Speeding up adiabatic passage with an optimal modified Roland–Cerf protocol

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Abstract

In this article we propose a novel method to accelerate adiabatic passage in a two-level system with only longitudinal field (detuning) control, while the transverse field is kept constant. The suggested method is a modification of the Roland–Cerf protocol, during which the parameter quantifying local adiabaticity is held constant. Here, we show that with a simple ‘on–off’ modulation of this local adiabaticity parameter, a perfect adiabatic passage can be obtained for every duration larger than the lower bound $\pi/\Omega$, where $\Omega$ is the constant transverse field. For a fixed maximum amplitude of the local adiabaticity parameter, the timings of the ‘on–off’ pulse-sequence which achieves perfect fidelity in minimum time are obtained using optimal control theory. The corresponding detuning control is continuous and monotonic, a significant advantage compared to the detuning variation at the quantum speed limit which includes non-monotonic jumps. The proposed methodology can be applied in several important core tasks in quantum computing, for example to the design of a high fidelity controlled-phase gate, which can be mapped to the adiabatic quantum control of such a qubit. Additionally, it is expected to find applications across all physics disciplines which exploit the adiabatic control of such a two-level system.

Keywords: quantum control, adiabatic passage, two-level systems, quantum gates

(Some figures may appear in colour only in the online journal)

1. Introduction

Controlling efficiently the fundamental quantum unit, the two-level quantum system, lies at the heart of many modern quantum technology applications [1, 2]. One of the most effective methods to address this problem is adiabatic passage (AP) [3, 4]. The system starts from an eigenstate of the initial Hamiltonian, then some parameter varies slowly with time and, if
the change is slow enough, it ends up to an eigenstate of the final Hamiltonian. The traditional setup for AP is a two-level system with constant transverse $x$-field and time-dependence restricted in the longitudinal $z$-field. This framework not only describes the setting of some classical applications, for example nuclear magnetic resonance, but is also pertinent to some modern applications, like some fundamental procedures in quantum computation [5–10]. As a concrete example we mention that the problem of designing a $cZ$ gate with high fidelity [5] can be formulated as controlling adiabatically the aforementioned qubit [6].

In the traditional AP, the slow change in the control parameter, $z$-field, is linear, and the process is called Landau–Zener (LZ) sweep [11, 12]. This protocol is robust to moderate changes in the model parameters, but its main drawback is the long required durations, which considerably reduce the final fidelity when the phenomena of decoherence and dissipation are present. Several methods have been suggested for speeding up the slow adiabatic evolution. For example, it has been shown that certain nonlinear LZ sweeps can achieve perfect fidelity for specific durations [13]. In a related work [6], the final error probability is minimized for durations larger than a certain threshold and high levels of fidelity (enabling fault-tolerant quantum computation) are obtained, even for durations as short as a few times the system timescale. Optimal control theory has also been exploited to find the quantum speed limit for the desired transfer [14, 15], but it requires infinite values of the control field in order to implement instantaneous rotations around $z$-axis. Other speed limits, restricted to bounded controls, have been found [16], but they include discontinuities and non-monotonic variations in the $z$-field. Finally, we mention the methods developed under the umbrella of Shortcuts to Adiabaticity [17–23], where the quantum system is driven at the same final adiabatic state, but it does not follow the intermediate adiabatic eigenstates. The common characteristic of these techniques when applied to two-level quantum systems [24–38] and in the AP framework is that, both the longitudinal ($z$) and transverse ($x$ and/or $y$) fields are used to accelerate adiabatic evolution, while here we focus on the case where the time-dependence is restricted only in the $z$-field.

The Roland–Cerf (RC) protocol was originally developed in order to accelerate quantum search in adiabatic quantum computation [39]. It relies on the fulfilment of a local (in time) adiabaticity condition, instead of a global one valid during the whole process. In the present work, we first apply the RC protocol with only detuning ($z$-field) control, as in [28, 29], and show that it can achieve perfect fidelity for specific durations, as the nonlinear LZ sweeps. During the application of this protocol, the parameter quantifying local adiabaticity is held constant. Next, we suggest a modified RC protocol, with ‘on–off’ modulation of the local adiabaticity parameter, which can achieve perfect fidelity for every duration larger than a lower bound. Compared to our recent related work [40], here we use optimal control theory to obtain an extra optimality condition, see section 4, which allows to determine the timings of the ‘on–off’ optimal control signal by solving a single transcendental equation. This is a significant improvement compared to [40], where the optimal timings are obtained through a numerical optimization with respect to the control amplitude. The suggested method takes advantage of the composite pulses characteristics [41–44], while the corresponding longitudinal field is continuous and monotonic with time. We emphasize that in previous studies where the longitudinal field is the only control used, its variation with time is non-monotonic and includes discontinuities [15, 16]. The present work is expected to find application in the wide spectrum of research fields where AP for two-level systems is exploited.

The article is structured as follows. In the next section we apply the classical RC protocol to the two-level system with detuning control. In section 3 we present the modification of the RC protocol and formulate the corresponding optimal control problem. In section 4 we derive
the optimality condition and use it in section 5 to determine the timings of the optimal pulse-sequences. Section 6 concludes this work.

2. Roland–Cerf protocol for a two-level system with detuning control

The Hamiltonian of a two-level system with constant Rabi frequency $\Omega$ (x-field) and time-dependent detuning $\Delta(t)$ (z-field) is

$$H(t) = \frac{\Delta(t)}{2} \sigma_z + \frac{\Omega}{2} \sigma_x = \frac{1}{2} \begin{bmatrix} \Delta(t) & \Omega \\ \Omega & -\Delta(t) \end{bmatrix},$$

(1)

where $\sigma_x, \sigma_z$ are the Pauli matrices. Another parameter which can be used as the control input, instead of the detuning, is the polar angle $\theta$ of the total field

$$\cot \theta(t) = \frac{\Delta(t)}{\Omega}.$$  

(2)

Using $\theta$, Hamiltonian (1) can be written as

$$H = \frac{\Omega}{2\sin \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}.$$  

(3)

If $|\psi\rangle = a_1|0\rangle + a_2|1\rangle$ is the system state, the probability amplitudes $a = (a_1, a_2)^T$ satisfy the following equation ($\hbar = 1$)

$$i\dot{a} = Ha.$$  

(4)

The adiabatic eigenstates of Hamiltonian (3) are

$$|\phi_+(t)\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad (5a)$$

$$|\phi_-(t)\rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} \end{pmatrix}, \quad (5b)$$

with corresponding eigenvalues

$$E_{\pm}(t) = \pm \frac{1}{2} \sqrt{\Delta^2 + \Omega^2} = \pm \frac{\Omega}{2\sin \theta}. \quad (6)$$

We consider that the detuning starts from a negative value, so the initial angle $\theta_i > \pi/2$ as displayed in figure 1, while the system starts from one of the eigenstates (5a), (5b), with $\theta = \theta_i$. For large initial negative detuning it is $\theta_i \approx \pi$ and $|\phi_+\rangle \approx (1 0)^T, |\phi_-\rangle \approx (0 1)^T$. In the traditional AP [11, 12], the detuning is varied linearly with time, until the angle obtains the final value $\theta_f < \pi/2$, see figure 1. For a sufficiently slow variation, corresponding to long enough time, the Bloch vector follows the eigenstate where it started initially. At the final time the detuning acquires a large positive value corresponding to $\theta_f \approx 0$ and $|\phi_+\rangle \approx (0 1)^T, |\phi_-\rangle \approx (0 \ 1)^T$. Since at the initial and final times the adiabatic states coincide with the original states, AP accomplishes perfect population inversion from state $|0\rangle$ to $|1\rangle$ and vice versa. In the present work we find control functions $\Delta(t)$ and $\theta(t)$ which drive the Bloch vector to the same final eigenstate without following the adiabatic eigenstates at intermediate times.

As a warm up example we present the Roland–Cerf protocol for the two-level system under consideration [28, 29]. In this protocol, the matrix element of the rate of change $dH/dt$ between the eigenstates $|\phi_\pm(t)\rangle$. 

3
\langle \phi_+ (t) | \frac{dH}{dt} | \phi_- (t) \rangle = - \frac{\Omega}{2 \sin \theta} \dot{\theta}, \quad (7)

is taken to be proportional to the square of the instantaneous energy gap,

\[ g(t) = E_+ (t) - E_- (t) = \sqrt{\Delta^2 + \Omega^2} = \frac{\Omega}{\sin \theta}, \quad (8) \]

I.e.

\[ \langle \phi_+ (t) | \frac{dH}{dt} | \phi_- (t) \rangle = \frac{1}{2} u g^2 (t), \quad (9) \]

where \( u \) is a constant parameter. For \( u \ll 1 \) the local adiabaticity condition \( \langle + | \dot{H} | - \rangle / g^2 \ll 1 \) is satisfied. For the two-level system, equation (9) becomes

\[ \dot{\theta} = - \frac{\Omega}{\sin \theta} u, \quad (10) \]

which can be easily integrated to give

\[ \theta (t) = \cos^{-1} (\cos \theta_i + u \Omega t). \quad (11) \]

The corresponding detuning can then be obtained from equation (2).

The performance of the RC protocol was evaluated numerically in [28, 29]. We can evaluate the performance analytically and for arbitrarily large \( u \) by working in the adiabatic reference frame. If we express the state of the two-level system in both the original and the adiabatic frames

\[ | \psi \rangle = a_1 | 0 \rangle + a_2 | 1 \rangle = b_1 | \phi_+ \rangle + b_2 | \phi_- \rangle, \quad (12) \]

we can find the following relation connecting the probability amplitudes in the two frames

\[ \text{Figure 1. Initial and final target states on the Bloch sphere, characterized by polar angles } \theta_i, \theta_f, \text{ respectively. The initial and final total fields are also aligned respectively.} \]
\[ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \]  

From equations (4), (13) we find the following equation for the probability amplitudes in the adiabatic frame

\[ i \dot{b} = H_{\text{ad}} b, \]  

where the Hamiltonian now is

\[ H_{\text{ad}} = \frac{1}{2} \begin{pmatrix} \frac{\Omega}{\sin \theta} & -i \dot{\theta} \\ i \dot{\theta} & -\frac{\dot{\theta}}{\sin \theta} \end{pmatrix}. \]  

The above equations are simplified if, inspired from equation (10), we use a dimensionless rescaled time \( \tau \) defined as

\[ d \tau = \frac{\Omega}{\sin \theta} dt. \]  

Note that \( \sin \theta > 0 \) since \( 0 < \theta < \pi \), thus equation (16) defines indeed a rescaling. Equation (14) takes the form

\[ i b' = H'_{\text{ad}} b, \]  

where

\[ H'_{\text{ad}} = \frac{1}{2} \sigma_z + \frac{\theta'}{2} \sigma_y = \frac{1}{2} \sigma_z - u \sigma_y, \]  

and \( b' = db/d\tau, \theta' = d\theta/d\tau = -u. \) Since Hamiltonian \( H'_{\text{ad}} \) is constant, from equations (17) and (18) we obtain at the final (rescaled) time \( \tau = T \) that \( b(T) = Ub(0) \), where the propagator \( U \) is

\[ U = e^{-iH'_{\text{ad}} T} = e^{-i\frac{\omega T}{2}(n_z \sigma_z - n_y \sigma_y)} = I \cos \frac{\omega T}{2} - i \sin \frac{\omega T}{2} (n_z \sigma_z - n_y \sigma_y), \]

and

\[ \omega = \sqrt{1 + u^2}, \quad n_y = \frac{u}{\omega} = \frac{u}{\sqrt{1 + u^2}}, \quad n_z = \frac{1}{\omega} = \frac{1}{\sqrt{1 + u^2}}. \]

If the initial state is \( |\phi_+\rangle \) then \( b(0) = (1 0)^T \). For a perfect AP the system should return to this state at the final time \( \tau = T \), thus it should be \( b_2(T) = 0 \). Equation (19) leads to the relation \( \sin (\omega T/2) = 0 \), thus \( U = \pm I \), and we obtain

\[ T \sqrt{1 + u^2} = 2k\pi, \quad k = 1, 2, \ldots \]

On the other hand, angle \( \theta \) changes from \( \theta_i \) to \( \theta_f \)

\[ \theta_f - \theta_i = -\int_0^T \theta' d\tau = uT. \]

Solving equations (21) and (22) we obtain

\[ u_k = \sqrt{\frac{\theta_f - \theta_i}{2k\pi}}, \]  

\[ (23a) \]
\[ T_k = 2k\pi \sqrt{1 - \left( \frac{\theta_i - \theta_f}{2k\pi} \right)^2}, \quad (23b) \]

for \( k = 1, 2, \ldots \). The corresponding durations in the original time \( t \) can be found from equation (11) and they are

\[ \tilde{T}_k = \frac{\cos \theta_f - \cos \theta_i}{u_k} \cdot \frac{1}{\Omega}. \quad (24) \]

We consider an example where the detuning \( \Delta \) changes from \(-10\Omega\) to \(10\Omega\), same as in [6], corresponding to \( \theta_f = \tan^{-1}(1/10), \theta_i = \pi - \theta_f \). We specifically examine the case of the second resonance \((k = 2)\), with duration \( \tilde{T}_2 \approx 2.630\pi/\Omega \) as can be obtained from equation (24).

In figure 2(a) the blue solid line displays the variation of the polar angle from \( \theta_i \) to \( \theta_f \). With red solid line is depicted the corresponding trajectory of the Bloch vector in the original reference frame. The same trajectory but in the adiabatic frame is displayed in figure 2(b), again with red solid line. The blue solid line corresponds to the total field, which in this frame points constantly in the \( \hat{z} \)-direction. Note that in this reference frame the system starts from the adiabatic state at the north pole, traverses the whole red circle and returns there at the half time \( \tilde{T}_2/2 \), corresponding to the point where the trajectory in the original frame passes through the equator. Then, the system in the adiabatic frame repeats this trajectory and at the final time returns to the north pole, while in the original reference frame arrives at the target point.

At this point it is worth mentioning that shortcuts to adiabaticity working for specific durations, like above, have been obtained for quantum teleportation [45] with two control fields (Stokes and pump fields in STIRAP notation) [46, 47], as well as for the quantum parametric oscillator [48, 49].

3. Modified Roland–Cerf protocol as an optimal control problem in the adiabatic reference frame

In the previous section we showed that the classical RC protocol, with constant control \( u = -d\theta/d\tau \) in the rescaled time, achieves perfect AP for specific durations \( T_k \) and amplitudes \( u_k \). In the present section we explain how we can generalize this procedure and obtain perfect fidelity for arbitrary durations larger than the lower bound \( T_0 = \pi \) in the rescaled time, which we derive below. The main idea is to apply a modified RC protocol with time-dependent bounded control \( 0 \leq u(\tau) \leq v \), and then use optimal control theory to obtain the minimum-time pulse-sequence which satisfies all the desired conditions, for specific maximum amplitude \( v \). Note that the nonnegativity of \( u(\tau) \) assures that the magnetic field angle \( \theta \) decreases monotonically from \( \theta_i \) to \( \theta_f \). On the other hand, as the upper bound \( v \) increases, the duration of the optimal pulse-sequence decreases, approaching the limit \( T_0 = \pi \).

In order to formulate the corresponding optimal control problem in the adiabatic frame, we will use the Bloch equations corresponding to the two-level system (17). If we define as new state variables the components of the Bloch vector

\[ s_x = b_1^* b_2 + b_1 b_2^*, \quad (25a) \]

\[ s_y = \frac{b_1^* b_2 - b_1 b_2^*}{i}, \quad (25b) \]

\[ s_z = |b_1|^2 - |b_2|^2, \quad (25c) \]
it is not hard to verify that they satisfy the following equations

\[ \dot{s}_x = -s_y - us_z, \]  
\[ \dot{s}_y = s_x, \]  
\[ \dot{s}_z = us_x, \]  

or, in a more compact form

\[ \dot{s} = (Z - uY)s, \]  

where \( s = (s_x, s_y, s_z)^T \) and

\[ X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]  

Since the matrices in equation (28) are antisymmetric, the system equation (27) can take the form

\[ \dot{s} = (\hat{z} - u\hat{y}) \times s, \]  

where \( \times \) denotes the vector cross product and

\[ \hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]  

are the axes unit vectors.

Figure 2. Trajectories on the Bloch sphere corresponding to the second resonance \( k = 2 \) of the traditional Roland–Cerf protocol and for detuning variation from \(-10\Omega\) to \(10\Omega\). The duration of the protocol is found to be \( T_2 \approx \frac{2.630\pi}{\Omega} \). (a) The red solid line displays the trajectory of the Bloch vector in the original reference frame. The blue solid line on the \( xz \)-plane shows the change in the polar angle \( \theta \) of the total field. (b) The red solid line displays again the state trajectory but in the adiabatic reference frame. Note that in this frame the system state returns to the north pole after traversing the red circle twice, while the total field, shown with blue solid line, points constantly in the up direction.
We can now formulate the optimal control problem for system (27) or (29). Starting from the north pole \( s = (0, 0, 1)^T \), we would like to find the bounded control \( 0 \leq u(\tau) \leq v \) with specified area \( \int_0^T u(\tau)d\tau = \theta_i - \theta_f \) which minimizes the time \( T = \int_0^T 1d\tau \) needed to return to the starting point. In the following section we analyze the solutions to this problem using optimal control theory. Before doing so, we explain how is obtained the lower bound to the starting point. In the following section we analyze the solutions to this problem using optimal control theory.

In the original reference frame the total field changes instantaneously from \( \theta = \theta_i \) to \( \theta = \theta_f = (\theta_i + \theta_f)/2 \), i.e. moves to the middle of the arc between the initial and final target states. The detuning which corresponds to this position of the total field is \( \Delta = \Omega \cot \bar{\theta} \), while the field strength is \( \sqrt{\Delta^2 + \Omega^2} = \Omega / \sin \bar{\theta} \). The total field remains fixed for a duration

\[
\bar{T}_0 = \sin \bar{\theta} \frac{\pi}{\Omega} = \sin \frac{\theta_i + \theta_f}{2} \frac{\pi}{\Omega}.
\]  

(31)

during which the Bloch vector rotates from the initial state \( (\phi = 0, \theta_i) \) to the final state \( (\phi = 0, \theta_f) \). Finally, the total field moves instantaneously from \( \theta = \bar{\theta} \) to \( \theta = \theta_f \) and is aligned with the final state of the Bloch vector. During the above described evolution it is \( \theta = \bar{\theta} \) except the (measure zero) initial and final instants, thus equation (16) gives \( d\tau = \Omega dt / \sin \bar{\theta} \) and the corresponding rescaled time duration is found to be

\[
T_0 = \frac{\Omega \bar{T}_0}{\sin \bar{\theta}} = \pi.
\]  

(32)

Note that the quantum speed limit (in original time) is \( T_{qsl} = (\theta_i - \theta_f)/\Omega \), as obtained in [14] and formally proved in [15], see also [28, 29, 50]. Its derivation makes use of infinite detuning values, corresponding to instantaneous rotations around \( z \)-axis, while angle \( \theta \) varies non-monotonically. Speed limits for bounded detuning have been found in [16], but the corresponding pulse-sequences include discontinuous and non-monotonic changes of the angle \( \theta \). On the other hand, the bounds in equations (31) and (32) are derived using finite values for the detuning while the angle \( \theta \) changes monotonically (decreases for \( \theta_i > \theta_f \)). For \( \theta_i \approx \pi \) and \( \theta_f \approx 0 \), we have \( T_{qsl} \approx \bar{T}_0 \approx \pi/\Omega \), as obtained in [51].

4. Analysis of the optimal solution

Let \( \lambda = (\lambda_x, \lambda_y, \lambda_z) \) be the time-dependent row vector of Lagrange multipliers corresponding to system equations and \( \mu \) the constant multiplier corresponding to the integral condition for the pulse area. The control Hamiltonian for the previously formulated problem incorporates the cost (time) both the integral condition and the system equation

\[
H_c = 1 + \mu u + \lambda \cdot \dot{s} = 1 + (\mu - \lambda \cdot Y)xu + \lambda \cdot Zs = 1 + (\mu + \lambda_x s_x - \lambda_z s_z)u + \lambda_y s_y - \lambda_x s_y.
\]  

(33)

Using Hamilton’s equations \( \dot{s}_\alpha = -\partial H_c / \partial s_{\alpha}, \alpha = x, y, z \), we find the following equation for the adjoint variables

\[
\dot{\lambda} = -\lambda(Z - uy).
\]  

(34)

Note that multiplier \( \mu \) is constant since the corresponding coordinate, angle \( \theta \), is cyclic.

According to Pontryagin Maximum Principle [52], the optimal control \( 0 \leq u(\tau) \leq v \) is chosen to minimize \( H_c \). If we define the functions
then the control Hamiltonian can be expressed as $H_c = 1 + \phi_y u + \phi_z$. Note that $\phi_y, \phi_z$ are the Hamiltonian functions corresponding to vector fields $-Ys, Zs$ appearing in state equation (27), while $\phi_x$ is the Hamiltonian function corresponding to their commutator, vector field $-Xs$. As we shall shortly see, they obey a closed set of differential equations which can be used to study the behavior of the quantity of interest, which is $\phi_y$ as we immediately explain. Obviously, $H_c$ is a linear function of the bounded control $0 \leq u \leq v$ with coefficient $\phi_y$, the so-called switching function. The optimal $u$ minimizing $H_c$ is $u = 0$ for $\phi_y > 0$ and $u = v$ for $\phi_y < 0$. If $\phi_y = 0$ for some finite time interval, then $u$ takes some intermediate value which cannot be found from Maximum Principle. However, if $\phi_y(\tau) = 0$ and $\dot{\phi}_y(\tau) \neq 0$, then at time $\tau$ the control switches between its boundary values and we call this a bang-bang switch. In the present article we concentrate on bang-bang solutions, i.e., pulse-sequences of the form ‘on-off-on...-on-off-on’, where $u(\tau)$ alternates between 0 and its maximum value $v$, as displayed in figure 3. For each value of parameter $v$ we will find the timings of the corresponding optimal pulse-sequence.

We start by showing geometrically that in the optimal bang-bang pulse-sequence all the ‘off’ pulses have the same duration, say $\tau_2$, and all the intermediate ‘on’ pulses (i.e. aside the first and the last) have the same duration, say $\tau_3$. Using the equations for the state and adjoint variables we can show that the vector $\Phi = (\phi_1, \phi_2, \phi_3)^T = (\phi_x, \phi_y - \mu, \phi_z)^T$ obeys the following equation

$$\dot{\phi} = (Z + uY)\phi,$$ (36)

or

$$\dot{\phi} = (\bar{z} + uw) \times \phi,$$ (37)

if we use vectors instead of antisymmetric matrices. From the last equation it is obvious that the motion of $\phi$ is restricted on a sphere,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = \phi_1^2 + (\phi_y - \mu)^2 + \phi_z^2 = \text{constant.}$$ (38)

Now suppose that at time $\tau$ there is a switching from $u = v$ to $u = 0$. This means that $\phi_y(\tau) = 0$, which also implies $\phi_2(\tau) = -\mu$, thus the switching point $P(\phi_1, -\mu, \phi_3)$ lies on the plane $\phi_2 = -\mu$, shown with green color in figure 4, while $\phi_1, \phi_3$ denote the other two coordinates of $P$. The control $u = 0$ is applied for duration $\tau_2$ and $\phi$ is rotated around $z$-axis along the horizontal black arc displayed in figure 4. Note that during this interval it is $\phi_2 > -\mu \Rightarrow \phi_y > 0$, thus $u = 0$ minimizes indeed the control Hamiltonian. At time $\tau + \tau_2$ the trajectory intersects the switching plane $\phi_2 = -\mu$ at point $Q(-\phi_1, -\mu, \phi_3)$, the symmetric of $P$ with respect to the $\phi_2\phi_3$-plane. Since we consider bang-bang pulse-sequences, the control switches from $u = 0$ to $u = v$. Vector $\phi$ is now rotated around the (red) axis $\mu = \bar{z} + uw$ for duration $\tau_3$, along the inclined red arc shown in figure 4. During this time interval it is $\phi_2 < -\mu \Rightarrow \phi_y < 0$, thus $u = v$ minimizes indeed the control Hamiltonian. At time $\tau + \tau_2 + \tau_3$ the trajectory meets again the switching plane $\phi_2 = -\mu$; we will show that this intersection takes place at point $P$. During the rotation around axis $\mu = \bar{z} + uw$, the inner product $\phi \cdot \mu = \phi_1 + v\phi_3$ is constant.

But $\phi_2(\tau + \tau_2 + \tau_3) = -\mu = \phi_2(\tau + \tau_2)$, thus $\phi_3(\tau + \tau_2 + \tau_3) = \phi_3(\tau + \tau_2) = \phi_3$. Since
Figure 3. Candidate optimal control \( u(\tau) \) in rescaled time \( \tau \). The ‘on’ pulses at the beginning and at the end have a common duration \( \tau_1 \), all the ‘off’ pulses have a common duration \( \tau_2 \), and all the intermediate ‘on’ pulses have a common duration \( \tau_3 \). The middle pulse can be ‘off’, as shown in this figure, or ‘on’. The total duration of the pulse-sequence is denoted by \( T \). Figure similar to figures from our recent work [40], © American Physical Society.

Figure 4. Trajectory of vector \( \phi \) for \( u = 0 \), black horizontal arc corresponding to a rotation around the vertical black axis, and \( u = v \), inclined red arc corresponding to a rotation around the tilted red axis. On the switching plane \( \phi_2 = -\mu \) the control changes from the one boundary value to the other and the evolution repeats itself.
the motion is restricted on the sphere (38), we easily deduce that \(\phi_1(\tau + \tau_2 + \tau_3) = \bar{\phi}_1\). The trajectory thus intersects the switching plane at the point \(P(\bar{\phi}_1, -\mu, \phi_1)\), and the evolution is repeated for all the subsequent ‘off’ and intermediate ‘on’ pulses. The conclusion is that all the ‘off’ pulses have a common duration \(\tau_2\), while all the intermediate ‘on’ pulses have a common duration \(\tau_3\).

The ‘on’ pulses at the beginning and at the end can have different durations than \(\tau_3\), corresponding to incomplete traversals of the red arc shown in figure 4. Since the system (27) starts from and comes back to the same point, the north pole, for symmetry reasons we take the initial and final ‘on’ pulses to have common duration \(\tau_1\). Thus, we consider candidate optimal pulse-sequences of the form shown in figure 3 and the optimization takes place within this subset. In the following we use geometric optimal control [53] to derive a relation between the pulse durations \(\tau_1, \tau_2, \tau_3\). This relation will be exploited in the next section, along with the integral condition for the pulse area and the final time condition that the system returns to the north pole, in order to obtain these durations when the maximum control amplitude \(\nu\) is given. In the rest of this section we particularly use the theory developed in [54], as specified for the two-level quantum system in [55, 56], while we adapt it to incorporate the pulse area condition.

Observe from the second line of equation (33) that the switching function can be expressed as \(\phi_\nu = \mu - \lambda \cdot Ys\), thus at a switching time \(\tau\) it holds

\[
\phi_\nu(\tau) = 0 \Rightarrow \lambda(\tau) \cdot Ys(\tau) = \mu. \tag{39}
\]

For the first three switchings at times \(\tau = \tau_1, \tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3\) we have

\[
\lambda(\tau_1) \cdot Ys(\tau_1) = \lambda(\tau_1 + \tau_2) \cdot Ys(\tau_1 + \tau_2) = \lambda(\tau_1 + \tau_2 + \tau_3) \cdot Ys(\tau_1 + \tau_2 + \tau_3) = \mu. \tag{40}
\]

We will express the first and third terms of the above equation at the middle time \(\tau = \tau_1 + \tau_2\). During the interval \(\tau_1 < \tau \leq \tau_1 + \tau_2\) the control is \(u = 0\). From equations (27), (34) for the state and adjoint variables \(s, \lambda\) we have

\[
s(\tau_1) = e^{-\tau_2Z} s(\tau_1 + \tau_2), \quad \lambda(\tau_1) = \lambda(\tau_1 + \tau_2) e^{\tau_2Z},
\]

thus

\[
\lambda(\tau_1) \cdot Ys(\tau_1) = \lambda(\tau_1 + \tau_2) \cdot Ys(\tau_1 + \tau_2), \tag{41}
\]

where

\[
Y_1 = e^{\tau_2Z} Ye^{-\tau_2Z} = \cos \tau_2 Y - \sin \tau_2 X. \tag{42}
\]

Note that in the derivation of the last equation we have used the commutation relations \([X, Y] = Z, [Y, Z] = X, [Z, X] = Y\). Analogously, during the interval \(\tau_1 + \tau_2 < \tau \leq \tau_1 + \tau_2 + \tau_3\) the control is \(u = \nu\), thus

\[
s(\tau_1 + \tau_2 + \tau_3) = e^{\nu(Z-Y)} s(\tau_1 + \tau_2), \quad \lambda(\tau_1 + \tau_2 + \tau_3) = \lambda(\tau_1 + \tau_2) e^{-\nu(Z-Y)}
\]

and

\[
\lambda(\tau_1 + \tau_2 + \tau_3) \cdot Ys(\tau_1 + \tau_2 + \tau_3) = \lambda(\tau_1 + \tau_2) \cdot Ys(\tau_1 + \tau_2), \tag{43}
\]

where

\[
Y_2 = e^{-\nu(Z-Y)} Ye^{\nu(Z-Y)} \tag{44}
\]

\[
= \frac{\sin \omega \tau_3}{\omega} X + \frac{u^2 + \cos \omega \tau_3}{\omega^2} Y - \frac{u(1 - \cos \omega \tau_3)}{\omega^2} Z \tag{45}
\]

and

\[
\omega = \sqrt{1 + \nu^2}. \tag{46}
\]
Using equations (41), (43), (40) becomes
\[ \lambda(\tau_1 + \tau_2) \cdot Ys(\tau_1 + \tau_2) = \lambda(\tau_1 + \tau_2) \cdot Ys(\tau_1 + \tau_2) = \lambda(\tau_1 + \tau_2) \cdot Ys(\tau_1 + \tau_2) = \mu. \] (47)

Since \( Y, Y_1, Y_2 \) are antisymmetric matrices, the above equation can be expressed using the corresponding vectors \( \hat{y}, \hat{y}_1, \hat{y}_2 \),
\[ y_1 = \cos \tau_2 \hat{y} - \sin \tau_2 \hat{x}, \] (48)
\[ y_2 = \frac{\sin \omega \tau_1}{\omega} \hat{x} + \frac{\mu^2 + \cos \omega \tau_2}{\omega^2} \hat{y} - \frac{u(1 - \cos \omega \tau_2)}{\omega^2} \hat{z}, \] (49)
as
\[ \lambda(\tau_1 + \tau_2) \cdot y_1 \times s(\tau_1 + \tau_2) = \lambda(\tau_1 + \tau_2) \cdot \hat{y} \times s(\tau_1 + \tau_2) = \lambda(\tau_1 + \tau_2) \cdot y_2 \times s(\tau_1 + \tau_2) = \mu, \] (50)
from which we obtain
\[ \lambda(\tau_1 + \tau_2) \cdot (\hat{y} - y_1) \times s(\tau_1 + \tau_2) = \lambda(\tau_1 + \tau_2) \cdot (\hat{y} - y_2) \times s(\tau_1 + \tau_2) = 0. \] (51)

Note that \( \lambda(\tau_1 + \tau_2) \neq \mathbf{0} \), since otherwise \( \mu = 0 \) from equation (50) and the homogeneous equation (34) would imply \( \lambda = \mathbf{0} \) for all times, i.e. all the multipliers would be zero, something which contradicts Maximum Principle [52]. Now, according to the above equation, the nonzero vector \( \lambda(\tau_1 + \tau_2) \) is perpendicular to the cross product \( (\hat{y} - y_1) \times s(\tau_1 + \tau_2) \), thus the vectors \( \lambda(\tau_1 + \tau_2), \hat{y} - y_1, s(\tau_1 + \tau_2) \) are coplanar. Analogously we show that the vectors \( \lambda(\tau_1 + \tau_2), \hat{y} - y_2, s(\tau_1 + \tau_2) \) are also coplanar. The conclusion is that the vectors \( \hat{y} - y_1, \hat{y} - y_2, s(\tau_1 + \tau_2) \) are coplanar, where
\[ s(\tau_1 + \tau_2) = e^{\tau_2}e^{\tau_1(\mathbf{Z} - \mathbf{W})} s(0) = \begin{pmatrix} -n_z \sin \omega \tau_1 \cos \tau_2 + n_z^2 (1 - \cos \omega \tau_1) \sin \tau_2 \\ -n_z \sin \omega \tau_1 \sin \tau_2 - n_z^2 (1 - \cos \omega \tau_1) \cos \tau_2 \\ n_z^2 + n_z^2 \cos \omega \tau_1 \end{pmatrix} \] (52)
and
\[ n_z = \frac{v}{\omega} = \frac{v}{\sqrt{1 + v^2}}, \quad n_z = \frac{1}{\omega} = \frac{1}{\sqrt{1 + v^2}}. \] (53)

Three coplanar vectors are linearly dependent, thus
\[ \det(\hat{y} - y_1, \hat{y} - y_2, s(\tau_1 + \tau_2)) = 0, \]
leading to
\[ A \sin \tau_2 + B(1 - \cos \tau_2) = 0, \] (54)
where
\[ A = (1 - \cos \omega \tau_3)[n_z + n_z^2(n_y - n_z)(1 - \cos \omega \tau_1)], \] (55a)
\[ B = n_z^2 \sin \omega \tau_1 + n_z^2 \sin \omega \tau_3 + n_z^2 \sin[\omega(\tau_3 - \tau_1)]. \] (55b)
Equation (54) is the optimality condition between the pulse durations \( \tau_1, \tau_2, \tau_3 \), for a given maximum control amplitude \( v \) included in \( \omega, n_y, n_z \).
5. Optimal pulse-sequences

In this section we use optimality condition (54), along with the pulse area condition \( \int_0^\tau u(\tau)d\tau = \theta_i - \theta_f \) and the final condition that the system returns to the north pole, in order to obtain the timings \( \tau_1, \tau_2, \tau_3 \) for the optimal pulse-sequences. Let us consider a pulse-sequence \( u(\tau) \) containing \( m \) ‘off’ pulses, where \( m = 1, 2, \ldots \) is a positive integer. All ‘on’ pulses have common amplitude \( \nu \), thus the total variation of \( \theta \) is

\[
\theta_i - \theta_f = \nu[2\tau_1 + (m - 1)\tau_3]
\]

and

\[
\tau_1 = \frac{1}{2} \left[ \frac{\theta_i - \theta_f}{\nu} - (m - 1)\tau_3 \right].
\]  

(56)

Next, observe that equation (54) can be solved with respect to \( \tau_2 \)

\[
\tau_2 = 2 \cot^{-1} \left( \frac{-B}{\lambda} \right),
\]

(57)

where note from equations (55a) and (55b) that \( A, B \) are functions of \( \tau_1, \tau_3 \) only. Since \( \tau_1 \) is expressed as a function of \( \tau_3 \) in equation (56), obviously \( \tau_2 \) can also be expressed as a function of \( \tau_3 \) only.

The last relation that we need is obtained from the final time condition that the system returns to the north pole. Instead of the Bloch system (29), it is more convenient to use system (17), for which the corresponding condition at the final time \( \tau = T \) is that it returns to the adiabatic state \( |\phi_+\rangle \). For the piecewise constant control \( u(\tau) \), the propagator \( U \) determining the final state from the starting state, \( b(T) = Ub(0) \), takes the form

\[
U = U_1W_2U_3 \ldots W_2 U_3 \ldots U_5W_2U_1,
\]

(58)

where \( U_j, j = 1, 3, \) is given by

\[
U_j = e^{-i\lambda_\tau \theta_j} = e^{-i\frac{\lambda}{2}\omega\tau_\sigma(\eta_1\sigma_z - \eta_3\sigma_y)} = I \cos \frac{\omega T_j}{2} - i \sin \frac{\omega T_j}{2}(\eta_1\sigma_z - \eta_3\sigma_y),
\]

(59)

and

\[
W_2 = e^{-i\lambda_\tau \sigma_z} = I \cos \frac{\tau_2}{2} - i \sin \frac{\tau_2}{2}\sigma_z.
\]

(60)

The propagator in the middle of (58) is \( W_2 \) or \( U_3 \), whether the middle pulse is ‘off’ or ‘on’. If we use the above equations for \( U_1, W_2, U_3 \) in equation (58) and additionally exploit the well-known property of Pauli matrices

\[
\sigma_a\sigma_b = \delta_{ab}I + i\epsilon_{abc}\sigma_c,
\]

(61)

with \( a, b, c \) any of \( x, y, z \), \( \delta_{ab} \) the Kronecker delta and \( \epsilon_{abc} \) the Levi–Civita symbol, then propagator \( U \) can be expressed as the following linear combination

\[
U = a_1I + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z,
\]

(62)

where coefficients \( a_x, a_y, a_z, a_l \) depend on the parameters of the pulse-sequence.

In the appendix we prove the relation \( a_x = 0 \) and, since the matrices \( I, \sigma_z \) are diagonal, if we additionally set \( a_x = 0 \) in equation (62) then propagator \( U \) becomes also diagonal. In this case, if we start from \( b(0) = (1 0)^T \), we obtain for the final state \( b(T) = Ub(0) \) the relation \( b_3(T) = 0 \), thus the system returns to the initial adiabatic state. Equation

\[
...
\[ a_{y,m}(\tau_1, \tau_2, \tau_3, v) = 0 \]  \hspace{1cm} (63)

is used, along with equations (56) and (57), for the determination of the timing parameters \(\tau_1, \tau_2, \tau_3\) in the optimal pulse-sequence. Note that subscript \(m\) in equation (63) indicates that coefficient \(a\) has a different functional dependence for controls with different number \(m\) of ‘off’ pulses. Following the procedure described in the appendix, we have found \(a_{y,m}\) for \(m = 1, 2, 3\),

\[
a_{y,1} = \frac{1}{2} \text{Tr}(\sigma, U) = \frac{1}{2} \text{Tr}(\sigma_1 U_1 W_2 U_1) = \frac{1}{2} \text{Tr}(U_1 \sigma_1 U_1 W_2)
\]

\[
= 2 i n_1 \sin (\omega \tau_1 / 2) \left[ \cos (\omega \tau_1 / 2) \cos (\tau_2 / 2) - n_2 \sin (\omega \tau_1 / 2) \sin (\tau_2 / 2) \right],
\]

\[
a_{y,2} = \frac{1}{2} \text{Tr}(\sigma, U) = \frac{1}{2} \text{Tr}(\sigma_1 U_2 W_2 U_1) = \frac{1}{2} \text{Tr}(U_2 \sigma_1 U_2 W_2)
\]

\[
= i n_1 \cos (\omega \tau_3 / 2) \left[ \sin \omega \tau_1 \cos \tau_2 - n_2 \sin \tau_2 (1 - \cos \omega \tau_1) \right]
\]

\[
+ i n_3 \sin (\omega \tau_3 / 2) \left\{ \cos \omega \tau_1 + n_2 \left[ - \sin \omega \tau_1 \sin \tau_2 + n_2 (1 - \cos \omega \tau_1) (1 - \cos \tau_2) \right] \right\}, \hspace{1cm} (64)
\]

\[
a_{y,3} = \frac{1}{2} \text{Tr}(\sigma, U) = \frac{1}{2} \text{Tr}(\sigma_1 U_2 W_3 U_3 W_2 U_3)
\]

\[
= i n_1 \left[ \cos (\tau_2 / 2) \cos \omega \tau_3 - n_2 \sin (\tau_2 / 2) \sin \omega \tau_3 \right]
\]

\[
\times \left[ \sin \omega \tau_1 \cos \tau_2 - n_2 \sin (1 - \cos \omega \tau_1) \right]
\]

\[
+ i n_3 \left[ n_2 \cos (\tau_2 / 2) \sin \omega \tau_3 + \sin (\tau_2 / 2) (n_2^2 + n_1^2 \cos \omega \tau_3) \right]
\]

\[
\times \left[ - \sin \omega \tau_1 \sin \tau_2 + n_2 (1 - \cos \omega \tau_1) (1 - \cos \tau_2) \right]
\]

\[
+ i n_3 \left[ \cos \omega \tau_1 \cos (\tau_2 / 2) \sin \omega \tau_3 - n_2 \sin (\tau_2 / 2) (1 - \cos \omega \tau_1 \cos \omega \tau_3) \right]. \hspace{1cm} (65)
\]

Observe that for \(m = 1\), i.e. the simplest ‘on–off–on’ pulse-sequence, there are no intermediate ‘on’ pulses, thus \(\tau_3 = 0\). In this case, equations (55a) and (55b) give \(A = B = 0\), and the optimality condition (54) is automatically satisfied. From equation (56) we have \(\tau_1 = (\theta_1 - \theta_f) / (2v)\), while equation \(a_{y,1} = 0\) becomes a transcendental equation for unknown \(\tau_2\). For \(m > 1\), using equations (56) and (57) in (63), we end up with a transcendental equation for \(\tau_3\). For each value of the maximum control amplitude \(v > 0\), the transcendental equations corresponding to different \(m\) may or may not have solutions. For each solution we find the total duration \(T\) of the corresponding pulse-sequence and compare the results. The pulse-sequence with the minimum \(T\) is the optimal one for the specific value of \(v\).

As in section 2, we consider a specific example where the detuning varies from \(\Delta_1 = -10\Omega\) to \(\Delta_2 = 10\Omega\), corresponding to \(\theta_f = \tan^{-1}(1/10)\), \(\theta_1 = \pi - \theta_f\). In figure 5 we plot the duration of the optimal pulse-sequence for a range of \(v\) values, both in the rescaled time, figure 5(a), and in the original time, figure 5(b). Note that the duration in the rescaled time is larger than the corresponding duration in the original time due to the sine factor in equation (16). The diagrams display a stairway-like form, where the circles separating the steps are the points \((u_i, T_i)\) obtained in section 2 where the original RC protocol, with constant control \(u(\tau) = u_0\), is optimal. We have obtained similar diagrams in our other works on optimal control of quantum systems [57, 58]. On the first step from the right (larger values of \(v\), the optimal pulse-sequence has the simple ‘on–off–on’ form, with \(m = 1\). Note that the solutions lying on this step are faster than the first resonance of the original RC protocol (first circle from the right). For large values of \(v\) the duration of these solutions tends to the limit \(T_0 = \pi\). On the second step, the optimal pulse-sequence changes to ‘on–off–on–off–on’, with \(m = 2\).
the third step becomes ‘on–off–on–off–on–off–on’ with \( m = 3 \), and so forth. Note that these solutions with more switchings may require longer times, but the corresponding maximum control amplitude \( v \) is smaller and thus the change in the total field angle \( \theta \) is less abrupt, a property which might be useful when designing a pulse-sequence.

In figure 6 we present a specific example of the optimal pulse-sequence for maximum control amplitude \( v = 0.35 \), the case highlighted with a red star in figure 5. Since this point lies on the second step of the stairway-like diagram, the corresponding optimal pulse-sequence has the ‘on–off–on–off–on–off–on’ form. In figure 6(a) we display the logarithmic error

\[
\log_{10} (1 - F) = \log_{10} |b_2(T)|^2 = \log_{10} |a_{1,2}|^2
\]

as a function of duration \( \tau \); the ‘resonance’ indicates the solution of the transcendental equation \( a_{1,2} = 0 \). Having found the duration \( \tau_3 \) of the intermediate ‘on’ pulse, we find the durations \( \tau_1 \) (of the ‘on’ pulses at the beginning and at the end) and \( \tau_2 \) (of ‘off’ pulses), using equations (56) and (57), respectively. In figure 6(b) we display the optimal control \( u(\tau) \) in rescaled time \( \tau \). In figure 6(c) we show the detuning \( \Delta(t) \) while in figure 6(d) the change in the polar angle \( \theta(t) \) of the field, both in original time \( t \). The red segments correspond to ‘on’ pulses, while the black segments to ‘off’ pulses. Note that the rescaled time duration is larger than the original time duration due to the sine factor in equation (16). In figure 6(e) the blue solid line shows the variation of the field polar angle \( \theta \), while the red-black solid line the trajectory of the Bloch vector in the original reference frame. The same convention is used here as in the previous figures, with the red segments corresponding to ‘on’ pulses and the black segments to ‘off’ pulses. The red-black solid line in figure 6(f) also displays the state trajectory but in the adiabatic frame. Here the Bloch vector starts and finally returns to the adiabatic state at the north pole, while the total field, indicated with blue solid line, always points in the up direction. Also, observe that the trajectory in the adiabatic frame includes a loop, something forbidden for the solution of a minimum-time optimal control problem. The resolution of this
Figure 6. Specific example for maximum control amplitude $v = 0.35$, corresponding to the case highlighted with a red star in figure 5. (a) Logarithmic error $\log_{10}(1 - F) = \log_{10}|\alpha_{t,2}|^2$ as a function of duration $\tau_i$; the ‘resonance’ indicates the solution of the transcendental equation $\alpha_{t,2} = 0$. (b) Optimal pulse-sequence in the rescaled time $\tau$. (c) Detuning $\Delta(t)$ in the original time $t$. The red segments correspond to ‘on’ pulses, the black segments to ‘off’ pulses. (d) Total field angle $\theta(t)$ in the original time $t$. (e) State trajectory (red-black solid line) on the Bloch sphere in the original reference frame. The blue solid line on the meridian lying on the $xz$-plane indicates the change in the total field angle $\theta$ (d) State trajectory (red-black solid line) on the Bloch sphere in the adiabatic frame. Observe that in this frame the state of the system returns to the north pole, while the total field points constantly in the $\hat{z}$-direction (blue solid line). Figure similar to figures from our recent work [40], © American Physical Society.
paradox is that the polar angle $\theta$, evolving from $\theta_i$ to $\theta_f$, is an extra state variable which is not displayed in the considered frame. The optimal trajectory contains no loop if we display it in the higher-dimensional space which includes $\theta$.

We next clarify the advantage of the present approach compared to our previous related work [40]. There, we fix the total duration $T = 2\tau_1 + m\tau_2 + (m - 1)\tau_3$ of the pulse-sequence in the rescaled time, while we take the amplitude $v$ as an unknown parameter. This relation, along with the pulse area condition (56) and the final condition (63), form a system of three equations with four unknowns, $\tau_1, \tau_2, \tau_3, v$. In order to tackle this problem, we calculate numerically the minimum amplitude $v$ which permits a solution of this system with respect to $\tau_1, \tau_2, \tau_3$. In the present article we follow a dual approach, where we fix amplitude $v$ and seek the pulse-sequence with minimum duration which satisfies the area and final conditions. The use of optimal control theory leads to the optimality condition (54) which, along with equations (56) and (63), form a system of three equations for the three unknowns $\tau_1, \tau_2, \tau_3$.

We close this section with some comments regarding the possible experimental implementation of the suggested methodology. In [28], the simple Roland–Cerf protocol described in section 2 and a minimum-time bang-bang protocol with jumps in $\Delta(t)$ are implemented using Bose–Einstein condensates in optical lattice potentials. In the present work, $\Delta(t)$ does not contain jumps but changes continuously in time. As can be observed from figures 6(c) and (d), there are intervals where the control changes continuously followed by intervals where it remains fixed. Based on the above observations we believe that the suggested control protocol is experimentally feasible. In the case where nevertheless one would like to avoid discontinuities in $\dot{\Delta}$ or $\dot{\theta}$, there is the excellent study [6] by Martinis and Geller. There, the derivative $d\theta/d\tau$ is expanded in a series of trigonometric functions of rescaled time (instead of the ‘on–off’ pulse-sequence that we use here), with coefficients chosen numerically to minimize the spin-flip error probability. The conclusion is that an error probability less than $10^{-4}$ can be obtained for large enough durations. The ‘on–off’ pulse-sequences in rescaled time that we use in the current work, permit the detailed analytical study that we present.

6. Conclusion

In this article, we presented a new method for speeding up adiabatic passage in a two-level system with only detuning ($z$-field) control. This technique is actually a modification of the Roland–Cerf protocol, where now the local adiabaticity parameter is not held constant but has a simple ‘on–off’ modulation. Using optimal control theory, we derived pulse-sequences which can obtain perfect fidelity for durations exceeding the speed limit $\pi/\Omega$, with $\Omega$ being the constant Rabi frequency (transverse field). The detuning (longitudinal field) varies continuously and monotonically with time. This work can be directly applied to the design of high fidelity controlled-phase gates for quantum computation, as well as to other research areas where adiabatic passage is exploited.

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Appendix

We first show that $a_x = 0$ in equation (62). From equations (58), (62) we obtain

\[ a_x = \frac{1}{2} \text{Tr}(\sigma_x U) \]
\[ = \frac{1}{2} \text{Tr}(\sigma_x U_1 W_2 U_3 \ldots W_2 or U_3 \ldots U_3 W_2 U_1) \]
\[ = \frac{1}{2} \text{Tr}(\ldots U_3 W_2 U_1 \sigma_x U_1 W_3 \ldots W_2 or U_3). \]  \hspace{1cm} (A.1)

But, using the explicit expressions (59), (60) for $U_1, W_2, U_3$ and relation (61), we find

\[ U_1 \sigma_x U_1 = W_2 \sigma_x W_2 = U_3 \sigma_x U_3 = \sigma_x. \]  \hspace{1cm} (A.2)

If we use these expressions repeatedly in equation (A.1), then calculating $a_x$ can be reduced to calculating $\text{Tr}(\sigma_x W_2)$ or $\text{Tr}(\sigma_x U_3)$, for ‘off’ or ‘on’ middle pulse, respectively. But $\text{Tr}(\sigma_x W_2) = \text{Tr}(\sigma_x U_3) = 0$, thus $a_x = 0$ as well.

Coefficient $a_y$ in equation (62) can obtained from a relation similar to equation (A.1),

\[ a_y = \frac{1}{2} \text{Tr}(\sigma_y U) \]
\[ = \frac{1}{2} \text{Tr}(\sigma_y U_1 W_2 U_3 \ldots W_2 or U_3 \ldots U_3 W_2 U_1) \]
\[ = \frac{1}{2} \text{Tr}(\ldots U_3 W_2 U_1 \sigma_y U_1 W_3 \ldots W_2 or U_3). \]  \hspace{1cm} (A.3)

using repeatedly the equations

\[ U_1 \sigma_y U_1 = in_y \sin \omega \tau_1 I + (n_x^2 + n_z^2 \cos \omega \tau_1) \sigma_y + n_z n_y (1 - \cos \omega \tau_1) \sigma_z, \]  \hspace{1cm} (A.4a)

\[ W_2 \sigma_y W_2 = \sigma_y, \]  \hspace{1cm} (A.4b)

\[ W_2 \sigma_y W_2 = -i \sin \tau_2 + \cos \tau_2 \sigma_z, \]  \hspace{1cm} (A.4c)

\[ U_3 \sigma_y U_3 = in_y \sin \omega \tau_3 I + (n_x^2 + n_z^2 \cos \omega \tau_3) \sigma_y + n_z n_y (1 - \cos \omega \tau_3) \sigma_z, \]  \hspace{1cm} (A.4d)

\[ U_3 \sigma_y U_3 = -in_y \sin \omega \tau_3 I + n_x n_z (1 - \cos \omega \tau_3) \sigma_y + (n_x^2 + n_z^2 \cos \omega \tau_3) \sigma_z, \]  \hspace{1cm} (A.4e)

which can be derived from expressions (59) for $U_1, U_3$ and (60) for $W_2$, as well as property (61).

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