On decoherence of cosmological perturbations and stochastic inflation

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Abstract. By making a suitable generalization of the Starobinsky stochastic inflation, we propose a classical phase space formulation of stochastic inflation which may be used for a quantitative study of decoherence of cosmological perturbations during inflation. The precise knowledge of how much cosmological perturbations have decohered is essential to the understanding of acoustic oscillations of cosmological microwave background (CMB) photons. In order to show how the method works, we provide the relevant equations for a self-interacting inflaton field. For pedagogical reasons and to provide a link to the field theoretical case, we consider the quantum stochastic harmonic oscillator.

Keywords: inflation, physics of the early universe, quantum cosmology

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1 Introduction

Decoherence [1–3] describes how a quantum system evolves into a state which most closely resembles a classical state through an interaction with the environment. As information about the system is lost to the environment as perceived by some observer, the entropy of the system increases. Recently quantitative studies have been done on decoherence for interacting quantum field theories, see Refs. [4–10]. These studies rely on the fact that neglecting observationally inaccessible, non-Gaussian correlators will give rise to an increase in Gaussian entropy of the state, which is then taken as a quantitative measure of decoherence. A useful application of this approach is to quantitatively study the decoherence of cosmological perturbations, i.e. cosmological decoherence. During inflation quantum fluctuations of the inflaton field – and thus scalar cosmological perturbations – generically decohere because of the interactions generated by the inflationary potential as well as because of the interactions between cosmological perturbations and other fields present during inflation, and as a result become more and more classical [11].

The ultimate goal of this project is to find out what the effect of cosmological decoherence on the cosmological microwave background (CMB) spectrum is for specific inflationary models, and thus to make a connection between inflationary models and late time cosmological observables.

In this paper we focus on cosmological decoherence in the framework of stochastic inflation [13–15]. In stochastic inflation the system is represented by the super-Hubble (IR) modes of the inflaton field, as these modes are amplified during inflation and they are the ones that ultimately induce temperature fluctuations in the CMB. The sub-Hubble (UV) modes form the environment and act as a (white) noise for the long-wavelength modes.

In Ref. [12] it is stated that even in the absence of interactions the quantum fluctuations become indistinguishable from classical fluctuations during inflation. The authors call this "decoherence without decoherence". Technically, the state becomes extremely squeezed, but the phase space area remains constant (pure state). Our notion of decoherence differs, in that we associate decoherence with growth of the phase space area and consequently the entropy of the state.
Central in the stochastic inflation scenario is the Starobinsky equation
\[
\dot{\phi}(t, \mathbf{x}) = -\frac{1}{3H} V'(\phi(t, \mathbf{x})) + F(t, \mathbf{x}).
\]

(1.1)
The field $\phi$ is the coarse-grained part of the complete field, i.e. it contains only the long wavelength modes with $k \ll H a$. $H = \dot{a}/a$ ($a$ is the scale factor) is the Hubble parameter in inflation and $V(\phi)$ is a potential, e.g. $V(\phi) = \lambda \phi^4/4!$ for a self-interacting scalar field in a chaotic inflationary scenario. The noise term $F$ originates from the short wavelength modes which keep crossing the Hubble radius due to the accelerated cosmic expansion. Interactions between UV modes are neglected, such that these modes are uncorrelated over time, thus the noise is white,
\[
\langle F(t, \mathbf{x}) F(t', \mathbf{x}) \rangle = \frac{H^3}{4\pi^2} \delta(t-t'),
\]

(1.2)
where $\langle \cdot \rangle$ denotes a (statistical) average. Although the coarse grained field $\phi$ is a quantum field with creation and annihilation operators, in inflation it nevertheless commutes with its first time derivative ($\propto$ the canonical momentum). In this sense the field $\phi$ may be called a classical stochastic field [15]. The Starobinsky equation is therefore a classical Langevin equation for the stochastic field $\phi$, in contrast to the quantum field equation for the complete quantum scalar field. From Eq. (1.1) a Fokker-Planck equation for the probability distribution of $\phi$ can be derived, which is then used to calculate correlators for $\phi$.

Perhaps the greatest success of stochastic inflation is that it recovers the leading infrared logarithms (i.e. $\ln^n(a)$ terms) of the field correlators at each order in perturbation theory [13, 15–19]. Originally it was believed that the time dependent separation between short and long wavelength modes would also lead to cosmological decoherence, even in a free field theory (see e.g. Refs. [20, 21]). However, Habib [22] showed that the free stochastic equations do not lead to any decoherence.

Although the Starobinsky equation (1.1) is excellent for describing the leading order behavior of the coincident field correlators, it fails when one is interested in the phase space of a state. Technically the Starobinsky equation cannot be derived from an action, thus no canonical momentum and Hamiltonian can be constructed. Moreover from the Starobinsky equation (1.1) no off-coincident two-point functions can be derived. This is especially troublesome for our approach to decoherence from the viewpoint of increased entropy due to incomplete knowledge of observationally inaccessible higher correlators [4–9]. The entropy of the system is in this case described by the Gaussian von Neumann entropy
\[
S_g = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\Delta_k + 1}{2} \ln \left( \frac{\Delta_k + 1}{2} \right) - \frac{\Delta_k - 1}{2} \ln \left( \frac{\Delta_k - 1}{2} \right) \right],
\]

(1.3)
where $\Delta_k$ is the phase space area ($\hbar = 1$) for a Gaussian state centered at the origin,
\[
\Delta_k^2 = 4 \left[ \langle \phi(t, \mathbf{k}) \rangle^2 \langle \pi_\phi(t, \mathbf{k}) \rangle^2 - \frac{1}{2} \langle \pi_\phi(t, \mathbf{k}) \rangle^2 - \frac{1}{2} \langle \phi(t, \mathbf{k}) \rangle \langle \pi_\phi(t, \mathbf{k}) \rangle \langle \phi(t, \mathbf{k}) \rangle \right],
\]

(1.4)
where $\phi(t, \mathbf{k})$ is a Fourier mode of the field $\phi(t, \mathbf{x})$ that describes the system, and $\pi(t, \mathbf{k})$ is its conjugate momentum. The phase space area is equal to 1 for a pure state, which consequently has zero entropy. For a mixed state the phase space area grows and entropy increases. Mixing occurs in general for an interacting quantum field theory, such as a self-interacting quartic potential for the inflaton field, used by Starobinsky and Yokoyama [14]. However, because the
canonical momentum is absent in the Starobinsky equation (1.1), the system will be in a ‘pencil state’ with zero phase space area. The entropy for such a state is not defined. Obviously we have to improve the Starobinsky equation in order to describe the growth of the phase space area and entropy.

The solution is a full Hamiltonian (phase space) formulation of stochastic inflation, which was first (for a different purpose) used by Habib [22] for the case of free quantum fields. The idea is to introduce coarse-graining for both the scalar field and its conjugate momentum. As a result there will be two noise terms in the Hamilton equations for the IR field and its momentum (see also Ref. [23] for a more recent discussion). This is very much like a quantum particle which experiences non-commuting noise in both the position and momentum direction. For illustrative purposes we treat the simpler model of this quantum stochastic particle in section 2.

We calculate the phase space area of a free particle moving under the influence of two types of noise: classical white noise and instant quantum noise. The latter is closely related to the Hamiltonian approach to stochastic inflation which we discuss in section 3 for a free scalar field. By calculating the phase space area we confirm that a pure state remains pure after the Starobinsky coarse graining. In section 4 we include a self-interacting potential for the coarse grained scalar field. We compare the stochastic Hamilton equations for a self-interacting field theory to the Starobinsky equation. A numerical method is proposed for solving the stochastic Hamilton equations which may present a first quantitative calculation of cosmological decoherence. Finally in appendices A and B we give basics of the quantisation procedure and the Wigner function approach to stochastic inflation, respectively.

2 Classical white noise versus instant quantum noise

In this section we study how the phase space area of a quantum particle changes under the influence of environmental noise by considering a simple toy model of a free quantum mechanical particle where both the position and momentum of the particle experience some environmental noise. These two noises induce kicks in the position and momentum direction. The Hamilton equations for a quantum particle moving under the influence of the noise terms are

\begin{align}
\dot{x} &= \frac{p}{m} + F_1(t) \tag{2.1} \\
\dot{p} &= -m\omega^2 x + F_2(t). \tag{2.2}
\end{align}

The time dependent noise terms $F_1(t)$ and $F_2(t)$ are environmental (white) noise, characterized by the noise correlators

\begin{align}
\langle F_i(t) \rangle &= 0 \tag{2.3} \\
\langle F_i(t)F_j(t') \rangle &= f_{ij}(t)\delta(t-t'), \quad \{i,j\} = 1,2. \tag{2.4}
\end{align}

The noise $F_2$ is the ordinary noise term for a classical Brownian particle. Physically it can be viewed as a (big) particle that exchanges small bits of momentum by moving through an environment of smaller particles. The other noise term $F_1$ has no classical analogue, as it represents a small infinitesimal displacement of the particle. However, it allows us to more easily draw an analogy with the stochastic scalar field in the Starobinsky picture. We will come back to this in the next section. By substituting Eq. (2.1) into Eq. (2.2) we find an equation for $x$

\[ \ddot{x} + \omega^2 x = \dot{F}_1 + \frac{1}{m} F_2. \tag{2.5} \]
We can now solve for $x$ and $p$ by considering the righthand side of (2.5) as a source term. The full solutions of Eqs. (2.1–2.2) are then

$$x(t) = x_0(t) + \int dt' G_{\text{ret}}(t,t') \left[ \partial_{t'} F_1(t') + \frac{1}{m} F_2(t') \right]$$

(2.6)

$$p(t) = p_0(t) + m \int dt' \partial_t G_{\text{ret}}(t,t') \left[ \partial_{t'} F_1(t') + \frac{1}{m} F_2(t') \right] - m F_1(t),$$

(2.7)

where $x_0(t)$ is the homogeneous solution of Eq. (2.5), $p_0(t) = m \dot{x}_0(t)$ and the retarded Green’s function is

$$G_{\text{ret}}(t,t') = \theta(t-t') \sin \left( \frac{\omega(t-t')}{\omega} \right).$$

(2.8)

So far we have not yet discussed canonical quantisation. The position $x$ and momentum $p$ are quantum operators which satisfy the canonical commutator $[x,p] = i$ (setting $\hbar = 1$). There are two ways to implement the quantum character of $x$ and $p$ into the solutions of the Hamilton equations (2.6–2.7): in Case 1 the homogeneous solutions $x_0$ and $p_0$ contain the quantum nature of $x$ and $p$. The noise terms $F_1$ and $F_2$ are consequently classical white noise terms. In Case 2 the quantumness of $x$ and $p$ originates from the noise terms, in the sense that $F_1$ and $F_2$ do not commute, i.e., $[F_1,F_2] \neq 0$. In order to satisfy the canonical commutator the noise terms have to be of the special type of instant quantum noise.

In both Case 1 and Case 2 we are interested in the phase space area $\Delta$ of the quantum particle,

$$\Delta^2 = 4 \left[ \langle x(t)^2 \rangle \langle p(t)^2 \rangle - \frac{1}{2} \langle \{x(t),p(t)\} \rangle^2 \right].$$

(2.9)

The phase space area can be used to calculate the Gaussian von Neumann entropy of the system (see for example Refs. [8, 24]),

$$S_g = \frac{\Delta + 1}{2} \ln \left( \frac{\Delta + 1}{2} \right) - \frac{\Delta - 1}{2} \ln \left( \frac{\Delta - 1}{2} \right).$$

(2.10)

For a pure state the phase space area is minimal, i.e., $\Delta = 1$, and consequently the entropy is zero. In the case of a free quantum mechanical particle without environmental noise the phase space area is minimal as there are no (environmental or internal) interactions that could lead to a growth of $\Delta$. On the other hand, the phase space area in general increases once we include environmental noise. Our goal is to find out how the phase space area grows in both Case 1 with classical white noise and Case 2 with instant quantum noise. We will now consider these two cases in more detail.

2.1 Case 1: Classical white noise

In the first case, the homogeneous solutions in Eqs. (2.6–2.7) are the usual quantum harmonic oscillator solutions expressed in terms of creation and annihilation operators.,

$$x_0(t) = \sqrt{\frac{1}{2m\omega}} \left( e^{-i\omega t} a + e^{i\omega t} a^\dagger \right)$$

(2.11)

$$p_0(t) = i \sqrt{\frac{m\omega}{2}} \left( -e^{-i\omega t} a + e^{i\omega t} a^\dagger \right),$$

(2.12)
with \([a, a^\dagger] = 1\) and \([a, a] = [a^\dagger, a^\dagger] = 0\). These solutions satisfy the commutation relation \([x_0, p_0] = i\). Provided that the noise terms \(F_1\) and \(F_2\) are classical, i.e. commuting, the full solutions \(x\) and \(p\) in (2.6–2.7) will satisfy the canonical commutator \([x, p] = i\). Now we can calculate the field correlators. The environmental noise and the free particle solutions are uncorrelated, i.e. \(<x_0 F_1> = <p_0 F_1> = 0\). Using Eqs. (2.6–2.8) and Eq. (2.4) we find

\[
\langle x(t)^2 \rangle = \frac{1}{2m\omega} + \int_0^t dt' \left[ c_\theta^2 f_{11}(t') + s_\theta^2 f_{22}(t') + s_\theta c_\theta \frac{(f_{12}(t') + f_{21}(t'))}{m\omega} \right]
\]

\[
\langle p(t)^2 \rangle = \frac{m\omega}{2} + m^2\omega^2 \int_0^t dt' \left[ s_\theta^2 f_{11}(t') + c_\theta^2 f_{22}(t') - s_\theta c_\theta \frac{(f_{12}(t') + f_{21}(t'))}{m\omega} \right]
\]

\[
\frac{1}{2} \left\{ x(t), p(t) \right\} = m\omega \int_0^t dt' \left[ -c_\theta s_\theta f_{11}(t') + c_\theta s_\theta \frac{f_{22}(t')}{m^2\omega^2} + \frac{1}{2} \left( c_\theta^2 - s_\theta^2 \right) \frac{(f_{12}(t') + f_{21}(t'))}{m\omega} \right],
\]

where \(s_\theta \equiv \sin \theta\), \(c_\theta \equiv \cos \theta\) and \(\theta \equiv \omega(t - t')\). We take the noise to be classical white noise. This means that the noise correlators in Eq. (2.4) are constant and that \([F_1, F_2] = 0\), i.e.

\[
f_{ij}(t) = f_{ij}, \quad f_{12} - f_{21} = 0, \quad (\text{classical white noise}).
\]

If we substitute these noise correlators (2.16) in the correlators Eqs. (2.13–2.15) we find the familiar result for ordinary Brownian motion that at late times \(\langle x(t)^2 \rangle \propto t\). Furthermore the variance of the momentum \(\langle p(t)^2 \rangle \propto t\). The phase space area (2.9) of the state becomes

\[
\Delta^2 = 1 + 2m\omega \left( f_{11} + \frac{f_{22}}{m^2\omega^2} \right) t
\]

\[
+ m^2\omega^2 \left\{ \left( f_{11} + \frac{f_{22}}{m^2\omega^2} \right)^2 t^2 - \frac{\sin^2 \omega t}{\omega^2} \left[ \left( f_{11} - \frac{f_{22}}{m^2\omega^2} \right)^2 + \frac{(f_{12} + f_{21})^2}{m^2\omega^2} \right] \right\}.
\]

At late times this implies

\[
\Delta \approx 1 + m\omega \left( f_{11} + \frac{f_{22}}{m^2\omega^2} \right) t.
\]

We clearly see that the phase space area of the initially pure state eventually grows linearly in time. As a consequence the late time entropy increases as \(S \sim \ln(t)\). These results are all well known for classical Brownian motion. The difference for the quantum Brownian particle is that the momentum as well as the position of the particle experiences a random environmental noise. As a result, the increase in phase space area depends equally on both noise correlators.

To draw a connection with classical Brownian motion, we present here the original Langevin equation for classical Brownian motion,

\[
m\ddot{x} + \zeta \dot{x} = F(t),
\]

where \(m\) is the mass of the particle and \(\zeta\) is a friction coefficient. The noise terms satisfy

\[
\langle F(t) \rangle = 0, \quad \langle F(t) F(t') \rangle = 2\zeta k_B T \delta(t - t'),
\]

with \(k_B\) the Boltzmann constant and \(T\) the temperature of the particle’s environment. The prefactor on the right-hand side of Eq. (2.20) is fixed by the fluctuation-dissipation relation. The classical Langevin equation (2.19) gives the familiar result that the displacement at late
times is proportional to \( \sqrt{t} \), specifically \( \langle (x - x_0)^2 \rangle \sim 2Dt \), where \( D = k_B T/\zeta \) is the Einstein diffusion constant. On the other hand the variance of the momentum is \( \langle p^2 \rangle = m^2 \langle \dot{x}^2 \rangle \sim mk_B T \). This is to be contrasted with the variance of the momentum for our model (2.1–2.2), where we find from Eq. (2.14) that \( \langle p^2 \rangle \propto t \). The difference originates from the friction term in the classical Langevin equation (2.19) which limits the growth of \( \langle p^2 \rangle \). Technically the friction term is always present in a thermal bath because a fluctuation-dissipation relation must be satisfied for a Brownian particle.

### 2.2 Case 2: Instant quantum noise

In the second case the quantumness of \( x \) and \( p \) in the general solutions (2.6–2.7) originates from the noise terms \( F_1 \) and \( F_2 \). The homogeneous solutions are the usual classical solutions for a free particle

\[
x_0(t) = x_{in} \cos[\omega(t - t_{in})] + \frac{p_{in}}{m\omega} \sin[\omega(t - t_{in})]
\]

\[
p_0(t) = -m\omega x_{in} \sin[\omega(t - t_{in})] + \frac{p_{in}}{\omega} \cos[\omega(t - t_{in})],
\]

where \( x_{in} \) and \( v_{in} \) are the position and velocity of the particle at the initial time \( t_{in} \). Now, using the complete solutions (2.6–2.7) we find that the canonical commutator is

\[
[x(t),p(t)] = \int dt' \left[ f_{12}(t') - f_{21}(t') \right].
\]

This suggests that our noise correlators (2.4) act only at one instant in time, and that the noise terms commute in a specific way, i.e.

\[
f_{ij}(t) = f_{ij}\delta(t - t_N), \quad f_{12} - f_{21} = i, \quad \text{(instant quantum noise)},
\]

where \( t_N \) is the time at which the noise acts. Physically, one can view this as a classical particle moving freely, then being kicked by an infinitesimal sheet of smaller particles at time \( t_N \). For classical Brownian motion with instant noise, the observer will see that the particle receives a small kick of momentum. On the other hand in the quantum mechanical case with noise in position and momentum (2.1–2.2), the particle experiences a kick of both its momentum and its position, i.e. a quantum kick. As a consequence of this kick, the phase space area of the particle becomes

\[
\Delta^2 = 4\theta(t - t_N) \left[ f_{11}f_{22} - \left( \frac{f_{12} + f_{21}}{2} \right)^2 \right].
\]

Physically the result can be explained in the following way (see Fig. 1): at a time \( t < t_N \) the particle follows a classical trajectory in phase space described by the classical homogeneous solutions (2.21–2.22). The phase space area, expressed in units of \( \hbar \), is a typical quantum mechanical object. For a classical state the phase space area is zero, which can easily be seen by substituting the classical solutions (2.21–2.22) into Eq. (2.9).

\[ \text{When } \langle x \rangle \neq 0 \text{ and/or } \langle p \rangle \neq 0, \text{ the correct definition of the phase space area is the following generalisation of (2.9):} \]

\[ \Delta^2 = 4 \left[ \langle \Delta x^2 \rangle \langle \Delta p^2 \rangle - \frac{1}{2} \langle \Delta x, \Delta p \rangle^2 \right], \]

where \( \Delta x = x - \langle x \rangle \) and \( \Delta p = p - \langle p \rangle \). Thus we must subtract any expectation value of \( x \) and \( p \), which in this case are the classical homogeneous solutions \( x_0 \) and \( p_0 \) in (2.21–2.22).
Figure 1: Trajectory of a particle influenced by instant quantum noise. From time $t_{in}$ the particle follows the classical trajectory of a free particle in (2.21–2.22). At time $t_N$ the quantum noise kicks in and the particle becomes aware of its quantumness. After $t_N$ the phase space area $\Delta$ is nonzero as described in (2.25). The classical trajectory is still shown as a dashed line.

Now, at the time $t_N$ the noise instantly kicks in. Through the noise the particle suddenly becomes aware of its quantumness, and its phase space area becomes nonzero. In this sense the instant quantum noise provides a way of writing quantum mechanics as a stochastic theory. For a consistent quantum mechanical nature of the particle, we find the condition

$$4 \left[ f_{11} f_{22} - \left( \frac{f_{12} + f_{21}}{2} \right)^2 \right] \geq 1,$$

such that entropy is well defined.

After the time $t_N$ the phase space area will generally be nonzero. Consequently, the entropy increases by some amount due to the quantum kick of the particle. The Case 1 of quantum white noise means that the particle continuously receives quantum kicks. As we have seen, the phase space area grows linearly in time in that case.

A special situation occurs when

$$4 \left[ f_{11} f_{22} - \left( \frac{f_{12} + f_{21}}{2} \right)^2 \right] = 1.$$

In that case the phase space area is minimal and the initially classical particle is now in a pure quantum mechanical state. As we will see in the next section, in stochastic inflation [13] the modes for a noninteracting field exhibit a similar behavior. In the phase space approach to stochastic inflation instant quantum noise is present in the Hamilton equations for the super-Hubble, or infrared (IR) modes of the inflaton field. The noise appears because of a special type of coarse graining, which is based on integrating out the sub-Hubble, or ultraviolet (UV) modes of the field. The properties of the noise are however extremely special, in the sense that the ultraviolet modes that cross the horizon are in a pure state, thus there is no entropy generation. We will examine this now in greater detail.

3 Correlators and entropy for free coarse-grained scalar field

We start with a field theory of a free massless scalar field in an expanding universe. Details of the theory and the quantized solutions of the fields and momenta can be found in Appendix A. In the phase space approach to stochastic inflation [22] the scalar field (A.8) and momentum (A.9) are separated into super- and sub-Hubble modes, or equivalently IR and UV modes,

$$\Phi = \phi + \varphi$$

$$\Pi = \pi_{\phi} + \pi_{\varphi}.$$
The super-Hubble modes \( \phi \) and \( \pi_\phi \) of the scalar field and momentum are those modes with \( k \ll H a \). We separate these modes from the rest by introducing a time dependent cut-off \( \theta(\epsilon a H - k) \), where \( \epsilon \lesssim 1 \), and

\[
\phi(t, \vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \theta(\epsilon a H - k) \left\{ u(t, k)e^{i\vec{k} \cdot \vec{x}} \alpha(\vec{k}) + u^*(t, k)e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger(\vec{k}) \right\} \tag{3.3}
\]

\[
\pi_\phi(t, \vec{x}) = a^3 \int \frac{d^3 \vec{k}}{(2\pi)^3} \theta(\epsilon a H - k) \left\{ \dot{u}(t, k)e^{i\vec{k} \cdot \vec{x}} \alpha(\vec{k}) + \dot{u}^*(t, k)e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger(\vec{k}) \right\} \tag{3.4}
\]

Similarly, the short wavelength expressions \( \varphi \) and \( \pi_\varphi \) are the other modes with \( k > \epsilon H a \). The time dependent momentum cut-off in the short wavelength terms leads to a stochastic noise term in the long wavelength equations. We can easily see this by substituting Eqs. (3.1–3.2) into the Hamilton equations (A.4–A.5) and using the field equation for the mode functions (A.11). We find

\[
\dot{\phi} = \frac{1}{a^3} \pi_\phi + F_1(t, \vec{x}) \tag{3.5}
\]

\[
\dot{\pi}_\phi = a^3 \nabla^2 \phi + F_2(t, \vec{x}) \tag{3.6}
\]

where the stochastic noise terms are

\[
F_1(t, \vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \delta(k - \epsilon a H) (\epsilon a H) \left\{ u(t, k)e^{i\vec{k} \cdot \vec{x}} \alpha(\vec{k}) + u^*(t, k)e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger(\vec{k}) \right\} \tag{3.7}
\]

\[
F_2(t, \vec{x}) = a^3 \int \frac{d^3 \vec{k}}{(2\pi)^3} \delta(k - \epsilon a H) (\epsilon a H) \left\{ \dot{u}(t, k)e^{i\vec{k} \cdot \vec{x}} \alpha(\vec{k}) + \dot{u}^*(t, k)e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger(\vec{k}) \right\} \tag{3.8}
\]

where \( u(t, k) \) are the scalar field mode functions defined in (A.8–A.9), and \( \alpha(\vec{k}) \) and \( \alpha^\dagger(\vec{k}) \) are the annihilation and creation operators (A.10). Note that the stochastic Hamilton equations in the field theoretical case (3.5–3.6) are (almost) the same as those in the quantum mechanical case (2.1–2.2). At first it was believed that the coarse-graining of the free field could lead to decoherence due to the time dependent split in long and short wavelength modes [20, 21]. Habib [22] however argued that the stochastic inflation type of coarse-graining for a free field does not lead to decoherence. Here we will put these arguments on a firmer footing by performing an explicit calculation of (potential) decoherence for coarse-grained free scalar fields. We will do this in a general accelerating FLRW universe without assuming a specific choice of the scale factor.

Our first goal is to show that coarse-graining through a time dependent cutoff does not lead to a growth of the phase space area (1.4) and Gaussian entropy (1.3) for the super-Hubble modes of the scalar field. In order to do so we will be interested in the momentum space correlators for \( \phi \) and \( \pi_\phi \) as they determine the phase space area for every mode \( k \). In principle we can proceed along the same lines as the quantum mechanical particle with quantum white noise in section 2. The (free) scalar field is namely nothing more than an infinite sum of simple harmonic oscillators. Analogously to (2.5), by combining (3.5) and (3.6) and transforming into a Fourier space, we can write the field equations for the modes \( \phi(t, \vec{k}) \) and \( \pi_\phi(t, \vec{k}) \):

\[
\ddot{\phi}(t, \vec{k}) + 3H \dot{\phi}(t, \vec{k}) + \frac{k^2}{a^2} \phi(t, \vec{k}) = \frac{1}{a^3} \left[ (a^3 F_1(t, \vec{k}))^* + F_2(t, \vec{k}) \right] \tag{3.9}
\]
and solve for $\phi(t, \vec{k})$ using the method of Green’s functions like in Eqs. (2.6) and (2.7). However, there is an important difference with respect to the quantum mechanical case. We obtained the stochastic Hamilton equations (3.5–3.6) by coarse graining the homogeneous solutions (A.8–A.9) and substituting the coarse grained field and momentum (3.3–3.4) back into the original Hamilton equations (A.4–A.5). It would therefore make no sense to add the homogeneous solution of Eq. (3.9) to the coarse-grained solution. Physically speaking the homogeneous solution is absent, as the IR quantum field only exists once the UV field has crossed the Hubble radius, which is described through the quantum noise terms (3.7–3.8) alone.

Mathematically speaking, we could solve Eqs. (3.5–3.6) without knowing the origin of the equations, but with knowledge of the noise terms (3.7–3.8). Because the quantumness of $\phi$ and $\pi_\phi$ is contained in the noise terms $F_1$ and $F_2$, we could still add a classical homogeneous solution. However, as we have also seen in section 2.2, the phase space area is defined in terms of quantum fluctuations on top of the classical solution. Therefore any classical solution does not appear in the phase space area and can for our purposes be neglected. Thus, the full (quantum) solutions for $\phi(t, \vec{k})$ and $\pi_\phi(t, \vec{k})$ are just

$$\phi(t, \vec{k}) = \int dt' G_{ret}(t, t', k) \left[ (a'^3 F_1(t', \vec{k}))' + F_2(t', \vec{k}) \right]$$

$$\pi_\phi(t, \vec{k}) = a(t)^3 \int dt' \left[ \partial_\delta G_{ret}(t, t', k) \right] \left[ (a'^3 F_1(t', \vec{k}))' + F_2(t', \vec{k}) \right] - a(t)^3 F_1(t, \vec{k}),$$

where $a' \equiv a(t')$. The retarded Green’s function is given in Eq. (A.15) of Appendix A. Of course, if one substitutes the actual noise terms (3.7–3.8) into the above solutions, we will recover again the coarse grained solutions in Eqs. (3.3–3.4). At this moment however we will keep using the solutions for $\phi(t, \vec{k})$ and $\pi_\phi(t, \vec{k})$ in terms of the noise terms. Here we wish to calculate the phase space area $\Delta_k$ in terms of the noise correlators. This allows us to draw an analogue with the quantum mechanical case in the previous section. First let us calculate the correlators for the noise terms in momentum space, which we can write in the following way,

$$\langle F_i(t, \vec{k}) F_j^*(t', \vec{k}) \rangle = f_{ij} \delta(t - t_H) \delta(t - t'),$$

with

$$f_{11} = |u(t_H, k)|^2$$

$$f_{22} = a^6(t_H)|\dot{u}(t_H, k)|^2$$

$$\frac{1}{2} (f_{12} + f_{21}) = \frac{1}{2} a^3(t_H) \left( u(t_H, k) \dot{u}^*(t_H, k) + \dot{u}(t_H, k) u^*(t_H, k) \right).$$

In these expressions $t_H$ is the time of the mode split, i.e. the instant of time when $k = \epsilon H a$. From these expressions one can also see that the noise terms $F_1$ and $F_2$ do not commute, since

$$[F_1(t, \vec{k}), F_2^*(t', \vec{k})] = (f_{12} - f_{21}) \delta(t - t_H) \delta(t - t') = i \delta(t - t_H) \delta(t - t'),$$

such that, strictly speaking, $F_1$ and $F_2$ are operator valued noises. We have used here the Wronskian condition (A.12). The crucial observation of the noise correlators in Eq. (3.12) is that they are precisely of the instant quantum noise type that we discussed in section 2.2, see Eq. (2.24). With this knowledge we can now calculate the phase space area (1.4) for the solutions (3.10–3.11). After various partial integrations and use of the Wronskian condition, we
\[ \Delta_k^2 = 4\theta(t - t_H) \left[ f_{11} f_{22} - \left( \frac{f_{12} + f_{21}}{2} \right)^2 \right]. \]  

(3.17)

Notice that \( \Delta_k \) does not depend on \( f_{12} - f_{21} \) and hence the operator character of the noises is not relevant for the calculation of the phase space area and thus neither for the calculation of the Gaussian entropy of the state. Not surprisingly, the phase space area of the IR modes (3.17) is precisely equal to the phase space area of the free particle influenced by instant quantum noise (2.25). Thus also here, once the UV modes cross the Hubble radius (i.e. at time \( t_H \)), the quantumness of the IR modes kicks in through a nonzero phase space area.

The question remains: how big is the phase space area for the super Hubble modes of an interacting scalar field under the influence of noise coming from the sub Hubble modes? Let us first consider the free scalar field theory from this section. In this case we know exactly what are the noise correlators from Eq. (3.12). We can therefore simply calculate the phase space area (3.17) of the coarse grained scalar field by inserting the noise correlators (3.12–3.15). By using the Wronskian condition (A.12) twice we obtain the result

\[ \Delta_k^2 = \theta(t - t_H) = \theta(\epsilon a H - k). \]  

(3.18)

Thus for the super-Hubble modes with \( k < \epsilon a H \) the phase space area is always exactly 1, i.e. the system is in a pure state. The Gaussian von Neumann entropy (1.3) is therefore

\[ S_g = \int_{k < \epsilon a H} d^3 \vec{k} \left[ \frac{\Delta_k + 1}{2} \ln \left( \frac{\Delta_k + 1}{2} \right) - \frac{\Delta_k - 1}{2} \ln \left( \frac{\Delta_k - 1}{2} \right) \right] = 0. \]  

(3.19)

This reflects the general fact that introducing a time dependent mode separation of a free scalar field by means of the Heaviside step function does not generate any entropy. Even though the Starobinsky coarse graining introduces noise terms for both the field and its conjugate momentum, the properties of the noise are such that, once the modes cross the Hubble radius, the phase space area is minimal. The reason is that the noise appears here not because of some coupling to an external environment, but because sub-Hubble modes are continuously crossing the Hubble radius.

The same results for the phase space area (3.18) and entropy (3.19) can be, of course, obtained by using the coarse-grained solutions (3.3–3.4). We emphasize that these solutions are the exact solutions of the Hamilton equations with noise (A.4–A.5). If we use these solutions from the start, we trivially find

\[ \langle |\phi(t, \vec{k})|^2 \rangle = \theta(\epsilon a H - k)|u(t, k)|^2 \]  

(3.20)

\[ \langle |\pi_\phi(t, \vec{k})|^2 \rangle = \theta(\epsilon a H - k)\alpha^3(t)|\dot{u}(t, k)|^2 \]  

(3.21)

\[ \left\langle \frac{1}{2} \left\{ \phi(t, \vec{k}), \pi_\phi^*(t, \vec{k}) \right\} \right\rangle = \theta(\epsilon a H - k)\alpha^3(t) \left( u(t, k)\dot{u}^*(t, k) + \dot{u}(t, k)u^*(t, k) \right). \]  

(3.22)

Substituting these propagators into the phase space area (1.4) gives as result again Eq. (3.18). The results here are thus general: once we have a free field and produce noise terms by introducing a time dependent cut-off for the modes, the entropy of the coarse-grained field will not grow. Here we considered a massless field, but even if we add a (time dependent) mass term, in principle we can still find normalized mode functions \( u(t, k) \) and we can express the coarse-grained fields as in Eqs. (3.3–3.4). In general adding or changing a mass term in the Hamilton equations
only changes the squeezing of a state, not its phase space area. Besides adding a mass term, we could also add a quartic potential $\lambda \phi^4/4!$ for the coarse-grained fields. In a mean field approach we can write the corresponding Hamilton equations for the super-Hubble field and momentum as

$$\dot{\phi} = \frac{1}{a^3} \pi_\phi + F_1(t, \vec{x}) \quad (3.23)$$
$$\dot{\pi}_\phi = a^3 \left( \frac{\nabla^2}{a^2} - m^2 - \frac{\lambda}{2} \langle \phi^2(t) \rangle \right) \phi + F_2(t, \vec{x}). \quad (3.24)$$

The term $m^2 - \lambda \langle \phi^2(t) \rangle/2$ is a time dependent mass term, and hence in a mean field approximation the field remains a free field and the entropy does not grow. We emphasize that, in the original Starobinsky’s approach, both interactions between the sub-Hubble fields as well as interactions between the sub-Hubble and super-Hubble fields are neglected. One can show [15] that, for a scalar field with an arbitrary potential in the stochastic formulation of Starobinsky, one correctly captures all leading order logarithms $[\ln(a)]^n$ of coincident correlators to all orders in perturbation theory. The stochastic approach has been extended to Yukawa theory [16] and to scalar electrodynamics [17–19], but no stochastic formulation of quantum gravity is as yet known.

An important question is then what can source decoherence within stochastic inflation. The simple answer is: one source of decoherence are interactions between the infrared fields treated beyond the mean field approximation. This source is present in our Hamiltonian formulation of stochastic inflation and it is the basis for the next section. There are, of course, other sources of decoherence in the full quantum theory, the most notable one are interactions between the sub-Hubble and super-Hubble fields. Including these would require a significant modification of the framework of stochastic inflation, and will not be considered in this work. Some aspects of these interactions are studied in Ref. [25]. However, Matacz (incorrectly) neglects the leading noise terms $F_1$ and $F_2$ (3.7–3.8) that are already present in the free theory (3.5–3.6). Finally, we remark that, based on the results of the study conducted in Ref. [7], we do not expect interactions between the sub-Hubble modes to generate any significant entropy.

### 4 Correlators and entropy for a self-interacting scalar field

Our main interest is to study decoherence of scalar cosmological perturbations during inflation. Here we model this problem by using a toy model (which captures some, but not all important features of cosmological perturbations): a self-interacting scalar field with a quartic potential $V(\phi) = \lambda \phi^4/4!$ in an accelerating universe. Since we are interested in the phase-space area, a full Hamiltonian analysis is necessary. The resulting classical stochastic equations are (cf. Eq. (3.24)),

$$\dot{\phi} = \frac{\pi_\phi}{a^3} + F_1(t, \vec{x}) \quad (4.1)$$
$$\dot{\pi}_\phi = a \nabla^2 \phi + F_2(t, \vec{x}) - a^3 \frac{\lambda}{3!} \phi^3. \quad (4.2)$$

We emphasize that here $F_1$ and $F_2$ are are not the quantum noises given in Eqs. (3.7–3.8), but they are classical noise terms, which can be obtained from (3.7–3.8) by replacing the creation and annihilation operators by classical random variables as follows,

$$\alpha(\vec{k}) \rightarrow \alpha_c(\vec{k}), \quad \alpha^\dagger(\vec{k}) \rightarrow \alpha^*_c(\vec{k}),$$

Thus, the classical noise terms act as corrections to the free theory.
where $\alpha^*_c(\vec{k})$ is simply the complex conjugate of $\alpha_c(\vec{k})$. If the initial state is taken to be pure and free, then the classical (commuting) random variables $\alpha_c(\vec{k}) = \Re[\alpha_c(\vec{k})] + i\Im[\alpha_c(\vec{k})]$ are drawn from a Gaussian phase space distribution, which can correspond to a pure state (with the surface area $\Delta = 1$), which can be squeezed and/or displaced, or to a mixed state with $\Delta_k > 1$ (at least for some $\vec{k}$). This prescription resembles in spirit the methods used for studying preheating after inflation [26]. Several comments are now in order.

Replacing the noises $F_{1,2}$ in (4.1–4.2) by their classical counterparts makes sense as long as correlators that involve anticommutators are large when compared with those that involve commutators of fields (and their momenta), and this is justified for the super-Hubble modes (see also Ref. [23] for a discussion about classicality). The spirit of the classical approximation becomes even clearer when one uses perturbative two-particle irreducible (2PI) methods to study out-of equilibrium field dynamics. From e.g. Refs. [7] and [9] it follows that evolution of the statistical Green function (which involves a field anticommutator) depends also on (an integral over) the causal Green function (involving a field commutator). Now, if the causal Green function is much smaller than the statistical Green function, one can neglect the former in the equation for the statistical Green function, resulting in a classical approximation. This classical approximation is the 2PI analogue of the 1PI classical scheme advocated here. Ultimately, the validity of our scheme can be tested by comparing with the full (quantum) 2PI methods, which would require suitable adaptations of methods in Refs. [6], [7] and [9].

For a quantitative assessment of decoherence in cosmological settings one needs the Gaussian von Neumann entropy (1.3–1.4), which is a function of the phase space area $\Delta$, and which in turn depends on correlators that involve only anticommutators. This then implies that solving the classical equations (4.1–4.2) should suffice.

The main purpose of this work was to propose a simple method suitable for studying decoherence in cosmological settings. We postpone an actual numerical investigation to a future paper, where more emphasis will be given to the important question: how are cosmological observables influenced by the amount of decoherence of cosmological perturbations.

Of course, the stochastic theory (4.1–4.2) must be discretized in order to solve these equations numerically. The theory will be discretized in both position and momentum space at the level of the action in order to keep unitarity of the resulting (discrete) equations of motion (4.1–4.2). This is important, since such a discretization method guarantees that discretization will not induce a new source of decoherence.

A further assumption in our stochastic approach is that there are no interactions between the UV and IR parts of the field. The validity of this approximation can be tested by making a comparison with a full 2PI evolution. This was firstly done by Giraud and Serreau [6] in Minkowsky space, and what remains to be done is to extend their work to inflationary spaces.

Finally we note that alternatively one could calculate the Gaussian correlators in the phase space area from the Wigner distribution. This Wigner distribution can be solved from a Liouville equation which can be obtained from the stochastic Hamiltonian. For illustrative purposes we derive the Liouville equation for the toy model of the stochastic quantum particle in appendix B. Although an alternative to our proposed method, calculating cosmological decoherence through the Wigner distribution method is much harder due to the non-local nature of the self-interacting field in momentum space.
5 Conclusion and Discussion

In this work we propose a generalised framework of stochastic inflation that is suitable for studies of decoherence of scalar cosmological perturbations during inflation. We take the Gaussian von Neumann entropy as a quantitative measure of decoherence. This means that the entropy is generated by neglecting observationally inaccessible higher order field correlators.

In sections 2 and 3 free theories are discussed. Section 2 is mostly pedagogical. There we first consider the quantum stochastic particle with (white) noise in the direction of position and momentum. The canonical commutator can be satisfied by having a free quantum mechanical particle moving under the influence of classical white noise. In that case the phase space area increases linearly in time at late times. However, the quantumness of position and momentum can also originate from the noise terms themselves. The noise in that case is instant quantum noise, with non-commuting noise terms in the position and momentum direction which only act at one instant of time. In that case the initially classical particle receives a quantum kick and becomes aware of its quantumness by having a nonzero phase space area. The latter is closely related to the stochastic noise present in the phase space approach to stochastic inflation, of which we discuss the free case in section 3. The origin of the (quantum) noise is in fact that the sub-Hubble modes continuously cross the Hubble radius. It has the special property that it is unitary, i.e. once the modes exit the Hubble radius, the phase space area remains minimal and constant, implying that the modes are in a pure state. This special property guarantees that there is no decoherence for a free stochastic inflationary theory. In addition, there is no decoherence when interactions are treated in a mean field approximation.

Next, in section 4, a classical Hamiltonian stochastic inflation for a self-interacting scalar field has been proposed, which should be suitable for studies of decoherence of cosmological perturbations during inflation. In our approach, the noises are taken to be classical stochastic commuting variables, such that the equations of motion are suitable for numerical treatment. The main advantage of our approach (when compared to more sophisticated 2PI methods) is its simplicity. Of course, it would be very nice to have an analytic argument concerning the accuracy of our method. In the absence of it, we point out that our approach to decoherence can be tested by the more sophisticated 2PI methods.

Finally, we note that understanding cosmological decoherence is important, since a highly de-cohered state of scalar cosmological perturbations will shift the first acoustic peak and reduce the amplitude of the secondary peaks in the CMB radiation (see, for example, Ref. [27]). The ultimate goal of this project is to find out how strong the cosmological decoherence is in specific inflationary models, such as a chaotic $\lambda \Phi^4$ theory, and to find out what are the observable effects on the CMB.

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A Free field quantization

Here we recall the physics of canonical quantisation of a free scalar field in inflation. We start with the action of a free scalar field on a general background

\[ S_{\text{free}} = -\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi. \]  

(A.1)

We use the FLRW metric \( g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2) \) with \( a = a(t) \) the scale factor. The canonical momentum for \( \Phi \) is

\[ \Pi_\Phi = \frac{\partial L}{\partial \dot{\Phi}} = a^3 \dot{\Phi}, \]  

(A.2)

where \( L \) is the Lagrangian density, \( S = \int d^4x L \). The Hamiltonian density becomes

\[ H = \frac{1}{2} \left\{ \frac{\Pi_\Phi^2}{a^3} + a^3 \left( \nabla \Phi \right)^2 \right\} \]  

(A.3)

The free Hamiltonian \( H \) is related to the Hamiltonian density as \( H = \int d^3x H \). The Hamilton equations are

\[ \dot{\Phi} = \frac{\partial H}{\partial \Pi_\Phi} = \frac{1}{a^3} \Pi_\Phi, \]
\[ \dot{\Pi}_\Phi = -\frac{\partial H}{\partial \Phi} = a^3 \nabla^2 \Phi. \]  

(A.4)

(A.5)

Combining these two equations gives the free field equation

\[ \ddot{\Phi} + 3H \dot{\Phi} - \frac{1}{a^2} \nabla^2 \Phi = 0. \]  

(A.6)

We now quantize the system by imposing the canonical commutator

\[ [\Phi(t, \vec{x}), \Pi_\Phi(t, \vec{x}')] = i\delta^3(\vec{x} - \vec{x}'). \]  

(A.7)

A general solution to the Hamilton equations (A.4) and (A.5) which satisfies this canonical commutation relation is

\[ \Phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ u(t, k) e^{i\vec{k} \cdot \vec{x}} \alpha(\vec{k}) + u^*(t, k) e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger(\vec{k}) \right\} \]  

(A.8)

\[ \Pi_\Phi(t, \vec{x}) = a^3(t) \int \frac{d^3\vec{k}}{(2\pi)^3} \left\{ \dot{u}(t, k) e^{i\vec{k} \cdot \vec{x}} \alpha(\vec{k}) + \dot{u}^*(t, k) e^{-i\vec{k} \cdot \vec{x}} \alpha^\dagger(\vec{k}) \right\}. \]  

(A.9)

The creation and annihilation operators satisfy the commutation relations

\[ [\alpha(\vec{k}), \alpha^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \quad [\alpha(\vec{k}), \alpha(\vec{k}')] = 0, \quad [\alpha^\dagger(\vec{k}), \alpha^\dagger(\vec{k}')] = 0. \]  

(A.10)

The mode functions \( u(t, k) \) obey the free field equation in momentum space

\[ \ddot{u} + 3H \dot{u} + \frac{k^2}{a^2} u = 0, \]  

(A.11)
and satisfy the Wronskian normalization condition
\[ W[u(t,k), u^*(t,k)] = u(t,k)\dot{u}(t,k) - u^*(t,k)\ddot{u}(t,k) = i \frac{\alpha}{a^3}. \] (A.12)

The retarded Green’s function for the field is
\[ G_{\text{ret}}(t, \vec{x}; t', \vec{x}') = i\theta(t - t') \langle \Phi(t, \vec{x}), \Phi(t', \vec{x}') \rangle \]
\[ = i\theta(t - t') \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\Delta\bar{x}} \left\{ u(t,k)u^*(t',k) - u^*(t,k)u(t',k) \right\}. \] (A.13)

which satisfies
\[ \left( \partial_t^2 + 3H \partial_t - \frac{1}{a^2} \nabla^2 \right) G_{\text{ret}}(t, \vec{x}; t', \vec{x}') = \delta(t - t') \frac{\delta^3(\vec{x} - \vec{x}')}{a^3}. \] (A.14)

Its Fourier transform is
\[ G_{\text{ret}}(t, t', k) = i\theta(t - t') \left\{ u(t,k)u^*(t',k) - u^*(t,k)u(t',k) \right\}. \] (A.15)

B Phase space area in stochastic quantum mechanics

An alternative approach to the calculation of the phase space area and entropy of stochastic fields is to solve the quantum Liouville equation. Here we focus on a simple quantum mechanical example, but the approach can in principle be extended to coarse-grained inflaton fields. We will follow mostly the approach by Habib [22], which is in turn based on Kubo’s approach [28]. We will clarify where necessary.

We start with the stochastic quantum mechanical equations of motion,
\[ \dot{x} = \frac{p}{m} + F_1 \] (B.1)
\[ \dot{p} = -m\omega^2 x - V'(x) + F_2, \] (B.2)

which in principle follow from the Hamiltonian
\[ H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + V(x) + F_1p - F_2x. \] (B.3)

\( x \) and \( p \) obey the canonical commutation relation \([x, p] = i\hbar\). The time dependent noises satisfy
\[ \langle F_i(t) \rangle = 0 \] (B.4)
\[ \langle F_i(t_1)F_j(t_2) \rangle = f_{ij}(t_1)\delta(t_1 - t_2), \] \( \{i, j\} = 1, 2 \). (B.5)

The density operator \( \hat{\rho} \) satisfies the von Neumann equation (the quantum Liouville equation)
\[ i\hbar\partial_t \hat{\rho} = [H, \hat{\rho}] \] (B.6)

In position space the density operator can be represented as
\[ \rho = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' |x\rangle \rho(x, x'; t) \langle x'|. \] (B.7)
The von Neumann equation projected onto position space becomes

\[ i\hbar \partial_t \rho(x, x'; t) = \left\{ -\frac{\hbar^2}{2m} (\partial_x^2 - \partial_{x'}^2) + \frac{m\omega^2}{2} (x^2 - x'^2) + V(x) - V(x') \right. \]

\[ \left. - i\hbar F_1(\partial_x + \partial_{x'}) - F_2(x - x') \right\} \rho(x, x'; t) \]  

(B.8)

where we have made use of that \( p|\rangle = -i\hbar \partial_x |\rangle \) using the canonical commutator \([x, p] = i\hbar\).

Now we perform a Wigner transform of the density matrix. This gives the Wigner function

\[ W(q, p; t) = \int_{-\infty}^{\infty} d\Delta x e^{-ip\Delta x/\hbar} \rho \left( q + \frac{\Delta x}{2}, q - \frac{\Delta x}{2}; t \right). \]  

(B.9)

The new coordinates are defined as

\[ q = \frac{x + x'}{2} \]  

(B.10)

\[ \Delta x = x - x'. \]  

(B.11)

Substituting these coordinates into the von Neumann equation (B.8) and using that

\[ (\partial_x^2 - \partial_{x'}^2) \rho(x, x'; t) = 2\partial_q \partial_{\Delta x} \rho \left( q + \frac{\Delta x}{2}, q - \frac{\Delta x}{2}; t \right) \]  

(B.12)

\[ \Delta x e^{-ip\Delta x/\hbar} = i\hbar \left( \partial_p e^{-ip\Delta x/\hbar} \right), \]  

we find that the equation for the Wigner function (B.9) becomes

\[ \left\{ \partial_t + \frac{p}{m} \partial_q - m\omega^2 q \partial_p + \frac{1}{i\hbar} \Delta V + 2F_1 \partial_q + F_2 \partial_p \right\} W(q, p; t) = 0, \]  

(B.14)

where

\[ \Delta V = V \left( q + \frac{i\hbar}{2} \partial_p \right) - V \left( q - \frac{i\hbar}{2} \partial_p \right). \]  

(B.15)

We write Eq. (B.14) schematically as

\[ \{\partial_t + L_0 + L_I + L_N(t)\} W(q, p; t) = 0, \]  

(B.16)

where

\[ L_0 = \frac{p}{m} \partial_q - m\omega^2 q \partial_p, \]  

(B.17)

\[ L_I = \frac{1}{i\hbar} \Delta V \]  

(B.18)

\[ L_N(t) = 2F_1(t) \partial_q + F_2(t) \partial_p. \]  

(B.19)

For specific potentials

\[ V = \frac{\sigma}{3!} x^3 \rightarrow L_{I,1} = \frac{\sigma}{3!} (3q^2 \partial_p - \frac{1}{4} \hbar^2 \partial_p^3) \]  

(B.20)

\[ V = \frac{\lambda}{4!} x^4 \rightarrow L_{I,2} = \frac{\lambda}{4!} (\hbar^2 q^2 \partial_p^3 - 4q^3 \partial_p). \]  

(B.21)
We now strip off the time dependence in Eq. (B.16) by defining

$$\sigma(q,p,t) = e^{(t-t_0)(L_0+L_I)} \mathcal{W}(q,p,t),$$

(B.22)

which leads to the equation

$$\partial_t \sigma = -e^{(t-t_0)(L_0+L_I)} L_0(t) e^{-(t-t_0)(L_0+L_I)} \sigma \equiv \Omega(t) \sigma$$

(B.23)

The formal solution of Eq. (B.16) is

$$\sigma(q,p,t) = T \left\{ \exp \left[ \int_{t_0}^t dt_1 \Omega(t_1) \right] \right\} \sigma(q,p;t_0)$$

$$= \left( 1 + \int_{t_0}^t dt_1 \Omega(t_1) + \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \Omega(t_1) \Omega(t_2) + \ldots \right) \sigma(q,p;t_0),$$

(B.25)

where

$$\sigma(q,p;t_0) = \mathcal{W}(q,p;t_0),$$

(B.26)

and \( T \) denotes the time ordering operation. Now we want to average over the noise in Eq. (B.31) by using the noise correlators (B.5). We assume that the initial Wigner distribution is uncorrelated with the noise,

$$\langle F_i(t) \mathcal{W}(q,p;t_0) \rangle = 0 = \langle F_i(t) \sigma(q,p;t_0) \rangle.$$  

(B.27)

We then find that

$$\langle \sigma(q,p;t) \rangle = \left\langle T \left\{ \exp \left[ \int_{t_0}^t dt_1 \Omega(t_1) \right] \right\} \right\rangle \langle \sigma(q,p;t_0) \rangle.$$  

(B.28)

The exponential can be expanded into powers of \( \Omega(t) \). We use that

$$\langle \Omega(t) \rangle = 0$$

(B.29)

$$\langle \Omega(t_1)\Omega(t_2) \rangle = \delta(t_1 - t_2) e^{(t_1-t_0)(L_0+L_I)} D^2(t_1) e^{-(t_1-t_0)(L_0+L_I)}.$$  

We use that

$$D^2(t_1) = 4 f_{11}(t_1) \partial_q^2 + 2 (f_{12}(t_1) + f_{21}(t_1)) \partial_q \partial_p + f_{22}(t_1) \partial_p^2.$$  

(B.30)

Without loss of generality we now set \( t_0 = 0 \). Furthermore we use some shorthand notation where

$$\int_{t_0}^t \Omega_1 \equiv \int_{t_0}^t dt_1 \Omega(t_1),$$

$$\int^{t_1} \int_{t_0}^t \Omega_1 \Omega_2 \equiv \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \Omega(t_1) \Omega(t_2), \text{ etc.}$$
In principle one could make an ansatz for the canonical commutator. We therefore find that
\[
\langle \sigma(q, p; t) \rangle = \bigg\langle T \left\{ \exp \left[ \int_0^t \Omega_1 \right] \right\} \bigg\rangle \langle \sigma(q, p; 0) \rangle \\
= \left( 1 + \int_0^t \int_0^{t_1} \langle \Omega_1 \Omega_2 \rangle + \int_0^t \int_0^{t_2} \int_0^{t_3} \langle \Omega_1 \Omega_2 \Omega_3 \Omega_4 \rangle + ... \right) \langle \sigma(q, p; 0) \rangle \\
= \left( 1 + \int_0^t \int_0^{t_1} \langle \Omega_1 \Omega_2 \rangle + \int_0^t \int_0^{t_2} \int_0^{t_3} \langle \Omega_1 \Omega_2 \rangle \rangle \langle \Omega_3 \Omega_4 \rangle + ... \right) \langle \sigma(q, p; 0) \rangle \\
= \left( 1 + \frac{1}{2} \int_0^t e^{(L_0 + L_I) t_1} D^2(t_1) e^{-(L_0 + L_I) t_1} + \int_0^t \int_0^{t_1} e^{(L_0 + L_I) t_1} D^2(t_1) e^{-(L_0 + L_I) t_1} + ... \right) \langle \sigma(q, p; 0) \rangle.
\]

In the third line we have performed Wick contractions of the four- and higher-point correlators. Notice that contractions other than the time ordered one vanish. The final line can be reexponentiated to the simple form
\[
\langle \sigma(q, p; t) \rangle = T \left\{ \exp \left[ \frac{1}{2} \int_0^t dt_1 e^{(L_0 + L_I) t_1} D^2(t_1) e^{-(L_0 + L_I) t_1} \right] \right\} \langle \sigma(q, p; 0) \rangle. \tag{B.31}
\]

Differentiation of the last line gives
\[
\partial_t \langle \sigma(q, p; t) \rangle = \frac{1}{2} e^{(L_0 + L_I) t} D^2(t) e^{-(L_0 + L_I) t} \langle \sigma(q, p; t) \rangle. \tag{B.32}
\]

As \(L_0\) and \(L_I\) only depend on the classical quantities \(q\) and \(p\), they can be taken out of the noise average of \(\sigma\). This allows us to reexpress Eq. (B.32) in terms of the Wigner function \(W\) by using Eq. (B.22),
\[
\left\{ \partial_t + L_0 + L_I - \frac{1}{2} D^2(t) \right\} W(q, p; t) = 0. \tag{B.33}
\]

This is the generalised Fokker-Planck equation in phase space variables. To remind the reader, \(L_0\), \(L_I\) and \(D^2\) are defined in Eqs. (B.17), (B.18) and (B.30), respectively. The Wigner distribution can be used to calculate field expectation values for operators of \(x\) and \(p\),
\[
\langle \mathcal{O}(x, p) \rangle = \int dqdp \mathcal{O}(q, p) W(q, p; t). \tag{B.34}
\]

We emphasize here that the \(x\) and \(p\) on the lefthand side are operators, whereas \(q\) and \(p\) on the righthand side are phase space variables. One has to be careful when calculating phase space correlators for which the ordering of \(x\) and \(p\) is important. The prescription is to move all the \(p\)’s to the right in the operator \(\mathcal{O}(x, p)\), and then apply Eq. (B.34). This is of no concern for our object of interest, the phase space area (2.9), which does not contain any correlators involving the canonical commutator.

In principle one could make an ansatz for \(W(q, p; t)\) which solves the Fokker-Planck equation (B.33). In the case of a free particle the solutions for the Gaussian Wigner distribution and related correlators can be obtained analytically, see for example [8] and [24]. For an interacting theory the non-Gaussian Wigner distribution cannot be solved exactly and one has to resort to perturbative methods.
We note that the above Wigner formalism for a quantum mechanical particle can be extended to quantum fields. This has been done by Habib [22] for free fields. In principle the Wigner distribution $W$ and the phase space area can be calculated numerically in a discretized version of Eq. (B.33) for interacting quantum fields. However, we warn the reader that it would be very challenging to find the Wigner function even numerically due to the non-local nature of the interacting potential in Wigner space. Thus, although Eq. (B.33) provides an alternative method, our proposed method in section 4 of solving the stochastic Hamilton equations (4.1–4.2) numerically provides a simpler way to model cosmological decoherence.

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