The Einstein equation should be divided by two

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Abstract

We present three reasons for rewriting the Einstein equation. The new version is physically equivalent but geometrically more clear. 1. We write \(4\pi\) instead of \(8\pi\) at the r.h.s, and we explain how this factor enters as surface area of the unit 2–sphere. 2. We define the Riemann curvature tensor and its contractions (including the Einstein tensor at the l.h.s.) with one half of its usual value. This compensates not only for the change made at the r.h.s., but it gives the result that the curvature scalar of the unit 2–sphere equals one, i.e., in two dimensions, now the Gaussian curvature and the Ricci scalar coincide. 3. For the commutator \([u,v]\) of the vector fields \(u\) and \(v\) we prefer to write (because of the analogy with the antisymmetrization of tensors)

\[ [u,v] = \frac{1}{2} (u \, v - v \, u) \]

which is one half of the usual value. Then, the curvature operator defined by

\[ \nabla_{[u \, v]} - \nabla_{[u,v]} \]

(where \(\nabla\) denotes the covariant derivative) is consistent with point 2, i.e., it equals one half of the usual value.
1 Introduction

In 1914, the foundation of General Relativity Theory was essentially finished [1]. It was completed by H. Lorentz and D. Hilbert in 1915 by noting that the Einstein tensor represents the variational derivative of the Riemannian curvature scalar [2]. Einstein himself was happy of having finished his work from the principal point of view, and he was aware, see e.g. [3], that there are yet misleading notational points.

I agree with the opinion formulated in the introduction of [4] that one should try "to develop gravitational theory in the most logical and straightforward way - in the way it would have developed without Einstein’s intervention." However, I would formulate it in a more respectful manner; and that goal seems not to be reached in [4], cf. [5].

The purpose of the present note is to rewrite the Einstein equation in a manner which is on the route Einstein went; it will be physically equivalent, but it gives more geometrical insight: We replace $8\pi$ at the r.h.s. by $4\pi$ and show, using [6], that this factor is just the surface area of a unit sphere. For comparison, we show, how several textbooks [7 - 11] deduce the factor $8\pi$.

As a byproduct of our form of the Einstein equation, two further inconsistencies of Riemannian geometry disappear.

Einstein always used the lower index position for coordinates; maybe, he was afraid of ambiguities when writing $x^2$. In [3], however, he already observed this to be an inconsistency ("Of course, according to this definition the $dx_{\nu}$ are components of a contravariant vector; however, here we continue to apply a beloved usage to write a subscript.") So it is consequential that now almost all textbooks on General Relativity e.g. [7 - 11] use the upper position: then the contravariant vector $dx^k$ carries, as all contravariant vectors do, one index in upper position. As one advantage of this, the Einstein sum convention can be made more rigorous: automatic summation is performed with all indices appearing in one upper and in one lower position. The line
element $ds$ is then defined via $ds^2 = g_{jk} dx^j dx^k$.

We mentioned this example to show that one can change the representation of the Einstein equation without damaging the ”spirit of General Relativity”.

Having seen this development we feel free rewriting the Einstein equation as follows.

\section{Einstein’s equation - right-hand side}

The right–hand side of the Einstein equation reads $\kappa T_{ij}$. In units where the light velocity $c = 1$, one takes usually $\kappa = 8 \pi G$; $G$ being Newton’s constant and $T_{ij}$ is the energy-momentum tensor. We want to give an argument why $\kappa = 4 \pi G$ is more natural.

Ludolf’s number $\pi$ is defined as the surface of a unit circle, and then $4 \pi$ turns out to be the surface of a unit sphere. Newton’s constant $G$ is defined by the acceleration $a = GM/r^2$ stemming from the gradient of the potential

\begin{equation}
\phi = - G M / r
\end{equation}

where $\phi$ is the potential of a point mass $M$ at distance $r$. This equation is equivalent to the Poisson equation

\begin{equation}
\Delta \phi = 4 \pi G \rho
\end{equation}

where $\rho$ represents the matter density. Looking into the proof that relates eq. (1) to eq. (2) one can see that the factor $4 \pi$ in eq. (2) is just the surface of the unit sphere.

This can be done in different ways: First, one can prove that in the sense of distributions

\begin{equation}
\Delta \left( -\frac{1}{r} \right) = 4\pi \delta
\end{equation}
and the proof uses the surface integral over a small sphere; second, one can approach the point mass by a sequence of spherical shells of matter with the same result. Third, one could look for higher dimensions whether the factor $4\pi$ is only accidentally equal to the surface area.

Let $\omega_n$ be the surface of the unit sphere $S^{n-1}$ in the Euclidean $\mathbb{R}^n$. It holds $\omega_n = 2\pi^{n/2}/\Gamma(\frac{n}{2})$.

Newton's constant $G$ in $n$ spatial dimensions is defined by the acceleration $a = GM/r^{n-1}$ stems from the gradient of the potential $\phi = -\frac{1}{n-2} GM/r^{n-2}$. (3)

This equation is equivalent (see [6] Ch. 5 and II Ch. 4) to the Poisson equation

$$\Delta \phi = \omega_n G \rho$$

(4)

Eq. (4) clearly shows that $4\pi$ in eq. (2) is not only by accident the surface of the sphere. Again, one can prove this in the sense of distributions by applying

$$\Delta\left(-\frac{1}{r^{n-2}}\right) = (n-2) \omega_n \delta.$$ 

Using the symbol $n!$ as usual (i.e., $0! = 1$, $n! = (n-1)! \cdot n$), we get

$$\omega_{2n} = 2\pi^n/(n-1)!$$

and

$$\omega_{2n+1} = 2\pi^n \cdot 4^n \cdot n!/(2n)!.$$ 

Einstein tried to generalize just this Poisson equation (2), and he used $\rho = T_{00}$. So it is natural to write the right–hand side of the Einstein equation as

$$4 \pi G T_{ij}$$

(5)

Let us now look how textbooks get the value $8\pi$: (we do not mention those books which do not comment this choice). In [7], chapter 9.1 one reads
for the deduction of the l.h.s. of the Einstein equation "There is only one tensor ... namely \( E_{ij} = R_{ij} - \frac{1}{2}g_{ij}R \). Looking into the details, one can see that this means "Up to a constant multiple, there is only one ...". Putting this constant to 1, the \( 8\pi \) at the r.h.s. is fixed. (The deduction in [9] via eq. (IV,3,3) turns out to be quite similar). In [10], the ansatz eq.(11.4) \( E_{ij} = R_{ij} + c_1 R g_{ij} + c_2 g_{ij} \) is called the only possible expression (where, of course, a factor \( c_0 \) in front of \( R_{ij} \) would be possible, too). In all these cases, an additional factor \( \frac{1}{2} \) in front of the l.h.s would lead to the r.h.s. \((5) = 4\pi GT_{ij}\). Such a r.h.s. is called an "apparently natural equation" ([11], after eq. (3.12), however, in a slightly different context).

How Einstein deduced his equation with the factor \( 8\pi \)? [1] page 1076 reads: "Wir setzen ..., indem wir über die Konstante willkürlich verfügen, ... \( H = -g^{ij}\Gamma^k_{il}\Gamma^l_{jk} \)" (We arbitrarily fix the constant such that ...). At that moment, it seems, he had already the knowledge about the sign, but not about the detailed consequences, and so he put the constant to 1. In eq. (73) he defines the object \( E_{ij} \) (now called the Einstein tensor) from the derivatives of \( H \), and again the factor in front of it was chosen to be 1 for simplicity. Then the famous equation (74) \( E_{ij} = \kappa T_{ij} \) follows. At page 1083 he deduces the Newtonian limit and writes \( \kappa^2 = 4\pi G \). Surely, he felt at that moment, that an additional factor \( \frac{1}{2} \) in the definition (73) of \( E_{ij} \) would more directly lead to the desired result \( \kappa = 4\pi G \). However, we do not know why he did not insert it.

3 The curvature tensor

The antisymmetrization brackets \( [\ ] \) are defined by

\[
S_{[ij]} = \frac{1}{2} (S_{ij} - S_{ji}).
\]
The factor $\frac{1}{2}$ follows from the natural requirement of idempotency of the antisymmetrization operator: antisymmetrization should not alter antisymmetric tensors.

The same kind of brackets $[ ]$ are used to express the commutator $[u, v]$ of the vector fields $u$ and $v$. We prefer to write

$$[u, v] = \frac{1}{2} ( u v - v u )$$

which is one half of the usual value. This can be motivated as follows: let $e_A$ be an $n$-bein, i.e., an anholonomic basis in the $n$-dimensional (Pseudo-) Riemannian manifold, then eqs. (6) and (7) imply the validity of

$$[e_A, e_B] = e_{[A e_B]}$$

which would be not only less aesthetic but also confusing if it needs an additional factor 2 at the right-hand side. (What happens if we write the commutator of fields as usual, e.g. in [8]? From (2.6) and (2.65) one gets an unexplained factor 2 in (2.68).)

Now, we define the curvature operator by

$$\nabla[u \nabla v] - \nabla[u, v]$$

where $\nabla$ denotes the covariant derivative. The first term most naturally defines curvature from the commutator of covariant derivatives, the second term is deduced as follows: It is the only multiple of $\nabla[u, v]$ which realizes the requirement that the curvature operator is linear with respect to multiplication of $u$ or $v$ with scalar functions.

In a coordinate basis eq. (9) reads

$$R^k_{mij} = \Gamma^k_{li} \Gamma^l_j m - \Gamma^k_{m[i,j]}$$

This represents just one half of its usual value, see e.g. eqs. (2.18) and (2.20) in [11] resp.
We keep the relations $R_{ij} = R_{imj}^m$ and $R = g^{ij} R_{ij}$. These formulas do not depend on the dimension, and they have the advantage that for $n = 2$, $R$ is equal to the Gaussian curvature $K$. This is more satisfactory than the usual equation $R = 2K$. In our convention, the unit sphere has $R = 1$.

4 Einstein’s equation - left-hand side

We keep the formula

$$E_{ij} = R_{ij} - \frac{R}{2} g_{ij}$$

for the Einstein tensor, but we use the definitions of sct. 3, i.e., one half of the usual values. This compensates for the factor $\frac{1}{2}$ introduced in sct. 2, and it has the advantage that now in the weak-field limit,

$$E_{00} = \Delta \phi = 4\pi G \rho = 4\pi G T_{00}$$

5 Summary

Poisson’s equation $\Delta \phi = 4\pi G \rho$ is generalized to the Einstein equation

$$E_{ij} = 4\pi G T_{ij}$$

where the Einstein tensor has one half of its usual value. In the weak-field limit we have: the relation of the right-hand sides is $\rho = T_{00}$, and for the left-hand sides $ds = (1 + \phi)dt$ for purely temporal distances, see [1, p. 1084]. So, a lot of superfluous and embarrassing factors 2 have been cancelled. Going this way, the Einstein tensor is unique, and not only uniquely defined up to a constant factor.

The differential form of energy–momentum conservation $T_{ij}^{\text{ij}} = 0$ can be proven to follow from the Bianchi identity, but a more lucid and direct proof
uses the fact that the Einstein tensor represents the variational derivative of a scalar.

The two byproducts promised in the introduction are: the consistency of eq. (8), and the definition of the curvature scalar such that it now coincides with the Gaussian curvature in two dimensions.

We do not need to fix any further sign conventions:

1. The metric signatures (- + + +) and (+ - - -) go into each other by the transformation $g_{ij} \rightarrow -g_{ij}$, and $E_{ij}$ and all other essential quantities are invariant by this transformation.

2. The mentioned condition that the curvature scalar of the 2–sphere equals 1 already fixes the sign conventions for $R$ and $R_{ij}$. (We need, of course, the additional, but always fulfilled, condition, that neither signature nor dimension explicitly enter the definition of curvature.)

3. A sign convention for the Weyl tensor is never necessary - it enters always with its square.

Let us conclude with a more general remark: The new version of the Einstein equation deduced here is the result of more than one decade of analysis of typical errors and typical barriers of understanding. The fact that the old version is in common use and should not be altered does not count: Nowadays even century–old mistranslations of the Holy Bible will be corrected, and the just now finished reform of the German language will hopefully cancel a lot of illogicalities, why should not also General Relativity Theory be freed from such burdens?

References
[1] A. Einstein, Sitzungsberichte der königlich preussischen Akademie der Wissenschaften, Berlin, pp. 1030 - 1085 ”Die formale Grundlage der allgemeinen Relativitätstheorie” vorgelegt (= submitted) am 29. October 1914, ausgegeben (= appeared) am 26. November 1914.
[2] ditto, pp. 1111 - 1116 "Hamiltonsches Prinzip und allgemeine Relativitätstheorie" 1916. Here, the contributions of Lorentz and Hilbert are acknowledged.

[3] Ref. [1], p. 1035 "Natürlich sind gemäß dieser Definition die $dx_\nu$ selbst Komponenten eines kontravarianten Vierervektors; trotzdem wollen wir hier, der Gewohnheit zuliebe, den Index unten belassen."

[4] Ohanian, H., Ruffini, R. (1994). *Gravitation and Spacetime* (Norton, New York).

[5] Krasiński, A. (1995). *Class. Quant. Grav.* 12, 2361.

[6] Courant, R., Hilbert, D. (1924/1937). *Methoden der mathematischen Physik I/II* (Springer, Berlin).

[7] Stephani, H. (1982). *General Relativity* (Cambridge University Press).

[8] Schmutzer, E. (Ed.) (1980) *Exact Solutions of Einstein’s Field Equations* (Verl. d. Wiss. Berlin).

[9] Schmutzer, E. (1968) *Relativistische Physik* (Teubner Leipzig).

[10] Möller, C. (1972) *The theory of relativity* (Clarendon Oxford).

[11] Hawking, S., Ellis, G. (1973) *The large scale structure of space–time* (Cambridge University Press).

9