Dynamics of entanglement and Bell non-locality for two stochastic qubits with dipole–dipole interaction

Ferdi Altintas and Resul Eryigit

Department of Physics, Abant Izzet Baysal University, Bolu, 14280, Turkey

E-mail: resul@ibu.edu.tr

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Abstract
We have studied the analytical dynamics of Bell non-locality as measured by the CHSH inequality and entanglement as measured by concurrence for two noisy qubits that have the dipole–dipole interaction. The non-local entanglement created by the dipole–dipole interaction is found to be protected from sudden death for certain initial states.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is a kind of quantum correlation and refers to separability of the states. The entangled states are the main resource for quantum information and computation applications, such as quantum teleportation [1], superdense coding [2] and quantum cryptography [3]. Entanglement and other types of quantum correlations are not easy to maintain when the system is in contact with an environment which leads to decoherence [4]. Thus, the study of entangled states under decoherence is one the most important aspects in theoretical as well as experimental areas [5, 6]. It was shown that the interaction between the single qubit and its environment leads to exponential decay of the qubit coherences while entanglement between two such qubits can cease to exist in a finite time [7, 8], a phenomenon which was named ‘entanglement sudden death’ (ESD) by Yu and Eberly [9]. Later Lopez et al revealed that when the bipartite entanglement suddenly disappears, the entanglement of the corresponding reservoir suddenly and necessarily appears, which is a counter phenomenon to ESD and termed ‘entanglement sudden birth’ (ESB) [10]. For bipartite systems, many theoretical efforts have been devoted to ESD [9, 11–16] as well as ESB [10, 17, 18], and also ESD is verified experimentally [6]. However, they need further clarifications for a deeper understanding.

On the other hand, it was demonstrated by Werner that there are some entangled mixed states whose correlations can be reproduced by a classical local model [19]. The only way
of identification of such states is to use violations of Bell inequalities. The violations of Bell inequalities identify the genuine multipartite entanglement which is useful for quantum computations [20, 21]. For a bipartite system, there are many Bell-type inequalities such as quadratic- and CHSH–Bell inequalities [22, 23]. The CHSH–Bell inequality is a good indicator of non-local correlations and the relation with the entanglement has already been known [24–31]. Among them Bellomo et al found that the Bell inequality might not be violated for a state with high entanglement for a two-qubit system subject to amplitude damping [24]. Kofman et al showed that the survival time for entanglement should be much longer than the Bell inequality violation under dephasing and energy relaxation-type channels at an arbitrary temperature [27]. Recently, Li et al studied the violations of the Bell inequality under classical Ornstein–Uhlenbeck noise and showed that the strong non-Markovian effect can protect the non-local entanglement identified by the violation of the CHSH inequality [26].

In this paper, we have analyzed the dynamics of Bell non-locality as measured by the CHSH inequality and entanglement as measured by concurrence for a system of two coupled qubits that interact via the dipole–dipole interaction. The effect of the environment on the qubit is modeled as a stochastic energy level with the Ornstein–Uhlenbeck-type correlation [11, 12]. The effects of non-Markovianity of the dynamics, purity of the initial states as well as the dipole–dipole interaction strength on Bell non-locality and entanglement have been investigated for a system initially prepared in extended Werner-like states.

The organization of this paper is as follows. In section 2, we introduce the model and its solution by solving the master equation for the two-qubit reduced density matrix. In section 3, we briefly discuss the Wootters concurrence as well as the CHSH–Bell inequality and the effect of purity, dipole–dipole interaction strength and the non-Markovianity on entanglement and Bell non-locality is studied for extended Werner-like initial states. In section 4, we conclude as a summary important results.

2. The model and its solution

In this paper, we consider two interacting qubits that are coupled to their independent environments which lead to fluctuating energy levels. The qubit–qubit interaction is assumed to be of the type Heisenberg XX model. This model can be thought of as the Kubo–Anderson model extended to two coupled qubits [32]. The typical Hamiltonian for this model can be given as [11, 12, 33] (we set ħ = 1)

\[ H_{\text{tot}}(t) = J \left( \hat{\sigma}_A^X \hat{\sigma}_B^X + \hat{\sigma}_A^Y \hat{\sigma}_B^Y \right) + \frac{\Omega_A(t)}{2} \hat{\sigma}_z^A + \frac{\Omega_B(t)}{2} \hat{\sigma}_z^B, \]

(1)

where \( J \) is the qubit–qubit interaction strength, \( \hat{\sigma}_i^A, \hat{\sigma}_i^B \) (\( i = x, y, z \)) are the usual Pauli spin operators and \( \Omega_A, \Omega_B(t) \) are the independent frequency fluctuations of the qubits with mean value properties that obey the non-Markovian approximation

\[ M[\Omega_i(t)] = 0, \]

(2)

\[ M[\Omega_i(t)\Omega_j(s)] = \alpha(t - s) = \frac{\Gamma_i \gamma}{2} e^{-\gamma|t - s|}, \quad i, j = A, B, \]

(3)

where \( M[\cdots] \) stands for the statistical mean over the noise \( \Omega_A(t) \) and \( \Omega_B(t) \). Here \( \Gamma_i \) (\( i = A, B \)) are the damping rates due to the coupling to the environments, \( \gamma \) is the noise bandwidth which determines the environment’s finite correlation time \( (\tau_c = \gamma^{-1}) \) and \( \alpha(t - s) \) is the reservoir correlation function. For simplicity, we will take the noise properties
to be the same for $A$ and $B$ (e.g. $\Gamma_A = \Gamma_B \equiv \Gamma$). Note that in the limit $\gamma \to \infty (\tau_c \to 0)$, Ornstein–Uhlenbeck noise reduces to the well-known Markovian case [13]:

$$\alpha(t - s) = \Gamma \delta(t - s).$$

(4)

For the total system described by the Hamiltonian (1), the stochastic Schrödinger equation is given by

$$i \frac{d}{dt} |\Psi(t)\rangle = \hat{H}_{tot}(t)|\Psi(t)\rangle,$$

(5)

with formal solution

$$|\Psi(t)\rangle = \hat{U}(t, \Omega_A, \Omega_B)|\Psi(0)\rangle,$$

(6)

where the stochastic propagator $\hat{U}(t, \Omega_A, \Omega_B)$ is given by

$$\hat{U}(t, \Omega_A, \Omega_B) = e^{-i \int_0^t \hat{H}_{tot}(s) ds}.$$  

(7)

The reduced density matrix for qubits $A$ and $B$ is then obtained from the statistical mean

$$\hat{\rho} = M\{ |\Psi(t)\rangle \langle \Psi(t)| \}.$$  

(8)

With the help of the raising and lowering operators, $\hat{\sigma}_\pm^{A,B} = (\sigma_x^{A,B} \pm i \sigma_y^{A,B})/2$, and the stochastic Schrödinger equation (5), the master equation for the reduced density matrix can be derived as [11, 34–36]

$$\frac{d}{dt} \hat{\rho} = -i[\hat{H}, \hat{\rho}] - \frac{G(t)}{2} (2\hat{\rho} - \hat{\sigma}_z^{A} \hat{\rho} \hat{\sigma}_z^{A} - \hat{\sigma}_z^{B} \hat{\rho} \hat{\sigma}_z^{B}),$$

(9)

where $G(t) = \int_0^t \alpha(t - s) ds = \frac{\gamma}{2} (1 - e^{-\gamma t})$ is a time-dependent coefficient which includes the memory information of the environmental noise and $\hat{H} = 2J (\hat{\sigma}_z^{A} \hat{\sigma}_z^{B} + \hat{\sigma}_z^{A} \hat{\sigma}_z^{B})$ is the interaction Hamiltonian which represents the dipole–dipole interaction between the qubits $A$ and $B$ [37]. Note that the first term on the right-hand side of equation (9) leads to oscillatory dynamics while the second term causes decay.

The differential equations governing the time evolution of the system in the standard basis \{|1\} \equiv |11\rangle, |2\} \equiv |10\rangle, |3\} \equiv |01\rangle, |4\} \equiv |00\rangle\} can be easily calculated:

$$\dot{\rho}_{ii} = 0 \quad (i = 1, 4),$$

$$\dot{\rho}_{22} = 2iJ(\rho_{23} - \rho_{23}^*),$$

$$\dot{\rho}_{33} = -2iJ(\rho_{23} - \rho_{23}^*),$$

$$\dot{\rho}_{12} = 2iJ\rho_{13} - G(t)\rho_{12},$$

$$\dot{\rho}_{13} = 2iJ\rho_{12} - G(t)\rho_{13},$$

$$\dot{\rho}_{14} = -2G(t)\rho_{14},$$

$$\dot{\rho}_{23} = 2iJ(\rho_{22} - \rho_{33}) - 2G(t)\rho_{23},$$

$$\dot{\rho}_{24} = -2iJ\rho_{34} - G(t)\rho_{24},$$

$$\dot{\rho}_{34} = -2iJ\rho_{24} - G(t)\rho_{34},$$

(10)

where the asterisk in the superscript of $\rho_{mn}^*$ denotes the complex conjugation of $\rho_{mn}$. After a simple calculation, the analytical solutions of the coupled first-order ordinary differential equations in equation (10) can be found. They are reported in the appendix. It should be noted that in the limit of $J \to 0$, solutions (A.1) have simple analytical forms which are analyzed in [11] and [26].
3. Entanglement and CHSH–Bell inequality dynamics

For two-qubit systems, Wootters concurrence can be used as a measure of entanglement [38]. The concurrence function varies from \( C = 0 \) for a separable state to \( C = 1 \) for a maximally entangled state. The concurrence function may be calculated from the density matrix \( \hat{\rho} \) for qubits \( A \) and \( B \) as

\[
C(\hat{\rho}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},
\]

where the quantities \( \lambda_i \) are the eigenvalues in decreasing order of the matrix

\[
\hat{\rho}_{\text{trans}} = \hat{\rho} (\hat{\sigma}_y^A \otimes \hat{\sigma}_y^B) \hat{\rho}^\dagger (\hat{\sigma}_y^A \otimes \hat{\sigma}_y^B).
\]

In the following, we consider entanglement dynamics of the qubits whose density matrix has a common X-form [13]:

\[
\hat{\rho} = \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
\rho_{41} & 0 & 0 & \rho_{44}
\end{pmatrix}.
\]

Such X-states arise in a wide variety of physical situations and include pure Bell states [39] as well as the well-known Werner mixed states [19]. Note that the time evolution of equation (10) preserves the form of the X-state. Then one can easily show that the concurrence function for the X-state (13) is given by [11]

\[
C(\hat{\rho}) = 2 \max\{0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\}.
\]

For the two-qubit case the CHSH-type Bell inequality can be written in the following form [23]:

\[
B(\hat{\rho}) = |\langle O_A O_B \rangle - \langle O_A'O_B' \rangle + \langle O_A'O_A' \rangle + \langle O_B'O_B' \rangle| \leq 2,
\]

where \( \langle O_A O_B \rangle = \text{Tr}(\hat{\rho} O_A O_B) \) is the correlation function and \( O_S = O_S \cdot \sigma_S \), wherein \( O_S = (\sin \theta_S \cos \phi_S, \sin \theta_S \sin \phi_S, \cos \theta_S) \) is the unit vector, \( \sigma_S = (\sigma_x^S, \sigma_y^S, \sigma_z^S) \) is the Pauli matrices vector and \( O'_S = O_S(\theta'_S, \phi'_S) \). Bellomo et al. showed that the maximum of the Bell function \( B(\hat{\rho}) \) for a X-structured density matrix (13) can be given as [24, 25]

\[
B_{\text{max}}(\hat{\rho}) = 2\sqrt{P^2(t) + Q^2(t)},
\]

where

\[
P(t) = \rho_{11} + \rho_{44} - \rho_{22} - \rho_{33},
Q(t) = 2(|\rho_{14}| + |\rho_{23}|).
\]

It should be noted that in the regions where \( B_{\text{max}}(\hat{\rho}) > 2 \), the CHSH inequality is violated. It implies that the state \( \hat{\rho} \) is genuinely bipartite Bell non-local; thus, the correlations cannot be accessible by any classical local model. Also note that the expression for \( B_{\text{max}}(\hat{\rho}) \) in equation (16) coincides with the one that would be obtained using the formal Horodecki criterion [40].
In the following, we consider two classes of initial states called extended Werner-like (EWL) states in the form
\[
\hat{\rho}_1(0) = \frac{1 - r}{4} I_4 + r |\Phi \rangle \langle \Phi | \\
= \begin{pmatrix}
\frac{1 - r}{4} & 0 & 0 & 0 \\
0 & \frac{1 - r}{4} + \alpha^2 r & \alpha \beta r & 0 \\
0 & \alpha \beta r & \frac{1 - r}{4} + \beta^2 r & 0 \\
0 & 0 & 0 & \frac{1 - r}{4}
\end{pmatrix},
\]
(18)
\[
\hat{\rho}_2(0) = \frac{1 - r}{4} I_4 + r |\Psi \rangle \langle \Psi | \\
= \begin{pmatrix}
\frac{1 - r}{4} + \alpha^2 r & 0 & 0 & \alpha \beta r \\
0 & \frac{1 - r}{4} & 0 & 0 \\
0 & 0 & \frac{1 - r}{4} & 0 \\
\alpha \beta r & 0 & 0 & \frac{1 - r}{4} + \beta^2 r
\end{pmatrix},
\]
where \( r \) is the purity of the initial states which ranges from 0 for maximally mixed states to 1 for pure states, \( I_4 \) is the \( 4 \times 4 \) identity matrix, \( |\Phi \rangle = \alpha |10 \rangle + \beta |01 \rangle \), \( |\Psi \rangle = \alpha |11 \rangle + \beta |00 \rangle \) are the Bell-like pure states and the parameter \( \alpha \) is sometimes called the degree of entanglement (here we set \( \alpha \) as real number and \( \beta = \sqrt{1 - \alpha^2} \) [24]. From equation (10) or (A.1), it can be easily noted that the initial states of equation (18) belong to ‘X’-states at any time \( t \). So the time-dependent concurrence and the maximum of the Bell function can be obtained from equations (14) and (16), respectively.

Before starting our qualitative analysis, we want to emphasize that the time-dependent concurrence and the maximum of the Bell function for \( \hat{\rho}_2(0) \) have simple analytical forms. According to equation (A.1), for this initial state the density matrix at any time can be found as
\[
\hat{\rho}_2(t) = \begin{pmatrix}
\frac{1 - r}{4} + \alpha^2 r & 0 & 0 & \alpha \beta r e^{-2 f(t)} \\
0 & \frac{1 - r}{4} & 0 & 0 \\
0 & 0 & \frac{1 - r}{4} & 0 \\
\alpha \beta r e^{-2 f(t)} & 0 & 0 & \frac{1 - r}{4} + \beta^2 r
\end{pmatrix},
\]
(19)
then the time-dependent concurrence and the maximum of the Bell function for the \( \hat{\rho}_2(0) \) initial state can be obtained as
\[
C^\Psi(\hat{\rho}) = 2 \max \left\{ 0, r \alpha \beta e^{-2 f(t)} - \frac{1 - r}{4} \right\},
\]
(20)
\[
B^\Psi_{\max}(\hat{\rho}) = 2 r \sqrt{1 + 4 (\alpha \beta e^{-2 f(t)})^2},
\]
where $f(t) = \frac{\Gamma}{2} \left( t + \frac{1}{\gamma} (e^{-\gamma t} - 1) \right)$. For the initial state $\tilde{\rho}_b(0)$, a similar expression to equation (20) can be written by using the solution in the appendix. But the elements are quite involved and we do not display them here for brevity; $\tilde{\rho}_b(t)$ for special values of $\alpha$ and $J$ are displayed and discussed below. One should note that the dynamics of the initial state $\tilde{\rho}_b(0)$ is undisturbed by the dipole–dipole interaction which is expected because of the single excitation nature of the dipole–dipole interaction and the form of the $|\Psi\rangle$ state which is $|\Psi\rangle = \alpha|11\rangle + \sqrt{1-\alpha^2}|00\rangle$. Furthermore, equation (20) also corresponds to the time-dependent analytical expressions of $C(\hat{\rho})$ and $B_{max}(\hat{\rho})$ for the initial states $\tilde{\rho}_b(0)$ and $\tilde{\rho}_b(0)$ in the absence of the dipole–dipole interaction (i.e. $J = 0$) [26]. The dynamics of entanglement and Bell non-locality for the initial state, $\tilde{\rho}_b(0)$, was extensively examined in [26]; thus, we will not cover this state in detail in the rest of this paper.

The dynamics of concurrence and Bell non-locality as a function of the dipole–dipole interaction strength and the dimensionless time are displayed in figure 1(a) and (b) for $\alpha = 1$ and in figure 1(c) for $\alpha = 1/\sqrt{4}$ for the initial states $\tilde{\rho}_b(0)$ with $r = 0.95$. All figures and quantities show a faster than exponentially decaying oscillatory behavior with $J$-dependent oscillation frequency. Based on equation (20), for $\alpha = 1$, it should be noted that in the absence of the dipole–dipole interaction (i.e. $J = 0$), the states $\tilde{\rho}_b(t)$ and $\tilde{\rho}_b(t)$ have no entanglement and do not violate the CHSH inequality at any time, while in the presence of the dipole–dipole interaction (i.e. $J \neq 0$), $\tilde{\rho}_b(t)$ possesses a high degree of entanglement and violates the CHSH–Bell inequality as can be seen from figure 1(a) and (b). On the other hand,
for $\alpha = 1/\sqrt{2}$, the initial state $\hat{\rho}_\Phi(0)$ is entangled at $t = 0$ and the entanglement as well as Bell non-locality for $J \neq 0$ lives longer compared to the $J = 0$ case (see figure 1(c)). The common feature of all figures is that Bell non-locality is found to die out before the entanglement does as also found by Kofman and Korotkov in the case of dephasing and dissipative dynamics of two qubits [27].

The non-Markovianity dependence of the entanglement and Bell inequality violation for $\hat{\rho}_\Phi(0)$ with $\alpha = 1$ is displayed in figure 2(a) and (b), respectively. For the relatively high purity ($r = 0.95$) considered in this case, the effect of non-Markovianity is found to be a prolonging of the lifetime of both concurrence and the Bell inequality violation. This finding is in line with many similar studies on the effect of non-Markovianity on the dynamics of entanglement; since non-Markovianity is a measure of memory effects, its increase may lead to a longer lived or protected entanglement [11, 12, 16, 24–26].

The effect of purity of the initial state on the dynamics of the $\hat{\rho}_\Phi(t)$ state for Markovian and non-Markovian environments is displayed in figure 3(a)–(d) for the concurrence and the Bell inequality with $\alpha = 1$. It should be noted that the dipole–dipole interaction creates a significantly wide purity range for non-zero entanglement as $0.4 < r \leq 1$ for Markovian and $0.35 < r \leq 1$ for non-Markovian dynamics. On the other hand, the purity dependence of non-zero Bell non-locality is narrower compared to the entanglement as $0.8 < r \leq 1$ and $0.7 < r \leq 1$ for Markovian and non-Markovian dynamics, respectively. The figures also show a pronounced difference between the dynamical behavior of both the quantities for the pure and the mixed states. While both entanglement and Bell non-locality decrease exponentially with time for pure states under Markovian and non-Markovian dynamics, there is a sudden death phenomenon for both quantities for the initially mixed states.

For the special case of $\alpha = 1/\sqrt{2}$, the initial states considered here become

$$\hat{\rho}_\Phi(0) = \frac{1 - r}{4} I_4 + r |\Phi\rangle \langle \Phi|,$$

$$\hat{\rho}_\Psi(0) = \frac{1 - r}{4} I_4 + r |\Psi\rangle \langle \Psi|,$$

(21)

where $|\Phi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and $|\Psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ are the pure Bell states. With the help of the solutions presented in the appendix, the time-dependent density matrix for $\hat{\rho}_\Phi(0)$ can be written as
Figure 3. $C(\hat{\rho})$ ((a) and (b)) and $B_{\text{max}}(\hat{\rho}) - 2$ ((c) and (d)) versus $\Gamma t$ and $r$ for $\hat{\rho}_\Phi(0)$ with $J = 0.5, \alpha = 1$. Here (a) and (c) correspond to Markovian dynamics with $\gamma/\Gamma = 10$ and (b) and (d) to non-Markovian dynamics with $\gamma/\Gamma = 0.1$.

\[
\hat{\rho}_\Phi(t) = \begin{pmatrix}
\frac{1-r}{4} & 0 & 0 & 0 \\
0 & \frac{1+r}{4} & \frac{r e^{-2f(t)}}{2} & 0 \\
0 & \frac{r e^{-2f(t)}}{2} & \frac{1+r}{4} & 0 \\
0 & 0 & 0 & \frac{1-r}{4}
\end{pmatrix};
\]  

then the concurrence and the maximum of the Bell function for the $\hat{\rho}_\Phi(t)$ of equation (22) can be expressed in a simple analytic form as

\[
C^\Phi(\hat{\rho}) = \max \left\{ 0, r e^{-2f(t)} - \frac{1-r}{2} \right\},
\]

\[
B^\Phi_{\text{max}}(\hat{\rho}) = 2r \sqrt{1 + e^{-4f(t)}}.
\]  

For the special value of $\alpha = 1/\sqrt{2}$, the dynamics of the entanglement and the Bell non-locality were investigated by Li and Liang who showed that $C(\hat{\rho})$ and $B_{\text{max}}(\hat{\rho})$ show exactly same
To further elucidate the role of the so-called degree of entanglement parameter $\alpha$ in the dynamics of $C(\hat{\rho})$ and $B_{\text{max}}(\hat{\rho})$ for the initial states $\hat{\rho}/\Phi_1(0)$ or $\hat{\rho}/\Psi_1(0)$, we consider the general initial states given by equation (18). It is easy to write down the time-dependent density matrix elements for the type of initial states of equation (18) by using the solution in the appendix. The dynamics of concurrence and Bell non-locality as a function of the parameter $\alpha$ of the initial state are presented in figures 4 and 5. In figure 4, we display the behavior of $C(\hat{\rho})$ and $B_{\text{max}}(\hat{\rho}) - 2$ for the non-interacting case for the initial state $\hat{\rho}/\Phi_1(0)$ or $\hat{\rho}/\Psi_1(0)$, while figure 5 presents the dynamics for the $J = 0.5$ value of the dipole–dipole interaction strength for $\hat{\rho}/\Phi_1(0)$. For the non-interacting case, the effect of purity and the $\alpha$ parameter on the dynamics of $C(\hat{\rho})$ and $B_{\text{max}}(\hat{\rho})$ is displayed in figure 4(a) and (b) and 4(c) and (d), respectively. As can be seen from these figures, the entanglement and Bell non-locality are maximum for $\alpha = 1/\sqrt{2}$ which corresponds to the initial Bell state which offers the longest time of survival for the mixed state case (figure 4(a) and (c)) in the absence of the dipole–dipole interaction. The volume of non-zero entanglement and Bell non-locality decreases as $r$ (purity) decreases as can be seen from a comparison of figure 4(a) and (b) and also 4(c) and (d). For $r = 1$, the entanglement and Bell non-locality have non-zero values for all $\alpha$ values with exponential decay, except for $\alpha = 0$ or $\alpha = 1$. One should also note that figure 4(a)–(d) are for non-Markovian dynamics; in the Markovian case figure 4(b) and (d) would be qualitatively similar, but the non-zero regions in figure 4(a) and (c) would shrink [11, 12, 26].

The effect of the dipole–dipole interaction on the dynamics $C(\hat{\rho})$ and $B_{\text{max}}(\hat{\rho}) - 2$ as a function of $\alpha$ and $\Gamma t$ is displayed in figure 5(a)–(d) for $\hat{\rho}/\Phi_1(0)$ with $J = 0.5$. Comparing to the corresponding sub-figures of figures 4 and 5, the most pronounced difference is in the...
dynamics of the initial state with $\alpha = 1/\sqrt{2}$ which suffers the quickest death among all the other probable initial mixed states. Also the $\alpha = 0$ and $\alpha = 1$ states which are initially non-entangled are found to have entanglement for the longest time under the dipole–dipole interaction. We should also mention that for the pure initial states ($r = 1$), the dipole–dipole interaction leads to non-zero entanglement and violation of the Bell inequality for $\alpha = 0$ and $\alpha = 1$ for all times as can be seen from figure 5(b) and (d).

4. Conclusion

We have investigated the dynamics of entanglement as measured by concurrence and Bell non-locality as measured by the CHSH inequality for two noisy qubits which are connected to each other by the dipole–dipole interaction. The considered noise have the Ornstein–Uhlenbeck-type correlation and the dynamics is investigated analytically for extended Werner-like initial states. We have found that the most important effects of the dipole–dipole interaction to the dynamics of the considered quantities are as follows: the dipole–dipole interaction can create non-zero entanglement and lead to violation of the Bell inequality for the initial state $\hat{\rho}_\Phi(0) = \frac{1}{\mathcal{N}} I + r |\Phi\rangle \langle \Phi|$ (where $|\Phi\rangle = \alpha |10\rangle + \sqrt{1 - \alpha^2} |01\rangle$) with $\alpha = 0$ or $\alpha = 1$ which can be protected from sudden death for a pure initial state ($r = 1$). On the other hand, for mixed initial states ($r < 1$) the dynamics of $\hat{\rho}_\Phi(0)$ with $\alpha = 1/\sqrt{2}$ is found to have longest life...
time for non-zero entanglement and Bell non-locality violation in the absence of the dipole–
dipole–J. Phys. A: Math. Theor.

\[ K = -\Gamma (\rho_{22}(0) - \rho_{33}(0))(\gamma (\gamma - \Gamma)(\epsilon - \Gamma) - 64 J^2 \gamma) F_1 \left( \frac{\epsilon}{\gamma} - \frac{\Gamma}{\gamma}; 2 - \frac{\epsilon}{\gamma}; -\frac{\Gamma}{\gamma} \right) \]

\[ \rho_{22} = \frac{Y + Z}{Y} , \]

\[ \rho_{33} = \frac{Y - Z}{2} , \]

\[ \rho_{12} = (\rho_{12}(0) \cos(2 J t) + i \rho_{13}(0) \sin(2 J t)) e^{-f(t)} , \]

\[ \rho_{13} = (\rho_{13}(0) \cos(2 J t) + i \rho_{12}(0) \sin(2 J t)) e^{-f(t)} , \]

\[ \rho_{14} = \rho_{14}(0) e^{-2 f(t)} , \]

\[ \rho_{23} = A + B \]

\[ \rho_{24} = (\rho_{24}(0) \cos(2 J t) - i \rho_{34}(0) \sin(2 J t)) e^{-f(t)} , \]

\[ \rho_{34} = (\rho_{34}(0) \cos(2 J t) - i \rho_{24}(0) \sin(2 J t)) e^{-f(t)} , \]

where

\[ f(t) = \int_{0}^{t} G(s) \, ds = \frac{\Gamma}{2} \left( t + \frac{1}{\gamma} (e^{-\gamma t} - 1) \right) , \]

\[ Y = \rho_{22}(0) + \rho_{33}(0) , \]

\[ Z = \frac{K \left( \frac{\gamma}{2J} \right)^{\eta} F_1 \left( \eta; 1 + \frac{\gamma}{2J} - \frac{\Gamma}{\gamma}; -\frac{\Gamma}{\gamma} \right) + L \left( \frac{1}{J^2} \right)^{\eta} \left( \frac{\gamma}{J^2} \right)^{\eta} F_1 \left( \eta; 0 + \frac{\gamma}{2J} - \frac{\Gamma}{\gamma}; -\frac{\Gamma}{\gamma} \right)}{M} , \]

\[ A = (\rho_{22}(0) + \rho_{33}(0)) e^{-2 f(t)} , \]

\[ B = \left( \frac{1}{J^2} \right)^{-\eta} \left( \frac{e^{-\gamma t}}{J^2} \right)^{\eta} \left( C + D \right) , \]

where

\[ K = \frac{1}{J^2} \left( \begin{array}{c} \eta \\ \eta \\ \eta \end{array} \right) \left( \frac{\gamma}{J^2} \right)^{\eta} F_1 \left( \eta; 1 + \frac{\gamma}{2J} - \frac{\Gamma}{\gamma}; -\frac{\Gamma}{\gamma} \right) + L \left( \frac{1}{J^2} \right)^{\eta} \left( \frac{\gamma}{J^2} \right)^{\eta} F_1 \left( \eta; 0 + \frac{\gamma}{2J} - \frac{\Gamma}{\gamma}; -\frac{\Gamma}{\gamma} \right) \]

\[ A = (\rho_{22}(0) + \rho_{33}(0)) e^{-2 f(t)} , \]

\[ B = \left( \frac{1}{J^2} \right)^{-\eta} \left( \frac{e^{-\gamma t}}{J^2} \right)^{\eta} \left( C + D \right) , \]

\[ K = -\Gamma (\rho_{22}(0) - \rho_{33}(0))(\gamma (\gamma - \Gamma)(\epsilon - \Gamma) - 64 J^2 \gamma) F_1 \left( \frac{\epsilon}{\gamma} - \frac{\Gamma}{\gamma}; 2 - \frac{\epsilon}{\gamma}; -\frac{\Gamma}{\gamma} \right) \]

\[ + \frac{\gamma (\gamma^2 - \epsilon^2)}{J^2} F_1 \left( \frac{\gamma}{J^2} - \frac{\Gamma}{J^2}; 2 - \frac{\epsilon}{\gamma}; -\frac{\Gamma}{\gamma} \right) \]

\[ - 8i \Gamma (\rho_{22}(0) - \rho_{33}(0)) \]
+ \Gamma(\gamma(\Gamma - \gamma)(\Gamma + \epsilon) - 64J^2\gamma_1F_1\left(\frac{\epsilon}{\gamma}; 2 + \frac{\epsilon}{\gamma}; \frac{\Gamma}{\gamma}\right) - \Gamma(\gamma(\Gamma - \Gamma)(\epsilon - \Gamma)) \right) \\
- 64J^2\gamma_1F_1\left(\eta_+; 1 + \frac{\epsilon}{\gamma}; -\Gamma; \frac{\epsilon}{\gamma}\right) + 1F_1\left(\frac{\epsilon}{\gamma}; 2 - \frac{\epsilon}{\gamma}; -\Gamma\right),
\end{equation}
\begin{equation}
C = \left(\frac{1}{J}\right)^2 iF_1\left(\frac{\epsilon}{\gamma}; 1 - \frac{\epsilon}{\gamma}; -\frac{\epsilon}{\gamma}\right) (\gamma^2 + \epsilon^2) F_1\left(\frac{\epsilon}{\gamma}; 1 + \frac{\epsilon}{\gamma}; -\Gamma\right) \\
- 64J^2\gamma_1F_1\left(\Delta_\gamma; 2 + \frac{\epsilon}{\gamma}; -\Gamma\right) + \gamma(\gamma^2 - \epsilon^2) F_1\left(\Delta_\gamma; 1 + \frac{\epsilon}{\gamma}; -\Gamma\right) \\
\times \left((\rho_\Delta(0) - \rho_\Delta(0))(\Gamma + \epsilon) - 8iJ(\rho_\Delta(0) - \rho_\Delta(0))\right),
\end{equation}
\begin{equation}
D = \left(\frac{e^{-\gamma/\gamma}}{J}\right)^2 iF_1\left(\frac{\epsilon}{\gamma}; 1 + \frac{\epsilon}{\gamma}; -\frac{\epsilon}{\gamma}\right) (\gamma^2 + \epsilon^2) F_1\left(\frac{\epsilon}{\gamma}; 1 + \frac{\epsilon}{\gamma}; -\Gamma\right) \\
+ 64J^2\gamma_1F_1\left(\Delta_\gamma; 2 - \frac{\epsilon}{\gamma}; -\Gamma\right) + \gamma(\gamma^2 - \epsilon^2) F_1\left(\Delta_\gamma; 1 - \frac{\epsilon}{\gamma}; -\Gamma\right) \\
\times \left((\rho_\Delta(0) - \rho_\Delta(0))(\Gamma - \epsilon) + 8iJ(\rho_\Delta(0) - \rho_\Delta(0))\right),
\end{equation}
\begin{equation}
E = iF_1\left(\Delta_\gamma; 1 - \frac{\epsilon}{\gamma}; -\Gamma\right) (\gamma(\gamma + \Gamma)(\Gamma - 2\gamma + \epsilon) - 64J^2) F_1\left(\Delta_\gamma; 2 + \frac{\epsilon}{\gamma}; -\Gamma\right) \\
- \gamma(\gamma + \Gamma)(2\gamma - \Gamma + \epsilon) + 64J^2 F_1\left(\Delta_\gamma; 2 - \frac{\epsilon}{\gamma}; -\Gamma\right) F_1\left(\frac{\epsilon}{\gamma}; 1 + \frac{\epsilon}{\gamma} - \Gamma\right),
\end{equation}
\begin{equation}
(A.3)
\end{equation}
where \(\epsilon = \sqrt{\Gamma^2 - 64J^2}\), \(\kappa_\pm = \frac{2\gamma + \Gamma \pm \epsilon}{2\gamma}\), \(\Delta_\pm = \frac{4\gamma + \Gamma \pm \epsilon}{2\gamma}\), \(\eta_\pm = \frac{\Gamma \pm \epsilon}{2\gamma}\) and \(F_1(a; b; z)\) is the Kummer confluent hypergeometric function [41]. These solutions are obtained by using the Mathematica program.

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