A SMALL IMPROVEMENT IN THE SMALL GAPS
BETWEEN CONSECUTIVE ZEROS OF THE RIEemann Zeta-Function

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Abstract. Feng and Wu introduced a new general coefficient sequence into Montgomery
and Odlyzko’s method for exhibiting irregularity in the gaps between consecutive zeros of
ζ(s) assuming the Riemann Hypothesis. They used a special case of their sequence to
improve upon earlier results on the gaps. In this paper we consider an equivalent form of
the general sequence of Feng and Wu, and introduce a somewhat less general sequence
{a_n} for which we write the Montgomery–Odlyzko expressions explicitly. As an application, we
give the following slight improvement of Feng and Wu’s result: infinitely often consecutive
non-trivial zeros of the Riemann zeta-function differ by at most 0.515396 times the average
spacing.

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1. Introduction. It is well known that the Riemann zeta-function ζ(s) has infinitely many
nontrivial zeros s = ρ = β + iγ, and all of them are in the critical strip 0 < Re s = σ < 1,
−∞ < Im s = t < ∞.

If N(T) denotes the number of zeros ρ = β + iγ (β and γ real), for which 0 < γ ⩽ γ', then

N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O \left( \frac{1}{T} \right),

with

S(T) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right)

and

S(T) = O(\log T).

This is the Riemann–von Mangoldt formula for N(T). Hence, if we let 0 < γ ⩽ γ' denote
consecutive ordinates of non-trivial zeros of ζ(s), the average size of γ' − γ is γ/N(γ) ~ 2π/ log γ.
Let

\lambda = \limsup_{\gamma > 0} (\gamma' - \gamma) \frac{\log \gamma}{2\pi}

and

\mu = \liminf_{\gamma > 0} (\gamma' - \gamma) \frac{\log \gamma}{2\pi}.

We note that μ ⩽ 1 ⩽ λ and it is expected that μ = 0 and λ = +∞.

Let N_0(T) be the number of zeros of ζ \left( \frac{1}{2} + it \right) when 0 < t ⩽ T, each zero counted with
multiplicity. The Riemann hypothesis is the conjecture that N_0(T) = N(T).

In this note we prove the following theorem.

1 Mathematics Subject Classification. Primary 11M26; Secondary 11M06.
Key words and phrases. Riemann zeta function, Zeros, Critical line, Gaps.
Theorem 1. Assume the Riemann Hypothesis. Then we have

\[ \mu < 0.515396. \]

We briefly describe the history of the problem, focusing mainly on \( \mu \).

- [7]: in 1946 Selberg remarked that \( \mu < 1 < \lambda \) unconditionally.

Now suppose that \( T \) is a large real number and \( K = T(\log T)^{-2} \). Let

\[ h(c) = c - \frac{\text{Re} \left( \sum_{nk \leq K} a_k \overline{a_n} g_c(n) \Lambda(n) n^{-1/2} \right)}{\sum_{k < K} |a_k|^2} \quad (1) \]

where

\[ g_c(n) = \frac{2 \sin \left( \pi c \frac{\log n}{\log T} \right)}{\pi \log n} \]

and \( \Lambda \) is the von Mangoldt’s function.

In the following results, the truth of the Riemann Hypothesis is assumed.

- [6]: in 1981 by an argument using the Guinand–Weil explicit formula, Montgomery and Odlyzko showed that if \( h(c) < 1 \) for some choice of \( c \) and \( \{a_n\} \), then \( \lambda \geq c \), and if \( h(c) > 1 \) for some choice of \( c \) and \( \{a_n\} \), then \( \mu \leq c \). They used the coefficients

\[ a_k = \frac{1}{k^2} f \left( \frac{\log k}{\log K} \right) \quad \text{and} \quad a_k = \frac{\lambda(k)}{k^2} f \left( \frac{\log k}{\log K} \right) \]

where \( f \) is a continuous function of bounded variation, and \( \lambda(k) \) is the Liouville function. With this choice of the coefficients they obtained \( \lambda > 1.9799 \) and \( \mu < 0.5179 \) by optimizing over the functions \( f \).

- [2]: in 1984 Conrey, Ghosh & Gonek chose the coefficients

\[ a_k = \frac{d_r(k)}{\sqrt{k}} \quad \text{and} \quad a_k = \frac{\lambda(k)d_r(k)}{\sqrt{k}} \]

where \( d_r(k) \) is a multiplicative function defined on integral powers of a prime \( p \) by

\[ d_r(p^k) = \frac{\Gamma(k + r)}{\Gamma(r) k!}. \]

The choice \( r = 1.1 \) with the latter \( a_k \) yields \( \mu < 0.5172 \) and the choice \( r = 2.2 \) with the former \( a_k \) yields \( \lambda > 2.337 \).

- [4]: in 2005, by making use of the Wirtinger’s inequality and the asymptotic formulae for the fourth mixed moments of the zeta-function and its derivative, R. R. Hall proved that \( \lambda > 2.6306 \).

- [1]: in 2010, Bui, Milinovich & Ng considered the coefficients of the form

\[ a_k = \frac{d_r(k)}{\sqrt{k}} f \left( \frac{\log K/k}{\log K} \right) \quad \text{and} \quad a_k = \frac{\lambda(k)d_r(k)}{\sqrt{k}} f \left( \frac{\log K/k}{\log K} \right) \]

for a polynomial \( f \) and obtained \( \lambda > 2.69 \) and \( \mu < 0.5155 \).
• In 2012, Feng & Wu introduced the coefficient
\[
a_k = \frac{d_r(k)}{k^2} \left( f_1 \left( \frac{\log K/k}{\log K} \right) + f_2 \left( \frac{\log K/k}{\log K} \right) \sum_{p_1 p_2 | k} \frac{\log p_1 \log p_2}{\log^2 K} \right. \\
+ f_3 \left( \frac{\log K/k}{\log K} \right) \sum_{p_1 p_2 p_3 | k} \frac{\log p_1 \log p_2 \log p_3}{\log^2 K} + \cdots \\
\left. + f_I \left( \frac{\log K/k}{\log K} \right) \sum_{p_1 p_2 \cdots p_I | k} \frac{\log p_1 \log p_2 \cdots \log p_I}{\log^2 K} \right),
\]
for any integer \( I \geq 2 \). Using \( I = 2 \) they obtained \( \lambda > 2.7327 \) and \( \mu < 0.5154 \), or, to higher precision, \( \mu < 0.515398 \).

We remark that the coefficient of Feng & Wu is equivalent to the coefficient
\[
a_k = \frac{d_r(k)}{k^2} \left( f_1 \left( \frac{\log K/k}{\log K} \right) + f_2 \left( \frac{\log K/k}{\log K} \right) \sum_{p_1 | k} \frac{\log^2 p_1}{\log^2 K} \right. \\
+ f_3 \left( \frac{\log K/k}{\log K} \right) \sum_{p_1 | k} \frac{\log^3 p_1}{\log^3 K} + \cdots \\
\left. + f_I \left( \frac{\log K/k}{\log K} \right) \sum_{p_1 | k} \frac{\log^I p_1}{\log^I K} \right),
\]
for which the calculations are simpler.

To prove Theorem 1, we choose the coefficients
\[
a_k = \frac{\lambda(k)d_r(k)}{k^2} f_1 \left( \frac{\log K/k}{\log K} \right) + \frac{\lambda(k)d_r(k)}{k^2} \sum_{p | k} P \left( \frac{\log p}{\log K} \right) f_1 \left( \frac{\log K/k}{\log K} \right),
\]
where \( f_1, f_1', P \) are some polynomials to be chosen later. The \( a_k \) given above are less general than the coefficients of Feng and Wu, but they are simpler, so we are able to write the Montgomery–Odlyzko expressions for our sequence explicitly.

2. Lemmas.

**Lemma 1** (Mertens Theorem).
\[
\sum_{p \leq y} \frac{\log p}{p} = \log y + O(1).
\]

**Lemma 2** (See Levinson [5]).
\[
\sum_{p | j} \frac{\log p}{p} = O(\log \log j).
\]

**Lemma 3.** For fixed \( r \geq 1 \),
\[
\sum_{k \leq x} \frac{d_r(k)^2}{k} = A_r(\log x)^r + O \left( (\log T)^{r-1} \right)
\]
uniformly for \( x \leq T \).
Lemma 4. Let $a_i$ be integer for $1 \leq i \leq m$, $D > 1$ and $f$ is a continuous function. Then
\[
\frac{\log^{a_1-1} x_1}{x_1} \frac{\log^{a_2-1} x_2}{x_2} \cdots \frac{\log^{a_m-1} x_m}{x_m} = \prod_{i=1}^{m} \left( a_i - 1 \right) \prod_{i=1}^{m} \frac{f(x_1 x_2 \cdots x_m)}{x} dx.
\]

Lemma 5. Let $a_i$ be integer for $1 \leq i \leq m$, and $g$ is a polynomial. Then we have
\[
\sum_{k \leq K} \frac{d_r(k)^2}{k} g \left( \frac{\log K/k}{\log K} \right) = A_r r^2 \int_1^K g \left( \frac{\log K/x}{\log K} \right) \left( \log x \right)^{r^2-1} \frac{dx}{x} + O \left( (\log K)^{r^2-1} \right)
\]
and
\[
\sum_{p_1 p_2 \cdots p_m \leq K} \prod_{i=1}^{m} \frac{\log^{p_i} (p_1 p_2 \cdots p_m)}{p_i} \sum_{k_0 \leq K/(p_1 p_2 \cdots p_m)} \frac{d_r(k_0)^2}{k_0} g \left( \frac{\log K/(p_1 p_2 \cdots p_m k_0)}{\log K} \right) = (1 + O \left( \log^{-1} K \right))
\]
\[
\times A_r r^2 \int_1^K \frac{dx_1}{x_1} \int_1^{x_1} \frac{dx_2}{x_2} \cdots \int_1^{x_1 \cdots x_m} \frac{dx_m}{x_m} g \left( \frac{\log K/(x_1 x_2 \cdots x_m)}{\log K} \right) \left( \log x \right)^{r^2-1} \frac{dx}{x}.
\]

For the proof of Lemma 5 using Lemmas 1-3, see [3].

3. Proof of Theorem 1. To give an upper bound for $\mu$, we evaluate $h(c)$ in (1) with the coefficients
\[
a_k = \frac{\lambda(k) d_r(k)}{k^r} f_1 \left( \frac{\log K/k}{\log K} \right) + \frac{\lambda(k) d_r(k)}{k^r} \sum_{p \mid k} P \left( \frac{\log p}{\log K} \right) \tilde{f}_1 \left( \frac{\log K/k}{\log K} \right),
\]
where $r \geq 1$ and $f_1$, $\tilde{f}_1$, $P$ are polynomials.

First, we evaluate the denominator in the ratio in the definition of $h(c)$.
\[
\sum_{k \leq K} \left| a_k \right|^2 = \sum_{k \leq K} \frac{d_r(k)^2}{k} f_1 \left( \frac{\log K/k}{\log K} \right)^2
\]
\[
+ 2 \sum_{k \leq K} \frac{d_r(k)^2}{k} f_1 \left( \frac{\log K/k}{\log K} \right) \tilde{f}_1 \left( \frac{\log K/k}{\log K} \right) \sum_{p \mid k} P \left( \frac{\log p}{\log K} \right)
\]
\[
+ \sum_{k \leq K} \frac{d_r(k)^2}{k} \tilde{f}_1 \left( \frac{\log K/k}{\log K} \right)^2 \sum_{p \mid k} P \left( \frac{\log p}{\log K} \right) \sum_{q \mid k} P \left( \frac{\log q}{\log K} \right)
\]
\[
= \hat{D}_1 + \hat{D}_2 + \hat{D}_3.
\]

Using Lemma 5 and recalling that $K = T(\log T)^{-2}$, we have
\[
\hat{D}_1 = A_r r^2 \int_1^K f_1 \left( \frac{\log K/x}{\log K} \right)^2 \left( \log x \right)^{r^2-1} \frac{dx}{x} + O \left( (\log T)^{r^2-1} \right)
\]
\[
= A_r r^2 (\log K)^2 \int_0^1 (1 - u)^{r^2-1} f_1(u)^2 du + O \left( (\log T)^{r^2-1} \right) \quad (2)
\]
\[
= A_r r^2 (\log T)^2 \int_0^1 (1 - u)^{r^2-1} f_1(u)^2 du + O \left( (\log T)^{r^2-1+\varepsilon} \right),
\]
where \( \varepsilon > 0 \) is arbitrarily small and the constant in the \( O \)-term depends on \( r, \varepsilon \) and \( f_1 \). By Lemma 5 we obtain that

\[
\tilde{D}_2 = \frac{2A_r^4}{\log K} \int_1^K P_1(\log x_1) \int_1^{\frac{K}{x_1}} f_1 \left( \frac{\log K}{x_1 x} \right) \times \tilde{f}_1 \left( \frac{\log K}{x_1 x} \right) (\log x)^{r-1} \frac{dx}{x} dx_1 + O \left( (\log T)^{r-1+\varepsilon} \right),
\]

where \( P_1(y) = \frac{p(y)}{y} \). By the variable changes \( u = 1 - \frac{\log x_1}{\log K}, v = 1 - \frac{\log x_1 x}{\log K}, \) we have

\[
\tilde{D}_2 = 2A_r^4 (\log K)^r \int_0^1 P_1(1-u) \int_0^u (u-v)^{r-1} f_1(v) \tilde{f}_1(v) dv du + O \left( (\log T)^{r-1+\varepsilon} \right) \tag{3}
\]

where the constant in the \( O \)-term depends on \( r, \varepsilon \) and \( f_1, \tilde{f}_1 \).

We have

\[
\tilde{D}_3 = \sum_{k \leq K} \frac{d_r(k)^2}{k} f_1 \left( \frac{\log K}{k} \right) \times \sum_{p | k} \mu^2(p_1 p_2) P \left( \frac{\log p_1}{\log K} \right) P \left( \frac{\log p_2}{\log K} \right) + \sum_{k \leq K} \frac{d_r(k)^2}{k} \tilde{f}_1 \left( \frac{\log K}{k} \right) \times \sum_{p | k} p^2 \left( \frac{\log p}{\log K} \right)
\]

\[
= \tilde{D}_{31} + \tilde{D}_{32}.
\]

Again by Lemma 5 we obtain that

\[
\tilde{D}_{31} = \frac{A_r^6}{\log^2 K} \int_1^K P_1(\log x_1) \int_1^{\frac{K}{x_1}} P_1(\log x_2) \int_1^{\frac{K}{x_1 x_2}} f_1 \left( \frac{\log K}{x_1 x_2 x} \right)^2 \times (\log x)^{r-1} \frac{dx}{x} dx_2 dx_1 + O \left( (\log T)^{r-1+\varepsilon} \right),
\]

where \( P_1(y) = \frac{p(y)}{y} \). We remark that by Lemma 4 we can reduce the number of the repeated integrations in the above expression. By the change of variables \( u = 1 - \frac{\log x_1}{\log K}, v = 1 - \frac{\log x_1 x_2}{\log K}, w = 1 - \frac{\log x_1 x_2 x}{\log K}, \)

\[
\tilde{D}_{31} = A_r^6 (\log K)^r \int_0^1 P_1(1-u) \int_0^{u+1} P_1(1-v) \int_0^{u+1} (u+v-w-1)^{r-1} \tilde{f}_1(w)^2 dw dv du + O \left( (\log T)^{r-1+\varepsilon} \right)
\]

\[
= A_r^6 (\log T)^r \int_0^1 P_1(1-u) \int_0^{u+1} P_1(1-v) \int_0^{u+1} (u+v-w-1)^{r-1} \tilde{f}_1(w)^2 dw dv du + O \left( (\log T)^{r-1+\varepsilon} \right), \tag{4}
\]

where the constant in the \( O \)-term depends on \( r, \varepsilon \) and \( \tilde{f}_1 \). Similarly,

\[
\tilde{D}_{32} = A_r^4 (\log K)^r \int_0^1 P_2(1-u) \int_0^u (u-v)^{r-1} \tilde{f}_1(v)^2 dv du + O \left( (\log T)^{r-1+\varepsilon} \right)
\]

\[
= A_r^4 (\log T)^r \int_0^1 P_2(1-u) \int_0^u (u-v)^{r-1} \tilde{f}_1(v)^2 dv du + O \left( (\log T)^{r-1+\varepsilon} \right), \tag{5}
\]

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where \( P_2(y) = \frac{p(y)^2}{y} \) and the constant in the \( O \)-term depends on \( r, \varepsilon \) and \( \tilde{f}_1 \).

We now proceed to evaluation of the numerator in the ratio in (1). If we let

\[
N(c) = \sum_{n \leq K} a_n a_n g_c(n) \Lambda(n)n^{-1/2},
\]

then

\[
N(c) = \frac{2}{\pi} \sum_{nk \leq K} \frac{\lambda(k)d_r(k)\lambda(nk)d_r(nk)\Lambda(n)}{kn \log n} \sin \left( \pi c \log \frac{n}{\log T} \right) \times \left( f_1 \left( \frac{\log K/k}{\log K} \right) f_1 \left( \frac{\log K/nk}{\log K} \right) + \right.
\]

\[
+ f_1 \left( \frac{\log K/k}{\log K} \right) \tilde{f}_1 \left( \frac{\log K/nk}{\log K} \right) \sum_{p|n} P \left( \frac{\log p_1}{\log K} \right) \left. \right)
\]

so we can write

\[
N(c) = N_1 + N_2 + N_3 + N_4.
\]

Using the distribution of \( \Lambda(n) \), we obtain

\[
N_1 = -\frac{2}{\pi} \sum_{pk \leq K} \frac{d_r(k)d_r(pk)}{kp} \sin \left( \pi c \frac{\log p}{\log T} \right) f_1 \left( \frac{\log K/k}{\log K} \right) f_1 \left( \frac{\log K/pk}{\log K} \right) + O \left( (\log T)^{r-1} \right)
\]

\[
= -\frac{2}{\pi} \sum_{p \leq K} \frac{\sin \left( \pi c \frac{\log p}{\log T} \right)}{p} \sum_{k \leq k/p} d_r(k)^2 f_1 \left( \frac{\log K/k}{\log K} \right) f_1 \left( \frac{\log K/pk}{\log K} \right) + O \left( (\log T)^{r-1} \right)
\]

By Lemma [3] we have

\[
N_1 = -\frac{2A_r r^3}{\pi} \sum_{\nu \leq K} \frac{\sin \left( \pi c \frac{\log \nu}{\log T} \right)}{\nu} \int_1^K \frac{f_1 \left( \frac{\log K/x}{\log K} \right) f_1 \left( \frac{\log K/px}{\log K} \right)}{x} (\log x)^{r-1} dx + O \left( (\log T)^{r-1} \right)
\]

From Lemma [4] and Abel’s summation,

\[
N_1 = -\frac{2A_r r^3}{\pi} \int_1^K \frac{\sin \left( \pi c \frac{\log x_1}{\log T} \right)}{x_1 \log x_1} \int_1^{x_1} \frac{f_1 \left( \frac{\log K/x}{\log K} \right) f_1 \left( \frac{\log K/xx_1}{\log K} \right)}{x} (\log x)^{r-1} dx d_{x_1} + O \left( (\log T)^{r-1} \right)
\]

Interchanging the order of integration and the names of the variables \( x \) and \( x_1 \), we find

\[
N_1 = -\frac{2A_r r^3}{\pi} \int_1^K \frac{\sin \left( \pi c \frac{\log x}{\log T} \right)}{x} \int_1^{x} \frac{f_1 \left( \frac{\log K/x}{\log K} \right) f_1 \left( \frac{\log K/xx_1}{\log K} \right)}{x_1} (\log x_1)^{r-1} dx_1 dx + O \left( (\log T)^{r-1} \right)
\]
Let $u = 1 - \frac{\log x}{\log K}$, $v = \frac{\log x}{\log K}$. Then

\[
N_1 = -\frac{2A_r r^3}{\pi}(\log K)^2 \int_0^1 (1 - u)^{r^2-1} f_1(u) \int_0^u \frac{\sin(\pi cv \log K)}{v} f_1(u-v) \, dv \, du \\
+ O\left((\log T)^{r^2-1}\right) \\
= -\frac{2A_r r^3}{\pi}(\log T)^2 \int_0^1 (1 - u)^{r^2-1} f_1(u) \int_0^u \frac{\sin(\pi cv)}{v} f_1(u-v) \, dv \, du \\
+ O\left((\log T)^{r^2-1+\varepsilon}\right),
\]

where the constant in the $O$-term depends on $r$, $\varepsilon$ and $f_1$.

In $N_2$, we can replace the product of the summation variables $nk$ by $pp_k$ to get

\[
N_2 = -\frac{2r^3}{\pi} \sum_{p_1 \leq K} \frac{P}{p_1} \sum_{p_k \leq K/p_1} \sin\left(\pi \frac{\log p}{\log T}\right) \frac{d_r(k_0)^2}{p_k} \\
\times f_1\left(\frac{\log K/(p_1x_2)}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(p_1x_2)}{\log K}\right) + O\left((\log T)^{r-2}\right).
\]

The inner sum $\sum_{p_k \leq K/p_1}$ in the expression above is the sum $\sum_{k \leq K/p_1} \sum_{p \leq K/(p_1k_0)}$. As in the calculation of $N_1$, we can show that this double sum is

\[
A_r r^2 \int_1^K \frac{f_1\left(\frac{\log K/(p_1x_2)}{\log K}\right) (\log x_2)^{r^2-1}}{x_2} \\
\times \int_1^K \frac{\sin\left(\pi \frac{\log x}{\log T}\right)}{\log x} f_1\left(\frac{\log K/(x_1x_2)}{\log K}\right) \frac{dx}{x} dx_2 + O\left((\log T)^{r^2-1}\right).
\]

By Lemma [1] we obtain

\[
N_2 = -\frac{2A_r r^5}{\pi (\log K)} \int_1^K \frac{P}{x_1} \int_1^K \frac{f_1\left(\frac{\log K/(x_1x_2)}{\log K}\right) (\log x_2)^{r^2-1}}{x_2} \\
\times \int_1^K \frac{\sin\left(\pi \frac{\log r}{\log T}\right)}{\log x} f_1\left(\frac{\log K/(x_1x_2)}{\log K}\right) \frac{dx}{x} dx_2 dx_1 + O\left((\log T)^{r^2-1}\right).
\]

Making the variable changes $u = 1 - \frac{\log x}{\log K}$, $v = 1 - \frac{\log x_1}{\log K}$, $w = \frac{\log x}{\log K}$, we get

\[
N_2 = -\frac{2A_r r^5}{\pi}(\log T)^2 \int_0^1 P_1(1 - u) \int_0^u (u - v)^{r^2-1} \tilde{f}_1(v) \int_0^v \frac{\sin(\pi cv)}{w} f_1(v-w) \, dw \, dv \, du \\
+ O\left((\log T)^{r^2-1+\varepsilon}\right),
\]

where the constant in the $O$-term depends on $r$, $\varepsilon$ and $f_1$, $\tilde{f}_1$.

As in $N_1$ and $N_2$, the terms with $n = p$ for the primes $p$ give the main contribution to $N_3$: \[N_3 = -\frac{2}{\pi} \sum_{p \leq K} \sin\left(\pi \frac{\log p}{\log T}\right) \frac{d_r(k) d_r(pk)}{kp} f_1\left(\frac{\log K/k}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right) \\
\times \sum_{p_1 | pk} P\left(\frac{\log p_1}{\log K}\right) + O\left((\log T)^{r^2-1}\right).\]
For \((p, k) = 1\) it follows that
\[
\sum_{p_1|p} P \left( \frac{\log p_1}{\log K} \right) = \sum_{p_1|k} P \left( \frac{\log p_1}{\log K} \right) + P \left( \frac{\log p}{\log K} \right).
\] (8)

Since the contribution of the terms with \((p, k) \neq 1\) in \(N_3\) is \(O \left( (\log T)^{r^2-1} \right)\), then, according to decomposition (8), we can write
\[
N_3 = N_{31} + N_{32} + O \left( (\log T)^{r^2-1} \right),
\]
where
\[
N_{31} = -\frac{2r^3}{\pi} \sum_{p_1 \leq K} P \left( \frac{\log p_1}{\log K} \right) \sum_{p_{k_0} \leq K/p_1} \sin \left( \frac{\pi \log p_{k_0}}{\log K} \right) d_r(k_0)^2
\times \tilde{f}_1 \left( \frac{\log K/(pp_1k_0)}{\log K} \right) f_1 \left( \frac{\log K/(p_1k_0)}{\log K} \right) + O \left( (\log T)^{r^2-1} \right)
\]
and
\[
N_{32} = -\frac{2r}{\pi} \sum_{p_k \leq K} \sin \left( \frac{\pi \log p}{\log K} \right) d_r(k)^2 P \left( \frac{\log p}{\log K} \right) \tilde{f}_1 \left( \frac{\log K/(pk)}{\log K} \right)
\times f_1 \left( \frac{\log K/k}{\log K} \right) + O \left( (\log T)^{r^2-1} \right).
\]

As in the calculation of \(N_2\) we get
\[
N_{31} = -\frac{2A_r}{\pi} r^5 (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v)
\times \int_0^v \frac{\sin(\pi cw)}{w} \tilde{f}_1(v-w) \, dw \, dv \, du + O \left( (\log T)^{r^2+1+\varepsilon} \right),
\]
and as in the calculation of \(N_1\),
\[
N_{32} = -\frac{2A_r}{\pi} r^3 (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u) \int_0^u \sin(\pi cv) P_1(v) \tilde{f}_1(u-v) \, dv \, du
+ O \left( (\log T)^{r^2+1+\varepsilon} \right),
\]
where \(P_1(y) = \frac{P(y)}{y}\). Thus,
\[
N_3 = -\frac{2A_r}{\pi} r^5 (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v)
\times \int_0^v \frac{\sin(\pi cw)}{w} \tilde{f}_1(v-w) \, dw \, dv \, du
- \frac{2A_r}{\pi} r^3 (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u) \int_0^u \sin(\pi cv) P_1(v) \tilde{f}_1(u-v) \, dv \, du
+ O \left( (\log T)^{r^2+1+\varepsilon} \right),
\] (9)
where the constant in the $O$-term depends on $r, \varepsilon$ and $f_1, \tilde{f}_1, P$.

Again, in the sum defining $N_4$ we can replace the integers $n \geq 2$ with the primes $p$:

$$N_4 = -\frac{2}{\pi} \sum_{p k \leq K} \sin \left( \frac{\pi c \log p}{\log T} \right) \frac{d_r(k) d_r(kp)}{kp} \tilde{f}_1 \left( \frac{\log K/k}{\log K} \right) \tilde{f}_1 \left( \frac{\log K/(kp)}{\log K} \right)$$

$$\times \sum_{p_1 | k} P \left( \frac{\log p_1}{\log K} \right) \sum_{q_1 | pk} P \left( \frac{\log q_1}{\log K} \right) + O \left( (\log T)^{r^2-1} \right).$$

For the two innermost sums, if $(k, p) = 1$, we have

$$\sum_{p_1 | k} P \left( \frac{\log p_1}{\log K} \right) \sum_{q_1 | pk} P \left( \frac{\log q_1}{\log K} \right)$$

$$= \sum_{p_1 q_1 | k} \mu^2(p_1 q_1) P \left( \frac{\log p_1}{\log K} \right) P \left( \frac{\log q_1}{\log K} \right)$$

$$+ \sum_{p_1 | k} P^2 \left( \frac{\log p_1}{\log K} \right) + P \left( \frac{\log p}{\log K} \right) \sum_{p_1 | k} P \left( \frac{\log p_1}{\log K} \right).$$

According to this decomposition, we write

$$N_4 = N_{41} + N_{42} + N_{43}.$$

As before, by Lemma 5 we find

$$N_{41} = -\frac{2 A_r r^7}{\pi (\log K)^2} \int_1^K P_1 \left( \frac{\log x_1}{\log K} \right) \int_1^{K/2} P_1 \left( \frac{\log x_2}{\log K} \right) \int_1^{K/2} \tilde{f}_1 \left( \frac{\log K/(x_1 x_2 x_3)}{\log K} \right) \frac{(\log x_3)^{r^2-1}}{x_3}$$

$$\times \int_1^{K/2} \frac{\sin \left( \pi c \log x \right)}{\log x} \frac{\tilde{f}_1 \left( \frac{\log K/(x x_1 x_2 x_3)}{\log K} \right)}{x} dx_3 dx_2 dx_1 + O \left( (\log T)^{r^2-1} \right).$$

Making the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_2}{\log K}$, $w = 1 - \frac{\log x_1 x_2 x_3}{\log K}$, $z = \frac{\log x}{\log K}$, we get

$$N_{41} = -\frac{2 A_r r^7}{\pi (\log K)^2} \int_0^1 P_1 (1 - u) \int_0^1 P_1 (1 - v)$$

$$\times \int_0^{u+v-1} (u + v - w - 1)^{r^2-1} \tilde{f}_1 (w) \int_0^w \frac{\sin (\pi c z)}{z} \tilde{f}_1 (w - z) \frac{dz}{dw} \frac{dw}{du}$$

$$+ O \left( (\log T)^{r^2-1+\varepsilon} \right),$$

where the constant in the $O$-term depends on $r, \varepsilon$ and $\tilde{f}_1, P$.

Next,

$$N_{42} = -\frac{2 A_r r^5}{\pi (\log K)^2} \int_1^K P_2 \left( \frac{\log x_1}{\log K} \right) \int_1^{K/2} \tilde{f}_1 \left( \frac{\log K/(x_1 x_2)}{\log K} \right) \frac{(\log x_2)^{r^2-1}}{x_2}$$

$$\times \int_1^{K/2} \frac{\sin \left( \pi c \log x \right)}{\log x} \frac{\tilde{f}_1 \left( \frac{\log K/(x x_1 x_2)}{\log K} \right)}{x} dx_2 dx_1 + O \left( (\log T)^{r^2-1} \right).$$
By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

\[
N_{42} = -\frac{2A_r x^5}{\pi (\log K)^2} \int_1^K \frac{P_1 \left( \frac{\log x_1}{\log K} \right)}{x_1} \int_1^{K x_1} \tilde{f}_1 \left( \frac{\log K/\left( x_1 x_2 \right)}{\log K} \right) \left( \frac{\log x_2}{x_2} \right)^{\epsilon - 1} x_1 dx_1 \int_v^u \sin \left( \frac{\log x}{\log T} \right) \frac{P_1 \left( \frac{\log x}{\log K} \right)}{x} \tilde{f}_1 \left( \frac{\log K/\left( x x_2 \right)}{\log K} \right) dx \ dx_2 \ dx_1 + O \left( \left( \log T \right)^{\epsilon - 1} \right),
\]

(11)

where the constant in the $O$-term depends on $r$, $\epsilon$ and $\tilde{f}_1$, $P$.

Finally,

\[
N_{43} = -\frac{2A_r x^5}{\pi (\log K)^2} \int_1^K \frac{P_1 \left( \frac{\log x_1}{\log K} \right)}{x_1} \int_1^{K x_1} \tilde{f}_1 \left( \frac{\log K/\left( x_1 x_2 \right)}{\log K} \right) \left( \frac{\log x_2}{x_2} \right)^{\epsilon - 1} x_1 dx_1 \int_v^u \sin \left( \frac{\log x}{\log T} \right) \frac{P_1 \left( \frac{\log x}{\log K} \right)}{x} \tilde{f}_1 \left( \frac{\log K/\left( x x_2 \right)}{\log K} \right) dx \ dx_2 \ dx_1 + O \left( \left( \log T \right)^{\epsilon - 1} \right).
\]

(12)

where the constant in the $O$-term depends on $r$, $\epsilon$ and $\tilde{f}_1$, $P$.

Using $D_i$, $N_i$ given by (2)–(12) we can evaluate

\[
h(c) = c - \frac{N_1 + N_2 + N_3 + N_4}{D_1 + D_2 + D_3}.
\]

The results of our numerical calculations are summarized in Table 1.

### Table 1: Numerically optimal polynomials in the coefficients $\{a_k\}$, for which $h(c) > 1$.

| Degrees | Value of $c$ | Value of $r$ | Polynomials | Value of $c$ | Value of $r$ | Polynomials |
|---------|--------------|--------------|-------------|--------------|--------------|-------------|
| $f_1$   | $f_1$ | $P$ | $f_1$ | $f_1$ | $P$ |
| 3 1 2   | 0.515398    | 1.18    | 1.95 + 1.47x - 1.07x^2 - 0.29x^3 | -0.7 - 1.92x | $x^2$ |
| 3 1 3   | 0.515397    | 1.18    | 1.655 + 1.25x - 0.886x^2 - 0.25x^3 | -0.57 - 1.6x | $x^2 + 0.036x^3$ |
| 6 2 3   | 0.515396    | 1.18    | 1.78 + 1.017x + 0.2x^2 - 1.56x^3 + 0.45x^4 - 0.06x^5 + 0.05x^6 | -0.629 - 0.88x - 1.799x^2 | $x^2 + 0.083x^3$ |

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