Precise Laplace approximation for mixed rough differential equation

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Abstract

This work focuses on the Laplace approximation for the rough differential equation (RDE) driven by mixed rough path \((B^H, W)\) with \(H \in (1/3, 1/2)\) as \(\varepsilon \to 0\). Firstly, based on geometric rough path lifted from mixed fractional Brownian motion (fBm), the Schilder-type large deviation principle (LDP) for the law of the first level path of the solution to the RDE is given. Due to the particularity of mixed rough path, the main difficulty in carrying out the Laplace approximation is to prove the Hilbert-Schmidt property for the Hessian matrix of the Itô map restricted on the Cameron-Martin space of the mixed fBm. To this end, we imbed the Cameron-Martin space into a larger Hilbert space, then the Hessian is computable. Subsequently, the probability representation for the Hessian is shown. Finally, the Laplace approximation is constructed, which asserts the more precise asymptotics in the exponential scale.

Keywords. Large deviation principle, Laplace approximation, Mixed rough path, fractional Brownian motion

Mathematics subject classification. 60F10, 60G22, 60H10.

1. Introduction

In this paper, we consider the rough differential equation (RDE) driven by mixed rough paths as follows,

\[ dY^\varepsilon_t = [\sigma(Y^\varepsilon_t)] \dot{\sigma}(Y^\varepsilon_t) \varepsilon d(B^H_t, W)_t + \beta(\varepsilon, Y^\varepsilon_t) dt, \quad Y^\varepsilon_0 = 0, \]

(1.1)

where \(\varepsilon > 0\) is a small parameter. And \([\sigma, \dot{\sigma}]\) denotes the block matrix with \(\sigma \in C^\infty_b(\mathbb{R}^n, \text{Mat}(n, d_1))\) and \(\dot{\sigma} \in C^\infty_b(\mathbb{R}^n, \text{Mat}(n, d_2))\), and \(\beta \in C^\infty_b([0, 1] \times \mathbb{R}^n, \mathbb{R}^n)\) with \(C^\infty_b\) being the set of bounded smooth functions with bounded derivatives. \((B^H_t, W)_t \in \mathcal{G}_{\beta}(\mathbb{R}^{d_1+d_2})\), where \(\mathcal{G}_{\beta}(\mathbb{R}^{d_1+d_2})\) is the geometric rough path space, represents the mixed geometric rough path, which will be introduced in Section 2. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a completely probability space, \((b^H_s, w_s)_{t \geq 0} \in \mathbb{R}^{d_1}\)-valued fractional Brownian motion (fBm) with Hurst index \(H \in (1/3, 1/2)\), \((w_t)_{t \geq 0} \in \mathbb{R}^{d_2}\)-valued Brownian motion (Bm). For each time \(t\), we denote by \(\mathcal{F}_t\) the \(\sigma\)-field generated by the random variables \(\{(b^H_s, w_s), 0 \leq s \leq t\}\) and all \(\mathbb{P}\)-null sets. The expectation with respect to \(\mathbb{P}\) is denoted by \(\mathbb{E}\). More details about fBm could see [\textsuperscript{1}] [\textsuperscript{2}].

Let \(Y^\varepsilon = \Phi_\varepsilon(\varepsilon(B^H, W), \lambda) : \mathcal{G}_{\beta}(\mathbb{R}^{d_1+d_2+1}) \to \mathcal{G}_{\beta}(\mathbb{R}^n)\) denote the Itô map corresponding to \([\textsuperscript{1}]\) with \(\lambda_t = t\). The purpose of this paper is to prove the Laplace approximation for \(Y^\varepsilon, 1\) under natural assumptions, the first level path of the solution map,

\[ J(\varepsilon) := \mathbb{E} \left[ \exp \left( - \frac{F(Y^\varepsilon, 1)}{\varepsilon^2} \right) \right], \]

(1.2)

where \(F\) is a suitable real-valued bounded continuous function. See Section 2 for precise assumptions.

Laplace approximation is devised by Pierre-Simon Laplace in 1977. For SDE theory, the origins can be traced back to the Azencott’s work on stochastic Taylor expansion [\textsuperscript{2}]. Based on this, Ben Arous gave

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some Laplace short time expansion for Wiener functional \([4]\). Since then, Laplace approximation has been a central topic in the probability field. Watanabe studied the precise asymptotics of the Schilder-type for some classes of generalized Wiener functionals by Malliavin calculus \([50,29]\). Besides, Kusuoka and Stroock presented the asymptotic expansion of certain Wiener functionals as the variance of the Wiener goes to 0 \([20]\). Osajima focused on the asymptotic expansion of the density function of Wiener functionals \([21]\). For stochastic partial differential equations (SPDEs), it could refer to \([22,3,13]\).

However, the abovementioned references focused on the Bm. Different from Bm, fBm is self-similar and possesses long-range dependence, which has been applied in some complex systems \([8]\). Since the fBm is neither a semi-martingale nor a Markov process, it can not be solved by conventional stochastic analysis \([26]\). Rough path, proposed by Terry Lyons in 1998, considers the path itself and the iterated integral together \([23]\). It does not need martingale integration theory, Markov property, and filtration theory, but focuses on the topology of geometric rough path space \([24]\). Up to now, there exist three main formulations to rough path theory, Lyons’ original formulation \([24]\), Gubinelli’s controlled path theory \([10]\), and Davie’s formulation \([12,4]\). With the aid of rough path theory, Gaussian processes, including Bm and fBm, can be lifted to geometric rough paths. Therefore, it has been applied to analyze the LDP for Gaussian rough paths, including the Brownian rough path and fBm \([22,23,11,14]\). Further, starting with Aida \([3]\), Inahama and Kawabi studied the Taylor expansion and Laplace approximation for Gaussian rough paths, where the fractional rough paths can be covered \([13,16,17]\).

Therefore, we wish to investigate the precise Laplace asymptotics for the first level path of \(Y^*_t\). Firstly, we prove that the mixed fBm can be lifted to the mixed geometric rough path. Then, we give the Schilder-type LDP for the laws of the first level path of the solution to the above RDE \([1,1]\), which provides the asymptotics of \(J(\varepsilon)\) with \(\varepsilon \to 0\) on logarithmic scale. Furthermore, it proceeds to show the Laplace approximation, providing more precise asymptotics in the exponential scale. Due to the particularity of mixed rough path, the main difficulty in carrying out the Laplace asymptotics is to analyze Hilbert-Schmidt property for the Hessian matrix of the Itô map restricted on the Cameron-Martin space. To this end, we imbed the Cameron-Martin space into a larger Hilbert space. We prove the Hilbert-Schmidt property with an orthonormal basis in this Hilbert space. Then, the probability representation for the Hessian is analyzed. Finally, the precise Laplace asymptotics for RDE driven by mixed rough path is constructed.

The plan of this paper is as follows. In the next section, we establish notation, give some precise conditions and main result. In Section 3, we state and prove the Schilder-type LDP for the laws of the first level path of the solution to the RDE \([1,1]\). We prove the Hilbert-Schmidt property and probability representation of the Hessian matrix in Section 4. Finally, Section 5 states the main proof. Throughout this paper, \(c, C, C_\star\) denote certain positive constants that may vary from line to line. Throughout this paper, we take \(t \in [0,1]\).

Analogous results hold for any finite time interval.

### 2. Preliminaries and main results

Before introducing the fBm and rough path, we illustrate some information for spaces. Let \(B\) be a Banach space with \(\text{dim}B < \infty\), such as \(B = \mathbb{R}^{d_1+d_2}\) or \(B = \text{Mat}(n, d_1+d_2)\). Denote that

\[
C = C([0,1],B) = \{k : [0,1] \to B \mid \text{continuous} \},
\]

the space of \(B\)-valued continuous functions with the usual sup-norm. For \(p \geq 1\), \(C_{p-\text{var}}\) is the set of \(k \in C\) such that

\[
\|k\|_{p-\text{var}} := |k_0| + \left( \sup_{P} \sum_{i=1}^{n} |k_{t_i} - k_{t_{i-1}}|^p \right)^{1/p} < \infty,
\]

where \(P\) runs over all the finite partition of \([0,1]\).

Denote \(W^d,p\) with \(p > 1, 0 < \delta < 1\) the Besov space, for a measurable function \(k : [0,1] \to B\),

\[
\|k\|_{W^d,p} = \|k\|_{L^p} + \left( \int_{[0,1]^2} \frac{|k_t - k_s|^p}{|t-s|^{1+\delta} \delta^p} ds dt \right)^{1/p}.
\]
Refer to [2], we can see that $W^{δ,p}$ is given by the interpolation of $W^{1,p}$ and $W^{0,p} = L^p$. When $p = 2$, $W^{δ,2}_0 ≅ L^{δ,2}_0 = [W^{1,2}, L^2]_{1−δ}$, which can be defined as follows,

$$L^{δ,2} = \{ f = c_0 + \sum_{n=1}^{∞} c_n \sqrt{2}\cos(nπx)|c_n \in \mathbb{C}; \sum_{n=0}^{∞} (1 + n^2)^δ |c_n|^2 < \infty \},$$

then $W^{δ,2}_0$ and $L^{δ,2}_0$ are equivalent Hilbert spaces.

### 2.1. FBm and Bm

Consider the $\mathbb{R}^{d_1}$-valued fBm $(b^H_t)_{t≥0}$ with Hurst parameter $H ∈ (1/3, 1/2)$,

$$b^H_t = (b^H_{1,t}, b^H_{2,t}, \cdots, b^H_{d_1,t}),$$

where $(b^H_{i,t})_{t≥0}, i ∈ \{1, \cdots, d_1\}$ are independent one-dimensional fBms. The above $\mathbb{R}^{d_1}$-valued fBm $(b^H_t)_{t≥0}$ is a centred Gaussian process, satisfying that

$$\mathbb{E}[b^H_t b^H_s] = \frac{1}{2} [t^{2H} + s^{2H} − |t − s|^{2H}] × I_{d_1}, \quad (s, t ≥ 0),$$

and

$$\mathbb{E}[b^H_t − b^H_s]^2 = |t − s|^{2H} × I_{d_1}, \quad (s, t ≥ 0),$$

where $I_{d_1}$ is the identity matrix in $\mathbb{R}^{d_1×d_1}$. When $H = 1/2$, it is a standard Bm in $\mathbb{R}^{d_1}$. The reproducing kernel Hilbert space for the fBm $(b^H_t)_{t≥0}$ denoted by $\mathcal{H}^{H,d_1}$.

Then, consider the $\mathbb{R}^{d_2}$-valued Bm $(w_t)_{t≥0}$,

$$w_t = (w^1_t, w^2_t, \cdots, w^{d_2}_t),$$

where $(w^i_t)_{t≥0}, i ∈ \{1, \cdots, d_2\}$ are independent one-dimensional Bms. The reproducing kernel Hilbert space for $(w_t)_{t≥0}$, denoted by $\mathcal{H}^{x,d_2}$, which is defined as follows,

$$\mathcal{H}^{x,d_2} := \{ \hat{k} ∈ P(\mathbb{R}^{d_2}) | \hat{k}_t = \int_0^t k^t_ds \text{ for all } t \text{ with } \|\hat{k}\|^2_{\mathcal{H}^{x,d_2}} := \int_0^1 |\hat{k}_t|^2_{\mathcal{H}^{x,d_2}} dt < \infty \},$$

where $P(\mathbb{R}^{d_2}) := \{ \hat{k} ∈ C([0, 1], \mathbb{R}^{d_2}) | \hat{k}_0 = 0 \}$. Due to [3], Proposition 3.4, it has $\mathcal{H}^{H,d_1} ⊆ \mathcal{H}^{x,d_2} ⊆ \mathcal{H}^{δ,2}$. Hence, $(k, \hat{k}) ∈ \mathcal{H}$ is of finite $q$-variation with $(H + 1/2)^{-1} < q < 2$.

### 2.2. Rough path

Next, we introduce the geometric rough path. Set $2 < p < 3$, and $\Delta = \{(s, t) | 0 ≤ s ≤ t ≤ 1\}$, for the continuous map $A$ from $\Delta$ to the Banach space $B$, define that

$$\|A\|_{p-var} = \left( \sup_{p} \sum_{i=1}^{n} |A_{t_{i-1},t}|_B^p \right)^{1/p},$$

where $\mathcal{P}$ stands for the finite partition of $[s, t]$.

Denote $B$ be the Banach space, a continuous map

$$X = (1, X^1, X^2) : \Delta → T^2(B) = \mathbb{R} ⊕ B ⊕ B^⊗2,$$

is said to be a $B$-valued rough path of roughness 2 if it satisfies the following conditions,

(Condition A): For any $s ≤ u ≤ t$, $X_{s,t} = X_{s,u} ⊗ X_{u,t}$ where $⊗$ stands for the tensor product.

(Condition B) For all $1 ≤ j ≤ [p]$, $\|X^j\|_{p/j-var} < \infty$.

The 0-th component 1 is omitted. Therefore, we denote the rough path by $X = (X^1, X^2)$. The set of all the $B$-valued rough paths of roughness $2 < p < 3$ is denoted by $\Omega_p(B)$. With the distance $d_p(X, Y) = \sum_{i=1}^{[p]} \|X^j - Y^j\|_{p/j-var}$, it is a complete space.
For rough path $X$, denote that

$$\|X\|_{p-\text{var}} = \|X^1\|_{p-\text{var}} + \|X^2\|_{p/2-\text{var}}^{1/2}.$$ 

Next, for rough paths $X$, $Y$, for $\kappa > p - 1$, define that

$$D_{j,p} (X, Y) = D_{j,p} (X^j, Y^j) = \left( \sum_{n=1}^{\infty} n^\kappa \sum_{l=1}^{2^n} \left| X_{t_{i_l-1}}^{j} - \right\|_{T}^{j/p} \right) \left( \sum_{n=1}^{\infty} n^\kappa \sum_{l=1}^{2^n} \left| X_{t_{i_l-1}}^{j} - \right\|_{T}^{j/p} \right),$$

(2.1)

and

$$d_p (X, Y) \leq C \max (D_{1,p} (X, Y), D_{1,p} (X, Y) (D_{1,p} (X) + D_{1,p} (Y)), D_{2,p} (X, Y)).$$

Then, from [24], Section 4.1, it has

$$\|X^1 - Y^1\|_{p-\text{var}}^p \leq c_1 D_{1,p} (X, Y)^p,$$

and

$$\|X^2 - Y^2\|_{p/2-\text{var}}^{p/2} \leq c_1 \left[ D_{2,p} (X, Y)^{p/2} + D_{1,p} (X, Y)^{p/2} (D_{1,p} (X)^p + D_{1,p} (Y)^p)^{1/2} \right],$$

where $c_1$ is a constant.

And $B$-valued continuous path $x$ with finite variation can be lifted to the rough path $X$, where the $j$-th level path is defined in the following way,

$$X^j_{s,t} = \int_{s \leq t_1 \leq \ldots \leq t_j \leq t} dx_{t_1} \otimes dx_{t_2} \otimes \cdots \otimes dx_{t_j}.$$

The rough path obtained in above way is called smooth rough path. A rough path constructed as the $d_p$-limit of a sequence of smooth rough path is called a geometric rough path. The set of all the geometric rough paths is denoted by $G\Pi_p (B)$, and it is a complete separable metric space [24].

For any $m \in \mathbb{N}$, consider the $m$-dyadic grid that $t_l^m = \frac{l}{2^m}$ and set $\Delta^m_t (b^H, w)^T = (b^H, w)_t^T - (b^H, w)_{t_{l-1}^m}^T$ for $0 \leq l \leq 2^m$. Then denote by $(b^H (m), w(m))$, the process obtained by linear interpolation of $(b^H, w)^T$ on the $m$th dyadic grid. So $(b^H (m), w(m))_0 = 0$ and for $t \in [t_{l-1}^m, t_l^m]$,

$$(b^H (m), w(m))_t^T = (b^H (m), w(m))_{t_{l-1}^m}^T + 2^m (t - t_{l-1}^m) \Delta^m_l (b^H, w)^T.$$ 

The corresponding smooth rough path $(B^H (m), W (m))_{s,t} = (1, (B^H (m), W (m))_{s,t}^1, (B^H (m), W (m))_{s,t}^2)$ is built by taking its iterated path integrals, that is

$$(B^H (m), W (m))_{s,t}^1 = \int_{s \leq t_1 \leq \ldots \leq t_1 \leq t} d(b^H (m), w(m))_{t_1}^T \otimes d(b^H (m), w(m))_{t_2}^T \otimes \cdots \otimes d(b^H (m), w(m))_{t_1}^T.$$ 

For $n \leq m, l = 1, \ldots, 2^n$

$$(B^H (m), W (m))_{s,t}^1 = \Delta^m_l (b^H, w)^T \quad \text{for } n \leq m,$$

and when $n \geq m$

$$(B^H (m), W (m))_{s,t}^1 = 2^{m-n} \Delta^m_l (b^H, w)^T \quad \text{for } n \geq m,$$

where $\tilde{l}$ is the unique integer $\tilde{l}$ among $1, \ldots, 2^m$.

The second level path $(B^H (m), W (m))_{s,t}^2$ could be defined as follows, when $n \leq m,$

$$(B^H (m), W (m))_{s,t}^2 = \frac{1}{2} \Delta^m_l (b^H, w)^T \otimes \Delta^m_l (b^H, w)^T.$$ 

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\[ + \frac{1}{2} \sum_{r,s=2^{m-n}(i-1)+1}^{2^{m-n}} (\Delta_r^m(b^H, w)^T \otimes \Delta_s^m(b^H, w)^T - \Delta_r^m(b^H, w)^T \otimes \Delta_s^m(b^H, w)^T). \]

While for the case of \( n \geq m \),

\[(B^H(m), W(m))_{t_{i-1}, t_i}^2 = 2^{2(m-n)-1} (\Delta_r^m(b^H, w)^T \otimes \Delta_s^m(b^H, w)^T). \]

The smooth rough path \((B^H(m), W(m))_{s,t} = (1, (B^H(m), W(m))_{s,t}^1, (B^H(m), W(m))_{s,t}^2)\) with order 2 is constructed.

In this paper, the mixed fBm \((b^H, w)^T \in \mathbb{R}^{d_1+d_2}\) can be lifted to a geometric rough path \((b^H, W) \in G_{p}(d_1+d_2)\) with roughness 2 < \( p < 3 \).

**Proposition 2.1.** Let \((b^H, w)_\cdot^T \in \mathbb{R}^{d_1+d_2}\) be the mixed fBm with Hurst parameter \( 1/3 < H < 1/2 \). Then, for any 2 < \( p < 3 \) such that \( hp > 1 \), the sequence of its dyadic polygonal approximations \((B^H(m), W(m)) = (1, (B^H(m), W(m))^1, (B^H(m), W(m))^2)\) converges to a unique geometric rough path \((b^H, W) = (1, (b^H, W)^1, (b^H, W)^2)\) almost surely according to the \( p \)-variation.

**Proof.** For \( i = 1, 2 \) and 0 < \( s \leq t \leq 1 \), we proceed to prove that

\[ \mathbb{E} \left[ \left\| (B^H(m), W(m))^j \right\|_{p/j}^{p/j} \right] \leq C \left( \frac{1}{2^m} \right)^{hp/2-1/2}, \]

for some constant \( C \). Since \( p > 2 \), this implies that

\[ \sum_{m=1}^{\infty} \left\| (B^H(m), W(m))^1 - (B^H, W)^1 \right\|_{p-\text{var}} < \infty, \quad a.s.. \]

In particular, the sequence of its dyadic polygonal approximations \((B^H(m), W(m))\) converges to a unique geometric rough path \((b^H, W)\) almost surely according to the \( p \)-variation.

The subsequent proof consists of two steps.

**Step 1.** Prove that \( (2.3) \) and \( (2.4) \) hold for \( i = 1 \). In particular, \((B^H(m), W(m))^1\) converges to \((b^H, W)^1\) in \( p \)-variation almost surely.

Note that for \( n \leq m, l = 1, \ldots, 2^n \), it has \((B^H(m), W(m))_{t_{l-1}, t_l}^1 - (b^H, W)^1_{t_{l-1}, t_l} = 0\). On the other hand, for \( n > m \) and \( \gamma > p/i - 1 \), we can conclude that

\[ \mathbb{E} \left[ \left\| (B^H(m), W(m))^1 - (b^H, W)^1 \right\|_{p-\text{var}} \right] \leq C \sum_{n=m+1}^{\infty} n^\gamma \sum_{l=1}^{2^n} \mathbb{E} \left[ \left\| (B^H(m), W(m))_{t_{l-1}, t_l}^1 - (b^H, W)^1_{t_{l-1}, t_l} \right\|^p \right] \leq C 2^{p-1} \sum_{n=m+1}^{\infty} n^\gamma \sum_{l=1}^{2^n} \mathbb{E} \left[ \left\| (B^H(m), W(m))_{t_{l-1}, t_l}^1 \right\|^p \right] + \mathbb{E} \left[ \left\| (b^H, W)^1_{t_{l-1}, t_l} \right\|^p \right] \leq C \left( \frac{1}{2^n} \right)^{hp-1} \sum_{n=m+1}^{\infty} n^\gamma \left( \frac{1}{2^n} \right)^{hp/2-1/2}, \]

the last series converges to a finite constant \( C \). Moreover, we can assert that \( (2.3) \) holds for \( i = 1 \).

Then, with Hölder inequality and \( (2.5) \), we have

\[ \sum_{m=1}^{\infty} \mathbb{E} \left[ \left\| (B^H(m), W(m))^1 - (b^H, W)^1 \right\|_{p-\text{var}} \right] \leq \sup_{\pi \in \Pi([0, 1])} \left( \sum_{l} \mathbb{E} \left[ \left\| (B^H(m), W(m))_{t_{l-1}, t_l}^1 - (b^H, W)^1_{t_{l-1}, t_l} \right\|^1/p \right] \right)^{1/p}, \]
\[ \sum_{m=1}^{\infty} \left[ \mathbb{E} \left[ \sup_{n \in \mathbb{N} \setminus \{0\}} \sum_{r} \left| (B^H(m), W(m))^1_{t_1, t_L} - (B^H, W)^1_{t_1, t_L} \right|^p \right] \right]^{1/p} \]

\[ \leq C \sum_{m=1}^{\infty} \left( \frac{1}{2^m} \right)^{h/2-1/2p}. \] (2.6)

Since \( h/2 - 1/2p > 0 \), the last series converges. Hence it deduces that (2.4) holds for \( i = 1 \). In particular, \((B^H(m), W(m))^1\) converges to \((B^H, W)^1\) in \( p \)-variation almost surely.

**Step 2.** Prove that (2.3) and (2.4) hold for \( i = 2 \). In particular, \((B^H(m), W(m))^2\) converges to \((B^H, W)^2\) in \( p \)-variation almost surely.

For \( n \geq m \),

\[ (B^H(m+1), W(m+1))_{t_1, t_L}^2 - (B^H(m), W(m))_{t_1, t_L}^2 = 2^{2(m+1-n)-1} \left( \Delta_{t_1}^{m+1}(b^H, w)^T \right)^{\otimes 2} - 2^{2(m-n)-1} \left( \Delta_{t_1}^{m}(b^H, w)^T \right)^{\otimes 2}. \]

Then by the triangle inequality, it has

\[ \sum_{l=1}^{2n} \mathbb{E} \left[ \left( (B^H(m+1), W(m+1))_{t_1, t_L}^2 - (B^H(m), W(m))_{t_1, t_L}^2 \right)^{p/2} \right] \]

\[ \leq C 2^{(2(m-n)+1)p/2} \sum_{l=1}^{2^{m+1}} \mathbb{E} \left[ \left( \left( \Delta_{t_1}^{m+1}(b^H, w)^T \right)^{\otimes 2} \right)^{p/2} \right] 
+ C 2^{(2(m-n)-1)p/2} \sum_{l=1}^{2^{m+1}} \frac{1}{2^p} \mathbb{E} \left[ \left( \left( \Delta_{t_1}^{m}(b^H, w)^T \right)^{\otimes 2} \right)^{p/2} \right] \]

\[ \leq C 2^{(2(m-n)+1)p/2} \sum_{l=1}^{2^{m+1}} \left( \frac{1}{2^{m+1}} \right)^{hp} + \frac{1}{2^p} \left( \frac{1}{2^m} \right)^{hp} \]

\[ \leq C \left( \frac{2m}{2n} \right)^{p-hp} \left( \frac{1}{2^m} \right)^{hp-1}. \] (2.7)

For \( n \leq m \),

\[ (B^H(m+1), W(m+1))_{t_1, t_L}^2 - (B^H(m), W(m))_{t_1, t_L}^2 \]

\[ = \frac{1}{4} \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}} \left( \Delta_{2r-1}^{m+1}(b^H, w)^T \otimes \Delta_{2r-1}^{m+1}(b^H, w)^T - \Delta_{2r-1}^{m+1}(b^H, w)^T \otimes \Delta_{2r-1}^{m+1}(b^H, w)^T \right). \]

Denote

\[ \Theta(n, m, l) = (B^H(m+1), W(m+1))_{t_1, t_L}^2 - (B^H(m), W(m))_{t_1, t_L}^2, \]

then

\[ \Theta(n, m, l) = \frac{1}{2} \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}} \left( \Delta_{2r-1}^{m+1}(b^H, w)^T \otimes \Delta_{2r-1}^{m+1}(b^H, w)^T - \Delta_{2r-1}^{m+1}(b^H, w)^T \otimes \Delta_{2r-1}^{m+1}(b^H, w)^T \right), \]

By the hypercontractivity inequality [11], it follows that

\[ \mathbb{E} \left[ \Theta(n, m, l)^{i,j} \right]^{p/2} \leq C \frac{p}{2} \mathbb{E} \left[ \Theta(n, m, l)^{i,j} \right]^{2}. \] (2.8)

Moreover,

\[ \mathbb{E} \left[ \Theta(n, m, l)^{i,j} \right] \leq C \left( \Theta_1(n, m, l)^{i,j} + \Theta_2(n, m, l)^{i,j} \right), \] (2.9)

where

\[ \Theta_1(n, m, l)^{i,j} = \sum_{r=2^{m-n}(l-1)+1}^{2^{m-n}} \mathbb{E} \left[ \left( \Delta_{2r-1}^{m+1}(b^H, w)^T \otimes \Delta_{2r-1}^{m+1}(b^H, w)^T - \Delta_{2r-1}^{m+1}(b^H, w)^T \otimes \Delta_{2r-1}^{m+1}(b^H, w)^T \right)^2 \right], \]
On the one hand, due to the property of fBm and standard Bm, it follows that

\[
\Theta_2 (n, m, l)^{i,j} = \sum_{i \neq j} \mathbb{E} \left[ (\Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} - \Delta_{2r-1}^n (b_H, w)^{T,j} \Delta_{2r}^m (b_H, w)^{T,i}) \right].
\]

(2.10)

On the other hand, due to the property of fBm and standard Bm, it follows that

\[
\Theta_1 (n, m, l)^{i,j} \leq C \sum_{r=2^{m-n(l-1)+1}}^{2^{m-n(l-1)+1}} \mathbb{E} \left[ |\Delta_{2r-1}^n (b_H, w)^{T,i}|^2 \right] \mathbb{E} \left[ |\Delta_{2r}^m (b_H, w)^{T,j}|^2 \right] \leq C 2^{-\varphi} 2^{-2m(H-\frac{1}{4})},
\]

(2.11)

On the other hand, let

\[
A = \mathbb{E} \left[ \sum_{r=2^{m-n(l-1)+1}}^{2^{m-n(l-1)+1}} \mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^2 \right] \right] - \mathbb{E} \left[ \sum_{r=2^{m-n(l-1)+1}}^{2^{m-n(l-1)+1}} \mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^2 \right] \right] + \mathbb{E} \left[ \sum_{r=2^{m-n(l-1)+1}}^{2^{m-n(l-1)+1}} \mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^2 \right] \right] - 2\mathbb{E} \left[ \sum_{r=2^{m-n(l-1)+1}}^{2^{m-n(l-1)+1}} \mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^2 \right] \right] + \mathbb{E} \left[ \sum_{r=2^{m-n(l-1)+1}}^{2^{m-n(l-1)+1}} \mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^2 \right] \right].
\]

(2.12)

When \(i \in \{1, \cdots, d_1\}\), \((b_H, w)^{T,i}\) are independent fBms for different \(i\). With the result in [17, Section 3.3], it has

\[
\mathbb{E} \left[ \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right] \leq (2h-1) C \frac{||H - l||^{2h-2}}{(2n)^{2h}}, \quad i \in \{1, \cdots, d_1\}.
\]

(2.13)

Besides, when \(i \in \{d_1 + 1, \cdots, d_1 + d_2\}\), \((b_H, w)^{T,i}\) are independent Bms for different \(i\), we have

\[
\mathbb{E} \left[ \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right] = 0,
\]

(2.14)

for \(l' \neq l\). Meanwhile, it has

\[
\mathbb{E} \left[ \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right] = 0, \quad i \neq j.
\]

(2.15)

From the above it follows that,

\[
\Theta_2 (n, m, l)^{i,j} \leq C 2^{-\varphi} 2^{-2m(H-\frac{1}{4})},
\]

(2.16)

for \(i, j \in \{1, \cdots, d_1 + d_2\}\).

Therefore, by the (2.11) and (2.16), it has

\[
\mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^{2h} \right] \leq C 2^{-\varphi} 2^{-2m(H-\frac{1}{4})},
\]

(2.17)

for \(i, j \in \{1, \cdots, d_1 + d_2\}\).

When \(n = m\), we get

\[
\mathbb{E} \left[ \left| \sum_{i,j} \Delta_{2r-1}^n (b_H, w)^{T,i} \Delta_{2r}^m (b_H, w)^{T,j} \right|^{2h} \right] \leq C 2^{-2h}.
\]

(2.18)

Combined with (2.2), (2.17), and (2.18), it deduces that

\[
\mathbb{E} \left[ \left| \sum_{i,j} (b_H, w)^{T,i} \Delta_{2r-1}^n (b_H, w)^{T,j} \right|^{2h} \right] \leq C \left( \frac{1}{2n} \right)^{hp/2-1/2} + C \left( \frac{1}{2m} \right)^{hp/2-1/2} \left( \frac{1}{2n} \right)^{hp-1}.
\]
\[ + C \left[ \sum_{n=m+1}^{\infty} n^R \left( \frac{1}{2^n} \right)^{(h_p-1)/2} \right]^{1/2} \]

\[ \leq C \left( \frac{1}{2^n} \right)^{(h_p-1)/2}, \quad (2.19) \]

hence, (2.20) holds for \( i = 2 \). By the straightforward computation, it follows that (2.21) holds for \( i = 2 \). Therefore, \((B^H(m), W(m))^2\) is a Cauchy sequence in \( p \)-variation, and the conclusion follows.

In above, the sequence of its dyadic polygonal approximations \((B^H(m), W(m))\) converges to a unique function \((B^H, W)\) almost surely in \( p \)-variation distance. According to the definition, \((B^H, W)\) is a geometric rough path. The proof is completed. \( \square \)

**Proposition 2.2.** Let \((b^H_t, w_t)_{t \geq 0} \in \mathbb{R}^{d_1+d_2}\) be the mixed fBm with \( \mathbb{R}^{d_1} \)-valued fBm \((b^H_t)_{t \geq 0} \) (1/3 < \( H < 1/2 \)) and \( \mathbb{R}^{d_2} \)-valued standard BM \((w_t)_{t \geq 0} \). Then,

\[ (B^H, W)^1_{st} = (b^H_{st}, w_{st})^T, \quad (B^H, W)^2_{st} = \left( \begin{array}{c} B^H_{st} \\ I[w, b^H]_{st} W^2_{st} \end{array} \right), \quad (2.20) \]

where the \((B^H)^1, B^H)^2\) is a canonical geometric rough path, and the \((W^1, W^2)\) is a geometric rough path in Stratonovich sense. Then

\[ I[b^H, w]_{st} \triangleq \int_s^t b^H_{su} \otimes d^1w_u, \quad (2.21) \]

\[ I[w, b^H]_{st} \triangleq w_{st} \otimes b^H_{st} - \int_s^t d^1w_u \otimes b^H_{st}, \quad (2.22) \]

where \( \int \cdots d^1w \) stands for the Itô integral.

**Proof.** \((B^H)^1, B^H)^2\) is a canonical geometric rough path (see [10], Section 10.3). And \((W^1, W^2)\) is a geometric rough path in Stratonovich sense (see [10], Section 3). It remains to show (2.21) and (2.22) hold. Take \( s = 0, t = 1 \) for simplicity. Then denote by \((b^H_t(m))_{0 \leq t \leq 1}\) the process obtained by linear interpolation of \((b^H_t)_{0 \leq t \leq 1}\) on the \( m \)-th dyadic grid. Similar for \((w_t(m))_{0 \leq t \leq 1}\). To show (2.21) hold, it turns to prove that

\[ \lim_{m \to \infty} \int_0^1 b^H_i (m) dw^j_t (m) = \int_0^1 b^H_i dw^j_t, \quad (2.23) \]

for \( i \in \{1, 2, \cdots, d_1\} \) and \( j \in \{1, 2, \cdots, d_2\} \). The right hand side of (2.23) is in the Itô sense.

To this end, define a step function \( \hat{b}^H_i = b^H_i |_{t^{0} \leq \cdot \leq t^{1}} \) for \( t \in [t^{0}_k, t^{1}_k] \) and \( 1 \leq k \leq 2^m \). Then it decudes that

\[ \mathbb{E} \left[ \int_0^1 (b^H_i - \hat{b}^H_i) dw^j_t \right] \to 0, \quad (2.24) \]

as \( m \to \infty \). Hence, the right hand side of (2.23) is in the Itô sense. According to the definition of dyadic approximation, the left hand side of (2.23) can be rewritten as follows,

\[ \int_0^1 b^H_i (m) dw^j_t (m) = \sum_{k=1}^{2^m} \int_{t^{0}_k}^{t^{1}_k} b^H_i (m) \left( \frac{w^j_{t^{m}_k} - w^j_{t^{m}_{k-1}}}{1/2^m} \right) du. \quad (2.25) \]

Next, we get that

\[ \mathbb{E} \left[ \sum_{k=1}^{2^m} \left( \int_{t^{0}_k}^{t^{1}_k} b^H_i (m) du - b^H_i (m) \right) \left( w^j_{t^{m}_k} - w^j_{t^{m}_{k-1}} \right)^2 \right] \]
Denote \( \varphi \) the function. Furthermore, there exist positive constants \( \gamma, \eta \). Then, we focus on the Laplace map  \( \hat{\Phi} \) of the solution map. Consider the solution map \( \hat{\Phi}_\varepsilon(\varepsilon(B^H, W), \lambda) \),

\[
d\hat{Y}^\varepsilon = [\sigma(\hat{Y}^\varepsilon) \hat{\sigma}(\hat{Y}^\varepsilon)](\varepsilon d(B^H, W) + (\gamma, \eta)^T + \beta(\varepsilon, \hat{Y}^\varepsilon) dt), \quad \hat{Y}^\varepsilon_0 = 0.
\]

Denote \( \phi^0 = \Phi(\gamma, \eta) \in C_0^{p'-\var} (\mathbb{R}^n) \) with \( 1 < q < 2 \), satisfying that

\[
d\phi^0_t = [\sigma(\phi^0) \hat{\sigma}(\phi^0)] d(\gamma, \eta)^T + \beta(0, \phi^0) dt, \quad \phi^0_0 = 0.
\]

We assume

A1. The function \( F \) is real-valued bounded continuous on \( C_0^{p'-\var} (\mathbb{R}^n) \) with \( p'>1/H \).

A2. The function \( F_\lambda := F \circ \Phi + ||(\cdot, \cdot)||^2_H/2 \) attains its minimum at a unique point \( (\gamma, \eta)^T \in \mathcal{H} \), with \( \Phi(\gamma, \eta) = \phi^0 \).

A3. The function \( F \) is \( m+3 \) times Fréchet differentiable on a neighborhood \( U(\phi^0) \) with \( \phi^0 \in C_0^{p'-\var} (\mathbb{R}^n) \). Furthermore, there exist positive constants \( M_1, M_2, \ldots \) such that

\[
|\nabla^j F(\eta)(z, \ldots, \varepsilon)| \leq M_j \|z\|_{p'-\var}^j (j = 1, \ldots, m+3),
\]

hold for any \( \eta \in U(\phi^0) \) and \( z \in C_0^{p'-\var} (\mathbb{R}^n) \).

A4. The bounded self-adjoint operator \( A \) on \( \mathcal{H} \), which is related to the Hessian matrix \( \nabla^2 F(\circ \Phi)(\gamma, \eta) |_{\mathcal{H} \times \mathcal{H}} \), is strictly larger than \(-Id_{\mathcal{H}}\).

Then, here follows our main result.

**Theorem 2.3.** Under Assumptions (A1)-(A4), we have the following asymptotic expansion with \( \varepsilon \to 0 \). There exist constants \( c, \alpha_0, \alpha_1, \ldots \) s.t.

\[
\mathbb{E}[\exp\{-(F(Y^{\varepsilon,1})/\varepsilon^2)\}] = \exp\left(-F_\lambda(\gamma, \eta)/\varepsilon^2\right)(-c/\varepsilon) \cdot (\alpha_0 + \alpha_1 \varepsilon + \cdots + \alpha_m \varepsilon^m + O(\varepsilon^{m+1}))
\]

for any \( m \geq 0 \).

**Remark 2.4.** Denote \( \tau_{p, p'} \) the injection from \( C^{p-\var} (\mathbb{R}^n) \) to \( C^{p'-\var} (\mathbb{R}^n) \). Then, we focus on the Laplace approximation for the first level path to the solution map, that is \( F(\tau_{p, p'}(Y^{\varepsilon,1})) \). But for simplicity, we write in the sense that \( F(Y^{\varepsilon,1}) \). The another way of saying that one views \( F \) as a function on \( C^{p-\var} (\mathbb{R}^n) \), even if it can be done and assumptions (A1)-(A4) remains equivalent.
3. Schilder-type large deviation principle

In this section, we prove the Schilder-type LDP for the law of the first level path of the solution map to the RDE \[ \text{RDE} \] with \( \varepsilon \to 0 \).

**Proposition 3.1.** With \( \varepsilon \to 0 \), the law of \( \varepsilon(B^H, W) \), \( \{\varepsilon_{\varepsilon_{\varepsilon}}\}_{\varepsilon > 0} \) satisfies the LDP with the good rate function \( I \), which is defined as follows,

\[
I(B^H, W) = \begin{cases} 
\frac{1}{2} \| (k, \hat{k}) \|_H^2 & \text{if } (B^H, W) \text{ is lying above } (k, \hat{k})^T \in H), \\
\infty & \text{otherwise}.
\end{cases}
\] (3.1)

**Proof.** The subsequent proof consists of several steps.

**Step 1.** Prove that the smooth rough paths \( (B^H(m), W(m)) \) are exponentially good approximations of \( (B^H, W) \) in the sense that, for every \( \varepsilon > 0 \),

\[
\lim_{m \to \infty} \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} (d_P(F(\varepsilon(B^H(m), W(m))), F(\varepsilon(B^H, W))) > \delta) = -\infty.
\] (3.2)

Let \( pH > 1 \), it will be enough to show that there is a sequence \( c(m) \to 0 \) such that for \( \varrho > 1 \),

\[
\mathbb{E} \left[ D_{1,p} \left( (B^H(m), W(m)), (B^H, W) \right)^{\varrho} \right]^{1/\varrho} \leq c(m) \sqrt{\varrho},
\] (3.3)

and

\[
\mathbb{E} \left[ D_{2,p} \left( (B^H(m), W(m)), (B^H, W) \right)^{\varrho} \right]^{1/\varrho} \leq C \sqrt{\varrho},
\] (3.4)

and

\[
\mathbb{E} \left[ D_{2,p} \left( (B^H(m), W(m)), (B^H, W) \right)^{\varrho} \right]^{1/\varrho} \leq C(m) \varrho,
\] (3.5)

where \( c(m) \) tends to zero as \( m \to \infty \).

Accordingly, by the Chebyshev inequality, for \( \varrho = \varrho(\varepsilon) = \varepsilon^{-2} \), and \( j = 1, 2 \), we can conclude

\[
\mathbb{P} \left( D_{j,p} \left( (B^H(m), W(m)), (B^H, W) \right) > \delta \varepsilon^{-j} \right) \leq \left( \delta^{-1} \varepsilon^{-j} \right)^{\varrho} c(m)^{\varrho} \varrho^{\varrho/2} \leq \left( \delta^{-1} c(m) \right)^{\varrho}.
\]

It deduces that

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left( D_{j,p} \left( (B^H(m), W(m)), (B^H, W) \right) > \delta \varepsilon^{-j} \right) \leq \log \left( \delta^{-1} c(m) \right).
\] (3.6)

Besides, since that for some constant \( C \), we have \( \mathbb{E} \left( D_{1,p}(B^H, W)^{\varrho} \right)^{1/\varrho} \leq C \sqrt{\varrho} \). Then by the Cauchy-Schwarz inequality and triangle inequalities, for every \( \varrho \), it has

\[
\mathbb{E} \left[ \left[ D_{1,p}((B^H(m), W(m)), (B^H, W)) \left( D_{1,p}(B^H(m), W(m)) + D_{1,p}(B^H, W) \right) \right]^{\varrho} \right]^{1/\varrho} \leq 2c(m)(2C + c(m)) \varrho,
\]

in consequence,

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left( D_{1,p}((B^H(m), W(m)), (B^H, W)) \left( D_{1,p}(B^H(m), W(m)) + D_{1,p}(B^H, W) \right) > \delta \varepsilon^{-2} \right)
\]

\[
\leq \log \left( \delta^{-1} 2c(m)(2C + c(m)) \right),
\] (3.7)

combined with \( 3.6 \), we can assert that \( 3.7 \) holds.

Therefore, we proceed to prove \( 3.8 \) and \( 3.9 \). First, for any \( \varrho > p \), it has

\[
\mathbb{E} \left[ D_{1,p} \left( (B^H(m), W(m)), (B^H, W) \right)^{\varrho} \right] \leq A(m, \varrho) \sum_{n=m+1}^{\infty} a_n^{\varrho/p} \sum_{l=1}^{2^n} \mathbb{E} \left[ \left| 2^{m-n} \Delta_l^m(b^H, w)^T - \Delta_l^m(b^H, w)^T \right|^{\varrho} \right].
\] (3.8)
Likewise, we can see \( 2^\kappa \mathbb{A} \) where the second inequality is due to Lemma 4 in [22] and \( a_n = 2^{np(H-\frac{1}{2}-\beta_1)} \) and \( n^\kappa \leq C2^{n\beta_2} \) with \( \beta_2 > 0 \) and \( \beta_1 + \beta_2 < 0, (H-1/p)/2 \), we get

\[
\mathbb{E}[D_{1,p} ((B^H (m), W(m)), (B^H, W))^\theta] \leq A(m, \varrho) \sum_{n=m+1} 2^{n(m-n)\varrho/2} 2^{m-n} (m-n)_2 \delta H + 2^{-n_\varrho H}
\]

where the second inequality is due to Lemma 4 in [22] and \( A(m, \varrho) \) is a bounded double-sequence in \( m \) and \( \varrho \).

Similarly, it has

\[
\mathbb{E}[D_{1,p}(B^H (m), W(m))^\varrho] \leq A(0, \varrho) (2d_1 + 2d_2)^\varrho 2^{m-n} (1-\varrho H) < \infty.
\]

Combined with (3.9) and the triangle inequality, we can prove that

\[
\mathbb{E}[D_{1,p}(B^H, W)^\varrho] < \infty.
\]

When \( j = 2 \), we proceed to assert that (3.5) holds. Consider the case of \( n \geq m \),

\[
(B^H (m), W(m))^2_{t_{i-1}, t_i} = 2^{2(m-n)-1} (\Delta^m_t (b^H, w))^\otimes 2.
\]

then due to the property of fBm and standard Bm, it follows that

\[
\mathbb{E}[(B^H (m), W(m))^2_{t_{i-1}, t_i}] \leq C_2^{-2n_2} 2^{-2m(H-1)},
\]

and

\[
\mathbb{E}[(B^H, W)^2_{t_{i-1}, t_i}] \leq C_2^{-2n_2} H.
\]

Likewise, we can see

\[
\mathbb{E}[(B^H, W)^2_{t_{i-1}, t_i} - (B^H (m), W(m))^2_{t_{i-1}, t_i}] \leq 2(\varrho-1) \left( \mathbb{E}[(B^H, W)^2_{t_{i-1}, t_i}] + (B^H (m), W(m))^2_{t_{i-1}, t_i} \right) \leq C_2^{-2(\varrho-1)} 2^{-2m_2 \varrho H} + C_2^{-2(\varrho-1)} 2^{-2(2m-n_2) \varrho H} \leq C_2^{-2n_2 \varrho H}.
\]

When \( n \leq m \),

\[
(B^H (m + 1), W(m + 1))^2_{t_{i-1}, t_i} - (B^H (m), W(m))^2_{t_{i-1}, t_i} = \frac{1}{2} \sum_{r=2}^{m-n+1} (\Delta^r_{2r-1} (b^H, w))^T \otimes (\Delta^r_{2r-1} (b^H, w))^T - (\Delta^r_{2r-1} (b^H, w))^T \otimes (\Delta^r_{2r-1} (b^H, w))^T.
\]

Take same manner from (2.15) to (2.15), it follows that

\[
\mathbb{E}[(B^H (m + 1), W(m + 1))^2_{t_{i-1}, t_i} - (B^H (m), W(m))^2_{t_{i-1}, t_i}] \leq C_2^{-2n_2} 2^{-2m(H-\frac{1}{2})}.
\]
Fix $M$ large enough, we can obtain that
\[
\mathbb{E} \left[ \left( (B^H(M), W(M))_{t_{n+1}}^2 - (B^H(m), W(m))_{t_{n+1}}^2 \right)_{t_{n+1}} \right]^{\frac{1}{p}} \leq C \varrho 2^{-\frac{n}{\varrho}} \sum_{n=0}^{M-1} 2^{-2n(N-H-\frac{1}{8})} \leq C \varrho 2^{-\frac{n}{\varrho}} 2^{-2m(H-\frac{1}{4})}.
\]

By the construction of the rough path lying above,
\[
\lim_{M \to \infty} (B^H(M), W(M))_{t_{n+1}}^2 = (B^H, W)_{t_{n+1}}^2.
\]

Hence, we get
\[
\mathbb{E} \left[ \left( (B^H, W)_{t_{n+1}}^2 - (B^H(m), W(m))_{t_{n+1}}^2 \right) | \theta \right]^{1/\theta} \leq C \varrho 2^{-\frac{n}{\varrho}} 2^{-2m(H-\frac{1}{4})}.
\]  

(3.16)

It means that
\[
\mathbb{E} \left[ D_{2,p} ((B^H(m), W(m)), (B^H, W)) \right] \leq A(\varrho) \sum_{n=1}^{\infty} \frac{a_n^{2p/\varrho}}{2^n} \sum_{l=1}^{2^n} \mathbb{E} \left[ \left( (B^H(m), W(m))_{t_{l-1}}^2 - (B^H, W)_{t_{l-1}}^2 \right) | \theta \right] \leq CA(\varrho) \varrho^\theta \sum_{n=1}^{\infty} \frac{a_n^{2p/\varrho}}{2^n} 2^{-n(2H-\frac{1}{4})-\frac{1}{4}},
\]

where $A(\varrho) = \sum_{n=1}^{\infty} 2^n (n^\varrho/a_n) \varrho^{p-\varrho}/p$.  

Then choose appropriate $a_n = 2^{-n(\beta_3 + H + \frac{1}{2} + \frac{1}{2})}$ and $n^\varrho \leq C 2^{np^\beta_5}$ with $\beta_3 > 0$, $\beta_4 \in (0, 2H - 1/2)$, and $\beta_3 + \frac{1}{2} + \beta_5 < H - \frac{1}{4}$, the series $\sum_n n^\varrho 2^{-n(2H-\beta_4-1)}$ converges. $A(\varrho)$ is a bounded double-sequence in $\varrho$. Moreover,
\[
\mathbb{E} \left[ D_{2,p} ((B^H(m), W(m)), (B^H, W)) \right]^{1/\theta} \leq C \varrho 2^{-m^\rho},
\]  

(3.17)

where $\rho > 0$.

In above, (3.13), (3.17) and (3.18) are proved. Furthermore, the smooth rough paths $(B^H(m), W(m))$ are exponentially good approximations of the geometric rough path $(B^H, W)$.

**Step 2.** Prove that the rough path $F(K, \hat{K})$ above any element $(k, \hat{k})$ in the Cameron-Martin space $\mathcal{H}$ is defined as the limit of $F(K(m), \hat{K}(m))$,
\[
\lim_{m \to \infty} \sup_{|\hat{K}(m)|_H \leq \alpha} d_p (F(K(m), \hat{K}(m)), F(K, \hat{K})) = 0.
\]  

(3.18)

That is to say proving
\[
\lim_{m, m' \to \infty} \sup_{|\hat{K}(m)|_H \leq \alpha} D_{j,p} ((K(m), \hat{K}(m)), (K(m'), \hat{K}(m'))) = 0,
\]  

(3.19)

for $j = 1, 2$, and
\[
\sup_{m \in H | (k, \hat{k})|_H \leq \alpha} \sup_{m' \in H | (k, \hat{k})|_H \leq \alpha} D_{j,p} (K(m), \hat{K}(m)) < \infty,
\]  

(3.20)

with $j = 1$.

For $j = 1$, when $m \geq n$, it has $D_{1,p} ((K(m), \hat{K}(m)), (K, \hat{K}))^p = 0$. Then turn to the case that $m \leq n$,
\[
\sup_{|\hat{K}(m)|_H \leq \alpha} D_{1,p} ((K(m), \hat{K}(m)), (K, \hat{K}))^p \leq \sup_{|\hat{K}(m)|_H \leq \alpha} \sum_{n=m+1}^{\infty} n^\varrho \sum_{l=1}^{2^n} 2^{m-n} \left| \Delta_l^m (k, \hat{k})^T - \Delta_l^m (k, \hat{k})^T \right|^p
\]

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\begin{align}
\leq C_p \sum_{n=m+1}^{\infty} n^\kappa \sum_{l=1}^{2^n} (2^{(m-n)p}2^{-mpH} + 2^{-pnH}) \\
\leq C \alpha^{p_2} 2^{-m(pH-1-\beta_0)},
\end{align}

(3.21)

in the final line, take suitable $\beta_0 > 0$ such that $pH - 1 - \beta_0 > 0$. Hence, (3.19) holds for $j = 1$.

In the same manner we can see that,

$$
\sup_{m \in \mathbb{N}} \sup_{|k-k| \leq \alpha} D_{1,p}(K, \hat{K}) < \infty,
$$

(3.22)

Together with (3.21), we can prove that (3.20) holds for $j = 1$.

For $j = 2$, when $m \leq n$,

$$
\sup_{|k-k| \leq \alpha} \left| (K(m+1), \hat{K}(m+1))_{t_{l-1}^n}^{2l^n} - (K(m), \hat{K}(m))_{t_{l-1}^n}^{2l^n} \right| \\
\leq \frac{1}{2} \sum_{l=2^{m-n}(l-1)+1}^{2^{m-n}} \left( \Delta_{2l-1}^m (k, \hat{k})^T \Delta_{2l}^m (k, \hat{k})^T - \Delta_{2l-1}^m (k, \hat{k})^T \Delta_{2l}^m (k, \hat{k})^T \right).
$$

(3.23)

Clearly,

$$
\left| (K(m+1), \hat{K}(m+1))_{t_{l-1}^n}^{2l^n} - (K(m), \hat{K}(m))_{t_{l-1}^n}^{2l^n} \right| \leq C \left( \Xi_{n,m,l}^{i,j} + \Xi_{n,m,l}^{i,i} \right).
$$

(3.24)

where $i, j \in \{1, \cdots, d_1 + d_2 \}$ and $i \neq j$.

$$
\Xi_{n,m,l}^{i,j} = \left| \sum_{l=2^{m-n}(l-1)+1}^{2^{m-n}} \Delta_{2l-1}^m (k, \hat{k})^T \Delta_{2l}^m (k, \hat{k})^T \right|.
$$

Therefore, due to the Hölder inequality, one can get

$$
\Xi_{n,m,l}^{i,j} \leq \left[ \sum_{l=2^{m-n}(l-1)+1}^{2^{m-n}} |\Delta_{2l-1}^m (k, \hat{k})^T|^2 \right]^{\frac{1}{2}} \left[ \sum_{l=2^{m-n}(l-1)+1}^{2^{m-n}} |\Delta_{2l}^m (k, \hat{k})^T|^2 \right]^{\frac{1}{2}}.
$$

With Lemma 5.1 in [11], it has $\sum_{l=2^{m-n}(l-1)+1}^{2^{m-n}} |\Delta_{2l-1}^m (k, \hat{k})^T|^2 \leq C_2^{-m(2H-1/2)-n/2}$ for $i \in \{1, \cdots, d_1 \}$, and $\sum_{l=2^{m-n}(l-1)+1}^{2^{m-n}} |\Delta_{2l}^m (k, \hat{k})^T|^2 \leq C_2^{-m/2-n/2}$ for $i \in \{d_1 + 1, \cdots, d_1 + d_2 \}$. Then it deduces

$$
\Xi_{n,m,l}^{i,j} \leq C_2^{-m(2H-1/2)-n/2},
$$

and similar results hold for the $\Xi_{n,m,l}^{i,j}$.

Together with (3.21), we have

$$
\sup_{|k-k| \leq \alpha} D_{2,p} ((K(m+1), \hat{K}(m+1)), (K(m), \hat{K}(m))) \leq C_2^{-m\beta_7},
$$

where $\beta_7 > 0$. This gives (3.18) combined with (3.21).

**Step 3.** Combine Step 1 and Step 2, with $\varepsilon \to 0$, the law of $\varepsilon (B^H, W)$, \{\varepsilon H\}_{\varepsilon > 0} satisfies the Schilder-type LDP with the good rate function $I$ defined in (3.1) by means of an extension of the contraction principle [3], Theorem 4.2.23.

The proof is completed.
Lemma 3.2. Let \((B^H, W)\) be the mixed geometric rough path with \(H \in (1/3, 1/2)\), there exists some constant \(c > 0\) such that
\[
\mathbb{E} \left[ \exp \left( c \| (B^H, W) \|^2_{p \text{-var}} \right) \right] = \int_{\mathcal{G}_{p, r}(\mathbb{R}^{d_1 + d_2})} \exp \left( c \| (B^H, W) \|^2_{p \text{-var}} \right) \mathbb{P}^H (B^H, W) < \infty.
\]

Proof. Proposition 3.3. shows that, there is a sequence \(c(m) \to 0\) such that for \(\varrho > 1\),
\[
\mathbb{E} \left[ D_{1,p} ((B^H (m), W(m)), (B^H, W))^\varrho \right]^{1/\varrho} \leq c(m) \sqrt{\varrho},
\]
and
\[
\mathbb{E} \left[ D_{1,p}(B^H (m), W(m))^\varrho \right]^{1/\varrho} \leq C \sqrt{\varrho},
\]
and
\[
\mathbb{E} \left[ D_{2,p} ((B^H (m), W(m)), (B^H, W))^\varrho \right]^{1/\varrho} \leq c(m) \varrho,
\]
where \(c(m)\) tends to zero as \(m \to \infty\).

Then, it deduces that
\[
\mathbb{E} \left[ e^{D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3}} \right] \leq \sum_{N=0}^{\infty} e^{C(N+1)} \mathbb{P} (N < D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3} \leq N + 1) \leq \left( e^c + \cdots + e^{4c} \right) + e^c \sum_{N=4}^{\infty} e^{CN} \mathbb{P} \left( N < D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3} \right) \leq \left( e^c + \cdots + e^{4c} \right) + e^c \sum_{N=4}^{\infty} e^{CN} \mathbb{E} \left[ (D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3})^N \right] \leq \left( e^c + \cdots + e^{4c} \right) + e^c \sum_{N=4}^{\infty} e^{CN} \mathbb{E} \left[ (D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3})^N \right] \leq \left( e^c + \cdots + e^{4c} \right) + e^c \sum_{N=4}^{\infty} e^{CN} \mathbb{E} \left[ (D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3})^N \right],
\]
for \(j = 1, 2\). We choose \(m_0\) large enough such that for any \(m > m_0\), it has \([N (c + \log c_2 + \log a_m)] < 0\), moreover, it has
\[
\sup_{m \geq m_0} \mathbb{E} \left[ e^{D_{j,p}((B^H (m), W(m)), (B^H, W))^{2j/3}} \right] < \infty.
\]

Besides, it is clear to see that for any fixed \(m_0\), there exists a constant \(C(m_0)\) such that \(D_{j,p}(B^H (m_0), W(m_0))^{1/3} \leq C(m_0) \left\| (b^H, w) \right\|_\infty\). By the conventional Fernique theorem for Gaussian measures, it follows that \(D_{j,p}(B^H (m_0), W(m_0))^{1/3}\) is square exponential integrable. Then with the triangle inequality, the conclusion follows.

The proof is completed. \(\square\)

Then, for simplicity, the RDE (1) is rewritten as follows,
\[
dY_t^\varepsilon = [\sigma (Y_t^\varepsilon) | \hat{\sigma} (Y_t^\varepsilon)] \varepsilon d(B^H, W)_t + \beta (Y_t^\varepsilon) \lambda^\varepsilon dt, \quad Y_0^\varepsilon = 0.
\]

Denote \(\delta_{\lambda^\varepsilon}\) the law of \(\lambda^\varepsilon_t\).

Proposition 3.3. With \(\varepsilon \to 0\), the law of \(Y^\varepsilon, 1\) satisfies the LDP with the good rate function \(I\),
\[
I(y) = \begin{cases} \inf \left\{ \| (k, \hat{k}) \|_H^2 / 2 \mid y = \hat{\Phi}_0 ((k, \hat{k}), \lambda) \right\} & \text{(if } y = \hat{\Phi}_0 ((k, \hat{k}), \lambda) \text{ for some } (k, \hat{k})^T \in \mathcal{H} \text{),} \\ \infty & \text{(otherwise).} \end{cases}
\]
**Proof.** The Lemma 3.3 and Lemma 3.2 state that the law of the geometric rough path \( \varepsilon(B^H, W) \), \( \{\mathbb{P}^H \}_{0 < \varepsilon \leq 1} \) is exponential tight and satisfies the LDP with the good rate function \( (3.31) \). Obviously, the deterministic family \( \{\delta_{\lambda^*} \}_{0 < \varepsilon \leq 1} \) is exponential tight and satisfies LDP on \( C^1_{0 \text{-var}}([0, 1], \mathbb{R}) \) with the good rate function \( +\infty \cdot 1_{\mathbb{F}} \) with the convention that \( +\infty \cdot 0 = 0 \). By the result for LDP of product measure \([3, \text{Exercise 4.2.7}]\), the product measure \( \{\mathbb{P}^H \otimes \delta_{\lambda^*} \}_{0 < \varepsilon \leq 1} \) satisfies the LDP on the \( G\Omega_p(\mathbb{R}^{d_1+d_2}) \times C^1_{0 \text{-var}}([0, 1], \mathbb{R}) \) with the good rate function as follows,

\[
\dot{I}_1(B^H, W) + \infty \cdot 1_{\{0\}}(\lambda) = \begin{cases} 
\frac{1}{2} \|k, \dot{k}\|^2_{\mathcal{H}} & (\text{if } (B^H, W) \text{ is lying above } (k, \dot{k})^T \in \mathcal{H} \text{ and } \lambda = 0), \\
\infty & (\text{otherwise}).
\end{cases}
\]

Clearly, the continuity theorem of Itô map \([24, \text{Section 6.3}]\), the contraction principle and the fact that \( \dot{I}_1(B^H, W) = 0 \) yield that the above proposition holds.

The proof is completed. \( \square \)

**Remark 3.4.** Consider the following RDE,

\[
d\tilde{Y}_t^\varepsilon = [\varepsilon^v\sigma(\tilde{Y}_t^\varepsilon)]\varepsilon^\nu d(B^H, W)_t + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)dt, \quad \tilde{Y}_0^\varepsilon = 0,
\]

where \( \nu, \nu' > 0 \). If \( \nu < \nu' \), the above RDE \( (3.31) \) can be rewritten as follows,

\[
d\tilde{Y}_t^\varepsilon = [\varepsilon^{\nu'-\nu}\sigma(\tilde{Y}_t^\varepsilon)]\varepsilon^{\nu'} d(B^H, W)_t + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)dt, \quad \tilde{Y}_0^\varepsilon = 0.
\]

Let \( \hat{\Psi}_\varepsilon(\varepsilon^v(B^H, W), \lambda) : G\Omega_p(\mathbb{R}^{d_1+d_2+1}) \to G\Omega_p(\mathbb{R}^n) \) denote the Itô map corresponding to \( (3.31) \) with \( \lambda_t = t \). Consider the solution map \( \hat{\Psi}_\varepsilon(\varepsilon^v(B^H, W) + (\gamma, \eta), \lambda) \),

\[
d\tilde{Y}_t^\varepsilon = [\varepsilon^{\nu'-\nu}\sigma(\tilde{Y}_t^\varepsilon)]\varepsilon^{\nu'} d(B^H, W)_t + (\gamma_t, \eta_t)^T + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)dt, \quad \tilde{Y}_0^\varepsilon = 0.
\]

Now \( \hat{\Psi}_0(\eta, \lambda) \) is the solution map lying above \( \psi^0 = \hat{\Psi}_0(\eta, \lambda) \in C^1_{0 \text{-var}}(\mathbb{R}^n) \) with \( 1 < q < 2 \), satisfying that

\[
d\psi_t^0 = \tilde{\sigma}(\psi_t^0) d\tilde{H}_t + \beta(0, \psi_t^0) dt, \quad \psi_0^0 = 0.
\]

The Lemma 3.3 and Lemma 3.2 state that the law of the geometric rough path \( \varepsilon^v(B^H, W) \), \( \{\mathbb{P}^H \}_{0 < \varepsilon \leq 1} \) is exponential tight and satisfies LDP with the good rate function \( (3.37) \). The deterministic family \( \{\delta_{\lambda^*} \}_{0 < \varepsilon \leq 1} \) is exponential tight and satisfies LDP on \( C^1_{0 \text{-var}}([0, 1], \mathbb{R}) \) with the good rate function \( +\infty \cdot 1_{\mathbb{F}} \) with the convention that \( +\infty \cdot 0 = 0 \). Similar to Lemma 3.3 with aid of the \([3, \text{Exercise 4.2.7}]\), it deduces that the product measure \( \{\mathbb{P}^H \otimes \delta_{\lambda^*} \}_{0 < \varepsilon \leq 1} \) satisfies the LDP on the \( G\Omega_p(\mathbb{R}^{d_1+d_2}) \times C^1_{0 \text{-var}}([0, 1], \mathbb{R}) \) with the good rate function as follows,

\[
\dot{I}_1(B^H, W) + \infty \cdot 1_{\{0\}}(\lambda) = \begin{cases} 
\frac{1}{2} \|k, \dot{k}\|^2_{\mathcal{H}} & (\text{if } (B^H, W) \text{ is lying above } (k, \dot{k})^T \in \mathcal{H} \text{ and } \lambda = 0), \\
\infty & (\text{otherwise}).
\end{cases}
\]

Clearly, the continuity theorem of Itô map \([24, \text{Section 6.3}]\), the contraction principle and the fact that \( \nu - \nu' > 0 \) yield that with \( \varepsilon \to 0 \) the law of \( \varepsilon^v \cdot \hat{\lambda} \) satisfies the LDP with the good rate function \( I \),

\[
I(\tilde{y}) = \inf \left\{ \frac{\|\tilde{k}\|^2_{\mathcal{H},d,1} / 2}{\|k\|^2_{\mathcal{H},d,1}} \mid \tilde{y} = \hat{\Psi}_0(\tilde{k}, \lambda) \right\} \quad (\text{if } \tilde{y} = \hat{\Psi}_0(\tilde{k}, \lambda) \text{ for some } \tilde{k} \in \mathcal{H}^{d_1,d_1}),
\]

(\text{otherwise}).

If \( \nu < \nu' \), the above RDE \( (3.37) \) can be rewritten as follows,

\[
d\tilde{Y}_t^\varepsilon = [\sigma(\tilde{Y}_t^\varepsilon)]\varepsilon^{\nu'-\nu}\sigma(\tilde{Y}_t^\varepsilon)\varepsilon^{\nu'} d(B^H, W)_t + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)dt, \quad \tilde{Y}_0^\varepsilon = 0.
\]

Let \( \hat{\Psi}_\varepsilon(\varepsilon^v(B^H, W), \lambda) : G\Omega_p(\mathbb{R}^{d_1+d_2+1}) \to G\Omega_p(\mathbb{R}^n) \) denote the Itô map corresponding to \( (3.37) \) with \( \lambda_t = t \). Consider the solution map \( \hat{\Psi}_\varepsilon(\varepsilon^v(B^H, W) + (\gamma, \eta), \lambda) \),

\[
d\tilde{Y}_t^\varepsilon = [\sigma(\tilde{Y}_t^\varepsilon)]\varepsilon^{\nu'-\nu}\sigma(\tilde{Y}_t^\varepsilon)\varepsilon^{\nu'} d(B^H, W)_t + (\gamma_t, \eta_t)^T + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)dt, \quad \tilde{Y}_0^\varepsilon = 0.
\]
Now $\hat{\Psi}_0(\gamma, \lambda)$ is the solution map lying above $\hat{\psi}^0 = \hat{\Psi}_0(\gamma, \lambda) \in C^0_{q,\var} (\mathbb{R}^n)$ with $1 < q < 2$, satisfying that
\[
d\hat{\psi}^0_t = \sigma(\hat{\psi}^0_t) d\gamma_t + \beta(0, \hat{\psi}^0_t) dt, \quad \hat{\psi}^0_0 = 0.
\] (3.39)
with $\varepsilon \to 0$, take the same manner as above, it can deduces that the law of $\hat{Y}^\varepsilon$ satisfies the LDP with the good rate function $I$,
\[ I(\hat{y}) = \inf \left\{ \|k\|^2_{\mathcal{H}^{d_2}}/2 \mid \hat{y} = \hat{\psi}_0^0(k, \lambda)^1 \right\} \] (if $\hat{y} = \hat{\psi}_0^0(k, \lambda)^1$ for some $k \in \mathcal{H}^{d_2}$),
(3.40)
(otherwise).

If $\nu = \nu'$, the results is shown in Lemma 3.3.

Here follows the result for the Taylor expansion of $\phi(\varepsilon) - \phi^0$ in (2.27) with $m$th term $\phi^m$ around $(\gamma, \eta)^T \in C^0_{q,\var} (\mathbb{R}^{d_2})$ with $1/p + 1/q > 1$.

Lemma 3.5. Let $p \geq 2, 1 \leq q < 2$, for any $m = 1, 2, \cdots$,
\[ \phi(\varepsilon) = \phi^0 + \varepsilon\phi^1 + \cdots + \varepsilon^m \phi^m + R_{\varepsilon}^{m+1}. \]
The maps $((B^H, W), (\gamma, \eta)^T) \in G\Omega_p (\mathbb{R}^{d_2}) \times C^0_{q,\var} (\mathbb{R}^{d_2}) \rightarrow \phi^k, R_{\varepsilon}^{m+1} \in C^0_{q,\var} (\mathbb{R}^n)$ are continuous for $0 \leq m' \leq m$. Moreover, we have the following properties:
(i) For any $r_1 > 0$, there exists $C_1 > 0$ depending on $r_1$ such that if $\|((\gamma, \eta))\|_{q,\var} \leq r_1$, then $\|\phi^m\|_{p,\var} \leq C_1(1 + \|(B^H, W)\|_{p,\var})^m$.
(ii) For any $r_2, r_3 > 0$, there exists $C_2 > 0$ depending on $r_2$ and $r_3$ such that if $\|((\gamma, \eta))\|_{q,\var} \leq r_2$ and $\|\varepsilon((B^H, W))\|_{p,\var} \leq r_3$, then $\|R_{\varepsilon}^{m+1}\|_{p,\var} \leq C_2(\varepsilon + \|(B^H, W)\|_{p,\var})^{m+1}$.

Proof. This lemma could be covered by reference [18].

Now, we give the Cameron-Martin theorem for mixed geometric rough path.

Theorem 3.6. (Cameron-Martin theorem for mixed rough path) For any $(k, \hat{k})^T \in \mathcal{H}$, the law of $(B^H, W), \{\mathbb{P}_x^H\}$ and $\{\mathbb{P}_x^H((\cdot, \cdot)+(K, \hat{K}))\}$ are mutually absolutely continuous, moreover, for any bounded Borel function $f$ on $G\Omega_p (\mathbb{R}^{d_2})$,
\[
\int_{G\Omega_p (\mathbb{R}^{d_2})} f((B^H, W) + (K, \hat{K})) \mathbb{P}_x^H (d(B^H, W)) = \int_{G\Omega_p (\mathbb{R}^{d_2})} f(B^H, W) \exp \left( k \langle (k, \hat{k}), (B^H, W)^1 \rangle - \frac{1}{2} \|k\|^2_{\mathcal{H}} \right) \mathbb{P}_x^H (d(B^H, W)).
\] (3.41)

Proof. $(B^H(m), W(m)) + (K(m), \hat{K}(m))$ is the smooth rough path constructed by $(b^H(m), w(m)) + (k(m), \hat{k}(m))$. Meanwhile, as $m \to \infty$, $(B^H(m), W(m)) \rightarrow (B^H, W)$ in $G\Omega_p (\mathbb{R}^{d_2})$ and $(k(m), \hat{k}(m)) \rightarrow (k, \hat{k})$ with $q$–norm, it is easy to see that (3.41) holds based on the Cameron-Martin theorem for fBm and Bm $(b^H, w_1)_{t \geq 0}$.

The proof is completed.

Next, we give the Fernique type theorem for the mixed geometric rough path for later use.

4. Computation of Hessian

In this section, we set conditions for parameters. Firstly, for the fBm, the Hurst parameter $H \in (1/3, 1/2)$. Then, choose $p$ and $q$ such that
\[
\frac{1}{p} + \frac{1}{q} > 1, \quad \frac{1}{p} + \frac{1}{q} < H, \quad \frac{1}{q} < H + 1, \quad 1 < \frac{1}{p} < H + \frac{1}{2},
\] (4.1)
For example, choose that $1/p = H - 2\varepsilon$ and $1/q = H + 1/2 - \varepsilon$ for small parameter $\varepsilon$, then the above condition (4.1) is satisfied.
4.1. Hilbert-Schmidt property of Hessian

In this subsection, we show the Hilbert-Schmidt property for the Hessian matrix of the Itô map restricted on the Cameron-Martin space $\mathcal{H}$. Throughout this section, set $\beta_0(y) = \beta(0, y)$.

For fixed $(\gamma, \eta)^T \in C^p_{0\text{-var}}(\mathbb{R}^{d_1+d_2})$, set

$$d\Omega_t = [\sigma(\phi_0^T)\hat{\sigma}(\phi_0^T)] \langle \cdot, d(\gamma_t, \eta_t)^T \rangle + \nabla \beta_0(\phi_0^T) \langle \cdot, 1 \rangle dt,$$

where $\phi_0$ is the solution to the ODE (4.2). And $M_t$ satisfies the ODE as follows,

$$dM_t = d\Omega_t \cdot M_t, \quad M_0 = \text{Id}_n,$$

where $\text{Id}_n$ is an identity matrix in $\mathbb{R}^{n \times n}$.

Similarly, its inverse exists and satisfies the ODE,

$$dM_t^{-1} = -M_t^{-1} \cdot d\Omega_t, \quad M_0^{-1} = \text{Id}_n.$$

More details about $M_t$, $M_t^{-1}$ and $\Omega_t$ refer to [18].

Set $\chi(k, \hat{k}) = (\nabla \Psi)(\gamma, \eta)((k, \hat{k})) \in C^p_{0\text{-var}}(\mathbb{R}^n)$, which satisfies the following ODE,

$$d\chi_t - [\nabla \sigma(\phi_0^T)\nabla \hat{\sigma}(\phi_0^T)] (\chi_t, d(\gamma_t, \eta_t)^T) - \nabla \beta_0(\phi_0^T) (\chi_t, 1) dt = [\sigma(\phi_0^T)\hat{\sigma}(\phi_0^T)]d(k_t, \hat{k}_t)^T, \quad \chi_0 = 0.$$

With aid of the results in [18], the solution could be written explicitly as follows,

$$\chi(k, \hat{k})_t = (\nabla \Psi)(\gamma, \eta)((k, \hat{k}))_t = M_t \int_0^t M_s^{-1} [\sigma(\phi_s^0)\hat{\sigma}(\phi_s^0)]d(k_s, \hat{k}_s)^T.$$

According to the Young integral theory, the above integral can be defined in the Young integral sense. And $(k, \hat{k})^T \in C^p_{0\text{-var}}(\mathbb{R}^{d_1+d_2}) \mapsto \chi(k, \hat{k}) \in C^p_{0\text{-var}}(\mathbb{R}^n)$ is a continuous map.

Likewise, $\psi^2((k, \hat{k}), (\gamma, \eta)) = \nabla^2 \Psi(\gamma, \eta)((f, \hat{f}), (k, \hat{k}))_t \in C^p_{0\text{-var}}(\mathbb{R}^n)$ satisfies the following ODE,

$$d\psi_t - [\nabla \sigma(\phi_0^T)\nabla \hat{\sigma}(\phi_0^T)](\psi_t, d(\gamma_t, \eta_t)^T) - \nabla \beta_0(\phi_0^T) (\psi_t, 1) dt = 2[\nabla \sigma(\phi_0^T)\nabla \hat{\sigma}(\phi_0^T)](\chi(k, \hat{k})_t, d(k_t, \hat{k}_t)^T)$$

$$+ [\nabla^2 \sigma(\phi_0^T)\nabla^2 \hat{\sigma}(\phi_0^T)](\chi(k, \hat{k})_t, d(\gamma_t, \eta_t)^T)$$

$$+ \nabla^2 \beta_0(\phi_0^T) (\chi(k, \hat{k})_t) d(k_t, \hat{k}_t)^T, \quad \psi_0 = 0.$$

Then, its solution can be written in the following sense,

$$\nabla^2 \Psi(\gamma)((f, \hat{f}), (k, \hat{k}))_t = M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)](\chi(f, \hat{f})_s, d(k_s, \hat{k}_s)^T)$$

$$\quad + [\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)](\chi(k, \hat{k})_s, d(f_s, \hat{f}_s)^T)$$

$$\quad + M_t \int_0^t M_s^{-1}[\nabla^2 \sigma(\phi_s^0)\nabla^2 \hat{\sigma}(\phi_s^0)](\chi(f, \hat{f})_s, \chi(k, \hat{k})_s, d(\gamma, \eta)_s^T)$$

$$\quad + \nabla^2 \beta_0(\phi_0^T) (\chi(f, \hat{f})_s, \chi(k, \hat{k})_s) ds.$$

Next, define $\nabla^2(F \circ \Psi)(\gamma, \eta)((\cdot, \cdot), (\cdot, \cdot))$ as follows,

$$\nabla^2(F \circ \Psi)(\gamma, \eta)((\cdot, \cdot), (\cdot, \cdot)) = \nabla^2(F(\Psi(\gamma, \eta)))((f, \hat{f}), (k, \hat{k}))$$

$$\quad + \nabla(F(\Psi(\gamma, \eta))) \cdot (\nabla \Psi(\gamma, \eta))((f, \hat{f}), \nabla \Psi(\gamma, \eta)((k, \hat{k}))).$$

where $\nabla \Psi(\gamma, \eta)((f, \hat{f})$ and $\nabla^2 \Psi(\gamma, \eta)((f, \hat{f}), (k, \hat{k}))$ are defined in [18] and [19] respectively.

**Theorem 4.1.** (Hilbert-Schmidt property of Hessian) $\nabla^2(F \circ \Psi)(\gamma, \eta)((\cdot, \cdot), (\cdot, \cdot))$ is a symmetric Hilbert-Schmidt bilinear form on $\mathcal{H}$ for all $(\gamma, \eta)^T \in \mathcal{H}$.

We have divided the proof for Theorem 4.1 into a sequence of lemmas.
Lemma 4.2. Choose appropriate $p$ and $q$. Then, if $p' \geq p$, $\nabla F(\varphi^0) \circ V_2$ is of trace class for a Fréchet differentiable function $F$. A similar fact holds for $\nabla^2 F(\varphi^0) \langle \chi(k, \hat{k}), \chi(k, \hat{k}) \rangle$.

Proof. According to the Young integral, $V_2$ can be defined in the Young integral sense. Then, applying Lemma 3.4 it follows that $\|M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)\|_{q-var} < \infty$. With aid of the Young integral,

$$\|\chi(k, \hat{k})\|_{q-var} \leq c_1 \|M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)\|_{q-var} \| (k, \hat{k})\|_{p-var},$$

for some constants $c_1, c_2$. Hence, $(k, \hat{k})^T \in C_0^{p-var}(\mathbb{R}^{d_1 + d_2}) \mapsto \chi(k, \hat{k}) \in C_0^{p-var}(\mathbb{R}^n)$ is a bounded linear map.

Moreover, $(f, \hat{f})^T, (k, \hat{k})^T \in C_0^{p-var}(\mathbb{R}^{d_1 + d_2}) \times C_0^{p-var}(\mathbb{R}^{d_1 + d_2}) \mapsto V_2 \in C_0^{p-var}(\mathbb{R}^n)$ is a bounded bilinear map. Due to the fact that the space $C_0^{p-var}(\mathbb{R}^{d_1 + d_2})$ is not separable, consider the abstract Wiener space $(\mathcal{X}, \mathcal{H}, \mu^{\mathcal{H}})$. The Goodman’s theorem (Theorem 4.6, [19]) shows that, its restriction on the Cameron-Martin space $\mathcal{H}$ is of trace class. The same reasoning applies to the case of $\nabla^2 F(\varphi^0) \langle \chi(k, \hat{k}), \chi(k, \hat{k}) \rangle$.

The proof is completed.

Our next concern will be $V_1(\langle f, \hat{f}, (k, \hat{k}) \rangle$, and it can be rewritten as follows,

$$V_1(\langle f, \hat{f}, (k, \hat{k}) \rangle = R_1(\langle f, \hat{f}, (k, \hat{k}) \rangle + R_1(\langle (k, \hat{k}), (f, \hat{f}) \rangle - R_2(\langle (f, \hat{f}, (k, \hat{k}) \rangle + R_2(\langle (k, \hat{k}), (f, \hat{f}) \rangle),$$

where

$$R_1(\langle f, \hat{f}, (k, \hat{k}) \rangle_t = M_t \int_0^t M_s^{-1}[\nabla \sigma(\varphi^0_s)]\nabla \vartheta(\varphi^0_s)(\langle [\sigma(\varphi^0_s)]\vartheta(\varphi^0_s)\rangle (f_s, \hat{f}_s)^T, (d(k_s, \hat{k}_s)^T \rangle, \quad (4.7)$$

and

$$R_2(\langle f, \hat{f}, (k, \hat{k}) \rangle_t = M_t \int_0^t M_s^{-1}[\nabla \sigma(\varphi^0_s)]\nabla \vartheta(\varphi^0_s)(\langle M_s \int_0^t d [M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)](f_u, \hat{f}_u)^T, (d(k_s, \hat{k}_s)^T \rangle \quad (4.8)$$

Lemma 4.3. As a bilinear form on Cameron-Martin space $\mathcal{H}$, $\nabla F(\varphi^0) \circ R_2$ is of trace class. Moreover, if $\vartheta_t$ is weak* convergent to $\vartheta$ as $t \to \infty$ in $C_0^{p-var}(\mathbb{R}^n)^*$, then $\vartheta_t \circ R_2$ converges to $\vartheta \circ R_2$ as $t \to \infty$ in the Hilbert-Schmidt norm.

Proof. Applying Lemma 3.4 it deduces that $\|M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)\|_{q-var} < \infty$ and $\|M_u^{-1}[\nabla \sigma(\varphi^0_u)]\nabla \vartheta(\varphi^0_u)\|_{q-var} < \infty$. Therefore

$$\left\| \int_0^t d [M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)](f_u, \hat{f}_u)^T \right\|_{q-var} \leq c_1 \left\| M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)\right\|_{q-var} \| (f, \hat{f})\|_{p-var},$$

for some constant $c_2$. Moreover,

$$\left\| M \int_0^t M_s^{-1}[\nabla \sigma(\varphi^0_s)]\nabla \vartheta(\varphi^0_s)\left\langle M_s \int_0^t d [M_u^{-1}[\sigma(\varphi^0_u)]\vartheta(\varphi^0_u)](f_u, \hat{f}_u)^T, (d(k_s, \hat{k}_s)^T \rangle \right\|_{p-var} \leq c_3 \| (f, \hat{f})\|_{p-var} \| (k, \hat{k})\|_{p-var},$$

for some constant $c_3$. Obviously, $(f, \hat{f})^T, (k, \hat{k})^T \in C_0^{p-var}(\mathbb{R}^{d_1 + d_2}) \times C_0^{p-var}(\mathbb{R}^{d_1 + d_2}) \mapsto R_2 \in C_0^{p-var}(\mathbb{R}^n)$ is a bounded bilinear map. In particular, $\vartheta \circ R_2$ is a bounded bilinear form on $C_0^{p-var}(\mathbb{R}^n)$. The Goodman’s theorem (Theorem 4.6, [19]) shows that, its restriction on the Cameron-Martin space $\mathcal{H}$ is of trace class.

Next, we intend to prove the second argument. Since $\mathcal{H}^{d_1, d_2} \hookrightarrow W_0^{d_1, d_2} \cong L_0^{d_1, d_2}$ [17], Proposition 3.4], it has $\mathcal{H} \hookrightarrow L_0^{d_1, d_2}(\mathbb{R}^{d_1}) \oplus H^{d_2, d_2}$. Then choose the ONB $(f, \hat{f})_{\alpha, i} = \cos(2\pi m \gamma) \cdot e_i$ and $(f, \hat{f})_{\alpha, i} = \sin(2\pi m \gamma) \cdot e_i$ for $i = 1, \cdots, d_1, (f, \hat{f})_{\alpha, i} = t \cdot e_i$, $(f, \hat{f})_{2m-1, i} = \cos(2m\pi \gamma) \cdot e_i$ and $(f, \hat{f})_{2m, i} = \sin(2m\pi \gamma) \cdot e_i$.
Fix $i = 1, \cdots, d_1$, $(f, \hat{f})_{m,i}^T = \sqrt{2 \cos(m \pi t)}$. For the jog-free function, it is easy to check that $\|\cos(n \pi t) - 1\|_{p-var} = 2n^{1/p}$. [17]. Therefore,

$$\| (f, \hat{f})_{m,i}^T \|_{p-var} \leq (1 + m^2)^{-\delta/2} \sqrt{2} \left( 1 + 2m^{1/p} \right) \leq c \left( \frac{1}{1 + m} \right)^{1/2 - 1/p},$$

(4.9)
due to $2(1/q - 1/p) > 1$, it leads to

$$\sum_{i,j=1}^{d_1} \sum_{m=0}^{\infty} \| R_2 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{m',j} \rangle \|_{p-var}^2 \leq c \sum_{i,j=1}^{d_1} \sum_{m=0}^{\infty} \| (f, \hat{f})_{m,i} \|_{p-var}^2 \| (f, \hat{f})_{m',j} \|_{p-var}^2 \leq c \sum_{m=0}^{\infty} \left( \frac{1}{1 + m} \right)^{2(1/q - 1/p)} \sum_{m'=0}^{\infty} \left( \frac{1}{1 + m'} \right)^{2(1/q - 1/p)}$$

Fix $i = d_1 + 1, \cdots, d_1 + d_2$. Likewise, it has $\| \sin(n \pi t) \|_{p-var} \leq c (1/2^p + n)^{1/p}$. Then

$$\| (f, \hat{f})_{2m,i}^T \|_{p-var} \leq c (2m \pi)^{-1} (1/2^p + m)^{1/p} \leq c \left( \frac{1}{1 + m} \right)^{1/2 - 1/p},$$

(4.10)
Fix $i = d_1 + 1, \cdots, d_1 + d_2, j = 1, \cdots, d_1$, since $2(1 - 1/p) > 1$ and $2(1/q - 1/p) > 1$, we have

$$\sum_{i=d_1+1}^{d_1+d_2} \sum_{j=1}^{d_1} \sum_{m,m'\in\mathbb{Z}} \left\| R_2 \langle (f, \hat{f})_{2m,i}, (f, \hat{f})_{m',j} \rangle \right\|_{p-var}^2 \leq c \sum_{i=d_1+1}^{d_1+d_2} \sum_{j=1}^{d_1} \sum_{m,m'\in\mathbb{Z}} \left\| (f, \hat{f})_{2m,i} \right\|_{p-var}^2 \left\| (f, \hat{f})_{m',j} \right\|_{p-var}^2 \leq c \sum_{m=0}^{\infty} \left( \frac{1}{1 + m} \right)^{2(1 - 1/p)} \sum_{m'=0}^{\infty} \left( \frac{1}{1 + m'} \right)^{2(1/q - 1/p)}$$

Fix $i = 1, \cdots, d_1, j = d_1, \cdots, d_1 + d_2$, in the same manner, it deduces that

$$\sum_{i=1}^{d_1} \sum_{j=d_1+1}^{d_1+d_2} \sum_{m,m'\in\mathbb{Z}} \left\| R_2 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{2m',-1, j} \rangle \right\|_{p-var}^2 < \infty,$$

and

$$\sum_{i=1}^{d_1} \sum_{j=d_1+1}^{d_1+d_2} \sum_{m,m'\in\mathbb{Z}} \left\| R_2 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{2m',-1, j} \rangle \right\|_{p-var}^2 < \infty.$$

Fix $i = d_1 + 1, \cdots, d_1 + d_2, j = d_1 + 1, \cdots, d_1 + d_2$, since $2(1 - 1/p) > 1$, we have

$$\sum_{i=d_1+1}^{d_1+d_2} \sum_{j=d_1+1}^{d_1+d_2} \sum_{m,m'\in\mathbb{Z}} \left\| R_2 \langle (f, \hat{f})_{2m-1,i}, (f, \hat{f})_{2m'-1,j} \rangle \right\|_{p-var}^2 \leq c \sum_{i=d_1+1}^{d_1+d_2} \sum_{j=d_1+1}^{d_1+d_2} \sum_{m,m'\in\mathbb{Z}} \left\| (f, \hat{f})_{2m-1,i} \right\|_{p-var}^2 \left\| (f, \hat{f})_{2m'-1,j} \right\|_{p-var}^2 \leq c \sum_{m=0}^{\infty} \left( \frac{1}{1 + m} \right)^{2(1 - 1/p)} \sum_{m'=0}^{\infty} \left( \frac{1}{1 + m'} \right)^{2(1/q - 1/p)}$$

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Lemma 4.4. As a bilinear form on Cameron-Martin space and similar for

Proof

\[
\|R_2(\langle f, \hat{f}\rangle_{m,i}, (f, \hat{f})_{m',j})\|_{p,\var}^2 \leq c \sum_{m=0}^{\infty} \left( \frac{1}{1 + m} \right)^{2(1-1/p)} \sum_{m'=0}^{\infty} \left( \frac{1}{1 + m'} \right)^{2(1-1/p)} < \infty.
\]

From the Banach-Steinhaus theorem, \(\|\vartheta - \vartheta\|_{C_0^{p,\var},*} \leq c\), then we have

\[
\| (\vartheta - \vartheta) \circ R_2(\langle f, \hat{f}\rangle_{m,i}, (f, \hat{f})_{m',j}) \|^2 \leq c^2 \|R_2(\langle f, \hat{f}\rangle_{m,i}, (f, \hat{f})_{m',j})\|_{p,\var}^2,
\]

similar results could be obtained for \(i = 1, \cdots, d_1, j = d_1 + 1, \cdots, d_1 + d_2, i = d_1 + 1, \cdots, d_1 + d_2, j = 1, \cdots, d_1\) and \(i = d_1 + 1, \cdots, d_1 + d_2, j = d_1 + 1, \cdots, d_1 + d_2\), and omitted for simplicity.

By the dominated convergence theorem, it is clear to see that

\[
\| \vartheta \circ R_2 - \vartheta \circ R_2 \|_{\text{HS},d_1,\hat{d}_1} \leq \|t\|_{\text{op}} \|t^*\|_{\text{op}} \| \vartheta \circ R_2 - \vartheta \circ R_2 \|_{\text{HS},L_2,F} \rightarrow 0,
\]

with \(l \rightarrow 0\). Hence, if \(\vartheta\) is weak* convergent to \(\vartheta\) as \(l \rightarrow \infty\) in \(C_0^{p,\var} (\mathbb{R}^n)^*\), then \(\vartheta \circ R_2\) converges to \(\vartheta \circ R_1\) as \(l \rightarrow \infty\) in the Hilbert-Schmidt norm. The proof is completed.

Lemma 4.4. As a bilinear form on Cameron-Martin space \(\mathcal{H}, \nabla F(\phi^0) \circ R_1\) is Hilbert-Schmidt. Moreover, if \(\vartheta\) is weak* convergent to \(\vartheta\) as \(l \rightarrow \infty\) in \(C_0^{p,\var} (\mathbb{R}^n)^*\), then \(\vartheta \circ R_1\) converges to \(\vartheta \circ R_1\) as \(l \rightarrow \infty\) in the Hilbert-Schmidt norm.

Proof. Similar to Lemma 4.3, it is sufficient to make the following observation,

\[
\sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \sum_{m,m'=0}^{\infty} \|R_1(\langle f, \hat{f}\rangle_{m,i}, (f, \hat{f})_{m',j})\|_{p,\var}^2 < \infty,
\]

and similar for \(i = 1, \cdots, d_1, j = d_1, \cdots, d_1 + d_2, i = d_1, \cdots, d_1 + d_2, j = 1, \cdots, d_1\) and \(i = d_1, \cdots, d_1 + d_2, j = d_1, \cdots, d_1 + d_2\).

We divide the rest of the proof into four steps.

Step 1. Fix \(i = 1, \cdots, d_1, j = 1, \cdots, d_1\).

Now, \(\langle f, \hat{f}\rangle_{m,i} = \sqrt{2 \cos(m\pi t)/(1 + m^2)} \sigma^{(m\pi t)}_{x} (f, \hat{f})_{m',j} = \sqrt{2 \cos(m'\pi t)/(1 + m'^2)} \sigma^{(m'\pi t)}_{x}, m \neq m'\),

\[
\sqrt{2 \cos(m\pi t)} d [\sqrt{2 \cos(m'\pi t)}] = m' d \left[ \frac{\cos((m' + m)\pi t)}{m' + m} + \frac{\cos((m' - m)\pi t)}{m' - m} \right].
\]

Then,

\[
R_1(\langle f, \hat{f}\rangle_{m,i}, (f, \hat{f})_{m',j}) = M \int_0^t M^{-1} [\nabla \sigma(\phi^0_x)] [\nabla \tilde{\sigma}(\phi^0_x)] [(\sigma(\phi^0_x)) [\tilde{\sigma}(\phi^0_x)] e_i, e_j] \sqrt{2 \cos(m\pi t)}/(1 + m^2)^{1/2}.
\]
\[
\begin{align*}
\sum_{m, m' \neq m} I_2^2 &\leq c \sum_{m \in \mathbb{Z}} \left( \frac{1}{1+|m|^2/q} \left( \frac{1}{1+|m' + m|^2/2q} \right) \right) \\
&\leq \sum_{m \in \mathbb{Z}} \frac{1}{|m|^2(1+|m|^2/2q)} < \infty.
\end{align*}
\]

Likewise, it has \( \sum_{d \leq m, m' \neq m'} I_2^2 \) \( p \)-var < \( \infty \).

Next, we turn to the case that \( m = m' \),

\[
\sqrt{2} \cos (m \pi t) d[\sqrt{2} \cos (m \pi t)] = d[\cos (2m \pi t)]/2,
\]

therefore

\[
R_1 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{m,j} \rangle_t = M_t \int_0^t M_s^{-1} \langle \nabla \sigma (\phi_s^0) \hat{\sigma} (\phi_s^0) \rangle (\langle \sigma (\phi_s^0) \hat{\sigma} (\phi_s^0) \rangle) e_i, e_j \rangle d[\sqrt{2} \cos (m \pi t)] / (1 + m^2)^{1/2q}.
\]

Since \( 4/q - 2/p > 1 \),

\[
\sum_{i,j=1} d_i \sum_{m=0}^{\infty} \| R_1 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{m,j} \rangle \|_{p \text{-var}}^2 \leq \frac{c}{(1 + m^2)^{2/q}} \| \cos (2m \pi t - 1) \|_{p \text{-var}}^2 \\leq \sum_{m=0}^{\infty} \frac{c}{(1 + m)^{1/2}} < \infty.
\]

Step 2. Fix \( i = 1, \ldots, d_1, j = d_1 + 1, \ldots, d_1 + d_2 \).

Consider that \( (f, \hat{f})^T_{m,i} = \sqrt{2} \cos (m \pi t) / (1 + m^2)^{1/2q} \), \( (f, \hat{f})^T_{2m - 1, j} = \cos (2m \pi t - 1) / 2m \pi \), and \( m \neq 2m' \), then

\[
\sqrt{2} \cos (m \pi t) d \left[ \frac{\cos (2m' \pi t - 1)}{2m' \pi} \right] = \frac{1}{\sqrt{2}} d \left[ \frac{\cos (2m' + m \pi t)}{2m' \pi} + \frac{\cos (2m' - m \pi t)}{(2m' - m) \pi} \right],
\]

then

\[
R_1 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{2m -1,j} \rangle_t = M_t \int_0^t M_s^{-1} \langle \nabla \sigma (\phi_s^0) \hat{\sigma} (\phi_s^0) \rangle (\langle \sigma (\phi_s^0) \hat{\sigma} (\phi_s^0) \rangle) e_i, e_j \rangle d[\sqrt{2} \cos (m \pi t)] / (1 + m^2)^{1/2q}.
\]
Likewise, it has
\[ \sum_{m,i} \text{now} \\leq c \sum_{m \in \mathbb{Z}} \left( \frac{1}{1 + |m|^2/q} \right) < \infty. \]

Next by the Young integral, and $2/q > 1$, we can obtain
\[ \sum_{0 \leq m, m' < \infty, m \neq 2m'} \|I_3\|^2_{p-\text{var}} \leq c \sum_{m \in \mathbb{Z}} \left( \frac{1}{|m|^{2/q}} \right) < \infty. \]

Likewise, it has $\sum_{0 \leq m, m' < \infty, m \neq 2m'} \|I_4\|^2_{p-\text{var}} < \infty.$

When $(f, \hat{f})_{m,i} = \sqrt{\frac{\cos(m\pi t)}{(1 + m^2)^{1/2q}}}$, $(f, \hat{f})_{2m'-1,j} = \frac{\cos(2m'\pi t) - 1}{2m'\pi}$, and $m = m'$, then
\[
\sqrt{2} \cos (m\pi t) \left[ \cos (m\pi t) - 1 \right] \left[ \frac{\cos (m\pi t)}{2m'\pi} \right].
\]

then
\[
R_1 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{2m'-1,j} \rangle = M_t \int_0^t \left[ \frac{\cos (2m'\pi s) - 1}{2m'\pi} \right] \times \left[ \frac{\cos (2m'\pi s)}{2m'\pi} \right] d \left[ \cos (2m'\pi s) \right] = I_5.
\]

In the same manner, we can deduce that $\sum_{0 \leq m < \infty} \|I_5\|^2_{p-\text{var}} < \infty.$

Consider that $(f, \hat{f})_{m,i} = \sqrt{\frac{\cos(m\pi t)}{(1 + m^2)^{1/2q}}}$, $(f, \hat{f})_{2m'-1,j} = \frac{\sin((2m'\pi t))}{2m'\pi}$, and $m \neq 2m'$, then
\[
\sqrt{2} \cos (m\pi t) \left[ \frac{\sin((2m'\pi t))}{2m'\pi} \right] = \frac{1}{\sqrt{2}} \left[ \frac{\sin((2m' + m)\pi t)}{(2m' + m)\pi} \right] + \frac{1}{\sqrt{2}} \left[ \frac{\sin((2m' - m)\pi t)}{(2m' - m)\pi} \right],
\]

it follows that
\[
R_1 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{2m'-1,j} \rangle = M_t \int_0^t \left[ \frac{\cos (2m'\pi s)}{2m'\pi} \right] \left[ \frac{\cos (2m'\pi s)}{2m'\pi} \right] d \left[ \cos (2m'\pi s) \right] = I_5.
\]
Likewise, it has \( \sum_{0 \leq m, m' < \infty, m \neq 2m'} \| I_6 \|^2_{p \text{-var}} < \infty. \)

Consider that \((f, \dot{f})_{m,i}^T = \sqrt{2} \cos (m \pi t), (\dot{f}, \ddot{f})_{2m',j}^T = \frac{\sin (2m' \pi t)}{2m' \pi}, \) and \(m = 2m',\) then

\[
\sqrt{2} \cos (m \pi t) \left[ \frac{\sin (m \pi t)}{m \pi} \right] = \frac{1}{\sqrt{2}} \left[ \frac{\sin (2m \pi t)}{2m \pi} \right] + \frac{1}{\sqrt{2}} dt.
\]

Therefore,

\[
R \langle (f, \dot{f})_{m,i}, (\dot{f}, \ddot{f})_{m',j} \rangle_t = M_t \int_0^t \frac{1}{2 \sqrt{2} (1 + m^2)^{1/2} m \pi} \left[ \frac{\sin (m \pi s)}{m \pi} \right] \left[ \frac{\sin (2m \pi s)}{2m \pi} \right] \frac{1}{\sqrt{2}} dt =: I_8 + I_9.
\]

By that \(2(1/q - 1/p) > 1,\) we can obtain

\[
\sum_{0 \leq m < \infty} \| I_8 \|^2_{p \text{-var}} \leq \frac{1}{m^2(1 + |m|)^{2(1/q - 1/p)}} < \infty.
\]

Likewise, it has \( \sum_{0 \leq m < \infty} \| I_9 \|^2_{p \text{-var}} < \infty. \)

**Step 3.** Fix \(i = d_1 + 1, \cdots, d_1 + d_2, j = 1, \cdots, d_i.\)

Consider that \((f, \dot{f})_{2m-1,i}^T = \frac{\cos (2m \pi t)}{2m \pi}, (\dot{f}, \ddot{f})_{2m',j}^T = \frac{\cos ((2m - m') \pi t)}{(2m - m') \pi}, \) and \(2m \neq m',\) then

\[
\frac{\cos (2m \pi t) - 1}{2m \pi} \left[ \frac{\sqrt{2} \cos (m \pi t)}{2m \pi} \right] = \frac{m'}{2 \sqrt{2} m'} \left[ \frac{\cos (2m \pi t)}{2m \pi} - \frac{\cos ((2m - m') \pi t)}{(2m - m') \pi} \right] - \frac{1}{2m} \left[ \frac{\sqrt{2} \cos (m \pi t)}{2m \pi} \right],
\]

hence

\[
R \langle (f, \dot{f})_{2m,i}, (\dot{f}, \ddot{f})_{m',j} \rangle_t = M_t \int_0^t \left[ \frac{\cos (2m \pi t) - 1}{2m \pi} \right] \left[ \frac{\sqrt{2} \cos (m \pi t)}{2m \pi} \right] \frac{1}{\sqrt{2}} dt = \frac{2 \sqrt{2} (1 + m'^2)^{1/2} (2m + m') \pi}{m'}
\]

\[
\times M_t \int_0^t M_s^{-1} \left[ \frac{\cos (2m \pi t)}{2m \pi} - \frac{\cos ((2m - m') \pi t)}{(2m - m') \pi} \right] \frac{1}{\sqrt{2}} \left[ \frac{\sqrt{2} \cos (m \pi t)}{2m \pi} \right] \frac{1}{\sqrt{2}} dt = \frac{2 \sqrt{2} (1 + m'^2)^{1/2} (2m + m') \pi}{m'}
\]

\[
\times M_t \int_0^t M_s^{-1} \left[ \frac{\cos (2m \pi t)}{2m \pi} - \frac{\cos ((2m - m') \pi t)}{(2m - m') \pi} \right] \frac{1}{\sqrt{2}} \left[ \frac{\sqrt{2} \cos (m \pi t)}{2m \pi} \right] \frac{1}{\sqrt{2}} dt = \frac{2 \sqrt{2} (1 + m'^2)^{1/2} (2m + m') \pi}{m'}
\]

\[
\times M_t \int_0^t M_s^{-1} \left[ \frac{\cos (2m \pi t)}{2m \pi} - \frac{\cos ((2m - m') \pi t)}{(2m - m') \pi} \right] \frac{1}{\sqrt{2}} \left[ \frac{\sqrt{2} \cos (m \pi t)}{2m \pi} \right] \frac{1}{\sqrt{2}} dt = \frac{2 \sqrt{2} (1 + m'^2)^{1/2} (2m + m') \pi}{m'}
\]
\[
- \frac{1}{\sqrt{2}(1 + m^2)^{1/2q}} \sum_{0 \leq m, m' < \infty, 2m \neq m'} M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \sigma(\phi_s^0)] \langle [\sigma(\phi_s^0) | \sigma(\phi_s^0)] e_i, e_j \rangle d [\cos (m^2 \pi s)]
\]
\[
=: \ I_{10} + I_{11} + I_{12}.
\]
Next by the Young integral, \(2/q > 1\) and \(2(1 - 1/p) > 1\), we have
\[
\sum_{0 \leq m, m' < \infty, 2m \neq m'} \|I_{10}\|^2_{p-var} \leq c \sum_{0 \leq m, m' < \infty, 2m \neq m'} \frac{m'(1 + |m' + 2m|)^{2/p}}{(1 + |m'|)^{2/q} |2m' + m|^2}
\]
\[
\leq c \sum_{m \in \mathbb{Z}} \frac{1}{(1 + |m|)^2} \left( \sum_{m' \in \mathbb{Z}} \frac{1}{(1 + |2m' + m|)^{2(1/q - 1/p)}} \right)
\]
\[
+ c \sum_{m \in \mathbb{Z}} \frac{1}{(1 + |m|)^{2/q}} \left( \sum_{m' \in \mathbb{Z}} \frac{1}{(1 + |2m' + m|)^{2(1 - 1/p)}} \right)
\]
\[
< \infty.
\]
Likewise, it has \(\sum_{0 \leq m, m' < \infty, 2m \neq m'} \|I_{11}\|^2_{p-var} < \infty\), and \(\sum_{0 \leq m, m' < \infty, 2m \neq m'} \|I_{12}\|^2_{p-var} < \infty\).
Consider that \((f, \hat{f})^T_{2m-1,i}, (f, \hat{f})^T_{m', j} = \sqrt{2} \cos(m' \pi t)\), and \(2m = m'\), then
\[
\frac{\cos (m^2 \pi t) - 1}{2m \pi d} \left[ \sqrt{2} \cos (m^2 \pi t) \right] = \frac{1}{2 \sqrt{2}} d \left[ \frac{\cos (2m\pi t)}{2m \pi} \right] - \frac{1}{2m \pi} d \left[ \sqrt{2} \cos (2m\pi t) \right],
\]
then
\[
R_1 \langle (f, \hat{f})^T_{2m-1,i}, (f, \hat{f})^T_{m', j} \rangle = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \sigma(\phi_s^0)] \langle [\sigma(\phi_s^0) | \sigma(\phi_s^0)] e_i, e_j \rangle \cos (m^2 \pi t) - 1 \frac{1}{m \pi} d \left[ \sqrt{2} \cos (m^2 \pi s) \right]
\]
\[
= \frac{1}{4 \sqrt{2}(1 + m^2)^{1/2q} m \pi}
\]
\[
\times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \sigma(\phi_s^0)] \langle [\sigma(\phi_s^0) | \sigma(\phi_s^0)] e_i, e_j \rangle d [\cos (2m^2 \pi s)]
\]
\[
= \frac{\sqrt{2}}{2(1 + m^2)^{1/2q} m \pi}
\times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \sigma(\phi_s^0)] \langle [\sigma(\phi_s^0) | \sigma(\phi_s^0)] e_i, e_j \rangle d [\cos (2m^2 \pi s)]
\]
\[
=: \ I_{13} + I_{14}.
\]
Next by the Young integral and \(2(1/q - 1/p) > 1\), we can obtain
\[
\sum_{0 \leq m < \infty} \|I_{13}\|^2_{p-var} \leq c \sum_{m \in \mathbb{Z}} \frac{1}{m^2(1 + m)^{2(1/q - 1/p)}} < \infty.
\]
Likewise, it has \(\sum_{0 \leq m < \infty} \|I_{14}\|^2_{p-var} < \infty\).
Consider that \((f, \hat{f})^T_{2m,i} = \sin(2m\pi t)\), \((f, \hat{f})^T_{m', j} = \sqrt{2} \cos(m' \pi t)\), and \(2m \neq m'\), then
\[
\frac{\sin (2m\pi t)}{2m \pi d} \left[ \sqrt{2} \cos (m' \pi t) \right] = \frac{\sqrt{2m'}}{4m} d \left[ \frac{\sin((2m + m')\pi t)}{(2m + m') \pi} - \frac{\sin((2m - m')\pi t)}{(2m - m') \pi} \right],
\]
then it has
\[
R_1 \langle (f, \hat{f})^T_{2m,i}, (f, \hat{f})^T_{m', j} \rangle = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \sigma(\phi_s^0)] \langle [\sigma(\phi_s^0) | \sigma(\phi_s^0)] e_i, e_j \rangle \sin (2m\pi t) \frac{1}{2m \pi} d \left[ \sqrt{2} \cos (m' \pi s) \right]
\]
\[
(1 + m^2)^{1/2q})
\]
\[
= 24
\]
Therefore,

\[
\frac{\sqrt{2}m'}{4(1 + m'^2)^{1/2q}(2m + m')\pi} \times M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)]\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j d[\sin((2m + m')\pi s)]
\]

\[
- \frac{\sqrt{2}m'}{4(1 + m'^2)^{1/2q}(2m - m')\pi} \times M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)]\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j d[\sin((2m - m')\pi s)]
\]

\[
= I_{15} + I_{16}.
\]

Similar to (4.14), we have $\sum_{0 \leq m, m' < \infty, m \neq m'} ||I_{15}||^2_{p \text{-var}} < \infty$, and $\sum_{0 \leq m, m' < \infty, m \neq m'} ||I_{16}||^2_{p \text{-var}} < \infty$.

Consider that $(f, \hat{f})^T_{2m,i} = \frac{\sin(2m\pi t)}{2m\pi}, (f, \hat{f})^T_{m',j} = \sqrt{2} \frac{\cos(m'\pi t)}{(1 + m'^2)^{1/2q}}$, and $2m = m'$, then

\[
\frac{\sin(2m\pi t)}{2m\pi} d[\sqrt{2} \cos(2m\pi t)] = \frac{\sqrt{2}}{8m\pi} d[\sin(4m\pi t)] - \frac{1}{\sqrt{2}} dt,
\]

Therefore,

\[
R_1\left(\langle f, \hat{f} \rangle_{2m,i}, \langle f, \hat{f} \rangle_{m',j}\right)_t = M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j \frac{\sin(2m\pi t)}{2m\pi} \times d\left[\sqrt{2} \frac{\cos(2m\pi t)}{(1 + (2m)^2)^{1/2q}}\right]
\]

\[
= \frac{\sqrt{2}}{4\sqrt{2}(1 + (2m)^2)^{1/2q}m\pi} \times M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j d[\sin(4m\pi s)]
\]

\[
- \frac{1}{\sqrt{2}(1 + (2m)^2)^{1/2q}} \times M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j dt
\]

\[
= I_{17} + I_{18}.
\]

Analysis similar to the proof in (4.14), we claim that $\sum_{0 \leq m < \infty} ||I_{17}||_{p \text{-var}}^2 < \infty$, and $\sum_{0 \leq m < \infty} ||I_{18}||_{p \text{-var}}^2 < \infty$.

**Step 4.** Fix $i = d_1 + 1, \ldots, d_1 + d_2, j = d_1 + 1, \ldots, d_1 + d_2$.

When $(f, \hat{f})^T_{2m-1,i} = \frac{\cos(2m\pi t)}{2m\pi}, (f, \hat{f})^T_{2m-1,j} = \frac{\cos(2m\pi t)}{2m\pi} - 1$ and $m \neq m'$,

\[
\left[\frac{\cos(2m\pi t) - 1}{2m\pi}\right] d\left[\frac{\cos(2m'\pi t) - 1}{2m'\pi}\right] = \frac{1}{4m\pi} d\left[\cos((2m + 2m')\pi t)\right] - \frac{1}{4m\pi} d\left[\cos((2m - 2m')\pi t)\right]
\]

\[
- \frac{1}{4m\pi} d\left[\cos(2m'\pi t)\right].
\]

then

\[
R_1\left(\langle f, \hat{f} \rangle_{2m-1,i}, \langle f, \hat{f} \rangle_{2m-1,j}\right)_t = M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j \left[\frac{\cos(2m\pi t) - 1}{2m\pi}\right] \times d\left[\frac{\cos(2m'\pi s) - 1}{2m'\pi}\right]
\]

\[
= \frac{1}{4m(2m + 2m')\pi^2} \times M_t \int_0^t M_s^{-1}[\nabla \sigma(\phi_s^0)\nabla \hat{\sigma}(\phi_s^0)][[\sigma(\phi_s^0)]\hat{\sigma}(\phi_s^0)]e_i, e_j d[\cos((2m + 2m')\pi s)]
\]

\[
- \frac{1}{4m(2m - 2m')\pi^2}
\]

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\[ \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle d[\cos((2m - 2m')\pi s)] \]

\[ \times \frac{1}{4m'm^2\pi^2} M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle d[\cos(2m'\pi s)] \]

\[ =: I_{19} + I_{20} + I_{21}. \]

The Young integral and \(2/q > 1\) yield that

\[ \sum_{0 \leq m, m' < \infty, m \neq m'} ||I_{19}||^2_{p\text{-var}} \leq c \sum_{m \in \mathbb{Z}} \frac{1}{(1 + m)^{2}} \left( \sum_{m' \in \mathbb{Z}} \frac{1}{(1 + m + m')^{2}} \right)^{2/q} < \infty. \quad (4.16) \]

Likewise, it has \( \sum_{0 \leq m, m' < \infty, m \neq m'} ||I_{20}||^2_{p\text{-var}} < \infty \), \( \sum_{0 \leq m, m' < \infty, m \neq m'} ||I_{21}||^2_{p\text{-var}} < \infty \).

When \((f, \hat{f})_{2m-1, i}^T = \frac{\cos(2m\pi t)}{2m\pi}, (f, \hat{f})_{2m'-1, j}^T = \frac{\cos(2m'\pi t) - 1}{2m'\pi}\) and \( m = m' \),

\[ \left[ \frac{\cos (2m\pi t) - 1}{2m\pi} \right] d \left[ \frac{\cos (2m\pi t) - 1}{2m\pi} \right] = \frac{1}{16m^2\pi^2} M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle d[\cos (4m\pi s)] \]

\[ \times \frac{1}{4m'\pi^2} M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle d[\cos (2m'\pi s)] \]

\[ =: I_{22} + I_{23}. \]

According to that \(2/q > 1\), we can prove

\[ \sum_{0 \leq m < \infty} ||I_{22}||^2_{p\text{-var}} \leq c \sum_{m \in \mathbb{Z}} \frac{1}{m^2(1 + m)^{2/q}} < \infty. \]

Likewise, it has \( \sum_{0 \leq m < \infty} ||I_{23}||^2_{p\text{-var}} < \infty \).

When \((f, \hat{f})_{2m-1, i}^T = \frac{\cos(2m\pi t)}{2m\pi}, (f, \hat{f})_{2m'-1, j}^T = \frac{\sin(2m'\pi t)}{2m\pi}\) and \( m \neq m' \),

\[ \left[ \frac{\cos (2m\pi t) - 1}{2m\pi} \right] d \left[ \frac{\sin (2m\pi t)}{2m'\pi} \right] = \frac{1}{4m\pi^2} \left[ \sin (2m + 2m')\pi t \right] \left( (2m + 2m')\pi t \right) + \frac{1}{4m'\pi} \left[ \sin (2m - 2m')\pi t \right] \left( (2m - 2m')\pi t \right) \]

\[ - \frac{1}{4mm'\pi^2} d[\sin(2m'\pi t)], \]

then it has

\[ R_1 \langle (f, \hat{f})_{2m-1, i}, (f, \hat{f})_{2m'-1, j} \rangle_t = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle \sqrt{2} \frac{\cos (m\pi s)}{(1 + m^2)^{1/2q}} d \left[ \frac{\sin (2m\pi s)}{2m'} \right] \]

\[ = \frac{1}{4m(2m + 2m')\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle d[\sin((2m + 2m')\pi s)] \]

\[ + \frac{1}{4m(2m - 2m')\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0)|\nabla \hat{\sigma}(\phi_s^0)] \langle \sigma(\phi_s^0)|\hat{\sigma}(\phi_s^0) \rangle e_i, e_j \rangle d[\sin((2m - 2m')\pi s)] \]
\[
\frac{-1}{4m'r\pi^2} \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] d \lbrack \sin(2m's) \rbrack = I_{24} + I_{25} + I_{26}.
\]

Similar arguments apply to the \( I_{14} \), we can prove
\[
\sum_{0 \leq m, m' < \infty, m \neq m'} \| I_{24} \|_{p-\text{var}}^2 < \infty, \quad \sum_{0 \leq m, m' < \infty, m \neq m'} \| I_{25} \|_{p-\text{var}}^2 < \infty, \quad \sum_{0 \leq m, m' < \infty, m \neq m'} \| I_{26} \|_{p-\text{var}}^2 < \infty.
\]

When \( (f, \dot{f})_{2m-1, i} = \frac{\cos(2m't)-1}{2m\pi} \), \( (f, \dot{f})_{2m', j} = \frac{\sin(2m't)}{2m\pi} \) and \( m = m' \),
\[
\left[ \frac{\cos(2m't) - 1}{2m\pi} \right] d \left[ \frac{\sin(2m't)}{2m\pi} \right] = \frac{1}{16m^2\pi^2} d \lbrack \sin(4mt) \rbrack + \frac{1}{4m\pi} dt - \frac{1}{4m^2\pi^2} d \lbrack \sin(2m't) \rbrack,
\]
then it has
\[
R_1 \langle (f, \dot{f})_{2m-1, i}, (f, \dot{f})_{2m', j} \rangle_t = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] \left[ \frac{\cos(2m't) - 1}{2m\pi} \right] d \left[ \frac{\sin(2m't)}{2m\pi} \right]
\]
\[
= \frac{1}{16m^2\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] d \lbrack \sin(4mt) \rbrack
\]
\[
+ \frac{1}{2m\pi} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] d s
\]
\[
- \frac{1}{4m^2\pi^2} M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] d \lbrack \sin(2m't) \rbrack
\]
\[
= I_{27} + I_{28} + I_{29}.
\]

Likewise, we can prove that \( \sum_{0 \leq m < \infty} \| I_{27} \|_{p-\text{var}}^2 < \infty, \sum_{0 \leq m < \infty} \| I_{28} \|_{p-\text{var}}^2 < \infty \) and \( \sum_{0 \leq m < \infty} \| I_{29} \|_{p-\text{var}}^2 < \infty \).

When \( (f, \dot{f})_{2m, i} = \frac{\sin(2m't)}{2m\pi} \), \( (f, \dot{f})_{2m', j} \cdot \) \( \frac{\cos(2m't)-1}{2m\pi} \) and \( m \neq m' \),
\[
\frac{\sin(2m't)}{2m\pi} d \left[ \frac{\cos(2m't) - 1}{2m\pi} \right] = \frac{1}{4m\pi} d \left[ \frac{\sin(2m + 2m't)}{2m + 2m't}\pi \right] - \frac{1}{4m\pi} d \left[ \frac{\sin((2m - 2m')t)}{2m - 2m'}\pi \right].
\]
then
\[
R_1 \langle (f, \dot{f})_{2m, i}, (f, \dot{f})_{2m', j} \rangle_t = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] \left[ \frac{\cos(2m't) - 1}{2m\pi} \right] d \left[ \frac{\cos(2m't)}{2m\pi} \right]
\]
\[
= \frac{1}{4m(2m + 2m')^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] d \lbrack \sin((2m + 2m')t) \rbrack
\]
\[
- \frac{1}{4m(2m - 2m')^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \dot{\sigma}(\phi_s^0) | (\sigma(\phi_s^0) | \dot{\sigma}(\phi_s^0)) e_i, e_j] d \lbrack \sin((2m - 2m')t) \rbrack
\]
\[
= I_{30} + I_{31}.
\]

Similar arguments apply to the \( I_{14} \), we can prove
\[
\sum_{0 \leq m, m' < \infty, m \neq m'} \| I_{30} \|_{p-\text{var}}^2 < \infty, \quad \sum_{0 \leq m, m' < \infty, m \neq m'} \| I_{31} \|_{p-\text{var}}^2 < \infty.
\]
When \((f, \hat{f})_{2m,i}^T = \frac{\sin(2m\pi t)}{2m\pi}, (f, \hat{f})_{2m',j}^T = \frac{\cos(2m'\pi t) - 1}{2m'\pi}\) and \(m = m'\),
\[
\left[ \frac{\sin(2m\pi t)}{2m\pi} \right] d \left[ \frac{\cos(2m\pi t) - 1}{2m\pi} \right] = \frac{1}{16m^2\pi^2} d [\sin(4m\pi t)] - \frac{1}{4m\pi} dt,
\]
then
\[
R_1 \langle (f, \hat{f})_{2m,i}, (f, \hat{f})_{2m-1,j} \rangle_t = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) d [\sin(2m\pi s) / (2m\pi)]
\]
\[
\times d \left[ \frac{\sin(2m\pi s)}{2m\pi} \right] = \frac{1}{4m^2(2m+2m')\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) d [\cos(2m+2m')\pi s]
\]
\[
- \frac{1}{4m^2(2m-2m')\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) d [\cos(2m-2m')\pi s]
\]
\[
= I_{32} + I_{33}.
\]

Similarly, we can show \(\sum_{0 \leq m < \infty} \|I_{32}\|_p < \infty\) and \(\sum_{0 \leq m < \infty} \|I_{33}\|_p < \infty\).

When \((f, \hat{f})_{2m,i}^T = \frac{\sin(2m\pi t)}{2m\pi}, (f, \hat{f})_{2m',j}^T = \frac{\sin(2m'\pi t)}{2m'\pi}\) and \(m \neq m'\),
\[
\left[ \frac{\sin(2m\pi t)}{2m\pi} \right] d \left[ \frac{\sin(2m\pi t)}{2m\pi} \right] = \frac{1}{4m^2} d [\cos((2m+2m')\pi t) / (2m+2m')\pi] - \frac{1}{4m^2} d [\cos((2m-2m')\pi t) / (2m-2m')\pi],
\]
then it has
\[
R_1 \langle (f, \hat{f})_{2m,i}, (f, \hat{f})_{2m'+j} \rangle_t = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) \sqrt{2} \frac{\cos(m\pi s)}{m^2} d [\sin(2m\pi s) / (1+m^2)^{1/2}]
\]
\[
\times d \left[ \frac{\sin(2m\pi s)}{2m\pi} \right] = \frac{1}{16m^2\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) d [\cos((2m+2m')\pi s)]
\]
\[
- \frac{1}{16m^2\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) d [\cos((2m-2m')\pi s)]
\]
\[
= I_{34} + I_{35}.
\]

Similarly, we can obtain that \(\sum_{0 \leq m, m' < \infty, m \neq m'} \|I_{34}\|_p < \infty\) and \(\sum_{0 \leq m, m' < \infty, m \neq m'} \|I_{35}\|_p < \infty\).

When \((f, \hat{f})_{2m,i}^T = \frac{\sin(2m\pi t)}{2m\pi}, (f, \hat{f})_{2m',j}^T = \frac{\sin(2m'\pi t)}{2m'\pi}\) and \(m = m'\),
\[
\left[ \frac{\sin(2m\pi t)}{2m\pi} \right] d \left[ \frac{\sin(2m\pi t)}{2m\pi} \right] = - \frac{1}{16m^2\pi^2} d [\cos(4m\pi t)],
\]
then it has
\[
R_1 \langle (f, \hat{f})_{2m,i}, (f, \hat{f})_{2m',j} \rangle_t = M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) \frac{\sin(2m\pi t)}{2m\pi} d [\sin(2m\pi s) / (2m\pi)]
\]
\[
\times d \left[ \frac{\sin(2m\pi s)}{2m\pi} \right] = \frac{1}{16m^2\pi^2} \times M_t \int_0^t M_s^{-1} [\nabla \sigma(\phi_s^0) | \nabla \hat{\sigma}(\phi_s^0)] ([\sigma(\phi_s^0) | \hat{\sigma}(\phi_s^0)] e_i, e_j) d [\cos(4m\pi s)]
\]
\[
= I_{36}.
\]

Similarly to (4.14), it has \(\sum_{0 \leq m < \infty} \|I_{36}\|_p < \infty\) with \(4 - 2/p > 1\).

According to estimates in Step 1 to Step 4, it deduces that
\[
\sum_{i,j=1}^d \sum_{m,m'=0}^\infty \| R_1 \langle (f, \hat{f})_{m,i}, (f, \hat{f})_{m',j} \rangle \|_p < \infty,
\]

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and similar estimates for \( i = 1, \cdots, d_1, j = d_1, \cdots, d_1 + d_2, i = d_1, \cdots, d_1 + d_2, j = 1, \cdots, d_1 \) and \( i = d_1, \cdots, d_1 + d_2, j = d_1, \cdots, d_1 + d_2 \).

The proof is completed. \( \square \)

**Proof of Theorem** Combine Lemma 4.2, Lemma 4.3, and Lemma 4.4. \( \nabla^2(F \circ \Psi)(\gamma, \eta) \langle (\cdot, \cdot), (\cdot, \cdot) \rangle \) is a symmetric Hilbert-Schmidt bilinear form on \( H \) for all \( (\gamma, \eta) \in H \). \( \square \)

4.2. A probabilistic representation of Hessian

Denote \( A_1 \) be a self-adjoint Hilbert-Schmidt operator \( H \) which corresponds to \( \nabla F (\phi^0) \langle V_i((\cdot, \cdot), (\cdot, \cdot)) \rangle \).

Let \( A - A_1 \) be a self-adjoint Hilbert-Schmidt operator \( H \) which corresponds to \( \nabla F (\phi^0) \langle V_2((\cdot, \cdot), (\cdot, \cdot)) \rangle + \nabla^2 F (\phi^0) \langle \chi((\cdot, \cdot)), \chi((\cdot, \cdot)) \rangle \).

Then combining with these all, it deduces that

\[
\langle k, \dot{k} \rangle \in H \mapsto \langle A(k, \dot{k}), (k, \dot{k}) \rangle_H = \nabla F (\phi^0) \langle \psi((k, \dot{k})_1, (k, \dot{k})) \rangle + \nabla^2 F (\phi^0) \langle \chi((k, \dot{k})), \chi((k, \dot{k})) \rangle
\]  

(4.17)

extends to a continuous map on \( GO_p (\mathbb{R}^{d_1 + d_2}) \).

Let \( \mathcal{X} \) be the closure of \( H \) with respect to the \( p \)-variation, then \( (\mathcal{X}, H, \mu^H) \) is the abstract Wiener space. Any \( \langle (k, \dot{k})^T, (\cdot, \cdot)^T \rangle \in H^* = (H^{d_1, d_1})^* \oplus (H^{d_2})^* \) extends to a measurable linear functional on \( \mathcal{X} \), which is denoted by \( \langle (k, \dot{k})^T, (b^H, w)^T \rangle \). Denote that \( C_n = C_n (\mu^H) \) \( (n = 0, 1, 2, \ldots) \) be the \( n \)th Wiener chaos of \( (b^H, w)^T \).

For a cylinder function \( F (b^H, w) = f \left( \langle (k, \dot{k})_1, (b^H, w)^T \rangle, \ldots, \langle (k, \dot{k})_{d_1 + d_2}, (b^H, w)^T \rangle \right) \), where \( f : \mathbb{R}^{d_1 + d_2} \rightarrow \mathbb{R} \) is a bounded smooth function with bounded derivatives, define that

\[
D_{(k, \dot{k})} F (b^H, w) := \sum_{j=1}^{d_1 + d_2} \partial_j f \left( \langle (k, \dot{k})_1, (b^H, w)^T \rangle, \ldots, \langle (k, \dot{k})_{d_1 + d_2}, (b^H, w)^T \rangle \right) \langle (k, \dot{k})_j, (k, \dot{k}) \rangle_H
\]

for \( (k, \dot{k})^T \in H \).

Firstly, we consider the stochastic integration of the kernel associated with \( A_1 \).

**Lemma 4.5.** For each fixed \( t \), \( V_i ((b^H(m), w(m)), (b^H(m), w(m)))_t \), \( i \in \{1, \cdots, n\} \) converges almost surely and in \( L^2(\mu^H) \) with \( m \rightarrow \infty \). Moreover,

\[
\lim_{m \rightarrow \infty} V_i ((b^H(m), w(m)), (b^H(m), w(m)))_t = \Theta_i^t + \Lambda_i^t,
\]

where \( \Lambda_i^t = \lim_{m \rightarrow \infty} \mathbb{E} [V_1 ((b^H(m), w(m)), (b^H(m), w(m)))_t] \) is of finite \( p \)-variation, and \( \Theta_i^t \in C_2 \) which corresponds to the symmetric Hilbert-Schmidt bilinear form \( V_1 ((\cdot, \cdot), (\cdot, \cdot))_t \).

**Proof.** Consider the rough path \( V'(b^H, W) \) of \( V_1 ((b^H, w), (b^H, w)) \), and \( V'(b^H, W)^1 = V_1 ((b^H, w), (b^H, w)) \) being the first level path. Moreover, \( V'(b^H, W) \) and \( V_1 ((b^H, w), (b^H, w)) \) is of second order, it follows that

\[
\| V'(b^H, W)^1 \|_{p \text{-var}} \leq c \left( 1 + \| (b^H, W) \|_{p \text{-var}}^2 \right),
\]

(4.18)

and

\[
\| V'(b^H, W)^1 - V'((\tilde{b}^H, \tilde{W})) \|_{p \text{-var}} \leq c \left( 1 + \| (b^H, W) \|_{p \text{-var}}^2 \right) \sum_{j=1}^{[p]} \| (b^H, W)^j - (\tilde{b}^H, \tilde{W})^j \|_{p/j \text{-var}}
\]

(4.19)
where $c$ is some constant, and $\|\langle B^H, W \rangle \|_{p-\text{var}}$ is defined in Section 3. With Proposition 3.1, it is easy to see that $V'(b^H(m), W(m))^1 = V_1((b^H(m), w(m)), (b^H(m), w(m)))$ converges to $V_1((b^H, w), (b^H, w))$ almost surely. Consequently, we also have

$$
\mathbb{E} \left[ \|V'(B^H(m), W(m))^1 - V'(B^H, W)^1\|_{p-\text{var}}^2 \right] \to 0 \quad \text{as } m \to \infty.
$$

(4.20)

Likewise, with (4.18) and Proposition 3.1, it shows that

$$
\mathbb{E} \left[ \|V'(B^H, W)^1\|_{p-\text{var}} \right] < \infty.
$$

(4.21)

Hence, it is clear to see that $\Lambda_i^1 = \lim_{m \to \infty} \mathbb{E} \left[ V_1((b^H(m), w(m)), (b^H(m), w(m)))^i \right]$ is of finite $p-$variation.

Our next objective is to prove that $\Theta_i^1 \in C_2$ which corresponds to the symmetric Hilbert-Schmidt bilinear form $V_1((\cdot, \cdot), (\cdot, \cdot))$. Due to the definition of derivative operator,

$$
D_{(k,k)} \chi(b^H(m), w(m))_t = M_t \int_0^t M_s^{-1} \left[ |\nabla \sigma(\phi_0)| \nabla \sigma(\phi_0) \right] D_{(k,k)} \chi(b^H(m), w(m)) |_{s} \left. \right| \left. \right|_{s} \|b^H(m), w(m)\|_{s}^T
$$

Similarly, we have

$$
\frac{1}{2} \left( D_{(k,k)} V_1((b^H(m), w(m)), (b^H(m), w(m))) - (b^H, w)_{p-\text{var}} \right) \to 0 \quad \text{almost surely and in } L^r \text{ for any } r > 0,
$$

and in $L^r$ for any $r > 0$, it is clear to see that $\mathbb{E} \left[ (b^H(m), w(m))^{i} \right] = V_1((k(m), \hat{k}(m)), (b^H(m), w(m))) \to V_1((k, \hat{k}), (b^H, w))$, almost surely as $m \to \infty$.

In the same manner, it deduces that

$$
\frac{1}{2} \left( D_{(k,k)} V_1((b^H(m), w(m)), (b^H(m), w(m))) - (b^H, w)_{p-\text{var}} \right) \to 0 \quad \text{almost surely as } m \to \infty.
$$

Meanwhile, by the definition of derivative operator, it has

$$
\frac{1}{2} D_{(k,k)} V'(B^H, W)^1_t = V_1((k, \hat{k}), (b^H, w))^i_t, \quad i \in \{1, \ldots, n\},
$$

and

$$
\frac{1}{2} D_{(k,k)} V'(B^H, W)^1_t = V_1((k, \hat{k}), (b^H, w))^i_t, \quad i \in \{1, \ldots, n\},
$$

which claims that all elements in the first Wiener chaos $C_1$ equal to 0. Furthermore, combined with Theorem 4.1 and the fact that the second Wiener chaos $C_2$ is unitarily isometric with the space of symmetric

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Theorem 4.8. Let $\Theta_1 \in \mathcal{C}_2$ which corresponds to the symmetric Hilbert-Schmidt bilinear form $V_1 ((\cdot, \cdot), (\cdot, \cdot))$. The proof is completed. \hfill $\square$

Lemma 4.6. Let $p' > p$ and $F : C_{0}^{p'-\text{var}}(\mathbb{R}^n)$ be a Fréchet differentiable function, then $\nabla F(\phi^0)(\Theta) \in \mathcal{C}_2(\mu^H)$ which is related to the symmetric Hilbert-Schmidt bilinear form $\nabla F(\phi^0) \circ V_1 = \nabla F(\phi^0)\langle V_1((\cdot, \cdot), (\cdot, \cdot))\rangle$ in Cameron-Martin space $\mathcal{H}$. \hfill $\square$

Proof. Denote $g_K$ the elements of $\mathcal{C}_2(\mu^H)$, where $K$ is symmetric Hilbert-Schmidt bilinear form, define that

$$M := \left\{ \hat{\vartheta} \in C_0^{p'-\text{val}}(\mathbb{R}^n)^* | \vartheta(\omega) = g_{\vartheta \circ V_1}(\omega) \text{ a.a. } \mu^H \right\}, \quad (4.22)$$

$M$ is a linear subspace. From Lemma 4.3 and Lemma 4.4, if $\hat{\vartheta}_1 \in M$ converges to $\hat{\vartheta} \in M$ in the weak$^*$-limit as $l \to \infty$, it deduces that $\vartheta_l \circ V_1$ converges to $\vartheta \circ V_1$ almost surely as $l \to \infty$. Therefore, $M$ is closed under weak$^*$-limit.

Since $y_{k/2^m}(1 \leq k \leq 2^m, 1 \leq i \leq n)$ is the dyadic approximation of $y \in C_0^{p'-\text{var}}(\mathbb{R}^n)$, $\nabla F(\phi^0)\langle y(m) \rangle$ is the linear combination of $y_{k/2^m}(1 \leq k \leq 2^m, 1 \leq i \leq n)$. Furthermore, $\nabla F(\phi^0) \circ \pi(m) \in M$. Due to that $\nabla F(\phi^0) \circ \pi(m) \to \nabla F(\phi^0)$ in the weak$^*$-limit, it follows that $\nabla F(\phi^0) \in M$. The proof is completed. \hfill $\square$

Next, we turn to the stochastic integration of the kernel associated with $A-A_1$.

Lemma 4.7. $A-A_1$ being a self-adjoint Hilbert-Schmidt operator $\mathcal{H}$ which corresponds to $\nabla F(\phi^0)\langle \chi((\cdot, \cdot)), (\cdot, \cdot) \rangle$ is of trace class, moreover,

$$\nabla F(\phi^0)\langle V_2((\cdot, \cdot), (\cdot, \cdot)) \rangle + \nabla^2 F(\phi^0)\langle \chi((\cdot, \cdot)), (\cdot, \cdot) \rangle.$$

is the sum of $\text{Tr}(A-A_1)$ and the second Wiener chaos corresponding to $A-A_1$ which is denoted by $\bar{K}_{A-A_1}((B^H, W)^1)$. \hfill $\square$

Proof. With the aid of Lemma 4.2, $A-A_1$, which is a self-adjoint Hilbert-Schmidt operator $\mathcal{H}$ is of trace class for a Fréchet differentiable function $F$. It yields that $\langle (A-A_1)((B^H, W), (B^H, W)) \rangle$ can be rewritten as the sum of $\text{Tr}(A-A_1)$ and the second Wiener chaos corresponding to $A-A_1$, \hfill $\square$

$$\langle (A-A_1)((B^H, W), (B^H, W)) \rangle = \bar{K}_{A-A_1}((B^H, W)^1) + \text{Tr}(A-A_1). \quad (4.24)$$

The proof is completed. \hfill $\square$

Theorem 4.8. Let $\alpha > 1$ be such that $\text{Id}_\mathcal{H} + \alpha A$ is strictly positive in the form sense, then

$$\int_{\mathcal{G}^d_\mathcal{H}(\mathbb{R}^d, +\varphi)} \exp \left( -\frac{\alpha}{2}(A(B^H, W), (B^H, W)) \right) d^H(d(B^H, W))$$

$$= \exp\left[ -\frac{\alpha}{2}(\text{Tr}(A-A_1) + \nabla F(\phi^0)\langle A \rangle) \right] \cdot \det \left( \text{Id}_\mathcal{H} + \alpha A \right)^{-1/2}, \quad (4.25)$$

where $\det_2$ denotes the Carleman-Fredholm determinant. \hfill $\square$

Proof. Firstly, we have

$$\nabla F(\phi^0)\psi(B^H, W) + \nabla^2 F(\phi)\langle \chi(B^H, W), (\cdot, (\cdot, \cdot)) \rangle = \nabla F(\phi)\langle V_1((B^H, W), (B^H, W)) \rangle + \langle (A-A_1)((B^H, W), (B^H, W)) \rangle,$$

where

$$\langle (A-A_1)((B^H, W), (B^H, W)) \rangle := \nabla F(\phi^0)\langle V_2((B^H, W), (B^H, W)) \rangle.$$
\[
+ \nabla^2 F(\phi^0) \langle \chi(B^H, W), \chi(B^H, W) \rangle,
\]

then by Lemma 4.7 and Lemma 4.6 it follows that \( \langle (A - A_1)(B^H, W), (B^H, W) \rangle \) is the sum of \( \text{Tr}(A - A_1) \) and the second Wiener chaos corresponding to \( A - A_1 \) which is denoted by \( \hat{K}_{A - A_1}((B^H, W)) \), and \( \nabla F(\phi^0)(\Theta) \in C_2(\mu^H) \). Hence, define that \( \{\lambda_j\} \) and \( \{\zeta_j\} \) are eigenvalues and corresponding (orthonormal) eigenvectors of \( A - A_1 \), and \( \{\lambda_j\} \) and \( \{\zeta_j\} \) are eigenvalues and corresponding (orthonormal) eigenvectors of \( A \), we have

\[
\nabla F(\phi)(V_1(B^H, W)) + \langle (A - A_1)(B^H, W), (B^H, W) \rangle \\
= \nabla F(\phi)(V_1(B^H, W)) + \sum_j \lambda_j \langle \epsilon_j, (B^H, W) \rangle^2 \\
= \nabla F(\phi)(V_1(B^H, W)) + \hat{K}_{A - A_1}((B^H, W)) + \text{Tr}(A - A_1) \\
= \sum_j \lambda_j \langle \zeta_j, (B^H, W) \rangle^2 - 1 + \text{Tr}(A - A_1) + \nabla F(\phi^0)(\Lambda),
\]

where \( \{\zeta_j\}_{j=1,2,...} \) are ONB of Cameron-Martin space \( \mathcal{H} \), and \( \Lambda \) defined in Lemma 4.5 is of finite \( p \)-variation. Take the above proof into consideration, (4.25) is obtained.

The proof is completed. \( \square \)

5. Main proof

Now, we give the proof of our main result.

Proof of Theorem 2.3 By Assumption (A2), \( F \) := \( F \circ \Phi + \| (\cdot, \cdot) \|^2_\mathbb{R} / 2 \) attains its minimum at a unique point \( (\gamma, \eta)^T \in \mathcal{H} \). In the neighborhood \( O \subset G\Omega_\rho(\mathbb{R}^{d_1 + d_2}) \) of \( (\gamma, \eta) \), there exists a constant \( C \) such that

\[
\int_{O^c} \exp \left( - F(\hat{\Phi}_{\varepsilon}((B^H, W), \lambda)^1) / \varepsilon^2 \right) \mathbb{P}^H(d(B^H, W)) \leq C e^{-a + 5} / \varepsilon^2,
\]

for \( \varepsilon \in (0, 1] \). The above term \( \mathbb{I} \) converges to zero as \( \varepsilon \to 0 \).

According to the Lemma 3.9 set \( O = \gamma + U_{\rho} \) where \( U_{\rho} = \left\{ (B^H, W) \in G\Omega_\rho(\mathbb{R}^{d_1 + d_2}) \mid \|(B^H, W)\|_{p-var} < \rho \right\} \) for \( \rho > 0 \),

\[
\int_{\gamma + U_{\rho}} \exp \left( - F(\hat{\Phi}_{\varepsilon}((B^H, W), \lambda)^1) / \varepsilon^2 \right) \mathbb{P}^H(d(B^H, W)) \\
= \int_{U_{\rho}} \exp \left( - F(\hat{\Phi}_{\varepsilon}((B^H, W) + (\gamma, \eta)^T, \lambda)^1) / \varepsilon^2 \right) \\
\quad \times \exp \left( - \frac{1}{\varepsilon^2} \langle (\gamma, \eta)^T, (B^H, W)^1 \rangle - \frac{1}{2 \varepsilon^2} \|(\gamma, \eta)\|^2_\mathcal{H} \right) \mathbb{P}^H(d(B^H, W)) \\
= \int_{\|(B^H, W)\|_{p-var} < \rho} \exp \left( - F(\phi^{(\varepsilon)}) / \varepsilon^2 \right) - \frac{1}{\varepsilon} \langle \gamma, \eta \rangle^T(B^H, W)^1 \\
- \frac{1}{2 \varepsilon^2} \|(\gamma, \eta)\|^2_\mathcal{H} \mathbb{P}^H(d(B^H, W)).
\]

(5.2)

Take the stochastic Taylor expansion for \( \phi^{(\varepsilon)} \) in the neighbourhood of \( \phi^{(0)} \), one can get

\[
F(\phi^{(\varepsilon)}) = F(\phi^{0}) + \nabla F(\phi^{0}) \langle \phi^{(\varepsilon)} - \phi^{0} \rangle + \frac{1}{2} \nabla^2 F(\phi^{0}) \langle \phi^{(\varepsilon)} - \phi^{0}, \phi^{(\varepsilon)} - \phi^{0} \rangle \\
+ \frac{1}{6} \int_0^1 d\theta \nabla^3 F(\theta \phi^{(\varepsilon)} + (1 - \theta) \phi^{0}) \langle \phi^{(\varepsilon)} - \phi^{0}, \phi^{(\varepsilon)} - \phi^{0}, \phi^{(\varepsilon)} - \phi^{0} \rangle \\
= F(\phi^{0}) + \nabla F(\phi^{0}) \langle \varepsilon \phi^1 + \varepsilon^2 \phi^2, \phi^{(\varepsilon)} - \phi^{0} \rangle + \frac{1}{2} \nabla^2 F(\phi^{0}) \langle \varepsilon \phi^1, \phi^{(\varepsilon)} - \phi^{0} \rangle + Q^3_{\varepsilon}.
\]

(5.3)

Due to the result in Lemma 5.3, there exists some constant \( C > 0 \) such that \( |Q^3_{\varepsilon}| \leq C \varepsilon + \|(B^H, W)\|_{p-var}^3 \)

on the set \( \{ \|(B^H, W)\|_{p-var} < \rho_0 \} \).

Consider the term with order \(-2\) in (5.2), due to the Assumption (A2)

\[
- \frac{1}{\varepsilon^2} (F(\phi^{0}) + \frac{1}{2} \|(\gamma, \eta)\|^2_\mathcal{H}) = - \frac{a}{\varepsilon^2}.
\]

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For the term of order $-1$ in (5.2), we have $\phi^t : \Omega_b^2(\mathbb{R}^{d_1 + d_2}) \rightarrow C_0^{\beta-\text{var}}(\mathbb{R}^n)$, and it satisfies the following differential equation,
\begin{equation}
\begin{aligned}
d\phi_t^2 - \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \phi_0^1, d(\gamma_t, \eta_t)^T \right) - \nabla_{y^1} \beta \left( 0, \phi_t^0 \right) \langle \phi_t^1, 1 \rangle dt \\
= \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \phi_0^1, d(k_s, \hat{k}_s)^T \right) + \nabla_{\hat{t}} \beta \left( 0, \phi_t^0 \right) dt, \quad \phi_0^1 = 0
\end{aligned}
\end{equation}

(5.4)

Then, set $\theta^1(k, \hat{k}) = \phi^1(k, \hat{k}) - \chi(k, \hat{k})$, where
\begin{equation}
\chi(k, \hat{k}) = M_t \int_0^t M_{s}^{-1} \left[ \nabla \phi_s^0 \right] \nabla \phi_s^0 \left( \phi_0^1, d(k_s, \hat{k}_s)^T \right). 
\end{equation}

(5.5)

By the Assumption (A2), we conclude that
\begin{equation}
\langle (k, \hat{k}), (\gamma, \eta) \rangle_{\mathcal{H}} + \nabla F \left( \phi^0 \right) \langle \chi(k, \hat{k}) \rangle = 0,
\end{equation}

(5.6)

Then, $\theta^1(k, \hat{k})$ satisfies the following differential equation,
\begin{equation}
\begin{aligned}
d\theta_t^1 - \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \theta_0^1, d(\gamma_t, \eta_t)^T \right) - \nabla_{y^1} \beta \left( 0, \phi_t^0 \right) \langle \theta_t^1, 1 \rangle dt \\
= \nabla_{\hat{t}} \beta \left( 0, \phi_t^0 \right) dt, \quad \theta_0^1 = 0,
\end{aligned}
\end{equation}

(5.7)

its solution can be rewritten as $\theta_t^1 = M_t \int_0^t M_{s}^{-1} \nabla \beta \left( 0, \phi_s^0 \right) ds$, then, $\theta^1(B^H, W)_t$ is independent of $(B^H, W)$ and of finite $q$-variation.

Hence, with (5.6) and straightforward computation, it deduces that
\begin{equation}
-\frac{1}{\varepsilon} \left[ \langle (k, \hat{k}), (\gamma, \eta) \rangle_{\mathcal{H}} + \nabla F \left( \phi^0 \right) \langle \chi(k, \hat{k}) \rangle \right] = -\frac{\nabla F \left( \phi^0 \right) \langle \theta^1(k, \hat{k}) \rangle}{\varepsilon}.
\end{equation}

Next, we focus on the term of order 0 in (5.2), it has
\begin{equation}
\begin{aligned}
d\phi_t^2 - \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \phi_0^1, d(\gamma_t, \eta_t)^T \right) - \nabla_{y^1} \beta \left( 0, \phi_t^0 \right) \langle \phi_t^1, 1 \rangle dt \\
= \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \phi_0^1, d(k_t, \hat{k}_t)^T \right) + \frac{1}{2} \left[ \nabla^2 \phi_t^0 \right] \nabla \phi_t^0 \left( \phi_0^1, d(\gamma_t, \eta_t)^T \right)
\end{aligned}
\end{equation}

+ \frac{1}{2} \nabla_{\hat{t}} \beta \left( 0, \phi_0^0 \right) \langle \phi_0^1, \phi_0^1 \rangle dt + \nabla \beta \left( 0, \phi_0^0 \right) \langle \phi_0^1, 1 \rangle dt
\end{equation}

+ \frac{1}{2} \nabla^2 \beta \left( 0, \phi_0^0 \right) dt, \quad \phi_0^0 = 0.

(5.8)

Then, $\chi$ and $\psi$ extend to continuous maps from $\Omega_b^2(\mathbb{R}^{d_1 + d_2})$, and rewrite that $\chi(B^H, W)$ and $\psi((B^H, W), (B^H, W))$.

Consequently, we set $\theta^2(k, \hat{k}) = \phi^2(k, \hat{k}) - \psi^2((k, \hat{k}), (k, \hat{k}))/2$, where
\begin{equation}
\begin{aligned}
d\psi_t - \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \psi_t, d(\gamma_t, \eta_t)^T \right) - \nabla_{y^1} \beta \left( 0, \phi_t^0 \right) \langle \psi_t, 1 \rangle dt \\
= 2 \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \chi(k, \hat{k}), d(k_t, \hat{k}_t)^T \right)
\end{aligned}
\end{equation}

+ \frac{1}{2} \nabla^2 \beta \left( 0, \phi_0^0 \right) \langle \chi(k, \hat{k}), \chi(k, \hat{k})^T \rangle dt + \nabla^2 \beta \left( 0, \phi_0^0 \right) \langle \chi(k, \hat{k}), \chi(k, \hat{k})^T \rangle dt
\end{equation}

+ \nabla^2 \beta \left( 0, \phi_0^0 \right) dt, \quad \psi_0 = 0.

(5.9)

And $\theta^2(k, \hat{k})$ satisfies the differential equation as follows,
\begin{equation}
\begin{aligned}
d\theta_t^2 - \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \theta_0^2, d(\gamma_t, \eta_t)^T \right) - \nabla_{y^1} \beta \left( 0, \phi_t^0 \right) \langle \theta_t^2, 1 \rangle dt \\
= \left[ \nabla \phi_t^0 \right] \nabla \phi_t^0 \left( \theta_0^2, d(k_t, \hat{k}_t)^T \right) + \frac{1}{2} \left[ \nabla^2 \phi_t^0 \right] \nabla \phi_t^0 \left( \theta_0^2, \theta_0^2, d(\gamma_t, \eta_t)^T \right)
\end{aligned}
\end{equation}

+ \frac{1}{2} \nabla^2 \beta \left( 0, \phi_0^0 \right) \langle \theta_0^2, \chi_t \rangle dt + \nabla^2 \beta \left( 0, \phi_0^0 \right) \langle \theta_0^2, \chi_t \rangle dt
\end{equation}

+ \nabla \beta \left( 0, \phi_0^0 \right) \langle \theta_0^2, 1 \rangle dt + \frac{1}{2} \nabla^2 \beta \left( 0, \phi_0^0 \right) dt, \quad \theta_0^2 = 0.

(5.10)

Equivalently, its solution can be rewritten as
\begin{equation}
\theta_t^2 = M_t \int_0^t M_{s}^{-1} \left[ \nabla \phi_s^0 \right] \nabla \phi_s^0 \left( \theta_s^2, d(k_s, \hat{k}_s)^T \right) + \frac{1}{2} \left[ \nabla^2 \phi_s^0 \right] \nabla \phi_s^0 \left( \theta_s^2, \theta_s^2, d(\gamma_s, \eta_s)^T \right)
\end{equation}
\begin{equation}
+ [\nabla^2 \sigma(\phi^0) | \nabla^2 \sigma(\phi^0)] \langle \theta^1, \chi_s, d(\gamma_s, \eta_s)^T \rangle \\
+ \frac{1}{2} \nabla_y \nabla \beta (\phi^0) \langle \theta^1_s, \theta^1_s \rangle ds + \nabla_y \nabla \beta (\phi^0) \langle \theta^1_s, \chi_s \rangle ds \\
+ \nabla_y \nabla \beta (\phi^0) \langle \theta^1_s + \chi_s \rangle ds + \frac{1}{2} \nabla^2 \beta (\phi^0, \phi^0) ds].
\end{equation}

(5.11)

So, \( \theta^2((k, \hat{k}), (k, \hat{k})) \) extends a map from \( G_{\Omega_\rho}(\mathbb{R}^{d_1 + d_2}) \) to \( C_{0}^{\rho - \text{var}}(\mathbb{R}^n) \). Moreover, from Lemma 3.5 for some constant \( C > 0 \), it has \( \| \theta^2(B^H, W) \|_{p-\text{var}} \leq C(1 + \| (B^H, W) \|_{p-\text{var}}) \) with \( (B^H, W) \in G_{\Omega_\rho}(\mathbb{R}^{d_1 + d_2}) \). By the Lemma 3.2, it follows that \( \theta^2(B^H, W) \) is exponential integrable.

By Lemma 3.3, Lemma 3.2 and Theorem 4.1, it deduces that

\[
\exp \left( - \nabla F (\phi^0) \langle \phi^2 \rangle - \frac{1}{2} \nabla^2 F (\phi^0) \langle \phi^1, \phi^1 \rangle \right) \in L^r \left( G_{\Omega_\rho}(\mathbb{R}^{d_1 + d_2}), \mathbb{P}^H \right), \quad r > 1.
\]

When \( \varepsilon \leq \rho \), it has

\[
\begin{align*}
1_{\{||\varepsilon (B^H, W)||_{p-\text{var}} < \rho\}} \exp \left( - \nabla F (\phi^0) \langle \phi^2 \rangle - \frac{1}{2} \nabla^2 F (\phi^0) \langle \phi^1, \phi^1 \rangle \right) & \exp \left( -\varepsilon^{-2}Q^2_2 \right) \\
& \leq \exp \left( - \nabla F (\phi^0) \langle \phi^2 \rangle - \frac{1}{2} \nabla^2 F (\phi^0) \langle \phi^1, \phi^1 \rangle \right) \exp \left[ 2C\rho (1 + \| (B^H, W) \|_{p-\text{var}}^2) \right].
\end{align*}
\]

(5.12)

The dominated convergence theorem yields that

\[
\lim_{\varepsilon \to 0} \int_{\{||\varepsilon (B^H, W)||_{p-\text{var}} < \rho\}} \exp \left( - \nabla F (\phi^0) \langle \phi^2 \rangle - \frac{1}{2} \nabla^2 F (\phi^0) \langle \phi^1, \phi^1 \rangle - \frac{1}{\varepsilon^2} Q^2_2 \right) \mathbb{P}^H (d(B^H, W)) \\
= \int_{G_{\Omega_\rho}(\mathbb{R}^{d_1 + d_2})} \exp \left( - \nabla F (\phi^0) \langle \phi^2 \rangle - \frac{1}{2} \nabla^2 F (\phi^0) \langle \phi^1, \phi^1 \rangle \right) \mathbb{P}^H (d(B^H, W)).
\]

(5.13)

By Lemma 4.8, the right-hand side of (5.13) exists, then the coefficient \( a_0 \) can be defined in the following sense,

\[
a_0 = \exp \left[ -\frac{1}{2} (\text{Tr} (A - A_1) + \nabla F (\phi^0) \langle A \rangle) \right] \cdot \det (\text{Id}_H + A)^{-1/2},
\]

(5.14)

where \( \Lambda \) is of finite \( p \)-variation.

The proof is completed. \( \square \)

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