A FUNCTIONAL ANALYTIC APPROACH TO PERTURBATIONS OF THE LORENTZ GAS

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ABSTRACT. We present a functional analytic framework based on the spectrum of the transfer operator to study billiard maps associated with perturbations of the periodic Lorentz gas. We show that recently constructed Banach spaces for the billiard map of the classical Lorentz gas are flexible enough to admit a wide variety of perturbations, including: movements and deformations of scatterers; billiards subject to external forces; nonelastic reflections with kicks and slips at the boundaries of the scatterers; and random perturbations comprised of these and possibly other classes of maps. The spectra and spectral projections of the transfer operators are shown to vary continuously with such perturbations so that the spectral gap enjoyed by the classical billiard persists and important limit theorems follow.

1. Introduction

The Lorentz gas is known to enjoy strong ergodic properties: both the continuous time dynamics and the billiard maps are completely hyperbolic, ergodic, K-mixing and Bernoulli (see [S, GO, SC, CH] and the references therein). Young [Y] proved exponential decay of correlations for billiard maps corresponding to the finite horizon periodic Lorentz gas using Markov extensions; this technique was subsequently extended to other dispersing billiards [Ch1] and used to obtain important limit theorems such as local large deviation estimates and almost-sure invariance principles [MN1, MN2, RY].

In this setting, it is natural to ask how the statistical properties of dispersing billiard maps vary with the shape and position of the scatterers. Alternatively, one may change the billiard dynamics by introducing an external force between collisions or by considering nonelastic reflections at the boundaries. Such perturbed dynamics lead to nonequilibrium billiards whose invariant measures are singular with respect to Lebesgue measure.

One of the first nonequilibrium physical models that was studied rigorously is the periodic Lorentz gas with a small constant electrical field [CELS1, CELS2] and the well-known Ohm’s law was proved for that case. More general external forces were handled in [Ch2, Ch4, CD2] and billiards with kicks at reflections have been studied in [MPS, Z]. Recently, Chernov and Dolgopyat [CD1] used coupling methods to study the motion of a point particle colliding with a moving scatterer. Locally perturbed periodic rearrangements of scatterers have also been the subject of recent studies [DSV]. Despite such successes, the study of perturbations of billiards has thus far been handled on a case by case basis, with methods adapted and developed for each specific type of perturbation considered.

In this paper, we propose a unified framework in which to study a large class of perturbations of dispersing billiards. This framework is based on the spectral analysis of the transfer operator associated with the billiard map and uses the recent work [DZ] which successfully constructed Banach spaces on which the transfer operator for the classical periodic Lorentz gas has a spectral gap.

We first present abstract conditions under which we have uniform control of spectral data for a given class of perturbed maps. We then prove that four broad classes of perturbations of billiards fit within this framework, namely:

(i) Tables with shifted, rotated or deformed scatterers;
(ii) Billiards under small external forces which bend trajectories during flight;
(iii) Billiards with kicks or twists at reflections, including slips along the disk;
(iv) Random perturbations comprised of maps with uniform properties (including any of the above classes, or a combination of them).

In particular, the results on random perturbations are a version of time-dependent billiards, in which scatterers are allowed to change positions between collisions. The fact that our main theorems, 2.2 and 2.3, are proved in an abstract setting will facilitate the application of this framework to other classes of perturbations as they arise in future works.

The present functional analytic approach uses the Banach spaces constructed in \[DZ\] as well as the perturbative framework of Keller and Liverani [KL] to prove that the spectral data and spectral projectors, including invariant measures, rates of decay of correlations, variance in the central limit theorem, etc, vary Hölder continuously for the classes of perturbations mentioned above (see \[B\] \[L\] for expositions of this approach). In addition, this approach yields new results for the perturbed billiard maps in terms of local limit theorems, in particular giving new information about the evolution of noninvariant measures in the context of these limit theorems. For example, applying Corollary 2.4 to billiards under external forces and kicks, we obtain a local large deviation estimate with a rate function that is the same for all probability measures in our Banach space. This implies in particular that Lebesgue measure and the singular SRB measure for the perturbed billiard have the same large deviation rate function.

The paper is organized as follows. In Section 2, we describe our abstract framework, state precisely the applications which serve as our model perturbations and formulate our main results. In Section 3, we lay out our common approach under the general conditions (H1)-(H5) which guarantee the required uniform Lasota-Yorke inequalities for Theorem 2.2, proved in Section 4; we also formulate conditions (C1)-(C4) to verify that a perturbation is small in the sense of our Banach spaces for Theorem 2.3, proved in Section 5. The investigations of the concrete models are provided in Sections 6 and 7.

2. Setting and Results

In this section, we describe the abstract framework into which we will place our perturbations and formulate precisely the classes of concrete deterministic perturbations to which our results apply. We also formulate a class of random perturbations with maps drawn from any mixture of the deterministic perturbations described below. We postpone until Section 3 a precise description of the Banach spaces and the formal requirements on the abstract class of maps \(F\).

2.1. Perturbative framework. We recall here the perturbative framework of Keller and Liverani [KL]. Suppose there exist two Banach spaces \((B, \| \cdot \|)\) and \((B_w, | \cdot |_w)\) with the unit ball of \(B\) compactly embedded in \(B_w, | \cdot |_w \leq \| \cdot \|_B, \) and a family of bounded linear operators \(\{L_\varepsilon\}_{\varepsilon \geq 0}\) defined on both \(B_w\) and \(B\) such that the following holds.\(^3\) There exist constants \(C, \eta > 0 \) and \(\sigma < 1\) such that for all \(\varepsilon \geq 0\) and \(n \geq 0,\)

\[
\begin{align*}
|L_\varepsilon^n h|_w &\leq C\eta^n |h|_w \quad \text{for all } h \in B_w, \\
\|L_\varepsilon^n h\|_B &\leq C\sigma^n \|h\|_B + C\eta^n |h|_w \quad \text{for all } h \in B.
\end{align*}
\]

If \(\sigma < \eta,\) the operators \(L_\varepsilon\) are quasi-compact with essential spectral radius bounded by \(\sigma\) and spectral radius at most \(\eta\) (see for example [B]). Suppose further that

\[
\|L_\varepsilon - L_0\| := \sup\{||L_\varepsilon h - L_0 h|_w : h \in B, \|h\|_B \leq 1\} \leq \rho(\varepsilon),
\]

where \(\rho(\varepsilon)\) is a non-increasing upper semicontinuous function satisfying \(\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0.\)

The main result of [KL] is the following. Let \(sp(L_0)\) denote the spectrum of \(L_0.\) For any \(\sigma_1 > \sigma,\) by quasi-compactness, \(sp(L_0) \cap \{z \in \mathbb{C} : |z| \geq \sigma_1\}\) consists of finitely many eigenvalues \(\varrho_1, \ldots, \varrho_k\) of finite multiplicity. Thus there exists \(t_* > 0\) and we may choose \(\sigma_1\) such that \(|\varrho_i - \varrho_j| > t_*\) for \(i \neq j\) and \(dist(sp(L_0), \{|z| = \sigma_1\}) > t_*\). For \(t \leq t_*\) and \(\varepsilon \geq 0,\) define the spectral projections,

\[
\Pi^{(j)}_\varepsilon := \frac{1}{2\pi i} \int_{|z-\varrho_j|=t} (z - L_\varepsilon)^{-1} dz \quad \text{and} \quad \Pi^{(\sigma_1)}_\varepsilon := \frac{1}{2\pi i} \int_{|z|=\sigma_1} (z - L_\varepsilon)^{-1} dz.
\]

\(^3\)The results of [KL] hold in a more general setting, but we only state the version we need for our purposes.
Theorem 2.1. \( (\text{KL}) \) Assume that (2.1) and (2.2) hold. Then for each \( t \leq t_e \) and \( 0 < s < 1 - \frac{\log \sigma}{\log \sigma} \), there exist \( \varepsilon_1, C > 0 \) such that for any \( 0 \leq \varepsilon < \varepsilon_1 \), the spectral projections \( \Pi_{\varepsilon}^{(j)} \) and \( \Pi_{\varepsilon}^{(\sigma_1)} \) are well defined and satisfy, for each \( j = 1, \ldots, k \),

1. \( |||\Pi_{\varepsilon}^{(j)} - \Pi_{0}^{(j)}||| \leq C \rho(\varepsilon)^s \) and \( |||\Pi_{\varepsilon}^{(\sigma_1)} - \Pi_{0}^{(\sigma_1)}||| \leq C \rho(\varepsilon)^s \);
2. \( \text{rank}(\Pi_{\varepsilon}^{(j)}) = \text{rank}(\Pi_{0}^{(j)}) \);
3. \( \|L_{\varepsilon} \Pi_{\varepsilon}^{(\sigma_1)}\| \leq C \sigma_1^n \), for all \( n \geq 0 \).

We say an operator \( L \) has a spectral gap if \( L \) has a simple eigenvalue of maximum modulus and all other eigenvalues have strictly smaller modulus. The above theorem implies in particular that if \( L_{\varepsilon} \) has a spectral gap, then so does \( L_{\varepsilon} \) for \( \varepsilon \) sufficiently small. In addition, the related statistical properties (for instance, invariant measures, rates of decay of correlations, variance of the Central Limit Theorem) are stable and vary Hölder continuously as a function of \( \rho(\varepsilon) \). This is the framework into which we will place our perturbations of the Lorentz gas.

2.2. An abstract result for a class of maps with uniform properties. We begin by fixing the phase space \( M \) of a billiard map associated with a periodic Lorentz gas. That is, we place finitely many (disjoint) scatterers \( \Gamma_i, i = 1, \ldots, d \), on \( T^2 \) which have \( C^3 \) boundaries with strictly positive curvature. The classical billiard flow on the table \( T^2 \setminus \bigcup_i \{ \text{interior } \Gamma_i \} \) is induced by a particle traveling at unit speed and undergoing elastic collisions at the boundaries. In what follows, we also consider particles whose motion between collisions follows slightly curved trajectories (due to external forces) as well as certain types of collisions which do not obey the usual law of reflection.

In all cases, the billiard map associated with the flow is the Poincaré map corresponding to collisions with the scatterers. Its phase space is \( M = \bigcup_{i=1}^d I_i \times [-\pi/2, \pi/2] \), where \( \ell(I_i) = |\partial \Gamma_i| \), i.e., the length of \( I_i \) equals the arclength of \( \partial \Gamma_i \), \( i = 1, \ldots, d \). \( M \) is parametrized by the canonical coordinates \((r, \varphi)\) where \( r \) represents the arclength parameter on the boundaries of the scatterers (oriented clockwise) and \( \varphi \) represents the angle an outgoing (postcollisional) trajectory makes with the unit normal to the boundary at the point of collision.

The phase space \( M \) and coordinates so defined are fixed for all classes of perturbations we consider; however, the configuration space (the billiard table on which the particles flow) and the laws which govern the motion of the particles may vary as long as all variations give rise to the same phase space \( M \), i.e., the number of \( \Gamma_i \) and the arclengths of their boundaries do not change. See Remark 2.7 for a way to relax this requirement on the arclength. For any \( x = (r, \varphi) \in M \), we define \( \tau(x) \) to be the first collision of the trajectory starting at \( x \) under the billiard flow. The billiard map is defined wherever \( \tau(x) < \infty \). We say that the billiard has finite horizon if there is an upper bound on the function \( \tau \). Otherwise, we say the billiard has infinite horizon. Notice that the function \( \tau \) depends on the placement of the scatterers in \( T^2 \), while \( M \) is independent of their placement.

We assume there exists a class of maps \( F \) on \( M \) satisfying properties (H1)-(H5) of Section 3.1 with uniform constants. For each \( T \in F \), in Section 3.2 we define the transfer operator \( L_T \) associated with \( T \) on an appropriate class of distributions \( h \) by

\[
L_T h(\psi) = h(\psi \circ T), \quad \text{for suitable test functions } \psi.
\]

In Section 3.3 we define Banach spaces of distributions \((B, \| \cdot \|_B)\) and \((B_w, \| \cdot \|_w)\), preserved under the action of \( L_T, T \in F \), such that the unit ball of \( B \) is compactly embedded in \( B_w \).

Theorem 2.2. Fix \( M \) as above and suppose there exists a class of maps \( F \) satisfying (H1)-(H5) of Section 3.1. Then \( L_T \) is well defined as a bounded linear operator on \( B \) for each \( T \in F \). In addition, there exist \( C > 0, \sigma < 1 \) such that for any \( T \in F \) and \( n \geq 0 \),

\[
\|L_T^n h\|_w \leq C \sigma^n \|h\|_w \quad \text{for all } h \in B_w,
\]

\[
\|L_T^n h\|_B \leq C \sigma^n \|h\|_B + C \sigma^n \|h\|_w \quad \text{for all } h \in B,
\]

where \( \eta \geq 1 \) is from (H5). This, plus the compactness of \( B \) in \( B_w \), implies that all the operators \( L_T, T \in F \), are quasi-compact with essential spectral radius bounded by \( \sigma \): i.e., outside of any disk of radius
greater than $\sigma$, their spectra contain finitely many eigenvalues of finite multiplicity. Moreover, for each $T \in \mathcal{F}$,

(i) the spectral radius of $L_T$ is 1 and the elements of the peripheral spectrum are measures absolutely continuous with respect to $\nu := \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} L_T^i$;

(ii) an ergodic, invariant probability measure $\nu$ for $T$ is in $\mathcal{B}$ if and only if $\nu$ is a physical measure for $T$;

(iii) there exist a finite number of $q_\varepsilon \in \mathbb{N}$ such that the spectrum of $L$ on the unit circle is $\cup_{k} \{ e^{2\pi i k/n} : 0 \leq k < q_\varepsilon, k \in \mathbb{N} \}$. The peripheral spectrum contains no Jordan blocks.

(iv) Let $S^{i\varepsilon}_{H,n}$ denote the $\varepsilon$-neighborhood of $S^{i\varepsilon}_{H,n}$, the singularity set for $T$ (with homogeneity strips). Then for each $\nu$ in the peripheral spectrum and $n \in \mathbb{N}$, we have $\nu(S^{i\varepsilon}_{H,n}) \leq C_n \varepsilon^\alpha$, for some constants $C_n > 0$.

(v) If $(T, \nu)$ is ergodic, then 1 is a simple eigenvalue. If $(T^n, \nu)$ is ergodic for all $n \in \mathbb{N}$, then 1 is the only eigenvalue of modulus 1, $(T, \nu)$ is mixing and enjoys exponential decay of correlations for Hölder observables.

Theorem 2.2 is proved in Section 4. In Section 3.4, we define a distance $d_{\mathcal{F}}(\cdot, \cdot)$ between maps in $\mathcal{F}$. Our next result shows that this distance controls the size of perturbations in the spectra of the associated transfer operators.

**Theorem 2.3.** Let $\beta > 0$ be from the definition of $(\mathcal{B}, \| \cdot \|_B)$ in Section 3.3. There exists $C > 0$ such that if $T_1, T_2 \in \mathcal{F}$ with $d_{\mathcal{F}}(T_1, T_2) \leq \varepsilon$, then

$$|||L_{T_1} - L_{T_2}||| \leq C \varepsilon^{\beta/2}, \quad \text{where } ||| \cdot ||| \text{ is from 2.2}. $$

We prove Theorem 2.3 in Section 5. According to Theorem 2.1, an immediate consequence of Theorems 2.2 and 2.3 is the following.

**Corollary 2.4.** If $T_0 \in \mathcal{F}$ has a spectral gap, then all $T \in X_\varepsilon(T_0) = \{ T \in \mathcal{F} : d_{\mathcal{F}}(T, T_0) < \varepsilon \}$ have a spectral gap for $\varepsilon$ sufficiently small. In particular, the maps in $X_\varepsilon$ enjoy the following limit theorems (among others), which follow from the existence of a spectral gap.

Fix $T \in \mathcal{F}$ with a spectral gap. Let $\gamma = \max\{ p, 2\beta + \delta \}$ for some $\delta > 0$, where $p$ and $\beta$ are from Sect. 3.3 and let $g \in C^\gamma(M)$. Define $S_n g = \sum_{k=0}^{n-1} g \circ T^k$.

(a) (Local large deviation estimate) For any (not necessarily invariant) probability measure $\nu \in \mathcal{B}$,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \nu \left( x \in M : \frac{1}{n} S_n g(x) \in [t - \varepsilon, t + \varepsilon] \right) = -I(t)$$

where the rate function $I(t)$ is independent of $\nu \in \mathcal{B}$ (but may depend on $T$), and $t$ is in a neighborhood of the mean $\nu(g)$.

(b) (Almost-sure invariance principle). Suppose $\nu(g) = 0$ and distribute $(g \circ T^j)_{j \in \mathbb{N}}$ according to a probability measure $\nu \in \mathcal{B}$. Then there exist $\lambda > 0$, a probability space $\Omega$ with random variables $\{X_n\}$ satisfying $S_n g \overset{\text{dist}}{=} X_n$, and a Brownian motion $W$ with variance $\varsigma^2 \geq 0$ such that

$$X_n = W(n) + O(n^{1/2-\lambda}) \quad \text{as } n \to \infty \text{ almost-surely in } \Omega.$$

The proof of the corollary is the same as that of [DZ, Theorem 2.6] and will not be repeated here.

### 2.3. Applications to concrete classes of deterministic perturbations.

In this section we describe precisely several types of perturbations of the Lorentz gas which fall under the abstract framework we have outlined above. In light of Theorems 2.2 and 2.3, it suffices to check two things for each class of perturbations we will introduce: (1) (H1)-(H5) hold uniformly in each class; (2) the perturbations are small in the sense of the distance $d_{\mathcal{F}}(\cdot, \cdot)$.

**A. Movements and Deformations of Scatterers.**

We fix the phase space $M = \bigcup_{i=1}^d I_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ associated with a billiard map corresponding to a periodic

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2Recall that a physical measure for $T$ is an ergodic, invariant probability measure $\nu$ such that $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} f(T^i x) = \int f \, d\nu$ for a positive Lebesgue measure set of $x \in M$. 
Lorentz gas with $d$ scatterers as described above. We assume that the billiard particle moves along straight lines and undergoes elastic reflections at the boundaries.

For given $I_1, \ldots, I_d$, we use the notation $Q = Q(\{\Gamma_i\}_{i=1}^{d}; \{I_i\}_{i=1}^{d})$ to denote the configuration of scatterers $\Gamma_1, \ldots, \Gamma_d$ placed on the billiard table such that $|\partial \Gamma_i| = I_i, i = 1, \ldots, d$. Since we have fixed $I_1, \ldots, I_d, M$ remains the same for all configurations $Q$ that we consider. For each such configuration, we define

$$\tau_{\min}(Q) := \inf \{\tau(x) : \tau(x) \text{ is defined for the configuration } Q\}.$$ 

Similarly, $K_{\min}(Q)$ and $K_{\max}(Q)$ denote the minimum and maximum curvatures respectively of the $\Gamma_i$ in the configuration $Q$. The constant $E_{\max}(Q)$ denotes the maximum $C^3$ norm of the $\partial \Gamma_i$ in $Q$.

For each fixed $\tau, K, E > 0$, define $Q(\tau, K, E)$ to be the collection of all configurations $Q$ such that $\tau_{\min}(Q) \geq \tau, K_{\min}(Q) \leq K_{\max}(Q) \leq K^{-1}$, and $E_{\max}(Q) \leq E$. The horizon for $Q \in Q(\tau, K, E)$ is allowed to be finite or infinite. Let $F(\tau, K, E)$ be the corresponding set of billiard maps induced by the configurations in $Q$. It follows from [DZ] that for any $T \in F(\tau, K, E), \mathcal{L}_T$ has a spectral gap in $\mathcal{B}$. We prove the following theorems in Section 6.

**Theorem 2.5.** Fix $I_1, \ldots, I_d$ and let $\tau, K, E > 0$. The family $F(\tau, K, E)$ satisfies (H1)-(H5) with uniform constants depending only on $\tau, K$ and $E$. As a consequence of Theorem 2.6, $\mathcal{L}_T$ is quasi-compact as an operator on $\mathcal{B}$ for each $T \in F(\tau, K, E)$ with uniform bounds on its essential spectral radius.

We fix an initial configuration of scatterers $Q_0 \in Q(\tau, K, E)$ and consider configurations $Q$ which alter each $\partial \Gamma_i$ in $Q_0$ to a curve $\partial \Gamma_i$ having the same arclength as $\partial \Gamma_i$. We consider each $\partial \Gamma_i$ as a parametrized curve $u_i : I_i \to M$ and each $\partial \Gamma_i$ as parametrized by $\tilde{u}_i$. Define $\Delta(Q, Q_0) = \sum_{i=1}^{d} |u_i - \tilde{u}_i|_{C^2(I_i, M)}$.

**Theorem 2.6.** Choose $\gamma \leq \min\{\tau/2, K/2\}$ and let $F_A(Q_0, E; \gamma)$ be the set of all billiard maps corresponding to configurations $Q$ such that $\Delta(Q, Q_0) \leq \gamma$ and $E_{\max}(Q) \leq E$. Then $F_A(Q_0, E; \gamma) \subset F(\tau/2, K/2, E)$ and $d_F(T_1, T_2) \leq C|\gamma|^{2/15}$ for any $T_1, T_2 \in F_A(Q_0, E; \gamma)$. If all $T_i \in F_A(Q_0, E; \gamma)$ have uniformly bounded finite horizon, then $d_F(T_1, T_2) \leq C|\gamma|^{1/3}$.

As a consequence, the eigenvalues outside a disk of radius $\sigma < 1$ and the corresponding spectral projectors of $\mathcal{L}_T$ vary Hölder continuously for all $T \in F_A(Q_0, E; \gamma)$ and all $\gamma$ sufficiently small.

**Remark 2.7.** (a) A remarkable aspect of this result is that it allows us to move configurations from finite to infinite horizon without interrupting Hölder continuity of the statistical properties such as the rate of decay of correlations and the variance in the CLT, among others.

(b) The requirement that all deformations of the initial configuration $Q_0$ maintain the same arclength can be relaxed. The purpose of this requirement is to define the corresponding transfer operators on fixed spaces $\mathcal{B}$ and $\mathcal{B}_u$. If a scatterer $\Gamma_i$ is deformed into $\Gamma'_i$ with a slight change in arclength, we can reparametrize $\Gamma'_i$ (no longer according to arclength) using the same interval $I_i$ as for $\Gamma_i$. This will change the derivatives of maps in the class $\mathcal{F}_B(Q_0, E; \gamma)$ slightly, but since properties (H1)-(H5) have some leeway built into the uniform constants, for small enough reparametrizations the same properties will hold with slightly weakened constants.

**B. Billiards Under Small External Forces with Kicks and Slips.**

As in part A, we fix $\tau, K$ and $E$ and choose a fixed $Q_0 \in Q(\tau, K, E)$. In this section, we consider the dynamics of the billiard map on the table $Q_0$, but subject to external forces both during flight and at collisions.

Let $q = (x, y)$ be the position of a particle in a billiard table $Q_0$ and $p$ be the velocity vector. For a $C^2$ stationary external force, $F : T^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, the perturbed billiard flow $\Phi$ satisfies the following differential equation between collisions:

$$\frac{dq}{dt} = p(t), \quad \frac{dp}{dt} = F(q, p).$$

At collision, the trajectory experiences possibly nonelastic reflections with slipping along the boundary:

$$q^+(t_i), p^+(t_i) = (\mathcal{R} p^-(t_i), G(q^- (t_i), p^- (t_i))$$

where $\mathcal{R} p^-(t_i) = p^- (t_i) + 2(n(q^-) \cdot p^-) n(q^-)$ is the usual reflection operator, $n(q)$ is the unit normal vector to the billiard wall $\partial Q_0$ at $q$ pointing inside the table $Q_0$, and $q^- (t_i), p^- (t_i), q^+ (t_i)$ and $p^+ (t_i)$ refer
to the incoming and outgoing position and velocity vectors, respectively. \( \mathbf{G} \) is an external force acting on
the incoming trajectories. Note that we allow \( \mathbf{G} \) to change both the position and velocity of the particle at the moment of collision. The change in velocity can be thought of as a kick or twist while a change in position can model a slip along the boundary at collision.

In [Ch2, Ch4], Chernov considered billiards under small external forces \( \mathbf{F} \) with \( \mathbf{G} = 0 \), and \( \mathbf{F} \) to be stationary. In [Z], a twist force was considered assuming \( \mathbf{F} = 0 \) and \( \mathbf{G} \) depending on and affecting only the velocity, not the position. Here we consider a combination of these two cases for systems under more general forces \( \mathbf{F} \) and \( \mathbf{G} \). We make four assumptions, combining those in [Ch2, Z].

(A1) (Invariant space) Assume the dynamics preserve a smooth function \( \mathcal{E}(\mathbf{q}, \mathbf{p}) \). Its level surface \( \Omega_c := \mathcal{E}^{-1}(c) \), for any \( c > 0 \), is a compact 3-d manifold such that \( \|\mathbf{p}\| > 0 \) on \( \Omega_c \) and for each \( \mathbf{q} \in \Omega \) and \( \mathbf{p} \in \mathbb{R}^2 \) the ray \( \{(\mathbf{q}, t\mathbf{p}), t > 0\} \) intersects the manifold \( \Omega_c \) in exactly one point.

Assumption (A1) specifies an additional integral of motion, so that we only consider restricted systems on a compact phase space. In particular, (A1) implies that the speed \( p = \|\mathbf{p}\| \) of the billiard along any typical trajectory at time \( t \) satisfies

\[
0 < p_{\min} \leq p(t) \leq p_{\max} < \infty
\]

for some constants \( p_{\min} \leq p_{\max} \). Under this assumption the particle will not become overheated, and its speed will remain bounded. For any phase point \( \mathbf{x} = (\mathbf{q}, \mathbf{p}) \in \Omega \) for the flow, let \( \tau(\mathbf{x}) \) be the length of the trajectory between \( \mathbf{x} \) and its next non-tangential collision.

(A2) (Finite horizon) There exist \( \tau_{\max} > \tau_{\min} > 0 \) such that free paths between successive reflections are uniformly bounded, \( \tau_s/2 \leq \tau_{\min} \leq \tau(\mathbf{x}) \leq \tau_{\max} \leq \tau_s^{-1} \), \( \forall \mathbf{x} \in \Omega \). Since \( \mathcal{Q}_0 \in \mathcal{Q}_1(\tau_s, K_s, E_s) \), the curvature \( K(\mathbf{r}) \) of the boundary is also uniformly bounded for all \( r \in \partial \mathcal{Q}_0 \).

(A3) (Smallness of the perturbation). We assume there exists \( \varepsilon_1 > 0 \) small enough, such that

\[
\|\mathbf{F}\|_{C^1} < \varepsilon_1, \|\mathbf{G}\|_{C^1} < \varepsilon_1.
\]

Let \( \mathbf{v} = (\cos \theta, \sin \theta) \) denote the unit velocity vector with \( \theta \in [0, 2\pi] \), and \( \mathcal{M} \) be a level surface \( \Omega_c \) with coordinates \( (\mathbf{q}, \theta) \), for some fixed \( c > 0 \). Denote \( T_{\mathbf{F}, \mathbf{G}} : \mathcal{M} \rightarrow \mathcal{M} \) as the billiard map associated to the flow on \( \mathcal{M} \), where \( \mathcal{M} \) is the collision space containing all post-collision vectors based at the boundary of the billiard table \( \mathcal{Q}_0 \).

(A4) We assume both forces \( \mathbf{F} \) and \( \mathbf{G} \) are stationary and that \( \mathbf{G} \) preserves tangential collisions. In addition, we assume that the singularity set of \( T_{\mathbf{F}, \mathbf{G}}^{-1} \) is the same as that of \( T_{\mathbf{F}, 0}^{-1} \).

The case \( \mathbf{F} = \mathbf{G} = 0 \) corresponds to the classical billiard dynamics. It preserves the kinetic energy \( \mathcal{E} = \frac{1}{2} \|\mathbf{p}\|^2 \). We denote by \( \mathcal{F}_B(\mathcal{Q}_0, \tau_s, \varepsilon_1) \) the class of all perturbed billiard maps defined by the dynamics

(2.4) and (2.5) under forces \( \mathbf{F} \) and \( \mathbf{G} \), satisfying assumptions (A1)-(A4).

Theorem 2.8. For any \( T \in \mathcal{F}_B(\mathcal{Q}_0, \tau_s, \varepsilon_1) \), the perturbed system \( T \) satisfies (H1)-(H5) with uniform constants depending only on \( \varepsilon_1, \tau_s, K_s, \) and \( E_s \).

Theorem 2.9. Within the class \( \mathcal{F}_B(\mathcal{Q}_0, \tau_s, \varepsilon_1) \), the change of either the force \( \mathbf{F} \) or \( \mathbf{G} \) by a small amount \( \delta \) yields a perturbation of size \( \mathcal{O}(|\delta|^{1/2}) \) in the distance \( d_T(\cdot, \cdot) \).

As a consequence, the spectral gap enjoyed by the classical billiard \( T_{0,0} \) persists for all \( T_{\mathbf{F}, \mathbf{G}} \in \mathcal{F}(\mathcal{Q}_0, \tau_s, \varepsilon_1) \) for \( \varepsilon_1 \) sufficiently small so that we may apply the limit theorems of Corollary [2.4] to any such \( T_{\mathbf{F}, \mathbf{G}} \).

The limit theorems implied by Theorem 2.9 are new even for the simplified maps \( T_{\mathbf{F}, 0} \) and \( T_{0, \mathbf{G}} \). We provide the proofs of Theorems 2.8 and 2.9 in Section 7.

\[\text{The assumption on the singularity set of } T_{\mathbf{F}, \mathbf{G}}^{-1} \text{ is not essential to our approach, but is made to simplify the proofs in Section 7 since the paper is already quite long and we include a number of distinct applications.}\]
2.4. Smooth random perturbations. We follow the expositions in [DL, DZ]. Suppose $\mathcal{F}$ is a class of maps satisfying (H1)-(H5) and let $d_\mathcal{F}(\cdot, \cdot)$ be the distance in $\mathcal{F}$ defined in Section 3.3. For $T_0 \in \mathcal{F}, \varepsilon > 0$, define

$$X_\varepsilon(T_0) = \{ T \in \mathcal{F} : d_\mathcal{F}(T, T_0) < \varepsilon \},$$


to be the $\varepsilon$-neighborhood of $T_0$ in $\mathcal{F}$.

Let $(\Omega, \nu)$ be a probability space and let $g : \Omega \times M \to \mathbb{R}^+$ be a measurable function satisfying: There exist constants $a, A > 0$ such that

1. $g(\omega, \cdot) \in C^1(M, \mathbb{R}^+)$ and $|g(\omega, \cdot)|_{C^1(M)} \leq A$ for each $\omega \in \Omega$;
2. $\int_\Omega g(\omega, x)d\nu(\omega) = 1$ for each $x \in M$;
3. $g(\omega, x) \geq a$ for all $\omega \in \Omega, x \in M$.

We define a random walk on $M$ by assigning to each $\omega \in \Omega$, a map $T \in X_\varepsilon(T_0)$. Starting at $x \in M$, we choose $T_\omega \in X_\varepsilon(T_0)$ according to the distribution $g(\omega, x)d\nu$. We apply $T_\omega$ to $x$ and repeat this process starting at $T_\omega x$. We say the process defined in this way has size $\Delta(\nu, g) \leq \varepsilon$.

Notice that if $\nu$ is the Dirac measure centered at $\omega_0$, then this process corresponds to the deterministic perturbation $T_{\omega_0}$ of $T_0$. If $g \equiv 1$, then the choice of $T_\omega$ is independent of the position $x$, while in general this formulation allows the choice of the next map to depend on the previous step taken.

The transfer operator $\mathcal{L}_{(\nu, g)}$ associated with the random process is defined by

$$\mathcal{L}_{(\nu, g)}h(x) = \int_\Omega \mathcal{L}_{T_\omega}h(x) g(\omega, T_\omega^{-1}x) d\nu(\omega)$$

for all $h \in L^1(M, m)$, where $m$ is Lebesgue measure on $M$.

**Theorem 2.10.** The transfer operator $\mathcal{L}_{(\nu, g)}$ satisfies the uniform Lasota-Yorke inequalities given by Theorem 2.2. Let $\varepsilon_0$ be given by (3.13) and let $\varepsilon \leq \varepsilon_0$. If $\Delta(\nu, g) \leq \varepsilon$, then there exists a constant $C > 0$ depending only on (H1)-(H5), such that $\| \mathcal{L}_{(\nu, g)} - \mathcal{L}_{T_0} \| \leq CA\varepsilon^{3/2}$.

It follows that all the operators $\mathcal{L}_{(\nu, g)}$ enjoy a spectral gap for $\varepsilon$ sufficiently small if $\mathcal{L}_{T_0}$ has a spectral gap and the limit theorems of Corollary 2.5 apply to $\mathcal{L}_{(\nu, g)}$.

2.5. Large perturbations: Large translations, rotations and deformations of scatterers. If we fix $\tau_*, K_*, E_*$ and $E_* < \infty$, then Theorems 2.2 and 2.5 imply that the transfer operator $\mathcal{L}_T$ corresponding to any $T \in \mathcal{F}_1(\tau_*, K_*, E_*)$ is quasi-compact with essential spectral radius bounded by $\sigma < 1$. In fact, [DZ, Theorem 2.5] implies that $\mathcal{L}_T$ has a spectral gap.

Now choose a compact interval $J \subset \mathbb{R}$ and parametrize a continuous path in $\mathcal{F}_1(\tau_*, K_*, E_*)$ according to the distance $d_\mathcal{F}(\cdot, \cdot)$. To each point $s \in J$ is assigned a map $T_s \in \mathcal{F}_1(\tau_*, K_*, E_*)$ and a corresponding transfer operator $\mathcal{L}_s$. Fix $\sigma_1 > \sigma$. Due to Theorem 2.3 there exists $\varepsilon_1 > 0$ such that the spectra and spectral projectors of $\mathcal{L}_{s'}$ outside the disk of radius $\sigma_1$ vary Hölder continuously for $s' \in (s - \varepsilon_1, s + \varepsilon_1) =: B(s, \varepsilon_1)$.

As these intervals overlap, as we move along the entire path from one end of $J$ to the other, the spectra and spectral projectors of $\mathcal{L}_s$ vary Hölder continuously in $s$. We have proved the following.

**Theorem 2.11.** Let $J \subset \mathbb{R}$ be a compact interval and let $\{T_s\}_{s \in J} \subset \mathcal{F}_1(\tau_*, K_*, E_*)$ be a continuously parametrized path according to the distance $d_\mathcal{F}(\cdot, \cdot)$. Then the spectra and spectral projectors of the associated transfer operators $\mathcal{L}_s$ vary Hölder continuously in the distance $d_\mathcal{F}(\cdot, \cdot)$.

As a consequence, the related dynamical properties of $T_s$, such as the rate of decay of correlations and variance in the Central Limit Theorem, vary Hölder continuously even across large movements and deformations of scatterers as long as the resulting maps remain in $\mathcal{F}_1(\tau_*, K_*, E_*)$. Indeed, since $J$ is compact, the continuity of the spectral data implies that the spectral gap is uniform along such paths even when the resulting configurations are no longer close to the original. This regularity holds as we move scatterers in such a way that the table changes from finite to infinite horizon.

**Remark 2.12.** One could just as well apply the above large movements of scatterers to billiards under external forces in the uniform families $\mathcal{F}_B(Q_s, \tau_*, \varepsilon_1)$ and allow the configurations $Q_s, s \in J$, to change over a continuously parametrized path in $\mathcal{F}_1(\tau_*, K_*, E_*)$ as long as the horizon along the path remains
bounded uniformly above by $\tau_*^{-1}$. Theorem 2.11 applies to such families of maps as well since they all possess spectral gaps by Theorem 2.9.

3. Common Approach

In this section, we describe the common approach we will take for each class of perturbations that we consider. We begin by formulating general conditions (H1)-(H5) under which the perturbations of a billiard map will satisfy the Lasota-Yorke inequalities (2.1) with uniform constants. We also introduce general conditions (C1)-(C4) to verify that a perturbation is small in the sense of (2.2). Theorems 2.2 and 2.3 show that these conditions are sufficient to establish the framework of [KL]. Once this is accomplished, we only need to check that these conditions are satisfied for each class of perturbations described above.

3.1. A class of maps with uniform properties. We fix the phase space $M = \cup_i I_i \times [-\pi/2, \pi/2]$ of a billiard map associated with a periodic Lorentz gas as in Section 2.2.

We define the set $S_0 = \{\varphi = \pm \pi/2\}$ and for a fixed $k_0 \in \mathbb{N}$, we define for $k \geq k_0$, the homogeneity strips,

$$H_k = \{(r, \varphi) : \pi/2 - k^{-2} < \varphi < \pi/2 - (k + 1)^2\}.$$ 

The strips $H_{-k}$ are defined similarly near $\varphi = -\pi/2$. We also define $H_0 = \{(r, \varphi) : -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}$. The set $S_{0,H} = S_0 \cup (\cup_{|k| \geq k_0} \partial H_{\pm k})$ is therefore fixed and will give rise to the singularity sets for the maps that we define below, i.e. for any map $T$ that we consider, we define $S_{T,n} = \cup_{i=0}^n T^{i+1} S_{0,H}$ to be the singularity sets for $T^{\pm n}$, $n \geq 0$.

Suppose there exists a class of invertible maps $F$ such that for each $T \in F$, $T : M \setminus S_{1} \to M \setminus S_{T,1}$ is a $C^2$ diffeomorphism on each connected component of $M \setminus S_1$. We assume that elements of $F$ enjoy the following uniform properties.

(H1) Hyperbolicity and singularities. There exist continuous families of stable and unstable cones $C^s(x)$ and $C^u(x)$, defined on all of $M$, which are strictly invariant for the class $F$, i.e., $DT(x)C^u(x) \subset C^u(Tx)$ and $DT^{-1}(x)C^s(x) \subset C^s(T^{-1}x)$ for all $T \in F$ wherever $DT$ and $DT^{-1}$ are defined.

We require that the cones $C^s(x)$ and $C^u(x)$ are uniformly transverse on $M$ and that $S_{-n}^T$ is uniformly transverse to $C^s(x)$ for each $n \in \mathbb{N}$ and all $T \in F$. We assume in addition that $C^s(x)$ is uniformly transverse to the horizontal and vertical directions on all of $M$.\footnote{This is not a restrictive assumption for perturbations of the Lorentz gas since the standard cones $\hat{C}^s$ and $\hat{C}^u$ for the billiard map satisfy this property (see for example [CM] Section 4.5); the common cones $C^s(x)$ and $C^u(x)$ shared by all maps in the class $F$ must therefore lie inside $\hat{C}^s(x)$ and $C^u(x)$ and therefore satisfy this property. In any case, a weaker formulation of this assumption is necessary: we use in the compactness argument that the lengths of stable curves in the homogeneity strips $H_k$, $k \geq k_0$, are proportional to the width of the strips. This is only true if stable curves are transverse to the horizontal direction in such strips.}

Moreover, there exist constants $C_e > 0$ and $\Lambda > 1$ such that for all $T \in F$,

$$\|DT^n(x)v\| \geq C_e^{-1} \Lambda^n \|v\|, \forall v \in C^u(x), \quad \text{and} \quad \|DT^{-n}(x)v\| \geq C_e^{-1} \Lambda^n \|v\|, \forall v \in C^s(x),$$

for all $n \geq 0$, where $\| \cdot \|$ is the Euclidean norm on the tangent space $T_x M$.

For any stable curve $W \in \hat{W}^s$ (see (H2) below), the set $W \cap TS_0$ (not counting homogeneity strips) is finite or countable and has at most $K$ accumulation points on $W$, where $K \geq 0$ is a constant uniform for $T \in F$. Let $x_\infty$ be one of them and let $\{x_n\}$ denote the monotonic sequence of points $W \cap TS_0$ converging to $x_\infty$. We denote the part of $W$ between $x_n$ and $x_{n+1}$ by $W_n$. We assume there exists $C_a > 0$ such that the expansion factor on $W_n$ satisfies

$$C_a n[\cos \varphi(T^{-1}x)][1-1]/\|v\| \leq \|DT^{-1}(x)v\| \leq C_a^{-1} n[\cos \varphi(T^{-1}x)][1-1]/\|v\|, \quad \forall x \in W_n, \forall v \in C^s(x),$$

where $\varphi(y)$ denotes the angle at the point $y = (r, \varphi) \in M$. Let $\exp_x$ denote the exponential map from $T_x M$ to $M$. We require the following bound on the second derivative,

$$C_a n^2[\cos \varphi(T^{-1}x)][3-3]/\|v\| \leq \|D^2 T^{-1}(x)v\| \leq C_a^{-1} n^2[\cos \varphi(T^{-1}x)][3-3], \quad \forall x \in W_n,$$

for all $v \in T_x M$ such that $T^{-1}(\exp_x(v))$ and $T^{-1}x$ lie in the same homogeneity strip.
We assume there exist constants $c, v_0 > 0$ such that if $x \in W$ and $T^{-1}x \in \mathbb{H}_k$, then
\[
k \geq c v_0.
\]
If $K = 0$ (i.e., the billiard has finite horizon) then the indexing scheme above based on $n$ is finite and (3.3), (3.4) and (3.5) hold with $n = 1$.

If $W \cap T\mathcal{S}_0$ has no accumulating points, we assume (3.3), (3.4) hold with $n = 1$.

**(H2)** *Families of stable and unstable curves.* We call $W$ a *stable curve* for a map $T \in \mathcal{F}$ if the tangent line to $W$, $T_x W$ lies in $C^\infty(x)$ for all $x \in W$. We call $W$ *homogeneous* if $W$ is contained in one homogeneity strip $\mathbb{H}_k$. Unstable curves are defined similarly.

We assume that there exists a family of smooth stable curves, $\widehat{\mathcal{W}}^s$, such that each $W \in \widehat{\mathcal{W}}^s$ is a $C^2$ stable curve with curvature bounded above by a uniform constant $B > 0$. The family $\widehat{\mathcal{W}}^s$ is required to be invariant under $\mathcal{F}$ in the following sense: For any $W \in \widehat{\mathcal{W}}^s$ and $T \in \mathcal{F}$, the connected components of $T^{-1}W$ are again elements of $\widehat{\mathcal{W}}^s$.

A family of unstable curves $\widehat{\mathcal{W}}^u$ is defined analogously, with obvious modifications: For example, we require the connected components of $TW$ to be elements of $\widehat{\mathcal{W}}^u$ for all $W \in \widehat{\mathcal{W}}^u$ and $T \in \mathcal{F}$.

**(H3)** *One-step expansion.* We formulate the one-step expansion in terms of an adapted norm $\| \cdot \|_s$, uniformly equivalent to $\| \cdot \|$, in which the constant $C_c$ in (3.2) can be taken to be 1, i.e., we have expansion and contraction in one step in the adapted norm. We assume such a norm exists for maps in the class $\mathcal{F}$.

Let $W \in \widehat{\mathcal{W}}^s$. For any $T \in \mathcal{F}$, we partition the connected components of $T^{-1}W$ into maximal pieces $V_i = V_i(T)$ such that each $V_i$ is a homogeneous stable curve in some $H_k$, $k \geq k_0$, or $H_0$. Let $|J_{V_i} T|_s$ denote the minimum contraction on $V_i$ under $T$ in the metric induced by the adapted norm $\| \cdot \|_s$. We assume that for some choice of $k_0$,
\[
\limsup_{T \in \mathcal{F}} \sup_{|W| < \delta} \sup_{i} |J_{V_i} T|_s < 1,
\]
where $|W|$ denotes the arclength of $W$.

In addition, the above sum converges even when the expansion on each piece is weakened slightly in the following sense: There exists $\varsigma_0 < 1$ such that for all $\varsigma > \varsigma_0$, there exists $C_\varsigma = C_\varsigma(\varsigma, \delta)$ such that for all $T \in \mathcal{F}$ and any $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$,
\[
\sum_i |J_{V_i} T|_{s(C_{\varsigma_0}(V_i))} < C_\varsigma,
\]
where $J_{V_i} T$ denotes the stable Jacobian of $T$ along the curve $V_i$ with respect to arc length. We formulate (3.7) in terms of the usual Euclidean Jacobian of $T$ along the curve $V_i$.

**(H4)** *Bounded distortion.* There exists a constant $C_d > 0$ with the following properties. Let $W' \in \widehat{\mathcal{W}}^s$ and for any $T \in \mathcal{F}$, $n \in \mathbb{N}$, let $x, y \in W'$ for some connected component $W \subset T^{-n}W'$ such that $T^n W$ is a homogeneous stable curve for each $0 \leq i \leq n$. Then,
\[
\left| \frac{J_{\mu T^n}(x)}{J_{\mu T^n}(y)} - 1 \right| \leq C_d d_W(x, y)^{1/3}, \quad \text{and} \quad \left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \leq C_d d_W(x, y)^{1/3},
\]
where $J_{\mu T^n}$ is the Jacobian of $T^n$ with respect to the smooth measure $d\mu = \cos \varphi \, dr \, d\varphi$.

We assume the analogous bound along unstable leaves: If $W \in \widehat{\mathcal{W}}^s$ is an unstable curve such that $T^n W$ is a homogeneous unstable curve for $0 \leq i \leq n$, then for any $x, y \in W$,
\[
\left| \frac{J_{\mu T^n}(x)}{J_{\mu T^n}(y)} - 1 \right| \leq C_d d(T^n x, T^n y)^{1/3}.
\]
Before defining the norms, we define the notion of a distance \(d\) and let \(\theta_s < 1\) be from \((3.10)\). Assume there exists a constant \(\eta < \min\{\Lambda^3, \Lambda^4, \theta_s^{-1}\}\) such that for any \(T \in \mathcal{F}\),

\[
(J_\mu T(x))^{-1} \leq \eta, \quad \text{wherever } J_\mu T(x) \text{ is defined.}
\]

### 3.2. Transfer operator

Recall the family of stable curves \(\hat{W}^s\) defined by \((H2)\). We define a subset \(W^s \subset \hat{W}^s\) as follows. By \((H3)\) we may choose \(\delta_0 > 0\) for which there exists \(\theta_s < 1\) such that

\[
\sup_{T \in \mathcal{F}} \sup_{|W| \leq \delta_0} \sum_i |J_i T| \leq \theta_s.
\]

We shrink \(\delta_0\) further if necessary so that the graph transform argument in Lemma \((3.3)\) holds. The set \(W^s\) comprises all those stable curves \(\hat{W}^s\) such that \(|W| \leq \delta_0\).

For any \(T \in \mathcal{F}\), we define scales of spaces using the set of stable curves \(W^s\) on which the transfer operator \(\mathcal{L}_T\) associated with \(T\) will act. Define \(T^{-n} W^s\) to be the set of homogeneous stable curves \(\hat{W}^s\) such that \(T^n\) is smooth on \(W\) and \(T^n W \in W^s\) for \(0 \leq i \leq n\). It follows from \((H2)\) that \(T^{-n} W^s \subset W^s\). We denote (normalized) Lebesgue measure on \(M\) by \(m\).

For \(W \in T^{-n} W^s\), a complex-valued test function \(\psi : M \rightarrow \mathbb{C}\), and \(0 < p \leq 1\) define \(H^p_W(\psi)\) to be the Hölder constant of \(\psi\) on \(W\) with exponent \(p\) measured in the Euclidean metric. Define \(H^p_W(\psi) = \sup_{\psi \in W^{T^{-n} W^s}} H^p_W(\psi)\) and let \(C^p(T^{-n} W^s) = \{\psi : M \rightarrow \mathbb{C} \mid H^p_W(\psi) < \infty\}\), denote the set of complex-valued functions which are Hölder continuous on elements of \(T^{-n} W^s\). The set \(C^p(T^{-n} W^s)\) equipped with the norm \(\|\psi\|_{C^p(T^{-n} W^s)} = \|\psi\|_\infty + H^p_W(\psi)\) is a Banach space. Similarly, we define \(\hat{C}^p(\hat{W}^u)\), the set of functions which are Hölder continuous with exponent \(p\) on unstable curves \(\hat{W}^u\).

It follows from \((H3)\) that if \(\psi \in \hat{C}^p(T^{-(n-1)} W^s)\), then \(\psi \circ T \in \hat{C}^p(T^{-n} W^s)\). Thus if \(h \in (\hat{C}^p(T^{-n} W^s))'\), is an element of the dual of \(\hat{C}^p(T^{-n} W^s)\), then \(\mathcal{L}_T : (\hat{C}^p(T^{-n} W^s))' \rightarrow (\hat{C}^p(T^{-(n-1)} W^s))'\) acts on \(h\) by

\[
\mathcal{L}_T h(\psi) = h(\psi \circ T) \quad \forall \psi \in \hat{C}^p(T^{-n} W^s).
\]

Recall that \(d\mu = c \cos \varphi \, dr \, d\varphi\) denotes the smooth invariant measure for the unperturbed Lorentz gas. If \(h \in L^1(M, \mu)\), then \(h\) is canonically identified with a signed measure absolutely continuous with respect to \(\mu\), which we shall also call \(h\), i.e., \(h(\psi) = \int_M \psi h \, d\mu\). With the above identification, we write \(L^1(M, \mu) \subset (C^p(T^{-n} W^s))'\) for each \(n \in \mathbb{N}\). Then restricted to \(L^1(M, \mu)\), \(\mathcal{L}_T\) acts according to the familiar expression

\[
\mathcal{L}_T^n h = h \circ T^{-n} (J_\mu T^n(T^{-n}))^{-1}\quad \text{for any } n \geq 0 \text{ and } h \in L^1(M, \mu).
\]

**Remark 3.1.** In \([DZ]\), we used Lebesgue measure as a reference measure to show that the functional analytic framework developed there did not need to assume the existence of a smooth invariant measure. Now that \(\mu\) has been established in our function space \(\mathcal{B}\), however, we find it more convenient to use it as a starting point in our study of the classes of perturbations considered here. It also simplifies our norms and estimates slightly since for example, it eliminates the need for the \(\cos W\) weight in our test functions that was used in \([DZ]\). We do not assume that \(\mu\) is an invariant measure for \(T \in \mathcal{F}\); indeed, the SRB measures for such \(T\) are in general singular with respect to Lebesgue measure.

### 3.3. Definition of the Norms

The norms are defined via integration on the set of stable curves \(W^s\). Before defining the norms, we define the notion of a distance \(d_{W^s}(\cdot, \cdot)\) between such curves as well as a distance \(d_{W^s}(\cdot, \cdot)\) defined among functions supported on these curves.

Due to the transversality condition on the stable cones \(C^s(x)\) given by \((H1)\), each stable curve \(W\) can be viewed as the graph of a function \(\varphi_W(r)\) of the arclength parameter \(r\). For each \(W \in W^s\), let \(I_W\) denote the interval on which \(\varphi_W\) is defined and set \(G_W(r) = (r, \varphi_W(r))\) to be its graph so that \(W = \{G_W(r) : r \in I_W\}\). We let \(m_W\) denote the unnormalized arclength measure on \(W\).

Let \(W_1, W_2 \in W^s\) and identify them with the graphs \(G_{W_i}\) of their functions \(\varphi_{W_i}\), \(i = 1, 2\). Suppose \(W_1, W_2\) lie in the same component of \(M\) and let \(I_{W_i}\) be the \(r\)-interval on which each curve is defined. Denote by \((I_{W_1} \cap I_{W_2})\) the length of the symmetric difference between \(I_{W_1}\) and \(I_{W_2}\). Let \(H_{W_i}\) be the homogeneity strip containing \(W_i\). We define the distance between \(W_1\) and \(W_2\) to be,

\[
d_{W^s}(W_1, W_2) = \eta(k_1, k_2) + \ell(I_{W_1} \triangle I_{W_2}) + |\varphi_{W_1} - \varphi_{W_2}||c^s(I_{W_1} \triangle I_{W_2})|
\]
where $\eta(k_1, k_2) = 0$ if $k_1 = k_2$ and $\eta(k_1, k_2) = \infty$ otherwise, i.e., we only compare curves which lie in the same homogeneity strip.

For $0 \leq p \leq 1$, denote by $\tilde{C}^p(W)$ the set of continuous complex-valued functions on $W$ with Hölder exponent $p$, measured in the Euclidean metric, which we denote by $\| \cdot \|_W$. We then denote by $C^p(W)$ the closure of $C^\infty(W)$ in the $\tilde{C}^p$-norm $\| \cdot \|_{\tilde{C}^p(W)}$. We define $C^p(M)$ similarly.

Given two functions $\psi_i \in C^q(W, \mathbb{C})$, $q > 0$, we define the distance between $\psi_1$, $\psi_2$ as

$$d_q(\psi_1, \psi_2) = \| \psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2} \|_{C^q(W)}.$$  

We will define the required Banach spaces by closing $C^1(M)$ with respect to the following set of norms. For $s, p \geq 0$, define the following norms for test functions,

$$|\psi|_{W,s,p} := |W|^s \cdot |\psi|_{C^p(W)}.$$  

Now fix $0 < p \leq \frac{1}{3}$. Given a function $h \in C^1(M)$, define the weak norm of $h$ by

$$|h|_w := \sup_{W \in W^s} \sup_{|\psi|_{W,0,p} \leq 1} \int_W h\psi \ dm_W.$$  

Choose $\alpha, \beta, q > 0$ such that $\alpha < 1 - s_0$, $q < p$ and $\beta \leq \min\{\alpha, p - q\}$. We define the strong stable norm of $h$ as

$$\|h\|_s := \sup_{W \in W^s} \sup_{\psi \in C^q(W)} \int_W h\psi \ dm_W$$

and the strong unstable norm as

$$\|h\|_u := \sup_{\varepsilon \leq \varepsilon_0} \sup_{W_1, W_2 \in W^s} \sup_{\psi \in C^q(W_1)} \frac{1}{\varepsilon^{\beta}} \left| \int_{W_1} h\psi_1 \ dm_W - \int_{W_2} h\psi_2 \ dm_W \right|$$

where $\varepsilon_0 > 0$ is chosen less than $\delta_0$, the maximum length of $W \in W^s$ which is determined by (3.10). We then define the strong norm of $h$ by

$$\|h\|_s = \|h\|_s + b\|h\|_u$$

where $b$ is a small constant chosen in Section 3.

We define $\mathcal{B}$ to be the completion of $C^1(M)$ in the strong norm and $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the weak norm.

3.4. **Distance in $\mathcal{F}$**. We define a distance in $\mathcal{F}$ as follows. Let $\varepsilon_0$ be from (3.13). For $T_1, T_2 \in \mathcal{F}$ and $\varepsilon \leq \varepsilon_0$, let $N_\varepsilon(S^1_{-1})$ denote the $\varepsilon$-neighborhood in $M$ of the singularity set $S^1_{-1}$ of $T^{-1}_i$, $i = 1, 2$. We say $d_\mathcal{F}(T_1, T_2) \leq \varepsilon$ if the maps are close away from their singularity sets in the following sense: For $x \notin N_\varepsilon(S^1_{-1} \cup S^2_{-1})$,

**C1** \hspace{1cm} $d(T^{-1}_1(x), T^{-1}_2(x)) \leq \varepsilon$;

**C2** \hspace{1cm} $\left| \frac{J_{\mu}T_i(x)}{J_{\mu}T_j(x)} - 1 \right| \leq \varepsilon$, $i, j = 1, 2$;

**C3** \hspace{1cm} $\left| \frac{J_{W}T_i(x)}{J_{W}T_j(x)} - 1 \right| \leq \varepsilon$, for any $W \in W^s$, $i, j = 1, 2$, and $x \in W$;
\[ \| DT_1^{-1}(x)v - DT_2^{-1}(x)v \| \leq \sqrt{c}, \text{ for any unit vector } v \in T_xW, W \in W^s. \]

3.5. Preliminary estimates. Before proving the Lasota-Yorke inequalities, we show how (H1)-(H5) imply several other uniform properties for our class of maps \( F \). In particular, we will be interested in iterating the one-step expansion relations given by (H3). We recall the estimates we need from \([DZ]\) Section 3.2.

Let \( T \in F \) and \( W \in W^s \). Let \( V_i \) denote the maximal connected components of \( T^{-1}W \) after cutting due to singularities and the boundaries of the homogeneity strips. To ensure that each component of \( T^{-1}W \) is in \( W^s \), we subdivide any of the long pieces \( V_i \) whose length is \( > \delta_0 \), where \( \delta_0 \) is chosen in (3.10).

This process is then iterated so that given \( W \) smooth on each \( F \), we construct the components of \( T^{-n}W \), which we call the \( n \)-th generation \( G_n(W) \), inductively as follows. Let \( G_0(W) = \{W\} \) and suppose we have defined \( G_{n-1}(W) \subset W^s \). First, for any \( W' \in G_{n-1}(W) \), we partition \( T^{-1}W' \) into at most countably many pieces \( W'_i \) so that \( T \) is smooth on each \( W'_i \) and each \( W'_i \) is a homogeneous stable curve. If any \( W'_i \) have length greater than \( \delta_0 \), we subdivide those pieces into pieces of length between \( \delta_0/2 \) and \( \delta_0 \). We define \( G_n(W) \) to be the collection of all pieces \( W'_i \subset T^{-n}W \) obtained in this way. Note that each \( W'_i \) is in \( W^s \) by (H2).

At each iterate of \( T^{-1} \), typical curves in \( G_n(W) \) grow in size, but there exist a portion of curves which are trapped in tiny homogeneity strips and in the infinite horizon case, stay too close to the infinite horizon points. In Lemma 3.2, we make precise the sense in which the proportion of curves that never grow to a fixed length decays exponentially fast.

For \( W \in W^s \), \( n \geq 0 \), and \( 0 \leq k \leq n \), let \( G_k(W) = \{W^k_i\} \) denote the \( k \)-th generation pieces in \( T^{-k}W \). Let \( B_k(W) = \{i : |W^k_i| < \delta_0/3\} \) and \( L_k(W) = \{i : |W^k_i| \geq \delta_0/3\} \) denote the index of the short and long elements of \( G_k(W) \), respectively. We consider \( \{G_k\}_{k=0}^n \) as a tree with \( W \) as its root and \( G_k \) as the \( k \)-th level.

At level \( n \), we group the pieces as follows. Let \( W^k_i \in G_n(W) \) and let \( W^k_j \in L_k(W) \) denote the most recent long “ancestor” of \( W^k_i \) i.e. \( k = \max\{0 \leq \ell \leq n : T^{n-\ell}(W^k_i) \subset W^\ell_j \) and \( j \in L_\ell \}. \)

If no such ancestor exists, set \( k = 0 \) and \( W^k_j = W \). Note that if \( W^k_i \) is long, then \( W^k_j = W^k_i \). Let

\[ I_n(W^k_j) = \{i : W^k_j \in L_k(W) \text{ is the most recent long ancestor of } W^k_i \in G_n(W)\}. \]

The set \( I_n(W) \) represents those curves \( W^n \in G_n(W) \) that belong to short pieces in \( G_k(W) \) at each time step \( 1 \leq k \leq n \), i.e. such \( W^n \) are never part of a piece that has grown to length \( \geq \delta_0/3 \).

We collect the results of \([DZ]\) Section 3.2 in the following lemma.

**Lemma 3.2.** \([DZ]\) Let \( W \in W^s \), \( T \in F \) and for \( n \geq 0 \), let \( I_n(W) \) and \( G_n(W) \) be defined as above. There exist constants \( C_1, C_2, C_3 > 0 \), independent of \( T \) and \( W \), such that for any \( n \geq 0 \),

(a) \( \sum_{i \in I_n(W)} |J_{W^n_i}T^n|_{C^0(W^n)} \leq C_1 \theta_n \); 
(b) \( \sum_{W^n \in G_n(W)} |J_{W^n}T^n|_{C^0(W^n)} \leq C_2 \); 
(c) for any \( 0 \leq \varsigma \leq 1 \), \( \sum_{W^n \in G_n(W)} \frac{|W^n|^{\varsigma}}{|W^n|} |J_{W^n}T^n|_{C^0(W^n)} \leq C_2^{1-\varsigma} \); 
(d) for \( \varsigma > \varsigma_0 \), \( \sum_{W^n \in G_n(W)} |J_{W^n}T^n|_{C^0(W^n)} \leq C_3^n \), where \( C_3 \) depends on \( \varsigma \).

**Proof.** The proofs of these items are combinatorial and require no more specific information about the maps than the uniform properties given by (H2), (H3) and (H4).

(a) This is Lemma 3.1 of \([DZ]\). The constant \( C_1 \) depends only on the constant relating the Euclidean norm \( \| \cdot \| \) to the adapted norm \( \| \cdot \|_s \). As such, \( C_1 \) is independent of \( T \in F \), \( W \in W^s \) and \( n \in \mathbb{N} \). 
(b) This statement is \([DZ]\) Lemma 3.2. The constant \( C_2 = C_2(\delta_0, \theta_\varsigma, C_1, C_d) \).
(c) This is \([DZ]\) Lemma 3.3. It follows from (b) by an application of Jensen’s inequality.
(d) This follows from (3.7) and is proved in \([DZ]\) Lemma 3.4. The constant \( C_3 = \delta_0^{-1}C_\varsigma(1 + C_d)^{2\varsigma} \) is uniform for \( T \in F \), but depends on \( \varsigma \). \(\square\)
Next we prove a distortion bound for the stable Jacobian of $T$ along different stable curves in the following context. Let $W^1, W^2 \in \mathcal{W}^s$ and suppose there exist $U^k \subset T^{-n}W^k$, $k = 1, 2$, such that for $0 \leq i \leq n$,

1. $T^i U^k \in \mathcal{W}^s$ and the curves $T^i U^1$ and $T^i U^2$ lie in the same homogeneity strip;
2. $U^1$ and $U^2$ can be put into a 1-1 correspondence by a smooth foliation $\{ \gamma_x \}_{x \in U^1}$ of curves $\gamma_x \in \hat{\mathcal{W}}^u$ such that $\{ T^n \gamma_x \} \subset \hat{\mathcal{W}}^u$ creates a 1-1 correspondence between $T^n U^1$ and $T^n U^2$;
3. $|T^i \gamma_x| \leq 2 \max(|T^i U^1|, |T^i U^2|)$, for all $x \in U^1$.

Let $J_{U^1} T^n$ denote the stable Jacobian of $T^n$ along the curve $U^k$ with respect to arclength.

**Lemma 3.3.** In the setting above, for $x \in U^1$, define $x^* \in \gamma_x \cap U^2$. There exists $C_0 > 0$, independent of $T \in \mathcal{F}$, $W \in \mathcal{W}^s$ and $n \geq 0$ such that

1. $d_{\mathcal{W}^s}(U^1, U^2) \leq C_0 \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$;
2. $|J_{U^1} T^n(x) - 1| \leq C_0 [d(T^n x, T^n x^*)^{1/3} + \theta(T^n x, T^n x^*)]$,

where $\theta(T^n x, T^n x^*)$ is the angle formed by the tangent lines of $T^n U^1$ and $T^n U_2$ at $T^n x$ and $T^n x^*$, respectively.

**Proof.** (a) This is essentially a graph transform argument adapted for this class of maps satisfying (H1). What we need to show here is that we do not need to cut curves lying in homogeneity strips any further in order to get the required contraction and control on distortion.

First notice that due to the uniform expansion of $\gamma_x$ under $T^n$ given by (3.2) of (H1), we have $|\gamma_x| \leq C_0 C_1 \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$, where $C_1$ is a constant depending only on the minimum angle between $C^u(x)$ and $C^s(x)$ and between $C^u(x)$ and the horizontal direction. Again by the transversality of $\gamma_x$ with $U^1$ and $U^2$, the $r$-intervals on which the functions $\varphi_{U^1}, \varphi_{U^2}$ describing the curves $U^1, U^2$ are defined can differ by no more than $C_0 C_1 \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$. Letting $I$ denote the intersection of intervals on which both functions are defined and recalling the definition of $d_{\mathcal{W}^s}((\cdot), (\cdot))$ from Section 3.3, it remains to estimate $|\varphi_{U^1} - \varphi_{U^2}|_{C^r(I)}$.

By the same observation as above, we have $|\varphi_{U^1} - \varphi_{U^2}|_{C^r(I)} \leq C_0 C_1 \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$. In order to show that the slopes of these curves also contract exponentially, we make the usual graph transform argument using charts in the adapted norm $\| \cdot \|_s$ from (H3).

Fix $x \in U^1$ and define charts along the orbit of $x$ so that $x_i := T^i x$, $0 \leq i \leq n$, corresponds to the origin in each chart with the stable direction at $x_i$ given by the horizontal axis and the unstable direction by the vertical axis in the charts. Let $\theta < 1$ denote the maximum absolute value of slopes of stable curves in the chart. Due to property (iii) before the statement of the lemma, we may choose the size of the charts to have stable and unstable diameters $\leq C |T^1 U^i|$ for each $i$, for some uniform constant $C$. The dynamics induced by $T^{-1}$ on these charts is defined by

$$\tilde{T}_{x_i}^{-1} = \chi_{x_{i-1}}^{-1} \circ T^{-1} \circ \chi_{x_i}$$

where $\chi_{x_i}$ are smooth maps with $|\chi_{x_i}|_{C^2}, |\chi_{x_i}^{-1}|_{C^2} \leq C$ for some uniform constant $C$.

Note that $D\tilde{T}_{x_i}^{-1}$ and $D^2\tilde{T}_{x_i}^{-1}$ satisfy (H1) with possibly larger $C_a$ and $C_c = 1$. In the chart coordinates, since $\tilde{T}_{x_i}^{-1}(0) = 0$, we have

$$\tilde{T}_{x_i}^{-1}(s, t) = (A_i s + \alpha_i(s,t), B_i t + \beta_i(s,t))$$

where $A_i$ is the expansion at $x_i$ in the stable direction and $B_i$ is the contraction at $x_i$ in the unstable direction given by $D\tilde{T}_{x_i}^{-1}(0)$. The nonlinear functions $\alpha_i, \beta_i$ satisfy $\alpha_i(0,0) = \beta_i(0,0) = 0$ and their Lipschitz constants are bounded by the maximum of

$$D\tilde{T}_{x_i}^{-1}(u) - D\tilde{T}_{x_i}^{-1}(v) \leq D^2\tilde{T}_{x_i}(z)||u - v||$$

where $u, v, z$ range over the chart at $x_i$.

We fix $i$ and let $\varphi_1, \varphi_2$ denote two Lipschitz functions whose graphs lie in the stable cone of the chart at $x_i$ and satisfy $\varphi_j(0) = 0$, $j = 1, 2$. Define $L(\varphi_1, \varphi_2) = \sup_{s \neq 0} \frac{|\varphi_2(s) - \varphi_1(s)|}{|s|}$. Let $\varphi_1' = \tilde{T}_{x_i}^{-1} \varphi_1$ and $\varphi_2' = \tilde{T}_{x_i}^{-1} \varphi_2$ denote the graphs of the images of these two curves in the chart at $x_{i-1}$. We wish to estimate $L(\varphi_1', \varphi_2')$. 


For $s$ on the horizontal axis in the chart at $x_i$, we write,
\[
|\varphi'_1(A_i s + \alpha_i(s, \varphi_1(s))) - \varphi'_2(A_i s + \alpha_i(s, \varphi_1(s)))| \leq |\varphi'_1(A_i s + \alpha_1(s, \varphi_1(s))) - \varphi'_2(A_i s + \alpha_1(s, \varphi_2(s)))| \\
+ |\varphi'_2(A_i s + \alpha_1(s, \varphi_2(s))) - \varphi'_2(A_i s + \alpha_1(s, \varphi_1(s)))| \\
\leq |B_i||\varphi_1(s) - \varphi_2(s)| + |\beta_i(s, \varphi_1(s)) - \beta_i(s, \varphi_2(s))| + \vartheta|\alpha_i(s, \varphi_1(s)) - \alpha_i(s, \varphi_2(s))| \\
\leq (|B_i| + \text{Lip}(\beta_i) + \vartheta\text{Lip}(\alpha_i))|\varphi_1(s) - \varphi_2(s)|
\]

On the other hand, by (3.3),
\[
|A_i s + \alpha_i(s, \varphi_1(s))| \geq (|A_i| - \text{Lip}(\alpha_i)(1 + \vartheta))|s|.
\]

Putting these together, we see that,
\[
L(\varphi'_1, \varphi'_2) \leq \sup_{s \neq 0} \frac{(|B_i| + \text{Lip}(\beta_i) + \vartheta\text{Lip}(\alpha_i))|\varphi_1(s) - \varphi_2(s)|}{(|A_i| - \text{Lip}(\alpha_i)(1 + \vartheta))|s|} \leq \frac{|B_i| + \text{Lip}(\beta_i) + \vartheta\text{Lip}(\alpha_i)}{|A_i| - \text{Lip}(\alpha_i)(1 + \vartheta)} L(\varphi_1, \varphi_2).
\]

Suppose that $x_{i-1}$ lies in the homogeneity strip $H_k$ and $x_i$ lies on a curve with index $n$ according to the index given by (H1). Then by (3.14) and (3.3) and (3.4) of (H1), the Lipschitz constants of $\alpha_i$ and $\beta_i$ are bounded by $C_a^{-n^2}k^6(C_a^{-1}n^{-1}k^{-5}) = C_a^{-2}nk$ since the size of the chart is taken to be on the order of the length of the curve $T^n U^i$ by property (iii) of the matching. Thus,
\[
L(\varphi'_1, \varphi'_2) \leq \frac{\Lambda^{-1} + C_a^{-2}nk(1 + \vartheta)}{C_a nk^2 - C_a^{-2}nk(1 + \vartheta)} L(\varphi_1, \varphi_2) \leq \frac{4C_a^{-3}}{k} L(\varphi_1, \varphi_2),
\]
for large $k$, which can be made smaller than $\Lambda^{-1}$. Note that since $k \geq c_s n^{\nu_0}$ by (H1), this bound is also small for large $n$. Thus we may choose $N_0, K_0 > 0$ such that the contraction is less than $\Lambda^{-1}$ on all curves with index $n \geq N_0$ or landing in homogeneity strip $H_k$, $k \geq K_0$. On the remainder of $M$, the first and second derivatives of $T^{-1}$ are uniformly bounded by constants depending on $N_0$ and $K_0$. For curves in this part of $M$, we choose $\delta_0$, the maximum length of stable curves in $W^s$, sufficiently small that the distortion given by (3.14) is less than $\frac{1}{2}(\Lambda^{-1/2} - \Lambda^{-1})$. Then by (3.15), since $\vartheta < 1$, the contraction on these pieces is less than $\Lambda^{-1}$ as well.

If $\varphi_1$ and $\varphi_2$ do not pass through the origin, the exponential contraction in the $C^0$ norm coupled with the above argument yields the required contraction.

(b) It is equivalent to estimate the ratio log $\frac{J_{T^n U_i} T^{-n}(T^n x)}{J_{T^n U_j} T^{-n}(T^n x)}$. We write
\[
\log \frac{J_{T^n U_i} T^{-n}(T^n x)}{J_{T^n U_j} T^{-n}(T^n x)} \leq \sum_{i=1}^{n} \frac{1}{A_i} \left| J_{T^n U_i} T^{-1}(T^n x) - J_{T^n U_j} T^{-1}(T^n x) \right|
\]
where $A_i = \min\{J_{T^n U_i} T^{-1}(T^n x), J_{T^n U_j} T^{-1}(T^n x)\}$.

We estimate the differences one term at a time and assume without loss of generality that the minimum for $A_i$ is attained at $T^n x_i$. Set $x_i = T^n x_i$, $x_i^* = T^n x_i^*$. Let $\tilde{u}_1(x_i)$ denote the unit tangent vector to $T^n U_i$ at $x_i$ and notice that $J_{T^n U_i} T^{-1}(x_i) = \|DT^{-1}(x_i)\tilde{u}_1\|$. Define $\tilde{u}_2(x_i^*)$ similarly. Then
\[
\|DT^{-1}(x_i)\tilde{u}_1 - DT^{-1}(x_i^*)\tilde{u}_2\| \leq \|DT^{-1}(x_i)\tilde{u}_1 - DT^{-1}(x_i)\tilde{u}_2\| + \|DT^{-1}(x_i)\tilde{u}_2 - DT^{-1}(x_i^*)\tilde{u}_2\| \\
\leq \|DT^{-1}(x_i)\|\|\tilde{u}_1 - \tilde{u}_2\| + \|DT^{-1}(x_i)\|\|d(x_i, x_i^*)\|
\]
where $z_i$ is some point on $T^n \gamma_x$.

Suppose $T^{-1} x_i$ lies in the homogeneity strip $H_k$ and $x_i$ lies on some curve $W_{n}$ according to the index given in (H1). Then $\|DT^{-1}(x_i)\|/\|DT^{-1}(x_i)\tilde{u}\| \leq C$ where $C$ is some uniform constant for all unit vectors $\tilde{u} \in C^s(x_i)$. Also by (H1), we have $|T^n U_j| \leq C_a/nk^5$, $j = 1, 2$, so that by property (iii) before the statement of the lemma, $d(x_i, x_i^*) \leq 2C_a/(nk^5)$. Thus
\[
\frac{\|DT^{-1}(x_i)\|\|d(x_i, x_i^*)\|}{\|DT^{-1}(x_i)\tilde{u}_1\|} \leq \frac{\left(C_a n^2 k^6 (2C_a/(nk^5))\right)}{C_a^{-1} nk^2} \leq \frac{2C_a^3}{k} \leq 2C_a^3 d(x_{i-1}, x_{i-1}^*)^{1/3}.
\]
Using these estimates in (3.16), we have
\[
\log \frac{J_{T^nW_1}T^{-n}(T^n\mathbf{x})}{J_{T^n\mathbf{y}_2}T^{-n}(T^n\mathbf{x}_2)} \leq C \sum_{i=1}^{n} \|\bar{u}_1(x_i) - \bar{u}_2(x_i^*)\| + d(x_{i-1}, x_{i-1}^*)^{1/3}.
\]
Now \(\|\bar{u}_1(x_i) - \bar{u}_2(x_i^*)\| \leq 2(x_i, x_i^*) \leq C_0 \Lambda^{-n}\theta(T^n\mathbf{x}, T^n\mathbf{x}^*)\) by part (a) of the lemma together with the fact that curves in \(\mathcal{W}^s\) have \(C^2\) norm uniformly bounded above. Finally, by (H1), \(d(x_{i-1}, x_{i-1}^*) \leq C\epsilon \Lambda^{-n-1}d(T^n\mathbf{x}, T^n\mathbf{x}^*)\), which completes the proof of the lemma. \(\square\)

3.6. Properties of the Banach spaces. We first prove that the weak and strong norms dominate distributional norms on \(M\) in the following sense.

**Lemma 3.4.** There exists \(C > 0\) such that for any \(h \in B_w, T \in F, n \geq 0\) and \(\psi \in C^0(T^{-n}\mathcal{W}^s)\),
\[
|h(\psi)| \leq C|\mathcal{L}(\psi)\| + H_{n}^{0}(\psi).
\]

This is the analogue of Lemma 3.9 of [DŻ], but it does not follow from the argument given there since condition (H5) and the weakened Lasota-Yorke inequalities in Theorem 2.2 suggest that the spectral radius of \(L_{T}\) can be as much as \(\eta > 1\). It is a consequence of Lemma 3.4 that the spectral radius is in fact 1 (see Section 1 proof of Theorem 2.2).

**Proof of Lemma 3.4.** On each \(M_{\ell} = \Gamma_{\ell} \times [-\pi/2, \pi/2]\), we partition the set \(\mathbb{H}_0\) into finitely many boxes \(B_j\) whose boundary curves are elements of \(\mathcal{W}^s\) and \(\mathcal{W}^u\) as well as the horizontal lines \(\pm \pi/2 \pm 1/k_0^2\). We construct the boxes so that each \(B_j\) has diameter \(\leq \delta_0\) and is foliated by a smooth family of stable curves \(\{W_{\xi}\}_{\xi \in \mathcal{E}_j} \subset \mathcal{W}^s\), each of whose elements completely crosses \(B_j\) in the approximate stable direction.

We decompose the smooth measure \(d\mu = \cos \varphi dm\) on \(B_j\) into \(d\mu = \hat{\mu}(d\xi)d\mu_{\xi}\), where \(\mu_{\xi}\) is the conditional measure of \(\mu\) on \(W_{\xi}\) and \(\hat{\mu}\) is the transverse measure on \(E_j\). We normalize the measures so that \(\hat{\mu}_{\xi}(W_{\xi}) = \int_{W_{\xi}} \cos \varphi dm_{W_{\xi}}\). Since the foliation is smooth, \(d\mu_{\xi} = \rho_{\xi} \cos \varphi dm_{W_{\xi}}\) where \(|\rho_{\xi}|C_{1}(W_{\xi}) \leq C\) for some constant independent of \(\xi\). Note that \(\hat{\mu}(E_j) \leq C\delta_0\) due to the transversality of curves in \(\mathcal{W}^s\) and \(\mathcal{W}^u\). Next we choose on each homogeneity strip \(\mathbb{H}_t, t \geq k_0\), a smooth foliation \(\{W_{\xi}\}_{\xi \in \mathcal{E}_t} \subset \mathcal{W}^s\) whose elements all have endpoints lying in the two boundary curves of \(\mathbb{H}_t\). We then decompose \(\mu\) on \(\mathbb{H}_t\) into \(d\mu = \hat{\mu}(d\xi)d\mu_{\xi}, \xi \in \mathcal{E}_t\), and \(d\mu_{\xi} = \rho_{\xi} \cos \varphi dm_{W_{\xi}}\) is normalized as above. By construction, \(\hat{\mu}(E_t) = \mathcal{O}(1)\). Given \(h \in C^1(M), \psi \in C^0(T^{-n}\mathcal{W}^s)\), since \(T^{-n}M = M \pmod{0}\), we have \(h(\psi) = \int_{M} h\psi d\mu = \int_{M} L^n h\psi \circ T^{-n} d\mu\). We split \(M = \bigcup_{\ell} M_{\ell}\) and integrate one \(\ell\) at a time.

\[
\int_{M_{\ell}} L^n h\psi \circ T^{-n} d\mu = \sum_{j} \int_{B_j} L^n h\psi \circ T^{-n} d\mu + \sum_{|t| \geq k_0} \int_{\mathbb{H}_t} L^n h\psi \circ T^{-n} d\mu
\]

\[
= \sum_{j} \int_{E_j} \int_{W_{\xi}} L^n h\psi \circ T^{-n} \rho_{\xi} d\mu_{\xi} d\hat{\mu}(\xi) + \sum_{|t| \geq k_0} \int_{E_t} \int_{W_{\xi}} L^n h\psi \circ T^{-n} \rho_{\xi} d\mu_{\xi} d\hat{\mu}(\xi)
\]

We change variables and estimate the integrals on one \(W_{\xi}\) at a time. Letting \(W_{\xi}^n\) denote the components of \(\mathcal{G}_n(W_{\xi})\) defined in Section 3.5 and recalling that \(J_{W_{\xi}} T^n\) denotes the stable Jacobian of \(T^n\) along the curve \(W_{\xi}^n\), we write

\[
\int_{W_{\xi}} L^n h\psi \circ T^{-n} \rho_{\xi} d\mu_{\xi} = \sum_{i} \int_{W_{\xi}^n} h\psi(J_{\mu} T^n)^{-1} J_{W_{\xi}^n} T^n \rho_{\xi} \circ T^n \cos \varphi \circ T^n d\mu_{\xi}
\]

\[
\leq \sum_{i} |h|_{W_{\xi}} |\cos |C_{\varphi}(W_{\xi}^n)| \cdot |(J_{\mu} T^n)^{-1} J_{W_{\xi}^n} T^n |C_{\varphi}(W_{\xi}^n)| \rho_{\xi} \circ T^n |C_{\varphi}(W_{\xi}^n)| \cos \varphi \circ T^n |C_{\varphi}(W_{\xi}^n)|.
\]

By (4.14), we have \(|\rho_{\xi} \circ T^n |C_{\varphi}(W_{\xi}^n)| \leq C|\rho_{\xi} |C_{\varphi}(W_{\xi})| \leq C\) for some uniform constant \(C\). The distortion bounds given by (H4), equation (3.15), imply that

(3.17) \(|(J_{\mu} T^n)^{-1} J_{W_{\xi}^n} T^n |C_{\varphi}(W_{\xi}^n)| \leq (1 + 2C_\theta) |(J_{\mu} T^n)^{-1} J_{W_{\xi}^n} T^n |C_{\varphi}(W_{\xi}^n)|\).

For \(W \in \mathcal{W}^s\), let \(\cos \psi\) denote the average value of \(\cos \varphi\) on \(W\). Note that there exists \(C_\epsilon > 0\), depending only on \(k_0\) and the uniform transversality of \(C^n(x)\) with the horizontal direction, such that \(C_\epsilon^{-1} \cos W \leq \cos \varphi(x) \leq C_\epsilon \cos W\) for all \(x \in W\).
Thus $|\cos \varphi T^n|_\infty \leq C \cos W_\xi$. Then since $|\cos \varphi T^n(x) - \cos \varphi T^n(y)| \leq d_W(T^n x, T^n y)$ for $x, y \in W^n_{\xi,i}$, we have $H^n_{\xi,i} \leq C \cos W_\xi^{1-p}$, where we have used (H1). If $W_\xi \subseteq \mathbb{H}_t$, then $\cos W_\xi \geq ct^{-2}$ while $|W_\xi| \leq C't^{-3}$ for uniform constants $c, C' > 0$, depending on the minimum angle between $C^s(x)$ and the horizontal. Thus since $p \leq 1/3$, we have $|\cos \varphi T^n|_{C^p(W^n_{\xi,i})} \leq C \cos W_\xi$ for some uniform constant $C$.

Gathering these estimates together, we have

\begin{equation}
\int_{W_\xi} \mathcal{L}^n h \psi \circ T^{-n} \rho_\xi \, d\mu_W \leq C|h|_w(|\psi|_\infty + H^n_{\psi}(\psi)) \cos(W_\xi) \sum_i |(J_\mu T^n)^{-1} J^n_{W_\xi,i} \, T^n|_{C^0(W^n_{\xi,i})},
\end{equation}

where $C$ is uniform in $T$ and $n$. We group the pieces $W^n_{\xi,i} \in G_\mathcal{L}(W_\xi)$ according to most recent long ancestor $W^n_{\xi,j} \in G_\mathcal{L}(W_\xi)$ as described in Section 3.5. Then splitting up the Jacobians according to times $k$ and using (H5), we have

\begin{equation}
\sum_i |(J_\mu T^n)^{-1} J^n_{W_\xi,i} \, T^n|_{C^0(W^n_{\xi,i})} \leq \sum_{i \in \mathcal{I}_n(W)} \eta^n |J^n_{W_\xi,i} \, T^n|_{C^0(W^n_{\xi,i})}
\end{equation}

where we have used Lemma 3.2(a) on each of the terms involving $\mathcal{I}_n(W^n_{\xi,j})$ from time $k$ to time $n$.

For each $k$, since $|W^n_{\xi,j}| \geq \delta_0/3$, we have by bounded distortion (H4),

\begin{equation}
\int_{W_\xi} \mathcal{L}^n h \psi \circ T^{-n} \rho_\xi \, d\mu_W \leq C|h|_w(|\psi|_\infty + H^n_{\psi}(\psi)) \left( \cos W_\xi + \sum_{k=1}^n (\eta s)^{n-k} \int_{W_\xi} (J_\mu T^n)^{-1} J^n_{W_\xi,j} \, T^n \, d\mu_W \right)
\end{equation}

for some uniform constant $C$. Thus

\begin{align*}
\int_{M_\ell} \mathcal{L}^n h \psi \circ T^{-n} \, d\mu &\leq C|h|_w(|\psi|_\infty + H^n_{\psi}(\psi)) \left( \sum_j \int_{E_j} \cos W_\xi \, \mu(d\xi) + \sum_{|t| \geq k_0} \int_{E_t} \cos W_\xi \, \mu(d\xi) ight) \\
&+ \sum_{j=1}^n (\eta s)^{n-k} \int_{B_j} (J_\mu T^n)^{-1} \, d\mu + \sum_{|t| \geq k_0} \sum_{k=1}^n (\eta s)^{n-k} \int_{B_t} (J_\mu T^n)^{-1} \, d\mu \\
&\leq C|h|_w(|\psi|_\infty + H^n_{\psi}(\psi)) \left( \sum_j \mu(E_j) + \sum_{|t| \geq k_0} t^{-2} \mu(E_t) + \sum_{k=1}^n (\eta s)^{n-k} \int_{M_\ell} (J_\mu T^n)^{-1} \, d\mu \right)
\end{align*}

where in the last line we have used the fact that $\cos W \leq C t^{-2}$ for $W \subseteq \mathbb{H}_t$. The first two sums are finite since there are only finitely many $E_j$ and $\mu(E_j)$ is of order $1$ for each $t$. Since there are only finitely many $M_\ell$, the two first sums remain finite when we sum over $\ell$. For the third sum, we sum over $\ell$ and use the fact that $\int_{M_\ell} (J_\mu T^n)^{-1} \, d\mu = 1$ for each $k \geq 1$. Thus the contribution from the third sum is uniformly bounded in $n$ using the fact that $\eta s < 1$ by (H5).

Several other properties of the spaces $\mathcal{B}$ and $\mathcal{B}_w$ proved in [DZ] do not need to be reproved since their proofs remain essentially unchanged. They are as follows.
Lemma 3.4 and items (i) and (ii) characterize the spaces $B$ in order to deduce the quasi-compactness of $L_{\gamma}$.

(i) ([DZ, Lemma 3.7]) $B$ contains piecewise Hölder continuous functions $h$ with exponent greater than $2\beta$ provided the discontinuities of $h$ are uniformly transverse to the stable cones $C^\alpha(x)$.

(ii) ([DZ, Lemma 2.1]) $\mathcal{L}$ is well-defined as a continuous linear operator on both $B$ and $B_w$. Moreover, there is a sequence of embeddings $C^\alpha(M) \hookrightarrow B \hookrightarrow B_w \hookrightarrow (C^\alpha(M))'$, for all $\gamma > 2\beta$.

(iii) ([DZ, Lemma 3.10]) The unit ball of $(B, \| \cdot \|_B)$ is compactly embedded in $(B_w, \cdot \|_w)$.

Lemma 3.4 and items (i) and (ii) characterize the spaces $B$ and $B_w$ as spaces of distributions containing all Hölder continuous and certain classes of piecewise Hölder continuous functions. The last item is necessary in order to deduce the quasi-compactness of $\mathcal{L}_T$ from the Lasota-Yorke inequalities given by Theorem 2.2.

There remains one final fact to establish. As mentioned earlier, since we identify $h \in C^1(M)$ with the measure $h\mu$ as an element of $B$, a priori Lebesgue measure may not be in $B$. The following Lemma shows that Lebesgue measure is in fact in $B$ and therefore so is $h\mu$ for any $h \in C^1(M)$.

**Lemma 3.5.** The function $(\cos \varphi)^{-1}$ is in $B$. Therefore, Lebesgue measure $\mu = (\cos \varphi)^{-1} \mu$ is also in $B$ and so is $h\mu$ for any $h \in C^1(M)$. Indeed, any piecewise Hölder continuous function as in item (i) above times Lebesgue belongs to $B$.

**Proof.** In order to show $(\cos \varphi)^{-1} \in B$, we must show that $(\cos \varphi)^{-1}$ can be approximated by functions $h \in C^1(M)$ in the $\| \cdot \|_B$ norm. Since $\| \cdot \|_B = \sup_h \| f \|_{C^0_k} \|_B$, our strategy will be to show that $\|(\cos \varphi)^{-1} \|_{C^0_k} \|_B \leq Ck^{-1/2}$ for some uniform constant $C$. We can then approximate $(\cos \varphi)^{-1}$ by 0 in homogeneity strips of sufficiently high index. More precisely, given $\varepsilon > 0$, we choose $K$ such that $CK^{-1/2} < \varepsilon$. Then on the remaining strips $k < K$, $(\cos \varphi)^{-1}$ has finite $C^1$-norm and satisfies the assumptions of [DZ, Lemma 3.7]. Thus we may find $f_\varepsilon \in C^1(M)$ as in the proof of that lemma such that

$$\|(\cos \varphi)^{-1} - f_\varepsilon\|_B \leq \sup_{k \geq K} \|(\cos \varphi)^{-1} \|_{C^0_k} \|_B + \sup_{k < K} \|(\cos \varphi)^{-1} - f_\varepsilon\|_{C^0_k} \|_B < 2\varepsilon,$$

proving that $(\cos \varphi)^{-1} \in B$.

It remains to prove the claim $\|(\cos \varphi)^{-1} \|_{C^0_k} \|_B \leq Ck^{-1/2}$. Choose $W \in \mathcal{W}^s$, $W \subset \mathbb{H}_k$, and let $\psi \in C^0(W)$ with $|\psi|_{W,\alpha,q} \leq 1$. Then

$$\int_W (\cos \varphi)^{-1} \psi dm_W \leq |(\cos \varphi)^{-1}|_{C^0(W)} |\psi|_{C^0(W)} |W| \leq |(\cos \varphi)^{-1}|_{C^0(W)} |W|^{1-\alpha}$$

since $|\psi|_{C^0(W)} \leq |W|^{-\alpha}$. On $\mathbb{H}_k$, we have $|W| \leq ck^{-3}$ and $|\cos \varphi|^{-1} \leq CK^2$ for some uniform constants $c$ and $C$ which depend only on the minimum angle of $C^\alpha(x)$ with the horizontal. Then, since $\alpha < 1/6$,

$$(3.20) \quad |(\cos \varphi)^{-1}|_{C^0(W)} |W|^{1-\alpha} \leq cCK^2k^{-3+3\alpha} \leq C'k^{-1/2}.$$

Taking the suprema over $W \subset \mathbb{H}_k$ and $\psi$ with $|\psi|_{W,\alpha,q} \leq 1$, we have $\|(\cos \varphi)^{-1} \|_{C^0_k} \|_B \leq C'k^{-1/2}$, completing the estimate on the strong stable norm.

To estimate the strong unstable norm, let $\varepsilon \leq \varepsilon_0$ and choose two curves in $\mathbb{H}_k$, $W^1, W^2 \in \mathcal{W}^s$, such that $d_W((W^1, W^2) \leq \varepsilon$. For $i = 1, 2$, let $\psi_i \in C^0(W^i)$ with $|\psi_i|_{C^0(W^i)} \leq 1$ and $d_q(\psi_1, \psi_2) \leq \varepsilon$.

Recalling the notation of Section 3.3 denote $W^i = \{G_{W^i}(r) = (r, \varphi_{W^i}(r)) : r \in I_{W^i}\}$, $i = 1, 2$ and note that by definition of $d_W((\cdot, \cdot)$, $W^1$ and $W^2$ can be put into one-to-one correspondence by a foliation of vertical line segments of length at most $\varepsilon$, except possibly near their endpoints. Denote by $U^i$ the single matched connected component of $W^i$ and by $V^i_j$ the at most 2 unmatched components of $W^i$. We let $\Theta : U^1 \rightarrow U^2$ denote the holonomy map along the vertical foliation. We estimate,

$$\int_{W^1} (\cos \varphi)^{-1} \psi_1 dm_W - \int_{W^2} (\cos \varphi)^{-1} \psi_2 dm_W = \sum_{i,j} \int_{V^i_j} (\cos \varphi)^{-1} \psi_1 dm_W$$

$$+ \int_{U^1} (\cos \varphi)^{-1} \psi_1 dm_W - \int_{U^2} (\cos \varphi)^{-1} \psi_2 dm_W.$$

We first estimate over the unmatched pieces $V^i_j$. Note that $|V^i_j| \leq C \varepsilon$ where $C$ depends only on the minimum angle of $C^\alpha(x)$ with the vertical. Recalling that $|\psi_i|_{C^0(W^i)} \leq 1$ and using (3.20) since $\beta \leq \alpha$, we
estimate

\begin{equation}
(3.22) \left| \sum_{i,j} \int_{V_j} \langle \cos \varphi \rangle^{-1} \psi_i \, dm_W \right| \leq \sum_{i,j} |V_j|^\beta |V_j|^{1-\beta} |\langle \cos \varphi \rangle^{-1}|_{C^0(V_j)} \leq C \varepsilon^\beta k^{-1/2}.
\end{equation}

To estimate the difference on the matched pieces \(U^j\), we change variables to \(U^1\) using \(\Theta\),

\[\int_{U^1} \langle \cos \varphi \rangle^{-1} \psi_1 \, dm_W - \int_{U^2} \langle \cos \varphi \rangle^{-1} \psi_2 \, dm_W = \int_{U^1} \langle \cos \varphi \rangle^{-1} \psi_1 - [(\cos \varphi)^{-1} \psi_2] \circ \Theta J \Theta \, dm_W \]

\[\leq \|U^1\| \|\langle \cos \varphi \rangle^{-1} \psi_1 - [(\cos \varphi)^{-1} \psi_2] \circ \Theta J \Theta\|_{C^0(U^1)}.
\]

To estimate the \(C^0\) norm of the test function, we split the difference into 3 terms and use the fact that \(|\psi_i|_{C^0} \leq 1,

\begin{equation}
(3.23) \|\langle \cos \varphi \rangle^{-1} \psi_1 - [(\cos \varphi)^{-1} \psi_2] \circ \Theta J \Theta\|_{C^0(U^1)} \leq \|\langle \cos \varphi \rangle^{-1} - (\cos \varphi)^{-1} \circ \Theta\|_{C^0(U^1)}
\end{equation}

+ \|\langle \cos \varphi \rangle^{-1}\|_{C^0(U^2)} \|\psi_1 - \psi_2 \circ \Theta\|_{C^0(U^1)} + \|\langle \cos \varphi \rangle^{-1}\|_{C^0(U^2)} \|1 - J \Theta\|_{C^0(U^1)}.
\]

For the first term above, note that for \(x \in U^1\), \(|\cos \varphi(x) - \cos \varphi \circ \Theta(x)| \leq d(x, \Theta(x)) \leq \min\{\varepsilon, Ck^{-3}\} \) for some uniform constant \(C > 0\). Thus

\[\|(\cos \varphi)^{-1}(x) - (\cos \varphi)^{-1} \circ \Theta(x)\| \leq \frac{d(x, \Theta(x))}{\cos \varphi(x) \cos \varphi \circ \Theta(x)} \leq C \varepsilon^\beta k^{-3(1-\beta)} k^{-4} \leq C' \varepsilon^\beta k^{-3/2},
\]

since \(\beta \leq 1/6\). To estimate the second term in (3.23), denote \(x \in U^1\) by \(x = G_{W^1}(r)\) for some \(r \in I_{W^1} \cap I_{W^2} =: I\). Then \(|\psi_1(x) - \psi_2 \circ \Theta(x)| = |\psi_1 \circ G_{W^1}(r) - \psi_2 \circ G_{W^2}(r)| \leq \varepsilon\) by definition of \(d_\eta(\cdot, \cdot)\). Thus

\[\|(\cos \varphi)^{-1}\|_{C^0(U^2)} \|\psi_1 - \psi_2 \circ \Theta\|_{C^0(U^1)} \leq Ck^2 \varepsilon.
\]

Finally, we estimate the third term of (3.23) by noting that

\[|1 - J \Theta| = \left| 1 - \frac{(\varphi_{W^1})^2}{\sqrt{1 + (\varphi_{W^2})^2}} \right| \leq |\varphi_{W^1} - \varphi_{W^2}| \leq \varepsilon,
\]

where we have used the fact that the derivative of \(\sqrt{1 + t^2}\), \(\frac{t}{\sqrt{1 + t^2}}\), is bounded by 1 for \(t \geq 0\).

Putting these 3 estimates together in (3.23), we estimate the norm on the matched pieces by

\[\int_{U^1} \langle \cos \varphi \rangle^{-1} \psi_1 \, dm_W - \int_{U^2} \langle \cos \varphi \rangle^{-1} \psi_2 \, dm_W \leq \|U^1\| C(\varepsilon^\beta k^{-3/2} + \varepsilon k^2 + \varepsilon k^2) \leq C' \varepsilon^\beta k^{-1},
\]

using the fact that \(|U^1| \leq Ck^{-3}\). This, combined with (3.22), yields the required estimate on the strong unstable norm. \(\square\)

4. Proof of Theorem 2.2

The proof of Theorem 2.2 relies on the following proposition.

**Proposition 4.1.** There exists \(C > 0\), depending only on (H1)-(H5), such that for any \(T \in \mathcal{F}, h \in \mathcal{B}\) and \(n \geq 0,\)

\begin{align*}
(4.1) &\quad \|L_T^nh\|_w \leq C \eta^n \|h\|_w \\
(4.2) &\quad \|L_T^nh\|_s \leq C (\theta_1^{(1-\alpha)n} + \Lambda^{-\eta n}) \eta^n \|h\|_s + C \eta^n \delta^{-\alpha} \|h\|_w \\
(4.3) &\quad \|L_T^nh\|_u \leq C \eta^n \Lambda^{-\beta n} \|h\|_u + C \eta^n C_3^n \|h\|_s
\end{align*}

**Proof of Theorem 2.2 given Proposition 4.1** Choose \(1 > \sigma > \eta \max\{\theta_1^{1-\alpha}, \Lambda^{-\eta}, \Lambda^{-\beta}\}\) and choose \(N \geq 0\) such that

\[\|L_T^N h\|_B = \|L_T^N h\|_s + b \|L_T^N h\|_u \leq \frac{\sigma^N}{2} \|h\|_s + C \delta_0^{-\alpha} \eta^N \|h\|_w + b \sigma^N \|h\|_u + b C \eta^N C_3^N \|h\|_s \leq \sigma^N \|h\|_B + C \delta_0 \eta^N \|h\|_w\]
providing \( b \) is chosen sufficiently small with respect to \( N \). This is the required inequality (2.3) for Theorem 2.2 which implies the essential spectrum of \( \mathcal{L}_T \) is less than \( \sigma \). Outside the disk of radius \( \sigma \), the spectrum of \( \mathcal{L}_T \) has finitely many eigenvalues, each with finite multiplicity. This follows using the compactness of the unit ball of \( B \) in \( \mathcal{B}_w \) [DZ, Lemma 3.10].

Despite the fact that \( \eta \) may be greater than 1, the spectral radius of \( \mathcal{L}_T \) equals 1. To see this, suppose \( z \in \mathbb{C}, |z| > 1 \), satisfies \( \mathcal{L}_T h = z h \) for some \( h \in \mathcal{B}, h \neq 0 \). For \( \psi \in \mathcal{C}^p(M) \), Lemma 3.4 implies that,

\[
|h(\psi)| = |z^{-n} \mathcal{L}_T^n h(\psi)| = |z^{-n} h(\psi \circ T^n)| \leq |z|^{-n} C |h|_\infty (|\psi|_\infty + H^p_n(\psi \circ T^n)) \to 0 \quad \text{as} \quad n \to \infty
\]

since \( H^p_n(\psi \circ T^n) \leq C_e \Lambda^{-pn} |\psi|_{\mathcal{C}^p(M)} \) by (4.6). Thus \( h = 0 \), contradicting the assumption on \( z \).

The characterization of the peripheral spectrum follows from Lemmas 5.1 and 5.2 of [DZ]. □

To prove Proposition 4.1, we fix \( T \in \mathcal{F} \) and prove the required Lasota-Yorke inequalities (4.1)-(4.3). It is shown in [DZ, Section 4] that \( \mathcal{L}_T \) is a continuous operator on both \( \mathcal{B} \) and \( \mathcal{B}_w \) so that it suffices to prove the inequalities for \( h \in \mathcal{C}^1(M) \). They extend to the completions by continuity. Since these estimates are similar to those in [DZ], our purpose in repeating them is to show how they depend explicitly on the uniform constants given by (H1)-(H5) and do not require additional information.

### 4.1. Estimating the weak norm.

Let \( h \in \mathcal{C}^1(M) \), \( W \in \mathcal{W}^s \) and \( \psi \in \mathcal{C}^p(W) \) such that \( |\psi|_{W,0,p} \leq 1 \). For \( n \geq 0 \), we write

\[
\int_W \mathcal{L}^n h \psi \, dm_W = \sum_{W_n \in \mathcal{G}_n(W)} \int_{W_n} h_{W_n} T^n \psi \circ T^n \, dm_W
\]

where \( h_{W_n} T^n \) denotes the Jacobian of \( T^n \) along \( W_n \).

Using the definition of the weak norm on each \( W_n \), we estimate (4.4) by

\[
\int_W \mathcal{L}^n h \psi \, dm_W \leq \sum_{W_n \in \mathcal{G}_n} |h|_w (|J_{\mu} T^n|^{-1} J_{W_n} T^n \psi \circ T^n |_{\mathcal{C}^p(W^n)}).
\]

For \( x, y \in W_n \), we use (H1) to estimate,

\[
\frac{|\psi(T^n x) - \psi(T^n y)|}{dW(T^n x, T^n y)^p} \leq |\psi|_{\mathcal{C}^p(W)} |J_{W_n} T^n \psi \circ T^n |_{\mathcal{C}^p(W^n)} \leq C_e \Lambda^{-pn} |\psi|_{\mathcal{C}^p(W)}
\]

so that \( |\psi \circ T^n |_{\mathcal{C}^p(W^n)} \leq C_e |\psi|_{\mathcal{C}^p(W)} \leq C_e \). We use this estimate together with (H5) and (3.17) to bound (4.5) by

\[
\int_W \mathcal{L}^n h \psi \, dm_W \leq C_e (1 + 2C_d \eta)^n |h|_w \sum_{W_n \in \mathcal{G}_n} |J_{W_n} T^n |_{\mathcal{C}^p(W^n)} \leq C' \eta^n |h|_w,
\]

where \( C' = C_e (1 + 2C_d) C_2 \) and we have used Lemma 3.2(b) for the last inequality. Taking the supremum over all \( W \in \mathcal{W}^s \) and \( \psi \in \mathcal{C}^p(W) \) with \( |\psi|_{W,0,p} \leq 1 \) yields (4.4) expressed with uniform constants given by (H1)-(H5).

### 4.2. Estimating the strong stable norm.

Let \( W \in \mathcal{W}^s \) and let \( W_n \) denote the elements of \( \mathcal{G}_n(W) \) as defined above. For \( \psi \in \mathcal{C}^q(W), |\psi|_{W,a,q} \leq 1 \), define \( \overline{\psi}_i = |W_i|^{-1} \int_{W_i} \psi \circ T^n \, dm_W \). Using equation (4.4), we write

\[
\int_W \mathcal{L}^n h \psi \, dm_W = \sum_i \int_{W_i} h_{W_i} \frac{J_{W_i} T^n}_{J_{\mu} T^n} (\psi \circ T^n - \overline{\psi}_i) \, dm_W + \overline{\psi}_i \int_{W_i} h_{W_i} \frac{J_{W_i} T^n}_{J_{\mu} T^n} \, dm_W.
\]

To estimate the first term of (4.7), we first estimate \( |\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)} \). If \( H^q_W(\psi) \) denotes the Hölder constant of \( \psi \) along \( W_i \), then equation (4.6) implies

\[
\frac{|\psi(T^n x) - \psi(T^n y)|}{dW(x, y)^q} \leq C_e \Lambda^{-qn} H^q_W(\psi)
\]
for any $x, y \in W^n_i$. Since $\overline{\psi}_i$ is constant on $W^n_i$, we have $H^q_{W^n_i}(\psi \circ T^n - \overline{\psi}_i) \leq C_\epsilon \Lambda^{-qn} H^q_W(\psi)$. To estimate the $C^0$ norm, note that $\overline{\psi}_i = \psi \circ T^n(y_i)$ for some $y_i \in W^n_i$. Thus for each $x \in W^n_i$,

$$|\psi \circ T^n(x) - \overline{\psi}_i| = |\psi \circ T^n(x) - \psi \circ T^n(y_i)| \leq H^q_{W^n_i}(\psi \circ T^n)|W^n_i|^q \leq C_\epsilon H^q_W(\psi) \Lambda^{-nq}.$$  

This estimate together with (4.8) and the fact that $|\varphi|_{W, \alpha, q} \leq 1$, implies

$$(4.9) \quad |\psi \circ T^n - \overline{\psi}_i|_{C^0(W^n_i)} \leq C_\epsilon \Lambda^{-nq}|\psi|_{C^0(W)} \leq C_\epsilon \Lambda^{-nq}|W|^{-\alpha}.$$  

We apply (3.17), (4.9) and the definition of the strong stable norm to the first term of (4.7),

$$\sum_k \int_{W^n_i} h \frac{J_{W^n_i}^n}{J_{\mu T^n}} (\psi \circ T^n - \overline{\psi}_i) \, dm_W \leq (1 + 2C_d) C_\epsilon \sum_k \|h\|_s \frac{|W^n_i|^{\alpha}}{|W|^{\alpha}} \frac{|J_{\mu T^n}|}{|J_{\mu T^n}|} \Lambda^{-nq}$$

$$(4.10) \quad \leq \eta^n (1 + 2C_d) C_\epsilon \Lambda^{-nq}\|h\|_s \sum_k \frac{|W^n_i|^{\alpha}}{|W|^{\alpha}} |J_{W^n_i}^n|_{C^0(W^n_i)} \leq C_4 \eta^n \Lambda^{-nq}\|h\|_s,$$

where $C_4 = (1 + 2C_d) C_\epsilon C_2^{-1-\alpha}$ and in the second line we have used (H5) and Lemma 3.2(c) with $\varsigma = \alpha$. For the second term of (3.7), we use the fact that $|\overline{\psi}_i| \leq |W|^{-\alpha}$ since $|\psi|_{W, \alpha, q} \leq 1$. Recall the notation introduced before the statement of Lemma 3.2. Grouping the pieces $W^n_i \in G_n(W)$ according to most recent long ancestors $W^k_j \in L_k(W)$, we have

$$\sum_i |W|^{-\alpha} \int_{W^n_i} h \frac{J_{W^n_i}^n}{J_{\mu T^n}} \, dm_W = \sum_{k=1}^n \sum_{j \in L_k(W)} \sum_{i \in L_n(W^k_j)} |W|^{-\alpha} \int_{W^n_i} h \frac{J_{W^n_i}^n}{J_{\mu T^n}} \, dm_W$$

$$+ \sum_{i \in L_n(W)} |W|^{-\alpha} \int_{W^n_i} h \frac{J_{W^n_i}^n}{J_{\mu T^n}} \, dm_W$$

where we have split up the terms involving $k = 0$ and $k \geq 1$. We estimate the terms with $k \geq 1$ by the weak norm and the terms with $k = 0$ by the strong stable norm. Using again (3.17) and (H5),

$$\sum_i |W|^{-\alpha} \int_{W^n_i} h \frac{J_{W^n_i}^n}{J_{\mu T^n}} \, dm_W \leq \eta^n (1 + 2C_d) \sum_{k=1}^n \sum_{j \in L_k(W)} \sum_{i \in L_n(W^k_j)} |W|^{-\alpha} |h|_w |J_{W^n_i}^n|_{C^0(W^n_i)}$$

$$+ \eta^n (1 + 2C_d) \sum_{i \in L_n(W)} \frac{|W^n_i|^{\alpha}}{|W|^{\alpha}} \|h\|_s |J_{W^n_i}^n|_{C^0(W^n_i)}.$$  

In the first sum above corresponding to $k \geq 1$, we write

$$|J_{W^n_i}^n|_{C^0(W^n_i)} \leq |J_{W^n_i}^n T^n-k|_{C^0(W^n_i)} |J_{W^k_j}^k T^k|_{C^0(W^k_j)}.$$  

Thus using Lemma 3.2(a) from time $k$ to time $n$,

$$\sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in L_n(W^k_j)} |W|^{-\alpha} |J_{W^n_i}^n|_{C^0(W^n_i)} \leq \sum_{k=1}^n \sum_{j \in L_k(W)} |J_{W^k_j}^k T^k|_{C^0(W^k_j)} |W|^{-\alpha} \sum_{i \in L_n(W^k_j)} |J_{W^n_i}^n T^n-k|_{C^0(W^n_i)}$$

$$\leq 3d_0^{-\alpha} \sum_{k=1}^n \sum_{j \in L_k(W)} |J_{W^k_j}^k T^k|_{C^0(W^k_j)} \frac{|W^k_j|^{\alpha}}{|W|^{\alpha}} C_1 \theta_*^{n-k},$$

since $|W^k_j| \geq \delta_0/3$. The inner sum is bounded by $C_2^{-\alpha}$ for each $k$ by Lemma 3.2(c) while the outer sum is bounded by $C_1/(1 - \theta_*)$ independently of $n$. Finally, for the sum corresponding to $k = 0$, since

$$|J_{W^n_i}^n T^n|_{C^0(W^n_i)} \leq (1 + C_d) |T^n W^n_i||W^n_i|^{-1} \leq (1 + C_d) |J_{W^n_i}^n T^n|_{C^0(W^n_i)},$$

we get

$$\sum_{i \in L_n(W)} |W|^{-\alpha} \int_{W^n_i} h \frac{J_{W^n_i}^n}{J_{\mu T^n}} \, dm_W \leq \eta^n \Lambda^{-nq}\|h\|_s + 3d_0^{-\alpha} \sum_{k=1}^n \sum_{j \in L_k(W)} |J_{W^k_j}^k T^k|_{C^0(W^k_j)} \frac{|W^k_j|^{\alpha}}{|W|^{\alpha}} C_1 \theta_*^{n-k},$$

and the proof of Lemma 2.6 is complete.
we use Jensen’s inequality and Lemma 3.2(a) to estimate,
\[
\sum_{i \in \mathcal{L}_n(W)} |W_i^n|^\alpha |J_{W_i^n} T^n|_{C^0(W^n)} \leq (1 + C_d) \left( \sum_{i \in \mathcal{L}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \right)^{1-\alpha} \leq (1 + C_d) C_1 \theta_n^{(1-\alpha)}.
\]

Gathering these estimates together, we have
\[
(W_{11}) \quad \sum_i |W|^{-\alpha} \left| \int_{W_i^n} h(J_{n \tilde{W}}^{-1} J_{W_i^n} T^n) \ dm_W \right| \leq C \| \eta \|_s \theta_n^{(1-\alpha)},
\]

where \(C_5 = 3(1 + 2C_d)C_1 \theta_n^{(1-\alpha)} / (1 - \theta_s)\) and \(C_6 = (1 + 2C_d)^2 C_1\). Putting together (4.10) and (4.11) proves (4.2),
\[
\| \mathcal{L} \|_s \leq C' \eta \left( \Lambda^{-q_s} + \theta_n^{(1-\alpha)} \right) \| \eta \|_s + C' \eta \| \eta \|_s \| \tilde{W} \|_s,
\]

with \(C' = \max\{C_4, C_5, C_6\}\), a uniform constant depending only on (H1)-(H5).

4.3. Estimating the strong unstable norm. Fix \(\varepsilon \leq \varepsilon_0\) and consider two curves \(W^1, W^2 \in W^s\) with \(d_{TV}(W^1, W^2) \leq \varepsilon\). For \(n \geq 1\), we describe how to partition \(T^{-n} W^\ell\) into “matched” pieces \(U^\ell_i\) and “unmatched” pieces \(V^\ell_i\), \(\ell = 1, 2\). In the what follows, we use \(C_\ell\) to denote a transversality constant which depends only on the minimum angle between various transverse directions: the minimum angle between \(\mathcal{C}^u(x)\) and \(\mathcal{C}^s(x)\), between \(\mathcal{S}^T_n\) and \(\mathcal{C}^s(x)\), and between \(\mathcal{C}^s(x)\) and the vertical and horizontal directions.

Let \(\omega\) be a connected component of \(W^1 \setminus \mathcal{S}^T_n\) such that \(T^{-n} \omega \in \mathcal{G}_n(W)\). To each point \(x \in T^{-n} \omega\), we associate a smooth curve \(\gamma_x \in \tilde{W}^u\) of length at most \(C_1 \mathcal{C}_v \Lambda^{-n} \varepsilon\) such that its image \(T^n \gamma_x\), if not cut by a singularity or the boundary of a homogeneity strip, will have length \(C_1 \varepsilon\). By (H2), \(T^n \gamma_x \in \tilde{W}^u\) for each \(i \geq 0\).

Doing this for each connected component of \(W^1 \setminus \mathcal{S}^T_n\), we subdivide \(W^1 \setminus \mathcal{S}^T_n\) into a countable collection of subintervals of points for which \(T^n \gamma_x\) intersects \(W^2 \setminus \mathcal{S}^T_n\) and subintervals for which this is not the case. This in turn induces a corresponding partition on \(W^2 \setminus \mathcal{S}^T_n\).

We denote by \(V^\ell_1\) the pieces in \(T^{-n} W^\ell\) which are not matched up by this process and note that the images \(T^n V^\ell_1\) occur either at the endpoints of \(W^\ell\) or because the curve \(\gamma_x\) has been cut by a singularity. In both cases, the length of the curves \(T^n V^\ell_1\) can be at most \(C_1 \varepsilon\) due to the uniform transversality of \(\mathcal{S}^T_n\) with \(C^u(x)\) and of \(C^s(x)\) with \(C^a(x)\).

In the remaining pieces the foliation \(\{T^n \gamma_x\}_{x \in T^{-n} W^1}\) provides a one to one correspondence between points in \(W^1\) and \(W^2\). We partition these pieces in such a way that the lengths of their images under \(T^{-i}\) are less than \(\delta_0\) for each \(0 \leq i \leq n\) and the pieces are pairwise matched by the foliation \(\{\gamma_x\}\). We call these matched pieces \(\tilde{U}^\ell_1\) and note that \(T^n \tilde{U}^\ell_1 \in \mathcal{G}_n(W^\ell)\) for each \(i = 0, 1, \ldots, n\). For convenience, we further trim the \(\tilde{U}^\ell_1\) to pieces \(U^\ell_1\) so that \(U^\ell_1\) and \(U^\ell_2\) are both defined on the same arclength interval \(I_j\). The at most two components of \(T^n (U^\ell_1 \setminus U^\ell_2)\) have length less than \(C_1 \varepsilon\) due to the uniform transversality of \(C^s(x)\) with the vertical direction. We attach these trimmed pieces to the adjacent \(U^\ell_i\) or \(V^\ell_k\) as appropriate so as not to create any additional components in the partition.

We further relabel any pieces \(U^\ell_j\) as \(V^\ell_j\) and consider them unmatched if for some \(i, 0 \leq i \leq n, |T^i \gamma_x| > 2|T^n U^\ell_j|\), i.e. we only consider pieces matched if at each intermediate step, the distance between them is at most of the same order as their length. We do this in order to be able to apply Lemma 3.3 to the matched pieces. Notice that since the distance between the curves at each intermediate step is bounded by \(C_1 C_v \varepsilon\) and due to the uniform contraction of stable curves going forward, we have \(|T^n V^\ell_k| \leq C_1 C_2 \varepsilon\) for all such pieces considered unmatched by this last criterion.

In this way we write \(W^\ell = (U^\ell \cup U^\ell_1) \cup (U^\ell_2 \cup V^\ell_k)\). Note that the images \(T^n V^\ell_k\) of the unmatched pieces must have length \(\leq C_v \varepsilon\) for some uniform constant \(C_v\) while the images of the matched pieces \(U^\ell_j\) may be long or short.
Recalling the notation of Section 3.3 we have arranged a pairing of the pieces $U_j^\ell$ with the following property:

$$U_j^1 = G_{U_j^1}(I_j) = \{(r, \varphi_{U_j^1}(r)) : r \in I_j\},$$

(4.12)

then $U_j^2 = G_{U_j^2}(I_j) = \{(r, \varphi_{U_j^2}(r)) : r \in I_j\},$

so that the point $x = (r, \varphi_{U_j^1}(r)) \in U_j^1$ can associated with the point $\bar{x} = (r, \varphi_{U_j^2}(r)) \in U_j^2$ by the vertical line $\{(r, s) \in [-\pi/2, \pi/2]\}$, for each $r \in I_j$. In addition, the $U_j^\ell$ satisfy the assumptions of Lemma 3.3.

Given $\psi_\ell$ on $W^\ell$ with $|\psi_\ell|_{W_{\ell,0,p}} \leq 1$ and $d_\eta(\psi_1, \psi_2) \leq \varepsilon$, with the above construction we must estimate

$$\left| \int_{W^1} \mathcal{L}^n h \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n h \psi_2 \, dm_W \right| \leq \sum_{j} \left| \int_{U_j^1} h(J_\mu T^n)^{-1} J_{V_j^1} T^n \psi_1 \, dm_W - \int_{U_j^2} h(J_\mu T^n)^{-1} J_{V_j^2} T^n \psi_2 \, dm_W \right|$$

(4.13)

$$+ \sum_{j} \left| \int_{U_j^2} h(J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \, dm_W - \int_{U_j^1} h(J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \, dm_W \right|$$

We do the estimate over the unmatched pieces $V_j^\ell$ first using the strong stable norm. Note that by (4.6), $|\psi_\ell \circ T^n|_{C^\alpha(W^\ell)} \leq C_\varepsilon |\psi_\ell|_{C^\alpha(W^\ell)} \leq C_\varepsilon$. We estimate as in Section 4.2, using the fact that $|T^n V_j^\ell| \leq C_\varepsilon$, as noted above,

$$\sum_{\ell,k} \left| \int_{V_j^\ell} h(J_\mu T^n)^{-1} J_{V_j^\ell} T^n \psi_\ell \, dm_W \right| \leq C_\varepsilon(1 + 2C_\varepsilon) \eta^n \left\| h \right\|_s \sum_{\ell,k} \left| V_j^\ell \right| \alpha \left\| J_{V_j^\ell} T^n \right\|_{C^\alpha(V_j^\ell)}$$

(4.14)

$$\leq C_\varepsilon \eta^n \left\| h \right\|_s \sum_{\ell,k} \left| J_{V_j^\ell} T^n \right|_{C^\alpha(V_j^\ell)} \leq 2C_\varepsilon \eta^n \left\| h \right\|_s C_3,$$

with $C_\varepsilon = C_\varepsilon(1 + 2C_\varepsilon)^2 C_\varepsilon$, where we have applied Lemma 3.2(d) with $\zeta = 1 - \alpha > \eta_0$ since there are at most two $V_j^\ell$ corresponding to each element $W_{i,n} \in G_n(W^\ell)$ as defined in Section 5.5 and $|V_j^\ell T^n|_{C^\alpha(V_j^\ell)} \leq |J_{W_{i,n}} T^n|_{C^\alpha(W_{i,n})}$ whenever $V_j^\ell \subset W_{i,n}$.

Next, we must estimate

$$\sum_{j} \left| \int_{U_j^1} h(J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \, dm_W - \int_{U_j^2} h(J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \, dm_W \right| .$$

We fix $j$ and estimate the difference. Define

$$\phi_j = ((J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \circ T^n) \circ G_{U_j^1} \circ G_{U_j^2}^{-1}.$$ 

The function $\phi_j$ is well-defined on $U_j^2$ and we can write,

$$\left| \int_{U_j^1} h(J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \, dm_W - \int_{U_j^2} h(J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \, dm_W \right|$$

(4.15)

$$\leq \left| \int_{U_j^1} h(J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \, dm_W - \int_{U_j^2} h(\phi_j) \right| + \left| \int_{U_j^2} h(\phi_j - (J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \circ T^n) \right| .$$

We estimate the first term on the right hand side of (4.15) using the strong unstable norm. Using (H5), 3.17 and (4.6),

$$|(J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \circ T^n|_{C^\alpha(U_j^1)} \leq C_\varepsilon(1 + 2C_\varepsilon) \eta^n |J_{U_j^1} T^n|_{C^\alpha(U_j^1)}.$$
Notice that

\[
|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{C^1(U_j^2)} \leq \sup_{r \in U_j^2} \frac{\sqrt{1 + (d\varphi_{U_j^1})^2}}{\sqrt{1 + (d\varphi_{U_j^2})^2}} \leq \sqrt{1 + \Gamma^2} =: C_g,
\]

where \( \Gamma \) is the maximum slope of curves in \( W^s \) given by (H2). Using this, we estimate as in (4.16),

\[
|\phi_j|_{C^0(U_j^2)} \leq C_g C_{e}(1 + 2C_d) \eta^0 |J_{U_j^1}^n T^n|_{C^0(U_j^2)}.
\]

By the definition of \( \phi_j \) and \( d_q(\cdot, \cdot) \),

\[
d_q((J_\mu T^n)^{-1} J_{U_j^1}^n \psi_1 \circ T^n, \phi_j) = \left| \left( (J_\mu T^n)^{-1} J_{U_j^1}^n \psi_1 \circ T^n \right) \circ G_{U_j^2} \right|_{C^0(I_j)} = 0.
\]

By Lemma 3.3(a), we have \( d_{W_*}(U_{j_1}^1, U_{j_2}^2) \leq C_g \Lambda^{-n} \varepsilon =: \varepsilon_1 \). In view of (4.16) and following, we renormalize the test functions by \( R_j = C_7 \eta^0 |J_{U_j^1}^n T^n|_{C^0(U_j^1)} \) where \( C_7 = C_g C_{e}(1 + 2C_d) \). Then we apply the definition of the strong unstable norm with \( \varepsilon_1 \) in place of \( \varepsilon \). Thus,

\[
\sum_j \left| \int_{U_j^1} h J_{U_j}^n T^n \psi_1 \circ T^n - \int_{U_j^2} h \phi_j \right| \leq C_7 \Lambda^{-n} \eta^0 \varepsilon_1 \sum_j |J_{U_j^1}^n T^n|_{C^0(U_j^1)}
\]

where the sum is \( \leq C_g \) by Lemma 3.2(b) since there is at most one matched piece \( U_{j_1}^1 \) corresponding to each element \( W_{j_1}^1 \in \mathcal{G}_n(W^1) \) and \( |J_{U_j^1}^n T^n|_{C^0(U_j^1)} \leq |J_{W_{j_1}^1, n} T^n|_{C^0(W_{j_1}^1)} \) whenever \( U_{j_1}^1 \subset W_{j_1}^1 \).

It remains to estimate the second term in (4.15) using the strong stable norm.

\[
\left| \int_{U_j^2} h \phi_j - (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \psi_2 \circ T^n \right| \leq \|h\|_{s} |U_j^2|^\alpha |\phi_j - (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \psi_2 \circ T^n|_{C^0(U_j^2)}.
\]

In order to estimate the \( C^0 \)-norm of the function in (4.19), we split it up into two differences. Since \( |G_{U_j^i}|_{C^1} \leq C_g \) and \( |G_{U_j^i}^{-1}|_{C^1} \leq 1, \ell = 1, 2, \) we write

\[
|\phi_j - ((J_\mu T^n)^{-1} J_{U_j^2}^n T^n) \cdot \psi_2 \circ T^n|_{C^0(U_j^2)} \leq \left| \left( (J_\mu T^n)^{-1} J_{U_j^1}^n T^n \right) \circ G_{U_j^1} - \left( (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \right) \circ G_{U_j^2} \right|_{C^0(I_j)}
\]

\[
\leq \left| \left( (J_\mu T^n)^{-1} J_{U_j^1}^n T^n \right) \circ G_{U_j^1} \left( \psi_1 \circ T^n G_{U_j^1} - \psi_2 \circ T^n G_{U_j^2} \right) \right|_{C^0(I_j)}
\]

\[
+ \left| \left( (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \right) \circ G_{U_j^2} \left( G_{U_j^1} - (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \right) \circ G_{U_j^2} \right|_{C^0(I_j)}
\]

\[
\leq C_g (1 + 2C_d) |(J_\mu T^n)^{-1} J_{U_j^1}^n T^n|_{C^0(U_j^1)} \left| \psi_1 \circ T^n G_{U_j^1} - \psi_2 \circ T^n G_{U_j^2} \right|_{C^0(I_j)}
\]

\[
+ C_g C_{e} \left| (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \right|_{C^0(U_j^2)} \left| G_{U_j^1} - (J_\mu T^n)^{-1} J_{U_j^2}^n T^n \right|_{C^0(I_j)}
\]

To bound the two differences above, we need the following lemma.

**Lemma 4.2.** There exist constants \( C_8, C_9 > 0 \), depending only on (H1)-(H5), such that,

(a) \( |(J_\mu T^n)^{-1} J_{U_j^1}^n T^n|_{C^0(U_j^1)} \leq C_8 |(J_\mu T^n)^{-1} J_{U_j^2}^n T^n|_{C^0(U_j^2)}^{\frac{1}{3} - q} \);

(b) \( |\psi_1 \circ T^n G_{U_j^1} - \psi_2 \circ T^n G_{U_j^2}|_{C^0(I_j)} \leq C_9 \varepsilon^{p-q} \).

We postpone the proof of the lemma to Section 4.3.1 and show how this completes the estimate on the strong unstable norm. Note that using (4.24), we may replace \( |J_{U_j^1}^n T^n|_{C^0(U_j^1)} \) by \( C |J_{U_j^1}^n T^n|_{C^0(U_j^1)} \) where it appears in our estimates for some uniform constant \( C \). Starting from (4.19), we apply Lemma 4.2 to (4.20)
to obtain,
\[
\sum_j \left| \int_{U^2_j} h(\phi_j - (J_\mu T^n)^{-1} J_{U^1_j} T^n \psi_2 \circ T^n) \, dm_W \right|
\]
(4.21)
\[
\leq C \|h\|_s \sum_j |U^2_j|^\alpha |(J_\mu T^n)^{-1} J_{U^1_j} T^n|_{C^0(U^2_j)} \varepsilon^{p-q} \leq \bar{C} \eta^n \|h\|_s \varepsilon^{p-q} \sum_j |J_{U^1_j} T^n|_{C^0(U^2_j)},
\]
for some uniform constant $\bar{C}$ where again the sum is finite as in (4.18). This completes the estimate on the second term in (4.15). Now we use this bound, together with (4.14) and (4.18) to estimate (4.13)
\[
\left| \int_{W^1} L^n h \psi_1 \, dm_W - \int_{W^2} L^n h \psi_2 \, dm_W \right| \leq C C_2 \eta^n \|h\|_s \varepsilon^{p-q} + C \|h\|_s L^{-\beta} \eta^n \varepsilon^\beta + C \eta^n \|h\|_s \varepsilon^{p-q},
\]
where again $C$ depends only on (H1)-(H5) through the estimates above. Since $p-q \geq \beta$ and $\alpha \geq \beta$, we divide through by $\varepsilon^\beta$ and take the appropriate suprema to complete the proof of (4.3).

4.3.1. **Proof of Lemma 4.2** First we prove the following general fact and then use it to prove Lemma 4.2

**Lemma 4.3.** Let $(N,d)$ be a metric space and let $0 < r < s \leq 1$. Suppose $g_1, g_2 \in C^s(N,\mathbb{R})$ satisfy $|g_1 - g_2|_{C^r(N)} \leq D_1 \varepsilon^s$ for some constant $D_1 > 0$. Then $|g_1 - g_2|_{C^r(N)} \leq 3 \varepsilon^{s-r} \max\{D_1, H^s(g_1) + H^s(g_2)\}$, where $H^s(\cdot)$ denotes the Hölder constant with exponent $s$ on $N$.

**Proof.** Since $| \cdot |_{C^r(N)} = | \cdot |_{C^0(N)} + H^r(\cdot)$, we must estimate $H^r(g_1 - g_2)$. Let $x, y \in N$. Then on the one hand, since $|g_1 - g_2| \leq D_1 \varepsilon^s$, we have
\[
\frac{|(g_1(x) - g_2(x)) - (g_1(y) - g_2(y))|}{d(x,y)^r} \leq 2D_1 \varepsilon^s d(x,y)^{-r}
\]
On the other hand, using the fact that $g_1, g_2 \in C^s(N)$, we have
\[
\frac{|(g_1(x) - g_2(x)) - (g_1(y) - g_2(y))|}{d(x,y)^r} \leq (H^s(g_1) + H^s(g_2))d(x,y)^{s-r}.
\]
These two estimates together imply that the Hölder constant of $g_1 - g_2$ is bounded by
\[
H^r(g_1 - g_2) \leq \sup_{x,y \in N} \min\{2D_1 \varepsilon^s d(x,y)^{-r}, (H^s(g_1) + H^s(g_2))d(x,y)^{s-r}\}.
\]
This expression is maximized when $2D_1 \varepsilon^s d(x,y)^{-r} = (H^s(g_1) + H^s(g_2))d(x,y)^{s-r}$, i.e., when $d(x,y) = \varepsilon \left( \frac{2D_1}{H^s(g_1) + H^s(g_2)} \right)^{1/s}$. Thus the Hölder constant of $g_1 - g_2$ satisfies
\[
H^r(g_1 - g_2) \leq \varepsilon^{s-r} \left(2D_1\right)^{1-\frac{s}{r}} \left(\frac{H^s(g_1) + H^s(g_2)}{2D_1}\right)^{\frac{s}{r}}.
\]

**Proof of Lemma 4.2(a).** Throughout the proof, for ease of notation we write $J^n_\mu$ for $(J_\mu T^n)^{-1} J_{U^1_j} T^n$.

For any $r \in I_j$, $x = G_{U^1_j}(r)$ and $\bar{x} = G_{U^2_j}(r)$ lie on a common vertical segment. By the construction at the beginning of Section 4.3, $U^1_j, U^2_j$ lie in two homogeneous stable curves $\bar{U}^1_j$ and $\bar{U}^2_j$ which are connected by the foliation $\{\gamma_x\}$. Thus $x^* := \gamma_x \cap \bar{U}^2_j$ is uniquely defined for all $x \in U^1_j$. Then $T^n(x)$ and $T^n(x^*)$ lie on the element $T^n \gamma_x \in W^u$ which intersects $W^1$ and $W^2$ and has length at most $C_t \varepsilon$. By (3.9) and Lemma 3.3(b),
\[
|J^n_\mu(x) - J^n_\mu(x^*)| \leq C_d C_0 |J^n_\mu|_{C^0(U^2_j)} (d_W(T^n x, T^n x^*))^{1/3} + \theta(T^n x, T^n x^*),
\]
where $\theta(T^n x, T^n x^*)$ is the angle between the tangent line to $W^1$ at $T^n x$ and the tangent line to $W^2$ at $T^n x^*$. Let $y \in W^2$ be the unique point in $W^2$ which lies on the same vertical segment as $T^n x$. Since by assumption $d_W(W^1, W^2) \leq \varepsilon$, we have $\theta(T^n x, y) \leq \varepsilon$. Due to the uniform transversality of curves in $W^u$ and $W^s$ and the fact that $W^2$ is the graph of a $C^2$ function with $C^2$ norm bounded by $B$ from (H2), we have $\theta(y, T^n x^*) \leq B C_t \varepsilon$ and so $\theta(T^n x, T^n x^*) \leq (1 + B C_t) \varepsilon$. Thus
\[
|J^n_\mu(x) - J^n_\mu(x^*)| \leq C_d C_0 (C_t + 1 + B C_t) \varepsilon^{1/3} |J^n_\mu|_{C^0(U^2_j)},
\]
(4.22)
Also, by (3.8), since \(x^*\) and \(\bar{x}\) are both on \(\overline{U^2_j}\), we have \(|J^n_2(x^*) - J^n_2(\bar{x})| \leq C_2 |J^n_2|_{C^0(U^2_j)} d_W(x^*, \bar{x})^{1/3}\). Putting this together with (4.22) and using the fact that \(d_W(x^*, \bar{x}) \leq C_1 \varepsilon\) by the transversality of \(\gamma_x\) with \(W^s\) yields,

\[
|J^n_1(x) - J^n_2(\bar{x})| \leq C|\varepsilon|^{1/3}J^n_2|_{C^0(U^2_j)},
\]

where \(C = C_2 C_0 (2C_1 + 1 + BC_1)\).

Now using the fact that \(|G_{U^j_i}|_{C^1(I_j)} \leq C_1\) from (4.17), we apply Lemma 4.3 with \(D_1 = C_1|J^n_2|_{C^0(U^2_j)}\) and \(g_i = J_i^n \circ G_{U^j_i}, i = 1, 2\). By (4.22), we have

\[
|J^n_1|_{C^0(U^j_i)} \leq (1 + C|\varepsilon|^{1/3})J^n_2|_{C^0(U^2_j)},
\]

and invoking (3.8), we complete the proof of (a).

Proof of (b). Let \(\varphi_{W^\ell}\) be the function whose graph is \(W^\ell\), for \(r \in I_{W^\ell}\), and set \(f_j = G_{W^\ell}^{-1} \circ T^n \circ G_{U^j_i}\), \(k = 1, 2\). Notice that since \(|G_{W^\ell}^{-1}|_{C^1} \leq 1\) and \(|G_{U^j_i}|_{C^1} \leq C_1\), and due to the uniform contraction along stable curves, we have \(\text{Lip}(f_j) \leq C_1\), where \(C_1\) is independent of \(W^\ell, T, j\). We may assume that \(f_j \subset I_{W^1} \cap I_{W^2}\) since if not, by the transversality of \(C^u(x)\) and \(C^s(x)\), we must be in a neighborhood of one of the endpoints of \(W^\ell\) of length at most \(C_1 \varepsilon\); such short pieces may be estimated as in (4.11) using the strong stable norm. Thus

\[
|\psi_1 \circ T^n \circ G_{U^j_i} - \psi_2 \circ T^n \circ G_{U^2_j}|_{C^0(I_j)} \leq |\psi_1 \circ G_{W^1} \circ f_j - \psi_2 \circ G_{W^2} \circ f_j|_{C^0(I_j)} + |\psi_2 \circ G_{W^2} \circ f_j - \psi_2 \circ G_{W^2} \circ f_j|_{C^0(I_j)}.
\]

Using the above observation about \(f_j\), we estimate the first term of (4.25) by

\[
|\psi_1 \circ G_{W^1} \circ f_j - \psi_2 \circ G_{W^2} \circ f_j|_{C^0(I_j)} \leq C_j |\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{C^0(f_j(I_j))} \leq C_j |\varepsilon|,
\]

since \(d_{\varepsilon}(\psi_1, \psi_2) \leq \varepsilon\). To estimate the second term of (4.25), notice that since \(U^1_j\) and \(U^2_j\) are joined by the transverse foliation \(\{\gamma_x\} \subset \tilde{W}^u\) and using the uniform contraction along stable curves under \(T^n\), we have \(|f_j - f_j|_{C^0(I_j)} \leq \tilde{C} |\varepsilon|\) for a constant \(\tilde{C}\) depending only on the uniform hyperbolicity of (H1) and the uniform transversality conditions in (H2). Thus for \(r \in I_j\),

\[
|\psi_2 \circ G_{W^2} \circ f_j(r) - \psi_2 \circ G_{W^2} \circ f_j(r)| \leq C_j |\psi_2|_{C^0} |f_j(r) - f_j(r)|^{p} \leq C_j \tilde{C} |\psi_2|_{C^p} \varepsilon^p.
\]

Now we again apply Lemma 4.3 to obtain

\[
|\psi_2 \circ G_{W^2} \circ f_j - \psi_2 \circ G_{W^2} \circ f_j|_{C^0(I_j)} \leq C|\psi_2|_{C^p} \varepsilon^{p-q},
\]

for a uniform constant \(C\). This estimate combined with (4.26) proves part (b) since \(|\psi_2|_{C^p(W^2)} \leq 1\).

5. Proof of Theorem 2.3

Fix \(\varepsilon < \varepsilon_0\) and suppose \(T_1, T_2 \in F\) with \(d_F(T_1, T_2) \leq \varepsilon\). We denote by \(S^\ell_i\) the singularity sets for \(T_1, \ell = 1, 2\), \(\|h\|_B \leq 1\), and \(W \in W^s\). Let \(\psi \in C^p(W)\) with \(\|\psi\|_{W,0,p} \leq 1\). We must estimate

\[
\int_W (L_1 h - L_2 h) \psi \, dm_W = \int_W L_1 h \psi \, dm_W - \int_W L_2 h \psi \, dm_W
\]

\[
= \int_{T^{-1}_1 W} h \psi \circ T_1 (J_{T_1} T^{-1}_1 W) T \, dm_W - \int_{T^{-1}_2 W} h \psi \circ T_2 (J_{T_2} T^{-1}_2 W) T \, dm_W.
\]

Notice that the estimate required is similar to that done in Section 4.3 except that instead of two close stable curves iterated under the same map, we have one stable curve iterated under two different maps.

We partition \(T^{-1}_1 W\) and \(T^{-1}_2 W\) into matched and unmatched pieces as in the beginning of Section 4.3.

Let \(G^\ell_1(W), \ell = 1, 2\), denote the elements of \(T^{-1}_1 W\) as described in Section 5.3. Let \(\omega \in G^\ell_1(W)\). Due to (C1), to each point \(x \in \omega\), we associate a curve \(\gamma_x \in \tilde{W}^u\) of length at most \(C_\ell \varepsilon\) which terminates on a
piece of $T_{2}^{-1}W$ that lies in the same homogeneity strip, if one exists. We also require that $\gamma_{x}$ is not cut by $S_{1}^{t} \cup S_{2}^{t}$.

We denote by $V_{\ell}^{\ell}$ those components of $T_{\ell}^{-1}W$ not matched by this process. We also include in the set of $V_{\ell}^{\ell}$ all images of connected components of $W \cap N_{\varepsilon}(S_{1}^{t} \cup S_{2}^{t})$ under $T_{\ell}^{-1}$. Note that the $T_{\ell}V_{\ell}^{\ell}$ occur either at the endpoints of $W$ or near a singularity or the boundary of $N_{\varepsilon}(S_{-1}^{t} \cup S_{-1}^{t})$. In all cases, the length of the curves $T_{\ell}V_{\ell}^{\ell}$ can be at most $C_{\ell}C_{s}\varepsilon$ due to the uniform transversality of $S_{-1}^{t}$ with $C_{s}$ and of $C_{s}$ with $C_{s}^{*}$.

In the remaining pieces the foliation $\{\gamma_{x}\}$ provides a one-to-one correspondence between points in $T_{1}^{-1}W$ and $T_{2}^{-1}W$. We further partition these pieces in such a way that their lengths are between $\delta_{0}/2$ and $\delta_{0}$ and the pieces are pairwise matched by the foliation $\{\gamma_{x}\}$. We call these matched pieces $\tilde{U}_{j}^{\ell}$. As in Section 4.3 we trim the $\tilde{U}_{j}^{\ell}$ to pieces $U_{j}^{\ell}$ so that $U_{1}^{\ell}$ and $U_{2}^{\ell}$ are defined on the same arclength interval $I_{j}$. The at most two components of $T_{\ell}(\tilde{U}_{j}^{\ell} \setminus U_{j}^{\ell})$ have length at most $C_{\ell}C_{s}A^{-1}\varepsilon$. We adjoin these trimmed pieces to the adjacent $U_{1}^{\ell}$ or $V_{\ell}^{\ell}$ as appropriate so as not to create more pieces in the partition of $T_{\ell}^{-1}W$.

In this way, we write $T_{\ell}^{-1}W = (\cup_{j}U_{1}^{\ell}) \cup (\cup_{j}V_{k}^{\ell})$ and note that the images $T_{\ell}V_{k}^{\ell}$ have length at most $C_{s}\varepsilon$ for some uniform constant $C_{s}$, $\ell = 1, 2$.

Now using (5.1), we have

$$
\int_{W} (\mathcal{L}_{1}h - \mathcal{L}_{2}h)\psi dm_{W} = \sum_{\ell,k} \int_{V_{k}^{\ell}} h \psi \circ T_{\ell} (J_{\mu}T_{\ell})^{-1} J_{V_{k}^{\ell}} T_{\ell} dm_{W} + \sum_{j} \int_{U_{j}^{1}} h \psi \circ T_{\ell} (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{1}} T_{\ell} dm_{W} - \int_{U_{j}^{2}} h \psi \circ T_{\ell} (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{2}} T_{\ell} dm_{W}.
$$

We estimate the integral on short pieces $V_{k}^{\ell}$ first using the strong stable norm. By (4.6), we have $|\psi \circ T_{\ell}|_{C^{\alpha}(V_{k}^{\ell})} \leq C_{s}|\psi|_{C^{\alpha}(W)} \leq C_{s}$. Following the estimate in (1.14), we have

$$
\sum_{\ell,k} \int_{V_{k}^{\ell}} h (J_{\mu}T_{\ell})^{-1} J_{V_{k}^{\ell}} T_{\ell} \psi \circ T_{\ell} dm \leq C_{s}^{\ell} \|h\|_{s} \sum_{\ell,k} |J_{V_{k}^{\ell}} T_{\ell}|^{1-\alpha}_{C^{\alpha}(V_{k}^{\ell})}.
$$

The sum is finite by (5.7) of (H3) with $\alpha = 1 - \alpha$ since there are at most two $V_{k}^{\ell}$ corresponding to each element $W_{i}^{\ell,1} \in G_{1}^{t}(W)$ as defined in Section 5.5 and $|J_{V_{k}^{\ell}} T_{\ell}|_{C^{\alpha}(V_{k}^{\ell})} \leq |J_{W_{i}^{\ell,1}} T_{\ell}|_{C^{\alpha}(W_{i}^{\ell,1})}$ whenever $V_{j}^{\ell} \subseteq W_{i}^{\ell,1}$. The constant $C$ above depends only on properties (H1)-(H5), but for brevity we do not write out the explicit dependence since these estimates are similar to those done in Section 4.3 and the constants are the same.

Next, we must estimate

$$
\sum_{j} \left| \int_{U_{j}^{1}} h (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{1}} T_{\ell} \psi \circ T_{\ell} dm_{W} - \int_{U_{j}^{2}} h (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{2}} T_{\ell} \psi \circ T_{\ell} dm_{W} \right|.
$$

Using notation analogous to (4.12), we fix $j$ and estimate the difference. Define $\phi_{j} = ((J_{\mu}T_{\ell})^{-1} J_{U_{j}^{1}} T_{\ell} \psi \circ T_{\ell}) \circ G_{U_{j}^{1}} \circ G_{U_{j}^{1}}^{-1}$.

The function $\phi_{j}$ is well-defined on $U_{j}^{2}$ and we can write,

$$
\left| \int_{U_{j}^{1}} h (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{1}} T_{\ell} \psi \circ T_{\ell} - \int_{U_{j}^{2}} h (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{2}} T_{\ell} \psi \circ T_{\ell} \right| \\
\leq \left| \int_{U_{j}^{1}} h (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{1}} T_{\ell} \psi \circ T_{\ell} - \int_{U_{j}^{2}} h \phi_{j} \right| + \left| \int_{U_{j}^{2}} h (\phi_{j} - (J_{\mu}T_{\ell})^{-1} J_{U_{j}^{2}} T_{\ell} \psi \circ T_{\ell}) \right|.
$$

To estimate the two terms above, we need the following adaptation of Lemma 4.1.

**Lemma 5.1.** There exists $\bar{C} > 0$, independent of $W \in W^{\ast}$ and $T_{1}, T_{2} \in \mathcal{F}$, such that for each $j$,
Note that by construction

\[ \int U_j^1 T_1 \circ G_{U_j^1} \leq \widetilde{C} \varepsilon^{1/2} ; \]

\[ |(J_\mu T_1)^{-1} J_{U_j^1} T_1 \circ G_{U_j^1} T_2 \circ G_{U_j^2}|_{C^0(I_j)} \leq \widetilde{C}|(J_\mu T_2)^{-1} J_{U_j^2} T_2|_{C^0(U_j^2)} \varepsilon^{1/3-q} ; \]

\[ |\psi \circ T_1 \circ G_{U_j^1} - \psi \circ T_2 \circ G_{U_j^2}|_{C^0(I_j)} \leq \widetilde{C} \varepsilon^{p-q} . \]

We estimate the first term in equation (5.4) using the strong unstable norm. The estimates (3.17) and (5.6) and property (H5) imply that

\[ |(J_\mu T_1)^{-1} J_{U_j^1} T_1 \cdot \psi \circ T_1|_{U_j^1,0,p} \leq \eta C \varepsilon |J_{U_j^1} T_1|_{C^0(U_j^1)} . \]

Similarly, since by (4.17), \(|G_{U_j^1} \cdot G_{U_j^2}|_{C^1} \leq C_g\), we have \(|\phi_j|_{U_j^2,0,p} \leq C_g \varepsilon |J_{U_j^1} T_1|_{C^0(U_j^1)}\). By the definition of \(\phi_j\) and \(d_\varepsilon(\cdot,\cdot)\),

\[ d_\varepsilon((J_\mu T_1)^{-1} J_{U_j^1} T_1 T_1, \phi_j) = \left| \int U_j^1 \psi \circ T_1 - \int U_j^2 \psi \circ T_2 \right| \leq C \varepsilon^{\beta/2} \|h\| \sum_j |J_{U_j^1} T_1|_{C^0(U_j^1)} , \]

where the sum is \(\leq C_2\) by Lemma 3.2(b) since there is at most one matched piece \(U_j^1\) corresponding to each curve \(W_i^1 \in G_1^1(W)\).

We estimate the second term in (5.4) using the strong stable norm.

\[ \left( (J_\mu T_2)^{-1} J_{U_j^2} T_2 \circ \psi \cdot T_1 \right) \leq C \|h\| \|U_j^2\|^{\alpha} \left| \phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \circ \psi \cdot T_2 \right|_{C^0(U_j^2)} . \]

In order to estimate the \(C^0\)-norm of the function in (5.7), we split it up into two differences. Following (4.20) line by line, we obtain

\[ |\phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \circ \psi \cdot T_2|_{C^0(U_j^2)} \]

\[ \leq C \left( (J_\mu T_1)^{-1} J_{U_j^1} T_1 \circ G_{U_j^2} \circ \psi \circ T_1 \circ G_{U_j^1} \circ \psi \circ T_2 \circ G_{U_j^2} \right) \]

\[ + C \left| \left( (J_\mu T_1)^{-1} J_{U_j^1} T_1 \circ G_{U_j^1} \circ \psi \circ T_1 \circ G_{U_j^2} \circ \psi \circ T_2 \circ G_{U_j^1} \circ \psi \circ T_2 \circ G_{U_j^2} \right) \right|_{C^0(I_j)} , \]

Notice that \(|J_{U_j^1} T_1|_{C^0(U_j^1)} \leq C |J_{U_j^2} T_2|_{C^0(U_j^2)}\) by (5.13). Then using Lemma 5.1(b) and (c) together with (5.8) yields by (5.7)

\[ \sum_j \left| \int U_j^2 h(\phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \circ \psi \circ T_2) \, dm \right| \leq C \|h\|_{s} \varepsilon^{p-q} \sum_j |J_{U_j^2} T_2|_{C^0(U_j^2)} , \]

where again the sum is finite by Lemma 3.2(b). This completes the estimate on the second term in (5.4).

Now we use this bound, together with (5.3) and (5.6) to estimate (5.2)

\[ \left| \int W L_1 h \psi \, dm - \int W L_2 h \psi \, dm \right| \leq C \|h\|_{s} \varepsilon^{\alpha} + C \|h\|_{u} \varepsilon^{\beta/2} + C \|h\|_{s} \varepsilon^{p-q} . \]

Since \(p - q \geq \beta\) and \(\alpha \geq \beta\), the theorem is proved.

5.1. Proof of Lemma 5.1

Proof of (a). Note that by construction \(U_j^1\) and \(U_j^2\) lie in the same homogeneity strip. Also, they are both defined on the same interval \(I_j\) so the length of the symmetric difference of their \(r\)-intervals is 0. Recalling the definition of \(d_{W^s}(U_j^1, U_j^2)\), we see that it remains only to estimate \(|\varphi_{U_j^1} - \varphi_{U_j^2}|_{C^1(I_j)}\) for their defining functions \(\varphi_{U_j^k}\).
For $x = (r, \varphi_{U_j}^1(r))$, define $\tilde{x} = (r, \varphi_{U_j}^2(r))$ and $x_\varepsilon = T_2^{-1} \circ T_1(x)$. We may assume that $x_\varepsilon$ lies on $\tilde{U}_j$ since otherwise, we would be $\varepsilon$-close to one of the $V_{\varepsilon j}$ and such short curves can be estimated as in (5.3) using the strong stable norm. Since $x$ and $x_\varepsilon$ are images of the same point $u \in W$ under $T_1^{-1}$ and $T_2^{-1}$ respectively, it follows from (C1) that $x$ and $x_\varepsilon$ are at most $\varepsilon$ apart. Then since all vectors in the stable cone have slope bounded away from $\pm \infty$, it follows that $x$ and $\tilde{x}$ are at most $C\varepsilon$ apart (and so by the triangle inequality, also $\tilde{x}$ and $x_\varepsilon$ are at most $C\varepsilon$ apart).

This proves that $|\varphi_{U_j}^1 - \varphi_{U_j}^2|_{C^0(I_j)} \leq C\varepsilon$. It remains to estimate $|\varphi_{U_j}^1 - \varphi_{U_j}^2|$, where $\varphi_{U_j}^1$ denotes the derivative of $\varphi_{U_j}^1$ with respect to $r$.

Let $\vec{v}_W(u)$ be the unit tangent vector to $W$ at $u := T_1(x) = T_2(x)$, as before. The tangent vector to $U_j^\ell$ is given by $DT_{\tilde{r}}^{-1}(u)\vec{v}_W(u)$, $\ell = 1, 2$. By (C4),

$$\|DT_{\tilde{r}}^{-1}(u)\vec{v}_W(u) - DT_r^{-1}(u)\vec{v}_W(u)\| \leq \varepsilon^{1/2}. \quad (5.10)$$

Then since $\|DT_{\tilde{r}}^{-1}(u)\vec{v}_W(u)\| \geq C^{-1} \varepsilon$ by (H1), we have $\theta(x, x_\varepsilon) \leq C\varepsilon^{1/2}$, where $\theta(x, x_\varepsilon)$ is the angle between the tangent vectors to $U_j^1$ and $U_j^2$ at $x$ and $x_\varepsilon$ respectively.

For $y \in U_j^\ell$, let $\phi(y)$ denote the angle that $G_{U_j}^\ell$ makes with the positive $r$-axis at $y$. Then

$$|\varphi_{U_j}^1(x) - \varphi_{U_j}^2(\tilde{x})| = |\tan \phi(x) - \tan \phi(\tilde{x})| \leq \left[ \sup_{z \in U_j^\ell} \sec^2 \phi(z) \right] |\phi(x) - \phi(\tilde{x})| = \left[ \sup_{z \in U_j^\ell} \sec^2 \phi(z) \right] \theta(x, \tilde{x}). \quad (5.11)$$

Since the slopes of curves in $C^3(x)$ are uniformly bounded away from $\pm \infty$, we have $\sec^2 \phi(z)$ uniformly bounded above for any $z \in U_j^\ell$. The proof of the lemma is completed by writing $\theta(x, \tilde{x}) \leq \theta(x, x_\varepsilon) + \theta(x_\varepsilon, \tilde{x})$. The first term is $\leq C\varepsilon^{1/2}$ using (5.10) and the second term is $\leq C\varepsilon$ since $x_\varepsilon$ and $\tilde{x}$ both lie on $\tilde{U}_j$ and stable curves have a uniform $C^2$ bound by (H2).

**Proof of (b).** We prove that the closeness condition (C3) implies the existence of a constant $C > 0$, independent of $W \in W^s$ and $T_1, T_2 \in \mathcal{F}$, such that

$$|J_{U_j^1}T_1 \circ G_{U_j^1}^1 - J_{U_j^1}T_2 \circ G_{U_j^1}^2|_{C^0(I_j)} \leq C|J_{U_j^2}T_2|_{C^0(U_j^2)}\varepsilon^{1/3 - q}. \quad (5.12)$$

The analogous statement concerning $(J_{U_j^k}T_k)^{-1}$ follows from condition (C2). Then since

$$|f_1g_1 - f_2g_2|_{C^\ell} \leq |f_1|_{C^\ell}|g_1|_{C^\ell} + |g_2|_{C^\ell}|f_1 - f_2|_{C^\ell},$$

for any $C^\ell$ functions $f_1, g_1, f_2, g_2$, part (b) of the lemma follows from these two estimates using the fact that $| \cdot |_{C^\ell} \leq (1 + C_4)| \cdot |_{C^0}$ by bounded distortion for the functions we are estimating. We proceed to prove (5.11).

For any $r \in I_j$, we write

$$|J_{U_j^1}T_1 \circ G_{U_j^1}^1(r) - J_{U_j^1}T_2 \circ G_{U_j^1}^2(r)| \leq |J_{U_j^1}T_1 \circ G_{U_j^1}^1(r) - J_{U_j^1}T_2 \circ G_{U_j^1}^1(r)| + |J_{U_j^1}T_2 \circ G_{U_j^1}^1(r) - J_{U_j^2}T_2 \circ G_{U_j^2}(r)| \quad (5.12)$$

The first term above is $\leq |J_{U_j^1}T_2|_{C^0(I_j)}\varepsilon$ by (C3).

Recall that $U_j^1, U_j^2$ lie inside the longer curves $\tilde{U}_j^1, \tilde{U}_j^2$ which are matched by the foliation $\{\gamma_x\}_{x \in \tilde{U}_j} \subset \tilde{W}^u$. Thus $|J_{U_j^1}T_2|_{C^0(I_j)} \leq C|J_{U_j^2}T_2|_{C^0(I_j)}$ by the same argument used to prove (4.24), completing the estimate on the first term of (5.12).

The second term of (5.12) is $\leq C\varepsilon^{1/3}|J_{U_j^2}T_2|_{C^0(I_j)}$ using (4.23) since it involves the Jacobian of a single map in $\mathcal{F}$ evaluated on two stable curves that are matched by a foliation of unstable curves. Thus

$$|J_{U_j^1}T_1 \circ G_{U_j^1}^1(r) - J_{U_j^1}T_2 \circ G_{U_j^1}^2(r)| \leq C\varepsilon^{1/3}|J_{U_j^2}T_2|_{C^0(U_j^2)} \quad (5.13)$$

This implies in particular that $|J_{U_j^1}T_1|_{C^0(U_j^1)} \leq C|J_{U_j^2}T_2|_{C^0(U_j^2)}$. Now we use (3.8) and the fact that $|G_{U_j^1}|_{C^1(I_j)} \leq C_9$ to apply Lemma (3.3) and complete the proof of (5.11). \qed
Proof of (c). Let $x = (r, \varphi_{U_j}(r))$ and as above, define $\tilde{x} = (r, \varphi_{U_j}(r))$ and $x_\varepsilon = T_2^{-1} \circ T_1(x)$. Since $\tilde{x}$ and $x_\varepsilon$ are at most $C\varepsilon$ apart and lie on $\tilde{U}_j^2$, we have $d_W(T_2\tilde{x}, T_2x_\varepsilon) \leq C\varepsilon$ by the uniform contraction given by (H1). Thus,

$$
(5.14) \quad |\psi \circ T_1 \circ G_{U_j}(r) - \psi \circ T_2 \circ G_{U_j}(r)| \leq |\psi|_{C^p(W)} d_W(T_1x, T_2\tilde{x})^p.
$$

Since $d_W(T_1x, T_2x_\varepsilon) = 0$, we may use the triangle inequality to conclude that the difference above is bounded by $C|\psi|_{C^p(W)}\varepsilon^p$.

Again applying Lemma 4.3 with $|\psi|_{C^p(W)} \leq 1$ completes the proof of part (c). □

6. Proofs of Applications: Movements and Deformations of Scatterers and Random Perturbations

In this section we prove Theorems 2.5, 2.6 and 2.10 and leave Theorems 2.8 and 2.9 regarding external forces and kicks to Section 7 since they require more background material.

6.1. Proof of Theorem 2.5. We fix constants $\tau_*, K_* > 0$ and $E_* < \infty$ and denote $F_1(\tau_*, K_*, E_*)$ as simply $F_1$ for brevity. Note that every $T \in F_1$ is a billiard map corresponding to a standard Lorentz gas with convex scatterers so that we may recall known facts about such maps to establish (H1)-(H5) with constants depending only on the three quantities $\tau_*, K_*$ and $E_*$. (H1). For $x \in M$, define

$$
C^s(x) = \{(dr, d\varphi) \in T_x M : -K_*^{-1} - \tau_*^{-1} \leq d\varphi/dr \leq -K_*\}
$$

and

$$
C^u(x) = \{(dr, d\varphi) \in T_x M : K_* \leq d\varphi/dr \leq K_*^{-1} + \tau_*^{-1}\}.
$$

Then for any $T \in F_1$, $DT_x C^s(x) \subset C^s(Tx)$ and $DT_x^{-1} C^s(u) \subset C^s(T^{-1}x)$ whenever $DT_x$ and $DT_x^{-1}$ are defined. Moreover, (3.2) is satisfied with $\Lambda = 1 + 2K_*\tau_*$ and

$$
C_\varepsilon = \frac{2\tau_* K_*}{\Lambda} \left(\sqrt{1 + K_*^2} \right) \left(\sqrt{1 + (K_*^{-1} + \tau_*^{-1})^2} \right).
$$

(see [CM] Section 4.4]). Notice that $C^s$ and $C^u$ are uniformly transverse to each other and to the vertical and horizontal directions in $M$ as required.

The bounds on the first and second derivatives of $T$ required by (3.3) and (3.4) are standard for such maps ([CM] Section 4.4). Here, the index $n$ corresponds to the free flight time $\tau(T^{-1}x)$. For finite horizon, this has a uniform upper bound, while for infinite horizon, the relation between $k$ and $n$ is satisfied with $v_0 = 1/4$ ([CM] Section 5.10)].

(H2). We say a $C^2$ curve $W$ in $M$ is stable if its tangent vectors $T_x W$ lie in $C^s(x)$ as defined above for each $x \in W$. We call a stable curve homogeneous if it is contained in a single homogeneity strip $H_k$. Since each stable curve $W$ has slope bounded away from infinity, we may identify $W$ with the graph of a function of $r$, which we denote by $\varphi_W(r)$.

By [CM] Proposition 4.29], we may choose $B$ depending only on $\tau_*, K_*$ and $E_*$ such that if $\frac{d^2 \varphi_W}{dr^2} \leq B$, then each smooth component $W'$ of $T^{-1}W$ satisfies $\frac{d^2 \varphi_W}{dr^2} \leq B$.

We define $\tilde{W}^s$ to be the set of all stable homogeneous curves $W$ such that $\frac{d^2 \varphi_W}{dr^2} \leq B$. The invariance of the family $C^s(x)$ as well as the choice of $B$ guarantee that $\tilde{W}^s$ is invariant as required. The set of unstable curves $\tilde{W}^u$ is defined similarly.

(H3). Following [CM] Section 5.10], we define the adapted norm in the tangent space at $x \in M$ by

$$
\|v\|_* = \frac{K(x) + |V|}{\sqrt{1 + V^2}} \|v\|, \quad \forall v \in C^s(x) \cup C^u(x)
$$

where, $v = (dr, d\varphi)$ is a tangent vector, $V = d\varphi/dr$ and $K(x)$ is the curvature of the scatterer at $x$. Since the slopes of vectors in $C^s(x)$ and $C^u(x)$ are bounded away from $\pm \infty$, we may extend $\|\cdot\|_*$ to all of $\mathbb{R}^2$ in
such a way that $\| \cdot \|_\ast$ is uniformly equivalent to $\| \cdot \|$. It is straightforward to check that for $v \in C^u(x)$,

$$\frac{\| DT(x)v \|_\ast}{\| v \|_\ast} \geq 1 + K_\ast \tau_\ast = \Lambda.$$ 

Uniform expansion in $C^s(x)$ under $DT^{-1}(x)$ follows similarly. Now (3.3) follows from [CM, Lemma 5.56] and (3.7) follows from [DZ, Sublemma 3.5] with $\zeta_0 = 1/6$. From this point forward, we consider $k_0$ to be fixed.

(H4). The bounded distortion constant $C_d$ in (3.3) and (3.9) depends only on the choice of $k_0$ from (H3) and the uniform hyperbolicity constants $C_e$ and $\Lambda$ ([CM, Lemma 5.27]).

(H5). For maps in $F_1$, $DT(x) \equiv 1$ so we may take $\eta = 1$.

6.2. Proof of Theorems 2.6. Fix constants $\tau_\ast, K_\ast > 0$ and $E_\ast < \infty$ and consider a configuration $Q_0 \in Q_1(\tau_\ast, K_\ast, E_\ast)$ with scatterers $\Gamma_1, \ldots, \Gamma_d$. Choose $\gamma \leq \frac{1}{2} \min \{ \tau_\ast, K_\ast \}$ and set $\tilde{Q} \in F_B(Q_0, E_\ast; \gamma)$ with scatterers $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_d$. Since $\ell(I_i) = |\partial \Gamma_i| = |\partial \tilde{\Gamma}_i|$ we may take the corresponding functions $u_i, \tilde{u}_i$ to be arclength parametrizations of $\partial \Gamma_i$ and $\partial \tilde{\Gamma}_i$ respectively. We denote by $u'_i$ and $u''_i$ the first and second derivatives of $u_i$ with respect to the arclength parameter $r$. Then the curvature of $\partial \Gamma_i$ is simply given by $K(r) = \| u''_i(r) \|$ at each point $u_i(r) \in \partial \Gamma_i$, and similarly for $\partial \tilde{\Gamma}_i$.

Thus on $\partial \tilde{\Gamma}_i$, we have by assumption on $\tilde{Q}$ and $\gamma$,

$$\tilde{K}(r) = \| \tilde{u}''_i \| = \| u''_i + \tilde{u}''_i - u''_i \| \geq K(r) - \gamma \geq K_\ast/2.$$ 

Also, $\tau_{\min}(\tilde{Q}) \geq \tau_{\min}(Q_0) - \gamma \geq \tau_\ast/2$ since $\| u_i - \tilde{u}_i \| \leq \gamma$. Thus $F_A(Q_0, E_\ast; \gamma) \subset F_1(\tau_\ast/2, K_\ast, E_\ast)$.

Let $T_0$ be the map corresponding to $Q_0$ and let $T_1$ be the map corresponding to $\tilde{Q}$. We fix $x = (r, y) \in I_1 \times [-\pi/2, \pi/2]$ and compare $T_0^{-1}x$ with $T_1^{-1}x$. To do this, we let $\Phi_0^i$ and $\Phi_1^i$ denote the flow on the tables $Q_0$ and $\tilde{Q}$ respectively. We denote by $\pi_0(x)$ the projection of $x$ onto the flow space $T^2 \times S^1$ corresponding to $Q_0$ and by $\pi_0^i$ and $\pi_1^0$ the projections onto the position and angular coordinates respectively. Let $\tau_0(x)$ denote the free flight time of $x$ under $\Phi_0^i$ and let $K_1(\cdot)$ denote the curvature of the scatterers in $Q_0$. The analogous objects, $\pi_1, \pi_1^0, \pi_0^0, \pi_1^i(\cdot)$ and $K_1(\cdot)$ are defined for the table $\tilde{Q}$.

First suppose that $T_0^{-1}x$ and $T_1^{-1}x$ lie on the same scatterer $\Gamma_j$. Notice that the trajectories $\Phi_{0,i}(\pi_0x)$ and $\Phi_{1,i}(\pi_1x)$ begin from two points in $\mathbb{T}^2$ at most $\gamma$ apart and make an angle of at most $\gamma$ with one another. We decompose this motion into the sum of (I) two parallel trajectories starting a distance $\gamma$ apart and (II) two trajectories starting at the same point and making an angle $\gamma$.

I. Parallel trajectories. It is an elementary estimate that two parallel lines a distance $\gamma$ apart will intersect a convex scatterer at a distance at most

$$d_{T^2}(\pi_0^0(T_0^{-1}x), \pi_1^0(T_1^{-1}x)) \leq \sqrt{3\gamma/K_{\min}(\Gamma_j)} \leq \sqrt{3\gamma/K_\ast},$$ 

where $d_{T^2}$ denotes distance on $T^2$.

II. Nonparallel trajectories making an angle $\gamma \neq 0$. After time $t$ under the flow, the two trajectories will be at most $\gamma t$ apart in $\mathbb{T}^2$. Let $\tau(x_{-1}) = \max \{ \tau_0(T_0^{-1}x), \tau_1(T_1^{-1}x) \}$. Then in the case of a finite horizon Lorentz gas, by the same estimate as in (6.1),

$$d_{T^2}(\pi_0^0(T_0^{-1}x), \pi_1^0(T_1^{-1}x)) \leq \sqrt{3\gamma \tau(x_{-1})/K_{\min}(\Gamma_j)} \leq \sqrt{3\gamma \tau_{\min}/K_\ast}.$$

In the infinite horizon case, define $\tilde{\gamma} = \gamma^{-1/3}$. If $\tau(x_{-1}) \leq \tilde{\gamma}$, then (6.2) implies $d_{T^2}(\pi_0^0(T_0^{-1}x), \pi_1^0(T_1^{-1}x)) \leq \sqrt{3/K_\ast \gamma^{1/3}}$. On the other hand, suppose $\tau_0(T_0^{-1}x) > \tilde{\gamma}$. Then $x$ lies in a cell $D_n$ such that $c^{-1}n \leq \tau_0(T_0^{-1}y) \leq cn$ for some $c > 0$ and all $y \in D_n$, and the width of $D_n$ in the stable direction is at most $C'/n$ (see [CM, Section 4.10]). Thus

$$d_M(x, S_{-1}^0) \leq C'n^{-1} \leq C'c\tau_0^{-1}(T_0^{-1}x) \leq C'c\tilde{\gamma}^{-1} \leq C'c_{\gamma}^{1/3}.$$
An identical estimate holds if \( \tau_1(T_1^{-1}x) > \hat{\tau} \). Thus either \( x \in N_{C\gamma^{1/3}}(S_{T_0}^{T_1} \cup S_{T_1}^{T_1}) \) or

\[
(6.4) \quad d_{\gamma^2}(\pi_0^\gamma(T_0^{-1}x, \pi_0^\gamma(T_1^{-1}x)) \leq \sqrt{3/\kappa_0^\gamma} \gamma^{1/3}.
\]

Concatenating these two estimates (I) and (II), we see that in terms of position coordinates, \( T_0^{-1}x \) and \( T_1^{-1}x \) in \( I_j \) are of order \( \gamma^{1/2} \) in the finite horizon case and of order \( \gamma^{1/3} \) in the infinite horizon case. Since the normal direction of \( \Gamma_j \) varies smoothly with the position, we have \( d_M(T_0^{-1}x, T_1^{-1}x) \) of the same order. Similar estimates hold when starting from \( x \in \Gamma_j \) and comparing images in \( \Gamma_1 \) and \( \Gamma_1 \).

In the case when \( T_0^{-1}x \) and \( T_1^{-1}x \) do not lie on the same scatterer \( \Gamma_j \), we must have \( x \in N_{C\gamma^{1/3}}(S_{T_0}^{T_1} \cup S_{T_1}^{T_1}) \) by the preceding arguments where \( C = 4\kappa_{\gamma}^{-3/2} \) is sufficient. We have thus shown (C1) holds with \( \varepsilon = C\gamma^{1/3} \). Indeed, (C1) holds with \( \varepsilon = C\gamma^b \) for any \( 0 < b \leq 1/3 \) by the same argument.

We can consider the deformation of \( d \) scatterers as the concatenation of errors induced by deforming one scatterer at a time. The preceding analysis holds with \( C \) increased by a factor of \( d \).

Condition (C2) is trivial to check since \( J_\mu T_i = 1 \) for \( i = 0, 1 \).

Next we prove (C4). By [CM, eq. (2.26)], \( DT_0^{-1}(x) = \frac{-1}{\cos \varphi(T_0^{-1}x)} A_0(x) \), where

\[
A_0(x) = \left[ -K_0(T_0^{-1}x)\kappa_0(x) + \cos \varphi(x) - K_0(T_0^{-1}x)\kappa_0(x) + \cos \varphi(T_0^{-1}x) \right],
\]

and \( DT_1^{-1}x = \frac{-1}{\cos \varphi(T_1^{-1}x)} A_1(x) \), with a similar definition for \( A_1(x) \). Thus

\[
(6.5) \quad \|DT_0^{-1}(x) - DT_1^{-1}x\| \leq \left| \frac{1}{\cos \varphi(T_0^{-1}x)} - \frac{1}{\cos \varphi(T_1^{-1}x)} \right| \|A_0(x)\| + \frac{1}{\cos \varphi(T_1^{-1}x)} \|A_0(x) - A_1(x)\|
\]

Note that \( \|A_i(x)\| \) is bounded by a uniform constant times \( \tau_i(T_i^{-1}x) \) and

\[
(6.6) \quad \|A_0(x) - A_1(x)\| \leq K \tau(x_{-1})(d_M(T_0^{-1}x, T_1^{-1}x) + \gamma)
\]

where \( K \) depends on \( \tau_\gamma, E_\gamma \) and \( \kappa_\gamma \) and the \( \gamma \) term is due to possible differences in the curvatures \( \kappa_0 \) and \( \kappa_1 \) at the same point. Notice that if \( W \in \mathcal{W}^a \), then \( |T_i^{-1}W| \leq C|W|^{1/3} \) in the infinite horizon case and \( |T_i^{-1}W| \leq C|W|^{1/2} \) in the finite horizon case. Thus for \( \delta < 1/k_0 \), if \( T_i^{-1}x \in N_\delta(S_0) \), then \( d_M(x, S_{T_1}^{T_1}) \leq C\delta^2 \) where \( C_\delta \) is a uniform constant depending on the transversality of \( C^a \) with the horizontal direction and of \( S_{T_1}^{T_1} \) with \( C^a \).

Now choose \( \varepsilon = \gamma^{a} \), where \( a \leq 1/3 \) will be determined shortly. Suppose \( x \notin N_\varepsilon(S_{T_1}^{T_1} \cup S_{T_1}^{T_1}) \). Then by the above observation, \( \cos \varphi(T_i^{-1}x) \geq C\gamma^{a/2}, i = 0, 1 \), and also by (6.3), \( \tau(x_{-1}) \leq C\varepsilon^{-1} \). Thus recalling that \( d_M(T_0^{-1}x, T_1^{-1}x) \leq C\gamma^{1/3} \), we estimate the first term of (6.5)

\[
(6.7) \quad \|A_0(x)\| \left| \frac{1}{\cos \varphi(T_0^{-1}x)} - \frac{1}{\cos \varphi(T_1^{-1}x)} \right| \leq \frac{K \tau(T_0^{-1}x)}{\cos \varphi(T_0^{-1}x) \cos \varphi(T_1^{-1}x)} \|\cos \varphi(T_1^{-1}x) - \cos \varphi(T_0^{-1}x)\| \\
\leq C\varepsilon^{-2} d_M(T_0^{-1}x, T_1^{-1}x) \leq C^a \gamma^{1/3-3a/2}.
\]

To estimate the second term of (6.5), we use (6.6) to estimate,

\[
\frac{1}{\cos \varphi(T_1^{-1}x)} \|A_0(x) - A_1(x)\| \leq C\varepsilon^{-3/2} \gamma^{1/3} = C\gamma^{1/3-3a/2}.
\]

Putting these estimates together, we have

\[
\|DT_0^{-1}(x) - DT_1^{-1}x\| \leq C^a \gamma^{1/3-2a}.
\]

Choosing \( a = 2/15 \) establishes (C4).

Condition (C3) follows similarly using the fact that the stable Jacobian along \( W \in \mathcal{W}^a \) is simply the norm of the tangent vector to \( W \) times \( DT_i(x) \), \( i = 0, 1 \). The improved estimate in (C3) comes from the fact that instead of estimating (6.7) as above, we must estimate instead

\[
\tau(x_{-1}) \left| \frac{\cos \varphi(T_0^{-1}x)}{\cos \varphi(T_1^{-1}x)} - 1 \right| \leq C\varepsilon^{-3/2} d_M(T_0^{-1}x, T_1^{-1}x) \leq C^a \gamma^{1/3-3a/2} = C^a \gamma^{2/15} = C^a \varepsilon
\]

with our choice of \( a = 2/15 \).
If we restrict perturbations to the finite horizon case with horizons uniformly bounded by some \( \tau_{\text{max}} < \infty \), then our estimates above improve by omitting a factor of \( \varepsilon^{-1} \) and \( d(T_0^{-1} x, T_1^{-1} x) \leq C \gamma^{1/2} \) by (6.2). In this case, the optimal choice of \( b = 1/3 \).

6.3. Proof of Theorem 2.10. We fix a class of maps \( \mathcal{F} \) for which (H1)-(H5) hold with uniform constants and choose \( T_0 \in \mathcal{F} \). Define \( X_T(T_0) = \{ T \in \mathcal{F} : d_T(T, T_0) \leq \varepsilon \} \). Recall the transfer operator \( L_{(\nu, g)} \) associated with the random process drawn from \( X_T(T_0) \) as defined in Section 2.4. Our first lemma is a generalization of Theorem 2.3 which shows that the transfer operator \( L_{(\nu, g)} \) is close to \( L_{T_0} \) in the norms we have defined.

Lemma 6.1. There exists \( C > 0 \) such that if \( \varepsilon \leq \varepsilon_0 \), then \( ||L_{(\nu, g)} - L_{T_0}|| \leq C A \varepsilon^{\beta} \).

Proof. Let \( h \in C^1(M) \), \( W \in W^s \) and \( \psi \in C^0(W) \) with \( |\psi|_{W,0,p} \leq 1 \). Then using (5.9),

\[
\left| \int_W L_{(\nu, g)} h \psi \, dm - \int_W L_{T_0} h \psi \, dm \right| = \int_{\Omega} \int_W (L_{T_0} h(x) - L_{T_0} h(x)) \psi(x) g(\omega, T_0^{-1} x) \, dmW \, dv \\
\leq \int_{\Omega} \int_W C b^{-1} \varepsilon^{\beta/2} \| h(\omega, \cdot) \|_{C^1(M)} \, dv(\omega) \leq C b^{-1} A \varepsilon^{\beta/2} \| h \|,
\]

where we have interchanged order of integration since \( \int_W L_{(\nu, g)} h \psi g(\omega, \cdot) \, dmW \) is uniformly and absolutely integrable for each \( \omega \in \Omega \) by Theorem 2.2.

It remains to prove the uniform Lasota-Yorke inequalities for \( L_{(\nu, g)} \). Let \( \omega_n = (\omega_1, \ldots, \omega_n) \in \Omega^n \) and define \( T_{\omega_n} = T_{\omega_n} \circ \cdots \circ T_{\omega_1} \). We first prove that the random compositions \( T_{\omega_n} \) have the same properties (H1)-(H5) as the maps \( T_{\omega} \in \mathcal{F} \), with possibly modified constants.

The singularity sets for \( T_{\omega_n} \) are \( S_n^{T_{\omega_n}} = \cup_{k=1}^n T_{\omega_k}^{-1} S_0 \), for \( n \geq 0 \), and similarly for \( S_n^{T_{\omega_n}} \). Thus the transversality properties (H1) of \( S_n^{T_{\omega_n}} \) hold with respect to \( C^s \) and \( C^u \) hold due to the uniformity of this transversality for all maps in \( \mathcal{F} \). The family \( W^s \) is preserved under \( T_{\omega_n}^{-1} \) since it is preserved by each map in the composition.

The uniform expansion given by (3.2) of (H1) also holds since \( DT_{\omega_n} = \prod_{k=1}^n DT_{\omega_k} \circ T_{\omega_k}^{-1} \) and in the adapted metric \( \| \cdot \|_* \) given by (H3), the expansion holds with \( C_e = 1 \) for each map in the composition. Translating to the Euclidean norm at the last step, we get (H1) with \( C_e \) depending only on the uniform constant relating the adapted and Euclidean metrics. Equations (3.3) and (3.4) also hold trivially since they concern only one iterate of a map drawn from \( \mathcal{F} \). (H5) follows for the same reason.

Due to the uniform expansion along stable and unstable leaves, (3.8) and (3.9) of (H4) hold with a possibly larger distortion constant \( C_q \), again using the bounded distortion of each map in the composition \( T_{\omega_n} \).

Finally, we establish that the iteration of the one-step expansion given in (H3) holds for random sequences of maps in the class \( \mathcal{F} \). As in Section 3.3, for \( W \in W^s \) we define the \( n \)-th generation \( \mathcal{G}_{n}^{T_{\omega_n}}(W) \subset W^s \) of smooth curves in \( T_{\omega_n}^{-1} W \). The elements of \( \mathcal{G}_{n}^{T_{\omega_n}}(W) \) are denoted by \( W_i^n \) as before and long and short pieces are defined similarly. Analogously, \( \mathcal{I}_{n}^{T_{\omega_n}}(W) \) denotes the set of indices \( i \) in generation \( n \) such that \( W_j^k \) is the most recent long ancestor of \( W_h^n \) under \( T_{\omega_n} \). Thus \( \mathcal{I}_{n}^{T_{\omega_n}}(W) \) denotes the set of curves that are never part of a curve that has grown to length \( \delta_0/3 \) at each time step \( 1 \leq k \leq n \).

Lemma 6.2. Let \( W \in W^s \) and for \( n \geq 0 \), let \( \mathcal{G}_{n}^{T_{\omega_n}}(W) \) and \( \mathcal{I}_{n}^{T_{\omega_n}}(W) \) be defined as above. There exist constants \( C_1, C_2, C_3 > 0 \), independent of \( W \in W^s \) and \( \omega_n \in \Omega^n \), such that for any \( n \geq 0 \),

(a) \( \sum_{i \in \mathcal{I}_{n}^{T_{\omega_n}}(W)} |JW_i^n T_{\omega_n}|_{C^0(W^n)} \leq C_1 \theta^n \); 

(b) \( \sum_{W_i^n \in \mathcal{G}_{n}^{T_{\omega_n}}(W)} |JW_i^n T_{\omega_n}|_{C^0(W^n)} \leq C_2 \); 

(c) for any \( 0 \leq \zeta \leq 1 \), \( \sum_{W_i^n \in \mathcal{G}_{n}^{T_{\omega_n}}(W)} \left| \frac{|W_i^n|}{|W|} \right| |JW_i^n T_{\omega_n}|_{C^0(W^n)} \leq C_2^{1-\zeta} \); 

(d) for \( \zeta > \zeta_0 \), \( \sum_{W_i^n \in \mathcal{I}_{n}^{T_{\omega_n}}(W)} |JW_i^n T_{\omega_n}|_{C^0(W^n)} \leq C_3^n \).


Proof. (a) Fix $W \in \mathcal{W}^n$ and for $\omega_n \in \Omega^n$, define $Z_n(W) = \sum_{i \in T^n_\omega} |J_{W_i}T_{\omega_i}|_*$, where $|J_{W_i}T_{\omega_i}|_*$ denotes the least contraction on $W^n_i$ under $T_{\omega_i}$ measured in the metric induced by the adapted norm. We will prove by induction on $n \in \mathbb{N}$ that $Z_n(W) \leq \theta_0^n$. Then, since $\| \cdot \|_*$ is equivalent to $\| \cdot \|$, statement (a) follows.

Note that at each iterate between 1 and $n$, every piece $W^n_i, i \in T^n_\omega(W)$, is created by genuine cuts due to singularities and homogeneity strips and not by any artificial subdivisions, since those are only made when a piece has grown to length greater than $\delta_0$. Thus we may apply the one-step expansion (3.6) to conclude,

\begin{equation}
(6.8)
Z_1(W) \leq \theta_0, \quad \forall W \in \mathcal{W}^n.
\end{equation}

Assume that $Z_n(W) \leq \theta_0^n$ is proved for some $n \geq 1$ and all $W \in \mathcal{W}^n$. We apply it to each component $W^1_i \in \mathcal{G}^n_1(W)$ such that $i \in T^n_\omega(W)$. Then $Z_n(W^1_i) \leq \theta_0^n$ since $W^1_i \in \mathcal{W}^n$. Given $\omega_n \in \Omega^n$, we use the notation $\omega_{n-k} = (\omega_n, \ldots, \omega_{n-k+1})$ so that we may split up compositions $\omega_n = (\omega_{n-k}, \omega_k)$ into two pieces. Given a sequence $\omega_{n+1}$, we group the components of $W^n_{i+1} \in \mathcal{G}^n_1(W)$ with $i \in T^n_\omega(W)$ according to elements with index in $T^n_\omega(W)$. More precisely, for $j \in T^n_\omega(W)$, let $A_j = \{ i : W^n_{i+1} \in \mathcal{G}^n_1(W), T_{\omega_i}W^n_{i+1} \subseteq W^n_j \}$. Note that $|J_{W^n_{i+1}}T_{\omega_{i+1}}|_* \leq |J_{W^n_{i+1}}T_{\omega_{i}}|_* |J_{W^n_j}T_{\omega_n}|_*$ whenever $T_{\omega_i}W^n_{i+1} \subseteq W^n_j$. Combining this and (6.8) with the inductive hypothesis, we get

\[
Z_{n+1}(W) = \sum_{j \in T^n_{\omega}(W)} \sum_{i \in A_j} |J_{W^n_{i+1}}T_{\omega_{i+1}}|_* \leq \sum_{j \in T^n_{\omega}(W)} \left( \sum_{i \in A_j} |J_{W^n_{i+1}}T_{\omega_{i}}|_* \right) |J_{W^n_j}T_{\omega_n}|_* = \sum_{j \in T^n_{\omega}(W)} Z_n(W^1_j) \cdot |J_{W^n_j}T_{\omega_n}|_* \leq \theta_0^{n+1}.
\]

(b) Fix $W \in \mathcal{W}^n$ and $\omega_n \in \Omega^n$. For any $0 \leq k \leq n$ and $W^n_i \in \mathcal{G}^n_i(W)$, we have

\begin{equation}
(6.9)
|J_{W^n_i}T_{\omega_i}|_{C^0(W^n_i)} \leq |J_{W^n_i}T_{\omega_i}|_{C^0(W^n_j)} |J_{W^n_j}T_{\omega_j}|_{C^0(W^n_j)},
\end{equation}

whenever $T_{\omega_{i-k}}W^n_i \subseteq W^n_j \in \mathcal{G}^n_k(W)$.

Now grouping $W^n_i \in \mathcal{G}^n_i(W)$ by most recent long ancestor $W^n_j \in L^n_k(W^n_k)$ as described in Section 3.5 and using (6.9), we have

\[
\sum_i |J_{W^n_i}T_{\omega_i}|_{C^0(W^n_i)} = \sum_{k=0}^n \sum_{W^n_k \in L^n_k(W^n_k)} \sum_{i \in \mathcal{Z}^n_k(W^n_k)} |J_{W^n_i}T_{\omega_i}|_{C^0(W^n_i)} \\
\leq \sum_{k=1}^n \sum_{W^n_k \in L^n_k(W^n_k)} \left( \sum_{i \in \mathcal{Z}^n_k(W^n_k)} |J_{W^n_i}T_{\omega_{i-k}}|_{C^0(W^n_i)} \right) |J_{W^n_j}T_{\omega_j}|_{C^0(W^n_j)} + \sum_{i \in \mathcal{Z}^n_k(W^n_k)} |J_{W^n_i}T_{\omega_i}|_{C^0(W^n_i)},
\]

where we have split off the terms involving $k = 0$ that have no long ancestor. We have

\[
|J_{W^n_j}T_{\omega_j}|_{C^0(W^n_j)} \leq (1 + C^*_d)|T_{\omega_k}W^n_j||W^n_k|^{-1} \leq 3\delta_0^{-1}(1 + C^*_d)|T_{\omega_k}W^n_j|
\]

since $|W^n_k| \geq \delta_0/3$. Since $T_{\omega_{i-k}}(W^n_j)$ and $T_{\omega_{i-k}}(W^n_j)$ correspond to the same set of short pieces in the $(n-k)$th generation of $W^n_j$, we apply part (a) of this lemma to each of these sums. Thus,

\[
\sum_i |J_{W^n_i}T_{\omega_i}|_{C^0(W^n_i)} \leq \sum_{k=0}^{n-1} \sum_{W^n_k \in L^n_k(W^n_k)} C_1\theta^n_{n-k} |J_{W^n_j}T_{\omega_j}|_{C^0(W^n_j)} + C_1\theta^n
\]

\[
\leq C\delta_0^{-1} \sum_{k=0}^{n-1} \sum_{W^n_k \in L^n_k(W^n_k)} \theta^n_{n-k} |T_{\omega_k}W^n_j| + C\theta^n \leq C\delta_0^{-1}|W| \sum_{k=0}^{n-1} \theta^n_{n-k} + C\theta^n,
\]

which is uniformly bounded in $n$. 
(c) follows from (b) by an application of Jensen’s inequality and (d) follows from (H3) using an inductive argument similar to the proof of (a).

We complete the proof of Theorem 2.10 via the following proposition. The uniform Lasota-Yorke inequalities of Theorem 2.2 then follow from the argument given at the beginning of Section 3.

**Proposition 6.3.** Choose ε ≤ ε₀ sufficiently small that σ(1 + ε) < 1 and let Δ(ν, g) ≤ ε. There exists a constant C, depending on a, A, and (H1)-(H5) such that for h ∈ B and n ≥ 0,

\[
|L_{ν,g}^n|_{H^w} \leq C\eta^n |h|_{H^w}
\]

(6.10)

\[
\|L_{ν,g}^n\|_s \leq C\eta^n (\|g\|_{L^∞}^1 + \Lambda^{-qn}) \|h\|_s + C\delta^{-\alpha} \eta^n |h|_{H^w}
\]

(6.11)

\[
\|L_{ν,g}^n\|_{H^u} \leq C\eta^n \Lambda^{-\beta n} \|h\|_{H^u} + C\eta^n C^3_\delta \|h\|_s
\]

(6.12)

**Proof.** We record for future use,

\[
L_{ν,g}^n(x) = \int_{\Omega^n} h \circ T_{ω_n}^{-1} (J_μ T_{ω_n} \circ T_{ω_n}^{-1})^{-1} \prod_{j=1}^n g(ω_j, T_{ω_j}^{-1} \circ \cdots \circ T_{ω_n}^{-1} x) dν^n(ω_n).
\]

(6.13)

The proofs of the inequalities are the same as in Section 3 except that we have the additional function g(ω, x). We show how to adapt the estimates of Section 4 to the operator L_{ν,g} in the case of the strong stable norm. The other estimates are similar.

**Estimating the Strong Stable Norm.** Following Section 4.2, we write,

\[
\int_W L_{ν,g}^n h \psi dm_W = \int_{\Omega^n} \sum_i \left\{ \int_{W_i^n} h(\psi \circ T_{ω_n} - \bar{ψ}_i)(J_μ T_{ω_n}^{-1})^{-1} J_μ^n T_{ω_n} \prod_{j=1}^n g(ω_j, T_{ω_j}^{-1} x) dm_W
\]

(6.14)

\[
+ \bar{ψ}_i \int_{W_i^n} h(J_μ T_{ω_n}^{-1})^{-1} J_μ^n T_{ω_n} \prod_{j=1}^n g(ω_j, T_{ω_j}^{-1} x) dm_W \right\} dν^n(ω_n),
\]

where \(\bar{ψ}_i = |W_i^n|^{-1} \int_{W_i^n} ψ \circ T_{ω_n} dm_W\). Since for each \(ω_n, T_{ω_n}\) satisfies properties (H1)-(H5) with uniform constants, we may use the estimates of Section 4. Accordingly, \(|ψ \circ T_{ω_n} - \bar{ψ}_i|_{C^0(W_i^n)} ≤ C\Lambda^{-qn}|W|^\alpha\) using (4.6). Define \(G_{ω_n}(x) = \prod_{j=1}^n g(ω_j, T_{ω_j}^{-1} x)\). We estimate the first term of (6.13) using (4.10)

\[
\sum_i \int_{W_i^n} h(\psi \circ T_{ω_n} - \bar{ψ}_i)(J_μ T_{ω_n}^{-1})^{-1} J_μ^n T_{ω_n} G_{ω_n} dm_W
\]

\[
\leq \sum_i C\|h\|_s |W_i^n|^{\alpha} |(J_μ T_{ω_n}^{-1})^{-1} J_μ^n T_{ω_n} G_{ω_n}|_{C^0(W_i^n)} |ψ \circ T_{ω_n} - \bar{ψ}_i|_{C^0(W_i^n)} G_{ω_n} |G_{ω_n}|_{C^0(W_i^n)}
\]

\[
\leq C\|h\|_s \Lambda^{-qn} \eta^n \sum_i \frac{|W_i^n|^{\alpha}}{|W|^\alpha} |J_μ^n T_{ω_n} G_{ω_n}|_{C^0(W_i^n)} G_{ω_n} |G_{ω_n}|_{C^0(W_i^n)}
\]

The only new term here is \(|G_{ω_n}|_{C^0(W_i^n)}\) which is addressed by the following lemma.

**Sublemma 6.4.** There exists \(C > 0\), independent of \(W\) and \(ω_n\), such that if \(W_i^n \in G_{ω_n}^n(W)\), then \(|G_{ω_n}|_{C^0(W_i^n)} ≤ CG_{ω_n}(x)\) for any \(x \in W_i^n\).

**Proof of Sublemma.** For \(x, y \in W_i^n\),

\[
\log \prod_{j=1}^n g(ω_j, T_{ω_j}^{-1} x) \prod_{j=1}^n g(ω_j, T_{ω_j}^{-1} y) \leq \sum_{j=1}^n a^{-1} |g(ω_j, \cdot)|_{C^0(M)} d(T_{ω_j}^{-1} x, T_{ω_j}^{-1} y)
\]

\[
\leq \sum_{j=1}^\infty a^{-1} A_\epsilon \Lambda^{-n} d(x, y) =: c_0 d(x, y),
\]

using properties (i) and (iii) of \(g\). The distortion bound yields the lemma with \(C = c_0 \epsilon^{a_0}\). \(\square\)
We estimate (6.14) using the sublemma and Lemma (6.2)(c),

\[(6.15) \sum_i \int_{W_i^n} h(\psi \circ T_{\mathcal{F}_n} - \overline{\psi}_i) (J_{\mu} T_{\mathcal{F}_n})^{-1} J_{W_i^n} T_{\mathcal{F}_n} G_{\mathcal{F}_n} dm_W \leq C\|h\|_s \eta^n \Lambda^{-qn} G_{\mathcal{F}_n}(x_0),\]

where \(x_0\) is some point in \(T_{\mathcal{F}_n}^{-1}W\).

Similarly, we estimate the second term in (6.13) using (4.11). In each term, \(G_{\mathcal{F}_n}\) plays the role of a test function and we replace the occurrences of \(|G_{\mathcal{F}_n}|_{C^j(W)}\) and \(|G_{\mathcal{F}_n}|_{C^j(W^*)}\) as appropriate according to Sublemma 6.4. Thus following (4.11), we write,

\[\sum_i \overline{\psi}_i \int_{W_i^n} h(J_{\mu} T_{\mathcal{F}_n})^{-1} J_{W_i^n} T_{\mathcal{F}_n} G_{\mathcal{F}_n} dm_W \leq C(\delta_0^{-\alpha} \eta^n |h|_w + \theta_s^{(1-\alpha)n} \|h\|_s)G_{\mathcal{F}_n}(x_0),\]

choosing the same \(x_0\) as in (6.15). Now combining this expression with (6.15) and (6.13), we obtain

\[\int_W \mathcal{L}_{T_{\mathcal{F}_n}} h\psi \ dm_W \leq C\eta^n (\|h\|_s \Lambda^{-qn} + \theta_s^{(1-\alpha)n}) + C\delta_0^{-\alpha} \eta^n |h|_w\]

which is the required inequality for the strong stable norm. The inequalities for the weak norm and the strong unstable norm follow similarly, always using Sublemma 6.4.

7. Proofs of Applications: External Forces with Kicks and Slips

In this section we prove Theorem 2.8 and 2.9 for the perturbed dispersing billiards with external forces with kicks and slips. To simplify the analysis, for any fixed force \(F\), we will consider our system, denoted as \(T_{F,G}\), as a perturbation of the map \(T_{F,0}\). We say a constant \(C\) is uniform if \(C = C(\xi_1, \tau_1, \kappa_1, E_1)\), where \(\xi_1, \tau_1, \kappa_1, E_1\) are from (A2) and (A3).

We begin by reviewing some properties of \(T_{F} = T_{F,0}\) proved in [Ch2] and proving some additional ones that we shall need.

7.1. Properties of \(T_{F}\). We assume the setup described in Section 2.3, B, which is the billiard flow given by \(2.4\) and \(2.5\) with \(G = 0\).

Let \(x = (\mathbf{q}, \theta) \in M\) be any phase point with position \(\mathbf{q}\) and \(V \in T_xM\) a tangent vector at \(x\). Pick a small number \(\delta_0 > 0\) and a \(C^3\) curve \(c_s(0) = (\mathbf{q}_s, \theta_s) \subset M\) tangent to the vector \(V\), such that \(c_0 = x\) and \(\frac{dc_s}{ds}|_{s=0} = V\), and \(s \in [0, \delta_0]\). Now we define \(c_s(t) = \tilde{F}^t c_s(0)\), for any \(t \geq 0\). Since \(\tau\) is the free path function, we have \(d\tau = pdt\). In the calculation below, we denote differentiation with respect to \(s\) by primes and that with respect to \(\tau\) by dots. In particular, \(c_\xi(t) = (\tilde{q}_\xi, \theta) = (\mathbf{v}, h)\), where \(\mathbf{v} = p/p = (\cos \theta, \sin \theta)\) and \(h = h(\mathbf{q}, \theta)\) is the geometric curvature of the billiard trajectory with initial condition \((\mathbf{q}, \theta)\) on the table.

If we assume \(t = t_s\) to be the time that the trajectory of \(c_s(0)\) hits the wall of the billiard table, then \(\{c_s(t) | t \in [0, t_s], s \in [0, \delta_0]\}\) is a \(C^3\) smooth 2-d manifold in \(M\). We introduce two quantities \(u = \mathbf{q}' \cdot \mathbf{v}\), and \(w = \mathbf{q}' \cdot \mathbf{v}^\perp\), where \(\mathbf{v}^\perp = (-\sin \theta, \cos \theta)\). Clearly \(\mathbf{q}' = uv + wv^\perp\). Now let \(\kappa = (\theta' - uh)/w\). We consider two vectors of the surface \(L = (v, h)\) and \(R = (v^\perp, \kappa)\). Clearly \(\dot{c}_u = U\) and \(\dot{c}_v = uU + wR\). Define \(p_U = \text{grad}(p) \cdot U\), \(p_R = \text{grad}(p) \cdot R\), and \(h_U = \text{grad}(h) \cdot U\), \(h_R = \text{grad}(h) \cdot R\), respectively. Then it is straightforward to check that

\[(7.1) \quad p' = \text{grad}(p) \cdot \dot{c}_u = p_U u + p_R w, \quad h' = h_U u + h_R w, \quad \text{and} \quad \theta' = \kappa w + hu.\]

In addition \(\dot{p} = p_U\) and \(\dot{h} = h_U\). The derivation of these formulas can be found in [Ch2]. The following lemma was proved in [Ch2] Lemmas 3.1, 3.2].
Lemma 7.1 ([Ch2]). The evolution of the quantities \( \kappa \) and \( w \) between collisions is given by the equations
\[
(7.2) \quad \dot{\kappa} = -\kappa^2 + a + b\kappa \quad \text{and} \quad \dot{w} = \kappa w,
\]
where \( a = a(h) \), \( b = b(h) \) are smooth functions whose \( C^0 \) norms are bounded by \( c_0\varepsilon_1 \) for some uniform \( c_0 > 0 \). Furthermore, at the moment of collision,
\[
(7.3) \quad u^+ = u^-, \quad w^+ = -w^- \quad \text{and} \quad \kappa^+ = \kappa^- + \frac{2\mathcal{K}(r) + (h^+ + h^-)\sin \varphi}{\cos \varphi}.
\]
In addition the derivative of \( r \) and \( \varphi \) satisfies
\[
(7.4) \quad \frac{dr}{ds} = \mp w^\pm / \cos \varphi \quad \text{and} \quad \frac{d\varphi}{dr} = \mp \mathcal{K}(r) + \kappa^\pm \cos \varphi \mp h^\pm \sin \varphi.
\]

We will calculate the differential of the map \( T_F \) (which is not contained in [Ch2]). It follows from (7.2) that
\[
(7.5) \quad \frac{d\dot{w}}{d\tau} = \frac{d}{d\tau}(\kappa w) = \dot{\kappa}w + \kappa \dot{w} = \kappa w - \kappa^2 w + (a + b\kappa) w = aw + \dot{w}.
\]
This implies that
\[
(7.6) \quad \begin{cases}
\dot{w}(\tau) = \dot{w}(0) + \int_0^\tau aw + \dot{w} d\gamma \\
\quad w(\tau) = w(0) + \dot{w}(0)\tau + \int_0^\tau \int_0^\xi aw + \dot{w} d\gamma d\xi.
\end{cases}
\]
At the moment of collision, (7.3) implies that
\[
(7.7) \quad \begin{cases}
w^+ = -w^- \\
\quad \dot{w}^+ = -\dot{w}^- - \frac{2\mathcal{K}(r) + (h^+ + h^-)\sin \varphi}{\cos \varphi} w^-.
\end{cases}
\]
In addition (7.4) implies that
\[
(7.8) \quad \frac{d\varphi}{ds} = \frac{\mathcal{K}(r) + h^\pm \sin \varphi}{\cos \varphi} w^\pm + \dot{w}^\pm.
\]

Lemma 7.2. For \( x = (r, \varphi) \), let \( \tau_1(x) \) denote the distance to the next collision under the flow. There exist constants \( \tilde{C}_1, \tilde{C}_2 > 0 \) independent of \( x \), such that \( |w(\tau)| \) and \( |\dot{w}(\tau)| \) are uniformly bounded from above by \( \tilde{C}_1|w^+(0)| + \tilde{C}_2|\dot{w}^+(0)| \) for \( \tau \in [0, \tau_1(x)] \).

Proof. We fix \( x \) and abbreviate \( \tau_1(x) \) as \( \tau_1 \). We begin by adapting [Ch2] Lemma 3.4, to show that if for some \( \tau_0 \in [0, \tau_1) \), \( \kappa(\tau_0) \) is bounded away from zero, then \( \kappa \) is bounded away from zero and infinity on \([\tau_0, \tau_1]\). More precisely, (7.2) implies that if \( \kappa > 0 \), then
\[
-(\kappa + \varepsilon_2)^2 \leq \kappa = -\kappa^2 + b\kappa + a = -(\kappa - \frac{b^2}{4})^2 + \frac{b^2}{4} + a \leq -(\kappa - c_0\varepsilon_1)^2 + \varepsilon_2^2
\]
where \( \varepsilon_2^2 = 2c_0\varepsilon_1 \).

So if we assume that for some \( \tau_0 \in [0, \tau_1) \), \( \kappa(\tau_0) > c_1 \) for a fixed \( c_1 > 5\sqrt{\varepsilon_0} \), then we may integrate these inequalities to obtain
\[
\frac{1}{(\kappa^+(\tau_0) + \varepsilon_2)^{-1} + (\tau - \tau_0)} - \varepsilon_2 \leq \kappa(\tau) \leq \frac{1}{(\kappa^+(\tau_0) + \varepsilon_2)^{-1} + (\tau - \tau_0)} + \varepsilon_2 - \varepsilon_2 + c_0\varepsilon_1,
\]
where \( A = (\kappa^+(\tau_0) - c_0\varepsilon_1 + \varepsilon_2)/(\kappa^+(\tau_0) - c_0\varepsilon_1 - \varepsilon_2) \). Then since \( \varepsilon_0 \) is small compared to \( \kappa^+(\tau_0) \), this reduces to
\[
(7.9) \quad \frac{1}{(\kappa^+(\tau_0) - \varepsilon_3)^{-1} + (\tau - \tau_0)} - \varepsilon_3 \leq \kappa(\tau) \leq \frac{1}{(\kappa^+(\tau_0) - \varepsilon_3)^{-1} + (\tau - \tau_0)} + \varepsilon_3
\]
where \( \varepsilon_3 = 2\varepsilon_2 + 2c_0\varepsilon_1 \).

Now (7.2) implies that for any \( 0 \leq \tau' < \tau \leq \tau_1 \),
\[
(7.10) \quad w(\tau) = w(\tau') \exp \left( \int_{\tau'}^\tau \kappa d\gamma \right).
\]
Also, (7.5) implies that $\frac{d}{d\tau} \log w = (a + b\kappa) d\tau$ and since $w = \kappa w$, we integrate this to obtain,

\begin{equation}
\dot{w}(\tau) = \dot{w}(0) \exp \left( \int_0^\tau \left( \frac{a}{\kappa} + b \right) d\gamma \right) \quad \text{for any } \tau \in [0, \tau_1].
\end{equation}

Integrating again, it follows that

\begin{equation}
w(\tau) = w(0) + \dot{w}(0) \int_0^\tau \exp \left( \int_0^\xi \left( \frac{a}{\kappa} + b \right) d\gamma \right) \, d\xi.
\end{equation}

This implies that both $w(\tau), \dot{w}(\tau)$ are functions of $(w^+(0), \dot{w}^+(0))$.

To show that $|w|$ and $|\dot{w}|$ are uniformly bounded, we consider three cases.

Case I: $\kappa$ is finite on $[0, \tau_1)$ and $\kappa(\tau) < 1/\tau_{\min}$ for all $\tau \in [0, \tau_1)$ ($\kappa$ can be positive or negative). Then by (7.10), $|w(\tau)| \leq |w(0)| e^{\gamma / \tau_{\min}} \leq |w(0)| e^{\tau_{\max} / \tau_{\min}}$ for all $\tau \in [0, \tau_1]$. Thus

\begin{equation}
|w(\tau)| \leq |w(0)| e^{(2\kappa_0 + 1) / \tau_{\min}} \leq |w(0)| e^{\tau_{\max} / \tau_{\min}} \leq C_1 \kappa(0) + C_2 |w^+(0)|.
\end{equation}

Case II: $\kappa$ is finite on $[0, \tau_1)$, $\kappa(\tau_0) > 1/\tau_{\min}$ for some $\tau_0 \in [0, \tau_1]$ and $\tau_0$ is the least $\tau$ in the interval with this property. Then by (7.9), $\kappa(\tau) \geq (\tau_{\min} + 2\tau_{\max})^{-1}$ for all $\tau \in [0, \tau_1]$. As a consequence, by (7.12),

\begin{equation}
|w(\tau)| \leq |w(\tau_0)| + |\dot{w}(\tau_0)| \tau_{\max} e^{(\kappa_0 + 1)(\tau_{\max} + 2\tau_{\max} + 1)}
\end{equation}

for each $\tau \in [\tau_0, \tau_1]$. On the other hand, for $\tau \in [0, \tau_0)$, we have $\kappa(\tau) \leq 1/\tau_{\min}$, so that both $|w(\tau)|$ and $|\dot{w}(\tau)|$ are uniformly bounded on this interval by Case I. This together with (7.15) proves Case II for $|w|$. The estimate for $|\dot{w}|$ follows again from (7.13) and (7.14).

Case III: $\kappa(\tau_0) = \pm \infty$ for some $\tau_0 \in (0, \tau_1)$. According to (7.2) and (7.9), the only way this case can occur is if $\kappa$ reaches $-\infty$ in finite time and changes from $-\infty$ to $\infty$ at $\tau_0$. (7.10) implies in particular that $w(\tau_0) = 0$.

On the interval $[0, \tau_0)$, $\kappa$ clearly satisfies the assumption of Case I so that both $|w|$ and $|\dot{w}|$ are uniformly bounded as in the statement of the lemma on this interval. Indeed, this is true on any interval in which $\kappa$ remains negative. Thus the only case left to consider is when $\kappa(\tau) > 0$ for $\tau \in (\tau_0, \tau_1]$.

In this case, (7.2) guarantees that $\kappa$ initially decreases and (7.9) guarantees that $\kappa(\tau) \geq \tau_{\min}^{-1}$ on this interval. Thus by (7.12), we estimate as in (7.15) to bound $|w|$ by a linear combination of $|w(\tau_0)|$ and $|\dot{w}(\tau_0)|$. But since these two quantities are in turn bounded by $|w^+(0)|$ and $|\dot{w}^+(0)|$ by the previous paragraph, the proof of Case III is complete for $|w|$. The estimate on $|\dot{w}|$ now follows again from (7.13) and (7.14).

Combining the above facts, we can show the following.

**Lemma 7.3.** If we denote $x_1 = (r_1, \varphi_1) = T_F x$, then there exists $C = C(\mathcal{K}_s, \tau_*) > 0$ such that for any unit vector $(dr/ds, d\varphi/ds)$,

\begin{equation}
\left\{ \begin{array}{l}
- \cos \varphi \frac{d\varphi}{ds} = (\cos \varphi + \tau K + a_1) \frac{dr}{ds} + (\tau + a_2) \frac{d\varphi}{ds} \\
- \cos \varphi \frac{d\varphi}{ds} = (K_1 K + K_1 \cos \varphi + K \cos \varphi_1 + b_1) \frac{dr}{ds} + (K_1 \tau + \cos \varphi_1 + b_2) \frac{d\varphi}{ds}
\end{array} \right.
\end{equation}

where $a_i \leq C \varepsilon_1$ and $b_i \leq C \varepsilon_1$, for $i = 1, 2$. In addition

\begin{equation}
(1 - C \varepsilon_1) \cos \varphi_1 \cos \varphi \leq |\det D_x T_F| \leq (1 + C \varepsilon_1) \cos \varphi_1.
\end{equation}

**Proof.** Let $x_1 = T_F x$, and $\tau_1(x)$ be the length of the free path of $x$. By (7.11) and (7.12), there exists a linear transformation $D_x$ such that

\begin{equation}
D_x(w^+, \dot{w}^+)^T = (w_1^-, \dot{w}_1^-)^T.
\end{equation}
where \( w_1^- = w^-(\tau_1) \) and \( \dot{w}_1^- = \dot{w}^- (\tau_1) \). Indeed, by Lemma 7.2 there exist smooth functions \( c_i, i = 1, \ldots, 4 \) with \( |c_i| \leq C \varepsilon_1 \) for some \( C = C(\mathcal{K}, \tau_\epsilon) > 0 \) such that

(7.19) \[ I := \int_0^\tau aw + b \dot{w} \, d\gamma = c_1 w^+(0) + c_2 \dot{w}^+(0), \quad II := \int_0^\tau aw + b \dot{w} \, d\xi = c_3 w^+(0) + c_4 \dot{w}^+(0), \]

so that using (7.6), we may write

\[
D_x = \begin{pmatrix}
1 + c_3 & \tau + c_4 \\
c_1 & 1 + c_2
\end{pmatrix}.
\]

Using (7.21) and (7.8), the differential of \( DT_F \) satisfies

(7.21) \[ DT_F = N_{x_1}^{-1} L_{x_1} D_x N_x \]

where

\[
N_x = - \begin{pmatrix}
\cos \varphi & 0 \\
\mathcal{K} + h^+ \sin \varphi & 1
\end{pmatrix}
\]

is the coordinate transformation matrix on \( T_x M \), such that \( (w^+(0), \dot{w}^+(0))^T = N_x (dr/ds, d\varphi/ds)^T \), and

(7.22) \[ L_{x_1} = \begin{pmatrix}
-1 & 0 \\
-2 K_1 + (h_+^+ + h_0) \sin \varphi_1 & -1
\end{pmatrix} \quad \text{and} \quad N_{x_1}^{-1} = \begin{pmatrix}
\frac{1}{\cos \varphi_1} & 0 \\
-\frac{K_1 h^+ \sin \varphi_1}{\cos \varphi_1} & -1
\end{pmatrix}.
\]

Now combining (7.6) with (7.19) and (7.21), we get

\[-\cos \varphi_1 \frac{dr_1}{ds} = (\cos \varphi + \tau \mathcal{K} + \tau h^+ \sin \varphi) \frac{dr}{ds} + \tau \frac{d\varphi}{ds} - II \]

\[= (\cos \varphi + \tau \mathcal{K} + \tau h^+ \sin \varphi) \frac{dr}{ds} + \tau \frac{d\varphi}{ds} - c_3 w^+ - c_4 \dot{w}^+ \]

\[= (\cos \varphi + \tau \mathcal{K} + 1) \frac{dr}{ds} + (\tau + a_2) \frac{d\varphi}{ds} \]

where \( a_1 = c_3 \cos \varphi + c_4 \mathcal{K} + h^+ \sin \varphi + \tau h^+ \sin \varphi \) and \( a_2 = c_4 \). Similarly we obtain

\[-\cos \varphi_1 \frac{d\varphi_1}{ds} = - (K_1 + h^- \sin \varphi_1) w^- - \cos \varphi_1 \dot{w}^- \]

\[= - (K_1 + h^- \sin \varphi_1) (w^+ + \dot{w}^+ \tau + II) - \cos \varphi_1 (\dot{w}^+ + I) \]

\[= \left[ (K_1 + h^- \sin \varphi_1) \cos \varphi + (\tau (K_1 + h^- \sin \varphi_1) + \cos \varphi_1) (K_1 + h^+ \sin \varphi) \right] \frac{dr}{ds} \]

\[+ (\tau K_1 + h^- \sin \varphi_1 + \cos \varphi_1) \frac{d\varphi}{ds} - II (K_1 + h^- \sin \varphi_1) - I \cos \varphi_1 \]

\[= (\tau K_1 \mathcal{K} + K_1 \cos \varphi + K \cos \varphi_1 + b_1) \frac{dr}{ds} + (K_1 \tau + \cos \varphi_1 + b_2) \frac{d\varphi}{ds} \]

where

\[b_1 = (\cos \varphi + \tau \mathcal{K}) h^- \sin \varphi_1 + \cos \varphi_1 (c_1 \cos \varphi + c_2 \mathcal{K} + (1 + c_2) h^+ \sin \varphi) \]

\[+ (c_3 \cos \varphi + h^+ \sin \varphi + c_4 (K + h^+ \sin \varphi)) (K_1 + h^- \sin \varphi) \]

and \( b_2 = (\tau + c_4) h^- \sin \varphi_1 + c_4 K_1 + c_2 \cos \varphi_1 \).

Now we use the assumption that the quantities \( K, \tau \) are uniformly bounded from above, and \( |h^\pm| = O(\varepsilon_1) \), to obtain that for any unit vector \( (dr/ds, d\varphi/ds) \), the quantities \( |a_i| \leq C \varepsilon_1 \) and \( |b_i| \leq C \varepsilon_1, i = 1, 2 \), for some uniform \( C > 0 \).

Finally we use (7.21) to calculate the determinant of the differential \( D_x T_F \),

\[
det D_x T_F = \det N_{x_1}^{-1} \cdot \det L_{x_1} \cdot \det D_x \cdot \det N_x = \frac{\cos \varphi}{\cos \varphi_1} \det D_x
\]

(7.23)

\[= \frac{\cos \varphi_1}{\cos \varphi_1} ((1 + c_2)(1 + c_3) - c_1 (\tau + c_4)) \]

which implies the last inequality (7.17). \qed
It follows from the above lemma that the differential $DT_F : T_xM \to T_{x_1}M$ at any point $x = (r, \varphi) \in M$ is the $2 \times 2$ matrix:

$$ DT_F(x) = -\frac{1}{\cos \varphi_1} \begin{pmatrix} \tau K + \cos \varphi & a_1 \\ K(r_1)(\tau K + \cos \varphi) + K \cos \varphi_1 + b_1 & \tau + a_2 \end{pmatrix} $$

where $x_1 = T_F(x) = (r_1, \varphi_1)$.

Furthermore, it was shown in [Ch01] that the map $T_F$ has two families of cones $\hat{C}^u(x)$ (unstable) and $\hat{C}^s(x)$ (stable) in the tangent spaces $T_xM$, for all $x \in M$. More precisely, the unstable cone $\hat{C}^u(x)$ contains all tangent vectors based at $x$ whose images generate dispersing wave fronts:

$$ \hat{C}^u(x) = \{ (dr, d\varphi) \in T_xM : B_0^{-1} \leq d\varphi/dr \leq B_0 \}. $$

The unstable cone $\hat{C}^u(x)$ is strictly invariant under $DT_F$. Similarly the stable cone

$$ \hat{C}^s(x) = \{ (dr, d\varphi) \in T_xM : -B_0^{-1} \geq d\varphi/dr \geq -B_0 \} $$

is strictly invariant under $DT_F^{-1}$. Here $B_0 = B_0(\varepsilon_1, \tau_*, \kappa_*) > 1$ is a uniform constant. Indeed, there exists a uniform constant $C > 0$ such that we can choose $B_0 = \kappa_*^{-1} + 2\tau_*^{-1} + C\varepsilon_1$ for all $\varepsilon_1$ sufficiently small.

Let $dx = (dr, d\varphi) \in T_xM$. Following [CM, Section 5.10], we define the adapted norm $\| \cdot \|_*$ by

$$ \| dx \|_* = \frac{\kappa(x) + |V|}{\sqrt{1 + V^2}} \| dx \|, \quad \forall dx \in C^s(x) \cup C^u(x), $$

where $\| dx \| = \sqrt{dr^2 + d\varphi^2}$ is the Euclidean norm. Since the slopes of vectors in $C^s(x)$ and $C^u(x)$ are bounded away from $\pm \infty$, we may extend $\| \cdot \|_*$ to all of $\mathbb{R}^2$ in such a way that $\| \cdot \|_*$ is uniformly equivalent to $\| \cdot \|$. It is straightforward to check that for $dx \in C^u(x)$,

$$ \frac{\| dx_1 \|_*}{\| dx \|_*} \geq \hat{A} := 1 + \kappa_{\min} \tau_{\min}/2. $$

Finally, a simple calculation using (7.24) shows that there exists a constant $B_1 = B_1(\kappa_*, \tau_{\min}, \tau_{\max}) > 0$ such that

$$ \frac{B_1^{-1}}{\cos \varphi(x)} \leq \frac{\| dx_1 \|}{\| dx \|} \leq \frac{B_1}{\cos \varphi(x_1)}, \quad \text{for all } dx \in C^u(x). $$

Uniform expansion in $C^s(x)$ under $DT_F^{-1}(x)$ follows similarly. (See also [Ch2 Sect. 3].)

### 7.2. Hyperbolicity of the perturbed map $T_{F,G}$

We are now ready to verify conditions (H1)-(H5) for the map $T_{F,G}$. We do this fixing $F, G$ satisfying assumptions (A1)-(A4) with $|F|_{C^1}, |G|_{C^1} \leq \varepsilon$ for some $\varepsilon \leq \varepsilon_1$. We then compare $T = T_{F,G}$ with the related map $T_F = T_{F,0}$.

Since $G$ preserves tangential collisions, the discontinuity set of $T$ is the same as that of $T_F$, which comprises the preimage of $S_0 := \{ \varphi = \pm \pi/2 \}$. Similarly, the singularity sets of $T^{-1}$ and $T_F^{-1}$ are the same due to (A4). But the singular sets for higher iterates are not the same. Let $S_{n+1}^T = \bigcup_{i=0}^n T_{r-H}^{-i} S_{0,H}^T$ with $n \in \mathbb{N}$. Then $T_{r-H}^{-n}$ is smooth on $M \setminus S_{n+1}^T$.

For any phase point $x = (r, \varphi) \in M$, let $Tx = (\bar{r}_1, \bar{\varphi}_1)$ and $T_Fx = (r_1, \varphi_1)$. According to (A3) and (A4) and since we are on a fixed integral surface, we may express $G$ in local coordinates via two smooth functions $g^1$ and $g^2$ such that $g^i(r, \pm \pi/2) = 0$, $i = 1, 2$, and

$$ \bar{r}_1 = r_1 + g^1(r_1, \varphi_1) \quad \text{and} \quad \bar{\varphi}_1 = \varphi_1 + g^2(r_1, \varphi_1) $$

where $g^i$ is a $C^2$ function with $C^1$ norm uniformly bounded from above by $c_9 \varepsilon$, for some uniform constant $c_9 > 0$.

According to (7.29), the differential of $T$ satisfies

$$ d\bar{r}_1 = (1 + g^1_1(r_1, \varphi_1)) dr_1 + g^1_2(r_1, \varphi_1) d\varphi_1 \quad \text{and} \quad d\bar{\varphi}_1 = g^2_1(r_1, \varphi_1) dr_1 + (1 + g^2_2(r_1, \varphi_1)) d\varphi_1 $$

where $g^1_1(r_1, \varphi_1) = \partial g^1/\partial r_1$ and $g^1_2(r_1, \varphi_1) = \partial g^1/\partial \varphi_1$. This implies

$$ DT(x) = \begin{pmatrix} 1 + g^1_1(r_1, \varphi_1) & g^1_2(r_1, \varphi_1) \\ g^2_1(r_1, \varphi_1) & 1 + g^2_2(r_1, \varphi_1) \end{pmatrix} DT_F(x) $$

where $x_1 = T_F(x) = (r_1, \varphi_1)$. Considering the case of $C^u(x)$, we can apply the same arguments as for $C^s(x)$, and thus we have verified that $T_{F,G}$ is hyperbolic.
Note that $T$ is not a $C^1$ perturbation of $T_F$ around the boundary of $M$. Furthermore, $T$ no longer preserves $\mu_F$, the SRB measure for $T_F$. However, it follows from (7.17) and (7.31) that

\begin{equation}
|\det DT(x)| \leq \frac{\cos \varphi(x) \cos \tilde{\varphi}_1(x)}{\cos \varphi_1(x) \cos \varphi_1(x)} (1 + C\varepsilon) \leq \frac{\cos \varphi(x)}{\cos \varphi_1(x)} (1 + C_1\varepsilon)
\end{equation}

since by (7.29),

\begin{equation}
\frac{\cos \tilde{\varphi}_1(x)}{\cos \varphi_1(x)} = \frac{\cos(\varphi_1(x) + g^2(x_1))}{\cos \varphi_1(x)} \leq (1 + C'\varepsilon)
\end{equation}

since $g^2(r, \pm \pi/2) = 0$ and $|\nabla g^2| \leq C\varepsilon$. Clearly this implies condition (H5).

The next proposition shows that although the perturbed maps do not have the same families of stable/unstable manifolds, they do share common families of stable and unstable cones.

**Proposition 7.4.** There exist two families of cones $C^u(x)$ (unstable) and $C^s(x)$ (stable) in the tangent spaces $T_x M$ and $\Lambda > 1$, such that for all $x \in M$:

1. $DT(C^u(x)) \subset C^u(Tx)$ and $DT(C^s(x)) \supset C^s(Tx)$ whenever $DT$ exists.
2. These families of cones are continuous on $M$ and the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero.
3. $\|D_x T(v)\|_* \geq \Lambda \|v\|_*, \forall v \in C^u(x)$ and $\|D_x T^{-1}(v)\|_* \geq \Lambda \|v\|_*, \forall v \in C^s(x)$.

**Proof.** For $x \in M$ and any unit vector $dx \in T_x M$, let $d\bar{x} = D_x T_F dx$. Then by (7.30) the slope $\bar{V}_1$ of the vector $d\bar{x}_1$ at $\bar{x}_1 := Tx = (\bar{r}_1, \bar{\varphi}_1)$ satisfies

\begin{equation}
\bar{V}_1 = \frac{g_1^2 + (1 + g_2^2)\bar{V}_1}{1 + g_1^2 + g_2^2\bar{V}_1} = V_1 + O(\varepsilon)
\end{equation}

So the cone $\tilde{C}^u(x)$ from (7.25) may not be invariant under $DT(x)$. Accordingly, we define a slightly bigger cone,

\begin{equation}
C^u(x) = \{(d\bar{r}, d\varphi) \in T_x M : B_0^{-1}(1 - c_1\varepsilon_1) \leq d\varphi/d\bar{r} \leq B_0(1 + c_2\varepsilon_1)\}
\end{equation}

for some constants $c_1, c_2 > 0$, and we use assumption (A2) to ensure that $c_i\varepsilon_1 < 1/2$, $i = 1, 2$. By (7.24), $D_T F$ maps the first and third quadrants strictly inside themselves and shrinks any cones larger than the unstable cones. More precisely, let $V$ be a unit vector on the upper boundary of $C^u(x)$, with slope $V = B_0(1 + c_2\varepsilon_1)$. Then by (7.24) the slope of $D_T V$ satisfies $V_1 = \frac{C + DB_{0}(1 + c_2\varepsilon_1)}{A + BB_{0}}$, where we denote

\begin{equation}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\tau K + \cos \varphi + a_1 \\
K(\varphi_1)(\tau K + \cos \varphi) + K \cos \varphi_1 + b_1 \\
\tau K(\varphi_1) + \cos \varphi_1 + b_2
\end{pmatrix}.
\end{equation}

It follows from the invariance of $\tilde{C}^u$ that $C + DB_{0}/A + BB_{0} < B_0$. One can easily check that

\begin{equation}
V_1 = \frac{C + DB_{0}(1 + c_2\varepsilon_1)}{A + BB_{0}(1 + c_2\varepsilon_1)} < V = B_0(1 + c_2\varepsilon_1)
\end{equation}

Similarly we can check the lower boundary of the cone is also mapped inside the cone $C^u$. Thus $C^u$ is invariant under $DT$.

Similarly we define the stable cone $C^s(x)$ as

\begin{equation}
C^s(x) = \{(d\bar{r}, d\varphi) \in T_x M : -B_0^{-1}(1 - c_1\varepsilon_1) \geq d\varphi/d\bar{r} \geq -B_0(1 + c_2\varepsilon_1)\}\,
\end{equation}

Then one can check that the stable cone $\tilde{C}^s$ is strictly invariant under $DT^{-1}$ whenever $DT^{-1}$ exists for any $T \in \mathcal{F}$. From the definitions of $C^s(x)$ and $C^u(x)$, it is clear that the angle between them is bounded away from 0 on $M$. Thus items (1) and (2) of the lemma are proved.

To prove (3), note that (7.26) implies,

\begin{equation}
\frac{\|dx\|_*}{\|dx\|_*} = \frac{\|d\bar{x}\|_*}{\|dx\|_*} = \frac{\|dx\|_* \|d\bar{x}\|_*}{\|dx\|_* \|dx\|_*} = \frac{\|dx\|_* \|d\bar{x}\|_*}{\|dx\|_* \|d\bar{x}\|_*} = \frac{\|dx\|_* \|d\bar{x}\|_*}{\|dx\|_* \|d\bar{x}\|_*}
\end{equation}
Using (7.29), (7.30), (7.27) and the fact that $\mathcal{K}(\cdot)$ is a $C^1$ function on $M$, we conclude that for $\epsilon_0 = 1$ small enough,

$$
(7.35) \quad \frac{\|d\bar{x}_1\|}{\|dx\|} \geq \Lambda := 1 + K_{\min} \tau_{\min}/3.
$$

Similarly, one can show property (3) for stable cones, which we will not repeat here.

Near grazing collisions, we have also using (7.28) and (7.33) along with (7.29) and (7.30),

$$
(7.36) \quad B_1^{-1}(1 - C \epsilon_1) \leq \frac{\|d\bar{x}_1\|}{\|dx\|} = \frac{\|d\bar{x}_1\|}{\|dx\|} \leq \frac{B_1(1 + C \epsilon_1)}{\cos \phi_1},
$$

which establishes (3.3) in (H1) since in the finite horizon case, there are only finitely many singularity curves so we may take $n$ in that formula to be 1.

The last formula (3.4) in (H1) (again with $n = 1$) follows directly from differentiating (7.31) and using (7.24) to recover this standard estimate for the unperturbed billiard (see [KS] or [Ch2] Sect. 9.9) for the classical result). This finishes the verification of (H1).

7.3. Regularity of stable and unstable curves. It follows from Proposition 7.4 that we may define common families of stable and unstable cones for all perturbations $T \in \mathcal{F}_B(Q_0, \tau_*, \epsilon_1)$. Recall the homogeneity strips $\mathbb{H}_k$ defined in Section 3.1 and that a homogeneous curve in $M$ is a curve that lies in a single homogeneity strip. In this subsection we will show that there is a class of $C^2$ smooth unstable homogeneous curves $\hat{\mathcal{W}}^{un}$ in $M$ which is invariant under any $T \in \mathcal{F}$. Furthermore these curves are regular in the sense that they have uniformly bounded curvature and distortion bounds. Similarly, there is an invariant class of homogeneous stable curves, $\hat{\mathcal{W}}^{us}$.

7.3.1. Curvature bounds. The next lemma, proved in $T_F$ in [Ch2], states that the images of an unstable curve are essentially flattened under the map $T_F$.

**Lemma 7.5.** Let $W \subset M$ be a $C^2$-smooth unstable curve with equation $\varphi_0 = \varphi_0(r_0)$ such that $T_F^iW$ is a homogeneous unstable curve for each $0 \leq i \leq n$. Then $T_F^iW$ has equation $\varphi_n = \varphi_n(r_n)$ which satisfies:

$$
(7.37) \quad \frac{d^2 \varphi_n}{dr_n^2} \leq C_1 + \theta^{3n} \frac{d^2 \varphi_0}{dr_0^2} \leq C_2
$$

where $C_i = C_i(Q)$, $i = 1, 2$ is a constant and $\theta \in (0, 1)$. Furthermore, for any regular unstable curve $W$, there exists $n_W \geq 1$, such that for any $n > n_W$, every smooth curve of $T^nW$ has uniformly bounded curvature.

One can obtain a similar bounded curvature property for the perturbed map $T$.

**Proposition 7.6.** (Curvature bounds) Let $W$ be any $C^2$ smooth unstable curve. Then there exists $n_W \geq 1$ and $C_b > 0$ such that every smooth curve $W' \subset T^nW$ with equation $\varphi_n = \varphi_n(r_n)$ satisfies

$$
(7.38) \quad \left| \frac{d^2 \varphi_n}{dr_n^2} \right| \leq C_b, \quad \text{for } n > n_W.
$$

**Proof.** We fix any phase point $\bar{x}_0 := x \in W$, denote $x_n = (r_n, \varphi_n) = T_F^n x$ and $\bar{x}_n = (\bar{r}_n, \bar{\varphi}_n) = T^n x$. According to (7.33), the slope of the vector $DT \bar{x}$ satisfies

$$
(7.39) \quad \frac{d\bar{\varphi}_1}{dr_1} = \frac{g_1^2 + (1 + g_2^2)V_1}{1 + g_1^2 + g_2^2 V_1} = \gamma_1 + \frac{g_1^2 + g_2^2 V_1 - g_1 V_1 - g_2 V_1}{1 + g_1^2 + g_2^2 V_1},
$$

where $V_1 = d\varphi_1/dr_1$, $\bar{V}_1 = d\bar{\varphi}_1/\bar{r}_1$. We differentiate the above equality with respect to $r_1$, using the fact that by (7.30) $\frac{d\bar{V}_1}{dr_1} = 1 + g_1^2 + g_2^2 V_1$. Now use the same notation as in Lemma 7.5 to get for some $C_0 > 0$ and $C_3 > 0$

$$
(7.40) \quad \left| \frac{d^2 \varphi_1}{dr_1^2} \right| \leq C_0 + (1 + C_3 \epsilon_1) \theta^3 \left| \frac{d^2 \varphi_0}{dr_0^2} \right|,
$$
since \( d^2 \varphi_0/dr_0^2 = d^2 \varphi_0/dr_0^2 \). By choosing \( \varepsilon_1 \) small one can make \( (1 + \varepsilon_1 C_3) \theta^2 < 1 \). Then we have for any \( n \geq 1 \),
\[
\left| \frac{d^2 \varphi_n}{dr_0^2} \right| \leq \frac{C_0}{1 - \theta^n} + \theta^n \left| \frac{d^2 \varphi_0}{dr_0^2} \right|
\]
Since \( W \) is \( C^2 \), there exists \( C_1 = C_1(W) > 0 \) such that \( \left| \frac{d^2 \varphi_0}{dr_0^2} \right| < C_1 \). We fix a constant \( C_b = C_b(Q) > 0 \) and define
\[
n_W = \frac{\ln(C_b/C_1)}{\ln \theta}.
\]
Then for any \( n > n_W \), connected components of \( T^n W \) have equation \( \varphi_n = \varphi_n(\tilde{r}_n) \) with second derivative bounded from above by \( C_b \).

We now fix the constant \( C_b > 0 \), then define \( \hat{W}^u \) be the class of all homogeneous unstable curves \( W \) whose curvature is uniformly bounded by \( C_b \). It follows from Propositions \( 7.4 \) and \( 7.5 \) that the class \( \hat{W}^u \) is invariant under any \( T \in \mathcal{F} \). Any unstable curve \( W \in \hat{W}^u \) is called a regular unstable curve. Similarly, one defines \( \hat{W}^s \). This verifies condition \((H2)\).

### 7.3.2. Distortion bounds

In this section, we establish the distortion bounds for \( T \) required by \((H4)\). For any stable curve \( W \in \hat{W}^s \) and \( x \in W \), denote by \( J_W T_{F}(x) \) (resp. \( J_W T(x) \)) the Jacobian of \( T_{F} \) (resp. \( T \)) along \( W \) at \( x \in W \). It was shown in \([\text{Ch2}]\) that there exists \( C_1 > 0 \), such that for any regular stable curve \( W \) for which \( T_{F} W \) is also a regular stable curve,
\[
(7.41) \quad |\ln J_W T_{F}(x) - \ln J_W T_{F}(y)| \leq C_1 d_W(x, y) \frac{1}{2}
\]
where \( d_W(x, y) \) is the arclength between \( x \) and \( y \) along \( W \). We show that \( T \) has the same properties on the set of all regular stable curves \( \hat{W}^s \).

**Lemma 7.7.** (Distortion bounds) Let \( T \in \mathcal{F} \) and \( W \in \hat{W}^s \) be such that \( T \) is smooth on \( W \) and \( TW \in \hat{W}^s \). There exists \( C_J > 0 \) independent of \( W \) and \( F \) such that
\[
|\ln J_W T(x) - \ln J_W T(y)| \leq C_J d_W(x, y) \frac{1}{2}.
\]

**Proof.** Fix \( T \in \mathcal{F} \) and \( W \in \hat{W}^s \) for which \( TW \in \hat{W}^s \). This implies in particular that both \( T \) and \( T_{F} \) are smooth on \( W \). For any \( x = (r, \varphi) \in W \), let \( x_1 := T_{F} x = (r_1, \varphi_1) \) and \( \tilde{x}_1 = T x = (\tilde{r}_1, \tilde{\varphi}_1) \). Similarly, let \( dx = (dr, d\varphi) \in T_{F} W \) be a unit vector and define \( dx_1 = DT_{F}(x)dx = (dr_1, d\varphi_1) \) and \( d\tilde{x}_1 = DT(x)dx = (d\tilde{r}_1, d\tilde{\varphi}_1) \). Then
\[
\frac{J_W T(x)}{J_W T_{F}(x)} = \sqrt{\frac{1 + V_1^2 |dr_1|}{1 + V_1^2 |dr_1|}}
\]
where \( V_1 = d\varphi_1/dr_1 \) and \( \tilde{V}_1 = d\tilde{\varphi}_1/d\tilde{r}_1 \). Then it follows from \((7.30)\) that
\[
(7.42) \quad \ln J_W T(x) = \ln J_W T_{F}(x) + \frac{1}{2} \ln(1 + V_1^2) - \frac{1}{2} \ln(1 + V_1^2) + \ln |1 + g_1^1 + g_2^1 V_1|.
\]
By the smoothness of \( W \) and the curvature bounds, there exists \( C > 0 \) such that for any \( x, y \in W \),
\[
|\ln(1 + V_1^2(x)) - \ln(1 + V_1^2(y))| \leq |V_1^2(x) - V_1^2(y)| \leq C d_W(x, y_1) \leq C' d_W(x, y),
\]
where \( y_1 = T_{F} y \), and similarly for \( \tilde{V}_1 \). Since \( G \) is \( C^2 \), the terms involving \( g_1^1 \) and \( g_2^1 \) satisfy a Lipschitz bound as well. Putting this together with \((7.41)\) and \((7.42)\) proves the lemma. \( \square \)

In general, for \( W \in \hat{W}^s \) and \( n \in \mathbb{N} \), suppose \( T^n \) is smooth on \( W \) and that \( T^k W \in \hat{W}^s \), \( 0 \leq k \leq n \). Define \( T^k W = W_k \) and for \( x, y \in W \), let \( x_k = T^k x \) and \( y_k = T^k y \). Then
\[
(7.43) \quad |\ln J_W T^n(x) - \ln J_W T^n(y)| \leq \sum_{k=0}^{n-1} |\ln J_{W_k} T(x_k) - \ln J_{W_k} T(y_k)|
\]
\[
\leq C \sum_{k=0}^{n-1} d_W(x_k, y_k)^{1/3} \leq C d_W(x, y)^{1/3} \sum_{k=0}^{\infty} \Lambda^{k/3},
\]
due to (7.35). This completes the required estimate on $J_W T$.

Finally, we prove the required bounded distortion estimate for $J_\mu T$. By (7.23) and (7.31), we have

$$
\det DT(x) = \frac{\cos \varphi}{\cos \varphi_1} \left( (1 + c_2)(1 + c_3) - c_1(1 + c_1) \right) \frac{\left( 1 + g_1^1 \right) \left( 1 + g_1^2 - g_2^1 g_2^3 \right)}{\cos \varphi_1},
$$

where $c_1, \ldots, c_4$ are defined by (7.19) and we have replaced $\cos \varphi$ with $\cos \varphi_1$ times a smooth function on $M \setminus S^1$ due to (7.33). Note that $A(x)$ is a smooth function of its argument wherever $T$ is smooth and has bounded $C^1$ norm on $M \setminus S^1$. It follows that $J_\mu T$ is a smooth function on $M \setminus S^1$ whose $C^1$-norm is bounded between $1 \pm C \varepsilon_1$ for some uniform constant $C$ depending on the table (recall that $d \mu = e \cos \varphi \, dm$ is the smooth invariant measure for the unperturbed billiard $T_0$). The required distortion estimates (3.8) and (4.9) for $J_\mu T$ follow using this smoothness and the uniform hyperbolicity of $T$ as in (7.43). Indeed, (7.43) holds with exponent $1$ rather than $1/3$ for $J_\mu T$. This completes the verification of (H4).

Distortion bounds for $\det DT$ with exponent $1/3$ follow from the above considerations in addition to recalling that $1/\cos \varphi$ is of order $k^2$ in $\mathbb{H}_k$, while the width of such a strip along a stable or unstable curve is $k^{-3}$. Similarly, one may prove absolute continuity of the holonomy map between unstable leaves as in (Ch2), but we do not do that here since we do not need this fact.

### 7.4. One step expansion

Since we have established the expansion factors given by (7.35) and (7.36), the one-step expansion condition (3.6) follows from an argument similar to the unperturbed case (see [CM, Lemma 5.56]) and fixes the choice of $k_0 \in \mathbb{N}$, the minimum index of the homogeneity strips. We will not reprove that lemma here. Instead, we focus on the second part of (H3), given by (3.7).

Fix $\delta_0 > 0$ and $k_0$ satisfying (3.10) and define $\mathcal{W}^s$ accordingly. For $W \in \mathcal{W}^s$, let $V_i$ denote the maximal homogenous connected components of $T^{-1}W$.

**Lemma 7.8.** For any $\zeta > 1/2$, there exists $C = C(\delta_0, \varepsilon, \varepsilon_0) > 0$ such that for any $W \in \mathcal{W}^s$, any $T \in F_B(Q_0, \tau_*, \varepsilon_1)$,

$$\sum_i \frac{|TV_i|}{|V_i|^\zeta} < C.
$$

**Proof.** According to the structure of singular curves, a stable curve of length $\leq \delta_0$ can be cut by at most $N \leq \tau_{\text{max}}/\tau_{\text{min}}$ singularity curves in $S^1_T$ (see [CM §5.10]). For each $s \in S^1_T$ intersecting $W$, $W$ is cut further by images of the boundaries of homogeneity strips $S^H_k$, $k \geq k_0$. For one such $s$, we relabel the components $V_i$ of $T^{-1}W$ on which $T$ is smooth by $V_k$, $k$ corresponding to the homogeneity strip $\mathbb{H}_k$ containing $V_i$. By (7.36), there exists $c_1 = c_1(\varepsilon_1) > 0$ such that on $TV_k$, the expansion under $T^{-1}$ is $\geq c_1 k^2$. So for all $\zeta > 1/2$,

$$\sum_{k \geq k_0} \frac{|TV_k|^\zeta}{|V_k|} \leq c_1 \sum_{k \geq k_0} \frac{1}{k^2} \leq \frac{c_1}{k_0^2} \zeta - 1.
$$

An upper bound for (7.45) in this case is given by $N$ times the bound in (7.46). \hfill $\Box$

This completes the verification of (H1)-(H5) and completes the proof of Theorem 2.8.

### 7.5. Smallness of the perturbation

In this section, we check that conditions (C1)-(C4) are satisfied for $\varepsilon_1$ sufficiently small. We will then be able to apply Theorem 2.8 to any map $T \in F_B(Q_0, \tau_*, \varepsilon_1)$.

We fix $\varepsilon \in (0, \varepsilon_1)$ and choose any $T := T_{F,G} \in F_B(Q_0, \tau_*, \varepsilon_1)$, such that $|F|_{C^1}, |G|_{C^1} \leq \varepsilon$. By the triangle inequality, it suffices to estimate $d_T(T_0, T)$ where $T_0 = T_{0,0}$ is the unperturbed billiard map.

Denote by $\Phi^t$ the flow corresponding to $T$ and by $\Phi_0^t$ the flow corresponding to $T_0$. Let $x \in M \setminus (S^1_T \cup S^1_{T_0})$. By the facts summarized in Section 7.1, $\Phi^t(x)$ and $\Phi_0^t(x)$ can be no further than a uniform constant times $\varepsilon t$ on the billiard table. Thus since $T$ has finite horizon bounded by $\tau_{\text{max}}$ and the scatterers have uniformly bounded curvature, $T(x)$ and $T_0(x)$ can be no more than a constant times $\sqrt{\varepsilon}$ apart if they lie on the same scatterer. By the smallness of $G$ and (7.29), we have $d_M(T_{F,G}(x), T_{0,F,G}) < C \varepsilon$ and thus by the triangle inequality, $d_M(T(x), T_0(x)) < C_f \sqrt{\varepsilon}$ for some uniform $C_f > 0$ as long as they lie on the same scatterer. A similar bound holds for $T^{-1}x$ and $T_0^{-1}x$.

Let $\epsilon = C_f \varepsilon^{1/3}$. It then follows that for any $x \notin N_t(S^1_T \cup S^1_{T_0})$, $d(T^{-1}(x), T_0^{-1}(x)) < \epsilon$. This is (C1).
To establish (C2), we use the fact that \( J_\mu T_0 = 1 \) while
\[
J_\mu T(x) = ((1 + c_2)(1 + c_3) - c_1(\tau + c_4))(1 + g_1^2)(1 + g_2^2) - g_2^2 \]
by (7.44). Since the functions here are all bounded by uniform constants times \( \varepsilon \) and our horizon is bounded by \( \tau_{\text{max}} \), (C2) is satisfied.

Next, we prove (C4). Inverting (7.31) and (7.24) and using (7.44), we have
\[
DT^{-1}(x) = -\frac{1}{A(T^{-1}x)\cos \varphi(T^{-1}x)} \left( \begin{array}{ccc} B + b_2 & C - a_2 \\ D - b_1 & E + a_1 \end{array} \right) \left( \begin{array}{cc} 1 + g_2^2 & -g_1^2 \\ -g_2^2 & 1 + g_1^2 \end{array} \right),
\]
where \( A \) is the smooth function from (7.44) and \( B = \tau(T^{-1}x)K(x) + \cos \varphi(x), C = -\tau(T^{-1}x), \)
\( D = -\mathcal{K}(T^{-1}x)(\tau(T^{-1}x)\mathcal{K}(x) + \cos \varphi(x)) - \mathcal{K}(x)\cos \varphi(T^{-1}x) \), and \( E = \tau(T^{-1}x)\mathcal{K}(T^{-1}x) + \cos \varphi(T^{-1}x) \)
match the corresponding entries of \( DT_0^{-1}x \) with \( T \) replaced by \( T_0 \).

We split the matrix as
\[
\left( \begin{array}{cc} B & C \\ D & E \end{array} \right) + \left( \begin{array}{cc} b_2 & -a_2 \\ -b_1 & a_1 \end{array} \right) \left( \begin{array}{cc} 1 + g_2^2 & -g_1^2 \\ -g_2^2 & 1 + g_1^2 \end{array} \right) =: F + R,
\]
where \( F = \left( \begin{array}{cc} B & C \\ D & E \end{array} \right) \) and \( R \) is a matrix whose entries are smooth functions, all bounded by a uniform constant times \( \varepsilon \). Now defining \( F_0 \) to be the matrix \( F \) with \( T_0 \) replacing \( T \), we write
\[
\|DT^{-1}(x) - DT_0^{-1}(x)\| = \left\| \frac{F + R}{A(T^{-1}x)\cos \varphi(T^{-1}x)} - \frac{1}{\cos \varphi(T_0^{-1}x)}F_0 \right\|
\leq \frac{\|F - F_0\|}{|A(T^{-1}x)\cos \varphi(T^{-1}x)|} + \|F_0\| \left| \frac{1}{A(T^{-1}x)\cos \varphi(T^{-1}x)} - \frac{1}{\cos \varphi(T_0^{-1}x)} \right| + \frac{\|R\|}{|A(T^{-1}x)\cos \varphi(T^{-1}x)|}.
\]
Notice that if \( x \notin N_\varepsilon(S_{T_1}^T \cup S_{T_0}^T) \), then due to the uniform expansion given by (7.38), and the uniform transversality of the stable cone with \( S_0 \), we have \( d_M(T^{-1}x, S_0) \geq C\sqrt{\varepsilon} \), for some uniform constant \( C \).
Thus \( \cos \varphi(T^{-1}x) \geq C'\sqrt{\varepsilon} \) for some uniform constant \( C' > 0 \). The same fact is true for \( T_0^{-1}x \).

Using this, plus the fact that the entries of \( F_0 \) are smooth functions of their arguments with uniformly bounded \( C^1 \) norms, we estimate the first term of (7.47) by
\[
\frac{\|F - F_0\|}{|A(T^{-1}x)\cos \varphi(T^{-1}x)|} \leq C\varepsilon^{-1/2}d_M(T^{-1}x, T_0^{-1}x) \leq C'\varepsilon^{-1/2}\varepsilon^{1/2} = C'C_f\varepsilon
\]
since the \( C^1 \) norm of \( A \) is bounded above and below by \( 1 \pm C\varepsilon \) by (7.44). Similarly, the third term of (7.47) is bounded by \( C\varepsilon \).

Since \( \|F_0\| \) is uniformly bounded, we split the middle term of (7.47) into the sum of two terms,
\[
\frac{1}{|A(T^{-1}x)\cos \varphi(T^{-1}x)|} - \frac{1}{\cos \varphi(T_0^{-1}x)} \leq \frac{1}{\cos \varphi(T^{-1}x)} \left| \frac{1}{A(T^{-1}x)} - 1 \right| + \frac{1}{\cos \varphi(T^{-1}x)} - \frac{1}{\cos \varphi(T_0^{-1}x)}.
\]
As noted earlier, the \( C^1 \) norm of \( A \) is bounded above and below by \( 1 \pm C\varepsilon \) so that the first difference above is bounded by \( C\varepsilon^{-1/2}\varepsilon \leq CC_f\varepsilon \). The second difference is bounded by \( C\varepsilon^{-1}d_M(T^{-1}x, T_0^{-1}x) \leq C'\varepsilon^{-1}\varepsilon^{1/2} = C'C_f\varepsilon^{1/2} \), similar to the estimate (6.7).

Putting these estimates together in (7.47) proves (C4) with \( \varepsilon = C\varepsilon^{1/3} \). Condition (C3) follows similarly using the fact that \( J_W T(x) = \|DT(x)v\| \) where \( v \in T_xW \) is a unit vector. The exponent of \( \varepsilon \) in (C3) is better than in (C4) by a factor of \( \varepsilon^{1/2} \) since we must estimate \( \frac{\cos \varphi(T^{-1}x)}{\cos \varphi(T_0^{-1}x)} - 1 \) in place of \( \frac{1}{\cos \varphi(T^{-1}x)} - \frac{1}{\cos \varphi(T_0^{-1}x)} \).
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