DERIVATION OF BELL POLYNOMIALS OF THE SECOND KIND

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Abstract

New methods for derivation of Bell polynomials of the second kind are presented. The methods are based on an ordinary generating function and its composita. The relation between a composita and a Bell polynomial is demonstrated. Main theorems are written and examples of Bell polynomials for trigonometric functions, polynomials, radicals, and Bernoulli functions are given.

1 Introduction

Bell polynomials are an important tool in solving various mathematical problems, among which is finding of higher derivatives of composite functions [1, 2, 3]. However, a general expression for Bell polynomials is rather difficult to derive. One of the main tools in computations of Bell polynomials is exponential generating functions [2, 3]. In this paper, it is proposed to use ordinary generating functions and their compositae [4] to derive expressions for Bell polynomials. Let us introduce the following notation. Let there be given a function $y(x)$ and an ordinary generating function $Y(x, z) = \sum_{n>0} \frac{y^{(n)}(x)}{n!} z^n$. By definition, the Bell polynomial of the second kind is written as

$$B_{n,k}(y^{(1)}, y^{(2)}, \ldots y^{(n-k+1)}) = \frac{1}{k!} \sum_{\pi_k \in C_n} \binom{n}{\lambda_1, \lambda_2, \ldots \lambda_k} y^{(\lambda_1)} y^{(\lambda_2)} \cdots y^{(\lambda_k)}$$

or as

$$B_{n,k} = \frac{n!}{k!} \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}(x)}{\lambda_1!} \frac{y^{(\lambda_2)}(x)}{\lambda_2!} \cdots \frac{y^{(\lambda_k)}(x)}{\lambda_k!}$$

where $y^{(i)}$ is the $i$-th derivative of the function $y(x)$, $C_n$ is the set of compositions of $n$, and $\pi_k$ is the composition of $n$ with $k$ parts exactly $\{\lambda_1 + \lambda_2 + \ldots \lambda_k = n\}$. 

The polynomial $B_{n,k}$ has the form of a triangle in which the left part contains all derivatives of the function $y(x)$ and the right part contains $[y'(x)]^n$.

| $y^{(1)}$ | $y^{(2)}$ | $y^{(3)}$ | $y^{(4)}$ | $\vdots$ | $y^{(n)}$ |
|-----------|-----------|-----------|-----------|--------|-----------|
| $y_1$     | $y_2$     | $B_{3,2}$ | $B_{4,2}$ | $\vdots$ | $B_{n,2}$ |
| $\frac{y_1}{1!}$ | $\frac{y_2}{2!}$ | $\frac{y_1}{1!} \frac{y_2}{2!}$ | $\frac{y_1}{1!} \frac{y_2}{2!} \frac{y_3}{3!}$ | $\vdots$ | $\frac{y_1}{1!} \frac{y_2}{2!} \cdots \frac{y_n}{n!}$ |

The generating function is $Y(x, z) = \sum_{n>0} \frac{y^{(n)}(x)}{n!} z^n = y(x + z) - y(x)$. Hence, we can introduce the composita of the generating function $Y(x, z)$ as 

$$Y^\Delta(n, k, x) = \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}(x)}{\lambda_1!} \frac{y^{(\lambda_2)}(x)}{\lambda_2!} \cdots \frac{y^{(\lambda_k)}(x)}{\lambda_k!},$$

and the generating function for $Y^\Delta(n, k, x)$ will have the expression:

$$[Y(x, z)]^k = (y(x + z) - y(x))^k = \sum_{n \geq k} Y^\Delta(n, k, x) z^n.$$ 

In view of the foregoing, we can write the relation for the Bell polynomial and composita of the ordinary generating function $Y(x, z)$:

$$B_{n,k} = \frac{n!}{k!} Y^\Delta(n, k, x). \quad (1)$$

Because there is a one-to-one relation between the composita and the Riordan array [5], the exponential Riordan array (1, $y(x)$) and the Bell polynomial $B_{n,k}(y_1, y_2, \ldots, y_{n-k+1})$, where $y(x) = \sum_{n>0} y_n \frac{x^n}{n!}$, are equivalent.

### 2 Expressions for Bell polynomials based on the composita of a generating function $Y(\alpha, z)$

Let us consider the problem of finding the Bell polynomial $B_{n,k}$ as the problem of finding coefficients of an ordinary generating function $Y(\alpha, z)^k$. This is possible if we represent the generating function $Y(x, z)$ as $F(g(x), h(z))$; then we can use the apparatus of compositae introduced in [4, 5]. Let us consider the following examples:

**Example 2.1.** Let there be given a function $y(x)$ with two derivatives $y'(x)$ and $y''(x)$. Let us find an expression for the composita of this function. By definition,

$$Y^\Delta(n, k, x) = \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}(x)}{\lambda_1!} \frac{y^{(\lambda_2)}(x)}{\lambda_2!} \cdots \frac{y^{(\lambda_k)}(x)}{\lambda_k!}.$$
Then, the generating function has the expression \( y(x + z) - y(x) = y'(x)z + \frac{y''(x)}{2}z^2 \). Hence, according to the formula of the composita for the polynomial \( ax + bx^2 \), we obtain

\[
Y^\Delta(n, k, x) = \binom{k}{n-k} [f'(x)]^{2k-n} \left( \frac{f''(x)}{2} \right)^{n-k}.
\] (2)

Thus, the Bell polynomial for the function with derivatives \( y'(x) \) and \( y''(x) \) is equal to

\[
B_{n,k} = \frac{n!}{k!} \binom{k}{n-k} [f'(x)]^{2k-n} \left( \frac{f''(x)}{2} \right)^{n-k}.
\] (3)

**Example 2.2.** Let there be given a function \( y(x) = x^m \), where \( m > 0 \). The generating function is \( Y(x, z) = (x + z)^m - x^m \). Let us find a composita of \( Y(x, z) \); for this purpose, we are to find the coefficients:

\[
x^{km} \left[ \left( 1 + \frac{z}{x} \right)^m - 1 \right]^k = x^{km} \sum_{j=0}^{k} \binom{k}{j} \left( 1 + \frac{z}{x} \right)^{jm} (-1)^{k-j};
\]

From whence, knowing that the coefficients for \( (1 + \frac{z}{x})^{jm} \) are equal to \( \binom{jm}{n} \frac{1}{x^n} \), we obtain the desired composita

\[
Y^\Delta(n, k, x) = x^{km} \sum_{j=0}^{k} \binom{k}{j} \binom{jm}{n} x^{-n} (-1)^{k-j}.
\]

Then the Bell polynomial is

\[
B_{n,k} = \frac{n!}{k!} x^{km-n} \sum_{j=0}^{k} \binom{k}{j} \binom{jm}{n} (-1)^{k-j}.
\]

**Example 2.3.** Let there be given a function \( y(x) = x^{-m} \), where \( m > 0 \). The generating function is \( Y(x, z) = \frac{1}{(x+z)^m} - \frac{1}{x^m} \). Let us find a composita of \( Y(x, z) \); for this purpose, we are to find the coefficients

\[
Y(x, z)^k = \frac{1}{x^{mk}} \left[ \frac{1}{(1 + \frac{z}{x})^m} - 1 \right]^k;
\]

from whence it follows that the composita is equal to

\[
\left( \sum_{j=1}^{k} \binom{k}{j} (-1)^{n+k-j} \binom{n+jm-1}{jm-1} \right) x^{-n-km}.
\]

Then the Bell polynomial is equal to

\[
B_{n,k} = \frac{n!}{k!} \left( \sum_{j=1}^{k} \binom{k}{j} (-1)^{n+k-j} \binom{n+jm-1}{jm-1} \right) x^{-n-km}.
\]
Example 2.4. Let us consider the example of use of the composita for the generating function \( f(z) = az + bz^2 + cz^3 \):

\[
F^n(k, n, x) = \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.
\]

Substitution of \( a = f'(x) \), \( b = f''(x) \), \( c = f'''(x) \) gives the Bell polynomial:

\[
B_n(k) = \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.
\]

Let us consider the example \( f(x) = x^3 + 2x, f'(x) = 3x^2 + 2, f''(x) = 6x, f'''(x) = 6; \) then, \( a = 3x^2 + 2, b = 3x, c = 1 \). Then the Bell polynomial is

\[
\frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.
\]

Presented below are the first terms of this polynomial

\[
\begin{align*}
3x^2 + 2 \\
6x, (3x^2 + 2)^2 \\
6, 18x(3x^2 + 2), (3x^2 + 2)^3 \\
0, 180x^2 + 48, 36x(3x^2 + 2)^2, (3x^2 + 2)^4
\end{align*}
\]

The same reasoning allows us to obtain Bell polynomials for functions whose generating functions \( y(x + z) - y(x) \) are expressed in polynomials. Expressions for the compositae of polynomials and methods of their derivation are described in [4].

Example 2.5. Let us find a Bell polynomial for the function \( \sin x \). For this purpose, we find the composita of the function \( \sin(x + z) - \sin x \). Then

\[
S(x, z) = \cos x \sin z + \sin x (\cos z - 1),
\]

where \( \sin z \) and \( \cos z \) are generating functions, and \( \sin x \) and \( \cos x \) are coefficients. Hence the composita of the function \( \cos x \sin z \) is

\[
F^n(k, n, x) = (\cos x)^k \left( 1 + (-1)^{n-k} \right) \sum_{m=0}^{k} \binom{k}{m} (2m-k)^n (-1)^{n-k-m}.
\]

Now, let us write the coefficients \( T_{n,k} \) for \( \cos^k(z) = \sum_{n\geq 0} T_{n,k} z^n \).

\[
T_{n,k} = \begin{cases} 
1, & n = 0 \\
0, & n \text{ odd} \\
\frac{1}{2^{k-1}} \sum_{i=0}^{k-1} \binom{k-1}{i} (k-2i)^n (-1)^{\frac{n}{2}}, & n \text{ even}.
\end{cases}
\]
Then we obtain the composita of the generating function \( \sin x (\cos (z) - 1) \)

\[
R^\Delta(n, k, x) = (\sin x)^k \frac{(-1)^n + 1}{n!} \sum_{j=1}^{k} \frac{(-1)^{\frac{n}{2} + k - j}}{2^j} \binom{k}{j} \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (j - 2i)^n \binom{j}{i}
\]

Next, from the theorem of the composita of the sum of generating functions [4], we obtain the desired composita

\[
S^\Delta(n, k, x) = F^\Delta(n, k, x) + R^\Delta(n, k, x) + \sum_{j=1}^{k-1} \binom{k}{n-k+j} \sum_{i=j}^{k} F^\Delta(i, j, x) R^\Delta(n-i, k-j, x).
\]

Presented below are the first terms of the Bell polynomial \( B_{n,k} = \frac{n!}{k!} S^\Delta(n, k, x) \) for the function \( \sin x \):

\[
\begin{align*}
&\cos x \\
&- \sin x, \cos^2 x \\
&- \cos x, -3 \cos x \sin x, \cos^3 x \\
&\sin x, 3 \sin^2 x - 4 \cos^2 x, -6 \cos^2 x \sin x, \cos^4 x \\
&\cos x, 15 \cos x \sin x, 15 \cos x \sin^2 x - 10 \cos^3 x, -10 \cos^3 x \sin x, \cos^5 x
\end{align*}
\]

Now the derivative \( f_1^{(4)}(x) \) for the function \( f_1(x) = e^{\sin x} \) is expressed as

\[
f_1^{(4)}(x) = e^{\sin x} \left( \sin x + 3 \sin^2 x - 4 \cos^2 x - 6 \cos^2 x \sin x + \cos^4 x \right).
\]

The derivative \( f_2^{(5)}(x) \) for \( f_2(x) = \sin^3 x \) is expressed as

\[
f_2^{(5)}(x) = 3 \sin^2 x (\cos x) + 6 \sin x (15 \cos x \sin x) + 6(15 \cos x \sin^2 x - 10 \cos^3 x) = 183 \sin^2 x \cos x - 60 \cos^3 x.
\]

In the same way, we can find a Bell polynomial for the function \( \cos x \); for this purpose, we are to find the composita of the generating function:

\[
C(x, z) = \cos x (\cos z - 1) - \sin x \sin z.
\]

**Example 2.6.** Let us consider the function \( y(x) = \sqrt[3]{x} \). The generating function is \( Y(x, z) = \sqrt[3]{x + z} - \sqrt[3]{x} \). Hence

\[
Y(x, z)^m = (-1)^m (\sqrt[3]{x})^m \left[ 1 - \sqrt[3]{1 + \left( \frac{z}{x} \right)} \right]^m.
\]

Given the composita of the generating function \( 1 - \sqrt[3]{1 - z} \) [5], we obtain the desired composita

\[
Y^\Delta(n, m, x) = \begin{cases} 
(\sqrt[3]{x})^m \frac{1}{3}^n, & n = m, \\
(\sqrt[3]{x})^m \frac{m}{n} \sum_{k=1}^{n-m-k} \binom{k}{n-m-k} 3^{-2n+m+k} (-1)^k \binom{n+k-1}{n-1} x^{-n}, & n > m.
\end{cases}
\]
3 Method based on operations on compositae \( Y^\Delta(n, k, x) \)

Let us consider peculiarities of the generating function \( Y(x, z) = \sum_{n \geq 0} \frac{y^{(n)}(x)}{n!} z^n = y(x+z) - y(x) \). For this purpose, we prove the following theorem.

**Theorem 3.1.** Let there be given a composition \( f(x) = g(y(x)) \) and functions \( g(x), y(x) \) with an infinite number of derivatives in the general case. Then the generating functions \( F(x, z) = \sum_{n \geq 0} \frac{f^{(n)}(x)}{n!} z^n, Y(x, z) = \sum_{n \geq 1} \frac{y^{(n)}(x)}{n!} z^n \) and \( G(x, z) = \sum_{n \geq 0} \frac{g^{(n)}(x)}{n!} z^n \) form the composition

\[
F(x, z) = G(y, Y(x, z)).
\]

**Proof.** Let us write the known Faa di Bruno formula \([1, 2, 3]\):

\[
f^{(n)}(x) = \sum_{k=1}^{n} \frac{n!}{k!} \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}(x) y^{(\lambda_2)}(x) \cdots y^{(\lambda_k)}(x)}{\lambda_1! \lambda_2! \cdots \lambda_k!}.
\]

Hence

\[
\frac{f^{(n)}(x)}{n!} = \sum_{k=1}^{n} \frac{g^{(k)}(y)}{k!} \sum_{\pi_k \in C_n} \frac{y^{(\lambda_1)}(x) y^{(\lambda_2)}(x) \cdots y^{(\lambda_k)}(x)}{\lambda_1! \lambda_2! \cdots \lambda_k!}.
\]

Thus, we obtain the formula for the composition of ordinary generating functions \([4]\). It is evident that the nonzero term of \( F(x, z) \) is equal to \( g(y(x)) \). \(\square\)

The peculiarity here is that in the operation of the composition of generating functions, the argument \( x \) in \( G(x, z) \) is replaced by \( y(x) \). Let us turn to the problem of finding compositae of the generating functions \( (y(x+z) - y(x)) \) using the operations of summation, product, and composition.

**Theorem 3.2.** Let there be generating functions \( F(x, z) = f(x+z) - f(x) = \sum_{n \geq 0} \frac{f^{(n)}(x)}{n!} z^n, G(x, z) = g(x+z) - g(x) = \sum_{n \geq 0} \frac{g^{(n)}(x)}{n!} z^n \) and their compositae \( F^\Delta(n, k, x), G^\Delta(n, k, x) \).

Then the generating function \( A(x, z) = F(x, z) + G(x, z) \) has the composita

\[
A^\Delta(n, k, x) = F^\Delta(n, k, x) + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^\Delta(i, j, x) G^\Delta(n-i, k-j, x) + G^\Delta(n, k, x).
\]

**Proof.** without proof \(\square\)

**Example 3.3.** Let there be \( f(x) = x^2, F(x, z) = 2xz + z^2, \) a composita \( F^\Delta(n, k, x) = \binom{n}{k} (2x)_2^{2k-n} \) and \( g(x) = \ln(x), G(x, z) = \ln(x+z) - \ln(x) = \ln(1 + \frac{z}{x}) \), and a composita \( G^\Delta(n, k, x) = \frac{k!}{n!} \binom{n}{k} x^{-n} \). Then for the function \( a(x) = x^2 + \ln(x) \), the Bell polynomial is

\[
B_{n,k} = \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=j}^{n-k+j} \frac{j!}{i!} \binom{i}{j} \frac{k-j}{n-i-k+j} 2^{(k-j)-n+i} x^{2(k-j)-n}.
\]

Now let us turn to finding of the composita \( Y^\Delta(n, k, x) \) of the function \( y(x) = f(x)g(x) \) expressed as the product of the functions \( f(x) \) and \( g(x) \). Let us prove the following theorem.
Theorem 3.4. Let there be a function \( a(x) = f(x)g(x) \); then the composita of the function \( Y(x, z) = f(x + z)g(x + z) - f(x)g(x) \) is equal to
\[
Y^\Delta(n, k, x) = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) \left( \sum_{i=0}^{n} F(i, j, x) G(n - i, j, x) \right) [f(x)g(x)]^{k-j}(-1)^{k-j}.
\]
where \( F(n, k) \) are coefficients of the generating function \([f(x + z)]^k\), and \( G(n, k) = [g(x + z)]^k\).

Proof. Here we have the second peculiarity: it is necessary to take into account the rule of finding a derivative of the product. According to the Leibniz rule, we can write
\[
\frac{y^{(n)}}{n!} = \sum_{i=0}^{n} \frac{f^{(n-i)}}{i!} (n-i)!.
\]
Hence
\[
y(x + z) = f(x + z)g(x + z)
\]
Now let us find coefficients for the expression \([f(x + z)g(x + z) - f(x)g(x)]^k\). By removing the brackets and substituting the expression for the coefficients of the generating functions \( f(x + z) \) and \( g(x + z) \), we obtain the desired formula. \( \square \)

Given the composita of the generating function \( f(x + z) - f(x) - F^\Delta(n, k, x) \), the coefficients of the generating function \([f(x + z)]^k\) are calculated by the formula:
\[
F(n, k, x) = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) F^\Delta(n, j, x) f(x)^{k-j}.
\]

Example 3.5. Let there be a function \( x^a \) (see the example in [3]). Let us find an expression for the \( n \)-th derivative of this function. Let us write it in the form \( \exp(ax \ln(x)) \). For this purpose, we find a composita of the function \((x + z) \ln(x + z) - x \ln(x)\) and expressions for coefficients of the generating functions \((x + z)^k\) and \(\ln(x + z)^k\). For the first function, \( F(n, k) = \left( \begin{array}{c} k \\ n \end{array} \right) x^{k-n} \); for the second function, \( G(n, k) = \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) \frac{n!}{j!} \left( \begin{array}{c} n \\ j \end{array} \right) x^{-n} \ln(x)^{k-j} \). Then the composita of the function \((x + z) \ln(x + z) - x \ln(x)\) is
\[
A^\Delta(n, k, x) = x^{k-n} \sum_{j=0}^{k} (-1)^{k-j} \left( \begin{array}{c} k \\ j \end{array} \right) \left( \sum_{i=0}^{n} \left( \begin{array}{c} j \\ i \end{array} \right) \frac{1}{(n-i)!} \sum_{m=0}^{j} \frac{m!}{m!} \left( \begin{array}{c} j \\ m \end{array} \right) \left( \begin{array}{c} n-i \\ m \end{array} \right) \ln(x)^{k-m} \right).
\]
From this it follows that the composita of the function \( ax \ln x \) is equal to \( a^k A^\Delta(n, k, x) \). Presented below are the first terms of this composita.
\[
a(\ln x + 1)
\]
\[
\frac{a}{2x}, \quad a^2 (\ln x + 1)^2
\]

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\[ a = \frac{a}{6x^2}, \quad a^2 \ln x + a^2, \quad a^3(\ln x + 1)^3 \]
\[ \frac{a}{12x^3}, \quad -\frac{4a^2 \ln x + a^2}{12x^2}, \quad \frac{3a^3 \ln^2 x + 6a^3 \ln x + 3a^3}{2x}, \quad a^4(\ln x + 1)^4 \]

Hence the expression for the \( n \)-th derivative of the generating function \( x^{a x} \) has the form:
\[
[x^{a x}]^{(n)} = x^{a x} \sum_{k=1}^{n} \frac{n!}{k!} a^k x^{k-n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left( \sum_{i=0}^{j} \sum_{m=0}^{j} \frac{m!}{(n-i)!} \left( \frac{j}{m} \right) \right) \left( \sum_{m=0}^{j} \frac{m!}{(n-i)!} \left( \frac{j}{m} \right) \right) (\ln x)^{k-m}.
\]

Now let us consider the operation of product of compositae. For this purpose, we prove the following theorem.

**Theorem 3.6.** Let there be functions \( f(x), g(x) \) and compositae of the generating functions \( F^\Delta(n, k, x) \) for \( f(x+z) - f(x) \) and \( G^\Delta(n, k, x) \) for \( g(x+z) - g(x) \). Then for the composition of the functions \( y(x) = g(f(x)) \), the composita of the generating function \( Y(x, z) = g(f(x+z) - f(x)) \) is
\[
Y^\Delta(n, m, x) = \sum_{k=m}^{n} F^\Delta(n, k, x) G^\Delta(k, m, f(x)).
\]

**Proof.**
\[
[f(x+z) - f(x)]^m = \sum_{n \geq m} F^\Delta(n, m, x) z^n
\]

From formula (1) we have
\[
Y^\Delta(n, m, x) = \sum_{k=m}^{n} F^\Delta(n, k, x) G^\Delta(k, m, f(x)).
\]

Given the expression for the coefficients \( Y(n, k, x) \) of the generating function \( y(x+z)^k \), the expression for the coefficients of the composition of the generating functions \( a(x+z) = y(f(x+z)) \) has the form:
\[
A(n, m, x) = \begin{cases} 
\frac{y(f(x))^m}{m!}, & n = 0 \\
\sum_{k=1}^{n} F^\Delta(n, k, x) Y(k, m, x), & n > 0.
\end{cases}
\]

\( \square \)

Note that this theorem holds true for Bell polynomials as well \([3]\), because
\[
B_{n,m}(x) = \sum_{k=m}^{n} \frac{n!}{k!} F^\Delta(n, k, x) k! \frac{m!}{m!} G^\Delta(k, m, f(x)) = \frac{n!}{m!} \sum_{k=m}^{n} F^\Delta(n, k, x) G^\Delta(k, m, f(x)).
\]
Example 3.7. Let us find a composita of the function \( f(x) = \frac{1}{x} \). The generating function for the composita is \( F(x, z) = \frac{1}{x^2} - \frac{1}{z} = \frac{1 - x}{x(1 + z)} \). Hence

\[
F^\Delta(n, k, x) = \binom{n - 1}{k - 1} (-1)^n x^{-n - k}.
\]

Now let us write the composition \( a(x) = \frac{1}{\ln(x)} \). The composita for the generating function \( \ln(x + z) - \ln(x) \) is \( \frac{k!}{n!} \binom{n}{k} x^n \). From this it follows that the desired composita is equal to

\[
A^\Delta(n, m) = \sum_{k=m}^{n} \frac{k!}{n!} \binom{n}{k} x^{-n} \left(\frac{k - 1}{m - 1}\right) (-1)^k (\ln(x))^{-n - k},
\]

and the Bell polynomial is

\[
B_{n,k} = m! \sum_{k=m}^{n} \frac{k!}{n!} \binom{n}{k} x^{-n} \left(\frac{k - 1}{m - 1}\right) (-1)^k (\ln(x))^{-n - k}.
\]

Example 3.8. Let us find a Bell polynomial for the function \( a(x) = g(f(x)) \), where \( g(x) = \frac{1}{x^2}, f(x) = x + x^2 \). The composita of the function \( f(x) \) is equal to \( F^\Delta(n, k, x) = \binom{k}{n-k} (2x + 1)^{2k-n} \) (see example No. 2.2). The composita of the function \( g(x) = \frac{1}{1-x^2} \) is equal to \( F^\Delta(n, k, x) = (\binom{n-1}{k-1}) (1-x)^{-k-n} \). Using theorem 3.6 we obtain the desired Bell polynomial:

\[
B_{n,m} = \frac{n!}{m!} \sum_{k=m}^{n} \binom{k-1}{m-1} \binom{k}{n-k} (2x+1)^{2k-n} (1-x-x^2)^{-m-k}.
\]

\[
\frac{2x+1}{(-x^2-x+1)^2} + \frac{2(2x+1)^2}{(-x^2-x+1)^3}, \quad \frac{2(2x+1)^2}{(-x^2-x+1)^3}, \quad \frac{(2x+1)^2}{(-x^2-x+1)^4},
\]

\[
\frac{12(2x+1)}{(-x^2-x+1)^3} + \frac{6(2x+1)}{(-x^2-x+1)^4}, \quad \frac{6(2x+1)}{(-x^2-x+1)^4} + \frac{6(2x+1)^3}{(-x^2-x+1)^5}, \quad \frac{(2x+1)^3}{(-x^2-x+1)^6}.
\]

Example 3.9. Let us find a Bell polynomial for the function \( \tan(x) \). For this purpose, we represent the generating function as

\[
A(x, z) = \tan(x + z) - \tan(x) = \frac{\tan(x) + \tan(z)}{1 - \tan(x) \tan(z)} - \tan(x) = \frac{\tan(z) \sec(x)^2}{1 - \tan(x) \tan(z)}.
\]

Hence \( A(x, z) = f(x, \tan(z)) \), where \( f(x, z) = \frac{\sec(x)^2 z}{1 - \tan(x) z} \). Then the composita of \( f(x, z) \) is equal to

\[
F^\Delta(n, k, x) = \frac{n-1}{k-1} x^n - k \sec(x)^{2k}.
\]
The composita of the generating function \( \tan(z) \) is
\[
G^\Delta(n, k) = \frac{1 + (-1)^{n-k}}{n!} \sum_{j=k}^{n} 2^{n-j-1} \binom{n}{j} j! (-1)^{\frac{n+k}{2}+j} \binom{j-1}{k-1}.
\]

Using the theorem of product of compositae [5], we obtain the composita of the desired function:
\[
G^\Delta(n, m) = \sum_{k=m}^{n} G^\Delta(n, k) F^\Delta(k, m) = \frac{1 + (-1)^{n-k}}{n!} \sum_{j=k}^{n} 2^{n-j-1} \binom{n}{j} j! (-1)^{\frac{n+k}{2}+j} \binom{j-1}{k-1} \binom{k-1}{m-1} \tan(x)^{k-m} \sec(x)^{2m}.
\]

Hence the Bell polynomial is equal to
\[
B_{n,m} = \frac{1 + (-1)^{n-k}}{n!} \sum_{k=m}^{n} \frac{1}{2} 2^{n-j-1} \binom{n}{j} j! (-1)^{\frac{n+k}{2}+j} \binom{j-1}{k-1} \binom{k-1}{m-1} \tan(x)^{k-m} \sec(x)^{2m} - 2 \sec(x)^2 \tan(x), \sec(x)^4 \quad 6 \sec(x)^2 \tan(x)^2 + 2 \sec(x)^2, \ 6 \sec(x)^4 \tan(x), \sec(x)^6] \quad 24 \sec(x)^2 \tan(x)^3 + 16 \sec(x)^2 \tan(x), \ 36 \sec(x)^4 \tan(x)^2 + 8 \sec(x)^4, \ 12 \sec(x)^6 \tan(x), \sec(x)^8.
\]

Given the composita of the function \( \tan(x) \), we can obtain the composita of \( \cot(x) \) by representing \( \cot(x) = \frac{1}{\tan(x)} \) (see example 3.7).

**Example 3.10.** Let us derive a Bell polynomial for the function \( \arctan(x) \). For this purpose, we write the generating function
\[
A(x, z) = \arctan(x + z) - \arctan(x) = \arctan\left(\frac{z}{1 + x^2 + xz}\right).
\]
Let us find a composita of the function \( \frac{z}{1 + x^2 + xz} \). We represent it as
\[
f(x, z) = \frac{1}{(1 + x^2) \left(1 + \frac{xz}{1 + x^2}\right)}.
\]

Hence, the composita of the function \( f(x, z) \) is equal to
\[
F^\Delta(n, k) = \binom{n-1}{k-1} (-1)^{n-k} \frac{x^{n-k}}{(1 + x^2)^n}.
\]

Given the composita of the generating function \( \arctan(z) \) [5]
\[
\left(\frac{(-1)^{\frac{n+k}{2}} + (-1)^{\frac{n-k}{2}}}{2^{k+1}}\right)^k \sum_{j=k}^{n} \frac{2^j}{j!} \binom{n-1}{j-1} \binom{j}{k},
\]

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we obtain the composita of the desired generating function $A(x, z)$:

$$A^\Delta(n, m) = \sum_{k=m}^{n} \frac{n - 1}{k - 1} \frac{(-x)^{n-k}}{(1 + x^2)^n} \left( (-1)^{\frac{3k+m}{2}} + (-1)^{\frac{k-m}{2}} \right) \frac{m!}{2^{m+1}} \sum_{j=m}^{k} \frac{2^j}{j!} \left( k - 1 \right) \left[ j \right].$$

Hence the desired Bell polynomial is equal to

$$B_{n,m} = n! \sum_{k=m}^{n} \frac{n - 1}{k - 1} \frac{(-x)^{n-k}}{(1 + x^2)^n} \left( (-1)^{\frac{3k+m}{2}} + (-1)^{\frac{k-m}{2}} \right) \frac{m!}{2^{m+1}} \sum_{j=m}^{k} \frac{2^j}{j!} \left( k - 1 \right) \left[ j \right].$$

Presented below are the first terms of the Bell polynomial for the function $\arctan(x)$

$$\frac{1}{x^2 + 1}$$

$$- \frac{2x}{(x^2 + 1)^2}$$

$$\frac{1}{(x^2 + 1)^2}$$

$$6 \left( \frac{x^2}{(x^2 + 1)^3} - \frac{1}{3 (x^2 + 1)^3} \right)$$

$$- \frac{6x}{(x^2 + 1)^3}$$

$$\frac{1}{(x^2 + 1)^3}$$

$$24 \left( \frac{x}{(x^2 + 1)^4} - \frac{x^3}{(x^2 + 1)^4} \right)$$

$$12 \left( \frac{3x^2}{(x^2 + 1)^4} - \frac{2}{3 (x^2 + 1)^4} \right)$$

$$- \frac{12x}{(x^2 + 1)^4}$$

$$\frac{1}{(x^2 + 1)^4}$$

**Example 3.11.** Let us find a Bell polynomial for the function $a(x) = \frac{x}{\sqrt{1 - x^2}}$. For this purpose, we represent this function in the form $g(h(g(f(x)))) = \frac{x}{\sqrt{1 - x^2}}$. Let us write the compositae for the functions $f(x) = x^2$ and $g(x) = \frac{1}{x}$

$$F^\Delta(n, k, x) = \binom{k}{n-k} (2x)^{2k-n}$$

$$G^\Delta(n, k, x) = \binom{n-1}{k-1} (-1)^n x^{-n-k}.$$
Note that the generating function in the brackets is the generating function for Catalan numbers [5]. Given the composita of the function, we obtain the composita for $H(x, z)$

$$H^\Delta(n, k, x) = \frac{k}{n} \binom{2n-k-1}{n-1} (-1)^{n-k} (\sqrt{x})^k 2^k 4^{-n}. $$

Hence the composita of the function $\frac{1}{\sqrt{x}}$ is equal to

$$(-1)^n (\sqrt{x})^m 4^{-n} \sum_{k=m}^{n} \frac{k}{n} \binom{2n-k-1}{n-1} 2^k \binom{k-1}{m-1}. $$

This result was obtained by L. Comtet [3]. Now from theorem 3.6, we obtain the composita of the function $a(x) = \frac{x}{e^x - 1}$

$$x^{m-n} \sum_{k=m}^{n} \frac{(-1)^k}{k!} \sum_{j=m}^{k} j^2 \binom{j-1}{m-1} (2k-j-1) \binom{i}{k-1} \binom{i}{n-1} (x-1)^{-n-k} (-1)^n. $$

**Example 3.12.** Let us find a Bell polynomial for the generating function of Bernoulli numbers $a(x) = \frac{x}{e^x - 1}$. For this purpose, we write the expressions for the coefficients of the generating functions $F(x, z) = (x+z)^{m}$ and $G(x, z) = \left(\frac{1}{e^x - 1}\right)^{k}$. Hence

$$F(n, k, x) = \binom{k}{n} x^{k-n}. $$

$$G(n, k, x) = \binom{n+k-1}{k-1} (x-1)^{-n-k} (-1)^n. $$

Using formula (4) for the composition of the generating functions $g(x+z)^{m}$ and $e^{x+z}$, we obtain expressions for the coefficients of $h(x+z) = \left[\frac{1}{e^{x+z} - 1}\right]^{m}$

$$H(n, m, x) = \left\{ \begin{array}{ll} \frac{1}{(e^x-1)^m}, & n = 0 \\ \frac{1}{m} \sum_{k=0}^{m} (-1)^k k! \binom{m+k-1}{m} \binom{n}{k} (e^x-1)^{-m-k} e^{kx}, & n > 0. \end{array} \right. $$

From theorem 3.9, we obtain the composita of the product $x \frac{1}{e^x - 1}$

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left( \frac{x}{e^x - 1} \right)^{m-j} \sum_{i=0}^{n} H(i, j, x) \binom{j}{n-i} x^{-n+j+i}. $$

Then the Bell polynomial for the generating function of Bernoulli numbers has the form:

$$B_{n,m} = \frac{n!}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left( \frac{x}{e^x - 1} \right)^{m-j} \sum_{i=0}^{n} H(i, j, x) \binom{j}{n-i} x^{-n+j+i}. $$

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4 Bell polynomials of inverse functions

Theorem 4.1. Let there be given a function \( f(x) \) and its composita \( F^\Delta(n, m, x) \). For the composita \( Y^\Delta(n, m, x) \) of the inverse function \( f^{-1}(x) = y(x) \), the following recurrent expressions hold true:

\[
Y^\Delta_1(n, m, x) = \begin{cases} 
\frac{1}{F^\Delta(m,m,g(x))} \sum_{k=m+1}^{n} Y^\Delta(n, k, x) F^\Delta(k, m, y(x)) & n = m, \\
- \frac{1}{F^\Delta(m,m,y(x))} \sum_{k=m+1}^{n} Y^\Delta(n, k, x) F^\Delta(k, m, y(x)) & n > 0. 
\end{cases}
\] (5)

\[
Y^\Delta_2(n, m, f(x)) = \begin{cases} 
\frac{1}{F^\Delta(n,n,x)} \sum_{k=m}^{n} F^\Delta(n, k, x) Y^\Delta(k, m, f(x)) & n = m, \\
- \frac{1}{F^\Delta(n,n,x)} \sum_{k=m}^{n-1} F^\Delta(n, k, x) Y^\Delta(k, m, f(x)) & n > 0. 
\end{cases}
\] (6)

Proof. For self-inverse functions, the condition \( f(f^{-1}(x)) = f^{-1}(f(x)) = x \) is fulfilled. Hence from theorem 3.6, we can write

\[
\sum_{k=m}^{n} Y^\Delta(n, k, x) F^\Delta(k, m, y(x)) = \sum_{k=m}^{n} F^\Delta(n, k, x) Y^\Delta(k, m, f(x)) = \delta(n, m).
\]

Simple transformations give us formulae (5) and (6).

Example 4.2. Let us consider a simple example. Let there be a function \( f(x) = x^2 \), its composita \( F^\Delta(n, k, x) = \binom{k}{n-k}(2x)^{2k-n} \), and inverse function \( g(x) = \sqrt{x} \). Let us find an expression for the Bell polynomial of the function \( \sqrt{x} \), given the composita of the function \( f(x) = x^2 \).

In view of expression (5), we obtain

\[
Z^\Delta_1(n, m, x) = \begin{cases} 
\frac{1}{(\sqrt{x})^m}, \\
- \frac{1}{2m \sqrt{x}} \sum_{k=m+1}^{n} Z^\Delta_1(n, k, x) \binom{m}{k-m}(2 \sqrt{x})^{2m-k}, 
\end{cases}
\]

\[
m = n, \\
> m.
\]

In view of expression (6), we derive

\[
Z^\Delta_2(n, m, x) = \begin{cases} 
\frac{1}{(2x)^m}, \\
- \frac{1}{2m x^n} \sum_{k=m}^{n-1} \binom{k}{n-k}(2x)^{2k-n} Z^\Delta_2(k, m, x), 
\end{cases}
\]

\[
m = n, \\
> m.
\]

Hence the Bell polynomial for the function \( \sqrt{x} \) is equal to

\[
B_{n,m} = \frac{n!}{m!} Z^\Delta_1(n, m, x) = \frac{n!}{m!} Z^\Delta_2(n, m, \sqrt{x}).
\]
Example 4.3. Let there be a function \( f(x) = x \exp(x) \) and Lambert function \( W(x) \). Let us find an expression for the n-derivative of the function \( W(f(x)) \). From theorem 3.4 the composita of the function \( f(x) \) is equal to

\[
F^\Delta(n, k, x) = e^{kx} \sum_{i=0}^{n} \frac{k^{n-i} \binom{k}{i} x^{k-i}}{(n-i)!}
\]

and the Bell polynomial is equal to

\[
B_{n,k} = \frac{n!}{k!} e^{kx} \sum_{i=0}^{n} \frac{k^{n-i} \binom{k}{i} x^{k-i}}{(n-i)!}.
\]

Hence from theorem 4 and in view of the fact that these are self-inverse functions we obtain

\[
W^{(n)}(f(x)) = \begin{cases} 
\frac{1}{B_{1,1}} & n = 1, \\
-\sum_{k=1}^{n-1} B_{n,k} W^{(k)}(f(x)) & n > 1.
\end{cases}
\]

Now we can write

\[
W^{(n)} = \begin{cases} 
\frac{1}{1+x} e^{-x} & n = 1, \\
-\frac{e^{-n x}}{(x+1)^{n+1}} n! \sum_{m=1}^{n-1} e^{m x} \frac{W^{(m)} W^{(m)} \sum_{j=1}^{m} (-1)^{j-m} \binom{m}{j} \sum_{i=0}^{n-i} j^{n-i} (i)! (n-i)!}}{(n-i)!} & n > 1
\end{cases}
\]

Presented below are the first terms for the derivative

\[
\begin{align*}
\frac{1}{1+x} e^{-x} \\
\frac{-x-2}{(1+x)^3} e^{-2x} \\
\frac{2x^2 + 8x + 9}{(1+x)^5} e^{-3x} \\
\frac{(-6x^3 - 36x^2 - 79x - 64)}{(1+x)^7} e^{-4x} \\
\frac{24x^4 + 192x^3 + 622x^2 + 974x + 625}{(1+x)^9} e^{-5x},
\end{align*}
\]

from whence we can obtain an expression for coefficients of the sequence A042977 [6].

5 Conclusion

For derivation of the Bell polynomial of the second kind for the generating function \( Y(x, z) = y(x+z) - y(x) \), it is necessary to use the composita of the generating function that can be obtained:
1) directly from the expression $Y(x, z)$ through transformations;
2) from theorem (3.1–4.1).
Next, using formula (1), the desired polynomial is derived. The numerous examples considered in the paper convincingly prove the efficiency of the proposed methods.

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