Calculations of Scattering Lengths in Four-Nucleon System on the Basis of Cluster Reduction Method for Yakubovsky Equations

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Abstract

The cluster reduction method for the Yakubovsky equations in configuration space is used for calculations of zero-energy scattering in four-nucleon system. The main idea of the method consists in making use of expansions for the Yakubovsky amplitudes onto the basis of the Faddeev components for the two-cluster sub-Hamiltonian eigenfunctions. The expansions reduce the original equations to ones for the functions depending on the relative coordinates between the clusters. On the basis of the resulting equations the \( N-(NNN) \) zero-energy scattering problems are solved numerically with the MT I–III model for \( N-N \) forces and neglecting the Coulomb interaction between protons.

1 Introduction

In this paper we continue our investigations of the four particle scattering problem making use the cluster reduction method for Yakubovsky equations in configuration space formulated in [1, 2]. The Yakubovsky differential equations (YDE) [3], being the direct generalization of the three body Faddeev approach, inherit all advantages of latter. Two main of them are the very simple way to impose the boundary conditions corresponding to all elastic and rearrangement channels and minimum inputs needed to formulate the equations and boundary conditions.

For the first time, YDE have been used for calculations of the four identical particles bound state in [4]. It was observed that even for the simple model \( N-N \) interactions the converged results within the usual numerical approximations of the YDE could be obtained with supercomputer facilities only. So,

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the progress in four nucleon calculations with YDE depends on a new numerical solution technique. For the bound state problem such a technique was elaborated in [6] and allowed to improve the results of [4] considerably. For the scattering problem one of the new solution method was introduced in [1, 2, 7].

Here, we employ the method for calculations of zero-energy \( N-(NNN) \) scattering neglecting the Coulomb interaction between protons. This system could be considered as a model for \( n-\text{He}, p-\text{H} \) or \( n-\text{H} \) scattering depending on a value of the total isospin.

2 Yakubovsky equations

In the Yakubovsky approach, the four particle wave function \( \Psi \) should be decomposed into components in one to one correspondence to all chains of partitions. The chains consist of two cluster \( a_2 \) (e.g., \((ijk)l\) or \((ij)(kl)\)) and three cluster \( a_3 \) (e.g., \((ij)kl\)) partitions obeying the relation \( a_3 \in a_2 \). The latter means that the partition \( a_3 \) can be obtained from partition \( a_2 \) by splitting of one subsystem. It is easy to see that there exist 18 chains of partitions for the four particle system.

The Yakubovsky wave function components can be defined by formulas [3, 4]

\[
\Psi_{a_3a_2} = R_{a_3}(E^+)V_{a_3}\sum_{(b, b_i) \in a_2} R_0(E^+)V_{b_i}\Psi,
\]

where

\[
R_{a_3}(E) = (H_0 + V_{a_3} - E)^{-1}, \quad R_0(E) = (H_0 - E)^{-1}, \quad E^+ = E + i0,
\]

and \( V_{a_3} \) stands for the two particle potential acting inside the two particle subsystem of a partition \( a_2 \). If the function \( \Psi \) is the solution of the Schroedinger equation

\[
(H_0 + \sum_{a_3} V_{a_3} - E)\Psi = 0,
\]

then the components \( \Psi_{a_3a_2} \) obey the Yakubovsky equations [3, 4]

\[
(H_0 + V_{a_3} - E)\Psi_{a_3a_2} + V_{a_3}\sum_{(c, c_i) \in a_2} \Psi_{c_3a_2} =

- V_{a_3}\sum_{d_2 \neq a_2} \sum_{(d, d_i) \in a_2} \Psi_{d_3d_2},
\]

There exists a remarkable rule which allows to construct the Schroedinger equation solution from the components \( \Psi_{a_3a_2} \) [3, 4]

\[
\Psi = \sum_{a_2} \sum_{a_3} \Psi_{a_3a_2}.
\]

Note that, the only inputs needed to formulate the YDE [3] are the inter-particle potentials.
The second advantage of the YDE consists in the structure of the boundary conditions. In contrast to the total wave function for the multichannel scattering, the asymptotics of the Yakubovsky component has the one channel form. Namely, $\Psi_{a_3a_2}$ involves in the asymptotical region the characteristics of the bound states for subsystems of the partitions $a_2$ only. As a consequence, the asymptotic behavior of $\Psi_{a_3a_2}$ can be described in terms of one set of the relative coordinates corresponding to the chain $a_3a_2$.

Let us turn to description of this asymptotics. To this end introduce the relative Jacobi coordinates for the four particle system. There exist 18 sets of Jacobi coordinates in one to one correspondence to 18 chains of partitions. Note, that among them only two sets are topologically different. One corresponds to chains with partitions $a_2$ of the type 3+1 (e.g., (123)4) and the other one does to chains with partitions $a_2$ of the type 2+2 (e.g., (12)(34)). These coordinates can be given by formulas:

\begin{equation}
\begin{align*}
x_{a_3} &= r_i - r_j \\
y_{a_3a_2} &= (r_i + r_j)/2 - r_k \\
z_{a_2} &= (r_i + r_j + r_k)/3 - r_l
\end{align*}
\end{equation}

for $a_2 = (ijk)l, a_3 = (ij)kl$ and

\begin{equation}
\begin{align*}
x_{b_3} &= r_i - r_j \\
y_{b_3b_2} &= r_k - r_l \\
z_{b_2} &= (r_i + r_j)/2 - (r_k + r_l)/2
\end{align*}
\end{equation}

for $b_2 = (ij)(kl), b_3 = (ij)kl$.

In this paper we are considering only binary processes, it assumes that only the two clusters channels are open. In this case, the components which correspond to the initial state with binding clusters of subsystems of the partition $l_2$ have the form

\begin{equation}
\begin{align*}
\Psi_{a_3a_2}(X, p_{l_2}) &= \delta_{a_2l_2} \psi^{a_3}_{a_2}(x_{a_2}) \exp\{i(p_{a_2}, z_{a_2})\} \\
&+ \psi^{a_3}_{a_2}(x_{a_2}) \langle U_{a_3a_2} p_{l_2}, z_{a_2} \rangle + \delta U_{a_3a_2}, l_2 (X, p_{l_2}).
\end{align*}
\end{equation}

Here $x_{a_2} = \{x_{a_3}, y_{a_3}, a_2\}$, $X = \{x_{a_2}, z_{a_2}\}$ and $p_{l_2}$ is the relative momentum conjugated to the vector $z_{l_2}$. The momentum $p_{l_2}$ and the energy of the system $E$ obey the equation

\[ E = \varepsilon_{l_2} + \delta_{l_2} \frac{\hbar^2}{m} (p_{l_2})^2. \]

$\varepsilon_{a_2}$ and $\psi^{a_3}_{a_2}$ are binding energies and the Faddeev components of the bound states for the two cluster Hamiltonians

\[ h_{a_2} = h^0_{a_2} + \sum_{a_3 \in a_2} V_{a_3} = -\frac{\hbar^2}{m} (\Delta x_{a_3} + \delta_{a_2} \Delta y_{a_3}) + \sum_{a_3 \in a_2} V_{a_3}(x_{a_3}), \]

where $\delta_{a_2} = 3/4$ for the partitions $a_2$ of the type (3+1) and $\delta_{a_2} = 1$ for the partitions $a_2$ of (2+2) type.
Amplitudes $U$ become the spherical waves

$$U_{a2l2}(z_{a2}, p_{l2}) \sim A_{a2l2} \frac{\exp\left\{i \sqrt{E - \varepsilon_{a2}}|z_{a2}|\right\}}{|z_{a2}|} \quad (|z_{a2}| \to \infty).$$  

(6)

Amplitudes $\delta U_{a2,a3,l2}$ are exponentially decreasing functions as $|X| \to \infty$ corresponding to virtual breakup processes.

3 Cluster reduction of YDE.

The cluster reduction of YDE \[2\] consists in expansion of the components $\Psi_{a3a2}$ onto the basis consisting of functions depending on the coordinates $x_{a2}$. The most suitable basis for that is the one of the eigenfunctions for the operators from the left hand side of the YDE \[2\]

$$(h_{a2}^0 + V_{a3})\psi_{a2,k}^{a3} + V_{a3} \sum_{(c3 \neq a3) \in a2} \psi_{a2,k}^{c3} \psi_{a2,k}^{a3} = \varepsilon_{a2}^k \psi_{a2,k}^{a3}.$$  

(7)

The expansion of the components has the form

$$\Psi_{a3a2} = \sum_{k=0}^{\infty} \psi_{a2,k}^{a3}(x_{a2}) F_{a2}^k(z_{a2}),$$  

(8)

where the coefficients $F_{a2}^k(z_{a2})$ depend on the relative coordinates between the clusters only.

The basis of the solutions of the eqs. \[7\] is not an orthogonal one, because the eqs. \[7\] are not Hermitean. Fortunately, it is easy to show that the Hermitean conjugated equations to the eqs. \[7\] are of the form \[8\]

$$(h_{a2}^0 + V_{a3})\phi_{a2,k}^{a3} + \sum_{(c3 \neq a3) \in a2} V_{c3} \phi_{a2,k}^{c3} = \varepsilon_{a2}^k \phi_{a2,k}^{a3}.$$  

(9)

In the papers \[8\] it is shown that equations \[7\] and \[8\] allow the solutions of two classes. They are the physical ones, i.e., the solutions of eqs. \[7\] which sum

$$\sum_{a3 \in a2} \psi_{a2,k}^{a3} = \psi_{a2,k}$$

is the solution of the Schroedinger equation

$$h_{a2} \psi_{a2,k} = \varepsilon_{a2}^k \psi_{a2,k},$$

and the solutions of eqs. \[7\] with the property

$$\phi_{a2,k}^{a3} = \psi_{a2,k}$$

for any $a3 \in a2$. The second class of solutions of eqs. \[8\] is formed by the spurious solutions for which

$$\sum_{a3 \in a2} \psi_{a2,k}^{a3} = 0.$$
Unfortunately, there is no a such simple formula for the spurious solutions of eqs. (9). It is important to note, that the complete basis is formed only by the physical and spurious solutions together. Complete bases of solutions of eqs. (7) and (9) are biorthogonal

\[ \sum_{a_3 \in a_2} \langle \phi_{a_2,l} | \psi_{a_2,k}^a \rangle = \delta_{lk}. \]

This property allows to apply the projection technique in order to obtain the equations for coefficients \( F_{a_2}^k \) from series (8) after substitution of eqs. (8) into eqs. (2). The result reads

\[ \left( -\frac{\hbar^2}{m} \delta_{a_2} \Delta z_{a_2} - E + \varepsilon_{a_2}^k \right) F_{a_2}^k (z_{a_2}) = \\
- \sum_{a_3 \in a_2} \langle \phi_{a_2,k}^a | V_{a_3} \sum_{d_3 \neq a_2} \sum_{d_2 \neq a_2} \sum_{l \geq 0} \psi_{d_2,l}^a F_{d_2}^l (z_{d_2}) \rangle, \]

(10)

where the brackets \( \langle . | . \rangle \) stand for the integration over \( x_{a_2} \).

The asymptotic boundary conditions for the coefficients \( F_{a_2}^0 (z_{a_2}) \) can be obtained from (3), (4) by projecting and have the following 'two body' form

\[ F_{a_2}^0 (z_{a_2}) \sim \delta_{a_2 l_2} \exp i (p_{a_2} z_{a_2}) + A_{a_2 l_2} z_{a_2}^{-1} \exp i \sqrt{E - \varepsilon_{a_2}^0 z_{a_2}} \]

for the open channels and

\[ F_{a_2}^k (z_{a_2}) \sim 0, \quad k \geq 1 \]

for the closed channels.

The equations (10) are the desired coupled channel equations for the two cluster collisions in the four body system.

4 Application to the \( N − NNN \) scattering problem.

The eqs. (10) can be applied to solving the \( N − NNN \) scattering problem after a suitable partial wave analysis. We have used the MTI-III model for the \( N − N \) forces and the isotopic spin formalism. Within this conditions and neglecting the Coulomb interaction the channels corresponding to total value of isospin \( T \) are independent.

The calculated values of scattering lengths in \( N − NNN \) system for a given values of the total spin \( S \) and the total isospin \( T \) are given in the Tab. 1 with data of direct solution of YDE from the paper [3]. As one can see the agreement of calculations is rather reasonable. In the Tab. 2 we collect the available data for \( n − ^3H \ (T = 1) \) scattering lengths calculated within various approaches. As one can see the agreement of the calculations is quite good. By the end in the Tab. 3, we give the results of calculations of scattering lengths in the channel \( T = 0 \) performed under conditions when the spurious part of basis functions is removed from expansions (5) in order to show how the presence of the spurious solutions affects on the results of calculations.
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Tab. 1

| S | T | 3 | present work |
|---|---|---|--------------|
| 0 | 0 | 14.75 | 14.7         |
| 1 | 0 | 3.25  | 2.9          |
| 0 | 1 | 4.13  | 4.0          |
| 1 | 1 | 3.73  | 3.6          |
| Refs.  | $A_{S=0}$ fm | $A_{S=1}$ fm |
|--------|--------------|--------------|
| Present work | 4.0          | 3.6          |
| [10]   | 4.09         | 3.61         |
| [11]   | 4.23         | 3.46         |
| [12]   | 3.905        | 3.597        |
| [13]   | 3.8          | 4.9          |
| [14]   | 3.38         | 3.25         |
| [15]   | 4.13         | 3.73         |
| exp. [15] | $3.91 \pm 0.12$ | $3.6 \pm 0.1$ |
Tab. 3

| S | A | 3 + 1 phys. | 3 + 1 spur. | 2 + 2 phys. | 2 + 2 spur |
|---|---|-------------|-------------|-------------|------------|
| 0 | 14.7 | +           | +           | +           | +          |
| 0 | 15.4 | +           | −           | +           | −          |
| 1 | 2.9  | +           | +           | +           | +          |
| 1 | 1.4  | +           | +           | +           | −          |
| 1 | 1.2  | +           | −           | +           | −          |