WEIGHTS OF EXPONENTIAL GROWTH AND DECAY FOR
SCHRÖDINGER-TYPE OPERATORS

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Abstract. Fix $d \geq 3$ and $1 < p < \infty$. Let $V : \mathbb{R}^d \to [0, \infty)$ belong to the reverse Hölder class $RH_d/2$ and consider the Schrödinger operator $L_V := -\Delta + V$. In this article, we introduce classes of weights $w$ for which the Riesz transforms $\nabla L_V^{-1/2}$, their adjoints $L_V^{-1/2} \nabla$ and the heat maximal operator $\sup_{t > 0} e^{-tL_V} |f|$ are bounded on the weighted Lebesgue space $L^p(w)$. The boundedness of the $L_V$-Riesz potentials $L_V^{-\alpha/2}$ from $L^p(w)$ to $L^{\nu}(w^{\nu/p})$ for $0 < \alpha \leq 2$ and $\frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d}$ will also be proved. These weight classes are strictly larger than a class previously introduced by B. Bongioanni, E. Harboure and O. Salinas in [5] that shares these properties and they contain weights of exponential growth and decay.

The classes will also be considered in relation to different generalised forms of Schrödinger operator. In particular, the Schrödinger operator with measure potential $-\Delta + \mu$, the uniformly elliptic operator with potential $-\text{div} A \nabla + V$ and the magnetic Schrödinger operator $(\nabla - ia)^2 + V$ will all be considered. It will be proved that, under suitable conditions, the standard operators corresponding to these second-order differential operators are bounded on $L^p(w)$ for weights $w$ in these classes.

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The Laplacian operator $\Delta$ is inextricably linked to classical weighted theory. The characterisation of the Muckenhoupt class $A_p$ for $1 < p < \infty$ in terms of various operators related to the Laplacian, such as the Riesz transforms $R_0 := \nabla (-\Delta)^{-\frac{1}{2}}$, the

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heat maximal operator $T^*_0 f := \sup_{t>0} e^{t\Delta} |f|$ and the Hardy-Littlewood maximal operator $M_0$, is widely regarded as the apex of classical weighted theory. This string of results, the culmination of work from a number of great mathematicians in the 1970’s, states that a weight $w$ will be contained in $A_p$ if and only if any one of these operators is bounded from the weighted $p$-Lebesgue space $L^p(w) := L^p(\mathbb{R}^d; w \, dx)$ to itself (see [14], [9] and [6] for sufficiency and [9] and [20] for necessity).

A current area of active research is the study of the harmonic analysis of differential operators other than the Laplacian. A natural question to ask in such a setting is whether it is possible to construct a Muckenhoupt-type class adapted to the underlying differential operator and whether this class can be characterised in terms of the corresponding operators such as the associated Riesz transforms and heat maximal operator. In this article, the differential operators of interest are Schrödinger operators.

Fix dimension $d \geq 3$. The Schrödinger operator with non-negative potential $V \in L^1_{loc}(\mathbb{R}^d)$ is defined to be the operator

$$L_V := -\Delta + V.$$ 

Let $R_V$ and $R_V^*$ denote the Riesz transforms associated with $L_V$ and their adjoints,

$$R_V := \nabla L_V^{-\frac{1}{2}} \quad \text{and} \quad R_V^* := L_V^{-\frac{1}{2}} \nabla.$$ 

Define the $L_V$-Riesz potential for $0 < \alpha \leq 2$ and heat maximal operator for $L_V$ respectively through

$$I^*_V := L_V^{-\frac{\alpha}{2}} \quad \text{and} \quad T^*_V f(x) := \sup_{t>0} e^{-tL_V} |f| (x),$$

for $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Recall the definition of the reverse Hölder class of potentials. A non-negative function $V \in L^1_{loc}(\mathbb{R}^d)$ is said to belong to the class $RH_q$ for $1 < q < \infty$ if there exists a constant $C > 0$ for which

$$\left( \frac{1}{|B|} \int_B V^q \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V \right)$$

for all open Euclidean balls $B \subset \mathbb{R}^d$. The smallest constant $C$ for which the above estimate is satisfied will be denoted $[V]_{RH_q}$. In the article [16], Z. Shen introduced the notion of the critical radius function $\rho_V$ for a potential $V \in RH_{\frac{d}{2}}$. This is the function $\rho_V : \mathbb{R}^d \to [0, \infty)$ defined through

$$\rho_V(x) := \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(x) \, dx \leq 1 \right\}$$

for $x \in \mathbb{R}^d$, where the notation $B(x,r)$ is used to denote the open Euclidean ball of radius $r$ centered at $x$. The critical radius function determines a scale below which the operators associated with the Schrödinger operator behave locally like their classical counterparts. This allowed for the analysis of operators such as $R_V$ to be split up into two regions; the local region where $R_V$ resembles the classical Riesz transforms $R_0$ and a global region where the singular kernel for $R_V$ will have substantially better decay properties than the kernel of $R_0$.

In the article [5], the authors B. Bongioanni, E. Harboure and O. Salinas initiated a study into the weighted theory of Schrödinger operators for $V \in RH_{\frac{d}{2}}$ by introducing a new class of weights, denoted by $A^V_{p,\infty}$, that was adapted to $L_V$. 


and whose definition was based on the Schrödinger operator machinery of Shen. What made the class $A_{p}^{V,\infty}$ so compelling was that not only were the operators $R_{V}$, $R_{V}^{*}$ and $T_{V}$ all bounded on $L^{p}(w)$ for any $w \in A_{p}^{V,\infty}$, but also that the class was strictly larger than the classical Muckenhoupt class $A_{p}$. In addition, in [5] it was also proved that the fractional integral operator $I_{V}^{\alpha}$ is bounded on appropriate weighted Lebesgue spaces that correspond to the classical case for weights in $A_{p}^{V,\infty}$.

Unfortunately, an inescapable deficiency with the class $A_{p}^{V,\infty}$ is that the reverse implication does not hold in general. That is, there exist weights not contained in $A_{p}^{V,\infty}$ for which $R_{V}$ and $T_{V}$ are bounded on $L^{p}(w)$. Such an example was found in [2] for the harmonic oscillator potential, $V(x) = |x|^{2}$. Indeed for polynomial potentials of order zero or higher, as will be shown in this article, there exist a wealth of such counterexamples that are non-doubling weights of exponential growth or decay. The existence of such weights demonstrates that the class $A_{p}^{V,\infty}$ is not characterised completely by the boundedness of $R_{V}$ or $T_{V}$ and thus the harmonic analytic aspects of $L_{V}$ are not fully captured by this class. The first aim of this article is thus to improve upon the class $A_{p}^{V,\infty}$ by introducing a strictly larger class that accounts for the counterexamples of exponential-type weights.

Let $d_{V}(x, y)$ denote the Agmon distance for the potential defined through

$$d_{V}(x, y) := \inf_{\gamma} \int_{0}^{1} \rho_{V}(\gamma(t))^{-1} |\gamma'(t)| \, dt,$$

where the infimum is taken over all curves $\gamma : [0, 1] \to \mathbb{R}^{d}$ connecting the points $x, y \in \mathbb{R}^{d}$. We introduce the notation $B_{V}(x, r)$ to denote the open ball of radius $r > 0$ centered at the point $x \in \mathbb{R}^{d}$ in the metric $d_{V}$,

$$B_{V}(x, r) := \{ y \in \mathbb{R}^{d} : d_{V}(x, y) < r \}.$$

Our weight class is defined as follows.

**Definition 1.1.** Let $1 < p < \infty$ and $c > 0$. $S_{p,c}^{V}$ is the class of all weights for which

$$[w]_{S_{p,c}^{V}} := \sup_{B_{V}} \left( \frac{1}{|B_{V}|} \int_{B_{V}} w \right)^{\frac{1}{p}} \left( \frac{1}{|B_{V}|} \int_{B_{V}} w^{-\frac{1}{p-1}} \right)^{\frac{p-1}{p}} < \infty,$$

where the supremum is taken over all balls $B_{V} = B_{V}(x, r) \subset \mathbb{R}^{d}$ in the metric $d_{V}$ with $x \in \mathbb{R}^{d}$ and $r > 0$.

One of the main results for this main article is the below theorem whose proof will be provided in Section 4.2.

**Theorem 1.1.** Let $V \in RH_{q}$ for some $q > \frac{d}{2}$. The following statements are true.

(i) Suppose that $q \geq d$. There must exist some $c_{1} > 0$ for which both $R_{V}$ and $R_{V}^{*}$ are bounded on $L^{p}(w)$ for $1 < p < \infty$ when $w \in S_{p,c_{1}}^{V}$.

(ii) Suppose instead that $\frac{d}{2} < q < d$ and let $s$ be defined through $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$. There exists a constant $c_{2} > 0$ for which the operator $R_{V}^{*}$ is bounded on $L^{p}(w)$ for $s' < p < \infty$ when $w \in S_{p'/s',c_{2}}^{V}$ and the operator $R_{V}$ is bounded on $L^{p}(w)$ for $1 < p < s$ when $w^{-\frac{1}{p-1}} \in S_{p'/s',c_{2}}^{V}$. 
(iii) For any \( q > \frac{d}{2} \) and \( 0 < \alpha \leq 2 \), there exists \( c_3 > 0 \) for which the operator \( I_p^V \) is bounded from \( L^p(w) \) to \( L^p(w^{\varepsilon/p}) \) for \( w^{\varepsilon/p} \in S_{1+\frac{\alpha}{2},c_3}^{V} \) and \( 1 < p < \frac{d}{\alpha} \), where
\[
\frac{1}{\alpha} = \frac{1}{p} - \frac{\alpha}{2}.
\]
In each of above statements, the constants \( c_1, c_2 \) and \( c_3 \) will depend on \( V \) only through \( [V]_{RH} \) and they will be independent of \( p \).

This theorem provides an improvement upon all existing weight classes that possess these properties since, as will be proved in Section 3, \( A_p^{V,\infty} \) is strictly contained in \( S_{p,c}^V \) for any \( c > 0 \). Indeed, our class will allow for exponential growth and decay in the weights as opposed to only polynomial growth and decay as in \( A_p^{V,\infty} \). Our class also has the advantage that it has a more natural geometric definition than other classes since it is defined in terms of the inherent geometry associated with the potential.

Let \( \Gamma_V \) denote the fundamental solution of the Schrödinger operator \( L_V \). This is a function defined on \( \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq y\} \) with the properties that \( \Gamma_V(\cdot, y) \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( L_V \Gamma_V(x, \cdot) = \delta_y \) for each \( y \in \mathbb{R}^d \), where \( \delta_y \) is the Dirac delta distribution with pole at \( y \). The proof of Theorem 1.1 relies heavily on sharp pointwise exponential decay estimates for \( \Gamma_V \) that were obtained by Shen in the article [17]. These estimates state that there must exist some \( C_1, C_2, \varepsilon_1, \varepsilon_2 > 0 \) for which
\[
C_1 \frac{e^{-\varepsilon_1 d_{V}(x,y)}}{|x-y|^{d-2}} \leq \Gamma_V(x, y) \leq C_2 \frac{e^{-\varepsilon_2 d_{V}(x,y)}}{|x-y|^{d-2}} \quad \forall \ x, y \in \mathbb{R}^d.
\]
It should be stressed that only the upper estimate is required for the proof of Theorem 1.1. However, the lower estimate is useful in that it hints towards the optimality of these weight classes in a way that will be stated explicitly in Conjecture 1.1.

One should also expect for the heat maximal operator to be bounded on \( L^p(w) \) for weights in the class \( S_{p,c}^V \). Unfortunately, it is still an open problem as to whether the heat kernel of \( L_V \) satisfies sharp estimates that are analogous to (2). The best known estimates were obtained by K. Kurata in [12] and are non-sharp. This indicates that the proof of the boundedness of \( T_v \) on \( L^p(w) \) for \( w \in S_{p,c}^V \), will be just beyond our reach and will remain so until sharp estimates for the heat kernel are proved. Therefore, instead of proving the boundedness of \( T_v \) on \( L^p(w) \) for \( w \in S_{p,c}^V \), a second smaller class of exponential-type weights will be introduced and boundedness will be proved for this class instead.

**Definition 1.2.** Let \( 1 < p < \infty \) and \( m, c > 0 \). Introduce the notation \( \Phi_{m,c}^V \) to denote the function
\[
\Phi_{m,c}^V(x,r) := \exp \left( c \left( 1 + \frac{r}{\rho_V(x)} \right)^m \right),
\]
for \( x \in \mathbb{R}^d \) and \( r > 0 \). Let \( H_{p,c}^{V,m} \) denote the class of all weights for which
\[
[w]_{H_{p,c}^{V,m}} := \sup_B \left( \frac{1}{|B|} \int_B w \right)^{\frac{1}{2}} \left( \frac{1}{|B|} \int_B w^{-\frac{1}{p'}} \right)^{\frac{p-1}{p}} < \infty,
\]
where the supremum is taken over all Euclidean balls \( B = B(x, r) \subset \mathbb{R}^d \) with radius \( r > 0 \) and center \( x \in \mathbb{R}^d \).

The class \( H_{p,c}^{V,m} \) is very similar in nature to \( A_p^{V,\infty} \) except that it will allow for exponential growth and decay (refer to Section 3 for a rigorous definition of \( A_p^{V,\infty} \)).
Due to the similarities between the two classes, it is not difficult to see that $A^{V,\infty}_p \subset H^{V,\infty}_{p,c}$ for any $c, m > 0$. An inclusion that is less obvious is that for any $c > 0$ we must have $H^{V,m}_{p,c} \subset S^{V}_{p,c}$ provided that $m \leq 2m_0$ for some $m_0 > 0$ that will be defined later in the article. This statement will be proved in Section 3.

One of our main results for this class of weights is the following theorem that will be proved in Section 4.4.

**Theorem 1.2.** Suppose that $V \in RH^2_d$ and let $1 < p < \infty$. There must exist $c, m_0 > 0$, independent of $p$, such that for any $w \in H^{V,m}_{p,c}$ with $m \leq m_0$ the operator $T_V^*$ is bounded on $L^p(w)$.

The second aim of this article is to extend the results for these freshly minted classes of weights to more general forms of Schrödinger operator. We will consider three different types of generalised Schrödinger operator and prove statements analogous to Theorems 1.1 and 1.2 for each of these forms. The first generalised form of Schrödinger operator that will be considered is the Schrödinger operator with measure potential. This is the operator defined through

$$L_\mu := -\Delta + \mu,$$

where $\mu$ is a non-negative Radon measure on $\mathbb{R}^d$ that satisfies the conditions

$$\mu(B(x,r)) \leq C_\mu \left( \frac{r}{R} \right)^{d-2+\delta_\mu} \mu(B(x,R))$$

and

$$\mu(B(x,2r)) \leq D_\mu (\mu(B(x,r)) + r^{d-2})$$

for all $x \in \mathbb{R}^d$ and $0 < r < R$, for some $\delta_\mu, C_\mu, D_\mu > 0$. In Section 4, it will be proved that the $L_\mu$-counterpart of Theorem 1.1 is true.

Next, we will consider uniformly elliptic operators with potential. Let $A$ be a $d \times d$ matrix-valued function with real-valued coefficients in $L^\infty(\mathbb{R}^d)$. Suppose that $A$ satisfies the ellipticity condition,

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2$$

for some $\lambda, \Lambda > 0$, for all $\xi \in \mathbb{R}^d$ and almost every $x \in \mathbb{R}^d$. Define for $V \in RH^2_d$ the operator

$$L_{A,V} := -\text{div} A \nabla + V.$$

In Section 5 it will be proved that if the critical radius function $\rho_V$ is bounded from above, as is the case for any polynomial potential of order zero or higher, and $A$ is Hölder continuous then the first and second parts of Theorem 1.1 will be true for $L_{A,V}$. It will also be proved that the $L_{A,V}$-counterpart of Theorem 1.2 is true subject to the constraint $A = A^*$. The third part of Theorem 1.1 will be proved subject to no additional constraints. For a rigorous statement of these results refer to Section 5.

**Remark 1.1.** Our result that $R_{A,V} := \nabla L_{A,V}^{-\frac{1}{2}}$ is bounded on $L^p(w)$ for all weights contained in a class strictly larger than the classical Muckenhoupt class $A_p$ when $V \in RH^2_d$ and $\rho_V$ is bounded from above sharply contrasts with the potential free case. Without the presence of the potential, it is well-known that there exists $A$ for which the operator $R_{A,0} := \nabla L_{A,0}^{-\frac{1}{2}}$ is unbounded on $L^p(w)$ for some $w \in A_p$. This follows by combining Remarks 1.7 and 1.8 of [18] for example. Thus, the perturbation $A$ has the tendency to decrease the size of the associated weight.
class. The inclusion of a sufficiently large potential \( V \) will have the opposite effect and increase the size of the weight class, effectively cancelling out the presence of \( A \).

Finally, for \( a = (a_1, \ldots, a_d) \) a vector of real-valued functions in \( C^1(\mathbb{R}^d) \), we will consider the magnetic Schrödinger operator

\[
L^a_V := (\nabla - ia)^* (\nabla - ia) + V.
\]

In Section 6 it will be proved that subject to additional constraints on \( a \) and \( V \) the third part of Theorem 1.1 and Theorem 1.2 will hold. A weaker form of the first part of Theorem 1.1 will also be proved.

In the article [13], the authors S. Mayboroda and B. Poggi proved that Shen’s sharp exponential decay estimates on the fundamental solution (2) could be generalised to the operators \( L_{A,V} \) and \( L^a_V \). Our weighted results for both the magnetic Schrödinger operator and the uniformly elliptic operator with potential will rely on this result. In addition, in order to prove the boundedness of the heat maximal operator for \( L_{A,V} \) and \( L^a_V \), we will also require the exponential decay estimates for the heat kernel provided by [12].

In the last part of this article, we will consider necessary conditions for a weight to satisfy in order for the operators \( R_V \) and \( T^*_V \) to be bounded on \( L^p(w) \). It will first be proved that for \( V \in RH_d \), if \( R_V \) is bounded on \( L^p(w) \) then the weight must be contained in the local Muckenhoupt class \( A_{V,\text{loc}} \) (refer to Section 3 for the definition of \( A_{V,\text{loc}} \)). Following this, we will discuss the optimality of the classes \( S^V_{p,c} \) and \( H^{V,m}_{p,c} \). In particular, the following conjecture will be considered.

**Conjecture 1.1.** Let \( V \in RH_d \). There exists \( c_1, c_2 > 0 \) for which the following chains of inclusions hold,

\[
S^V_{p,c_1} \subset \{ w : \| R_V \|_{L^p(w)} < \infty \} \subset S^V_{p,c_2}
\]

and

\[
S^V_{p,c_1} \subset \{ w : \| T^*_V \|_{L^p(w)} < \infty \} \subset S^V_{p,c_2}.
\]

In Section 7, it will be proved that the first chain of inclusions of the previous conjecture is true for constant potentials and the second chain of inclusions is true for potentials that are bounded both from above and below.

It will be proved in Section 3 that for any \( c > 0 \) there exists \( c_1, c_2, m_1, m_2 > 0 \) for which \( H^{V,m_1}_{p,c_1} \subset S^V_{p,c} \subset H^{V,m_2}_{p,c_2} \). The following is therefore a weaker form of Conjecture 1.1.

**Conjecture 1.2.** Let \( V \in RH_d \). There exists \( c_1, c_2, m_1, m_2 > 0 \) for which the following chains of inclusions hold,

\[
H^{V,m_1}_{p,c_1} \subset \{ w : \| R_V \|_{L^p(w)} < \infty \} \subset H^{V,m_2}_{p,c_2}
\]

and

\[
H^{V,m_1}_{p,c_1} \subset \{ w : \| T^*_V \|_{L^p(w)} < \infty \} \subset H^{V,m_2}_{p,c_2}.
\]

In Section 7, the second chain of inclusions in Conjecture 1.2 will be proved for the harmonic oscillator potential \( V(x) = |x|^2 \).
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2. **Critical Radius Function and Agmon Distance**

Throughout this article, the notation $A \lesssim B$ will be used to denote that there exists a constant $c > 0$ for which the inequality $A \leq cB$ is satisfied. Similarly, the notation $A \simeq B$ will denote that there exists $c > 0$ for which $c^{-1}B \leq A \leq cB$. The dependence of the constant $c$ on the various parameters should be clear from the context. To emphasise the dependence of the constant on a particular parameter subscript notation will be used. For example, $A \lesssim_b B$ will indicate that the implicit constant $c$ depends on $b$.

For a weight $w$ on $\mathbb{R}^d$ and measurable set $E \subset \mathbb{R}^d$, the below notation will frequently be used

$$w(E) := \int_E w(x) \, dx.$$ 

In this section, the function $\rho_V$ from (1) will be generalised and the inherent geometry attached to such a function will be discussed.

**Definition 2.1.** A function $\rho : \mathbb{R}^d \to [0, \infty)$ will be called a critical radius function if there exist constants $k_0, B_0 > 1$ for which

$$B_0^{-1} \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq B_0 \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{k_0}$$

for all $x, y \in \mathbb{R}^d$.

The following result from [16] confirms that the above definition is indeed a generalisation of the function $\rho_V$.

**Lemma 2.1** ([16, Lem. 1.4]). For $V \in RH_{\frac{d}{2}}$, the function $\rho_V$, as defined in (1), is a critical radius function in the sense of Definition 2.1. In particular, (3) is satisfied with constants $k_0$ and $B_0$ that depend on $V$ only through $[V]_{RH_{\frac{d}{2}}}$.

For any critical radius function $\rho : \mathbb{R}^d \to [0, \infty)$, it is possible to construct a corresponding Agmon distance through

$$d_\rho(x, y) := \inf_\gamma \int_0^1 \rho(\gamma(t))^{-1} |\gamma'(t)| \, dt,$$

where the infimum is taken over all possible curves $\gamma : [0, 1] \to \mathbb{R}^d$ connecting the points $x, y \in \mathbb{R}^d$. At a local scale, the Agmon distance $d_\rho$ will be comparable to the Euclidean distance.

**Lemma 2.2.** Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a critical radius function. There exists $D_0 > 1$ so that for any $x, y \in \mathbb{R}^d$ with $|x - y| \leq 2\rho(x)$ we have

$$D_0^{-1} |x - y| \rho(x)^{-1} \leq d_\rho(x, y) \leq D_0 |x - y| \rho(x)^{-1}.$$
Let $x, y \in \mathbb{R}^d$ with $|x - y| \leq 2\rho(x)$. Let’s first prove the upper estimate. Let $\gamma$ denote the straight line starting at $x$ and ending at $y$. Then, on applying (3),

$$
\int_0^1 \rho(\gamma(t))^{-1} |\gamma'(t)| \, dt \leq B_0 \rho(x)^{-1} \int_0^1 \left(1 + \frac{|x - \gamma(t)|}{\rho(x)}\right)^{k_0} |\gamma'(t)| \, dt
$$

$$
\leq B_0 \rho(x)^{-1} 3^{k_0} \int_0^1 |\gamma'(t)| \, dt
$$

$$
= B_0 \rho(x)^{-1} 3^{k_0} |x - y|,
$$

where the second to last line follows from the fact that $|x - \gamma(t)| \leq 2\rho(x)$ for all $t \in [0, 1]$.

Let’s now prove the lower bound. Let $\gamma$ be a curve for which $d_\rho(x, y) \geq \frac{1}{2} \int_0^1 \rho(\gamma(t))^{-1} |\gamma'(t)| \, dt$.

The definition of a critical radius function leads to

$$
d_\rho(x, y) \geq \frac{1}{2} B_0^{-1} \rho(x)^{-1} \int_0^1 \left(1 + \frac{|x - \gamma(t)|}{\rho(x)}\right)^{-\frac{k_0}{\kappa_0 + 1}} |\gamma'(t)| \, dt.
$$

First suppose that the curve is entirely contained within $B(x, 2\rho(x))$. Then

$$
d_\rho(x, y) \geq \frac{1}{2} B_0^{-1} \rho(x)^{-1} 3^{\frac{k_0}{\kappa_0 + 1}} \int_0^1 |\gamma'(t)| \, dt
$$

$$
\geq \frac{1}{2} B_0^{-1} \rho(x)^{-1} 3^{\frac{k_0}{\kappa_0 + 1}} |x - y|.
$$

Next, suppose that the curve $\gamma$ leaves the ball $B(x, 2\rho(x))$. Due to the continuity of the curve $\gamma$, there must then exist $a \in (0, 1)$ for which $|x - \gamma(a)| = 2\rho(x)$ but $|x - \gamma(t)| < 2\rho(x)$ for all $t \in [0, a)$. Then

$$
d_\rho(x, y) \geq \frac{1}{2} B_0^{-1} \rho(x)^{-1} \int_0^a \left(1 + \frac{|x - \gamma(t)|}{\rho(x)}\right)^{-\frac{k_0}{\kappa_0 + 1}} |\gamma'(t)| \, dt
$$

$$
\geq \frac{1}{2} B_0^{-1} 3^{\frac{k_0}{\kappa_0 + 1}} \rho(x)^{-1} \int_0^a |\gamma'(t)| \, dt
$$

$$
\geq \frac{1}{2} B_0^{-1} 3^{\frac{k_0}{\kappa_0 + 1}} \rho(x)^{-1} |x - y|.
$$

The following lemma will allow us to compare the Agmon distance $d_\rho(x, y)$ with the quantity $\left(1 + \frac{|x - y|}{\rho(x)}\right)$ at a global scale.

**Lemma 2.3.** Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a critical radius function. There exists $D_1 > 1$, dependent on $\rho$ only through $B_0$ and $k_0$, such that

$$
d_\rho(x, y) \leq D_1 \left(1 + \frac{|x - y|}{\rho(x)}\right)^{k_0 + 1}
$$

for all $x, y \in \mathbb{R}^d$ and

$$
d_\rho(x, y) \geq D_1^{-1} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{\frac{1}{\kappa_0 + 1}}.
$$
for all $x, y \in \mathbb{R}^d$ satisfying $|x - y| \geq \rho(x)$.

Proof. This was proved for the case $\rho = \rho_V$ in [17]. For a general critical radius function, the estimates follow using an identical proof.

Define the constant

\begin{equation}
\beta := \max (B_0, D_0, D_1, 2).
\end{equation}

It is obvious that (3) and Lemmas 2.2 and 2.3 will all hold with the constant $\beta$ replacing $B_0$, $D_0$ and $D_1$ respectively. The constant $\beta$ will thus be used as a way of simplifying notation by consolidating the three constants $B_0$, $D_0$ and $D_1$ into the single constant $\beta$.

We introduce the notation $B_\rho(x, r)$ to denote the open ball in the metric $d_\rho$ centered at the point $x \in \mathbb{R}^d$ and of radius $r > 0$. That is,

$\begin{align*}
B_\rho(x, r) := \{ y \in \mathbb{R}^d : d_\rho(x, y) < r \}.
\end{align*}$

As a consequence of Lemmas 2.2 and 2.3, we are able to compare balls in the Euclidean metric with balls in the metric $d_\rho$.

Lemma 2.4. Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a critical radius function. Let $r > 0$ and $x \in \mathbb{R}^d$. Suppose that $r \leq 2$. Then

$B(x, r \rho(x)) \subset B_\rho(x, \beta r)$.

Suppose instead that $r > 2$. Then

$B(x, r \rho(x)) \subset B_\rho(x, \beta (1 + r)^{k_0 + 1})$.

Proof. First suppose that $r \leq 2$ and let $y \in B(x, r \rho(x))$. Then from Lemma 2.2,

$\begin{align*}
d_\rho(x, y) &\leq \beta |x - y| \rho(x)^{-1} \\
&< \beta r,
\end{align*}$

which implies that $y \in B_\rho(x, \beta r)$.

Next, suppose that $r > 2$ and let $y \in B(x, r \rho(x))$. Lemma 2.3 implies

$\begin{align*}
d_\rho(x, y) &\leq \beta \left(1 + \frac{|x - y|}{\rho(x)}\right)^{k_0 + 1} \\
&< \beta (1 + r)^{k_0 + 1}.
\end{align*}$

\[\Box\]

Lemma 2.5. Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a critical radius function. There exists a constant $A_0 > 1$, dependent on $\rho$ only through $B_0$ and $k_0$, such that for all $r > 0$ with $r \leq \beta$ and $x \in \mathbb{R}^d$,

$B_\rho(x, r) \subset B(x, A_0 r \rho(x))$.

Also, for $x \in \mathbb{R}^d$ and $r > \beta$,

$B_\rho(x, r) \subset B(x, ((r \beta)^{k_0 + 1} - 1) \rho(x))$.

Proof. Fix $A_0 > 1$ to be a constant large enough so that

$\frac{A_0}{2 \beta (1 + A_0 \beta)^{k_0 + 1}} \geq 1.$
Suppose first that $r \leq \beta$ and fix $y \in B(x, A_0 r \rho(x))^c$. Let $\gamma$ be a curve connecting the points $x$ and $y$ such that

$$d_\rho(x, y) \geq \frac{1}{2} \int_0^1 \rho(\gamma(t))^{-1} |\gamma'(t)| \, dt.$$ 

On applying (3),

$$d_\rho(x, y) \geq \frac{1}{2} \beta^{-1} \rho(x)^{-1} \int_0^1 \left( 1 + \left| \frac{x - \gamma(t)}{\rho(x)} \right| \right)^{-\frac{k_0}{k_0+1}} |\gamma'(t)| \, dt.$$ 

Let $a \in (0, 1)$ be the unique point in the interval that satisfies $|x - \gamma(a)| = A_0 r \rho(x)$ and $|x - \gamma(t)| < A_0 r \rho(x)$ for all $t \in [0, a)$. Then

$$d_\rho(x, y) \geq \frac{1}{2} \beta^{-1} (1 + A_0 \beta)^{-\frac{k_0}{k_0+1}} \rho(x)^{-1} \int_0^a |\gamma'(t)| \, dt \geq \frac{1}{2} \beta^{-1} (1 + A_0 \beta)^{-\frac{k_0}{k_0+1}} A_0 r \geq r,$$

which completes the proof of the first inclusion.

For the second inclusion, suppose that $r > \beta$ and fix $y \in B(x, ((r \beta)^{k_0+1} - 1) \rho(x))^c$. Since $r > \beta$, it follows that

$$|x - y| \geq (r \beta)^{k_0+1} - 1 \rho(x) \geq (4^{k_0+1} - 1) \rho(x) > \rho(x).$$

This allows us to apply the second part of Lemma 2.3 to obtain

$$d_\rho(x, y) \geq \beta^{-1} \left( 1 + \left| \frac{x - y}{\rho(x)} \right| \right)^{\frac{1}{k_0+1}} \geq r,$$

which tells us that $y \in B_\rho(x, r)^c$. \hfill \Box

**Corollary 2.1.** Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a critical radius function. For all $x \in \mathbb{R}^d$ and $r > 0$,

$$|B_\rho(x, 2r)| \lesssim (1 + r)^{(k_0+1)d} |B_\rho(x, r)|.$$

**Proof.** Suppose first that $r \leq \beta/2$. Successively applying Lemmas 2.5 and 2.4,

$$|B_\rho(x, 2r)| \leq |B(x, 2A_0 r \rho(x))| \leq |B(x, r \rho(x))| \leq |B_\rho(x, r)| \leq (1 + r)^{(k_0+1)d} |B_\rho(x, r)|.$$
Next suppose that $\beta/2 < r \leq 2^{(k_0 + 1)} \beta$. Then Lemmas 2.5 and 2.4 lead to

\[
|B_\rho(x, 2r)| \leq |B(x, (2r\beta)^{k_0 + 1}\rho(x))|
\leq r^{(k_0 + 1)d} \left| B\left(x, \frac{r\rho(x)}{2^{(k_0 + 1)} \beta}\right) \right|
\leq r^{(k_0 + 1)d} |B_\rho(x, r)|.
\]

Finally, suppose that $r > 2^{(k_0 + 1)} \beta$. We would then have

\[
|B_\rho(x, 2r)| \leq \left| B\left(x, \frac{r\rho(x)}{(k_0 + 1)^\beta}\right) \right|
\leq r^{(k_0 + 1)d} |B_\rho(x, r)|,
\]

where the last line follows from Lemma 2.4.

For an operator $S$ acting on functions in $L^1_{\text{loc}}(\mathbb{R}^d)$, we define the local and global components through

\[
S_{\text{loc}} f(x) := S \left( f \cdot 1_{B(x, \rho(x))} \right)(x), \quad S_{\text{glob}} f(x) := S \left( f \cdot 1_{B(x, \rho(x))_c} \right)(x)
\]

for $x \in \mathbb{R}^d$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. In order to prove that an operator $S$ is bounded on some weighted space $L^p(w)$, it is sufficient to prove that both $S_{\text{loc}}$ and $S_{\text{glob}}$ are bounded on $L^p(w)$. The below proposition will be of vital importance for proving the $L^p(w)$-boundedness of our operators.

**Proposition 2.1** ([5]). There exists a sequence of points $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^d$ that satisfies the following two properties,

(i) $\mathbb{R}^d = \bigcup_{j \in \mathbb{N}} B(x_j, \rho(x_j))$,

(ii) There exists $C, N_1 > 0$ such that for every $\sigma \geq 1$

\[
\sum_{j \in \mathbb{N}} 1_{B(x_j, \sigma \rho(x_j))} \leq C \sigma^{N_1}.
\]

3. The Adapted Weight Classes

Throughout this section, let $\rho : \mathbb{R}^d \to [0, \infty)$ be a critical radius function satisfying (3) with constants $B_0, k_0 > 1$. The Agmon distance corresponding to this critical radius function allows one to define classes $S^\rho_{p,c}$ and $H^\rho_{p,m}$ in an identical manner to the classes $S^V_{p,c}$ and $H^V_{p,m}$.

**Definition 3.1.** Let $1 < p < \infty$ and $c > 0$. $S^\rho_{p,c}$ is the class of all weights $w$ on $\mathbb{R}^d$ for which

\[
[w]_{S^\rho_{p,c}} := \sup_{B_\rho} \left( \frac{1}{|B_\rho|} \int_{B_\rho} w \right)^{\frac{1}{p}} \left( \frac{1}{|B_\rho|} \int_{B_\rho} w^{-\frac{1}{p'}} \right)^{\frac{p'-1}{p}} < \infty,
\]

where the supremum is taken over all balls $B_\rho = B_\rho(x, r) \subset \mathbb{R}^d$ in the metric $d_\rho$ with $x \in \mathbb{R}^d$ and $r > 0$. 
Definition 3.2. Let $1 < p < \infty$ and $m, c > 0$. Let $\Phi_{m,c}^\rho : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ be the function defined by

$$\Phi_{m,c}^\rho(x, r) := \exp \left( c \left( 1 + \frac{r}{\rho(x)} \right)^m \right),$$

for $x \in \mathbb{R}^d$ and $r > 0$. Let $H_{p,c}^{\rho,m}$ denote the class of all weights $w$ on $\mathbb{R}^d$ for which

$$[w]_{H_{p,c}^{\rho,m}} := \sup_B \left( \frac{1}{|B|} \frac{1}{\Phi_{m,c}^\rho(x, r)} \int_B w \right)^\frac{1}{p} \left( \frac{1}{|B|} \frac{1}{\Phi_{m,c}^\rho(x, r)} \int_B w^{-\frac{1}{p'}} \right)^{\frac{p-1}{p'}} < \infty,$$

where the supremum is taken over all Euclidean balls $B = B(x, r) \subset \mathbb{R}^d$ with radius $r > 0$ and center $x \in \mathbb{R}^d$.

We clearly have $S_{p,c}^\rho = S_{p,c}^V$ and $H_{p,c}^{\rho,m} = H_{p,c}^{V,m}$. Let $A_{p,\rho,loc}$ be as defined in [5]. That is, $A_{p,\rho,loc}$ is the collection of all weights $w$ for which

$$w(B) \frac{1}{p} w^{-\frac{1}{p'}} (B) \frac{p-1}{p'} \lesssim |B|$$

for all balls $B = B(x, r)$ with $r \leq \rho(x)$. 

Proposition 3.1. For any $1 < p < \infty$ and $c > 0$,

$$S_{p,c}^\rho \subset A_{p,\rho,loc}.$$

Proof. Fix $w \in S_{p,c}^\rho$, for some $1 < p < \infty$ and $c > 0$. Let $B := B(x, r\rho(x))$ be a ball in the Euclidean metric with $r \leq 1$. On successively applying Lemma 2.4, the condition that $w \in S_{p,c}^\rho$ and finally the bound $r \leq 1$,

$$w(B) \frac{1}{p} w^{-\frac{1}{p'}} (B) \frac{p-1}{p'} \lesssim w(B\rho(x, \beta r)) \frac{1}{p} w^{-\frac{1}{p'}} (B\rho(x, \beta r)) \frac{p-1}{p'}$$

$$\lesssim \rho(x, \beta r)$$

Since $\beta r \leq \beta$, Lemma 2.5 can be applied to give

$$w(B) \frac{1}{p} w^{-\frac{1}{p'}} (B) \frac{p-1}{p'} \lesssim |B(x, A_0 \beta r \rho(x))|$$

$$\simeq |B(x, r\rho(x))|. $$

This proves that $w \in A_{p,\rho,loc}$.

Recall the definition of the class $A_{p,\rho,\infty}$ introduced in [5].

Definition 3.3 ([5]). For $1 < p < \infty$ and $\theta \geq 0$, a weight $w$ on $\mathbb{R}^d$ is said to belong to the class $A_{p,\rho,\theta}$ if there exists a constant $C > 0$ for which

$$w(B) \frac{1}{p} w^{-\frac{1}{p'}} (B) \frac{p-1}{p'} \leq C |B| \left( 1 + \frac{r}{\rho(x)} \right)^\theta$$

for all balls $B = B(x, r)$ with center $x \in \mathbb{R}^d$ and radius $r > 0$. Define

$$A_{p,\rho,\infty} := \bigcup_{\theta \geq 0} A_{p,\rho,\theta}.$$

For $V \in \mathbb{R}H_d$, the class $A_{p,\rho,\infty}^V$ discussed in the introductory section is then defined by $A_{p,\rho,\infty}^V = A_{p,\rho,\infty}$. 
Proposition 3.2. Let $1 < p < \infty$. For any $c_1, c_2, c_3 > 0$, $m_1 < (k_0 + 1)^{-1}$ and $m_2 > (k_0 + 1)$ we have

$$A_p^{\rho, \infty} \subset H_{\rho, c_1}^p \subset S_{\rho, c_2}^p \subset H_{\rho, c_3}^p.$$ 

Proof. The first inclusion is almost trivial. Fix $w \in A_p^{\rho, \infty}$. Then there must exist some $\theta \geq 0$ for which $w \in A_p^{\rho, \theta}$. Since $\theta \lesssim c_1 x^{m_1}$ we have

$$w(B)^{\frac{1}{\theta}} w^{-\frac{1}{\rho(x)}} (B)^{\frac{p-1}{\rho}} \lesssim \left(1 + \frac{r}{\rho(x)}\right)^{\theta} |B|$$

$$\lesssim \exp \left(c_1 \left(1 + \frac{r}{\rho(x)}\right)^{m_1}\right) |B|$$

$$= \Phi_{m_1, c_1} (x, r) |B|$$

for all balls $B := B(x, r) \subset \mathbb{R}^d$.

Let’s prove the second inclusion. Fix $w \in H_{\rho, c_1}^p$ for some $c_1 > 0$ and $m_1 < (k_0 + 1)^{-1}$. Also fix $c_2 > 0$. Let $B_\rho := B_\rho (x, r) \subset \mathbb{R}^d$ be a ball in the metric $\varrho_\rho$ for some $x \in \mathbb{R}^d$ and $r > 0$. First suppose that $r > \beta$. Then

$$w (B_\rho)^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B_\rho)^{\frac{p-1}{\rho}} \leq w (B(x, r_\rho(x)))^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B(x, r_\rho(x)))^{\frac{p-1}{\rho}}$$

by Lemma 2.5, where $r_\rho := \left((r\beta)^{k_0+1} - 1\right)$. Applying the condition $w \in H_{\rho, c_1}^p$ gives

$$w (B_\rho)^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B_\rho)^{\frac{p-1}{\rho}} \lesssim |B(x, r_\rho(x))| \exp \left(c_1 \left(1 + \frac{r_\rho(x)}{\rho(x)}\right)^{m_1}\right)$$

$$= |B(x, r_\rho(x))| \exp \left(c_1 (r\beta)^{m_1 (k_0+1)}\right)$$

$$\lesssim \rho(x)^d (r\beta)^{(k_0+1)d} \exp \left(c_1 (r\beta)^{m_1 (k_0+1)}\right)$$

$$\lesssim \rho(x)^d \exp \left(c' r^{m_1 (k_0+1)}\right),$$

for any $c' > c_1$. Since $m_1 < (k_0 + 1)^{-1}$,

$$w (B_\rho)^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B_\rho)^{\frac{p-1}{\rho}} \lesssim \rho(x)^d e^{c' r},$$

for any $c_2 > 0$. Lemma 2.4 tells us that we must have the inclusion $B(x, \rho(x)) \subset B_\rho (x, r)$ and therefore

$$|B_\rho (x, r)| \geq |B(x, \rho(x))| = \rho(x)^d.$$

This gives

$$w(B_\rho)^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B_\rho)^{\frac{p-1}{\rho}} \lesssim |B_\rho| e^{c_2 r}.$$ 

We must now consider the case $r \leq \beta$. Lemma 2.5 implies that

$$w (B_\rho)^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B_\rho)^{\frac{p-1}{\rho}} \leq w (B(x, A_0 \rho(x)))^{\frac{1}{p}} w^{-\frac{1}{\rho(x)}} (B(x, A_0 \rho(x)))^{\frac{p-1}{\rho}}$$

$$\lesssim |B(x, A_0 \rho(x))| \exp (c_1 (1 + A_0 r)^{m_1})$$. 

Since \( r \leq \beta \),
\[
    w(B_p)^{\frac{1}{r}} w^{-\frac{1}{p'}} (B_p)^{\frac{p-1}{p}} \lesssim |B(x, A_0 \rho p(x))| \\
    \leq |B(x, A_0 \rho p(x))| e^{c_2 r} \\
    \simeq |B(x, \beta^{-1} \rho p(x))| e^{c_2 r}.
\]

Lemma 2.4 then gives
\[
    w(B_p)^{\frac{1}{r}} w^{-\frac{1}{p'}} (B_p)^{\frac{p-1}{p}} \lesssim |B_p(x, r)| e^{c_2 r}.
\]

This completes the proof of \( w \in S_{p,c_2}^p \).

Finally, let’s prove the last inclusion. Let \( w \in S_{p,c_2}^p \) with \( c_2 > 0 \). Fix \( c_3 > 0 \) and \( m_2 > (k_0 + 1) \). Let \( r > 0 \) and \( x \in \mathbb{R}^d \). First consider the case \( r \leq 1 \). On consecutively applying Lemma 2.4, the hypothesis \( w \in S_{p,c_2}^p \) and Lemma 2.5,
\[
    w(B(x, r \rho p(x)))^{\frac{1}{r}} w^{-\frac{1}{p'}} (B(x, r \rho p(x)))^{\frac{p-1}{p}} \leq w(B_p(x, \beta r \rho p(x)))^{\frac{1}{p'}} w^{-\frac{1}{p'}} (B_p(x, \beta r \rho p(x)))^{\frac{p-1}{p}} \\
    \lesssim |B_p(x, \beta r \rho p(x))| e^{c_2 \beta r} \\
    \lesssim |B(x, A_0 \beta r \rho p(x))| e^{c_2 \beta r} \\
    \lesssim r^d \rho(x)^d e^{c_3 (1 + r)^{m_2}}.
\]

Next consider \( r \geq 1 \). On successively applying Lemma 2.4, the hypothesis \( w \in S_{p,c_2}^p \) and Lemma 2.5,
\[
    w(B(x, r \rho p(x)))^{\frac{1}{r}} w^{-\frac{1}{p'}} (B(x, r \rho p(x)))^{\frac{p-1}{p}} \\
    \leq w(B_p(x, \beta (1 + r)^{k_0 + 1})^{\frac{1}{p'}} w^{-\frac{1}{p'}} (B_p(x, \beta (1 + r)^{k_0 + 1}))^{\frac{p-1}{p}} \\
    \lesssim |B_p(x, \beta (1 + r)^{k_0 + 1})| e^{c_2 \beta (1 + r)^{k_0 + 1}} \\
    \lesssim |B(x, \beta^2 (k_0 + 1) (1 + r) (1 + k_0)^2 \rho p(x))| e^{c_2 \beta (1 + r)^{k_0 + 1}} \\
    \lesssim r^d \rho(x)^d e^{c_3 (1 + r)^{m_2}}.
\]

\[ \square \]

**Definition 3.4.** Let \( c > 0 \). For each \( t > 0 \), define the averaging operator
\[
    A_{p,c}^t f(x) := \frac{1}{|B_p(x, t)| e^{ct}} \int_{B_p(x, t)} f(y) \, dy
\]
for \( f \in L_{loc}^1(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \). Define the centered Hardy-Littlewood operator associated with \( \rho \) through
\[
    M_{p,c} f(x) := \sup_{t > 0} A_{p,c}^t |f| (x).
\]

Similarly, define the uncentered Hardy-Littlewood operator by
\[
    \mathcal{M}_{p,c} f(x) := \sup_{B_{p,2x}} \frac{1}{e^{ct} |B_p|} \int_{B_p} |f(y)| \, dy,
\]
where the supremum is taken over all balls \( B_p = B_p(x', r) \subset \mathbb{R}^d \) in the metric \( d_\rho \) that contain \( x \).

For any \( x \in \mathbb{R}^d \) and \( f \in L_{loc}^1(\mathbb{R}^d) \), the inequality \( M_{p,c} f(x) \leq \mathcal{M}_{p,c} f(x) \) is trivial. The below proposition states that a weak converse will hold.
Proposition 3.3. For any $c_1, c_2 > 0$ with $c_1 > 2c_2$ we have
\[ M_{p,c_1} f(x) \lesssim M_{p,c_2} f(x) \]
for all $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

**Proof.** Fix $c_1, c_2 > 0$ with $c_1 > 2c_2$ and let $x \in \mathbb{R}^d$ and $f \in L^1_{loc}(\mathbb{R}^d)$. Let $B_{\rho} = B_{\rho}(x', r) \subset \mathbb{R}^d$ for $x' \in \mathbb{R}^d$ and $r > 0$ be a ball in the metric $d_{\rho}$ that contains the point $x$. We have by Corollary 2.1,
\[
|B_{\rho}(x, 2r)| \leq e^{(\frac{1}{2} - c_2)^r} |B_{\rho}(x, r)| \leq e^{(\frac{1}{2} - c_2)^r} |B_{\rho}(x', 2r)| \leq e^{(c_1 - 2c_2)^r} |B_{\rho}(x', r)|.
\]
Therefore,
\[
\frac{1}{e^{c_1 r}} |B_{\rho}(x', r)| \int_{B_{\rho}(x', r)} |f(y)| \, dy \lesssim \frac{e^{(c_1 - 2c_2)^r}}{e^{c_1 r} |B_{\rho}(x, 2r)|} \int_{B_{\rho}(x, 2r)} |f(y)| \, dy \leq \frac{1}{e^{2c_2 r} |B_{\rho}(x, 2r)|} \int_{B_{\rho}(x, 2r)} |f(y)| \, dy \leq \frac{1}{e^{c_1 r} |B_{\rho}(x, 2r)|} \int_{B_{\rho}(x, 2r)} |f(y)| \, dy \leq \frac{1}{e^{c_1 r} |B_{\rho}(x, 2r)|} \int_{B_{\rho}(x, 2r)} |f(y)| \, dy \leq M_{c_2} f(x).
\]
Taking the supremum over all $B_{\rho} = B_{\rho}(x', r)$ then proves the proposition. \(\square\)

Proposition 3.4. Fix $1 < p < \infty$. For any $c_1, c_2 > 0$ with $c_1 > 2c_2$,
\[
\left\{ w : \|M_{p,c_2}\|_{L^p(w)} < \infty \right\} \subset S_{p,c_1}^p.
\]

**Proof.** Fix $1 < p < \infty$ and $c_1, c_2 > 0$ with $c_1 > 2c_2$. Let $w$ be a weight for which \(\|M_{p,c_2}\|_{L^p(w)} < \infty\). The proof that $w \in S_{p,c_1}^p$ is similar to the standard classical proof that can be found in [8] for example. Fix $B_{\rho} := B_{\rho}(x, r)$ for some $x \in \mathbb{R}^d$ and $r > 0$. The boundedness of $M_{p,c_2}$ on $L^p(w)$ together with Proposition 3.3 implies that
\[
w(B_{\rho}) \left( \frac{1}{e^{c_1 r} |B_{\rho}|} \int_{B_{\rho}} |f| \right)^p \lesssim \int_{B_{\rho}} M_{p,c_1} (f 1_{B_{\rho}})(y) w(y) \, dy \lesssim \int_{B_{\rho}} M_{p,c_2} (f 1_{B_{\rho}})(y) w(y) \, dy \lesssim \int_{B_{\rho}} |f|^p w(y) \, dy.
\]
For $\epsilon > 0$, take $f := (w + \epsilon)^{-\frac{1}{p-1}}$ in the above inequality to obtain
\[
\frac{1}{e^{c_1 r} |B_{\rho}|^p} w(B_{\rho}) \left( \int_{B_{\rho}} (w + \epsilon)^{-\frac{1}{p-1}} \right)^p \lesssim \left( \int_{B_{\rho}} (w + \epsilon)^{-\frac{1}{p-1}} w \right) \leq \left( \int_{B_{\rho}} (w + \epsilon)^{-\frac{1}{p-1}} \right).
\]
Which leads to
\[
\left( \frac{1}{e^{c_1 r} |B_{\rho}|} \int_{B_{\rho}} w \right) \left( \frac{1}{e^{c_1 r} |B_{\rho}|} \int_{B_{\rho}} (w + \epsilon)^{-\frac{1}{p-1}} \right)^{p-1} \lesssim C
\]
for some constant $C > 0$. The monotone convergence theorem then allows us to conclude that $w \in S_{p,c_1}^p$. \(\square\)
Definition 3.5. Let \( c, m > 0 \). For each \( t > 0 \), define the averaging operator

\[
\tilde{A}_{t,c}^{\rho,m} f(x) := \frac{1}{\Phi_{\rho,m,c}(x,t)} \int_{B(x,t)} f(y) \, dy
\]

for \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \). Define the corresponding centered Hardy-Littlewood operator through

\[
\tilde{M}_{\rho,c}^{m} f(x) := \sup_{t > 0} \tilde{A}_{t,c}^{\rho,m} |f|(x).
\]

Similarly, define the uncentered Hardy-Littlewood operator by

\[
\tilde{M}_{\rho,c}^{m} f(x) := \sup_{B \ni x} \frac{1}{\Phi_{m,c}(x',r)} |B| \int_B |f(y)| \, dy,
\]

where the supremum is taken over all Euclidean balls \( B = B(x',r) \subset \mathbb{R}^d \) that contain the point \( x \).

For these operators, an analogue of the pointwise bound from Proposition 3.3 will hold.

Proposition 3.5. For any \( c_1, c_2, m_1, m_2 > 0 \) with \( m_1 > (k_0 + 1)m_2 \) we have

\[
\tilde{M}_{\rho,c_1}^{m_1} f(x) \lesssim \tilde{M}_{\rho,c_2}^{m_2} f(x)
\]

for all \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \).

Proof. Fix \( c_1, c_2, m_1, m_2 > 0 \) with \( m_1 > (k_0 + 1)m_2 \) and let \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \). Let \( B(x',r) \subset \mathbb{R}^d \) be a ball in the Euclidean metric that contains the point \( x \). We have by Lemma 2.1

\[
1 + \frac{2r}{\rho(x)} \leq 1 + 2B_0 \left( 1 + \frac{r}{\rho(x')} \right)^{k_0} \leq 2B_0 \left( 1 + \frac{r}{\rho(x')} \right)^{k_0 + 1}.
\]

Therefore,

\[
\Phi_{m_2,c_2}(x,2r) = \exp \left( c_2 \left( 1 + \frac{2r}{\rho(x')} \right)^{m_2} \right) \leq \exp \left( 2c_2B_0 \left( 1 + \frac{r}{\rho(x')} \right)^{(k_0+1)m_2} \right) \lesssim \exp \left( c_1 \left( 1 + \frac{r}{\rho(x')} \right)^{m_1} \right) = \Phi_{m_1,c_1}(x',r).
\]

Which gives

\[
\frac{1}{\Phi_{m_1,c_1}(x',r) |B(x',r)|} \int_{B(x',r)} |f(y)| \, dy \lesssim \frac{1}{\Phi_{m_2,c_2}(x,2r) |B(x,2r)|} \int_{B(x,2r)} |f(y)| \, dy \leq \tilde{M}_{\rho,c_2}^{m_2} f(x).
\]

Taking the supremum over all \( B(x',r) \) that contains \( x \) then proves the proposition.

The pointwise estimate from the previous proposition then allows us to deduce the following inclusion. The proof is identical to that of Proposition 3.4.
Proposition 3.6. Fix $1 < p < \infty$. For any $c_1, c_2, m_1, m_2 > 0$ with $m_1 > (k_0 + 1)m_2$,
\[
\left\{ w : \| \tilde{M}_{p,c_2}^m \|_{L^p(w)} < \infty \right\} \subset H_{p,c_1}^{m_1}.
\]

4. Schrödinger Operators

In this section, a proof of Theorems 1.1 and 1.2 will be provided. Theorem 1.1 will be proved by demonstrating that the statements actually hold for a more general form of operator, the Schrödinger operator with measure potential $-\Delta + \mu$.

4.1. Schrödinger Operators with Measure Potential. Schrödinger operators with measure potential, or generalised Schrödinger operators, were considered by Z. Shen in the article [17]. For this form of Schrödinger operator, the scalar potential $V$ is replaced by a non-negative Radon measure $\mu$ on $\mathbb{R}^d$. It is assumed that the measure $\mu$ satisfies the property that there exists $\mu_0, C_\mu, D_\mu > 0$ such that
\begin{align}
  \mu(B(x,r)) &\leq C_\mu \left( \frac{r}{R} \right)^{d-2+\delta_\mu} \mu(B(x,R)) \\
  \mu(B(x,2r)) &\leq D_\mu \left( \mu(B(x,r)) + r^{d-2} \right)
\end{align}
for all $x \in \mathbb{R}^d$ and $0 < r < R$.

Remark 4.1. For $V \in RH_q$ with $q > \frac{d}{2}$, the measure $d\mu(x) = V(x) \, dx$ will satisfy both properties (7) and (8) with constants $C_\mu, D_\mu > 0$, dependent on $V$ only through $|V|_{RH_q}$, and $\delta_\mu = 2 - \frac{d}{q}$ (c.f. [16, Lem. 1.2]). This tells us that standard Schrödinger operators are instances of this generalised form. If $V \in RH_d$ then, due to the self-improvement property of the reverse Hölder classes, $V \in RH_{q'}$ for some $q' > d$ and therefore (7) will be satisfied with $\delta_\mu = 2 - \frac{d}{q'} > 1$.

A proof of the following lemma can be found in [3, Lem. 2.6] for the case $d\mu(x) = V(x) \, dx$ with $V \in RH_{\frac{d}{2}}$. The proof for general $\mu$ is identical.

Lemma 4.1. Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^d$ that satisfies (7) and (8) with constants $C_\mu, D_\mu, \delta_\mu > 0$. For all $x \in \mathbb{R}^d$ and $R > 0$,
\begin{align}
  \int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{d-2}} &\lesssim \frac{\mu(B(x,R))}{R^{d-2}} \\
  \text{If } \delta_\mu > 1, \text{ then we will also have} \int_{B(x,R)} \frac{d\mu(y)}{|y-x|^{d-1}} &\lesssim \frac{\mu(B(x,R))}{R^{d-1}}
\end{align}
for all $x \in \mathbb{R}^d$ and $R > 0$.

We consider the operator $L_\mu := -\Delta + \mu$. This can be defined rigorously through its corresponding sesquilinear form as a non-negative unbounded operator on $L^2(\mathbb{R}^d)$ with maximal domain. Define the operators
\[
  R_\mu := \nabla L_\mu^{-\frac{1}{2}}, \quad R^*_\mu := L_\mu^{-\frac{1}{2}} \nabla, \quad I_\mu := L_\mu^{-\frac{1}{2}}
\]
for $0 < \alpha \leq 2$ and
\[ T_\mu^* f(x) := \sup_{t > 0} e^{-tL_\mu} |f| (x) \]
for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. For this measure form of the electric potential, the corresponding critical radius function is defined through
\[ \rho_\mu(x) := \sup \left \{ r > 0 : \frac{\mu(B(x,r))}{r^{d-2}} \leq 1 \right \}, \quad x \in \mathbb{R}^d. \]

**Remark 4.2.** It follows directly from the definition of $\rho_\mu$ that for any $x \in \mathbb{R}^d$
\[ \frac{\mu(B(x,\rho_\mu(x)))}{\rho_\mu(x)^{d-2}} \simeq 1. \]

In [17] it was proved that $\rho_\mu$ is indeed a critical radius function in the sense of Definition 2.1.

**Lemma 4.2 ([17, Prop. 1.8, Rmk. 1.9]).** The function $\rho_\mu$ is a critical radius function in the sense of Definition 2.1. In particular, (3) is satisfied with constants $B_0$ and $k_0$ depending on $\mu$ only through $C_\mu$, $D_\mu$, and $\delta_\mu$.

Through the function $\rho_\mu$, we can define a corresponding Agmon distance $d_\mu := d_{\rho_\mu}$ and balls $B_\mu(x,r) := B_{\rho_\mu}(x,r)$ for $x \in \mathbb{R}^d$ and $r > 0$. This then allows us to define appropriate analogues of our weight classes $S_{\mu,c} := S_{\mu,c}^\rho$ and $H_{\mu,c} := H_{\mu,c}^\rho$. For these classes of weights, the following theorem will be proved.

**Theorem 4.1.** Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^d$ that satisfies (7) and (8) with constants $C_\mu$, $D_\mu$, $\delta_\mu > 0$.

(i) Suppose that $\delta_\mu > 1$. There exists $c_1 > 0$ for which both $R_\mu$ and $R_\mu^*$ are bounded on $L^p(w)$ for all $w \in S_{\mu,c_1}$ and $1 < p < \infty$.

(ii) Suppose instead that $0 < \delta_\mu < 1$ and let $\eta \in (2, (2 - \delta_\mu)/(1 - \delta_\mu))$. There exists $c_2 > 0$ for which the operator $R_\mu$ is bounded on $L^p(w)$ for $\eta' < p < \infty$ when $w \in S_{\mu/c_2}$ and the operator $R_\mu^*$ is bounded on $L^p(w)$ for $1 < p < \eta$ when $w^{-\frac{1}{\eta' - \eta}} \in S_{\mu/c_2}$.

(iii) If $\delta_\mu > 0$ and $0 < \alpha \leq 2$, there must exist a constant $c_3 > 0$ for which the operator $R_\mu^*$ is bounded from $L^p(w)$ to $L^\nu(w^{\nu/p})$ for $w^{\nu/p} \in S_{1+\frac{\alpha}{\nu},c_3}$ and
\[ 1 < p < \frac{d}{\alpha}, \quad \text{where} \quad \frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d}. \]
The constants $c_1$, $c_2$, and $c_3$ are independent of $p$ and depend on $\mu$ only through $C_\mu$, $D_\mu$, and $\delta_\mu$.

Let $\Gamma_\mu$ denote the fundamental solution of the operator $L_\mu$. Refer to [17] for further information and properties. The proof of Theorem 4.1 will rely heavily on the following exponential decay estimates that were proved by Shen in [17].

**Theorem 4.2 ([17, Thm. 0.8, Thm. 0.17]).** Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^d$ that satisfies (7) and (8) with constants $C_\mu$, $D_\mu$, $\delta_\mu > 0$. There exist constants $C_1, C_2, \varepsilon_1, \varepsilon_2 > 0$ for which
\[ C_1 \frac{e^{-\varepsilon_1 d_\mu(x,y)}}{|x - y|^{d-2}} \leq \Gamma_\mu(x,y) \leq C_2 \frac{e^{-\varepsilon_2 d_\mu(x,y)}}{|x - y|^{d-2}} \]
for all \( x, y \in \mathbb{R}^d \). There will also exist \( C_3, \varepsilon_3 > 0 \) for which
\[
(11) \quad |\nabla \Gamma_\mu(x, y)| \leq C_3 e^{-\varepsilon_3 d_\mu(x,y)} \left( \int_{B(x, |x-y|/2)} \frac{d\mu(z)}{|z-x|^d} + \frac{1}{|x-y|} \right)
\]
for all \( x, y \in \mathbb{R}^d \), where the gradient is taken with respect to the first variable. If \( \delta_\mu > 1 \) then \( C_4, \varepsilon_4 > 0 \) can be chosen so that
\[
|\nabla \Gamma_\mu(x, y)| \leq C_4 e^{-\varepsilon_4 d_\mu(x,y)} \quad \forall \ x, y \in \mathbb{R}^d.
\]
The constants \( \varepsilon_i, C_i \) for \( i = 1, \ldots, 4 \) will depend on \( \mu \) only through \( C_\mu, D_\mu \) and \( \delta_\mu \).

It should be noted that although the estimate (11) is not explicitly stated in [17], its proof is essentially contained within the proof of [17, Thm. 0.17].

We will not attempt to prove the measure potential version of Theorem 1.2 since, to the best of the author’s knowledge, heat kernel estimates for the general operator \( L_\mu \) have not yet been proved. The best known result for heat kernel estimates can be found in [12] and requires the assumption that \( d\mu(x) = V(x) \, dx \) with \( V \in RH_\frac{d}{2} \).

Therefore, the boundedness of \( \mathcal{T}_\mu^* \) will only be considered for this case.

4.2. The Riesz Transforms. Let’s consider the boundedness of the operators \( R_\mu \) and \( R^*_\mu \) on the weighted Lebesgue space \( L^p(w) \). The Riesz transforms \( R_\mu \) can be expressed as

\[
R_\mu f(x) = \nabla \left( -\Delta + \mu \right)^{-\frac{1}{2}} f(x)
\]

\[
= \frac{1}{\pi} \int_0^{\infty} \lambda^{-\frac{1}{2}} \nabla \left( -\Delta + \mu + \lambda \right)^{-1} f(x) \, d\lambda
\]

\[
= \frac{1}{\pi} \int_0^{\infty} \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^d} \nabla \Gamma_{\mu+\lambda}(x, y) f(y) \, dy \, d\lambda,
\]

where \( \Gamma_{\mu+\lambda} \) is the fundamental solution of \( (-\Delta + \mu + \lambda) \) (c.f. [11, pg. 282]). Fubini’s Theorem then gives

\[
R_\mu f(x) = \int_{\mathbb{R}^d} \frac{1}{\pi} \int_0^{\infty} \lambda^{-\frac{1}{2}} \nabla \Gamma_{\mu+\lambda}(x, y) f(y) \, dy \, d\lambda
\]

\[
= \int_{\mathbb{R}^d} K_\mu(x, y) f(y) \, dy,
\]

where \( K_\mu \) is the singular kernel of \( R_\mu \) given by

\[
K_\mu(x, y) = \frac{1}{\pi} \int_0^{\infty} \lambda^{-\frac{1}{2}} \nabla \Gamma_{\mu+\lambda}(x, y) \, d\lambda.
\]

The adjoints \( R^*_\mu \) will then be given by

\[
R^*_\mu f(x) = \int_{\mathbb{R}^d} K^*_\mu(x, y) f(y) \, dy = \int_{\mathbb{R}^d} K_\mu(y, x) f(y) \, dy.
\]

In particular, the singular kernel of \( R^*_\mu \), denoted \( K^*_\mu \), satisfies \( K^*_\mu(x, y) = K_\mu(y, x) \) for all \( x, y \in \mathbb{R}^d \).
Lemma 4.3. There exists $\varepsilon > 0$, independent of $p$ and depending on $\mu$ only through $C_\mu$, $D_\mu$ and $\delta_\mu$, for which

\begin{equation}
\left| K^*_\mu(x,y) \right| \lesssim e^{-\varepsilon d_\mu(x,y)} \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|} \right)
\end{equation}

for all $x, y \in \mathbb{R}^d$.

Proof. First note that from the definition of the Agmon distance

\begin{equation}
d_{\mu+\lambda}(x,y) \geq \frac{1}{2} (d_\mu(x,y) + d_\lambda(x,y))
\end{equation}

for all $x, y \in \mathbb{R}^d$. Combining this with (11) from Theorem 4.2 gives

\begin{equation}
|\nabla \Gamma_{\mu+\lambda}(y,x)| \lesssim e^{-\varepsilon d_\mu(x,y)} e^{-\varepsilon d_\lambda(x,y)} \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|} \right)
\end{equation}

for some $\varepsilon > 0$ independent of $p$ and depending on $\mu$ only through $C_\mu$, $D_\mu$ and $\delta_\mu$. Since $d_\lambda(x,y) = \lambda^\frac{1}{2} |x-y|$, we then have by (12) and (13)

\begin{align*}
|K^*_\mu(x,y)| & \lesssim e^{-\varepsilon d_\mu(x,y)} \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|} \right) \int_0^\infty \lambda^{-\frac{1}{2}} e^{-\varepsilon \lambda^\frac{1}{2} |x-y|} d\lambda \\
& \lesssim e^{-\varepsilon d_\mu(x,y)} \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|} \right) \int_0^\infty \lambda^{-\frac{1}{2}} e^{-\varepsilon \lambda^\frac{1}{2}} d\lambda \\
& \lesssim e^{-\varepsilon d_\mu(x,y)} \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|} \right).
\end{align*}

As stated previously, in order to prove the $L^p(w)$-boundedness of an operator it is sufficient to prove the boundedness of the global and local components separately. The following proposition is a measure generalisation of the local boundedness result proved in [5, Thm. 1].

Proposition 4.1. Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^d$ that satisfies (7) and (8) with constants $C_\mu$, $D_\mu$, $\delta_\mu > 0$.

(i) If $\delta_\mu > 1$ then the operators $R_{\mu,\text{loc}}$ and $R^*_{\mu,\text{loc}}$ are bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A^\mu_{\text{loc}}$.

(ii) Suppose instead that $0 < \delta_\mu < 1$ and let $\eta \in (2, (2-\delta_\mu)/(1-\delta_\mu))$. Then $R^*_{\mu,\text{loc}}$ will be bounded on $L^p(w)$ for $\eta' < p < \infty$ when $w \in A^{\mu/\eta'}_{\text{loc}}$ and $R_{\mu,\text{loc}}$ will be bounded on $L^p(w)$ for $1 < p < \eta$ when $w^{-\frac{1}{\eta'}} \in A^\mu_{\text{loc}}$.

Proof. For any weight $w$ on $\mathbb{R}^d$, $1 < p < \infty$ and $f \in L^p(w)$ we have

$$
\| R^*_{\mu,\text{loc}} f \|_{L^p(w)} \lesssim \| (R^*_{\mu,\text{loc}} - R^*_{0,\text{loc}}) f \|_{L^p(w)} + \| R^*_{0,\text{loc}} f \|_{L^p(w)}.
$$

As the operator $R^*_{0,\text{loc}}$ is bounded on $L^p(w)$ for any $w \in A^\mu_{\text{loc}}$ by [5, Thm. 1], it suffices to prove that the difference term $R^*_{\mu,\text{loc}} - R^*_{0,\text{loc}}$ is bounded on $L^p(w)$. In the proof of [17, Lem. 7.13], it was proved that for $x, y \in \mathbb{R}^d$ satisfying $|x-y| \leq \rho_\mu(x)$,

$$
|K^*_\mu(x,y) - K_0^*(x,y)| \lesssim \frac{1}{|x-y|^{d-1}} \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} + \frac{1}{|x-y|^d} \left( \frac{|x-y|}{\rho_\mu(x)} \right)^{\delta_\mu}
$$
for all \( x, y \in \mathbb{R}^d \). This gives

\[
\| (R^*_{\mu,loc} - R^*_0,loc) f \|_{L^p(w)}^p \lesssim \int_{\mathbb{R}^d} \left( \int_{B(x,\rho_\mu(x))} |K^*_\mu(x,y) - K_0^*(x,y)| |f(y)| \, dy \right)^p w(x) \, dx
\]

\[
\lesssim \int_{\mathbb{R}^d} \left( \int_{B(x,\rho_\mu(x))} \frac{1}{|x-y|^d} \left( \frac{|x-y|}{\rho_\mu(x)} \right)^\delta \mu |f(y)| \, dy \right)^p w(x) \, dx
\]

\[
+ \int_{\mathbb{R}^d} \left( \int_{B(x,\rho_\mu(x))} \frac{1}{|x-y|^{d-1}} \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} |f(y)| \, dy \right)^p w(x) \, dx
\]

\[
=: J_1 + J_2.
\]

Using the same argument that is used to bound the function \( h_1(x) \) in [5, Thm. 3], it is clearly true that

\[
J_1 \lesssim \| f \|_{L^p(w)}
\]

for any \( w \in A^\mu_{p,loc} \). It therefore suffices to estimate the term \( J_2 \).

**Proof of Part (i).** Suppose that \( \delta_\mu > 1 \). For this case, Lemma 4.1, (7) and Remark 4.2 imply that for \( y \in B(x,\rho_\mu(x)) \),

\[
\int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} \lesssim \frac{\mu(B(y,|x-y|/2))}{|x-y|^{d-1}}
\]

\[
\lesssim \frac{\mu(B(x,2|x-y|))}{|x-y|^{d-1}}
\]

\[
\lesssim \left( \frac{|x-y|}{\rho_\mu(x)} \right)^{d-2+\delta} \frac{\mu(B(x,2\rho_\mu(x)))}{|x-y|^{d-1}}
\]

\[
\lesssim \frac{1}{|x-y|} \left( \frac{|x-y|}{\rho_\mu(x)} \right)^\delta
\]

and therefore \( J_2 \lesssim J_1 \). Since we already know that \( J_1 \) is bounded, this implies that \( R^*_{\mu,loc} \) is bounded on \( L^p(w) \) for \( w \in A^\mu_{p,loc} \). The boundedness of \( R_{\mu,loc} \) on \( L^p(w) \) follows from duality. This proves the first part of our proposition.

**Proof of Part (ii).** Suppose that \( 0 < \delta_\mu < 1 \). For this case, Lemma 4.1 can no longer be applied to bound the term \( J_2 \). Instead, \( J_2 \) will be handled by adapting the argument from [5, Thm. 3]. Let \( \eta \in (0,(2-\delta_\mu)/(1-\delta_\mu)) \), \( \eta' < p < \infty \) and assume that \( w \in A^\mu_{p'/q'} \). Let \( B_j \) be a covering of \( \mathbb{R}^d \) by balls as given in Proposition 2.1. For each \( j, k \in \mathbb{N} \), there exists \( 2^{dk} \) balls \( B_j^{l,k} = B(x^{l,k}_j,2^{-k} \rho_\mu(x_j)) \), \( l = 1, \ldots, 2^{dk} \), with the properties that \( B_j \subset \bigcup_{l=1}^{2^{dk}} B_j^{l,k} \subset 2B_j \) and \( \sum_{l=1}^{2^{dk}} \chi_{B_j^{l,k}} \leq 2d \). Set \( B_j^{l,k} = 10B_0 \cdot B_j^{l,k} \) with \( B_0 \) as given in Lemma 4.2. The construction can be done in such a way so that

\[
\sum_j \sum_{l=1}^{2^{dk}} \chi_{B_j^{l,k}} \leq C
\]

for some \( C \geq 0 \) independent of \( k \). Define the function

\[
h_2(x) := \int_{B(x,\rho_\mu(x))} |f(y)| \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} \right) \, dy.
\]
Then we have
\[
(19) \quad h_2(x) \lesssim \sum_{k=0}^{\infty} 2^{k(d-1)} h_{2,k}(x),
\]
where
\[
h_{2,k}(x) := \rho_{\mu}(x)^{-d+1} \int_{B(x,2^{-k} \rho_{\mu}(x))} |f(y)| \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} \right) \, dy.
\]
If \( x \in B_{i}^{j,k} \) then
\[
h_{2,k}(x) \lesssim \rho_{\mu}(x_j)^{-d+1} \int_{B_{i}^{j,k}} |f(y)| \left( \int_{B_{i}^{j,k}} \frac{d\mu(z)}{|z-y|^{d-1}} \right) \, dy \lesssim \rho_{\mu}(x_j)^{-d+1} \left\| \frac{d\mu(z)}{|z-y|^{d-1}} \right\|_{L^\infty(B_{i}^{j,k})} \left\| f \right\|_{L^p(B_{i}^{j,k},w)} \left( \int_{B_{i}^{j,k}} w^{-\gamma/p} \right)^{\frac{1}{p}},
\]
where the second line follows from Hölder’s inequality and \( \frac{1}{\gamma} := 1 - \frac{1}{p} - \frac{1}{\eta}. \) Lemma 7.9 of [17], (7) and Remark 4.2 imply that
\[
\left\| \int_{B_{i}^{j,k}} \frac{d\mu(z)}{|z-y|^{d-1}} \right\|_{L^\infty(B_{i}^{j,k})} \lesssim \frac{\mu(3B_{i}^{j,k})}{(2^{-k} \rho_{\mu}(x_j))^\frac{1}{\eta}-1} \lesssim \rho_{\mu}(x_j)^{-d+1} + \frac{1}{\gamma} \rho_{\mu}(x_j)^{-d+\frac{1}{\gamma}}.
\]
Since \( w \in A^\mu_{p,q} \) we have
\[
w(B_{i}^{j,k})^\frac{1}{\gamma} \left( \int_{B_{i}^{j,k}} w^{-\frac{2}{\gamma}} \right)^{\frac{1}{2}} \lesssim \left| B_{i}^{j,k} \right| \frac{1}{\sigma}
\]
\[
\lesssim (2^{-k} \rho_{\mu}(x_j))^\frac{1}{\gamma}.
\]
Therefore,
\[
\left\| h_{2,k} \right\|_{L^p(w)}^p \lesssim \sum_{j,l} \left( \rho_{\mu}(x_j)^{-d+1} \left| B_{i}^{j,k} \right| \frac{1}{\sigma} \right)^p \left\| f \right\|_{L^p(B_{i}^{j,k},w)} \left( \int_{B_{i}^{j,k}} w^{-\frac{2}{\gamma}} \right)^{\frac{1}{2}} \left( \int_{B_{i}^{j,k}} \frac{d\mu(z)}{|z-y|^{d-1}} \right) \, dx
\]
\[
\lesssim \sum_{j,l} \left( \rho_{\mu}(x_j)^{-d+1} \left| B_{i}^{j,k} \right| \frac{1}{\sigma} \right)^p \left\| f \right\|_{L^p(B_{i}^{j,k},w)} \left( \int_{B_{i}^{j,k}} w^{-\frac{2}{\gamma}} \right)^{\frac{1}{2}} \left( \int_{B_{i}^{j,k}} \frac{d\mu(z)}{|z-y|^{d-1}} \right) \, dx
\]
\[
\lesssim \sum_{j,l} \left( \rho_{\mu}(x_j)^{-d+1} \rho_{\mu}(x_j)^{d-1} + \frac{1}{\gamma} (2^{-k})^{d-1} + \frac{1}{\gamma} (2^{-k} \rho_{\mu}(x_j)) \frac{1}{\gamma} \right)^p \left\| f \right\|_{L^p(B_{i}^{j,k},w)}
\]
\[
\lesssim 2^{-kp(d-1+\delta_\mu)} \left\| f \right\|_{L^p(w)}^p,
\]
where (18) was used to obtain the final line. Referring back to (19),

\[
J_2 = \|h_2\|_{L^p(w)} \lesssim \sum_{k=0}^{\infty} 2^{k(d-1)} \|h_{2,k}\|_{L^p(w)} \\
\lesssim \|f\|_{L^p(w)} \left( \sum_{k=0}^{\infty} 2^{k(d-1)} 2^{-k(d-1+\delta_\mu)} \right) \\
= \|f\|_{L^p(w)} \left( \sum_{k=0}^{\infty} 2^{-k\delta_\mu} \right) \\
\lesssim \|f\|_{L^p(w)}.
\]

This proves that \(R_{\mu,\text{loc}}^*\) is bounded on \(L^p(w)\) for \(\eta' < p < \infty\) and \(w \in A_{p/\eta'}^{\mu,\text{loc}}\). The \(L^p(w)\)-boundedness of \(R_{\mu,\text{loc}}^*\) for \(1 < p < \eta\) and \(w^{-\frac{1}{p'}} \in A_{p'/\eta'}^{\mu,\text{loc}}\) follows by using duality.

With the boundedness of the local component of our operators established, it now suffices to consider the boundedness of the global components.

**Theorem 4.3.** Let \(\mu\) be a non-negative Radon measure on \(\mathbb{R}^d\) that satisfies (7) and (8) with constants \(C_\mu, D_\mu, \delta_\mu > 0\).

(i) Suppose that \(\delta_\mu > 1\). There exists \(c_1 > 0\) such that \(R_{\mu,\text{glob}}^*\) and \(R_{\mu,\text{glob}}^*\) are both bounded on \(L^p(w)\) for any \(w \in S_{\mu,c_1}^p\) and \(1 < p < \infty\).

(ii) Suppose instead that \(0 < \delta_\mu < 1\) and let \(\eta \in (2, (2-\delta_\mu)/(1-\delta_\mu))\). There exists \(c_2 > 0\) for which the operator \(R_{\mu,\text{glob}}^*\) is bounded on \(L^p(w)\) for \(\eta' < p < \infty\) when \(w \in S_{p',\eta',c_2}^\mu\) and the operator \(R_{\mu}^*\) is bounded on \(L^p(w)\) for \(1 < p < \eta\) when \(w^{-\frac{1}{p'}} \in S_{p'/\eta',c_2}^\mu\).

The constants \(c_1\) and \(c_2\) are independent of \(p\) and depend on \(\mu\) only through \(C_\mu, D_\mu\) and \(\delta_\mu\).

**Proof.** For \(1 < p < \infty\), weight \(w\) on \(\mathbb{R}^d\) and \(f \in L^p(w)\),

\[
\| R_{\mu,\text{glob}}^* f \|_{L^p(w)}^p = \int_{\mathbb{R}^d} \left| \int_{B(x,\rho_\mu(x))} |K_\mu^*(x,y)| |f(y)| \, dy \right|^p w(x) \, dx \\
\leq \int_{\mathbb{R}^d} \left( \int_{B(x,\rho_\mu(x))} |K_\mu^*(x,y)| |f(y)| \, dy \right)^p w(x) \, dx.
\]

Let \(\{x_j\}_{j \in \mathbb{N}}\) be a collection of points in \(\mathbb{R}^d\) as given in Proposition 2.1. Introduce the notation \(B_j := B(x_j, \rho_\mu(x_j))\) for \(j \in \mathbb{N}\). Since the collection of balls \(\{B_j\}_{j \in \mathbb{N}}\) forms a cover for \(\mathbb{R}^d\),

\[
\| R_{\mu,\text{glob}}^* f \|_{L^p(w)}^p \leq \sum_j \int_{B_j} \left( \int_{B(x,\rho_\mu(x))} |K_\mu^*(x,y)| |f(y)| \, dy \right)^p w(x) \, dx.
\]
The estimate (14) from Lemma 4.3 then leads to
\[
\| R^\ast_{\mu, \text{glob}} f \|_{L^p(w)} \lesssim J_1 + J_2 := \sum_j \int_{B_j} \left( \int_{B(x, \rho_\mu(x))} e^{-\varepsilon d_\mu(x,y)} f(y) \, dy \right)^p w(x) \, dx \\
+ \sum_j \int_{B_j} \left( \int_{B(x, \rho_\mu(x))} e^{-\varepsilon d_\mu(x,y)} \frac{d\mu(z)}{|x-y|^{d-1}} \int_{B(y, |x-y|/2)} |y-z|^{-d} f(y) \, dy \right)^p w(x) \, dx,
\]
for some \( \varepsilon > 0 \) independent of \( p \) and depending on \( \mu \) only through \( C_\mu, D_\mu \) and \( \delta_\mu \).

**Proof of Part (i).** Assume that \( \delta_\mu > 1 \). Fix \( 1 < p < \infty \) and \( w \in S_{p,c_1}^V \) for some \( c_1 > 0 \). Lemma 4.1 followed by (7) implies that
\[
e^{-\varepsilon d_\mu(x,y)} \frac{d\mu(z)}{|x-y|^{d-1}} \int_{B(y, |x-y|/2)} |y-z|^{-d} f(y) \, dy \lesssim \frac{e^{-\varepsilon d_\mu(x,y)} \mu(B(y, |x-y|/2))}{|x-y|^{d-1}}
\]
\[
\lesssim \frac{e^{-\varepsilon d_\mu(x,y)} (|x-y|)^{d-2+\delta_\mu} \mu(B(y, \rho(y)))}{|x-y|^{d-1}}.
\]
Remark 4.2 together with Lemma 2.3 gives
\[
e^{-\varepsilon d_\mu(x,y)} \frac{d\mu(z)}{|x-y|^{d-1}} \int_{B(y, |x-y|/2)} |y-z|^{-d} f(y) \, dy \lesssim \frac{e^{-\varepsilon d_\mu(x,y)} (|x-y|)^{d-2+\delta_\mu} \mu(B(y, \rho(y)))}{|x-y|^{d-1}}.
\]
This proves that \( J_2 \) is bounded from above by a term that is identical to \( J_1 \) except that the exponent of the exponential is \( \varepsilon/2 \) instead of \( \varepsilon \). Due to this similarity, in order to prove our claim it will then suffice to show that \( c_1 \) can be set small enough so that \( J_1 \lesssim \| f \|_{L^p(w)} \).

Notice that Lemma 2.5 tells us that \( B(x, \rho_\mu(x))^c \subseteq B_\mu(x, A_0^{-1})^c \). Let’s use the shorthand notation \( B_{\mu,x} \) to denote the ball \( B_\mu(x, A_0^{-1}) \). Then
\[
J_1 \leq \sum_j \int_{B_j} \left( \int_{B_{\mu,x}} e^{-\varepsilon d_\mu(x,y)} |f(y)| \, dy \right)^p w(x) \, dx
\]
\[
= \sum_j \int_{B_j} \left( \sum_{k=1}^j \int_{(k+1)B_{\mu,x} \setminus kB_{\mu,x}} e^{-\varepsilon d_\mu(x,y)} |f(y)| \, dy \right)^p w(x) \, dx,
\]
where \( kB_{\mu,x} := B_\mu(x, kA_0^{-1}) \) for \( k \geq 1 \). Fix \( x \in B_j \) for some \( j \in \mathbb{N} \) and \( y \in (k+1)B_{\mu,x} \setminus kB_{\mu,x} \) for some \( k \geq 1 \). Suppose first that \( k \leq 2A_0\beta \). Lemma 2.4 will then imply that
\[
(kB_{\mu,x})^c = B_\mu(x, kA_0^{-1})^c \subset B \left( x, \frac{k}{\beta A_0} \rho_\mu(x) \right)^c.
\]
Therefore,
\[
|x-y|^{-d} \lesssim k^{-d} \rho_\mu(x)^{-d} \leq \rho_\mu(x)^{-d}.
\]
Next, suppose that \( k > 2\beta A_0 \). For this case, Lemma 2.4 implies that
\[
(kB_{\mu,x})^c \subset B(x, 2\rho_\mu(x))^c
\]
and therefore
\[
|x-y|^{-d} \lesssim \rho_\mu(x)^{-d}.
\]
This produces the estimate,

\[ \frac{e^{-\varepsilon d_\mu(x,y)}}{|x-y|^d} \approx e^{-\delta k \rho_\mu(x)^d} \]

for any \( k \geq 1 \), where \( \delta := \varepsilon A_0^{-1} \). Since \( x \in B_j \), we have by Lemma 4.2

\[ \rho_\mu(x)^d \approx \rho_\mu(x_j)^d \left(1 + \frac{|x - x_j|}{\rho_\mu(x_j)}\right)^{d \kappa_0} \approx \rho_\mu(x_j)^d, \]

which implies that

\[ \frac{e^{-\varepsilon d_\mu(x,y)}}{|x-y|^d} \approx e^{-\delta k \rho_\mu(x_j)^d}. \]

Applying this estimate to \( J_1 \) gives

\[
J_1 \lesssim \sum_j \int_{B_j} \left( \sum_{k=1}^{\infty} e^{-\delta k \rho_\mu(x_j)^d} \int_{(k+1)B_{\mu,x}} |f(y)| \, dy \right)^p w(x) \, dx
\]

(21) Define \( B_{\mu,j} := B_\mu(x_j, \beta + A_0^{-1}) \) for \( j \in \mathbb{N} \). Notice that for \( x \in B_j \), Lemma 2.2 implies that \( d_\mu(x,x_j) \leq \beta \). Therefore, for \( k \geq 1 \) and \( y \in (k+1)B_{\mu,x} \),

\[
d_\mu(x_j,y) \leq d_\mu(x,x_j) + d_\mu(x,y) \leq \beta + A_0^{-1}(k+1) \leq (\beta + A_0^{-1})(k+1),
\]

which implies that \( (k+1)B_{\mu,x} \subset (k+1)B_{\mu,j} \). Therefore,

\[
J_1 \lesssim \sum_j \int_{B_j} \left( \sum_{k=1}^{\infty} e^{-\delta k \rho_\mu(x_j)^d} \int_{(k+1)B_{\mu,j}} |f(y)| \, dy \right)^p w(x) \, dx
\]

\[
\lesssim \sum_j \int_{B_{\mu,j}} \left( \sum_{k=1}^{\infty} e^{-\delta k \rho_\mu(x_j)^d} \int_{(k+1)B_{\mu,j}} |f(y)| \, dy \right)^p w(x) \, dx,
\]

where the last line follows from the inclusion \( B_j \subset B_{\mu,j} \) by Lemma 2.4. Hölder’s inequality then leads to,

\[
J_1 \lesssim \sum_j \int_{B_{\mu,j}} \left( \sum_{k=1}^{\infty} e^{-\delta k \rho_\mu(x_j)^d} \|f\|_{L^p((k+1)B_{\mu,j},w)} w^{-\frac{1}{p-1}} ((k+1)B_{\mu,j})^{\frac{p-1}{p}} \right)^p w(x) \, dx
\]

(22) Since \( w \in S^\mu_{p,c_1} \), we have the estimate

\[
w^{-\frac{1}{p-1}} ((k+1)B_{\mu,j})^{\frac{p-1}{p}} w(B_{\mu,j})^{\frac{1}{p}} \leq w^{-\frac{1}{p-1}} ((k+1)B_{\mu,j})^{\frac{p-1}{p}} w((k+1)B_{\mu,j})^{\frac{1}{p}}
\]

\[
\lesssim ((k+1)B_{\mu,j})^{e'((k+1))},
\]

where \( c' := c_1(\beta + A_0^{-1}) \). For \( j, k \in \mathbb{N} \), let \( B_{j,k} \) denote ball

\[ B_{j,k} := B \left(x_j, (\beta(k+1)(\beta + A_0^{-1}))^{k_0+1} \rho_\mu(x_j)\right). \]
Lemma 2.5 can then be applied to obtain

\[ (k + 1)B_{\mu,j} \subset B_{j,k} \]

and therefore

\[ |(k + 1)B_{\mu,j}| \lesssim (k + 1)^{d(k_0 + 1)}p(x_j)^d. \]

Combining this with (23) and (22) leads to

\[ J_1 \lesssim \sum_j e^{-\delta k} \rho(x_j)^{-d} \|f\|_{L^p((k+1)B_{\mu,j},w)} (k + 1)^{d(k_0 + 1)}p(x_j)^d e^{c'k} \]

\[ \lesssim \left( \sum_{k=1}^\infty e^{(c'-\delta)k} \|f\|_{L^p(B_{j,k},w)} \right)^{\frac{1}{p}} \]

so long as we choose \( c' = c_1(\beta + A_0^{-1}) < \varepsilon A_0^{-1} = \delta \). It remains to prove the boundedness of the global part of the Riesz transforms \( R_\mu \). However, this follows from the boundedness of \( R_{\mu,\text{glob}} \) using duality.

Proof of Part (ii). Assume that \( 0 < \delta_\mu < 1 \). The term \( J_1 \) can be handled in an identical manner to the case \( \delta_\mu > 1 \). It therefore suffices to demonstrate that there exists \( c_2 > 0 \) for which the term \( J_2 \) is bounded on \( L^p(w) \) for \( \eta' < p < \infty \) when \( w \in S_{\mu/\eta', c_2} \). From reasoning identical to the case of \( J_1 \),

\[ J_2 \leq \sum_j \int_{B_j} \left( \sum_{k=1}^\infty e^{-\delta k} \rho(x_j)^{-(d-1)} \int_{(k+1)B_{\mu,j}} \frac{d\mu(z)}{|z - y|^{d-1}} |f(y)| \, dy \right)^p w(x) \, dx \]

\[ \lesssim \sum_j \int_{B_j} \left( \sum_{k=1}^\infty e^{-\delta k} \rho(x_j)^{-(d-1)} \int_{(k+1)B_{\mu,j}} \frac{d\mu(z)}{|z - y|^{d-1}} |f(y)| \, dy \right)^p w(x) \, dx. \]
For any \( x \in B_j \), Hörder’s inequality implies that for \( \frac{1}{\gamma} := 1 - \frac{1}{p} - \frac{1}{q} \),
\[
(25) \quad \int_{(k+1)B_{\mu,x}} \left( \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} \right) |f(y)| \, dy \\
\lesssim \left\| \int_{B(y,|x-y|/2)} \frac{d\mu(z)}{|z-y|^{d-1}} \right\|_{L^q(B_{j,k})} \|f\|_{L^p(B_{j,k},w)} \left( \int_{(k+1)B_{\mu,x}} w^{-\gamma/p} \right)^{\frac{1}{\gamma}}.
\]
where we have used the inclusions \((k+1)B_{\mu,x} \subset (k+1)B_{\mu,j} \subset B_{j,k}\) and \(B(y,|x-y|/2) \subset 2B_{j,k}\) for \( y \in B_{j,k}\). For the first term in (25), successively apply [17, Lem. 7.9], the property (8) and Remark 4.2 to obtain
\[
(26) \quad \left\| \int_{2B_{j,k}} \frac{d\mu(z)}{|z-x|^{d-1}} \right\|_{L^q(2B_{j,k},dx)} \lesssim \frac{\mu(6B_{j,k})}{\rho_{\mu}(x_j)^{\frac{d}{q}-1}} \lesssim k^{M'} \rho_{\mu}(x_j)^{d-1-\frac{d}{q}},
\]
for some \( M' > 0 \). This implies that
\[
(27) \quad J_2 \lesssim \sum_j \left( \sum_{k=1}^{\infty} e^{-\delta k} \rho_{\mu}(x_j)^{-(d-1)k} k^{M'} \rho_{\mu}(x_j)^{d-1-\frac{d}{q}} \|f\|_{L^p(B_{j,k},w)} w^{-\gamma} ((k+1)B_{\mu,j})^{-\frac{1}{\gamma}} w(B_j)^{\frac{1}{p}} \right)^{p}.
\]
Our assumption \( w \in S_{p/q',c_2} \) then allows us to bound the term involving the weights by
\[
w^{-\gamma} ((k+1)B_{\mu,j})^{-\frac{1}{\gamma}} w(B_j)^{\frac{1}{p}} \leq w^{-\gamma} ((k+1)B_{\mu,j})^{-\frac{1}{\gamma}} w((k+1)B_{\mu,j})^{\frac{1}{\gamma}} \lesssim ((k+1)B_{\mu,j})^{-\frac{1}{\gamma}} e^{\frac{\gamma}{\beta}(k+1)(\beta+A_0^{-1})} \lesssim k^{M''} \rho_{\mu}(x_j)^{\frac{d}{q}} e^{\frac{\gamma}{\beta}(k+1)(\beta+A_0^{-1})} \lesssim k^{M''} \rho_{\mu}(x_j)^{\frac{d}{q}} e^{2c_2k},
\]
for some \( M'' > 0 \), where \( c_2 = c_2(\beta + A_0^{-1}). \) On applying this to (27),
\[
J_2 \lesssim \sum_j \left( \sum_{k=1}^{\infty} e^{-\delta_k} k^{M'} \rho_{\mu}(x_j)^{-\frac{d}{q}} \|f\|_{L^p(B_{j,k},w)} k^{M''} \rho_{\mu}(x_j)^{\frac{d}{q}} e^{2c_2k} \right)^{p}.
\]
The bounded overlap property of the balls \( \{ B_j \}_{j \in \mathbb{N}} \) will then imply that \( J_2 \lesssim \|f\|_{L^p(w)} \) provided that we choose \( c_2 \) small enough so that \( c_2 = c_2(\beta + A_0^{-1}) < \varepsilon A_0^{-1} = \delta \).

Notice that in Theorem 4.1, when \( \delta_{\mu} \in (0,1) \), the range of \( p \) for which \( R_\mu \) is bounded must be restricted to the interval \( (1, (2-\delta_{\mu})/(1-\delta_{\mu})) \). When \( d\mu(x) =
Suppose that Proposition 4.2. \(d\mu\) in [5, Thm. 3]. The following proposition improves the range of \(p\) for the case \(d\mu(x) = V(x) \, dx\). This proposition, when taken with the boundedness of \(R_{V,loc}\) by [5, Thm. 3], completes the proof of the second part of Theorem 1.1.

**Proposition 4.2.** Suppose that \(V \in RH_q\) for some \(\frac{d}{2} < q < d\) and define \(s\) through \(s = \frac{1}{q} - \frac{1}{d}\). There exists \(c > 0\) for which the operator \(R_{V,\text{glob}}\) is bounded on \(L^p(w)\) for \(\frac{d}{s} < p < \infty\) when \(w \in \mathcal{S}_{p'/s',c}\) and the operator \(R_V\) is bounded on \(L^p(w)\) for \(1 < p < s\) when \(w^{-\frac{1}{p'}} \in \mathcal{S}_V\). The constant \(c\) will depend on \(V\) only through \([V]_{RH_\frac{d}{2}}\) and will be independent of \(p\).

**Proof.** The proof is essentially identical to that of Theorem 4.3.(ii). The only difference is that in (26), Lemma 7.9 of [17] should no longer be used since this leads to a restricted range of \(p\). Instead, the boundedness of the classical fractional integral operator of order one from \(L^q\) into \(L^s\) should be exploited as in the proof of [5, Thm. 3]. Specifically replace (26) with

\[
\left\| \int_{2B_{j,k}} \frac{V(z)}{|z-x|^{d-1}} \, dz \right\|_{L^s(2B_{j,k},dx)} = \| I_0^1 (\mathbb{1}_{2B_{j,k}} V) \|_{L^s(2B_{j,k})} \lesssim \| I_{2B_{j,k}} V \|_q,
\]

where \(I_0^1\) is the classical fraction integral operator of order 1 and the well-known property that \(I_0^1\) is bounded from \(L^q\) into \(L^s\) is exploited. On applying the property \(V \in RH_q\), followed by (8) and Remark 4.2,

\[
\left\| \int_{2B_{j,k}} \frac{V(z)}{|z-x|^{d-1}} \, dz \right\|_{L^s(2B_{j,k},dx)} \lesssim |2B_{j,k}|^{-\frac{d}{s'}} \int_{2B_{j,k}} V \lesssim k^{M'} |2B_{j,k}|^{-\frac{d}{s'}} \int_{B_j} V \lesssim k^{M'} p_\rho(x_j)^{-\frac{d}{s'}} p_\rho(x_j)^{d-2} = k^{M'} p_\rho(x_j)^{d-1-\frac{d}{s'}},
\]

for some \(M' > 0\). The rest of the proof proceeds identically to Theorem 4.3. \(\square\)

4.3. **The Riesz Potentials.** Next, let’s consider the Riesz potentials for the operator \(L_\mu\) and prove the third part of Theorem 4.1. Notice that the pointwise estimate

\[
|I_\mu^\alpha f(x)| \leq I_0^\alpha |f|(x)
\]

holds for all \(f \in L^1_{\text{loc}}(\mathbb{R}^d)\) and \(x \in \mathbb{R}^d\). Therefore, in order to prove the boundedness of the operator \(I_\mu^\alpha,\text{loc}\) from \(L^p(w)\) to \(L^p(w^{\rho_\mu/p})\) for \(w^{\rho_\mu/p} \in \mathcal{S}_{1+\frac{d}{s},c}\) with \(c > 0\) and \(1 < p < \frac{d}{s}\) it is sufficient to prove the boundedness of \(I_0^\alpha,\text{loc}\). This follows on noting
that $S_{1+\frac{2}{\nu}}^{\mu} \subset A_{1+\frac{2}{\nu}}^{\mu,loc}$ for any $c > 0$ by Proposition 3.1 and $I_0^{\alpha,loc}$ is bounded from $L^p(w)$ to $L^p(w^{\nu/p})$ for $w^{\nu/p} \in A_{1+\frac{2}{\nu}}^{\mu,loc}$ and $1 < p < \frac{d}{\alpha}$ by [5, Thm. 1]. It remains to prove the boundedness of the global operators $I_0^{\mu,\text{glob}}$.

**Theorem 4.4.** Fix $0 < \alpha \leq 2$. There exists a constant $c > 0$ for which $I_0^{\mu,\text{glob}}$ is bounded from $L^p(w)$ to $L^p(w^{\nu/p})$ for all weights $w$ with $w^{\nu/p} \in S_{1+\frac{2}{\nu}}^{\mu,\text{glob}}$ and $1 < p < \frac{d}{\alpha}$, where $\frac{1}{p} = \frac{1}{p} - \frac{\alpha}{\nu}$. Moreover, the constant $c$ will be independent of $p$ and will depend on $\mu$ only through $C_\mu$, $D_\mu$ and $\delta_\mu$.

**Proof.** Let $K_{\mu,\alpha}(x,y)$ denote the singular integral kernel of the operator $I_0^{\mu}$. It will first be proved that $K_{\mu,\alpha}$ satisfies the following pointwise bound

$$K_{\mu,\alpha}(x,y) \lesssim e^{-\varepsilon d_\mu(x,y)} \frac{1}{|x-y|^{d-\alpha}} \quad \forall \, x, y \in \mathbb{R}^d,$$  

for some $\varepsilon > 0$ that depends on $\mu$ only through $C_\mu$, $D_\mu$ and $\delta_\mu$. The estimate is obviously satisfied for $\alpha = 2$ owing to the pointwise bound on the fundamental solution given by Theorem 4.2. Consider the case $\alpha \in (0, 2)$. The functional calculus (c.f. [11, pg. 286]) implies that

$$I_0^{\mu} f(x) = \frac{1}{\pi} \sin \left( \frac{\pi \alpha}{2} \right) \int_0^\infty \lambda^{-\alpha/2} (-\Delta + \mu + \lambda)^{-1} f(x) d\lambda$$

$$= \frac{1}{\pi} \sin \left( \frac{\pi \alpha}{2} \right) \int_0^\infty \lambda^{-\alpha/2} \int_{\mathbb{R}^d} \Gamma_{\mu + \lambda}(x,y) f(y) dy d\lambda$$

$$= \int_{\mathbb{R}^d} \left( \frac{1}{\pi} \sin \left( \frac{\pi \alpha}{2} \right) \int_0^\infty \lambda^{-\alpha/2} \Gamma_{\mu + \lambda}(x,y) d\lambda \right) f(y) dy$$

$$=: \int_{\mathbb{R}^d} K_{\mu,\alpha}(x,y) f(y) dy.$$  

The inequality (15) together with the pointwise bound on our fundamental solution implies

$$K_{\mu,\alpha}(x,y) \lesssim e^{-\varepsilon d_\mu(x,y)} \int_0^\infty \lambda^{-\alpha/2} e^{-\frac{\alpha}{2} \frac{\lambda^2}{\nu} |x-y|^2} d\lambda$$

$$\lesssim e^{-\varepsilon d_\mu(x,y)} \frac{1}{|x-y|^{d-\alpha}}$$

for some $\varepsilon > 0$, for all $x, y \in \mathbb{R}^d$. This completes the proof of the pointwise bound (28) for any $0 < \alpha \leq 2$.

For $j \in \mathbb{N}$, $k \geq 1$ and $x \in \mathbb{R}^d$, let the balls $B_j$, $B_{j,k}$, $B_{\mu,x}$ and $B_{\mu,j}$ be as defined in the proof of the Theorem 4.3. On expanding out $\|I_0^{\mu,\text{glob}} f\|_{L^p(w^{\nu/p})}^\nu$:

$$\|I_0^{\mu,\text{glob}} f\|_{L^p(w^{\nu/p})}^\nu \leq \sum_j \int_{B_j} \left( \int_{B_{\mu,x}} K_{\mu,\alpha}(x,y) |f(y)| dy \right)^\nu w^{\nu/p}(x) dx$$

$$= \sum_j \int_{B_j} \left( \sum_{k=1}^\infty \int_{(k+1)B_{\mu,x} \setminus kB_{\mu,x}} K_{\mu,\alpha}(x,y) |f(y)| dy \right)^\nu w^{\nu/p}(x) dx.$$
For \( j \in \mathbb{N}, k \geq 1 \) and \( x \in B_j \), Lemma 2.4 gives
\[
(kB_{\mu,x})^c \subset B(x, \frac{k}{\beta A_0} \rho_\mu(x))^c \cup B(x, 2\rho_\mu(x))^c.
\]
Therefore, for \( y \in (k+1)B_{\mu,x} \setminus kB_{\mu,x} \),
\[
K_{\mu,\alpha}(x,y) \lesssim \frac{e^{-\varepsilon d_\mu(x,y)}}{|x-y|^{d-\alpha}} \lesssim e^{-\delta k} \rho_\mu(x)^{-(d-\alpha)},
\]
where \( \delta := \varepsilon A_0^{-1} \). Since \( x \in B_j \) we then have by Lemma 4.2,
\[
(29) \quad K_{\mu,\alpha}(x,y) \lesssim e^{-\delta k} \rho(x)^{-(d-\alpha)}.
\]
Recall that it was proved in the proof of Theorem 4.3 that the inclusion \((k+1)B_{\mu,x} \subset (k+1)B_{\mu,j}\) holds for any \( j \in \mathbb{N}, x \in B_j \) and \( k \geq 1 \). Applying the kernel estimate \((29)\), followed by this inclusion and finally Hölder’s inequality gives
\[
\|I_{\mu,\text{glob}}^\alpha f\|_{L^\nu(w^{\nu/p})}^\nu \lesssim \sum_j \int_{B_j} \left( \sum_{k=1}^\infty e^{-\delta k} \rho_\mu(x)^{-(d-\alpha)} \int_{(k+1)B_{\mu,x} \setminus kB_{\mu,x}} |f(y)| \, dy \right)^\nu w^{\nu/p}(x) \, dx
\]
\[
\lesssim \sum_j \left( \sum_{k=1}^\infty e^{-\delta k} \rho_\mu(x)^{-(d-\alpha)} \|f\|_{L^\nu((k+1)B_{\mu,j},w)} w^{-\frac{1}{\nu'}} \left( (k+1)B_{\mu,j} \right)^{\frac{\nu-1}{\nu}} w^\frac{\nu}{\nu'}(B_j) \right)^\nu.
\]
Since \( B_j \subset B_{\mu,j} \) for each \( j \in \mathbb{N} \), we then obtain
\[
(30) \quad \|I_{\mu,\text{glob}}^\alpha f\|_{L^\nu(w^{\nu/p})}^\nu \lesssim \sum_j \left( \sum_{k=1}^\infty e^{-\delta k} \rho_\mu(x)^{-(d-\alpha)} \|f\|_{L^\nu((k+1)B_{\mu,j},w)} w^{-\frac{1}{\nu'}} \left( (k+1)B_{\mu,j} \right)^{\frac{\nu-1}{\nu}} w^\frac{\nu}{\nu'}(B_{\mu,j}) \right)^\nu.
\]
Using the assumption that \( w^\frac{\nu}{\nu'} \in S^{1,\frac{\nu}{\nu'},c}_1 \),
\[
w^{-\frac{1}{\nu'}} \left( (k+1)B_{\mu,j} \right)^{\frac{1}{\nu'}} w^\frac{\nu}{\nu'}(B_{\mu,j}) \frac{1}{\nu'} \leq w^{-\frac{1}{\nu'}} \left( (k+1)B_{\mu,j} \right)^{\frac{1}{\nu'}} w^\frac{\nu}{\nu'}((k+1)B_{\mu,j}) \frac{1}{\nu'}
\]
\[
\lesssim |(k+1)B_{\mu,j}|^{(1+\frac{\nu}{\nu'})} \frac{1}{\nu'} e(1+\frac{\nu}{\nu'}) (k+1)^{\frac{1}{2}(\beta + A_0^{-1})}.
\]
The estimate \((24)\) then gives
\[
w^{-\frac{1}{\nu'}} \left( (k+1)B_{\mu,j} \right)^{\frac{1}{\nu'}} w^\frac{\nu}{\nu'}(B_{\mu,j}) \frac{1}{\nu'}
\]
\[
\lesssim (k+1)^{\frac{d(k_0 + 1)}{2}(1+\frac{\nu}{\nu'})} \rho_\mu(x)^{d(1+\frac{\nu}{\nu'})} e(1+\frac{\nu}{\nu'}) (k+1)^{\frac{1}{2}(\beta + A_0^{-1})}
\]
\[
\lesssim \rho_\mu(x)^{d(1+\frac{\nu}{\nu'})} e^{c' k}
\]
\[
= \rho_\mu(x)^{d-\alpha} e^{c' k},
\]
where \( c' := 2c \left( 1 + \frac{\nu}{\nu'} \right) \frac{\beta}{\nu} \) and the final line follows from the equality \( d \left( 1 + \frac{\nu}{\nu'} \right) \frac{1}{\nu'} = d - \alpha \). Note that this equality also implies that \( c' = 2c\beta \left( \frac{d-\alpha}{d} \right) \) and therefore \( c' \) is
independent of $p$. Applying this estimate to (30) implies
\[
\|f_{\mu,\text{glob}}\|_{L^p(w^{\nu/p})} \lesssim \left( \sum_j \left( \sum_{k=1}^{\infty} \epsilon^{(c'-\delta)k} \|f\|_{L^p((k+1)B_{\mu,j},w)} \right)^{\nu} \right)^{\frac{1}{\nu}}
\]
\[
\lesssim \sum_{k=1}^{\infty} \epsilon^{(c'-\delta)k} \left( \sum_j \|f\|_{L^p((k+1)B_{\mu,j},w)} \right)^{\frac{1}{\nu}}
\]
\[
\lesssim \sum_{k=1}^{\infty} \epsilon^{(c'-\delta)k} \left( \sum_j \|f\|_{L^p(B_{j,k},w)} \right)^{\frac{1}{\nu}},
\]
where the inclusion $(k + 1)B_{\mu,j} \subset B_{j,k}$, as proved in Theorem 4.3, was used to obtain the final line. The bounded overlap property of the balls $\{B_j\}$ was used and the fact that $\nu \geq p$ will then complete our proof provided that we set $c$ small enough so that $c' < \delta$.

4.4. The Heat Maximal Operator. Let’s now move onto the boundedness of the heat maximal operator for $L_V$ and the proof of Theorem 1.2. Let $k_V^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the kernel of the operator $e^{-tL_V}$ so that
\[
e^{-tL_V} f(x) = \int_{\mathbb{R}^d} k_V^t(x, y) f(y) \, dy
\]
for all $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. This function is called the heat kernel for the operator $L_V$. We will require the following pointwise estimate for the heat kernel proved by K. Kurata in [12].

**Proposition 4.3** ([12, Thm. 1]). Suppose that $V \in RH_\frac{1}{2}$. There exist constants $D_0, D_1, D_2 > 0$ such that
\[
0 \leq k_V^t(x, y) \leq D_0 \cdot e^{-D_1 t \left( 1 + \frac{1}{\mu \nu^{1/2}} \right)^{\frac{3}{2}} \left( \frac{1}{t^2} e^{-D_2 t \frac{|x-y|^2}{t^2}} \right)}
\]
for all $x, y \in \mathbb{R}^d$ and $t > 0$, where $k_0$ is the constant from Lemma 2.1 corresponding to $\rho_V$.

Notice that the previous proposition implies the pointwise estimate
\[
T^*_V f(x) \leq T^*_0 f(x)
\]
for all $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Therefore, in order to prove the boundedness of the operator $T^*_V$ on $L^p(w)$ for $w \in \mathcal{H}_{p,c}^{V,m}$ with $m, c > 0$ it is sufficient to prove the boundedness of $T^*_0$ on $L^p(w)$. This follows on noting that $\mathcal{H}_{p,c}^{V,m} \subset \mathcal{A}_{p,\text{loc}}^{V,m}$ by Propositions 3.1 and 3.2 and that $T^*_0$ is bounded on $L^p(w)$ for weights in $\mathcal{A}_{p,\text{loc}}^{V,m}$ by [5, Thm. 1]. It remains to establish the boundedness of $T^*_V$, as proved in Proposition 4.3, for $L^p(w)$.

**Proposition 4.4.** There exists $c > 0$ such that the global part of the heat maximal operator $T^*_V$ is bounded on $L^p(w)$ for any $w \in \mathcal{H}_{p,c}^{V,m_0}$ where $1 < p < \infty$ and $m_0 := (2(k_0 + 1))^{-1}$. 

\[\]
Proof. Let \( c > 0, w \in \mathcal{H}_{p; \infty}^{V;} \) and fix \( f \in L^p(w) \). Let’s first obtain a pointwise estimate for \( T_{V;\text{glob}}^* f \). Let \( \{ B_j \} \), \( j \in \mathbb{N} = \{ B(x_j, \rho_V(x_j)) \} \), \( j \in \mathbb{N} \) be a cover of balls of \( \mathbb{R}^d \) as given in Proposition 2.1. Let \( B_x \) denote the critical ball \( B_x := B(x, \rho_V(x)) \). For \( x \in B_j \) with \( j \in \mathbb{N} \), Proposition 4.3 implies that

\[
T_{V;\text{glob}}^* f(x) \leq \sup_{t > 0} \int_{B_x^c} k_t^V (x, y) |f(y)| \, dy \leq \sup_{t > 0} \int_{B_x^c} \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} e^{-D_2 \frac{|x-y|^2}{t}} |f(y)| \, dy.
\]

Notice that since \( x \in B_j \) we have by Lemma 2.1,

\[
\Phi_{j, D_1}^V(\sqrt{t}, x) \lesssim \Phi_{j, D_1}^V(\sqrt{t}, x)
\]

for some \( D_1' > 0 \), for all \( t > 0 \). This then gives

\[
T_{V;\text{glob}}^* f(x) \lesssim \sup_{t > 0} \int_{B_x^c} \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} \sum_{k=1}^\infty \int_{(k+1)B_x \setminus kB_x} e^{-D_2 \frac{|x-y|^2}{t}} |f(y)| \, dy
\]

\[
\lesssim \sup_{t > 0} \sum_{k=1}^\infty \int_{(k+1)B_x \setminus kB_x} \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} e^{-D_2 \frac{|x-y|^2}{t}} |f(y)| \, dy.
\]

Since \( x \in B_j \) we must have \( \rho_V(x_j) \simeq \rho_V(x) \) by Lemma 2.1 and therefore

\[
e^{-D_2 \frac{|x-y|^2}{t}} \lesssim e^{-D_2 \frac{|x-y|^2}{t}}
\]

for some \( D_2' > 0 \). On successively applying this, the inclusion \( B_j \subset \tilde{B}_j := 2\sigma B_j \) where \( \sigma := \beta/\max(\rho_V) \) and Hölder’s inequality,

\[
T_{V;\text{glob}}^* f(x) \lesssim \sup_{t > 0} \sum_{k=1}^\infty \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} e^{-D_2 \frac{|x-y|^2}{t}} \int_{(k+1)B_x \setminus kB_x} |f(y)| \, dy
\]

\[
\lesssim \sup_{t > 0} \sum_{k=1}^\infty \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} e^{-D_2 \frac{|x-y|^2}{t}} \int_{(k+1)B_x} |f(y)| \, dy
\]

\[
\lesssim \sup_{t > 0} \sum_{k=1}^\infty \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} e^{-D_2 \frac{|x-y|^2}{t}} \left\| f \right\|_{L^p((k+1)\tilde{B}_j, w)} w^{-\frac{1}{p'}} \left( (k+1)\tilde{B}_j \right)^{\frac{p-1}{p}}.
\]

This pointwise estimate then allows us to estimate our norm by

(31)

\[
\left\| T_{V;\text{glob}}^* f \right\|_{L^p(w)}^p \leq \sum_j \int_{B_j} T_{V;\text{glob}}^* f(x)^p w(x) \, dx
\]

\[
\lesssim \sum_j \left( \sup_{t > 0} \sum_{k=1}^\infty \Phi_{j, D_1}^V(\sqrt{t}, x)^{-1} e^{-D_2 \frac{|x-y|^2}{t}} \left\| f \right\|_{L^p((k+1)\tilde{B}_j, w)} w^{-\frac{1}{p'}} \left( (k+1)\tilde{B}_j \right)^{\frac{p-1}{p}} w(B_j)^{\frac{1}{p'}} \right)^p.
\]
The condition \( w \in H^{\nu,\mu}_0 \) implies that
\[
(w(B_j)^{\frac{1}{2}}w^{-\frac{1}{2}}(k+1)B_j)^{\frac{p-1}{p}} \leq w\left((k+1)\tilde{B}_j\right)^{\frac{p-1}{p}} \leq e^{c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d} \]
\[
\leq e^{c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d}.
\]

Applying this estimate to (31) then gives
\[
(32) \quad \left\| T_{V,\text{glob}} f \right\|^p_{L_p(w)} \leq \sum_j \left( \sup_{t>0} \sum_{k=1}^{\infty} \Phi_{2m_0,D'}(\sqrt{t},x_j)^{-1} e^{-D_2' k^2 \rho_{V}(x_j)^2} \frac{t}{2} \right)^d \left\| f \right\|_{L_p((k+1)\tilde{B}_j, w)} e^{c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d}.
\]

Define, for \( t > 0, j \in \mathbb{N} \) and \( k \in \mathbb{N}^* \),
\[
F(t, j, k) := \Phi_{2m_0,D'}(\sqrt{t},x_j)^{-1} e^{-D_2' k^2 \rho_{V}(x_j)^2} e^{c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d} \left( \frac{k \rho_{V}(x_j)}{t^{\frac{1}{2}}} \right)^d
\]
\[
= \left( \frac{k \rho_{V}(x_j)}{t^{\frac{1}{2}}} \right)^d \exp \left( -D_1' \left( 1 + \frac{\sqrt{t}}{\rho_{V}(x_j)} \right)^{2m_0} - D_2' \frac{k^2 \rho_{V}(x_j)^2}{2t} + c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d \right).
\]

It will now be proved that if \( c > 0 \) is small enough then there will exist \( \varepsilon > 0 \) for which
\[
(33) \quad F(t, j, k) \leq e^{-\varepsilon k^m_0}
\]
for all \( t > 0, j \in \mathbb{N} \) and \( k \in \mathbb{N}^* \). First note that from the estimate \( x^d \leq e^{ax^2} \) for any \( a > 0 \)
\[
F(t, j, k) \leq \exp \left( -D_1' \left( 1 + \frac{\sqrt{t}}{\rho_{V}(x_j)} \right)^{2m_0} - D_2' \frac{k^2 \rho_{V}(x_j)^2}{2t} + c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d \right)
\]
\[
\leq \exp \left( -D_1' \left( \frac{t}{\rho_{V}(x_j)^2} \right)^{m_0} - D_2' \frac{k^2 \rho_{V}(x_j)^2}{2t} + c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d \right).
\]

Consider the case \( t \geq k \rho_{V}(x_j)^2 \). For this case, we will have the estimate
\[
F(t, j, k) \leq \exp \left( -D_1' k^{m_0} + c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d \right).
\]
If we let \( c \) be small enough, namely \( c < \frac{D_1'}{(8\sigma)^{m_0}} \), then we will have
\[
(34) \quad F(t, j, k) \leq e^{-\varepsilon_1 k^{m_0}}
\]
for some \( \varepsilon_1 > 0 \). Next, consider the case \( t < k \rho_{V}(x_j)^2 \). In this situation, we will have
\[
F(t, j, k) \leq \exp \left( -D_2' \frac{k}{2} + c(8\sigma)^{m_0}k^d \rho_{V}(x_j)^d \right).
\]
Since \( m_0 < 1 \) we will then have
\[
F(t, j, k) \leq e^{-\varepsilon_2 k}
\]
for some $\varepsilon_2 > 0$. Putting this together with (34) then gives (33) for all $t > 0$, $j \in \mathbb{N}$ and $k \in \mathbb{N}^*$. Referring back to (32), we can apply (33) to obtain
\[
\|T_{\text{glob}}^* f\|_{L^p(w)} \lesssim \left( \sum_{k=1}^{\infty} e^{-\varepsilon k^{m_0}} \|f\|_{L^p((k+1)\tilde{B}_j, w)} \right)^{\frac{1}{p}}.
\]

It then follows from the bounded overlap property of the balls $B_j$, Proposition 2.1, that
\[
\|T_{\text{glob}}^* f\|_{L^p(w)} \lesssim \left( \sum_{k=1}^{\infty} k^{N_1} e^{-\varepsilon k^{m_0}} \|f\|_{L^p(R^d, w)} \right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(w)}.
\]

5. Uniformly Elliptic Operators with Potential

Let $A$ be a $d \times d$ matrix-valued function with real-valued coefficients in $L^\infty (\mathbb{R}^d)$. Suppose that $A$ satisfies the ellipticity condition
\[
\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2
\]
for some $\lambda, \Lambda > 0$, for all $\xi \in \mathbb{R}^d$ and for almost every $x \in \mathbb{R}^d$. In this section we consider the uniformly elliptic operator with potential $V \in RH^2_{d/2}$,
\[
L_{A,V} := -\text{div} A\nabla + V.
\]
This operator is defined through its corresponding sesquilinear form as an unbounded operator on $L^2(\mathbb{R}^d)$ with maximal domain. Similar to the perturbation free case, appropriate analogues of the usual operators can be defined,
\[
R_{A,V} := \nabla L_{A,V}^{-\frac{\alpha}{2}}, \quad R_{A,V}^* := L_{A,V}^{-\frac{\alpha}{2}} \nabla, \quad R_{A,V}^\alpha := L_{A,V}^{-\frac{\alpha}{2}},
\]
for $0 < \alpha \leq 2$ and
\[
T_{A,V}^\alpha f(x) := \sup_{t>0} e^{-tL_{A,V}} |f| (x) \quad \text{for } f \in L^1_{\text{loc}}(\mathbb{R}^d), \ x \in \mathbb{R}^d.
\]
In this section we will prove weighted estimates for these operators that are analogous to Theorems 1.1 and 1.2. Before attempting to do so we must first discuss exponential decay estimates for the associated kernels.

5.1. Exponential Decay Estimates. In a similar manner to the perturbation free case of Section 4, the weighted results for $L_{A,V}$ will be proved using exponential decay estimates for the relevant kernels.

Let $\Gamma_{A,V}$ denote the fundamental solution of the operator $L_{A,V}$. This is a function defined on $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq y\}$ with the properties that $\Gamma_{A,V} (\cdot, y) \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $L_{A,V} \Gamma_{A,V} (\cdot, y) = \delta_y$ for each $y \in \mathbb{R}^d$, where $\delta_y$ is the Dirac delta distribution with pole at $y$. Refer to [7] for the construction of this object. The following exponential decay estimates for $\Gamma_{A,V}$ were proved by S. Mayboroda and
B. Poggi in [13]. These estimates are a generalisation of the perturbation free case proved by Shen in [17] (see Theorem 4.2).

**Theorem 5.1** ([13, Cor. 6.16]). Let $A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d))$ have real-valued coefficients and assume that it satisfies the ellipticity condition (35). Fix $V \in RH_{d/2}$.

There exist constants $C_1, C_2, \varepsilon_1, \varepsilon_2 > 0$ for which

$$C_1 \frac{e^{-\varepsilon_1 dV(x,y)}}{|x-y|^{d-2}} \leq \Gamma_{A,V}(x,y) \leq C_2 \frac{e^{-\varepsilon_2 dV(x,y)}}{|x-y|^{d-2}}$$

for all $x, y \in \mathbb{R}^d$. The constants $C_1, C_2, \varepsilon_1$ and $\varepsilon_2$ will depend on $V$ only through $[V]_{RH_{d/2}}$.

Using the previous theorem, it is then possible to estimate the derivative of $\Gamma_{A,V}$ from above provided that $A$ is Hölder continuous.

**Proposition 5.1.** Let $A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d))$ have real-valued coefficients and assume that it satisfies the ellipticity condition (35). Suppose also that $A$ is $\gamma$-Hölder continuous for some $\gamma \in (0, 1)$. Fix $V \in RH_q$ for some $q > d/2$.

(i) There exist constants $C_1, \varepsilon_1 > 0$ for which

$$|\nabla \Gamma_{A,V}(x,y)| \leq C_1 \frac{e^{-\varepsilon_1 dV(x,y)}}{|x-y|^{d-2}} \left(1 + \frac{1}{|x-y|} + \int_{B(x,|x-y|/2)} \frac{V(z)}{|x-z|^{d-1}} d\mu(z)\right)$$

for all $x, y \in \mathbb{R}^d$.

(ii) Suppose that $q \geq d$. Then $C_2, \varepsilon_2 > 0$ can be chosen so that

$$|\nabla \Gamma_{A,V}(x,y)| \leq C_2 \frac{e^{-\varepsilon_2 dV(x,y)}}{|x-y|^{d-2}} \left(1 + \frac{1}{|x-y|}\right)$$

for all $x, y \in \mathbb{R}^d$.

The constants $\varepsilon_1$ and $\varepsilon_2$ will depend on $V$ only through $[V]_{RH_{d/2}}$.

**Proof.** The estimate for $q \geq d$ was proved in [3, Thm. 3.2] which itself was an adaptation of the perturbation free arguments from [17, Lem. 2.20]. Let’s apply this argument to the case $q > d/2$. Fix $x, y \in \mathbb{R}^d$ with $x \neq y$ and define $R := |x-y|/2$. Let $r \leq R$. Define

$$u(\xi) := \Gamma_{A,V}(\xi, y)$$

for $\xi \in B(x,R)$. Clearly $u$ is a weak solution to $L_{A,V} u = 0$ on the ball $B(x,R)$. Next define

$$v(\xi) := u(\xi) + \int_{B(x,r)} \Gamma_{A,0}(\xi,z) u(z) V(z) dz$$

for $\xi \in B(x,R)$. From reasoning identical to that of [3, Thm. 3.2], $v$ will be $L_{A,0}$-harmonic in $B := B(x,r)$. Theorem 2.1 of [3] then implies that

$$\|\nabla v\|_{L^\infty(B)} \lesssim \frac{1}{r} \|v\|_{L^\infty(B)}.$$
Therefore,

$$|\nabla \Gamma_{A,V}(x,y)| = |\nabla u(x)| \leq |\nabla v(x)| + \int_B |\nabla \Gamma_{A,0}(x,z)| |u(z)| V(z) \, dz$$

$$\lesssim \frac{1}{r} \|v\|_{L^\infty(B)} + \|u\|_{L^\infty(B)} \int_B |\nabla \Gamma_{A,0}(x,z)| V(z) \, dz$$

$$\lesssim \frac{1}{r} \|u\|_{L^\infty(B)} \left(1 + \sup_{\xi \in B} \int_B |\Gamma_{A,0}(\xi,z)| V(z) \, dz\right)$$

$$+ \|u\|_{L^\infty(B)} \int_B |\nabla \Gamma_{A,0}(x,z)| V(z) \, dz.$$ 

For $\xi \in B$, by [3, Lem. 2.2], Lemma 4.1 and the inclusion $B \subset B(\xi, 2r) \subset B(x, 4r)$,

$$\int_B |\Gamma_{A,0}(\xi,z)| V(z) \, dz \lesssim \int_B \frac{V(z)}{|\xi - z|^{d-2}} \, dz \leq \int_{B(\xi,2r)} \frac{V(z)}{|\xi - z|^{d-2}} \, dz$$

$$\lesssim V(B(\xi,2r)) \lesssim \frac{V(B(x,4r))}{r^{d-2}}$$

$$\lesssim \frac{V(B(x,r))}{r^{d-2}},$$

where the last line follows from the doubling property of $V$. Lemma 2.3 of [3] also gives

$$\int_B |\nabla \Gamma_{A,0}(x,z)| V(z) \, dz \lesssim \int_B \frac{V(z)}{|x - z|^{d-1}} \, dz + \int_B \frac{V(z)}{|x - z|^{d-2}} \, dz$$

$$\lesssim \int_B \frac{V(z)}{|x - z|^{d-1}} \, dz + \frac{V(B(x,r))}{r^{d-2}}.$$

Putting everything together implies that for any $r \leq |x - y|/2$,

$$|\nabla \Gamma_{A,V}(x,y)| \lesssim \|u\|_{L^\infty(B(x,r))} \left(\frac{1}{r} + \frac{V(B(x,r))}{r^{d-1}} + \frac{V(B(x,r))}{r^{d-2}} + \int_{B(x,r)} \frac{V(z)}{|z - x|^{d-1}} \, dz\right).$$

Then, by setting $r = |x - y|/2$ for the case $|x - y| \leq 2\rho_V(x)$ and $r = \rho_V(x)$ for the case $2\rho_V(x) < |x - y|$ as in [3, Thm. 3.2], this estimate will imply part (i) of our proposition.

Consider the fundamental solution of the potential free operator $-\text{div} A \nabla$, $\Gamma_{A,0}$. An interesting consequence of the presence of the perturbation $A$ is that, in contrast to the Laplacian case, the derivative of the fundamental solution is no longer guaranteed to be bounded universally from above by a constant multiple of $|x - y|^{-(d-1)}$. Instead, $|\nabla \Gamma_{A,0}(x,y)|$ is only guaranteed to satisfy this estimate locally. At a global scale, we can only assert that

\begin{equation}
|\nabla \Gamma_{A,0}(x,y)| \lesssim \frac{1}{|x - y|^{d-2}} \quad \text{for all } x, y \in \mathbb{R}^d \text{ s.t. } |x - y| \geq 1.
\end{equation}

See [3] for a proof of this bound. A corollary of this decrease in the strength of the global decay is that the Riesz transform operator $R_{A,0} := \nabla (-\text{div} A \nabla)^{-\frac{1}{2}}$ will no longer necessarily be Calderón-Zygmund and therefore it will not necessarily be bounded on $L^p(w)$ for all weights $w$ in the classical Muckenhoupt class $A_p$ (see [18], Remarks 1.7 and 1.8). The perturbation $A$ and potential $V$ therefore have two directly opposing effects on the underlying Muckenhoupt class. The inclusion of
the perturbation $A$ decreases the size of the associated weight class to weights that have less global decay, while the potential $V$ will increase the size of the class to include weights of greater global decay. These two effects can be played against each other and, for large enough $V$, the effect of the perturbation $A$ can be effectively cancelled out by the potential. This interaction between the perturbation $A$ and the potential $V$ is precisely what is responsible for the validity of the below corollary. This corollary, and therefore the cancellation effect, will play an important role when we come to consider the boundedness of the Riesz transforms $R_{A,V}$.

**Corollary 5.1.** Let $A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d))$ have real-valued coefficients and assume that it satisfies the ellipticity condition (35). Suppose also that $A$ is $\gamma$-Hölder continuous for some $\gamma \in (0, 1)$. Fix $V \in RH_q$ for some $q > \frac{d}{2}$. Suppose that there exists $D_V > 0$ for which $\rho_V(x) \leq D_V$ for all $x \in \mathbb{R}^d$.

(i) There must exist constants $C_1, \varepsilon_1 > 0$ for which

$$|\nabla \Gamma_{A,V}(x,y)| \leq C_1 e^{-\varepsilon_1 d_V(x,y)} \left( \frac{V(z)}{|x-y|^{d-2}} \int_{B(x, |x-y|/2)} \frac{1}{|z-x|^{d-1}} dz + \frac{1}{|x-y|} \right)$$

for all $x, y \in \mathbb{R}^d$.

(ii) Assume that $q \geq d$. Then $C_2, \varepsilon_2 > 0$ can be chosen so that

$$|\nabla \Gamma_{A,V}(x,y)| \leq C_2 e^{-\varepsilon_2 d_V(x,y)}$$

for all $x, y \in \mathbb{R}^d$.

The constants $\varepsilon_1$ and $\varepsilon_2$ will depend on $V$ only through $[V]_{RH_\frac{d}{2}}$.

**Proof.** Let $x, y \in \mathbb{R}^d$. Suppose first that $|x-y| \leq 2\rho_V(x)$. Then

$$\frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-2}} = \frac{|x-y|}{|x-y|^{d-1}} \frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-1}} \leq 2\rho_V(x) \frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-1}} \lesssim D_V \frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-1}}.$$

Next, suppose that $|x-y| > 2\rho_V(x)$. For this case,

$$\frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-2}} = \rho_V(x) \frac{|x-y|}{|x-y|^{d-1}} \frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-1}} \lesssim D_V \frac{|x-y|}{\rho_V(x)} \frac{e^{-\varepsilon_d V(x,y)}}{|x-y|^{d-1}} \lesssim D_V \frac{e^{-\varepsilon^\prime d_V(x,y)}}{|x-y|^{d-1}}.$$

for any $\varepsilon^\prime \in (0, \varepsilon)$, where we applied Lemma 2.3 in the second to last line. The result then follows from Proposition 5.1. $\blacksquare$
The condition $\rho_V \leq D_V$ ensures that the size of the potential is sufficiently large to cancel out the negative effect of the perturbation $A$. Without this size condition the corollary will not necessarily be valid and as a result the operator $R_{A,V}$ might not be bounded on $L^p(w)$ for all $w \in A_p$.

Finally, the following exponential decay estimate for the heat kernel $k^A,V$ of $L_{A,V}$ will be used in the proof of Theorem 5.5.

**Theorem 5.2 ([12, Thm. 1]).** Suppose that $V \in RH_{\frac{d}{2}}$. Let $A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d))$ have real-valued coefficients. Assume that $A$ satisfies the ellipticity condition (35) and $A = A^*$. There exist constants $D_0, D_1, D_2 > 0$ such that

$$0 \leq k^A,V(x,y) \leq D_0 \cdot e^{-D_1 \left(1 + \frac{\rho^2 V(x,y)}{p + 1} \right) \frac{1}{1 + \frac{\rho^2 V(x,y)}{p + 1}} \left( \frac{1}{1 + \frac{\rho^2 V(x,y)}{p + 1}} \right)^{\frac{1}{2}} e^{-D_2 |x-y|^2}}$$

for all $x,y \in \mathbb{R}^d$ and $t > 0$, where $k_0$ is the constant from Lemma 2.3 corresponding to $\rho_V$.

5.2. **Riesz Transform.** For the Riesz transforms $R_{A,V}$ and their adjoints $R^*_{A,V}$, our main result is as follows.

**Theorem 5.3.** Let $A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d))$ have real-valued coefficients and assume that it satisfies the ellipticity condition (35). Suppose also that $A$ is $\gamma$-Hölder continuous for some $\gamma \in (0,1)$. That is, there exists some $\tau > 0$ so that for any $x,y \in \mathbb{R}^d$

$$|A(x) - A(y)| \leq \tau |x-y|^\gamma.$$

Fix $V \in RH_{\frac{d}{2}}$ for some $q > \frac{d}{2}$. Assume that $\rho_V(x) \leq D_V$ for all $x \in \mathbb{R}^d$, for some $D_V > 0$. The following statements are true.

(i) If $q \geq d$ then there exists $c_1 > 0$ for which both $R_{A,V}$ and $R^*_{A,V}$ are bounded on $L^p(w)$ for all $w \in S^V_{p,c_1}$ with $1 < p < \infty$.

(ii) Suppose instead that $\frac{d}{2} < q < d$ and let $s$ be defined through $\frac{1}{s} = \frac{1}{q} - \frac{1}{2}$. Then there exists a constant $c_2 > 0$ for which the operator $R^*_{A,V}$ is bounded on $L^p(w)$ for $s' < p < \infty$ when $w \in S^V_{p',s',c_2}$ and the operator $R_{A,V}$ is bounded on $L^p(w)$ for $1 < p < s$ when $w^{-\frac{1}{1-s}} \in S^V_{p',s',c_2}$.

In each of the above statements, the constants $c_1$ and $c_2$ are independent of $p$ and depend on $V$ only through $[V]_{RH_{\frac{d}{2}}}$ and $D_V$.

**Remark 5.1.** The condition $\rho_V(x) \leq D_V$ for all $x \in \mathbb{R}^d$ is a size condition on the potential. From equation (0.13) of [16], we know that if $V = P$ is a non-negative polynomial of degree $k \in \mathbb{N}$ then

$$\rho_V(x)^{-1} = \sum_{|\alpha| \leq k} |\partial^\alpha_k P(x)|^{\frac{1}{s+1}}.$$

Therefore any non-negative polynomial will satisfy the size condition $\rho_V \leq D_V$.

Throughout this section fix $A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d))$ with real-valued coefficients that satisfies (35) and assume that $A$ is $\gamma$-Hölder continuous for some $\gamma \in (0,1)$. The notation $K_{A,0}$ and $K_{A,V}$ will be used to denote the singular kernels of the
operators $R_{A,0}$ and $R_{A,V}$ respectively. Similarly, let $K_{A,0}^\ast$ and $K_{A,V}^\ast$ be the kernels for $R_{A,0}^\ast$ and $R_{A,V}^\ast$ respectively. The operator $L_{A,V}^{-\frac{1}{2}}$ can be expressed as

$$L_{A,V}^{-\frac{1}{2}} f(x) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} L_{A,V}^{-1} f(x) d\lambda.$$ 

Refer to [11, pg. 281] for a proof of this formula. Fubini’s Theorem then implies

$$R_{A,V} f(x) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \nabla L_{A,V}^{-1} f(x) d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^d} \Gamma_{A,V} f(y) g(y) d\lambda$$

$$= \int_{\mathbb{R}^d} \left( \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \nabla \Gamma_{A,V} f(y) d\lambda \right) g(y) dy.$$ 

This leads to the expression

$$K_{A,V}(x,y) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \nabla \Gamma_{A,V} f(y) d\lambda$$

for all $x, y \in \mathbb{R}^d$. It follows from duality that

$$R_{A,V}^\ast f(x) = \int_{\mathbb{R}^d} K_{A,V}^\ast(x,y) f(y) dy = \int_{\mathbb{R}^d} K_{A,V}(y,x) f(y) dy.$$ 

Then, since $K_{A,V}$ is real-valued,

$$K_{A,V}^\ast(x,y) = K_{A,V}(y,x)$$

for all $x, y \in \mathbb{R}^d$. Similarly we will have

$$K_{A,0}^\ast(x,y) = K_{A,0}(y,x)$$

for all $x, y \in \mathbb{R}^d$.

As usual, to prove the boundedness of the operators $R_{A,V}$ and $R_{A,V}^\ast$ it is sufficient to consider the local and global behaviour separately. For the local behaviour, a few preliminary results must first be proved.

**Proposition 5.2.** There exists a constant $\varepsilon > 0$ such that for fixed $R > 0$ we have

$$|\nabla \Gamma_{A,\lambda}(x,y) - \nabla \Gamma_{A,\lambda}(x',y)| \lesssim_R \left( \frac{|x-x'|}{|x-y|} \right)^\gamma e^{-\varepsilon \sqrt{\lambda|x-y|}} |x-y|^{\frac{d-1}{2}}$$

for all $\lambda > 0$, $y \in \mathbb{R}^d$ and $x, x' \in B(y,R)$ with $|x-x'| \leq \frac{1}{2} |x-y|$. An identical estimate will hold for $|\nabla \Gamma_{A,\lambda}(y,x) - \nabla \Gamma_{A,\lambda}(y,x')|$.

**Proof.** Fix $R > 0$. In [3, Prop. 3.3], it was proved that

$$|\nabla \Gamma_{A,\lambda}(x,y) - \nabla \Gamma_{A,\lambda}(x',y)| \lesssim_R \left( \frac{|x-x'|}{|x-y|} \right)^\gamma \sup_{\xi \in B(x,\frac{1}{2}|x-y|)} |\Gamma_{A,\lambda}(\xi,y)| \left( 1 + \lambda |x-y|^2 \right)$$

(38)
for all \( \lambda > 0 \), \( y \in \mathbb{R}^d \) and \( x, x' \in B(y, R) \) with \( |x - x'| \leq \frac{1}{2} |x - y| \). Theorem 5.1 then implies that there exists \( \varepsilon > 0 \) such that for any \( \xi \in B(x, \frac{1}{3} |x - y|) \)

\[
|\Gamma_{A, \lambda}(\xi, y)| \lesssim \frac{e^{-\varepsilon \sqrt{|\lambda|}|\xi - y|}}{|\xi - y|^{d-2}} \lesssim \frac{e^{-\frac{4}{7} \sqrt{|\lambda|}|\xi - y|}}{|\xi - y|^{d-2}},
\]

where we have used the identity \( d_A(\xi, y) = \lambda^{\frac{2}{7}} |\xi - y| \). This together with (38) produces our result.

\[\square\]

**Corollary 5.2.** The kernel \( K_{A,0} \) is locally Calderón-Zygmund. That is, for each \( R > 0 \)

\[
|K_{A,0}(x, y)| \lesssim_R |x - y|^{-d}
\]

for all \( x, y \in \mathbb{R}^d \) with \( |x - y| \leq R \) and

\[
|K_{A,0}(x, y) - K_{A,0}(x', y)| + |K_{A,0}(y, x) - K_{A,0}(y, x')| \lesssim \frac{|x - x'|^\gamma}{|x - y|^{d+\gamma}}
\]

for all \( x, x', y \in \mathbb{R}^d \) with \( x, x' \in B(y, R) \) and \( 2|x - x'| \leq |x - y| \).

**Proof.** For the size estimates, Proposition 5.1 combined with the expression (37) lead to

\[
|K_{A,0}(x, y)| \lesssim \int_0^\infty \lambda^{-\frac{1}{2}} |\nabla \Gamma_{A, \lambda}(x, y)| d\lambda
\]

\[
\lesssim_R \int_0^\infty \lambda^{-\frac{1}{2}} e^{-\varepsilon \sqrt{|\lambda|}|x - y|} \frac{1}{|x - y|^{d-1}} d\lambda
\]

\[
\lesssim \frac{1}{|x - y|^d}.
\]

For the regularity estimate, the expression (37) implies that

\[
|K_{A,0}(x, y) - K_{A,0}(x', y)| \lesssim \int_0^\infty \lambda^{-\frac{1}{2}} |\nabla \Gamma_{A, \lambda}(x, y) - \nabla \Gamma_{A, \lambda}(x', y)| d\lambda.
\]

Taken with Proposition 5.2, this gives

\[
|K_{A,0}(x, y) - K_{A,0}(x', y)| \lesssim \int_0^\infty \lambda^{-\frac{1}{2}} |\nabla \Gamma_{A, \lambda}(x, y) - \nabla \Gamma_{A, \lambda}(x', y)| d\lambda
\]

\[
\lesssim_R \left( \frac{|x - x'|}{|x - y|} \right)^\gamma \frac{1}{|x - y|^{d-1}} \int_0^\infty \lambda^{-\frac{1}{2}} e^{-\varepsilon \sqrt{|\lambda|}|x - y|} d\lambda
\]

\[
\lesssim \left( \frac{|x - x'|}{|x - y|} \right)^\gamma \frac{1}{|x - y|^d}.
\]

The difference \( |K_{A,0}(y, x) - K_{A,0}(y, x')| \) is estimated similarly. \[\square\]

**Lemma 5.1.** Let \( V \in RH_\gamma \) for some \( q > \frac{d}{2} \). For fixed \( R > 0 \),

\[
|K_{A,0}^*(x, y) - K_{A,0}^*(x, y)| \lesssim_R \frac{1}{|x - y|^{d-1}} \int_{B(x, |x-y|/2)} \frac{V(z)}{|z - x|^{d-1}} dz + \left( \frac{|x - y|}{\rho V(x)} \right)^{2-d} \frac{1}{|x - y|^d}.
\]

\[\square\]
for all \(x, y \in \mathbb{R}^d\) with \(|x - y| \leq \min (\rho_V(x), R)\).

**Proof.** The perturbation free case of \(-\Delta + V\) was shown in the proof of [17, Lem. 7.13]. This argument will be generalised to our setting. It is straightforward to prove using the uniqueness property of the fundamental solution and the property \(L_A^V \Gamma_{A,V}(x,y) = \delta_y\) for all \(y \in \mathbb{R}^d\), where \(\delta_y\) is the Dirac delta distribution, that

\[
\Gamma_{A,\lambda}(y, x) = \Gamma_{A, V + \lambda}(y, x) + \int_{\mathbb{R}^d} \Gamma_{A, \lambda}(y, z) \Gamma_{A, V + \lambda}(z, x) V(z) \, dz
\]

for almost every \((x,y) \in \mathbb{R}^d \times \mathbb{R}^d\), for any \(\lambda > 0\). Theorem 5.1, Proposition 5.1.(ii) and (15) then imply

\[
|\nabla \Gamma_{A, V + \lambda}(y, x) - \nabla \Gamma_{A, \lambda}(y, x)| \leq \int_{\mathbb{R}^d} |\nabla \Gamma_{A', \lambda}(z, y)| \Gamma_{A, V + \lambda}(z, x) V(z) \, dz
\]

\[
\lesssim \int_{\mathbb{R}^d} \frac{e^{-\varepsilon \sqrt{\lambda}|y - z|}}{|y - z|^{d-2}} \left(1 + \frac{1}{|y - z|}\right) \frac{e^{-\varepsilon \sqrt{\lambda}|x - z|} e^{-\varepsilon d V(z, x)}}{|x - z|^{d-2}} V(z) \, dz
\]

\[
\lesssim \mathcal{R} \left( \int_{B(y, \mathcal{R})} \frac{e^{-\varepsilon \sqrt{\lambda}|y - z|} e^{-\varepsilon \sqrt{\lambda}|x - z|} e^{-\varepsilon d V(z, x)}}{|y - z|^{d-1}} V(z) \, dz + \int_{\mathbb{R}^d \setminus B(y, \mathcal{R})} \frac{e^{-\varepsilon \sqrt{\lambda}|y - z|} e^{-\varepsilon \sqrt{\lambda}|x - z|} e^{-\varepsilon d V(z, x)}}{|y - z|^{d-2}} V(z) \, dz \right)
\]

\[
:= J_1 + J_2.
\]

The term \(J_1\) is precisely the quantity obtained in the proof of Lemma 7.13 of [17] and can therefore be estimated from above by

\[
J_1 \lesssim e^{-\varepsilon \sqrt{\lambda} r} \left( \frac{1}{r^{d-2}} \int_{B(y, r)} \frac{V(z) \, dz}{|z - y|^{d-1}} + \left( \frac{r}{\rho_V(x)} \right)^{2 - \frac{d}{q}} \left( \frac{1}{r^{d-1}} \right) \right),
\]

where \(r := |x - y|/2\). As for the term \(J_2\), if \(z \in \mathbb{R}^d \setminus B(y, \mathcal{R})\) we will have \(r := |x - y| \leq |z - y|\). Therefore,

\[
J_2 \lesssim e^{-\varepsilon \sqrt{\lambda} r} \int_{\mathbb{R}^d \setminus B(y, \mathcal{R})} \frac{1}{|y - z|^{d-2}} \frac{e^{-\varepsilon \sqrt{\lambda}|x - z|} e^{-\varepsilon d V(z, x)}}{|x - z|^{d-2}} V(z) \, dz.
\]

Using the argument from Lemma 4.8 of [17], this term can be estimated by

\[
J_2 \lesssim \frac{e^{-\varepsilon \sqrt{\lambda} r}}{r^{d-2}} \left( \frac{r}{\rho_V(x)} \right)^{2 - \frac{d}{q}} \lesssim \mathcal{R} \left( \frac{r}{\rho_V(x)} \right)^{2 - \frac{d}{q}}.
\]

Combining this with (41) produces

\[
|\nabla \Gamma_{A, V + \lambda}(y, x) - \nabla \Gamma_{A, \lambda}(y, x)| \lesssim \mathcal{R} e^{-\varepsilon \sqrt{\lambda} r} \left( \frac{1}{r^{d-2}} \int_{B(y, r)} \frac{V(z) \, dz}{|z - y|^{d-1}} + \left( \frac{r}{\rho_V(x)} \right)^{2 - \frac{d}{q}} \left( \frac{1}{r^{d-1}} \right) \right),
\]

for some \(\varepsilon > 0\). This estimate can then be used together with the expressions

\[
K_{A,V}^*(x, y) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{d}{2}} \nabla \Gamma_{A, V + \lambda}(y, x) d\lambda, \quad K_{A,0}^*(x, y) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{d}{2}} \nabla \Gamma_{A, \lambda}(y, x) d\lambda
\]

to obtain (39).

\[
\square
\]
Let \( \{ B_j \}_{j \in \mathbb{N}} = \{ B(x_j, \rho_V(x_j)) \}_{j \in \mathbb{N}} \) be a collection of balls as given in Proposition 2.1. Define \( \sigma = \beta 2^{-\frac{\kappa_0}{\rho_0}} \), with \( \kappa_0 \) as in Lemma 2.1 and \( \beta \) as in (6). Set \( \tilde{B}_j := 2\sigma B_j \) for each \( j \in \mathbb{N} \). For each \( j \in \mathbb{N} \), let \( \chi_j : \mathbb{R}^d \to [0,1] \) be a smooth function that is identically equal to one on \( \tilde{B}_j \), vanishes outside of \( 2\tilde{B}_j \) and \( \| \nabla \chi_j \|_{\infty} \lesssim \frac{1}{\sigma \rho_V(x_j)} \).

For each \( j \in \mathbb{N} \), define the operator
\[
R_{A,0}^{*,j} f(x) := \chi_j(x) \cdot R_{A,0}^j (f \chi_j)(x) \quad \text{for } f \in L^1_{\text{loc}}(\mathbb{R}^d), \ x \in \mathbb{R}^d.
\]

**Lemma 5.2.** Let \( V \in RH_q \) for some \( q > \frac{d}{2} \) and suppose that \( \rho_V(x) \leq D_V \) for some \( D_V > 0 \), for all \( x \in \mathbb{R}^d \). For \( 1 < p < \infty \) and \( w \in A_p \),
\[
\left\| R_{A,0}^{*,j} f \right\|_{L^p(w)} \lesssim [w]_{A_p}^{\max(1,1/(p-1))} \| f \|_{L^p(w)}
\]
for all \( f \in L^p(w) \) and \( j \in \mathbb{N} \), where the implicit constant is independent of \( j \in \mathbb{N} \) and \( w \).

**Proof.** This will be proved by demonstrating that \( R_{A,0}^{*,j} \) is a Calderón-Zygmund operator that is bounded on \( L^2(\mathbb{R}^d) \) with constant independent of \( j \in \mathbb{N} \) and whose kernel satisfies size and regularity estimates with constants independent of \( j \). Corollary 5.2 states that the kernel of \( R_{A,0} \) is locally Calderón-Zygmund. This implies that \( K_{A,0}^* \) is itself locally Calderón-Zygmund since \( K_{A,0}^*(x,y) = K_{A,0}(y,x) \) is satisfied for all \( x, y \in \mathbb{R}^d \). Set \( R := 20\sigma D_V \). Then
\[
\left| K_{A,0}^*(x,y) \right| \lesssim_R |x-y|^{-d}
\]
for all \( x, y \in \mathbb{R}^d \) with \( |x-y| \leq R \) and
\[
\left| K_{A,0}^*(x,y) - K_{A,0}^*(x',y) \right| \lesssim_R \frac{|x-x'|^\gamma}{|x-y|^{d+\gamma}}
\]
for all \( x, x', y \in \mathbb{R}^d \) with \( x, x' \in B(y, R) \) and \( |x-x'| \leq \frac{1}{2} |x-y| \).

Let’s first prove the size estimate for \( K_{A,0}^{*,j}(x,y) := \chi_j(x)K_{A,0}^*(x,y)\chi_j(y) \). Fix \( x, y \in \mathbb{R}^d \). If either \( x \notin 2\tilde{B}_j \) or \( y \notin 2\tilde{B}_j \) then \( K_{A,0}^{*,j}(x,y) \) vanishes completely and the size estimate will be trivially satisfied. Suppose that \( x \in 2\tilde{B}_j \) and \( y \in 2\tilde{B}_j \). It then follows from the boundedness of the critical radius function that
\[
|x-y| \leq |x-x_j| + |x_j-y| \\
\leq 4\sigma \rho_V(x_j) + 4\sigma \rho_V(x_j) \\
\leq 8\sigma D_V \\
\leq R.
\]

Therefore,
\[
K_{A,0}^{*,j}(x,y) = \chi_{\tilde{B}_j}(x)K_{A,0}^{*}(x,y)\chi_{\tilde{B}_j}(y) \lesssim_R |x-y|^{-d}.
\]

Next, let’s prove the regularity estimate for \( K_{A,0}^{*,j} \). Let \( x, x', y \in \mathbb{R}^d \) with \( |x-x'| \leq \frac{1}{2} |x-y| \). If either \( y \notin 2\tilde{B}_j \) or \( x \in 2\tilde{B}_j \) and \( x' \notin 2\tilde{B}_j \) then the regularity estimate will be trivially satisfied. It can therefore be assumed that \( y \in 2\tilde{B}_j \) and either \( x \) or \( x' \in 2\tilde{B}_j \). This will imply that both \( x \) and \( x' \) are contained in \( 8\tilde{B}_j \) and therefore
Let $\chi_i(x)K_{A,0}^*(x,y)\leq |\chi_i(x)K_{A,0}^*(x,y)| + |K_{A,0}^*(x,y)| + |x - x'|^{\gamma}$
\[\leq R \|\nabla \chi_i\|_\infty |x - x'| |K_{A,0}^*(x,y)| + |x - x'|^{\gamma} \frac{|x - y|^{d+\gamma}}{|x - y|^{d+\gamma}} \leq R \|\nabla \chi_i\|_\infty |x - x'|^{\gamma} \frac{|x - y|^{d+\gamma}}{|x - y|^{d+\gamma}} \leq \sigma \rho_V(x) |x - y|^{d+\gamma} \leq \sigma |x - y|^{d+\gamma} \leq |x - y|^{d+\gamma},\]
where the second to last line follows from the fact that $|x - y| \leq \rho_V(x)$. Similar reasoning can be applied to obtain the estimate
\[|K_{A,0}^*(y,x) - K_{A,0}^*(y,x')| \leq R \||x - x'|^{\gamma} \frac{|x - y|^{d+\gamma}}{|x - y|^{d+\gamma}} \leq \sigma \rho_V(x) |x - y|^{d+\gamma} \leq \sigma |x - y|^{d+\gamma} \leq |x - y|^{d+\gamma},\]
for all $x, y \in \mathbb{R}^d$. Since it is obvious that the operators $R_{A,0}^*$ are bounded on $L^2(\mathbb{R}^d)$ with constant independent of $j \in \mathbb{N}$, it follows that the operators $R_{A,0}^*$ are Calderón-Zygmund with constants independent of $j \in \mathbb{N}$. Our result then follows from the well-known $A_2$-conjecture that was proved in [10].

**Proposition 5.3.** Let $V \in RH_q$ for some $q > \frac{d}{2}$ and assume there exists $D_V > 0$ for which $\rho_V(x) \leq D_V$ for all $x \in \mathbb{R}^d$.

(i) Suppose that $q \geq d$ and let $1 < p < \infty$. The operators $R_{A,V}^{\text{loc}}$ and $R_{A,V}^{*,\text{loc}}$ are bounded on $L^p(w)$ for any $w \in A_p^{V,\text{loc}}$.

(ii) Suppose that $q < d$ and let $s$ be defined through $\frac{1}{q} = \frac{1}{s} - \frac{1}{q}$. The operator $R_{A,V}^{*,\text{loc}}$ is bounded on $L^p(w)$ for any $s' < p < \infty$ with $w \in A_p^{V,\text{loc}}$ and the operator $R_{A,V}^{\text{loc}}$ is bounded on $L^p(w)$ for any $1 < p < s$ with $\frac{1}{p} = \frac{1}{s'} - \frac{1}{s} \in A_p^{V,\text{loc}}$.

**Proof.** For $1 < p < \infty$, weight $w$ on $\mathbb{R}^d$ and $f \in L^p(w)$, the triangle inequality allows us to estimate the $L^p(w)$-norm of $R_{A,V}^{*,\text{loc}} f$ from above by
\[
\left\| R_{A,V}^{*,\text{loc}} f \right\|_{L^p(w)} \leq \left\| R_{A,V}^{*,\text{loc}} (R_{A,V}^{*,\text{loc}} - R_{A,0}^{*,\text{loc}}) f \right\|_{L^p(w)} + \left\| R_{A,0}^{*,\text{loc}} f \right\|_{L^p(w)}.
\]
In the proof of [5, Thm. 3], the authors prove the boundedness of the operator difference $R_{V,\text{loc}}^{\text{loc}} - R_{0,\text{loc}}^{\text{loc}}$ using only the kernel estimate
\[
|K_{V}^*(x,y) - K_{0}^*(x,y)| \leq \frac{1}{|x - y|^{d-1}} \int_{B(x,|x-y|/2)} \frac{V(z)}{|z - x|^{d-1}} dz + \left( \frac{|x - y|}{\rho_V(x)} \right)^{2 - \frac{d}{q}} \frac{1}{|x - y|^{d}},
\]
for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq \rho_V(x)$. By setting $R = D_V$ in Lemma 5.1, it is clear that this estimate also holds for the difference $|K_{A,V}^*(x,y) - K_{A,0}^*(x,y)|$. Therefore, an argument identical to that of [5, Thm. 3] can be used. This will imply
the boundedness of the difference \( R_{A,0}^{s,loc} - R_{A,0}^{s,loc} \) on \( L^p(w) \) for any \( w \in A_{p'}^{V,loc} \) when \( q \geq d \) and for \( w \in A_{p'/s'}^{V,loc} \) with \( s' < p < \infty \) when \( \frac{d}{2} < q < d \).

To complete the proof of our proposition, it is then sufficient to show that the operator \( R_{A,0}^{s,loc} \) is bounded on \( L^p(w) \) for \( 1 < p < \infty \) and \( w \in A_{p}^{V,loc} \) when \( q > \frac{d}{2} \). Assume that \( q > \frac{d}{2} \) and fix \( 1 < p < \infty \) and \( w \in A_{p}^{V,loc} \). We have

\[
\left\| R_{A,0}^{s,loc} f \right\|_{L^p(w)}^p \leq \sum_j \int_{B_j} \left| R_{A,0}^{s,loc} f \right|_{L^p(w)}^p w(x) \, dx
\]

\[
\lesssim \sum_j \int_{B_j} \left| R_{A,0}^{s,loc} f(x) - R_{A,0}^{s,j} f(x) \right|^p w(x) \, dx + \int_{B_j} \left| R_{A,0}^{s,j} f(x) \right|^p w(x) \, dx.
\]

For any \( x \in B_j \), it follows from Lemma 2.1 that \( B(x, \rho V(x)) \subset \tilde{B}_j \). The fact that \( K_{A,0}^s \) is locally Calderón-Zygmund implies that for any \( x \in B_j \),

\[
\left| R_{A,0}^{s,loc} f(x) - R_{A,0}^{s,j} f(x) \right| \leq \int_{2\tilde{B}_j \setminus B(x, \rho V(x))} \left| K_{A,0}^s(x,y) \right| |f(y)| \, dy
\]

\[
\lesssim_{D_V} \int_{2\tilde{B}_j \setminus B(x, \rho V(x))} \frac{|f(y)|}{|x - y|a} \, dy
\]

\[
\lesssim \frac{1}{|2\tilde{B}_j|} \int_{2\tilde{B}_j} |f(y)| \, dy,
\]

where the last line follows from Lemma 2.1. This will lead to

\[
\sum_j \int_{B_j} \left| R_{A,0}^{s,loc} f(x) - R_{A,0}^{s,j} f(x) \right|^p w(x) \, dx \lesssim_{D_V} \sum_j \int_{B_j} \left( \frac{1}{|2\tilde{B}_j|} \int_{2\tilde{B}_j} |f(y)| \, dy \right)^p w(x) \, dx
\]

\[
\lesssim \sum_j w(2\tilde{B}_j)w^{-\frac{p}{q'}} \left( 2\tilde{B}_j \right)^{p-1} \left( \frac{1}{|2\tilde{B}_j|} \int_{2\tilde{B}_j} |f(y)|^p w(y) \, dy \right)
\]

On exploiting the property that \( w \in A_p^{v,loc} = A_{p}^{4pV,loc} \) (c.f. [5, Cor. 1]) and the bounded overlap property of the balls \( \{B_j\}_{j \in \mathbb{N}} \)

\[
\sum_j \int_{B_j} \left| R_{A,0}^{s,loc} f(x) - R_{A,0}^{s} (f \chi_j)(x) \right|^p w(x) \, dx \lesssim \sum_j \int_{2\tilde{B}_j} |f(y)|^p w(y) \, dy
\]

\[
\lesssim \int_{\mathbb{R}^d} |f(y)|^p w(y) \, dy.
\]

For \( j \in \mathbb{N} \), let \( w_j \in A_p \) denote the extension of \( w|_{2\tilde{B}_j} \) to all of \( \mathbb{R}^d \) with \( [w_j]_{A_p} \leq [w]_{A_p^{v,loc}} \). The existence of such a weight is given in [5, Lem. 1]. For the second
term in (43), Lemma 5.2 implies
\[
\sum_j \int_{B_j} \left| R_{A,0}^{s,j} f(x) \right|^p w(x) \, dx \leq \sum_j \int_{\mathbb{R}^d} \left| R_{A,0}^{s,j} \left( f \mathbb{1}_{2B_j} \right) (x) \right|^p w_j(x) \, dx
\]
\[
\lesssim \sum_j \max_{j=1}^{\infty} \int_{2B_j} |f(x)|^p w_j(x) \, dx
\]
\[
\lesssim \sum_j \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx.
\]
This completes our proof of the $L^p(w)$-boundedness of $R_{A,V}^{s,\text{loc}}$. The boundedness of $R_{A,V}^{s,\text{loc}}$ follows from this by duality.

It remains to consider the global behaviour of $R_{A,V}$ and $R_{A,V}^s$.

**Proposition 5.4.** Let $V \in RH_q$ for some $q > \frac{d}{2}$ and assume that $\rho_V(x) \leq D_V$ for all $x \in \mathbb{R}^d$, for some $D_V > 0$.

(i) Suppose that $q \geq d$. There must exist a constant $c_1 > 0$ for which the operators $R_{A,V}^{s,\text{global}}$ and $R_{A,V}^{s,\text{global}}$ are bounded on $L^p(w)$ for all $w \in S_{p,c_1}$ with $1 < p < \infty$.

(ii) Suppose instead that $\frac{d}{2} < q < d$ and let $s$ be defined through $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$. There must exist $c_2 > 0$ for which the operator $R_{A,V}^{s,\text{global}}$ is bounded on $L^p(w)$ for any $s' < p < \infty$ with $w \in S_{p,s',c_2}$ and the operator $R_{A,V}^{s,\text{global}}$ is bounded on $L^p(w)$ for any $1 < p < s$ with $w^{-\frac{1}{s-1}} \in S_{p,s',c_2}$.

The constants $c_1$ and $c_2$ will be independent of $p$ and depend on $V$ only through $[V]_{RH_q}$ and $D_V$.

**Proof.** From identical reasoning as in the proof of Lemma 4.3, Corollary 5.1 implies that the the singular kernel of $R_{A,V}$ will satisfy
\[
|K_{A,V}(x,y)| \lesssim e^{-c \rho_V(x,y)} \left( \int_{B(y,|x-y|/2)} \frac{V(z)}{|z-x|^{d-1}} \, dz + \frac{1}{|x-y|} \right)
\]
for all $x, y \in \mathbb{R}^d$, for some $\varepsilon > 0$. The proof of the perturbation free analogue of the statement that we wish to prove, Theorem 4.3, relied entirely on the estimate provided by Lemma 4.3. Since this estimate is also true in the perturbation dependent case, it follows that the proof from Theorem 4.3 can be repeated verbatim to give us our result.

Combining Propositions 5.3 and 5.4 completes the proof of Theorem 5.3.

### 5.3. Riesz Potentials.

For the Riesz potentials $I_{A,V}^s$ we have the following theorem.

**Theorem 5.4.** Fix $V \in RH_q$. Let $A \in L^\infty \left( \mathbb{R}^d; \mathcal{L}(\mathbb{C}^d) \right)$ have real-valued coefficients and assume that it satisfies the ellipticity condition (35). For any $0 < \alpha \leq 2$, there must exist $c > 0$ for which the operator $I_{A,V}^s$ is bounded from $L^p(w)$ to
Let \( L^\nu(w^{\nu/p}) \) for \( w^{\nu/p} \in S_{1,1}^{V,\frac{\nu}{p},c} \) and \( 1 < p < \frac{d}{\alpha} \), where \( \frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d} \). Moreover, the constant \( c \) is independent of \( p \) and depends on \( V \) only through \([V]_{RH_{\frac{d}{2}}} \).

**Proof.** Let \( 1 < p < \frac{d}{\alpha} \) and \( \frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d} \). Let’s first prove the boundedness of the local component \( \Gamma_{\alpha, V}^{\nu} \). Notice that the pointwise estimate

\[
|I_{\alpha, V}^\nu f(x)| \leq I_0^\nu |f|(x)
\]

holds for all \( f \in L_{1,loc}^1(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \). Therefore, in order to prove the boundedness of the operator \( \Gamma_{\alpha, V}^{\nu} \) from \( L^p(w) \) to \( L^\nu(w^{\nu/p}) \) for \( w^{\nu/p} \in S_{1,1}^{V,\frac{\nu}{p},c} \) and \( c > 0 \) it is sufficient to prove the boundedness of \( I_0^{\alpha, loc} \). This follows on noting that \( S_{1,1}^{V,\frac{\nu}{p},c} \subset A_{1,1}^{V,loc} \) for any \( c > 0 \) by Proposition 3.1 and \( I_0^{\alpha, loc} \) is bounded from \( L^p(w) \) to \( L^p(w^{\nu/p}) \) for \( w^{\nu/p} \in A_{1,1}^{V,loc} \) by [5, Thm. 1].

It remains to consider the boundedness of the global component \( \Gamma_{\alpha, V}^{\nu, glob} \). The proof of Theorem 4.4 relied entirely on the estimate

\[
\Gamma_V(x,y) \lesssim e^{-\epsilon d_V(x,y)} \frac{1}{|x-y|^{d-2}}.
\]

Since an estimate of this form holds for the case \( L_{A, V} \) by Theorem 5.1, it follows that the proof of Theorem 4.4 can be repeated verbatim. This argument will show that \( c > 0 \), independent of \( p \) and depending on \( V \) only through \([V]_{RH_{\frac{d}{2}}} \), so that \( \Gamma_{\alpha, V}^{\nu, glob} \) is bounded from \( L^p(w) \) to \( L^\nu(w^{\nu/p}) \) for any \( w^{\nu/p} \in S_{1,1}^{V,\frac{\nu}{p},c} \).

**5.4. Heat Maximal Operator.** For the heat maximal operator \( T_{A, V}^* \), our weighted result is given below.

**Theorem 5.5.** Fix \( V \in RH_{\frac{d}{2}} \). Let \( A \in L^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^d)) \) have real-valued coefficients and assume that it satisfies the ellipticity condition (35). Suppose also that \( A = A^* \). Then there exists \( c > 0 \) for which the heat maximal operator \( T_{A, V}^* \) is bounded on \( L^p(w) \) for all \( w \in H^{V,m_0}_{p,c} \) with \( m_0 := (2(k_0 + 1))^{-1} \) and \( 1 < p < \infty \). The constant \( c \) is independent of \( p \).

**Proof.** Let \( 1 < p < \infty \). Consider the heat maximal operator for \( L_{A, V} \),

\[
T_{A, V}^* f(x) := \sup_{t > 0} e^{-t L_{A, V}} |f|(x) \quad f \in L_{1, loc}^1(\mathbb{R}^d), \ x \in \mathbb{R}^d.
\]

From Theorem 5.2, we have the pointwise estimate

\[
T_{A, V}^* f(x) \leq T_0^* f(x),
\]

for any \( f \in L_{1, loc}^1(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \). Therefore, in order to prove the boundedness of the operator \( T_{A, V}^{\nu, loc} \) on \( L^p(w) \) for \( w \in H^{V,m_0}_{p,c} \) and \( c > 0 \) it is sufficient to prove the boundedness of \( T_0^{\nu, loc} \) on \( L^p(w) \). This follows on noting that \( H^{V,m_0}_{p,c} \subset A_{p}^{V,loc} \) for any \( c > 0 \) by Propositions 3.1 and 3.2 and that \( T_0^{\nu, loc} \) is bounded on \( L^p(w) \) for weights in \( A_{p}^{V,loc} \) by [5, Thm 1].
It remains to establish the boundedness of $T^{*, \text{glob}}_{A,V}$. The proof of Theorem 1.2 in Section 4.4 relied entirely on the pointwise estimate

$$k^V_t(x,y) \lesssim \frac{1}{t^{\frac{d}{2}}} e^{-D_1(1+\sqrt{\frac{D_2}{t}}) \frac{|x-y|^2}{t} - D_2 |x-y|^2}.$$ 

Since, by Theorem 5.2, this estimate also holds for the heat kernel $k^1_{A,V}$, the entire proof of Proposition 4.4 can be repeated verbatim. This will allow us to show that there exists $c > 0$, independent of $p$, such that $T^{*, \text{glob}}_{A,V}$ is bounded on $L^p(w)$ for any $w \in H^{V,m}_{p,c}$.

6. Magnetic Schrödinger Operators

The final form of Schrödinger operator that will be considered in this article is the magnetic Schrödinger operator. Let $a = (a_1, \cdots, a_d)$ be a vector of real-valued functions in $C^1(\mathbb{R}^d)$ that will be referred to as the magnetic potential. The magnetic field, denoted by $B$: $\mathbb{R}^d \to \mathbb{R}^d$, is then defined through $B := \text{curl} a$. The magnetic Schrödinger operator with electric potential $V$ and magnetic potential $a$ is the operator 

$$L^a_V := (\nabla - ia)^* (\nabla - ia) + V.$$ 

The standard operators associated with $L^a_V$ are defined through 

$$R^a_V := (\nabla - ia) (L^a_V)^{-\frac{1}{2}}, \quad R^a_{V,*} := (L^a_V)^{-\frac{1}{2}} (\nabla + ia), \quad I^{a,\alpha}_V := (L^a_V)^{\frac{\alpha}{2}}$$ 

for $0 < \alpha \leq 2$ and 

$$T^a_{V,*} f(x) := \sup_{t > 0} e^{-tL^a_V} |f|(x), \quad f \in L^1_{\text{loc}}(\mathbb{R}^d), \ x \in \mathbb{R}^d.$$ 

As usual, in order to prove the weighted boundedness of our operators, we will require exponential decay estimates for the associated fundamental solution, denoted by $\Gamma^a_V$ (refer to [7] for the construction of $\Gamma^a_V$), and heat kernel, denoted by $h^a_{V,t}$ (see [12]).

**Theorem 6.1** ([13, Cor. 6.16]). Suppose that $V + |B| \in RH^\frac{d}{2}$ and that there exists $D, D' > 0$ for which

$$0 \leq V \leq D \cdot \rho_{V+|B|}^{-2} \quad \text{and} \quad |\nabla B| \leq D' \cdot \rho_{V+|B|}^{-3}.$$ 

There exist constants $\varepsilon, C > 0$ for which

$$|\Gamma^a_V(x,y)| \leq C e^{-\varepsilon d_{V+|B|}(x,y)} |x-y|^{d-2}$$

for all $x, y \in \mathbb{R}^d$. The constant $\varepsilon$ will depend on $V$ and a only through $[V + |B|]_{RH^\frac{d}{2}}$, $D$ and $D'$.

Let $\nabla^a := \nabla - ia$. Using Theorem 6.1 and the work of B. Ben Ali from [4], it is then not too difficult to prove exponential decay estimates for the derivative of the fundamental solution of $L^a_V$. 


Proposition 6.1. Suppose that $V + |B| \in RH_d$ and that there exists $D, D' > 0$ for which (44) is satisfied. There exist constants $\varepsilon, C > 0$ for which

$$|\nabla^a \Gamma^\varepsilon(x, y)| \leq C e^{-\varepsilon d_{V+\|B\|} (x, y)} |x - y|^{-d-1}$$

for all $x, y \in \mathbb{R}^d$. The constant $\varepsilon$ will depend on $V$ and $\|B\|$ only through $[V + |B|]_{RH_d}$. Lemma 4.10 and Theorem 6.1 then imply that $Q$ for all $x, y \in \mathbb{R}^d$ where

$$Q = \left\{ \xi \in \mathbb{R}^d : |\nabla^a \Gamma^\varepsilon(x, y)| \leq C e^{-\varepsilon d_{V+\|B\|} (x, y)} |x - y|^{-d-1} \right\}$$

for $\xi \in Q$. It is obvious that $u$ is a weak solution to $L^\varepsilon u = 0$ on $Q$.

**Case 1:** Suppose that $\rho_{V+\|B\|}(x) \leq |x - y|/2$. Set $R := \rho_{V+\|B\|}(x)/\sqrt{d}$. Consider the cube $Q := Q(x, R)$ centered at $x$ with side-length $R$ and define the function $u$ on $Q$ through

$$u(\xi) := \Gamma^\varepsilon(\xi, y)$$

for $\xi \in Q$. It is obvious that $u$ is a weak solution to $L^\varepsilon u = 0$ on $Q$.

**Proof.** Let $x, y \in \mathbb{R}^d$ and $0 < R \leq |x - y|/2\sqrt{d}$. Consider the cube $Q := Q(x, R)$ centered at $x$ with side-length $R$ and define the function $u$ on $Q$ through

$$u(\xi) := \Gamma^\varepsilon(\xi, y)$$

for $\xi \in Q$. It is obvious that $u$ is a weak solution to $L^\varepsilon u = 0$ on $Q$.

**Case 2:** Suppose that $\rho_{V+\|B\|}(x) > |x - y|/2$. Set $R := |x - y|/2\sqrt{d}$. Consider the cube $Q := Q(x, R)$ centered at $x$ with side-length $R$ and define the function $u$ on $Q$ through

$$u(\xi) := \Gamma^\varepsilon(\xi, y)$$

for $\xi \in Q$. It is obvious that $u$ is a weak solution to $L^\varepsilon u = 0$ on $Q$.

The estimates $|x - y| \lesssim |\xi - y|$ and $d_{V+\|B\|}(x, y) \leq d_{V+\|B\|}(\xi, y) + C$ for some $C > 0$ and for all $\xi \in Q$ then allow us to conclude the proof of our proposition.
**Theorem 6.2** ([12, Thm. 1]). Suppose that $V + |B| \in RH_{\frac{d}{2}}$ and (44) is satisfied with constants $D$, $D' > 0$. There exist constants $D_0$, $D_1$, $D_2 > 0$ such that

$$
\left| h_t^{\alpha,V}(x,y) \right| \leq D_0 \cdot e^{-D_1 \left( \frac{1}{t^2} e^{\frac{\sqrt{t}}{r_{V+|B|}(x)}} \right)^{\frac{1}{\alpha}}} \left( \frac{1}{t^2} e^{-D_2 \frac{x-y}{t}} \right)^{\frac{1}{\alpha}}
$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$.

For the global components of the Riesz transforms $R_V^\alpha$ and their adjoints $R_V^{\alpha,*}$ our weighted result is as given below.

**Theorem 6.3.** Let $1 < p < \infty$. Suppose that $V + |B| \in RH_d$ and (44) is satisfied with constants $D$, $D' > 0$. Then there exists $c > 0$ for which $R_{V,\text{glob}}^\alpha$ and $R_{V,\text{glob}}^{\alpha,*}$ are bounded on $L^p(w)$ for any $w \in S_{p,c}^{V+|B|}$. Moreover, $c$ will depend on $V$ and $a$ only through $|V + |B||_{RH_{\frac{d}{2}}}$, $D$ and $D'$.

**Proof.** The proof of Theorem 4.3 is entirely reliant on the estimate

$$
| \nabla I_V(x,y)| \lesssim e^{-c_{DV}(x,y)} |x-y|^{d-1}.
$$

Since the appropriate magnetic analogue for this estimate is true by Proposition 6.1, the proof of Theorem 4.3 can be reapplied verbatim to this case. This will be enough to tell us that there exists $c > 0$ such that $R_{V,\text{glob}}^{\alpha,*}$ is bounded on $L^p(w)$ for all $w \in S_{p,c}^{V+|B|}$. The boundedness of $R_{V,\text{glob}}^\alpha$ follows by duality.

In order to prove that the operators $R_V^\alpha$ and $R_V^{\alpha,*}$ are bounded on $L^p(w)$ for weights in the class $S_{p,c}^{V+|B|}$ it is then sufficient to prove the boundedness of the local components. A condition that can be used to guarantee this is the rather strong hypothesis that $R_V^\alpha$ and $R_V^{\alpha,*}$ are both bounded on $L^p(w)$ for any weight $w$ in the standard Muckenhoupt class $A_p$. Investigating when exactly this occurs is unfortunately beyond the scope of this article.

**Theorem 6.4.** Let $1 < p < \infty$. Suppose that $V + |B| \in RH_d$ and (44) is satisfied with constants $D$, $D' > 0$. Suppose also that for any $w \in A_p$

$$
\| R_V^\alpha \|_{L^p(w)} \cdot \| R_V^{\alpha,*} \|_{L^p(w)} \lesssim [w]^l_{A_p}
$$

for some $l \geq 1$. Then there exists $c > 0$ for which $R_V^\alpha$ and $R_V^{\alpha,*}$ are bounded on $L^p(w)$ for any $w \in S_{p,c}^{V+|B|}$. Moreover, $c$ will depend on $V$ and $a$ only through $|V + |B||_{RH_{\frac{d}{2}}}$, $D$ and $D'$.

**Proof.** Let $c > 0$ be as given in Theorem 6.3 and fix $w \in S_{p,c}^{V+|B|}$. For the local boundedness of $R_V^{\alpha,*}$, let $B_j = B(x_j, \rho_{V+|B|}(x_j))$ for $j \in \mathbb{N}$ be as given in Proposition 2.1 and set $B_j := 2\sigma B_j$. We have

$$
\left\| R_{V,\text{loc}}^{\alpha,*} f \right\|_{L^p(w)}^p \leq \sum_j \int_{B_j} \left| R_{V,\text{loc}}^{\alpha,*} f(x) \right|_{L^p(w)}^p w(x) \, dx
$$

$$
\lesssim \sum_j \int_{B_j} \left| R_{V,\text{loc}}^{\alpha,*} f(x) - R_{V,\text{loc}}^{\alpha,*} (f \cdot \mathbb{1}_{2B_j}) \right|_{L^p(w)}^p w(x) \, dx
$$

$$
+ \int_{B_j} \left| R_{V,\text{loc}}^{\alpha,*} (f \cdot \mathbb{1}_{2B_j}) \right|_{L^p(w)}^p w(x) \, dx.
$$
The first term can be handled in an identical manner to how the difference term from (43) was handled by making use of the derivative estimates, Proposition 6.1. The second term can also be handled in a similar manner to how the second term was handled in (43). Namely, let \( w_j \in A_p \) denote the extension of \( w|_{2B_j} \) to all of \( \mathbb{R}^d \) with \([w_j]_{A_p} \leq [w]_{A_p^{V+|B|,loc}} \) (c.f. [5, Lem. 1]). Then on applying our hypothesis,

\[
\sum_j \int_{B_j} \left| R_{V,loc}^{a,*} (f \mathbb{1}_{2B_j})(x) \right|^p w(x) \, dx \leq \sum_j \int_{\mathbb{R}^d} \left| R_{V,loc}^{a,*} (f \mathbb{1}_{2B_j})(x) \right|^p w_j(x) \, dx
\]

\[
\leq \sum_j \left| w_j \right|_{A_p} \int_{2B_j} |f(x)|^p w_j(x) \, dx
\]

\[
\leq \sum_j \left| w \right|_{A_p^{V+|B|,loc}} \int_{2B_j} |f(x)|^p w(x) \, dx
\]

\[
\leq \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx.
\]

Duality will imply that \( R_{V,loc}^{a,*} \) is also bounded on \( L^p(w) \).

Our weighted result for the magnetic Riesz potentials and heat maximal operator is as follows.

**Theorem 6.5.** Let \( 1 < p < \infty \). Suppose that \( V + |B| \in RH_{\frac{d}{2}} \) and that there exists \( D, D' > 0 \) for which (44) is satisfied. The following statements are true.

(i) For \( 0 < \alpha \leq 2 \), there must exist a constant \( c_1 > 0 \) for which the operator \( T_{V,loc}^{\alpha} \) is bounded from \( L^p(w) \) to \( L^{p,V(\nu/p)} \) for \( w^{\nu/p} \in S_{1+1/p} \), and \( 1 < p < \frac{d}{\alpha} \), where

\[
\frac{1}{p} = \frac{1}{p} - \frac{d}{2}.
\]

(ii) There exists \( c_2 > 0 \) for which \( T_{V,loc}^{\alpha,*} \) is bounded on \( L^p(w) \) for all \( w \in H_{p,c_2}^{V+|B|,m_0} \) with \( m_0 := (2(k_0 + 1))^{-1} \).

**Proof.** The weighted boundedness of the operators \( T_{V}^{\alpha} \) and \( T_{V}^{\alpha,*} \) is proved in an identical manner to Theorem 5.4 and 5.5 respectively by making use of Theorems 6.1 and 6.2.

7. Necessary Conditions

Let us now investigate necessary conditions for a weight \( w \) to satisfy in order for \( R_V \) and \( T_V \) to be bounded on \( L^p(w) \). We will begin by proving that \( w \in A_{p}^{V,loc} \) if \( \|R_{V,loc}\|_{L^p(w)} < \infty \) for \( V \in RH_d \), thereby demonstrating that at a local scale the operator \( R_V \) characterises \( A_{p}^{V,loc} \). Following this, it will be shown that the first chain of inclusions in Conjecture 1.1 is true for constant potentials and the second chain of inclusions is true for any potential that is bounded both from above and below. Finally, we will prove that the second chain of inclusions in Conjecture 1.2 is true for the harmonic oscillator potential \( V(x) = |x|^2 \).

7.1. Characterisation of \( A_{p}^{V,loc} \) in terms of \( R_{V,loc} \). In this section it will be proved that the condition \( w \in A_{p}^{V,loc} \) is necessary in order for the operator \( R_{V,loc} \) to be bounded on \( L^p(w) \). To keep the result as general as possible, we will prove this statement for generalised Schrödinger operators with measure potentials \( \mu \) in the sense of Section 4.1. When taken together with Proposition 4.1, this result
will complete the proof of a localized Hunt-Muckenhoupt-Wheeden type theorem for measure potentials $\mu$ satisfying (7) and (8) with constants $C_\mu$, $D_\mu$ and $\delta > 1$. In particular, the below theorem proves that the class $A_p^{\mu, loc}$ is characterised completely by the boundedness of the operator $R_{\mu, loc}$ on $L^p(w)$.

**Theorem 7.1.** Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^d$ that satisfies (7) and (8) with constants $C_\mu$, $D_\mu > 0$ and $\delta_\mu > 1$. Fix $1 < p < \infty$ and weight $w$ on $\mathbb{R}^d$. Suppose that the operator $R_{\mu, loc}$ is bounded on $L^p(w)$. Then it must be true that $w \in A_p^{\mu, loc}$.

**Proof.** Assume that $\|R_{\mu, loc}\|_{L^p(w)} < \infty$. Let $0 < \kappa_0 < 1$ be some constant whose value will be determined at a later time. Since $\rho_\mu$ is a critical radius function in the sense of Definition 2.1, it follows easily that the function $\kappa_0 \rho_\mu$ will also satisfy (3) and therefore will also be a critical radius function. Moreover, it follows from [5, Cor. 1] that $A_p^{\kappa_0 \rho_\mu, loc} = A_p^{\rho_\mu, loc}$. Therefore, in order to prove that $w \in A_p^{\rho_\mu, loc}$ it is sufficient to prove that $w \in A_p^{\kappa_0 \rho_\mu, loc}$. It must therefore be proved that

$$w(B) \frac{1}{\rho} w^{-\frac{1}{p-1}}(B)^{\frac{2}{p-1}} \lesssim |B|$$

for all Euclidean balls $B = B(c, r) \subset \mathbb{R}^d$ with $r \leq \kappa_0 \cdot \rho_\mu(c)$.

Consider the operator

$$R_{\mu}^{(1)} f(x) := \partial_1 (-\Delta + \mu)^{-\frac{1}{2}} f(x).$$

On combining the expression

$$\Gamma_{\mu + \lambda}(x, y) = \Gamma_{\lambda}(x, y) - \int_{\mathbb{R}^d} \Gamma_{\lambda}(x, z) \Gamma_{\mu + \lambda}(z, y) d\mu(z) \quad \forall \ x, y \in \mathbb{R}^d$$

with (12) we obtain the identity

$$K_{\mu}^{(1)}(x, y) = K_{\lambda}^{(1)}(x, y) - \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^d} \partial_1 \Gamma_{\lambda}(x, z) \Gamma_{\mu + \lambda}(z, y) d\mu(z) \, d\lambda$$

for all $x, y \in \mathbb{R}^d$, where $K_{\mu}^{(1)}$ and $K_{\lambda}^{(1)}$ are the first components of the singular kernels for $R_{\mu}$ and $R_{\lambda}$ respectively.

Fix $B = B(c, r) \subset \mathbb{R}^d$ with $r \leq \kappa_0 \cdot c$. Let $B'$ be the Euclidean ball in $\mathbb{R}^d$ that has center $c' = c + 4r = (c_1 + 4r, \ldots, c_d + 4r)$ and radius $r$. Let’s estimate the size of the second term in (46) for $x \in B'$ and $y \in B$. Theorem 4.2 states that there must exist some $\varepsilon > 0$ such that

$$\int_{\mathbb{R}^d} |\partial_1 \Gamma_{\lambda}(x, z)| \Gamma_{\mu + \lambda}(z, y) d\mu(z) \lesssim \int_{\mathbb{R}^d} e^{-\varepsilon \sqrt{x-z}} e^{-\varepsilon \sqrt{y-z}} e^{-\varepsilon d_\mu(z, y)} \frac{d\mu(z)}{|z-y|^{d-2}},$$

where the argument used to obtain equation (7.15) from [17] was used to obtain the final line. Since $x \in B$ and $y \in B'$, we will clearly have $|x - y| \leq 8\sqrt{d} r \leq 8\sqrt{d} \kappa_0 \rho_\mu(c) \leq 8\sqrt{d} \rho_\mu(c)$. On successively applying Lemma 4.1, (7) and the inclusion
\( B(x, 4\sqrt{d}\rho_\mu(c)) \subset B(c, 12\sqrt{d}\rho_\mu(c)) \), the estimate (47) will lead to

\[
\int_{\mathbb{R}^d} |\partial_1 \Gamma_\lambda(x,z)| \Gamma_{\mu+\lambda}(z,y) \, d\mu(z) \\
\leq e^{-\frac{\kappa}{2}\sqrt{|x-y|}} \left( \frac{1}{|x-y|^{d-2}} \right) \frac{\mu(B(x,|x-y|/2))}{\rho_\mu(y)} + \left( \frac{|x-y|}{\rho_\mu(y)} \right) \frac{1}{|x-y|^{d-1}}
\]

where the final line follows from (8) and Remark 4.2. Lemma 4.2 and the bound

\[
|y| \leq \delta \rho_\mu(c)
\]

then imply

\[
\int_{\mathbb{R}^d} |\partial_1 \Gamma_\lambda(x,z)| \Gamma_{\mu+\lambda}(z,y) \, d\mu(z) \leq e^{-\frac{\kappa}{2}\sqrt{|x-y|}} \frac{\kappa_0^{\delta_\mu}}{|x-y|^{d-1}}
\]

and therefore there must exist some \( C > 0 \), independent of \( \kappa_0 \), for which

\[
(48) \quad \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^d} |\partial_1 \Gamma_\lambda(x,z)| \Gamma_{\mu+\lambda}(z,y) \, d\mu(z) \, d\lambda \leq \frac{C\kappa_0^{\delta_\mu}}{|x-y|^d}.
\]

For \( x \in B' \) and \( y \in B \), it is clear that \( x_1 \geq y_1 \) and therefore

\[
K_0^{(1)}(x,y) = -c \frac{(x_1 - y_1)}{|x-y|^{d+1}} \leq -\frac{2c'}{|x-y|^d},
\]

for some \( c, c' > 0 \). This can be combined with (46) and (48) to produce the estimate

\[
K_\mu^{(1)}(x,y) \leq \frac{-2c' + C\kappa_0^{\delta_\mu}}{|x-y|^d}
\]

If we now set \( \kappa_0 = \left( \frac{c'}{C} \right)^{1/\delta} \) we will then obtain the estimate

\[
K_\mu^{(1)}(x,y) \leq -\frac{c'}{|x-y|^d}
\]

for all \( x \in B' \) and \( y \in B \). Let \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) be a non-negative function with support contained in \( B \) that satisfies

\[
\int_B f > 0.
\]
Then for \( x \in B' \),
\[
\left| R^{(1)}_\mu f(x) \right| = \left| \int_B K^{(1)}_\mu (x,y)f(y) \, dy \right| \\
\geq c' \int_B \frac{f(y)}{|x-y|^d} \, dy \\
\geq c'' \int_B f(y) \, dy,
\]
for some \( c'' > 0 \). This proves that for all \( 0 < \alpha < c'' \int_B f \) we must have
\[
B' \subset \left\{ x \in \mathbb{R}^d : \left| R^{(1)}_\mu f(x) \right| > \alpha \right\}.
\]
The \( L^p(w) \)-boundedness of the operator \( R^{(1)}_\mu \) can then be exploited in order to obtain the estimate
\[
w(B') \lesssim \frac{1}{\alpha^p} \int_B f(x)^p w(x) \, dx
\]
for all \( \alpha < c'' \int_B f \). On letting \( \alpha \to c'' \int_B f \),
\[
(49) \quad \left( \int_B f \right)^p \lesssim \frac{1}{w(B')} \int_B f(x)^p w(x) \, dx.
\]
By reversing the roles of \( B \) and \( B' \) in the previous argument, we will also find that the estimate
\[
\left( \int_{B'} g \right)^p \lesssim \frac{1}{w(B')} \int_{B'} g(x)^p w(x) \, dx
\]
must be valid for all non-negative \( g \) supported in \( B' \) with \( \int_{B'} g > 0 \). By setting \( g = 1_{B'} \) in this estimate we find that \( w(B) \lesssim w(B') \). On applying this to (49),
\[
(49) \quad \left( \int_B f \right)^p \lesssim \frac{1}{w(B)} \int_B f(x)^p w(x) \, dx.
\]
Setting \( f := (w + \varepsilon)^{-\frac{1}{p-1}} \mathbb{1}_B \) then leads to
\[
w(B) \left( \int_B (w + \varepsilon)^{-\frac{1}{p-1}} \right)^p \lesssim \int_B (w + \varepsilon)^{-\frac{1}{p-1}} w(x) \, dx \leq \int_B (w + \varepsilon)^{-\frac{1}{p-1}} (x) \, dx.
\]
By letting \( \varepsilon \to 0 \), the monotone convergence theorem then gives us (45).

7.2. Constant Potentials. Throughout this section, set \( V \equiv N \) for some \( N > 0 \). Notice that for this case the Agmon distance will be given by
\[
d_V(x, y) = N^{\frac{1}{2}} |x - y| \quad \forall x, y \in \mathbb{R}^d.
\]
This straightforward expression for the Agmon distance leads to a simpler characterisation of our weight class \( S^{V}_{p,c} \).

**Proposition 7.1.** Let \( 1 < p < \infty \) and \( c > 0 \). A weight \( w \) will be contained in the class \( S^{N}_{p,c} \) if and only if
\[
(50) \quad \sup_B \left( \frac{1}{|B|^c N^{\frac{1}{2}r}} \int_B w \right)^{\frac{1}{p}} \left( \frac{1}{|B|^c N^{\frac{1}{2}r}} \int_B w^{-\frac{1}{p-1}} \right)^{\frac{p-1}{p}} < \infty,
\]
where the supremum is taken over all Euclidean balls \( B = B(x, r) \in \mathbb{R}^d \) with \( x \in \mathbb{R}^d \) and \( r > 0 \).
**Proof.** The proof follows entirely from the fact that

$$B(x, r) = B_V(x, N^{\frac{1}{2}} r)$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Indeed, first suppose that $w \in S_{p,c}^N$. Let $B(x, r) \subset \mathbb{R}^d$ be a ball in the Euclidean metric. Then

$$w(B(x, r))^{\frac{1}{p}} w^{-\frac{1}{p-1}} (B(x, r))^\frac{p-1}{p} = w \left( B_V(x, N^{\frac{1}{2}} r) \right)^{\frac{1}{p}} w^{-\frac{1}{p-1}} \left( B_V(x, N^{\frac{1}{2}} r) \right)^\frac{p-1}{p}$$

$$\lesssim e^{N^{\frac{1}{2}} r} |B_V(x, N^{\frac{1}{2}} r)|$$

$$= e^{N^{\frac{1}{2}} r} |B(x, r)|.$$ 

Let’s now prove the converse. Suppose that (50) is finite. Then for any ball $B_V(x, r) \subset \mathbb{R}^d$ in the metric $d_V$,

$$w(B_V(x, r))^{\frac{1}{p}} w^{-\frac{1}{p-1}} (B_V(x, r))^\frac{p-1}{p} = w \left( B(x, N^{\frac{1}{2}} r) \right)^{\frac{1}{p}} w^{-\frac{1}{p-1}} \left( B(x, N^{\frac{1}{2}} r) \right)^\frac{p-1}{p}$$

$$\lesssim e^{cr} |B(x, N^{\frac{1}{2}} r)|$$

$$= e^{cr} |B_V(x, r)|,$$

which proves that $w \in S_{p,c}^N$.

In order to prove the first chain of inclusions of Conjecture 1.1, we need to know more about the behaviour of the singular kernel of $R_V = R_N$. This is provided by the following lemma.

**Lemma 7.1.** For $1 \leq j \leq d$, the singular kernel of the Riesz transform $R_N^{(j)} := \partial_j (-\Delta + N)^{-\frac{1}{2}}$ is given by

$$K_N^{(j)}(x, y) = -c_N \frac{(x_j - y_j)}{|x - y|} e^{-N^{\frac{1}{2}} |x - y|} s(N^{\frac{1}{2}} |x - y|),$$

for all $x, y \in \mathbb{R}^d$, where $c_N > 0$ and $s : (0, \infty) \to [0, \infty)$ is the function defined through

$$s(a) := \int_0^\infty e^{-at} \left( t + \frac{t^2}{2} \right)^{\frac{d-2}{2}} dt + \int_0^\infty te^{-at} \left( t + \frac{t^2}{2} \right)^{\frac{d-2}{2}} dt.$$

**Proof.** The operator $(-\Delta + N)^{-\frac{1}{2}}$ is given by convolution with the function

$$G_N(z) = c_N \int_0^\infty e^{-Nt} e^{-\frac{|z|^2}{4t}} \frac{4t}{z} dt,$$

for some constant $c_N > 0$ (c.f. [8, pg. 7]). Through a change of variables, it is easy to see that

$$G_N(z) \simeq G^1(N^{\frac{1}{2}} z)$$

for all $z \in \mathbb{R}^d$. The function $G^1$ is the well-known Bessel kernel of order one and has the representation

$$G^1(z) = ce^{-|z|} \int_0^\infty e^{-|z|t} \left( t + \frac{t^2}{2} \right)^{\frac{d-2}{2}} dt$$

for all $z \in \mathbb{R}^d$. The proof of this representation can be found in [1]. This, together with (51) and simple differentiation then proves our lemma. 

Notice that for the constant potential $V \equiv N$, the associated Hardy-Littlewood operator for $c > 0$, as defined in Definition 3.4, is given by

$$M_{N,c} f(x) := \sup_{t > 0} A^N_{t,c} |f| (x) := \sup_{t > 0} \frac{1}{|B(x,N^{-\frac{1}{2}} t)|} \int_{B(x,N^{-\frac{1}{2}} t)} |f(y)| \, dy,$$

where we are using the shorthand notation $M_{N,c} = M_{\rho N,c}$ and $A^N_{t,c} = A^c_{t,c}$. Let $R_{0,loc}, T^*_0,loc$ and $M^0,loc$ denote the $\rho_N$-localized parts of the classical Riesz transform, heat and Hardy-Littlewood maximal operators respectively. That is,

$$R_{0,loc} f(x) := \nabla (-\Delta)^{-\frac{1}{2}} \left( f \cdot 1_{B(x,\rho N(x))} \right) (x), \quad T^*_0,loc f(x) := \sup_{t > 0} e^{t\Delta} \left| f \cdot 1_{B(x,\rho N(x))} \right| (x)$$

and

$$M^0,loc f(x) := \sup_{t > 0} A^0_{t} \left| f 1_{B(x,\rho N(x))} \right| (x) := \sup_{t > 0} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left| f \cdot 1_{B(x,\rho N(x))} \right|.$$

The following proposition is an embodiment of the idea that the operators attached to the Schrödinger operator $-\Delta + N$ should behave, at a local scale, like their classical counterparts. In particular, a Muckenhoupt-type equivalence will hold at a local level between all of the operators listed.

**Proposition 7.2.** For any $c > 0$, the following statements are equivalent:

(i) $w \in A^\rho_{p,loc}$;
(ii) $T^*_0,loc$ is bounded on $L^p(w)$;
(iii) $T^*_N,loc$ is bounded on $L^p(w)$;
(iv) $M^0,loc$ is bounded on $L^p(w)$;
(v) $M^c,loc$ is bounded on $L^p(w)$;
(vi) $R_{0,loc}$ is bounded on $L^p$;
(vii) $R^N,loc$ is bounded on $L^p$.

**Proof.** (i) $\Rightarrow$ (ii). This is proved in [5, Thm. 1].

(ii) $\Rightarrow$ (iii). This follows trivially from the fact that the heat kernel of the operator $-\Delta + N$ is pointwise bounded from above by the heat kernel of $-\Delta$.

(iii) $\Rightarrow$ (iv). Fix a ball $B := B(x,r) \subset \mathbb{R}^d$ with $r \leq \rho_N(x) = N^{-\frac{1}{2}}$. For any other point $y \in B$ we will clearly have

$$\exp \left( -\frac{|x-y|^2}{4r^2} \right) \simeq 1.$$

Also note that since $r \leq N^{-\frac{1}{2}}$,

$$e^{-Nr^2} \simeq 1.$$

This gives

$$\frac{1}{|B|} \int_B |f(y)| \, dy \lesssim \frac{1}{e^{Nr^2}} \frac{1}{r^n} \int_B \exp \left( -\frac{|x-y|^2}{4r^2} \right) |f(y)| \, dy.$$
Set $t = r^2$ to obtain
\[
\frac{1}{|B|} \int_B |f(y)| \ dy \lesssim \frac{1}{e^{Nt/2}} \int_B \exp \left( -\frac{|x-y|^2}{4t} \right) |f(y)| \ dy \\
\lesssim e^{-t((N-\Delta)} |f1_B(x,\rho_N(x))| (x) \\
\lesssim T^*_N,loc f(x).
\]
This demonstrates that $\|T^*_N,loc\|_{L^p(w)} < \infty$ implies $\|M_{\theta,loc}\|_{L^p(w)} < \infty$.

$(iv) \implies (v)$. This implication follows trivially from the fact that the kernel of $A_{t,c}^N$ is pointwise bounded from above by the kernel of $A_0^N,\frac{1}{2}t$, for all $t > 0$.

$(v) \implies (i)$. Suppose that $M_{N,c}^{loc}$ is bounded on $L^p(w)$. First note that from the argument at the beginning of Theorem 7.1, in order to show that $w \in A_{p^*N,loc}^p$ it is sufficient to prove that $w \in A_{p,\rho_N,loc}^N$. The proof that $w \in A_{p,\rho_N,loc}^N$ is essentially identical to the classical proof that can be found in [8, pg. 280]. Fix $B := B(x,\rho) \subset \mathbb{R}^d$ with $\rho \leq \frac{1}{2}\rho_N(x) = \frac{1}{2}N^{-\frac{1}{2}}$. Let $M_{N,c} = M_{\rho_N,c}$ be the uncentered Hardy-Littlewood operator as defined in Definition 3.4. Then since $B \subset B(y,\rho_N(y))$ for all $y \in B$,

\[
w(B) \left( \frac{1}{|B|} \int_B |f| \right)^p \lesssim w(B) \left( \frac{1}{e^{cN^2t} |B|} \int_B |f| \right)^p \\
= \int_B \left( \frac{1}{e^{cN^2t} |B|} \int_B 1_B(y,\rho_N(y)) |f| \right)^p w(y) \ dy \\
\lesssim \int_B M_{N,c}^{loc} (f1_B)(y)^p w(y) \ dy \\
\lesssim \int_B M_{N,c}^{loc} (f1_B)(y)^p w(y) \ dy,
\]

for $c' < \frac{1}{2}$, where the last line follows from Proposition 3.3. It is not difficult to see that since the operator $M_{N,c}^{loc}$ is localized we must have

\[
M_{N,c}^{loc} (f1_B)(y) \lesssim M_{N,c}^{loc} (f1_B)(y).
\]

The boundedness of $M_{N,c}^{loc}$ then implies that

\[
w(B) \left( \frac{1}{|B|} \int_B |f| \right)^p \lesssim \int_B M_{N,c}^{loc} (f1_B)(y)^p w(y) \ dy \\
\lesssim \int_B |f(y)|^p w(y) \ dy.
\]

For $\epsilon > 0$, take $f := (w + \epsilon)^{-\frac{1}{p-1}}$ in the above inequality to obtain

\[
\frac{1}{|B|^p} w(B) \left( \int_B (w + \epsilon)^{-\frac{1}{p-1}} \right)^p \lesssim \left( \int_B (w + \epsilon)^{-\frac{1}{p-1}} \right) \leq \left( \int_B (w + \epsilon)^{-\frac{1}{p-1}} \right).
\]

Which leads to

\[
\left( \frac{1}{|B|} \int_B w \right) \left( \frac{1}{|B|} \int_B (w + \epsilon)^{-\frac{1}{p-1}} \right)^{p-1} \leq C,
\]
for some constant $C > 0$. The monotone convergence theorem then completes the proof of this implication.

$(i) \Rightarrow (vi)$, $(i) \Rightarrow (vii)$. The proof of these two implication is essentially contained in [5, Thm. 1] and [5, Thm. 3].

$(vi) \Rightarrow (i)$. This implication can be proved using a classical argument that can be found, for example, in [8, Thm. 9.4.8].

$(vii) \Rightarrow (i)$. This is proved in Theorem 7.1.

Let’s now prove the first chain of inclusions in Conjecture 1.1 for the case $V \equiv N$.

**Theorem 7.2.** Let $1 < p < \infty$. There exists $c_1, c_2 > 0$, independent of $p$ and $N$, such that

$$S_{p,c_1}^N \subset \left\{ w : \|R_N\|_{L^p(w)} < \infty \right\} \subset S_{p,c_2}^N$$

**Proof.** The first inclusion has already been proved in Theorem 1.1 and so it suffices to consider the second inclusion. Suppose that $w$ is a weight on $\mathbb{R}^d$ for which $\|R_N\|_{L^p(w)} < \infty$. We will adapt the classical proof of [8, Thm. 9.4.9].

Consider the operator $Wf(x) := \sum_{j=1}^d R_N^{(j)} f(x)$. Let $B = B(c,r)$ be a ball in $\mathbb{R}^d$ and $f \in L^1_{loc}(\mathbb{R}^d)$ a non-negative function with support contained in $B$ that satisfies

$$\int_B f > 0.$$ 

Let $B'$ be the ball in $\mathbb{R}^d$ that has center $c' = c + 2r = (c_1 + 2r, \cdots, c_d + 2r)$ and radius $r$. Clearly $B'$ will satisfy $x_j \geq y_j$ for each $1 \leq j \leq d$ when $x = (x_1, \cdots, x_d) \in B'$ and $y = (y_1, \cdots, y_d) \in B$. Lemma 7.1 then implies that for $x \in B'$,

$$|Wf(x)| \approx \sum_{j=1}^d \int_B \frac{x_j - y_j}{|x - y|} e^{-N^\frac{1}{2}|x-y|s(N^{\frac{1}{2}}|x-y|)} f(y) \, dy \geq \int_B e^{-N^\frac{1}{2}|x-y|s(N^{\frac{1}{2}}|x-y|)} f(y) \, dy.$$ 

It isn’t too difficult to check that the function $s$ satisfies the lower bound

$$s(a) \gtrsim \frac{1}{a^d}.$$
for all $a > 0$. Indeed, this follows from

$$s(a) \geq \int_0^\infty t e^{-at} \left( t + \frac{t^2}{2a^2} \right)^{\frac{d-2}{2}} dt$$

$$= \int_0^\infty \frac{t}{a} e^{-t} \left( \frac{t}{a} + \frac{t^2}{2a^2} \right)^{\frac{d-2}{2}} dt$$

$$\geq \frac{1}{a^2} \int_0^\infty t e^{-t} \left( \frac{t^2}{2a^2} \right)^{\frac{d-2}{2}} dt$$

$$\sim \frac{1}{a^d} \int_0^\infty t^{d-1} e^{-t} dt$$

$$\sim \frac{1}{a^d}.$$  

Therefore, for $x \in B'$ we will have

$$|Wf(x)| \geq D' \int_B e^{-N^{\frac{1}{2}} |x-y|} f(y) dy$$

$$\geq D e^{-4\sqrt{d}rN^{\frac{1}{2}}} \int_B f(y) dy,$$  

for some constants $D, D' > 0$. This implies that for any $0 < \alpha < De^{-4\sqrt{d}rN^{\frac{1}{2}} f_B f}$ we will have

$$B' \subset \{ x \in \mathbb{R}^d : |Wf(x)| > \alpha \}.$$  

The $L^p(w)$-boundedness of the operator $W$ will imply

$$w(B') \lesssim \frac{1}{\alpha^p} \int_B f(x)^p w(x) dx$$  

for all $\alpha < De^{-4\sqrt{d}rN^{\frac{1}{2}} f_B f}$ which then gives

$$\left( \int_B f \right)^p e^{-4p\sqrt{d}rN^{\frac{1}{2}}} \lesssim \frac{1}{w(B')} \int_B f(x)^p w(x) dx.$$  

The roles of $B$ and $B'$ can be reversed to obtain

$$\left( \int_{B'} g \right)^p e^{-4p\sqrt{d}rN^{\frac{1}{2}}} \lesssim \frac{1}{w(B)} \int_{B'} g(x)^p w(x) dx$$  

for all non-negative $g$ supported in $B'$ with $f_{B'} g > 0$. Setting $g = 1_{B'}$ in the above estimate then gives $w(B) e^{-4p\sqrt{d}rN^{\frac{1}{2}}} \lesssim w(B')$. Applying this to (52),

$$w(B) \left( \int_B f \right)^p e^{-8p\sqrt{d}rN^{\frac{1}{2}}} \lesssim \int_B f(x)^p w(x) dx.$$  

For $\varepsilon > 0$, set $f := (w + \varepsilon)^{-\frac{1}{p-1}} 1_B$ in the above estimate to obtain

$$w(B) \left( \frac{1}{|B|} \int_B (w + \varepsilon)^{-\frac{1}{p-1}} \right)^p e^{-8\sqrt{d}prN^{\frac{1}{2}}} \lesssim \int_B (w + \varepsilon)^{-\frac{p}{p-1}} w$$

$$\leq \int_B (w + \varepsilon)^{-\frac{1}{p-1}}.$$
The monotone convergence theorem then yields
\[ w(B)^{\frac{1}{p}} w^{-\frac{1}{p}}(B)^{\frac{p-1}{p}} \lesssim e^{8\sqrt{d}N^{\frac{1}{2}}r} |B| . \]
Proposition 7.1 then implies that \( w \in S_{p,8\sqrt{d}}^N \).

7.3. Potentials Bounded from Above and Below. In this section, it will be proved that the second chain of inclusions in Conjecture 1.1 holds for potentials that are bounded both from above and from below. Suppose that there exists \( N, M > 0 \) for which \( M \leq V(x) \leq N \) for all \( x \in \mathbb{R}^d \). We will require the following lemma.

Lemma 7.2. The heat kernel for the Schrödinger operator \(-\Delta + V\) satisfies the estimate
\[ e^{-Nt} e^{-\frac{|x-y|^2}{4t}} \lesssim k_t^V(x,y) \lesssim e^{-Mt} e^{-\frac{|x-y|^2}{4t}} \]
for a.e. \( x, y \in \mathbb{R}^d \) and \( t > 0 \).

Proof. Let \( V_1 \) and \( V_2 \) be two potentials with \( 0 \leq V_1(x) \leq V_2(x) \) for all \( x \in \mathbb{R}^d \). To prove our lemma it is sufficient to show that \( k_t^{V_2}(x,y) \leq k_t^{V_1}(x,y) \) for each \( t > 0 \) and a.e. \( x, y \in \mathbb{R}^d \). Theorem 2.24 of [15] states that the semigroup of \((V_2 - \Delta)\) will be dominated by the semigroup of \((V_1 - \Delta)\). That is,
\[ (53) \quad |e^{-t(V_2 - \Delta)} f| \leq e^{-t(V_1 - \Delta)} |f| \]
for all \( f \in L^2(\mathbb{R}^d) \). Fix \( (x, t) \in \mathbb{R}^d \times (0, \infty) \) and suppose that there exists compactly supported \( E \subset \mathbb{R}^d \) such that \( k_t^{V_1}(x,y) < k_t^{V_2}(x,y) \) for all \( y \in E \). Then applying (53) to the function \( f(y) = 1_E \) yields
\[ \int_E k_t^{V_2}(x,y) dy \leq \int_E k_t^{V_1}(x,y) dy, \]
implying \( |E| = 0 \). Therefore \( k_t^{V_2}(x,y) \leq k_t^{V_1}(x,y) \) for each \( t > 0 \), for almost every \( x, y \in \mathbb{R}^d \).

Lemma 7.3. For any \( c > 0 \) and \( 1 < p < \infty \), the following chain of inclusions holds,
\[ S_{p,c}^M \subset S_{p,c}^V \subset S_{p,c}^N. \]

Proof. It is clear from the definition of the Agmon distance that the inequality \( M \leq V(x) \leq N \) will imply \( d_M(x,y) \leq d_V(x,y) \leq d_N(x,y) \) for any \( x, y \in \mathbb{R}^d \). Therefore, \( B_N(x,r) \subset B_V(x,r) \subset B_M(x,r) \) for all \( x \in \mathbb{R}^d \) and \( r > 0 \).
Suppose that $w \in S^M_{p,c}$. Then for any $x \in \mathbb{R}^d$ and $r > 0$,
\[
    w(B_V(x,r))^{\frac{1}{p}} \max_{x \neq y} \frac{1}{r} (B_V(x,y))^{\frac{p-1}{p}} \leq w(B_M(x,r))^{\frac{1}{p}} \max_{x \neq y} \frac{1}{r} (B_M(x,y))^{\frac{p-1}{p}} \\
    \lesssim e^{c\max_{x \neq y} |B_M(x,y)|} \\
    \lesssim e^{c\max_{x \neq y} |B_N(x,y)|} \\
    \lesssim e^{c\max_{x \neq y} |B_V(x,y)|}.
\]

Suppose instead that $w \in S^V_{p,c}$. Then for any $x \in \mathbb{R}^d$ and $r > 0$,
\[
    w(B_N(x,r))^{\frac{1}{p}} \max_{x \neq y} \frac{1}{r} (B_N(x,y))^{\frac{p-1}{p}} \leq w(B_V(x,r))^{\frac{1}{p}} \max_{x \neq y} \frac{1}{r} (B_V(x,y))^{\frac{p-1}{p}} \\
    \lesssim e^{c\max_{x \neq y} |B_V(x,y)|} \\
    \lesssim e^{c\max_{x \neq y} |B_M(x,y)|} \\
    \lesssim e^{c\max_{x \neq y} |B_N(x,y)|}.
\]

Proposition 7.3. Let $1 < p < \infty$. We have the following chain of inclusions for some $c_1, c_2, c_3 > 0$ with $c_1 < c_2 < c_3$,
\[
S^V_{p,c_1} \subset \left\{ w : \|T^*_V w \|_{L^p(w)} < \infty \right\} \subset \left\{ w : \|M_{N,c_2} w \|_{L^p(w)} < \infty \right\} \subset S^V_{p,c_3}.
\]
The constants $c_1, c_2$ and $c_3$ will be independent of $p$.

Proof. First let’s consider the last inclusion in the above chain. The implication that $\|M_{N,c_2} w \|_{L^p(w)} < \infty$ gives $w \in S^N_{p,3c_2}$ is asserted by Proposition 3.4. It is obvious that there must exist some $c_3 > 3c_2$ for which $S^N_{p,3c_2} \subset S^M_{p,c_3}$. The last inclusion of our proposition then follows from Lemma 7.3.

Next, let’s prove the second inclusion. Let $w$ be a weight on $\mathbb{R}^d$ for which $\|T^*_V w \|_{L^p(w)} < \infty$. Set $c_2 = N^{\frac{1}{2}} + N^{-\frac{1}{4}}$ and fix $f \in L^p(w)$. Then for $0 < t < N^{-\frac{1}{2}}$ we have
\[
A^N_{N^{\frac{1}{2}} t, c_2} |f| (x) = \frac{1}{|B(x,t)| e^{c_2 N^{\frac{1}{2}}}} \int_{B(x,t)} |f(y)| \, dy \\
\leq \frac{1}{t^d} \int_{B(x,t)} |f(y)| \, dy \\
\lesssim \frac{1}{t^d e^{N^{\frac{1}{2}}}} \int_{B(x,t)} |f(y)| \, dy.
\]
For $y \in B(x,t)$ we have
\[
\exp\left( -\frac{|x-y|^2}{4t^2} \right) \approx 1
\]
and therefore, on applying Lemma 7.2,
\[
A^N_{N^{\frac{1}{2}} t, c_2} |f| (x) \lesssim \frac{1}{t^d e^{N^{\frac{1}{2}}}} \int_{B(x,t)} \exp\left( -\frac{|x-y|^2}{4t^2} \right) |f(y)| \, dy \\
\lesssim \int_{B(x,t)} k_{c_2}^V(x,y) |f(y)| \, dy \\
\lesssim T^*_V f(x).
\]
Next, consider the case \( t \geq N^{-\frac{1}{2}} \). We have
\[
A_{N^2 t, c_2}^N |f|(x) = \frac{1}{|B(x,t)|^{c_2 N^{\frac{2}{3}}} t} \int_{B(x,t)} |f(y)| \, dy
\]
\[
\simeq \frac{1}{t^d e^{N^{t}}} \int_{B(x,t)} |f(y)| \, dy.
\]
For \( y \in B(x,t) \) we must have
\[
\exp \left( -\frac{t}{4} \right) \leq \exp \left( -\frac{|x - y|^2}{4t} \right).
\]
Therefore,
\[
A_{N^2 t, c_2}^N |f|(x) \lesssim \frac{1}{t^d e^{N^{t}}} \int_{B(x,t)} e^{-\frac{|x - y|^2}{4t}} |f(y)| \, dy.
\]
Notice that since \( t \geq N^{-\frac{1}{2}} \), (54) gives
\[
A_{N^2 t, c_2}^N |f|(x) \lesssim \frac{1}{t^d e^{N^{t}}} \int_{B(x,t)} e^{-\frac{|x - y|^2}{4t}} |f(y)| \, dy
\]
\[
\lesssim e^{-t(V-\Delta)} |f|(x)
\]
\[
\lesssim T^*_p f(x),
\]
where Lemma 7.2 was applied in the second line. This proves that
\[
M_{N,c_2} f(x) \lesssim T^*_p f(x)
\]
for all \( x \in \mathbb{R}^d \) and thus completes the proof of the second inclusion.

Finally, let’s prove the first inclusion. Suppose that \( w \in S^V_{p,c_1} \) for some \( c_1 > 0 \), \( 1 < p < \infty \) and fix \( f \in L^p(w) \). Proposition 3.1 tells us that \( S^V_{p,c_1} \subset A_{p}^{V,\text{loc}} \). It follows from the fact that \( T^*_p \) is pointwise bounded from above by the heat maximal operator for the Laplacian \( T^*_0 \) and [5, Thm 1] that \( \|T^*_p\|_{L^p(w)} < \infty \). It remains to show that the global part of \( T^*_p \) is bounded on \( L^p(w) \).

Let \( B_j := B(x_j, \rho_V(x_j)) \) be as given in Proposition 2.1 and set \( B_z := B(x, \rho_V(x)) \) for each \( x \in \mathbb{R}^d \). On expanding the \( L^p(w) \)-norm of \( T^*_V g_{\text{glob}}f \) and applying Lemma 7.2,
\[
\|T^*_V g_{\text{glob}}f\|_{L^p(w)} \lesssim \left( \sum_j \int_{B_j} \left( \sup_{t \geq 0} \frac{e^{-Mt}}{t^\frac{d}{2}} \int_{B_z^*} e^{-\frac{|x-y|^2}{4t}} |f(y)| \, dy \right)^p w(x) \, dx \right)^{\frac{1}{p}}
\]
\[
= \left( \sum_j \int_{B_j} \left( \sup_{t \geq 0} \sum_{k=1}^{\infty} \frac{e^{-Mt}}{t^\frac{d}{2}} \int_{(k+1)B_z \setminus kB_z} e^{-\frac{|x-y|^2}{4t}} |f(y)| \, dy \right)^p w(x) \, dx \right)^{\frac{1}{p}}.
\]
Observe that the critical radius function satisfies \( N^{-\frac{1}{2}} \leq \rho_V(x) \leq M^{-\frac{1}{2}} \) for all \( x \in \mathbb{R}^d \). Therefore, for \( j \in \mathbb{N}, k \geq 1, x \in B_j \) and \( y \notin kB_z \),
\[
|x - y| \geq k\rho_V(x) \geq kN^{-\frac{1}{2}}.
\]
Applying this bound to (55) gives

\[ \| T_{V,\text{glob}} f \|_{L^p_w} \lesssim \left( \sum_j \left( \sup_{t>0} \sum_{k=1}^\infty \frac{e^{-M t}}{t^2} e^{-\frac{k^2}{M^2 t}} \int_{(k+1)B_x \setminus kB_x} |f(y)| \, dy \right)^p \, w(x) \, dx \right)^{\frac{1}{p}}. \]

For \( y \in (k+1)B_x \), Lemma 2.1 implies that

\[ |x_j - y| \leq |x - y| + |x_j - x| \leq (k + 1)\rho \nu(x) + \rho \nu(x_j) \leq \beta 2^{\frac{k_0}{\sigma \rho}} (k + 1)\rho \nu(x) + \rho \nu(x_j) \leq 4\sigma k \rho \nu(x_j), \]

where \( \sigma := \beta 2^{\frac{k_0}{\sigma \rho}} \). Therefore \((k + 1)B_x \subset 4\sigma k B_x \). This inclusion together with \( B_j \subset B(x_j, M^{-\frac{1}{2}}) \) and Hölder’s inequality then leads to

\[ \| T_{V,\text{glob}} f \|_{L^p_w} \lesssim \left( \sum_j \left( \sup_{t>0} \sum_{k=1}^\infty \frac{e^{-M t}}{t^2} e^{-\frac{k^2}{M^2 t}} (B(x_j, 4\sigma k M^{-\frac{1}{2}}))^{\frac{p-1}{p}} \| f \|_{L^p(4\sigma k B_j, w)} \right)^p \right)^{\frac{1}{p}}. \]

Lemma 7.3 tells us that \( w \in S_{p, c_1}^N \). This condition can be applied to the term involving the weights to give

\[ w(B(x_j, 4\sigma k N^{\frac{1}{2}} M^{-\frac{1}{2}}))^{\frac{1}{p}} \| f \|_{L^p(4\sigma k B_j, w)} \lesssim e^{c_1 4\sigma k N^{\frac{1}{2}} M^{-\frac{1}{2}}}. \]

Define, for \( t > 0 \) and \( k \in \mathbb{N}^* \), the quantity

\[ F(k, t) := \frac{e^{-Mt}}{t^2} e^{-\frac{k^2}{Mt}} e^{c_1 k} k^d. \]

It will be proved that if we set \( c_1 \) small enough then there must exist some \( C, \epsilon > 0 \) such that

\[ F(k, t) \leq C \cdot e^{-\epsilon k} \]

for all \( k \in \mathbb{N}^* \) and \( t > 0 \). First note that \( k^d/t^2 \leq e^{\delta k^2/t} \) for any \( \delta > 0 \). We will therefore have

\[ F(k, t) \lesssim e^{-M t} e^{-\frac{k^2}{Mt}} e^{c_1 k} \]

for all \( k \in \mathbb{N}^* \) and \( t > 0 \).
Suppose first that \( t \geq k \). Then
\[
F(k, t) \lesssim e^{-Mk}e^{c_1k}.
\]
If we let \( c_1 \) be small enough so that \( c'_1 = 4\sigma c_1 N^{\frac{1}{2}} M^{-\frac{1}{2}} < M \) then (59) will hold for this case.

Next, suppose that \( t < k \). Then
\[
F(k, t) \lesssim e^{-\frac{k}{N}}e^{c_1k}.
\]
Setting \( c_1 \) small enough so that \( c'_1 = 4\sigma c_1 N^{\frac{1}{2}} M^{-\frac{1}{2}} < \frac{1}{N} \) will ensure that (59) holds for this case as well.

Applying estimate (59) to (58) gives
\[
\|T^*_{V, \text{glob}}\|_{L^p(w)} \lesssim \left( \sum_j \left( \sum_{k=1}^{\infty} e^{-ck} \|f\|_{L^p(4\sigma k B_j \cdot w)} \right)^p \right)^{\frac{1}{p}}
\]
(60)
\[
\leq \sum_{k=1}^{\infty} e^{-ck} \left( \sum_j \|f\|_{L^p(4\sigma k B_j \cdot w)}^p \right)^{\frac{1}{p}}.
\]

From the bounded overlap property of the balls \( B_j \), Proposition 2.1, there exists \( N_1 > 0 \) for which
\[
\sum_j \|f\|_{L^p(4\sigma k B_j \cdot w)} \lesssim k^{N_1} \|f\|_{L^p(w)}.
\]
Applying this to (60) then gives us our result.

7.4. The Harmonic Oscillator. In this section we consider the harmonic oscillator potential, \( V(x) = |x|^2 \), and prove the second chain of inclusions in Conjecture 1.2. We will require the following lemma. It states the exact form of the heat kernel corresponding to this potential. Its proof can be found in [19] in dimension \( d = 1 \). Higher dimensions follow from this case by taking tensor products of Hermite functions.

Lemma 7.4. For \( t > 0 \), define the map \( \tilde{k}_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) through
\[
\tilde{k}_t(x, y) = \frac{1}{(2\pi t)^{d/2}} \exp \left( -\frac{|x-y|^2}{2t} \right) \cdot \exp \left( -\alpha(t) \left( |x|^2 + |y|^2 \right) \right),
\]
where \( \alpha \) is defined by
\[
\alpha(t) := \frac{\sqrt{1+t^2} - 1}{2t}
\]
for all \( x, y \) in \( \mathbb{R}^d \) and \( t > 0 \). The operator \( T^*_{|x|^2} \) is then given by
\[
T^*_{|x|^2} f(x) := \sup_{t > 0} \int_{\mathbb{R}^d} \tilde{k}_t(x, y) |f(y)| \, dy
\]
for \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \).

Let \( \tilde{A}^m_{V,c} = \tilde{A}^{\text{Hermite}}_{V,c} \) and \( \tilde{M}^m_{V,c} = \tilde{M}^{m_{\text{Hermite}}} \) be as defined in Definition 3.5.

Proposition 7.4. Let \( V(x) = |x|^2 \). There exists \( c, m > 0 \) such that if \( \|T^*_{|x|^2}\|_{L^p(w)} < \infty \) for \( 1 < p < \infty \) then \( w \in H^m_{p,c} \).
implies that \( \rho \) where the last line follows from

\[ H_0 \text{ for which } c > 0 \text{ and } m > 0 \text{ for which the operator} \]

\[ \tilde{M}_{V,c}^m \| f \| (x) := \sup_{t > 0} \tilde{A}^{V,m}_{t,c} \| f \| (x) := \sup_{r > 0} \frac{1}{\Phi_{m,c}^V(x,r) |B(x,r)|} \int_{B(x,r)} |f(y)| \, dy \]

is bounded on \( L^p\). Proposition 3.6 will then allow us to conclude that \( w \in H_{p,c}^{V,2m(k_0+1)} \). To prove the boundedness of \( \tilde{M}_{V,c}^m \) on \( L^p\) it is sufficient to prove the pointwise bound \( \tilde{A}^{V,m}_{t,c} \| f \| (x) \lesssim T^*_V f(x) \) for any \( t > 0 \) and \( x \in \mathbb{R}^d \).

Fix \( x \in \mathbb{R}^d \) and \( t > 0 \). For \( y \in B(x,t) \),

\[ \alpha(t^2) \left( |x|^2 + |y|^2 \right) \lesssim \alpha(t^2) \left( |x|^2 + |x-y|^2 \right) \lesssim \alpha(t^2) \left( |x|^2 + t^2 \right). \]

It then follows from the simple estimates \( \alpha(t^2) \lesssim t^2 \) and \( \alpha(t^2) \lesssim 1 \),

\[ \alpha(t^2) \left( |x|^2 + |y|^2 \right) \lesssim t^2 |x|^2 + t^2 \lesssim t^2 (1 + |x|)^2 \approx \left( \frac{t}{\rho_{|x|^2}(x)} \right)^2, \]

where the last line follows from \( \rho_{|x|^2}(x) \simeq (1 + |x|)^{-1} \). This proves that there exists \( c > 0 \) for which

\[ \Phi^{2}_{c,2}(x,t) \exp \left( -c \left( 1 + \frac{t}{\rho_{|x|^2}(x)} \right)^2 \right) \leq \exp(-\alpha(t^2)(|x|^2 + |y|^2)). \]

This estimate combined with the trivial estimate \( e^{-\frac{|x-y|^2}{2t^2}} \simeq 1 \) for \( y \in B(x,t) \) implies that

\[ \tilde{A}^{V,2}_{t,c} \| f \| (x) = \frac{1}{\Phi^{2}_{c,2}(x,t) |B(x,t)|} \int_{B(x,t)} |f(y)| \, dy \lesssim \int_{B(x,t)} \frac{1}{t^{d+2}} \exp\left( -\alpha(t^2)(|x|^2 + |y|^2) \right) \exp\left( -\frac{|x-y|^2}{2t^2} \right) |f(y)| \, dy \lesssim T^*_V f(x). \]

Therefore \( \| \tilde{M}_{V,c}^2 \|_{L^p\left( w \right)} < \infty \) and Proposition 3.6 allows us to conclude that \( w \in H_{p,c}^{V,4(k_0+1)} \). \qed

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