COLOR IMAGE PROCESSING BY VECTORIAL TOTAL VARIATION WITH GRADIENT CHANNELS COUPLING

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ABSTRACT. We study a regularization method for color images based on the vectorial total variation approach along with channel coupling for color image processing, which facilitates the modeling of inter channel relations in multidimensional image data. We focus on penalizing channel gradient magnitude similarities by using $L^2$ differences, which allow us to explicitly couple all the channels along with a vectorial total variation regularization for edge preserving smoothing of multichannel images. By using matched gradients to align edges from different channels we obtain multichannel edge preserving smoothing and decomposition. A detailed mathematical analysis of the vectorial total variation with penalized gradient channels coupling is provided. We characterize some important properties of the minimizers of the model as well as provide geometrical results regarding the regularization parameter. We are interested in applying our model to color image processing and in particular to denoising and decomposition. A fast global minimization based on the dual formulation of the total variation is used and convergence of the iterative scheme is provided. Extensive experiments are given to show that our approach obtains good decomposition and denoising results in natural images. Comparison with previous color image decomposition and denoising methods demonstrate the advantages of our approach.

1. Introduction. Techniques based on variational and partial differential equations (PDEs) are widely used in various image processing tasks. In particular, classical ill-posed problems such as image restoration and denoising can be effectively tackled by variational energy minimization models. One of the famous method is the total variation ($TV$) regularization which was original proposed for edge preserving image restoration [58]. The well-known Rudin-Osher-Fatemi (ROF) image
A denoising model seeks a minimizer of the sum of a fidelity term measured in the square of $L^2$-norm along with the TV regularization,

$$
\min_{u \in BV(\Omega, \mathbb{R})} \left\{ \int_{\Omega} |\nabla u| \, dx + \lambda \int_{\Omega} |u - f|^2 \, dx \right\}
$$

where $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is the given noisy image, $BV(\Omega, \mathbb{R})$ is space of bounded variation functions, and $\lambda \geq 0$ is the regularization parameter. The TV term acts as an important regularizer penalizing the function’s gradient [63, 36]:

- Sharp edges are recovered without noise and avoids discriminate smoothing associated with $L^2$ regularization functionals.
- Edge locations are preserved, and under certain conditions edge locations kept exactly. This is important as dislocations of edges while smoothing is detrimental to further higher level image processing tasks.

Despite its usefulness in image restoration, TV regularization is known to exhibit blocky or staircasing artifacts in solutions [47]. To avoid such artifacts various modifications has been studied in the recent years like higher order regularization [38, 41] and adaptive or weighted TV regularization [64, 62, 56, 52, 57]. For example, consider the weighted TV regularization,

$$
\min_{u \in BV(\Omega, \mathbb{R})} \left\{ \int_{\Omega} g(x) |\nabla u| \, dx + \lambda \int_{\Omega} |u - f|^2 \, dx \right\}
$$

with function $g$, an edge indicator function that vanishes at object boundaries, $g(x) = (1 + \beta |\nabla f(x)|^2)^{-1}$ where $\beta > 0$ is a constant parameter. Recently, other non-smooth data fidelity terms [45] have found to be useful in various image processing applications [65, 25, 72, 5, 44, 6, 40, 73, 34, 29, 35, 33, 43] where the properties of corresponding minimization formulations are very well stablished [42, 46].

Bresson et al [14] have considered a similar regularization model with a $L^1$ data fidelity term,

$$
\min_{u \in BV(\Omega, \mathbb{R})} \left\{ \int_{\Omega} g(x) |\nabla u| \, dx + \lambda \int_{\Omega} |u - f| \, dx \right\}.
$$

We refer to [22, 30] for the geometric motivation and properties of $L^1$ fidelity term. The minimization of the $gTV + L^1$ model given by (3) yields a contrast invariant filter [22, 27, 61] and well preserves contrasted features at different scales. The weighted TV norm allows a better preservation of corners and sharp angles in shape denoising. Moreover the introduction of such as function allows to establish a link between $gTV$ and the geodesic active contours model introduced by [17, 50, 19] as an improvement of the original snakes model [39].

The variational minimization problem (3) involves the non-differentiable total variation regularization. Nevertheless, it can be solved by standard calculus of variations and Euler-Lagrange equations toward a smooth approximation of the $L^1$-norm driving the function $u$ towards a minimum [9, 23, 14]. Following the work in [24, 20, 5, 8, 7] a convex regularization of the variational model is implemented in [14] by applying a dual formulation of the TV norm. While most existing work involving TV focuses on scalar valued functions, the generalization to vector-valued images remains an important challenge. The main idea of the vectorial total variation ($VTV$) approach is to extend the TV definition for vector-valued functions $u = (u_1, \ldots, u_N) : \Omega \to \mathbb{R}^N$, also referred as multispectral images, with $u_i \in BV(\Omega, \mathbb{R})$ such that in the scalar case, $N = 1$, both definitions coincide. Two important notions of vectorial total variation can be considered:
• **Channel by channel summation:** Total variation for vector (multichannel) functions can simply be channel by channel summation \([2]\). This simple modelling does not assume any coupling between channels and the multidimensional TV consist of summing up the contributions of separate channels. Therefore, chromatic edges can be lost and color smearing can occur between inter-channel edges, see \([13, 31]\) for discussions. The \(VTV\) expression is given by,

\[
VTV_1(u) := \sum_{i=1}^N \int_{\Omega} |\nabla u_i| \, dx.
\]

(4)

• **Implicit coupling with Euclidean norm:** In this approach, an implicit coupling between channels is considered such that each channel uses information coming from other channels \([11, 13]\). The \(VTV\) definition is given by,

\[
VTV_2(u) := \int_{\Omega} \sqrt{\sum_{i=1}^N |\nabla u_i|^2} \, dx.
\]

(5)

Coupling channels in a variational/PDE setting for improving denoising and other image processing performance has been done by many researchers in the past. One of the earlier works is by Blomgren and Chan \([11]\) who utilized the implicit coupling via the \(VTV_2\) above. However as noted in \([66]\) color smearing can still appear near edges and the staircasing of flat regions can persist. Bresson and Chan \([13]\) used a dual formulation to decrease the computational expense via TV minimization. A multiscale approach along with Moreau-Yosida based primal-dual approach was considered by Dong and Hintermuller \([28]\). Brook et al \([15]\) used the inter-channel term \(E_c(u) = \sum_{i=1}^3 |\nabla x u_i \times \nabla y u_i|^2\), where the derivatives \((\nabla x u, \nabla y u)\) are computed in their corresponding \((x,y)\) axis directions. This helps in aligning the color edges as in our model. Moreover the intrachannel term is chosen to be isotropic, \(\phi(|\nabla u_i|) = |\nabla u_i|^2\). The corresponding wellposedness of the minimization problem and its numerical convergence are not proven. Also, the term \(|\nabla x u \times \nabla y u|^2\) is of order \(|\nabla u|^4\) and cannot be bounded below by \(c |\nabla u|^4\) for some \(c > 0\), thus we cannot prove a similar result such as Theorem 2.4 in our work. More recently Ehrhardt and Arridge \([32]\) use a similar alignment for coupling parallel level sets. A closely related approach denoise color or multichannel is to use anisotropic diffusion \([59, 66, 12, 55]\). They typically involve similar coupling as in the variational approaches or selectively denoise color images in pure chromaticity spaces. Tschumperle and Deriche \([68]\) unified such vector valued diffusion PDEs into one common framework via tensor diffusion. Goldluecke et al \([36]\) use implicit coupling via a natural vectorial total variation motivated from geometric measure theory. Ono and Yamada \([48]\) decorrelate the different channels along with vectorial total variation.

In this paper, we study \(VTV\) with an explicit channel coupling term which uses component gradients of a multispectral image. Our aim here is to demonstrate that it is possible to perform an efficient minimization of the \(VTV\) with proposed coupling term. This paper, besides developing new coupling models, also proposes a new numerical scheme based on Chambolle’s dual minimization \([20]\) to perform the algorithm evolution in an efficient and fast way. The aims of this paper is to
introduce a regularizing term allowing to generalize the $TV$ term to multispectral images. The main contributions of this paper are summarized as follows:

1. Introduction of the new vectorial total variation with coupling channels ($VTV$) approach which penalizes different image gradient components in order to perform edge preservation across channels.

2. Presentation of the different properties related to the image processing schemes involving the new $VTV$ norm. In this context, we show the existence of solutions to the decomposition model.

3. Extensive numerical implementation to our variational scheme over different image databases show the improvement of our $VTV$ term in relation to other approaches based on functions of bounded variation.

We also consider the $L^1$ data fidelity term along with $VTV$ regularization with coupling for image decomposition. The coupling terms provide better edge alignment and can also be used in a PDE formulation [55, 51, 53]. Here we utilize weighted $TV$ regularization along with coupling gradient channels for obtaining decomposition and denoising of images. Our aim here is to provide a detailed mathematical analysis of the variational minimization method and provide some applications in decomposition and denoising of noisy images. Experimental results with other related variational/PDE color image denoising schemes indicate that the proposed approach consistently outperforms them in terms of different error metrics. The rest of the paper is organized as follows. Section 2 explains the model in detail using the theory of level sets, and in Section 3 we describe the implementation of the proposed model using the dual minimization. Section 4 provides some experimental results. Section 5 concludes the paper.

2. Vectorial total variation with channel coupling.

2.1. Coupling channels. For motivating our $VTV$ scheme, we first consider a scenario where two 1-D spatially varying edges meet (Figure 1, First Row). Let us consider the signals $u_1$ and $u_2$ (Figure 1(a)) satisfying the following relation in terms of gradient magnitudes: $|\nabla u_1| > |\nabla u_2|$ (Figure 1(b)). In order to define a unique correlated edge we will penalize the expression $\nabla u_1 - \nabla u_2$ (Figure 1(c)) in terms of $L^2$ norm. Caselles et al [18] have shown that the geometry is constrained by the intensity channel along and the image level lines contain all the salient information. Then, the main idea of the coupling terms is to constrain the regularization near discontinuities by penalizing large derivations in gradients between different channels, therefore making them to look different to the normal Red, Green and Blue image gradient magnitudes. Second row in Figure 1 shows the behavior of the proposed coupling term over a synthetic image defined by the intersection of three different components. This synthetic image is composed of discs determined by pure chromatic colors of red, green, and blue. The gradient differences for each channel clearly differentiates from channel-wise gradients ($|\nabla u_i|^2$) since the coupling term is able to align edges over different channels. This in turn helps to keep the edge contrast during the further vectorial total variation smoothing process. Intersected edges over different colors are penalized by minimizing the terms $J(u_k, u_l)$. From the geometrical point of view, the proposed coupling term is well suited as it does not degrade the edge contrast quality of non-correlated channels, as shown in third row of Figure 1. In this case, a synthetic image where the green color circle does not intersect with the red and blue circles.
Gradient magnitudes and coupling term for two 1-D crossing edges

Behavior of the coupling channels for a color image with crossing edges between different channels

Behavior of the coupling channels for a color image with crossing edges over two color channels

Figure 1. First Row: Displays the behaviour of the proposed model for two intersected signals. (a) 1-D edges \((u_1, u_2)\) intersection. (b) Gradients \((\nabla u_1, \nabla u_2)\). (c) Coupling term \(\nabla u_1 - \nabla u_2\). Second-Third Rows: Show the edge contrast quality due to the proposed coupling terms over two color images, and the differences with the gradient magnitude components.
In general, we propose the coupling term $J(u_k, u_l) = \int_{\Omega} |\nabla u_k - \nabla u_l|^2 \, dx$ for two given channels $u_k$ and $u_l$. Then, the penalizing term extending the $TV$ term for a multispectral image $u : \Omega \to \mathbb{R}^N$, with $\Omega \subset \mathbb{R}^M$ a bounded open domain, is given by the following regularization term,

$$\sum_{k=1}^{N} \sum_{l=1}^{N} J(u_k, u_l) = \int_{\Omega} \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla u_k - \nabla u_l|^2 \, dx,$$

where it is expected to penalize large differences between gradient images [55, 51].

We further define the coupling-$VTV$ of $u$ to be the following,

$$VTV(u) := \sum_{k=1}^{N} \int_{\Omega} |D u_k| + \alpha \int_{\Omega} \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla u_k - \nabla u_l|^2 \, dx,$$

where $\alpha \geq 0$ is a parameter for the cross-coupling term. Thus, the above model unifies the weighted $TV$ regularization with coupling terms for multichannel image processing. We next provide detailed mathematical analysis of the coupled $VTV$ method along with $L^1$ fidelity case. Note that the $L^2$-norm is used in the proposed term to smooth common salient edges within the different channels as well as to distinguish between scaled edge features with nonuniform intensities. A general framework with $J(u_k, u_l) = \int_{\Omega} \psi(|\nabla u_k - \nabla u_l|) \, dx$ for a convex function $\psi$ can also be accommodated within our framework studied here and we restrict ourselves to the canonical example of $L^2$ differences.

2.2. Analysis of the coupling $VTV$ model. Under definition (7) and following the original $TV$ theory in [4, 2] we define for $N \geq 1$ the space $BV(\Omega, \mathbb{R}^N)$ of vector valued functions as the set of functions $u \in L^1(\Omega, \mathbb{R}^N)$ such that $VTV(u) < \infty$ according to equation (7). The space $BV(\Omega, \mathbb{R}^N)$ endowed with the following norm:

$$\|u\|_{L^1(\Omega, \mathbb{R}^N)} + \|u\|_{BV(\Omega, \mathbb{R}^N)},$$

where $\|u\|_{BV(\Omega, \mathbb{R}^N)} := VTV_1(u)$, is a Banach space. We recall further properties of $VTV$ and a decomposition property which extends to our coupling with $VTV$.

- **Lower Semicontinuity**: For a given sequence $\{u_n\}$ of functions in $BV(\Omega, \mathbb{R}^N)$ converging to a function $u \in L^1(\Omega, \mathbb{R}^N)$

$$\liminf_{n \to \infty} \|u_n\|_{BV(\Omega, \mathbb{R}^N)} \leq \|u\|_{BV(\Omega, \mathbb{R}^N)},$$

- **Compactness**: Every sequence $\{u_n\} \in BV(\Omega, \mathbb{R}^N)$ such that $\|u_n\|_{L^\infty(\Omega, \mathbb{R}^N)} \leq M$, for all $n \in \mathbb{N}$, admits a subsequence $\{u_{nk}\}$ converging in $L^1(\Omega, \mathbb{R}^N)$ to a function $u \in BV(\Omega, \mathbb{R}^N)$.

- **Decomposition of $\|u\|_{BV(\Omega, \mathbb{R}^N)}$:** The functional $VTV$ can be decomposed as follows

$$VTV(u) = \sum_{k=1}^{N} \int_{\Omega} |\nabla u_k| \, dx + |D^c u|(\Omega) + \int_{J_0 \cap \Omega} |u^+ - u^-| \, dH^{M-1}$$

$$+ \alpha \int_{\Omega} \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla u_k - \nabla u_l|^2 \, dx,$$

where $D^c u$ is the Cantor part of $Du$, $J_0$ is the set of all jump points of $u$, $u^+_k$, $u^-_k$ are the jump functions and $H^{M-1}$ denotes the $(M-1)$-dimensional Hausdorff measure in $\mathbb{R}^M$ with $\Omega \subset \mathbb{R}^M$. 

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Given \( u \in BV(\Omega, \mathbb{R}) \), the \( gTV \) term is defined as:

\[
gTV(u) = \int_{\Omega} g |Du| \, dx := \sup_{p \in \Phi_g} \int_{\Omega} u(x) \text{div} p(x) \, dx,
\]

where \( \Phi_g = \{ p \in C_c^1(\Omega, \mathbb{R}^2) : |p(x)| \leq g \text{ for all } x \in \mathbb{R}^M \} \). We also adopt the notation \([14]\)

\[
\int_{\Omega} g |Du| = \int_{\Omega} g |\nabla u| \, dx.
\]

We recall that the perimeter of a Borel measurable set \( \Gamma \subset \Omega \) is defined by

\[
\text{Per}_g(\Sigma) := gTV(\mathbb{1}_\Sigma) = \int_{\Omega} g |\nabla \mathbb{1}_\Sigma| \, dx,
\]

where \( \mathbb{1}_\Sigma \) is the characteristic function of \( \Sigma \). For a given image \( u \) and any \( \xi \in \mathbb{R} \), we define the upper level set

\[
\Gamma_u := \Gamma(u, \xi) = \{ x \in \Omega : u(x) > \xi \}.
\]

We should also use the co-area formula

\[
gTV(u) = \int_{-\infty}^{+\infty} \left( \int_{\gamma_\xi} g \, ds \right) \, d\xi = \int_{-\infty}^{+\infty} \text{Per}_g(\Gamma(u, \xi)) \, d\xi = \int_{-\infty}^{+\infty} \int_{\gamma_\xi} g \, ds \, d\xi,
\]

where \( \gamma_\xi \) is the boundary of the set \( \Gamma(u, \xi) \).

The following result shows that our proposed \( VTV \) term decouples the level sets of the given multichannel image for each channel, making it a geometry problem for each channel’s level set. The proof relies the perimeter (13) and its connection to the \( TV \) regularization.

**Theorem 2.1.** For \( u \in BV(\Omega, \mathbb{R}^N) \)

\[
VTV(u) = \int_{-\infty}^{+\infty} \sum_{k=1}^{N} \text{Per}(\Gamma_{u_k}) \, d\xi_k
\]

\[
+ \alpha \int_{-\infty}^{+\infty} \int_0^1 \left( \sum_{i=1}^{M} \sum_{k=1}^{N} \left( \int_{\Gamma_{\nabla x_i u_k \setminus \Gamma_{\nabla x_i u_l}}} (\xi_k - \mathbb{1}_{[0,\nabla x_i u_l]}) \, dx \right) d\mu_l \right) d\xi_k
\]

\[
+ \left( \int_{\Gamma_{\nabla x_i u_k \setminus \Gamma_{\nabla x_i u_k}}} (\xi_l - \mathbb{1}_{[0,\nabla x_i u_k]}) \, dx \right) d\mu_k \, d\xi_l
\]

\[
+ \left( \int_{\Gamma_{\nabla x_i u_k \setminus \Gamma_{\nabla x_i u_k}}} (\mathbb{1}_{[0,\nabla x_i u_k]} - \xi_l) \, dx \right) d\mu_k \, d\xi_l.
\]

**Proof.** From the standard total variation based co-area formula (15),

\[
\int_{\Omega} |\nabla u_k| \, dx = \int_{-\infty}^{+\infty} \text{Per}(\Gamma_{u_k}) \, d\xi_k.
\]
Now, let \( f_k = \nabla^{x_i} u_k \) and \( f_l = \nabla^{x_i} u_l \). We have that
\[
\int_{\Omega} ( \nabla^{x_i} u_k - \nabla^{x_i} u_l )^2 \, dx = \frac{1}{2} \int_{\Omega} (f_k - f_l)^2 \, dx + \frac{1}{2} \int_{\Omega} (f_l - f_k)^2 \, dx
\]
\[
= \frac{1}{2} \left( \int_{\{f_k > f_l\}} (f_k - f_l)^2 \, dx + \int_{\{f_l > f_k\}} (f_l - f_k)^2 \, dx \right)
\]
\[
+ \frac{1}{2} \left( \int_{\{f_l > f_k\}} (f_l - f_k)^2 \, dx + \int_{\{f_k > f_l\}} (f_k - f_l)^2 \, dx \right)
\]
\[
= \left( \int_{\{f_k > f_l\}} f_k \, d\xi_k + \int_{\{f_l > f_k\}} f_l \, d\xi_k \right) - \left( \int_{\{f_l > f_k\}} f_k \, d\xi_l + \int_{\{f_k > f_l\}} f_l \, d\xi_l \right)
\]
\[
= \left( \int_{\{f_k > f_l\}} f_k \, d\xi_k \right) - \left( \int_{\{f_l > f_k\}} f_l \, d\xi_l \right)
\]
\[
= \int_{\{f_k > f_l\}} f_k \, d\xi_k - \int_{\{f_l > f_k\}} f_l \, d\xi_l.
\]
Let us observe that
\[
\int_{\Omega} \mathbb{I}_{[\nabla^{x_i} u_k, \nabla^{x_i} u_l]} 1_{\Gamma_{\nabla^{x_i} u_k \setminus \Gamma_{\nabla^{x_i} u_l}}} \, dx = \int_{\Gamma_{\nabla^{x_i} u_k \setminus \Gamma_{\nabla^{x_i} u_l}}} \, dx,
\]
and similarly
\[
\int_{\Omega} \mathbb{I}_{[\nabla^{x_i} u_l, \nabla^{x_i} u_k]} 1_{\Gamma_{\nabla^{x_i} u_l \setminus \Gamma_{\nabla^{x_i} u_k}}} \, dx = \int_{\Gamma_{\nabla^{x_i} u_l \setminus \Gamma_{\nabla^{x_i} u_k}}} \, dx.
\]
We also set
\[
\nabla^{x_i} u_l = \int_0^1 \mathbb{I}_{[0, \nabla^{x_i} u_l]} \, d\mu_l, \quad \nabla^{x_i} u_k = \int_0^1 \mathbb{I}_{[0, \nabla^{x_i} u_k]} \, d\mu_k.
\]
By using the identities in (18), (19), (20) and (21), we get
\[
\int_{\Omega} (\nabla^{x_i} u_k - \nabla^{x_i} u_l)^2 \, dx = \int_{\Omega} \mathbb{I}_{[0, \nabla^{x_i} u_l]} \, dx
\]
\[
+ \int_{\Gamma_{\nabla^{x_i} u_k \setminus \Gamma_{\nabla^{x_i} u_l}}} (\mathbb{I}_{[0, \nabla^{x_i} u_l]} - \xi_k) \, d\mu_l \, d\xi_k
\]
\[
+ \int_{\Gamma_{\nabla^{x_i} u_l \setminus \Gamma_{\nabla^{x_i} u_k}}} (\mathbb{I}_{[0, \nabla^{x_i} u_k]} - \xi_l) \, d\mu_k \, d\xi_l.
\]
A combination of (17) and (22) leads to the desired result.
2.3. Vectorial TV with $L^1$ fidelity term. The $gTV + L^1$ model is one of the most influential variational PDE-based image decomposition models in image processing [22, 14]. This decomposition model separates a grayscale image in cartoon and texture components while preserving main features such as edges. For multispectral images, $f: \Omega \to \mathbb{R}^N$, we extend the $gTV + L^1$ model to the vectorial case called $gVTV + L^1$ by considering the following variational problem:

$$
\text{(23)} \min_{u \in BV(\Omega, \mathbb{R}^N)} \left\{ E_{gVTV}^1(u) = \sum_{k=1}^N \left( \int_{\Omega} g_k(x) |\nabla u_k| \, dx + \lambda \int_{\Omega} |u_k - f_k| \, dx \right) + \alpha \int_{\Omega} \sum_{k=1}^N \sum_{l=1}^N |\nabla u_k - \nabla u_l|^2 \, dx \right\},
$$

where $g_k(x) = (1 + |\nabla f_k(x)|^2)^{-1}$ is an edge indicator function. Note that in the edge indicator function the noisy image $f$ is smoothed with a Gaussian kernel. When $g \equiv 1$ we refer to our model as $VTV + L^1$.

Remark 1. Let us note that (23) with $L^2$ image fidelity term is straightforward [54, 55],

$$
\text{(24)} \min_{u \in BV(\Omega, \mathbb{R}^N)} \left\{ E_{gVTV}^2(u) = \sum_{k=1}^N \left( \int_{\Omega} g_k(x) |\nabla u_k| \, dx + \lambda \int_{\Omega} |u_k - f_k|^2 \, dx \right) + \alpha \int_{\Omega} \sum_{k=1}^N \sum_{l=1}^N |\nabla u_k - \nabla u_l|^2 \, dx \right\}.
$$

By following [22] and the result from Theorem 2.1, we show that the $gVTV + L^1$ based functional (23) can also be expressed in terms of level sets. That is, the following theorem proves the equivalence of the energy functional in (23) based on $VTV$ with $L^1$ data fidelity term and each channel level sets via the co-area formula.

Theorem 2.2. Given a noise image $f$ and a function $u \in BV(\Omega, \mathbb{R}^N)$ we have

$$
E_{gVTV}^1(u) = \int_{-\infty}^{+\infty} \left( \sum_{k=1}^N \left( \text{Per}_{g_k} (\Gamma u_k) + \lambda \int_{\Gamma u_k \setminus \Gamma f_k} d\xi_k \right) + \alpha \int_{0}^{+\infty} \sum_{i=1}^M \sum_{k=1}^N \left( \int_{\Gamma \psi_{\xi_k \psi_{\xi_k}}} (\xi_k - 1_{[0, \psi_{\xi_k} \psi_{\xi_k}]} \right) d\mu_{\xi_k} d\xi_k 
\right. 
+ \int_{\Gamma \psi_{\xi_k \psi_{\xi_k}}} (1_{[0, \psi_{\xi_k} \psi_{\xi_k}} - \xi_k) \, d\mu \right. 
\left. + \int_{\Gamma \psi_{\xi_k \psi_{\xi_k}}} (\xi_k - 1_{[0, \psi_{\xi_k} \psi_{\xi_k}}) \, d\mu \right) 
\text{d}\xi_k.
$$

1We used a small amount of pre-smoothing to avoid smearing out possible edges, that is we used $\sigma = 1$ for the Gaussian kernel variance.
In order to prove (31) we observe the following identities,

$$
\int_\Omega |u_k - f_k| \, dx = \int_{-\infty}^{+\infty} \left( \int_{\Gamma_{uk} \setminus \Gamma_{f_k}} dx + \int_{\Gamma_{f_k} \setminus \Gamma_{uk}} dx \right) \, d\xi_k.
$$

Following the level set formulation of the functional $E_{gTV}^1$ given in Theorem 2.2, and the convexity of the minimization problem (23), we can consider the following geometric sub-problems

$$
\min_{\Gamma_k \subset \Omega} \left\{ G(\Gamma_k, \Gamma_{f_k}) = G_1(\Gamma_k) + G_2(\Gamma_k, \Gamma_{f_k}) + \sum_{i=1}^M \sum_{l=1}^N \alpha G_{3,l}(\nabla^{\xi_i}, \Gamma_k) \right\},
$$

where

$$
G_1(\Gamma_{uk}) = \text{Per}_{g_k}(\Gamma_{uk})
$$

and

$$
G_2(\Gamma_{uk}, \Gamma_{f_k}) = \lambda \left( \int_{\Gamma_{uk} \setminus \Gamma_{f_k}} \, dx + \int_{\Gamma_{f_k} \setminus \Gamma_{uk}} \, dx \right)
$$

and

$$
G_{3,l}(\nabla^{\xi_i}, \Gamma_{uk}) = \int_{0}^{1} \left( \int_{\Gamma_{\nabla^{\xi_i} u_k} \setminus \Gamma_{\nabla^{\xi_i} l_k}} (\xi_k - 1_{[0, \nabla^{\xi_i} u_k]}) \, dx \right. \right.
$$

$$
+ \left. \int_{\Gamma_{\nabla^{\xi_i} u_k} \setminus \Gamma_{\nabla^{\xi_i} l_k}} (1_{[0, \nabla^{\xi_i} u_k]} - \xi_k) \, dx \right) \mu_k
$$

$$
+ \int_{0}^{1} \left( \int_{\Gamma_{\nabla^{\xi_i} l_k} \setminus \Gamma_{\nabla^{\xi_i} u_k}} (\xi_l - 1_{[0, \nabla^{\xi_i} u_k]}) \, dx \right. \right.
$$

$$
+ \left. \int_{\Gamma_{\nabla^{\xi_i} u_k} \setminus \Gamma_{\nabla^{\xi_i} l_k}} (1_{[0, \nabla^{\xi_i} u_k]} - \xi_l) \, dx \right) \mu_k.
$$

With the geometrical minimization formulation (27), we can prove the monotonicity property, which is in turn a generalization of the property called the submodularity of Per functional [1].

**Theorem 2.3.** (Monotonicity Property) Let $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1 < \xi_2$. Given an image $f_k$ and two minimizers $\Gamma^1_{f_k}$ and $\Gamma^2_{f_k}$ with finite parameter of the minimization problem (27) corresponding to $\Gamma^1_{f_k} = \Gamma(f_k, \xi_1)$ and $\Gamma^2_{f_k} = \Gamma(f, \xi_2)$ respectively, the following identities are satisfied

$$
G(\Gamma^2, \Gamma^1_{f_k}) - G(\Gamma^1 \cap \Gamma^2, \Gamma^1_{f_k}) \geq G(\Gamma^1 \cup \Gamma^2, \Gamma^1_{f_k}) - G(\Gamma^1, \Gamma^1_{f_k}),
$$

and

$$
0 \geq G(\Gamma^2, \Gamma^2_{f_k}) - G(\Gamma^1 \cap \Gamma^2, \Gamma^2_{f_k}) \geq G(\Gamma^2, \Gamma^2_{f_k}) - G(\Gamma^1 \cap \Gamma^2, \Gamma^2_{f_k}).
$$

**Proof.** In order to prove (31) we observe the following identities,

$$
(\Gamma^2_k \setminus \Gamma_{f_k}) \setminus (\Gamma^1_k \cap \Gamma^2_k) \setminus \Gamma_{f_k} = (\Gamma^2_k \setminus \Gamma_{f_k}) \setminus \Gamma_{f_k},
$$

and

$$
((\Gamma^2_k \cup \Gamma^1_k) \setminus \Gamma_{f_k}) \setminus (\Gamma^1_k \setminus \Gamma_{f_k}) = (\Gamma^2_k \setminus \Gamma_{f_k}) \setminus \Gamma_{f_k}.
$$
As \((\Gamma_k^1 \cap \Gamma_k^2) \subset \Gamma_k^2\), we have

\[
(\Gamma_{f_k}^1 \setminus (\Gamma_k^1 \cap \Gamma_k^2)) \setminus (\Gamma_{f_k}^1 \setminus \Gamma_k^2) = (\Gamma_{f_k}^1 \setminus \Gamma_k^2) \setminus \Gamma_k^1,
\]

and

\[
(\Gamma_{f_k}^1 \setminus \Gamma_k^1) \setminus (\Gamma_{f_k}^1 \setminus (\Gamma_k^1 \cup \Gamma_k^2)) = (\Gamma_{f_k}^1 \setminus \Gamma_k^2) \setminus \Gamma_k^1.
\]

From the identities (33), (34), (35) and (36) combined all together, it follows

\[
\mathcal{G}_2(\Gamma^2, \Gamma_{f_k}^1) - \mathcal{G}_2(\Gamma^1 \cap \Gamma^2, \Gamma_{f_k}^1) - \mathcal{G}_2(\Gamma^1 \cup \Gamma^2, \Gamma_{f_k}^1) + \mathcal{G}_2(\Gamma^1, \Gamma_{f_k}^1) = 0.
\]

Similarly, the same previous reasoning can be applied to coupling term \(\mathcal{G}_3\), and we get

\[
\mathcal{G}_3(\nabla \xi, \Gamma^2) - \mathcal{G}_3(\nabla \xi, \Gamma^1 \cap \Gamma^2) - \mathcal{G}_3(\nabla \xi, \Gamma^1 \cup \Gamma^2) + \mathcal{G}_3(\nabla \xi, \Gamma^1) = 0.
\]

By considering the sub modularity property of \(\text{Per}\) functional and the identities (37) and (38) that

\[
\mathcal{G}(\Gamma^2, \Gamma_{f_k}^1) - \mathcal{G}(\Gamma^1 \cap \Gamma^2, \Gamma_{f_k}^1) - \mathcal{G}(\Gamma^1 \cup \Gamma^2, \Gamma_{f_k}^1) + \mathcal{G}(\Gamma^1, \Gamma_{f_k}^1) =
\]

\[
= \mathcal{G}_1(\Gamma^2) - \mathcal{G}_1(\Gamma^1 \cap \Gamma^2) - \mathcal{G}_1(\Gamma^1 \cup \Gamma^2) + \mathcal{G}_1(\Gamma^1) \geq 0.
\]

To prove the identity (32), we apply a similar reasoning as before to the term

\[
\mathcal{G}(\Gamma^2, \Gamma_{f_k}^1) - \mathcal{G}(\Gamma^1 \cap \Gamma^2, \Gamma_{f_k}^1) - \mathcal{G}(\Gamma^2, \Gamma_{f_k}^1) + \mathcal{G}(\Gamma^1 \cap \Gamma^2, \Gamma_{f_k}^1).
\]

The following theorem proves that the our \(gVTV + L^1\) energy minimization problem (23) is well-posed. The proof involves the direct methods of calculus of variations.

**Theorem 2.4.** Given an image \(f \in L^\infty(\Omega, \mathbb{R}^N)\), there exists a minimizer of (23) in \(BV(\Omega, \mathbb{R}^N)\).

**Proof.** For the proof of this theorem see Appendix A. \(\square\)

The parameter \(\lambda\) in our minimization functional (23) is crucial in experimental results (see Section 4.2 which shows the importance of \(\lambda\) with respect to image denoising and different error metrics). The value of \(\lambda\) determines how much of important geometrical features are retained. We extend the work of Scherzer et al [60] to the vectorial case and study \(G(\Omega)\) color space à la Y. Meyer [42] for obtaining important quantitative guidelines, see Appendix B.

### 3. Dual minimization based implementation.

Implementation of \(VTV\) with coupling in (23) can be done in a variety of ways. The dual minimization technique for total variation introduced by Chambolle [20] is an attractive one as it is proven to be very efficient. We adapt dual minimization to our model and provide a convergence result for the alternative minimizations. Following [14] we use a splitting scheme by introducing an auxiliary variable \(v\),

\[
\min_{u, v} \left\{ \tilde{E}^3_{gVTV}(u, v) \right\}
\]

\[
= \sum_{k=1}^N \left( \int_{\Omega} g_k(x) |\nabla u_k| \, dx + \frac{1}{2\theta_k} \int_{\Omega} (u_k + v_k - f_k)^2 \, dx + \lambda \int_{\Omega} |v_k| \, dx \right).
\]
or equivalently the subproblems

\[
V_k \left( u_k, v_k \right) = \int_\Omega g_k(x) |\nabla u_k| \, dx + \frac{1}{2\theta_k} \int_\Omega (u_k + v_k - f_k)^2 \, dx + \lambda \int_\Omega |v_k| \, dx
\]

where \( \theta \) is chosen to be small so that \( f_k \sim u_k + v_k \), where \( (u_k, v_k) \) represents the cartoon and texture components respectively. Since the functional \( V_k \) is convex, (42) can be split into two alternating minimization problems and a fast numerical scheme based on Chambolle’s dual minimization [20] can be used.

The solution of the minimization problem (42) is given by the following alternating minimization steps:

1. Fixing \( v_k \), the minimization problem in \( u_k \) is:

\[
\min_{u_k} \left\{ \int_\Omega g_k(x) |\nabla u_k| \, dx + \frac{1}{2\theta_k} \|u_k + v_k - f_k\|_{L^2(\Omega, \mathbb{R})}^2 \right\}.
\]

The solution of (43) is given by

\[
u_k = f_k - v_k - \theta_k \text{div} p_k\]

where \( p_k = (p_{k1}, p_{k2}) \) satisfies \( g_k(x) \nabla (\theta_k \text{div} p_k - (f_k - v_k)) - |\nabla (\theta_k \text{div} p_k - (f_k - v_k))| p_k = 0 \), which is solved using a fixed point method: \( p_k^0 = 0 \) and

\[
p_k^{n+1} = p_k^n + \frac{\Delta t}{\theta_k} \nabla (\text{div} (p_k^n) - (f_k - v_k)/\theta_k) - \frac{\Delta t}{\theta_k} \nabla (\text{div} (p_k^n) - (f_k - v_k)/\theta_k)
\]

We remark that (43) can also be implemented using algorithms which solve TV minimization problems of the form \( \min_u \{ gTV(u) + \lambda \|u - f\|_2^2 \} \), such as the split Bregman [37], fast gradient based (FISTA) [10], and the first-order primal-dual (due to Chambolle and Pock) [21] algorithms, by considering \( u_k = u \) and \( f = f_k - v_k \).

2. Fixing \( u_k \), the minimization problem in \( v_k \) is:

\[
\min_{v_k} \left\{ \frac{1}{2\theta_k} \|u_k + v_k - f_k\|_{L^2(\Omega, \mathbb{R})}^2 + \lambda \|v_k\|_{L^1(\Omega, \mathbb{R})} + \alpha \sum_{i=1}^N \|\nabla v_i - \nabla v_k\|_{L^2(\Omega, \mathbb{R})}^2 \right\},
\]

and the solution is found as

\[
v_k = 2\alpha \sum_{i=1}^N (\Delta v_i - \Delta v_k) + \begin{cases}
  f_k - u_k - \theta_k \lambda_k & \text{if } f_k - u_k \geq \theta_k \lambda_k, \\
  f_k - u_k + \theta_k \lambda_k & \text{if } f_k - u_k \leq -\theta_k \lambda_k,
\end{cases}
\]

By implementing the algorithm, we get the sequences,

\[(u_0, v_0, u_1, v_1, \ldots, u_k, v_k, \ldots).\]
The following theorem provides a proof of convergence of these sequences based on the coerciveness of the functionals in (42). This result is supported by experimental results, see for example Figure 7 energy plots.

**Theorem 3.1.** The sequence \((u_k, v_k)\) given by the minimization problem (42) converges to coordinatewise minimum on \(BV(\Omega, \mathbb{R}^N) \times BV(\Omega, \mathbb{R}^N)\).

**Proof.** For the proof of this theorem see Appendix A.

4. **Experimental results.** All the images are normalized to the range [0, 1] and we implement the proposed \(gVTV + L^1\) model (23) using the dual minimization implementation described in section (3). The algorithm is implemented in MATLAB2012a on a Mac laptop with Intel Core i7 CPU 2.3GHz, 8GB RAM CPU. The edge indicator function \(g = 1/(1 + |\nabla I|^2)\) is used for all the results reported here. The \(\alpha = 1/2, \delta t = 1/8\) and \(\theta = 1\) are fixed, and the best results according to the \(\max\{||u_n^{k+1} - u_n^n||, ||v_n^{k+1} - v_n^n|| \leq \epsilon\}\) are shown. A quantitative evaluation of the proposed decomposition model is performed for both our approach and other related methods in color images\(^2\). The comparative performance analysis is done over the following databases: Mosaic art images\(^3\), Kodak Lossless True Color Image Suite\(^4\), Color Test Images Database\(^5\), USC-SIPI Image Database\(^6\) and Berkeley segmentation dataset (BSDS500)\(^7\). Our model takes < 10 sec for 50 iterations for 3 channels image of size 256 × 256. To reduce the computational cost one can utilize faster alternative algorithmic implementations such as split Bregman [37], FISTA [10] or Chambolle-Pock first order primal-dual [21]. We consider here color image processing, that is, \(N = 3\), for RGB channels and extension of the cross-channel term (6) to cases \(N > 3\) is straightforward [51].

4.1. **Decomposition and multiscale decomposition results.** Image decomposition provides a separation of a given image into cartoon (piecewise constant) and texture components. For obtaining decomposition results we choose \(\lambda = 1/2\) in our scheme, since with this value we are able to obtain nice cartoon (smooth) regions along with edge preservation. A better justification for a practical choice of the \(\lambda\) parameter is displayed in Tables 1-3 for the denoising results.

In Figure 2 we use two images from the Mosaic art database to first show the cartoon component result using the proposed \(VTV + L^1\) scheme, and \(L^2\) fidelity terms. We also display the result incorporating the weight function \(g\) in the proposed model. By comparing the cartoon component of the proposed \(gVTV + L^1\) against the \(gVTV + L^2\) scheme for the input image Figure 2(a) we see that they behave qualitatively differently. For example, different shapes of the input image are preserved well in our scheme using the \(L^1\) fidelity terms whereas the result using the implementation with the \(L^2\) fidelity terms looks blurred in the final result. Canny edge map computed with all the three channels using the MATLAB’s command `edge(Input, ‘canny’)` for both different results (implementing \(gVTV + L^1\) and \(gVTV + L^2\) schemes) indicates that chromatic edges are preserved coherently and the spatial location is intact. Figure 2(b) shows the difference of implementing the

\(^2\)More results are available online: http://sites.google.com/site/suryaiit/research/spectral/vtv
\(^3\)http://www.cse.cuhk.edu.hk/leojia/projects/texturesep/
\(^4\)http://r0k.us/graphics/kodak
\(^5\)http://www.hlevkin.com/TestImages/classic.htm
\(^6\)http://sipi.usc.edu
\(^7\)http://www.eecs.berkeley.edu/Research/Projects/CS/vision/bdss/
Figure 2. Cartoon component results showing the differences between using the $L^1$ and $L^2$ norms in the fitting term, as well as the differences of using a weight for the proposed regularized term (stopping parameter $\epsilon = 10^{-2}$). Results show better edge preservation using $L^1$ norm in the fitting term and $g_k = 1/(1 + |\nabla f_k|^2)$ with our proposed VTV method.
proposed VTV with coupling channels approach by using edge indicator functions $g_k = (1 + |\nabla f_k|)^{-1}$ and $g_k = 1$. As can be seen from the corresponding cartoon and edge map components, the result using edge indicator function contains smaller scale chromatic edges compared with setting $g_k = 1$ in our implementation.

Figure 3 shows different texture components for two color images (stopping time parameter $\epsilon = 10^{-2}$). First Row: For Boats image we show the (a) input image (b) cartoon (smooth), (c) texture component, and (d) scaled texture component. Second Row: For Goldhill image as input (e), we show the texture components from (f) Red (g) Green, and (h) Blue texture channels.

Figure 4 shows the proposed color coupling channels for the cartoon components of close-up from Motorcycle & Lighthouse images. As can be seen, different channel edges are captured by the coupling terms (bottom row) and the final cartoon (smooth) components are devoid of small scale texture details. Moreover, the smoothing is edge preserving due to the weighted $TV$ regularization. Figure 5 provides cartoon and texture components for a variety of natural color images using our scheme as stopping parameter increases from $\epsilon = 10^{-4}$ to $\epsilon = 10^{-2}$. We notice that when the stopping time decreases texture details are gradually lost in the cartoon components and they are added to the texture component.

We next show in Figure 6 a comparison of the cartoon and texture components for the Lighthouse image using different vectorial total variation terms along with $L^1$ data fidelity: $gVTV_1 + L^1$ and $gVTV_2 + L^1$ with our coupling VTV method. We can observe that cartoon component results look visually similar for all three
Figure 4. Cartoon component results using the proposed $gVTV + L^1$ model together with their Red, Green and Blue gradient magnitudes and the coupling terms (stopping parameter $\epsilon = 10^{-2}$). We show original inputs and their cartoon components, gradient magnitudes for Red ($u_1$), Green ($u_2$), Blue ($u_3$), and corresponding coupling terms.
Figure 5. Cartoon and texture decomposition for different color images by implementing the proposed model and different choices of the stopping parameter $\epsilon$. (a) Original color RGB images. (b)-(d) Decomposition results with $\epsilon = 10^{-2}$, $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$, respectively.
Original color image together with patches and color histograms of the full image

(a) Input  (b) Patch 1  (c) Patch 2  (d) Patch 3

Decomposition components and color histograms of the full image for the $gVTV + L^1$ model

Decomposition components and color histograms of the full image for the $gVTV_1 + L^1$ model

Decomposition components and color histograms of the full image for the $gVTV_2 + L^1$ model

Figure 6. Cartoon and texture decomposition of a natural image together with the corresponding R, G, B histograms (stopping parameter $\epsilon = 10^{-4}$). Decomposition results of small patches are also displayed to highlight the cartoon and texture separation of different methods.
models, although our scheme retains the structures better. This can be observed in the color distribution of cartoon components using the corresponding histograms. It can also be noted from scaled texture components that edges are smoother for our proposed $gVTV + L^1$ scheme.

Figure 7 depicts cartoon-texture decomposition experiments using different natural images from Berkeley Segmentation data set (BSDS500), using our $gVTV + L^1$ model, compared with $gVTV_1 + L^1$ and $gVTV_2 + L^1$ schemes. A close look at the cartoon components reveals that our scheme obtains better preservation of small scale geometrical structures. For both the Butterfly, Lady images we show corresponding energy decrease against number of iterations of the dual minimization implementation for different $VTV$ schemes (see Theorem 3.1).

Following Yin et al [73] and Tang et al [67] we can make the $gVTV + L^1$ model with multiscale parameter $\lambda_k = \lambda_0/2^k$ in the $L^1$-fidelity terms with $k = 1, 2, \ldots$ and $\lambda_0$ a given value. Figure 8 displays 5 scales of our decomposition for the close-up of Barbara color image. Cartoon components (Figure 8(b-e)) retain salient edges during increasing steps and corresponding (scaled) texture components (Figure 8(g-j)) start to show progressive capture from small scale textures to big scales.

4.2. Denoising results. Note that the decomposition provides cartoon - piecewise cartoon component which is obtained using a weighted $TV$ ($gVTV$) regularization in an edge preserving way. This allows us to apply our model to color image denoising as well, with $u$ being the denoised image and $v$ noise/texture part. To compare the schemes quantitatively in denoising we report our results using three standard error metrics: mean squared error (MSE), peak signal to noise ratio (PSNR) [4], and mean structural similarity measure (MSSIM) [71]. We also fix the stopping parameter $\epsilon = 10^{-2}$ for all the results.

1. MSE measures the average of the squares of the errors, i.e., the difference between the estimator and what is estimated. Lower MSE value indicates better denoising result. For an estimated RGB image $u$:

\[
\text{MSE}(u) := \sum_{x \in \Omega} (u - u_0)^2/|\Omega|,
\]

where $|\Omega|$ is the size of the image domain and $u_0$ is the original noise-free color image.

2. PSNR is given in decibels (dB). A difference of $0.5 \text{dB}$ can be identified visually. Higher PSNR value indicates optimum denoising capability.

\[
\text{PSNR}(u) := 10 \log_{10} \left( \frac{1}{\sqrt{\text{MSE}}} \right) \text{dB}.
\]

3. MSSIM index is in the range $[0, 1]$. MSSIM value near one implies the optimal denoising capability of the scheme [71] and is mean value of the SSIM metric. SSIM uses a combined product of luminance, contrast and structure comparison measures. The SSIM is calculated between two windows $\omega_1$ and $\omega_2$ of common size $N \times N$,

\[
\text{SSIM}(\omega_1, \omega_2) = \frac{(2\mu_{\omega_1}\mu_{\omega_2} + c_1)(2\sigma_{\omega_1\omega_2} + c_2)}{\left(\mu_{\omega_1}^2 + \mu_{\omega_2}^2 + c_1\right)\left(\sigma_{\omega_1}^2 + \sigma_{\omega_2}^2 + c_2\right)},
\]

where $\mu_{\omega_i}$ the average of $\omega_i$, $\sigma_{\omega_i}^2$ the variance of $\omega_i$, $\sigma_{\omega_1\omega_2}$ the covariance, $c_1, c_2$ stabilization parameters, see [71] for more details.
Figure 7. Cartoon + Texture decomposition results with corresponding energy versus iterations (stopping parameter $\epsilon = 10^{-4}$). Third Row: $gVTV + L^1$ proposed model and $gVTV_1 + L^1$ approach. Fourth Row: $gVTV + L^1$ proposed model and $gVTV_2 + L^1$ approach.

Since the main application of our proposed model is to study denoising based on chromatic edges through the different channels, we determine the MSE, PSNR and MSSIM error metrics on each channel and use the average as a final value for the
Figure 8. Multiscale cartoon and texture decomposition results for the close-up Barbara with 5 steps by implementing our proposed approach (stopping parameter $\epsilon = 10^{-4}$).

The full Berkeley segmentation dataset (BSDS500) where different noise levels (with Gaussian noise of standard deviation $\sigma = 20$ and $\sigma = 30$) have been added.

Tables 1, 2 and 3 compare the MSE, PSNR and MSSIM performance measures for the $gVTV_1 + L^1$, $gVTV_2 + L^1$ and the proposed $gVTV + L^1$ schemes by using different $\lambda$ values weighting the $L^1$ fidelity term averaged over the complete BSDS500, respectively. The idea behind choosing different $\lambda$ values is to determine which value is able to capture important geometrical features (see Theorem B.5 in Appendix B related to the choice of $\lambda$). The proposed vectorial total variation approach with coupling channels scheme performed better than other vectorial total variation approaches over different $\lambda$ values. We also see that MSE, PSNR and MSSIM value change over different $\lambda$ values, with decreasing MSE values and increasing PSNR and MSSIM values. We note that $\lambda = 1$ is the best choice for multichannel edge preservation property followed by $\lambda = 1/2^k$ with $k \neq 0$ which tends to over-smooth the images. Note that the MSSIM is a better error metric than MSE and PSNR as it provides a quantitative way of measuring the structural similarity of denoised image against the original noise-free image. Different images for visualizing the improvements of the proposed $gVTV + L^1$ scheme are shown in Figure 9, and Figure 10 where we can see the effect of different VTV regularizations corresponding to $\lambda$ choices.

Next we use the digitized color images taken from the USC-SIPI standard image dataset to compare the performance of the proposed scheme (23) (Our) together with the multiscale case (M-Our) along with following schemes: variational regularization (BKS, gradient descent with finite differences) [15], vector valued regularization with PDEs of (TD, spatial discretization with centered finite differences) [68], regularization of vector signals based on vectorial TV norm (BC, Chambolle’s dual approach [20]) [13], Perona and Malik with a coupling term [55] (PS, finite differences), multiscale vectorial $L^7$-TV scheme for color images (DHC, using a unified...
Table 1. Average MSE values for the $gVTV + L^1$ proposed model and the $gVTV_1 + L^1$, $gVTV_2 + L^1$ schemes (see equations (4) and (5)) over different Gaussian noise levels and $\lambda$ parameters on the complete Berkeley segmentation database (BSDS500). Lower value indicates better result.

| Method      | Noise 20% | MSE (Average) $\lambda = 1/2^0$ | MSE (Average) $\lambda = 1/2^1$ | MSE (Average) $\lambda = 1/2^2$ | MSE (Average) $\lambda = 1/2^3$ | MSE (Average) $\lambda = 1/2^4$ |
|-------------|-----------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $gVTV + L^1$ | 101.4588  | 109.3552                        | 116.6560                        | 122.3441                        | 126.6188                        |
| $gVTV_1 + L^1$ | 119.0799  | 135.8866                        | 149.9856                        | 160.6470                        | 168.5642                        |
| $gVTV_2 + L^1$ | 117.7041  | 132.0977                        | 145.2602                        | 155.4844                        | 163.1516                        |
| $gVTV + L^1$ | $\sqrt{}$ | 156.6409                        | 164.6322                        | 173.2940                        | 180.2798                        | 185.5986                        |
| $gVTV_1 + L^1$ | 173.2375  | 188.5963                        | 203.7099                        | 215.6114                        | 224.5914                        |
| $gVTV_2 + L^1$ | 174.1293  | 185.4584                        | 198.8797                        | 209.9324                        | 218.4031                        |

Table 2. Average PSNR values for the $gVTV + L^1$ proposed model and the $gVTV_1 + L^1$, $gVTV_2 + L^1$ schemes (see equations (4) and (5)) over different Gaussian noise levels and $\lambda$ parameters on the complete Berkeley segmentation database (BSDS500). Higher value indicates better result.

| Method      | Noise 20% | PSNR (dB) (Average) $\lambda = 1/2^0$ | PSNR (dB) (Average) $\lambda = 1/2^1$ | PSNR (dB) (Average) $\lambda = 1/2^2$ | PSNR (dB) (Average) $\lambda = 1/2^3$ | PSNR (dB) (Average) $\lambda = 1/2^4$ |
|-------------|-----------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $gVTV + L^1$ | 29.0622   | 28.7418                         | 28.4856                         | 28.3011                         | 28.1685                         |
| $gVTV_1 + L^1$ | 28.3252   | 27.7356                         | 27.2851                         | 26.9635                         | 26.7322                         |
| $gVTV_2 + L^1$ | 28.3311   | 27.8423                         | 27.4229                         | 27.1128                         | 26.8868                         |
| $gVTV + L^1$ | $\sqrt{}$ | 27.6266                         | 27.4016                         | 27.2122                         | 27.0823                         | 26.9892                         |
| $gVTV_1 + L^1$ | 27.0800   | 26.5864                         | 26.1756                         | 25.8667                         | 25.6422                         |
| $gVTV_2 + L^1$ | 27.0315   | 26.6478                         | 26.2724                         | 25.9839                         | 25.7706                         |

Table 3. Average MSSIM values for the $gVTV + L^1$ proposed model and the $gVTV_1 + L^1$, $gVTV_2 + L^1$ schemes (see equations (4) and (5)) over different Gaussian noise levels and $\lambda$ parameters on the complete Berkeley segmentation database (BSDS500). Values close to 1 indicates better result.

| Method      | Noise 20% | MSSIM (Average) $\lambda = 1/2^0$ | MSSIM (Average) $\lambda = 1/2^1$ | MSSIM (Average) $\lambda = 1/2^2$ | MSSIM (Average) $\lambda = 1/2^3$ | MSSIM (Average) $\lambda = 1/2^4$ |
|-------------|-----------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $gVTV + L^1$ | 0.7991    | 0.7858                          | 0.7740                          | 0.7653                          | 0.7589                          |
| $gVTV_1 + L^1$ | 0.7721    | 0.7475                          | 0.7286                          | 0.7151                          | 0.7053                          |
| $gVTV_2 + L^1$ | 0.7682    | 0.7527                          | 0.7363                          | 0.7237                          | 0.7143                          |
| $gVTV + L^1$ | $\sqrt{}$ | 0.7302                          | 0.7206                          | 0.7100                          | 0.7019                          | 0.6959                          |
| $gVTV_1 + L^1$ | 0.7068    | 0.6990                          | 0.6749                          | 0.6634                          | 0.6550                          |
| $gVTV_2 + L^1$ | 0.6954    | 0.6999                          | 0.6790                          | 0.6688                          | 0.6610                          |

Moreau-Yosida based primal-dual approach) [28] which uses the $VTV_2$ regularization, vectorial total variation deriviation from the generalized Jacobians from geometric measure (GSC, primal-dual method [21]) [36], multichannel gradients combined by parallel level sets (EA, central differencing scheme) [32], and separate gradient measurement of the luminance component and that of the chrominance (OY, primal-dual method [21]) [48]. In these results, we do not average the PSNR and the MSSIM error metrics over the RGB channels. Instead, we apply directly the PSNR error metric for multichannel images given by equation (46) and (47), respectively. In case of the MSSIM error metric we adapt it to color images by converting the color image into a grayscale (for example, using MATLAB’s command rgb2gray) and then compute MSSIM for the converted grayscale image. Results are displayed in Table 4 for seventeen standard test images of size $256 \times 256$ and $512 \times 512$. Our proposed vectorial total variation with coupling tends to perform better than
| Noisy M-SSIM | BKS | TD | BC | PS | DHC | GSC | EA | OY | Our M-Our | BCM | DFRE |
|--------------|-----|----|----|----|-----|-----|----|----|----------|-----|------|
| 0.520†       | 27.24 | 29.56 | 29.42 | 27.99 | 30.66 | 29.75 | 30.03 | 31.10 | 31.05 | 31.37 | 31.04 | 32.23 |
| 0.524†       | 26.97 | 29.69 | 29.02 | 29.21 | 30.35 | 29.41 | 30.24 | 23.36 | 30.20 | 31.29 | 31.51 | 33.18 |
| 0.381†       | 29.55 | 32.59 | 31.75 | 23.25 | 33.51 | 31.97 | 33.52 | 34.83 | 33.99 | 34.25 | 34.69 | 36.62 |
| 0.505†       | 27.68 | 31.04 | 30.87 | 24.49 | 31.75 | 31.14 | 30.15 | 28.80 | 33.68 | 32.85 | 32.32 | 34.10 |
| 0.487†       | 26.85 | 30.18 | 27.43 | 24.81 | 31.53 | 27.67 | 29.85 | 31.46 | 31.78 | 32.53 | 32.32 | 33.78 |
| 0.663†       | 23.10 | 26.64 | 25.09 | 23.67 | 28.54 | 25.39 | 27.41 | 28.77 | 28.72 | 29.15 | 29.37 | 29.70 |
| 0.393†       | 29.13 | 32.72 | 31.22 | 27.97 | 32.84 | 31.64 | 31.93 | 33.56 | 32.51 | 33.40 | 34.09 | 35.52 |
| 0.465†       | 27.19 | 31.13 | 29.74 | 28.34 | 31.56 | 30.03 | 31.01 | 32.09 | 31.59 | 32.21 | 32.98 | 33.59 |
| 0.764†       | 31.11 | 33.52 | 33.07 | 24.90 | 34.28 | 33.42 | 31.36 | 22.51 | 33.78 | 34.41 | 34.27 | 35.62 |
| 0.798†       | 28.91 | 30.88 | 30.00 | 17.78 | 32.62 | 30.24 | 29.24 | 22.91 | 32.79 | 32.85 | 32.20 | 33.08 |
| 0.910†       | 20.53 | 23.53 | 20.82 | 19.51 | 26.02 | 21.09 | 24.96 | 25.67 | 26.19 | 26.41 | 26.75 | 26.97 |
| 0.808†       | 27.39 | 29.84 | 29.77 | 22.52 | 31.69 | 30.12 | 31.01 | 31.15 | 31.74 | 32.27 | 31.86 | 33.01 |
| 0.785†       | 24.65 | 26.50 | 27.48 | 26.78 | 31.66 | 27.85 | 29.71 | 30.35 | 31.53 | 32.13 | 32.16 | 33.75 |
| 0.845†       | 23.34 | 26.85 | 26.30 | 23.90 | 29.24 | 26.72 | 28.07 | 28.43 | 29.35 | 29.22 | 29.23 | 29.48 |
| 0.849†       | 27.81 | 30.07 | 28.84 | 29.09 | 30.90 | 29.09 | 28.93 | 29.48 | 30.52 | 31.14 | 30.84 | 31.63 |
| 0.860†       | 24.94 | 27.27 | 25.28 | 24.47 | 28.68 | 25.56 | 28.25 | 25.94 | 28.55 | 29.78 | 30.93 | 33.03 |
| 0.861†       | 0.796 | 0.870 | 0.836 | 0.850 | 0.932 | 0.849 | 0.935 | 0.941 | 0.945 | 0.953 | 0.955 | 0.974 |

Table 4. PSNR (dB) & M-SSIM values for results of various schemes with noise level $\sigma = 20$ for standard images of size $f = 256 \times 256$ (Noisy PSNR = 25.27) and $g = 512 \times 512$ (Noisy PSNR = 25.83). Higher PSNR & M-SSIM values indicate better denoising results. The best results for each image is indicated with **boldface** (within variational/PDE based methods only). The last two columns are nonlocal means [16] and color-BM3D [26] results.
\( \lambda = 1/2^0 \): Noisy Image (Left), Proposed (Center - PSNR=30.17), \( gVTV + L^1 \) (Right - PSNR=29.23)

\( \lambda = 1/2^1 \): Noisy Image (Left), Proposed (Center - PSNR=28.51), \( gVTV_1 + L^1 \) (Right - PSNR=27.66)

\( \lambda = 1/2^2 \): Noisy Image (Left), Proposed (Center - PSNR=27.05), \( gVTV_2 + L^1 \) (Right - PSNR=25.23)

\( \lambda = 1/2^3 \): Noisy Image (Left), Proposed (Center - PSNR=29.12), \( gVTV_1 + L^1 \) (Right - PSNR=28.03)

\( \lambda = 1/2^4 \): Noisy Image (Left), Proposed (Center - PSNR=31.46), \( gVTV_2 + L^1 \) (Right - PSNR=29.22)

**Figure 9.** Denoising results for different images corrupted with 20\% of Gaussian noise. The values of \( \lambda = 1/2^0, \lambda = 1/2^1, \lambda = 1/2^2, \lambda = 1/2^3, \lambda = 1/2^4 \) are used for comparing the proposed \( gVTV + L^1 \) scheme with \( gVTV_1 + L^1 \) and \( gVTV_2 + L^1 \) models.

other variational/PDEs base diffusion methods by improving color enhancement properties along with edge preservation. Although, we are not comparing with other state-of-the art schemes (non variational-PDE methods) in terms of PSNR and MSSIM, the last two columns in Table 4 show the results for non-local means (BCM [16]) and block matching 3D filtering (DFKE [26]). These results evidence that the proposed model is able to remove noise while preserving edges (according to PSNR (dB) & MSSIM values, which are relatively close to these state of the art schemes).
Figure 10. Denoising results for different images corrupted with 30% of Gaussian noise. The values of $\lambda = 1/2^0, \lambda = 1/2^1, \lambda = 1/2^2, \lambda = 1/2^3, \lambda = 1/2^4$ are used for comparing the proposed $gVT + L^1$ scheme with $gVTV_1 + L^1$ and $gVTV_2 + L^1$ models.

Finally, Figure 11 shows a visual comparison of different methods on sub-images (size 330 × 360) of the noisy Lena test image with $\sigma = 20$ for the TV based schemes OY [48], GSC [36] and DHC [28]. Visual comparison of various regions show that color smearing artifacts occur in chromatic edges in other schemes due to channel mixing and diffusion transfer and also some staircasing artifacts in flat regions are visible. The method noise comparison shows that our multiscale method leaves no visible structure in the residue in comparison with other schemes. The parameters
Figure 11. Denoising results for the close-up Lena image corrupted with Gaussian noise ($\sigma = 20$) together with neighbour regions. (a) Original image, input noise and different sub-regions of the original image. (b)-(c) results of our $gVTV + L^1$ (with multiscale) and $gVTV + L^1$ (with $\lambda = 1$) schemes respectively, displaying denoised images, residual noise/method noise components ($u - u_0$) and different sub-regions of the denoised images.

In all the comparison schemes are chosen based on maximum peak signal-to-noise ratio (PSNR).

5. **Conclusion.** In this paper, we study a multichannel image restoration scheme with total variation regularization and $L^1$ fidelity. By using matched gradients along with cross channels to align the edges our scheme performs multichannel smoothing and restoration. We provided a complete mathematical characterization of our vectorial total variation along with channel coupling model using its geometrical formulation, which provides some insights for the original model. Utilizing the notion of $G$-sets and $G$-values, we give a characterization of a solution of the proposed minimization problem in terms of the $\lambda$ parameter, as well as the ability of such a parameter to decouple image features at different level sets. A fast dual minimization based implementation was used and experimental results indicate the proposed scheme provides better smoothing than other related models. Experimental results with other related color image denoising schemes indicate that the proposed approach consistently outperforms other color based restoration methods in terms of the perceptually guided PSNR (dB), MSSIM and MSE error metrics.
Appendix A. Existence of solution and convergence of the dual minimization based scheme.

Proof of Theorem 2.4. Let $C := \inf E_{gVTV}^1(u)$, and let $\{u^n\}_{n=1}^\infty \subseteq BV(\Omega, \mathbb{R}^N)$ be a minimizer sequence for the energy $E_{gVTV}^1$, i.e.,

$$E_{gVTV}^1(u^n) \xrightarrow{n \to \infty} C.$$  

Then, for a given $k = 1, \ldots, N$, it follows

$$\int_\Omega |\nabla u^n_k| \, dx \leq E_{gVTV}^1(u^n),$$

and consequently

$$\sup_{n \in \mathbb{N}} \int_\Omega |\nabla u^n_k| \, dx < \infty.$$  

Moreover,

$$\|u^n_k\|_{L^1(\Omega)} \leq \int_\Omega dx < \infty.$$  

Then, we have that $\{u^n\}_{n=1}^\infty$ is bounded in $BV(\Omega, \mathbb{R}^N)$. By compactness property of the $BV(\Omega, \mathbb{R}^N)$ space there is a subsequence also denoted by $\{u^n\}_{n=1}^\infty$, strongly convergent to an element $u^* = (u^*_1, \ldots, u^*_N) \in L^1(\Omega, \mathbb{R}^N)$. Therefore, it follows that

Figure 11. (Continued) Denoised image, method noise $u_0 - u$, and different sub-regions of the denoised image for schemes (d) OY [48], (e) GSC [36] and (f) DHC [28].
\[ u^* \in BV(\Omega, \mathbb{R}^N) \]

\[\int_{\Omega} |\nabla u_k^*| \, dx \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla u^n_k| \, dx.\]

We also notice that the fidelity term and the coupling term in \( E_{gVTV}^1 \) are convex in \( u \), then both term are lower semicontinuous with respect to the \( BV(\Omega, \mathbb{R}^N) \) weak* topology and therefore

\[\int_{\Omega} |u_k^* - u_l^*|^2 \, dx \leq \liminf_{n \to \infty} \int_{\Omega} |u^n_k - u^n_l|^2 \, dx.\]

and

\[\int_{\Omega} |u_k^* - f_k| \, dx \leq \liminf_{n \to \infty} \int_{\Omega} |u^n_k - f_k| \, dx.\]

Finally, combining inequalities (49), (50) and (51) on a suitable sequence \( \{u^n\}_{n=1}^{\infty} \), we get

\[ E_{gVTV}^1(u^*) \leq \liminf_{n \to \infty} E_{gVTV}^1(u^n) = C, \]

i.e., \( u^* \) is a minimizer of \( E_{gVTV}^1 \).

**Proof of Theorem 3.1.** Since we are solving a splitting scheme defined by the subproblems (43) and (45), the following inequalities hold

\[\tilde{E}_{gVTV}^1(u^{k+1}, v^{k+1}) \leq \tilde{E}_{gVTV}^1(u^k, v^{k+1}) \leq \tilde{E}_{gVTV}^1(u^k, v^k).\]

We also have that the sequence \( \tilde{E}_{gVTV}^1(u^k, v^k) \) is non increasing and bounded by zero. Then, there exist \( n \in \mathbb{R} \) such that \( n = \lim_{k \to \infty} \tilde{E}_{gVTV}^1(u^k, v^k) \). Since \( \tilde{E}_{gVTV}^1 \) is coercive, we have that the sequence \( (u^k, v^k) \) is bounded in \( BV(\Omega, \mathbb{R}^N) \times BV(\Omega, \mathbb{R}^N) \) allowing us to extract a subsequence \( (u^{kr}, v^{kr}) \) converging to \( (u^*, v^*) \), i.e., \( (u^{kr}, v^{kr}) \xrightarrow{k \to \infty} (u^*, v^*) \) and by (44) we have

\[u^{kr} = v^{kr-1} - \theta div(p) \xrightarrow{k \to \infty} u^* = v^* - \theta div(p).\]

Considering that

1. \( \tilde{E}_{gVTV}^1(u^{k+1}, v^{kr}) \leq \tilde{E}_{gVTV}^1(u, v^{kr}) \) for all \( u \in BV(\Omega, \mathbb{R}^N) \), and
2. \( \tilde{E}_{gVTV}^1(u^{k}, v^{kr}) \leq \tilde{E}_{gVTV}^1(u^{kr}, v) \) for all \( v \in BV(\Omega, \mathbb{R}^N) \),

it follows from (53) that for all \( u \in BV(\Omega, \mathbb{R}^N) \)

\[\tilde{E}_{gVTV}^1(u^*, v^*) \leq \tilde{E}_{gVTV}^1(u^*, v^*)\]

and for all \( v \in BV(\Omega, \mathbb{R}^N) \)

\[\tilde{E}_{gVTV}^1(u^*, v^*) \leq \tilde{E}_{gVTV}^1(u^*, v)\]

allowing us to conclude that \( (u^*, v^*) \) is a minimum with \( \tilde{E}_{gVTV}^1(u^*, v^*) = n. \)

**Appendix B. The color \( G(\Omega) \) space.** Characterizing the minimizers of non-differentiable regularization functional has been analyzed by many authors. For grayscale images, Y. Meyer [42] defined the space \( G(\mathbb{R}^2) \) to model texture using the ROF scheme [58]. Here, we extend the notion of \( G \)-sets and \( G \)-values studied in [60] to the vectorial case. For an observed color image \( \mathbf{f} \), we show a relation between the solutions of the minimization problem (23) and the parameter \( \lambda \) which balances the \( gVTV \) regularization term and \( L^1 \) fitting term. This will help us to correctly model the cartoon and texture decomposition, i.e., \( \mathbf{f} \approx \mathbf{u} + \mathbf{v} \). A quantitative study using
MSE, PSNR (db) and MSSIM error metrics is given on different selections for the \( \lambda \) in denoising, see Section 4.2 and Tables 1, 2 and 3. See also [3, 70, 69, 49, 73, 31, 61] for related efforts to study TV models in this context.

The following definition is for vectorial \( G \)-values using measurable functions.

**Definition B.1.** Let \( \Psi = (\Psi_1, \ldots, \Psi_N) \) where \( \Psi_k : \Omega \to \mathbb{R} \) and \( k = 1, \ldots, N \) be a set valued function and let

\[
\Psi := \{ \psi : \Omega \to \mathbb{R}^N \text{ is a measurable and } \psi(x) \in \Psi(x) \text{ almost everywhere} \} \neq \emptyset.
\]

We define the \( G \)-value of \( \Psi \) as follows:

\[
G(\Psi) := \sup_{\{h \in C_0^\infty(\Omega, \mathbb{R}^N) : gVTV(h) = 1\}} \psi \left( \sum_{k=1}^N \int_{\Omega} \psi_k h_k \, dx \right)
\]

\[
= \sup_{\{h \in C_0^\infty(\Omega, \mathbb{R}^N) : gVTV(h) = 1\}} \inf_{\psi \in \Psi} \sum_{k=1}^N \int_{\Omega} \psi_k h_k \, dx.
\]

**Remark 2.** There is no distinction between the set \( \Psi \) and the function \( \Psi \). In case \( \Psi \) is a single value and measurable then \( G(\Psi) \) is the \( G \)-norm of \( \Psi \) as defined in [31].

**Definition B.2.** The \( G_\lambda \)-values of \( \Psi \) are defined as

\[
G_\lambda(\Psi) := \sup_{\{h \in C_0^\infty(\Omega, \mathbb{R}^N) : \|h\|_{\lambda} = 1\}} \psi \left( \sum_{k=1}^N \int_{\Omega} \psi_k h_k \, dx \right).
\]

We next derive the definition of \( G \)-sets for the vectorial case. Let \( \partial |g| \) denote the set-valued subdifferential of \( \|g\|_1 \), i.e.,

\[
\partial |g| := \begin{cases} 
\text{sgn}(g) & \text{if } g \neq 0, \\
[-1, 1] & \text{if } g = 0.
\end{cases}
\]

An important \( G \)-value property of \( \partial |g| \) is given by the following lemma which relates the parameter \( \lambda \) and our \( gVTV \) energy.

**Lemma B.3.** For \( g \in L^1(\Omega, \mathbb{R}^N), \) \( G(\partial |g|) \leq \lambda \) if and only if

\[
\lambda \sum_{k=1}^N \left( \left| \int_{\{g_k \neq 0\}} \text{sgn}(g_k) h_k \, dx \right| - \int_{\{g_k = 0\}} |h_k| \, dx \right) \leq gVTV(h)
\]

for all \( h \in BV(\Omega, \mathbb{R}^N) \). Moreover

\[
G(\partial |g|) = \sup_{\{h \in BV(\Omega, \mathbb{R}^N) : gVTV(h) = 1\}} \psi \left( \sum_{k=1}^N \int_{\Omega} \psi_k h_k \, dx \right).
\]

**Definition B.4.** Let \( f \in L^1(\Omega, \mathbb{R}^N) \) and \( u \in BV(\Omega, \mathbb{R}^N) \). Assume that for every \( h \in BV(\Omega, \mathbb{R}^N) \)

\[
\lambda \sum_{k=1}^N \left( -\int_{\{u_k \neq f_k\}} \text{sgn}(u_k - f_k) h_k \, dx - \int_{\{u_k = f_k\}} |h_k| \, dx \right) \leq gVTV(u + h) - gVTV(u).
\]
We define the $G$-set for the vectorial case as
\begin{equation}
G_u(\partial(u - f)) := \{ \lambda \in \mathbb{R} : \lambda \text{satisfies } (62) \}.
\end{equation}

Note that for $\lambda \in G_u(\partial(u - f))$ it follows that for every $h \in BV(\Omega, \mathbb{R}^N)$
\begin{equation}
\lambda \sum_{k=1}^{N} \left( \left| \int_{\{u_k \neq f_k\}} \text{sgn}(u_k - f_k)h_k \, dx \right| - \int_{\{u_k = f_k\}} |h_k| \, dx \right) \leq gTV(u + h) - gTV(u),
\end{equation}
and thus
\begin{equation}
G(\partial(u - f)) \leq \frac{1}{\lambda}.
\end{equation}

An equivalence relation between finding a solution of the minimization problem (23) and choosing an appropriate parameter $\lambda$ for the $gTV + L^1$ model is stated in the next theorem.

**Theorem B.5.** For $\lambda > 0$, $u_\lambda$ is a minimizer of the vectorial $gTV+L^1$ model (23), if only if $\lambda$ satisfies
\begin{equation}
\lambda \sum_{k=1}^{N} \left( - \int_{\Omega} \text{sgn}(u_k^\lambda - f_k)h_k \, dx - \int_{\Omega} |h_k| \, dx \right) \leq gTV(u^\lambda + h) - gTV(u^\lambda),
\end{equation}
for all $h \in BV(\Omega, \mathbb{R}^N)$, i.e., $\lambda \in G_u(\partial(u - f))$.

**Proof.** If $u_\lambda = (u_1^\lambda, \ldots, u_N^\lambda)$ is a minimizer of (23), we have that for any $h \in BV(\Omega, \mathbb{R}^N)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ with $\epsilon_k > 0$ for every $k = 1, \ldots, N$ the following inequalities
\begin{equation}
gTV(u_k^\lambda) + \alpha \sum_{l=1}^{N} J_{k,l}(u_k^\lambda, u_l^\lambda) + \lambda \int_{\Omega} |u_k^\lambda - f_k| \, dx \\
\leq gTV(u_k^\lambda + \epsilon_k h_k) + \alpha \sum_{l=1}^{N} J_{k,l}(u_k^\lambda + \epsilon_k h_k, u_l^\lambda + \epsilon_l h_l) + \lambda \int_{\Omega} |u_k^\lambda + \epsilon_k h_k - f_k| \, dx \\
\leq gTV(u_k^\lambda + \epsilon_k h_k) + \alpha \sum_{l=1}^{N} J_{k,l}(u_k^\lambda + \epsilon_k h_k, u_l^\lambda + \epsilon_l h_l) + \lambda \int_{\Omega} |u_k^\lambda - f_k| \, dx \\
+ \epsilon_k \lambda \int_{\{u_k \neq f_k\}} \text{sgn}(u_k^\lambda - f_k) \, dx + \lambda \eta(\epsilon_k, u_k^\lambda, h_k),
\end{equation}
where $\eta(\epsilon_k, u_k^\lambda, h_k) = o(\epsilon_k)$. This implies
\begin{equation}
\lambda \left( - \int_{\{u_k \neq f_k\}} \text{sgn}(u_k^\lambda - f_k)h_k \, dx - \int_{\{u_k = f_k\}} |h_k| \, dx \right) \leq gTV(u_k^\lambda + \epsilon_k h_k) + \alpha \sum_{l=1}^{N} J_{k,l}(u_k^\lambda + \epsilon_k h_k, u_l^\lambda + \epsilon_l h_l) - gTV(u_k^\lambda) - \sum_{l=1}^{N} J_{k,l}(u_k^\lambda, u_l^\lambda)
\end{equation}
\begin{equation}
+ \lambda \left( \eta(\epsilon_k, u_k^\lambda, h_k) - \int_{\{u_k = f_k\}} |h_k| \, dx \right).
\end{equation}
The convexity of \( gTV(u^1_k) + \alpha \sum_{l=1}^N J_{k,l}(u^1_k, u^1_l) \) implies
\[
gTV(u^1_k + \epsilon_k h_k) + \alpha \sum_{l=1}^N J_{k,l}(u^1_k + \epsilon_k h_k, u^1_l + \epsilon_l h_l) - (gTV(u^1_k) + \alpha \sum_{l=1}^N J_{k,l}(u^1_k, u^1_l))
\]
\[
\leq gTV(u^1_k + h_k) + \alpha \sum_{l=1}^N J_{k,l}(u^1_k + h_k, u^1_l + \epsilon_l h_l) - gTV(u^1_k) + \alpha \sum_{l=1}^N J_{k,l}(u^1_k, u^1_l)
\]
for all \( 0 < \epsilon_k < 1 \). Since \( \eta(\epsilon_k, u^1_k, h_k) \to \int_{\{u^1_k = f_k\}} |h_k| \, dx \), we found by using (69) and letting \( \epsilon \to 0 \) that
\[
\lambda \sum_{k=1}^N \left( - \int_\Omega sgn(u^1_k - f_k) h_k \, dx - \int_\Omega |h_k| \, dx \right) \leq gTV(u^1 + h) - gTV(u^1),
\]
Conversely by (66) and the convexity of \(| \cdot |\) it follows that
\[
\lambda \sum_{k=1}^N \int_\Omega |u^1_k + h_k - f_k| \, dx + gTV(u^1 + h)
\]
\[
\geq gTV(u^1) + \lambda \sum_{k=1}^N \left( \int_\Omega |u^1_k - f_k| \, dx + \int_{\{u^1_k \neq f_k\}} sgn(u^1_k - f_k) h_k \, dx + \int_{\{u^1_k = f_k\}} |h_k| \, dx \right)
\]
\[
+ gTV(u^1 + h) - gTV(u^1)
\]
\[
\geq \lambda \sum_{k=1}^N \int_\Omega |u^1_k - f_k| \, dx + gTV(u^1).
\]

The proposed \( gTV + L^1 \) model is expected to discriminate between different features in an image during regularizing while preserving strong geometrical features such as edges and corners. For this purpose, it is essential to get bounds on the values for the parameter \( \lambda \) that can provide a guideline in experiments. In the next theorem, it is shown that for some particular selections of the parameter \( \lambda \), we can separate different level sets of an image at multiple scales.

**Theorem B.6.** Let \( N = N_1 \cup N_2 \subset \Omega \) with \( N_1 \) and \( N_2 \) disjoint, and let \( f : \Omega \to \mathbb{R}^N \) such that \( f_\mathcal{N} = f_{N_1} \mathbb{1}_{N_1} + f_{N_2} \mathbb{1}_{N_2} \) (being \( f_{N_i} \) the restriction of \( f \) in \( N_i \)). Given a parameter \( \lambda \) satisfying
\[
\frac{1}{G(\partial f_{N_1})} < \lambda < \frac{1}{G(\partial f_{N_2})}
\]
with \( G(\partial f_{N_1}) > G(\partial f_{N_2}) \), if \( u \) is a minimizer of the \( gTV + L^1 \) model (23), it follows \( u_{N_1} = 0 \) and \( u_{N_2} \neq 0 \).

**Proof.** Let \( \epsilon = (\epsilon_1, \ldots, \epsilon_N) \) with \( 0 < \epsilon_k < 1 \) for every \( k = 1, \ldots, N \), and \( h \in BV(N, \mathbb{R}^N) \). By contradiction, we prove that \( u_{N_2} \neq 0 \) i.e., we suppose \( u_{N_2} = 0 \). From the convexity of \( gTV(\cdot) \) and \(| \cdot |\), the following inequalities hold
\[
\epsilon_k \left( gTV(h_k) + \sum_{l=1}^N J_{k,l}(h_k, h_l) \right) \geq gTV((u_k)_{N \setminus N_2} + \epsilon_k h_k)
\]
By properly combining identities (73), (74) and (75), we get

\[
\sum_{l=1}^{N} J_{k,l}((u_k)_{\mathcal{N}\setminus\mathcal{N}_2} + \epsilon_k h_k, (u_l)_{\mathcal{N}\setminus\mathcal{N}_2} + \epsilon_l h_l)
\]

and

\[
\int_{\mathcal{N}_2} |(f_k)_{\mathcal{N}_2}| \, dx - \int_{\mathcal{N}_2} |(f_k)_{\mathcal{N}_2} - \epsilon_k h_k| \, dx
\]

\[
\geq \epsilon_k \left( \int_{\{(f_k)_{\mathcal{N}_2} \neq \epsilon_k h_k\}} sgn((f_k)_{\mathcal{N}_2} - \epsilon_k h_k) h_k \, dx - \int_{\{(f_k)_{\mathcal{N}_2} = \epsilon_k h_k\}} h_k \, dx \right).
\]

We also have from the minimizer \( u \) that

(75) \quad gTV(u_{\mathcal{N}\setminus\mathcal{N}_2}) + \lambda \sum_{k=1}^{N} \left( \int_{\mathcal{N}_2} |(f_k)_{\mathcal{N}_2}| \, dx + \int_{\mathcal{N}\setminus\mathcal{N}_2} |(u_k)_{\mathcal{N}\setminus\mathcal{N}_2} - (f_k)_{\mathcal{N}\setminus\mathcal{N}_2}| \, dx \right)

\leq gTV(u_{\mathcal{N}\setminus\mathcal{N}_2} + \epsilon h) + \lambda \sum_{k=1}^{N} \left( \int_{\mathcal{N}_2} |(f_k)_{\mathcal{N}_2} - \epsilon_k h_k| \, dx + \int_{\mathcal{N}\setminus\mathcal{N}_2} |(u_k)_{\mathcal{N}\setminus\mathcal{N}_2} - (f_k)_{\mathcal{N}\setminus\mathcal{N}_2}| \, dx \right).

By properly combining identities (73), (74) and (75), we get

\[
\sum_{k=1}^{N} \epsilon_k \left( gTV(h_k) + \sum_{l=1}^{N} J_{k,l}(h_k, h_l) \right)
\]

\[
\geq gTV(u_{\mathcal{N}\setminus\mathcal{N}_2}) - gTV(u_{\mathcal{N}\setminus\mathcal{N}_2} + \epsilon h)
\]

(76)

\[
\geq \lambda \sum_{k=1}^{N} \left( \int_{\mathcal{N}_2} |(f_k)_{\mathcal{N}_2}| \, dx - \int_{\mathcal{N}_2} |(f_k)_{\mathcal{N}_2} - \epsilon_k h_k| \, dx \right)
\]

\[
\geq \lambda \sum_{k=1}^{N} \epsilon_k \left( \int_{\{(f_k)_{\mathcal{N}_2} \neq \epsilon_k h_k\}} sgn((f_k)_{\mathcal{N}_2} - \epsilon_k h_k) h_k \, dx - \int_{\{(f_k)_{\mathcal{N}_2} = \epsilon_k h_k\}} h_k \, dx \right),
\]

and with \( \epsilon \to 0^+ \) in (76), it happens

(77) \quad gTV(h) \geq \lambda \sum_{k=1}^{N} \left( \int_{\{(f_k)_{\mathcal{N}_2} \neq 0\}} sgn((f_k)_{\mathcal{N}_2}) h_k \, dx - \int_{\{(f_k)_{\mathcal{N}_2} = 0\}} h_k \, dx \right).

On the other hand, from the assumed inequalities in (72) we conclude from Lemma B.3 that there exists \( \tilde{h} \in BV(\mathcal{N}, \mathbb{R}^N) \) satisfying

(78) \quad \lambda \sum_{k=1}^{N} \left( \int_{\{(f_k)_{\mathcal{N}_2} \neq 0\}} sgn((f_k)_{\mathcal{N}_2}) \tilde{h}_k \, dx - \int_{\{(f_k)_{\mathcal{N}_2} = 0\}} \tilde{h}_k \, dx \right) \geq gTV(\tilde{h})

contradicting (77), and we conclude that \( u_{\mathcal{N}_2} \neq 0 \).
In order to prove that \( u_{N_1} = 0 \), we proceed by contradiction by assuming that \( u_{N_1} \neq 0 \). Given \( N_3 = N \setminus (N_1 \cup N_2) \) with \( u_{N_3} \neq 0 \), we notice that

\[
\begin{align*}
gTV((u_k)_{N_3} + (u_k)_{N_3'}, + \sum_{l=1}^{N} J_{k,l}((u_k)_{N_3}, (u_k)_{N_3'}, (u_l)_{N_3} + (u_l)_{N_3'}) = \\
&= \sup_{p \in \Phi_g} \left\{ \int_{\Omega} (u_k)_{N_3}\text{div} p dx + \int_{\Omega} (u_k)_{N_3'}\text{div} p dx \right\} \\
&+ \int_{\Omega} \sum_{l=1}^{N} |(\nabla (u_k)_{N_3} - \nabla (u_l)_{N_3}) + (\nabla (u_k)_{N_3'} - \nabla (u_l)_{N_3'})|^2 dx \\
&\geq \sup_{p \in \Phi_g} \left\{ \int_{\Omega} (u_k)_{N_3}\text{div} p dx \right\} + \int_{\Omega} \sum_{l=1}^{N} |(\nabla (u_k)_{N_3} - \nabla (u_l)_{N_3})|^2 dx \\
&= gTV((u_k)_{N_3} + \sum_{l=1}^{N} J_{k,l}((u_k)_{N_3}, (u_l)_{N_3'}).
\end{align*}
\]

Therefore,

\[
gVT(V(u_{N_3} + u_{N_3'}) + \lambda \sum_{k=1}^{N} \int_{\Omega} |(u_k)_{N_3}| dx \geq gVT(V(u_{N_3}'),
\]

On the other hand, since \( u \) is a minimizer

\[
gVT(V(u_{N_3} + u_{N_3'}) + \lambda \sum_{k=1}^{N} \left( \int_{\Omega} |(u_k)_{N_3} - (f_k)_{N_3}| dx + \int_{\Omega} |(u_k)_{N_3'} - (f_k)_{N_3'}| dx \right) \\
\leq gVT(V(u_{N_3'} + \lambda \sum_{k=1}^{N} \int_{\Omega} |(u_k)_{N_3}| dx \leq gVT(V(u_{N_3}'),
\]

which is different from the identity in (80). Then, considering the definition of the subset \( N_3 \), it necessarily happens that \( u_{N_3} = 0 \), and we get

\[
\begin{align*}
E^1_{gVT(V(u_{N_3} \setminus N_1)} \\
\geq E^2_{gVT(V(u_{N_3} \setminus N_1) + u_N)} \\
= \sum_{k=1}^{N} \int_{N_3 \setminus N_1} g_k(x)|\nabla (u_k)_{N_3 \setminus N_1}| dx \\
\end{align*}
\]

\[
\begin{align*}
+ \alpha \int_{N_3 \setminus N_1} \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla (u_k)_{N_3 \setminus N_1} - \nabla (u_l)_{N_3 \setminus N_1}|^2 dx \\
+ \sum_{k=1}^{N} \int_{N_1} g_k(x)|\nabla (u_k)_{N_1}| dx + \alpha \int_{N_1} \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla (u_k)_{N_1} - \nabla (u_l)_{N_1}|^2 dx
\end{align*}
\]
\begin{equation}
\sum_{k=1}^{N} \left( \lambda \int_{\mathcal{N} \setminus \mathcal{N}_1} |(u_k)_{\mathcal{N} \setminus \mathcal{N}_1} - (f_k)_{\mathcal{N} \setminus \mathcal{N}_1}| \, dx + \lambda \int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1} - (u_k)_{\mathcal{N}_1}| \, dx \right) \\
= E_{gTV}(u_{\mathcal{N} \setminus \mathcal{N}_1}) + \sum_{k=1}^{N} \int_{\mathcal{N}_1} g_k(x) \nabla (u_k)_{\mathcal{N}_1} \, dx \\
+ \alpha \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla (u_k)_{\mathcal{N}_1} - \nabla (u_l)_{\mathcal{N}_1}|^2 \, dx \\
+ \lambda \sum_{k=1}^{N} \left( \int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1} - (u_k)_{\mathcal{N}_1}| \, dx - \int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1}| \, dx \right)
\end{equation}

and therefore

\begin{equation}
\sum_{k=1}^{N} \int_{\mathcal{N}_1} g_k(x) \nabla (u_k)_{\mathcal{N}_1} \, dx + \alpha \sum_{k=1}^{N} \sum_{l=1}^{N} |\nabla (u_k)_{\mathcal{N}_1} - \nabla (u_l)_{\mathcal{N}_1}|^2 \, dx \\
\leq \lambda \sum_{k=1}^{N} \left( \int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1} - (u_k)_{\mathcal{N}_1}| \, dx - \int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1}| \, dx \right).
\end{equation}

We also have from the convexity of $|\cdot|$ that

\begin{equation}
\int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1}| \, dx - \int_{\mathcal{N}_1} |(f_k)_{\mathcal{N}_1} - u_k| \, dx \\
\leq \left( \int_{\{(f_k)_{\mathcal{N}_1} \neq 0\}} \text{sgn}((f_k)_{\mathcal{N}_1}) u_k \, dx \right) - \left( \int_{\{(f_k)_{\mathcal{N}_1} = 0\}} u_k \, dx \right).
\end{equation}

From inequalities in (84) and (85) we get

\begin{equation}
g_{TV}(u_{\mathcal{N}_1}) \leq \lambda \sum_{k=1}^{N} \left( \int_{\{(f_k)_{\mathcal{N}_1} \neq 0\}} \text{sgn}((f_k)_{\mathcal{N}_1}) h_k \, dx \right) - \left( \int_{\{(f_k)_{\mathcal{N}_1} = 0\}} h_k \, dx \right),
\end{equation}

and at the same time from Lemma B.3, it happens

\begin{equation}
g_{TV}(u_{\mathcal{N}_1}) \geq \lambda \sum_{k=1}^{N} \left( \int_{\{(f_k)_{\mathcal{N}_1} \neq 0\}} \text{sgn}((f_k)_{\mathcal{N}_1}) h_k \, dx \right) - \left( \int_{\{(f_k)_{\mathcal{N}_1} = 0\}} h_k \, dx \right),
\end{equation}

which is a contradiction with (86). Then, we have $u_{\mathcal{N}_1} = 0$. \hfill \Box

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