A Discrete-Time Clark-Ocone Formula and its Application to an Error Analysis

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Abstract
In this paper, we will establish a discrete-time version of Clark-Ocone-Haussmann formula, which can be seen as an asymptotic expansion in a weak sense. The formula is applied to the estimation of the error caused by the martingale representation. In the way, we use another distribution theory with respect to Gaussian rather than Lebesgue measure, which can be seen as a discrete Malliavin calculus.

1 Introduction
Let $T > 0$, $(W_t)_{0 \leq t \leq T}$ be a Brownian motion starting from 0, and $(\mathcal{F}_t)_{0 \leq t \leq T}$ be its natural filtration. Let $X \in L^2(\mathcal{F}_T)$ be differentiable in the sense of Malliavin, for which we may write $X \in D_{2,1}$ (see e.g. [8]). Then, it holds that

$$X = E[X] + \int_0^T E[D_sX|\mathcal{F}_s]dW_s, \quad (1.1)$$

where $D_s$ means the Malliavin derivative (evaluated at $s$).
The formula (1.1) is known as Clark-Ocone formula though there are many variants; Clark [3] obtained (1.1) for some Fréchet differentiable functionals and Ocone [11] related it to Malliavin derivatives, while Haussmann [7] extended it to functionals of a solution to a stochastic differential equation. There are yet much more contexts, which we omit here.

In the context of mathematical finance, the formula gives an alternative description of the hedging portfolio in terms of Malliavin derivatives. However, explicit expressions of the Malliavin derivatives of a Wiener functional are not available in general (except for some special cases: see [14]). In the paper we will introduce a finite dimensional approximation of (1.1) and discuss the “order of the convergence” in a finance-oriented model.

Let us be more precise. Put \( \Delta W_k = W_k \Delta t - W_{(k-1)\Delta t} \) for \( k \in \mathbb{N} \), where \( \Delta t \) is a fixed constant. Then, for fixed \( n \), the random variable \( (\Delta W_1, \cdots, \Delta W_n) \) is distributed as \( \mathcal{N}(0, \Delta t I) \). Let \( G_k, k = 1, \cdots, N \), be the sigma-algebra generated by \( (\Delta W_1, \cdots, \Delta W_k) \). Note that \( \mathcal{G} := \{G_k\}_{k=0}^\infty \) is a filtration, and

\[
L^2(\mathcal{G}_N, P) \simeq L^2(\mathbb{R}^N, \mu^N),
\]

where

\[
\mu^N(dx) = \frac{1}{(2\pi \Delta t)^{N/2}} e^{-|x|^2/2\Delta t} dx.
\]

With the filtration \( \mathcal{G} \), we can discuss “stochastic integral” (which is in fact a Riemannian sum) with respect to the process (random walk) \( W^{\Delta t} = \sum \Delta W \). On the other hand, we can naturally define (a precise formulation will be given in section 2.1) a finite dimensional version of the Malliavin derivative \( D_s \) by the weak partial derivatives such as

\[
\partial_l X(x_1, \cdots, x_N)|_{x_k = \Delta W_k, k=1, \cdots, N}.
\]

Then one might well guess that a discrete version of the Clark-Ocone formula could be

\[
X \overset{?}{=} \mathbb{E}[X] + \sum_{l=1}^N \mathbb{E}[\partial_l X|\mathcal{G}_{l-1}] \Delta W_l
\]

but this is not true since the random walk \( W^{\Delta t} \) does not have the

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1Actually, this kind of finite-dimensional approximation or something similar is commonly used in financial practice. Hence the results presented in this paper might be more insightful and useful for the practitioners in the field.
martingale representation property\(^2\).

We should instead ask how much the (martingale representation) error,

\[
\text{Mart.Err} := X - E[X] - \sum_{l=1}^{N} E[\partial_l X | \mathcal{G}_{t-1}] \Delta W_l,
\]

measured by a norm, (in this paper we concentrate on the estimation with respect to \(L^2(\mathbb{R}^N)\)-one), is. Further, its asymptotic behaviour as \(N \to \infty\) with \(N \Delta t = \text{time horizon} T\). This is closely related to the problem of so-called tracking error of the delta hedge. If one has a nice finite dimensional approximation \(X^N\) of a Wiener functional \(X\), both defined on the same probability space, then the tracking error can be controlled by the (supremum in \(N\) of) Mart.Err plus the error caused by the discretization (finite-dimensional approximation) as we see from:

\[
\begin{align*}
\text{Tra.Err} &:= X - E[X] - \sum_{l=1}^{N} E[\partial_l X | \mathcal{F}_{t\Delta t}] \Delta W_l \\
&= X - X^N + E[X - X^N] \\
&\quad - \sum_{l=1}^{N} \left( E[D_1 X | \mathcal{F}_{(l\Delta t)}] - E[\partial_l X^N | \mathcal{G}_{t-1}] \right) \Delta W_l + \text{Mart.Err} \\
&=: \text{Disc.Err} + \text{Mart.Err}.
\end{align*}
\]

There are considerably many studies on the subject of the tracking error as well. It at least dates back to the paper by Rootzen \(^3\), where the weak convergence of the scaled error was studied. The problem is reformulated as “tracking error of the delta hedge” in Bertsimas, Kogan, and Lo \(^2\), where the error was also measured by \(L^2\)-norm. Hayashi and Mykland \(^1\) further developed the argument from financial perspectives.

Notable results in this topic are summarized as follows.

1. The scaled tracking error \(N^{-1/2}\text{Tra.Err}\) converges weakly to \(B_\tau\) with\(^3\)

\[
\tau = \frac{1}{2} \int |E[D^2_s X | \mathcal{F}_s]|^2 ds,
\]

where \(B\) is a Brownian motion independent of \(\tau\).

\(^2\) If the martingale representation property holds for a random walk, then we can establish a precise discrete-time Clark-Ocone formula if we define “differentiation” properly. For the binary case, N. Privault \(^2\) has made a detailed study on the discrete Clark-Ocone formula and related discrete Malliavin calculus.

\(^3\) Here actually the differentiability is not required. The expression \(E[D_s X | \mathcal{F}_s]\) should be understood as simply the integrand of the martingale representation of \(X\) and the meaning of \(E[D^2_s X | \mathcal{F}_s]\) will be clarified later.
2. The tracking error estimated with $L^2$-norm is in $O(N^{-1/2})$ in the cases of $X = F(S)$ with “ordinary pay-off” $F$ and the solution $S$ of an SDE, while it is in $O(N^{-1/4})$ when $F$ is “irregular” like Heaviside function (Gobet and Temam [5], Temam [17]). Later the irregularity is associated with differentiability in the fractional order $s \in (0, 1)$ by Geiss and Geiss [4]; it is in $O(N^{-s/2})$ for $s$-differentiable $F$.

In this paper, we shall establish the corresponding results for the Mart.Err, which almost parallel with the above.

After introducing the Discrete Clark-Ocone formula (Theorem 2.1, section 2.2), we will show, by using the formula, a multi-level central limit theorem for the error (Theorem 3.2). This corresponds to the result 1 above. Since we will be working on a sequence of discrete Wiener functionals unlike the situations concerning tracking error, we need to some discussions on the finite-dimensionality. An answer is given in section 3.3 and under the condition it is proven that the convergence order is related to a fractional smoothness (Theorem 3.5). This corresponds to the result 2 above. Section 3.4 is devoted to a study of the asymptotics of the error of the additive functionals. As a case study, we give a detailed estimate of the martingale representation error of the Riemann-sum approximation of Brownian occupation time (Theorem 3.9).

The proofs given in this paper is largely based on elementary calculus with a bit of classical Fourier analysis and distribution theory, but nonetheless our methods can be, in spirit, a finite-dimensional reduction of Malliavin-Watanabe’s distribution theory. Some detailed discussions on this point of view will be given in sections 2.1, 2.3, and 3.1. We have restricted ourselves to one-dimensional Wiener space case, but this is only for simplicity for the notations.

2 A Discrete Version of Clark-Ocone Formula

2.1 Generalized Wiener Functional in Discrete Time

Throughout this section we fix $N \in \mathbb{N}$ and work on the canonical probability space $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mu^N)$ though we will abuse the notations like $\Delta W$ as the coordinate map.

Let $\mathcal{S}_N \equiv \mathcal{S}(\mathbb{R}^N)$ be the Schwartz space: the space of all rapidly decreasing functions and $\mathcal{S}_N'$ be its dual; the space of all tempered
distributions (see, e.g. [16]). We (may) call \( X \in S'_{N} \) a “discrete generalized Wiener functional” and its generalized expectation is defined to be the coupling \( s'_{N}(X, p^{N})_{S'_{N}} \), where \( p^{N} \) is the density of \( \mu^{N} \), which is of course in \( S_{N} \).

The conditional expectation \( \mathbf{E}[X|G_{k}] \) for \( X \in S'_{N} \) is then defined as follows. We first note that the inclusion \( G_{k} \subset G_{N} \) induces those of \( S(\mathbb{R}^{k}) \subset S(\mathbb{R}^{N}) \) and \( S'(\mathbb{R}^{k}) \subset S'(\mathbb{R}^{N}) \). In this sense we write \( S_{k} \) and \( S'_{k} \) for the Schwartz space and the space of generalized Wiener functionals with respect to \( G_{k} \), \( k = 1, \cdots, N \). Then \( Y = \mathbf{E}[X|G_{k}] \) in \( S'_{k} \) is defined in terms of the relation

\[
\mathbf{E}[XZ] = \mathbf{E}[YZ], \quad \forall Z \in S_{k},
\]

which should be understood as

\[
s'_{N}(X, Zp^{N})_{S_{N}} = s_{k}(Y, Zp^{k})_{S_{k}}, \quad \forall Z \in S_{k}.
\]

In particular, we see that the conditional expectation is well-defined by du Bois-Reymond lemma (see e.g. [16]). Note that this generalized conditional expectation reduces to the standard one on \( L^{1}(\mu^{N}) \), which is included in \( S'_{N} \) unlike the \( L^{1} \) space with respect to the Lebesgue measure. Furthermore, differentiations of \( X \in S'_{N} \) are defined as usual, namely,

\[
\partial_{k}X = Y \iff s'_{N}(Y, Z)_{S_{N}} = -s'_{N}(X, \partial_{k}Z)_{S_{N}} \quad \forall Z \in S_{N},
\]

which imply

\[
\mathbf{E}[\partial_{k}X] = \mathbf{E}[X\partial_{k}\log p^{N}],
\]

and so on.

### 2.2 Clark-Ocone Formula in Discrete Time

We have the following series expansion in \( \Delta t \):

**Theorem 2.1** (A Discrete Version of Clark-Ocone Formula). For \( X \in L^{2}(G_{N}) \approx L^{2}(\mu^{N}) \), we have the following \( L^{2} \)-convergent series expansion:

\[
X - \mathbf{E}[X] = \sum_{m=1}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}[\partial^{m}X|G_{l-1}]H_{m} \left( \frac{\Delta W_{l}}{\sqrt{\Delta t}} \right)
\]

where \( H_{m} \) is the \( m \)-th Hermite polynomial for \( m \in \mathbb{Z}_{+} \);

\[
H_{m}(x) = \frac{(-1)^{m}}{\sqrt{m!}} e^{x^{2}} \frac{d^{m}}{dx^{m}} e^{-x^{2}} \quad (m \in \mathbb{Z}_{+}).
\]

Here the differentiations are understood in the distribution sense, as explained in the previous section.
Proof. Since \( \{ \prod_{i=1}^{N} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \}_{k_1, \ldots, k_N \in \mathbb{Z}_+} \) is an orthonormal basis of \( L^2(\mathbb{R}^N, \mu^N) \), we have the following orthogonal expansion of \( X \in L^2(\mathbb{R}^N, \mu^N) \):

\[
X(\Delta W_1, \ldots, \Delta W_N) = \sum_{k_1, \ldots, k_N} c_{(k_1, \ldots, k_N)} \prod_{i=1}^{N} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right).
\]

(2.3)

where we denote

\[
c_{(k_1, \ldots, k_N)} := \langle X, \prod_{i=1}^{N} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \rangle = \mathbb{E} \left[ X \prod_{i=1}^{N} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \right].
\]

Let us “sort” the series according as the “highest” non-zero \( k_i \):

\[
X(\Delta W_1, \ldots, \Delta W_N) = \mathbb{E}[X] + \sum_{l=1}^{N} \sum_{k_1, \ldots, k_{l-1}, k_l \geq 1} \sum_{c_{(k_1, \ldots, k_l, 0, \ldots, 0)}} \prod_{i=1}^{l} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right). \quad (2.4)
\]

Here we claim that

\[
\sum_{l=1}^{N} \sum_{k_1, \ldots, k_{l-1}, k_l \geq 1} \sum_{c_{(k_1, \ldots, k_l, 0, \ldots, 0)}} \prod_{i=1}^{l} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) = \mathbb{E} \left[ X H_{k_l} \left( \frac{\Delta W_l}{\sqrt{\Delta t}} \right) \mid G_{l-1} \right]. \quad (2.5)
\]

In fact, from the expansion (2.3) we have

\[
\mathbb{E}[X H_{k_l} \left( \frac{\Delta W_l}{\sqrt{\Delta t}} \right) \mid G_{l-1}]
\]

\[
= \mathbb{E} \left[ \sum_{k_1', \ldots, k_{l-1}'} c_{(k_1', \ldots, k_{l-1}', 0, \ldots, 0)} \prod_{i=1}^{l-1} H_{k_i'} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) H_{k_l} \left( \frac{\Delta W_l}{\sqrt{\Delta t}} \right) \mid G_{l-1} \right]
\]

\[
= \sum_{k_1', \ldots, k_{l-1}'} c_{(k_1', \ldots, k_{l-1}') \mid H_{k_1'} \left( \frac{\Delta W_{k_1'}}{\sqrt{\Delta t}} \right) \prod_{i=1}^{l-1} H_{k_i'} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \mathbb{E} \left[ H_{k_l} \left( \frac{\Delta W_l}{\sqrt{\Delta t}} \right) \prod_{i=l}^{N} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \right],
\]

and we confirm the claim since \( \mathbb{E}[H_{k_l} \left( \frac{\Delta W_l}{\sqrt{\Delta t}} \right) \prod_{i=l}^{N} H_{k_i} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right)] = 0 \) unless \( k'_l = k_l \) and \( k'_i = 0 \) for \( i > l \).

We further claim that

\[
\mathbb{E} \left[ X H_{k_l} \left( \frac{\Delta W_l}{\sqrt{\Delta t}} \right) \mid G_{l-1} \right] = \left( \frac{\Delta}{\sqrt{\Delta t}} \right)^{k_l/2} \mathbb{E} \left[ \partial_{\Delta}^{k_l} X \mid G_{l-1} \right], \quad (2.6)
\]

which, together with (2.4) and (2.5), will prove the expansion (2.1) in the \( L^2 \) case. Here, the conditional expectation should be understood
in the generalized sense. Following the definition we have made, it suffices to show that
\[
E \left[ XH_{k_{i}} \left( \frac{\Delta W_{i}}{\sqrt{\Delta t}} \right) f(\Delta W_{1}, \ldots, \Delta W_{l-1}) \right] = \frac{(\Delta t)^{k/2}}{\sqrt{k!}} E[\partial_{k}^{k} Xf]
\]
for any \( f \in S_{l-1} \) but this is easy to see if we write down the generalized expectation as the coupling of \( S \) and \( S' \):
\[
S' \langle X, H_{k}(x/\sqrt{\Delta t})fp \rangle_{S} = \frac{(\Delta t)^{k/2}}{\sqrt{k!}} S' \langle \partial_{k}^{k} X, fp \rangle_{S}.
\]

2.3 Comment on Discrete Generalized Wiener Functionals

In this subsection, we remark that our discrete generalized Wiener functionals is slightly broader than that of the direct finite dimensional reduction; there is a gap. For simplicity, we let \( \Delta t = 1 \) in this subsection.

We know that (see e.g. [13, Appendix to V.3]) the orthogonal expansion in \( L^{2}(\mathbb{R}^{N}, \text{Leb}) \) with respect to the Hermite functions:
\[
\phi_{N}(x) := \frac{1}{\sqrt{N!}} H_{N}(x)(p^{N})^{1/2}
\]
gives so-called \( \mathcal{N} \)-representation of \( S \) and \( S' \); the series for \( f \in S \) (resp. \( \in S' \))
\[
\sum_{S'} \langle f, \phi_{N} \rangle_{S} \phi_{N}
\]
converges to \( f \) in \( S \) (resp. \( \in S' \)). In our context, it then follows that if \( X(p^{N})^{1/2} \in S \) (resp. \( \in S' \)), then the convergence of the expansion \( (2.1) \) is in \( S \) (resp. \( \in S' \)) as well. It should be further noted that we have the following equivalences:

**Proposition 2.2.** It holds that
\[
X(p^{N})^{1/2} \in S \iff X \in D_{2,\infty}^{(N)} = \cap_{s>0} D_{2,s}^{(N)}
\]
and
\[
X(p^{N})^{1/2} \in S' \iff X \in D_{2,-\infty}^{(N)} = \cup_{s<0} D_{2,s}^{(N)}.
\]
where $\mathbb{D}_{2,s}^{(N)}$ is the completion of $L^2(\mu^N)$ by the norm $\|f\|_{2,s} = \|(1 + L)^{s/2} f\|_{L^2(\mu^N)}$. Here $L$ is the Ornstein-Uhlenbeck operator on $\mathbb{R}^N$;

$$L = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}.$$ 

**Proof.** Let $\{\phi_n : n \in \mathbb{Z}\}$ be norms defined by

$$\phi_n(f) = \|(1 + S)^n f\|_{L^2(\text{Leb})},$$

where $S$ is the following Schrödinger operator of the harmonic oscillator:

$$S := -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{4} |x|^2 - \frac{1}{2}.$$ 

We know that $S$ is a Fréchet space by the seminorms $\{\phi_n\}$. In fact, both $L$ and $S$ are the number operators respectively in that:

$$L \prod_{i=1}^N H_{k_i}(x_i) = \left( \sum_{i=1}^N k_i \right) \prod_{i=1}^N H_{k_i}(x_i)$$

and

$$S \prod_{i=1}^N \phi_{k_i}(x_i) = \left( \sum_{i=1}^N k_i \right) \prod_{i=1}^N \phi_{k_i}(x_i).$$

We also have

$$L(f)(p^N)^{1/2} = S(f(p^N)^{1/2}),$$

which implies

$$\|f\|_{2,n} = \phi_n(f).$$

This proves (2.7).

The equivalence (2.8) follows from the following equivalence of the duality:

$$\mathbb{D}_{2,-\infty}^{(N)} \langle X, Y \rangle_{\mathbb{D}_{2,-\infty}^{(N)}} = S' \langle X(p^N)^{1/2}, Y(p^N)^{1/2} \rangle_S.$$ 

**Corollary 2.3.** For $X \in \mathbb{D}_{2,s}^{(N)}, s \in \mathbb{R}$, the convergence of (2.1) is also attained in $\mathbb{D}_{2,s}^{(N)}$.

**Proof.** It follows from the fact that, by the assumption, the partial sums

$$X_n := \sum_{k_1 + \cdots + k_N \leq n} c_{(k_1, \ldots, k_N)} \prod_{i=1}^N H_{k_i}(x_i), \quad n \in \mathbb{N}$$

form a Cauchy sequence in $\mathbb{D}_{2,s}^{(N)}$. 


3 Asymptotic Analysis of Martingale Representation Errors

In this section, we will consider the asymptotic behavior of the error term when \( N \to \infty \) with \( N \Delta t = T \). For this purpose, to make explicit the dependence on \( N \) we redefine some of the notations.

\[ t_k := t_k^{(N)} := \frac{kT}{N} \quad \text{for each } k = 0, 1, \ldots, N. \]

We also write \( \Delta W_k^N = W_k^{(N)} - W_{k-1}^{(N)} \) for each \( k \) and \( N \), and \( G_k^N := \sigma(\Delta W_k^N; t = 1, \ldots, k) \). Further, to facilitate the discussion in the limit, we implement our discrete Malliavin-Watanabe calculus into the classical one in the first subsection.

3.1 Consistency with the Classical Malliavin Calculus

First, we review briefly the Malliavin calculus over the one-dimensional classical Wiener space to introduce notations which we will use in the following sections devoted to asymptotic analyses, and then will show how our framework, established in the previous sections, is “embedded” to the classical Malliavin calculus (Proposition 3.1).

Let \((\mathcal{W}, \mathbb{P})\) be the one-dimensional Wiener space on \([0, T]\). We consider the canonical process \( w = (w(t))_{0 \leq t \leq T} \) starting from zero a.s. In this context, the Hilbert space

\[ H = \left\{ h \in \mathcal{W} : h(0) = 0 \text{ and } h \text{ is absolutely continuous with square-integrable derivative} \right\} \]

equipped with the inner product defined by

\[ \langle h_1, h_2 \rangle_H = \int_0^T \dot{h}_1(t)\dot{h}_2(t)dt, \quad h_1, h_2 \in H \]
is called the Cameron-Martin subspace of \( \mathcal{W} \). For each complete orthonormal system (CONS, in short) \( \{h_i\}_{i=1}^\infty \) of \( H \), it is known that

\[ \left\{ \prod_{i=1}^\infty H_{a_i} \left( \int_0^T \dot{h}_i(t)dw(t) \right) : a \in \Lambda \right\} \]
forms a CONS in \( L^2(\mathcal{W}) \) (see e.g., [8] Proposition 8.1), where \( \Lambda \) is the set of all sequence \( a = (a_i)_{i=1}^\infty \) of nonnegative integers except for a finite number of \( i \)'s and \( H_n \) is the \( n \)-th Hermite polynomial defined in (2.1). We also denote by \( J_n : L^2(\mathcal{W}) \to \mathcal{C}_n \) the orthogonal projection, where \( \mathcal{C}_n \) is the \( L^2(\mathcal{W}) \)-closure of the subspace spanned by \( \left\{ \prod_{i=1}^\infty H_{a_i} \left( \int_0^T \dot{h}_i(t)dw(t) \right) : \sum_{i=1}^\infty a_i = n \right\} \) over \( \mathbb{R} \). Each \( \mathcal{C}_n \) is called the subspace of \( n \)-th Wiener’s homogeneous chaos.
For each $s \in \mathbb{R}$, a Sobolev-type Hilbert space $D_{2,s} = D_{2,s}(\mathbb{R})$ is defined as the completion of \( \{ F \in L^2(\mathcal{W}) : \| F \|_{D_{2,s}} < \infty \} \) under the seminorm $\| \cdot \|_{D_{2,s}}$ on $L^2(\mathcal{W})$ defined by

$$
\| F \|_{D_{2,s}}^2 = \sum_{n=0}^{\infty} (1 + n)^s \| J_n F \|_{L^2}^2, \quad F \in L^2(\mathcal{W})
$$

which may be infinite in general.

In the following, for any two separable Hilbert space $H_1$ and $H_2$, we denote by $H_1 \otimes H_2$ the completion of the algebraic tensor product of $H_1$ and $H_2$ under the Hilbert-Schmidt norm.

It is known that one can define a (continuous) linear operator $D : D_{2,1} \to L^2(\mathcal{W}) \otimes H$ such that

$$
\langle DF, h \rangle_H = D_h F \in L^2(\mathcal{W})
$$

for every $h \in H$ and $F \in D_{2,1}$, where $D_h F$ is defined by

$$
(D_h F)(w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ F(w + \varepsilon h) - F(w) \right\} \quad \text{for a.e. } w \in \mathcal{W},
$$

which is well-defined due to the Cameron-Martin theorem (see e.g., [S] Theorem 8.5). For each $t \in [0,T]$, let $e_t : \mathcal{W} \to \mathbb{R}$ denote the evaluation map defined by $e_t(w) = w(t)$. Then a linear operator $D_t : D_{2,1} \to L^2(\mathcal{W})$ is defined by

$$
D_tF = \frac{d}{dt}(\text{id}_{L^2(\mathcal{W})} \otimes e_t)(DF), \quad F \in D_{2,1}
$$

for a.a. $t \in [0,T]$.

Under these notations, we can state the relationship between our framework established in section 2 and that of Malliavin calculus. We omit the proof because it is immediate from the definition.

**Proposition 3.1.** For each $X \in D_{2,1}^{(N)}$, we have

$$
(D_tX)(w) = \sum_{i=1}^{N} 1_{\{t_{i-1} \leq t < t_i\}}(\partial h X)(w)
$$

for a.a. $(t, w) \in [0,T] \times \mathcal{W}$.

For each $F \in D_{2,1}$, one can prove that $E[F|G_N^N] \in D_{2,1}^{(N)}$ and $\lim_{N \to \infty} E[F|G_N^N] = F$ in $D_{2,1}$ (consult e.g., [M] Theorem 1.10). By using also the fact that $e_t(h) = 1_{[0,t)}(h)_H$ for each $h \in H$, one can obtain

$$
(D_t F)(w) = \lim_{N \to \infty} \sum_{i=1}^{N} 1_{\{t_{i-1} \leq t < t_i\}} \partial h E[F|G_N^N](w)
$$
for a.a. \((t, w) \in [0, T] \times \mathcal{W}\). Note that in [9], the derivative \(D\) on the path space \(\mathcal{W}\) is defined directly by (3.2) with \(N = 2^n\). Following this approach in [9], we define \(D^{k}X \in L^2[0, T] \otimes L^2(\mathcal{P})\) as the \(L^2\)-limit of the sequence \((D^{k}E[X|G_{N}^{N}])_{N=1}^{\infty}\) if it exists (see [9], Theorem 1.10 to consult what condition is enough to get this limit).

By the above discussions, we may write
\[
D^{k}tX := \partial^{k}lX \text{ if } t_{l-1} \leq t < t_{l}
\]
for \(X \in D_{2,n}^{(N)}\), \(t \in [0, T]\), and \(k = 1, 2, \cdots, n\).

### 3.2 A Central Limit Theorem for the Errors

Suppose that we are given a sequence \((X^{N})_{N=1}^{\infty}\) of finite dimensional Wiener functionals \(X^{N} \in L^2(G_{N}^{N})\) for each \(N\).

We put, for \(n \geq 0\),
\[
Err_{N}(n) := X^{N} - \sum_{m=0}^{n} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{m!} E[D_{m/N}^{m}X^{N}|G_{l-1}^{-1}]H_{m}\left(\frac{\Delta W^{N}_l}{\sqrt{\Delta t}}\right).
\]

**Theorem 3.2.** Let \(n \in \mathbb{N}\). Suppose that \(X^{N} \in D_{2,n+2}^{(N)}\) for each \(N = 1, 2, \cdots\) and for some Wiener functional \(X \in D_{2,n+1}(\mathbb{R})\), we have
- \(X^{N} \rightarrow X \text{ in } L^2(\mathcal{P})\),
- \(\int_{0}^{T} \|D_{t}^{p+1}X^{N} - D_{t}^{p+1}X\|_{L^2}^2dt \rightarrow 0\)

as \(N \rightarrow \infty\) for each \(p = 0, 1, \cdots, n\) and
- \(\sup_{N} \int_{0}^{T} \|D_{t}^{n+2}X^{N}\|_{L^2}^2dt < \infty\).

Then we have
\[
\begin{pmatrix}
Err_{N}(0) \\
(\Delta t)^{-1/2}Err_{N}(1) \\
\vdots \\
(\Delta t)^{-n/2}Err_{N}(n)
\end{pmatrix} \rightarrow \begin{pmatrix}
\int_{0}^{T} E[D_{t}X|G_{t}]dW_{t} \\
\frac{1}{\sqrt{2}} \int_{0}^{T} E[D_{t}^{2}X|G_{t}]dB_{1}^{1} \\
\vdots \\
\frac{1}{\sqrt{(n+1)!}} \int_{0}^{T} E[D_{t}^{n+1}X|G_{t}]dB_{n}^{n}
\end{pmatrix}
\]
in probability on an extended probability space as \(N \rightarrow \infty\), where \((B_{1}, \cdots, B_{n}) = (B_{1}^{1}, \cdots, B_{1}^{n})_{0 \leq t \leq T}\) is an \(n\)-dimensional Brownian motion independent of \(W = (W_{t})_{0 \leq t \leq T}\).
Remark 3.3. Although the Brownian motion $B = (B^1, \cdots, B^n)$ above is not adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, the above stochastic integrals make sense because it is an $(\mathcal{F}_t \vee \sigma(B_s : 0 \leq s \leq t))_{0 \leq t \leq T}$ Brownian motion.

Proof. By Theorem 2.1, we have,

$$(\Delta t)^{-p/2} \text{Err}_N(p) = \sum_{m=p+1}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{(m-p)/2}}{\sqrt{m!}} \mathbb{E}[D_{iT/N}^m X_N^N | \mathcal{G}_{i-1}] H_m \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right).$$

For $m \geq p+2$, by using the integration by parts formula (2.6), we see that

$$\left\| \sum_{m=p+2}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{(m-p)/2}}{\sqrt{m!}} \mathbb{E}[D_{iT/N}^m X_N^N | \mathcal{G}_{i-1}] H_m \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \right\|^2_{L^2} \leq (\Delta t)^2 \sum_{k=0}^{\infty} \frac{k!}{(k+p+2)!} \left\| \mathbb{E}[D_{iT/N}^{p+2} X_N^N | \mathcal{G}_{i-1}] H_k \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \right\|^2_{L^2} \Delta t$$

$$= (\Delta t)^2 \sum_{k=1}^{\infty} \frac{1}{k^{p+2}} \sum_{l=1}^{N} \| D_{iT/N}^{p+2} X_N^N \|_{L^2}^2 \Delta t$$

$$= (\Delta t)^2 \sum_{k=1}^{\infty} \frac{1}{k^{p+2}} \int_0^T \| D_{tT/N}^{p+2} X_N^N \|_{L^2}^2 dt$$

which goes to zero as $N \to \infty$ for each $p = 0, 1, \cdots, n$ by the assumption.

Let us consider the case $m = p+1$. For each $p = 0, 1, \cdots, n$, we define a right-continuous process $L_{l}^{p,N} = (L_{l}^{p,N})_{0 \leq t \leq T}$ with left-hand side limits by

$$L_{t}^{p,N} := \sum_{l=1}^{k} H_{p+1} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right) \quad \text{if} \ t_{k-1} \leq t < t_k$$

for $k = 1, 2, \cdots, N$, and $L_T^{p,N} := L_T^{p,N}$.

Since $H_{p+1} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right)$, $l = 1, 2, \cdots, N$ are i.i.d. random variables and $H_{p+1} \left( \frac{\Delta W_i}{\sqrt{\Delta t}} \right)$, $p = 0, 1, \cdots, n$ are orthogonal to each other for each $l = 1, 2, \cdots, N$, the central limit theorem of finite dimensional distributions of $(\Delta t)^{1/2} L_{l}^{p,N}$, $N = 1, 2, \cdots$ follows as for each $0 \leq s <
little-o-notation is with respect to the asymptotics when \( N \rightarrow \infty \) that the distribution of \((n)\)
filtration generated by a stochastic process
and for each \( \varepsilon > 0 \),

They imply the tightness of \( \{\Delta t\}^1/2L_{t}^{p,N} \}_{p=0}^\infty \) (see Billingsley \[\Xi\], Theorem 13.2). Therefore,

\[
\{\{(\Delta t)^{1/2}L_{0,N},(\Delta t)^{1/2}L_{1,N}, \ldots , (\Delta t)^{1/2}L_{n,N}\}\}_{N=1}^\infty
\]
also forms a tight family. Hence we have

\[
(\sqrt{\Delta t}L_{0,N}, \sqrt{\Delta t}L_{1,N}, \ldots , \sqrt{\Delta t}L_{n,N}) \rightarrow (B^0, B^1, \ldots , B^n)
\]

for each \( \xi = (\xi_0, \xi_1, \ldots , \xi_n) \in \mathbb{R}^{n+1} \), where \((\mathcal{F}_t^Z)_{0 \leq t \leq T}\) denotes the filtration generated by a stochastic process \( Z = (Z_t)_{0 \leq t \leq T} \) and the little-o-notation is with respect to the asymptotics when \( N \rightarrow \infty \) (so that \( k - j \rightarrow \infty \)). This implies that every finite dimensional distribution of \((n + 1)\)-dimensional process \((\Delta t)^{1/2}L^{p,N}\) for each \( \varepsilon > 0 \),

\[
\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \left| (\Delta t)^{1/2}L_{t}^{p,N} \right| \geq K
\]
and for each \( \varepsilon > 0 \),

\[
\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \left| (\Delta t)^{1/2}L_{t}^{p,N} - L_{s}^{p,N} \right| \geq \varepsilon
\]

for each \( \varepsilon > 0 \),

\[
\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \left| (\Delta t)^{1/2}L_{t}^{p,N} - L_{s}^{p,N} \right| \geq \varepsilon
\]

They imply the tightness of \( \{\Delta t\}^1/2L_{t}^{p,N} \}_{p=0}^\infty \) (see Billingsley \[\Xi\], Theorem 13.2). Therefore,

\[
\{\{(\Delta t)^{1/2}L_{0,N},(\Delta t)^{1/2}L_{1,N}, \ldots , (\Delta t)^{1/2}L_{n,N}\}\}_{N=1}^\infty
\]
also forms a tight family. Hence we have

\[
(\sqrt{\Delta t}L_{0,N}, \sqrt{\Delta t}L_{1,N}, \ldots , \sqrt{\Delta t}L_{n,N}) \rightarrow (B^0, B^1, \ldots , B^n)
\]
in law as $N \to \infty$. By the Skorohod representation theorem (see Ikeda-Watanabe [8], Theorem 2.7), we may assume that the above convergence is realized as an almost sure convergence on an extended probability space. Note that on the probability space we still have $B_0 = W$ a.s. Hence we have

$$
\frac{(\Delta t)^{(p+1)-p/2}}{\sqrt{(p+1)!}} \sum_{l=1}^{N} \mathbb{E}[D_{T/N}^{p+1} X|\mathcal{G}_{l-1}] H_{p+1} \left( \frac{\Delta W_{l}^{N}}{\sqrt{\Delta t}} \right) \\
= \frac{1}{\sqrt{(p+1)!}} \sum_{l=1}^{N} \mathbb{E}[D_{T}^{p+1} X|\mathcal{G}_{l-1}] \left\{ (\Delta t)^{1/2} L_{p,0}^{N} - (\Delta t)^{1/2} L_{p,0}^{N-1} \right\} \\
\to \frac{1}{\sqrt{(p+1)!}} \int_{0}^{T} \mathbb{E}[D_{T}^{p+1} X|\mathcal{G}_{l}] dB_{l}^{p}
$$

in probability as $N \to \infty$ simultaneously for $p = 0, 1, \cdots, n$. \hfill \Box

Substituting $p = 0$ into the inequality (3.3) in the proof of Theorem 3.2, we also obtain the following

**Corollary 3.4.** If $\sup_{N} \int_{0}^{T} \|D_{T}^{2} X\|_{L_{2}}^{2} dt < \infty$ then we have

$$
\|X^{N} - \left\{ \mathbb{E}[X^{N}] + \mathbb{E}[D_{T/N} X|\mathcal{G}_{l-1}^{N}] \Delta W_{l}^{N} \right\} \|_{L_{2}} = O(N^{-1/2})
$$

as $N \to \infty$.

### 3.3 The Cases with “Finite Dimensional” Functionals

We have seen that the martingale representation error is of an order $1/2$ for a smooth functional. In this section, we will observe that for a non-smooth functional, the order is related to its fractional differentiability if it behaves eventually like a finite dimensional functional. This parallels with the corresponding results in the cases of the tracking error as we have pointed out in Introduction.

Let us start with one-dimensional cases. Let $F \in L^{2}(\mathbb{R}, \mu_{T})$, where $\mu_{T}$ is the Gaussian measure with variance $T > 0$. Then, since

$$
\frac{\partial^{k}}{\partial x_{1}^{k}} F(x_{1} + \cdots + x_{N}) = F^{(k)}(x_{1} + \cdots + x_{N}),
$$


\footnote{On the space of all right-continuous functions with left-hand side limits, one can endow so-called the Skorohod topology which is metrizable and makes the space a complete separable metric space. For details, see Billingsley [11], Chapter 5.}
we have, for $k_1 + \cdots + k_N = n$,
\[
E[D^{k_1}_{t_{l_1}^{(1)}}} \cdots D^{k_N}_{t_{l_N}^{(N)}} F(W_T)]^2 = E[F^{(N)}(W_T)]^2
\]
\[
= \frac{n!}{T^n} E[F(W_T)H_n \left( \frac{W_T}{\sqrt{T}} \right)]^2 = \frac{n!}{T^n} \|J_n F(W_T)\|^2_{L^2},
\]
irrespective of $l$ and $N$. Here $J_n$ is the projection to the $n$-th chaos. With this observation in mind, we understand the following property as a finite-dimensionality of a sequence; let $\{F^N\}$ be such that each $F^N$ being $G^N$-measurable and that
\[
\sup_{k_1 + \cdots + k_N = n} (E[D^{k_1}_{t_{l_1}^{(1)}}} \cdots D^{k_N}_{t_{l_N}^{(N)}} F])^2 = O \left( \frac{n! \|J_n F^N\|^2}{T^n} \right)
\]
uniformly in $n = 2, 3, \cdots$ as $N \rightarrow \infty$.

Note that a sequence composed of a one dimensional functional $F(W_T)$ satisfies the above property trivially. Furthermore, the multi dimensional case where $F^N = F, N = 1, 2, \cdots$ for some $F \in L^2(G_m^m)$, $2 \leq m < \infty$ satisfies (3.4) as well. In fact, for arbitrary non-negative integers $k_1, \cdots, k_N$ with $k_1 + \cdots + k_N = n$, the relation
\[
(E[D^{k_1}_{t_{l_1}^{(N)}}} \cdots D^{k_N}_{t_{l_N}^{(N)}} F])^2 = (E[D^{l_1}_{t_{l_1}^{(m)}}} \cdots D^{l_m}_{t_{l_m}^{(m)}} F])^2,
\]
where $l_j = k_N(j-1)+1 + \cdots + k_N(j-1)+N_j$, implies
\[
(E[D^{k_1}_{t_{l_1}^{(N)}}} \cdots D^{k_N}_{t_{l_N}^{(N)}} F])^2 \leq \frac{l_1! \cdots l_m!}{T^n} m^n \|J_n F\|^2
\]
\[
= \frac{l_1! \cdots l_m!}{T^n} \sum_{l_1 + \cdots + l_m = n} \frac{n!}{l_1! \cdots l_m!} \|J_n F\|^2
\]
\[
\leq \frac{n!}{T^n} \|J_n F\|^2.
\]

**Theorem 3.5.** Suppose that we are given a sequence of $F^N \in D_2^{(N)}$, $N = 1, 2, \cdots$ satisfying
\[
\sup_{N} \|F^N\|^2_{D_2,s} < \infty
\]
for some $0 \leq s \leq 1$ and the “finite-dimensional property” (3.4). Then
\[
\|1\text{-Mart.Err}(F^N)\|^2_{L^2} = O(N^{-s/2}) \quad \text{as } N \rightarrow \infty.
\]

**Proof.** By observing (2.4), we notice that
\[
\|1\text{-Mart.Err}(F^N)\|^2_{L^2}
\]
\[
= \sum_{l=1}^{N} \sum_{k_1 + \cdots + k_l = n} \frac{n!}{k_1! \cdots k_l!} (\Delta t)^n E[\partial^{k_1}_{t_1} \cdots \partial^{k_l}_{t_l} F^N]_2^2
\]
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for each \( n = 2, 3, \cdots \). By the assumption, there is a constant \( C > 0 \) such that
\[
\sup_{k_1 + \cdots + k_l = n} \mathbb{E} \left[ \partial_t^{k_1} \cdots \partial_t^{k_l} F^N \right]^2 \leq C \frac{n! \| J_n F^N \|_2^2}{T^n}
\]
for each \( n = 2, 3, \cdots \) and \( N = 1, 2, \cdots \) and the multinomial theorem yields that
\[
\sum_{k_1 + \cdots + k_l = n} \frac{n!}{k_1! \cdots k_l!} (\Delta t)^n
\]
\[
= \left( \frac{t}{N} \right)^n - \left( \frac{(l-1)T}{N} \right)^n - n \left( \frac{t}{N} \right) \left( \frac{(l-1)T}{N} \right) \left( \frac{t}{N} \right)^{n-1}.
\]
Putting them together, we have
\[
\|1\text{-Mart.Err}(F^N)\|_{L^2}^2
\]
\[
\leq C \sum_{n=2}^{\infty} \left\{ 1 - n \frac{1}{N} \sum_{l=0}^{N-1} \left( \frac{l}{N} \right)^{n-1} \right\} \| J_n F^N \|_2^2
\]
\[
= C N^{-s} \sum_{n=2}^{\infty} \frac{N^s}{n^{s-1}} \left\{ 1 - n \frac{1}{N} \sum_{l=0}^{N-1} \left( \frac{l}{N} \right)^{n-1} \right\} n^s \| J_n F^N \|_2^2
\]
for each \( s \in \mathbb{R} \).
On the other hand, since we have
\[
I_{n,N} := \frac{1}{n} - \frac{1}{N} \sum_{l=0}^{N-1} \left( \frac{l}{N} \right)^{n-1}
\]
\[
= \sum_{l=0}^{N-1} \int_{l/N}^{(l+1)/N} \left\{ x^{n-1} - \left( \frac{l}{N} \right)^{n-1} \right\} dx > 0,
\]
\( I_{n,N} \leq 1/n \), and
\[
I_{n,N} \leq \frac{1}{N} \sum_{l=0}^{N-1} \left\{ \left( \frac{l+1}{N} \right)^{n-1} - \left( \frac{l}{N} \right)^{n-1} \right\} = \frac{1}{N},
\]
we notice that
\[
I_{n,N} = \int_{n,N}^{n+s} \left( \frac{1}{N} \right)^s \left( \frac{1}{k} \right)^{1-s}
\]
for every \( 0 \leq s \leq 1 \).

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By (3.5) and (3.6), we finally have
\[
\|1\text{-Mart.Err}(F^N)\|_{L^2}^2 
\leq CN^{-s} \sum_{n=2}^{\infty} n^s \|J_n F^N\|_{L^2}^2 \leq CN^{-s} \sup_N \|F^N\|_{\mathcal{B}_{2,s}^1}^2.
\]

\[\square\]

### 3.4 A Study on Additive Functionals

In this subsection, we study sequences of “additive functionals”,
\[
F^N := \sum_{i=1}^{N} f_N(t_i, W_{i(N)}) \Delta t
\]
where \(f_N(t_i, \cdot), i = 1, \ldots, N\) is a sequence in \(\mathbb{D}_{2,-\infty}^{(1)}\).

We are interested in the conditions for the sequence to be “finite-dimensional” in the sense of (3.4).

We define an index to control the finite-dimensionality. Let
\[
A_l := (\sum_{i=1}^{N} i^{-n/2} E[f_N(t_i, W_{i(N)}) H_n(W_{t_i}/\sqrt{t_i})])^2
\]
and
\[
\alpha_{N,n}(F^N) := \begin{cases} 
0 & \text{if } \sum_{l=1}^{N} A_l \{l^n - (l-1)^n\} = 0 \\
\frac{N^n}{\sum_{l=1}^{N} A_l \{l^n - (l-1)^n\}} & \text{otherwise}.
\end{cases}
\]

Then, we have the following criterion.

**Proposition 3.6.** The sequence \(\{F_N\}\) of (3.7) satisfies (3.4) if and only if
\[
\sup_n \sup_N \alpha_{n,N}(F^N) < \infty.
\]

**Proof.** For arbitrary non-negative integers \(k_1, \ldots, k_N\) with \(k_1 + \cdots + k_N = n\), we have
\[
E[D_{i_1}^{k_1} \cdots D_{i_N}^{k_N} F^N] = \sum_{i=1}^{N} 1_{\{k_{i+1}=\cdots=k_N=0\}} E[f_N^{(n)}(t_i, W_{t_i})] \Delta t
\]
\[
= (n!)^{1/2} (\Delta t)^{(2-n)/2} \sum_{i=1}^{N} 1_{\{k_{i+1}=\cdots=k_N=0\}} i^{-n/2} E[f_N(t_i, W_{t_i}) H_n(W_{t_i}/\sqrt{t_i})].
\]
If further $k_t \geq 1$ and $k_{t+1} = \cdots k_N = 0$ for some $l$, then

$$E[D_{t_1}^{k_1} \cdots D_{t_l}^{k_l} F^N]$$

$$= (n!)^{1/2} (\Delta t)^{(2-n)/2} \sum_{i=1}^{N} i^{-n/2} E[f_N(t_i, W_{t_i}) H_n(W_{t_i}/\sqrt{\Delta t})]$$

$$= (n!)^{1/2} (\Delta t)^{(2-n)/2} A_l^{1/2}.$$

Therefore,

$$\sup_{k_1 + \cdots + k_N = n} (E[D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N])^2 = n! (\Delta t)^{(2-n)} \sup_{l=1, \cdots, N} A_l$$ (3.8)

On the other hand, we have

$$\|J_n F^N\|^2$$

$$= \sum_{l=1}^{N} \sum_{k_1 + \cdots + k_l = n \atop k_i \geq 1} (E[F^N H_{k_1}(\Delta W_1/\sqrt{\Delta t})] \cdots H_{k_l}(\Delta W_l/\sqrt{\Delta t})])^2$$

$$= \sum_{l=1}^{N} \sum_{k_1 + \cdots + k_l = n \atop k_i \geq 1} \frac{(\Delta t)^n}{k_1! \cdots k_l!} (E[D_{t_1}^{k_1} \cdots D_{t_l}^{k_l} F^N])^2$$ (3.9)

$$= \sum_{l=1}^{N} A_l \sum_{k_1 + \cdots + k_l = n \atop k_i \geq 1} \frac{(\Delta t)^{2n}}{k_1! \cdots k_l!} = (\Delta t)^2 \sum_{l=1}^{N} A_l \{l^n - (l-1)^n\}.$$

Putting (3.8) and (3.9) together, we have

$$\sup_{k_1 + \cdots + k_N = n} \frac{(E[D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N])^2}{\|J_n F^N\|^2} = \frac{n!}{T^n} \frac{N^n \sup_l A_l}{\sum_{l=1}^{N} A_l \{l^n - (l-1)^n\}}$$

$$= \frac{n!}{T^n} \alpha_{N,n}(F^N).$$

Note that $\|J_n F^N\|^2 = 0$ implies both $\alpha_{N,n}(F^N) = 0$ and

$$\sup_{k_1 + \cdots + k_N = n} E[D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N])^2 = 0.$$

\[\square\]

**Corollary 3.7.** If

$$\sup_N \frac{\sup_l A_l}{\inf_l A_l} < \infty,$$

then $\{F^N\}$ is finite-dimensional.
Proof. Since
\[ \sum_{l=1}^{N} A_l \{ l^n - (l-1)^n \} \geq \inf_{l} A_l \sum_{l=1}^{N} \{ l^n - (l-1)^n \} = N^n \inf_{l} A_l, \]
we see that
\[ \alpha_{n,N}(F^N) \leq \sup_l A_l \frac{\inf_l A_l}{\inf_l A_l}. \]

3.5 Asymptotic Analysis of the Martingale Representation Error of a Discretization of Brownian Occupation Time

The sequence of Riemann sum approximations
\[ F^N := \sum_{i=1}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t, \quad N \in \mathbb{N} \] (3.10)
of the Brownian occupation time \( \int_0^T 1_{[0,\infty)}(W_s) \, ds \) is an interesting example where an explicit calculation is possible. We first prove that the sequence is not finite-dimensional in the sense of (3.4). However, it is rather difficult to check if the condition for Corollary 3.4 is satisfied. Instead, by a direct calculation the martingale representation error of the sequence is proven to be of order \( 1/2 \).

Proposition 3.8. The index \( \alpha_{n,N}(F^N) \) of the sequence (3.10) is not bounded.

Proof. First, we observe that
\[ A_l = \left( \sum_{i=l}^{N} i^{-n/2} \mathbb{E}[1_{[0,\infty)}(W_{t_i}) H_n(W_{t_i}/\sqrt{t_i})] \right)^2 \]
\[ = \left( \sum_{i=l}^{N} i^{-n/2} t_i^{1/2} n^{-1/2} \mathbb{E}[\delta_{0}(W_{t_i}) H_{n-1}(W_{t_i}/\sqrt{t_i})] \right)^2 \]
\[ = (2\pi n)^{-1} (H_{n-1}(0))^2 \left( \sum_{i=l}^{N} i^{-n/2} \right)^2. \]

Then, we now see that
\[ \alpha_{n,N}(F^N) = \frac{N^n \left( \sum_{i=1}^{N} i^{-n/2} \right)^2}{\sum_{l=1}^{N} \left( \sum_{i=l}^{N} i^{-n/2} \right)^2 \{ l^n - (l-1)^n \}}. \] (3.11)
First, we estimate the numerator of (3.11). We let \( n \geq 5 \). Then

\[
N^n \left( \sum_{i=1}^{N} i^{-n/2} \right)^2 = N^2 \left( \sum_{i=1}^{N} \left( \frac{i}{N} \right)^{-n/2} \right)^2 \geq N^2 \left( \int_{1/N}^{1} x^{-n/2} \, dx \right)^2 = N^2 \left\{ \frac{2}{n-2} \left( N^{(n-2)/2} - 1 \right) \right\}^2.
\]  

(3.12)

Next, the denominator is estimated as follows:

\[
\sum_{l=1}^{N} \left( \sum_{i=l}^{N} i^{-n/2} \right)^2 \{ l^n - (l-1)^n \}
\]

\[
= N^2 \sum_{l=1}^{N} \left( \sum_{i=l}^{N} \left( \frac{i}{N} \right)^{-n/2} \right)^2 \{ \left( \frac{l}{N} \right)^n - \left( \frac{l-1}{N} \right)^n \}
\]

\[
\leq N^2 \sum_{l=1}^{N} \left( \frac{2}{n-2} \left\{ \left( \frac{l}{N} \right)^{(2-n)/2} - 1 \right\} + \left( \frac{l}{N} \right)^{-n/2} \right)^2 \{ \left( \frac{l}{N} \right)^n - \left( \frac{l-1}{N} \right)^n \}
\]

\[
= N^2 \{ J_1^N + J_2^N + J_3^N \}
\]

where

\[
J_1^N := (n-2)^{-2} \sum_{l=1}^{N} \left\{ \left( \frac{l}{N} \right)^{(2-n)/2} - 1 \right\}^2 \left\{ \left( \frac{l}{N} \right)^n - \left( \frac{l-1}{N} \right)^n \right\},
\]

\[
J_2^N := \frac{2(n-2)^{-1}}{N} \sum_{l=1}^{N} \left\{ \left( \frac{l}{N} \right)^{1-n} - \left( \frac{l}{N} \right)^{-n/2} \right\} \left\{ \left( \frac{l}{N} \right)^n - \left( \frac{l-1}{N} \right)^n \right\}
\]

and

\[
J_3^N := \frac{1}{N^2} \sum_{l=1}^{N} \left( \frac{l}{N} \right)^{-n} \left\{ \left( \frac{l}{N} \right)^n - \left( \frac{l-1}{N} \right)^n \right\}.
\]

It is easy to see that \( \sup_N J_2^N < \infty \) and \( \lim_{N \to \infty} J_3^N = 0 \). Since \( J_1^N \) behaves like

\[
(n-2)^{-2} \int_{0}^{1} \left\{ x^{(2-n)/2} - 1 \right\}^2 x^n \, dx < \infty
\]

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as $N \to \infty$, it is also seen that $\sup_N J_i^N < \infty$. Therefore, there is a constant $C_n$ independent of $N$ but possibly dependent on $n$ such that

$$\sum_{l=1}^{N} \left( \sum_{i=l}^{N} i^{-n/2} \right)^2 \{l^n - (l-1)^n\} \leq N^2 C_n. \tag{3.13}$$

From (3.12) and (3.13), we see that $\sup_N \alpha_{n,N} = \infty$. \hfill \Box

Our main result in this subsection is the following.

**Theorem 3.9.** It holds that

$$\|1\text{-Mart.Err}(F_N)\|_{L^2} = O(N^{-1/2}).$$

**Proof.** By Theorem 2.1, we have

$$\|\text{Err}_N\|_{L^2}^2 = \sum_{l=1}^{N} \sum_{k=2}^{\infty} \mathbb{E} \left[ \mathbb{E} \left[ \sum_{i=1}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t H_k \left( \frac{\Delta W_i^N}{\sqrt{\Delta t}} \right) | \mathcal{G}_{l-1}^N \right]^2 \right] \tag{3.14}$$

$$= \sum_{l=1}^{N} \sum_{k=2}^{\infty} \frac{\Delta t^k}{k!} \mathbb{E} \left[ \mathbb{E} \left[ \sum_{i=1}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t | \mathcal{G}_{l-1}^N \right]^2 \right].$$

For $l \geq 2$, by the Hermite expansion in $L^2(\mathbb{R}, \mu_{t_{l-1}})$,

$$\mathbb{E} \left[ \sum_{i=l}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t | \mathcal{G}_{l-1}^N \right] = \sum_{n=0}^{\infty} \frac{(t_{l-1})^{n/2}}{\sqrt{n!}} \mathbb{E} \left[ \sum_{i=l}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t \right] H_n \left( \frac{W_{t_{l-1}}}{\sqrt{t_{l-1}}} \right),$$

and by Parseval’s identity we have

$$\mathbb{E} \left[ \mathbb{E} \left[ \sum_{i=l}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t | \mathcal{G}_{l-1}^N \right]^2 \right] = \sum_{n=0}^{\infty} \frac{(t_{l-1})^{n}}{n!} \mathbb{E} \left[ \sum_{i=l}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t \right]^2. \tag{3.15}$$

Note that (3.15) is also valid for $l = 1$ with the conventions $t_0 = 0$ and $t_0^1 = 1$. Plugging (3.15) into (3.14), we have

$$\|\text{Err}_N\|_{L^2}^2 = \sum_{l=1}^{N} \sum_{k=2}^{\infty} \sum_{n=0}^{\infty} \frac{\Delta t^k}{k!} \frac{(t_{l-1})^{n}}{n!} \mathbb{E} \left[ \sum_{i=l}^{N} 1_{[0,\infty)}(W_{t_i}) \Delta t \right]^2.$$
By the renumbering \((n + k, n) \mapsto (k, n)\), we have

\[
\| \text{Err}_N \|^2_{L^2} = \sum_{l=1}^{N} \sum_{k=2}^{\infty} \frac{1}{k!} \mathbb{E} \left[ \sum_{i=l}^{N} \mathbb{1}_{[0, \infty)}(W_{t_i}) \Delta t \right]^2 \sum_{n=0}^{k-2} \frac{k!}{(k-n)!n!} (\Delta t)^k (t_{l-1})^n,
\]

by keeping the conventions on \(t_0\). With a use of the binomial theorem,

\[
\| \text{Err}_N \|^2_{L^2} = \sum_{l=1}^{N} \sum_{k=2}^{\infty} \frac{1}{k!} \mathbb{E} \left[ \sum_{i=l}^{N} \mathbb{1}_{[0, \infty)}(W_{t_i}) \Delta t \right]^2 \times \left\{ (t_l)^k - (t_{l-1})^k - k(\Delta t)(t_{l-1})^{k-1} \right\}.
\]

Then, on one hand, for \(l \geq 1\) and \(k \geq 2\),

\[
\mathbb{E} \left[ \sum_{i=l}^{N} \mathbb{1}_{[0, \infty)}(W_{t_{i-1}}) \Delta t \right]^2 = \left\{ \sum_{i=l}^{N} \frac{\sqrt{(k-1)!}}{(t_i)^{k-1/2}} \mathbb{E} \left[ \delta_0(W_{t_i}) H_{k-1} \left( \frac{W_{t_i}}{\sqrt{t_i}} \right) \right] \Delta t \right\}^2
\]

\[
= \left\{ \sum_{i=l}^{N} \frac{\sqrt{(k-1)!}}{(t_i)^{k-1/2}} H_{k-1}(0) \frac{1}{\sqrt{2\pi t_i}} \Delta t \right\}^2
\]

\[
= k! \cdot \frac{H_{k-1}(0)^2}{2\pi k} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{n/2}} \right\}^2.
\]

By a similar argument, we find

\[
\mathbb{E} \left[ \mathbb{1}_{[0, \infty)}(W_T) H_k \left( \frac{W_T}{\sqrt{T}} \right) \right] = \frac{H_{k-1}(0)}{\sqrt{2\pi k}}
\]

and therefore

\[
\| \text{Err}_N \|^2_{L^2} = \sum_{k=2}^{\infty} Z_{N,k} \mathbb{E} \left[ \mathbb{1}_{[0, \infty)}(W_T) H_k \left( \frac{W_T}{\sqrt{T}} \right) \right]^2
\]

where

\[
Z_{N,k} := \sum_{l=1}^{N} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \left\{ (t_l)^k - (t_{l-1})^k - k(\Delta t)(t_{l-1})^{k-1} \right\}.
\]

(3.16)
On the other hand, by Lemma 3.10 below, we know that there exists a constant $K > 0$ such that

$$Z_{N,k} \leq \frac{K}{N}$$

for each $k = 2, 3, \ldots$ and $N = 3, 4, \ldots$. Hence we have

$$\|\text{Err}_N\|^2_{L^2} \leq \frac{2K}{N}\|1_{[0,\infty)}(W_T)\|^2_{L^2}.$$
Using this,

\[ Z_{N,k} = (\Delta t)^2 + 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \left( \frac{t_{l-1}}{t_i} \right)^{k/2} (\Delta t)^2 \]

\[- N(\Delta t)^2 - k \sum_{l=2}^{N} \left\{ \sum_{i=l}^{N} \frac{(t_{l-1})(k-1/2)}{(t_i)^{k/2}} \Delta t \right\}^2 \Delta t \]

\[ \leq 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \left( \frac{t_{l-1}}{t_i} \right)^{k/2} (\Delta t)^2 - k \sum_{l=1}^{N} \left\{ \sum_{i=l}^{N} \frac{(t_{l-1})(k-1/2)}{(t_i)^{k/2}} \Delta t \right\}^2 \Delta t. \]

\[ (3.18) \]

We observe that

\[ 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \left( \frac{t_{l-1}}{t_i} \right)^{k/2} (\Delta t)^2 \]

behaves like

\[ 2 \int_0^T \int_t^T \left( \frac{t}{s} \right)^{k/2} ds \, dt \]

and

\[ k \sum_{l=1}^{N} \left\{ \sum_{i=l}^{N} \frac{(t_{l-1})(k-1/2)}{(t_i)^{k/2}} \Delta t \right\}^2 \Delta t \]

behaves like

\[ k \int_0^T \left\{ \int_t^T \frac{T(t(k-1/2)}{s^{k/2}} ds \right\}^2 \, dt \]

as \( N \to \infty \) respectively. We note that

\[ 2 \int_0^T \int_t^T \left( \frac{t}{s} \right)^{k/2} ds \, dt = n \int_0^T \left\{ \int_t^T \frac{T(t(k-1/2)}{s^{k/2}} ds \right\}^2 \, dt = \begin{cases} T^2 \frac{k^2}{2} & \text{if } k = 2, \\ \frac{T^2}{2} & \text{if } k \geq 2. \end{cases} \]

Based on the observations, we estimate \( Z_{N,k} \) by separating it into two terms;

\[ Z_{N,k} \leq Z_{N,k}^1 + Z_{N,k}^2 \]

where

\[ Z_{N,k}^1 := 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \left( \frac{t_{l-1}}{t_i} \right)^{k/2} (\Delta t)^2 - 2 \int_0^T \int_t^T \left( \frac{t}{s} \right)^{k/2} ds \, dt, \]

\[ Z_{N,k}^2 := k \int_0^T \left\{ \int_t^T \frac{T(t(k-1/2)}{s^{k/2}} ds \right\}^2 \, dt - k \sum_{l=1}^{N} \left\{ \sum_{i=l}^{N} \frac{(t_{l-1})(k-1/2)}{(t_i)^{k/2}} \Delta t \right\}^2 \Delta t. \]

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We estimate each of them. Firstly, we have

\[ Z_{N,n} \leq 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N-1} \int_{t_{i-1}}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \left\{ \left( \frac{t_{i-1}}{t_i} \right)^{k/2} - \left( \frac{t_{i-2}}{t_{i+1}} \right)^{k/2} \right\} ds dt + 2 \sum_{l=2}^{N} \left( \frac{t_{l-1}}{t_N} \right)^{k/2} (\Delta t)^2 \]

\[ \leq 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N-1} \int_{t_{i-1}}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \left\{ \left( \frac{t_{i-1}}{t_i} \right)^{k/2} - \left( \frac{t_{i-2}}{t_{i+1}} \right)^{k/2} \right\} ds dt \]

\[ = 2(\Delta t)^2 \sum_{l=2}^{N} \sum_{i=l-1}^{N-1} \left\{ \left( \frac{l-1}{i} \right)^{k/2} - \left( \frac{l-2}{i} \right)^{k/2} \right\} \]

\[ + 2(\Delta t)^2 \sum_{l=2}^{N} \sum_{i=l-1}^{N-1} \left\{ \left( \frac{l-2}{i+1} \right)^{k/2} - \left( \frac{l-2}{i} \right)^{k/2} \right\} \]

\[ + 2 \sum_{l=2}^{N} \left( \frac{t_{l-1}}{t_N} \right)^{k/2} (\Delta t)^2. \]

(3.19)

By a bit of algebra, the last term in (3.19) is seen to be

\[ 2(\Delta t)^2 \sum_{l=2}^{N} \left\{ 1 + \left( \frac{l-1}{i} \right)^{k/2} \right\}, \]

(3.20)

which is bounded above by \(4T^2/N\).

Next, we estimate \(Z_{N,k}^2\). We set

\[ I = \sum_{l=1}^{N} \int_{t_{i-1}}^{t_i} t^{k-1} \left\{ \int_{t_i}^{T} \frac{ds}{s^{k/2}} \right\}^2 dt - \sum_{l=1}^{N} \int_{t_{i-1}}^{t_i} t^{k-1} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 dt \]

and

\[ II = \sum_{l=1}^{N} \int_{t_{i-1}}^{t_i} t^{k-1} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 dt - \sum_{l=1}^{N} \int_{t_{i-1}}^{t_i} t^{k-1} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 dt. \]
Note that $Z_{N,k}^2 = k(I + II)$. For $t_{l-1} \leq t \leq t_l$, $l = 1, \cdots, N$, we have

$$
\int_t^T \frac{ds}{s^{k/2}} - \sum_{i=l+1}^{N} \frac{\Delta t}{(t_i)^{k/2}} = \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left( \frac{1}{s^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) ds + \int_t^{t_l} \frac{ds}{s^{k/2}} - \frac{\Delta t}{(t_l)^{k/2}} \geq 0, \tag{3.21}
$$

and

$$
\sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left( \frac{1}{s^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) ds \leq \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left( \frac{1}{(t_i)^{k/2}} - \frac{1}{(t_{i-1})^{k/2}} \right) ds = \Delta t \left( \frac{1}{(t_i)^{k/2}} - \frac{1}{(t_{i-1})^{k/2}} \right).
$$

Combining these two, we have

$$
\left\{ \int_t^T \frac{ds}{s^{k/2}} \right\}^2 - \left\{ \sum_{i=l+1}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \leq \int_t^{t_l} \frac{ds}{s^{k/2}} \left( \int_t^T \frac{ds}{s^{k/2}} + \sum_{i=l+1}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right) \leq 2 \int_t^{t_l} \frac{ds}{s^{k/2}} \left( \int_t^T \frac{ds}{s^{k/2}} \right)
$$

$$
= \left\{ \frac{4}{k-2} \left( t^{1-\frac{k}{2}} - T^{1-\frac{k}{2}} \right) \right\} \int_t^{t_l} \frac{ds}{s^{k/2}} \leq \frac{4}{k-2} \int_t^{t_l} \frac{ds}{s^{k/2}} \text{ if } k \geq 3,
$$

$$
= \left\{ 2 \int_t^{t_l} \frac{ds}{s} \log \frac{T}{t} \right\} \text{ if } k = 2.
$$

Then for $k \geq 3$,

$$
I \leq \frac{4}{k-2} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_t^{t_l} \left( \frac{1}{s} \right)^{k/2} ds dt
$$

$$
\leq \frac{4}{k-2} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_t^{t_l} ds dt = \frac{2}{k-2} \sum_{i=1}^{N} (t_l - t_{i-1})^2 = \frac{2}{k-2} \frac{T^2}{N}, \tag{3.22}
$$

and for $k = 2$, we have

$$
I \leq 2 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_t^{t_l} s^{-1} \log \frac{T}{t} ds dt \leq 2 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_t^{t_l} ds dt \log \frac{T}{t} dt
$$

$$
\leq 2 \Delta t \sum_{i=1}^{N} \left\{ \Delta t \log T - \left[ t \log t - t \frac{t_{i-1}}{t_{i-1} + \Delta t} \right] \right\} = \frac{2T^2}{N}. \tag{3.23}
$$
Now we turn to the estimate of $II$. By (3.21), for $k \geq 3$,

$$II \leq \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \{t^{k-1} - (t_{l-1})^{k-1}\} \left(\int_{t}^{T} \frac{ds}{sk/2}\right)^2 dt$$

$$\leq \frac{4}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \{t^{k-1} - (t_{l-1})^{k-1}\} t^{2-k} dt$$

$$= \frac{4(k-1)}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t} \left(\frac{s}{t}\right)^{k-2} ds dt$$

$$\leq \frac{4(k-1)}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t} ds dt = \frac{2(k-1)}{(k-2)^2} \frac{T^2}{N}. \quad (3.24)$$

For $k = 2$, we have

$$II \leq \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} (t - t_{l-1}) \left(\int_{t}^{T} \frac{ds}{s}\right)^2 dt$$

$$\leq \Delta t \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left(\frac{T}{t}\right)^2 dt = \frac{2T^2}{N}. \quad (3.25)$$

By (3.22), (3.23), (3.24) and (3.25), we have

$$Z_{N,k}^2 \leq \frac{5T^2}{N}. \quad (3.26)$$

Combining (3.20) and (3.26), we obtained (3.17).

Remark 3.11. A result by Ngo-Ogawa ([10], Theorem 2.2.) tells us that the sequence of processes

$$\left\{n^{3/4} \left(\frac{1}{N} \sum_{i=0}^{[Nt]} \mathbb{1}_{[0,\infty)}(X_{i/N}) - \int_{0}^{t} \mathbb{1}_{[0,\infty)}(X_{s}) ds\right)\right\}_{t \geq 0}$$

is tight for a diffusion $X = (X_t)_{t \geq 0}$ although their results are more general. Moreover they say that this is optimal in $L^2$-sense in the case where $X$ is the standard Brownian motion (see [10], Proposition 2.3).

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