Theta correspondence and Arthur packets: on the Adams conjecture

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Abstract

The Adams conjecture predicts that the local theta correspondence should respect Arthur packets. In this paper, we revisit the Adams conjecture for the symplectic–even orthogonal dual pair. Our results provide a precise description of all situations in which the conjecture holds.

Introduction

The theory of theta correspondence is important in the study of automorphic forms. This paper is an attempt to understand this theory in the language of Arthur packets.

The systematic study of theta correspondence was initiated by Roger Howe in the 1970’s [11], building on the work of Andre Weil [30]. Since then, it has motivated a large body of work by numerous authors. This sustained interest stems from the fact that theta correspondence is one of the few ways to explicitly construct automorphic forms and representations. Notably, it has been used to construct many instances of Langlands functoriality. In this paper, we study the local version of this correspondence.

To recall the basic idea, we need two main ingredients. The first one is a reductive dual pair inside a symplectic group. Let $F$ be a non-archimedean local field of characteristic zero. We fix $\epsilon = \pm 1$ and let

$W_n = a (\epsilon)$-Hermitian space of even dimension $n$ over $F$,
$V_m = an \epsilon$-Hermitian space of even dimension $m$ over $F$.

We let $G = G_n$ (resp. $H = H_m$) denote the isometry group of $W_n$ (resp. $V_m$). Tensoring the two bilinear forms, we get a natural symplectic form on $W \otimes V$. The groups, $G$ and $H$, both preserve the symplectic form on $W \otimes V$. In fact, the crucial fact is that $G$ and $H$ form a reductive dual pair inside $\text{Sp}(W \otimes V)$: each one is the centralizer of the other. The second ingredient we need is the so-called Weil representation. The symplectic group has a unique non-trivial double cover, called the metaplectic group. The Weil representation $\omega$ is a representation of $\text{Mp}(W \otimes V)$, the metaplectic group associated with $W \otimes V$. A key

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For an irreducible representation \( \pi \) of \( G \), the maximal \( \pi \)-isotypic quotient of \( \omega_{W,V} \) is of the form

\[
\pi \otimes \Theta(\pi),
\]

for some smooth representation \( \Theta(\pi) \) of \( H \). The representation \( \Theta(\pi) \) is called the big theta lift of \( \pi \); an early result of Kudla \cite{Kudla} shows that it has finite length. When non-zero, it has a unique irreducible quotient, denoted \( \theta(\pi) \) and called the small theta lift of \( \pi \). This crucial result — known as the Howe duality — was first formulated by Howe \cite{Howe}, proven by Waldspurger \cite{Waldspurger} (for odd residue characteristic) and by Gan and Takeda \cite{GanTakeda} in full generality. The map \( \pi \mapsto \theta(\pi) \) is called the local theta correspondence.

There are two basic questions concerning this construction:

1) For which \( \pi \) is \( \theta(\pi) \neq 0 \)?

2) What is \( \theta(\pi) \) (when it is non-zero)?

To even attempt an answer, one needs a way to list all irreducible representations of \( G \) and \( H \). Of course, a good way to list all representations of a reductive group is provided by the local Langlands classification (LLC). Given an \( L \)-parameter corresponding to \( \pi \), can we tell whether \( \theta(\pi) \) is non-zero? Furthermore, can we determine the \( L \)-parameter of \( \theta(\pi) \)?

An early result in this direction was proved by Mœglin, Vigneras and Waldspurger in \cite{MWV} for cuspidal representations. Similar results were obtained by Muić \cite{Muc} for discrete series representations. However, these results were obtained much before LLC was available.

Nowadays, complete answers to questions 1) and 2) are known: in \cite{AG}, Atobe and Gan described the theta lifts of tempered representations; in \cite{AMT}, we extend these results to obtain answers for general irreducible representations.

Although these results offer complete answers, in some sense they could never form the entire story. The reason is that theta correspondence does not respect \( L \)-packets: two representations from the same packet can lift to representations belonging to different \( L \)-packets; so much was known from the early days of theta correspondence. In his 1989 paper \cite{Adams}, Adams proposed a solution to this problem: instead of \( L \)-packets, consider Arthur packets! Indeed, by enlarging the packets, one could hope to obtain a nicer formula for theta correspondence. To explain this idea, we briefly recall the notion of (local) Arthur packets; we refer the reader to Section 1.6 for a more detailed overview.

In his work \cite{Arthur}, Arthur proposed a classification of square-integrable automorphic representations of \( G \). The central notion is that of an Arthur parameter. Each parameter corresponds to a (global) Arthur packet of automorphic representations. For each local place, the global parameter restricts to a local Arthur parameter, which is a map

\[
\psi : L_F \times SL_2(\mathbb{C}) \to LG
\]

subject to certain technical requirements. Here \( L_F \) denotes the Weil–Deligne group of \( F \), and \( LG \) is the (complex) dual group of \( G \):

\[
LG = \begin{cases} 
\text{SO}(n+1, \mathbb{C}), & \text{if } G \text{ is symplectic (i.e. if } \epsilon = 1) \\
\text{O}(n, \mathbb{C}), & \text{if } G \text{ is even orthogonal (i.e. if } \epsilon = -1) 
\end{cases}
\]
To each such parameter, one attaches a local Arthur packet, i.e. a packet $\Pi_\psi$ of irreducible representations of $G$. The representations in $\Pi_\psi$ are precisely those which appear as local constituents of the corresponding global Arthur packet. Composing $\psi$ with the standard representation of $L_G$, one may view $\psi$ as a representation of $L_F \times \text{SL}_2(\mathbb{C})$.

A critical contribution to this theory was that of Mœglin [16, 17, 18, 20, 22], who constructed the local Arthur packets $\Pi_\psi$ and showed that they are multiplicity free. To tell apart the different irreducible representations in $\Pi_\psi$, Mœglin uses additional data $(\eta, t)$ (see §3.1). Given a parameter $\psi$, each choice of $(\eta, t)$ yields a representation $\pi_{(\eta, t)}$. This representation is either irreducible or 0, and the non-zero representations obtained this way constitute the local $A$-packet $\Pi_\psi$. The key technical question of whether $\pi_{(\eta, t)}$ is non-zero has been resolved by Xu [32] and more recently Atobe [3].

We now explain the Adams conjecture. Let $\pi$ be an irreducible representation of $G = G_n$. Recall that we are interested in computing the lift to $H = H_m$; we usually denote this lift by $\theta_{-\alpha}(\pi)$, where $\alpha = m - n - \epsilon$. Let $S_\alpha$ denote the irreducible (algebraic) $\alpha$-dimensional representation of $\text{SL}_2$. We now have

\textbf{Conjecture (Adams [1]).} Assume that $\pi$ is parametrized by $\psi$. Then $\theta_{-\alpha}(\pi)$ is parametrized by

$$\psi_\alpha = \psi \oplus 1 \otimes S_\alpha.$$ 

Note that this indeed gives a homomorphism into $L_H$; moreover, this is the simplest way to obtain a map into $L_H$ using $\psi$ as the input!

Adams himself verified the conjecture in [1] for all examples of theta correspondence available at the time. However, subsequent work on the local theta correspondence led to new examples against which the conjecture could be tested. In her paper [21], Mœglin revisited the work of Adams. She showed that

— the above conjecture is true for large $\alpha$ (see Proposition 2.2);
— the above conjecture fails in many examples.

This failure of the Adams conjecture is the point of departure for the present paper. To be precise, given a representation $\pi$ in $\Pi_\psi$, we consider the set

$$\mathcal{A}(\pi, \psi) = \{ \alpha \geq 0, \ \alpha \equiv 1(\text{mod } 2) : \theta_{-\alpha}(\pi) \in \Pi_{\psi_\alpha} \}$$

and the following questions suggested by Mœglin (see [21 Section 6.3]):

1) Can we describe $\mathcal{A}(\pi, \psi)$? In particular, is it true that $\alpha \in \mathcal{A}(\pi, \psi)$ implies $\alpha + 2 \in \mathcal{A}(\pi, \psi)$?
2) If so, can we find $a(\pi, \psi) := \min \mathcal{A}(\pi, \psi)$ explicitly? 

\[^{1}\text{See [22] for a fully precise statement; to simplify the exposition, we are not mentioning the characters needed to fix the splitting } G \times H \to \text{Sp}(W \otimes V)\]
In this paper, we answer both questions. In particular, we show that \( \alpha \in \text{A}(\pi, \psi) \) does imply \( \alpha + 2 \in \text{A}(\pi, \psi) \), and we develop a method for computing \( a(\pi, \psi) \).

We provide a rough overview of our results here, and leave the details for Section 2. Using Mœglin’s results on lifts for \( \alpha \gg 0 \) as a starting point, we define a representation \( \pi_\alpha \) for every odd \( \alpha > 0 \). This \( \pi_\alpha \) is either 0, or an element of \( \Pi_\psi \); moreover, for \( \alpha \gg 0 \) we have \( \pi_\alpha = \theta_{-\alpha}(\pi) \). Our first result is

**Theorem A.** Assume \( \pi_\alpha = \theta_{-\alpha}(\pi) \) and \( \pi_{\alpha-2} \neq 0 \). Then \( \pi_{\alpha-2} = \theta_{-(\alpha-2)}(\pi) \); in particular, \( \theta_{-(\alpha-2)}(\pi) \) is in \( \Pi_\psi \).

Thus Theorem A shows that the Adams conjecture holds for all \( \alpha \geq d(\pi, \psi) \), where

\[
d(\pi, \psi) = \min\{\alpha_0 : \pi_\alpha \neq 0 \text{ for all } \alpha \geq \alpha_0\}.
\]

To utilize the so-called conservation relation (see §), we simultaneously look at two towers of lifts. On one of the towers (dubbed the going-down tower) the lifts start appearing early; on the other (which we call the going-up tower), they appear late. We prove

**Theorem B.** On the going-up tower, the Adams conjecture is true for all non-zero lifts. In other words, \( d^{\text{up}}(\pi, \psi) \) corresponds to the first occurrence of \( \pi \):

\[
d^{\text{up}}(\pi, \psi) = \min\{\alpha > 0 : \theta^{\text{up}}_{-\alpha}(\pi) \neq 0\}.
\]

Moreover, \( d^{\text{down}}(\pi, \psi) < d^{\text{up}}(\pi, \psi) \).

Finally, we prove that the Adams conjecture never holds for \( \alpha < d(\pi, \psi) \) (see Theorem C and Corollary D). This shows that \( \alpha \in \text{A}(\pi, \psi) \) implies \( \alpha + 2 \in \text{A}(\pi, \psi) \), and that \( a(\pi, \psi) = d(\pi, \psi) \).

The methods we use are rather elementary. Considering Mœglin’s construction of Arthur packets, it is not surprising that we need to have good control of Jacquet modules of the representations appearing in this work. We thus make frequent use of Kudla’s filtration (which describes the Jacquet modules of the Weil representation), but also some general results on Jacquet modules for representations of Arthur type. In a few places, we use the terminology developed by Atobe and Minguez. These results are reviewed in §. In recent years, there has been considerable progress in understanding local Arthur packets. We mention some relevant work. Xu describes an algorithm which determines whether a given representation \( \pi(\eta,t) \) is non-zero (this answers the crucial question in Mœglin’s construction). The work of Atobe reinterprets those results, and constructs local A-packets explicitly. Finally, Atobe and Hazeltine–Liu–Lo compute the set of A-packets which contain a given irreducible representation (in particular, given an irreducible representation, these results determine whether or not it is of Arthur type). Given that an explicit description of theta correspondence is available in terms of the Local Langlands classification, one might conceivably use these results to study the Adams conjecture in a roundabout manner: find the \( L \)-parameter of a representation \( \pi \in \Pi_\psi \), compute \( \theta_{-\alpha}(\pi) \), and check whether it is in \( \Pi_\psi \). Using a direct approach and staying within the Arthur world seemed to us the easier of the two possible approaches. Of course, our results do rely on those of Xu (and Atobe): recall that we show the Adams conjecture is valid if and only if \( \alpha \geq d(\pi, \psi) \).
But $d(\pi, \psi)$ is defined precisely by the non-vanishing of a certain representation $\pi_\alpha$. The algorithm developed by Xu is what makes this a satisfactory criterion.

Although we work with the symplectic–even orthogonal dual pairs, our methods are applicable to other dual reductive pairs, provided the local Arthur packets are defined. Indeed, our proofs rely only on Mœglin’s construction of Arthur packets and on results about theta correspondence which hold for all dual pairs of type I. Assuming Arthur’s (local) parametrization of representations in terms of $\Lambda$-packets (thus assuming all the issues concerning endoscopy, transfers, fundamental lemmas are resolved), Mœglin [22] describes the classes of groups for which her explicit construction of $\Lambda$-packets applies. We thus expect the same results to hold for the metaplectic–odd orthogonal dual pair, as well as for unitary dual pairs.

There are still some related questions that are not addressed by the present paper. For example, one could ask for a description of theta correspondence in terms of Arthur packets when lifting to groups of smaller rank (i.e. for $\alpha < 0$). Although this question is not studied by Adams [1], it would be interesting to have a description of lifts, or at least a criterion for non-vanishing of lifts that does not involve translating the question to the language of $L$-parameters. We suspect the answers in this case are unlikely to be as tidy as the ones provided by the Adams conjecture. Another interesting question is the following: if the Adams conjecture fails and the lift is not in the expected Arthur packet, could $\theta_{-\alpha}(\pi)$ still be a representation of Arthur type (just lying in some other, unexpected packet)? Finally, we mention a question related to the work of Hazeltine–Liu–Lo [10]. Given a representation $\pi$ of Arthur type, one could consider all the parameters $\psi$ such that $\pi \in \Pi_\psi$. Our results attach an index $d(\pi, \psi)$ to each of these parameters. Notice that, on the going-up tower, the various $d(\pi, \psi)$ are independent of $\psi$ (they depend only on the first occurrence of $\pi$). However, on the going down-tower, they might vary with $\psi$. We suspect that $d(\pi, \psi)$ will be lower if $\psi$ is “more tempered”, suggesting a connection between theta correspondence and the notion of “the Arthur packet” discussed in [10]. Exploring this connection would be an interesting problem. These and similar questions fall beyond the scope of the present paper.

Adams’s original motivation for stating the conjecture was explaining the theta correspondence. However, it turned out that an explicit description of theta correspondence in terms of $L$-parameters was found before the conjecture was fully investigated. It is fair to ask why one would bother obtaining a less precise description (using Arthur packets) if a fully precise recipe in terms of $L$-parameters is already available. The reasons for this are twofold. The first reason is technical: as shown by the work of Atobe [3], going from Arthur- to $L$-parameters and back is possible, but quite involved. For most applications involving Arthur packets, it is therefore valuable to have a recipe for theta correspondence that bypasses these translation issues. The second reason is perhaps even more important. The nature of local Arthur packets remains relatively mysterious, as they are defined by endoscopic character identities. However, the simplicity of Adams’s formula suggests a possibly deep connection between theta correspondence and Arthur packets, that one could exploit in either direction. It is our hope that this work sheds some light on representations of Arthur type.

We give a brief summary of the contents. In Section 1 we introduce the objects and the notation we use in the paper. We give an overview of the theta correspondence as well as
Mœglin’s construction of local Arthur packets. In Section 2 we state the Adams conjecture and the relevant questions; we provide a detailed description of our results. Most of the technical results we use are collected in Section 3. We prove the main results — Theorems A, B, and C — in Sections 4, 5, and 6, respectively. Finally, Section 7 contains examples which illustrate the main results.

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1 Preliminaries and Notation

In this section we introduce the objects and the notation we use throughout the paper.

1.1 Groups

Let $F$ be a nonarchimedean local field of characteristic 0 and let $| \cdot |$ be the absolute value on $F$ with the usual normalization. We fix $\epsilon = \pm 1$ and let
\[
\begin{cases}
W_n = a (-\epsilon)-\text{Hermitian space of even dimension } n \text{ over } F, \\
V_m = a \epsilon-\text{Hermitian space of even dimension } m \text{ over } F.
\end{cases}
\]

We let $G = G_n$ (resp. $H = H_m$) denote the isometry group of $W_n$ (resp. $V_m$). Thus $G$ is a symplectic group when $\epsilon = 1$, and a (full) orthogonal group when $\epsilon = -1$. If $W$ is a symplectic group, we shall also consider the metaplectic group $\text{Mp}(W)$; this is the unique non-trivial two-fold cover of $\text{Sp}(W)$; cf. [14], [23]. If $X$ is a vector space over $F$, we use $\text{GL}(X)$ to denote the general linear group of $X$. All the groups defined here are totally disconnected locally compact topological groups.

1.2 Witt towers

Every $\epsilon$-Hermitian space $V_m$ has a Witt decomposition
\[
V_m = V_{m_0} + V_{r,r} \quad (m = m_0 + 2r),
\]
where $V_{m_0}$ is anisotropic and $V_{r,r}$ is split (i.e. a sum of $r$ hyperbolic planes). The space $V_{m_0}$ is unique up to isomorphism, and so is the number $r \geq 0$ (the so-called Witt index of $V_m$). The collection of spaces
\[
\mathcal{V} = \{V_{m_0} + V_{r,r} : r \geq 0\}
\]
is called a Witt tower.

1.3 Parabolic subgroups

Let $V_m$ be an $\epsilon$-Hermitian space of Witt index $r$. We fix a set of maximal standard parabolic subgroups $Q_t$, $t = 1, \ldots, r$, of $H(V_m)$ like in [7, 2.3]. The parabolic subgroup $Q_t$ has a Levi decomposition $Q_t = M_t N_t$ with Levi factor $M_t$ isomorphic to $\text{GL}_t(F) \times H(V_{m-2y})$. By further partitioning $t$ we get the rest of the standard parabolic subgroups. The standard maximal parabolic subgroups of $G(W_n)$ are denoted $P_t$. 
1.4 Representations

Let $G = G(W_n)$ be one of the groups introduced in §1.1. We work in the category of smooth complex representations of $G$. The set of equivalence classes of irreducible representations of $G$ will be denoted by $\text{Irr}(G)$. We denote the contragredient of $\pi$ by $\pi^\vee$.

For each parabolic subgroup $P = MN$ of $G$ we have the (normalized) induction and Jacquet functors, denoted by $\text{Ind}^G_P$ and $r_P$. These are related by the standard Frobenius reciprocity

$$\text{Hom}_G(\pi, \text{Ind}^G_P(\pi')) \cong \text{Hom}_M(r_P(\pi), \pi')$$

and by the second (Bernstein) form of Frobenius reciprocity,

$$\text{Hom}_G(\text{Ind}^G_P(\pi'), \pi) \cong \text{Hom}_M(\pi', r_P(\pi)).$$

Here $P = MN$ is the parabolic subgroup opposite to $P$. We will often use alternative notation for these two functors:

1.4.1 Notation for induction and restriction

If $P = MN$ is the standard parabolic subgroup of $G(W_n)$ with Levi factor $M = \text{GL}_{t_1}(F) \times \cdots \times \text{GL}_{t_k}(F) \times G(W_n-2t)$, we use

$$\tau_1 \times \cdots \times \tau_k \times \pi_0$$

to denote $\text{Ind}^G_P(\tau_1 \otimes \cdots \otimes \tau_k \otimes \pi_0)$, where $\tau_i$ is a representation of $\text{GL}_{t_i}(F)$, $i = 1, \ldots, k$, and $\pi_0$ is a representation of $G(W_n-2t)$ (with $t = t_1 + \cdots + t_k$). We use the analogous (Zelevinsky) notation for parabolic induction in general linear groups.

The construction of local Arthur packets involves repeatedly applying various Jacquet functors. To simplify the exposition, we introduce a convenient way to denote the Jacquet modules in question. Variations of this notation have become standard in the literature on local Arthur packets. Given $\pi \in \text{Irr}(G(W_n))$ and a standard maximal parabolic $P$ with Levi factor $\text{GL}_t(F) \times G(W_n-2t)$, we may write the semisimplification of $r_P(\pi)$ as a direct sum

$$\bigoplus_{i} \tau_i \otimes \sigma_i$$

with $\tau_i \in \text{Irr}(\text{GL}_t(F))$ and $\sigma_i \in \text{Irr}(G(W_n-2t))$. Fix $\rho$, an irreducible unitary supercuspidal representation of $\text{GL}_t(F)$. For any real number $x$ we define

$$\text{Jac}^\rho_x(\pi) := \bigoplus_{\tau_i = |\rho|^x} \sigma_i.$$

When $\rho$ is clear from the context, we omit it from the notation and simply write $\text{Jac}_x(\pi)$. We extend this notation as follows. For a tuple $(x_1, \ldots, x_k)$ of real numbers, we set

$$\text{Jac}_{x_1,\ldots,x_k} := \text{Jac}_{x_k} \circ \text{Jac}_{x_{k-1}} \circ \cdots \circ \text{Jac}_{x_1}.$$  

Finally, let $[B, A]$ be a segment (cf. §1.4.3) and $T > 0$ an integer. We set

$$\text{Jac}_{[B+T,A+T]} := \text{Jac}_{B+1,A+1} \circ \cdots \circ \text{Jac}_{B+T-1,A+T-1} \circ \text{Jac}_{B+T,A+T}.$$  

See also Remark 3.2 for additional notation related to Jacquet modules.
1.4.2 The Langlands classification

We briefly recall the Langlands classification. Let $\delta_i \in \text{GL}_{t_i}(F), i = 1, \ldots, k$ be irreducible discrete series (unitarizable) representations, and let $\tau$ be an irreducible tempered representation of $G(W_{n-2t})$, where $t = t_1 + \cdots + t_k$. Any representation of the form

$$\nu^{s_k} \delta_k \times \cdots \times \nu^{s_1} \delta_1 \times \tau,$$

where $s_k \geq \cdots \geq s_1 > 0$ (and where $\nu$ denotes the character $|\det|$ of the corresponding general linear group) is called a standard representation (or a standard module). It possesses a unique irreducible quotient (the so-called Langlands quotient), denoted by $L(\nu^{s_k} \delta_k, \ldots, \nu^{s_1} \delta_1; \tau)$. Conversely, every irreducible representation of $G(W_n)$ can be represented as the Langlands quotient of a unique standard representation. In this way, we obtain a complete description of $\text{Irr}(G(W_n))$.

1.4.3 Segments

Let $\rho$ be an irreducible unitary supercuspidal representation of $\text{GL}_k(F)$ and let $x, y$ be real numbers such that $y - x$ is a positive integer. Any tuple of representations of the form

$$(\rho) \cdot [^x, \rho] \cdot [^x+1, \ldots, \rho] \cdot [^y]$$

is called a segment. We denote such a segment by $[x, y]_\rho$; when $\rho$ is clear from the context, we further simplify the notation and use $[x, y]$. The representation

$$\rho) \cdot [^x \times \rho] \cdot [^x+1 \times \cdots \times \rho] \cdot [^y]$$

has a unique irreducible quotient, and a unique irreducible subrepresentation. We denote the quotient by $\delta([x, y]_\rho)$, and the subrepresentation by $\zeta([x, y]_\rho)$. When $\rho$ is trivial these are usually shortened to $\delta(x, y)$ and $\zeta(x, y)$, respectively. When $y = x - 1$, this is to be interpreted as the trivial representation of the trivial group.

Recall that the segments $[x, y]_\rho$ and $[x', y']_{\rho'}$ are said to be linked if neither contains the other, and their union is itself a segment. Similarly, they are said to be juxtaposed if their intersection is empty, but their union is a segment. We use the following well known result throughout the paper.

**Lemma 1.1.** Let $[x, y]_\rho$ and $[x', y']_{\rho'}$ be two segments.

(i) The representation $\delta([x, y]_\rho) \times \delta([x', y']_{\rho'})$ reduces if and only if the segments $[x, y]_\rho$ and $[x', y']_{\rho'}$ are linked. The same holds if we replace $\delta$ by $\zeta$.

(ii) The representation $\delta([x, y]_\rho) \times \zeta([x', y']_{\rho'})$ reduces if and only if the segments $[x, y]$ and $[x', y']$ are juxtaposed.

**Proof.** Part (i) is shown in [33], and (ii) is Lemma I.6.3 of [24]. See also Section 3.1 of [7] for a more detailed analysis of the situation in (ii). \qed

This construction generalizes to multisegments. Fix $\zeta = \pm 1$. A multisegment is a matrix

$$\begin{pmatrix}
    x_{11} & \cdots & x_{1n} \\
    \vdots & \ddots & \vdots \\
    x_{m1} & \cdots & x_{mn}
\end{pmatrix}_{\rho}$$
with \( x_{i+1,j} = x_{ij} - \zeta \) and \( x_{i,j+1} = x_{ij} + \zeta \), for all \( i,j \). Let \( \sigma_i \) be the unique irreducible subrepresentation of \( \rho \cdot |^{x_{i1}} \times \cdots \times \rho| \cdot |^{x_{in}} \), for \( i = 1, \ldots, m \). The representation attached to the above multisegment is defined as the unique irreducible subrepresentation of \( \sigma_1 \times \cdots \times \sigma_m \). We abuse notation and simply use the above matrix to denote the corresponding representation. As before, when \( \rho \) is clear from the context, we suppress it from the notation. We note that the transpose of the matrix corresponds to the same representation.

The reducibility criterion also extends to multisegments. Let \([x_{ij}]_{\rho}\) and \([y_{ij}]_{\rho'}\) be multisegments of dimensions \( m \times n \) and \( m' \times n' \), respectively. Let \( \zeta = x_{11} - x_{21} \), \( \zeta' = y_{11} - y_{21} \). If \( \zeta = \zeta' \), we say that the multisegments are linked if

- \([x_{m1},x_{1n}]_{\rho}\) and \([y_{n'1},y_{1n'}]_{\rho'}\) are linked; and
- the four segments corresponding to the sides of the rectangle \([x_{ij}]_{\rho}\) do not contain, nor are contained in, the corresponding segments of \([y_{ij}]_{\rho'}\).

If \( \zeta = -\zeta' \), we say that the multisegments are linked if \([x_{ij}]_{\rho}\) and \([y_{ij}]_{\rho'}\) are linked.

**Lemma 1.2** (Mœglin–Waldspurger [24]). The representation \([x_{ij}]_{\rho} \times [y_{ij}]_{\rho'}\) reduces if and only if the corresponding multisegments are linked.

### 1.5 Theta correspondence

We recall the basic facts about the local theta correspondence and describe the setting in which we work.

We begin by fixing, once and for all, an additive character \( \psi \) of \( F \). The space \( W_n \otimes V_m \) has a natural symplectic structure, and the choice of \( \psi \) determines a Weil representation of \( \text{Mp}(W_n \otimes V_m) \). We fix a pair of characters \((\chi_W, \chi_V)\) attached to the spaces \( W_n \) and \( V_m \) as in Section 3.2 of [8]; note that the characters depend only on the Witt tower, and not on the dimension of the space. These characters determine a splitting of the metaplectic cover:

\[
G(W_n) \times H(V_m) \twoheadrightarrow \text{Mp}(W_n \otimes V_m).
\]

Pulling back the Weil representation, we obtain the Weil representation of the dual pair \( G(W_n) \times H(V_m) \). We denote this representation by \( \omega_{m,n} \); since \( \chi_W, \chi_V \) and \( \psi \) are fixed throughout the paper, we omit them from the notation. In fact, if the dimensions are known (or irrelevant), we further simplify the notation and denote it by \( \omega \).

For any \( \pi \in \text{Irr}(G(W_n)) \), the maximal \( \pi \)-isotypic quotient of \( \omega_{m,n} \) is of the form

\[
\pi \otimes \Theta(\pi, V_m)
\]

for a certain smooth representation \( \Theta(\pi, V_m) \) of \( H(V_m) \), called the full theta lift of \( \pi \). When the target Witt tower is fixed, we will denote it by \( \Theta_\alpha(\pi) \), where \( \alpha = n + \epsilon - m \) (recall that \( \epsilon \) is defined in [11]). Note that \( \alpha \) is an odd integer.

A classic result of Kudla [13] shows that \( \Theta(\pi, V_m) \) is either zero, or an admissible representation of finite length. The following theorem establishes the theta correspondence:

**Theorem 1.3** (Howe duality). If \( \Theta(\pi, V_m) \) is non-zero, it possesses a unique irreducible quotient, denoted by \( \theta(\pi, V_m) \).
Originally conjectured by Howe in [11], this was first proven by Waldspurger [29] when the residual characteristic of $F$ is different from 2, and by Gan and Takeda [9] in general. The representation $\theta(\pi, V_m)$ is called the (small) theta lift of $\pi$; like the full lift, we will also denote it by $\theta_\alpha(\pi)$. The resulting bijection $\pi \leftrightarrow \theta(\pi)$ between representations appearing as quotients of $\omega$ is called the (local) theta correspondence.

A standard approach to the theory of theta correspondence involves considering towers of lifts (see Propositions 4.1 and 4.3 of [14]):

**Proposition 1.4** (Tower property). Let $\pi$ be an irreducible representation of $G(W_n)$. Fix a Witt tower $V = (V_m)$.

(i) If $\Theta(\pi, V_m) \neq 0$, then $\Theta(\pi, V_{m+2r}) \neq 0$ for all $r \geq 0$.

(ii) For $m$ large enough, we have $\Theta(\pi, V_m) \neq 0$.

The above proposition allows us to define, for any Witt tower $V = (V_m)$, $m_V(\pi) := \min\{m \geq 0 : \Theta(\pi, V_m) \neq 0\}$.

This number—also denoted $m(\pi)$ when the choice of $V$ is implicit—is called the first occurrence index of $\pi$. Note the slight abuse of terminology: we are using the term “index” even though $m(\pi)$ is the dimension.

A refinement of the tower property is the so-called conservation relation. When $\epsilon = 1$, the Witt towers of quadratic spaces we consider can be appropriately organized into pairs, with the towers comprising a pair denoted $V^+$ and $V^-$; a complete list of pairs of dual towers can be found in [14, Chapter V]. Thus, instead of observing just one target tower, we simultaneously look at two of them. This way, each $\pi \in \text{Irr}(G(W_n))$ gives us two first occurrence indices, $m^+(\pi)$ and $m^-(\pi)$.

If $\epsilon = -1$, there is only one tower of $\epsilon$-hermitian (i.e. symplectic) spaces. In this case $W_n$ is a quadratic space, and we proceed as follows: since $G(W_n)$ is now equal to $O(W_n)$, any $\pi \in \text{Irr}(G(W_n))$ is naturally paired with its twist, $\det \otimes \pi$. This allows us to define

$$m^+(\pi) = \min\{m(\pi), m(\det \otimes \pi)\},$$

$$m^-(\pi) = \max\{m(\pi), m(\det \otimes \pi)\}.$$

Thus, regardless of whether $\epsilon = 1$ or $-1$, we may set

$$m^{\text{down}}(\pi) = \min\{m^+(\pi), m^-(\pi)\}, \quad m^{\text{up}}(\pi) = \max\{m^+(\pi), m^-(\pi)\}.$$

Note that when $V_m$ is a symplectic space, we have $m^{\text{down}}(\pi) = m^+(\pi)$ and $m^{\text{up}}(\pi) = m^-(\pi)$. The Conservation relation (first conjectured by Kudla and Rallis in [15], and ultimately proven by Sun and Zhu in [27]) states that

$$m^{\text{up}}(\pi) + m^{\text{down}}(\pi) = 2n + 2\epsilon + 2.$$

The tower in which $m(\pi) = m^{\text{down}}(\pi)$ (resp. $m^{\text{up}}(\pi)$) is called the going-down (resp. going-up) tower.
Remark 1.5. This labeling of towers is slightly imprecise. Indeed, the conservation relation implies
\[ m_{\text{down}}(\pi) \leq n + \epsilon + 1 \quad \text{and} \quad m_{\text{up}}(\pi) \geq n + \epsilon + 1. \]
However, it may well happen that \( m_{\text{down}}(\pi) = m_{\text{up}}(\pi) = n + \epsilon + 1 \), in which case the “going-up” and “going-down” designations are ambiguous. This ambiguity can be resolved; see e.g. Theorem 4.1 (2) in [5]. However, we do not have to worry about this: the questions we study in this paper simplify dramatically when \( \pi \) satisfies the above equality; in this case we can describe the results without specifying the target tower.

1.6 Arthur packets

We now recall the notion of a local A-packet. Let \( W_F \) denote the Weil group of \( F \); then \( L_F = W_F \times \text{SL}_2(\mathbb{C}) \) is the Weil–Deligne group. Let \( G = G(W) \) be one of the groups introduced in §1.1 and let \( G^\circ \) denote the identity component of \( G \):
\[ G^\circ = \begin{cases} \text{Sp}(W), & \text{for } \epsilon = 1; \\ \text{SO}(W), & \text{for } \epsilon = -1. \end{cases} \]
Furthermore, when \( \epsilon = -1 \) (so that \( G = \text{O}(W) \)), we let \( \sigma_0 \) denote the outer automorphism of \( G^\circ \) given by conjugation by a fixed element \( \varepsilon \in \text{O}(W) \setminus \text{SO}(W) \); to allow for uniform notation we set \( \sigma_0 = \text{id} \) when \( \varepsilon = 1 \). We let \( \Sigma_0 \) denote the group generated by \( \sigma_0 \).

Attached to \( G^\circ \) we have \( \hat{G}^\circ = \text{the complex dual group of } G^\circ \), \( L^{G^\circ} = \text{the Langlands dual group of } G^\circ \).

An Arthur parameter for \( G^\circ \) is a \( \hat{G}^\circ \)-conjugacy class of admissible homomorphisms
\[ \psi : L_F \times \text{SL}_2(\mathbb{C}) \to L^{G^\circ} \]
such that the image of \( W_F \) is bounded. We abuse the notation and use \( \psi \) to denote both the representative and its conjugacy class. We let \( \Psi(G^\circ) \) denote the set of Arthur parameters. In [2], Arthur attaches to each \( \psi \in \Psi(G) \) a multiset \( \Pi_\psi(G^\circ) \) of elements in \( \text{Irr}(G^\circ) \). We call \( \Pi_\psi(G^\circ) \) the Arthur packet (or A-packet, for short) attached to \( \psi \). To define A-packets for the (possibly) disconnected group \( G \), recall that \( G = G^\circ \rtimes \Sigma_0 \). The group \( \Sigma_0 \) acts on \( \Psi(G^\circ) \) through the dual automorphism \( \hat{\sigma} \), and we let \( \Psi(G) \) denote the set of \( \Sigma_0 \)-orbits in \( \Psi(G^\circ) \).

We now define \( \Pi_\psi(G) \) to be the set of all irreducible representations of \( G \) whose restrictions have irreducible constituents in \( \Pi_\psi(G^\circ) \). When \( G \) is fixed and there is no ambiguity, we will often write \( \Pi_\psi \) instead of \( \Pi_\psi(G) \).

Arthur packets for \( G \) have been constructed by Moeglin [16], [17], [18], [20], who also showed that the packets are in fact multiplicity-free [22]. We provide a rough outline of this construction.

Let \( \psi \in \Pi_\psi(G) \). Composing \( \psi \) with the standard representation of \( \hat{G} \), we can view it as a representation of \( W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \). Thus \( \psi \) can be decomposed as
\[ \psi = \bigoplus_i m_i(\rho_i \otimes S_{a_i} \otimes S_{b_i}). \]
Here $\rho_i$ is an equivalence class of irreducible unitary representations of $W_F$; each such $\rho_i$ can be identified (via the local Langlands correspondence) with an irreducible unitary cuspidal representation of $GL_{\dim \rho_i}(F)$. Furthermore, $S_n$ denotes the irreducible algebraic $n$-dimensional representation of $SL_2(\mathbb{C})$, and $m_i$ is the multiplicity of the corresponding irreducible summand. We call the triple $(\rho, a, b)$ a Jordan block of $\rho$ if the corresponding representation $\rho \otimes S_a \otimes S_b$ occurs in $\psi$. The set of all Jordan blocks will be denoted by $\text{Jord}(\rho)$.

The first step of the construction of $A$-packets reduces the problem to the case of good parity. We say that a Jordan block $(\rho, a, b) \in \text{Jord}(\rho)$ is of good parity if $\rho \otimes S_a \otimes S_b$ factors through a group of the same type (symplectic/orthogonal) as $G$. Otherwise, we say that $(\rho, a, b) \in \text{Jord}(\rho)$ is of bad parity. Following Moeglin, we denote the sub(multi)set of elements of good (resp. bad) parity by $\text{Jord}(\psi_{bp})$ (resp. $\text{Jord}(\psi_{mp})$). The parameter $\psi$ can then be written as

$$\psi = \psi_{mp} + \psi_{bp} + \psi_{mp}^\vee.$$ 

Here $\psi_{bp}$ corresponds to $\psi$ in an obvious manner, while the rest of the terms (which correspond to blocks in $\text{Jord}(\psi_{mp})$) can be grouped as $\psi_{mp} + \psi_{mp}^\vee$, with $\psi_{mp}^\vee$ denoting the dual of $\psi_{mp}$. There is a bijection between $\Pi_\psi$ and $\Pi_{\psi_{bp}}$: $\psi_{mp}$ determines an irreducible representation $\pi_{mp}$ of a general linear group; for any representation $\pi_0 \in \Pi_{\psi_{bp}}$, the representation $\pi = \pi_{mp} \otimes \pi_0$ is irreducible, and is contained in $\Pi_\psi$. Conversely, any $\pi \in \Pi_\psi$ is of the form $\pi_{mp} \otimes \pi_0$ for some $\pi_0 \in \Pi_{\psi_{bp}}$. Applying Kudla’s filtration (Lemma 3.8 and Remark 3.10) to the map $\pi_{mp} \otimes \pi_0 \rightarrow \pi$, one shows that the Adams conjecture (Conjecture 2.1) holds for $\pi$ if and only if it holds for $\pi_0$. This allows us to focus on parameters of good parity: from now on, we assume $\psi = \psi_{bp}$.

Now let $\psi = \psi_{bp}$ be a parameter of good parity. Moeglin’s construction of $\Pi_\psi$ involves the following data:

— an admissible order on $\text{Jord}(\psi)$

— a function $t : \text{Jord}(\psi) \rightarrow \mathbb{Z}_{\geq 0}$ such that $t(\rho, a, b) \in [0, \min(a, b)/2]$ for every $(\rho, a, b) \in \text{Jord}(\psi)$

— a function $\eta : \text{Jord}(\psi) \rightarrow \{\pm 1\}$

Note that we allow $t$ and $\eta$ to attain different values on different copies of the same Jordan block $(\rho, a, b)$. The functions $t$ and $\eta$ satisfy additional requirements. First, $\eta(\rho, a, b) = 1$ whenever $t(\rho, a, b) = \min(a, b)/2$. To express the second requirement, we set $\epsilon_{t, \eta}(\rho, a, b) = \eta(\rho, a, b)^{\min(a, b)}(-1)^{\min(a, b)/2 + t(\rho, a, b)}$. Then

$$\prod_{(\rho, a, b) \in \text{Jord}_\psi} \epsilon_{t,\eta}(\rho, a, b) = \epsilon_G.$$ 

Here $\epsilon_G = 1$ if $\psi$ is of odd dimension; when $\psi$ is of even dimension, $\epsilon_G$ is the Hasse invariant of the orthogonal group $G$.

To explain the admissibility condition for orders on $\text{Jord}(\psi)$, we introduce the following notation we will use throughout the paper. Given $(\rho, a, b) \in \text{Jord}(\psi)$, we let

$$A = \frac{a + b - 1}{2}, \quad B = \frac{|a - b|}{2}, \quad \zeta = \text{sgn}(a - b) \in \{\pm\}.$$ 

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and we write \((\rho, A, B, \zeta)\) instead of \((\rho, a, b)\). When \(\rho\) is clear from the context, we suppress it from the notation altogether. Now an order \(>\) on \(\text{Jord}(\psi)\) is said to be admissible if for all \((\rho, A, B, \zeta)\) and \((\rho, A', B', \zeta')\) \(\in\) \(\text{Jord}(\psi)\) we have

\[
A > A' \text{ and } B > B' \text{ and } \zeta = \zeta' \implies (\rho, A, B, \zeta) > (\rho, A', B', \zeta').
\]

Recall that a parameter \(\psi\) is said to have discrete diagonal restriction (DDR, for short) if the segments \([B, A]_\rho\) and \([B', A']_\rho\) are disjoint for any two blocks \((\rho, A, B, \zeta), (\rho, A', B', \zeta')\) \(\in\) \(\text{Jord}(\psi)\). Each choice of \((t, \eta)\) (subject to conditions described above) then corresponds to an irreducible representation in \(\Pi_\psi\). DDR parameters play an important role in this paper. Although they are much simpler than general parameters of good parity, constructing the corresponding packets is non-trivial. We do not describe this construction here; instead, we refer the reader to Proposition 3.3 of \cite{21}. We point out that there is more than one way to parametrize the representations inside \(\Pi_\psi\); see Remark 1.7 below. When talking about DDR parameters, we often use the notation \((\rho, A, B, \zeta)\) interchangeably with the segment notation \([B, A]_\rho\).

One may reduce the construction of \(\Pi_\psi\) for a general (good parity) parameter \(\psi\) to the DDR case, as follows: Let \(\text{Jord}(\psi) = \{(\rho_i, A_i, B_i, \zeta_i) : i = 1, \ldots, k\}\); here we assume that the Jordan blocks are enumerated with respect to some admissible order. We consider a new parameter \(\psi_{\gg}\) (of a larger group) with \(\text{Jord}(\psi_{\gg}) = \{(\rho_i, A_i + T_i, B_i + T_i, \zeta_i) : i = 1, \ldots, k\}\), where \(T_i, i = 1, \ldots, n\) are positive integers such that \(\psi_{\gg}\) is DDR. For each choice of \((t, \eta)\), we get a representation \(\pi_{t,\eta,\psi_{\gg}}\) in \(\Pi\). We set

\[
\pi_{t,\eta} = \text{Jac}[A_k + T_k, B_k + T_k] \circ \cdots \circ \text{Jac}[A_1 + T_1, B_1 + T_1] \circ \text{Jac}[(\pi_{\gg}, t, \eta)](\pi_{\gg, t, \eta}).
\]

Then \(\pi_{t,\eta}\) is either 0, or is an irreducible element of the packet \(\Pi_\psi\); the packet \(\Pi_\psi\) consists of all non-zero representations obtained this way. We will say more about this construction in \cite{3}. When there is no fear of confusion, we sometimes abuse notation (and terminology) by referring to the triple \((\psi, \eta, t)\) as just \(\psi\).

**Remark 1.6.** We use the above notation throughout the paper. Any time we use \(\psi_{\gg}\) it is implied that the numbers \(T_i\) are defined; we will freely refer to them whenever we deal with \(\psi_{\gg}\). Moreover, this notation will be used functorially with respect to decorations: for instance, the DDR parameter which dominates \(\psi_{\gg}\) is clear from the context, we suppress \(\psi_{\gg}\), etc.

Although this reduction to the DDR case allows a simple description, it is far from being easy to work with. For instance, it is not a priori clear whether the representation \(\pi_{t,\eta}\) is non-zero for a given \((\eta, t)\). This question is of course crucial for our considerations in this paper, and it turns out to be highly non-trivial. In \cite{32}, Xu gives an algorithm to determine whether \(\pi_{t,\eta}\) is non-zero. The recent work of Atobe \cite{3} also contains a criterion which answers this question.

**Remark 1.7.** As mentioned above, there are in fact two standard ways to parametrize representations inside \(\Pi_\psi\). In her work, Mœglin uses a parametrization which differs from the one originally used by Arthur. The comparison between the two has been conducted by Mœglin herself; more recently, this has been addressed by Xu \cite{31}. In this paper, we use the parametrization introduced by Mœglin. However, there is another choice to be made, related (roughly) to the parametrization of supercuspidal representations. Here we use the parametrization of supercuspidals provided by Arthur. This combination of Mœglin’s parametrization with Arthur’s “initial conditions” is precisely the one studied by Xu in \cite{31}.
2 The Adams conjecture

In his 1989 paper [1], Adams proposed the following

Conjecture 2.1. Suppose $\pi$ is an irreducible representation of $G$ contained in the Arthur packet attached to the parameter $\psi$. Then $\theta_{-\alpha}(\pi)$ is contained in the A-packet parametrized by

$$\psi_{\alpha} = (\chi W \chi^{-1} \otimes \psi) \oplus \chi W \otimes S_1 \otimes S_{\alpha}$$

When $\psi$ and $\alpha$ are fixed, we often refer to $\Pi_{\psi_{\alpha}}$ as the A-A (Adams–Arthur) packet.

Adams himself verified this conjecture for all the examples of theta correspondence available at the time. However, subsequent work on the local theta correspondence led to new findings related to the above conjecture. In particular, in her paper [21], Mœglin revisited the work of Adams. She showed that

— Conjecture 2.1 is true for large $\alpha$ (see Proposition 2.2 for a more precise statement);
— Conjecture 2.1 fails in many examples.

In view of these findings, our goal in this paper is to investigate the extent to which Conjecture 2.1 holds. To be precise, we are interested in the following questions suggested by Mœglin (see [21, Section 6.3]):

1) Given a representation $\pi$ in $\Pi_{\psi_{\alpha}}$, consider the set

$$\mathcal{A}(\pi, \psi) = \{ \alpha \geq 0, \alpha \equiv 1\text{ (mod 2)} : \theta_{-\alpha}(\pi) \in \Pi_{\psi_{\alpha}} \}.$$ 

Is it true that $\alpha \in \mathcal{A}(\pi, \psi)$ implies $\alpha + 2 \in \mathcal{A}(\pi, \psi)$?

2) If so, can we find $a(\pi, \psi) := \min \mathcal{A}(\pi, \psi)$ explicitly?

This paper provides complete answers to both of these questions. We briefly explain our approach and results here. To do that, we start with the following result of Mœglin:

**Proposition 2.2** ([21], Theorem 5.1). Let $\alpha \gg 0$. Then $\theta_{-\alpha}(\pi)$ is parametrized by $(\psi_{\alpha}, t', n')$, where $t = t'$ and $n' = -n$ for all blocks that appear in $\psi$. The order on $\text{Jord}(\psi_{\alpha})$ is the same as the one on $\text{Jord}(\psi)$, with the added block being the largest.

**Remark 2.3.** When lifting from the orthogonal group, this leaves us just one possibility for $n'$ on the newly added block in $\psi_{\alpha}$. However, when lifting from the symplectic group, the two possible choices for $n'$ correspond to the lifts onto two different towers of orthogonal groups.

Proposition 2.2 gives us the necessary starting point and allows us to identify a candidate parameter for each lift:

**The Recipe.** First, for each odd $\alpha \gg 0$, we define a candidate representation $\pi_{\alpha}$ in $\Pi_{\psi_{\alpha}}$ using the formula from the above proposition. Now, starting from $\psi_{\alpha}$ for any $\alpha \gg 0$, we define parameters $\psi_{\alpha-2}$, $\psi_{\alpha-4}$, etc. using the following recipe: to go from $\psi_{\alpha}$ to $\psi_{\alpha-2}$, we simply shift the added block from $\chi W \otimes S_1 \otimes S_\beta$ to $\chi W \otimes S_1 \otimes S_{\beta-2}$. By default, the order
on $\text{Jord}_\chi W$ and the functions $\eta$ and $t$ do not change; the exception is the case when $\text{Jord}_\chi V$ contains a segment $[B, A]$ with $B = \frac{\beta - 1}{2}$. In this case, going from $\psi_\beta$ to $\psi_{\beta - 2}$, we change the order so that the added block becomes smaller than $[B, A]$, and use the formulas described in 3.4 to update $\eta$ and $t$. In case there are multiple segments with $B = \frac{\beta - 1}{2}$, we change the order and $\eta, t$ for each such segment.

This defines a candidate representation $\pi_\alpha$ for each odd $\alpha > 0$. We do not know a priori whether $\pi_\alpha$ is non-zero; however, this representation has the property that $\pi_\alpha = \theta_{-\alpha}(\pi)$ for $\alpha \gg 0$ (so in particular, $\pi_\alpha \neq 0$). Our first result is then

**Theorem A.** Assume $\pi_\alpha = \theta_{-\alpha}(\pi)$ and $\pi_{\alpha - 2} \neq 0$. Then $\pi_{\alpha - 2} = \theta_{-(\alpha - 2)}(\pi)$; in particular, $\theta_{-(\alpha - 2)}(\pi)$ is in the A-A packet.

With this result in mind, we define

$$d(\pi, \psi) = \min\{\alpha_0 : \pi_\alpha \neq 0 \text{ for all } \alpha \geq \alpha_0\}.$$ 

Thus Theorem A shows that the Adams conjecture holds for all $\alpha \geq d(\pi, \psi)$. As explained in §1.5, it is fruitful to look at two towers of lifts simultaneously. Therefore, we consider $d_{\text{up}}(\pi, \psi)$ and $d_{\text{down}}(\pi, \psi)$, corresponding to the going-up and the going-down tower, respectively. We prove

**Theorem B.** On the going-up tower, the Adams conjecture is true for all non-zero lifts. In other words, $d_{\text{up}}(\pi, \psi)$ corresponds to the first occurrence of $\pi$:

$$d_{\text{up}}(\pi, \psi) = \min\{\alpha > 0 : \theta_{-\alpha}(\pi) \neq 0\}.$$ 

Moreover, $d_{\text{down}}(\pi, \psi) < d_{\text{up}}(\pi, \psi)$.

Notice that Theorem B explains why we don’t have to worry about the situation $m_{\text{up}}(\pi) = m_{\text{down}}(\pi)$ addressed in Remark 1.5. In this case, $d(\pi, \psi) = 1$ on both towers, and so Conjecture 2.1 holds for all $\alpha > 0$.

Finally, we show that question 1) posed above has a positive answer. We use

**The Epicer** Given a representation $\pi^\alpha \in \Pi_\psi_\alpha$, one may reverse the Recipe to obtain a representation $\pi^{\alpha + 2} \in \Pi_\psi_{\alpha + 2}$: in other words, replace $\chi W \otimes S_1 \otimes S_\alpha$ with $\chi W \otimes S_1 \otimes S_{\alpha + 2}$ and perform the necessary updates on $\eta, t$.

Of course, it is not a priori clear that $\pi^{\alpha + 2} \neq 0$. However, we show

**Theorem C.** Assume $\theta_{-\alpha}(\pi) = \pi^\alpha \in \Pi_\psi_\alpha$. Then $\theta_{-(\alpha + 2)}(\pi) = \pi^{\alpha + 2} \neq 0$; in particular, $\theta_{-(\alpha + 2)}(\pi)$ is in the A-A packet.

An immediate consequence is

**Corollary D.** Conjecture 2.1 is false for $\alpha < d(\pi, \psi)$.
Proof. To simplify notation, we set \( d = d(\pi, \psi) \). Assume the contrary, i.e. that \( \theta_{-\alpha_0}(\pi) \) is in the A-A packet for some \( \alpha_0 < d \); let \( \pi^{\alpha_0} = \theta_{-\alpha_0}(\pi) \). Applying the Epicer, we form a representation \( \pi^\beta \in \Pi_{\psi_\beta} \) for any \( \beta \geq \alpha_0 \). By Theorem \( \square \) we have \( \pi^\beta = \theta_{-\beta}(\pi) \) for \( \beta \geq \alpha_0 \). In particular,
\[
\pi^d = \theta_{-d}(\pi) = \pi_d,
\]
that is, the representation \( \pi^d \) obtained by ascending from \( \pi^{\alpha_0} \) is isomorphic to the representation \( \pi_d \) obtained by descending from \( \pi_\alpha \) for some \( \alpha \gg 0 \). The packets are multiplicity-free, so \( \pi^d \) and \( \pi_d \) must be given by the same data \( \eta, t \). But since Epicer is precisely the inverse of Recipe, \( \pi^d = \pi_d \) implies \( \pi^{d-2} = \pi_{d-2} \). This is impossible, since \( \pi^{d-2} \neq 0 \) (by Theorem \( \square \)) and \( \pi_{d-2} = 0 \) by definition of \( d = d(\pi, \psi) \).

Thus Theorems \( \square \) and \( \square \) answer both questions 1) and 2) posed above: we have \( A(\pi, \psi) = \{ \alpha \geq d(\pi, \psi), \ \alpha \equiv 1(\text{mod } 2) \} \); in particular, \( \alpha \in A(\pi, \psi) \) implies \( \alpha + 2 \in A(\pi, \psi) \). Furthermore, \( a(\pi, \psi) = d(\pi, \psi) \).

On the going-up tower, this is equivalent to finding the first occurrence of \( \pi \). The question of non-vanishing has been completely resolved in \([7]\), where the answer is stated in terms of the L-parameter of the representation. In this paper, we start with representation given as a member of an Arthur packet, and the Recipe enables us to identify the going-up tower and determine the first occurrence index in terms of Arthur packet data. As one irreducible representation can belong to multiple Arthur packets, this leads to certain necessary compatibility conditions among them.

On the going-down tower, the question boils down to establishing the (non-)vanishing of the candidate representation \( \pi_i \). This criterion is satisfactory thanks to the recent work by Xu \([32]\) and Atobe \([3]\).

3 Technical results

This section contains certain auxiliary results we use throughout the paper. These are mostly (variations of) known results, but we list them here in an attempt to keep the paper self-contained. We advise the reader to merely skim this section on initial reading.

3.1 Mœglin’s construction

Recall that the representations parametrized by a general (non-DDR) good parity parameter \( \psi \) are constructed by passing to a DDR parameter \( \psi_\gg \) and then shifting the blocks by taking Jacquet modules. We give a more detailed description of this process. Fixing \( \rho \), let \( \text{Jord}_\rho(\psi) = \{(A_i, B_i, \zeta_i) : i = 1, \ldots, r\} \). Let \( \pi \) be an irreducible representation in \( \Pi_{\psi_\gg} \), and let \( \pi_\gg \) denote the corresponding representation in \( \Pi_{\psi_\gg} \). We wish to describe intermediate steps in going from \( \pi_\gg \) to \( \pi \).

For each \( i \), denote by \( \pi_{(i)} \) the \( i \)-th intermediate step in going from \( \pi_\gg \) to \( \pi \): the lowest \( i \) blocks in \( \text{Jord}_\rho \) have been returned to the original position they had in \( \psi \), while the higher blocks remain in the position they have in \( \psi_\gg \). In particular, \( \pi_0 = \pi_\gg \). We let \( L_i \) be the representation which corresponds to the \( \zeta_i \)-multisegment given by segments \([B_i + T_i, A_i + T_i - 1], \ldots, [B_i + 1, A_i + 1] \). We have

Lemma 3.1 (2.8 in \([20]\)).
Remark 3.2. Again, we use the above notation throughout the paper: We will freely refer to the representations $L_i$ anytime we deal with $\pi_{\geq}$, and we will use $\text{Jac}_{L_1,\ldots,L_r}$ to denote $\text{Jac}_{[B_1,T_1,A_1,T_1]\to[B_1,A_1]} \circ \cdots \circ \text{Jac}_{[B_1,T_1,A_1,T_1]\to[B_1,A_1]}$.

The following result is a special case of Proposition 3.3 (2) in [32] (see also Proposition 3.2. of [19] for DDR representations).

Lemma 3.3. Let $\alpha \in \mathbb{R}$ be such that $\frac{\alpha - 1}{2} \neq \zeta_i B_i$, for all $i = 1,\ldots,r$. Then $\text{Jac}_{\alpha, \frac{\alpha - 1}{2}} (\pi) = 0$.

Now let $\psi$ be a parameter with $\text{Jord}_\rho (\psi) = \{(A_i, B_i, \zeta_i)\}$; assume that the lowest $j$ blocks of $\text{Jord}_\rho (\psi)$ are DDR (i.e. there are no overlapping segments).

Definition. We say that the lowest $j$ blocks of $\text{Jord}_\rho (\psi, t, \eta)$ form a cuspidal sequence if

(i) $t(A_i, B_i, \zeta_i) = 0$ for $i = 1,\ldots,j$;

(ii) $\eta(A_{i+1}, B_{i+1}, \zeta_{i+1}) = -\eta(A_i, B_i, \zeta_i)(-1)^{A_i - B_i}$;

(iii) $B_{i+1} = A_i + 1$ for for $i = 1,\ldots,j - 1$.

We sometimes describe this situation by saying that $\psi$ begins with a cuspidal sequence of length (at least) $A_j$. Note that in this case $\text{Jac}_x \pi = 0$ for any $x \leq A_j$.

3.2 Change of order formulas

A key technical result in Xu’s work [32] are the so-called change of order formulas. Let $\pi$ be parametrized by $(\psi, \eta, t)$. Assume that the blocks $(A, B, \zeta)$ and $(A', B', \zeta')$ in $\text{Jord}_\rho (\psi)$ are adjacent under some fixed order $\prec$. Now consider the order $\prec'$ obtained by swapping $(A, B, \zeta)$ and $(A', B', \zeta')$ and keeping all other blocks in their positions. If $\prec'$ is admissible, one would like to know $(\eta', t')$ in order for $\pi$ to be parametrized by $(\psi, \eta', t')$ using the order $\prec'$ on $\text{Jord}_\rho$. Section 6.1 of [32] contains formulas for $(\eta', t')$ in this situation. The formulas in question are somewhat complicated, so we only record them in the special case that we need, namely, when one of the segments is a singleton, contained in the other segment.

Lemma 3.4. Let $\psi$ be a parameter such that $\text{Jord}(\psi)$ contains $(A, B, \zeta)$ as well as the singleton segment $(c, c, \zeta = -)$ for $c \in [B, A]$. Let $\succ$ be an order on $\text{Jord}(\psi)$ under which the singleton segment is the immediate successor of $(A, B, \zeta)$. Using this order, let $\pi$ be parametrized by $(\psi, \eta, t)$. Let $\succ'$ be the order obtained from $\succ$ by swapping the two segments in question. Let $(\psi, \eta', t')$ be the parameter for $\pi$ with respect to $\succ'$. If $\zeta = -$, then
If \( \eta(A, B, -) = (-1)^{A-B} \eta(c, c, -) \) and \( t(A, B, -) < \lfloor (A - B)/2 \rfloor \), then
\[
\begin{align*}
\eta'(c, c, -) &= (-1)^{A-B} \eta(c, c, -) \\
\eta'(A, B, -) &= -\eta(A, B, -)
\end{align*}
\]

If \( \eta(A, B, -) = (-1)^{A-B} \eta(c, c, -) \) and \( t(A, B, -) = \lfloor (A - B)/2 \rfloor \), then
\[
\begin{align*}
\eta'(c, c, -) &= (-1)^{A-B} \eta(c, c, -) \\
\eta'(A, B, -) &= -\eta(A, B, -)
\end{align*}
\]

If \( \eta(A, B, -) \neq (-1)^{A-B} \eta(c, c, -) \), then
\[
\begin{align*}
\eta'(A, B, -) &= -\eta(A, B, -)
\end{align*}
\]

If \( \zeta = + \), then \( t' = t \), \( \eta'(c, c, -) = (-1)^{A-B+1} \eta(c, c, -) \), and \( \eta'(A, B, +) = -\eta(A, B, +) \).

### 3.3 Computing Jacquet modules

Considering the way Arthur packets are constructed, it is not surprising that we often need to compute various Jacquet modules. A very useful tool here is Tadić’s formula \[28\]. For a representation \( \pi \) of \( G_n \) we let \( \mu^*(\pi) \) denote the sum of (the semi-simplifications of) Jacquet modules taken with respect to the maximal standard parabolics. We let \( m^* \) denote the analogous construction for general linear groups. Then
\[
(1) \quad \mu^*(\delta \times \pi) = M^*(\delta) \times \mu^*(\pi).
\]

In place of a precise (but lengthy) description of \( M^* \), we offer a rough (but brief) outline:
\[
(2) \quad M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*.
\]

where
\[
\begin{align*}
m & : x \otimes y \to x \times y \\
s & : x \otimes y \to y \otimes x \\
\sim & = \text{contragredient} \\
1 & = \text{identity mapping}
\end{align*}
\]

We will need the following special case of the above formula:
\[
(3) \quad M^*(\zeta(a, b)) = \sum_{i=a-1}^{b} \sum_{j=1}^{b} \zeta(-b, -(j + 1)) \times \zeta(a, i) \otimes \zeta(i + 1, j).
\]

We refer the reader to Theorem 5.4 of \[28\] for further details.
3.4 On standard modules

Recall that any $\pi \in \text{Irr}(G)$ is the unique irreducible quotient of a (unique) standard module $\nu^n \delta_1 \times \cdots \times \nu^n \delta_1 \times \tau$. We will use this quotient form of the Langlands classification interchangeably with the subrepresentation form, by means of the Gelfand-Kazhdan results for general linear groups and the Mœglin–Vigneras–Waldspurger involution through the following lemma (see [5, Lemma 2.2]):

Lemma 3.5. Let $\tau_i \in \text{Irr}(\text{GL}_i(F))$, $i = 1, \ldots, k$ and $\pi_0 \in \text{Irr}(G(W_{n_0}))$. Let $P$ be a standard parabolic subgroup of $G(W_n)$ ($n = n_0 + 2\sum t_i$) with Levi component equal to $\text{GL}_i(F) \times \cdots \times \text{GL}_k(F) \rtimes G(W_{n_0})$. Then, for any $\pi \in \text{Irr}(G(W_n))$, the following statements are equivalent:

(i) $\pi \hookrightarrow \tau_1 \times \cdots \times \tau_k \times \pi_0$;

(ii) $\tau^\vee_1 \times \cdots \times \tau^\vee_k \times \pi_0 \twoheadrightarrow \pi$.

Because each $\nu^n \delta_i$ is given by a segment, we often view the (GL part of the) standard module simply as a collection of segments. In a few places in this paper we refer to the standard module of $\pi$ above (resp. below) $z$ for an irreducible representation $\pi$ and some fixed $z > 0$. This just means the collection of segments $[x, y]$ that appear in the standard module, for which $y \geq z$ (resp. $y < z$). In §1.4.2 the segments were sorted by their midpoints; we point out that we can also sort them lexicographically with respect to their endpoints. Thus we can write any irreducible representation $\pi$ as the unique irreducible quotient of

$$\Sigma_{\geq z} \times \Sigma_{< z} \rtimes \tau,$$

where $\Sigma_{\geq z}$ (resp. $\Sigma_{< z}$) denotes the irreducible GL-representation obtained as the unique irreducible quotient of the standard module above (resp. below) $z$. We will need:

Lemma 3.6. Let $z > 0$. Then $\text{Jac}_{-z}(\pi) \neq 0$ if and only if $m^*(\Sigma_{\geq z})$ contains a subquotient of the form $\xi \otimes \nu^z$.

Proof. This follows from Tadić’s formula (1) and (2). First, assume $\text{Jac}_{-z}(\pi) \neq 0$. Then $\text{Jac}_{-z}(\Sigma_{\geq z} \times \Sigma_{< z} \rtimes \tau) \neq 0$. Because the cuspidal support of $\Sigma_{< z}$ is bounded, and because $\tau$ is tempered, $\mu^*(\Sigma_{< z} \rtimes \tau)$ cannot contain a subquotient of the form $\nu^{-z} \otimes \cdots$. Thus $\text{Jac}_{-z}(\Sigma_{\geq z} \times \Sigma_{< z} \rtimes \tau) \neq 0$ implies $M^*(\Sigma_{\geq z})$ contains a subquotient of this form. Now we use formula (2): $M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*$ shows the subquotients of $M^*(\Sigma_{\geq z})$ are of the form

$$\tilde{B} \times A_1 \otimes A_2,$$

where $A \otimes B \leq m^*(\Sigma_{\geq z})$ and $A_1 \otimes A_2 \leq m^*(A)$. But $\tilde{B} \times A_1 = \nu^{-z}$ if and only if $A_1 = 1$ (trival) and $B = \nu^z$. This shows $A \otimes \nu^z \leq m^*(\Sigma_{\geq z})$, which we needed to prove.

Conversely, assume $\xi \otimes \nu^z \leq m^*(\Sigma_{\geq z})$. This implies that the Jacquet module of $\Sigma_{\geq z}$ (with respect to the appropriate parabolic) contains a quotient of this form. But $r_P(\Sigma_{\geq z}) \rightarrow \xi \otimes \nu^z$ implies, using Frobenius, that $\Sigma_{\geq z} \hookrightarrow \xi \times \nu^z$. Equivalently, we have $\nu^z \times \xi \hookrightarrow \Sigma_{\geq z}$, which in turns shows

$$\nu^z \times \xi \times \Sigma_{< z} \rtimes \tau \twoheadrightarrow \pi.$$

This implies $\text{Jac}_{-z}(\pi) \neq 0$, which we needed to show. \qed
One of the recurring technical questions in this paper is the following: suppose $\pi'$ is the unique irreducible quotient of $\nu_{\frac{a-1}{2}} \rtimes \pi$. How are the standard modules of $\pi$ and $\pi'$ related? At least in one important case, the situation is simple:

**Lemma 3.7.** Suppose $\nu_{\frac{a-1}{2}} \rtimes \pi$ is irreducible. Then its standard module is obtained by simply adding the singleton segment $\nu_{\frac{a+1}{2}}$ to the standard module of $\pi$.

**Proof.** The fact that $\nu_{\frac{a-1}{2}} \rtimes \pi$ is irreducible implies $\text{soc}(\nu_{\frac{a+1}{2}} \rtimes \pi) = \text{soc}(\nu_{\frac{a+1}{2}} \rtimes \pi)$. Now the result follows from the descriptions of these socles in Proposition 6.1 and Theorem 7.1. of [6].

### 3.5 Kudla’s filtration

One of the basic tools we use is Kudla’s filtration, which describes the Jacquet modules (with respect to maximal parabolics) of the Weil representation. The result was originally proved as Theorem 2.8 in Kudla’s paper [13]. We omit the full statement here; instead, we record the corollaries we use most often.

**Lemma 3.8.** Let $\pi \in \text{Irr}(G_n)$, $\pi_0 \in \text{Irr}(G_{n-k})$. Let $\alpha = n - m + \epsilon$ and let $\sigma$ be a multisegment representation such that the multisegment defining $\sigma$ does not contain $\frac{a-1}{2}$. Then $\chi_V \sigma \rtimes \pi_0 \rightarrow \pi$ implies $\chi_W \sigma \rtimes \Theta_\alpha(\pi_0) \rightarrow \Theta_\alpha(\pi)$.

**Proof.** This is a straightforward application of Kudla’s filtration; we omit the details.

**Remark 3.9.** We often use the above lemma to show (non-)vanishing. Indeed, assume $\chi_V \sigma \rtimes \pi_0 \rightarrow \pi$, and let $\Theta_\alpha(\pi_0) = 0$. If we can apply Lemma 3.8 we get $\chi_W \sigma \rtimes \Theta_\alpha(\pi_0) \rightarrow \Theta_\alpha(\pi)$ which shows $\Theta_\alpha(\pi) = 0$.

If we can apply the lemma with $-\alpha$ (instead of $\alpha$), we get the converse as well: suppose $\Theta_\alpha(\pi_0) \neq 0$. Then the Conservation relation shows $\Theta'_\alpha(\pi_0) = 0$, where $\Theta'$ denotes the lift to the other tower. Applying the lemma with $-\alpha$, we get $\Theta'_\alpha(\pi) = 0$. But then the Conservation relation shows $\Theta_\alpha(\pi) \neq 0$.

**Remark 3.10.** We often use Lemma 3.8 to deduce the same about the small lifts:

$$\chi_W \sigma \rtimes \theta_\alpha(\pi_0) \rightarrow \theta_\alpha(\pi).$$

This stronger claim follows whenever the multisegment defining $\sigma$ does not contain $\frac{a-1}{2}$ and

a) $r\varphi(\theta_\alpha(\pi))$ has only one irreducible subquotient on which $\text{GL}_k(F)$ acts by $\chi_W \sigma$; or

b) $r\varphi(\pi)$ has only one irreducible subquotient on which $\text{GL}_k(F)$ acts by $\chi_V \sigma$, and the multisegment defining $\sigma$ does not contain $-\frac{a+1}{2}$.

(Here $P$ is the suitable maximal parabolic.) To see this, note that Lemma 3.8 implies $\chi_W \sigma \rtimes \Theta_\alpha(\pi_0) \rightarrow \Theta_\alpha(\pi) \rightarrow \theta_\alpha(\pi)$. Bernstein’s Frobenius reciprocity then shows there is a non-zero map $\chi_W \sigma \otimes \Theta_\alpha(\pi_0) \rightarrow r\varphi(\theta_\alpha(\pi))$. Since $\Theta_\alpha(\pi_0)$ has a unique irreducible quotient, this implies that $\chi_W \sigma \otimes \theta_\alpha(\pi_0)$ is an irreducible subquotient of $r\varphi(\theta_\alpha(\pi))$. Assuming a), we find that it is a subrepresentation, and using Frobenius again, the claim follows.

In case b), we can argue as follows. The lemma shows $\chi_W \sigma \rtimes \Theta_\alpha(\pi_0) \rightarrow \Theta_\alpha(\pi) \rightarrow \theta_\alpha(\pi)$, which means that there is an irreducible subquotient $\pi'_0$ of $\Theta_\alpha(\pi_0)$ such that $\chi_W \sigma \rtimes \pi'_0 \rightarrow$
θ_α(π). Because of the extra assumption in b), we can now use the lemma in reverse (to compute Θ_−α); we get χ_Vσ ⊗ Θ_−α(π'_0) → π, i.e. χ_Vσ ⊗ Θ_−α(π'_0) → r_π(π) follows. Since Θ_−α(π'_0) has a unique irreducible quotient, we must have χ_Vσ ⊗ Θ_−α(π'_0) → r_π(π). But if r_π(π) only has one subquotient on which GL_k(F) acts by χ_Vσ, this implies θ_−α(π'_0) = π_0. Therefore π'_0 = θ_α(π_0), so χ_Vσ × θ_α(π_0) → θ_α(π).

Remark 3.11. We typically use Lemma 3.8 and Remarks 3.9, 3.10 to go from

\[ \pi_\gg \ni \chi_V L_1 \times \cdots \times \chi_V L_r \times \pi \]

to

\[ \theta_\alpha(\pi_\gg) \ni \chi_W L_1 \times \cdots \times \chi_W L_r \times \theta_\alpha(\pi). \]

We explain this. Let \( \pi \in \Pi_\psi \) be a representation of Arthur type parametrized by \( \psi \), and let \( \pi_\gg \) be a DDR representation parametrized by \( \psi_\gg \) which dominates \( \psi \). We recall the setting of Lemma 3.1 we let \( \pi(i) \) denote the \( i \)-th intermediate step in going from \( \pi_\gg \) to \( \pi \), so that \( \pi(i_{i-1}) \) is the unique irreducible subrepresentation of \( L_i \times \pi(i) \).

Now assume none of the multisegments defining the \( L_i \)'s contain \( \frac{1-\alpha}{2} \). Instead of \( \pi(i_{i-1}) \ni \chi_V L_i \times \pi(i) \), we may write

\[ \chi_V L'_i \times \pi(i) \ni \pi(i_{i,1}). \]

Since the multisegment corresponding to \( L'_i \) does not contain \( \frac{1-\alpha}{2} \), we may safely apply Lemma 3.8. In fact, because of the uniqueness of the appropriate Jacquet modules (see Lemma 3.1) we may apply Remark 3.10 to get \( \chi_W L'_i \times \theta_\alpha(\pi(i)) \ni \theta_\alpha(\pi(i_{i-1})), \) or equivalently,

\[ \theta_\alpha(\pi(i_{i-1})) \ni \chi_W L_i \times \theta_\alpha(\pi(i)). \]

Of course, one can apply Remark 3.9 as well, if necessary. Repeating this reasoning for each \( i \), we get the desired result.

The following two results complement Lemma 3.8. The proofs are a direct application of Kudla’s filtration.

Lemma 3.12. Let \( \pi \in \text{Irr}(G_n), \pi_0 \in \text{Irr}(G_{n-2k}) \) be representations such that

\[ \chi_V \delta(\frac{\beta+1}{2} - k, \frac{\beta-1}{2}) \ni \pi_0 \ni \pi. \]

If \( \alpha \neq \beta \), then \( \chi_W \delta(\frac{\beta+1}{2} - k, \frac{\beta-1}{2}) \ni \Theta_\alpha(\pi_0) \ni \Theta_\alpha(\pi) \)

Lemma 3.13. Let \( \pi \in \text{Irr}(G_n), \pi_0 \in \text{Irr}(G_{n-2k}) \) be representations such that

\[ \chi_V \zeta(\frac{\alpha+1}{2} - k, \frac{\alpha-1}{2}) \ni \pi_0 \ni \pi. \]

Then one of the following is true:

(i) \( \chi_W \zeta(\frac{\alpha+1}{2} - k, \frac{\alpha-1}{2}) \ni \Theta_\alpha(\pi_0) \ni \Theta_\alpha(\pi); \)

(ii) \( \theta_\alpha(\pi) = \theta_{\alpha-2k}(\pi_0). \)
Finally, we will often use the following corollary of Kudla’s filtration (cf. Lemma 5.1 of [25]):

**Lemma 3.14.** Let \( \pi \in \text{Irr}(G_n) \). Assume \( \theta_{-\alpha}(\pi) \neq 0 \) for some \( \alpha > 0 \). If \( \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) = 0 \), but \( \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) \neq 0 \), then \( \theta_{2-\alpha}(\pi) \neq 0 \). Moreover, if \( \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) \) is irreducible, then \( \theta_{2-\alpha}(\pi) = \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) \).

**Proof.** The non-vanishing of \( \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) \) implies that there exists an irreducible representation \( \xi \) such that \( \theta_{-\alpha}(\pi) \mapsto \chi_W \cdot |\frac{\alpha}{\alpha+1}| \times \xi \). We then have

\[
\rho \mapsto \text{Hom}_{G_n}(\omega_{m,n}, \theta_{-\alpha}(\pi))
\]

\[
\mapsto \text{Hom}_{G_n}(\omega_{m,n}, \chi_W \cdot |\frac{\alpha}{\alpha+1}| \times \xi)
\]

\[
\cong \text{Hom}_{G_n}(r \rho(\omega_{m,n}), \chi_W \cdot |\frac{\alpha}{\alpha+1}| \otimes \xi),
\]

where \( P \) is the appropriate maximal standard parabolic. This proves that the last Hom-space is non-zero. Now Kudla’s filtration shows that this implies one of the following:

- \( \text{Jac}^{\chi_W}_{\alpha}(\pi) \neq 0 \): or
- \( \pi \) is a quotient of \( \omega_{m-2,n} \).

Since we are assuming the first option is not true, the second must hold; this is equivalent to \( \theta_{2-\alpha}(\pi) \neq 0 \). The same calculation shows that \( \Theta_{\alpha-2}(\xi) \mapsto \pi \), so \( \theta_{\alpha-2}(\xi) \cong \pi \). If \( \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) \) is irreducible (and thus equal to \( \xi \)), this shows \( \theta_{2-\alpha}(\pi) \cong \xi = \text{Jac}^{\chi_W}_{\alpha}(\theta_{-\alpha}(\pi)) \).

**Remark 3.15.** Notice that the characters \( \chi_V \) and \( \chi_W \) — which are required to fix the splitting above the dual pair — feature prominently in this section. To simplify the exposition in the remainder of the paper, we choose to omit them from the notation. So, for example, we write \( \text{Jac}^{\chi_W}_{\alpha} \) and \( \nu^{\frac{\alpha}{\alpha+1}} \times \pi \) instead of \( \text{Jac}^{\chi_W}_{\alpha} \) and \( \chi_V \nu^{\frac{\alpha}{\alpha+1}} \times \pi \). This slight abuse of notation is compensated by a drastic improvement in readability.

### 3.6 Higher lifts

Finally, we occasionally make use of the results of [7], which provide a description of lifts in terms of L-parameters. We do not need the full extent of these results, but we record three useful observations:

**Lemma 3.16.**

a) Suppose \( \theta_{-\alpha}(\pi) \neq 0 \) for some \( \alpha > 0 \). Then the standard module of \( \theta_{-\alpha-2}(\pi) \) is obtained by adding the singleton segment \( \nu^{\frac{\alpha}{\alpha+1}} \) to the standard module of \( \theta_{-\alpha}(\pi) \).

b) Let \( z > \frac{\alpha-1}{2} \). Let \( \xi \) be the representation whose standard module is obtained by adding the singleton segment \( \nu^z \) to the standard module of \( \pi \). Then the standard module of \( \theta_{-\alpha}(\xi) \) is obtained by adding the singleton segment \( \nu^z \) to the standard module of \( \theta_{-\alpha}(\pi) \).
Atobe and Minguez define an order on $A$ where $x$ is a certain subset of $A$. A key property of this bijection is that every segment of one, given in subsection §1.4.2, uses the subrepresentation form of the Langlands classification, as opposed to the quotient $a$ representation.

We collect here some of the results of Atobe and Minguez from the sixth section of [6], which describes the derivatives and socles of the representation $s$ of the form $\nu^x \rho \times \pi$, where $\pi$ is an irreducible representation of $G_n$, $x$ is a negative real number and $\rho$ is (this is the case relevant for us) an irreducible, cuspidal, self-dual representation of $GL_t(F)$ for some $t$. We use the subrepresentation form of the Langlands classification, as opposed to the quotient one, given in subsection §1.4.2 (see also §1.4.3). Thus, $\pi$ is a unique subrepresentation of a representation $\delta([-y_1, -x_1]_{\rho_1}) \times \cdots \times \delta([-y_k, -x_k]_{\rho_k}) \times \tau$, where $x_1 + y_1 \geq x_2 + y_2 \geq \cdots \geq x_k + y_k > 0$ and $\tau$ is tempered. For any $x < 0$, let $A_{\nu^x \rho} = \{i \in \{1, 2, \ldots, k\} : \rho_i \cong \rho, -x_i = x\}$.

Atobe and Minguez define an order on $A_{\nu^x \rho}$ and $A_{\nu^{x-1} \rho}$, as well as a bijection between a certain subset of $A_{\nu^{x-1} \rho}$ (denoted by $A^0_{\nu^{x-1} \rho}$) and a certain subset of $A_{\nu^x \rho}$ (denoted by $A^0_{\nu^x \rho}$). A key property of this bijection is that every segment of $A^0_{\nu^{x-1} \rho}$ is linked to the corresponding segment of $A^0_{\nu^x \rho}$. The description of the socle of $\nu^x \rho \times \pi$ (which is irreducible, by Proposition 3.3. of [6]) depends only on these two sets. We shall just need the following result, which is part of Proposition 6.1 of [6]:

**Proposition 3.17.** The Langlands datum of the socle of the representation $\nu^x \rho \times \pi$ is obtained from that of $\pi$ by inserting $\nu^{-x} \rho$ if and only if $A^0_{\nu^{x-1} \rho} = A^0_{\nu^x \rho}$ (i.e. every segment in $A_{\nu^{x-1} \rho}$ is matched with some segment in $A_{\nu^x \rho}$).

### 4 Theorem A

Let $\pi$ be parameterized by

$$\psi = \bigoplus_{i=1}^{r} \chi V \otimes S_{\alpha_i} \otimes S_{\beta_i} \bigoplus_{\rho \not\in \chi V} \rho \otimes S_{\alpha} \otimes S_{\beta}$$

Proposition 2.2 shows that $\theta_{-\alpha}(\pi)$ is in the A-A packet for $\alpha \gg 0$, with parameter

$$\psi_{\beta} = \bigoplus_{i=1}^{r} \chi W \otimes S_{\alpha_i} \otimes S_{\beta_i} \otimes \chi W \otimes S_{\delta_i} \otimes S_{\epsilon_i} \bigoplus_{\rho \not\in \chi V} \chi W \chi V^{-1} \rho \otimes S_{\alpha} \otimes S_{\beta}$$

Note that $\epsilon$ determines the target tower. From now on, we omit the part $\bigoplus_{\rho \not\in \chi V}$ from the parameter of $\pi$ since it remains unchanged in — and does not affect — the theta correspondence. As explained in Remark 8.15, we also drop $\chi V$ and $\chi W$ from the notation.

Throughout this section we fix an admissible order $>$ on $\text{Jord}(\psi) = \{(A_i, B_i, \zeta_i) : i = 1, \ldots, r\}$. The precise nature of this order is non-important, but we require that there exist an index $k \in \{1, \ldots, r\}$ such that $i > k$ if and only if
\[ B_i > \frac{a_i - 1}{2} \text{ or } \]
\[ B_i = \frac{a_i - 1}{2} \text{ and } \zeta_i = -1. \]

We now prove Theorem A. Assuming \( \pi \) (obtained by the Recipe) is equal to \( \theta_{-\alpha}(\pi) \), we need to prove that \( \pi_{\alpha-2} = \theta_{-(\alpha-2)}(\pi) \). We first prove this in a special case:

**Lemma 4.1.** Assume \( \zeta_i B_i \neq -\frac{a_i - 1}{2} \) for \( i = 1, \ldots, r \). Suppose \( \pi_\alpha = \theta_{-\alpha}(\pi) \). If \( \pi_{\alpha-2} \neq 0 \) then \( \pi_{\alpha-2} = \theta_{-(\alpha-2)}(\pi) \).

**Proof.** Extend the order we have fixed on Jord(\( \psi \)) to Jord(\( \psi_\alpha \)) by inserting \( 1 \otimes S_1 \otimes S_\alpha \) between \( (A_k, B_k, \zeta_k) \) and \( (A_{k+1}, B_{k+1}, \zeta_{k+1}) \). Let \( \pi_{\alpha,>} \) be a DDR representation which dominates \( \pi_\alpha \) with respect to this order. We have,

\[ \pi_{\alpha,>} : L_1 \times \cdots \times L_k \times \zeta((-\frac{\alpha}{2}, -\frac{\alpha + 1}{2}) \times \nu^{-\frac{\alpha}{2}} \times L_{k+1} \times \cdots \times L_r \times \pi_{\alpha-2}. \]

Our assumption about the order ensures that \( \nu^{-\frac{\alpha}{2}} \) can change places with \( L_{k+1}, \ldots, L_r \). After applying \( \text{Jac}_{L_1, \ldots, L_k, \zeta((-\frac{\alpha}{2}, -\frac{\alpha + 1}{2}), L_{k+1}, \ldots, L_r} \) we get,

\[ \pi_\alpha : \nu^{-\frac{\alpha}{2}} \times \pi_{\alpha-2}. \]

This shows \( \text{Jac}_{-\frac{\alpha}{2}}(\theta_{-\alpha}(\pi)) \) is irreducible and isomorphic to \( \pi_{\alpha-2} \). Moreover, \( \text{Jac}_{-\frac{\alpha}{2}}(\pi) = 0 \) by assumption (and Lemma 3.3). The result now follows from Lemma 3.14.

We can now deduce Theorem A for general \( \pi \) from Lemma 4.1. Keeping the orders we have fixed on Jord(\( \psi \)) and Jord(\( \psi_\alpha \)), we let \( \pi_\alpha \) and \( \pi_{\alpha,>} \) denote the representations obtained from \( \pi \) and \( \pi_{\alpha,>} \) by shifting up all the blocks \( (A_i, B_i, \zeta_i) \) for \( i > k \). Note that \( \pi_\alpha \) then satisfies the conditions of Lemma 4.1.

We show that \( \pi_\alpha = \theta_{-\alpha}(\pi) \) implies \( \pi_{\alpha,>} = \theta_{-\alpha}(\pi_\alpha) \). Our construction implies

\[ \pi_{\alpha,>} : L_{k+1} \times \cdots \times L_r \times \pi_\alpha. \]

Since \( \theta_\alpha(\pi_\alpha) = \pi \neq 0 \), we may apply Remark 3.9 (repeatedly) to show \( \theta_\alpha(\pi_{\alpha,>}) \neq 0 \). Indeed, notice that none of the multisegments \( L_i \) contain \( \frac{\alpha-1}{2} \) (though they might contain \( -\frac{\alpha+1}{2} \)). Furthermore, since none of them contain \( \frac{\alpha-1}{2} \), we may apply Remark 3.11 to the above embedding to compute \( \theta_\alpha \). We get

\[ \theta_\alpha(\pi_{\alpha,>) : L_{k+1} \times \cdots \times L_r \times \pi. \]

Since \( \pi_\alpha \) is precisely the unique irreducible subrepresentation of the representation on the right, we conclude \( \theta_\alpha(\pi_{\alpha,>) = \pi_\alpha \).

This shows \( \theta_{-\alpha}(\pi_\alpha) = \pi_{\alpha,>} \). We are now in position to apply Lemma 4.1 to \( \pi_\alpha \): we get that \( \pi_{\alpha-2}> \) is isomorphic to \( \theta_{2-\alpha}(\pi_\alpha) \). By construction,

\[ \theta_{2-\alpha}(\pi_\alpha) = \pi_{\alpha-2,>} : L_{k+1} \times \cdots \times L_r \times \pi_{\alpha-2}. \]

It remains to compute \( \theta_{\alpha-2} \). We apply Remark 3.11 again (none of the \( L_i \)’s contain \( -\frac{\alpha+3}{2} \)) to get

\[ \pi_\alpha : L_{k+1} \times \cdots \times L_r \times \theta_{\alpha-2}(\pi_{\alpha-2}). \]

Since \( \text{Jac}_{L_{k+1}, \ldots, L_r} \pi_\alpha = \pi \), we conclude \( \theta_{\alpha-2}(\pi_{\alpha-2}) = \pi \). Equivalently, \( \pi_{\alpha-2} = \theta_{2-\alpha}(\pi) \), which we needed to prove.
5 Theorem B

We recall the setting of Theorem B. We are simultaneously lifting $\pi$ to two different towers, $\mathcal{V}^+$ and $\mathcal{V}^-$; we denote the corresponding lifts by $\theta^\pm(\pi)$. For $\alpha \gg 0$, the lifts $\theta^\pm_\alpha(\pi)$ are given by parameters

$$\psi^\pm_\alpha = \psi \oplus 1 \otimes S_1 \otimes S_\alpha$$

To obtain the lower lifts, we descend using the Recipe. At some point, the Recipe will give a parameter $\psi_\alpha$ such that $\pi_\alpha = 0$; we let $d^\pm(\pi, \psi)$ denote the last level above this (see 32).

It is possible that $\pi^\pm_\alpha \neq 0$ for all $\alpha > 0$. (If that is the case, we set $d^\pm(\pi, \psi) = 1$.) From the point of view of this paper, this is uninteresting, albeit nice: it implies (using Theorem A) that $\theta^-_\alpha(\pi)$ is non-zero and contained in the A-A packet for all $\alpha > 0$. Because of this, we assume that at least one of the indices $d^\pm(\pi, \psi)$ is greater than 1.

Without loss of generality, let $d^+(\pi, \psi) \geq d^-(\pi, \psi)$. To simplify notation, let $\alpha = d^+(\pi, \psi)$. Thus the assumption is $\pi^\pm_\alpha \neq 0$, but $\pi^\pm_{\alpha-2} = 0$ (recall $\alpha \geq 3$). Proving Theorem B then amounts to proving the following claims:

a) $\theta^+_{2^{-\alpha}}(\pi) = 0$. (This shows that $\mathcal{V}^+$ is the going-up tower and that $\theta^+_{-\alpha}(\pi)$ is the first occurrence of $\pi$ on $\mathcal{V}^+$.)

b) $\pi^{-\alpha}_{\alpha-2} \neq 0$.

To prove this, we use the same reduction we used in the proof of Theorem A. Namely, we let $\pi^\pm_{\alpha,>}$ (resp. $\pi_{>}$) be the expanded version of $\pi^\pm_{\alpha}$ (resp. $\pi$) constructed there. The proof from Section 4 now shows $\theta^\pm_{-\alpha}(\pi_{>}) = \pi^\pm_{\alpha,>} \neq 0$. The proof of Theorem A also shows that $\pi^\pm_{\alpha-2,>} = 0$. Indeed, if we assume that $\pi^\pm_{\alpha-2,>} \neq 0$, it follows that $\pi^\pm_{\alpha,>} \hookrightarrow \nu^{-\frac{\alpha}{2}} \times \pi^\pm_{\alpha-2,>}$. On the other hand, $\text{Jac}_{L_{k+1}, \ldots, L_r}(\pi^\pm_{\alpha,>}) = \pi^\pm_{\alpha} \neq 0$. This forces $\text{Jac}_{L_{k+1}, \ldots, L_r}(\pi^\pm_{\alpha-2,>}) = \pi^\pm_{\alpha-2} \neq 0$, a contradiction. Our plan is to prove:

**Lemma 5.1.** $\pi^{-\alpha}_{\alpha-2,>} \neq 0$.

The proof in Section 4 shows that this implies b). Lemma 5.1 will also imply the following:

**Lemma 5.2.** $\theta^+_{2^{-\alpha}}(\pi_{>}) = 0$.

We will then prove that this implies a), completing the proof of Theorem B.

**Proof of Lemma 5.2.** This can be verified using the algorithm developed by Xu [32]. However, going through the entire algorithm is quite involved, so we take a shortcut. By construction, for any block $(A, B, \zeta) \in \text{Jord}(\psi^\pm_{\alpha,>})$ with $B > \frac{\alpha+1}{2}$, we have $B \gg \frac{\alpha+1}{2}$ (in other words, the only blocks above $\alpha$ are far away). One immediate consequence of the purely combinatorial algorithm of [32] is that the question of (non-)vanishing of $\pi^\pm_{\alpha-2,>}$ is independent of these far-away blocks (provided $\pi^\pm_{\alpha,>}$ is nonzero). Thus, it suffices to prove that $\pi^{-\alpha}_{\alpha-2,>} \neq 0$ assuming that there are no blocks in $\text{Jord}(\psi^\pm_{\alpha,>})$ with $B > \frac{\alpha+1}{2}$.

Under this assumption, we have $\text{Jac}_{-x}(\pi^\pm_{\alpha,>}) = 0$ for any $x > \frac{\alpha+1}{2}$. This implies that there are no segments $[x, y]$ in the standard module of $\pi^\pm_{\alpha,>}$ with $x > \frac{\alpha+1}{2}$. Since $\pi^\pm_{\alpha-2,>} = 0$ we also have $\text{Jac}_{-x}(\pi^\pm_{\alpha,>}) = 0$ which then implies that there the singleton segment $\nu^{\alpha-1}$
does not appear in the standard module of \( \pi_{\alpha,>}'s \). Lemma 3.16 a) now shows that \( \pi_{\alpha,>}'s \) is the first lift of \( \pi >'s \) on \( V^+ \) tower; in particular, \( V^+ \) is the going-up tower for \( \pi >'s \).

But then \( V^- \) is the going-down tower for \( \pi >'s \), which implies that the standard module of \( \pi_{\alpha,>} = \theta_{-\alpha}(\pi_{\alpha,>}) \) contains the singleton segment corresponding to \( \nu^{\frac{\alpha_1}{2}} \) (and no segments \([x,y] \) with \( x = \frac{\alpha_1+1}{2} \)). Thus \( \pi_{\alpha-2,>} = \text{Jac}_{-\frac{\alpha_1}{2}}(\pi_{\alpha,>}) \neq 0 \), which we needed to show.

**Proof of Lemma 5.2.** Assume the contrary: \( \theta_{\alpha-2,0}(\pi_<) \neq 0 \). Lemma 3.16 c) then shows that the standard modules of \( \pi_{\alpha,0}^+ \) and \( \pi_{\alpha,0}^- \) are equal above \( \frac{\alpha_1-1}{2} \). But then Lemma 3.6 shows that \( \text{Jac}_{-\frac{\alpha_1}{2}}(\pi_{\alpha,0}^-) \) and \( \text{Jac}_{-\frac{\alpha_1}{2}}(\pi_{\alpha,0}^+) \) are either both non-zero, or both zero. This contradicts Lemma 5.1. We conclude that \( \theta_{\alpha-2,0}(\pi_<) = 0 \).

We have thus shown \( \theta_{-\alpha}(\pi_<) = 0 \), and it remains to show that this implies \( \theta_{-\alpha}(\pi) = 0 \). Recall the construction of \( \pi_\alpha \) from \[ \pi \]. We have

\[
\pi_\alpha \rightarrow L_{i+1} \times \cdots \times L_r \times \pi
\]

where none of the (segments corresponding to) \( L_i \)'s contain \(-\frac{\alpha_1}{2} \). Now Remark 3.9 shows that \( \theta_{-\alpha}(\pi_\alpha) = 0 \) implies \( \theta_{-\alpha}(\pi) = 0 \). This concludes the proof of Theorem B.

### 6 Theorem C

We retain the notation from the previous section: \( \pi \) is an irreducible representation parametrized by a DDR parameter (cf. Remark 3.13)

\[
\psi = \bigoplus_{i=1}^r 1 \otimes S_{\alpha_i} \otimes S_{\beta_i}.
\]

In this section we prove Theorem C for \( \pi \).

#### 6.1 The DDR case

We first prove Theorem C for DDR representations. The proof will be divided into several cases, depending on the position of the added block relative to the other blocks in \( \text{Jord}(\psi) \). The first step is

**Lemma 6.1.** Assume that \( \theta_{-\alpha}(\pi) \) is isomorphic to a representation \( \pi_{\alpha} \) in the A-A packet. Let \( \pi_{\alpha+2} \in \Pi_{\nu_{\alpha+2}} \) be the representation obtained by applying Epicer to \( \pi_{\alpha} \). If \( \alpha \notin \text{Jord}(\pi) \) then \( \pi_{\alpha+2} \neq 0 \) and \( \theta_{-(\alpha+2)}(\pi) = \pi_{\alpha+2} \).

**Proof.** Under our assumption, the representation \( \nu^{\frac{\alpha_1+1}{2}} \times \pi \) is irreducible (cf. [12], 6.3.1). According to Lemma 3.7, the Langlands parameter of \( \nu^{\frac{\alpha_1+1}{2}} \times \pi \) is obtained by adding \( \nu^{\frac{\alpha_1}{2}} \) to the Langlands parameter of \( \pi \). This means, by Proposition 3.17, that \( A_{\nu^{\frac{\alpha_1}{2}}}(\nu^{\frac{\alpha_1+1}{2}} \times \pi) = A_{\nu^{\frac{\alpha_1+1}{2}}}(\pi) \). By Theorem 6.7 of [17], we also have \( A_{\nu^{\frac{\alpha_1}{2}}}(\pi) = A_{\nu^{\frac{\alpha_1+1}{2}}}(\theta_{-\alpha}(\pi)) \). It follows that \( A_{\nu^{\frac{\alpha_1}{2}}}(\theta_{-\alpha}(\pi)) = A_{\nu^{\frac{\alpha_1+1}{2}}}(\theta_{-\alpha}(\pi)) \). This, thus, by Lemma 3.10 a) and Proposition 3.17,

\[
\theta_{-(\alpha+2)}(\pi) = \text{soc}(\nu^{\frac{\alpha_1+1}{2}} \times \theta_{-\alpha}(\pi)).
\]
The above discussions shows that it suffices to prove
\[ (4) \quad \pi_{\alpha+2} \hookrightarrow \nu^{-\alpha+1} \times \theta_\alpha(\pi). \]
We have two cases:

Case (i): $\alpha + 2 \notin \mathrm{Jord}(\pi)$. In this case $\pi_{\alpha+2}$ is DDR (so in particular non-zero) and (4) holds, so we are done.

Case (ii): $\alpha + 2 \in \mathrm{Jord}(\pi)$. Thus, we assume that $\alpha = |a_i - b_j| - 1$ for some $i$. Now we let $\pi_{\alpha+2,\triangleright}$ (resp. $\pi_{\alpha,\triangleright}$) be the DDR representation obtained from $\pi_{\alpha+2}$ (resp. $\pi_{\alpha}$) by shifting up the segments $[A_j, B_j]$ for $j \geq i$. Now $\alpha + 2 \notin \mathrm{Jord}(\pi_{\alpha,\triangleright})$, so we are back in case (i).

Therefore
\[ \pi_{\alpha+2,\triangleright} \hookrightarrow \chi W \nu^{-\alpha+1} \times \pi_{\alpha,\triangleright} \hookrightarrow \chi W \nu^{-\alpha+1} \times L_i \times \cdots \times L_r \times \pi_{\alpha}. \]

Now assume $\zeta = 1$ (i.e. $a_i \geq b_i$). Since $\frac{\alpha+1}{2} = B_i$, Lemma [12] shows that $\chi W \nu^{-\alpha+1}$ and all the $L_j$’s can exchange places in the embedding above:
\[ \pi_{\alpha+2,\triangleright} \hookrightarrow L_i \times \cdots \times L_r \times \chi W \nu^{-\alpha+1} \times \pi_{\alpha}. \]

Applying the appropriate Jacquet modules, we get $0 \neq \pi_{\alpha+2} \hookrightarrow \chi W \nu^{-\alpha+1} \times \pi_{\alpha}$, as claimed.

If $\zeta = -1$, $\chi W \nu^{-\alpha+1} \times L_i$ will be reducible, but we can get (4) using Kudla’s filtration. We have
\[ \pi_{\alpha,\triangleright} \hookrightarrow L_i \times \cdots \times L_r \times \pi_{\alpha}. \]

Since none of the multisegments which correspond to $L_i$ contain $-\frac{\alpha+1}{2}$, we may apply Remark [3.11] to compute $\theta_\alpha$. Since $\theta_\alpha(\pi_{\alpha}) = \pi$, we get
\[ \theta_\alpha(\pi_{\alpha,\triangleright}) \hookrightarrow L_i \times \cdots \times L_r \times \pi. \]

This shows $\theta_\alpha(\pi_{\alpha,\triangleright}) = \pi_{\triangleright}$. Equivalently, $\theta_{-\alpha}(\pi_{\triangleright}) = \pi_{\alpha,\triangleright}$; again, this brings us back to case (i) so we conclude that $\theta_{-\alpha-2}(\pi_{\triangleright}) = \pi_{\alpha+2,\triangleright}$. Now, by definition of $\pi_{\triangleright}$ we have
\[ \pi_{\triangleright} \hookrightarrow L_i \times \cdots \times L_r \times \pi. \]

Notice that the multisegments which correspond to $L_i, \ldots, L_r$ do not contain $B_i + 1 = \frac{\alpha+3}{2}$; the cuspidal support of $L_i$ is negative, and the cuspidal support of other $L_j$’s is bounded from below by $B_i + 3$. Therefore we may apply Remark [3.11] again, to compute $\theta_{-(\alpha+2)}$. We get
\[ \pi_{\alpha+2,\triangleright} = \theta_{-(\alpha+2)}(\pi_{\triangleright}) \hookrightarrow L_i \times \cdots \times L_r \times \theta_{-(\alpha+2)}(\pi). \]

By Moeglin’s construction, $\pi_{\alpha+2} = \mathrm{Jac}_{L_i, \ldots, L_r}(\pi_{\alpha+2,\triangleright})$ is either irreducible or zero. The above embedding shows that it is non-zero, and in fact equal to $\theta_{-(\alpha+2)}(\pi)$. Thus $\pi_{\alpha+2} = \theta_{-(\alpha+2)}(\pi)$, which we needed to show.

Now we treat the case $\alpha \in \mathrm{Jord}(\pi)$. The following result treats the case in which the added block is contained within a segment $\mathrm{Jord}(\pi)$ and stays withing the segment when shifted up:

**Lemma 6.2.** Assume that $\theta_{-\alpha}(\pi)$ is isomorphic to a representation $\pi_\alpha$ in the $A$-$A$ packet. Let $\pi_{\alpha+2} \in \Pi_{\nu_{\alpha+2}}$ be the representation obtained by applying Epicer to $\pi_{\alpha}$. If $\alpha \in \{ |a_i - b_j| + 1, a_i + b_j - 3 \}$ then $\pi_{\alpha+2} \neq 0$ and $\theta_{-(\alpha+2)}(\pi) = \pi_{\alpha+2}$.
Since \( \nu \frac{a+1}{2} \times \pi \) is irreducible (for all \( \alpha \in [|a_i - b_i| + 1, a_i + b_i - 3] \)) and that \([4]\) holds.

The latter is not complicated in this case: Since the added block is just shifting within the same segment, we may view \( \pi_{\alpha + 2} \) as the last stage in going from \( \pi_{\alpha, \gg} \) to \( \pi_\alpha \). Thus \( \pi_\alpha \neq 0 \) implies \( \pi_{\alpha + 2} \neq 0 \) and \([4]\) holds.

However, the fact that \( \nu \frac{a+1}{2} \times \pi \) is irreducible is no longer obvious, so we prove it. \( \square \)

**Lemma 6.3.** Let \( \alpha \in [|a_i - b_i| + 1, a_i + b_i - 3] \). Then the representation \( \nu \frac{a+1}{2} \times \pi \) is irreducible.

**Proof.** Let \( \psi_\gg \) be the DDR parameter obtained from \( \psi \) by shifting up the segments \([A_j, B_j]\) for \( j = i, \ldots, r \). We assume \( \alpha, \alpha + 2 \notin \text{Jord}(\psi_\gg) \). As usual, we have \( \pi_\gg \hookrightarrow \pi \), and thus

\[
\nu \frac{a+1}{2} \times \pi_\gg \hookrightarrow \nu \frac{a+1}{2} \times L_i \times \cdots \times L_r \times \pi. 
\]

Notice that \( \nu \frac{a+1}{2} \times L_j \) is irreducible for every \( j \). Indeed, the cuspidal support of \( L_j \) for \( j > i \) is far from \( \frac{a+1}{2} \), for \( L_i \), the statement follows from Lemma \([1, 2]\). We may thus write

\[
\nu \frac{a+1}{2} \times \pi_\gg \hookrightarrow \nu \frac{a+1}{2} \times L_i \times \cdots \times L_r \times \nu \frac{a+1}{2} \times \pi.
\]

Since \( \nu \frac{a+1}{2} \times \pi_\gg \) is irreducible, there is an irreducible subquotient \( \pi' \) of \( \nu \frac{a+1}{2} \times \pi \) such that the image of the above map lands in \( L_i \times \cdots \times L_r \times \pi' \).

Now if \( \nu \frac{a+1}{2} \times \pi \) were reducible (and thus strictly larger than \( \pi' \)), we would have either \( \text{Jac}_{\frac{a+1}{2}}(\pi') = 0 \) or \( \text{Jac}_{\frac{a+1}{2}}(\pi') = 0 \); the crucial input here is Lemma \([3, 4]\) which shows that \( \text{Jac}_{\frac{a+1}{2}}(\pi) = 0 \). We now use formula \([1]\). Note that \( M^*(L_j) \) cannot contribute with a factor of the form \( \nu \frac{a+1}{2} \oplus \xi \); for \( j > i \) this is obvious because of the cuspidal support, and for \( L_i \) it follows from e.g. \([12]\). It follows that the representation on the right hand side has either \( \text{Jac}_{\frac{a+1}{2}} = 0 \) or \( \text{Jac}_{\frac{a+1}{2}} = 0 \). However, this is impossible because it contains the irreducible representation \( \nu \frac{a+1}{2} \times \pi_\gg \), for which both of these Jacquet modules are non-zero. This shows that \( \nu \frac{a+1}{2} \times \pi \) is irreducible. \( \square \)

It remains to consider the case \( \frac{a+1}{2} = A_i \). We first assume that \( \pi_{\alpha + 2} \) is DDR:

**Lemma 6.4.** Assume that \( \theta_{-\alpha}(\pi) \) is isomorphic to a representation \( \pi_\alpha \) in the A-A packet. Let \( \pi_{\alpha + 2} \in \Pi_{\psi_{\alpha + 2}} \) be the representation obtained by applying Epicer to \( \pi_\alpha \); assume \( \psi_{\alpha + 2} \) is DDR. If \( \alpha = a_i + b_i - 1 \) then \( \pi_{\alpha + 2} \neq 0 \) and \( \theta_{-\alpha + 2}(\pi) = \pi_{\alpha + 2} \).

**Proof.** The additional assumption that \( \psi_{\alpha + 2} \) is DDR automatically implies \( \pi_{\alpha + 2} \neq 0 \); we will later show that this is true even in the non-DDR case. Let the representation \( \theta_{-\alpha}(\pi) = \pi_\alpha \) be parametrized by

\[
\psi_\alpha = (\text{lower blocks}) \oplus [A_i, B_i] \oplus \chi_W \otimes S_{1} \otimes S_{a_i+b_i-1} \oplus (\text{higher blocks}).
\]

(We are assuming the order on \( \psi_\alpha \) under which \( \chi_W \otimes S_{1} \otimes S_{a_i+b_i-1} \) is bigger than \( \chi_W \otimes S_{a_i} \otimes S_{b_i} \)). The proof is lengthy, so we split it into three cases:

1. \( \zeta_i = - \), \( t > 0 \)
2. $\zeta_t = -t = 0$

3. $\zeta_t = +$

**Case 1.** For $\beta \in \{\alpha, \alpha + 2\}$ let $\pi_{\beta}$ be the representation parametrized by

$$\psi_{\beta} = (\text{lower blocks}) \oplus [A_i - 1, B_i + 1] \oplus \chi_W \otimes S_i \oplus S_{\beta} \oplus (\text{higher blocks})$$

We have $\pi_{\alpha^2} \hookrightarrow \zeta(-B_i, A_i)$ and $\pi_{\alpha^2} \hookrightarrow \nu^{-\frac{\alpha + 1}{2}} \times \pi_{\alpha}^-$, so

$$\pi_{\alpha^2} \hookrightarrow \zeta(-B_i, A_i) \times \nu^{-\frac{\alpha + 1}{2}} \times \pi_{\alpha}^-.$$

Note that $\zeta(-B_i, A_i) \times \nu^{-\frac{\alpha + 1}{2}}$ is irreducible, so we have

$$\pi_{\alpha^2} \hookrightarrow \nu^{-\frac{\alpha + 1}{2}} \times \zeta(-B_i, A_i) \times \pi_{\alpha}^-.$$

Since $\text{Jac}_{-\frac{\alpha + 1}{2}}(\pi_{\alpha^2}) = \pi_{\alpha}$, the above shows

$$\pi_{\alpha} \hookrightarrow \zeta(-B_i, A_i) \times \pi_{\alpha}^-.$$

Since $\zeta(-B_i, A_i)$ does not contain $-A_i$ (note that we are assuming $t_i > 0$, and thus $A_i > B_i$), we may use Remarks 3.9 and 3.10 to conclude

$$\pi = \theta_\alpha(\pi_{\alpha}) \hookrightarrow \zeta(-B_i, A_i) \times \theta_\alpha(\pi_{\alpha}^-).$$

Note that Remark 3.10 b) applies because $\text{Jac}_{-B_i, \ldots, A_i}(\pi)$ is irreducible. In fact, this Jacquet module is equal to the representation $\pi^-$ whose parameter $\psi^-$ is obtained from $\psi$ by replacing $[A_i, B_i]$ with $[A_i - 1, B_i + 1]$ (with $t_i - 1$). Thus the above embedding shows $\theta_\alpha(\pi^-) = \pi^-$, i.e. $\pi^- = \theta_{-\alpha}(\pi^-)$.

This shows $\theta_{-\alpha}(\pi^-)$ is in the A-A packet. Furthermore, since $\alpha \notin \text{Jord}(\psi^-)$, we may now use Lemma 6.1 to get $\theta_{-\alpha - 2}(\pi^-) = \pi_{\alpha^2}$. By construction,

$$\pi_{\alpha^2} \hookrightarrow \zeta(-B_i, A_i) \times \pi_{\alpha^2},$$

and $\text{Jac}_{-B_i, \ldots, A_i}(\pi_{\alpha^2}) = \pi_{\alpha^2}^{-1}$. This shows we can apply 3.10 to compute $\theta_{\alpha + 2}$. We get

$$\theta_{\alpha + 2}(\pi_{\alpha^2}) \hookrightarrow \zeta(-B_i, A_i) \times \theta_{\alpha + 2}(\pi_{\alpha^2}^-) = \zeta(-B_i, A_i) \times \pi^-.$$

But we know the right-hand side of the above embedding has a unique irreducible subrepresentation, namely $\pi$. This shows $\theta_{\alpha + 2}(\pi_{\alpha^2}) = \pi$, which we needed to prove.

**Case 2.** Here we have $\pi_{\alpha} \hookrightarrow \zeta(-A_i, A_i) \times \pi^-$, where $\pi^-$ is parametrized by

$$\psi^- = \text{(lower blocks)} \oplus [A_i - 1, B_i] \oplus (\text{higher blocks})$$

We begin by showing that $\pi = \theta_{-\alpha}(\pi^-)$. Applying Lemma 3.13 to the above embedding, we get

- $\pi = \theta_{-\alpha}(\pi_{\alpha}) = \theta_{-\alpha}(\pi^-)$ or
\begin{itemize}
    \item $\zeta(-A_i, A_i) \times \Theta_\alpha(\pi^-) \to \theta_\alpha(\pi_\alpha) = \pi$.
\end{itemize}

The second bullet would imply $\text{Jac}_{-A_i,\ldots,A_i}(\pi) \neq 0$. This is clearly impossible if $A_i > B_i$ since $\pi$ is a DDR representation. If $A_i = B_i$ the (by assumption) non-zero representation $\text{Jac}_{-A_i,\ldots,-1}(\pi)$ is, again, of Arthur type and does not have $A_i$ in its Jordan blocks and this contradicts Proposition 2.3 (ii) of [32]. This leaves us with the first bullet, i.e. $\text{Jac}_{-\alpha,\pi}(\theta) / \pi$ packet.

The second bullet would imply $\text{Jac}_{-\pi}(\theta) \neq 0$. Lemma 3.16 shows that the standard module for $\pi$ may apply Kudla's filtration (in the form of Lemma 3.8 and Remark 3.10) to compute $\nu$, we get $\nu^{\pi+} \times \pi \to \theta_{\alpha}(\pi_{\alpha+2})$. This shows $\theta_{\alpha}(\pi_{\alpha+2}) = \pi^+$, that is, $\pi_{\alpha+2} = \theta_{-\alpha}(\pi^+)$. Finally, we show that $\pi_{\alpha+2} = \theta_{-\alpha}(\pi^-)$. We do this by using our results on the explicit description of lifts [7]. Recall that we showed $\pi = \theta_{-\alpha}(\pi^-)$ and $\pi^+ = \theta_{-\alpha}(\pi^-)$. Thus Lemma 3.16 shows that the standard module for $\pi^+$ is obtained by simply adding $\nu^{\pi+}$ (corresponding to the singleton tower segment) into the standard module for $\pi$. Note that $\pi^+$ and $\pi$ have the same going-down tower, and we are currently lifting on the going-down tower for $\pi$. In turn, that means that the standard module for $\pi_{\alpha+2} = \theta_{-\alpha}(\pi^+)$ is obtained by adding $\nu^{\pi+}$ to the standard module of $\pi_{\alpha} = \theta_{-\alpha}(\pi^-)$. But this means precisely $\pi_{\alpha+2}$ is the next lift in the same tower, i.e. $\pi_{\alpha+2} = \theta_{-\alpha}(\pi^-)$, which is what we needed to show.

Case 3. In this case, $\zeta = 1$. For simplicity, suppose that $[A_{i-1}, B_{i-1}]$ and $[A_i, B_i]$ are not juxtaposed (i.e. $B_i > A_{i-1} + 1$). Then, for $\beta \in \{\alpha, \alpha+2\}$ let $\pi_{\beta}^\prec$ be the DDR representation parametrized by

$$\psi_{\beta}^\prec = (\text{lower blocks}) \oplus [A_i - 1, B_i - 1] \oplus \oplus S_1 \otimes S_\beta \oplus (\text{higher blocks}).$$

Then $\pi_\alpha \mapsto \zeta(B_i, A_i) \times \pi_{\beta}^{\prec}$, and we can use Kudla's filtration (Lemma 3.8 and Remarks 3.9 3.10) to compute $\theta_{\alpha}$. We get $\pi \mapsto \zeta(B_i, A_i) \times \theta_{\alpha}(\pi_{\alpha}^\prec)$. This shows that $\theta_{\alpha}(\pi_{\alpha}^\prec)$ is equal to the representation $\pi^<$ whose parameter is obtained from $\psi$ by shifting down the block $[A_i, B_i]$. Thus $\theta_{-\alpha}(\pi^<) = \pi_{\alpha}^\prec$ is in the A-A packet, so we can apply Lemma 6.1 to conclude $\theta_{-\alpha}(\pi^<) = \pi_{\alpha+2}^\prec$. By construction, $\pi_{\alpha+2} \mapsto \zeta(B_i, A_i) \times \pi_{\alpha+2}^{\prec}$. Again, using Kudla's filtration, we conclude

$$\theta_{\alpha+2}(\pi_{\alpha+2}) \mapsto \zeta(B_i, A_i) \times \theta_{\alpha+2}(\pi_{\alpha+2}^\prec) = \zeta(B_i, A_i) \times \pi^<$$

But this last embedding shows that $\theta_{\alpha+2}(\pi_{\alpha+2}) = \pi$, which is what we needed to show.

We still have to explain the non-DDR. Even if $\pi_{\alpha}^\prec$ (or equivalently, $\pi^<$) is not DDR, it may define a non-zero representation, in which case the above proof applies without modification. In fact, suppose that there exists an index $j \in \{1, \ldots, i\}$ such that the representation obtained from $\pi_{\alpha}$ by replacing $[A_j, B_j]$ with $[A_j - 1, B_j - 1]$ is non-zero. Now use $\pi_{\alpha}^\prec$ to denote the representation obtained from $\pi_{\alpha}$ by shifting all the blocks $[A_j, B_j], \ldots, [A_i, B_i]$ down one position. Then the above proof applies again, mutatis mutandis. (The only difference is that we need to write $\zeta(B_j, A_j), \ldots, \zeta(B_i, A_i)$ instead of just $\zeta(B_i, A_i)$.)

If no such index $j$ exists, it follows that the first $i$ blocks of $\pi_{\alpha}$ form a cuspidal sequence. In this case, we may flip the sign $\zeta_i$ from $+$ to $-$ without changing the representation $\pi_{\alpha}$.
(Recall that we are using the Mœglin’s parameterization). With this transformation, we are back in case 2 of this proof, which shows that \( \pi_{\alpha+2} = \theta_{-(\alpha+2)}(\pi) \). Flipping \( \zeta_i \) back to + again, we obtain the desired result.

It remains to show that the DDR assumption in the above lemma is superfluous.

**Lemma 6.5.** Lemma 6.4 is true even without the assumption that \( \psi_{\alpha+2} \) is DDR.

**Proof.** The proof of this lemma is entirely analogous to case (ii) in the proof of Lemma 6.1; we omit the details.

### 6.2 The general case

In this subsection we prove Theorem C for general \( \pi \). Thus, we assume that \( \theta_{-\alpha}(\pi) \) is isomorphic to a representation \( \pi_\alpha \) in the A-A packet. We let \( \pi_{\alpha+2} \in \Pi_{\psi_{\alpha+2}} \) be the representation obtained by applying Epicer to \( \pi_\alpha \); we then wish to show that \( \pi_{\alpha+2} \neq 0 \) and \( \theta_{-\alpha-2}(\pi) = \pi_{\alpha+2} \).

We first relate this with the DDR case. We fix an admissible order on \( \text{Jord}(\psi) \) with the following property: there exists an index \( i \) such that \( j < i \iff B_j \leq \frac{\alpha-1}{2} \). We extend this order to \( \text{Jord}(\psi_{\alpha+2}) \) by inserting \( 1 \otimes S_1 \otimes S_\alpha \) between \( (A_{i-1}, B_{i-1}, \zeta_{i-1}) \) and \( (A_i, B_i, \zeta_i) \).

**Lemma 6.6.** Suppose \( \theta_{-\alpha}(\pi) \) is a representation \( \pi_\alpha \) in the A-A packet. Let \( \pi_\gamma \) and \( \pi_{\alpha,\gamma} \) be DDR representations which dominate \( \pi \) and \( \pi_\alpha \). Assume that \( \pi_{\alpha,\gamma} = \theta_{-\alpha}(\pi_\gamma) \) is in the A-A packet. Then, \( \theta_{-\beta}(\pi) = \pi_\beta \) for all \( \beta \geq \alpha \).

**Proof.** We let \( \pi_{i,\gamma} \) and \( \pi_{\alpha,\gamma,i,\gamma} \) denote the intermediate representations obtained from \( \pi \) and \( \pi_\alpha \) by shifting the segments above \( (A_{i-1}, B_{i-1}, \zeta_{i-1}) \). Then

\[
\pi_\gamma \hookrightarrow L_1 \times \cdots \times L_{i-1} \rtimes \theta_{-\alpha}(\pi_{i,\gamma})
\]

and we may apply Remark 3.11 to obtain

\begin{equation}
\theta_{-\alpha}(\pi_\gamma) \hookrightarrow L_1 \times \cdots \times L_{i-1} \rtimes \theta_{-\alpha}(\pi_{i,\gamma}).
\end{equation}

(two are no obstructions because we take \( \alpha_\gamma \) to be greater than anything appearing in \( L_1, \ldots, L_{i-1} \)). Combining this with the assumption that \( \theta_{-\alpha}(\pi_\gamma) \) is in the A-A packet, we get that \( \theta_{-\alpha}(\pi_{i,\gamma}) = \pi_{\alpha,\gamma,i,\gamma} \) is in the A-A packet as well. Now, from Lemma 3.14 it follows that

\[
0 \neq \pi_{\alpha+2,i,\gamma} = \theta_{-(\alpha+2)}(\pi_{i,\gamma}) \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \pi_{\alpha,i,\gamma},
\]

where \( \pi_{\alpha,i,\gamma} = \theta_{-\alpha}(\pi_{i,\gamma}) \). Now the following Lemma 6.7 (whose proof we postpone to the end of the section) shows that \( \theta_{-(\alpha+2)}(\pi) \) is in the A-A packet.

**Lemma 6.7.** We retain the notation of Lemma 6.6. Then

\[
0 \neq \pi_{\alpha+2,i,\gamma} = \theta_{-(\alpha+2)}(\pi_{i,\gamma}) \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \pi_{\alpha,i,\gamma}
\]

implies

\[
0 \neq \pi_{\alpha+2} = \theta_{-(\alpha+2)}(\pi) \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \pi_\alpha.
\]
Note that \( \Theta \) shows that \( \theta_{-\beta}(\pi_\alpha) \) is in the A-A packet for all \( \beta \geq \alpha \). This enables us to apply the above discussion inductively; it follows that \( \theta_{-\beta}(\pi) \) in the A-A packet for every \( \beta \geq \alpha \).

We will prove Theorem \( \mathbf{C} \) by reducing the proof to the special case described by the following Lemma. We keep the same order on \( \text{Jord}(\psi) \) and \( \text{Jord}(\psi_\alpha) \); in particular, the index \( i \) is still defined as above.

**Lemma 6.8.** Suppose \( \theta_\alpha(\pi) \) is a representation \( \pi_\alpha \) in the A-A packet. Assume additionally that \( A_j \leq \frac{a_j}{2} \), for all \( j \leq 1 \). Then, \( \theta_\beta(\pi) = \pi_\beta \) for all \( \beta \geq \alpha + 2 \).

**Proof.** The proof of Lemma \( \mathbf{6.6} \) shows \( \theta_\alpha(\pi_{i,>}^\gamma) = \pi_{\alpha,i,>} \). By first passing to a corresponding DDR representation, and then taking the appropriate Jacquet modules, it follows that there exist real numbers \( \beta_1, \ldots, \beta_t \) with \( |\beta_j| \leq \frac{a_j}{2} \) for \( j = 1, 2, \ldots, t \), and a DDR representation \( \Pi_\gamma \) such that

\[
\pi_{i,>} \leq \nu^{\beta_1} \times \nu^{\beta_2} \times \cdots \times \nu^{\beta_t} \times \Pi_\gamma;
\]

here the lowest blocks in a parameter of \( \Pi_\gamma \) form a cuspidal sequence, whereas the remaining \( r - i + 1 \) blocks are the same as in \( \pi_{i,>} \). By a result of Mœglin \( \mathbf{17} \), we can change the order and the signs of the \( \beta_j \)'s to obtain an embedding; abusing the notation we again write

\[
\pi_{i,>} \hookrightarrow \nu^{\beta_1} \times \nu^{\beta_2} \times \cdots \times \nu^{\beta_t} \times \Pi_\gamma.
\]

Applying \( \Theta_\alpha \) to the embedding just above, Lemma \( \mathbf{3.8} \) (together with Frobenius reciprocity) shows that the irreducible representation

\[
(7) \quad \nu^{\beta_1} \otimes \nu^{\beta_2} \otimes \cdots \otimes \nu^{\beta_t} \otimes \theta_\alpha(\Pi_\gamma)
\]

is a subquotient in the appropriate Jacquet module of \( \theta_\alpha(\pi_{i,\\gamma}) \). Now we apply the same procedure directly to \( \theta_\alpha(\pi_{i,>}) = \pi_{\alpha,i,>} \): We get \( \gamma_1, \ldots, \gamma_k \) such that

\[
(8) \quad \theta_\alpha(\pi_{i,>}) \hookrightarrow \nu^{\gamma_1} \times \nu^{\gamma_2} \times \cdots \times \nu^{\gamma_k} \times \Pi_{\alpha,>}^t,
\]

where \( \Pi_{\alpha,>}^t \) is a DDR representation whose lowest blocks form a cuspidal sequence, while the remaining \( r - i + 1 \) blocks are the same as in \( \pi_{\alpha,i,>} \).

Note that \( \text{Jac}_\beta(\Pi_\gamma) = 0 \), for each \( \beta \in \{\pm 0, \pm 1, \ldots, \frac{a_j}{2} \} \). Now Lemma \( \mathbf{3.8} \) gives that the same is true for \( \theta_\alpha(\Pi_\gamma) \), except possibly for \( \beta = \frac{\alpha_j}{2} \). In this case, we get an embedding \( \theta_\alpha(\Pi_\gamma) \hookrightarrow \nu^{\frac{\alpha_j}{2}} \times \theta_{-\beta}(\Pi_\gamma) \). We continue in the same vein and get an embedding

\[
\theta_\alpha(\Pi_\gamma) \hookrightarrow \nu^{\frac{a_j}{2}} \times \nu^{\frac{a_j}{2}} \times \cdots \times \nu^{\frac{a_j}{2}} \times \theta_{-\beta}(\Pi_\gamma),
\]

for some \( \delta - 2 \leq \alpha \) and \( \text{Jac}_{\beta}(\theta_{-\beta}(\Pi_\gamma)) = 0 \). Thus, from \( \mathbf{7} \), we get that

\[
(9) \quad \nu^{\beta_1} \otimes \nu^{\beta_2} \otimes \cdots \otimes \nu^{\beta_t} \otimes \nu^{\frac{a_j}{2}} \otimes \cdots \otimes \nu^{\frac{a_j}{2}} \otimes \theta_{-\beta}(\Pi_\gamma)
\]

belongs to the appropriate Jacquet module of \( \theta_\alpha(\pi_{i,>}) \), and — by \( \mathbf{8} \) — of \( \nu^{\gamma_1} \times \nu^{\gamma_2} \times \cdots \times \nu^{\gamma_k} \times \Pi_{\alpha,>}^t \) as well. Using the conditions on \( \beta, \gamma_j \) and the fact that \( \text{Jac}_\beta(\Pi_{\alpha,>}^t) = 0 \) for each \( \beta \in \{\pm 0, \pm 1, \ldots, \frac{a_j}{2} \} \), we get that \( \theta_{-\beta}(\Pi_\gamma) = \Pi_{\alpha,>}^t \). This means that \( \theta_{-\beta}(\Pi_\gamma) \) is
in the A-A packet, and so are \( \theta_{-\alpha}(\Pi_\geq) \) and \( \theta_{-(\alpha+2)}(\Pi_\geq) \), by Theorem 3 in the DDR case. Furthermore,

\[
\theta_{-(\alpha+2)}(\Pi_\geq) \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \theta_{-\alpha}(\Pi_\geq).
\]

By Lemma 3.16, the Langlands datum of \( \theta_{-(\alpha+2)}(\Pi_\geq) \) is obtained by adding \( \nu^{-\frac{\alpha+1}{2}} \) to the Langlands datum of \( \theta_{-\alpha}(\Pi_\geq) \). This means, using the embedding above and Lemma 3.17, that \( A^0_{\nu^{-\frac{\alpha+1}{2}}} (\Pi_\geq) = A_{\nu^{-\frac{\alpha+1}{2}}} (\Pi_\geq) \) since the same is true for the representation \( \theta_{-\alpha}(\Pi_\geq) \). (the representations \( \Pi_\geq \) and \( \theta_{-\alpha}(\Pi_\geq) \) share the same sets \( A_{\nu^{-\frac{\alpha+1}{2}}} \) and \( A_{\nu^{-\frac{\alpha+1}{2}}} \)). Since \( |\beta_j| \leq \frac{\alpha-1}{2} \) for all \( j \), this implies (see (6))

\[
A^0_{\nu^{-\frac{\alpha+1}{2}}} (\pi_{i,\geq}) = A_{\nu^{-\frac{\alpha+1}{2}}} (\pi_{i,\geq}).
\]

Using Lemma 3.17 again, this leads to

\[
\theta_{-(\alpha+2)} (\pi_{i,\geq}) \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \theta_{-\alpha} (\pi_{i,\geq}).
\]

By Lemma 6.7, this implies

\[
0 \neq \pi_{\alpha+2} = \theta_{-(\alpha+2)} (\pi) \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \theta_{-\alpha} (\pi),
\]

which we needed to show.

Moreover, the fact that \( \nu^{-\frac{\alpha+1}{2}} \times \Pi_\geq \) is irreducible implies the irreducibility of \( \nu^{-\frac{\alpha+1}{2}} \times \pi_{i,\geq} \). It follows that \( \theta_{-(\alpha+4)} (\pi_{i,\geq}) \hookrightarrow \nu^{-\frac{\alpha+3}{2}} \times \theta_{-(\alpha+2)} (\pi_{i,\geq}) \), which shows \( \theta_{-(\alpha+4)} (\pi_{i,\geq}) \) is in the A-A packet. Repeatedly applying this argument, we get that \( \theta_{-\alpha} (\pi_{i,\geq}) \) is in the A-A packet. Then Remark 3.11 shows that \( \theta_{-\alpha} (\pi_{i,\geq}) \) is in the A-A packet. Now Lemma 6.6 implies that \( \theta_{-\beta} (\pi) = \pi_\beta \) for all \( \beta \geq \alpha \).

Now we are ready to prove the general case.

**Proof of Theorem 3.** We begin by adding another requirement to the order on \( \text{Jord}(\psi) \) we fixed at the beginning of the section: in addition to \( i \) (such that \( j < i \iff B_j \leq \frac{\alpha-1}{2} \)), we require that there exist an index \( t \) such that \( j \leq t \iff A_j < \frac{\alpha+1}{2} \). Note that \( t + 1 \leq i \). We extend this order to \( \text{Jord}(\psi_\alpha) \) by inserting \( 1 \otimes S_1 \otimes S_\alpha \) between \( (A_t, B_t, \zeta_t) \) and \( (A_{t+1}, B_{t+1}, \zeta_{t+1}) \).

Let \( \pi_{t+1,\geq} \) (resp. \( \pi_{\alpha,t+1,\geq} \)) be the expanded version of \( \pi \) (resp. \( \pi_\alpha \)) obtained by shifting the blocks above \( (A_t, B_t, \zeta_t) \) we define \( \pi_{i,\geq} \) (resp. \( \pi_{\alpha,i,\geq} \)) analogously. In Lemma 3.10 we showed that \( \pi_{\alpha,i,\geq} = \theta_{-\alpha} (\pi_{\alpha,i,\geq}) \) is in the A-A packet (this follows from 3.11). We now claim that \( \pi_{\alpha,t+1,\geq} = \theta_{-\alpha} (\pi_{\alpha,t+1,\geq}) \); this will imply the desired result.

To simplify the notation, we assume that \( t + 1 = i - 1 \); if there are more blocks between \( (A_t, B_t, \zeta_t) \) and \( (A_{t+1}, B_{t+1}, \zeta_{t+1}) \), the same argument can be repeated inductively. Now suppose that \( \zeta_{t+1} = \cdots = \zeta_{i-1} = -1 \). Then the claim follows by applying Remark 3.11. Indeed, since \( \zeta = -1 \), the multisegment that appears here does not contain \( \frac{\alpha+1}{2} \), so there are no obstructions when computing \( \Theta_{-\alpha} \); when computing \( \Theta_{\alpha} \), we use Lemma 3.12. This shows that we may assume \( \zeta_{t+1} = \cdots = \zeta_{i-1} = + \). In that case, we may also assume that \( t_{i-1} = 0 \); this follow by an application of Lemma 3.11.

With all these reductions in mind, we now proceed to prove that \( \pi_{\alpha,t+1,\geq} = \theta_{-\alpha} (\pi_{t+1,\geq}) \) assuming \( t + 1 = i - 1 \), \( \zeta_{i-1} = + \), and \( t_{i-1} = 0 \). Setting \( T = T_{t+1} \), we then have

\[
\zeta ((A_{t+1} + T), -(B_{t+1} + T) \times \cdots \times \zeta ((A_{t+1} + 1), -(B_{t+1} + 1) \times \pi_{i,\geq}) \rightarrow \pi_{t+1,\geq}.
\]

(10)
We now use Lemma 3.13. If option (i) from the Lemma applies to each of the segments, \( \Theta_{-a} \) then we may apply \( \Theta_a \) like in Remark 3.11 to get
\[
\zeta(-(A_{t+1} + T), -(B_{t+1} + T)) \times \cdots \times \zeta(-(A_{t+1} + 1), -(B_{t+1} + 1)) \times \theta_{-a}(\pi_{t, >})
\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.

We show that the other option is not possible. Assume the contrary: when computing \( \Theta_{-a} \), option (ii) of Lemma 3.13 applies to one of the segments in (10). In that case, only that segment is affected, and the rest of the segments remain unchanged. Thus, the part \( \Theta_{-a} \) as well, and the claim is proved.

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\[
\zeta(-(A_{t+1} + T), -(B_{t+1} + T)) \times \cdots \times \zeta(-(A_{t+1} + 1), -(B_{t+1} + 1)) \times \theta_{-a}(\pi_{t, >})
\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.

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\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.

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\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.

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\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.

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\[
\zeta(-(A_{t+1} + T), -(B_{t+1} + T)) \times \cdots \times \zeta(-(A_{t+1} + 1), -(B_{t+1} + 1)) \times \theta_{-a}(\pi_{t, >})
\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.

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\[
\zeta(-(A_{t+1} + T), -(B_{t+1} + T)) \times \cdots \times \zeta(-(A_{t+1} + 1), -(B_{t+1} + 1)) \times \theta_{-a}(\pi_{t, >})
\rightarrow \theta_{-a}(\pi_{t+1, >}).
\]
Since we know \( \theta_{-a}(\pi_{t, >}) \) is in the A-A packet, this shows \( \theta_{-a}(\pi_{t+1, >}) \) is in the A-A packet as well, and the claim is proved.
as
\[
\pi_{t+1,>} \leftrightarrow \begin{bmatrix}
B_{t+1} + T & \ldots & \frac{\alpha}{2} + T \\
\vdots & \ddots & \vdots \\
B_{t+1} + 1 & \ldots & \frac{\alpha}{2} + 1 + 1
\end{bmatrix} \times \begin{bmatrix}
\frac{\alpha}{2} + T & \ldots & A_{t+1} + T \\
\vdots & \ddots & \vdots \\
\frac{\alpha}{2} + 3 & \ldots & A_{t+1} + 1
\end{bmatrix} \times \pi_{i,>}
\]

This shows that \( \pi' = \text{Jac}_{L_{t+1}'}(\pi_{t+1,>}) \neq 0 \), and moreover, \( \pi' \leftrightarrow L_{t+1}^2 \times \pi_{i,>} \). This means

\[
(12) \quad \delta[-(\frac{\alpha}{2} + 1 + T), \frac{\alpha}{2} + 3] \times \cdots \times \delta[-(A_{t+1} + T), -(A_{t+1} + 1)] \times \pi_{i,>} \rightarrow \pi';
\]

the point is that we can apply Remark 3.11 to \((12)\) to compute both \(\Theta_{-(2\alpha+3)}\) and \(\Theta_{-\alpha}\). Computing \(\Theta_{-\alpha}\), we get \(\theta_{-\alpha}(\pi') \leftrightarrow L_{t+1}^2 \times \theta_{-\alpha}(\pi_{i,>})\), which shows that \(\theta_{-\alpha}(\pi')\) is in the A-A packet. Now we can apply Lemma 6.8 to \(\pi'\) to conclude that \(\theta_{-(2\alpha+3)}(\pi') = \pi'_{A+3}\) is also in the A-A packet. Finally, applying \(\theta_{-(2\alpha+3)}\) to \((12)\) we get \(\theta_{-(2\alpha+3)}(\pi') \leftrightarrow L_{t+1}^2 \times \theta_{-(2\alpha+3)}(\pi_{i,>})\), which shows that \(\theta_{-(2\alpha+3)}(\pi_{i,>})\) is in the A-A packet. We now use this to show the same about \(\theta_{-(\alpha+2)}(\pi_{i,>})\).

We now have
\[
\theta_{-(2\alpha+3)}(\pi') \leftrightarrow \zeta(-C + 1, -\frac{\alpha+1}{2}) \times \theta_{-\alpha}(\pi')
\]
as well as \(\theta_{-(2\alpha+3)}(\pi') \leftrightarrow L_{t+1}^2 \times \theta_{-(2\alpha+3)}(\pi_{i,>})\). We thus get

\[
(13) \quad \zeta(-(A + 1), -\frac{\alpha+1}{2}) \otimes \theta_{-\alpha}(\pi') \leq \mu^*(L_{t+1}^2 \times \theta_{-(2\alpha+3)}(\pi_{i,>}))
\]

One can analyze the right-hand side of \((13)\) using Tadić’s formula \((11)\) and the formula of Kret and Lapid for Jacquet modules of ladder representations \((12)\). It follows that \(\mu^*(\theta_{-(2\alpha+3)}(\pi_{i,>}))\) necessarily contains \(\nu \otimes \xi\), where \(\xi\) is irreducible and \(\gamma\) is either \((-C+1)\) or an element of \(\{-\frac{\alpha+1}{2}, \ldots, -A_{t+1}\}\). This is the \(\gamma\) which was alluded to in \((11)\): we thus get \(\text{Jac}_\gamma(\theta_{-\alpha}(\pi_{t+1,>})) \neq 0\), and consequently \(\text{Jac}_\gamma(\pi_{t+1,>}) \neq 0\). This is impossible by Lemma 3.6.

We have thus arrived at a contradiction with the assumption that option (ii) of Lemma 3.6 applies to one of the segments in \((10)\). This proves that \(\theta_{-\alpha}(\pi_{t+1,>})\) is in the A-A packet. We can apply Lemma 6.8 to \(\theta_{-\alpha}(\pi_{t+1,>})\) to show that \(\theta_{-\beta}(\pi_{t+1,>})\) is in the A-A packet for every \(\beta \geq \alpha\). Using this for \(\beta\) which satisfies \(2(A_{t+1}+T_{t+1})+1 < \beta < 2(A_{t+1}+T_{t+1})+1\) we get
\[
\pi_{\beta,t+1,>} = \theta_{-\beta}(\pi_{t+1,>}) \leftrightarrow L_{t+1} \times \cdots \times L_{i-1} \times \pi_{\beta,t,>}
\]
Applying Remark 3.11 to compute \(\theta_{\beta}\), we get that \(\pi_{\beta,t,>} = \theta_{-\beta}(\pi_{t,>})\) is in the A-A packet. As in \((13)\), we get that \(\pi_{\beta,>} = \theta_{-\beta}(\pi_{>})\) is in the A-A packet and then Lemma 6.8 gives the required conclusion.

Only one thing remains:

\textbf{Proof of Lemma 6.7.} We order the last \(r - i + 1\) blocks in \(\text{Jord}(\psi)\) in such a way that \(\xi_1 = \cdots = \xi_k = 1\), and \(\xi_{k+1} = \cdots = \xi_{r} = -1\). We define \(\pi_{k+1,>}\) (resp. \(\pi_{\alpha+2,k+1,>}\)) by shifting up the blocks above \((A_k, B_k, \xi_k)\). Thus, we have

\[
(14) \quad \theta_{-(\alpha+2)}(\pi_{i,>}) = \pi_{\alpha+2,i,>} \leftrightarrow L_{i} \times \cdots \times L_{k} \times \pi_{\alpha+2,k+1,>}
\]
provided $\pi_{\alpha+2,k+1,>}$ is non-zero. But we know it is; indeed:

$$\pi_{\alpha+2,i,>} \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times \pi_{\alpha,i,>} \hookrightarrow \nu^{-\frac{\alpha+1}{2}} \times L_i \times \cdots \times L_k \times L_{k+1} \cdots \times L_r \times \pi_{\alpha,>}$$

$$\cong L_i \times \cdots \times L_k \times \nu^{-\frac{\alpha+1}{2}} \times L_{k+1} \cdots \times L_r \times \pi_{\alpha,>}$$

shows that $\text{Jac}_{L_i,\ldots,L_k}(\pi_{\alpha+2,i,>}) \neq 0$. Since the multisegments $L_i, \ldots, L_k$ are all positive, and since $\text{Jac}_{L_i,\ldots,L_k}(\pi_{i,>})$ is irreducible (if non-zero), we can apply Remark 3.11 to get $\pi_{\alpha+2,k+1,>} = \theta_{-(\alpha+2)}(\pi_{k+1,>})$.

We deal with negative multisegments in a similar way. We have

$$\pi_{k+1,>} \hookrightarrow L_{k+1} \times \cdots \times L_r \times \pi.$$ 

Here the multisegments are negative, and we know $\text{Jac}_{L_{k+1},\ldots,L_r}(\pi_{\alpha+2,k+1,>})$ is irreducible (if non-zero), so we can apply Remark 3.11 to compute $\theta_{-(\alpha+2)}$. We get

$$0 \neq \pi_{\alpha+2} = \text{Jac}_{L_{k+1},\ldots,L_r}(\theta_{-(\alpha+2)}(\pi_{k+1,>})) = \theta_{-(\alpha+2)}(\pi),$$

which we needed to prove.

\[ \square \]

7 Examples

We demonstrate the results on a few examples. Note that the Arthur classification is still conjectural in the non-quasi-split case (so in particular, for orthogonal groups of trivial discriminant and Hasse invariant equal to $-1$); however, to simplify our examples, we momentarily ignore this fact and work with $\chi_V = \chi_W = 1$.

Example 7.1. We let $\pi$ be the Steinberg representation of $\text{Sp}_6$. Then $\pi$ is given by

$$\psi = 1 \otimes \tilde{S}_7 \otimes S_1$$

For $\alpha > 7$ the lift $\theta_{-\alpha}(\pi)$ is given by

$$\psi_{\alpha} = 1 \otimes \tilde{S}_7 \otimes S_1 \oplus 1 \otimes S_1^\epsilon \otimes S_\alpha$$

The choice of $\epsilon \in \{\pm 1\}$ corresponds to the choice of target tower of orthogonal groups. In this case, $d(\pi, \psi) = 1$ on both towers; in particular, $\theta_{-1}(\pi) \neq 0$ on both towers. The lifts for $\alpha \in \{1, 3, 5, 7\}$ are given by

$$\psi_{\alpha} = 1 \otimes \tilde{S}_1 \otimes S_\alpha \oplus 1 \otimes S_7^\epsilon \otimes S_1.$$ 

Here (and everywhere else) the blocks are written in the order that corresponds to the order on $\text{Jord}(\psi_{\alpha})$.

Example 7.2. We let $\pi$ be the trivial representation of $\text{Sp}_6$. Then $\pi$ is given by

$$\psi = 1 \otimes \tilde{S}_1 \otimes S_7$$
For $\alpha > 7$ the lift $\theta_{-\alpha}(\pi)$ is given by

$$\psi_\alpha = 1 \otimes \tilde{S}_1 \otimes S_7 \oplus 1 \otimes S^\epsilon \otimes S_\alpha$$

Again, the choice of $\epsilon \in \{\pm 1\}$ corresponds to the choice of target tower of orthogonal groups.

If $\epsilon = 1$, then $\pi_7 = 0$. This shows that $d(\pi, \psi) = 9$ on the corresponding tower. In other words, this is the going-up tower, and $\theta_{-9}(\pi)$ is the first occurrence of $\pi$.

If $\epsilon = -1$, then $d(\pi, \psi) = 1$. The lifts for $\alpha \in \{1, 3, 5, 7\}$ on this tower are given by

$$\psi_\alpha = 1 \otimes \tilde{S}_1 \otimes S_\alpha \oplus 1 \otimes S_1 \otimes S_7.$$  

**Example 7.3.** Let $\pi$ be the representation of $\text{Sp}_{10}$ given by the Arthur parameter

$$\psi = 1 \otimes \tilde{S}_1 \otimes S_1 \oplus 1 \otimes \tilde{S}_3 \otimes S_1 \oplus 1 \otimes S_1 \otimes S_7$$

Then $\pi$ is the Langlands quotient of $\nu^3 \times \sigma$, where $\sigma$ is the supercuspidal representation of $\text{Sp}_8$ parametrized by

$$1 \otimes \tilde{S}_1 \otimes S_1 \oplus 1 \otimes \tilde{S}_3 \otimes S_1 \oplus 1 \otimes S_5 \otimes S_1$$

For $\alpha > 7$ the lift $\theta_{-\alpha}(\pi)$ is given by

$$\psi_\alpha = 1 \otimes S^\epsilon_1 \otimes S_\alpha \oplus 1 \otimes \tilde{S}_1 \otimes S_7 \oplus 1 \otimes S^\epsilon \otimes S_\alpha$$

If $\epsilon = -1$, then $\pi_7 = 0$. Indeed, $\pi_9 = \theta_{-9}(\pi)$ is the Langlands quotient of $\delta(3, 4) \times \theta_{-7}(\sigma)$; note that $\theta_{-7}(\sigma)$ is supercuspidal because it is the first lift of a supercuspidal representation $\sigma$. Thus $\pi_7 = \text{Jac}_{-4}(\pi_9) = 0$. In other words, $\epsilon = -1$ corresponds to the going-up tower, and $\theta_{-9}(\pi)$ is the first occurrence on this tower.

If $\epsilon = 1$, then $\pi_7$ and $\pi_5$ are non-zero, given by

$$\psi_\alpha = 1 \otimes \tilde{S}_1 \otimes S_1 \oplus 1 \otimes \tilde{S}_3 \otimes S_1 \oplus 1 \otimes S_1 \otimes S_7 \oplus 1 \otimes S_1 \otimes S_7$$

for $\alpha \in \{5, 7\}$. However, we have $\pi_3 = 0$. Indeed, $\pi_5 = \theta_{-5}(\pi)$ on the going-down tower. This means that $\pi_3$ is the Langlands quotient of

$$\nu^3 \times \nu^2 \times \nu^1 \times \theta_{-1}(\sigma),$$

where $\theta_{-1}(\sigma)$ is tempered. Thus $\pi_3 = \text{Jac}_{-2}(\pi_5) = 0$, even though $\theta_{-3}(\pi) \neq 0$.

**Example 7.4.** Let $V_{10}$ be the 10-dimensional quadratic space of discriminant 1 and Hasse invariant $-1$. Let $\pi$ be the representation of $\text{O}(V_{10})$ parametrized by

$$\psi = 1 \otimes \tilde{S}_1 \otimes S_3 \oplus 1 \otimes S_1 \otimes S_7$$

Then $\pi$ is the Langlands quotient of $\nu^3 \times \delta(1, 2) \times \sigma$, where $\sigma$ is the supercuspidal representation of $\text{O}(V_4)$ parametrized by

$$1 \otimes \tilde{S}_1 \otimes S_1 \oplus 1 \otimes \tilde{S}_3 \otimes S_1.$$
Note that $\pi \otimes \text{det}$ is parametrized by

$$
\psi = 1 \otimes S_1^+ \otimes S_3 + 1 \otimes \tilde{S}_1 \otimes S_7.
$$

We consider the lifts of both $\pi$ and $\text{det} \otimes \pi$ to the symplectic tower; we abuse the terminology in the standard way and refer to $\theta(\pi)$ and $\theta(\pi \otimes \text{det})$ as the lifts of $\pi$ to two different towers.

For $\alpha > 7$ the lifts $\theta_{-\alpha}$ are given by

$$
\psi_\alpha = 1 \otimes S_1^{\eta} \otimes S_3 + 1 \otimes \tilde{S}_1^{\eta} \otimes S_7 + 1 \otimes \tilde{S}_1 \otimes S_\alpha
$$

where $\eta = 1$ (resp. $\eta = -1$) corresponds to the lift of $\pi$ (resp. $\pi \otimes \text{det}$).

If $\eta = -1$, then $\pi_7 = 0$. Indeed, $\pi_9 = \theta_{-9}(\pi \otimes \text{det})$ is the Langlands quotient of $\delta(3,4) \times \delta(1,3) \times \theta_{-5}(\sigma \otimes \text{det})$; note that $\theta_{-5}(\sigma \otimes \text{det})$ is supercuspidal because it is the first lift of a supercuspidal representation $\sigma \otimes \text{det}$. Thus $\pi_7 = \text{Jac}_{-4}(\pi_9) = 0$. In other words, $\pi \otimes \text{det}$ is the going-up tower, and $\theta_{-9}(\pi \otimes \text{det})$ is its first occurrence.

If $\eta = 1$, then $\pi_7$ and $\pi_5$ are non-zero, given by

$$
\psi_\alpha = 1 \otimes S_1^{\eta} \otimes S_3 + 1 \otimes \tilde{S}_1^{\eta} \otimes S_7 + 1 \otimes \tilde{S}_1 \otimes S_7
$$

for $\alpha \in \{5,7\}$. However, we have $\pi_3 = 0$. Indeed, $\pi_5 = \theta_{-5}(\pi)$ is the Langlands quotient of

$$
\nu^3 \times \nu^2 \times \delta(1,2) \times \nu^1 \times \theta_{-1}(\sigma),
$$

where $\theta_{-1}(\sigma)$ is tempered. Thus $\pi_3 = \text{Jac}_{-2}(\pi_5) = 0$, even though $\theta_{-3}(\pi) \neq 0$.

It is interesting to consider theorem C through the lens of this example. Theorem C states that $\pi_\alpha$ cannot belong to the expected Arthur packet if $\alpha < d(\pi,\psi)$ (in this case, when $\alpha < 5$). What would happen if that were the case? Suppose, for example, that $\theta_{-3}(\pi)$ is in the expected Arthur packet, parametrized by

$$
\psi_3 = 1 \otimes S_1 \otimes S_3 + 1 \otimes S_1 \otimes S_3 + 1 \otimes S_1 \otimes S_7.
$$

Since $1 \otimes S_1 \otimes S_7$ is the highest block in this parameter, Theorem A implies that (regardless of the signs) $\theta_{-3}(\pi)$ is isomorphic to $\theta_{-7}(\pi')$ for some representation $\pi'$ of an orthogonal group parametrized by

$$
\psi_3 = 1 \otimes S_1 \otimes S_3 + 1 \otimes S_1 \otimes S_3.
$$

In particular, $\theta_7(\theta_{-3}(\pi)) \neq 0$ on one of the towers. However, using the results of [7], it is easy to see that this is impossible. Indeed, $\theta_{-3}(\pi)$ is the Langlands quotient of

$$
\nu^3 \times \delta(1,2) \times \nu^1 \times \theta_{-1}(\sigma).
$$

Now [7] shows that $\theta_7(\theta_{-3}(\pi)) \neq 0$ is possible only if $\theta_3(\theta_{-1}(\sigma)) \neq 0$, but this is not true (by the results of [9]). Thus we arrive at a contradiction.

In short, the failure of the Adams conjecture for $\alpha < d(\pi,\psi)$ can be interpreted using the vanishing of certain theta lifts. In fact, it is possible to prove Theorem C for elementary parameters simply by using the arguments sketched above. Of course, this interpretation becomes more complicated with non-elementary parameters, but it still gives some intuition as to why Theorem C is true.
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