Metric spaces in which many triangles are degenerate

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Abstract

Richmond and Richmond (American Mathematical Monthly 104 (1997), 713–719) proved the following theorem: If, in a metric space with at least five points, all triangles are degenerate, then the space is isometric to a subset of the real line. We prove that the hypothesis is unnecessarily strong: In a metric space on \( n \) points, \( \binom{n}{3} - n + 5 \) arbitrarily placed or \( 3\binom{n-2}{2} + 1 \) suitably placed degenerate triangles suffice.

1 Results.

Given a metric space \((V, \text{dist})\), we follow [1] in writing \([rst]\) to signify that \( r, s, t \) are pairwise distinct points of \( V \) and \( \text{dist}(r, s) + \text{dist}(s, t) = \text{dist}(r, t) \). Following [15], we refer to three-point subsets of \( V \) as triangles; if \([rst]\), then the triangle \( \{r, s, t\} \) is called degenerate.

Now let \((V, \text{dist})\) be a metric space. Trivially, if there is a linear order \( \preceq \) on \( V \) such that \( r \prec s \prec t \Rightarrow [rst] \), then all triangles in \( V \) are degenerate. Richmond and Richmond [15] proved the converse under a mild lower bound on \(|V|\):

**Theorem 1 ([15]).** Let \((V, \text{dist})\) be a metric space such that \(|V| \geq 5\). If all triangles in \( V \) are degenerate, then there is a linear order \( \preceq \) on \( V \) such that \( r \prec s \prec t \Rightarrow [rst] \).
Here, the lower bound on $|V|$ cannot be reduced: consider $V = \{a, b, c, d\}$ and $\operatorname{dist}(a, b) = \operatorname{dist}(b, c) = \operatorname{dist}(c, d) = \operatorname{dist}(d, a) = 1$, $\operatorname{dist}(a, c) = \operatorname{dist}(b, d) = 2$.

The purpose of this note is to prove that the hypothesis of Theorem 1 can be relaxed as soon as $|V| = 6$ and that it can be relaxed further and further as $|V|$ gets larger and larger. To state these results, let us call a set $E$ of three-point subsets of a set $V$ an anchor in $V$ if, for every metric space $(V, \operatorname{dist})$, the assumption that all triangles in $E$ are degenerate implies a linear order $\preceq$ on $V$ such that $r \prec s \prec t \Rightarrow [rst]$. In this terminology, Theorem 1 asserts that whenever $|V| = n \geq 5$, the set of all $\binom{n}{3}$ three-point subsets of $V$ is an anchor in $V$.

**Theorem 2.** If $|V| = n$, then every set of $\binom{n}{3} - n + 5$ three-point subsets of $V$ is an anchor in $V$.

The $\binom{n}{3} - n + 5$ in Theorem 2 cannot be replaced by $\binom{n}{3} - n + 4$. To see this, consider the graph with vertices $1, 2, \ldots, n$ and edges $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$ and $\{i, i + 1\}$ with $i = 4, 5, \ldots, n - 1$. In the metric space induced by this graph (in the usual way, where edges have unit lengths), the only nondegenerate triangles are $\{1, 2, i\}$ with $i = 5, 6, \ldots, n$.

**Theorem 3.** If $|V| = n \geq 5$, then there is an anchor in $V$ consisting of $3\binom{n-2}{2} + 1$ three-point subsets of $V$.

We do not know whether or not the $3\binom{n-2}{2} + 1$ in Theorem 3 can be reduced.

## 2 Proofs.

Our arguments involve the following algorithm that, given a set $E$ of triangles in $V$, produces a certain sequence $T_1, T_2, \ldots, T_m$ of pairwise distinct triangles outside $E$:

Set $m = 0$.

While some six-point subset $S$ of $V$ contains precisely 19 members of $E \cup \{T_i : 1 \leq i \leq m\}$, increment $m$ by one and then let $T_m$ be the 20th three-point subset of $S$.

In an iteration of this algorithm, more than one $S$ may be available, and so there may be more than one candidate for $T_m$. Nevertheless, candidates that are rejected now remain available in the next iteration and so, when the
If the algorithm terminates, the set \( \{ T_i : 1 \leq i \leq m \} \) is uniquely determined. We let \( \text{cl} \mathcal{E} \) denote its union with \( \mathcal{E} \). The role of this notion is explained by the following lemma.

**Lemma 4.** Let \((V, \text{dist})\) be a metric space and let \( \mathcal{E} \) be a set of triangles in \( V \). If all triangles that belong to \( \mathcal{E} \) are degenerate, then all triangles that belong to \( \text{cl} \mathcal{E} \) are degenerate.

Most of the work involved in proving Lemma 4 is subsumed in its following special case:

**Lemma 5.** Let \((V, \text{dist})\) be a metric space such that \(|V| = 6\). If 19 triangles in \( V \) are degenerate, then all 20 triangles in \( V \) are degenerate.

**Proof.** By assumption, there is a triangle \( T \) in \( V \) such that the 19 triangles distinct from \( T \) are degenerate. In particular, for each of the three elements \( \lambda \) of \( T \), all triangles in \( V - \{ \lambda \} \) are degenerate and so, by Theorem 1 there is a linear order \( \preceq_\lambda \) on \( V - \{ \lambda \} \) such that
\[
 r \prec_\lambda s \prec_\lambda t \Rightarrow [rst].
\] (1)

Having chosen a \( \mu \) in \( T \), let us label the elements of \( V - T \) as \( u, v, w \) in such a way that \( u \prec_\mu v \prec_\mu w \), and so \([uvw]\). Now for each of the remaining two elements \( \nu \) of \( T \), property (1) implies \( u \prec_\nu v \prec_\nu w \) or \( w \prec_\nu v \prec_\nu u \); reversing the orders if necessary, we may assume that \( u \prec_\nu v \prec_\nu w \), and so
\[
 u \prec_\lambda v \prec_\lambda w \text{ for all three } \lambda \text{ in } T. \quad (2)
\]

Next, let us label the elements of \( T \) temporarily as \( a, b, c \) in such a way that \( a \prec_c b \) and then permanently as \( x, y, z \): If \( b \prec_a c \), then \( x = a, y = b, z = c \); else either \( x = a, y = c, z = b \) (in case \( a \prec_b c \)) or \( x = c, y = a, z = b \) (in case \( c \prec_b a \)). Now we have
\[
 x \prec_z y \quad \text{and} \quad y \prec_x z. \quad (3)
\]

For future reference, let us note that
\[
 \text{the restrictions of } \leq_x \text{ and } \leq_z \text{ on } \{ u, v, w, y \} \text{ are identical.} \quad (4)
\]
(Analogous statements apply to \( x, y \) in place of \( x, z \) and to \( y, z \) in place of \( x, z \), but we will not need these variations.) To see this, observe that, by virtue of (2) and (1), our metric space determines the rank of \( y \) in the restriction of \( \leq_x \) on \( \{ u, v, w, y \} \) in the same way as it determines the rank of \( y \) in the restriction of \( \leq_z \) on \( \{ u, v, w, y \} \).
The remainder of the proof relies on the fact that

\[ [\alpha\beta\gamma] \text{ and } [\alpha\gamma\delta] \implies [\alpha\beta\delta] \text{ and } [\beta\gamma\delta], \]

(5)

which has been pointed out by Menger [13]. (By the way, extensions of (5) are discussed in [6, Section 6].)

Finally, we are ready to prove that the triangle \{x, y, z\} is also degenerate. Actually, we are going to prove \[xyz\]. For this purpose, let us distinguish between three cases.

**Case 1:** \( u \preceq z \). By (3), we have \( u \preceq x \preceq z \), and so \([uxy]\); by (4) and (3), we have \( u \preceq x \preceq y \preceq x \), and so \([uyz]\). Now we have \([xyz]\) by (5).

**Case 2:** \( z \preceq x \). By (3), we have \( y \preceq x \preceq z \), and so \([yzw]\); by (3) and (4), we have \( x \preceq z \preceq w \), and so \([xyw]\). Now we have \([xyz]\) by (5).

**Case 3:** \( x \preceq u \preceq z \) and \( u \preceq x \preceq w \). In particular,

\([xuv],[xvw],[vwz],[uvz]\).

Here we have neither \([vxz]\) (else \([uvz]\) and (5) would imply \([uvx]\), contradicting \([xuv]\)) nor \([xvz]\) (else \([xvw]\) and (5) would imply \([zvw]\), contradicting \([vwz]\)); since \(\{x, v, z\}\) is degenerate, we must have \([xvz]\).

If \(v \preceq y\), then \([vyz]\) by (3); else \(y \preceq v\) by (4), and so \([xyv]\) by (5); in either case, this implies \([xyz]\) by \([xvz]\) and (5).

**Proof of Lemma 4.** By definition, members of \(\text{cl} \mathcal{E} - \mathcal{E}\) can be enumerated as \(T_1, T_2, \ldots T_m\) in such a way that, for each \(i = 1, 2, \ldots , m\), some six-point subset \(S_i\) of \(V\) contains precisely 19 members of \(\mathcal{E} \cup \{T_1, T_2, \ldots T_{i-1}\}\) and \(T_i\). Induction on \(i\) (= 1, 2, \ldots m) shows that all triangles belonging to \(\mathcal{E} \cup \{T_1, T_2, \ldots T_i\}\) are degenerate; the induction step relies on Lemma 5.

The remaining proofs fit the following framework. Berge [3] defined a 3-uniform hypergraph as an ordered pair \((V, \mathcal{E})\) such that \(V\) is a set and \(\mathcal{E}\) is a set of three-point subsets of \(V\); elements of \(V\) are called vertices and elements of \(\mathcal{E}\) are called hyperedges. Bollobás [4] coined the term \(m\)-saturated to designate certain graphs and hypergraphs and later [5] he introduced a related notion of weakly \(m\)-saturated graphs (which applies to hypergraphs, too). In the established terminology with notation far from unified, a 3-uniform hypegraph \((V, \mathcal{E})\) is said to be weakly \(K_6^3\)-saturated [8, p. 97] or
weakly $K_6^3$-saturated \cite{14} p. 484 or weakly $K_6^{(3)}$-saturated \cite{10} if and only if $cl \mathcal{E}$ consists of all three-point subsets of $V$. We will adopt the notation of \cite{14}. The following lemma is reminiscent of the theorem asserting that all weakly $(d+2)$-saturated graphs are rigid \cite{12} Theorem 1, where “rigid” has a geometric meaning that pertains to embedding these graphs into $\mathbb{R}^d$.

**Lemma 6.** All weakly $K_6^3$-saturated 3-uniform hypergraphs are anchors.

**Proof.** Concatenation of Lemma 4 and Theorem 1. □

**Proof of Theorem 2.** Concatenation of the following lemma with Lemma 6. □

**Lemma 7.** Every 3-uniform hypergraph with $n$ vertices and at least $(\frac{n}{3})^3 - n + 5$ hyperedges is weakly $K_6^3$-saturated.

**Proof.** Consider a 3-uniform hypergraph $(V, \mathcal{E})$ such that $|\mathcal{E}| \geq (\frac{n}{3})^3 - n + 5$, where $n = |V|$. Since $|\mathcal{E}| \leq (\frac{n}{3})^3$, we have $n \geq 5$; if $n = 5$, then $\mathcal{E}$ consists of all three-point subsets of $V$; now let us assume that $n \geq 6$. Given any three-point subset $T$ of $V$, we shall show that $T \in cl \mathcal{E}$. This is trivial when $T \in \mathcal{E}$; to prove it when $T \notin \mathcal{E}$, it suffices to find a six-point subset $S$ of $V$ such that $T$ is the unique three-point subset of $S$ not belonging to $\mathcal{E}$. For this purpose, take one vertex from each $T' - T$ such that $T'$ is a three-point subset of $V$ not belonging to $\mathcal{E}$ and $T' \neq T$. With $W$ standing for the set of all these vertices, $T$ is the unique three-point subset of $V - W$ not belonging to $\mathcal{E}$; since $|W| \leq (\frac{n}{3}) - |\mathcal{E}| - 1$, we have $|V - W| \geq n - (\frac{n}{3}) + |\mathcal{E}| + 1 \geq 6$. □

The remark that follows Theorem 2 shows that the $(\frac{n}{3})^3 - n + 5$ in Lemma 7 cannot be reduced.

**Proof of Theorem 3.** Concatenation of the following lemma with Lemma 6. □

**Lemma 8.** For every integer $n$ greater than four there is a weakly $K_6^3$-saturated 3-uniform hypergraph with $n$ vertices and $3(\frac{n-2}{2}) + 1$ hyperedges.

**Proof.** Take a three-point subset $S$ of $V$ and let $\mathcal{E}$ consist of all three-point subsets of $V$ that have a nonempty intersection with $S$. Note that $\binom{n}{3} - \binom{n-3}{3} = 3(\frac{n-2}{2}) + 1$. □

It is a known fact that the $3(\frac{n-2}{2}) + 1$ in Lemma 8 cannot be reduced: see, for instance, \cite{9} or \cite{11} Theorem 5.5.
3 Concluding remarks

1. Theorems[1][2] and[3] extend to the context of pseudometric betweenness defined in [2] p. 643], which is more general than the context of metric spaces.

2. The proof of Lemma[7] extends to show that every r-uniform hypergraph with n vertices and at least \(^{(n)_r} - n + k - 1\) hyperedges is weakly \(K_r^k\)-saturated.

3. Following [2], let us refer to a 3-uniform hypergraph \((V,E)\) as metric if there is a metric space \(M\) such that a triangle in \(M\) is degenerate if and only if it belongs to \(E\). Lemma[5] can be reformulated as the statement that the 3-uniform hypergraph with 6 vertices and 19 hyperedges is non-metric. Actually, this hypergraph is minimal non-metric in the sense that the deletion of an arbitrary vertex from it produces a metric hypergraph. To verify this, observe first that the deletion produces a hypergraph with 5 vertices and 10 or 9 hyperedges. The former hypergraph is obviously metric. To see that the latter hypergraph is metric, set \(n = 5\) in the comment that follows Theorem[2] (By the way, infinitely many minimal non-metric hypergraphs have been constructed in [7].)

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