A new spectral method using Legendre wavelets for shallow water model in limited-area

Fukang Yin, Junqiang Song, Jianping Wu and Xiaoqun Cao
Academy of Ocean and Engineering, National University of Defense Technology, Changsha, P.R. China, 410073
yinfukang@nudt.edu.cn

Abstract. This paper presents a new spectral method using Legendre wavelets (named LWSTCM), which complete the stepping in spectral space while deal with boundary conditions in grid-point space by collocation method, for the numerical solution of shallow water model in limited-area. In order to deal with the overlapping boundaries, some proper schemes are considered for exchanging the information on the boundaries between sub-domains. 1-D advection equation is used to analysis the exponential convergence property and error characteristics of LWSTCM. Finally, we study LWSTCM on 2-D shallow water equations for a more realistic application. The numerical results are compared with existing numerical solutions found in the literature and demonstrate the validity and applicability of the presented method.

1. Introduction
The most common numerical methods used for atmospheric and oceanic models are finite difference method (FDM), finite elements method (FEM) and spectral method. Moreover, FDM is an approximation to the differential equation while others are an approximation to its solution. Comparing with FDM and FEM, spectral method can eliminate pole problems and has high accuracy due to its “exponential-convergence” property [1]. The spectral methods also offer the discrete conservations of kinetic energy and entropopy, which are very important for the 2-D turbulence modeling [2].

As a relatively new and an emerging area in mathematical research, wavelet theory has been applied in many different fields of science and engineering. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [3]. Therefore, various wavelets [4-13] have been used for studying problems with greater computational complexity and proved to be powerful tools to explore a new direction in solving differential equations. Recently, spectral method using Legendre wavelets is successfully used to obtain the numerical solution of IVP on large intervals [14] and Klein/Sine-Gordon equations [15]. The main attraction of this method is that it can exploit multi-level parallelism by employing the multi-scale analysis and hierarchy structure of Legendre wavelets.

The research of the spectral methods for limited-area model always attracts many attentions. Tatsumi [16] developed a spectral limited-area model by employing sinusoidal-subtracted Fourier sine-cosine series-expansion method. Hoyer [17] divided the basic variables into background field and perturbation components and deal with the perturbation part with spectral methods through double trigonometric series. Fulton, S. R. and Schubert, W. H. [18,19] proposed a Chebyshev spectral method...
for the limited-area atmospheric and oceanic modeling. Haugen and Machenhauer [20] used Fourier series with cyclic boundary conditions in high resolution limited-area model; then, harmonic-sine series and cosine series expansion methods for limited-area model were develop by Chen and Kuo in [21] and Chen and Bai in [22] respectively. Hereafter, Juang and Kanamitsu [23] developed NCEP regional spectral model and Chen and Bai (1996) [24] developed a harmonic-Fourier spectral limited-area model with an external wind lateral boundary condition. Kuo [25] discussed the scale-dependent accuracy associated with the regional spectral model variables expanded by sine-cosine series. Katsuyuki V.Ooyama [26] gave some tests in 1-D single domain using cubic-spline spectral transform method for the time integration of nonlinear meteorological equations. The major issue in large scale problems arising from scientific and engineering computing is that of reducing the computational cost while preserving numerical accuracy. The purpose of this paper is to develop a new effectively and exponential convergent method, which combining the spectral tau method with collocation method using the Legendre wavelets as the basis, for the numerical solution of limited-area models. Section 2 introduces a new spectral method a using Legendre wavelets. A simple linear advection equation is adopted to explore the numerical accuracy of proposed method in section 3. In section 4, LWSTCM is applied to shallow water model. Conclusions and summary are made in section 5.

2. A new spectral method using legendre wavelets

Because of that Legendre wavelets basis doesn’t satisfy the boundary conditions and the locality of Legendre wavelets, Legendre wavelets spectral method needs to exchange the information on the boundaries between sub-domains. The overset method is employed at those overlapped boundaries. Let $k$ and $M$ is the scale and order of Legendre wavelets, respectively. Let $i$ denoted the $i$th sub-domains $(i=1, 2, \ldots, 2^{k-1} - 1)$. Five simple overset schemes are tested, including (1) $u(iM) = u(iM + 1) = (u(iM + 2) + u(iM - 1))/2$, (2) $u(iM + 1) = u(iM)$ or $u(iM) = u(iM + 1)$ (depending on the propagated direction of wave) (3) spline interpolation (one in each side) (4) spline interpolation (two in each side). The above four schemes is denoted by $ib=1$, $ib=2$, $ib=3$ and $ib=4$, respectively. Without loss of generality, the method which uses the exact solutions at the overlapped boundaries for comparing is noted as $ib=5$. The information exchange is show in figure 1 at the overlapped boundary, where the green points denote the points on overlapping boundaries, and the purple points denote the points that participate in the information exchange.

![Figure 1. The information exchange at the overlapped boundary.](image1)

As Fulton and Schubert point out, spectral collocation method is more stable than spectral Tau method. This is duo to the inaccuracy of treatment for the boundary conditions. What’s more, it’s nearly impossible to conduct the information exchange on the boundaries between sub-domains in spectral...
space. So we propose a hybrid method which complete time stepping in spectral space while deal with boundary conditions and information exchange in physical space by using collocation method. It should be noted that the collocation method is employed to evaluate the coefficient of Legendre wavelets expansion. Figure 2 gives the schematic description of the spectral transform for LWSTCM.

3. Test case

In this section, we give some computational results of one-dimensional linear advection equation to validation with the method based on preceding section, to support our theoretical discussion. Consider the one-dimensional linear advection equation [1, 18, 25]
\[
\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = 0,
\]
(1)
in the domain \([-1,1]\). The exact solution of Eq. (1) is
\[
u(x,t) = A \exp \left[ -\left( \frac{x-x_0-t}{L} \right)^2 \right].
\]
(2)
This is the simplest model involving wave or advective processes. The incoming boundary condition (1) is specified according to the exact solution at \(x = -1\). No boundary condition is needed at right end point. This is an open boundary situation in the sense that any wave should propagated out of the domain without any deformation. In the case of that exact solution is Eq. (2), we assume \(A = 1.0\) and \(x_0 = -0.5\). In order to discuss the accuracy of LWSTCM, Chebyshev spectral collocation method (named CSCM) is introduced to do the comparison with it. It should be noted that the CSCM program used here is based on the burgers’ code from the home page of Jie Shen (http://lsec.cc.ac.cn/~hyu/teaching/SHONM2013.html).

Figure 3 gives the \(L_2\) errors (in log10 form) of single-scale LWSTCM as a function of the spatial-scale parameter \(L\) (Gaussian e-folding distance [1]) at \(t = 1.0\) for \(M = 48\). The single-scale LWSTCM clearly gives a much better approximation than CSCM method as \(L\) bigger than 0.2. In particular, the LWSTCM method reaches machine accuracy more slowly than CSCM. Figure 4 shows the corresponding \(L_2\) errors of LWSTCM at \(t = 1.0\) as a function of \(M\) for different Legendre wavelets scale \(k\) and overlapped boundaries scheme. The exponential convergence of the two methods is obvious, while rapidly approach their asymptotic rate of decrease for \(L = 2.0\). From the results of multiple-scale LWSTCM in figure 4, it can be concluded that (1) the accuracy of overlapped boundaries scheme 2 is nearly the same as scheme 5 which using the exact solution, (2) the accuracy of overlapped boundaries scheme 1 is the same as scheme 3, (3) the overlapped boundaries scheme 2 is the best while the schemes 1 and 3 is the worst, (4) multiple-scale LWSTCM using scheme 2 is convergence fast with the increase of scale \(k\).

Figure 5 presents the \(L_2\) errors of single-scale LWSTCM as a function of \(M\) and \(dt\) at \(t = 1.0\). From the figure 5, it can be found that single-scale LWSTCM has good exponential convergence property and error characteristics.

Like the second test described in [18], we also consider the test case with following analytical solution
\[
u(x,t) = \cos \left[ \Omega \pi (x-t) \right],
\]
(3)
where \(\Omega\) is the wave number in the solution.

Figure 6 shows the \(L_2\) error with \(dt = 0.0001\) at \(t = 1.0\) as a function of \(M\) and \(\Omega\) for the LWSTCM and CSCM. From figure 6, it can be found that LWSTCM requires less grid points than CSCM to achieve the same error for the same wave number while multiple-scale LWSTCM converges faster than single-scale LWSTCM.

The above results indicate that when the solution is smooth enough, LWSTCM needs less grid points than CSCM for the same accuracy and that this advantage increase dramatically as the desired accuracy increases. Even in the case of not smooth enough, LWSTCM has almost the same accuracy as CSCM in the same degrees of freedom. Moreover, LWSTCM can save more storage and
computing time profit from the sparseness of the derivative matrix and the previous mentioned advantages.

Figure 3. $L_2$ errors in the numerical solutions of the model problem 1 as a function of $L$ at $t=1.0$ with $dt=0.0001$ for $M=48$ and $k=0$.

Figure 4. $L_2$ errors of LWSTCM as a function of $M$ for different Legendre wavelets scale $k$ and overlapped boundaries scheme at $t=1.0$ with $dt=0.0001$.
The boundary conditions are set by a periodic overset condition along the east/west boundary and a wall-condition is applied at along the south and north boundaries. In other words, 
\[ u(1,1:1,N_y) = \bar{u}(N_x - 1,1:1,N_y), \quad \bar{v}(1,1:1,N_y) = \bar{v}(N_x - 1,1:1,N_y), \quad \bar{\phi}(1,1:1,N_y) = \bar{\phi}(N_x - 1,1:1,N_y), \]
while in the \( y \)-direction
\[ \bar{v}(1:N_x,1) = \mathbf{0}, \quad \bar{v}(1:N_x,N_y) = \mathbf{0}, \]
Note that no boundary conditions are necessary for \( u \) and \( \phi \) in the \( y \)-direction. The integration time window was 24h with \( dt = 10s \). The time step can be increased by employing some technologies including more stable time discrete scheme and well-posed boundary conditions.

The initial conditions given by Grammeltvedt in reference [28] are
\[ h(x,y) = H_0 + H_1 \tanh \left( \frac{D/2 - y}{2D} \right) + H_2 \text{sech}^2 \left( \frac{D/2 - y}{2D} \right) \sin \left( \frac{2\pi x}{L} \right), \]
The initial velocity fields can be derived from the following relationship
\[ u = -\frac{g}{f} \frac{\partial h}{\partial y} , \quad v = \frac{g}{f} \frac{\partial h}{\partial x}. \]
The constants are listed as
\[ L = 4400 \text{ km}, \quad g = 10 \text{m/s}^2, \quad D = 6000 \text{ km}, \quad H_2 = 133 \text{ m}, \]
\[ H_0 = 2000 \text{ m}, \quad \bar{f} = 10^{-4} \text{s}^{-1}, \quad H_1 = 220 \text{ m}, \quad \beta = 1.5 \times 10^{-11} \text{s}^{-1} \text{ m}^{-1}, \]

Figures 7-9 show the geopotential field \( h/c \) at \( t = 24, 48, \) and 72 hours for different \((M1, M2, k1, k2)\). From figure 7, it can be found that the result of LWSTCM is the same as that obtained by using ADI finite-difference scheme [29]. By comparison with the results of single-scale LWSTCM, multi-scale LWSTCM for Houghton’s shallow water model shows some oscillations in geopotential field. These may be caused by the inaccuracy of the overlapping boundaries. From figures 7-9, it can be found that the results of multi-scale LWSTCM are good agree with those obtained by single scale LWSTCM when use the same number of grid points while have more degree of parallelism.

**Figure 7.** The geopotential field at \( T=24h \) for difference \((M1, M2, k1, k2)\).

**Figure 8.** The geopotential field at \( T=48h \) for difference \((M1, M2, k1, k2)\).
5. Conclusions

In this paper, we presented a hybrid spectral Tau method and collocation method for shallow water model in limited-area. The exponential convergence property and error characteristics are shown in the test of advection equation. LWSTCM clearly gives a much better approximation than CSCM as L (Gaussian e-folding distance) increases. In other words, LWSTCM needs less grid points than CSCM for the same accuracy and that this advantage increase dramatically as the desired accuracy increases. Further, LWSTCM appear to have better stability than CSCM for multiple-waves solutions. Finally, LWSTCM is applied to the solution of 2-D shallow water model for a more realistic application. Numerical results demonstrate that the LWSTCM can reduce the freedom of model and increase the parallelism of model while preserving numerical accuracy.

6. References

[1] Kuo, H. C., Williams, R. T., Boundary effects in regional spectral models. Mon. Wea. Rev., 120 (1992), 2986-2992.
[2] Yung-Chieh Chang, Chebyshev Collocation Method for Shallow Water Models with Domain Decomposition, PhD thesis, National Chiao Tung University, 2009.
[3] Beylkin, G., Coifman R., Rokhlin V., Fast wavelet transforms and numerical algorithms I. Commun. Pur. Appl. Math., 44 (1991), 141-183.
[4] Yousefi, S. A., Legendre wavelets method for solving differential equations of Lane-Emden type. Appl. Math. Comput., 181 (2006), 1417-1422.
[5] Lepik, Ü., Numerical solution of evolution equations by the Haar wavelet method. Appl. Math. Comput., 185 (2007), 695-704.
[6] Hariharan, G., Kannan K., Sharma K. R., Haar wavelet method for solving Fisher’s equation. Appl. Math. Comput., 211 (2009), 284-292.
[7] Mohammadi, F., Hosseini M. M., A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations. J. Franklin I., 348 (2011), 1787-1796.
[8] Venkatesh, S. G., Ayyaswamy S. K., Raja Balachandar S., The Legendre wavelet method for solving initial value problems of Bratu-type. Comput. Math. Appl., 63 (2012), 1287-1295.
[9] Yang, C., and Hou J., Chebyshev wavelets method for solving Bratu's problem. Bound. Value. Probl., 2013 (2013), 142-146.
[10] Fukang Yin, Junqiang Song, and Fengshun Lu, A Coupled Method of Laplace Transform and Legendre Wavelets for Nonlinear Klein-Gordon Equations, Math. Method. Appl. Sci., 37 (2014) 781-792.
[11] Pandey R K, Kumar N, Bhardwaj A, et al. Solution of Lane-Emden type equations using legendre operational matrix of differentiation. Appl. Math. Comput., 218 (2012) 7629-7637.
[12] Jafari H, Yousefi S, Firoozjaee M, et al. Application of Legendre wavelets for solving fractional differential equations. Compum. Math. Appl., 62 (2011) 1038-1045.
[13] Liu N and Lin E-B, Legendre wavelet method for numerical solutions of partial differential equations, Numer. Meth. Part. D. E., 26 (2010) 81-94.
[14] A. Karimi Dizicheh, F. Ismail, M. Tavassoli Kajani, Mohammad Maleki, A Legendre wavelet spectral collocation method for solving oscillatory initial value problems, J. Appl. Math., 2013 (2013)1-5.
[15] Fukang Yin, Tian Tian, Junqiang Song, and Min Zhu, Spectral methods using Legendre wavelets for nonlinear Klein/Sine-Gordon equations, J. Comput. Appl. Math., 275 (2015) 321-334.
[16] Tatsumi, Y., A spectral limited-area model with time-dependent lateral boundary conditions and its application to a multi-level primitive equation model, J. Meteor. Soc. Japan, 64 (1986) 637-664.
[17] Hoyer, J. M., The ECMWF spectral limited-area model. Proc. ECMWF Workshop on Techniques for Horizontal Discretization in Numerical Weather Prediction Models, Berkshire, Reading, United Kingdom, ECMWF, 1987, 343-359.
[18] Fulton, S. R., and W. H. Schubert, Chebyshev spectral methods for limited-area models. Part I: Model problem analysis. Mon. Wea. Rev., 115 (1987) 1940-1953.
[19] Fulton, S. R., and W. H. Schubert, Chebyshev spectral methods for limited-area models. Part II: Shallow water model. Mon. Wea. Rev., 115 (1987) 1954-1965.
[20] Haugen, J. E., and Machenhauer, B., A spectral limited-area model formulation with time-dependent boundary conditions applied to the shallow-water equations. Mon. Wea. Rev., 121 (1993), 2618-2630.
[21] Chen Q. S., Kuo Y. H., A harmonic-sine series expansion and its application to partitioning and reconstruction problems in a limited area Mon. Wea. Rev., 120 (1992) 91-112.
[22] Chen Q. S., Kuo Y. H., A consistency condition for wind-field reconstruction in a limited area and a harmonic-cosine series expansion. Mon. Wea. Rev., 120 (1992) 2653-2670.
[23] Juang, H. M. H. and Kanamitsu M., The NMC nested regional spectral model. Mon. Wea. Rev., 122 (1994) 3-26.
[24] Chen, Q., Bai L., and Bromwich, D. H., A Harmonic-Fourier Spectral Limited-Area Model with an External Wind Lateral Boundary Condition. Mon. Wea. Rev., 125 (1997) 143-167.
[25] Kuo, H. C., and Williams, R. T., Scale-dependent accuracy in regional spectral methods. Mon. Wea. Rev., 126 (1998) 2640-2647.
[26] Ooyama K. V., The cubic-spline transform method: Basic definitions and tests in a 1D single domain. Mon. Wea. Rev., 130 (2002) 2392-2415.
[27] Houghton D, Kaahara A, Washington W. Long-Term Integration of the Barotropic Equations by the Lax-Wendroff Method. Mon. Wea. Rev., 94 (1966) 141-150.
[28] Grammeltvedt A. A survey of Finite-Difference schemes for the primitive equations for a barotropic fluid. Mon. Wea. Rev., 97 (1969) 384-404.
[29] R. Ştefănescu and I.M. Navon, POD/DEIM nonlinear model order reduction of an ADI implicit shallow water equations model, J. Comput. Phys., 237 (2013) 95-114.

Acknowledgments
This work is supported by National Natural Science Foundation of China (Grant No. 61379022 and 41105063).