THE EXISTENCE OF SUPERLUMINAL PARTICLES IS CONSISTENT WITH RELATIVISTIC DYNAMICS

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Abstract. Within an axiomatic framework, we prove that the existence of faster than light particles is consistent with (does not contradict) the dynamics of Einstein’s special relativity.

1. Introduction

Time to time short-lived experiments (such as OPERA 2011, MINOS 2007, etc.) appear suggesting that there may be faster than light (FTL) particles. All of these experiments turned out to be erroneous so far. However, this gives us no guarantee that there will be no experiment in the future justifying the existence of FTL particles. If we have a reliable experiment showing the existence of FTL particles, we have to rebuild all the theories inconsistent with (contradicting) FTL motion. That is why it is important to know which “layers” (parts, subtheories) of Einstein’s relativity theory are consistent with FTL motion. Here we investigate this question within an axiomatic framework.

In [26] it is shown that the existence of FTL inertial particles does not contradict (i.e. it is consistent with) special relativistic kinematics, because the existence of FTL particles is logically independent of special relativistic kinematics. This means that the existing theory implies neither the nonexistence nor the existence of FTL particles, or equivalently both the existence and the nonexistence of FTL particles are consistent with the existing theory. This result is completely analogous to the fact that Euclid’s postulate of parallels is logically independent of the rest of its axioms (in this case two different consistent theories extending the theory of absolute geometry are Euclidean geometry and hyperbolic geometry). In the present paper, we show that the existence of FTL inertial particles is logically independent of special relativistic dynamics, too. Therefore it is consistent with special relativistic dynamics.

The investigation of FTL motion goes back to pre-relativistic times, see, e.g., Fröman [15] and Recami [21, §3]. Since 1905 it is generally believed that the nonexistence of FTL particles is a direct consequence of special theory of relativity. Since Tolman’s antitelephone paradox

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based on his 1917 observation [27] there appear several paradoxes concerning causality violations and FTL particles. Since the 1950s many papers were published on theories for FTL particles as well as on possible resolutions of the paradoxes, see, e.g., Arntzenius [7], Bilanik-Deshpande-Sudarshan [8], Chashchina-Silagadze [10], Feinberg [13], Geroch [16], Jentschura–Wundt [18], Nikolić [20], Recami [22, 21, 23], Sudarshan [25], and Selleri [24]. As far as we know none of the theories for FTL particles in the literature is truly axiomatic. Moreover, the properties (e.g., the connection between energy, speed and rest mass) of FTL particles are not derived from some basic assumptions. They are taken to be analogous or similar to the properties of slower than light particles.

The only framework for investigating the consistency of a statement with a theory is the axiomatic framework of mathematical logic. We show that a statement is consistent with a theory by showing that its negation does not follow from the theory. For this we use model theory: we construct a model of the theory in which the statement is true.

Therefore, we investigate the question of consistency and independence of the existence of FTL particles in the framework of mathematical logic, and we use model theory to show the existence of the required models.

Based on Einstein’s original postulates, we formalize the dynamics of special relativity within an axiomatic framework. We chose first-order logic to formulate axioms of special relativity because experience (e.g., in geometry and set theory) shows that this logic is an adequate logic for providing axiomatic foundations for a theory.

To create any theory of FTL particles, we have to deal with the following phenomenon implied already by the kinematics of special relativity. If an observer sees a fusion of two particles in which an FTL particle participates, then a fast enough (but slower than light) observer sees this fusion as a decay, see Fig.1. The same example also appears, e.g., in [8, 13, 21, 25] and in connection with the phenomenon [8] says: “... according to the original criteria, various observers must agree on the identity of physical laws, and not the description of any given phenomenon ...”. So the existence of FTL particles adds new concepts to the already long list of observer dependent concepts of relativity theory, namely it is also observer dependent whether a particle participates in a decay or a fusion.

The structure of this paper is as follows. In Section 2, we explain our result and axiomatic framework without going into the details of formalization. In Section 3 we give the intuitive idea of the proof of our result. In Section 4 we recall an axiomatic framework for dynamics from [4], In Section 5 we recall an axiom system and some theorems for kinematics of special relativity relevant to our present investigation.
In Section 6 we present an axiom system for dynamics of special relativity theory. The axioms for dynamics are some trivial assumptions on collisions of inertial particles, e.g., conservation of relativistic mass and linear momentum. In Section 7 within this axiomatic framework, we formulate and prove our main result, namely that the existence of FTL inertial particles is independent of dynamics of special relativity, i.e., we prove that neither the existence nor the nonexistence of FTL inertial particles follows from the theory, see Theorem 7.1. Consequently, it is consistent with dynamics of special relativity that there are FTL particles. In Section 8 we show an experimental prediction of Einstein’s special relativity on FTL particles, namely that the relativistic mass and momentum of an FTL particle decrease with the speed, see also [8, 17, 21].

2. INFORMAL STATEMENT OF THE MAIN RESULT

To prove our statement on the existence of FTL inertial particles, we present an axiom system \textit{SRDyn} which is a formalized version of Einstein’s special relativistic dynamics, see p.16. \textit{SRDyn} contains the following axioms for kinematics (see Fig.4 on p.10):

- Principle of relativity (Einstein’s first postulate): Inertial observers (reference frames) are indistinguishable from each other by physical experiments (see \textit{SPR} on p.9).
- The light axiom (Einstein’s second postulate): There is an inertial observer, according to whom light signals move with the same velocity (see \textit{AxLight} on p.9).
- Axioms which were implicitly assumed by Einstein, as well as by all approaches to special relativity theory:
  - Physical quantities satisfy some algebraic properties of real numbers (see \textit{AxEField} on p.10).
Inertial observers coordinatize the same events (see AxEv on p.11).
- Inertial observers are stationary according to their own coordinate systems (see AxSelf on p.11).
- Inertial observers (can) use the same units of measurements (see AxSymD on p.11).

In the axioms of SRDyn concerning dynamics, we use the notion of collision of particles. Intuitively, by a possible collision according to an inertial observer at a coordinate point we mean a set of incoming and outgoing inertial particles such that the relativistic mass and linear momentum are conserved, i.e., the sum of the relativistic masses of the incoming particles coincides with that of the outgoing ones and the same holds for the linear momenta of the particles, see Fig.2. So the conservations of relativistic mass and linear momentum are built into the definition of the possible collisions. Inelastic collisions are defined as collisions in which there is only one outgoing particle.

Now we list the axioms of SRDyn concerning dynamics (see Fig.7):

- The notion of possible collision does not depend on the observers (see Coll on p.15). By the definition of possible collisions, this assumption basically states the conservation of relativistic mass and linear momentum.
- Particles (with given velocities and relativistic masses) can be collided inelastically at any coordinate point (see Ax\textit{\&}inecoll on p.15).
- Relativistic masses of slower than light inertial particles depend only on their speeds (see AxSpeed on p.16).
If the velocities and relativistic masses of two particles coincide for one observer then they coincide for all the other observers, too (see AxMass on p.16).

Inertial observers and inertial particles of arbitrary positive relativistic masses can move with any speed slower than light (see AxThEx+ on p.16).

The main result of this paper is the following, see Thm.7.1 (p.17):

The existence of FTL particles is consistent with special relativistic dynamics SRDyn. The nonexistence of FTL particles is also consistent with SRDyn. Therefore the existence of FTL particles is logically independent of SRDyn.

3. The idea of constructing a model for FTL particles

The main result says that the existence of FTL inertial particles is independent of special relativistic dynamics SRDyn. To prove this statement, we construct two models (solutions of the axioms) of SRDyn such that there are FTL inertial particles in one model and there are no FTL inertial particles in the other one. The interesting case is the construction of the model in which there are FTL inertial particles. Now we turn to explaining the intuitive idea of the construction of this model. The key idea is similar to the ideas of Sudarshan [25], Recami [21, 22], Bilaniuk et al. [8] and Arntzenius [7] using the “switching-reinterpretation” principle.

To simplify the proof we use the notion of four-momentum, which is a defined concept in our framework. As it is known, the four-momentum of an inertial particle according to an observer is a spacetime vector whose time component is the relativistic mass and space component is the linear momentum of the particle since we assume that the speed of light is 1, see Fig.8 on p.18. Thus in possible collisions four-momentum is preserved, see Fig.2.

First we construct the worldview of a distinguished observer such that there are FTL inertial particles. For every coordinate point and every spacetime vector with positive time component we include an incoming and an outgoing inertial particle such the four-momenta of the particles are the given vector. We also include particles for spacetime vectors with zero time component. Clearly there are particles with arbitrary speeds in the worldview of the distinguished observer, thus there are FTL ones.

Constructing the worldview of one observer is easy. The nontrivial part of our construction is to construct a worldview of observers moving with respect to this observer and associating relativistic masses to all the possible particles in the moving frame such that all the axioms
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distinguished observer 
$A, B, C$ 4mom’s 
new observer 
$A_{\text{new}}, B_{\text{new}}, C_{\text{new}}$ 4mom’s 

![Diagram of FTL particles and their transformations](image)

Figure 3. Illustration for model construction.

of SRDyn are satisfied. By Theorem 5.1 the worldviews are transformed by Poincaré transformations. The relativistic masses of slower than light particles also have to be transformed in accordance with the corresponding Poincaré transformation. So the question whether our construction can or cannot be finished depends on whether we can associate appropriate relativistic masses to FTL particles.

We concentrate on defining the four-momenta of inertial particles, since relativistic masses are definable by using four-momenta. The four-momenta of the inertial particles according to the distinguished observer are already given. To define the four-momenta according to the other observers let us choose a Poincaré transformation corresponding to the new observer, see Fig 3. Now we turn to define the four-momenta of inertial particles according to this new observer. For the idea of the definition let us consider the following situation. Let $a$, $b$ and $c$ be inertial particles and let their four-momenta be vectors $A$, $B$ and $C$ according to the distinguished observer as in the left-hand side of Fig 3. Let us note that $A + B = C$, particle $c$ is obtained by “fusion” of particles $a$ and $b$, and particle $a$ is FTL. Then particles $a$, $b$ and $c$ form a possible collision according to the distinguished observer. In the worldview of the new observer particles $a$ and $c$ are obtained by “decay” of particle $b$, see the middle of Fig 3. One of the main axioms of special relativistic dynamics SRDyn is that possible collisions do not depend on the observers, i.e., relativistic mass and linear momentum have to be conserved according to all observers. Thus the four-momenta $A_{\text{new}}$, $B_{\text{new}}$ and $C_{\text{new}}$ of particles $a$, $b$ and $c$ according to the new observer...
have to be such that
\[ B_{\text{new}} = A_{\text{new}} + C_{\text{new}} \]  
(1)
since \(a\) and \(c\) are obtained by “decay” of \(b\). Let us try to define \(A_{\text{new}}\), \(B_{\text{new}}\) and \(C_{\text{new}}\) as the images of \(A\), \(B\) and \(C\) by the linear part of the Poincaré transformation. So let \(A', B'\) and \(C'\) be the images of \(A\), \(B\) and \(C\) by the linear part of the Poincaré transformation. Therefore \(A' + B' = C'\) since \(A + B = C\) and \(A', B', C'\) are obtained by using a linear transformation, see the middle of Fig.3. Since \(B' \neq A' + B'\), equation (1) does not hold automatically. Thus, if \(A_{\text{new}}, B_{\text{new}}\) and \(C_{\text{new}}\) are \(-A', B'\) and \(C'\), equation (1) is satisfied, see the right-hand side of Fig.3. This gives the idea to define the four-momentum \(P_{\text{new}}\) of an arbitrary inertial particle \(p\) according to the new observer the following way. Let \(P\) be the four-momentum of \(p\) according to the distinguished observer and let \(P'\) be the image of \(P\) by the linear part of the chosen Poincaré transformation. \(P_{\text{new}}\) is defined to be \(P'\) if the time component of \(P'\) is positive and \(P_{\text{new}}\) is defined to be \(-P'\) if the time component of \(P'\) is negative (and undefined otherwise). This is basically the “switching-reinterpretation” principle used in [8, 21, 22, 25, 7]. It can be seen that possible collisions do not depend on the observer, and relativistic masses remain positive. It remains to check that all the other axioms of SRDyn hold in our model. For example, Einstein’s first postulate, the principle of relativity holds basically because the worldviews of all the observers are “alike.” For a precise proof, see p.19.

4. THE LANGUAGE OF OUR AXIOM SYSTEM

To make the informal assumptions listed in Section 2 precise, we need a formal language containing a set of basic symbols for the theory, i.e., what objects and relations between them we use as basic concepts.

Here we use the following two-sorted1 language of first-order logic parameterized by a natural number \(d \geq 2\) representing the dimension of spacetime:

\[ \{ B, Q ; \text{IOb, Ph, +, ·, <, W, M} \}, \]

where \(B\) (bodies) and \(Q\) (quantities) are the two sorts, \(\text{IOb}\) (inertial observers) and \(\text{Ph}\) (light signals or photons) are one-place relation symbols of sort \(B\), \(+\) and \(\cdot\) are two-place function symbols and \(<\) is a two-place relation symbol of sort \(Q\), \(W\) (the worldview relation) is a \(d + 2\)-place relation symbol the first two arguments of which are of sort \(B\) and the rest are of sort \(Q\), \(M\) (the mass relation) is a 3-place relation symbol the first two arguments of which are of sort \(B\) and the third argument is of sort \(Q\).

1That our theory is two-sorted means only that there are two types of basic objects (bodies and quantities) as opposed to, e.g., Zermelo–Fraenkel set theory where there is only one type of basic objects (sets).
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Relations \( \text{IOb}(k) \) and \( \text{Ph}(p) \) are translated as “\( k \) is an inertial observer,” and “\( p \) is a light signal or a photon,” respectively. To speak about coordinatization, we translate \( \text{W}(k, b, x_1, x_2, \ldots, x_d) \) as “body (observer) \( k \) coordinatizes body \( b \) at spacetime location \( \langle x_1, x_2, \ldots, x_d \rangle \) (i.e., at space location \( \langle x_2, \ldots, x_d \rangle \) and instant \( x_1 \)). Finally we use the mass relation to talk about the relativistic masses of bodies according to inertial observers by reading \( \text{M}(k, b, q) \) as “the mass of body \( b \) is \( q \) according to body (observer) \( k \).”

**Quantity terms** are the variables of sort \( Q \) and what can be built from them by using the two-place operations + and \( \cdot \), **body terms** are only the variables of sort \( B \). \( \text{IOb}(k) \), \( \text{Ph}(p) \), \( \text{W}(k, b, x_1, \ldots, x_d) \), \( \text{M}(k, b, x) \), \( x = y \) and \( x < y \) where \( k, p, b, x, y, x_1, \ldots, x_d \) are arbitrary terms of the respective sorts are so-called atomic formulas of our first-order logic language. The **formulas** are built up from these atomic formulas by using the logical connectives not \( (\neg) \), and \( (\land) \), or \( (\lor) \), implies \( (\rightarrow) \), if-and-only-if \( (\leftrightarrow) \) and the quantifiers exists \( (\exists) \) and for all \( (\forall) \).

We use the notation \( Q^n \) for the set of all \( n \)-tuples of elements of \( Q \). If \( \bar{x} \in Q^n \), we assume that \( \bar{x} = \langle x_1, \ldots, x_n \rangle \), i.e., \( x_i \) denotes the \( i \)-th component of the \( n \)-tuple \( \bar{x} \). Specially, we write \( \text{W}(k, b, \bar{x}) \) in place of \( \text{W}(k, b, x_1, \ldots, x_d) \), and we write \( \forall \bar{x} \) in place of \( \forall x_1 \ldots \forall x_d \), etc.

The **models** of this language are of the form

\[ \mathfrak{M} = \langle B, Q; \text{IOb}_{\mathfrak{M}}, \text{Ph}_{\mathfrak{M}}, +_{\mathfrak{M}}, \cdot_{\mathfrak{M}}, <_{\mathfrak{M}}, \text{W}_{\mathfrak{M}}, \text{M}_{\mathfrak{M}} \rangle, \]

where \( B \) and \( Q \) are nonempty sets, \( \text{IOb}_{\mathfrak{M}} \) and \( \text{Ph}_{\mathfrak{M}} \) are unary relations on \( B \), \( +_{\mathfrak{M}} \) and \( \cdot_{\mathfrak{M}} \) are binary operations and \( <_{\mathfrak{M}} \) is a binary relation on \( Q \), \( \text{W}_{\mathfrak{M}} \) is subset of \( B \times B \times Q^d \) and \( \text{M}_{\mathfrak{M}} \) is a subset of \( B \times B \times Q \). Formulas are interpreted in \( \mathfrak{M} \) in the usual way. For precise definition of the syntax and semantics of first-order logic, see, e.g., [9, §1.3], [12, §2.1, §2.2].

We denote that formula \( \varphi \) is **valid** in model \( \mathfrak{M} \) by \( \mathfrak{M} \models \varphi \). Formula \( \varphi \) is **logically implied** by set of formulas, in symbols \( \Sigma \models \varphi \), iff (if and only if) \( \varphi \) is valid in every model of \( \Sigma \).

To make our axioms and definitions easier to read, we usually omit the outermost universal quantifiers from our axioms and sometimes we omit them from the definitions, too, i.e., all the free variables are universally quantified.
5. Axioms for kinematics

Here we axiomatize the kinematics of special relativity in our first-order logic language of Section 4. Einstein has assumed two postulates in his 1905 paper [11], the principle of relativity and the light postulate. The principle of relativity roughly states that inertial observers are indistinguishable from each other by physical experiments, see, e.g., Friedman [14, §5].

To formalize the principle of relativity let $P$ be the set of formulas of our language with at most one free variable of sort $B$. Elements of $P$ play the role of potential “laws of physics” in the formulation of the principle of relativity theory. The free variable of sort $B$ is used to evaluate these formulas on observers and to check whether they are valid or not according to the observer in question. Now we can formulate the strong principle of relativity as the following axiom schema:

$\text{SPR}^+$: Every potential law of nature $\varphi \in P$ is either true for all the inertial observers or false for all of them:

$$\{ \text{IOb}(k) \land \text{IOb}(h) \rightarrow [\varphi(k, \bar{x}) \leftrightarrow \varphi(h, \bar{x})] : \varphi \in P \}.$$ 

$P$ contains formulas which may not counted as laws of nature. Therefore, $\text{SPR}^+$ may be stronger than Einstein’s Principle of Relativity. However, this fact does not concern us now because we show here that something does not follow from special relativity, and if something does not follow if we use the possibly stronger assumption $\text{SPR}^+$ it does not follow if we use Einstein’s principle. Let us note here that the difficulty of formulating Einstein’s principle precisely comes from the fact that the notion of “laws of nature” is non well-defined.

The second postulate of Einstein states that “Any ray of light moves in the stationary system of co-ordinates with the determined velocity $c$, whether the ray be emitted by a stationary or by a moving body,” see [11]. We can easily formulate this statement in our first-order logic frame. To do so, let us introduce the following two concepts. The time difference of coordinate points $\bar{x}, \bar{y} \in Q^d$ is defined as:

$$\text{time}(\bar{x}, \bar{y}) := |x_1 - y_1|.$$ 

The spatial distance of $\bar{x}, \bar{y} \in Q^d$ is defined as:

$$\text{space}(\bar{x}, \bar{y}) := \sqrt{(x_2 - y_2)^2 + \ldots + (x_d - y_d)^2}.$$ 

$\text{AxLight}$: There is an inertial observer, according to whom, any light signal moves with the same speed $c$ (independently of the fact that which body emitted the signal). Furthermore, it is

\footnote{Since we will assume that the quantity part is a Euclidean field in $\text{AxEField}$ below, $\text{time}(\bar{x}, \bar{y})$ and $\text{space}(\bar{x}, \bar{y})$ are definable in the language of Section 4.}
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\[ \exists c \exists k \ \exists \bar{p} \ \exists \bar{x}, \bar{y} \ \exists \bar{p} \iff \text{space} = c \cdot \text{time} \]

| AxLight | AxSelf |
|---------|--------|
| \[ \exists \bar{p} \\exists \bar{x}, \bar{y} \text{ time}(\bar{x}, \bar{y}) \] | \[ k \] |
| \[ \text{space}(\bar{x}, \bar{y}) \] | \[ \] |

| AxEv |
|------|
| \[ \exists \bar{x} \iff \text{ev}_k(\bar{x}) = \text{ev}_h(\bar{y}) \] |

| AxSymD |
|--------|
| \[ \exists \bar{x}, \bar{y} \iff \text{space}(\bar{x}, \bar{y}) = c \cdot \text{time}(\bar{x}, \bar{y}) \] |

\[ \exists k c [ \text{Ob}(k) \land 0 < c \land \forall \bar{x} \bar{y} \left( \exists \bar{p} [ \text{Ph}(\bar{p}) \land \text{W}(k, \bar{p}, \bar{x}) \land \text{W}(k, \bar{p}, \bar{y}) ] \leftrightarrow \text{space}(\bar{x}, \bar{y}) = c \cdot \text{time}(\bar{x}, \bar{y}) \right) ] \], \ (2) \]

Einstein assumed without postulating it explicitly that the structure of quantities is the field of real numbers. We make this postulate more general by assuming only the most important algebraic properties of real numbers for the quantities.

**AxEField**: The quantity part \( \langle Q, +, \cdot, \prec \rangle \) is a Euclidean field, i.e.,

- it is a field in the sense of abstract algebra;
- the relation \( \prec \) is a linear ordering on \( Q \) such that
  
  i) \( x < y \rightarrow x + z < y + z \) and

**Figure 4.** Illustration for the axioms of kinematics.

possible to send out a light signal in any direction everywhere (see Fig.4):
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ii) \(0 < x \land 0 < y \rightarrow 0 < xy\) holds; and

iii) every positive element has a square root, i.e.,

\[0 < x \rightarrow \exists y \ x = y^2.\]

Let us note that SPR\(^+\), AxLight and AxField imply that the speed of light is the same for every inertial observer in every direction, i.e. formula (2) holds if we replace “\(\exists kc\)” with “\(\exists c \forall k\)” in it, see [26, Prop.4.2.]

As any other approach to relativity theory, we also assume that observers coordinatize the same “external” reality (the same set of events). By the event occurring for observer \(k\) at coordinate point \(\bar{x}\), we mean the set of bodies \(k\) coordinatizes at \(\bar{x}\):

\[ev_k(\bar{x}) := \{b : W(k, b, \bar{x})\}.

\textbf{AxEv}: All inertial observers coordinatize the same set of events (see Fig.4):

\[IOb(k) \land IOb(h) \rightarrow \exists \bar{y} \forall b [W(k, b, \bar{x}) \leftrightarrow W(h, b, \bar{y})].\]

From now on, we use \(ev_k(\bar{x}) = ev_h(\bar{y})\) to abbreviate the subformula \(\forall b [W(k, b, \bar{x}) \leftrightarrow W(h, b, \bar{y})]\) of AxEv.

Basically we are ready for formulating the kinematical part of Einstein’s special relativity theory within our axiomatic framework. Nevertheless, let us introduce two more simplifying axioms.

\textbf{AxSelf}: Any inertial observer is stationary according to its own coordinate system (see Fig.4):

\[IOb(k) \rightarrow \forall \bar{x} [W(k, k, \bar{x}) \leftrightarrow x_2 = \ldots = x_d = 0].\]

Axiom AxSelf makes it easier to speak about the motion of inertial observers since it identifies the observers with their time-axes. So instead of always referring to the time-axes of inertial observers we can speak about their motion directly.

Our last axiom on kinematics is a symmetry axiom saying that all observers use the same units of measurement.

\textbf{AxSymD}: Any two inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them; and the speed of light is 1 for all observers (see Fig.4):

\[IOb(k) \land IOb(h) \land x_1 = y_1 \land x_1' = y_1' \land ev_k(\bar{x}) = ev_h(\bar{x}') \land ev_k(\bar{y}) = ev_h(\bar{y}') \rightarrow \text{space}(\bar{x}, \bar{y}) = \text{space}(\bar{x}', \bar{y}'), \text{ and} \]

\[IOb(k) \rightarrow \exists p [Ph(p) \land W(k, p, 0, \ldots, 0) \land W(k, p, 1, 1, 0, \ldots, 0)].\]

Axiom AxSymD simplifies the formulation of our theorems because we do not have to consider situations such as when one observer measures distances in meters while another observer measures them in feet.
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Let us now introduce an axiom system $\text{SR}$ for kinematics of special relativity as the collection of the axioms above:

$$\text{SR} := \text{SPR}^+ \cup \{\text{AxLight}, \text{AxEField}, \text{AxEv}, \text{AxSelf}, \text{AxSymD}\}.$$ 

Let us note that usually we use a more general axiom system called $\text{SpecRel}$ for special relativity, which does not contain $\text{SPR}^+$, see, e.g., [3], [5], [26] because even $\text{SpecRel}$ captures the kinematics of special relativity, i.e., it implies that the worldview transformations are Poincaré ones. It is proved in [26] that $\text{SpecRel}$ is more general than $\text{SR}$.

To characterize the possible relations between the worldviews of inertial observers, let us introduce the worldview transformation between observers $k$ and $h$ (in symbols, $w_{kh}$) as the binary relation on $Q^d$ connecting the coordinate points where $k$ and $h$ coordinatize the same events:

$$w_{kh}(\vec{x}, \vec{y}) \overset{\text{def}}{\iff} \text{ev}_k(\vec{x}) = \text{ev}_h(\vec{y}).$$

Map $P : Q^d \to Q^d$ is called a Poincaré transformation iff it is an affine bijection having the following property:

$$\text{time}(\vec{x}, \vec{y})^2 - \text{space}(\vec{x}, \vec{y})^2 = \text{time}(\vec{x}', \vec{y}')^2 - \text{space}(\vec{x}', \vec{y}')^2$$

for all $\vec{x}, \vec{y}, \vec{x}', \vec{y}' \in Q^d$ for which $P(\vec{x}) = \vec{x}'$ and $P(\vec{y}) = \vec{y}'$.

Theorem 5.1 formally confirms that the postulates of Einstein’s imply that the worldview transformations between inertial observers are Poincaré transformations.

**Theorem 5.1.** Let $d \geq 3$. Assume $\text{SR}$. Then $w_{kh}$ is a Poincaré transformation if $k$ and $h$ are inertial observers.

We note that Thm 5.1 also holds, if we replace $\text{SR}$ with the more general axiom system $\text{SpecRel}$, see, e.g., [6]. For versions of Theorem 5.1 using a similar but different axiom systems of special relativity, see, e.g., [1], [2], [3].

Let $\text{FTL}(k, b)$ be the following formula saying that body $b$ moves FTL according to inertial observer $k$:

$$\text{FTL}(k, b) \overset{\text{def}}{\iff} \text{IOb}(k) \land \exists \vec{x}\vec{y}[W(k, b, \vec{x}) \land W(k, b, \vec{y}) \land \text{time}(\vec{x}, \vec{y}) < \text{space}(\vec{x}, \vec{y})].$$ (3)

Let $\exists FTL\text{IOb}$ be the following formula saying that there is an FTL inertial observer:

$$\exists FTL\text{IOb} \overset{\text{def}}{\iff} \exists kh[\text{IOb}(h) \land \text{FTL}(k, h)].$$

By Thm 5.1, $\text{SR}$ implies that there are no FTL inertial observers:

**Corollary 5.2.** Assume $d \geq 3$. Then $\text{SR} \models \lnot \exists FTL\text{IOb}$. 

We note that, by Thm 7.1 on p.17, $\text{SR}$ does not imply that there are no FTL inertial particles.
We need the following concepts of kinematics in our axioms for dynamics. The \textbf{world-line} of body $b$ according to observer $k$ is defined as:

$$\text{wl}_k(b) := \{\vec{x} : W(k, b, \vec{x})\}.$$ 

Body $b$ is called \textbf{inertial} if for every inertial observer the world-line of body $b$ is at least two element subset of a straight-line, formally:

$$\text{IOb}(k) \rightarrow \exists \vec{x} \vec{y} [\vec{x} \neq \vec{y} \land W(k, b, \vec{x}) \land W(k, b, \vec{y}) \land (W(k, b, \vec{z}) \rightarrow \exists q [Q(q) \land \vec{z} = q \cdot (\vec{x} − \vec{y})])].$$

The \textbf{velocity} $v_k(b)$ and the \textbf{speed} $v_k(b)$ of inertial body $b$ according to observer $k$ are defined as follows. Let $\vec{x}, \vec{y}$ be such that $W(k, b, \vec{x}), W(k, b, \vec{y})$ and $x_1 \neq y_1$. Then

$$v_k(b) := \frac{x_2 - y_2, \ldots, x_d - y_d}{x_1 - y_1} \quad \text{and} \quad v_k(b) := \frac{\text{space}(\vec{x}, \vec{y})}{\text{time}(\vec{x}, \vec{y})}.$$ 

and if there are no such $\vec{x}$ and $\vec{y}$, then $v_k(b)$ and $v_k(b)$ are undefined.

For \textit{inertial} bodies these are well defined concepts since they do not depend on the choice of $\vec{x}$ and $\vec{y}$. $v_k(b) < \infty$ abbreviates that $v_k(b)$ is defined, i.e., $v_k(b) < \infty$ iff $\exists \vec{x} \vec{y} [W(k, b, \vec{x}) \land W(k, b, \vec{y}) \land x_1 \neq y_1]$.

We say that the speed of body $b$ according to observer $k$ is finite iff $v_k(b) < \infty$.

\section*{6. Axioms for dynamics}

In this section, we introduce axioms for dynamics of special relativity, which are some natural assumptions on collisions of inertial particles and they concern FTL particles, too.

To introduce the notion of collisions of particles we need some definitions. The relativistic mass of body $b$ according to inertial observer $k$, in symbols $m_k(b)$, is defined to be $q$ if $M(k, b, q)$ holds and there is only one such $q \in Q$; otherwise $m_k(b)$ is undefined. Here we are interested in inertial bodies having relativistic masses. Body $b$ is called \textbf{inertial particle}, in symbols $lp(b)$, iff $b$ is an inertial body and $m_k(b)$ is defined and is nonzero for every inertial observer $k$ according to whom the speed of $b$ is finite (i.e., $\text{IOb}(k) \land v_k(b) < \infty \rightarrow m_k(b) \neq 0$).

Body $b$ is \textbf{incoming (outgoing)} at coordinate point $\vec{x}$ according to inertial observer $k$, in symbols $\text{in}_k(b, \vec{x})$ (\text{out}_k(b, \vec{x}))), iff $b$ is an inertial particle, $\vec{x}$ is on the world-line of $b$, and the time component of each coordinate point on the world-line of $b$ different from $\vec{x}$ is less than (greater than) the time component of $\vec{x}$ (see the left-hand side of Fig.5).
FTL particles are consistent with relativistic dynamics

\[ \text{coll}_k(b_1 \ldots b_5) \iff A + B + C = D + E \]

**Figure 5.** Illustration for incoming, outgoing, possible collision and inelastic collision of bodies. The vectors \((A, B, C, D, E \in Q^d)\) in the figure are the four-momenta of inertial particles, i.e., \(\langle m_k(b_i), m_k(b_i) \cdot v_k(b_i) \rangle\), cf. (6) and Fig. 8.

Let us define the possible collisions of bodies as follows. Bodies \(b_1, \ldots, b_n\) form a possible collision according to observer \(k\) if there is a coordinate point such that all the bodies are incoming or outgoing in that coordinate point and the sum of the relativistic masses of the incoming bodies coincides with that of the outgoing ones, and the same holds for the linear momenta of the bodies (see Fig. 5):

\[
\text{coll}_k(b_1 \ldots b_n) \iff \exists \vec{x} \left[ \bigwedge_{i=1}^{n} [\text{in}_k(b_i, \vec{x}) \lor \text{out}_k(b_i, \vec{x})] \land \\
\sum_{\{i : \text{in}_k(b_i, \vec{x})\}} m_k(b_i) = \sum_{\{i : \text{out}_k(b_i, \vec{x})\}} m_k(b_i) \land \\
\sum_{\{i : \text{in}_k(b_i, \vec{x})\}} m_k(b_i) \cdot v_k(b_i) = \sum_{\{i : \text{out}_k(b_i, \vec{x})\}} m_k(b_i) \cdot v_k(b_i) \right]. \tag{4}
\]

Let us note that, if bodies \(b_1, \ldots, b_n\) form a possible collision, then they are inertial particles by the definition of incoming and outgoing particles.

For every natural number \(n\) we introduce an axiom saying that possible collisions formed by \(n\) bodies do not depend on the observers.
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\[ \forall k \exists a' \exists b' \bar{x} - \text{inecoll}(a'b') \ldots \]

**Figure 6.** Illustration for Ax\(\text{inecoll}\).

Thus conservations of relativistic mass and linear momentum do not depend on the observers.

**AxColl\(_n\):** If bodies \(b_1, \ldots, b_n\) form a possible collision for an inertial observer, they form a possible collision for every inertial observer according to whom the speed of each of them is finite (see Fig.7):

\[ \text{IOb}(k) \land \text{IOb}(h) \land \bigwedge_{i=1}^{n} v_{h}(b_i) < \infty \land \text{coll}_k(b_1 \ldots b_n) \rightarrow \text{coll}_h(b_1 \ldots b_n). \]

Let Coll below be the axiom schema containing AxColl\(_n\) for every natural number \(n\):

**Coll:** Possible collisions do not depend on the observers:

\[ \text{Coll} := \{ \text{AxColl}_n : n \text{ is a natural number} \}. \]

Bodies \(a\) and \(b\) collide inelastically according to inertial observer \(k\) at coordinate point \(\bar{x}\), in symbols \(\bar{x}\)-inecoll\(_k\)(ab), iff there is a body \(c\) such that \(a, b, c\) form a possible collision, \(a, b\) are incoming and \(c\) is outgoing at \(\bar{x}\) (see the right-hand side of Fig.5):

\[ \bar{x} - \text{inecoll}_k(ab) \overset{\text{def}}{=} \exists c [\text{coll}_k(abc) \land \text{in}_k(a, \bar{x}) \land \text{in}_k(b, \bar{x}) \land \text{out}_k(c, \bar{x})]. \quad (5) \]

**Ax\(\text{inecoll}\):** Every potential inelastic collision can be realized. That is, for every inertial observer, every coordinate point and every two inertial particles \(a\) and \(b\), if the sum of their relativistic masses is nonzero and their speeds are finite, there are inertial particles \(a'\) and \(b'\) such that they collide inelastically at the given coordinate point and the relativistic masses and velocities of \(a'\) and \(b'\) coincide with those of \(a\) and \(b\), respectively
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Fig[6] and Fig[7]:

\[
\text{IOb}(k) \land \text{lp}(a) \land \text{lp}(b) \land v_k(a) < \infty \land v_k(b) < \infty \land \\
m_k(a) + m_k(b) \neq 0 \implies \exists a' [\bar{x}-\text{inecoll}_k(a'b') \land \\
m_k(a') = m_k(a) \land m_k(b') = m_k(b) \land v_k(a') = v_k(a) \land v_k(b') = v_k(b)].
\]

We assume that relativistic masses of slower than light inertial particles depend only on their speeds.

**AxSpeed:** If an inertial particle is moving with the same slower than light speed according to two inertial observers, then the relativistic masses of the particle are the same for them (see Fig[7]):

\[
\text{IOb}(k) \land \text{IOb}(h) \land \text{lp}(b) \land v_k(b) = v_h(b) < 1 \implies m_k(b) = m_h(b).
\]

We also assume that, if two inertial particles have the same velocities and relativistic masses according to an inertial observer, then they have the same relativistic masses according to every inertial observer.

**AxMass:** If the relativistic masses and velocities of two inertial particles coincide for an observer, then their relativistic masses coincide for every observer (see Fig[7]):

\[
\text{IOb}(k) \land \text{IOb}(h) \land \text{lp}(a) \land \text{lp}(b) \land \\
m_k(a) = m_k(b) \land v_k(a) = v_k(b) \implies m_h(a) = m_h(b).
\]

To avoid trivial models, we also assume that there are inertial observers moving relative to each other and there are slower than light inertial particles with arbitrary positive relativistic mass.

**AxThEx⁺:** Inertial observers and inertial particles of arbitrary positive relativistic mass can move along any straight line of speed less than the speed of light:

\[
\text{IOb}(k) \land \text{space}(\bar{x}, \bar{y}) < \text{time}(\bar{x}, \bar{y}) \land 0 < q \implies \\
(\exists h \cdot \text{IOb}(h) \land W(k, h, \bar{x}) \land W(k, h, \bar{y})] \land \\
\exists b \cdot \text{lp}(b) \land W(k, b, \bar{x}) \land W(k, b, \bar{y}) \land m_k(b) = q).
\]

Let Dyn be the collection of our axioms for dynamics, and let us now introduce an axiom system SRDyn for dynamics of special relativity as the collection of all the axioms of kinematics and dynamics above:

\[
\text{Dyn} := \text{Coll} \cup \{\text{Ax∀inecoll}, \text{AxSpeed}, \text{AxMass}, \text{AxThEx⁺}\}, \text{and}
\]

\[
\text{SRDyn} := \text{SR} \cup \text{Dyn}.
\]

7. Independence of FTL Inertial Particles of SRDyn

Now we show that the existence of FTL inertial particles is independent of SRDyn. To formulate this statement we need some definitions.
Let us recall that formula $\text{FTL}(k, b)$ states that body $b$ moves FTL according to inertial observer $k$, see (3) on p.12.

Let $\exists \text{FTLIp}$ be the following formula saying that there is an FTL inertial particle:

$$\exists \text{FTLIp} \overset{def}{=} \exists k b \left[ lp(b) \land \text{FTL}(k, b) \right].$$

Let $\Sigma$ be a set of formulas and $\varphi$ be a formula. $\Sigma \not\models \varphi$ denotes that $\varphi$ is not implied by $\Sigma$, i.e., there is a model of $\Sigma$ in which $\varphi$ is not valid. Statement $\varphi$ is called independent of $\Sigma$ if neither $\varphi$ nor its negation $\neg \varphi$ is implied by $\Sigma$, i.e., $\Sigma \not\models \varphi$ and $\Sigma \not\models \neg \varphi$. Let us note that $\varphi$ is independent of $\Sigma$ if there are two models of $\Sigma$ such that $\varphi$ is valid in one model and $\neg \varphi$ is valid in the other one.

The main result of the present paper is Theorem 7.1 below. It says that the existence of FTL inertial particles is independent of relativistic dynamics.

**Theorem 7.1.** $\exists \text{FTLIp}$ is independent of $\text{SRDyn}$, that is

$$\text{SRDyn} \not\models \exists \text{FTLIp}, \text{ and}$$

$$\text{SRDyn} \not\models \neg \exists \text{FTLIp},$$

equivalently, both $\exists \text{FTLIp}$ and $\neg \exists \text{FTLIp}$ are consistent with $\text{SRDyn}$. 

**Figure 7.** Illustration for the axioms of dynamics.
\textbf{Proof}. The theorem is a corollary of Thm\[7.2\] below.

The following theorem is a stronger form of Thm\[7.1\] above.

\textbf{Theorem 7.2}. For every \(d \geq 2\) and for every Euclidean field \(\mathcal{Q}\), there are models \(\mathcal{M}_1\) and \(\mathcal{M}_2\) of SRDyn such that \(\mathcal{M}_1 \models \exists \text{FTLIp}\), \(\mathcal{M}_2 \models \neg \exists \text{FTLIp}\), and \(\mathcal{Q}\) is the field reduct of both models.

Based on the intuitive idea in Section \[3\] we give a formal proof here using the following concepts.

Let \(f : Q^d \to Q^d\) and \(g : Q^d \to Q^d\) be maps. \(f \circ g\) denotes the composition of the two maps, i.e., \((f \circ g)(\bar{x}) = f(g(x))\). \(f^{-1}\) denotes the inverse map of \(f\). Let \(H\) be a subset of \(Q^d\). The \textbf{image} of set \(H\) is defined as: \(f[H] := \{ f(\bar{x}) : \bar{x} \in H \}\). The \textbf{identity map} is defined as: \(\text{Id}(\bar{x}) := \bar{x}\) for all \(\bar{x} \in Q^d\). Let \(\bar{x}, \bar{y} \in Q^d\). Then \(\text{ray} \bar{x} \bar{y}\) denotes the closed ray (or half-line) with initial point \(\bar{x}\) and containing \(\bar{y}\), i.e., \(\text{ray} \bar{x} \bar{y} := \{ \bar{x} + q \cdot (\bar{y} - \bar{x}) : 0 \leq q \}\).

The \textbf{time-axis} is defined as \(t\) - \text{axis} := \{ \bar{x} \in Q^d : x_2 = \ldots = x_d = 0 \}\).

Let \(k \in \text{IOb}\) and \(b \in B\). The \textbf{four-momentum} \(P_k(b)\) of body \(b\) according to inertial observer \(k\) is defined as the element of \(Q^d\) whose time component is the relativistic mass and space component is the linear momentum of \(b\) according to \(k\) if \(b\) is an inertial particle and the speed of \(b\) is finite (see Fig.\[8\]), i.e.:

\[ P_k(b)_1 = m_k(b) \quad \text{and} \quad \langle P_k(b)_2, \ldots, P_k(b)_d \rangle = m_k(b) \cdot v_k(b), \] (6)

if \(\text{lp}(b) < \infty\), and \(P_k(b)\) is undefined otherwise. It is not difficult to prove that \(P_k(b)\) is parallel to the world-line of \(b\).

\[\begin{align*}
\text{m}_k(b) \quad \text{(mass)} \\
\text{m}_k(b) \cdot \text{v}_k(b) \quad \text{(linear momentum)} \\
\text{P}_k(b) \quad \text{(four-momentum)} \\
\text{wl}_k(b)
\end{align*}\]

\textbf{Figure 8}. Illustration for four-momentum.
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Using the concept of four-momentum definition (4) of possible collision of bodies can be written in a simpler form:

\[ \text{coll}_k(b_1 \ldots b_n) \iff \exists \bar{x} \bigwedge_{i=1}^n [\text{in}_k(b_i, \bar{x}) \lor \text{out}_k(b_i, \bar{x})] \land \]

\[ \sum_{\{i : \text{in}_k(b_i, \bar{x})\}} P_k(b_i) = \sum_{\{i : \text{out}_k(b_i, \bar{x})\}} P_k(b_i). \quad (7) \]

**Proof.** The idea of the proof is in Section 3 on p.5.

Let \( \Omega = \langle Q; + , , < \rangle \) be a Euclidean field. We are going to prove our statement by constructing two models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) of SRDyn such that in \( \mathcal{M}_1 \) there are FTL inertial particles, in \( \mathcal{M}_2 \) there are no FTL inertial particles and the ordered field reduct of both models is \( \Omega \). There is only a slight difference in the two constructions. Therefore, we are going to construct the two models simultaneously. By \( \bar{x}, \bar{y} \) below we denote the ordered pair \( \langle \bar{x}, \bar{y} \rangle \).

\( \text{IOb} := \{ \text{Poincaré transformations of } Q^d \} \), \quad (8)

\( \text{lp}_1 := \{ \bar{x}\bar{y} \in Q^d \times Q^d : \bar{x} \neq \bar{y} \} \), \quad (9)

\( \text{lp}_2 := \{ \bar{x}\bar{y} \in \text{lp}_1 : \text{space}(\bar{x}, \bar{y}) \leq \text{time}(\bar{x}, \bar{y}) \} \), and \( \text{lp}_2 \), \quad (10)

\( \text{Ph} := \{ \bar{x}\bar{y} \in \text{lp}_1 : \text{space}(\bar{x}, \bar{y}) = \text{time}(\bar{x}, \bar{y}) \} \). \quad (11)

Let us note that \( \text{Ph} \subseteq \text{lp}_2 \subseteq \text{lp}_1 \). The only difference between the construction of models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) is in the definition of the set of bodies:

\[ B_1 := \text{IOb} \cup \text{lp}_1 \quad \text{and} \quad B_2 := \text{IOb} \cup \text{lp}_2. \quad (12) \]

Throughout the proof \( B \) and \( \text{lp} \) denote \( B_1 \) and \( \text{lp}_1 \) in the case of \( \mathcal{M}_1 \) and denote \( B_2 \) and \( \text{lp}_2 \) in the the case of \( \mathcal{M}_2 \). Furthermore, by “inertial particles” we mean the members of \( \text{lp}_1 \) in the case of \( \mathcal{M}_1 \) and the members \( \text{lp}_2 \) in the case of \( \mathcal{M}_2 \).

The model construction is illustrated in Fig.9 and its intuitive idea is the following: The worldview transformation between inertial observers identity \( \text{Id} \) and \( k \) will be Poincaré transformation \( k \). First we define the world-lines of bodies according to observer \( \text{Id} \). In particular, the world-line of particle \( \bar{x}\bar{y} \) is \( \text{ray}_{\bar{x}\bar{y}} \) for every \( \bar{x}\bar{y} \). We transform the world-lines by transformation \( k \) to obtain world-lines according to arbitrary observer \( k \), cf. Fig.9. Thus the world-line of particle \( \bar{x}\bar{y} \) is \( \text{ray}_k(\bar{x})k(\bar{y}) \) according to observer \( k \).

We define relativistic masses such that (i)-(iii) holds. (i) The time components of the four-momenta are positive, thus the relativistic masses of particles of finite speeds are positive. (ii) The four-momentum \( P(\bar{x}\bar{y}) \) of particle \( \bar{x}\bar{y} \), according to observer \( \text{Id} \), is one of the two vectors connecting \( \bar{x} \) and \( \bar{y} \), i.e., one of \( \bar{x} - \bar{y} \) and \( \bar{y} - \bar{x} \), if the speed of \( \bar{x}\bar{y} \) is
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Finite. (iii) According to observer $k$, the four-momentum of particle $\bar{x}\bar{y}$ with finite speed is one of the two vectors connecting $k(\bar{x})$ and $k(\bar{y})$.

Now we construct our models based on the intuitive idea above. See Fig.9. In the worldview of observer $I_d$, we define the world-line of inertial observer $h$ and the world-line of inertial particle $\bar{x}\bar{y}$ as:

$$\text{wl}(h) := h^{-1}[\text{t-axis}], \quad \text{and} \quad \text{wl}(\bar{x}\bar{y}) := \text{ray}\bar{x}\bar{y},$$

see Fig.9. We define the world-line of body $b$ and the relativistic mass of inertial particle $\bar{x}\bar{y}$ in the worldview of observer $k$ as:

$$\text{wl}_k(b) := k[\text{wl}(b)] \quad \text{and} \quad m_k(\bar{x}\bar{y}) := \text{time}(k(\bar{x}), k(\bar{y})), \quad (14)$$

cf. Fig.9.

Finally we define the worldview relation $W$ and the mass relation $M$ as:

$$W(k, b, \bar{x}) \overset{def}{=} k \in \text{IOb} \wedge b \in B \wedge \bar{x} \in \text{wl}_k(b), \quad \text{and} \quad (15)$$

$$M(k, b, q) \overset{def}{=} k \in \text{IOb} \wedge b \in \text{Ip} \wedge m_k(b) = q. \quad (16)$$

Now models $\mathcal{M}_1$ and $\mathcal{M}_2$ are given. It is easy to see that the set of inertial particles in $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\text{Ip}_1$ and $\text{Ip}_2$, respectively.

For every inertial observers $k$ and $h$, and inertial particle $\bar{x}\bar{y}$, by (14), it is easy to see that

$$\text{wl}_k(\bar{x}\bar{y}) = \text{ray}k(\bar{x})k(\bar{y}), \quad (17)$$

$$P_k(\bar{x}\bar{y}) = \begin{cases} k(\bar{x}) - k(\bar{y}) & \text{if } k(\bar{y})_1 < k(\bar{x})_1, \\ k(\bar{y}) - k(\bar{x}) & \text{if } k(\bar{x})_1 < k(\bar{y})_1, \\ \text{undefined} & \text{if } k(\bar{x})_1 = k(\bar{y})_1, \end{cases} \quad (18)$$

$$v_k(\bar{x}\bar{y}) = \begin{cases} \text{space}(k(\bar{x}), k(\bar{y})) & \text{if } k(\bar{y})_1 \neq k(\bar{x})_1, \\ \text{time}(k(\bar{x}), k(\bar{y})) & \text{if } k(\bar{x})_1 = k(\bar{y})_1, \\ \text{undefined} & \text{if } k(\bar{x})_1 = k(\bar{y})_1, \end{cases} \quad (19)$$

$$w_{hk} = k \circ h^{-1} \text{ (is a Poincaré transformation)}, \quad (20)$$

$$\text{wl}_k(h) = k \circ h^{-1}[\text{t-axis}]. \quad (21)$$

By (7)–(21), it is not difficult to prove that $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of $\text{SRDyn} \setminus \text{SPR}^+$, there are FTL inertial particles in $\mathcal{M}_1$ and there are...
no FTL inertial particles in $\mathcal{M}_2$. For details of the proof see below. By [26, Prop.5.1.], to prove that axiom schema $\text{SPR}^+$ is valid in models $\mathcal{M}_1$ and $\mathcal{M}_2$, it is enough to show that, for every inertial observer $k$, there is an automorphism fixing the quantities and taking observer $k$ to observer $\text{Id}$. For fixed observer $k$, let $\alpha$ be the following map:

$$\alpha(h) = h \circ k^{-1}, \quad \alpha(q) = q \quad \text{and} \quad \alpha(\vec{x}\vec{y}) = k(\vec{x})k(\vec{y})$$

(22)

for every inertial observer $h$, quantity $q$, and inertial particle $\vec{x}\vec{y}$. Clearly $\alpha$ takes observer $k$ to observer $\text{Id}$. It is not difficult to prove that $\alpha$ is an automorphism of our model, see below. Thus $\text{SPR}^+$ is valid in the models.

Details of the proof: Now we are going to show in detail that $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of axiom system $\text{SRDyn}$ and there are FTL inertial particles in $\mathcal{M}_1$ and there are no FTL inertial particles in $\mathcal{M}_2$. We are going to prove this simultaneously for the two models.

The field reduct of both models is the Euclidean field $Q$. Thus $\text{AxEField}$ is valid in the models.

By (20), world-view transformations are Poincaré transformations, hence they are affine transformations and bijections. Thus $\text{AxEv}$ is valid in models $\mathcal{M}_1$ and $\mathcal{M}_2$.

By (21), we have that $wl_k(k) = \text{t-axis}$. Thus, $\text{AxSelf}$ is valid in the two models, by (15).

We say that ray $\vec{x}\vec{y}$ is light-like iff $\vec{x} \neq \vec{y}$ and space$(\vec{x}, \vec{y}) = \text{time}(\vec{x}, \vec{y})$ and it is time-like iff space$(\vec{x}, \vec{y}) < \text{time}(\vec{x}, \vec{y})$.

By (17) and by the properties of Poincaré transformations, the world-lines of bodies in $\text{Ph}$ are the light-like rays according to any observer. Thus the “speed of light” is 1 for any observer. Thus $\text{AxLight}$ and the second part of $\text{AxSymD}$ holds.

Any Poincaré transformation $P$ preserves the spatial distance of points $\vec{x}, \vec{y} \in Q^d$ for which $x_1 = y_1$ and $P(\vec{x})_1 = P(\vec{y})_1$. Therefore, inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them. We have already shown that the speed of light is 1 according to each inertial observer in models $\mathcal{M}_1$ and $\mathcal{M}_2$. Consequently, axiom $\text{AxSymD}$ is also valid in these models.

To prove that axiom schema $\text{SPR}^+$ is valid in models $\mathcal{M}_1$ and $\mathcal{M}_2$, it is enough to show that, for observer $k$, map $\alpha$ given in (22) is an automorphism. It is clear that $\alpha$ leaves the elements of $Q$ fixed and it is a permutation on sets $B$, $\text{Ob}$, $\text{Ph}$ and $\text{lp}$ since $\text{Ob}$ is the set of Poincaré transformations. Thus $B(b) \Leftrightarrow B(\alpha(b))$, $\text{Ob}(b) \Leftrightarrow \text{Ob}(\alpha(b))$ and $\text{Ph}(p) \Leftrightarrow \text{Ph}(\alpha(p))$. To prove that $\alpha$ is an automorphism it remains to prove that $W(k, b, \vec{x}) \Leftrightarrow W(\alpha(k), \alpha(b), \vec{x})$ and $M(k, b, q) \Leftrightarrow M(\alpha(k), \alpha(b), q)$. By (15) and (16), it is sufficient to prove that for
every inertial observers $h$ and $o$, and inertial particle $\bar{x}\bar{y}$,

$$\text{wl}_h(o) = \text{wl}_{\alpha(h)}(\alpha(o)), \quad \text{wl}_h(\bar{x}\bar{y}) = \text{wl}_{\alpha(h)}(\alpha(\bar{x}\bar{y})), \quad \text{and}$$

$$m_h(\bar{x}\bar{y}) = m_{\alpha(h)}(\alpha(\bar{x}\bar{y})). \quad (23)$$

By (13), (14) and (22), we have

$$\text{wl}_{\alpha(h)}(\alpha(o)) = \alpha(h)[\text{wl}(\alpha(o))], \quad \text{wl}_{\alpha(h)}(\alpha(\bar{x}\bar{y})) = h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] =$$

$$h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] = h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] =$$

$$h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] = h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] =$$

$$h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] = h \circ k^{-1}[\text{wl}(\alpha(\bar{x}\bar{y}))] =$$

$$m_{\alpha(h)}(\alpha(\bar{x}\bar{y})) = m_{\alpha(h)}(\alpha(\bar{x}\bar{y})), \quad \text{and}$$

Thus (23) above holds.

Therefore axiom schema $\text{SPR}^+$ is valid in the models. We have proved that $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of the axiom system $\text{SR}$ of kinematics of special relativity.

By (17) and by the properties of Poincaré transformations, the world-lines of bodies in $\mathfrak{lp}_1$ are the rays, the world-lines of bodies in $\mathfrak{lp}_2$ are the time-like and the light-like rays according to any observer. Thus there are FTL inertial particles in $\mathcal{M}_1$ and there are no FTL inertial particles in $\mathcal{M}_2$. Thus we have proved that $\mathcal{M}_1 \models \exists \text{FTLp}$ and $\mathcal{M}_2 \models \neg \exists \text{FTLp}$.

It remains to prove that axioms of dynamics contained in $\text{Dyn}$ are also valid in the models.

First we turn proving that, for every natural number $n$, $\text{AxColl}_n$ is valid in the model. The proof is illustrated in Fig[10] Throughout $\bar{o} := (0, \ldots, 0) \in Q^d$ denotes the origin of the coordinate system. Recall that, by (17), $\text{wl}_k(\bar{x}\bar{y}) = ray_k(\bar{x}\bar{y})$ for every inertial particle $\bar{x}\bar{y}$ and inertial observer $k$. Therefore

$$\text{in}_k(\bar{x}\bar{y}, k(\bar{x})) \iff k(y)_1 < k(x)_1 \quad \text{and}$$

$$\text{out}_k(\bar{x}\bar{y}, k(\bar{x})) \iff k(x)_1 < k(y)_1. \quad (24)$$

Recall that, by (18), $\text{P}_k(\bar{x}\bar{y}) = k(\bar{x}) - k(\bar{y})$ iff $k(\bar{y})_1 < k(\bar{x})_1$ and $\text{P}_k(\bar{x}\bar{y}) = k(\bar{y}) - k(\bar{x})$ iff $k(\bar{x})_1 < k(\bar{y})_1$. Now, by (18) and (24), for every inertial observer $k$ and inertial particles $\bar{x}_1^1 \bar{y}_1^1, \ldots, \bar{x}_n^1 \bar{y}_n^1$ with
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Figure 10. Illustration for proving that \( \text{AxColl}_n \) is valid in the models.
Let \( \bigwedge_{i=1}^{n} v_k(\bar{x}^i \bar{y}^i) < \infty \), we get
\[
\sum_{\{i : \text{in}(\bar{x}^i \bar{y}^i, k(\bar{x}^i))\}} P_k(\bar{x}^i \bar{y}^i) = \sum_{\{i : \text{out}(\bar{x}^i \bar{y}^i, k(\bar{x}^i))\}} P_k(\bar{x}^i \bar{y}^i)
\]

\[
\sum_{\{i : k(\bar{y}^i) < k(\bar{x}^i)_{1}\}} P_k(\bar{x}^i \bar{y}^i) = \sum_{\{i : k(\bar{x}^i)_{1} < k(\bar{y}^i)\}} P_k(\bar{x}^i \bar{y}^i)
\]

\[
\sum_{\{i : k(\bar{y}^i)_{1} < k(\bar{x}^i)_{1}\}} (k(\bar{x}^i) - k(\bar{y}^i)) = \sum_{\{i : k(\bar{x}^i)_{1} < k(\bar{y}^i)\}} (k(\bar{y}^i) - k(\bar{x}^i)) \iff \sum_{i=1}^{n} (k(\bar{x}^i) - k(\bar{y}^i)) = \bar{o}. \tag{25}
\]

Therefore, by (25) and the equivalent form (7) of the definition of \( \text{coll} \), we get that
\[
\text{coll}_k(\bar{x}^1 \bar{y}^1 \ldots \bar{x}^n \bar{y}^n) \iff \bar{x}^1 = \ldots = \bar{x}^n \land \bigwedge_{i=1}^{n} v_k(\bar{x}^i \bar{y}^i) < \infty \land \sum_{i=1}^{n} (k(\bar{x}^i) - k(\bar{y}^i)) = \bar{o}. \tag{26}
\]

for every \( n \), inertial observer \( k \), and inertial particles \( \bar{x}^1 \bar{y}^1, \ldots, \bar{x}^n \bar{y}^n \).

To prove that \( \text{AxColl}_n \) is valid in the models let \( k \) and \( h \) be inertial observers and let \( \bar{x}^1 \bar{y}^1, \ldots, \bar{x}^n \bar{y}^n \) be inertial particles such that \( \text{coll}_k(\bar{x}^1 \bar{y}^1 \ldots \bar{x}^n \bar{y}^n) \) and \( v_k(\bar{x}^i \bar{y}^i) < \infty \) for every \( i \). Then, by (26), \( \bar{x}^1 = \ldots = \bar{x}^n \) and \( \sum_{i=1}^{n} (k(\bar{x}^i) - k(\bar{y}^i)) = \bar{o} \). Therefore \( \sum_{i=1}^{n} (h(\bar{x}^i) - h(\bar{y}^i)) = \bar{o} \), since \( h \circ k^{-1} \) is an affine transformation taking \( k(\bar{x}) \) to \( h(\bar{x}) \) for every \( \bar{x} \). Thus, by (26), \( \text{coll}_{h_k}(\bar{x}^1 \bar{y}^1 \ldots \bar{x}^n \bar{y}^n) \) holds. Therefore \( \text{AxColl}_n \) is valid in the models for every \( n \). Thus the axiom schema \( \text{Coll} \) is valid in the models.

Next we turn proving that \( \text{Ax\forall\text{inecoll}} \) is valid in the models. Let \( \text{fm}_1 \subseteq Q^d \) be the set of vectors with positive time components and let \( \text{fm}_2 \subseteq Q^d \) be the set of non FTL vectors with positive time components, i.e.,
\[
\text{fm}_1 := \{ \bar{x} \in Q^d : 0 < x_1 \}, \text{and} \quad \text{fm}_2 := \{ \bar{x} \in \text{fm}_1 : \text{space}(\bar{x}, \bar{o}) \leq \text{time}(\bar{x}, \bar{o}) \}.
\]

Let \( \text{fm} \) denote \( \text{fm}_1 \) in the case of \( \mathcal{M}_1 \) and denote \( \text{fm}_2 \) in the case of \( \mathcal{M}_2 \).

Recall that, by (17), \( \text{wl}_k(\bar{x} \bar{y}) = \text{ray}k(\bar{x})k(\bar{y}) \). By (15), \( P_k(\bar{x} \bar{y}) = k(\bar{x}) - k(\bar{y}) \) iff \( \bar{x} \bar{y} \) is incoming at \( k(\bar{x}) \) according to \( k \), and \( P_k(\bar{x} \bar{y}) = k(\bar{y}) - k(\bar{x}) \) iff \( \bar{x} \bar{y} \) is outgoing at \( k(\bar{x}) \). By the above, by the fact that the observers are Poincaré transformations, and by (8)–(10), it is easy to see that for every inertial observer at every coordinate point, the set of four-momenta of the incoming bodies is \( \text{fm} \) and the same holds for the outgoing bodies. Formally, for every inertial observer \( k \) and
coordinate point $\bar{x}$,
\[ \text{fm} = \{P_k(b) : l_p(b) \wedge \text{in}_k(b, \bar{x})\} = \{P_k(b) : l_p(b) \wedge \text{out}_k(b, \bar{x})\}. \tag{27} \]
By (14), for every inertial observer, every inertial particle of finite speed is an incoming or outgoing body at some coordinate point. Therefore (27) implies that
\[ \text{fm} = \{P_k(b) : l_p(b) \wedge v_k(b) < \infty\} \tag{28} \]
for every inertial observer $k$. It can be easily seen that fm is closed under addition, i.e.,
\[ P, P' \in \text{fm} \implies P + P' \in \text{fm}. \tag{29} \]

To prove that Ax\text{\text{\textbackslash{}vinecoll}} is valid, let $\bar{x}$ be a coordinate point, let $k$ be an inertial observer and let $a$ and $b$ be inertial particles with finite speeds according to $k$ such that the sum of their relativistic masses is nonzero. We have to prove that there are inertial particles $a'$ and $b'$ such that $\bar{x}$-\text{\textbackslash{}inecoll}_k(a'b') and $P_k(a') = P_k(a)$ and $P_k(b') = P_k(b)$. By (28) and (29), we have that $P_k(a), P_k(b), P_k(a) + P_k(b) \in \text{fm}$. By (27), there are inertial particles $a', b'$ and $c$ such that $P_k(a') = P_k(a), P_k(b') = P_k(b), P_k(c) = P_k(a) + P_k(b), \text{in}_k(a', \bar{x}), \text{in}_k(b', \bar{x})$ and $\text{out}_k(c, \bar{x})$. Thus $\text{coll}_k(a'b'c)$ by (7). Now, by (5), $\bar{x}$-\text{\textbackslash{}inecoll}_k(a'b')$. Therefore Ax\text{\textbackslash{}vinecoll} is valid in the models.

To prove that Ax\text{\textbackslash{}Speed} is valid let $k, h \in \text{\textbackslash{}lo}b, \bar{x}y \in l_p$ and $q \in Q$ be such that $v_k(\bar{x}y) = v_h(\bar{x}y) = q < 1$. It is enough to prove that $m_k(\bar{x}y) = m_h(\bar{x}y)$. By (19),
\[ \text{space}(k(\bar{x}), k(\bar{y})) = q \cdot \text{time}(k(\bar{x}), k(\bar{y})), \quad \text{and} \tag{30} \]
\[ \text{space}(h(\bar{x}), h(\bar{y})) = q \cdot \text{time}(h(\bar{x}), h(\bar{y})). \tag{31} \]

Since $h \circ k^{-1}$ is a Poincaré transformation taking $k(\bar{x})$ and $k(\bar{y})$ to $h(\bar{x})$ and $h(\bar{y})$, respectively, we get that
\[ \text{time}(k(\bar{x}), k(\bar{y}))^2 - \text{space}(k(\bar{x}), k(\bar{y}))^2 = \text{time}(h(\bar{x}), h(\bar{y}))^2 - \text{space}(h(\bar{x}), h(\bar{y}))^2. \tag{32} \]
By (31) - (32), $(1 - q^2)\text{time}(k(\bar{x}), k(\bar{y}))^2 = (1 - q^2)\text{time}(h(\bar{x}), h(\bar{y}))^2$. Thus \[ \text{time}(k(\bar{x}), k(\bar{y})) = \text{time}(h(\bar{x}), h(\bar{y})). \]
Now, by (14), \[ m_k(\bar{x}y) = \text{time}(k(\bar{x}), k(\bar{y})) = \text{time}(h(\bar{x}), h(\bar{y})) = m_h(\bar{x}y). \]
Then \[ m_k(\bar{x}y) = m_h(\bar{x}y). \] Therefore Ax\text{\textbackslash{}Speed} is valid in the models.

To prove that Ax\text{\textbackslash{}Mass} is valid, let $k$ and $h$ be inertial observers and let $b$ and $b'$ be inertial particles such that their velocities and their relativistic masses coincide according to observer $k$. Then, by definition (6) of four-momentum, the four-momenta of $b$ and $b'$ coincide according to observer $k$. Assume that the speed of $b$ is finite according to $h$. By (18) and by the fact that $h \circ k^{-1}$ is an affine transformation, it is easy to prove that the four-momenta of $b$ and $b'$ coincide according
to inertial observer \(h\), too. Then the relativistic masses of \(b\) and \(b'\) coincide according to \(h\) since relativistic mass is the time component of the four-momentum. Therefore \(AxMass\) is valid in the models.

Now we turn proving that \(AxThEx^+\) is valid in the models. We say that (straight) line \(\{\bar{x} + q \cdot (\bar{y} - \bar{x}) : Q(q)\}\) is \textbf{time-like} iff \(ray \bar{x}\bar{y}\) is time-like. The world-lines of inertial observers are the time-like lines according to observer \(Id\) by (13) and (14) since Poincaré transformations take the \(t\)-axis to time-like lines and for any time-like line \(\ell\) there is an orthocronous Poincaré transformation taking \(\ell\) to \(t\)-axis. Poincaré transformations take the set of time-like lines onto the set of time-like lines. Therefore the world-lines of inertial observers are the time-like lines according to any observer by (14). Therefore the first part of \(AxThEx^+\) holds. It is easy to see that the second part of \(AxThEx^+\) holds, because of (i)–(iv) below. (i) The set of four-momenta of the incoming bodies contains set \(fm_2\) by (27). (ii) The time components of the four-momenta are the relativistic masses. (iii) Four-momenta are parallel to the world-lines. (iv) World-lines of inertial particles are rays. Therefore \(AxThEx^+\) is valid in the models.

By the above axioms of dynamics are also valid in our models. Therefore both \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are models of \(SRDyn\). This completes the proof.

\[\square\]

Remark 7.3. We note that in the proof of Thm 7.2 the relativistic masses in both \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are positive. This means that \(\exists FTLp\) is also independent of \(SRDyn \cup \{\text{PosMass}\}\), where \(\text{PosMass}\) is the formula saying that the relativistic masses are positive, i.e.,

\[
\text{PosMass} : \ IOb(k) \land Ip(b) \land v_k(b) < \infty \rightarrow 0 < m_k(b).
\]

8. Concluding remarks

Paper [26] shows that the existence of FTL particles is logically independent of special relativistic kinematics. In this paper we have seen that the existence of FTL inertial particles is independent of special relativistic dynamics. So there are at least two “layers” (parts, sub-theories) of relativity theory which do not contradict faster than light particles. However, there are other “layers” to be investigated in a similar manner. For example, a next “layer” could be one in which fields interacting with particles are also considered.

In our forthcoming paper [19] we will show that \(SRDyn\) gives new predictions on relativistic masses of FTL inertial particles. In more

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3 This is so because of the following. Let \(b = \bar{x}\bar{y}\) and \(b' = \bar{x}'\bar{y}'\). Then, by [13], \(\pm(k(\bar{x}) - k(\bar{y})) = P_h(\bar{x}\bar{y}) = P_h(\bar{x}'\bar{y}') = \pm(k(\bar{x'}) - k(\bar{y}'))\). But then \(h(\bar{x}) - h(\bar{y}) = \pm(h(\bar{x'}) - h(\bar{y}'))\) since \(h \circ k^{-1}\) is an affine transformation taking \(k(\bar{x}), k(\bar{y}), k(\bar{x'}), k(\bar{y}')\) to \(h(\bar{x}), h(\bar{y}), h(\bar{x'}), h(\bar{y}')\), respectively. By [13], we conclude that \(P_h(\bar{x}\bar{y}) = \pm P_h(\bar{x}'\bar{y}')\). The time-components of the four-momenta are positive since relativistic masses are positive. Therefore \(P_h(\bar{x}\bar{y}) = P_h(\bar{x}'\bar{y}')\).
FTL particles are consistent with relativistic dynamics in detail. \textbf{SRDyn} implies that
\[ m_k(b) \sqrt{1 - v_k(b)^2} = m_h(b) \sqrt{1 - v_h(b)^2}, \] (33)
where \( b \) is a possibly FTL particle and \( k \) and \( h \) are (ordinary slower than light) inertial observers. Equation (33) gives back the usual mass-increase theorem for slower than light particles, and predicts that the relativistic mass and momentum of an FTL particle decrease with the speed, see Fig.11.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{Illustration for equation (33).}
\end{figure}

Similar predictions on FTL particles appear in Bilaniuk-Deshpande-Shudarshan [8], Sudarshan [25], Recami [21, 22], and Hill-Cox [17] without deriving them from basic assumptions.

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References

[1] H. Andréka, J. X. Madarász, and I. Németi, with contributions from: A. Andai, G. Sági, I. Sain, and Cs. Tőke. \textit{On the logical structure of relativity theories}. Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, 2002. http://www.math-inst.hu/pub/algebraic-logic/Contents.html.
[2] H. Andréka, J. X. Madarász, and I. Németi. Logical axiomatizations of space-time. Samples from the literature. In A. Prékopa and E. Molnár, editors, \textit{Non-Euclidean geometries}, pages 155–185. Springer-Verlag, New York, 2006.
[3] H. Andréka, J. X. Madarász, and I. Németi. Logic of space-time and relativity theory. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, \textit{Handbook of spatial logics}, pages 607–711. Springer-Verlag, Dordrecht, 2007.
FTL PARTICLES ARE CONSISTENT WITH RELATIVISTIC DYNAMICS

[4] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. Axiomatizing relativistic dynamics without conservation postulates. *Studia Logica*, 89(2):163–186, 2008. arXiv:0801.4870.

[5] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. A logic road from special relativity to general relativity. *Synthese*, 186(3):633–649, 2012. arXiv:1005.0960.

[6] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. What are the numbers in which spacetime?, 2012. arXiv:1204.1350v1 [gr-qc].

[7] Frank Arntzenius. Causal paradoxes in special relativity. *British J. Philos. Sci.*, 41(2):223–243, 1990.

[8] O.M.P. Bilaniuk, V.K. Deshpande, and E.C.G. Sudarshan. “meta” relativity. *Amer. J. Phys.*, 30:718–723, 1962.

[9] C. C. Chang and H. J. Keisler. *Model theory*. North-Holland Publishing Co., Amsterdam, 1990.

[10] O.I. Chashchina and Z.K. Silagadze. Breaking the light speed barrier, 2011. arXiv:1112.4714.

[11] A. Einstein. Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, 17:891–921, 1905.

[12] H. B. Enderton. *A mathematical introduction to logic*. Academic Press, New York, 1972.

[13] G. Feinberg. Possibility of faster-than-light particles. *Phys. Rev.*, 159:1089–1105, Jul 1967.

[14] M. Friedman. *Foundations of Space-Time Theories. Relativistic Physics and Philosophy of Science*. Princeton University Press, Princeton, 1983.

[15] P.O. Fröman. Historical background of the tachyon concept. *Archive for history of exact sciences*, 48(3-4):373–380, 1994.

[16] Robert Geroch. Faster than light?, 2010. arXiv:1005.1614v1 [gr-qc].

[17] James M. Hill and Barry J. Cox. Einstein’s special relativity beyond the speed of light. *Proc. R. Soc. A*, 3 October 2012. published online.

[18] Ulrich Jentschura and Benedikt Wundt. An infinitesimally superluminal neutrino is left-handed, conserves lepton number and solves the autobahn paradox (illustrative discussion). conference paper, First International Conference on Logic and Relativity: honoring Istvn Nmeti’s 70th birthday, September 8 -12, 2012, Budapest.

[19] J. X. Madarász and G. Székely. Why do the moments of superluminal particles decrease with their velocities?, 2012. in preparation.

[20] H. Nikolić. Causal paradoxes: A conflict between relativity and the arrow of time. *Foundations of Physics Letters*, 19(3), 2006.

[21] E. Recami. Classical tachyons and possible applications. *Rivista del nuovo cimento*, 9(6):1–178, 1986.

[22] E. Recami. Tachyon kinematics and causality: a systematic thorough analysis of the tachyon causal paradoxes. *Found. Phys.*, 17(3):239–296, 1987.

[23] E. Recami. The tolmann-regge antitelephone paradox: Its solution by tachyon mechanics. *Electronic Journal of Theoretical Physics*, 6(21):1–8, 2009.

[24] F. Selleri. Superluminal signals and the resolution of the causal paradox. *Foundations of Physics*, 36:443–463, 2006.

[25] E.C.G. Sudarshan. The theory of particles traveling faster than light i. In A. Ramakrishnan, editor, *Symposia on Theoretical Physics and Mathematics 10*, pages 129–151. Plenum Press, New York, 1970.

[26] G. Székely. The existence of superluminal particles is consistent with the kinematics of einstein’s special theory of relativity, 2012. arXiv:1204.1773.
[27] R. C. Tolman. *The Theory of the Relativity of Motion*. University of California, Berkely, 1917.

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