BACKWARD COMPACT AND PERIODIC RANDOM ATTRACTORS FOR NON-AUTONOMOUS SINE-GORDON EQUATIONS WITH MULTIPLICATIVE NOISE

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Abstract. A non-autonomous random attractor is called backward compact if its backward union is pre-compact. We show that such a backward compact random attractor exists if a non-autonomous random dynamical system is bounded dissipative and backward asymptotically compact. We also obtain both backward compact and periodic random attractor from a periodic and locally asymptotically compact system. The abstract results are applied to the sine-Gordon equation with multiplicative noise and a time-dependent force. If we assume that the density of noise is small and that the force is backward tempered and backward complement-small, then, we obtain a backward compact random attractor on the universe consisted of all backward tempered sets. Also, we obtain both backward compactness and periodicity of the attractor under the assumption of a periodic force.

1. Introduction. In order to analyze random dynamics of a non-autonomous stochastic PDE, an important concept of a non-autonomous random attractor was first introduced by Wang[24], with notable applications, see [1, 13, 14, 22, 23, 25] and the references therein.

Such an attractor is a bi-parametric set \( \mathcal{A} = \{ A(t, \omega) \} \), whose components/fibers depend on both time \( t \in \mathbb{R} \) and sample \( \omega \in \Omega \), where \( \Omega \) is a probability space. However, some properties (such as compactness, attraction, dimensions) of a non-autonomous random attractor were investigated with respect to each time-fiber only, see the above and more references [30, 32, 34, 35, 37] etc. Such properties of time-fibers for a deterministic pullback attractor \( A = \{ A(t) \} \) were investigated in some earlier papers, see [2, 5, 12] and the references therein. The fibrous property of a non-autonomous attractor is just an analogue of an autonomous attractor.

In this paper, we develop a new subject on non-autonomous random attractors to reveal various \textit{time-dependent compactness} such as: local, backward, forward and global compactness. There are some examples of non-autonomous ODEs (cf.[20, 21]) indicating that a non-autonomous attractor may not be globally compact. So, we fix attention on backward compactness (the forward compactness can be treated similarly).

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A non-autonomous random attractor $\mathcal{A}$ is called **backward compact** if the backward union $\cup_{s \leq t} \mathcal{A}(s, \omega)$ is pre-compact for each $t \in \mathbb{R}$ and $\omega \in \Omega$. Such backward compactness describes that a non-autonomous random dynamical system (cocycle) $\Phi$ has a more concentrative attractor in the past. For such time-dependent compactness, there is not any referenced methods from the autonomous case because the autonomous attractor is time-independent.

Some criteria will be established to ensure the backward compactness of a non-autonomous random attractor. An important criterion is that the non-autonomous cocycle $\Phi$ is **backward asymptotically compact**, which means that the usual asymptotic compactness is uniform in the past. Then, we establish an abstract result: a non-autonomous cocycle $\Phi$ has a backward compact random attractor if $\Phi$ is bounded dissipative and backward asymptotically compact, see Theorem 2.8.

In order to prove the above abstract result, we need to introduce a new concept of a **backward limit set**, which is a backward version of the usual omega-limit set. We show that the backward limit set is compact if $\Phi$ is backward asymptotically compact. Because the backward limit set contains each backward union of the usual omega-limit set, we obtain the backward compactness of the attractor, in view of the fact that the attractor is just the omega-limit set of an absorbing set.

Another issue is to consider the relationship between periodicity and backward compactness of a random attractor. It is shown that a periodic random attractor is backward compact if and only if it is local compact.

In the non-random case, a pullback attractor for a continuous dynamical system must be locally compact as proved in a recent paper (see [19]), and so a periodic deterministic attractor must be backward compact, see [36].

The random case is different from the deterministic case in view of the varying of the sample. Although we can show that $t \to \mathcal{A}(t, \theta_t \omega)$ is automatically locally compact, we can not prove the local compactness of $t \to \mathcal{A}(t, \omega)$. So, we obtain backward compactness from periodicity of a random attractor under some additional conditions that $\Phi$ is **locally asymptotically compact**, see Theorem 2.6.

The second part in this paper is to apply the abstract result to the following non-autonomous stochastic damped sine-Gordon equation:

$$
\begin{align*}
\left\{ 
\begin{array}{l}
u_{tt} + u_t - \Delta u + \lambda u + \beta \sin u &= f(x, t) + \epsilon u \circ \frac{dW}{dt}, \
\quad \quad t \geq \tau, \\
u(\tau) &= u_0, \quad u_t(\tau) = u_1, \quad x \in Q, \quad u|_{\partial Q} = 0,
\end{array}
\right.
\end{align*}
$$

(1)

where $\lambda > 0, \beta \in \mathbb{R}$ and $Q$ is a bounded domain in $\mathbb{R}^n$ with arbitrary dimensions. The noise is multiplicative in the Stratonovich integrals sense, see [7, 10, 15, 17, 18, 27].

The existence of a random attractor for 3D sine-Gordon equation had been discussed in the literature [11, 29], for the 3D stochastic wave equation with other nonlinearity, see [26, 28].

In order to obtain a backward compact random attractor for Eq.(1), we need to make some new assumptions that the time-dependent force $f$ is backward tempered and backward complement-small (see Hypotheses I, III later), which is stronger than the usual tempered assumption. Also, we need to assume that the density $\epsilon$ of noise is small (see Hypotheses II).

The usual tempered universe (see [4, 6, 16, 33]) cannot be suitable when we prove that the backward asymptotic compactness on an attracted universe $\mathcal{D}$. So,
we take the universe $\mathcal{D}$ by the collection of all backward tempered set, which is backward-closed. While the tempered universe cannot be backward-closed.

In this case, an difficulty arises from the measurability of the absorbing set, where the absorbing radius is the maximum of some random functions over an uncountable index set. The difficulty will be overcome by splitting both time and sample parameters.

By using the spectrum method and carefully analyzing the differential of the sine function, we can prove the $\mathcal{D}$-backward asymptotic compactness, see Proposition 7. So, we obtain a backward compact random attractor, see Theorem 5.1.

Finally, we establish both backward compactness and periodicity from the periodic force assumption, see Theorem 5.2. Although a periodic random attractor may not be backward compact, the periodicity of $f$ can imply that $f$ is backward tempered and backward complement-small.

2. Abstract results on backward compactness and periodicity. Let $(X,||\cdot||)$ be a separable Banach space equipped with a Borel algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, P, \theta)$ be a metric dynamical system, where $\Omega, \mathcal{F}, P$ be a probability space and $\theta = \{\theta_t : t \in \mathbb{R}\}$ be a group of measure-preserving self-transformations on $\Omega$.

We need to introduce the concept of a non-autonomous cocycle in the sense of Wang [24].

**Definition 2.1.** A mapping $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$ is said to be a non-autonomous cocycle if

(i) it is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ measurable,

(ii) it satisfies the cocycle property: $\Phi(0, \tau, \omega) = \text{id}_X$, and

$$\Phi(t + s, \tau, \omega) = \Phi(t, \tau + s, \theta_s \omega) \Phi(s, \tau, \omega), \quad t, s \geq 0.$$ (2)

We always assume that a non-autonomous cocycle $\Phi(t, \tau, \omega)x$ is continuous in $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $x \in X$ respectively.

2.1. Time-dependent compactness of random attractors. A set-valued mapping $\mathcal{D}: \mathbb{R} \times \Omega \rightarrow 2^X \setminus \{\emptyset\}$ is called a bi-parametric set.

**Definition 2.2.** A bi-parametric set $\mathcal{D} = \{\mathcal{D}(\tau, \omega)\}$ is said to be

(i) measurable (or random) if $\omega \rightarrow d(x, \mathcal{D}(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^+))$ measurable for each $x \in X$ and $\tau \in \mathbb{R}$,

(ii) compact if $\mathcal{D}(\tau, \omega)$ is compact in $X$ for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(iii) locally compact if it is compact, and $\cup_{a \leq s \leq b} \mathcal{D}(s, \omega)$ is pre-compact for any compact interval $[a, b]$ and $\omega \in \Omega$,

(iv) backward compact if it is compact, and $\cup_{s \leq \tau} \mathcal{D}(s, \omega)$ is pre-compact for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

(v) $T$-periodic if there is a $T > 0$ such that $\mathcal{D}(\tau + T, \omega) = \mathcal{D}(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Let $\mathcal{D}$ be a universe of some bi-parametric sets in $X$. We always assume that $\mathcal{D}$ is neighborhood-closed in the sense of [24], that is, for each $\mathcal{D} \in \mathcal{D}$, there is $\eta = \eta(\mathcal{D}) > 0$ such that $\mathcal{D}_0 \in \mathcal{D}$ as soon as $\mathcal{D}_0(\tau, \omega) \subset N_\eta \mathcal{D}(\tau, \omega)$ for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, where

$$N_\eta \mathcal{D}(\tau, \omega) = \{x \in X : d(x, \mathcal{D}(\tau, \omega)) \leq \eta\}.$$ Note that neighborhood-closedness implies inclusion-closedness, the latter means that $\mathcal{D}_1 \in \mathcal{D}$ whenever $\mathcal{D} \in \mathcal{D}$ and $\mathcal{D}_1(\tau, \omega) \subset \mathcal{D}(\tau, \omega)$. 
Remark 1. Recently, Yin et al. [36] have proved that a periodic pullback attractor

Definition 2.3. A compact bi-parametric set \( \mathcal{A} = \{ \mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is said to be a **\( \mathcal{D} \)-random attractor** for a non-autonomous cocycle \( \Phi \) if

1. \( \mathcal{A} \in \mathcal{D} \) and \( \mathcal{A} \) is a random set,
2. \( \mathcal{A} \) is invariant, that is, \( \Phi(t, \tau, \omega)\mathcal{A}(\tau, \omega) = \mathcal{A}(t + \tau, \theta_\tau \omega) \) for \( t \geq 0 \),
3. \( \mathcal{A} \) is attracting under the Hausdorff semi-distance, that is, for each \( \mathcal{D} \in \mathcal{D} \),

\[
\lim_{t \to +\infty} \text{dist}_X(\Phi(t, \tau - t, \theta_{-t} \omega)\mathcal{D}(\tau - t, \theta_{-t} \omega), \mathcal{A}(\tau, \omega)) = 0. \tag{3}
\]

We first establish the following simple relationship between periodicity and backward compactness of a random attractor.

Lemma 2.4. A periodic random attractor \( \mathcal{A} \) is backward compact if and only if it is locally compact.

Proof. Let \( T > 0 \) be a period of the periodic attractor \( \mathcal{A} \). For each \( s \leq \tau \), we can write \( s = mT + s_0 \) with \( m \in \mathbb{Z} \) and \( 0 \leq s_0 < T \), then, the periodicity of \( \mathcal{A} \) implies that

\[
\bigcup_{s \leq \tau} \mathcal{A}(s, \omega) = \bigcup_{mT + s_0 \leq \tau} \mathcal{A}(mT + s_0, \omega) \subset \bigcup_{s_0 \in [0, T]} \mathcal{A}(s_0, \omega).
\]

Therefore, \( \mathcal{A} \) is backward compact if it is locally compact.

On the contrary, for any compact interval \([a, b]\), we have

\[
\bigcup_{s \in [a, b]} \mathcal{A}(s, \omega) \subset \bigcup_{s \leq b} \mathcal{A}(s, \omega),
\]

which means that backward compactness must imply local compactness. \( \square \)

Remark 1. Recently, Yin et al. [36] have proved that a periodic pullback attractor \( \mathcal{A} = \{ \mathcal{A}(t) \} \) must be backward compact. The reason is that a pullback attractor \( \mathcal{A} = \{ \mathcal{A}(t) \} \) is automatically locally compact. However, due to the variety of the sample in the invariance, we cannot prove the local compactness of a random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau, \omega) \} \). So, we have to find some additional conditions.

2.2. Local compactness and periodicity of a random attractor. Recall that a bi-parametric set \( \mathcal{K} \) is said to be a **\( \mathcal{D} \)-pullback absorbing** set (briefly, an absorbing set) if for each \( (\mathcal{D}, \tau, \omega) \in \mathcal{D} \times \mathbb{R} \times \Omega \) there is a \( T := T(\mathcal{D}, \tau, \omega) \) such that

\[
\Phi(t, \tau - t, \theta_{-t} \omega)\mathcal{D}(\tau - t, \theta_{-t} \omega) \subset \mathcal{K}(\tau, \omega), \forall t \geq T. \tag{4}
\]

Definition 2.5. A non-autonomous cocycle \( \Phi \) is said to be **locally asymptotically compact** if, for any compact interval \([a, b]\) and \( (\omega, \mathcal{D}) \in \Omega \times \mathcal{D} \),

\[
\{ \Phi(t_n, \tau_n - t_n, \theta_{-t_n} \omega)x_n \}_{n=1}^\infty
\]

is pre-compact in \( X \), whenever \( \tau_n \in [a, b] \), \( t_n \to +\infty \) and \( x_n \in \mathcal{D}(\tau_n - t_n, \theta_{-t_n} \omega) \).

Proposition 1. Suppose that a non-autonomous cocycle \( \Phi \) has a random attractor \( \mathcal{A} \) from \( \mathcal{D} \). Then, \( \mathcal{A} \) is locally compact if one of the following conditions holds true:

(a) \( \Phi \) is locally asymptotically compact.

(b) \( \Phi \) has a locally compact absorbing set \( \mathcal{K} \).

Proof. Let the sequence \( \{ y_n \} \) be taken from \( \cup_{\tau \in [a, b]} \mathcal{A}(\tau, \omega) \). Then, there are \( \tau_n \in [a, b] \) such that \( y_n \in \mathcal{A}(\tau_n, \omega) \). Given \( t_n \to +\infty \), by the invariance of \( \mathcal{A} \in \mathcal{D} \), there are \( x_n \in \mathcal{A}(\tau_n - t_n, \theta_{-t_n} \omega) \) such that

\[
y_n = \Phi(t_n, \tau_n - t_n, \theta_{-t_n} \omega)x_n.
\]
Suppose the condition (a) holds true, that is, \( \Phi \) is locally asymptotically compact. Definition 2.5 implies that \( \{y_n\} \) is pre-compact as required.

Suppose the condition (b) holds true, that is, \( K \) is a locally compact absorbing set. Then, for each \( y_n \in A(\tau_n, \omega) \), it follows from the invariance of \( A \) that

\[
y_n \in A(\tau_n, \omega) = \Phi(t, \tau_n - t, \theta_{-t} \omega)A(\tau_n - t, \theta_{-t} \omega) \subseteq K(\tau_n, \omega),
\]

provided \( t \) is large enough. Hence,

\[
\{y_n\}_{n=1}^{\infty} \subseteq \bigcup_{n} K(\tau_n, \omega) \subseteq \bigcup_{\tau \in [a,b]} K(\tau, \omega).
\]

Therefore, the local compactness of \( K \) implies that \( \{y_n\} \) is still pre-compact.

Although we cannot show the local compactness of the set-mapping \( \tau \to A(\tau, \omega) \), we can show that \( s \to A(s, \theta_s \omega) \) is automatically locally compact.

**Proposition 2.** \( \bigcup_{s \in [a,b]} A(s+\tau, \theta_s \omega) \) is compact for any compact interval \([a, b]\), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \).

**Proof.** Since \( \Phi \) is assumed to be a continuous cocycle, we can define a continuous operator by

\[
\Pi : \mathbb{R}^+ \times X \mapsto X, \ (t, x) \to \Phi(t, a, \theta_a \omega)x,
\]

where \( a \) and \( \theta_a \omega \) are fixed. We first prove the special case of \( \tau = 0 \). By invariance of \( A \), we have

\[
A(s, \theta_s \omega) = \Phi(s - a, a, \theta_a \omega)A(a, \theta_a \omega), \ \forall s \in [a, b],
\]

which implies that

\[
\bigcup_{s \in [a,b]} A(s, \theta_s \omega) = \bigcup_{s \in [a,b]} \Phi(s - a, a, \theta_a \omega)A(a, \theta_a \omega) = \bigcup_{t \in [0, b-a]} \Phi(t, a, \theta_a \omega)A(a, \theta_a \omega) = \Pi([0, b-a] \times A(a, \theta_a \omega)).
\]

Since \([0, b-a] \times A(a, \theta_a \omega)\) is a compact set in \( \mathbb{R}^+ \times X \), the continuity of \( \Pi \) implies the local compactness of \( A(s, \theta_s \omega) \) in \( s \).

We replace \( \omega \) by \( \theta_{-\tau} \omega \) and \([a, b]\) by \([a + \tau, b + \tau]\) to find

\[
\bigcup_{s \in [a,b]} A(s+\tau, \theta_s \omega) = \bigcup_{s \in [a+\tau,b+\tau]} A(s, \theta_s \theta_{-\tau} \omega),
\]

which is compact in \( X \). \( \square \)

We recall some terminologies and criteria for a periodic attractor given by Wang[24]. A non-autonomous cocycle \( \Phi \) is called \( T \)-periodic if

\[
\Phi(t, \tau + T, \omega) = \Phi(t, \tau, \omega), \ \text{for all} \ t \in \mathbb{R}^+, \ \tau \in \mathbb{R}, \ \omega \in \Omega.
\]

A universe \( D \) of some bi-parametric sets is called \( T \)-translation closed if \( D_T \in D \) whenever \( D \in D \), where

\[
D_T(\tau, \omega) = D(\tau + T, \omega), \ \text{for all} \ \tau \in \mathbb{R}, \ \omega \in \Omega.
\]

While, \( D \) is called \( T \)-translation invariant if it is \( T \)-translation closed and \(-T\)-translation closed.

Now, by using Proposition 1 and [24, theorem 2.24], we obtain the following result immediately.
Theorem 2.6. Let $A = \{A(\tau, \omega)\}$ be a $\mathcal{D}$-random attractor for a non-autonomous cocycle $\Phi$. Then, $A$ is $T$-periodic if $\Phi$ is $T$-periodic and $\mathcal{D}$ is $T$-translation invariant.

Moreover, this periodic attractor $A$ is backward compact if $\Phi$ is locally asymptotically compact, or if $\Phi$ has a locally compact absorbing set $K$.

Remark 2. By Proposition 2, it is easy to show that the set mapping $s \to A(s, \theta_\omega)$ is backward compact if it is periodic. However, we cannot expect that $s \to A(s, \theta_\omega)$ is periodic, because the transformation $s \to \theta_\omega$ is not generally periodic in the realistic model.

2.3. Backward compact random attractors. In this subsection, we establish some criteria to ensure the backward compactness of a random attractor.

Definition 2.7. A non-autonomous cocycle $\Phi$ is said to be backward asymptotically compact on a bi-parametric set $\mathcal{D}$ if, for each $(\tau, \omega) \in \mathbb{R} \times \Omega$, the sequence 
\[
\{\Phi(t_n, \tau_n - t_n, \theta_{-\tau_n}\omega)x_n\}_{n=1}^{\infty}
\]
whenever $\tau_n \leq \tau, t_n \to +\infty$ and $x_n \in \mathcal{D}(\tau_n - t_n, \theta_{-\tau_n}\omega)$. Furthermore, $\Phi$ is said to be $\mathcal{D}$-backward asymptotically compact if it is backward asymptotically compact on each $\mathcal{D} \in \mathcal{D}$.

We require a new concept of a backward limit set: Given a bi-parametric set $\mathcal{D}$,
\[
\Gamma(\tau, \omega, \mathcal{D}) := \bigcap_{T > 0} \bigcup_{t \geq T} \bigcup_{s \leq \tau} \Phi(t, s - t, \theta_{-t}\omega)D(s - t, \theta_{-t}\omega), \ \forall (\tau, \omega) \in \mathbb{R} \times \Omega,
\]
which is the backward version of the usual omega-limit set:
\[
W(\tau, \omega, \mathcal{D}) := \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, \tau - t, \theta_{-t}\omega)D(\tau - t, \theta_{-t}\omega), \ \forall (\tau, \omega) \in \mathbb{R} \times \Omega.
\]

Proposition 3. The backward limit set has the following properties:

(i) Characteristics. $y \in \Gamma(\tau, \omega, \mathcal{D})$ if and only if there are $\tau_n \leq \tau, t_n \uparrow +\infty$ and $x_n \in \mathcal{D}(\tau_n - t_n, \theta_{-t_n}\omega)$ such that
\[
\Phi(t_n, \tau_n - t_n, \theta_{-t_n}\omega)x_n \to y \text{ in } X.
\]

(ii) Increasing properties. $\Gamma(\tau, \omega, \mathcal{D})$ is increasing in $\tau$ and in $\mathcal{D}$ respectively:
\[
\Gamma(\tau_1, \omega, \mathcal{D}) \subset \Gamma(\tau_2, \omega, \mathcal{D}) \text{ if } \tau_1 \leq \tau_2;
\]
\[
\Gamma(\tau, \omega, \mathcal{D}_1) \subset \Gamma(\tau, \omega, \mathcal{D}_2) \text{ if } \mathcal{D}_1 \subset \mathcal{D}_2.
\]

(iii) Backward inclusion. The backward limit set contains the backward union of the omega-limit set:
\[
\bigcup_{s \leq \tau} W(s, \omega, \mathcal{D}) \subset \Gamma(\tau, \omega, \mathcal{D}) \subset \bigcup_{s \leq \tau} \Gamma(s, \omega, \mathcal{D}).
\]

(iv) Backward compactness. Both $W(\tau, \omega, \mathcal{D})$ and $\Gamma(\tau, \omega, \mathcal{D})$ are backward compact if the cocycle $\Phi$ is backward asymptotically compact on $\mathcal{D}$.

(v) Backward equi-attraction. If $\Phi$ is backward asymptotically compact on $\mathcal{D}$, then, $\Gamma(\tau, \omega, \mathcal{D})$ backward equi-attracts $\mathcal{D}$ in the following sense:
\[
\lim_{t \to +\infty} \sup_{s \leq \tau} \text{dist}_X(\Phi(t, s - t, \theta_{-t}\omega)D(s - t, \theta_{-t}\omega), \Gamma(\tau, \omega, \mathcal{D})) = 0.
\]
Proof. (i) Assume $y \in \Gamma(\tau, \omega, D)$. By (6),

$$y \in \bigcup_{t \geq n} \bigcup_{s \leq \tau} \Phi(t, s - t, \theta_{-t}\omega)D(s - t, \theta_{-t}\omega), \quad \forall n \in \mathbb{N}.$$ 

For each $n \in \mathbb{N}$, we sequentially choose $t_n > \max\{n, t_{n-1}\}$, $\tau_n \leq \tau$ and $x_n \in D(\tau_n - t_n, \theta_{-t_n}\omega)$ such that $\|\Phi(t_n, \tau_n - t_n, \theta_{-t_n}\omega)x_n - y\| \leq 1/n$, which implies (7).

Conversely, let $\tau_n \leq \tau$, $t_n \uparrow +\infty$ and $x_n \in D(\tau_n - t_n, \theta_{-t_n}\omega)$ such that (7) holds true. Then, for each $T > 0$, there is a $n_0 \in \mathbb{N}$ such that $t_{n_0} \geq T$, and thus

$$y \in \bigcup_{n \geq n_0} \{\Phi(t_n, \tau_n - t_n, \theta_{-t_n}\omega)x_n\} \subset \bigcup_{t \geq T} \bigcup_{s \leq \tau} \Phi(t, s - t, \theta_{-t}\omega)D(s - t, \theta_{-t}\omega),$$

which deduces that $y \in \Gamma(\tau, \omega, D)$.

(ii) Both increasing properties follow from the definition given by (6) immediately.

(iii) The first inclusion in (8) follows from the following calculation:

$$\bigcup_{s \leq \tau} \mathcal{W}(s, \omega, D) = \bigcup_{s \leq \tau} \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, s - t, \theta_{-t}\omega)D(s - t, \theta_{-t}\omega) \subset \bigcap_{T > 0} \bigcup_{s \leq \tau} \Phi(t, s - t, \theta_{-t}\omega)D(s - t, \theta_{-t}\omega) = \Gamma(\tau, \omega, D).$$

Since $\Gamma(\tau, \omega, D)$ is increasing in $\tau$, we know that $\bigcup_{s \leq \tau} \Gamma(s, \omega, D) \subset \Gamma(\tau, \omega, D)$ and so it is an equality, which proves (8).

(iv) Assume that $\Phi$ is backward asymptotically compact on $D$, then it is asymptotically compact on $\mathcal{D}$. It is well-known that $\mathcal{W}(\tau, \omega, D)$ is nonempty. Hence, by (iii), $\Gamma(\tau, \omega, D)$ is nonempty.

We take any sequence $\{y_n\}_{n=1}^{\infty}$ from $\Gamma(\tau, \omega, D)$. Then, by (i), there are $\tau_n \leq \tau$, $t_n \uparrow +\infty$ and $x_n \in D(\tau_n - t_n, \theta_{-t_n}\omega)$ such that

$$\|\Phi(t_n, \tau_n - t_n, \theta_{-t_n}\omega)x_n - y_n\| \leq \frac{1}{n}.$$ 

By the backward asymptotical compactness on $D$, there is a subsequence such that

$$\Phi(t_{n_k}, \tau_{n_k} - t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k} \to y_0, \quad \text{in } X.$$ 

By (i), $y_0 \in \Gamma(\tau, \omega, D)$. Also, we have $y_{n_k} \to y_0$ in $X$, and so $\Gamma(\tau, \omega, D)$ is compact.

Moreover, by the backward inclusion given in (8), we know both limit-sets $\mathcal{W}$ and $\Gamma$ are backward compact.

(v) It remains to show that $\Gamma(\tau, \omega, D)$ backward equi-attracts $\mathcal{D}$ in the sense of (9). If it is not true, then there are $\eta > 0$, $\tau_0 \in \mathbb{R}$, $\omega_0 \in \Omega$ and $t_n \uparrow +\infty$ such that

$$\sup_{s \leq \tau_0} \text{dist}_{\mathcal{X}} \left(\Phi(t_n, s - t_n, \theta_{-t_n}\omega_0)D(s - t_n, \theta_{-t_n}\omega_0), \Gamma(\tau_0, \omega_0, D)\right) \geq 2\eta, \quad \forall n \in \mathbb{N}.$$ 

Furthermore, there exist $\tau_n \leq \tau_0$ and $x_n \in D(\tau_n - t_n, \theta_{-t_n}\omega_0)$ such that

$$\text{dist}_{\mathcal{X}} \left(\Phi(t_n, \tau_n - t_n, \theta_{-t_n}\omega_0)x_n, \Gamma(\tau_0, \omega_0, D)\right) \geq \eta, \quad \forall n \in \mathbb{N}. \quad (10)$$

By the backward asymptotical compactness of $\Phi$, there is a subsequence such that

$$\Phi(t_{n_k}, \tau_{n_k} - t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k} \to z, \quad \text{in } X.$$ 

By (i), $z \in \Gamma(\tau_0, \omega_0, D)$, which contradicts with (10). 

\qed
Proof. Li et al. [21] had given an example of ODE: \( \dot{x} = -x \) if \( t \leq 0 \), and \( \dot{x} = tx(1 - x^2)/2 \) if \( t > 0 \). The pullback attractor is given by \( A(\tau) = W(\tau, K(\tau)) \equiv \{0\} \) for \( \tau \in \mathbb{R} \), while the backward limit set \( \Gamma(t, K(t)) \equiv [-1, 1] \) for \( t > 0 \), where \( K(t) \equiv [-1/2, 1/2] \) is an absorbing set. So, the inclusion as given in (8) is strict. This example also indicates that a non-autonomous attractor is periodic even if the system is not periodic.

Now, we can establish the existence theorem of a backward compact random attractor.

**Theorem 2.8.** Let \( \Phi \) be a non-autonomous cocycle with an inclusion-closed universe \( \mathcal{D} \). Then, \( \Phi \) has a unique backward compact random attractor \( A \) from \( \mathcal{D} \) if \( \Phi \) satisfies the following two conditions:

(a) \( \Phi \) has a closed random absorbing set \( K \in \mathcal{D} \);

(b) \( \Phi \) is backward asymptotically compact on \( K \).

Moreover, there is an increasing compact set \( \mathfrak{A} \) which backward equi-attracts \( K \). In this case, \( A(\tau, \omega) = W(\tau, \omega, K) \) and \( \mathfrak{A}(\tau, \omega) = \Gamma(\tau, \omega, K) \).

**Proof.** Firstly, We show that \( \Phi \) is \( \mathcal{D} \)-asymptotically compact. Given \( t_n \to +\infty \) and \( x_n \in \mathcal{D}(\tau - t_n, \theta_{-t_n} \omega) \) with \( \mathcal{D} \in \mathcal{D} \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), we need to show that the sequence

\[
y_n := \Phi(t_n, \tau - t_n, \theta_{-t_n} \omega)x_n \quad \text{is pre-compact.}
\]

For each \( k \in \mathbb{N} \), by the absorption of \( K \), there is a \( T_k > 0 \) such that for all \( t \geq T_k \),

\[
\Phi(t, \tau - k - t, \theta_{-t} \theta_{-k} \omega)D(\tau - k - t, \theta_{-t} \theta_{-k} \omega) \subset K(\tau - k, \theta_{-k} \omega).
\]

By \( t_n \to +\infty \), we can take a subsequence \( \{t_{n_k}\} \) of \( \{t_n\} \) such that \( t_{n_k} - k \geq T_k \). Since \( x_{n_k} \in D(\tau - k - (t_{n_k} - k), \theta_{-(t_{n_k} - k)} \theta_{-k} \omega) \), we have

\[
z_k := \Phi(t_{n_k} - k, \tau - k - (t_{n_k} - k), \theta_{-(t_{n_k} - k)} \theta_{-k} \omega)x_{n_k} \in K(\tau - k, \theta_{-k} \omega).
\]

By the cocycle property of \( \Phi \), we have the corresponding subsequence

\[
y_{n_k} = \Phi(k, \tau - k, \theta_k \omega)\Phi(t_{n_k} - k, \tau - k - (t_{n_k} - k), \theta_{-(t_{n_k} - k)} \theta_{-k} \omega)x_{n_k} = \Phi(k, \tau - k, \theta_k \omega)z_k.
\]

Now, the condition (b) implies that \( \{y_{n_k}\} \) has a convergent subsequence.

Secondly, by using the \( \mathcal{D} \)-asymptotic compactness as proved above, we apply [24, Theorem 2.23] to obtain a unique \( \mathcal{D} \)-random attractor \( A \in \mathcal{D} \) given by \( A(\tau, \omega) = W(\tau, \omega, K) \).

Thirdly, the backward compactness of \( A \) follows from (b) and (iv) in Proposition 3. The equi-atraction of \( \Gamma(\tau, \omega, K) \) follows from (v) in Proposition 3. □

For a compact system, we have the following simple criterion.

**Proposition 4.** Let \( \Phi \) be a non-autonomous cocycle with an inclusion-closed universe \( \mathcal{D} \). Then, \( \Phi \) has a unique backward compact random attractor \( A \) from \( \mathcal{D} \) if

(c) \( \Phi \) has an increasing, compact and random absorbing set \( K \in \mathcal{D} \).

**Proof.** Let \( \mathcal{D} \in \mathcal{D} \) and \( t_n \to +\infty \). By the absorption of \( K \), we have

\[
\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega)D(\tau - t_n, \theta_{-t_n} \omega) \subset K(\tau, \omega)
\]
provided \( t_n \) is large enough. By (c), \( K_\tau(\tau, \omega) \) is compact, and so \( \Phi \) is asymptotic compact. Hence, by [24, Theorem 2.23], \( \Phi \) has a unique \( D \)-random attractor \( A \) given by \( A(\tau, \omega) = \mathcal{W}(\tau, \omega, K) \).

As \( K \in D \), it can be absorbed by itself, we have, for all \( s \leq \tau \) and some \( t_0(s) > 0 \),

\[
\bigcup_{t \geq t_0(s)} \Phi(t, s - t, \theta_{-t} \omega) K(s - t, \theta_{-t} \omega) \subset K(s, \omega) = \mathcal{K}(s, \omega).
\]

Noting that \( K \) is assumed to be increasing, we further have

\[
\bigcup_{s \leq \tau} A(s, \omega) = \bigcup_{s \leq \tau} \mathcal{W}(s, \omega, K) = \bigcup_{s \leq \tau} \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, s - t, \theta_{-t} \omega) K(s - t, \theta_{-t} \omega)
\]

\[
\subset \bigcup_{s \leq \tau} \mathcal{K}(s, \omega) = \mathcal{K}(\tau, \omega),
\]

which implies that \( A \) is backward compact in view of the compactness of \( K \).

In fact, the existence of an increasing, bounded absorbing set is a necessary condition for the existence of a backward compact random attractor.

**Proposition 5.** Suppose \( \Phi \) has a backward compact random attractor \( A \) from \( D \). Then, \( \Phi \) has an increasing, closed, bounded and absorbing set \( K \).

**Proof.** Given \( \eta > 0 \), we define a bi-parametric set \( K \) by

\[
K(\tau, \omega) := \bigcup_{s \leq \tau} N_\eta A(s, \omega) = N_\eta \left( \bigcup_{s \leq \tau} A(s, \omega) \right),
\]

where \( N_\eta(\cdot) \) denotes the closed \( \eta \)-neighborhood of a set. Since \( \bigcup_{s \leq \tau_1} A(s, \omega) \subset \bigcup_{s \leq \tau_2} A(s, \omega) \) for \( \tau_1 \leq \tau_2 \), we have \( K(\tau_1, \omega) \subset K(\tau_2, \omega) \) and so \( K \) is increasing. Since \( A \) is backward compact, \( \bigcup_{s \leq \tau} A(s, \omega) \) is pre-compact and thus bounded. Hence, its \( \eta \)-neighborhood \( K \) is bounded still.

Finally, the absorption of \( K \) follows from the \( D \)-pullback attraction of \( A \). In fact, we can prove that \( K \) is \textit{backward absorbing} in the following sense: for each \( s \leq \tau \) with \( \tau \in \mathbb{R} \) and \( D \in D \), there is a \( T(s, D) > 0 \) such that for all \( t \geq T(s, D) \),

\[
\Phi(t, s - t, \theta_{-t} \omega) D(s - t, \theta_{-t} \omega) \subset K(\tau, \omega).
\]

Indeed, by the \( D \)-pullback attraction of \( A \) at each \( s \) with \( s \leq \tau \), we have

\[
\lim_{t \to +\infty} \text{dist}_X(\Phi(t, s - t, \theta_{-t} \omega) D(s - t, \theta_{-t} \omega), A(s, \omega)) = 0.
\]

Hence, there is a \( T = T(s) > 0 \) such that for all \( t \geq T \),

\[
\Phi(t, s - t, \theta_{-t} \omega) D(s - t, \theta_{-t} \omega) \subset N_\eta(A(s, \omega)) \subset K(\tau, \omega),
\]

which means that \( K \) absorbs \( D \) backwards (or in the past).

**Remark 4.** We do not know whether \( K \) given by (11) is a random set. Although we can show that each \( N_\eta A(s, \cdot) \) is measurable, their union over an uncountable set \((-\infty, \tau]\) may not be measurable. On the other hand, we do not know whether \( K \) given by (11) is a member in the universe \( D \). Those difficulties will be overcome in the application part.
3. Non-autonomous stochastic sine-Gordon equation. Throughout the application part, the metric dynamical system \((\Omega, \mathcal{F}, \mathcal{P}, \theta)\) is defined by \(\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}\), \(\mathcal{F}\) is the Borel algebra under the Frechét topology of \(\Omega\), \(\mathcal{P}\) is the Wiener measure. The Wiener process \(W(t, \omega)\) is identified with \(\omega(t)\) for all \(t \in \mathbb{R}\) and \(\omega \in \Omega\).

3.1. Translation of variables. Let \(z(\theta, \omega) = -\int_{-\infty}^{0} e^s(\theta, \omega)(s)ds\), which solves the stochastic Ornstein-Uhlenbeck equation \(dz + z dt = dW(t)\). By [3, 11], we have the following limits:

\[
\lim_{t \to \pm \infty} \frac{z(\theta, t)}{t} = \lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^{t} z(\theta, s)ds = 0, \tag{12}
\]

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^{t} |z(\theta, s)|^m ds = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi}}, \forall m > 0, \tag{13}
\]

where, \(\Gamma\) denotes the Gamma function.

We take the state space by \(X = H^1_0(Q) \times L^2(Q)\), which is equipped with the following norm

\[
\|\varphi\|_X := (\delta^2 - \delta + \lambda)\|u\|^2 + \|\nabla u\|^2 + \|v\|^2, \quad \forall \varphi = (u, v) \in X, \tag{14}
\]

where \(\delta\) is a suitable constant such that \(\delta^2 - \delta + \lambda > 0\) with \(\delta \in (0, 1/3)\), and \(\|\cdot\|\) denotes the \(L^2\)-norm.

Let \(v(t, \omega) = u_t(t, \omega) + \delta u(t, \omega) - \varepsilon z(\theta, \omega) u(t, \omega)\). We can rewrite the second-order equation (1) as the following first-order equations:

\[
\begin{cases}
u_t = \varepsilon z(\theta, \omega) u - \delta u + v, \\
u + (1 - \delta) v + \delta u - \Delta u + \beta \sin u = (2\delta - \varepsilon) \varepsilon z u - \varepsilon z v + f(t), \\
u(\tau) = u_0, v(\tau) = v_0 := u_0 + \delta u_0 - u_0 \varepsilon z(\theta, \omega).
\end{cases} \tag{15}
\]

3.2. Hypotheses and non-autonomous cocycle. We make the assumptions on the time-dependent force \(f\).

**Hypothesis I.** Backward tempered. The force \(f \in L^2_{\text{loc}}(\mathbb{R}, L^2(Q))\) satisfies the following condition:

\[
F_\gamma(\tau) := \sup_{s \leq \tau} \int_{-\infty}^{s} e^{\gamma(r-s)} \|f(r)\|^2 dr < +\infty, \text{ for some } \gamma > 0, \quad \tau \in \mathbb{R}.
\]

**Lemma 3.1.** If \(f \in L^2_{\text{loc}}(\mathbb{R}, L^2(Q))\), then the following two conditions are equivalent.

A. \(F_{\gamma_0}(\tau_0) < +\infty\) for some \(\gamma_0 > 0\) and some \(\tau_0 \in \mathbb{R}\).

B. \(F_\gamma(\tau) < +\infty\) for all \(\gamma > 0\) and all \(\tau \in \mathbb{R}\).

**Proof.** It suffices to prove that A implies B. Assume that \(F_{\gamma_0}(\tau_0) < +\infty\) for some \(\gamma_0 > 0\) and \(\tau_0 \in \mathbb{R}\). We first prove the backward translation boundedness at \(\tau_0\):

\[
\sup_{s \leq \tau_0} \int_{s-1}^{s} \|f(r)\|^2 dr \leq e^{\gamma_0} \sup_{s \leq \tau_0} \int_{s-1}^{s} e^{\gamma_0(r-s)} \|f(r)\|^2 dr \leq e^{\gamma_0} F_{\gamma_0}(\tau_0) < +\infty.
\]

Let \(\tau\) be an arbitrary real number. We then prove the backward translation boundedness at \(\tau\):

\[
\sup_{s \leq \tau} \int_{s-1}^{s} \|f(r)\|^2 dr \leq \sup_{s \leq \tau_0} \int_{s-1}^{s} \|f(r)\|^2 dr + \int_{\tau_0-1}^{\tau} \|f(r)\|^2 dr
\leq e^{\gamma_0} F_{\gamma_0}(\tau_0) + \int_{\tau_0-1}^{\tau} \|f(r)\|^2 dr < +\infty
\]
in view of the local integrability of $f$. Finally, let $\gamma$ be an arbitrary positive number, we have

$$F_{\gamma}(r) = \sup_{s \leq r} \sum_{n=0}^{\infty} e^{\gamma(r-s)} \|f(r)\|^2 dr \leq \sup_{s \leq \tau} \sum_{n=0}^{\infty} e^{-n\gamma} \int_{s-n-1}^{s-n} \|f(r)\|^2 dr$$

$$\leq \sum_{n=0}^{\infty} e^{-n\gamma} \cdot \sup_{s_1 \leq \tau} \int_{s_1-1}^{s_1} \|f(r)\|^2 dr$$

$$\leq \frac{1}{1 - e^{-\gamma}} \left( e^{\gamma_0} F_{\gamma_0}(\tau_0) + \int_{\tau_0-1}^{\tau_0} \|f(r)\|^2 dr \right) < +\infty.$$ 

The proof is complete. 

By the proof of Lemma 3.1, the backward tempered condition is equivalent to the backward translation boundedness, see [9, 31]. In particular, the backward tempered condition implies the usual tempered condition:

$$\int_{-\infty}^{\tau} e^{\delta(r-\tau)} \|f(r)\|^2 dr < +\infty, \text{ for all } \tau \in \mathbb{R}. \quad (16)$$

So, we can obtain the following existence and uniqueness of solutions by the same method for the 3D sine-Gordon equation given in [11, 29].

**Lemma 3.2.** For each $\varphi_0 = (u_0, v_0) \in X$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, Eq. (15) has a unique solution

$$\varphi(\cdot, \tau, \omega, \varphi_0) := (u(\cdot, \tau, \omega, u_0), v(\cdot, \tau, \omega, v_0)) \in C([\tau, \infty), X)$$

such that $\varphi(\tau, \tau, \omega, \varphi_0) = \varphi_0$. Moreover, $\varphi$ is continuous in $\varphi_0$.

By the same method as given in Cui et al. [8], one can show that the mapping $\omega \mapsto \varphi(t, \tau, \omega, \varphi_0)$ is Lusin continuous from $\Omega$ to $X$ and thus $(\mathcal{F}, \mathcal{B}(X))$-measurable.

So, we can define a continuous non-autonomous cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$, given by

$$\Phi(t, \tau, \omega, \varphi_0) = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_0) \quad (17)$$

for every $(t, \tau, \omega, \varphi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X$.

Now, we specify the universe $\mathcal{D}$ of all backward tempered sets (instead of the usual tempered sets), where a bi-parametric set $\mathcal{D} = \{ D(t, \omega) \}$ in $X$ is called backward tempered if

$$\lim_{t \to +\infty} e^{-\delta t} \sup_{s \leq \tau} \|D(s - t, \theta_{-t} \omega)\|_X^2 = 0, \forall \tau \in \mathbb{R}, \omega \in \Omega, \quad (18)$$

where $\delta > 0$ is given in (14). This universe is backward-closed, that is, $\hat{\mathcal{D}} \in \mathcal{D}$ whenever $\mathcal{D} \in \mathcal{D}$ and $\mathcal{D}(\tau, \omega) = \sup_{s \leq \tau} D(s, \omega)$.

In order to obtain a $\mathcal{D}$-pullback random absorbing set, we need the following hypothesis of small noise.

**Hypothesis II.** Small noise. The density of noise is small: $\epsilon \in (0, \epsilon_0]$, where

$$\epsilon_0 = \frac{\sqrt{\pi} \delta}{2(2 + \delta_2)(6 + \sqrt{\pi})}, \quad \delta_2 = \max\{1, \frac{1}{\delta_1}\}, \quad \delta_1 = \delta^2 - \delta + \lambda. \quad (19)$$
3.3. Backward uniform absorption. In order to prove that the cocycle is backward asymptotically compact, we need to show that the absorption is uniform in the past.

In this case, the absorbing radius may be a maximum of some random function over an uncountable index set \((-\infty, \tau]\). We will overcome the difficulty of measurability for the absorbing set, by expanding the absorbing radius and by splitting both time and sample variables.

We will frequently impose a result on the quadratic polynomial of the Ornstein-Uhlenbeck process:

\[
y(\omega) := 2z(\omega) + \delta_2(5|z(\omega)| + z^2(\omega)), \quad \forall \omega \in \Omega.
\]  

(20)

Lemma 3.3. There is a random time \(t_0 = t_0(\omega) > 0\) such that

\[
e^\epsilon \int_{-\tau}^0 y(\theta_\tau, \omega) \, d\tau \leq \frac{\delta}{2} \tau, \quad \forall \tau \geq t_0(\omega), \quad \epsilon \leq \epsilon_0  
\]  

(21)

\[
e^{\epsilon^*} \int_{-\tau}^0 y(\theta_\tau, \omega) \, d\tau \leq e^{R_0(\omega)} e^{\frac{\delta}{2} \tau}, \quad \forall \tau \geq 0, \quad \epsilon \leq \epsilon_0,  
\]  

(22)

where \(R_0\) is a positive random variable, given by

\[
R_0(\omega) := \epsilon_0(2 + \delta_2) \int_{-t_0(\omega)}^0 (5|z(\theta_\tau, \omega)| + z^2(\theta_\tau, \omega)) \, d\tau.
\]

Proof. By (13) and (19), there is a \(t_0 = t_0(\omega) > 0\) such that for all \(t \geq t_0\) and \(\epsilon \leq \epsilon_0\),

\[
e\left| \int_{-t}^0 y(\theta_\tau, \omega) \, d\tau \right| \leq \epsilon_0(2 + \delta_2) \int_{-t}^0 (5|z(\theta_\tau, \omega)| + z^2(\theta_\tau, \omega)) \, d\tau
\]

\[
\leq \epsilon_0(2 + \delta_2) \left( \frac{6\Gamma(1)}{\sqrt{\pi}} + \frac{2\Gamma(3/2)}{\sqrt{\pi}} \right) t
\]

\[
\leq \epsilon_0(2 + \delta_2) \left( \frac{6}{\sqrt{\pi}} + 1 \right) t \leq \frac{\delta}{2} t,
\]

which proves (21). By this, we have, for all \(t \geq t_0\) and \(\epsilon \leq \epsilon_0\),

\[
e^{\epsilon^*} \int_{-t}^0 y(\theta_\tau, \omega) \, d\tau \leq e^{\epsilon^*} \int_{-t}^0 y(\theta_\tau, \omega) \, d\tau \leq e^{\frac{\delta}{2} t} \leq e^{R_0(\omega)} e^{\frac{\delta}{2} t}.
\]

For all \(t \in [0, t_0]\) and \(\epsilon \leq \epsilon_0\), we have

\[
e^{\epsilon^*} \int_{-t}^0 y(\theta_\tau, \omega) \, d\tau \leq e^{\epsilon^*} \int_{-t}^0 y(\theta_\tau, \omega) \, d\tau
\]

\[
\leq e^{\epsilon^* (2 + \delta_2)} \int_{-t}^0 (5|z(\theta_\tau, \omega)| + z^2(\theta_\tau, \omega)) \, d\tau \leq e^{R_0(\omega)} e^{\frac{\delta}{2} t}.
\]

Therefore, (22) holds true for all \(t \geq 0\).

\[
\square
\]

Proposition 6. Let the hypotheses I and II be satisfied. Then, for each \((\tau, \omega, D) \in \mathbb{R} \times \Omega \times D\), there is a \(T := T(\tau, \omega, D) > 0\) such that

\[
\sup_{s \leq \tau} \sup_{t \geq T} \sup_{\varphi_0 \in D(s-t, \theta_{-t} \omega)}\|\varphi(s, s-t, \theta_{-t} \omega, \varphi_0)\|_X^2 \leq c_0 e^{R_0(\omega)} (F_\delta(\tau) + 1),
\]

(23)

and for all \(s \leq \tau\), \(t \geq T\), \(\varphi_0 \in D(s-t, \theta_{-t} \omega)\),

\[
\int_s^\tau e^{\epsilon^* (\frac{\delta}{2} - \epsilon y(\theta_{-t} \omega))} \, d\tau \|\varphi(r, s-t, \theta_{-t} \omega, \varphi_0)\|_X^2 \, dr \leq \frac{2c_0}{\delta} e^{R_0(\omega)} (F_\delta(\tau) + 1),
\]

(24)
where \( c_0 = \max\{\frac{\beta^2}{4}\|Q\|, \frac{1}{1-3\delta}\}\), \( R_0(\omega) \) is the random variable given in Lemma 3.3, and \( F_\delta(\cdot) \) is an increasing function given by

\[
F_\delta(\tau) = \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\delta r} \|f(r+s)\|^2 dr < +\infty, \forall \tau \in \mathbb{R}.
\]

In this case, the cocycle \( \Phi \) has an increasing random absorbing set \( \mathcal{K} \in \mathcal{D} \), given by

\[
\mathcal{K}(\tau, \omega) = \left\{ \varphi = (u, v) \in H^1_0(Q) \times L^2(Q) : \|\varphi\|_X^2 \leq c_0e^{R_0(\omega)}(F_\delta(\tau)+1) \right\}. \tag{25}
\]

Moreover, the absorption is uniform in the past.

**Proof.** Let \( s \leq \tau \) with \( \tau \in \mathbb{R}, t > 0 \) and \( \omega \in \Omega \), which are fixed. Taking the inner product of the second equation in (15) with \( v(r) = v(r, s-t, \theta_{r-s}\omega, \nu_0) \), \( r \geq s-t \), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dr} \|v\|^2 + (1 - \delta) \|v\|^2 + \delta_1(u, v) - (\Delta u, v) + \beta(\sin u, v) \\
= (2\delta - \epsilon z(\theta_{r-s}\omega))\epsilon z(\theta_{r-s}\omega)(u, v) - \epsilon z(\theta_{r-s}\omega) \|v\|^2 + (f(x, r), v),
\end{align*}
\]

where we recall \( \delta_1 = \delta^2 - \delta + \lambda > 0 \). We multiply the first equation of (15) with \( u \) and \( \Delta u \) respectively, the results are

\[
\begin{align*}
\delta_1(u, v) &= \frac{1}{2} \frac{d}{ds} (\delta_1 \|u\|^2) + (\delta - \epsilon z(\theta_{r-s}\omega)) (\delta_1 \|u\|^2), \quad \text{and} \tag{27} \\
- (\Delta u, v) &= \frac{1}{2} \frac{d}{ds} \|\nabla u\|^2 + (\delta - \epsilon z(\theta_{r-s}\omega)) \|\nabla u\|^2. \tag{28}
\end{align*}
\]

By \( |\sin u| \leq 1 \) and \( \delta < 1/3 \), the Young inequality gives

\[
\begin{align*}
- (\beta \sin u, v) &\leq \int_Q (\frac{\beta^2}{4\delta} + \delta v^2) dx \leq \delta \|v\|^2 + \frac{\beta^2}{4\delta}\|Q\|. \tag{29} \\
(f(x, r), v) &\leq (1 - 3\delta) \|v\|^2 + \frac{1}{4(1 - 3\delta)} \|f(r)\|^2. \tag{30}
\end{align*}
\]

We substitute (27)-(30) into (26) to find

\[
\begin{align*}
\frac{d}{dr} \|\varphi\|_X^2 &+ 2(\delta - \epsilon z(\theta_{r-s}\omega)) \|\varphi\|_X^2 + 4\epsilon z(\theta_{r-s}\omega) \|v\|^2 \\
&\leq 2(2\delta - \epsilon z(\theta_{r-s}\omega)) \epsilon z(\theta_{r-s}\omega)(u, v) + \frac{c_0}{2} \|f(r)\|^2 + 1, \tag{31}
\end{align*}
\]

where \( c_0 = \max\{\frac{\beta^2}{4}\|Q\|, \frac{1}{1-3\delta}\} \) and \( \|\varphi\|_X^2 = \delta_1 \|u\|^2 + \|\nabla u\|^2 + \|v\|^2 \).

On the other hand, note that \( \epsilon \leq c_0 < 1 \) and so \( \epsilon^2 \leq \epsilon \), which along with \( \delta < 1/3 \) implies that

\[
\begin{align*}
2(2\delta - \epsilon z(\theta_{r-s}\omega)) \epsilon z(\theta_{r-s}\omega)(u, v) - 4\epsilon z(\theta_{r-s}\omega) \|v\|^2 \\
&\leq \epsilon (|z(\theta_{r-s}\omega)| + z^2(\theta_{r-s}\omega))(\|u\|^2 + \|v\|^2) + 4\epsilon |z(\theta_{r-s}\omega)| \|v\|^2 \\
&\leq \epsilon (5|z(\theta_{r-s}\omega)| + z^2(\theta_{r-s}\omega)) \delta_2 \|\varphi\|_X^2,
\end{align*}
\]

where we recall that \( \delta_2 = \max\{1/\delta_1, 1\} \). We substitute the above inequality into (31) to find

\[
\begin{align*}
\frac{d}{dr} \|\varphi\|_X^2 + \left( 3\delta - \epsilon y(\theta_{r-s}\omega) \right) \|\varphi\|_X^2 + \frac{1}{2} \delta \|\varphi\|_X^2 &\leq \frac{c_0}{2} \|f(r)\|^2 + 1. \tag{32}
\end{align*}
\]
where \( y(\omega) = 2z(\omega) + \delta_2(5|z(\omega)| + z^2(\omega)) \) as given in Lemma 3.3. Multiplying (32) by \( e^{\int_{s-t}^s (\frac{4}{3} \delta - \epsilon y(\theta_s - \omega)) ds} \) and then integrating in \( r \in (s-t, s) \) with \( s \leq \tau \), we obtain that

\[
\| \varphi(s, s-t, \theta_{s-t}\omega, \varphi_0) \|^2_X + \frac{\delta}{2} \int_{s-t}^s e^{\int_r^s (\frac{4}{3} \delta - \epsilon y(\theta_r - \omega)) ds} \| \varphi(r, s-t, \theta_{s-t}\omega, \varphi_0) \|^2_X dr \leq e^{-\frac{2}{3} \epsilon t + e_x^0 y(\theta, \omega) dr} \sup_{t \leq \tau} \| \varphi_0 \|^2_X + \frac{C_0}{2} \int_{s-t}^s e^{\int_r^s (\frac{4}{3} \delta - \epsilon y(\theta_r - \omega)) ds} \| f(r) \|^2 + 1) dr.
\]

(33)

We use (22) in Lemma 3.3 to obtain

\[
e^{-\frac{2}{3} \epsilon t + e_x^0 y(\theta, \omega) dr} \leq e^{R_0(\omega)} e^{-\delta t}, \quad \forall t \geq 0, \quad \forall \epsilon \leq \epsilon_0.
\]

(34)

Since the initial value \( \varphi_0 \in \mathcal{D}(s-t, \theta_{s-t}\omega) \) depending on \( s \leq \tau \), the first term in (33) is bounded by

\[
e^{-\frac{2}{3} \epsilon t + e_x^0 y(\theta, \omega) dr} \sup_{t \leq \tau} \| \varphi_0 \|^2_X \leq e^{R_0(\omega)} e^{-\delta t} \sup_{s \leq \tau} \| \mathcal{D}(s-t, \theta_{s-t}\omega) \|^2_X \to 0
\]

(35)

as \( t \to +\infty \), where the backward tempered property is applied. By (34) again, the last term in (33) is bounded by

\[
\frac{C_0}{2} \sup_{s \leq \tau} \int_{s-t}^s e^{\int_r^s (\frac{4}{3} \delta + e \epsilon y(\theta_r - \omega)) ds} \| f(r) \|^2 + 1) dr \leq \frac{C_0}{2} e^{R_0(\omega)} (F_\delta(\tau) + 1),
\]

(36)

which together with (35) and (33) imply both formulas (23) and (24).

From (23), we see that \( \mathcal{K} \) is a backward-uniformly absorbing set. Since \( F_\delta(\cdot) \) is an increasing function, it follows that \( \tau \to \mathcal{K}(\tau, \omega) \) is increasing according to the inclusion relationship. In particular, \( \mathcal{K} \in \mathcal{D} \) because \( \cap_{s \leq \tau} \mathcal{K}(s, \omega) = \mathcal{K}(\tau, \omega) \).

4. Further estimates via spectrum decomposition. It is well know that there is a countable family of eigenvalues: \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to +\infty \) for the unbounded \( -\Delta \) with the Dirichlet boundary condition. We know that the corresponding eigenfunctions \( \{ e_n \}_{n=1}^\infty \subset H_0^1(Q) \) and constitute a complete orthogonal basis of \( L^2(Q) \).

We then consider the composite projection operator \( \mathcal{P}_n = P_n \times P_n \) from \( L^2(Q) \times L^2(Q) \) into \( X_n \times X_n \), where \( X_n = \text{span} \{ e_1, e_2, \ldots, e_n \} \subset H_0^1(Q) \) and \( P_n : L^2(Q) \to X_n \) is the canonical projection.

In this case, the solution \( \varphi = (u, v) \) of Eq.(15) has the following decomposition

\[
\varphi = (u, v) = \mathcal{P}_n(u, v) \oplus (I - \mathcal{P}_n)(u, v) =: (u_{n1}, v_{n1}) \oplus (u_{n2}, v_{n2}), \quad \forall n \in \mathbb{N}.
\]

(36)

Note that \( u \in H_0^1(Q) \) and so \( (I - \mathcal{P}_n)u \in H_0^1(Q) \). We have the Poincaré inequality:

\[
\| \nabla u_{n2} \|^2 \geq \lambda_{n+1} \| u_{n2} \|^2.
\]

However, we have not the similar inequality for \( v \) because \( v \) has not the smoothness.

We also require an additional assumption on the force.

**Hypothesis III.** Backward complement-smallness. The force \( f \) satisfies

\[
\lim_{n \to \infty} \sup_{s \leq \tau} \int_{-\infty}^0 e^{\frac{2}{3} \epsilon r} \| (I - \mathcal{P}_n)f(r + s) \|^2 dr = 0, \quad \forall \tau \in \mathbb{R}.
\]
Proposition 7. Let the hypothesis I, II, III be satisfied. Then, for each \((r, \omega, D) \in \mathbb{R} \times \Omega \times \mathcal{D}\), we have
\[
\lim_{n \to +\infty} \sup_{s \leq r} \sup_{\varphi_0 \in \mathcal{D}(s-t, \theta_{-s}\omega)} \| (I - P_n) \varphi(s, s-t, \theta_{-s}\omega, \varphi_0) \|^2_X = 0. \tag{37}
\]

Proof. For each \(n \in \mathbb{N}\), we take the inner product of the second equation in (15) with \(v_{n2}(r) = v_{n2}(r, s-t, \theta_{-s}\omega, \varphi_0, v_{0,n2}), \ r \geq s-t\), the result is
\[
\frac{1}{2} \frac{d}{dr} \| v_{n2} \|^2 + (1 - \delta) \| v_{n2} \|^2 + \delta_1(u_{n2}, v_{n2}) - (\Delta u_{n2}, v_{n2}) + \beta(\sin u, v_n) = (2\delta - \epsilon \zeta(\theta_{r-s}\omega)) \zeta(\theta_{r-s}\omega)(u_{n2}, v_{n2}) - \epsilon \zeta(\theta_{r-s}\omega) \| v_{n2} \|^2 + (f(r), v_{n2}). \tag{38}
\]
Applying \(I - P_n\) to the first equation in (15) with respect to the function \(u(r) = u(r, s-t, \theta_{-s}\omega, \varphi_0, v_0)\), we have
\[
v_{n2} = \frac{du_{n2}}{dr} + \delta u_{n2} - \epsilon \zeta(\theta_{r-s}\omega)u_{n2}, \tag{39}
\]
which further implies that
\[
\delta_1(u_{n2}, v_{n2}) = \frac{1}{2} \frac{d}{dr} (\delta_1 \| u_{n2} \|^2) + (\delta - \epsilon \zeta) \delta_1 \| u_{n2} \|^2, \tag{40}
\]
\[-(\Delta u_{n2}, v_{n2}) = \frac{1}{2} \frac{d}{dr} \| \nabla u_{n2} \|^2 + (\delta - \epsilon \zeta) \| \nabla u_{n2} \|^2. \tag{41}
\]
Since \(v\) has not enough smoothness, we need to translate \(v_{n2}\) in the sine function term into \(u_{n2}\). Indeed, by (39), we have
\[
(\beta \sin u, v_{n2}) = (\beta \sin u, \frac{du_{n2}}{dr}) + (\delta - \epsilon \zeta)(\beta \sin u, u_{n2}) = \frac{d}{dr}(\beta \sin u, u_{n2}) - \beta(u_r \cos u, u_{n2}) + (\delta - \epsilon \zeta)(\beta \sin u, u_{n2}). \tag{42}
\]
By the Young inequality, we have
\[
(f(r), v_{n2}) \leq (1 - 2\delta) \| v_{n2} \|^2 + c \| (I - P_n)f(r) \|^2. \tag{43}
\]
Substituting (40)-(43) into (38), we obtain
\[
\frac{d}{dr} \psi(r) + 2(\delta - \epsilon \zeta) \psi(r) \leq 2\beta(u_r \cos u, u_{n2}) + 2(\delta - \epsilon \zeta)\beta(\sin u, u_{n2}) + 2(\delta - \epsilon \zeta)\epsilon \zeta(u_{n2}, v_{n2}) - 4\epsilon \| v_{n2} \|^2 + c \| (I - P_n)f(r) \|^2, \tag{44}
\]
where \(\psi\) is defined by
\[
\psi := \delta_1 \| u_{n2} \|^2 + \| \nabla u_{n2} \|^2 + \| v_{n2} \|^2 + 2\beta(\sin u, u_{n2}) = \| \varphi_{n2} \|^2_X + 2\beta(\sin u, u_{n2}).
\]
The terms involving \(v_{n2}\) in (44) is bounded by
\[
2(\delta - \epsilon \zeta)\epsilon \zeta(u_{n2}, v_{n2}) - 4\epsilon \| v_{n2} \|^2 \\
\leq(2\delta \epsilon |z| + \epsilon^2 z^2)\| u_{n2} \|^2 + \| v_{n2} \|^2 + 4\epsilon |z| \| v_{n2} \|^2 \\
\leq(5|z| + z^2)\| \varphi_{n2} \|^2_X = \epsilon (5|z| + z^2)\delta_2 \psi - \epsilon (5|z| + z^2)\delta_2 \cdot 2\beta(\sin u, u_{n2}).
\]
So, by recalling \(y = 2z + \delta_2(5|z| + z^2)\) in Lemma 3.3, we can rewrite (44) as follows.
\[
\frac{d}{dr} \psi(r) + (2\delta - \epsilon y) \psi(r) \leq 2\beta(u_r \cos u, u_{n2}) + 2(\delta - \epsilon \zeta - 5\delta_2\epsilon |z| - \delta_2\epsilon z^2)\beta(\sin u, u_{n2}) + c \| (I - P_n)f(r) \|^2. \tag{45}
\]
We estimate each term on the right-hand side of (45). By the first equation in Eq. (15), we have \( u_r = \varepsilon u - \delta u + v \). Hence, by \(|\cos u| \leq 1\), we have

\[
2\beta(u_r \cos u, u_{n2}) \leq 2|\beta||z| + \delta + 1)(||u|| + ||v||) \cdot \lambda_n^{-1/2} ||u_{n2}||
\]

\[
\leq c\lambda_n^{-1/2}(||u|| + ||v||)\nabla u || \leq c\lambda_n^{-1/2}(||u|| + ||v||)\nabla u || \leq c\lambda_n^{-1/2}(||u|| + ||v||)\nabla u || \leq c\lambda_n^{-1/2}(\|\varphi\|^2 + 1).
\]

Similarly, by \(|\sin u| \leq 1\) and \(\epsilon \leq \epsilon_0\), we have

\[
2(\delta - \varepsilon z - 5\delta_2 |z| - \delta_2 \varepsilon z^2)(\beta \sin u, u_{n2}) \leq c\lambda_n^{-1/2}(\|z^2 + \|z + 1\|\nabla u|| \leq c\lambda_n^{-1/2}(\|z^2 + |z + 1\|\nabla u|| \leq c\lambda_n^{-1/2}(\|\varphi\|^2 + 1).
\]

Therefore, we can rewrite (45) as follows.

\[
\frac{d}{dr}\psi(r) + (2\delta - \varepsilon y(\theta_r - \omega))\psi(r) \leq c \left\| (I - P_n)f(r) \right\|^2 + c\lambda_n^{-1/2} z^2(\theta_r - \omega) + |z(\theta_r - \omega)| + 1)\|\|\varphi\|^2 + 1).
\]

(46)

Multiplying (46) by \( e^{\int_t^r (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \) and then integrating in \( r \in (s - t, s) \) with \( s \leq \tau \), we obtain that

\[
\psi(s, s - t, \theta - s, \omega, \psi_0) \leq e^{-\int_t^r (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \psi_0 + c\lambda_n^{-1/2} I_1(s, t) + c \int_{s-t}^s e^{\int_t^r (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r) \right\|^2 dr.
\]

(47)

where \( I_1 \) is defined as follows.

\[
I_1(s, t) := \int_{s-t}^s e^{\int_t^r (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r) \right\|^2 dr.
\]

\[
= \int_{s-t}^s e^{\int_t^r (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r) \right\|^2 dr,
\]

where \( \hat{z} = z^2 + |z + 1\). By the tempered property (12) of \( z \), we know that

\[
e^{-\int_{s-t}^s (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r) \right\|^2 dr \rightarrow 0 \text{ as } \hat{r} \rightarrow -\infty.
\]

So, there is a random variable \( C(\omega) \) such that for all \( r \in (s - t, s) \), i.e. \( \hat{r} := r - s \leq 0 \),

\[
e^{-\int_{s-t}^s (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r) \right\|^2 dr \leq C(\omega) e^{R_0(\omega)(F_\delta(\tau) + 1)} < +\infty, \forall s \leq \tau.
\]

(48)

For the last term in (47), by the hypothesis \( \text{III} \), we know

\[
I_2(s, t, n) := \int_{s-t}^s e^{\int_t^r (2\delta - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r) \right\|^2 dr
\]

\[
= \int_{s-t}^s e^{2\delta r - \varepsilon y(\theta_r - \omega))d\sigma} \left\| (I - P_n)f(r + s) \right\|^2 dr
\]

\[
\leq e^{R_0(\omega)} \sup_{s \leq \tau} \int_{s-t}^s e^{2\delta r} \left\| (I - P_n)f(r + s) \right\|^2 dr \rightarrow 0
\]

(49)

as \( n \rightarrow \infty \), uniformly in \( s \leq \tau \) and \( t \geq 0 \).
For the second term in (47), we know
\[ \psi = \| \varphi_{0,n2} \|_X^2 + 2\beta (\sin u_0, u_{0,n2}) \leq c(\| \varphi_0 \|_X^2 + 1), \]
So, by the backward tempered property of \( D \) and \( \varphi_0 \in D(s-t, \theta_{-t}\omega) \), we have
\[ I_3(s,t) := e^{-\int_{s}^{t}(2\delta - \epsilon y(\theta_{-t}\omega))}d\tau \psi_0 \leq ce^{-\frac{1}{2}\delta t}(\sup_{s \leq \tau} \| \varphi_0 \|_X^2 + 1) \]
\[ \leq ce^{-\delta t}(\sup_{s \leq \tau} \| D(s-t, \theta_{-t}\omega) \|_X^2 + 1) \to 0, \quad \text{as } t \to +\infty. \quad (50) \]
We substitute (48)-(50) into (47) to find
\[ \sup_{s \leq \tau} \psi(s, s-t, \theta_{-t}\omega, \psi_0) \leq I_3(s,t) + c\lambda_{n+1}^{-1/2}I_1(s,t) + cI_2(s,t,n) \to 0, \]
as \( n, t \to +\infty \), in view of \( \lambda_{n+1} \to +\infty \). Therefore, by the relationship between \( \varphi \) and \( \psi \), we have
\[ \sup_{s \leq \tau} \| (I - \mathcal{P}_n)\varphi(s, s-t, \theta_{-t}\omega, \varphi_0) \|_X^2 \]
\[ \leq \sup_{s \leq \tau} \psi(s, s-t) + 2|\beta| \sup_{s \leq \tau} |(\sin u, u_{n2}(s, s-t))| \]
\[ \leq \sup_{s \leq \tau} \psi(s, s-t) + c\lambda_{n+1}^{-1/2} \left( \sup_{s \leq \tau} \| \varphi(s, s-t) \|_X^2 + 1 \right), \]
which tends to zero as \( n, t \to +\infty \), in view of the backward-uniform absorption as given in Proposition 6.

5. Backward compact and periodic random attractors. This section gives all application results for the non-autonomous stochastic sine-Gordon equation.

5.1. Backward compactness. Applying the backward-uniform estimates as given in Proposition 6 and Proposition 7, we obtain the following existence of a backward compact random attractor.

Theorem 5.1. Let the hypotheses I, II, III be satisfied. Then, the non-autonomous cocycle \( \Phi \), induced by the stochastic sine-Gordon equation, possesses a backward compact random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( X = H^0_0(Q) \times L^2(Q) \). Moreover, \( \mathcal{A} \) is a backward tempered set, that is, \( \mathcal{A} \in \mathcal{D} \).

Proof. By Proposition 6, the cocycle \( \Phi \) has a \( \mathcal{D} \)-pullback random absorbing \( \mathcal{K} \) as defined by (25). Moreover, \( \mathcal{K} \) is tempered and increasing. So, \( \mathcal{K} \) is backward tempered, that is, \( \mathcal{K} \in \mathcal{D} \).

By the abstract result as given in Theorem 2.8, it suffices to prove that \( \Phi \) is \( \mathcal{D} \)-asymptotically compact. For this end, we let \((\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathcal{D} \) be fixed, and take arbitrary sequences \( \tau_k \leq \tau, t_k \to +\infty \) and \( \varphi_{k,0} \in D(\tau_k - t_k, \theta_{-t_k}\omega) \). We then define
\[ \varphi_k := (u_k, v_k) \colon \Phi(t_k, \tau_k - t_k, \theta_{-t_k}\omega)\varphi_{k,0} = \varphi(\tau_k, \tau_k - t_k, \theta_{-t_k}\omega, \varphi_{k,0}). \]
It suffices to prove that the sequence \( \{ \varphi_k : k \in \mathbb{N} \} \) is pre-compact in \( X \).

Indeed, by Proposition 6, the sequence \( \{ \varphi_k : k \in \mathbb{N} \} \) is bounded in \( X \). Hence, passing to a subsequence, we have
\[ \varphi_k \rightharpoonup \bar{\varphi} \in X, \quad \text{weakly in } X, \quad \text{as } k \to \infty. \]
Given small \( \eta > 0 \). By Proposition 7, there are \( N \in \mathbb{N} \) and \( k_1 \in \mathbb{N} \) such that
\[ \| (I - \mathcal{P}_N)\varphi_k \|_X^2 < \eta, \quad \forall k \geq k_1, \quad \text{and } \| (I - \mathcal{P}_N)\bar{\varphi} \|_X^2 < \eta. \]
Note that \( \{ \mathcal{P}_N \varphi_k : k \in \mathbb{N} \} \) is bounded in the finitely dimensional subspace \( \mathcal{P}_N X \) of \( X \). We see that there exists a convergence subsequence of \( \{ \mathcal{P}_N \varphi_k : k \in \mathbb{N} \} \), the limit must be \( \mathcal{P}_N \hat{\varphi} \). Hence, there is \( k_2 \geq k_1 \) such that
\[
\| \mathcal{P}_N \varphi_k - \mathcal{P}_N \hat{\varphi} \|_X^2 < \eta, \quad \forall k \geq k_2.
\]
Therefore, for all \( k \geq k_2 \),
\[
\| \varphi_k - \hat{\varphi} \|_X^2 = \| \mathcal{P}_N (\varphi_k - \hat{\varphi}) \|_X^2 + \| (I - \mathcal{P}_N) (\varphi_k - \hat{\varphi}) \|_X^2 \\
\leq \eta + 2 \| (I - \mathcal{P}_N) \varphi_k \|_X^2 + 2 \| (I - \mathcal{P}_N) \hat{\varphi} \|_X^2 < 5 \eta,
\]
which proves that \( \{ \varphi_k \} \) has a convergent subsequence as required. \( \square \)

5.2. Establish backward compactness from periodicity. In this subsection, we need to assume that the non-autonomous force \( f \) is \( T \)-periodic.

**Hypothesis IV. Periodicity.** \( f \in L^2_{loc}(\mathbb{R}, L^2(\Omega)) \) is \( T \)-periodic with \( T > 0 \):
\[
f(x, t + \tau) = f(x, \tau), \quad \forall x \in Q, \tau \in \mathbb{R}.
\]

**Theorem 5.2.** Under the hypothesis II of small noise \( \epsilon \leq \epsilon_0 \) and the hypothesis IV of the periodic forcing, the non-autonomous cocycle \( \Phi \), induced by the stochastic sine-Gordon equation, possesses a \( T \)-periodic random attractor \( A \) in \( X = H^1_0(Q) \times L^2(Q) \). Moreover, \( A \) is backward compact.

**Proof.** Periodicity. For each \( \varphi_0 \in X \), we know that the mapping
\[
t \to \Phi(t, t + T, \omega) \varphi_0 = \varphi(t + T, t, \theta_{-T} \tau \omega, \varphi_0)
\]
satisfies the equation (15) with respect to the OU process \( z = z(t, t + T, \theta_{-T} \tau \omega, \varphi_0) \) and the force \( f = f(x, t + T) \). While, the mapping
\[
t \to \Phi(t, \tau, \omega) \varphi_0 = \varphi(t + \tau, t, \theta_{-\tau} \omega, \varphi_0)
\]
satisfies the equation (15) with respect to the OU process \( z = z(t, t + \tau, \theta_{-\tau} \omega) \) and the force \( f = f(x, t + \tau) \). We see that they have the same sample \( \theta_\omega \). On the other hand, by the \( T \)-periodicity of \( f \), we have \( f(x, t + \tau + T) = f(x, t + \tau) \), which means that both mappings satisfy the same equation. So, the uniqueness of the solution implies that
\[
\Phi(t, t + T, \omega) \varphi_0 = \Phi(t, \tau, \omega) \varphi_0,
\]
that is, the cocycle \( \Phi \) is \( T \)-periodic.

We then prove that the universe \( \mathcal{D} \) of all backward tempered sets is \( T \)-translation invariant. Let \( \mathcal{D} \in \mathcal{D} \). By (18), we have
\[
\lim_{t \to +\infty} e^{-\epsilon t} \sup_{s \leq \tau} \| \mathcal{D}(s - t, \theta_{-t} \omega) \|_X^2 = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
\]
We denote \( \mathcal{D}_T \) by \( \mathcal{D}_T(\tau, \omega) = \mathcal{D}(\tau + T, \omega) \). By replacing \( \tau \) by \( \tau + T \) in the above supremum, we have
\[
\lim_{t \to +\infty} e^{-\epsilon t} \sup_{s \leq \tau} \| \mathcal{D}_T(s - t, \theta_{-t} \omega) \|_X^2 = \lim_{t \to +\infty} e^{-\epsilon t} \sup_{s \leq \tau} \| \mathcal{D}(s - t + T, \theta_{-t} \omega) \|_X^2 = \lim_{t \to +\infty} e^{-\epsilon t} \sup_{s \leq \tau + T} \| \mathcal{D}(s - t, \theta_{-t} \omega) \|_X^2 = 0,
\]
which means \( \mathcal{D}_T \in \mathcal{D} \). Similarly, we can prove \( \mathcal{D}_{-T} \in \mathcal{D} \) and thus \( \mathcal{D} \) is \( T \)-translation invariant.
Note that the periodicity hypothesis implies the usual tempered condition (16). So, the standard method shows the existence of a random attractor $A$. By Theorem 2.6, we know that $A$ is $T$-periodic.

**Backward compactness.** We show the backward compactness of $A$ by verifying directly that $f$ is backward tempered (Hypothesis I) and backward complement-small (Hypothesis III).

Indeed, since $f$ is $T$-periodic, we know that

$$
\|f(s+T)\|^2 = \|f(s)\|^2, \quad \int_{kT}^{kT+T} \|f(r+s)\|^2 \, dr = \int_s^T \|f(r)\|^2 \, dr, \quad \forall s \in \mathbb{R}, \ k \in \mathbb{Z}.
$$

Hence, for each $\gamma > 0$ and $\tau \in \mathbb{R}$,

$$
F_\gamma(\tau) = \sup_{s \leq \tau} \int_{-\infty}^s e^{\gamma(r-s)} \|f(r)\|^2 \, dr = \sup_{s \leq \tau} \int_{-\infty}^s e^{\gamma r} \|f(r+s)\|^2 \, dr
$$

$$
= \sup_{s \leq \tau} \sum_{n=0}^\infty \int_{-kT}^{-kT-T} e^{\gamma r} \|f(r+s)\|^2 \, dr
$$

$$
\leq \sum_{k=0}^\infty e^{-kT\gamma} \sup_{s \leq \tau} \int_{-kT-T}^{-kT} \|f(r+s)\|^2 \, dr = \frac{1}{1 - e^{-\gamma T}} \int_0^T \|f(r)\|^2 \, dr.
$$

Therefore, by $f \in L^2_\text{loc}(\mathbb{R}, L^2(Q))$, we see that $F_\gamma(\tau) < +\infty$ and thus $f$ is backward tempered.

On the other hand, for each $n \in \mathbb{N}$, by the same method as given above, we have

$$
\sup_{s \leq \tau} \int_{-\infty}^0 e^{2\delta r} \|(I-P_n)f(r+s)\|^2 \, dr \leq \frac{1}{1 - e^{-2\delta T}} \int_0^T \|(I-P_n)f(r)\|^2 \, dr,
$$

where $P_n : L^2(Q) \to \text{span}\{e_1, e_2, \cdots, e_n\}$ is the canonical projection. Note that

$$
\|(I-P_n)f(r)\|^2 \leq \|f(r)\|^2, \quad \forall n \in \mathbb{N}, r \in [0,T], \quad \text{and} \quad \int_0^T \|f(r)\|^2 \, dr < +\infty.
$$

By the Lebesgue controlled convergence theorem,

$$
\lim_{n \to \infty} \sup_{s \leq \tau} \int_{-\infty}^0 e^{2\delta r} \|(I-P_n)f(r+s)\|^2 \, dr \leq \frac{1}{1 - e^{-2\delta T}} \int_0^T \|(I-P_n)f(r)\|^2 \, dr
$$

$$
= \frac{1}{1 - e^{-2\delta T}} \int_0^T \lim_{n \to \infty} \|(I-P_n)f(r)\|^2 \, dr = 0.
$$

Therefore, $f$ is backward complement-small. By Theorem 5.1, the attractor $A$ is backward compact as required.

**Remark 5.** As discussed in the section 2, the periodic random attractor may not be backward compact. This is different from the non-random case: a periodic pullback attractor must be backward compact, as proved in [36].

However, in Theorem 5.2, we can obtain backward compactness from periodicity of the random attractor for the sine-Gordon equation. In fact, the local integrability assumption of the force $f$ plays an important role.
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