Dimensionally continued multi-loop gauge theory

D. J. Broadhurst

Physics Department, Open University, Milton Keynes MK7 6AA, UK

http://physics.open.ac.uk/~dbroadhu
D.Broadhurst@open.ac.uk

Abstract A dimensionally continued background-field method makes the rationality of the 4-loop quenched QED beta function far more reasonable than had previously appeared. After 33 years of quest, dating from Rosner’s discovery of 3-loop rationality, one finally sees cancellation of zeta values by the trace structure of individual diagrams. At 4-loops, diagram-by-diagram cancellation of $\zeta(5)$ does not even rely on the values of integrals at $d = 4$. Rather, it is a property of the rational functions of $d$ that multiply elements of the full $d$-dimensional basis. We prove a lemma: the basis consists of slices of wheels. We explain the previously mysterious suppression of $\pi^4$ in massless gauge theory. The 4-loop QED result $\beta_4 = -46$ is obtained by setting $d = 4$ in a precisely defined rational polynomial of $d$, with degree 11. The other 5 rational functions vanish at $d = 4$. 
1. Introduction

Connes and Kreimer [1] have recently shown that dimensional regularization of Feynman diagrams is underwritten by the uniqueness of the solution of Hilbert’s 21st problem [2]. Here, we celebrate this welcome legitimization of current practice in perturbative quantum field theory, by finding a rational function of the spacetime dimension $d$ that yields, at $d = 4$, the 4-loop term [3, 4] in the beta function of quenched QED. For the first time, suppression of zeta values is seen diagram by diagram. We prove the suppression of $\pi^4$ in the $\overline{\text{MS}}$-renormalized 3-loop single-scale Green functions of any massless gauge theory.

There are 6 methods of calculating the 3-loop term $\beta_3 = -2$ in the beta function of quenched QED, with a coupling $a := \alpha/4\pi$. As described in [3], they are as follows.

M1: Dyson-Schwinger skeleton expansion [4, 5, 6, 7, 8].
M2: Integration by parts of massive bubble diagrams [9, 10].
M3: Integration by parts of massless two-point diagrams [11].
M4: Infrared rearrangement of massless bubble diagrams [3, 12, 13, 14].
M5: Propagation in a background field [15, 16, 17].
M6: Crewther connection to deep-inelastic processes [18].

For the 4-loop term, one does not know how to use M3 directly: there is as yet no algorithm for 4-loop 2-point functions. The historical progression through other methods was as follows. Method M4 first gave $\beta_4 = -46$ in [3]. Then it was noted in [18] that the result was consistent with deep-inelastic results [19] by virtue of the exactness of the Crewther [20] connection M6 in the quenched abelian case. Progress with 4-loop massive bubbles in M2 led to the 4-loop beta function of a general [14] gauge theory, confirming the particular case of QED. Recently [4] we used the Dyson-Schwinger method M1, which was shown to be very efficient. However, in none of these 4 analyses does one gain an understanding of how the rationality of $\beta_4 = -46$ comes about; all 4 involve intricate cancellations of zeta values between diagrams with quite different momentum flows.

The attentive reader will have noticed that one stone lay unturned: the background-field method M5. Very recently we attempted it, and found it to be wonderfully user-friendly: 8 lines of code suffice. This is because it gives $\beta_4$ directly in terms of 8 three-loop diagrams, whereas in [3, 4] a reduction to 3 loops was achieved indirectly, via nullification of four-loop diagrams, which are much more numerous. Moreover, we find that in a background field the cancellation of zeta values is much less obscure.

Our method follows immediately from the telling observation [21] that $d(\beta(a)/a)/da = \sum_{n>1}(n - 1)\beta_n a^{n-2}$ is given by the radiative corrections to the photon self-energy of massless quenched QED, in a background field. Consider the momentum-space correlator

$$i \int dx \, e^{ik\cdot x} \langle 0|T(J_\mu(x)J_\nu(0))|0\rangle = (k_\mu k_\nu - k^2 g_\mu\nu) \left\{ \Pi_0(k^2) + \Pi_2(k^2)F^2 + O(F^4) \right\}$$  \hspace{1cm} (2)
of the electromagnetic current \( J_\mu := \overline{\psi} \gamma_\mu \psi \) in a background electromagnetic field \( F_{\mu\nu} \), with \( F^2 := \langle F_{\mu\nu} F^{\mu\nu} \rangle \). After struggling \[\text{[4]}\] to decode the rationality of \( \beta_4 \) via the 4-loop contribution to \( \Pi_0 \), I judged it more prudent to tackle the finite 3-loop contribution in

\[
\Pi_2(k^2) = \frac{\beta_3 a}{6k^4} \left\{ 1 + \left( \frac{2\beta_3}{\beta_2} \right) a + \left( \frac{3\beta_4}{\beta_2} \right) a^2 + O(a^3) \right\}
\]  

(3)

2. Three-loop beta function

First we dimensionally continue Rosner’s result, \( 2\beta_3/\beta_2 = -1 \). Consider the \( O(F^2) \) term of (2) in \( d := 4 - 2\epsilon \) dimensions, with a dimensionless coupling \( \overline{a} := (4\pi)^\epsilon g(k^2)a \), where

\[
g(k^2) := \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{k^{2\epsilon}\Gamma(1 - 2\epsilon)}
\]

(4)

absorbs the \( \Gamma \) functions and momentum dependence of one-loop integration. Let the \( d \)-dimensional form of (3) be written as

\[
\Pi_2(k^2) = \frac{\beta_3 a}{2(d-1)k^4} \left\{ 1 + \left( \frac{2\beta_3}{\beta_2} \right) \overline{a} + \left( \frac{3\beta_4}{\beta_2} \right) \overline{a}^2 + O(\overline{a}^3) \right\}
\]

(5)

with bars denoting analytic continuation in \( d \). Then the one-loop self-energy yields

\[
\overline{\beta}_2 = \frac{(6 - d)(d - 2)}{d} \text{Tr}(1)
\]

(6)

where \( \text{Tr}(1) \) is whatever one chooses to take for the trace of the unit matrix in the Clifford algebra of \( d \) dimensions. At \( d = 4 \), where certainly \( \text{Tr}(1) = 4 \), we get \( \overline{\beta}_2 = 4 \).

Now consider the \( d \)-dimensional analysis for \( \overline{\beta}_3 \). Integration by parts \[\text{[11]}\] gives the first radiative correction in (5) in terms of \( \Gamma \) functions:

\[
\overline{\beta}_3 := \frac{(d - 3)k^4}{6(g(k^2)\pi^{d/2})} \int \frac{dP_1dP_2dP_3 \ \delta(k - P_1)}{P_1^2P_2^2P_3^2(P_1 - P_2)^2(P_2 - P_3)^2(P_3 - P_1)^2}
\]

(8)

is a slice, via a \( \delta \) function, of the wheel with three spokes: the tetrahedron. It is a template for our later construction of a 3-loop basis. At 2 loops, one easily \[\text{[11]}\] evaluates

\[
\overline{\beta}_3 = \frac{1}{6\epsilon^3} \left[ 1 - \sec(\pi\epsilon)\Gamma(1 - 2\epsilon) \right] = \zeta(3) + \frac{3}{4}\zeta(4)\epsilon + 7\zeta(5)\epsilon^2 + O(\epsilon^3)
\]

(9)
which is the sole origin of $\zeta(3) := \sum_{n>0} 1/n^3$ in 2-loop 2-point functions at $d = 4$. Exact $d$-dimensional analysis gives the rational functions

$$R_{2,0} = -4 \frac{d^6 - 37d^5 + 554d^4 - 4280d^3 + 17826d^2 - 37728d + 31608}{(d-1)(d-3)^2(d-6)^3(d-8)}$$  (10)

$$R_{2,3} = 6 \frac{(d-2)(d-4)(d-5)(d^4 - 25d^3 + 230d^2 - 920d + 1376)}{(d-1)(d-6)^3(d-8)}$$  (11)

This massless abelian analysis is far simpler than our massive non-abelian work [17, 22]. The rationality of $\beta_3 = -2$ is seen in the vanishing of (14) at $d = 4$; the precise value comes from setting $d = 4$ in (13), which gives $2\beta_3/\beta_2 = -1$. Before proceeding to $\beta_4$, we prove a lemma, which lies at the heart of the method and has wider import.

3. Suppression of $\pi^4$ in massless 3-loop gauge theory

It is well understood [23] why $\zeta(3)$ is the first irrational number to appear in single-scale processes, renormalized in the $\overline{\text{MS}}$ scheme. The suppression of a single power of $\pi^2$ is a property of any massless Lorentz-covariant theory: every bare 2-loop 2-point diagram, with spacelike momentum $k$, gives a result of the form $(R_{2,0} + R_{2,3}\zeta_3)\pi^2$, where $R_{2,0}$ may have $1/\varepsilon^2$ and $1/\varepsilon$ singularities, while $R_{2,3}$ is regular at $d = 4$. In heavy-quark effective theory, where the worldline of the heavy quark breaks Lorentz symmetry, one finds 2-loop combinations of $\Gamma$ functions that lack [24] the $\pi^2$ suppression of (9).

It is the experience of several workers that if – and usually not until – one combines the 3-loop diagrams specified by a massless gauge theory, $\pi^4$ vanishes as well [14, 18]. Multi-loop colleagues have asked me why this happens. The answer lies in the output format of the program slicer [25], which gives the results of subsequent sections. Following the example of [8], we define 4 three-loop integrals that depend only on $d$:

$$\bar{\zeta}_{3,5} := \frac{(d-3)k^4}{20(g(k^2)^{\pi d/2})^3} \int \frac{dP_1dP_2dP_3dP_4}{P_1^3P_2^3P_3^3P_4^3(P_1-P_2)^2(P_2-P_3)^2(P_3-P_4)^2(P_4-P_1)^2}$$  (12)

with 4 distinct slices of the 4-spoke wheel, encoded by $S = L, M, N, Q$, corresponding to $P_L := P_1-P_3$, $P_M := P_1$, $P_N := P_1-P_2+P_3-P_4$, $P_Q := P_1-P_2$, for the external momentum, $k$. Then each integral evaluates to $\zeta(5)$ at $d = 4$, since it comes from slicing the wheel with 4 spokes and we have proved [26] that the wheel with $n + 1$ spokes gives $\binom{2n}{n}\zeta(2n - 1)$, for $n > 1$. The coding was suggested by letters in [11]: the L (ladder) slice is at the 4-point hub, in the $s$-channel; M (Mercedes-Benz) is at the rim; N (non-planar) is at the hub, in the $u$-channel; Q (?) is at a spoke. Remarkably, this completes a basis.

**Lemma:** A bare 3-loop 2-point diagram with external momentum $k$ gives

$$\left\{ R_{3,0} + R_{3,3}\zeta_3 + \sum_{S=L,M,N,Q} R_{3,S}\tilde{\zeta}_{3,5,S} \right\}\pi^3$$  (13)

with coefficients $\{ R_{3,S} | S = 0, 3, L, M, N, Q \}$ that are rational functions of $d$. 

3
Proof: This is an extension of the fine analysis of Chetyrkin and Tkachov [11]. Consider the 5 one-particle-irreducible 4-loop bubble diagrams of Fig. 1. Generate all tadpole-free trivalent 3-loop 2-point diagrams by slicing these, to obtain the 12 distinct cases of Fig 2, where each slice is marked. Whatever the numerators, and whatever the integer powers of the propagators, integration by parts [11] allows one to eliminate all and only those diagrams that contain at least one unsliced triangle. Thus we may eliminate 5 of the 12 cases, namely C2, C3, C4, D1, D2. The final observation is that B3, the sole case with a sliced chord in Fig. 2, is rationally related to B2. The basic one-loop integral in $d$ dimensions is

$$G(a, b) := \frac{\Gamma(a + b - d/2) \Gamma(d/2 - a) \Gamma(d/2 - b)}{\Gamma(a) \Gamma(b) \Gamma(d - a - b)}$$

(14)

with propagators raised to powers $a$ and $b$. Let $\Phi(X)$ be the representative of class $X$ in $d$-dimensional $\phi^3$ theory. Using only $\Gamma(z + 1) = z\Gamma(z)$, we obtain

$$\frac{\Phi(B3)}{\Phi(B2)} = \frac{G(1, 1)G(1 + \varepsilon, 1 + \varepsilon)}{G(1, 1 + \varepsilon)G(1, 1 + 2\varepsilon)} = \frac{3d - 10}{d - 3}$$

(15)

Then (13) is a basis for the 6 remaining cases, since (8,12) are independent. $\square$.

Two comments are in order. First, the factor $d - 3$ in (12) ensures that the $\varepsilon$ expansions of $\zeta_{5,S}$ are all pure: at order $\varepsilon^n$ one encounters multiple zeta values, exclusively of weight $n + 5$. All mixing derives from the rational coefficients, of which only $R_{3,0}$ and $R_{3,3}$ may be singular at $d = 4$. Secondly, there are two combinations of $\{\zeta_{5,S} \mid S = L, M, Q\}$ that may be reduced to $\Gamma$ functions. No purpose is served by making such reductions: one would need 5th order Taylor series for those $\Gamma$ functions to arrive back at a result for $d = 4$ that is immediately available from (13). Moreover, such reduction would obscure our physical conclusion.

Corollary: In massless gauge theory, $\overline{\text{MS}}$-renormalized 3-loop single-scale Green functions do not involve $\pi^4$.

Proof: The lemma shows that $\zeta(4) = \pi^4/90$ may arise only from (1). This would occur only if $R_{3,3}$ is singular. But such a singularity, in the sum of diagrams, would require a counterterm $\zeta(3)/\varepsilon$ in the coupling. It is proven that the 3-loop $\beta$ function of any gauge theory is rational [28], hence no such counterterm is available. $\square$

Now one sees why cancellations of $\pi^4$ were regarded as surprising in computations such as [14, 19]. The bare $\zeta(4)$ terms generated by Mincer [29] appeared to have several sources: Taylor expansions of $\Gamma$ functions from diagrams in classes A1, A2, B1, B2, B3. By contrast, Slicer [25] outputs the 6 exact $d$-dimensional rational functions of the lemma. In our approach, one is bound to find that $R_{3,3}$ is regular in the sum of diagrams of a massless gauge theory; Slicer encodes the suppression of $\pi^4$, ab initio.

Note, however, that the 3-loop anomalous mass dimension [30] involves $\zeta(3)$. Thus if one expands in the fermion mass, $\pi^4$ may emerge. Indeed, one finds $\zeta(4)$ in non-abelian
terms of the 4-loop anomalous mass dimension \cite{31, 32}. Its absence from the 4-loop beta function is guaranteed by the lemma and serves as a check of the full result \cite{10}.

4. Four-loop beta function

For \( \beta_4 \), we need the finite 3-loop term in (3). Instead of the 7 \( \times \) 5 \( \times \) 3 = 105 four-loop diagrams of [3], there are only 5 \( \times \) 3 = 15 three-loop diagrams. These are reduced to 8, by symmetries. Each contains a single fermion loop: a hamiltonian circuit with 6 vertices. Consider a bare fermion propagator, with momentum \( p \). Without a background field, it would simply be \( S(p) = i/p \). To take account of the abelian background field, \( F_{\mu\nu} \), one replaces this by

\[
S(p + iD) = S(p) - S(p) \slashed{D} S(p) + S(p) \slashed{D} S(p) \slashed{D} S(p) + O(D^3)
\]  

(16)

where \( D_\mu \) is an operator in a Fock-Feynman-Schwinger formalism \cite{14} that is oblivious to the momentum integrations, feeling only the order of \( \gamma \) matrices. All one needs to know is that \([D_\mu, D_\nu] = -ieF_{\mu\nu}\), since \( D_\mu \) acts as a gauge-covariant derivative in the external configuration space. For a modern review of the non-abelian case, see \cite{33}. Here, we need only obtain the coefficient of \( F^2 := \langle F_{\mu\nu}F^{\mu\nu} \rangle \), using

\[
\langle D_\alpha D_\beta D_\mu D_\nu \rangle = (g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\mu\nu}) \frac{e^2 F^2}{2d(d-1)}
\]

(17)

for the expectation value of 4 external derivations. This is contracted with all ways of making 4 ordered insertions of gamma matrices in the 6 fermion propagators on a hamiltonian circuit that begins and ends at an external vertex of the 3-loop Feynman diagram. The tensor in (17) implies that the second-order expansion (16) suffices for any single propagator: insertion of more than 2 gamma matrices in any one propagator leads to a vanishing contraction. Thus no infrared divergence is produced. Simple counting shows that each diagram produces 180 traces, containing 20 gamma matrices.

This astoundingly user-friendly method is CPU-intensive. In the Dyson-Schwinger method \cite{4}, it was possible to take traces before double differentiation w.r.t. an external photon momentum, thus limiting the traces to 16 gamma matrices. Moreover, for the most demanding integrals one needed only traces taken at \( d = 4 \). Now, the exact handling of two non-commutative algebras – Dirac’s gamma matrices and Schwinger’s covariant derivatives – requires more machine time, while the physicist relaxes. Within hours of reading \cite{1}, I had typed 8 lines into SLICER, one for each of the 8 diagrams, added a short procedure to generate the 8 \( \times \) 180 = 1440 traces, and farmed the problem out, using a cluster of DecAlpha machines. Since SLICER, using REDUCE, is wildly uncompetitive with Jos Vermaseren’s lightning-fast implementation of Mincer in FFORM, it was 2 days later when the answer \(-46\) appeared on screen. However, the object was not speed. Rather it was a better understanding of cancellation of zeta values, which was mysterious in the faster Dyson-Schwinger method \cite{4}. 

5
5. Fock-Feynman-Schwinger anatomy

The Dyson-Schwinger anatomy of $\beta_3$ exhibits cancellation of $\zeta(3)$ between diagrams in Landau gauge [5], in Feynman gauge [4, 21], and indeed in any gauge [6]. In the background-field method, $\beta_3$ comes from two 2-loop diagrams, each of which is free of $\zeta(3)$, as $d \to 4$, in any combination of internal and external gauges. We used a gauge $(q^2 g_{\mu\nu} + (\xi_{\text{int}} - 1)q_{\mu}q_{\nu})/q^4$, for the internal photon propagator, and contracted (2) with $k^2 g_{\mu\nu} + (\xi_{\text{ext}} - 1)k_{\mu}k_{\nu}$, where $k$ is the external photon momentum. As $\varepsilon \to 0$, we found

$$C_3(\text{PE}) = -\frac{1}{2} + \xi_{\text{int}} \left( \frac{2}{\varepsilon} + 1 \right) - \xi_{\text{ext}} \left( \frac{1}{\varepsilon} + 2 \right) - \xi_{\text{int}} \xi_{\text{ext}}$$

(18)

$$C_3(\text{DF}) = -\frac{1}{2} - \xi_{\text{int}} \left( \frac{2}{\varepsilon} + 1 \right) + \xi_{\text{ext}} \left( \frac{1}{\varepsilon} + 2 \right) + \xi_{\text{int}} \xi_{\text{ext}}$$

(19)

$$2\beta_3/\beta_2 = -1$$

(20)

where PE and DF identify the photon-exchange and dressed-fermion contributions. There is no sign of $\zeta(3)$ in either diagram.

This was no accident, as witnessed by the 4-loop result, obtained in Feynman gauge, for the sake of economy. As $\varepsilon \to 0$, the contributions and total for $3\beta_4/\beta_2$ are

$$C_4(A1) = \frac{2}{3\varepsilon^2} - \frac{7}{3\varepsilon} - \frac{58}{3}$$

(21)

$$C_4(B2) = \frac{2}{3\varepsilon^2} - \frac{4}{3\varepsilon} + \frac{5}{3}$$

(22)

$$C_4(B3) = -\frac{3}{2\varepsilon} - \frac{21}{4}$$

(23)

$$C_4(C2) = -64\zeta(3) - \frac{4}{3\varepsilon^2} + \frac{26}{3\varepsilon} + 79$$

(24)

$$C_4(C4) = -\frac{32}{3}\zeta(3) - \frac{4}{3\varepsilon^2} + \frac{6}{\varepsilon} + \frac{209}{9}$$

(25)

$$C_4(D1) = \frac{320}{3}\zeta(3) + \frac{4}{3\varepsilon^2} - \frac{9}{\varepsilon} - \frac{2695}{18}$$

(26)

$$C_4(D2) = 64\zeta(3) - \frac{9}{2\varepsilon} - \frac{1177}{12}$$

(27)

$$C_4(E1) = -96\zeta(3) + \frac{4}{\varepsilon} + 134$$

(28)

$$3\beta_4/\beta_2 = -\frac{69}{2}$$

(29)

with diagrams identified by Fig. 2. There is no sign of $\zeta(5)$ in any diagram. This, again, is radically different from the Dyson-Schwinger method.

Even more impressive is the distribution of $\zeta(5)$ between the 4 distinct slices in (12). By construction, slicer is oblivious to the contingency that these integrals happen to be equal at $d = 4$. Thus it records that each is cancelled separately, in each diagram, by
the Dirac-Schwinger traces that produce 4 exact rational functions of $d$, each vanishing at $d = 4$. Thus, in the background-field method, the absence of $\zeta(5)$ relies neither on conspiracies between different diagrams nor on any happenstance of 4-dimensional analysis. This is a reasonable type of rationality, that one may realistically hope to understand better, by essentially combinatoric methods.

It will be noted that the subleading zeta value, $\zeta(3)$, is seen in results (24–28), for individual diagrams that are slices of the C, D, E topologies of Fig. 1. However, it should also be noted that this subleading irrational occurs only at subleading order in $\varepsilon$. Topologies C and D produce $\zeta(3)/\varepsilon$ singularities, at the level of individual Dirac-Schwinger traces. These conspicuously cancel, diagram by diagram. Thus background-field trace algebra accounts for 6 of the 7 features of rationality up to 4 loops: the cancellation of $\zeta(3)$ at 3 loops; of all 4 species of $\zeta(5)$, separately, at 4-loops; of $\zeta(3)/\varepsilon$ at 4 loops. Only the final cancellation – subleading irrational, subleading in $\varepsilon$ – entails conspiracy between Feynman-gauge diagrams. A MINCER analysis for all $\xi_{\text{int}}$ and $\xi_{\text{ext}}$ would be fascinating.

6. Exercise and conclusion

We have completed an instructive exercise in dimensional continuation of a finite quantity.

1. Evaluate the exact $d$-dimensional 3-loop radiative correction in (1). By the lemma of section 3, this amounts to finding 6 precisely defined rational functions of $d$ in

$$
\frac{3\beta_4}{\beta_2} = R_{3,0} + R_{3,3} + \sum_{S=L,M,N,Q} R_{3,S} \zeta_{S,5} \tag{30}
$$

with wheel-slice basis (8,12). Forget $\pi^4$; in massless gauge theory it never happens.

2. Verify that $\{R_{3,S} | S \neq 0\}$ contain the factor $d-4$. Dissect this, diagram by diagram. Conclusion: rationality appears far more reasonable than in any other method.

3. Find the numerator and denominator of $R_{3,0} = N(d)/D(d)$. Solution:

$$
N(d) = 1215d^1 - 53433d^10 + 1072059d^9 - 12995191d^8 + 105924166d^7 \\
- 609433848d^6 + 2520429944d^5 - 7469717936d^4 + 15495188128d^3 \\
- 21364053504d^2 + 17580978560d - 6532684800 \tag{31}
$$

$$
D(d) = 2(d-1)(d-3)^3(d-5)^2(d-6)^3(3d-8)(3d-10)^2 \tag{32}
$$

Hence, in 4-loop quenched QED, $\beta_4 = N(4)/2^83^2$. Check that the answer is $-46$.

I conclude that the key to Jonathan Rosner’s fine puzzle [5] was given by Marshall Baker and Kenneth Johnson, in Eq (3.3) of [21]. Noting the profound work of Alain Connes and Dirk Kreimer [1], one arrives at the nub of the rationality of quenched QED: dimensional continuation of the derivative of the scheme-independent single-fermion-loop Gell-Mann–Low function, via the Fock–Feynman–Schwinger formalism (16,17). It remains to be seen whether this can tell us what comes after $\beta_4 = -46$. Hope has risen.
Acknowledgements: Encouragement from Marshall Baker and Jonathan Rosner kept me believing that there was a key to be found. Recollection of enjoyable work with Sotos Generalis and Andrey Grozin reminded me of (17). Jos Vermaseren encouraged inclusion of my explanation of the suppression of $\pi^4$. Alain Connes and Dirk Kreimer provided the vital impetus for an exact $d$-dimensional analysis, by showing me an early version of [1].

References

[1] A. Connes, D. Kreimer, J. High Energy Phys. 9 (1999) 024, in press. hep-th/9909126

[2] D. Hilbert, Lecture on Mathematical Problems, International Congress of Mathematicians (Paris, 1900). http://aleph0.clarku.edu/~djoyce/hilbert

[3] S.G. Gorishny, A.L. Kataev, S.A. Larin, L.R. Surguladze, Phys. Lett. B256 (1991) 81.

[4] D.J. Broadhurst, to appear in Phys. Lett. B. hep-ph/9909336

[5] J.L. Rosner, Phys. Rev. Lett. 17 (1966) 1190; Ann. Phys. 44 (1967) 11.

[6] D.J. Broadhurst, R. Delbourgo, D. Kreimer, Phys. Lett. B366 (1996) 421. hep-ph/9509296

[7] K. Johnson, R. Willey, M. Baker, Phys. Rev. 163 (1967) 1699.

[8] E. de Rafael, J.L. Rosner, Ann. Phys. 82 (1974) 369.

[9] D.J. Broadhurst, Zeit. Phys. C54 (1992) 599.

[10] T. van Ritbergen, J.A.M. Vermaseren, S.A. Larin, Phys. Lett. B400 (1997) 379. hep-ph/9701390

[11] K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B192 (1981) 159; F.V. Tkachov, Phys. Lett. B100 (1981) 65.

[12] A.A. Vladimirov, Teor. Mat. Fiz. 43 (1980) 210.

[13] K.G. Chetyrkin, F.V. Tkachov, Phys. Lett. B114 (1982) 340.

[14] S.G. Gorishny, A.L. Kataev, S.A. Larin, Phys. Lett. B259 (1991) 144.

[15] V.A. Fock, Sov. Phys. 12 (1937) 404; R.P. Feynman, Phys. Rev. 80 (1950) 440; J. Schwinger, Phys. Rev. 82 (1951) 664.
[16] K.G. Chetyrkin, S.G. Gorishny, V.P. Spiridonov, Phys. Lett. B160 (1985) 149; G.T. Loladze, L.R. Surguladze, F.V. Tkachov, Phys. Lett. B162 (1985) 363; L.R. Surguladze, F.V. Tkachov, Nucl. Phys. B331 (1990) 35.

[17] D.J. Broadhurst, P.A. Baikov, V.A. Ilyin, J. Fleischer, O.V. Tarasov, V.A. Smirnov, Phys. Lett. B329 (1994) 103. hep-ph/9403274

[18] D.J. Broadhurst, A.L. Kataev, Phys. Lett. B315 (1993) 179. hep-ph/9308274

[19] S.A. Larin, J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345.

[20] R.J. Crewther, Phys. Rev. Lett. 28 (1972) 1421.

[21] K. Johnson, M. Baker, Phys. Rev. D8 (1973) 1110.

[22] D.J. Broadhurst, J. Fleischer, O.V. Tarasov, Zeit. Phys. C60 (1993) 287. hep-ph/9304303

[23] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Phys. Lett. B85 (1979) 277; Nucl. Phys. B174 (1980) 345.

[24] D.J. Broadhurst, A.G. Grozin, Phys. Lett. B267 (1991) 105; B274 (1992) 421; Phys. Rev. D52 (1995) 4082. hep-ph/9410240

[25] D.J. Broadhurst, A.L. Kataev, O.V. Tarasov, Phys. Lett. B298 (1993) 445. hep-ph/9210255

[26] D.J. Broadhurst, Phys. Lett. B164 (1985) 356.

[27] D.J. Broadhurst, J.A. Gracey, D. Kreimer, Zeit. Phys. C75 (1997) 559. hep-th/9607174

[28] O.V. Tarasov, A.A. Vladimirov, A.Y. Zharkov, Phys. Lett. B93 (1980) 429.

[29] S.G. Gorishny, S.A. Larin, L.R. Surguladze, F.V. Tkachov, Comput. Phys. Commun. 55 (1989) 381.

[30] O.V. Tarasov, JINR P2-82-900 (1982); S.A. Larin, NIKHEF-H/92-18, in Proc. int. Baksan School on Particles and Cosmology, ed. E.N. Alexeev, V.A. Matveev, Kh.S. Nirov, V.A. Rubakov, (World Scientific, 1994). hep-ph/9302240

[31] K.G. Chetyrkin, Phys. Lett. B404 (1997) 161. hep-ph/9703278

[32] J.A.M. Vermaseren, S.A. Larin, T. van Ritbergen, Phys. Lett. B405 (1997) 327. hep-ph/9703284

[33] A.G. Grozin, Int. J. Mod. Phys. A10 (1995) 3497. hep-ph/9412238
Fig 1: The 5 trivalent four-loop bubble diagrams, presented as chord diagrams

A
B
C
D
E

Fig 2: The 12 trivalent 3-loop 2-point diagrams, from slices of Fig. 1

A1
A2
B1
B2
B3
C1
C2
C3
C4
D1
D2
E1