STRUCTURAL STABILITY OF INTERIOR SUBSONIC STEADY-STATES TO HYDRODYNAMIC MODEL FOR SEMICONDUCTORS WITH SONIC BOUNDARY

YUE-HONG FENG\textsuperscript{1,4}, HAIFENG HU\textsuperscript{2,4} AND MING MEI\textsuperscript{3,4*}

\textsuperscript{1}College of Mathematics, Faculty of Science, Beijing University of Technology, Beijing 100022, China
\textsuperscript{2}School of Science, Changchun University, Changchun 130022, China
\textsuperscript{3}Department of Mathematics, Champlain College Saint-Lambert, Quebec, J4P 3P2, Canada
\textsuperscript{4}Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada

*Corresponding author. E-mail: ming.mei@mcgill.ca
Contributing authors. E-mails: fyh@bjut.edu.cn; huhf@ccu.edu.cn

\textsuperscript{†}These authors contributed equally to this work.

Abstract. For the stationary hydrodynamic model for semiconductors with sonic boundary, represented by Euler-Poisson equations, it possesses the various physical solutions including interior subsonic solutions/interior supersonic solutions/shock transonic solutions/$C^1$-smooth transonic solutions. However, the structural stability for these physical solutions is challenging and has remained open as we know. In this paper, we investigate the structural stability of interior subsonic solutions when the doping profiles are restricted in the subsonic region. The main result is proved by using the local (weighted) singularity analysis and the monotonicity argument. Both the result itself and techniques developed here will give us some truly enlightening insights into our follow-up study on the structural stability of the remaining types of solutions.

Keywords. Euler-Poisson equations, semiconductor effect, sonic boundary, interior subsonic solutions, structural stability.

AMS Subject Classification. 35B35, 35J70, 35L65, 35Q35

1. Introduction

The hydrodynamic model was first derived by Bløtekjær [2] for electrons in a semiconductor. After appropriate simplification the one-dimensional time-dependent system in the isentropic case reads:

\begin{equation}
\begin{cases}
    n_t + (nu)_x = 0, \\
    (nu)_t + (nu^2 + p(n))_x = nE - \frac{nu}{\tau}, \\
    E_x = n - b(x),
\end{cases}
\end{equation}

where $n(x,t)$, $u(x,t)$ and $E(x,t)$ denote the electron density, velocity, and electric field respectively. The given function $p = p(n)$ is the pressure-density relation on which a commonly used
hypothesis is
\[ p(n) = T n^\gamma, \]
where \( T > 0 \) is Boltzmann’s constant and \( \gamma \geq 1 \) is the adiabatic exponent. The constant parameter \( \tau > 0 \) is the momentum relaxation time. The given background density \( b(x) > 0 \) is called the doping profile. The hydrodynamic model (1.1) is also called Euler-Poisson equations with semiconductor effect. For more details we refer to treatises [18, 25] and references therein.

In this paper, the main focus is on the isothermal steady-state flows satisfying equations
\[
\begin{cases}
J = \text{constant}, \\
\left( \frac{J^2}{n} + p(n) \right)_x = nE - \frac{J}{\tau}, \\
E_x = n - b(x),
\end{cases}
\]
where \( J = nu \) stands for the current density, and \( p(n) = Tn \) corresponds to the isothermal ansatz. By the terminology from gas dynamics, we call \( c := \sqrt{P'(n)} = \sqrt{T} > 0 \) the speed of sound. The flow is referred to as subsonic, sonic or supersonic provided the velocity satisfies
\[ u < c, \quad u = c \quad \text{or} \quad u > c, \]
respectively.

For convenience of notation, we introduce
\[ \alpha = \frac{1}{\tau}, \]
the reciprocal of the momentum relaxation time.

Without loss of generality, we set
\[ T = 1 \text{ and } J = 1, \]
thus the system (1.2) is equivalently reduced to the system
\[
\begin{cases}
\left( 1 - \frac{1}{n^2} \right) n_x = nE - \alpha, \\
E_x = n - b(x).
\end{cases}
\]

From (1.3) and (1.5), it is easy to see that the flow is subsonic if \( n > 1 \), sonic if \( n = 1 \), or supersonic if \( 0 < n < 1 \). By virtue of (1.4), we call the system (1.6) the Euler-Poisson equations with the semiconductor effect if \( \alpha > 0 \), and without the semiconductor effect if \( \alpha = 0 \), respectively. Throughout this paper, we are interested in the system (1.6) in the open interval \((0, 1)\), which is subjected to the sonic boundary condition
\[ n(0) = n(1) = 1. \]

We also assume that the doping profile \( b(x) \) is of class \( C[0, 1] \), satisfying the subsonic condition \( b(x) > 1 \) on \([0, 1]\). For simplicity of notation, its infimum and supremum over \([0, 1]\) is denoted by
\[
\bar{b} := \inf_{x \in [0,1]} b(x) \quad \text{and} \quad \bar{b} := \sup_{x \in [0,1]} b(x),
\]
respectively.

Over the past three decades, major advances in the mathematical theory of steady-state Euler-Poisson equations with/without the semiconductor effect have been made by many authors. In what follows, we just list several results which are closely linked to the present paper.
For the purely subsonic steady-state flows, in 1990, Degond et al. [7] first proved the existence of the subsonic solution to the one-dimensional steady-state Euler-Poisson with the semiconductor effect when its boundary states belongs to the subsonic region. Subsequently, Degond et al. [8] further showed the existence and local uniqueness of irrotational subsonic flows to the three-dimensional steady-state semiconductor hydrodynamic model under a smallness assumptions on the data. Along this line of research, the steady-state subsonic flows with and without the semiconductor effect were investigated in various physical boundary conditions and different dimensions [9, 13, 20, 3]. As for the purely supersonic steady-state flows, Peng et al. [21] established the existence and uniqueness of the supersonic solutions with the semiconductor effect, which correspond to a large current density.

Note that the system (1.2) or (1.6) will be degenerate at the sonic state, thus the study on the transonic solutions and various steady states satisfying the sonic boundary condition becomes very difficult. Ascher et al. [1] first examined the existence of the transonic solution to the one-dimensional isentropic Euler-Poisson equations without and with the semiconductor effect when the doping profile is a supersonic constant, and then Rosini [22] extended this work to the non-isentropic case by the analysis of phase plane. When the doping profile is non-constant, Gamba [11, 12] investigated the one-dimensional and two-dimensional transonic solutions with shocks, respectively. However, these transonic solutions yield boundary layers because they are constructed as the limits of vanishing viscosity. Luo et al. [17, 16] further considered the one-dimensional Euler-Poisson equations without the semiconductor effect, under the restriction that boundary data are far from the sonic state and the doping profile is either a subsonic constant or a supersonic constant, a comprehensive analysis on the structure and classification of steady states was carried out in [17] by using the analysis of phase plane. Meanwhile, both structural and dynamical stability of steady transonic shock solutions was obtained in [16].

What if the sonic state appears in the solutions? As we have seen, all the existing works introduced above cannot answer this question. Even the works regarding transonic shocks cannot radically answer it either because the two different phase states are connected by the jump of shocks satisfying the Rankine-Hugoniot condition and entropy condition, avoiding the degeneracy caused by the sonic state. Recently, Li et al. [14, 15] systematically explored the critical case, that is, the one-dimensional semiconductor Euler-Poisson equations with the sonic boundary condition. The existence, nonexistence and classification of all types of physical steady states to this critical boundary-value problem was obtained for the subsonic doping profile in [14] and supersonic doping profile in [15]. More precisely, in [14], the authors proved that the critical boundary-value problem admits a unique subsonic solution, at least one supersonic solution, infinitely many transonic shocks if \( \alpha \ll 1 \), and infinitely many transonic \( C^1 \)-smooth solutions if \( \alpha \gg 1 \); in [15], the authors showed the nonexistence of all types of physical steady states to the critical boundary-value problem assuming that the doping profile is small enough and \( \alpha \gg 1 \), and they also discussed the existence of supersonic and transonic shock solutions under the hypothesis that the doping profile is close to the sonic state and \( \alpha \ll 1 \). Inspired by the groundbreaking works [14, 15], there is a series of interesting generalizations into the transonic doping profile case in [4], the case of transonic \( C^\infty \)-smooth steady states in [24], the multi-dimensional cases in [5, 6], and even the bipolar case [19].
Compared with the existence theory in the critical case, there are rather few works on the study of stability (both structural and dynamical stability) in the critical case due to the interior or boundary degeneracy. To the best of our knowledge, Feng et al. [10] first demonstrated the structural stability of the \( C^1 \)-smooth transonic steady states with respect to the small perturbation of both the supersonic constant doping profiles and non-degenerate boundary data. In [10], for supersonic doping profile, the authors also discussed the structural stability and linear dynamic instability of the transonic steady states with the non-degenerate boundary data under some appropriate hypotheses. Thus far, the most difficult part of the critical case, namely, the problem about the structural and dynamical stability of all types of physical solutions with the sonic boundary condition for the non-constant subsonic doping profile (for the existence theory, see [14]), is still open. To thoroughly solve this problem is full of challenges, owing to the boundary degeneracy. However, in the present paper, we intend to shed new light on this problem.

The purpose of this paper is to show that interior subsonic solutions to the system (1.6) with the sonic boundary condition (1.7) are structurally stable if we propose a monotonicity restriction on subsonic doping profiles (see Theorem 2.1).

This paper is organized as follows. Some necessary preliminaries and the main result are stated in Section 2. The proof of the main result, Theorem 2.1, is given in Section 3.

2. Preliminaries and the main result

In this section we shall present the main result. Before proceeding, we first give the important preliminaries from the foregoing research [14]. First of all, we recall the definition of the interior subsonic solution.

**Definition 2.1.** We say a pair of functions \((n, E)(x)\) is an interior subsonic solution of the boundary value problem (1.6)\(^{[1]}\)\( (1.7)\) provided (i) \((n−1)^2 \in H_0^1(0,1)\), (ii) \(n(x) > 1\), for all \(x \in (0, 1)\), (iii) \(n(0) = n(1) = 1\), (iv) the following equality holds for all test functions \(\varphi \in H_0^1(0,1)\),

\[
\int_0^1 \left( \frac{1}{n} - \frac{1}{n^3} \right) n_x \varphi_x dx + \alpha \int_0^1 \frac{\varphi_x}{n} dx + \int_0^1 (n-b) \varphi dx = 0,
\]

and (v) \(E(x)\) is given by

\[
E(x) = \alpha + \int_0^x (n(y) - b(y)) dy.
\]

In addition, we continue to recall the existence and uniqueness of interior subsonic solutions, which is excerpted from the first part of Theorem 1.3 in [14].

**Proposition 2.1** (Existence theory in [14]). Suppose that the doping profile \(b \in L^\infty(0,1)\) is subsonic such that \(b > 1\). Then for any \(\alpha \in [0, \infty)\) the boundary value problem (1.6)\( (1.7)\) admits a unique interior subsonic solution \((n, E) \in C^2[0, 1] \times H^1(0,1)\) satisfying the boundedness

\[
1 + m \sin(\pi x) \leq n(x) \leq \bar{b}, \quad x \in [0, 1],
\]

and the boundary behavior at endpoints

\[
E(0) = \alpha, \quad E(1) < \alpha,
\]
globally structural stability over the entire interval $[0, 1]$. Due to the boundary degeneracy of interior subsonic solutions, the study of their structure is sophisticated and challenging. Therefore, we shall have to divide the whole interval $[0, 1]$ into three domains as follows:

$$[0, 1] = [0, \delta] \cup [\delta, 1 - \delta] \cup (1 - \delta, \delta],$$

where the intrinsic segmentation constant $\delta > 0$ would be appropriately determined (see Lemma 3.6), and we will also have to establish structural stability estimates separately on their respective domains in the following order: (i) near the left endpoint $x = 0$; (ii) near the right endpoint $x = 1$; (iii) on the middle domain. This strategy is feasible because we have discovered the following facts:

1. the local singularity analysis reveals that the plausible singularity at the left endpoint $x = 0$ is removable (see Lemma 3.2); based on this, we are able to establish the local structural stability estimate on an inherent neighborhood $[0, \delta_0)$ by the monotonicity argument. The main point is that both the radius $\delta_0$ and the positive estimate constant are independent of $\|b_1 - b_2\|_{C[0, 1]}$ (see Lemma 3.3). This is the reason why this type of neighborhood is referred to as “to be intrinsic or inherent”. This sort of tacit convention will be used throughout the present paper.

2. the local weighted singularity analysis discloses that the genuine singularity at the right endpoint $x = 1$ can be well controlled by the $(1 - x)^{\frac{1}{2}}$-weight (see Lemma 3.4); thus, the monotonicity argument further ensures that the local weighted structural stability holds on an intrinsic neighborhood $(1 - \delta_1, 1]$ (see Lemma 3.5).

3. the remaining part constitutes the middle domain, which is regular as to the structural stability (see Lemma 3.6).
It is worth mentioning that the monotonicity argument has been playing a crucial role in establishing structural stability estimates near both endpoints. The principle behind the monotonicity argument is given by Lemma 3.1. From a technical point of view, the monotonicity argument is useful, but only at the cost of adding an extra restriction \( b_1(x) \geq b_2(x) \) on \([0, 1]\). How to get rid of this restriction is a tough question, and we will explore it in the future study.

3. Proof of Theorem 2.1

This section is devoted to proving our main result. In order to make the line of reasoning accessible to the reader, the proof will be divided into a sequence of lemmas.

We let \((n_i, E_i)(x)\) denote the interior subsonic solution corresponding to the subsonic doping profile \(b_i(x) > 1\), satisfying the sonic boundary value problem

\[
\begin{cases}
\left(1 - \frac{1}{n_i^2}\right)n_{ix} = n_iE_i - \alpha, \\
E_{ix} = n_i - b_i(x), \quad x \in (0, 1), \quad \text{for } i = 1, 2, \text{ respectively.} \\
n_i(0) = n_i(1) = 1,
\end{cases}
\]

(3.1)

First of all, we adapt the comparison principle in [14](Lemma 2.2, P4773) for use with two doping profiles and their corresponding interior subsonic solutions, which is the basis of the monotonicity argument in studying the structural stability near endpoints.

**Lemma 3.1 (Comparison principle).** Let the doping profiles \(b_1, b_2 \in C[0, 1]\). If \(b_1(x) \geq b_2(x) > 1\) on \([0, 1]\). Then

\[
n_1(x) \geq n_2(x), \text{ on } [0, 1].
\]

**Proof.** According to the relevant arguments from [14](Equation (17), P4773), for \(i = 1, 2\), since \((n_i, E_i)\) is the interior subsonic solution, thereby having the approximate solution sequence \(\{n_{ij}\}_{0 < j < 1} \subset C^1[0, 1]\) satisfying the weak form

\[
\int_0^1 A(n_{ij}, n_{ijx})\varphi_x dx + \int_0^1 (n_{ij} - b_i)\varphi dx = 0, \quad \forall \varphi \in H_0^1(0, 1),
\]

(3.3)

where

\[
A(z, p) := \left(1 - \frac{j^2}{z^2}\right)p + \alpha \frac{j}{z},
\]

Subtracting \((3.3)\big|_{i=1}\) from \((3.3)\big|_{i=2}\), for all nonnegative test functions \(\varphi \in H_0^1(0, 1)\), we have

\[
\int_0^1 (A(n_{2j}, n_{2jx}) - A(n_{1j}, n_{1jx}))\varphi_x dx + \int_0^1 (n_{2j} - n_{1j})\varphi dx = \int_0^1 (b_2 - b_1)\varphi dx \leq 0,
\]

(3.4)

where we have used the assumption that \(b_1(x) \geq b_2(x)\) on \([0, 1]\) in the last inequality. This is exactly the crucial Equation (19) in [14], the same result therefore applies to (3.4) provided we simply imitate the remaining arguments in Lemma 2.2 of [14]. That is,

\[
n_{1j}(x) \geq n_{2j}(x), \text{ on } [0, 1], \text{ for } 0 < j < 1.
\]

(3.5)

Now the monotonicity relation (3.1) follows after a passage to the limit as \(j \to 1^-\) on both sides of the inequality (3.5).
In addition, for \( i = 1, 2 \), we set about analyzing the boundary behavior of the first-order derivative of \( n_i(x) \) at the left endpoint \( x = 0 \). It seems plausible that the singularity should have appeared there, as a matter of fact this “fake” singularity at \( x = 0 \) is removable because of \( E_i(0) = \alpha \).

**Lemma 3.2.** Suppose that \( b_i, i = 1, 2 \) satisfy the same conditions in Lemma 3.1, and \( \alpha \geq 2\sqrt{2} \max\{\sqrt{b_1(0)} - 1, \sqrt{b_2(0)} - 1\} \). Then

\[
\lim_{x \to 0^+} n_{ix}(x) = \frac{1}{4} \left( \alpha - \sqrt{\alpha^2 - 8(b_i(0) - 1)} \right) =: A_i > 0, \quad i = 1, 2.
\]

**Proof.** In much the same way as in [14](Theorem 5.6, P4802), owing to \( n_i(0) = 1 \) and \( E_i(0) = \alpha \), it is easy to see that \( \lim_{x \to 0^+} n_{ix}(x) \) exists by the monotone convergence argument. Then from the first equation of (3.1), we have

\[
n_{ix} = \frac{E_i n_i^2}{n_i + 1} + \frac{(E_i - \alpha) n_i^2}{(n_i - 1)(n_i + 1)} \quad \text{in } (0, 1).
\]

Noting that \( n_i(0) = 1 \) and \( E_i(0) = \alpha \), it follows from the L’Hospital Rule that

\[
A_i = \lim_{x \to 0^+} n_{ix}(x) = \lim_{x \to 0^+} \frac{E_i n_i^2}{n_i + 1} + \lim_{x \to 0^+} \frac{(E_i - \alpha) n_i^2}{(n_i - 1)(n_i + 1)} = \frac{\alpha}{2} + \frac{1}{2} \lim_{x \to 0^+} \frac{E_{ix}}{n_{ix}^2}
\]

\[
= \frac{\alpha}{2} + \frac{1}{2} \lim_{x \to 0^+} \frac{n_i(x) - b_i(x)}{n_{ix}} = \frac{\alpha}{2} + \frac{1}{2A_i} - b_i(0),
\]

which in turn implies that

\[
A_i = \frac{1}{4} \left( \alpha - \sqrt{\alpha^2 - 8(b_i(0) - 1)} \right) = \frac{2(b_i(0) - 1)}{\alpha + \sqrt{\alpha^2 - 8(b_i(0) - 1)}} = O\left(\frac{1}{\alpha}\right),
\]

or

\[
A_i = \frac{1}{4} \left( \alpha + \sqrt{\alpha^2 - 8(b_i(0) - 1)} \right) = O(\alpha).
\]

According to the local singularity analysis in [14](Lemma 5.3, P4796), we know that in a small neighborhood of \( x = 0 \), the drastic change of the density component \( n_i(x) \) of the interior subsonic solution is impossible when \( \alpha \) is suitably large. Therefore, we have to choose the former root as the limit value of \( \lim_{x \to 0^+} n_{ix}(x) \), and the latter one is the extraneous root. \( \square \)

Based on Proposition 2.1 and Lemmas 3.1~3.2, we are now preparing to establish the local structural stability of interior subsonic solutions to the boundary value problem (1.6)&(1.7) on an intrinsic neighborhood of the left endpoint \( x = 0 \).

**Lemma 3.3** (Local structural stability estimate near \( x = 0 \)). Under the same conditions in Lemma 3.2. There exist two positive constants \( \delta_0 \in (0, \frac{1}{2}] \) and \( C > 0 \) independent of \( \|b_1 - b_2\|_{C[0,1]} \) such that

\[
\|n_1 - n_2\|_{C^1[0,\delta_0]} + \|E_1 - E_2\|_{C^1[0,\delta_0]} \leq C\|b_1 - b_2\|_{C[0,1]}.
\]
Proof. Firstly, in light of Lemma 3.1, it is clear that the following monotonicity relation holds,

\[ \frac{n_1^3}{n_1 + 1} \geq \frac{n_2^3}{n_2 + 1}, \quad \forall x \in [0, 1]. \tag{3.8} \]

Next, for simplicity, we set \( \tilde{E}_i := E_i - \frac{\alpha_i}{n_i} \). Multiplying Equation (3.1) \(_1\) by \( \frac{n_i^2}{n_i^2 - 1} \), we have

\[ n_{ix} = \frac{\tilde{E}_i n_i^3}{n_i^2 - 1}, \quad i = 1, 2. \tag{3.9} \]

Taking the difference of Equations (3.9)\(_{i=1}\) and (3.9)\(_{i=2}\), near \( x = 0 \), we compute together with the monotonicity relation (3.8) that

\[ (n_1 - n_2)_x = \frac{\tilde{E}_1 n_1^3}{n_1^2 - 1} - \frac{\tilde{E}_2 n_2^3}{n_2^2 - 1} \]

\[ = \frac{n_1^3}{n_1 + 1} \frac{\tilde{E}_1}{n_1 - 1} - \frac{n_2^3}{n_2 + 1} \frac{\tilde{E}_1}{n_1 - 1} + \frac{n_2^3}{n_2 + 1} \frac{\tilde{E}_1}{n_1 - 1} - \frac{n_2^2}{n_2 + 1} \frac{\tilde{E}_2}{n_2 - 1} \]

\[ = \frac{\tilde{E}_1}{n_1 - 1} \left( \frac{n_1^3}{n_1 + 1} - \frac{n_2^3}{n_2 + 1} \right) + \frac{n_2^3}{n_2 + 1} \left( \frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1} \right) \]

\[ \leq M_0 \alpha \left( \frac{n_1^3}{n_1 + 1} - \frac{n_2^3}{n_2 + 1} \right) + \frac{n_2^3}{n_2 + 1} M_0 \|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0), \]

where we have used the fact that there exist two positive constants \( \delta_0 \in (0, \frac{1}{2}) \) and \( M_0 > 0 \) independent of \( \|b_1 - b_2\|_{C[0,1]} \) such that

\[ \frac{\tilde{E}_1}{n_1 - 1} (x) \leq M_0 \alpha, \quad \text{and} \quad \left( \frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1} \right) (x) \leq M_0 \|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0). \tag{3.11} \]

To prove that the crucial estimate (3.11) on a certain intrinsic neighborhood \([0, \delta_0)\) holds, we assume for the sake of contradiction that for any \( \delta \in (0, \frac{1}{2}) \) and \( M > 0 \), there exists \( x_\delta \in [0, \delta) \) such that

\[ \frac{\tilde{E}_1}{n_1 - 1} (x_\delta) > M \alpha, \quad \text{or} \quad \left( \frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1} \right) (x_\delta) > M \|b_1 - b_2\|_{C[0,1]} \tag{3.12} \]

Particularly, we take \( \delta = \frac{1}{k}, k = 3, 4, 5, \ldots \), for any \( M > 0 \), there is \( x_k \in [0, \frac{1}{k}) \) such that

\[ \frac{\tilde{E}_1}{n_1 - 1} (x_k) > M \alpha, \quad \text{or} \quad \left( \frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1} \right) (x_k) > M \|b_1 - b_2\|_{C[0,1]}, \]

which implies that

\[ \lim_{x_k \to 0^+} \frac{\tilde{E}_1}{n_1 - 1} (x_k) \geq M \alpha, \tag{3.13} \]

or

\[ \lim_{x_k \to 0^+} \left( \frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1} \right) (x_k) \geq M \|b_1 - b_2\|_{C[0,1]} \tag{3.14} \]
Combining the boundary behavior (2.4), the L’Hospital Rule, Equation (3.1) and Lemma 3.2, we calculate

\[
\lim_{x \to 0^+} \frac{\tilde{E}_i}{n_i - 1} (x) = \lim_{x \to 0^+} \frac{E_i - \frac{\alpha}{n_i}(x)}{n_i - 1} = \lim_{x \to 0^+} \frac{E_i(x) - E_i(0)}{n_i(x) - 1} + \lim_{x \to 0^+} \frac{\alpha}{n_i(x)} = \lim_{x \to 0^+} \frac{n_i - b_i}{n_i(x)} + \alpha\]

(3.15)

and

\[
\lim_{x \to 0^+} \left( \frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1} \right) (x) = \frac{b_2(0) - 1}{A_2} - \frac{b_1(0) - 1}{A_1} = \frac{1}{2} \left( \sqrt{\alpha^2 - 8(b_2(0) - 1)} - \sqrt{\alpha^2 - 8(b_1(0) - 1)} \right) \leq \tilde{C}_0 \| b_1 - b_2 \|_{C[0,1]},
\]

(3.16)

where \( \eta \in (b_2(0), b_1(0)) \). Furthermore, we note that the constant \( M \) in (3.13) and (3.14) can be chosen arbitrarily. Consequently, if we take \( M = 2 \) in (3.13), together with (3.15), we obtain the contradiction that \( 2\alpha < \alpha \); if we take \( M = 2\tilde{C}_0 \) in (3.14), combined with (3.16), we have the contradiction \( 2 \leq 1 \).

Based on the local estimate (3.10), we continue establishing the structural stability locally on the intrinsic neighborhood \([0, \delta_0)\). To this end, we multiply through the inequality (3.10) by \( n_1 - n_2 \) and calculate

\[
\frac{d}{dx}(n_1 - n_2)^2(x) \leq C(n_1 - n_2)^2(x) + C\|b_1 - b_2\|_{C[0,1]}^2, \quad x \in [0, \delta_0),
\]

(3.17)

where we have used Lemma 3.1 and Cauchy’s inequality. By Gronwall’s inequality and the sonic boundary condition \( n_1(0) = n_2(0) = 1 \), we get

\[
(n_1 - n_2)^2(x) \leq C\|b_1 - b_2\|_{C[0,1]}^2, \quad x \in [0, \delta_0),
\]

(3.18)

which in turn implies that

\[
|n_1 - n_2| + |(n_1 - n_2)_x| \leq C\|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0).
\]

(3.19)

with the aid of the foregoing local estimate (3.10) again.

Finally, from Equation (2.2) in Definition 2.1, we have

\[
E_i(x) = \alpha + \int_0^x (n_i(y) - b_i(y)) dy, \quad i = 1, 2.
\]

(3.20)
Taking the difference of \((3.20)|_{i=1}\) and \((3.20)|_{i=2}\), we compute that

\[
|E_1 - E_2|(x) \leq \int_0^x |n_1 - n_2|(y)\,dy + \int_0^x |b_1 - b_2|(y)\,dy \\
\leq C\|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0),
\]

and

\[
|(E_1 - E_2)_x|(x) = |n_1 - n_2 - (b_1 - b_2)|(x) \\
\leq C\|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0).
\]

Hence, the local structural stability estimate (3.7) follows immediately from Equations (3.19), (3.21) and (3.22).

We now turn to analyzing the refined boundary behavior of the first-order derivative of \(n_i(x)\) at the right endpoint \(x = 1\). From the boundary estimate displayed in the second line of (2.5), we know that \(\lim_{x \to 1^-} n_i(x) = -\infty\). This means the “genuine” singularity will occur at the right endpoint \(x = 1\). Inspired by (2.5), we are able to implement the local “weighted” singularity analysis. The result is summarized as follows.

**Lemma 3.4.** Assume that \(b_i, i = 1, 2\) satisfy the same conditions in Lemma 3.1. Then

\[
\lim_{x \to 1^-} (1 - x)^{\frac{1}{2}} n_i(x) = -\frac{1}{2} \sqrt{\int_0^1 (b_i - n_i)\,dx} =: B_i, \quad i = 1, 2.
\]

**Proof.** For \(i = 1, 2\), from the boundary estimate (2.5), we know that the coefficient \(1 - \frac{1}{n_i^2}\) in the degenerate principal part of Equation (3.1)_1 is comparable to \((1 - x)^{\frac{1}{2}}\) near the right endpoint \(x = 1\). Thus the regularity theory of boundary-degenerate elliptic equations in one dimension (e.g. [23]) ensures that \((1 - x)^{\frac{1}{2}} n_i(x)\) is continuous up to the right endpoint \(x = 1\).

We now proceed to calculate the exact limit value of \(\lim_{x \to 1^-} (1 - x)^{\frac{1}{2}} n_i(x)\). For convenience, we set

\[
B_i := \lim_{x \to 1^-} (1 - x)^{\frac{1}{2}} n_i(x).
\]

Thereupon, multiplying through Equation (3.1)_1 by \((1 - x)^{\frac{1}{2}} n_i^n / n_i - 1\), we have

\[
(1 - x)^{\frac{1}{2}} n_i = \frac{n_i^2}{n_i + 1} \left( E_i - \frac{\alpha}{n_i} \right) \frac{(1 - x)^{\frac{1}{2}}}{n_i - 1}.
\]
By virtue of the sonic boundary condition $n_i(1) = 1$, the known boundary behavior (2.4) and the L’Hospital Rule, we compute

\begin{equation}
(3.24) \quad B_i = \lim_{x \to 1^-} (1 - x)^{1/2} n_{ix}
\end{equation}

\begin{align*}
&= \lim_{x \to 1^-} \frac{n_i^3}{n_i + 1} \lim_{x \to 1^-} (E_i - \alpha) \lim_{x \to 1^-} \frac{(1 - x)^{1/2}}{n_i - 1} \\
&= \frac{1}{2} (E_i(1) - \alpha) \lim_{x \to 1^-} \frac{-1}{2} (1 - x)^{-1/2} n_{ix} \\
&= \frac{1}{4} (\alpha - E_i(1)) \lim_{x \to 1^-} \frac{1}{n_{ix} (1 - x)^{1/2}} \\
&= \frac{1}{4} (E_i(0) - E_i(1)) \frac{1}{B_i},
\end{align*}

which implies from Equation (3.1) that

$$
B_i = -\frac{1}{2} \sqrt{E_i(0) - E_i(1)} = -\frac{1}{2} \sqrt{\int_0^1 (b_i - n_i) dx} < 0,
$$

where the boundary estimate (2.5) has been employed to uniquely determine the value of $B_i$, which is strictly negative. \qed

Proposition 2.1 alongside Lemmas 3.1 and 3.4 now enable us to demonstrate the local weighted structural stability of interior subsonic solutions to the boundary value problem (1.6)&(1.7) on an intrinsic neighborhood of the right endpoint $x = 1$.

**Lemma 3.5** (Local weighted structural stability estimate near $x = 1$). Under the same conditions in Lemma 3.2. There exist two positive constants $\delta_1 \in (0, \frac{1}{2})$ and $C > 0$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that

\begin{equation}
(3.25) \quad \left\| (1 - x)^{1/2} (n_1 - n_2) \right\|_{C(1-\delta_1,1]} + \left\| (1 - x)^{1/2} (n_{1x} - n_{2x}) \right\|_{C(1-\delta_1,1]} + \|E_1 - E_2\|_{C^1(1-\delta_1,1]} \leq C \|b_1 - b_2\|_{C[0,1]}.
\end{equation}

**Proof.** For $i = 1, 2$, from (2.5), we have known that $\frac{n_i - 1}{(1-x)^{1/2}}$ possesses the uniform positive upper and lower bounds near $x = 1$, and so does its reciprocal $\frac{(1-x)^{1/2}}{n_i - 1}$. This property will be used repeatedly hereafter.

Owing to the fact that $n_{ix}(x)$ has the genuine singularity at $x = 1$, we are compelled to establish the structural stability estimate near $x = 1$ only in the weighted manner as follows.
Firstly, multiplying through Equation (3.1) by \((1 - x)^{\frac{1}{2}}\frac{n_i^2}{n_i^2 - 1}\) and taking the difference of resultant equations for \(i = 1, 2\), we calculate that

\[
(3.26) \quad (1 - x)^{\frac{1}{2}}(n_{1x} - n_{2x})
\]

\[
= \frac{n_i^2}{n_i + 1}(n_iE_1 - \alpha)(1 - x)^{\frac{1}{2}} - \frac{n_i^2}{n_i + 1}(n_2E_2 - \alpha)(1 - x)^{\frac{1}{2}}
\]

\[
= \frac{n_i^2}{n_i + 1}(n_iE_1 - \alpha)\left(\frac{(1 - x)^{\frac{1}{2}}}{n_i - 1} - \frac{(1 - x)^{\frac{1}{2}}}{n_i - 1}\right)
\]

\[
+ \left(\frac{n_i^2}{n_i + 1}(n_iE_1 - \alpha) - \frac{n_i^2}{n_i + 1}(n_2E_2 - \alpha)\right)\frac{(1 - x)^{\frac{1}{2}}}{n_2 - 1}
\]

\[
h(n_i, E_i) = (1 - x)^{\frac{1}{2}}(1 - x)^{\frac{1}{2}} \frac{n_2 - n_i}{n_2 - 1} \frac{(1 - x)^{\frac{1}{2}}}{n_2 - 1} + (h(n_1, E_1) - h(n_2, E_2))\frac{(1 - x)^{\frac{1}{2}}}{n_2 - 1}
\]

\[
=: I_1 + I_2,
\]

where

\[
h(n_i, E_i) := \frac{n_i^2}{n_i + 1}(n_iE_i - \alpha), \quad i = 1, 2.
\]

In what follows, near \(x = 1\), we shall estimate \(I_1\) and \(I_2\), respectively. But first, we claim that the following estimates

\[
|E_i(x)| \leq \alpha + 2\bar{b}_i, \quad x \in [0, 1], \quad i = 1, 2,
\]

\[
|E_1(1) - E_2(1)| \leq C|b_1 - b_2|_{C[0,1]},
\]

hold, where the estimate constant \(C > 0\) is independent of \(\|b_1 - b_2\|_{C[0,1]}\), and the proof of which is deferred to Lemma 3.7 at the end of this paper.

As for \(I_1\), it is clear from (2.5) and (3.27) that

\[
|I_1| = \left|h(n_1, E_1)(1 - x)^{\frac{1}{2}}(1 - x)^{\frac{1}{2}} \frac{n_2 - n_i}{n_2 - 1} \frac{(1 - x)^{\frac{1}{2}}}{n_2 - 1}\right| \leq C\frac{|n_1 - n_2|}{(1 - x)^{\frac{1}{2}}}
\]

However, as far as \(I_2\) is concerned, the situation becomes more complicated because of the factor \(h(n_1, E_1) - h(n_2, E_2)\). Next, we are taking it step by step. Precisely, a straightforward computation gives

\[
(3.30) \quad h(n_1, E_1) - h(n_2, E_2) = \frac{n_i^2}{n_i + 1}(n_iE_1 - \alpha) - \frac{n_i^2}{n_i + 1}(n_2E_2 - \alpha)
\]

\[
= \alpha \left(\frac{n_i^2}{n_i + 1} - \frac{n_i^2}{n_i + 1}\right) + \left(\frac{n_i^3E_1}{n_i + 1} - \frac{n_i^3E_2}{n_i + 1}\right)
\]

\[
=: R_1 + R_2.
\]

From (2.3) and Lemma 3.1, we know that

\[
1 \leq 1 + m(\alpha, \bar{b}_2) \sin(\pi x) \leq n_2(x) \leq n_1(x) \leq \bar{b}_1, \quad x \in [0, 1].
\]

Consequently, it follows from the mean-value theorem of differentials that

\[
|R_1| = \alpha \left(\frac{n_i^2}{n_i + 1} - \frac{n_i^2}{n_i + 1}\right) \leq C|n_1 - n_2| \leq C\frac{|n_1 - n_2|x}{(1 - x)^{\frac{1}{2}}}.
\]
We now turn to estimating $R_2$ near $x = 1$. Combining (3.31), (3.27), (3.28), the mean-value theorem of differentials, and the mean-value theorem of integrals, we have

\[
(3.33) \quad |R_2| = \left| \frac{n_1^3 E_1}{n_1 + 1} - \frac{n_2^3 E_2}{n_2 + 1} \right| \\
= \left| E_1 \left( \frac{n_1^3}{n_1 + 1} - \frac{n_2^3}{n_2 + 1} \right) + \frac{n_2^3}{n_2 + 1} (E_1 - E_2) \right| \\
\leq C |n_1 - n_2| (x) + C \left| (E_1(1) - E_2(1)) - \left( \int_x^1 (n_1 - n_2) - (b_1 - b_2)dy \right) \right| \\
\leq C |n_1 - n_2| (x) + C |E_1(1) - E_2(1)| + C \int_x^1 |n_1 - n_2| dy + C \|b_1 - b_2\|_{C[0,1]} \\
\leq C \|b_1 - b_2\|_{C[0,1]} + C \left( \frac{|n_1 - n_2|(x)}{(1 - x)\frac{1}{2}} + \frac{|n_1 - n_2|(\xi)}{(1 - \xi)\frac{1}{2}} \right), \quad \exists \xi \in [x, 1],
\]

where we have used the formula

\[
(3.34) \quad E_i(x) = E_i(1) - \int_x^1 (n_i - b_i)(y)dy, \quad i = 1, 2.
\]

Substituting (3.32) and (3.33) into (3.30), we have

\[
|h(n_1, E_1) - h(n_2, E_2)| \leq C \|b_1 - b_2\|_{C[0,1]} + C \left( \frac{|n_1 - n_2|(x)}{(1 - x)\frac{1}{2}} + \frac{|n_1 - n_2|(\xi)}{(1 - \xi)\frac{1}{2}} \right), \quad \exists \xi \in [x, 1],
\]

which further implies that

\[
(3.35) \quad |I_2| = \left| (h(n_1, E_1) - h(n_2, E_2)) \frac{(1 - x)\frac{1}{2}}{n_2 - 1} \right| \\
\leq C \|b_1 - b_2\|_{C[0,1]} + C \left( \frac{|n_1 - n_2|(x)}{(1 - x)\frac{1}{2}} + \frac{|n_1 - n_2|(\xi)}{(1 - \xi)\frac{1}{2}} \right), \quad \exists \xi \in [x, 1].
\]

Inserting (3.29) and (3.35) into (3.26), near $x = 1$, we obtain

\[
(3.36) \quad (1 - x)\frac{1}{2}|(n_1 - n_2)_x|(x) \\
\leq C \left( \frac{|n_1 - n_2|(x)}{(1 - x)\frac{1}{2}} + \frac{|n_1 - n_2|(\xi)}{(1 - \xi)\frac{1}{2}} \right) + C \|b_1 - b_2\|_{C[0,1]}, \quad \exists \xi \in [x, 1].
\]

It is worth mentioning that the generic constant $C > 0$ in (3.36) is independent of $\|b_1 - b_2\|_{C[0,1]}$. Moreover, the term $\frac{|n_1 - n_2|(x)}{(1 - x)\frac{1}{2}} + \frac{|n_1 - n_2|(\xi)}{(1 - \xi)\frac{1}{2}}$ on the right-hand side of (3.36) can be bounded by an appropriate constant multiple of $\|b_1 - b_2\|_{C[0,1]}$ in an intrinsic neighborhood of the right endpoint $x = 1$. Precisely, we claim that there exist two positive constants $0 < \delta_1 < \frac{1}{2}$ and $M_1 > 0$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that

\[
(3.37) \quad \frac{|n_1 - n_2|(x)}{(1 - x)\frac{1}{2}} \leq M_1 \|b_1 - b_2\|_{C[0,1]}, \quad x \in (1 - \delta_1, 1).
\]
Aiming for a contradiction, suppose that for any \( \delta \in (0, \frac{1}{3}) \) and \( M > 0 \), there is \( x_\delta \in (1 - \delta, 1] \) such that

\[
(3.38) \quad \frac{|n_1 - n_2|(x_\delta)}{(1 - x_\delta)^\frac{1}{2}} > M \| b_1 - b_2 \|_{C[0,1]}.
\]

By the arbitrariness, we could take \( \delta = \frac{1}{k}, k = 3, 4, 5, \cdots \), for arbitrary \( M > 0 \), there exists \( x_k \in (1 - \frac{1}{k}, 1] \) such that

\[
(3.39) \quad \frac{|n_1 - n_2|(x_k)}{(1 - x_k)^\frac{1}{2}} > M \| b_1 - b_2 \|_{C[0,1]},
\]

which implies that

\[
(3.40) \quad \lim_{x_k \to 1^-} \frac{|n_1 - n_2|(x_k)}{(1 - x_k)^\frac{1}{2}} \geq M \| b_1 - b_2 \|_{C[0,1]}.
\]

Besides, combining Lemma 3.1, the L’Hospital Rule and Lemma 3.4, we calculate that

\[
(3.41) \quad \lim_{x \to 1^-} \frac{|n_1 - n_2|(x)}{(1 - x)^\frac{1}{2}} = \lim_{x \to 1^-} \frac{(n_1 - n_2)(x)}{(1 - x)^\frac{1}{2}} = \lim_{x \to 1^-} \frac{(n_1 - 1)(x)}{(1 - x)^\frac{1}{2}} - \lim_{x \to 1^-} \frac{(n_2 - 1)(x)}{(1 - x)^\frac{1}{2}}
\]

\[= -2 \lim_{x \to 1^-} (1 - x)\frac{1}{2} n_1 + 2 \lim_{x \to 1^-} (1 - x)\frac{1}{2} n_2 \]

\[= \sqrt{\int_0^1 (b_1 - n_1)dx - \int_0^1 (b_2 - n_2)dx} \]

\[\leq \sqrt{\int_0^1 (b_1 - b_2)dx + \int_0^1 (b_2 - n_2)dx} \]

\[\leq C_1 \| b_1 - b_2 \|_{C[0,1]}.
\]

Moreover, we note that the constant \( M > 0 \) in (3.40) is arbitrary. Therefore, together with (3.41), taking \( M = 2C_1 \) in (3.40) leads to the contradiction that \( 2 \leq 1 \).

Applying (3.37) to (3.36), we have

\[
(3.42) \quad (1 - x)^\frac{1}{2} |(n_1 - n_2)_x(x) | \leq C \| b_1 - b_2 \|_{C[0,1]}, \quad x \in (1 - \delta_1, 1].
\]

Similarly to (3.33), we are able to compute that

\[
(3.43) \quad |E_1 - E_2|(x) \leq C \| b_1 - b_2 \|_{C[0,1]} + \left| \frac{n_1 - n_2(\xi)}{(1 - \xi)^\frac{1}{2}} \right|, \quad \exists \xi \in [x, 1]
\]

\[\leq C \| b_1 - b_2 \|_{C[0,1]}, \quad x \in (1 - \delta_1, 1],
\]

and

\[
(3.44) \quad |(E_1 - E_2)_x(x)| = |n_1 - n_2 - (b_1 - b_2)|(x) \leq |n_1 - n_2|(x) + |b_1 - b_2|(x)
\]

\[\leq \frac{|n_1 - n_2|(x)}{(1 - x)^\frac{1}{2}} + \| b_1 - b_2 \|_{C[0,1]}
\]

\[\leq C \| b_1 - b_2 \|_{C[0,1]}, \quad x \in (1 - \delta_1, 1].
\]
Finally, putting results (3.37), (3.42), (3.43) and (3.44) together, we obtain the desired local weighted estimate (3.25).

Up to now, we have obtained two intrinsic small domains $[0, \delta_0)$ and $(1 - \delta_1, 1]$ distributed around the two endpoints $x = 0$ and $x = 1$, respectively. This fact enables us to establish the structural stability estimate on a certain regular domain $[\delta, 1 - \delta]$, where $0 < \delta := \min\{\delta_0, \delta_1\} < 1/2$.

**Lemma 3.6.** Under the same conditions in Lemma 3.2. Let $\delta := \min\{\delta_0, \delta_1\}$. Then there is a positive constant $C > 0$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that

\[
\|n_1 - n_2\|_{C^1[\delta, 1 - \delta]} + \|E_1 - E_2\|_{C^1[\delta, 1 - \delta]} \leq C\|b_1 - b_2\|_{C[0,1]}.
\]

**Proof.** We are now able to work with Equations (3.1) on a regular closed interval $[\delta, 1 - \delta]$ away from singularities, where $\delta$ has been defined in the hypothesis of the present lemma.

Firstly, we rewrite the estimate (3.31) on the regular interval as follows.

\[
1 < l := 1 + m(\alpha, \beta_0) \sin(\pi\delta) \leq n_2(x) \leq n_1(x) \leq \bar{b}_1, \quad x \in [\delta, 1 - \delta].
\]

Secondly, subtracting (3.1)_{i=2} from (3.1)_{i=1}, for $x \in [\delta, 1 - \delta]$, we thus get

\[
(n_1 - n_2)_x = \frac{n_1^3 E_1 - \alpha n_2}{n_1^3 - 1} - \frac{n_2^3 E_2 - \alpha n_2}{n_2^3 - 1} = E_1 (f(n_1) - f(n_2)) + f(n_2)(E_1 - E_2) - \alpha (g(n_1) - g(n_2)) = (E_1 f'(\tilde{n}) - \alpha g'(\tilde{n}))(n_1 - n_2) + f(n_2)(E_1 - E_2), \quad \exists \tilde{n}, \bar{n} \in (n_2, n_1),
\]

and

\[
(E_1 - E_2)_x = (n_1 - n_2) - (b_1 - b_2),
\]

where

\[
f(n) := \frac{n^3}{n^3 - 1}, \quad g(n) := \frac{n^2}{n^2 - 1}, \quad \forall n \in [l, \bar{b}_1],
\]

and we have used the mean-value theorem of differentials in the third line of Equation (3.47).

Thirdly, multiplying through (3.47) by $n_1 - n_2$, and using (3.27), (3.46) and Cauchy’s inequality together, we have

\[
((n_1 - n_2)^2)_x \leq C(\alpha, l, \bar{b}_1) ((n_1 - n_2)^2 + (E_1 - E_2)^2), \quad x \in [\delta, 1 - \delta].
\]

Similarly, multiplying through (3.48) by $E_1 - E_2$, and employing Cauchy’s inequality, we obtain

\[
((E_1 - E_2)^2)_x \leq (n_1 - n_2)^2 + 2(E_1 - E_2)^2 + \|b_1 - b_2\|_{C[0,1]}^2, \quad x \in [\delta, 1 - \delta].
\]

And then, summing estimates (3.49) and (3.50) gives

\[
((n_1 - n_2)^2 + (E_1 - E_2)^2)_x (x) \leq C \left( (n_1 - n_2)^2 + (E_1 - E_2)^2 \right) (x) + \|b_1 - b_2\|_{C[0,1]}^2, \quad x \in [\delta, 1 - \delta].
\]
Applying the Gronwall inequality to (3.51), we have
\begin{equation}
(n_1 - n_2)^2 + (E_1 - E_2)^2 \leq e^{\int_{b_{2\alpha}} C_{[0,1]} dy} \left[ (n_1 - n_2)^2 + (E_1 - E_2)^2 \right] \leq C \left( (n_1 - n_2)^2 + (E_1 - E_2)^2 \right) + \|b_1 - b_2\|_{C^{[0,1]}}^2, \quad x \in [\delta, 1 - \delta].
\end{equation}
Noting that \( \delta \leq \delta_0 \) and the continuity of the error function pair \((n_1 - n_2, E_1 - E_2)(x)\) at \(x = \delta_0\), from Lemma 3.3 we see
\begin{equation}
(n_1 - n_2)^2 + (E_1 - E_2)^2 \leq C \|b_1 - b_2\|_{C^{[0,1]}},
\end{equation}
which along with (3.52) implies
\begin{equation}
|n_1 - n_2|(x) + |E_1 - E_2|(x) \leq C \|b_1 - b_2\|_{C^{[0,1]}}, \quad x \in [\delta, 1 - \delta].
\end{equation}
Finally, from Equations (3.47)&(3.48), and the estimate (3.54), we directly calculate
\begin{equation}
|(n_1 - n_2)_x|(x) + |(E_1 - E_2)_x|(x) \leq C \|b_1 - b_2\|_{C^{[0,1]}}, \quad x \in [\delta, 1 - \delta].
\end{equation}
Combining estimates (3.54) and (3.55) yields the desired structural stability estimate (3.45) on the regular domain \([\delta, 1 - \delta]\).

**Lemma 3.7.** Under the same conditions in Lemma 3.5. Then there exists a positive constant \(C\) independent of \(\|b_1 - b_2\|_{C^{[0,1]}}\) such that estimates (3.27) and (3.28) hold, that is,
\[
|E_i(x)| \leq \alpha + 2\bar{b}_i, \quad x \in [0, 1], \quad i = 1, 2,
\]
and
\[
|E_1(1) - E_2(1)| \leq C \|b_1 - b_2\|_{C^{[0,1]}},
\]
respectively.

**Proof.** From Equation (2.2) in Definition 2.1, we have
\begin{equation}
E_i(x) = \alpha + \int_0^x (n_i - b_i)(y) dy, \quad \forall x \in [0, 1], \quad i = 1, 2.
\end{equation}
First of all, in light of the lower and upper bounds (2.3) of \(n_i(x)\), a straightforward computation gives
\begin{equation}
|E_i(x)| = \left| \alpha + \int_0^x (n_i - b_i) dy \right| \leq \alpha + \int_0^1 (n_i + b_i) dy \leq \alpha + 2\bar{b}_i, \quad \forall x \in [0, 1], \quad i = 1, 2.
\end{equation}
Next, taking the value \(x = 1\) in Equation (3.56), we have
\begin{equation}
E_i(1) = \alpha + \int_0^1 (n_i - b_i)(y) dy, \quad i = 1, 2.
\end{equation}
Furthermore, taking the difference of Equations (3.58)|_{i=1} and (3.58)|_{i=2}, we calculate

\begin{equation}
|E_1(1) - E_2(1)| \leq \left| \int_0^1 \left( (n_1 - b_1) - (n_2 - b_2) \right) dy \right|
\leq \int_0^1 |n_1 - n_2|(y)dy + \|b_1 - b_2\|_{C[0,1]}
= |n_1 - n_2|(\xi) + \|b_1 - b_2\|_{C[0,1]}, \quad \exists \xi \in [0,1],
\end{equation}

where we have used the mean-value theorem of integrals in the last line.

Finally, we claim that there is a positive constant $C$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that

\begin{equation}
|n_1 - n_2|(\xi) \leq C\|b_1 - b_2\|_{C[0,1]},
\end{equation}

wherever the point $\xi$ is located in the whole interval [0,1]. In fact, take $\delta$ the same as in Lemma 3.6, and if $\xi \in [0,1 - \delta]$, it is clear from estimates (3.7) and (3.45) that (3.60) is true; if $\xi \in (1 - \delta,1]$, the intrinsic local estimate (3.37) guarantees that (3.60) is true as well. Consequently, substituting (3.60) into (3.59), we obtain the desired estimate (3.28). \hfill \square

We end this section by a summary for the proof of Theorem 2.1 because the previous lemmas have already made the proof evident.

**Proof of Theorem 2.1.** Obviously, putting all the estimates we have established in Lemmas 3.3, 3.6 and 3.5 together, we have the globally structural stability estimate (2.6). \hfill \square

**Acknowledgments:** This work was commenced while the first two authors were visiting McGill University from 2021 to 2022. They would like to express their gratitude to McGill University for its hospitality. The research of Y. H. Feng was supported by China Scholarship Council for the senior visiting scholar program (202006545001). The research of H. Hu was partially supported by National Natural Science Foundation of China (Grant No.11801039), China Scholarship Council (No.202007535001) and Scientific Research Program of Changchun University (No.ZKP202013). The research of M. Mei was partially supported by NSERC grant RGPIN 354724-2016.

**References**

[1] U.M. Ascher, P.A. Markowich, P. Pietra and C. Schmeiser, A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model, *Math. Models Methods Appl. Sci.*, 1(3) (1991), 347-376.
[2] K. Bløtekjær, Transport equations for electrons in two-valley semiconductors, *IEEE Trans. Electron Devices*, 17 (1970), 38-47.
[3] M. Bae, B. Duan and C.J. Xie, Subsonic solutions for steady Euler-Poisson system in two-dimensional nozzles, *SIAM J. Math. Anal.*, 46 (2014), 3455-3480.
[4] L. Chen, M. Mei, G. Zhang and K. Zhang, Steady hydrodynamic model of semiconductors with sonic boundary and transonic doping profile, *J. Differential Equations*, 269 (2020), 8173-8211.
[5] L. Chen, M. Mei, G. Zhang and K. Zhang, Radial solutions of the hydrodynamic model of semiconductors with sonic boundary, *J. Math. Anal. Appl.*, 501 (2021), 125187.
[6] L. Chen, M. Mei, G. Zhang and K. Zhang, Transonic steady-states of Euler-Poisson equations for semiconductor models with sonic boundary, *SIAM J. Math. Anal.*, 54 (2022), no. 1, 363-388.
[7] P. Degond and P.A. Markowich, On a one-dimensional steady-state hydrodynamic model for semiconductors, *Appl. Math. Lett.*, 3(3) (1990), 25-29.
[8] P. Degond and P.A. Markowich, A steady state potential flow model for semiconductors, *Ann. Mat. Pura Appl.*, 165(4) (1993), 87-98.
[9] W. Fang and K. Ito, Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors, *J. Differential Equations*, 133 (1997), 224-244.

[10] Y.H. Feng, M. Mei and G. Zhang, Nonlinear structural stability and linear dynamic instability of transonic steady-states to a hydrodynamic model for semiconductors, *arXiv:2202.03475*, submitted.

[11] I.M. Gamba, Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors. *Commun. Partial Differ. Equ.*, 17(3-4) (1992), 553-577.

[12] I.M. Gamba and C.S. Morawetz, A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: existence theorem for potential flow, *Commun. Pure Appl. Math.*, 49(10) (1996), 999-1049.

[13] Y. Guo and W. Strauss, Stability of semiconductor states with insulating and contact boundary conditions, *Arch. Rational Mech. Anal.*, 179 (2005), 1-30.

[14] J. Li, M. Mei, G. Zhang and K. Zhang, Steady hydrodynamic model of semiconductors with sonic boundary: (I) Subsonic doping profile, *SIAM J. Math. Anal.*, 49 (2017), 4767-4811.

[15] J. Li, M. Mei, G. Zhang and K. Zhang, Steady hydrodynamic model of semiconductors with sonic boundary: (II) Supersonic doping profile, *SIAM J. Math. Anal.*, 50 (2018), 718-734.

[16] T. Luo, J. Rauch, C.J. Xie and Z.P. Xin, Stability of transonic shock solutions for one-dimensional Euler-Poisson equations, *Arch. Rational Mech. Anal.*, 202 (2011), 787-827.

[17] T. Luo and Z. Xin, Transonic shock solutions for a system of Euler-Poisson equations, *Commun. Math. Sci.*, 10 (2012), 419-462.

[18] P.A. Markowich, C.A. Ringhofer and C. Schmeiser, *Semiconductor equations*, Springer, 1990.

[19] P. Mu, M. Mei and K. Zhang, Subsonic and supersonic steady-states of bipolar hydrodynamic model of semiconductors with sonic boundary. *Commun. Math. Sci.*, 18 (2020), no. 7, 2005-2038.

[20] S. Nishibata and M. Suzuki, Asymptotic stability of a stationary solution to a hydrodynamic model of semiconductors, *Osaka J. Math.*, 44 (2007), 639-665.

[21] Y. Peng and I. Violet, Example of supersonic solutions to a steady state Euler-Poisson system, *Appl. Math. Lett.*, 19(12) (2006), 1335-1340.

[22] M. Rosini, A phase analysis of transonic solutions for the hydrodynamic semiconductor model. *Q. Appl. Math.*, 63(2) (2005), 251-268.

[23] R. Schreiber, Regularity of singular two-point boundary value problems. *SIAM J. Math. Anal.*, 12 (1981), no. 1, 104-109.

[24] M. Wei, M. Mei, G. Zhang and K. Zhang, Smooth transonic steady-states of hydrodynamic model for semiconductors, *SIAM J. Math. Anal.*, 53(4), (2021), 4908-4932.

[25] K. Zhang and H. Hu, *Introduction to Semiconductor Partial Differential Equations (Chinese Edition)*, Science Press, Beijing, 2016.