BEYOND EXPANSION II: TRACES OF THIN SEMIGROUPS

JEAN BOURGAIN AND ALEX KONTOROVICH

Abstract. We continue our study of particular instances of the Affine Sieve, producing levels of distribution beyond those attainable from expansion alone. Motivated by McMullen’s Arithmetic Chaos Conjecture regarding low-lying closed geodesics on the modular surface defined over a given number field, we study the set of traces for certain sub-semi-groups of $SL_2(\mathbb{Z})$ corresponding to absolutely Diophantine numbers (see §1.2). In particular, we are concerned with the level of distribution for this set. While the standard Affine Sieve procedure, combined with Bourgain-Gamburd-Sarnak’s resonance-free region for the resolvent of a “congruence” transfer operator, produces some exponent of distribution $\alpha > 0$, we are able to produce the exponent $\alpha < 1/4$ unconditionally. Furthermore, we obtain the improved exponent $\alpha < 1/3$, conditioned on a conjecture about the additive energy of $SL_2(\mathbb{Z})$.

Contents

1. Introduction 2
2. Levels of Distribution and Ingredients 14
3. Preliminaries 17
4. The Main Term Analysis and Initial Manipulations 20
5. Proof of Theorem 1.30 24
6. Proof of Theorem 1.34 30
7. Proof of Theorem 1.38 34
Appendix A. Construction of $\aleph$ 35
Appendix B. Proof of Proposition 3.12 37
References 40

Date: October 29, 2013. 
JB is partially supported by NSF grant DMS-0808042.
AK is partially supported by an NSF CAREER grant DMS-1254788, an Alfred P. Sloan Research Fellowship, and a Yale Junior Faculty Fellowship. Some of this work was carried out thanks to support from the Ellentuck Fund at IAS.
1. Introduction

In this paper, we reformulate McMullen’s (Classical) Arithmetic Chaos Conjecture (see Conjecture 1.6) as a local-global problem for the set of traces in certain thin semigroups, see Conjecture 1.13. Our main goal is to make some partial progress towards this conjecture by establishing strong levels of distribution for this trace set, see §1.5.

1.1. Low-Lying Closed Geodesics With Fixed Discriminant.

This paper is motivated by the study of long closed geodesics on the modular surface defined over a given number field, which do not have high excursions into the cusp. Let us make this precise.

To set notation, let $H$ denote the upper half plane, and let $X = T^1(SL_2(\mathbb{Z}) \setminus \mathbb{H}) \cong SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R})$ be the unit tangent bundle of the modular surface. A closed geodesic $\gamma$ on $X$ corresponds to a hyperbolic matrix $M \in SL_2(\mathbb{Z})$ (more precisely its conjugacy class). Let $\alpha_M \in \partial H$ be one of the two fixed points of $M$, the other being its Galois conjugate $\overline{\alpha_M}$; then $\gamma$ is the projection mod $SL_2(\mathbb{Z})$ of the geodesic connecting $\alpha_M$ and $\overline{\alpha_M}$. We will say that $\gamma$ is defined over the (real quadratic) field $K = \mathbb{Q}(\alpha_M)$ and has discriminant $\Delta_M$, where $\Delta_M$ is the discriminant of $K$. This discriminant is (up to factors of 4) the square-free part of

$$D_M := (\text{tr } M)^2 - 4. \quad (1.1)$$

To study excursions into the cusp, let $\mathcal{Y}(\gamma)$ denote the largest imaginary part of $\gamma$ in the standard upper-half plane fundamental domain for the modular surface. Given a “height” $C > 1$, we say that the closed geodesic $\gamma$ is low-lying (of height $C$) if $\mathcal{Y}(\gamma) < C$. By the well-known connection [Art24, Ser85] between continued fractions and the cutting sequence of the geodesic flow on $X$, the condition that $\gamma$ be low-lying can be reformulated as a Diophantine property on the fixed point $\alpha_M$ of $M$, as follows. Write the (eventually periodic) continued fraction expansion

$$\alpha_M = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \ldots],$$

as usual, where the numbers $a_j$ are called partial quotients. Given any $A \geq 1$, we say that $\alpha_M$ is Diophantine (of height $A$) if all its partial quotients $a_j$ are bounded by $A$. Then $\alpha_M$ being Diophantine of height $A$ is essentially equivalent to $\gamma$ being low-lying of height $C = C(A)$. 


Figure 1. The closed geodesic on the modular surface corresponding to the matrix $M$ in (1.4).

**Question 1.2.** Given a real quadratic field $K$ and a height $C$, can one find longer and longer primitive closed geodesics defined over $K$ which are low-lying of height $C$? Equivalently, given a fixed fundamental discriminant $\Delta > 0$ and a height $A \geq 1$, we wish to find larger and larger (non-conjugate) matrices $M$ so that their fixed points $\alpha_M$ are Diophantine of height $A$, and so that $t = \text{tr}(M)$ solves the Pell equation $t^2 - \Delta s^2 = 4$; cf. (1.1). If solutions exist, how rare/ubiquitous are they?

**Example 1.3.** An example with the field $K = \mathbb{Q}(\sqrt{3})$ of discriminant $\Delta = 12$, and heights $C = 2$, $A = 3$ is illustrated in Figure 1. Here the closed geodesic $\gamma$ corresponds to

$$M = \begin{pmatrix} 80198051 & 50843528 \\ 33895684 & 21489003 \end{pmatrix},$$

with fixed point

$$\alpha_M = \left\lfloor 2, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, 3, 1, 2, 1, 2, 1, 2, 2, 1, 1, 2, 1, 1, 1 \right\rfloor$$

$$= \frac{2521}{2911} (1 + \sqrt{3}) \in K,$$

and

$$D_M = \text{tr}(M)^2 - 4 = 10340256951198912 = \Delta \times (2 \times 41 \times 71 \times 2521)^2.$$
Observe that $\alpha_M$ is Diophantine: the partial quotients of $\alpha_M$ are all at most $A = 3$. Moreover, it is evident from Figure 1 that $\gamma$ is low-lying: we clearly have $\mathcal{Y}(\gamma) < C = 2$.

Slightly changing the matrix $M$ to

$$
M = \begin{pmatrix}
80198051 & 52617107 \\
32753150 & 21489001
\end{pmatrix}
$$

gives a fixed point

$$
\alpha_M = [2, 2, 4, 2, 1, 3, 2, 62, 2, 5, 5, 1, 9, 1, 1, 1] \in \mathbb{Q}(\sqrt{2110256437643}),
$$

with corresponding geodesic $\gamma$ reaching a height of about $\mathcal{Y}(\gamma) \approx 31.5$. This is the more typical situation.

1.2. Arithmetic Chaos.

On one hand, the answer to Question 1.2 is, on average, negative. Indeed, Duke’s equidistribution theorem [Duk88] forces generic closed geodesics to have arbitrarily high excursions into the cusp. On the other hand, McMullen’s (Classical) Arithmetic Chaos Conjecture (see [McM09, McM12] for the dynamical perspective and origin of this problem) predicts that solutions exist, and while not of positive proportion, they should be of positive entropy:

**Conjecture 1.5** (Arithmetic Chaos [McM12]). There exists an absolute height $A \geq 2$ so that, for any fixed real quadratic field $K$, the cardinality of the set

$$
\left\{ [a_0, a_1, \ldots, a_\ell] \in K : 1 \leq a_j \leq A \right\}
$$

(1.6)

grows exponentially, as $\ell \to \infty$.

**Remark 1.7.** Though we have stated the conjecture with some absolute height $A$, McMullen formulated this problem with $A = 2$ (of course $A = 1$ only produces the golden mean). He further suggested it should also hold whenever the corresponding growth exponent exceeds $1/2$, see Remark 1.15.

**Remark 1.8.** As pointed out to us by McMullen, one can also formulate a $\text{GL}_n(\mathbb{Z})$ version of Arithmetic Chaos by strengthening [McM09, Conjecture 1.7 (3)] so as to postulate exponential growth of periodic points and positive entropy, instead of just infinitude.

It is not currently known whether the following much weaker statement is true: for some $A$ and every $K$, the cardinality of the set (1.6) is unbounded. Even worse, it is not known whether (1.6) is eventually non-empty, that is, whether there exists an $A \geq 2$ so that any $K$ contains at least one element which is Diophantine of height $A$. 
Some progress towards Conjecture 1.5 appears in [Woo78, Wil80, McM09, Mer12], where special periodic patterns of partial quotients are constructed to lie in certain prescribed real quadratic fields. (In particular, we constructed our example (1.4) by following Wilson’s algorithm [Wil80, p. 139].) These results prove that for any $K$, there exists an $A = A(K)$ so that the cardinality of (1.6) is unbounded with $\ell$; but exponential growth is not known in a single case.

In light of this conjecture, we call a number absolutely Diophantine if it is Diophantine of height $A$ for some absolute constant $A \geq 1$. That is, when we speak of a number being absolutely Diophantine, the height $A$ is fixed in advance. In the next subsection, we describe a certain “local-global” conjecture which has the Arithmetic Chaos Conjecture as a consequence.

1.3. The Local-Global Conjecture.

First we need some more notation. Consider a finite subset $\mathcal{A} \subset \mathbb{N}$, which we call an alphabet, and let

$$\mathcal{C}_\mathcal{A} = \{[a_0, a_1, \ldots] : a_j \in \mathcal{A}\}$$

denote the set of all $\alpha \in \mathbb{R}$ with all partial quotients in $\mathcal{A}$. If $\max \mathcal{A} \leq A$ for an absolute constant $A$, then every $\alpha \in \mathcal{C}_\mathcal{A}$ is clearly absolutely Diophantine. Assuming $2 \leq |\mathcal{A}| < \infty$, each $\mathcal{C}_\mathcal{A}$ is a Cantor set (see Figure 2) of some Hausdorff dimension

$$0 < \delta_\mathcal{A} < 1,$$

and by choosing $\mathcal{A}$ appropriately (for example, $\mathcal{A} = \{1, 2, \ldots, A\}$ with $A$ large), one can make $\delta_\mathcal{A}$ arbitrarily close to 1 [Hen92].
It is easy to see that the matrix
\[
M = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix}
\]
(of determinant ±1) fixes the quadratic irrational
\[\alpha_M = [a_0, a_1, \ldots, a_\ell],\]
so we introduce the semi-group\(^1\)
\[
\mathcal{G}_\mathcal{A} := \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} : a \in \mathcal{A} \right\rangle^+ \subset GL_2(\mathbb{Z})
\]
of all such matrices whose fixed points \(\alpha_M\) lie in \(\mathcal{C}_\mathcal{A}\). Preferring to work in \(SL_2\), we immediately pass to the even-length (determinant-one) sub-semi-group
\[
\Gamma_\mathcal{A} := \mathcal{G}_\mathcal{A} \cap SL_2(\mathbb{Z}),
\]
which is (finitely) generated by the products \(\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}\), for \(a, b \in \mathcal{A}\).

Having accounted for the “low-lying” (or Diophantine) criterion, we must study the discriminants, or what is essentially the same, the set
\[
\mathcal{T}_\mathcal{A} := \{\text{tr } M : M \in \Gamma_\mathcal{A}\} \subset \mathbb{Z}
\]
of traces in \(\Gamma_\mathcal{A}\). Borrowing language from Hilbert’s 11\textsuperscript{th} problem on numbers represented by quadratic forms, we call an integer \(t\) admissible (for the alphabet \(\mathcal{A}\)) if for every \(q \geq 1,
\[t \in \mathcal{T}_\mathcal{A}(\text{mod } q),\]
that is, if \(t\) passes all finite local obstructions.

Remark 1.12. If \(\{1, 2\} \subseteq \mathcal{A}\), then, allowing inverses, the group \(\langle \Gamma_\mathcal{A} \rangle\) generated by the semigroup \(\Gamma_\mathcal{A}\) is all of \(SL_2(\mathbb{Z})\), and hence every integer is admissible. In general, Strong Approximation \([MVW84]\) shows that admissibility can be checked using a single modulus \(q(\mathcal{A})\).

We say \(t\) is represented if \(t \in \mathcal{T}_\mathcal{A}\), and let \(\mathcal{M}_\mathcal{A}(t)\) denote its multiplicity,
\[
\mathcal{M}_\mathcal{A}(t) := \#\{M \in \Gamma_\mathcal{A} : \text{tr } M = t\}.
\]
Since the entries of \(\Gamma_\mathcal{A}\) are all positive, the multiplicity is always finite.

The following conjecture seems plausible.

---

\(^1\)The superscript + in (1.10) denotes generation as a semigroup, that is, no inverses.
Conjecture 1.13 (Local-Global Conjecture). If the dimension $\delta_A$ exceeds $1/2$, then the set $\mathcal{T}_A$ of traces contains every sufficiently large admissible integer. Moreover, the multiplicity $\mathcal{M}_A(t)$ of an admissible $t \in [N, 2N)$ is at least

$$\mathcal{M}_A(t) > N^{2\delta_A - 1 - o(1)}.$$  \hfill (1.14)

Remark 1.15. It is now clear how to generalize Conjecture 1.5; the same should hold for $a_j$ restricted to any alphabet $\mathcal{A}$, as long as $\delta_A > 1/2$.

Our interest in this conjecture stems from the following

Lemma 1.16. The Local-Global Conjecture implies the Arithmetic Chaos Conjecture.

Sketch of proof. Let $A = 2$ and $\mathcal{A} = \{1, 2\}$ with dimension $\delta_A$ known [Goo41] to exceed $1/2$. Fix a fundamental discriminant $\Delta > 0$ corresponding to a real quadratic field $K = \mathbb{Q}(\sqrt{\Delta})$. It is easy to see that if $M \in \Gamma_A$ has wordlength $\ell = \ell(M)$ in the generators (that is, if $M$ is of the form (1.9)), then

$$\log \|M\| \asymp \ell(M),$$  \hfill (1.17)

where $\|M\|^2 = \text{tr}(M^t M)$. That is, the log-norm and wordlength metrics are commensurable. Choose a large parameter $N$, and find a solution $t \asymp \Delta N$ to the Pell equation $t^2 - \Delta s^2 = 4$. Every $t$ is admissible for this alphabet $\mathcal{A}$, so (1.14) implies the existence of at least $\mathcal{M}_A(t) > N^c > c^\ell$ matrices $M \in \Gamma_A$ with trace $t$, as desired. \hfill \Box

Remark 1.18. It is clear from the proof that to establish the Arithmetic Chaos Conjecture, it would be enough to demonstrate Conjecture 1.13 with the exponent $2\delta_A - 1$ in (1.14) replaced by any constant $c > 0$. The reason behind predicting this particular exponent is clarified below.

Remark 1.19. The multiplicity bound (1.14) with the stated exponent also has [McM09, Conjecture 6.1] as an easy corollary (with a similar proof), since $2\delta_A - 1$ approaches 1 as $\delta_A \to 1$.

In light of the lemma, we switch our focus henceforth to the local-global problem. In the next subsection, we present some evidence for the conjecture, before stating our partial results.

1.4. Evidence for Conjecture 1.13.

1.4.1. Hensley’s Theorem.

A reformulation of a theorem of Hensley’s [Hen89] shows that the traces up to $N$, counted with multiplicity, satisfy

$$\sum_{t < N} \mathcal{M}_A(t) \asymp N^{2\delta_A}. $$  \hfill (1.20)
If one expects the admissible traces of size $t \approx N$ to be roughly uniformly distributed, then each should appear with multiplicity about $N^{2\delta_A - 1}$. This is some evidence for the exponent in the multiplicity bound (1.14). When $\delta_A > 1/2$, this exponent is positive, so the multiplicity should eventually be positive; that is, sufficiently large admissible $t$’s should be represented, as claimed in Conjecture 1.13.

The estimate (1.20) also shows that the number $1/2$ in Conjecture 1.13 cannot be reduced; if $\delta_A < 1/2$, then the set of traces is certainly a thin subset of the integers.

1.4.2. The Circle Method.

A circle method approach as in [BK11] would predict that for $t \approx N$,

$$\mathcal{M}_A(t) \approx \mathcal{G}(t) N^{2\delta_A - 1},$$

where $\mathcal{G}(t) \geq 0$ is a certain “singular series” which vanishes for non-admissible $t$, and otherwise can fluctuate by factors of $\log \log N$.

Some numerical evidence for this behavior is presented in Figure 3. Here we have taken the alphabet $A = \{1, 2, \ldots, 10\}$ with dimension $\delta_A \approx 0.9257$ [Jen04], and plotted $t$ versus the ratio of the multiplicity $\mathcal{M}_A(t)$ to the expected size $t^{2\delta_A - 1}$. All numbers are admissible for this alphabet, and the value $t = 49$ is the largest known to not be represented.

**Figure 3.** A plot of $t$ versus $\mathcal{M}_A(t)/t^{2\delta_A - 1}$, in the range $1 \leq t \leq 1000$ for the alphabet $A = \{1, \ldots, 10\}$. 
1.4.3. A Lower Bound.

For another piece of evidence, we have the following trivial

**Lemma 1.21. Counting without multiplicity, we have**

\[ \# \mathcal{T}_A \cap [1, N] > N^{2\delta_A - 1 - o(1)}. \quad (1.22) \]

**Proof.** We first bound the multiplicity of \( t < N \) by

\[ \mathcal{M}_A(t) \ll N^{1+\varepsilon}. \quad (1.23) \]

Indeed, if \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_A \) has trace \( t \), then there are at most \( N \) choices of \( a \), whence \( d = t - a \) is determined. Then there are \( \ll N^\varepsilon \) choices for \( b \) and \( c \), divisors of \( bc = ad - 1 \), giving the multiplicity bound. Combined with (1.20), (1.23) immediately implies (1.22). \( \Box \)

In particular, choosing \( A \) so that \( \delta_A \) is sufficiently near 1, one can produce \( \gg N^{1-\varepsilon} \) traces in \( \mathcal{T}_A \), for any fixed \( \varepsilon > 0 \). Of course this is not even a positive proportion of numbers, so is still very far from the Local-Global Conjecture 1.13. The exponent \( 2\delta_A - 1 \) in (1.22) can be improved by methods similar in spirit to those going into [BK11, Theorem 1.22].

1.4.4. Zaremba’s Conjecture.

Perhaps our most convincing evidence is the similarity of Conjecture 1.13 to Zaremba’s Conjecture on absolutely Diophantine rational numbers, see [BK11]. Here one considers the same semi-groups \( \Gamma_A \) for a fixed finite alphabet \( A \), but instead of studying the traces, \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d \), one looks at the first entries,

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a, \]

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_A \). To make this more precise, let

\[ \mathcal{A} := \{ a \in \mathbb{Z} : \exists \begin{pmatrix} a & * \\ * & * \end{pmatrix} \in \Gamma_A \} \]

be the set of top-left entries in question and let

\[ \mathcal{B} := \{ n \in \mathbb{Z} : n \in \mathcal{A} \pmod{q}, \forall q \geq 1 \} \]

denote the set of “admissible” numbers in this context. Building on Hensley’s Conjecture [Hen96, Conjecture 3], the authors proposed in [BK11, Conjecture 1.7] the conjecture that, if \( \delta_A > 1/2 \), then every sufficiently large member of \( \mathcal{B} \) also belongs to \( \mathcal{A} \). Moreover, they showed the following density-one version of this conjecture, together with a multiplicity bound, under a more stringent restriction on the dimension \( \delta_A \).
Theorem 1.24 ([BK11]). If $\delta_A > 0.984$ (e.g., if $A = \{1, \ldots, 50\}$), then
\[
\frac{\#A \cap [1, N]}{\#B \cap [1, N]} = 1 + O \left( N^{-\epsilon/\log \log N} \right),
\]
as $N \to \infty$. Moreover, the multiplicity of an admissible $a \in A$ of size $a \asymp N$ produced in the above is at least\(^2\) of the order $N^{2\delta_A - 1 - \epsilon}$.

Remark 1.26. To make the connection to Conjecture 1.13 more clear, Theorem 1.24 is a translation of the former in which

1. the set $T_A$ of traces is replaced by the set $A$ of top-left entries,
2. a new version of “admissibility” is encoded in the set $B$,
3. “contains every sufficiently large” is replaced by “contains almost every,” and
4. the condition $\delta_A > 1/2$ is replaced by the more stringent condition\(^3\) $\delta_A > 0.984$.

Even a density-one version of Conjecture 1.13 (as conjectured by McMath [McM12]) would not, as far as we know, have the Arithmetic Chaos Conjecture 1.5 as a consequence, since solutions to Pellian equations are exponentially rare. Nevertheless, proving such a result may be an important first step, and it is not even known for a single finite alphabet $A$ that $T_A$ contains a positive proportion of numbers.

Results of this strength seem out of reach of current technology. Therefore we shift our focus to the study of the arithmetical properties of the trace set, more specifically to its equidistribution along progressions, with applications to almost primes. Our main goal is to make some progress in this direction.

1.5. Statements of the Main Theorems.

In this subsection, we state our main theorems, though we defer the precise (and somewhat technical) definitions to the next section.

For several applications, an important barometer of our understanding of a sequence is its level of distribution, defined roughly as follows. In our context, we wish to know that the traces in $T_A$ up to some growing parameter $N$ are equi-distributed along multiples of integers $q$, with $q$ as large as possible relative to $N$. That is, the quantity
\[
\#\{t \in T_A : t < N, \ t \equiv 0(q)\},
\]
\(^2\)For ease of exposition, the theorem is stated in [BK11] with $\epsilon = 1/1000$, but can be replaced by an arbitrary $\epsilon > 0$.

\(^3\)Recently Shinyih Huang [Hua13] has relaxed this further to $\delta_A > 5/6$, following improvements by Frolenkov-Kan [FK13].
counted with multiplicity, should be “close” to
\[ \frac{1}{q} \times \#\{t \in T_A : t < N\}, \]
in the sense that their difference should be much smaller than the total number of \( t \in T_A \) up to \( N \). This proximity cannot be expected once \( q \) is as large as \( N \), say, but perhaps can be established with \( q \) of size \( N^{1/2} \) or more generally \( N^\alpha \) for some \( \alpha > 0 \). If this is the case, in an average sense, then \( N^\alpha \) is called a level of distribution for \( T_A \), and \( \alpha \) is called an exponent of distribution. Let us make matters a bit more precise.

Looking at traces up to \( N \) counted with multiplicity is essentially the same as looking at matrices in the semigroup \( \Gamma_A \) of norm at most \( N \). Writing
\[ r_q(N) := \sum_{\substack{\gamma \in \Gamma_A \\ \|\gamma\| < N}} 1_{\{\text{tr} \gamma \equiv 0(q)\}} - \frac{1}{q} \sum_{\substack{\gamma \in \Gamma_A \\ \|\gamma\| < N}} 1 \]
for the “remainder” terms, we will say, again roughly, that \( T_A \) has level of distribution \( Q \) if
\[ \sum_{q < Q} |r_q(N)| = o \left( \sum_{\substack{\gamma \in \Gamma_A \\ \|\gamma\| < N}} 1 \right). \quad (1.27) \]

In applications it is enough to consider only square-free \( q \) in the sum. If (1.27) can be established with \( Q \) as large as \( N^\alpha \), then we will say \( T_A \) has exponent of distribution \( \alpha \). See §2.1 for a precise definition of level and exponent of distribution.

Remark 1.28. Note that a level and exponent of distribution is not a quantity intrinsic to \( T_A \), but rather a function of what one can prove about \( T_A \). The larger this exponent, the more control one has on the distribution of \( T_A \) on such arithmetic progressions.

Remark 1.29. The set \( T_A \) is of Affine Sieve type; see [BK13] for a definition. As such, the general Affine Sieve procedure introduced in [BGS06, BGS10], combined with the “expansion property” established in [BGS11], shows that \( T_A \) has some exponent of distribution \( \alpha > 0 \), see §2.2.

Our main goal in this paper is to make some partial progress towards Conjecture 1.13 by establishing levels of distribution for \( T_A \) beyond those available from expansion alone.
Theorem 1.30. For any small \( \eta > 0 \), there is an effectively computable \( \delta_0 = \delta_0(\eta) < 1 \) so that, if the dimension \( \delta_A \) of the alphabet \( A \) exceeds \( \delta_0 \), then the set \( T_A \) has exponent of distribution
\[
\alpha = \frac{1}{4} - \eta. \tag{1.31}
\]

We can make further progress, assuming the following

Conjecture 1.32 (Additive Energy in \( \text{SL}_2(\mathbb{Z}) \)). For any \( \varepsilon > 0 \),
\[
\# \{ \gamma_1, \ldots, \gamma_4 \in \text{SL}_2(\mathbb{Z}) : \|\gamma_j\| < N, \quad \text{and} \quad \gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 \} \ll N^{4+\varepsilon}, \tag{1.33}
\]
as \( N \to \infty \).

If true, this conjecture is sharp, in the sense that the exponent 4 in (1.33) cannot be replaced by a smaller number (since the diagonal \( \gamma_1 = \gamma_3, \gamma_2 = \gamma_4 \) already contributes at least \( N^4 \) terms to the count). Conditioned on this conjecture, we have the following

Theorem 1.34. Assume Conjecture 1.32. Then Theorem 1.30 holds with (1.31) replaced by
\[
\alpha = \frac{1}{3} - \eta. \tag{1.35}
\]

Applying standard sieve theory [Gre86], these levels of distribution have the following immediate corollary on almost primes. Recall that a number is \( R \)-almost-prime if it has at most \( R \) prime factors.

Corollary 1.36. There exists an effectively computable \( \delta_0 < 1 \) so that, if the dimension \( \delta_A \) of the alphabet \( A \) exceeds \( \delta_0 \), then the set \( T_A \) of traces contains an infinitude of \( R \)-almost-primes, with \( R = 5 \). Assuming Conjecture 1.32, the same holds with \( R = 4 \).

As an afterthought, we explore what can be said about \( R \)-almost-primes, not in the set \( T_A \) of traces, but in the set of discriminants which arise. To this end, recalling (1.1), we define
\[
\mathcal{D}_A := \{ \text{sqf}(t^2 - 4) : t \in T_A \}, \tag{1.37}
\]
where \( \text{sqf}(\cdot) \) denotes the square-free part. As explained in §7, an easy consequence of Mercat’s thesis [Mer12], combined with Theorem 1.24 and Iwaniec’s theorem [Iwa78], gives the following

Theorem 1.38. For the alphabet \( A = \{1, \ldots, 50\} \), the set \( \mathcal{D}_A \) contains an infinitude of \( R \)-almost-primes with \( R = 2 \).
1.6. Organization.

In §2, we give precise definitions of level and exponent of distribution, thus making unambiguous the statements of Theorems 1.30 and 1.34. There we also discuss the main ingredients involved in the proofs. In §3, we give some preliminaries needed in the analysis, the proofs of which are reserved for the two appendices. We spend §4 constructing the sequence \( \mathcal{A} \) and executing the main term analysis. The error analysis is handled separately: Theorem 1.30 is proved in §5 and Theorem 1.34 is proved in §6. Finally, Theorem 1.38 is proved quickly in §7. Some technical calculations are reserved for the appendices.

1.7. Notation.

We use the following notation throughout. Set \( e(x) = e^{2\pi ix} \) and \( e_q(x) = e(\frac{x}{q}) \). We use the symbol \( f \sim g \) to mean \( f/g \to 1 \). The symbols \( f \ll g \) and \( f = O(g) \) are used interchangeably to mean the existence of an implied constant \( C > 0 \) so that \( f(x) \leq Cg(x) \) holds for all \( x > C \); moreover \( f \asymp g \) means \( f \ll g \ll f \). The letters \( c, C \) denote positive constants, not necessarily the same in each occurrence. Unless otherwise specified, implied constants may depend at most on \( \mathcal{A} \), which is treated as fixed. The letter \( \varepsilon > 0 \) is an arbitrarily small constant, not necessarily the same at each occurrence. When it appears in an inequality, the implied constant may also depend on \( \varepsilon \) without further specification. The symbol \( 1\{\cdot\} \) is the indicator function of the event \( \{\cdot\} \). The trace of a matrix \( \gamma \) is denoted \( \text{tr} \gamma \). The greatest common divisor of \( n \) and \( m \) is written \( (n,m) \) and their least common multiple is \([n,m]\). The function \( \nu(n) \) denotes the number of distinct prime factors of \( n \). The cardinality of a finite set \( S \) is denoted \( |S| \) or \#\( S \). The transpose of a matrix \( g \) is written \( t^g \). When there can be no confusion, we use the shorthand \( r(q) \) for \( r(\text{mod } q) \). The prime symbol \( ' \) in \( \Sigma ' \) means the range of \( r(\text{mod } q) \) is restricted to \( (r,q) = 1 \). The set of primitive vectors in \( \mathbb{Z}^4 \) (ones with coprime coordinates) is denoted \( \mathcal{P}(\mathbb{Z}^4) \).

Acknowledgments.

It is our pleasure to thank Curt McMullen for many detailed comments and suggestions on an earlier version of this paper, and Tim Browning, Zeev Rudnick, and Peter Sarnak for illuminating conversations. Thanks also to Michael Rubinstein for numerics in support of Conjecture 1.32. The second-named author would like to thank the hospitality of the IAS, where much of this work was carried out.
2. Levels of Distribution and Ingredients

2.1. Levels of Distribution.

In this subsection, we give precise definitions of level and exponent of distribution. Fix the alphabet \( \mathcal{A} \) and let \( \mathcal{T}_A \) be the set of traces of \( \Gamma_A \). First we assume that the set of traces is primitive, that is, \( \gcd(\mathcal{T}_A) = 1 \).

If not, then replace \( \mathcal{T}_A \) by \( \mathcal{T}_A / \gcd(\mathcal{T}_A) \). Given a large parameter \( N \), let \( \mathfrak{A} = \{a_N(n)\} \) be a sequence of non-negative numbers supported on \( \mathcal{T}_A \cap [1, N] \), and set

\[
|\mathfrak{A}| = \sum_n a_N(n).
\]

We require that \( \mathfrak{A} \) is well-distributed on average over multiples of square-free integers \( q \). More precisely, setting

\[
|\mathfrak{A}_q| := \sum_{n \equiv 0(q)} a_N(n),
\]

we insist that

\[
|\mathfrak{A}_q| = \beta(q)|\mathfrak{A}| + r(q), \tag{2.2}
\]

where

1. the “local density” \( \beta \) is a multiplicative function assumed to satisfy the “linear sieve” condition

\[
\prod_{w \leq p < z} (1 - \beta(p))^{-1} \leq C \cdot \frac{\log z}{\log w}, \tag{2.3}
\]

for some \( C > 1 \) and any \( 2 \leq w < z \); and

2. the “remainders” \( r(q) \) are small on average, in the sense that

\[
\sum_{q \leq Q} |r(q)| \ll K \frac{1}{(\log N)^K} |\mathfrak{A}|, \tag{2.4}
\]

for some \( Q \geq 1 \) and any \( K \geq 1 \). That is, we require an arbitrary power of log savings.

If a sequence \( \mathfrak{A} \) exists for which the conditions (2.2)–(2.4) hold, then we say that \( \mathcal{T}_A \) has a level of distribution \( Q \). If (2.4) can be established with \( Q \) as large as a power,

\[
Q = N^\alpha, \quad \alpha > 0, \tag{2.5}
\]

then we say that \( \mathcal{T}_A \) has an exponent of distribution \( \alpha \).

\[\text{In fact, since the identity matrix has trace 2, the set of traces is not primitive if and only if the alphabet } \mathcal{A} \subset 2\mathbb{Z} \text{ consists entirely of even numbers (in which case the traces are all even and should be halved).}\]
2.2. The Main Ingredients.

This subsection is purely heuristic and expository. First we recall how the “standard” Affine Sieve procedure applies in this context, explaining Remark 1.29. Since $\delta_A$ is assumed to be large, we must have $\{1, 2\} \subset A$, whence for all $q \geq 1$, the reduction $\Gamma_A(\text{mod } q)$ is all of $\text{SL}_2(q)$; cf. Remark 1.12. Initially, we could construct the sequence $\mathfrak{A}$ by setting

$$a_N(n) := \sum_{\gamma \in \Gamma_A, \| \gamma \| < N} 1_{\{\text{tr} \gamma = n\}},$$

which is clearly supported on $n \in T_A, n \ll N$. Then (1.20) gives

$$|\mathfrak{A}| = \#\{\gamma \in \Gamma_A : \| \gamma \| < N\} \asymp N^{2\delta_A},$$

and $|\mathfrak{A}_q|$ can be expressed as

$$|\mathfrak{A}_q| = \sum_{\gamma \in \Gamma_A, \| \gamma \| < N} 1_{\{\text{tr} \equiv 0(q)\}} = \sum_{\gamma_0 \in \text{SL}_2(q)} 1_{\{\text{tr} \equiv 0(q)\}} \left( \sum_{\gamma \in \Gamma_A, \| \gamma \| < N} 1_{\{\gamma \equiv \gamma_0(q)\}} \right),$$

where we have decomposed the $\gamma$ sum into residue classes mod $q$. A theorem of Bourgain-Gamburd-Sarnak [BGS11] in this context states very roughly (see Theorem 3.2 for a precise statement) that

$$\#\{\gamma \in \Gamma_A : \| \gamma \| < N, \gamma \equiv \gamma_0(q)\} \sim \frac{1}{|\text{SL}_2(q)|} \#\{\gamma \in \Gamma_A : \| \gamma \| < N\} + O(q C N^{2\delta - \Theta}),$$

for some $\Theta > 0$. (We reiterate that the error in (2.9) is heuristic only; a statement of this strength is not currently known. That said, the true statement serves the same purpose in our application.) This is the “spectral gap” or “expander” property of $\Gamma_A$, and follows from a resonance-free region for the resolvent of a certain “congruence” transfer operator, see §3.1.

Inserting the expander property (2.9) into $|\mathfrak{A}_q|$ in (2.8) gives the desired decomposition (2.2), with local density

$$\beta(q) = \frac{1}{|\text{SL}_2(q)|} \sum_{\gamma_0 \in \text{SL}_2(q)} 1_{\{\text{tr} \equiv 0(q)\}},$$

and error

$$|r(q)| \ll q^C N^{2\delta - \Theta}.$$
requires (a condition weaker than)
\[ \sum_{q < Q} |r(q)| \ll Q^C N^{2\delta - \Theta} < N^{2\delta - \epsilon}, \]
or
\[ Q = N^\alpha < N^{\Theta/C - \epsilon}. \]
In this way, one can prove some exponent of distribution \( \alpha > 0 \), cf. Remark 1.29, but without making numeric the error term in (2.9), there is not more one can currently say. While the constants \( C \) and \( \Theta \) are in principle effectively computable, if one were to estimate them numerically, the known methods would lead to an astronomically small exponent \( \alpha \).

The novel technique employed here, used in some form already in [BK10, BK11, BK12, BK13], is to take inspiration from Vinogradov’s method, developing a “bilinear forms” approach, as follows. Instead of (2.6), let \( X \) and \( Y \) be two more parameters, each a power of \( N \), with \( XY = N \), and set (roughly)
\[ a_N(n) := \sum_{\gamma \in \Gamma, \|\gamma\| < X} \sum_{\xi \in \Gamma, \|\xi\| < Y} 1_{\{ \text{tr}(\gamma \xi) = n \}}. \tag{2.10} \]
This sum better encapsulates the group structure of \( \Gamma_A \), while still only being supported on the traces \( T_A \) of \( \Gamma_A \). Again, this is still an oversimplification; see §4 for the actual construction of \( \mathfrak{A} \).

Instead of directly appealing to expansion as in (2.8), we first invoke finite abelian harmonic analysis, writing
\[ |\mathfrak{A}_q| = \sum_{n \equiv 0(q)} a_N(n) = \sum_n \left[ \frac{1}{q} \sum_{r(q)} e_q(rn) \right] a_N(n). \tag{2.11} \]
After some manipulations, we decompose our treatment according to whether \( q \) is “small” or “large”. For \( q \) small, we apply expansion as before. For \( q \) large, the corresponding exponential sum already has sufficient cancellation (on average over \( q \) up to the level \( Q \)) that it can be treated as an error term in its entirety. It is in this range of large \( q \) that we exploit the bilinear structure of (2.10). On several occasions we replace the deficient group \( \Gamma_A \) by all of \( \text{SL}_2(\mathbb{Z}) \); this perturbation argument only works when \( \delta_A \) is near 1, at least some \( \delta_0 \). Assuming Conjecture 1.32, we are able to more efficiently estimate the resulting exponential sums, giving the improvement from Theorem 1.30 to Theorem 1.34.
3. Preliminaries

3.1. Expansion.

Let \( \mathcal{A} \subset \mathbb{N} \) be a finite alphabet with dimension \( \delta_{\mathcal{A}} \) sufficiently near 1. As such, it must contain the sub-alphabet \( \mathcal{A}_0 := \{1, 2\} \subset \mathcal{A} \). This has the consequence that for all \( q \geq 1 \),

\[
\Gamma(\text{mod } q) \cong \text{SL}_2(q),
\]

(3.1)
cf. Remark 1.12. Furthermore, we will only require expansion for the fixed alphabet \( \mathcal{A}_0 \), so as to make the expansion constants absolute, and not dependent on \( \mathcal{A} \); see Remark 5.24.

To this end, let \( \Gamma_0 \subset \text{SL}_2(\mathbb{Z}) \) be the semigroup as in (1.11) corresponding to \( \mathcal{A}_0 \). It is easy to see that \( \Gamma_0 \) is free, that every non-identity matrix \( \gamma \in \Gamma_0 \) is hyperbolic, and that

\[
\text{tr} \, \gamma \asymp \| \gamma \|.
\]

The following theorem is a consequence of the general expansion theorem proved by Bourgain-Gamburd-Sarnak in [BGS11].

**Theorem 3.2 ([BGS11]).** Let \( \Gamma = \Gamma_0 \) be the semigroup above. There exists an absolute square-free integer

\[
\mathfrak{B} \geq 1
\]

(3.3)

absolute constants \( c, \ C > 0 \), and an absolute “spectral gap”

\[
\Theta = \Theta(\mathcal{A}) > 0,
\]

(3.4)

so that, for any square-free \( q \equiv 0(\mathfrak{B}) \) and any \( \omega \in \text{SL}_2(q) \), as \( Y \to \infty \), we have

\[
\# \{ \gamma \in \Gamma : \| \gamma \| < Y, \ \gamma \equiv \omega(\text{mod } q) \} = \frac{\text{SL}_2(\mathfrak{B})}{\text{SL}_2(q)} \left\lfloor \gamma \in \Gamma : \| \gamma \| < Y, \ \gamma \equiv \omega(\text{mod } \mathfrak{B}) \right\rfloor + O(\# \{ \gamma \in \Gamma : \| \gamma \| < Y \} \mathcal{E}(Y; q)),
\]

(3.5)

where

\[
\mathcal{E}(Y; q) := \begin{cases} 
Y^{-c/\log \log Y}, & \text{if } q \leq C \log Y, \\
q^{cY^{-\Theta}}, & \text{if } q > C \log Y.
\end{cases}
\]

(3.6)

**Remark 3.7.** This theorem is proved in [BGS11, see Theorem 1.5] under the assumption that \( \Gamma \) is a convex-cocompact subgroup of \( \text{SL}_2(\mathbb{Z}) \), but the proof is the same when the group is replaced by our free semigroup \( \Gamma_0 \); we emphasize again that \( \Gamma_0 \) has no parabolic elements. The error
term (3.6) is the consequence of a Tauberian argument applied to a resonance-free region [BGS11, see Theorem 9.1] of the form
\[ \sigma > \delta A_0 - C \min \left\{ 1, \frac{\log q}{\log(1 + |t|)} \right\}, \quad \sigma + it \in \mathbb{C}, \quad (3.8) \]
for the resolvent of a certain “congruence” transfer operator, see [BGS11, §12] for details. For small \( q \), we only obtain a “Prime Number Theorem”-quality error (given here in crude form), while for larger \( q \), (3.8) is as good as a resonance-free strip.

We have stated the result only for the case \( \mathfrak{B} | q \). The distribution modulo \( \mathfrak{B} \) cannot be obtained directly from present methods, even though all reductions of \( \Gamma \) are surjective; cf. (3.1). Nevertheless, one can construct a set which has the desired equidistribution for all \( q \), as claimed in the following

**Proposition 3.9.** Given any \( Y \gg 1 \), there is a non-empty subset \( \mathfrak{N} = \mathfrak{N}(Y) \subset \Gamma_0 \) so that

1. for all \( a \in \mathfrak{N} \), \( \|a\| < Y \), and
2. for all square-free \( q \) and \( a_0 \in \text{SL}_2(q) \),
\[ \left| \frac{\# \{ a \in \mathfrak{N} : a \equiv a_0(q) \}}{|\mathfrak{N}|} - \frac{1}{|\text{SL}_2(q)|} \right| \ll \mathfrak{E}(Y; q). \quad (3.10) \]

Here \( \mathfrak{E} \) is given in (3.6).

Note that we do not have particularly good control on the cardinality of \( \mathfrak{N} \); regardless, the estimate (3.10) is only nontrivial if \( q < Y^{\Theta/C} \). The construction of the set \( \mathfrak{N} \) proceeds in a similar way to [BK11, §8]; we sketch a proof of Proposition 3.9 for the reader’s convenience in Appendix A.

### 3.2. An Exponential Sum over \( \text{SL}_2(\mathbb{Z}) \).

In this subsection, we state an estimate, showing roughly that there is cancellation in a certain exponential sum over \( \text{SL}_2(\mathbb{Z}) \) in a ball. We identify \( \mathbb{Z}^4 \) with \( M_{2\times 2}(\mathbb{Z}) \), and observe that for \( A, B \in M_{2\times 2}(\mathbb{Z}) \),
\[ \text{tr}(AB) = A \cdot B, \quad (3.11) \]
where the operation on the right is the dot product in \( \mathbb{Z}^4 \). Recall that \( \mathcal{P}(\mathbb{Z}^4) \) denotes the set of primitive vectors in \( \mathbb{Z}^4 \), that is, those for which the coordinates are coprime.

**Proposition 3.12.** Let \( X \gg 1 \) be a growing parameter and for a fixed non-negative, smooth, even function \( \phi : \mathbb{R} \to \mathbb{R}_{\geq 0} \) of compact support,
which is assumed to be at least 1 on $[-10, 10]$, let $\varphi_X : M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ be given by

$$\varphi_X : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \varphi \left( \frac{a + d}{X} \right) \varphi \left( \frac{a - d}{X} \right) \varphi \left( \frac{b + c}{X} \right) \varphi \left( \frac{b - c}{X} \right).$$

(Then $\varphi_X(\gamma) \geq 1$ if $\|\gamma\| < X$.) For any $q \geq 1$, any a primitive vector $s \in \mathcal{P}(\mathbb{Z}^4)$, and any $\varepsilon > 0$,

$$\left| \sum_{\xi \in \text{SL}(2, \mathbb{Z})} \varphi_X(\xi) e_q(\xi \cdot s) \right| \ll X^\varepsilon \left( q^{-3/2} X^2 + X^{3/2} + qX \right). \quad (3.13)$$

The proof is an application of Kloosterman’s version of the circle method. Since it is of a more classical nature, we give a sketch in Appendix B.
4. The Main Term Analysis and Initial Manipulations

4.1. Construction of \( \mathfrak{A} \).

The first goal in this subsection is to construct the appropriate sequence \( \mathfrak{A} = \{a_N(n)\} \). Let \( \mathcal{A} \subset \mathbb{N} \) be our fixed alphabet with corresponding dimension \( \delta_A \) near 1, and let \( \Gamma_A \) be the semigroup in (1.11). Since \( \mathcal{A} \) is fixed, we drop the subscripts, writing \( \Gamma = \Gamma_A \) and \( \delta = \delta_A \).

Let \( N \) be the main growing parameter, and let

\[
X = N^x, \quad Y = N^y, \quad Z = N^z, \quad x, y, z > 0, \quad x + y + z = 1, \quad (4.1)
\]

be some parameters to be chosen later; in particular,

\[
N = XYZ. \quad (4.2)
\]

We think of \( X \) as large, \( X > N^{1/2} \), and \( Y \) as tiny. The final choices of the parameters depend on the treatment, that is, whether we are proving Theorem 1.30 or the conditional Theorem 1.34.

Let \( \aleph = \aleph(Y) \subset \Gamma_0 \subset \Gamma \) be the set constructed in Proposition 3.9, and let

\[
\Xi := \{\xi \in \Gamma : \|\xi\| < X\}, \quad \Omega := \{\omega \in \Gamma : \|\omega\| < Z\} \quad (4.3)
\]

be norm balls in \( \Gamma \). Recall that

\[
|\Xi| \asymp X^{2\delta}, \quad |\Omega| \asymp Z^{2\delta}, \quad (4.4)
\]

while we do not have good control on the size of \( |\aleph| \).

Then we define

\[
a_N(n) := \sum_{\xi \in \Xi} \sum_{a \in \aleph} \sum_{\omega \in \Omega} 1_{\{n = \text{tr}(\xi a\omega)\}}, \quad (4.5)
\]

which is supported on \( n \ll N \) by (4.2). We have that

\[
|\mathfrak{A}| = |\Xi| \cdot |\aleph| \cdot |\Omega| \gg |\aleph|(XZ)^{2\delta}. \quad (4.6)
\]

Next for parameters \( 1 \ll Q_0 < Q \) and any square-free \( q < Q \), we decompose

\[
|\mathfrak{A}_q| = \sum_{n \equiv 0(q)} a_N(n) = \sum_n \frac{1}{q} \sum_{q \mid q} \sum_{r(q)} e_q(rn)a_N(n) = \mathcal{M}_q + r(q), \quad (4.7)
\]

say, according to whether or not \( q < Q_0 \). Here

\[
\mathcal{M}_q := \sum_n \frac{1}{q} \sum_{q \mid q} \sum_{r(q)} e_q(rn)a_N(n) \quad (4.8)
\]

will be treated as a “main” term, the remainder \( r(q) \) being an error.
4.2. Main Term Analysis.

We now analyze the $M_q$ term, proving the following

**Proposition 4.9.** Let $\beta$ be the multiplicative function given at primes by

$$
\beta(p) := \frac{1}{p} \left( 1 + \frac{\chi_4(p)}{p} \right) \left( 1 - \frac{1}{p^2} \right)^{-1},
$$

where $\chi_4$ is the Dirichlet character mod 4. There is a decomposition

$$
M_q = \beta(q) |A| + r^{(1)}(q) + r^{(2)}(q),
$$

where

$$
\sum_{q < Q} |r^{(1)}(q)| \ll |A| \log Q \left( \frac{1}{Y^{c/\log \log Y}} + Q^C Y^{-\Theta} \right),
$$

and

$$
\sum_{q < Q} |r^{(2)}(q)| \ll |A| Q^\varepsilon Q_0^\varepsilon.
$$

**Proof.** Inserting the definition (4.5) of $a_N$ into (4.8) gives

\[
M_q = \sum_{\xi \in \Xi} \sum_{a \in A} \sum_{\omega \in \Omega} \sum_{q \mid q < Q_0} \frac{1}{q} \sum_{r(q)} e_q(r \text{tr}(\xi a \omega))
\]

\[
= \sum_{\xi \in \Xi} \sum_{\omega \in \Omega} \sum_{q \mid q < Q_0} \frac{1}{q} \sum_{r(q)} \sum_{a_0 \in SL_2(q)} e_q(r \text{tr}(\xi a_0 \omega)) \left[ \sum_{a \in N} \frac{1}{q} \right].
\]

Apply (3.10) to the innermost sum, giving

$$
M_q = M_q^{(1)} + r^{(1)}(q),
$$

say, where

$$
M_q^{(1)} := |A| \frac{1}{q} \sum_{q \mid q < Q_0} \sum_{r(q)} \sum_{a \in N} e_q(r \text{tr}(\gamma)),
$$

and

$$
|r^{(1)}(q)| \ll |A| \frac{1}{q} \sum_{q \mid q < Q_0} q^4 E(Y; q).
$$
The error $E$ is as given in (3.6). We estimate

$$\sum_{q<Q} |r^{(1)}(q)| \ll |A| \sum_{q<Q_0} q^4 E(Y; q) \sum_{q<Q} \frac{1}{q},$$

$$\ll |A| \log Q \left[ (\log Y)^C Y^{-c/\log \log Y} + Q_0^C Y^{-\Theta} \right],$$

thus proving (4.12).

Returning to $M_q^{(1)}$, we add back in the large divisors $q | q$, writing

$$M_q^{(2)} = M_q^{(1)} + r^{(2)}(q),$$

say, where

$$M_q^{(2)} := |A| \sum_{q|q} \sum'_{r(q)} \frac{1}{|SL_2(q)|} \sum_{\gamma \in SL_2(q)} e_q(r \text{tr}(\gamma)).$$

Let $\rho(q)$ be the multiplicative function given at primes by

$$\rho(p) : = \frac{1}{|SL_2(p)|} \sum_{\gamma \in SL_2(p)} \sum'_{r(p)} e_p(r \text{tr}(\gamma)),$$

so that

$$M_q^{(2)} = |A| \cdot \beta(q),$$

with $\beta$ as given in (4.10).

Lastly, we deal with $r^{(2)}$. It is easy to see from the above that $|\rho(p)| \ll 1/p$, so $|\rho(q)| \ll q^\epsilon / q$, giving the bound

$$|r^{(2)}(q)| \ll |A| \sum_{q|q} \frac{q^\epsilon}{q} \ll |A| \frac{q^\epsilon}{q} \frac{1}{Q_0}.$$

The estimate (4.13) follows immediately, completing the proof. □

Remark 4.14. Since $Y$ in (4.1) is a small power of $N$, the first error term in (4.12) saves an arbitrary power of $\log N$, as required in (2.4). For the rest of the paper, all other error terms will be power savings. In particular, setting

$$Q_0 = N^{\alpha_0}, \quad \alpha_0 > 0,$$  \hspace{1cm} (4.15)
the error in (4.13) is already a power savings, while the second term in (4.12) requires that

$$\alpha_0 < \frac{y \Theta}{C}. \quad (4.16)$$

It is here that we crucially use the expander property for $\Gamma$, but the final level of distribution will be independent of $\Theta$.

### 4.3. Initial Manipulations.

Returning to (4.7), it remains to control the average error term

$$E := \sum_{q < Q} |r(q)| = \sum_{q < Q} \left| \sum_{\xi \in \Xi} \sum_{a \in A} \sum_{\omega \in \Omega} \frac{1}{q} \sum_{q \geq q_0} \sum' e_q(r \text{ tr}(\xi a \omega)) \right|. \quad (4.17)$$

We first massage $E$ into a more convenient form.

Let $\zeta(q) := |r(q)|/r(q)$ be the complex unit corresponding to the absolute value in (4.17), and rearrange terms as:

$$E = \sum_{Q_0 \leq q < Q} \frac{\zeta_1(q)}{q} \sum_{\xi \in \Xi} \sum_{a \in A} \sum_{\omega \in \Omega} \sum' e_q(r \text{ tr}(\xi a \omega)),$$

where we have set

$$\zeta_1(q) := \sum_{q < Q/q} \frac{\zeta(q q')}{q}.$$ 

Leaving the special set $A$ alone, we break the $q$ sum into dyadic pieces and estimate $\zeta_1(q) \ll \log Q$. We obtain

$$E \ll \log Q \sum_{a \in A} \sum_{Q_0 \leq q < Q \text{ dyadic}} \frac{1}{Q} |E_1(Q; a)|, \quad (4.18)$$

where we have defined

$$E_1(Q; a) := \sum_{q = Q} \sum_{\xi \in \Xi} \sum_{\omega \in \Omega} \sum' e_q(r \text{ tr}(\xi a \omega)). \quad (4.19)$$

It remains to estimate $E_1(Q; a)$. In the next two sections, we give two different treatments, depending on whether or not we allow ourselves to use the Additive Energy Conjecture 1.32.
5. Proof of Theorem 1.30

In this section, we analyze $E_1(Q; a)$ in (4.19) unconditionally, that is, without use of the Additive Energy Conjecture 1.32. Our first main result is the following

**Theorem 5.1.** For any $\varepsilon > 0$, and any $1 \ll Q_0 < Q < N \to \infty$, with

$$Z = o \left( Q_0^{3/(3+4\delta)} \right),$$

(5.2)

we have

$$|E_1(Q; a)| \ll N^\varepsilon Q \Xi^{1/2} |\Omega| X \left[ \frac{1}{|\Omega|^{1/6}} + \frac{Q}{X^{1/4}} + \frac{Q^2}{X^{1/2}} \right].$$

(5.3)

To begin the proof, we apply Cauchy-Schwarz in the “long” variable $\xi$ in (4.19), giving

$$|E_1(Q; a)|^2 \ll |\Xi| \sum_{\xi \in SL_2(\mathbb{Z})} \varphi_X(\xi) \left| \sum_{q \asymp Q} \sum_{\omega \in \Omega} \sum_{r(q)} e_q(\text{tr}(\xi \omega)) \right|^2.$$

Here we have extended the $\xi$ sum to all of $SL_2(\mathbb{Z})$, and inserted the weighting function $\varphi_X$ from Proposition 3.12. Since the trace of a product is a dot-product (on identifying $\mathbb{Z}^4$ with $M_{2 \times 2}(\mathbb{Z})$ as in (3.11)), it is linear, and hence when we open the square, we obtain

$$|E_1(Q; a)|^2 \ll |\Xi| \sum_{q,q' \asymp Q} \sum_{\omega} \sum_{r(q)} \sum_{r'(q')} \varphi_X(\xi) e \left( \frac{r}{q} \omega - \frac{r'}{q'} \omega' \right).$$

(5.4)

Write the bracketed expression in lowest terms as

$$s = \frac{s}{q_0} = \frac{r}{q} \omega - \frac{r'}{q'} \omega',$$

(5.5)

with $s = s(q, q', r, r', \omega, \omega', a) \in \mathcal{P}(\mathbb{Z}^4)$ a primitive vector and $q_0 \geq 1$ depending on the same parameters as $s$. To study this expression in greater detail, we introduce some more notation. All variables labelled $q$, however decorated, denote square-free numbers.

Write

$$\tilde{q} := (q, q'), \quad q = q_1 \tilde{q}, \quad q' = q_1' \tilde{q}, \quad \tilde{q} := [q, q'] = q_1 q_1' \tilde{q},$$

and observe from (5.5) that $q_1 q_1' \mid q_0$ and $q_0 \mid \tilde{q}$. Hence we can furthermore write

$$\tilde{q}_0 := (q_0, \tilde{q}), \quad \tilde{q} = q_0 \tilde{q}_0 = q_1 q_1' \tilde{q}_0 \tilde{q}_0,$$

whence $q_0 = q_1 q_1' \tilde{q}_0$. Note also that $Q \ll \tilde{q} \ll Q^2$. 

Observe further that (5.5) implies
\[ q'_1 r \omega \equiv q_1 r' \omega' \mod \hat{q}_0, \]
and using \( \det \omega = \det \omega' = 1 \) gives
\[ (q'_1 r)^2 \equiv (q_1 r')^2 \mod \hat{q}_0. \]
Since \((q_1 r', \hat{q}_0) = 1 = (q'_1 r, \hat{q}_0)\), we obtain
\[ q'_1 r \equiv u q_1 r' \mod \hat{q}_0, \quad (5.6) \]
where \( u \equiv 1(\hat{q}_0) \). There are at most \( 2^{\nu(\hat{q}_0)} \ll N^\varepsilon \) such \( u(\mod \hat{q}_0) \), where \( \nu(m) \) is the number of distinct prime factors of \( m \). It follows that
\[ \omega \equiv u \omega' \mod \hat{q}_0. \quad (5.7) \]
Returning to (5.4), we will only get sufficient cancellation in the \( \xi \) sum when \( q_0 \) is not too small. So we introduce another parameter
\[ Q_1 < Q_0 \quad (5.8) \]
and break the sum according to whether or not \( q_0 \leq Q_1 \), writing
\[ |E_{\leq}(Q; a)|^2 \ll |\Xi| \cdot \left[ |E_{\leq}| + |E_{>}| \right]. \quad (5.9) \]
Here for \( \diamond \in \{\leq, >\} \), we have written
\[
E_{\diamond} := \sum_{Q \leq q \leq Q^2} \sum_{q \equiv q'_0 \mod \hat{q}_0} \sum_{r(q')} \sum_{r'(q')} \sum_{s \in \Omega}
\sum_{\omega, \omega' \in \Omega} \varphi_X(\xi) e_{q_0}(\xi \cdot s). \quad (5.10)
\]
We give a separate analysis for the values of \( \diamond \) in the following two subsections.

5.1. Analysis of \( E_{\leq} \).
In this subsection, we prove the following

**Proposition 5.11.** With notation as above,
\[ |E_{\leq}| \ll N^\varepsilon Q^2 |\Omega|^2 X^2 \left( \frac{Q_1}{|\Omega|} \right). \quad (5.12) \]
Proof. Since \( q_0 \) is small, we may not have any cancellation in the \( \xi \) sum and instead save by turning the modular restriction (5.7) into an archimedean one, as follows.

Observe that we have
\[
q_0 \leq Q_1 < Q \ll \hat{q} = q_0 \hat{q}_0,
\]
which forces
\[
\hat{q}_0 \gg \frac{Q}{q_0} > \frac{Q_0}{Q_1}.
\]
Then choosing
\[
Z \ll \frac{Q_0}{Q_1}
\]
with implied constants small enough forces
\[
\omega = u\omega'
\]
from (5.7) and (4.3) that \( \|\omega\|, \|\omega'\| < Z \).

Collecting these facts and returning to \( E_\leq \) in (5.10), we estimate
\[
|E_\leq| \ll \sum_{\hat{q} \ll Q^2} \sum_{q_1 \hat{q}_0 \equiv \hat{q}} \sum_{q' \equiv q \bmod \hat{q}_0} \sum_{r(q)} \sum'_{r'(q') \equiv q_1 r \bmod \hat{q}_0} \sum'_{r''(q'')} \sum_{\omega, \omega' \in \Omega} \sum_{\xi \in \text{SL}_2(\mathbb{Z})} \sum_{\|\xi\| \ll X} 1.
\]

Working from the inside out, the innermost \( \xi \) sum contributes \( X^2 \). Since \( \omega \) is determined from \( u \) and \( \omega' \), the \( \omega, \omega' \) sum contributes \( |\Omega| \). There are at most \( q'/\hat{q}_0 \) values for \( r' \), and at most \( q \) values for \( r \); note that
\[
\frac{qq'}{\hat{q}_0} = \frac{qq'q_0}{\hat{q}} \ll \frac{Q^2Q_1}{\hat{q}}.
\]
The \( u \) sum contributes \( N^\varepsilon \), as does the sum on divisors of \( \hat{q} \). Putting everything together gives our final estimate
\[
|E_\leq| \ll \sum_{\hat{q} \ll Q^2} N^\varepsilon \frac{Q^2Q_1}{\hat{q}} |\Omega| X^2 \ll N^\varepsilon Q^2Q_1 |\Omega| X^2.
\]
The claim easily follows. \( \square \)
5.2. Estimate of $E_\succ$.

Now we give the following estimate.

**Proposition 5.14.** Keeping notation as above, we have

$$|E_\succ| \ll N^\varepsilon (Q|\Omega|X)^2 \left( \frac{1}{Q_1^{1/2}} + \frac{Q^2}{X^{1/2}} + \frac{Q^4}{X} \right). \quad (5.15)$$

**Proof.** Returning to (5.10), we are now in the large $q_0$ range, so (5.7) cannot be effectively used; on the other hand, we are in position to exploit cancellation in the $\xi$ sum using Proposition 3.12. Applying the estimate (3.13) and using (5.6) gives

$$|E_\succ| \ll N^\varepsilon \sum_{\hat{q} \ll Q} \sum_{q_1 q_0 \hat{q} = \hat{q}, \ \hat{q} \equiv 1(\hat{q}_0)} \sum_u \sum_{r(q)} \sum_{r'(q')} \sum_{q_1 r = u q_0 r'(\hat{q}_0)} \sum_{u, \omega, \omega' \in \Omega} \left( \frac{X^2}{q_0^{3/2}} + X^{3/2} + q_0 X \right).$$

Now we argue as follows. We estimate the $\omega, \omega'$ sum trivially. The $r, r'$ sums again contribute at most $qq'/\hat{q}_0 = qq'/q_0 \hat{q}$, and the $u$ sum at most $N^\varepsilon$. Then we have

$$|E_\succ| \ll N^\varepsilon \sum_{\hat{q} \ll Q} \sum_{q_1 q_0 \hat{q} = \hat{q}, \ q = q_0 \hat{q}, \ q' = q_0 \hat{q}, \ q, q' \ll Q, \ \hat{q}_0 = q_1 q_0 > Q_1} \frac{qq'}{\hat{q}} |\Omega|^2 \left( \frac{X^2}{q_0^{1/2}} + q_0 X^{3/2} + q_0^2 X \right)$$

$$\ll N^\varepsilon Q_1^2 |\Omega|^2 \left( \frac{X^2}{Q_1^{1/2}} + Q^2 X^{3/2} + Q^4 X \right),$$

from which the claim follows. \qed

5.3. **Proof of Theorem 5.1.**

Combining (5.9) with (5.12) and (5.15) gives

$$|E_1(Q; a)| \ll N^\varepsilon |\Xi|^{1/2} Q |\Omega| X \left[ \frac{Q_1^{1/2}}{|\Omega|^{1/2}} + \frac{1}{Q_1^{1/4}} + \frac{Q}{X^{1/4}} + \frac{Q^2}{X^{1/2}} \right].$$

We choose $Q_1$ to balance the first two errors, setting

$$Q_1 = |\Omega|^{2/3} \asymp Z^{48/3},$$
by (4.4). The assumption (5.2) implies that both (5.8) and (5.13) are certainly satisfied. Then (5.3) follows immediately, completing the proof.

5.4. Proof of Theorem 1.30.

Let \( A = \{a_N(n)\}_{n \geq 1} \) be as constructed in (4.5). Combining (4.7) and (4.11) gives the decomposition

\[
|A_q| = \beta(q)|A| + r^{(1)}(q) + r^{(2)}(q) + r(q),
\]

as in (2.2), with \( \beta \) given by (4.10). It is classical that (2.3) holds, so it remains to verify (2.4) with \( Q \) as large as possible. Write

\[
X = N^x, \quad Y = N^y, \quad Z = N^z, \quad Q = N^\alpha, \quad Q_0 = N^{\alpha_0},
\]

with

\[
x + y + z = 1.
\] (5.16)

The bounds (4.12) and (4.13) are sufficient as long as

\[
y > 0, \quad \alpha_0 < \frac{y\Theta}{C}, \quad \text{and} \quad \alpha_0 > 0.
\] (5.17)

To bound the average of \( r(q) \), assume (5.2) and insert (5.3) into (4.18), giving

\[
E \ll N^\varepsilon |A| \cdot X^{(1-\delta)} \left[ \frac{1}{Z^{\delta/3}} + \frac{Q}{X^{1/4}} + \frac{Q^2}{X^{1/2}} \right],
\] (5.18)

where we used (4.4) and (4.6). The assumption (5.2) is satisfied if

\[
z < \frac{3\alpha_0}{3 + 4\delta},
\] (5.19)

while the three error terms in (5.18) are sufficiently controlled if

\[
z\delta/3 > x(1 - \delta),
\] (5.20)

\[
x/4 > \alpha + x(1 - \delta), \quad \text{and}
\] (5.21)

\[
x/2 > 2\alpha + x(1 - \delta).
\] (5.22)

Remark 5.23. Of course (5.21) implies (5.22), so the latter condition may be dropped. Taking \( y, \alpha_0, \) and \( z \) very small and \( x \) and \( \delta \) very near 1, it is clear that (5.21) will not allow us to do better than \( \alpha < 1/4 \); this is what we achieve below.

Let \( \eta > 0 \) be given, and set

\[
\alpha = \frac{1}{4} - \eta,
\]
as claimed in (1.31). We may already assume that

\[
\delta > 1 - \frac{\eta}{2},
\]
(more stringent restrictions on $\delta$ follow), and set

$$x = 1 - \frac{\eta}{2}.$$  

Then

$$x \left(\frac{1}{4} - (1 - \delta)\right) > \left(1 - \frac{\eta}{2}\right) \left(\frac{1}{4} - \frac{\eta}{2}\right) > \frac{1}{4} - \frac{5\eta}{8} > \alpha,$$

whence (5.21) is satisfied.

Next we replace $C$ by $C + 1$, say, so that we can make the choice

$$\alpha_0 = \frac{y \Theta}{C},$$

and satisfy (5.17). Making the choice

$$z = \frac{3y \Theta}{7C} = \frac{3\alpha_0}{7} < \frac{3\alpha_0}{3 + 4\delta}$$

will satisfy (5.19), whence (5.16) requires

$$\frac{\eta}{2} = 1 - x = y + z = \left(1 + \frac{3\Theta}{7C}\right) y.$$

In other words, we set

$$y = \frac{\eta}{2} \cdot \frac{7C}{7C + 3\Theta}, \quad z = \frac{3\Theta \eta}{2(7C + 3\Theta)}.$$

Lastly, to ensure (5.20), we impose the stronger restriction (assuming $\delta > 3/4$) that

$$\frac{3\Theta \eta}{2(7C + 3\Theta)} = z > 4(1 - \delta),$$

or

$$\delta > 1 - \frac{3\eta \Theta}{8(7C + 3\Theta)}.$$

Setting

$$\delta_0 := 1 - \frac{\eta \Theta}{C},$$

completes the proof.

**Remark 5.24.** It is here that we need $\aleph$ to come from the fixed alphabet $\{1, 2\} \subset \mathcal{A}$. Indeed, the constants $\Theta$ and $C$, and hence $\delta_0$, are then absolute, and do not depend on $\mathcal{A}$. 
6. Proof of Theorem 1.34

Returning to (4.19), we devote this section to proving an even stronger (but conditional) bound for \( E_1(Q; a) \), by now allowing ourselves to use the Additive Energy Conjecture 1.32. Our main result is the following

**Theorem 6.1.** Assume Conjecture 1.32. For any \( \varepsilon > 0 \), and any 
\[ 1 \ll Q_0 < Q < Q < N \to \infty, \]
with 
\[ Q < Z, \quad Q^2 = o(X), \] 
we have

\[
|E_1(Q; a)| \ll N^\varepsilon Q|\Omega|^{1/2} X^2 Z \left[ \frac{Q^{1/2}}{Z^{1/2}} + \frac{1}{Q^{1/8}} \right].
\]  

(6.3)

**Proof.** This time, start the proof by applying Cauchy-Schwarz in \( q, r \), and the “short” variable \( \omega \) to (4.19). This opens the “long” variable \( \xi \) into a pair of such, as follows.

\[
|E_1(Q; a)|^2 \ll \left( \sum_{q \approx Q} \sum_{\omega \in \Omega} \sum_{r(q)} \right) \left( \sum_{q \approx Q} \sum_{\omega \in \text{SL}_2(\mathbb{Z})} r(q) \sum_{\xi \in \Xi} e_q(r \text{ tr}(\xi a \omega)) \right)^2.
\]

Here we have extended both the \( \omega \) and \( \xi, \xi' \) sums to all of \( \text{SL}_2(\mathbb{Z}) \) (after inserting absolute values). Collect the difference of \( \xi \) and \( \xi' \) into a single variable, writing

\[ \xi - \xi' = M \in M_{2 \times 2}(\mathbb{Z}) \cong \mathbb{Z}^4, \]

and setting

\[
\mathcal{N}_M(X) := \sum_{\xi, \xi' \in \text{SL}_2(\mathbb{Z})} 1_{\{M = \xi - \xi'\}}.
\]

(6.4)

The Additive Energy Conjecture 1.32 is then the assertion that

\[
\sum_{M \in \mathbb{Z}^4} \mathcal{N}_M(X)^2 \ll X^{4+\varepsilon}.
\]

So writing

\[
|E_1(Q; a)|^2 \ll Q^2|\Omega| \sum_{M \in \mathbb{Z}^4 \atop \|M\| \ll X} \mathcal{N}_M(X) \left| \sum_{q \approx Q} \sum_{r(q)} \sum_{\omega \in \text{SL}_2(\mathbb{Z})} e_q(r \text{ tr}(M a \omega)) \right|,
\]

(6.4)
we apply Cauchy-Schwarz in the $M$ variable, giving

$$|E_1(Q; a)|^4 \ll Q^4|\Omega|^2X^{4+\varepsilon} \sum_{M \in \mathbb{Z}^4} \Psi \left( \frac{M}{X} \right) \left| \sum_{q \gg Q} \sum_{r(q)} \sum' \sum' \prod_{e(q) \in \mathbb{SL}_2(\mathbb{Z})} \psi(M \cdot Xa \omega) \right|^2$$

$$\ll N^\varepsilon Q^4|\Omega|^2X^4 \sum_{q,q' \gg Q} \sum_{r(q)} \sum_{r'(q')} \sum' \sum' \prod_{\omega, \omega' \in \mathbb{SL}_2(\mathbb{Z})} e\left( M \cdot (\frac{r}{q}a \omega - \frac{r'}{q'}a \omega') \right)$$

where we have inserted a suitable bump function $\Psi$ and applied Poisson summation. Assuming $qq' \ll Q^2 = o(X)$ as in (6.2), the innermost condition implies

$$q'rq \equiv qr'\omega' \pmod{qq'}.$$ 

Taking determinants gives

$$(q'r)^2 \equiv (r'q)^2 \pmod{qq'},$$

whence reducing mod $q$ gives

$$(q'r)^2 \equiv 0 \pmod{q}.$$ 

But this implies $(q')^2 \equiv 0 \pmod{q}$, since $(r, q) = 1$. Because $q$ is square-free, we have thus forced $q' \equiv 0 \pmod{q}$. By symmetry, we also have $q \equiv 0 \pmod{q'}$, and hence

$$q = q', \quad r \equiv ur' \pmod{q}, \quad \text{and} \quad \omega \equiv u\omega' \pmod{q},$$

where $u^2 \equiv 1(q)$; again there are at most $2^{\nu(q)} \ll N^\varepsilon$ such $u$’s. We then have

$$|E_1(Q; a)|^4 \ll N^\varepsilon Q^4|\Omega|^2X^8 \sum_{q \ll Q} \sum_{u^2 \equiv 1(q)} \sum_{r, r' \in \mathbb{SL}_2(\mathbb{Z})} \sum' \sum' \prod_{\omega, \omega' \in \mathbb{SL}_2(\mathbb{Z})} 1\{\omega \equiv u\omega' \pmod{q}\},$$
We dispose of \( u \) in the last summation via Cauchy-Schwarz:

\[
\sum_{\omega, \omega' \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{\omega \equiv u \omega' \pmod{q}\}} = \sum_{\gamma \in \text{SL}_2(q)} \left[ \sum_{\omega \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{\omega \equiv \gamma(q)\}} \right] \left[ \sum_{\omega' \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{u \omega' \equiv \gamma(q)\}} \right]
\]

\[
\leq \left( \sum_{\gamma \in \text{SL}_2(q)} \left[ \sum_{\omega \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{\omega \equiv \gamma(q)\}} \right] \right)^2 \left[ \sum_{\omega' \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{u \omega' \equiv \gamma(q)\}} \right]^{1/2}
\]

\[
= \sum_{\omega, \omega' \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{\omega \equiv \omega' \pmod{q}\}},
\]

since \((u, q) = 1\). Applying this estimate gives

\[
|\mathcal{E}_1(Q; a)|^4 \ll N^5 Q^5 |\Omega|^2 X^8 \sum_{q \approx Q} \sum_{\omega, \omega' \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{\omega \equiv \omega' \pmod{q}\}}
\]

\[
= N^5 Q^5 |\Omega|^2 X^8 \sum_{q \approx Q} \sum_{M \in \mathbb{Z}^4, M \equiv 0(q)} N_M(Z),
\]

where we have again used the notation

\[
N_M(Z) := \sum_{\omega, \omega' \in \text{SL}_2(\mathbb{Z})} \mathbf{1}_{\{\omega - \omega' = M\}}.
\]

We first isolate the \( M = 0 \) term, writing

\[
|\mathcal{E}_1(Q; a)|^4 \ll N^5 Q^5 |\Omega|^2 X^8 (QZ^2 + \mathcal{E}_2),
\]

say, where

\[
\mathcal{E}_2 := \sum_{q \approx Q} \sum_{M \in \mathbb{Z}^4, M \equiv 0(q), M \neq 0} N_M(Z).
\]
Next apply Cauchy-Schwarz yet again, now to $E_2$ in the $q$ and $M$ variables. Assuming (6.2) that $Q < Z$ gives

$$E_2^2 \ll Q \left( \frac{Z}{Q} \right)^4 \sum_{q\leq Q} \sum_{M\in \mathbb{Z}^4} N_M(Z)^2$$

$$\ll \frac{Z^4}{Q^3} \sum_{M\in \mathbb{Z}^4} \sum_{q | M} N_M(Z)^2 \ll \frac{Z^4}{Q^3} \sum_{M\in \mathbb{Z}^4} N^\varepsilon N_M(Z)^2$$

$$\ll N^\varepsilon \frac{Z^4}{Q^3} Z^4,$$

where we used the Additive Energy Conjecture 1.32 a second time.

Combining (6.6) with (6.5) gives (6.3), as claimed.

6.1. **Proof of Theorem 1.34.**

The proof is now nearly identical to that in §5.4, so we give a brief sketch. Again let $A$ be constructed as in (4.5). Assuming (6.2) and Conjecture 1.32, insert (6.3) into (4.18). Together with (4.4), this gives

$$E \ll N^\varepsilon |A| (X^2Z)^{1-\delta} \left[ \frac{Q^{1/2}}{Z^{1/2}} + \frac{1}{Q_0^{1/8}} \right].$$

With $\delta$ very near 1, the second term is a power savings as long as $Q_0$ is some tiny power, which requires $Y = N_y$ to be some tiny power, $y \approx \varepsilon$. Writing $Q = N^\alpha$, $X = N^x$ and $Z = N^z$ with

$$1 = x + y + z \approx x + z$$

the first term is a power savings if $z = \alpha - \varepsilon$, while (6.2) is satisfied if $x = 2\alpha - \varepsilon$. Since $x + z \approx 1$, this gives a maximal value of $\alpha \approx \frac{1}{3} - \varepsilon$.

We leave the details to the reader.
7. Proof of Theorem 1.38

Recall from (1.37) that $\mathcal{D}_A$ is the set of discriminants which arise from the alphabet $\mathcal{A}$. Set $\mathcal{A} = \{1, \ldots, A\}$ with $A = 50$. In his thesis, Mercat connects the Arithmetic Chaos Conjecture with Zaremba’s, by proving the following

**Theorem 7.1** ([Mer12]). *If the reduced rational $m/n$ has all partial quotients bounded by $A$, and if the denominator $n$ arises as a solution to the Pellian equation $n^2 - \Delta r^2 = \pm 1$, then $\mathbb{Q}[\sqrt{\Delta}] \cap \mathcal{C}_A$ is non-empty.*

In fact, he exhibits a periodic continued fraction in $\mathbb{Q}[\sqrt{\Delta}]$ via an explicit construction involving the partial quotients of $m/n$.

With his theorem, we can now sketch a

**Proof of Theorem 1.38.** Iwaniec’s theorem [Iwa78] states that the number of $n$ up to $N$ with $\Delta = n^2 + 1$ having at most 2 prime factors is at least $CN/\log N$. Taking the alphabet $\mathcal{A} = \{1, \ldots, 50\}$ in Theorem 1.24, the error term in the estimate (1.25) is much smaller than $N/\log N$, and hence 100% of such denominators $n$ have a coprime numerator $m$ with $m/n$ having all partial quotients bounded by $A = 50$. Clearly setting $r = 1$ gives a solution to $n^2 - \Delta r^2 = -1$, whence $\Delta \in \mathcal{D}_A$ by Mercat’s theorem.  \[\square\]
Appendix A. Construction of $\aleph$

In this section, we sketch a proof of Proposition 3.9, constructing the special set $\aleph \subset \Gamma_0$ with good modular distribution properties, and all its elements having archimedean size at most a given parameter $Y$. Recall that $\Gamma_0$ is the semigroup corresponding to the fixed alphabet $\mathcal{A}_0 = \{1, 2\}$. The constants $\mathcal{B}$, $c$, $C$, and $\Theta$ in Theorem 3.2 depend only on $\mathcal{A}_0$, and thus are absolute. As we no longer need the original alphabet $\mathcal{A}$, we drop the subscript 0 from $\Gamma_0$, writing just $\Gamma$; similarly write $\delta$ for $\delta_{\mathcal{A}_0}$.

Let $T$ be a parameter to be chosen later relative to $Y$. Let

$$R := |\text{SL}_2(\mathcal{B})| \asymp 1,$$

and let

$$S(T) := \{ \gamma \in \Gamma : \|\gamma\| < T \},$$

with

$$\#S(T) \gg T^{2\delta}.$$

Then by the pigeonhole principle, there exists some $s_T \in S(T)$ so that

$$S'(T) := \{ \gamma \in \Gamma : \|\gamma\| < T, \gamma \equiv s_T(\mathcal{B}) \}$$

has cardinality at least

$$\#S'(T) \geq \frac{1}{R}\#S(T) \gg T^{2\delta}.$$

Observe that the elements in

$$S'(T) \cdot s_T^{R-1}$$

are all congruent the identity mod $\mathcal{B}$. Write $\text{SL}_2(\mathcal{B}) = \{\gamma_1, \ldots, \gamma_R\}$, and find $x_1, \ldots, x_R \in \Gamma$ with

$$x_j \equiv \gamma_j(\text{mod } \mathcal{B}).$$

Such $x_j$ can be found of size

$$\|x_j\| \ll 1.$$

For each $j = 1, \ldots, R$, let

$$S'_j(T) := S'(T) \cdot s_T^{R-1} \cdot x_j.$$

This is a subset of $\Gamma$, in which each element $s \in S'_j(T)$ has size as most

$$\|s\| \ll T^R.$$

Choose $T \asymp Y^{1/R}$ so that all elements $s \in \bigcup_j S'_j(T)$ satisfy

$$\|s\| < Y.$$
Applying Theorem 3.2 gives that for each \( j = 1, \ldots, R \), any \( q < Q_0 \) with \( q \equiv 0(\mathfrak{B}) \), and any \( \omega \in \text{SL}_2(q) \) with \( \omega \equiv \tau_j(\mathfrak{B}) \), we have

\[
\# \{ s \in S_j'(T) : s \equiv \omega(q) \} = \# \{ s \in S(T) : s \equiv \omega(s_T^{-1}\tau_j)^{-1}(q) \} \\
= \# \{ s \in S(T) : s \equiv \omega(s_T^{-1}\tau_j)^{-1}(q) \} \\
= \frac{|\text{SL}_2(\mathfrak{B})|}{|\text{SL}_2(q)|} \# \{ s \in S(T) : s \equiv s_T(\mathfrak{B}) \} + O(\mathcal{E}(T;q)) \\
= \frac{|\text{SL}_2(\mathfrak{B})|}{|\text{SL}_2(q)|} \# S_j'(T) + O(\mathcal{E}(Y;q)). \tag{A.1}
\]

Then the sets \( S_j'(T) \) each have good modular distribution properties for distinct residues mod \( \mathfrak{B} \). Note that they also all have the same cardinality, namely that of \( S'(T) \). Moreover after renaming constants, \( \mathcal{E}(T;q) \ll \mathcal{E}(Y;q) \). Hence setting

\[
\mathfrak{N} := \bigsqcup_{j=1}^{R} S_j'(T)
\]

gives the desired special set. The equidistribution (3.10) is now clear for any \( q \equiv 0(\mathfrak{B}) \), while the same for other \( q \) it is obtained by summing over suitable arithmetic progressions. This completes the proof.
Appendix B. Proof of Proposition 3.12

In this section, we sketch a proof of Proposition 3.12. Making the change of variables

\((a, b, c, d) \mapsto (x_1, x_2, x_3, x_4) = (a + d, a - d, b + c, b - c)\),

we have that the left hand side of (3.13),

\[
\sum_{\xi = (a, b, c, d) \in \mathbb{Z}^4 \atop ad - bc = 1} \varphi_X(\xi) e_q(\xi \cdot s),
\]

is essentially the same as

\[
\sum_{x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \atop x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4} \varphi\left(\frac{x_1}{X}\right) \cdots \varphi\left(\frac{x_4}{X}\right) e_q(x \cdot s), \quad (B.1)
\]
after renaming \(s = (s_1, \ldots, s_4)\). To capture the determinant condition \(\sum \delta_j x_j^2 = 4\) (with \(\delta_j \in \{\pm 1\}\)), we apply the circle method, writing

\[
(B.1) = \int_{\theta \in [0, 1]} \prod_{j=1}^4 G_X(\delta_j \theta; \frac{s_j}{q}) e(-4\theta) d\theta,
\]

where

\[
G_X(\theta; \lambda) := \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{X}\right) e\left(\theta x^2 + \lambda x\right). \quad (B.2)
\]

Write \(\theta = \frac{a}{r} + \beta\), with \(r < X\), \((a, r) = 1\), and \(|\beta| < 1/(rX)\). For given \(r < X\) and \(K < X/r\), break the circle \([0, 1]\) into dyadic regions of the form

\[V_{r,K} := \left\{ \theta = \frac{a}{r} + \beta : (a, r) = 1, \ |\beta| \gg \frac{K}{X^2} \right\}.\]

By Poisson summation, we have

\[
G_X\left(\frac{a}{r} + \beta; \lambda\right) = \sum_{k \in \mathbb{Z}} S_r(a; k) \cdot J_X\left(\beta; \lambda - \frac{k}{r}\right), \quad (B.3)
\]

where

\[
S_r(a; k) := \frac{1}{r} \sum_{y(r)} c_r(ay^2 + ky),
\]

and

\[
J_X(\beta; z) := \int_{x \in \mathbb{R}} \varphi\left(\frac{x}{X}\right) e\left(\beta x^2 + zx\right) dx.
\]
From stationary phase, we estimate $|J_X(\beta; z)| \ll \min(X, |\beta|^{-1/2})$; moreover only $\ll X^\varepsilon$ values of $k$ contribute to (B.3). A standard argument (see, e.g. [Bou93]) then gives the bound

$$\left| \int_{V_{r,K}} \prod_{j=1}^4 G_X \left( \delta_j \theta; \frac{s_j}{q} \right) e(-4\theta) d\theta \right| \ll \frac{X^\varepsilon X^2}{K} \frac{1}{r^{3/2}}, \quad (B.4)$$

which is insufficient for our purposes, since it has no decay in the $q$ aspect. Therefore we must extract more cancellation from the fact that the $\lambda$-variables in (B.4) are actually rational numbers, $\lambda = s_j/q$.

Assume for simplicity that $(s_1, q) = 1$. Returning to (B.2) with $\lambda = s/q$, $(s, q) = 1$, apply van der Corput’s shifting trick, averaging (in smooth form) over an auxiliary parameter $|\ell| < L$, with $L$ given by

$$L = L_{r,K} = \frac{X^{1-\varepsilon}}{Kr}. \quad (B.5)$$

Thus shifting $x \mapsto x + \ell r$ and writing $\theta = \frac{a}{r} + \beta$ gives

$$G_X \left( \theta; \frac{s}{q} \right) = \frac{1}{L} \sum_{\ell \in \mathbb{Z}} \psi \left( \frac{\ell}{L} \right) \sum_{x \in \mathbb{Z}} \varphi \left( \frac{x}{X} \right) e \left( \theta x^2 + \frac{s}{q} x \right)$$

$$= \frac{1}{L} \sum_{\ell \in \mathbb{Z}} \psi \left( \frac{\ell}{L} \right) \sum_{x \in \mathbb{Z}} \varphi \left( \frac{x + \ell r}{X} \right) e \left( \theta x^2 + \frac{s}{q} x \right) e \left( \beta \ell r (2x + \ell r) + \frac{s}{q} \ell r \right),$$

where $\psi$ is an appropriate bump function. Then $G_X$ may be replaced by

$$\sum_{x \in \mathbb{Z}} \varphi_1 \left( \frac{x}{X} \right) e \left( \theta x^2 + \frac{s}{q} x \right) \left[ \frac{1}{L} \sum_{\ell \in \mathbb{Z}} \psi \left( \frac{\ell}{L} \right) e \left( \frac{s}{q} \ell r \right) \right],$$

where we have replaced $e(\beta \ell r (2x + \ell r))$ by 1 using (B.5), and $\varphi_1$ has the same properties as $\varphi$. Applying Poisson summation allows us essentially to replace the bracketed term by $1_{\{\|sr/q\| < 1/L\}}$, where now $\| \cdot \|$ is the distance to the nearest integer. Having inserted this factor into the first copy of $G_X$, we proceed as before, replacing (B.4) with

$$\left| \int_{V_{r,K}} \cdots \right| \ll \frac{X^\varepsilon X^2}{K} \frac{1}{r^{3/2}} 1_{\{\|s\|/r < \frac{1}{L}\}},$$

where $s = s_1$ is coprime to $q$.

To summarize, we have estimated

$$| (B.1) | \ll \sum_{r < X} X^\varepsilon \sum_{K < X/r} \frac{X^2}{K} \frac{1}{r^{3/2}} 1_{\{\|s\|/r < \frac{1}{L}\}}, \quad (B.6)$$
Group the contributions according to whether or not $L_{r,K} > q$. In the former range, $\|sr/q\| = 0$, and hence $q \mid r$ since $(s,q) = 1$; in particular, $r \geq q$. The contribution in this range to (B.6) is

$$\ll X^\varepsilon \sum_{q < r < X \atop r = 0(q)} \sum_{K < X/r \atop \text{dyadic}} \frac{X^2}{K} \frac{1}{r^{3/2}} \ll X^\varepsilon \frac{X^2}{q^{3/2}},$$

giving the first term on the right hand side of (3.13).

When $L_{r,K} \leq q$, break $r$ into dyadic regions, $r \asymp R < X$, and in each, write $sr = uq + v$, $-q/2 \leq v < q/2$, so that $\|sr/q\| = |v|/q < 1/L$. Then by (B.5), there are at most $q/L \asymp qX^\varepsilon KR/X$ choices for $v$, and at most $R/q + 1$ values for $u$, giving the contribution

$$\ll X^\varepsilon \sum_{R,K \atop \text{dyadic}} \frac{X^2}{K} \frac{1}{R^{3/2}} \mathbf{1}_{\{\|sr/q\| < \frac{1}{L_{r,K}}\}}$$

$$\ll X^\varepsilon \sum_{R < X \atop \text{dyadic}} \frac{qR}{X} \left(\frac{R}{q} + 1\right) X^2 \frac{1}{R^{3/2}} \ll X^\varepsilon \left(X^{3/2} + qX\right).$$

These constitute the last two terms in (3.13), thus completing the proof.
References

[Art24] E. Artin. Ein mechanisches System mit quasi-ergodischen Bahnen. *Abh. Math. Sem. Hamburg*, 3:170–175, 1924. 2

[BGS06] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Sieving and expanders. *C. R. Math. Acad. Sci. Paris*, 343(3):155–159, 2006. 11

[BGS10] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Affine linear sieve, expanders, and sum-product. *Invent. Math.*, 179(3):559–644, 2010. 11

[BGS11] J. Bourgain, A. Gamburd, and P. Sarnak. Generalization of Selberg’s 3/16th theorem and affine sieve. *Acta Math.*, 207:255–290, 2011. 11, 15, 17, 18

[BK10] J. Bourgain and A. Kontorovich. On representations of integers in thin subgroups of SL(2, Z). *GAFA*, 20(5):1144–1174, 2010. 16

[BK11] J. Bourgain and A. Kontorovich. On Zaremba’s conjecture, 2011. To appear, *Annals Math.*, arXiv:1107.3776, 96 pp. 8, 9, 10, 16, 18

[BK12] J. Bourgain and A. Kontorovich. On the local-global conjecture for integral Apollonian gaskets, 2012. To appear, *Invent. Math.*, arXiv:1205.4416v1, 63 pp. 16

[BK13] Jean Bourgain and Alex Kontorovich. The affine sieve beyond expansion I: thin hypotenuses, 2013. Submitted, arXiv:1307.3535. 11, 16

[Bou93] Jean Bourgain. Eigenfunction bounds for the laplacian on the n-torus. *IMRN*, (3):61–66, 1993. 38

[Duk88] W. Duke. Hyperbolic distribution problems and half-integral weight Maass forms. *Invent. Math.*, 92(1):73–90, 1988. 4

[FK13] D. Frolenkov and I. D. Kan. A reinforcement of the Bourgain-Kontorovich’s theorem by elementary methods II, 2013. Preprint, arXiv:1303.3968. 10

[Goo41] I. J. Good. The fractional dimensional theory of continued fractions. *Proc. Cambridge Philos. Soc.*, 37:199–228, 1941. 7

[Gre86] G. Greaves. The weighted linear sieve and Selberg’s λ²-method. *Acta Arith.*, 47(1):71–96, 1986. 12

[Hen89] Doug Hensley. The distribution of badly approximable numbers and continuants with bounded digits. In *Théorie des nombres (Quebec, PQ, 1987)*, pages 371–385. de Gruyter, Berlin, 1989. 7

[Hen92] Doug Hensley. Continued fraction Cantor sets, Hausdorff dimension, and functional analysis. *J. Number Theory*, 40(3):336–358, 1992. 5

[Hen96] Douglas Hensley. A polynomial time algorithm for the Hausdorff dimension of continued fraction Cantor sets. *J. Number Theory*, 58(1):9–45, 1996. 9

[Hua13] S. Huang. An improvement on Zaremba’s conjecture, 2013. Preprint, arXiv:1310.3772. 10

[Iwa78] Henryk Iwaniec. Almost-primes represented by quadratic polynomials. *Invent. Math.*, 47:171–188, 1978. 12, 34

[Jen04] Oliver Jenkinson. On the density of Hausdorff dimensions of bounded type continued fraction sets: the Texan conjecture. *Stoch. Dyn.*, 4(1):63–76, 2004. 8

[McM09] Curtis T. McMullen. Uniformly Diophantine numbers in a fixed real quadratic field. *Compos. Math.*, 145(4):827–844, 2009. 4, 5, 7
[McM12] C. McMullen. Dynamics of units and packing constants of ideals, 2012. Online lecture notes, \url{http://www.math.harvard.edu/~ctm/expositions/home/text/papers/cf/slides/slides.pdf}. 4, 10

[Mer12] P. Mercat. Construction de fractions continues périodiques uniformément bornées, 2012. To appear, \textit{J. Théor. Nombres Bordeaux}. 5, 12, 34

[MVW84] C. Matthews, L. Vaserstein, and B. Weisfeiler. Congruence properties of Zariski-dense subgroups. \textit{Proc. London Math. Soc}, 48:514–532, 1984. 6

[Ser85] Caroline Series. The modular surface and continued fractions. \textit{J. London Math. Soc. (2)}, 31(1):69–80, 1985. 2

[Wil80] S. M. J. Wilson. Limit points in the Lagrange spectrum of a quadratic field. \textit{Bull. Soc. Math. France}, 108:137–141, 1980. 5

[Woo78] A. C. Woods. The Markoff spectrum of an algebraic number field. \textit{J. Austral. Math. Soc. Ser. A}, 25(4):486–488, 1978. 5

\textit{E-mail address}: bourgain@ias.edu

\textbf{School of Mathematics, IAS, Princeton, NJ}

\textit{E-mail address}: alex.kontorovich@yale.edu

\textbf{Department of Mathematics, Yale University, New Haven, CT}