Semi-Parametric Contextual Bandits with Graph-Laplacian Regularization

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Abstract

Non-stationarity is ubiquitous in human behavior and addressing it in the contextual bandits is challenging. Several works have addressed the problem by investigating semi-parametric contextual bandits and warned that ignoring non-stationarity could harm performances. Another prevalent human behavior is social interaction which has become available in a form of a social network or graph structure. As a result, graph-based contextual bandits have received much attention. In this paper, we propose \textbf{SemiGraphTS}, a novel contextual Thompson-sampling algorithm for a graph-based semi-parametric reward model. Our algorithm is the first to be proposed in this setting. We derive an upper bound of the cumulative regret that can be expressed as a multiple of a factor depending on the graph structure and the order for the semi-parametric model without a graph. We evaluate the proposed and existing algorithms via simulation and real data example.

Keywords: Contextual multi-armed bandits, Graph Laplacian, Semi-parametric reward model.
1 Introduction

In contextual multi-armed bandits (MAB), a learning agent sequentially chooses actions while balancing to maximize the reward (exploitation) and to learn the reward mechanism as a function of contexts with higher precision (exploration). Algorithms for contextual MAB problems have demonstrated their usefulness in many applications including recommendations of news articles, advertisements, or behavioral interventions (Li et al., 2010; Tang et al., 2013; Tewari and Murphy, 2017). Thompson sampling (TS)-based algorithms randomly choose an action from repeatedly updated posterior, and have been widely used among other bandit algorithms (Agrawal and Goyal, 2013; Kaufmann, Korda, and Munos, 2012; Scott, 2010).

The semi-parametric contextual bandit (Greenewald et al., 2017; Kim and Paik, 2019; Krishnamurthy, Wu, and Syrgkanis, 2018) models the mean of the reward by a linear function of the contexts and a time-varying intercept. The algorithms for semi-parametric models allow the reward distribution to change over time in a non-stationary manner. For example, behavior may change over time depending on the user’s circumstances or preference for a shopping item may change according to a time trend. These may not be captured in the context vectors. In the single-user setting, the semi-parametric bandits have demonstrated success in accommodating non-stationarity in mobile health and product recommendation (Greenewald et al., 2017; Kim and Paik, 2019; Liao et al., 2020; Peng et al., 2019).

In many real-life settings, there are multiple users and the relationships among the users in a social network are often available as side information. Such graph information has been utilized in recommendation (Delporte et al., 2013; Li et al., 2010; Rao et al., 2015). Several graph-based contextual MAB algorithms have been proposed to take the graph information into account under the ordinary linear reward assumption (Casa-bianchi et al., 2013; Gentile, Li, and Zappella, 2014; Li et al., 2019; Vaswani, Schmidt, and Lakshmanan, 2017; Yang, Toni, and Dong, 2020). The aforementioned graph-based methods have shown to take advantages of a graph structure and perform well, but may be restrictive in real-life settings when the rewards tend to change over time.

Our goal is to construct a semi-parametric bandit algorithm that accommodates multiple users equipped with a network, with practically feasible computational cost. To the best of our knowledge, our algorithm is the first algorithm proposed in this setting. The main contributions of the work presented in this paper are as follows.

- We propose *SemiGraphTS* (semi-parametric-graph-Thompson-sampling), a novel TS algorithm for a setting in which each user’s reward follows the semi-parametric model and user-specific parameters are regularized by the given graph.
- We derive an upper bound of the cumulative regret for *SemiGraphTS*, which be expressed as a multiple of a factor depending on the graph structure and the bound from the semi-parametric model without a graph.
We study the semi-parametric contextual bandit problem for multiple users although with the single-user semi-parametric bandit problem. Whatever gradually and abruptly, when entering, and we omit the subscript. Regret at time $t$ is formally described as filtration $\mathcal{F}_{t-1} = \{j_t, \{b_i(t)\}_{i=1}^N\} \cup \left(\bigcup_{\tau=1}^{t-1} \{j_\tau, \{b_i(\tau)\}_{i=1}^N, a(\tau), r_{a(\tau), j_\tau}(\tau)\}\right)$.

The multiple-user semi-parametric reward model is described as below:

$$r_{i,j}(t) = \nu_j(t) + b_i(t)^T \mu_j + \eta_{i,j}(t),$$

for $i = 1, \ldots, N$, $j = 1, \ldots, n$, and $t = 1, \ldots, T$. Here, $\mu_j \in \mathbb{R}^d$ denotes the unknown user-specific parameter that represents the preference of the $j$-th user for a given context. The intercept $\nu_j(t)$ indicates the baseline reward for user $j$ at time $t$. We do not impose any parametric assumption on the functional form of $\nu_j(t)$; we allow the baseline to arbitrarily change over time and users, whatever gradually and abruptly. When $\nu_j(t) = 0$ for all $j$, (1) is reduced to the standard linear reward model. Without loss of generality, we assume a uniform boundedness of the contexts and true parameters, i.e., $|\nu_j(t)| \leq 1, \|b_i(t)\| \leq 1$ and $\|\mu_j\| \leq 1$ for all $i, j$ and $t$, where $\|\cdot\|$ denotes the vector $\ell_2$ norm. This assumption can be satisfied by rescaling the data. We assume that the random error $\eta_{i,j}(t)$ satisfies $\mathbb{E}(\eta_{i,j}(t)|\mathcal{F}_{t-1}) = 0$. If $n = 1$, (1) coincides with the single-user semi-parametric bandit problem.

The optimal arm $a^*(t)$ is defined as the arm that maximizes the expected reward for the $j_t$-th user given the history, that is, $a^*(t) = \arg\max_i \mathbb{E}(r_{i,j_t}(t)|\mathcal{F}_{t-1}) = \arg\max_i \{\nu_j(t) + b_i(t)^T \mu_j\} = \arg\max_i \{b_i(t)^T \mu_j\}$. Although $a^*(t)$ may be different across users, in each round $t$, only one user enters, and we omit the subscript. Regret at time $t$ is defined by the difference between the expected rewards from the optimal arm and the chosen arm,

$$\text{regret}(t) = \mathbb{E}(r_{a^*(t), j_t}(t)|\mathcal{F}_{t-1}) - \mathbb{E}(r_{a(t), j_t}(t)|\mathcal{F}_{t-1}) = b_{a^*(t)}(t)^T \mu_j - b_{a(t)}(t)^T \mu_j.$$
The goal of the agent is to minimize the cumulative regret, $R(T) = \sum_{t=1}^{T} \text{regret}(t)$.

In graph-based bandit settings, the user network is given a priori as the side information. Without any information on the user network, the problem reduces to learning $n$ independent instances. Let $G = (V, E)$ be an undirected simple graph, where a node $j \in V = \{1, \ldots, n\}$ corresponds to a user and an edge $\{j, k\} \in E$ represents the link between users. There are several ways to uniquely represent $G$ as a Laplacian matrix $L = (l_{jk}) \in \mathbb{R}^{n \times n}$. We employ the random-walk normalized Laplacian defined by

$$l_{jj} = 1, \quad l_{jk} = \begin{cases} -1/\text{deg}(j) & \text{if } \{j, k\} \in E, \\ 0 & \text{otherwise}, \end{cases}$$

for $j, k = 1, \ldots, n$ with $j \neq k$. In addition, let $\Delta_j = \sum_{k=1}^{n} l_{jk} \mu_k = \mu_j - \sum_{k: \{j, k\} \in E} \mu_k / \text{deg}(j)$. The choice of random-walk normalized Laplacian is particularly useful in the regret analysis and discussed after the proof sketch.

Our working assumption is that $\|\Delta_j\|$ is small for all $j$, i.e., the edges encode the affinity of user preferences. Without loss of generality, we assume that $G$ is connected. If not, each connected component of users do not share any information of parameters and it suffices to learn each connected component separately.

In addition, let $\|x\|_A = \sqrt{x^T A x}$ for $x \in \mathbb{R}^d$ and a positive semi-definite $A \in \mathbb{R}^{d \times d}$. A matrix-valued inequality $A \geq B$ ($A > B$) denotes that $A - B$ is positive semi-definite (positive definite).

### 2.1 Related work

Since linear contextual MAB problems for single users were investigated (Abbasi-Yadkori, Pal, and Szepesvári, 2011; Agrawal and Goyal, 2013), there has been a rich line of works on contextual bandits in recent years. For conciseness, we focus on works that consider either the semi-parametric model for single user or the linear model for multiple user equipped with graph.

**Semi-parametric contextual MABs for single user.**

The semi-parametric reward model for a single user (Greewewald et al., 2017; Kim and Paik, 2019; Krishnamurthy, Wu, and Syrgkanis, 2018) assumes, say

$$r_i(t) = \nu(t) + b_i(t)^T \mu + \eta_i(t),$$

which is a special case of our model (1) with $n = 1$. Greewewald et al. (2017) first proposed (3). A novel challenge in the semi-parametric bandit problem is to mitigate the confounding effect from the baseline reward. Greewewald et al. (2017) considered a two-stage TS algorithm that fixes a random base action and contrasts the base and other actions. Krishnamurthy, Wu, and Syrgkanis (2018) proposed another TS algorithm that contrasts every pair of actions repeatedly.
Kim and Paik (2019) proposed a single-step TS algorithm and arguably the state-of-the-art in this setting. Specifically, for each time $t$, they estimate $\mu$ in (3) by

$$\hat{\mu}(t) = B(t)^{-1} \sum_{\tau=1}^{t-1} 2X_{\tau}r(a(\tau), k(\tau)),$$

where $X_{\tau} = b(a(\tau)) - \mathbb{E}(b(a(\tau)|F_{\tau-1})$ and $B(t) = \hat{\Sigma}_t + \Sigma_t + I_d$ where $\hat{\Sigma}_t = \sum_{\tau=1}^{t-1} X_{\tau}X_{\tau}^T$ and $\Sigma_t = \sum_{\tau=1}^{t-1} \mathbb{E}(X_{\tau}X_{\tau}^T|F_{\tau-1}).$

Compared with Agrawal and Goyal (2013), a TS algorithm under the standard linear reward model, the context vector and covariance part were centered by $\mathbb{E}(b(a(\tau)|F_{\tau-1})$, which is crucial for ruling out the confounding effect of $\nu(t)$. The regret bound derived in Kim and Paik (2019) has the same order with that in Agrawal and Goyal (2013).

**Linear graph-based bandit algorithms for multiple users.**

Algorithms for graph-based linear contextual bandits have been proposed under the following model (Casa-bianchi et al., 2013; Gentile, Li, and Zappella, 2014; Li, Wu, and Wang, 2021; Li et al., 2019; Vaswani, Schmidt, and Lakshmanan, 2017; Yang, Toni, and Dong, 2020):

$$r_{i,j}(t) = b_i(t)\mu_j(t) + \eta_{i,j}(t),$$

which coincides with a special case of (1) when $\nu_j(t) = 0$. Gentile, Li, and Zappella (2014) proposed an algorithm utilizing the given graph for clustering users, where those in the same cluster are represented by the same parameter. Li et al. (2019) generalized Gentile, Li, and Zappella (2014)’s algorithm to address non-uniform user frequencies. Li, Wu, and Wang (2021) proposed another clustering-based algorithm that allows each $\mu_j$ to change abruptly over time. The regret bound proposed in this work depends on the number of abrupt shifts and can be linear in $T$ if the shifts occur proportionally to $T$. On the other hand, Casa-bianchi et al. (2013) and Vaswani, Schmidt, and Lakshmanan (2017) proposed UCB- and TS-based algorithms with regret bound $\tilde{O}(dn\sqrt{T})$, where the entire parameters for all users are estimated under regularization by a graph Laplacian. However, this led to scalability issues as a result of solving an equation involving $nd$ by $nd$ matrix. Yang, Toni, and Dong (2020) proposed a local version of the Casa-bianchi et al. (2013) with an improved regret bound $\tilde{O}(\Phi d\sqrt{nT})$, where $\Phi \in (0, 1)$ depends on $G$. It updates only the parameter associated with the user to serve at each round. Specifically, Yang, Toni, and Dong (2020) first calculates the ordinary least squares estimator $\tilde{\mu}_k(t)$ for each user $k$ as if running $n$ bandits independently. Then, $\mu_j$ is estimated by adjusting $\tilde{\mu}_j$ for $\tilde{\mu}_k(t)$ weighted by the Laplacian, particularly

$$\hat{\mu}_j(t) = \tilde{\mu}_j(t) - \lambda C_{ji}(t)^{-1} \sum_{k=1}^{n} l_{ji,k} \tilde{\mu}_k(t),$$

where $\lambda$ is a tunable parameter and $C_{ji}$ is the gram matrix of the selected arm features for user $j_t$ up to time $t$.

**3 Proposed Algorithm**

We observe rewards that are correlated with neighbors defined from the given graph structure and yet whose conditional mean changes over time. Our main challenge is to incorporate the network information in estimating $\mu_j$ while
handling the confounding by \( \nu_j(t) \). Our strategy is to handle non-stationarity for each individual by conditioning, while simultaneously accommodating information from neighbors. The key idea of conditioning is based on that the non-stationarity does not change across the arms, hence centering the context around the mean for the arms does not alter the problem of finding the maximum reward across the arms. This allows us to construct an estimator of \( \mu_j \) that is robust to the effect of \( \nu_j(t) \) while exploiting the user affinity information via graph.

The proposed SemiGraphTS algorithm is described in Algorithm 1. Key steps include parameter estimation and Thompson sampling steps.

**Algorithm 1 Proposed algorithm (SemiGraphTS)**

1: Fix \( \lambda > 0 \). Set \( B_j(1) = \lambda l_{jj}I_d, \) \( y_j(1) = 0_d \) and \( v_j = (4R + 12)\sqrt{d \log \{ (24 T^4 / \delta) (1 + \lambda^{-1}) \} + \sqrt{X(1 + \| \Delta_j \|) } } \) for \( j = 1, \ldots, n \).
2: for \( t = 1, 2, \ldots, T \) do
3: Observe \( j_t \).
4: for \( j = 1, 2, \ldots, n \) do
5: if \( j \neq j_t \) then
6: Update \( B_j(t + 1) \leftarrow B_j(t), \mu_j(t + 1) \leftarrow \mu_j(t), \) and \( y_j(t + 1) \leftarrow y_j(t). \)
7: else
8: \( \mu_j(t) \leftarrow \mu_j(t) - B_j(t)^{-1} \sum_{k \neq j} \lambda_{jk} \mu_k(t). \)
9: \( \Gamma_j(t) \leftarrow B_j(t) + \lambda^2 \sum_{k \neq j} l_{jk}^2 B_k(t)^{-1} \)
10: Sample \( \mu_j(t) \) from \( N(d(\mu_j(t), v_j^2 \Gamma_j(t)^{-1})}. \)
11: \( \pi_i(t) \leftarrow \mathbb{P}(a(t) = i | F_{t-1}), i = 1, \ldots, N. \)
12: \( b(t) \leftarrow \sum_{i=1}^{N} \pi_i(t) b_i(t) \) and \( X_t \leftarrow b_{a(t)}(t) - b(t). \)
13: Update \( B_j(t + 1) \leftarrow B_j(t) + X_t X_t^T + \sum_{i=1}^{N} \pi_i(t) (b_i(t) - b(t)) (b_i(t) - b(t))^T \), \( y_j(t + 1) \leftarrow y_j(t) + 2X_t r_{a(t), j}(t), \) and \( \mu_j(t + 1) \leftarrow B_j(t + 1)^{-1} y_j(t + 1). \)
14: end if
15: end for
16: end for

In the parameter estimation step, we propose a novel estimator \( \hat{\mu}_j(t) \) for the \( j_t \)-th user, which is constructed as follows. Define \( T_{j,t} = \{ \tau : j_{\tau} = j, 1 \leq \tau \leq t \} \), i.e, \( T_{j,t} \) collects time indices when user \( j \) is served up to time \( t \). We first calculate an unadjusted user-specific estimator \( \mu_k(t) (k = 1, \ldots, n) \) proposed by

\[
\mu_k(t) = B_k(t)^{-1} \sum_{\tau \in T_{k,t-1}} 2X_{\tau} r_{a(\tau), k}(\tau), \tag{5}
\]

where

\[
B_k(t) = \Sigma_{k,t} + \Sigma_{k,t} + \lambda I_d, \tag{6}
\]
The choice of the tuning parameter $\lambda$ controls the influence of the graph structure. For a larger $\lambda$, (7) indicates that adjacent nodes more profoundly affect on $\tilde{\mu}_{jt}$. Our regret analysis does not make any assumptions based on $\lambda$, except for $\lambda > 0$.

In the Thompson sampling step, we propose to sample $\tilde{\mu}_{jt}(t)$ from $\mathcal{N}_{d}(\tilde{\mu}_{jt}(t), v_{jt}^2, \Gamma_{jt}(t)^{-1})$, where

$$\Gamma_{jt}(t) = B_{jt}(t) + \lambda^2 \sum_{k \neq j} l_{jk}^2 B_{kt}(t)^{-1}. \quad (8)$$

The choice of $\Gamma_{jt}(t)$ in the variance part replaces a conventional choice $B_{jt}(t)$. Since each $B_{kt}(t)$ is positive definite, it holds that $\Gamma_{jt}(t)^{-1} < B_{jt}(t)^{-1}$. This intuitively means that $\Gamma_{jt}(t)$ contains more information than $B_{jt}(t)$ by incorporating the neighborhood information. As a result, our proposed sampling searches over narrower region around $\tilde{\mu}_{jt}(t)$ than the sampling with variance $v_{jt}^2 B_{jt}(t)^{-1}$. This leads to an improvement of regret up to a factor less than one compared to an algorithm without graph, as we will see in the next Section. Finally, we select the arm $a(t)$ that satisfies $a(t) = \arg\max_i \{b_i(t)^T \tilde{\mu}_{jt}(t)\}$.

It is worth mentioning that the proposed estimator $\tilde{\mu}_{jt}$ and Thompson sampling step are local, in a sense that we run the procedure only for user $j_t$ at each time, not for the entire users. The idea of local update appears natural because we have no updated information about the other nodes at time $t$. 
The terms related to the conditional expectation can be calculated as follows. We define $\pi_i(t)$ as the probability of choosing the $i$-th arm at time $t$, that is, $\pi_i(t) = \mathbb{P}(a(t) = i|\mathcal{F}_{t-1})$. This is determined by the posterior distribution of $\tilde{\mu}_{ji}(t)$, which calls for the evaluation of an integral of a multivariate normal density on a polytope. One may employ well-known approximation algorithms for the integral, for example, Wilhelm and Manjunath (2010) and Botev (2017). In our experiments on both synthetic and real data, the Monte Carlo approximation performed well. Once $\pi_i(t)$ is obtained, we can calculate $\mathbb{E}(b_{a(t)}(t)|\mathcal{F}_{t-1}) = \mathbb{E}(\sum_{i=1}^{N} I(a(t) = i)b_i(t)|\mathcal{F}_{t-1}) = \sum_{i=1}^{N} \pi_i(t)b_i(t)$. Similarly, $\mathbb{E}(X_iX_i^T|\mathcal{F}_{t-1}) = \sum_{i=1}^{N} \pi_i(t)(b_i(t) − \bar{b}(t))(b_i(t) − \bar{b}(t))^T$, where $\bar{b}(t) = \mathbb{E}(b_{a(t)}(t)|\mathcal{F}_{t-1})$.

The computation complexity of the proposed algorithm is $O(d^2N + d^2\text{deg}(j_t) + M(d^2 + dN))$ if we use the Monte Carlo approximation for evaluating $\pi_i(t)$, where $M$ is the number of Monte Carlo samples. Note that the complexity does not depend on $n$; thus, the proposed algorithm is scalable for large graphs, provided that the average degree of nodes is in a moderate range. To see why, first, $\tilde{\mu}_{ji}(t)$ and $\Gamma_{ji}(t)$ in (7) requires $O(d^2\text{deg}(j_t))$ computations given $\tilde{\mu}_k(t)$. As for $\tilde{\mu}_k(t)$ and $B_k(t)$, note that $B_j(t) = B_j(t-1)$ and $\tilde{\mu}_j(t) = \tilde{\mu}_j(t-1)$ if $j \neq j_t$. Thus, $\tilde{\mu}_k(t)$ and $B_k(t)$ is computed only for $k = j_t$, which requires $O(d^2N)$ operations. In addition, the Thompson sampling step and the approximation for $\pi_i(t)$ cost $O(M(d^2 + dN))$. To compare with the fastest algorithms in similar settings, Kim and Paik (2019) and Yang, Toni, and Dong (2020) require $O(d^2N + M(d^2 + dN))$ and $O(d^2\text{deg}(j_t))$ operations, respectively. Although the proposed algorithm has slightly increased order, in the Experiments Section, we demonstrate that the actual runtime of the proposed method is comparable to those fastest algorithms.

4 Regret Analysis

We present the high-probability regret upper bound for the proposed SemiGraphTS algorithm. A sketch of proof is provided for a key step. The complete proof can be found in Appendices B and C in the Supplement Material. We assume that the noise term $\eta_{i,j}(t)$ given $\mathcal{F}_{t-1}$ is $R$-sub-Gaussian, that is, for every $c \in \mathbb{R}$,

$$\mathbb{E}[\exp\{c\eta_{i,j}(t)\}|\mathcal{F}_{t-1}] \leq \exp(c^2R^2/2), \quad (9)$$

for all $i, j, t$, which is a common assumption in the literature for theoretical derivations. The regret bound for SemiGraphTS is described in the following theorem.

**Theorem 1** Assume (9) and $\delta \in (0, 1)$. Under the semi-parametric linear reward model (1), with probability $1 - \delta$, the cumulative regret from SemiGraphTS (Algorithm 1) achieves

$$R(T) \leq \sum_{j=1}^{n} O\left(\Psi_{j,T}\left\{\sqrt{d\log(|T_{j,T}|) + \sqrt{\lambda}\|\Delta_j\|}\right\}\times (10)$$
\[
\min\left\{ \sqrt{d \log(dT)}, \sqrt{\log(NT)} \right\} \sqrt{d|T_j, t| \log(|T_j, t|)} ,
\]
where \( \Psi_{j,T} = \sum_{t \in T_j, t} \| X_t \|_{\Gamma_j(t)^{-1}} / \sum_{t \in T_j, t} \| X_t \|_{B_j(t)^{-1}} . \)

We note that \( \Psi_{j,T} \in (0, 1) \) due to \( \Gamma_{j,t}^{-1} < B_{j,t}^{-1} \). A simpler representation of our regret is \( \tilde{O}(\max_j \Psi_{j,T} \cdot d \sqrt{nT \min\{ \sqrt{d}, \sqrt{\log(N)} \}}) \), if we assume \( |T_j, t| \approx T/n \) (each \( j \) is uniformly chosen at random). Compared to the regret bound derived in Yang, Toni, and Dong (2020) for the linear graph bandit, our model have additional nonparametric intercept \( \nu_j(t) \). Running Kim and Paik (2019) for each user independently under the same setting leads to the same form of regret bound with (10), except for the term \( \Psi_{j,T} \{ \sqrt{d \log(|T_j, t|)} + \sqrt{\lambda}\| \Delta_j \| \} \) is replaced with \( \sqrt{d \log(|T_j, t|)} + \sqrt{\lambda}\| \mu_j \| \). Since \( \Psi_{j,T} \in (0, 1) \), the regret bound of the propose algorithm is strictly lower than that from running Kim and Paik (2019) independently, provided \( \| \Delta_j \| \leq \| \mu_j \| \).

The outline of the proof for Theorem 1 follows Agrawal and Goyal (2013) and Kim and Paik (2019). Major modifications are made at establishing a high-probability bound for \( \tilde{\mu}_{j,t} - \mu_{j,t} \), as stated in the theorem below.

**Theorem 2** Assume that the settings for the semi-parametric linear reward model (1) holds along with (9). Let \( E^{\tilde{\mu}}(t) \) be an event satisfying
\[
E^{\tilde{\mu}}(t) = \left\{ \forall i : |b^c_i(t)^T (\tilde{\mu}_{j,t} - \mu_{j,t})| \leq s^c_{i,j,t}(t) \alpha(t) \right\} ,
\]
where \( b^c_i(t) = b_i(t) - \bar{b}(t) \), \( s^c_{i,j,t}(t) = \| b^c_i(t) \|_{\Gamma_j(t)^{-1}} \) and
\[
\alpha(t) = (4R + 12) \sqrt{2d \log \left( \frac{24t^4}{\delta} \right) \left( 1 + \frac{1}{\lambda} \right)} + \sqrt{2\lambda(1 + \| \Delta_j \|)} .
\]
For all \( \delta \in (0, 1) \) and \( t \geq 1 \), \( \mathbb{P}(E^{\tilde{\mu}}(t)) \geq 1 - \delta/t^2 \).

The proof for Theorem 2 carefully leverages the structures of \( B_{j,t} \) and \( \Gamma_{j,t} \). First, the lemma below enables us to induce \( s^c_{i,j,t}(t) \) from \( |b^c_i(t)^T (\tilde{\mu}_{j,t} - \mu_{j,t})| \) while encapsulating the other terms into quadratic forms associated with \( B_{j,t}^{-1} \).

**Lemma 3** For any \( x, y \in \mathbb{R}^d \) and \( j = 1, \ldots, n \),
\[
x^T B_{j,t}(t)^{-1} y \leq \sqrt{2}\| x \|_{\Gamma_j(t)^{-1}} \| y \|_{B_j(t)^{-1}} .
\]

**Proof** For simplicity, let \( B_j = B_{j,t}(t) \) and \( \Gamma_j = \Gamma_{j,t} \) for all \( j \). By the Cauchy-Schwartz inequality, \( x^T B_j^{-1} y = x^T \Gamma_j^{-\frac{1}{2}} B_j^{-\frac{1}{2}} B_j^{-1} \Gamma_j B_j^{-1} y \leq \| x \|_{\Gamma_j^{-1}} \sqrt{y^T B_j^{-1} \Gamma_j B_j^{-1} y} . \) Note that \( B_j^{-1} \preceq (\lambda_{j,j})^{-1} I_d \). Then, by (6) and (8), \( B_j^{-1} \Gamma_j = I_d + \sum_{k \neq j} \lambda_{j,k}^2 I_{k,k} B_j^{-1} B_k^{-1} \leq \)
Then, we utilize the lemma below to simplify random quadratic forms caused by neighboring users’ intermediate estimators \(\bar{\mu}_k(t)\) \((k \neq j_i)\).

**Lemma 4** For any \(x \in \mathbb{R}^d\) and \(j, k = 1, \ldots, n\),
\[
\|B_k(t)^{-1}x\|_{B_j(t)^{-1}} \leq \|x\|_{B_k(t)^{-1}}/\sqrt{\lambda^2 I_{jj}l_{kk}}.
\]

**Proof** By (6), if suffices to show \((uId + A)(vId + B)(uId + A) \geq uv(uId + A)\) for any scalars \(u, v > 0\) and positive semi-definite matrices \(A, B\). Observe that \((uId + A)(vId + B)(uId + A) = v(uId + A)^2 + (uId + A)B(uId + A) \geq v(uId + A)^2 = u^2v(Id + u^{-1}A) \geq u^2v(Id + u^{-1}A) = uv(uId + A)\), which completes the proof. \(\square\)

Finally, we separately bound each of the simplified terms by employing the technique of Abbasi-Yadkori, Pal, and Szepesvári (2011). We apply a union bound argument to obtain a uniform bound.

**Sketch of proof for Theorem 2** Detailed derivations for key inequalities are provided in Appendix B in the Supplementary Material. Suppose that the semi-parametric reward model (1) holds. Fix \(t\) and \(\delta\). Let \(\bar{\mu}_{j_i}(t), B_k(t)\) and \(\tilde{\mu}_k(t)\) be as in (7), and (5). For simplification, we write as \(b_\tau = b_{a(\tau)}(\tau)\) and \(\eta_{\tau} = \eta_{a(\tau),j_r}(\tau)\) for \(\tau = 1, \ldots, t - 1\), and \(j = j_t\) with slight abuse of notation. By algebra and Lemma 3,
\[
|b_i^c(t)^T(\tilde{\mu}_j(t) - \mu_j)| \leq \sqrt{2}s_{i,j}(t)\sum_{l=1}^{6} C_l,
\]
where
\[
C_1 = \|\sum_{k=1}^{n} \lambda_{jk} \mu_k\|_{B_j(t)^{-1}}, \quad C_2 = \|\sum_{k \neq j} \lambda_{jk} B_k(t)^{-1} \lambda_{kk} \mu_k\|_{B_j(t)^{-1}},
\]
\[
C_3 = \|\sum_{\tau \in T_{j_i,t-1}} X_{\tau} \eta_{\tau}\|_{B_j(t)^{-1}}, \quad C_4 = \|\sum_{k \neq j} \lambda_{jk} B_k(t)^{-1} \sum_{\tau \in T_{j_i,t-1}} X_{\tau} \eta_{\tau}\|_{B_j(t)^{-1}},
\]
\[
C_5 = \|A_j(t)\|_{B_j(t)^{-1}}, \quad C_6 = \|\sum_{k \neq j} \lambda_{jk} B_k(t)^{-1} A_k(t)\|_{B_j(t)^{-1}},
\]
with
\[
A_k(t) = \sum_{\tau \in T_{k,t-1}} D_\tau \mu_k + \sum_{\tau \in T_{k,t-1}} 2X_{\tau} \left(\nu_k(\tau) + \bar{b}(\tau)^T \mu_k\right),
\]
k = 1, \ldots, n, and \(D_\tau = X_{\tau} X_{\tau}^T - E(X_{\tau} X_{\tau}^T|F_{\tau-1})\).

For \(C_1\), we have \(C_1 \leq \sqrt{\lambda} \|\Delta_j\| \text{ from } B_j(t)^{-1} \leq (\lambda I_{jj})^{-1}I_d\). For \(C_2\), from Lemma 4, we have \(C_2 \leq \sqrt{\lambda} \sum_{k \neq j} (|l_{jk}|/\sqrt{l_{kk}})\|\mu_k\|\) and so \(C_2 \leq \sqrt{\lambda} \|\mu_k\| \leq 1\) and (2).

To bound \(C_3\) and \(C_4\), we first observe that applying Lemma 4 to \(C_4\) yields
\[
C_3 + C_4 \leq 2 \sum_{k=1}^{n} \frac{|l_{jk}|}{\sqrt{l_{kk}}} \left\|\sum_{\tau \in T_{j_i,t-1}} X_{\tau} \eta_{\tau}\right\|_{B_k(t)^{-1}}.
\]
Next, for each $k$, Lemma A.1 in Appendix A of the Supplementary Material yields the following with probability at least $1 - \delta((|T_{k,t-1}| + 1/n)/3t^3$:

$$\left\| \sum_{\tau \in T_{k,t-1}} X_\tau \eta_\tau \right\|_{B_k(t)^{-1}} \leq R \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda_{kk}} \right) \right\}}.$$  \hfill (14)

Since $\sum_{k=1}^{n}(|T_{k,t-1}| + 1/n) = t$, a union bound argument shows that event (14) holds for all $k = 1, \ldots, n$ with probability at least $1 - \delta/t^2$. Under this event, (13) and along with (2) yields

$$C_3 + C_4 \leq 4R \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda} \right) \right\}}.$$  \hfill (15)

Now, for $C_5$ and $C_6$, applying Lemma 4 to $C_6$ leads to

$$C_5 + C_6 \leq \sum_{k=1}^{n} \frac{|l_{jk}|}{\sqrt{l_{jj}l_{kk}}} \|A_k(t)\|_{B_k(t)^{-1}}.$$  \hfill (16)

To bound $\|A_k(t)\|_{B_k(t)^{-1}}$, we first use the definition for a fixed $k$,

$$\|A_k(t)\|_{B_k(t)^{-1}} \leq 2 \left\| \sum_{\tau \in T_{k,t-1}} X_\tau \left( \nu_k(\tau) + \bar{b}(\tau)^T \mu_k \right) \right\|_{B_k(t)^{-1}} + \left\| \sum_{\tau \in T_{k,t-1}} D_\tau \mu_k \right\|_{B_k(t)^{-1}}.$$  \hfill (17)

Using the fact that $X_\tau$ and $D_\tau$ are mean-zero random variables given $F_{\tau-1}$, we can follow the techniques in Theorem 4.2 of Kim and Paik (2019) to bound each term in the right-hand side of the equation above. Then, by a union bound argument,

$$\|A_k(t)\|_{B_k(t)^{-1}} \leq 6 \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda_{kk}} \right) \right\}}.$$  \hfill (17)

uniformly for all $k = 1, \ldots, n$ with probability at least $1 - 2\delta/(3t^2)$. Combining (16), (17) and the definition of random-walk Laplacian (2), we have with probability at least $1 - 2\delta/(3t^2)$

$$C_5 + C_6 \leq 12 \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda} \right) \right\}}.$$  \hfill (18)

Finally, plugging the bounds of $C_1, C_2, (15)$, and (18) into (11) completes the proof. \hfill $\square$

**Remark 1** Our proof used the definition of the random-walk normalized Laplacian to obtain $\sum_{k=1}^{n} |l_{jk}|/\sqrt{l_{jj}l_{kk}} = 2$. This property does not hold in general in other Laplacian representations; see also Yang, Toni, and Dong (2020) for further discussion.

**Remark 2** In deriving the regret bound in Theorem 1, we assumed that that $\pi_i(t)$ can be exactly computed, as in Kim and Paik (2019). This assumption appears reasonable since we can choose arbitrary precision to approximate $\pi_i(t)$. The additional regret caused by the uncertainty of finite Monte Carlo samples can be absorbed in the current bound; detailed discussion is provided in Appendix D of the Supplementary Material.
5 Experiments

We compared the proposed SemiGraphTS with algorithms for (i) semi-parametric bandits without exploiting graph, (ii) linear bandits exploiting graph, and (iii) linear bandits without graph. For (i), we included running Kim and Paik (2019) independently on $n$ users to fully personalize recommendations (“SemiTS-Ind”), running a single instance of Kim and Paik (2019) for all users to synchronize recommendations across users (“SemiTS-Sin”). For (ii), we considered a Laplacian regularization-based method (Yang, Toni, and Dong 2020, namely “GraphUCB”) and clustering-based methods (Li et al. 2019, “SCLUB”; Li, Wu, and Wang 2021, “DyClu”). For (iii), we included “LinTS-Ind” and “LinTS-Sin”, running Agrawal and Goyal (2013) in “independent” and “single” fashions. Every bandit algorithm involves a hyperparameter that controls the degree of exploration, either through the variance of $\tilde{\mu}(t)$ in the TS-type algorithms (e.g. $\nu_j$ in our algorithm) or through the confidence width in the UCB-type algorithms. In graph-based and independent bandit algorithms, we use the same value across users, i.e., $\nu_j = \nu$. Another hyperparameter is $\lambda$, which controls the strength incorporating the graph structure. We tuned $(\nu, \lambda)$ by a grid search for first $t_0$ rounds, with $\nu \in \{10^{-3}, 10^{-2}, 10^{-1}, 10^0, 10^1\}$ and $\lambda \in \{5^{-3}, 5^{-2}, 5^{-1}, 5^0, 5^1\}$. Then, with the best combination of hyperparameters, we assessed each algorithm for over next $T$ rounds. Other hyperparameters were set as default for each algorithm. All computations were conducted in a workstation with AMD Ryzen 3990X CPU and 256GB RAM. All results were generated over five replications. In all Figures, we report the average in solid line and the confidence band (average $\pm 1.96 \times$ (standard deviation)/$\sqrt{5}$) in light band.

**Synthetic dataset.**

We generated data under (1). We considered $\nu_j(t)$ as $\nu_j(t) = -b_{\ast j(t)}(t)^T \mu_j$ to simulate a non-stationary scenario and $\nu_j(t) = 0$ for a stationary scenario. We fixed $n = 30, N = 10, d = 40$. For each time $t$, we chose $j_t$ uniformly at random. We constructed the item features as $b_i(t) = (I(i = 1)z_1(t)^T, I(i = 2)z_2(t)^T, \ldots, I(i = N)z_N(t)^T)^T$, where $z_i(t)$ follows a uniform distribution on $d'$-dimensional sphere ($d' = d/N$). A random error $\eta_{i,j}(t)$ was generated from $N(0, 0.1^2)$. Next, the user network $G$ was generated following the Erdős-Rényi (ER) model, in which the edges were generated independently and randomly with probability $p$. We set $p = 0.4$. Then we constructed the true user-specific parameters $\mu \in \mathbb{R}^{nd}$ according to $\mu = \arg\min_{\mu' \in \mathbb{R}^{nd}} \left[ ||\mu' - \mu_0||^2 + \gamma \mu'^T(L \otimes I_d)\mu' \right]$, where $\mu_0 \in \mathbb{R}^{nd}$ is randomly initialized, $L$ is the random-walk graph Laplacian of $G$, and $\gamma \geq 0$ (Yanekelevsky and Elad, 2016). We put $t_0 = 5,000$ and $T = 50,000$.

Figure 1 displays the result for the non-stationary scenario with $\gamma = 5$. This scenario satisfies all of our assumptions. As expected, the proposed SemiGraphTS outperformed other algorithms. Compared to SemiTS-Ind that was the second-best, SemiGraphTS additionally exploited the graph structure, which might
have led to the final cumulative regret decreased by 11.5 percent. The third best was SemiTS-Sin, although it performed the best in early rounds. Since SemiTS-Sin estimates only a small number of parameters, the fitted coefficients may have been converging fast to a biased target. Another observation is that SemiGraphTS outperformed the linear graph-based methods. This may suggest that our method could robustly leverage the graph structure when non-stationarity exists. As a next experiment, we tested the same setting but under the stationary scenario $\nu_j(t) = 0$. Note that both linear and semi-parametric algorithms have theoretical guarantees for this case. The result is reported in the right panel of Figure 1. We see that the linear graph-based algorithms (GraphUCB and SCLUB) outperformed SemiGraphTS. Similarly, LinTS-Ind outperformed SemiTS-Ind. We hypothesize that accommodating the nuisance terms in semi-parametric algorithms may delay convergence of fitted coefficients, which is a price to pay for robustness.

For sensitivity analysis, we tested the performances of the algorithms against graph strength and graph misspecification. In the left panel of Figure 2, we tracked the final cumulative regrets for varying $\gamma$ from $\gamma = 0$ through $\gamma = 15$, under the non-stationary scenario. A larger $\gamma$ indicates a stronger similarity between $\mu_j$’s. For large-$\gamma$ cases, SemiGraphTS was between those of SemiTS-Ind and SemiTS-Sin. For small-$\gamma$ cases, SemiGraphTS was comparable to SemiTS-Ind and outperformed SemiTS-Sin with a large margin. The right panel of Figure 2 shows the results for misguided graphs, where we varied the proportion of node $j$s in which the signs of $\mu_j$ were reversed. When the proportion was large, SemiGraphTS behaved comparably to SemiTS-Ind, while SemiTS-Sin performed poorly.
Fig. 2: Final cumulative regrets under the non-stationary scenario, while varying $\gamma$ (left) and the proportion of sign-reversed nodes (right). All regrets are relative to that of the random selection.

Fig. 3: Average runtimes of the algorithms over varying $n$ (left), $d$ (middle), and $N$ (right).

**Scalability.**

Figure 3 reports the average runtime per step of each algorithm, varying the number of users $n$ (left panel), the number of features $d$ (middle panel), and the number of arms $N$ (right panel), fixing other settings the same as in the non-stationary synthetic experiment. SemiGraphTS was slightly slower than SemiTS-Ind. This difference is expected; the construction of $\hat{\mu}_{jt}(t)$ and $\Gamma_{jt}(t)$ depends on the degree of the node (user) to serve, which increases linearly with $n$ in the ER graph we tested. A comparison of the semi-parametric methods with the linear methods revealed that each of the semi-parametric methods costed more time than its linear counterparts, mainly due to the Monte Carlo approximation of the arm selection probability. One exception was that
Fig. 4: Current cumulative rewards for the LastFM dataset, normalized by the random selection policy.

SemiGraphTS was faster than GraphUCB as $n$ increases. Overall, SemiGraphTS demonstrated comparable efficiency for large graphs when $d$ and $N$ are in a moderate range.

**Real data example.**

The LastFM dataset\(^1\) is from a music streaming service last.fm, released by Cantador, Brusilovsky, and Kuflik (2011). The dataset consists of $n = 1,892$ nodes (users) connected by $|E| = 12,717$ edges, and 17,632 items (artists) described by 11,946 tags. It contains an aggregated table for the frequencies of (user, artist) pairs, representing the number of times a user listened to any music of an artist. We generated an artificial history of $t_0 = 5,000$ and $T = 50,000$ rounds following Casa-bianchi et al. (2013) and Gentile, Li, and Zappella (2014). In short, we randomly sampled one user to serve and $N = 25$ artists for each round. As item features, we used the first $d = 25$ principal component scores resulting from a term-frequency-inverse-document-frequency (TF-IDF) matrix of artists versus tags, treating artists as “documents” and tags as “words.” We set the reward to 1 if the selected user ever listened to a selected artist and 0 otherwise.

Figure 4 displays the cumulative rewards of the considered algorithms, relative to that of the random selection policy. SemiGraphTS produced the best final cumulative reward, 16.7 percent higher value compared to the second-best algorithms. In particular, SemiGraphTS uniformly outperformed SemiTS-Ind and SemiTS-Sin, which we believe that the proposed method might have exploited the graph structure successfully. Compared to the linear graph-based algorithms, SemiGraphTS underperformed GraphUCB in early stages but eventually outperformed them. This result is somewhat anticipated from the synthetic experiment; the presence of nuisance term might have slowed down

\(^1\)URLs: [https://last.fm/](https://last.fm/), [http://ir.ii.uam.es/hetrec2011/](http://ir.ii.uam.es/hetrec2011/)
the learning process of the proposed method but enhanced the robustness of against the change of timely trends. To summary the synthetic and real-data experiments, Proposed appears to robustly achieve desirable performances.

6 Concluding Remarks

This study proposes SemiGraphTS, the first algorithm for the semi-parametric contextual bandit MAB problem for multiple users equipped with a graph encoding similarity between user preferences. SemiGraphTS is well suited to more realistic problems in which individual baseline rewards change over time. Experiments demonstrate the potential advantage of SemiGraphTS.
Supplementary Material

In Section A, we introduce lemmas for theoretical derivation. In Appendix B, we complete the proof for Theorem 2. In Appendix C, we provide the proof for Theorem 1. Finally, in Appendix D, we discuss the derivation of the regret bound that addresses the approximation to exact $\pi_i(t)$ by Monte Carlo sampling.

Appendix A Auxiliary Lemmas

**Lemma 5** (Simplified version of Corollary 4.3 in de la Peña, Klass, and Lai (2004)) Let $X_\tau \in \mathbb{R}^d$ and $c_\tau \in \mathbb{R}$ be random variables for $\tau = 1, \ldots, t$. Let $A(t) \in \mathbb{R}^{d \times d}$ be a symmetric and positive semi-definite matrix. Suppose that, for all $u \in \mathbb{R}^d$,
$$
E \left[ \exp \left\{ u^T \sum_{\tau=1}^{t} X_\tau c_\tau - \frac{1}{2} u^T A(t) u \right\} \right] \leq 1.
$$

Then, for any $\delta \in (0, 1)$ and any symmetric positive definite matrix $Q \in \mathbb{R}^{d \times d}$, the following holds with probability at least $1 - \delta$:
$$
\left\| \sum_{\tau=1}^{t} X_\tau c_\tau \right\|_2^2 \leq \log \left\{ \frac{\det(Q + A(t))}{\det(Q)} \right\}^\frac{1}{2} \lesssim \frac{\log \left\| A(t) \right\|}{\delta^2}.
$$

The lemma below is Lemma 7 in de la Peña, Klass, and Lai (2009). See also Lemma A.3 of Kim and Paik (2019) for proof.

**Lemma 6** Let $\{F_\tau\}_{\tau=1}^{t}$ be a filtration. Let $X_\tau \in \mathbb{R}^d$ and $c_\tau \in \mathbb{R}$ be $F_\tau$-measurable random variables such that $E(X_\tau | F_{\tau-1}) = 0$, $X_\tau \perp c_\tau | F_{\tau-1}$, $\|X_\tau\| \leq B$, and $\|c_\tau\| \leq 1$ for some constant $B$, $\tau = 1, \ldots, t$. Then, for any $u \in \mathbb{R}^d$,
$$
E \left[ \exp \left\{ u^T \sum_{\tau=1}^{t} X_\tau c_\tau - \frac{1}{2} u^T \left( \sum_{\tau=1}^{t} X_\tau X_\tau^T + \sum_{\tau=1}^{t} E(X_\tau X_\tau^T | F_{\tau-1}) \right) u \right\} \right] \leq 1.
$$

**Lemma 7** (Azuma-Hoeffding inequality) If $\{M_t\}_{t=0}^{T}$ is a supermartingale satisfying $|M_t - M_{t-1}| \leq c_t$ for all $t$ almost surely, then for any $a > 0$,
$$
P(|M_T - M_0| \geq a) \leq \exp \left( -\frac{a^2}{2 \sum_{t=1}^{T} c_t^2} \right).
$$

**Lemma 8** (Abramowitz and Stegun 1964) If $Z$ is a standard normal random variable, then for any $u \geq 1$,
$$
\frac{1}{2\sqrt{\pi u}} \exp \left( -\frac{u^2}{2} \right) \leq P(|Z| > u) \leq \frac{1}{\sqrt{\pi u}} \exp \left( -\frac{u^2}{2} \right).
$$

Appendix B Proof of Theorem 2

The proof of Theorem 2 follows the sketch in the Regret Analysis Section.
B.1 Proof of (11)

By the semi-parametric reward assumption, for \( k = 1, \ldots, n \),

\[
\bar{\mu}_k(t) = B_k(t)^{-1} \sum_{\tau \in T_{k,t-1}} 2X_\tau \{ \nu(\tau) + b^T_\tau \mu_k + \eta_\tau \}
\]

\[
= B_k(t)^{-1} \left\{ \sum_{\tau \in T_{k,t-1}} 2X_\tau \nu(\tau) + \sum_{\tau \in T_{k,t-1}} 2X_\tau X^T_\tau \mu_k \right. \\
+ \left. \sum_{\tau \in T_{k,t-1}} 2X_\tau \bar{b}(\tau)^T \mu_k + \sum_{\tau \in T_{k,t-1}} 2X_\tau \eta_\tau \right\}
\]

\[
= B_k(t)^{-1} \left\{ \left( \tilde{\Sigma}_{k,t} + \Sigma_{k,t} \right) + \left( \tilde{\Sigma}_{k,t} - \Sigma_{k,t} \right) + \lambda l_{kk} I_d - \lambda l_{kk} I_d \right\} \mu_k \\
+ B_k(t)^{-1} \left\{ \sum_{\tau \in T_{k,t-1}} 2X_\tau \left( \nu(\tau) + \bar{b}(\tau)^T \mu_k \right) + \sum_{\tau \in T_{k,t-1}} 2X_\tau \eta_\tau \right\}
\]

\[
= \mu_k - \lambda l_{kk} B_k(t)^{-1} \mu_k + B_k(t)^{-1} A_k(t) + B_k(t)^{-1} \sum_{\tau \in T_{k,t-1}} 2X_\tau \eta_\tau.
\]

By the relation above, the proposed estimator satisfies

\[
\hat{\mu}_j(t) - \mu_j
\]

\[
= - \left[ B_j(t)^{-1} \sum_{k=1}^n \lambda l_{jk} \mu_k \right] + \left[ B_j(t)^{-1} \sum_{k \neq j} \lambda^2 l_{jk} l_{kk} B_k(t)^{-1} \mu_k \right]
\]

\[
+ 2 \left[ B_j(t)^{-1} \sum_{\tau \in T_{k,t-1}} X_\tau \eta_\tau \right] - 2 \left[ B_j(t)^{-1} \sum_{k \neq j} \lambda l_{jk} B_k(t)^{-1} \sum_{\tau \in T_{k,t-1}} X_\tau \eta_\tau \right]
\]

\[
+ \left[ B_j(t)^{-1} A_j(t) \right] - \left[ B_j(t)^{-1} \sum_{k \neq j} \lambda l_{jk} B_k(t)^{-1} A_k(t) \right].
\]

Now, left-multiply \( b^c_j(t)^T \) on the each side of the equation above and applying Lemma 3 on the six terms in the right-hand side yields the desired result. \( \square \)

B.2 Proof of (15)

Fix \( k = 1, \ldots, n \). Note that when \( \mathcal{F}_{\tau-1} \) and \( a(\tau) \) are given, \( X_\tau \) is fixed and \( \eta_\tau \) is \( R \)-sub-Gaussian. Then, from (7), for all \( u \in \mathbb{R}^d \),

\[
\mathbb{E} \left[ \exp \left\{ u^T X_\tau \left( \frac{\eta_\tau}{R} \right) - \frac{1}{2} u^T X_\tau X^T_\tau u \right\} \bigg| \mathcal{F}_{\tau-1}, a(\tau) \right] \leq 1.
\]
This leads to
\[
\mathbb{E} \left[ \exp \left\{ u^T \sum_{\tau \in \mathcal{T}_{k,t-1}} X_{\tau} c_{\tau} - \frac{1}{2} u^T \hat{\Sigma}_{k,t} u \right\} \right] \leq 1,
\]
which satisfies the assumption of Lemma 5 with the choice of \(X_{\tau} = X_{\tau},
\]
c\(c_{\tau} = n_{\tau} / R, Q = \lambda l_{kk} I_d + \Sigma_{k,t}\) and \(A(t) = \hat{\Sigma}_{k,t}\). Then, for any \(0 < \delta < 1\), with probability at least \(1 - \delta(|\mathcal{T}_{k,t-1}| + 1/n)/(3t^3)\),
\[
\left\| \sum_{\tau \in \mathcal{T}_{k,t-1}} X_{\tau} \eta_{\tau} \right\|_{B_k(t)^{-1}} \leq R \sqrt{\log \left\{ \frac{\det(B_k(t))}{\det(\lambda l_{kk} I_d + \Sigma_{k,t})} \right\} \left( \frac{1 + 8|\mathcal{T}_{k,t-1}|}{d \lambda l_{kk}} \right) d}.
\]
We may assume \(|\mathcal{T}_{k,t-1}| \geq 1\), otherwise the left-hand side of (B1) is zero. The determinant-trace inequality for \(\det(B_k(t))\) yields
\[
\det(B_k(t)) \leq \left( \frac{\text{tr}(B_k(t))}{d} \right)^d = \left( \frac{\text{tr}(\lambda l_{kk} I_d) + \sum_{\tau \in \mathcal{T}_{k,t-1}} \text{tr}(X_{\tau} X_{\tau}^T + \mathbb{E}(X_{\tau} X_{\tau}^T | F_{\tau-1}))}{d} \right)^d \leq \left( \lambda l_{kk} + \frac{8|\mathcal{T}_{k,t-1}|}{d} \right)^d,
\]
where we used \(\|X_{\tau}\| \leq 2\). On the other hand, since \(\Sigma_{k,t}\) is positive semi-definite, we have \(\det(\lambda l_{kk} I_d + \Sigma_{k,t}) \geq \det(\lambda l_{kk} I_d) = (\lambda l_{kk})^d\). Then, for \(d \geq 2\) and \(t \geq 1\),
\[
\frac{\det(B_j(t))}{\det(\lambda l_{kk} I_d + \Sigma_{k,t})} \leq \left( 1 + \frac{8|\mathcal{T}_{k,t-1}|}{d \lambda l_{kk}} \right)^d \leq 8^d |\mathcal{T}_{k,t-1}|^d \left( 1 + \frac{1}{\lambda l_{kk}} \right)^d.
\]
Since \(d \geq 2\), \(t > 1\), \(0 < \delta < 1\) and \(1 \leq |\mathcal{T}_{k,t-1}| \leq t\), the right-hand side of (B1) is further simplified by
\[
R \sqrt{\log \left\{ \frac{\det(B_k(t))}{\det(\lambda l_{kk} I_d)} \right\} \left( \frac{1 + 8|\mathcal{T}_{k,t-1}|}{d \lambda l_{kk}} \right)^d} \leq R \sqrt{\log \left\{ \frac{24t^3}{\delta^2} \left| \mathcal{T}_{k,t-1} \right|^d \left( 1 + \frac{1}{\lambda l_{kk}} \right)^d \right\} \left( \frac{1 + 8|\mathcal{T}_{k,t-1}|}{d \lambda l_{kk}} \right)^d} \leq R \sqrt{\log \left\{ \frac{24t^3}{\delta} \left| \mathcal{T}_{k,t-1} \right| \left( 1 + \frac{1}{\lambda l_{kk}} \right) \right\} \left( \frac{1 + 8|\mathcal{T}_{k,t-1}|}{d \lambda l_{kk}} \right)^d}.
\]
\[ \leq R \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda l_{kk}} \right) \right\}}. \]

Combining (B1) and the result above, for any \( 0 < \delta < 1 \), we have
\[
\left\| \sum_{\tau \in T_{k,t-1}} X_\tau \eta_\tau \right\|_{B_k(t)^{-1}} \leq R \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda l_{kk}} \right) \right\}}
\]

with probability at least \( 1 - \delta(\varepsilon_{\varepsilon}) + 1/n/(3t^3) \). This concludes the derivation. \( \square \)

### B.3 Proof of (16)

Fix \( k (k = 1, \ldots, n) \). Recall the definition of \( A_k(t) \),
\[
\|A_k(t)\|_{B_k(t)^{-1}} \leq 2 \left\| \sum_{\tau \in T_{k,t-1}} X_\tau (\nu(\tau) + \bar{b}(\tau)^T \mu_k) \right\|_{B_k(t)^{-1}} + \left\| \sum_{\tau \in T_{k,t-1}} D_\tau \mu_k \right\|_{B_k(t)^{-1}}.
\]

For the first term of the right-hand side of (B2), Lemma 6 yields
\[
\mathbb{E} \left[ \exp \left\{ u^T \sum_{\tau \in T_{k,t-1}} X_\tau c_\tau - \frac{1}{2} u^T \left( \hat{\Sigma}_{k,t} + \Sigma_{k,t} \right) u \right\} \right] \leq 1
\]
for any \( u \in \mathbb{R}^d \), where \( c_\tau = (\nu(\tau) + \bar{b}(\tau)^T \mu_k)/2 \). Then, we can apply Lemma 5 with \( A(t) = \hat{\Sigma}_{k,t} + \Sigma_{k,t} \) and \( Q = \lambda l_{kk} I_d \) to obtain the following inequality with probability at least \( 1 - \delta(\varepsilon_{\varepsilon}) + 1/n/(3t^3) \):
\[
\left\| \sum_{\tau \in T_{k,t-1}} X_\tau (\nu(\tau) + \bar{b}(\tau)^T \mu_k) \right\|_{B_k(t)^{-1}} \leq 2 \sqrt{\log \frac{\det(B_k(t))}{\det(\lambda l_{kk} I_d)}} \left( \frac{\varepsilon_{\varepsilon}}{\delta(\varepsilon_{\varepsilon}) + 1/n/(3t^3)^2} \right)^2.
\]

(B3)

We can bound (B3) similarly as in bounding the right-hand side of (B1). Therefore, with probability at least \( 1 - \delta(\varepsilon_{\varepsilon}) + 1/n/(3t^3) \),
\[
\left\| \sum_{\tau \in T_{k,t-1}} X_\tau (\nu(\tau) + \bar{b}(\tau)^T \mu_k) \right\|_{B_k(t)^{-1}} \leq 2 \sqrt{d \log \left\{ \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda l_{kk}} \right) \right\}}.
\]

(B4)

For the second term of the right-hand side of (B2), we let \( Y_{k,\tau} = D_\tau \mu_k \) and observe \( Y_{k,\tau} \in \mathbb{R}^d, \mathbb{E}(Y_{k,\tau}|F_{\tau-1}) = 0 \). It is straightforward from Lemma 4.4
and its proof in Kim and Paik (2019) to derive
\[
\mathbb{E} \left[ \exp \left\{ u^T \sum_{\tau \in T, t-1} \frac{1}{\sqrt{2}} Y_{k, \tau} - \frac{1}{2} u^T \left( \tilde{\Sigma}_{k,t} + \Sigma_{k,t} \right) u \right\} \right] \leq 1
\]
for any \( u \in \mathbb{R}^d \), which again satisfies the assumption of Lemma 5 with the choice of \( X_\tau = Y_{k, \tau}, c_\tau = 1/\sqrt{2} \) and \( A(t) = \Sigma_{k,t} + \hat{\Sigma}_{k,t} \). Then, putting \( Q = \lambda I_d \), we have with probability at least \( 1 - \delta(|T_{k,t-1}| + 1/n) / 3t^3 \),
\[
\left\| \sum_{\tau \in T_{k,t-1}} D_{\tau} \mu_k \right\|_{B_k(t)-1} \leq \sqrt{2} \log \left\{ \frac{\det(B_k(t))}{\det(\lambda I_d)} \frac{\delta(|T_{k,t-1}| + 1/n)}{3t^3} \right\}
\]
\[
\leq \sqrt{2} \log \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda} \right) \}. \quad (B5)
\]
Plugging (B4) and (B5) into (B2) yields bounds for each user:
\[
\mathbb{P} \left[ \right. \| A_k(t) \|_{B_k(t)-1} \leq 6 \sqrt{d \log \left( \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda} \right) \right)} \left. \right] \geq 1 - \frac{2\delta(|T_{k,t-1}| + 1/n)}{3t^3} .
\]
Finally, applying the union bound argument yields
\[
\mathbb{P} \left[ \forall k = 1, \ldots, n : \| A_k(t) \|_{B_k(t)-1} \leq 6 \sqrt{d \log \left( \frac{24t^4}{\delta} \left( 1 + \frac{1}{\lambda} \right) \right)} \right] \geq 1 - \frac{2\delta}{3t^2} , \quad (B6)
\]
which completes the proof.

Appendix C  Proof of Theorem 1

The proof incorporates the lines of Agrawal and Goyal (2013) and Kim and Paik (2019) with the proposed estimation and Thompson sampling steps. Throughout the Section, we write as \( j = j_t, b_\tau = b_{a(\tau)}(\tau) \) and \( \eta_\tau = \eta_{a(\tau), j_\tau}(\tau) \) for brevity. We reserve \( k (k = 1, \ldots, k) \) to denote user index. The proof has six steps:

(a) (Theorem 2) To establish a high-probability upper bound of \(|b^j(t)(\mu_j(t) - \mu_j)|\).

(b) (Lemma 9) To establish a high-probability upper bound of \(|b^j(t)(\hat{\mu}_j(t) - \hat{\mu}_j(t))|\) given \( F_{t-1} \).

(c) (Definition 1) To divide arms at each time \( t \) into saturated arms and unsaturated arms.

(d) (Lemma 10) To bound the probability of playing saturated arms by a function of playing unsaturated arms.

(e) (Lemma 11) To bound regret(t) given \( F_{t-1} \) for each \( t \).

(f) To bound \( R(T) \) and complete the proof.
We begin with step (b).

**Lemma 9** Let $E\tilde{\mu}(t)$ be an event defined by

$$E\tilde{\mu}(t) = \left\{ \forall i : |b_i^c(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t))| \leq v_j s_{i,j}^c(t) \min\{\sqrt{4d\log(2dT)}, \sqrt{4\log(2NT)}\} \right\}.$$

for all $t \geq 1$, $P(E\tilde{\mu}(t)|F_{t-1}) \geq 1 - 1/T^2$.

**Proof** We first show $|b_i^c(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t))| \leq v_j s_{i,j}^c(t) \sqrt{4d\log(2dT)}$. Given $F_{t-1}$, the values of $b_i^c(t)$, $\Gamma_j(t)$, and $\hat{\mu}_j(t)$ are fixed. Then, for $i = 1, \ldots, N$, we have

$$|b_i^c(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t))| = \left| v_j b_i^c(t)^T \Gamma_j(t)^{-\frac{1}{2}} \frac{1}{v_j} \Gamma_j(t)^{\frac{1}{2}} (\tilde{\mu}_j(t) - \hat{\mu}_j(t)) \right|_2 \leq v_j s_{i,j}^c(t) \sum_{t=1}^{d} Z_l(t)^2,$$

(C7)

where $Z_l(t)|F_{l-1}$ ($l = 1, \ldots, d$) identically and independently follow the standard normal distribution. We apply Lemma 8 with the choice of $u = \sqrt{2\log(2dT^2)}$. Noting $\sqrt{2\log(2dT^2)} \leq \sqrt{2\log(2^2d^2T^2)} = \sqrt{4\log(2dT)}$,

$$P\left( |Z_l(t)| > \sqrt{4\log(2dT)}|F_{l-1}\right) \leq P\left( |Z_t| > \sqrt{2\log(2dT^2)}|F_{t-1}\right) \leq \frac{1}{\sqrt{2\pi} \log(2dT^2)} \cdot \frac{1}{2dT^2} \leq \frac{1}{2dT^2},$$

for each $l = 1, \ldots, d$. Then, by a union bound argument,

$$P\left( \forall t = 1, \ldots, d : |Z_t| > \sqrt{4\log(2dT)}|F_{t-1}\right) \leq \frac{1}{2T^2}. \quad (C8)$$

Therefore, combining (C7) and (C8) yields

$$P\left( \forall i : |b_i^c(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t))| \leq v_j s_{i,j}^c(t) \sqrt{4d\log(2dT)} \right) \geq 1 - \frac{1}{2T^2}.$$

On the other hand, by the observation that $b_i^c(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t))|F_{l-1}$ ($i = 1, \ldots, N$) identically and independently follow the standard normal distribution, one can apply a similar technique to derive $|b_i^c(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t))| \leq v_j s_{i,j}^c(t) \sqrt{4\log(2NT)}$ with probability at least $1 - 1/(2T^2)$ given $F_{t-1}$. Combining the two bounds, we obtain the desired result. 

In step (c), we divide arms at each time $t$ into saturated arms and unsaturated arms. Note that $C(t)$ implicitly depends on $j_t$.

**Definition 1** Define $C(t)$, the set of saturated arms, by

$$C(t) = \{ i : b_i^c(t)^T \mu_j + g_j(T) s_{i,j}^c(t) < b_{a^*}(t)^T \mu_j \},$$

where $g_k(T) = \alpha_k(T) + v_k \min\{\sqrt{4d\log(2dT)}, \sqrt{4\log(2NT)}\}$ and $\alpha_k(T) = (4R+12) \cdot \sqrt{d \log \{(24T^4/\delta)(1+\lambda^{-1})\} + \sqrt{\lambda(1+\|\Delta_k\|)}}$, $k = 1, \ldots, n$. 


In step (d), we establish that the probability of playing saturated arms is bounded by the probability of playing unsaturated arms up to constant multiplication and addition.

**Lemma 10** Given $\mathcal{F}_{t-1}$ such that $E\tilde{\mu}(t)$ is true,

$$\mathbb{P}(a(t) \in C(t)|\mathcal{F}_{t-1}) \leq \frac{1}{p} \mathbb{P}(a(t) \notin C(t)|\mathcal{F}_{t-1}) + \frac{1}{pT^2},$$

where $p = 1/(4\varepsilon\sqrt{\pi})$.

**Proof** Since $a(t) = \arg\max_{1 \leq i \leq N} \{b_i^t(t)^T\tilde{\mu}_j(t)\}$ by definition, if $b_{a^*(t)}(t)^T\tilde{\mu}_j(t) > b_i^t(t)^T\tilde{\mu}_j(t)$ for every $i \in C(t)$, then $a(t) \notin C(t)$. This implies

$$\mathbb{P}(a(t) \notin C(t)|\mathcal{F}_{t-1}) \geq \mathbb{P}\left(\forall i \in C(t) : b_{a^*(t)}(t)^T\tilde{\mu}_j(t) > b_i^t(t)^T\tilde{\mu}_j(t)|\mathcal{F}_{t-1}\right). \quad (C9)$$

On the other hand, when $E\tilde{\mu}(t)$ is additionally true,

$$b_i^t(t)^T\tilde{\mu}_j(t) \leq b_i^t(t)^T\mu_j + g_j(T)s_{i,j}^c(t) \quad \text{(Def. of } E\tilde{\mu}(t) \text{ & } E\tilde{\mu}(t)\text{)}$$

which implies that

$$\mathbb{P}\left(b_{a^*(t)}(t)^T\mu_j < b_{a^*(t)}(t)^T\tilde{\mu}_j(t)|\mathcal{F}_{t-1}\right)$$

$$\leq \mathbb{P}\left(\forall i \in C(t) : b_i^t(t)^T\tilde{\mu}_j(t) < b_{a^*(t)}(t)^T\tilde{\mu}_j(t)|\mathcal{F}_{t-1}\right) + \left(1 - \mathbb{P}\left(E\tilde{\mu}(t)|\mathcal{F}_{t-1}\right)\right) \quad (C10)$$

The left-hand side of (C10) can be lower-bounded, because the normality of $\tilde{\mu}_j(t)$ and Lemma 8 yields

$$\mathbb{P}\left(b_{a^*(t)}(t)^T\tilde{\mu}_j(t) > b_{a^*(t)}(t)^T\mu_j|\mathcal{F}_{t-1}\right)$$

$$= \mathbb{P}\left(b_{a^*(t)}(t)^T(\tilde{\mu}_j(t) - \hat{\mu}_j(t)) > b_{a^*(t)}(t)^T(\mu_j - \hat{\mu}_j(t))|\mathcal{F}_{t-1}\right)$$

$$\geq \mathbb{P}\left(Z(t) > \frac{\alpha_j(T)}{v_j}|\mathcal{F}_{t-1}\right)$$

$$\geq \frac{1}{4\sqrt{\pi}} \exp\left(-\frac{u^2}{2}\right),$$

where $u = \alpha_j(T)/v_j$ and $Z(t)|\mathcal{F}_{t-1}$ is a standard normal random variable. Note that $u \leq 1$ by the construction. Therefore,

$$\mathbb{P}\left(b_{a^*(t)}(t)^T\tilde{\mu}_j(t) > b_{a^*(t)}(t)^T\mu_j|\mathcal{F}_{t-1}\right) \geq \frac{1}{4\varepsilon\sqrt{\pi}} = p. \quad (C11)$$

Combining (C9), (C10), (C11) and Lemma 9, we have

$$\mathbb{P}(a(t) \notin C(t)|\mathcal{F}_{t-1}) + \frac{1}{T^2} \geq p,$$

which implies

$$\frac{\mathbb{P}(a(t) \in C(t)|\mathcal{F}_{t-1})}{\mathbb{P}(a(t) \notin C(t)|\mathcal{F}_{t-1})} \leq \frac{1}{p}.$$

This completes the proof. \qed
Before proceeding to bound the cumulative regret, we bound each \( \text{regret}(t) \) given \( \mathcal{F}_{t-1} \) in step (e).

**Lemma 11** Given \( \mathcal{F}_{t-1} \) such that \( E\tilde{\mu}(t) \) is true,

\[
\mathbb{E}(\text{regret}(t)|\mathcal{F}_{t-1}) \leq \frac{5g_j(T)}{p}\mathbb{E}\left(s_{a(t),j}^c(t)|\mathcal{F}_{t-1}\right) + \frac{4g_j(T)}{pT^2}.
\]

**Proof** Let \( \bar{a}(t) = \arg\min_{a \notin C(t)} s_{i,j}^c(t) \). If \( \mathcal{F}_{t-1} \) is given, then \( \bar{a}(t) \) is deterministic. This value is also well-defined due to \( a^*(t) \notin C(t) \). Under \( \mathcal{F}_{t-1} \) such that both \( E\tilde{\mu}(t) \) and \( E\tilde{\mu}(t) \) holds,

\[
\begin{align*}
\quad & b_{a^*(t)}(t)^T \mu_j \\
& = b_{a^*(t)}(t)^T \mu_j - b_{\bar{a}(t)}(t)^T \mu_j + b_{\bar{a}(t)}(t)^T \mu_j \\
& \leq g_j(T)s_{\bar{a}(t),j}^c(t) + b_{\bar{a}(t)}(t)^T \mu_j \quad (\bar{a}(t) \notin C(t) \& \text{def. of } C(t)) \\
& \leq g_j(T)s_{\bar{a}(t),j}^c(t) + b_{\bar{a}(t)}(t)^T \tilde{\mu}_j(t) + g_j(T)s_{\bar{a}(t),j}^c(t) \quad \text{(Def. of } E\tilde{\mu}(t) \text{ and } E\tilde{\mu}(t)) \\
& \leq 2g_j(T)s_{\bar{a}(t),j}^c(t) + b_{\bar{a}(t)}(t)^T \tilde{\mu}_j(t) \quad \text{(Def. of } a(t)) \\
& \leq 2g_j(T)s_{\bar{a}(t),j}^c(t) + b_{\bar{a}(t)}(t)^T \mu_j + g_j(T)s_{\bar{a}(t),j}^c(t) \quad \text{(Def. of } E\tilde{\mu}(t) \text{ and } E\tilde{\mu}(t)),
\end{align*}
\]

which yields

\[
\text{regret}(t) \leq 2g_j(T)s_{\bar{a}(t),j}^c(t) + g_j(T)s_{\bar{a}(t),j}^c(t).
\]

Then, under \( \mathcal{F}_{t-1} \) such that \( E\tilde{\mu}(t) \) holds, the following holds from inequality above, Lemma 9 and \( |\text{regret}(t)| \leq 2 \):\[\begin{align*}
\mathbb{E}(\text{regret}(t)|\mathcal{F}_{t-1}) \\
& = \mathbb{E} \left( \text{regret}(t) I(E\tilde{\mu}(t))|\mathcal{F}_{t-1} \right) + \mathbb{E} \left( \text{regret}(t)\{1 - I(E\tilde{\mu}(t))\}|\mathcal{F}_{t-1} \right) \\
& \leq 2g_j(T)s_{\bar{a}(t),j}^c(t) + g_j(T)\mathbb{E} \left( s_{\bar{a}(t),j}^c(t)|\mathcal{F}_{t-1} \right) + 2 \left( 1 - \mathbb{P} \left( E\tilde{\mu}(t)|\mathcal{F}_{t-1} \right) \right) \\
& \leq 2g_j(T)s_{\bar{a}(t),j}^c(t) + g_j(T)\mathbb{E} \left( s_{\bar{a}(t),j}^c(t)|\mathcal{F}_{t-1} \right) + \frac{2}{T^2}. \tag{C12}
\end{align*}\]

We now further bound \( s_{\bar{a}(t),j}^c(t) \). Observe that

\[
\begin{align*}
& s_{\bar{a}(t),j}^c(t) \\
& = s_{\bar{a}(t),j}^c(t) \left\{ \mathbb{P}(a(t) \in C(t)|\mathcal{F}_{t-1}) + \mathbb{P}(a(t) \notin C(t)|\mathcal{F}_{t-1}) \right\} \\
& = s_{\bar{a}(t),j}^c(t) \left\{ \frac{2}{p} \mathbb{P}(a(t) \notin C(t)|\mathcal{F}_{t-1}) + \frac{1}{pT^2} \right\} \tag{Lemma 10} \\
& = \frac{2}{p} \mathbb{E} \left( s_{\bar{a}(t),j}^c(t)|\mathcal{F}_{t-1} \right) I(a(t) \notin C(t)) + \frac{s_{\bar{a}(t),j}^c(t)}{pT^2} \\
& \leq \frac{2}{p} \mathbb{E} \left( s_{\bar{a}(t),j}^c(t)|\mathcal{F}_{t-1} \right) I(a(t) \notin C(t)) + \frac{s_{\bar{a}(t),j}^c(t)}{pT^2} \tag{Def. of } \bar{a}(t) \\
& \leq \frac{2}{p} \mathbb{E} \left( s_{\bar{a}(t),j}^c(t)|\mathcal{F}_{t-1} \right) + \frac{1}{pT^2}. \quad (s_{i,k}^c(t) \leq 1 \text{ for any } i,k,t)
\end{align*}
\]

Combining the inequality above and (C12) conclude the proof. \( \square \)
In step (f), we complete the proof.

Proof for Theorem 1 Let
\[ M_t := \text{regret}(t)I(E^{\hat{\mu}}(t)) - \frac{5g_j(t)}{p}s_{a(t),j}(t) - \frac{4g_j(t)}{pT^2}, \quad t = 1, \ldots, T, \]
with \( M_0 = 0 \).

We apply martingale arguments for each user \( k = 1, \ldots, n \), and aggregate them by union bound. Fix \( k \) and let \( T_k = |T_{k,T}| \). Due to Lemma 11 and \( s_{a(t),k}^c(t) \leq 1 \), \( \{M_t\}_{t \in \{0\} \cup T_{k,T}} \) is a supermartingale process satisfying \( |M_t| \leq 10g_k(T)/p \). We apply Lemma 7 with the choice of \( c_t = 10g_k(T)/p \) and \( a = (10g_k(T)/p)\sqrt{2T_k \log(2T/(\delta T_k))} \) that satisfies \( \exp(-a^2/(2\sum_t c_t^2)) = \delta T_k/(2T) \). This yields
\[
\sum_{t \in T_{k,T}} \text{regret}(t)I(E^{\hat{\mu}}(t)) \leq \frac{5g_k(T)}{p} \sum_{t \in T_{k,T}} s_{a(t),k}^c(t) + \frac{4g_k(T)}{pT} + \frac{10g_k(T)}{p} \sqrt{2T_k \log(2T/(\delta T_k))} \tag{13}\]
with probability at least \( 1 - \delta T_k/(2T) \). Since \( T_1 + \ldots + T_n = T \), a union bound argument over \( k = 1, \ldots, n \) leads to
\[
\sum_{t=1}^T \text{regret}(t)I(E^{\hat{\mu}}(t)) \leq \sum_{k=1}^n \left[ \frac{5g_k(T)}{p} \sum_{t \in T_{k,T}} s_{a(t),k}^c(t) + \frac{4g_k(T)}{pT} + \frac{10g_k(T)}{p} \sqrt{2T_k \log(2T/(\delta T_k))} \right] \tag{14}\]
with probability at least \( 1 - \delta/2 \).

On the other hand, we apply a union bound argument to Theorem 2 over \( t = 1, \ldots, T \) and replace \( \delta \) with \( 3\delta/2\pi^2 \), which yields \( P(E^{\hat{\mu}}(t)) \) for all \( t = 1, \ldots, T \) \( \geq 1 - \delta/2 \). Then, \( \text{regret}(t)I(E^{\hat{\mu}}(t)) = \text{regret}(t) \) for every \( t \) with probability at least \( 1 - \delta/2 \).

Therefore, with probability at least \( 1 - \delta \),
\[
\sum_{t=1}^T \text{regret}(t) \leq \sum_{k=1}^n \left[ \frac{5g_k(T)}{p} \sum_{t \in T_{k,T}} s_{a(t),k}^c(t) + \frac{4g_k(T)}{pT} + \frac{10g_k(T)}{p} \sqrt{2T_k \log(2T/(\delta T_k))} \right].
\]

Now, by Lemma 12 below and the definitions of \( g_k(T) \) and \( p \),
\[
R(T) \leq \sum_{k=1}^n O \left( \Psi_{k,T} \left\{ \sqrt{d \log(|T_{k,T}|)} + \sqrt{\lambda k} \right\} x \min \left\{ \sqrt{d \log(d|T_{k,T}|)}, \sqrt{\log(N|T_{k,T}|)}, \sqrt{d|T_{k,T}| \log(|T_{k,T}|)} \right\} \right)
\]
with probability at least \( 1 - \delta \), which completes the proof. \( \square \)

Lemma 12
\[
\sum_{t \in T_{k,T}} s_{a(t),k}^c(t) = O \left( \Psi_{k,T} \sqrt{d|T_{k,T}| \log(|T_{k,T}|)} \right).
\]
Proof} We recall that \( \Psi_{k, T} = \sum_{t \in T_{k, T}} \| X_t \| \Gamma_{k(t)} - 1 \| / \sum_{t \in T_{k, T}} \| X_t \| B_{k(t)} - 1 \) and that \( \Psi_{k, T} \in (0, 1) \) due to \( \Gamma_{k(t)} - 1 \| < B_{k(t)} - 1 \) for all \( j, t \). Since \( \sum_{t \in T_{k, T}} s_{a(t), k(t)} = \sum_{t \in T_{k, T}} \| X_t \| \Gamma_{k(t)} - 1 \) by the definitions of \( s_{a(t), k(t)} \) and \( X_t \), we have
\[
\sum_{t \in T_{k, T}} s_{a(t), k(t)} = \Psi_{k, T} \sum_{t \in T_{k, T}} \| X_t \| B_{k(t)} - 1 .
\]

We now claim \( \sum_{t \in T_{k, T}} \| X_t \| B_{k(t)} - 1 \| = O \left( \sqrt{d | T_{k, T} | | \log (| T_{k, T} |) \right) \). This has been proved in similar settings (Abbasi-Yadkori, Pal, and Szepesvári, 2011; Agrawal and Goyal, 2013; Kim and Paik, 2019; Vaswani, Schmidt, and Lakshmanan, 2017); for completeness, we present the proof. Define \( s_{i, k(t)}(t) = ||b_i^f(t)||B_{k(t)} - 1 \). Note that \( s_{a(t), i, j}(t) = X_t \) and \( \sum_{t \in T_{k, T}} X_t \| B_{k(t)} - 1 \| = \text{Then}, \| X_t \| B_{k(t)} - 1 \| = \sum_{t \in T_{k, T}} s_{a(t), k(t)}(t) \). Following the lines for equation 60 of Vaswani, Schmidt, and Lakshmanan (2017), we can derive
\[
\log \det (B_{k(t + 1)}) \| \geq \log \det (\lambda_{kk} I_d) \| + \sum_{\tau \in T_{k, t}} \log \left( 1 + s_{a(\tau), k(\tau)}^2 \right). \quad (C15)
\]

On the other hand, the trace of \( B_{k(t + 1)} \) is
\[
\text{tr} \left( B_{k(t + 1)} \right) \leq 8 | T_{k, T} | + \lambda_{kk} d, \quad (C16)
\]
where we used \( \| X_t \| \| \leq 2 \) by construction. Plugging (C15) and (C16) into the determinant-trace inequality \{ \text{tr} \left( B_{k(t + 1)} \right) / d \} \| \geq \det (B_{k(t + 1)}) \| , equivalently \( d \log \left( \text{tr} \left( B_{k(t + 1)} \right) / d \right) \| \geq \log \det (B_{k(t + 1)}) \| , \) we obtain
\[
d \log \left( \frac{8 | T_{k, T} |}{d} + \lambda_{kk} \right) \| \geq d \log (\lambda_{kk}) + \sum_{\tau \in T_{k, t}} \log \left( 1 + s_{a(\tau), k(\tau)}^2 \right),
\]

or,
\[
\sum_{\tau \in T_{k, t}} \log \left( 1 + s_{a(\tau), k(\tau)}^2 \right) \leq d \log \left( 1 + \frac{8 | T_{k, T} |}{d \lambda_{kk}} \right).
\]

Now, we bound \( \sum_{\tau \in T_{k, t}} s_{a(\tau), k(\tau)}^2 \) by the result above. First, we have \( s_{a(\tau), k(\tau)}^2 \in [0, 1 / (\lambda_{kk})] \) because
\[
s_{a(\tau), k(\tau)}^2 = b_i^f(t) B_{k(t)} - 1 b_\tau \leq b_i^f(t) (\lambda_{kk} I_d) - 1 b_\tau \leq (\lambda_{kk}) - 1 .
\]

Considering a function \( f(t) = \log (1 + t) / \left[ \lambda_{kk} \log \left( 1 + (\lambda_{kk}) - 1 \right) \right] \), \( f \) satisfies \( t \leq f(t) \) for all \( t \in [0, 1 / (\lambda_{kk})] \). Therefore,
\[
\sum_{\tau \in T_{k, t}} s_{a(\tau), k(\tau)}^2 \leq \frac{1}{\lambda_{kk} \log (1 + (\lambda_{kk}) - 1)} \sum_{\tau \in T_{k, t}} \log \left( 1 + s_{a(\tau), k(\tau)}^2 \right)
\]
\[
\leq \frac{d}{\lambda_{kk} \log (1 + (\lambda_{kk}) - 1)} \log \left( 1 + \frac{| T_{k, T} |}{d \lambda_{kk}} \right).
\]

Finally, from the Cauchy-Schwartz inequality and the result above,
\[
\sum_{\tau \in T_{k, t}} s_{a(\tau), k(\tau)} \leq \sqrt{8 | T_{k, T} |} \sqrt{\sum_{\tau \in T_{k, t}} s_{a(\tau), k(\tau)}^2}
\]
\[
\leq \sqrt{\frac{d | T_{k, T} |}{\lambda_{kk} \log (1 + (\lambda_{kk}) - 1)}} \log \left( 1 + \frac{| T_{k, T} |}{d \lambda_{kk}} \right).
\]

Since \( l_{kk} = 1 \) by the definition of the random-walk Laplacian,
\[
\sum_{\tau \in T_{k, t}} s_{a(\tau), k(\tau)} \leq \sqrt{\frac{d | T_{k, T} |}{\lambda \log (1 + \frac{1}{\lambda})}} \log \left( 1 + \frac{8 | T_{k, T} |}{d \lambda} \right),
\]
which proves the claim and concludes the proof. \( \square \)
Algorithm 2: A special case of the SemiGraphTS algorithm that approximates $\pi_i(t)$ by the Monte Carlo sampling (SemiGraphTS-MC)

1: Fix $\lambda > 0$ and $M$. Set $B_j(1) = \lambda l_{jj}I_d$, $y_j(1) = 0_d$ and $v_j = (4R + 12)\sqrt{d\log \{((24T^3/\delta)(1 + \lambda^{-1}) + \sqrt{\lambda(1 + \Delta_j)}\} / \delta}$ for $j = 1, \ldots, n$.
2: for $t = 1, 2, \ldots, T$ do
3: Observe $j_t$.
4: for $j = 1, 2, \ldots, n$ do
5: if $j \neq j_t$ then
6: Update $B_j(t + 1) \leftarrow B_j(t)$, $\mu_j(t + 1) \leftarrow \hat{\mu}_j(t)$, and $y_j(t + 1) \leftarrow y_j(t)$.
7: else
8: $\hat{\mu}_j(t) \leftarrow \hat{\mu}_j(t) - B_j(t)^{-1}\sum_{k \neq j} \lambda l_{jk}\hat{\mu}_k(t)$.
9: $\Gamma_j(t) \leftarrow B_j(t) + \lambda^2 \sum_{k \neq j} l_{jk}^2 B_k(t)^{-1}$
10: for $m = 1, 2, \ldots, M$ do
11: Sample $\bar{\mu}_j^m(t)$ from $N_d(\hat{\mu}_j(t), v_j^2 \Gamma_j(t)^{-1})$
12: end for
13: for $i = 1, 2, \ldots, N$ do
14: Compute $\bar{\pi}_i(t) = \frac{1}{M} \sum_{m=1}^M I\{i = \arg\max_k \{b_k(t)^T \bar{\mu}_j^m(t)\}\}$
15: end for
16: Sample $a(t)$ from Multinom($\bar{\pi}_1(t), \ldots, \bar{\pi}_N(t)$).
17: $b(t) \leftarrow \sum_{i=1}^N \bar{\pi}_i(t)b_i(t)$ and $X_t \leftarrow b_{a(t)}(t) - \bar{b}(t)$.
18: Update $B_j(t + 1) \leftarrow B_j(t) + X_tX_t^T + \sum_{i=1}^N \bar{\pi}_i(t)(b_i(t) - \bar{b}(t))(b_i(t) - \bar{b}(t))^T$, $y_j(t + 1) \leftarrow y_j(t) + 2X_tr_{a(t), j}(t)$, and $\hat{\mu}_j(t + 1) \leftarrow B_j(t + 1)^{-1}y_j(t + 1)$.
19: end if
20: end for
21: end for

Appendix D: Regret bound when $\pi_i(t)$ is approximated by Monte Carlo sampling

In this section, we analyze the additional regret induced by approximation and show that the regret upper bound of the alternative algorithm has the same order as the bound of SemiGraphTS.

Our discussion is based on Algorithm 2, a special case of the SemiGraphTS algorithm (Algorithm 1), that explicitly states that we use the Monte Carlo approximated values of $\pi_i(t)$ for action selection. Before action selection, Algorithm 2 computes first the Monte Carlo approximates of $\pi_i(t)$. We denote the approximated value as $\hat{\pi}_i(t)$. Then, Algorithm 2 samples the arm from a multinomial distribution with size 1, say Multinom($\hat{\pi}_1(t), \ldots, \hat{\pi}_N(t)$). In comparison, Algorithm 1 samples $a(t) \sim$ Multinom($\pi_1(t), \ldots, \pi_N(t)$).

We now discuss the regret bound for Algorithm 2. We highlight the key differences from following the lines of Section C. Let the filtration $\mathcal{F}_{t-1}$ further include all Monte Carlo samples up to time $t - 1$. 
For step (a), Theorem 2 directly holds with $\pi_i(t)$’s replaced with $\hat{\pi}_i(t)$’s since the approximated values $\hat{\pi}_i(t)$’s are now the true probabilities of the arm selection.

Steps (b)-(e) in Section C exploited that the arm is selected form the exact probability. In other words, those results were derived if we select arm according to $\tilde{a}(t) = \text{argmax}_{1 \leq i \leq N}\{b_i(t)^T \mu_{j_i}(t)\}$ (i.e., $\tilde{a}(t) \sim \text{Multinom}(\pi_1(t), \ldots, \pi_N(t))$). Now we show through an inductive argument that the remaining proofs are still valid with the new arm selection $a(t) \sim \text{Multinom}(\tilde{\pi}_1(t), \ldots, \tilde{\pi}_N(t))$.

Suppose that until round $t - 1$, we have sampled arms $a(\tau) \sim \text{Multinom}(\tilde{\pi}_1(\tau), \ldots, \tilde{\pi}_N(\tau))$, $\tau = 1, \ldots, t - 1$. Then we have the desired high-probability upper bound for the estimate $\hat{\mu}_j(t)$ for every $j = 1, \ldots, n$ (Theorem 2). Now suppose that at round $t$, we sample the arm $\tilde{a}(t) = \text{argmax}_{1 \leq i \leq N}\{b_i(t)^T \tilde{\mu}_{j_i}(t)\}$. Then the proofs (b)-(e) go through, and by Lemma 13 we have,

$$
\mathbb{E}\left((b_{a^*(t)}(t)^T \mu_{j_i} - b_{\tilde{a}(t)}(t)^T \mu_{j_i})I(E^\tilde{a}(t))|\mathcal{F}_{t-1}\right) \\
\leq \frac{5g_{j_i}(T)}{p} \mathbb{E}(b_{a^*(t)}(t)^T \mu_{j_i}|\mathcal{F}_{t-1}) + \frac{4g_{j_i}(T)}{pT^2}. \tag{D17}
$$

Then, given $\mathcal{F}_{t-1}$ such that $E^\tilde{a}(t)$ is true,

$$
\mathbb{E}(\text{regret}(t)|\mathcal{F}_{t-1}) \\
= \mathbb{E}(b_{a^*(t)}(t)^T \mu_{j_i} - b_{a(t)}(t)^T \mu_{j_i}|\mathcal{F}_{t-1}) \\
= \mathbb{E}\left((b_{a^*(t)}(t)^T \mu_{j_i} - b_{\tilde{a}(t)}(t)^T \mu_{j_i}) + (b_{\tilde{a}(t)}(t)^T \mu_{j_i} - b_{a(t)}(t)^T \mu_{j_i})|\mathcal{F}_{t-1}\right) \\
\leq \frac{5g_{j_i}(T)}{p} \mathbb{E}(s^c_{a^*(t),j_i}(t)|\mathcal{F}_{t-1}) + \frac{4g_{j_i}(T)}{pT^2} + \mathbb{E}(b_{\tilde{a}(t)}(t)^T \mu_{j_i} - b_{a(t)}(t)^T \mu_{j_i}|\mathcal{F}_{t-1}) \\
= \frac{5g_{j_i}(T)}{p} \mathbb{E}(s^c_{a(t),j_i}(t)|\mathcal{F}_{t-1}) + \frac{4g_{j_i}(T)}{pT^2} + \mathbb{E}(b_{\tilde{a}(t)}(t)^T \mu_{j_i} - b_{a(t)}(t)^T \mu_{j_i}|\mathcal{F}_{t-1}) \\
+ \frac{5g_{j_i}(T)}{p} \mathbb{E}(s^c_{\tilde{a}(t),j_i}(t) - s^c_{a^*(t),j_i}(t)|\mathcal{F}_{t-1}) .
$$

As compared to Lemma 13 for Algorithm 1, we have two additional terms to bound; for step (f), those terms appear in the final cumulative regret. We claim below that the cumulative sum of the two additional terms have lower order than the original regret bound of Algorithm 1. We first have,

$$
\mathbb{E}(b_{\tilde{a}(t)}(t)^T \mu_{j_i} - b_{a(t)}(t)^T \mu_{j_i}|\mathcal{F}_{t-1}) \\
= \mathbb{E}\left(\sum_{i=1}^{N} b_i(t)^T \mu_{j_i} I(\tilde{a}(t) = i) - \sum_{i=1}^{N} b_i(t)^T \mu_{j_i} I(a(t) = i)|\mathcal{F}_{t-1}\right) \\
= \sum_{i=1}^{N} b_i(t)^T \mu_{j_i} \pi_i(t) - \sum_{i=1}^{N} b_i(t)^T \mu_{j_i} \mathbb{E}(\hat{\pi}_i(t)|\mathcal{F}_{t-1})
$$

\[ = \sum_{i=1}^{N} b_i(t)^T \mu_j, \pi_i(t) - \sum_{i=1}^{N} b_i(t)^T \mu_j, \pi_i(t) = 0, \]

which is due to unbiasedness of the Monte-Carlo estimate \( \hat{\pi}_i(t) \). Since we also have \( b_{\tilde{a}(t)}(t)^T \mu_j - b_{a(t)}(t)^T \mu_j \leq 2 \), we can show from the Azuma-Hoeffding inequality, with high probability,

\[
\sum_{t=1}^{T} \{ b_{\tilde{a}(t)}(t)^T \mu_j - b_{a(t)}(t)^T \mu_j \} \leq O(\sqrt{T}). \tag{D18}
\]

Similarly, we have

\[
\frac{5g_j(T)}{p} \mathbb{E} \left( s^c_{\tilde{a}(t), j_t}(t) - s^c_{a(t), j_t}(t) | \mathcal{F}_{t-1} \right)
\]

\[
= \frac{5g_j(T)}{p} \mathbb{E} \left( \sum_{i=1}^{N} s^c_{i, j_t}(t) I(\tilde{a}(t) = i) - \sum_{i=1}^{N} s^c_{i, j_t}(t) I(a(t) = i) | \mathcal{F}_{t-1} \right)
\]

\[
= \frac{5g_j(T)}{p} \sum_{i=1}^{N} s^c_{i, j_t}(t) (\pi_i(t) - \pi_i(t)) = 0.
\]

Hence with high probability,

\[
\sum_{t=1}^{T} \frac{5g_j(T)}{p} \left\{ s^c_{\tilde{a}(t), j_t}(t) - s^c_{a(t), j_t}(t) \right\} \leq \frac{5 \max_j g_j(T)}{p} O(\sqrt{T}). \tag{D19}
\]

We remark that the right-hand sides of (D18) and (D19) does not depend on \( d, n, N \) nor the graph structure. Therefore, our claim holds.

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