A \((p, q)\)-ANALOG OF POLY-EULER POLYNOMIALS AND SOME RELATED POLYNOMIALS

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We introduce a \((p, q)\)-analog of the poly-Euler polynomials and numbers by using the \((p, q)\)-poly-logarithm function. These new sequences are generalizations of the poly-Euler numbers and polynomials. We present several combinatorial identities and properties of these new polynomials and also show some relations with \((p, q)\)-poly-Bernoulli polynomials and \((p, q)\)-poly-Cauchy polynomials. The \((p, q)\)-analogs generalize the well-known concept of \(q\)-analog.

1. Introduction

The Euler numbers are defined by the generating function

\[
\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.
\]

The sequence \((E_n)_n\) counts the numbers of alternating \(n\)-permutations. An \(n\)-permutation \(\sigma\) is alternating if the \(n - 1\) differences \(\sigma(i+1) - \sigma(i)\) for \(i = 1, 2, \ldots, n - 1\) have alternating signs. Thus, \((1324)\) and \((3241)\) are alternating permutations (cf. [10]).

The Euler polynomials are given by the generating function

\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]

Note that \(E_n = 2^n E_n(1/2)\).

Numerous kinds of generalizations of these numbers and polynomials can be found in the literature (see, e.g., [39]). In particular, we are interested in the poly-Euler numbers and polynomials (cf. [12, 15, 16, 32]).

The poly-Euler polynomials \(E_n^{(k)}(x)\) are defined by the following generating function:

\[
\frac{2\text{Li}_k(1-e^{-t})}{1+e^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}, \quad k \in \mathbb{Z},
\]

where

\[
\text{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}
\]

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is the $k$th polylogarithm function. Note that if $k = 1$, then $\text{Li}_1(t) = -\log(1 - t)$. Hence,

$$E_n^{(1)}(x) = E_{n-1}(x) \quad \text{for} \quad n \geq 1.$$ 

It is also possible to define the poly-Bernoulli and poly-Cauchy numbers and polynomials by using the $k$th polylogarithm function. Thus, in particular, the poly-Bernoulli numbers $B_n^{(k)}$ were introduced by Kaneko [17] by using the following generating function:

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z}.$$ 

If $k = 1$, then we get

$$B_n^{(1)} = (-1)^n B_n \quad \text{for} \quad n \geq 0,$$

where $B_n$ are the Bernoulli numbers. Remember that the Bernoulli numbers $B_n$ are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$ 

The poly-Bernoulli numbers and polynomials were studied in several papers; among other references, see [3, 4, 7, 8, 21, 22, 25–27].

The poly-Cauchy numbers of the first kind $c_n^{(k)}$ were introduced by the first author in [19]. They are defined as follows:

$$c_n^{(k)} = \int_0^1 \cdots \int_0^1 (t_1 \cdots t_k)_n \ dt_1 \cdots dt_k,$$

where

$$(x)_n = x(x - 1) \cdots (x - n + 1)(n \geq 1) \quad \text{with} \quad (x)_0 = 1.$$ 

Moreover, the exponential generating function has the form

$$\text{Li}_k(\ln(1 + t)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z},$$

where

$$\text{Li}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}$$

is the $k$th polylogarithm factorial function. For more properties of these numbers see, e.g., [8, 20–24]. If $k = 1$, then we recover the Cauchy numbers $c_n^{(1)} = c_n$. The Cauchy numbers $c_n$ were introduced in [10] by the generating
function
\[ \frac{t}{\ln(1 + t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}. \]

The sequences presented above have been recently generalized in [21] by using the \( k \)th \( q \)-polylogarithm function and Jackson’s integral. In particular, the \( q \)-poly-Bernoulli numbers are defined as follows:
\[
\frac{\text{Li}_{k,q}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z}, \quad n \geq 0, \quad 0 \leq q < 1,
\]
where
\[
\text{Li}_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}
\]
is the \( k \)th \( q \)-polylogarithm function (cf. [29]) and
\[
[n]_q = \frac{1 - q^n}{1 - q}
\]
is the \( q \)-integer (cf. [39]). Note that
\[
\lim_{q \to 1} [x]_q = x, \quad \lim_{q \to 1} B_{n,q}^{(k)} = B_{n}^{(k)}, \quad \text{and} \quad \lim_{q \to 1} \text{Li}_{k,q}(x) = \text{Li}_{k}(x).
\]
The \( q \)-poly-Cauchy numbers of the first kind \( c_{n,q}^{(k)} \) are defined by using Jackson’s \( q \)-integral (cf. [1]) as follows:
\[
c_{n,q}^{(k)} = \int_0^1 \cdots \int_0^1 (t_1 \cdots t_k)_n d_q t_1 \cdots d_q t_k,
\]
where
\[
\int_0^x f(t) d_q t = (1 - q)x \sum_{n=0}^{\infty} f(q^n x)q^n.
\]
Moreover, its exponential generating function is
\[
\text{Lif}_{k,q}(\ln(1 + t)) = \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{t^n}{n!}, \quad k \in \mathbb{Z},
\]
where
\[
\text{Lif}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! [n + 1]_q^k}
\]
is the $k$th $q$-polylogarithm factorial function (cf. [18, 21]). Note that
\[
\lim_{q \to 1} c_{n,q}^{(k)} = c_n^{(k)} \quad \text{and} \quad \lim_{q \to 1} \text{Li}_{k,q}(t) = \text{Li}_k(t).
\]

In the present paper, we introduce a $(p, q)$-analog of the poly-Euler polynomials by the formula
\[
\frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} e^{xt} = \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x) \frac{t^n}{n!}, \quad k \in \mathbb{Z},
\]
where $p$ and $q$ are real numbers such that $0 < q < p \leq 1$ and
\[
\text{Li}_{k,p,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_{p,q}^k}
\]
is an extension of the $q$-polylogarithm function, which is called the $(p, q)$-polylogarithm function. The polynomials
\[
E_{n,p,q}^{(k)}(0) := E_{n,p,q}^{(k)}
\]
are called $(p, q)$-poly-Euler numbers. The polynomial
\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}
\]
is the $n$th $(p, q)$-integer (cf. [13, 14, 37]). It was introduced in the context of set partition statistics (cf. [40]). Note that
\[
\lim_{p \to 1} [n]_{p,q} = [n]_q \quad \text{and} \quad \lim_{p \to 1} \text{Li}_{k,p,q}(t) = \text{Li}_{k,q}(t).
\]

As already indicated, the $(p, q)$-analogs serve as an extension of the $q$-analogs and coincide with the latter in the limit as $p$ tends to 1. The $(p, q)$-calculus was studied in [9] in connection with quantum mechanics. The properties of the $(p, q)$-analogs of the binomial coefficients were studied in [11]. The $(p, q)$-analogs of hypergeometric series, special functions, Stirling numbers and their generalizations, Hermite polynomials, and Volkenborn integration have been studied earlier; see, e.g., [2, 14, 30, 31, 33, 34, 36, 38].

The present paper is split in two parts. In Section 2, we show several combinatorial identities for the $(p, q)$-poly-Euler polynomials. In particular, Theorem 2 shows a relationship between the $(p, q)$-poly-Euler polynomials and the classical Euler polynomials. Some of them involve the classical Euler polynomials and another special numbers and polynomials, such as the Stirling numbers of the second kind, Bernoulli polynomials of order $s$, etc. In Section 3, we introduce the $(p, q)$-poly-Bernoulli polynomials and $(p, q)$-poly-Cauchy polynomials of both kinds. We also generalize some well-known identities for the classical Bernoulli and Cauchy numbers and polynomials.

2. Some Identities for the Poly-Euler Polynomials

In this section, we present several identities for the $(p, q)$-poly-Euler polynomials. In particular, Theorem 2 shows a relationship between the $(p, q)$-poly-Euler polynomials and the classical Euler polynomials.

It is possible to give the first values of the $(p, q)$-polylogarithm function for $k \leq 0$. Thus,
\[
\text{Li}_{0,p,q}(x) = \frac{x}{1 - x},
\]
\[
\text{Li}_{-1,p,q}(x) = \frac{x}{(1 - px)(1 - qx)},
\]

\[
\text{Li}_{-2,p,q}(x) = \frac{x(1 + pqx)}{(1 - p^2x)(1 - q^2x)(1 - pqx)},
\]

\[
\text{Li}_{-3,p,q}(x) = \frac{x\left(p^3q^3x^2 + 2p^2qx + 2pq^2x + 1\right)}{(1 - p^3x)(1 - q^3x)(1 - p^2qx)(1 - pq^2x)}.
\]

In general, the \((p, q)\)-polylogarithm function for \(k \leq 0\) is a rational function. Indeed, let \(k\) be a nonnegative integer. Then

\[
\text{Li}_{-k,p,q}(x) = \sum_{n=1}^{\infty} \left[ n \right]_{p,q}^{-k} x^n = \sum_{n=1}^{\infty} \left[ n \right]_{p,q}^k x^n = \sum_{n=1}^{\infty} \left( \frac{p^n - q^n}{p - q} \right)^k x^n
\]

\[
= \frac{1}{(p - q)^k} \sum_{n=1}^{\infty} \sum_{l=0}^{k} \binom{k}{l} p^n (-q^n)^{k-l} x^n = \frac{1}{(p - q)^k} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \frac{p^l q^{k-l}}{1 - p^l q^{k-l}} x.
\]

Note that, from (3), we conclude that \(\left\{ E_{n,p,q}(x) \right\}_{n \geq 0}\) is an Appel sequence [35]. Therefore, we have the following basic relations:

**Theorem 1.** If \(n \geq 0\) and \(k \in \mathbb{Z}\), then

\(i\) \quad \(E_{n,p,q}^{(k)}(x) = \sum_{i=0}^{n} \binom{n}{i} E_{i,p,q}^{(k)} x^{n-i},\)

\(ii\) \quad \(E_{n,p,q}^{(k)}(x + y) = \sum_{i=0}^{n} \binom{n}{i} E_{i,p,q}^{(k)}(x) y^{n-i},\)

\(iii\) \quad \(E_{n,p,q}^{(k)}(m x) = \sum_{i=0}^{n} \binom{n}{i} E_{i,p,q}^{(k)}(x) (m - 1)^{n-i} x^{n-i}, m \geq 1,\)

\(iv\) \quad \(E_{n,p,q}^{(k)}(x + 1) - E_{n,p,q}^{(k)}(x) = \sum_{i=0}^{n-1} \binom{n}{i} E_{i,p,q}^{(k)}(x).\)

**Theorem 2.** If \(n \geq 1\), then

\[
E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{\infty} \frac{1}{[\ell + 1]_{p,q}} \sum_{j=0}^{\ell+1} \binom{\ell + 1}{j} (-1)^j E_n(x - j).
\]

**Proof.** It follows from (1) and (3) that

\[
\frac{2 \text{Li}_{k,p,q}(1 - e^{-t}) e^t}{1 + e^t} = \sum_{\ell=0}^{\infty} \frac{(1 - e^{-t})^{\ell+1}}{[\ell + 1]_{p,q}} 2 e^t.
\]
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\[
\begin{align*}
&= \sum_{\ell=0}^{\infty} \frac{1}{[\ell + 1]_p^k} \sum_{j=0}^{\ell+1} \binom{\ell + 1}{j} (-1)^j \frac{2e^{(x-j)t}}{1 + e^t} \\
&= \sum_{\ell=0}^{\infty} \frac{1}{[\ell + 1]_p^k} \sum_{j=0}^{\ell+1} \binom{\ell + 1}{j} (-1)^j \sum_{n=0}^{\infty} E_n(x - j)t^n/n!.
\end{align*}
\]

Comparing the coefficients on both sides, we get the desired result.

Theorem 2 is proved.

**Theorem 3.** If \(n \geq 1\), then

\[
E^{(k)}_{n,p,q}(x) = \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \left( \sum_{j=0}^{i} \frac{(1 - e^{-t})^{\ell-i-j}}{[i + 1]_p^k} \binom{i + 1}{j} (\ell - i - j + x)^n \right).
\]

**Proof.** By using the binomial series, we get

\[
\frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} e^{xt} = 2 \left( \sum_{\ell=0}^{\infty} (-1)^\ell e^{\ell t} \right) \left( \sum_{\ell=0}^{\infty} \frac{(1 - e^{-t})^{\ell+1}}{[\ell + 1]_p^k} \right) e^{xt}
\]

\[
= 2 \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \frac{(-1)^{\ell-i} e^{(\ell-i)t}}{[i + 1]_p^k} (1 - e^{-t})^{\ell+1} e^{xt}
\]

\[
= \left( 2 \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} \frac{(-1)^{\ell-i} e^{(\ell-i)t}}{[i + 1]_p^k} \right) \left( \sum_{j=0}^{i+1} \binom{i + 1}{j} (-1)^j e^{-tj} e^{xt} \right)
\]

\[
= 2 \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell+1} \frac{(-1)^{\ell-i+j} e^{(\ell-i-j+x)t}}{[i + 1]_p^k} \binom{i + 1}{j}
\]

\[
= 2 \sum_{n=0}^{\infty} \sum_{i=0}^{\ell} \sum_{j=0}^{i+1} \frac{(-1)^{\ell-i+j} e^{(\ell-i-j+x)t}}{[i + 1]_p^k} \binom{i + 1}{j} \sum_{n=0}^{\infty} (\ell - i - j + x)^n t^n/n!
\]

Comparing the coefficients on both sides, we arrive at the desired result.

Theorem 3 is proved.

**2.1. Some Relations with Other Special Polynomials.** Jolany, et al. [15] discovered several combinatorial identities involving generalized poly-Euler polynomials in terms of the Stirling numbers of the second kind \(S_2(n, k)\), rising factorial functions \((x)^{(m)}\), falling factorial functions \((x)_m\), Bernoulli polynomials \(B^{(s)}_n(x)\) of
order $s$, and Frobenius–Euler functions $H_n^{(s)}(x; u)$. We present similar expressions in terms of $(p, q)$-poly-Euler polynomials.

Recall that the Stirling numbers of the second kind are defined as follows:

$$\frac{(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n, m) \frac{x^n}{n!}. \tag{4}$$

**Theorem 4.** The following identity is true:

$$E_{n, p, q}^{(k)}(x) = \sum_{\ell=0}^{\infty} \sum_{i=\ell}^{n} \binom{n}{i} S_2(i, \ell) E_{n-i, p, q}^{(k)}(-\ell)(x)^{i}, \tag{5}$$

where

$$(x)^m = x(x + 1) \ldots (x + m - 1), \quad m \geq 1, \quad \text{with} \quad (x)^0 = 1.$$

**Proof.** By using (3), (4), and the binomial series

$$\frac{1}{(1 - x)^c} = \sum_{n=0}^{\infty} \binom{c + n - 1}{n} x^n,$$

we get

$$\frac{2\text{Li}_{k, p, q}(1 - e^{-t})}{1 + e^t} e^{xt} = \frac{2\text{Li}_{k, p, q}(1 - e^{-t})}{1 + e^t} (1 - (1 - e^{-t}) - x$$

$$= \frac{2\text{Li}_{k, p, q}(1 - e^{-t})}{1 + e^t} \sum_{\ell=0}^{\infty} \binom{x + \ell - 1}{\ell} (1 - e^{-t})^{\ell}$$

$$= \sum_{\ell=0}^{\infty} \frac{(x)^{\ell}}{\ell!} (1 - e^{-t})^{\ell} \frac{2\text{Li}_{k, p, q}(1 - e^{-t})}{1 + e^t}$$

$$= \sum_{\ell=0}^{\infty} \frac{(x)^{\ell}}{\ell!} \frac{(e^t - 1)^{\ell}}{\ell!} \left( \frac{2\text{Li}_{k, p, q}(1 - e^{-t})}{1 + e^t} e^{-t\ell} \right)$$

$$= \sum_{\ell=0}^{\infty} \frac{(x)^{\ell}}{\ell!} \left( \sum_{n=0}^{\infty} S_2(n, \ell) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{E_{n, p, q}^{(k)}(-\ell)(x)^{n}}{n!} \right)$$

$$= \sum_{\ell=0}^{\infty} \frac{(x)^{\ell}}{\ell!} \left( \sum_{n=0}^{\infty} \sum_{i=\ell}^{n} \binom{n}{i} S_2(i, \ell) E_{n-i, p, q}^{(k)}(-\ell)(x)^{i} \right) \frac{t^n}{n!}.$$
Comparing the coefficients on both sides, we arrive at (5). Note that we have used the following relation:

\[ \binom{x + \ell - 1}{s} = \frac{(x)^{(\ell)}}{s!}. \]

Theorem 4 is proved.

**Theorem 5.** The following identity is true:

\[
E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{\infty} \sum_{i=\ell}^{n} \binom{n}{i} S_2(i, \ell) E_{n-i,p,q}^{(k)}(x)^{\ell},
\]

(6)

where

\[
(x)_m = x(x-1)\ldots(x-m+1), \quad m \geq 1, \quad \text{with} \quad (x)_0 = 1.
\]

**Proof.** From (3) and (4), we obtain

\[
\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} ((e^t - 1) + 1)^x
\]

\[
= \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \sum_{\ell=0}^{\infty} \binom{x}{\ell} (e^t - 1)^\ell
\]

\[
= \sum_{\ell=0}^{\infty} \frac{(x)^{\ell}}{\ell!} (e^t - 1)^\ell \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}
\]

\[
= \sum_{\ell=0}^{\infty} (x)^{\ell} \sum_{n=0}^{\infty} S_2(n, \ell) \frac{t^n}{n!} \left( \sum_{n=0}^{\infty} E_{n,p,q}^{(k)} \frac{t^n}{n!} \right)
\]

\[
= \sum_{\ell=0}^{\infty} (x)^{\ell} \sum_{n=0}^{\infty} \binom{n}{i} S_2(i, \ell) E_{n-i,p,q}^{(k)}(x)^{\ell} \frac{t^n}{n!}
\]

Comparing the coefficients on both sides, we get (6). Note that we have used the following relation:

\[
\binom{x}{s} = \frac{(x)^{s}}{s!}.
\]

Theorem 5 is proved.
The Bernoulli polynomials $B_n^{(s)}(x)$ of order $s$ are defined by

$$ \left( \frac{t}{e^t - 1} \right)^s e^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(s)}(x)}{n!} t^n. $$

(7)

It is clear that if $s = 1$, then we get the classical Bernoulli polynomials. Some explicit formulas for these polynomials can be found, e.g., in [28].

**Theorem 6.** The following identity is true:

$$ E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} S_2(\ell + s, s) \sum_{i=0}^{n-\ell} \binom{n-\ell}{i} B_i^{(s)}(x) E_{n-\ell-i,p,q}^{(k)}. $$

(8)

**Proof.** From (3) and (7), we get

$$ \frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} e^{xt} = \frac{(e^t - 1)^s}{s!} \frac{t^s e^{xt}}{(e^t - 1)^s} \left( \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n \right) \frac{s!}{t^s} $$

$$ = \left( \sum_{n=0}^{\infty} S_2(n + s, s) \frac{t^{n+s}}{(n+s)!} \right) \left( \sum_{n=0}^{\infty} \frac{B_n^{(s)}(x)}{n!} \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} \frac{t^n}{n!} \right) \frac{s!}{t^s} $$

$$ = \left( \sum_{n=0}^{\infty} S_2(n + s, s) \frac{t^{n+s}}{(n+s)!} \right) \sum_{i=0}^{n} \binom{n}{i} B_i^{(s)}(x) E_{n-i,p,q}^{(k)} \frac{t^n}{n!} \frac{s!}{t^s} $$

$$ = \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n} \binom{n}{\ell} S_2(\ell + s, s) \frac{t^{\ell+s}}{\ell!} \sum_{i=0}^{n-\ell} \binom{n-\ell}{i} B_i^{(s)}(x) E_{n-\ell-i,p,q}^{(k)} \frac{t^{n-\ell-i}}{(n-\ell)!} \right) \frac{s!}{t^s} $$

$$ = \sum_{n=0}^{\infty} \binom{n}{\ell} S_2(\ell + s, s) \sum_{i=0}^{n-\ell} \binom{n-\ell}{i} B_i^{(s)}(x) E_{n-\ell-i,p,q}^{(k)} \frac{t^n}{n!}. $$

Comparing the coefficients on both sides, we obtain (8).

Theorem 6 is proved.

The Frobenius–Euler functions $H_n^{(s)}(x; u)$ are defined by

$$ \left( \frac{1 - u}{e^t - u} \right)^s e^{xt} = \sum_{n=0}^{\infty} \frac{H_n^{(s)}(x; u)}{n!} t^n. $$

(9)
Theorem 7. The following identity is true:

\[ E_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \left( \frac{n}{\ell} \right) \left( \frac{1}{1-u} \right)^{s} \sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) (-u)^{s-i} H_{\ell}^{(s)}(x; u) E_{n-\ell,p,q}^{(k)}(i). \]  

(10)

Proof. From (3) and (9), we obtain

\[
\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} = \frac{(1-u)^s (e^t - u)^s}{(1-u)^s} 2\text{Li}_{k,p,q}(1-e^{-t})
\]

\[
= \frac{1}{(1-u)^s} \left( \sum_{n=0}^{\infty} H_n^{(s)}(x; u) \frac{t^n}{n!} \right) \sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) (-u)^{s-i} \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(i) \frac{t^n}{n!}
\]

\[
= \frac{1}{(1-u)^s} \sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) (-u)^{s-i} \left( \sum_{n=0}^{\infty} H_n^{(s)}(x; u) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(i) \frac{t^n}{n!} \right)
\]

\[
= \frac{1}{(1-u)^s} \sum_{i=0}^{s} \left( \begin{array}{c} s \\ i \end{array} \right) (-u)^{s-i} \sum_{n=0}^{\infty} \left( \begin{array}{c} n \\ \ell \end{array} \right) H_{\ell}^{(s)}(x; u) E_{n-\ell,p,q}^{(k)}(i) \frac{t^n}{n!}
\]

Comparing the coefficients on both sides, we get (10).

Theorem 7 is proved.

3. The \((p, q)\)-Poly-Bernoulli Polynomials and \((p, q)\)-Poly-Cauchy Polynomials

In this section, we introduce the \((p, q)\)-poly-Bernoulli polynomials in terms of the \((p, q)\)-polylogarithm function and \((p, q)\)-poly-Cauchy polynomials by using the \((p, q)\)-integral. In general, it is not difficult to extend the results obtained in [21].

The \((p, q)\)-derivative of the function \(f\) is defined as follows (cf. [5, 13]):

\[
D_{p,q}f(x) = \begin{cases} 
\frac{f(px) - f(qx)}{(p-q)x}, & \text{if } x \neq 0, \\
 f'(0), & \text{if } x = 0.
\end{cases}
\]

In particular, if \(p \to 1\), then we get the \(q\)-derivative [1].
The \((p, q)\)-integral of the function \(f\) is defined as

\[
\int_0^x f(t) \, dp,q \, t = \begin{cases} 
(q - p) x \sum_{n=0}^{\infty} \frac{p^n}{q^{n+1}} f \left( \frac{p^n}{q^{n+1}} x \right), & \text{if } |p/q| < 1, \\
(p - q) x \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f \left( \frac{q^n}{p^{n+1}} x \right), & \text{if } |p/q| > 1.
\end{cases}
\]

Thus,

\[
\int_0^t t^\ell \, dp,q \, t = \frac{1}{\ell + 1}_{p,q}.
\]

We introduce the \((p, q)\)-poly-Bernoulli polynomials by

\[
\frac{\text{Li}_{k,p,q}(1 - e^{-t})}{1 - e^{-t}} e^{-xt} = \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x) \frac{t^n}{n!}, \quad k \in \mathbb{Z}.
\]

In particular, we have

\[
\lim_{p \to 1} B_{n,p,q}^{(k)}(x) = B_{n,q}^{(k)}(x).
\]

These are the \(q\)-poly-Bernoulli polynomials recently studied in [21].

The following theorem establishes the relationship between the \((p, q)\)-poly-Bernoulli polynomials and the \((p, q)\)-poly-Euler polynomials.

**Theorem 8.** If \(n \geq 1\), then

\[
E_{n,p,q}^{(k)}(x) + E_{n,p,q}^{(k)}(x + 1) = 2B_{n,p,q}^{(k)}(-x) - 2B_{n,p,q}^{(k)}(1 - x).
\]

**Proof.** It follows from the equality

\[
\frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} (1 + e^t) e^{xt} = \frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 - e^{-t}} (1 - e^{-t}) e^{xt}
\]

that

\[
\sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x + 1) \frac{t^n}{n!}
\]

\[
= 2 \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(-x) \frac{t^n}{n!} - 2 \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(1 - x) \frac{t^n}{n!}.
\]

Comparing the coefficients on both sides, we arrive at the desired result.

Theorem 8 is proved.
The weighted Stirling numbers of the second kind, \( S_2(n, m, x) \), were defined by Carlitz [6] as follows:

\[
\frac{e^{xt}(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n, m, x) \frac{t^n}{n!}.
\]

**Theorem 9.** If \( n \geq 1 \), then

\[
B_{n,p,q}^{(k)}(x) = \sum_{m=0}^{n} \frac{(-1)^{m+n}m!}{[m+1]^k_{p,q}} S_2(n, m, x).
\]

**Proof.** We obtain

\[
\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_{p,q}(1-e^{-t})}{1-e^{-t}} e^{-xt} \sum_{m=0}^{\infty} \frac{(1-e^{-t})^m}{[m+1]^k_{p,q}} e^{-xt}.
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^{m+n}m! (e^{-t} - 1)^m}{m!} e^{-xt}.
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^{m+n}m!}{[m+1]^k_{p,q}} \sum_{n=m}^{\infty} S_2(n, m, x) \frac{(-t)^n}{n!}.
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} \frac{(-1)^{m+n}m!}{[m+1]^k_{p,q}} S_2(n, m, x) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients on both sides, we get the desired result.

Theorem 9 is proved.

The \((p, q)\)-poly-Cauchy polynomials of the first kind are defined by the equality

\[
C_{n,p,q}^{(k)}(x) = \int_0^1 \ldots \int_0^1 (t_1 \ldots t_k - x)^n d_{p,q,t_1} \ldots d_{p,q,t_k}.
\] (11)

Note that

\[
\lim_{p \to 1} C_{n,p,q}^{(k)}(x) = C_{n,q}^{(k)}(x),
\]

i.e., we get the \(q\)-poly-Cauchy polynomials [18, 21].

Recall that the (unsigned) Stirling numbers of the first kind are defined by

\[
\frac{(\ln(1 + x))^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} S_1(n, m) \frac{x^n}{n!}.
\] (12)
Moreover, they satisfy the following equality (cf. [10]):

\[ x^{(n)} = x(x + 1) \ldots (x + n - 1) = \sum_{m=0}^{n} S_1(n, m)x^m. \]  

(13)

The weighted Stirling numbers of the first kind \( S_1(n, m, x) \) are defined as follows [6]:

\[
\frac{(1-t)^{-x}(-\ln(1-t))^m}{m!} = \sum_{n=m}^{\infty} S_1(n, m, x) \frac{t^n}{n!}.
\]

(12)

**Theorem 10.** If \( n \geq 1 \), then

\[
C_{n,p,q}^{(k)}(x) = \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^{\ell}}{m - \ell + 1}_{p,q}^{k}
\]

(14)

\[
= \sum_{m=0}^{n} S_1(n, m, x) (-1)^{n-m} \frac{(-x)^{m}}{[m + 1]_{p,q}}.
\]

(15)

**Proof.** By (11), (13), and \((x)_n = (-1)^n (-x)^{(n)}\), we obtain

\[
C_{n,p,q}^{(k)}(x) = \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \int_{0}^{1} \ldots \int_{0}^{1} (t_1 \ldots t_k - x)^m d_{p,q} t_1 \ldots d_{p,q} t_k
\]

\[
= \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \sum_{\ell=0}^{m} \binom{m}{\ell} (-x)^{m-\ell} \int_{0}^{1} \ldots \int_{0}^{1} t_1^{\ell} \ldots t_k^{\ell} d_{p,q} t_1 \ldots d_{p,q} t_k
\]

\[
= \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^{\ell}}{[\ell + 1]_{p,q}}
\]

\[
= \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m, x) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^{\ell}}{[m - \ell + 1]_{p,q}}.
\]

Comparing the coefficients on both sides, we get (14). Finally, by virtue of the following relation from [6] [Eq. (5.2)]:

\[ S_1(n, m, x) = \sum_{i=0}^{n} \binom{m + i}{i} x^i S_1(n, m + i), \]

we find

\[
C_{n,p,q}^{(k)}(x) = \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^{\ell}}{[m - \ell + 1]_{p,q}}
\]
\[
\sum_{\ell=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \binom{m}{\ell} \frac{(-x)^\ell}{[m - \ell + 1]^k_{p,q}} = \sum_{\ell=0}^{n} \sum_{m=0}^{n+m} (-1)^{n-m} S_1(n, m) \binom{m}{\ell} \frac{(-x)^\ell}{[m - \ell + 1]^k_{p,q}}
\]

\[
\sum_{\ell=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \binom{m+\ell}{\ell} \frac{(-x)^\ell}{[m+1]^k_{p,q}} \]

\[
\sum_{m=0}^{n} \frac{(-1)^n}{[n+1]^k_{p,q}} S_1(n, m) x^\ell
\]

Theorem 10 is proved.

It is not difficult to give a \((p, q)\)-analog of (2).

**Theorem 11.** The exponential generating function of the \((p, q)\)-poly-Cauchy polynomials \(C_{n,p,q}^{(k)}(x)\) is

\[
\frac{\text{Lif}_{k,p,q}(\ln(1+t))}{(1+t)^x} = \sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x) \frac{t^n}{n!},
\]

where

\[
\text{Lif}_{k,p,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n![n+1]^k_{p,q}}
\]

is the \(k\)th \((p, q)\)-polylogarithm factorial function.

**Proof.** From Theorem 10, we obtain

\[
\sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m) \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^\ell}{[m - \ell + 1]^k_{p,q}} \frac{t^n}{n!}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (-1)^{n-m} S_1(n, m) \frac{t^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} \frac{(-x)^\ell}{[m - \ell + 1]^k_{p,q}}
\]

\[
= \sum_{\ell=0}^{\infty} \frac{(\ln(1+t))^m}{m!} \sum_{\ell=0}^{m} \frac{(-x)^\ell}{[m - \ell + 1]^k_{p,q}}
\]

\[
= \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{\ell!} \sum_{m=\ell}^{\infty} \frac{(\ln(1+t))^m}{(m - \ell)![m - \ell + 1]^k_{p,q}}
\]
\[
\begin{align*}
&= \sum_{\ell=0}^{\infty} \frac{(-x)^\ell}{\ell!} \sum_{n=0}^{\infty} \frac{(\ln(1 + t))^{n+\ell}}{n!(n + 1)^k_{p,q}} \\
&= \frac{1}{(1 + t)^x} \sum_{n=0}^{\infty} \frac{(\ln(1 + t))^n}{n!(n + 1)^k_{p,q}} \\
&= \frac{\text{Lif}_{k,p,q}(\ln(1 + t))}{(1 + t)^x}.
\end{align*}
\]

Theorem 11 is proved.

Similarly, we can define the \((p, q)\)-poly-Cauchy polynomials of the second kind by the equality

\[
\hat{C}_{n,p,q}^{(k)}(x) = \int_0^1 \cdots \int_0^1 (-t_1 \cdots t_k + x)_{n} d_{p,q} t_1 \cdots d_{p,q} t_k.
\]

We can also establish expressions analogous to (14)–(16).

**Theorem 12.** If \(n \geq 1\), then

\[
\hat{C}_{n,p,q}^{(k)}(x) = (-1)^n \sum_{m=0}^{n} S_1(n, m) \sum_{\ell=0}^{m} \frac{(-x)^\ell}{m - \ell + 1}_{p,q},
\]

\[
= (-1)^n \sum_{m=0}^{n} S_1(n, m, -x) \frac{1}{[m + 1]_{p,q}}.
\]

Moreover, the exponential generating function of the \((p, q)\)-poly-Cauchy polynomials \(\hat{C}_{n,p,q}^{(k)}(x)\) is

\[
(1 + t)^x \text{Lif}_{k,p,q}(\ln(1 + t)) = \sum_{n=0}^{\infty} \hat{C}_{n,p,q}^{(k)}(x) \frac{t^n}{n!}.
\]

**3.1. Some Relations between \((p, q)\)-Poly-Bernoulli Polynomials and \((p, q)\)-Poly-Cauchy Polynomials.**

The weighted Stirling numbers satisfy the following orthogonality relation [6]:

\[
\sum_{\ell=m}^{n} (-1)^{n-\ell} S_2(n, \ell, x) S_1(\ell, m, x) = \sum_{\ell=m}^{n} (-1)^{\ell-m} S_1(n, \ell, x) S_2(\ell, m, x) = \delta_{m,n},
\]

where \(\delta_{m,n} = 1\) if \(m = n\) and 0, otherwise. From these relations, we obtain the following inverse relation:

\[
f_n = \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m, x) g_m \iff g_n = \sum_{m=0}^{n} S_2(n, m, x) f_m.
\]
Theorem 13. The \((p, q)\)-poly-Bernoulli polynomials and \((p, q)\)-poly-Cauchy polynomials of both kinds satisfy the following relations:

\[
\sum_{m=0}^{n} S_1(n, m, x) B_{m,p,q}^{(k)}(x) = \frac{n!}{[n+1]_{p,q}},
\]

(17)

\[
\sum_{m=0}^{n} S_2(n, m, x) C_{m,p,q}^{(k)}(x) = \frac{1}{[n+1]_{p,q}},
\]

(18)

\[
\sum_{m=0}^{n} S_2(n, m, -x) \tilde{C}_{m,p,q}^{(k)}(x) = (-1)^n. \]

(19)

Proof. From Theorem 9 and the inverse relation for the weighted Stirling numbers with

\[
f_m = \frac{(-1)^m m!}{[m+1]_{p,q}}, \quad g_n = (-1)^n B_{n,p,q}^{(k)}(x),
\]

we arrive at the identity (17). The remaining relations can be verified in a similar way by using Theorems 10 and 12.

Theorem 13 is proved.

Note that if \( p \to 1 \), then we get Theorem 6 in [21].

Theorem 14. The \((p, q)\)-poly-Bernoulli polynomials and \((p, q)\)-poly-Cauchy polynomials of both kinds satisfy the following relations:

\[
B_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! S_2(n, m, x) S_2(m, \ell, y) C_{\ell,p,q}^{(k)}(y),
\]

(20)

\[
B_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! S_2(n, m, x) S_2(m, \ell, y) \tilde{C}_{\ell,p,q}^{(k)}(y),
\]

(21)

\[
C_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} S_1(n, m, x) S_1(m, \ell, y) B_{\ell,p,q}^{(k)}(y),
\]

(22)

\[
\tilde{C}_{n,p,q}^{(k)}(x) = \sum_{\ell=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} S_1(n, m, -x) S_1(m, \ell, y) B_{\ell,p,q}^{(k)}(y).
\]

(23)

Proof. We restrict ourselves to the proof of (22). The proofs of the remaining identities are similar. By using equations (15) and (17), we find

\[
\sum_{\ell=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} S_1(n, m, x) S_1(m, \ell, y) B_{\ell,p,q}^{(k)}(y)
\]
\[
= \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} S_1(n, m, x) \sum_{\ell=0}^{m} S_1(m, \ell, y) B^{(k)}_{\ell, p, q}(y)
\]
\[
= \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} S_1(n, m, x) \frac{m!}{(m+1)^{k}_{p, q}} = C^{(k)}_{n, p, q}(x).
\]

Theorem 14 is proved.

Finally, we establish some relations between the \((p, q)\)-poly-Cauchy polynomials of both kinds.

**Theorem 15.** If \(n \geq 1\), then

\[
(-1)^n \frac{C^{(k)}_{n, p, q}(x)}{n!} = \sum_{m=1}^{n} \left(\frac{n-1}{m-1}\right) \frac{C^{(k)}_{m, p, q}(x)}{m!},
\]  
(24)

\[
(-1)^n \frac{\tilde{C}^{(k)}_{n, p, q}(x)}{n!} = \sum_{m=1}^{n} \left(\frac{n-1}{m-1}\right) \frac{C^{(k)}_{m, p, q}(x)}{m!}.
\]  
(25)

**Proof.** From definition of the \((p, q)\)-poly-Cauchy polynomials of the first kind, we get

\[
(-1)^n \frac{C^{(k)}_{n, p, q}(x)}{n!} = (-1)^n \int_{0}^{1} \cdots \int_{0}^{1} \left(\frac{t_1 \cdots t_k - x}{n!}\right)^n d_{p, q} t_1 \cdots d_{p, q} t_k
\]

\[
= (-1)^n \int_{0}^{1} \cdots \int_{0}^{1} \left(\frac{t_1 \cdots t_k - x}{n}\right) d_{p, q} t_1 \cdots d_{p, q} t_k
\]

\[
= \int_{0}^{1} \cdots \int_{0}^{1} \left(\frac{x - t_1 \cdots t_k + n - 1}{n}\right) d_{p, q} t_1 \cdots d_{p, q} t_k.
\]

By using the Vandermonde convolution

\[
\sum_{k=0}^{n} \left(\begin{array}{c} r \\ k \end{array}\right) \left(\begin{array}{c} s \\ n - k \end{array}\right) = \left(\begin{array}{c} r + s \\ n \end{array}\right)
\]

with \(r = x - t_1 \cdots t_k\) and \(s = n - 1\), we obtain

\[
(-1)^n \frac{C^{(k)}_{n, p, q}(x)}{n!} = \int_{0}^{1} \cdots \int_{0}^{1} \sum_{\ell=0}^{n} \left(\frac{x - t_1 \cdots t_k}{\ell}\right) \left(\frac{n-1}{n-\ell}\right) d_{p, q} t_1 \cdots d_{p, q} t_k
\]
The proof of (25) is similar.

Theorem 15 is proved.

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