On the spectrum of critical almost Mathieu operators in the rational case

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Abstract

We derive a new Chambers-type formula and prove sharper upper bounds on the measure of the spectrum of critical almost Mathieu operators with rational frequencies.

1 Introduction

The Harper operator, a.k.a. the discrete magnetic Laplacian\(^1\), is a tight-binding model of an electron confined to a 2D square lattice in a uniform magnetic field orthogonal to the lattice plane and with flux \(2\pi\alpha\) through an elementary cell. It acts on \(\ell^2(\mathbb{Z}^2)\) and is usually given in the Landau gauge representation

\[
(H(\alpha)\psi)_{m,n} = \psi_{m,n-1} + \psi_{m,n+1} + e^{-i2\pi\alpha n}\psi_{m-1,n} + e^{i2\pi\alpha n}\psi_{m+1,n},
\]

first considered by Peierls \[18\], who noticed that it makes the Hamiltonian separable and turns it into the direct integral in \(\theta\) of operators on \(\ell^2(\mathbb{Z})\) given by:

\[
(H_{\alpha,\theta}\varphi)(n) = \varphi(n-1) + \varphi(n+1) + 2\cos 2\pi(\alpha n + \theta)\varphi(n), \quad \alpha, \theta \in [0,1). \quad (2)
\]

In physics literature, it also appears under the names Harper’s or the Azbel-Hofstadter model, with both names used also for the discrete magnetic Laplacian \(H(\alpha)\). In mathematics, it is universally called the critical almost Mathieu operator.\(^2\) In addition to importance in physics, this model is of special interest, being at the boundary of two reasonably well understood regimes: (almost) localization and (almost) reducibility, and not being amenable to methods of either side. Recently, there has been some progress in the study of the fine structure of its spectrum \[7, 8, 10, 14, 16\].

Denote the spectrum of an operator \(H\), as a set, by \(\sigma(H)\). An important object is the union of \(\sigma(H_{\alpha,\theta})\) over \(\theta\), which coincides with the spectrum of \(H(\alpha)\). We denote it \(S(\alpha) := \sigma(H(\alpha)) = \cup_{\theta \in [0,1]} \sigma(H_{\alpha,\theta})\). Note that by the general theory of ergodic operators, if \(\alpha\) is irrational, \(\sigma(H_{\alpha,\theta})\) is independent of \(\theta\). We denote the Lebesgue measure of a set \(A\) by \(|A|\).

For irrational \(\alpha\), the Lebesgue measure \(|S(\alpha)| = 0\), and \(S(\alpha)\) is a set of Hausdorff dimension no greater than 1/2 \[15, 2, 9\]. The proof of the Hausdorff dimension result in \[9\]

\(^1\)The name “discrete magnetic Laplacian” was first introduced by M. Shubin in \[19\].

\(^2\)This name was originally introduced by Barry Simon \[20\].
(which was a conjecture of D. J. Thouless) is based on upper bounds of the measure of the spectrum for \( \alpha \in \mathbb{Q} \) and a strong continuity. For rational \( \alpha = \frac{p_0}{q_0} \), where \( p_0, q_0 \) are coprime positive integers, last obtained the bounds [15, Lemma 1]:

\[
\frac{2(\sqrt{5} + 1)}{q_0} < |S\left(\frac{p_0}{q_0}\right)| < \frac{8\pi}{q_0},
\]

where \( e = \exp(1) = 2.71 \ldots \). While the upper bound in (3) was sufficient for the argument of [9], the measure of the spectrum is subject to another conjecture of Thouless [21, 22]: that in the limit \( p_n/q_n \rightarrow \alpha \), we have \( q_n|S(p_n/q_n)| \rightarrow c \), where \( c = 32C_c/\pi = 9.32 \ldots \), 
\( C_c = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2} \) being the Catalan constant. Thouless provided a partly heuristic argument in the case \( p_n = 1, q_n \rightarrow \infty \). A rigorous proof for \( \alpha = 0 \) and \( p_n = 1 \) or \( p_n = 2, q_n \) odd, was given in [6].

The purpose of this note is to present a sharper upper bound, for all \( \alpha \in \mathbb{Q} \):\n
**Theorem 1.** For all positive coprime integers \( p_0 \) and \( q_0 \),

\[
|S\left(\frac{p_0}{q_0}\right)| \leq \frac{4\pi}{q_0}.
\]

Thus, the upper bound is reduced from \( 8e = 21.74 \ldots \) to \( 4\pi = 12.56 \ldots \). The way we prove Theorem 1 is very different from that of [15]; we use the chiral gauge representation [9] and Lidskii’s inequalities. The chiral gauge representation of the almost Mathieu operator also leads to a new type of Chambers’ relation (equations (14), (15) below).

## 2 Proof of Theorem 1

Consider the following operator on \( \ell^2(\mathbb{Z}) \):

\[
(H_{\alpha,\theta}\varphi)(n) = 2\sin 2\pi(\alpha(n-1)+\theta)\varphi(n-1) + 2\sin 2\pi(\alpha n+\theta)\varphi(n+1), \quad \alpha, \theta \in [0, 1),
\]

and define \( \tilde{S}(\alpha) := \cup_{\theta \in [0, 1]} \sigma(H_{\alpha,\theta}) \). It was shown in [9, Theorem 3.1] that the operators 
\( M_{2\alpha} := \oplus_{\theta \in [0, 1]} H_{2\alpha,\theta} \) and \( \tilde{M}_{\alpha} := \oplus_{\theta \in [0, 1]} \tilde{H}_{\alpha,\theta} \) are unitarily equivalent, so that \( S(\alpha) = \tilde{S}(\alpha/2) \).

(Note that \( \sigma(H_{2\alpha,\theta}) \neq \sigma(\tilde{H}_{\alpha,\theta}) \), in general.) See also related partly non-rigorous considerations in [17, 11, 23, 12, 13], and an application of the rational case in [14]. Operator (4) corresponds to the chiral gauge representation of the Harper operator.

From now on, we always consider the case of rational \( \alpha \). Furthermore, the analysis below for \( q_0 = 1, q_0 = 2 \) becomes especially elementary, and gives \( |S(1)| = 8 \), \( |S(1/2)| = 4\sqrt{2} \), so that Theorem 1 obviously holds in these cases. From now on, we assume \( q_0 \geq 3 \).

If \( p_0 \) is even, define \( p := \frac{p_0}{2} \) and \( q := q_0 \) (note that \( q \) is necessarily odd in this case). This corresponds to case I below. If \( p_0 \) is odd, define \( p := p_0 \) and \( q := 2q_0 \). This corresponds to case II below. We note that in either case \( p \) and \( q \) are coprime and \( S(p_0/q_0) = \tilde{S}(p/q) \).

Let \( b(x) := 2\sin(2\pi x) \), and further identify \( b_n(\theta) := b((p/q)n+\theta) \). For the operator \( \tilde{H}_{\frac{q}{p},\theta} \), Floquet theory states that \( E \in \sigma(\tilde{H}_{\frac{q}{p},\theta}) \) if and only if the equation \( (\tilde{H}_{\frac{q}{p},\theta}\varphi)(n) = E\varphi(n) \) has
a solution \( \{ \varphi(n) \}_{n \in \mathbb{Z}} \) satisfying \( \varphi(n + q) = e^{ikq}\varphi(n) \) for all \( n \), and for some real \( k \). Therefore, for a fixed \( k \), there exist \( q \) values of \( E \) satisfying the eigenvalue equation

\[
B_{\theta,k,\ell} \begin{pmatrix} \varphi(\ell) \\ \vdots \\ \varphi(\ell + q - 1) \end{pmatrix} = E \begin{pmatrix} \varphi(\ell) \\ \vdots \\ \varphi(\ell + q - 1) \end{pmatrix}
\]

(5)

for any \( \ell \), where

\[
B_{\theta,k,\ell} := \begin{pmatrix} 0 & b_\ell & 0 & 0 & \cdots & 0 & 0 & e^{-ikq}b_{\ell+q-1} \\ b_\ell & 0 & b_{\ell+1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_{\ell+1} & 0 & b_{\ell+2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{\ell+q-3} & 0 & b_{\ell+q-2} \\ e^{ikq}b_{\ell+q-1} & 0 & 0 & 0 & \cdots & 0 & b_{\ell+q-2} & 0 \end{pmatrix}
\]

(6)

Thus, the eigenvalues of \( B_{\theta,k,\ell} \) are independent of \( \ell \).

### 2.1 Chambers-type formula

The celebrated Chambers’ formula presents the dependence of the determinant of the almost Mathieu operator with \( \alpha = p_0/q_0 \) restricted to the period \( q_0 \) with Floquet boundary conditions, on the phase \( \theta \) and quasimomentum \( k \). In the critical case it is given by (see, e.g., [15])

\[
\det(A_{\theta,k,\ell} - E) = \Delta(E) - 2(-1)^{q_0}(\cos(2\pi q_0 \theta) + \cos(kq_0)),
\]

(7)

where

\[
A_{\theta,k,\ell} := \begin{pmatrix} a_\ell & 1 & 0 & 0 & \cdots & 0 & 0 & e^{-ikq} \\ 1 & a_{\ell+1} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & a_{\ell+2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{\ell+q-2} & 1 \\ e^{ikq} & 0 & 0 & 0 & \cdots & 0 & 1 & a_{\ell+q-1} \end{pmatrix}, \quad \ell \in \mathbb{Z},
\]

(8)

\[
a(x) := 2\cos(2\pi x), \quad a_\theta(\theta) := a((p_0/q_0)n + \theta),
\]

(9)

and \( \Delta \), the discriminant\(^3\), is independent of \( \theta \) and \( k \). An immediate corollary of this formula is that \( S\left(\frac{p_0}{q_0}\right) = \Delta^{-1}([-4, 4]) \), e.g., [15].

Here we obtain a formula of this type for \( \det(B_{\theta,k,\ell} - E) \). Indeed, as usual, separating the terms containing \( k \) in the determinant, we obtain, for the characteristic polynomial \( D_{\theta,k}(E) := \det(B_{\theta,k,\ell} - E) \):

\[
D_{\theta,k}(E) = D_g^{(0)}(E) - (-1)^q b_0 \cdots b_{q-1} \cdot 2\cos(kq),
\]

(10)

where \( D_g^{(0)}(E) \) is independent of \( k \) and equal therefore to \( D_{\theta,k=\frac{p_0}{q_0}}(E) \).

For the product of \( b_j \)'s we have:

\(^3\)In [15], the discriminant differs from \( \Delta(E) \) by the factor \((-1)^{q_0}\).
Lemma 1.

\[ b_0 \cdots b_{q-1} = \prod_{j=0}^{q-1} 2 \sin 2\pi \left( \frac{p}{q} j + \theta \right) \]
\[ = 4 \sin(\pi q \theta) \sin \pi q(\theta + 1/2) = 2(\cos(\pi q/2) - \cos \pi q(2\theta + 1/2)). \] (11)

Proof. To evaluate the product of \( b_j \)'s, we expand sine in terms of exponentials and use the formula\[ 1 - z^{-q} = \prod_{j=0}^{q-1} (1 - z^{-1} e^{2\pi i q j}). \] An alternative derivation can go along the lines of the proof of Lemma 9.6 in [1].

Substituting (11) into (10), we have

\[ D_{\theta,k}(E) = D_{\theta,0}^{(0)}(E) - 8(-1)^q \sin(\pi q \theta) \sin \pi q(\theta + 1/2) \cos(kq). \] (12)

We can further obtain the dependence of \( D_{\theta,0}^{(0)}(E) \) on \( \theta \):

Lemma 2.

\[ D_{\theta,0}^{(0)}(E) = \widetilde{\Delta}(E) + \begin{cases} 0, & q \text{ odd} \\ 4(\cos(2\pi q \theta) - 1), & q \text{ even}, \end{cases} \]

where the discriminant \( \widetilde{\Delta}(E) := D_{\theta,0}^{(0)}(E) \) is independent of \( \theta \).

Proof. Since \( D_{\theta,k}(E) \) is independent of \( \ell \), it is \( 1/q \) periodic in \( \theta \), i.e., \( D_{\theta,k}(E) = D_{\theta+1/q,k}(E) \), and by (10) so is \( D_{\theta,0}^{(0)}(E) \). Therefore, since, clearly, \( D_{\theta,0}^{(0)}(E) = \sum_{n=-q}^{q} c_n(E)e^{2\pi i q n} \), the terms \( c_k \) other than \( k = mq \) vanish, and \( D_{\theta,0}^{(0)}(E) \) has the following Fourier expansion:

\[ D_{\theta,0}^{(0)}(E) = c_0(E) + c_q e^{2\pi i q \theta} + c_{-q} e^{-2\pi i q \theta}. \]

It is easily seen that the \( c_q \) and \( c_{-q} \) can be obtained from the expansion of the determinant and that, moreover, they do not depend on \( E \). Expanding \( D_{\theta,0}^{(0)}(E) \) with \( E = 0 \) in rows and columns (cf. [14]), we obtain

\[ D_{\theta,0}^{(0)}(0) = D_{\theta,k=\frac{q-2}{2}}(0) = \begin{cases} 0, & q \text{ odd} \\ (-1)^{q/2}(b_0^2 b_2^2 \cdots b_{q-2}^2 + b_1^2 b_3^2 \cdots b_{q-1}^2), & q \text{ even}, \end{cases} \] (13)

This gives \( c_q = c_{-q} = 0 \) for \( q \) odd, and \( c_q = \prod_{j=0}^{q-2} e^{8\pi i j} + \prod_{j=0}^{q-2} e^{4\pi i j(2j+1)} = 2 = c_{-q}, \) for \( q \) even. It remains to denote \( \widetilde{\Delta}(E) = c_0(E) \) for \( q \) odd, and \( \widetilde{\Delta}(E) = c_0(E) + 4 \) for \( q \) even, and the proof is complete.

We therefore have, by (12) and Lemma 2:

Lemma 3 (Chambers-type formula).

\[ D_{\theta,k}(E) = \widetilde{\Delta}(E) + 4(-1)^{(q-1)/2} \sin(2\pi q \theta) \cos(kq), \quad q \text{ odd.} \] (14)
\[ D_{\theta,k}(E) = \widetilde{\Delta}(E) - 4(1 - \cos(2\pi q \theta))(1 + (-1)^{(q-1)/2} \cos(kq)), \quad q \text{ even.} \] (15)
Note that $\tilde{\Delta}(E)$ is a polynomial of degree $q$ independent of $k \in \mathbb{R}$ and $\theta \in [0, 1)$. By Floquet theory, the spectrum $\sigma(\tilde{H}_{q, \theta})$ is the union of the eigenvalues of $B_{\theta, k, \ell}$ over $k$, a collection of $q$ intervals.

We make the following observations.

Case I: $q$ is odd.

By (14), $D_{\theta, k}(E) \equiv \det(B_{\theta, k, \ell} - E) = 0$ if and only if $\tilde{\Delta}(E) = 4(-1)^{(q+1)/2} \sin(2\pi q \theta) \cos(k q)$. Thus, $\sigma(\tilde{H}_{q, \theta})$ is the preimage of $[-4| \sin(2\pi q \theta)|, 4| \sin(2\pi q \theta)|]$ under the mapping $\tilde{\Delta}(E)$. If $\theta = m/(2q)$, $m \in \mathbb{Z}$, $\sigma(\tilde{H}_{q, \theta})$ is a collection of $q$ points where $\tilde{\Delta}(E) = 0$. (In this case, $b_0(m/(2q)) = 0$, so that $\tilde{H}$ splits into the direct sum of an infinite number of copies of a $q$-dimensional matrix.)

We note that the spectra $\sigma(\tilde{H}_{q, \theta})$ for different $\theta$ are nested in one another as $\theta$ grows from $0$ to $1/(4q)$; in particular, for each $\theta \in [0, 1)$,

$$\sigma(\tilde{H}_{q, \theta}) = \tilde{\Delta}^{-1}([-4| \sin(2\pi q \theta)|, 4| \sin(2\pi q \theta)|]) \subseteq \sigma(\tilde{H}_{q, \theta, \psi_{q/4}}) = \tilde{\Delta}^{-1}([-4, 4]).$$

This implies that all the maxima of $\tilde{\Delta}(E)$ are no less than $4$, and all the minima are no greater than $-4$. Moreover, taking the union over all $\theta \in [0, 1)$ gives:

$$\tilde{S}\left(\frac{p}{q}\right) = \sigma(\tilde{H}_{q, \theta, \psi_{q/4}}) = \tilde{\Delta}^{-1}([-4, 4]).$$

Clearly, it is sufficient to consider only $\theta \in [0, 1/(4q)]$.

Case II: $q$ is even. This case is similar to case I, so we omit some details for brevity. By (15), $D_{\theta, k}(E) = 0$ if and only if $\tilde{\Delta}(E) = 4(1 - \cos(2\pi q \theta))(1 + (-1)^{(q)/2} \cos(k q))$. Considering the cases $k = 0, \frac{\pi}{q}$, it is easy to see that $\sigma(\tilde{H}_{q, \theta})$ is the preimage of $[0, 8 - 8 \cos(2\pi q \theta)]$ under the mapping $\tilde{\Delta}(E)$. If $\theta = m/q$, $m \in \mathbb{Z}$, $\sigma(\tilde{H}_{q, \theta})$ is a collection of $q$ points where $\tilde{\Delta}(E) = 0$.

We note that the spectra $\sigma(\tilde{H}_{q, \theta})$ for different $\theta$ are nested in one another as $\theta$ grows from $0$ to $1/(2q)$; in particular, for each $\theta \in [0, 1)$,

$$\sigma(\tilde{H}_{q, \theta}) = \tilde{\Delta}^{-1}([0, 8 - 8 \cos(2\pi q \theta)]) \subseteq \sigma(\tilde{H}_{q, \theta, \psi_{q/4}}) = \tilde{\Delta}^{-1}([0, 16]).$$

This implies that all the maxima of $\tilde{\Delta}(E)$ are no less than $16$, and all the minima are no greater than $0$. Moreover, taking the union over all $\theta \in [0, 1)$ gives:

$$\tilde{S}\left(\frac{p}{q}\right) = \sigma(\tilde{H}_{q, \theta, \psi_{q/4}}) = \tilde{\Delta}^{-1}([0, 16]).$$

Clearly, it is sufficient to consider only $\theta \in [0, 1/(2q)]$.

In this case of even $q$ we can say more about the form of $\tilde{\Delta}(E)$. Note that $b_0(0) = b_{q/2}(0) = 0$ and $b_k(0) = b_{-k}(0)$. Recall that by Floquet theory, $D_{\theta, k}(E) = \det(B_{\theta, k, \ell} - E)$ is independent of the choice of $\ell$. For convenience, choose $\ell = -q/2 + 1$. It is easily seen that $B_{\theta=0,k,\ell=-q/2+1}$ decomposes into a direct sum, and moreover $\tilde{\Delta}(E) = D_{\theta=0,k}(E) = (-1)^{q/2} P_{q/2}(-E) P_{q/2}(E)$, where $P_{q/2}(E)$ is a polynomial of degree $q/2$, odd if $q/2$ is odd, and even if $q/2$ is even (as
it is a characteristic polynomial of a tridiagonal matrix with zero main diagonal). Thus \( \Delta(E) = P_{q/2}(E)^2 \) is a square.

The discriminants \( \tilde{\Delta}(E) \equiv \Delta_{p/q}(E) \) and \( \Delta(E) \equiv \Delta_{p_0/q_0}(E) \) are related in the following way:

**Lemma 4.** For \( q \) odd,

\[
\tilde{\Delta}_{p/q}(E) = \Delta_{p_0/q_0}(E), \quad p_0 = 2p, \quad q_0 = q.
\]  

For \( q \) even,

\[
\tilde{\Delta}_{p/q}(E) = \Delta^2_{p_0/q_0}(E), \quad p_0 = p, \quad q_0 = q/2.
\]

**Proof.** Case I: \( q \) is odd. Here, by our definitions at the start of the section, \( p_0 = 2p \) and \( q_0 = q \). \( \Delta_{p/q}(E) \) and \( \Delta_{p_0/q_0}(E) \) are polynomials in \( E \) of degree \( q \) with the same coefficient \(-1\) of \( E^q \). Since \( \tilde{\Delta}(E) = \Delta(E) = \pm 4 \) at the \( 2q \geq q + 1 \) distinct edges of the bands (cf. [5, 3.3]), these polynomials coincide: \( \tilde{\Delta}(E) = \Delta(E) \) for each \( E \).

Case II: \( q \) is even. Here, \( p_0 = p \) and \( q_0 = q/2 \). \( \tilde{S}\left(\frac{q}{q}\right) = S\left(\frac{p_0}{q_0}\right) \) is the preimage of \([0, 16]\) under \( \tilde{\Delta}_{p/q} \) and of \([-4, 4]\) under \( \Delta_{p_0/q_0} \), hence also of \([0, 16]\) under \( \Delta^2_{p_0/q_0} \). On the other hand, we have seen above that \( \tilde{\Delta}(E) = P_{q/2}^2(E) \) for some polynomial \( P_{q/2}(E) \) of degree \( q/2 = q_0 \). Thus, \( P_{q/2}^2(E) \) and \( \Delta^2(E) \) coincide at the \( 2q_0 \geq q_0 + 1 \) (for \( q_0 \) odd) and \( 2q_0 - 1 \geq q_0 + 1 \) (for \( q_0 \) even) distinct edges of the bands (cf. [5, 3.3]; the central bands merge for \( q_0 \) even), so these polynomials of degree \( q \) are equal: \( \tilde{\Delta}(E) = \Delta^2(E) \) for each \( E \).

### 2.2 Measure of the spectrum

The rest of the proof follows the argument of [3], namely it uses Lidskii’s inequalities to bound \( |\tilde{S}(E)| \). The key observation is that choosing \( \ell \) appropriately, we can make the corner elements of the matrix \( B_{q,k,\ell} \) very small, of order \( 1/q \) when \( q \) is large. This is not possible to do in the standard representation for the almost Mathieu operator. Here are the details.

Case I: \( q \) is odd. Assume without loss of generality that \((-1)^{(q+1)/2} > 0, \; \theta \in (0, 1/(4q)]\). (If \((-1)^{(q+1)/2} < 0, \) the analysis is similar.) Then the eigenvalues \( \{\lambda_i(\theta)\}_{i=1}^q \) of \( B_{q,k=0,\ell} \) labelled in decreasing order are the edges of the spectral bands where \( \Delta(E) \) reaches its maximum \( 4 \sin(2\pi q \theta) \) on the band; and the eigenvalues \( \{\tilde{\lambda}_i(\theta)\}_{i=1}^q \) of \( B_{q,k=\pi/q,\ell} \) labelled in decreasing order are the edges of the spectral bands where \( \tilde{\Delta}(E) \) reaches its minimum \( -4 \sin(2\pi q \theta) \) on the band. Then

\[
|\sigma(\tilde{H}_{q,\ell})| = \sum_{j=1}^q (-1)^{q-j} (\tilde{\lambda}_j(\theta) - \lambda_j(\theta)) = \sum_{j=1}^{(q+1)/2} (\hat{\lambda}_{2j-1}(\theta) - \lambda_{2j-1}(\theta)) + \sum_{j=1}^{(q-1)/2} (\lambda_{2j}(\theta) - \hat{\lambda}_{2j}(\theta));
\]

\[
\tilde{\lambda}_j(\theta) - \lambda_j(\theta) > 0, \quad \text{if } j \text{ is odd}; \quad \hat{\lambda}_j(\theta) - \lambda_j(\theta) < 0, \quad \text{if } j \text{ is even}.
\]  

(22)
Now we view $B_{\theta,k=\pi/q,\ell}$ as $B_{\theta,k=0,\ell}$ with the added perturbation

$$B_{\theta,k=\pi/q,\ell} - B_{\theta,k=0,\ell} = \begin{pmatrix} -2b_{\ell+q-1} \\ -2b_{\ell+q-1} \end{pmatrix},$$

which has the eigenvalues $\{E_i(\theta)\}_{i=1}^q$ given by:

$$E_q(\theta) = -2|b_{\ell+q-1}| < 0 = E_{q-1}(\theta) = \cdots = E_2(\theta) = 0 < 2|b_{\ell+q-1}| = E_1(\theta).$$

The Lidskii inequalities (e.g., [4]) are:

**Theorem 2.** For any $q \times q$ self-adjoint matrix $M$, we denote its eigenvalues by $E_1(M) \geq E_2(M) \geq \cdots \geq E_q(M)$. For $q \times q$ self-adjoint matrices $A$ and $B$, we have:

$$E_{i_1}(A+B) + \cdots + E_{i_m}(A+B) \leq E_{i_1}(A) + \cdots + E_{i_m}(A) + E_1(B) + \cdots + E_m(B);$$

$$E_{i_1}(A+B) + \cdots + E_{i_m}(A+B) \geq E_{i_1}(A) + \cdots + E_{i_m}(A) + E_{q-m+1}(B) + \cdots + E_q(B),$$

for any $1 \leq i_1 < \cdots < i_m \leq q$.

Applying these inequalities with $A = B_{\theta,k=0,\ell}$, $B = B_{\theta,k=\pi/q,\ell} - B_{\theta,k=0,\ell}$ gives:

$$\sum_{j=1}^{(q+1)/2} (\hat{\lambda}_{2j-1}(\theta) - \lambda_{2j-1}(\theta)) \leq \sum_{j=1}^{(q+1)/2} E_j(\theta) = E_1(\theta);$$

$$\sum_{j=1}^{(q-1)/2} (\lambda_{2j}(\theta) - \hat{\lambda}_{2j}(\theta)) \leq - \sum_{j=(q-1)/2}^{q} E_j(\theta) = -E_q(\theta).$$

Substituting these into (22), we obtain:

$$|\sigma(\tilde{H}_{\theta,\ell})| \leq E_1(\theta) - E_q(\theta) = 4|b_{\ell+q-1}|. \quad (23)$$

Moreover, by the invariance of $D_{\theta,k}(E)$ under the mapping $b_n \mapsto b_{n+m}$, for $n = 0, 1, \ldots, q-1$ and any $m$, we can choose any $\ell$ in (23), so that

$$|\sigma(\tilde{H}_{\theta,\ell})| \leq 4 \min_{\ell} |b_{\ell+q-1}|. \quad (24)$$

In particular,

$$\left| \tilde{S}\left(\frac{p}{q}\right) \right| = |\sigma(\tilde{H}_{\theta,\ell})| \leq 4 \min_{\ell} \left| b_{\ell+q-1} \left( \frac{1}{4q} \right) \right| = 4 \cdot 2 \left| \sin 2\pi \left( \frac{1}{4q} \right) \right| \leq \frac{4\pi}{q}. \quad (25)$$

Therefore, $|S(\frac{p}{q})| = |\tilde{S}\left(\frac{p}{q}\right)| \leq \frac{4\pi}{q} = \frac{4\pi}{q_0}$, as required.

Case II: $q$ is even. This case is similar to case I, so we omit some details for brevity. This time, the Lidskii equations of Theorem 2 show that $|\tilde{S}(\frac{p}{q})| \leq \frac{8\pi}{q}$. Indeed, as in (24), we have (note the doubling of the eigenvalues for $\tilde{\Delta}(E) = 0$)

$$|\sigma(\tilde{H}_{\theta,\ell})| \leq 4 \min_{\ell} |b_{\ell+q-1}|. \quad (26)$$
In particular,
\[
\left| \tilde{S}(\frac{p}{q}) \right| = |\sigma(\tilde{H}_{\frac{p}{q}, \theta = 1/2q})| \leq 4 \min_{\ell} \left| b_{\ell+q-1}\left(\frac{1}{2q}\right) \right| = 4 \cdot 2 \left| \sin(\pi \frac{1}{2q}) \right| \leq \frac{8\pi}{q}. \tag{27}
\]

Therefore, \( |S(\frac{p_0}{q_0})| = |\tilde{S}(\frac{p_0}{q_0})| \leq \frac{8\pi}{q_0} = \frac{4\pi}{q_0} \), as required. This completes the proof of Theorem 1.

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References

[1] A. Avila and S. Jitomirskaya. The Ten Martini Problem. Ann. of Math. 170, 303–342 (2009).
[2] A. Avila and R. Krikorian. Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles. Ann. of Math. 164, 911–940 (2006).
[3] S. Becker, R. Han and S. Jitomirskaya. Cantor spectrum of graphene in magnetic fields. Invent. Math. 218.3, 979–1041 (2019).
[4] R. Bhatia. Perturbation bounds for matrix eigenvalues. Pitman Research Notes in Mathematics Series 162, Essex: Longman 1987.
[5] M.-D. Choi, G.A. Elliott and N. Yui. Gauss polynomials and the rotation algebra. Invent. Math. 99.2, 225–246 (1990).
[6] B. Helffer, P. Kerdelhue, On the total bandwidth for the rational Harper’s equation. Comm. Math. Phys. 173, 2335–356 (1995).
[7] B. Helffer, Q. Liu, Y. Qu and Q. Zhou. Positive Hausdorff Dimensional Spectrum for the Critical Almost Mathieu Operator. Commun. Math. Phys., to appear.
[8] S. Jitomirskaya. On point spectrum at critical coupling. Adv. Math., to appear.
[9] S. Jitomirskaya and I. Krasovsky. Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum. Preprint (2019). arXiv:1909.04429.
[10] S. Jitomirskaya and S. Zhang. Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators, arXiv:1510.07086 (2015).
[11] M. Kohmoto, Y. Hatsugai. Peierls stabilization of magnetic-flux states of two-dimensional lattice electrons. Phys. Rev. B 41, 9527–9529 (1990).

[12] I. V. Krasovsky. Bethe ansatz for the Harper equation: solution for a small commensurability parameter. Phys. Rev. B 59, 322–328 (1999).

[13] I. V. Krasovsky. On the discriminant of Harper’s equation. Lett. Math. Phys. 52, 155–163 (2000).

[14] I. Krasovsky. Central spectral gaps of the almost Mathieu operator. Commun. Math. Phys. 351, 419–439 (2017).

[15] Y. Last. Zero measure spectrum for the almost Mathieu operator. Commun. Math. Phys. 164, 421–432 (1994).

[16] Y. Last and M. Shamis. Zero Hausdorff dimension spectrum for the almost Mathieu operator. Commun. Math. Phys. 348, 729–750 (2016).

[17] V. A. Mandelshtam, S. Ya. Zhitomirskaya. 1D-quasiperiodic operators. Latent symmetries. Commun. Math. Phys. 139, 589–604 (1991).

[18] R. Peierls. Zur Theorie des Diamagnetismus von Leitungselektronen. Zeitschrift für Physik A: Hadrons and Nuclei 80, 763–791 (1933).

[19] M.A. Shubin. Discrete Magnetic Laplacian, Commun. Math. Phys. 164, 259–275 (1994).

[20] B. Simon. Almost periodic Schrödinger operators: a review. Adv. Appl. Math. 3, 463–490 (1982).

[21] D.J. Thouless. Bandwidths for a quasiperiodic tight-binding model. Phys. Rev. B 28, 4272–4276 (1983).

[22] D.J. Thouless. Scaling for the discrete Mathieu equation. Commun. Math. Phys. 127, 187–193 (1990).

[23] P. B. Wiegmann and A. V. Zabrodin. Quantum group and magnetic translations Bethe ansatz for the Azbel-Hofstadter problem. Nucl. Phys. B 422, 495–514 (1994).