AN ALMOST SPLITTING THEOREM FOR A WARPED PRODUCT SPACE

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Abstract. We prove an almost splitting theorem in the sense of [3] for the warped product space with warped function \( f(r) = \cosh \left( r \sqrt{\frac{\lambda}{n-2}} \right) \).

1. Introduction

The classical splitting theorem of Cheeger-Gromoll [4] states that a Riemannian manifold with non-negative Ricci curvature that contains a line is isometric to a product of the real line with a submanifold. Here a line is a geodesic which is the image of the real line such that each segment is length minimizing between its end points.

The above theorem is an example of rigidity results in Riemannian geometry. The assumptions of rigidity results involve inequalities and equalities of certain geometric quantities of the manifold and they conclude that the manifold is isometric to certain model space.

In [3], a theory of almost rigidity is developed. It is shown that if an inequality on the Ricci curvature holds and the volume or the diameter is approximately equal to that of the model space, then the manifold is close to the model space in the Gromov-Hausdorff topology. In particular, an almost splitting theorem is shown. It states that if the manifold has approximately non-negative curvature, there are two points \( q_0 \) and \( q_1 \) which are far enough away from a point \( p \), and the excess function \( e(x) = d(x, q_0) + d(x, q_1) - d(q_0, q_1) \) is small at \( p \), then the ball centered at \( p \) of large radius is close in the Gromov-Hausdorff sense to the corresponding ball in the model. Here the model is the product of the real line and a metric space.

In this paper, we prove a version of the almost splitting theorem for the warped product space with warped function \( f \) given by

\[ f(r) = \cosh \left( r \sqrt{\frac{\lambda}{n-2}} \right). \]

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The corresponding rigidity result is obtained in [11, 12, 9]. In order to state the result, let us introduce some notations. Let us fix a positive function $V$ and let $c$ be the cost function defined by

\begin{equation}
(1.1) \quad c(x, y) = \inf_{\gamma \in \Gamma} \int_0^T V(\gamma(t)) \, dt,
\end{equation}

where infimum is taken over all time $T > 0$ and all Lipschitz curves $\gamma : [0, T] \to M$ which begins at $x$ and ends at $y$ such that $|\dot{\gamma}(t)| \leq 1$ for Lebesgue almost all $t$ in $[0, T]$. Here $| \cdot |$ denotes the norm defined by the given Riemannian metric.

Let $g$ be a positive eigen-function of the Laplace-Beltrami operator with eigenvalue $\lambda$. Let $q_0$ and $q_1$ be two points on the manifold $M$, let $e$ be the excess function corresponding to the cost $c$ defined by $e(x) = c(x, q_0) + c(x, q_1) - c(q_0, q_1)$ with $V = \sqrt{n-2}$. Let $p$ be another point in $M$. For $i = 0, 1$, let $b_i(x) = c(x, q_i) - c(p, q_i)$.

The model is defined by the warped product $\mathbb{R} \times_\gamma b_i^{-1}(0)$, where $b_i(0)$ is equipped with the distance function induced by that of $M$. Recall that given two points $x$ and $y$ on a warped product space $\mathbb{R} \times_\gamma N$, the distance between $(r_0, x_0)$ and $(r_1, x_1)$ depends only $r_0, r_1$, and the distance of $x_0$ and $x_1$ in $N$. Therefore, this defines a distance function on $\mathbb{R} \times_\gamma b_i^{-1}(0)$.

**Theorem 1.1.** Let $p$ be a point in $M$ and let $v > 0$ be a constant such that $B_R(p) \geq v R^n$. For each $\epsilon > 0$, there are constants $\epsilon_0 > 0$ and $L > 0$ such that if the followings hold

1. the distance from $q_i$ to the ball $B_R(p)$ of radius $R$ centered at $p$ is greater than $LR$, where $i = 0, 1$,
2. $|f(G^{-1}(b_i))^{2-n} - g| < \epsilon$ on $\partial B_R(p)$, where $i = 0, 1$ and $G(r) = \int_0^r f^{1-n}$,
3. the maximum of $g$ is achieved at a point in $B_R(p)$,
4. the Ricci curvature $Rc$ satisfies $Rc \geq -\frac{\lambda(n-1)}{n-2} - \frac{\epsilon}{R^2}$,
5. the sectional curvature is bounded by $C \lambda$ for some positive constant $C$,
6. $e(p) < \epsilon R$.

Then there is $R_0(K, \lambda, n) > 0$ such that the ball $B_{R_0}(p)$ is $k(\epsilon)$-close in the Gromov-Hausdorff distance to the corresponding ball in the model of radius $R$ centered at $(p, 0)$, where $k(\epsilon) \to 0$ as $\epsilon \to 0$ and it depends on $n, C, \epsilon, \lambda, \max_{x \in M} g(x)$, and $v$ but not on $R$.

The proof of Theorem 1.1 relies heavily on the ideas from [3]. In section 2 we prove an Abresch-Gromoll type inequality using the cost function (1.1). This is motivated by the work in [9]. In section 3
similar to the almost splitting theorem in [3], we develop several estimates on the harmonic approximations of $b_i$. The Hessian estimates, in our case, are more complicated due to the involvement of the eigenfunction. Using the estimates established in section 3.1, we show that the distance function of the Riemannian manifold is close to that of the model. Finally, we finish the proof in section 5.

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Notations

Throughout this paper, there are different constants depending on $K$, $c_0$, $n$, $\lambda$, and $v$. These dependencies will be suppressed throughout the paper. The symbols $k_i(\epsilon)$ and $c_i(\epsilon)$ denote continuous family of constants such that $k_i(\epsilon), c_i(\epsilon) \to 0$ as $\epsilon \to 0$.

2. Eikonal Type Equations and Mechanical Hamiltonian Systems

In this section, we discuss various facts concerning the cost function (1.1) which are needed for this paper. First, the cost function is a viscosity solution of the following eikonal type equation

$$|\nabla f|_x - V(x) = 0.$$  

where $| \cdot |$ and $V$ denote, respectively, a Riemannian metric and a positive potential function of a manifold $M$.

In the Euclidean case, this is shown in [13]. One can also consider the above equation as a usual eikonal equation with Riemannian metric given by $\frac{1}{V} | \cdot |$. Therefore, $c$ is also given by the distance function corresponding to this Riemannian metric. However, for later discussion, it is more convenient and natural to think of the cost $c$ as the above optimal control problem.

An application of the Pontryagin maximum principle (see, for instance, [8]) gives

\textbf{Theorem 2.1.} Let $(T, \gamma)$ be a minimizer of the above minimization problem. Then there is a path $(\gamma(t), p(t))$ in the cotangent bundle $T^*M$ of the manifold $M$ which is a solution to the Hamiltonian system of the Hamiltonian

$$H(x, p) = |p|_x - V(x).$$
In particular, $|\dot{\gamma}(t)| = 1$ and

$$
\frac{D^2}{dt^2} \gamma = \nabla \log V(\gamma) - \langle \nabla \log V, \dot{\gamma} \rangle \dot{\gamma}.
$$

Here $\frac{D}{dt}$ denotes the covariant derivative with respect to the given Riemannian metric $| \cdot |$.

For convenience, we consider the following map $\Psi_t$ defined by

$$
\Psi_0(v) = x, \quad \frac{d}{dt} \Psi_t(v) \bigg|_{t=0} = v,
$$

$$
\frac{D^2}{dt^2} \Psi_t(v) = \frac{1}{2} \nabla^2 V(\Psi_t(v)),
$$

where $|v|_x = V(x)$. Then minimizers of (1.1) are of form $\Psi_{r-1(t)}(v)$, where $r(t) = \int_0^t V(\Psi_s(v)) \, ds$.

The following facts can be obtained using arguments similar to the Riemannian case.

**Lemma 2.2.** Assume that $s \in [0,T] \mapsto \gamma(s) := \Psi_{r-1(s)}(v)$ is a minimizer between its end-points. Then

1. $\gamma|_{[0,t]}$ is the unique minimizer connecting its end-points $\gamma(0)$ and $\gamma(t)$ for each $t < T$,
2. there is a neighborhood $U$ of $x$ such that $c_x$ is smooth on $U \setminus \{x\}$,
3. $c_x$ is smooth at $\gamma(s)$ for each $s$ in $[0,T)$,
4. $d(\Psi_{r-1(s)})$ is invertible for each $s$ in $[0,T)$.

Next, we state a Laplacian comparison type theorem for the cost function $c$.

**Lemma 2.3.** Assume that the Ricci curvature $\text{Rc}$ of the manifold is bounded below by a constant $-\frac{\lambda(n-1)}{n-2} - \epsilon$ and the function $g$ satisfies

$$
\Delta g \leq -\lambda g \text{ and } g \leq K.
$$

Suppose that $V = g^{\frac{n-2}{n-1}}$. Then the Laplacian of $c_x$ satisfies

$$
\Delta c_x(\Psi_t(v)) \leq \sqrt{\epsilon(n-1)K^{\frac{n-1}{n-2}}} \text{coth} \left( \sqrt{\frac{\epsilon}{n-1}} K^{\frac{n-3}{n-2}} u^{-1}(t) \right),
$$

where $u^{-1}(t) = \int_0^t g(\Psi_s(v))^{\frac{n-2}{n-1}} \, ds$.

We also need the following volume growth estimate.

**Lemma 2.4.** Under the assumptions of Lemma 2.3 and that $\Psi_{r-1(s)}$ is contained in $B_1(p)$ for each $s$ in $[\frac{t}{2}, t]$, there is a constant $C(\epsilon, K, D, n) > 0$ such that

$$
\frac{\det(d(\Psi_{r-1(t)}(v)))}{\det(d(\Psi_{r-1(s)}(v)))} \leq C(\epsilon, K, D, n, \lambda)
$$
for each $\frac{1}{2} < s < t$, where $D = \sup_{x,y \in B_1(p)} \tau(x,y)$.

As a consequence, we obtain a version of the Abresch-Gromoll inequality \[1, 3] in our setting assuming that the potential $V$ is given by $V = \frac{1}{2} g \frac{2n - 2}{n - 2}$, where $g$ is a positive function satisfying $\Delta g = -\lambda g$.

**Theorem 2.5.** Assume that the conditions in Lemma 2.3 hold. Let $q_0$ and $q_1$ be two points in the manifold $M$ and let $e$ be the excess function defined by

$$e(y) = c(y, q_0) + c(y, q_1) - c(q_0, q_1).$$

Assume that, given $\epsilon > 0$, there are constants $L(K, \epsilon)$ and $\epsilon_0(K, n, \epsilon) > 0$ such that the followings hold:

1. $d(q_i, B_1(p)) \geq L$, where $i = 0, 1$,
2. $Rc \geq -\frac{\lambda(n-1)}{n-2} - \epsilon_0$,
3. $e(p) < \epsilon_0$.

Then $e(x) < \epsilon$ for all $x$ in $B_1(p)$.

The proof of Theorem 2.5 follows closely that of the corresponding result in \[1, 3\]. We give the proof here for the purpose of introducing notations and results needed for later sections. The rest of this section is devoted to the proofs.

**Proof of Lemma 2.3.** Let $w_1, \ldots, w_{n-1}$ be an orthonormal frame of the space $\{v \in T_x M \mid |v_x| = V(x)\}$ and let $w_0 = \partial_t$. Let $B(t)$ be the matrix defined by

$$d\Psi_{(t,v)}(w_i) = \sum_{j=0}^{n-1} B_{ij}(t)v_j(t),$$

where $v_0(t) = \frac{\Psi_t(v)}{|\Psi_t(v)|}$ and $\{v_1(t), \ldots, v_{n-1}(t)\}$ is an orthonormal frame of $v_0(t)^\perp$ such that $\tilde{v}_i(t)$ is contained in $\mathbb{R}v_0(t)$, $i = 1, \ldots, n - 1$.

Let $E(t) = (v_0(t), \ldots, v_{n-1}(t))^T$ and let $\dot{E}(t) = \left(\frac{\partial}{\partial t} v_0(t), \ldots, \frac{\partial}{\partial t} v_{n-1}(t)\right)^T$. It follows that $\dot{E}(t) = A(t)E(t)$, where

$$A(t) = \begin{pmatrix} 0 & A_1(t) \\ -A_1(t)^T & 0 \end{pmatrix},$$

and $A_1(t) = (\langle \nabla V(\Psi_t(v)), v_1(t) \rangle \ldots \langle \nabla V(\Psi_t(v)), v_{n-1}(t) \rangle)$. Therefore,

$$\frac{D}{dt} d\Psi_{(t,v)}(w_i) = \sum_{j=0}^{n-1} \left( \dot{B}_{ij}(t) + \sum_{k=0}^{n-1} B_{ik}(t) A_{kj}(t) \right) v_j(t).$$
By differentiating the above equation again with respect to $t$, it follows that
\[
\ddot{B}(t) + 2\dot{B}(t)A(t) + B(t)\dot{A}(t) + B(t)A(t)^2 = -B(t)R(t) + B(t)W(t)
\]
where $W_{ij}(t) = \frac{1}{2} \nabla^2 V^2(v_i(t), v_j(t))$.

Let $s(t)$ be the trace of the matrix $B(t)^{-1}\dot{B}(t) + A(t)$. A computation as in [9] Section 3 shows that
\[
\dot{s}(t) + \frac{s(t)^2}{n-1} - \frac{2s(t)}{n-2} \frac{\left< \nabla g(\Psi_t(v)), \dot{\Psi}_t(v) \right>}{g(\Psi_t(v))} - \epsilon g(\Psi_t(v)) \frac{2^{n-2}}{n-2} \leq 0.
\]

Let $w(t) = g(\Psi_t(v))^{-\frac{2}{n-2}} s(t)$ and $\dot{u}(t) = \frac{1}{g(\Psi_{u(t)}(v))^{\frac{2}{n-2}}}$. Another computation shows that
\[
\frac{d}{dt} w(u(t)) + \frac{w(u(t))^2}{n-1} - \epsilon K^{\frac{2(n-3)}{n-2}} \leq 0.
\]

It follows that $w(u(t)) \leq \sqrt{\epsilon(n-1)K^{\frac{n-3}{n-2}}} \coth \left( \sqrt{\frac{\epsilon}{n-1}K^{\frac{n-3}{n-2}}} t \right)$. \hfill \(\square\)

**Proof of Lemma 2.4.** We use the same notations as that of the proof of Lemma 2.3. The function $b(t) = \det(B(t))$ satisfies
\[
\frac{d}{dt} \log b(u(t)) = s(u(t))g(\Psi_{u(t)}(v))^{-\frac{2}{n-2}} = w(u(t)).
\]

Let $\tilde{b}(t) = \sinh^{n-1} \left( \sqrt{\frac{\epsilon}{n-1}K^{\frac{n-3}{n-2}}} t \right)$. Then
\[
\frac{d}{dt} \log \tilde{b}(t) = \sqrt{\epsilon(n-1)K^{\frac{n-3}{n-2}}} \coth \left( \sqrt{\frac{\epsilon}{n-1}K^{\frac{n-3}{n-2}}} t \right)
\geq \frac{d}{dt} \log b(u(t)).
\]

It follows that
\[
\frac{b(r^{-1}(t))}{b(r^{-1}(s))} \leq \frac{\tilde{b}(u^{-1}(r^{-1}(t)))}{\tilde{b}(u^{-1}(r^{-1}(s)))} \leq \frac{\tilde{b}(u^{-1}(r^{-1}(t)))}{\tilde{b}(u^{-1}(r^{-1}(t/2)))} \leq C(\epsilon, K, D, n).
\]

Therefore,
\[
\frac{\det(\Psi_{r^{-1}(t)})^{\frac{n-1}{n-2}}}{\det(\Psi_{r^{-1}(s)})^{\frac{n-1}{n-2}}} \leq C(\epsilon, K, D, n).
\]

The result follows from the Harnack inequality for $g$ (see, for instance, [10]). \hfill \(\square\)
Proof of Theorem 2.5. Let $\gamma$ be a minimizer such that $\gamma(0) = q_i$ and $\gamma(u(t))$ is contained in $B_1(p)$. A computation shows that

$$\left| \frac{d}{dt}\gamma(u(t)) \right| \leq \frac{\left| \gamma(u(t)) \right|}{g(\gamma(u(t)))^{\frac{n-3}{2}}} = g(\gamma(u(t)))^{\frac{n-3}{2}} \leq K^{\frac{n-3}{2}}.$$

Assume that $L \geq \frac{K^{\frac{n-3}{2}}}{\sqrt{\epsilon}}$. It follows from the assumptions that

$$\frac{K^{\frac{n-3}{2}}}{\sqrt{\epsilon}} \leq d(q_i, \gamma(u(t))) \leq K^{\frac{n-3}{2}}t.$$

Therefore, by Lemma 2.3, $\Delta e(\gamma(u(t))) \leq \sqrt{\epsilon} C''(K, n)$.

Let $s_k(t) = \sinh(kt)$ and

$$\varphi_{n,k}(r, l) = \int_r^l \int_t^l \left( \frac{s_k(\tau)}{s_k(t)} \right)^{n-1} d\tau dt.$$

The function $\varphi_{n,k}$ satisfies

$$\partial_r \varphi_{n,k}(r, l) = -\int_r^l \left( \frac{s_k(\tau)}{s_k(r)} \right)^{n-1} d\tau,$$

$$\partial_r^2 \varphi_{n,k}(r, l) = 1 + (n-1) \int_r^l \left( \frac{s_k(\tau)}{s_k(r)} \right)^{n-1} \left( \frac{s_k'(\tau)}{s_k(r)} \right) d\tau,$$

$$\partial_r^2 \varphi_{n,k}(r, l) + \frac{(n-1)s_k'(r)}{s_k(r)} \partial_r \varphi_{n,k}(r, l) = 1.$$

Let $\varphi_{s,n,k,\epsilon}$ be the $C^1$ function which is decreasing linearly on $[0, s]$ and equal to $\sqrt{\epsilon} C'' \varphi_{n,k}$ on $[s, l]$. It follows that $r \mapsto -\varphi_{n,k}(r, l)$ is increasing and concave. So $y \mapsto -\varphi(d(y, x), l)$ is locally semi-concave on $M - \{x\}$ (see [2]).

Let $x$ be a point in $B_1(p)$ such that $e(x) \geq \epsilon_2$ and let $h_{x,s,l,\epsilon}$ be the locally semi-convex function defined on $B_l(x) - \{x\}$ by $h_{x,s,l,\epsilon}(y) = \varphi_{s,n,k,\epsilon}(d(y, x), l)$. By choosing $\epsilon$ and $s$ small enough, we can assume that $e > h_{x,s,l,\epsilon}$ on $B_s(x)$.

On the other hand, the above computation together with the Laplacian comparison theorem shows that the followings hold in the distributional sense on $B_l(x) - B_s(x)$

$$\nabla h_{x,s,l,\epsilon}(y) = \partial_r \varphi_{n,k}(d(y, x), l) \nabla d_x,$$

$$\Delta h_{x,s,l,\epsilon}(y) = \partial_r^2 \varphi_{n,k}(d(y, x), l) + \partial_r \varphi_{n,k}(d(y, x), l) \Delta d_x \geq 1,$$
if \( k = \sqrt{\frac{\lambda}{n-2} + \frac{\epsilon}{n-1}} \). Therefore, \( \Delta (e - \sqrt{\epsilon} C' h_{x,s,t,\epsilon}) < 0 \) on \( B_t(x) - B_s(x) \). If there is a point \( y_0 \) which satisfies \( d(x, y_0) = l \) and \( e - h_{x,s,t,\epsilon} \) achieves the infimum at \( y_0 \) among all points \( y \) in \( B_t(x) - B_s(x) \), then
\[
e(y_0) - h_{x,s,t,\epsilon}(y_0) \leq e(y) - h_{x,s,t,\epsilon}(y)
\]
and
\[
\sqrt{\epsilon} C'' \phi_{n,k}(1, l) \leq \sqrt{\epsilon} C'' \phi_{n,k}(d(x, p), l) \leq h_{x,s,t,\epsilon}(p) - h_{x,s,t,\epsilon}(y_0) \leq e(p) - e(y_0) \leq e(p) < \epsilon.
\]
This gives a contradiction if \( \epsilon \) is sufficiently small and the assertion follows.

\[\square\]

3. Harmonic approximations and their estimates

Recall that \( b_i(y) = c(y, q_i) - c(p, q_i) \). In this section, we discuss the key estimates involving \( b_i \) and its harmonic approximation \( \bar{b}_i \) defined to be the harmonic function which is equal to \( b_i \) on the boundary of the ball of radius 1 centered at \( p \).

**Theorem 3.1.** Under the assumptions of Theorem 2.5, the followings hold:

1. \( |b_i - \bar{b}_i| < k_1(\epsilon) \) on \( B_1(p) \),
2. \( \frac{1}{\text{vol}(B_1(p))} \int_{B_1(p)} |\nabla (b_i - \bar{b}_i)|^2 < k_2(\epsilon) \),
3. \( |\bar{b}_i| < k_3(\epsilon) \) on \( B_{1/2}(p) \),

where \( k_i(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

Let \( f(r) = \cosh \left( r \sqrt{\frac{\lambda}{n-2}} \right) \) and \( G(r) = \int_0^r f^{1-n} \). If \( \gamma \) is a unit speed geodesic starting from \( p \), then
\[
\frac{d}{dt} b_1(\gamma(t)) \leq |\nabla \bar{b}_i|_{\gamma(t)} \leq K^{\frac{n-1}{n-2}}.
\]
Since \( \bar{b}_i \) is harmonic and equal to \( b_i \) on the boundary of \( B_1(p) \), \( \bar{b}_i \) is bounded above by \( K^{\frac{n-1}{n-2}} \). If \( K \) is chosen so that

\[
K^{\frac{n-1}{n-2}} < \int_0^\infty f^{1-n},
\]
then \( f(G^{-1}(\bar{b}_i)) \) is well-defined. Finally, we also assume that there is a constant \( v > 0 \) such that

\[
\text{vol}(B_1(p)) \geq v
\]
and the constants below could depend on \( v \).
Theorem 3.2. Let \( u = f(G^{-1}(\bar{b}_i))^{2-n} - g \). Under the assumptions of Theorem 2.5, (3.1), and (3.2),

\[
\sup_{B_1(p)} |u| \leq k(\epsilon) + \sup_{\partial B_1(p)} |u|
\]

for some positive constant \( k(\epsilon) \) which goes to 0 as \( \epsilon \rightarrow 0 \).

As a consequence,

Corollary 3.3. Under the assumptions of Theorem 3.2 and the condition that \( \sup_{\partial B_1(p)} |u| < \epsilon \),

\[
\int_{B_1(p)} |\nabla u|^2 \leq k(\epsilon)
\]

for some positive constant \( k(\epsilon) \) which goes to 0 as \( \epsilon \rightarrow 0 \).

Let \( F(r) = \int_0^r f \) and \( F_i = F(G^{-1}(\bar{b}_i)) \). Finally, we obtain the following Hessian estimates.

Theorem 3.4. Under the assumptions of Theorem 3.2,

\[
\int_{B_1^2(p)} \left| \nabla^2 F_i - \frac{\Delta F_i}{n} I \right|^2 \leq k(\epsilon)
\]

for some positive constant \( k(\epsilon) \) which goes to 0 as \( \epsilon \rightarrow 0 \).

As a consequence,

Corollary 3.5. Under the assumptions of Theorem 3.2,

\[
\int_{B_1^2(p)} \left| \nabla^2 F_i - f'(G^{-1}(\bar{b}_i)) I \right|^2 \leq k(\epsilon)
\]

for some positive constant \( k(\epsilon) \) which goes to 0 as \( \epsilon \rightarrow 0 \).

The rest of this section is devoted to the proof of the above theorems.

Proof of Theorem 3.1. We use the same notations as that of the proof of Theorem 2.5. It follows as in the proof of Theorem 2.5 that

\[
\Delta \left( b_1 - \bar{b}_1 - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} \right) < 0,
\]

\[
\Delta \left( b_0 + \bar{b}_1 + e(p) - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} \right) < 0,
\]

where \( h_{z,1}(y) = \varphi_{n,k}(d(y, z), 1) \) and \( z \) is a point outside \( B_1(p) \).

Since \( b_1 - \bar{b}_1 - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} > -\frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} \rightarrow 0 \) on the boundary of \( B_1(p) \) as \( \epsilon \rightarrow 0 \), it follows that

\[
b_1 - \bar{b}_1 \geq -c_1(\epsilon)
\]
for some $c_1(\epsilon) > 0$.

On the other hand,

\[ b_0 + \bar{b}_1 + e(p) - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} = -b_1 + \bar{b}_1 + e - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} \]

is sub-harmonic. Moreover, on the boundary of $B_1(p)$,

\[ -b_1 + \bar{b}_1 + e - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} \geq -c_2(\epsilon) \]

for some $c_2(\epsilon) > 0$.

So, by the maximum principle,

\[ b_1 - \bar{b}_1 \leq c_2(\epsilon) + e - \frac{1}{2} \sqrt{\epsilon} C'' h_{z,1} < c_3(\epsilon). \]

The first assertion follows.

The function $b_1$ is locally semi-concave. Let $\Delta_D b_1$ denotes its distributional Laplacian which is a measure (see [6]). Let $\Delta b_1$ be the absolutely continuous part of $\Delta_D b_1$. Since the singular part of $\Delta_D b_1$ is non-positive,

\[ -\int_{B_1(p)} \Delta b_1 \leq -\int_{B_1(p)} \Delta_D b_1 \leq \text{vol}(\partial B_1(p)) \leq C_1(\epsilon) \text{vol}(B_1(p)) \]

for some constant $C_1 > 0$.

On the other hand, if $\Delta b_1 = (\Delta b_1)_+ - (\Delta b_1)_-$, where $(\Delta b_1)_+$ and $(\Delta b_1)_-$ are the positive and negative parts of $\Delta b_1$, respectively, then

\[ \int_{B_1(p)} (\Delta b_1)_- \leq C_1(\epsilon) \text{vol}(B_1(p)) + \int_{B_1(p)} (\Delta b_1)_+ \]

\[ \leq C_1(\epsilon) \text{vol}(B_1(p)) + \epsilon_3 \int_{B_1(p) \cap \Delta b_1 > 0} 1 \leq C_2(\epsilon) \text{vol}(B_1(p)). \]

It follows that

\[ \frac{1}{\text{vol}(B_1(p))} \int_{B_1(p)} |\Delta b_1| \leq C_2(\epsilon). \]

Since $|\nabla b_1| = g^{\frac{n-1}{2}}$ a.e. and $\bar{b}_1 - b_1$ vanishes on the boundary of $B_1(p)$,

\[ \int_{B_1(p)} |\nabla (\bar{b}_1 - b_1)|^2 = \int_{B_1(p)} \Delta_D b_1 (\bar{b}_1 - b_1) \]

\[ = \int_{B_1(p)} \Delta_D b_1 (\bar{b}_1 - b_1 + \epsilon_2 + \epsilon_3) - (\epsilon_2 + \epsilon_3) \int_{B_1(p)} \Delta_D b_1 \]

\[ \leq (\epsilon_2 + 2 \epsilon_3) \int_{B_1(p)} |\Delta b_1| + (\epsilon_2 + \epsilon_3) C_1(\epsilon) \text{vol}(B_1(p)) \]

\[ \leq (\epsilon_2 + 2 \epsilon_3) C_3(\epsilon) \text{vol}(B_1(p)). \]
By the first assertion and $|b_0 + b_1 - \bar{b}_0 - \bar{b}_1| < k_1(\epsilon)$. By the gradient estimate for harmonic functions \[10\] and Theorem 2.5, $|\nabla \bar{b}_0 + \nabla \bar{b}_1| \leq C(\epsilon)|\bar{b}_0 + \bar{b}_1| < k_2(\epsilon)$ on $B_{1/2}(p)$.

Proof of Theorem 3.2. A computation shows that
\[
\Delta f(G^{-1}(\bar{b}_i))^{2-n} = -\lambda f(G^{-1}(\bar{b}_i))^n|\nabla \bar{b}_i|^2
\]
and so
\[
\Delta(f(G^{-1}(\bar{b}_i))^{2-n} - g) = -\lambda f(G^{-1}(\bar{b}_i))^n(|\nabla \bar{b}_i|^2 - g^{\frac{2n-2}{n-2}})
\]
\[
+ \lambda f(G^{-1}(\bar{b}_i))^n g\frac{(f(G^{-1}(\bar{b}_i))^{2-n})^{\frac{n}{n-2}} - g^{\frac{n}{n-2}}}{f(G^{-1}(\bar{b}_i))^{2-n} - g} (f(G^{-1}(\bar{b}_i))^{2-n} - g).
\]

By Theorem 3.1,
\[
\int_{B_1(p)} \lambda^p f(G^{-1}(\bar{b}_i))^{pn}(|\nabla \bar{b}_i|^2 - g^{\frac{2n-2}{n-2}})^p
\]
\[
\leq \lambda^p \int_{B_1(p)} f(G^{-1}(\bar{b}_i))^{pn}(|\nabla \bar{b}_i| - g^{\frac{1}{n-2}})^2(|\nabla \bar{b}_i| + g^{\frac{1}{n-2}})^{2p-2}
\]
\[
\leq k(\epsilon)\text{vol}(B_1(p)).
\]

We also have
\[
0 \leq \lambda f(G^{-1}(\bar{b}_i))^n g\frac{(f(G^{-1}(\bar{b}_i))^{2-n})^{\frac{n}{n-2}} - g^{\frac{n}{n-2}}}{f(G^{-1}(\bar{b}_i))^{2-n} - g} \leq C.
\]

Therefore, the function $u = f(G^{-1}(\bar{b}_i))^{2-n} - g$ satisfies a differential equation of the form
\[
\Delta u = F_1 + F_2 u
\]
where $F_1$ satisfies $\frac{1}{\text{vol}(B_1(p))} \int_{B_1(p)} F_1 < k(\epsilon)$ for each fixed $p \geq 1$ and $F_2$ is non-negative and bounded by a constant depending on $\lambda$, $K$, and $n$. Here $k(\epsilon) \to 0$ as $\epsilon \to 0$.

An argument using Moser iteration as in \[7, Theorem 8.16\] using the Sobolev inequality \[10, Theorem 14.2\] gives the result.

Proof of Theorem 3.4. Let $F_i = h(\bar{b}_i)$ and $h = F \circ G^{-1}$. A computation using Bochner formula shows that
\[
\frac{1}{2} \Delta (h'(\bar{b}_i)^2 v) \geq \left| \nabla^2 F_i - \frac{\Delta F_i}{n} \right|^2 + \left( \frac{h''(\bar{b}_i)^2}{n} + h'(\bar{b}_i)h'''(\bar{b}_i) \right) |\nabla \bar{b}_i|^2 v \\
+ h'(\bar{b}_i)h''(\bar{b}_i) \left< \nabla \bar{b}_i, \nabla v \right> - \frac{\lambda(n-1)}{n-2} h'(\bar{b}_i)^2 v - \epsilon h'(\bar{b}_i)^2 |\nabla \bar{b}_i|^2 v \\
- \frac{n(n-1)h'(\bar{b}_i)^2 g^{2n-2}}{(n-2)^2} |\nabla u + (n-2)f'(G^{-1}(\bar{b}_i))f(G^{-1}(\bar{b}_i))^{-2} u \nabla \bar{b}_i|^2,
\]

where \( u = f(G^{-1}(\bar{b}_i))^{2-n} - g \) and \( v = |\nabla \bar{b}_i|^2 - g^{\frac{2n-2}{n-2}} \).

By multiplying the above inequality by a cut-off function, which equals to 1 on \( B_{\frac{1}{2}}(p) \) and supported in \( B_1(p) \), integrating over \( B_1(p) \), and applying Theorem 3.2 Corollary 3.3 and Theorem 3.1, the result follows.

Proof of Corollary 3.5. Let \( H(r) = r^{\frac{2n-2}{n-2}} \)
\[
\Delta F_i = h''(\bar{b}_i)v + nf'(G^{-1}(\bar{b}_i)) \\
+ nf(G^{-1}(\bar{b}_i))^{2n-2}f'(G^{-1}(\bar{b}_i))(H(g) - H(\bar{f}(G^{-1}(\bar{b}_i))^{2-n}))
\]

The result follows from this, Theorem 3.4, Theorem 3.1 and Theorem 3.2.

4. Distance Estimate

In this section, we prove that the distance function is close to that of the model space using the estimates obtained from the previous sections. More precisely, let \( x_m, y_m, z_m \) be three points in the unit ball \( B_{1,m}(p) \) of the model space such that \( c_m(x, z) = b_{1,m}(z) - b_{1,m}(x) \). Here \( c_m \) and \( b_{1,m} \) denote the functions \( c \) and \( b_1 \) in the case of the model. For the rest of this work, the subscript \( m \) will be reserved for quantities in the model space.
Theorem 4.1. Let \( x, y, z \) be points in the ball \( B_1(p) \) such that \( b_1(z) - b_1(x) = c(x, z) = c_m(x_m, z_m) \), \( b_1(x) = b_{1,m}(x_m) \), \( b_1(y) = b_{1,m}(y_m) \), and \( b_1(z) = b_{1,m}(z_m) \). Then, under the assumptions of Theorem 3.2, that the sectional curvature on \( B_1(p) \) is bounded by \( C \lambda \) for some positive constant \( C \), and that the maximum \( K \) of \( g \) is achieved in \( B_1(p) \), there is a constant \( k(\epsilon) > 0 \) such that \( k(\epsilon) \to 0 \) as \( \epsilon \to 0 \) and \( |d(y, z) - d_m(y_m, z_m)| < k(\epsilon) \).

In the above theorem, \( k(\epsilon) \) depends on the bounds of the sectional curvature which is suppressed.

For the proof of Theorem 4.1, we need the following lemmas.

Lemma 4.2. Suppose that the assumptions of Theorem 4.1 hold. Let

\[
Q_i(y, z) = \left| (\nabla d_y, \nabla F_i)_z + \frac{F_i(z) - F_i(y)}{d(y, z)} \right|
\]

\[
+ \frac{1}{d(y, z)} \int_0^{d(y, z)} \int_s^{d(y, z)} f'(G^{-1}(\tilde{b}_i(y, z)(s'))) \, ds' \, ds,
\]

Then there is a constant \( k(\epsilon) > 0 \) such that

\[
\text{vol}(V_1) > (1 - k(\epsilon)^{1/2}) \text{vol}(B_{1/2}(p)),
\]

where

\[
D_1(y, y_0) = \left\{ y_1 \in B_r^c(p) \left| \int_0^{r(y_0, y_1)} Q_i(y, \gamma_{y_0, y_1}(t)) \, dt < k(\epsilon)^{1/6} \right\} \right.,
\]

\[
Q_1(y) = \{ y_0 \in B_r^c(p) \left| \text{vol}(D_1(y, y_0)) \geq (1 - k(\epsilon)^{1/6}) \text{vol}(B_r^c(p)) \right\},
\]

\[
V_1 = \{ y \in B_1(p) \left| \text{vol}(Q_1(y)) \geq (1 - k(\epsilon)^{1/6}) \text{vol}(B_r^c(p)) \right\},
\]

\( k(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

Here \( B_r^c(p) \) denotes the ball of radius \( r = cK^{-n-1}8 \) centered at \( p \) defined using the cost function \( c \).

Let \( U_0 \) and \( U_1 \) be two subsets of \( M \). Let \( U \) be the set of points of the form \( \gamma(t) \), where \( \gamma \) is a minimizer of (1.1) which starts from a point in \( U_0 \) and ends at \( U_1 \). Let \( \gamma_{y_0, y_1}(t) \) be the minimizer of (1.1) connecting \( y_0 \) and \( y_1 \) which is defined Lebesgue almost everywhere on \( M \times M \). The proof of the following lemma can be proved using Lemma 2.4 and the arguments in 3.1, Theorem 2.11.
Lemma 4.3. Assume that $P$ is a non-negative measurable function on $M$. Then
\[ \int_{U_0 \times U_1} \int_0^{\tau(x,y)} P(\gamma_{x,y}(s)) \, ds \, dx \, dy \leq C(\varepsilon, D)(T(U_0, U_1)\vol(U_0) + T(U_1, U_0)\vol(U_1)) \int_U P, \]
where $\gamma_{y_0, y_1}$ is the minimizer connecting $y_0$ and $y_1$, $\tau(y_0, y_1)$ is the length of $\gamma_{y_0, y_1}$, $T(\gamma_{y_0, y_1}(0))$ is the set of time $t$ such that $\Psi_{r^{-1}(t)}(v) \in U_1$ and $s \in [0, t] \mapsto \Psi_{r^{-1}(s)}(v)$ is a minimizer, $T(U_0, U_1) = \sup_{x \in U_0, |v| = 1} |T(v)|$, and $D = \sup_{x_i \in U_i} \tau(x_0, x_1)$.

We will need the following lemma which can be proved using an argument in [5].

Lemma 4.4. Let $\Psi_t(x, v)$ be the geodesic flow defined on the unit tangent bundle $SM$. Assume that $P$ is a non-negative measurable function on the unit tangent bundle $SM$ and $\frac{1}{\vol(SU)} \int_{SU} \sup_{|v| = 1} P(x, v) \, dx < \epsilon$, where $U$ is a subset of $M$ and $B_1(U)$ is a neighborhood of $U$ of radius $l$. Then
\[ \frac{1}{l \vol(SU)} \int_{SU} \int_0^l P(\Psi_t(x, v)) \, dt \, dx \, dv < \epsilon. \]

Similar fact holds for the Hamiltonian flow $\Psi^c_t$ of the Hamiltonian $H(x, v) = |v|_x - g(x) \frac{n-1}{n-2}$ restricted to the set $S^cU = \{(x, v) \in TU | H(x, v) = 0\}$.

Lemma 4.5. Assume that $P$ is a non-negative measurable function on the unit tangent bundle $SM$ and $\frac{1}{\vol(S^cU)} \int_{S^cU} \sup_{|v| = 1} P(x, v) \, dx < \epsilon$, where $U$ is a subset of $M$ and $B_l(U)$ is a neighborhood of $U$ of radius $l$. Then, there is a constant $C > 0$ such that
\[ \frac{1}{l \vol(S^cU)} \int_{S^cU} \int_0^l P(\Psi^c_t(x, v)) \, dt \, dx \, dv < C\epsilon. \]

The rest of this section is devoted to the proofs.

Proof of Lemma 4.2. Since $c(p, y) \leq K \frac{n-1}{n-2} d(p, y)$, $B_r(p)$ is contained in $B^c_{\frac{n-1}{K \frac{n-1}{n-2}}} (p)$. On the other hand, by the Harnack inequality, $g^{\frac{n-1}{n-2}} \geq C(\lambda, n, \epsilon) K^{\frac{n-1}{n-2}}$ on $B_4(p)$. It follows that $B^c_r(p)$ is contained in $B_{\frac{cr}{\lambda}}(p)$.

Let $\gamma_{y_0, y_1}$ and $\tilde{\gamma}_{y_0, y_1}$ denote the minimizer and geodesic which connect $y_0$ and $y_1$, respectively, which are well-defined Lebesgue almost
everywhere. It follows that
\[ Q(y, z) = \frac{1}{t} \int_0^t - \langle \nabla F_i(\hat{\gamma}_{y,z}(t)), \dot{\hat{\gamma}}_{y,z}(t) \rangle \\
+ \langle \nabla F_i(\hat{\gamma}_{y,z}(s)), \dot{\hat{\gamma}}_{y,z}(s) \rangle + \int_s^t f'(G^{-1}(b(\hat{\gamma}_{y,z}(s')))) ds' ds \\
= -\frac{1}{t} \int_0^t \int_s^t \langle \nabla^2 F_i(\hat{\gamma}_{y,z}(s')) \dot{\hat{\gamma}}_{y,z}(s'), \dot{\hat{\gamma}}_{y,z}(s') \rangle - f'(G^{-1}(b(\hat{\gamma}_{y,z}(s')))) ds' ds \\
\leq \int_0^t |\nabla^2 F_i - f'(G^{-1}(\bar{b}_i))|_{\gamma_{y,z}(s)} ds.
\]

Therefore, by Lemma 4.3 and Corollary 3.5
\[ \frac{1}{\text{vol}(B_{c_i}(p))^2} \int_{B_{c_i}(p)^2} \int_{B_{\frac{c_i}{4}(p)^2}} \int_{\tau(y_0,y_1)} Q(y, \gamma_{y_0,y_1}(t)) \, dt \, dy_0 \, dy_1 \\
\leq \frac{C_1}{\text{vol}(B_{\frac{c_i}{4}}(p))^2} \int_{B_{\frac{c_i}{4}}(p)^2} Q(y, z) \, dy \, dz \\
\leq \frac{C_2}{\text{vol}(B_{\frac{c_i}{2}}(p))} \int_{B_{\frac{c_i}{2}}(p)} |\nabla^2 F_i - f'(G^{-1}(\bar{b}_i))|_{\gamma_{y,z}(s)} \, dz < k(\epsilon).
\]
The assertion follows.

Proof of Theorem 4.1. Recall that \( c_i(\epsilon) \) and \( k_i(\epsilon) \) are positive continuous functions of \( \epsilon \) which \( \to 0 \) as \( \epsilon \to 0 \). The ball \( B_{R_0}(p) \) with \( R_0 = \frac{r}{K^{n-2}} \) is contained in the ball \( B_{c_i}(p) \). By Theorem 4.2, Lemma 4.3, Theorem 3.1 the volume comparison theorem, and the assumption (3.2), there are positive \( c_i(\epsilon) \to 0 \) as \( \epsilon \to 0 \) \((i = 1, 2, 3)\), such that the followings hold: given any three points \( x, y, z \) in \( B_{R_0}(p) \) there are points \( x', y', z' \) in \( B_{c_1(\epsilon)}(x), B_{c_2(\epsilon)}(y), B_{c_3(\epsilon)}(z) \), respectively, such that

small1 (4.1) \[ \int_0^{\tau(x',z')} |\nabla b_i - \nabla \bar{b}_i|_{\gamma_{x',z'}(t)}^2 \, dt < k_1(\epsilon), \]

small2 (4.2) \[ \int_0^{d(x',z')} Q(y', \gamma_{x',z'}(t)) \, dt < k_2(\epsilon). \]

small3 (4.3) \[ \int_0^{d(x',z')} |\nabla^2 F_i - f'(G^{-1}(\bar{b}_i))|_{\gamma_{x',z'}(t)} \, dt < k_3(\epsilon), \]
(4.4) \[ \int_0^{\tau(x',z')} |\nabla^2 \mathcal{F}_i - f'\left(G^{-1}(\tilde{b}_i)\right)|_{\gamma_{x',z'}(t)} dt < k_4(\varepsilon). \]

Let us fix a time \( s \) in \([0, \tau(x',z')]\) and let \((y',v')\) be the tangent vector such that \(\Psi_t(y', v') = \tilde{\gamma}_{x',z'}(s)\). By Corollary 3.5 and Lemma 4.4, there is a point \((y'',v'')\) in \(B_{c_4(\varepsilon)}(y',v')\) such that

\[ \int_0^{1/4} \left| U''(t) - \frac{\lambda}{n-2} U(t) \right| dt \]

\[ \leq \int_0^{1/4} |\nabla^2 \mathcal{F}_i - f'\left(G^{-1}(\tilde{b}_i)\right)|_{\tilde{\gamma}_{y'',v''}(t)} dt < k_4(\varepsilon) \]

where \(U(t) = F\left(G^{-1}(\tilde{b}_i(\Psi_t(y'', v'')))\right)\). Here the ball \(B_{c_4(\varepsilon)}(y',v')\) is defined by the distance on the unit tangent bundle \(SM\) induced by the Riemannian metric on \(M\) and its Levi-Civita connection.

Let \(\bar{U}_{a_1,a_2,T}\) be the solution of the equation

(4.5) \[ \bar{U}''(t) = \frac{\lambda}{n-2} \bar{U}(t) \]

with boundary conditions \(\bar{U}(0) = a_1\) and \(\bar{U}(T) = a_2\).

Since the sectional curvature is bounded and the two points \((y',v')\) and \((y'',v'')\) are \(c_4(\varepsilon)\)-close, \(\Psi_t(y', v')\) and \(\Psi_t(y'', v'')\) are \(k_8(\varepsilon)\)-close. It follows from an argument using Gronwall’s inequality that \(|U(t) - \bar{U}_{U(0),U(t)}(t)| < k_5(\varepsilon)\) and \(|U'(t) - \bar{U}'_{U(0),U(t)}(t)| < k_6(\varepsilon)\). It also follows that

(4.6) \[ |\bar{U}_{U(0),U(t)}(t) - F\left(G^{-1}(\tilde{b}_i(\tilde{\gamma}_{y',\tilde{\gamma}_{x',z'}(t)})\right))| < k_7(\varepsilon). \]

By the same argument applied to (4.3), we also have

(4.7) \[ |\bar{U}_{F_i(x'),F_i(z'),\tau(x',z')}(t) - F_i(\gamma_{x',z'}(t))| < k_8(\varepsilon) \]

and

(4.8) \[ |\bar{U}_{F_i(x'),F_i(z'),\tau(x',z')}(t) - \left\langle \nabla F_i(\gamma_{x',z'}(t)), \tilde{\gamma}_{x',z'}(t) \right\rangle| < k_9(\varepsilon). \]

By Theorem 3.2,

\[ \frac{d}{dt} \left(G^{-1}(\tilde{b}_i(\gamma_{x,z}(t))) - t\right) \]

\[ = \frac{\left|\nabla \tilde{b}_i(\gamma_{x,z}(t))\right|}{G'(G^{-1}(\tilde{b}_i(\gamma_{x,z}(t))))} - 1 \]

\[ < k_3(\varepsilon). \]

It follows from this and Theorem 3.1 that

(4.9) \[ |G^{-1}(\tilde{b}_i(\gamma_{x,z}(t))) - G^{-1}(\tilde{b}_i(x)) - t| < k_{10}(\varepsilon)t \]
and so

\[ |F_i(\gamma_{x',z}(t)) - F(G^{-1}(\bar{b}_i(x)) + t)| < k_{11}(\epsilon)t. \]

Since \( F(G^{-1}(\bar{b}_i(x)) + t) \) is a solution of (4.5), and by (4.10), the boundary values are close to \( F_i(x') \) and \( F_i(z') \), it follows from (4.7) and (4.8) that

\[ |F^{-1}(\bar{b}_i(x')) + t) - F_i(\gamma_{x',z'}(t))| < k_{12}(\epsilon), \]

and

\[ |f(G^{-1}(\bar{b}_i(x')) + t) - \langle \nabla F_i(\gamma_{x',z}(t)), \gamma_{x',z}(t)\rangle| < k_{13}(\epsilon). \]

By combining (4.11) and (4.12),

\[ |f(G^{-1}(\bar{b}_i(x')) + t) - \langle \nabla F_i(\gamma_{x',z}(t)), \gamma_{x',z}(t)\rangle| < k_{14}(\epsilon). \]

It follows from this and (4.2) that

\[ \left\| \nabla \bar{b}_i|_{\gamma_{x',z}(t)} - \left\langle \nabla \bar{b}_i|_{\gamma_{x',z}(t)}, \gamma_{x',z}(t)\right\rangle \right\| < k_{15}(\epsilon). \]

Therefore, by (4.1),

\[ \int_0^{\tau(x',z')} \left\| \nabla \bar{b}_i|_{\gamma_{x',z}(t)} - \left\langle \nabla \bar{b}_i|_{\gamma_{x',z}(t)}, \gamma_{x',z}(t)\right\rangle \right\| dt < k_{16}(\epsilon). \]

It follows that

\[ \int_0^{\tau(x',z')} \left| \dot{\gamma}_{x',z}(t) - \frac{\nabla \bar{b}_i|_{\gamma_{x',z}(t)}}{\left\| \nabla \bar{b}_i|_{\gamma_{x',z}(t)} \right\|} \right| dt < k_{17}(\epsilon). \]

Let

\[ G(t, r_1, r_2, l) = \frac{F(r_1 + t) - F(r_2)}{l f^n(r_1 + t)} + \frac{1}{l f^n(r_1 + t)} \int_0^l \int_s^l f'(F^{-1}(\bar{U}(s'))) ds' ds. \]

Here the dependencies of \( \bar{U} \) on \( r_1 \) and \( \tau(x, z) \) are suppressed.

Assume that the distance from \( \gamma \) to \( y \) is greater than \( c_3(\epsilon) \). It follows from (4.2), (4.6), (4.11), and (4.13) that if \( c_5 \) is appropriately chosen, then

\[ \int_0^{\tau(x',z')} \left| L'(t) + G(t, G^{-1}(\bar{b}_i(x'))), G^{-1}(\bar{b}_i(y')), L(t)\right| dt < k_{18}(\epsilon). \]

Let \( L_m \) be the corresponding quantity in the model. It follows that

\( L'_m(t) + G(t, G^{-1}(\bar{b}_i(x'))), G^{-1}(\bar{b}_i(y')), L_m(t) = 0 \) and
\[ |L(t) - L_m(t)| \leq \int_0^t |L'(s) - L'_m(s)| ds \]
\[ \leq \int_0^t |L'(s) - G(s, L(s)) + G(s, L(s)) - G(s, L_m(s))| ds \]
\[ \leq k_{11}(\varepsilon) + C \int_0^t |L(s) - L_m(s)| ds \]

It follows from Gronwall's inequality that \(|L(t) - L_m(t)| < k_{12}(\varepsilon)|.

Suppose the distance from \(\gamma\) to \(y\) is less than \(c_5(\varepsilon)\). In this case, it is enough to show that \(|\tau(x, z) - d(x, z)| < k_{13}(\varepsilon)|. Since \(\tau(x, z)\) is the length of the minimizer \(\gamma_{x,z}\), \(\tau(x, z) \geq d(x, z)\).

It also follows from Theorem 3.2, (4.10), and (4.11) that
\[ \int_{\tau(x', z')}^{\tau(x', z')} f(G^{-1}(\bar{b}_i(\gamma_{x', z'}(t))))^{2-n} dt \]
\[ = c(x', z') \leq \int_0^{d(x', z')} g(\bar{\gamma}_{x', z'}(t)) dt \]
\[ \leq k(\varepsilon) + \int_0^{d(x', z')} f(G^{-1}(\bar{b}_i(\gamma_{x', z'}(t))))^{2-n} dt \]
\[ \leq \bar{k}(\varepsilon) + \int_0^{d(x', z')} f(G^{-1}(\bar{b}_i(\gamma_{x', z'}(t))))^{2-n} dt \]

It follows that \(|d(x, z) - \tau(x, z)| < k(\varepsilon)| as claim.

\[ \Box \]

5. Proof of Theorem 1.1

In this section, we finish the proof of Theorem 1.1. By scaling, it is enough to consider the case when \(R = 1\). Let \(R_0 > 0\) be a small enough constant such that all the estimates in the previous sections hold on \(B_{R_0}(p)\).

By the third assertion of Theorem 3.1 we can find a subset \(W = \{x_1, \ldots, x_N\}\) in \(B_{R_0}(p)\) which is \(c_1(\varepsilon)\)-dense. Suppose that \(b_1(x_i) > 0\). Let \(y_i\) be the point in \(b_1^{-1}(0)\) such that \(b_1(x_i) - b_1(y_i) = c(x_i, y_i)\). Let \(x_{i,m}\) be the point in the model \(\mathbb{R} \times f b_1^{-1}(0)\) such that \(b_m(x_i) - b_m(y_i) = c(x_i, y_i)\) \((b_1 = -b_0\) in the case of the model and it is denoted by \(b_m\)). If \(b_1(x_1) < 0\) instead, then one can move \(x_i\) along the flow of \(-\nabla b_0\) (recall that \(b_0\) is differentiable at \(x_i\)). It follows that there is a point \(y_i\) in \(b_1^{-1}(0)\) such that \(b_0(x_i) - b_0(y_i) = c(x_i, y_i)\).
It remains to show that \( \{x_1, \ldots, x_N, m\} \) is \( c_2(\epsilon) \)-dense in \( B_{R_0, m}(p) \). It follows from this, Theorem 4.1, and \([15, 10.1.1]\) that \( B_{R_0, m}(p) \) and \( B_{R_0, m}(p) \) are \( c_3(\epsilon) \)-close in the Gromov-Hausdorff distance.

Let \( x_m = (b_m(x_m), y_m) \) be a point in \( B_{R_0, m}(p) \) of the warped product model. Assume that \( b_m(x_m) > 0 \). By assumption \( b_1(y_m) = 0 \). Let \( y \) be a point in \( W \) which is \( c_1(\epsilon) \)-close to \( y_m \) in \( M \). Let \( x \) be the point in \( M \) which satisfies \( b_0(y) - b_0(x) = b_m(x_m) = c(x, y) \). Let \( x_i \) be a point in \( W \) which is \( c_1(\epsilon) \)-close to \( x \). By applying Theorem 4.1 twice, it follows that \( x_{i,m} \) is \( c_2(\epsilon) \)-close to \( x_m \) as claimed. Similar procedure works if \( b_m(x_m) < 0 \).

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