Convergence and error estimates of a viscosity-splitting finite-element scheme for the Primitive Equations

F. Guillén-González† M.V. Redondo-Neble‡

Abstract

The purpose of this paper is the numerical analysis of a first order fractional-step time-scheme, using decomposition of the viscosity, and “inf-sup” stable finite element space-approximations for the Primitive Equations of the Ocean. The aim of the paper is twofold. Firstly, we prove that the scheme is unconditionally stable and convergent towards weak solutions of the Primitive Equations. Secondly, optimal error estimates for velocity and pressure are provided of order $O(k + h^l)$ for $l = 1$ or $l = 2$ when either first or second order finite-element approximations are considered ($k$ and $h$ being the time step and the mesh size, respectively). In both cases, these error estimates are obtained under the same constraint $k \leq h^2$.

Subject Classification 35Q35, 65M12, 65M15, 76D05

Keywords: Primitive Equations, finite elements, anisotropic estimates, time-splitting schemes, stability, convergence, error estimates.

Introduction

Assuming some simplifications (basically hydrostatic pressure and “rigid lid” hypothesis), the 3D Navier-Stokes equations derive to the so-called “Primitive Equations” (or Hydrostatic Navier-Stokes equations). These equations arise a general mathematical problem in the field of geophysical fluids ([11, 29, 32]). In particular, they describe the large-scale motions in the ocean [30].

The rigid lid hypothesis (no vertical displacements of the free surface of the ocean) is usually assumed in Oceanography, except in the case when fast surface waves are of interest [11].

For simplicity, we take constant density, Cartesian coordinates ($x$ in the easterly direction, $y$ in the northerly direction and $z$ perpendicular to the surface of the Earth) and we assume that the effects due to temperature and salinity can be decoupled from the flow dynamic. Then, the

---

*The authors have been partially supported by MINECO (Spain), Grant MTM2012–32325 and the second author is also partially supported by the research group FQM-315 of Junta de Andalucía.

†Departamento de Ecuaciones Diferenciales y Análisis Numérico and IMUS. Universidad de Sevilla. Aptdo. 11690, 41080 Sevilla (Spain), email: guillen@us.es, phone: ++ 34 954559907.

‡Departamento de Matemáticas. Universidad de Cádiz. C.A.S.E.M. Polígono Río San Pedro S/N, 11510 Puerto Real. Cádiz (Spain), email: victoria.redondo@uca.es, phone: ++ 34 956016058.
Primitive Equations model can be written as \([28, 29, 30]\):

\[
\begin{aligned}
\partial_t u + (U \cdot \nabla) u - \nu \Delta u + b(u) + \nabla_x p &= f & \text{in } \Omega \times (0, T), \\
\partial_z p &= -\rho g, & \nabla \cdot U &= 0 & \text{in } \Omega \times (0, T), \\
\mathbf{u} &= u_3 n_3 = 0 & \text{on } \Gamma_b \times (0, T), \\
\mathbf{u} &= 0 & \text{on } \Gamma_l \times (0, T), \\
\nu \partial_z \mathbf{u} &= \mathbf{g}_s, & u_3 &= 0 & \text{on } \Gamma_s \times (0, T), \\
\mathbf{u}|_{t=0} &= \mathbf{u}_0 & \text{in } \Omega,
\end{aligned}
\]

\((P)\)

where \(\Omega = \{(x, z) \in \mathbb{R}^3 : x = (x, y) \in S, -D(x) < z < 0\}\) is the 3D domain filled by the water, with \(S \subset \mathbb{R}^2\) the surface domain (a regular bounded 2D domain) and \(D : S \rightarrow \mathbb{R}_+\) (with \(D > 0\) in \(S\)) the bottom function. Then, the different boundaries of \(\Omega\) are denoted as \(\Gamma_s = \mathbb{S} \times \{0\}\) the surface, \(\Gamma_b = \{(x, -D(x)) : x \in S\}\) the bottom and \(\Gamma_l = \{(x, z) : x \in \partial S, -D(x) < z < 0\}\) the lateral walls (with outwards normal vector \((n_x, n_3)\)). Note that if \(D \in C(\mathbb{S})\) then the bottom has no steps and the condition \(u_3 n_3 = 0\) on \(\Gamma_b \times (0, T)\) derives to \(u_3 = 0\) on \(\Gamma_b \times (0, T)\). In fact, \(\Omega\) is the non-dimensional domain obtained after a vertical scaling and problem \((P)\) appears as asymptotic limit from the anisotropic Navier-Stokes equations when the aspect ratio (vertical/horizontal) goes to zero \([11, 2, 5]\).

The unknowns of the problem are \(\mathbf{U} = (\mathbf{u}, u_3) : \Omega \times (0, T) \rightarrow \mathbb{R}^3\) the 3D velocity field (with \(\mathbf{u} = (u_1, u_2)\) the horizontal velocity and \(u_3\) the vertical one) and \(p : \Omega \times (0, T) \rightarrow \mathbb{R}\) the pressure.

Also, \(b(u) = f u^\perp\) represents the effect of the Coriolis Forces, with \(u^\perp = (-u_2, u_1)^t\) and \(f = 2|w| \sin \theta\), where \(w\) is the angular velocity of the Earth and \(\theta = \theta(y)\) is the latitude, \(\rho \in \mathbb{R}_+\) is the water density (that it is assumed a positive constant), \(g \in \mathbb{R}_+\) is the gravity acceleration (another positive constant), \(f : \Omega \times (0, T) \rightarrow \mathbb{R}^2\) is a field of external horizontal forces (depending for instance on the salinity and temperature) and \(\mathbf{g}_s : \Gamma_s \times (0, T) \rightarrow \mathbb{R}^2\) represents the stress of the wind on the surface.

Finally, \(\nabla = (\nabla_x, \partial_z)^t\) stands for the three-dimensional gradient operator (with \(\nabla_x = (\partial_x, \partial_y)^t\) its horizontal component) \(\Delta\) stands for the three-dimensional Laplacian operator.

The problem \((P)\) have been vertically scaled (see \([2]\)) such that the horizontal and vertical dimensions in \(\Omega\) are of the same order.

**Remark 1** When variations in the surface are important in the problem, it is usual to consider the general Navier Stokes equations with hydrostatic pressure, introducing the free surface as a new unknown. In this case, one has to change the boundary condition of “rigid lid” \((u_3 = 0\) on \(\Gamma_s)\) for the equation of the free surface, arriving at the so-called three-dimensional Shallow Water model. Some numerical approximations of this model can be seen in \([8, 9, 11]\).

We will give two reformulations of problem \((P)\) leading to different spatial approximations.

The vertical dependence of the pressure can be decomposed by using the hydostatic pressure \(\rho g z\), defining \(p_s(t; \mathbf{x}) = p(t; \mathbf{x}, z) - \rho g z\), where \(p_s : S \times (0, T) \rightarrow \mathbb{R}\) is a new unknown (defined only on the surface \(S\)), that it will be called *surface pressure*.

Notice that incompressibility equation \(\nabla \cdot U = 0\) in \(\Omega \times (0, T)\) and boundary condition \(u_3 = 0\) on \(\Gamma_s \times (0, T)\) are equivalent to the following integral formula for the vertical velocity:

\[
u_3(t; \mathbf{x}, z) = \int_z^0 \nabla_x \cdot \mathbf{u}(t; \mathbf{x}, s) \, ds. \tag{1}\]
Moreover, the following equality holds:
\[
\int_{-D(x)}^{0} \nabla \cdot U \, dz = \nabla_x \cdot \langle u \rangle - (u, u_3)(x, -D(x)) \cdot \langle \nabla_x D(x), 1 \rangle = 0 \quad \text{in } S \times (0, T), \quad (2)
\]
where \( \langle u \rangle \) denotes the total vertical flux of the horizontal velocity:
\[
\langle u \rangle(t; x) = \int_{-D(x)}^{0} u(t; x, z) \, dz.
\]
Therefore, since \( \langle \nabla_x D(x), 1 \rangle \) is parallel to the normal vector \( (n_x, u_3) \) on \( \Gamma_b \), assuming \( \nabla \cdot U = 0 \) in \( \Omega \times (0, T) \), the so-called slip condition \( u \cdot n_x + u_3 n_3 = 0 \) on \( \Gamma_b \times (0, T) \) is equivalent to the constraint \( \nabla_x \cdot \langle u \rangle = 0 \) in \( S \times (0, T) \) \([28, 29, 30]\).

Then, problem \((P)\) can be reformulated as the following integro-differential problem:
\[
\begin{cases}
\partial_t u + (U \cdot \nabla)u - \nu \Delta u + b(u) + \nabla_x p_s = f & \text{in } \Omega \times (0, T), \\
\nabla_x \cdot \langle u \rangle = 0 & \text{in } S \times (0, T), \\
\nu \partial_z u = g_s & \text{on } \Gamma_s \times (0, T), \\
u_3 = 0 & \text{on } (\Gamma_b \cup \Gamma_l) \times (0, T), \\
u|_{t=0} = u_0 & \text{in } \Omega,
\end{cases}
\]
\( (Q) \)

where \( U = (u, u_3) \) with \( u_3 \) depending on the \( \nabla_x \cdot u \) by the integral formula \( (1) \).

Instead of problem \((Q)\), other reformulation of problem \((P)\) can be done, based on the following equivalence: assuming the slip-condition \( u \cdot n_x + u_3 n_3 = 0 \) on \( \Gamma_b \), one has
\[
\frac{\partial_z}{\partial_z} \langle \nabla \cdot U \rangle = 0 \quad \text{in } \Omega \quad \left\{ \begin{array}{c}
\nabla_x \cdot \langle u \rangle = 0 \quad \text{in } S \end{array} \right\} \iff \nabla \cdot U = 0 \quad \text{in } \Omega.
\]

Indeed, from the equation \( \partial_z \langle \nabla \cdot U \rangle = 0 \), one has \( \nabla \cdot U = a(x) \). By integrating in vertical this equality and using \( (2) \) (taking into account \( \nabla_x \cdot \langle u \rangle = 0 \) and the slip-condition on \( \Gamma_b \)), one has
\[
0 = \int_{-D(x)}^{0} g(x) \, dz = D(x) a(x) \quad \text{in } S.
\]

Then \( a \equiv 0 \) in \( S \) and \( \nabla \cdot U = 0 \). Conversely, since \( \nabla \cdot U = 0 \) then \( \partial_z \langle \nabla \cdot U \rangle = 0 \). Moreover, \( \nabla_x \cdot \langle u \rangle = 0 \) is deduced again from \( (2) \) integrating in vertical \( \nabla \cdot U = 0 \) and taking into account the slip-condition on \( \Gamma_b \).

Therefore, the second reformulation of problem \((P)\) is:
\[
\begin{cases}
\partial_t u + (U \cdot \nabla)u - \nu \Delta u + b(u) + \nabla_x p_s = f & \text{in } \Omega \times (0, T), \\
\partial_z^2 u_3 + \partial_z \nabla_x \cdot u = 0 & \text{in } \Omega \times (0, T), \\
\nabla_x \cdot \langle u \rangle = 0 & \text{in } S \times (0, T), \\
\nu \partial_z u = g_s & \text{on } \Gamma_s \times (0, T), \\
u_3 = 0 & \text{on } (\Gamma_s \cup \Gamma_b) \times (0, T), \\
u|_{t=0} = u_0 & \text{in } \Omega.
\end{cases}
\]
\( (R) \)

Notice that in \((R)\) the vertical velocity \( u_3 \) is uniquely defined by the \( z \)-elliptic problem
\[
\partial_z^2 u_3 = -\partial_z \nabla_x \cdot u \quad \text{in } \Omega \times (0, T), \quad u_3|_{\Gamma_s \cup \Gamma_b} = 0. \quad (3)
\]
From the numerical analysis point of view, the convergence of some Finite Element (FE) schemes for the stationary problem related to \((Q)\), has been proved in [12], where the so-called hydrostatic Inf-Sup stability condition appears. To approximate the time-dependent problem, a stabilized FE scheme was used by Chacón-Rodríguez in [13,14], and Bermejo in [3] and Bermejo-Galán in [4] used a semi-lagrangian projection time-scheme together with finite elements in space. On the other hand, Chacón-Gómez-Sánchez in [15] have derived the numerical approximation of this model by the Orthogonal Sub-Scales FE method, obtaining stability, convergence and error estimates (optimal for 2D flows) for a steady linearized model and they have performed some numerical tests for the non-linear case.

The goal of this paper is to design numerical schemes associated to both formulations \((R)\) and \((Q)\), based on a fractional-step time-scheme and FE in space, which satisfies analytical results into two directions: on one hand, unconditional stability and convergence towards weak solutions of \((R)\) and, on the other hand, error estimates with respect to a sufficiently regular solution, under the constraint

\[
(\text{H}) \quad k \leq h^2.
\]

These results could be seen as an extension of the numerical analysis done for a viscosity-splitting scheme applied to the time-dependent Navier-Stokes Equations in [6] and [22, 24, 25]. Nevertheless, error estimates for the Navier-Stokes case have been deduced in [22, 25], using the corresponding time-discrete scheme as intermediate problem to obtain error estimates for the fully discrete scheme, under the constraint

\[
h \leq C k.
\]

Since this constraint has contrary sense that \((\text{H})\), it is not clear how the argument done in [22, 25] for Navier-Stokes could be extended for the Primitive Equations.

In the scheme studied in this paper, three subproblems must be solved at every time step \(m\). Indeed, given \((u^m_h, p^m_{s,h})\), firstly the vertical velocity \(u^m_{3,h}\) is computed in function of \(\nabla_x \cdot u^m_h\), afterwards an intermediate horizontal velocity \(u^{m+1/2}_h\) and finally a pair \((u^{m+1}_h, p^{m+1}_{s,h})\) is computed solving an Hydrostatic Stokes problem.

The rest of this paper is organized as follows. After giving some preliminaries in Section 1, the fully discrete scheme related to \((R)\) is described in Section 2, obtaining in Section 3 some stability a priori estimates and convergence (by subsequences) as \((k, h) \to 0\) towards weak solutions of \((R)\).

In Section 4, some error estimates for velocity and pressure are deduced. Firstly, we obtain \(O(\sqrt{k} + h^l)\) error estimates for both velocities \(u^{m+1/2}_h\) and \(u^{m+1}_h\), improving to optimal accuracy \(O(k + h^l)\) for the “end of step” velocity \(u^{m+1}_h\), for \(l = 1, 2\) the order of the FE approximation. Afterwards, \(O(\sqrt{k} + h^l)\) for the discrete time derivative of \(u^{m+1}_h\) and for the pressure will be deduced.

On the other hand, only when \(l = 2\), we obtain optimal \(O(k + h^2)\) error estimates for the discrete time derivative of \(u^{m+1}_h\) and for the pressure.

In order to also deduce optimal accuracy for the pressure when \(l = 1\), in Section 5 we consider a modified scheme associated to \((Q)\), where the vertical velocity is approximated via an integral computation like \([\text{1}]\). For this scheme, optimal accuracy in the \(L^2(\Omega)\)-norm; \(O(k + h^{l+1})\) for \(u^{m+1}_h\) and \(O(k + h^l)\) for the pressure are obtained. Finally, some comments about the treatment of the Coriolis term are given in Section 6, and some conclusions in Section 7.
1 Preliminaries

1.1 The discrete Gronwall Lemma

In this paper, the following discrete Gronwall lemma will be frequently used (see [26, p. 369]):

Lemma 1 Let $k$, $B$ and $a_m$, $b_m$, $c_m$, $\gamma_m$ be nonnegative numbers.

a) (Discrete Gronwall inequality) We assume

$$a_{r+1} + k \sum_{m=0}^{r} b_m \leq k \sum_{m=0}^{r} \gamma_m a_m + k \sum_{m=0}^{r} c_m + B \quad \forall r \geq 0.$$ 

Then, one has

$$a_{r+1} + k \sum_{m=0}^{r} b_m \leq \exp \left( k \sum_{m=0}^{r} \gamma_m \right) \left( k \sum_{m=0}^{r} c_m + B \right) \quad \forall r \geq 0.$$ 

b) (Generalised discrete Gronwall inequality) We assume

$$a_{r} + k \sum_{m=0}^{r} b_m \leq k \sum_{m=0}^{r} \gamma_m a_m + k \sum_{m=0}^{r} c_m + B \quad \forall r \geq 0$$

such that $k\gamma_m < 1$ for all $m$. Then, setting $\sigma_m \equiv (1 - k\gamma_m)^{-1}$, one has

$$a_{r} + k \sum_{m=0}^{r} b_m \leq \exp \left( k \sum_{m=0}^{r} \sigma_m \gamma_m \right) \left( k \sum_{m=0}^{r} c_m + B \right) \quad \forall r \geq 0.$$ 

1.2 Space of functions and weak solutions

To define the notion of weak solution of problem (R), we introduce the following Hilbert spaces:

$$H^1_{b,l}(\Omega) = \{ v \in H^1(\Omega) / v|_{\Gamma_b,\Gamma_l} = 0 \},$$

$$H = \{ v \in L^2(\Omega)^2 / \nabla \times \langle v \rangle = 0 \text{ in } S, \langle v \rangle \cdot n_{\partial S} = 0 \},$$

$$V = \{ v \in H^1_{b,l}(\Omega)^2 / \nabla \times \langle v \rangle = 0 \text{ in } S \},$$

being $n_{\partial S}$ the normal outward unitary vector of $\partial S$. Observe that spaces $H$ and $V$ are the “hydrostatic version” of the classical spaces used for the Navier-Stokes equations.

We denote $H^1_{b,l}(\Omega) = H^1_b(\Omega)^2$, etc. The norm and scalar product in $L^2(\Omega)$ will be denoted by $| \cdot |$ and $\langle \cdot, \cdot \rangle$, whereas in $H^1_{b,l}(\Omega)$ we denote by $\| \cdot \|$ the norm of the gradient in $L^2(\Omega)$, that is $\| u \| = |\nabla u|$. On the other hand, we denote by $H^1_{b,l}(\Omega)$ and $H^{-1/2}(\Gamma_s)$ the dual spaces of $H^1_{b,l}(\Omega)$ and $H^{1/2}(\Gamma_s)$ respectively, with duality products $\langle \cdot, \cdot \rangle_{\Omega}$ and $\langle \cdot, \cdot \rangle_{\Gamma_s}$. Spaces defined in $\Omega$ will be frequently abbreviate as $L^2$ instead of $L^2(\Omega)$ etc.

The space related to the surface pressure will be:

$$L^2_0(S) = \left\{ q \in L^2(S) / \int_S q = 0 \right\}.$$
The vertical velocity \( u_3 \) can be obtained from \( \nabla_x \cdot u \) by means of either the integral formulation (11) or the differential formulation (3). In this process, the \( L^2(\Omega) \) regularity for the horizontal derivatives of \( u_3 \) is not obtained, hence the following “anisotropic” Hilbert spaces should be defined:

\[
H(\partial_z) = \{ v \in L^2(\Omega) / \partial_2 v \in L^2(\Omega) \} \quad \text{(resp. } H^k(\partial_z) = \{ v \in H^k(\Omega) / \partial_2 v \in H^k(\Omega) \}),
\]

and \( H_0(\partial_z) = \{ v \in H(\partial_z) / v = 0 \text{ on } \Gamma_s \cup \Gamma_b \} \). The inner products are defined by \( (v, w)_{H(\partial_z)} = \langle v, w \rangle + \langle \partial_2 v, \partial_2 w \rangle \) in \( H(\partial_z) \) and by \( (v, w)_{H_0(\partial_z)} = \langle \partial_2 v, \partial_2 w \rangle \) in \( H_0(\partial_z) \), owing to a vertical Poincaré inequality (see (4) below).

Notice that, given \( u \in H^1_{b,l}(\Omega) \), the weak solution \( u_3 \) of problem (3) can be defined by:

\[
u_3 \in H_0(\partial_z) \quad \text{such that} \quad (u_3, w)_{H_0(\partial_z)} = -\left( \nabla_x \cdot u, \partial_2 w \right) \quad \forall w \in H_0(\partial_z).
\]

Due to the loss of regularity of \( u_3 \) (\( u_3 \in L^2(\Omega) \) but \( u_3 \notin H^1(\Omega) \)), the vertical convection term \( u_3 \partial_2 u \) does not belong to \( H^1_{b,l}(\Omega) \), hence more regular test functions must be introduced in the variational formulation of \( (R) \). For instance, it suffices to consider \( v \in H^1_{b,l}(\Omega) \) such that \( \partial_2 v \in L^3(\Omega) \), because in this case one has (see [12]):

\[
\left\| \left( (U \cdot \nabla) u, v \right)_\Omega \right\| = \left| \int_\Omega (U \cdot \nabla) v \cdot u \right| < +\infty.
\]

Another possibility is to assume \( v \in H^1_{b,l}(\Omega) \cap L^\infty(\Omega) \), and then \( \int_\Omega (U \cdot \nabla) u \cdot v < +\infty \).

For fully discrete schemes, it is usual to use the following skew-symmetric of the convective terms: for each \( U \in H^1_{b,l} \times H_0(\partial_z), v \in H^1, w \in H^1 \) with either \( w \in L^\infty \) or \( \partial_2 w \in L^3 \),

\[
c(U, v, w) = \int_\Omega \left\{ (U \cdot \nabla) v \cdot w + \frac{1}{2} (\nabla \cdot U) v \cdot w \right\} \quad \text{if } w \in L^\infty
\]

\[
= -\int_\Omega \left\{ (U \cdot \nabla) w \cdot v + \frac{1}{2} (\nabla \cdot U) v \cdot w \right\} \quad \text{if } \partial_2 w \in L^3.
\]

Obviously, \( c(U, v, w) = \int_\Omega (U \cdot \nabla) v \cdot w \) whether \( \nabla \cdot U = 0 \). By simplicity, we denote the vertical part of these trilinear forms in the same manner, i.e.

\[
c(u_3, v, w) = \int_\Omega \left\{ u_3 \partial_2 v \cdot w + \frac{1}{2} (\partial_2 u_3) v \cdot w \right\} = -\int_\Omega \left\{ u_3 \partial_2 w \cdot v + \frac{1}{2} (\partial_2 u_3) v \cdot w \right\}
\]

Previous equalities hold even for discrete spaces, hence in the sequel, we will use any of these two possibilities.

Let us to define the weak solutions of \( (R) \) in \( (0, T) \), solving the following variational formulation (in a reduced form without pressure): Find \( \bar{U} = (u, u_3) \in L^2(0, T; V \times H_0(\partial_z)) \) with \( u \in L^\infty(0, T; H) \) and \( u|_{t=0} = u_0 \) in \( \Omega \), such that a.e. \( t \in (0, T) \):

\[
\begin{cases}
(u_t, v) + c(U, u, v) + \nu \left( \nabla u, \nabla v \right) = \left( f, v \right)_\Omega + \left( g_s, v \right)_{\Gamma_s}, & \forall v \in V \cap W^{1,3}_{b,l} \cap L^\infty,
\end{cases}
\]

\[
\begin{cases}
\left( \partial_2 u_3, \partial_2 w \right) = -\left( \nabla_x \cdot u, \partial_2 w \right), & \forall w \in H_0(\partial_z).
\end{cases}
\]

Here the Coriolis term has not been considered, because it does not add important difficulties to the arguments of this paper. In Section 6, we briefly analyze some possibilities to approximate the Coriolis term.
1.3 Known analytical results

The existence of weak solutions \((\mathbf{u}, p_s)\) of problem \((Q)\) is well known, see Lions-Teman-Wang \cite{30} and Lewandowski \cite{28}, always in domains with side-walls (i.e. \(D \geq D_{\text{min}} > 0\) in \(\mathcal{S}\)). In these works, a compactness method is used to obtain the velocity \(\mathbf{u}\) in a space with the restriction \(\nabla \cdot (\mathbf{u}) = 0\) and afterwards, the surface pressure \(p_s\) is obtained by means of a specific De Rham’s lemma on the surface \(S\). In domains without side-walls (i.e. when the depth function \(D\) can degenerate to zero), the existence of weak solutions \((\mathbf{u}, \nu_3, p)\) of \((P)\) is obtained by an asymptotic limit applied to the Navier-Stokes equations with anisotropic viscosity when the ratio depth over horizontal diameter (of the domain) goes to zero; see Besson-Laydi \cite{5} for the stationary case and Azerad-Guillén \cite{1, 2} for the time-dependent one. The existence of weak solutions of the stationary problem related to \((Q)\) in domains without side-walls is proved in Chacón-Guillén \cite{12} by an internal approximation argument; a mixed (velocity-pressure) variational formulation is approximated by a conformed Finite Element method verifying the so-called “hydrostatic Inf-Sup condition”. On the other hand, Ortegón in \cite{31} obtained a generalization of De Rham’s Lemma to general domains without side-walls.

Respect to regularity results for the Primitive Equations, the existence of strong solutions (with \(H^2(\Omega)\)-regularity for the horizontal velocity) is treated by Ziane in \cite{35} for the linear stationary problem associated to \((Q)\). This result is extended in \cite{20} to the linear evolutive case. For the nonlinear problem, the existence (and uniqueness) of local in time strong solutions for 2D domains (global in time for small enough data), is proved in \cite{20}. The extension (and improvement) of this kind of results to 3D domains can be seen in \cite{18}. Finally, assuming flat bottom and Neumann boundary condition on the bottom, the existence of global in time regular solutions without constraints is proved in \cite{7}. In \cite{27}, this result is also obtained with Dirichlet boundary conditions on the bottom.

1.4 Some 3D anisotropic spaces and related estimates

Given \(p, q \in [1, +\infty]\), it will be said that a function \(u \in L^q_{x}L^p_{\nu} (\Omega)\) (or simply \(L^q_{x}L^p_{\nu}\)) if:
\[
\begin{align*}
\mathbf{u} (\cdot, z) \in L^p (S_z) \quad \text{and} \quad \| \mathbf{u} (\cdot, z) \|_{L^p (S_z)} \in L^q (-D_{\text{max}}, 0),
\end{align*}
\]
where \(S_z = \{ x \in S : (x, z) \in \Omega \}\), and its norm is given by \(\| \mathbf{u} (\cdot, z) \|_{L^p (S_z)} \|_{L^q (-D_{\text{max}}, 0)}\). Some anisotropic norms frequently used in this paper will be:
\[
\begin{align*}
\| u \|_{L^q_x L^p_{\nu} (\Omega)} &= \left( \int_{-D_{\text{max}}}^0 \| u (\cdot, z) \|_{L^p (S_z)}^2 dz \right)^{1/2}, \\
\| u \|_{L^q_x L^p_{\nu} (\Omega)} &= \sup_{z \in (-D_{\text{max}}, 0)} \| u (\cdot, z) \|_{L^2 (S_z)},
\end{align*}
\]
In a similar way, we define the spaces
\[
H^1_x L^2_{\nu} \equiv H^1 (-D_{\text{max}}, 0; L^2 (S_z)), \quad L^2_x H^1_{\nu} \equiv L^2 (-D_{\text{max}}, 0; H^1 (S_z)).
\]
Notice that \(H^1_x L^2_{\nu} = H (\partial_z)\).

On the other hand, we will use frequently the following inequalities (see \cite{18}):

- Horizontal Gagliardo-Nirenberg inequality (related to 2D domains):
\[
\begin{align*}
\| u \|_{L^q_x L^p_{\nu}} &\leq C \| u \|_{H^1_{\nu}}^{1/2} \| \nabla_x u \|_{L^2_{\nu}}^{1/2} \quad \forall u \in L^2_x H^1_{\nu} \quad \text{such that} \quad u|_{\Gamma_{\nu} \cup \Gamma_l} = 0, \\
\| u \|_{L^q_x L^p_{\nu}} &\leq C \| u \|^{1/2} \| u \|^{1/2} \quad \forall u \in H^1.
\end{align*}
\]

7
• Vertical Poincaré Inequality (related to 1D domains):

\[ |v| \leq D_{\text{max}}^{1/2} |\partial_z v|, \quad \forall v \in H^1_{z} L^2_{\mathbf{x}} \text{ such that } v|_{\Gamma_b} = 0 \text{ or } v|_{\Gamma_s} = 0. \quad (4) \]

• Vertical Gagliardo-Nirenberg inequality (related to 1D domains):

\[ \|v\|_{L^p_{z} L^q_{\mathbf{x}}} \leq C (\|v\| + |\partial_z v|^{1/2}) \quad \forall v \in H^1_{z} L^2_{\mathbf{x}}. \quad (5) \]

Moreover, if \( v|_{\Gamma_b} = 0 \) or \( v|_{\Gamma_s} = 0 \), then \( \|v\|_{L^p_{z} L^q_{\mathbf{x}}} \leq C |\partial_z v|^{1/2}. \)

In particular, from (4) and (5), one has

\[ \|v\|_{L^p_{z} L^q_{\mathbf{x}}} \leq C |\partial_z v|, \quad \forall v \in H^1_{z} L^2_{\mathbf{x}} \text{ such that } v|_{\Gamma_b} = 0 \text{ or } v|_{\Gamma_s} = 0. \quad (6) \]

## 2 Description of the scheme

The time interval \([0, T]\) is divided into \( M \) subintervals of equal length \( k = T/M \), arising the partition of \([0, T]\), \( \{t_m = mk\}_{m=0}^{M} \). For simplicity and without loss of generality, we fix the viscosity constant \( \nu = 1. \)

We consider a time-scheme, where in each time step \( m \), given \((f^m, g^m)^M_{m=1}\) approximations of data \((f, g_s)\) at \( t = t_m \), a sequence \((u^m_{h, m}, u^m_{3, h}, p^m_{s, h})_m\) will be computed, as approximations to a regular solution \((u, u_3, p_s)\) of \((R)\) at \( t = t_m \). We are going to consider a fractional-step scheme, splitting the three main difficulties of the problem, namely:

• the computation of the vertical velocity,

• the non linear convective terms, \((U \cdot \nabla)u\) (in particular, the vertical convection \(u_3 \partial_z u\) is more singular than in the Navier-Stokes case),

• the restriction \(\nabla_x \cdot (u) = 0\) in \(S \times (0, T)\) related to the surface pressure \(p_s\).

Indeed, given \((u^m_{h, m}, p^m_{s, h})\), firstly the vertical velocity \(u^m_{3, h}\) is computed as function of \(\nabla_x \cdot u^m_{h}\), afterwards an intermediate horizontal velocity \(u^{m+1/2}_{h}\) is obtained by using convective terms, and finally \((u^{m+1}_{h}, p^{m+1}_{s, h})\) is computed solving a linear hydrostatic Stokes problem (associated to the restriction \(\nabla_x \cdot (u^{m+1}_{h}) = 0\)). This method can be called a “viscosity-splitting” scheme, because the diffusion terms are considered in the last two steps (see Sub-steps 1 and 2 below).

Let \(X_h, Y_h\) and \(Q_h\) be three adequate families of FE spaces which it will be precised below. Then, the fully discrete scheme is defined as follows:

**Initialization:** Let \(u^0_h \in X_h\) be an approximation of \(u_0\).

**Step of time \(m + 1\):**

**Sub-step 0:** Given \(u^m_h \in X_h\), compute \(u^m_{3, h} \in Y_h\) such that,

\[
(S_0)^m_h \quad \left( \partial_z u^m_{3, h}, \partial_z y_h \right) = -\left( \nabla_x \cdot u^m_{h}, \partial_z y_h \right) \quad \forall y_h \in Y_h.
\]
Sub-step 1: Given $U_h^m = (u_h^m, u_{3h}^m) \in X_h \times Y_h$, compute $u_h^{m+1/2} \in X_h$ such that,

$$(S_1)_h^{m+1} \left\{ \begin{array}{l}
\frac{1}{k} (u_h^{m+1/2} - u_h^m, v_h) + c (U_h^m, u_h^{m+1/2}, v_h) + \left( \nabla u_h^{m+1/2}, \nabla v_h \right) \\
= \left( f^{m+1}, v_h \right)_\Omega + \left( g_{s}^{m+1}, v_h \right)_{\Gamma_s} \quad \forall v_h \in X_h.
\end{array} \right.$$ 

Sub-step 2: Given $u_h^{m+1/2} \in X_h$, compute $(u_h^{m+1}, p_h^{m+1}) \in X_h \times Q_h$, such that

$$(S_2)_h^{m+1} \left\{ \begin{array}{l}
\frac{1}{k} (u_h^{m+1} - u_h^{m+1/2}, v_h) + \left( \nabla (u_h^{m+1} - u_h^{m+1/2}), \nabla v_h \right) - \left( p_h^{m+1}, \nabla \cdot (v_h) \right)_S = 0, \\
\left( \nabla \cdot (u_h^{m+1}), q_h \right)_S = 0 \quad \forall (v_h, q_h) \in X_h \times Q_h.
\end{array} \right.$$ 

A linear $z$-elliptic problem must be computed in Sub-step 0, a decoupled linear convection-diffusion problem in Sub-step 1 and a (generalized) Hydrostatic Stokes problem in Sub-step 2, which will be well-defined if a particular Inf-Sup stability condition holds, see (H1) below.

On the other hand, in order to do the effective computation of the integrals on the $2D$ surface $S$ given in $(S_2)_h^{m+1}$, $(p_h^{m+1}, \nabla \cdot (v_h))_S$, it will be necessary to use vertically structured grids.

With respect to the consistency of this scheme, adding $(S_1)_h^{m+1}$ and $(S_2)_h^{m+1}$ one has:

$$(S)_h^{m+1} \left\{ \begin{array}{l}
\frac{1}{k} (u_h^{m+1} - u_h^m, v_h) + c (U_h^m, u_h^{m+1/2}, v_h) + \left( \nabla u_h^{m+1/2}, \nabla v_h \right) \\
- \left( p_h^{m+1}, \nabla \cdot (v_h) \right)_S = \left( f^{m+1}, v_h \right)_\Omega + \left( g_{s}^{m}, v_h \right)_{\Gamma_s}.
\end{array} \right.$$ 

This formulation will be used to prove the convergence of the scheme.

Although viscosity-splitting schemes solve a mixed method which request higher computational cost than classical projection (segregated) schemes, they present some advantages because viscosity-splitting schemes have not numerical boundary layer for the pressure due to diffusion terms are also included in the free-divergence projection step (here Sub-step 2), which let to impose exact boundary conditions for the velocity.

On the other hand, viscosity-splitting schemes improve the numerical treatment of Euler (or semi-implicit) mixed schemes [34], because Sub-step 2 is a symmetric problem which can be formulated as a minimization problem, and then it can be approximated by using many numerical optimization solvers, as the Uzawa’s method, the Augmented Lagrangian method, etc. [17].

2.1 Choice of adequate Finite Element spaces

We restrict ourselves to the case where the surface domain $S \subset \mathbb{R}^2$ has a polygonal boundary and the bottom function $D$ is globally continuous and locally $P_1$, hence the $3D$ domain $\Omega$ has polygonal boundary. Moreover, the following hypothesis will be imposed about $\Omega$:

(H0) Regularity of the Domain: Assume $\Omega \subset \mathbb{R}^3$ such that the Hydrostatic Stokes Problem has $H^2(\Omega) \times H^1(S)$ regularity for (horizontal) velocity and pressure respectively. For this, the following hypothesis of existence of sidewalks should be imposed (see [35]):

$$D \geq D_{\text{min}} > 0 \quad \text{in } S.$$
To discretize the domain $\Omega$, let $\mathcal{T}_h(\Omega)$ be a regular and quasi-uniform triangulation of $\Omega$ (with elements $K \in \mathcal{T}_h(\Omega)$) and $\mathcal{T}_h(S)$ its associated triangulation of $S$ with elements $T \in \mathcal{T}_h(S)$. Assume that $\mathcal{T}_h(\Omega)$ is a vertically structured mesh, and then each element $K \in \mathcal{T}_h(\Omega)$ is projected onto only one element $T \in \mathcal{T}_h(S)$. Some references about how to construct vertically structured meshes can be seen in [12], [13] and [14] by using the so-called iso-$\sigma$ layers or in [19] by using a $P_0$ approximation on the bottom.

We consider three families of FE spaces: $X_h \subset \mathbf{H}^1_{b,l}(\Omega)$ for the horizontal velocity, $Y_h \subset H_0(\partial_\varepsilon)$ for the vertical velocity and $Q_h \subset L^2_0(S)$ for the pressure. Functions of $X_h$ are globally continuous, whereas functions in $Y_h$ must be globally continuous only with respect to vertical direction and $Q_h$ could be furnished by discontinuous functions.

The following hypotheses are required about $(X_h, Y_h, Q_h)$:

**H1** $(X_h, Q_h)$ satisfies the so called “hydrostatic Inf-Sup” condition ([12]): There exists $\beta > 0$ (independent of $h$) such that, for all $h > 0$,

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{(q_h, \nabla x \cdot (v_h))_S}{|\nabla x v_h| + \|\partial_y v_h\|_{L^3}} \geq \beta \|q_h\|_{L^2(S)}, \quad \forall q_h \in Q_h \setminus \{0\}.$$  

**H2** The following inverse inequalities hold: for each $v_h \in X_h$,

$$\|v_h\|_{L^2(L^2_k)} \leq C h^{-1/2} |v_h|, \quad \|v_h\|_{L^\infty(L^\infty_k)} \leq C h^{-1/2} \|v_h\|_{L^\infty(L^2_k)}, \quad \|v_h\|_{L^2(L^\infty_k)} \leq C h^{-1} |v_h|,$$

$$\|v_h\|_{L^3} \leq C h^{-1/2} |v_h|, \quad \|v_h\| \leq C h^{-1} |v_h|, \quad \|v_h\|_{W^{1,6}} \leq C h^{-1} \|v_h\|.$$  

**H3** The approximation properties of order $O(h^l)$ (for $l = 1$ or 2):

$$h^{-1} |v - I_h v| + \|v - I_h v\| \leq C h^l \|v\|_{H^{l+1}} \quad \forall v \in H^{l+1}(\Omega) \cap V,$$

$$|v - I_h v| \leq C h^l \|v\|_{H^l} \quad \forall v \in H^l \cap V,$$

$$\|q - J_h q\|_{L^2(S)} \leq C h^l \|q\|_{H^l(S)} \quad \forall q \in H^l(S) \cap L^2_0(S),$$

$$\|v_3 - K_h v_3\|_{H(\partial_\varepsilon)} \leq C h^l \|v_3\|_{H^{l+1}} \quad \forall v_3 \in H^{l+1}(\Omega) \cap H_0(\partial_\varepsilon),$$  

where $(I_h, J_h) : V \times L^2_0(S) \to X_h \times Q_h$ is the hydrostatic Stokes projector defined as:

$$(I_h v, J_h q) \in X_h \times Q_h : \left\{ \begin{array}{ll}
(\nabla (I_h v - v), \nabla v_h) - (J_h q - q, \nabla x \cdot (v_h))_S = 0 & \forall v_h \in X_h, \\
(\nabla x \cdot (I_h v), q_h)_S = 0 & \forall q_h \in Q_h,
\end{array} \right.$$  

and $K_h : H_0(\partial_\varepsilon) \to Y_h$ is the $H_0(\partial_\varepsilon)$-projector onto $Y_h$ defined as:

$$K_h v_3 \in Y_h : \left( \partial_y (K_h v_3 - v_3), \partial_y y_h \right) = 0 \quad \forall y_h \in Y_h.$$  

There are some possibilities to define $(X_h, Y_h, Q_h)$ satisfying (H1)-(H3). For instance, to approximate the pressure, we can consider

$$Q_h = \{q_h \in C^0(S) : q_h|_T \in P_1[x], \forall T \in T_h(S) \} \cap L^2_0(S).$$  

To choose $(X_h, Y_h)$ there are at least two possibilities ([12], [21]):
1. *Taylor-Hood* \((O(h^2)\) approximation, \(l = 2\)); locally \(P_2[x, z]\) by tetrahedrons and globally continuous FE.

2. *Mini-element* \((O(h)\) approximation, \(l = 1\)). Let \(\mathcal{P}(K) = P_1[x, z] \oplus \alpha_K \lambda_1 \lambda_2 \lambda_3 \lambda_4\) with \(\alpha_K \in \mathbb{R}\) and \(\lambda_i \in P_1(K)\) such that \(\lambda_i(a_j) = \delta_{ij}\), with \(a_j\) the vertices of the tetrahedron \(K\). Then, we consider

\[
X_h = \{ \mathbf{v}_h \in C^0(\overline{\Omega}) : \mathbf{v}_h|_K \in \mathcal{P}(K), \forall K \in \mathcal{T}_h \}^2 \cap H_{div}^1(\Omega),
\]

\[
Y_h = \{ \mathbf{y}_h \in C^0(\overline{\Omega}) : \mathbf{y}_h|_K \in P_1[x, z], \forall K \in \mathcal{T}_h(\Omega) \} \cap H_0(\partial \varepsilon).
\]

For vertically structured grids furnished by prisms, there are other possibilities to choose \(X_h\). For instance, using a bubble by prism or a bubble by each column of vertical prisms as in [21]. Notice that these possibilities are not stables for the Navier-Stokes problem.

On the other hand, considering triangulations where each element is a right prism, other possibilities to choose \(Y_h\) are ([19]):

- locally \(P_0[x] \otimes P_1[z]\) and globally \(z\)-continuous (for \(l = 1\)),
- locally \(P_1[x] \otimes P_2[z]\) and globally continuous and \(z-C^1\) (for \(l = 2\)).

For these anisotropic FE approximations of \(Y_h\), we have not seen in the literature any approximation result like hypothesis [7].

### 3 Stability and convergence towards weak solutions of \((R)\)

In this Section, we are going to study stability properties of scheme \((S_0)_{h-h}(S_2)_{h}\) of \((R)\). For this, we will obtain some a priori (stability) estimates that let us pass to the limit (convergence), where compactness results must be applied to “control” the limit in the (nonlinear) convective terms.

Fixed the (uniform) partition of \([0, T]\) of diameter \(k = T/M\): \(\{ t_m = mk \}_{m=0}^M\), for a given vector \(u = (u^m)^M_{m=0}\) with \(u^m \in X\) (\(X\) being a Banach space), let us to introduce the following notation for discrete in time norms:

\[
\| u \|_{l^2(X)} = \left( k \sum_{m=0}^M \| u^m \|_X^2 \right)^{1/2} \text{ and } \| u \|_{l^\infty(X)} = \max_{m=0,...,M} \| u^m \|_X.
\]

In this section, we consider the following weak regularity on the data

\[
(WR) \quad f \in L^2(0, T; H^{-1}_0(\Omega)), \quad g_s \in L^2(0, T; H^{-1/2}(\Gamma_s)) \quad \text{and} \quad u_0 \in H,
\]

and we choose

\[
f^{m+1} = \frac{1}{k} \int_{t_m}^{t_{m+1}} f(t) \, dt, \quad g^{m+1}_s = \frac{1}{k} \int_{t_m}^{t_{m+1}} g_s(t) \, dt.
\]

**Lemma 2** (Stability) Assume \((WR)\) and \((H1)\). If \((u^0_h)\) is bounded in \(L^2\), then the following estimates hold:

\[
\| u_{h}^{m+1} \|_{l^\infty(L^2) \cap l^{2}(H^1)} + \| u_{h}^{m+1/2} \|_{l^\infty(L^2) \cap l^2(H^1)} \leq C,
\]

\[
\| u_{h}^{m+1} - u_{h}^{m+1/2} \|_{l^2(L^2)} + \| u_{h}^{m+1/2} - u_{h}^{m} \|_{l^2(L^2)} \leq C k^{1/2},
\]

\[
\| u_{3,h}^{m+1} \|_{l^2(H(\partial \varepsilon) \cap L^\infty(L^2))} \leq C.
\]
Proof. Estimates (8) and (9) can be deduced making
\[ (S_1)_h^{m+1}, u_h^{m+1/2} + (S_2)_h^{m+1}, u_h^{m+1} \]
and using that
\[ c(U_h^m, u_h^{m+1/2}, u_h^{m+1/2}) = 0 \quad \text{and} \quad \left( p_h^{m+1}, \nabla \cdot (u_h^{m+1}) \right)_S = 0. \]
Indeed, one arrives at
\[ \frac{1}{2k}(u_h^{m+1/2} - u_h^m)^2 + \frac{1}{2k}(u_h^{m+1/2} - u_h^m)^2 = \left( \mathbf{m}^{m+1}, u_h^{m+1/2} \right) + \left( \mathbf{g}_h^{m+1}, u_h^{m+1/2} \right)_S. \]
Then, adding from \( m = 0 \) to \( r \) (with any \( r < M \)), we obtain the desired estimates (8) and (9).

On the other hand, taking \( y_h = u_{3,h}^{m+1} \in Y_h \) as test function in \((S_0)_h^{m+1}\), one has \( |\partial_x u_{3,h}^{m+1}| \leq |\nabla x \cdot u_{h}^{m+1}| \). Therefore, (10) is a consequence of estimate (8) and inequality (6).

Now, we define the following sequences of functions (defined for all \( t \in [0, T] \)):
- \( u_{k,h}^{(i)} : [0, T] \to H_{b,f}^1(\Omega) \), such that \( u_{k,h}^{(i)}(t) = u_{h}^{m+i/2} \) if \( t \in (t_m, t_{m+1}) \), \( i = 0, 1, 2 \).
- \( u_{3,k,h}^{(0)} : [0, T] \to L^2(\Omega) \), such that \( u_{3,k,h}^{(0)}(t) = u_{3,h}^{m} \) if \( t \in (t_m, t_{m+1}) \).
- \( u_{k,h}^{(0)} : [0, T] \to H_{b,f}^1(\Omega) \), continuous, linear by subintervals and \( u_{k,h}(t_m) = u_{h}^{m} \).

**Theorem 3 (Convergence)** Assume (WR) and (H0)-(H2), then there exists a subsequence \((k', h')\) of \((k, h)\), with \((k', h') \downarrow 0\), and a weak solution \( U = (u, u_3) \) of (R) in \((0, T)\), such that:
- \( (u_{k',h'}^{(i)}) \) for each \( i = 0, 1, 2 \) and \( u_{k',h'} \) converge to \( u \) strongly in \( L^2(0, T; \mathbf{L}^2(\Omega)) \), weakly-star in \( L^\infty(0, T; \mathbf{L}^2(\Omega)) \) and weakly in \( L^2(0, T; \mathbf{H}_{b,f}^1(\Omega)) \), whereas \( u_{3,k',h'} \) converges to \( u_3 \) weakly in \( L^2(0, T; H_0(\partial_\delta)) \).

**Proof.** Owing to definition of the functions \( u_{k,h}^{(i)}, u_{3,k,h}^{(0)} \) and \( u_{k,h}^{(0)} \), Lemma 2 says:

\begin{equation}
(u_{k,h}^{(i)})_{k,h} \quad \text{and} \quad (u_{k,h})_{k,h} \quad \text{are bounded in} \quad L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}_{b,f}^1), \quad \forall i = 0, 1, 2,
\end{equation}

\begin{equation}
(u_{3,k,h}^{(0)})_{k,h} \quad \text{is bounded in} \quad L^2(H_0(\partial_\delta)).
\end{equation}

On the other hand, from (9), there exists \( C = C(\nu, u_0, f, g_\nu) > 0 \) such that, \( \forall i, j = 0, 1, 2 \),
\[ \| u_{k,h}^{(i)} - u_{k,h}^{(j)} \|_{L^2(\mathbf{L}^2)} \leq C k, \quad \| u_{k,h}^{(i)} - u_{k,h} \|_{L^2(\mathbf{L}^2)} \leq C k. \]

Therefore, there exist subsequences of \( (u_{k,h})_{k,h} \) and \( (u_{k,h})_{k,h} \) (denoted in the same way) and a limit function \( u \) verifying the following weak convergences as \((h, k) \to 0\):
\[ (u_{k,h})_{k,h} \to u, \quad (u_{k,h})_{k,h} \to u \quad \text{in} \quad \begin{cases} L^2(0, T; \mathbf{H}_{b,f}^1(\Omega)) & \text{weak} \\ L^\infty(0, T; \mathbf{L}^2(\Omega)) & \text{weak*} \end{cases} \]
Note that, thanks to (12), the uniqueness of the limits \(u^{(i)}_{k,h} = u_i\) for each \(i = 0, 1, 2\) hold. Moreover, from (11)
\[
u^{(0)}_{3,k,h} \to u_3 \quad \text{in } L^2(H_0(\partial z)).
\]
Finally, \((S)^{m+1}_h\) can be rewritten (eliminating the pressure) as follows:
\[
\left(\partial_t u_{k,h}, v_h\right) + c\left(U^{(0)}_{k,h}, u^{(1)}_{k,h}, v_h\right) + \left(\nabla u^{(2)}_{k,h}, \nabla v_h\right) = \left(f^{m+1}, v_h\right) + \left(g^{m+1}_s, v_h\right)_{\Gamma_s} \tag{13}
\]
for each \(v_h \in X_h \cap V\).
On the other hand, \((S^m_h)\) is rewritten as
\[
\left(\partial_z u^{(0)}_{3,k,h}, \partial_z w_h\right) = -\left(\nabla_x \cdot u^{(0)}_{k,h}, \partial_z w_h\right) \quad \forall w_h \in Y_h \tag{14}
\]
To take limits in (13), we need for instance compactness of \((u^{(1)}_{k,h})_{k,h}\) in \(L^2(L^2(\Omega))\). But, owing to (12), it suffices to obtain compactness of \((u^{(2)}_{k,h})_{k,h}\) in \(L^2(L^2(\Omega))\). Assuming this compactness, taking (13)-(14) in \((k', h')\), the pass to the limit when \((k', h') \to 0\) can be realized by a standard way, concluding that \((u, u_3)\) is a weak solution of the continuous problem \((R)\).
Therefore, it suffices to get compactness of \((u^{(2)}_{k,h})_{k,h}\) in \(L^2(L^2(\Omega))\). For this, let us introduce
\[
V_h = \{v_h \in X_h / \left(\nabla_x \cdot v_h\right)_{S} = 0, \forall q_h \in Q_h\}
\]
and the operator \(A_h^{-1} : V_h \to V_h\) defined as the inverse of the discrete “hydrostatic” Stokes problem: given \(u_h \in V_h\),
\[
A_h^{-1} u_h \in V_h \quad \text{such that} \quad \left(\nabla A_h^{-1} u_h, \nabla v_h\right) = \left(u_h, v_h\right) \quad \forall v_h \in V_h \tag{15}
\]
Taking \(v_h = A_h^{-1} u_h \in V_h\) in (15),
\[
|\nabla A_h^{-1} u_h|^2 = \left(u_h, A_h^{-1} u_h\right) \leq C \|u_h\|_{V_h'} |\nabla A_h^{-1} u_h|, \quad \forall v_h \in V_h \tag{16}
\]
hence
\[
|\nabla A_h^{-1} u_h| \leq C \|u_h\|_{V_h'}.
\]
To obtain the inverse bound, we take any \(v_h \in V_h\) in (15), then
\[
\left(u_h, v_h\right) = \left(\nabla A_h^{-1} u_h, \nabla v_h\right) \leq |\nabla A_h^{-1} u_h| |\nabla v_h| \quad \forall v_h \in V_h,
\]
hence
\[
\|u_h\|_{V_h'} \leq |\nabla A_h^{-1} u_h|.
\]
Now, to obtain compactness of \(u^{(2)}_{k,h}\) we follow an argument of [23]. First of all, we prove the following result.

**Lemma 4** Under hypothesis of Theorem 3, one has
\[
\int_0^{T-\delta} \|u^{(2)}_{k,h}(t + \delta) - u^{(2)}_{k,h}(t)\|^2_{V_h'} dt \leq C \delta, \quad \forall \delta : 0 < \delta < T, \tag{17}
\]
where \(C > 0\) is independent of \(k, h\) and \(\delta\).
Proof. Since \( u_{k,h}^{(2)} \) is a piecewise constant function, it suffices to suppose that \( \delta \) is proportional to time step \( k \), i.e., \( \delta = rk \) for any \( r = 1, \ldots, M \). In fact, to obtain (17), it suffices to prove

\[
k \sum_{n=0}^{M-r} \| u_h^{n+r} - u_h^n \|_{V_h}^2 \leq C(rk), \quad \forall r = 1, \ldots, M.
\]  

(18)

Taking \( kv_h \), for any \( v_h \in V_h \), as test functions in \((S_1)_h^{m+1}\) and adding from \( m = n \) to \( n + r - 1 \),

\[
\left( u_h^{n+r} - u_h^n, v_h \right) = -k \sum_{m=n}^{n+r-1} c(U_h^m, u_h^{m+1/2}, v_h) - k \sum_{m=n}^{n+r-1} \left( \nabla u_h^{m+1}, \nabla v_h \right) + k \sum_{m=n}^{n+r-1} \left\{ \left( p_h^m, \nabla \cdot (v_h) \right)_S + \left( f^{m+1}, v_h \right)_\Omega + \left( g^m, v_h \right)_{\Gamma_s} \right\}.
\]  

(19)

Now, taking \( v_h = kA_h^{-1}(u_h^{n+r} - u_h^n) \) as test function in (19), using (16) and adding from \( n = 0 \) to \( M - r \),

\[
k \sum_{n=0}^{M-r} \| u_h^{n+r} - u_h^n \|_{V_h}^2 = -k^2 \sum_{n=0}^{M-r} \sum_{m=n}^{n+r-1} c(U_h^m, u_h^{m+1/2}, A_h^{-1}(u_h^{n+r} - u_h^n))
- k^2 \sum_{n=0}^{M-r} \sum_{m=n}^{n+r-1} \left( \nabla u_h^{m+1}, \nabla A_h^{-1}(u_h^{n+r} - u_h^n) \right)
+ k^2 \sum_{m=n}^{n+r-1} \left\{ \left( f^{m+1}, A_h^{-1}(u_h^{n+r} - u_h^n) \right)_\Omega + \left( g^m, A_h^{-1}(u_h^{n+r} - u_h^n) \right)_{\Gamma_s} \right\}
\]

:= J_1 + J_2 + J_3.

Now, we have to bound the \( J_i \) terms. The bound of \( J_3 \) is rather standard. Since \( J_2 \) is easier to bound than \( J_1 \), we only analyze the more complicate term of \( J_1 \), which is the vertical convection:

\[
c(U_h^m, u_h^{m+1/2}, A_h^{-1}(u_h^{n+r} - u_h^n)) = -C(u_h^{m+1/2}, A_h^{-1}(u_h^{n+r} - u_h^n))
- \left( u_h^{m+1/2}, \partial_z A_h^{-1}(u_h^{n+r} - u_h^n) \right) = I_1 + I_2.
\]

Since bound \( I_1 \) is easier than \( I_2 \) (in fact, the \( I_1 \) term is the classical isotropic term appearing in the Navier-Stokes framework, see [23]), we only bound \( I_2 \) using inequality (3) as follows:

\[
I_2 \leq \| u_h^{m+1/2} \|_{L^2} \| \partial_z A_h^{-1}(u_h^{n+r} - u_h^n) \|_{L^2} \leq C \| u_h \| \| u_h^{m+1/2} \| \| A_h^{-1}(u_h^{n+r} - u_h^n) \|_{W^{1,2}}.
\]

To bound the term \( \| A_h^{-1}(u_h^{n+r} - u_h^n) \|_{W^{1,3}} \), we are going to use the following Lemma (see Appendix for a proof):

Lemma 5 Assuming (H0) and the inverse inequality \( \| v_h \|_{W^{1,6}} \leq C_1 \| v_h \| \) given in (H2), then

\[
\| A_h^{-1} v_h \|_{W^{1,6}} \leq C |v_h|, \quad \forall v_h \in V_h.
\]

In particular, \( \| A_h^{-1}(u_h^{n+r} - u_h^n) \|_{W^{1,3}} \leq C(u_h^{n+r} - u_h^n) \) hence the following bound holds:

\[
J_1 \leq C k^2 \sum_{n=0}^{M-r} \sum_{m=n}^{n+r-1} \| u_h^m \| \| u_h^{m+1/2} \| \| u_h^{n+r} - u_h^n \|
\]

14
Then, applying Fubini’s discrete rule, we obtain

\[ J_1 \leq C k^2 \sum_{m=0}^{M-1} \|u_h^m\| \|u_h^{m+1}\| \sum_{n=m-r+1}^{M} |u_h^{n+r} - u_h^n| \]

where

\[ m = \begin{cases} 
0 & \text{if } m < 0, \\
m & \text{if } 0 \leq m \leq M - r, \\
M - r & \text{if } m > M - r. 
\end{cases} \]

Since \(|m - m - r + 1| \leq r\), then \(\sum_{n=m-r+1}^{m} |u_h^{n+r} - u_h^n| \leq C r\). Finally, since \(\sum_{n=0}^{M-1} \|u_h^m\| \|u_h^{m+1}\| \leq C\), one arrives at \(J_1 \leq C (r k)\). On the other hand, one also has \(J_2 + J_3 \leq C (r k)\) and the proof of Lemma 4 is finished. 

Note that the bound for the fractional derivative in time (17) has been obtained in the norm \(V_h'\) which moves with respect to the space parameter \(h\). But, the compactness results (see for instance J. Simon [33]) does not work in these conditions. Then, we will use the already cited argument of [23] in order to find a fixed norm where the time fractional derivative can be bounded. For this, we consider the orthogonal projector onto \(V\):

\[ R_h : V_h \rightarrow V \text{ defined as } \left( \nabla (R_h v_h - v_h), \nabla w \right) = 0, \quad \forall w \in V, \]

which has the following properties (arguing as in [23]):

\[ \|R_h u_h\|_{H^1} \leq \|u_h\|_{H^1} \quad (H^1\text{-stability}), \]

\[ |R_h u_h - u_h| \leq C h \|\nabla X \cdot \langle u_h \rangle\|_{L^2(S)} \quad (L^2\text{-error estimate}), \]

and

\[ \|R_h u_h\|_{V'} \leq \|u_h\|_{V_h'} + C h. \]

For the second estimate, the \(H^2\) regularity of the hydrostatic Stokes problem with second member \(R_h u_h - u_h\) must be used, and for the last estimate, it uses the orthogonal projector onto \(V_h\)

\[ P_h : V \rightarrow V_h \text{ defined as } \left( P_h v - v, v_h \right) = 0, \quad \forall v_h \in V_h \text{ (see [23] for more details)}. \]

From here, using (18), one has

\[ k \sum_{n=0}^{M-r} \|R_h (u_h^{n+r} - u_h^n)\|_{V_h}' \leq C k \sum_{n=0}^{M-r} \|u_h^{n+r} - u_h^n\|_{V_h}' + C h \leq C(r k + h). \]

The above inequality can be written as

\[ \int_0^{T-\delta} \|R_h u_{h,k}^{(2)}(t + \delta) - R_h u_{h,k}^{(2)}(t)\|_{V_h'}^2 dt \leq C (\delta + h). \]

Now, we can apply a compactness (by perturbations) result due to P. Azérad and F. Guíllén [2], obtaining that \(R_h u_{h,k}^{(2)} \rightarrow u\) in \(L^2(0, T; L^2)\)-strong. From here, arguing again as in [23], one can deduce \(u_{h,k}^{(2)} \rightarrow u\) in \(L^2(0, T; L^2)\)-strong, and the proof of Theorem 3 is finished. \(\blacksquare\)
4 Error estimates with respect to problem (R)

In this section, we will obtain optimal error estimates (for the velocity and pressure) with respect to a sufficiently regular solution \( \{u, u_3, p_s\} \) of problem (R).

In order to obtain these error estimates, the following constraint between the time step \( k \) and the mesh size \( h \) will be assumed:

\[
(H) \quad k \leq h^2.
\]

4.1 Regularity hypotheses

The following regularity hypotheses for the solution \( (U = (u, u_3), p_s) \) of (R) will be imposed:

- To obtain order \( O(\sqrt{k} + h^l) \) in \( l^\infty(L^2) \cap l^2(H^1) \) for both velocities:
  \[
  (R1) \quad U \in L^\infty(H^{l+1}), \quad u_t \in L^2(H^l), \quad U_t \in L^2(L^2), \quad p_s \in L^2(H^l), \quad \sqrt{t} u_{tt} \in L^2(H^{-1}_{b,l}).
  \]

- To obtain order \( O(k + h^l) \) in \( l^\infty(L^2) \cap l^2(H^1) \) for the end-of-step velocity:
  \[
  (R2) \quad u_{tt} \in L^2(V').
  \]

- To get order \( O(\sqrt{k} + h^l) \) in \( l^2(L^2) \) for the time discrete derivative of end-of-step velocity, in \( l^\infty(H^1) \) for end-of-step velocity and in \( l^2(L^2(S)) \) for pressure,
  \[
  (R3) \quad U_t \in L^2(H^1), \quad u_{tt} \in L^2(L^2).
  \]

- To get order \( O(\sqrt{k} + h^2) \) (\( l = 2 \)) in \( l^\infty(L^2) \cap l^2(H^1) \) for the time discrete derivative of velocities,
  \[
  (R4) \quad \partial_t p_s \in L^2(H^l), \quad U_t \in L^\infty(L^3) \cap L^2(H^{l+1}), \quad u_t \in L^\infty(H^2), \quad u_{tt} \in L^2(H^l), \quad U_{tt} \in L^2(L^2), \quad \sqrt{t} u_{ttt} \in L^2(H^{-1}_{b,l}).
  \]

- To obtain order \( O(k + h^2) \) (\( l = 2 \)) in \( l^\infty(L^2) \cap l^2(H^1) \) for the time discrete derivative of end-of-step velocity and in \( l^2(L^2(S)) \) for the pressure,
  \[
  (R5) \quad U_t \in L^\infty(H^1), \quad u_{ttt} \in L^2(V').
  \]

4.2 Problems related to the spatial errors

We will present an error analysis for the fully discrete scheme \( (u_h^{m+1/2}, u_h^{m+1}, p_h^{m+1}) \) as an approximation of \( (u(t_{m+1}), u(t_{m+1}), p(t_{m+1})) \). Consequently, we consider the following errors:

\[
\begin{align*}
  e^{m+1/2} &= u(t_{m+1}) - u_h^{m+1/2}, \\
  e^{m+1} &= u(t_{m+1}) - u_h^{m+1}, \\
  e_p^{m+1} &= p_s(t_{m+1}) - p_h^{m+1}, \\
  e_3^{m+1} &= u_3(t_{m+1}) - u_{3,h}^{m+1}, \\
  E^{m+1} &= (e^{m+1}, e_3^{m+1}).
\end{align*}
\]

These errors can be decomposed as follows (splitting interpolation and discrete parts):

\[
\begin{align*}
  e^{m+1/2} &= e_i^{m+1} + e_h^{m+1/2}, \\
  e^{m+1} &= e_i^{m+1} + e_h^{m+1}, \\
  e_p^{m+1} &= e_{p,i}^{m+1} + e_{p,h}^{m+1}, \\
  e_3^{m+1} &= e_{3,i}^{m+1} + e_{3,h}^{m+1}, \\
  E^{m+1} &= E_h^{m+1} + E_i^{m+1}.
\end{align*}
\]
Concretely

\[ e_i^{m+1} = u(t_{m+1}) - I_h u(t_{m+1}) \quad \text{and} \quad e_h^{m+1} = I_h u(t_{m+1}) - u_h^{m+1}, \]

\[ e_h^{m+1/2} = I_h u(t_{m+1}) - u_h^{m+1/2}, \]

\[ e_{p,h}^{m+1} = p(t_{m+1}) - J_h p(t_{m+1}) \quad \text{and} \quad e_{p,h}^{m+1} = J_h p(t_{m+1}) - p_h^{m+1}, \]

\[ e_{u_3,h}^{m+1} = u_3(t_{m+1}) - K_h u_3(t_{m+1}) \quad \text{and} \quad e_{u_3,h}^{m+1} = K_h u_3(t_{m+1}) - u_{3,h}^{m+1}, \]

\[ E_t^{m+1} = (e_t^{m+1}, e_{3}^{m+1}) \quad \text{and} \quad E_h^{m+1} = (e_h^{m+1}, e_{3,h}^{m+1}). \]

Using the following Taylor expansion with integral rest of a function \( \phi = \phi(t) \):

\[ \phi(t + k) - \phi(t) = \phi'(t + k) k - \int_t^{t+k} (s - t)\phi''(s) \, ds, \]

and the variational problem verified for an exact solution \((U, p_s)\) at \( t = t_{m+1} \) of \((R)^{m+1}\), one has:

\[(R)^{m+1}\]

\[ \left\{ \begin{array}{l}
\left( \frac{1}{k} (u(t_{m+1}) - u(t_m)), v \right) + c(q(t_m), \nabla u(t_{m+1}), v) + (\nabla u(t_{m+1}), \nabla v) \\
-p_s(t_{m+1}), \nabla_v \cdot (v) \right)_S = \left( f(t_{m+1}), v \right)_\Omega + \left( g_s(t_{m+1}), v \right)_{\Gamma_s} + (E^{m+1}, v), \quad \forall v \in W_{s,h}^{1,3} \cap L^\infty, \\
\left( \nabla \cdot (u(t_{m+1})), q \right)_S = 0, \quad \forall q \in L^2_0(S), \\
\left( \partial_z u_3(t_{m+1}), \partial_z v_3 \right) = -\left( \nabla_v \cdot u(t_{m+1}), \partial_z v_3 \right), \quad \forall v_3 \in H_0(\partial_z),
\end{array} \right. \]

where \( E^{m+1} := -\frac{1}{k} \int_{t_m}^{t_{m+1}} (t - t_m) u_t(t) \, dt - \left( \int_{t_m}^{t_{m+1}} U_t \cdot \nabla \right) u(t_{m+1}) \) is the consistency error.

For simplicity, we assume \( f \in C([0, T]; H^{1,3}_h) \) and \( g_s \in C([0, T]; H^{-1/2}(\Gamma_s)) \) and we choose

\[ f^{m+1} = f(t_{m+1}) \quad \text{and} \quad g_s^{m+1} = g_s(t_{m+1}). \]

Then, the data errors \( f(t_{m+1}) - f^{m+1} \) and \( g_s(t_{m+1}) - g_s^{m+1} \) vanish.

Comparing \((R)^{m+1}\) with \((S_0)^m_h\) and \((S_1)^{m+1}_h\), the following variational problems for the spatial errors \( e_{3,h}^{m+1} \) and \( e_h^{m+1/2} \) hold:

\[(E_0)^m_h \quad \left( \partial_z e_{3,h}^{m}, \partial_z y_h \right) = -\left( \nabla_v \cdot (e_h^m + e_{i}^m), \partial_z y_h \right), \quad \forall y_h \in Y_h, \]

\[(E_1)^{m+1}_h \quad \left\{ \begin{array}{l}
\frac{1}{k} (e_{h}^{m+1/2} - e_h^m, v_h) + (\nabla e_h^{m+1/2}, v_h) + (p_s(t_{m+1}), \nabla_v \cdot (v_h))_S \\
= NL^{m+1}(v_h) + (E^{m+1}, v_h) - (\delta_t e_{i}^{m+1}, v_h) - (\nabla e_{i}^{m+1}, v_h), \quad \forall v_h \in X_h,
\end{array} \right. \]

where \( \delta_t e_{i}^{m+1} = \frac{1}{k} (e_{i}^{m+1} - e_i^m) \) and

\[ NL^{m+1}(v_h) = -c\left( E^m_h + E_{i}^m, u(t_{m+1}), v_h \right) - c\left( U^m_e, e_h^{m+1/2} + e_{i}^{m+1}, v_h \right). \]

On the other hand, adding and subtracting \( I_h u(t_{m+1}) \) to \((S_2)^{m+1}_h\) one has

\[(E_2)^{m+1}_h \quad \left\{ \begin{array}{l}
\frac{1}{k} (e_{i}^{m+1} - e_{i}^{m+1/2}, v_h) + (\nabla (e_{i}^{m+1} - e_{i}^{m+1/2}), v_h) = - (p_{h}^{m+1}, \nabla_v \cdot (v_h))_S \\
(\nabla \cdot (e_{h}^{m+1}), q_h)_S = 0,
\end{array} \right. \]
for each \((v_h, q_h) \in X_h \times Q_h\).

Due to the choice of the projector \(K_h\), \(\partial_x e_{3,h}^{m+1}, \partial_y y_h \) = 0, hence it is not appear in \((E_0)^{m+1}_h\). Since the same discrete space for \(u_h^{m+1/2}\) and \(u_h^{m+1}\) has been chosen, the interpolation error depending on \(e_h^{m+1/2} - e_h^m\) in \((E_1)^{m+1}_h\) is \(e_i^{m-1} - e_i^m\) and the interpolation error depending on \(e_i^{m+1} - e_i^{m+1/2}\) in \((E_2)^{m+1}_h\) is zero. Finally, since \(\left( \nabla_x \cdot (I_h u(t_m+1)), q_h \right)_S = 0\), the corresponding interpolation error \(\left( \nabla_x \cdot (e_i^{m+1}), q_h \right)_S = 0\), hence it is not appear in \((E_2)^{m+1}_h\).

Therefore, adding \((E_1)^{m+1}_h\) and \((E_2)^{m+1}_h\), one arrives at:

\[
\begin{aligned}
\tag{E}_{h}^{m+1} &= \left\{ \begin{array}{l}
\frac{1}{k} \left( e_h^{m+1} - e_h^m, v_h \right) + \left( \nabla e_h^{m+1}, \nabla v_h \right) - \left( e_p^{m+1}, \nabla \cdot (v_h) \right)_S \\
- \frac{1}{k} \left( e_i^{m+1} - e_i^m, v_h \right) + NL^{m+1}(v_h) + \left( e_i^{m+1}, v_h \right), \\
\left( \nabla_x \cdot (e_i^{m+1}), q_h \right)_S & = 0.
\end{array} \right.
\end{aligned}
\]

Due to the choice of the interpolation operator \((I_h, J_h)\) related to the hydrostatic Stokes problem, the interpolation error \(\left( \nabla e_i^{m+1}, \nabla v_h \right) + \left( e_{p,i}^{m+1}, \nabla \cdot (v_h) \right)_S = 0\), hence it does not appear in \((E)^{m+1}_h\).

### 4.3 Estimates for the vertical velocity

From \((E_0)^{m}_h\), the following estimate holds:

\[
|\partial_x e_{3,h}^m| \leq C(|\nabla_x \cdot e_h^m| + |\nabla_x \cdot e_i^m|).
\]

Therefore, by using inequality \((6)\),

\[
\|e_{3,h}^m\|^2_{L^\infty_2 L^2_2} \leq C(\|e_h^m\|^\alpha + \|e_i^m\|^\beta).
\]

On the other hand, the approximation property \((7)\) and the regularity \(u_3 \in L^\infty(0, T; H^{l+1})\) imply:

\[
|\partial_x e_{3,1}^m| \leq C h^l \|u_3(t_m)\|_{H^{l+1}} \leq C h^l.
\]

Therefore, by using again \((6)\),

\[
\|e_{3,1}^m\|_{L^\infty_2 L^2_2} \leq C h^l.
\]

### 4.4 \(O(\sqrt{k} + h^l)\) error estimates for both velocities in \(L^\infty(L^2) \cap L^2(H^1)\)

**Theorem 6** We assume \((H0)-(H3), \ (R1)\) and \(|e_h^0| \leq C h^l\). Then, there exists \(k_0 > 0\) such that for any \(k \leq k_0\), the following error estimates hold

\[
\|e_h^{m+1/2}\|_{L^\infty(L^2) \cap L^2(H^1)} + \|e_h^{m+1}\|_{L^\infty(L^2) \cap L^2(H^1)} \leq C(\sqrt{k} + h^l),
\]

\[
\|e_i^{m+1/2} - e_i^m\|_{L^2(L^2)} + \|e_i^{m+1} - e_i^{m+1/2}\|_{L^2(L^2)} \leq C \sqrt{k}(\sqrt{k} + h^l).
\]

Notice that in this result, the constraint \((H)\) on parameters \((k, h)\) is not necessary although \(k\) small enough must be imposed.

**Proof.** The main idea is to make \(2k \sum_{m=0}^{M-1} \left\{ \left( (E_1)^{m+1}_h, e_h^{m+1/2} \right) + \left( (E_2)^{m+1}_h, e_h^{m+1} \right) \right\} \).
In fact, making $2k\left((E_1)^{m+1}_h, e^{m+1/2}_h\right)$ and taking into account that $c\left(U^m_h, e^{m+1/2}_h, e^{m+1/2}_h\right) = 0$, we arrive at

$$|e^{m+1/2}_h|^2 - |e^m_h|^2 + 2k\|e^{m+1/2}_h\|^2 + 2k\left(p_s(t_{m+1}), \nabla_x \cdot \epsilon^{m+1/2}_h\right) + 2k\left(E^m_h + E^m_i, u(t_{m+1}), e^{m+1/2}_h\right)$$

$$+ 2k\left(U^m_h, e^{m+1}_i, e^{m+1/2}_h\right) + 2k\left(e^{m+1} - \delta_i e^{m+1}_i, e^{m+1/2}_h\right) - \left(\nabla e^{m+1}_i, \nabla e^{m+1/2}_h\right) := \sum_{i=1}^5 I_1$$

For brevity, we only bound the more difficult terms of the RHS of (26) (for more details, see [25], where these type of bounds have been made for the Navier Stokes case). We bound term $I_1$, by using that $(J_h p_s(t_{m+1}), \nabla_x \cdot e^m_h) = 0$, as

$$I_1 = 2k\left(p_s(t_{m+1}), \nabla_x \cdot (e^{m+1/2}_h - e^m_h)\right) + 2k\left(e^{m+1}_i, \nabla \cdot e^m_h\right)$$

$$= -2k\left(\nabla_x p_s(t_{m+1}), e^{m+1/2}_h - e^m_h\right) + 2k\left(e^{m+1}_i, \nabla \cdot e^m_h\right)$$

$$\leq \varepsilon|e^{m+1/2}_h - e^m_h|^2 + C k^2\|p_s(t_{m+1})\|^2 + \varepsilon k\|e^m_h\|^2 + C k h^2 \|p_s(t_{m+1})\|^2_{H^t(S)}$$

The interpolation part of $I_4$ is

$$2k\left(\delta_i e^{m+1}_i, e^{m+1/2}_h\right) = 2k\left(e^{m+1}_i, \delta_i u(t_{m+1})\right) \leq \varepsilon k\|e^{m+1/2}_h\|^2 + C h^2 \int_{t_m}^{t_{m+1}} \|u\|_{H^t}^2$$

The discrete vertical part of $I_2$ is

$$2k\left(e^{m}_3, u(t_{m+1}), e^{m+1/2}_h\right) = 2k\left(e^{m}_3, \partial_z u(t_{m+1}), e^{m+1/2}_h\right) + k\left(\partial_z e^{m}_3, u(t_{m+1}), e^{m+1/2}_h\right) := L_1 + L_2.$$ 

By using that $u \in L^{\infty}(H^{1+1})$ and (21),

$$L_1 \leq 2k\|e^m_3\|_{L^\infty\|L^2\|} \|\partial_z u(t_{m+1})\|_{L^2} \|e^{m+1/2}_h\|_{L^2}$$

$$\leq C k\left(\|e^m_3\|_1 + \|e^m_3\|_{L^2}\right) \|e^{m+1/2}_h\|_{H^1}$$

$$\leq \varepsilon k\|e^m_3\|^2 + C k h^2 + \varepsilon k\|e^{m+1/2}_h\|^2 + C k |e^{m+1/2}_h|^2$$

By using that $u \in L^{\infty}(H^2)$ and (20),

$$L_2 \leq k\|\partial_z e^{m}_3\| \|u(t_{m+1})\|_{L^\infty} \|e^{m+1/2}_h\| \leq k\left(\|e^m_3\|^2 + \|e^m_3\|^2\right) + C k |e^{m+1/2}_h|^2$$

$$\leq \varepsilon k\|e^m_3\|^2 + C k |e^{m+1/2}_h|^2 + C k h^2.$$ 

By a similar way, by using that $u \in L^{\infty}(H^2), u_3 \in L^{\infty}(H^{1+1})$ and (22), the vertical interpolation part of $I_2$ is bounded as:

$$2k\left(e^{m}_3, u(t_{m+1}), e^{m+1/2}_h\right) = 2k\left(e^{m}_3, \partial_z u(t_{m+1}), e^{m+1/2}_h\right) + k\left(\partial_z e^{m}_3, u(t_{m+1}), e^{m+1/2}_h\right)$$

$$\leq C k\|e^{m}_3\|_{H^{1+1}} \|u(t_{m+1})\|_{H^2} \|e^{m+1/2}_h\| \leq \varepsilon k\|e^{m+1/2}_h\|^2 + C k h^2.$$ 

Now, we decompose $I_3$ as

$$I_3 = 2k\left(U^m_h, e^{m+1}_i, e^{m+1/2}_h\right) = 2k\left(E^m, e^{m+1}_i, e^{m+1/2}_h\right) - 2k\left(U(t_{m+1}), e^{m+1}_i, e^{m+1/2}_h\right),$$

\[19\]
and their more difficult terms can be bounded, using (20), (21) and the inverse inequalities
\[ \|e_{h}^{m+1/2}\|_{L_2} \leq C h^{-1}\|e_{h}^{m+1/2}\|_{L^1} \leq C h^{-1/2}\|e_{h}^{m+1/2}\|_{L^1}, \] as follows:

\[
2k\left( e_{3,h}^m + e_{3,i}^m, e_{i}^{m+1}, e_{h}^{m+1/2} \right) = 2k \left( (e_{3,h}^m + e_{3,i}^m) \partial_2 e_{i}^{m+1}, e_{h}^{m+1/2} \right) \\
+ k \left( (\partial_2 e_{3,h}^m + \partial_2 e_{3,i}^m) e_{i}^{m+1}, e_{h}^{m+1/2} \right) \\
\leq 2k \|e_{3,h}^m + e_{3,i}^m\|_{L_2} \|\partial_2 e_{i}^{m+1}\|_{L_2} \|e_{h}^{m+1/2}\|_{L_2} \\
+ k \|\partial_2 e_{3,h}^m + \partial_2 e_{3,i}^m\|_{L_2} \|e_{i}^{m+1}\|_{L_2} \|e_{h}^{m+1/2}\|_{L_2} \\
\leq C k (\|e_{h}^m\| + h^l) h^{-1} \|e_{h}^{m+1/2}\| \\
\leq C k (\|e_{h}^m\| + h^l) h^{-1} \|e_{h}^{m+1/2}\| + 2 C k h^2 l + C k \|e_{h}^{m+1/2}\|^2.
\]

Therefore, applying previous estimates to (29) and making \(\|e_{h}^{m+1/2}\|^2 \leq 2(\|e_{h}^m\|^2 + \|e_{h}^{m+1/2} - e_{h}^m\|^2)\), we get

\[
\|e_{h}^{m+1/2}\| - \|e_{h}^{m+1/2} - \sum_{m=0}^{k} \left( (e_{h}^{m+1} - e_{h}^{m+1/2})^2 + \frac{1}{2} \|e_{h}^{m+1/2} - e_{h}^m\|_2 \right) \\
+ k \sum_{m=0}^{r} \left( \frac{1}{2} \|e_{h}^{m+1}\|^2 + \|e_{h}^{m+1} - e_{h}^{m+1/2}\|^2 + \frac{1}{2} \|e_{h}^{m+1/2}\|_2 \right) \\
\leq C k \sum_{m=0}^{r} \|e_{h}^{m}\|^2 + \varepsilon k \|e_{h}^0\|^2 + C k h^2 l.
\]

On the other hand, making \(2k \left( (E_{2,h})^{m+1}, e_{h}^{m+1} \right)\), we arrive at

\[
\|e_{h}^{m+1}\|^2 - \|e_{h}^{m+1} - \sum_{m=0}^{k} \left( (e_{h}^{m+1} - e_{h}^{m+1/2})^2 + \frac{1}{2} \|e_{h}^{m+1/2} - e_{h}^m\|_2 \right) \\
+ k \sum_{m=0}^{r} \left( \frac{1}{2} \|e_{h}^{m+1}\|^2 + \|e_{h}^{m+1} - e_{h}^{m+1/2}\|^2 + \frac{1}{2} \|e_{h}^{m+1/2}\|_2 \right) \\
\leq C k \sum_{m=0}^{r} \|e_{h}^{m}\|^2 + \varepsilon k \|e_{h}^0\|^2 + C(k + h^2 l).
\]

Therefore, applying discrete Gronwall’s Lemma, we can get (24) and (25).

4.5 \(O(k + h^l)\) error estimates for \(e_{h}^{m+1}\) in \(l^\infty(L^2) \cap l^2(H^1)\).

**Theorem 7** Under hypotheses of Theorem 6 (R2) and (H), the following error estimate holds

\[
\|e_{h}^{m+1}\|_{l^\infty(L^2) \cap l^2(H^1)} \leq C (k + h^l).
\]
Note that, from (20) and (29), we also have
\[ \| e_{3,h}^{m+1} \|_{L^2(H, \partial \Omega)} \leq C (k + h^l). \]

**Proof.** The main idea is to make \(2 k \sum_{m=0}^{M-1} \langle (E_h^{m+1}, e_h^{m+1}) \rangle\).

In fact, making \(2 k \left( (E_h^{m+1}, e_h^{m+1}) \right)\), the pressure term vanish, and we arrive at
\[
\begin{align*}
|e_h^{m+1}|^2 - |e_h^{m+1} + e_h^{m+1} - e_h^{m+1}|^2 + 2 k ||e_h^{m+1}||^2 \\
= -2 k \left( \delta e_i^{m+1}, e_h^{m+1} \right) - 2 k c \left( E_i, u(t_{m+1}), e_h^{m+1} \right) - 2 k c \left( U_h, u(t_{m+1}), e_h^{m+1} \right)
\end{align*}
\]

\(= 2 k c \left( (U_h, e_h^{m+1/2}, e_h^{m+1}) \right) = 0\), but using that \(c \left( (U_h, e_h^{m+1/2}, e_h^{m+1}) \right) = 0\), this term \(I_4\) is decomposed as
\[ I_4 = -2 k c \left( (U_h, e_h^{m+1/2}, e_h^{m+1}) \right) = 0. \]

The vertical part of \(I_5\) is bounded as
\[ 2 k c \left( e_{3,h}^{m}, e_i^{m+1}, e_h^{m+1} \right) \leq C \| e_{3,h}^{m} \|_{L^\infty L^2} \| e_i^{m+1} \|_{H^1} \| e_h^{m+1} \|_{L^2 L^\infty} \leq C \| e_h^{m} \| h^{-1} |e_h^{m+1}|. \]

The term \(I_4 = c \left( (U_h, e_h^{m+1/2}, e_h^{m+1}) \right) \neq 0\), but using that \(c \left( (U_h, e_h^{m+1/2}, e_h^{m+1}) \right) = 0\), this term \(I_4\) is decomposed as
\[ I_4 = -2 k c \left( (U_h, e_h^{m+1/2}, e_h^{m+1}) \right) = J_1 + J_2. \]

The more complicate terms of \(J_1\) are the vertical parts:
\[ 2 k \left( (e_{3,h}^{m} + e_{3,i}^{m}) \left( e_h^{m+1/2} - e_h^{m+1}, \partial e_h^{m+1} \right) + 2 k \left( \partial e_i^{m+1/2} + e_{3,i}^{m} \right) \left( e_h^{m+1/2} - e_h^{m+1}, e_h^{m+1} \right) \right) = J_{1,1} + J_{1,2}. \]

Since \(J_1,2\) is easier to bound than \(J_{1,1}\), we only bound \(J_{1,1}\):
\[ J_{1,1} \leq 2 k \left( e_{3,h}^{m} + e_{3,i}^{m} \right) \left( e_h^{m+1/2} - e_h^{m+1}, \partial e_h^{m+1} \right) \leq C h^{-1} |e_h^{m+1/2} - e_h^{m+1}| \leq C h^{-1} |e_h^{m+1/2} - e_h^{m+1}|. \]

Here, we have used (21) and (23) and the inverse inequalities
\[ |e_h^{m+1/2} - e_h^{m+1}| \leq C h^{-1} |e_h^{m+1/2} - e_h^{m+1}| \text{ and } |e_h^{m}| \leq C h^{-1} |e_h^{m}|. \]

Moreover, \(J_2\) can be bounded as
\[ J_2 \leq C h^{-1} \|e_h^{m+1}\|^2 + C k |e_h^{m+1/2} - e_h^{m+1}|^2, \]

Finally, adding from \(m = 0\) to \(r\) (with any \(r < M\)) and taking into account the bound
\[ C k h^{-1} \sum_{m=0}^{M-1} |e_h^{m+1/2} - e_h^{m+1}|^2 \leq C h^{-2}(k + h^2) \leq C, \]

(where estimate (25) of Theorem 6 and (H) have been used) we can apply the discete Gronwall’s Lemma, obtaining the desired estimates.
4.6 \(O(\sqrt{k} + h^l)\) error estimates for \(\delta_t e_h^{m+1}\) in \(l^2(L^2)\) and for \(e_h^{m+1}\) in \(l^\infty(H^1)\)

We will use the following notations for the discrete derivative of errors

\[
\delta_t e_h^{m+1} := \frac{e_h^{m+1} - e_h^m}{k}, \quad \delta_t e_h^{m+1/2} := \frac{e_h^{m+1/2} - e_h^{m-1/2}}{k}.
\]

**Theorem 8** Assume hypotheses of Theorem 4 (R3), (H) and \(\|e_h^m\| \leq C h^l\). Then, the following error estimate holds

\[
\|e_h^{m+1}\|_{l^\infty(H^1)} + \|\delta_t e_h^{m+1}\|_{l^2(L^2)} \leq C (\sqrt{k} + h^l).
\]

**Proof.** Making \(2k(\langle E \rangle_h^{m+1}, \delta_t e_h^m)\) the pressure term vanish, and we arrive at

\[
\|e_h^{m+1}\|^2 - \|e_h^m\|^2 + \|e_h^{m+1} - e_h^m\|^2 + 2k \|\delta_t e_h^{m+1}\|^2 = -2k \left( \delta_t e_h^{m+1}, \delta_t e_h^{m+1} \right)
\]

\[
-2k c \left( E_h^m, u(t_{m+1}), \delta_t e_h^{m+1} \right) - 2k c \left( E_t^m, u(t_{m+1}), \delta_t e_h^{m+1} \right)
\]

\[
-2k c \left( U_h^m, e_h^{m+1/2}, \delta_t e_h^{m+1} \right) - 2k c \left( U_t, e_h^{m+1/2}, \delta_t e_h^{m+1} \right) + 2k \left( e^{m+1}, \delta_t e_h^{m+1} \right)
\]

\[
=: \sum_{i=1}^6 I_i
\]

We must bound the \(I_i\) terms:

\[
I_1 = -2k \left( e_i (\delta_t u(t_{m+1})), \delta_t e_h^{m+1} \right) \leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C h^l \int_{t_m}^{t_{m+1}} \|u_t\|_{L^2}^2 dt.
\]

Now, the term \(I_4 = -2k c \left( U_h^m, e_h^{m+1/2}, \delta_t e_h^{m+1} \right)\) does not vanish,

\[
I_4 = 2k c \left( E_h^m, e_h^{m+1/2}, \delta_t e_h^{m+1} \right) - 2k c \left( U(t_m), e_h^{m+1/2}, \delta_t e_h^{m+1} \right) := J_1 + J_2.
\]

The more complicate terms to bound are the vertical parts of \(J_1:\)

\[
2k \left( (e_{3,h}^m + e_{3,i}^m) \partial_z e_h^{m+1/2}, \delta_t e_h^{m+1} \right) + k \left( \partial_z (e_{3,h}^m + e_{3,i}^m) e_h^{m+1/2}, \delta_t e_h^{m+1} \right) := J_{1,1} + J_{1,2}.
\]

By using (21) and (23) and the inverse inequality \(\|\partial_z e_h^{m+1/2}\|_{L^2 L^\infty} \leq C h^{-1} \|e_h^{m+1/2}\|,\)

\[
J_{1,1} \leq 2k \left\| e_{3,h}^m + e_{3,i}^m \right\|_{L^\infty L^2} \left\| \partial_z e_h^{m+1/2} \right\|_{L^2 L^\infty} \left\| \delta_t e_h^{m+1} \right\|_{L^2 L^2}
\]

\[
\leq C k \left( \|e_h^m\| + h^l \|u(t_m)\|_{H^1} + h^l \|u_3(t_m)\|_{H^1} \right) h^{-1} \|e_h^{m+1/2}\| \|\delta_t e_h^{m+1}\|
\]

\[
\leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k h^{-2} \|e_h^{m+1/2}\|^2 \|e_h^m\|^2 + C k \|e_h^{m+1/2}\|^2.
\]

By using the inverse inequality \(\|e_h^{m+1/2}\|_{L^\infty} \leq C h^{-1/2} \|e_h^{m+1/2}\|,\)

\[
J_{1,2} \leq k \left\| \partial_z (e_{3,h}^m + e_{3,i}^m) \right\|_{L^2} \left\| e_h^{m+1/2} \right\|_{L^\infty} \left\| \delta_t e_h^{m+1} \right\|_{L^2} \leq k \left( \|e_h^m\| + h^l \right) h^{-1/2} \|e_h^{m+1/2}\| \|\delta_t e_h^{m+1}\|
\]

\[
\leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k h^{-1} \|e_h^{m+1/2}\|^2 \|e_h^m\|^2 + C k \|e_h^{m+1/2}\|^2.
\]

The \(J_2\)-term is bounded as

\[
J_2 \leq C k \|e_h^{m+1/2}\| \|\delta_t e_h^{m+1}\| \leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k \|e_h^{m+1/2}\|^2.
\]
On the other hand, the vertical part of $I_2$ and $I_3$ are bounded as

$$I_2 \leq C k \left( \|e_h^m\| + h^l \right) \|u(t_{m+1})\|_{H^3} \|\delta_t e_h^{m+1}\| \leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k \|e_h^m\|^2 + C k h^2,$$

$$I_3 \leq \|e_{3,i}^m\|_{H(\partial_\nu)} \|u(t_{m+1})\|_{H^3} \|\delta_t e_h^{m+1}\| \leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k h^2.$$

We write $I_5$ as

$$I_5 = 2 k c \left( e_i, e_i^{m+1}, \delta_t e_h^{m+1} \right) - 2 k c \left( U(t), e_i^{m+1}, \delta_t e_h^{m+1} \right)$$

and its vertical part as

$$2 k \left( (e_{3,h}^m + e_{3,i}^m) \delta_x e_i^{m+1}, \delta_t e_h^{m+1} \right) - k \left( \partial_x (e_{3,h}^m + e_{3,i}^m) e_i^{m+1}, \delta_t e_h^{m+1} \right) = K_1 + K_2$$

We bound both terms as

$$K_1 \leq 2 k \left( \|e_{3,h}^m + e_{3,i}^m\|_{L^\infty L^2} \|\partial_x e_i^{m+1}\|_{L^2 L^2} \|\delta_t e_h^{m+1}\|_{L^2 L^2} \right) \leq C k \left( \|e_h^m\| + \|e_i^m\| + h^l \right) \|u(t_{m+1})\|_{H^3} h^{-1} \|\delta_t e_h^{m+1}\| \leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k \|e_h^m\|^2 + C k h^2,$$

$$K_2 \leq k \|\partial_x (e_{3,h}^m + e_{3,i}^m)\|_{L^2} \|e_i^{m+1/2}\|_{L^6} \|\delta_t e_h^{m+1}\|_{L^3} \leq C k \left( \|e_h^m\| + \|e_i^m\| + h^l \right) \|u(t_{m+1})\|_{H^3} h^{-1/2} \|\delta_t e_h^{m+1}\| \leq \varepsilon k \|\delta_t e_h^{m+1}\|^2 + C k \|e_h^m\|^2 + C k h^2.$$

Finally, taking into account the above estimates and adding (30) from $m = 0$ to $r$ (with any $r < M$), since from estimates of Theorem (6) and (H),

$$k h^{-2} \sum_m \|e_h^{m+1/2}\|^2 \leq C h^{-2} (k + h^2) \leq C, \quad (l \geq 1),$$

we can apply the discrete Gronwall’s Lemma obtaining the desired estimates.

\[ \square \]

4.7 $O(\sqrt{k} + h^l)$ error estimates for $e_{p,h}^{m+1}$ in $l^2(L^2)$

Corollary 9 Assuming hypotheses of Theorem 8 one has

$$\|e_{p,h}^{m+1}\|_{L^2(L^2)} \leq C(\sqrt{k} + h^l).$$

The proof is rather standard, starting from estimates of previous Theorems and applying the hydrostatic Inf-Sup condition (H1).

It is important to remark that, up this moment, the order obtained is $O(\sqrt{k} + h^l) = O(h + h^l)$ under the constraint $k \leq h^2$, then this order is optimal for $O(h)$ approximation (i.e. $l = 1$). In the next Section, we study an argument to arrive at optimal order $O(k + h^l)$ for the case $l = 2$. 

23
4.8 An alternative way for $O(h^2)$ accuracy ($l = 2$)

4.8.1 $O(\sqrt{k} + h^2)$ error estimates for $\delta_i e_{h}^{m+1}$ and $\delta_i e_{h}^{m+1/2}$ in $l^\infty(L^2) \cap l^2(H^1)$.

Making $\delta_t(E_1)^{m+1}$ and $\delta_t(E_2)^{m+1}$ for each $m \geq 1$, one obtains $\forall \mathbf{v}_h \in X_h$:

$$(D_1)^{m+1}_{h} \begin{cases} 
\frac{1}{k} \left( \delta_t e_{h}^{m+1/2} - \delta_t e_{h}^{m}, \mathbf{v}_h \right) + \left( \nabla \delta_t e_{h}^{m+1/2}, \nabla \mathbf{v}_h \right) - \left( \delta_t p_s(t_{m+1}), \nabla x \cdot \langle \mathbf{v}_h \rangle \right)_S \\
= \left( \delta_t \mathcal{E}^{m+1}, \mathbf{v}_h \right) + \delta_t \text{NL}_{h}^{m+1}(\mathbf{v}_h) - \frac{1}{k} \left( \langle \delta_t e_{i}^{m+1} - \delta_t e_{i}^{m}, \mathbf{v}_h \rangle - \nabla \delta_t e_{i}^{m+1}, \nabla \mathbf{v}_h \right)
\end{cases}$$

where

$$\delta_t \text{NL}_{h}^{m+1}(\mathbf{v}_h) = -c \left( \delta_t \mathcal{E}^{m}, \mathbf{u}(t_{m+1}), \mathbf{v}_h \right) - c \left( \delta_t \mathcal{E}_{h}^{m+1/2}, \mathbf{v}_h \right)$$

and, for all $(\mathbf{v}_h, q_h) \in X_h \times Q_h$,

$$(D_2)^{m+1}_{h} \begin{cases} 
\frac{1}{k} \left( \delta_t e_{h}^{m+1} - \delta_t e_{h}^{m+1/2}, \mathbf{v}_h \right) + \left( \nabla (\delta_t e_{h}^{m+1} - \delta_t e_{h}^{m+1/2}), \nabla \mathbf{v}_h \right) - \left( \delta_t p_{s,h}^{m+1}, \nabla x \cdot \langle \mathbf{v}_h \rangle \right)_S \\
\left( \nabla x \cdot \langle \delta_t e_{h}^{m+1} \rangle, q_h \right)_S = 0
\end{cases}$$

Finally, adding $(D_1)^{m+1}_{h}$ and $(D_2)^{m+1}_{h}$ we obtain, for all $(\mathbf{v}_h, q_h) \in X_h \times Q_h$:

$$(D_3)^{m+1}_{h} \begin{cases} 
\frac{1}{k} \left( \delta_t e_{h}^{m+1} - \delta_t e_{h}^{m}, \mathbf{v}_h \right) + \left( \nabla \delta_t e_{h}^{m+1}, \nabla \mathbf{v}_h \right) + \left( \delta_t e_{h}^{m+1}, \nabla x \cdot \langle \mathbf{v}_h \rangle \right)_S \\
= \left( \delta_t \mathcal{E}^{m+1}, \mathbf{v}_h \right) + \delta_t \text{NL}_{h}^{m+1}(\mathbf{v}_h) - \frac{1}{k} \left( \langle \delta_t e_{i}^{m+1} - \delta_t e_{i}^{m}, \mathbf{v}_h \rangle - \nabla \delta_t e_{i}^{m+1}, \nabla \mathbf{v}_h \right)
\end{cases}$$

$$\left( \nabla x \cdot \langle \delta_t e_{h}^{m+1} \rangle, q_h \right)_S = 0.$$

**Theorem 10** Under hypotheses of Theorem 7 for $l = 2$ (i.e. $O(h^2)$ FE approximation), (R4) and assuming the following hypothesis for the first step of the scheme

$$|\delta_t e_{h}^{1}| \leq C (\sqrt{k} + h^2),$$

then there exists $k_1 > 0$ such that for any $k \leq k_1$, 

$$\|\delta_t e_{h}^{m+1} \|_{l^\infty(L^2) \cap l^2(H^1)} + \|\delta_t e_{h}^{m+1/2} \|_{l^\infty(L^2) \cap l^2(H^1)} \leq C (\sqrt{k} + h^2),$$

$$\|\delta_t e_{h}^{m+1} - \delta_t e_{h}^{m+1/2} \|_{l^2(L^2)} + \|\delta_t e_{h}^{m+1/2} - \delta_t e_{h}^{m+1/2} \|_{l^2(L^2)} \leq C \sqrt{k} (\sqrt{k} + h^2).$$

**Proof.** Since the initial estimate $|\delta_t e_{h}^{1}| \leq C (\sqrt{k} + h^2)$ has been assumed, it suffices to prove the generic estimate for $\delta_t e_{h}^{m+1}$ and $\delta_t e_{h}^{m+1/2}$, for each $m \geq 1$.

Taking $2k \delta_t e_{h}^{m+1/2} \in X_h$ as test function in $(D_1)^{m+1}_{h}$, one has

$$|\delta_t e_{h}^{m+1/2} - \delta_t e_{h}^{m+2} + |\delta_t e_{h}^{m+1/2} - \delta_t e_{h}^{m+1/2} + 2k \|\delta_t e_{h}^{m+1/2} \|^2$$

$$= 2k \left( \frac{1}{k} \langle \delta_t e_{h}^{m+1/2} - \delta_t e_{h}^{m+1/2}, \delta_t e_{h}^{m+1} \rangle - 2k \left( \nabla \delta_t e_{h}^{m+1/2}, \nabla \delta_t e_{h}^{m+1/2} \right) \\
+ 2k \left( \nabla \delta_t e_{h}^{m+1/2}, \nabla x \cdot \langle \delta_t e_{h}^{m+1/2} \rangle \right) \right) + 2k \left( \delta_t \mathcal{E}^{m+1}, \delta_t e_{h}^{m+1/2} \right) + 2k \delta_t \text{NL}_{h}^{m+1}(\delta_t e_{h}^{m+1/2})$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$
We bound the RHS of (31) as in Theorem 6 (recalling that now one has $O(h^2)$ approximation)

$$I_1 \leq \varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C h^4 \int_{t_m}^{t_{m+1}} \| u_t \|^2_{H^2}$$

$$I_2 \leq \varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C h^4 \int_{t_m}^{t_{m+1}} \| u_t \|^2_{H^3}$$

$$I_3 \leq \varepsilon k \| \delta_t e_h^{m+1/2} - \delta_t e_h^m \|^2 + C k^2 \| \delta_t p_h(t_{m+1}) \|^2_{H^1(S)} + \varepsilon k \| \delta_t e_h^m \|^2 + C k h^4 \| \delta_t p_h(t_{m+1}) \|^2_{H^2}$$

The bound of $I_1$ depending on the consistency error $\delta_t e^{m+1}$ is not problematic.

Now, we bound the more complicate terms of $I_5$, again as in the proof of Theorem 6

$$2 k c \left( \delta_t e_{3,i}^m, u(t_{m+1}), \delta_t e_h^{m+1/2} \right) \leq \varepsilon k \left( \| \delta_t e_h^m \|^2 + \| \delta_t e_i^m \|^2 \right) + \varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C k \| \delta_t e_h^{m+1/2} \|^2$$

$$\leq \varepsilon k \| \delta_t e_h^m \|^2 + C h^4 \int_{t_m}^{t_{m+1}} \| u_t \|^2_{H^3} + \varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C k \| \delta_t e_h^{m+1/2} \|^2,$$

$$2 k c \left( \delta_t e_{3,i}^m, u(t_{m+1}), \delta_t e_h^{m+1/2} \right) \leq \varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C k \| \delta_t e_{3,i}^m \|^2_{H(\partial_\gamma)}$$

$$\leq \varepsilon k \| \delta_t e_h^m \|^2 + C h^4 \int_{t_m}^{t_{m+1}} \| \delta_t u_3 \|^2_{H^3} + C k \| \delta_t e_h^{m+1/2} \|^2.$$

The vertical part of $2 k c \left( e^{m-1}, \delta_t u(t_{m+1}), \delta_t e_h^{m+1/2} \right)$ is decomposed as follows:

$$2 k \left( e_{3,3}^{m-1}, e_{3,i}^{m-1}, \delta_z \delta_t u(t_{m+1}), \delta_t e_h^{m+1} \right) + k \left( \delta_z(e_{3,3}^{m-1} + e_{3,i}^{m-1}), \delta_t u(t_{m+1}), \delta_t e_h^{m+1} \right) := L_1 + L_2$$

Since $L_2$ is easier to bound than $L_1$, we only bound $L_1$:

$$L_1 \leq \| e_{3,h}^{m-1} + e_{3,i}^{m-1} \|_{L_t^\infty L_x^2} \| \partial_z \delta_t u(t_{m+1}) \|_{L_t^2 L_x^4} \| \delta_t e_h^{m+1/2} \|_{L_t^4 L_x^4}$$

$$\leq \varepsilon k \| e_h^m \|^2 + C k h^{2l} + \varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C k \| \delta_t e_h^{m+1/2} \|^2.$$

On the other hand, we bound the other terms of $I_5$ which have not similar terms in the proof of Theorem 6

$$c \left( \delta_t U_h^m, e_h^{m+1/2}, \delta_t e_h^{m+1/2} \right) = -c \left( \delta_t e_h^m, e_h^{m+1/2}, \delta_t e_h^{m+1/2} \right) + c \left( \delta_t U(t_m), e_h^{m+1/2}, \delta_t e_h^{m+1/2} \right).$$

The second term of the RHS is bounded by $\varepsilon k \| \delta_t e_h^{m+1/2} \|^2 + C k \| e_h^{m+1/2} \|^2$. With respect to the first term on the RHS, the more complicate term to bound is the vertical part:

$$-2 k c \left( \delta_t e_{3,i}^{m}, e_h^{m+1/2}, \delta_t e_h^{m+1/2} \right)$$

$$= -2 k \left( \delta_t e_{3,i}^{m}, e_h^{m+1/2}, \delta_z \delta_t e_h^{m+1/2} \right) + k \left( \partial_z(\delta_t e_{3,i}^{m}, e_h^{m+1/2}, \delta_z e_h^{m+1/2}) := J_1 + J_2 \right.$$
Since $J_2$ is easier to bound than $J_1$, we only bound $J_1$ (by using the inverse inequalities $\|v_h\|_{H_x^2} \leq C h^{-1} \|v_h\|_{L_x^2}$ and $\|v_h\|_{L_x^\infty} \leq C h^{-1} \|v_h\|_{L_x^2}$):

\[
J_1 \leq 2 k \|\delta_t e_h^{m+1/2} \|_{L_x^\infty} \|e_h^{m+1/2} \|_{L_x^2} \mathcal{E} + \mathcal{K}_1 + \mathcal{K}_2
\]

Notice that to apply the discrete Gronwall’s Lemma with the term $C k \frac{1}{r^2} \|e_h^{m/2} \|_{L_x^2} \|\delta_t e_h^{m/2} \|_{L_x^2}$, it is necessary that $k \frac{1}{m \sum_m} \|e_h^{m/2} \|_{L_x^2} \leq C$ and this is true for $l = 2$. In Section 6 below, we will see a modified scheme where it is possible to obtain optimal error estimates $O(k + h^l)$ for $l = 1$. On the other hand, we have to bound

\[
c\left(U_h^{m-1} - \delta_t e_h^{m+1/2} + \delta_t e_h^{m+1/2}\right) = c\left(U_h^{m-1} + \delta_t e_h^{m+1} + \delta_t e_h^{m+1/2}\right)
\]

(here we have used that $c\left(U_h^{m-1}, \delta_t e_h^{m+1}, \delta_t e_h^{m+1/2}\right) = 0$). The more complicate term is the vertical part:

\[
2 k c\left(e_h^{m-1} + \delta_t e_h^{m+1}, \delta_t e_h^{m+1}\right) = 2 k \left(e_h^{m-1} + \delta_t e_h^{m+1}, \delta_t e_h^{m+1/2}\right) - k \left(\partial_z (e_h^{m-1} + \delta_t e_h^{m+1}, \delta_t e_h^{m+1/2}\right) := K_1 + K_2
\]

Since $K_2$ is easier to bound than $K_1$, we only bound $K_1$:

\[
K_1 \leq 2 k \|e_h^{m-1} + \delta_t e_h^{m+1} \|_{L_x^\infty} \mathcal{E} + \mathcal{K}_1
\]

On the other hand, making $2 k \left((D_2) e_h^{m+1} \right)$, we arrive at

\[
|\delta_t e_h^{m+1/2} |^2 - |\delta_t e_h^{m+1/2} |^2 + |\delta_t e_h^{m+1} - \delta_t e_h^{m+1/2} |^2 + k \left(\|\delta_t e_h^{m+1/2} |^2 - |\delta_t e_h^{m+1/2} |^2 + |\delta_t e_h^{m+1} - \delta_t e_h^{m+1/2} |^2 \right) = 0.
\]

Reasoning as in Theorem 6 adding (31) and (32) from $m = 0$ to $r$ (with any $r < M$), taking into account the previous estimates and choising $\varepsilon$ and $k$ small enough, we can apply the discrete Gronwall’s Lemma obtaining the desired estimates.
4.8.2 $O(k + h^2)$ error estimates for $\delta t e_h^{m+1}$ in $l^\infty(L^2) \cap l^2(H^1)$

**Theorem 11** Under hypotheses of Theorem 10 and (R5), assuming the following hypothesis for the first step of the scheme

$$|\delta t e_h^{m+1}| \leq C (k + h^2),$$

then

$$\|\delta t e_h^{m+1}\|_{l^\infty(L^2) \cap l^2(H^1)} \leq C (k + h^2).$$

**Proof.** The main idea is to make $2k \left( (D_{\delta t})_{h}^{m+1}, \delta t e_h^{m+1} \right)$, now, the pressure term vanish but the term $c \left( U^{m-1}, \delta t e_h^{m+1/2}, \delta t e_h^{m+1} \right)$, and can be decomposed as:

$$2k \left( U^{m-1}, \delta t e_h^{m+1/2}, \delta t e_h^{m+1} \right) = 2k \left( U^{m-1}, \delta t e_h^{m+1/2} - \delta t e_h^{m+1}, \delta t e_h^{m+1} \right) + 2k \left( U(t_{m-1}), \delta t e_h^{m+1/2} - \delta t e_h^{m+1}, \delta t e_h^{m+1} \right) := I_1 + I_2.$$

The more complicate term in $I_1$ is the vertical part:

$$k \left( (e_{3,i}^{m-1} + e_{3,i}^{m-1}) (\delta t e_h^{m+1/2} - \delta t e_h^{m+1}), \delta t e_h^{m+1} \right)$$

$$\leq k \|e_{3,i}^{m-1} + e_{3,i}^{m-1}\|_{L^\infty L^2} \|\delta t e_h^{m+1/2} - \delta t e_h^{m+1}\|_{L^2 L^\infty} \|\delta t e_h^{m+1}\|_{L^2 L^\infty}$$

$$\leq k \|e_{3,i}^{m-1}\|_{L^\infty L^2} \|\delta t e_h^{m+1/2} - \delta t e_h^{m+1}\|_{L^2 L^\infty} \|\delta t e_h^{m+1}\|_{L^2 L^\infty}$$

$$+ C k \frac{\|e_{3,i}^{m-1}\|_{L^\infty L^2}}{h} \|\delta t e_h^{m+1/2} - \delta t e_h^{m+1}\|_{L^2 L^\infty} \|\delta t e_h^{m+1}\|_{L^2 L^\infty}$$

$$\leq \varepsilon k \|\delta t e_h^{m+1}\|_{L^2 L^\infty}^2 + C k \frac{\|e_{3,i}^{m-1}\|_{L^\infty L^2}}{h} \|\delta t e_h^{m+1/2} - \delta t e_h^{m+1}\|_{L^2 L^\infty}^2 |e_{h}^{m-1}|^2$$

$$+ C k h^2 \|\delta t e_h^{m+1/2} - \delta t e_h^{m+1}\|_{L^2 L^\infty}^2$$

(here, we have used the inverse inequalities $\|v\|_{L^\infty} \leq C h^{-1} |v|_{L^2}$ and $\|v\| \leq C h^{-1} |v|$ and the estimates (21) and (22)). Since $|e_{h}^{m-1}|^2 \leq C (k^2 + h^4)$, adding the third term of the RHS of the previous inequality,

$$C k h^{-4} \sum_{m} |\delta t e_h^{m+1/2} - \delta t e_h^{m+1}|^2 |e_{h}^{m-1}|^2 \leq C k h^{-4}(k + h^4)(k^2 + h^4) \leq C (k^2 + h^4),$$

where (H) and estimates of Theorem 10 have been used.

The more complicate term in $I_2$ is the vertical part:

$$2k \left( u_3(t_{m-1}), \delta t e_h^{m+1/2} - \delta t e_h^{m+1}, \delta t e_h^{m+1} \right) \leq \varepsilon k \|\delta t e_h^{m+1}\|_{L^2 L^\infty}^2 + C k \|\delta t e_h^{m+1/2} - \delta t e_h^{m+1}\|_{L^2 L^\infty}^2.$$

Adding from $m = 0$ to $r$ (with any $r < M$), we can apply the discrete Gronwall’s Lemma obtaining the desired estimates. ■

Again, from estimates of previous Theorems and applying the hydrostatic Inf-Sup condition (H1), we arrive at the following optimal error estimate for the pressure.

**Corollary 12** Assuming hypotheses of Theorem 11, one has

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \leq C (k + h^2).$$

Notice that the previous estimate for the pressure is not obtained in the $l^\infty(L^2)$ norm, due to the convective term depending on the intermediate error $e_{h}^{m+1/2}$ which has not optimal approximation in $l^\infty(L^2)$, only in $l^2(L^2)$.
5 A modified scheme with integral computation for the vertical velocity.

In this section, we will approximate the problem (Q). We consider the variational formulation of (Q) satisfied for the exact solution \((u, p)\) at \(t = t_{m+1}\):

\[
(Q)_{m+1}^{t+1} \quad \begin{cases}
\left( \frac{1}{k}(u(t_{m+1}) - u(t_m)), v \right) + c \left(U(t_m), u(t_{m+1}), v \right) + \left( \nabla u(t_{m+1}), \nabla v \right) \\
- \left(p_s(t_{m+1}), \nabla_x \cdot \langle v \rangle \right)_S = \langle f(t_{m+1}), v \rangle + \langle g_s(t_{m+1}), v \rangle_{\Gamma_s} + \left( c_{m+1}, v \right)
\end{cases}
\]

for each \((v, q) \in \left(W^{1,3}_{b,l} \cap L^\infty \right) \times L^2_0(S)\), where

\[
u_3(t_m; x, z) = \int_z^0 \nabla_x \cdot u(t_m; x, s) \, ds.
\]

5.1 \(O(k + h^{l+1})\) for \(e_h^{m+1}\) in \(L^2(\Omega)\)

We change in the scheme the computation of the vertical velocity \(u_{3,h}^m\) in \((S_0)_h^{m+1}\), replacing Sub-step 0 by the vertical integral computation

\[
u_{3,h}^m(x, z) = \int_z^0 \nabla_x \cdot u_h^m(x, s) \, ds.
\]

Accordingly, we will change the vertical interpolation operator as follows

\[
\overline{K}_h \nu_3(x, z) = \int_z^0 \nabla_x \cdot I_h u(x, s) \, ds,
\]

hence \(e_h^m = \int_0^z \nabla_x \cdot e_h^m\). This interpolation operator \(\overline{K}_h\) conserves the properties already used for the interpolation \(K_h\) and consequently, properties (20)-(23) can be used in this context.

**Theorem 13** Assuming hypotheses of Theorem 7 (R3) and \(\|A^{-1}_h e_h^0\| \leq C h^{l+1}\) (recall that \(A_h\) is the discrete hydrostatic Stokes operator used in (15), then there exists \(k_0 > 0\) such that for any \(k \leq k_0\), the following error estimates hold

\[
\|e_h^{m+1}\|_{L^2(\Omega)} \leq C(k + h^{l+1}).
\]

**Proof.** Taking \(v_h = A^{-1}_h e_h^{m+1} \in V_h \in (E)^{m+1}_h\), we obtain

\[
\|A^{-1}_h e_h^{m+1}\|^2 - \|A^{-1}_h e_h^m\|^2 + \|A^{-1}_h e_h^{m+1} - A^{-1}_h e_h^m\|^2 + 2 k \|e_h^{m+1}\|^2 \\
\leq 2 k \|c e_i^{m+1} - A^{-1}_h e_h^{m+1}\| + 2 k \left( c_{m+1}, e_h^{m+1} \right) \cdot A^{-1}_h e_h^{m+1} := I_1 + I_2 + I_3.
\]

We bound the more complicate terms of \(I_1\):

\[
2 k c \left( u_{3,h}^m, e_h^{m+1/2} \right) + 2 k c \left( u_3(t_m), e_h^{m+1/2} \right) := L_1 + L_2
\]
By using the inverse inequalities $\|e_h\|_{L^3} \leq h^{-1/2}|e_h|$ in 3D-domains and $\|e_h\|_{L^\infty} \leq h^{-1}\|e_h\|_{L^2}$ in 2D-domains:

$$L_1 = k \left( \partial_z e_{3,h}^m e_h^{m+1/2}, A_h^{-1} e_h^{m+1} \right) - 2 k \left( e_{3,h}^m e_h^{m+1/2}, \partial_z A_h^{-1} e_h^{m+1} \right)$$

$$\leq k \left( \|\partial_z e_{3,h}^m \|_{L^2} \|e_h^{m+1/2}\|_{L^3} \|A_h^{-1} e_h^{m+1}\|_{L^6} + 2 \|e_{3,h}^m \|_{L^3} \|L^\infty \|_{L^2} \|\partial_z A_h^{-1} e_h^{m+1}\|_{L^2} \|L^\infty \|_{L^2} \right)$$

$$\leq C k \left( \|e_h^m\| + \|e_i^m\| \right) \left( h^{-1/2} + h^{-1} \right) \left( |e_h^{m+1/2} - e_h^{m+1}| + |e_h^{m+1}| \right) \|A_h^{-1} e_h^{m+1}\|$$

$$\leq \varepsilon k |e_h^{m+1}|^2 + C k \|A_h^{-1} e_h^{m+1}\|^2 + C \left( k^2 \|e_h^m\|^2 + h^2 \right) \|A_h^{-1} e_h^{m+1}\|^2,$$

$$L_2 \leq C k \left( \|u_3(t_m)\|_{L^\infty} + \|\partial_z u_3(t_m)\|_{L^3} \right) \left( |e_h^{m+1/2} - e_h^{m+1}| + |e_h^{m+1}| \right) \|A_h^{-1} e_h^{m+1}\|$$

$$\leq \varepsilon k \left( |e_h^{m+1}|^2 + |e_h^{m+1/2} - e_h^{m+1}|^2 \right) + C k \|A_h^{-1} e_h^{m+1}\|^2.$$
In a similar way, by using that $\partial_x e^{m}_{3,h} = -\nabla_x \cdot \mathbf{e}^m_h$,

$$J_2 \leq k |e^m_h| (\|\nabla_x u(t_{m+1})\|_{L^3} \|A^{-1}_h e^{m+1}_h\|_{L^6} + \|u(t_{m+1})\|_{L^\infty} \|\nabla_x A^{-1}_h e^{m+1}_h\|)$$

$$\leq \varepsilon k |e^m_h|^2 + C k \|A^{-1}_h e^{m+1}_h\|^2.$$

On the other hand, we bound the $I_2$-term as follows

$$I_2 = -2k (\delta t e^{m+1}_h \cdot A^{-1}_h e^{m+1}_h) \leq C k \|A^{-1}_h e^{m+1}_h\|^2 + \varepsilon k |e^m_h| (\delta t u^{m+1})^2$$

$$\leq C k \|A^{-1}_h e^{m+1}_h\|^2 + \varepsilon k h^{2(l+1)} \|\delta t u^{m+1}\|_{H^{l+1}}^2$$

$$\leq C k \|A^{-1}_h e^{m+1}_h\|^2 + \varepsilon k h^{2(l+1)} \int_{t_m}^{t_{m+1}} \|u_t\|_{H^{l+1}}^2.$$

Finally, the $I_3$-term is easy to bound by using that $u_{tt} \in L^2(L^2)$ given by hypothesis (R3).

Adding from $m = 0$ to $r$ (for any $r < M$), since one has $\|e^m_h\|^2 \leq C h^2$ and $k \sum_{m=0}^{r} |e^{m+1}_h| - e^m_h|^2 \leq C (k + h^2)$, we can apply the generalized discrete Gronwall’s Lemma, obtaining the desired estimates for $k$ small enough.

5.2 $O(k + h^l)$ for $e^{m+1}_{p,h}$ in $L^2(L^2)^n$

Owing to the improved error estimate obtained in the above Subsection, now we can prove the same error estimates obtained in Theorems 10 and 11 also for $l = 1$, that is, using $O(h)$ FE-approximation.

Indeed, in the proof of Theorem 10, we can apply the discrete Gronwall’s Lemma, since the term $C k h^{-4} \sum_{m=0}^{r} |e^m_h|^2 \|\delta t e^m_h\|^2$ appear, and now for $l = 1$, owing to (33) and (H), we have

$$C k h^{-4} \sum_{m=0}^{r} |e^m_h|^2 \leq C.$$

The rest of the proof is similar, arriving at the following result:

**Corollary 14** Assuming hypotheses of Theorem 11 and Theorem 13, one has

$$\|e^{m+1}_{p,h}\|_{L^2} \leq C (k + h^l).$$

6 Approximation of the Coriolis term

Looking at the results obtained in previous Sections, we consider that the more convenient forms to introduce the Coriolis term in the scheme can be the following (the Coriolis term will be always refereed at the end-of-step velocity, either $u^{m+1}_h$ or $u^m_h$, because it is the better approximation in the scheme):

1. Considering in $(S_1)^{m+1}_h$ the explicit term $b(u^m_h)$. In the proof of Lemma 2 this term introduces the extra-term

$$\left(b(u^m_h), u^{m+1/2}_h - u^m_h\right) = \left(b(u^m_h), u^{m+1/2}_h - u^m_h\right)$$
which produces an artificial exponential bound in time in the stability estimates. Indeed, bounding as
\[
\left( b(u_h^{m_k}), u_h^{m_k+1/2} - u_h^m \right) \leq C |u_h^m|^2 + \varepsilon |u_h^{m_k+1/2} - u_h^m|^2
\]
and applying the discrete Gronwall inequality, a new exponential bound appears. With respect to the error estimates, some new terms in \((E_1)_h^{m+1}\) and \((E_2)_h^{m+1}\) appear, although these terms do not add new difficulties.

2. Considering the following Coriolis correction strategy: to introduce in \((S_1)_h^{m+1}\) the explicit term \(b(u_h^m)\) and in \((S_2)_h^{m+1}\) the correction term \(b(u_h^{m+1} - u_h^m)\). This correction scheme works in a similar way to scheme given in point 1 with respect to the stability and error estimates.

Notice that in the two previous cases, the computation of the two components of velocity \(u_h^{m+1/2}\) is decoupled. Finally, the implementation of Hydrostatic Stokes step (Sub-step 2) is simpler in the case 1.

7 Conclusions

In this paper we have developed two new ways of handling stable and convergent approximations for the Primitive Equations, based on the reformulations (Q) and (R) of the problem, being (Q) an integral-differential formulation and (R) a fully differential one.

In both cases, vertically structured meshes are needed, and the error estimates are deduced under the same constraint \(k \leq h^2\).

With respect to computational implementation, scheme based on (R) is simpler than scheme related to (Q), although the latter satisfies more analytical error estimates. In fact, (Q)-scheme satisfies optimal accuracy \(O(k + h^{l+1})\) in the \(L^2(\Omega)\)-norm for the velocity and \(O(k + h^l)\) in the \(H^1(\Omega) \times L^2(\Omega)\)-norm for the velocity and pressure, whereas for the (R)-scheme this optimal accuracy can be proved only in the \(H^1(\Omega) \times L^2(\Omega)\)-norm for the velocity and pressure when \(l = 2\), that is, using \(O(h^2)\) FE-approximation.

Nevertheless, the constraint \(k \leq h^2\), imposed in both schemes, is compatible with the accuracy \(O(k + h^2)\) which is satisfied in the \(H^1 \times L^2\)-norm for velocity and pressure when \(l = 2\) and only in the \(L^2\)-norm for the velocity when \(l = 1\). Therefore, by using \(O(h^2)\) FE-approximation \((l = 2)\) is more convenient the (R)-scheme because is simpler to implement.

Appendix

**Proof** [of Lemma 5]: Let \(A^{-1}v \in V\) be the solution of the hydrostatic Stokes Problem with second member \(v\). This solution verifies (see [16] for the Stokes case)

\[
\|A^{-1}v_h - A_h^{-1}v_h\| \leq Ch|v_h|. \tag{34}
\]

On the other hand,

\[
\|A_h^{-1}v_h\|_{W^{1,6}} \leq \|A_h^{-1}v_h - \bar{I}_h A^{-1}v_h\|_{W^{1,6}} + \|\bar{I}_h A^{-1}v_h\|_{W^{1,6}}
\]

where \(\bar{I}_h\) is an interpolator with respect to \(X_h\).

We bound the RHS as follows, using hypothesis (H0), the stability property \(\|\bar{I}_h v_h\|_{W^{1,6}} \leq C\|v_h\|_{W^{1,6}}\) and the inverse inequality \(\|v_h\|_{W^{1,6}} \leq C h^{-1}\|v_h\|\):

\[
\|\bar{I}_h A^{-1}v_h\|_{W^{1,6}} \leq C \|A^{-1}v_h\|_{W^{1,6}} \leq C \|A^{-1}v_h\|_{H^2} \leq C |v_h|,
\]

\[\\]

31
\[ \|A_h^{-1}v_h - \tilde{I}_h A^{-1}v_h\|_{W^{1,6}} \leq \frac{C}{h} \left( \|A_h^{-1}v_h - \tilde{I}_h A^{-1}v_h\| + \|A^{-1}v_h - \tilde{I}_h A^{-1}v_h\| \right). \]

Finally, applying (34) and the error interpolation inequality
\[ \|A^{-1}v_h - \tilde{I}_h A^{-1}v_h\| \leq C h \|A^{-1}v_h\|_{H^2} \leq C h |v_h|, \]
we arrive at \( \|A_h^{-1}v_h - \tilde{I}_h A^{-1}v_h\|_{W^{1,6}} \leq C |v_h| \). Therefore, we conclude
\[ \|A_h^{-1}v_h\|_{W^{1,6}} \leq C |v_h|. \]

References

[1] P. Azérad, F. Guillén. Équations de Navier-Stokes en bassin peu profond: l’approximation hydrostatique. C. R. Acad. Sci. Paris, Série I 329 (1999), 961-966.

[2] P. Azérad, F. Guillén. Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. Siam J. Math. Anal., 33 (4) (2001), 847-859.

[3] R. Bermejo. Velocity Error Estimates for a Semi-Lagrangian Ocean General Circulation Model. Actas de las II Jornadas de Análisis de Variables y Simulación Numérica del Intercambio de Masas de Agua a través del Estrecho de Gibraltar, Cádiz, (2000), 19-34.

[4] R. Bermejo, P. Galán del Sastre. Long-Term Behavior of the Wind Stress Circulation of a Numerical North Atlantic Ocean Circulation Model. European Congress on Computational Methods in Applied Sciences and Engineering, ECCOMAS (2004), 1-21.

[5] O. Besson, M. R. Laydi. Some Estimates for the Anisotropic Navier-Stokes Equations and for the Hydrostatic Approximation. M2AN-Mod. Math. Ana. Num., 7 (1992) 855-865.

[6] J. Blasco, R Codina, A. Huerta. A fractional-step method for the incompressible Navier-Stokes equations related to a predictor-multicorrector algorithm. Internat. J. Numer. Methods Fluids, 28 (10) (1998), 1391–1419.

[7] C. Cao, E.S. Titi. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. Annals of Mathematics, 166(1) (2007), 245-267.

[8] V. Casulli, R.T. Cheng. Semi-implicit finite difference methods for three-dimensional sahllow water flow. Internat. J. Numer. Methods Fluids, 15, (1992) 629-648.

[9] V. Casulli, E. Cattani. Stability, Accuracy and Efficiency of a semi-implicit method for tree-dimensional shallow water flow. Comp. Math. Applic., 27, 4 (1994) 99-112.

[10] V. Casulli, R.A.. Walters. A robust finite element model for hydrostatic surface water flows. Communications in Numerical Methods in Engineering, 14, (1998) 931-940.

[11] B.Cushman-Roisin, J.M. Beckers. Introduction to Geophysical Fluid Dynamics - Physical and Numerical Aspects. Academic Press, 2009.

[12] T. Chacón, F. Guillén. An intrinsic analysis of existence of solutions for the hydrostatic approximation of Navier-Stokes equations. C. R. Acad. Sci. Paris, Série I 329 (2000), 841-846.

[13] T. Chacón; D. Rodríguez-Gómez. A stabilized space-time discretization for the primitive equations in oceanography. Numer. Math., 98 (3) (2004), 427-475.

[14] T. Chacón; D. Rodríguez-Gómez. A numerical solver for the primitive equations of the ocean using term-by-term stabilization. Appl. Numer. Math., 55 (1) (2005), 1-31.
[15] T. Chacón Rebollo, M. Gómez Mármol, I. Sánchez Muñoz. Numerical solution of the Primitive Equations of the ocean by the Orthogonal Sub-Scales VMS method. Appl. Numer. Math. 62 (2012) 342-359.

[16] V. Girault, P.A. Raviart. Finite Element Methods for Navier-Stokes Equations. Springer-Verlag, 1986.

[17] R. Glowinski, Numerical Methods for Fluids (Part 3), Handbook of Numerical Analysis, vol. IX, P. G. Ciarlet and J. L. Lions, eds., North-Holland, 2003.

[18] F. Guillén-González, N. Masmoudi, M.A. Rodríguez-Bellido. Anisotropic Estimates and strong solutions of the Primitive Equations. J. Diff. Integral Equations, 14 (11) (2001), 1381-1408.

[19] F. Guillén-González, M. V. Redondo-Neble, J. R. Rodríguez-Galván Análisis Numérico y resolución efectiva de las Ecuaciones Primitivas con esquemas de tipo proyección. Actas del XVII CEDYA/ VII CMA, Universidad de Salamanca (2001).

[20] F. Guillén-González, M.A. Rodríguez-Bellido. On the strong solutions of the Primitive Equations in 2D domains. Nonlinear Analysis: Serie A, Theory and Methods, 50 (5) (2002), 621-646.

[21] F. Guillén-González, D. Rodríguez-Gómez. Bubble finite elements for the primitive equations of the ocean. Num. Math., 101 (4) (2005), 689-728.

[22] F. Guillén-González, M.V. Redondo-Neble. Sharp error estimates for a fractional-step method applied to the 3D Navier-Stokes equations. C. R. Acad. Sci. Paris, Ser. I 345 (2007), 359-362.

[23] F. Guillén-González, J.V. Gutiérrez-Santacreu. Conditional stability and convergence of a fully discrete scheme for 3D viscous fluids models with mass diffusion. SIAM J. Num. Anal., 46 (5) (2008), 2276-2308.

[24] F. Guillén-González, M.V. Redondo-Neble. New error estimates for a viscosity-splitting scheme in time for the 3D Navier-Stokes equations.IMA J. Numer. Anal. (2011) 31 (2), 556-579.

[25] F. Guillén-González, M.V. Redondo-Neble. Spatial error estimates for a finite element viscosity-splitting scheme for the Navier-Stokes equations. Int. J. Numer. Anal. Mod. 10 (4) (2013), 826-844.

[26] J.G. Heywood, R. Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second order time discretization. SIAM J. Numer. Anal., 27 (1990), 353-384.

[27] I. Kukavica, M. Ziane. On the regularity of the primitive equations of the ocean. Nonlinearity, 20 (2007), 2739-2753.

[28] R. Lewandowski. Analyse Mathématique et Océanographie. Masson (1997).

[29] J.L. Lions, R. Teman, S. Wang. New formulations of the primitives equations of the atmosphere and applications. Nonlinearity, 5 (1992), 237-288.

[30] J.L. Lions, R. Teman, S. Wang. On the equations of the large scale Ocean. Nonlinearity, 5 (1992), 1007-1053.

[31] F. Ortegón Gallego. On distributions independent of $x_N$ in certain non-cylindrical domains and a de Rham lemma with a non-local constraint. Nonlinear Analysis, 59 (2004), 335-345.

[32] J. Pedlosky. Geophysical fluid dynamics. Springer-Verlag (1987).

[33] J. Simon. Compact sets in $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146 (1987), 65–97.

[34] R. Temam, Navier-Stokes equations. Theory and Numerical Analysis, North-Holland, 1984.

[35] M. Ziane. Regularity Results for Stokes Type Systems. Applicable Analysis, 58 (1995), 263-292.