Models with non-Hermitian Hamiltonian and optical theorem

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Abstract

The applicability of the optical theorem in the models with the non-Hermitian Hamiltonian is studied. By way of example we consider the \( n\bar{n} \) transition in a medium followed by annihilation. It is shown that an application of optical theorem for the non-unitary \( S \)-matrix leads to the qualitative error in the result. The lower limit on the free-space \( n\bar{n} \) oscillation time \( \tau \) calculated by means of the model with Hermitian Hamiltonian lies in the range \( 10^{16} \text{ yr} > \tau > 1.2 \cdot 10^9 \text{ s} \).

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1 Introduction

The importance of the unitarity condition is well known. Optical theorem should be applied for the unitary $S$-matrix. It is frequently used for the non-Hermitian Hamiltonians as well. In this case the $S$-matrix should be unitarized. However, this requirement breaks down for a number of well-known models. In this paper the possible consequences are studied by the example of $n\bar{n}$ transitions [1-3] in a medium followed by annihilation. It is shown that an application of optical theorem for the non-unitary $S$-matrix leads to the qualitative error in the result.

The second purpose of this paper is to calculate the lower limit on the free-space $n\bar{n}$ oscillation time $\tau$ by means of the model with the Hermitian Hamiltonian. The matter is that in the standard calculations (see [4-9], for example) the optical theorem is applied in the models with the essentially non-unitary $S$-matrix.

We consider the simple model of the $n\bar{n}$ transition in a medium followed by annihilation (see Fig. 1) based on Hermitian Hamiltonian. In [8,9] the model with bare propagator (see Fig. 1b) has been studied only. In this paper the model with dressed propagator (see Fig. 1a) is considered as well. It turns out that the results depend critically on the model. Due to this the value of $\tau$ lies in the broad range $10^{16}$ yr > $\tau$ > $1.2 \cdot 10^9$ s.

2 Optical theorem and unitarity

We recall that unitarity condition

$$ (SS^+)_{fi} = \delta_{fi}, $$

$S = 1 + iT$, gives

$$ 2\text{Im}T_{ii} = \sum_f |T_{fi}|^2. $$

From this equation the optical theorem and expression for the decay width

$$ \Gamma_{opt} = \frac{1}{T_0}(1 - |S_{ii}|^2) \approx \frac{1}{T_0}2\text{Im}T_{ii} $$

are obtained. Here $T_0$ is the normalization time, $T_0 \rightarrow \infty$. The non-unitarity of $S$-matrix implies that $(SS^+)_{fi} \neq \delta_{fi}$ or, what is the same

$$ (SS^+)_{fi} = \delta_{fi} + \alpha_{fi}, $$

$\alpha_{fi} \neq 0$, resulting in

$$ 2\text{Im}T_{ii} = \sum_f |T_{fi}|^2 - \alpha_{fi} \neq \sum_f |T_{fi}|^2 $$

since the value $\sum_f |T_{fi}|^2$ can be very small. Instead of (2) we have (5) and eq. (3) and optical theorem are inapplicable. Also eq. (4) means the probability non-conservation: $\sum_f |S_{fi}|^2 \neq 1$. 

2
3 Models with the Hermitian Hamiltonian

Let us consider the $n\bar{n}$ transition in a medium followed by annihilation:

\[(n - \text{medium}) \rightarrow (\bar{n} - \text{medium}) \rightarrow (f - \text{medium}), \quad (6)\]

where $f$ are the annihilation mesons. This is a simplest process which allows a result in the analytical form for unitary and non-unitary models. We calculate directly the off-diagonal matrix element and demonstrate the consequences of the incorrect application of eqs. (2) and (3) for the non-unitary $S$-matrix.

The background potential of neutron-medium interaction $U_n$ is included in the neutron wave function: $n(x) = \Omega^{-1/2} \exp(-i\epsilon_n t + i\mathbf{p}_n \cdot \mathbf{x})$, $\epsilon_n = \mathbf{p}_n^2/2m + U_n$. The interaction Hamiltonian is

\[
\mathcal{H}_I = \mathcal{H}_{n\bar{n}} + \mathcal{H}, \quad (7)
\]

\[
\mathcal{H}_{n\bar{n}} = \epsilon \bar{\Psi}_n \Psi_n + \text{H.c.}, \quad (8)
\]

\[
\mathcal{H} = V \bar{\Psi}_n \Psi_{\bar{n}} + \mathcal{H}_a. \quad (9)
\]

Here $\mathcal{H}_{n\bar{n}}$ and $\mathcal{H}$ are the Hamiltonians of $n\bar{n}$ conversion [4,5] and $\bar{n}$-medium interaction, respectively; $\mathcal{H}_a$ and $V$ are the effective annihilation Hamiltonian and the residual scalar field, respectively; $\epsilon$ is a small parameter.

3.1 Model a

We consider the model shown in Fig. 1a (model $a$). The amplitude of antineutron annihilation in the medium $M_a$ is given by

\[
<f|0\bar{n}_p> = N(2\pi)^4\delta^4(p_f - p_i)M_a. \quad (10)
\]

Here $|0\bar{n}_p>$ is the state of the medium containing the $\bar{n}$ with the 4-momentum $p = (\epsilon, \mathbf{p}); <f|$ denotes the annihilation mesons, $N$ includes the normalization factors of the wave functions. $M_a$ includes the all orders in $\mathcal{H}_a$. The antineutron annihilation width $\Gamma$ is expressed through $M_a$ (see (15)).

In the lowest order in $\mathcal{H}_{n\bar{n}}$ the amplitude of process (6) is uniquely determined by the Hamiltonians (7)-(9):

\[
M = \epsilon G_V M_a, \quad (11)
\]

where the antineutron Green function $G_V$ is

\[
G_V = G + GVG + ... = \frac{1}{(1/G) - V} = -\frac{1}{V}, \quad (12)
\]
Figure 1: a $n\bar{n}$ transition in the medium followed by annihilation. b Same as a but the annihilation amplitude is given by (17). The blocks $M_a$ and $M'_a$ involve the all orders in $\mathcal{H}_a$ (see (10) and (17)). c The on-diagonal matrix element $T_{ii}$ (see text)

\[ G = \frac{1}{\epsilon_{\bar{n}} - p_{\bar{n}}^2/2m - U_{\bar{n}} + i0} \sim \frac{1}{0}, \]  
\[ \text{(13)} \]

since $p_{\bar{n}} = p$, $\epsilon_{\bar{n}} = \epsilon$. The Hamiltonian $\mathcal{H}_a$ acts in the block $M_a$ only and so $G_V$ is completely determined by $V$.

For the total process width $\Gamma_a$ one obtains

\[ \Gamma_a = N_1 \int d\Phi |M|^2 = \frac{\epsilon^2}{V^2} N_1 \int d\Phi |M_a|^2 = \frac{\epsilon^2}{V^2} \Gamma, \]  
\[ \text{(14)} \]

\[ \Gamma = N_1 \int d\Phi |M_a|^2, \]  
\[ \text{(15)} \]

where $\Gamma$ is the annihilation width of $\bar{n}$. The normalization multiplier $N_1$ is the same for $\Gamma_a$ and $\Gamma$.

The time-dependence is determined by the exponential decay law:

\[ W_a(t) = 1 - e^{-\Gamma_a t} \approx \frac{\epsilon^2}{V^2} \Gamma t. \]  
\[ \text{(16)} \]

### 3.2 Model b

We consider the model shown in Fig. 1b (model b). If $M_a$ is determined by (10), the process amplitude (11) follows uniquely from (7)-(9). On the other hand, for the one-step process of the antineutron annihilation in the medium ($\bar{n}$ - medium) $\rightarrow$ (annihilation mesons - medium), the annihilation amplitude $M'_a$ can be defined through the Hamiltonian $\mathcal{H}$ and not $\mathcal{H}_a$:

\[ \langle f|0 \rangle T \exp(-i \int dx \mathcal{H}(x)) = N(2\pi)^4 \delta^4(p_f - p_i) M'_a. \]  
\[ \text{(17)} \]
$M'_a$ contains all $\bar{n}$-medium interactions including antineutron rescattering in the initial state. In this case the amplitude of process (6) is (see Fig. 1b)

$$M' = \epsilon GM'_a.$$  (18)

The definition of annihilation amplitude through eq. (17) is natural since it corresponds to the observable values. There are many physical arguments in support of the model (18). However, this model contains infrared singularity $M' \sim 1/0$ since $G \sim 1/0$. This problem has been considered in [8,9]. (In [8,9] only model b has been studied.) For the purposes of this paper it is essential that model (18) gives linear $\Gamma$-dependence $\Gamma_b \sim \int d\Phi |M'|^2 \sim \Gamma$, as well as model (11). Consequently, the model with Hermitian Hamiltonian gives linear $\Gamma$-dependence at any definition of annihilation amplitude.

If $V \to 0$, model a converts to model b.

### 4 Model with the non-Hermitian Hamiltonian

In the standard calculation [4-9] of the process (6) the $\bar{n}$-medium interaction is described by optical potential (potential model). The interaction Hamiltonian is given by (7), where

$$\mathcal{H} \to \mathcal{H}_{opt} = (U_{\bar{n}} - U_n) \bar{\Psi}_{\bar{n}} \Psi_{\bar{n}} = (V - i\Gamma/2) \bar{\Psi}_{\bar{n}} \Psi_{\bar{n}},$$  (19)

where $U_{\bar{n}}$ is the antineutron optical potential. In eq. (19) we have put $\text{Re}U_{\bar{n}} - U_n = V$, $\text{Im}U_{\bar{n}} = -\Gamma/2$.

The full in-medium antineutron propagator $G_m$ is

$$G_m = \frac{1}{\epsilon_{\bar{n}} - \mathbf{p}_{\bar{n}}^2/2m - U_{\bar{n}} + i0}.$$  (20)

The on-diagonal matrix element $T_{ii}$ is shown in Fig. 1c. For the total decay width $\Gamma_{opt}$ eq. (3) gives the well-known result [4-9]:

$$\Gamma_{opt} = -2\text{Im} \epsilon G_m \epsilon = 2\epsilon^2 \frac{\Gamma/2}{V^2 + (\Gamma/2)^2} \approx 4\epsilon^2 / \Gamma.$$  (21)

### 5 Lower limits on the free-space $n\bar{n}$ oscillation time

Let $\tau_a$, $\tau_b$ and $\tau_{pot}$ be the lower limits on the free-space $n\bar{n}$ oscillation time obtained by means of model a, model b and model with the non-Hermitian Hamiltonian, respectively; $T_{n\bar{n}}$ is the oscillation time of neutron bound in a nucleus. We use the experimental bound on the neutron lifetime in oxygen $T_{n\bar{n}} > 1.77 \cdot 10^{32}$ yr obtained by Super-Kamiokande collaboration [7].
The limit derived by means of potential model [7] (see eq. (21)) is

$$\tau_{\text{pot}} = 2.36 \cdot 10^8 \text{ s.}$$  \hspace{1cm} (22)

For the model a (see eq. (16)) we take $\Gamma = 100$ MeV, and $V = 10$ MeV and obtain

$$\tau_a = 5\tau_{\text{pot}} = 1.2 \cdot 10^9 \text{ s.}$$  \hspace{1cm} (23)

For the model b it was obtained [8,9]

$$\tau_b = 10^{16} \text{ yr.}$$  \hspace{1cm} (24)

6 Discussion

The $\Gamma$-dependence of the results differs fundamentally: $\Gamma_{\text{opt}} \sim 1/\Gamma$, whereas $\Gamma_{a,b} \sim \Gamma$. At the same time the annihilation is the main effect which defines the process speed. One of two models is wrong.

We assert that model with non-Hermitian Hamiltonian $H_{\text{opt}}$ is wrong since (21) follows from (3) which is inapplicable for non-unitary $S$-matrix (see (5)). Besides $\text{Im} T_{ii}$ is unknown (see below). Notice that the result (21) takes place in the all standard calculations (see, for example, refs. [4-7]) because they are based on the optical potential.

We compare (14) and (21):

$$r = \frac{\Gamma_a}{\Gamma_{\text{opt}}} = \frac{\Gamma^2}{4V^2}. \hspace{1cm} (25)$$

For the $n\bar{n}$ transition in the nuclear matter the realistic set of parameters is $\Gamma = 100$ MeV, and $V = 10$ MeV. Then $r = 25$. When $V = 0$ as well as in the case of the model b, eqs. (14) and (25) are invalid. However, in that event $r \gg 1$ as well [9].

On the other hand, for small $\Gamma$ eq. (21) coincides with (14):

$$2\epsilon^2 \frac{\Gamma/2}{V^2 + (\Gamma/2)^2} \approx \frac{\epsilon^2}{V^2} \Gamma. \hspace{1cm} (26)$$

This is because the Hamiltonian $H_{\text{opt}}$ is practically Hermitian in this case. Also we believe that the Hamiltonian (19) describes correctly the $n\bar{n}$ transition with $\bar{n}$ in the final state ($n$ – medium) $\rightarrow$ ($\bar{n}$ – medium) since eq. (3) is not used in this case [10].

So in the first approximation the optical theorem can be used for the non-unitary model if

$$| \text{Im} U_{\bar{n}} | \ll | \text{Re} U_{\bar{n}} - U_n |. \hspace{1cm} (27)$$

This is not the case for the $n\bar{n}$ transition in the nuclear matter. Because of this we performed the calculations in the framework of unitary models.
If the optical potential is used for the problems described by Schrödinger-type equation (optical model), the unitarization takes place [10]: the matrix elements and optical potential are fitted to $\bar{p}$-atom ($\pi^-$-atom, $K^-$-atom) and low energy scattering data. However, the optical potential is the effective one. The $n\bar{n}$ transition is described by the system of coupled equations [5,8-10]. The corresponding $S$-matrix differs principally [10]. There are no experimental data and unitarization in this case. Im$T_{ii}$ is unknown since it depends on Im$U_{\bar{n}}$ which was defined and fitted for the principally different problem. So eq. (2) accomplishes nothing. Besides, eqs. (2) and (3) cannot be used for the non-unitary $S$-matrix. (We also note that it is meaningless to impose the condition of probability conservation $\sum_f |S_{fi}|^2 = 1$ since $S_{ii}$ is unknown.) In the model described in sect. 4 the reverse takes place: equation (3) is used where $T_{ii}$ is unknown. The consequences are illustrated by eqs. (14), (21) and (25).

If the optical theorem is not used, the range of applicability of optical potential is considerably wider. It describes the $n\bar{n}$ transitions with $\bar{n}$ in the final or intermediate states [10]. In these cases the off-diagonal matrix elements are calculated directly without use of optical theorem.

In view of the uncertainty in the annihilation amplitude we cannot decide between models a and b. The same is true for the value of the antineutron self-energy $V$ in (12). These problems are general in the theory of reactions. However, in the problem under study the values of $\Gamma_a$ and $\tau_a$ are extremely sensitive to $V$ (see (14)). This is because the amplitude (18) is in the peculiar point $M' \sim 1/0$. The small change of $V$ affects the result vastly: $\Gamma_a \sim 1/V^2$. Owing to this $\tau_a$ and $\tau_b$ differ greatly. Further investigations are desirable.

7 Conclusion

The main results are as follows.

(a) For the non-unitary models the optical theorem can be used for the estimations if the absorption is small. For the models with the essentially non-Hermitian Hamiltonians the optical theorem can be applied only if $S$-matrix is unitarized. The same is true for the condition of probability conservation.

(b) At present it is impossible to decide between models a and b as well as to determine the value of $V$ exactly. So the values $\tau_a = 1.2 \cdot 10^9$ s and $\tau_b = 10^{16}$ yr are interpreted as the estimations from below (conservative limit) and from above, respectively. The realistic limit $\tau$ can be in the range

$$10^{16} \text{ yr} > \tau > 1.2 \cdot 10^9 \text{ s.}$$

The estimation from below $\tau_a = 1.2 \cdot 10^9$ s exceeds the restriction given by the Grenoble reactor experiment [11] by a factor of 14 and the lower limit given by potential model by a factor of 5.
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