On the running coupling constant in QCD

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Abstract

We try to review the main current ideas and points of view on the running coupling constant in QCD. We begin by recalling briefly the classic analysis based on the Renormalization Group Equations with some emphasis on the exact solutions for a given number of loops, in comparison with the usual approximate expressions. We give particular attention to the problem of eliminating the unphysical Landau singularities, and of defining a coupling that remains significant at the infrared scales. We consider various proposal of couplings directly related to the quark-antiquark potential or to other physical quantities (effective charges) and discuss optimization in the choice of the scale parameter and of the renormalization scheme. Our main focus is, however, on dispersive methods, their application, their relation with non-perturbative effects. We try also to summarize the main results obtained by Lattice simulations in particular in various MOM schemes. We conclude briefly recalling the traditional comparison with the experimental data.

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1 Introduction

The renormalization procedure in field theory consists in a redefinition of the unrenormalized constants that appear in the Lagrangian, in such a way that...
the observable quantities can be kept finite when the ultraviolet cut off $\Lambda_{UV}$ is removed. In the framework of any given renormalization scheme (RS), dimensional reasons require the introduction of a new quantity $\mu$ with the dimension of a mass. Note that observables must be independent of the particular RS and of $\mu$ by definition. Intermediate quantities, like renormalized coupling constants, masses and field normalization factors depend on RS and on $\mu$ by construction and a change in RS amounts to a redefinition of such quantities. Obviously, approximate expressions of the observables depend in general on $\mu$ and RS.

In QED the masses of the charged particles have a direct physical meaning, so there exists a natural scale referring to which $\mu$ can be fixed. In QCD, due to confinement, such a natural scale does not exist and RS and $\mu$ have to be chosen with other criteria, those of simplicity and of convergence.

In QCD we have a single coupling constant $g_s$, or the usually more convenient $\alpha_s = \frac{g^2_s}{4\pi}$, and various quark masses $m_f$ with $f = u, d, \ldots, t$. We refer to their dependence on $\mu$ in the framework of a given RS $(\alpha_s(\mu^2), m_f(\mu^2), \ldots)$ as to the running coupling constant, to the running masses and so on. MS is the simplest and most commonly used scheme but, being essentially perturbative, alternative definitions of $\alpha_s(\mu^2)$ may be more appropriate in many cases.

Even in QED perturbative expansion is believed to be only asymptotic. As known this means that the approximation to the considered quantity improves as the number of terms included increases, until a maximum number $N_*$ is reached. After this the single terms become progressively larger and the series loses any meaning. In QED $\alpha \sim 1/137$ and $N_* \sim 1/\alpha \sim 137$, so in practice no problem arises from this lack of convergence. However in QCD $\alpha_s$ is nearly two orders of magnitude larger and $N_*$ is of order 1. An appropriate choice of $\mu$ and possibly of RS becomes therefore essential.

The $\mu$ dependence of the renormalized quantities is controlled by the renormalization group (RG) equations. Let us concentrate on coupling constant $\alpha_s(\mu^2)$ which is related to the unrenormalized constant, $\alpha_{s}^{ur}$, by the usual equation

$$\alpha_s(\mu^2) Z_\alpha \left( \alpha_s(\mu^2), \frac{\mu^2}{\Lambda_{UV}^2} \right) = \alpha_{s}^{ur} \tag{1}$$

where $\Lambda_{UV}^2$ is the cut-off parameter. The corresponding differential equation

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s), \tag{2}$$
can be obtained differentiating eq.(1). \( \beta(\alpha_s) \) remains obviously finite as \( \Lambda_{UV} \to \infty \) and in perturbation theory takes the form

\[
\beta(\alpha_s) = -\alpha_s^2(\beta_0 + \beta_1 \alpha_s + \beta_2 \alpha_s^2 + \ldots).
\]  

(3)

As is known, the various terms in \( \beta_0, \beta_1, \beta_2, \ldots \) correspond to one loop, two loops, three loops ... contributions; \( \beta_0 \) and \( \beta_1 \) are universal in the mass independent schemes.

Note that, for a general RS, \( Z_\alpha, \beta(\alpha_s), \beta_0, \beta_1, \ldots \) depend also on quark masses through the variables \( m_f^2/\mu^2 \), not explicitly indicated. However, according to the decoupling theorem, all quarks with masses much larger than the energy scale of interest (in particular \( m_f \gg \mu \)) can be ignored. On the contrary, if \( m_f \ll \mu \), we can often neglect \( m_f \). Then, the discussion can be greatly simplified if for every \( \mu \) we divide the quarks in active quarks with \( m_f = 0 \) and inactive ones, which we simply ignore. Within this framework \( \beta_0, \beta_1, \ldots \) depend on \( \mu \) only through the number of active quarks \( n_f \), which changes by \( \pm 1 \) any time \( \mu \) crosses a quark threshold \( m_f \). Furthermore the first two coefficients, \( \beta_0 \) and \( \beta_1 \), are RS independent, while all the others depend on the scheme. In the one loop approximation (i.e. keeping only the first term in (3)) eq.(2) gives

\[
\alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + \beta_0 \alpha_s(\mu_0^2) \ln(\mu^2/\mu_0^2)} = \alpha_s(\mu_0^2) \sum_{n=0}^{\infty} \left( -\beta_0 \alpha_s(\mu_0^2) \frac{\mu^2}{\mu_0^2} \right)^n,
\]  

(4)

which explicitly expresses \( \alpha_s \) at the \( \mu \) scale as a function of the same quantity at the \( \mu_0 \) scale. Eq. (4) clearly shows that a change in the value of \( \mu \) consists in a reorganization of the perturbative expansion of any observable or, what is the same thing, in a resummation of various contributions. Setting

\[
\Lambda^2 = \mu_0^2 \exp \left[ -\frac{1}{\beta_0} \frac{1}{\alpha_s(\mu_0^2)} \right],
\]  

(5)

\( \alpha_s(\mu^2) \) can be written in terms of the overall scale \( \Lambda \), without any reference to a specific \( \mu_0 \)

\[
\alpha_s(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)}.
\]  

(6)

As concerns the best choice of \( \mu^2 \) in a specific calculation, let us consider the perturbative expansion of an amplitude or observable \( G(q,x) \) of canonical dimension 0. We assume \( G \) written in terms of an overall momentum \( q \) and
a set of dimensionless variables (angles, Bjorken variables and so on) which we collectively denote by $x$. According to the process we are considering $q$ may be space-like, $q^2 < 0$ (a momentum transfer), or time-like, $q^2 > 0$ (an energy). We shall often set $q^2 = \mp Q^2$ ($Q^2$ being as a rule positive) or also $q^2 = s$ in the time-like case.

Under the above mentioned assumption that the active quark masses can be neglected, we can write

$$G = g_0 \left( \frac{Q^2}{\mu^2}, x \right) + g_1 \left( \frac{Q^2}{\mu^2}, x \right) \frac{\alpha_s(\mu^2)}{\pi} + g_2 \left( \frac{Q^2}{\mu^2}, x \right) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 + \ldots. \quad (7)$$

As we mentioned, $G$ must be independent of $\mu^2$, so

$$\mu^2 \frac{d}{d\mu^2} G = \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) G = 0. \quad (8)$$

Then by replacing (7) and (3) in (8), we obtain a system of differential equation in the coefficients of (7), whose solution gives (see e.g. sec. 3.3 for details)

$$g_0 = \overline{g}_0(x) \quad g_1 = \overline{g}_1(x) \quad g_2 = \overline{g}_2(x) - \pi \beta_0 \overline{g}_1(x) \ln \frac{Q^2}{\mu^2} \quad (9)$$

Let us assume that $\overline{g}_0(x), \overline{g}_1(x), \ldots$ decrease sufficiently fast in order (7) to become significant when $Q^2 \sim \mu^2$. This would clearly no longer be true for very different values of $Q^2$. We must obviously modify $\mu^2$ choosing it always on the scale of interest. In particular it may be convenient to chose exactly $\mu^2 = Q^2$ and then (7) takes the form

$$G(q, x) = \overline{g}_0(x) + \overline{g}_1(x) \frac{\alpha_s(Q^2)}{\pi} + \overline{g}_2(x) \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 + \ldots, \quad (10)$$

showing that $\alpha_s(Q^2)$ is actually the most convenient expansion parameter for calculating the quantity $G(q, x)$.

Now let us go back to eq. (4). We have $\beta_0 = \frac{1}{4\pi} (11 - 2n_f/3)$ and, since the number of presently known flavors is 6, this remains positive at least
in the entire range so far accessible. Then, for $\mu \rightarrow \infty$ we have $\alpha_s(\mu^2) \rightarrow 0$ (asymptotic freedom) and as we can see (see sec. 2) this remains true even if a larger number of terms is taken into account in (3).

Conventionally the mass of the $Z_0$ boson, $M_Z \sim 91.2$ GeV, is used as $\mu_0$ reference value. The world average of all determination is currently $\alpha_s(M_Z^2) = 0.1189 \pm 0.0010$ [6, 18].

With diminishing $Q^2$ we may expect expansion (7) or (10) to remain meaningful up to a few GeV if $q$ is space-like. The situation requires more attention for time-like $q$, $q^2 = s > 0$, as discussed in sec. 2.5. Then, in fact, in (3) $\beta_0 \ln \frac{Q^2}{\mu^2} \rightarrow \beta_0 \ln \frac{s-i\delta}{\mu^2} = \beta_0 (\ln \frac{s}{\mu^2} - i\pi)$ and, even if we take $\mu^2 = s$ to control the logarithms, now the coefficients $f_2, f_3, \ldots$ are modified for terms proportional to powers of $\beta_0 \pi$, which become rather large as the order increases (see sec. 2.5). Then the “space-like” coupling $\alpha_s(s)$ remains a good expansion parameter for large $s$, but for some intermediate $s$ may be more appropriate to define a new “time-like” coupling $\tilde{\alpha}(s)$ in which the terms in $\beta_0 \pi$ are reabsorbed, so that the coefficients remain small. This can be easily done starting from eq. (10) rather than (7) and considering the analytic continuation of the coupling rather than the coefficients. Naturally using the time-like coupling rather than the space-like one can be interpreted as a change of RS and, as such, the two expressions become identical for large $s$.

Note furthermore that $\alpha_s(\mu^2)$, as given by (6), has a pole in $\mu^2 = \Lambda^2$. In the $\overline{\text{MS}}$ scheme, the expression remains singular even if more loops are taken into account in (3), and the singularity structure changes. Such singularities are obviously related to lack of convergence of the series occurring in (11) but they are clearly non physical. In fact in a generic scheme and in a generic gauge $\alpha_s(\mu^2)$ is a kind of intermediate quantity only indirectly related to actual observables and it is often chosen with criteria of formal simplicity. It is just the case for the mentioned $\overline{\text{MS}}$ scheme and for the Landau gauge to which as a rule we shall refer. On the contrary we know that on general grounds $G(q, x)$ is analytic in the entire $q^2$ complex plane apart from a cut on the real positive axis from some threshold to $+\infty$ and in its exact expression the above non physical singularities must cancel. Correspondingly, eq. (10) suggests that it is possible to define schemes in which $\alpha_s(Q^2)$ can be extended again to the entire complex $Q^2$ plane apart from a cut on the real negative axis. To solve the problem of the elimination of the above spurious singularities various strategies have been proposed. We remind here for instance the Brodsky-Lepage-Mackenzie (BLM) criterion [3], which fixes the scale $\mu$
such that the next-to-leading order (NLO) coefficient of perturbative series is independent of $n_f$, and also the so-called “Fastest Apparent Convergence” (FAC) technique \[4\], that amounts to setting the scale $\mu$ so that the NLO coefficient and all the higher ones are zero. The latter is related to the effective charges \[4\], \[64\]-\[66\], which consist in defining new coupling constants $\alpha_{\text{eff}}(Q^2)$ in more strict connection with some physical observable (see sec. 3.2). Typically let us assume that the quantity $G$ in (10) is independent of $x$ (or $x$ has been fixed on some special value) and set exactly

$$G(Q^2) = g_0 + g_1 \frac{\alpha_{\text{eff}}(Q^2)}{\pi}.$$  

(11)

Effective charges are obviously well defined everywhere in principle and easily extracted from the data. They depend on the particular observables chosen and, accordingly, can be either of the time-like or space-like type. However, they can be related to each other by referring back to the MS scheme. In fact, comparing (11) and (10), we have

$$\frac{\alpha_{\text{eff}}(Q^2)}{\pi} = \frac{\alpha_{\text{MS}}(Q^2)}{\pi} + \frac{g_2}{g_1} \left( \frac{\alpha_{\text{MS}}(Q^2)}{\pi} \right)^2 + \ldots .$$  

(12)

Another device is the Optimized Perturbation Theory (OPT) \[72\]-\[78\], which consists in improving the convergence of the perturbative expansion by choosing the value of $\mu^2$ and of some of the RS parameters (like $\beta_2$, $\beta_3$ themselves) at every order according to a criterion of minimum sensitivity. This has been discussed as a matter of example in some details in sec. 3.3, with reference to the theoretical predictions it yields on the IR behavior of the coupling.

Finally another method, summarized in sec. 4, consists in requiring ab initio the desired analyticity properties for $\alpha_s(\mu^2)$, by rewriting this as a dispersion relation and applying the perturbative theory to the spectral function $\text{Im} [\alpha_s(-m^2-i0)]$ rather than to $\alpha_s(\mu^2)$ itself \[79\]-\[84\]. In this way Landau singularities are suppressed at their very roots. In the one loop approximation from eq.(11) we find

$$\text{Im} [\alpha_s(-m^2-i0)] = \frac{\pi \beta_0 \alpha_s^2(\mu_0^2)}{[1 + \beta_0 \alpha_s(\mu_0^2) \ln(m^2/\mu_0^2)]^2 + (\pi \beta_0 \alpha_s(\mu_0^2))^2},$$  

(13)

which is finite everywhere on the positive $m^2$ axis and can be considered in some way as a reorganization of the series occurring in the last term in eq. (11).
after evaluation of the imaginary part.

It must be stressed that all these techniques lead to a finite $\alpha_s(0)$ and are essentially consistent. Furthermore all of them use as input the coefficients of the power expansion of $\alpha_s(\mu^2)$ itself or of specific observables in terms of $\alpha_s(\mu_0^2)$. In this sense we may call them perturbative \(^1\).

However, we believe that there are intrinsically non perturbative effects in the theory. These are in part related to the Gribov problem of the multiple solutions of the gauge fixing equation [5], and more seriously with the occurrence of the singularities in $\alpha_s = 0$ which can not be obtained by any manipulation of perturbative expansions. Confinement and string tension, condensates, topologically non trivial classical solutions are of this type. In this context, unfortunately, we can not rely on any rigorous methods in the continuum. We have achieved some intuitions and very partial results, but we are far from a comprehensive understanding of the all the matter. Within this framework, what we can do is to parametrize such non perturbative effects in terms of some universal quantities that occur in different types of phenomena. In particular, according to perturbation theory, in high energy processes observables should vanish logarithmically with the energy (apart from mass corrections). Non perturbative effects are expected to appear in terms of powers of the $\frac{\Delta_{\nu}}{Q^2}$ and $\frac{A'_{\nu}}{Q^2} \ln Q^2$ form (see sec. 4.5). Such terms should in principle be detected at somewhat intermediate energies even if the experimental situation is not sufficiently clear. In this context the coupling is written as the sum of a perturbative $\alpha_s^{PT}(Q^2)$ and a non perturbative part $\alpha_s^{NP}(Q^2)$. The latter is supposed to vanish fast enough out of the infrared region and the coefficients $A_{\nu}$, $A'_{\nu}$ can be expressed in terms of moments of $\alpha_s^{NP}(Q^2)$ or of its time-like counterpart $\tilde{\alpha}_s^{NP}(m^2)$ [64].

The only really first principle method that we have to handle non perturbative effects is to approximate the continuum with a lattice and then rely on numerical simulations. As applied to the running coupling this method delivers results in substantial agreement with those obtained in the continuum. The significance of the latter result is however uncertain, due to finite size limitations.

In this review the emphasis is put on the IR behavior of the coupling con-

\(^1\)However, at the one loop level, the analytic coupling can be written explicitly in terms of the ordinary coupling in the form (see later) $\alpha_{an} = \alpha_s + \frac{1}{3\pi} \left(1 - e^{\frac{\pi}{\nu s}}\right)^{-1}$ that has an essential singularity in $\alpha_s = 0$ and in this sense can be said non perturbative. On the contrary the relation between $\alpha_{an}(\mu^2)$ and $\alpha_{an}(\mu_0^2)$ is analytic.
stant and in particular on the analytic methods. We have tried to give a unified presentation of the matter. To this end we often took the liberty of modifying notations used in the original papers. Due to the extent of the literature on the subject, and to space limitation, we had to make appropriate choices. In this sense we do not claim our paper to be exhaustive or complete. The choice we have made is obviously due to our personal taste or to the specific point of view we have adopted. In particular in some cases we have discussed with a greater detail some specific papers as an example of a more general methodology. In such a context we apologize in advance for giving less attention to some papers than to others, even very significant. As a rule, we have kept to a general theoretical level and referred to application only occasionally. Apart from a brief summary of significant results in the concluding remarks, no attempt was made to discuss the actual methods by which the coupling is extracted from experiments at the various energies. For this we refer interested people to specific reviews on the subject such as [6] and [18].

The plan of the paper is as follows. In sec. 2 we have summarized the main results of the usual perturbation theory as reference. For details and bibliography on this subject we refer to standard books and reviews (see e.g. [1, 2, 15, 31]). We have tried to emphasize the importance of exact solutions to renormalization group equations with a given number of loops in comparison with the usual iterative solutions appropriate for high energies. This was due to our interest in the IR region $^2$. Some attention has also been given to the need to define a specific time-like coupling. In sec. 3 we have considered some purely phenomenological modifications of the coupling constant expression, suggested essentially by the quark-antiquark potential theory. We have also discussed some significant examples of physical couplings, a classic application of the optimized perturbation theory. Sec. 4 is entirely devoted to the dispersive approach, analytic coupling constant and relation between space-like and time-like expressions, the parametrization of non-perturbative effects and the so called Analytic Perturbation Theory. In sec. 5 we have tried to summarize the present state of the studies on the running coupling on the lattice. In sec. 6 we outline our conclusive remarks and also discuss briefly some information on the coupling under 1 GeV that can be obtained

$^2$In this connection it should be recalled that the expansion in numbers of loops is an expansion in the Planck constant $\hbar$ and is conceptually different, even if related, to a power coupling expansion.
from relativistic studies of the light quark-antiquark spectrum and from a parametrization of other non-perturbative contributions.

2 Perturbation Theory

2.1 QCD $\beta$ function

As we mentioned, eq. (2) is derived applying the operator $\mu^2(d/d\mu^2)$ to eq. (1). Thus one needs to compute the renormalization factor for the coupling $Z_{\alpha}$ and this can be accomplished in several ways. One can start from the quark-gluon vertex $Z_{\bar{q}qg}$ together with renormalization factors of quark and gluon propagators $Z_q$ and $Z_g$ to obtain $Z_{\alpha} = Z_{\bar{q}qg}^2 Z_q^{-2} Z_g^{-1}$, but other choices, as ghost-ghost-gluon vertex or trilinear and quartic gluon interactions, equally work. If dimensional regularization is used the limit $\Lambda_{UV} \to \infty$ can be anticipate as long as the space dimension $D = 4 - \varepsilon$ is kept different from 4. Taking into account the physical dimension of the fields in $D$-dimensional space the coupling constant acquires a not vanishing physical dimension, thus it is usual the parametrization $\alpha^u_s = \mu^\varepsilon \alpha_s Z_{\alpha}$, where in the $\overline{\text{MS}}$ renormalization scheme

$$Z_{\alpha} = 1 + \sum_{n=1}^{\infty} \varepsilon^{-n} Z_{\alpha}^{(n)}(\alpha_s)$$

(14)

and then

$$\beta(\alpha_s) = \frac{1}{2} \alpha_s^2 \frac{d}{d\alpha_s} Z_{\alpha}^{(1)}(\alpha_s).$$

(15)

From this equation and the explicit perturbative form of $Z_{\alpha}^{(1)}$ we have

$$\beta_0 = \frac{1}{4\pi} \left[ 11 - \frac{2}{3} n_f \right]$$

$$\beta_1 = \frac{1}{(4\pi)^2} \left[ 102 - \frac{38}{3} n_f \right]$$

$$\beta_2 = \frac{1}{(4\pi)^3} \left[ 2857 \frac{1}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right]$$

$$\beta_3 = \frac{1}{(4\pi)^4} \left[ \left( \frac{149753}{6} + 3564\zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f \right.$$  

$$+ \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3 \right]$$

(16)
where $\zeta$ is the Riemann zeta-function, $\zeta_3 \simeq 1.202057$. The coefficients $\beta_j$ generally depend on the RS employed, but the first two are universal among the massless schemes. Moreover, in the MS-scheme the $\beta$-function is gauge-independent at any order [4], and in an arbitrary mass-dependent scheme this feature is preserved only at the first order.

As well known, the universal one-loop coefficient [8] has a positive sign provided there is a small enough number of quark fields ($n_f \leq 33/2$), thus the theory is asymptotically free, that is the $\beta$-function has a stable UV fixed point as its argument approaches zero; indeed this coefficient is the sum of two contributions, the relevant one with respect to asymptotic freedom property being the first, which arises from pure gauge field effects i.e. from the nonlinear Yang-Mills interaction terms. The two-loop coefficient has been computed for the first time in [9] and is positive up to $n_f = 8$.

Higher order approximations are scheme-sensitive, and it is common practice to perform computation within MS or $\overline{\text{MS}}$ procedures which have the same the $\beta$-function. The first calculation of three-loop coefficient is due to [10], where the ghost-ghost-gluon combination in the Feynman gauge was used. In a more recent work [11] the quark-gluon vertex was instead employed, providing an independent check in an arbitrary covariant gauge of the previous result and its gauge-independence. Finally, the original four-loop calculation [12] has been performed using the ghost-ghost-gluon vertex in a arbitrary covariant gauge, and for a generic semi-simple compact Lie symmetry group. The result turns out to be gauge-independent as expected within MS procedure, and involves higher order group invariants such as quartic Casimir operators; specialized to the standard SU(3) symmetry, the four-loop coefficient is a positive number for every positive $n_f$ (see also [13]). Finally note that all four coefficients are positive up to $n_f = 6$ except for $\beta_2$ which is negative for $6 \leq n_f \leq 40$.

2.2 Running coupling

Evolution of QCD running coupling can be gained integrating the differential equation (2), that can be rewritten as

$$
\ln \frac{\mu^2}{\mu_0^2} = \int_{\alpha_0(\mu_0^2)}^{\alpha_0(\mu^2)} \frac{d\alpha}{\beta(\alpha)}
$$

The exact one-loop solution [11] or [13] is obtained by straightforward integration retaining only the first term in [4]. We report for reference the second
As yet noted the dimensional scale $\Lambda$ keeps track of the initial parametrization $(\mu_0, \alpha_s(\mu_0^2))$, and it is scale-invariant; its value is not predicted by the theory but must be extracted from a measurement of $\alpha_s$ at a given reference scale. Emergence of a scale parameter, sometimes referred to as dimensional transmutation, breaks naive scale invariance of the massless theory, and it is commonly believed to be associated with the typical hadron size i.e. to the energy range where confinement effects set in. Roughly speaking, $\Lambda$ is the scale at which the (one-loop) coupling diverges (Landau ghost), and perturbation theory becomes meaningless. Further, it is scheme-dependent and receives further corrections at each loop level, but for simplicity we use the same notation throughout.

In the next loop level the integration of (17) leads to a transcendental equation, that is, starting from the two-loop approximation to the $\beta$-function in (3), straightforward integration in (17) yields

$$\ln \frac{\mu^2}{\mu_0^2} = C + \frac{1}{\beta_0 \alpha_s} + B_1 \ln \alpha_s - B_1 \ln \left(1 + \frac{\beta_1}{\beta_0 \alpha_s}\right)$$

(19)

where $B_1 = \beta_1/\beta_0^2$ and the constant term from the lower end points can be again reabsorbed into the $\Lambda$-parametrization, with the commonly adopted prescription (see e.g. [14, 15])

$$\ln \frac{\Lambda^2}{\mu_0^2} = C - B_1 \ln \beta_0$$

(20)

which fixes a specific choice for $\Lambda$. Thus we get the two-loop implicit solution

$$\ln \frac{\mu^2}{\Lambda^2} = \frac{1}{\beta_0 \alpha_s} - B_1 \ln \left(1 + \frac{1}{\beta_0 \alpha_s}\right),$$

(21)

from which the two-loop scaling constant is immediately read with $\mu = \mu_0$. To achieve an explicit expression for the running coupling at this level one should resort to the many-valued Lambert function $W(\zeta)$ defined by

$$W(\zeta) \exp[W(\zeta)] = \zeta,$$

(22)
which has an infinite number of branches $W_k(\zeta) \; k = 0, \pm1, \pm2 \ldots$ such that $W^*_n(\zeta) = W_{-n}(\zeta^*)$ (for more details we refer to [16]). The exact solution to eq. (21), being $B_1$ positive, for $0 \leq n_f \leq 8$ reads$^3$ (see e.g. [17])

$$\alpha^{(2)}_{\text{ex}}(z) = -\frac{1}{\beta_0 B_1} - \frac{1}{1 + W_{-1}(\zeta)} \quad \zeta = -\frac{1}{eB_1} \left(\frac{1}{\mu}\right)^{1/B_1}$$

where $z = \mu^2/\Lambda^2$, and $W_{-1}(\zeta)$ is the “physical” branch of the Lambert function, i.e. it defines a regular real values function for $\zeta \in (-e^{-1}, 0)$ which fulfills the asymptotic freedom constraint. Indeed, $W_{-1}(\zeta)$ as a function of complex variable has a branch cut along the negative real axis (actually a superimposition of two cuts starting from $-\infty$ to $-e^{-1}$ and to 0 respectively), and defining it on the cut coming from the upper complex plane it assumes real values in the interval $( -e^{-1}, 0)$, with the asymptotics

$$W_{-1}(-\epsilon) = \ln \epsilon + O(\ln |\ln \epsilon|)$$

$$W_{-1}\left(-\frac{1}{e} + \epsilon\right) = -1 - \sqrt{2e\epsilon} + O(\epsilon)$$

as $\epsilon \to 0^+$. Outside this region of the real axis it takes on complex values, indeed it is not a real analytic function. Though not easy for practical aims, eq. (23) yields the most accurate expression for investigating the IR behavior of the running coupling, since it has not been derived by means of deep perturbative approximations (aside from the truncation of the two-loop $\beta$ function). Actually, a frequently used two-loop approximate solution, known as the iterative solution ([79]), is obtained starting from eq. (19) together with a single iteration of the one-loop formula (18), that is

$$\alpha^{(2)}_{\text{it}}(z) = \frac{\beta_0^{-1}}{\ln z + B_1 \ln(1 + B_1^{-1} \ln z)} \; \beta_0,$$

where $z = \mu^2/\tilde{\Lambda}^2$, and $\tilde{\Lambda}$ now defined by

$$\ln \frac{\tilde{\Lambda}^2}{\mu_0^2} = C - B_1 \ln \frac{\beta_1}{\beta_0}$$

$^3$Note that if $9 \leq n_f \leq 16$ the principal branch $W_0$ is involved, but here and throughout our discussion is focused on the physical values $n_f \leq 6$.
\[ \ln(\Lambda/\tilde{\Lambda}) = \frac{1}{2} B_1 \ln B_1. \]  

(28)

However, the commonly used two-loop solution is an asymptotic formula which strictly relies on the smallness of \( \alpha_s \) for fairly large \( \mu^2 \), since it amounts on solving eq. (19) (with the choice (20)) where the last term in the r.h.s. has been neglected. Again after one iteration of the one-loop formula, the result is then re-expanded in powers of \( 1/L \), where \( L = \ln z \) and \( z \) as before

\[ \alpha_s^{(2)}(z) = \frac{1}{\beta_0 \ln z} \left[ 1 - \frac{\beta_1}{\beta_0^2} \ln \left( \frac{\ln (\ln z)}{\ln z} \right) \right]. \]  

(29)

Eq. (29) is known as the standard two-loop running coupling and works only in the deep UV regime, i.e. for \( L \gg 1 \) (see fig.1(a)).

Under the same assumptions one can easily derive the three and four-loop approximate formulas; at any loop level asymptotic RG solutions are obtained as a rule through a recursive recipe involving the previous order result as an input in the transcendental equation arising from integration of (17).

Starting with the approximate implicit solution at four-loop level (obtained by expanding the integrand on the r.h.s. of eq.(17))

\[ \ln \left(\frac{\mu^2}{\mu_0^2}\right) = C + \frac{1}{\beta_0 \alpha_s} + \frac{\beta_1}{\beta_0^2} \ln \alpha_s + \frac{\beta_2 \beta_0 - \beta_1^2}{\beta_0^3} \alpha_s + \frac{\beta_1^3 - 2 \beta_0 \beta_1 \beta_2 + \beta_0^2 \beta_3}{2 \beta_0^3} \alpha_s^2 + O(\alpha_s^3), \]  

(30)

the analogous trick which led to eq. (29) yields the UV asymptotic four-loop running coupling in the standard form of an expansion in inverse powers of the logarithm \( L \) for \( L \gg 1 \) (see e.g. [27])

\[ \alpha_s^{(4)}(\mu^2) = \frac{1}{\beta_0 \mu^2} \left\{ 1 - \frac{\beta_1}{\beta_0^2} \ln \mu \left[ \frac{\beta_1}{\beta_0^2} \ln \left( \frac{\ln L}{\ln \mu} \right) + \frac{\beta_1}{\beta_0^2} \ln \left( \frac{\ln L}{\ln \mu} - 1 \right) + \frac{\beta_2}{\beta_0^2} \right] \right\} + \frac{1}{\beta_0^3 \mu^2} \left[ \frac{\beta_1^2}{\beta_0^3} \left( - \ln^3 L + \frac{5}{2} \ln^2 L + 2 \ln L - \frac{1}{2} \right) - 3 \frac{\beta_1 \beta_2}{\beta_0^3} \ln L + \frac{\beta_3}{2 \beta_0^3} \right] \} \]  

(31)

which turns out to be nearly indistinguishable from the three-loop curve (see fig.1(b)). Here the one-loop solution eq. (18) has been emphasized, and being the leading UV term in (31), it defines the asymptotic behavior of the perturbative running coupling, i.e. it clearly exhibits the asymptotic freedom property. On the other hand, the two and three-loop asymptotic
Figure 1: (a) We show the fractional difference between eq. (23) and eqs. (29), (26) (solid and dashed line respectively), with $\Lambda = 350\text{ MeV}$ and $n_f = 4$. (b) From [6]: it is displayed the fractional difference between the 4-loop and the 1-, 2-, 3-loop approximations to eq. (31), with $n_f = 5$ and normalizing conditions for all curves at $\alpha_s(M_Z^2) = 0.119$.

expressions are easily read from eq. (31) by keeping only the first two or three terms respectively inside the curly bracket. We recall that exact integration of the truncated four (or three)-loop $\beta$-function leads to a more involved structure than eq. (30), which poses serious difficulties in finding its inverse; nevertheless, in [17] a useful solution has been still worked out at three-loop level via the real branch $W_{-1}(\zeta)$ of the Lambert function together with the Pade’ Approximant of the related $\beta$-function. Moreover, in [19] a reliable approximation to higher order running coupling has been suggested, via a power expansion in the two-loop exact coupling eq. (23), of the form

$$\alpha_s^{(k)}(\mu^2) = \sum_{n\geq 1} p_n^{(k)} [\alpha_{sc}^{(2)}(\mu^2)]^n$$

(32)

with $k \geq 3$ the loop order, and $p_n^{(k)}$ proper functions of the coefficients $\beta_j$. Comparison with these multi-loop approximants to the coupling, with the relative asymptotic formulas, eq. (31) and the three-loop analogue, reveals the better agreement of the former ones with the higher-loop exact coupling numerically estimated (i.e. starting from the exact implicit solution), even at low scales (see also [119]).

Finally, being the definition of the scaling constant $\Lambda$ not completely unambiguous, few comments are in order. As yet pointed out, starting from the
two-loop level an arbitrary constant has to be fixed for $\Lambda$ being uniquely defined; beside the commonly accepted convention (20) (or (27)), we just mention that other prescriptions have been proposed [40] in order to optimize the $1/L$-expansion for the running coupling, while (20) does remain the preferred one as no further terms of order $1/L^2$ appear in the two-loop asymptotic formula (29). Thus, in the higher-loop levels the scaling constant is analogously related to the initial parameterization, and the four (and three) loop value reproducing the world average, roughly $\alpha(M_Z^2) = 0.119$, is $\Lambda_{\overline{\text{MS}}}^{(n_f=5)} = 220$ MeV with consistently five active flavors [6].

A last remark concerns the scheme-dependence of the coupling and the scale parameter. Restricting ourselves to mass-independent RS (as MS-like schemes or trivially any prescription in the massless theory), renormalized coupling constants (see eq.(11)) in two such different schemes can be related perturbatively at any fixed scale

\[
\alpha'_s = \alpha_s \left[ 1 + v_1 \frac{\alpha_s}{\pi} + v_2 \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots \right]. \tag{33}
\]

Then, it is easy to verify that the first two coefficients of the relative $\beta$-functions do not change as the renormalization prescription is changed, while, for instance, the third ones are related by $\beta'_2 = \beta_2 - v_1 \beta_1 + v_2 \beta_0 - \beta_0 v_1^2$. As a result the running coupling at each loop-level (e.g. eq.(31)) depends on the RS, through the coefficients $\beta_j$ with $j \geq 2$ and the initial parameterization as well. The latter obviously amounts in suitably adjusting the scaling constant $\Lambda$, and the relation is given exactly by one-loop calculation (20)

\[
\ln \left( \frac{\Lambda'_s}{\Lambda} \right) = \frac{v_1}{2\pi \beta_0}, \tag{34}
\]

which works through all orders; e.g. $\Lambda_{\overline{\text{MS}}}/\Lambda_{\overline{\text{MS}}} = \exp \left( (\ln 4\pi - \gamma_e)/2 \right) \simeq 2.66$, and with $n_f = 4$ we roughly have $\Lambda_{\text{MOM}}/\Lambda_{\overline{\text{MS}}} \simeq 4.76$ where $\Lambda_{\text{MOM}}$ refers to a scheme in which the subtraction of the relevant green functions is performed in the symmetric point and the renormalized coupling is defined through the three gluon vertex.

## 2.3 Threshold matching

Quark mass effects, till now ignored, reveal themselves in explicit corrections within higher order perturbation theory (see e.g. [21]), and in the energy
dependence of the effective (running) quark masses as a result of the RG improvement \[22\], but we will not go into further details being these topics beyond the scope of this report. A direct effect of the quark masses on the evolution of the coupling is, as yet noted, through the dependence of the $\beta$ coefficients on the number $n_f$ of active quarks. A quark is active if $m_f \ll \mu$, where $\mu$ is the renormalization scale, and $m_f$ the $\overline{\text{MS}}$ quark mass (see e.g. \[13\]); the definition can be also formulated in terms of the pole mass $M_f$ that can be related to the former \[23\]. Indeed, within MS-like RS, decoupling of heavy quarks \[24\] is made explicit by constructing beside the full $n_f$ flavors QCD an $(n_f - 1)$ effective theory below a heavy quark threshold \[25\]. Then, to have a unique theory on the whole range, the two couplings $\alpha_s^{(n_f)}$ and $\alpha_s^{(n_f-1)}$ must be matched at each heavy quark mass scale $\mu^{(n_f)} = O(m_f)$. As a result the scaling constant $\Lambda$ depends also on $n_f$ (see e.g. \[20, 27, 28\]).

The most straightforward way is to impose continuity of $\alpha_s$ by means of the matching condition $\alpha_s^{(n_f-1)}(m_f^2) = \alpha_s^{(n_f)}(m_f^2)$, which works up to next-to-leading order. At the one-loop level e.g., it translates into

$$
\Lambda^{(n_f)} = \Lambda^{(n_f-1)} \left[ \frac{\Lambda^{(n_f-1)}}{m_f} \right]^{2/(33 - 2n_f)},
$$

which can be extended up to two-loop, and exhibits explicit dependence on the $m_f$ values. Since trivial matching does not generally hold in higher orders within $\overline{\text{MS}}$ scheme, a more accurate formula is required in this case (see \[27\] and refs. therein); specifically to obtain the global evolution of the four-loop coupling the proper matching condition reads \[27\]

$$
\alpha_s^{(n_f-1)} = \alpha_s^{(n_f)} \left[ 1 + k_2 \left( \frac{\alpha_s^{(n_f)}}{\pi} \right)^2 + k_3 \left( \frac{\alpha_s^{(n_f)}}{\pi} \right)^3 \right]
$$

with

$$
k_2 = \frac{11}{72}, \quad k_3 = \frac{564731}{124416} - \frac{82043}{27648} \zeta_3 - \frac{2633}{31104} (n_f - 1)
$$

if $\mu^{(n_f)} = m_f$ is exactly assumed. With this convention, eq.\[36\] yields the relationship between the scaling constants $\Lambda^{(n_f-1)}$ and $\Lambda^{(n_f)}$ in the $\overline{\text{MS}}$ scheme (see \[27\]). Note that one can equally fix $\mu^{(n_f)} = M_f$, that amounts to a proper adjustment of the coefficients in \[36\]; for instance \[6\] starting with \[31\] and $\Lambda^{(n_f=5)} = 220$ MeV, the values $\Lambda^{(n_f=4)} = 305$ MeV and $\Lambda^{(n_f=3)} = 346$ MeV.
are obtained, with threshold fixed at the pole masses $M_b = 4.7 \text{ GeV}$ and $M_c = 1.5 \text{ GeV}$. Finally, we observe that eq. (36) clearly spoils continuity of $\alpha_s$; thereby, whenever continuity of the global coupling and of its first derivative is mandatory, one can resort to a more sophisticated technique [29], which relying upon mass-dependent RS yields a smooth transition across thresholds.

### 2.4 Landau singularities

The running coupling is not an observable by itself but plays the role of the expansion parameter for physical quantities. At one loop-level (cf. eq.(11)), it resums an infinite series of leading-logs UV contributions and similarly the two-loop solution yields the next-to-leading-logs approximation; thus RG formalism provides an iterative recipe for improving perturbative results.

However, as a result, the RG resummation significantly modifies the analytical structure of the series in the complex plane. As yet noted the one-loop coupling (18) is clearly affected by a spacelike pole at $\Lambda$ with residue $1/\beta_0$ (Landau ghost). Adding multi-loop corrections does not overcome the hurdle. Rather the singularity structure of the higher order solutions is more involved due to the log-of-log dependence, so that a branch cut adds on to the one-loop single pole in the IR domain of the spacelike axis. Moreover, at a given loop level Landau singularities sensibly depend upon the approximation used. For instance, the two-loop iterative formula (26) has a pole at $z = 1$ ($\mu = \tilde{\Lambda}$, see eqs. (27) and (28)) with residue $1/(2\beta_0)$, and a cut for $0 < z < \exp(-B_1)$ due to the double logarithm. On the other hand, when considering the same loop approximation (29), we note how the singularity in $z = 1$ already acquires a stronger character

$$\alpha_s^{(2)}(z) \simeq -\frac{B_1}{\beta_0} \ln(z - 1) \ln(z - 1)^2 \quad z \to 1$$

(38)

with $z = \mu^2/\Lambda^2$, and the cut now runs from 0 to 1. Analogously the three and four-loop asymptotic solutions, as given by eq.(31), nearby the Landau ghost respectively becomes

$$\alpha_s^{(3)}(z) \simeq \frac{B_1^2}{\beta_0} \ln^2(z - 1) \ln(z - 1)^3, \quad \alpha_s^{(4)}(z) \simeq -\frac{B_1^3}{\beta_0} \ln^3(z - 1) \ln(z - 1)^4 \quad z \to 1$$

(39)

and are equally affected by an unphysical cut. However, the cumbersome singularity structure of the leading Landau ghost, and of the unphysical cut
as well, are an artifact of the UV approximations introduced. Therefore to deal with the IR behavior of the running coupling, e.g. at two-loop level, it is necessary to face with the exact solution (23); clearly it is singular when \( W_{-1}(\zeta) = -1 \) that is at \( z = B_1^{-B_1}(\mu^2 = B_1^{-B_1}\Lambda^2) \), corresponding to the branch point \( \zeta = -1/e \) of the Lambert function. Nearby this point, due to the asymptotic (25) of \( W_{-1}(\zeta) \), we have

\[
a_{\text{ex}}^{(2)}(z) \simeq \frac{1}{\beta_0} \sqrt{\frac{B_1^{-B_1}-1}{2}} \left[ z - B_1^{-B_1} \right]^{-1/2}.
\]

i.e., an integrable singularity (note that the awkward fact or in front of \( \Lambda \) in the singular point can be reabsorbed into a proper redefinition of the integration constant, through (28)).

A more detailed investigation about IR singularity structure of higher-order perturbative running coupling is performed on the ground of eq.(32) in the recent work [119], where in particular the location of Landau singularities is determined as a function of \( n_f \).

### 2.5 Time-like coupling

Until now we have worked exclusively in the space-like region to derive the evolution of coupling, namely we have implicitly admitted the theory renormalized at momentum scale with negative squared invariant mass. However, in perturbation theory one needs to parameterize observables depending on time-like arguments by means of an effective parameter, to improve perturbative expansions. While this poses no special problem in large energy processes, at any finite energy the issue of which should be the most suitable parameter in the s-channel must be carefully considered. For this purpose we briefly sketch the key points of this subject, and start by noting that the standard practice is to merely take over to the time-like domain the space-like form at any loop level, regardless of the crossing between two disconnected regions, thus importing the same IR singular structure in a specular way. Nevertheless, from many early works based upon analysis of \( e^+e^- \)-annihilation data (see e.g. [30] and refs. therein), it is known that this should not be the case but far in the asymptotic regime. This is because of the appearance of not negligible corrections (\( \pi^2 \)-terms) to higher order coefficients of the \( \alpha_s(s) \)-expansions, due to analytic continuation from space-like to time-like axis. The problem has not yet been univocally solved, as it is
strongly related to the IR non analyticity of perturbative running coupling, 
though it receives a well satisfactory answer in the framework of Analytic 
Perturbation Theory (see secs.4.3 and 4.6).

Referring for definiteness to $e^+e^-$-annihilation into hadrons, the issue can be 
stated as follows. Firstly recall that the main way (see e.g. \[31\]) to deal with 
the ratio

$$ R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} $$

(41)

where $s = q^2 > 0$, is to start with the hadron contribution to photon polar-
ization tensor in (space-like) momentum space

$$ \Pi^\mu\nu_h(q) = (g^\mu\nu q^2 - q^\mu q^\nu)\Pi_h(-q^2) \propto \int d^4xe^{iqx} < 0|T(j^\mu(x)j^{\nu}(0))|0 > \quad (42) $$

$j^\mu$ being the quark electromagnetic current operator. Then, as known, optical theorem ensures the ratio $R(s)$ to be straightforwardly related to the 
absorptive part of the forward scattering amplitude $e^+e^- \rightarrow e^+e^-$. Indeed, 
being the analytical properties of the two-point correlation function (42) in 
the cut complex plane $\mathbb{C} - \{q^2 > 0\}$ well established, this amounts to taking 
the discontinuity of $\Pi_h(-q^2)$ across the cut

$$ R(s) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left[ \Pi_h(-s+i\epsilon) - \Pi_h(-s-i\epsilon) \right] , \quad (43) $$

having computed the RG improved expansion for $\Pi_h(-q^2)$ on the space-like 
axis ($q^2 < 0$). To this end one formally works with its first logarithmic 
derivative (thus avoiding subtraction constants), the Adler $D$-function \[32\], 
with the same space-like argument

$$ D(-q^2) = -q^2 \frac{d\Pi_h(-q^2)}{dq^2}, \quad (44) $$

which in the $\overline{\text{MS}}$ reads

$$ D_{\text{PT}}(Q^2) = 3 \sum_f Q_f^2 \left[ 1 + \frac{\alpha_s(Q^2)}{\pi} + d_2 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 + d_3 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^3 + \ldots \right] \quad (45) $$

being $Q^2 = -q^2$, $Q_f$ the quark charges, and \[33\], \[34\]

$$ d_2 \simeq 1.986 - 0.115n_f $$

$$ d_3 \simeq 18.244 - 4.216n_f + 0.086n_f^2 - 1.24 \left( \sum_f Q_f^2 \right)^{-1} \left( \sum_f Q_f \right)^2. \quad (46) $$
Then by integration $\Pi_h(Q^2)$ is readily obtained. One should be aware that in the massless theory the cut spreads over the whole positive axis, and when taking into account quark masses the cut starts at the two-pion threshold $4m^2_p$.

Whatever the loop order, the result for $R(s)$ is usually recasted as a series in the effective parameter $\alpha_s(s)$, naively obtained by specular reflection, that is by replacing the space-like argument $Q^2 = -q^2 > 0$ straightway with the time-like one $s = q^2 > 0$ in the coupling at a given loop order (e.g. eq. (31)). This final step displays nontrivial correction terms starting from $O(\alpha^3_s)$, which are proportional to powers of $\pi$ and are the drawback of the analytic continuation of hadronic tensor nearby the time-like axis. We then have the ordinary perturbative expansion for $R(s)$ (see e.g. (35))

$$R_{PT}(s) = 3 \sum_f Q_f^2 \left[ 1 + \frac{\alpha_s(s)}{\pi} + r_2 \left( \frac{\alpha_s(s)}{\pi} \right)^2 + r_3 \left( \frac{\alpha_s(s)}{\pi} \right)^3 + \ldots \right] \quad (47)$$

$$r_2 = d_2; \quad r_3 = d_3 - \delta_3; \quad \delta_3 = \frac{\pi^2 b_0^2}{48} \quad (48)$$

with $d_2$ and $d_3$ the same as in (45), and we have used the shortcut $b_j = (4\pi)^{j+1}b_j$ to emphasize the $\pi$-powers. The number $\delta_3$ gives to the $O(\alpha^3_s)$ coefficient a strongly negative contribution for each $n_f$, e.g. roughly $\simeq 14.3$ for $n_f = 4$. Higher order $\pi^2$-terms were analyzed in detail in [36], for instance the fourth order correction is

$$\delta_4 \equiv d_4 - r_4 = \frac{\pi^2 b_0^2}{16} \left( r_2 + \frac{5b_1}{24b_0} \right) \quad (49)$$

roughly $\delta_4 \simeq 120$ for $n_f = 4$; analogous (but more cumbersome) behavior is found for still higher orders, from which it becomes patent the remarkable growth in these correction terms.

A similar treatment also holds for other s-channel observables [35], as the normalized rate for $\tau$-decay into hadrons, showing that the effects of analytical continuation make the perturbative expansions in the time-like region deeply different from Euclidean ones.

Since the $\pi^2$-terms play a dominant role in higher order coefficients, expansion (47) works only asymptotically at large $s$ (that is when the smallness of $\alpha_s(s)$ scales down these large coefficients); thus the space-like coupling is not a reliable expansion parameter in the s-channel at finite energy, and it
is not yet clear which one is instead to be used. Actually, as yet noted in pioneer works \cite{37,38}, the expression of $R(s)$, resulting from eq. (43) together with the improved perturbative series for $\Pi_h$, exhibits no natural expansion parameter (since both real and imaginary parts of $\alpha_s(Q^2)$ enter into the form of $R$), and the choice of such a suitable parameter is essentially a matter of expediency, that is it should be selected the one which yields better convergence properties.

Alongside less meaningful attempts, we mention here the analysis \cite{37} of the use of $|\alpha_s(-s)|$ as expansion parameter for $R(s)$. By sensibly reducing higher order terms it yields faster convergence than $\alpha_s(s)$; it further possesses the relevant feature of IR freezing, in agreement with contemporary phenomenological models (e.g. \cite{39}), thus avoiding the hurdle of Landau singularities on the time-like domain. Indeed for low $s$, $|\alpha_s(-s)|$ is anyway less than 0.33 for $n_f = 3$. According to this prescription the one-loop running coupling in this region should read \cite{37}

$$|\alpha_s(-s)| = \frac{1}{\beta_0} \left[ \frac{1}{\ln^2(s/\Lambda^2) + \pi^2} \right]^{1/2} \tag{50}$$

and asymptotically, i.e. for $s \gg \Lambda^2 e^\pi$

$$|\alpha_s(-s)| = \frac{1}{\beta_0 \ln(s/\Lambda^2)} \left[ 1 - \frac{\pi^2}{2} \frac{1}{\ln^2(s/\Lambda^2)} + \ldots \right] \tag{51}$$

resembling the UV behavior of the relative space-like coupling \cite{18}. Aside from these nice features, this function cannot entirely sum up the $\pi^2$-terms. In order to deal with these corrections a somewhat different approach, known as RKP (Radyushkin-Krasnikov-Pivovarov) procedure, has been suggested \cite{40,41} (see also \cite{42}), which is firmly based upon the analytical properties of the polarization tensor $\Pi_h(-q^2)$ and of the related $D$-function \cite{44}, summarized by the dispersion relations, respectively

$$\Pi_h(-q^2) = \int_0^\infty ds \frac{R(s)}{s-q^2}, \tag{52}$$

$$D(-q^2) = -q^2 \int_0^\infty ds \frac{R(s)}{(s-q^2)^2} \tag{53}$$

where $R(s)$ is given by \cite{43}, and $q^2$ lying in $\mathbb{C} - \{q^2 = s > 0\}$. The key point here is the inverse of $\Pi_h$ given by the contour integral

$$R(s) = \frac{i}{2\pi} \int_{s-i\varepsilon}^{s+i\varepsilon} \frac{dq^2}{q^2} D(-q^2) \tag{54}$$
to be computed along a path in the analyticity region for the $D$-function. Eq. (54) can be then generalized to an integral transform mapping a space-like argument function into a time-like one

$$R(s) = \Phi[D(-q^2)] ,$$  

(55)

that can be straightforwardly applied to the perturbative expansion (45), provided that the integration contour is always kept far enough from IR space-like singularities. This yields $R(s)$ as an expansion into the images of $\alpha_s(Q^2)$ and of its powers, through the map $\Phi$

$$R(s) = 3 \sum_f Q_f^2 \left\{ 1 + \sum_{n \geq 1} d_n \Phi \left[ \left( \frac{\alpha_s(Q^2)}{\pi} \right)^n \right] \right\} ,$$  

(56)

where $d_n$ are the same as in (45). Eq. (56) is to be compared with the standard perturbative expansion (47); here the $\pi^2$-terms are entirely summed up, with the drawback that within this framework there is no uniquely defined expansion parameter. However, it is useful to work out its behavior at $O(\alpha_s)$-approximation for $R(s)$, i.e.

$$\tilde{\alpha}^{(1)}(s) \equiv \Phi[\alpha_s^{(1)}(Q^2)] = \frac{1}{\beta_0} \left\{ 1 + \frac{1}{2} - \frac{1}{\pi} \arctan \left[ \frac{\ln(s/\Lambda^2)}{\pi} \right] \right\}$$  

(57)

easily obtained by applying the integral transformation (54) to the one-loop space-like coupling (18). The related time-like form (57) is once more free of unphysical singularities at low $s$, and for $s \gg \Lambda^2 e^\pi$ as above can be expanded into powers of $\pi/\ln(s/\Lambda^2)$

$$\tilde{\alpha}^{(1)}(s) = \frac{1}{\pi \beta_0} \left[ \frac{\pi}{\ln(s/\Lambda^2)} - \frac{\pi^3}{3 \ln^3(s/\Lambda^2)} + \ldots \right] ,$$  

(58)

Then eq. (58) can be recasted as a power series in the one-loop space-like coupling

$$\tilde{\alpha}^{(1)}(s) = \alpha_s^{(1)}(s) \left[ 1 - \frac{\pi^2 b_0^2}{48} \left( \frac{\alpha_s^{(1)}(s)}{\pi} \right)^2 + \ldots \right] ,$$  

(59)

emphasizing that the two couplings differ in three-loop level. By comparing (59) with eqs. (47) and (48), it becomes clear how RKP resummation of the $\pi^2$-terms into the time-like coupling (57) does work. The main shortcoming
of this recipe is that by applying to (57) the inverse transformation to (55), i.e. relation (53), we are not back to the original input (18). Obviously, this is because integral transformations (54) and its inverse are well behaved as long as the integrand possesses the correct analytical properties in the cut complex plane; actually this is not the case for the space-like coupling (18) and its higher-loop approximations, and we are then forced to compute the integral along a path large enough to avoid the IR space-like singularities.

3 Infrared behavior

As we told, Landau singularities are an artifact of the formalism and are related to a breakdown of the perturbative expansion even as an asymptotic expansion. On the basis of general principles of field theory \( \alpha_s(\mu^2) \) should have singularities in the \( \mu^2 \) complex plane only on the negative real axis, from a threshold to \(-\infty\). This circumstance is irrelevant as long as we consider high energy processes in which the energy scale is much larger than \( \Lambda \), but leads to serious difficulties in other problems in which much smaller values of \( \mu \) are involved. Of this type are the bound state calculations, the decay of particles, the annihilation processes and even the deep inelastic collisions in particular geometries. Let us consider e.g. the ratio

\[
R_\tau = \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau + e + \bar{\nu}_e)}
\]

between the hadronic and leptonic decay width of the \( \tau \) lepton. The non-strange contribution to this quantity can be written in the form

\[
R_{\tau}^{\text{inst}} = \frac{12\pi S_{\text{EW}} |V_{ud}|^2}{m_\tau^2} \int_{m_\tau^2}^{m_\pi^2} dt \left( 1 - \frac{t}{m_\tau^2} \right)^2 \left\{ \left( 1 + \frac{2t}{m_\tau^2} \right) \text{Im} \Pi^{(1)}_{ud}(t) + \text{Im} \Pi^{(0)}_{ud}(t) \right\},
\]

where \( S_{\text{EW}} \) is the electro-weak factor, \( V_{ud} \) the relevant CKM matrix element, \( \Pi^{(1)}_{ud}(t) \) and \( \Pi^{(0)}_{ud}(t) \) are the transversal and the longitudinal part of the appropriate current-current correlator respectively. In this expression \( \alpha_s(\mu^2) \) should be known in principle from the threshold \( m_\pi \sim 0.14 \text{ GeV} \) to \( m_\tau = 1.78 \text{ GeV} \). As we told, to extrapolate \( \alpha_s(\mu^2) \) to the infrared region, various proposals have been advanced. In this section we shall consider some purely phenomenological attempts, few examples of the so-called physical couplings and an optimization procedure.
3.1 Potential inspired approaches

One of the first attempts to modify the expression for $\alpha_s(\mu^2)$ in the infrared region was made in the framework of the quark-antiquark potential. The potential $V(r)$ between two infinitely heavy quark and antiquark (static potential) can be defined by the equation

$$V(r) = \lim_{T \to \infty} \frac{i}{T} \ln W[\Gamma]$$  \hspace{1cm} (62)

$\Gamma$ being a rectangular Wilson loop of size $r \times T$ and

$$W[\Gamma] = \left\langle \frac{1}{3} \text{Tr} \mathcal{P} \exp \left\{ ig \oint_\Gamma dx^\mu A_\mu(x) \right\} \right\rangle,$$  \hspace{1cm} (63)

$\mathcal{P}$ being the ordering prescription on the path $\Gamma$ and $\text{Tr}$ the trace over the color indices. The simplest assumption is to write $\ln W$ as the sum of its perturbative expression and a non perturbative term proportional to the area $S = rT$ delimited by $\Gamma$

$$i \ln W = (i \ln W)_{PT} + \sigma S.$$  \hspace{1cm} (64)

If $(i \ln W)_{PT}$ is evaluated at the first order in the coupling one obtains

$$V(r) = -\frac{4}{3} \frac{\alpha_s(\mu^2)}{r} + \sigma r,$$  \hspace{1cm} (65)

where $\mu$ should be taken of the order of the typical $\langle 1/r \rangle$. This is the so called Cornell potential, expressed as the sum of a Coulombian and linear term. As well known, by solving the corresponding Schroedinger equation a reasonable reproduction of the spin averaged bottomonium and charmonium spectrum can already be obtained. In the momentum representation (65) can be written

$$\tilde{V}(Q) \equiv \langle k'|V(r)|k \rangle = \frac{1}{(2\pi)^3} \int d^3r e^{-iQ \cdot r} V(r) =$$

$$= -\frac{1}{2\pi^2 3} \frac{4 \alpha_s(\mu^2)}{Q^2} - \frac{1}{\pi^2 Q^4},$$  \hspace{1cm} (66)

where $k, k'$ denote the initial and final center of mass momentum of the quark and $Q = k' - k$ the momentum transfer \footnote{More properly $\frac{1}{Q^2}$ stays for the limit of $\frac{1}{(Q^2+\epsilon^2)^2} - \frac{4\epsilon^2}{(Q^2+\epsilon^2)^3}$ (the Fourier transform of $-\pi^2 re^{-r}$) for infinitesimal $\epsilon$. The same recipe should be applied to regularize eqs. (67-69).}. According to the general rule we
should identify in (66) the scale $\mu$ with $Q$. Since however in heavy quarkonia ($b\bar{b}$ and $c\bar{c}$) $Q$ typically ranges between 200 MeV and 2 GeV, we come close to Landau singularities and need to use some kind of regularization. In this order of ideas many years ago it was proposed [43] to write the entire potential as

$$V(Q) = -\frac{1}{2\pi^2} \frac{4}{3} \alpha_V(Q^2)$$

(67)

and to take

$$\alpha_V(Q^2) = \frac{1}{\beta_0 \ln(1 + Q^2/\Lambda^2)}.$$  

(68)

Note that, defined in this way, $\alpha_V(Q^2)$ reproduces the ordinary one-loop expression (18) for $Q \to \infty$, while for $Q \to 0$ it gives

$$\alpha_V(Q^2) \to \frac{\Lambda^2}{\beta_0 Q^2}.$$  

(69)

Replacing (69) in (67) and comparing it with (66) we see that (68) incorporate the confining part of the potential (65) with

$$\sigma = \frac{2\Lambda^2}{3\beta_0}.$$  

(70)

In (65) the perturbative part of the potential was evaluated at the first order in $\alpha_s(\mu^2)$. However, higher orders corrections have also been considered starting from (62) or by other methods [45]. As proposed in [44], such corrections can be taken into account assuming (67) as the definition of a new coupling constant $\alpha_V(\mu^2)$ and re-expressing this in term the ordinary MS constant. We can write

$$\alpha_V(\mu^2) = \alpha_{\text{MS}}(\mu^2) \left[ 1 + u_1 \frac{\alpha_{\text{MS}}(\mu^2)}{\pi} + u_2 \left( \frac{\alpha_{\text{MS}}(\mu^2)}{\pi} \right)^2 + ... \right].$$  

(71)

with

$$u_1 = \frac{31}{12} - \frac{5}{18} n_f; \quad u_2 \cong 28.538 - 4.145 n_f + 0.077 n_f^2$$  

(72)

i. e. $u_1 \cong 1.472$, $u_2 \cong 13.190$ for $n_f = 4$. Alternatively one can use ordinary one or two-loop equations for the running $\alpha_V(\mu^2)$ but with $\Lambda_V$ redefined as (cf. 34)

$$\Lambda_V = \Lambda_{\text{MS}} e^{u_1/2\pi \beta_0} \cong 1.42 \Lambda_{\text{MS}}$$  

(73)

\footnote{Note that at three loops even terms of the form $(\frac{\alpha_{\text{MS}}}{\pi})^4 \ln \alpha_{\text{MS}}$ occur [40].}
and this should be the value to be used in Eq. (68). Eq. (68) can be re-obtained from (17) by the modified \( \beta \)-function [44]

\[
\beta_V(\alpha) = -\beta_0 \alpha^2 \left(1 - e^{-\frac{1}{\beta_0 \alpha}}\right).
\]

More generally one can take

\[
\frac{1}{\beta_V(\alpha)} = -\frac{1}{\beta_0 \alpha^2 \left(1 - e^{-\frac{1}{\beta_0 \alpha}}\right)} + \frac{\beta_1}{\beta_0^2} \frac{1}{\alpha} e^{-l\alpha},
\]

\( l \) being an adjustable parameter. Eq. (75) reduces to (74) for \( l \to \infty \), while for finite \( l \) produce the following asymptotic behavior

\[
\beta_V(\alpha) \sim -\beta_0 \alpha^2 - \beta_1 \alpha^3 - \ldots \quad \text{for} \quad \alpha \to 0
\]

\[
\beta_V(\alpha) \sim -\alpha + \ldots \quad \text{for} \quad \alpha \to \infty
\]

which corresponds for \( \alpha_s(Q^2) \) to a two-loop expression of the type (26) (or (29)) for \( Q \to \infty \) or of the form (38) for \( Q \to 0 \).

Eqs. (67-70), or the more general corresponding to (75), are very appealing. They do not introduce any additional parameters in the theory and for \( \Lambda_V \sim 0.5 \text{ GeV} \) (and \( l = 24 \)) reproduce well the spin averaged first excited states in the \( b\bar{b} \) and \( c\bar{c} \) systems. However, they are not satisfactory from other points of view. They correspond to assume that all forces including confinement are due to the exchange of some effective vectorial object. On the contrary, if eq. (14) is generalized to Wilson loops with distorted contour (more sophisticate ansatzs also exist [50, 51]) and some more elaborate method is applied, spin dependent and velocity dependent (relativistic) corrections can be obtained, which differ from those that would be derived from the exchange of such a vectorial particle alone and seem to be preferred by the data [17, 18, 19]. The majority of the analysis seems rather in favor of assuming a finite limit for \( \alpha_s(\mu^2) \) for \( \mu \to 0 \) and adding a separate term for confinement as in [63]. Besides hadron spectroscopy there is other abundant phenomenology that seems to be consistent with the existence of a finite IR limit for \( \alpha_s(\mu^2) \). This concerns the hadron-hadron scattering and the hadron form factors, the properties of jets, the transversal momentum spectrum in \( W \) and \( Z \) production etc.

Coming to the explicit form of \( \alpha_s(Q^2) \) the simplest modification of the one-loop expression that has been considered is the “hard freeze” assumption

\[
\alpha_s(Q^2) = \left\{ \begin{array}{ll}
\frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} & \text{for } Q^2 \geq Q_0^2 \\
\frac{1}{H \equiv \frac{1}{\beta_0 \ln(Q_0^2/\Lambda^2)}} & \text{for } Q^2 \leq Q_0^2
\end{array} \right.
\]

(78)
This equation has been used in hadron spectrum calculations, in a model for hadron-hadron scattering and in studies on nucleon structure function \( [52] \) (see also \( [53] \)); the values \( Q_0 = 0.44 \) GeV and \( \Lambda \) corresponding to \( \frac{\mu}{\pi} = 0.28 \) GeV have been found appropriate for the last two applications, a little smaller value \( \frac{\mu}{\pi} = 0.26 \) GeV comes from other phenomenology. Other convenient interpolation formulas between the large \( Q \) perturbative expression and a finite \( \alpha_s(0) \) have been used again in hadron spectrum studies \( [54] \) with \( \frac{\alpha_s(0)}{\pi} \sim 0.19 - 0.25 \). In a fully relativistic treatment in \( [55] \) it was found that in order to obtain a \( \pi \) mass so much lighter than the \( \rho \) mass a value \( \frac{\alpha_s(0)}{\pi} = 0.265 \) was necessary. A similar results was obtained in \( [165] \) with a one-loop analytic coupling (see later) with \( \Lambda = 0.18 \) GeV corresponding to \( \frac{\alpha_s(0)}{\pi} = 0.44 \) but, what is really relevant, to an average value of \( \frac{\alpha_s(Q^2)}{\pi} \) in the interval between 0 and 0.5 GeV about 0.22. Finally in many analysis the successful phenomenological hypothesis was adopted that the gluon acquires an effective dynamical mass \( m_g \) \( [57, 58, 59] \). We shall come back in the following to this point, for the moment let us observe that to the leading order the following equation generalizes naively \( [68] \)

\[
\alpha_s(Q^2) = \frac{1}{\beta_0 \ln \left( \frac{\mu^2 + Q^2}{\Lambda^2} \right)}. \tag{79}
\]

In particular the mentioned hadron-hadron scattering model gives in this case \( \Lambda = 0.3 \) GeV and \( m_g = 0.37 \) GeV (\( \frac{\alpha_s(0)}{\pi} = 0.26 \)) \( [56] \).

All the above mentioned attempts are essentially in agreement for what concerns the qualitative behavior of \( \alpha_s(Q^2) \) in the infrared region, even if they differ in the details. The point is that for many purposes we can simply introduce an infrared-ultraviolet matching point in the range of variability of \( Q \); let us say \( Q_1 = 1 \) or 2 GeV. Then one can use ordinary perturbative expressions of the type \( [18] \) for \( Q > Q_1 \) and treat the quantities

\[
\langle \frac{\alpha_s}{\pi} \rangle = \frac{1}{Q_1} \int_0^{Q_1} dQ \frac{\alpha_s(Q^2)}{\pi} \tag{80}
\]

or corresponding higher moments as adjustable parameters. Various event shape in \( e^+e^- \) annihilation can be reproduced simply assuming \( Q_1 = 1 \) GeV and \( \langle \frac{\alpha_s}{\pi} \rangle \approx 0.2 \) \( [63] \). Such procedure is believed to be a way to parametrize and derive from experience truly non-perturbative effects (see 4.5).
3.2 Physical couplings

By physical couplings we mean any type of effective charges \cite{4,60}, defined by (11), and in general any other observable related coupling. The advantage of such quantities is that they are in principle well defined at every scale and that can be easily extracted from experience. The disadvantage is that they are observable dependent. However, as we told they can be expanded in terms of the MS coupling (cf.(12)) and related to each other \cite{61}. First we may consider the typical case of the quantity $R_{e^+ e^-}(s)$, eq.(41), and the related Adler function eq.(44). Their perturbative expansions in the MS scheme are given by (47) and (45), respectively. Correspondingly two different effective charges $\alpha_R(s)$ and $\alpha_D(Q^2)$ can be defined

$$R_{e^+ e^-}(s) = 3 \sum_f Q_f^2 \left[ 1 + \frac{\alpha_R(s)}{\pi} \right], \quad D(Q^2) = 3 \sum_f Q_f^2 \left[ 1 + \frac{\alpha_D(Q^2)}{\pi} \right] \quad (81)$$

The first of these is of time-like, the second of space-like type. As apparent from eq.(53), they are related by the transformation (55), i.e. $\alpha_R(s) = \Phi[\alpha_D(Q^2)]$, and are actually the model over which (57) is written.

A third effective charge, that has been used in analysis of the $\tau$ decay, can be defined with reference to the quantity $R_\tau$, eq.(60) and (61), which we discuss in some more detail as an example. We can consider separately the contributions due to the vector and the axial currents and set \cite{65,66}

$$R_{\tau}^{V/A}(s) = S E W |V_{ud}|^2 \int_0^s \frac{dt}{s} \left( 1 - \frac{t}{s} \right)^2 \left\{ \left( 1 + \frac{2t}{s} \right) \text{Im} \Pi^{(1)}_{V/A}(t) + \text{Im} \Pi^{(0)}_{V/A}(t) \right\}. \quad (82)$$

For $s = m_\tau^2$ eq.(82) gives the ordinary $R_{\tau}^{V/A}$ for the $\tau$ lepton, for $s < m_\tau^2$ gives the corresponding expression for a fictitious $\tau'$ with mass $m_{\tau'} = \sqrt{s}$. In analogy with (81) we can define a vector and axial coupling $\alpha_{\tau}^{V/A}(s)$ by

$$R_{\tau}^{V/A}(s) = \frac{R_\tau^0}{2} \left[ 1 + \frac{\alpha_{\tau}^{V/A}(s)}{\pi} \right], \quad (83)$$

where $R_\tau^0$ denotes the same quantity at zero order in the strong coupling. It is also defined a global $\alpha_\tau(s)$ by

$$R_\tau(s) = R_{\tau}^V(s) + R_{\tau}^A(s) = R_\tau^0 \left[ 1 + \frac{\alpha_\tau(s)}{\pi} \right] \quad (84)$$
and $\alpha_\tau(s) = \frac{1}{2} \left[ \alpha^V(s) + \alpha^A(s) \right]$. The quantity $\text{Im} \Pi^{(0)}_V(s)$ may be assumed to vanish for small quark masses and $\text{Im} \Pi^{(0)}_A(s)$ to be given only by the pion pole, $\text{Im} \Pi^{(0)}_A(s) = \frac{s}{m^2} \delta(s - m^2)$. Under the same hypothesis the quantities $\text{Im} \Pi^{(1)}_V(s)$ and $\text{Im} \Pi^{(1)}_A(s)$ should be identical at the perturbative level and they are expected to differ asymptotically only for powers of $1/s$ which have a non-perturbative origin. The same must be true for $\alpha^V(s)$ and $\alpha^A(s)$.

Note that, due to isospin invariance $\text{Im} \Pi^{(1)}_V$ is proportional to the isovectorial component $\text{Im} \Pi^{(1)}_{I=1,em}$, and $\alpha^V(s)$ and $\alpha_R(s)$ are related by

$$
\alpha^V(s) = 2 \int_0^s \frac{dt}{s} \left( 1 - \frac{t}{s} \right)^2 \left( 1 + \frac{2t}{s} \right) \alpha_R(t). \tag{85}
$$

Since the first coefficients of the perturbative expansion of $R_{e^+e^-}$ and $R_\tau$ in the $\overline{\text{MS}}$ scheme are known, even the coefficients $v_1, v_2, v_3$ of (33) and $\beta^2_2, \beta^3_3$ can be calculated. Therefore the couplings $\alpha_\tau(s)$ and $\alpha_R(s)$ that can be extracted directly from experience can be immediately translated in terms of a $\alpha_{\overline{\text{MS}}}(s)$ value. Note, however, that in the expressions of $v_3$ and $\beta_3$, which we do not report, there appears a quantity $K_4$ that has been only estimated, $K_4 = 25 \pm 50 \quad [68]$. We report in fig 2(a) taken from [65] the experimental value for $\alpha^V(s)$, $\alpha^A(s)$ and $\alpha_\tau(s)$ as extracted from the data of OPAL collaboration [66]. The results are confronted with the resolution of the RG equation for the appropriate $\beta_\tau(\alpha_\tau)$. As it can be seen the 3-loop $\alpha_\tau(s)$ and 4-loop for $K_4 = \pm 25$ have a finite limit for $s \to 0$; the latter fits data very well for $K_4 = 25$ down to $s \sim 1 \text{ GeV}^2$. The strong enhancement of the experimental $\alpha_\tau(s)$ below such value can be related to the pion pole that has not been included in the RG treatment.

A last definition of effective charge has been given recently [69] in the context of the Sudakov resummation formalism. Let us consider, e.g., the Mellin transform $\hat{F}_2(Q^2,N)$ of the structure function $F_2(Q^2,x)$ in DIS ($x$ being the Bjorken variable). Even after the mutual cancellation of the infrared singularities due to the soft real and virtual gluons, $F_2(Q^2,x)$ has a logarithmic singularity at any order for $x \to 1$ which makes ordinary perturbation theory inapplicable. Such a singularity which is translated in the Mellin $N$-space in $\ln N$ power behavior for $N \to \infty$, can however be resummed [70]. One can then obtain the following asymptotic equation

$$
Q^2 \frac{\partial \ln \hat{F}_2(Q^2,N)}{\partial Q^2} = \tag{86}
$$
\[ C_F \pi \left[ \int_0^1 dx x^{N-1} - \frac{1}{1-x} A_S[(1-x)Q^2] + H(Q^2) + O \left( \frac{1}{N} \right) \right], \]

where \( A_S \) and \( H \) have the usual expansions

\[ A_S[Q^2] = \alpha_s(Q^2)[1 + a_1 \alpha_s(Q^2) + \ldots]. \quad (87) \]

Then \( A_S[Q^2] \) is assumed as an effective charge (Sudakov charge). For large \( n_f \) the Borel transform of (87) \( B[A_S](t) \) can be given as an analytic expression without singularities. The corresponding \( A_S[Q^2] \) turns out to be free of Landau singularities but for \( Q \to 0 \) behaves as \(-\frac{1}{2\pi} \frac{\Lambda^4}{Q^2} \). This embarrassing singularity can be compensated with a non-perturbative term and leads to corrections in \( 1/Q \) (see sec 4.5).

Another very interesting physical coupling is introduced in [64]. This is a generalization to QCD of the Gell-Mann Low effective coupling for QED. It can be considered a generalization of the coupling \( \alpha_V(Q^2) \) discussed in sec. 3.1 and it is also related to the pinch scheme discussed in [71], which is somewhat more involved but explicitly gauge-independent.

Let us isolate in a Feynman integrand the factor corresponding to the exchange of one dressed gluon between two quark lines

\[ \frac{-i\alpha_s(\mu^2)}{q^2 + i0} \frac{1}{1 - \Pi[q^2; \alpha_s(\mu^2), \mu^2]}, \quad (88) \]

and set, e.g. in the \( \overline{\text{MS}} \) scheme

\[ \alpha_{\text{SGD}}(Q^2) = \left[ \frac{Z_{\text{qg}}^{\text{NA}}(\mu^2)}{Z_{\text{qg}}^{\text{NA}}(Q^2)} \right]^2 \alpha_s(\mu^2) \frac{1}{1 - \Pi[-Q^2; \alpha_s(\mu^2), \mu^2]}, \quad (89) \]

where \( Z_{\text{qg}}^{\text{NA}}(\mu^2) \) is the appropriate renormalization factor that makes the definition independent of the initial scale \( \mu^2 \). Obviously

\[ Z_{\text{qg}}^{\text{NA}}(\mu^2) = [Z\alpha(\mu^2)Z_g(\mu^2)]^{\frac{1}{2}} = Z_{\text{qg}}(\mu^2)Z_{qg}^{-1}(\mu^2), \quad (90) \]

where we have used the same notation as in sec. 2.1. Note that in an abelian theory, like QED, \( Z_{\text{qg}}^{\text{NA}} = 1 \) due to Ward identity. In QCD the quark factor \( Z_q \) cancels only the “abelian part” of the vertex factor \( Z_{\text{qg}} \). This is the meaning of the superscript \( \text{NA} \) (non abelian part). The index SGD in eq. (89) means “single gluon dressing”.

From (89), if \( \Pi \) is evaluated in the \( \overline{\text{MS}} \) scheme, we can write, setting \( \mu = Q \),

\[ \alpha_{\text{SGD}}(Q^2) = \alpha_{\overline{\text{MS}}}(Q^2) \left[ 1 + k_1 \frac{\alpha_{\overline{\text{MS}}}(Q^2)}{\pi} + \ldots \right]. \quad (91) \]
Here $k_1$ depends on the constant terms occurring in the renormalized $\Pi$ and it is so gauge dependent; e.g. in the Feynman gauge $k_1 = (31 - 10n_f/3)/12$. Note, however, that eq. (89) can be related to the pinch scheme and $k_1$ made gauge independent simply including in the definition of $\Pi$ additional constants coming from the pinch parts (selfenergy like parts originating from contraction of internal lines as a consequence of Ward identities) of the vertex and the box diagrams [71]. In this way we obtain $k_1 = (67 - 10n_f/3)/12$. On the contrary, if we absorb the pertinent soft gluon corrections in a redefinition of $\alpha_{\text{SGD}}(Q^2)$, we find $k_1 = (67 - 3\pi^2 - 10n_f/3)/12$ [62, 64]. It is the latter value which is used in connection with the considerations of sec. 4.5.

### 3.3 Optimized perturbation theory

By optimized perturbation theory it is generally meant some kind of perturbative expansion in which the expansion variable, or the splitting of the Lagrangian in an unperturbed and a perturbation part, is chosen in dependence on a number of arbitrary parameters. Such parameters should not appear in the exact result, but obviously they occur in any expansion which is stopped to a certain maximum term $n$. However, it is immediately shown even by very simple examples, that the convergence of the series is greatly improved if for every $n$ the parameters are chosen at some stationary value (optimized choice) that depends on $n$.

In QCD, due to the arbitrariness in the choice of the RS, the optimization can be required in the variables that control such a scheme, e.g. the subtraction point $\mu$ and the scheme dependent coefficients $\beta_2$, $\beta_3$, .... Obviously every choice corresponds to a different definition of the coupling constant $\alpha_s$. Let us e.g. consider the quantity $R_{e^+e^-}(s)$ defined by eq. (41), and rewrite expansion (47) with an arbitrary choice of $\mu^2$ (for the moment different from the total energy $s$) and in a arbitrary RS

$$R_{e^+e^-}(s) = 3 \sum_f Q_f^2 \left[ 1 + \frac{\alpha_s(\mu^2)}{\pi} + r_2(s) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 + r_3(s) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^3 + \ldots \right]$$

(92)

The quantity $R_{e^+e^-}(s)$ must be RS independent and, if we neglect the masses of active quarks, we can write

$$\left( \frac{\partial}{\partial \tau} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) R_{e^+e^-} = 0$$
\[
\left( \frac{\partial}{\partial \beta_j} - \beta(\alpha_s) \int_0^{\alpha_s} d\alpha' \frac{\alpha^{j+2}}{[\beta(\alpha')]^2} \frac{\partial}{\partial \alpha_s} \right) R_{e^+e^-} = 0, \quad (93)
\]

where \( j = 2, 3, \ldots \), \( \tau = \ln(\mu^2/\Lambda^2) \) (recall eq. (28)) and we have used
\[
\frac{\partial \alpha_s}{\partial \beta_j} = \beta(\alpha_s) \int_0^{\alpha_s} d\alpha' \frac{\alpha^{j+2}}{[\beta(\alpha')]^2} = \alpha_s^{j+1} \left( \frac{1}{j-1} - \frac{\beta_1}{\beta_0} \frac{j-2}{j-1} \alpha_s + \ldots \right), \quad (94)
\]
as it can be seen deriving eq. (17). Eq. (93) can be used, first to obtain \( r_2, r_3, \ldots \) in an arbitrary RS when we know this quantities in a specific RS (cf. eq. (9)), and then to make the optimal choice for \( \tau, \beta_2, \beta_3, \ldots \), when we have stopped expansions (92) and (3) to a certain maximum order; let us say to the term in \( r_3 \) in (92) and to three-loop (i.e. to the term in \( \beta_2 \) ) in (3).

From now on we will use the notations \( b_j = (4\pi)^{j+1} \beta_j \) and \( a = \alpha_s/\pi \). Replacing (92) in (93), asking that this equations for a given \( \mu^2 \) are satisfied for an arbitrary value of \( a \) we obtain differential equations for \( r_2, r_3, \ldots \).

Restricting to \( r_2 \) and \( r_3 \) and \( j = 2 \) we have
\[
\frac{\partial r_2}{\partial \tau} = \frac{1}{4} b_0 \quad \frac{\partial r_2}{\partial b_2} = 0 \quad \frac{\partial r_3}{\partial \tau} = \frac{1}{2} b_0 r_2 + \frac{1}{16} b_1 \quad \frac{\partial r_3}{\partial b_2} = -\frac{1}{16} b_0 \quad (95)
\]

Integrating the above equations, we obtain
\[
r_2 = \frac{1}{4} b_0 \tau + \rho_2 \\
r_3 = \frac{1}{16} b_0^2 \tau^2 + \frac{1}{2} b_0 \rho_2 \tau + \frac{1}{16} b_1 \tau - \frac{1}{16} b_2 + \rho_3
\]

\[
= \left( r_2 + \frac{1}{8} b_1 \right)^2 - \frac{1}{16} b_0 \rho_3, \quad (96)
\]

where \( \rho_2 \) and \( \rho_3 \) are integration constants and so quantities independent of \( \tau, b_2, \ldots \) and RS independent. They can be calculated e.g. equating \( b_2, r_2, r_3 \) to their expressions \( b_2^{\text{MS}}, r_2^{\text{MS}}, r_3^{\text{MS}} \) in the \( \overline{\text{MS}} \) scheme as given by equations (48) and (46) after setting \( \mu^2 = s \). We have
\[
\rho_2 = r_2^{\overline{\text{MS}}} - \frac{1}{4} b_0 \ln \frac{s}{\Lambda^2}, \quad \rho_3 = r_3^{\overline{\text{MS}}} - \left( r_2^{\overline{\text{MS}}} + \frac{1}{8} b_1 \right)^2 + \frac{1}{16} b_2^{\overline{\text{MS}}}. \quad (97)
\]
Note that \(\rho_3\) turns out to be independent of \(s\), and \(r_2\) has the form
\[
r_2 = -\frac{1}{4} b_0 \ln \frac{s}{\Lambda^2} + \frac{1}{4} b_0 \tau + r_2^{\overline{\text{MS}}} ,
\]
(98)
while \(r_3\) depends on \(s\) and \(\tau\) only through \(r_2\). Let us now replace \(\beta(a)\) in (17) with its 3-loop expression \(\beta(3)(a) = -\pi a^2 (\frac{b_1}{4} + \frac{1}{16} a + \frac{b_2}{64} a^2)\). We have
\[
\tau = \frac{4}{b_0 a} + \frac{b_1}{b_0^2} \ln \left( \frac{b_1 a}{b_0} \right) - \frac{b_1}{2b_0^2} \ln \left( \frac{16b_0 + 4b_1 a + b_2 a^2}{b_0} \right) + \frac{2b_2 b_0 - b_1^2}{2b_0^2} f(a, b_2)
\]
(99)
with
\[
f(a, b_2) = \frac{1}{\sqrt{D}} \ln \left( \frac{4b_0 + \frac{1}{2} a(b_1 + \sqrt{D})}{4b_0 + \frac{1}{2} a(b_1 - \sqrt{D})} \right)
\]
(100)
and \(D = b_1^2 - 4b_2 b_0\). Let us make the same replacement in (93) and stop (92) at the \(a^3\) term. By requiring that eq.(93) is now exactly satisfied, we obtain the following equations
\[
3b_0 r_3 + \frac{1}{2} b_1 r_2 + \frac{1}{16} a + (3b_1 r_3 + \frac{1}{2} b_2 r_2) a + 3b_2 r_3 \frac{a^2}{16} = 0
\]
\[
\left[ 1 + \left( \frac{b_1}{4b_0} + 2r_2 \right) a \right] I(a, b_2) - a = 0 ,
\]
(101)
with
\[
I(a, b_2) = \frac{4b_0}{D} \left[ \frac{(4b_1^2 - 8b_2 b_0) a + b_2 b_1 a^2}{16b_0 + 4b_1 a + b_2 a^2} - 2b_2 b_0 f(a, b_2) \right]
\]
(102)
Eq. (99) gives \(a\) as a function of \(\tau\) (or \(\mu\)) and then (102) and (101) become equations in \(\tau\) and \(b_2\) that determine the optimal choice \(\tau(s)\) and \(b_2(s)\) of such quantities for every \(s\). In this way we obtain an optimized running coupling \(\alpha(s) = \alpha_{\text{OPT}}(s)\) which together with the optimized values \(\tau_2(s)\), \(\tau_3(s)\), can be used in (93) to evaluate the quantity \(R_{e^+e^-}(s)\). The resulting \(\alpha_{\text{OPT}}(s)\) can be evaluated numerically and is reported in fig.2(b) taken from [72] as a function of \(q = \sqrt{s}\) for \(\Lambda_{\text{MS}} = 230\text{ MeV} (\overline{\Lambda}_{\text{MS}} = 264\text{ MeV})\). Note that \(\alpha_{\text{OPT}}(q^2)\) stays finite as \(q\) decreases and attains a maximum for \(q \sim 200\text{ MeV}\), after which it remains practically constant. Such a maximum value can be shown to be given by the equation [72]
\[
\frac{\alpha_{\text{OPT}}^*}{\pi} = \frac{-b_1 + \sqrt{b_1^2 - 336b_0^2 \rho_3}}{24b_0 \rho_3} .
\]
(103)
Taking $n_f = 2$, as appropriate to the range of energy, we have from (97) $\rho_3 = -10.911$ and so $\frac{\alpha_{\text{OPT}}}{\pi} = 0.263$. Note that the behavior of $\alpha_{\text{OPT}}(q^2)$ for $0 < q < 1$ GeV is perfectly consistent with what follows eq. (80). In fig 2(b) are also reported the second and third order results for the quantity $R$ defined by $R_{e^+e^-} = 3 \sum f Q_f^2 (1 + R)$ and expressing the QCD correction to $R_{e^+e^-}$. The results compare favorably with the experimental data appropriately smeared to wash the irregularity due to the resonances.

4 The dispersive approach

As mentioned in the introduction we can dispose of the Landau singularities simply exploiting the general analyticity properties expected for $\alpha_s(Q^2)$ and applying perturbation theory directly to the spectral function \cite{79, 80}. This idea generalizes to QCD a method originally introduced in QED \cite{81}. Defining the spectral density

$$
\rho(\sigma) = \text{Im} \alpha_s(-\sigma - i0) = \frac{1}{2i} \left[ \alpha_s(-\sigma - i0) - \alpha_s(-\sigma + i0) \right], \quad (104)
$$
where $\sigma > 0$ and $\alpha_s(-\sigma)$ is the perturbative RG solution at a given loop level, the analytically improved running coupling is thus given by [79]

$$
\alpha_{\text{an}}(Q^2) = \frac{1}{\pi} \int_0^\infty d\sigma \frac{\rho(\sigma)}{\sigma + Q^2}
$$

(105)

whose argument $Q^2 = -q^2 > 0$ now runs over the whole space-like axis (we observe here and in the foregoing the identification $\mu^2 = Q^2$), that is $\alpha_{\text{an}}(Q^2)$ is free of any space-like unphysical singularities by construction; moreover, due to the asymptotically free nature of the perturbative input the spectral integral (105) needs no subtractions.

Note that different strategies to incorporate analyticity into the RG formalism, or even to implement the above device, exist as well, and they will be briefly reminded in sec. 4.4.

### 4.1 One-loop analytic coupling

At one-loop level this trick works quite straightforwardly; starting from the leading-logs expression (18) the related spectral density (see also eq. (13))

$$
\rho^{(1)}(\sigma) = \frac{\pi \beta_0^{-1}}{\ln^2(\sigma/\Lambda^2) + \pi^2}
$$

(106)

turns out to be explicitly integrable, and eq. (105) yields [79]

$$
\alpha_{\text{an}}^{(1)}(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right].
$$

(107)

The analytically generated non-perturbative contribution in (107) subtracts the pole in a minimal way, yielding a ghost-free behavior which avoids any adjustable parameter. Obviously eq. (5) for the scaling constant does not work anymore and, at one-loop, it has now to be changed to

$$
\Lambda^2 = \mu_0^2 \exp \left[ -\phi \left( \beta_0 \alpha_s(\mu_0^2) \right) \right],
$$

(108)

where the function $\phi$ is related to the formal inverse of (107) that is, with $x = \beta_0 \alpha_{\text{an}}^{(1)}$ and $Q^2/\Lambda^2 = \exp \phi(x)$, it satisfies

$$
\frac{1}{\phi(x)} + \frac{1}{1 - \exp \phi(x)} = x.
$$

(109)
Among the main features of the analytically improved space-like coupling, it should be firstly stressed its agreement with asymptotic freedom constraint, being the pure perturbative contribution ruling in the deep UV region over the "non-perturbative" one. Indeed the latter for \( Q^2 > \Lambda^2 \) can be rewritten as the sum of the series

\[
\Delta_b^{(1)}(Q^2) = -\frac{1}{\beta_0} \sum_{1}^{\infty} \left( \frac{\Lambda^2}{Q^2} \right)^n, \tag{110}
\]

yielding power correction terms to the one-loop perturbative coupling \( (18) \) (see sec 4.5). On the other hand, in the extreme opposite domain eq.(107) exhibits the infrared freezing value \( \alpha^{(1)}_{\text{an}}(0) = 1/\beta_0 \simeq 1.396 \) choosing consistently \( n_f = 3 \) at low scales; this value turns out to be independent of \( \Lambda \) and universal with respect to higher-loop corrections, i.e. the analytic coupling \( (105) \) has a remarkably stable IR behavior.

The beta-function for the one-loop coupling \( (107) \) reads \[79\]

\[
\beta^{(1)}_{\text{an}}(x) = -\frac{1}{\phi^2(x)} + \frac{\exp \phi(x)}{[\exp \phi(x) - 1]^2}, \tag{111}
\]

with \( \phi \) satisfying eq.(109). Despite the implicit form of (111), its symmetry property \( \beta^{(1)}_{\text{an}}(x) = \beta^{(1)}_{\text{an}}(1 - x) \) reveals the existence of a IR fixed point at \( x = 1 \), corresponding to \( \alpha_{\text{an}}(0) = 1/\beta_0 \) (see also \[88\]).

Finally, we just mention here that the relation between the \( \beta \)-function structure and the analytical properties of the running coupling has been investigated in ref.\[84\] within a pure perturbative approach; some resummation tricks for the asymptotic \( \beta \)-function have been there outlined, which can cure the Landau ghost problem, and rely upon freedom in choosing the RS in higher orders. Another viewpoint is given in \[17\], where the hypothesis of pure perturbative freezing has been explored by analyzing the \( n_f \)-dependence of the \( \beta \)-coefficients (see also eq.(87) and the following).

### 4.2 Two-loop and higher orders

Actually, as discussed in sec. 2.2, the two and higher-loop RG equation for invariant coupling has no simple exact solution, so that rough UV approximations are commonly used (eq. (31)), leading to a cumbersome IR nonanalytical structure. Nevertheless, to go further inside analytization and its features we follow ref.\[82\] and choose the two-loop iterative solution as given by eq.(26).
By applying now the analytization recipe the main difficulty arises from the integration of the related spectral density \[ \rho^{(2)}_{it}(-\sigma) = \frac{1}{\beta_0} \frac{I(t)}{T^2(t) + R^2(t)} \quad t = \ln(\sigma/\Lambda^2) \] (112)

\[ I(t) = \pi + B_1 \arccos \frac{B_1 + t}{\sqrt{(B_1 + t)^2 + \pi^2}} \quad R(t) = t + B_1 \ln \frac{\sqrt{(B_1 + t)^2 + \pi^2}}{B_1} \]

which does not lead to an explicit final formula, though the relative dispersion integral (105) can be handled by numerical tools. However, recalling the singularity structure of (26) (sec. 2.4), entirely subtracted by analytization, the two-loop analytic coupling can be recovered by merely adding to (26) two compensating terms, cancelling respectively the pole and the cut [82]

\[ \alpha_{\text{an.it}}^{(2)}(z) = \alpha_{\text{it}}^{(2)}(z) + \Delta_{\text{p}}^{(2)} + \Delta_{\text{c}}^{(2)} \] (113)

\[ \Delta_{\text{p}}^{(2)}(z) = \frac{1}{2\beta_0} \frac{1}{1 - z} \]

\[ \Delta_{\text{c}}^{(2)}(z) = \frac{1}{\beta_0} \int_0^{\exp(-B_1)} \frac{d\xi}{\xi - z} \frac{B_1}{\xi + B_1 \ln \left(-1 - B_1^{-1} \ln \xi\right)^2 + \pi^2B_1^2} \]

with the dimensionless variable \( \xi = \sigma/\Lambda^2 \). Despite the little handiness of (113), one can still readily verify its limit \( \alpha_{\text{an.it}}^{(2)}(0) = 1/\beta_0 \); for more general arguments concerning universality of the IR freezing value through all orders see for instance [82, 83] and [87]. Moreover, the non-perturbative UV tail of analytized coupling can be estimated by expanding the two compensating terms in (113) into inverse powers of \( z = Q^2/\Lambda^2 \) for large \( z \) [85]

\[ \Delta_{\text{p}}^{(2)}(z) + \Delta_{\text{c}}^{(2)}(z) = \frac{1}{\beta_0} \sum_{n=1}^{\infty} \frac{c_n}{z^n} \]

(114)

\[ c_n = -\frac{1}{2} - \int_0^{\infty} d\xi \frac{\exp[-nB_1(1 + \xi)]}{(1 + \xi - \ln \xi)^2 + \pi^2} \]

Thus for \( z > 1 \) non-perturbative contributions can be recasted analogously to (110) as a convergent power series with all negative coefficients, approaching \(-1/2\) (e.g. \( c_1 = 0.535 \), see refs. [85] and [86]).

As yet pointed out, to get the most accurate result at two-loop in the IR domain, one needs to start with the exact RG solution (23). To this aim first
of all one has to extrapolate eq.(23) in such a way to define a real analytic
function regular for $-e^{-1} \leq \zeta \leq 0$; the two branches $W_1(\zeta)$ and $W_{-1}(\zeta)$
merge continuously on $(-e^{-1}, 0)$, so that the correct recipe is to use eq.(23)
for $\text{Im}(\zeta) \geq 0$ and the same equation with $W_{-1}(\zeta)$ replaced by $W_1(\zeta)$ for
$\text{Im}(\zeta) < 0$. The discontinuity of the two loop exact solution across the
time-like cut defines the spectral density \[90\] (see also \[88\])
\[
\rho^{(2)}_{\text{ex}}(\sigma) = -\frac{1}{\beta_0 B_1} \text{Im} \left[ \frac{1}{1 + W_1(\zeta(t))} \right] t = \ln(\sigma/\Lambda^2) \quad (115)
\]
\[
\zeta(t) = \frac{1}{eB_1} \exp \left[ -\frac{t}{B_1} + i\pi \left( \frac{1}{B_1} - 1 \right) \right].
\]
Dispersion integral \[105\] with $t = \ln(\sigma/\Lambda^2)$ now leads to the exact two-loop analytic coupling
\[
\alpha^{(2)}_{\text{an}}(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{e^t}{e^t + z} \rho^{(2)}_{\text{ex}}(t) \quad (116)
\]
where $\rho^{(2)}_{\text{ex}}(t) = \rho^{(2)}_{\text{ex}}(\sigma)$. Numerical estimates of \[116\] as well as for the
analytic iterative coupling \[113\] have been performed at low scales with
$n_f = 3$ in \[88\] (see Tab.1), both normalized at the \(\tau\) mass $M_\tau = 1.777$ GeV,
$\alpha_s(M_\tau^2) = 0.36$; comparison reveals the relative error for the analytized solution \[113\] to be around 1.8% in the IR region.
In spite of its accuracy eq.\[116\] cannot be easily handled, even though it is
a source of numerical information to which simpler expressions have to be
compared. Actually, for practical aims, many useful two-loop approximate formulas have been suggested; among them we recall here the “one-loop-like” model \[82\]
\[
\tilde{\alpha}^{(2)}_{\text{an}}(l) = \frac{1}{\beta_0} \left[ \frac{1}{l} - \frac{1}{\exp l - 1} \right], \quad l = \ln \left( \frac{Q^2}{\Lambda^2} \right) + B_1 \ln \sqrt{\ln^2 \left( \frac{Q^2}{\Lambda^2} \right) + 4\pi^2} \quad (117)
\]
suitable for analysis of rather low energy phenomena, since it approximates
the exact two-loop analytic coupling with 1% precision for $Q \geq 1$ GeV, and
correctly reproduces both the universal freezing value and the UV two-loop asymptotic behavior. However, its accuracy breaks down when taking into
account flavor thresholds. To this end it has been suggested \[117\] to use eq.\[117\] provided that the scaling constant and the coefficient $B_1$ are replaced by adjustable parameters (respectively $\Lambda_{21}$ and $c_{21}^{\text{fit}}$ listed in tab. III
of ref. [117] for different initial $\Lambda^{(n_f=3)}$, as a result of an interpolation procedure; this yields an accuracy within 1% in the whole space-like region. A “one-loop-like” model employing the scaling constant as a fitting parameter, suitable from 2 to 100 GeV, has been recently developed in [89]. Increasing difficulties arise when dealing with even higher-loop level and it becomes prohibitive to achieve useful explicit formulas. Starting e.g. with the standard three or four-loop asymptotic solution (31), one has to face with the leading singularity in $z = 1$ of the form (39), beside the IR log-of-log generated cut; terms accounting for these divergences acquire the form of cumbersome finite limits integral as in (113). Nonetheless, the effects of non-perturbative contributions have been widely investigated up to four-loop [86, 87], both in IR (where they play the most prominent role) and UV region, by using asymptotic solution (31) as a perturbative input. While confirming at once IR stability due to the universal freezing value of the analytized coupling, its UV tail has been reduced [87] in the form of power type corrections analogous to (114). Within this approximation the $n_f$-dependent coefficients $c_n$ are all negative up to four-loop, and their absolute values monotonously increase with $n$. In the large $Q^2$ limit, however, there is no need to sum a high number of terms, and truncation of e.g. the three-loop non-perturbative series to first term yields the approximate expression [87]

$$\bar{\alpha}_{an}^{(3)}(z) = \alpha_s^{(3)}(z) + \frac{1}{\beta_0} c_1 \Lambda^2 Q^2$$ (118)

$$c_1 = -1 + B_1(1 - \gamma_E) - \frac{B_2}{2} \left[ B_2 - \frac{\pi^2}{6} + (1 - \gamma_E)^2 \right]$$

for the analytic coupling, with 1% accuracy yet at $Q^2 \approx 5\Lambda^2$. Here $\gamma_E$ is the Euler constant, $B_2 = \beta_0\beta_2/\beta_1^2$ and $\alpha_s^{(3)}(z)$ is the perturbative counterpart as given by (31); if $n_f = 6$ we roughly have $c_1 \approx -0.52$.

By normalizing the three-loop analytized coupling at $\alpha_s(M_Z^2) \approx 0.118$ (with perturbative input from eq. (31) and numerical evaluation of dispersion integral (105)), one can extract the three-loop scaling constant $\Lambda^{(n_f=5)} \approx 210 \text{ MeV}$ [87], which lies within the errors of the pure perturbative estimate (see sec. 2.3), being the non-perturbative tail negligible around the normalization point. Obviously the main discrepancies emerge in the low energy region, where non-perturbative contributions slow down the rise of the curve. By using continuous matching up to three-loop, at the $\overline{\text{MS}}$ quark masses $m_b = 4.3 \text{ GeV}$ and $m_c = 1.3 \text{ GeV}$, for the three-loop analytized case
one finds roughly \[ \Lambda^{(n_f=5)} \simeq 210 \text{ MeV} \rightarrow \Lambda^{(n_f=4)} \simeq 299 \text{ MeV} \rightarrow \Lambda^{(n_f=3)} \simeq 382 \text{ MeV} \] (119)

to be compared with the relative perturbative estimates\(^6\)

\[ \Lambda^{(n_f=5)} \simeq 210 \text{ MeV} \rightarrow \Lambda^{(n_f=4)} \simeq 290 \text{ MeV} \rightarrow \Lambda^{(n_f=3)} \simeq 329 \text{ MeV} \] (120)

These discrepancies at low scales can be translated into the values of analytic and perturbative coupling at the \( \tau \) mass \( M_\tau = 1.777 \text{ GeV} \), given respectively by \( \alpha_{\text{an}}^{(3)}(M^2_\tau) \simeq 0.294 \) and \( \alpha_{\text{s}}^{(3)}(M^2_\tau) \simeq 0.318 \). Note that normalization at \( \alpha_{\text{s}}(M_\tau) = 0.35 \) is sometimes adopted, and this leads to considerably higher values for the scaling constant. Last, it should be stressed the nearly indistinguishability of the two to four-loop analytic curves (fig. 3a)), confirming higher order stability on the whole spacelike axis \[79, 87\]. Thus we argue the two-loop correction to be a well satisfactory improvement of the one-loop result, so that it is reasonable to resort to approximate formulas as given e.g. by (117). However, in recent works \[91, 119\] the multi-loop approximation (32) as a power series in the two-loop exact coupling (23) has been exploited as an input in the dispersion integral (105), to yield high-accuracy three and four-loop analytic coupling of the form \[91\]

\[
\alpha_{\text{an}}^{(k)}(Q^2) = \sum_{n \geq 1} p_{n}^{(k)} \left[ \alpha_{\text{an}}^{(2)}(Q^2) \right]^n
\] (121)

with \( \alpha_{\text{an}}^{(2)} \) given by (116) and coefficients the same as in (32). Numerical values at low scales can be found in [91], to which we refer for more details.

### 4.3 The time-like coupling in the analytic approach

Coming back to the issue of defining a reasonable expansion parameter in the time-like domain \( (q^2 = s > 0) \), we start by noting that such a definition naturally arises in a self-consistent way within the framework of the analytic approach \[92, 93\]. Further, it can be regarded as the final step in the procedure RKP for \( \pi^2 \)-resummation outlined in sec. 2.5.

As yet noted, eq. (53) and its formal inverse (54) can be generalized to the

\(^6\)The shifts of \( \Delta s \) in (120) w.r.t. sec. 2.3 are mainly due to the different normalization value, and partly to continuous matching used here; indeed there is little sensitivity to the implementation to trivial matching given by eq. (36).
proper tool for relating s- and t-channel observables, that is for crossing the
two distinct regions. On this ground, it has then been suggested \cite{42,92} to
use the same integral transformations in order to connect the time-like and
space-like couplings, employing as a suitable regularization the analytically
improved coupling \cite{105}, free of unphysical singularities at any loop level.

Then \cite{92}

\[ \tilde{\alpha}(s) = \frac{i}{2\pi} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dq^2}{q^2} \alpha_{an}(-q^2) \] \hspace{1cm} (122)

\[ \alpha_{an}(-q^2) = -q^2 \int_0^\infty ds \frac{\tilde{\alpha}(s)}{(s-q^2)^2} \] \hspace{1cm} (123)

with \( q^2 \) as usual in \( \mathbb{C} - \{ q^2 = s > 0 \} \). Specifically, eq.\,(122) can be assumed as
a definition of the time-like coupling. Once the space-like singularities have
been washed out the ambiguity about the path for (122) disappears, and we
find a one-to-one relation between the t- and s-channel couplings. Note also
that \( \tilde{\alpha}(s) \) can be equally defined by the differential equation \cite{64}

\[ s \frac{d}{ds} \tilde{\alpha}(s) = -\frac{1}{\pi} \rho(s) \quad \text{with} \quad \tilde{\alpha}(\infty) = 0, \] \hspace{1cm} (124)

as can be immediately checked, e.g. by differentiating eq.\,(123). Then, eq.\,(124) immediately yields

\[ \tilde{\alpha}(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho(\sigma) \] \hspace{1cm} (125)

and \( \rho(\sigma) \) as given by \cite{104}. Moreover, eq.\,(124) emphasizes the straightforward relation between the “time-like \( \beta \)-function” and the spectral density, thus reviving an old hypothesis due to Schwinger \cite{94}. With the use of \cite{124}, eq.\,(123) can be also formally inverted as \cite{64}

\[ \tilde{\alpha}(s) = \frac{\sin(\pi P)}{\pi P} \alpha_{an}(s) = \alpha_{an}(s) - \frac{1}{3!} (\pi P)^2 \alpha_{an}(s) + \ldots \] \hspace{1cm} (126)

\[ = \alpha_{an}(s) \left[ 1 - \frac{\pi^2 \beta_0^2}{3} \alpha_{an}^2(s) - \frac{5\pi^2 \beta_0 \beta_1}{6} \alpha_{an}^3(s) + \ldots \right] \]

where \( P = s(d/ds) \). Of course at one-loop \cite{126} coincides with \cite{59}, and
from eqs.\,(125) and \cite{106} one finds again eq.\,(57); however this now leads
through \cite{123} to the starting space-like coupling \cite{107}, being the required
analytic properties preserved within this framework (they are both plotted in fig.3(b)). Furthermore, at two-loop level inserting into eq. (125) the spectral density (115) computed on the two-loop exact RG solution (23′), unlike the space-like case, integral can be taken analytically [91]

\[
\tilde{\alpha}^{(2)}(s) = \frac{-\beta_0}{\beta_1} + \frac{1}{\pi \beta_0} \text{Im} [1 + W_1 (\zeta(s))] \tag{127}
\]

\[
\zeta(s) = \frac{1}{e B_1} \exp \left[ -\frac{1}{B_1} \ln \left( \frac{s}{\Lambda^2} \right) + i \pi \left( \frac{1}{B_1} - 1 \right) \right].
\]

The main feature [92] of (125) and (105) is the common freezing value at the origin \(\tilde{\alpha}(0) = \alpha_{\text{an}}(0) = 1/\beta_0\), independent of the loop level and of any adjustable parameter. Moreover, they exhibit similar leading UV behavior, constrained by asymptotic freedom, as can be seen by (126) (and recalling that asymptotically (105) reduces to the pure perturbative coupling \(\alpha_s\)). Nevertheless this approximate “mirror symmetry” is broken in the intermediate region, the discrepancy being about 9% at one-loop, and slightly less at two and three-loop (see ref. [92] for numerical comparisons).
We finally just note that ref. [95] exploits an argument against a possible exact symmetry ruling the “t-s dual” couplings, (105) and (125), on the ground of causality principle.

4.4 Related models

Since the last ten years there have been a number of different attempts to avoid Landau singularities invoking as well analyticity of the coupling in the space-like momentum region. Within the dispersive approach, it is remarkable the existence of models suggesting IR enhancement of the QCD coupling, whose most attractive feature is supposed to be a straightforward relation with quark confining potential within the framework of one-gluon exchange (see sec. 3.1).

To this class belongs for instance the “synthetic coupling” model recently developed in [96, 97], which amounts to modify by hand the analytically improved coupling (105) by additional non-perturbative pole-type terms; at one-loop it reads

\[ \alpha_{\text{syn}}(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{c\Lambda^2}{Q^2} + \frac{(1-c)\Lambda^2}{Q^2 + m_g^2} \right]. \]  

(128)

where \( m_g = \Lambda/\sqrt{c-1} \). Infrared enhancement, due to the pole term at \( Q^2 = 0 \) (cf. eq.(69)), is governed by one dimensionless parameter \( c \in (1, \infty) \), which is meant to relate the scaling constant \( \Lambda \) to the string tension \( \sigma \) of potential models, through the same eq. (70) but with \( \Lambda \) replaced by \( \sqrt{c} \Lambda \). The pole term at \( Q^2 = -m_g^2 < 0 \) corresponds to a non vanishing dynamical gluon mass, while leaving the analytical structure of eq.(107) along the space-like axis unchanged. The value of the \( m_g \) parameter has been estimated [97] as 400 – 600MeV (see sec. 3.1). Eq.(128) can be derived, analogously to (107), from a dispersion relation with a spectral density of the form (106) plus two \( \delta \)-terms properly accounting for the poles. Along with the singular behavior as \( 1/Q^2 \) at \( Q^2 \rightarrow 0 \), reproducing the linear confining part of the potential (67), construction of (128) is mainly motivated by the UV asymptotic [97] of its non-perturbative contribution of the form \( 1/(Q^2)^3 \), decreasing faster than (110) as \( Q^2 \rightarrow \infty \).

Quite analogous result has been achieved in [98, 99], merging analyticity and IR slavery at zero momentum in the one-loop formula

\[ \alpha_N(Q^2) = \frac{1}{\beta_0} \frac{z - 1}{z \ln z}. \]  

(129)
where \( z = Q^2/\Lambda^2 \). The trick undertaken here is to impose analyticity ab initio on the whole perturbative expansion of the QCD \( \beta \)-function, and then to solve the ensuing “analytized RG equation” for the running coupling. This could be done in principle at any loop-level, formally \[98\]

\[
\mu^2 d\ln\alpha_N(\mu^2) = - \left\{ \sum_{j=0}^{l-1} \beta_j \alpha^{j+1}(\mu^2) \right\}_an,
\]

where the r.h.s. of \[130\] is achieved through the usual dispersion integral \[104\], starting from its spectral density now given by the discontinuity of the expression of the l-loop \( \beta \)-function as a whole across the time-like cut. Note that in the one-loop case the r.h.s. of \[130\] is merely proportional to the one-loop analytic coupling \[107\], so that further integration leads to the singularity at zero momentum. This “new analytic invariant charge” possesses the universal (i.e. loop-independent) asymptotic \( \simeq \Lambda^2/Q^2 \) as \( Q^2 \to 0 \), which again results in confining quark-antiquark potential \[99\]. Besides, this variant of the dispersive approach, while disclosing in some measure the ambiguities suffered by the method, shares appealing features with the IR finite analytic coupling \[105\]; namely, it displays no unphysical singularities and no adjustable parameters, and exhibits good higher-loop and RS stability (see e.g. \[99\] for technical details). Analogous IR behavior of the QCD coupling has been found in \[100\] on the ground of different reasoning (see also \[98, 99\] and refs. therein for similar proposal and results). It is worth noting that in the most recent developments \[101\] of the model in hand, eqs. \[130\] and \[129\], inclusion of the lightest hadron masses (\( \pi \) meson) is accomplished consistently with the dispersive approach, and relations with chiral symmetry breaking phenomena are also investigated \[102\]. It has been shown there how nonzero pion mass substantially affects low energy behavior of the invariant charge \[129\], by slowing down the IR enhancement of the massless case to an IR finite value, depending in turn on the pion mass. As a result, this massive running coupling displays a plateau-like freezing on the IR time-like axis, specifically for \( \sqrt{s} \) in the interval between 0 and the two-pion threshold (see also \[103\]), in qualitative agreement with results of OPT \[72\] (sec. 3.3).

Finally we remind another attempt \[104\] to modify eq. \[107\] in order to estimate non-perturbative power corrections (see sec 4.5). This is dictated by the need to cancel somehow the unwanted behavior \[110\], and further to slow down the too large value of \[107\] at the origin (see sec. 5). This has been done by requiring no power corrections faster than \( 1/Q^{2p} \), and a number of
adjustable parameters to remodel the IR behavior of the one-loop analytic coupling \((107)\). Thus an useful generalization reads

\[
\alpha_W(z) = \frac{1}{\beta_0} \left[ \frac{1}{\ln z} + \frac{z + b}{(1 - z)(1 + b)} \left( \frac{1 + c}{z + c} \right)^p \right].
\] (131)

Setting \(b = 1/4\) and \(c = p = 4\) and \(n_f = 3\), \(\Lambda = 250\) MeV eq.\((131)\) has a maximum at \(0.4\) GeV and then freezes to a considerably lower value than \((107)\), fitting data on power corrections (see \([104]\) and refs. therein).

### 4.5 Power suppressed non-perturbative corrections

As mentioned in the introduction, intrinsically non-perturbative effects would manifest themselves in power-type corrections \(A_\nu/Q^\nu\) or \((A_\nu/Q^\nu) \ln(Q^2/Q^2_1)\), to the expression of various observables. We want to discuss the subject in the context of ref. \([64]\), in which it was originally proposed, by making use of the coupling \(\alpha_{SGD}(Q^2)\) we have introduced by eq. \((89)\).

Note that, as consequence of its definition, \(\alpha_{SGD}(Q^2)\) must have only physical singularities in the \(Q^2\) complex plane and the entire dispersive formalism discussed in this section can be applied to it. Actually such formalism sprang from a common source \([105]\) and developed along parallel lines from references \([79]\) and \([64]\). According to eq. \((105)\) we can write

\[
\alpha_{SGD}(Q^2) = \frac{1}{\pi} \int_0^\infty dm^2 \frac{\rho(m^2)}{m^2 + Q^2} = \alpha_{SGD}(0) - \frac{Q^2}{\pi} \int_0^\infty dm^2 \frac{\rho(m^2)}{m^2(m^2 + Q^2)}. \tag{132}
\]

Then, the factor corresponding to a dressed gluon line in a Feynman integral can be written

\[
\frac{-i \alpha_{SGD}(-k^2)}{k^2 + i0} = \frac{-i \alpha_{SGD}(0)}{k^2 + i0} - \frac{1}{\pi} \int_0^\infty dm^2 \frac{\rho(m^2)}{m^2/k^2 - m^2 + i0}. \tag{133}
\]

Let us now consider some hard process initiated by a quark and an inclusive infrared and collinear safe observable \(V(Q^2, x)\) related to it. Let us assume, to be specific, the observable has a zero order expression \(V_0(Q^2, x)\) in terms of parton model and consider the first order QCD correction \(V_1(Q^2, x)\) in which dressed gluon lines are inserted in the original skeleton graph. Let us denote by \(F_1(\epsilon, x)\) the Feynman integral (or the sum of Feynman integrals)
that gives such correction but in which formally a mass \( m \) is given to the
gluon and it is set \( \epsilon = \frac{m^2}{Q^2} \). Due to (133) the insertion gives

\[
V_1(Q^2, x) = \alpha_{\text{SGD}}(0)F_1(0, x) - \frac{1}{\pi} \int_0^\infty \frac{dm^2}{m^2} \rho(m^2)F_1(\epsilon, x) = \\
= \int_0^\infty \frac{dm^2}{m^2} \tilde{\alpha}_{\text{SGD}}(m^2) \hat{F}_1(\epsilon, x), \quad \text{with} \quad \hat{F}_1(\epsilon, x) = -\epsilon \frac{\partial}{\partial \epsilon} F_1(\epsilon, x),
\]

where we have introduced the time-like coupling \( \tilde{\alpha}_{\text{SGD}}(m^2) \), related to \( \alpha_{\text{SGD}}(m^2) \) by the equation \( m^2 \left( d \tilde{\alpha}_{\text{SGD}}(m^2) / dm^2 \right) = -\rho(m^2) / \pi \) (cf. eq. (124)).

Eq. (134) gives a kind of weighted average of \( F_1(\epsilon, x) \) and shows that the
effect of the running coupling can be simulated giving an appropriate effective
mass \( m_g \), and justifies the phenomenological application based on such
idea that we mentioned in sec. 3.1. Let us now write \( \alpha_{\text{SGD}}(Q^2) \) and corre-
spondingly \( \tilde{\alpha}_{\text{SGD}}(Q^2) \) as the sum of a perturbative and a non perturbative
part

\[
\alpha_{\text{SGD}}(Q^2) = \alpha_{\text{SGD}}^{\text{PT}}(Q^2) + \alpha_{\text{SGD}}^{\text{NP}}(Q^2),
\]

Since, however, perturbative theory appears to work very well down to \( Q \sim \)
1 or 2 GeV and in some particular case even better, \( \alpha_{\text{SGD}}^{\text{NP}}(Q^2) \) must vanish
sufficiently fast as \( Q \) increases, let us say at least as \( 1/Q^6 \). Consequently ,
looking at eq.(123) we must assume that at least the two first integer moments
of \( \tilde{\alpha}_{\text{SGD}}(m^2) \) vanish,

\[
\int_0^\infty \frac{dm^2}{m^2} m^2 \tilde{\alpha}_{\text{SGD}}^{\text{NP}}(m^2) = 0, \quad \int_0^\infty \frac{dm^2}{m^2} m^4 \tilde{\alpha}_{\text{SGD}}^{\text{NP}}(m^2) = 0.
\]

Let us now restrict to collinear and infrared safe observables. Then \( \hat{F}_1(\epsilon, x) \)
must vanish conveniently for \( \epsilon \to 0 \) or \( \infty \). Let us assume for \( \epsilon \to 0 \)

\[
\hat{F}_1(\epsilon, x) \to \\
\to \frac{C_F}{2\pi} \epsilon^p \left[ (f_0 + f_1 \ln \epsilon + f_2 \ln^2 \epsilon) + \epsilon(g_0 + g_1 \ln \epsilon + g_2 \ln^2 \epsilon) + \ldots \right]
\]

where \( p \) can be integer or half integer and \( f_q, g_q, \ldots \) depend on the particular
process. For the various jet shape variables in \( e^+e^- \) annihilation (thrust, jet
mass, \( C \) parameter) e.g., \( \hat{F}_1(\epsilon, x) \to \frac{C_F}{2\pi} f_V \sqrt{\epsilon} \) with \( f_V = 4, 2, 6\pi \) respectively.

\footnote{Note that in the original papers the quantity \( \tilde{\alpha}_{\text{SGD}}(m^2) \) was denoted as \( \alpha_{\text{eff}}(m^2) \).}
As a consequence we have for \( \frac{Q}{\to} \to \infty \)

\[
\int_0^\infty \frac{dm^2}{m^2} \tilde{\alpha}^{\text{NP}}_{\text{SGD}}(m^2) \hat{F}_1(\epsilon, x) \to f_0 \frac{A_{2p}}{Q^{2p}} + f_1 \left( \frac{A'_{2p}}{Q^{2p}} - \frac{A_{2p}}{Q^{2p}} \ln Q^2 \right) + \ldots \tag{138}
\]

or

\[
V(Q^2, x) = V_{\text{PT}}(Q^2, x) + \frac{1}{Q^{2p}} [C_1 A_{2p} + C_2 A'_{2p} + C_3 A''_{2p}] \tag{139}
\]

where we have introduced the moments

\[
A_{2p} = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} m^{2p} \tilde{\alpha}^{\text{NP}}_{\text{SGD}}(m^2),
\]

\[
A'_{2p} = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} m^{2p} \ln m^2 \tilde{\alpha}^{\text{NP}}_{\text{SGD}}(m^2),
\]

\[
A''_{2p} = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} m^{2p} \ln^2 m^2 \tilde{\alpha}^{\text{NP}}_{\text{SGD}}(m^2).
\]

The coefficients \( C_k \) are dimensionless, are in practice at most linear in \( \ln Q^2 \)
and are calculable but process dependent, \( A_{2p}, A'_{2p}, \ldots \) are theoretical unknown but should be universal. They could be determined in principle by studying the dependence on the hard scale \( Q \) of appropriate observables, \( V(Q^2, x) \).

Terms in \( 1/Q \) should emerge for what we have seen in various \( e^+e^- \) jet shape variables; correction of the type \( 1/Q^2 \) in the DIS structure function (see \[106\]). The results can be equivalently expressed in terms of the moments of \( \tilde{\alpha}^{\text{NP}}_{\text{SGD}}(Q^2) \) rather then of \( \tilde{\alpha}^{\text{NP}}_{\text{SGD}}(m^2) \). The experimental situation is not completely clear but the data seem to be consistent with

\[
A_1 = \frac{C_F}{2\pi} \int_0^\infty \frac{dQ^2}{Q^2} \frac{Q}{Q^2} \alpha^{\text{NP}}_{\text{SGD}}(Q^2) = \frac{2}{\pi} A_1 \approx 0.2 - 0.25 \text{GeV} \tag{141}
\]

and

\[
A_2 = \frac{C_F}{2\pi} \int_0^\infty \frac{dQ^2}{Q^2} Q^2 \alpha^{\text{NP}}_{\text{SGD}}(Q^2) = -A'_2 \approx 0.2 \text{GeV}^2. \tag{142}
\]

Note that in principle the quantity \( V_{\text{PT}}(Q^2, x) \) is well defined, since \( \alpha_{\text{SGD}} \) has no unphysical singularities, due to its definition \[89\]. To be consistent up two-loop one should use equations for \( \alpha^{\text{NP}}_{\text{SGD}} \) of the type considered in sec. 4.1 or 4.2 with the appropriate value of \( \Lambda_{\text{SGD}} \). However, often an infrared cutoff \( Q_I \) has been introduced in the application and \( A_1 \) related to the quantity
(\langle \alpha_s \pi \rangle) considered in sec. 3.1.

Relations involving OPE and the non trivial IR structure of the theory have been considered in the 2D Gross-Neveu model in [107].

To conclude this section we should mention that the same physical problem we have briefly discussed from the point of view of the running coupling is the object of another very alive line of research that operates in the framework of the Borel summability and the renormalon singularities. Since there are obviously connections between the two perspectives, we must invite the interested reader to consult some of the existing excellent general reviews, e.g. [108], and among the most recent works we remind ref. [109].

4.6 Analytic Perturbation Theory

As explained at length, analytization displays a suitable method to get rid of unphysical singularities which affect the standard expansion parameter in t- and s-channel. Thus the issue of how perturbation theory should be accordingly modified naturally rises, and it has been investigated from phenomenological and theoretical point of view (e.g. [110]). We briefly recall here a consistent way to incorporate the ghost-free model for the IR finite couplings eqs. (105) and (125) within perturbation theory, known as Analytic Perturbation Theory (APT), and developed in [111]-[114]. The main requirement here is the subtraction of unphysical singularities in the RG improved series for physical observables as a whole, by computing their discontinuity across the time-like cut, as it has been done for the space-like coupling itself eqs. (104) and (105). Specifically [113], given a space-like observable perturbatively known

\[ D_{PT}(Q^2) = 1 + \sum_{k \geq 1} d_k \alpha_s^k(Q^2), \] (143)

one can define the k-th spectral density

\[ \rho_k(\sigma) = \text{Im} \left[ \alpha_s^k(-\sigma) \right]. \] (144)

Then eq. (143) is to be substituted by the ghost-free expansion

\[ D_{APT}(Q^2) = 1 + \sum_{k \geq 1} d_k A_k(Q^2) \] (145)

where, leaving the loop level understood,

\[ A_k(Q^2) = \frac{1}{\pi} \int_0^{\infty} \frac{d\sigma}{\sigma + Q^2} \rho_k(\sigma). \] (146)
Then a standard power expansion is converted into a non power one. Clearly, there is now no unique expansion parameter, but an entire set of ghost-free expansion functions \((146)\) at any loop level, each defined by the analytization of subsequent powers of the perturbative coupling. Obviously from \((146)\) with \(k = 1\) the analytic coupling \((105)\) is recovered at each loop level. This recipe, working for space-like observables, is manifestly quite analogous to the RKP non-power expansion \((56)\) of a time-like observable; this in turn, within this framework, can be reexpressed via the \(k\)-th spectral density \((144)\) \[113\]

\[ R_{\text{APT}}(s) = 1 + \sum_{k \geq 1} d_k A_k(s) \] \hspace{1cm} (147)

and

\[ A_k(s) = \frac{1}{\pi} \int_s^{\infty} \frac{d\sigma}{\sigma} \rho_k(\sigma) \] \hspace{1cm} (148)

which yields for \(k = 1\) the analytic time-like coupling itself \((125)\). The key point here \[113\] is that, due to the forced analyticity of the coupling and its analytized powers \((146)\), the two sets \((146)\) and \((148)\) are put into one-to-one relation by the linear integral transformations \((53)\) and \((54)\), in this context usually renamed \(R\) and \(D\) respectively, that is

\[ A_k(Q^2) = D[A_k(s)] \], \hspace{1cm} A_k(s) = R[A_k(Q^2)] \] \hspace{1cm} (149)

This yields a closed theoretical scheme for representing observables of any real argument, both space-like and time-like (for a quite recent review see \[114\] to which we also refer for technical details). The main features of these functional sets are illustrated in fig.4 taken from \[114\]. As yet noted the first function in both sets coincides with the relative analytic coupling, respectively \((105)\) and \((125)\), while \(A_{k \geq 2}\) and \(A_{k \geq 2}\) start with an IR zero and oscillate in the IR domain around \(k - 1\) zeros; furthermore they all obey the UV asymptotic \(1/\ln^k z\) resembling the corresponding powers \(\alpha_k^k(z)\) \[114\].

The differential recursion relations \[89\]

\[ \frac{1}{k} \frac{dA_k^{(n)}(Q^2)}{d\ln Q^2} = - \sum_{j=1}^{n} \beta_j - 1 A_{k+j}^{(n)}(Q^2) \], \hspace{1cm} \frac{1}{k} \frac{dA_k^{(n)}(s)}{d\ln s} = - \sum_{j=1}^{n} \beta_j - 1 A_{k+j}^{(n)}(s) \] \hspace{1cm} (150)

that hold in all orders \[91\], allow to relate different analytized powers within each set, tough explicit expressions are reliable in a simple form only in the
Figure 4: (a) Space-like and time-like global analytic couplings in a few GeV domain with \( n_f = 3 \) and \( \Lambda = 350 \text{ MeV} \); (b) “Distorted mirror symmetry” for global expansion functions, corresponding to exact two-loop solutions.

one-loop case (e.g., [89]). Meanwhile, \( \mathcal{A}_k \) and \( \mathfrak{A}_k \) for \( k = 1, 2, 3 \) have been tabulated in [90] up to three-loop, starting from the exact two-loop solution [23], and the Pade’ approximant technique in the three-loop case; further improved approximations as eq. [82] have been exploited at low scales in [91] (see also [119]). These numerical values are reproduced quite well in the range (2, 100) GeV by “one-loop” inspired approximate formulas, derived for practical aims in [89], and depending on an effective scaling constant \( \Lambda_{\text{eff}}^{n_f} \) as a fitting parameter (see Tab.1 in ref. [89]). Computation of the functional sets has been recently extended to include analytic images of any real power of the coupling (Fractional Analytic Perturbation Theory) on the grounds of properties of the transcendental Lerch function [120].

The APT algorithm can also include thresholds effects [113] in a real description, by modifying the \( k \)-th spectral density [144] discontinuously at the heavy quark thresholds \( m_f \)

\[
\rho_k(\sigma) = \rho_k(\sigma, 3) + \sum_{n_f \geq 4} \theta(\sigma - m_f^2) \left[ \rho_k(\sigma, n_f) - \rho_k(\sigma, n_f - 1) \right], \quad (151)
\]

descending from the trivial matching condition (cf. sec. 4.2). The global functions resulting from densities [151], \( \mathcal{A}_k \) and \( \mathfrak{A}_k \), can be obtained from the local ones with \( n_f \) fixed, by adding specific shift constants \( c_k(n_f) \), not negligible in the \( n_f = 3, 4 \) region (see [113] for numerical evaluations; for instance, in both t- and s-channel it has been estimated \( c_1(3) \simeq 0.02 \)).

The main tests of APT being obviously at low and intermediate scales, a number of applications to specific processes, both in the space-like and
time-like (low and high energy) domain, have been quite recently performed.
For instance, Bjorken and Gross-Llywellin-Smith sum rules [111], \( \tau \) -lepton
[112] (see also [115]) and Ypsilon decay [89], \( e^+e^- \)-annihilation into hadrons
[93, 82, 114] and hadronic form factors [116, 117]. As a result (see tab.1 in
ref. [118]), the main advantages of the APT approach are better convergence
properties of the ghost-free non-power expansion than the usual power one,
since the three-loop term is always strongly suppressed (even less than data
errors). This feature yields a reduced scheme and loop-dependence. Finally,
transition from Euclidean (space-like momentum) to distance picture has
been also developed in [121] to which we refer for details, involving a suit-
able modified sine-Fourier transformation, consistently with the APT logic.

5 Lattice Theory

In all development above one starts from manipulation on the perturbative
expansion and try to remedy to its lack of convergence, but one can never
attain to really non perturbative effects related to the singularity of the the-
ory in \( \alpha_s = 0 \).
Up to now, the only general technique to handle (even if with many limita-
tions) non perturbative problems is Lattice Theory.

We want here to mention essential ideas and summarize the present status
of art for what concerns the running coupling.
We refer to the standard formulation as due to Wilson [122], a classical text-
book is e.g. [123].

On the Euclidean lattice the spacing \( a \) plays the role of the UV cutoff of the
continuum formulations. In momenta space the fields are then defined on
the Brillouen zone but, for practical purposes, the lattice is finite, thus an
IR cutoff appears too and the momenta become finite. The quantum theory
is obtained via path integral quantization and the result is similar to a sta-
tistical mechanics formulation in \( D = 4 \), at an inverse temperature \( \beta \) which
turns out to be \( 2N/g_0^2 \), for the SU(N) gauge theory and where \( g_0 \) is the bare
coupling constant. The lattice is a gauge invariant regulator but for finite
lattice the functional integral is perfectly convergent, and no gauge-fixing
procedure is in principle necessary to evaluate gauge invariant quantities.

Very used are improved actions, namely actions with extra terms of physi-
cal dimensions higher than four, with coefficients suitable chosen in order to
have a better approach to the continuum limit (see [124] for a review). A
complication of the lattice formulation is the well known doubling problem
of the fermions, in order to have in the continuum limit the correct number of fermions one has to use an action which breaks the chiral symmetry for finite $a$. The chiral symmetry can be restored by considering a critical value for the hopping parameter for any lattice spacing. Recipes for the fermionic action which differ from the Wilsonian one are widely used, each one with its own problems. To get answers from the lattice usually one does not compute the functional integral but perform Monte-Carlo simulations. The computation of the running coupling constant on the lattice starts from its non perturbative definition in the low energy region; there are a lot of definitions which have been actually used, and in this region they can have a very different behavior with the energy scale, but extrapolated in some way at high energy they must behave similarly, and it has to be possible to correlate one to another using perturbation theory. That this happens is a consistency check of the theory on all the energy scales. In particular at high energy it is possible to commute to the $\overline{\text{MS}}$ scheme in dimensional regularization, defined only in perturbation theory. An exhaustive review on the various possible approaches is in [125].

From a physical point of view the framework of the computation is fixed by setting the scale. In the pure gauge theory one needs, from the experiments, the knowledge of a physical quantity of mass dimension $M_H$, and on the lattice, starting from a reasonable $g_0$, or equivalently $\beta$, one has to be able to compute the number $aM_H$, and this determines the spacing $a$ in physical units. The window of energies on the lattice, in which $M_H$ has to fall is then $[1/L, 1/a]$. $L/a$ is the number of sites for side of the lattice and so is fixed by the computer power, typical values are $16−64$.

The asymptotic freedom can be formulated also in terms of bare coupling constant which vanishes for $a \to 0$, for instance we have the two loop formula

$$\frac{1}{\alpha_0} = \beta_0 \ln(a^{-2} \Lambda^{-2}_L) + (\beta_1/\beta_0) \ln(\ln(a^{-2} \Lambda^{-2}_L))$$

(152)

(the ratios $\Lambda_L/\Lambda$, where $\Lambda$ refers to a suitable continuum scheme, e.g. $\overline{\text{MS}}$, have been evaluated in lattice perturbation theory long time ago [126, 127]).

Being interested in the continuum limit one would like to have $a$ and $g_0$ as small as possible but one needs to take into account the limitations of the window energy. The dependence on the chosen observable should be not important as far as we are “close” to the continuum limit (scaling region), used quantities are the string tension, the mass splitting in heavy quarkonium or the hadronic radius $r_0$ [128]. In the nineties there have been several
determinations of the running coupling constant in the pure gauge theory or in quenched QCD (129–136).

Among the papers, 135 and 136 are closer in the formalism to the other parts of this review, indeed the definition of the renormalized coupling constant uses a condition on the trigluon vertex, therefore very similar to that employed in usual perturbative procedures. In principle such a definition of the coupling can however be used in the full infrared region, the only limitations being due to the lattice size effects. The disadvantages of such an approach are that $A_\mu$ on the lattice is a rather unnatural quantity, defined in conventional way through the fundamental link gauge variables; moreover, to compute the Green functions, the gauge fixing is needed. For the implementation of the gauge fixing on the lattice a good review is 137. It is worth to mention that even the lattice formulation of the Landau condition suffers for the problem of the Gribov ambiguity. Actually one could pick out a unique element in each gauge orbit by finding the absolute minimum of a suitable functional with respect to gauge transformations but, from a numerical point of view, to distinguish the absolute minimum for instance from the local ones is a very difficult task. Beyond 137 a clear analysis of the numerical situation is in 138, relations with other approaches are in the review 139, devoted to the infrared behavior of the QCD Green functions. On the lattice, by Monte Carlo average, after performing the Fourier transform and in the Landau gauge, one computes the unrenormalized Green functions

$$G_{U\mu_1\ldots\mu_n}(p_1,\ldots,p_n) = \langle A_{\mu_1}(p_1)\ldots A_{\mu_n}(p_n) \rangle.$$  \hspace{1cm} (153)

In 135 and 136 the scale is set by a previous determination of the string tension 129, then $\beta = 6$ and correspondingly $a^{-1} = 1.9 \pm 0.1$ GeV in a hypercubic lattice are taken in 135, more lattices are used in 136. Assuming that finite-volume effects and discretization errors are under control, for suitable small momenta one can adopt the formalism of the continuum QCD and then to write for the gluonic propagator

$$G_{U\mu\nu}(p) = G_U(p^2)(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}).$$  \hspace{1cm} (154)

and for the three point function (after factorization of the color tensor $f^{abc}$)

$$G_{U\mu\nu\rho}(p_1, p_2, p_3) = \Gamma_{U\alpha\beta\gamma}(p_1, p_2, p_3) G_{U\alpha\mu}(p_1) G_{U\beta\nu}(p_2) G_{U\gamma\rho}(p_3).$$  \hspace{1cm} (155)

Where $p_1 + p_2 + p_3 = 0$. On the lattice of course one measures the Green functions, so differently from the usual perturbative approach the renormal-
ization conditions are expressed on the Green functions and not on the 1-P-I functions. Thus
\[ G_R(p)|_{p^2=\mu^2} = Z_g^{-1}(a\mu)G_U(pa)|_{p^2=\mu^2} = \frac{1}{\mu^2} \] (156)
is the condition which fixes the wave-function renormalization constant in non perturbative way. For the vertex the most obvious approach would be to impose a condition on the most symmetric point (MOM scheme), but from a numerical point of view it turns out that an asymmetrical condition in which one of the momenta vanish works better (MOM in continuum QCD [126]). Defining the vertex renormalization constant according to
\[ \sum_{\mu=1}^{4} G_U^{(3)}(pa,0,pa) G_U(pa)^2 G_U(0) = 6iZ_V^{-1}(pa)g_0p_\nu, \] (157)
the vertex renormalization condition reads
\[ g(\mu) = Z_g^{3/2}(a\mu)Z_V^{-1}(a\mu)g_0. \] (158)

In a massless theory the Green functions at exceptional momenta are potentially divergent, but in this case all is finite [140], [141]. Note that the tensorial structure of the previous formulae is not obvious because on the lattice O(4) symmetry no longer holds; a test of consistency of their numerical computation performed by the authors is to check the previous structures for some values of \( pa \). As a matter of fact the parametrization of the formulae (154) and (155) cannot be the most suitable in order to extract the coupling, lattice perturbation theory would suggest using \( p_\mu = \frac{2}{a}\sin(\frac{p_a}{2}) \) instead of \( p_\mu \), indeed the authors from [136] claim that only in this case they are able to find evidence of the running of the coupling. In order to prove the running of the coupling one fits the data of the simulation with the curve which express \( \Lambda \) through \( g(\mu) \), see Fig.5 using the two loop formula (but in [136] they control the three loop effect)
\[ \Lambda_{\text{MOM}} = \mu \exp\left(-\frac{1}{2\beta_0\alpha(\mu)}(\beta_0\alpha(\mu))^{\frac{\beta_1}{2\beta_0}}\right) \] (159)
in both the papers one finds an evidence of plateau close to 2 GeV, see Fig.5(b), beyond \( \Lambda(g) \) decreases due to cutoff effect. The effects of finite size are controlled by performing the computation using more lattices with
Figure 5: (a) The QCD coupling constant $\alpha_{\text{MOM}}$ from [136], $\mu$ is in GeV. The full line is the three-loop running. (b) $\Lambda$ from [136].

Having computed $\Lambda_{\text{MOM}}$ one can get $\Lambda_{\text{MS}}$. To this end one has to do a one-loop [20] calculation which expresses perturbatively the relation between $g_{\text{MS}}(\mu)$ and $g_{\text{MOM}}(\mu)$. Indeed a discrete renormalization group transformation which connects the Green functions computed on the lattice and those computed in $\overline{\text{MS}}$ scheme must exist. Being this scheme on the lattice specified by conditions on the Green functions, the transformation is easily determined by the one loop $\overline{\text{MS}}$ continuum calculation of the trigluon vertex and of the propagator at the $\overline{\text{MOM}}$ point and in Landau gauge. More difficult computations in lattice perturbative theory are not needed in this approach. In [135] it is obtained $\Lambda_{\text{MS}} = 0.34 \pm 0.05$ GeV; the authors of [136], with a writing which distinguishes the possible errors coming from the lattice spacing, instead get $\Lambda_{\text{MS}} = (0.303 \pm 0.05 \text{GeV}) \frac{a^{-1}}{1.97 \text{GeV}}$.

The same renormalization scheme has been applied to measure the coupling using the gluon-quark vertex in quenched simulations in [142]. In order to avoid the major source of systematic errors, in spite of the computer cost, simulations with dynamical quarks have been addressed. The task has been undertaken using NRQCD [143] in the past; also the authors of [136] published a computation with $n_f = 2$ dynamical quarks [144] in the MOM scheme previously described. Interestingly the matching with the perturba-
tive behavior of the coupling, and then the extraction of \( \Lambda \), was made taking into account an effect of OPE condensate arising from the gluonic propagator; therefore the data computed on the lattice for the coupling where compared not with \( \alpha_{s,\text{pert}}(\mu^2) \) but with \( \alpha_{s,\text{pert}}(\mu^2)(1 + \frac{c}{\mu^2}) \), where \( c \) has to be adjusted in a combined fit. The form of this correction, similar to those introduced in eq. (128), is based on previous lattice computations and already utilized in order to improve the determination of \( \alpha_s \) in quenched simulations [145]. They also perform the extrapolation of the coupling to \( n_f = 3 \) and then the renormalization group evolution to the value of \( M_Z \) of the energy through the thresholds, Fig. 6(a) quotes their result.

A recent review [146] analyzes the results from lattice calculations up to 2004 with emphasis to the theoretical issues in dynamical quark simulations. In particular a comparison among determinations of the various coupling constant is presented, besides the already cited papers, Fig. 6(a) includes some newer [147], [148] and some older result [150]. The most recent result in Fig. 6(a), obtained by the HPQCD collaboration (2004), is \( \alpha_s(M_Z) = 0.1175(15) \) in good agreement with the PDG 2004 average 0.1182(20) [18]. It has be noted that the results obtained using Wilson fermions [148], [149] are significantly lower than those obtained using staggered fermions [147].

An original approach to the lattice computations has been developed by the ALPHA collaboration which recently performed a determination of the strong coupling constant with two dynamical flavors [151]. Exposition of the method is in [152], a detailed discussion of the mass renormalization in [153]. One of the computational problems in the usual approaches is that the window energy must be wide to contain both the matching to the hadronic scale and the asymptotic behavior of the coupling. In the approach of the ALPHA collaboration one can determine a recurrence relation between quantities referring to double energy each step, and thus one constructs a ladder from the hadronic to the perturbative regime.

The technical tool is an intermediate renormalization scheme in which the renormalized coupling constant is defined by derivative with respect to parameters characterizing the boundary conditions of the Schroedinger functional, namely the amplitude for a transition from a given field configuration at a given time, \( t = 0 \), to another one at \( t = T \). Moreover the boundary conditions yield a natural infrared cut-off so it is possible to perform simulations at vanishing quark mass [153] (using Wilson fermions the mass renormalization is additive, and imposed using the PCAC relation). The renormalization of the coupling is defined on the massless theory thus a non
perturbative mass independent scheme is obtained. In other computations the light quark masses are actually rather large, and chiral extrapolations are needed in order to compute the coupling to be compared to the $\overline{\text{MS}}$ one. Taking $T = L$, being $a$ and $L$ the only dimensional quantities, the following relation between bare and renormalized couplings holds $\overline{g}_L = \overline{g}(a/L, g_0)$.

Here $1/L$ plays the role of the finite scale of energy. The same procedure can be applied by doubling the size, $L \rightarrow 2L$, and after elimination of $g_0$ one finds a relation between the two renormalized coupling constants. At the end, by extrapolating the limit $a/L \rightarrow 0$, they define the ”step scaling function” which connects the two couplings:

$$\sigma(u_L/2) = u_L.$$  \hspace{1cm} (160)

In (160) $u = \overline{g}^2$ and the two couplings refer to lattices of size $L/2$ and $L$ at fixed Physics, namely bare parameters. The previous equation is actually a discrete renormalization group transformation, indeed a relation between the $\beta$ and the $\sigma$ functions is determined. Of course the step scaling function is constructed by a numerical procedure; consistency with perturbation theory requires $\sigma(u) \sim u$ as $u \rightarrow 0$, for larger $u$, $\sigma(u)$ is computed by a suitable interpolation starting from the set of points determined on the lattice (a discussion of the systematic error arising from this interpolation, which it is claimed to be negligible in a suitable range of coupling, is in [154]).

Starting with a given coupling and a given size $(u_{\text{max}}, L_{\text{max}})$ using the step scaling function one computes $u_i = \overline{g}^2(2^{-i}L_{\text{max}})$ and for sufficient high energy, with the identification $\mu = 2^{i}/L_{\text{max}}$, one matches the perturbative regime evaluating $\Lambda$ (the ratio between this $\Lambda$ and $\Lambda_{\overline{\text{MS}}}$ is known).

The starting point of the iteration is a priori unknown, it is determined by matching some hadronic quantity, as a practical approach it was computed $r_0$ in the chiral limit; assuming $r_0 = 0.5 \text{ fm}$ it turns out $\Lambda_{\overline{\text{MS}}}^{r_0=2} = 245(16)(16) \text{ MeV}$. For the low energy behavior of this coupling see [157].

The most recent determination of the running coupling constant has been obtained by a HPQCD – UKQCD collaboration [155]. Using a new discretization of the light quark action the authors perform the first computation with three dynamical quarks. After having fixed the lattice parameters ($m_u = m_d$, $m_s$, $m_c$, $m_b$ and $a$) matching five experimental quantities the determination of the coupling is obtained by an optimization of lattice perturbation theory, in which the coupling must fit 28 short-distance quantities calculated on the lattice. The final estimate is $\alpha_s(M_Z) = 0.1170(12)$, which slightly corrects the already reported result of [147].
The figure 6(b), from the cited paper, shows the remarkable effect of the vacuum polarization of the three light quarks on the coupling $\alpha_V$ introduced in eq. (67) (the momentum is represented in units of the inverse lattice spacing). Among the many lattice results, we would like now to pay a little attention to those concerning the behavior of the coupling in the very low energy region. The MOM definition of the coupling is likely the best suited to this aim, indeed the already cited authors using this approach addressed this task in [156]. In order to reach low momenta also lattices with a rather large lattice spacing, and then low $\beta$, have been used. The results for the pure gauge theory are shown in Fig. 7. The number of data in fig. 7(b) is sufficient to check the theoretical model of the vacuum as instanton liquid. According to this model the coupling must have a $p^4$ dependence for small energy, the data in Fig. 7(a) are indeed fitted by this power low, the proportionality coefficient yields the instanton density $n$:

$$\alpha_{\text{MOM}}(p) = \frac{1}{18\pi} n^{-1} p^4. \quad (161)$$

Such a behavior is not a universal feature, from a glance at the figures 8 and 9 in the second paper of [142], in which the coupling is defined in MOM scheme but using the quark-gluon vertex, one realizes that there the coupling is still vanishing for small momenta but, in spite of the rather little number
of data, it does not seem with the $p^4$ low. On the contrary the coupling $\alpha_V(Q)$ introduced in eq. (67) should instead diverge for $Q \to 0$.

In the last years infrared features of QCD has been addressed using the Dyson-Schwinger equations. This approach, which starts from suitable truncations of the DSE, seems in particular effective in the study of the low energy behavior of the propagators (see [139] for a review). In this way infrared exponents for the gluon-gluon and ghost-ghost two point functions have been determined. In the Landau gauge using the ghost and the gluon propagators one can give a definition of the invariant coupling (which then resembles the familiar definition of QED) and it turns out that it has a finite infrared limit [158]. It is found evidence for this fixed point at least in the context of pure SU(2) lattice gauge theory [159], however this conclusion does not seem supported in other recent lattice SU(3) simulations [160]-[161]. The two results are compared and discussed in [162], very recent results also in [163].

As yet noted, such a lack of universality of the running coupling constant in the low energy regime is somewhat expected, however on such different behaviors, which are a little disturbing, some comment is in order. In [164] a comparison of various approaches is made with reference to singular RG transformations. The RG transformations are associated to a change of variable $g \to g' = f(g)$, which in general depends on the dimensional parameters too; in the perturbative framework $f(g)$ is usually supposed to be an analytic function in the origin $g = 0$. Now considering the MOM schemes, it is well
known that in perturbation theory the definitions of the coupling are vertex
dependent [20] but, at least in the massless case, this dependence turns out
to be weak, namely the function $f$ is close to the identity. It is conceivable
that in the infrared region, all more reason taking into account the different
dependence of the coupling on the quark light masses, the function $f$ can
have non trivial consequences.

Some proposals that we have examined which tries to extend the perturbative
series in the low energy region can be seen as a not trivial RG transfor-
mation, at the beginning defined in full perturbative region, but such that the
perturbative series in the new variable does not suffer from some pathology
of the first one. Thus formula (107), which avoids the Landau pole, can be
viewed, as already noticed, as the result of the non-analytic transformation
$\alpha_s \rightarrow \alpha_s' = \alpha_s + \frac{1}{\beta_0}(1 - \exp(\frac{1}{\beta_0 \alpha_s}))^{-1}$.
Every devisable transformation should share with the previous one the two
nice properties to enjoy the asymptotic freedom, $\alpha_s \approx 0^+ \rightarrow \alpha_s' \approx 0^+$, and to
be free of ghost singularity, $\alpha_s = \infty \rightarrow \alpha_s' = \text{finite}$. A priori there are not
other compelling requirements, and one can imagine a situation in which,
with respect to a given problem, a definition, and then a consequent pertur-
bative expansion, has better properties than another. The considerations in
[164] show that with other transformations one can induce various infrared
behaviors in the new coupling. The conclusion is that “there is not direct
physical sense in attempts to establish some “correct IR behavior” of the
perturbative QCD invariant coupling”.

6 Comparison with the data and conclusive
remarks

For completeness in this concluding section we want to comment briefly on
the experimental situation as continuously kept updated by [158] and other
specialized reviews (see e.g. [6]). Tab.1 in the contribution of S.Bethke in
this issue summarizes the determination of $\alpha_s(Q^2)$ at various energies be-
tween 1.58 and 206 GeV extracted by various types of experiments: deep
inelastic scattering (DIS), $e^+e^-$ annihilation (hadron cross section, structure
function, jets and jet shapes), hadron-hadron collision, $Z$, $\Upsilon$ and $\tau$ decay,
heavy quark bound states.
In fig. 8(a) from [6] such results are compared with the 4-loop perturbative
expression of $\alpha_{\overline{\text{MS}}}(Q^2)$ as given by (31) together with threshold matching.
Figure 8: (a) From [6]: summary of measurements of $\alpha_s(Q^2)$. (b) From [16]: comparison between the 1-loop analytic coupling (107) with $\Lambda(n_f=3)=206$ MeV, and the values of $\alpha_s$ fitting experimental data on quarkonium spectrum within Salpeter formalism (circles, pentagrams and squares refer to light-light states, diamonds and crosses to heavy-heavy, plus signes and asterisks to light-heavy states).

As it can be seen the agreement is very good inside the errors and this constitutes a very significant test for QCD. Note, however, that assuming all expressions equally normalized at $M_Z = 91.2$ GeV the difference between the 3-loop and the 4-loop expressions is of order 1/10000 in the interval from 10 to 200 GeV, it becomes of the order 1/1000 between 2-loop and 4-loop and of few per cent between 1-loop and 4-loop. We have in this way the proof that at three loops the theory is practically at convergence in comparison with the precision of the experimental data and that two loops (naturally after the appropriate rescaling of $\Lambda$) is already a very good approximation in the considered range.\(^8\) The choice of a particular RS or another is essentially immaterial. However, under a scale of few $\Lambda(n_f=3)$ (let us say 2 GeV), the MS becomes useless, due to the Landau singularities, and we have to refer to any alternative scheme free of unphysical singularities and to follow it very consistently. A comparison with the data becomes RS dependent and would have no meaning out of the well defined framework.

\(^8\)Obviously the difference explodes near the singularity.
Note that the analytic scheme discussed in sec. 4 seems particularly convenient at this aim. It shares the simplicity and the universality of the $\overline{\text{MS}}$ scheme at a large extent. It provides a coupling which is regular in the entire interval $0 < Q^2 < \infty$ and has a finite limit $\frac{1}{\alpha_{\text{an}}}$ for $Q \to 0$ independent of the number of loops used. As a consequence of this fact its expression converges much faster as the number of loops increases. The difference between the 2-loop and the 3-loop expression is again of order $1/1000$ but this time more or less in the entire range from 0 to $+\infty$. Even the difference between the 1-loop and the 3-loop is of order of $1/100$ and may be a fraction of this in limited ranges.

For construction $\alpha_{\text{an}}(Q^2)$ coincides asymptotically with $\alpha_{\overline{\text{MS}}}(Q^2)$ for large $Q$ and $\alpha_{\text{an}}^{(2)}(Q^2)$ fits the data of Tab.1 from [6] nearly as well as $\alpha_{\overline{\text{MS}}}^{(3)}(Q^2)$. Note however that, as seen in sec. 4.2, we have more precisely

$$\alpha_{\text{an}}^{(p)}(Q^2) = \alpha_{\overline{\text{MS}}}^{(p)}(Q^2) + \frac{c_1^{(p)}}{Q^2}. \quad (162)$$

The coefficients $c_1^{(p)}$ seem to decrease as the number $p$ of the loops increases. However for the value of $p$ we actually use the power term in (162) can be non negligible at some intermediate scale and may be disturbing in particular with reference to the problematic discussed in sec 4.5. As we have seen they may be eliminated for every given $p$ by choosing an appropriate non perturbative term $\alpha_{\text{NP}}^{(p)}(Q^2)$ to be added to the “perturbative” $\alpha_{\text{an}}^{(p)}(Q^2)$. As we have seen phenomenologically modified expressions for $\alpha_{\text{an}}(Q^2)$ have also been proposed to eliminate such term.

Information on the QCD interaction under few GeV can be obtained from power corrections to various observable at some intermediate energy, as we have seen. Other information can come from relativistic calculation of the spectrum of light-light, light-heavy and from the highly excited heavy-heavy quarkonium states.

In [165], e.g., a reasonable reproduction of the entire calculable spectrum and in particular of $\pi - \rho$, the $K - K^*$ and $\eta_c - J/\psi$ was obtained. Such analysis was performed in the framework of a Salpeter formalism constructed using as input only a generalization of the ansatz (64) and a 1-loop analytic coupling with $n_f = 3$ and $\Lambda^{(n_f=3)} = 180$ MeV. Note that such $\alpha_{\text{an}}^{(1)}(Q^2)$ differs from $\alpha_{\text{an}}^{(2)}(Q^2)$ with $\Lambda^{(n_f=3)} = 375$ MeV by less than 0.5 %.

Actually the developments reported in sec 4.5 were made in the framework of the SGD scheme and this scheme may be the most natural even for bound
states problems. In fact the inclusion of certain second order corrections would simply amount to a rescaling of $\Lambda$. However the SGD and the analytic scheme are strictly related and have even a common root in a sense [105]. They are both based on dispersive techniques and can be made to coincide up to two loops.

In any way according to a more recent preliminary analysis, made in the context of the mentioned relativistic determination of the quarkonium spectrum [166], it seems that the curve $\alpha_{an}^{(1)}(Q^2)$ (with $\Lambda(n_f=3) = 206\text{MeV}$) is in very good agreement with the data up to 200 MeV corresponding e.g. to the $c\bar{c}(1D)$ state (see fig.8(b)). On the contrary for higher excited states, corresponding to $100 < Q < 200\text{MeV}$, the values of $\alpha_s$ that reproduce the data seem to be somewhat lower than $\alpha_{an}^{(1)}(Q^2)$ and this could confirm the need of a non perturbative contribution to be added to $\alpha_{an}(Q^2)$ as given in sec. 4.5 [166]. However, the experimental situation is so uncomplete and confused in this range and the applicability of the theoretical method so questionable that no real significant conclusion can be drawn.

In conclusion, the use in QCD of a RS, in which the running coupling constant is free from Landau singularities, seems to offer a framework in which all existing phenomenology can be discussed and non perturbative corrections usefully parametrized. It is even a framework in which a comparison with Lattice Theory results is easier and more natural.

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