Holographic Principle during Inflation and
a Lower Bound on Density Fluctuations

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ABSTRACT. We apply the holographic principle during the inflationary stage of our universe. Where necessary, we illustrate the analysis in the case of new and extended inflation which, together, typify generic models of inflation. We find that in the models of extended inflation type, and perhaps of new inflation type also, the holographic principle leads to a lower bound on the density fluctuations.

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The holographic principle is simple and yet profound: it implies that the degrees of freedom in a spatial region can all be encoded on its boundary, with a density not exceeding one degree of freedom per Planck area. Accordingly, the entropy in a spatial region does not exceed its boundary area in Planck units. Also, for example, the physics of the bulk is describable by the physics on the boundary. This has, indeed, been shown recently for some anti-de Sitter spaces.

Fischler and Susskind have proposed how to apply the holographic principle in cosmology, and showed that our universe has evolved in the past in accordance with this principle and will continue to do so if it is flat or open. This principle has recently been applied in the context of pre-big bang scenario in, and in the context of singularity problem in.

Our universe is believed to have gone through an inflationary stage in the past. Among other things, an enormous amount of entropy is released into the universe at the end of inflation. Is this amount of entropy consistent with the holographic principle?

We apply the holographic principle during the inflationary stage and study this issue. The relevant details are often model dependent. Hence, where necessary, we illustrate the analysis in the case of new and extended inflation where the reheating and entropy production is due to inflaton decay and bubble wall collisions respectively. These two models, together, typify generic models of inflation. For more details, see for new and for extended inflation.

We find that in the models of extended inflation type, and perhaps of new inflation type also, the holographic principle leads to an upper bound on the inflation factor. This, in turn, leads to a lower bound on the density fluctuations in the universe, which seed the large scale structure formation. These consequences, although obtained explicitly for new and extended inflation, are expected to be valid generally. Considering the approximations involved, the lower bound on density fluctuations we find is remarkably close to the observed value if inflation takes place when the temperature of the universe is. To our knowledge, this is the first instance where a lower bound on the density fluctuations is obtained theoretically. Such a bound, if established rigorously and in a model independent way, could be taken as a prediction of the holographic principle.

In the models of new inflation type, the inflaton decay is often modelled by that of a massive scalar field interacting with other fields.
graphic principle is automatically satisfied in these models with no further consequences.

2. In the context of cosmology, the Fischler-Susskind (FS) proposal for the application of holographic principle is as follows. Let $\Gamma$ be a spherical spatial region of coordinate size $r$ with boundary $B$ and let $L$ be the light-like surface formed by past light rays from $B$ towards the center of $\Gamma$. Then according to FS proposal, the holographic principle implies that the entropy passing through $L$ never exceeds the area of $B$ \[4\].

Let the metric of the $3+1$ dimensional homogeneous isotropic flat universe be given by the line element

$$ds^2 = -dt^2 + R^2(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) .$$  \(1\)

Let $\rho$, $p$, and $T$ be the density, pressure, and temperature of the universe respectively. Also, let

$$r_H = \int_0^t \frac{dt}{R} \quad \text{and} \quad d_H = r_H R$$

be the coordinate and the physical size of the horizon respectively. The (constant) comoving and the (varying) physical entropy densities, $\sigma$ and $s$ respectively, are then given by

$$\sigma = \frac{\rho + p}{T} R^3 \equiv sR^3 .$$  \(2\)

The entropy $S$ contained within $\Gamma$, and the area $A$ of the boundary $B$ are

$$S = \frac{4\pi}{3} s d_H^3$$ \(3\)

$$A = 4\pi d_H^2 .$$ \(4\)

According to FS proposal, the holographic principle implies that

$$S \lesssim A .$$ \(5\)

The approximate order of magnitude values of various quantities in our universe at different epochs are tabulated below \[10\]. All quantities, here and in the following, are in planck units, unless mentioned otherwise.
$T_0 = 2.75K$ is the present temperature of the universe and $T_{pl} = 1.2 \times 10^{19} GeV$ is the planck temperature. Note that the constant comoving entropy density is given, for our choice of $r_H(\text{present}) = 1$, by

$$\sigma_0 \simeq 10^{87}.$$  

It is clear from the above table that the holographic principle is obeyed in our universe from planckian time upto the present. It is obeyed indefinitely in the future too if our universe is flat or open [4].

3. Our universe is believed to have gone through an inflationary stage at a temperature $T_b$ (usually taken to be $10^{14} GeV \simeq 10^{-5}$ in planck units) [11]. Generically, a small, causally connected patch of the universe inflates from say time $t = 0$ to $t = t_e$. The scale factor $R$ grows by a factor of $e^N$ and the universe supercools. At the end of inflation, the universe reheats to a temperature $T_R \lesssim T_b$, releasing an enormous amount of entropy. Is the amount of entropy produced consistent with the holographic principle?

For the region with $r_H = 1$, the natural values of the entropy $S$ and the area $A$ at the beginning of inflation are

$$S_b \approx \sigma_b T_b^{3/4} T_b^{-3} \approx (10^{15})$$
$$A_b \approx t_b^2 \approx T_b^{-4} (10^{20}),$$

where the numbers in the bracket are the values if $T_b = 10^{-5}$. So, $S_b < A_b$ and the holographic principle is obeyed at the beginning of inflation.

At the end of inflation we get, for the region with $r_H = 1,$

$$S_e \approx \sigma_e T_b^{-3}$$
$$A_e \approx T_b^{-4}$$
where we have taken $T_R \simeq T_b$. In order to account for the observed entropy of the universe, we require [7, 10]

$$\sigma_e \gtrsim \sigma_0 \simeq 10^{87}.$$  

Hence,

$$e^N \gtrsim \sigma_0^\frac{4}{9} T_b \left(10^{24}\right).$$

This is the required 60 e-folding of the inflationary scenario [7]. For these values,

$$\frac{S_e}{A_e} \simeq e^N T_b \left(10^{19}\right)$$

which clearly violates the holographic principle. The required entropy production will not violate the holographic principle, as applied above, only if $T_b \lesssim 10^{-15} \simeq 10^4 GeV$ with $e^N T_b \lesssim 1$. However, such a low value is unsatisfactory for other reasons [10].

More importantly, the above application of the holographic principle is naive and is precisely the one Fischler and Susskind admonished against. The spatial region $\Gamma$ and the boundary $B$, which evolve along the light-like surface $L$ into the present ones, are marked at the end of inflation, when $T = T_R \simeq T_b$, by (see the table)

$$r_H \simeq 3 \times 10^{-30} T_b^{-1} \left(10^{-25}\right).$$

For such a region, we get

$$\frac{S_e}{A_e} \simeq r_H T_b e^N \simeq 10^{-30} e^N.$$

The holographic principle is then obeyed if the inflation factor

$$e^N \lesssim 10^{30},$$

which is sufficient to solve all the problems in Guth’s original proposal for inflation [7].

4. Typically, however, $e^N$ is of the order of $10^{100} - 10^{300}$ in extended inflation [8] and of the order of $10^{10^3} - 10^{10^7}$ in new inflation [8, 10]. So, the above bound is a severe constraint on inflationary models and achieving it is
likely to be unnatural, if possible at all. Also, the above application of holographic principle is in the era immediately following the entropy production, and not when the entropy is actually being produced.

The entropy is produced at the end of inflation during the reheating process and the universe reheats to a temperature $T_R \lesssim T_b$, where $T_b$ is the temperature at the beginning of inflation. The physical entropy density $s_R$ during the entropy production can be taken, on an average, to be

$$s_R \simeq T_R^3.$$  

The holographic principle, applied during this process to a suitable region, to be identified below, of physical size $\simeq d$ (hence of volume $d^3$ with boundary area $d^2$), implies that

$$S_R \lesssim A_R \quad \rightarrow \quad T_R^3 d \lesssim 1. \quad (6)$$

The actual details of the reheating and the entropy production are model dependent \cite{10,11,12}. However, the relevant physical process falls broadly in one of the two categories where the reheating and the entropy production are due to (1) bubble wall collisions - typified by extended inflation \cite{3}, or (2) the decay of the ‘slow rolling’ inflaton - typified by new inflation \cite{8}. We now identify the size $d$ in each of these cases.

(1) The true vacuum bubbles nucleate during inflation, expand with the speed of light, and eventually percolate the universe, thus ending the inflation. Upon percolation, the bubble walls collide and release the energy and entropy into the interior of the bubbles, thereby reheating the universe to a temperature $T_R \lesssim T_b$. Typically, the reheating time is of the order of the time required for light to cross the bubble \cite{12}. Thus, it is natural to apply the holographic principle to the interior of each bubble. On an average, the time between the bubble nucleation and collision is less than or of the order of $t_e$, the duration of inflation. The interior of the bubble is in a true vacuum state and, thus, its size $d \simeq t_e$. With no further condition on $T_R$, the holographic principle implies that

$$T_R^3 t_e \lesssim 1. \quad (7)$$

(2) The inflaton slowly rolls down to its minimum and begins to oscillate, thus ending the inflation. The oscillating inflaton decays into other particles, releasing the energy and entropy into the universe and, thereby, reheating
it to a temperature $T_R \lesssim T_b$. The entropy is produced simultaneously and everywhere in the inflated region. Thus, it is natural to apply the holographic principle to any region covered by a light ray starting from a point and travelling for a time $\simeq t_R$, where $t_R$ is the duration of reheating. The universe is in a true vacuum state during reheating and, thus, the size of this region $d \simeq t_R$.

The universe reheats within a few Hubble time $H_e^{-1} \simeq t_e$ at the end of inflation. Taking $t_R \lesssim t_e$, and with no further condition on $T_R$, the holographic principle implies the relation (7), same as in the previous case.

Often, the inflaton decay is modelled by that of a massive scalar field interacting with other fields - scalars, fermions, photons, gravitons, etc. [10, 11]. In typical models, the reheating time $t_R \simeq \gamma_d^{-1}$, where $\gamma_d$ is the decay rate which, for $T_b = 10^{14}$ GeV, is $\mathcal{O}(10^{-6} - 10^{-12})$ in planck units depending on the model and the decay products. Moreover, the reheating temperature $T_R$ is related to $\gamma_d$ by

$$T_R \simeq \sqrt{\gamma_d}$$

in planck units. Note that, in these models, the reheating time $t_R$ (the reheating temperature $T_R$) has no relation to the Hubble time $t_e$ at the end (the temperature $T_b$ at the beginning) of inflation. The holographic relation (7) then implies

$$\sqrt{\gamma_d} \lesssim 1,$$

a condition well satisfied in these models.

5. We now explore the consequences of the relation (7). The reheating temperature $T_R \lesssim T_b$ depends only on at what temperature the inflation sets in. The duration of inflation $t_e$ then satisfies an upper bound given by (7). Such an upper bound on $t_e$ can be expected, among other things, to lead to an upper bound on the inflation factor $e^N$. This is simply because longer the duration of inflation, larger is the expansion factor.

An upper bound on $e^N$ can, in turn, be expected to lead to a lower bound on the density fluctuations in the universe, which seed the large scale structure formation. This is because, essentially the inflation dampens the quantum fluctuations of the fields, which reenter as density fluctuations in the later era. Hence, larger the inflation factor, more the damping of quantum fluctuations, and thus smaller the resulting density fluctuations.
Although the above physical reasoning is direct and simple, the actual calculations of $t_e$ and of the density fluctuations are quite involved. Also, to our knowledge, there is no model independent formula which relates the density fluctuations to the duration or the amount of inflation. Hence, we illustrate these consequences explicitly in the context of new and extended inflation. However, following the above reasoning, they are expected to be valid generally.

5 a. Consider the duration of inflation $t_e$ and the expansion factor $e^N$. (For details about various expressions used below, see [3, 10, 11] for new and [9, 12, 13, 14] for extended inflation.)

**New Inflation:** Let the inflaton potential be

$$V = V_0 - \frac{\lambda}{n} \phi^n, \quad n \geq 4,$$

where $V_0 \equiv \frac{3H^2}{8\pi} = M^4 \simeq T_b^4$, and $\lambda$ is a coupling constant. Equation (7) implies that the inflation factor $e^N$ is restricted by an upper bound given by

$$N \simeq H_b t_e < \left( \frac{T_b}{T_R} \right)^3 T_b^{-1}.$$

Note that with $T_R \approx T_b = 10^{14} \text{GeV}$ and $T_R \approx 0.1T_b$, we have $N \lesssim 10^8$, which conforms well with the amount of inflation occurring in these models.

The duration of inflation $t_e$ is related to the coupling constant $\lambda$ by

$$t_e \simeq 4\pi^2 H_b^{\frac{2n}{3}} \left( \frac{3}{8\pi^2 \lambda} \right)^{\frac{1}{2}}.$$

Equation (7) then implies a lower bound

$$\lambda > T_R^{\frac{3n}{2}} H_b^{4-\frac{3n}{2}}.$$

**Extended Inflation:** The model is specified by a parameter $\omega = 10 - 20 \lesssim 25$. Equation (7) implies that the inflation factor $e^N$ is restricted by an upper bound

$$e^N \simeq t_e^{\omega+\frac{1}{2}} < T_R^{-3(\omega+\frac{1}{2})}.$$

Note that with $T_R \approx T_b = 10^{14} \text{GeV}$, and with $\omega = 10$, we have $e^N \lesssim 10^{160}$, which conforms well with the amount of inflation occurring in these models.
5 b. Consider the density fluctuations on a scale $\lambda_0 (\simeq 10^{60} \text{ for the horizon})$ today. Let $T_0 = 2.75K$ be the present temperature of the universe.

**New Inflation:** The inflaton potential is given by (9). The density fluctuations are then given by

$$\frac{\delta \rho}{\rho} \simeq A_{ni} H_b^{\frac{n-4}{3}} \lambda_0^{\frac{1}{n-2}}$$  \hspace{1cm} (11)

where $n \geq 4$ and

$$A_{ni} \simeq \frac{16}{3} \left( \frac{2}{3} \ln \frac{H_b \lambda_0 T_0}{T_R} \right)^{\frac{n-1}{n-2}} .$$

With $H_b \simeq M^2 \simeq T_b^2$, equation (10) then implies a lower bound on the density fluctuations:

$$\frac{\delta \rho}{\rho} \gtrsim A_{ni} \left( \frac{T_R}{T_b} \right)^{\frac{3n}{2(n-2)}} T_b^{\frac{n-2}{2(n-2)}} .$$  \hspace{1cm} (12)

For $T_b = 10^{14}GeV, T_R \simeq 0.1T_b$, and $n = 4$, $A_{ni} \simeq \mathcal{O}(10^2)$ and the above bound gives

$$\frac{\delta \rho}{\rho} \gtrsim \mathcal{O}(10^{-6}) .$$

Considering the approximations involved, the above lower bound on the density fluctuations implied by equation (7) is remarkably close to the observed value $\simeq 10^{-6}$ if inflation takes place at $T_b \simeq 10^{14}GeV$.

**Extended Inflation:** The density fluctuations are given by

$$\frac{\delta \rho}{\rho} \simeq A_{ei} \left( T_b \lambda_0 \sqrt{2\omega + 1} \right)^{\frac{4}{2\omega - 1}} (2\omega + 1)^{\frac{3}{2}} t_e^{\frac{2\omega + 1}{2\omega - 1}}$$  \hspace{1cm} (13)

where

$$A_{ei} \simeq \frac{\sqrt{\pi}}{3} \left( \frac{8\pi}{9} \right)^{\frac{3}{2\omega - 1}} \left( \frac{6\omega + 9}{6\omega + 5} \right)^{\frac{2\omega + 3}{2\omega - 1}} .$$

With $\lambda_0 \simeq 10^{60}, T_0 \simeq 10^{-32}$, and $\omega \simeq 10$, equation (11) then implies a lower bound on the density fluctuations:

$$\frac{\delta \rho}{\rho} \gtrsim 10^8 A_{ei} T_R^{\frac{3(2\omega + 1)}{2\omega - 1}} .$$  \hspace{1cm} (14)
For $T_R \simeq T_b \simeq 10^{14} GeV$ and $\omega \simeq 10$, $A_{nl} \simeq \mathcal{O}(1)$ and the above bound gives
\[ \frac{\delta \rho}{\rho} \gtrsim \mathcal{O}(10^{-7}) . \]

Considering the approximations involved, the above lower bound on the density fluctuations implied by equation (7) is remarkably close to the observed value $\simeq 10^{-6}$ [15] if inflation takes place at $T_b \simeq 10^{14} GeV$.

6. We conclude with a few remarks. We have shown, in the case of new and extended inflation which, together, typify generic models of inflation, that equation (7) leads to an upper bound on the inflation factor. This, in turn, leads to a lower bound on the density fluctuations. As discussed before, these consequences are expected to be valid generally. Considering the approximations involved, the lower bound on the density fluctuations, obtained in specific cases here, is remarkably close to the observed value if inflation takes place at $T \simeq 10^{14} GeV$. To our knowledge, this is the first instance where a lower bound on the density fluctuations is obtained theoretically. Such a bound, if established rigorously and in a model independent way, could be taken as a prediction of the holographic principle.

Equation (7), which led to these bounds, arises as a consequence of the holographic principle in those models, typified by extended inflation, where the reheating and the entropy production are due to bubble wall collisions. The reheating temperature is taken to depend only on at what temperature the inflation sets in. Thus, in such models, the lower bound on density fluctuations is a consequence of the holographic principle.

Equation (7) also arises as a consequence of the holographic principle in those models, typified by new inflation, where the reheating and the entropy production are due to the decay of the ‘slow rolling’ inflaton decay, if the reheating time is of the order of a few Hubble time at the end of inflation and if the reheating temperature depends only on at what temperature the inflation sets in. Then, in these models also, the lower bound on the density fluctuations is a consequence of the holographic principle.

Often the inflaton decay, relevent in the models of new inflation type, is modelled by that of a massive scalar field interacting with other fields. In such models, the reheating time (the reheating temperature) has no relation to the Hubble time at the end (the temperature at the beginning) of inflation.
Typically, the holographic principle is automatically satisfied in these models with no further consequences.

Perhaps, it is that such models of inflaton decay may be specific possibilities only, while the generic possibilities have the reheating time (the reheating temperature) of the order of the Hubble time at the end (the temperature at the beginning) of inflation. If so then, in the models of new inflation type also, the holographic principle is likely to lead to a lower bound on the density fluctuations.

Conversely, and just as likely, the inflaton decay models are the generic models of reheating. Moreover, it may also be that similar generic models exist for bubble wall collisions also, in which the reheating time (the reheating temperature) has no relation to the Hubble time at the end (the temperature at the beginning) of inflation. If so then, in the models of extended inflation type also, the holographic principle is likely to be automatically satisfied with no further consequences. It is desirable to settle this issue definitively, but it is beyond the scope of the present work.

In closing, we mention an interesting application of the present analysis: Note that by an appropriate coordinate transformation [10], the inflating universe can be cast as a static de Sitter one. One can then translate the present analysis and compare the results with those obtained for some anti de Sitter spaces in [3]. This might provide some insights into the holographic principle in static universes.

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