Abstract

The gravitational interaction is scale-free in both Newtonian gravity and general theory of relativity. The concept of self-similarity arises from this nature. Self-similar solutions reproduce themselves as the scale changes. This property results in great simplification of the governing partial differential equations. In addition, some self-similar solutions can describe the asymptotic behaviors of more general solutions. Newtonian gravity contains only one dimensional constant, the gravitational constant, while the general relativity contains another dimensional constant, the speed of light, besides the gravitational constant. Due to this crucial difference, incomplete similarity can be more interesting in general relativity than in Newtonian gravity. Kinematic self-similarity has been defined and studied as an example of incomplete similarity in general relativity, in an effort to pursue a wider application of self-similarity in general relativity. We review the mathematical and physical aspects of kinematic self-similar solutions in general relativity.
1 Introduction

Scale-invariance is one of the most fundamental characteristics of gravitational interaction in both Newtonian gravity and general relativity. This implies that if we consider appropriate matter fields, the governing partial differential equations are invariant under scale transformation. Due to this feature of the governing equations, there are self-similar solutions, which are invariant under the scale transformation. Self-similarity assumption enables us to simplify the governing equations. Self-similar solutions have a wide range of applications in astrophysics. See [14] for a recent review of self-similar solutions in general relativity. See [1] for self-similarity in more general contexts.

When a theory has no characteristic scale, we can expect scale-invariance of the theory. In Newtonian gravity, the gravitational constant $G$, with dimension $M^{-1}L^3T^{-2}$, is the only dimensional physical constant in the field equations, where $M$, $L$ and $T$ denote the dimensions of mass, length and time, respectively. It is impossible to construct a physical scale only from $G$. In general relativity, there exists another physical constant $c$, which is the speed of light, with dimension $LT^{-1}$. In spite of these two dimensional constants, no characteristic length scale can be constructed from these physical constants. However, due to the existence of these two dimensional constants, general relativity is qualitatively different from Newtonian gravity with respect to scale invariance. If we consider quantum gravity, the Planck constant $h$ appears, with dimension $ML^2T^{-1}$, so that there exists a characteristic scale $l_{\text{pl}} \equiv G^{1/2}h^{1/2}/c^{3/2}$, which is called the Planck length. Therefore, in the quantum theory of gravity, it is plausible that the scale invariance of the theory is broken down. Hereafter in this review we focus on Newtonian gravity and general relativity. We follow the sign conventions of [53] for the metric, Riemann and Einstein tensors.

2 Self-similarity in Newtonian gravity

Since Newtonian gravity postulates an absolute system of space and time, we can directly apply the general formulation of self-similarity to this system [1]. A solution is called self-similar, if a dimensionless quantity $Z(t, \vec{x})$ made of the solution is of the form

$$Z(t, \vec{x}) = Z\left(\frac{\vec{x}}{a(t)}\right),$$

where $\vec{x}$ and $t$ are independent space and time coordinates, respectively, and $a(t)$ is a function of $t$. This implies that the spatial distribution of the characteristics of motion remains similar to itself at all times during the motion. If the function $a(t)$ is derived from dimensional considerations alone, i.e., if it is uniquely determined so that $\vec{x}/a(t)$ is dimensionless, the self-similarity is called complete similarity or similarity of the first kind [1]. In more general situations, the characteristic length or time scale may be constructed by the dimensional constants in the system. Then, the function $a(t)$ cannot be uniquely determined from dimensional considerations alone. In such cases, self-similarity is called incomplete similarity or similarity of the second kind [1]. For example, when we have the constant sound speed $c_s$ and no characteristic scale, then $a(t)$ is uniquely determined as $a(t) = c_st$. In this case, the similarity is called complete. However, when we have a characteristic length scale $l$ besides the sound speed $c_s$, then $a(t) = l^{1-\alpha}/(c_st)^\alpha$ is possible and the constant $\alpha$ may not be determined from the governing equations. In this case, the similarity is called incomplete. The constant $\alpha$ may be determined by boundary conditions. It should be noted that the dimensional constant could appear not only from governing equations but also from boundary conditions.

Here, we give two important examples of completely self-similar solutions in Newtonian self-gravitating fluid mechanics. The basic field equations for spherically symmetric hydrodynamics of
a self-gravitating ideal gas in Eulerian description are given by
\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0, \tag{2.2}
\]
\[
\frac{\partial}{\partial t} (\rho v) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v^2) + \frac{\partial \rho}{\partial r} + \rho \frac{GM}{r^2} = 0, \tag{2.3}
\]
\[
\frac{\partial M}{\partial t} + v \frac{\partial M}{\partial r} = 0, \tag{2.4}
\]
\[
\frac{\partial M}{\partial r} = 4 \pi r^2 \rho, \tag{2.5}
\]
where \( \rho, v, M \) and \( G \) denote the mass density, radial velocity, total mass inside the radial coordinate \( r \), and gravitational constant, respectively.

2.1 Isothermal gas
First we consider an isothermal gas as a gravitational source. Since the isothermal gas is a relevant description of cold molecular clouds in galaxies, self-similar solutions have been intensively studied in Newtonian gravity in modeling the star formation process \([15, 61, 92, 117, 76]\). It has been revealed that self-similar solutions play important roles in the gravitational collapse of an isothermal gas \([27, 72, 89]\). The stability of these self-similar solutions have been studied \([50, 82, 83, 34]\). A new insight has been obtained in this system in the context of critical behavior in gravitational collapse \([47, 39]\).

For an isothermal gas that obeys \( p = \frac{c_s^2}{2} \rho \), where \( c_s \) is the constant speed of sound with dimension \( LT^{-1} \), it is impossible to construct a characteristic scale from \( c_s \) and \( G \). We introduce the dimensionless self-similar coordinate
\[
z = \frac{c_s t}{r}, \tag{2.6}
\]
for self-similar solutions. Then we also introduce the dimensionless functions \( U, P \) and \( m \):
\[
v(r, t) = -c_s U(r, t), \tag{2.7}
\]
\[
\rho(r, t) = \frac{c_s^2 P(r, t)}{4 \pi G r^2}, \tag{2.8}
\]
\[
M(r, t) = \frac{c_s^3 t m(r, t)}{G}. \tag{2.9}
\]
We assume that the above-defined functions \( U, P \) and \( m \) depend only on \( z \). From this assumption, equations \((2.2) - (2.5)\) become
\[
U' = \frac{(zU + 1)[P(zU + 1) - 2]}{(zU + 1)^2 - z^2}, \tag{2.10}
\]
\[
P' = \frac{zP[2 - P(zU + 1)]}{(zU + 1)^2 - z^2}, \tag{2.11}
\]
\[
m = P(U + 1/z), \tag{2.12}
\]
\[
-z^2 m' = P, \tag{2.13}
\]
where the prime denotes the derivation with respect to \( z \). The self-similar solutions for an isothermal gas are obtained from these ordinary differential equations. Self-similar solutions scale for the scale transformations \( \bar{t} = at, \bar{r} = ar \) as
\[
v(\bar{r}, \bar{t}) = v(r, t), \tag{2.14}
\]
\[
\rho(\bar{r}, \bar{t}) = \frac{\rho(r, t)}{a^2}, \tag{2.15}
\]
\[
M(\bar{r}, \bar{t}) = aM(r, t), \tag{2.16}
\]
where \( a \) is a constant. The basic equations for self-similar solutions are singular at the center and at the point at which \((zU + 1)^2 - z^2 = 0\) is satisfied, which is called a sonic point.

### 2.2 Polytropic gas

Next, we consider a polytropic gas as a gravitational source. A polytropic gas obeys the equation of state \( p = K\rho^\gamma \), where \( \gamma \) is the dimensionless constant called the adiabatic exponent and \( K \) is a constant with dimension \( M^{1-\gamma}L^{3\gamma-1}T^{-2} \). As in the isothermal gas system, it is impossible to construct a characteristic scale only from \( G \) and \( K \) if \( \gamma \neq 2 \). For the exceptional case, \( \gamma = 2 \), the system has a characteristic length scale \( l = \sqrt{K/G} \) but even in this case the self-similar variable \( z \) is uniquely constructed. Then, complete similarity is applicable to this system. Self-similar solutions in this system have been studied [77, 65]. The stability of these solutions have been studied [33, 34].

For the polytropic case, we introduce the dimensionless self-similar coordinate

\[
z = \frac{\sqrt{K}(-t)^{2-\gamma}}{(4\pi G)^{(\gamma-1)/2} r}.
\]

Then we also introduce the dimensionless functions \( U \), \( P \) and \( m \):

\[
v(r, t) = -(4\pi G)^{(1-\gamma)/2} \sqrt{K}(-t)^{1-\gamma}U(r, t),
\]

\[
\rho(r, t) = \frac{K^{1/(2-\gamma)}P(r, t)}{(4\pi G)^{1/(2-\gamma)}r^{2/(2-\gamma)}},
\]

\[
M(r, t) = \frac{K^{3/2}(-t)^{4-3\gamma}m(r, t)}{(4\pi)^{3\gamma-1/2}G^{3\gamma-1}/2}.
\]

We assume that the above-defined functions \( U \), \( P \) and \( m \) depend only on \( z \). In the polytropic case, the sonic point is defined by \((2 - \gamma - zU)^2 - \gamma z^2/(2-\gamma) = 0\). Self-similar solutions scale for the scale transformations \( \bar{t} = at \), \( \bar{r} = a^{2-\gamma}r \), as

\[
v(\bar{r}, \bar{t}) = a^{1-\gamma}v(r, t),
\]

\[
\rho(\bar{r}, \bar{t}) = \frac{\rho(r, t)}{a^{2/(2-\gamma)}},
\]

\[
M(\bar{r}, \bar{t}) = a^{1-3\gamma}M(r, t),
\]

where \( a \) is a constant. In this case, the scaling rates for \( r \) and \( t \), which keep \( z \) constant, are different from each other.

It should be again emphasized that in both the isothermal and polytropic cases, the self-similarity is complete since the self-similar variable \( z \) can be obtained from dimensional considerations alone. This is because there are only two dimensional constants in the system, while there are three independent dimensions \( M \), \( L \) and \( T \).

### 3 Self-similarity in general relativity

#### 3.1 Homothety

In general relativity, the concept of self-similarity is not so straightforward because general relativity has general covariance against coordinate transformation. This implies that the definition should be made covariantly in general relativity. In the following, we use units where the speed of light \( c \) is unity. In this choice of units, \( T = L \) is obtained and the velocity is dimensionless.

In general relativity, the term self-similarity can be used in two ways. One is for the properties of spacetimes, the other is for the properties of matter fields. These are not equivalent in general. The
self-similarity in general relativity was defined for the first time by Cahill and Taub [11]. Self-similarity is defined by the existence of a homothetic vector $\xi$ in the spacetime, which satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 2\alpha g_{\mu\nu},$$  \hspace{1cm} (3.1)$$

where $g_{\mu\nu}$ is the metric tensor, $\mathcal{L}_\xi$ denotes Lie differentiation along $\xi$ and $\alpha$ is a constant [11]. This is a special type of conformal Killing vectors. This self-similarity is called homothety. If $\alpha \neq 0$, then it can be set to be unity by a constant rescaling of $\xi$. If $\alpha = 0$, i.e. $\mathcal{L}_\xi g_{\mu\nu} = 0$, then $\xi$ is a Killing vector.

Homothety is a purely geometric property of spacetime so that the physical quantity does not necessarily exhibit self-similarity such as $\mathcal{L}_\xi Z = dZ$, where $d$ is a constant and $Z$ is, for example, the pressure, the energy density and so on. From equation (3.1) it follows that

$$\mathcal{L}_\xi R_{\mu\nu\sigma\rho} = 0,$$  \hspace{1cm} (3.2)$$

and hence

$$\mathcal{L}_\xi R_{\mu\nu} = 0,$$  \hspace{1cm} (3.3)$$

$$\mathcal{L}_\xi G_{\mu\nu} = 0.$$  \hspace{1cm} (3.4)$$

A vector field $\xi$ that satisfies equations (3.2), (3.3) and (3.4) is called a curvature collineation, a Ricci collineation and a matter collineation, respectively. It is noted that equations (3.2), (3.3) and (3.4) do not necessarily mean that $\xi$ is a homothetic vector. We consider the Einstein equations

$$G_{\mu\nu} = 8\pi GT_{\mu\nu},$$  \hspace{1cm} (3.5)$$

where $T_{\mu\nu}$ is the energy-momentum tensor. If the spacetime is homothetic, the energy-momentum tensor of the matter fields must satisfy

$$\mathcal{L}_\xi T_{\mu\nu} = 0,$$  \hspace{1cm} (3.6)$$

through equations (3.3) and (3.4). For a perfect fluid case, the energy-momentum tensor takes the form of

$$T_{\mu\nu} = (p + \mu)u_\mu u_\nu + pg_{\mu\nu},$$  \hspace{1cm} (3.7)$$

where $p$ and $\mu$ are the pressure and the energy density, respectively. Then, equations (3.1) and (3.6) result in

$$\mathcal{L}_\xi u^\mu = -\alpha u^\mu,$$  \hspace{1cm} (3.8)$$

$$\mathcal{L}_\xi \mu = -2\alpha \mu,$$  \hspace{1cm} (3.9)$$

$$\mathcal{L}_\xi p = -2\alpha p.$$  \hspace{1cm} (3.10)$$

As shown above, for a perfect fluid, the self-similarity of the spacetime and that of the physical quantity coincide. However, this fact does not necessarily hold for more general matter fields.

For spherically symmetric homothetic spacetimes, we can assume that there is a coordinate system $t$ and $r$ such that all dimensionless variables are functions of a single dimensionless self-similar variable $\xi \equiv r/t$. The solution is invariant under scale transformation $\bar{t} = at$, $\bar{r} = ar$ for any constant $a$. Thus the self-similar variables can be determined from dimensional considerations in the case of homothety. Therefore, we can conclude homothety as the general relativistic analogue of complete similarity.

From the constraints (3.9) and (3.10), we can show that if we consider the barotropic equation of state, i.e., $p = f(\mu)$, then the equation of state must have the form $p = K\mu$, where $K$ is a constant. This class of equations of state contains a dust fluid ($K = 0$), a radiation fluid ($K = 1/3$) and a stiff fluid ($K = 1$) as special cases. Other important matter fields that are compatible with homothety are a massless scalar field and a scalar field with an exponential potential.
3.2 Kinematic self-similarity

Although homothetic solutions can contain several interesting matter fields, the matter fields compatible with homothety are rather limited. In more general situations, matter fields will have intrinsic dimensional constants. For example, when we consider a polytropic equation of state, such as \( p = K \gamma^\gamma \), the constant \( K \) has dimension \( M^{1-\gamma} L^{3(\gamma-1)} \), where we should be reminded that we have chosen the light speed \( c \) to be unity. We can also consider a massive scalar field, where the mass of the scalar field has dimension \( M \).

In such cases, it is impossible to assume homothety because the system has a characteristic scale. By analogy, we can consider the general relativistic counterpart of incomplete similarity. From comparison with self-similarity for a polytropic gas in Newtonian gravity, kinematic self-similarity has been defined in the context of relativistic fluid mechanics as an example of incomplete similarity \([19, 20, 23]\). It should be noted that the introduction of incomplete similarity to general relativity is not unique. For example, partial self-similarity has been defined and applied to inhomogeneous cosmological solutions \([67, 68, 71]\).

A spacetime is said to be kinematic self-similar if it admits a kinematic self-similar vector \( \xi \) which satisfies the conditions

\[
\mathcal{L}_\xi h_{\mu\nu} = 2\delta h_{\mu\nu}, \tag{3.11}
\]
\[
\mathcal{L}_\xi u_\mu = \alpha u_\mu, \tag{3.12}
\]

where \( u^\mu \) is the four-velocity of the fluid and \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) is the projection tensor, and \( \alpha \) and \( \delta \) are constants \([19, 20, 23]\). If \( \delta \neq 0 \), the similarity transformation is characterized by the scale-independent ratio \( \alpha/\delta \), which is referred to as the similarity index. If the ratio is unity, \( \xi \) turns out to be a homothetic vector. In the context of kinematic self-similarity, homothety is referred to as self-similarity of the first kind. If \( \alpha = 0 \) and \( \delta \neq 0 \), it is referred to as self-similarity of the zeroth kind. If the ratio is not equal to zero or one, it is referred to as self-similarity of the second kind. If \( \alpha \neq 0 \) and \( \delta = 0 \), it is referred to as self-similarity of the infinite kind. If \( \delta = \alpha = 0 \), \( \xi \) turns out to be a Killing vector.

From the Einstein equation (3.5), we can derive

\[
\mathcal{L}_\xi G_{\mu\nu} = 8\pi G \mathcal{L}_\xi T_{\mu\nu}. \tag{3.13}
\]

This equation is called the integrability condition. Now we can rewrite the integrability conditions (3.13) in terms of kinematic quantities of the fluid. The covariant derivative of the fluid four velocity is decomposed into the following form:

\[
u_{\mu;\nu} = \sigma_{\mu\nu} + \frac{1}{3} \theta h_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_{\nu}, \tag{3.14}
\]

where

\[
\theta_{\mu\nu} \equiv h_(\mu h_\nu)^\lambda u_{\kappa;\lambda}, \tag{3.15}
\]
\[
\theta \equiv g^{\mu\nu} \theta_{\mu\nu}, \tag{3.16}
\]
\[
\sigma_{\mu\nu} \equiv \theta_{\mu\nu} - \frac{1}{3} \theta h_{\mu\nu}, \tag{3.17}
\]
\[
\omega_{\mu\nu} \equiv h_(h_\mu h_\nu)^\lambda u_{\kappa;\lambda}, \tag{3.18}
\]
\[
\omega^2 \equiv \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu}, \tag{3.19}
\]
\[
\dot{u}_\mu \equiv u_{\mu;\nu} u^\nu, \tag{3.20}
\]
where the semicolon denotes the covariant derivative. Using the above quantities, the integrability condition (3.13) is rewritten as follows (cf. [23]):

\[(\delta - \alpha)(-8\omega^2 - 2\dot{u}_\kappa;\kappa) = 8\pi G \left[\frac{1}{2}(\mathcal{L}\xi\mu + 2\alpha\mu) + \frac{3}{2}(\mathcal{L}\xi\nu + 2\alpha\nu)\right],\] (3.21)

\[2(\delta - \alpha)(\dot{\theta} + \theta^2 - 4\omega^2) = 8\pi G \left[\frac{3}{2}(\mathcal{L}\xi\mu + 2\delta\mu) - \frac{3}{2}(\mathcal{L}\xi\nu + 2\delta\nu)\right],\] (3.22)

\[2\omega\lambda\mu\dot{u}^\mu + 2\omega\kappa\lambda;\kappa = 4\omega^2 u^\lambda = 0,\] (3.23)

\[\dot{\sigma}\lambda\rho - u_\rho\sigma\lambda\nu\dot{u}^\nu - u_\lambda\sigma\rho\nu\dot{u}^\nu + \theta\sigma\lambda\rho + \sigma\lambda\kappa\omega^\kappa\rho + \sigma\rho\kappa\omega^\kappa\lambda + 2\omega^\kappa\kappa\omega^\kappa\lambda + \frac{4}{3}h_{\lambda\rho}\omega^2 = 0.\] (3.24)

For the first-kind case, in which \(\alpha = \delta \neq 0\), equations (3.9) and (3.10) are obtained from equations (3.21) and (3.22). When a perfect fluid is irrotational, i.e., \(\omega_{\mu\nu} = 0\), the Einstein equations and the integrability conditions (3.21)–(3.24) give [23, 50]

\[(\alpha - \delta)R_{\mu\nu} = 0,\] (3.25)

where \(R_{\mu\nu}\) is the Ricci tensor on the hypersurface orthogonal to \(u^\mu\). This means that if a solution is kinematic self-similar but not homothetic and if the fluid is irrotational, then the hypersurface orthogonal to fluid flow is flat.

4 Spherically symmetric self-similar solutions

4.1 Spherically symmetric solutions

Although self-similar solutions can play important roles even in nonspherically symmetric solutions, such as homogeneous cosmological models [40, 73], we focus in the rest of this article on spherically symmetric spacetimes. The line element in a spherically symmetric spacetime is given by

\[ds^2 = -e^{2\Phi(t,r)}dt^2 + e^{2\Psi(t,r)}dr^2 + R(t,r)^2d\Omega^2,\] (4.1)

where \(d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\). We consider a perfect fluid as a matter field, for which the energy-momentum tensor is given by equation (3.7). We adopt the comoving coordinates, where the four-velocity of the fluid \(u^\mu\) has the components

\[u_\mu = (-e^\Phi, 0, 0, 0).\] (4.2)

Then, the Einstein equations and the equations of motion for the perfect fluid are reduced to the following simple form:

\[(\mu + p)\Phi_t = -p_r,\] (4.3)

\[(\mu + p)\Psi_t = -\mu_t - 2(\mu + p)\frac{R_t}{R},\] (4.4)

\[m_r = 4\pi \mu R_t R^2,\] (4.5)

\[m_t = -4\pi p R_t R^2,\] (4.6)

\[0 = -R_{tr} + \Phi_t R_t + \Psi_t R_r,\] (4.7)

\[2Gm = R(1 + e^{-2\Phi}R_t^2 - e^{-2\Psi}R_r^2),\] (4.8)

where the subscripts \(t\) and \(r\) denote the partial derivatives with respect to \(t\) and \(r\), respectively, and \(m(t,r)\) is called the Misner-Sharp mass. When a perfect fluid obeys an equation state \(p + \mu = 0\),
which is equivalent to a cosmological constant, the first two equations are trivially satisfied. In this case, one can use the following equations:

\[
-\frac{e^{2\Phi}}{R^2} - \left( \frac{R_t}{R} \right)^2 + 2 \frac{R_t R_t}{R} \Psi_t + e^{2\Phi - 2\Psi} \left[ \frac{2 R_{rr}}{R} - 2 R_r R_r \Psi_r + \left( \frac{R_r}{R} \right)^2 \right] = -8\pi G \mu e^{2\Phi},
\]

\[
\frac{e^{2\Psi}}{R^2} + e^{2\Psi - 2\Phi} \left[ \frac{2 R_{tt}}{R} - 2 \frac{R_t}{R} \Phi_t + \left( \frac{R_t}{R} \right)^2 \right] - \left[ \frac{1}{(R_t/R)^2} \right] = -8\pi G p e^{2\Psi},
\]

\[
e^{-2\Phi} \left( \Psi_{tt} + \Psi_t^2 - \Phi_t \Psi_t + \frac{R_{tt}}{R} + \frac{R_t \Psi_t}{R} - \frac{R_t \Phi_t}{R} \right) - e^{-2\Psi} \left( \Phi_{rr} + \Phi_r^2 - \Phi_r \Psi_r + \frac{R_{rr}}{R} + \frac{R_r \Phi_r}{R} - \frac{R_r \Psi_r}{R} \right) = -8\pi G p,
\]

which are \((tt)\), \((rr)\) and \((\theta\theta)\) components of the Einstein equations, respectively. Five of the above nine equations are independent.

### 4.2 Spherically symmetric homothetic solutions

There is a large variety of spherically symmetric homothetic solutions. The pioneering work in this area was done by Cahill and Taub [11]. The application contains primordial black holes [18, 5, 6], cosmological voids [4, 69, 70, 71], cosmic censorship [56, 57, 58, 74, 75, 60, 44, 26, 37] and critical behavior [21, 42, 25]. See [36] and [30] for recent reviews of cosmic censorship and critical behavior, respectively. The classification of all spherically symmetric homothetic solutions with a perfect fluid has been made [28, 29, 12, 13, 15, 16]. The spacetime structure possible for homothetic solutions has been studied [17]. The special case where the homothetic vector is orthogonal or parallel to the fluid flow has also been studied [52, 22]. It has been revealed that a homothetic solution describes the dynamical properties of more general solutions in spherically symmetric gravitational collapse [37]. The stability of homothetic solutions has been studied [42, 51, 43, 55, 37, 35, 9, 38].

When the spacetime admits a homothetic vector, which is neither parallel nor orthogonal to the fluid flow, the homothetic vector \(\xi\) can be written as

\[
\xi = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r},
\]

and the self-similar variable \(\xi\) is given by

\[
\xi = \frac{r}{t}.
\]

Homothety implies that the metric functions can be written

\[
ds^2 = -e^{2\Phi(\xi)} dt^2 + e^{2\Psi(\xi)} dr^2 + r^2 S(\xi)^2 d\Omega^2.
\]

As we have seen, the equation of state must be of the form \(p = K \mu\) for homothetic spacetimes. Then the governing equations for homothetic solutions are written as

\[
e^{2\Phi} = a_\sigma \xi^{\frac{4K}{K+1}} \eta \frac{2K}{K+1},
\]

\[
e^{2\Psi} = a_\omega \eta^{-\frac{2}{K+1}} S^{-4},
\]

\[
M + M' = \eta S^2 (S + S'),
\]

\[
M' = -K \eta S^2 S',
\]

\[
\frac{M}{S} = 1 + a_\sigma^{-1} \left( \eta \xi^{-2} \right)^{\frac{2K}{K+1}} \xi^2 S'^2 - \eta \frac{2}{K+1} S^4 (S + S')^2.
\]
where $a_\sigma$ and $a_\omega$ are integration constants and the prime denotes the derivative with respect to $\ln \xi$. The dimensionless functions $\eta(\xi)$ and $M(\xi)$ are defined by

\begin{align}
8\pi G\mu &= \frac{\eta}{r^2}, \\
2Gm &= rM.
\end{align}

(4.20) (4.21)

The above formulation is based on [11, 5, 6]. It is possible to choose another function in the same comoving coordinates, as adopted in [18, 13, 14]. In the comoving coordinates, the dynamical properties of the fluid elements are very clear.

There are other useful formulations in analyzing homothetic solutions. One of the most natural coordinate systems for homothetic spacetimes is the so-called homothetic coordinates. In terms of this coordinate system, the dynamical systems theory has been applied to homothetic solutions with a perfect fluid for classification [7, 28, 29]. In the homothetic coordinates, the self-similar variable is chosen to be the spatial or time coordinate, depending on whether the homothetic vector is timelike or spacelike. If the homothetic vector is timelike, the line element is written as

$$d\sigma^2 = e^{2t} \left[ -D_1^2(x)dt^2 + dx^2 + D_2^2(x)d\Omega^2 \right].$$

(4.22)

If the homothetic vector is spacelike, the line element is written as

$$d\sigma^2 = e^{2x} \left[ -dt^2 + D_1^2(t)dx^2 + D_2^2(t)d\Omega^2 \right].$$

(4.23)

If the homothetic vector is timelike in one region and spacelike in another region of the same spacetime, the above two charts must be patched on the hypersurface on which the homothetic vector is null.

Another coordinate system is that of area coordinates, in which the physical properties of the spacetime are clear. The area coordinate system has been adopted [57, 58, 60]. In this coordinate system, the line element in homothetic spacetimes is written as

$$d\sigma^2 = -e^{2\Phi(\xi)}dt^2 + e^{2\Psi(\xi)}dr^2 + r^2 S(\xi)^2d\Omega^2,$$

(4.24)

where $u^t$ and $u^r$ are also to be determined.

### 4.3 Spherically symmetric kinematic self-similar solutions

A kinematic self-similar vector may be parallel, orthogonal or tilted, i.e., neither parallel nor orthogonal, to the fluid flow. Spherically symmetric kinematic self-similar perfect fluid solutions have been recently explored by several authors [2, 3, 63, 10, 48, 49, 50].

In a spherically symmetric spacetime, the kinematic self-similar vector field $\xi$ is written in general as

$$\xi = h_1(t,r)\frac{\partial}{\partial t} + h_2(t,r)\frac{\partial}{\partial r},$$

(4.27)

in the comoving coordinates, where $h_1(t,r)$ and $h_2(t,r)$ are functions of $t$ and $r$. When $h_2 = 0$, $\xi$ is parallel to the fluid flow, while when $h_1 = 0$, $\xi$ is orthogonal to the fluid flow. When both $h_1$ and $h_2$ are nonzero, $\xi$ is tilted.

When the kinematic self-similar vector $\xi$ is tilted to the fluid flow and not of the infinite kind, $\xi$ and the metric tensor $g_{\mu\nu}$ are written in appropriate comoving coordinates as

$$\xi = (\alpha t + \beta)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r},$$

(4.28)

$$d\sigma^2 = -e^{2\Phi(\xi)}dt^2 + e^{2\Psi(\xi)}dr^2 + r^2 S(\xi)^2d\Omega^2,$$

(4.29)
where \( \alpha \) is the index of self-similarity. For \( \alpha = 1 \), i.e., homothety or self-similarity of the first kind, we can set \( \beta = 0 \) and then \( \xi \) is given by \( \xi = r/t \). For \( \alpha = 0 \), i.e., self-similarity of the zeroth kind, we can set \( \beta = 1 \) and then \( \xi \) is given by \( \xi = r/e^t \). For \( \alpha \neq 0 \) and \( \alpha \neq 1 \), i.e. self-similarity of the second kind, we can set \( \beta = 0 \) and then \( \xi \) is given by \( \xi = r/(at)^{1/\alpha} \). If the kinematic self-similar vector \( \xi \) is tilted to the fluid flow and of the infinite kind, \( \xi \) and the metric tensor \( g_{\mu\nu} \) are written in appropriate comoving coordinates as

\[
\xi = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r},
\]

\[
ds^2 = -e^{2\Phi(\xi)} dt^2 + \frac{e^{2\Psi(\xi)}}{r^2} dr^2 + S(\xi)^2 d\Omega^2,
\]

where the self-similar variable is given by \( \xi = r/t \).

If the kinematic self-similar vector \( \xi \) is parallel to the fluid flow and of the infinite kind, we have in appropriate coordinates

\[
\xi = t \frac{\partial}{\partial t},
\]

\[
ds^2 = -t^{2(\alpha-1)} e^{2\Phi(r)} dt^2 + t^2 dr^2 + t^2 S(r)^2 d\Omega^2,
\]

where \( \alpha \) is the index of self-similarity and the self-similar variable is given by \( \xi = r \). If the kinematic self-similar vector \( \xi \) is parallel to the fluid flow and of the infinite kind, we have in appropriate coordinates

\[
\xi = t \frac{\partial}{\partial t},
\]

\[
ds^2 = -e^{2\Phi(r)} dt^2 + dr^2 + S(r)^2 d\Omega^2,
\]

where the self-similar variable is given by \( \xi = r \).

If the kinematic self-similar vector \( \xi \) is orthogonal to the fluid flow and of the infinite kind, we have in appropriate coordinates

\[
\xi = r \frac{\partial}{\partial r},
\]

\[
ds^2 = -r^{2\alpha} dt^2 + e^{2\Psi(t)} dr^2 + r^2 S(t)^2 d\Omega^2,
\]

where \( \alpha \) is the index of self-similarity and the self-similar variable is given by \( \xi = t \). If the kinematic self-similar vector \( \xi \) is orthogonal to the fluid flow and of the infinite kind, we have in appropriate coordinates

\[
\xi = r \frac{\partial}{\partial r},
\]

\[
ds^2 = -r^2 dt^2 + e^{2\Psi(t)} dr^2 + S(t)^2 d\Omega^2,
\]

where the self-similar variable is given by \( \xi = t \).

Not as homothetic solutions in the tilted case, kinematic self-similar solutions have a characteristic structure. We now show an example of them in the case of self-similarity of the second kind, where a kinematic self-similar vector is tilted to the fluid flow. In this case, the Einstein equations imply that the quantities \( m, \mu \) and \( p \) must be of the following form:

\[
\frac{2Gm}{r} = M_1(\xi) + \frac{r^2}{t^2} M_2(\xi),
\]

\[
8\pi G\mu r^2 = W_1(\xi) + \frac{r^2}{t^2} W_2(\xi),
\]

\[
8\pi Gpr^2 = P_1(\xi) + \frac{r^2}{t^2} P_2(\xi),
\]
where \( \xi = r/(\alpha t)^{1/\alpha} \). In other words, dimensionless quantities on the left hand side are decomposed into two parts, one remains constant and the other behaves as \((r/t)^2 \propto \xi^{2(1-\alpha)}\) as \(\xi\) is fixed. Then, the original partial differential equations are satisfied when and only when the Einstein equations and the equations of motion for the matter field are satisfied for each of the \(O(1)\) and \(O((r/t)^2)\) terms. The equations (4.3)–(4.10) for a perfect fluid then reduce to the following:

\[
\begin{align*}
M_1 + M_1' &= W_1 S^2(S + S'), \\
3M_2 + M_2' &= W_2 S^2(S + S'), \\
M_1' &= -P_1 S^2 S', \\
2\alpha M_2 + M_2' &= -P_2 S^2 S', \\
M_1 &= S[1 - e^{-2\Psi}(S + S')^2], \\
\alpha^2 M_2 &= SS'e^{-2\Phi}, \\
(P_1 + W_1)\Phi' &= 2P_1 - P_1', \\
(P_2 + W_2)\Phi' &= -P_2', \\
W_1 S &= -(P_1 + W_1)(\Psi' S + 2S'), \\
(2\alpha W_2 + W_2')S &= -(P_2 + W_2)(\Psi' S + 2S'), \\
S'' + S' &= SS' + (S + S')\Psi', \\
S'(S' + 2\Psi' S) &= \alpha^2 W_2 S^2 e^{2\Phi}, \\
2S(S'' + 2S') - 2\Psi'S(S + S') &= -S'^2 - S^2 + e^{2\Psi}(1 - W_1 S^2), \\
2S(S'' + \alpha S' - \Phi'S') + S'^2 &= -\alpha^2 P_2 S^2 e^{2\Phi}, \\
(S + S')(S + S' + 2\Phi' S) &= (1 + P_1 S^2)e^{2\Psi},
\end{align*}
\]

where we have omitted the bars of \(\Phi\) and \(\Psi\) in (1.23) for simplicity and the prime denotes the derivative with respect to \(\ln \xi\). A similar structure of basic equations can be found for kinematic self-similar solutions of the second, zeroth and infinite kinds both in the tilted and orthogonal cases and of the second and zeroth kind in the parallel case. The exceptions are the first kind in the tilted, parallel and orthogonal cases and the infinite kind in the parallel case. See [49] [50] for the basic equations for spherically symmetric self-similar solutions for all cases.

It is interesting to consider the spherically symmetric self-similar solutions of the infinite kind with a kinematic self-similar vector parallel to the fluid flow. The metric form demanded by this self-similarity, which is given by equation (1.35), is nothing but the general form of the line element in spherically symmetric static spacetimes when the chosen radial coordinate is the radial physical length. Therefore, all static solutions have a kinematic self-similar vector of the infinite kind that is parallel to the fluid flow. Inversely, all spherically symmetric solutions with a kinematic self-similar vector of the infinite kind parallel to the fluid flow are static. The equation of state is not restricted at all.

### 4.4 Equation of state

It is obvious that equations (4.41) and (4.42) strongly restrict the form of the possible equations of state. The detailed analysis shows the following restriction on the equation of state [49]. Suppose we have the barotropic equation of state, i.e., \(p = f(\mu)\). The self-similarity of the second kind with the index \(\alpha\) can admit only the following equation of state:

\[
k_1 x + k_2 x^\alpha = f(C_1 x + C_2 x^\alpha),
\]

where \(k_1, k_2, C_1\) and \(C_2\) are arbitrary constants. For the self-similarity of the zeroth and infinite kinds, we cannot determine the equation of state alone from the decomposed form. It should be noted that
the above discussion is based not on the whole equations but only on the decomposed form of $p$ and $\mu$ such as equations (4.41) and (4.42).

For later convenience, we introduce and focus on the following equations of state:

- **Equation of state (1) (EOS1)**
  
  $$p = K \mu^\gamma,$$
  
  where $K$ and $\gamma$ are constants. Here we assume that $K \neq 0$ and $\gamma \neq 0, 1$.

- **Equation of state (2) (EOS2)**
  
  $$\begin{align*}
  p &= Kn^\gamma, \\
  \mu &= m_b n + \frac{p}{\gamma - 1},
  \end{align*}$$
  
  where the constant $m_b$ and $n(t, r)$ correspond to the mean baryon mass and the baryon number density, respectively. Here we assume that $K \neq 0$ and $\gamma \neq 0, 1$. In the literature, this equation of state is sometimes called a relativistic polytrope.

- **Equation of state (3) (EOS3)**
  
  $$p = K \mu,$$
  
  where we assume that $-1 \leq K \leq 1$.

EOS1 and EOS2 are two kinds of polytropic equations of state. These equations of state are incompatible with homothety. For $0 < \gamma < 1$, both EOS1 and EOS2 are approximated by a dust fluid in the high-density regime. For $1 < \gamma$, EOS2 is approximated by EOS3 with $K = \gamma - 1$ in the high-density regime. For $2 < \gamma$ for EOS2 and $1 < \gamma$ for EOS1, the dominant energy condition can be violated in the high-density regime, which would be unphysical.

## 5 Exact spherically symmetric self-similar solutions

### 5.1 Vacuum

In a vacuum, the only spherically symmetric solutions are the Minkowski solution and the Schwarzschild solution from Birkhoff’s theorem. Both solutions have kinematic self-similar vectors. Although there are no fluids, we can introduce a unit timelike vector $u^\mu$.

The Minkowski solution has seven kinematic self-similar vectors including a homothetic vector in the tilted case. The Minkowski solution is represented by

\[ ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \]
\[ 2Gm = 0, \]
\[ 8\pi G p = 8\pi G \mu = 0. \]

The metric can be represented in the Milne form

\[ ds^2 = -d\tau^2 + \tau^2 d\rho^2 + \tau^2 \sinh^2 \rho d\Omega^2, \]

where $t = \tau \cosh \rho$ and $r = \tau \sinh \rho$, or in another form

\[ ds^2 = -d\omega^2 d\nu^2 + d\omega^2 + \omega^2 \cosh^2 \nu d\Omega^2, \]

where $t = \omega \sinh \nu$ and $r = \omega \cosh \nu$. This spacetime has the following kinematic self-similar vectors:
• First kind, tilted

\[ \frac{\partial}{\partial t} + \frac{r}{r} \frac{\partial}{\partial r}, \]  

(5.6)

• First kind, parallel

\[ \frac{\tau}{\partial \tau}, \]  

(5.7)

• First kind, orthogonal

\[ \frac{\partial}{\partial \omega}, \]  

(5.8)

• Second kind with any \( \alpha \), tilted

\[ \alpha \frac{\partial}{\partial t} + \frac{r}{r} \frac{\partial}{\partial r}, \]  

(5.9)

• Zeroth kind, tilted

\[ \frac{\partial}{\partial t} + \frac{r}{r} \frac{\partial}{\partial r}, \]  

(5.10)

• Zeroth kind, orthogonal

\[ \frac{r}{\partial r}, \]  

(5.11)

• Infinite kind, parallel

\[ \frac{t}{\partial t}. \]  

(5.12)

The Schwarzschild solution has two kinematic self-similar vectors but does not have a homothetic vector. The Schwarzschild solution is represented by

\[ ds^2 = -\left(1 - \frac{2Gm_0}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2Gm_0}{r}} + \frac{r^2}{3} d\Omega^2, \]  

(5.13)

\[ 2Gm = 2Gm_0, \]  

(5.14)

\[ 8\pi Gp = 8\pi G\mu = 0, \]  

(5.15)

where \( m_0 \) is a constant. This spacetime can be represented in the following choice of coordinates:

\[ ds^2 = -d\tau^2 + (2Gm_0)^{2/3} \left( \frac{d\rho^2}{\left[\frac{3}{2}(\rho - \tau)\right]^{2/3}} + \left[\frac{3}{2}(\rho - \tau)\right]^{4/3} d\Omega^2 \right). \]  

(5.16)

This spacetime has the following kinematic self-similar vectors:
• Second kind with $\alpha = 3/2$, tilted
\[ \frac{\tau}{\partial \tau} + \frac{\rho}{\partial \rho}, \] (5.17)

• Infinite kind, parallel
\[ t \frac{\partial}{\partial t}. \] (5.18)

5.2 Cosmological constant

Since the cosmological constant introduces a length scale $1/\sqrt{|\Lambda|}$, solutions cannot be homothetic. However, the de Sitter solution, the Schwarzschild-de Sitter solution and the Nariai solution admit kinematic self-similar vectors.

The de Sitter solution is represented by
\[ ds^2 = -dt^2 + e^{2\sqrt{\Lambda/3t}} (d\tau^2 + r^2 d\Omega^2), \] (5.19)
\[ 2Gm = \frac{\Lambda}{3} r^3 e^{3\sqrt{\Lambda/3t}}, \] (5.20)
\[ 8\pi Gp = -8\pi G\mu = -8\pi G\Lambda, \] (5.21)

where $\Lambda$ is a cosmological constant. This solution is represented in the static coordinates as
\[ ds^2 = -\left( 1 - \frac{1}{3} \Lambda \rho^2 \right) d\tau^2 + \left( 1 - \frac{1}{3} \Lambda \rho^2 \right)^{-1} d\rho^2 + \rho^2 d\Omega^2, \] (5.22)
\[ 2Gm = \frac{1}{3} \Lambda \rho^3, \] (5.23)
\[ 8\pi Gp = -8\pi G\mu = -8\pi G\Lambda. \] (5.24)

When $\Lambda$ is negative, the solution is called the anti de Sitter solution. The de Sitter solution has the following kinematic self-similar vectors:

• Zeroth kind, tilted
\[ \frac{\partial}{\partial t} + \lambda r \frac{\partial}{\partial r}, \] (5.25)

where $\lambda$ is a non-zero constant.

• Zeroth kind, parallel
\[ \frac{\partial}{\partial t}, \] (5.26)

• Zeroth kind, orthogonal
\[ r \frac{\partial}{\partial r}. \] (5.27)
The Schwarzschild-de Sitter solution is an exact solution with a cosmological constant, which is represented by
\[ ds^2 = - \left( 1 - \frac{2GM_0}{\rho} - \frac{1}{3} \Lambda \rho^2 \right) d\tau^2 + \left( 1 - \frac{2GM_0}{\rho} - \frac{1}{3} \Lambda \rho^2 \right)^{-1} d\rho^2 + \rho^2 d\Omega^2, \]  
(5.29)
where \( M_0 \) is a constant and \( \Lambda \) is a cosmological constant. When \( \Lambda \) is negative, the solution is called the Schwarzschild-anti de Sitter solution. The Schwarzschild-de Sitter solution has the following kinematic self-similar vector:

\[ \tau \frac{\partial}{\partial \tau}. \]  
(5.28)

The Nariai solution \([54]\) is an exact solution with a cosmological constant, which is represented by
\[ ds^2 = - \left[ a(t) \sin \left( \ln(\sqrt{\Lambda} r) \right) + b(t) \cos \left( \ln(\sqrt{\Lambda} r) \right) \right]^2 dt^2 + \frac{1}{\Lambda r^2} (dr^2 + r^2 d\Omega^2), \]  
(5.33)
where \( a \) and \( b \) are arbitrary functions of \( t \). With the choice
\[ a = \frac{t^{1/(c_1 \sqrt{\Lambda}) - 1}}{c_1 \sqrt{\Lambda}} (A \cos(\ln t) + B \sin(\ln t)), \]  
(5.36)
\[ b = \frac{t^{1/(c_1 \sqrt{\Lambda}) - 1}}{c_1 \sqrt{\Lambda}} (-A \sin(\ln t) + B \cos(\ln t)), \]  
(5.37)
where \( A \) and \( B \) are constants, and the coordinate transformation
\[ r = \frac{r'}{\sqrt{\Lambda}}, \]  
(5.38)
this metric is written as
\[ ds^2 = - \frac{1}{c_1^2 \Lambda^{2/(c_1 \sqrt{\Lambda}) - 2}} \left[ A \sin \left( \ln \frac{r'}{t} \right) + B \cos \left( \ln \frac{r'}{t} \right) \right]^2 dt^2 + \frac{1}{\Lambda r^2} (dr^2 + r^2 d\Omega^2). \]  
(5.39)
This spacetime is also written in the static coordinates as
\[ ds^2 = - \left[ \tilde{A} \sin \left( \ln(\sqrt{\Lambda} r) \right) + \tilde{B} \cos \left( \ln(\sqrt{\Lambda} r) \right) \right]^2 d\tau^2 + \frac{1}{\Lambda r^2} (dr^2 + r^2 d\Omega^2), \]  
(5.40)
where \( \tilde{A} \) and \( \tilde{B} \) are constants. This spacetime has the following kinematic self-similar vectors:

\[ \tau \frac{\partial}{\partial \tau}. \]  
(5.32)
• Infinite kind, tilted

\[ t \frac{\partial}{\partial t} + r' \frac{\partial}{\partial r'}, \quad (5.41) \]

• Infinite kind, parallel

\[ \tau \frac{\partial}{\partial \tau}. \quad (5.42) \]

5.3 Dust fluid

Without assumption of self-similarity, the general solution for a spherically symmetric dust fluid is exactly obtained, which is called the Lemaitre-Tolman-Bondi (LTB) solution [46, 66, 8]. Therefore, spherically symmetric self-similar solutions with a dust fluid are a subclass of the LTB solutions. Homothetic solutions in the tilted case was completely classified [12]. The homothetic LTB solutions have been discussed in the context of cosmic censorship [56]. The homothetic LTB solution is represented by

\[ ds^2 = -dt^2 + \frac{(S + \xi S')^2}{1 + 2E} dr^2 + r^2 S^2 d\Omega^2, \quad (5.43) \]

\[ 2Gm = 2\sqrt{1 + 2E}, \quad (5.44) \]

\[ 8\pi G \mu = \frac{2\xi \Gamma}{r^2 S^2 (\xi S \pm \sqrt{2E + 2\Gamma/S})}, \quad (5.45) \]

where \( \xi = r/t \), where a subscript \( \xi \) means the derivative with respect to \( \xi \), and where \( E \) and \( \Gamma \) are constants with a relation

\[ E = \frac{1}{2} (\Gamma^2 - 1). \quad (5.46) \]

\( S \) is given by

\[ D = \frac{1}{\xi} \begin{cases} \frac{\sqrt{ES^2 + \Gamma S}}{\sqrt{2E}} - \frac{2\Gamma}{(2E)^{\frac{3}{2}}} \sin^{-1} \left( \frac{\sqrt{ES^2 + \Gamma S}}{\sqrt{2E}} \right) & \text{for } E > 0, \\ \frac{\sqrt{3}}{2} S^{\frac{3}{2}} & \text{for } E = 0, \\ \frac{2\Gamma}{(-2E)^{\frac{3}{2}}} \sin^{-1} \left( \frac{-ES}{\Gamma} \pm \frac{\sqrt{ES^2 + \Gamma S}}{\sqrt{2E}} \right) & \text{for } -\frac{1}{2} < E < 0, \end{cases} \quad (5.47) \]

where \( D \) is a constant. This solution has a homothetic vector

\[ t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \quad (5.48) \]

\( E \) can be interpreted as the sum of the kinetic energy and the potential energy per unit mass. When \( E = 0 \), each shell is marginally bound. This solution is a two-parameter family of solutions of \( E \) and \( D \) and reduces to the flat Friedmann-Robertson-Walker (FRW) solution when \( E = D = 0 \). It is noted that there are nonmarginally bound LTB solutions with a homothetic vector.

In addition, there are kinematic self-similar solutions of the second, zeroth and infinite kinds. From equation (3.25), any three-surface orthogonal to the fluid flow in a kinematic self-similar solution to the Einstein equations of the second, zeroth or infinite kind that contains only irrotational dust as a
matter field is flat. Because the flatness of the three-surface implies that these solutions are marginally
bound, a spherically symmetric kinematic self-similar solution to the Einstein equations of the second,
zereth or infinite kind that contains only a dust fluid is described by the marginally bound LTB
solutions. These solutions have been investigated by several authors [19, 2, 10, 50].

The second-kind kinematic self-similar LTB solution is represented by

\[ ds^2 = -dt^2 + \frac{9[kr^{3\alpha/(3-2\alpha)} + (2\alpha/3 - 1)t^2]{2/3} dr^2 + r^2[kr^{3\alpha/(3-2\alpha)} - t|^{4/3} d\Omega^2,} \]

(5.49)

\[ 8\pi G\mu = \frac{4(3 - 2\alpha)}{9[kr^{3\alpha/(3-2\alpha)} + (2\alpha/3 - 1)t|^{3-2\alpha} - t].} \]

(5.50)

\[ 2Gm = \frac{4}{9} r^3. \]

(5.51)

where \( \kappa \) is an arbitrary dimensional constant and \( \alpha \neq 3/2 \). For \( \alpha = 3/2 \), the solution turns out to be
the flat FRW solution. This spacetime has the following kinematic self-similar vector:

\[ \alpha t \frac{\partial}{\partial t} + \frac{3 - 2\alpha}{3} r \frac{\partial}{\partial r}. \]

(5.52)

The zeroth-kind kinematic self-similar LTB solution is represented by

\[ ds^2 = -dt^2 + \frac{(t - 2\lambda/3 - \lambda \ln r)^2}{|t - \lambda \ln r|^{2/3}} dr^2 + r^2|t - \lambda \ln r|^{4/3} d\Omega^2, \]

(5.53)

\[ 8\pi G\mu = \frac{4}{3(t - \lambda \ln r)(t - 2\lambda/3 - \lambda \ln r)}; \]

(5.54)

\[ 2Gm = \frac{4}{9} r^3. \]

(5.55)

This spacetime has the following kinematic self-similar vector:

\[ \lambda \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \]

(5.56)

where \( \lambda \) is an arbitrary dimensional constant.

The infinite kind kinematic self-similar LTB solution is represented by

\[ ds^2 = -dt^2 + t^2|\sigma r^{-3/2} - t|^{-2/3} dr^2 + r^2|\sigma r^{-3/2} - t|^{4/3} d\Omega^2, \]

(5.57)

\[ 8\pi G\mu = \frac{-4}{3 t|\sigma r^{-3/2} - t|}; \]

(5.58)

\[ 2Gm = \frac{4}{9} r^3. \]

(5.59)

This spacetime has the following kinematic self-similar vector:

\[ t \frac{\partial}{\partial t} - \frac{2}{3} r \frac{\partial}{\partial r}, \]

(5.60)

where \( \sigma \) is an arbitrary dimensional constant.

The flat FRW solution with a dust fluid will be discussed together with those with a perfect fluid.

5.4 Perfect fluid

5.4.1 Homothetic solutions

As examples of homothetic exact solutions with a perfect fluid obeying the equation of state \( p = K\mu \),
we discuss the power-law flat FRW solution, the homothetic static perfect fluid solution, and the
Kantowski-Sachs solution. The power-law flat FRW solution and the homothetic static solution have kinematic self-similar vectors as well as a homothetic vector. The Kantowski-Sachs solution has no kinematic self-similar vector except a homothetic vector.

The flat FRW solution has the following form:

\[ ds^2 = -dt^2 + \frac{1}{3(1+K)}(dr^2 + r^2d\Omega^2), \]
\[ 2Gm = \frac{4}{9(1+K)^2}t^{-2K/(1+K)}, \]
\[ 8\pi Gp = \frac{4K}{3(1+K)^2r^2}. \]

This solution has the following kinematic self-similar vectors:

• First kind, tilted \((K \neq -1/3)\)

\[ \frac{t}{3(1+K)} \frac{\partial}{\partial t} + \frac{1 + 3K}{3(1+K)} \frac{\partial}{\partial r}, \]

• First kind, parallel \((K = -1/3)\)

\[ \frac{t}{3} \frac{\partial}{\partial t}, \]

• Second kind for \(\alpha \neq 3(1 + K)/2\), tilted

\[ \alpha t \frac{\partial}{\partial t} + \left(1 - \frac{2\alpha}{3(1 + K)}\right) \frac{\partial}{\partial r}, \]

• Second kind with \(\alpha = 3(1 + K)/2\), parallel

\[ \alpha t \frac{\partial}{\partial t}, \]

• Zeroth kind, orthogonal

\[ r \frac{\partial}{\partial r}, \]

• Infinite kind, tilted

\[ \frac{t}{3(1+K)} \frac{\partial}{\partial t} - \frac{2}{3(1+K)} r \frac{\partial}{\partial r}. \]

The homothetic static perfect fluid solution is represented as the following:

\[ ds^2 = -r^{4K/(1+K)}dt^2 + \frac{K^2 + 6K + 1}{(1+K)^2}dr^2 + r^2d\Omega^2, \]
\[ 2Gm = \frac{4K}{K^2 + 6K + 1}r, \]
\[ 8\pi Gp = \frac{4K^2}{(K^2 + 6K + 1)r^2}. \]

Since the center \(r = 0\) is singular and timelike, it must be a naked singularity. This solution has the following kinematic self-similar vectors:
• First kind, tilted \((K \neq 1)\)

\[
\frac{1 - K}{1 + K} t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}.
\]

(5.73)

• First kind, orthogonal \((K = 1)\)

\[
r \frac{\partial}{\partial r},
\]

(5.74)

• Second kind for \(\alpha \neq 2K/(1 + K)\), tilted

\[
\left(\alpha - \frac{2K}{1 + K}\right) t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r},
\]

(5.75)

• Second kind with \(\alpha = 2K/(1 + K)\), orthogonal

\[
r \frac{\partial}{\partial r},
\]

(5.76)

• Zeroth kind, tilted

\[
- \frac{2K}{1 + K} t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}.
\]

(5.77)

The Kantowski-Sachs solution is represented by

\[
ds^2 = -dt^2 + t^{-4K/(1+K)}dr^2 + \frac{(1 + K)^2}{(1 + 3K)(K - 1)} t^2 d\Omega^2,
\]

(5.78)

\[
2Gm = \frac{4(1 + K)K^2 t}{(1 + 3K)^{3/2}(K - 1)^{3/2}},
\]

(5.79)

\[
8\pi Gp = 8\pi G\mu = -\frac{4K^2}{(1 + K)^2 t^2},
\]

(5.80)

with a homothetic vector

\[
t \frac{\partial}{\partial t} + \frac{1 + 3K}{1 + K} r \frac{\partial}{\partial r}.
\]

(5.81)

\(-1 < K < -1/3\) must be satisfied for this solution to be physical and in that case the above homothetic vector is tilted.

5.4.2 Nonhomothetic kinematic self-similar solutions

As examples of nonhomothetic kinematic self-similar solutions, we discuss the general FRW solutions and the Gutman-Bespal’ko solution.
The general FRW solution has kinematic self-similar vectors and does not have homothetic vectors if it is not power-law or if it is not flat. The general FRW solutions are given by

\[ ds^2 = -dt^2 + a(t)^2(dr^2 + S(r)^2d\Omega^2), \tag{5.82} \]
\[ 2Gm = aS(1 - S^2 + \dot{a}^2S^2), \tag{5.83} \]
\[ 8\pi Gp = -\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2}, \tag{5.84} \]
\[ 8\pi G\mu = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2}, \tag{5.85} \]
\[ S(r) = \begin{cases} 
\sin r, & \text{for } k = 1 \\
 r, & \text{for } k = 0 \\
\sinh r, & \text{for } k = -1
\end{cases} \tag{5.86} \]

where a dot and a prime denote the derivatives with respect to \( t \) and \( r \), respectively.

The non-power-law flat FRW solution \((k = 0)\) has the following kinematic self-similar vector independent of the form of the equation of state:

- Zeroth kind, orthogonal

\[ r \frac{\partial}{\partial r}. \tag{5.87} \]

The closed FRW solution \((k = 1)\) has the following kinematic self-similar vector for EOS2 with \( \gamma = 2/3 \):

- Second kind with \( \alpha = 3/2 \), parallel

\[ t \frac{\partial}{\partial t}. \tag{5.88} \]

The curved FRW solutions \((k = \pm 1)\) with the equation of state \( p = -\mu/3 \) have the following additional kinematic self-similar vector:

- First kind, parallel

\[ t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \tag{5.89} \]

Only for a stiff fluid \( p = \mu \), the Gutman-Bespal’ko solution exists, which is represented by [31, 64, 50].

\[ ds^2 = -\frac{1}{4}r^2dt^2 + dr^2 + \frac{1}{2}r^2(1 + a_1e^t + a_2e^{-t})d\Omega^2, \tag{5.90} \]
\[ 2Gm = \frac{1}{2\sqrt{2}}(1 - 4a_1a_2)(1 + a_1e^t + a_2e^{-t})^{-3/2}r, \tag{5.91} \]
\[ 8\pi Gp = 8\pi G\mu = (1 - 4a_1a_2)(1 + a_1e^t + a_2e^{-t})^{-2}r^{-2}, \tag{5.92} \]

where \( a_1 \) and \( a_2 \) are arbitrary constants. In this spacetime the physical center \( r = 0 \) is singular. This spacetime has the following kinematic self-similar vector:
• First kind, orthogonal

\[ \frac{\partial}{\partial r}. \]  

(5.93)

This solution includes the homothetic static perfect fluid solution for a stiff fluid as a special case \( a_1 = a_2 = 0 \).

The results in this section are summarized in Table 1. As we will see in section 6, it can be shown that this table provides the complete list of kinematic self-similar solutions compatible with EOS1, EOS2 and EOS3.

6 Nonexistence of kinematic self-similar solutions with a polytropic equation of state

Among kinematic self-similarities, homothety in the tilted case includes a large variety of solutions and has been intensively investigated. As we have seen, the equation of state is restricted to be of the form \( p = K \mu \) for homothetic solutions.

In this section, based on [48, 49, 50], we will briefly see that there are no kinematic self-similar solutions with nontrivial polytropic equations of state although we first expected that the generalization from homothety to kinematic self-similarity would enable us to analyze a wider class of physical solutions. As we have already mentioned, it is possible to construct a characteristic length scale from given dimensional constants in the general relativistic system of a perfect fluid with a polytropic equation of state. According to a usual procedure, we have introduced incomplete similarity into this system. Kinematic self-similarity is a natural generalization of incomplete similarity into general relativity. Therefore, it is highly nontrivial that there are no kinematic self-similar solutions with a kinematic self-similar vector tilted to the flow of a perfect fluid with a polytropic equation of state.

6.1 Tilted cases

6.1.1 Second kind

We consider the second-kind kinematic self-similar solutions in the tilted case, in which the energy density and the pressure of a perfect fluid are of the form (4.41) and (4.42), respectively. The equation of state gives the relation among the functions \( P_1, P_2, W_1 \) and \( W_2 \). EOS1 admits the following two cases:

\[ \alpha = \gamma, \quad P_1 = W_2 = 0, \quad P_2 = \frac{K}{(8\pi G)^{\gamma-1}\gamma^2} \xi^{-2\gamma} W_1^\gamma, \quad (A) \]  

(6.1)

\[ \alpha = \frac{1}{\gamma}, \quad P_2 = W_1 = 0, \quad P_1 = \frac{K}{(8\pi G)^{\gamma-1}\gamma^2} \xi^2 W_2^\gamma, \quad (B) \]  

(6.2)

while EOS2 admits the following two cases:

\[ \alpha = \gamma, \quad P_1 = 0, \quad P_2 = \frac{K}{m_b(8\pi G)^{\gamma-1}\gamma^2} \xi^{-2\gamma} W_1^\gamma = (\gamma - 1)W_2, \quad (C) \]  

(6.3)

\[ \alpha = \frac{1}{\gamma}, \quad P_2 = 0, \quad P_1 = \frac{K}{m_b(8\pi G)^{\gamma-1}\gamma^2} \xi^2 W_2^\gamma = (\gamma - 1)W_1. \quad (D) \]  

(6.4)

We can show that none of these cases satisfies the Einstein equations although they are compatible with the decomposition given by equations (4.41) and (4.42). Subtracting equation (4.57) from (4.55) and eliminating \( S'' \) by use of equation (4.53), we obtain

\[ 2\Phi' = (P_1 + W_1)e^{2\Phi}. \]  

(6.5)
Then equations (4.49) and (4.50) result in
\[ e^{2\Psi}(P_1 + W_1)^2 = 4P_1 - 2P_1', \quad (6.6) \]
\[ e^{2\Psi}(P_1 + W_1)(P_2 + W_2) = -2P_2'. \quad (6.7) \]

It is obvious that \( P_1 = 0 \) implies \( W_1 = 0 \) from equation (6.6) while \( P_2 = 0 \) implies \( W_2 = 0 \) or \( P_1 + W_1 = 0 \) from equation (6.7). Therefore, it is concluded that all cases (A)–(D) result in vacuum spacetimes.

### 6.1.2 Zeroth kind

Next, we consider the zeroth-kind kinematic self-similar solutions in the tilted case. In this case, the Einstein equations imply that the quantities \( \mu, p \) and \( m \) are of the forms
\[ \frac{2Gm}{r} = M_1(\xi) + r^2M_2(\xi), \quad (6.8) \]
\[ 8\pi G\mu r^2 = W_1(\xi) + r^2W_2(\xi), \quad (6.9) \]
\[ 8\pi Gpr^2 = P_1(\xi) + r^2P_2(\xi), \quad (6.10) \]
where \( \xi = r/e^\ell \). A set of ordinary differential equations is obtained when it is stipulated that the Einstein equations and the equations of motion for the matter field be satisfied for the \( O(1) \) and \( O(r^2) \) terms separately. See [49, 50] for the complete set of the ordinary differential equations. Both from EOS1 and EOS2 with equations (6.9) and (6.10), \( P_2 = 0 \) implies \( W_1 = 0 \) and \( P_1 = 0 \) implies \( W_2 = 0 \), for EOS1, while
\[ P_1 = W_1 = 0, \quad P_2 = K(8\pi G)^{1-\gamma}W_2^{\gamma}, \quad (A) \]
for EOS1, while
\[ P_2 = 0 \]
for EOS2. From equations (4.5), (4.6) and (4.8), we obtain
\[ \exp(\Phi) = c_0 \]
where \( c_0 \) is a positive constant. Then \( P_2 = p_0 \) is obtained from equation (4.3), where \( p_0 \) is a constant, which implies that \( W_2 = w_0 \), where \( w_0 \) is a constant. Then, equation (4.18) gives the evolution equation for \( S \):
\[ \frac{3}{c_0^2} \left( \frac{S'}{S} \right)^2 - (p_0 + w_0)\frac{S'}{S} - w_0 = 0. \quad (6.17) \]
The solution to this equation is \( S = s_0 e^q \), where \( s_0 \) and \( q \) are constants. \( q \neq -1 \) must be satisfied because of equation (4.13). Equation (4.13) with the fact \( P_2 + W_2 \neq 0 \) gives
\[ \Psi' + 2S' = 0. \quad (6.18) \]
Then, the equality \( q = 0 \) can be obtained from equation (6.13) and (6.18), which implies that \( S = s_0 \). Finally, equations (6.15) and (6.17) give \( p_0 = 0 \) and \( w_0 = 0 \), respectively. Therefore, it is concluded that cases (A) and (B) result in vacuum spacetimes.
Finally, we consider the infinite-kind kinematic self-similar solutions in the tilted case. In this case, the Einstein equations imply that the quantities $\mu, p$ and $m$ are of the forms

\begin{align}
2Gm &= M_1(\xi)/t^2 + M_2(\xi), \\
8\pi G\mu &= W_1(\xi)/t^2 + W_2(\xi), \\
8\pi Gp &= P_1(\xi)/t^2 + P_2(\xi),
\end{align}

(6.19) (6.20) (6.21)

where $\xi = r/t$. A set of ordinary differential equations is obtained when it is demanded that the Einstein equations and the equations of motion for the matter field be satisfied for the $O(1)$ and $O(t^{-2})$ terms separately. See [49, 50] for the complete set of ordinary differential equations. Both from EOS1 and EOS2 with equations (6.20) and (6.21), $P_1 = W_1 = 0$ is concluded such that

\begin{align}
P_1 &= W_1 = 0, \quad P_2 = K(8\pi G)^{1-\gamma}W_2^\gamma, \quad (A) \\
\end{align}

(6.22)

for EOS1, while

\begin{align}
P_1 &= W_1 = 0, \quad P_2 = \frac{K}{(8\pi G)^{1-\gamma/2}} \left( W_2 - \frac{P_2}{\gamma - 1} \right)^\gamma, \quad (B) \\
\end{align}

(6.23)

for EOS2. From equations (6.24), (6.25) and (6.26), we obtain

\begin{align}
S'' &= S'(\Phi' + \Psi',) \\
(1 - W_2 S^2)e^{2\Psi} &= 2SS'' + S'^2 - 2\Psi'S'S, \\
(1 + P_2 S^2)e^{2\Psi} &= S'(2\Phi'S + S'),
\end{align}

(6.24) (6.25) (6.26)

where we have omitted the bars of $\Phi$ and $\Psi$ in (4.31) for simplicity the prime denotes the derivative with respect to $\ln \xi$. From equations (6.24), (6.25) and (6.26),

\begin{align}
P_2 + W_2 &= 0, \\
\end{align}

(6.27)

is obtained, which implies that $p = -\mu$ and gives a contradiction. Therefore, it is concluded that cases (A) and (B) result in vacuum spacetimes.

As a result, it is shown that there is no kinematic self-similar solutions with a nontrivial polytropic equation of state in the tilted case.

### 6.2 Nontilted cases

Even when the parallel and orthogonal cases are considered, except for the infinite-kind kinematic self-similar solutions in the parallel case which include all static solutions, the only possible solutions are the flat FRW solution as a zeroth-kind kinematic self-similar solution in the orthogonal case both for EOS1 and EOS2 and the closed FRW solution as a second-kind kinematic self-similar solution with an index $\alpha = 3/2$ in the orthogonal case for EOS2 with $\gamma = 1/\alpha = 2/3$.

### 7 Summary

Self-similarity has been applied to many aspects of physics and other scientific fields. The introduction of self-similarity into Newtonian gravity is straightforward because it postulates absolute space and time. Since Newtonian gravity has only one dimensional constant, i.e. the gravitational constant, we can incorporate a polytropic gas as well as an isothermal gas into the framework of complete similarity. The introduction of self-similarity into general relativity is, however, not so straightforward.
because there is no preferred coordinate system in the theory. The covariant definition of complete similarity in general relativity is homothety. Moreover, since two dimensional physical constants, the gravitational constant and the speed of light, are included in general relativity, it is impossible to incorporate many physically interesting matter fields, such as a polytropic equation of state, into the framework of homothety. This naturally leads to the introduction of incomplete similarity in general relativity. One of the most natural definitions of incomplete similarity in the fluid system in general relativity is kinematic self-similarity. Many known exact solutions turn out to be kinematic self-similar. At first glance it seems possible to construct kinematic self-similar solutions with a polytropic equation of state. However, more comprehensive study of the Einstein equations reveals that there are no such solutions. Although the present discussion implies somewhat limited application of kinematic self-similarity, there still remains a large possibility that kinematic self-similar solutions describe interesting gravitational phenomena of physically important matter fields, such as a double fluid system [24].

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Table 1: The complete list of kinematic self-similar solutions with a perfect fluid in the spherically symmetric spacetime for a perfect fluid with EOS1, EOS2 and EOS3. It is assumed that the energy density of the perfect fluid is not negative. See text and references therein.

| Matter field     | Kind                                      | Solution                  |
|------------------|-------------------------------------------|---------------------------|
| Vacuum           | first, parallel                           | Minkowski                 |
|                  | first, orthogonal                         | Minkowski                 |
|                  | second, tilted, any $\alpha$              | Minkowski                 |
|                  | second, tilted, $\alpha = 3/2$            | Schwarzschild             |
|                  | zeroth, tilted                            | Minkowski                 |
|                  | zeroth, orthogonal                        | Minkowski                 |
|                  | infinite, parallel                        | Schwarzschild             |
| Cosmological constant | zeroth, tilted                           | de Sitter                 |
|                  | zeroth, parallel                          | de Sitter                 |
|                  | zerother, orthogonal                      | de Sitter                 |
|                  | infinite, tilted                          | Nariai                    |
|                  | infinite, parallel                        | de Sitter                 |
|                  |                                           | Nariai                    |
|                  |                                           | Schwarzschild-de Sitter   |
| Dust             | first, tilted                             | see Ref. [12]             |
|                  | second, tilted                            | KSS LTB                   |
|                  | second, parallel, $\alpha = 3/2$          | Flat FRW                  |
|                  | zeroth, tilted                            | KSS LTB                   |
|                  | zeroth, orthogonal                        | Flat FRW                  |
|                  | infinite, tilted                          | KSS LTB                   |
| Perfect fluid: EOS1 | zeroth, orthogonal                    | Flat FRW                  |
|                  | infinite, parallel                        | All static solutions      |
| Perfect fluid: EOS2 | second, parallel, $\alpha = 3/2$        | Closed FRW with $\gamma = 2/3$ |
|                  | zeroth, orthogonal                        | Flat FRW                  |
|                  | infinite, parallel                        | All static solutions      |
| Perfect fluid: EOS3 | first, tilted                            | see Refs. [60] [13]      |
|                  | first, parallel                           | FRW ($K = -1/3$)          |
|                  | first, orthogonal                         | Gutman-Bespalko ($K = 1$) |
|                  | second, tilted, $\alpha \neq 3(1 + K)/2$  | Flat FRW                  |
|                  | second, tilted, $\alpha \neq 2K/(1 + K)$  | Homothetic static         |
|                  | second, parallel, $\alpha = 3(1 + K)/2$   | Flat FRW                  |
|                  | second, orthogonal, $\alpha = 2K/(1 + K)$ | Homothetic static         |
|                  | zeroth, orthogonal, infinite, tilted      | Flat FRW                  |
|                  | infinite, parallel                        | Flat FRW                  |
|                  | infinite, parallel                        | All static solutions      |
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