A statistical-mechanical approach to CDMA multiuser detection: propagating beliefs in a densely connected graph

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Abstract

The task of CDMA multiuser detection is to simultaneously estimate binary symbols of $K$ synchronous users from the received $N$ base-band CDMA signals. Mathematically, this can be formulated as an inference problem on a complete bipartite graph. In the research on graphically represented statistical models, it is known that the belief propagation (BP) can exactly perform the inference in a polynomial time scale of the system size when the graph is free from cycles in spite that the necessary computation for general graphs exponentially explodes in the worst case [1]. In addition, recent several researches revealed that BP can also serve as an excellent approximation algorithm even if the graph has cycles as far as they are relatively long [2, 3, 4]. However, as there exist many short cycles in a complete bipartite graph, one might suspect that the BP would not provide a good performance when employed for the multiuser detection.

The purpose of this paper is to make an objection to such suspicion. More specifically, we will show that appropriate employment of the central limit theorem and the law of large numbers to BP, which is one of the standard techniques in statistical mechanics, makes it possible to develop a novel multiuser detection algorithm the convergence property of which is considerably better than that of the conventional multistage detection [5] without increasing the computational cost significantly. Furthermore, we will also provide a scheme to analyse the dynamics of the proposed algorithm, which can be naturally linked to the equilibrium analysis recently presented by Tanaka in [6].

1 Multiuser detection

We will focus on a CDMA system using binary shift keying (BPSK) symbols and $K$ random binary spreading codes of the spreading factor $N$ with unit energy over an additive white Gaussian noise (AWGN) channel. For simplicity, we assume the power is completely controlled to unit energy; but the extension to the case of distributed powers is straightforward. Under these assumptions, a received base-band CDMA signal is expressed as

\[ y_\mu = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_{\mu k} b_k + \sigma_0 n_\mu, \]

where $\mu \in \{1,2,\ldots,N\}$ and $k \in \{1,2,\ldots,K\}$ are indices for samples and users, respectively. $s_{\mu k} \in \{-1,1\}$ is the spreading code with unit energy independently generated from the identical unbiased distribution $P(s_{\mu k} = +1) = P(s_{\mu k} = -1) = 1/2$ and $b_k$ is the bit signal of user $k$. $n_\mu$ is a Gaussian white noise sample with zero mean and unit variance and $\sigma_0$ is the standard deviation of AWGN. Using these normalisations, the signal to noise ratio is defined as $SNR = \beta/(2\sigma_0^2)$ where $\beta = K/N$. In the following, we assume a situation where both of $N$ and $K$ are large keeping $\beta$ finite.

The goal of multiuser detection is to simultaneously infer the bit signals $b_1, b_2, \ldots, b_K$ after receiving the base-band signals $y_1, y_2, \ldots, y_N$. The Bayesian approach offers a useful framework
for such purposes. Assuming that the bit signals are independently generated from the unbiased distribution, the posterior distribution given the base-band signals is provided as

$$P(b|y) = \frac{\prod_{\mu=1}^{N} P(y_\mu|b)}{\sum_b \prod_{\mu=1}^{N} P(y_\mu|b)}$$, \hspace{1cm} (2)

where

$$P(y_\mu|b) = \frac{1}{\sqrt{2\pi\sigma^2_0}} \exp \left[-\frac{1}{2\sigma^2_0} (y_\mu - \Delta_\mu)^2\right],$$ \hspace{1cm} (3)

and $\Delta_\mu \equiv \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_{\mu k} b_k$. Following the Bayesian framework, one can systematically derive the optimal inference strategy from the posterior distribution (2) for various cost functions. For instance, it can be shown that the bit error rate (BER), which is the cost function that we will focus on in this paper, is minimised by the maximiser of the posterior marginal (MPM) estimator

$$\hat{b}_k = \underset{b_k \in \{+1, -1\}}{\text{argmax}} \sum_{b_{l \neq k}} P(b|y).$$ \hspace{1cm} (4)

2 Graphical expression and belief propagation

Unfortunately, the necessary cost for exactly computing the MPM estimator explodes exponentially with respect to the number of users $K$ in the current system, which implies that one has to resort to an approximation in practice. The belief propagation (BP), or the sum-product algorithm, is known as one of the most promising approaches to such tasks although its performance for densely connected systems, including complete bipartite graphs, has not been sufficiently examined yet [7]. We here investigate the efficacy of BP in densely connected systems employing it to the present CDMA multiuser detection problem.

In order to introduce this algorithm to the current system, let us denote the base-band and bit signals by two kinds of nodes and connect them with an edge when they are related. Since the conditional probability of $y_\mu$ (3) depends on all of $b_1, b_2, \ldots, b_k$, this implies that the posterior distribution (2) can be expressed as a complete bipartite graph as shown in Figure 1.

Then, BP can be defined as an algorithm passing messages between the two kinds of nodes through edges as

$$P^{t+1}(y_\mu|b_k, \{y_\nu \neq \mu\}) \propto \hat{\alpha}_{\mu k}^{t+1} \sum_{b_{l \neq k}} P(y_\mu|b) \prod_{l \neq k} P^t(b_l|\{y_\nu \neq \mu\}),$$ \hspace{1cm} (5)

$$P^t(b_k|\{y_\nu \neq \mu\}) = \alpha_{\mu k}^t \prod_{\nu \neq \mu} P^t(y_\nu|b_k, \{y_\sigma \neq \nu\}),$$ \hspace{1cm} (6)

where $t = 1, 2, \ldots$ is an index for counting the number of updates, $\hat{\alpha}_{\mu k}^t$ and $\alpha_{\mu k}^t$ are constants for normalisation constraints $\sum_{b_k = \pm 1} P^t(y_\mu|b_k, \{y_\nu \neq \mu\}) = 1$ and $\sum_{b_k = \pm 1} P^t(b_k|\{y_\nu \neq \mu\}) = 1$. 

Figure 1: Graphical expression of the CDMA multiuser detection problem. Each edge corresponds to a component of spreading codes $s_{\mu k}$.
respectively. The marginalised posterior at $t$th update is evaluated from $P^t(y_t|b_t, \{y_{\tau \neq \mu}\})$ as $P^t(b_t|y) = \alpha_k \prod_{i=1}^N P^t(y_{\mu_i}|b_t, \{y_{\tau \neq \mu}\})$, where $\alpha_k$ is a normalisation constant.

As $b_t$ is a binary variable, one can parameterise the above functions as $P^t(y_{\mu_i}|b_t, \{y_{\tau \neq \mu}\}) \propto (1 + \hat{m}^t_{\mu_k} b_t)/2$, $P^t(b_t|\{y_{\tau \neq \mu}\}) = (1 + m^t_{\mu_k} b_t)/2$ and $P^t(b_t|y) = (1 + m^t_{\mu_k} b_t)/2$ without loss of generality, which simplifies the expressions (5) and (6) as

$$ m^t_{\mu_k} = \tanh \left( \sum_{\tau \neq \mu} \tanh^{-1} \hat{m}^t_{\tau_{\mu_k}} \right). $$

Employing these variables, the approximated posterior average of $b_k$ at $t$th update can be computed as $m_k^t = \tanh \left( \sum_{\mu=1}^N \tanh^{-1} \hat{m}^t_{\mu_k} \right)$.

### 3 Propagating beliefs in a large complete bipartite graph

Reflecting the fact that each base-band signal $y_{\mu}$ is connected with every bit signal $b_t$, evaluating eq. (7) brings about a computational explosion when $K$ is large, which implies exactly performing BP becomes hopeless in the current system. However, appropriately employing the central limit theorem and the law of large numbers, which is a standard procedure in statistical-mechanical analysis [8, 9], makes it possible to approximately carry out the belief updates (7) and (8) in a practical time scale.

Since $s_{\mu_k} b_t/\sqrt{N}$ is small for large $N$, we expand the conditional probability as

$$ P(y_{\mu_i}|b) \simeq \frac{1}{\sqrt{2\pi}\sigma_0^2} \exp \left[ -\frac{(y_{\mu_i} - \Delta_{\mu_k})^2}{2\sigma_0^2} + \frac{s_{\mu_k}(y_{\mu_i} - \Delta_{\mu_k})}{\sqrt{N}\sigma_0^2} b_t \right], $$

where $\Delta_{\mu_k} \equiv \sum_{\tau \neq \mu} s_{\mu_k} b_t/\sqrt{N}$ in eq. (7). As the spreading codes are generated independently, $s_{\mu_k}$ and $b_t$ would be uncorrelated when $b_t$ is generated from $P^t(b_t|\{y_{\tau \neq \mu}\}) = (1 + m^t_{\mu_k} b_t)/2$. This, in conjunction with the central limit theorem, implies that $\Delta_{\mu_k} \equiv \sum_{\tau \neq \mu} s_{\mu_k} b_t/\sqrt{N}$ obeys a normal distribution $N\left( \langle \Delta^t_{\mu_k} \rangle_{\mu}, \beta(1 - Q^t_{\mu_k}) \right)$, where $\langle \Delta^t_{\mu_k} \rangle_{\mu} \equiv \sum_{\tau \neq \mu} s_{\mu_k} m^t_{\mu_k}/\sqrt{N}$ and $Q^t_{\mu_k} \equiv (1/K)\sum_{k=1}^K (m^t_{\mu_k})^2$. Furthermore, due to the law of large numbers, $Q^t_{\mu_k}$ is highly likely to be well approximated by $Q^t_{\mu_k} \equiv (1/K)\sum_{k=1}^K (m^t_{\mu_k})^2$. Substituting these, one can evaluate eq. (7) as

$$ \hat{m}^{t+1}_{\mu_k} = A^t \left( \frac{y_{\mu_i} s_{\mu_k}}{\sqrt{N}} - \beta \left( P_{\mu} - \frac{I}{K} \right) m^t_{\mu_k} \right), $$

where $s_{\mu_k} \equiv \langle s_{\mu_k} \rangle$, $m^t_{\mu_k} \equiv \langle m^t_{\mu_k} \rangle$ and $A^t \equiv (\sigma_0^2 + \beta(1 - Q^t_{\mu_k}))^{-1}$. Here, we also introduced the projection and the identity matrices $P_{\mu} \equiv (1/K)\langle s_{\mu_k} s_{\mu_l} \rangle$ and $I \equiv (\delta_{kl})$, respectively. ($\dots)_k$ denotes $k$th component of the vector $\cdots$. Eq. (10) can be evaluated by $O(K)$ computations per pair $(\mu_k)$, which implies that $O(NK^2)$ computations are totally required per update.

The computational cost can be further reduced to $O(K^2)$ when $N$ is large employing eq. (8). As $\hat{m}^{t}_{\mu_k}$ typically scales as $O(N^{-1/2})$, eq. (8) can be expanded as $m^t_{\mu_k} \simeq m^t_{\mu_k} - (\partial m^t_{\mu_k}/\partial \hat{m}^{t}_{\mu_k}) \hat{m}^{t}_{\mu_k} = m^t_{\mu_k} - (1 - (m^t_{\mu_k})^2) \hat{m}^{t}_{\mu_k}$. Plugging this into eq. (10) provides a recursive equation with respect to
\[ \hat{m}_{\mu} = (\hat{m}_{\mu}^t) \] as
\[ \hat{m}_{\mu}^t+1 = A^t \frac{y_{\mu}s_{\mu}}{\sqrt{N}} - \beta A^t \left( P_{\mu} - \frac{1}{K} \right) m^t + \beta A^t P_{\mu} C^t \hat{m}_{\mu}^t, \] (11)
where \( C^t \equiv ((1 - (\hat{m}_{\mu}^t)^2)\delta_{kl}) \). Employing useful relations \( P_{\mu} C^t s_{\mu} = (1 - Q^t) s_{\mu} \) and \( P_{\mu} C^t P_{\mu} = (1 - Q^t) P_{\mu} \) and omitting negligible terms, the solution of eq. (11) can be expressed as
\[ \hat{m}_{\mu}^t+1 = R^t \frac{y_{\mu}s_{\mu}}{\sqrt{N}} - U_{\mu}^t + \frac{1}{K} \beta A^t m^t, \] (12)
where \( R^t \) and \( U^t \) are obtained from recursive equations
\[
R^t = A^t + A^t \beta (1 - Q^t) R^{t-1}, \]
\[ U_{\mu}^t = A^t \beta P_{\mu} m^t + A^t \beta (1 - Q^t) U_{\mu}^{t-1}. \] (14)

Since \( \hat{m}_{\mu k} \) typically scales as \( O(N^{-1/2}) \), the posterior average can be expressed as \( m_k^t = \tanh \left( \sum_{\mu=1}^{N} \tanh^{-1} \hat{m}_{\mu k}^t \right) \simeq \tanh \left( \sum_{\mu=1}^{N} m_{\mu k}^t \right) \). This implies that the belief updates (5) and (6) are finally summarised into
\[
h^{t+1} = R^t h^0 - U^t + A^t m^t, \]
\[ U^t = A^t \beta W m^t + A^t \beta (1 - Q^t) U^{t-1}, \] (16)
and eq. (13), where \( m_k^t = \tanh(h_k^t), h^0 \equiv (h_k^0) = (\sum_{\mu=1}^{N} y_{\mu} s_{\mu} / \sqrt{N}), h^t \equiv (h_k^t) \) and \( W \equiv (W_{kl}) \equiv (\sum_{\mu=1}^{N} s_{\mu k} s_{\mu l} / N) \). From the posterior average \( m_k^t \), the MPM estimator at \( t \)th update is evaluated as \( \hat{b}_{k}^t = \text{sign}(m_k^t) \) where \( \text{sign}(x) \equiv \lim_{\epsilon \rightarrow +0} x / |x + \epsilon| \).

Two points are worthy of noticing. Firstly, the most time-consuming operation in eqs. (13), (15) and (16) is \( W m^t \), which totally requires \( O(K^2) \) computations. This implies that the computational cost for performing the current scheme is similar to that of the conventional multistage detection [5]
\[ \hat{b}_{k}^{t+1} = \text{sign} \left( h_k^0 - \sum_{l \neq k} W_{kl} b_l^t \right), \] (17)
Secondly, as the fixed point condition, coupled nonlinear equations
\[ m_k = \tanh \left( \sigma_0^{-2} \left( h_k^0 - \sum_{l \neq k} W_{kl} m_l \right) - \frac{\beta (1 - Q) m_k}{\sigma_0^2 (\sigma_0^2 + \beta (1 - Q))} \right), \] (18)
are obtained from our update scheme, where \( Q = (1/K) \sum_{k=1}^{K} m_k^2 \). This is identical to the Thouless-Anderson-Palmer (TAP) equation for the current system known in statistical mechanics [10]\(^1\). However, it should be emphasised here that, the naive iteration of eq. (18) does not serve as a useful detection algorithm as finding the fixed point by it from a reasonable initial state is difficult. This will be illustrated by numerical experiments in the final section.

4 Density evolution and equilibrium analysis

The density evolution is a framework to analyse the dynamical property of BP pursuing a macroscopic distribution of messages [11, 12]. In the current system, this analysis is considerably
\(^1\)In statistical physics, pattern ratio \( \tilde{\alpha} = N / K = \beta^{-1} \) and inverse temperature \( \tilde{\beta} = \beta \sigma_0^{-2} \) are usually employed for characterising a system instead of \( \beta \) and \( \sigma_0^2 \).
simplified as the aligned field $b_k h_k^t$ is likely to obey a normal distribution as a result of the central limit theorem.

Let us assume that $b_k h_k^t = b_k \sum_{\mu=1}^{N} \hat{m}_{\mu k}^t$ is independently sampled from a normal distribution the average and variance of which are $E^t$ and variance $F^t$, respectively. This implies that the overlap $M^t \equiv \sum_{k=1}^{K} b_k m_{\mu k}^t / K$ and $Q^t$ are evaluated as

$$M^t = \int Dz \tanh(\sqrt{F^t}z + E^t), \quad Q^t = \int Dz \tanh^2(\sqrt{F^t}z + E^t),$$

where $Dz \equiv dz \exp[-z^2/2]/\sqrt{2\pi}$. Since the MPM estimator is given as $\hat{b}_k^t = \text{sign}(h_k^t)$, BER is provided as $P_b^t = (1/K) \sum_{k=1}^{K} (1 - \text{sign}(b_k h_k^t))/2 = \int_{-\infty}^{-E^t/\sqrt{F^t}} Dz$.

On the other hand, as far as $b_k h_k^t$ is independently sampled, $b_k \hat{m}_{\mu k}^{t+1}$ evaluated from eq. (10) is uncorrelated for a given $k$ since spreading codes $s_\mu$ are almost orthogonal with each other when $N$ is large as $s_\mu \cdot s_\nu / N \simeq O(N^{-1/2})$ holds for $\mu \neq \nu$. This implies that the central limit theorem holds for $b_k h_k^{t+1}$, which, in conjunction with the statistical uniformness with respect to indices $\mu$ and $k$, provides the average and the variance at $t + 1$ update as $E^{t+1} = (1/K) \sum_{k=1}^{K} \sum_{\mu=1}^{N} b_k \hat{m}_{\mu k}^{t+1} = (1/K) \sum_{k=1}^{K} b \cdot \hat{m}_\mu^t$ and $F^{t+1} = (1/K) \sum_{k=1}^{K} \sum_{\mu=1}^{N} (b_k \hat{m}_{\mu k}^t)^2 = (1/K) \sum_{\mu=1}^{N} \hat{m}_\mu^t \cdot \hat{m}^t_\mu$, respectively. Evaluating these employing eqs. (1) and (10), $E^{t+1}$ and $F^{t+1}$ are obtained as

$$E^{t+1} = \frac{1}{\sigma_0^2 + \beta(1-Q^t)}, \quad F^{t+1} = \frac{\beta(1-2M^t + Q^t) + \sigma_0^2}{[\sigma_0^2 + \beta(1-Q^t)]^2},$$

where we assumed that $(1/K) \sum_{k=1}^{K} b \cdot \hat{m}_\mu^t \simeq M^t$ holds as a result from the law of large numbers. Eqs. (19) and (20) express the density evolution with respect to the current algorithm.

It should be noticed that the obtained expression of the density evolution directly links the proposed algorithm to the equilibrium analysis presented in [6] since eqs. (19) and (20) can be regarded as the naive iteration dynamics of the saddle point equations provided by the replica method [13]. This implies that our algorithm can practically calculate the MPM estimator (4) in $O(K^2)$ computations obtaining the fixed point solution when $K$ is large since the replica analysis is likely to evaluate the exact performance for $K \to \infty$, which, however, has not been rigorously proved yet.

5 Method comparison and discussion

In order to validate the obtained results, we performed numerical experiments in a system $N = 2000$ and $\beta = 0.5$. Figure 2 shows time evolution of BER obtained from 10000 experiments for the proposed algorithm (eqs. (13), (15) and (16): PA), the conventional multistage detection (eq. (17): MSD), the iteration of the TAP equation (eq. (18): TAP) and the density evolution (eqs. (19) and (20): DE).

Firstly, it is clear that PA converges to the fixed point considerably faster than MSD, which is a highly preferred property in practical use. Secondly, PA and DE exhibit excellent consistency as we speculated in the previous section, which implies that employment of the central limit theorem and the law of large numbers for deriving eqs. (19) and (20) is fully validated. Finally, TAP does not serve as a useful detection algorithm. This is because the iteration of eq. (18) does not correctly approximate BP and, therefore, can not sufficiently cancel self-reactions from the past states in the transient dynamics although it does provide the correct fixed point condition in the stationary state.

In summary, we have developed a novel algorithm for the CDMA multiuser detection from the belief propagation appropriately employing the central limit theorem and the law of large numbers. The new algorithm exhibits considerably faster convergence than the conventional multistage detection without increasing the computational cost significantly and is likely to practically provide the optimal MPM estimator when the spreading factor $N$ is large. We have also clarified the relation between the obtained algorithm and the existing equilibrium analysis presented in [6] employing the density evolution scheme.
Figure 2: Time evolution of BER for the proposed algorithm (PA:□), the conventional multistage detection (MSD:×), the naive iteration of the TAP equation (TAP:+) and the density evolution (DE: lines) in the case of $N = 2000$, $\beta = 0.5$ and $SNR = 4, 9$ (data of TAP are shown only for $SNR = 9$). Each marker represents the averaged BER at $t$th update evaluated from 10000 experiments. PA exhibits the fastest convergence and excellent consistency with DE.

We have here assumed randomly generated spreading codes for simplicity, which might not be suitable for practical use. Extension of the current scheme to other methods of code generation is under way.

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