Electromagnetic mass-models in general relativity reexamined

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Abstract

The problem of constructing a model of an extended charged particle within the context of general relativity has a long and distinguished history. The distinctive feature of these models is that, in some way or another, they require the presence of negative mass in order to maintain stability against Coulomb’s repulsion. Typically, the particle contains a core of negative mass surrounded by a positive-mass outer layer, which emerges from the Reissner-Nordström field. In this work we show how the Einstein-Maxwell field equations can be used to construct an extended model where the mass is positive everywhere. This requires the principal pressures to be unequal inside the particle. The model is obtained by setting the “effective” matter density, rather than the rest matter density, equal to zero. The Schwarzschild mass of the particle arises from the electrical and gravitational field (Weyl tensor) energy. The model satisfies the energy conditions of Hawking and Ellis. A particular solution that illustrates the results is presented.

1 Introduction

A point charge is incompatible with classical electrodynamics, because it leads to the well-known self-energy and stability problems as well as to the occurrence of “runaway” solutions of the Lorentz-Dirac equations. One way to overcome these problems is to assume that charged particles are built as singularity-free concentrations of fields that, however small, have finite size.

The first model of an “extended” charged particle was studied by Abraham [1]. This model ascribed the entire mass of the particle to the interaction energy with its own electromagnetic field. This would make the particle radius \( R \) equal to

\[
R = \frac{Q^2}{Me^2}.
\]

where \( Q \) and \( M \) are the charge and mass of the particle. This quantity is commonly known as the “classical electron radius” [2].

Shortly after, it was realized that this model was unstable and inconsistent with the Lorentz transformations of special relativity. A mechanism to overcome the electrostatic repulsion was suggested by Poincaré. He postulated the existence of non-electromagnetic cohesive forces that would hold the charge together, and make the model compatible with special relativity. Since these forces provide a phenomenological, rather than a fundamental, description of the particle, this mechanism is not really satisfactory.

Today, the Abraham-Lorentz-Poincaré model for an extended charge belongs to the history of physics. And the problems associated with the point charge theory are overcome in quantum electrodynamics via renormalization, without the necessity of introducing extended particles.

However, this does not mean that the interest in the description of extended charged particles has been lost. And although charged particles obviously belong to the quantum domain, today it is understood that the concept

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of particle structure does not negate the notion of “elementarity” \cite{3}. Rather, extended particles are still in use to model the actual particle structure and extract relevant physical predictions.

In general relativity extended models have been used by several authors to discuss some important aspects in the theory. For example, the role of gravitation in charged-particle formation has been analyzed by Cooperstock and Rosen \cite{3}. The relevance of the equation of state of “false vacuum” $\rho = -p$ to relativistic electromagnetic mass models has been discussed by Grøn \cite{4} and by Tiwari, Rao and Kanakamedala \cite{5}. The phenomenon of gravitational repulsion around elementary particles like electrons has been investigated by a number of authors \cite{3}-\cite{9}. Also, the validity of singularity theorems inside the electron has been discussed by Bonnor and Cooperstock \cite{10} as well as by the present author \cite{11}.

All the models for extended charged particles, used in the literature, exhibit the following “peculiar” feature: They need the presence of some negative mass to maintain stability against Coulomb’s repulsion. That is, independently of the working assumptions of the specific model, the picture of a (classical) charged particle is always the same: the particle should consist of a core of negative mass surrounded by a positive-mass outer layer, which emerges from the Reissner-Nordström field.

However, in “conventional” physics, the mass is always positive. And, although one can invoke that macroscopic physics does not hold within charged particles, it is natural to ask whether it is possible or not to avoid the use of negative masses in the structure of charged particles. That is, without the introduction of a negative mass, can one construct an extended model for a charged particle?

The object of this work is to show that the answer to this question is positive. In Sec. 2, we show how the Einstein-Maxwell equations can be used to construct a model of a charged particle whose gravitational and inertial masses are nowhere negative. In Sec. 3, we discuss the condition for the mass to be of electromagnetic origin. In Sec. 4, we give a simple example that illustrates the fact that the model satisfies the energy conditions of Hawking and Ellis. Sec. 5 is a summary and discussion.

## 2 Structure of the Abraham-Lorentz-Poincaré particle

In its rest frame, the charge will be described by a static, spherically symmetric distribution of matter, which is assumed to be governed by the Einstein-Maxwell equations.

We choose the line element in curvature coordinates

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $\nu$ and $\lambda$ are functions of $r$ alone. In these coordinates the energy-momentum tensor $T_{\mu\nu}$ is diagonal, viz.,

$$T_{\nu}^\mu = diag (M_0^\mu + \frac{E^2}{8\pi}, M_1^\mu + \frac{E^2}{8\pi}, M_2^\mu - \frac{E^2}{8\pi}, M_3^\mu - \frac{E^2}{8\pi}),$$

where $(0, 1, 2, 3) \equiv (t, r, \theta, \phi)$, $E$ is the usual electric field intensity, $M_{\mu\nu}$ represents the energy-momentum tensor associated with the “matter” contribution, and $M_2^\mu = M_3^\mu$ because of the spherical symmetry (We note that the symmetry does not require $M_1^\mu = M_2^\mu$).

The electrovacuum region around the particle is described by the Reissner-Nordström field, which, in curvature coordinates, has the form\footnote{In what follows we use gravitational units: $c = G = 1$.}

$$ds^2 = (1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 - (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

The charge inside a sphere of radius $r$ is given by

$$q(r) = 4\pi \int_0^r \rho_e r^2 dr,$$
where \( \rho_e \) is the charge density\(^2\). Therefore, the total charge is \( Q \equiv q(R) \).

The “effective” gravitational mass inside a sphere of radius \( r \) is given by the Tolman-Whittaker formula, viz.,

\[
M_G(r) = 4\pi \int_0^r (T_0^0 - T_1^1 - T_2^2 - T_3^3) r^2 e^{(\nu + \lambda)/2} dr
\]

By analogy with (5), the quantity

\[
\mu(r) = [(T_0^0 - T_1^1 - T_2^2 - T_3^3)e^{(\nu + \lambda)/2}],
\]

can be interpreted as an “effective” gravitational mass density.

The total mass \( M \) in (4) is then

\[
M \equiv 4\pi \int_0^R \mu_{in} r^2 dr + 4\pi \int_R^\infty \mu_{out} r^2 dr,
\]

where the subscripts “in” and “out” mean inside and outside the particle, respectively.

Outside the particle \( M_{\mu\nu} = 0 \), and \( E^2 = Q^2/r^4 \). Therefore, using (3), (4) and (7), we find

\[
\mu_{out} = \frac{Q^2}{(4\pi R^4)}.
\]

Therefore, the second term in (8) can integrated to get

\[
M = 4\pi \int_0^R \mu_{in} r^2 dr + \frac{Q^2}{R}.
\]

Now, in order to construct the relativistic version of the old Abraham-Lorentz-Poincaré model for an extended charge \[1, 2\], we set \( R = Q^2/M \). Thus, from (9) we find

\[
\int_0^R \mu_{in} r^2 dr = 0.
\]

We now assume that the effective gravitational mass density is nowhere negative, viz.,

\[
\mu(r) \geq 0.
\]

Consequently, from (10) and (11) it follows that \( \mu_{in}(r) \) must vanish everywhere within the source, viz.,

\[
\mu_{in}(r) = 0.
\]

In general relativity, because of the linear relation between the curvature tensor and \( T_{\mu\nu} \), the strong energy condition requires \( R_{\mu\nu}V^\mu V^\nu \geq 0 \) for an arbitrary non-spacelike vector \( V^\mu \). Therefore, our assumption (11) is equivalent to assuming that the “strong” energy condition is applicable within the particle.

Let us now write the Einstein-Maxwell equations associated to (2)

\[
8\pi \rho + E^2 = -e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2},
\]

\[
- 8\pi p_r + E^2 = -e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) + \frac{1}{r^2},
\]

\[
- 8\pi p_\perp - E^2 = \frac{e^{-\lambda}}{2} \left( \nu'' + \frac{\nu'^2}{r^2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right),
\]

\[
E(r) = \frac{q(r)}{r^2},
\]

where \( \rho \equiv M_0^0 \), \( p_r \equiv -M_1^1 \) and \( p_\perp \equiv -M_3^3 = -M_2^2 \) denote the rest energy density and the principal pressures of the matter present, respectively. The primes denote differentiation with respect to \( r \). In this notation the condition (12) reduces to

\[
\mu_{in} = (\rho + p_r + 2p_\perp + \frac{E^2}{4\pi}) e^{(\nu + \lambda)/2} = 0,
\]

\( ^2\)This quantity is related to the “proper” charge density \( \hat{\rho}_e \) by \( \rho_e = e^{\lambda/2}\hat{\rho}_e \).
2.1 Unequal principal pressures

Let us immediately note that if \( p_r \) were equal to \( p_\perp \), then the particle would contain some negative “rest (or inertial)” mass density. Indeed, at the boundary \( r = R \), (17) would reduce to \( \rho(R) = -Q^2/(4\pi R^4) < 0 \), because the continuity of \( E, \nu, \lambda, \) and \( \nu' \), requires \( E^2(R) = Q^2/R \) and \( \rho(R) = 0 \).

The conclusion, therefore, is that to construct an Abraham-Lorentz-Poincaré model for an extended charge with (I) Everywhere non-negative gravitational mass and (II) Everywhere positive rest mass density, the particle must have unequal principal pressures.

2.2 \( M_G = 0 \) inside the particle

From the field equations, we find that (17) is equivalent to
\[
(\nu^2e^{(\nu-\lambda)/2})'=0.
\]
(18)

The regularity conditions as well as the condition of local flatness at the center demand \( \nu' \to 0 \) as \( r \to 0 \). Therefore, from (18) it follows that \( \nu' = 0 \) and \( M_G = 0 \) throughout the source (although the “inertial” mass \( 4\pi \int_0^R T_0^0 r^2 dr > 0 \)).

On the other hand, the boundary conditions require continuity of \( \nu \), and \( \nu' \) across the boundary, defined as \( r = R \). Consequently, from (4) we get
\[
e'= (1 - \frac{M}{R}).
\]
(19)

3 The pure-field condition

The purpose of this section is to construct a model of a charged particle as a non-singular concentration of fields. With this aim we now introduce the “purely gravitational field energy”, which is represented by the Weyl tensor.

In a spherically symmetric space-time all the components of the Weyl tensor are proportional to the quantity \( W \), defined by [12, 13]
\[
W = \frac{r}{6} e^{\nu-\lambda} \frac{\nu''}{6} + \frac{\nu'^2}{4} - \frac{\nu'-\lambda'}{2r} - \frac{\nu'\lambda'}{4} + \frac{1}{r^2}.
\]
(20)

Now using the Einstein-Maxwell equations (13)-(15), one can show that
\[
M_G = [W + \frac{4\pi r^3}{3} (\rho + 2p_r + p_\perp)]e^{(\nu+\lambda)/2}.
\]
(21)

This expression is interesting because it gives the effective mass as the sum of two parts only; \( W \) and \( (\rho + 2p_r + p_\perp) \), for the “purely gravitational field” and matter contribution, respectively. It suggests that the quantity \( (\rho + 2p_r + p_\perp)e^{(\nu+\lambda)/2} \) can be interpreted as a kind of “average” effective density of the matter inside a sphere of radius \( r \).

Thus, an extended particle consisting of “pure-field” is obtained by setting the matter terms equal to zero.

\[
(\rho + 2p_r + p_\perp) = 0
\]
(22)

The effective gravitational mass arises completely from the Weyl tensor, viz.,
\[
M_G(r) = e^{(\nu+\lambda)/2}W(r)
\]
(23)

In this sense, the “equation of state” generates a model wherein the particle is composed only of charge and gravitational energy and, consequently, can be interpreted as a singularity-free concentration of fields.
Let us now focus on the properties of the model. Substituting (19) and (22) into the field equations we get

$$e^{-\lambda} \left( \frac{1}{r^2} + \frac{\lambda'}{2r} \right) - \frac{1}{r^2} = 0. \tag{24}$$

This equation can be easily integrated as

$$e^{-\lambda} = 1 + Cr^2, \tag{25}$$

where $C$ is a constant of integration, to be defined from the boundary conditions.

The final form of the interior metric is then

$$ds^2 = \left( 1 - \frac{M}{R} \right) dt^2 - \left( 1 - \frac{Mr^2}{R^3} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{26}$$

The corresponding “equations of state” are

$$\rho = -p_r + M \frac{4\pi R^3}{8}, \tag{27}$$

$$\rho = p_\perp + M \frac{2\pi R^3}{8}. \tag{28}$$

Note that $|dp_r/d\rho| = |dp_\perp/d\rho| = 1$. Which means that the distribution is consistent with the “causality condition” $|dp/d\rho| \leq 1$ (See for example Ref. [14]).

### 4 Uniform charge density

In order to illustrate our model, we assume that the charge density $\rho_e(r)$ is constant throughout the sphere. This is equivalent to assuming that the proper charge density $\hat{\rho}_e$ varies as

$$\hat{\rho}_e(r) = \hat{\rho}_e(0) e^{-\lambda(r)/2}, \tag{29}$$

where $\hat{\rho}_e(0)$ is the constant charge density at $r = 0$.

The final form of the matter distribution inside the charge is as follows

$$\rho(r) = -p_r(r) + M \frac{4\pi R^3}{8} = \frac{3M}{8\pi R^3} \left( 1 - \frac{r^2}{3R^2} \right), \quad p_r(R) = 0, \tag{30}$$

$$E^2(r) = \frac{M}{R^3} r^2. \tag{31}$$

It is not difficult to see that the resulting model is “physically reasonable”, in the sense that it is free of singularities, $p_r = p_\perp$ at $r = 0$, and $\rho > 0$ as well as $\rho \geq |p_i|$ throughout the distribution.

### 5 Discussion and conclusions

We have presented here a general-relativistic version of the old Abraham-Lorentz-Poincaré model for an extended charged particle.

In contrast to other models in the literature where $\rho = 0$, in our model the particle contains matter with positive rest density $\rho$ and positive proper “inertial” mass $4\pi \int_0^R r^2 e^{\lambda/2} dr$. Therefore, the matter and charge that make up the particle also have a positive density, viz., $T_{00} = (\rho + E^2/8\pi)$.

The fact that $T_{00}$ and $\mu$ are different can be understood from the following argument. In any volume element there is not only matter (with positive density) but also certain amount of binding energy - which is the energy necessary to maintain stability and keep the charge together. The effective matter density $\mu$ can be interpreted as the sum of

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6This assumption was used by Tiwari, Rao and Kanakamedala in their study of electromagnetic mass models [5].
the positive density $T^0_0$ and the binding energy, which is negative. Condition $\mu_{in}(r) = 0$, in Eq. (17), expresses that the binding energy, in our model, exactly balances the positive contribution from the matter and the electrical field. Because of this, in our model, the effective gravitational mass is nowhere negative. The “weak”, “dominant” and “strong” energy conditions are satisfied and there is no gravitational repulsion anywhere.

The equation of state inside the particle is (what we call the pure-field condition) $\rho + 2p_r + p_\perp = 0$, which is the anisotropic generalization of $(\rho + 3p) = 0$. This condition replaces the usual $\rho = 0$ requirement. We note that the equation of state $(\rho + 3p) = 0$ has been considered in different contexts by several authors. Notably, in discussions of cosmic strings [15]. Also, it is the only equation of state consistent with the existence of zero-point fields [16]. Another important feature in our model is that the particle must have unequal principal pressures, otherwise it would contain some negative rest mass.

From a mathematical viewpoint, the pure-field model discussed here (with $\rho + 2p_r + p_\perp = 0$) can be generalized in several ways. For example, one can assume $\mu_{in}(r) = \text{const}$, instead of $\mu_{in}(r) = 0$ as in (12). In this case the “bare (or intrinsic) mass” of the particle will be different from zero, i.e., $M_G(R) = W(R) \neq 0$, and consequently $R > Q^2/M$. All these models share similar properties in the sense that the tensions $p_r \neq p_\perp$ are responsible for holding the charge together. However, from a physical and historical viewpoint, they are different. In the Abraham-Lorentz-Poincaré model the mass is entirely of electromagnetic origin. While, if we modify (12) this is no longer so. The mass of the charged particle is now the sum of the bare “pure gravitational” mass $W(R)$ and the electromagnetic mass $Q^2/R$.

In summary, without the introduction of negative masses, here we have been able to construct a simple model where a charged particle can be visualized as a concentration of fields. The positiveness of energy inside the source, as well as the energy conditions, require the electron to be an “extreme” Reissner-Nordström source of gravity [11].

References

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