Multivariate nonparametric regression by least squares Jacobi polynomials approximations

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Abstract — In this work, we introduce and study a random orthogonal projection based least squares estimator for the stable solution of a multivariate nonparametric regression (MNPR) problem. More precisely, given an integer \(d \geq 1\) corresponding to the dimension of the MNPR problem, a positive integer \(N \geq 1\) and a real parameter \(\alpha \geq -\frac{1}{2}\), we show that a fairly large class of \(d\)-variate regression functions are well and stably approximated by its random projection over the orthonormal set of tensor product \(d\)-variate Jacobi polynomials with parameters \((\alpha, \alpha)\).

The associated uni-variate Jacobi polynomials have degree at most \(N\) and their tensor products are orthonormal over \(U = [0, 1]^d\), with respect to the associated multi-variate Jacobi weights. In particular, if we consider \(n\) random sampling points \(X_i\) following the \(d\)-variate Beta distribution, with parameters \((\alpha + 1, \alpha + 1)\), then we give a relation involving \(n, N, \alpha\) to ensure that the resulting \((N+1)^d \times (N+1)^d\) random projection matrix is well conditioned. This is important in the sense that unlike most least squares based estimators, no extra regularization scheme is needed by our proposed estimator. Moreover, we provide squared integrated as well as \(L^2\)-risk errors of this estimator. Precise estimates of these errors are given in the case where the regression function belongs to an isotropic Sobolev space \(H^s(I^d)\), with \(s > \frac{d}{2}\). Also, to handle the general and practical case of an unknown distribution of the \(X_i\), we use Shepard’s scattered interpolation scheme in order to generate fairly precise approximations of the observed data at \(n\) i.i.d. sampling points \(X_i\) following a \(d\)-variate Beta distribution. Finally, we illustrate the performance of our proposed multivariate nonparametric estimator by some numerical simulations with synthetic as well as real data.

Keywords: Multivariate nonparametric regression, least squares, Jacobi polynomials, orthogonal projection, generalized polynomial chaos, condition number of a random matrix.

1 Introduction

For an integer \(d \geq 1\) and for sufficiently large positive integer \(n\), we consider the \(d\)-dimensional multivariate nonparametric regression (MNR) model given by

\[
Y_i = f(X_i) + \varepsilon_i, \quad 1 \leq i \leq n.
\]

Here, the \(Y_i\) are the \(n\) random responses and \(f : [0, 1]^d \to \mathbb{R}\) is the real valued \(d\)-variate regression function. The \(X_i = (X_{i,1}, \ldots, X_{i,d})\) are the \(n\) random sampling vectors following a given joint probability distribution over \([0, 1]^d\). The \(\varepsilon_i\) are the \(n\) i.i.d. centered random variables with variance \(E[\varepsilon_i^2] = \sigma^2\). In the sequel, we adopt the notation

\[
\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d, \quad \|\mathbf{k}\|_{\infty} = \max_i |k_i|, \quad [(0, N)]^d = \{0, 1, \ldots, N\}^d, \quad \gamma_{\alpha,d} = (\beta(\alpha + 1, \alpha + 1))^d.
\]

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Here, $\alpha \geq -\frac{1}{2}$ is a real number and $\beta(\cdot, \cdot)$ is the usual beta function. For an integer $k \geq 0$, we let $\tilde{P}_k^{(\alpha, \alpha)}(x)$ denote the normalized Jacobi polynomial of degree $k$ and parameters $(\alpha, \alpha)$. The $\tilde{F}_k^{(\alpha, \alpha)}(x)$, $k \geq 0$ and satisfy the orthonormality relation

$$
\int_I \tilde{P}_j^{(\alpha, \alpha)}(x) \tilde{P}_k^{(\alpha, \alpha)}(x) \omega_\alpha(x) \, dx = \delta_{j,k}, \quad I = [0,1], \quad \omega_\alpha(x) = x^\alpha (1-x)^\alpha.
$$

(3)

Under this notation, it is easy to check that the $d$–variate tensor product Jacobi polynomials family

$$
\Phi_\alpha^\alpha(x) = \prod_{j=1}^d \tilde{P}_{m_j}^{(\alpha, \alpha)}(x_j), \quad x = (x_1, \ldots, x_d), \quad m = (m_1, \ldots, m_d) \in \{0,1,\ldots\}^d = \mathbb{N}_0^d,
$$

(4)

forms an orthonormal basis of $L^2(I^d, \omega_\alpha)$, where $\omega_\alpha(x) = \prod_{j=1}^d \omega_\alpha(x_j)$. For a convenient positive integer $N$, our proposed scheme is based on the approximation of the $d$–variate regression function $f$ by its approximate projection over the finite dimensional Hilbert subspace $\mathcal{H}_N$ of $L^2(I^d, \omega_\alpha)$, given by

$$
\mathcal{H}_N = \text{Span} \{ \Phi_\alpha^\alpha(x), \, m \in [[0,N]]^d \}.
$$

(5)

We first assume that the $n$ i.i.d. random sampling vectors $X_i$ follow the $d$–variate Beta distribution with density function $h_{\alpha+1}(\cdot)$, given by

$$
h_{\alpha+1}(x) = \frac{1}{\beta(\alpha + 1, \alpha + 1)} \prod_{j=1}^d \omega_\alpha(x_j) 1_{[0,1]^d}(x),
$$

(6)

where $\beta(\cdot, \cdot)$ is the usual Beta function. Nonetheless, we will see how our proposed multivariate estimator $\hat{f}_N^\alpha$ can be adapted in order to handle the more general and practical case where the $X_i$ follow an unknown sampling distribution. By assuming that $Y_i$ is approximated by the function $f_N^\alpha(X_i)$, $1 \leq i \leq n$, using an approximate of $f$ with the help of (5), the estimator $\hat{f}_N^\alpha(\cdot)$ of $f$ is given by

$$
\hat{f}_N^\alpha(x) = \sum_{m \in [[0,N]]^d} \hat{C}_m \Phi_\alpha^\alpha(x), \quad x \in [0,1]^d.
$$

(7)

Here, the expansion coefficients vector $\hat{C} = (\hat{C}_m)_{m \in [[0,N]]^d}^T$ is computed in a stable way by the following formula,

$$
\hat{C} = (G_{d,N}^\alpha)^{-1} \cdot \left( (F_{d,N}^\alpha)^T \cdot \frac{(\beta(\alpha + 1, \alpha + 1))^{d/2}}{n^{1/2}} \right) \left[ Y_i^T \right]_{i=1}^n.
$$

(8)

The $n \times (N+1)^d$ random matrix $F_{d,N}^\alpha$ and the $(N+1)^d \times (N+1)^d$ positive definite random matrix $G_{d,N}^\alpha$ are given by

$$
G_{d,N}^\alpha = (F_{d,N}^\alpha)^T F_{d,N}^\alpha, \quad F_{d,N}^\alpha = \frac{(\beta(\alpha + 1, \alpha + 1))^{d/2}}{n^{1/2}} \left[ \Phi_\alpha^\alpha(X_i) \right]_{m \in [[0,N]]^d}^{1\leq i \leq n}.
$$

(9)

We should mention that in practice $N \ll n$. Moreover, the scheme given by (7)–(9) is nothing but the least squares of the over-determined system $\hat{f}_N^\alpha(X_i) = Y_i$, $1 \leq i \leq n$. In other words, the estimator $\hat{f}_N^\alpha$ is a solution of the minimization problem

$$
\hat{f}_N^\alpha = \arg \min_{f \in \mathcal{H}_N} \sum_{i=1}^n (f(X_i) - Y_i)^2.
$$

(10)
Note that our proposed estimator $\hat{f}_N^\alpha$ belongs to a larger class of multivariate orthogonal polynomials based least-squares estimators for multivariate nonparametric regression problems, see for example [8, 12, 22, 28]. Moreover, $\hat{f}_N^\alpha$ is closely related to the generalized Polynomial Chaos (gPC) or the Polynomial Chaos Expansions (PCE) class of nonparametric regression estimators in higher dimensions, see for example [8, 12, 10, 22, 23, 26, 28]. The gPC or the PCE techniques aim to approximate a $d$–variate function $f$ via $d$–orthogonal polynomials, where the orthogonality is defined by a probability measure on the input space $\mathcal{X} \subset \mathbb{R}^d$. This technique is widely used in the area of parametric uncertainty quantification, where one is faced with the challenge to approximate functions in high dimension $d$ and via its point evaluations. In the literature, there exist various techniques for gPC based multivariate regression. Among these techniques, we cite least-squares, weighted discrete least-squares, sparse grids approximations, $l_1$–minimization sparse approximation. For more details, the reader is refereed to [28] and the references therein. In general, three types of finite dimensional multivariate polynomials spaces are used by gPC type regression schemes. More precisely, for a positive integer $N$, these spaces are:

- $\mathcal{P}_{N,T}^\alpha = \text{Span}\{x_i \mid i \leq N \}$: the tensor product space of degree $N$.
- $\mathcal{P}_{N,T}^D = \text{Span}\{x_i \mid i \leq N \}$: the total degree space of degree $N$.

For $0 < q < 1$, $\mathcal{P}_{q,N}^{HC} = \text{Span}\{x_i \mid i \leq N \}$: the hyperbolic cross space of degree $N$.

Note that due to the blow-up of its cardinality with respect to the dimension $d$, the tensor product space $\mathcal{P}_{N,T}^\alpha$ is practical only for small values of the dimension $d$. For moderate large values of $d$, one has to apply dimension reductions techniques, such as sparsity and/or more optimal sampling techniques, see for example [5, 12, 16, 22, 28]. Also, a popular class of nonparametric regression estimators adapted for moderate values of the dimension $d$ are based on the combination of a model selection and smoothing (regularization) techniques, see for example [18] and the references therein. Perhaps, the stability problem is one of the most important issues related to the multivariate least-squares regression schemes. In the literature, only very few references have dealt with this issue so far, see for example [8, 28]. In particular, in [28] the authors have studied the stability of weighted least-squares with random sub-sampling of tensor Gauss points. Under our notation, they have shown that for any $\mu \geq 2$, the previous scheme is stable with probability at least $1 - \frac{2}{n^{\mu-1}}$, provided that

$$\frac{n}{\log n} \geq \frac{4\mu}{c} C_w C_b \text{dim}(\mathcal{P}_{\Lambda N}). \quad (11)$$

Here, $\mathcal{P}_{\Lambda N}$ is one of the three polynomial spaces, $\Lambda_N$ is the associated set of indices, $c$ is a uniform positive constant, $C_w$ is a constant depending on the Gauss weights and $C_b = \max_{i \leq \Lambda_N} \psi_i(\mathbf{z})$, where the $\psi_i$ are the different multivariate polynomials and the $\mathbf{z}$ are the different tensor product Gauss nodes. The quantities $C_b$ and $\text{dim}(\mathcal{P}_{\Lambda N})$ have the largest contributions to the previous lower bound of the stability condition. Precise estimates of these two quantities and consequently of the stability condition have been given in [28] for the special cases of the tensor product Legendre, as well as the Chebyshev polynomials. One of the main results of this work is to prove that under a condition relating the parameters $N, d$ and $\alpha \geq -\frac{1}{2}$, our least-squares polynomials regression, with random sampling following a multivariate beta distribution is stable. More precisely, if $\kappa_2(G_{d,N}^\alpha) = \lambda_{\max}(G_{d,N}^\alpha) / \lambda_{\min}(G_{d,N}^\alpha)$ is the 2–norm condition number of the
random matrix $G^2_{d,N}$, then for $\alpha \geq -\frac{1}{2}$ and for any $0 < \delta < 1$, we have
\[
P\left( \kappa_2(G^2_{d,N}) \leq \frac{1 + \delta}{1 - \delta} \right) \geq 1 - 2(N + 1)^d \exp \left( - \frac{n \delta^2}{3(B_\alpha(N + 1)^{2\alpha+2})^d} \right). \tag{12}
\]
Here, $B_\alpha \leq 1$ is a constant depending only on $\alpha$. From the previous estimate, it can be easily checked that our estimator is stable with high probability whenever $n$, the total number of sampling points satisfies
\[
n \geq \frac{3}{\delta^2} \log (2(N + 1)^d) \left( B_\alpha(N + 1)^{2\alpha+2} \right)^d. \tag{13}
\]
Note that for the case of the multivariate tensor product space based on Jacobi polynomials, the previous stability is a refinement of the more general purpose stability condition \[11\]. Moreover, from the inequality \[13\], the special value of $\alpha = -\frac{1}{2}$ is convenient in the sense that it ensures the stability of the estimator $\hat{f}_N^\alpha$ with the smaller values of $n$.

A second main result of this work is the following weighted $L^2(I^d)$—error of the estimator $\hat{f}_N^\alpha$. For this purpose, we use the fairly usual assumption on the i.i.d. random noises $(\varepsilon_i)$, that for a given probability value $0 < p_\epsilon \ll 1$, there exists a moderate positive constant $M_\epsilon$ so that
\[
P\left( |\varepsilon_i| > M_\epsilon \right) = p_\epsilon, \quad 0 < p_\epsilon \ll 1. \tag{14}
\]
Let $\pi_N f$ be the orthogonal projection of $f$ over $\mathcal{H}_N$ and let $\| \cdot \|_\alpha$ be the usual 2–norm of $L^2(I^d, \omega_\alpha)$. Then, for any $0 < \delta < \frac{1}{\|\pi_N(f)\|_\alpha}$, we have with high probability depending on $\delta$,
\[
\|f - \hat{f}_N^\alpha\|_\alpha \leq \|f - \pi_N(f)\|_\alpha + \sqrt{\kappa_2(G^2_{d,N})} \left( \|f - \pi_N(f)\|_\alpha^2 + \sigma^2 + \delta \right)^{1/2} \frac{1}{\sqrt{1 - \frac{\delta}{\|\pi_N(f)\|_\alpha}}}, \tag{15}
\]
Precise estimates for the errors $\|f - \pi_N(f)\|_\alpha$ and $\|f - \pi_N(f)\|_\infty$ will be given for those regression function $f$ belonging to an isotropic Sobolev space $H^s(I^d)$, with $s > d(\alpha + \frac{3}{2})$. More importantly, for $0 < \delta < 1$, we give the following $L^2$—risk error of $\hat{f}_N$ (see equation \[54\]), a truncated version of the estimator $\hat{f}_N^\alpha$,
\[
\mathbb{E}\left[ \|f - \hat{f}_N\|^2_\alpha \right] \leq \frac{(N + 1)^d}{n(1 - \delta)^2} \left( \sigma^2 + \left( \eta_\alpha^2(N + 1)^{2\alpha+1} \right)^d \right) + \|f - \pi_N f\|^2_\alpha 
+ 4M_\epsilon^2(N + 1)^d (\beta(\alpha + 1, \alpha + 1))^d \exp \left( - \frac{n \delta^2}{2(B_\alpha(N + 1)^{2\alpha+2})^d} \right), \tag{16}
\]
where $\eta_\alpha, B_\alpha$ are constants depending only on $\alpha$ and $\|f(x)\| \leq M_f$, a.e. $x \in I^d$.

It is interesting to note that the curse of dimension does not only affect the computational load that grows drastically with the dimension $d$, but also has a negative effect on the convergence rate for the multivariate regression estimators. In \[14\], the authors have given a detailed study of this issue. In particular, for the case of a multivariate nonparametric problem with the $n$ random sampling vectors $X_i$ belonging to a compact subset $\chi$ of $\mathbb{R}^d$, and for $L > 0$ and a positive integer $\alpha \geq 1$, they have considered the Hölder class of $d$—variante functions, defined by
\[
H_d(\alpha, L) = \left\{ g : |D^s g(x) - D^s g(y)| \leq L \|x - y\|, \quad x, y \in \chi, \quad \|s\|_1 \leq \alpha - 1 \right\}.
\]
Here, $D^s g$ denotes the partial derivatives of $g$ of order $\|s\|_1$ and associated with $s \in \mathbb{N}^d_0$. Then, it has been shown in \[14\], see also \[21\], that for most of the usual nonparametric regression estimator $\hat{g}$, the min-max
convergence rate over the functional space $H_d(\alpha, L)$ is given by

$$\inf_{\tilde{g}} \sup_{g_0 \in H_d(\alpha, L)} \mathbb{E}\left( \| \tilde{g} - g_0 \|^2 \right) \geq n^{-2\alpha/(2\alpha + d)}.$$ 

That is the optimal convergence rate $O\left(n^{-2\alpha/(2\alpha + d)}\right)$ decays in a significant manner as the dimension $d$ grows.

In this work, we prove that our proposed estimator has a similar optimal convergence rate under the condition that the regression function belongs to an isotropic Sobolev space $H^s(I^d)$, for some $s > 0$. More precisely, we prove that in this case, the $L_2$-risk of the proposed estimator is of order $O\left(n^{-2s/(2s + d)}\right)$.

This work is organized as follows. In section 2, we give some mathematical preliminaries that will be frequently used to prove the different results of this work. In section 3, we first prove the stability property of our proposed nonparametric regression estimator. Then, we study the convergence rate of our estimator. Section 4 is devoted to various numerical simulations (on synthetic data as well as real data) that illustrate the different results of this work. Finally, in section 5, we give some concluding remarks concerning this work.

## 2 Mathematical preliminaries

In this paragraph, we provide the reader with some mathematical preliminaries that are frequently used to describe and prove the different results of this work. We first give the following fairly known definitions and properties related to the uni-variate Jacobi polynomials.

It is well known, see for example [2], that for any two real parameters $\alpha, \beta > -1$, the classical Jacobi polynomials $P_k$ are defined for $x \in [-1, 1]$, by the following Rodrigues formula,

$$P_k^{(\alpha, \beta)}(x) = \frac{(-1)^k}{2^k k!} \cdot \frac{1}{\omega_{\alpha, \beta}(x)} \cdot \frac{d^k}{dx^k} \left( \omega_{\alpha, \beta}(x)(1-x^2)^k \right), \quad \omega_{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta,$$

with

$$P_k^{(\alpha, \beta)}(1) = \left( k + \max(\alpha, \beta) \right) / k \Gamma(1 + \max(\alpha, \beta)).$$

Here, $\Gamma(\cdot)$ denotes the usual Gamma function. More importantly and as others families of classical orthogonal polynomials, the Jacobi polynomials $P_k$ are given by the following practical three term recursion formula

$$P_{k+1}^{(\alpha, \beta)}(x) = (a_k x + b_k) P_k^{(\alpha, \beta)}(x) - c_k P_{k-1}^{(\alpha, \beta)}(x), \quad x \in [-1, 1],$$

with $P_0^{(\alpha, \beta)}(x) = 1$, $P_1^{(\alpha, \beta)}(x) = \frac{1}{2}(\alpha + \beta + 2)x + \frac{1}{2}(\alpha + \beta)$. Here,

$$a_k = \frac{(2k + \alpha + \beta + 1)(2k + \alpha + \beta + 2)}{(k+1)(k + \alpha + \beta + 1)}, \quad b_k = \frac{(\alpha^2 - \beta^2)(2k + \alpha + \beta + 1)}{2(k+1)(k + \alpha + \beta + 1)(2k + \alpha + \beta)},$$

$$c_k = \frac{(k + \alpha)(k + \beta)(2k + \alpha + \beta + 2)}{(k+1)(k + \alpha + \beta + 1)(2k + \alpha + \beta)}.$$

These Jacobi polynomials satisfy the following orthogonal relation

$$\int_{-1}^{1} P_k^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) \omega_{\alpha, \beta}(y) \, dy = h_k^{\alpha, \beta} \delta_{k,m}, \quad h_k^{\alpha, \beta} = \frac{2^{\alpha + \beta + 1} \Gamma(k + \alpha + 1) \Gamma(k + \beta + 1)}{k!(2k + \alpha + \beta + 1) \Gamma(k + \alpha + \beta + 1)},$$

$$\int_{-1}^{1} P_k^{(\alpha, \beta)}(x) \omega_{\alpha, \beta}(y) \, dy = \frac{1}{2} \frac{\Gamma(k + \alpha + 1) \Gamma(k + \beta + 1)}{\Gamma(k + \alpha + \beta + 1)}.$$
where \( \delta_{k,m} \) is the usual Kronecker delta function. Note that the set \( \{ \frac{1}{\sqrt{h_k}} P_k^{(\alpha,\beta)}(x), k \geq 0 \} \) is an orthonormal basis of the weighted \( L^2([-1,1], \omega_{\alpha,\beta}) \), which is a Hilbert space associated with the inner product 
\[ \langle \cdot, \cdot \rangle_\omega = \int_{-1}^{1} f(t)g(t) \omega_{\alpha,\beta}(t) \, dt. \]

In the sequel, we will only consider the case of \( \beta = \alpha \geq -\frac{1}{2}. \) We let \( \tilde{P}_k^{(\alpha,\alpha)} \) denote the normalized Jacobi polynomial over \( I = [0,1] \) of degree \( k \) and given by

\[ \tilde{P}_k^{(\alpha,\alpha)}(x) = \frac{2^{\alpha+1/2}}{\sqrt{h_k}} P_k^{(\alpha,\alpha)}(2x - 1), \quad x \in I. \]  

The following useful upper bounds for the normalized Jacobi polynomials \( \tilde{P}_k^{(\alpha,\alpha)} \), for \( k \geq 2 \) is borrowed from [4]

\[ \max_{x \in [-1,1]} |\tilde{P}_k^{(\alpha,\alpha)}(x)| \leq \eta_\alpha k^\alpha \sqrt{k + \alpha + \frac{1}{2}}, \quad \forall k \geq 2, \]

where

\[ \eta_\alpha = \frac{\sqrt{2}}{\Gamma(\alpha + 1)} \exp \left( \frac{\max(0,\alpha)}{6} + \frac{\alpha^2}{4} \right). \]

Next, we briefly describe the original as well as modified Shepard’s algorithms for the interpolation of multivariate scattered data. This type of interpolation is needed in order to get convenient interpolations of the observations at some appropriate random sampling points. The original Shepard’s interpolation of multivariate scattered data can be described as follows, see for example [11]. Let \( \mu > 0 \) be a positive real number and let \( X = \{x_1, \ldots, x_n\} \) be a set of \( n \) distinct points of a domain \( D \subset \mathbb{R}^d, d \geq 1 \), then for the \( n \) associated real valued function evaluations \( f_i = f(x_i) \), the Shepard’s interpolation operator is given by

\[ S_\mu(f)(x) = \sum_{i=1}^{n} A_{\mu,i}(x)f_i, \quad A_{\mu,i}(x) = \left( \frac{d(x, x_i)}{\sum_{i=1}^{n} d(x, x_i)} \right)^{-\mu}, \]

where \( d(\cdot, \cdot) \) is a distance on \( \mathbb{R}^d \). The previous interpolation has the drawback that the basis functions \( A_{\mu,i} \) have significant values at those points \( x_i \), which are far from the considered interpolation point \( x \). To overcome this drawback, see for example [11], the \( A_{\mu,i} \) are substituted by compactly supported basis functions \( W_{\mu,i} \), so that for a given radius of influence \( R > 0 \), the modified Shepard’s algorithm is given by

\[ S_\mu^R(f)(x) = \sum_{i=1}^{n} W_{\mu,i}(x)f_i, \quad W_{\mu,i}(x) = \left( \frac{1}{\sum_{i=1}^{n} \frac{1}{d(x, x_i)}} \right)^{\frac{\mu}{R^+}} \]

where, \( (t)^+ = \max(t, 0) \). Also, the original Shepard’s algorithm has been further developed by considering a combined Shepard-Multivariate Taylor interpolation polynomial \( S_{\mu,r} \), where for an integer \( r \geq 0 \),

\[ S_{\mu,r}(f)(x) = \sum_{i=1}^{n} A_{\mu,i}(x)T_{r,x_i}(f)(x), \quad T_{r,x_i}(f)(x) = \sum_{\nu=0}^{r} \frac{D_\nu f(x_i)}{\nu!}(x - x_i)^\nu. \]
It has been showed that
\[
\|S_{\mu,r}(f) - f\| = \begin{cases} O(h^{r+1}) & \text{if } \mu - d > r + 1 \\ O(h^{\mu-d} \log h) & \text{if } \mu - d = r + 1. \end{cases} \tag{27}
\]
Here, \(h\) is the mesh step size that is the largest distance between the neighboring points \(x_i\). For more details, the reader is referred to [11] or to [19]. In particular, the last reference is a comprehensive review of several other types of scattered multivariate interpolation techniques.

Next, we give the following highly useful matrix Chernoff theorem, see for example [24], p.10, that provides us with upper and lower bounds for the smallest and largest eigenvalues of a sum of random Hermitian matrices.

**Matrix Chernoff Theorem:** Consider a finite sequence of \(n\) independent \(D \times D\) random Hermitian matrices \(\{Z_k\}\). Assume that for some \(B > 0\), we have
\[
0 \preceq Z_k \preceq BI_D.
\]
Let
\[
A = \sum_{k=1}^n Z_k, \quad \mu_{\min} = \lambda_{\min}(\mathbb{E}(A)), \quad \mu_{\max} = \lambda_{\max}(\mathbb{E}(A)).
\]
Then, for any \(\delta \in (0,1]\), we have
\[
\mathbb{P}\left(\lambda_{\min}(A) \leq (1-\delta)\mu_{\min}\right) \leq D \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu_{\min}/B}, \quad \mathbb{P}\left(\lambda_{\max}(A) \geq (1+\delta)\mu_{\max}\right) \leq D \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu_{\max}/B}. \tag{28}
\]
As a consequence of the previous two inequalities, one can check, see for example [24], p.12, that for \(\delta \in (0,1]\), we have
\[
\mathbb{P}\left(\lambda_{\min}(A) \leq (1-\delta)\mu_{\min}\right) \leq D \exp\left(-\frac{\delta^2\mu_{\min}}{2B}\right), \quad \mathbb{P}\left(\lambda_{\max}(A) \geq (1+\delta)\mu_{\max}\right) \leq D \exp\left(-\frac{\delta^2\mu_{\max}}{3B}\right). \tag{29}
\]
Finally, the following Gershgorin circle theorem, see for example [15], will be needed to prove one of the main theoretical results of this work which is an estimate for an upper bound of the random projection matrix.

**Gershgorin circle Theorem:** Let \(A = [a_{ij}]\) be a complex \(n \times n\) matrix. For \(1 \leq i \leq n\), let \(R_i = \sum_{j \neq i} |a_{ij}|\).

Then, every eigenvalue of \(A\) lies within at least one of the discs \(D(a_{ii}, R_i)\).

### 3 Stability and convergence rates of the estimator

In this paragraph, we first describe our orthogonal projection based scheme for solving the MNPR problem [11]. Then, we prove the inequality [12]. That is with high probability, the random projection matrix \(G_{d,N}\), given by [9] is well conditioned. Our random orthogonal projection based scheme is described as follows. We first substitute in (7), \(x\) by \(X_i\), \(i = 1, \ldots, n\), where the \(X_i\) are i.i.d random samples following the \(d\)-dimensional Beta probability distribution \(h_{\alpha+1}(\cdot)\), given by [6]. Note that the beta distribution is widely used in the framework of various models from mathematical statistics, see for example [13] [16] [28]. Then, after rescaling by the factor \(n(\alpha(\alpha + 1) + 1)^{-d/2}\), one obtains the following overdetermined system of \(n\) equations in the \((N+1)^d\) unknown expansion coefficients vectors \(\widehat{C}_m\),

\[
\left(\frac{\beta(\alpha + 1)}{n^{1/2}}\right)^{d/2} Y_i = F^\alpha_{d,N} \cdot \widehat{C}, \quad \widehat{C} = (\widehat{C}_m)_{m \in [0,N]^d}^T \tag{30}
\]
where the random matrix $F_{d,N}^\alpha$ is given by (8). Note that since the basis functions $\Phi_m^\alpha(\cdot), m \in [0,N]d$ form an orthonormal basis of the finite dimensional subspace $\mathcal{H}_N$ of $E = L^2(I^d,\omega_\alpha(x)dx)$, then for $f(\cdot) \in E$, its orthogonal projection over $\mathcal{H}_N$ denoted by $\pi_N f$ is uniquely defined. The expansion coefficients of $\pi_N f$ are given by

$$C_m = < f, \Phi_m^\alpha > = \int_{I^d} f(x)\Phi_m^\alpha(x)\omega_\alpha(x)dx, \quad m \in [0,N]d.$$  

That is under the hypothesis of a noise free regression model with a piecewise continuous regression function, it can be easily checked that for sufficiently large value of the total sampling points $n$, the least square norm solution of (30) doest not depend on the considered sampling set $\{X_i, i \in [1,n_d] = [[1,n]]\}$. Multiplying the system (30) from both sides by $(F_{d,N}^\alpha)^T$. As we show in the sequel, with high probability, the $(N+1)^d$-dimensional random matrix $G_{d,N}^\alpha = (F_{d,N}^\alpha)^T \cdot F_{d,N}^\alpha$ is positive definite and hence invertible. Consequently, by applying $(G_{d,N}^\alpha)^{-1}$ to the previous intermediate system (30), one gets the reduced Cramer system (8).

The following theorem is one of the main results of this work. It gives us with high probability, a relatively small upper bound for $\kappa_2(G_{d,N}^\alpha)$, the 2-norm condition number of the positive definite random projection matrix, associated to the Jacobi system. This theorem allows us to use the inverse of $G_{d,N}^\alpha$ in computing the estimator $\delta_x(\cdot)$, given by (7) and (8).

**Theorem 1.** Under the previous notation and assumption, for any $\alpha \geq -1/2$ and any $0 < \delta < 1$, we have

$$\mathbb{P}\left( \kappa_2(G_{d,N}^\alpha) \leq \frac{1 + \delta}{1 - \delta} \geq 1 - 2(N+1)^d \exp\left(-\delta^2 \frac{n}{3(B_\alpha (N+1)^{2\alpha+2})^\alpha}\right) \right).$$  

Here, $B_\alpha \leq 1$ is a constant depending only on $\alpha$.

**Proof:** To alleviate notation, we consider the correspondence $g : [0,N]^d \rightarrow [1,(N+1)^d]$, given by

$$g(m_1, \ldots, m_d) = 1 + \sum_{k=1}^d m_k (N+1)^{k-1}, \quad m = (m_1, \ldots, m_d) \in [0,N]^d.$$  

Also, we shall use the notation

$$\Psi_m^\alpha(x) = \Phi_{g^{-1}(m)}^\alpha(x), \quad m \in [1,(N+1)^d].$$

So that we have

$$G_{d,N}^\alpha = \left[ \frac{(\beta(\alpha + 1, \alpha + 1))^d}{n} \sum_{i=1}^n \Psi_{m_1}^\alpha(X_i)\Psi_{m_2}^\alpha(X_i) \right]_{1 \leq m_1 \leq (N+1)^d, \atop 1 \leq m_2 \leq (N+1)^d}.$$  

It is easy to see that for $i = 1, \ldots, n$ and $1 \leq j \leq d$, we have

$$\mathbb{E}\left( \tilde{P}_k^{(\alpha,\alpha)}(X_{i,j})\tilde{P}_l^{(\alpha,\alpha)}(X_{i,j}) \right) = \int_I \tilde{P}_k^{(\alpha,\alpha)}(x)\tilde{P}_l^{(\alpha,\alpha)}(x) \frac{\omega_\alpha(x)}{\beta(\alpha + 1, \alpha + 1)} dx = \frac{1}{\beta(\alpha + 1, \alpha + 1)} \delta_{k,l}.$$  

Consequently, we have

$$\mathbb{E}\left( G_{d,N}^\alpha \right) = I_{(N+1)^d},$$

the $(N+1)^d$-dimensional identity matrix. On the other hand, we write $G_{d,N}^\alpha$ as follows

$$G_{d,N}^\alpha = \sum_{i=1}^n H_i^\alpha, \quad H_i^\alpha = \left[ \frac{(\beta(\alpha + 1, \alpha + 1))^d}{n} \Psi_{m_1}^\alpha(X_i)\Psi_{m_2}^\alpha(X_i) \right]_{1 \leq m_1 \leq (N+1)^d, \atop 1 \leq m_2 \leq (N+1)^d}.$$
We check that there exists a constant $B$ such that
\[ 0 < H_i^\alpha \preceq BI_{(N+1)^d}, \quad i = 1, \ldots, n, \tag{34} \]
that is $H_i^\alpha$ and $BI_{(N+1)^d}$ are positive semi-definite. Since $H_i^\alpha = A_i^T A_i$, where $A_i$ is the $(N + 1)^d \times 1$ matrix given by $A_i = \frac{(\beta(\alpha + 1, \alpha + 1)/2)}{n^{1/2}} \left[ \Psi_i^\alpha(X_i) \cdots \Psi_i^{(N+1)^d}(X_i) \right]$, then its different eigenvalues $\lambda_k(H_i^\alpha)$ are non-negative, that is $H_i^\alpha \succeq 0$. To prove the second inequality of (34), we note that from Gershgorin circle theorem, we have
\[ \lambda_{\max}(H_i^\alpha) \leq \left( \frac{\beta(\alpha + 1, \alpha + 1)/2}{n} \right)^{d} \max_{1 \leq m_1 \leq (N+1)^d} \left( |\Psi_{m_1}(X_i)|^2 + \sum_{m \neq m_1, 1 \leq m \leq (N+1)^d} |\Psi_{m}(X_i)\Psi_{m_1}(X_i)| \right). \tag{35} \]

On the other hand, from the upper bound for the normalized Jacobi polynomials $\tilde{P}_k^{(\alpha, \alpha)}(\cdot)$, given by (22), we have
\[ \max_{1 \leq m \leq (N+1)^d} |\Psi_{m}(X_i)| \leq \|\Psi_{m}(X_i)\|_{\infty} = \prod_{k=1}^{d} \|\tilde{P}_{N+1}^{(\alpha, \alpha)}\|_{\infty} \leq \eta_\alpha^{d} \left( N^\alpha \sqrt{N + \alpha + \frac{1}{2}} \right)^d \leq \left( \eta_\alpha (N + 1)^{\alpha/2} \right)^d \]
That is for $i = 1, \ldots, n$, we have
\[ \max_{m} |\Psi_{m}(X_i)|^2 + \sum_{m \neq m_1} |\Psi_{m}(X_i)\Psi_{m_1}(X_i)| \leq (N + 1)^d \left( \eta_\alpha^d (N + 1)^{2\alpha + 2} \right)^d = \left( \eta_\alpha^d (N + 1)^{2\alpha + 2} \right)^d \tag{36} \]
By combining (35) and (36), one gets the second inequality of (34) with
\[ B = \left( \frac{\beta(\alpha + 1, \alpha + 1)/2}{n} \right)^{d} \left( \eta_\alpha (N + 1)^{2\alpha + 2} \right)^d. \tag{37} \]
Next, we apply the following estimate from (23), for the minimum and the maximum eigenvalue of a sum of positive semi definite random matrices. If $A = \sum_{k=1}^{n} Z_k$, where the $Z_k$ are $D \times D$ random matrices satisfying $0 \ll Z_k \ll BI_n$, for some positive constant $B$ and if $\mu_{\min} = \lambda_{\min}(E(A))$, $\mu_{\max} = \lambda_{\max}(E(A))$, then from (29), we have
\[ \mathbb{P}\left( \lambda_{\min}(A) \geq (1 - \delta)\mu_{\min} \right) \geq 1 - D \exp \left( - \frac{\delta^2 \mu_{\min}}{2B} \right), \quad \forall \delta \in (0, 1] \tag{38} \]
and
\[ \mathbb{P}\left( \lambda_{\max}(A) \leq (1 + \delta)\mu_{\max} \right) \geq 1 - D \exp \left( - \frac{\delta^2 \mu_{\max}}{3B} \right), \quad \forall \delta > 0. \tag{39} \]
In the special case where $Z_k = H_k^\alpha$, are the $n$ previous $(N + 1)^d \times (N + 1)^d$ positive semi-definite random matrices, we have already shown that $\sum_{k=1}^{n} H_k^\alpha = I_{(N+1)^d}$. Consequently, we have $\mu_{\min} = \mu_{\max} = 1$. Moreover, in this case, the constant $B$ is given by (37). Hence, by applying (38) and (39) with $0 < \delta < 1$, $D = (N + 1)^d$, $B$ as given by (37) and by combining the obtained both inequalities, one gets the desired estimate (32).
Remark 1. Note that instead of using Chernoff theorem and its consequence to get with high probability an estimate for the deviations of the largest and smallest eigenvalues of the random positive definite matrix $\textbf{A}$ from those of $\text{E}(\textbf{A})$, one can use the McDiarmid’s concentration inequality, together with the techniques developed in [10].

Remark 2. It is easy to see that for $0 < \delta < 1$, the estimate (32) implies that our estimator is stable with high probability whenever the number of sampling points $n$ satisfies

$$n \geq \frac{3}{\delta^2} \log \left(2(N + 1)^d \right) \left( B_\alpha(N + 1)^{2\alpha + 2} \right)^d. \quad (40)$$

Moreover, from the previous inequality, one concludes that the special value of $\alpha = -\frac{1}{2}$ (corresponding to the tensor product of Chebyshev polynomials) is a convenient choice. In fact, this choice ensures the stability of the estimator $\hat{f}_N^\alpha$ with the smaller values of $n$.

A second main result of this work is the following theorem that provides us with a weighted $L^2$-error of our estimator $\hat{f}_N^\alpha$.

Theorem 2. For fixed real number $\alpha \geq -\frac{1}{2}$ and a positive integer $N \geq 1$, let $f \in L^2(I^d, \omega_\alpha)$ be as given by (1). We assume that $\| \pi_N f \|_\alpha > 0$ and

$$\text{ess sup}_{x \in I^d} |\pi_N f(x)| \leq M_N, \quad \text{ess sup}_{x \in I^d} |f(x)| = \| f \|_\infty < +\infty, \quad \mathbb{P}(|\varepsilon_i| > M_\varepsilon) = p_\varepsilon, \quad 0 < p_\varepsilon \ll 1. \quad (41)$$

Then, under the hypotheses of Theorem 1, for any $0 < \delta < \| \pi_N(f) \|_\alpha$, we have with probability at least

$$(1 - p_\varepsilon)^n - \exp \left(\frac{-2n\delta^2}{\gamma_{\alpha,d} M_N^4}\right) - \exp \left(\frac{-n\delta^2}{\gamma_{\alpha,d} \max(\| f - \pi_N f \|^2, M_\varepsilon^2)}\right),$$

$$\| f - \hat{f}_N^\alpha \|_\alpha \leq \| f - \pi_N(f) \|_\alpha + \sqrt{2\gamma_2(G_{d,N}^\alpha)} \left( \| f - \pi_N(f) \|^2_\infty + \sigma^2 + \delta \right)^{1/2} \frac{1}{\sqrt{1 - \| \pi_N f \|_\alpha}}, \quad (42)$$

where $\gamma_{\alpha,d} = (\beta(\alpha + 1, \alpha + 1))^d$.

Proof: Let $f \in L^2(I^d, \omega_\alpha)$, then for an integer $N \geq 1$, let $\pi_N(f)$ denote the orthogonal projection of $f$ over $\mathcal{H}_N = \text{Span}\{\Psi_m, m = 0, \ldots, N_d\}$. That is

$$\pi_N(f)(x) = \sum_{m=0}^{N_d} < f, \Psi_m^\alpha > \Psi_m^\alpha(x), \quad < f, \Psi_m^\alpha > = \int_{I^d} f(x) \Psi_m^\alpha(x) \omega_\alpha(x) dx. \quad (43)$$

Let $c_m(f) = < f, \Psi_m^\alpha >$ and let $C_N = [c_m(f)]_{0 \leq m \leq N_d}^T$. From the uniqueness of the expansion coefficients of $f$ with respect to the orthonormal basis $\{\Psi_m^\alpha, m \in \mathbb{N}_0\}$ and by substituting $x$ with the sampling points $X_i$, one concludes that the finite length expansion coefficients vector $C_N$ satisfies the identity

$$F_{d,N}^\alpha C_N = \frac{\sqrt{\gamma_{\alpha,d}}}{\sqrt{n}} \left[ \pi_N f(X_1), \ldots, \pi_N f(X_n) \right]^T, \quad (44)$$

where the matrix $F_{d,N}^\alpha$ is as given by (9). On the other hand, our multivariate nonparametric estimator $\hat{f}_N^\alpha$ is given by

$$F_{d,N}^\alpha \hat{C}_N = \frac{\sqrt{\gamma_{\alpha,d}}}{\sqrt{n}} \left[ Y_i \right]_{1 \leq i \leq n} = \frac{\sqrt{\gamma_{\alpha,d}}}{\sqrt{n}} \left[ f(X_i) + \varepsilon_i \right]_{1 \leq i \leq n}. \quad (45)$$
Hence, by comparing (44) and (45), one concludes that the least square norm solution of system (45) can be viewed as a perturbation of the least square solution of system (44). More precisely, we have

$$F_{d,N}^\alpha \hat{C}_N = F_{d,N}^\alpha C_N + F_{d,N}^\alpha (\hat{C}_N - C_N),$$

where

$$F_{d,N}^\alpha (\hat{C}_N - C_N) = \sqrt{\frac{\gamma_{\alpha,d}}{n}} \left[ (f - \pi_N(f))(X_i) + \varepsilon_i \right]_{1 \leq i \leq n}.$$  

From the classical perturbation theory of least square norm solution of perturbed overdetermined system of linear equations, see for example [17], one has

$$\| C_N - \hat{C}_N \|^2_{F_{d,N}} \leq \kappa_2(G_{d,N}^\alpha) \frac{\| F_{d,N}^\alpha (C_N - \hat{C}_N) \|^2_{F_{d,N}}}{\| F_{d,N}^\alpha C_N \|^2_{F_{d,N}}}.$$  

That is

$$\| C_N - \hat{C}_N \|^2_{F_{d,N}} \leq \kappa_2(G_{d,N}^\alpha) \| F_{d,N}^\alpha (C_N - \hat{C}_N) \|^2_{F_{d,N}} \| F_{d,N}^\alpha C_N \|^2_{F_{d,N}}.$$  

(46)

Next, since $E[\gamma_{\alpha,d}(\pi_N f(X_i))^2] = \| \pi_N f \|^2_\alpha$, then we have

$$E\left[ \frac{\gamma_{\alpha,d}}{n} \sum_{i=1}^n (\pi_N f(X_i))^2 \right] = \| \pi_N f \|^2_\alpha = \| C_N \|^2_{F_{d,N}}.$$  

The last equality is a consequence of Parseval’s equality. Assume that $\text{ess sup}_{x \in \mathcal{D}} |\pi_N f(x)| \leq M_N$, then by using Hoeffding’s inequality, for any $\delta > 0$, we have

$$\mathbb{P} \left( \frac{\gamma_{\alpha,d}}{n} \sum_{i=1}^n (\pi_N f(X_i))^2 - E \left[ \frac{\gamma_{\alpha,d}}{n} \sum_{i=1}^n (\pi_N f(X_i))^2 \right] \geq \delta \right) \leq \exp \left( \frac{-2n\delta^2}{\gamma_{\alpha,d}^2 M_N^2} \right).$$  

That is

$$\| F_{d,N}^\alpha C_N \|^2_{F_{d,N}} \geq \| C_N \|^2_{F_{d,N}} - \delta,$$  

(47)

with probability at least $1 - \exp \left( \frac{-2n\delta^2}{\gamma_{\alpha,d}^2 M_N^2} \right)$. On the other hand, we have

$$\| F_{d,N}^\alpha (\hat{C}_N - C_N) \|^2_{F_{d,N}} = \frac{\gamma_{\alpha,d}}{n} \sum_{i=1}^n \left( (f - \pi_N f)(X_i) + \varepsilon_i \right)^2.$$  

Since $E[\varepsilon_i] = 0$ and since the $X_i$ and $\varepsilon_i$ are independent, then it is easy to see that

$$E\left[ \| F_{d,N}^\alpha (\hat{C}_N - C_N) \|^2_{F_{d,N}} \right] = \| f - \pi_N f \|^2_\alpha + \sigma^2.$$  

(48)

Next, consider the tensor product set $\mathcal{D} = \prod_{i=1}^n \mathcal{F}^d \times \prod_{i=1}^n \mathcal{A}$, where $\mathcal{A}$ is the support of the $\varepsilon_i$ which is a subset of $\mathbb{R}$, that might be unbounded. Let $h$ be the real valued function $h$ defined on $\mathcal{D}$ by

$$h_{\varepsilon}(x_1, \ldots, x_n) = \frac{\gamma_{\alpha,d}}{n} \sum_{i=1}^n \left( (f - \pi_N f)(x_i) + \varepsilon_i \right)^2.$$  

Note that if $(X_1, \ldots, X_n, \varepsilon_1, \ldots, \varepsilon_n), (X'_1, \ldots, X'_n, \varepsilon'_1, \ldots, \varepsilon'_n) \in \mathcal{D}$ differ only in the $k$–th coordinate, then the following bounded difference condition holds with high probability $1 - p_{\varepsilon}$,

$$\left| h_{\varepsilon}(X_1, \ldots, X_n) - h_{\varepsilon'}(X'_1, \ldots, X'_n) \right| \leq \frac{\gamma_{\alpha,d}}{n} \max \left( \| f - \pi_N f \|^2_\infty, M_{\varepsilon}^2 \right).$$  

(49)
From [9], for any $\delta > 0$ and due to the tensor product structure of the set $\mathcal{D}_1 = \prod_{i=1}^{n} I^d \times \prod_{i=1}^{n} [-M_\varepsilon, M_\varepsilon]$, one has

$$P(\hat{h}_\varepsilon(X_1, \ldots, X_n) - (\|f - \pi_N f\|_\alpha + \sigma^2) \geq \delta, (X_1, \ldots, X_n, \varepsilon_1, \ldots, \varepsilon_n) \notin \mathcal{D}_1) \leq 1 - (1 - p_\varepsilon)^n. \tag{50}$$

Moreover, on $\mathcal{D}_1$, McDiarmid’s inequality gives us

$$P(\hat{h}_\varepsilon(X_1, \ldots, X_n) - (\|f - \pi_N f\|_\alpha + \sigma^2) \geq \delta, (X_1, \ldots, X_n, \varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{D}_1) \leq \exp \left( \frac{-n\delta^2}{\gamma^2_{\alpha,d} \max (\|f - \pi_N f\|_{\infty}^4, M_\varepsilon^4)} \right). \tag{51}$$

By combining (50) and (51), one gets

$$\|F_{\alpha,N}(\hat{C}_N - C_N)\|_{\ell_2}^2 \leq \|f - \pi_N f\|_\alpha^2 + \sigma^2 + \delta \tag{52}$$

with probability at least $(1 - p_\varepsilon)^n - \exp \left( \frac{-n\delta^2}{\gamma^2_{\alpha,d} \max (\|f - \pi_N f\|_{\infty}^4, M_\varepsilon^4)} \right)$. By combining (46), (47) and (52), one concludes that for any $0 < \delta < \|\pi_N f\|_\alpha^2$, we have

$$\|\hat{C}_N - C_N\|_{\ell_2}^2 \leq \kappa_2(G_{d,N}^\alpha) \left( \|f - \pi_N f\|_\alpha^2 + \sigma^2 + \delta \right) \|\pi_N f\|_\alpha^2 \|\pi_N f\|_\alpha^2 - \delta, \tag{53}$$

with probability at least $(1 - p_\varepsilon)^n - \exp \left( \frac{-n\delta^2}{\gamma^2_{\alpha,d} \max (\|f - \pi_N f\|_{\infty}^4, M_\varepsilon^4)} \right) - \exp \left( \frac{-2n\delta^2}{\gamma^2_{\alpha,d} \max (\|f - \pi_N f\|_{\infty}^4, M_\varepsilon^4)} \right)$. Finally, we note that by Parseval’s equality, we have $\|\hat{C}_N - C_N\|_{\ell_2}^2 = \|\pi_N f - \hat{f}_N^\alpha\|_\alpha^2$. Hence, to conclude the proof, it suffices to combine (53) with the inequality

$$\|f - \hat{f}_N^\alpha\|_\alpha \leq \|f - \pi_N f\|_\alpha + \|\pi_N f - \hat{f}_N^\alpha\|_\alpha.$$

**Remark 3.** It is interesting to note that in practice, the probability $p_\varepsilon$ is fairly small for moderate values of the truncation bound $M_\varepsilon$. For example, for the largely used Gaussian white noise model with variance $\sigma^2$, for any fixed $K > 0$ and for any $i \in \mathbb{N}$, we have $|\varepsilon_i| \geq K\sigma$ with probability at most $\text{erf}(\frac{K\sqrt{2}}{\varepsilon}) \approx e^{-K^2/2} K\sqrt{\pi/2}$. This last quantity is very close to 0 even for small positive values of $K$.

Note that unless $\sigma^2 = 0$ in the previous theorem vanishes (that is the very special case of noiseless nonparametric regression model), the integrated error bound (42) has the drawback to lack of a convergence rate to zero, in terms of the parameters $n, N$. To overcome this problem, we give in the sequel an estimate of the $L_2$-risk error of a truncated version of our estimator $\hat{f}_N^\alpha$. The techniques used to get this $L_2$-risk are similar to those used in [4] in the univariate case. We assume that the regression function $f$ is almost everywhere bounded by a constant $M_f$, that is

$$|f(x)| \leq M_f, \quad \text{a.e.} \quad x \in I^d.$$ 

Let $\hat{F}_N$ be the truncated version of the estimate $\hat{f}_N^\alpha$ given by

$$\hat{F}_N(x) = \text{sign}(\hat{f}_N^\alpha(x)) \min(M_f, |\hat{f}_N^\alpha(x)|), \quad x \in I^d. \tag{54}$$

Under the usual assumption that the $\varepsilon_i$ are the i.i.d. centered random noises with variance $\sigma^2$, we have the following theorem that provides us with an estimate of the $L_2$-risk error of the estimator $\hat{F}_N$. The proof of this theorem is partly inspired from the techniques developed in [5].
Theorem 3. Let \( \alpha \geq -\frac{1}{2} \) and let \( 0 < \delta < 1 \). Then, under the previous notations and hypotheses, we have
\[
\mathbb{E}
\left[
\| f - \hat{F}_N \|^2 \alpha \right] \leq \frac{(N + 1)^d}{n(1 - \delta)^2} \left( \sigma^2 + (\eta^2_\alpha(N + 1)^{2\alpha+1})^d \| f - \pi_N f \|^2 \alpha \right) + \| f - \pi_N f \|^2 \alpha
\]
\[
+ 4M^2(N + 1)^d \left( \beta(\alpha + 1, \alpha + 1) \right)^d \exp \left( -\frac{n\delta^2}{2(B_\alpha(N + 1)^{2\alpha+2})^d} \right),
\tag{55}
\]
where \( B_\alpha \) is a constant depending only on \( \alpha \).

Proof. Recall that from (38), we have for any \( \delta \in (0, 1] \)
\[
\mathbb{P}
\left( \lambda_{\min}(G_{d,N}^\alpha) \geq 1 - \delta \right) \geq 1 - (N + 1)^d \exp \left( -\frac{n\delta^2}{2(B_\alpha(N + 1)^{2\alpha+2})^d} \right)
\]
where \( B_\alpha \leq 1 \) is a constant depending only on \( \alpha \). As it is done in [8], see also [4], let \( \Omega_+ \) and \( \Omega_- \) be the subsets of \((I^d)^n\) given by all possible draw \((X_1, \ldots, X_n)\) with \( \lambda_{\min}(G_{d,N}^\alpha) \geq 1 - \delta \) and \( \lambda_{\min}(G_{d,N}^\alpha) < 1 - \delta \), respectively. Let \( d\rho_n \) be the probability measure on \( \mathcal{U}^n \), given by the tensor product
\[
d\rho_n = \prod_{i=1}^n dh_{\alpha+1}(x_i),
\]
where \( h_{\alpha+1}(x_i) \) is as given by (6). Then, we have
\[
\int_{\Omega_-} d\rho_n = \mathbb{P}\{(X_1, \ldots, X_n) \in \mathcal{U}^n; \lambda_{\min}(G_{d,N}^\alpha) < 1 - \delta \} \leq (N + 1)^d \exp \left( -\frac{n\delta^2}{2(B_\alpha(N + 1)^{2\alpha+2})^d} \right).
\tag{56}
\]
Next, by using (54), the truncated estimator \( \hat{F}_N \) satisfies
\[
|f(x) - \hat{F}_N(x)| \leq |f(x) - \hat{f}_N^\alpha(x)| \leq |f(x)| + |\hat{F}_N(x)| \leq 2M_f, \quad \forall x \in I^d.
\tag{57}
\]
Hence, we have
\[
\mathbb{E}(\|f - \hat{F}_N\|^2 \alpha) = \int_{\Omega_+} \|f - \hat{F}_N\|^2 \alpha d\rho_n + \int_{\Omega_-} \|f - \hat{F}_N\|^2 \alpha d\rho_n.
\tag{58}
\]
By using (57), one gets
\[
\int_{\Omega_-} \|f - \hat{F}_N\|^2 \alpha d\rho_n \leq 4M^2(N + 1)^d \left( \beta(\alpha + 1, \alpha + 1) \right)^d \exp \left( -\frac{n\delta^2}{2(B_\alpha(N + 1)^{2\alpha+2})^d} \right).
\tag{59}
\]
On the other hand, from (57), we have
\[
\int_{\Omega_+} \|f - \hat{F}_N\|^2 \alpha d\rho_n \leq \int_{\Omega_+} \|f - \hat{f}_N^\alpha\|^2 \alpha d\rho_n
\]
\[
\leq \int_{\Omega_+} \|f - \pi_N f\|^2 \alpha d\rho_n + \int_{\Omega_+} \|\pi_N f - \hat{f}_N^\alpha\|^2 \alpha d\rho_n.
\]
Note that by Parseval’s equality, we have on \( \Omega_+ \),
\[
\|\pi_N f - \hat{f}_N^\alpha\|^2 \alpha = \|\hat{C}_N - C_N\|^2 \alpha \leq \| (G_{d,N}^\alpha)^{-1} \|^2 \alpha \| (F_{d,N}^\alpha)^T \Delta P \|^2 \alpha
\]
\[
\leq \frac{1}{(1 - \delta)^2} \| (F_{d,N}^\alpha)^T \Delta P \|^2 \alpha.
\]
where \( \Delta P = \frac{1}{\sqrt{n}} \left[ (f - \pi_N(f))(X_i) + \varepsilon_i \right]_{1 \leq i \leq n} \). This last inequality implies

\[
E \left[ \| \pi_N f - \hat{f}_N \|^2 \right] \leq \frac{1}{(1 - \delta)^2} E \left[ \| F_{d,N} \|^2 \right] \left\| \Delta P \right\|_{\ell_2}^2.
\]

Straightforward computation gives us

\[
\left\| (F_{d,N})^T \Delta P \right\|_{\ell_2}^2 = \frac{\beta(\alpha + 1, \alpha + 1)}{n^2} \sum_{k \in [0, N]^d} \sum_{j=1}^n \left( \Phi_k^\alpha(X_j)(\theta_N(X_j) + \varepsilon_j)\Phi_k^\alpha(X_l)(\theta_N(X_l) + \varepsilon_l) \right)
\]

where \( \theta_N(\cdot) = (f - \pi_N f)(\cdot) \perp \Phi_k^\alpha(\cdot), \forall k \in [0, N]^d \). Since the \( \varepsilon_j \)'s are independent of the \( X_j \)'s, and since \( E[\varepsilon_j] = 0, E[\varepsilon_j^2] = \sigma^2 \), then we have

\[
E \left[ \left\| (F_{d,N})^T \Delta P \right\|_{\ell_2}^2 \right] = \frac{\beta(\alpha + 1, \alpha + 1)}{n^2} \sum_{k \in [0, N]^d} \sum_{j=1}^n E \left[ \varepsilon_j^2 (\Phi_k^\alpha(X_j))^2 \right] + \frac{\beta(\alpha + 1, \alpha + 1)}{n^2} \sum_{k \in [0, N]^d} \left[ \sum_{j=1}^n (\Phi_k^\alpha(X_j))^2 (\theta_N(X_j))^2 \right].
\]

(60)

Since

\[
E \left[ \varepsilon_j^2 (\Phi_k^\alpha(X_j))^2 \right] = E[\varepsilon_j^2] E \left[ (\Phi_k^\alpha(X_j))^2 \right] = \sigma^2 \frac{1}{\beta(\alpha + 1, \alpha + 1)^d}
\]

and since

\[
E \left[ (\theta_N(X_j))^2 \right] = \frac{1}{\beta(\alpha + 1, \alpha + 1)^d} \| \theta_N \|^2_{\alpha},
\]

then by using the fact \( \sum_{k \in [0, N]^d} (\Phi_k^\alpha(X_j))^2 \leq \left( \eta_{\alpha}^2(N + 1)^{2\alpha + 2} \right)^d \), one gets

\[
\frac{\beta(\alpha + 1, \alpha + 1)}{n^2} \sum_{k \in [0, N]^d} \left[ \sum_{j=1}^n (\Phi_k^\alpha(X_j))^2 (\theta_N(X_j))^2 \right] \leq \frac{\left( \eta_{\alpha}^2(N + 1)^{2\alpha + 2} \right)^d}{n} \| \theta_N \|^2_{\alpha}.
\]

(61)

To conclude for the proof of the theorem, it suffices to combine (57), (60) and get (55).

\[\square\]

Next, we show that if \( f \) belongs to the functional space of 2–norm isotropic Soblev space \( H^s(I^d) \), with an appropriate \( s > 0 \), then \( f \) satisfies condition (41). Moreover, for such a function, one also gets an estimate for the quantity \( \| f - \pi_N f \|_{\alpha} \) given in (42). The 2–norm isotropic Soblev space \( H^s(I^d) \) is given by, see for example [7]

\[H^s(I^d) = \left\{ f \in L^2(I^d), \sum_{k \in \mathbb{Z}^d} \left( 1 + \sum_{j=1}^d |k_j|^2 \right)^s | \langle e^{2\pi i k \cdot x}, f \rangle |^2 < \infty \right\}, \]

(62)

where, \( \langle e^{2\pi i k \cdot x}, f \rangle = \int_{I^d} f(x)e^{-2\pi i k \cdot x} dx \). Also, we recall that if \( f \in L^2(I^d) \), then by using the notation \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d, \| m \|_{\infty} = \max_{1 \leq j \leq d} |m_j| \), the projection \( \pi_N f \) is given by

\[
\pi_N f(x) = \sum_{\| m \|_{\infty} \leq N} C_m \Phi_m^\alpha(x), \quad C_m = \langle f, \Phi_m^\alpha(x) \rangle_{\alpha},
\]

\[\| \| m \|_{\infty} \leq N \| \]
where $\Phi^\alpha_m(x)$ is as given by (4). In the sequel, we use the notation $\lesssim_{s,d}$ to say that the inequality holds up to a constant depending only on $s$ and $d$. This last constant is generic and may change from one line to another line.

**Theorem 4.** Under the previous notations, let $s > 0$ and $\alpha \geq -\frac{1}{2}$. For any integer $N \geq 2$ satisfying
\[
\frac{N}{\log N} \geq \frac{1}{2} \left(s + d + \frac{1}{2}\right)
\]
and for $f \in H^{s+d}(I^d)$, we have
\[
\|f - \pi_N f\|_\alpha \lesssim_{s,d} \sqrt{\frac{d}{2s}} N^{-s}.
\]

Moreover if $s > d(\alpha + 1)$, then we have
\[
\|f - \pi_N f\|_\infty \lesssim_{s,d} \frac{d}{s - d(\alpha + 1)} N^{-s+d(\alpha+1)}.
\]

**Proof:** Since the family of multivariate trigonometric exponentials $\{e^{2i\pi k \cdot x}, k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(I^d)$, then we have
\[
f(x) = \sum_{k \in \mathbb{Z}^d} a_k(f) e^{2i\pi k \cdot x}, \quad x \in I^d, \quad a_k(f) = \langle f(\cdot), e^{-2i\pi k \cdot \cdot} \rangle.
\]

It is not hard to see that if $m \in \mathbb{N}_0^d$, and $C_m(f) = \langle f, \Phi^\alpha_m \rangle$, then for $f \in H^s(I^d)$ with $s > d$ we have
\[
C_m(f) = \sum_{k \in \mathbb{Z}^d} d_{k,m} a_k(f), \quad d_{k,m} = \langle \Phi^\alpha_m(x), e^{2i\pi k \cdot x} \rangle.
\]

Taking into account that
\[
\Phi^\alpha_m(x) = \prod_{j=1}^d \tilde{P}^{(\alpha,\alpha)}_{m_j}(x_j), \quad x = (x_1, \ldots, x_d), \quad m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d,
\]

one gets
\[
d_{k,m} = \prod_{j=1}^d e^{2i\pi k_j x_j} \tilde{P}^{(\alpha,\alpha)}_{m_j}(x_j) > \alpha = \prod_{j=1}^d d_{k_j,m_j}, \quad d_{k_j,m_j} = \int_I e^{2i\pi k_j x_j} \tilde{P}^{(\alpha,\alpha)}_{m_j}(x_j) \omega_\alpha(x_j) dx_j.
\]

On the other hand, it is known that, see for example [20]
\[
\int_{-1}^1 e^{ixy} \tilde{P}^{(\alpha,\alpha)}_{m}(y) \omega_\alpha(y) dy = i^m \sqrt{\pi} \sqrt{2m + 2\alpha + 1} \frac{\Gamma(m + 2\alpha + 1)}{\Gamma(m + 1)} \frac{J_{m+\alpha+1/2}(x)}{x^{\alpha+1/2}}, \quad x \in \mathbb{R}.
\]

Here, the $\tilde{P}^{(\alpha,\alpha)}_{m}$ are the orthonormal Jacobi on $[-1,1]$, with $\omega_\alpha(y) = (1 - y^2)^\alpha$, $J_a$ is the Bessel function of the first kind and order $a > -1$ and $\Gamma(x)$ is the usual Gamma function. The Gamma and Bessel functions $\Gamma(\cdot)$ and $J_a(\cdot)$ satisfy the following useful inequalities that can be found in the literature,
\[
\sqrt{2e} \left(\frac{x + 1/2}{e}\right)^{x+1/2} \leq \Gamma(x + 1) \leq \sqrt{2\pi} \left(\frac{x + 1/2}{e}\right)^{x+1/2}, \quad x > 0.
\]

and
\[
|J_\mu(x)| \leq \frac{|x|^\mu}{2^\mu \Gamma(\mu + 1)}, \quad \mu > -1, \quad x \in \mathbb{R}.
\]
By using (66) and (67), one gets
\[
d_{k_j,m_j} = \frac{(-1)^{k_j}}{2} i^{m_j} \sqrt{\pi} \sqrt{2m_j + 2\alpha + 1} \sqrt{\frac{\Gamma(m_j + 2\alpha + 1)}{\Gamma(m_j + 1)}} J_{m_j + \alpha + 1/2}(\pi k_j). \tag{69}
\]
Note that since the function \(x \rightarrow \frac{J_{m_j + \alpha}(x)}{x^{\alpha}}\) has same parity as \(m\), then one can only consider the case \(k_j \geq 0\) in (69). The value of \(d_{k_j,m_j}\) for \(k_j < 0\) is simply given by \(d_{k_j,m_j} = (-1)^{m_j} d_{-k_j,m_j}\). Hence, by using (66), (67) and (68) together with some straightforward computations, one gets the useful inequality
\[
|d_{k_j,m_j}| \lesssim_{\alpha,d} \sqrt{m_j} \left(\frac{e\pi |k_j|}{2m_j + 2\alpha}\right)^{m_j}, \quad m_j \geq 1. \tag{70}
\]
Moreover, from Cauchy-Schwarz inequality, we also have \(|d_{k_j,m_j}| \leq 1\) for any integers \(k_j, m_j\). Consequently, if \(|k| \leq \frac{|m|}{e\pi}\), then one concludes that
\[
|d_{k,m}| \lesssim_{\alpha,d} \|m\|_\infty 2^{-\|m\|_\infty}, \quad \forall \|k\|_\infty \leq \frac{|m|}{e\pi}. \tag{71}
\]
Next, we write the multivariate Jacobi coefficient expansion of \(f \in H^s(I^d)\), as follows
\[
C_m(f) = \sum_{\|k\|_\infty \leq |m|_\infty/e\pi} d_{k,m} a_k(f) + \sum_{\|k\|_\infty > |m|_\infty/e\pi} d_{k,m} a_k(f) = S_1 + S_2, \tag{72}
\]
where, \(a_k(f) = \langle f(\cdot), e^{-2\pi k \cdot} \rangle\). To bound \(|S_1|\), we first note that in \(\mathbb{Z}^d\), there exist at most \(\left\lfloor \frac{2}{\pi\|m\|_\infty + 1}\right\rfloor^d\) different \(k\) satisfying \(|k|_\infty \leq \|m\|_\infty/e\pi\), where \([x]\) denotes the integer part of \(x\). Moreover, from Bessel’s inequality, we have
\[
\sum_{\|k|_\infty \leq |m|_\infty/e\pi} |a_k(f)|^2 \leq \|f\|_2^2. \tag{73}
\]
Consequently, by using (71) and Cauchy-Schwarz inequality, one concludes that
\[
|S_1| \lesssim_{\alpha,d} \|m\|_\infty^{1+d/\pi} 2^{-\|m\|_\infty} \|f\|_2. \tag{74}
\]
On the other hand, since \(f \in H^{s+d/2}(I^d)\) and since by Bessel’s inequality, we have
\[
\sum_{\|k|_\infty > |m|_\infty/e\pi} |d_{k,m}|^2 \leq \|\Phi_m^\alpha\|_{H^s}^2 = 1,
\]
then a simple Cauchy-Schwarz inequality gives us
\[
|S_2|^2 \leq \sum_{\|k|_\infty > |m|_\infty/e\pi} |a_k(f)|^2 \leq \left(\frac{e\pi}{\|m\|_\infty}\right)^{2s+d} \sum_{\|k|_\infty > |m|_\infty/e\pi} \left(1 + \sum_{j=1}^d |k_j|^2\right)^{s+d/2} |a_k(f)|^2.
\]
By combining (73) and (74), one concludes that
\[
|C_m| \lesssim_{\alpha,d} \left(\|m\|_\infty^{1+d/\pi} 2^{-\|m\|_\infty} + \|m\|_\infty^{-s+d/2}\right)(\|f\|_2 + \|f\|_{H^s}). \tag{75}
\]
Note that \(\|m\|_\infty^{1+d/\pi} 2^{-\|m\|_\infty} \leq \|m\|_\infty^{-s-d/2}\), whenever \(\|m\|_\infty \geq N\), with \(\frac{N}{\log N} \geq \frac{1}{\log 2} (s + d + \frac{1}{2})\). Hence, in this case, (75) is simply written as
\[
|C_m| \lesssim_{\alpha,d} \|m\|_\infty^{-s-d/2}(\|f\|_2 + \|f\|_{H^s}). \tag{76}
\]
Next, since
\[ f(x) - \pi_N f(x) = \sum_{m \in \mathbb{Z}^d, \|m\|_{\infty} \geq N+1} C_m \Phi_m^\alpha(x) = \sum_{n=N+1}^{\infty} \sum_{\|m\|_{\infty} = n} C_m \Phi_m^\alpha(x). \]  
\tag{77}
Again, since there exist \( d(n + 1)^{d-1} \) \( d \)-tuples \( m \in \mathbb{N}^d \) with \( \|m\|_{\infty} = n \), then by using (76) and by Parseval’s equality applied to (77) (which is due to the orthonormality of the \( \Phi_m^\alpha \) derivatives being Hölder continuous. In this case, the optimal rate of convergence is of \( -C \) where \( C \) belongs to an isotropic Sobolev space \( H^{s} \). Hence, by using the previous technique we have used to bound \( \sup_{x \in I} |\Phi_m^\alpha(x)| \) for some constant \( \tilde{C} \), one gets
\[ \|f - \pi_N f\|_2^2 \lesssim_{\alpha,d} d \left( \sum_{n=N+1}^{\infty} (n + 1)^{d-1} n^{-2s-d} \right) (\|f\|_2 + \|f\|_{H^s})^2 \]
\[ \lesssim_{\alpha,d} \frac{d}{2s} N^{-2s} (\|f\|_2 + \|f\|_{H^s})^2. \]  
\tag{78}
This concludes the proof of inequality (63). Finally to prove (64), we recall the following known upper bound for the Jacobi polynomials \( \tilde{P}_m^{(\alpha,\alpha)}(x) \) with \( \alpha \geq -\frac{1}{2} \), see for example
\[ \sup_{x \in I} |\tilde{P}_m^{(\alpha,\alpha)}(x)| \leq M_\alpha m^{\alpha + \frac{d}{2}}, \]
for some constant \( M_\alpha \). Consequently, we have
\[ \sup_{x \in I^d} |\Phi_m^\alpha(x)| \lesssim_{\alpha,d} \|m\|_2^{d(\alpha+1/2)}. \]
Hence, by using the previous technique we have used to bound \( \|f - \pi_N f\|_\alpha \), one gets
\[ \sup_{x \in I^d} \|f(x) - \pi_N f(x)\| \lesssim_{\alpha,d} \frac{d}{2s} \sum_{n=N+1}^{\infty} (n + 1)^{d-1} n^{-s-d/2} n^{d(\alpha+1/2)} \]
\[ \lesssim_{\alpha,d} N^{-s+d(\alpha+1)} \frac{d}{s - d(\alpha + 1)}. \]
This concludes the proof of the Theorem.

As a consequence of the previous two theorems, we have the following corollary that provides us with a convergence rate for our truncated estimator \( \tilde{F}_N \), when the regression function belongs to an isotropic Sobolev space.

**Corollary 1.** Let \( s > 0 \) and let \( \alpha \geq -\frac{1}{2} \) be such \( s > d \left( \alpha + \frac{1}{2} \right) \). Assume that the regression function \( f \) belongs to an isotropic Sobolev space \( H^{s+d/2}(I^d) \), then the convergence rate of the estimator \( \tilde{F}_N \) is of order \( O \left( n^{-2s/(2s+d)} \right) \).

**Proof.** Straightforward computations show that if \( s > d \left( \alpha + \frac{1}{2} \right) \), then the first and the third quantity in the sum of the left hand side of (55) are of order \( O \left( \frac{(N+1)^d}{n} \right) \). Hence, by using (63) and (55), the fastest rate of convergence of the estimator \( \tilde{F}_N \) is obtained when \( N = O(n^{1/(2s+d)}) \). In this case, we have
\[ \mathbb{E} \left[ \|f - \tilde{F}_N\|_\alpha^2 \right] = O \left( n^{-2s/(2s+d)} \right). \]

**Remark 4.** It is interesting to note that the previous convergence rate of our estimator \( \tilde{F}_N \) is too similar to the theoretical optimal convergence rate of min-max nonparametric estimator. This last optimal convergence rate is given in \([27]\), see also \([3]\). It is given in the case where the regression function \( f \) belongs to the class \( \mathcal{C} - p \) functions, that is the set of \( d \)-variate functions of class \( \mathcal{C}^p \), with their different \( p \)-th order partial derivatives being Hölder continuous. In this case, the optimal rate of convergence is of \( O \left( n^{-2p/(2p+d)} \right) \), as for kernel regression estimate.
4 Computational analysis

In this section, we check the performance of our estimator \( \hat{f}_{\alpha N}(\cdot) \), by applying it to synthetic data as well as to real data.

4.1 Numerical simulations on synthetic data

In this paragraph, we give three numerical examples that illustrate the results of this work. The first example is an illustration of the first main Theorem 1, while the other two examples illustrate the performance of our estimator when applied to synthetic data.

Example 1: In this first example, we check numerically the important result given by Theorem 1. To this end, we have considered the dimension \( d = 2 \), then we have computed an average for the true condition number \( \kappa_2(G_{d,N}^\alpha) \) over 10 realizations and for different values of \( \alpha = -0.5, 0.0, N = 5, 10 \) and \( n = n_1^2, n_1 = 60, 80, 100 \). The obtained numerical results are given by Table 1 and they are fairly coherent with the theoretical result of Theorem 1. As it is stated by formula (32) of Theorem 1, the value of \( \alpha = -\frac{1}{2} \) seems to be the optimal value that provides the smallest condition number for the random matrix \( G_{d,N}^\alpha \). Moreover, for a fixed value of the parameter \( \alpha \), a smaller value of \( N \) and/or a larger value of \( n \), give us a smaller condition number \( \kappa_2(G_{d,N}^\alpha) \).

| \( \alpha \) | \( N \) | \( n_1 \) | \( \kappa_2(G_{d,N}^\alpha) \) | \( N \) | \( n_1 \) | \( \kappa_2(G_{d,N}^\alpha) \) |
|---|---|---|---|---|---|---|
| -0.5 | 5 | 60 | 7.45 | 10 | 60 | 35.41 |
| | | | | | | |
| | 80 | 5.62 | | 80 | 19.97 |
| | 100 | 3.94 | | 100 | 11.41 |
| 0.0 | 5 | 60 | 13.51 | 10 | 60 | 6596.05 |
| | | | | | | |
| | 80 | 10.17 | | 80 | 369.43 |
| | 100 | 7.57 | | 100 | 158.53 |

Table 1: Illustration of Theorem 1's results with \( n = n_1^2 \).

Example 2: In this second example, we illustrate the performance of our estimator \( \hat{f}_{\alpha N}(\cdot) \), given by (7)–(9), by applying it to data, generated by a synthetic 2-variate regression function, given by

\[
f(u, v) = 1 + 2u - 4v + 3v^2 - 2uv + 3uv^2 - v^3 + u^4 + 2u^5 + \sin(2\pi u) - \cos(3\pi v), \quad u, v \in [0, 1].
\]

For this purpose, we have considered the special case of \( \alpha = -\frac{1}{2}, N = 5, 10 \) and \( n = n_1^2, n_1 = 60, 80, 100 \). Then we considered the regression problem (1) with i.i.d. white gaussian noises associated to the standard deviation \( \sigma = 0.05 \) and \( \sigma = 0.15 \). To assess the performance of the regression estimate, we have computed the mean squared error (MSE), which is given by

\[
MSE = \frac{1}{n} \sum_{i=1}^{n} \left( f(X_i) - \hat{f}_{N}(X_i) \right)^2.
\]

Here, the \( X_i \) are i.i.d. bi-variate random sampling points with each of the two components following a Beta distribution on \([0, 1]\) and associated with the parameter \( \alpha = -0.5 \). From these numerical results and as it has been stated by Theorem 4 for fixed values of the parameters \( \alpha, \sigma, n \) the largest the value of \( N \), the smallest is the associated MSE. Also, it is interesting to note that according to Theorem 1 our estimator \( \hat{f}_{\alpha N}(\cdot) \) is surprisingly stable in the sense that it behaves well in the presence of data perturbation by a white noise.
Table 2: The mean squared error for the proposed method and the nonparametric kernel method on the considered case of Example 2 with \( n = n_1^2 \). Kernel denotes the mean squared errors of the kernel regression estimate while \( OM \) is for the proposed method. The best mean squared errors are in bold. The last two columns give the computing time (in minutes) of the proposed (TOM) and for the kernel (TK) methods.

Moreover, we have pushed forward the previous numerical simulations associated to the previous synthetic example by comparing our proposed regression estimator to other well known parametric and nonparametric regression estimators. These estimators are given by Kernel (the kernel regression with optimal bandwidth selection by cross-validation); SVM (support vector machine) and RF (the Random Forest regression estimator). For this second set of simulations, we have used 80% of the sample size \( n \) for the construction of the estimators and the remaining 20% of the sample size are used to validate the estimators by computing the resulting mean squared, the mean absolute errors as well as the coefficient of adjustment \( R^2 \). The obtained numerical results are given in Table 3. These results indicate that for larger \( N \), sample size \( n \) and error variance, the proposed method is competitive to the kernel method, which outperforms. In fact, our method is less time consuming than the kernel method as shown in Table 2 and outperforms compare to random forest and support vector machine methods.

| \( N \) | \( \sigma \) | \( n_1 \) | \( OM \) MSE | \( R^2 \) | MAE | \( Kernel \) MSE | \( R^2 \) | MAE | SVM MSE | \( R^2 \) | MAE | RF MSE | \( R^2 \) | MAE |
|-------|--------|------|--------|---------|------|---------|---------|------|-------|---------|------|-------|---------|------|
| 10    | 0.05   | 60   | 2.62e-3 | 1.4e-3  | 0.15 | 60      | 2.64e-2 | 6.0e-3 | 1.55  | 1.10   |
|       | 80     | 2.60e-3 | 3.0e-3  | 80    | 2.30e-2 | 2.4e-2  | 2.51  | 4.33  |
|       | 100    | 2.44e-3 | 2.6e-3  | 100   | 2.25e-2 | 2.3e-2  | 3.52  | 10.25 |
| 5     | 0.05   | 60   | 4.67e-3 | 1.4e-3  | 0.15 | 60      | 2.47e-2 | 6.0e-3 | 0.18  | 1.10   |
|       | 80     | 6.40e-3 | 3.0e-3  | 80    | 2.70e-2 | 2.4e-2  | 0.29  | 4.33  |
|       | 100    | 5.11e-3 | 2.6e-3  | 100   | 2.66e-2 | 2.3e-2  | 0.38  | 10.25 |

Table 3: Validation of the regression model estimation on a testing sample of size 80% of the sample of size \( n = n_1^2 \) given in Example 2. The mean squared error (MSE), mean absolute error (MAE) and \( R^2 \) for the proposed method (OM), the nonparametric kernel (Kernel), Support Vector Machine (SVM) and Random Forest (RF) methods are given.

Example 3: In this example, we illustrate the performance of our estimator by applying it to a classification
problem, generated by the synthetic 2−variate sample \((X_i, Y_i)_{i=1}^{n_1}, n = n_1^2, X_i = (X_{i,1}, X_{i,2})^\top\) of Example 2. The classification data is generated by the rule \(Y_i = 1\) if \(f(X_{i,1}, X_{i,2}) > c\), \(Y_i = 0\) otherwise, \(c\) is the taken as the mean of the \(f(X_{i,1}, X_{i,2})\). We consider as in Example 2, a gaussian white noise associated to the two values of \(\sigma = 0.05\) and \(\sigma = 0.15\). Then, we have constructed our estimator \(\hat{f}_N(\cdot)\), with \(\alpha = -0.5\) \(N = 5, 10\) by using 80% of the samples data with size \(n = n_1^2\) for the different values of \(n_1 = 60, 80, 100\). The remaining 20% of the data are used for testing the classification performance. Note that we have used the standard classification rule \(\hat{Y}_i = 0\) if \(\hat{f}_N(X_i) \leq 0.5\), otherwise \(\hat{Y}_i = 1\). Moreover, we have compared our proposed classifier (OM) with other three estimators frequently used in the literature for classification purposes. These estimators are the LDA (Linear Discriminant Analysis), SVM (Support Vector Machine) and NN (the Neural Network based classifier). The obtained numerical results summarized in Table 4 show that proposed classification rule and neural networks method outperform the linear discriminant and support vector machines classifiers. As mentioned before, the main advantages of the proposed method are its stability, convergence rate and fairly low computation time.

| \(N\) | \(\sigma\) | \(n_1\) | OM  | LDA  | SVM  | NN  |
|---|---|---|---|---|---|---|
| 5  | .05 | 60 | 95 | 65  | 79  | 96  |
|    |     | 80 | 95 | 74  | 82  | 97  |
|    |     | 100| 96 | 73  | 82  | 97  |
| 10 | .05 | 60 | 99 | 65  | 79  | 96  |
|    |     | 80 | 97 | 74  | 82  | 97  |
|    |     | 100| 98 | 73  | 82  | 97  |
| 5  | .15 | 60 | 93 | 76  | 84  | 95  |
|    |     | 80 | 93 | 83  | 90  | 97  |
|    |     | 100| 93 | 72  | 81  | 94  |
| 10 | .15 | 60 | 95 | 76  | 84  | 95  |
|    |     | 80 | 94 | 83  | 90  | 97  |
|    |     | 100| 95 | 72  | 81  | 94  |

Table 4: Prediction results on a testing sample of size 80% of the sample size of data generated using Example 3. The correct classification rate (CCR) in % for the proposed method (OM), Linear Discriminant Analysis (LDA), Support Vector Machine (SVM) and Neural Network (NN). The bold values highlight the best classification rates

4.2 Application to breast cancer cell lines regression and classification

Most cancer patient die due to metastasis, and the early onset of this multi-step process is usually missed by current staging tumor modalities. Advanced techniques exist to enrich disseminated tumor cells from patient blood and bone marrow as cancer progression marker. However, these cells present high heterogeneity, only some of them can exhibit stem cell phenotype and tumor development potential, others can have plasticity potential to reprogram into cancer stem cells. So, detection and characterization are challenging due to lack of clear phenotypic markers. Therefore, there is a critical need to find new ways to anticipate and predict metastasis development at an early stage of patient care. Cancer progression involves many cellular morphological effects, which have been revealed by biophysical studies. The relevance of the characterization of cancer cells by their bio-mechanical phenotype is attested by reports pointing out their physical alteration as reduced cell stiffness with invasiveness for lung, breast and colon cancers, while the deformability of circulating lymphocytes is reduced in the case of acute lymphoblastic leukemia. The physical properties even allow identifying different malignant breast epithelial cell lines by their deform-ability and their viscoelastic behavior. Even though the analysis capability of cancer cells by their physical characteristcs has been demonstrated, reports mostly compare different known states of cancer cells. So far to our knowledge, no prediction capability has been reported to detect cancer cells and evaluate their invasiveness only by bio-mechanical characterization. This application aims to use cell physical phenotyping to detect and
categorize disseminating cells population by physical parameters (electrical measurements) using MEMS (Microelectromechanical systems) technology performing electrical single cell measurements. The MEMS devices capture a cell for stimulation and provides the information on the mechanical or electrical properties of the captured cell. They performed a compression protocol on each cell and measured changes in the resonance frequency and amplitude values for single-cell biophysical properties as a function of time in addition to the initial measurement on the cell dimension.

The dataset analyzed here are derived from MEMS devices, they are composed of electrical properties (maximum values during the compression period, at 1 and 5 Mega hertz) of single cells from three different breast cancer cell lines in a controlled environment. The compression assays on different breast cancer cell lines, U937, MCF7, SUM159-PT, give four electrical measurements (real and imaginary, at 1 and 5 Mega hertz, Figure 2). The size is obtained from 1 Mega hertz parameter (Figure 2). The three cell lines has potential metastatic. The SUM159-PT cell line has higher metastatic potential compared to the two other cell lines, SUM159-PT showed that cancer cells exhibit softer characteristics compared to their benign counterparts. The comparison of the average size (amplitude) between the cell lines (Figure 2) indicated that SUM159-PT cell line (very aggressive and highly metastatic) was softer than MCF7 cell line (having lower metastatic potential). For the four electrical parameters and size (amplitude) the cell lines showed significant differences (Kruskal-Wallis comparison test have been done) between metastatic cell lines.

We apply the developed methodology to the cell dataset. To run our proposed estimator, we have first transformed (by usual dilation and translation techniques), the set of 4-variate real data corresponding to the 1 and 5 Mega real and imaginary parameters values of the different cells into the square $[0,1]^2$. Then, we have used these transformed data with total size almost equal to 3000, together with the standard Shepard scattered interpolation algorithm with $p = 3$ and $n = n_1^2$, for three couples $(N,n_1) = (5,30), (10,50), (15,70)$ and we have computed fairly accurate numerical approximations of the values of cells at $n = n_1^2$ random i.i.d. sampling points in the $2$-dimensional unit square and following the two-dimensional Beta distribution with parameters $(\alpha + 1, \alpha + 1)$, $\alpha = -0.5$. First regression analysis has been done to explain the cell size (response variable) with the 5 Mega electrical parameters using the three cell types. We run the proposed model on 80% of the sample size for the construction of the estimators and the remaining 20% to validate the regression estimation by computing the resulting mean squared and as well as the $R^2$. We compare our results with that of the kernel, the support vector machine and the Random Forest regression estimations. The results given in Table 5 show the same behavior as the results based on the simulated data. The proposed method is competitive compare to the above mentioned methods in particular the kernel method. However our method is less time consuming than the kernel regression estimate.

| OM | Kernel | SVM | RF |
|----|--------|-----|----|
| 5  | 5.97e-2| 3.62e-2| 9.87e-1 | 2.03e-1| 9.98e-1| 2.04e-1| 9.88e-1 |
| 10 | 4.95e-2| 3.62e-2| 9.87e-1 | 2.03e-1| 9.98e-1| 2.04e-1| 9.88e-1 |
| 15 | 4.99e-2| 9.86e-1| 3.62e-2| 9.87e-1 | 2.03e-1| 9.98e-1| 2.04e-1| 9.88e-1 |

Table 5: Validation of the cell size regression model estimation on a test sample of size 20% of the sample of size $n = 2926$ cells using a training sample with the remaining 80%. The mean squared error (MSE) and $R^2$ on the test sample for the proposed method, the kernel, Support Vector Machine (SVM) and Random Forest (RF) methods.

In the other hand, we run the proposed method (OM) and three other supervised learning methods to predict the population the cells belong to. These last methods are the Linear Discriminant Analysis (LDA), Generalized Additive Models (GAM) and Generalized Linear Models (GLM, logit). For classification purpose, the cells of the classes MCF7 and U937 were assigned the integer values 1 and 2, respectively. We have run our proposed estimator by using the same way as in the previous regression example with the couple $(N,n_1) = (5,30)$ and transformed data with total size almost 1700. We have computed our associated estimator $\hat{f}_{\alpha}^N(\cdot)$, given by (7)–(9). The values are given in Table 6 where the overall classification accuracy.
(CR), precision (PR: the fraction of correct predictions for a certain class), recall (R: the fraction of instances of a class that were correctly predicted), the F1 (harmonic mean of precision and recall), are given on a test sample based on 20% of the two different sample cells (MCF7 and U937) while the remaining 80% cells data are used for training. The results show that the proposed method has the second best overall correct classification rate compare to the best GAM model. When looking at the precision, the proposed method and GAM outperform when predicting the less metastatic cells (U937).

|      | OM | LDA | GLM | GAM |
|------|----|-----|-----|-----|
| N    | PR | R   | F1  | PR | R   | F1 | PR | R   | F1 |
| 5    | MCF | 94 | 94 | 94 | 99 | 92 | 96 | 92 | 94 | 96 | 95 | 96 |
| 5    | U937 | 91 | 91 | 91 | 68 | 97 | 80 | 83 | 93 | 87 | 90 | 94 | 92 |

Table 6: Validation of the classification model estimation on a testing sample of size 20% of the MCF7 and U937 cells. The overall classification accuracy (CR), precision (PR); the fraction of correct predictions for a certain class; the recall (R); fraction of instances of a class that were correctly predicted, the F1; harmonic mean of precision; for the proposed method, LDA, GAM and GLM

The finite sample properties of this section shows that the proposed methodology is competitive to the well known parametric (GAM); nonparametric (kernel) and SVM methods and less time consuming than the kernel regression and does not require any extra regularization or conditioning step.

5 Concluding remarks

We have proposed a least-squares multivariate nonparametric regression estimator by using the gPC (generalized Polynomial Chaos) principle. This estimator is given in terms of the tensor product of univariate Jacobi polynomials with parameters \( \alpha = \beta \geq -\frac{1}{2} \). In particular, by using some spectral analysis results from the theory of positive definite random matrices, we have shown that this estimator is stable under the condition that the i.i.d. random training sampling points \( X_i \) follow a \( d \)-variate Beta distribution with parameters \( (\alpha + 1, \alpha + 1) \) for each variable. Note that unlike many other least-squares nonparametric regression based estimators, the stability of our estimator does not require any extra regularization or conditioning step. Also, we have performed an error analysis of our proposed estimator. More precisely, we have given an \( L_2 \)-error as well as the \( L_2 \)-risk error of this later. Moreover, we have studied its convergence rate, when the regression function is assumed to belong to \( d \)-dimensional isotropic Sobolev space \( H^s(I^d) \), where \( I = [0,1] \) and \( s > 0 \) is the associated Sobolev smoothness exponent. In this case, we have shown that our estimator has the optimal min-max type convergence rate for \( d \)-dimensional regression problems and under the hypothesis that the regression function belongs to a functional space with some smoothness property. Moreover, in the case of the frequently encountered case where the \( n \) i.i.d. training sampling data points follow an unknown distribution, we have proposed to first apply a Shepard’s type scattered interpolation technique to get fairly accurate approximations of the outputs at \( n \) neighboring random sampling sampling points following a \( d \)-variate Beta distribution. Numerical evidences indicate that even in the case of interpolated data, our proposed estimator still provides good results. We have performed numerical simulations on synthetic as well as real data. The results of these simulations indicate that the proposed estimator is competitive with some of popular multivariate regression estimators from the literature, such as the smoothing kernel estimator, see for example [25]. Finally, we should mention that due to the its fairly heavy computational load, the estimator we have proposed in this work is rather adapted for small values of the dimension \( d \). Its extension/adaptation to the case of moderate or large values of \( d \), will be the subject of a future work.
Figure 1: Boxplot of the four electrical parameters; 5 Mega Hertz and 1 Mega Hertz, real and imaginary. Kruskal-Wallis test has been done to compare the parameters of the different cell types.
Figure 2: Boxplot of the amplitude (size) of the different cell types with Kruskal-Wallis comparison test.

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