conformal Killing forms which are sometimes referred as twistor forms or conformal Killing-Yano tensors.

**Definition.** A conformal Killing-Yano tensor of rank $p$ on an $n$-dimensional Riemannian manifold $(M, g)$ is a $p$-form $\Psi$ which satisfies

$$\nabla_X \Psi = \frac{1}{p+1} X \lhd d\Psi - \frac{1}{n-p+1} X^* \wedge d^* \Psi,$$

for any vector field $X$ on $M$.

Here we used the standard conventions: $\nabla$ is the Levi-Civita connection with respect to the metric $g$, $X^*$ is the 1-form dual to the vector field $X$, $\lhd$ is the operator dual to the wedge product and $d^*$ is the adjoint of the exterior derivative $d$.

In fact, considering the left hand side of the equation (1) as a natural first order differential operator (called twistor operator), one can also describe the set of conformal Killing-Yano tensors as the kernel of this operator [8]. If $\Psi$ is co-closed in (1), then we obtain the definition of a Killing-Yano tensor [9]. Examples of Killing and conformal Killing form can be found on standard sphere, on Sasakian manifolds, on nearly Kähler manifolds [8].

A particular class of Killing forms is represented by the special Killing forms:

**Definition.** A Killing form $\Psi$ is said to be a special Killing form if it satisfies for some constant $c$ the additional equation [10, 8]

$$\nabla_X (d\Psi) = cX^* \wedge \Psi,$$

for any vector field $X$ on $M$.

At the first sight this definition seems to be rather restrictive having in mind that for a space of constant curvature any Killing form is special with $c = R/n(n-1)$ where $R$ is the scalar curvature [10]. Nevertheless, all known examples of Killing forms on Sasaki manifolds are special. Moreover, any Killing form of degree at least 3 on a compact Sasaki-Einstein manifold is actually special and the above result extends on compact 3-Sasakian spaces [11].

In the next Section we see how one can extract these Killing forms in the framework represented by Sasaki-Einstein manifolds.
3. Special Killing forms on Sasaki-Einstein manifolds

3.1. Sasaki-Einstein and Kähler structures

A $(2n-1)$-dimensional manifold $Y$ is a contact manifold if there exists a 1-form $\eta$ (called a contact 1-form) on $M$ such that

$$\eta \wedge (d\eta)^{n-1} \neq 0.$$  \hfill (3)

For every choice of contact 1-form $\eta$ there exists a unique vector field $K_\eta$, called Reeb vector field, that satisfies

$$\eta(K_\eta) = 1 \quad \text{and} \quad K_\eta \cdot d\eta = 0.$$  \hfill (4)

There are many equivalent definitions of the Sasakian structures. A simple and direct definition is the following:

**Definition.** A compact Riemannian manifold $(Y, g)$ is Sasakian if and only if its metric cone $(X = C(Y) \cong \mathbb{R}_+ \times Y, \bar{g} = dr^2 + r^2 g)$ is Kähler.

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold $(Y, g)$ is naturally isometrically embedded into the metric cone via the inclusion $Y = \{r = 1\} = \{1\} \times Y \subset C(Y)$.

Let us denote $\tilde{K} = J \left( r \frac{\partial}{\partial r} \right)$, \hfill (5)

where $J$ is the complex structure on the cone manifold. $\tilde{K} - iJ\tilde{K}$ is a holomorphic vector field on $C(Y)$ and the restriction $K$ of $\tilde{K}$ to $Y \subset C(Y)$ is the Reeb vector field on $Y$.

Now, let $Y$ be a Sasaki-Einstein manifold of dimension $\dim \mathbb{R} Y = 2n - 1$ and its Kähler cone $X = C(Y)$ is of dimension $\dim \mathbb{C} X = 2n$, $(\dim \mathbb{C} X = n)$. Sasaki-Einstein geometry is naturally “sandwiched” between two Kähler-Einstein geometries as shown in the following proposition [2]:

**Proposition.** Let $(Y, g)$ be a Sasaki manifold of dimension $2n - 1$. Then, the following statements are equivalent:

(i) $(Y, g)$ is Sasaki-Einstein with $\text{Ric}_g = 2(n-1)g$;

(ii) The Kähler cone $(C(Y), \bar{g})$ is Ricci-flat $(\text{Ric}_{\bar{g}} = 0)$, i.e. Calabi-Yau manifold;

(iii) The transverse Kähler structure to the Reeb foliation $\mathcal{F}_K$ is Kähler-Einstein with $\text{Ric}_{\mathcal{T}}^T = 2ng^T$.

The Kähler form $\omega$ is a 2-form which is exact and homogeneous of degree 2 under the Euler angle $r \frac{\partial}{\partial r}$

$$\omega = -\frac{1}{2} d(r^2 \eta) = -r dr \wedge \eta - \frac{1}{2} r^2 d\eta, \quad \mathcal{L}_{\frac{\partial}{\partial r}} \omega = 2\omega.$$  \hfill (6)

Here $\eta$ is the Sasakian 1-form of $Y$. It lifts to $C(Y)$ as

$$\eta = J \left( \frac{dr}{r} \right) = i(\partial - \bar{\partial}) \log r.$$  \hfill (7)

We use the same letter $\eta$ by the abuse of notation. From (5) and (7) it results that $\tilde{K}$ is dual to the 1-form $r^2 \eta$. The Kähler form $\omega$ can be written as

$$\omega = \frac{1}{2} i \partial \bar{\partial} r^2,$$  \hfill (8)

which means that

$$F = \frac{r^2}{4},$$  \hfill (9)

is the Kähler potential.
3.2. Special Killing forms and complex coordinates

There is a correspondence between special Killing forms defined on the Sasaki-Einstein manifold \( Y \) and the parallel forms defined on the metric cone \( C(Y) \). Namely, a \( p \)-dimensional differential form \( \Psi \) is a special form on \( Y \) if and only if the corresponding form

\[
\Psi_{\text{cone}} := r^p dr \wedge \Psi + \frac{r^{p+1}}{p+1} d\Psi ,
\]

is parallel on \( C(M) \).

In particular on a \((2n-1)\)-dimensional Sasaki manifold with the contact 1-form \( \eta \) there are the following special Killing forms:

\[
\Psi_k = \eta \wedge (d\eta)^k, \quad k = 0, 1, \ldots, n-1.
\]

Besides these Killing forms, there are \( n-1 \) closed conformal Killing forms (also called \(*\)-Killing forms)

\[
\Phi_k = (d\eta)^k, \quad k = 1, \ldots, n-1.
\]

Moreover, in the case of the Calabi-Yau cone the holonomy is \( SU(n) \) and there are two additional parallel forms of degree \( n \). In order to write explicitly the additional Killing forms which correspond to these parallel forms, we shall express the volume form of the metric cone in terms of the Kähler form (6)

\[
d\mathcal{V} = \frac{1}{n!} \omega^n .
\]

Here \( \omega^n \) is the wedge product of \( \omega \) with itself \( n \) times. The volume of a Kähler manifold can be also written as [8, 12]

\[
d\mathcal{V} = \frac{i^n}{2^n} (-1)^{(n-1)/2} \Omega \wedge \overline{\Omega} ,
\]

where \( \Omega \) is the complex volume holomorphic \((n,0)\) form of \( C(Y) \). The additional (real) parallel forms are given by the real and (respectively) the imaginary part of the complex volume form.

Having in mind the connection between the special Killing forms and the complex volume holomorphic form of the metric cone, it is necessary to consider in the following the complex coordinated defined on a toric Sasaki-Einstein manifold.

To be more specific, let us consider that the \( n \)-dimensional metric cone \((X,\omega)\) is toric. This means that the real torus \( T^n \) acts effectively on \( X \), preserving the Kähler form \( \omega \). We introduce the symplectic coordinates \((y^i, \phi^i)\) where \( \phi^i \) are the angular coordinates along the orbits of the torus action. The corresponding Kähler metric on \( X = C(Y) \) is [13]

\[
ds^2 = G_{ij} dy^i dy^j + G^{ij} d\phi^i d\phi^j ,
\]

where the metric coefficients are obtained using the symplectic potential \( G \)

\[
G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j} ,
\]

and \( (G^{ij}) = (G_{ij})^{-1} \).

From symplectic coordinates we can pass to complex ones \( z^i := x^i + i\phi^i \), and the metric is written in the form

\[
ds^2 = F_{ij} dx^i dx^j + F_{ij} d\phi^i d\phi^j .
\]

In this setting, the metric coefficients are again obtained using the Hessian of the Kähler potential \( F \), i.e.

\[
F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} ,
\]
The symplectic potential $G$ and the Kähler potential $F$ are related by the Legendre transform

$$F(x) = \left( y^i \frac{\partial G}{\partial y^j} - G \right) \quad (y = \partial F/\partial x). \quad (19)$$

From the above considerations it follows that $F_{ij} = G^{ij}$ $(y = \partial F/\partial x)$.

Our interest for complex coordinates on Sasakian toric manifolds is motivated by the fact that using this particular type of coordinates it is possible to describe a complex volume form in a very convenient way. More exactly, on these manifolds the complex volume form $\Omega$ can be written as [13]

$$\Omega = e^{z_1} dz_1 \wedge \ldots \wedge dz_n.$$

The above relation makes possible the calculation of the special Killing forms on manifolds of Sasaki-Einstein type.

3.3. Complex volume form and foliated coordinates on geometric cone

Within this subsection, we show that working with foliated coordinates we can locally extract the special Killing forms on a Sasaki-Einstein manifold.

In order to study the foliated structure on the metric cone $C(Y)$, we consider from now on the existence of (local) foliated coordinates $(r; f^2, \ldots, f^n, \phi^1, \ldots, \phi^n)$. $r$ will stand for the transverse coordinate, while $f^2, \ldots, f^n, \phi^1, \ldots, \phi^n$ will stand for leafwise coordinates; for general topics concerning foliated structures we refer to [14]. As the coordinates $(x^i, \phi^j)$ are assumed independent, the initial local coordinates $(x_i, \phi^i)$ and the foliated coordinates $(r; f^i, \phi^j)$ are related in the following manner

$$\begin{cases} x^i = x^i(r, f^2, \ldots, f^n), \\ \phi^i = \phi^i. \end{cases} \quad (20)$$

Consequently, for the coframes $(dx^i, d\phi^i)$ and $(dr; df^i, d\phi^j)$ we get

$$\begin{cases} dx^i = \frac{\partial x^i}{\partial r} dr + \frac{\partial x^i}{\partial f^j} df^j, \\ d\phi^i = d\phi^i. \end{cases} \quad (21)$$

Using foliated coordinates, the complex volume form $\Omega$ is:

$$\Omega = e^{z_1} \left( \frac{\partial x^1}{\partial r} dr + \frac{\partial x^1}{\partial f^j} df^j + i d\phi^1 \right) \wedge \ldots \wedge \left( \frac{\partial x^n}{\partial r} dr + \frac{\partial x^n}{\partial f^j} df^j + i d\phi^n \right). \quad (22)$$

With this preparative, and using (10), we are ready to extract the complex differential form $\omega^Y$ which corresponds to the complex volume form $\Omega$:

$$\Omega = r^{n-1} dr \wedge \omega^Y + \frac{r^n}{n} d\omega^Y. \quad (23)$$

The explicit expression of $\omega^Y$ in the foliated coordinates approach is given in [15]. In the next Section we shall exemplify this construction in the case of the 5-dimensional Sasaki-Einstein space $Y^{p,q}$. 
4. An application: Special Killing 2-forms on $Y^{p,q}$ space

The special Killing 2-forms on $Y^{p,q}$ space were previously obtained in [16] using a different approach. In what follows we shall construct these special Killing forms working with foliated coordinates.

For the beginning, we consider the explicit local metric of the 5-dimensional $Y^{p,q}$ manifold given by the line element [17]

$$ds^2 = \frac{1 - cy}{6}(d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{1}{w(y)q(y)} \, dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta \, d\phi)^2$$

$$+ w(y) \left[ d\alpha + \frac{ac - 2y + cy^2}{6(a - y^2)} (d\psi - \cos \theta \, d\phi) \right]^2,$$  

(24)

where

$$w(y) = \frac{2(a - y^2)}{1 - cy},$$

$$q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2}.$$  

(25)

The constant $c$ can be rescaled by a diffeomorphism, so in what follows we take $c = 1$. For $0 < \alpha < 1$ we can take the range of the angular coordinates $(\theta, \phi, \psi)$ to be $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi$. Choosing $0 < a < 1$, the range of the coordinate $y$ is taken between the negative and the smallest positive roots of the cubic equation

$$a - 3y^2 + 2y^3 = 0.$$  

(26)

Following [17, 13], for this particular space we take the complex coordinates

$$z^1 := \log \left( r^3 \sin \theta \sqrt{\frac{p(y)(1 - y)}{2}} e^{i\psi} \right)$$

$$= 3 \ln r + \ln \sin \theta + \frac{1}{2} \ln \frac{p(y)(1 - y)}{2} + i\psi',$$

$$z^2 := \frac{1}{3\sqrt{3}} \log \left( \tan \frac{\theta}{2} e^{i\psi} \right)$$

$$= \frac{1}{3\sqrt{3}} \ln \tan \frac{\theta}{2} + i\phi',$$

$$z^3 := \frac{1}{6} \log \left( \frac{1}{\sin \theta} \sqrt{(y - y_1) - \frac{1}{\sigma^2} (y_2 - y) - \frac{1}{\sigma^2} (y_3 - y) - \frac{1}{\sigma^2}} e^{-6i\alpha - i\psi} \right)$$

$$= -\frac{1}{6} \ln \sin \theta - \frac{1}{2} \ln \left( (y - y_1) - \frac{1}{\sigma^2} (y_2 - y) - \frac{1}{\sigma^2} (y_3 - y) - \frac{1}{\sigma^2} \right)$$

$$+ i\beta',$$

(27)

where we denote (see also [16])

$$p(y) := w(y) \cdot q(y) = \frac{2(a - 3y^2 + 2y^3)}{a - y^2},$$

$$\beta' := -\alpha - \frac{1}{6} \psi,$$

$$\phi' := \frac{1}{3\sqrt{3}} \phi,$$

$$\psi' := \psi.$$  

(28)
From (27) we can extract the real parts of the complex coordinates:

\[ x^1 = 3 \ln r + \ln \sin \theta + \frac{1}{2} \ln \frac{p(y)(1 - y)}{2}, \]
\[ x^2 = \frac{1}{3\sqrt{3}} \ln \tan \frac{\theta}{2}, \]
\[ x^3 = -\frac{1}{6} \ln \sin \theta - \frac{1}{2} \ln \left( (y - y_1)^{-\frac{1}{\sqrt{3}}} (y_2 - y)^{-\frac{1}{\sqrt{3}}} (y_3 - y)^{-\frac{1}{\sqrt{3}}} \right), \]

while for the toric coordinates we have

\[ \phi^1 = \psi', \]
\[ \phi^2 = \phi', \]
\[ \phi^3 = \beta'. \] (30)

From the relation (29) and (30) it follows that we can consider the foliated coordinates \((r; \theta, y, \psi', \phi', \beta').\) We regard \(r\) as transverse coordinate and \(\theta, y, \psi', \phi', \beta'\) as leafwise coordinates, describing the immersed submanifolds.

It is easy to verify that

\[ \Omega = e^{\psi'} dz_1 \wedge dz_2 \wedge dz_3, \] (31)

is the correct complex volume 3-form.

Using the procedure described in the previous Section, after some calculations we finally get the 2-form \(\omega^Y\) on the Sasaki-Einstein space \(Y^{p,q}\):

\[
\omega^Y = e^{iv'} \sqrt{\frac{1-y}{6p(y)}} \left( d\theta \wedge dy + \frac{i\sqrt{3}}{2} \cos \theta p(y) d\theta \wedge d\phi' \right.
\]
\[ - i3\sqrt{3} \sin \theta dy \wedge d\phi' + ip(y) d\theta \wedge d\beta' \]
\[ -3\sqrt{3} p(y) \sin \theta d\phi' \wedge d\beta' \). \] (32)

Calculating the real and imaginary parts, we end up with the desired special Killing 2-forms on the Sasaki-Einstein \(Y^{p,q}\) space:

\[
\Xi = \sqrt{\frac{1-y}{6p(y)}} \left( \cos \psi' \left[-dy \wedge d\theta + 3\sqrt{3} p(y) \sin \theta d\beta' \wedge d\phi' \right] \right.
\]
\[ - \sin \psi' \left[-3\sqrt{3} \sin \theta dy \wedge d\phi' - p(y) d\beta' \wedge d\theta \right. \]
\[ + \frac{\sqrt{3}}{2} \cos \theta p(y) d\theta \wedge d\phi' \left] \right) \), \] (33)

and respectively

\[
\Upsilon = \sqrt{\frac{1-y}{6p(y)}} \left( \cos \psi' \left[-3\sqrt{3} \sin \theta dy \wedge d\phi' - p(y) d\beta' \wedge d\theta \right.ight.
\]
\[ + \frac{\sqrt{3}}{2} \cos \theta p(y) d\theta \wedge d\phi' \left] + \sin \psi' \left[-dy \wedge d\theta \right. \]
\[ + 3\sqrt{3} p(y) \sin \theta d\beta' \wedge d\phi' \left] \right), \] (34)
Finally, we calculate the Killing forms described in (11-12) using the contact 1-form $\eta$. We obtain

$$\eta = \frac{1}{3} \left( d\psi' - (1 - y) \cos \theta d\phi' + y d\beta' \right) ,$$

$$d\eta = \frac{1}{3} \left( (1 - y) \sin \theta d\theta \wedge d\phi' + dy \wedge d\beta' + \cos \theta dy \wedge d\phi' \right) .$$

Then, using (11), (12) and (35), for $k = 1$ and 2 we get (see also [16])

$$\Psi_1 = \frac{1}{9} \left( (1 - y) \sin \theta d\theta \wedge d\phi' \wedge d\psi' + dy \wedge d\beta' \wedge d\psi' 
+ \cos \theta dy \wedge d\phi' \wedge d\psi' - \cos \theta dy \wedge d\beta' \wedge d\phi' 
+ (1 - y) y \sin \theta d\beta' \wedge d\theta \wedge d\phi' \right) ,$$

$$\Psi_2 = \frac{2}{27} \left( (1 - y) \sin \theta dy \wedge d\beta' \wedge d\theta \wedge d\phi' \wedge d\psi' \right) ,$$

and respectively

$$\Phi_1 = \frac{1}{3} \left( (1 - y) \sin \theta d\theta \wedge d\phi' + dy \wedge d\beta' + \cos \theta dy \wedge d\phi' \right) ,$$

$$\Phi_2 = \frac{2}{9} (1 - y) \sin \theta dy \wedge d\beta' \wedge d\theta \wedge d\phi' .$$

5. Conclusions

In this expository paper we present the complete set of Killing forms on 5-dimensional Sasakian-Einstein spaces $Y^{p,q}$. Furthermore, a general scheme to construct the special Killing forms on toric Sasakian-Einstein manifolds is described. In fact, using toric geometry many examples of Sasakian-Einstein manifolds can be constructed, and these spaces are highly relevant for the predictions of the AdS/CFT correspondence [13].

Concerning Killing forms, let us stress the fact that a family of Stäckel-Killing tensors can be associated with these differential forms, implying the superintegrability of the geodesic motions. Also, using the third rank Killing-Yano tensors on $Y^{p,q}$ a geometrical interpretation of the Lax representation can be provided [18, 19, 20].

On the other hand, concerning the potential of the general scheme presented in the paper, we emphasize the fact that many non-trivial examples of toric Sasakian-Einstein manifolds occur in the literature; we only refer here to [21], where authors study two infinite families $Y^{p,q}$ of Sasakian-Einstein 7-manifolds; these are lens-space bundles $S^3/\mathbb{Z}_p$ over $CP^2$ and $CP^1 \times CP^1$, respectively, and it turns out that the metric cones of these Sasakian-Einstein manifolds are toric.

Acknowledgments

The work of M. V. has been partly supported by the joint Romanian-LIT, JINR, Dubna Research Project, theme no. 05-6-1119-2014/2016 and partly by the program NUCLEU PN-09370102, Romania. G. E. V. was supported by National Research Council - Executive Agency for Higher Education Research and Innovation Funding (CNCS-UEFISCODI), project number PN-II-ID-PCE-2011-3-0118.

References

[1] Boyer C and Galicki K 2010 Sasakian geometry, holonomy, and supersymmetry Handbook of pseudo-Riemannian geometry and supersymmetry (IRMA Lect. Math. Theor. Phys.) vol 16 (Zürich: Eur. Math. Soc.) p 39 (Preprint math/0703231)

[2] Sparks J 2011 Sasaki-Einstein Manifolds Surv. Diff. Geom. 16 265 (Preprint 1004.2461)
[3] Gauntlett J P, Martelli D, Sparks J and Waldram D 2004 Supersymmetric $AdS_5$ solutions of $M$-theory Class. Quant. Grav. 21 4335 (Preprint hep-th/0402153)
[4] Gauntlett J P, Martelli D, Sparks J and Waldram D 2004 Sasaki-Einstein metrics on $S^2 \times S^3$ Adv. Theor. Math. Phys. 8 711 (Preprint hep-th/0403002)
[5] Maldacena J M 1998 The large N limit of superconformal field theories and supergravity Adv. Theor. Math. Phys. 2 231
[6] Kubizňák D 2009 On the supersymmetric limit of Kerr-NUT-AdS metrics Phys. Lett. B 675 110 (Preprint 0902.1999)
[7] Hashimoto Y, Sakaguchi M and Yasui Y 2004 Sasaki-Einstein twist of Kerr-AdS black holes Phys. Lett. B 600 270 (Preprint 1103.5573)
[8] Slesar V, Visinescu M and Vîlcu G E 2014 Special Killing forms on Sasaki-Einstein manifolds Physica Scripta (in press) (Preprint 1403.1015)
[9] Rosquist K and Goliath M 1998 Lax pair tensors and integrable spacetimes Gen. Rel. Grav. 30 1521 (Preprint gr-qc/9707003)
[10] Goliath M, Karlovini M and Rosquist K 1999 Lax pair tensors in arbitrary dimensions J. Phys. A: Math. Gen. 32 3377 (Preprint solv-int/9810011)
[11] Martelli D and Sparks J 2008 Notes on toric Sasaki-Einstein seven-manifolds and AdS4/CFT3 J. High Energy Phys. 11 016 (Preprint 0808.0904)