Chern-Simons Currents and Chiral Fermions on the Lattice

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Abstract

We compute the Chern-Simons current induced by Wilson fermions on a $d = 2n + 1$ dimensional lattice, making use of a topological interpretation of the momentum space fermion propagator as a map from the torus to the sphere, $T^d \rightarrow S^d$. These mappings are shown to fall in different homotopy classes depending on the value of $m/r$, where $m$ is the fermion mass and $r$ is the Wilson coupling. As a result, the induced Chern-Simons term changes discontinuously at $d + 1$ different values for $m$, unlike in the continuum. This behavior is exactly what is required by the peculiar spectrum found for a recently proposed model of chiral lattice fermions as zeromodes bound to a domain wall.
Recently a method for simulating chiral fermions on the lattice was proposed by one of us [1]. The idea is to implement an odd dimensional $d = 2n + 1$ theory of Wilson fermions with a mass coupling of the fermions to a domain wall. The low energy effective theory consists of massless chiral fermions bound to the $d - 1$ dimensional domain wall, without there being “doubler” modes [2] from the Brillouin zone boundary. The anomalous Ward identities for these chiral fermions in the presence of background gauge fields can be directly measured on a finite lattice; this has already been done numerically for $d = 3$ and the results are in agreement with the continuum anomaly in $1 + 1$ dimensions [3]. In this system, the anomalous divergence of the $d - 1$ dimensional zeromode flavor currents is due to charge flow in the direction normal to the domain wall, even though there is a mass gap off the wall—an effect discussed by Callan and Harvey [4] for continuum fermions coupled to a domain wall. They pointed out that the Chern-Simons action induced by integrating out the heavy fermion modes [5], being proportional to $m/|m|$, has opposite signs on the two sides of the domain wall. This gives rise to a Chern-Simons current in the presence of background gauge fields with a nonzero divergence at the domain wall. Furthermore, the divergence exactly reproduces the even $(d - 1)$ dimensional anomaly for the single chiral fermion zeromode that is bound to the domain wall. In this Letter, we perform the Callan–Harvey (CH) analysis for the lattice theory in Euclidian space, where the zeromode spectrum is more complicated than in the continuum.

It is far from obvious that the lattice theory should follow the CH continuum analysis; after all, the coefficient of the Chern-Simons action gets $O(1)$ contributions from arbitrarily heavy fermion modes, and the heavy spectrum on the lattice looks nothing like in the continuum. In fact, we know the induced Chern-Simons operator must have a coefficient very different from the continuum result. While ref. [1] analyzed the spectrum of the theory for a Wilson coupling $r = 1$ and a domain wall height $0 < m_0 < 2$ and found a single chiral mode, a recent paper by Jansen and Schmaltz [6] analyzes the same model for general parameters and shows that the spectrum bound to the domain wall changes discontinuously with varying $m_0/r$ [7]. They find that for $2k < |m_0/r| < 2k + 2$—where $k$ is an integer in the range $0 \leq k \leq (d - 1)$—there are $\binom{d - 1}{k}$ chiral modes bound to the domain wall with chirality $(-1)^k \times \text{sign}(m_0)$; there are no chiral fermions for $|m_0/r| > 2d$. This is quite different than the continuum theory, for which there is a single chiral mode for any $m_0 \neq 0$. If the induced Chern-Simons action on the lattice is to correctly account for the anomalous divergences of the chiral fermion currents on the domain wall, then
evidently its coefficient must also depend discontinuously on \( m_0/r \) in a very particular way. We show in this Letter that that does indeed happen \[\text{[8]}\].

The Abelian Chern-Simons action in \( d = (2n + 1) \) continuous Euclidian dimensions is given by

\[
\Gamma^{(d)}_{CS} = \epsilon_{\alpha_1 \cdots \alpha_{2n+1}} \int d^{2n+1}x \ A_{\alpha_1} \partial_{\alpha_2} A_{\alpha_3} \cdots \partial_{\alpha_{2n}} A_{\alpha_{2n+1}}.
\] (1)

When a massive fermion is integrated out of the theory it generates a contribution to the effective action of the form \( S_{\text{eff}} = c_n \Gamma_{CS} \). The coefficient \( c_n \) can be computed by calculating the relevant portion of the graph in fig. [1]. This is true on the lattice as well in the weak field, long wavelength limit for the gauge fields. Denoting the fermion propagator and photon vertex as \( S(p) \) and \( i\Lambda_\mu(p, p') \) respectively, the graph of fig. [1] yields a value for \( c_n \) which may be expressed as

\[
c_n = \frac{i\epsilon_{\mu_1 \cdots \mu_2n+1} (n + 1)(2n + 1)!}{(p)} \left( \frac{\partial}{\partial (q_1)_{\beta_1}} \right) \cdots \left( \frac{\partial}{\partial (q_n)_{\beta_n}} \right) \times 
\int_{BZ} \frac{d^{2n+1}p}{(2\pi)^{2n+1}} \ Tr \left[ S(p) \Lambda_{\alpha_1}(p, p - q_1) S(p - q_1) \cdots \Lambda_{\alpha_{2n}}(p + q_{2n}, p) \right] \bigg|_{q_i = 0}.
\] (2)

The factor of \((n + 1)\) in equ. (2) is due to the symmetry factor of the graph fig. [1]; the factor of \( i \) is the product of \( i^{n+1} \) from the photon vertices and \((-i)^n\) from relating the derivatives in equ. (1) to powers of momenta. The \( p \)-integration is over the Brillouin zone of a hypercubic lattice with lattice spacing \( a = 1 \).

The integral (2) looks very difficult to compute on the lattice for arbitrary \( n \), as both \( S(p) \) and \( \Lambda_\mu(p, p') \) are in general rather complicated functions. It is made quite tractable, however, by exposing its topological properties. Gauge invariance implies that the photon coupling satisfies the Ward identity \[\text{[9]}\]

\[
\Lambda_\mu(p, p) = -i \frac{\partial}{\partial p_\mu} S^{-1}(p),
\] (3)

allowing \( c_n \) in equ.(2) to be reexpressed as

\[
c_n = \frac{(-i)^n \epsilon_{\mu_1 \cdots \mu_{2n+1}}}{(n + 1)(2n + 1)!} \int \frac{d^{2n+1}p}{(2\pi)^{2n+1}} \ Tr \left\{ \left[ S(p) \partial_{\mu_1} S(p)^{-1} \right] \cdots \left[ S(p) \partial_{\mu_{2n}} S(p)^{-1} \right] \right\}
\] (4)
where the differentiation $\partial_i$ is with respect to $p_i$. The fermion propagator may be written in the generic form

$$S^{-1}(p) = a(p) + i\vec{b}(p) \cdot \vec{\gamma}$$

$$= N(p) \left( \cos |\vec{\theta}(p)| + i\vec{\theta}(p) \cdot \vec{\gamma} \sin |\vec{\theta}(p)| \right)$$

$$\equiv N(p)V(p)$$

where $N(p) \equiv \sqrt{a^2 + \vec{b} \cdot \vec{b}},$ \( \vec{\theta}(p) = \hat{b} \arctan(|\vec{b}|/a), \) and $V(p)$ is seen to be a $2^n \times 2^n$ unitary matrix. Provided that $S^{-1}(p)$ doesn’t vanish for any $p$, equ. (4) is independent of $N(p)$, allowing $S^{-1}(p)$ and $S(p)$ to be replaced everywhere by $V(p)$ and $V^\dagger(p)$ respectively. This matrix $V(p)$ is seen to describe a mapping from momentum space—which on the hypercubic lattice is the torus $T^d$—onto the sphere $S^d$ defined by the vector $\vec{\theta}(p)$. The homotopy classes of such maps are identified by integers, and so the integral in equ. (4) has a simple topological interpretation: it is, up to a normalization constant, the winding number of the map described by the fermion propagator.

We now proceed to compute this winding number for the Wilson fermion propagator. Although we are ultimately interested in the effective action for lattice fermions in the presence of a domain wall, we can compute the effective action far from the mass defect on either side by treating the fermion mass as a constant $m$. Thus we can use the standard Wilson propagator

$$S^{-1}(p) = \sum_{\mu=1}^d i\gamma_{\mu} \sin p_{\mu} + m + r \sum_{\mu=1}^d \left[ \cos p_{\mu} - 1 \right].$$

Continuous changes in the mass and Wilson coupling, $m$ and $r$, cannot change the value of the winding number except at points for which $S^{-1}$ has a zero for some momentum $p$. Such singular points of the mapping occur only at momenta corresponding to the corners of the Brillouin zone, and then only for $m/r = 0, 2, \cdots, 2d$. The Chern-Simons coefficient $c_n$ as a function of $m/r$ must therefore be piecewise constant, changing only at these critical values. Furthermore, for fixed $r$, $V(p) \to \pm 1$ as $m \to \pm \infty$, so we may deduce that

$$c_n(m, r) = 0 \quad \text{for } m/r < 0, \quad m/r > 2d.$$  

To compute $c_n$ for $0 < m/r < 2d$, we need only evaluate the derivative of the integral in equ. (4) with respect to $m$ across the critical values $m/r = 0, 2, \cdots, 2d$. This task is simplified by the fact that $c_n(m+dm, r)$ is unchanged as one deforms $dm$ in a $p$-dependent
way so that $dm(p)$ vanishes for all $p$ except in the vicinity of the Brillouin zone corners; these points are denoted by $\vec{p} = \vec{\xi}_\alpha^{(k)}$, the $\alpha = 1, \ldots, \binom{d}{k}$ vectors with $k$ nonvanishing components equal to $\pi$. Therefore we need only evaluate the integrals in infinitesimal regions near the Brillouin zone corners $\vec{\xi}_\alpha^{(k)}$. After some algebra this yields \[ \frac{dc_n(m, r)}{dm} = \frac{i(-1)^n2^n}{(n+1)} \sum_{k, \alpha} (-1)^k \frac{d}{dm} \int d\Omega \frac{d^{2n+1}p}{(2\pi)^{2n+1}} \frac{(m-2rk)}{[|\vec{p}|^2+(m-2rk)^2]^{n+1}} \] (8)

where the integration $d\Omega$ is over the infinitesimal region $|\vec{p}| < \epsilon \to 0$. This may be trivially integrated with respect to $m$, given the boundary values (7), to yield the Chern-Simons coefficient for Wilson fermions:

$$c_n(m, r) = \frac{i(-1)^n}{2(n+1)(2\pi)^n n!} \sum_{k=0}^{d} (-1)^k \binom{d}{k} \frac{m-2rk}{|m-2rk|}.$$  (Wilson fermions) (9)

This is to be compared to the continuum result for a fermion of mass $m$, computed by inserting the continuum free propagator in equ. (2) and integrating over continuum momentum space:

$$c_n(m) = \frac{i(-1)^n}{2(n+1)(2\pi)^n n!} \frac{m}{|m|}.$$  (continuum result) (10)

Our computation is exact in the limit of small and adiabatic external gauge fields, even for $m$ of $O(1)$ in lattice units; it is readily generalized to non-Abelian gauge fields.

So far we have only considered lattice fermions with a constant mass; our primary interest though is in studying current flow in the presence of a domain wall mass term for the fermion. Following Callan and Harvey, we assume that sufficiently far away from the domain wall on either side, we can treat the mass as constant and compute the induced current by varying the Chern-Simons action,

$$J^CS_\mu = \frac{\delta S_{eff}}{\delta A_\mu} = \frac{(n+1)}{2^n} c_n(m(s), r) \epsilon_{\mu\alpha_1\alpha_2\ldots\alpha_{2n+1}} F_{\alpha_1\alpha_2} F_{\alpha_2\ldots\alpha_{2n-1}2n}.$$  (11)

where the factor of $2^n$ arises from replacing $\partial A$ by the field strength $F$. Since $c_n$ depends on the mass $m(s)$ which is space dependent, $J^CS_\mu$ has a nonzero divergence corresponding to current flow normal to the domain wall. In their continuum analysis, Callan and Harvey...
pointed out that with $c_n$ given by equ. (10), there was equal current flow toward the wall from each side that exactly compensated for the anomalous divergence of the chiral fermion current along the wall’s surface. For lattice fermions in the presence of the domain wall, we must use the value (9) for $c_n(m, r)$. Thus on the side of the domain wall for which $m(s)/r$ is negative, the Chern-Simons current vanishes. On the other side, where $m(s)/r \to |m_0/r|$, the current either vanishes if $|m_0/r| > 2(d + 1)$, or else, for $2\ell < |m_0/r| < 2(\ell + 1)$, the current is given by

$$J_{CS}^\mu({\text{continuum}}) = \sum_{k=0}^{\ell} (-1)^k \binom{d}{k} - \sum_{k=\ell+1}^{d} (-1)^k \binom{d}{k}$$

For example, for $\ell = 0$ (a domain wall height satisfying $0 < |m_0/r| < 2$) the lattice current vanishes on one side of the wall, while having twice the continuum magnitude on the other side; in this case the total divergence of the continuum and lattice currents are the same. In general the lattice Chern-Simons current vanishes on the side of the domain wall for which $m(s)/r$ is negative, while on the other side has the correct magnitude to compensate for the anomaly of not one positive chirality zeromode at the domain wall, but rather $\binom{d-1}{\ell}$ zeromodes with chirality $(-1)^\ell \times \text{sign}(m_0/r)$. This result agrees precisely with the zeromode spectrum bound to the lattice domain wall found by Jansen and Schmaltz \[6\]. It also agrees with a numerical computation of the Chern-Simons current on a $d = 3$ finite lattice that we have performed (see ref. \[3\]), which exhibits the behaviour peculiar to the lattice that the Chern-Simons current flows on only one side of the domain wall, as well as the discontinuous dependence of its magnitude on $m_0/r$.

Our analysis confirms that the model of ref. \[1\] correctly reproduces the continuum anomaly for chiral fermions on a finite lattice in the presence of weak gauge fields, even for a domain wall height $m_0 = \mathcal{O}(1)$ in lattice units.
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References

[1] D.B. Kaplan, *A Method for Simulating Chiral Fermions on the Lattice*, UCSD-PTH-92-16, to appear in Phys. Lett. B.

[2] K.G. Wilson, in “New Phenomena in Subnuclear Physics”, ed. A. Zichichi (Plenum, New York, 1977) (Erice, 1975); L.H. Karsten and J. Smit, Nucl. Phys. B183, 103 (1981).

[3] K. Jansen, *Chiral Fermions and Anomalies on a Finite Lattice*, UCSD-PTH-92-18, (hep-lat/9209002) to appear in Phys. Lett. B.

[4] C.G. Callan, Jr., and J.A. Harvey, Nucl. Phys. B250, 427 (1985).

[5] J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47, 986 (1981); S. Deser, R. Jackiw and S. Templeton, Ann. Phys. 140, 372 (1982); *ibid.* 185, 406E (1988); N. Redlich, Phys. Rev. Lett. 52, 18 (1984); A.J. Niemi and G. Semenoff, Phys. Rev. Lett. 51, 2077 (1984).

[6] K. Jansen and M. Schmaltz, *Critical Momenta of Lattice Chiral Fermions*, UCSD-PTH-92-29, submitted to Physics Letters B.

[7] All dimensionful parameters are given in lattice units. By a domain wall of height \( m_0 \) we mean a spatially dependent mass term \( m(s) \to \pm m_0 \) as \( s \to \pm \infty \), where \( s \) is the coordinate transverse to the domain wall.

[8] The dependence of the induced Chern-Simons action on the Wilson coupling \( r \) has been previously discussed for three dimensions in the continuum limit (spatially constant \( m \to 0 \)) in H. So, Prog. Theor. Phys. 73, 528 (1985); 74, 585 (1985); and for \( |m| < 1 \) by A. Coste and M. Lüscher, Nucl. Phys. B323, 631 (1989). Some of the techniques used in this Letter are similar to those found in the latter work.

[9] We expand the lattice gauge field as \( U_\mu(x) = 1 - iA_\mu(x) + \ldots \).

[10] We work in \( d = (2n+1) \) dimensions, and our gamma matrix conventions are \( \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}; \gamma_\mu = \gamma_\mu^\dagger; (\gamma_1 \cdots \gamma_d) = i^n \).
Figure Captions

Fig. 1. The Feynman diagram in $2n + 1$ dimensions contributing to the induced Chern-Simons action for Abelian gauge fields; $\sum_{i=1}^{n+1} q_i = 0$. Graphs with multiple photon vertices peculiar to the lattice do not contribute, having the wrong Lorentz structure.