Local risk-minimization for Barndorff-Nielsen and Shephard models

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Abstract We obtain explicit representations of locally risk-minimizing strategies for call and put options in Barndorff-Nielsen and Shephard models, which are Ornstein–Uhlenbeck-type stochastic volatility models. Using Malliavin calculus for Lévy processes, Arai and Suzuki (Int. J. Financ. Eng. 2:1550015, 2015) obtained a formula for locally risk-minimizing strategies for Lévy markets under many additional conditions. Supposing mild conditions, we make sure that the Barndorff-Nielsen and Shephard models satisfy all the conditions imposed in (Arai and Suzuki in Int. J. Financ. Eng. 2:1550015, 2015). Among others, we investigate the Malliavin differentiability of the density of the minimal martingale measure. Moreover, we introduce some numerical experiments for locally risk-minimizing strategies.

Keywords Local risk-minimization · Barndorff-Nielsen and Shephard models · Stochastic volatility models · Malliavin calculus · Lévy processes

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1 Introduction

Our objective is to obtain explicit representations of locally risk-minimizing (LRM) strategies for call and put options in Barndorff-Nielsen and Shephard (BNS) models; these are Ornstein–Uhlenbeck-(OU-)type stochastic volatility models developed by Barndorff-Nielsen and Shephard [4, 5]. On the other hand, local risk-minimization is a very well-known quadratic hedging method for contingent claims in incomplete financial markets. Although its theoretical aspects have been well developed, less is known about its explicit representations. Accordingly, Arai and Suzuki [3] have analyzed this problem for Lévy markets using Malliavin calculus for Lévy processes. They gave in Theorem 3.7 of their paper an explicit formula for LRM strategies including some Malliavin derivatives. Here, Lévy markets mean models for which the asset price process is described by a solution to the stochastic differential equation (SDE)

\[ dS_t = S_t \left( \alpha t dt + \beta_t dW_t + \int_{\mathbb{R}\setminus\{0\}} \gamma_{t,z} \tilde{N}(dt, dz) \right), \quad S_0 > 0, \]  

where \( W \) is a one-dimensional Brownian motion, \( \tilde{N} \) is a compensated Poisson random measure, and \( \alpha, \beta, \) and \( \gamma \) are predictable processes. If \( \alpha, \beta, \) and \( \gamma \) are deterministic, a representation for LRM strategies is given simply under some mild conditions. Indeed, [3] calculated explicitly LRM strategies for call options, Asian options, and lookback options for the deterministic coefficient case. However, according to [3, Theorem 3.7], one needs to impose many additional conditions on models with random coefficients. Thus, concrete calculations for such models were set aside.

In this paper, we obtain explicit LRM strategies for BNS models, which are popular examples for the random coefficient case. In particular, various empirical studies confirm that BNS models capture well important stylized features of financial time series. In a BNS model, the squared volatility process \( \sigma^2 \) is given by an OU process driven by a subordinator, that is, a nondecreasing Lévy process. More precisely, \( \sigma^2 \) is given as a solution to the SDE

\[ d\sigma^2_t = -\lambda \sigma^2_t dt + dH_{\lambda t}, \quad \sigma^2_0 > 0, \]  

where \( \lambda > 0 \), and \( H \) is a subordinator without drift. Now, the asset price process \( S \) of a BNS model is described as

\[ S_t = S_0 \exp \left( \int_0^t \left( \mu - \frac{1}{2} \sigma^2_s \right) ds + \int_0^t \sigma_s dW_s + \rho H_{\lambda t} \right), \]  

where \( S_0 > 0, \rho \leq 0, \mu \in \mathbb{R} \). Note that the last term \( \rho H_{\lambda t} \) accounts for the leverage effect, which is a stylized fact such that the asset price declines at the moment when volatility increases. Moreover, defining \( J_t := H_{\lambda t} \), we denote by \( N \) the Poisson random measure of \( J \). Hence, we have

\[ J_t = \int_0^\infty x N([0, t], dx). \]
Denoting by $\nu$ the Lévy measure of $J$, we find that

$$\tilde{N}(dt, dx) := N(dt, dx) - \nu(dx)dt$$

is the compensated Poisson random measure. Then, the asset price process $S$ given in (1.3) is a solution to the SDE

$$dS_t = S_t \left( \alpha dt + \sigma_t dW_t + \int_0^\infty (e^{\rho x} - 1) \tilde{N}(dt, dx) \right), \quad (1.4)$$

where $\alpha := \mu + \int_0^\infty (e^{\rho x} - 1) \nu(dx)$. Therefore, BNS models correspond to instances where $\beta$ in (1.1) is random.

We use Theorem 3.7 of [3] in this paper to derive LRM strategies for BNS models described as in (1.3). Therefore, the primary part of our discussion lies in confirming all the conditions imposed in [3, Theorem 3.7]. In particular, we need to investigate the Malliavin differentiability of the density of the minimal martingale measure (MMM), which is an indispensable equivalent martingale measure to discuss LRM strategies. To the best of our knowledge, the literature on LRM strategies for BNS models is very limited. Arai [1] studied this problem for a different setting from ours. In [1], the volatility risk premium is taken into account, but $\rho$ is restricted to 0. Hence, $S$ is described as

$$S_t = S_0 \exp \left( \int_0^t (\mu + \beta \sigma_s^2) ds + \int_0^t \sigma_s dW_s \right),$$

where $\beta \in \mathbb{R}$ is called the volatility risk premium. Note that $S$ is continuous. Employing Malliavin calculus under the MMM, [1] gave an explicit representation of LRM strategies. On the other hand, there is some previous research on mean–variance hedging, which is an alternative quadratic hedging method, for BNS models. Cont et al. [9] and Kallsen and Pauwels [13] studied this problem assuming that $S$ is a martingale. Kallsen and Vierthauer [14] treated the case where $\rho = 0$. Recently, Benth and Detering [6] dealt with the BNS model framework to represent a futures price process on electricity assuming that $S$ is a martingale and $\rho = 0$.

In addition, we also develop in this paper a numerical scheme for LRM strategies for call options using the method of Arai et al. [2], which is a numerical scheme for LRM strategies in exponential Lévy models. Their scheme is based on the so-called Carr–Madan approach [7], which is based on the fast Fourier transform (FFT). Moreover, we compare LRM strategies with so-called delta-hedging strategies, which are given as the partial derivative of the option price with respect to the asset price.

The outline of this paper is as follows. After giving preliminaries in Sect. 2, we address the main results in Sect. 3. Theorem 3.1 gives an explicit representation of LRM strategies for put options. LRM strategies for call options are provided as a corollary. The proof of Theorem 3.1 is discussed in Sect. 4. Section 5 is devoted to the Malliavin differentiability of the density of the MMM. Numerical experiments for LRM strategies are illustrated in Sect. 6. Conclusions are given in Sect. 7. The statement of [3, Theorem 3.7] for our setting and some additional calculations are provided in the Appendix.
2 Preliminaries

We consider a financial market model in which only one risky asset and one riskless asset are tradable. For simplicity, we assume the interest rate to be 0. Let $T$ be a finite time horizon. The fluctuation of the risky asset is described as a process $S$ given by (1.3). We adopt the same mathematical framework as in [3]. The structure of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be discussed in Sect. 2.3. Notice that the Poisson random measure $N$ and the Lévy measure $\nu$ of $J$ are defined on $[0, T] \times (0, \infty)$ and $(0, \infty)$, respectively, and that

$$\int_0^\infty (x \wedge 1) \nu(dx) < \infty$$

by Proposition 3.10 of Cont and Tankov [8]. Let $\nu^H$ be the Lévy measure of $H$; we then have $\nu(dx) = \lambda \nu^H(dx)$. Denoting

$$A_t := \int_0^t S_{s-} \alpha ds \quad \text{and} \quad M_t := S_t - S_0 - A_t,$$

we have $S = S_0 + M + A$, which is the canonical decomposition of $S$. Further, we denote $L_t := \log(S_t/S_0)$ for $t \in [0, T]$, that is,

$$L_t = \int_0^t \left( \mu - \frac{1}{2} \sigma^2_s \right) ds + \int_0^t \sigma_s dW_s + \rho J_t.$$  \hfill (2.1)

Remark 2.1 Noting that $\sigma_t = \sigma_t$ a.s. for any $t \in [0, T]$, we can regard $\sigma_t$ and $\sigma_t^2$ as predictable processes. For example, we may identify $\sigma_t dW_t$ in (1.4) with $\sigma_t^- dW_t$ if necessary.

Now, we define three constants and one function by

$$C_\rho := \int_0^\infty (e^{\rho x} - 1)^2 \nu(dx), \quad C_\alpha := \max \left\{ \frac{||\alpha|| e^{\frac{1}{2}T \sigma_0^2}}{C_\rho}, \frac{||\alpha||}{C_\rho} \right\}, \quad C_0 := \max \left\{ \frac{||\alpha||}{C_\rho}, 1 \right\},$$

and $B(t) := \frac{1 - e^{-\lambda t}}{\lambda}$ for $t \in [0, T]$.

Next, we state our standing assumptions.

Assumption 2.2 1. $\int_1^\infty \exp(2(B(T) \vee |\rho|)x) \nu(dx) < \infty$.

2. $e^{-\lambda T \sigma_0^2 + C_\rho} \frac{\alpha}{\sigma_0^2 + C_\rho} > -1$.

Remark 2.3 1) Item 1 in Assumption 2.2 ensures $\int_0^\infty x^2 \nu(dx) < \infty$, which means $E[J_T^2] < \infty$. In addition, we have

$$\int_0^\infty (e^{\rho x} - 1)^2 \nu(dx) \leq \int_0^\infty \rho^2 x^2 \nu(dx) < \infty$$

because $0 \leq 1 - e^{\rho x} \leq -\rho x$.

\[ \square \]
Table 1 Estimated parameters for IG-OU and gamma-OU processes

|       | IG-OU |       |       | gamma-OU |       |       |       |
|-------|-------|-------|-------|----------|-------|-------|-------|
|       | ρ     | λ     | a     | b        | σ²₀   |       |       |
| [15]  | −4.7039 | 2.4958 | 0.0872 | 11.9800  | 0.0041 |       |       |
| [17]  | −0.1926 | 0.0636 | 6.2410 | 0.7995   | 0.0156 |       |       |

2) As seen in [3, Sect. 2.3], the so-called condition (SC) is satisfied under Assumption 2.2. For more details on the condition (SC), see Schweizer [18, 19]. Moreover, Lemma 2.11 of [3] implies that $E [\sup_{t \in [0,T]} |S_t|^2] < \infty$.

3) By (A.2) in the Appendix, item 2 ensures that $\frac{\alpha}{\sigma^2 + C_\rho} > -1$ for any $t \in [0,T]$.

Remark 2.4 We present two important examples of $\sigma^2$ introduced by Nicolato and Venardos [15] that fulfil Assumption 2.2 under certain conditions on the involved parameters. For more details on this topic, see also Schoutens [17, Sects. 5.5 and 7.1].

1) The first concerns $\nu^H$ given by

$$
\nu^H(dx) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}} (1 + b^2 x) e^{-\frac{1}{2} b^2 x} I_{(0,\infty)}(x) dx,$$

where $a > 0$ and $b > 0$. In this case, the invariant distribution of the squared volatility process $\sigma^2$ follows an inverse Gaussian distribution with parameters $a > 0$ and $b > 0$. $\sigma^2$ is called an IG-OU process. If $\frac{b^2}{T} > 2 (B(T) \vee |\rho|)$, then item 1 of Assumption 2.2 is satisfied.

2) The second example is what we call a gamma-OU process, where the invariant distribution of $\sigma^2$ is given by a gamma distribution with parameters $a > 0$ and $b > 0$. In this case, $\nu^H$ is described as

$$
\nu^H(dx) = a be^{-bx} I_{(0,\infty)}(x) dx.
$$

Like in the IG-OU case, item 1 of Assumption 2.2 is satisfied whenever we have $b > 2 (B(T) \vee |\rho|)$.

3) Here we discuss item 2 of Assumption 2.2. Since we can rewrite this condition as $C_\rho > -\alpha - e^{-\lambda T} \sigma_0^2$, it is satisfied when $C_\rho$ is sufficiently large. For example, $C_\rho$ for the gamma-OU case is given as

$$
C_\rho = ab\lambda \left( \frac{1}{b - 2\rho} - \frac{2}{b - \rho} + \frac{1}{b} \right) = \frac{2a\lambda \rho^2}{(b - \rho)(b - 2\rho)},
$$

from which a sufficient condition on $a$ and $b$ for item 2 of Assumption 2.2 is given.

4) [15] and Sect. 7 in [17] estimated the parameter sets for the above two models using real data. Note that the discounted asset price process is assumed to be a martingale in both [15] and [17]. Hence, the value of $\mu$ is automatically determined, and $\alpha$ is given by 0, which means that item 2 of Assumption 2.2 is satisfied. For any $T > 0$, the parameter set for IG-OU in [17] does not satisfy item 1 of Assumption 2.2. In contrast, the other estimated parameter sets listed in Table 1 satisfy the condition.
2.1 Locally risk-minimizing strategies

In this section, we give a definition of LRM strategies based on Theorem 1.6 of [19].

**Definition 2.5**

1. \( \Theta_S \) denotes the space of all \( \mathbb{R} \)-valued predictable processes \( \xi \) satisfying
   \[
   \mathbb{E}\left[ \int_0^T \xi_s^2 d\langle M \rangle_s + (\int_0^T |\xi_s dA_s|)^2 \right] < \infty. 
   \]

2. An \( L^2 \)-strategy is given by a pair \( \varphi = (\xi, \eta) \), where \( \xi \in \Theta_S \), and \( \eta \) is an adapted process such that \( V(\varphi) := \xi S + \eta \) is a right-continuous process with \( \mathbb{E}[V^2(\varphi)] < \infty \) for every \( t \in [0, T] \). Note that \( \xi_t \) (resp., \( \eta_t \)) represents the number of units of the risky asset (resp., the risk-free asset) an investor holds at time \( t \).

3. For a claim \( F \in L^2(\mathbb{P}) \), the process \( C^F(\varphi) \) defined by
   \[
   C^F_t(\varphi) := F 1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s 
   \]
   is called the cost process of \( \varphi = (\xi, \eta) \) for \( F \).

4. An \( L^2 \)-strategy \( \varphi \) is said to be locally risk-minimizing (LRM) for the claim \( F \) if \( V_T(\varphi) = 0 \) and \( C^F(\varphi) \) is a martingale orthogonal to \( M \), that is, \([C^F(\varphi), M] \) is a uniformly integrable martingale.

5. An \( F \in L^2(\mathbb{P}) \) admits a Föllmer–Schweizer (FS) decomposition if it can be described by
   \[
   F = F_0 + \int_0^T \xi^F_t dS_t + L^F_T, \quad (2.3) 
   \]
   where \( F_0 \in \mathbb{R} \), \( \xi^F \in \Theta_S \), and \( L^F \) is a square-integrable martingale orthogonal to \( M \) with \( L^F_0 = 0 \).

For more details on LRM strategies, see [18, 19]. We now introduce Proposition 5.2 of [19].

**Proposition 2.6** (Proposition 5.2 of [19]) Under Assumption 2.2, an LRM strategy \( \varphi = (\xi, \eta) \) for \( F \) exists if and only if \( F \) admits an FS decomposition, and their relationship is given by
   \[
   \xi_t = \xi^F_t, \quad \eta_t = F_0 + \int_0^t \xi^F_s dS_s + L^F_t - F 1_{\{t=T\}} - \xi^F_t S_t. 
   \]

Therefore, it suffices to derive a representation of \( \xi^F \) in (2.3) to obtain the LRM strategy for a claim \( F \). Henceforth, we identify \( \xi^F \) with the LRM strategy for \( F \).

2.2 Minimal martingale measure

To discuss the FS decomposition, we first need to study the MMM. A probability measure \( \mathbb{P}^* \sim \mathbb{P} \) is called an MMM if \( S \) is a \( \mathbb{P}^* \)-martingale and any square-integrable \( \mathbb{P} \)-martingale orthogonal to \( M \) remains a martingale under \( \mathbb{P}^* \). Next, we consider
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the SDE

\[ dZ_t = -Z_t \Lambda_t \, dM_t, \quad Z_0 = 1, \quad (2.4) \]

where \( \Lambda_t := \frac{1}{s_t} - \frac{\alpha}{s_t^2 + C_p} \). The solution to (2.4) is given by the stochastic exponential of \(- \int_0^t \Lambda_t \, dM_t\). More precisely, denoting

\[ u_s := \Lambda_s S_s - \sigma_s = \frac{\alpha \sigma_s}{s_s^2 + C_p} \quad \text{and} \quad \theta_{s,x} := \Lambda_s S_s - (e^{\rho x} - 1) = \frac{\alpha (e^{\rho x} - 1)}{s_s^2 + C_p} \]

for \( s \in [0, T] \) and \( x \in (0, \infty) \), we have \( \Lambda_t \, dM_t = u_t \, dW_t + \int_0^\infty \theta_{t,z} \tilde{N}(dt, dz) \) and

\[ Z_t = \exp \left( -\int_0^t u_s \, dW_s - \frac{1}{2} \int_0^t u_s^2 \, ds + \int_0^t \int_0^\infty \log(1 - \theta_{s,x}) \tilde{N}(ds, dx) \right. \]

\[ + \left. \int_0^t \int_0^\infty \left( \log(1 - \theta_{s,x}) + \theta_{s,x} \right) \nu(dx) \, ds \right). \quad (2.5) \]

We remark here that

\[ \int_0^T \int_0^\infty \left( |\log(1 - \theta_{s,x})|^2 + \theta_{s,x}^2 \right) \nu(dx) \, ds \leq 2 T C_{\theta}^2 \rho^2 \int_0^\infty x^2 \nu(dx) < \infty \]

by Lemma A.7. Noting the boundedness of \( u \) by Lemma A.7 and the inequality

\[ (1 - \theta_{s,x}) \log(1 - \theta_{s,x}) + \theta_{s,x} \leq (1 - \theta_{s,x})(-\theta_{s,x}) + \theta_{s,x} = \theta_{s,x}^2, \]

we have the martingale property of \( Z \) by Theorem 1.4 of Ishikawa [12]. Now, we get the following:

**Proposition 2.7**

1. \( Z_T \in L^2(\mathbb{P}) \).

2. The probability measure defined by \( \frac{d\mathbb{P}^*}{d\mathbb{P}} = Z_T \) is the MMM.

**Proof** We first demonstrate item 1. Here (2.5) and Lemma A.7 imply that

\[ Z_T^2 = \exp \left( -\int_0^T 2u_s \, dW_s - \frac{1}{2} \int_0^T 4u_s^2 \, ds + \int_0^T \int_0^\infty \log(1 - \delta_{s,x}) \tilde{N}(ds, dx) \right. \]

\[ + \left. \int_0^T \int_0^\infty \left( \log(1 - \delta_{s,x}) + \delta_{s,x} + \theta_{s,x}^2 \right) \nu(dx) \, ds + \int_0^T u_s^2 \, ds \right) \]

\[ \leq \exp \left( -\int_0^T 2u_s \, dW_s - \frac{1}{2} \int_0^T 4u_s^2 \, ds + \int_0^T \int_0^\infty \log(1 - \delta_{s,x}) \tilde{N}(ds, dx) \right. \]

\[ + \left. \int_0^T \int_0^\infty \left( \log(1 - \delta_{s,x}) + \delta_{s,x} \right) \nu(dx) \, ds + T (C_{\theta}^2 C_p + C_{\theta}^2) \right), \]

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where \( \delta_{s,x} := 2\theta_{s,x} - \theta_{s,x}^2 \), and \( C_u \) and \( C_\theta \) are the constants defined in (2.2). That is, denoting

\[
Y_t := \exp \left( -\int_0^t 2usdW_s - \frac{1}{2} \int_0^t 4u_s^2 ds + \int_0^t \int_0^\infty \log(1 - \delta_{s,x}) \tilde{N}(ds, dx) 
+ \int_0^t \int_0^\infty (\log(1 - \delta_{s,x}) + \delta_{s,x}) v(dx)ds \right)
\]

for \( t \in [0, T] \), we have

\[
Z_T^2 \leq Y_T \exp \left( T \left( C^2_\theta C^2_\rho + C^2_u \right) \right).
\]

Therefore, we only need to show that the process \( Y \) is a martingale. First, the Brownian part of \( Y \) is a martingale as \( u \) is bounded. Lemma A.7 again yields

\[
\int_0^T \int_0^\infty |\log(1 - \delta_{s,x})|^2 v(dx)ds \leq \int_0^T \int_0^\infty 4C^2_\rho^2 \theta^2 x^2 v(dx)ds < \infty,
\]

and \( \delta_{s,x} = \theta_{s,x}^2 (2 - \theta_{s,x})^2 \leq C^2_\theta^2 \rho^2 x^2 (2 + C_\theta)^2 \), that is, \( \int_0^T \int_0^\infty \delta_{s,x}^2 v(dx)ds < \infty \). In addition, we have

\[
\int_0^T \int_0^\infty ((1 - \delta_{s,x}) \log(1 - \delta_{s,x}) + \delta_{s,x}) v(dx)ds \leq \int_0^T \int_0^\infty \delta_{s,x}^2 v(dx)ds < \infty.
\]

Hence, all the conditions in [12, Theorem 1.4] are satisfied, and so \( Y \) is a martingale.

We proceed to item 2. Applying the integration-by-parts formula to the product process \( ZS \), we obtain that \( ZS \) is a \( \mathbb{P} \)-local martingale. Thus, \( S \) is a \( \mathbb{P}^* \)-martingale because \( \sup_{t \in [0,T]} |S_t| \) and \( ZT \) are in \( L^2(\mathbb{P}) \). Moreover, letting \( L \) be a square-integrable \( \mathbb{P} \)-martingale null at 0 and orthogonal to \( M \), we have that \( LZ \) is a \( \mathbb{P} \)-local martingale. By the square-integrability of \( L \), \( L \) remains a martingale under \( \mathbb{P}^* \). Therefore, \( \mathbb{P}^* \) is the MMM. This completes the proof of Proposition 2.7. \( \square \)

### 2.3 Malliavin calculus

In this section, we introduce Malliavin calculus based on the canonical Lévy space framework developed by Solé, Utzet, and Vives [21]. The underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is assumed to be given by

\[
(\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J),
\]

where \((\Omega_W, \mathcal{F}_W, \mathbb{P}_W)\) is a one-dimensional Wiener space on \([0, T]\) with coordinate mapping process \( W \), and \((\Omega_J, \mathcal{F}_J, \mathbb{P}_J)\) is the canonical Lévy space for \( J \), that is, \( \Omega_J = \bigcup_{n=0}^\infty ([0, T] \times (0, \infty))^n \) and \( J_t(\omega_J) = \sum_{i=1}^n z_i 1_{\{t_i \leq t\}} \) for \( t \in [0, T] \) and \( \omega_J = ((t_1, z_1), \ldots, (t_n, z_n)) \in ([0, T] \times (0, \infty))^n \). Note that \(([0, T] \times (0, \infty))^0 \) rep-...
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resents an empty sequence. Let \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]} \) be the canonical filtration completed for \( \mathbb{P} \). For more details, see Delong and Imkeller [10] and [21].

To begin, we define measures \( q \) and \( Q \) on \([0, T] \times [0, \infty)\) as

\[
q(E) := \int_E \delta_0(dz)dt + \int_E z^2 \nu(dz)dt
\]

and

\[
Q(E) := \int_E \delta_0(dz)dW_t + \int_E z\tilde{N}(dt, dz),
\]

where \( E \in \mathcal{B}([0, T] \times [0, \infty)) \), and \( \delta_0 \) is the Dirac measure at 0. For \( n \in \mathbb{N} \), we denote by \( L^2_{T,q,n} \) the set of product-measurable functions \( h : ([0, T] \times [0, \infty])^n \to \mathbb{R} \) satisfying

\[
\|h\|_{L^2_{T,q,n}}^2 := \int_{([0,T] \times [0,\infty])^n} |h((t_1, z_1), \ldots, (t_n, z_n))|^2 q(dt_1, dz_1) \cdots q(dt_n, dz_n) < \infty.
\]

For \( n \in \mathbb{N} \) and \( h \in L^2_{T,q,n} \), we define

\[
I_n(h) := \int_{([0,T] \times [0,\infty])^n} h((t_1, z_1), \ldots, (t_n, z_n)) Q(dt_1, dz_1) \cdots Q(dt_n, dz_n).
\]

Formally, we set \( L^2_{T,q,0} := \mathbb{R} \) and \( I_0(h) := h \) for \( h \in \mathbb{R} \). Under this setting, any \( F \in L^2(\mathbb{P}) \) has the unique representation \( F = \sum_{n=0}^{\infty} I_n(h_n) \) with functions \( h_n \in L^2_{T,q,n} \) that are symmetric in the \( n \) pairs \((t_i, z_i), 1 \leq i \leq n\), and we have

\[
\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|h_n\|_{L^2_{T,q,n}}^2.< \infty.
\]

We define a Malliavin derivative operator.

**Definition 2.8**

1. Let \( \mathbb{D}^{1,2} \) denote the set of all random variables \( F \in L^2(\mathbb{P}) \) with \( F = \sum_{n=0}^{\infty} I_n(h_n) \) satisfying \( \sum_{n=1}^{\infty} nn! \|h_n\|_{L^2_{T,q,n}}^2 < \infty \).

2. For any \( F \in \mathbb{D}^{1,2} \), the Malliavin derivative \( DF : [0, T] \times [0, \infty) \times \Omega \to \mathbb{R} \) is defined as

\[
D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(h_n((t, z), \cdot))
\]

for \( q \)-a.e. \((t, z) \in [0, T] \times [0, \infty)\), \( \mathbb{P} \)-a.s.

### 3 Main results

Using the framework of [3, Theorem 3.7], we introduce in this section explicit representations of LRM strategies for call and put options as the main results of this paper. Note that the statement of [3, Theorem 3.7] for our setting is introduced in the Appendix as Theorem A.1. To use this, denoting by \( F \) the underlying contingent
claim, we need $Z_T F \in L^2(\mathbb{P})$ (Condition AS1 in Theorem A.1). If $F$ is a call option, this condition is not necessarily satisfied in our setting. On the other hand, because put options are bounded, we need not care about any integrability condition for them. Therefore, we treat put options first and derive LRM strategies for call options from the put–call parity. With this procedure, we can compute call prices without any additional assumptions.

**Theorem 3.1** For any $K > 0$, the LRM strategy $\xi^{(K-S_T)}_t$ of the put option $(K-S_T)^+$ is represented as

$$
\xi^{(K-S_T)}_t = \frac{1}{S_t - (\sigma^2 + C)} \left( \sigma^2 \mathbb{E}_{\mathbb{P}^*}[ -1_{\{S_T < K\}} S_T | \mathcal{F}_{t-}] + \int_0^\infty \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+(H^*_t, z) - 1] \right. 
+ \left. zH^*_t D_{t,z}(K - S_T)^+ | \mathcal{F}_{t-}] d\nu(dz) \right), \quad (3.1)
$$

where $D_{t,z}(K - S_T)^+$ is given by Proposition 4.1, and

$$
H^*_t := \exp \left( zd_{t,z} \log Z_T - \log(1 - \theta_{t,z}) \right)
$$

for $(t, z) \in [0, T] \times (0, \infty)$. Note that $D_{t,z} \log Z_T$ is provided in Proposition A.11.

**Remark 3.2** To obtain a more explicit representation of $\xi^{(K-S_T)}_t$, we calculate the conditional expectation in the second term of (3.1), for $z \in (0, \infty)$, as

$$
\mathbb{E}_{\mathbb{P}^*}[ (K - S_T)^+(H^*_t, z) - 1 + zH^*_t D_{t,z}(K - S_T)^+ | \mathcal{F}_{t-}] 
= \mathbb{E}_{\mathbb{P}^*}[H^*_t ((K - S_T)^+ + zD_{t,z}(K - S_T)^+) - (K - S_T)^+ | \mathcal{F}_{t-}] 
= \mathbb{E}[Z_T H^*_t ((K - S_T)^+ + zD_{t,z}(K - S_T)^+) | \mathcal{F}_{t-}] - \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ | \mathcal{F}_{t-}] 
= \frac{\mathbb{E}[Z_T H^*_t (K - S_T \exp(zD_{t,z}L_T)) | \mathcal{F}_{t-}]}{Z_{t-}} - \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ | \mathcal{F}_{t-}],
$$

where $D_{t,z}L_T$ is given explicitly by Proposition A.6. Note that the last equality is implied by Proposition 4.1.

We now calculate $\frac{Z_T H^*_t}{Z_{t-}}$ and investigate its properties for later use. For $t \in [0, T]$, $z \in (0, \infty)$, $s \in [t, T]$, and $x \in (0, \infty)$, we obtain by Lemmas A.8 and A.9 that

$$
A_{t,z,s}^u := u_s + zD_{t,z}u_s = f_u \left( \sqrt{\sigma^2 + ze^{-\lambda(s-t)}} \right) = \frac{\alpha \sqrt{\sigma^2 + ze^{-\lambda(s-t)}}}{\sigma^2 + ze^{-\lambda(s-t)} + C}.
$$
and

\[ A_{t,z,s,x}^\theta := \theta_{s,x} + zD_{t,z}\theta_{s,x} = f_\theta(\sqrt{\sigma^2_s + ze^{-\lambda(s-t)}})(e^{\rho x} - 1) \]

\[ = \frac{\alpha(e^{\rho x} - 1)}{\sigma^2_s + ze^{-\lambda(s-t)} + C_\rho}, \quad (3.2) \]

where \( f_u(r) := \frac{\alpha r}{r^2 + C_\rho} \) and \( f_\theta(r) := \frac{\alpha}{r^2 + C_\rho} \) for \( r \in \mathbb{R} \). We obtain then by (2.5), Lemmas A.8–A.10, and Proposition A.11 that

\[
\frac{Z_T H_{t,z}^*}{Z_{t-}} = \exp\left( -\int_t^T (u_s + zD_{t,z}u_s)dW_s - \frac{1}{2} \int_t^T (u_s + zD_{t,z}u_s)^2 ds \right.

+ \left. \int_{t-}^T \int_0^\infty \left( \log(1 - \theta_{s,x}) + zD_{t,z} \log(1 - \theta_{s,x}) \right) \tilde{N}(ds,dx) \right.

+ \left. \int_t^T \int_{t-}^\infty \left( \log(1 - \theta_{s,x}) + zD_{t,z} \log(1 - \theta_{s,x}) + \theta_{s,x} \right. \right.

+ \left. zD_{t,z}\theta_{s,x} \right) v(dx)ds \bigg)

= \exp\left( -\int_t^T A_{t,z,s}^u dW_s - \frac{1}{2} \int_t^T (A_{t,z,s}^u)^2 ds \right.

+ \left. \int_{t-}^T \int_0^\infty \log(1 - A_{t,z,s,x}^\theta) \tilde{N}(ds,dx) \right.

+ \left. \int_t^T \int_{t-}^\infty \left( \log(1 - A_{t,z,s,x}^\theta) + A_{t,z,s,x}^\theta \right) v(dx)ds \bigg). \]

Note that \( A_{t,z,s}^u \) is bounded. Moreover, (3.2) and (A.8) yield \( A_{t,z,s,x}^\theta \leq 1 - e^{\rho x} \) and

\[ \int_0^\infty (A_{t,z,s,x}^\theta)^2 v(dx) < C^2_\theta C_\rho. \]

We then have

\[ |\log(1 - A_{t,z,s,x}^\theta)|^2 \leq \left\{ \begin{array}{ll}
\rho^2 x^2 & \text{if } A_{t,z,s,x}^\theta > 0, \\
(A_{t,z,s,x}^\theta)^2 & \text{otherwise}, 
\end{array} \right. \]

which implies that \( \int_0^\infty |\log(1 - A_{t,z,s,x}^\theta)|^2 v(dx) < \infty. \) As a result, we have

\[ \mathbb{E}\left[ \frac{Z_T H_{t,z}^*}{Z_{t-}} \mid \mathcal{F}_{t-} \right] = 1 \quad (3.3) \]

in view of [12, Theorem 1.4].

**Corollary 3.3** For any \( K > 0 \), the LRM strategy for the call option \((S_T - K)^+\) is given as \( \xi(S_T - K)^+ = 1 + \xi(K - S_T)^+. \)
Proof Note that $S$ is a $\mathbb{P}^*$-martingale by Remark 2.3 and Proposition 2.7. We then obtain

$$(S_T - K)^+ = S_T - K + (K - S_T)^+$$

$$= S_0 + \int_0^T dS_t - K + \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+] + \int_0^T \xi_t^{(K - S_T)^+} dS_t + L_T^{(K - S_T)^+}$$

$$= \mathbb{E}_{\mathbb{P}^*}[S_T - K + (K - S_T)^+] + \int_0^T (1 + \xi_t^{(K - S_T)^+}) dS_t + L_T^{(K - S_T)^+}$$

$$= \mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+] + \int_0^T (1 + \xi_t^{(K - S_T)^+}) dS_t + L_T^{(K - S_T)^+},$$

where $L_T^{(K - S_T)^+}$ is defined in (2.3). This is the FS decomposition of $(S_T - K)^+$ because $1 \in \Theta_S$ by the condition (SC) since [3, Eq. (2.4)] holds in our setting as seen in [3, Sect. 2.3].

\[\square\]

4 Proof of Theorem 3.1

We begin with the Malliavin derivatives of put options.

Proposition 4.1 For $K > 0$, we have $(K - S_T)^+ \in \mathbb{D}^{1,2}$ and

$$D_{t,z}(K - S_T)^+ = -1_{(S_T < K)} S_T (D_{t,0} L_T) 1_{(0,\infty)}(z)$$

$$+ \frac{(K - S_T) e^{z D_{t,z} L_T} - (K - S_T)^+}{z} 1_{(0,\infty)}(z).$$

Proof First, note that $S_T = S_0 e^{L_T}$ and $L_T \in \mathbb{D}^{1,2}$ by Proposition A.6. However, $S_T$ is not necessarily Malliavin-differentiable. Hence, we regard $(K - S_T)^+$ as a functional of $L_T$ rather than of $S_T$ to calculate its Malliavin derivative. To this end, noting the boundedness of $(K - S_T)^+$, we introduce the function

$$f_K(r) := \begin{cases} S_0 e^r & \text{if } r \leq \log(K/S_0), \\ Kr + K(1 - \log(K/S_0)) & \text{if } r > \log(K/S_0). \end{cases}$$

Then $f_K \in C^1(\mathbb{R})$ and $0 < f'_K(r) \leq K$ for any $r \in \mathbb{R}$. We also note

$$(K - S_T)^+ = (K - f_K(L_T))^+.$$

Proposition 2.6 in [22] implies that $f_K(L_T) \in \mathbb{D}^{1,2}$ and

$$D_{t,z} f_K(L_T) = f'_K(L_T)(D_{t,0} L_T) 1_{(0,\infty)}(z) + \frac{f_K(L_T + z D_{t,z} L_T) - f_K(L_T)}{z} 1_{(0,\infty)}(z).$$
The same argument as for Theorem 4.1 of [3] implies that

\[
D_{t,z}(K - S_T)^+ = D_{t,z}(K - f_K(L_T))^+
\]

\[
= -\mathbf{1}_{\{f_K(L_T) < K\}} (D_{t,0} f_K(L_T)) \mathbf{1}_{[0]}(z) + \frac{(K - f_K(L_T)) - z D_{t,z} f_K(L_T))^+}{z} (K - f_K(L_T))^+ \mathbf{1}_{(0,\infty)}(z)
\]

\[
= -\mathbf{1}_{\{S_T < K\}} S_T (D_{t,0} L_T) \mathbf{1}_{[0]}(z) + \frac{(K - f_K(L_T) + z D_{t,z} L_T))^+}{z} (K - f_K(L_T))^+ \mathbf{1}_{(0,\infty)}(z)
\]

\[
= -\mathbf{1}_{\{S_T < K\}} S_T (D_{t,0} L_T) \mathbf{1}_{[0]}(z) + \frac{(K - S_T e^{z D_{t,z} L_T})^+ - (K - S_T)^+}{z} \mathbf{1}_{(0,\infty)}(z)
\]

for \(q\text{-a.e. } (t, z) \in [0, T] \times [0, \infty)\). \(\square\)

We now prove Theorem 3.1 using Theorem A.1 (Theorem 3.7 of [3]). To this end, we only need to make sure of Conditions AS2 and AS3 in Theorem A.1. Note that Condition AS1 is ensured by Proposition 2.7 and the boundedness of \(\theta_{s,x} < (K - S_T)^+ \mathbf{1}_{[0]}(z)\).

\[C1: u, u^2 \in \mathbb{L}_0^{1,2}, \text{ and } 2us D_{t,z} us + z(D_{t,z} us)^2 \in \mathbb{L}^2(q \otimes \mathbb{P}) \text{ for a.e. } s \in [0, T].\]

\[C2: \theta + \log(1 - \theta) \in \mathbb{U}_1^{1,2}, \text{ and } \log(1 - \theta) \in \mathbb{U}_1^{1,2}.\]

\[C3: \text{For } q\text{-a.e. } (s, x) \in [0, T] \times (0, \infty), \text{ there is } \varepsilon_{s,x} \in (0, 1) \text{ such that we have } \theta_{s,x} < 1 - \varepsilon_{s,x}.\]

\[C4: Z_T (D_{t,0} \log Z_T) \mathbf{1}_{[0]}(z) + \frac{\exp(z D_{t,z} \log Z_T - 1)}{z} \mathbf{1}_{(0,\infty)}(z) \in \mathbb{L}^2(q \otimes \mathbb{P}).\]

\[C5: \mathbb{F} \in \mathbb{L}_0^{1,2}, \text{ and } Z_T D_{t,z} \mathbb{F} + F D_{t,z} Z_T + z(D_{t,z} \mathbb{F}) D_{t,z} Z_T \in \mathbb{L}^2(q \otimes \mathbb{P}).\]

\[C6: FH_{t,z}^*, H_{t,z}^* D_{t,z} \mathbb{F} \in \mathbb{L}^1(\mathbb{P}^*), \text{ for } q\text{-a.e. } (t, z) \in [0, T] \times (0, \infty).\]

Here \(\mathbb{L}_0^{1,2}, \mathbb{U}_1^{1,2}, \text{ and } \mathbb{P}_1^{1,2}\) are defined as follows:

- \(\mathbb{L}_0^{1,2}\) denotes the space of all \(G : [0, T] \times \Omega \to \mathbb{R}\) satisfying
  
  (a) \(G_s \in \mathbb{D}_1^{1,2}\) for a.e. \(s \in [0, T],\)

(b) \(E \int_{[0,T]} |G_s|^2 ds < \infty,\)

(c) \(E \int_{[0,T] \times [0,\infty)} \gamma_{\mathbb{U}_1^{1,2}}^T \{D_{t,z} G_s\}^2 (dt, dz) < \infty.\)

- \(\mathbb{U}_1^{1,2}\) is defined as the space of all \(G : [0, T] \times (0, \infty) \times \Omega \to \mathbb{R}\) such that

(d) \(G_{s,x} \in \mathbb{D}_1^{1,2}\) for \(q\text{-a.e. } (s, x) \in [0, T] \times (0, \infty),\)

(e) \(E \int_{[0,T] \times (0,\infty)} |G_{s,x}|^2 m(dx) ds < \infty,\)

(f) \(E \int_{[0,T] \times (0,\infty)} \int_{[0,T] \times (0,\infty)} |D_{t,z} G_{s,x}|^2 v(dx) ds (dt, dz) < \infty.\)

- \(\mathbb{P}_1^{1,2}\) is defined as the space of all \(G \in \mathbb{L}_1^{1,2}\) such that

(g) \(E \int_{[0,T] \times (0,\infty)} |G_{s,x}| v(dx) ds < \infty,\)

(h) \(E \int_{[0,T] \times (0,\infty)} \int_{[0,T] \times (0,\infty)} |D_{t,z} G_{s,x}| v(dx) ds (dt, dz) < \infty.\)
**Condition C1:** First, we show that $u \in L_{0,1}^{1,2}$. To this end, we check items (a)–(c) in the definition of $L_{0,1}^{1,2}$. Lemmas A.8 and A.7 ensure items (a) and (b), respectively. For item (c), Lemma A.8 implies

$$
\mathbb{E} \left[ \int_{[0,T] \times [0,\infty)} \int_0^T |D_{t,z} u_s|^2 ds \, q(dt, dz) \right] 
\leq \int_{[0,T] \times [0,\infty)} (T-t) C_2^2 u z \, z^2 \nu(dz) dt < \infty,
$$

from which it follows that $u \in L_{0,1}^{1,2}$.

Next, we show $2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2 \in L^2(q \otimes \mathbb{P})$; this holds because

$$
\mathbb{E} \left[ \int_{[0,T] \times [0,\infty)} \left( 2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2 \right)^2 q(dt, dz) \right] 
\leq 2C_4^2 \int_{[0,T] \times [0,\infty)} \left( \frac{4}{z} + 1 \right) z^2 \nu(dz) dt < \infty \quad (4.1)
$$

by Lemmas A.7 and A.8.

Finally, we prove that $u^2 \in L_{0,1}^{1,2}$. Item (b) holds by Lemma A.7. Since $u_s \in D_{1,2}^{1,2}$ and $u^2_s \in L^2(\mathbb{P})$, Propositions 5.1 and 5.4 of [21], together with (4.1), imply item (a) and $D_{t,z} u^2_s = 2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2$. Moreover, a similar calculation as (4.1) gives item (c) via

$$
\mathbb{E} \left[ \int_{[0,T] \times [0,\infty)} \int_0^T (D_{t,z} u^2_s)^2 ds \, q(dt, dz) \right] 
= \mathbb{E} \left[ \int_{[0,T] \times [0,\infty)} \int_0^T (2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2)^2 ds \, q(dt, dz) \right] < \infty. \quad \square
$$

**Condition C2:** We first demonstrate that $\log(1-\theta) \in \tilde{L}_{1,2}^{1,2}$. Items (d) and (e) in the definition of $\tilde{L}_{1,2}^{1,2}$ are given by Lemmas A.10 and A.7, respectively. As for item (f), Lemmas A.9 and A.10 imply

$$
|D_{t,z} \log(1-\theta_{s,x})|^2 \leq \left( \frac{C_\theta'}{z} \right)^2 e^{-2\rho x} (1-e^{\rho x})^2.
$$

Because

$$
\int_0^\infty e^{-2\rho x} (1-e^{\rho x})^2 \nu(dx) \leq \int_0^1 e^{-2\rho \rho^2 x^2} \nu(dx) + \int_1^\infty e^{-2\rho x} \nu(dx) < \infty
$$

by Assumption 2.2, item (f) follows.

Next, we show that $\theta + \log(1-\theta) \in \tilde{L}_{1,2}^{1,2}$. Note that we can demonstrate that $\theta \in \tilde{L}_{1,2}^{1,2}$ in the same manner as in the proof of condition C1. Hence, we only have to verify items (g) and (h) in the definition of $\tilde{L}_{1,2}^{1,2}$. Because

$$
|\theta_{s,x} + \log(1-\theta_{s,x})| \leq 2C_\theta |\rho|x,
$$

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item (g) follows. Next, Lemmas A.10 and A.9 and Assumption 2.2 imply
\[ \int_{[0,T] \times (0,\infty)} |D_{t,z}(\theta_{s,x} + \log(1 - \theta_{s,x}))| v(dx) ds \leq \int_{[0,T] \times (0,\infty)} |D_{t,z}\theta_{s,x}| (1 + e^{-\rho x}) v(dx) ds \leq \frac{C'}{\sqrt{z}} \]
for some \( C > 0 \), from which item (h) follows. \( \square \)

Condition C3: This is given by Lemma A.7. \( \square \)

Condition C4: Proposition A.11 implies that \( \log Z_T \in \mathbb{D}^{1,2} \) and \( D_{t,0} \log Z_T = u_t \), from which \( \mathbb{E}\left[\int_0^T (Z_T D_{t,0} \log Z_T)^2 dt\right] < \infty \) follows by Lemma A.7 and Proposition 2.7. Next, let \( \Psi_{t,z} \) be the increment quoting operator defined in [21], that is, for any random variable \( F, \omega_W \in \Omega_W \), and \( \omega_J = ((t_1, z_1), \ldots, (t_n, z_n)) \in \Omega_J \), we define
\[ \Psi_{t,z}F(\omega_W, \omega_J) := \frac{F(\omega_W, \omega_{t,z}^j) - F(\omega_W, \omega_J)}{z}, \]
where \( \omega_{t,z}^j := ((t, z), (t_1, z_1), \ldots, (t_n, z_n)) \). Since \( Z_T \in \mathbb{D}^{1,2} \) by Sect. 5, Proposition 5.4 of [21] yields that, for \( z > 0 \),
\[ D_{t,z}Z_T = \Psi_{t,z}Z_T = \Psi_{t,z} \exp(\log Z_T) = \exp(\log Z_T(\omega_W, \omega_{t,z}^j)) - \exp(\log Z_T(\omega_W, \omega_J)) \]
\[ = \frac{\exp(\log Z_T(\omega_W, \omega_{t,z}^j)) - \exp(\log Z_T(\omega_W, \omega_J))}{z} \]
\[ = \frac{\exp(\log Z_T + z \frac{\log Z_T(\omega_W, \omega_{t,z}^j) - \log Z_T(\omega_W, \omega_J)}{z}) - \exp(\log Z_T)}{z} \]
\[ = \frac{\exp(\log Z_T + z \Psi_{t,z} \log Z_T) - \exp(\log Z_T)}{z} \]
\[ = \frac{\exp(\log Z_T + z D_{t,z} \log Z_T) - \exp(\log Z_T)}{z} \]
\[ = Z_T \frac{\exp(z D_{t,z} \log Z_T) - 1}{z}. \quad (4.2) \]
As a result, condition C4 follows. \( \square \)

Condition C5: Noting that \( |F + z D_{t,z} F| \leq K \) by Theorem 4.1, we have
\[ FD_{t,z}Z_T + z(D_{t,z}F)D_{t,z}Z_T \in L^2(q \otimes \mathbb{P}) \]
since $Z_T \in \mathbb{D}^{1,2}$. Therefore, it suffices to show that $Z_T D_{t,z} F \in L^2(q \otimes \mathbb{P})$. To this end, we first prove that

$$E \left[ \int_0^T (Z_T D_{t,0} F)^2 dt \right] < \infty.$$  

Because $D_{t,0} F = -\mathbf{1}_{\{S_T < K\}} S_T D_{t,0} L_T = -\mathbf{1}_{\{S_T < K\}} S_T \sigma_t$ by Propositions 4.1 and A.6, we have $E[\int_0^T (Z_T D_{t,0} F)^2 dt] \leq E[Z_T^2 K^2 \int_0^T \sigma_t^2 dt]$. Hence, we only have to show that $E[Z_T^2 J_T] < \infty$ in view of (A.3). Now, as seen in the proof of Proposition 2.7, $Y$ defined in (2.6) is a positive martingale. Therefore, we can define a probability measure $\mathbb{P}_Y$ as $d\mathbb{P}_Y = Y_T d\mathbb{P}$, and we have

$$E[Y_T J_T] = E_{\mathbb{P}_Y}[J_T] = E_{\mathbb{P}_Y} \left[ \int_0^T \int_0^\infty (1 - \delta_{s,x}) x \nu(dx) ds \right] < \infty$$

since $(1 - \delta_{s,x}) x = (1 - \theta_{s,x})^2 x \leq (1 + C_0)^2 x$. Hence, we have $E[Z_T^2 J_T] < \infty$ by (2.7).

Next, we show $E[\int_0^T \int_0^\infty (Z_T D_{t,z} F)^2 z^2 \nu(dz) dt] < \infty$. Note that

$$E \left[ \int_0^T \int_1^\infty (Z_T D_{t,z} F)^2 z^2 \nu(dz) dt \right] \leq E \left[ \int_0^T \int_1^\infty \left( Z_T \frac{K}{z} \right)^2 z^2 \nu(dz) dt \right]$$

$$\leq K^2 E \left[ Z_T^2 \int_0^T \int_1^\infty \nu(dz) dt \right] < \infty.$$  

We have therefore only to show that $E[\int_0^T \int_0^1 Z_T^2 |D_{t,z} F|^2 z^2 \nu(dz) dt] < \infty$. If we have

$$|D_{t,z} F| \leq K |D_{t,z} L_T|$$  

and there is $C > 0$ such that

$$E[Z_T^2 |D_{t,z} L_T|^2] < \frac{C}{z}$$  

for any $z \in (0, 1)$, then we obtain

$$E \left[ \int_0^T \int_0^1 Z_T^2 |D_{t,z} F|^2 z^2 \nu(dz) dt \right]$$

$$\leq K^2 \int_0^T \int_0^1 E[Z_T^2 |D_{t,z} L_T|^2] z^2 \nu(dz) dt$$

$$\leq K^2 C \int_0^T \int_0^1 z \nu(dz) dt < \infty.$$  

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It remains to show (4.3) and (4.4). Inequality (4.3) follows from
\[ |D_{t,z}F| = \left| (K - f_K(L_T + zD_{t,z}L_T))^+ - (K - f_K(L_T))^+ \right| / |z| \]
\[ \leq \left| f_K(L_T + zD_{t,z}L_T) - f_K(L_T) \right| / |z| \leq K |zD_{t,z}L_T| / |z| \]
\[ = K |D_{t,z}L_T|. \]

Next, to prove (4.4), it suffices to show that
\[ \mathbb{E}_{P_Y} \left[ |D_{t,z}L_T|^2 \right] < C z^{-1} \text{ for some } C > 0. \]
The process \( W^Y \) defined as \( dW^Y_t := dW_t + 2\mu_s ds \) is a Brownian motion under \( \mathbb{P}_Y \).
Noting that \( \sqrt{\sigma_s^2 + z e^{-\lambda(s-t)} - \sigma_s} \leq \sqrt{z} \) for \( s \in [t, T] \), we have
\[ |D_{t,z}L_T| \leq C_1 + \left| \int_t^T \sqrt{\sigma_s^2 + z e^{-\lambda(s-t)} - \sigma_s} dW^Y_s \right| + 2C u(T-t) \sqrt{z} \]
for some \( C_1 > 0 \) by Proposition A.6. Hence, we have
\[ \mathbb{E}_{P_Y} \left[ |D_{t,z}L_T|^2 \right] \leq 3C_1^2 + 3\mathbb{E}_{P_Y} \left[ \int_t^T \frac{1}{z} ds \right] + \frac{12C^2 u(T-t)^2}{z} \leq C / z \]
for some \( C > 0 \) as \( 0 < z < 1 \). \( \square \)

**Condition C6** To see that \( FH_{t,z}^* \in L^1(\mathbb{P}^*) \) for \( q \)-a.e. \( (t, z) \in [0, T] \times (0, \infty) \), it suffices to show \( \mathbb{E}[Z_T H_{t,z}^*] < \infty \) since \( F \) is bounded. Now, we have
\[ Z_T H_{t,z}^* = Z_T e^{zD_{t,z} \log Z_T} = \frac{zD_{t,z}Z_T + Z_T}{1 - \theta_{t,z}} \leq \hat{C}_\theta(zD_{t,z}Z_T + Z_T) \]
by (4.2) and item 5 of Lemma A.7. Since \( Z_T \in \mathbb{D}^{1,2} \) by Sect. 5, we have \( D_{t,z}Z_T \in L^1(\mathbb{P}) \) for \( q \)-a.e. \( (t, z) \in [0, T] \times (0, \infty) \). Hence, \( \mathbb{E}[Z_T H_{t,z}^*] < \infty \). Moreover, because \( D_{t,z}F \leq \frac{K}{z} \), we have \( H_{t,z}^* D_{t,z} F \in L^1(\mathbb{P}^*) \) for \( q \)-a.e. \( (t, z) \). \( \square \)

**Condition AS3 in Theorem A.1**: As the last part of the proof of Theorem 3.1, we make sure of Condition AS3, which is given as
\[ \mathbb{E} \left[ \int_0^T \left( (h^0_t)^2 + \int_0^\infty (h_{t,z}^1)^2 v(dz) \right) dt \right] < \infty, \quad (4.5) \]
where \( h_{1,t,z} := \mathbb{E}_{P^*}[F(H_{t,z}^* - 1) + zH_{t,z}^*D_{t,z}F|\mathcal{F}_t^-] \) and

\[
h_t^0 := \mathbb{E}_{P^*}\left[ D_{t,0}F - F\left( \int_0^T D_{t,0}u_s dW_s^{P^*} + \int_0^T \int_0^\infty \frac{D_{t,0}s,x_1}{1 - \theta_{s,x}} \tilde{N}_P^{P^*}(ds, dx) \right) |\mathcal{F}_t^- \right]
\]

\[= -\mathbb{E}_{P^*}[\mathbf{1}_{\{S_T < K\}} S_T \sigma_t |\mathcal{F}_t^-].\]

Here \( dW_s^{P^*} := dW_t + u_t^t dt \) and \( \tilde{N}_P^{P^*}(dt, dz) := \tilde{N}(dt, dz) + \theta_{t,z} \nu(dz)dt \) are a Brownian motion and the compensated Poisson random measure of \( N \) under \( P^* \), respectively.

First, we have \( \mathbb{E}\left[ \int_0^T (h_t^0)^2 dt \right] \leq K^2 \mathbb{E}\left[ \int_0^T \sigma_t^2 dt \right] < \infty \) by (A.3). Next, we show that \( \mathbb{E}\left[ \int_0^T \int_0^\infty (h_{1,t,z})^2 \nu(dz)dt \right] < \infty \). Noting that \( h_{1,t,z} = \mathbb{E}_{P^*}\left[ (F + zD_{t,z}F)H_{t,z}^* - F|\mathcal{F}_t^- \right] \), we have

\[
h_{1,t,z} \leq \mathbb{E}_{P^*}\left[ (F + zD_{t,z}F)H_{t,z}^*|\mathcal{F}_t^- \right] \leq K \mathbb{E}_{P^*}[H_{t,z}^*|\mathcal{F}_t^-] = K
\]

since \( F \) and \( H_{t,z}^* \) are nonnegative, \( 0 \leq F + zD_{t,z}F \leq K \) by Proposition 4.1, and \( \mathbb{E}_{P^*}[H_{t,z}^*|\mathcal{F}_t^-] = 1 \) by (3.3). In addition, we have

\[
h_{1,t,z} \geq -\mathbb{E}_{P^*}[F|\mathcal{F}_t^-] \geq -K.
\]

As a result, \( h_{1,t,z} \) is bounded. Hence, we obtain \( \mathbb{E}\left[ \int_0^T \int_1^\infty (h_{1,t,z})^2 \nu(dz)dt \right] < \infty \).

Next, we show that \( \mathbb{E}\left[ \int_0^T \int_0^1 (h_{1,t,z})^2 \nu(dz)dt \right] < \infty \). To this end, we rewrite \( h_{1,t,z} \) as

\[
h_{1,t,z} = \mathbb{E}_{P^*}\left[ (F + zD_{t,z}F)(H_{t,z}^* - 1) + zD_{t,z}F|\mathcal{F}_t^- \right].
\]

Because \( |zD_{t,z}F| \leq K \), we have \( (\mathbb{E}_{P^*}[zD_{t,z}F|\mathcal{F}_t^-])^2 \leq K^2 \). Thus, it suffices to prove

\[
\mathbb{E}\left[ \int_0^T \int_0^1 \left( \mathbb{E}_{P^*}\left[ (F + zD_{t,z}F)(H_{t,z}^* - 1)|\mathcal{F}_t^- \right] \right)^2 \nu(dz)dt \right] < \infty. \tag{4.6}
\]

Now (3.3) implies

\[
(\mathbb{E}_{P^*}[F + zD_{t,z}F|\mathcal{F}_t^-])^2 \leq K^2 \mathbb{E}_{P^*}[H_{t,z}^* - 1]^2|\mathcal{F}_t^-] \\
\leq K^2 \mathbb{E}_{P^*}[(H_{t,z}^* - 1)^2|\mathcal{F}_t^-] \\
\leq K^2 (\mathbb{E}_{P^*}[(H_{t,z}^* - 1)^2|\mathcal{F}_t^-] - 2\mathbb{E}_{P^*}[H_{t,z}^*|\mathcal{F}_t^-] + 1) \\
= K^2(\mathbb{E}_{P^*}[(H_{t,z}^* - 1)^2|\mathcal{F}_t^-] - 1). \tag{4.7}
\]
Next, we calculate \((H_{t,z}^*)^2\). By the definition of \(H_{t,z}^*\) in Theorem 3.1 and by Proposition A.11, we have

\[
(H_{t,z}^*)^2 = \exp \left( -2z \int_0^T D_{t,z} u_s dW_s - 2z \int_0^T u_s D_{t,z} u_s ds - z^2 \int_0^T (D_{t,z} u_s)^2 ds \\
+ 2z \int_0^T \int_0^\infty D_{t,z} \log(1 - \theta_{s,x}) \tilde{N}(ds, dx) \\
+ 2z \int_0^T \int_0^\infty (D_{t,z} \log(1 - \theta_{s,x}) + D_{t,z} \theta_{s,x}) \nu(dx) ds \right)
\]

\[
= \exp \left( -2z \int_0^T D_{t,z} u_s dW_s - 2z \int_0^T u_s D_{t,z} u_s ds - \frac{1}{2} \int_0^T (2z D_{t,z} u_s)^2 ds \\
+ \int_0^T (z D_{t,z} u_s)^2 ds + \int_0^T \int_0^\infty \log(1 - y_{t,z,s,x}) \tilde{N}(ds, dx) \\
+ \int_0^T \int_0^\infty (\log(1 - y_{t,z,s,x}) + y_{t,z,s,x}) \nu(dx) ds \right) \\
- \int_0^T \int_0^\infty y_{t,z,s,x} \theta_{s,x} \nu(dx) ds + \int_0^T \int_0^\infty \frac{(z D_{t,z} \theta_{s,x})^2}{1 - \theta_{s,x}} \nu(dx) ds \right),
\]

(4.8)

where

\[
y_{t,z,s,x} := \frac{2 z D_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} - \left( \frac{z D_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right)^2.
\]

We remark that Lemma A.10 implies that

\[
z D_{t,z} \log(1 - \theta_{s,x}) = \log(1 - \theta_{s,x} - z D_{t,z} \theta_{s,x}) - \log(1 - \theta_{s,x})
\]

\[
= \log \left( 1 - \frac{z D_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right),
\]

that is, \(2z D_{t,z} \log(1 - \theta_{s,x}) = \log(1 - y_{t,z,s,x})\). Now, we have \((z D_{t,z} u_s)^2 \leq z C_\theta^2\) by Lemma A.8, and

\[
\int_0^\infty \frac{(z D_{t,z} \theta_{s,x})^2}{1 - \theta_{s,x}} \nu(dx) \leq z (C_\theta)^2 \tilde{C}_\theta C_\rho
\]
by Lemmas A.7 and A.9. Therefore, we have

\[
\text{RHS of (4.8)} \leq \exp\left(-2z \int_0^T D_{t,z,u_s}dW_s - 2z \int_0^T u_s D_{t,z,u_s}ds - \frac{1}{2} \int_0^T (2z D_{t,z,u_s})^2 ds \right. \\
\left. + \int_0^T \int_0^\infty \log(1 - \gamma_{t,z,s,x}) \tilde{N}(ds,dx) \right. \\
\left. + \int_0^T \int_0^\infty (\log(1 - \gamma_{t,z,s,x}) + \gamma_{t,z,s,x}) v(dx)ds \\
- \int_0^T \int_0^\infty \gamma_{t,z,s,x} \theta_{s,x} v(dx)ds + Cz \right)
\] (4.9)

for some \( C > 0 \). Hence, Lemma 4.2 implies that

\[
\mathbb{E}_{P^*}[H_{t,z}^2 | \mathcal{F}_{t-}] \leq \mathbb{E}_{P^*}[X_{T,z}^t | \mathcal{F}_{t-}] e^{Cz} = X_{t,z}^t e^{Cz} = e^{Cz}. \] (4.10)

Consequently, we have

\[
\text{RHS of (4.7)} \leq K^2(e^{Cz} - 1) \leq K^2z(e^C - 1)
\]

for any \( z \in (0, 1) \). Hence (4.6) follows, from which we obtain (4.5). This completes the proof of Theorem 3.1. \( \square \)

To see (4.10), we show the following lemma.

**Lemma 4.2** Given \((t, z) \in [0, T) \times (0, \infty)\), we consider the SDE

\[
dX_{s}^{t,z} = -X_{s}^{t,z} \left(2z D_{t,z,u_s}dW_s + 2z u_s D_{t,z,u_s}ds + \int_0^\infty \gamma_{t,z,s,x} \tilde{N}(ds,dx) \right. \\
\left. + \int_0^\infty \gamma_{t,z,s,x} \theta_{s,x} v(dx)ds \right).
\] (4.11)

The solution \( X_{t,z} \) is a martingale under \( \mathbb{P}^* \) with \( X_{0,z}^t = 1 \) for any \( s \in [0, t) \). In particular, the right-hand side of (4.9) is equal to \( X_{t,z}^t e^{Cz} \).

**Proof** First, note that \( z D_{t,z,u_s} \) and \( z u_s D_{t,z,u_s} \) are bounded. In addition, we have

\[
\left| \frac{z D_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right| < 2C_\theta \hat{C}_\theta (1 - e^{\rho x}) < 2C_\theta \hat{C}_\theta
\] (4.12)
by Lemmas A.7 and A.9. Therefore, Lemma A.7 yields
\[
\int_0^\infty |\gamma_{t,z,s,x} \theta_{s,x} \nu(dx) = \int_0^\infty \left| \frac{zD_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right| \left( 2 - \frac{zD_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right) \theta_{s,x} \nu(dx) \leq 2C_\theta \hat{C}_\theta (2 + 2C_\theta \hat{C}_\theta) \theta_{s,x} \nu(dx) < \infty.
\]
Moreover, (4.12) again implies
\[
\int_0^\infty \gamma_{t,z,s,x}^2 \nu(dx) = \int_0^\infty \left( \frac{zD_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right)^2 \left( 2 - \frac{zD_{t,z} \theta_{s,x}}{1 - \theta_{s,x}} \right)^2 \theta_{s,x} \nu(dx) \leq 4C_\theta^2 \hat{C}_\theta^2 C_\rho \nu(dx) < \infty.
\]
As a result, we can apply Theorem 117 of Situ [20] to (4.11); we then conclude that (4.11) has a solution \(X_t^z\) satisfying
\[
E \left[ \sup_{0 \leq s \leq T} |X_t^z|^2 \right] < \infty,
\]
which implies that \(E_{\mathbb{P}^*} \left[ \sup_{0 \leq s \leq T} |X_t^z| \right] < \infty\) by the \(L^2(\mathbb{P})\)-property of \(Z_T\). Now, \(X_t^z\) is a local martingale under \(\mathbb{P}^*\) because we can rewrite (4.11) as
\[
dX_t^z = -X_t^z \left( 2zD_{t,z} u_s dW^*_s + \int_0^\infty \gamma_{t,z,s,x} \tilde{N}(ds, dx) \right).
\]
Consequently, Theorem I.51 of Protter [16] implies that \(X_t^z\) is a \(\mathbb{P}^*\)-martingale satisfying \(X_t^z = 1\) for any \(s \in [0, t)\). Moreover, by Example 9.6 of Di Nunno et al. [11], the right-hand side of (4.9) is expressed by \(X_T^z e^{C_z}\).

\section{Malliavin differentiability of \(Z\)}

This section is devoted to showing that \(Z_t \in D^{1,2}\) for any \(t \in [0, T]\). To this end, for \(t \in [0, T]\), we define \(Z_t^{(0)} := 1\) and
\[
Z_t^{(n+1)} := 1 - \int_0^t Z_s^{(n)} u_s dW_s - \int_0^t \int_0^\infty Z_s^{(n)} \theta_{s,x} \tilde{N}(ds, dx)
\]
for \(n \geq 0\). Furthermore, we denote, for \(n \geq 0\),
\[
\phi_n(t) := E \left[ \int_{[0,t] \times [0,\infty]} (D_{r,z} Z_t^{(n)})^2 q(dr, dz) \right].
\]
Note that \(\phi_0(t) \equiv 0\).

\begin{lemma}
We have \(Z_t^{(n)} \in D^{1,2}\) for every \(n \geq 0\) and any \(t \in [0, T]\). Moreover, there exist constants \(k_1 > 0\) and \(k_2 > 0\) such that
\[
\phi_{n+1}(t) \leq k_1 + k_2 \int_0^t \phi_n(s) ds
\]
for every \(n \geq 0\) and any \(t \in [0, T]\).
\end{lemma}
Thanks to Lemma 5.1, we have
\[
\phi_{n+1}(t) \leq k_1 + k_2 \int_0^t \phi_n(s) ds \leq k_1 + k_2 \int_0^t \left( k_1 + k_2 \int_0^s \phi_{n-1}(s) ds \right) ds
\]
\[
\leq \cdots \leq k_1 \sum_{j=0}^n \frac{k_1^j t^j}{j!} < k_1 e^{k_2 t}
\]
for any \( t \in [0, T] \). Hence, \( \sup_{n \geq 1} \phi_n(t) < \infty \). As \( Z_t^{(n)} \to Z_t \) in \( L^2(\mathbb{P}) \), [11, Lemma 17.1] implies that \( Z_t \in \mathbb{D}^{1,2} \) for \( t \in [0, T] \). Note that the Malliavin derivative in [11] is defined in a way different from ours. Denoting by \( \hat{D} \) the Malliavin derivative operator in [11], we have
\[
\hat{D}_t z F = z D_t z F \quad \text{for } z \neq 0 \text{ and } F \in \mathbb{D}^{1,2}.
\]

**Proof of Lemma 5.1** We take an arbitrary integer \( n \geq 0 \). Suppose that \( Z_t^{(n)} \) is in \( \mathbb{D}^{1,2} \) and \( \int_0^t \phi_n(s) ds < \infty \) for any \( t \in [0, T] \). Lemma 5.2 below and Lemma 3.3 of [10] imply that \( Z_t^{(n+1)} \in \mathbb{D}^{1,2} \) for any \( t \in [0, T] \) and that for any \( t \in [r, T] \) and any \( z \in (0, \infty) \),
\[
D_{r,0} Z_t^{(n+1)}
\]
\[
= -D_{r,0} \int_{[0,T] \times [0,\infty)} Z_{s-}^{(n)} \left( u_s 1_{(0)}(x) + \frac{\theta_{s,x}}{x} 1_{(0,\infty)}(x) \right) 1_{[0,t]}(s) Q(ds,dx)
\]
\[
= -Z_{r-}^{(n)} u_r - \int_r^t D_{r,0} (Z_{s-}^{(n)} u_s) dW_s - \int_r^t \int_0^\infty D_{r,0} \left( Z_{s-}^{(n)} \frac{\theta_{s,x}}{x} \right) x \tilde{N}(ds,dx)
\]
\[
= -Z_{r-}^{(n)} u_r - \int_r^t u_s D_{r,0} Z_{s-}^{(n)} dW_s - \int_r^t \int_0^\infty \theta_{s,x} D_{r,0} Z_{s-}^{(n)} \tilde{N}(ds,dx) \quad (5.1)
\]
and
\[
D_{r,z} Z_t^{(n+1)} = -Z_{r-}^{(n)} \frac{\theta_{r,z}}{z} - \int_r^t D_{r,z} (Z_{s-}^{(n)} u_s) dW_s
\]
\[
= \int_r^t \int_0^\infty D_{r,z} \left( Z_{s-}^{(n)} \frac{\theta_{s,x}}{x} \right) x \tilde{N}(ds,dx). \quad (5.2)
\]

Next, we fix an arbitrary \( t \in [0, T] \). We have then
\[
\phi_{n+1}(t) = \mathbb{E} \left[ \int_0^t (D_{r,0} Z_t^{(n+1)})^2 dr \right] + \mathbb{E} \left[ \int_0^t \int_0^\infty (D_{r,z} Z_t^{(n+1)})^2 z^2 v(dz) dr \right]. \quad (5.3)
\]
Equality (5.1) implies that
\[
\text{the first term on RHS of (5.3)}
\]
\[
\leq 3 \mathbb{E} \left[ \int_0^t (Z_{r-} u_r)^2 dr \right] + 3 \mathbb{E} \left[ \int_0^t \left( \int_r^t u_s D_{r,0} Z_{s-} dW_s \right)^2 dr \right]
\]
\[
+ 3 \mathbb{E} \left[ \int_0^t \left( \int_0^t \theta_{s,x} D_{r,0} Z_{s-} \tilde{N}(ds,dx) \right)^2 dr \right]. \quad (5.4)
\]
We evaluate each term on the right-hand side of (5.4). Lemma A.7 implies
\[
E \left[ \int_0^t (Z_{r-}^{(n)} u_r)^2 dr \right] \leq C_u^2 \left[ \int_0^t (Z_{r-}^{(n)})^2 dr \right] \leq C_u^2 T \left[ \sup_{0 \leq s \leq T} (Z_s^{(n)})^2 \right]
\]
and
\[
E \left[ \int_0^t \left( \int_r^t u_s D_{r,0} Z_s^{(n)} dW_s \right)^2 dr \right] \leq C_u^2 \int_0^t \left[ \int_r^t (D_{r,0} Z_{r-}^{(n)})^2 ds \right] dr.
\]
The same argument implies that
\[
E \left[ \int_0^t \left( \int_r^t \int_0^\infty \theta_{s,x} D_{r,0} Z_s^{(n)} \tilde{N}(ds,dx) \right)^2 dr \right]
= \int_0^t E \left[ \int_r^t \int_0^\infty \theta_{s,x} D_{r,0} Z_s^{(n)} v(dx) ds \right] dr
\leq C_\theta^2 C_\rho \int_0^t \left[ \int_r^t (D_{r,0} Z_s^{(n)})^2 ds \right] dr.
\]
As a result, we obtain that
\[
\text{the first term on RHS of (5.3) } \leq 3 C_u^2 T \left[ \sup_{0 \leq s \leq T} (Z_s^{(n)})^2 \right]
+ 3 (C_u^2 + C_\theta^2 C_\rho) \int_0^t \left[ \int_r^t (D_{r,0} Z_{r-}^{(n)})^2 ds \right] dr.
\]
(5.5)
Next, (5.2) yields
\[
\text{the second term on RHS of (5.3)}
\leq 3 E \left[ \int_0^t \int_0^\infty \left( \frac{\theta_{r,z}}{z} \right)^2 z^2 v(dz) dr \right]
+ 3 E \left[ \int_0^t \int_0^\infty \left( \int_r^t D_{r,z} (Z_{r-}^{(n)} u_s) dW_s \right)^2 z^2 v(dz) dr \right]
+ 3 E \left[ \int_0^t \int_0^\infty \left( \int_0^t \int_0^\infty D_{r,z} \left( Z_{r-}^{(n)} \frac{\theta_{s,x}}{x} \right) x^{\tilde{N}(ds,dx)} \right)^2 z^2 v(dz) dr \right].
\]
(5.6)
We now calculate each term on the right-hand side of (5.6). First,
\[
\text{the first term on RHS of (5.6)} \leq 3 C_\theta^2 C_\rho \left[ \int_0^t (Z_{r-}^{(n)})^2 dr \right]
\leq 3 C_\theta^2 C_\rho T \left[ \sup_{0 \leq s \leq T} (Z_s^{(n)})^2 \right].
\]
Next, Lemma A.8 implies that

the second term on RHS of (5.6)

\[
= 3 \int_0^t \int_0^\infty \mathbb{E} \left[ \int_r^t \left( D_{r,z} \left( Z_{s-}^{(n)} u_s \right) \right)^2 ds \right] z^2 v(dz) dr
\]

\[
= 3 \int_0^t \int_0^\infty \mathbb{E} \left[ \int_r^t \left( u_s D_{r,z} Z_{s-}^{(n)} + Z_{s-}^{(n)} D_{r,z} u_s + z(D_{r,z} Z_{s-}^{(n)}) D_{r,z} u_s \right)^2 ds \right] z^2 v(dz) dr
\]

\[
\leq 9 \int_0^t \int_0^\infty \left( C_u^2 \mathbb{E} \left[ \int_r^t (D_{r,z} Z_{s-}^{(n)})^2 ds \right] + \frac{C_u^2}{z} \mathbb{E} \left[ \int_r^t (Z_{s-}^{(n)})^2 ds \right] \right) z^2 v(dz) dr
\]

\[
+ 9 \left( C_u^2 + (C_u')^2 \right) \int_0^t \int_0^\infty \mathbb{E} \left[ \int_r^t (D_{r,z} Z_{s-}^{(n)})^2 ds \right] z^2 v(dz) dr.
\]

Finally, we evaluate the third term of (5.6). By Lemma A.9 we obtain

the third term on RHS of (5.6)

\[
= 3 \int_0^t \int_0^\infty \mathbb{E} \left[ \int_r^t \int_0^\infty \left( D_{r,z} Z_{s-}^{(n)} \frac{\theta_{s,x}}{x} + Z_{s-}^{(n)} \frac{\theta_{s,x}}{x} \right) \right. \]

\[
\left. + z(D_{r,z} Z_{s-}^{(n)}) D_{r,z} \frac{\theta_{s,x}}{x} \right)^2 x^2 v(dx) ds \right] z^2 v(dz) dr
\]

\[
\leq 9 \int_0^t \int_0^\infty \left( C_\theta^2 C_\rho \mathbb{E} \left[ \int_r^t (D_{r,z} Z_{s-}^{(n)})^2 ds \right] + \frac{(C_\theta')^2 C_\rho}{z} \mathbb{E} \left[ \int_r^t (Z_{s-}^{(n)})^2 ds \right] \right) z^2 v(dz) dr
\]

\[
+ z^2 4 C_\theta^2 C_\rho \mathbb{E} \left[ \int_r^t (D_{r,z} Z_{s-}^{(n)})^2 ds \right] z^2 v(dz) dr
\]

\[
\leq 9 (C_\theta')^2 C_\rho \int_0^\infty z v(dz) T^2 \mathbb{E} \left[ \sup_{0 \leq s \leq T} (Z_s^{(n)})^2 \right]
\]

\[
+ 45 C_\theta^2 C_\rho \int_0^t \int_0^\infty \mathbb{E} \left[ \int_r^t (D_{r,z} Z_{s-}^{(n)})^2 ds \right] z^2 v(dz) dr.
\]
Consequently, by (5.3), (5.5)–(5.9), and Lemma 5.3 below, there are constants $k_1 > 0$ and $k_2 > 0$ such that

$$
\phi_{n+1}(t) \\
\leq k_1 \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left( Z_s^{(n)} \right)^2 \right] + k_2 \int_0^t \mathbb{E} \left[ \int_r^T (D_{r,z} Z_s^{(n)})^2 ds \right] q(dr, dz) \\
\leq k_1 \sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left( Z_s^{(n)} \right)^2 \right] + k_2 \int_0^t \mathbb{E} \left[ \int_{[0,s] \times [0,\infty)} (D_{r,z} Z_s^{(n)})^2 q(dr, dz) \right] ds \\
= k_1 \sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left( Z_s^{(n)} \right)^2 \right] + k_2 \int_0^t \mathbb{E} \left[ \int_{[0,s] \times [0,\infty)} (D_{r,z} Z_s^{(n)})^2 q(dr, dz) \right] ds \\
\leq k_1 + k_2 \int_0^t \phi_n(s) ds,
$$

where $k_1$ and $k_2$ may vary from line to line. □

Now, we prove two lemmas, which are used in the proof of Lemma 5.1.

**Lemma 5.2** Fix $n \geq 0$ arbitrarily. Assume that $Z_t^{(n)}$ is in $D^{1,2}$ and $\int_0^t \phi_n(s) ds < \infty$ for any $t \in [0, T]$. We then have $Z_t^{(n)} u \in L^{1,2}_0$ and $Z_t^{(n)} \theta \in L^{1,2}_1$.

**Proof** We show that $Z_t^{(n)} u \in L^{1,2}_0$. Since $Z_t^{(n)} \in D^{1,2}$, by Lemmas A.7 and A.8 we have $Z_t^{(n)} D_{t,z} u_s + u_s D_{t,z} Z_t^{(n)} + z(D_{t,z} Z_t^{(n)}) D_{t,z} u_s \in L^2(q \otimes \mathbb{P})$ for any $s \in [0, T]$. Hence, item (a) in the definition of $L^{1,2}_0$ is given by Propositions 5.1 and 5.4 of [21]. Next, item (b) is satisfied by Lemma A.7. As for item (c), there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$
\left( D_{t,z} (Z_t^{(n)} u_s) \right)^2 \leq C_1 (Z_t^{(n)})^2 + C_2 (D_{t,z} Z_t^{(n)})^2.
$$

In addition, we have

$$
\mathbb{E} \left[ \int_{[0,T] \times [0,\infty)} \int_0^T (D_{t,z} Z_s^{(n)})^2 ds q(dt, dz) \right] \\
= \int_0^T \mathbb{E} \left[ \int_{[0,T] \times [0,\infty)} (D_{t,z} Z_s^{(n)})^2 q(dt, dz) \right] ds = \int_0^T \phi_n(s) ds < \infty.
$$

As a result, item (c) follows. This completes the proof of $Z_t^{(n)} u \in L^{1,2}_0$. Moreover, $Z_t^{(n)} \theta \in L^{1,2}_1$ is shown similarly. □

**Lemma 5.3** $\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left( Z_s^{(n)} \right)^2 \right] < \infty.$
Proof. First, we can see inductively that $Z^{(n)}$ is a martingale with $Z^{(n)}_T \in L^2(\mathbb{P})$. Denoting $\zeta_n(t) := \mathbb{E}[\sup_{0 \leq s \leq t} (Z^{(n)}_s)^2]$ for $t \in [0, T]$ and $n \geq 1$, we have

$$
\zeta_n(T) \leq 4 \mathbb{E}\left[\left(1 - \int_0^T Z^{(n-1)}_s u_s dW_s - \int_0^T \int_0^\infty Z^{(n-1)}_s \theta_{s,x}\tilde{N}(ds,dx)\right)^2\right]
$$

$$
\leq 4 \left(1 + \mathbb{E}\left[\int_0^T (Z^{(n-1)}_s)^2 (u_s^2 + \int_0^\infty \theta_{s,x}^2 \nu(dx)) ds\right]\right)
$$

$$
\leq 4 + 4(C_u^2 + C_\theta^2 C_\rho) \int_0^T \zeta_{n-1}(s) ds \leq 4 \exp\left(4(C_u^2 + C_\theta^2 C_\rho)T\right)
$$

by Doob’s inequality and Lemma A.7. \hfill \Box

6 Numerical experiments

In this section, we illustrate LRM strategies for the BNS models with numerical experiments. [2] developed a numerical scheme for LRM strategies in exponential Lévy models using the Carr–Madan approach [7], which is a numerical method for option prices based on the fast Fourier transform (FFT). In the following, we compute (6.1) numerically for the call options using the method developed in [2]. Moreover, we compare LRM strategies with delta-hedging strategies, which are given as the partial derivative of the option price with respect to the asset price.

We treat the gamma-OU model in which the Lévy measure $\nu$ is given as

$$
\nu(dx) = ab \lambda e^{-bx} \mathbf{1}_{(0,\infty)}(x)dx,
$$

where $a > 0$, $b > 0$. Moreover, we use the parameter set estimated in [17] (see Table 1). To this end, we need to adopt the same setting as [17]. Hence, we need to take into account the interest rate $r > 0$ and the continuous dividend rate $q > 0$; that is, the discount factor is given by $r - q$. Moreover, suppose that the discounted asset price process $(e^{-(r-q)t}S_t)$ is a martingale. Hence, $\mu$ appearing in (1.3) is given as

$$
\mu = r - q + \int_0^\infty (1 - e^{\rho x}) \nu(dx) = r - q - \frac{a\lambda \rho}{b - \rho}.
$$

We consider a call option with strike price $K$. From Theorem 3.1, Corollary 3.3 and Proposition 4.1, we have

$$
\xi_t(S_T - K)^+
$$

$$
= \frac{e^{-(r-q)(T-t)}}{S_t - (\sigma_t^2 + C_\rho)} \left(\sigma_t^2 \mathbb{E}[S_T \mathbf{1}_{S_T \geq K} | \mathcal{F}_{t-}] + \int_0^\infty \mathbb{E}[(S_T e^{zD_t - LT} - K)^+ - (S_T - K)^+ | \mathcal{F}_{t-}](e^{\rho z} - 1) \nu(dz)\right)\quad (6.1)
$$
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since \( H^*_{t,z} = 1 \) and \( Z_T = 1 \). Therefore, denoting

\[
I_1 := e^{-(r-q)(T-t)} \mathbb{E}[S_T \mathbf{1}_{\{S_T \geq K\}} | S_t, \sigma^2_t]
\]

and

\[
I_2 := e^{-(r-q)(T-t)} \int_0^\infty \mathbb{E}[(S_T e^{zD_{t,z}}L_T - K)^+ - (S_T - K)^+ | S_t, \sigma^2_t](e^{\rho z} - 1) \nu(dz),
\]

we can rewrite (6.1) as

\[
\xi(S_T - K)^+ = \frac{\sigma^2 I_1 + I_2}{S_t - (\sigma^2_t + C\rho)}.
\]

We next develop numerical schemes for \( I_1 \) and \( I_2 \) separately.

Denoting by \( \phi \) the characteristic function of \( L_T \) given \( S_t \) and \( \sigma^2_t \), we have

\[
\phi(\vartheta) := \mathbb{E}[\exp(i\vartheta L_T) | S_t, \sigma^2_t]
\]

\[
= \exp\left( i\vartheta (L_t + \mu(T-t)) - (\vartheta^2 + i\vartheta) \frac{B(T-t)}{2} \sigma^2_t \right.
\]

\[
+ \frac{a}{b - f_2} \left( b \log \frac{b - f_1}{b - i\vartheta \rho} + f_2 \lambda(T-t) \right) \right) \quad (6.2)
\]

for \( \vartheta \in \mathbb{C} \) from [17, Sect. 7.1.1], where

\[
f_1 := i\vartheta \rho - \frac{1}{2}(\vartheta^2 + i\vartheta) \lambda B(T-t) \quad \text{and} \quad f_2 := i\vartheta \rho - \frac{1}{2}(\vartheta^2 + i\vartheta).
\]

Recall that \( B(t) = \frac{1-e^{-\lambda t}}{\lambda} \) for \( t \in [0,T] \). As for \( I_1 \), Proposition 2.1 of [2] implies

\[
I_1 = \frac{e^{-(r-q)(T-t)}}{\pi} \int_0^\infty K^{i\xi + 1} \frac{\phi(\xi)}{i\xi - 1} d\nu,
\]

where \( \xi := \nu - i\delta \), and \( \delta \) is a real number satisfying

\[
\sup_{t \leq s < T} \left\{ \frac{1}{2} - \frac{B}{B(T-s)} - \sqrt{\Xi_s} \right\} < \delta < \inf_{t \leq s < T} \left\{ \frac{1}{2} - \frac{B}{B(T-s)} + \sqrt{\Xi_s} \right\} \quad (6.4)
\]

by [15, Theorem 2.2]. Here,

\[
\Xi_s := \left( -\frac{1}{2} + \frac{\rho}{B(T-s)} \right)^2 + \frac{2\hat{\vartheta}}{B(T-s)}
\]

and

\[
\hat{\vartheta} := \sup \left\{ \vartheta \in \mathbb{R} : \int_0^\infty (e^{\vartheta x} - 1) \nu(dx) < \infty \right\}.
\]
which is ensured to be positive by Assumption 2.2. Note that the right-hand side of (6.3) is independent of the choice of \( \delta \). As a result, since the integrand of (6.3) is given by the product of \( K^{-i\zeta+1} \) and a function of \( \zeta \), we can compute \( I_1 \) through the FFT.

Next, we calculate \( I_2 \). First, Proposition A.6 implies

\[
\frac{S_T}{S_t} \exp(zD_{t,z}L_T) = \exp \left( \mu(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s + \rho \int_t^T dJ_s \right. \\
- \frac{z}{2} B(T-t) + \int_t^T \left( \sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} - \sigma_s \right) dW_s + \rho z \\
= \exp \left( \mu(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds \right. \\
+ \int_t^T \sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} dW_s + \rho \int_t^T dJ_s + \rho z \\
= \exp \left( \mu(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_{s,z} dW_s \right. \\
+ \rho \int_t^T dJ_s + \rho z \\
\]

for \( t \in [0, T] \) and \( z \in (0, \infty) \), where \( B(T-t) = \int_t^T e^{-\lambda(s-t)} ds \) for \( t \in [0, T] \), and \( \sigma_{s,z}^2 := \sigma_s^2 + ze^{-\lambda(s-t)} \) for \( (s, z) \in [t, T] \times (0, \infty) \). Denoting

\[
L_s^{(z)} := \int_t^s \left( \mu - \frac{1}{2} \sigma_u^2 \right) du + \int_t^s \sigma_{u,z} dW_u + \rho \int_t^s dJ_u
\]

for \( (s, z) \in [t, T] \times (0, \infty) \), we have

\[
S_T \exp(zD_{t,z}L_T) = S_t \exp(L_s^{(z)} + \rho z).
\]

In addition, since the process \( (\sigma_{s,z}^2)_{t \leq s \leq T} \) is a solution to the SDE (1.2) with \( \sigma_{t,z}^2 = \sigma_t^2 + z \), (6.2) implies that the characteristic function of \( \log S_t + L_T^{(z)} \) given \( S_t \) and \( \sigma_t^2 \) is given as

\[
\phi^{(z)}(\vartheta) := \mathbb{E}[\exp(i \vartheta L_T^{(z)}) | S_t, \sigma_t^2] S_t^{i \vartheta} = \mathbb{E}[\exp(i \vartheta L_T) | S_t, \sigma_t^2 + z] \\
= \phi(\vartheta) \exp \left( -(\vartheta^2 + i \vartheta) \frac{B(T-t)}{2} \right).
\]
Proposition 2.3 of [2] implies

\[ e^{(r-q)(T-t)} I_2 = \int_0^\infty \mathbb{E}\left[ (S_t \exp(L_T(z) + \rho z) - K)^+ - (S_T - K)^+ | S_t, \sigma_t^2 \right] (e^{\rho z} - 1) \nu(dz) \]

\[ = \int_0^\infty \left( \frac{e^{\rho z}}{\pi} \int_0^\infty (Ke^{-\rho z})^{-i\zeta + 1} \left( \frac{\phi(z)}{(i\zeta - 1)i\zeta} dv \right) - \frac{1}{\pi} \int_0^\infty \frac{K^{-i\zeta + 1}\phi(z)}{(i\zeta - 1)i\zeta} dv \right) \]

\[ \times (e^{\rho z} - 1) \nu(dz) \]

\[ = \int_0^\infty \frac{1}{\pi} \int_0^\infty \frac{K^{-i\zeta + 1}\phi(z)}{(i\zeta - 1)i\zeta} \left( e^{\rho z} \mathbb{E}\left[ (e^{\eta z} - 1)(e^{\rho z} - 1) \nu(dz) \right] \right) \]

\[ \times (e^{\rho z} - 1) \nu(dz) \]

\[ = \int_0^\infty \frac{1}{\pi} \int_0^\infty \frac{K^{-i\zeta + 1}\phi(z)}{(i\zeta - 1)i\zeta} \int_0^\infty (e^{\eta z} - 1)(e^{\rho z} - 1) \nu(dz) dv, \]

where \( \eta := i\rho - (\zeta^2 + i\zeta) \frac{B(T-t)}{2} \), which is a function of \( \zeta \). Note that as in the proof of Theorem 2.2 of [15], \( \Re(\eta) \leq 0 \) when \( 0 < \delta < 1 - \frac{2\rho}{B(T)} \), which is a subinterval of (6.4) for any \( t \in [0, T] \). Therefore, taking such \( \delta \), we have

\[ \int_0^\infty (e^{\eta z} - 1)(e^{\rho z} - 1) \nu(dz) = ab\lambda \left( \frac{1}{b - \eta - \rho} - \frac{1}{b - \eta} - \frac{1}{b - \rho} + 1 \right), \]

from which we can compute \( I_2 \) using the FFT.

Next, we discuss the delta-hedging strategy \( \Delta_t^{(S_T - K)^+} \) for a call option with strike price \( K \), which is given as the partial derivative of the option price with respect to \( S_t \), that is,

\[ \Delta_t^{(S_T - K)^+} := e^{-(r-q)(T-t)} \frac{\partial}{\partial S_t} \mathbb{E}[ (S_T - K)^+ | S_t, \sigma_t^2 ]. \]

Noting that

\[ \mathbb{E}[ (S_T - K)^+ | S_t, \sigma_t^2 ] = \frac{1}{\pi} \int_0^\infty K^{-i\zeta + 1} \frac{\phi(z)}{(i\zeta - 1)i\zeta} dv, \]

we have

\[ \Delta_t^{(S_T - K)^+} = \frac{e^{-(r-q)(T-t)}}{\pi} \int_0^\infty \frac{K^{-i\zeta + 1}}{(i\zeta - 1)i\zeta} \frac{\partial \phi(z)}{\partial S_t} dv \]

\[ = \frac{e^{-(r-q)(T-t)}}{\pi} \int_0^\infty \frac{K^{-i\zeta + 1}\phi(z)S_t^{-1}}{i\zeta - 1} dv = \frac{I_1}{S_t}. \]
Hence, the delta-hedging strategy is given from $I_1$. In addition, when the LRM strategy $\xi_t(S_T - K)^+$ is realized, the corresponding amount invested in the risk-free asset, denoted by $\eta_t(S_T - K)^+$, is given as

$$\eta_t(S_T - K)^+ = e^{-(r-q)t}E[(S_T - K)^+ | S_t, \sigma_t^2] - e^{-(r-q)t}\xi_t(S_T - K)^+ S_t$$

for $t < T$, and $\eta_T(S_T - K)^+ = -e^{-(r-q)t}\xi_T(S_T - K)^+ S_T$ by Proposition 2.6.

We show numerical results on LRM strategies $\xi_t(S_T - K)^+$, delta-hedging strategies $\Delta_t(S_T - K)^+$, and $\eta_t(S_T - K)^+$, using the parameter set estimated in [17]. We fix $T = 1$, $r = 0.019$, and $q = 0.012$. The asset price and the squared volatility at time $t$ are fixed to $S_t = 1124.47$ and $\sigma_t^2 = 0.0145$, respectively. Recall from Table 1 that $\rho = -1.2606$, $\lambda = 0.5783$, $a = 1.4338$, $b = 11.6641$. Moreover, just like [17], we take $\delta = 1.75$. Note that the parameter $\alpha$ on p. 20 of [17] corresponds to $\delta - 1$ in our setting, and $1 - \frac{2\rho}{B(T)}$ is greater than 1.75. In Figs. 1 and 2, red crosses and blue circles represent the values of $\xi_t(S_T - K)^+$ and $\Delta_t(S_T - K)^+$, respectively. We implement the following two types of experiments: First, for fixed strike price $K$, we compute $\xi_t(S_T - K)^+$ and $\Delta_t(S_T - K)^+$ for times $t = 0, 0.02, \ldots, 0.98$. Note that we fix $K$ to 900, 1124.47, 1300, which correspond to “out of the money”, “at the money,” and “in the money,” respectively. Second, $t$ is fixed to 0, 0.5 and 0.9, and we instead vary $K$ from 200 to 2000 at steps of 25 and compute $\xi_t(S_T - K)^+$ and $\Delta_t(S_T - K)^+$. Moreover, numerical results on the values of $\eta_t(S_T - K)^+$ for the same pattern as Figs. 1(b) and 2(b) are shown in Fig. 3.

Now, we discuss the implications from Figs. 1–3. First, $\xi_t(S_T - K)^+$ is always less than or equal to $\Delta_t(S_T - K)^+$. This suggests that local risk-minimization is more risk-averse than the delta-hedge. Second, Fig. 1 shows that both $\xi_t(S_T - K)^+$ and $\Delta_t(S_T - K)^+$ are increasing functions of $t$ when the option is “in the money”, and decreasing when “at the money” or “out of the money”. Third, Fig. 2 implies that both $\xi_t(S_T - K)^+$ and $\Delta_t(S_T - K)^+$ tend to 1 when the option is “deep in the money” and to 0 when “deep out of the money.” In addition, the values of strategies decrease from 1 to 0 around “at the money,” and the gradient is steep when the time to maturity is near to 0. Fourth, the spread between $\xi_t(S_T - K)^+$ and $\Delta_t(S_T - K)^+$ in Fig. 2 is wider when the option is “in the money” than “out of the money”. Finally, Fig. 3(b) shows that the values of $\eta_t(S_T - K)^+$ are approximately $-K$ when the option is “deep in the money,” 0 when “deep out of the money,” and increasing rapidly around “at the money.”

Remark 6.1 As another well-known quadratic hedging method, we discuss mean–variance hedging strategies. For simplicity, we do not take the discount factor into account. Now, the mean–variance hedging strategy for a claim $F$ is defined as the self-financing strategy minimizing the mean-squared hedging error

$$\mathbb{E} \left[ (F - c - \int_0^T \xi_t dS_t)^2 \right]$$
Fig. 1  Values of $\xi_t^{(ST-K)^+}$ and $\Delta_t^{(ST-K)^+}$ for fixed $K$ vs. times $t = 0, 0.02, \ldots, 0.98$.

(a) Values of $\xi_t^{(ST-K)^+}$ and $\Delta_t^{(ST-K)^+}$ when $K$ is fixed to 900 vs. times $t$ from 0 to 0.98 at steps of 0.02. In this case, the option is “in the money” at time $t$. Red crosses and blue circles represent the values of $\xi_t^{(ST-K)^+}$ and $\Delta_t^{(ST-K)^+}$, respectively. (b) Example where the option is “at the money” at time $t$, that is, $K$ is fixed to 1124.47. (c) Example where $K$ is fixed to 1300, that is, the option is “out of the money” at time $t$.

over $c \in \mathbb{R}$ and $\xi \in \Theta_S$. When the asset price process is a martingale under $\mathbb{P}$, LRM strategies coincide with mean–variance hedging strategies, that is, the solution $(c^*, \xi^*)$ to the above minimization problem is given as $c^* = \mathbb{E}[F]$ and $\xi^* = \xi_F$, respectively, where $\varphi_F = (\xi^F, \eta^F)$ represents the LRM strategy for $F$. The optimal
Fig. 2 Values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$ at fixed $t$ vs. strike price $K$ from 200 to 2000 at steps of 25. (a) Values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$ at $t = 0$ vs. strike price $K$ from 200 to 2000 at steps of 25. Red crosses and blue circles represent the values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$, respectively. (b) Example where $t = 0.5$. (c) Example where $t = 0.9$.

Mean-squared hedging error is then given as $\mathbb{E}[(F - \mathbb{E}[F] - \int_0^T \xi_t^F dS_t)^2] = \mathbb{E}[(L_T^F)^2]$, where $L_T^F$ is the orthogonal martingale appearing in the FS decomposition (2.3). This value $\mathbb{E}[(L_T^F)^2]$ is equivalent to the risk of $\varphi^F$ at the maturity, which is given as $\mathbb{E}[(C_T^F (\varphi^F) - C_0^F (\varphi^F))^2]$, where $C_T^F (\varphi^F)$ is the cost process defined in Definition 2.5. On the other hand, denoting by $\eta^*$ the amount invested in the risk-free asset.
Fig. 3 Values of $\eta_t (S_T - K)^+$.

(a) Values of $\eta_t (S_T - K)^+$ when $K$ is fixed to 1124.47 vs. times $t = 0, 0.02, \ldots, 0.98$. In this case, the option is “at the money” at time $t$.

(b) Values of $\eta_t (S_T - K)^+$ at $t = 0.5$ vs. strike price $K$ from 200 to 2000 at steps of 25.

In the mean–variance hedging strategy, we can see that it is different from $\eta^F$ since

$$\eta_t^s = \mathbb{E}[F] + \int_0^t \xi_t^F dS_s - \xi_t^F S_t = \eta_t^F - L_t^F.$$

7 Conclusions

We obtain explicit representations of LRM strategies for call and put options in the BNS models given by (1.2) and (1.3), and implement related numerical experiments. We impose only Assumption 2.2 as the standing assumptions. Recall that Assumption 2.2 includes the two important examples IG-OU and gamma-OU, although parameters are restricted. Our discussion is based on the framework of [3]. We verify all the additional conditions imposed in [3]. Above all, we need some integrability conditions on the underlying contingent claim $F$. For example, we need $Z_T F \in L^2(\mathbb{P})$, which is almost equivalent to $Z_T S_T \in L^2(\mathbb{P})$ if $F$ is a call option. However, $Z_T S_T$ is not in $L^2(\mathbb{P})$ in our setting, which means that an additional condition is needed to treat call options directly in the framework of [3]. Thus, we first consider put options in this paper since they are bounded. LRM strategies for call options are then given as a corollary. With this simple idea, we do not need to impose any additional condition.
Moreover, to demonstrate condition C4, we need to investigate the Malliavin differentiability of the process $Z$. Note that $Z$ is a solution to the SDE (2.4). [11] showed the Malliavin differentiability of solutions to Markovian-type SDEs with the Lipschitz condition. However, the SDE (2.4) is not Markovian because $u_s$ and $\theta_{s,x}$ are random. In Sect. 5, as an extension of Sect. 17 in [11], we show that $Z_t \in \mathbb{D}^{1,2}$. This result should be a valuable mathematical contribution in its own right. Recall that $u_s$ and $\theta_{s,x}$ are bounded by Lemma A.7, and the Malliavin derivatives of $u_s$ and $\theta_{s,x}$ are equivalent to $O(1/z)$ and $O(1/\sqrt{z})$ simultaneously by Lemmas A.8 and A.9. These facts play a vital role in the demonstration of the Malliavin differentiability of $Z$.

We consider throughout the paper BNS models for which the asset price process is given by (1.3). Actually, the general form of BNS models is

$$S_t = S_0 \exp \left( \int_0^t (\mu + \beta \sigma_s^2) ds + \int_0^t \sigma_s dW_s + \rho J_t \right),$$

where the parameter $\beta \in \mathbb{R}$ is called the volatility risk premium. In other words, we restrict $\beta$ to $-1/2$. If $\beta \neq -1/2$, then the boundedness of $u_s$ and $\theta_{s,x}$ no longer holds, from which it is not easy to show that $Z_T \in \mathbb{D}^{1,2}$. Thus, formulating a Malliavin calculus under the MMM, [1] took a different approach to study the case of $\beta \in \mathbb{R}$ and $\rho = 0$. On the other hand, some new ideas are needed to treat the fully general case. This is left for future research.

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**Appendix**

**A.1 Theorem 3.7 of [3]**

Theorem 3.7 of [3], which provides an explicit representation formula for LRM strategies in Lévy markets, is frequently referred to in this paper. Therefore, we give its statement for BNS models under Assumption 2.2. Note that although Assumption 2.1 of [3] is imposed in Theorem 3.7 of [3], it is satisfied under Assumption 2.2. For more details, see Remark 2.3.

**Theorem A.1** (Theorem 3.7 of [3]) *Let $F$ be an $L^2(\mathbb{P})$ random variable satisfying the following three conditions:*

**AS1:** (Assumption 2.6 in [3]) $Z_T F$ is in $L^2(\mathbb{P})$.

**AS2:** (Assumption 3.4 in [3]) Conditions (C1)–(C6) for $F$ are satisfied.

**AS3:** ((3.1) in [3]) We have

$$\mathbb{E} \left[ \int_0^T \left( h_t^0 \right)^2 + \int_0^\infty \left( h_{t,z}^1 \right)^2 \nu(dz) \right] dt < \infty.$$
where \( h_{t,z}^1 = \mathbb{E}_{P^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F|\mathcal{F}_{t-}] \) and

\[
h_t^0 = \mathbb{E}_{P^*}\left[ D_{t,0}F - F\left( \int_0^T D_{t,0}s dW_s^{P^*} \right) + \int_0^T \int_0^\infty \frac{D_{t,0}\theta_{s,x}}{1 - \theta_{s,x}} \tilde{N}^{P^*}(ds,dx) \right| \mathcal{F}_{t-} \].

Then the LRM strategy \( \xi_t^F \) for the claim \( F \) is given by

\[
\xi_t^F = \frac{1}{S_t - (\sigma_t^2 + C)^t} \left( h_t^0 \sigma_t + \int_0^\infty h_{t,z}^1 (e^{\rho z} - 1) \nu(dz) \right).
\]

### A.2 Properties of \( \sigma_t \) and related Malliavin derivatives

The squared volatility process \( (\sigma_t^2) \), given as a solution to the SDE (1.2), is represented as

\[
\sigma_t^2 = e^{-\lambda t} \sigma_0^2 + \int_0^t e^{-\lambda (t-s)} dJ_s. \tag{A.1}
\]

Remark that we have

\[
\sigma_t^2 \geq e^{-\lambda t} \sigma_0^2 \geq e^{-\lambda T} \sigma_0^2 \tag{A.2}
\]

and

\[
\int_0^t \sigma_s^2 ds = \frac{1}{\lambda} (J_t - \sigma_t^2 + \sigma_0^2) \leq \frac{1}{\lambda} (J_t + \sigma_0^2). \tag{A.3}
\]

Next, we calculate some related Malliavin derivatives.

**Lemma A.2** For any \( s \in [0, T] \), we have \( \sigma_s^2 \in \mathcal{D}^{1,2} \) and

\[
D_{t,z} \sigma_s^2 = e^{-\lambda (s-t)} 1_{[0,s] \times (0,\infty)}(t,z) \tag{A.4}
\]

for \( (t,z) \in [0, T] \times [0, \infty) \).

**Proof** We can rewrite (A.1) as

\[
\sigma_s^2 = e^{-\lambda s} \sigma_0^2 + \int_0^s \int_0^\infty e^{-\lambda (s-u)} x \nu(dx)du + \int_{[0,T] \times [0,\infty)} e^{-\lambda (s-u)} 1_{[0,s] \times (0,\infty)}(u,x) Q(du,dx).
\]

Moreover, we have \( \int_{[0,T] \times [0,\infty)} e^{-2\lambda (s-u)} 1_{[0,s] \times (0,\infty)}(u,x) q(du,dx) < \infty \). The lemma follows by Definition 2.8. \( \square \)
Lemma A.3 For any \( s \in [0, T] \), we have \( \sigma_s \in \mathbb{D}^{1,2} \) and

\[
D_{t,z} \sigma_s = \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)} - \sigma_s}}{z} 1_{[0,s]}(t, z) \]

for \((t, z) \in [0, T] \times [0, \infty)\). Furthermore, we have \( 0 \leq D_{t,z} \sigma_s \leq \frac{1}{\sqrt{z}} 1_{[0,s]}(t) \) for \( z > 0 \).

Proof Taking a \( C^1 \)-function \( f \) such that \( f'(r) = \sqrt{r} \) for \( r \geq e^{-\lambda T} \sigma_0^2 \), we have \( \sigma_s = f(\sigma_s^2) \) by (A.2). Proposition 2.6 in [22] implies that we have \( \sigma_s \in \mathbb{D}^{1,2} \), \( D_{t,0} \sigma_s = f'(\sigma_s^2)D_{t,0} \sigma_s^2 = 0 \), and

\[
D_{t,z} \sigma_s = \frac{f(\sigma_s^2 + zD_{t,z} \sigma_s^2) - f(\sigma_s^2)}{z} = \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)} - \sigma_s}}{z} 1_{[0,s]}(t)
\]

for \( z > 0 \) since \( D_{t,z} \sigma_s^2 \) is nonnegative by (A.4). In addition, we have

\[
D_{t,z} \sigma_s \leq \frac{\sqrt{ze^{-\lambda(s-t)}}}{z} 1_{[0,s]}(t) \leq \frac{1}{\sqrt{z}} 1_{[0,s]}(t)
\]

for \( z > 0 \). \qed

Lemma A.4 We have \( \int_0^T \sigma_s^2 ds \in \mathbb{D}^{1,2} \) and

\[
D_{t,z} \int_0^T \sigma_s^2 ds = B(T-t) 1_{(0,\infty)}(z)
\]

for \((t, z) \in [0, T] \times [0, \infty)\), where the function \( B \) is defined just before Assumption 2.2.

Proof First, we have

\[
\int_0^T \sigma_s^2 ds = \sigma_0^2 \int_0^T e^{-\lambda s} ds + \int_0^T \int_0^s e^{-\lambda(s-u)} dJ_u ds
\]

\[
= \sigma_0^2 \frac{1 - e^{-\lambda T}}{\lambda} + \int_0^T \int_u^T e^{-\lambda(s-u)} ds dJ_u
\]

\[
= \sigma_0^2 B(T) + \int_0^T B(T-u) dJ_u.
\]

From Definition 2.8 we obtain \( \int_0^T \sigma_s^2 ds \in \mathbb{D}^{1,2} \) and

\[
D_{t,z} \int_0^T \sigma_s^2 ds = B(T-t) 1_{(0,\infty)}(z)
\]

for \((t, z) \in [0, T] \times (0, \infty)\). \qed
Lemma A.5 We have $\int_0^T \sigma_s dW_s \in \mathbb{D}^{1,2}$ and

$$D_{t,z} \int_0^T \sigma_s dW_s = \sigma_t 1_{[0]}(z) + \int_t^T \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)} - \sigma_s}}{z} dW_s 1_{(0,\infty)}(z)$$

for $(t,z) \in [0, T] \times [0, \infty)$. 

Proof To begin, we show that $\sigma \in \mathbb{L}^{1,2}_0$. Lemma A.3 implies that $\sigma_s \in \mathbb{D}^{1,2}$ for any $s \in [0, T]$. We have $\mathbb{E}[\int_0^T \sigma_s^2 ds] < \infty$ by (A.3) and the integrability of $J_T$. Since $|D_{t,z} \sigma_s|^2 \leq \frac{1}{z}$ by Lemma A.3, item (c) of the definition of $\mathbb{L}^{1,2}_0$ is satisfied. Hence, Lemma 3.3 in [10] provides $\int_0^T \sigma_s dW_s \in \mathbb{D}^{1,2}$ and

$$D_{t,z} \int_0^T \sigma_s dW_s = D_{t,z} \int_{[0,T] \times [0,\infty)} \sigma_s 1_{[0]}(x) Q(ds, dx) = \sigma_t 1_{[0]}(z) + \int_0^T D_{t,z} \sigma_s dW_s$$

$$= \sigma_t 1_{[0]}(z) + \int_t^T \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)} - \sigma_s}}{z} dW_s 1_{(0,\infty)}(z)$$

for $(t,z) \in [0, T] \times [0, \infty)$ by Lemma A.3. $\square$

Finally, we calculate $D_{t,z} L_T$ as follows.

Proposition A.6 $L_T \in \mathbb{D}^{1,2}$, and for $(t,z) \in [0, T] \times [0, \infty)$, we have

$$D_{t,z} L_T$$

$$= \sigma_t 1_{[0]}(z) + \left( -\frac{1}{2} B(T - t) + \int_t^T \frac{\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)} - \sigma_s}}{z} dW_s + \rho \right) 1_{(0,\infty)}(z).$$

Proof By (2.1) we have $L_T = \mu T - \frac{1}{2} \int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s + \rho J_T$. Since $J_T \in \mathbb{D}^{1,2}$ and $D_{t,z} J_T = 1_{(0,\infty)}(z)$, we obtain the result by Lemmas A.4 and A.5. $\square$

A.3 Properties of $u_s$ and $\theta_{s,x}$ and related Malliavin derivatives

We begin with recalling the two constants defined in (2.2):

$$C_u := \max \left\{ \frac{\left| \alpha \right| e^{\frac{\lambda T}{2}}}{\sigma_0}, \frac{\left| \alpha \right|}{C_\rho} \right\} \quad \text{and} \quad C_\theta := \max \left\{ \frac{\left| \alpha \right|}{C_\rho}, 1 \right\}.$$

The next lemma is cited often throughout the paper.
Lemma A.7  For any $s \in [0, T]$ and any $x \in (0, \infty)$, the following hold:
1. $|u_s| \leq C_u$.
2. $|\theta_{s,x}| \leq C_\theta$ and $|\theta_{s,x}| \leq C_\theta (1 - e^{\rho x}) \leq C_\theta |\rho| x$.
3. $\theta_{s,x} < 1 - e^{\rho x}$.
4. $|\log (1 - \theta_{s,x})| \leq C_\theta |\rho| x$.
5. $\frac{1}{1 - \theta_{s,x}} < \tilde{C}_\theta$ for some $\tilde{C}_\theta > 0$.

Proof 1. We have $|u_s| \leq \frac{|\alpha|}{\sigma_s} \leq \frac{|\alpha| e^{\frac{T}{2} / \sigma_0}}{\sigma_0}$ for any $s \in [0, T]$ by (A.2).
2. $|\theta_{s,x}| \leq \frac{|\alpha|}{C_\rho} (1 - e^{\rho x}) \leq C_\theta$ and $1 - e^{\rho x} \leq |\rho| x$ for any $x > 0$.
3. As seen in Remark 2.3, $\alpha \sigma_s^2 + C_\rho > -1$ for any $s \in [0, T]$. We then have $\theta_{s,x} < 1 - e^{\rho x}$.
4. When $\theta_{s,x} \geq 0$, we have $0 \geq \log (1 - \theta_{s,x}) > \log \left(1 - (1 - e^{\rho x})\right) = \rho x \geq C_\theta |\rho| x$.

On the other hand, if $\theta_{s,x} < 0$, then $0 < \log (1 - \theta_{s,x}) \leq -\theta_{s,x} \leq C_\theta |\rho| x$.
5. If $\theta_{s,x} \leq 0$, then $\frac{1}{1 - \theta_{s,x}} \leq 1$; otherwise, if $\theta_{s,x} > 0$, equivalently $\alpha < 0$, then

$$1 - \theta_{s,x} = 1 + \frac{\alpha}{\sigma_s^2 + C_\rho} (1 - e^{\rho x}) \geq 1 + \frac{\alpha}{\sigma_s^2 + C_\rho} \geq 1 + \frac{\alpha}{e^{-\lambda T} \sigma_0^2 + C_\rho} > 0$$

by Assumption 2.2. This completes the proof.

Next, we calculate some Malliavin derivatives related to $u_s$ and $\theta_{s,x}$.

Lemma A.8  For any $s \in [0, T]$, we have $u_s \in \mathbb{D}^{1,2}$ and

$$D_{t,z} u_s = f_u(\sigma_s + z D_{t,z} \sigma_s) - f_u(\sigma_s) \frac{\partial}{\partial z} I_{[0,s] \times (0,\infty)}(t,z)$$

$$= \frac{f_u(\sqrt{\sigma_s^2 + ze^{-(s-t)}}) - f_u(\sigma_s)}{z} \frac{\partial}{\partial z} I_{[0,s] \times (0,\infty)}(t,z) \quad (A.5)$$

for $(t,z) \in [0, T] \times [0, \infty)$, where $f_u(r) := \frac{ar}{r^2 + C_\rho}$ for $r \in \mathbb{R}$. Moreover, we have

$$|D_{t,z} u_s| \leq \frac{C_u}{\sqrt{z}} I_{[0,s]}(t) \quad \text{and} \quad |D_{t,z} u_s| \leq \frac{C'_u}{z} I_{[0,s]}(t)$$

for some $C'_u > 0$.

Proof  Note that $f'_u(r) = \alpha \frac{C_u - r^2}{(r^2 + C_\rho)^2}$ and $|f'_u(r)| \leq \frac{|\alpha|}{C_\rho} \leq C_u$. Since $u_s = f_u(\sigma_s)$ and $\sigma_s \in \mathbb{D}^{1,2}$, Proposition 2.6 in [22], together with Lemma A.3, implies $u_s \in \mathbb{D}^{1,2}$ and (A.5). In particular, we have $D_{t,0} u_s = f'_u(\sigma_s) D_{t,0} \sigma_s = 0$. Further, Lemma A.3 again yields $|D_{t,z} u_s| \leq \frac{1}{2} |z D_{t,z} \sigma_s| C_u \leq \frac{1}{\sqrt{z}} I_{[0,s]}(t) C_u$. Moreover, since $f_u(r)$ is bounded, we can find a $C'_u > 0$ such that $|D_{t,z} u_s| \leq \frac{C'_u}{z}$.

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Lemma A.9 For any \((s, x) \in [0, T] \times (0, \infty)\), we have \(\theta_{s,x} \in \mathbb{D}^{1,2}\) and
\[
D_{t,z} \theta_{s,x} = \frac{f_\theta(\sigma_s + zD_{t,z}\sigma_s) - f_\theta(\sigma_s)}{z}(e^{\rho x} - 1)1_{[0,s] \times (0,\infty)}(t, z)
\]
\[
= \frac{f_\theta(\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}}) - f_\theta(\sigma_s)}{z}(e^{\rho x} - 1)1_{[0,s] \times (0,\infty)}(t, z)
\]
(A.6)
for \((t, z) \in [0, T] \times [0, \infty)\), where \(f_\theta(r) := \frac{\alpha}{r^2 + C_\rho}\) for \(r \in \mathbb{R}\). Moreover, we have
\[
|D_{t,z} \theta_{s,x}| \leq \frac{C'_\theta}{\sqrt{z}}(1 - e^{\rho x})1_{[0,s]}(t) \quad \text{and} \quad |D_{t,z} \theta_{s,x}| \leq \frac{2C_\theta}{z}(1 - e^{\rho x})1_{[0,s]}(t)
\]
(A.7)
for some \(C'_\theta > 0\).

Proof Note that \(\theta_{s,x} = f_\theta(\sigma_s)(e^{\rho x} - 1)\) and \(f'_\theta(r) = \frac{-2\alpha r}{(r^2 + C_\rho)^2}\). Hence, \(|f'_\theta(r)|\) is bounded. Therefore, the same argument as for Lemma A.8 implies (A.6). In addition, (A.7) is given by the boundedness of \(f_\theta\) and \(f'_\theta\). \(\square\)

Lemma A.10 For any \((s, x) \in [0, T] \times (0, \infty)\), we have \(\log(1 - \theta_{s,x}) \in \mathbb{D}^{1,2}\) and
\[
D_{t,z} \log(1 - \theta_{s,x}) = \frac{\log(1 - \theta_{s,x} - zD_{t,z}\theta_{s,x}) - \log(1 - \theta_{s,x})}{z}1_{(0,\infty)}(z)
\]
for \((t, z) \in [0, T] \times [0, \infty)\). Moreover, we have
\[
|D_{t,z} \log(1 - \theta_{s,x})| \leq |D_{t,z} \theta_{s,x}| e^{-\rho x}.
\]

Proof For \(x > 0\), we denote
\[
g_x(r) := \begin{cases} 
\log(1 - r), & r < 1 - e^{\rho x}, \\
-e^{-\rho x}r + e^{-\rho x} - 1 + \rho x, & r \geq 1 - e^{\rho x}.
\end{cases}
\]
Note that \(g_x\) is a \(C^1\)-function satisfying \(|g_x'(r)| \leq e^{-\rho x}\) for all \(r \in \mathbb{R}\). Because \(\theta_{s,x} \in \mathbb{D}^{1,2}\) and \(\log(1 - \theta_{s,x}) = g_x(\theta_{s,x})\) by item 3 of Lemma A.7, we have
\[
D_{t,z} \log(1 - \theta_{s,x}) = \frac{g_x(\theta_{s,x} + zD_{t,z}\theta_{s,x}) - g_x(\theta_{s,x})}{z}1_{(0,\infty)}(z).
\]

Lemma A.9 implies, for \(t \in [0, s]\) and \(z \in (0, \infty)\),
\[
\theta_{s,x} + zD_{t,z}\theta_{s,x} = f_\theta(\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}})(e^{\rho x} - 1)
\]
\[
= \frac{\alpha(e^{\rho x} - 1)}{\sigma_s^2 + ze^{-\lambda(s-t)} + C_\rho} < 1 - e^{\rho x}.
\]
(A.8)
We have then \(g_x(\theta_{s,x} + zD_{t,z}\theta_{s,x}) = \log(1 - \theta_{s,x} - zD_{t,z}\theta_{s,x})\). \(\square\)
A.4 On $D_{t,z} \log Z_T$

We show that $\log Z_T \in D^{1,2}$ and calculate $D_{t,z} \log Z_T$. Equality (2.5) implies that

$$
\log Z_T = -\int_0^T u_s dW_s - \frac{1}{2} \int_0^T u_s^2 ds + \int_0^T \int_0^\infty \log(1 - \theta_{s,x}) \tilde{N}(ds, dx) + \int_0^T \int_0^\infty \left( \log(1 - \theta_{s,x}) + \theta_{s,x} \right) v(dx) ds.
$$

(A.9)

We discuss each term of (A.9) separately. As seen in Sect. 4, we have $u \in L^{1,2}_0$. Therefore, Lemma 3.3 of [10] implies that

$$
D_{t,0} \int_0^T u_s dW_s = u_t + \int_0^T D_{t,0} u_s dW_s = u_t
$$

and $D_{t,z} \int_0^T u_s dW_s = \int_0^T D_{t,z} u_s dW_s$ for $z > 0$. Similarly, we have

$$
D_{t,0} \int_0^T \int_0^\infty \log(1 - \theta_{s,x}) \tilde{N}(ds, dx) = 0
$$

and

$$
D_{t,z} \int_0^T \int_0^\infty \log(1 - \theta_{s,x}) \tilde{N}(ds, dx) = \frac{\log(1 - \theta_{t,z})}{z} + \int_0^T \int_0^\infty D_{t,z} \log(1 - \theta_{s,x}) \tilde{N}(ds, dx)
$$

for $z > 0$. As for $D_{t,z} \int_0^T u_s^2 ds$, because $u^2 \in L^{1,2}_0$ by Sect. 4, Lemma 3.2 of [10] yields

$$
D_{t,z} \int_0^T u_s^2 ds = \int_0^T D_{t,z} u_s^2 ds = 2 \int_0^T u_s D_{t,z} u_s ds + z \int_0^T (D_{t,z} u_s)^2 ds
$$

for $z \geq 0$. In particular, $D_{t,0} \int_0^T u_s^2 ds = 0$. For the fourth term of (A.9), because $\log(1 - \theta) + \theta \in \tilde{L}^{1,2}_1$, Proposition 3.5 of [22] implies

$$
D_{t,z} \int_0^T \int_0^\infty \left( \log(1 - \theta_{s,x}) + \theta_{s,x} \right) v(dx) ds = \int_0^T \int_0^\infty \left( D_{t,z} \log(1 - \theta_{s,x}) + D_{t,z} \theta_{s,x} \right) v(dx) ds
$$

for $z \geq 0$. Collectively, we conclude the following:
Proposition A.11  We have \( \log Z_T \in D^{1,2} \), \( D_{t,0} \log Z_T = u_t \), and

\[
D_{t,z} \log Z_T = -\int_0^T D_{t,z} u_s d W_s - \int_0^T u_s D_{t,z} u_s ds - \frac{z}{2} \int_0^T (D_{t,z} u_s)^2 ds
+ \int_0^T \int_0^\infty D_{t,z} \log(1 - \theta_{s,x}) \tilde{N}(ds, dx)
+ \int_0^T \int_0^\infty \left( D_{t,z} \log(1 - \theta_{s,x}) + D_{t,z} \theta_{s,x} \right) v(dx) ds + \frac{\log(1 - \theta_{t,z})}{z}
\]

for \( z > 0 \).

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