On a problem involving the squares of odd harmonic numbers

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Abstract
We introduce a full solution to a problem considered by Wang and Chu concerning series involving the squares of finite sums of the form \(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\). Our proof involves techniques from the theory of colored multiple zeta values.

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1 Introduction
As Adamchik has expressed [1] (cf. [8]), there is a great history concerning the study of series of the following form:

\[
S(r) = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{2k}{k}\right)^2 \frac{1}{k+r}.
\]  

(1)

As in [1, 8], we recall that Ramanujan was interested in the \(S\)-function shown above [4, p. 39], and we record Ramanujan’s identity whereby

\[
S(r) = \frac{16^r}{\pi^2 (2r)^2} \sum_{k=0}^{r-1} \left(\frac{1}{16}\right)^k \left(\frac{2k}{k}\right)^2
\]  

(2)

for \(r \in \mathbb{N}_0\). Ramanujan’s discoveries concerning the \(S\)-function in (1), together with the large amount of research, over the years, concerning this function [1] [4, pp. 39–40], have inspired the development of Ramanujan-like series given by introducing summand factors, such as harmonic or harmonic-type numbers, in the series shown in (1) [5, 21]. In this

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article, we solve a problem recently considered by Wang and Chu [22] concerning a family of series resembling (1) and involving harmonic-type numbers, and we introduce many related evaluations.

We write \( O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \) to denote the \( n \)th odd harmonic number, by analogy with the usual notation \( H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \) for the \( n \)th entry in the classical sequence of harmonic numbers. In the recent article [22], Wang and Chu left the problem of evaluating series of the form

\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \left( \frac{2n}{n} \right)^2 \frac{O_n^2}{(1+2n-2\lambda)^2}
\]

(3)
as an open problem, letting \( \lambda \in \mathbb{N} \). Even for the base case such that \( \lambda = 1 \), evaluating the series in (3) is quite difficult, and it appears that the coefficient-extraction methods employed by Chu et al. [13, 21, 22] do not apply to (3). We offer a complete solution to the problem of evaluating (3) for an arbitrary parameter \( \lambda \in \mathbb{N} \).

2 The base case

We adopt notation from [9], writing

\[
G := \Im \left( \text{Li}_3 \left( \frac{i+1}{2} \right) \right)
\]

(4)
to denote the Catalan-like constant explored in [9], letting \( G := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \) denote the “original” Catalan constant, and writing \( \text{Li}_k(z) = z + \frac{z^2}{2^k} + \frac{z^3}{3^k} + \cdots \) to denote the polylogarithm function. As discussed in [9], the trilogarithmic expression in (4) has been considered as an irreducible constant in a variety of different applications and by a variety of different authors, and is a naturally occurring, higher-order analogue of both \( G = \Im(\text{Li}_2(i)) \) and the constant \( \mathcal{G} = \Im(\text{Li}_2(\sqrt{-1})) \) known as Gieseking’s constant [2]. The uses of the irreducible constant in (4) related to both quantum field theory [14] and the study of Euler-like sums [10, 19] further motivate Theorem 1 below, which is the base case for our full solution to Wang and Chu’s problem concerning (3). Our full solution is highlighted as Theorem 5 below.

**Theorem 1.** The infinite series

\[
\sum_{k=1}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k^2}{(2k-1)^2}
\]

admits the symbolic form

\[-\frac{8G}{\pi} - \frac{8G \ln(2)}{\pi} - \frac{48G}{\pi} + \frac{9\pi^2}{8} + \frac{\pi}{6} + \frac{4 \ln^2(2)}{\pi} + \frac{3 \ln^2(2)}{2}.
\]
Proof. Using the Wilf–Zeilberger method [18], Campbell [7] had recently proved the following equality, noting the appearance of a copy of the base case of (3) for \( \lambda = 1 \):

\[
4 \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k}{2k + 1} - \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k^2}{2k - 1} - 2 \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k^2}{(2k - 1)^2} = \frac{12G}{\pi} + \frac{32G}{\pi} - \frac{6 \ln^2(2)}{\pi} - \frac{3\pi^2}{4} - \frac{\pi}{4} - \ln^2(2).
\]

Using a recursive proof approach, Campbell [7] had also evaluated the “central” series in (5):

\[
\sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k^2}{2k - 1} = \frac{4G}{\pi} - \frac{\pi}{12} - \frac{2 \ln^2(2)}{\pi},
\]

giving us that the following equality holds:

\[
4 \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k}{2k + 1} - 2 \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k^2}{(2k - 1)^2} = \frac{16G}{\pi} + \frac{32G}{\pi} - \frac{3\pi^2}{4} - \frac{8 \ln^2(2)}{\pi} - \ln^2(2).
\]

So, the problem of evaluating the base case for (3) is equivalent to the problem of evaluating

\[
\sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k}{2k + 1}.
\]

In this direction, we claim that the following equality holds:

\[
\sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left( \frac{2k}{k} \right)^2 \frac{O_k - H_k}{2k + 1} = -\frac{16G}{\pi} + \frac{\pi^2}{8} + \frac{8G \ln(2)}{\pi} + \frac{\ln^2(2)}{2}.
\]

To prove this, we need the following integrals:

\[
\int_0^{\pi/4} (\ln(\sin(t)) - \ln(\cos(t)))dt = -G,
\]

\[
\int_0^{\pi/4} (\ln^2(\sin(t)) - \ln^2(\cos(t)))dt = \frac{5\pi^3}{64} + \frac{1}{16} \pi \ln^2(2) - 2G + G \ln(2).
\]

The first is a well known expression for Catalan’s constant, whereas the second one can be found in [9]. We are to also use the following result [17]:

\[
\int_0^{\arcsin(x)} \ln^n(u)du = \sum_{i=0}^{n} \left( -1 \right)^i i! \binom{n}{i} \ln^{n-i}(x) \sum_{k=0}^{\infty} \frac{\left( \frac{2k}{k} \right)^{2k+1} x^{2k+1}}{\left( 2k + 1 \right)^{i+1} 2^{2k}}.
\]
For $n = 1$, (11) reduces to:
\[
\int_0^{\arcsin(x)} \ln(\sin u) \, du = \ln(x) \sum_{k=0}^{\infty} \frac{(2k)_k x^{2k+1}}{(2k+1) 2^{2k}} - \sum_{k=0}^{\infty} \frac{(2k)_k x^{2k+1}}{(2k+1)^2 2^{2k}}.
\]
We divide by $x$, replace $x$ by $\sin \nu$, and integrate the result between 0 and $\frac{\pi}{2}$:
\[
\int_0^{\pi/2} \left( \frac{1}{\sin \nu} \int_0^{\nu} \ln(\sin u) \, du \right) \, d\nu = \sum_{k=0}^{\infty} \frac{2k}{(2k+1) 2^{2k}} \int_0^{\pi/2} \ln(\sin \nu) \sin^{2k} \nu \, d\nu - \sum_{k=0}^{\infty} \frac{2k}{(2k+1)^2 2^{2k}}. \tag{12}
\]
In the double integral on the left-hand side, we change the order of integration:
\[
\int_0^{\pi/2} \left( \frac{1}{\sin \nu} \int_0^{\nu} \ln(\sin u) \, du \right) \, d\nu = \int_0^{\pi/2} \left( \ln(\sin u) \int_u^{\pi/2} \frac{1}{\sin \nu} \, d\nu \right) \, du = -\int_0^{\pi/2} \ln(\sin \nu) \ln(\tan \frac{\nu}{2}) \, d\nu.
\]
Using the substitution $u = 2t$ the integral can be rewritten as:
\[
-2\int_0^{\pi/4} \left[ \ln 2 \cdot (\ln(\sin(t)) - \ln(\cos(t))) + \ln^2(\sin(t)) - \ln^2(\cos(t)) \right] \, dt \tag{13}
\]
\[
= -\frac{5}{32} \pi^3 - \frac{1}{8} \pi \ln^2 2 + 4G
\]
using (9) and (10). The integral in the right-hand side of (12) is given in [15, 4.241.1]. Explicitly, writing $I_k := \int_0^{\pi/2} \ln(\nu) \sin^{2k} \nu \, d\nu$, we may evaluate $I_k$ so that $I_k = \frac{\pi}{2} \frac{1}{2^{2k}} \binom{2k}{k} \left( O_k - \frac{1}{2} H_k - \ln 2 \right)$. Hence, the right-hand side of (12) reduces to:
\[
\frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1}{16} \binom{2k}{k}^2 \left( O_k - \frac{1}{2} H_k - \ln 2 \right) \frac{1}{2k+1} - \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1}{16} \binom{2k}{k}^2 \frac{1}{(2k+1)^2}.
\]
Multiplying both sides of (12) by $\frac{4}{\pi}$ and using (9), (10) and (13) we obtain an equivalent form of (8).

So, it remains to evaluate the following series:
\[
\sum_{k=0}^{\infty} \frac{1}{16} \binom{2k}{k}^2 H_k \left(\frac{2k}{2k+1}\right)^2. \tag{14}
\]
Using a reindexing argument, we obtain the following:
\[
\sum_{k=0}^{\infty} \frac{1}{16} \binom{2k}{k}^2 H_k \left(\frac{2k}{2k+1}\right)^2 = \sum_{k=1}^{\infty} \frac{1}{16} \binom{2(k-1)}{k-1}^2 \frac{H_{k-1}}{2k-1}.
\]
\[-\frac{4(2G - 1)}{\pi} + \sum_{k=1}^{\infty} \left( \frac{1}{16} \right)^k \frac{(2k)^2 H_k}{(2k - 1)^3} + \frac{2}{(2k - 1)^2} + \frac{1}{2k - 1} \].

Using previously known Ramanujan-like series introduced in [5, 12], we obtain that the series in (14) equals:
\[
\frac{24}{\pi} - \frac{8G}{\pi} - \frac{24 \ln(2)}{\pi} + \sum_{k=1}^{\infty} \left( \frac{1}{16} \right)^k \frac{(2k)^2 H_k}{(2k - 1)^3}.
\]

We claim that
\[
\sum_{k=1}^{\infty} \left( \frac{1}{16} \right)^k \frac{(2k)^2 H_k}{(2k - 1)^3}
\]
equals
\[
\frac{8G}{\pi} - \frac{16G \ln(2)}{\pi} - \frac{16G}{\pi} + \frac{5\pi^2}{8} - \frac{24}{\pi} + \frac{\ln^2(2)}{2} + \frac{24 \ln(2)}{\pi}.
\]
To show this, we use an iterated integral-based approach, using results from [23]. We recall that for any real numbers \(a, b\) and 1-forms \(f_j(t) dt\) \((j = 1, \ldots, d)\) with \(d > 1\) we may recursively define the iterated integral
\[
\int_a^b f_1(t) dt f_2(t) dt \cdots f_d(t) dt := \int_a^b f_1(u) \left( \int_a^u f_2(t) dt \cdots f_d(t) dt \right) du.
\]
By the proof of [23, Theorem 6.1] we may use iterated integrals to compute (15) as follows:
\[
I_s := \sum_{k>m>0} \left( \frac{1}{16} \right)^k \frac{(2k)^2 H_k}{(2k - 1)^3+2(2m)} \frac{1}{(2k - 1)^s+2(2m)}
\]
\[
= \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^s \cot^2 t dt d(\sec t) \left( \csc t - \cot t \right) dt
\]
\[
= \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^s (\csc^2 t - 1) dt d(\sec t) \left( \csc t - \cot t \right) dt = X_s - Y_s - I_{s-1},
\]
where
\[
X_s := \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^s \sec t dt \left( \csc t - \cot t \right) dt,
\]
\[
Y_s := \frac{2}{\pi} \int_0^{\pi/2} \sin t dt (\cot t dt)^s dt d(\sec t) \left( \csc t - \cot t \right) dt.
\]
Set 1-forms \(a = dt/t, x_\mu = dt/(\mu - t), d_{\mu,\nu} = x_\mu - x_\nu\) for any roots of unity \(\mu\) and \(\nu\), and \(y = x_i + x_{-i} - x_1 - x_{-1}\). Then
\[
X_0 = \frac{2}{\pi} \int_0^{\pi/2} dt \left( \csc t - \cot t \right) dt = \frac{2}{\pi} \int_0^1 i (2x_{-1} - x_i - x_{-i}) d_{-i,i},
\]
Similarly, by the change of variables $t \to \sin^{-1}(1 - t^2)/(1 + t^2)$ at the last step for both $X_0$ and $X_1$. Similarly,

\[
X_1 = 2 \pi \int_{0}^{\pi/2} (\cos t \cot t) \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= 2 \pi \int_{0}^{\pi/2} (\csc t - \sin t) \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= - X_0 + 2 \pi \int_{0}^{\pi/2} \csc t \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= - X_0 - (-1)^s \frac{2 \pi}{\pi} \int_{0}^{1} \left( 2x_{-1} - x_{i} - x_{-i} \right) \text{ad}_{-1,1},
\]

\[
Y_0 := 2 \pi \int_{0}^{\pi/2} \sin t \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= 2 \pi \int_{0}^{\pi/2} \cos t \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= 2 \pi \int_{0}^{\pi/2} \csc t \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= - Y_0 + 2 \pi \int_{0}^{\pi/2} \csc t \ dt \ \sec t \ dt \left( \csc t - \cot t \right) \ dt
\]

\[
= - Y_0 + 2 \pi \int_{0}^{\pi/2} \csc t \ dt \left[ \sec t \ dt \left( \csc t - \cot t \right) \ dt - \csc t \ dt \left( \csc t \sec t - \csc t \right) \ dt \right]
\]

\[
= - Y_0 + 2 \pi \int_{0}^{1} \left( 2x_{-1} - x_{i} - x_{-i} \right) \text{ad}_{-1,1} - \frac{2 \pi}{\pi} \int_{0}^{\pi/2} i \left( a + 2x_{-1} \right) \text{d}_{-i,1} d_{-1,1}.
\]

Thus

\[
I_1 = X_1 - Y_1 - I_0 = Y_0 - X_0 - I_0 + \frac{2 \pi}{\pi} \int_{0}^{1} \left( 2x_{-1} - x_{i} - x_{-i} \right) \text{ad}_{-1,1}
\]

\[
- \frac{2 \pi}{\pi} \int_{0}^{1} \left( 2x_{-1} - x_{i} - x_{-i} \right) \text{ad}_{-1,1} + \frac{2 \pi}{\pi} \int_{0}^{\pi/2} i \left( a + 2x_{-1} \right) \text{d}_{-i,1} d_{-1,1}
\]

\[
= - I_0 + \frac{2 \pi}{\pi} \int_{0}^{1} i \left( 2x_{-1} - x_{i} - x_{-i} \right) \text{d}_{-i,1} i + \frac{2 \pi}{\pi} \int_{0}^{\pi/2} i \left( a + 2x_{-1} \right) \text{d}_{-i,1} d_{-1,1}.
\]
Note that $I_0$ is given by [23, Example B.8]. Hence we obtain the following formula by using Au’s Mathematica package of colored multiple zeta values [3]:

$$
\sum_{k,m > 0} \frac{1}{(2k-1)^3(2m)} = \frac{2}{\pi} \left( \frac{5}{32} \pi^3 - 4G - 2G(1 + 2 \ln 2) + 6 \ln 2 + \frac{\pi}{8} \left( 8 \ln 2 + \ln^2 2 - 12 \right) \right).
$$

Now, using partial fraction decomposition, we see that

$$
\sum_{k > 0} \frac{1}{16} \frac{2k^2}{(2k)^2} \frac{1}{(2k-1)^3(2k)} = \sum_{k > 0} \left( \frac{\pi}{2} + 2G - 3 \right) - \frac{2}{\pi} \left( 2 - \frac{\pi}{2} \right) + 2 \left( \frac{\pi}{2} - 1 \right) - \frac{2}{\pi} (\pi \ln 2 - 2G) = \frac{2}{\pi} \left( \frac{3\pi}{2} + 4G - \pi \ln 2 - 6 \right).
$$

by Examples B.1 and B.3 of [23]. Therefore,

$$
\sum_{k > 0} \frac{1}{16} \frac{(2k)^2}{(2k)} \frac{H_k}{(2k-1)^3} = 2 \sum_{k > 0} \frac{1}{16} \frac{(2k)^2}{(2k-1)^3} + 2 \sum_{k,m > 0} \frac{1}{(2k-1)^3(2m)} = \frac{4}{\pi} \left( \frac{5}{32} \pi^3 - 4G + 2G(1 - 2 \ln 2) + 6 \ln 2 + \frac{\pi}{8} \ln^2 2 - 6 \right).
$$

So, we obtain that:

$$
\sum_{k=1}^{\infty} \frac{1}{16} \frac{(2k)^2}{k} \frac{H_k}{2k+1} = \frac{16G \ln(2)}{\pi} - \frac{16G}{\pi} + \frac{5\pi^2}{8} + \frac{\ln^2(2)}{2}.
$$

(16)

Thus, according to (8), we find that:

$$
\sum_{k=1}^{\infty} \frac{1}{16} \frac{(2k)^2}{k} \frac{O_k}{2k+1} = -\frac{4G \ln(2)}{\pi} - \frac{16G}{\pi} + \frac{3\pi^2}{8} + \frac{\ln^2(2)}{2}.
$$

(17)

So, according to our relation for (7), as derived from [7], we obtain the desired result. □
3 The general case

We will now derive an explicit evaluation for (3), for an arbitrary parameter $\lambda \in \mathbb{N}$. We also take care of the open problem from [22] of evaluating

$$\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(1 + 2n - 2\lambda)^2}. \quad (18)$$

Chu [13] had recently employed a hypergeometric approach to evaluate

$$\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{1 + 2n - 2\lambda}, \quad (19)$$

and we will also need an evaluation for (19), in our solution to the main problem of evaluating (3). We will also need an evaluation for

$$\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2\lambda}, \quad (20)$$

noting that series of the form indicated in (20) were also evaluated by Wang and Chu in [22], again via coefficient extractions.

Defining

$$A(m) := \begin{cases} \frac{16^m}{2\pi m^2 (2m)^2} \sum_{k=0}^{m-1} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 & \text{if } m \in \mathbb{N} \\ \frac{4G}{\pi} & \text{if } m = 0 \end{cases}$$

and

$$B(m) := \begin{cases} \frac{16^m}{m^2 (2m)^2} \sum_{k=0}^{m-1} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 k A(k) + \frac{1}{2k + 1} & \text{if } m \in \mathbb{N} \\ \frac{-16G}{\pi} + \frac{3\pi^2}{8} + \frac{\ln^2(2)}{2} & \text{if } m = 0, \end{cases}$$

this leads us toward the following Lemma. The $m = 0$ case for (22) has been established in [9, 10].

**Lemma 1.** For $m \in \mathbb{N}_0$, we have that

$$\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{2n + 1 - 2m} = A(m), \quad (21)$$

$$\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(2n + 1 - 2m)^2} = B(m). \quad (22)$$
The proof of these two results is similar to that of (2). See also [16]. With regard to our applying the A- and B-sequences above toward our fully solving open problems from [22], our results as in Theorem 5 are also of interest due to how the Landau constants $\sum_{i=0}^{n} \left( \frac{1}{16} \right)^i \left( \frac{2i}{i} \right)^2$ are heavily involved in our work; see [8] and references therein for related uses of the Landau constants. We see that the $A$-expression defined in (21) is, up to a scalar multiple by a closed-form expression, equal to a Landau constant for all $m$, and similarly with the Ramanujan summation in (1). The history of Ramanujan’s $S$-function [1], the importance of the Landau constants in complex analysis, and the ongoing interest in computational problems concerning approximating the Landau constants all contribute to the interest in how our explicit, finite sum evaluations as in Theorem 5 may be formulated in a natural way in terms of the above indicated $A$-sequence.

**Theorem 2.** For all $\lambda \in \mathbb{N}$ we have

$$\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2\lambda} = -\ln(2)A(\lambda) + \frac{16^\lambda}{2\lambda^2 (2^\lambda)} \sum_{k=0}^{\lambda-1} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 kA(k). \quad (23)$$

**Proof.** To determine a recursion for (20), we start from

$$\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2\lambda} = \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_{n+1} - \frac{1}{2n+1}}{1 + 2n - 2\lambda} \quad (24)$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_{n+1}}{1 + 2n - 2\lambda} - \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(2n+1)(1 + 2n - 2\lambda)}.$$

For the second term at the right-hand side we use (21) and the well known series

$$\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{2n+1} = \frac{4G}{\pi}.$$

Using partial fractions, we then obtain:

$$\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(2n+1)(1 + 2n - 2\lambda)} = \frac{1}{2\lambda}A(\lambda) - \frac{1}{2\lambda} \frac{4G}{\pi}. $$

For the first term at the right-hand side we use reindexing:

$$\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_{n+1}}{1 + 2n - 2\lambda} = 4 \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 O_n \frac{n^2}{(2n-1)^2(1 + 2n - 2\lambda - 2)}.$$

The last factor has the following partial fraction expansion:

$$\frac{n^2}{(2n-1)^2(1 + 2n - 2\lambda - 2)} = \frac{1}{16\lambda^2} \frac{1 + 2\lambda}{2n + 1 - 2\lambda - 2} - \frac{1}{16\lambda^2} \frac{1 + 4\lambda}{2n - 1} - \frac{1}{8\lambda} \frac{1}{(2n-1)^2}.$$
leading to the following two known series (see [22])

\[- \frac{1 + 4\lambda}{16\lambda^2} \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{2n-1} = -\frac{1 + 4\lambda \ 2 \ln(2)}{16\lambda^2 \pi},\]

\[- \frac{1}{8\lambda} \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(2n-1)^2} = -\frac{1}{2\lambda} \frac{G - \ln(2)}{\pi},\]

and the series

\[\frac{(1 + 2\lambda)^2}{16\lambda^2} \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2\lambda - 2}.\]

Setting

\[C(m) := \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2m}\]

(24) can be rewritten as:

\[C(\lambda + 1) = \frac{4\lambda^2}{(1 + 2\lambda)^2} C(\lambda) + \frac{2}{(1 + 2\lambda)^2} \left( \lambda A(\lambda) + \frac{\ln(2)}{\pi} \right) \quad \text{with} \quad C(1) = \frac{2\ln(2)}{\pi}.\]

To solve this recurrence, note that

\[z(\lambda) := 2 \left( \frac{1}{16} \right)^{\lambda} \left( \frac{2\lambda}{\lambda} \right)^2 \lambda^2 C(\lambda)\]

is a solution of the recurrence

\[z(\lambda + 1) = z(\lambda) + \left( \frac{1}{16} \right)^{\lambda} \left( \frac{2\lambda}{\lambda} \right)^2 \left( \lambda A(\lambda) + \frac{\ln(2)}{\pi} \right) \quad \text{with} \quad z(0) = 0.\]

An explicit evaluation for this solution is given by

\[z(\lambda) = \sum_{k=0}^{\lambda-1} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \left( k A(k) + \frac{\ln(2)}{\pi} \right),\]

leading to:

\[C(\lambda) = \frac{16\lambda}{2\lambda^2 (2\lambda)^2} \sum_{k=0}^{\lambda-1} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \left( k A(k) + \frac{\ln(2)}{\pi} \right),\]

which is equivalent with (23).

\[\square\]

**Theorem 3.** For all \(\lambda \in \mathbb{N}\) we have

\[\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(1 + 2n - 2\lambda)^2} = \frac{16\lambda}{\lambda^2 (2\lambda)^2} \left( G - \ln(2) \right) \frac{\pi}{\lambda} + \sum_{k=1}^{\lambda-1} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 D(k)\]

where \(C(m)\) is given by (25) and

\[D(m) := \frac{1}{2m + 1} \left( mC(m) - \frac{2m - 1}{4} A(m) - \frac{\ln(2)}{\pi} \right) + \frac{m}{2} B(m).\]
Proof. In this case we start from:

\[ \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(1 + 2n - 2\lambda)^2} = \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_{n+1} - \frac{1}{2n+1}}{(1 + 2n - 2\lambda)^2} \]  

(27)

= \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_{n+1}}{(1 + 2n - 2\lambda)^2} - \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(2n+1)(1 + 2n - 2\lambda)^2}.

The rest of the proof is very similar to the proof of the previous theorem, but in this case we need the following two results:

\[ \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2\lambda} = B(\lambda) \]

and

\[ \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{1 + 2n - 2\lambda} = C(\lambda). \]

Let us define

\[ E(m) := \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(1 + 2n - 2m)^2} \]

then the corresponding recurrence relation can be written as:

\[ (\lambda + 1)^2 \left( \frac{1}{16} \right)^{\lambda+1} \binom{2\lambda}{\lambda+1} \binom{2\lambda}{\lambda} E(\lambda+1) = \lambda^2 \left( \frac{1}{16} \right)^\lambda \binom{2\lambda}{\lambda} \binom{2\lambda}{\lambda} E(\lambda) + \left( \frac{1}{16} \right)^\lambda \binom{2\lambda}{\lambda} D(\lambda) \]

with \( D(m) \) as in the statement of the theorem and with \( E(1) = \frac{4G - 4\ln(2)}{\pi} \).

**Theorem 4.** For all \( \lambda \in \mathbb{N} \) we have

\[ \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{1 + 2n - 2\lambda} = \frac{16\lambda}{\lambda^2(2\lambda)^2} \left( \frac{48G - \pi^2 - 24\ln^2(2)}{48\pi} \sum_{k=1}^{\lambda-1} \frac{1}{16} \binom{2k}{k} F(k) \right) \]

(28)

where \( C(m) \) is given by (25) and

\[ F(m) := mC(m) + \frac{1}{4} A(m) - \frac{\pi^2 + 24\ln^2(2)}{48\pi}. \]

**Proof.** We start with

\[ \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{1 + 2n - 2\lambda} = \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_{n+1} - \frac{1}{2n+1}}{(1 + 2n - 2\lambda)^2}. \]

We proceed in the same way as before. After expanding using partial fractions, we need the sums of some additional series:

\[ \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(2n+1)^2} = -\frac{16G}{\pi} + \frac{3\pi^2}{8} + \frac{\ln^2(2)}{2} \quad \text{(see (31))}, \]
\[
\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{(2n-1)^2} = \frac{8G}{\pi} - \frac{8G \ln(2)}{\pi} - \frac{48G}{\pi} + \frac{9\pi^2}{8} + \frac{\pi}{6} + \frac{4\ln^2(2)}{\pi} + \frac{3\ln^2(2)}{2}
\]  
(see Theorem 1), and
\[
\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(2n-1)^3} = -\frac{4G(\ln(2) + 2)}{\pi} - \frac{32G}{\pi} + \frac{6\ln(2)}{\pi} + \frac{3\pi^2}{4} + \ln^2(2).
\]

The sum of this last series can be found by reindexing:
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(2n-1)^3} = \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \frac{4(2n-1)^2}{n^2} \frac{(2n-2)^2}{(n-1)^2} \frac{O_{n-1} + \frac{1}{2n-1}}{(2n-1)^3}
\]
\[
= \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(n+1)^2(2n+1)} + \frac{1}{4} \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(n+1)^2(2n+1)^2}.
\]

In addition to the series used before, here we also need these:
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{1}{(n+1)^i} (i = 1, 2) \quad (\text{see [16]}),
\]
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{(n+1)^i} (i = 1, 2) \quad (\text{see [22]}),
\]
and
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n}{2n+1},
\]
which is given in (17). Defining
\[
G(m) := \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{1 + 2n - 2m}
\]
we find the same type of recurrence for \( G \).

The following is a main result in our work, as it provides a full solution to an especially difficult open problem from [22], recalling the nontrivial nature of the base case covered in Section 2.

**Theorem 5.** For \( \lambda \in \mathbb{N} \),
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{(1 + 2n - 2\lambda)^2} = \frac{16^\lambda}{\lambda^2} \binom{2}{\lambda}^2 \left( K + \sum_{k=1}^{\lambda-1} \binom{1}{16} \binom{2k}{k}^2 H(k) \right)
\]
with
\[
K := \frac{1}{4} \left( -\frac{8G}{\pi} - \frac{8G \ln(2)}{\pi} - \frac{48G}{\pi} + \frac{9\pi^2}{8} + \frac{\pi}{6} + \frac{4\ln^2(2)}{\pi} + \frac{3\ln^2(2)}{2} \right)
\]
and

\[ H(m) := \frac{2m(G(m) - C(m)) + C(m) - A(m)}{2(2m + 1)} + \frac{\pi^2 + 24 \ln^2(2)}{24(2m + 1)\pi} + \frac{1}{4} B(m) + mE(m). \]

**Proof.** The method is the same as in the proof of the previous theorems. We start with:

\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{O_n^2} \frac{O_n^2}{(1 + 2n - 2\lambda)^2} = \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{O_{n+1}^2} \frac{O_{n+1}^2}{(1 + 2n - 2\lambda - 2)^2}.
\]

Expanding the square in the numerator and reindexing the series containing \(O_n\) leads to a sum of three series:

\[
4 \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{O_n^2} \frac{n^2}{(2n - 1)^2(1 + 2n - 2\lambda - 2)^2},
\]

\[
-8 \sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{O_n^2} \frac{n^2}{(2n - 1)^3(1 + 2n - 2\lambda - 2)^2},
\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{O_n^2} \frac{1}{(2n + 1)^2(1 + 2n - 2\lambda - 2)^2}.
\]

We now use partial fractions as before. One of the series we obtain is the series (29) with \(\lambda\) replaced by \(\lambda + 1\). All the other ones we have already encountered in the previous theorems. Setting

\[
J(m) := \sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{O_n^2} \frac{O_n^2}{(1 + 2n - 2m)^2},
\]

we again find that \(J\) satisfies a first order recurrence:

\[
(\lambda + 1)^2 \left( \frac{1}{16} \right)^{\lambda+1} \left( \frac{2\lambda + 2}{\lambda +1} \right)^2 J(\lambda + 1) = \lambda^2 \left( \frac{1}{16} \right)^{\lambda} \left( \frac{2\lambda}{\lambda} \right)^2 J(\lambda) + \left( \frac{1}{16} \right)^{\lambda} \left( \frac{2\lambda}{\lambda} \right)^2 H(\lambda)
\]

with \(J(1) = 4K\) by Theorem 1. Solving this recurrence in the same way as in Theorem 2 leads to the desired result.

**Example 1.** In addition to the base case covered in Section 2, we obtain, for the \(\lambda = 2\), \(\lambda = 3\), and \(\lambda = 4\) cases, the following:

\[
\sum_{k=1}^{\infty} \left( \frac{1}{16} \right)^k \frac{(2k)^2}{(2k - 3)^2} = \frac{32G}{27\pi} - \frac{32G\ln 2}{9\pi}
\]

\[
- \frac{64G}{3\pi} + \frac{\pi^2}{2} + \frac{11\pi}{162} + \frac{16}{27\pi} + \frac{44\ln^2 2}{27\pi} + \frac{2\ln^2 2}{3} - \frac{52\ln 2}{27\pi},
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{(2n - 5)^2} = \frac{32\ln^2(2)}{75} + \frac{8\pi^2}{25} + \frac{3388 \ln^2(2)}{3375\pi} - \frac{512G \ln(2)}{225\pi}
\]

13
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{O_n^2}{(2n-7)^2} = \frac{384 \ln^2(2)}{1225} + \frac{288 \pi^2}{1225} + \frac{92804 \ln^2(2)}{128625\pi} - \frac{2048G \ln(2)}{1225\pi}
\]

\[
+ \frac{23201\pi}{771750} - \frac{65284 \ln(2)}{42875\pi} + \frac{188144}{231525\pi}.
\]

\section{Further evaluations and concluding remarks}

Our methods have involved the evaluation of series involving summand factors of the form

\[
\left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{1}{2k+1}
\]

for \( k \in \mathbb{N}_0 \), so it seems worthwhile to consider further evaluations for such sums and related summations. In this regard, setting \( \beta(4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} \), let us reproduce the following identities from [9, 10]:

\[
\sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{1}{2k+1} = \frac{4G}{\pi}, \quad (30)
\]

\[
\sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{1}{(2k+1)^2} = -\frac{16G}{\pi} + \frac{3\pi^2}{8} + \frac{\ln^2(2)}{2}, \quad (31)
\]

\[
\sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{1}{(2k+1)^3} = -\frac{64}{\pi} \left( \text{Li}_4(1-i) \right) - \frac{\pi^2}{2} \ln^2(2) - \frac{2}{3} \ln^2(2) - \frac{48}{\pi} \frac{\beta(4)}{\pi}. \quad (32)
\]

Such evaluations are closely related to the evaluation of series of the form \( \sum_{k=1}^{\infty} \frac{(2k)^2}{k^2 16^k} \). For example, consider the following analogue of (31).

\textbf{Theorem 6.} The evaluation

\[
\sum_{k=1}^{\infty} \frac{(2k)^2}{k^2 16^k} = \frac{3\pi^2}{2} - \frac{64G}{\pi} - 6\ln^2(2)
\]

holds true.

\textit{Proof.} For our proof, we have to calculate the following:

\[
\sum_{k=1}^{\infty} \frac{(2k)^2}{k^2 16^k} \cdot \frac{\pi}{2} = \int_0^{\pi/2} 2 \text{Li}_2\left(\frac{1-\cos(t)}{2}\right)dt - 4 \int_0^{\pi/2} \ln^2\left(\cos\left(\frac{t}{2}\right)\right)dt. \quad (33)
\]

The second integral in (33) is equivalent with

\[
8 \int_0^{\pi/4} \ln^2(\cos(t)) dt = -\frac{7\pi^3}{48} + \frac{7}{4} \pi \ln^2(2) + 8G - 4G \ln(2), \quad (34)
\]
which can be found in [9]. For the first one, we use partial integration:

\[
\int_0^{\pi/2} 2 \text{Li}_2\left(\frac{1-\cos(t)}{2}\right) dt = \frac{1}{12} \pi^3 - \frac{1}{2} \pi \ln^2(2) + 2 \int_0^{\pi/2} t \cot(t) \ln(\cos^2\left(\frac{t}{2}\right)) dt \\
= \frac{1}{12} \pi^3 - \frac{1}{2} \pi \ln^2(2) + 8 \int_0^{\pi/4} t \cot(t) \ln(\cos^2(t)) dt.
\]

For this integral we use the substitution \( u = \tan(t) \):

\[
\int_0^{\pi/4} t \cot(t) \ln(\cos^2(t)) dt = -\int_0^1 \arctan(u) \ln(1 + u^2) \frac{1}{u(1 + u^2)} du \\
= -\int_0^1 \arctan(u) \ln(1 + u^2) \left( \frac{1}{u} - \frac{u}{1 + u^2} \right) du.
\]

The first integral can be found in [9]. The second one requires one partial integration in combination with [9]:

\[
\int_0^{\pi/4} \ln^2(\sin(t)) dt = -G + \frac{1}{2} G \ln(2) + \frac{23}{384} \pi^3 + \frac{9}{32} \pi \ln^2(2)
\]

and (10). Bringing everything together, we obtain the desired result.

Remark 1. Another method to evaluate the series in Theorem 6 is given in [23, Example B.1] using different basis elements. Moreover, Xu and Zhao proved in loc. cit. that

\[
\sum_{k=1}^{\infty} \frac{(2k)^2}{k^3 16^k} = \frac{512}{\pi^3} \left( \text{Li}_4\left(\frac{1+i}{2}\right) \right) - 6\pi^2 \ln(2) + 8 \ln^3(2) + 384 \frac{\beta(4)}{\pi} + 4\zeta(3) \quad (35)
\]

Noting that

\[
\sum_{k=1}^{\infty} \frac{(2k)^2}{k^n 16^k} = \sum_{k=0}^{\infty} \frac{4(2k+1)^2(2k)^2}{(k+1)^{n+2} 16^{k+1}} = \sum_{k=0}^{\infty} \frac{4(2k+1-1)^2(2k)^2}{(k+1)^{n+2} 16^{k+1}},
\]

we obtain, using previous results, that

\[
\sum_{k=0}^{\infty} \frac{(2k)^2}{(k+1)^3 16^k} = \frac{48}{\pi} - 16 - \frac{32G}{\pi} - 16 \ln(2),
\]

\[
\sum_{k=0}^{\infty} \frac{(2k)^2}{(k+1)^4 16^k} = \frac{128}{\pi} - 48 - \frac{128G}{\pi} + 64 \ln(2) + 6\pi^2 - \frac{256G}{\pi} - 24 \ln^2(2),
\]

\[
\sum_{k=0}^{\infty} \frac{(2k)^2}{(k+1)^5 16^k} = \frac{320}{\pi} - 128 - \frac{384G}{\pi} + 192 \ln(2) + 24\pi^2 + \frac{1024G}{\pi} - 96 \ln^2(2)
\]

\[
- \frac{2048}{\pi} \Im\left( \text{Li}_4\left(\frac{1+i}{2}\right) \right) - 24\pi^2 \ln(2) + 32 \ln^3(2) + \frac{1536\beta(4)}{\pi} + 16\zeta(3).
\]

Our applications, as above, of series as in (30)–(32) motivate the study and application of identities as below.
Theorem 7. For any integer $p \geq 0$, we have
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{(2n+1)^p+1} = \frac{\theta \ln^p(2 \sin \theta)}{2p!} + \frac{1}{4p!} \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} \ln^{p-j}(2 \sin \theta) L_{s,j+1}(2\theta),
\]
where $\theta := \arcsin(2z)$ and
\[
L_{s,j}(\theta) := -\int_{0}^{\theta} \ln^{j-1} \left(2 \sin \frac{t}{2}\right) dt.
\]

Proof. We begin with the elementary Maclaurin series expansion
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{2n+1} = \frac{1}{2} \arcsin(2z),
\]
which is the $p = 0$ case. Now, from (38), one deduces for $p > 0$ that
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{(2n+1)^p+1} = \frac{1}{2(p-1)!} \int_{0}^{z} \frac{\ln^{p-1} \left(\frac{z}{t}\right) \arcsin(2t)}{t} d(4t)
\]
\[
= \frac{1}{2(p-1)!} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \ln^{p-1-j}(4z) \int_{0}^{z} \frac{\ln^j(4t) \arcsin(2t)}{4t} d(4t)
\]
\[
= \frac{1}{2(p-1)!} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \ln^{p-1-j}(4z) \int_{0}^{\theta} \arcsin(2t) d \ln^{j+1}(2 \sin x)
\]
\[
= \frac{1}{2(p-1)!} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \frac{1}{j+1} \ln^{p-1-j}(4z) \left(\theta \ln^{j+1}(2 \sin \theta) - \int_{0}^{\theta} \ln^{j+1}(2 \sin x) dx\right)
\]
\[
= \frac{1}{2(p-1)!} \theta \ln^p(2 \sin \theta) \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \frac{(-1)^j}{j+1} + \frac{1}{4p!} \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} \ln^{p-j}(2 \sin \theta) L_{s,j+1}(2\theta).
\]
Then, noting the fact that
\[
\sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \frac{(-1)^j}{j+1} = \frac{1}{p},
\]
we obtain the desired evaluation. \qed
Setting $p \leq 3$ in (36), we deduce (with $\theta := \arcsin(2z)$)

\[
\sum_{n=0}^{\infty} \frac{(2n)}{n} \frac{z^{2n+1}}{2n+1} = \frac{1}{2} \theta,
\]

\[
\sum_{n=0}^{\infty} \frac{(2n)}{n} \frac{z^{2n+1}}{(2n+1)^2} = \frac{1}{2} \theta \ln(2 \sin \theta) + \frac{1}{4} \text{Cl}_2(2\theta),
\]

\[
\sum_{n=0}^{\infty} \frac{(2n)}{n} \frac{z^{2n+1}}{(2n+1)^3} = \frac{1}{4} \theta \ln^2(2 \sin \theta) + \frac{1}{4} \ln(2 \sin \theta) \text{Cl}_2(2\theta) - \frac{1}{8} \text{Ls}_3(2\theta),
\]

where we used the relation $\text{Ls}_2(\theta) = \text{Cl}_2(\theta)$ and for any positive integer $m$ the Clausen function $\text{Cl}_m(\theta)$ is defined as follows, for all $\theta \in [0, \pi]$:

\[
\text{Cl}_{2m-1}(\theta) := \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^{2m-1}} \quad \text{and} \quad \text{Cl}_2m(\theta) := \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^{2m}}.
\]

If we replace $z$ by $\frac{\sin \theta}{2}$ in (39), divide by $\sin \theta$, multiply by $\ln(\sin \theta)$ and integrate between 0 and $\frac{\pi}{2}$, we get the following result:

\[
\int_{0}^{\pi/2} \frac{\theta}{\sin \theta} \ln(\sin \theta) \, d\theta = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1}{16^k} \binom{2k}{k}^2 \left( O_k - \frac{1}{2} H_k - \ln 2 \right) \frac{1}{2k+1}
\]

\[
= -4G + \frac{\pi^3}{32} + \frac{\pi \ln^2(2)}{8}
\]

using (8). Note that using integration techniques from [5] in combination with (16), we can prove the related result:

\[
\int_{0}^{\pi/2} \frac{\theta}{\sin \theta} \ln(\cos \theta) \, d\theta = 2G \ln(2) + 4G - \frac{5\pi^3}{32} - \frac{\pi \ln^2(2)}{8}
\]

**Remark 2.** The series in (3) may be reduced to a $\mathbb{Q}$-combination of the series treated in [23]. We offer a sketch of a proof, as below, based on this alternate approach.

First, for all $k \in \mathbb{N}$ and $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$ we define the multiple $t$-sums as

\[
t_k(s) := \sum_{k > k_1 > \ldots > k_d > 0} \frac{1}{(2k_1 - 1)^{s_1} \cdots (2k_d - 1)^{s_d}}.
\]

They satisfy the stuffle relations. For example,

\[
t_k(1)^2 = 2t_k(1, 1) + t_k(2).
\]

Since $O_k = t_k(1)$ the $\lambda = 1$ case of (2) is

\[
\sum_{k>0} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{O_k^2}{(2k-1)^2} = 2 \sum_{k>0} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{t_k(1, 1)}{(2k-1)^2} + \sum_{k>0} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{t_k(2)}{(2k-1)^2}.
\]

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Now, for all $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$ and $m \in \mathbb{N}$,

$$
\sum_{k > 0} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{t_k(\mathbf{s})}{(2k - 1)^m} = \sum_{k > k_2 > \cdots > k_d > 0} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{1}{(2k - 1)^{s_1 + m}(2k_2 - 1)^{s_2} \cdots (2k_d - 1)^{s_d}} + \sum_{k > k_1 > \cdots > k_d > 0} \left( \frac{1}{16} \right)^k \binom{2k}{k}^2 \frac{1}{(2k - 1)^{m}(2k_1 - 1)^{s_1} \cdots (2k_d - 1)^{s_d}},
$$

and this can be handled by the approach in [23]. It is not hard to see that this above procedure can be generalized to treat arbitrary powers of $O_k$, not only its square.

Our last result, Theorem 7, differs noticeably from the others in two aspects: First, the binomial coefficients appear as a first power instead of a square; second, the series involves a variable $z$ so that its specializations may offer many series identities involving log-sine integrals. For example, a harmonic analog of Theorem 7 has been found in Charlton et al. [11, Lemma 2.1] who use them to prove the following identity

$$
\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n + 1)^3 16^n} \left( 9H_{2n+1} + \frac{32}{2n + 1} \right) = 40\beta(4) + \frac{5}{12} \pi \zeta(3)
$$

first conjectured by Z.-W. Sun [20, Conjecture 10.60]. It is highly likely that Theorem 7 can be further generalized to evaluate other Apéry-like sums in closed forms.

**Competing interests statement**

There are no competing interests to declare.

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**References**

[1] V. S. Adamchik, A certain series associated with Catalan’s constant, Z. Anal. Anwendungen 21 (2002), 817–826.
[2] C. C. Adams, The newest inductee in the number hall of fame, Math. Mag. 71 (1998), 341–349.

[3] K. C. Au, Evaluation of one-dimensional polylogarithmic integral, with applications to infinite series, arXiv:2007.03957. A companion Mathematica package available at researchgate.net/publication/357601353.

[4] B. Berndt, Ramanujan’s Notebooks, Vol. 2. New York: Springer-Verlag, 1989.

[5] J. M. Campbell, Ramanujan-like series for $\frac{1}{\pi}$ involving harmonic numbers, Ramanujan J. 46 (2018), 373–387.

[6] J. M. Campbell, Series containing squared central binomial coefficients and alternating harmonic numbers, Mediterr. J. Math. 16 (2019), Paper No. 37, 7.

[7] J. M. Campbell, A Wilf-Zeilberger-based solution to the Basel problem with applications, Discrete Math. Lett. 10 (2022), 21–27.

[8] J. M. Campbell and K.-W. Chen, Explicit identities for infinite families of series involving squared binomial coefficients, J. Math. Anal. Appl. 513 (2022), Paper No. 126219, 23.

[9] J. M. Campbell, P. Levrie, and A. S. Nimbran, A natural companion to Catalan’s constant, J. Class. Anal. 18 (2021), 117–135.

[10] M. Cantarini and J. D’Aurizio, On the interplay between hypergeometric series, Fourier-Legendre expansions and Euler sums, Boll. Unione Mat. Ital. 12 (2019), 623–656.

[11] S. Charlton, H. Gangl, L. Lai, C. Xu, and J. Zhao, On two conjectures of Sun concerning Apéry-like series, to appear: Forum Math. arXiv preprint arXiv:2210.14704, (2022).

[12] H. Chen, Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers, J. Integer Seq. 19 (2016), Article 16.1.5, 11.

[13] W. Chu, Infinite series on quadratic skew harmonic numbers, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM 117 (2023), Paper No. 75, 12.

[14] M. W. Coffey, Evaluation of a ln tan integral arising in quantum field theory, J. Math. Phys. 49 (2008), 093508, 15.

[15] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, 6th ed. Academic Press, San Diego, CA, 2000.

[16] A. S. Nimbran, Deriving Forsyth-Glaisher type series for $\frac{1}{\pi}$ and Catalan’s constant by an elementary method, Math. Student 84 (2015), 69–86.
[17] A. S. Nimbran and P. Levrie, Series of the form \( \sum a_n \binom{2n}{n} \), submitted to Math. Student.

[18] M. Petkovšek, H. S. Wilf, and D. Zeilberger, \( A = B \). A K Peters, Ltd., Wellesley, MA, 1996.

[19] A. Sofo and A. S. Nimbran, Euler-like sums via powers of log, arctan and arctanh functions, Integral Transforms Spec. Funct. 31 (2020), 966–981.

[20] Z.-W. Sun, New Conjectures in Number Theory and Combinatorics, Harbin Institute of Technology Press, (in Chinese) 2021.

[21] X. Wang and W. Chu, Further Ramanujan-like series containing harmonic numbers and squared binomial coefficients, Ramanujan J. 52 (2020), 641–668.

[22] X. Wang and W. Chu, Series with harmonic-like numbers and squared binomial coefficients, Rocky Mountain J. Math. (2022).

[23] C. Xu and J. Zhao, Apéry-type series with summation indices of mixed parities and colored multiple zeta values, II, (2022). arXiv:2203.00777.