A direct calculation of the free energy from the Bethe ansatz equation for the Heisenberg model.

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Abstract

Thermodynamics of the XXX Heisenberg model is studied. The trace of the Boltzmann weight with respect to the Hilbert space is taken in the thermodynamic limit with the number of up-spins being fixed. The expression of the trace gives an explanation why the correct thermodynamic quantities are derived from the string hypothesis. Combining this with the previous result, we conclude that the free energy can be calculated only by assuming the Bethe ansatz equation. The method is more direct than other known methods which were used to derive the free energy.

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1 Introduction

There are two well-known methods to calculate the free energy for quantum integrable systems. One is the Thermodynamic Bethe Ansatz (TBA) method [1], and the other is the Quantum Transfer Matrix (QTM) method [2, 3, 4, 5, 6]. Both methods, however, are still unsatisfactory. In TBA case, we assume the form of the entropy. And we use the string hypothesis [7] for some models, whose validity is not yet proved. While QTM is a general formulation, it is difficult to analyze the resultant equations. We solve them asymptotically or numerically. In this paper, we present a direct method whereby the free energy is derived without logical jumps, assumptions and numerical supports.

We treat the spin-half XXX Heisenberg model. The Hamiltonian of the system is

\[ H = -J \sum_{j=1}^{L} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z \right) - h \sum_{j=1}^{L} S_j^z + \text{constant}, \tag{1.1} \]

where \( L \) is the number of sites, \( J \) is the coupling constant and \( h \) expresses the external field. This model is interesting since it has bound states, and correspondingly the Bethe equations have complex solutions. Throughout the paper, we assume periodic boundary condition, and use a unit which makes \( J = 1 \).

We briefly summarize our previous work [8] on this system. We suppose the expression of \( Z_M \), the trace of the Boltzmann weight with respect to the Hilbert space in which the number of up-spins is fixed to be \( M \). An infinite sum \( \sum_M Z_M \) defined by the expression is analyzed by use of combinatorial relations. Then, it is proved that the free energy \( -\beta \log \sum_M Z_M \) can be expressed in term of the function which satisfies an integral equation. The result perfectly agrees with the free energy derived from a different method [9].

Our purpose is to derive directly the expression;

\[ Z_M \equiv \text{Tr} e^{-\beta H_M} = e^{hM} \sum_{\theta \in \Theta_M} \left[ \prod_{\sigma \in \theta} N_\sigma! \right] \sum_{\zeta \in \Theta(\theta)} \mu \left( \theta, \zeta \right) \left[ \prod_{\theta' \in \zeta} \int_{-\infty}^{\infty} dx_{\theta'} \left| \frac{\partial I}{\partial x_{L}} \right|_{L, \zeta} \right] e^{-\beta E(\zeta)} \tag{1.3} \]

in the thermodynamic limit \( L \to \infty \). All the notation in (1.3) are explained in the following sections. Here, \( H_M \) denotes the Hamiltonian (1.1) acting on the Hilbert space in which the number of up-spins is restricted to a constant \( M \). In this paper, we use only the Bethe
Ansatz; the eigen energy $E$ of $H_M$ is given by
\[ E + \hbar M = \sum_{m=1}^{M} \frac{2}{x_m^2 + 1}, \tag{1.4} \]
where \( \{x_m\} \) satisfies the Bethe equations,
\[ \left[ \frac{x_m + i}{x_m - i} \right]^L = \prod_{m' \neq m} \frac{x_m - x_{m'}}{x_m - x_{m'} - 2i}, \tag{1.5} \]
which are due to periodic boundary condition. Note that (1.3) is the expression of $Z_M$ supposed in the previous paper [8]. We thus complete a new method to derive the free energy of this system independent of TBA and QTM.

The derivation goes as follows. First, we express the trace in a series with respect to solutions of the Bethe equations, which constitute a complete set of the system. Next, the series are replaced by integrals over pseudo-momenta. This replacement is justified in the same way as that in the calculation of the free energy for a $\delta$-function bose system [10, 11, 12]. The replacement, however, has a difficulty, because the Bethe equations have complex solutions. The difficulty is resolved by taking a “good” integral path. Finally, using combinatorial relations, $Z_M$ (1.3) is obtained.

We also show that the expression of $Z_M$ gives a reason why the string hypothesis, that is, the substitution of the Bethe equations for string center equations, is appropriate. The substitution is a critical procedure when we apply the thermodynamic Bethe Ansatz to an integrable system which has bound states.

The outline of this paper is the following. In §2, we derive the expression (1.3) of $Z_M$ only from the Bethe ansatz. In §3, we examine the string hypothesis by use of thus proved expression of $Z_M$. The last section is devoted to the concluding remarks. Technical details of calculations and proofs are summarized in Appendices A-H.

2 Direct derivation of $Z_M$ from Bethe equations

In order to explain our analysis of (1.2), we begin with several symbols and notations. We divide the procedure into 9 steps; definitions of symbols are placed on the top of each step where they first appear.
Step 1

With the eigen energies $E(1.4)$, $Z_M$ becomes

$$M!Z_M e^{-\beta h M} = \sum \exp \left( -\beta \sum_{m=1}^{M} \frac{2}{x_m^2 + 1} \right).$$  \hspace{1cm} (2.1)

Here, $\sum$ without subscript means a summation over all the “different” “physical solutions” of the Bethe equations. It is interpreted as a summation with respect to a set of integers $\{I_m\}$ corresponding to a “physical solution”, where $\{I_m\}$ is related to $\{x_m\}$ by

$$\left[ \frac{x_m + i}{x_m - i} \right]^L = e^{-2\pi i I_m} \prod_{m' \in \theta, m' \neq m} \left[ \frac{x_m - x_{m'} + 2i}{x_m - x_{m'} - 2i} \right]^{N_{m'}}. \hspace{1cm} (2.2)$$

The “physical solution” means a solution of the Bethe equations which contains some pseudomomenta $x_m$ of the “same value”. The symbol $\theta$ and the words “different”, “same value” will be explained later.

Step 2

• definition: $\Theta(\sigma)$

Let $\sigma$ be a set which has a finite number of elements. $\Theta(\sigma)$ denotes all the patterns of division of a set $\sigma$. A pattern of division $\theta$ is a set with elements each of which is a cluster. The cluster $\sigma'$ is one of the pieces into which a set $\sigma$ is divided, and the cluster is also regarded as a set;

$$\Theta(\sigma) = \left\{ \theta \right| \sigma = \bigoplus_{\sigma' \in \theta} \sigma' \right\}. \hspace{1cm} (2.3)$$

In case of $\sigma = \{1, 2, \cdots, n\}$, we write $\Theta_n$ for $\Theta(\{1, 2, \cdots, n\})$ for simplicity.

• definition: $N_\sigma$

Let $\sigma$ be a set or a sequence which has a finite number of elements. The number of elements of a set or a sequence $\sigma$ is denoted by $N_\sigma$.

Using the above symbols, (2.1) can be rewritten in the thermodynamic limit as

$$M!Z_M e^{-\beta h M} = \sum_{\theta \in \Theta_M} \mu(\hat{0}_M, \theta) \int e^{-\beta E_\theta} \prod_{\sigma \in \theta} dI_{\sigma}, \hspace{1cm} (2.4)$$
where $\mu(\hat{0}_M, \theta)$ and $E_\theta$ are respectively defined as

$$
\mu(\hat{0}_M, \theta) \equiv \prod_{\sigma \in \theta} (-)^{N_\sigma - 1} (N_\sigma - 1)!, \quad (2.5)
$$

$$
E_\theta \equiv \beta \sum_{\sigma \in \theta} \frac{2N_\sigma}{x^2_\sigma + 1}, \quad \theta \in \Theta_M. \quad (2.6)
$$

This rewriting is justified in the same way as we have done for the $\delta$-function boson case [10, 11, 12]. Roughly speaking, the sum can be replaced with a multiple integral in taking the thermodynamic limit, and extinction of non-physical states from the sum is realized by the coefficients $\frac{1}{M!} \sum_{\theta \in \Theta_M} \mu(\hat{0}_M, \theta)$. The relation between $\{x_\sigma\}$ and $\{I_\sigma\}$ is defined by

$$
\left[ \frac{x_\sigma + i}{x_\sigma - i} \right]^L = e^{-2\pi iI_\sigma} \prod_{\sigma' \in \theta, \sigma' \neq \sigma} \left[ \frac{x_\sigma - x_{\sigma'} + 2i}{x_\sigma - x_{\sigma'} - 2i} \right]^{N_{\sigma'}}, \quad \sigma \in \theta. \quad (2.7)
$$

We note that eqs. (2.7) are eqs. (2.2) restricted by the conditions;

$$
x_m = x_\sigma \quad I_m = I_\sigma \quad m \in \sigma. \quad (2.8)
$$

The function $\mu(\hat{0}_M, \theta)$ is a special case of the Möbius function $\mu(\theta', \theta)$ which comes from a natural definition of partial order among elements in $\Theta_M$. The most important thing in (2.4) is the definition of the paths of the multiple integrals. We define the path of the multiple integral corresponding to $\theta$. The integral path is a manifold on which $\{I_\sigma\}$ is a set of real numbers and $\{x_\sigma\}$ are continuously distributed. Concretely, the integral path is an $N_{\theta}$-dimensional surface defined by conditions,

$$
\left| \frac{x_\sigma - i}{x_\sigma + i} \right|^L \prod_{\sigma' \in \theta, \sigma' \neq \sigma} \left[ \frac{x_\sigma - x_{\sigma'} + 2i}{x_\sigma - x_{\sigma'} - 2i} \right]^{N_{\sigma'}} = 1. \quad (2.9)
$$

The orientation of the integral path is chosen so that the value of an integral $\int \prod dI_\sigma$ for any part of the integral path is positive. Here and hereafter, we consider conditions such as (2.9) for an integral path.

For rewriting (2.1) into (2.4), it is necessary to define the words “different” and “same value” used in step 1. We regard $\{x_m\}$ in eq. (2.2) as a multivalued vector function of $\{\exp(2\Pi i I_m) \neq 0, \infty\}$. Whether $x_{m'}$ and $x_{m''}$ take the “same value” or not depends on neighborhoods of the point $\{x_m\}$ on the Riemann surface defined by the function. We say that $x_{m'}$ and $x_{m''}$ take the “same value” if and only if there is a point $\{x_{m'}\}$ in any
neighborhood where \( x'_{m'} = x'_{m''} \) and \( x_m' \neq x_m' \). In case \( x_m' = x_m'' \neq \infty, \pm i \), this definition indicates that \( x_m' \) and \( x_m'' \) take the same value. Therefore, in almost all the region of variables \( \{x_m\} \), this definition leads to a valid definition of the “physical solution”. On the other hand, we have to prove that the “physical solution” is a real physical solution when some variables \( x_m \) are equal to \( \infty \) or \( \pm i \), where a real physical solution means that the Bethe vector corresponding to the solution expresses a non-trivial eigen vector. Next, “different” is defined as follows. Solutions of (2.9) corresponding to different points on the Riemann surface are “different”, where we regard (2.9) as an equation for \( \{x_m\} \) fixing \( \{\exp(2\Pi i I_m)\} \) to be constants. At any point on the Riemann surface, the number of “different” solutions is

\[
\lim_{\epsilon \to 0} \max \text{ number of points on } R_\epsilon \text{ corresponding to the same } \{\exp(2\Pi i I_m)\}, \quad (2.10)
\]

where \( R_\epsilon \) is an \( \epsilon \)-neighborhood of the point on the Riemann surface. For example, in case \( M = 2 \), the point on the Riemann surface \( x_1, x_2 = \infty, I_1, I_2 \) = integer corresponds to two “different” solutions, and to two “physical solutions”. But, there is only one eigen state corresponding to the point. We interpret this as follows. The “values” \( x_1 \) and \( x_2 \) are not the “same value”, and one of the “physical solutions” is given by exchanging the “values” \( x_1, x_2 \) of the other “physical solution”.

**Step 3**

In [Appendix A], we prove that the value of the integral

\[
\int e^{-\beta E_\theta} \prod_{\sigma \in \theta} dI_\sigma, \quad (2.11)
\]

along the integral path

\[
\left| \left( \frac{x_\sigma - i}{x_\sigma + i} \right)^L \prod_{\sigma' \in \theta, \neq \sigma} \left( \frac{x_\sigma - x_{\sigma'} + 2i}{x_\sigma - x_{\sigma'} - 2i} \right)^{N_{\sigma'}} \right| = A_\sigma, \quad \sigma \in \theta, \quad (2.12)
\]

does not depend on \( A_\sigma \) in case that all \( A_\sigma \) are finite and positive. The relation between \( x_\sigma \) and \( I_\sigma \) is defined by (2.7) and \( E_\sigma \) is defined by (2.6). Note that we define the orientation of the path so that an integral \( \int \prod dI_\sigma \) for any part of the path is positive.
In the multiple integral (2.4), we thus change the integral path (2.9) into (2.12). We prove in Appendix B that the expression (2.4) can be rewritten as

\[
(2.4) = \sum_{\theta \in \Theta_M} \mu(\hat{0}_M, \theta) \left[ \prod_{\sigma \in \theta} \int_{|x_\sigma - i| = 0^+} dx_\sigma \right] \left| \frac{\partial I_\sigma}{\partial x_\sigma} \right|_{L, \theta} e^{-\beta E_\theta},
\]

(2.13)

by defining \( A_\sigma \) as a set of sufficiently small real number set, where \( |x_\sigma - i| = 0^+ \) indicates the integral path that \( x_\sigma \) turns around a point \( i \) anticlockwise.

**Step 4**

- definition: \( \Lambda(\theta) \)

Let \( \theta \) be a set with a finite number of elements. We denote by \( \Lambda(\theta) \) all the patterns of connection of a set \( \theta \). What we call a pattern of connection satisfies the following two conditions. 1) Any two elements of \( \theta \) are connected or not. Simply, there is no multiple connection. 2) There is no closed path in the connections. Then, a pattern of connection \( \lambda \) is a set of elements each of which corresponds to a connection \( \eta \). To summarize, \( \Lambda(\theta) \) satisfies conditions,

\[
\text{if } \eta \in \lambda \in \Lambda(\theta), \text{ then } \eta = \{\sigma, \sigma'\}, \quad \sigma, \sigma' \in \theta,
\]

\[
\text{if } \{\sigma_1, \sigma_2\}, \{\sigma_2, \sigma_3\}, \cdots, \{\sigma_{m-1}, \sigma_m\} \in \lambda \in \Lambda(\theta), \text{ then } \{\sigma_1, \sigma_m\} \not\in \lambda, \quad (2.14)
\]

and has the most elements of all sets which satisfy the above conditions.

- definition: \( G_\theta(\lambda) \)

Let \( \theta \) be a set with a finite number of elements, and \( \lambda \in \Lambda(\theta) \). \( G_\theta(\lambda) \) is an element of \( \Theta(\theta) \). In other words, \( G_\theta(\lambda) \) is a pattern of division of \( \theta \). We call that \( \sigma \) and \( \sigma' \) in \( \theta \) are indirectly connected by \( \lambda \) when \( \sigma \) is linked to \( \sigma' \) through one or some connections in \( \lambda \). Two elements in \( \theta \) are indirectly connected by \( \lambda \) if and only if there is a cluster \( \theta' \in G_\theta(\lambda) \) containing the two elements. Precisely, \( G_\theta(\lambda) \) satisfies the conditions,

\[
G_\theta(\lambda) \in \Theta(\theta)
\]

\[
\text{if } \{\sigma, \sigma'\} \in \lambda, \text{ then } \sigma, \sigma' \in \theta' \in G_\theta(\lambda), \quad (2.15)
\]

and has the most elements of all sets which satisfy the conditions.
• definition: $\Theta(\theta|\lambda)$

Let $\theta$ be a set with a finite number of elements, and $\lambda \in \Lambda(\theta)$. $\Theta(\theta|\lambda)$ consists of all elements $\zeta$ satisfying the following conditions: $\zeta$ is in $\Theta(\theta)$, and any $\theta' \in G_\theta(\lambda)$ is a union of some sets in $\zeta$. Then, $\Theta(\theta|\lambda)$ is

$$
\Theta(\theta|\lambda) = \{ \zeta \mid \zeta = G_\theta(\lambda'), \ \lambda' \subseteq \lambda \}.
$$

(2.16)

• definition: $\tilde{\zeta}$

Let $\zeta \in \Theta(\theta)$, and $\theta$ be a set with a finite number of elements. We denote by $\tilde{\zeta}$ a $\zeta$-dependent equivalent relation with respect to elements in $\theta$,

$$
\sigma \sim \sigma' \quad \text{when} \quad \sigma, \sigma' \in \theta' \in \zeta.
$$

(2.17)

• definition: $l_\lambda(\sigma, \sigma')$

Let $\sigma, \sigma' \in \theta$, and $\lambda \in \Lambda(\theta)$. We define $l_\lambda(\sigma, \sigma')$ in case that $\sigma$ and $\sigma'$ are indirectly connected by $\lambda$, or $\sigma = \sigma'$. In case $\sigma = \sigma'$, $l_\lambda(\sigma, \sigma) = 0$. In the other case, $l_\lambda(\sigma, \sigma)$ is the number of connections from $\sigma$ to $\sigma'$.

• definition: $\theta[\zeta, \sigma]$

Let $\sigma$ be an element of a set which belongs to $\zeta$. We denote by $\theta[\zeta, \sigma]$ a set which is in $\zeta$ and includes $\sigma$. That is, $\theta[\zeta, \sigma]$ is the set which satisfies a condition,

$$
\sigma \in \theta[\zeta, \sigma] \subseteq \zeta.
$$

(2.18)

• definition: $\Theta(\theta|\lambda|\sigma_1, \ldots, \sigma_m)$

Let $\theta = \{\sigma_1, \ldots, \sigma_m, \ldots\}$ be a finite set and $\lambda$ be in $\Lambda(\theta)$. The symbol $\Theta(\theta|\lambda|\sigma_1, \ldots, \sigma_m)$ is defined as follows; it consists of all the elements $\zeta$ in $\Theta(\theta|\lambda)$ which satisfy the condition $\theta[\zeta, \sigma_k] \neq \theta[\zeta, \sigma_{k'}]$ when $k \neq k'$ and $k, k' \leq m$. Then, $\Theta(\theta|\lambda|\sigma_1, \ldots, \sigma_m)$ satisfies the condition,

$$
\text{if} \quad \zeta \in \Theta(\theta|\lambda|\sigma_1, \ldots, \sigma_m) \quad \text{and} \quad k, k' \leq m \quad \text{then} \quad \zeta \in \Theta(\theta|\lambda), \quad \sigma_k \ncong \sigma_{k'},
$$

(2.19)

and has the most elements of all sets satisfying the condition.
Using these symbols, we can express the change of the integral path from \(|x_\sigma - i| = 0+\) into \((-\infty, \infty)\). Then, (2.13) becomes

\[
(2.20) = \sum_{\theta \in \Theta_M} \mu \left(0_M, \theta \right) \sum_{\lambda \in \Lambda(\theta)} \left[ \prod_{\sigma \in \theta} N_{\sigma} \right]^{-1} \left[ \prod_{\{\sigma, \sigma'\} \in \lambda} N_{\sigma} N_{\sigma'} \right] \sum_{\sigma \in \theta} \sum_{\{\sigma, \sigma'\} \in \lambda} \sum_{\zeta \in \Theta(\theta \lambda)} \left( \sum_{\{\sigma(\theta), \lambda\} \in \{\theta_k \in \zeta\}} 2 N_{\sigma} L \right)
\]

\[
\left[ \prod_{\theta' \in \zeta} \int_{(-\infty, \infty)} \frac{dx_{\theta'}}{2\pi} \right] \left[ \prod_{\theta' \in G_\theta(\lambda)} \left( \sum_{\sigma \in \theta'} \left( x_{\theta' \sigma} + 2 l_\lambda \left( \sigma, \sigma(\theta[\zeta, \sigma]) \right) i^2 \right) + 1 \right) \right]
\]

\[
\prod_{\{\sigma, \sigma'\} \in \lambda} \frac{-4 N_{\sigma} N_{\sigma'}}{\left( x_\sigma - x_{\sigma'} \right)^2 + 4} e^{-\beta \sum_{\sigma \in \theta} \frac{2 N_{\sigma}}{x_\sigma^2 + 1}}
\]

We have used two simplified notations:

\[
\sum_{\{\sigma(\theta) \} \in \{\theta_k \in \zeta\}} \equiv \sum_{\sigma(\theta_1) \in \theta_1} \sum_{\sigma(\theta_2) \in \theta_2} \cdots \sum_{\sigma(\theta_N) \in \theta_N}, \quad \theta_1, \theta_2 \cdots \theta_N \in \zeta, \quad (2.21)
\]

and \(\min(\sigma) \equiv \min_{n \in \sigma} n\) where \(\sigma \subset \mathbb{N}\).

A remark is in order. In (2.20), the series \(\sum_{\zeta}^{\infty}\) with respect to \(\zeta\) can be divided into two parts. One is a set in which all elements have only one element, and the other is a set which consists of the other elements. The former is a set of terms corresponding to (2.13) with an integral path \((-\infty, \infty)\). The latter is a set of terms corresponding to residues which the modification of the integral path generates.

From now on, we prove the equivalence of (2.13) and (2.20). Equivalently, we prove that

\[
\prod_{\sigma \in \theta} \int_{|x_\sigma - i| = 0+} dx_\sigma \left[ \prod_{\sigma \in \theta} N_{\sigma} \right]^{-1} \left[ \prod_{\theta' \in G_\theta(\lambda)} \left( \sum_{\sigma \in \theta'} \frac{2 N_{\sigma} L}{x_\sigma^2 + 1} \right) \right]
\]

\[
\prod_{\{\sigma, \sigma'\} \in \lambda} \frac{-4 N_{\sigma} N_{\sigma'}}{\left( x_\sigma - x_{\sigma'} \right)^2 + 4} e^{-\beta \sum_{\sigma \in \theta} \frac{2 N_{\sigma}}{x_\sigma^2 + 1}}
\]

\[
= \left[ \prod_{\sigma \in \theta} N_{\sigma} \right]^{-1} \left[ \prod_{\{\sigma, \sigma'\} \in \lambda} N_{\sigma} N_{\sigma'} \right] \sum_{\zeta \in \Theta(\theta \lambda)} \sum_{\{\sigma(\theta), \lambda\} \in \{\theta_k \in \zeta\}} \sum_{\{\sigma, \sigma'\} \in \lambda} \sum_{\zeta \in \Theta(\theta \lambda)} \left( \sum_{\{\sigma(\theta), \lambda\} \in \{\theta_k \in \zeta\}} 2 N_{\sigma} L \right)
\]

\[
\left[ \prod_{\theta' \in \zeta} \int_{(-\infty, \infty)} \frac{dx_{\theta'}}{2\pi} \right] \left[ \prod_{\theta' \in G_\theta(\lambda)} \left( \sum_{\sigma \in \theta'} \left( x_{\theta' \sigma} + 2 l_\lambda \left( \sigma, \sigma(\theta[\zeta, \sigma]) \right) i^2 \right) + 1 \right) \right]
\]
where \( \theta \in \Theta_M \) and \( \zeta \in \Lambda (\theta) \), since both sides of this equation become (2.13) and (2.20) respectively when we apply \( \sum_{\theta \in \Theta_M} \mu (\bar{0}_M, \theta) \sum_{\lambda \in \Lambda (\theta)} \) to both sides. Here, we have substituted an explicit expression of the Jacobian,

\[
(2\pi)N_{\theta} \left| \frac{\partial I_{\sigma}}{\partial x_{\sigma}} \right|_{L,\theta} = \prod_{\sigma \in \theta} N_{\sigma}^{-1} \prod_{\lambda \in \Lambda (\theta)} \left[ \sum_{\{\sigma, \sigma'\} \in \lambda} \sum_{\zeta \in \Theta (\theta|\sigma_{1}, \ldots, \sigma_{m}, \theta_{m})} \prod_{k=1}^{m} \int_{i-x_{\theta_k}}^{i} \frac{dx_{\theta_k}}{2\pi} \right] \prod_{\theta' \in G_{\theta}(\lambda)} \left[ \sum_{\sigma \in \theta'} \frac{2N_{\sigma} L}{x_{\sigma}^2 + 1} \right].
\]  

A sufficient condition of (2.22) is that an expression does not depend on \( m \), where we demand that the constants \( \delta_{\sigma_k} \) and \( \bar{\delta}_{\sigma_k} \) satisfy

\[
1 >> \delta_{\sigma_1} > \delta_{\sigma_2} > \cdots > 0, \quad 0 < \bar{\delta}_{\sigma_1} < \bar{\delta}_{\sigma_2} < \cdots < \frac{1}{\sqrt{2}}.
\]

A proof of this fact is shown in Appendix C. This fact is a sufficient condition, because (2.24) becomes the l.h.s.(r.h.s.) of eq.(2.22) in case \( m \) is \( N_{\theta}(0) \). In (2.24), we have used a simplified notation \( \sum_{\{\sigma_{\theta_k}\} \in \{\theta_{k} \in \zeta\}_{k>m}} \) where \( \zeta \) is in \( \Theta (\theta|\sigma_{1}, \ldots, \sigma_{m'}) \): it is defined as

\[
\sum_{\{\sigma_{\theta_k}\} \in \{\theta_{k} \in \zeta\}_{k>m}} F \left( \{\sigma_{\theta_k}\} \right) = \sum_{\sigma_{\theta_{m+1}}} \sum_{\sigma_{\theta_{m+2}}} \cdots \sum_{\sigma_{\theta_{N_{\zeta}}}} F \left( \{\sigma_{\theta_k}\} \right) |_{\sigma_{\theta_{k}} = \sigma_{k} \leq m}, \quad (2.26)
\]

\[
\theta_1, \theta_2 \cdots \theta_{N_{\zeta}} \in \zeta.
\]
We have numbered the elements \( \sigma_1, \ldots, \sigma_{N_\theta} \) in \( \theta \in \Theta_M \) so that \( \{\sigma_m\} \) satisfy the condition \( \min(\sigma_1) > \min(\sigma_2) > \cdots \). And when \( \zeta \) is in \( \Theta(\theta|\sigma_1, \ldots, \sigma_m) \) and \( 0 < k \leq N_\zeta \), elements \( \theta' \in \zeta \) have been named \( \theta_k \) so that \( \theta_k \) is equal to \( \theta[\zeta, \sigma_k] \) in case \( k \leq m \). These notations are used only in (2.24) and Appendix C.

**Step 5**

- **definition: \( \tilde{\Theta}(\theta) \)**

  Let \( \theta \) be a set with a finite number of elements. A set \( \tilde{\Theta}(\theta) \) consists of all elements \( \tilde{\zeta} \) satisfying the following conditions. First, \( \tilde{\zeta} \) is a set of sequences as elements. Second, all the elements in the sequences are in \( \theta \). Third, a set of sets \( \zeta \) derived from \( \tilde{\zeta} \) is in \( \Theta(\theta) \). Here, \( \zeta \) is derived from \( \tilde{\zeta} \) when we replace sequences in \( \tilde{\zeta} \) with sets by ignoring the order of the sequences. Note that the number of \( \tilde{\zeta} \)'s which become a \( \zeta \) by the above procedure is \( \prod_{\theta' \in \zeta} N_{\theta'} ! \).

Using these definitions, we change an expression of summations in (2.20) with respect to permutations. Then, (2.20) becomes

\[
\begin{align*}
&= \sum_{\theta \in \Theta_M} \sum_{\tilde{\zeta} \in \tilde{\Theta}(\theta)} \sum_{\lambda \in \Lambda(\tilde{\zeta})} \left[ \prod_{(\sigma_1, \sigma_2, \ldots)} \left( \sum_{\theta \in \tilde{\zeta}} \sum_{\sigma_m \in \theta = (\sigma_1, \ldots)} \sum_{\delta \in \Lambda(\tilde{\zeta})} \int_{-\infty+i \min(\sigma_1)\delta}^{+\infty+i \min(\sigma_1)\delta} \frac{dx_{\tilde{\zeta}}}{2\pi} \right) \right] \\
&\quad \left[ \prod_{\tilde{\zeta} \in \tilde{\Theta}(\lambda)} \left( \sum_{\theta \in \tilde{\zeta}} \sum_{\sigma_m \in \theta = (\sigma_1, \ldots)} \sum_{\beta \in \Lambda(\tilde{\zeta})} \left( \sum_{\theta' \in \tilde{\zeta}} \sum_{\sigma_m' \in \theta' = (\sigma'_1, \ldots)} \left( \sum_{\delta' \in \Lambda(\tilde{\zeta})} \int_{-\infty+i \min(\sigma_1)\delta'}^{+\infty+i \min(\sigma_1)\delta'} \frac{dx_{\tilde{\zeta}}}{2\pi} \right) \right) \right] \\
&\quad \left[ \prod_{\theta \in \Theta_M} \left( (-)^{N_{\sigma_1} - 1} (N_{\sigma_1} - 1)! \sum_{\sigma_{m+1} \in \theta} \left( \sum_{\theta' \in \tilde{\Theta}(\sigma_m)} \prod_{\sigma_m' \in \theta' = (\sigma'_1, \ldots)} (-)^{N_{\sigma}} (N_{\sigma} - 1)! \right) \right) \right].
\end{align*}
\]

The “change of an expression of summations with respect to permutations” means a change from a series \( A \) to a series \( B \) satisfying the following two conditions: First, the value
of $A$ and $B$ are the same. Second, the equivalence can be shown by making a correspondence between some elements in $A$ and in $B$.

For example, in case of the simple relations,

\[ \sum_{m \geq 0} \sum_{n \geq 0} a^m b^n = \sum_{m \geq 0} \sum_{n=0}^m a^{m-n} b^n, \quad (2.28) \]
\[ \sum_{m \geq 0} \sum_{n \geq 0} a^m b^n = \sum_{m \geq 0} a^m \frac{1 - \left(\frac{b}{a}\right)^{m+1}}{1 - \frac{b}{a}}, \quad (2.29) \]

we say that the r.h.s. is made by means of changing an expression of summation $\sum_{m \geq 0} \sum_{n \geq 0}$ in the l.h.s. with respect to permutations. In case of (2.28), when we write terms in $\sum_{m \geq 0} \sum_{n \geq 0}$ and $\sum_{m \geq 0} \sum_{n=0}^m$ as $(m, n)_{l}$ and $(m, n)_{r}$, $(m, n)_{l}$ and $(m + n, n)_{r}$ are the same. And, in case of (2.29), when we write terms in $\sum_{m \geq 0} \sum_{n \geq 0}$ and $\sum_{m \geq 0}$ as $(m, n)_{l}$ and $(m)_{r}$, a sum of terms $(m, 0)_{l}, \cdots, (m, m)_{l}$ is equal to $(m)_{r}$.

We introduce the words “correspond” and “correspondence” as follows. When we say “an element $a$ corresponds to $b$” where $a$ and $b$ are elements under $\sum$, e.g. $(m, n)_{l}$ and $(m, n)_{r}$, we demand the following two facts. Values of both $a$ and $b$ are the same, and the correspondence makes a one-to-one correspondence between $A$ and $B$ where $a \in A$ and $b \in B$. In such case, we say “there is a one-to-one correspondence between $A$ and $B$”. Remark that we have given the extra meaning to the word “correspond” and “correspondence”. In case of (2.28) there is a one-to-one correspondence between $\{(m, n)_{l}\}$ and $\{(m, n)_{r}\}$, and $(m, n)_{l}$ corresponds to $(m + n, n)_{r}$. And, when we say “a subset $\{a\}$ corresponds to an element $b$” where $a \in \{a\}$ and $b$ are elements under $\sum$, e.g. $(m, n)_{l}$ and $(m)_{r}$, we demand the following two facts. The summation of values over $a \in \{a\}$ is equal to a value of $b$ and the correspondence makes a many-to-one correspondence between $A$ and $B$ where $a \in A$ and $b \in B$. In such case, we say that there is a many-to-one correspondence between $A$ and $B$. Therefore, in case of (2.29) there is a many-to-one correspondence between $\{(m, n)_{l}\}$ and $\{(m)_{r}\}$, and $(m, 0)_{l}, \cdots, (m, m)_{l}$ corresponds to $(m)_{r}$. “One-to-many correspondence”, “an element corresponds to a set”, “many-to-many correspondence” and “a set corresponds to a set” should be understood in the same way.

We note that when we change an expression of summations in (2.20) into (2.27), there is
a many-to-one correspondence between \( \{ \lambda, \zeta, \{ \sigma(\theta_k) \} \} \) and \( \{ \tilde{\zeta}, \lambda \} \) associated with

\[
\sum_{\theta \in \Theta_M} \sum_{\lambda \in \Lambda(\theta)} \sum_{\zeta \in \Theta(\theta|\lambda)} \sum_{\{\sigma(\theta_k)\} \in \{\theta_k \in \zeta\}} \leftrightarrow \sum_{\theta \in \Theta_M} \sum_{\tilde{\zeta} \in \tilde{\Theta}(\theta)} \sum_{\lambda \in \Lambda(\tilde{\zeta})}.
\]

(2.30)

A subset \( A \) of \( \{ \lambda, \zeta, \{ \sigma(\theta_k) \} \} \) corresponds to \( \tilde{\zeta}', \lambda' \) in \( \{ \tilde{\zeta}, \lambda \} \), where \( \lambda'', \zeta'', \{ \sigma(\theta_k) \} \) in \( A \) and \( \tilde{\zeta}', \lambda' \) satisfy the following conditions. There is a bijection \( f \) from the set \( \zeta'' \) to the set \( \tilde{\zeta}' \) which satisfies

\[
\bigoplus_{\sigma \in \theta'' \setminus \Lambda(\theta, \sigma(\theta'')) = m-1} \sigma = \sigma'_{m},
\]

(2.31)

where \( f(\theta'') = \tilde{\theta}' \) and \( (\sigma'_1, \ldots) = \tilde{\theta}' \). And, \( \lambda' \) can be written as

\[
\lambda' = \left\{ \{ \tilde{\theta}', \tilde{\theta}'' \} \bigg| \{ \sigma', \sigma'' \} \in \zeta'', \sigma' \in f^{-1}(\tilde{\theta}'), \sigma'' \in f^{-1}(\tilde{\theta}'') \right\}.
\]

(2.32)

It is clear that this correspondence is a “many-to-one correspondence”.

**Step 6**

- **definition: \( \theta_>(\tilde{\theta}) \)**

Let \( \tilde{\theta} \) be a sequence of sets with a finite number of elements. \( \theta_>(\tilde{\theta}) \) is defined by

\[
\theta_>(\tilde{\theta}) \equiv \prod_{\sigma_{m+1} \in \tilde{\theta} = (\sigma_1, \ldots)} \theta \left( N_{\sigma_{m-1}} - N_{\sigma_m} + \frac{1}{2} \right),
\]

(2.33)

where the \( \theta \)-function is defined as

\[
\theta(n) \equiv \begin{cases} 
1 & \text{in case of } n > 0 \\
\frac{1}{2} & \text{in case of } n = 0 \\
0 & \text{in case of } n < 0.
\end{cases}
\]

(2.34)

- **definition: \( \Lambda_c(\theta) \)**

Let \( \theta \) be a set with a finite number of elements. \( \Lambda_c(\theta) \) consists of all elements \( \lambda \) satisfying following two condition. First, \( \lambda \) is in \( \Lambda(\theta) \). Second, any two of elements in \( \theta \) are indirectly connected by \( \lambda \), that is to say, \( N_{G_{\theta}(\lambda)} = 1 \). Then,

\[
\Lambda_c(\theta) = \left\{ \lambda | \lambda \in \Lambda(\theta), N_{G_{\theta}(\lambda)} = 1 \right\}.
\]

(2.35)
• definition: \( D(\tilde{\theta}) \)

Let \( \tilde{\theta} \) be a sequence of sets, where the number of elements in each set decreases in order of the sequence. \( D(\tilde{\theta}) \) consists of elements \( \tilde{\zeta} \) satisfying the following conditions. First, \( \tilde{\zeta} \) is a set of sequences \( \tilde{\theta}' \) which consist of sets \( \sigma \). Second, all sets \( \sigma \) in any sequence \( \tilde{\theta}' \) have the same number of elements. Third,

\[
\sigma_k = \bigoplus_{(\sigma_1, \ldots) = \tilde{\theta}' \in \tilde{\zeta}} \sigma'_k,
\]

where \((\sigma_1, \sigma_2, \ldots) = \tilde{\theta}\).

• definition: \( \delta(\tilde{\zeta}, \lambda, n) \)

Let \( \tilde{\zeta} \) be in \( D(\tilde{\theta}) \), \( \lambda \) be in \( \Lambda_c(\tilde{\zeta}) \), \( n \) be in a set \( \sigma \) and \( \tilde{\theta} \) be in \( \tilde{\Theta}(\sigma) \). \( \delta(\tilde{\zeta}, \lambda, n) \) is a function whose value is 1 in case \( \tilde{\zeta}, \lambda, n \) satisfy the following conditions and is 0 otherwise. First, there is only one sequence \( \tilde{\theta}^{(0)} \) in \( \tilde{\zeta} \) which has the smallest number of elements in \( \tilde{\zeta} \). Second, when we write the sequence \( \tilde{\theta}^{(0)} \) as \( (\sigma_1^{(0)}, \ldots) \), \( n \) is in \( \sigma_1^{(0)} \).

Third,

\[
\text{if } l_\lambda \left( \tilde{\theta}^{(0)}, \tilde{\theta}' \right) + 1 = l_\lambda \left( \tilde{\theta}^{(0)}, \tilde{\theta}'' \right), \text{ then } N_{\tilde{\theta}'} < N_{\tilde{\theta}''},
\]

for any \( \tilde{\theta}', \tilde{\theta}'' \in \tilde{\zeta} \). This function is sometimes referred to as \( \delta \)-function, but is rather close to the Kronecker’s \( \delta \).

• definition: \( M_\theta \)

Let \( \theta \) be a set or sequence of sets which have the same number of elements. Then, \( M_\theta \) means \( N_\sigma \) where \( \sigma \in \theta \).

In terms of those notation, we change the expression (2.27) into

\[
\begin{align*}
\text{(2.4)} & = \sum_{\tilde{\theta} \in \Theta_M} \sum_{\tilde{\zeta} \in \tilde{\Theta}(\tilde{\theta})} \left[ \prod_{\tilde{\theta} \in \tilde{\zeta}} \delta_{>}(\tilde{\theta}) \right] \sum_{\lambda \in \Lambda(\tilde{\zeta})} \left[ \prod_{(\sigma_1, \sigma_2, \ldots) = \tilde{\theta} \in \tilde{\zeta}} \int_{-\infty+i \min(\sigma_1) \delta}^{+\infty+i \min(\sigma_1) \delta} \frac{d \bar{x}_{\tilde{\theta}}}{2\pi} \right] \\
& \quad \left[ \prod_{\tilde{\zeta}' \in \tilde{G}_{\tilde{\zeta}}(\lambda)} \left( \sum_{\bar{\theta} \in \tilde{\zeta}'} \sum_{\sigma_m \in \Theta(\sigma_1, \ldots)} 2 \left( N_{\sigma_m} - N_{\sigma_{m+1}} \right) mL \frac{x_{\bar{\theta}} + (m - 1)i}{\bar{\theta} + (m - 1)i + m^2} \right) \right].
\end{align*}
\]
In Appendix E, we give a proof of a relation, where the function 

\[ K_{m,m'}(x_\tilde{\theta} - x_\tilde{\theta'} + (m - m')i) \]

is defined as

\[
\prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda} \left( \sum_{\sigma_m \in \tilde{\theta} = (\sigma_1, \ldots)} \sum_{\sigma_m' \in \tilde{\theta}' = (\sigma'_1, \ldots)} \left( - (N_{\sigma_m} - N_{\sigma_m+1}) (N_{\sigma_m'} - N_{\sigma_m'+1}) \right) \right)
\]

\[
\exp \left[ -\beta \sum_{\tilde{\theta} \in \zeta} \sum_{\sigma_m \in \tilde{\theta} = (\sigma_1, \ldots)} \frac{2 (N_{\sigma_m} - N_{\sigma_m+1}) m}{(x_\tilde{\theta} + (m - 1) i)^2 + m^2} \right]
\]

\[
\prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda'} \left[ \sum_{\tilde{\zeta}' \in \mathcal{D}(\tilde{\theta})} \sum_{\lambda' \in \Lambda_c(\tilde{\zeta})} \delta \left( \tilde{\zeta}', \lambda', \min(\sigma_1) \right) \prod_{\tilde{\theta}' \in \tilde{\zeta}'} \left( (-)^{M_{\tilde{\theta}'} - 1} (M_{\tilde{\theta}'} - 1) ! M_{\tilde{\theta}'} (M_{\tilde{\theta}'} !)^{N_{\tilde{\theta}'} - 1} \right) \right]
\]

\[
\prod_{\{\tilde{\theta}', \tilde{\theta}''\} \in \lambda'} \left( -M_{\tilde{\theta}'} M_{\tilde{\theta}''} \right) \right) ,
\]

(2.38)

where the function \( K_{n,m}(x) \) is defined as

\[
K_{n,m}(x) \equiv \begin{cases} 
\kappa_{[n-m]}(x) + 2\kappa_{[n-m]+2}(x) + \cdots + 2\kappa_{n+m-2}(x) + \kappa_{n+m}(x) & n \neq m \\
2\kappa_2(x) + \cdots + 2\kappa_{n-2}(x) + \kappa_n(x) & n = m
\end{cases}
\]

(2.39)

and \( \kappa_n(x) \) is defined as

\[
\kappa_n(x) \equiv \frac{2n}{x^2 + n^2}.
\]

(2.40)

In Appendix E, we give a proof of a relation,

\[
(-)^{N_{\sigma_1} - 1} (N_{\sigma_1} - 1) ! N_{\sigma_1}^{-1} \prod_{\sigma_m > 1 \in \tilde{\theta}} \left[ \sum_{\tilde{\theta}' \in \Theta(\sigma_m)} N_{\sigma_m'}^{N_{\sigma_m'}} \prod_{\sigma'_m \in \tilde{\theta}'} (-)^{N_{\sigma'_m} - 1} (N_{\sigma'_m} - 1) ! \right] 
\]

\[
= \sum_{\tilde{\zeta}' \in \mathcal{D}(\tilde{\theta})} \sum_{\lambda' \in \Lambda_c(\tilde{\zeta})} \delta \left( \tilde{\zeta}', \lambda', \min(\sigma_1) \right) \prod_{\tilde{\theta}' \in \tilde{\zeta}'} \left( (-)^{M_{\tilde{\theta}'} - 1} (M_{\tilde{\theta}'} - 1) ! M_{\tilde{\theta}'} (M_{\tilde{\theta}'} !)^{N_{\tilde{\theta}'} - 1} \right) 
\]

\[
\prod_{\{\tilde{\theta}', \tilde{\theta}''\} \in \lambda'} \left( -M_{\tilde{\theta}'} M_{\tilde{\theta}''} \right) \right) ,
\]

(2.41)

where \( (\sigma_1, \ldots) = \tilde{\theta} \in \tilde{\Theta}(\theta \in \Theta_M) \). The equality of (2.27) and (2.38) is proved by using this relation and executing some elementary calculations.

Step 7

- **definition:** \( \tilde{\Theta}(\theta) \)
Let $\theta$ be a set which has a finite number of sets as elements. $\tilde{\Theta}(\theta)$ consists of all elements $\tilde{\zeta}$ satisfying the following two conditions. First, $\tilde{\zeta}$ is in $\tilde{\Theta}(\theta)$. Second, any sequence in $\tilde{\zeta}$ satisfies the condition that all sets as elements in the sequence have the same number of elements. Then,

$$\tilde{\Theta}(\theta) = \left\{ \tilde{\zeta} \in \tilde{\Theta}(\zeta) \mid N_\sigma = N_{\sigma'}, \quad \sigma, \sigma' \in \tilde{\theta} \in \tilde{\zeta} \right\}. \quad (2.42)$$

Here and hereafter, we use the symbol $\in$ for an inclusion relation between a sequence and its element like a set and its element.

- definition: $\lambda[\lambda', \tilde{\zeta}]$

Let $\tilde{\zeta}$ be a subset of a set $\tilde{\zeta}$ and $\lambda$ be an element of $\Lambda(\tilde{\zeta})$. We write $\lambda[\lambda', \tilde{\zeta}]$ as

$$\lambda[\lambda', \tilde{\zeta}] \equiv \left\{ \{\tilde{\theta}, \tilde{\theta}'\} \in \lambda \mid \tilde{\theta}, \tilde{\theta}' \in \tilde{\zeta}' \right\}. \quad (2.43)$$

By using those symbols, we change an expression of summations in $\{2.38\}$ with respect to permutation. Then, $\{2.38\}$ becomes

$$\sum_{\delta \in \Theta} \sum_{\zeta \in \tilde{\Theta}(\delta)} \sum_{\lambda \in \Lambda(\zeta)} \sum_{\lambda' \subseteq \lambda} \left[ \prod_{\zeta' \in G(\lambda')} \delta \left( \zeta', \lambda \left[ \lambda', \tilde{\zeta} \right], \min_1 \left( \tilde{\zeta}' \right) \right) \int_{-\infty+i \min_1}^{\infty+i \min_1} (\zeta')^\delta \frac{dx_{\zeta'}}{2\pi} \right]$$

$$\left[ \prod_{\tilde{\zeta}' \in G(\lambda')} \left( \sum_{\tilde{\zeta}'' \in G(\lambda')} \sum_{\tilde{\zeta}'' \subseteq \lambda'} \frac{2M_{\tilde{\theta}'} N_{\tilde{\theta}'} L}{(x_{\tilde{\zeta}''} + (N_{\tilde{\theta}'} - 1)i)^2 + N_{\tilde{\theta}'}^2} \right) \right] \prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda'} (-M_{\tilde{\theta}} M_{\tilde{\theta}'})$$

$$\prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda'' - \lambda', \tilde{\theta}' \supseteq \tilde{\theta}, \tilde{\theta}' \supseteq \tilde{\theta}'} (-M_{\tilde{\theta}'} M_{\tilde{\theta}'}) K_{N_{\tilde{\theta}'}, N_{\tilde{\theta}'}}, \left( x_{\tilde{\zeta}'} - x_{\tilde{\zeta}''} + (N_{\tilde{\theta}'} - N_{\tilde{\theta}'})i \right) \right]$$

$$e^{-\beta \sum_{\tilde{\zeta}' \in G(\lambda')} \sum_{\tilde{\theta}'' \in \tilde{\zeta}'} \frac{2M_{\tilde{\theta}''} N_{\tilde{\theta}''}}{\left( x_{\tilde{\theta}''} + (N_{\tilde{\theta}''} - 1)i \right)^2 + N_{\tilde{\theta}''}^2} \prod_{\tilde{\theta} \in \tilde{\zeta}} \left( (-M_{\tilde{\theta}} - 1)! M_{\tilde{\theta}}^{-1} (M_{\tilde{\theta}'} - 1)! M_{\tilde{\theta}'}^{-1} \right) \right]. \quad (2.44)$$

Here, we have simplified the function $\min(\sigma_1, \cdots) = \tilde{\theta} \in \tilde{\zeta} (\min(\sigma_1))$ as $\min_1(\tilde{\zeta})$ where $\tilde{\zeta}$ is in $\tilde{\Theta}(\theta)$ and $\theta$ is in $\Theta_M$. In the change from $\{2.38\}$ into $\{2.44\}$, there is a one-to-many correspondence between $\{\tilde{\zeta}, \lambda, \{\lambda'\}\}$ and $\{\tilde{\zeta}, \lambda, \lambda'\}$ associated with

$$\sum_{\delta \in \Theta} \sum_{\zeta \in \tilde{\Theta}(\delta)} \sum_{\lambda \in \Lambda(\zeta)} \sum_{\lambda' \subseteq \lambda} \left[ \prod_{\delta \in \tilde{\Theta}(\delta)} \delta_{\tilde{\theta}_1} \right] \sum_{\lambda \in \Lambda(\zeta)} \left[ \prod_{\lambda' \subseteq \lambda} \sum_{\tilde{\zeta}' \in G(\lambda')} \right].$$
can be written as
\[ \leftrightarrow \sum_{\theta \in \Theta_M} \sum_{\zeta \in \tilde{\Theta}(\theta)} \sum_{\lambda \in \Lambda(\zeta)} \sum_{\lambda' \leq \lambda}. \] (2.45)

Each \( \tilde{\zeta}, \lambda, \{ \lambda' \} \) in \( \{ \tilde{\zeta}, \lambda, \{ \lambda' \} \} \) corresponds to a subset \( A \) of \( \{ \tilde{\zeta}, \lambda, \{ \lambda' \} \} \) and \( \tilde{\zeta}', \lambda', \{ \lambda' \} \) can be written as
\begin{align*}
\tilde{\zeta}' &= \left\{ \tilde{\theta} \mid (\sigma_k) = \left( \oplus (\sigma_1, \ldots, \sigma_k) = \tilde{\theta} \right) \sigma_k \right\}, \quad \tilde{\zeta}'' \in G_{\tilde{\zeta}''}(\lambda'') \\
\{ \lambda' \} &= \left\{ \lambda \mid \lambda = \lambda \left[ \lambda'', \tilde{\zeta}'' \right], \tilde{\zeta}'' \in G_{\tilde{\zeta}''}(\lambda'') \right\}. \quad (2.46)
\end{align*}

Second, \( \lambda_0 \) can be written as
\[ \lambda_0 = \left\{ \left\{ \tilde{\theta}, \tilde{\theta}' \right\} \mid \tilde{\theta} = f \left( \tilde{\theta}' \right), \tilde{\theta}' = f \left( \tilde{\theta}'' \right), \{ \tilde{\theta}'', \tilde{\theta}'' \} \in \lambda'' \right\}, \quad (2.47)\]
where \( f \) is an onto-mapping from the set \( \tilde{\zeta}'' \) to the set \( \tilde{\zeta}' \) satisfying a condition: For any \( \tilde{\theta}'' \in \tilde{\zeta}'' \) there is a \( \tilde{\zeta} \in D \left( f(\tilde{\theta}'') \right) \) in which \( \tilde{\theta}'' \) exists. It is clear that this correspondence is a “one-to-many correspondence”.

**Step 8**

We change the integral path of a variable \( x_{\tilde{\theta}} \) in (2.44) from \( (-\infty + i \min(\sigma_1) \delta, +\infty + i \min(\sigma_1) \delta) \) to \( (-\infty - (N_{\tilde{\theta}} - 1)i, +\infty - (N_{\tilde{\theta}} - 1)i) \). Then, we get
\[ (2.49) \]
\[ = \sum_{\theta \in \Theta_M} \sum_{\zeta \in \tilde{\Theta}(\theta)} \sum_{\lambda \in \Lambda(\zeta)} \left[ \prod_{\theta \in \zeta} \int_{-\infty - iN_{\tilde{\theta}} \delta}^{+\infty - iN_{\tilde{\theta}} \delta} \frac{dx_{\tilde{\theta}}}{2\pi} \right] \left[ \prod_{\theta \in \zeta} \left( \sum_{\tilde{\theta} \in G_{\Lambda}(\zeta)} \frac{2M_{\tilde{\theta}} N_{\tilde{\theta}} L}{(x_{\tilde{\theta}} + (N_{\tilde{\theta}} - 1)i)^2 + N_{\tilde{\theta}}^2} \right) \right] \]
\[ \prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda} \left( -M_{\tilde{\theta}} M_{\tilde{\theta}'} \right) K_{N_{\tilde{\theta}}, N_{\tilde{\theta}'} \left( x_{\tilde{\theta}} - x_{\tilde{\theta}} + (N_{\tilde{\theta}} - N_{\tilde{\theta}'}) i \right)} \]
\[ e^{-\beta \sum_{\delta \in \zeta} \frac{2M_{\tilde{\theta}} N_{\tilde{\theta}}}{(x_{\tilde{\theta}} + (N_{\tilde{\theta}} - 1)i)^2 + N_{\tilde{\theta}}^2}} \prod_{\tilde{\theta} \in \zeta} \left[ (-)^{M_{\tilde{\theta}} - 1} (M_{\tilde{\theta}} - 1)! M_{\tilde{\theta}}^{-1} (M_{\tilde{\theta}}!)^{N_{\tilde{\theta}} - 1} \right]. \quad (2.48)\]

There are cancellations of terms corresponding to \( \tilde{\zeta} \notin \tilde{\Theta}(\theta) \) and all effects of residues. The equality of (2.44) and (2.48) is proved in Appendix F.
By further changing variables $x_{\tilde{\theta}}$ into $x_{\tilde{\theta}} + (N_{\tilde{\theta}} - 1)i$, (2.48) becomes

\[ (2.49) = \sum_{\theta \in \Theta} \sum_{\tilde{\zeta} \in \tilde{\Theta}(\theta)} \sum_{\lambda \in \Lambda(\tilde{\zeta})} \left[ \prod_{\tilde{\theta} \in \tilde{\zeta}} \int_{-\infty}^{\infty} \frac{dx_{\tilde{\theta}}}{2\pi} \right] \left[ \prod_{\tilde{\theta} \in \tilde{\zeta}} \left( \sum_{x_{\tilde{\theta}}} \frac{2M_{\tilde{\theta}}N_{\tilde{\theta}}L}{x_{\tilde{\theta}}^2 + N_{\tilde{\theta}}^2} \right) \right] \left[ \prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda} (-M_{\tilde{\theta}}M_{\tilde{\theta}'})K_{N_{\tilde{\theta}}, N_{\tilde{\theta}'}}(x_{\tilde{\theta}} - x_{\tilde{\theta}'}) \right] \left[ \prod_{\theta' \in \zeta} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \right] \left[ \prod_{\theta' \in \zeta} \int_{-\infty}^{\infty} dx_{\theta'} \left( \prod_{\theta' \in \zeta} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \right) \right] \left[ \prod_{\theta' \in \zeta} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \right] \right]. \]

**Step 9**

- **definition**: $\tilde{\Theta}(\theta)$

Let $\theta$ be a set which has a finite number of sets as elements. $\tilde{\Theta}(\theta)$ consists of all elements $\zeta$ satisfying the following two conditions. First, $\zeta$ is in $\Theta(\theta)$. Second, any set $\theta'$ in $\zeta$ satisfies the condition that all sets as elements in the set $\theta'$ have the same number of elements. Then,

\[ \tilde{\Theta}(\theta) = \{ \zeta \in \Theta(\zeta) \mid N_{\sigma} = N_{\sigma'}, \sigma, \sigma' \in \theta' \in \zeta \}. \]  

(2.50)

We rewrite the expression of summations in (2.49) with respect to permutations. Then, eq.(2.49) becomes

\[ (2.51) = \sum_{\theta \in \Theta_{M}} \sum_{\tilde{\zeta} \in \tilde{\Theta}(\theta)} \sum_{\lambda \in \Lambda(\tilde{\zeta})} \left[ \prod_{\tilde{\theta} \in \tilde{\zeta}} \int_{-\infty}^{\infty} \frac{dx_{\tilde{\theta}}}{2\pi} \right] \left[ \prod_{\tilde{\theta} \in \tilde{\zeta}} \left( \sum_{x_{\tilde{\theta}}} \frac{2M_{\tilde{\theta}}N_{\tilde{\theta}}L}{x_{\tilde{\theta}}^2 + N_{\tilde{\theta}}^2} \right) \right] \left[ \prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda} (-M_{\tilde{\theta}}M_{\tilde{\theta}'})K_{N_{\tilde{\theta}}, N_{\tilde{\theta}'}}(x_{\tilde{\theta}} - x_{\tilde{\theta}'}) \right] \left[ \prod_{\theta' \in \zeta} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \right] \left[ \prod_{\theta' \in \zeta} \int_{-\infty}^{\infty} dx_{\theta'} \left( \prod_{\theta' \in \zeta} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \right) \right] \left[ \prod_{\theta' \in \zeta} \left( -\frac{1}{N_{\theta'} - 1} \right)^{M_{\theta'} - 1} \right] \right]. \]

In this rewriting, we observe a many-to-many correspondence between $\{\tilde{\zeta}, \lambda\}$ and $\{\zeta, \lambda\}$.
associated with
\[
\sum_{\theta \in \Theta_M} \sum_{\zeta \in \tilde{\Theta}^{(\theta)}} \sum_{\lambda \in \Lambda^{(\zeta)}} \leftrightarrow \sum_{\theta \in \Theta_M} \sum_{\zeta \in \tilde{\Theta}^{(\theta)}} \sum_{\lambda \in \Lambda^{(\zeta)}} .
\] (2.52)

A subset \( A \) of \( \{\tilde{\zeta}, \lambda\} \) corresponds to a subset \( B \) of \( \{\zeta, \lambda\} \), where \( \tilde{\zeta}, \lambda \) in \( A \) and \( \zeta', \lambda' \) in \( B \) satisfy the following conditions: There is a bijection \( f \) from the set \( \tilde{\zeta} \) to the set \( \zeta' \) which satisfies

\[
\bigoplus_{\sigma \in \tilde{\Theta}} \sigma = \bigoplus_{\sigma' \in \Theta'} \sigma', \quad N_{\tilde{\sigma}} = M_{\theta'},
\] (2.53)

where \( f(\tilde{\sigma}) = \theta' \). And, \( \{\tilde{\theta}', \tilde{\theta}''\} \) is in \( \lambda \), if and only if \( \{f(\tilde{\theta}'), f(\tilde{\theta}'')\} \) is in \( \lambda' \). It is clear that this correspondence is a “many-to-many correspondence”.

We newly introduce three functions,
\[
\mu(\hat{\theta}, \zeta) \equiv \prod_{\theta' \in \zeta} (-N_{\theta'}^{-1} (N_{\theta'} - 1)!,
\] (2.54)
\[
E(\zeta) \equiv \sum_{\theta' \in \zeta} \frac{2N_{\theta'} M_{\theta'}}{x_{\theta'}^2 + M_{\theta'}^2}
\] (2.55)

and
\[
(2\pi)^N \left| \frac{\partial I}{\partial x} \right|_{L, \zeta} \equiv \left[ \prod_{\theta' \in \zeta} N_{\theta'}^{-1} \right] \sum_{\lambda \in \Lambda^{(\zeta)}} \left[ \prod_{\zeta' \in G^{(\lambda)}} \left( \sum_{\theta' \in \zeta'} \frac{2N_{\theta'} M_{\theta'}}{x_{\theta'}^2 + M_{\theta'}^2} \right) \right]
\] (2.56)

where \( \theta \in \Theta_M \) and \( \zeta \in \tilde{\Theta}(\theta) \). In terms of these functions, the expression (2.51) can be simplified as

\[
M! Z_M e^{-hM} = \sum_{\theta \in \Theta_M} \left[ \prod_{\sigma \in \theta} N_{\sigma}! \right] \sum_{\zeta \in \tilde{\Theta}^{(\theta)}} \mu(\hat{\theta}, \zeta) \left[ \prod_{\theta' \in \zeta} \int_{-\infty}^{\infty} dx_{\theta'} \right] \left| \frac{\partial I}{\partial x} \right|_{L, \zeta} e^{-\beta E(\zeta)}.\) (2.57)

In this way, we have proved the equivalence of (2.4) and (2.57). Remark that (2.57) is the expression which we suppose \( Z_M \) to be in the previous paper [8].
3 A relation between $Z_M$ and the string hypothesis

The function (2.56) is the Jacobian between $\{x_\theta\}$ and $\{I_\theta\}$ defined by relations,

$$\left(\begin{array}{c}
  x_{\theta'} + M_{\theta'} i \\
  x_{\theta'} - M_{\theta'} i
\end{array}\right)^L = e^{-2\pi i I_{\theta'}} \prod_{\theta'' \in \zeta, \neq \theta'} E_{M_{\theta'}, M_{\theta''}} (x_{\theta'} - x_{\theta''})^{N_{\theta''}},$$  

(3.1)

where $\theta \in \Theta_M$, $\zeta \in \bar{\Theta}(\theta)$ and

$$E_{n,m}(x) = \prod_{k=1}^{\min(n,m)} \left[ \frac{x - (n + m - 2k) i}{x - (n + m + 2k) i} \right]^2.$$  

(3.2)

We emphasize the following: We regard all the variable $\{I_\theta\}$ as integer and interpret the value $x_{\theta'}$ as an $M_{\theta'}$-string center. The relations (3.1) are the equations which are introduced by the string hypothesis in case of any $N_{\theta'\in\zeta} = 1$. And, each term of (2.55) is the energy corresponding to $M_{\theta'}$-string in the string hypothesis.

We study eq. (2.57) again. As we have shown in [8], the expression (2.57) can be derived by summing up the string center equation formally. The derivation is summarized as follows. From the string hypothesis, the free energy is written as

$$M! Z_M e^{-hM} = \sum_{\theta \in \Theta_M} \left[ \prod_{\sigma \in \Theta} N_{\sigma} \right] \sum_{\zeta \in \bar{\Theta}(\theta)} \mu (\tilde{\theta}, \zeta) \sum_{\{I_\theta\}} e^{-\beta E(\zeta)},$$  

(3.3)

where $\sum_{\{I_\theta\}}$ means a summation over all the real number solutions of (3.1) on condition that $\{I_\theta\}$ are integers. Note that the coefficients are due to the symmetry of the quasi-particles. Using a relation

$$\sum_{\theta \in \Theta_M} f (\{x_j (\{n_i\})\}) = \int \left| \frac{dn}{dx} \right| f (\{x_j\}) \prod dx_j$$  

(3.5)

where the limit means the thermodynamic limit, and the integral path in the r.h.s. is the region containing points $\{x_j\}$ summed up in the l.h.s. A simple example of (3.5) is

$$\lim_{n_1 = -\infty} \cdots \lim_{n_1 = -\infty} \exp \left( -\beta \sum_{i=1}^{N} k_i^2 \right) = \prod_{i=1}^{N} \left( \int_{-\infty}^{\infty} \frac{Ldk_i}{2\pi} \right) \exp \left( -\beta \sum_{i=1}^{N} \frac{k_i^2}{2\pi n_i} \right),$$  

(3.6)

$$2\pi n_i = k_i L,$$
which is a partition function of the free particles. We have used this modification from (2.1) to (2.4).

There remains a problem with this derivation of (2.57) from the string hypothesis. When we use eq.(3.5), a condition is required. That is, the orientation of the integral path in the r.h.s. of (3.5) has the following property: the value of an integral $\int |\frac{dn}{dx}| \prod dx_j$ for any part of the integral path is positive. Note that we have defined the orientation of the integral path (2.4) by this condition. The integral path in (3.4), however, does not have the property. The value of the integral is $O(L^M)$ and the difference between the r.h.s. of (3.4) and the value of the integral using the truly oriented integral path is $O(L^{M-1.5})$ where $M$ is the number of up-spins. A further research is required to conclude whether this causes the difference in the thermodynamic quantities or not.

Note that in [8], we do not claim that the derivation of (2.57) using the string hypothesis is complete, but claim that (2.57) is related to the string hypothesis and the free energy is properly derived when we assume the relation (2.57). On the contrary, in this paper, we have derived (2.57) step by step only assuming the Bethe ansatz equations.

4 Conclusion

In this paper, we have calculated $\text{Tr} e^{-\beta H_M}$ for the XXX Heisenberg model, which is the trace of the Boltzmann weight under the restriction that the number of up-spin $M$ is fixed. This method relies only on the Bethe ansatz equations. Using this method and the result in [8], we have obtained the free energy, whose expression is perfectly agree with TBA. In a sense, we have generalized the direct method or the Bethe ansatz cluster expansion method [10, 11] into models with bound states. We emphasize that this derivation of the free energy is independent of QTM, TBA and is free from the string hypothesis. We can replace the Boltzmann weight with some other functions in case that the Boltzmann weight and the functions have the same analyticity. Therefore, it may be possible to calculate some other thermodynamic quantities $<A>$ in the same way by replacing the Boltzmann weight $e^{-\beta H}$ with $Ae^{-\beta H}$. 

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Appendix A Path-independence of the integrals

We prove that the value of the integral (2.11) along the path (2.12) does not depend on $A_{\sigma}$ in case all $A_{\sigma}$ are finite positive numbers, where $E_{\theta}$ is defined by (2.6) and the relation between $x_{\sigma}$ and $I_{\sigma}$ is defined by (2.7). In other words, when we suppose that $\{A_{\sigma}\}$ are continuous functions with respect to $0 \leq t \leq 1$ and none of $\{A_{\sigma}(t) \in \mathbb{R}_{>0}\}$ is equal to $\infty$, the integrals for the paths specified by $\{A_{\sigma}(0)\}$ and $\{A_{\sigma}(1)\}$ have the same value.

To prove this, we define the integral (2.11) more precisely. The reason is that, there is some ambiguity in the treatment of the region where the relation between $\{x_{\sigma}\}$ and $\{I_{\sigma}\}$ is not analytic. We define the integral path (2.12) and the integrand of (2.11) as follows. We denote by $\Omega$ a metric space $(y_1 \in \mathbb{C}, y_2, \cdots)$ which is topologically equivalent
to a direct product of one-point compactified complex plane \( \otimes S^2 \). We need not specify the distance function definitely. Let \( \Omega_I \) be a subspace of \( \Omega \). The points on \( \Omega_I \) are written as \( \langle \{x_\sigma\}, \{e^{2\pi I_\sigma i}\} \rangle \), where they satisfy the relations \( (2.17) \), \( x_\sigma \neq \pm i, \infty \) and \( x_\sigma' - x_\sigma'' \neq \pm 2i \). Here, we regard \( \{x_\sigma\} \) as \( x_\sigma, x_\sigma', x_\sigma'', \ldots \) where \( \sigma, \sigma', \sigma'', \ldots \in \theta \). We define \( \bar{\Omega}_I \) as a subspace of the closure of \( \Omega_I \) in \( \Omega \) where \( |\Im\theta| \) is not equal to \( \infty \). The topological space \( \partial \Omega_I \) is defined to be a subset of \( \bar{\Omega}_I \) where \( \{x_\sigma\} \) as a function of \( \{I_\sigma\} \) is non-analytic or the integrand is non-analytic with respect to \( \{I_\sigma\} \). The integral path \( (2.12) \) represents the space \( \Omega_I \) under the restriction \( |e^{2\pi I_\sigma i}| = A_\sigma \). The orientation of the integral path is defined so that the value of an integral \( \int \prod dI_\sigma \) for any part of the integral path is positive.

We shall prove in [A.1] the following two facts: 1) The dimension of \( \partial \Omega_I \) under the restriction \( |e^{2\pi I_\sigma i}| = A_\sigma \) is not more than \( N_\sigma - 1 \). 2) The dimension of \( \partial \Omega_I \) is not more than \( 2N_\sigma - 2 \). Here, \( 2N_\sigma \) is the dimension of the space \( \Omega_I \). The first indicates that the orientation is defined for all the region on the space \( \bar{\Omega}_I \) under the restriction \( |e^{2\pi I_\sigma i}| = A_\sigma \). Therefore, the space can be considered as an oriented manifold or \( N_\sigma \)-chain. Here and hereafter, we identify the symbol \( \{A_\sigma\} \) with the \( N_\sigma \)-chain. And, the second indicates that the boundary \( \partial\{A_\sigma\} \) is equal to zero as \( N_\theta - 1 \)-chain. In [A.2] we shall prove that 3) the integrand of \( (2.11) \) converges to a finite value in the limit on any sequence of points on \( \Omega_I \) which approaches to a point on \( \bar{\Omega}_I \).

Then, we define the value of the integrand of \( (2.11) \) as the limit at any point on \( \bar{\Omega}_I \). Thus, the integral \( (2.11) \) has been defined.

Next, we define three \( N_\theta \)-chains \( \{A'_\sigma(t, \delta, \delta')\}, \{A''_\sigma(t, \delta)\} \) and \( \Delta A''(\delta, \delta') \) embedded in \( \Omega_I \) as integral paths of \( (2.11) \). The \( \{A'_\sigma(t, \delta)\} \) is continuously changed by \( 0 \leq t \leq 1 \) and \( 0 \leq \delta << 1 \). We may choose so that \( \{A_\sigma(t)\} - \{A'_\sigma(t, 0)\} \) is equal to zero, where we regard \( \{A_\sigma(t)\} \) as an \( N_\sigma \)-chain. The dimension of the region \( \{A'_\sigma(t, \delta)\} \cap \partial \Omega_I \) is not more than \( N_\theta - 2 \) for any \( t \), in case of \( \delta \neq 0 \). And, \( \partial\{A'_\sigma(t, \delta)\} \) is equal to zero. The existence of the \( N_\theta \)-chain \( \{A'_\sigma(t, \delta)\} \) is assured by 1) and the fact \( \partial\{A_\sigma(t)\} = 0 \). Note that the expression \( \{A''_\sigma(t, \delta)\} \) is merely a symbol of \( N_\sigma \)-chain and does not indicate that the integral path satisfies the condition \( (2.12) \). \( \{A''_\sigma(t, \delta, \delta')\} \) is a part of \( \{A''_\sigma(t, \delta)\} \) where the distance between all of the points and \( \partial \Omega_I \) is more than \( \delta' \). We can also regard \( \{A''_\sigma(t, \delta, \delta')\} \) as an \( N_\theta + 1 \)-chain depending on \( \delta \) and \( \delta' \) by regarding \( 0 \leq t \leq 1 \) as a variable. And, we define the orientation of the \( N_\theta + 1 \)-chain so that \( \partial\{A''_\sigma(t, \delta, 0)\} \) is equal to \( \{A''_\sigma(1, \delta, 0)\} - \{A''_\sigma(0, \delta, 0)\} \). Then,
$\Delta A''(\delta, \delta')$ is defined to be $\partial\{A''_\sigma(t, \delta, \delta')\} - \{A''_\sigma(1, \delta, \delta')\} + \{A''_\sigma(0, \delta, \delta')\}$.

Finally, we evaluate the difference between the values of the integral (2.11) for the integral paths, $\{A_\sigma(0)\}$ and $\{A_\sigma(1)\}$. The fact 3) indicates that the integrals along the paths $\{A'_\sigma(t, +0)\}$ and $\{A_\sigma(t)\}$ are the same, and the integrals along the paths $\{A''_\sigma(t, \delta, +0)\}$ and $\{A''_\sigma(1, \delta)\}$ are the same. The difference between the values of the integral for the paths $\{A''_\sigma(0, \delta, \delta')\}$ and $\{A''_\sigma(1, \delta, \delta')\}$ is the integral along the path $\Delta A''(\delta, \delta')$ when $\delta' > 0$, because (2.11) is a multiple complex integral and all the point on $\{A_\sigma(t, \delta, \delta')\}$ is regular. The integral for the path $\Delta A''(\delta, +0)$ is equal to zero when $\delta > 0$, because the dimension of $\Delta A''(\delta, +0)$ is $N_\sigma - 1$ and the integrand is finite. Thus, the values of the integral (2.11) do not depend on the integral paths, $\{A_\sigma(0)\}$ and $\{A_\sigma(1)\}$.

A. 1

We prove that the dimension of $\partial \Omega_I$, which is defined in Appendix A, is not more than $2N_\theta - 2$ where $2N_\theta$ is the dimension of $\Omega_I$. As a corollary, the number of solution $\{x_\sigma\}$ of eqs. (2.7) does not depend on $\{e^{I_\theta}\}$ when $\{e^{I_\theta}\}$ is in $\Omega_I - \partial \Omega_I$. We also prove that the dimension of the non-analytic region is not more than $N_\theta - 1$ in any subspace $|e^{I_\theta}| = \text{constant}$. As a corollary, the dimension of $\bar{\Omega}_I$ under the restriction $|e^{I_\theta}| = \text{constant}$ is $N_\theta$.

The first is proved by mathematical induction as follows.

In case of $N_\theta = 1$, $\partial \Omega_I$ contains only one point ($x_\sigma = \infty, e^{I_\theta} = 1$). Then, the dimension of $\partial \Omega_I$ is zero.

There are two sufficient conditions when any point is in $\partial \Omega_I$. One is that $\{x_\sigma\}$ regarded as a function of $\{I_\theta\}$ is not analytic. The other is that the integrand is not analytic when we regard the argument of the function as $\{x_\sigma\}$. The inverse mapping theorem indicates that one of the following three relations holds when the first condition holds. One is that the Jacobian (2.23) is equal to 0, and the other are $x_\sigma = \pm i, \infty$ or $x_{\sigma'} - x_{\sigma''} = \pm 2i$. The second relation is equivalent to the second condition. Therefore, we shall show that the dimension of the space which satisfies the previous three relations is not more than $2N_\theta - 2$. Note that, in A. 3 we show the following fact. In case $\{x_\sigma\}, \{e^{I_\theta}\}$ are in $\partial \Omega_I$, the condition that some variables satisfy $x_\sigma = \pm i$ or $x_{\sigma'} - x_{\sigma''} = \pm 2i$ is equivalent to the condition that there are subsets $\theta_\pm \subseteq \theta$ which satisfy $x_{\sigma_\pm} = \pm i, x_{\sigma_\pm} - x_{\sigma'} \neq \pm 2i$, $x_{\sigma'} \neq \pm i, \infty$ and $x_{\sigma'} - x_{\sigma''} \neq \pm 2i$.
where $\sigma', \sigma'' \in \theta - \theta_+ - \theta_-$ and $\sigma_+ \in \theta_+$. 

We divide the space $\partial \Omega_I$ into three subspaces: The first subspace is $\partial \Omega_I$ under the restrictions $x_{\sigma \sigma' \theta'} = \infty$ and $x_{\sigma \sigma' \theta'} \neq \infty$ where $\theta' \subseteq \theta$, the second under the restriction, $x_{\sigma} \neq \infty, x_{\sigma \pm \theta_\pm} = \pm i, x_{\sigma \theta_\pm} \neq \pm i$ where $\theta_\pm \subseteq \theta$, and the third under the restriction $x_{\sigma} \neq \pm i, \infty$ and $x_{\sigma'} - x_{\sigma''} \neq \pm 2i$.

In case that $\{x_{\sigma}\}, \{e^{I_{\sigma}}\}$ are in the first subspace, eq.(2.7) are reduced to

\[
\prod_{\sigma' \neq \sigma, \sigma \in \theta'} \left[ \frac{x_{\sigma} - x_{\sigma'} + 2i}{x_{\sigma} - x_{\sigma'} - 2i} \right]^{N_{\sigma'}} = e^{2\pi i I_{\sigma}}, \quad \sigma \in \theta', \quad (A.1)
\]

\[
\prod_{\sigma' \in \theta - \theta', \sigma \in \theta} \left[ \frac{x_{\sigma} - i}{x_{\sigma} + i} \right]^L \prod_{\sigma' \in \theta - \theta', \sigma \in \theta} \left[ \frac{x_{\sigma} - x_{\sigma'} + 2i}{x_{\sigma} - x_{\sigma'} - 2i} \right]^{N_{\sigma'}} = e^{2\pi i I_{\sigma}}, \quad \sigma \in \theta - \theta'. \quad (A.2)
\]

Eq.(A.1) requires a condition $\sum_{\sigma \in \theta'} N_{\sigma} I_{\sigma} \in \mathbb{Z}$. And, eq.(A.2) is the Bethe equation with respect to $N_{\sigma} - 1$ variables. These indicate that the dimension of the first subspace is not more than $2N_{\theta} - 2$.

In case that $\{x_{\sigma}\}, \{e^{I_{\sigma}}\}$ are in the second subspace, eq.(2.7) becomes

\[
\prod_{\sigma \in \theta - \theta_+ - \theta_-} \left[ \frac{(x_{\sigma} + 3i)(x_{\sigma} + 1)}{(x_{\sigma} - 3i)(x_{\sigma} - 1)} \right]^{N_{\sigma}} \sum_{\sigma' \subseteq \theta_+} N_{\sigma'} = e^{2\pi i \sum_{\sigma \in \theta_+ + \theta_-} N_{\sigma} I_{\sigma}} \quad (A.3)
\]

\[
\prod_{\sigma' \in \theta, \sigma \in \theta} \left[ \frac{x_{\sigma} - x_{\sigma'} + 2i}{x_{\sigma} - x_{\sigma'} - 2i} \right]^{N_{\sigma'}} \prod_{\sigma' \in \theta_+} \frac{x_{\sigma} + 1}{x_{\sigma} - 1} \prod_{\sigma' \in \theta_-} \frac{x_{\sigma} + 3i}{x_{\sigma} - i} = e^{2\pi i I_{\sigma}}, \quad (A.4)
\]

The first equation is obtained as follows. We raise both sides of eq.\[(2.7)\] with respect to $\sigma \in \theta_+ + \theta_-$ to the power of $N_{\sigma}$. Then, the product of these equations is the first equation in case of

\[
\lim_{\sigma \in \theta_+} \frac{\prod_{\sigma' \in \theta_+} (x_{\sigma} - i)^{N_{\sigma}}}{\prod_{\sigma' \in \theta_-} (x_{\sigma} + i)^{N_{\sigma}}} = 1, \quad (A.5)
\]

where $\lim$ means a limit on a sequence of points on $\Omega_I$ converging to a point where $x_{\sigma \pm \theta_\pm} = \pm i, x_{\sigma \theta_\pm} \neq \pm i$ and $\Im I_{\sigma} \neq \infty$. (A.5) is proved in A.4. We define a projection from $\bar{\Omega}_I$ onto the $4N_{\theta - \theta_+ - \theta_-}$-dimensional space (\{(x_{\sigma} \in \theta - \theta_+ - \theta_-), \{e^{I_{\sigma}} \in \theta - \theta_+ - \theta_-}\}). The projection is the elimination of the $2N_{\theta_+ + \theta_-}$ variables $\{x_{\sigma} \in \theta_+ + \theta_-\}$, $\{e^{I_{\sigma}} \in \theta_+ + \theta_-\}$. Then, using the projection, we define two spaces; 1) the union of the second subspace and the inverse image of a point, 2) the direct image of the second subspace. Eq.(A.3) indicates that the dimension of the union
is not more than $2N_{\theta_+ + \theta_-} - 2$. Eq. (A.4) indicates that the dimension of the direct image is not more than $2N_{\theta - \theta_+ - \theta_-}$.

The dimension of the third subspace is not more than the dimension of $\{x_\sigma\}$ in $\Omega_I$ where the Jacobian is equal to 0, that is $2N_\theta - 2$, because $\{e^{I_\sigma}\}$ regarded as a function of $\{x_{\sigma'}\}$ are injections in this subspace.

Thus, we have proved that the dimension of $\partial\Omega_I$ is not more than $2N_\theta - 2$.

The second fact for the non-analytic region is proved by re-evaluating the dimension of the three subspace under the restriction $|e^{I_\sigma}|=$constant.

A. 2

We prove that the integrand of (2.11) converges in the limit on any sequence of points on $\Omega_I$ which converges to a point on $\Omega_I$. The sufficient condition is that the integrand on any sequences of points satisfying the following condition is convergent. The sequence of points converges to a point where some of $x_\sigma$ are equal to $i$ or $-i$.

Then, all we have to prove is that the equation

$$\lim \left[ \sum_{\sigma \in \theta_+} \frac{N_\sigma}{x_\sigma - i} - \sum_{\sigma \in \theta_-} \frac{N_\sigma}{x_\sigma + i} \right] = 0$$

(A.6) holds. Here, lim means a sequence on $\Omega_I$ converging to a point where $x_{\sigma \in \theta_\pm} = \pm i$ and $x_{\sigma \not\in \theta_\pm} \neq \pm i$ and $|\Im I_\sigma| \neq \infty$. This relation is proved in A. 4.

A. 3

We prove the following: In case $\{x_\sigma\}, \{e^{I_\sigma}\}$ are in $\partial\Omega_I$, the condition that some variables satisfy $x_\sigma = \pm i$ or $x_{\sigma'} - x_{\sigma''} = \pm 2i$ is equal to the condition that there are subsets $\theta_\pm \subseteq \theta$ which satisfies $x_{\sigma_\pm} = \pm i, x_{\sigma_\pm} - x_{\sigma'} \neq \pm 2i, x_{\sigma'} \neq \pm i, \infty$ and $x_{\sigma'} - x_{\sigma''} \neq \pm 2i$ where $\sigma', \sigma'' \in \theta - \theta_+ - \theta_-$ and $\sigma_\pm \in \theta_\pm$.

In case $x_{\sigma'} - x_{\sigma''} = 2i$, eq.(2.7) with respect to $\sigma', \sigma''$ and a condition $|\Im (I_{\sigma'}), |\Im (I_{\sigma''})| < \infty$ require that $x_{\sigma'}$ is equal to $i$ or $x_{\sigma''} - 2i$ and $x_{\sigma''}$ is equal to $-i$ or $x_{\sigma'''} + 2i$. This result and the fact that the number of the variables is finite lead to $x_{\sigma'} = i$ and $x_{\sigma''} = -i$
A. 4

We prove two relations,

\[
\lim \left[ \sum_{\sigma \in \theta_+} \frac{N_\sigma}{x_\sigma - i} - \sum_{\sigma \in \theta_-} \frac{N_\sigma}{x_\sigma + i} \right] = 0, \quad (A.7)
\]

\[
\lim \frac{\prod_{\sigma \in \theta_+} (x_\sigma - i)^{N_\sigma}}{\prod_{\sigma \in \theta_-} (x_\sigma + i)^{N_\sigma}} = 1, \quad (A.8)
\]

where \( \lim \) means a limit on a sequence of points on \( \Omega_I \) converging to a point where \( x_{\sigma \in \theta_\pm} = \pm i, \) \( x_{\sigma \notin \theta_\pm} \neq \pm i, |\Im I_{\sigma}| \neq \infty \) and \( \theta_\pm \subseteq \theta. \)

To prove them, we define two topological spaces \( \Omega_{I_p} \) and \( \partial \Omega_{I_p}. \) They are defined by subsets \( \theta_\pm. \) The topological space \( \Omega_{I_p} \) is a subspace on which all the points can be written as \( (\{x_\sigma\}, \{e^{I_\sigma}\}, \{\delta/\delta'\}, \{\delta^{-1} - \delta'^{-1}\}) \) under the restrictions \( x_\sigma \neq \pm i, \infty, x_{\sigma \pm 2i}, |\Im I_{\sigma}| \neq \infty, \) and \( (2.7), \) where \( \delta, \delta' = x_\sigma \in \theta_\pm. \) The topological space \( \partial \Omega_{I_p} \) is a subspace of the closure of \( \Omega_{I_p}, \) and a point in the closure is in \( \partial \Omega_{I_p} \) if and only if the point satisfies \( x_{\sigma \in \theta_\pm} = \pm i, x_{\sigma \notin \theta_\pm} \neq \pm i \) and \( |\Im I_{\sigma}| \neq \infty. \)

A sufficient condition of these relation is that there is a set \( \zeta \in \Theta(\theta_+ + \theta_-) \) any element \( \theta' \) of which satisfies

\[
\lim \frac{1}{x_{\sigma_+} - i} - \frac{1}{x_{\sigma_-} + i} = 0 \quad \sigma_+ \in \theta'_+, \sigma_- \in \theta'_-, 
\]

\[
\sum_{\sigma \in \theta' \cap \theta'_+} N_\sigma = \sum_{\sigma \in \theta' \cap \theta'_-} N_\sigma, \quad (A.9)
\]

where \( \theta'_+ = \theta' \cap \theta_+ \) and \( \lim \) means a limit on an sequence of points on \( \Omega_{I_p} \) which converges to a point on \( \partial \Omega_{I_p}. \) The reason is that, any sequence on \( \Omega_I \) converging to a point where \( x_{\sigma \in \theta_\pm} = \pm i \) and \( x_{\sigma \notin \theta_\pm} \neq \pm i \) and \( |\Im I_{\sigma}| \neq \infty \) corresponds to a sequence on \( \Omega_{I_p} \) converging to a subspace of a compact space in \( \partial \Omega_{I_p}. \)

First, we introduce an equivalence relation on a set \( \theta_+ + \theta_- \),

\[
\lim \frac{1}{x_{\sigma\pm} \mp i} - \frac{1}{x_{\sigma'\pm} \mp i} = 0, \quad \text{then} \quad \sigma_\pm \sim \sigma'_{\pm}, 
\]

\[
\lim \frac{1}{x_{\sigma\pm} \mp i} - \frac{1}{x_{\sigma'\pm} \mp i} = 0, \quad \text{then} \quad \sigma_\pm \sim \sigma'_{\pm} \quad (A.10)
\]

where \( \sigma_\pm, \sigma'_{\pm} \in \theta_\pm. \) Therefore, we define \( \zeta_0 \in \theta_+ + \theta_- \) as a set any element of which belongs to the equivalence class. It is clear that \( \zeta_0 \) satisfies the first condition of \( (A.9). \) In the following, we show that \( \zeta_0 \) satisfies the second condition of \( (A.9). \)
From (2.7), it follows that
\[
\lim \frac{(x_{\sigma} \pm i)^L}{\prod_{\sigma \in \theta^\pm}(x_{\sigma \pm} - x_{\sigma \mp} + 2i)^{N_{\sigma}}} \neq 0, \infty,
\] (A.11)
where \(\sigma \pm \in \theta \cap \theta'\), \(\sigma' \in \theta^\prime\). Now, we define \(\delta_{\theta'}\) as a variable on \(\Omega_{\theta'}\) which satisfies \(\lim \frac{1}{x_{\sigma \pm} \mp 2i} = \theta\), where \(\theta' \subset \zeta\) and \(\sigma \pm \in \theta' \cap \theta_{\mp}\). From this definition, \(\lim \frac{\delta_{\theta'}(x_{\sigma_{\pm}} \mp i)}{(x_{\sigma_{\pm}} - x_{\sigma_{\mp}} + 2i)} = 1\), \(\lim \frac{\delta_{\theta'}^3(x_{\sigma_{\pm}} \mp i)}{(x_{\sigma_{\pm}} - x_{\sigma_{\mp}} + 2i)} = 0\) are shown to hold when \(\sigma_{\mp} \in \theta_{\pm}, \sigma' \in \theta_{\pm}\). We raise both sides of the equations (A.11) to the power of \(\pm N_{\sigma_{\pm}}\). Then, the product of these equations is
\[
\lim \delta_{\theta'}^{(N_{+} - N_{-})(x_{\sigma_{\mp}} \mp i)} \prod_{\sigma \in \theta_{\pm} \cap \theta_{\prime}} \frac{(x_{\sigma_{\pm}} - x_{\sigma_{\mp}} + 2i)^{N_{\sigma_{\pm}}}}{(x_{\sigma_{\pm}} - x_{\sigma_{\mp}} + 2i)^{N_{\sigma_{\pm}}}} \neq 0, \infty,
\] (A.12)
where
\[
N_{+} \equiv \sum_{\sigma \in \theta_{+} \cap \theta'} N_{\sigma}, \quad N_{-} \equiv \sum_{\sigma \in \theta_{-} \cap \theta'} N_{\sigma}.
\] (A.13)
Therefore, it holds that \(N_{+} = N_{-}\). Here we have used that \(L\) is much larger than \(N_{\sigma_{\pm}}\).

**Appendix B  Definition of the integral path**

We show that (2.4) is equal to (2.13), where \(E_{\theta}\) is defined by (2.6), the relation between \(\{x_{\sigma}\}\) and \(\{I_{\sigma}\}\) is defined by (2.7) and the integral path in (2.4) is defined by (2.12).

In **Appendix A**, we have proved that the integrals in the l.h.s. of (2.12) do not depend on \(A_{\sigma}\). Then, we examine the case \(A_{\sigma} \leq \epsilon\). In this case, all the points on the integral path satisfy the condition, for any \(x_{\sigma}\), there exists \(n_{\sigma} \in 2\mathbb{Z}_{\leq 0} + 1\) such that
\[
|x_{\sigma} - n_{\sigma}i| < \epsilon^{L(M+1)}.
\] (B.1)
because eq. (2.12) indicates that one of terms \(\frac{|x_{\sigma} - n_{\sigma}i|^L}{|x_{\sigma} - n_{\sigma}i + 2i|^{N_{\sigma}'}}\) is less than \(\epsilon^{L(M+1)}\). In other words, the integral path (2.12) is divided into several pieces, which are classified by the vector \((n_{\sigma}i, n_{\sigma}'i, \cdots), n_{\sigma}, n_{\sigma}' \in \mathbb{Z}\). In the following, we consider a piece of the integral path characterized by the vector, and regard \(D\) as a set of variables \(x_{\sigma_{\pm}} \pm i, x_{\sigma_{n}} - x_{\sigma_{n-2}} - 2i\), where \(\sigma_{n} \in \theta\) is a set which satisfies \(\lim_{n \to 0} |x_{\sigma} - ni| = 0\) on the piece of the integral path. Remark that the variables are linearly dependent. Before we change the integral path, we
like to derive several values from (2.7), and prove several facts. For a while, we fix \( \{n_\sigma\} \), equivalently \( D \), and consider the corresponding part of the integral path.

We define \( A_\sigma \) which depends on \( \epsilon \),

\[
A_\sigma = \epsilon^{m_\sigma},
\]

(B.2)

where \( m_\sigma \in \mathbb{R}_{\geq 1} \) are defined as below. We choose an element \( D \) in \( \Theta(D) \) which satisfies \( N_D \leq N_\theta \). Using \( D \), we modify eq.(2.7) by replacing terms in the l.h.s. of equation as follows. We replace all the terms which converge to non-zero numbers in the limit \( \epsilon \to 0 \) with the numbers, and replace all the terms which converge to zero in the limit \( \epsilon \to 0 \), any of which is made of a value in \( D \), with variables \( \delta_i \) under the restriction that all variables in any set \( D_i \in \mathbb{D} \) are substituted for the same variable. We name this modification the limitation of (2.7) using \( D \). These equations can be solved with respect to \( \{\delta_i\} \). Note that, \( \{m_\sigma\} \) defines the \( \{\Re I_\sigma\} \). Using these equations, we demand that \( \{m_\sigma\} \) satisfy the following conditions: In case \( N_\theta > N_D \) there is no solution. In case \( N_\theta = N_D \) there are finite number of solutions. The value of different variables as a solution of the equations has different powers of \( \epsilon \). Then, we fix \( \{m_\sigma\} \) so that the above three conditions are satisfied for all \( \mathbb{D} \in \Theta(D) \) under the restriction of \( N_D \leq N_\theta \).

We also define a subset \( \mathbb{D} \) of \( \Theta(D) \). Any element \( \mathbb{D} \) in \( \Theta(D) \) which satisfies the following three conditions is in \( \mathbb{D} \). First, \( N_D = N_\theta \). Second, any value \( \delta_i \) has the positive power of \( \epsilon \). Here, we regard the variables \( \{\delta_i\} \) as a solution of the limitation of (2.7) using \( D \). To write the third condition, we need some preparations. We denote elements in \( \mathbb{D} \) by \( D_1, D_2, \ldots, D_{N_\theta} \) so that \( m_i \) is not larger than \( m_{i+1} \). Here, \( m_i \) is the powers of \( \epsilon \) with respect to the value \( \delta_i \) corresponding to \( D_i \). We regard any element in \( D \) as an oriented connection between two element in a set \( \{0\}, \sigma \in \theta \). Here, \( x_{\sigma \pm_1} \equiv i \) is considered as an oriented connection from \( \sigma_{\pm_1} \) to \( \{0\} \) and \( x_{\sigma_n} - x_{\sigma_{n-2}} - 2i \) is considered as an oriented connection from \( \sigma_n \) to \( \sigma_{n-2} \). Then, a subset of \( D \) defines a division \( \zeta \in \Theta(\{0\} + \theta) \) as follows: Any two elements in a set \( \theta' \in \zeta \) are directly or indirectly connected by connections in the subset, and any two elements in different sets \( \theta', \theta'' \in \zeta \) are not connected by connections in the subset. We write \( \zeta_n \) for a division defined by \( \bigcup_{m \geq n} D_m \). The third condition is that any element in \( D_n \) connects two elements which are in different clusters in \( \zeta_{n+1} \). Note that from the definition of \( \zeta_n \), there are only two clusters in \( \zeta_{n+1} \) whose elements are linked by connections in \( D_n \).
Using $D \in \mathcal{D}$, we shall define $\{m_\delta\}$ and $\{r_\delta\}$ where $\delta \in D$. We modify the set $D$ by multiplying some elements by $-1$. The modification causes a reflection on $\mathcal{D}$. We have regarded an elements in $D$ as an oriented connection; An element multiplied by $-1$ is the oppositely oriented connection. Then, $D$ is modified so that all the connections in the modified $D_i$ connect elements in the same cluster in $\zeta_i+1$ to other elements. We define $\{r_i\}$ and $\{m_i\}$ so that $\{m_\delta r_{-\delta} = m_i \}$ is a solution of the limitation of (2.7) using the modified $\mathcal{D}$.

Then, $m_\delta = m_i$ and $r_\delta = r_i$ where $\delta$ is in the modified $D_i$, and $m_{-\delta} = m_\delta$, $r_{-\delta} = -r_\delta$.

We remark that while $\{m_\delta\}$ is uniquely defined, $\{r_\delta\}$ is not uniquely defined, due to the ununiqueness of the solutions for the limitation of (2.7).

We define $\Omega_R$ as a topological space embedded in $\Omega$: Any point on $\Omega_R$ is written as $(\{m_\delta\}, \{r_\delta\})$ where $\{m_\delta\}, \{r_\delta\}$ are given by the above procedure, and $\mathcal{D} \in \mathcal{D}$ and $\{\mathcal{RI}_\sigma\}$ are not fixed. Note that, $\Omega_R$ depends on $D$ and the number of elements in $\Omega$ in this case is $2N_D$.

The topological space $\Omega_R$ has the following properties. Given sufficiently small $\epsilon'$ and $\epsilon$ which depends on $\epsilon'$ (for simplicity, $\epsilon(\epsilon')$), there is a one-to-one correspondence between a point on $\Omega_R$ and a point on a part of the integral path defined by (2.12) and (B.2) which satisfies $|x_\sigma - n_\sigma i| < 3\epsilon^{\epsilon \epsilon' + \epsilon' + 1}$. We denote by $C$ this piece of the integral path. The correspondence can be regarded as a homeomorphic mapping. Furthermore, the relation

$$\left| \frac{\delta}{r_\delta \epsilon^{m_\delta}} - 1 \right| < \epsilon'$$

holds where $\{\delta\} = D$ is the value on a point on $C$, $(\{m_\delta\}, \{r_\delta\})$ is a point on $\Omega_R$ and the point on $C$ corresponds to the point on $\Omega_R$. These facts are proved in [B.1].

Next, we shall define an integral path $C'$ which consists of $N_D$ connected integral paths. There is a one-to-one correspondence between elements in $\mathcal{D}$ and connected spaces of $C'$. From $\mathcal{D} \in \mathcal{D}$, we define the corresponding connected space as follows. We select one variable from each element $D_i \in \mathcal{D}$. Then, the space is $\{x_\sigma\}$ with the restriction of $N_\theta$ conditions $|\delta| = |r_\delta \epsilon^{m_\delta}|$ where $\delta$ is one of the elements we choose. The integral path $C'$ has the following property. Given $\epsilon'$ and a sufficiently small $\epsilon(\epsilon')$, there is a one-to-one correspondence between a point on $\Omega_R$ and a point on the integral path $C'$. The correspondence can be regarded as a homeomorphic mapping. Furthermore, the relation (B.3) holds. The point on $C'$ corresponding to a point $(\{m_\delta\}, \{r_\delta\})$ on $\Omega_R$ is given by $N_\theta$ conditions $\delta = r_{-\delta} \epsilon^{m_\delta}$ where $\delta$ is the element we choose from $D_i \in \mathcal{D}$.
From the previous two homeomorphic mappings, we make a homeomorphic mapping from $C$ to $C'$. This causes a continuous modification of the integral path from $C$ to $C'$ by means of moving each point on $C$ to the mapped point on $C'$. For sufficiently small $\epsilon'$ and $\epsilon(\epsilon')$, this modification does not change the value of the integral in (2.4). The reason is that $C'$ has no boundary, $C''$ is compact and the integrand in (2.4) is analytic on (B.3), in which the corresponding points on $C$ and $C'$ are. Using $C'$, we can evaluate the integral (2.4), because the integral path $C'$ is a sum of direct products of integral paths with respect to a single integral.

Using these procedures, we can check the equality of (2.4) and (2.13) for any $M$ case. Explicit check is done in B.2.

B.1

We prove the following: For sufficiently small $\epsilon' > 0$ and $\epsilon(\epsilon') > 0$, there is a one-to-one correspondence between a point on the piece $C$ of the integral path (2.12) and a point on $\Omega_R$, where $C$ is defined using a vector $(n, \cdot \cdot \cdot)$ and $\Omega_R$ is defined using $D$ made from the vector. The correspondence can be considered as a homeomorphic mapping. Furthermore, the relation (B.3) holds where $\{\delta\} = D$ is a value on a point on $C$, $\{(m, \{r\})\}$ is a point on $\Omega_R$, and the point on $C$ corresponds to the point on $\Omega_R$.

We define a subset $\Omega_{RI}$ of $\Omega_R$. $\Omega_{RI}$ is a set of points which correspond to $\{m, r\}$ generated by a fixed set of phases $\{RI\}$ where $D$ are freely moved. There are a finite number of elements in $\Omega_{RI}$. For a while, we fix a set of phases $\{RI\}$.

First, we prove the following: lemma 1. For $\epsilon' > 0$ and a sufficiently small $\epsilon(\epsilon') > 0$, any solution of (2.7), (2.12), (B.2) which satisfies $|x - n| < 3\epsilon^{L(M+1)}$ exists in (B.3) generated by a point $\{(m, \{r\})\}$ on $\Omega_{RI}$. We can choose $\epsilon$ independently on $\{RI\}$.

We define a phase space $\Omega_F(\epsilon)$ embedded in $\Omega$,

$$\left(\left\{\frac{\delta}{r \epsilon^{m}}\right\}, \epsilon\right), \quad \delta \in D, \quad \epsilon > 0,$$

where $\{(r, m)\}$ are all the points on $\Omega_{RI}$. For $\Omega$ in this case, note that the number of elements is $N\Omega_{RI}N_D + 1$, and that $\delta$ and $\epsilon$ do not need to satisfy the conditions (2.12) and (B.2). We define a subspace $\Omega_C(\epsilon)$ of $\Omega_F(\epsilon)$ where $\{\delta\}$ and $\epsilon$ satisfy the conditions (2.7), (2.12), (B.2), $x_\sigma \neq \pm i$ and $|x_\sigma - n_\sigma| < 3\epsilon^{L(M+1)}$. We also define a subspace $\Omega_B(\epsilon, \epsilon')$ of
\[ \Omega_F(\epsilon) \text{ where } \{r_\delta\} \text{ and } \{m_\delta\} \text{ satisfy } (B.3) \text{ with respect to a } (\{r_\delta\}, \{m_\delta\}) \in \Omega_{RI}. \] Next, we define a sequence \((P_{Bn})\) of points on \(\Omega_C(\epsilon)\) as follows, where \(\epsilon\) is freely moved. The series of \(\epsilon\) which corresponds to \((P_{Bn})\) is decreasing and converges to 0. A necessary and sufficient condition of the lemma 1 is that there is a subsequence of any \((P_{Bn})\) defined previously which converges to a point on \(\lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} \Omega_B(\epsilon, \epsilon')\). We note that since the parameter \(\{\Re I_\sigma(\text{mod 1})\}\) is compact we can choose \(\epsilon\) independently on \(\{\Re I_\sigma\}\).

We define a subspace \(\Omega_D(\epsilon)\) of \(\Omega\) as
\[
\left( \left\{ \frac{\delta}{\delta'} \right\}, \{\delta\}, \epsilon \right), \quad \delta, \delta' \in D,
\] where the variables satisfy (2.7), (2.12), (B.2), \(x_\sigma \neq \pm 2\) and \(|x_\sigma - n_\sigma i| < 3\epsilon \frac{1}{\sqrt{\pi}}\). The number of variables in \(\Omega\) is \(N_D^2 + 1\) in this case. We denote the closure of \(\Omega_D(\epsilon)\) by \(\bar{\Omega}_D(\epsilon)\). We define a sequence \((P_{Dn})\) of points on \(\Omega_D(\epsilon)\), where \(\epsilon\) is freely moved: The series of \(\epsilon\) which corresponds to \((P_{Dn})\) is decreasing and converges to 0. Then, we can choose a subsequence of \((P_{Dn})\) which converges to a point \(P_D\) on \(\bar{\Omega}_D(0)\), because \(\bar{\Omega}_D(\epsilon)\) is a compact space. Using this subsequence, we define a partial order, \(<\), for \(D\
\[
\lim \frac{\delta}{\delta'} = 0 \rightarrow \delta \prec \delta' \quad \text{for} \quad \delta, \delta' \in D,
\] where \(\lim\) means a limit on the subsequence. We define \(D_1\) as the set of maximal elements in \(D\), and \(D_2\) as the set of maximal elements in \(D - D_1\), and in the same way define \(D_3, D_4, \cdots\). Then, \(D\) is defined as \(\{D_1, D_2, D_3, \cdots\}\). Eqs. (2.7), (2.12), (B.2) are solved in the meaning that \(\{r_\delta \neq 0\}\) and \(\{m_\delta\}\) which satisfy
\[
\lim \frac{\delta}{e^{m_\delta}} = r_\delta
\] are given using \(P_D\), and therefore using the limit of variable’s ratios. From the definition of \(\{m_\sigma\}, \{m_\delta\}\) corresponding to elements in different clusters of \(D\) are distinct positive numbers, and \(N_D\) is equal to \(N_\theta\). Therefore, the limit of variable’s ratios is either of 0, ±1, \(\infty\). Then, \(\{r_\delta\}, \{m_\delta\}\) are given without any ambiguity. It is clear that \(\{r_\delta\}, \{m_\delta\}\) derived here are in \(\Omega_{RI}\). In fact, the number of solutions \((\{r_\delta\}, \{m_\delta\})\) is not always one but a finite number. We can therefore choose a subsequence for the subsequence previously defined which converges to a single solution in the meaning of (B.7). We note that there is a one-to-one correspondence between \((P_{Bn})\) and \((P_{Dn})\). Therefore, there is a subsequence of \((P_{Bn})\)
corresponding to the subsequence of \((P_{Dn})\) and the subsequence of \((P_{Bn})\) converges to a point on \(\lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \Omega_B(\epsilon, \epsilon')\). Thus, the lemma \(1\) is proved.

Second, the following lemma \(2\) holds: For \(\epsilon' > 0\) and a sufficiently small \(\epsilon(\epsilon') > 0\), there are one or more solutions of (2.7), (2.12), (B.2) in (B.3) generated by any element \((\{m_\delta\}, \{r_\delta\})\) on \(\Omega_{RI}\). In other words, \(\{I_\sigma\}\) is defined by means of \(\epsilon\) and is in the image of any region (B.3) where (2.7) defines \(\{I_\sigma\}\) as a function of \(\{x_\delta\}\). We can choose \(\epsilon\) independently on \(\{\Re I_\sigma\}\).

Third, we prove the following lemma \(3\): For sufficiently small \(\epsilon' > 0\) and \(\epsilon(\epsilon') > 0\), there exists one or no solution of (2.7), (2.12), (B.2) in (B.3) generated by any element \((\{m_\delta\}, \{r_\delta\})\) on \(\Omega_{RI}\). We can choose \(\epsilon\) independently on \(\{\Re I_\sigma\}\).

We say that an analytic vector function \(f(x)\) is “monotonic” on \(A\) when \(A\) is a convex domain and an analytic vector function \(f(x)\) satisfies a relation; for any \(x \in A\) and \(a \neq 0\), there exists \(b\) such that

\[
\Re \left( b, \frac{df}{dx} \cdot a \right) > 0
\]

where \(,\) is the inner product on an \(M\)-dimensional complex linear space, \(a\) is an \(N\)-dimensional vector, \(b\) is an \(M\)-dimensional vector, \(N\) and \(M\) are dimensions of vector \(x\) and \(f\). Then, an equation \(f(x) = b\) has one or no solution on \(A\) when \(f(x)\) is a monotonic analytic vector function.

A sufficient condition of the lemma \(3\) is that (2.7) is a monotonic analytic vector function on any region (B.3) where (2.7) defines \(\{I_\sigma\}\) as a function of \(\{x_\sigma\}\). It is evident that (B.3) generated by any element on \(\Omega_{RI}\) is a convex domain. In the following, we fix \((\{r_\delta\}, \{m_\delta\})\) to an element in \(\Omega_{RI}\) and \(D\) to an element in \(D\) which gives the element \((\{r_\delta\}, \{m_\delta\})\). Now, we express \(b\) in terms of \(a\) which satisfies (B.8). We derive the limitation of (2.7) using \(D\). We replace any variable with an element in \(D_i\) corresponding to the variable. Then, the Jacobi matrix of this functions at \(\delta = r_\delta \epsilon^{m_\delta}\) is written as \(\partial I_\sigma / \partial \delta_0\) where \(\delta\) are selected variables in the previous modification. We choose \(b\) to be \(\partial I_\sigma / \partial \delta_0 \cdot a\) for any \(a\), and then (B.8) holds when \(\epsilon'\) and \(\epsilon\) are sufficiently small. The \(\epsilon'\) and \(\epsilon\) can be given independently on \(\{\Re I_\sigma\}\) and the element of \(\Omega_{RI}\).

Fourth, we point out the following: Two regions (B.3) corresponding to different elements on \(\Omega_{RI}\) have no common element when \(\epsilon\) and \(\epsilon'\) are sufficiently small.
Three lemmas and the above fact indicate the following. Given a sufficiently small $\epsilon$ and a sufficiently small $\epsilon'(\epsilon)$, there is only one solution of $\text{(2.7)}$, $\text{(2.12)}$, $\text{(B.2)}$ in $\text{(B.3)}$ generated by any element on $\Omega_{RI}$. Then, the correspondence between $\Omega_{RI}$ and solutions of $\text{(2.7)}$, $\text{(B.2)}$ which satisfy $|x_{\sigma} - n_\sigma i| < 3 \epsilon^{\frac{1}{L}}$ is one-to-one. This correspondence can be regarded as a one-to-one correspondence between $\Omega_{R}$ and solutions of $\text{(2.12)}$ which satisfy $|x_{\sigma} - n_\sigma i| < 3 \epsilon^{\frac{1}{L(M+1)}}$, that is, $C$. And, any point on $C$ and the corresponding element $(\{m_\delta\}, \{r_\delta\})$ on $\Omega_{R}$ satisfy $\text{(B.3)}$.

Finally, we prove that this correspondence is a homeomorphic mapping. This is due to the following three facts: $\Omega_{R}$ is a compact space. The correspondence is a bijection. And the correspondence is a continuous mapping from to $\Omega_{R}$ to $C$.

**B. 2 An example**

In the case $M$ of $\Theta_{M}$ is equal to 1, a set $\Theta_1$ has one element $\theta = \{1\}$. There is only a case that $n_1 = 1$, and therefore $D$ has only one element $x_1 - i = \delta_1$. We can choose $m_1$ to be 1. $D$ has only one element $D$, and $D$ has only one element $D$. Then, $\Omega_{R}$ is a space $(|r_{\delta_1}| = 2, m_{\delta_1} = 1/L)$. Therefore, the integral path $C'$ becomes $|x_1 - i| = 2 \epsilon^{1/L}$. Then, the value of $\text{(2.4)}$ is equal to the value of $\text{(2.13)}$ in the case $M = 1$.

In case that $\theta$ is arbitrary, there is a case $n_\sigma = 1$ for all $\sigma \in \theta$. For that case, all the elements in $D$ is $x_\sigma - i = \delta_\sigma$. We can choose $m_\sigma$ arbitrarily. $D$ has only one element $D$, and all the elements in $D$ is $\{\delta_\sigma\}$. Then, $\Omega_{R}$ is a space $(|r_{\delta_\sigma}| = 2, m_{\delta_\sigma} = m_\sigma / L)$. Therefore, the integral path $C'$ becomes $|x_\sigma - i| = 2 \epsilon^{m_\sigma/L}$. Then, if a sum of the integrals using the paths corresponding to the case $n_\sigma \neq 1$ for some $\sigma \in \theta$ is zero, the value of $\text{(2.4)}$ is equal to the value of $\text{(2.13)}$.

In the case $M$ of $\Theta_{M}$ is equal to 2, it is only possible that $n_\sigma$ is not equal to 1 for some $\sigma \in \theta$. Note that, we judge whether the case $\{n_\sigma\}$ occurs or not by checking the number of $D$ corresponding to $\{n_\sigma\}$ is non-zero or not. Then, the values of $\text{(2.4)}$ and $\text{(2.13)}$ are the same in the case $M = 2$. This procedure can be summarized in the following table:
| $\theta$ | 1–2 | 1, 2 |
|---------|------|------|
| $\{n_\sigma\}$ | $n_{1-2} = 1$ | $n_1 = 1, n_2 = 1$ |
| $\{m_\sigma\}$ | $m_{1-2}$ | $m_1, m_2$ |
| $\{\delta_{1-2}\}$ | $\{\delta_1\}$, $\{\delta_2\}$ |
| $\Omega_R$ | $r_{\delta_{1-2}} = 2$ | $r_{\delta_1} = |r_{\delta_1}| = 2$ |
| $m_{\delta_{1-2}} = m_{1-2}/L$ | $m_{\delta_1} = m_1/L$ | $m_{\delta_2} = m_2/L$ |

where $x_\sigma - i$ is equal to $\delta_\sigma$.

In the same way, the procedure in case of $M = 3$ becomes

| $\theta$ | 1–2–3 | 1–2, 3 | 1, 2, 3 |
|---------|-------|--------|--------|
| $\{n_\sigma\}$ | $n_{1-2-3} = 1$ | $n_{1-2} = 1$ | $n_1 = 1$ |
| $\{m_\sigma\}$ | $m_{1-2}$ | $m_1 = 1$ | $m_1 = 3$ |
| $\{\delta_{1-2}\}$ | $\{\delta_1\}$ | $\{\delta_1\}$ | $\{\delta_1\}$ |
| $\Omega_R$ | $|r_{\delta_{1-2}}| = 2$ | $|r_{\delta_1}| = 2$ | $|r_{\delta_1}| = 2$ |
| $m_{\delta_{1-2}} = m_{1-2}/L$ | $m_{\delta_1} = 3/L$ | $m_{\delta_1} = 6/L$ | $m_{\delta_1} = 6/L$ |

where $x_\sigma - x_\sigma' - 2i$ is equal to $\delta_\sigma, \delta_\sigma'$. Note that we have other choices of values $\{m_\sigma\}$. In the table, we have omitted the cases which correspond to $(\theta = 2–3, 1)$, $(\theta = 3–1, 2)$, $(n_2 = n_3 = 1, n_1 = -1)$ and $(n_3 = n_1 = 1, n_2 = -1)$. The columns for $(\theta = 2–3, 1)$ and $(\theta = 3–1, 2)$ are given by replacing 1, 2, 3 of the subscripts in the column $(\theta = 1–2, 3)$ with 2, 3, 1 or 3, 1, 2.
respectively. Similarly, the columns for \((n_2 = n_3 = 1, n_1 = -1)\) and \((n_3 = n_1 = 1, n_2 = -1)\) are given by replacing 1, 2, 3 of the subscripts in the column \((n_1 = n_2 = 1, n_3 = -1)\) with 2, 3, 1 or 3, 1, 2 respectively. Then, the integral path corresponding to \(\theta = 1 \rightarrow 2, 3\) becomes a union of two connected integral paths;

\[
|x_{1-2} - i| = 2e^{m_{1-2}/L}, \quad |x_3 - i| = 2e^{m_3/L}, \quad (B.9)
\]

and

\[
|x_{1-2} - i| = 2e^{5/L}, \quad |x_3 - x_2 - 2i| = 4e^2. \quad (B.10)
\]

And, the integral path corresponding to \(\theta = 1, 2, 3\) becomes a union of four connected integral paths;

\[
|x_1 - i| = 2e^{m_1/L}, \quad |x_2 - i| = 2e^{m_2/L}, \quad |x_3 - i| = 2e^{m_3/L}, \quad (B.11)
\]

\[
|x_1 - i| = 2e^{6/L}, \quad |x_1 - x_3 - 2i| = 4e^3, \quad |x_2 - x_3 - 2i| = 4e^4 \quad (B.12)
\]

and the integral paths which are given by replacing 1, 2, 3 of the subscripts in the second integral path \((3,12)\) with 2, 3, 1 or 3, 1, 2 respectively.

Using the part of the integral path corresponding to \(n_{1-2} = 1, n_3 = -1\), we evaluate the integral which is a term in \((2.4)\) corresponding to \(\theta = \{1-2, 3\}\),

\[
\int \left| \frac{\partial I}{\partial x} \right| \exp \left( \frac{2}{x_{1-2}^2 + 1} + \frac{1}{x_3^2 + 1} \right) d(x_{1-2} - i)d(x_{1-2} - x_3 - 2i) = -\int_{|x-i|=+0} L \left( \frac{2}{x + i} - \frac{1}{x - i} - \frac{1}{x - 3i} \right) e^{-\beta(\frac{x+i}{\epsilon} - \frac{x-i}{\epsilon})} \frac{d(x - i)}{2\pi i}. \quad (B.13)
\]

The orientation of the path is defined so that the value of the integral is a positive integer when \(e^{-\beta E}\) is replaced with 1. This definition is consistent with the orientation of integrals in \((2.4)\).

Using the part of the integral path corresponding to \(n_1 = n_2 = 1, n_3 = -1\), we evaluate the integral which is a term in \((2.4)\) corresponding to \(\theta = \{1, 2, 3\}\),

\[
\int \left| \frac{\partial n}{\partial x} \right| \exp \left( \frac{1}{x_1^2 + 1} + \frac{1}{x_2^2 + 1} + \frac{1}{x_3^2 + 1} \right) d(x_1 - i)d(x_1 - x_3 - 2i)d(x_2 - x_3 - 2i) = -\int_{|x-i|=+0} L \left( \frac{2}{x + i} - \frac{1}{x - i} - \frac{1}{x - 3i} \right) e^{-\beta(\frac{x+i}{\epsilon} - \frac{x-i}{\epsilon})} \frac{d(x - i)}{2\pi i}. \quad (B.14)
\]

The coefficient of the integrals in \((2.4)\) are \(\mu(\hat{\theta}_n, \{1-2, 3\}) = -1\) and \(\mu(\hat{\theta}_n, \{1, 2, 3\}) = 1\). Therefore, in \((2.4)\), these terms cancel out. Then, \((2.4)\) is equal to \((2.13)\) in the case \(M = 3\).
We have checked that all the values of integrals corresponding to \( n_\sigma \neq 1 \) for some \( \sigma \in \theta \) cancel out in the case of \( M = 4 \). It is adequate to suppose that the value of (2.13) is equal to the value of (2.13) in case of \( M > 4 \).

Appendix C Modifications of integral paths 1

We prove that the value of (2.24) does not change when \( m \) is decreased by 1.

First, we change the integral path into \((-\infty, \infty)\) with respect to \( x_{\theta_m} \). Then (2.24) becomes

\[
(2.24)
\]

\[
\begin{align*}
&= \left[ \prod_{\sigma \in \Theta} N_\sigma \right]^{-1} \left[ \prod_{\{\sigma, \sigma'\} \subseteq \lambda} N_\sigma N_{\sigma'} \right] \sum_{\zeta \subseteq \Theta(\theta) \backslash \{\sigma_1, \ldots, \sigma_m\}} \sum_{\{\theta_k \in \zeta \} \subseteq \{\theta_{k_1} \in \zeta \}} \left[ \prod_{k=1}^{m-1} \int_{|i-x_{\theta_k}| = \lambda_{\theta_k}} \frac{dx_{\theta_k}}{2\pi} \right] \\
&\quad \times \left[ \prod_{\{\sigma, \sigma'\} \subseteq \lambda, \omega \neq \omega'} \left( x_{\theta[\zeta, \sigma]} - x_{\theta[\zeta, \sigma']} + 2l_\lambda (\sigma, \sigma^{(\theta[\zeta, \sigma])}) i - 2l_\lambda (\sigma', \sigma^{(\theta[\zeta, \sigma'])}) i \right)^2 + 4 \right] \\
&\quad \exp \left[ -\beta \sum_{\sigma \in \Theta} \frac{2N_\sigma}{(x_{\theta[\zeta, \sigma]} + 2l_\lambda (\sigma, \sigma^{(\theta[\zeta, \sigma])}) i)^2 + 1} \right] \\
&\quad + \left[ \prod_{\sigma \in \Theta} N_\sigma \right]^{-1} \left[ \prod_{\{\sigma, \sigma'\} \subseteq \lambda} N_\sigma N_{\sigma'} \right] \sum_{\zeta \subseteq \Theta(\theta) \backslash \{\sigma_1, \ldots, \sigma_m\}} \sum_{\{\theta_k \in \zeta \} \subseteq \{\theta_{k_1} \in \zeta \}} \left[ \prod_{k=1}^{m-1} \int_{|i-x_{\theta_k}| = \lambda_{\theta_k}} \frac{dx_{\theta_k}}{2\pi} \right] \\
&\quad \times \left[ \prod_{\{\sigma, \sigma'\} \subseteq \lambda, \omega \neq \omega'} \left( x_{\theta[\zeta, \sigma]} - x_{\theta[\zeta, \sigma']} + 2l_\lambda (\sigma, \sigma^{(\theta[\zeta, \sigma])}) i - 2l_\lambda (\sigma', \sigma^{(\theta[\zeta, \sigma'])}) i \right)^2 + 4 \right] \\
&\quad \times \sum_{\{\sigma, \sigma'\} \subseteq \lambda, \omega \neq \omega'} 2\pi s \theta \left( l_\lambda (\sigma', \sigma^{(\theta[\zeta, \sigma'])}) - l_\lambda (\sigma, \sigma_m) - \frac{1}{2} + s \right) \\
&\quad \delta \left( x_{\theta_m} - x_{\theta[\zeta, \sigma']} + 2l_\lambda (\sigma, \sigma_m) i - 2l_\lambda (\sigma', \sigma^{(\theta[\zeta, \sigma'])}) i \right) - 2si \right)
\end{align*}
\]

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\[
\prod_{\{\sigma'', \sigma''\} \in \lambda, \sigma'' \in \xi \sigma'' \in \xi \sigma''} \left( x_{\theta m} - x_{\theta[\zeta, \sigma'' \zeta]} + 2l_\lambda (\sigma'', \sigma_m) i - 2l_\lambda (\sigma'', (\theta[\zeta, \sigma''])) i \right)^2 + 4 \]

exp \left[ -\beta \sum_{\sigma \in \theta} \frac{2N_\sigma}{x_{\theta[\zeta, \sigma]} + 2l_\lambda (\sigma, (\theta[\zeta, \sigma])) i} \right].

(C.1)

The second term comes from the residues. In the second term, the \( \delta \)-function and the integral with respect to \( x_{\theta m} \) mean the following procedure: Before integrations with respect to the other variables, the variable \( x_{\theta m} \) is replaced with the other variables and constants so that a value of the expression which is the independent variable of the \( \delta \)-function becomes 0. And, \( \frac{1}{2} \) in the expression which is the independent variable of the \( \theta \)-function is caused by the relation (2.25).

There occur cancellations in the second term. We recognize the term as a series with respect to sets \( (\zeta, \{\sigma(\theta_k)\}, \sigma, \sigma', s) \),

\[
\sum_{\zeta \in \Theta(\theta[\lambda], \sigma_1, \ldots, \sigma_m)} \sum_{\{\sigma(\theta_k)\} \in \{\theta_k \in \zeta \}_{k>m}} \sum_{\{\sigma', \sigma'' \} \in \lambda, \xi \sigma'' \in \xi \sigma''} \sum_{s=\pm 1}.
\]

(C.2)

Then, there are cancellations between any term corresponding to \( (\zeta, \{\sigma(\theta_k)\}, \sigma, \sigma', -) \) which has a non-zero value and a term corresponding to \( (\zeta', \{\sigma(\theta_k)\}, \sigma'', \sigma', +) \) which is defined as

\[
\theta'' = \{\sigma'' \} \in \Theta[\lambda, \sigma'], l_\lambda (\sigma'', (\theta[\zeta, \sigma''])) = l_\lambda (\sigma', (\theta[\zeta, \sigma''])) + l_\lambda (\sigma', \sigma''), \theta'' = \{\sigma', \sigma'' \} \in \Theta[\lambda, \sigma'], l_\lambda (\sigma', \sigma'') = l_\lambda (\sigma', (\theta[\zeta, \sigma''])) + l_\lambda (\sigma', \sigma'') + 1.
\]

(C.3)

We note that the term \( (\zeta', \{\sigma(\theta_k)\}, \sigma'', \sigma', +) \) exists for the condition \( \sigma'' \neq \sigma(\theta[\zeta, \sigma'']) \). This condition is satisfied automatically when the term \( (\zeta, \{\sigma(\theta_k)\}, \sigma, \sigma', -) \) has a non-zero value, in other words, the \( \theta \)-function is not equal to 0 in the term \( (\zeta, \{\sigma(\theta_k)\}, \sigma, \sigma', -) \).

Then, the second term in (C.1) becomes

\[
\int \frac{dx_{\theta m}}{2\pi} \left[ \prod_{k=m+1}^{N_\zeta} \left( x_{\theta[\xi, \sigma(\theta_k)]} + 2l_\lambda (\sigma(\theta[\xi, \sigma(\theta_k)])) i \right)^2 + 1 \right] \frac{2N_\sigma L}{2\pi} \]

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Then, (C.4) is modified as

\[
\Pi_{\{\sigma,\sigma\}'\in\lambda, \sigma'\neq \sigma_m} \left( x_{\theta[\zeta,\sigma]} - x_{\theta[\zeta,\sigma']} + 2l_{\lambda} (\sigma, \sigma(\theta[\zeta,\sigma]) ) i - 2l_{\lambda} (\sigma', \sigma(\theta[\zeta,\sigma]) ) i \right)^2 + 4 \]

\sum_{\{\sigma,\sigma\}'\in\lambda, \sigma'\neq \sigma_m} 2\pi \delta \left( x_{\theta_m} - x_{\theta[\zeta,\sigma]} + 2l_{\lambda} (\sigma, \sigma_m ) i - 2l_{\lambda} (\sigma', \sigma(\theta[\zeta,\sigma]) ) i - 2si \right)

\[
\Pi_{\{\sigma',\sigma\'''\in\lambda, \sigma''\neq \sigma_m, \sigma''\neq \sigma'\}} \left( x_{\theta_m} - x_{\theta[\zeta,\sigma''']} + 2l_{\lambda} (\sigma''', \sigma_m ) i - 2l_{\lambda} (\sigma'''', \sigma(\theta[\zeta,\sigma''']) ) i \right)^2 + 4 \]

\exp \left[ -\beta \sum_{\sigma \in \Theta} \frac{2N_\sigma}{x_{\theta[\zeta,\sigma]} + 2l_{\lambda} (\sigma, \sigma(\theta[\zeta,\sigma]) ) i} \right] \cdot (C.4)

This expression is the sum with respect to terms corresponding \((\zeta, \{\sigma(\theta_k)\}, \sigma, \sigma', +)\) which avoids the cancellation, equivalently, which satisfies \(\sigma = \sigma_m\).

Next, we execute the replacement which is indicated by the \(\delta\)-function and the integral with respect to \(x_{\sigma_m}\) and change an expression of summations with respect to permutations. Then, (C.4) is modified as

\[
\left[ \prod_{\sigma \in \Theta} N_\sigma \right]^{-1} \left[ \prod_{\{\sigma,\sigma\}'\in\lambda} N_\sigma N_{\sigma'} \right] \left[ \sum_{\zeta \in \Theta(\theta|\lambda)\{\sigma_1, \ldots, \sigma_{m-1}\}} \sum_{\{\theta_k \in \zeta\}_{k=m-1}} \sum_{\sigma_m \neq \sigma(\theta_k)} \left[ \prod_{k=1}^{m-1} \int_{|i-x_{\theta_k}| = \delta_{\theta_k}} \frac{dx_{\theta_k}}{2\pi} \right] \right]

\[
\prod_{\{\sigma,\sigma\}'\in\lambda, \sigma'\neq \sigma_m} \left( x_{\theta[\zeta,\sigma]} - x_{\theta[\zeta,\sigma']} + 2l_{\lambda} (\sigma, \sigma(\theta[\zeta,\sigma]) ) i - 2l_{\lambda} (\sigma', \sigma(\theta[\zeta,\sigma]) ) i \right)^2 + 4 \]

\exp \left[ -\beta \sum_{\sigma \in \Theta} \frac{2N_\sigma}{x_{\theta[\zeta,\sigma]} + 2l_{\lambda} (\sigma, \sigma(\theta[\zeta,\sigma]) ) i} \right] \cdot (C.5)

Here, we have used a simple relation,

\[
l_{\lambda} \left( \sigma'(\theta[\zeta,\sigma']), \sigma \right) = l_{\lambda} \left( \sigma'(\theta[\zeta,\sigma']), \sigma' \right) + l_{\lambda} (\sigma_m, \sigma)
\]

for \(\zeta \in \Theta(\theta|\lambda), \{\sigma_m, \sigma'\} \in \lambda, \sigma_m \nless \sigma', \sigma \nless \sigma_m\) (C.6)

In this change, there is a one-to-one correspondence between \(\{\zeta, \{\sigma(\theta_k)\}\}, \sigma'\) and \(\{\zeta, \{\sigma(\theta_k)\}\}\)
associated with
\[
\sum_{\zeta \in \Theta(\theta|\lambda_1, \ldots, \lambda_m)} \sum_{\sigma' \in \zeta_{m+1}} \sum_{\zeta' \in \sigma'_{m+1}} \leftrightarrow \sum_{\zeta \in \Theta(\theta|\lambda_1, \ldots, \lambda_{m-1})} \sum_{\sigma_{m+1} \neq \sigma'_{m+1}} \sum_{\zeta' \in \sigma'_{m+1}} (C.7)
\]
for \( \zeta, \{\sigma_k\}, \sigma' \) and \( \zeta', \{\sigma'_k\} \) which satisfy conditions,

\[
\theta_m + \theta[\zeta, \sigma'] = \theta[\zeta', \sigma_m], \quad \zeta - \theta_m - \theta[\zeta, \sigma'] = \zeta - \theta[\zeta', \sigma_m], \quad \{\sigma(\theta_k)\} - \sigma(\theta_m) = \{\sigma'(\theta_k)\}, \quad (C.8)
\]
where \( \zeta, \{\sigma(\theta_k)\}, \sigma' \) correspond to \( \zeta', \{\sigma'(\theta_k)\} \). This correspondence is a ”one-to-one correspondence”.

In this way, the second term in (C.1) becomes the expression (C.5). Using this expression (C.9) is

\[
\left[ \prod_{\sigma \in \Theta} N_{\sigma} \right]^{-1} \left[ \prod_{\{\sigma, \sigma'\} \in \lambda} N_{\sigma} N_{\sigma'} \right] \sum_{\zeta \in \Theta(\theta|\lambda_1, \ldots, \lambda_{m-1})} \sum_{\{\sigma(\theta_k)\} \in \{\zeta_k \} \in \{\theta_k \} \in \zeta_{m+1}} \left[ \prod_{k=1}^{m-1} \int_{|i-x_{\theta_k}| = \delta_{\sigma_k}} \frac{dx_{\theta_k}}{2\pi} \right]
\]

\[
\left[ \prod_{\{\sigma, \sigma'\} \in \lambda} N_{\sigma} N_{\sigma'} \right] \sum_{\{\sigma, \sigma'\} \in \lambda} \sum_{\{\sigma, \sigma'\} \in \lambda} \left[ \prod_{\sigma \in \Theta} \left( x_{\theta[\zeta, \sigma']} - x_{\theta[\zeta', \sigma']} + \frac{2N_{\sigma}L}{\sigma(\theta[\zeta, \sigma'])} \frac{2N_{\sigma}L}{\sigma(\theta[\zeta', \sigma'])} \right) \right]^{\frac{1}{2}}
\]

\[
\exp \left[ -\beta \sum_{\sigma \in \Theta} \frac{2N_{\sigma}}{\sigma(\theta[\zeta, \sigma])} \left( x_{\theta[\zeta, \sigma]} - x_{\theta[\zeta', \sigma']} + 2\lambda (\sigma, \sigma(\theta[\zeta, \sigma]) \right) \right] \cdot (C.9)
\]

Here, we have used a simple relation,

\[
\sum_{\zeta \in \Theta(\theta|\lambda_1, \ldots, \lambda_{m-1})} \sum_{\{\sigma(\theta_k)\} \in \{\zeta_k \} \in \{\theta_k \} \in \zeta_{m+1}} + \sum_{\zeta \in \Theta(\theta|\lambda_1, \ldots, \lambda_{m-1})} \sum_{\{\sigma(\theta_k)\} \in \{\zeta_k \} \in \{\theta_k \} \in \zeta_{m+1}} \sum_{\sigma_m \neq \sigma(\theta_k)} = \sum_{\zeta \in \Theta(\theta|\lambda_1, \ldots, \lambda_{m-1})} \sum_{\{\sigma(\theta_k)\} \in \{\zeta_k \} \in \{\theta_k \} \in \zeta_{m+1}} \sum_{\sigma_m \neq \sigma(\theta_k)} \cdot (C.10)
\]

When \( m \) is replaced with \( m - 1 \), (2.24) becomes (C.9). And (2.24) has been shown to be equivalent to (C.8). Thus, we have proved that the value of (2.24) does not depend on \( m \).

**Appendix D A proof with respect to a permutation 1**

We prove

\[
\sum_{\theta \in \Theta_n} \left( \prod_{\sigma \in \Theta} (-1)^{N_{\sigma-1}} (N_{\sigma} - 1)! \right) x_{N_{\theta}} = x (x-1) \cdots (x-n+1). \quad (D.1)
\]
For the purpose, we introduce an expression
\[ \prod_{i=2}^{n} \left( 1 - \sum_{j=1}^{i-1} y_{i,j} \right), \quad \text{(D.2)} \]
and expand this product. We interpret \( y_{i,j} \) as a connection between elements \( i \) and \( j \), and a product of \( y_{i,j} \) as a set of connections. Then, the expansion can be considered as a linear combination with respect to a set \( \lambda \) of connections, and the coefficient of each term is \((-1)^{N_{\lambda}}\). In fact, all the set \( \lambda \) is an element of \( \Lambda(\{1, \cdots, n\}) \). Any two elements in \( \{1, \cdots, n\} \) are connected or not, and there is no closed path in the connections. That is to say, there is no term like \( y_{2,1}y_{2,1} \) or \( y_{2,1}y_{3,2}y_{3,1} \).

By replacing of \( \lambda \) with \( \theta = G_{\{1, \cdots, n\}}(\lambda) \), the expansion can be considered as a linear combination with respect to \( \theta \in \Theta_n \). And, it is easily shown that the coefficient of \( \theta \in \Theta_n \) is \( \prod_{\sigma \in \theta} (-1)^{N_{\sigma}} (N_{\sigma} - 1)! \). The reason is that \( \prod_{\sigma \in \theta} (-1)^{N_{\sigma}} \) is caused by \((-1)^{N_{\lambda}}\) and \( \prod_{\sigma \in \theta} (N_{\sigma} - 1)! \) counts the number of terms when \( \lambda \) is replaced with the \( \theta \). Due to this fact and a relation \( N_{\theta} = n - N_{\lambda} \) where \( \theta = G_{\{1, \cdots, n\}}(\lambda) \), the l.h.s. of (D.1) is equal to
\[ x^n \prod_{i=2}^{n} \left( 1 - \sum_{j=1}^{i-1} x^{-1} \right). \quad \text{(D.3)} \]
It is easy to see that this expression is equivalent to the r.h.s. of (D.1).

**Appendix E  A proof with respect to a permutation 2**

Here, we prove eq. (2.41).

Using a relation (D.1), the function with respect to \( \sigma_m \) and \( \sigma_{m-1} \)
\[ \sum_{\theta' \in \Theta(\sigma_m)} N_{\sigma_m^\theta}^{N_{\sigma_m^\theta}} \prod_{\sigma' \in \theta'} (-1)^{N_{\sigma'}-1} (N_{\sigma'} - 1)! \quad \text{(E.1)} \]
in the l.h.s of (2.41) is simplifies as
\[ \theta \left( N_{\sigma_m-1} - N_{\sigma_m} + \frac{1}{2} \right) \frac{\prod_{\sigma_m-1}^{N_{\sigma_m}!}}{(N_{\sigma_m-1} - N_{\sigma_m})!}, \quad \text{(E.2)} \]
and therefore the l.h.s of (2.41) becomes
\[ \theta \left( \tilde{\theta} \right) (-1)^{N_{\sigma_1}-1} (N_{\sigma_1} - 1)! \prod_{\sigma_m > 1}^{N_{\sigma_m}!} \frac{\prod_{\sigma_m > 1}^{N_{\sigma_m}!}}{(N_{\sigma_m-1} - N_{\sigma_m})!}. \quad \text{(E.3)} \]
A remaining task is to prove the relation

\[
\sum_{\tilde{\zeta} \in D(\tilde{\theta})} \sum_{\lambda' \in \Lambda_{\zeta}(\tilde{\zeta})} \left[ \prod_{\tilde{\theta} \in \tilde{\zeta}'} (-M_{\tilde{\theta}'}^{-1} (M_{\tilde{\theta}'} - 1)M_{\tilde{\theta}'}^{-1} (M_{\tilde{\theta}'}!)^{N_{\tilde{\theta}'} - 1}) \right] 
\prod_{\{\tilde{\theta}', \tilde{\theta}''\} \in \lambda'} (-M_{\tilde{\theta}'} M_{\tilde{\theta}''})
\]

where \( \tilde{\theta} \) is written as \((\sigma_1, \sigma_2, \ldots, \sigma_m)\), and the number of elements in each set \( \sigma_k \) is decreasing in the order of the sequence \( \tilde{\theta} \), and \( l \) is an element of \( \sigma_1 \).

We use a symbol \( \sum_{\tilde{\theta}' \leftarrow \tilde{\theta}} \) defined as follows for a convenience’ sake. The summation \( \sum_{\tilde{\theta}' \leftarrow \tilde{\theta}} \) with respect to \( \tilde{\theta}' \) satisfies conditions related to \( \tilde{\theta} \),

\[
\sigma_k' \subset \sigma_k \quad l \in \sigma_1' \quad N_{\sigma_k'} = N_{\sigma_k} - N_{\sigma_{m,s}}
\]

where \( (\sigma_1, \ldots, \sigma_m) = \tilde{\theta} \) and \( (\sigma_1', \ldots, \sigma_{m-1}') = \tilde{\theta}' \). We change an expression of summations in the l.h.s. of (E.4) with respect to permutations. The l.h.s. of (E.4) becomes

\[
\sum_{\tilde{\theta}' \leftarrow \tilde{\theta}} \left[ (-)^{N_{\sigma_m}} \sum_{\theta \in \Theta(\sigma_m)} (N_{\sigma_1} - N_{\sigma_m}) N_{\sigma} \left[ \frac{N_{\sigma_{m'}}!}{\prod_{\sigma \in \theta} N_{\sigma}!} \right]^{m-1} \prod_{\sigma \in \theta} (N_{\sigma} - 1)! (N_{\sigma}!)^{m-1} \right] \sum_{\tilde{\zeta} \in D(\tilde{\theta})} \sum_{\lambda' \in \Lambda_{\zeta}(\tilde{\zeta})} \delta \left( \tilde{\zeta}', \lambda', m \right) \left[ \prod_{\tilde{\theta}'' \in \tilde{\zeta}'} (-M_{\tilde{\theta}''}^{-1} (M_{\tilde{\theta}''} - 1)M_{\tilde{\theta}''}^{-1} (M_{\tilde{\theta}''}!)^{N_{\tilde{\theta}''} - 1}) \right] \left[ \prod_{\{\tilde{\theta}'', \tilde{\theta}'''\} \in \lambda'} (-M_{\tilde{\theta}''} M_{\tilde{\theta}'''}) \right].
\]

In this change, there is a one-to-many correspondence between \( \{\tilde{\zeta}', \lambda'\} \) and \( \{\theta, \tilde{\zeta}', \lambda'\} \) associated with

\[
\sum_{\tilde{\zeta} \in D(\tilde{\theta})} \sum_{\lambda' \in \Lambda_{\zeta}(\tilde{\zeta})} \leftrightarrow \sum_{\tilde{\theta}' \leftarrow \tilde{\theta}} \sum_{\theta \in \Theta(\sigma_m)} \sum_{\tilde{\zeta} \in D(\tilde{\theta})} \sum_{\lambda' \in \Lambda_{\zeta}(\tilde{\zeta})}
\]

When \( \tilde{\zeta}', \lambda' \) correspond to a subset \( A \) of \( \{\theta, \tilde{\zeta}', \lambda'\} \), \( \tilde{\zeta}', \lambda' \) and \( \theta'', \tilde{\zeta}'', \lambda'' \) in \( A \) satisfy the following conditions. There is a bijection \( f \) from a set \( \{\tilde{\theta} \in \tilde{\zeta}, N_{\tilde{\theta}} = m\} \) to the set \( \theta'' \) which satisfies \( \sigma_m' = \sigma'' \); \( f(\tilde{\theta}') = (\sigma_1', \ldots) = \sigma'' \). The conditions for \( \tilde{\zeta}'' \) and \( \lambda'' \) are

\[
\tilde{\zeta}'' = \left\{ \tilde{\theta}' | \tilde{\theta}' \in \tilde{\zeta}, N_{\tilde{\theta}} \neq m \right\}, \quad \lambda'' = \lambda \left[ \lambda', \tilde{\zeta}'' \right].
\]
This correspondence is a “one-to-many correspondence”.

In fact, the l.h.s. of (E.4) depends only on a decreasing sequence \( (N_{\sigma_1}, N_{\sigma_2}, \cdots, N_{\sigma_m}) \) where \( \tilde{\theta} \) is written as \( (\sigma_1, \sigma_2, \cdots, \sigma_m) \) which is a sequence defining the expression (E.4). Then, we can regard the l.h.s. as a function \( F(N_{\sigma_1}, N_{\sigma_2}, \cdots, N_{\sigma_m}) \). This fact indicates that elements which are summed up by means of \( \sum_{\tilde{\theta} \in \tilde{\theta}} \) in (E.6) have the same value. We note that the function \( F \) is defined for any number of independent variables, e.g. \( F(n_1), F(n_1, n_2) \).

Using the function \( F \), the equivalence between the l.h.s. of (E.4) and (E.6) is written as

\[
F(n_1, n_2, \cdots, n_m) = \left( - \right)^{n_m} \sum_{\tilde{\theta} \in \tilde{\theta}_{n_m}} (n_1 - n_m)^{N_{\tilde{\theta}}} \left[ \frac{n_m!}{\prod_{\sigma \in \tilde{\theta}} N_{\sigma}} \right]^{m-1} \left[ \prod_{\sigma \in \tilde{\theta}} \left( N_{\sigma} - 1 \right)! (N_{\sigma}!)^{m-1} \right] \]

\[
F(n_1 - n_m, n_2 - n_m, \cdots, n_{m-1} - n_m) \left[ \prod_{k=2}^{m-1} \frac{n_k!}{(n_k - n_m)!(n_k)!} \right] \frac{(n_1 - 1)!}{(n_1 - n_m - 1)!(n_m)!}. \tag{E.9}
\]

The factor \( \left[ \prod_{k=2}^{m-1} \frac{n_k!}{(n_k - n_m)!(n_k)!} \right] \frac{(n_1 - 1)!}{(n_1 - n_m - 1)!(n_m)!} \) is the number of elements which is summed up with respect to \( \sum_{\tilde{\theta} \in \tilde{\theta}} \). Using the relation (D.4), we simplify (E.9) as

\[
F(n_1, n_2, \cdots, n_m) = \left( - \right)^{n_m} F(n_1 - n_m, \cdots, n_{m-1} - n_m) \left[ \frac{(n_1 - 1)!}{(n_1 - n_m - 1)!} \right]^{2} \left[ \prod_{k=2}^{m-1} \frac{n_k!}{(n_k - n_m)!} \right] \tag{E.10}
\]

In case \( m = 1 \), eq.(E.4) obviously holds, which gives

\[
F(n_1) = \left( - \right)^{n_1-1} (n_1 - 1)!n_1^{-1}. \tag{E.11}
\]

On the other hand, eq.(E.10) can be regarded as an inductive relation of \( F \). Therefore, all we have to do for a proof of the relation (E.4) is to show

\[
F(n_1, n_2, \cdots, n_m) = \left( - \right)^{n_1-1} (n_1 - 1)!n_1^{-1} \prod_{k=2}^{m} \frac{n_k!}{(n_k - n_{k-1})!(n_k)!} \tag{E.12}
\]

from (E.10) and (E.11). It is sufficient to derive (E.12) from (E.10) by supposing an expression of \( F(n_1 - n_m, \cdots, n_{m-1} - n_m) \) like (E.12), i.e.

\[
\left( - \right)^{n_1-n_m-1} \frac{(n_1 - n_m - 1)!}{n_1 - n_m} \prod_{k=2}^{m-1} \frac{(n_k - n_{k-1})!}{(n_k - n_{k-1})!}. \tag{E.13}
\]

It is clear that (E.12) is given when we substitute (E.13) for \( F(n_1 - n_m, n_2 - n_m, \cdots, n_{m-1}) \) in (E.10).
Appendix F Modifications of integral paths 2

We prove the equivalence of (2.44) and (2.48).

A sufficient condition for the equivalence is

\[
\sum_{\lambda' \subseteq \lambda, \xi \equiv G_{\xi}(\lambda')} \left[ \prod_{\xi' \in \xi} \delta \left( \xi', \lambda' \left[ \lambda', \tilde{\xi}' \right], \min_{1} \left( \tilde{\xi}' \right) \right) \right] \left[ \prod_{\{\tilde{\theta}, \tilde{\theta}'\}} f_{N_{\tilde{\theta}}, N_{\tilde{\theta}'}} \left( x_{\tilde{\xi}[\xi, \tilde{\theta}]} - x_{\tilde{\xi}[\xi, \tilde{\theta}']} \right) \right] g \left( x_{\tilde{\xi}[\xi, \tilde{\theta}_{1}], x_{\tilde{\xi}[\xi, \tilde{\theta}_{2}], \cdots} \right)
\]

\[
= \left[ \prod_{\tilde{\theta} \in \tilde{\xi}} f_{N_{\tilde{\theta}}, N_{\tilde{\theta}}} \left( x_{\tilde{\theta}} - x_{\tilde{\theta}'} \right) \right] g \left( x_{\tilde{\theta}_{1}, x_{\tilde{\theta}_{2}}, \cdots} \right)
\]

(F.1)

where \( \theta \in \Theta(\sigma \subset \mathbb{N}), \tilde{\xi} \in \tilde{\Theta}(\theta), \lambda \in \Lambda \left( \tilde{\xi} \right) \) and \( \{\tilde{\theta}_{1}, \tilde{\theta}_{2}, \cdots\} = \tilde{\xi} \). Both sides of the equation are functions of \( \tilde{\xi} \) and \( \lambda \). Here, we have used several functions defined below. The definition of \( \tilde{\xi}[\xi, \tilde{\theta}] \) is the same as \( \theta[\zeta, \sigma] \).

Let \( \xi \) be a set with a finite number of sets as elements, and \( \tilde{\theta} \) be an element in a set which is an element of \( \xi \). Then, \( \tilde{\xi}[\xi, \tilde{\theta}] \) is a set which is in \( \xi \) and includes \( \tilde{\theta} \),

\[
\tilde{\theta} \in \tilde{\xi}[\xi, \tilde{\theta}] \subset \xi.
\]

(F.2)

The definition of the function \( f_{n,m} \) for natural numbers \( n, m \) is

\[
f_{n,m}(x) \equiv \frac{2\theta(n-m)-1}{2\pi i x} + \tilde{f}_{n,m}(x)
\]

(F.3)

where \( \tilde{f}_{n,m}(x) \) is a function of \( x \) which satisfies the following conditions. First, the function is analytic on neighborhood of the real axis. Second, the function does not diverge in the limit \( x \to \infty \). Finally, the function satisfies \( \tilde{f}_{n,m}(x) = \tilde{f}_{m,n}(-x) \). In (F.1), \( g(\cdots) \) is an analytic function on neighborhood of the real axis. And, the multiple integral of \( g \) from \( -\infty \) to \( \infty \) with respect to the all variables has a finite value. These definitions imply that the relation such as

\[
\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx f_{n,m}(x - y) g(x, y)
\]

\[
= \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx f_{n,m}(x - y) g(x, x) - \int_{-\infty}^{+\infty} dx g(x, x),
\]

44
holds for \( n < m \).

The reason why the condition (F.1) is a sufficient condition of the equivalence between (2.44) and (2.48) is as follows. When we apply \( \sum_{\theta \in \Theta_M} \sum_{\xi \in \Theta_{(\theta)}} \) and \( \sum_{\lambda \in \Lambda(\xi)} \) to both sides of (F.1) and define functions \( f \) and \( g \) as

\[
f_{n,m}(x) \equiv \frac{1}{2\pi} K_{n,m}(x + (n - m)i)
\]

\[
g(x_{\bar{\theta}_1}, x_{\bar{\theta}_2}, \ldots) \equiv \frac{1}{2\pi} \prod_{\xi \in G_{\zeta}(\lambda)} \left( \sum_{\theta \in \xi} \frac{2M_{\bar{\theta}}N_{\bar{\theta}}}{(x_{\bar{\theta}} + (N_{\bar{\theta}} - 1)i)^2 + N_{\bar{\theta}}^2} \right) \left[ \prod_{(\theta, \theta') \in \lambda} \frac{-M_{\bar{\theta}}M_{\bar{\theta}'}}{2\pi} \right] \]

which satisfy the previous conditions, both sides of this equation become (2.44) and (2.48). Therefore, we may prove (F.1) by means of the mathematical induction with respect to \( N_{\zeta} \).

It is clear that (F.1) holds in case of \( N_{\zeta} = 1 \). We suppose that (F.1) holds for \( \bar{\zeta}, \lambda \) which satisfy

\[
\begin{align*}
\theta_+ &\in \Theta(\sigma_+ \subset \mathbb{N}), \quad \bar{\zeta} = \bar{\xi}(\theta_+), \quad \lambda_+ \in \Lambda(\bar{\zeta}_+), \\
\bar{\theta}_0 &\in \bar{\zeta}_+, \quad \bar{\zeta} = \bar{\xi} - \bar{\theta}_0, \quad \lambda = \lambda[\lambda_+, \bar{\zeta}], \quad N_{\lambda_+} - N_{\lambda} = 0 \text{ or } 1.
\end{align*}
\]

We note that for any \( \bar{\zeta}_+ \) and \( \lambda_+ \) satisfying (F.5), there are \( \bar{\zeta} \) and \( \lambda \) which satisfy (F.6). From now on we prove that (F.1) holds for the sets \( \bar{\zeta}_+, \lambda_+ \). Hereafter in this appendix, (F.1) for the sets \( \bar{\zeta}, \lambda \) is just referred to as (F.1), while (F.1) for the sets \( \bar{\zeta}_+, \lambda_+ \) is referred to as (F.1)_+.

In case of \( N_{\lambda} = N_{\lambda_+} \), we replace \( g(x_1, x_2 \ldots) \) in (F.1) with

\[
g(x_1, x_2 \ldots) \leftrightarrow \int_{-\infty}^{+\infty} dx_{\bar{\theta}_0} g(x_0, x_1, \ldots).
\]

Then, (F.1) becomes (F.1)_+. Therefore, (F.1)_+ holds in case (F.1) does.

Next we analyze (F.1)_+ in the case of \( N_{\lambda_+} + 1 = N_{\lambda_+} \), which means that \( \bar{\theta}_0 = (\sigma^{(0)}_1, \ldots) \) connects a sequence \( \bar{\theta}_1 = (\sigma^{(1)}_1, \ldots) \in \bar{\zeta}_+ \) by \( \lambda_+ \). We subdivide the case into three,

\[
\begin{align*}
\text{Case 1: } N_{\bar{\theta}_0} &= N_{\bar{\theta}_1}, \quad \text{Case 2: } N_{\bar{\theta}_0} > N_{\bar{\theta}_1}, \quad \text{Case 3: } N_{\bar{\theta}_0} < N_{\bar{\theta}_1}.
\end{align*}
\]

Case 1: \( N_{\bar{\theta}_0} = N_{\bar{\theta}_1} \)

We replace \( g(x_1, x_2 \ldots) \) in (F.1) with

\[
g(x_1, x_2 \ldots) \leftrightarrow \int_{-\infty}^{+\infty} dx_{\bar{\theta}_0} g(x_0, x_1, \ldots) f_{N_{\bar{\theta}_0}, N_{\bar{\theta}_1}}(x_{\bar{\theta}_0} - x_{\bar{\theta}_1}).
\]
Then, (F.1) becomes (F.1)_+. Therefore, (F.1)_+ holds in case (F.1) does.

Case 2: \( N_{\tilde{\theta}_0} > N_{\tilde{\theta}_1} \)

We replace \( g(x_1, x_2 \cdots) \) in (F.1) with

\[
g(x_1, x_2 \cdots) \leftarrow \int_{-\infty - iN_{\tilde{\theta}_0} \delta}^{+\infty - iN_{\tilde{\theta}_0} \delta} dx_{\tilde{\theta}_0} g(x_0, x_1, \cdots) f_{N_{\tilde{\theta}_0}, N_{\tilde{\theta}_1}}(x_{\tilde{\theta}_0} - x_{\tilde{\theta}_1}). \quad (F.10)
\]

We apply this relation to the r.h.s. of (F.1)_+. The integral path with respect to \( x_{\tilde{\theta}_0} \) is changed into \(-\infty + i \min(\sigma_1^{(0)}) \delta, +\infty + i \min(\sigma_1^{(0)}) \delta\). Then, the r.h.s. of (F.1)_+ becomes

\[
\sum_{\lambda' \subseteq \lambda, \xi \equiv G_{\xi_+}(\lambda')} \left[ \prod_{\tilde{\xi} \in \xi} \delta \left( \tilde{\xi}', \lambda \left[ \lambda', \tilde{\xi}' \right], \min_1 \left( \tilde{\xi}' \right) \right) \int_{-\infty + i \min_1 \left( \tilde{\xi}' \right) \delta}^{+\infty + i \min_1 \left( \tilde{\xi}' \right) \delta} dx_{\tilde{\xi}'} \right]
\]

\[
\sum_{\lambda' \subseteq \lambda, \lambda' \equiv \lambda' + \{\tilde{\theta}_0, \tilde{\theta}_1\}, \xi_+ \equiv G_{\xi_+}(\lambda')} \left[ \prod_{\tilde{\xi} \in \xi_+} \delta \left( \tilde{\xi}', \lambda \left[ \lambda', \tilde{\xi}' \right], \min_1 \left( \tilde{\xi}' \right) \right) \int_{-\infty + i \min_1 \left( \tilde{\xi}' \right) \delta}^{+\infty + i \min_1 \left( \tilde{\xi}' \right) \delta} dx_{\tilde{\xi}'} \right]
\]

The second term is a sum of the residues generated when the integral path with respect to \( x_{\tilde{\theta}_0} \) is moved from \( \Im x_{\tilde{\theta}_0} = -N_{\tilde{\theta}_0} \delta \) to \( \min(\sigma_0^{(1)}) \delta \). The residues are due to poles at \( x_{\tilde{\xi} \left[ G_{\xi_+}(\lambda') \right], \tilde{\theta}_1} = x_{\tilde{\theta}_0} \). Note that a residue characterized by a set \( \lambda' \) in \( \sum_{\lambda' \subseteq \lambda} \) of the second term is generated by a term characterized by the same set \( \lambda' \) in \( \sum_{\lambda' \subseteq \lambda} \) of the first term, and, the following fact enable us to write these residues like the second terms. In case

\[
\delta \left( \tilde{\zeta}_0, \lambda \left[ \lambda', \tilde{\zeta}_0 \right], \min_1 \left( \tilde{\zeta}_0 \right) \right) = 1 \quad (F.12)
\]

and

\[
\min \left( \sigma_1^{(0)} \right) > \min_1 \left( \tilde{\zeta} \left[ G_{\xi_+}(\lambda') \right], \tilde{\theta}_1 \right), \quad (F.13)
\]

which means that the pole is in a region surrounded by integral paths \( \Im x_{\tilde{\theta}_0} = -N_{\tilde{\theta}_0} \delta \) and \( \min(\sigma_0^{(1)}) \delta \), it follows that

\[
\delta \left( \tilde{\zeta}_1, \lambda \left[ \lambda', \tilde{\zeta}_1 \right], \min_1 \left( \tilde{\zeta}_1 \right) \right) = 1 \iff \delta \left( \tilde{\zeta}_1, \lambda \left[ \lambda'_+, \tilde{\zeta}_1 \right], \min_1 \left( \tilde{\zeta}_1 \right) \right) = 1, \quad (F.14)
\]
and in the other case,

\[
\delta \left( \tilde{\zeta}_1', \lambda \left[ \lambda_+ \right], \min_1 \left( \tilde{\zeta}_1' \right) \right) = 0,
\]

where we suppose \( N_{\tilde{\theta}_0} > N_{\tilde{\theta}_1} \), \( \lambda' \subseteq \lambda \), \( \lambda_+ \equiv \lambda' + \{ \tilde{\theta}_0, \tilde{\theta}_1 \} \), \( \xi \equiv G_{\tilde{\lambda}_+} (\lambda') \), \( \xi_+ \equiv G_{\tilde{\lambda}_+} (\lambda_+) \), \( \tilde{\zeta}_1 = \tilde{\zeta} \left[ \xi, \tilde{\theta}_0 \right] \), \( \tilde{\zeta}_0 = \tilde{\zeta} \left[ \xi, \tilde{\theta}_0 \right] \), and \( \tilde{\zeta}_1' = \tilde{\zeta} \left[ \xi_+, \tilde{\theta}_1 \right] \).

It is clear that (F.11) is equal to the l.h.s. of (F.1). A term in the l.h.s. of (F.1) characterized by \( \lambda_1' \) in \( \sum_{\lambda' \subseteq \lambda_+} \) which \( \{ \tilde{\theta}_0, \tilde{\theta}_1 \} \) belongs to is the same as a term in the second term of (F.11) characterized by \( \lambda_2' = \lambda_1' - \{ \tilde{\theta}_0, \tilde{\theta}_1 \} \) in \( \sum_{\lambda' \subseteq \lambda_+} \). And, a term in (F.1) characterized by \( \lambda_1' \) which \( \{ \tilde{\theta}_0, \tilde{\theta}_1 \} \) does not belong to is the same as a term in the first term of (F.11) characterized by \( \lambda_2' = \lambda_1' \).

Case 3: \( N_{\tilde{\theta}_0} < N_{\tilde{\theta}_1} \)

We shall modify the r.h.s. of (F.1) \( + \). First, we move the integral path with respect to \( x_{\{ \tilde{\theta}_0 \}} \) from \( \Im x_{\{ \tilde{\theta}_0 \}} = -N_{\tilde{\theta}_0} \delta \) to \( (\max(\sigma_+) + 1)\delta \). Next, we apply the relation (F.1) in the same way that we have done in the case 1 and 2. Finally, we move the integral path with respect to \( x_{\{ \tilde{\theta}_0 \}} \) from \( \Im x_{\{ \tilde{\theta}_0 \}} = (\max(\sigma_+) + 1)\delta \) to \( \min(\sigma_1(0)) \delta \). Then, the r.h.s. of (F.1) becomes

\[
= \sum_{\lambda' \subseteq \lambda_+} \left[ \prod_{\tilde{\zeta}' \in \xi} \delta \left( \tilde{\zeta}', \lambda \left[ \lambda_+ \right], \min_1 \left( \tilde{\zeta}' \right) \right) \int_{-\infty + i \min_1 \left( \tilde{\zeta}' \right) \delta}^{+\infty + i \min_1 \left( \tilde{\zeta}' \right) \delta} dx_{\tilde{\zeta}'} \right] \left[ \prod_{\{ \tilde{\theta}, \tilde{\vartheta} \} \in \lambda_+ - \lambda'} f_{\tilde{\lambda}_0, N_{\tilde{\theta}_0}} \left( x_{\tilde{\zeta}[\xi, \tilde{\theta}]} - x_{\tilde{\zeta}[\xi, \tilde{\vartheta}]} \right) g \left( x_{\tilde{\zeta}[\xi, \tilde{\theta}]} \right) \right] + \sum_{\lambda' \subseteq \lambda_+} \left[ \prod_{\{ \tilde{\theta}, \tilde{\vartheta} \} \in \lambda_+ - \lambda'} f_{\tilde{\lambda}_0, N_{\tilde{\theta}_0}} \left( x_{\tilde{\zeta}[\xi, \tilde{\theta}]} - x_{\tilde{\zeta}[\xi, \tilde{\vartheta}]} \right) g \left( x_{\tilde{\zeta}[\xi, \tilde{\theta}]} \right) \right] \sum_{\lambda' \subseteq \lambda_+} \left[ \prod_{\{ \tilde{\theta}, \tilde{\vartheta} \} \in \lambda_+ - \lambda'} f_{\tilde{\lambda}_0, N_{\tilde{\theta}_0}} \left( x_{\tilde{\zeta}[\xi, \tilde{\theta}]} - x_{\tilde{\zeta}[\xi, \tilde{\vartheta}]} \right) g \left( x_{\tilde{\zeta}[\xi, \tilde{\theta}]} \right) \right],
\]

(4.16)
where $Sgn[\tilde{\zeta}, \tilde{\zeta}', 1]$ is defined as

$$Sgn[\tilde{\zeta}, \tilde{\zeta}', 1] = \begin{cases} 
1 & \text{In Case } m < m' \text{ and } \min_{\theta \in \tilde{\zeta}} N_\theta > \min_{\tilde{\zeta}} N_\tilde{\theta} \\
-1 & \text{In Case } m < m' \text{ and } \min_{\theta \in \tilde{\zeta}} N_\theta < \min_{\tilde{\zeta}} N_\tilde{\theta} \\
0 & \text{Otherwise}
\end{cases}$$  

(F.17)

and $Sgn[\tilde{\zeta}', \tilde{\zeta}, -1] \equiv Sgn[\tilde{\zeta}, \tilde{\zeta}', 1]$ in case of $\theta \in \Theta_n$, $\tilde{\zeta}, \tilde{\zeta}' \subseteq \tilde{\Theta}(\theta)$, $m = \min_1 \left(\tilde{\zeta}\right)$ and $m' = \min_1 \left(\tilde{\zeta}'\right)$. The second term in (F.16) is a sum of residues characterized by sets $\lambda_1'$ and $\lambda''_1$ in $\sum_{\lambda' \subseteq \lambda} \sum_{\lambda'' \subseteq \lambda} \lambda$ of the first term. Each residue is generated by a term which is characterized by $\lambda_2' = \lambda_1' - \lambda \left[\lambda_1', \tilde{\zeta}'\right] + \lambda''_1$ in $\sum_{\lambda' \subseteq \lambda} \lambda'$ of the first term. We note two facts. First, we move the integral path with respect to $\tilde{\zeta}'$ in the second term; as a result, a term in the first term as a sum generates not one but several residues in the second term. Second, a condition that a pole in the first term corresponding to a residue is in a region where the integral path passes is the same as a condition that $\theta$-function and all the $Sgn$-function in the residue are not zero.

Using a relation proved in Appendix G, the second term of (F.16) becomes

$$\sum_{\lambda' \subseteq \lambda, \lambda'_1 = \lambda' \{\tilde{\theta}_0, \tilde{\theta}_1\}, \xi'_1 = G_{\tilde{\zeta}'}(\lambda' + \{\tilde{\theta}_0, \tilde{\theta}_1\})} \left[ \prod_{\tilde{\zeta}' \in \xi'} \delta \left(\tilde{\zeta}', \lambda_1' [\lambda_1', \tilde{\zeta}'], \min_1 \left(\tilde{\zeta}'\right)\right) \int_{-\infty + i \min_1(\tilde{\zeta}) \delta}^{+ \infty + i \min_1(\tilde{\zeta}) \delta} dx_{\tilde{\zeta}'} \right]$$

$$\prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda_1' - \lambda_1''} f_{N_{\tilde{\theta}}, N_{\tilde{\theta}'}} \left(x_{\tilde{\zeta}[\xi', \tilde{\theta}]} - x_{\tilde{\zeta}[\xi', \tilde{\theta}']}\right) g \left(x_{\tilde{\zeta}[\xi', \tilde{\theta}]}; x_{\tilde{\zeta}[\xi', \tilde{\theta}']}; \ldots\right).$$  

(F.18)

By the same reason as the case 2, the first term of (F.16) plus (F.18) equals the l.h.s. of (F.17) +.

**Appendix G** A proof with respect to a permutation 3

We prove that

$$\theta \left(\min_1 \left(\tilde{\zeta}'\right) - \min \left(\sigma_1^{(0)}\right) + \frac{1}{2}\right) \sum_{\lambda'' \subseteq \lambda', \xi' \equiv G_{\tilde{\zeta}'}(\lambda'')} \left[ \prod_{\tilde{\zeta}' \in \xi'} \delta \left(\tilde{\zeta}', \lambda'' [\lambda'', \tilde{\zeta}'], \min_1 \left(\tilde{\zeta}'\right)\right) \right]$$

$$\prod_{\{\tilde{\theta}, \tilde{\theta}'\} \in \lambda' - \lambda''} Sgn \left[\tilde{\zeta} [\xi', \tilde{\theta}], \tilde{\zeta} [\xi', \tilde{\theta}'], l_{\lambda'} [\tilde{\theta}, \tilde{\theta}_1] - l_{\lambda'} [\tilde{\theta}', \tilde{\theta}_1]\right]$$

$$= \delta \left(\tilde{\zeta}_1', \lambda_1', \min_1 \left(\tilde{\zeta}_1\right)\right),$$  

(G.1)
where \( Sgn \)-function is defined in Appendix F, and \( \theta \in \Theta (\sigma \subset \mathbb{N}) \), \( \zeta' \in \bar{\Theta} (\theta) \), \( \tilde{\theta}_0, \tilde{\theta}_1 \in \zeta'_1 \), \( N_{\tilde{\theta}_0} < N_{\tilde{\theta}_1} \), \( \lambda' \in \Lambda_c \left( \zeta'_1 - \tilde{\theta}_0 \right) \) and \( \lambda'_+ = \lambda' + \{ \tilde{\theta}_0, \tilde{\theta}_1 \} \). Note that the assumption \( \tilde{\theta}_0 \in \zeta'_1 \) leads to \( \min_1 (\zeta'_1) \leq \min (\sigma^{(0)}_1) \).

It is evident that in the following two case, both sides of (G.1) are equal to zero. One is a case of \( \min_1 (\zeta'_1) < \min (\sigma^{(0)}_1) \). The other is a case that there are two elements \( \tilde{\theta}, \tilde{\theta}' \) which satisfy \( N_{\tilde{\theta}} = N_{\tilde{\theta}'} \) and \( \{ \tilde{\theta}, \tilde{\theta}' \} \in \lambda' \). Therefore, we shall suppose \( \min_1 (\tilde{\zeta}) = \min (\sigma^{(0)}_1) \) and \( N_{\tilde{\theta}} \neq N_{\tilde{\theta}'} \) in case of \( \{ \tilde{\theta}, \tilde{\theta}' \} \in \lambda' \).

Case 1: the r.h.s. of (G.1) is equal to 1. We define a subset of \( \lambda' \) as

\[
\lambda''_1 \equiv \left\{ \left\{ \tilde{\theta}, \tilde{\theta}' = (\sigma'_1, \cdots) \right\} \in \lambda' \left| N_{\tilde{\theta}} < N_{\tilde{\theta}'}, \min_1 \left( \left\{ \tilde{\theta}_1, \sim, \tilde{\theta} \right\} \right) < \min \sigma'_1 \right. \right\} \quad \text{G.2}
\]

where

\[
\left\{ \tilde{\theta}, \sim, \tilde{\theta}' \right\} \equiv \left\{ \tilde{\theta}'' \in \tilde{\zeta} \left| l_\lambda \left( \tilde{\theta}, \tilde{\theta}' \right) = l_\lambda \left( \tilde{\theta}, \tilde{\theta}'' \right) = l_\lambda \left( \tilde{\theta}'', \tilde{\theta}' \right) \right. \right\} \quad \text{G.3}
\]

In the l.h.s. of (G.1), any term corresponding to \( \lambda''_1 \) which satisfies \( \lambda''_1 \subseteq \lambda''_1 \) is equal to 0, because \( Sgn \)-function with respect to \( \left\{ \tilde{\theta}, \tilde{\theta}' \right\} \in \lambda''_1 - \lambda''_2 \), where \( \tilde{\theta}'' \notin \left\{ \tilde{\theta}_1, \sim, \tilde{\theta} \right\} \) in case \( \{ \tilde{\theta}'' \} \in \lambda''_1 \) is equal to 0, because \( \delta \)-function with respect to \( \zeta [\xi', \tilde{\theta}] \), where \( \{ \tilde{\theta}, \tilde{\theta}' \} \in \lambda' \cap \lambda''_1 \), is equal to 0. Therefore, only one term, which corresponds to \( \lambda''_1 \), survives. Each function in the term satisfies

\[
\begin{align*}
\delta \left( \zeta', \lambda \left[ \lambda''_1, \zeta' \right], \min_1 \left( \zeta' \right) \right) &= 1 \\
\text{Sgn} \left[ \zeta \left[ \xi', \tilde{\theta} \right], \zeta' \left[ \xi', \tilde{\theta} \right], l_{\chi'} \left( \tilde{\theta}, \tilde{\theta}_1 \right) - l_{\chi'} \left( \tilde{\theta}_1, \tilde{\theta} \right) \right] &= 1
\end{align*} \quad \text{G.4}
\]

where \( \{ \tilde{\theta}, \tilde{\theta}' \} \in \lambda \left[ \lambda', \zeta'_1 \right] - \lambda''_1 \), \( \xi' = G_{\zeta'_1} (\lambda''_1) \) and \( \zeta' \in \xi' \). Thus, the relation (G.1) holds in the case 1.

Case 2: the r.h.s. of (G.1) is equal to 0. In this case, we prove that the l.h.s. of (G.1) equals to 0. First, we introduce an equivalence relation on a set which consists of subsets of \( \lambda' \) corresponding to non-zero terms in the l.h.s. of (G.1). Next, we prove that there are two or no terms in an equivalence class. Finally, two terms in any equivalence classes cancel out each other.
To define the equivalence relation, we define several sets by use of $\lambda, \tilde{\zeta}, \tilde{\theta}_1$. First, a set $\tilde{\zeta}_*$ consists of elements $\tilde{\theta} \in \tilde{\zeta}$ which satisfy the following two conditions. There is $\tilde{\theta}' \in \zeta$ which satisfies
\[
\left\{ \tilde{\theta}, \tilde{\theta}' \right\} \in \lambda', \quad N_{\tilde{\theta}} > N_{\tilde{\theta}'} , \quad l_{\lambda'} \left( \tilde{\theta}_1, \tilde{\theta} \right) + 1 = l_{\lambda'} \left( \tilde{\theta}_1, \tilde{\theta}' \right). \tag{G.5}
\]
And,
\[
\left\{ \tilde{\theta}'', \tilde{\theta}''' \right\} \in \lambda', \quad N_{\tilde{\theta}''} < N_{\tilde{\theta}'''} , \quad \tilde{\theta}'', \tilde{\theta}''' \in \left\{ \tilde{\theta}_1, \sim, \tilde{\theta} \right\} \Rightarrow l_{\lambda'} \left( \tilde{\theta}_1, \tilde{\theta}'' \right) + 1 = l_{\lambda'} \left( \tilde{\theta}_1, \tilde{\theta}''' \right). \tag{G.6}
\]
It is assured that there are elements in $\tilde{\zeta}_*$ by the assumption that the r.h.s. of (G.1) is equal to 0. Next, we define $\tilde{\theta}_*$ as one of sets in $\tilde{\zeta}_*$ which satisfy
\[
\tilde{\theta} \in \tilde{\zeta}, \quad \tilde{\theta} \neq \tilde{\theta}_* \Rightarrow \theta_* \notin \left\{ \tilde{\theta}_1, \sim, \tilde{\theta} \right\}. \tag{G.7}
\]
We also define sets $\lambda_{*0}, \tilde{\theta}_{*0}, \lambda_{*-}, \tilde{\zeta}_{*-}, \lambda_{**}$ and $\lambda_e$
\[
\lambda_{*0} \equiv \left\{ \left\{ \tilde{\theta}, \tilde{\theta}_* \right\} \in \lambda \right| \tilde{\theta} \in \left\{ \tilde{\theta}_1, \sim, \tilde{\theta}_* \right\} \right\} = \left\{ \left\{ \tilde{\theta}_{*0}, \tilde{\theta}_* \right\} \right\}
\lambda_{*-} \equiv \left\{ \left\{ \tilde{\theta}, \tilde{\theta}_* \right\} \in \lambda \right| N_{\tilde{\theta}} < N_{\tilde{\theta}_*} , \tilde{\theta} \notin \left\{ \tilde{\theta}_1, \sim, \tilde{\theta}_* \right\} \right\}
\tilde{\zeta}_{*-} \equiv \left\{ \tilde{\theta} \in \tilde{\zeta}_1 \right| \left\{ \tilde{\theta}, \tilde{\theta}_* \right\} \in \lambda_{*-} \right\}
\lambda_{**} \equiv \lambda \left[ \lambda' \left| \left\{ \tilde{\theta} \in \tilde{\zeta}_1 \right| \tilde{\theta}_* \in \left\{ \tilde{\theta}_1, \sim, \tilde{\theta} \right\} , \left\{ \tilde{\theta}_1, \sim, \tilde{\theta} \right\} \cap \tilde{\zeta}_{*-} = \emptyset \right\} \right]
\lambda_e \equiv \lambda' - \lambda_{*0} - \lambda_{**} - \lambda_{*-} . \tag{G.8}
\]
Note that $\lambda_{*0}$ contains only one element. We introduce an equivalence relation on a set which consists of subsets of $\lambda'$ corresponding to non-zero terms in the l.h.s. of (G.1) as
\[
\lambda'' \sim \lambda''' \iff \lambda'' \cap \lambda_e = \lambda''' \cap \lambda_e . \tag{G.9}
\]
Then, $\lambda'' \cap \lambda_e$ is a character of an equivalence class. Hereafter, we shall study in the case of a term corresponding to $\lambda''$ in an equivalence class $\lambda''_e$. We define $\xi''_e$ and $\tilde{\theta}_{*-}$ as
\[
\xi''_e \equiv G_{\tilde{\zeta}_1} \left[ \lambda''_e \right] , \quad \tilde{\theta}_{*-} = \left( \sigma_1^{(*)}, \cdots \right) \in \tilde{\zeta}_{*-}, \quad \min \left( \sigma_1^{(*)} \right) = \max_{\tilde{\theta} \in \tilde{\zeta}_{*-}} \left( \min_1 \left( \tilde{\zeta} \left[ \xi''_e, \tilde{\theta} \right] \right) \right) . \tag{G.10}
\]
We prove that there are two or no terms in an equivalence class.
Case 2.1 \( \min(\sigma_1^{(s)}) < \min_1(\tilde{\zeta}^\prime \xi_0^\prime, \tilde{\theta}_n) \)

In this case, the assumptions that all \( \delta \)-functions and \( Sgn \)-functions in the term are non-zero request that

\[
\lambda'' \cap (\lambda_{s0} \cup \lambda_{s-}) = \left\{ \{\tilde{\theta}_{s-}, \tilde{\theta}_s\} \right\} \quad \text{or} \quad \emptyset,
\]

\[
\min \left( \sigma_1^{(s)} \right) > \min_1 \left( \tilde{\zeta} \left[ \xi_0^\prime, \tilde{\theta}_{am} \right] \right) \quad (G.11)
\]

In case \( \lambda'' \cap (\lambda_{s0} \cap \lambda_{s-}) \neq \emptyset \), \( \lambda'' \) is non-zero restricts \( \lambda'' \) to be

\[
\lambda'' = \lambda_e'' + \left\{ \{\tilde{\theta}_{s-}, \tilde{\theta}_s\} \right\} + \left\{ \{\tilde{\theta}, \tilde{\theta}' = (\sigma_1', \ldots)\} \in \lambda_{s+} | N_{\tilde{\theta}} < N_{\tilde{\theta}'}, \right. \\
\min \left( \min_1 \left( \tilde{\zeta} \left[ \xi_e'' - \tilde{\theta}_{s-} \right] \right), \min_1 \left( \{\tilde{\theta}_s, \sim, \tilde{\theta}\} \right) \right) < \min (\sigma_1') \right\}. \quad (G.12)
\]

In case \( \lambda'' \cap (\lambda_{s0} \cap \lambda_{s-}) = \emptyset\), \( \lambda'' \) is restricted as

\[
\lambda'' = \lambda_o'' + \left\{ \{\tilde{\theta}, \tilde{\theta}' = (\sigma_1', \ldots)\} \in \lambda_{s+} | N_{\tilde{\theta}} < N_{\tilde{\theta}'}, \right. \\
\min_1 \left( \{\tilde{\theta}_s, \sim, \tilde{\theta}\} \right) < \min (\sigma_1') \right\}. \quad (G.13)
\]

The condition that the term corresponding to \((G.12)\) is non-zero and the condition \((G.13)\) is non-zero request the same condition with respect to \( \lambda_e'' \)

Case 2.2 \( \min(\sigma_1^{(s)}) > \min_1(\tilde{\zeta}^\prime \xi_0^\prime, \tilde{\theta}_n) \)

In this case, the assumptions that all \( \delta \)-functions and \( Sgn \)-functions in the term are non-zero request that

\[
\lambda'' \cap (\lambda_{s0} \cup \lambda_{s-}) = \left\{ \{\tilde{\theta}_{s-}, \tilde{\theta}_s\} \right\} \quad \text{or} \quad \left\{ \{\tilde{\theta}_{s0}, \tilde{\theta}_s\} \right\},
\]

\[
\min \left( \sigma_1^{(s)} \right) > \min_1 \left( \tilde{\zeta} \left[ \xi_0^\prime, \tilde{\theta}_{am} \right] \right) \quad (G.14)
\]

In case \( \lambda'' \cap (\lambda_{s0} \cap \lambda_{s-}) \neq \emptyset \), \( \lambda'' \) is restricted as

\[
\lambda'' = \lambda_e'' + \left\{ \{\tilde{\theta}_{s-}, \tilde{\theta}_s\} \right\} + \left\{ \{\tilde{\theta}, \tilde{\theta}' = (\sigma_1', \ldots)\} \in \lambda_{s+} | N_{\tilde{\theta}} < N_{\tilde{\theta}'}, \right. \\
\min \left( \min_1 \left( \tilde{\zeta} \left[ \xi_e'' - \tilde{\theta}_{s-} \right] \right), \min_1 \left( \{\tilde{\theta}_s, \sim, \tilde{\theta}\} \right) \right) < \min (\sigma_1') \right\}. \quad (G.15)
\]

In case \( \lambda'' \cap (\lambda_{s0} \cap \lambda_{s-}) \neq \emptyset \), \( \lambda'' \) is restricted as

\[
\lambda'' = \lambda_e'' + \left\{ \{\tilde{\theta}_{s0}, \tilde{\theta}_s\} \right\} + \left\{ \{\tilde{\theta}, \tilde{\theta}' = (\sigma_1', \ldots)\} \in \lambda_{s+} | N_{\tilde{\theta}} < N_{\tilde{\theta}'}, \right. \\
\min \left( \min_1 \left( \tilde{\zeta} \left[ \xi_e'' - \tilde{\theta}_{s0} \right] \right), \min_1 \left( \{\tilde{\theta}_s, \sim, \tilde{\theta}\} \right) \right) < \min (\sigma_1') \right\}. \quad (G.16)
\]
The condition that the term corresponding to (G.13) is non-zero and the condition (G.16) is non-zero requests the same condition with respect to \( \lambda''_e \).

Thus, we have proved that there are two or no terms in an equivalence class.

Finally, we consider about a sign of the term corresponding to \( \lambda'' \). The sign depends on only \( Sgn\)-functions. And, each \( Sgn\)-function depends on a connection in \( \lambda' - \lambda'' \) when \( \lambda', \tilde{\zeta}_1' \) and \( \tilde{\theta}_1 \) are fixed. Moreover, a sign of \( Sgn\)-function corresponding to any connection in \( \lambda_{s+} \) and \( \lambda_{s0} \) is positive, and a sign of \( Sgn\)-function corresponding to any connection in \( \lambda_{s-} \) is negative. Therefore, in both the case 2.1 and the case 2.2, terms corresponding to the two \( \lambda'' \) classes cancel out each other.