Quantum information-geometry of dissipative quantum phase transitions

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A general framework for analyzing the recently discovered phase transitions in the steady state of dissipation-driven open quantum systems is still missing. In order to fill this gap we extend the so-called fidelity approach to quantum phase transitions to open systems whose steady state is a Gaussian Fermionic state. We endow the manifold of correlations matrices of steady-states with a metric tensor $g$ measuring the distinguishability distance between solutions corresponding to different set of control parameters. The phase diagram can be then mapped out in terms of the scaling behavior of $g$ and connections with the Liouvillean gap and the model correlation functions unveiled. We argue that the fidelity approach, thanks to its differential-geometric and information-theoretic nature, provides novel insights on dissipative quantum critical phenomena as well as a general and powerful strategy to explore them.

Introduction:-- The occurrence of typical equilibrium phenomena in out equilibrium driven condensed matter systems (e.g. long range order, topological order, quantum phase transitions) has recently been discovered [1–4]. This poses new, fascinating and challenging problems both at the theoretical and at the experimental level. Indeed, it has been shown that dissipation processes can in principle be controlled and tailored in order to compete with systems free evolution and to realize fundamental protocols such as quantum state preparation [5], quantum simulation [6], and computation [7]. The general approach has been so far successfully applied to a large variety of ground state QPTs (GS-QPTs) [8–11] and quantum chaos [12]. In the following we show how the scheme can be fruitfully applied to NESS-QPTs. In this context the set of (control) parameters $\lambda \in \mathcal{M}$ defines a Liouvillian superoperator $\mathcal{L}(\lambda)$ which drives the system, independently of the chosen initial state, to the corresponding (unique) NESS $\rho(\lambda)$. Depending on $\lambda$ the NESS can exhibit quite different properties and the system can exhibit NESS-QPTs. The main idea behind the fidelity approach is the following: when dramatic structural changes occur in $\rho(\lambda)$, e.g. approaching a critical point, the geometric-statistical distance $d[\rho(\lambda), \rho(\lambda + \delta \lambda)]$ between two infinitesimally close states grows as they become more and more statistically distinguishable. Since $\rho(\lambda)$ is in general mixed, one has to resort to the Bures distance $d^2_B = 2[1 - \mathcal{F}(\rho, \rho + d\rho)]$, where $\mathcal{F}$ is the Uhlmann fidelity [13]. The infinitesimal distance $d^2_B$, when expressed in terms of the parameters $\lambda$, provides a metric $g$ onto the parameters manifold $\mathcal{M}$. The tensor $g$ is the fundamental tool of the fidelity approach: it has been shown that the study of its scaling behavior (extensive vs. superextensive) allows a systematic study of GS-QPTs [10, 14] and it is therefore the natural candidate for the investigation of NESS-QPTs.

In this paper we concentrate on an important set of Fermionic systems whose dissipative Markovian evolution is governed by a Lindblad type master equation [15]. Their NESS are Gaussian Fermionic (GF) states and specific models belonging to this class indeed display rich non-equilibrium features and NESS-QPTs, which have been characterized by studying long range magnetic correlations (LRMC) and the Liouvillean spectral gap $\Delta_L [1]$. In order to extend the fidelity approach to such systems: i) we derive a general formula for the Bures distance and the metric tensor $g$ for quadratic Fermionic (mixed) states; ii) we derive the general expression of Gaussian NESS for Fermionic Liouvillean evolutions, a general bound on the scaling behavior of $g$ and we discuss its relation with two-point correlations and with $\Delta_L$; iii) we specialize our results to an exactly solvable master equation modelling an open-ended XY spin chain coupled to two reservoirs at the boundaries. Our analysis demonstrates that the NESS phase diagram can be accurately mapped by studying the (finite-size) scaling behavior of the metric tensor $g$; critical lines can be identified and the different phases distinguished.

Bures metric for Gaussian Fermionic states:-- The first step of our information-geometric analysis is based on the pull-back of the Bures metric for GF-states onto the manifold of the two point correlation functions. Let us consider a system of $n$ Fermion modes described by a set of $2n$ Majorana operators $w_i$. These operators are hermitian, linearly depend on the Fermionic creation and annihilation operators via $w_i = f_i + f_i^\dagger$, $w_{n+i} = i(f_i - f_i^\dagger)$, $i = 1 \ldots n$, and satisfy the algebra $[w_i, w_j] = 2\delta_{ij}$. Let $\rho$ be a GF-state, i.e. a Gaussian state in terms of the operators $w_j$. Owing to the Wick’s theo-
parameters the statistical distinguishability between two infinitesimally manifolds of GF-states. In this respect, Theorem 1 together with the pseudo-inverse. In particular, when \( \rho \) is pure, Sp(\( C \)) = \{ ±1 \} and the above equation reduces to \( d s^2_{\text{pure}} = \|dC\|_2^2/2 \).

The proof is given in the SM. In terms of the external parameters \( \{ \lambda_p \} \in M \) of the model: \( dC = \sum_p d\lambda_p d\lambda_r C \) and

\[
d s^2 = \sum_{\mu, r} g_{\mu r} d\lambda_\mu d\lambda_r \ , \quad g_{\mu r} = \sum_{s} \frac{C_{\mu r} C_{\mu s}}{1 - c_r s_c} \ ,
\]

where \( C = \sum_r c_r \vert r \rangle \langle r \vert \), with \( c_r \in \mathbb{R} \) and \( \{ \partial C \}_{rs} = \langle r \vert \partial C \vert s \rangle \), i.e. the sum in the above equation is performed in the basis in which \( C \) is diagonal and it is restricted over the elements such that \( c_r c_s \neq 1 \). The infinitesimal distance \( d s^2 \) encodes the statistical distinguishability between two infinitesimally close Gaussian Fermionic states; this result is completely general and it can be used to study the geometrical properties of manifolds of GF-states. In this respect, Theorem 3 together with (2) provide the basic tool for studying the phase transitions occurring when the NESS are GF-states. For GS-QPTs a superextensive behaviour of \( d s^2 \) implies criticality (10). A first qualitative indication that an analogue result may hold for NESS-QPT is suggested by the following inequality (see SM):

\[
d s^2 \leq 2n P_C \|dC\|_\infty^2 , \quad P_C = \| (I + C^\otimes 2) \|_\infty \ ,
\]

where \( \|A\|_\infty \) refers to the maximum singular value of \( A \). If \( P_C = O(1) \) a superextensive behaviour of \( d s^2 \) implies some sort of singularity in the correlation functions that may reflect the occurrence of a phase transition. In the following we analyse the NESS-QPTs occurring in quadratic Fermionic models.

Dissipative solvable model: We consider a Markovian dissipative open quantum system evolution \([16]\) governed by the Lindblad master equation

\[
\frac{d\rho}{dt} = \mathcal{L}\rho := -i[\mathcal{H}, \rho] + \sum_{\mu} (2L_{\mu} \rho L_{\mu}^\dagger - (L_{\mu}^\dagger L_{\mu}) \rho) \ ,
\]

with a quadratic Hamiltonian \( \mathcal{H} = \sum_{ij} H_{ij} w_i w_j \) and linear Lindblad operators \( L_{\mu} = \sum_{ij} \ell_{\mu ij} w_i \), where the matrices \( H \) and \( \ell \) depends on the parameters \( \lambda \in \mathcal{M} \) defining the specific model. In the following we diagonalize \( \mathcal{L} \) and obtain the (Gaussian) steady state \( \Omega \) of the Liouvillian, namely the state for which \( d\Omega/dt = \mathcal{L}\Omega = 0 \). We call \( \mathcal{R} \) the 4-dimensional operator spaces generated by \( \prod_{j} w_{j}^n \), \( (s_j \in \{ 0, 1 \}) \), and we use the notation \( \{ s \} \) for referring to the elements of \( \mathcal{R} \), normalized with respect to the Hilbert-Schmidt inner product, i.e. \( \langle \{ s \} \vert \{ s' \} \rangle = \mathbb{1} \) for \( \{ s \} \in \mathcal{R} \). The Liouvillean \( \mathcal{L}: \mathcal{R} \rightarrow \mathcal{R} \) can be written as a quadratic form in terms of the following set of 2n creation and annihilation superoperators

\[
a_j^\dagger = -i \frac{W}{2} \{ w_j \} , \quad a_j = -i \frac{W}{2} [ w_j , \} \ ,
\]

where \( W = \mathcal{R} \prod_{j=1}^{2n} w_j \) is a Hermitian idempotent operator which anti-commutes with all the \( w_j \). The superoperator \( a_j^\dagger \) is the Hermitian conjugate of \( a_j \) in \( \mathcal{R} \). A direct calculation proves that the operators defined in Eq. (6) satisfy the canonical auto-commutation relations (CAR), \( \{ a_j^\dagger , a_k \} = \delta_{jk} \), and that \( \mathcal{L} = \sum_{ij} a_j^\dagger a_j^\dagger a_j + Y_j a_j^\dagger a_j / 2 \), where \( X = 4(iH + \mathbb{R} M) \equiv X^\dagger \), \( Y = -8i3M \equiv -Y^\dagger \equiv Y^T \), with \( M_{ij} = \sum_{k} \ell_{ik} \ell_{kj} \equiv M^\dagger \). This result was derived in \([15]\), but thanks to our definition (5), complex transformations \([17]\) for unifying the different parity sectors are avoided. In order to diagonalized the Liouvillian and to develop our theoretical analysis we assume the real matrix \( X \) to be diagonalizable, i.e. \( X = U X^T U^{-1} \) for \( x = diag(\{ x_i \}) \), \( x_i \in \mathbb{C} \), though generalizing the present derivation with the singular value decomposition is straightforward. We consider also the anti-symmetric imaginary matrix \( C \) solution of the following Sylvester equation

\[
X C + C X^T = Y \ .
\]

One can show that the transformation \( \mathbf{d} = U^{-1} (a + C a^\dagger) \), \( d^\dagger = U^T a^\dagger \), realizes a non-unitary Bogoliubov transformations (NuBT), i.e. the operators \( d_i \) and \( d_i^\dagger \) satisfies the CAR-algebra but \( d_i^\dagger \neq d_i \), and brings \( \mathcal{L} \) to the diagonal form \( \mathcal{L} = -\sum_{k} x_k d_k^\dagger d_k \). It is now possible to express the steady state \( \Omega \) as the \( \mathbf{d} \)-vacuum, i.e. \( d_j \vert \Omega \rangle = 0 \), and obtain its operator from the NuBT. Indeed, the identity operator, i.e. the element \( \{ 0 \} \in \mathcal{R} \) is the \( \mathbf{a} \)-vacuum, i.e. \( a_j \vert 0 \rangle = 0 \), \( \forall j = 1, \ldots, 2n \), and in particular \( \{ 0 \}, \mathcal{L} = 0 \). Using textbook results, by writing the NuBT via a quadratic superoperator \( \mathcal{V} \) such that \( d_j = \mathcal{V} a_j^\dagger d_j^{-1} \), then the \( \mathbf{d} \)-vacuum is readily obtained

\[
\vert \Omega \rangle = \mathcal{V} \{ 0 \} = e^{-i u^\dagger C a^\dagger} \{ 0 \} \ .
\]

Owing to the explicit form of the superoperators (5), in the SM we show that the above state is a GF-state and that its two point correlation functions \( \langle \{ w_i , w_j \} \rangle \) are given by \( C_{ij} \), i.e. by the solution of (6). The physical conditions for the existence and uniqueness of the steady state are given in \([18]\): if \( \Delta := 2 \min \mathcal{R} x_i > 0 \) then the solution of (6) is unique and every initial state converges for \( t \rightarrow \infty \) to the unique steady state (7). The gap \( \Delta \) represents both the inverse of the timescale for reaching the steady state and the gap of the Liouvillian: \( \min \{ \sum_i x_i n_i \} \equiv \Delta \). If \( \Delta > 0 \) the steady state \( \vert \Omega(\lambda) \rangle \) is unique and, since \( \mathcal{L} \) smoothly depends on the parameters \( \lambda \in \mathcal{M} \), it is smooth function of \( \lambda \) (19). If the gap \( \Delta (n) \rightarrow 0 \) for \( n \rightarrow \infty \) the steady state
Parameters | Phase | Parameters | $\Delta$ | $|g|$ | Quality of fit
--- | --- | --- | --- | --- | ---
Critical (⋆) | $h = 0$ | $n^{-3}$ | $n^6$ | good
Long-range | $0 < |h| < h_c$ | $n^{-3}$ | $n^3$ | average
Critical | $|h| = h_c$ | $n^{-5}$ | $n^6$ | bad
Short-range | $|h| > h_c$ | $n^{-3}$ | $n$ | good
Critical (⋆) | $\gamma = 0, |h| < h_c$ | $n^{-3}$ | $n^2$ | good

TABLE I. Scaling analysis of the gap $\Delta$ and of the maximum eigenvalue of the fidelity metric $g_{\Gamma}$. These laws do not depend on the particularly chosen ratio $\Gamma_{\alpha \gamma}^*$. The lines $h = 0$ and $\gamma = 0$ consists of a SRMC region embedded in the LRMC phase; one finds (see discussion in the text) $|g| \approx g_{ab}$ for $h = 0$ and $|g| \approx g_{\gamma \gamma}$ for $\gamma = 0$.

gap ruling ground state QPT, the Liouvillean gap $\Delta$ closes for $n \to \infty$ both at the critical point and for $h \neq h_c$, both in the LRMC and SRMC phase. As the reservoirs acts only at the boundaries of the spin chain the eigenvalues $x_i$ of the matrix $X$ for $n \gg 1$ are a small perturbation of the $n \to \infty$ case where $x_c = \pm 4i \omega_k$, being $\omega_k = \sqrt{(\cos k - h)^2 + \gamma^2 \sin^2 k}$ the quasi-particle dispersion relation of the Hamiltonian (5). In particular $x_c$ gains a small real part and one finds a gap $\Delta = O(n^{-3})$ for $h \neq h_c$ and $\Delta = O(n^{-3})$ for $h = h_c$. Therefore the scaling of the Liouvillean gap allows one to identify the transition form the SRMC phase to the LRMC phase only along the critical line $h = h_c$, while the transition occurring at the $h = 0$ (or $\gamma = 0$) line can only be appreciated by evaluating the long-rangeness of the magnetic correlations. The question that naturally arises is how the different phases and transitions can be precisely characterized in a way similar to what happens for GS-QPTs. This question becomes more compelling if one compares the above results with the scaling of the geometric tensor $g_{\Gamma}$, and in particular of its largest eigenvalue $|g|$, see Table II and Fig. I for specific values of the parameters.

A first important result is that the tensor $g$ is able to identify the transitions between SRMC and LRMC phases. On the “transition lines” $h = 0$ and $h = h_c$ one has that $|g| = O(n^0)$, while in the rest of the phase diagram $|g| < O(n^0)$. Furthermore, a closer inspection of the elements of $g$ shows that while $g_{ab}(h = 0, \gamma) = O(n^0)$, one has that $g_{\gamma \gamma}(h = 0, \gamma) = O(n)$: the scaling is superextensive only if one moves away from the line $h = 0$ ($g_{ab}$) and enters in the LRMC phase, while if one moves along the $h = 0$ line ($g_{\gamma \gamma}$) i.e., if one remains in the SRMC phase, the scaling is simply extensive and it matches the scaling displayed in the other SRMC phase $h > h_c$. On the other hand, the transition occurring at $\gamma = 0$ has a different scaling: $g_{\gamma \gamma} = O(n^2)$ while $g_{bb} \approx 0$. These findings can be further confirmed by a detailed study [2] based on the analytical results available for $\gamma \ll 1$ [17]. It turns out that the introduction of the magnetic field or the anisotropy drives different transitions whose specificity is accounted for by the different superextensive scalings.

Another important result shown in Table II is that the metric tensor is able to signal the presence of long-range correlations: within the LRMC phase $d^2$ scales superextensively as $|g| = O(n^3)$, and this superextensive behaviour is different
The red line is the linear fit, whose results are summarized in Table I. The same of Fig. 2. Blue curves represent the numerical data, while

\[ \gamma = \text{field relative amplitude of the fluctuations increase close to the critical} \]

has to be taken for fitting the data for \( h > h_c \). Due to finite size effects and to the differential nature of the geometric tensor, the value where \( |g| \) takes its maximum is slightly smaller than \( h_c \), and this difference depends on \( n \). Thus special care has to be taken for fitting the data for \( h \leq h_c \).

![FIG. 1. Scaling of \(|g|\) for \( \gamma = 0.6 \) and \( h \in [0,0.8] \) (left) and for \( \gamma = 0.5 \) and \( h \in [0.735,0.755] \) (right). The Lindblad parameters are the same of Fig. 2. Blue curves represent the numerical data, while the red line is the linear fit, whose results are summarized in Table I. \( |g| \) slightly fluctuates as a function of \( n \) in the LRMC phase and the relative amplitude of the fluctuations increase close to the critical field \( h_c \). Due to finite size effects and to the differential nature of the geometric tensor, the value where \( |g| \) takes its maximum is slightly smaller than \( h_c \), and this difference depends on \( n \). Thus special care has to be taken for fitting the data for \( h \leq h_c \).](image1)

![FIG. 2. Maximum eigenvalue \(|g|\) of the fidelity metric (2) for \( n = 250 \) and \( \Gamma_f = 0.3, \Gamma_g = 0.5, \Gamma_x = 0.1, \Gamma_y = 0.5 \). The larger value of \(|g|\) close to the phase transition line \( h = h_c(\gamma) \) is not evident in Fig. 2 because of the numerical mesh and because, the actual values of \(|g|\) for \( h = h_c \) can be comparable to those of the LRMC phase, depending on \( n \) (e.g. see Fig. 1). The qualitative form of Fig. 2 is not affected by different values of the Lindblad parameters \( \Gamma_{L,L} \) and by the dimension \( n \).](image2)

from that displayed at the transition lines. One is therefore led to conjecture that whole LRMC phase have a critical character, due to the presence of long range correlations.

The findings discussed in the above demonstrate that the metric tensor \( g \) being directly linked to the correlations properties of the Gaussian NESS, encodes all the relevant information about the dissipative phase transition featured by the model (9); in particular, the specificity of the different phases (SRMC vs LRMC), and the information about the physical relevant parameters, being them the magnetic field or the anisotropy, that drive the different transitions are properly accounted for. As shown in Fig. 2, the complete phase diagram can indeed be reconstructed with the study of the single function \( g \). While these results are specific to the model examined, the connection established in (2) roots the behaviour of \( g \) in the correlations properties of the general class of GF-states. Accordingly, one expects the fidelity approach to have a broader scope of application. We would like to stress that there are compelling questions that are still unanswered. In the first place the relation between \( g \) and other relevant quantities that have been used so far to characterize NESS-QPT; For the model (9), these are the range of correlations, and the finite-size scaling of Liouvillian gap \( \Delta \). The latter does not entirely capture the criticality phenomenon, and further investigation of the relation between criticality in NESS-QPT and geometrical and dynamical aspects is in order [22]. For example, it would be interesting to find a direct connection between criticality and superextensivity of \( ds^2 \), and to understand whether the scaling exponents of \( ds^2 \) at different lines can be related to different universality classes. Notice also, that, in the XY model, different type of symmetries (discrete vs. continuous) are broken moving away from the \( h = 0 \) or \( \gamma = 0 \) line.

Conclusions: In this Letter we developed an information-geometric framework for studying dissipative critical phenomena exhibit by the non-equilibrium steady states of Markovian evolutions described by quadratic Fermionic Liouvillian. Our strategy is an extension of the Fidelity approach developed for zero-temperature quantum phase transitions [8–10]. We first derived a general formula for the infinitesimal Bures distance between Gaussian Fermionic (mixed) states. This in turn allows one to define a metric tensor \( g \) on the manifold of steady states corresponding to different sets of control parameters. The intuitive idea underlying is that a transition between two structurally different phases should be reflected by the statistical distinguishability of pairs of infinitesimally close steady states. The method does not require the knowledge or the existence of any order parameters, as the tensor \( g \) is directly connected to the two-point correlation functions which define the Gaussian Fermionic steady states. We have shown that a superextensive behaviour of the tensor \( g \), implies some singularity for \( n \to \infty \) in the derivative of the correlation functions. We have applied the method to a specific Fermionic (XY) model and shown that the scaling of the geometric tensor enables one to identify both the critical lines and to distinguish between different phases characterized by short or long ranged correlations. The metric tensor encodes also for the direction of maximal distinguishability in the parameter manifold, thus allowing a detailed study of the sensitivity of the steady state to small variations of some control parameters. This is a crucial point for experimental applications of dissipative evolution. The scope of the information-geometric
approach extends well beyond the important quadratic case analyzed in this paper and may pave the way to the systematic study of general non-equilibrium critical phenomena. This in turn would allow the investigation of a broad class of systems and processes which are natural candidates for the preparation of desired quantum states and realization of quantum protocols.

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SUPPLEMENTARY MATERIAL

PROOF OF THEOREM 1

We consider a Gaussian Fermionic state written in the following form
\[ ρ = e^{-2\sum_i G_i w_i} / Z , \]
where the matrix \( G \) has to be real and antisymmetric. Accordingly \( G \) can be cast in the canonical form by an orthogonal matrix \( Q \), i.e.
\[ G = Q^T \sum_{k=1}^{n} \begin{pmatrix} 0 & g_k \\ -g_k & 0 \end{pmatrix} Q = Q^{-1} , \]
and has eigenvalues \( \pm ig_k \). Moreover let \( z_i = \sum_j Q_{ij} w_j \) be the new Majorana operators. Hence
\[ \rho = \frac{1}{Z} \prod_k \cosh \left( \frac{g_k}{2} \right) - i \sinh \left( \frac{g_k}{2} \right) z_{k-1} z_k , \]
\[ Z = \prod_k 2 \cosh \left( \frac{g_k}{2} \right) = \sqrt{\det \left[ 2 \cosh \left( \frac{i G}{2} \right) \right]} , \]
where we used the fact that the eigenvalues of \( iG \) are \( \pm g_k \). As \( C_{ij} = \frac{1}{2} (|w_i \rangle w_j \rangle \langle \langle w_j \langle w_i |) \) one can show that
\[ C = \tanh \left( \frac{i G}{2} \right) . \]
The correlation matrix \( C = C^\dagger = -C^T \) is diagonal in the same basis of \( G \) and its eigenvalues read \( c_k = \tanh(g_k/2) \). Hence
\[ \rho = \prod_k \frac{1 - ic_k z_{k-1} z_k}{2} , \]
where \( |c_k| \leq 1 \). Note that for \( c_k = \pm 1 \), one has \( g_k = \pm \infty \), making the ansatz \( (10) \) not well defined, unlike Eq. \( (15) \). The latter possibility occurs for instance for pure states, as it is clear from the following explicit expression for the purity of the states \( (10) \) and the states \( (10) \) and \( (13) \):
\[ \text{Tr}[\rho^2] = \frac{\det \left[ 2 \cosh \left( i \frac{G}{2} \right) \right]^2}{\det \left[ 2 \cosh \left( i \frac{G}{2} \right) \right]} = \sqrt{\det \left( 1 + C^2 \right)} . \]
We now derive the proof of Theorem 1 dividing the different steps into three lemmas. At first we assume \( c_k \neq \pm 1 \) and then we extend the result for including pure states.

**Lemma 1.** Let \( \rho, \rho' \) two GF-states \( (10) \) parametrized by \( G, G' \) respectively. Then
\[ \mathcal{F}(\rho, \rho') = \text{Tr} \sqrt{\sqrt{\rho} \sqrt{\rho'} \sqrt{\rho}} = \frac{\det \left[ \mathbb{I} + \sqrt{e^{G^2} G e^{G^2} G'} \right]^\dagger}{\det \left[ \mathbb{I} + e^{G^2} \right]^\dagger \det \left[ \mathbb{I} + e^{G'} \right]^\dagger} . \]

**Proof.** This lemma is a direct consequence of the fact the quadratic Majorana operators form a Lie algebra:
\[ \left[ w \cdot A w, w \cdot B w \right] = \frac{w \cdot [A, B] w}{4} , \]
and accordingly
\[ e^{w \cdot A w} e^{w \cdot B w} = e^{w \cdot D w} \]
Thanks to the above identity
\[ \sqrt{\sqrt{\rho} \sqrt{\rho'}} \sqrt{\rho} = \frac{\det \left[ \mathbb{I} + \sqrt{e^{G^2} G e^{G^2} G'} \right]^\dagger}{\det \left[ \mathbb{I} + e^{G^2} \right]^\dagger \det \left[ \mathbb{I} + e^{G'} \right]^\dagger} \]
and using \( (13) \) we find
\[ \mathcal{F}(\rho, \rho') = \frac{\det \left[ \mathbb{I} + \sqrt{e^{G^2} G e^{G^2} G'} \right]^\dagger}{\sqrt{\det \left[ \mathbb{I} + e^{G^2} \right] \det \left[ \mathbb{I} + e^{G'} \right]}^\dagger} \]
which is equivalent to \( (18) \).

A convenient parametrization of Eq. \( (18) \) is obtained in terms of the correlation function by defining the new matrix \( T = e^{G} \). Then
\[ C = \frac{T - \mathbb{I}}{T + \mathbb{I}} \quad T^T = T^{-1} , \]
\[ \mathcal{F}(\rho, \rho') =: \mathcal{F}(T, T') = \frac{\det \left[ \mathbb{I} + \sqrt{TT' T^T} \right]^\dagger}{\det \left[ \mathbb{I} + T \right]^\dagger \det \left[ \mathbb{I} + T' \right]^\dagger} , \]
The following lemma conveys the metric pull back with in the manifold of states parametrized by \( T \):

**Lemma 2.** Let \( ds^2 = 8 ds_B^2 = 16 \left[ 1 - \mathcal{F}(T, T + dT) \right] \) the fidelity metric around the state \( (10) \) pulled back in the space of the matrices \( T \) and let \( dT = \partial_T dT d\lambda \) where \( \lambda \in \mathcal{M} \) are the parameters of the model. Then the fidelity metric can be cast into the form \( ds^2 = \sum_{\mu \nu} g_{\mu \nu} d\lambda_\mu d\lambda_\nu \) where the geometric tensor is
\[ g_{\mu \nu} = 2 \sum_{ij} \frac{\partial_T T}_{ij} (\partial_T T)_{ji} \]
In \( (24) \) the sum is performed in the basis in which \( T \) is diagonal, i.e. we set \( T = \sum_i |i\rangle \langle i| \) and \( (\partial_T T)_{ij} = \langle i| \partial_T |j\rangle \).

**Proof.** Proceeding along the same lines of Section 3 of \( (23) \) we obtain for \( T' = T + dT \)
\[ \sqrt{TT' T^T} = T + \sum_{ij} |i\rangle \langle j| \sqrt{T_{ij}} \frac{dT_{ij}}{T_{ij} + t_{ij}} + O(dT)^3 \]
and
\[ \sqrt{T_{ij}^2} \frac{dT_{ij}}{T_{ij} + t_{ij}} (t_{ij} + t_k)(t_{ij} + t_k) + O(dT) \]

\[ - \sum_{ijk} |i\rangle \langle k| dT_{ij} dT_{jk} \frac{\sqrt{t_j^{2}}}{t_{ij} + t_{jk}} + O(dT)^3 \]
Owing to the above expression and to Eq. (24) the fidelity \( \mathcal{F}(T, T + dT) \) can be written in terms of some infinitesimal operators \( \delta, \partial \)

\[
\mathcal{F}(T, T + dT) \approx \frac{\det([\mathbb{I} + T](\mathbb{I} + \delta))]^\frac{1}{2}}{\det([\mathbb{I} + T]^2) \det([\mathbb{I} + T](\mathbb{I} + \delta))]^\frac{1}{2}}
= \frac{\det[\mathbb{I} + \delta)]^\frac{1}{2}}{\det[\mathbb{I} + \delta]} = e^{\frac{1}{2} \text{Tr}(\log(1 + \delta) - \frac{1}{2} \text{Tr}(\log(1 + \delta))}
\approx e^{\frac{1}{2} \text{Tr}(\delta/2) - \frac{1}{4} \text{Tr}(\partial^2/2)},
\]

(27)

where

\[
\delta = (1 + T)^{-1} dT = \sum_{ij} |i\rangle\langle j| \frac{1}{1 + t_i} dT_{ij},
\]

(28)

\[
\partial = (1 + T)^{-1} \left( \sqrt{TT'} - T \right)
= \sum_{ij} |i\rangle\langle j| \frac{1}{t_i + t_j + 1} dT_{ij} - \sum_{ijk} |i\rangle\langle k| dT_{ij} dT_{jk} \frac{1}{t_i + t_j + t_k + 1}.
\]

(29)

The elements of Eq. (27) become

\[
\text{Tr}(\partial - \delta/2) = -\frac{1}{4} \sum_{ij} |dT_{ij}|^2 \frac{1}{(t_i + t_j)(1 + t_i + t_j)},
\]

(30)

\[
\text{Tr}(\delta) = \sum_{ij} |dT_{ij}|^2 \frac{1}{(1 + t_i)(1 + t_j)},
\]

(31)

\[
\text{Tr}(\partial^2) \approx \sum_{ij} |dT_{ij}|^2 \frac{t_i t_j}{(t_i + t_j)^2 (1 + t_i)(1 + t_j)},
\]

(32)

so that

\[
\mathcal{F}(T, T + dT) \approx 1 - \frac{1}{8} \sum_{ij} \frac{|dT_{ij}|^2}{(1 + t_i)(1 + t_j)(t_i + t_j)},
\]

(33)

which completes the proof.

Before proving Theorem 11 we introduce the following lemma which will be used for analytical continuations to the pure state manifold:

**Lemma 3.** Let \( f(x, y) := (x - y)^2/(1 - xy)^{-1} \) be a function defined in \([-1, 1]^2\].\([z^+, z^-], z^\pm := (\pm 1, \pm 1)\). Then \( f(x, y) \leq 4 \) and \( \lim_{(x,y)\to z^\pm} f(x, y) = 0 \).

**Proof.** The upper bound is found thanks to \( 1 - xy = 1 - [(x + y)^2 - (x - y)^2]/4 \geq (x - y)^2/4 \). In order to show that \( \lim_{(x,y)\to z^\pm} f(x, y) = 0 \) let us restrict \( f \) to the \( x \geq 0, y \geq 0 \) part of the domain to analyse the limit to \( z^+ \). The limit \( z^- \) follows because of the \((x, y) \to (-x, -y)\) symmetry of \( f \). One can write \( y = 1 + m(x - 1) \) or \( x = 1 + m(y - 1) \) with \( m \in [0, 1] \). Because of the \((x, y) \to (y, x)\) symmetry of \( f \) we can consider just the first case. One obtains \( f(x, y) = (1 - x)(1 - m(x - 1))^2 \leq 1 - x \) this quantity in a disk of radius \( \delta \) centered on \( z^+ \) is upper bounded by \( \delta \). This shows that \( \forall \epsilon > 0, \exists \delta = \delta(\epsilon) \) s.t \( ||(x, y) - z^+|| \leq \delta \Rightarrow f(x, y) \leq \epsilon \) (with \( \delta(\epsilon) = \epsilon \)), i.e., the claim.

We are finally ready for

**Proof of Theorem 11** Eq. (2) is obtained directly from lemma 2. Indeed, from Eq. (23)

\[
dC = dT \frac{T - 1}{T + 1} - dT \frac{1}{T + 1} = 2 \frac{1}{T + 1} dT \frac{T - 1}{T + 1}.
\]

(34)

Inserting the above equation in (25), and noting that \( C \) and \( T \) are diagonal in the same basis, \( c_i = \frac{1}{2} e^{i \phi} \), one obtains

\[
g_{\mu\nu} = \sum_{ij} (\partial_{\phi} C)_{ij}(\partial_{\phi} C)_{ji} \frac{1}{1 - c_i c_j}.
\]

(35)

The singular behaviour of (35) for \( c_i = \pm 1 \) is just apparent. Indeed, let \( iG(j) = g_{ij} \) \( (j = 1, \ldots, 2n) \), \( Sp(G) = \{ g_i \} \subset \mathbb{R} \) then \( C = \sum_j c_j |j\rangle\langle j|, c_j := \text{tanh}(g_j/2) \). By differentiation \( dC \) is \( \sum_j \left( (1 - c_i^2)\frac{d^2}{d\phi^2} + c_i (d\phi)^2 |j\rangle\langle j| + |j\rangle\langle j| (d\phi)^2 \right) \). One has therefore the following matrix elements \( (dC)_{ij} = (1 - c_i^2) d\phi \) and \( (dC)_{ij} = c_i c_j (d\phi)^2 \). Plugging these in (35)

\[
d\hat{s} = \frac{1}{4} \sum_j (1 - c_i^2) d\phi \frac{g_{ij}}{1 - c_i c_j} + \sum_{ij} f(c_i, c_j) \frac{(d\phi)^2}{1 - c_i c_j}.
\]

(36)

Now one sees easily that for \( c_i \to \pm 1 \) the first (diagonal) contribution in (35) vanishes while the second, thanks to lemma 3, is upper bounded by \( 4 \sum_{ij} |d\phi|^2 \) for all \( c_i, c_j \in (-1, 1) \) and vanishes for \((c_i, c_j) \to z^\pm \): even if (35) has been derived for \( C \) such that \( c_i \neq \pm 1 \), we can perform the limit \( \lim_{c_i \rightarrow 1} \), \((\forall i)\) and, in this way, extend the metric to the pure state manifold just by setting \( c_i c_j \) to \(-1 \) (as for the case \( c_i c_j = 1 \) gives vanishing contribution).

The basis independent expression Eq. (11) follows from (35) and from the definition of \( Ad_{\mathcal{C}} \), the zero contribution to the sum (35) for \( c_i c_j = 1 \) is considered thanks to the pseudo-inverse.

One can show that Eq. (11) reduces to the known expressions when \( \rho \) is a thermal state [11] and when \( \rho \) is a pure state [24], provided that the appropriate matrices \( T \) or \( C \) are used. In the next section, this theorem is applied to NESS-QPT where \( C \) is given by the solution of the Sylvester equation (6).

**LIouvillian Steady State**

Following the notation introduced in the Letter, the Liouvillian (4) can be written as

\[
\mathcal{L} = -\frac{1}{2} \left( a^\dagger a \right) \begin{pmatrix} X & Y \\ 0 & -X^T \end{pmatrix} \left( a^\dagger \right) - \frac{1}{2} \text{Tr} X.
\]

(37)
If $C$ is the matrix solution of (6) then
\[
\begin{bmatrix}
  X & Y \\
  0 & -X^T
\end{bmatrix} = \begin{bmatrix}
  U & -CU^{-T} \\
  0 & U^{-T}
\end{bmatrix} \begin{bmatrix}
  x & 0 \\
  0 & -x
\end{bmatrix} \begin{bmatrix}
  U^{-1} & U^{-1}C \\
  0 & U^T
\end{bmatrix}.
\] (38)

We show now that the latter transformation is non-unitary Bogoliubov transformation \cite{2} and that everything is consistent. It is known that non-unitary Bogoliubov transformations are isomorphic to the group of orthogonal complex matrices $O(4n, \mathbb{C})$. This condition can be expressed in a simple way thanks to Eq. (2.6) of \cite{2}, i.e.
\[
\hat{V} \Sigma_e \hat{V}^T = \Sigma_e^x, \quad \Sigma_e = \sigma^x \otimes \mathbb{1}_{2n}.
\] (39)

It is simple to show that the transformation $\hat{V}$
\[
\hat{V} = \begin{bmatrix}
  U^{-1} & U^{-1}C \\
  0 & U^T
\end{bmatrix}
\] (40)
satisfies that condition. Moreover, using $\left( \begin{array}{c} a^\dagger \\ a \end{array} \right) \Sigma_e^x$ then in terms of the new diagonal creation and annihilation operators
\[
\begin{bmatrix}
  d \\
  d^x
\end{bmatrix} = \hat{V} \begin{bmatrix}
  a \\
  a^\dagger
\end{bmatrix},
\] (41)

it is simple to show that
\[
\mathcal{L} = -\frac{1}{2} \begin{bmatrix}
  \tilde{d}^x \\
  \tilde{d}
\end{bmatrix} \begin{bmatrix}
  x & 0 \\
  0 & -x
\end{bmatrix} \begin{bmatrix}
  \tilde{d} \\
  \tilde{d}^x
\end{bmatrix} - \frac{1}{2} \text{Tr} X,
\] (42)
i.e.,
\[
\mathcal{L} = -\sum_j x_j d^j \tilde{d}_j.
\] (43)

Note also that the transformation (41) can be written thanks to Eq (2.16) of \cite{2} into the form
\[
\begin{align*}
  d_j &= \mathcal{V} a_j \mathcal{V}^{-1}, \\
  d_j^x &= \mathcal{V} a_j^\dagger \mathcal{V}^{-1},
\end{align*}
\] (44)
where
\[
\mathcal{V} = \exp \left( -\frac{1}{2} a^\dagger C a^\dagger + a^\dagger (U - 1) a \right):
\] (45)

and $\exp(\cdot)$ refers to the normal ordering of the exponential. The stationary state of the Liouvillean, i.e. the state $\Omega$ such that $d_j|\Omega\rangle = 0$ can be written accordingly
\[
|\Omega\rangle = \mathcal{V} |0\rangle = e^{-\frac{i}{2} a^\dagger C a} |0\rangle.
\] (46)

Indeed, as $a_j|0\rangle = 0$, one has $d_j|\Omega\rangle = \mathcal{V} a_j \mathcal{V}^{-1} |\Omega\rangle = 0$. We now show that the transformation $Q$ defined in (11) and the direct relation
\[
\frac{1}{2} a^\dagger C a^\dagger \rho = \frac{1}{8} (w \cdot C \rho w + 2w \cdot C \rho w + \rho w \cdot C w)
\]
\[
= \frac{1}{4} \sum_k c_k (z_{2k-1} z_{2k} + z_{2k-1}^{-1} z_{2k} - z_{2k} z_{2k-1}^{-1} + z_{2k}^{-1} z_{2k-1})
\]
\[
=: \sum_k \mathcal{G}_k(\rho).
\] (47)

As
\[
\mathcal{G}_k(\mathbb{1}) = i c_k z_{2k-1} z_{2k}, \quad \mathcal{G}_k(z_{2k-1} z_{2k}) = 0,
\] (48)
it is clear that
\[
\Omega \propto e^{-\frac{i}{2} z^\dagger C z^\dagger} |0\rangle \propto \prod_k e^{-\mathcal{G}_k} \mathbb{1} = \prod_k (1 - i c_k z_{2k-1} z_{2k})
\] (49)
thus recovering Eq. (10).

The conditions for the existence and uniqueness of (49) are given in \cite{18}. We now study those conditions and express them in terms of the spectral gap. The correlation matrix matrix $C \in M_{2n}(\mathbb{C})$ is the matrix solution of Eq. (6).

To study the solution of that equation it is useful to consider the (non-canonical) “vectorising” isomorphism $\phi: M_{2n}(\mathbb{C}) \rightarrow (\mathbb{C}^{2n})^{\otimes 2}$, i.e., a Hilbert-space isomorphism, namely $\langle \phi(A), \phi(B) \rangle = \langle A, B \rangle = \text{Tr}(A^\dagger B)$. One can directly check that if $R_X(C) := XC$ and $L_X(C) := XC$ then $\phi(R_X(C)) = (\phi \circ R_X \circ \phi^{-1} \circ \phi)(C) = (1 \otimes X^T)\phi(C)$, and $\phi(L_X(C)) = (\phi \circ L_X \circ \phi^{-1} \circ \phi)(C) = (X \otimes 1)\phi(C)$. Applying $\phi$ to both sides of (6) one then obtains $\tilde{C} := \phi(C)$, $\tilde{Y} := \phi(Y)$
\[
(X \otimes 1 \otimes X)\tilde{C} = \tilde{X}\tilde{C} = \tilde{Y},
\] (50)

where $\tilde{C}, \tilde{Y} \in (\mathbb{C}^{2n})^{\otimes 2}$, $\tilde{X} \in \text{End}(\mathbb{C}^{2n})^{\otimes 2} \cong M_{4n^2}(\mathbb{C})$. There are three key operator formalisms in the formalism for obtaining the steady state:

1. The Liouvillean $\mathcal{L}: \text{End}(\mathbb{C}^{2n})^{\otimes n} \rightarrow \text{End}(\mathbb{C}^{2n})^{\otimes n}$, a $2^{2n} \times 2^{2n}$ matrix. Its complex spectrum, from (43), is given by
\[
\text{Sp}(\mathcal{L}) = \{-x_n := \sum_{j=1}^{2n} x_j n_j / n_j = 0, 1, x_j \in \text{Sp}(X)\}.
\]

Notice that $0 \in \text{Sp}(\mathcal{L})$ i.e., $\mathcal{L}$ is always non-invertible and that the steady state(e.g., our Gaussian one $\mathbf{n} = 0$) are in the kernel of $\mathcal{L}$. If this latter is one-dimensional (unique steady state) the gap of $\mathcal{L}$ can be defined as $\Delta_{\mathcal{L}} := \min_{x_n \neq 0} |x_n|$.

2. The map $X: \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$, a $2 \times 2$ real diagonalizable matrix. Its spectrum is $\{x_j \}_{j=1}^{2n} \subset \mathbb{C}$ and (because of reality) is invariant under complex conjugation. On physical grounds (stability) we must have $\Re x_j \geq 0, \forall j$. Indeed, the time-scale for convergence $\rho(t) \rightarrow \rho(\infty)$ is dictated by $\Delta^{-1}$ where $\Delta = \min_{x_n \neq 0} \Re x_n$. 
3. The map \( \hat{X} = X \otimes 1 + 1 \otimes X: \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \), a 4\( n^2 \times 4n^2 \) matrix. It spectrum is \( \{ x_i + x_j \}_{i,j=1}^{2n} \subset \mathbb{C} \) and the minimum (in modulus) is given by \( \Delta_{\hat{X}} := \min_{i,j} |x_i + x_j| \).

For the uniqueness of the steady state we must have \( \hat{X} \) invertible i.e., \( \Delta_{\hat{X}} > 0 \).

Proposition 1. If \( \Delta = \min_j 2 \Re(x_j) > 0 \) then

\[
\Delta_{\hat{X}} = \Delta_{\hat{X}} = \Delta. \quad (51)
\]

Proof. \( |x_n| = |\sum_{j=1}^{2n} x_j n_j| \geq |\Re(\sum_{j=1}^{2n} n_j x_j)| \). The first bound can be saturated by choosing the \( n_j \)'s in such a way that only a set \( P \) of complex conjugated pairs \( x^p \) of eigenvalues are present. In this case \( |\Re(\sum_{j=1}^{2n} n_j x_j)| = 2 \sum_{p \in P} |\Re x_p| \). Where we used the assumption \( \forall p \Re x_p \geq 0 \). Using again positivity of all the terms, this sum can be made as small as possible by choosing \( |P| = 1 \) and minimizing over \( p = 1, \ldots, n \). This shows that \( \Delta_{\hat{X}} = \min_n |x_n| = 2 \min(\Re x_p) \). It is clear now that a similar argument shows that \( \Delta_{\hat{X}} = \min_i |x_i + x_j|^{2n}_{j=1} \) is given by the same expression i.e. \( \Delta_{\hat{X}} = \Delta_{\hat{X}} \). Finally \( \Delta = 2 \min_n \Re x_n \equiv 2\Delta = 2 \min_p \Re x_p = \Delta_{\hat{X}} \). \( \Box \)

UPPER BOUNDS

In order to derive some bounds to the fidelity metric \( ds^2 \) let us express Eq. (1) in a convenient form thanks to the vectorization isomorphism. As \( \text{Ad}_C(X) = (L_C \circ R_C)(X) \) one has \( \phi \circ (L_C \circ R_C) \circ \phi^{-1} = C \otimes C_T = -C^\otimes 2 \) and Eq. (1) becomes

\[
ds^2 = ((1 + C^\otimes 2)^{-1}(d\tilde{C}), d\tilde{C}) = \|(1 + C^\otimes 2)^{-1/2}(d\tilde{C})\|^2, \quad (52)
\]

where \( d\tilde{C} = \phi(dC) \). Using the Cauchy-Schwarz inequality and the definition of operator norm one obtains

\[
ds^2 \leq \|(1 + C^\otimes 2)^{-1}(d\tilde{C})\| d\tilde{C} \| \leq P_C \| d\tilde{C} \|^2 \leq 2nP_C \| d\tilde{C} \|^2, \quad (53)
\]

where we have exploited the fact that, by construction, \( \|\tilde{A}\| = \|\phi(A)\| = \|A\|_2 \) and \( \|A\|_2 \leq \sqrt{2n} |A|_\infty \). Now \( \text{Sp}(C^\otimes 2) = \{ c_i c_j \mid c_i, c_j \in \text{Sp}(C) \} \) and, from \( C = -C_T \), the spectrum of \( C \) is invariant under \( c_i \rightarrow -c_i \), it follows that \( \|\tilde{1} + C^\otimes 2\|_\infty = (1 + \min_i c_i c_j)^{-1} = (1 - \max_i c_i^2)^{-1} = (1 - |\text{Sp}(C)|_\infty^{-1}) \). The bound (53) is not specific to dissipative quadratic Liouvillean. In order to connect Eq. (53) with the properties of the Liouvillean (43) we differentiate Eq. (50)

\[
d\tilde{C} = \hat{X}^{-1} d\hat{Y} - \hat{X}^{-1} d\hat{X} C. \quad (54)
\]

As \( d \equiv \sum_p d\lambda_p \partial_{\lambda_p} \) the above equation can be conveniently calculated via

\[
X (\partial_{\lambda_i} C) + (\partial_{\lambda_j} C) X^T = \partial_{\lambda_j} Y - (\partial_{\lambda_j} X) C - C (\partial_{\lambda_j} X^T), \quad (55)
\]

i.e. the matrices \( \partial_{\lambda_j} C \) entering in (35) can be obtained by solving a new Sylvester equation where the matrices \( X, Y, \partial_{\lambda_j} X, \partial_{\lambda_j} Y \) are given by the model. Taking norms in \( C^\otimes 2 \)

\[
\|d\tilde{C}\|^2 \leq \|X^{-1}\|_\infty (\|d\hat{Y}\|_2 + \|d\hat{X}\|_\infty \|C\|_2) \\
\leq \sqrt{2n\|X^{-1}\|_\infty (\|d\hat{Y}\|_\infty + \|d\hat{X}\|_\infty \|C\|_\infty) \\
\leq \sqrt{2n\|X^{-1}\|_\infty (\|d\hat{Y}\|_\infty + \|d\hat{X}\|_\infty) \quad (56)
\]

where, among other things, we used the inequality \( \|C\|_\infty \leq 1 \) which follows from the anastaz (10). In summary we have the following upper bound on the squared Hibert-Schmidt norm of \( dC \) in terms of the control parameters and their differentials i.e., \( X, dX \) and \( Y, dY \)

\[
\|d\tilde{C}\|^2 \leq 2n\|X^{-1}\|_\infty (\|d\hat{Y}\|_\infty + 2\|d\hat{X}\|_\infty^2) \quad (57)
\]

where we also used \( \|d\hat{X}\|_\infty = \|dX \otimes 1 + 1 \otimes dX\|_\infty \leq 2\|dX\|_\infty \). Pluggin the above equation in (53) one then obtains the bound (8).