STRICT POLYNOMIAL FUNCTORS
AND COHERENT FUNCTORS

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Abstract. We build an explicit link between coherent functors in the sense of Auslander \cite{Auslander} and strict polynomial functors in the sense of Friedlander and Suslin \cite{Friedlander-Suslin}. Applications to functor cohomology are discussed.

1. Introduction

Since the foundational work of Schur \cite{Schur}, the representation theory of general linear groups has been closely related to the representation theory of symmetric groups. Especially fruitful has been the study, for all integers $n$, of tensor products $T^n(V) := V^\otimes n$ of a vector space $V$, endowed with the commuting actions of the general linear group $GL(V)$ (diagonally on each factor by linear substitution) and of the symmetric group $S_n$ (by permutation of the factors). For many purposes, the mysterious group ring of the general linear group can thus be replaced by the more manageable Schur algebra of $S_n$-equivariant linear maps of $V^\otimes n$.

The use of functors in representation theory, maybe first promoted by Auslander, is practical and efficient for formalizing the relations between symmetric groups and general linear groups. The classical work of Green \cite{Green} on representations of the Schur algebras pushes these ideas quite far and in great generality. In the example of interest to us, Green associates to every additive functor $f$, defined on representations of the symmetric group, the representation over the Schur algebra given by $f(V^\otimes n)$. The main problem of constructing reverse correspondences is solved naturally by Green. It is one of the purpose of this paper to shed new light on these correspondences.

A few years later, Friedlander and Suslin \cite{Friedlander-Suslin} introduced strict polynomial functors, which are equivalent to representations of the Schur algebra when the dimension of $V$ is at least $n$. This new formalization is aimed at cohomology computations in positive characteristic, and it has numerous applications, including a proof of finite generation of the cohomology of finite group schemes in the same paper \cite{Friedlander-Suslin}. The pleasing properties of the category $\mathcal{P}$ of strict polynomial functors lead to impressive cohomological computations \cite{Friedlander-Suslin}, following the fundamental computation of $\text{Ext}_{\mathcal{P}}(Id^{(r)}, Id^{(r)})$ given in \cite{Friedlander-Suslin} (the decoration $(r)$ indicates Frobenius twists, that is extension of scalars through the $r$-th power of the Frobenius isomorphism). More recently, Chałupnik proves \cite{Chalupnik} elegant formulae computing functor cohomology, for various fundamental functors $F$ of the form

$$fT^n: V \mapsto F(V) = f(V^\otimes n).$$

He succeeds in comparing the groups $\text{Ext}_\mathcal{P}(G^{(r)}, F^{(r)}) = \text{Ext}_\mathcal{P}(fT^n^{(r)}, gT^n^{(r)})$ with $f g (\text{Ext}_\mathcal{P}(T^n^{(r)}, T^n^{(r)}))$ for many important families of functors $F$ and $G$. [When the functors $f$ and $g$ are given by idempotents in the group ring of the symmetric group, the two terms are easily seen to be isomorphic. However, this is rarely the case in positive characteristic smaller than $n$.] To this end, Chałupnik considers for each strict polynomial functor $F$ certain adequate choices of functors

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\( f \), defined on representations of the symmetric group \( \mathfrak{S}_n \), such that \( f(V^\otimes n) = F(V) \), thus informally rediscovering the correspondences set up by Green. These methods motivated and inspired the present work.

Green’s correspondences are best expressed in terms of adjoint functors and recollements of categories. The category \( \mathcal{P} \) thus appears as a quotient category of the category of all additive functors defined on representations of the symmetric groups. Unfortunately, the latter is very large and stays quite mysterious. Representable functors, such as the functor \( \mathbb{H}^0(\mathfrak{S}_n, -) \) taking invariants, are examples of functors obtained through the reverse correspondences of Green, but they are many more. However, all strict polynomial functors arise from \textit{coherent functors}, which are functors that are presented by representable functors (see Definition 2.2.1). The resulting category, if still quite rich, is much better behaved. For instance, the global dimension of the category of coherent functors is two. This comes in sharp contrast with the rich functor cohomology obtained through homological algebra in the category of strict polynomial functors.

We revisit in this setting some of the properties of functors which make the category \( \mathcal{P} \) much more tractable than coarse representations: tensor product, composition (or plethysm), linearization etc. and we try and find corresponding constructions for coherent functors. We also apply our insight to functor cohomology, and obtain Chalupnik’s constructions in a natural way.

Section 2 develops the general properties of coherent functors. Although we do not claim much originality, it contains a few results which we could not find in the literature. We believe Section 2.7 to be new. In Section 3.1 we present strict polynomial functors to fit our purpose. Section 3.2 contains our main results. It compares coherent functors and strict polynomial functors. Since the comparison is best stated in terms of recollements of abelian categories, we recall in an appendix \( \text{A.1} \) what is needed from this theory. Section 3.3 lifts the tensor product of polynomial functors to the level of coherent functors. For later use, the final section \( \text{A.2} \) does the same for the composition of polynomial functors. Section 4 applies this new setting to functor cohomology and obtains natural versions of Chalupnik’s results. Section 4 enjoys its own introduction, which hopefully makes clear the implications of our results for [2], and further developments.

Notations. We fix a field \( \mathbb{K} \) of positive characteristic \( p \). All vector spaces are considered over \( \mathbb{K} \), and Hom and \( \otimes \) are taken over \( \mathbb{K} \), unless otherwise decorated. Let \( \mathcal{V} \) be the category of finite-dimensional vector spaces. For a finite group \( G \), we let \( _G\mathcal{V} \) denote the category of finite dimensional \( G \)-modules.

## 2. Coherent functors

Most results in this section are known to experts. A good reference for the first subsections is a recent survey by Harthorne [9].

### 2.1. Finite-dimensional representations as functors

In this section, we fix a finite group \( G \). Let \( _G\mathcal{V} \) be the category of all finite dimensional \( G \)-modules and let \( \mathfrak{A}(G) \) be the category of all covariant \( \mathbb{K} \)-linear functors from \( _G\mathcal{V} \) to the category \( \mathcal{V} \) of finite dimensional vector spaces. Any \( M \) in \( _G\mathcal{V} \) yields a functor \( t_M \) defined by:

\[
 t_M = M \otimes_G (-) .
\]

For instance, the functor \( t_{\mathbb{K}[G]} \) is the forgetful functor to \( \mathcal{V} \). Dually, let \( h_M \) be the functor represented by \( M \):

\[
 h_M = \text{Hom}_G(M, -) .
\]

Note that the functor \( t_{\mathbb{K}[G]} \) is (isomorphic to) the forgetful functor as well. We shall use that \( h_M(\mathbb{K}[G]) \) is isomorphic to the \( \mathbb{K} \)-dual vector space \( M^\vee \). This isomorphism
is precisely defined as follows. Let \( \tau \) be the element of \( \mathbb{K}[G] \) given by

\[
\tau(g) = 0, \quad \text{if} \quad g \neq 1 \quad \text{and} \quad \tau(1) = 1.
\]

By the Yoneda lemma, the function \( \tau \) yields a natural morphism

\[
\tau_X : \text{Hom}_G(X, \mathbb{K}[G]) \to X^\vee
\]

\[
\theta \mapsto \tau \circ \theta.
\]

The homomorphism \( \tau_X \) is an isomorphism when \( X = \mathbb{K}[G] \). Since \( \mathbb{K}[G] \) is a self-injective algebra, \( \text{Hom}_G(-, \mathbb{K}[G]) \) is an exact functor, and \( \tau \) is a natural transformation between exact functors. It results that \( \tau_X \) is an isomorphism for all \( X \) in \( G\mathcal{V} \). We shall use this fact without further reference.

The category \( \mathcal{A}(G) \) is an abelian category. We state below its elementary properties.

**Proposition 2.1.1.** For all object \( f \) of the category \( \mathcal{A}(G) \), let \( \mathbb{D}(f) \) be the object of \( \mathcal{A}(G) \) defined by

\[
(\mathbb{D}f)(M) = (f(M^\vee))^\vee.
\]

The resulting functor \( \mathbb{D} \) is a duality in \( \mathcal{A}(G) \).

**Proposition 2.1.2.** For any \( M \) in \( G\mathcal{V} \), the Yoneda lemma yields a natural isomorphism

\[
\text{Hom}_{\mathcal{A}(G)}(h_M, f) \cong f(M).
\]

Thus the functor \( h_M \) is a projective object in the category \( \mathcal{A}(G) \). Moreover, for all \( M, N \) in \( G\mathcal{V} \), there is a natural isomorphism:

\[
\text{Hom}_{\mathcal{A}(G)}(h_M, h_N) \cong \text{Hom}_G(N, M).
\]

**Proposition 2.1.3.** For all \( M \) in \( G\mathcal{V} \), there is a natural isomorphism: \( \mathbb{D}(h_M) \cong t_M \). Hence the functor \( t_M \) is an injective object in the category \( \mathcal{A}(G) \). Moreover, for all \( f \) in \( \mathcal{A}(G) \), there is a natural isomorphism

\[
\text{Hom}_{\mathcal{A}(G)}(f, t_M) \cong \mathbb{D}f(M).
\]

In particular, there is a natural isomorphism:

\[
\text{Hom}_{\mathcal{A}(G)}(t_N, t_M) = \text{Hom}_G(N, M).
\]

**Proof.** For \( X \) in \( \mathcal{V} \) we define

\[
\theta_X : \text{Hom}_G(M, X) \to (X^\vee \otimes_G M)^\vee
\]

by: \( \theta_X(\alpha) = \{ \xi \otimes m \mapsto \xi(\alpha(m)) \} \).

It defines a natural transformation \( \theta : h_M \to \mathbb{D}(t_M) \). Since \( \theta_{\mathbb{K}[G]} \) is an isomorphism and both \( h_M \) and \( \mathbb{D}(t_M) \) are left exact functors, it follows that \( \theta \) is an isomorphism. The rest follows because \( \mathbb{D} \) is a duality.

**2.2. Coherent functors.** An object \( f \) in \( \mathcal{A}(G) \) is finitely generated if it is a quotient of \( h_M \) for some \( M \) in \( G\mathcal{V} \). Among finitely generated functors, coherent functors are defined by further requiring finiteness of the relations.

**Definition 2.2.1.** An object \( f \) in \( \mathcal{A}(G) \) is coherent [9], or finitely presented [8] p. 251 [11] p. 204, if it fits in an exact sequence

\[
(1) \quad h_N \to h_M \to f \to 0
\]

for some \( M, N \) in \( G\mathcal{V} \).

By the Yoneda lemma, the morphism \( h_N \to h_M \) is of the form \( h_\alpha \) for a uniquely defined \( \alpha : M \to N \). It follows that, for every coherent functor \( f \), there is an exact sequence:

\[
(2) \quad 0 \to h_{\text{Coker } \alpha} \to h_N \to h_\alpha \to h_M \to f \to 0.
\]
We let \( \mathcal{C}(G) \) be the category of all coherent functors and natural transformations between them. It is a classical fact due to Auslander [1] that the category \( \mathcal{C}(G) \) is an abelian category with enough projective and injective objects. Moreover the inclusion \( \mathcal{C}(G) \subset \mathcal{A}(G) \) is an exact functor.

We now give examples of coherent functors.

**Proposition 2.2.2.**
(i) For any \( M \) in \( G^V \), the functor \( t_M = D(h_M) \) is an injective object in \( \mathcal{C}(G) \);
(ii) If \( f \) is a coherent functor, then \( Df \) is also a coherent functor;
(iii) For any integer \( i \geq 0 \), the homology and cohomology functors \( H^i(G, -) \), \( H_i(G, -) \) are coherent on \( G^V \);
(iv) For any integer \( i \geq 0 \), the Tate homology and cohomology functors \( \hat{H}^i(G, -) \) and \( \hat{H}_i(G, -) \) are coherent on \( G^V \).

**Proof.** Examples (i) to (iii) are also in [9, §2].

(i) If \( M \) is free and finite dimensional, there is a natural isomorphism: \( h_M \cong t_M \). For a general \( M \), choose a presentation \( K \to N \to M \to 0 \) in the category \( F^V \), with free and finite dimensional \( K \) and \( N \). Then \( t_M \) is the cokernel of \( t_K \to t_N \) and hence it is coherent. It is injective in \( \mathcal{C}(G) \) because it is so even in a \( \mathcal{A}(G) \) by Proposition 2.1.3.

(ii) Assume \( f \) is a coherent functor. By definition, it is a cokernel of a morphism \( h_N \to h_M \). Then \( Df \) is the kernel of the dual morphism \( t_M \to t_N \), hence it is also a coherent functor.

(iii) Since \( H^0(G, -) = h_K \) and \( H_0(G, -) = t_K \),

\[
\text{they are coherent. For a general } i, \text{ choose a projective resolution } P_\ast \text{ of } K \text{ with finite dimensional } P_i, i \geq 0. \text{ Then } H^i(G, -) \text{ is the } i\text{th homology of the cochain complex } h_{P_\ast} \text{ of coherent functors, therefore it is also a coherent functor.}
\]

(iv) The functors \( \hat{H}^0(G, -) \) and \( \hat{H}^{-1}(G, -) \) are respectively the cokernel and the kernel of the norm homomorphism \( H_0(G, -) \to H^0(G, -) \), so they are also coherent. We then proceed as for (iii).

\[ \square \]

2.3. **Homological dimension.** For any \( M \) in \( G^V \), the functor \( h_M \in \mathcal{C}(G) \) is projective, because it is projective even in \( \mathcal{A}(G) \). The exact sequence (2) shows that the category \( \mathcal{C}(G) \) has global dimension at most 2. In fact the global dimension is never 1, it is either 2 or 0. The last possibility happens if, and only if, \( G^V \) is semi-simple \([1]\) i.e. when the order of \( G \) is invertible in \( \mathbb{K} \).

The following result characterizes objects of projective or injective dimension smaller than two.

**Proposition 2.3.1.** Let \( f \) be a coherent functor.

(i) The following are equivalent:
   (a) \( f \) is projective;
   (b) The functor \( f \) is left exact;
   (c) \( f \) is of the form \( h_M \) for some \( M \).

(ii) The following are equivalent:
   (a) \( f \) is injective;
   (b) The functor \( f \) is right exact;
   (c) \( f \) is of the form \( t_M \) for some \( M \).

(iii) \( pd(f) \leq 1 \) if, and only if, \( f \) respect monomorphisms.

(iv) \( id(f) \leq 1 \) if, and only if, \( f \) respect epimorphisms.
Proof. We prove only (ii) and (iii). The rest follows by duality. Statement (i) is also proved in [9, Proposition 3.12 & 4.9].

If \( f \) is an injective object, then \( \mathbb{D}(f) \) is projective, hence it is a direct summand of an object of the form \( h_M \). It follows that \( f \) is a direct summand of \( t_M \) for some \( M \). The corresponding projector of \( t_M \) has the form \( t_{h_M} \), where \( \alpha \) is a projector of \( M \). Thus \( f \cong t_{h_M(\alpha)} \). In particular any injective object is a right exact functor.

Conversely, assume that \( f \) is right exact. The \( G \)-module \( M = f(\mathbb{K}[G]) \) is finitely generated and therefore one can consider the functor \( t_M \). There is a well-defined transformation \( \alpha : t_M \rightarrow f \) given by: \( \alpha_X(m \otimes x) = f(\hat{x})(m) \), where, for \( x \) in \( X \), \( \hat{x} : \mathbb{K}[G] \rightarrow X \) is the \( G \)-homomorphism defined by \( \hat{x}(1) = x \). By construction this map is an isomorphism when \( X = \mathbb{K}[G] \). Since \( f \) is right exact \( \alpha_X \) is an isomorphism for all \( X \) in \( G \mathcal{V} \). Hence \( f \) is injective in \( \mathcal{C}(G) \).

Suppose that \( f \) respects monomorphisms. Consider an exact sequence of functors

\[
0 \rightarrow f' \rightarrow h_M \rightarrow f \rightarrow 0.
\]

We want to show that \( f' \) is projective. By (i), we need to prove that \( f' \) is left exact. For any short exact sequence in \( G \mathcal{V} \)

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,
\]

there is a commutative diagram with exact columns:

\[
\begin{array}{ccc}
0 & \rightarrow & f'(A) \\
\downarrow & & \downarrow \\
f'(B) & \rightarrow & f'(C) \\
\downarrow & & \downarrow \\
0 & \rightarrow & h_M(A) \\
\downarrow & & \downarrow \\
f(A) & \rightarrow & f(B) \\
\downarrow & & \downarrow \\
f(C) & \rightarrow & f(C) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

In this diagram the middle row is also exact and a diagram chase shows that \( \alpha \) is a monomorphism. It follows that \( f' \) is left exact and hence projective. This shows that \( pd(f) \leq 1 \).

Conversely, suppose that the projective dimension of \( f \) is \( \leq 1 \). There is a short exact sequence of functors

\[
0 \rightarrow h_N \rightarrow h_M \rightarrow f \rightarrow 0.
\]

If

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]
is a short exact sequence in \( \mathcal{G}^\vee \), there is a commutative diagram with exact columns:

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
h_N(A) & \longrightarrow & h_N(B) & \longrightarrow & h_N(C) \\
\downarrow & & \downarrow & & \downarrow \\
h_M(A) & \longrightarrow & h_M(B) & \longrightarrow & h_M(C) \\
\downarrow & & \downarrow & & \downarrow \\
f(A) & \longrightarrow & f(B) & \longrightarrow & f(C) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

The first two rows in this diagram are also exact, and \( \alpha \) is a monomorphism. This shows that \( f \) respects monomorphisms. □

2.4. Coherent functors and recollement. Following Auslander [1, 8], we relate the category of coherent functors \( \mathcal{C}(\mathcal{G}) \) with the category \( \mathcal{G}^\vee \). The best way to formulate the result is to use the language of recollement of categories (see [10], [3], or Appendix A).

One considers the functors \( t^*: \mathcal{C}(\mathcal{G}) \to \mathcal{G}^\vee \), \( t_*: \mathcal{G}^\vee \to \mathcal{C}(\mathcal{G}) \) and \( t_1: \mathcal{G}^\vee \to \mathcal{C}(\mathcal{G}) \) given respectively by

\[
t^*(f) = f(\mathbb{K}[G]), \quad t_*(M) = h_{M^\vee} \quad \text{and} \quad t_!(M) := t_*(M).
\]

**Proposition 2.4.1.**

(i) The functor \( t_! \) is left adjoint to \( t^* \) and, for all \( M \) in \( \mathcal{G}^\vee \), the \( \mathcal{G} \)-module \( t^* t_!(M) \) is naturally isomorphic to \( M \).

(ii) The functor \( t_* \) is right adjoint to \( t^* \) and, for all \( M \) in \( \mathcal{G}^\vee \), the \( \mathcal{G} \)-module \( t^* t_*(M) \) is naturally isomorphic to \( M \).

**Proof.**

(i) For any coherent functor \( f \), the functor \( \mathbb{D}f \) is also coherent, so one can assume as in (2) that \( \mathbb{D}f = \text{Coker}(h_\alpha: h_N \to h_M) \), for some linear map \( \alpha: M \to N \). Applying \( \mathbb{D} \), we get an exact sequence:

\[
0 \longrightarrow f \longrightarrow t_M \longrightarrow t_!(M) \longrightarrow 0.
\]

It shows that: \( t^*(f) = f(\mathbb{K}[G]) = \text{Ker}(\alpha) \). Moreover, there are natural isomorphisms:

\[
\text{Hom}_{\mathcal{C}(\mathcal{G})}(t_M, f) = \text{Hom}_{\mathcal{C}(\mathcal{G})}(t_M, \text{Ker}(t_\alpha)) = \text{Ker}(\text{Hom}_{\mathcal{C}(\mathcal{G})}(t_M, t_\alpha)) = \text{Ker}(\text{Hom}_{\mathcal{G}}(M, \alpha)) = \text{Hom}_{\mathcal{G}}(M, \text{Ker} \alpha) = \text{Hom}_{\mathcal{G}}(M, t^*(f)).
\]

From this follows the first statement of (i). The second one follows from the natural isomorphism: \( t^*_! t!(M) = t_!(\mathbb{K}[G]) \cong M \).

(ii) We use the duality: \( \text{Hom}_{\mathcal{G}}(M, \mathbb{K}[G]) \cong M^\vee. \) Take \( f \) as in the exact sequence (2): \( f = \text{Coker}(h_\alpha) \) for some linear map \( \alpha \). Because \( \mathbb{K}[G] \) is a self-injective algebra, it follows:

\[
t^*(f) = f(\mathbb{K}[G]) = \text{Coker}(\text{Hom}_{\mathcal{G}}(\alpha, \mathbb{K}[G])) \cong \text{Hom}_{\mathcal{G}}(\text{Ker}(\alpha), \mathbb{K}[G]) \cong (\text{Ker} \alpha)^\vee.
\]
Moreover, there are natural isomorphisms:

\[
\text{Hom}_\mathcal{C}(f, h_X^\vee) = \text{Ker}(h_X^\vee(\alpha)) \\
= \text{Ker}(\text{Hom}_G(X^\vee, \alpha)) \\
= \text{Hom}_G(X^\vee, \text{Ker} \alpha) \\
\cong \text{Hom}_G((\text{Ker} \alpha)^\vee, X) \\
\cong \text{Hom}_G(t^*(f), X)
\]

Finally, for \( M \in \mathcal{G} \mathcal{V} \):

\[
t^* t_*(M) = t^*(h_M^\vee) = \text{Hom}_G(M^\vee, \mathbb{K}[G]) \cong M.
\]

We let \( \mathcal{E}^0(G) \) be the full subcategory of \( \mathcal{E}(G) \) whose objects \( f \) are such that:

\( t^*(f) = 0 \). Thus the category \( \mathcal{E}^0(G) \) consists exactly of coherent functors which
vanish on projective objects. Since \( t^* \) is exact, the subcategory \( \mathcal{E}^0(G) \) is abelian. Indeed, it is a Serre subcategory of \( \mathcal{E}(G) \).

We let \( r^*: \mathcal{E}^0(G) \to \mathcal{E}(G) \)
be the inclusion. It is an exact functor. It is a consequence of Proposition A.1.2
that the functors \( r_*, t^*, t_*, t_! \) are part of a recollement situation

\[
\mathcal{E}^0(G) \supseteq r^* \mathcal{E}(G) \supseteq \mathcal{G} \mathcal{V}.
\]

where \( r^* \) and \( r^! \), left and right adjoint to \( r_* \), are defined by the following exact sequences:

\[
0 \to r_* r^!(f) \to f \to t_*(f), \quad tt^*(M) \to M \to r_* r^*(f) \to 0.
\]

The following Proposition gives another description of \( r^* \) and \( r^! \).

**Proposition 2.4.2.**

(i) For a functor \( f : \mathcal{G} \mathcal{V} \to \mathcal{V} \), let \( \tau : L_0 f \to f \) be the natural transformation from the 0-th left derived functor. There is an isomorphism

\[
r^*(f) \cong \text{Coker}(\tau).
\]

(ii) For \( M \in \mathcal{G} \mathcal{V} \), let \( \Sigma M \) be a finitely generated \( G \)-module which fits in a short exact sequence

\[
0 \to M \to P \to \Sigma M \to 0
\]

where \( P \) is a projective \( G \)-module. There is an isomorphism

\[
r^!(t_M) \cong \text{Tor}^G_1(-, \Sigma M).
\]

(iii) For a functor \( f : \mathcal{G} \mathcal{V} \to \mathcal{V} \), write \( f = \text{Ker}(\alpha) \) for some linear map \( \alpha : M \to N \), as in the exact sequence (3). There is an isomorphism

\[
r^!(f) \cong \text{Ker}(\text{Tor}^G_1(-, \Sigma(\alpha))).
\]

**Proof.**

(i) Let us consider a natural transformation

\[
\xi : f \to g
\]

where \( f \) is a coherent functor and \( g \) is in \( \mathcal{E}^0(G) \). We have to prove that \( \xi \) factors through \( \text{Coker}(\tau) \). In other words we have to show that the composite \( \xi \circ \tau : L_0 f \to g \) is zero. To this end, for an object \( M \) in \( \mathcal{G} \mathcal{V} \) choose an
exact sequence $0 \to N \to P \to M \to 0$ with projective $P$. The following commutative diagram with exact top row implies the result:

\[
\begin{array}{c}
\begin{array}{c}
L_0 f(P) \\ \approx \\
\downarrow \\
f(P) \\
\downarrow \\
0 = g(P)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
L_0 f(M) \\
\downarrow \\
f(M)
\end{array}
\end{array}
\to \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\]

(ii) The long exact sequence for Tor-groups on $0 \to M \to P \to \Sigma M \to 0$ yields an exact sequence

\[
0 \to \text{Tor}_1^G(-, \Sigma M) \to t_M \to t_P \to t_{\Sigma M} \to 0.
\]

Let $\psi : t_M \to h_{M^\vee}$ be the natural transformation defined by:

\[
\psi_X : X \otimes_G M \to \text{Hom}_G(M^\vee, X)
\]

\[
x \otimes m \mapsto (\xi \mapsto \xi(m)x).
\]

It is an isomorphism when $M$ is a projective object in $G\mathcal{V}$. There is a commutative diagram with exact rows

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
h_{M^\vee} \\
\approx
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
h_{P^\vee} \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
h_{\Sigma M^\vee}
\end{array}
\end{array}
\to \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Tor}_1^G(-, \Sigma M) \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
t_M \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
t_P \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
t_{\Sigma M} \\
\end{array}
\end{array}
\to \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\]

It follows that there is an exact sequence

\[
0 \to \text{Tor}_1^G(-, \Sigma M) \to t_M \to h_{M^\vee}.
\]

The result follows from the comparison with the exact sequence

\[
0 \to r_s(t_M) \to t_M \to t_s(t_M) = t_s(M) \cong h_{M^\vee}.
\]

(iii) Apply $r_s$ to the exact sequence $\mathcal{3}$. Because the functor $r_s$ is left exact:

\[
r_s(f) = r_s(\text{Ker}(t_\alpha)) = \text{Ker}(r_s(\text{Ker}(t_\alpha))),
\]

and the result follows from (ii).

\[\square\]

2.5. Induction and restriction for coherent functors. Let $H$ be a subgroup of the finite group $G$. The induction functor $\text{Ind}_H^G : G\mathcal{V} \to H\mathcal{V}$ is left and right adjoint to the restriction functor $\text{Res}_H^G : G\mathcal{V} \to H\mathcal{V}$. Both functors are exact. By pre-composition, we obtain exact functors

\[
\begin{array}{c}
\begin{array}{c}
\uparrow_H^G : \mathcal{A}(H) \to \mathcal{A}(G)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow_H^G : \mathcal{A}(G) \to \mathcal{A}(H)
\end{array}
\end{array}
\]

which are given by

\[
(\uparrow_H^G \mathcal{g})(M) = \mathcal{g}(\text{Res}_H^G(M)) \quad \text{and} \quad (\downarrow_H^G \mathcal{f})(N) = \mathcal{f}(\text{Ind}_H^G(N))
\]

for $M \in G\mathcal{V}$, $N \in H\mathcal{V}$, $\mathcal{f} \in \mathcal{A}(G)$ and $\mathcal{g} \in \mathcal{A}(H)$. From the adjunction of induction and restriction, it is formal that $\downarrow_H^G$ is left and right adjoint to the functor $\uparrow_H^G$.

Proposition 2.5.1. If $\mathcal{f}$ is coherent in $\mathcal{C}(G)$, then $\downarrow_H^G(\mathcal{f})$ is coherent in $\mathcal{C}(H)$.

If $\mathcal{g}$ is coherent in $\mathcal{C}(H)$, then $\uparrow_H^G(\mathcal{g})$ is coherent in $\mathcal{C}(G)$. Thus, induction and restriction define adjoint pairs of functors

\[
\begin{array}{c}
\begin{array}{c}
\uparrow_H^G : \mathcal{C}(H) \to \mathcal{C}(G)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow_H^G : \mathcal{C}(G) \to \mathcal{C}(H)
\end{array}
\end{array}
\]
Proof. Take $M \in G\mathcal{V}$ and $N \in H\mathcal{V}$. Since Ind is left adjoint to Res we have
\[ \uparrow_H^G(h_N) \cong h_{\text{Ind}}^G h_N. \]
Similarly, since Ind is also right adjoint to Res:
\[ \downarrow_H^G(h_M) = h_{\text{Res}}^G h_M. \]
The rest follows from the exactness of $\uparrow$ and $\downarrow$ and the exact sequence (1). \( \square \)

2.6. **Universal property of the category of coherent functors.** The following elementary proposition stresses the relevance of the category of coherent functors.

**Proposition 2.6.1.** Let $\mathcal{E}$ be an abelian category and assume additive functors $U: G\mathcal{V} \to \mathcal{E}$ and $V: (G\mathcal{V})^{\text{op}} \to \mathcal{E}$ are given. Then

(i) There exists a left exact functor $U^c: \mathcal{E}(G) \to \mathcal{E}$, unique up to an isomorphism, such that $U^c(t_\alpha) = U(\alpha)$ for all $\alpha$ in $\mathcal{E}(G)$. Moreover $U^c$ is exact when $U$ is right exact.

(ii) There exists a right exact functor $V^c: \mathcal{E}(G) \to \mathcal{E}$, unique up to an isomorphism such that $V^c(h_\alpha) = V(\alpha)$ for all $\alpha$ in $\mathcal{E}(G)$. Moreover $V^c$ is exact when $V$ is left exact.

(iii) Suppose that $U$ is right exact, that $V$ is left exact and suppose that natural isomorphisms $\alpha_P: U(P) \to V(P^{\text{op}})$ are given for all projective objects $P$ of the category $G\mathcal{V}$. Then there exists an isomorphism of functors $\alpha^c: U^c \to V^c$ such that for any projective object $P \in G\mathcal{V}$ the morphism $\alpha^c(t_P)$ is the same as the following composite:
\[ U^c(t_P) = U(P) \to V(P^{\text{op}}) = V^c(h_{P^{\text{op}}}) \cong V^c(t_P), \]
where the last isomorphism is induced by the isomorphism $t_P \cong h_{P^{\text{op}}}$. \( \square \)

Proof. Deriving the additive functor provides the desired extension. The details are as follows.

i) Let $f$ be a coherent functor. Write $f = \text{Ker}(t_\alpha)$ as in (3) to get an injective resolution of $f$:
\[
0 \to f \to t_M \xrightarrow{t_\alpha} t_N \xrightarrow{t_{\text{Coker}(\alpha)}} 0.
\]
Thus $U^c(f)$ is necessarily the kernel of $U(\alpha): U(M) \to U(N)$. Not only this proves the uniqueness of the functor $U^c$, but it also gives the construction of $U^c$ as the 0-th right derived functor of the restriction of $U^c$ on injective objects. If $U$ is right exact then
\[ 0 \to U^c(f) \to U(M) \to U(N) \to U(\text{Coker}(\alpha)) \to 0 \]
is an exact sequence. Thus the right 1-st (and therefore any higher) derived functor vanishes and hence $U^c$ is an exact functor. Similarly for ii).

iii) For $M \in G\mathcal{V}$ choose a free presentation, an exact sequence
\[ K \to L \to M \to 0 \]
with free $K$ and $L$ in $G\mathcal{V}$. This gives an exact sequence of coherent functors
\[ t_K \to t_L \to t_M \to 0. \]
Since both functors $U^c$ and $V^c$ are exact, there exists a unique morphism $\alpha^c(t_M)$ which fits in a commutative diagram with $\alpha^c(t_K)$ and $\alpha^c(t_L)$. Since any coherent functor $f$ has a resolution $0 \to f \to t_M \to t_N$ and $U^c$, $V^c$ are exact functors, there exists a unique morphism $\alpha^c(f)$ which fits in a commutative diagram with $\alpha^c(t_M)$ and $\alpha^c(t_N)$. The claim follows. \( \square \)
2.7. External tensor products of coherent functors. Let $G$ and $H$ be finite groups. For $M$ in $G \mathcal{V}$ and $N$ in $H \mathcal{V}$, let $M \boxtimes N$ denote the vector space $M \otimes N$ with $G \times H$-action. It yields an exact bifunctor

$$\boxtimes : G \mathcal{V} \times H \mathcal{V} \to G \times H \mathcal{V}.$$ 

We wish to extend this bifunctor to coherent functors.

We first discuss a concrete elementary way of extending the product, inspired by [2, p. 781]. Note that for $X$ in $G \times H \mathcal{V}$ and $g$ in $\mathcal{A}(H)$, the vector space $g(\text{Res}_{1 \times H}^{G \times H}(X))$ has a natural $G$-structure, because $G \times 1$ commutes with $1 \times H$ in $G \times H$. Thus, for $f$ in $\mathcal{A}(G)$, the evaluation $f(g(\text{Res}_{1 \times H}^{G \times H}(X)))$ makes sense. Denote it by $(f \circ g)(X)$. We have thus defined an additive bifunctor

$$\circ : \mathcal{A}(G) \times \mathcal{A}(H) \to \mathcal{A}(G \times H).$$

Proposition 2.7.1. Let $G$ and $H$ be finite groups.

(i) For all coherent $f$ in $\mathcal{C}(G)$ and $g$ in $\mathcal{C}(H)$, the functor $f \circ g$ is coherent in $\mathcal{C}(G \times H)$. This defines a bifunctor

$$\circ : \mathcal{C}(G) \times \mathcal{C}(H) \to \mathcal{C}(G \times H).$$

(ii) For all $M$ in $G \mathcal{V}$ and $N$ in $H \mathcal{V}$, there are natural isomorphisms:

$$h_M \circ h_N \cong h_{M \boxtimes N}, \quad t_M \circ t_N \cong t_{M \boxtimes N}.$$

(iii) The bifunctor $\circ$ is exact with respect to the first argument.

(iv) The bifunctor $\circ$ commutes with duality.

Proof. The last two points are easy. We prove the first two.

(ii) For $X$ in $G \times H \mathcal{V}$, one has:

$$h_{M \boxtimes N}(X) = \text{Hom}_{G \times H}(M \boxtimes N, X)$$

$$= \text{Hom}_G(M, \text{Hom}_H(N, \text{Res}_{1 \times H}^{G \times H}(X)))$$

$$= h_M \circ h_N(X).$$

and the other isomorphism follows by duality (or directly as easily).

(i) By (iii), it is enough to prove that $h_M \circ g$ is coherent for all $g$ in $\mathcal{A}(H)$ and $M$ in $G \mathcal{V}$. Taking a free presentation of $M$ reduces further to the case when $M$ is free. Because $h_M \circ g$ is then exact in $g$, the case $g = h_N$ is enough. This is (ii).

\[ \square \]

Example 2.7.2. Note that: $f \circ h_K(X) = f(X)^H$ but: $h_K \circ f(X) = f(X)^H$, so that $h_K \circ f$ is not even right exact in $f$, even for $H = S_2$.

As a direct application of the method of Proposition 2.6.1, we obtain symmetric exact alternatives to this extension of the external tensor.

Proposition 2.7.3. There is a right exact (in each argument) functor:

$$\boxtimes : \mathcal{C}(G) \times \mathcal{C}(H) \to \mathcal{C}(G \times H),$$

unique up to isomorphism, such that:

$$h_M \boxtimes h_N = h_{M \boxtimes N}$$

for all $M$ in $G \mathcal{V}$ and $N$ in $H \mathcal{V}$. This bifunctor is symmetric and balanced.
Proof. For $M$ in $\mathcal{V}_{H \times G}$, let $M'$ be the $G \times H$-module with same underlying vector space and action given by: $(g, h)_m = (h, g)_m$. Symmetry means that the following diagram commutes up to natural isomorphism:

\[
\begin{array}{ccc}
\mathcal{C}(G) \times \mathcal{C}(H) & \xrightarrow{\otimes} & \mathcal{C}(G \times H) \\
\downarrow_{\text{tw}} & & \downarrow_{\text{tw}} \\
\mathcal{C}(H) \times \mathcal{C}(G) & \xrightarrow{\otimes} & \mathcal{C}(H \times G)
\end{array}
\]

where $\text{tw}$ is twisting of factors, while $\text{tw}$ is given by $\text{tw}(f)(M) = f(M')$, for $f$ in $\mathcal{C}(G \times H)$. To prove this property, it suffices to note that both functors $\text{tw} \circ \hat{\otimes}$ and $\hat{\otimes} \circ \text{tw}$ are right exact and take isomorphic values on projective generators; for projective generators, it reduces to the symmetry of the external tensor product.

To prove that the bifunctor is balanced, it is enough, by symmetry, to prove that $- \hat{\otimes} h_N$ is exact for each $N$ in $\mathcal{V}_H$. This functor is the extension to $\mathcal{C}(G)$ of the left exact functor $M \mapsto h_{M \otimes N}$. It is therefore exact by Proposition 2.6.1(ii). $\square$

We now compare these constructions.

**Proposition 2.7.4.** There is a natural transformation:

\[ f \hat{\otimes} g \to f \circ g \]

for all $f$ in $\mathcal{C}(G)$ and $g$ in $\mathcal{C}(H)$, which is an isomorphism when $g$ is projective.

**Proof.** Since both sides are right exact on $f$, the isomorphism for a projective generator $g = h_N$ follows from the case of projective generators $f = h_M$. $\square$

**Example 2.7.5.** Note that the two products do not always coincide. For instance, using Example 2.7.2 for $G = H$, $h_K \hat{\otimes} f \cong f \hat{\otimes} h_K \cong f \circ h_K$ is not always equal to $h_K \circ f$.

**Proposition 2.7.6.** There is a left exact (in each argument) functor:

\[ \hat{\otimes} : \mathcal{C}(G) \times \mathcal{C}(H) \to \mathcal{C}(G \times H) \]

unique up to isomorphism, such that:

\[ t_M \hat{\otimes} t_N = t_{M \otimes N} \]

for all $M$ in $\mathcal{V}_G$ and $N$ in $\mathcal{V}_H$. This bifunctor is symmetric and balanced.

**Proposition 2.7.7.** There is a natural transformation:

\[ f \circ g \to f \hat{\otimes} g \]

for all $f$ in $\mathcal{C}(G)$ and $g$ in $\mathcal{C}(H)$, which is an isomorphism when $g$ is injective.

These can be deduced from the natural isomorphism:

\[ D(f \hat{\otimes} g) \cong Df \hat{\otimes} Dg. \]

**Remark 2.7.8.** Indeed, the $\hat{\otimes}$-product coincide with Chaupnik's mysterious $(-, -)$ construction [2, p. 785]. For, choose an embedding of $f$ in some $f'$ and of $g$ in some $g'$. Because the $\hat{\otimes}$-product is left exact, $f \hat{\otimes} g$ embeds in $f' \hat{\otimes} g'$. More precisely, it is contained both in the image of $f' \hat{\otimes} g$ and in the image of $f \hat{\otimes} g'$ in $f' \hat{\otimes} g'$. A simple diagram chase shows that $f \hat{\otimes} g$ indeed equals the intersection of these images in $f' \hat{\otimes} g'$. Choosing $f'$ and $g'$ to be injectives gives a description in term of the more explicit $\circ$-product and recovers Chaupnik's definition of $(f, g)$ as well as the definition of $\hat{\otimes}$ as a 0-th right derived functor of $\circ$. 

2.8. Extension of the domain of coherent functors. In this section, we show that a coherent functor $f$ in $\mathcal{C}(G)$ can be evaluated on any object in an abelian category which is equipped with a $G$-action.

Let $\mathcal{E}$ be a $\mathbb{K}$-linear abelian category. There exists a unique (up to a unique isomorphism) exact bifunctor

$$\otimes : \mathcal{V} \times \mathcal{E} \to \mathcal{E}$$

such that $\mathbb{K} \otimes (-) : \mathcal{E} \to \mathcal{E}$ is isomorphic to the identity functor $Id_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$. It is obtained as follows. On the full subcategory of $\mathcal{V}$ with objects $\mathbb{K}^n$, $n \geq 0$, it is defined by: $\mathbb{K}^n \otimes E := E^n$, naturally in $E$. We are left with defining the functor $(-) \otimes E : \mathcal{V} \to \mathcal{E}$ on morphisms, for a fixed object $E$. For a matrix $(k_{ij})$, $k_{ij} \in \mathbb{K}$, the corresponding morphism $E^n \to E^m$ is given by the same matrix, but where $k_{ij}$ is considered as an elements of $\text{End}_{\mathcal{E}}(E)$ via the $K$-linear structure on $\mathcal{E}$.

Putting

$$\text{Hom}(V, A) := V^\vee \otimes A$$

defines an exact bifunctor

$$\text{Hom} : (\mathcal{V})^{op} \times \mathcal{E} \to \mathcal{E}$$

such that $\text{Hom}(\mathbb{K}, -) : \mathcal{E} \to \mathcal{E}$ is isomorphic to the identity functor $Id_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$.

It is clear that one has a natural isomorphism

$$\text{Hom}(V, W \otimes A) \cong W \otimes \text{Hom}(V, A),$$

for $V,W$ in $\mathcal{V}$ and $A$ in $\mathcal{E}$. Thus $\mathcal{E}$ is tensored and co-tensored over $\mathcal{V}$.

Remark 2.8.1. More generally, for any $\mathbb{K}$-linear functor $T : \mathcal{E} \to \mathcal{E}'$ between $\mathbb{K}$-linear categories, there are natural isomorphisms: $V \otimes T(A) \cong T(V \otimes A)$ and $\text{Hom}(V, TA) \cong T(\text{Hom}(V, A))$ for $V$ in $\mathcal{V}$ and $A$ in $\mathcal{E}$. Indeed, the second isomorphism is a consequence of the first one, while the first one is obvious for $V = \mathbb{K}$ and follows from the additivity of $T$ for all $V$’s.

We are now ready to define invariants and co-invariants of a finite group $G$ in an abelian category $\mathcal{E}$. A left $G$-object in $\mathcal{E}$ is a pair $(A, \alpha)$, where $A$ is in $\mathcal{E}$ and $\alpha : G \to \text{Aut}_{\mathcal{E}}(A)$ is a group homomorphism. Equivalently it is an object $A$ equipped with a map $\lambda : \mathbb{K}[G] \otimes A \to A$ satisfying obvious properties. We let $\text{Rep}(G, \mathcal{E})$ be the abelian category of left $G$-objects in $\mathcal{E}$. If $A$ is a $G$-object, its structural map $\mathbb{K}[G] \otimes A \to A$ has an adjoint $A \to \text{Hom}(\mathbb{K}[G], A)$. One defines:

$$H^0(G, A) := \text{Ker}(A \to \text{Hom}(\mathbb{K}[G], A)).$$

If $M$ is a $G$-module, one defines $\text{Hom}_G(M, A)$ to be $H^0(G, \text{Hom}(M, A))$.

Similarly, the reader will define $M \otimes_G A$ as a coequalizer of two canonical maps $M \otimes \mathbb{K}[G] \otimes A \to M \otimes A$. We also put: $H_0(G, A) := \mathbb{K} \otimes_G A$.

Proposition 2.8.2. There exists a bifunctor

$$< -, - > : \mathcal{C}(G) \times \text{Rep}(G, \mathcal{E}) \to \mathcal{E}$$

unique up to an isomorphism, such that the functor $< -, A > : \mathcal{C}(G) \to \mathcal{E}$ is exact for any $G$-object $A$ and

$$< t_M, A > = M \otimes_G A$$

for any $M \in G\mathcal{V}$. Moreover, one has a natural isomorphism

$$< h_M, A > \cong \text{Hom}_G(M, A).$$

Proof. Fix a $G$-object $A$. Put $R(M) = M \otimes_G A$ in Proposition 2.6.1 to get the existence and uniqueness of the pairing. To show the last assertion, observe that both sides of the isomorphism are left exact additive functors on $M$ and therefore it suffices to prove the isomorphism for $M = \mathbb{K}[G]$. Since $h_{\mathbb{K}[G]} \cong t_{\mathbb{K}[G]}$ is the forgetful functor, both sides in this case are isomorphic to $A$. $\square$
It is clear that for $\mathcal{E} = G\mathcal{V}$ this pairing is just evaluation: $<f, M> = f(M)$. The pairing allows to evaluate a coherent functor on graded $G$-objects, for instance.

3. Application to strict polynomial functors

3.1. The category of strict polynomial functors. Strict polynomial functors were introduced by Friedlander and Suslin [7, §2]. We now explain what they are in a given degree, in a way suitable for our purpose (see also [14, §4]).

Let us fix a positive integer $n$. We start with the $n$-th divided power of a vector space $V$, defined by:

$$\Gamma^n(V) := \mathcal{H}^0(\mathcal{S}_n, V^\otimes n) = (V^\otimes n)^{\mathcal{S}_n},$$

where the symmetric group on $n$-letters $\mathcal{S}_n$ acts on $V^\otimes n$ by permuting the factors. For any $x$ in some vector space $X$, we let $\gamma(x)$ be the element $x^\otimes n$ in $\Gamma^n(X)$. This defines a natural set map $\gamma_X : X \to \Gamma^n(X)$. Reordering the factors

$$A^\otimes n \otimes B^\otimes n \to (A \otimes B)^\otimes n$$

induces a $K$-linear natural transformation

$$\Gamma^n(A) \otimes \Gamma^n(B) \to \Gamma^n(A \otimes B)$$

sending $\gamma_A(a) \otimes \gamma_B(b)$ to $\gamma_{A \otimes B}(a \otimes b)$. Together with the composition law in $\mathcal{V}$, these maps define a composition map:

$$\Gamma^n(\text{Hom}(V, W)) \otimes \Gamma^n(\text{Hom}(U, V)) \to \Gamma^n(\text{Hom}(V, W) \otimes \text{Hom}(U, V)) \to \Gamma^n(\text{Hom}(U, W)).$$

This defines a category $\Gamma^n\mathcal{V}$, with the same objects as $\mathcal{V}$, and with morphisms

$$\text{Hom}_{\Gamma^n\mathcal{V}}(V, W) := \Gamma^n(\text{Hom}(V, W)).$$

The following Lemma describes the category $\Gamma^n\mathcal{V}$ as a full subcategory of $\mathcal{S}_n \mathcal{V}$.

Lemma 3.1.1. For a positive integer $n$, the functor

$$\iota : \Gamma^n\mathcal{V} \to \mathcal{S}_n \mathcal{V}$$

$$V \mapsto V^\otimes n$$

is a full embedding.

Proof. This follows from the natural isomorphism:

$$\text{Hom}_{\mathcal{S}_n}(V^\otimes n, W^\otimes n) = (\text{Hom}(V^\otimes n, W^\otimes n))^{\mathcal{S}_n} \cong (\text{Hom}(V, W)^\otimes n)^{\mathcal{S}_n} = \text{Hom}_{\Gamma^n\mathcal{V}}(V, W).$$

Reformulating [7, §2], a homogeneous strict polynomial functor of degree $n$ defined on $\mathcal{V}$ is a $K$-linear functor $\Gamma^n\mathcal{V} \to \mathcal{V}$. We let $\mathcal{P}_n$ be the category of homogeneous strict polynomial functors of degree $n$. It is known [7, §3] that the category $\mathcal{P}_n$ is equivalent to the category of finite dimensional modules over the Schur algebra $S(n, n)$.

The collection of maps $\gamma_X : X \to \Gamma^n(X)$ yields a (nonlinear) functor $\gamma : \mathcal{V} \to \Gamma^n\mathcal{V}$. Pre-composition with $\gamma$ associates to any strict polynomial functor defined on $\mathcal{V}$ an usual functor on $\mathcal{V}$; it is called the underlying functor of the strict polynomial functor. It is usual to denote by the same letter a strict polynomial functor and its underlying functor. For example, the composite

$$\begin{array}{ccc}
\Gamma^n\mathcal{V} & \xrightarrow{\iota} & \mathcal{S}_n \mathcal{V} \\
& \xrightarrow{\mathcal{H}^0(\mathcal{S}_n, \_)} & \mathcal{V}
\end{array}$$

is denoted by $\Gamma^n$, since its underlying functor is the $n$-th divided power functor, and $S^n$ denotes the composite

$$\begin{array}{ccc}
\Gamma^n\mathcal{V} & \xrightarrow{\iota} & \mathcal{S}_n \mathcal{V} \\
& \xrightarrow{\mathcal{H}_0(\mathcal{S}_n, \_)} & \mathcal{V}
\end{array},$$

where the symmetric group $\mathcal{S}_n$ acts on $\mathcal{S}_n \mathcal{V}$ by permuting the factors.
because its underlying functor is the $n$-th symmetric power. Similarly the composite

$$\Gamma^n V \xrightarrow{\text{forget}} S_n V \xrightarrow{\text{forget}} V$$

is denoted by $T^n$, because the underlying functor is the $n$-th tensor power. We now recall from [1, §3] the basic properties of the category $P_n$.

There is a well-defined tensor product of strict polynomial functors which corresponds to the usual tensor product of underlying functors, and it yields a bifunctor

$$- \otimes - : P_n \times P_m \to P_{n+m}.$$  

For example: $T^n = T^1 \otimes \cdots \otimes T^1$ ($n$-factors).

There is also a duality in $P_n$. For an object $F$ in $P_n$, we let $D F$ be the homogeneous strict polynomial functor given by

$$(D F)(V) = (F(V^\vee))^\vee$$

where $W^\vee$ denotes the dual vector space of $W$. Since the values of any homogeneous strict polynomial functor are finite dimensional, $D$ is an involution and defines an equivalence of categories $D : P_n^{\text{op}} \to P_n$. The functor $D F$ is called the dual of $F$.

The category $P_n$ has enough projective and injective objects. A set of generators is indexed by partitions of $n$, that is decreasing sequences of positive integers adding up to $n$. For a partition $\lambda = (n_1 \geq n_2 \geq \cdots \geq n_k)$, put

$$\Gamma^\lambda := \Gamma^{n_1} \otimes \cdots \otimes \Gamma^{n_k}.$$  

The functors $\Gamma^\lambda$, when $\lambda$ runs through all partitions of the integer $n$, form a set of projective generators. Indeed, $\text{Hom}_{P_n}(\Gamma^\lambda, F)$ is the evaluation on the base field of the cross-effect of the functor $F$ of homogeneous multidegree $\lambda$ (i.e. the component of weight $\lambda$ under the action of $G_m^k$ on the cross-effect of $F$). Dually, the functors

$$S^\lambda := S^{n_1} \otimes \cdots \otimes S^{n_k}$$

form a set of injective cogenerators. In particular, the functor $T^n$ is projective and injective in $P_n$. Moreover, the action of $S_n$ by permuting factors yields an exact functor

$$c^* : P_n \to \mathcal{S}_n V$$

$$F \mapsto \text{Hom}_{P_n}(T^n, F).$$

The representation $c^*(F)$ is often called the linearization of the functor $F$; we use the letter $c$ for cross-effect. The functor $c^*$ has both a left and a right adjoint functor given respectively by

$$(c_!(M))(V) = (M \otimes V^\otimes n)_{S_n}, \quad c_*(M) = (M \otimes V^{\otimes n})_{S_n}.$$  

Let $P_n^0$ be the full subcategory of $P_n$ whose objects are the strict polynomial functors $F$ such that $c^*(F) = 0$. This condition means that the underlying functor has degree less than $n$ in the additive sense of Eilenberg and MacLane. Let $d_* : P_n^0 \to P_n$ be the inclusion and let $d^*$ and $d^!$ be the left and right adjoint of $d_*$. By Proposition A.1.2 this defines a recollement situation:

$$\begin{array}{ccc}
P_n^0 & \xrightarrow{d_*} & P_n \\
\downarrow d^* & & \downarrow c^* \\
\mathcal{S}_n V & \xrightarrow{c_!} & \mathcal{S}_n V \\
& \downarrow c_* & \\
& & P_n^0 \\
\end{array}$$

More on this recollement can be found in [11].
3.2. The relation between strict polynomial and coherent functors. For simplicity, we write $\mathcal{A}_n$ and $\mathcal{C}_n$ instead of $\mathcal{A}(\mathfrak{S}_n)$ and $\mathcal{C}(\mathfrak{S}_n)$. For an object $f$ in $\mathcal{A}_n$, the composite
\[ \Gamma^n f \longrightarrow \varepsilon_n f \longrightarrow f \]
defines a strict polynomial functor, which is denoted by $j^*(f)$. This construction defines a functor
\[ j^*: \mathcal{A}_n \rightarrow \mathcal{P}_n \]
\[ j^*(f)(V) = f(V \otimes d). \]
The same functor was constructed, in terms of Schur algebras, by Green [8, §5, pp 275–276]. The functor $j^*$ was also considered by Chalupnik [2, §3] in his work on functor cohomology.

Lemma 3.2.1. The functor $j^*$ respects duality: $\mathbb{D} \circ j^* \cong j^* \circ \mathbb{D}$.

Proof. $\mathbb{D}j^*(f)(V) = (j^*(f)(V^\vee))^\vee = (f(V^\vee \otimes n))^\vee \cong (\mathbb{D}f)(V^\otimes n) = j^* \mathbb{D}(f)(V)$. □

Proposition 3.2.2. The functor $j^*: \mathcal{A}_n \rightarrow \mathcal{P}_n$ has a right adjoint functor defined by:
\[ j_*(F)(M) = \text{Hom}_{\mathcal{A}_n}(j^*(h_{M^\vee}), F) = \text{Hom}_{\mathcal{P}_n}(\text{Hom}_{\mathfrak{S}_n}(M, (-)^\otimes n), F) \]
where $M$ is representation of $\mathfrak{S}_n$. It has also a left adjoint functor defined by:
\[ j^!(F) = \mathbb{D}(j_*(\mathbb{D}F)). \]

In other words:
\[ (j_!(F)(M))^\vee = \text{Hom}_{\mathcal{P}_n}(F, (-)^\otimes n \otimes \mathfrak{S}_n M^\vee). \]

Proof. Since hom’s in the category $\mathcal{P}_n$ are finite dimensional vector spaces, we see that $j_!(F)$ belongs to $\mathcal{A}_n$. The fact that it is right adjoint of $j^*$ follows from the Yoneda lemma. The dual formula is formal:
\[ \text{Hom}_{\mathcal{A}_n}(\mathbb{D}(j_*(\mathbb{D}F)), f) \cong \text{Hom}_{\mathcal{A}_n}(\mathbb{D}f, j_*(\mathbb{D}F)) \cong \text{Hom}_{\mathcal{P}_n}(j^*(\mathbb{D}f), \mathbb{D}F) \cong \text{Hom}_{\mathcal{P}_n}(\mathbb{D}j^!(f), \mathbb{D}F) \cong \text{Hom}_{\mathcal{P}_n}(F, j^!(f)). \]

To check the last formula, observe that:
\[ j_*(\mathbb{D}F)(M^\vee) = \text{Hom}_{\mathcal{P}_n}(j^*(h_{M^\vee}), \mathbb{D}F) \cong \text{Hom}_{\mathcal{P}_n}(F, j^*(\mathbb{D}h_{M^\vee})) \]
and
\[ j^!(\mathbb{D}h_{M^\vee}) \cong j^* h_{M^\vee} = (-)^\otimes n \otimes \mathfrak{S}_n M^\vee. \]

Remark 3.2.3. In particular $j_*$ and $j^!$ are a functorial choice of, respectively, an injective and projective symmetrization of [2, Section 3].

Remark 3.2.4. The existence of adjoints of a precomposition functor is quite a general phenomenon, see Example [A.1.4].

We now study these adjoint functors. For a partition $\lambda$ of a positive integer $n$, we let $\mathfrak{S}_\lambda$ be the corresponding Young subgroup of $\mathfrak{S}_n$.

Lemma 3.2.5. Let $\lambda$ be a partition of $n$ there are natural isomorphisms:
\[ j_*(S^\lambda) \cong H_0(\mathfrak{S}_\lambda, -) = t_{\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda]}, \]
\[ j^!(\Gamma^\lambda) \cong H^0(\mathfrak{S}_\lambda, -) = h_{\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda]}. \]

In particular
\[ \text{Hom}_{\mathcal{P}_n}(S^\lambda, S^\mu) = \text{Hom}_{\mathfrak{S}_n}(\mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\lambda], \mathbb{K}[\mathfrak{S}_n/\mathfrak{S}_\mu]) = \text{Hom}_{\mathcal{P}_n}(\Gamma^\mu, \Gamma^\lambda). \]
Proposition 3.2.6. \[8, 2.14\] The values of the functors \( j_* \) and \( j_! \) are coherent functors.

Proof. By duality, it is enough to consider the functor \( j_* \). By Lemma 3.2.10, the statement is true for injective cogenerators of \( \mathcal{P}_n \). Since \( j_* \) is left exact the result follows by a resolution argument.

Proposition 3.2.7. The unit \( \text{Id}_{\mathcal{P}_n} \to j^* j_! \) and the counit \( j^* j_* \to \text{Id}_{\mathcal{P}_n} \) are isomorphisms.

Proof. We prove only the second isomorphism, the first one follows by duality. It is clear that \( j^* (H_0(\mathcal{G}_\lambda, -)) \cong S^\lambda \). Thus Lemma 3.2.10 shows that the statement is true for injective cogenerators of \( \mathcal{P}_n \). Since \( j^* \) is exact and \( j_* \) is left exact, the result follows by taking resolutions.

Remark 3.2.8. Since the functor \( j_* \) is a full embedding, this gives a new proof of [2, Lemma 3.4].

Proposition 3.2.9. Let \( \mathcal{C}^\vee_n \) be the full subcategory of \( \mathcal{C}_n \) whose objects are the coherent functors \( f \) such that, for all partitions \( \lambda \) of \( n \):

\[
f(K[\mathcal{G}_n/\mathcal{G}_\lambda]) = 0.
\]

The functors \( j^* \) and its adjoints \( j_* \) are part of a recollement of abelian categories:

\[
\begin{array}{cccc}
\mathcal{C}^\vee_n & \downarrow j^* & \mathcal{C}_n & \downarrow j_* \\
\downarrow i^* & & \downarrow i \\
\mathcal{C}_n & & \mathcal{P}_n
\end{array}
\]

Proof. According to Proposition 3.1.2, the functor \( j^* \) and its adjoints give rise to a recollement situation. To determine the kernel category, it is enough to notice that every Young-permutation representation \( k[\mathcal{G}_n/\mathcal{G}_\lambda] \) occurs as a direct factor in the tensor product \( V \otimes^n \) as soon as the dimension of \( V \) is \( n \).

Example 3.2.10. \( \mathcal{C}^\vee_n = 0 \) if \( p > n \). Moreover \( \mathcal{C}^\vee_n = 0 \) for \( p = 2 \) and \( n = 2 \) or 3.

Proposition 3.2.11. The counit \( j_! j^* (f) \to f \) is an isomorphism when \( f = t_M \). Dually, the unit \( f \to j_* j^* (f) \) is an isomorphism when \( f = h_M \).

Proof. We prove only the first assertion. Since both \( j_! j^*(t_M) \) and \( t_M \) are right exact functors of \( M \), it is enough to consider the case when \( M = k[\mathcal{G}_n] \). In this case \( t_M \) is the forgetful functor. Therefore:

\[
j_! j^* (t_M) = \otimes^n = \Gamma^{11\cdots 1} = H^0(\mathcal{G}_{11\cdots 1}, -) = t_{k[\mathcal{G}_n]}.
\]

Proposition 3.2.12. The norm transformation (see Appendix 3.1) for the previous recollement situation is an isomorphism on projective and injective objects.

Proof. By Lemma 3.2.6 we have \( j_!(\Gamma^\lambda) = h_M \), for \( M = k[\mathcal{G}_n/\mathcal{G}_\lambda] \). By Proposition 3.2.11 we have also \( j_!(\Gamma^\lambda) = j_* j^* (h_M) = h_M \), thus the norm is an isomorphism on projective objects. By duality it is also an isomorphism on injective objects.
Proposition 3.2.16. There are commutative diagrams of categories and functors:

\[
\begin{array}{ccc}
\mathcal{C}_n & \xrightarrow{j_*} & \mathcal{P}_n^0 \\
\mathcal{C}_n & \xrightarrow{t_*} & \mathcal{C}_n \\
\mathcal{E}_n & \xrightarrow{c_*} & \mathcal{E}_n \\
\mathcal{C}_n & \xrightarrow{j^*} & \mathcal{P}_n \\
\mathcal{C}_n & \xrightarrow{t^*} & \mathcal{E}_n \\
\mathcal{E}_n & \xrightarrow{c^*} & \mathcal{E}_n
\end{array}
\]

Example 3.2.13. The relations of Proposition 3.2.11

\[j_*(j^*t_M) \cong t_M \cong j_*(j^*t_M)\]

apply in particular when \(M\) is the signature, or when \(M\) is induced from the signature of a Young subgroup \(\mathfrak{S}_p\). This shows that, when \(p\) is odd, the norm is an isomorphism on a tensor product of exterior powers \(\Lambda^p\).

Example 3.2.14. Let \(Id^{(1)}\) be the Frobenius twist in \(\mathcal{P}_p\), that is extension of scalars through the Frobenius [7 p. 212]. It is related to the norm \(N\) by an exact sequence:

\[
0 \longrightarrow Id^{(1)} \longrightarrow S^p \xrightarrow{N} \Gamma^p \longrightarrow Id^{(1)} \longrightarrow 0
\]

It follows that there are exact sequences:

\[
0 \rightarrow j_*(Id^{(1)}) \rightarrow \mathcal{H}_0(\mathfrak{S}_p, -) \rightarrow \mathcal{H}_0(\mathfrak{S}_p, -)
\]

\[
\mathcal{H}_0(\mathfrak{S}_p, -) \rightarrow \mathcal{H}_0(\mathfrak{S}_p, -) \rightarrow j_!(Id^{(1)}) \rightarrow 0.
\]

Thus \(j_*(Id^{(1)}) = \hat{H}^{-1}(\mathfrak{S}_p, -)\) and \(j_!(Id^{(1)}) = \hat{H}^0(\mathfrak{S}_p, -)\) (where as usual \(\hat{H}\) denotes Tate cohomology).

Example 3.2.15. Assume \(p = 2\). Let \(S\) be a set with \(n\) elements. For each \(0 \leq k \leq n\) we let \(B_k\) be the vector space spanned on the set of all subsets of \(S\) with exactly \(k\)-elements. Define \(d : B_k \rightarrow B_{k+1}\) by

\[
d(X) = \sum_{X \subset Y \in B_{k+1}} Y
\]

Then \(d^2 = 0\) and \(B_*\) is a cochain complex of \(\mathfrak{S}_n\)-modules. One checks that it is acyclic. For an integer \(m \geq 1\) and \(n = 2^{m+1}\), the explicit injective resolution of \(S^{2m(1)}\) of [5] [2] and [5] [8] allows to compute:

\[
R^k j_* (S^{2m(1)}) \cong H^k(t_B,*).
\]

In particular, \(j_*(S^{2m(1)})\) is the kernel of the obvious map \(H_0(\mathfrak{S}_n, -) \rightarrow H_0(\mathfrak{S}_{n-1,1}, -)\). Another consequence of [5] is the fact that \(R^k j_* (S^{2m(1)}) = 0\) when \(k \geq m\).

We now show the compatibility of the different recollement situations.

Proposition 3.2.16. There are commutative diagrams of categories and functors:

\[
\begin{array}{ccc}
\mathcal{C}_n & \xrightarrow{j_*} & \mathcal{P}_n^0 \\
\mathcal{C}_n & \xrightarrow{t_*} & \mathcal{C}_n \\
\mathcal{E}_n & \xrightarrow{c_*} & \mathcal{E}_n \\
\mathcal{C}_n & \xrightarrow{j^*} & \mathcal{P}_n \\
\mathcal{C}_n & \xrightarrow{t^*} & \mathcal{E}_n \\
\mathcal{E}_n & \xrightarrow{c^*} & \mathcal{E}_n
\end{array}
\]

Proof. To show that \(c^* \circ j^* = t^*\), note that the three functors involved are exact. It is therefore enough to check that they coincide on \(h_M\) for each \(M\) in \(\mathcal{E}_n\). This means that we need to show that:

\[
\text{Hom}_{\mathcal{E}_n}(T^n, \text{Hom}_{\mathcal{E}_n}(M, (-)^{\otimes n})) \cong M^{\vee}
\]

Since \(T^n\) is projective, the left hand side of the expected isomorphism is left exact as a functor of \(M\). So it suffices to consider the case when \(M\) is injective, and it reduces to the case when \(M = K[\mathfrak{E}_n]\). In this case it is a well-known isomorphism:

\[
\text{Hom}_{\mathcal{E}_n}(T^n, T^n) \cong K[\mathfrak{E}_n].
\]
To show that $j_c! = !t$, note that both sides are right exact. It is therefore enough to check that they coincide on $K[\mathfrak{S}_n]$. In this case, $j_c!(K[\mathfrak{S}_n]) = j_!(T^n)$ has already been seen (see the proof of Proposition 3.2.11) to be the forgetful functor $t_{K[\mathfrak{S}_n]}$.

The rest is quite similar.

3.3. Tensor products of coherent functors. The aim of this section is to lift the bifunctor $\otimes : \mathcal{P}_m \times \mathcal{P}_m \to \mathcal{P}_{m+n}$ given by $(F \otimes G)(V) = F(V) \otimes G(V)$ at the level of coherent functors. Not surprisingly, it involves the induction functors

$$\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} : \mathbb{S}_m \times \mathbb{S}_n \mathcal{V} \to \mathbb{S}_{m+n} \mathcal{V}.$$  

For $M$ in $\mathbb{S}_m \mathcal{V}$ and $N$ in $\mathbb{S}_n \mathcal{V}$, we let $M \boxtimes N$ denote the vector space $M \otimes N$ with $\mathbb{S}_m \times \mathbb{S}_n$-action, and simply denote by $M \otimes N$ the module $\text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}(M \boxtimes N)$. The operation $\otimes$ yields a symmetric monoidal structure on the category $\bigoplus_{n \geq 0} \mathbb{S}_n \mathcal{V}$.

Using Section 2.7, one defines biadditive functors $\mathbb{S}_m \times \mathbb{S}_n \to \mathbb{S}_{m+n}$ by:

$$f \otimes g = \text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}(f \otimes g) \quad f \otimes g = \text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}(f \otimes g).$$

**Theorem 3.3.1.**

(i) For all $M$ in $\mathfrak{S}_m \mathcal{V}$ and $N$ in $\mathfrak{S}_n \mathcal{V}$, there are natural isomorphisms

$$h_M \otimes h_N \cong h_{M \otimes N} = h_M \cdot h_N.$$  

(ii) For all $f$ in $\mathfrak{S}_m$ and all $g$ in $\mathfrak{S}_n$, there is a natural transformation:

$$f \otimes g \to f \cdot g$$

which is an isomorphism when $g$ is projective.

(iii) The bifunctor $\otimes$ equips the category $\bigoplus_{n \geq 0} \mathfrak{S}_n \mathcal{V}$ with a symmetric monoidal structure.

(iv) The bifunctor $\otimes$ is right exact (in each argument) and balanced.

(v) For all $f$ in $\mathfrak{S}_m$ and $g$ in $\mathfrak{S}_n$, there are natural isomorphisms

$$j^*(f) \otimes j^*(g) \cong j^*(f \otimes g) \cong j^*(f \cdot g);$$

(vi) For all $F$ in $\mathcal{P}_m$ and $G$ in $\mathcal{P}_n$, there is a natural isomorphism

$$j_!(F) \otimes j_!(G) \cong j_!(F \otimes G)$$

**Proof.** Right exactness of $\otimes$ and (i) follow from Proposition 2.7.1. As in Proposition 2.6.1 these properties characterize $\otimes$ up to isomorphism, and (iii) follows. We next show the first isomorphism in (v). Since $j^*$ and $\otimes$ are right exact functors, it is enough to consider the case when $f = h_M$ and $g = h_N$. We need to prove that

$$h_M(V^\otimes n) \otimes h_N(V^\otimes n) \cong h_{M \otimes N}(V^\otimes m+n).$$

This follows from the isomorphisms:

$$\text{Hom}_{\mathfrak{S}_m}(M, V^\otimes m) \otimes \text{Hom}_{\mathfrak{S}_n}(N, V^\otimes n) = \text{Hom}_{\mathfrak{S}_m \times \mathfrak{S}_n}(M \boxtimes N, V^\otimes m \boxtimes V^\otimes n) = \text{Hom}_{\mathfrak{S}_m \times \mathfrak{S}_n}(M \boxtimes N, \text{Res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} V^\otimes m+n) = \text{Hom}_{\mathfrak{S}_{m+n}}(M \otimes N, V^\otimes m+n).$$

To show the second isomorphism in (v), one observes that for all $W \in \mathcal{V}$ and $N \in \mathfrak{S}_n \mathcal{V}$, there is a natural isomorphism (as in Remark 2.8.1):

$$g(W \otimes N) \cong W \otimes g(N),$$

where $\mathfrak{S}_n$ acts on the second factor of $W \otimes N$. Now: $V^\otimes m+n = V^\otimes m \otimes V^\otimes n$ as $\mathfrak{S}_n$-modules, with trivial action on the first factor, so:

$$g(\text{Res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} V^\otimes m+n) = V^\otimes m \otimes g(V^\otimes n).$$

Similarly, the group $\mathfrak{S}_m$ acts trivially on $g(V^\otimes n)$, and we obtain:

$$j^*(f \cdot g)(V) = f(g(\text{Res}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} V^\otimes m+n)) \cong f(V^\otimes m) \otimes g(V^\otimes n).$$
Finally, we show the isomorphism in (vi). Let us denote by $\lambda \cup \mu$ the concatenation of two partitions $\lambda$ and $\mu$. There is an isomorphism:

\[
\text{Ind}_{S_{\lambda \cup \mu}} \mathbb{K} \otimes \text{Ind}_{S_{\lambda}} \mathbb{K} = \text{Ind}_{S_{\lambda \mu}} \mathbb{K}.
\]

Now to (vi). It is enough to assume that $F$ and $G$ are projective generators: $F = \Gamma^\lambda$ and $G = \Gamma^\mu$. By Lemma 3.2.6, we have

\[
j_r(\Gamma^\lambda) = H^r(S_{\lambda}, -) = h_{\text{Ind}_{S_{\lambda}} \mathbb{K}}, \quad j_r(\Gamma^\mu) = h_{\text{Ind}_{S_{\mu}} \mathbb{K}}.
\]

So the isomorphism (vi) in this case follows from the isomorphism (6).

We leave to the reader the statement dual to Theorem 3.3.1.

4. Application to functor cohomology

4.1. Introduction. The first computation of Ext-groups between strict polynomial functors in [7] states that the graded vector space $A_r = \text{Ext}^r_{\mathcal{P}}(I_d^{(r)}, I_d^{(r)})$ is one-dimensional in every degree smaller than $2p^r$ and zero else. A lot more basic computations were carried out in [6, §4 - 5], based on these results, M. Chałupnik [2, §4 - 5] has succeeded in extending them to a lot more basic functors. His calculations eventually rely on the very special form of the fundamental but elementary computation of $\text{Ext}^*_{\mathcal{P}_{d}^r}(I_{d}^{(r)} \otimes \mathbb{K}, I_{d}^{(r)} \otimes \mathbb{K})$ as $B_r = A_{\mathcal{P}}^d \otimes \mathbb{K}[S_d]$, a $S_n^d \times S_n$-permutation graded module.

The present paper’s setting allows to better formulate the tools Chałupnik used and the results he obtained. This includes his notion of symmetrization [2, §3], which we proved in Section 3.2 to be functorial, or his $(-, -)$-product [2, Theorem 4.4, p. 785], which coincides with our $\mathbb{E}$-product of Section 2.7 by Remark 2.7.8. We thus explain key results showing exactness and symmetry in the contravariant/covariant variables, for example. We apply our insight to prove naturality of [2, Theorem 5.3], a result which does not follow from the methods in [2], and we further extend Chałupnik’s results. It has to be noted however, that the results in [2] do not follow formally from our considerations, but they rather are the elementary calculations to build on from.

4.2. Cohomology of Frobenius twists. We start (as does Chałupnik [2]) by recalling [6, Theorem 4.5].

Let us observe that in the category $\mathcal{P} := \bigoplus_{d \geq 0} \mathcal{P}_d$, a tensor product of two projectives is projective. This allows a Künneth morphism:

\[
\text{Ext}^*_{\mathcal{P}}(A, B) \otimes \text{Ext}^*_{\mathcal{P}}(C, D) \to \text{Ext}^*_{\mathcal{P}}(A \otimes C, B \otimes D).
\]

There is thus a natural map:

\[
\text{Ext}^*_{\mathcal{P}}(A, B) \otimes \mathbb{K} \to \text{Ext}^*_{\mathcal{P}}(A \otimes \mathbb{K}, B \otimes \mathbb{K})
\]

which defines a natural homomorphism

\[
S^d(\text{Ext}^*_{\mathcal{P}}(A, B)) \to \text{Ext}^*_{\mathcal{P}}(\Gamma^d \circ A, \mathbb{K}^d(\circ B)).
\]

Taking $A = B = I_d^{(r)}$, the $r$-th Frobenius twist of the identity functor, we get the map

\[
S^d(\text{Ext}^*_{\mathcal{P}}(I_d^{(r)}, I_d^{(r)})) \to \text{Ext}^*_{\mathcal{P}_{d}^r}(\Gamma^d, S_d^{(r)})
\]

which, by a special case of Theorem 4.5 in [6], is an isomorphism.

From this, Chałupnik [2, Theorem 4.3] deduces that, when $G = \Gamma^\mu$, there is a natural in $G$ isomorphism:

\[
\text{Ext}^*_{\mathcal{P}}(G^{(r)}, F^{(r)}) \cong < j_r F \circ j_r^* G, B_r >.
\]
Proof. First consider the special case when \(R\) where
\[
\text{Proposition 4.2.1.}
\]
We readily obtain the following natural version of \([2, \text{Theorem 5.3}]\):
\[
\text{Let } j : (9)
\]
by Proposition 2.7.7:
On the latter formulation, left exactness in both \(F\) and \(G\), as well as symmetry, become transparent.
We apply the formula for \(G = \Gamma^{d}\), knowing the \(\mathcal{G}_{d}\)-bimodule structure on \(B_{r}\).
We readily obtain the following natural version of \([2, \text{Theorem 5.3}]\):

**Proposition 4.2.1.** For all \(F\) in \(\mathcal{P}_{d}\), there is a natural isomorphism:
\[
\text{Ext}_{\mathcal{P}_{d}}(\Gamma^{d}(i), F^{(i)}) \cong F(\text{Ext}_{\mathcal{P}_{d}}(Id^{(i)}, Id^{(i)})).
\]

More generally:

**Proposition 4.2.2.** Let \(F\) and \(G\) be strict polynomial functors of degree \(d\). There exists a natural spectral sequence:
\[
E_{2}^{ij} = < R^{i}T(G, F), B_{r} > \Rightarrow \text{Ext}_{\mathcal{P}_{d}}^{i+j}(G^{(r)}, F^{(r)}),
\]
where \(R^{i}T\) denotes the \(i\)-th right derived functor of the left exact functor
\[
T : \mathcal{P}_{d}^{\text{op}} \times \mathcal{P}_{d} \rightarrow \mathcal{G}_{d}^{\text{op}} \times \mathcal{G}_{d}
\]
\[
(G, F) \mapsto j_{*}G \bar{\otimes} j_{*}F.
\]

**Proof.** First consider the special case when \(G\) is projective. In this case we have to show that
\[
< j_{*}G \bar{\otimes} j_{*}F, B_{r} > \cong \text{Ext}_{\mathcal{P}_{d}}^{i+j}(G^{(r)}, F^{(r)})
\]
Since both sides are additive in \(G\), it is enough to check the isomorphism when \(G = \Gamma^{d}\). This case results from \([8]\) and \([9]\).

For the general case, we take a projective resolution of \(G_{*}\) of \(G\). After twisting, we obtain a (non-projective) resolution \(G_{*}^{(r)}\) of \(G^{(r)}\). The hypercohomology spectral sequence obtained by applying \(\text{Hom}_{\mathcal{P}}(-, F^{(r)})\) to \(G_{*}^{(r)}\) has the form
\[
E_{1}^{ij} = \text{Ext}_{\mathcal{P}}^{i}(G_{*}^{(r)}, F^{(r)}) \Rightarrow \text{Ext}_{\mathcal{P}}^{i+j}(G^{(r)}, F^{(r)}).
\]

Since \(G_{*}\) is projective we can apply the previous computation to obtain:
\[
E_{1}^{i} = < j_{*}G \bar{\otimes} j_{*}F, B_{r} >
\]
and the \(E_{2}\) term has the expected form. Left exactness of the \(\bar{\otimes}\)-product ensures that in the first column appears \(R^{0}T = T\). \qed

**Appendix A.**

**Recollement of abelian categories.** To reveal the relationship between the different abelian categories, we use the language of recollements (see \([10]\) and \([3]\)).

A recollement of abelian categories consists of a diagram of abelian categories and additive functors
\[
\begin{align*}
\mathcal{A}' & \xrightarrow{i} \mathcal{A} & \xrightarrow{j} & \mathcal{A}''
\end{align*}
\]
satisfying the following conditions:

(i) the functor \(j_{*}\) is left adjoint to \(j^{*}\) and the functor \(j^{*}\) is left adjoint to \(j_{*}\);

(ii) the unit \(\text{Id}_{\mathcal{A}'} \rightarrow j^{*}j_{*}\) and the counit \(j_{*}j^{*} \rightarrow \text{Id}_{\mathcal{A}'}\) are isomorphisms;

(iii) the functor \(i^{*}\) is left adjoint of \(i_{*}\) and \(i_{*}\) is left adjoint of \(i^{*}\);

(iv) the unit \(\text{Id}_{\mathcal{A}'} \rightarrow i^{*}i_{*}\) and the counit \(i^{*}i_{*} \rightarrow \text{Id}_{\mathcal{A}'}\) are isomorphisms;

(v) the functor \(i_{*} : \mathcal{A}' \rightarrow \text{Ker}(j^{*})\) is an equivalence of categories.
Example A.1.1. The following example is the paradigm of a recollement situation. Let \( X \) be a space, \( C \) is a closed subset in \( X \) and \( U = X \setminus C \) its open complement. Extension and restriction yield a recollement of sheaves categories:

\[
\begin{array}{ccc}
\text{Sh}(C) & \overset{i^*}{\longrightarrow} & \text{Sh}(X) \overset{j^*}{\longrightarrow} \text{Sh}(U) \\
\downarrow j_! & & \downarrow j_* \\
\text{Sh}(X) & \overset{i_*}{\longrightarrow} & \text{Sh}(C)
\end{array}
\]

The list of properties (i)-(v) can be somewhat shortened.

**Proposition A.1.2.** Let \( j^*: \mathcal{A} \to \mathcal{A}'' \) be an exact functor which satisfies (i) and (ii): it admits both a left adjoint \( j_l \) and a right adjoint \( j_* \), and the unit \( \text{Id}_{\mathcal{A}''} \to j^* j_l \) and counit \( j^* j_* \to \text{Id}_{\mathcal{A}''} \) are isomorphisms. Let \( \mathcal{A}' \) be the full subcategory of \( \mathcal{A} \) with objects those \( A \) such that \( j^* A = 0 \). Then the full embedding \( i_*: \mathcal{A}' \to \mathcal{A} \) has adjoint functors \( (i^*, i_*') \) and the unit \( \text{Id}_{\mathcal{A}'} \to i^* i_*' \) and counit \( i_*' i_* \to \text{Id}_{\mathcal{A}'} \) are isomorphisms. In other words we have a recollement situation.

**Proof.** Let \( A \) in \( \mathcal{A} \) and let \( \epsilon_A: j_! j^* A \to A \) be the counit of the adjoint pair \((j_!, j^*)\). Because \( \text{Id}_{\mathcal{A}''} \to j^* j_l \) is an isomorphism, we have \( j^* (\text{Coker} \epsilon_A) = 0 \). It follows that \( \text{Coker}(\epsilon_A) \) lies in the subcategory \( \mathcal{A}' \). So there is a well-defined functor \( i^*: \mathcal{A} \to \mathcal{A}' \) such that \( \text{Coker}(\epsilon_A) = i_* i^* A \). The rest follows, using the short exact sequence of natural transformations:

\[
j_! j^* \xrightarrow{\eta} \text{Id}_{\mathcal{A}'} \to i_* i^* \to 0
\]

and the dual study of the unit of adjunction \( \eta \) which sums up in the following exact sequence:

\[
0 \to i_* i^* \to \text{Id}_{\mathcal{A}'} \xrightarrow{\epsilon} j_! j^* .
\]

\[\square\]

**Remark A.1.3.** Actually, if \( \mathcal{A} \) is a category of modules over a ring (or, more generally, if \( \mathcal{A} \) is a Grothendieck category), then it is enough to assume that \( j^* \) is an exact functor which has a left adjoint functor \( j_l \) such that the unit of adjunction: \( \text{Id}_{\mathcal{A}''} \to j^* j_l \) is an isomorphism. The existence of \( j_* \) follows from [13 Proposition 2.2].

**Example A.1.4.** Recollements arise naturally when relating functor categories through precomposition. Indeed, starting with a functor \( i: \mathcal{A} \to \mathcal{B} \), precomposition is an exact functor:

\[
j^*: \mathcal{Y}^{\mathcal{B}} \to \mathcal{Y}^{\mathcal{A}} \\
F \mapsto F \circ i.
\]

A classic result of D. Kan tells that it always admits adjoint functors, called the left and the right Kan extension. By [12 §X.3, Corollary 3], the unit and the counit of adjunction are isomorphisms when the functor \( i \) is a full embedding. A recollement situation then arises by Proposition A.1.2.

In the case when \( i \) is a full embedding of \( \mathcal{K} \)-linear categories, the functor \( j^* \) and its adjoints restrict to the subcategories of \( \mathcal{K} \)-linear functors. Proposition 3.2.9 describes the resulting recollement when the functor \( i \) is the full embedding of Lemma 3.1.1.

Another useful functor arises from a recollement: the functor \( j_{1*}: \mathcal{A}'' \to \mathcal{A} \) is the image of the norm \( N: j_l \to j_* \), the natural transformation which corresponds to \( 1_X \), under the isomorphism

\[
\text{Hom}_{\mathcal{A}}(j_l X, j_{1*} X) \cong \text{Hom}_{\mathcal{A}''}(X, j^* j_{1*} X) \cong \text{Hom}_{\mathcal{A}''}(X, X).
\]

The functor \( j_{1*} \) preserves simple objects, and every simple object in \( \mathcal{A} \) is either the image of a simple in \( \mathcal{A}' \) by the functor \( i_* \), or the image of a simple in \( \mathcal{A}'' \) by the functor \( j_{1*} \).
We close with an immediate consequence of the Grothendieck spectral sequence for a composite functor.

**Proposition A.1.5.** Assume in a recollement situation all abelian categories have enough projective objects. For \( X \) in \( \mathcal{A}' \) and \( B \) in \( \mathcal{A} \), there are spectral sequences:

\[
E_2^{pq} = \text{Ext}_{\mathcal{A}}^p(L_qj_!(X), B) \Rightarrow \text{Ext}_{\mathcal{A}''}^{p+q}(X, j^*B)
\]

and

\[
E_2^{pq} = \text{Ext}_{\mathcal{A}}^p(B, R^qj_*(X)) \Rightarrow \text{Ext}_{\mathcal{A}''}^{p+q}(j^*B, X).
\]

**A.2. Composition and coherent functors.** The composite of two strict polynomial functors is a strict polynomial functor \([7]\). For completeness we lift the resulting bifunctor:

\[
\circ : \mathcal{P}_n \times \mathcal{P}_m \to \mathcal{P}_{nm}
\]

\[
(F, G) \mapsto F \circ G
\]

at the level of coherent functors.

Composition of functors is exact with respect to the first variable. Although the functor \( G \mapsto F \circ G \) is not additive for \( n > 1 \), it still has some exactness properties.

**Definition A.2.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories. For any short exact sequence

\[
0 \to A_1 \xrightarrow{\alpha} A \xrightarrow{\beta} A_2 \to 0
\]

in \( \mathcal{A} \), define \( \delta_1, \delta_2 : A \oplus A_1 \to A \) by

\[
\delta_1(a, a_1) = a + \alpha(a_1), \quad \delta_2(a, a_1) = a.
\]

A covariant functor \( T : \mathcal{A} \to \mathcal{B} \) preserves reflective coequalizers if for every short exact sequence \((10)\), the following sequence is exact:

\[
T(A \oplus A_1) \xrightarrow{T(\delta_1)-T(\delta_2)} T(A) \xrightarrow{T(\beta)} T(A_2) \to 0.
\]

Observe that when \( T \) is additive, then it preserves reflective coequalizers if, and only if, \( T \) is right exact. Let us observe also that if the exact sequence \((10)\) splits then the sequence \((11)\) is exact for any functor \( T \). If \( \mathcal{A} \) has enough projective objects, then any (possibly nonadditive) functor, from the category of projective objects in \( \mathcal{A} \) to the category \( \mathcal{B} \), has a unique (up to unique isomorphism) extension to a functor \( \mathcal{A} \to \mathcal{B} \) which preserves reflective coequalizers.

**Lemma A.2.2.** For any \( F \) in \( \mathcal{P}_n \), the functor

\[
\mathcal{P}_m \to \mathcal{P}_{nm}
\]

\[
G \mapsto F \circ G
\]

preserves reflective coequalizers and coreflective equalizers.

**Proof.** Take any short exact sequence in \( \mathcal{P}_m \):

\[
0 \to G_1 \xrightarrow{\alpha} G \xrightarrow{\beta} G_2 \to 0.
\]

After evaluating at \( V \in \mathcal{V} \), the corresponding sequence

\[
0 \to G_1(V) \to G(V) \to G_2(V) \to 0
\]

splits. Therefore for any \( F \), the sequence

\[
F(G(V) \oplus G_1(V)) \to F(G(V)) \to F(G_2(V)) \to 0
\]

is exact. This shows that \( F \circ (-) \) respects reflective coequalizers. Similarly for coreflective equalizers. \( \square \)
For a natural number $m$ and a group $G$, let $\mathcal{G}_m \lhd G$ be the wreath product, which by definition is the semi-direct product $G^m \rtimes \mathcal{G}_m$. For $M$ in $\mathcal{C}_m\mathcal{V}$ and $N$ in $G\mathcal{V}$, it acts on $M \otimes N^{\otimes m}$. In particular, for $G = \mathcal{G}_n$, let

$$M \cdot N := \text{Ind}^{\mathcal{G}_m \rtimes \mathcal{G}_n}_{\mathcal{G}_m \rtimes \mathcal{G}_n}(M \otimes N^{\otimes m})$$

It defines a functor:

$$\bullet : \mathcal{C}_m\mathcal{V} \times \mathcal{C}_n\mathcal{V} \to \mathcal{C}_m\mathcal{V}$$

**Proposition A.2.3.** There is a unique (up to isomorphism) bifunctor

$$\circ : \mathcal{C}_m \times \mathcal{C}_n \to \mathcal{C}_{mn}$$

with the following properties

(i) The functor $\circ$ is exact with respect of the first variable and preserves reflective coequalizers with respect to the second variable;

(ii) $h_M \circ h_N = h_{M \cdot N}$. We define another bifunctor

$$\bar{\circ} : \mathcal{C}_m \times \mathcal{C}_n \to \mathcal{C}_{mn}.$$ 

Because $\otimes$ is a symmetric monoidal structure on $\oplus_{d \geq 0} \mathcal{C}_d$, the functor $g^{\otimes m}$ has a natural action of $\mathcal{G}_m$ for $g$ in $\mathcal{C}_n$. We put (compare with [4]):

$$f \circ g := \langle f, g^{\otimes m} \rangle.$$ 

**Proposition A.2.4.** For all $f$ in $\mathcal{C}_m$ and $N$ in $\mathcal{C}_n\mathcal{V}$, there is a natural isomorphism:

$$f \circ h_N \cong f \circ h_N.$$ 

**Proof.** Since $f \circ g$ and $f \circ g$ are exact on $f$, it is enough to consider the case $f = h_M$. Then:

$$h_{M \cdot N}(X) = \text{Hom}_{\mathcal{C}_m}(\text{Ind}_{\mathcal{G}_m \rtimes \mathcal{G}_n}^{\mathcal{G}_m \rtimes \mathcal{G}_n}(M \otimes N^{\otimes m}), X)$$

$$= \text{Hom}_{\mathcal{C}_m}(M \otimes N^{\otimes m}, \text{Res}_{\mathcal{G}_m \rtimes \mathcal{G}_n}^{\mathcal{G}_m \rtimes \mathcal{G}_n}(X))$$

$$= \text{Hom}_{\mathcal{C}_m}(M, \text{Hom}(\mathcal{G}_m)^m(N^{\otimes m}, \text{Res}_{\mathcal{G}_n \rtimes \mathcal{G}_n}^{\mathcal{G}_n}(X)))$$

$$= h_M(h_{N^m})(X)$$

$$= (h_m \circ h_N)(X).$$

□

**Proposition A.2.5.** (i) For all $f$ in $\mathcal{C}_m$ and $g$ in $\mathcal{C}_n$, there is a natural isomorphism

$$j^*(f) \circ j^*(g) \cong j^*(f \circ g).$$

(ii) For all $F$ in $\mathcal{P}_m$ and $G$ in $\mathcal{P}_n$, there is a natural isomorphism

$$j_!(F) \circ j_!(G) \cong j_!(F \circ G).$$

(iii) for all $F$ in $\mathcal{P}_m$ and $G$ in $\mathcal{P}_n$, there is a natural isomorphism

$$j_*(F) \circ j_*(G) \cong j_*(F \circ G),$$

(iv) $t_M \circ t_N = t_{M \cdot N};$

**Proof.** (i) Since $j^*$ is right exact and $\circ$ respects reflective coequalizers, it suffices to consider the case when $f = h_M$ and $g = h_N$. Then we have:

$$j^*(f) \circ j^*(g) \cong \text{Hom}_{\mathcal{C}_m}(M, (\text{Hom}_{\mathcal{C}_n}(N, V^{\otimes n}))^{\otimes m})$$

$$= \text{Hom}_{\mathcal{C}_m}(M, \text{Hom}(\mathcal{G}_m)^m(N^{\otimes m}, \text{Res}_{\mathcal{G}_n \rtimes \mathcal{G}_n}^{\mathcal{G}_n}(V^{\otimes mn})))$$

By the previous computation the last group is isomorphic to $h_{M \cdot N}(V^{\otimes mn})$. 

□
(ii) It is enough to assume that $F$ and $G$ are projective generators: $F = \Gamma^u$, $G = \Gamma^v$. We set $M = \mathbb{K}[S_m/S_{\mu}]$ and $N = \mathbb{K}[S_n/S_{\nu}]$. Then we have $j_!(F) = h_M$ and $j_!(T) = h_N$. Therefore $j_!(F) \circ j_!(G) \cong h_{M \otimes N}$. On the other hand

$$F \circ G(V) = \Gamma^u \mu(\Gamma^v(V)) \text{Hom}_{\mathbb{E}_n}(M, (\text{Hom}_{\mathbb{E}_n}(N, V^\otimes n)) \otimes m)$$

$$= \text{Hom}_{\mathbb{E}_n}(M, \text{Hom}_{\mathbb{E}_n} \times \cdots \times \text{Hom}_{\mathbb{E}_n}(N^\otimes m, \text{Res}_{\mathbb{E}_n} \times \cdots \times \text{Res}_{\mathbb{E}_n}(V^\otimes mn)))$$

$$= \text{Hom}_{\mathbb{E}_n}(M \otimes N^\otimes m, \text{Res}_{\mathbb{E}_n} \times \cdots \times \text{Res}_{\mathbb{E}_n}(V^\otimes mn))$$

$$= \text{Hom}_{\mathbb{E}_n}(N \otimes M, V^\otimes mn)$$

It follows that: $j_!(F \circ G) = h_{M \otimes N}$.

\[\square\]

References

[1] M. Auslander. Coherent functors. Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) Springer (1966), 189–231.

[2] M. Chalupnik. Extensions of strict polynomial functors. Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 5, 773–792.

[3] V. Franjou & T. Pirashvili. Comparison of abelian categories recollements. Doc. Math. 9 (2004), 41–56.

[4] B. Fresse. On the homotopy of simplicial algebras over an operad. Trans. Amer. Math. Soc. 352 (2000), no. 9, 4113–4141.

[5] V. Franjou, J. Lannes & L. Schwartz. Autour de la cohomologie de MacLane des corps finis. Invent. Math. 115 (1994), no. 3, 513–538.

[6] V. Franjou, E. Friedlander, A. Scorichenko & A. Suslin. General linear and functor cohomology over finite fields. Ann. of Math. (2) 150 (1999), 663–728.

[7] E. Friedlander & A. Suslin. Cohomology of finite group schemes over a field. Invent. Math. 127 (1997), no. 2, 209–270.

[8] J.A. Green. On three functors of M. Auslander’s. Comm. Algebra 15 (1987), 241–277.

[9] R. Harthorne. Coherent functors. Adv. in Math. 140 (1998), no. 1, 44–94.

[10] N. J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. II. K-Theory 8 (1994), no. 4, 395–428.

[11] N. J. Kuhn. A stratification of generic representation theory and generalized Schur algebras. K-Theory 26 (2002), no. 1, 15–49.

[12] S. MacLane. Categories for the Working Mathematician. Graduate Texts in Mathematics 5, Springer, New York-Berlin, 1971. ix+262 pp.

[13] C. Nastasescu & B. Torrecillas. Co localization on Grothendieck categories with applications to coalgebras. J. Algebra, 185 (1996), 108–124.

[14] T. Pirashvili. Introduction to functor cohomology. In Rational Representations, the Steenrod Algebra, and Functor Homology, Panor. Synthèses 16, Soc. Math. France, Paris, 2003.

[15] I. Schur. Thesis (1903). In Gesammelte Abhandlungen, Band I, 5-76. Springer, 1973.