AMBARZUMYAN TYPE THEOREM FOR A MATRIX QUADRATIC
STURM-LIOUVILLE EQUATION WITH ENERGY-DEPENDENT POTENTIAL

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Abstract: Ambarzumyan’s theorem for quadratic Sturm-Liouville problem is extended to second order differential systems of dimension $d \geq 2$. It is shown that if the spectrum is the same as the spectrum belonging to the zero potential, the matrix valued functions $P(x)$ and $Q(x)$ are both zero.

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1. Introduction

From an historical viewpoint, Ambarzumyan’s paper may be thought of as the first paper in the theory of inverse spectral problems associated with Sturm-Liouville operators. Ambarzumyan proved the following theorem:

If \( \{ n^2 : n = 0, 1, 2, \ldots \} \) is the spectral set of the boundary value problem

\[
y'' + (\lambda - q(x))y = 0, \quad y'(0) = y'(\pi) = 0
\]

then \( q = 0 \) in \([0, \pi]\), where \( q(x) \in L^2 [0, \pi] \) \cite{1}. Ambarzumyan’s theorem was extended to the second differential systems of two dimensions in \cite{2}, to Sturm-Liouville differential systems of any dimension \cite{3}, to the Sturm–Liouville equation (which is concerned only with Neumann boundary conditions) with general boundary conditions by imposing an additional condition on the potential function \cite{4}, and to multi-dimensional Dirac operator in \cite{5}. In addition, some different results of Ambarzumyan’s theorem have been obtained in \cite{6, 7, 8, 9, 10}.

Ambarzumyan’s theorem was extended to the following boundary value problem by imposing the condition for \( p \)

\[
- y'' + [2\lambda p(x) + q(x)] y = \lambda^2 y, \quad x \in [0, \pi]
\]

with the homogeneous Neumann boundary conditions

\[
y'(0) = y'(\pi) = 0,
\]
where $\lambda$ is a spectral parameter and $p \in W^2_2[0, \pi]$ and $q \in W^1_2[0, \pi]$ by Koyunbakan, Lesnic and Panakho [11]. This problem is called the quadratic pencil of the Schrödinger operator. If $p(x) = 0$ the classical Sturm-Liouville operator is obtained. Some versions of the eigenvalue problem (1.2)-(1.3) were studied extensively in [12], [13], [14], [15], [16].

Before giving the main results, we mention some physical properties of the quadratic equation. The problem of describing the interactions between colliding particles is of fundamental interest in physics. It is interesting in collisions of two spinless particles, and it is supposed that the $s-$wave scattering matrix and the $s-$wave binding energies are exactly known from collision experiments. For a radial static potential $V(E, x)$ and $s-$wave, the Schrödinger equation is written as

$$y'' + [E - V(E, x)]y = 0,$$

where

$$V(E, x) = 2\sqrt{Ep(x) + q(x)}.$$  

We note that with the additional condition $q(x) = -p^2(x)$, the above equation reduces to the Klein-Gordon $s$-wave equation for a particle of zero mass and energy $\sqrt{E}$ [17].

2. Matrix Differential Equations

For simplicity, $A_{ij}$ denotes entry of matrix $A$ at the $i-th$ row and $j-th$ column and $I_d$ is a $d \times d$ identity matrix and $0_d$ is a $d \times d$ zero matrix.

We are interested in the eigenvalue problem

$$-\phi'' + [2\lambda P(x) + Q(x)]\phi = \lambda^2 \phi$$

$$A\phi(0) + B\phi'(0) = C\phi(\pi) + D\phi'(\pi) = 0,$$

where $P(x) = \text{diag}[p_1(x), p_2(x), ... , p_d(x)]$ and $Q(x)$ are $d \times d$ real symmetric matrix-valued functions for those $n \times n$ matrices $A, B, C$ and $D$ satisfy the following conditions

$$DC^* : \text{Self-Adjoint}$$

$$BA^* = 0$$

$$\text{rank}[A, B] = \text{rank}[C, D] = n.$$  

To study (2.1)-(2.2), we introduce the following matrix differential equation

$$-Y'' + [2\lambda P(x) + Q(x)]Y = \lambda^2 Y, x \in [0, \pi]$$

$$Y(0, \lambda) = I_d, Y'(0, \lambda) = 0_d$$
where $\lambda$ is a spectral parameter, $Y(x) = [y_k(x)]$, $k = 1, d$ is a column vector, $P(x) = \text{diag}[p_1(x), p_2(x), \ldots]$ and $Q(x)$ are $d \times d$ real symmetric matrix-valued functions, $P \in W^2_2[0, \pi]$ and $Q \in W^1_2[0, \pi]$, where $W^k_2 \,(k = 1, 2)$ denotes a set whose element is a $k$-th order continuously differentiable function in $L^2_2[0, \pi]$, and $\mu = \sigma + it \in \mathbb{C}$. Then, $\lambda$ is an eigenvalue of (2.1)-(2.2), if the matrix which is called characteristic function

$$W(\mu) = CY(\pi, \mu) + DY'(\pi, \mu)$$

is singular.

In order to describe $W(\mu)$ explicitly, we must know how to express the solution $Y(x, \mu)$. The solution $Y(x, \mu)$ of (2.6)-(2.7) can be expressed as

$$Y(x, \mu) = \cos(\lambda I_d x - \alpha(x)) + \int_0^x A(x, t) \cos(\lambda t) dt + \int_0^x B(x, t) \sin(\lambda t) dt$$

where $A(x, t)$ and $B(x, t)$ are symmetric matrix-valued functions whose entries have continuous derivatives up to order two respect to $t$ and $x$, and it can described by the following lemma.

**Lemma 2.1.** [18] Let $A$ and $B$ be as in (2.8). Then, $A$ and $B$ satisfy following conditions

$$\frac{\partial^2 A(x, t)}{\partial x^2} - 2P(x) \frac{\partial B(x, t)}{\partial t} - Q(x) A(x, t) = \frac{\partial^2 A(x, t)}{\partial t^2}$$

$$\frac{\partial^2 B(x, t)}{\partial x^2} + 2P(x) \frac{\partial A(x, t)}{\partial t} - Q(x) B(x, t) = \frac{\partial^2 B(x, t)}{\partial t^2}$$

and

$$A(0, 0) = h_d, \quad B(x, 0) = 0_d, \quad \frac{\partial A(x, t)}{\partial t} \bigg|_{t=0} = 0_d,$$

with $\alpha(x) = \int_0^x P(t) dt$. Moreover, there holds

$$2[\cos \alpha(x) A(x, x) + \sin \alpha(x) B(x, x)] = 2h + \int_0^x T_1(t) dt$$

and

$$2[\sin \alpha(x) A(x, x) - \cos \alpha(x) B(x, x)] = P(x) - P(0) + \int_0^x T_2(t) dt$$

where

$$T_1(x) = P^2(x) + \cos \alpha(x) Q(x) \cos \alpha(x) + \sin \alpha(x) Q(x) \sin \alpha(x)$$

and

$$T_2(x) = \sin \alpha(x) Q(x) \cos \alpha(x) - \cos \alpha(x) Q(x) \sin \alpha(x).$$
Lemma 2.2. [18] The eigenvalues of the operator \( L(P, Q; h, H) \) are

\[
\lambda_n = n + \frac{\alpha_j}{\pi}, n = 0, \pm 1, \pm 2, \pm 3, \ldots, j = \frac{1}{d} \text{ and } \alpha_j = \int_0^\pi p_j(x)dx.
\]

3. Main Results

In this section, some uniqueness theorems are given for the equation (2.6) with the Robin boundary conditions. It is shown that an explicit formula of eigenvalues can determine the functions \( Q(x) \) and \( \alpha(\pi) \) to be zero both. Results are some generalizations of [11].

Consider the second matrix quadratic initial value problem

\[
-Y'' + \left[ 2\lambda P(x) + \tilde{Q}(x) \right] Y = \lambda^2 Y, x \in [0, \pi]
\]

\[
Y(0, \lambda) = I_d, Y'(0, \lambda) = 0_d
\]

where \( \tilde{Q} \) has the same properties of \( Q \).

The problems (2.6)-(2.7) and (3.1)-(3.2) will be denoted by \( L(P, Q; h, H) \) and \( \tilde{L}(P, \tilde{Q}; h, H) \) and the spectra of these problems will be denoted by \( \sigma(P, Q) \) and \( \tilde{\sigma}(P, \tilde{Q}) \), respectively.

**Theorem 3.1.** Suppose that \( \sigma(P, Q) = \tilde{\sigma}(P, \tilde{Q}) \) and \( \alpha(\pi) = 0 \), then

\[
\int_0^\pi \left[ Q(x) - \tilde{Q}(x) \right] dx = 0
\]

everywhere on \([0, \pi]\).

**Proof:** Since \( \sigma(P, Q) = \tilde{\sigma}(P, \tilde{Q}) \), it follows that \( \lambda_n \in \sigma(P, Q) \) are large eigenvalues. Then we can write from (2.8) that

\[
Y'(\pi, \lambda_n) = - (\lambda_n I_d - P(\pi)) \sin [\lambda_n I_d \pi - \alpha(\pi)] + A(\pi, \pi) \cos (\lambda_n \pi) + B(\pi, \pi) \sin (\lambda_n \pi)
\]

\[
+ \int_0^x A_x(\pi, t) \cos (\lambda_n t) dt + \int_0^x B_x(\pi, t) \sin (\lambda_n t) dt
\]

and similarly for the problem (3.1)-(3.2), we can write

\[
\tilde{Y}'(\pi, \lambda_n) = - (\lambda_n I_d - \tilde{P}(\pi)) \sin [\lambda_n \pi - \alpha(\pi)] + \tilde{A}(\pi, \pi) \cos (\lambda_n \pi) + \tilde{B}(\pi, \pi) \sin (\lambda_n \pi)
\]

\[
+ \int_0^x \tilde{A}_x(\pi, t) \cos (\lambda_n t) dt + \int_0^x \tilde{B}_x(\pi, t) \sin (\lambda_n t) dt.
\]

By subtracting, \( Y'(\pi, \lambda_n) \) and \( \tilde{Y}'(\pi, \lambda_n) \), we get

\[
0 = \left[ A(\pi, \pi) - \tilde{A}(\pi, \pi) \right] \cos (\lambda_n \pi) + \left[ B(\pi, \pi) - \tilde{B}(\pi, \pi) \right] \sin (\lambda_n \pi)
\]

\[
+ \int_0^x \left[ A_x(\pi, t) - \tilde{A}_x(\pi, t) \right] \cos (\lambda_n t) dt + \int_0^x \left[ B_x(\pi, t) - \tilde{B}_x(\pi, t) \right] \sin (\lambda_n t) dt.
\]
By using Riemann-Lebesque lemma and for \( \lambda_n \to \infty \) in Lemma 2.2., we obtain \( A(\pi, \pi) = \tilde{A}(\pi, \pi) \). On the other hand by Lemma 2.1., we know the following equalities,

\[
2 \frac{d}{dx} \left[ \cos \alpha(x)A(x,x) + \sin \alpha(x)B(x,x) \right] = P^2(x) + Q(x)
\]

\[
2 \frac{d}{dx} \left[ \cos \alpha(x)\tilde{A}(x,x) + \sin \alpha(x)\tilde{B}(x,x) \right] = P^2(x) + \tilde{Q}(x).
\]

After substracting (3.3),(3.4) and integrating, we get

\[
\int_0^\pi \left[ Q(x) - \tilde{Q}(x) \right] dx = 2 \left\{ \left[ A(\pi, \pi) - \tilde{A}(\pi, \pi) \right] \cos \alpha(\pi) + \left[ B(\pi, \pi) - \tilde{B}(\pi, \pi) \right] \sin \alpha(\pi) \right\}
\]

and

\[
\int_0^\pi \left[ Q(x) - \tilde{Q}(x) \right] dx = 0
\]

where \( \alpha(0) = \alpha(\pi) = 0 \) and \( A(\pi, \pi) = \tilde{A}(\pi, \pi) \). This completes the proof.

**Theorem 3.2.** Let \( P(x) = \text{diag}[p_1(x), p_2(x), \ldots, p_d(x)] \) and \( Q(x) \) are two \( d \times d \) real symmetric matrix-valued functions, and \( \alpha(\pi) = 0 \). If \( \{0\} \cup \{m_j : j = 1, 2, \ldots\} \) is a subset of the spectrum of the following \( d \)-dimensional second order differential system

\[
-\phi'' + [2\lambda P(x) + Q(x)] \phi = \lambda^2 \phi, \quad \phi'(0) = \phi'(\pi) = 0
\]

where 0 is the first eigenvalue of (3.5), \( m_j \) is a strictly ascending infinite sequence of positive integers, 0 and \( m_j \) are multiplicity of \( n \), then \( P(x) = Q(x) = 0 \).

**Proof:** Suppose for the (3.5) Neumann problem, there are infinitely many eigenvalues of the form \( m_j \) are positive integers, \( j = 1, 2, \ldots \) and each \( m_j \) is of multiplicity \( n \). For such a case, Let \( A = C = B = D = I_d \) in (2.1). Then, we get

\[
Y'(\pi, m_j) = 0.
\]

On the other hand, by (2.8), we have

\[
Y'(x, \lambda_n) = - (\lambda_n I_d - P(x)) \sin [\lambda_n x - \alpha(x)] + A(x,x) \cos (\lambda_n x) + B(x,x) \sin (\lambda_n x) + \int_0^x A_\lambda(x,t) \cos (\lambda_n t) dt + \int_0^x B_\lambda(x,t) \sin (\lambda_n t) dt.
\]

Equations (3.6) and (3.7) imply

\[
A(\pi, \pi) \cos (m_j \pi) + \int_0^\pi A_\lambda(\pi,t) \cos (\lambda_n t) dt + \int_0^\pi B_\lambda(\pi,t) \sin (\lambda_n t) dt = 0.
\]
We have from (3.8) and Riemann Lebesque lemma that $A(\pi, \pi) = 0$. Then, by integration of

$$Q(x) + P^2(x) = 2\frac{d}{dx} [\cos \alpha A(x, x) + \sin \alpha B(x, x)],$$

we get

$$\cos \alpha A(x, x) + \sin \alpha B(x, x) = \frac{1}{2} \int_0^x [Q(t) + P^2(t)] \, dt.$$  

Since $\alpha(\pi) = 0$, we have

$$0 = A(\pi, \pi) = \frac{1}{2} \int_0^\pi [Q(t) + P^2(t)] \, dt$$

and

$$\int_0^\pi Q(x) \, dx = - \int_0^\pi P^2(x) \, dx.$$  

By using the reality of 0 being the ground state of the eigenvalue problem (3.5), we may find $n-1$ independent constant vectors corresponding to the same eigenvalue 0 by the variational principle and them by $\varphi_j$, $j = 1, 2, ..., d$. Since they should the following equation

$$-\varphi''_j + [2\lambda_0 P(x) + Q(x)] \varphi_j = \lambda^2_0 \varphi_j.$$  

Then, we obtain

$$Q(x)\varphi_j = 0, \ 0 \leq x \leq \pi.$$  

Thus $Q(x) = 0$. If we consider (3.9) and diagonally of $P$, we get $P(x) = 0$. This completes the proof.

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