RATIONAL POINTS ON $X_0^+(125)$

VISHAL ARUL AND J. STEFFEN MÜLLER

Abstract. We compute the rational points on the Atkin–Lehner quotient $X_0^+(125)$ using the quadratic Chabauty method. Our work completes the study of exceptional rational points on the curves $X_0^+(N)$ of genus between 2 and 6. Together with the work of several authors, this completes the proof of a conjecture of Galbraith.

1. Introduction

For an integer $N > 1$, let $w_N$ denote the Atkin–Lehner involution on the modular curve $X_0(N)$. Then the Atkin–Lehner quotient of level $N$

$$X_0^+(N) := X_0(N)/\langle w_N \rangle$$

is a smooth projective curve over $\mathbb{Q}$. A rational point on $X_0^+(N)$ lifts to a quadratic point on $X_0(N)$, which provides motivation for the computation of $X_0^+(N)(\mathbb{Q})$; see for instance [Box21].

A non-cuspidal point on $X_0^+(N)$ corresponds to an unordered pair $\{E, E'\}$ of elliptic curves, together with a cyclic isogeny $E \to E'$ of degree $N$. In particular, a non-cuspidal point in $X_0^+(N)(\mathbb{Q})$ such that the associated elliptic curves do not have complex multiplication corresponds to a quadratic $\mathbb{Q}$-curve [Elk04] without complex multiplication. Following Galbraith [Gal02], we call such a rational point exceptional. It turns out that exceptional rational points are quite rare. Guided by extensive numerical computations, Galbraith made the following conjecture in [Gal02]. We write $g^+_N$ for the genus of $X_0^+(N)$.

**Conjecture 1.1.** Let $2 \leq g^+_N \leq 5$. Then $X_0^+(N)$ contains exceptional rational points if and only if

$$N \in \{73, 91, 103, 125, 137, 191, 311\}.$$

The main result of this short note is the following modest contribution to the study of the rational points on $X_0^+(N)$.

**Proposition 1.2.** There are precisely 6 rational points on $X_0^+(125)$. One of these is a cusp, four are CM points with respective discriminants $D = -19, -16, -11, -4$, and one is exceptional, corresponding to a quadratic $\mathbb{Q}$-curve with $j$-invariant satisfying

$$11^5 j = -2140988208276499951039156514868631437312$$

$$\pm 9489763389784109284120033467612564480\sqrt{509}.$$

These points, the discriminants and the $j$-invariant of the exceptional rational point were already found by Galbraith [Gal02, Section 10]. To show that there are no other rational points on this curve, we use the quadratic Chabauty method [BD18, BD21, BDM+19, EL21, BDM+21, BMS21], combined with the Mordell–Weil sieve [BS10], as
explained in [BBB⁺21, BDM⁺21]. The quadratic Chabauty method makes Kim’s non-abelian Chabauty program [Kim05, Kim09] explicit in the simplest non-abelian case using $p$-adic heights.

It was already known by work of Momose [Mom87], Galbraith [Gal02] and Arai–Momose [AM10] that there are no composite integers $N$ such that $2 \leq g_N^+ \leq 6$ and $X_0^+(N)$ has an exceptional rational point, except for $N = 91, 125$ (where an exceptional point was found by Galbraith) and possibly for $N = 169$. See Section 2 below. The rational points (in fact the $\mathbb{Q}(i)$-points) on $X_0^+(91)$ were computed by Balakrishnan, Besser, Bianchi, and the second-named author, see [BBBM21, Example 7.1]. Balakrishnan et al. showed in [BDM⁺19] that $X_0^+(169)$ contains no exceptional rational points. For both of these computations, the quadratic Chabauty method was used. We obtain:

Theorem 1.3. The only composite levels $N$ such that $X_0^+(N)$ has genus $g_N^+ \in \{2, \ldots, 6\}$ and contains an exceptional rational point are $N = 91$ and $N = 125$. There are two exceptional rational points on $X_0^+(91)$, with $j$-invariants satisfying

$$2^{14} j = -27048390693611915236875 \pm 6908504215856136863265\sqrt{-87}$$

and

$$2^{92} j = -8366877442964720618049886816125 \pm 32028251460268098916979319375\sqrt{-87}.$$

The $j$-invariant of the exceptional rational point on $X_0^+(125)$ is given in Proposition 1.2.

The $j$-invariants were already computed by Galbraith [Gal02].

For prime levels $N$ such that $2 \leq g_N^+ \leq 6$, the rational points on $X_0^+(N)$ were computed in [BBB⁺21, BDM⁺21, AAB⁺21]. More precisely:

- Balakrishnan et al. [BBB⁺21] computed $X_0^+(N)(\mathbb{Q})$ for $N \in \{67, 73, 103\}$.
- Balakrishnan et al. [BDM⁺21] computed $X_0^+(N)(\mathbb{Q})$ for $N \in \{107, 167, 191\}$ (the remaining genus 2 prime levels) and for all prime $N$ such that $g_N^+ = 3$.
- Adžaga et al. [AAB⁺21] computed $X_0^+(N)(\mathbb{Q})$ for all prime $N$ such that $g_N^+ \in \{4, 5, 6\}$.

All exceptional points that occurred in these computations had already been found by Galbraith. Together with Theorem 1.3, this implies:

Theorem 1.4. Conjecture 1.1 holds.

Of course, the upper bound 5 on the genus in Galbraith’s conjecture seems rather arbitrary and is likely due to computational limitations. According to Elkies [Elk98, p. 44], one would expect that there are no exceptional rational points on $X_0^+(N)$ for $N \gg 0$. The explicit methods of this paper are not suitable to prove such a statement. However, there are some known results that hold for infinitely many levels. For instance, Dogra and Le Fourn [DF21] have shown that the Chabauty–Kim set $X_0^+(N)(\mathbb{Q}_p)_2$ is finite for all prime $N$ such that $g_N^+ > 1$. Furthermore, the results of Momose [Mom86, Mom87] and Arai-Momose [AM10] apply to infinitely many composite $N$.

It would also be interesting to compute the rational points on quotients of $X_0(N)$ by more general groups of Atkin–Lehner involutions. Adžaga, Chidambaram, Keller and Padurariu [ACKP22] have recently completed the computation of the rational points on the quotient $X_0^*(N)$ by the full Atkin–Lehner subgroup whenever this curve is hyperelliptic of genus $> 1$. See also [BGX21].
Acknowledgements. We thank Timo Keller for comments on an earlier version of this paper, Jennifer Balakrishnan, Netan Dogra, Nikola Adžaga, Lea Beneish, Mingjie Chen, Shiva Chidambaram, Timo Keller and Boya Wen for helpful discussions and the anonymous referee for useful suggestions. SM was supported by DFG grant MU 4110/1-1 and by an NWO Vidi grant.

2. Composite level

We first list all composite levels $N$ such that $2 \leq g_N^+ \leq 6$:

$g_N^+ = 2 : \quad N = 42, 46, 52, 57, 62, 68, 69, 72, 74, 77, 80, 87, 91, 98, 111, 121, 125, 143$

$g_N^+ = 3 : \quad N = 58, 60, 66, 76, 85, 86, 96, 99, 100, 104, 128, 169$

$g_N^+ = 4 : \quad N = 70, 82, 84, 88, 90, 92, 93, 94, 108, 115, 116, 117, 129, 135, 147, 155, 159,$

$\quad \quad \quad \quad \quad 161, 215$

$g_N^+ = 5 : \quad N = 78, 105, 106, 110, 112, 122, 123, 134, 144, 145, 146, 171, 175, 185, 209$

$g_N^+ = 6 : \quad N = 118, 124, 136, 141, 152, 153, 163, 164, 183, 197, 203, 211, 221, 223, 269,$

$\quad \quad \quad \quad \quad 271, 299, 359$

The list is taken from Table 2 of [AAB+21], and follows from a lower bound for $g_N^+$ in terms of $N$ given in Proposition 4.4 of [AAB+21]. This bound was obtained by expressing $g_N^+$ in terms of the genus of $X_0(N)$ and the class number of certain quadratic orders. Bounds on the former are classical, and the latter can be bounded using the Dirichlet class number formula.

We now discuss previous work on the computation of $X_0^+(N)({\mathbb Q})$ for composite $N$. In [Mom87], Momose showed that there are no rational points on $X_0^+(N)$ if $N$ is composite and contains a sufficiently large prime factor, in the following sense.

Theorem 2.1. (Momose, [Mom87, Theorem 0.1, Proposition 2.11]) The curve $X_0^+(N)$ does not contain an exceptional rational point when $N$ is composite and one of the following conditions holds:

1. There is a prime $p \mid N$ such that $p \geq 11$, $p \notin \{13, 37\}$ and the isogeny factor $J^0_0(p)$ of $J_0(p)$ has Mordell–Weil rank 0 over $\mathbb{Q}$. The latter holds for $p = 11$ and all primes $p \in \{17, \ldots, 300\} - \{151, 199, 227, 277\}$.

2. $g_N^+ \geq 1$ and at least one of the following: $26 \mid N$, $27 \mid N$, $35 \mid N$.

3. $g_N^+ \geq 1$ and $49 \mid N$; moreover $N/49$

- is divisible by 7 or 9, or
- is divisible by a prime $q \equiv 2 \mod 3$ or
- is not divisible by 7 but satisfies $(\frac{-7}{m}) = -1$.

Momose’s techniques do not apply to multiples of 37. This case was resolved much later by Arai and Momose.

Theorem 2.2. (Arai-Momose, [AM10, Theorem 1.2]) The curve $X_0^+(N)$ does not contain an exceptional rational point when $N$ is composite and divisible by 37.
The only composite levels $N$ such that $2 \leq g_N^+ \leq 6$ and such that $N$ is not covered by Theorem 2.1 or Theorem 2.2 are
\[
\begin{align*}
g_N^+ &= 2 : \quad 42, 72, 80, 91, 125 \\
g_N^+ &= 3 : \quad 60, 96, 100, 128, 169 \\
g_N^+ &= 4 : \quad 84, 90, 117 \\
g_N^+ &= 5 : \quad 112, 144 \\
g_N^+ &= 6 : \quad \text{none}.
\end{align*}
\]
For all of these, Galbraith constructs explicit models in [Gal02] using the techniques of [Gal99]. For $N \notin \{91, 125, 169\}$, he also computes the rational points using a morphism to an elliptic curve with rank 0. Galbraith gives no details, but it is not hard to find such a morphism, for instance by computing the automorphisms of the curve. We provide code to check these computations at [AM]. Note that for some of these levels, such as $N = 128$, [Mom86, Theorem 0.1] also implies that there are no exceptional rational points.

3. The remaining cases

In [Gal02], Galbraith computes the rational points of small height on the curves $X_0^+(N)$ for $N \in \{91, 125, 169\}$ and conjectures that these are all rational points. The techniques mentioned at the end of Section 2 cannot be used here, because these curves do not cover a curve with finitely many rational points. Moreover, they have Mordell–Weil rank equal to the genus, so that the method of Chabauty and Coleman [MP12] is not applicable.

Fortunately, all three curves have Jacobians with real multiplication, and one may apply the quadratic Chabauty method [BD18, BDM+19, BD21] to find a finite set of $p$-adic points containing the rational points. When the genus is 2, one typically needs to combine this with the Mordell–Weil sieve [BS10] as in [BBB+21, BDM+21], to identify the rational points among the solutions. We describe this computation for $N = 125$ in Section 4 below.

The curve $X_0^+(91)$ is a bielliptic genus 2 curve of Mordell–Weil rank 2. Its label in the LMFDB [LMF22] is 8281.a.8281.1 and it can be described by the equation $y^2 = x^6 - 3x^4 + 19x^2 - 1$. Hence, the explicit methods developed by Balakrishnan and Dogra in [BD18, Section 8] apply. In fact, the $\mathbb{Q}(i)$-points on this curve were found in [BBBM21, Example 7.1] using an extension of these techniques to curves over number fields and the Mordell–Weil sieve. This shows that that $\#X_0^+(91)(\mathbb{Q}) = 10$, as predicted by Galbraith.

The curve $X_0^+(169)$ is a non-hyperelliptic curve of genus 3. It is isomorphic to both the split Cartan modular curve $X^+_0(13)$ and the nonsplit Cartan modular curve $X^+_m(13)$. In [BDM+19] the quadratic Chabauty method was applied to show that $X_0^+(169)$ has precisely 7 rational points, as predicted by Galbraith. There was no need for the Mordell–Weil sieve because it was possible to construct two 17-adic functions whose common zero set is precisely the set of rational points (it turns out that one can actually work 3-adically, see [BDM+]).

4. Quadratic Chabauty for $X_0^+(125)$

The final curve to consider is $X := X_0^+(125)$, with LMFDB label 15625.a.15625.1. Galbraith [Gal02, Section 10] finds the model
\[
y^2 = x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 8x + 1,
\]
and the six rational points \( \infty_\pm, (0, \pm 1), (-2, \pm 5) \). Of these, \( \infty_-, (0, 1), (0, -1), (-2, 5) \) are Heegner points, \( \infty_+ \) is a cusp, and \( (-2, -5) \) is exceptional.

Since the Galois group of the polynomial \( x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 8x + 1 \) is non-abelian of order 60, it is not feasible to apply elliptic curve Chabauty, as introduced by Bruin [Bru03] and used, for instance, to find the rational points on many genus 2 Atkin-Lehner quotients \( X_0^+(N) \) by Bars, González and Xarles [BGM21].

We compute the rational points on \( X \) by applying quadratic Chabauty for \( p = 29 \). Our code [AM] is written in Magma [BCP97], and is based on the package QCMod, available from [BDM+]. We describe the computation below, referring to [BD18,BD21] for the theoretical background and to [BDM+19,BBB+21,BDM+21] for details on the explicit methods. Alternative approaches to the quadratic Chabauty method have been introduced by Edixhoven and Lido [EL21] and by Besser, Srinivasan and the second-named author [BMS21].

The quadratic Chabauty method for rational points, described explicitly in [BDM+21], requires (in its simplest form) that \( \text{rk} J(\mathbb{Q}) = g \), that \( \text{rk} \text{NS}(J) > 1 \) and that the closure of \( J(\mathbb{Q}) \) in \( J(\mathbb{Q}_p) \) has finite index, where \( p \) is a prime of good reduction for \( X \). As discussed above, the first two conditions are satisfied for \( X \). The third condition means that the method of Chabauty–Coleman is not applicable; it is satisfied for all Atkin-Lehner quotients \( X_0^+(N) \) and all \( p \) by [DF21, Lemma 7].

The quadratic Chabauty method also requires an explicit description of the semi-stable reduction. Clearly our curve \( X \) has good reduction away from 5. For many modular curves of level \( N \), the semi-stable reduction at primes \( \ell | N \) is known. For instance, when \( \ell \) is a prime and \( N = \ell \) or \( N = \ell^2 \), Edixhoven [Edi89,Edi91] has described the semi-stable model of \( X_0(\ell^2) \) and the action of \( \omega_N \) on it, from which one may deduce the semi-stable reduction of \( X_0^+(N) \). More recently, Edixhoven and Parent [EP21] found the semi-stable reduction of the modular curve associated to any maximal subgroup of \( \mathrm{GL}_2(\mathbb{F}_\ell) \).

However, we are not aware of any statements in the literature that would allow us to find the semi-stable reduction of \( X_0^+(125) \) at 5 explicitly using a modular approach, though the techniques of Weinstein [Wei16] may be of use here. Instead, we apply Qing Liu’s code genus2reduction (for instance through Part/GP or SageMath). This shows that \( X_0^+(125) \) has potentially good reduction at 5. While potentially good reduction is not strictly necessary (see [BDM+21, §5.4]) for the quadratic Chabauty method, it greatly simplifies its application in practice; see [BDM+21, §3.1] and [BD19, Section 12] for a discussion.

We use the prime \( p = 29 \) of good reduction and the model
\[
y^2 = -1487x^6 - 3238x^5 - 2915x^4 - 1390x^3 - 370x^2 - 52x - 3
\]
of \( X \). While this has larger coefficients than (4.1), it has the advantage that there are no \( \mathbb{F}_{29} \)-rational points at infinity; hence we have \( X(\mathbb{Q}) = Y(\mathbb{Q}) \), where \( Y \) is the affine plane curve defined by (4.2). Because there are also no Weierstrass points in \( X(\mathbb{F}_{29}) \), all rational points are contained in non-Weierstrass residue disks of \( Y(\mathbb{Q}_{29}) \). We fix the base-point \( b = (-\frac{1}{7}, \frac{1}{8}) \in X(\mathbb{Q}) \) for the Abel-Jacobi map \( X 

We compute the Frobenius matrix \( F_{29} \) on \( H^1_{\text{dR}}(X_{\mathbb{Q}_{29}}) \) using the algorithm of Tuitman [Tui16,Tui17]. From \( F_{29} \), we find the Hecke operator \( T_{29} \) using Eichler-Shimura, and form a nice correspondence \( Z \) on \( X \times X \) (corresponding to a nontrivial element of \( \ker(\text{NS}(J) \to \text{NS}(X)) \)) as in [BDM+19, §6.4]. The correspondence \( Z \) induces a fundamental group quotient \( U = U_Z \) whose Chabauty–Kim set \( X(\mathbb{Q}_{29})_{U} \) is finite by [BD18, Lemma 3.1]. From this, we may find a global (respectively local) 29-adic Galois representation \( A_Z(x) = A_Z(b, x) \) with graded pieces \( \mathbb{Q}_{29}, H^1_{\text{dR}}(X, \mathbb{Q}_p)^\vee \) and \( \mathbb{Q}_{29}(1) \) for each
which completes the proof of Theorem 1.3.

REFERENCES

[AAB+21] N. Adžaga, V. Arul, L. Beneish, M. Chen, S. Chidambaram, T. Keller, and B. Wen, Quadratic Chabauty for Atkin-Lehner quotients of modular curves of prime level and genus 4, 5, 6, ArXiv preprint arXiv:2105.04811 (2021).

[ACKP22] N. Adžaga, S. Chidambaram, T. Keller, and O. Padurariu, Rational points on hyperelliptic Atkin-Lehner quotients of modular curves and their coverings, Res. Number Theory 8 (2022), no. 87. ↑1

$x \in X(\mathbb{Q})$ (respectively $x \in X(\mathbb{Q}_{29})$), see [BD18, Section 5] and [BDM+19, §3.4]. By [BD18, Section 5], the local height $x \mapsto h_{29}(A_Z(x))$ constructed by Nekovář [Nek93] is a locally analytic function on all of $X(\mathbb{Q}_{29})$, and the global 29-adic height $h(A_Z(x))$ extends to a locally analytic function $h: X(\mathbb{Q}) \to \mathbb{Q}_{29}$ as well. Since our curve has potentially good reduction, we have $h(x) = h_{29}(A_Z(x))$ for $x \in X(\mathbb{Q})$ by [BDM+19, Lemma 3.2]. This equality only holds for finitely many points in $X(\mathbb{Q}_{29})$, so we obtain a locally analytic quadratic Chabauty function

$$\rho: X(\mathbb{Q}_{29}) \to \mathbb{Q}_{29}; \quad x \mapsto h(x) - h_{29}(A_Z(x))$$

that vanishes along $X(\mathbb{Q})$ and has only finitely many zeros.

On the model $(\mathbb{Q}, \mathbb{Q}_{29})$, the 6 small rational points found by Galbraith are:

$$X(\mathbb{Q})_{\text{known}} = \left\{ \left( -\frac{1}{2}, \frac{1}{8} \right), \left( -\frac{1}{3}, \frac{1}{27} \right), \left( -\frac{1}{4}, \frac{5}{64} \right) \right\} \subset Y(\mathbb{Q}).$$

The curve $X$ has sufficiently many rational points in the sense of [BDM+21, §3.3] so that we may determine the global height pairing $h$ as a bilinear pairing on the tangent space $H^0(X_{\mathbb{Q}_{29}}, \Omega^1)^\vee$ using the values $h(A_Z(x))$ for $x \in X(\mathbb{Q})_{\text{known}}$. In our example, one needs three rational points with pairwise distinct $x$-coordinates. This suffices to extend $x \mapsto h(A_Z(x))$ to a locally analytic function $h: X(\mathbb{Q}_{29}) \to \mathbb{Q}_{29}$, see [BDM+21, §3.3].

For a point $x \in X(\mathbb{Q}_{29})$, the local height $h_{29}(A_Z(x))$ can be expressed in terms of the filtered $\phi$-module $D_{\text{cris}}(A_Z(x))$, where $D_{\text{cris}}$ is Fontaine’s functor. While $x \mapsto h_{29}(A_Z(x))$ is locally analytic on all of $X(\mathbb{Q}_{29})$, the explicit methods for its computation discussed in [BDM+19, BDM+21] require that the Frobenius lift of Tuitman [Tui16, Tui17] is defined at $x$. This holds if $x$ lies in an affine non-Weierstrass disk; by construction, all points in $Y(\mathbb{Q}_{29})$ and in $X(\mathbb{Q})$ are contained in such a disk. Using the techniques of [BDM+19, Sections 4, 5], we may compute the Hodge filtration and the Frobenius structure of $D_{\text{cris}}(A_Z(x))$ for $x \in Y(\mathbb{Q}_{29})$, and we obtain an expansion of $\rho$ as a power series on every residue disk in $Y(\mathbb{Q}_{29})$.

We compute that the function $\rho$ indeed vanishes along $X(\mathbb{Q})_{\text{known}}$; it also vanishes in 22 additional points $x \in Y(\mathbb{Q}_{29}) - X(\mathbb{Q})_{\text{known}}$. The points are provably correct to $O(29^5)$, since the code incorporates the precision analysis in [BDM+21, §4]. Following [BDM+19, §3.4], we apply the Mordell–Weil sieve with the prime $v = 1399$ to show that none of these are rational. More precisely, suppose that $x \in Y(\mathbb{Q}_{29}) - X(\mathbb{Q})_{\text{known}}$ were rational. Then we could write $|x - b| = a_1 P_1 + a_2 P_2$ with $a_1, a_2 \in \mathbb{Z}$, where $J(\mathbb{Q}) = \langle P_1, P_2 \rangle \cong \mathbb{Z}^2$. Under this assumption, we find $a_i \mod 29$ from $x$ for $i \in \{1, 2\}$ using linearity of Coleman integrals of holomorphic differentials and we compute the image of the putative point $[x - b]$ in $J(\mathbb{F}_{1399}) \cong (\mathbb{Z}/(29 \cdot 50))^2$. We find that none of the images of the points $x \in Y(\mathbb{Q}_{29}) - X(\mathbb{Q})_{\text{known}}$ are in the image of $X(\mathbb{F}_{1399})$ under our Abel–Jacobi map, so we derive a contradiction. Since $X(\mathbb{Q}) \subset Y(\mathbb{Q})$, this proves that we indeed have

$$X(\mathbb{Q}) = X(\mathbb{Q})_{\text{known}},$$

which completes the proof of Theorem 1.3.
[AM10] K. Arai and F. Momose, Rational points on $X_0^+ (37 M)$, J. Number Theory 130 (2010), no. 10, 2272–2282. ¶ 1, 1, 2.2

[AM] V. Arul and J. S. Müller, Magma code. https://github.com/steffenmueller/ArulMueller. ¶ 2

[BBBM21] J. S. Balakrishnan, A. Besser, F. Bianchi, and J. S. Müller, Explicit quadratic Chabauty over number fields, Israel J. Math. 243 (2021), no. 1, 185–232. ¶ 1, 3

[BBB+21] J. S. Balakrishnan, A. J. Best, F. Bianchi, B. Lawrence, J. S. Müller, N. Triantafillou, and J. B. Vonk, Two recent p-adic approaches towards the (effective) Mordell conjecture, Regulators IV: An international conference on arithmetic L-functions and differential geometric methods, 2021, pp. 31–74. ¶ 1, 1, 3, 4

[BCP97] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system I: The user language, J. Symb. Comput. 24 (1997), no. 3-4, 235–265. ¶ 14

[BD18] J. S. Balakrishnan and N. Dogra, Quadratic Chabauty and rational points I: p-adic heights, Duke Math. J. 167 (2018), no. 11, 2081–2088. ¶ 1, 3, 4

[BD19] A. Betts and N. Dogra, The local theory of unipotent Kummer maps and refined Selmer schemes, ArXiv preprint arXiv:1909.05734 (2019). ¶ 14

[BD21] J. S. Balakrishnan and N. Dogra, Quadratic Chabauty and rational points II: Generalised height functions on Selmer varieties, Int. Math. Res. Not. IMRN 15 (2021), 11923–12008. ¶ 1, 3, 4

[BDM+19] J. S. Balakrishnan, N. Dogra, J. S. Müller, J. Tuitman, and J. Vonk, Explicit Chabauty–Kim for the split Cartan modular curve of level 13, Annals of Math. 189 (2019), no. 3. ¶ 1, 1, 3, 4

[BDM+21] J. S. Balakrishnan, N. Dogra, J. S. Müller, J. Tuitman, and J. B. Vonk, Quadratic chabauty for modular curves: Algorithms and examples, ArXiv preprint arXiv:2101.01862 (2021). ¶ 1, 3, 4

[BGM01] ———, Magma code. https://github.com/steffenmueller/QMod. ¶ 1, 4

[BGX21] F. Bars, J. González, and X. Xarles, Hyperelliptic parametrizations of curves, Ramanujan J. 56 (2021), no. 1, 103–120. ¶ 1, 4

[BMS21] Amnon Besser, J. Steffen Müller, and Padmavathi Srinivasan, p-adic adelic metrics and quadratic chabauty, ArXiv preprint https://arxiv.org/abs/2112.03873 (2021). ¶ 1, 4

[Box21] J. Box, Quadratic points on modular curves with infinite Mordell-Weil group, Math. Comp. 90 (2021), no. 327, 321–343. ¶ 1

[Bru03] N. Bruin, Chabauty methods using elliptic curves, J. Reine. Angew. Math. 562 (2003), 27–49. ¶ 14

[BS10] N. Bruin and M. Stoll, The Mordell-Weil sieve: proving non-existence of rational points on curves, LMS J. Comput. Math. 13 (2010), 272–306. ¶ 1, 3

[DF21] N. Dogra and S. Le Fourn, Quadratic Chabauty for modular curves and modular forms of rank one, Math. Ann. 380 (2021), no. 1-2, 393–448. ¶ 1, 4

[Edi89] B. Edixhoven, Stable models of modular curves and applications, Ph.D. Thesis, 1989. ¶ 14

[Edi91] ———, L'action de l'algèbre de Hecke sur les groupes de composantes des jacobiennes des courbes modulaires est Eisenstein, Astérisque 196-197 (1991), 159–170. ¶ 14

[EL21] Bas Edixhoven and Guido Lido, Geometric quadratic chabauty, Journal of the Institute of Mathematics of Jussieu (2021), 1–55. ¶ 1, 4

[Elik04] N. D. Elkies, On elliptic K-curves, Modular curves and abelian varieties, 2004, pp. 81–91. ¶ 1

[Elik98] ———, Elliptic and quadratic chabauty over finite fields and related computational issues, Computational perspectives on number theory (Chicago, IL, 1995), 1998, pp. 21–76. ¶ 1

[EP21] B. Edixhoven and P. Parent, Semistable reduction of modular curves associated with maximal subgroups in prime level, Doc. Math. 26 (2021), 231–269. ¶ 14

[Gal02] S. D. Galbraith, Rational points on $X_0^+(N)$ and quadratic Q-curves, J. Théor. Nombres Bordeaux 14 (2002), no. 2, 205–219. ¶ 1, 1, 3

[Gal09] ———, Rational points on $X_0^+(p)$, Experiment. Math. 8 (1999), no. 4, 311–318. ¶ 2

[Kim05] M. Kim, The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel, Invent. Math. 161 (2005), 629–656. ¶ 1

[Kim09] ———, The unipotent Albanese map and Selmer varieties for curves, Publ. RIMS 45 (2009), 89–133. ¶ 1

[LMF22] The LMFDB Collaboration, The L-functions and modular forms database, 2022. [Online; accessed 27 May 2022]. ¶ 3

[Mom86] F. Momose, Rational points on the modular curves $X_0^+(p^r)$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), no. 3, 441–466. ¶ 1, 2

[Mom87] ———, Rational points on the modular curves $X_0^+(N)$, J. Math. Soc. Japan 39 (1987), no. 2, 269–286. ¶ 1, 2, 2.1
[MP12] W. McCallum and B. Poonen, The method of Chabauty and Coleman, Explicit methods in number theory, 2012, pp. 99–117. ↑3
[Nek93] J. Nekovář, On p-adic height pairings, Séminaire de Théorie des Nombres, Paris 1990–1991, 1993, pp. 127–202. ↑4
[Tui16] J. Tuitman, Counting points on curves using a map to \( \mathbb{P}^1 \), Math. Comp. 85 (2016), no. 298, 961–981. ↑4
[Tui17] ———, Counting points on curves using a map to \( \mathbb{P}^1 \), II, Finite Fields Appl. 45 (2017), 301–322. ↑4
[Wei16] J. Weinstein, Semistable models for modular curves of arbitrary level, Invent. Math. (2016).

Email address: varul.math@gmail.com

Vishal Arul, Department of Mathematics, University College London, United Kingdom

Email address: steffen.muller@rug.nl

J. Steffen Müller, Bernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands