Synthesis of the relay-linear control with the hysteresis region for the second order object example

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Abstract. The purpose of this work is a relay-linear control synthesis with using of a hysteresis region, that contains a switching surface and is build with piecewise linear approximation of the switching surface. An application of the hysteresis region together with linear control region allows to avoid sliding mode. A second order object is discussed for better visual presentation of many geometrical concepts that are used in relay-linear control theory.

Introduction

The relay control principles are widely used within constraints of control value [1-5]. Relay-linear control laws description and synthesis are described in [6-8]. The purpose of this work is a relay-linear control synthesis and its application to a second order object, that becomes classical [9-12] in describing optimal control laws for the Pontryagin maximum principle [13,14] and allows visual presentation of many geometrical concepts that are used in relay-linear control theory. For example, state space for a system (1) is a 2-d order phase plane, and switching surface for some time optimal control law (3) is transformed into some curve on that plane.

\[ \dot{x} = Ax + bu, \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |u| \leq 1 \quad (1) \]

The aim is a time-optimal control that leads \( x \) to its origin. The ideal time-optimal control law for (1) is [13]:

\[ u = -\text{sign}(x_2 - \sqrt{2|x_1| \cdot \text{sign}(x_1))}. \quad (2) \]

The optimal control law (2) is defined by an ideal analytical switching curve [13]:

\[ x_2 = -\sqrt{2|x_1| \cdot \text{sign}(x_1)}, \quad (3) \]

For practical realization of a suboptimal control we will represent the switching curve (3) by its piecewise linear approximation (Fig. 1). For that we will define some set of the interpolated points \( \omega \) of the switching curve inside some considered \( x \) domain area.
The switching curve in the phase plane for the system

The interpolated points \( \omega_i \) can be found by solving the system (1) in inverse time [15] from origin \( x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( u = 1 \) or \( u = -1 \); or by solving an equation (5):

\[
x = - \int_0^r \Phi(r,\varphi)b \cdot \text{sign}(g^T \cdot \Phi(r,\varphi)b) d\varphi, (5)
\]

where \( \Phi(r,\varphi) = e^{A(r-\varphi)} \) – is a fundamental matrix of the system (1), \( r \) – corresponds to the expected transient time and the requirement conditions of \( g \) [6]:

\[
g^T b = 0, \|g\| = 1. \text{ Then } g \in \{g_1, g_2\}, \text{ where } g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.
\]

The solutions (5) for our suggested case \( (x_i \in [-2, 2], i = 1, 2) \) is shown in Fig.1: a switching curve \( AO \) – for a case \( g = g_1 \); a switching curve \( BO \) – for a case \( g = g_2 \).

Set of interpolated points \( \omega = \{-\omega_2, -\omega_1, \omega_0, \omega_1, \omega_2\} \) are chosen, where

\[
\omega_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \omega_1 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad \omega_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.
\]

The relay-linear control synthesis basics

In a common case of \( n \) – dimensional space the switching surface linear approximation can be produced by the \( (n-1) \) dimensional hyperplanes, every of which can be defined by an equation [6]:

\[
p^T \cdot x - d = 0.
\]

For \( d \neq 0 \) \( \dot{\hat{p}} = \frac{1}{d} \), \( \hat{p}^T \cdot x = 1 \);

and for \( d = 0 \) \( \dot{\hat{p}} = \frac{p}{|p|} \), \( \hat{p}^T \cdot x = 0 \).
In a case of a unit length vector $p$, $d$ can be regarded as a distance along the vector $p$ from the origin to the hyperplane that perpendicular to the vector $p$. According to the linear approximation of the switching curve a suboptimal [6] relay control law is:

$$u_{cr} = -\text{sign} \left( l \hat{p}^T x + \text{sign} \left( \hat{p}^T x \right) \right),$$  

(6)

where $l$ — is a quantity of approximating hyperplanes.

For our case quantity of approximating hyperplanes is two: $l = 2$. Then

$$u_{cr} = -\text{sign} \left( 2 \hat{p}^T x + \text{sign} \left( \hat{p}^T x \right) \right),$$

with $\hat{p} = \begin{bmatrix} 0.894 & 0.447 \end{bmatrix}^T$, $\hat{p} = [1 \ 1.5]^T$.

Switching to the linear control law near the origin is intended to exclude the self-oscillation mode. In works [6,7] a law of switching between relay and linear control (Fig.2) is mathematically described as below (for a second order object in our case).

**Fig. 2.** “Classical” relay-linear control regions

**Relay control law:**

$$F^R (x) = -\text{sign} \left( 2 \hat{p}^T x + \text{sign} \left( \hat{p}^T x \right) \right)$$

applicable at $y_1 \cup y_2 = 1$,

where

$$y_i = \begin{cases} 1 : & \left| \hat{p}^T x \right| < 1 \\ 0 : & \left| \hat{p}^T x \right| \geq 1 \end{cases}, \quad i \in [1,2] .$$

**Linear control law:**

$$F^L (x) = -k \cdot \hat{p}^T x$$

applicable at $y_1 \cap y_2 = 1$,

where $k$ is a coefficient of linear control law in the linear control law region.

Then final relay-linear control law can then be written as:

$$u = F^R (x) \cdot (\overline{y_1} \cup \overline{y_2}) + F^L (x) \cdot (y_1 \cap y_2) =$$

$$= -\text{sign} \left( 2 \hat{p}^T x + \text{sign} \left( \hat{p}^T x \right) \right) \cdot (\overline{y_1} \cup \overline{y_2}) - k \cdot \hat{p}^T x \cdot (y_1 \cap y_2).$$  

(7)
A usage of the piecewise linear approximation of the switching curve can lead to a sliding mode on some sections of object phase trajectory (Fig.3).

**Fig. 3.** A sliding mode section on the piecewise linear approximation of the switching curve for a given initial conditions.

**The relay-linear control synthesis with the hysteresis region**

If we want to avoid the sliding mode at all during movement we can introduce a hysteresis region that contains the switching curve. The hysteresis region is bounded by piecewise linear approximation of the switching curve and by the lines of support which are parallel to appropriate pieces of the switching curve linear approximation (Fig.3).

As a boundary of the linear law applicability a constant energy line may be chosen. Then an equation, that defines a region of the linear law applicability in our case (region, bounded by the ellipse):

$$\frac{k_1x_1^2}{2} + \frac{x_2^2}{2} < c.$$  

To define $k_1$ and $c$, an additional requirement to minimize the number of switches can be entered.

The belonging of a point in the phase plane to the regions: linear, hysteresis or relay, will be determined by a sequential check for the truth of the corresponding entered Boolean variables $CL, CN, CR$ defined by expressions (for the second order case):

$$CL = \left(\frac{k_1x_1^2}{2} + \frac{x_2^2}{2} < c\right); (8)$$
\[ CN = (F_{R1} \cdot F_{R2} < 0), \quad (9) \]

\[ F_{R1} = -\text{sign}\left(2 \hat{P}^T x + \text{sign}\left(1 \hat{P}^T x + \text{sign}\left(0 \hat{P}^T x\right)\right)\right), \]

\[ F_{R2} = -\text{sign}\left(2 \hat{P}^T x + \text{sign}\left(1 \hat{P}^T x\right)\right), \]

where \(i\hat{P}^T, \quad i = 1,2\) is accordingly parallel to \(i\hat{P}^T\) and tangent to the switching line;

\[ CR = \overline{CN \cap CL}. \]

**Fig. 4.** The linear, relay and hysteresis control regions.

And control law with the hysteresis region defines as follows:

In the linear law region:

\[ u = -Kx; \quad (10) \]

In the relay law region:

\[ u = -\text{sign}\left(2 \hat{P}^T x + \text{sign}\left(1 \hat{P}^T x + \text{sign}(x_1)\right)\right); \quad (11) \]

In the hysteresis region:

The control is remaining the same as a value just before entering the hysteresis region and defined by the relay law. After \(x\) got out of the hysteresis region, the control will be defined by the appropriate relay or linear control laws.

**Simulation results and discussion**

The relay-linear control simulation results for the studied system mathematical model (1) was produces in Matlab with both control laws (7) and (8-11). The same coefficients were implemented for the corresponding regions of the linear control laws. The phase trajectories for the system with two control laws: with the hysteresis region and without it are considered (Fig. 5).

In Figure 5:

1. – the linear control region for the control law (7) (without hysteresis);
2. – the hysteresis region border defined by (9);
3. – the linear region defined by (8);
4. – the phase trajectory of the system (1) with implemented (7) control law;
5. – the phase trajectory of the system (1) with implemented (8)-(11) control law.
Fig. 5. The phase trajectories for the system with two control laws: with the hysteresis region and without it.

The phase trajectory 4 in Fig.5 contains sliding mode section. That is resulted in specific changes of control value during simulation (Fig. 6).

Fig. 6. The control $u$ time dependence (a) and the transient processes (b) for the system with two control laws: without the hysteresis region (1) and with it (2).

The simulation results show the effectiveness of the hysteresis zone introduction in the linear-linear control law from the point of view of the absence of a large number of switches caused by the sliding mode section. In addition, the introduction of hysteresis in the formation of the control law does not lead to a deterioration of the transition process in this case.

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