SIR Distribution in downlink Poisson point cellular network with $\kappa - \mu$ shadowed fading

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Abstract—The downlink coverage probability of a cellular network, when the base station locations are modelled by a Poisson point process, is known when the channel power is Gamma distributed with integer shape parameter. However, for many interesting fading distributions such as Rician, Rician shadowing, $\kappa - \mu$, $\eta - \mu$, etc., the coverage probability is unknown. $\kappa - \mu$ shadowed fading is a generic fading distribution whose special cases are many of these popular distributions known so far. In this letter, we derive the coverage probability when the desired base station experiences $\kappa - \mu$ shadowed fading. Using numerical simulations, we verify the analytical expressions.

I. INTRODUCTION

The downlink coverage probability of a single-tier cellular network with distance-dependent interference was analyzed in [1] using tools from stochastic geometry. This was followed by new results for multi-tier cellular networks in single antenna [2] and multi-antenna systems [3]. An important assumption in these works is that the fading of the nearest (desired) base station (BS) is Rayleigh. Rayleigh fading is an important assumption as the channel power will follow exponential distribution. This allows the distribution of signal-to-interference ratio (SIR) to be expressed in terms of the Laplace transform of interference which can easily be computed using standard tools from stochastic geometry. In a multi-antenna system that uses maximal ratio combining, if the fading distribution of all the links are i.i.d. Rayleigh distributed, the channel power is Gamma distributed with integer shape parameter (equal to the number of antenna terminals). Coverage probability of such a system can be computed by using the standard Laplace transform trick.

In [4], the coverage probability of a two-tier cellular network was obtained when the desired signal experiences Rician fading. This approach assumes that the complementary cumulative distribution function (CCDF) of a random variable, which is non-central chi-squared distributed (square of Rician), can be approximated as a weighted sum of exponentials. The weights and abscissas are found by minimizing the mean squared error between the CCDF and this approximation. As the function minimized is not convex, the weights and abscissas obtained are locally optimal and are highly dependent on the initial points assumed. Also, there are no closed-form expressions available for these weights and abscissas and have to be computed numerically.

In [5], coverage probability for a single-tier network with any arbitrary fading distribution was derived using Gil-Pelaez inversion theorem. The coverage probability expression in [5] requires an integration of the imaginary part of the moment generating function (MGF) of the desired channel’s power. Also, this approach is not useful for a heterogeneous $L$-tier network or multi-antenna system as it requires numerical evaluation of multi-dimensional integrals which could result in stability issues. In [6], the authors derive the the coverage probability of arbitrary fading channels, if the moment of channel power $\mathbb{E}(h^2) < 1$, where $\alpha > 2$. The coverage probability derived is not in closed form and requires numerical evaluation of integrals. In [7], [8] average rate was derived for generalized fading channels and fading channels with dominant specular components. But this approach cannot be used for finding the coverage probability.

In this letter we derive an approximate closed form expression for coverage probability of a single tier network when the desired and the interfering links experience $\kappa - \mu$ shadowed fading [9], using the standard Laplace trick [1]. As the popular fading distributions such as Rician, Rayleigh, Nakagami, Rician shadowing, $\kappa - \mu$, $\eta - \mu$ are special cases of $\kappa - \mu$ shadowed fading, the analytically tractable coverage probability expression found for this is generic. Our analysis assumes that the interferers fade independently and identically. The analysis can be easily extended to a multi tier heterogeneous network by using the association probability based method as proposed in [2]. We also show that for $\kappa - \mu$ shadowed fading as the threshold approaches 0, asymptotic coverage probability is similar to that of Nakagami fading.

II. SYSTEM MODEL

We assume a cellular network in which the base stations are modelled by a homogeneous Poisson point process $\Phi \subseteq \mathbb{R}^2$ of intensity $\lambda$. We assume that the all the base stations transmit with unit power. The signal from a base station located at $x \in \mathbb{R}^2$, experiences a path loss $||x||^{-\alpha}$, where $\alpha > 2$. Without loss of generality, a typical user is assumed to be at the origin and is associated with the nearest base station located at a distance ‘$r$’. From [1], the nearest neighbour distance is Rayleigh distributed, i.e., $f(r) = 2\pi \lambda e^{-\pi \lambda r^2}$. We assume the system to be interference limited and hence neglect noise.
The signal to interference ratio of a typical user at distance \( r \) from its associated base station is \( \text{SIR} = \frac{m_\mu}{m + \kappa + m\mu} \), where \( I = \sum_{i \in \Phi \setminus B_0} g_i |x_i|^{-\alpha} \) and \( B_0 \) is the base station that the typical user is associated with. Here, \( g_i \) is the channel power from the \( i \)-th base station to the typical user.

The coverage probability of a typical user is
\[
P_c = \mathbb{P}(\text{SIR} > T) = \int_0^\infty \mathbb{P}(\text{SIR} > T|r)f(r)dr,
\]
where \( f(r) \) is the probability density function of the distance to the nearest base station.

Our objective is to come up with a method to express the CCDF of SIR \([4]\) in terms of the Laplace transform of interference when the desired link experiences \( \kappa-\mu \) shadowed fading, as the Laplace transform of interference can be computed using the probability generating functional (PGFL) of the underlying node distribution. This is possible by expressing the PDF of signal power \( g_0 \) as a weighted sum of gamma distribution’s density functions.

### III. \( \kappa-\mu \) SHADOWED FADING

\( \kappa-\mu \) shadowed fading is represented by three parameters viz. \( \kappa, \mu \) and \( m \). Let \( \gamma \) denote the signal power. The probability density function of signal power \( f(\gamma) \) when the channel experiences \( \kappa-\mu \) shadowed fading is \([9]\)
\[
\frac{\mu^\mu m^{m(1+\kappa)}\gamma^{\mu-1}}{\Gamma(\mu)\Gamma(m+\mu+m)\gamma} e^{-\frac{\mu\gamma}{\mu + \kappa + m\mu}} \Gamma(\mu + m + \gamma),
\]
where \( \Gamma(\mu + m + \gamma) \) is the confluent hypergeometric function.

Hypergeometric function
\[
\mathcal{F}_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{l=0}^{\infty} \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)} \frac{z^l}{l!},
\]
where \( (a)_l \) is the Pochhammer symbol defined as \((a)_0 = 1 \) and \((a)_l = \frac{(a)_l}{(a)_1}\) for \( l \geq 1 \),
\[
f(\gamma) = \sum_{l=0}^{\infty} w_l \frac{e^{-cl\gamma}(\mu + \kappa + m + \gamma)^l}{\Gamma(l + \mu + m)\gamma},
\]
where \( c = \frac{\mu(1+\kappa)}{\mu + \kappa + m\mu}, \)
\[
w_l = \frac{\Gamma(l + \mu + m + \gamma)}{\Gamma(l + \mu)} \frac{\Gamma(m + \mu + \kappa + m\mu)}{\Gamma(m + \mu + \kappa + m\mu + l + \mu + m)}.
\]

So the PDF of channel power \( f(\gamma) \) has been now represented as a weighted sum of Gamma densities of parameters \( (l + \mu, 1) \).

\[
\sum_{l=0}^{\infty} w_l \frac{e^{-cl\gamma}(\mu + \kappa + m + \gamma)^l}{\Gamma(l + \mu + m + \gamma)} = \frac{\Gamma(k + \frac{\mu}{\kappa})}{\Gamma(k)} = (k + \frac{\mu}{\kappa} - \frac{1}{2})^\frac{\mu}{2},
\]
we get
\[
\sum_{l=0}^{\infty} \frac{w_l}{\Gamma(l + \mu)} \frac{(m + \mu + \kappa)^l}{\Gamma(l + \mu + m + \gamma)} = \frac{(m + \mu + \kappa)^l}{\Gamma(l + \mu + m + \gamma)}.
\]

So the PDF is \( \sum_{l=0}^{\infty} w_l \frac{e^{-c(\mu + \kappa + m + \gamma)}(\mu + \kappa + m + \gamma)^l}{\Gamma(l + \mu + m + \gamma)} \).

### IV. COVERAGE PROBABILITY

Rayleigh, Rician, Rician shadowed, \( \kappa-\mu \), Nakagami-\( m \) (integer parameter) are special cases of \( \kappa-\mu \) shadowed fading where \( \mu \) is an integer \([9]\). In the following theorem we derive the coverage probability when \( \mu_0 \) is an integer.

**Theorem 1.** If \( \mu_0 \) is an integer, then
\[
P_c = \sum_{l=0}^{N_0} \sum_{n=0}^{l+\mu_0-1} \frac{w_l(-1)^n F_{l+n}}{n!},
\]
where \( F_{l+n} = \sum_{s=1}^{\infty} \frac{1}{\sum_{q=0}^{\infty} v_q F_1(q + \mu_0, -\frac{\mu}{\mu_0}, 1 - \frac{\mu}{\mu_0}, -\frac{v_q}{\mu_0})} \) is the Gauss-Hypergeometric function.

**Proof:** Substituting for SIR in \([2]\), coverage probability
\[
P_c = \int_0^\infty \mathbb{P}(g_0 > T | \text{SIR})f(r)dr,
\]
where \( I = \sum_{i \in \Phi \setminus B_0} g_i |x_i|^{-\alpha} \). As the PDF of \( g_0 \) is \( f(g_0) \approx \frac{w_l e^{-c_l g_0} (m + \mu_0 + 1)^l}{\Gamma(l + \mu_0)} \), the CCDF is
\[
\mathbb{P}(g_0 > T | \text{SIR}) \approx \sum_{l=0}^{N_0} \frac{w_l}{\Gamma(l + \mu_0, c_l T \mu_0)}.
\]
Let $Y = c_0 T I r^\alpha$. So

$$\mathbb{P}(g_0 > TI r^\alpha) = \mathbb{E}_Y \left( \sum_{l=0}^{N_0} w_l \frac{\Gamma(l + \mu_0, Y)}{\Gamma(l + \mu_0)} \right)$$  \hspace{1cm} (5)

(\textit{a})

$$\mathbb{E}
\left( \sum_{l=0}^{N_0} w_l \frac{l!}{\sum_{n=0}^{l+\mu_0-1} \left( -1 \right)^n n! \frac{\partial^n}{\partial s^n} L_Y(s) \bigg|_{s=1} \right)$$  \hspace{1cm} (6)

$$= \sum_{l=0}^{N_0} \sum_{n=0}^{l+\mu_0-1} \left( -1 \right)^n \frac{n!}{n!} \frac{\partial^n}{\partial s^n} L_Y(s) \bigg|_{s=1}$$  \hspace{1cm} (7)

(\textit{a}) as $\mu_0$ is an integer.

$$L_Y(s) = \mathbb{E}(e^{-s Y}) = L_I(s c_0 T I r^\alpha),$$  \hspace{1cm} (8)

by substituting $Y = c_0 T I r^\alpha$.

From (11),

$$L_I(s) = \exp \left( -2\pi \lambda \int_0^\infty \left( 1 - \mathbb{E}_\gamma(\exp(-s g v^{-\alpha})) \right) v \gamma \right)$$  \hspace{1cm} (9)

\hspace{1cm} (a)

$$\approx \exp \left( -2\pi \sum_{q=0}^{N_1} v_q \int_0^\infty \frac{1}{1 + \left( \frac{2}{\pi} s \gamma \right)^{\frac{1}{\alpha}}} v \gamma \right)$$  \hspace{1cm} (10)

$$= \exp \left( -2\pi \sum_{q=0}^{N_1} v_q \left( \gamma^2 \int_0^\infty \left( 1 + \left( \frac{2}{\pi} s \gamma \right)^{\frac{1}{\alpha}} \right) v \gamma \right) \right),$$  \hspace{1cm} (11)

(a) as the PDF of interfering signal can be expressed as a weighted sum of Gamma density functions and the weights approximately sum to 1.

Combining (4), (7), (8), (11), and by using the fact that the weights $v_q$ sum to 1, $P_c$ is

$$\sum_{l=0}^{N_0} \sum_{n=0}^{l+\mu_0-1} \frac{\partial^n}{\partial s^n} \left( \frac{2\pi \lambda w_l}{n!} \right) \bigg|_{s=1}$$

On integration, coverage probability is obtained.

If $\mu_0$ is not an integer, it is possible to proceed as in (5)-(7) and express the distribution of SIR in terms of fractional derivatives of Laplace transform of interference which leads to intractable expressions. Another approach is to express each of the weighted Gamma PDF in turn as a weighted sum of Erlang PDF (Erlang is a special case of Gamma PDF with integer shape parameters). The parameters of the Erlang density functions and weights can be obtained only through an iterative expectation maximization procedure [10]. So we come up with a technique to express the PDF of Gamma distribution of non integer shape parameters as a weighted sum of Erlang PDF using Rician approximation of Nakagami. The advantage of this method is that the weights and Erlang parameters can be computed offline. As bounding the approximation error analytically is not possible, we will later show that the coverage probability obtained through this approximation is very close to simulation results and serves as a good lower bound.

Square root of a Gamma distributed random variable of parameters $(l + \mu_0, \frac{\mu_0}{\gamma_c})$ is Nakagami distributed of parameters $(l + \mu_0, \frac{(l + \mu_0)^2}{\gamma_c})$. Nakagami random variable of parameters $(l + \mu_0, \frac{(l + \mu_0)^2}{\gamma_c})$ is approximated well by Rician distribution of parameters $(K_i, \frac{l + \mu_0}{\gamma_c})$ through moment matching where $l + \mu_0 = (\frac{K_i}{\gamma_c^2})^2 \forall l + \mu_0 \geq 1$ [11]. Rician fading is a special case of $\kappa - \mu$ shadowed fading with $\mu = 1, \kappa = K_i, m \to \infty$ [12]. So the PDF of power of a Rician faded channel can be expressed as a weighted sum of Erlang PDF (as $\mu = 1$ is an integer). Hence using this approximate equivalence, Gamma density of non integer shape parameter can be expressed as a weighted sum of Erlang PDF.

$$\sum_{l=0}^{N_0} \sum_{n=0}^{l+\mu_0-1} \frac{\partial^n}{\partial s^n} \left( \frac{2\pi \lambda w_l}{n!} \right) \bigg|_{s=1}$$

V. ASYMPTOTICS OF COVERAGE PROBABILITY

It was observed in [12] that coverage probability of different cellular networks looks similar in shape and can be approximated by a horizontal shift of the coverage probability of baseline PPP model. The quantity $G$ that characterizes the shift is obtained using the asymptotic characteristics of coverage probability as the threshold approaches 0.

$$G \triangleq G(1) = \lim_{p \to 1} G(p),$$

where $G(p) \triangleq \frac{F_S I R(p)}{F_S I R(1)}$ is the CDF of the baseline model. The asymptotic gain $G$ has been derived for Nakagami fading [12] but is not known even for popular fading distributions like Rician. Here we show that the asymptotic coverage probability for $\kappa - \mu$ shadowed fading is similar to that of Nakagami fading. In fact for Rician and Rician shadowed fading, asymptotic coverage probability is similar to Rayleigh fading and hence has the same asymptotic gain.

The CDF of $\kappa - \mu$ shadowed fading [9] is given by

$$F_\gamma(\theta) = \frac{\mu^{\mu-1} m^m (1 + \kappa)^\mu \theta^\mu}{\Gamma(\mu + m)}$$

$$\Phi_2(\mu, m, \mu + 1; \frac{\mu + 1 + \kappa}{\gamma}, \frac{\mu + m}{\gamma(\mu + m)}).$$

As $\theta \to 0$, $F_\gamma(\theta) \sim c \theta^\mu$, where

$$c = \frac{\gamma - \mu}{\gamma(\mu + m) + \mu^2(1 + \kappa)^{\mu-1}(m + \kappa)^{\mu-2} \Gamma(\mu + m)}.$$
So \( F_\gamma \) is similar to the characteristics of the power of a Nakagami faded channel as the threshold approaches zero. Hence from [12]

\[
G = \left( \frac{\mathbb{E}(\bar{I}_SR_{1}^\mu)}{\mathbb{E}(ISR_{2}^\mu)} \right)^{1/\mu},
\]

where \( ISR \) is interference to average signal ratio. It is defined as \( ISR = \frac{I}{\mathbb{E}(S)} \), where \( I \) is the interference power, \( \mathbb{E}(S) \) is the signal power averaged over the fading. Rician, Rayleigh and Rician shadowed fading are special cases of \( \kappa - \mu \) shadowed fading for \( \mu = 1 \) [9]. Hence \( \bar{G} = MISR_1 / MISR_2 \), where \( MISR = \mathbb{E}(ISR) \). If PPP is the baseline model, then \( MISR_1 = \frac{2}{\alpha} \) [12]. Nakagami-\( m \) fading is a special case of \( \kappa - \mu \) shadowed fading for \( \mu = m \) and hence asymptotic gain is \( \frac{\mathbb{E}(ISR^m)}{\mathbb{E}(ISR_2^m)} \) as given in [12]. Using the asymptotic gain \( G \), coverage probability of any network of same diversity can be obtained from the baseline model as \( F_{SR_2}(\theta) = F_{SR_1}(\bar{G}) \) [12].

### VI. Results

Fading distributions such as Rician, Nakagami, Rician shadowing, \( \kappa - \mu \), \( \eta - \mu \) etc. are particular cases of \( \kappa - \mu \) shadowed fading distribution [9], [13]. The parameters \( \kappa, \mu \) and \( m \) of \( \kappa - \mu \) shadowed fading for which these different fading distributions are obtained are given in [9], [13]. In Table 1 we provide the weights \( \omega_i \) and parameters of the weighted Gamma distribution \( (1 + \mu, 1/c) \) for all these distributions. The parameter \( \mu \) in \( \eta - \mu \) distribution will be denoted as \( \mu_1 \) in this letter to avoid confusion with the parameter \( \mu \) in \( \kappa - \mu \) shadowed fading. The parameter \( m \) in Nakagami-\( m \) fading is denoted as \( m_1 \) to avoid confusion with parameter \( m \) in \( \kappa - \mu \) shadowed fading.

| Fading Channels | Parameters | Weights |
|-----------------|------------|---------|
| Rician          | \( \frac{\alpha}{\bar{I}} \) | \( c + \frac{\mu}{\mu} \) |
| \( \kappa - \mu \) | \( \frac{\bar{I} \sqrt{\Gamma(1 + \mu)}}{\bar{I} \sqrt{\Gamma(1 + \mu)}} \) | \( \frac{\mu_1 + 1}{\mu_1 + 1 + \kappa} \) |
| \( \eta - \mu \) | \( \frac{\rho_1}{\rho_1 + \mu_1 + 1 + \kappa} \) | \( \frac{\mu_1 + 1}{\mu_1 + 1 + \kappa + \mu} \) |
| Rician Shadowing| \( K(m - m_1) \) | \( \frac{\lambda_0 + m}{\lambda_0 + m} \) |
| Nakagami-\( m \) | \( \delta(1) \) | \( m_{1,2} \) |

The coverage probability plots for different fading distributions are provided in Fig. 1. We assume identical and independent fading distribution in the desired and interferer links (though the desired link and interferers can experience non identical fading). All the plots are for unit mean power. From the plots we can see that in \( \kappa - \mu \) shadowed fading when \( \kappa \) or \( \mu \) increases, coverage probability increases. We do not observe significant difference in coverage probability with change in \( m \) though. We also plot the theoretical and simulation results for \( \kappa - \mu \) fading and verify the expression of coverage probability derived. From the plot of Nakagami fading we observe that the Rician approximation is very tight and its accuracy increases with SIR threshold.

![Fig. 1: Coverage Probability](image)

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