Properties of branching exponential flights in bounded domains

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Abstract – In a series of recent works, important results have been reported concerning the statistical properties of exponential flights evolving in bounded domains, a widely adopted model for finite-speed transport phenomena (Blanco S. and Fournier R., Europhys. Lett., 61 (2003) 168; Mazzolo A., Europhys. Lett., 68 (2004) 350; Bénichou O. et al., Europhys. Lett., 70 (2005) 42). Motivated by physical and biological systems where random spatial displacements are coupled with Galton-Watson birth-death mechanisms, such as neutron multiplication, diffusion of reproducing bacteria or spread of epidemics, in this letter we extend those results in two directions, via a Feynman-Kac formalism. First, we characterize the occupation statistics of exponential flights in the presence of absorption and branching, and give explicit moment formulas for the total length travelled by the walker and the number of performed collisions in a given domain. Then, we show that the survival and escape probability can be derived as well by resorting to a similar approach.

Introduction. – Branching random walks basically originate from coupling a stochastic transport process in space with a Galton-Watson birth-death mechanism, which thus results in a branched structure, as illustrated in fig. 1. As such, branching random walks lie at the heart of physical and biological modeling, and are key to the description of neutron transport in multiplying media and nucleon cascades, fragmentation phenomena, spread of epidemics, diffusion of reproducing bacteria, and mutation-propagation of genes, just to name a few [1–3]. A central question for random walks is to determine the occupation statistics of the stochastic paths in a given region \( V \): for instance, when the underlying transport process is a Brownian motion, such an issue is intimately related to residence times and first-passage properties, as well as survival probabilities [4–10]. In the presence of branching, exact results are seldom available (even for simple one-dimensional geometries), and deriving precise asymptotic estimates often demands a great amount of ingenuity [11–15]. For an overview of the mathematical aspects, see, e.g., the recent monograph [16].

In specific real-world applications (such as neutron transport or bacterial growth), assuming that random displacements obey a plain Brownian motion is questionable: this is typically the case when finite-speed effects dominate, so that the diffusion limit is not attained. In this context, a more faithful description of the walker dynamics might be achieved by resorting to Pearson random walks, where particles undergo a sequence of displacements at constant speed, separated by collisions with the surrounding medium [17]. When the scattering centers encountered by the travelling particles are spatially uniform, the intercollision lengths are exponentially distributed [2,18], with mean free path \( \lambda \).

For these exponential flights, the occupation statistics is naturally defined in terms of two observables: the number of occurred collisions in the volume \( V \) and the total length travelled in \( V \) [2]. In principle, one would be interested in assessing the full distribution of these two quantities: however, similarly as for the case of Brownian motion, this
unfortunately turns out to be a formidable task [19–22]. A somewhat simpler approach consists in deriving formulas for the moments of the distributions, which are sometimes amenable to exact results. For exponential flights without branching, important findings have been reported in recent years concerning trajectories starting either on the boundary or inside a given bounded volume [23–27]. In particular, it has been shown that the average travelled length for walkers starting from the boundary depends only on the ratio of the volume over the surface, which is a rather counterintuitive result. In this letter we extend these findings in two directions: first, we include more generally absorption and branching mechanisms; second, we show that other physical observables associated to the process, such as escape and survival probabilities, can be easily derived within the same formalism.

**Travelled lengths.** – Consider a bounded domain of non-zero volume \( V \) (with finite diameter) and non-zero measurable surface \( \Sigma = \partial V \). An exponential flight is started in \( V \) or on its boundary: at each collision, the incident particle disappears, and \( k \) particles (the descendants) are emitted with probability \( p_k \), whose directions are randomly redistributed according to a given probability density [2]. Each descendant will then behave as the mother particle, and undergo a new sequence of displacements (at constant speed \( v \)) and collisions, until all descendants have been either absorbed in \( V \) or escaped from \( \Sigma \).

In order to keep notation simple, yet retaining the key features of the process, we will assume in the following that descendants are emitted isotropically and independently from each other, and the total cross-section \( \sigma = 1/\lambda \) is constant over the volume \( V \).

We begin our analysis of the occupation statistics by considering the total length travelled in \( V \) by the particle and all its descendants when observed up to time \( t \), starting from a single walker at \( r_0 \) with direction \( \omega_0 \) at time \( t = 0 \). This is the finite-speed analog of the residence time for Brownian motion. A systematic and far-reaching approach to the study of residence times for simple Brownian motion has been originally proposed by Kac (based on Feynman path integrals) in a series of seminal papers [28], and later extended to more general Markovian and non-Markovian processes [29]. By closely following the same arguments, a Feynman-Kac formalism for branching exponential flights has been very recently developed [22], which allows deriving a recursive formula for the \( m \)-moment \( L^m(r_0, \omega_0, t) \) of the travelled length, namely,

\[
\frac{1}{\nu} \frac{\partial}{\partial t} L^m(r_0, \omega_0, t) = \sigma \sum_{j=2}^m \nu_j B_{m, j} \left( \langle L^j \rangle \Omega(r_0, t) \right) + \mathcal{L}^* L^m(r_0, \omega_0, t) + m \mathbb{1}_V L^{m-1}(r_0, \omega_0, t),
\]

for \( m \geq 1 \), starting with \( L^m(r_0, \omega_0, 0) = 0 \) and \( L^0(r_0, \omega_0, t) = 1 \) from normalization. Here

\[
\mathcal{L}^* = \omega_0 \cdot \nabla - \sigma + \nu_1 \int \frac{d\omega_0}{\Omega_d}.
\]

is the backward transport operator [2], \( \nu_1 = \langle k \rangle = \sum_k kp_k \) being the average number of secondary particles per collision, and \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \) the surface of the unit sphere in dimension \( d \). Furthermore, we denote by

\[
\langle h \rangle_\Omega(r_0, t) = \int \frac{d\omega_0}{\Omega_d} h(r_0, \omega_0, t)
\]

the average over directions, and by \( \mathbb{1}_V \) the marker function of the volume \( V \), i.e., \( \mathbb{1}_V = 1 \) when \( r_0 \) belongs to \( V \), and \( \mathbb{1}_V = 0 \) elsewhere. Finally, \( B_{m, j} \) and \( \nu_j \) stand for Bell’s polynomials [30], \( \nu_j = \langle k(k-1)\ldots(k-j+1) \rangle \) being the falling factorial moments of the descendant number, starting with \( \nu_0 = 1 \). When trajectories are observed up to a time \( t \) much larger than the characteristic time scale of the system dynamics, we can define the stationary moments \( L^m(r_0, \omega_0) = \lim_{t \to \infty} L^m(r_0, \omega_0, t) \), provided that the limit exists. Intuitively, this condition is satisfied when the particle losses due to absorptions and leakages from the boundaries are larger than the gain due to population growth, which is always the case if \( \nu_1 \leq 1 \). When \( \nu_1 > 1 \), this typically amounts to further requiring that \( V \) be below some critical size \( V_c \). In the following we will assume that \( V < V_c \), unless differently specified: the time derivative in eq. (1) then vanishes at large times, and we get

\[
-L^* L^m(r_0, \omega_0) + m \mathbb{1}_V L^{m-1}(r_0, \omega_0) + \sigma \sum_{j=2}^m \nu_j B_{m, j} \left( \langle L^j \rangle \Omega(r_0) \right) = 0,
\]

with the boundary conditions \( L^m(r_0 \in \Sigma, \omega_0) = 0 \) when \( \omega_0 \) is directed outward.

Following [26], our aim is now to average eq. (4) over the starting position and direction of the walker. As the domain \( V \) is bounded, we can safely define the probability measures for trajectories born in the domain and for those starting on the surface. Choosing the starting coordinates uniformly distributed inside \( V \) imposes the uniform volume probability measure

\[
\frac{d\Omega}{\Omega_d} \frac{dV}{V},
\]

where \( d\Omega \) is the solid angle element\(^1\). Similarly, an isotropic incident flux uniformly distributed on the frontier \( \Sigma \) imposes the surface probability measure

\[
\frac{d\Omega}{\Omega_d} \frac{d\Sigma}{\Sigma} (\Omega \cdot n),
\]

where \( \eta_d = \sqrt{\pi(d-1)} \Gamma((d-1)/2)/\Gamma(d/2) \) is a dimension-dependent normalization constant, equal to twice the inverse of the average height of the \( d \)-dimensional unit

\(^1\)A \( \mu \)- or equivalently \( \nu \)-randomness in the language of stochastic geometry [23,31].
properties of branching exponential flights in bounded domains

shell, and \( n \) is the normal entering the surface.\(^2\) By means of such probability measures, for any function \( h(r_0, \omega_0) \) we define its volume average \( \langle h \rangle_V \) and its surface average \( \langle h \rangle_S \), respectively, by

\[
\langle h \rangle_V = \int \frac{d\omega_0}{\Omega} \int \frac{dr_0}{V} h(r_0, \omega_0),
\]

\[
\langle h \rangle_S = \eta_d \int \frac{d\Sigma}{\sum} \int \frac{d\omega_0}{\Omega} (\omega_0 \cdot n) h(r_0, \omega_0).
\]

(7)

We integrate then eq. (4) uniformly over all possible initial positions and directions (taking into account the isotropy property), and apply the Gauss divergence theorem. This yields the recursive formula

\[
\langle L^m \rangle_S = \eta_d \frac{V}{\Sigma} \left[ m \langle L^{m-1} \rangle_V + \sigma (\nu_1 - 1) \langle L^m \rangle_V \right] \\
+ \sigma \sum_{j=2}^m \nu_j \langle L^{m-j} (\Omega) \rangle_V \right].
\]

(8)

Equation (8) relates the \( m \)-th moment of trajectories starting on the surface to the different moments (up to order \( m \)) of trajectories born inside the volume, and as such extends to branching random flights the formulas

\[
\langle L \rangle_S = \eta_d \frac{V}{\Sigma} \left[ 1 + \sigma (\nu_1 - 1) \langle L \rangle_V \right],
\]

(9)

previously obtained for Pearson random walks [24–26]. In particular, from eq. (8) the average length \( (m = 1) \) reads

\[
\langle L \rangle_S = \eta_d \frac{V}{\Sigma} \left[ 1 + \sigma (\nu_1 - 1) \langle L \rangle_V \right],
\]

(10)

which generalizes the celebrated Cauchy’s formula \( \langle L \rangle_S = \eta_d \frac{V}{\Sigma} \) (also known as the mean chord length property), originally established for random straight lines drawn from the surface of the volume [31] and recently shown to rather surprisingly apply also to Pearson random flights [24–27].

The term \( O = (\langle L \rangle_S)^{1/\lambda} \) is a measure of the opacity of the volume, in that it expresses the ratio between the average length travelled in \( V \) when an isotropic particle flux is imposed at the surface \( \Sigma \) and the mean free path. Another quantity of interest is \( \chi = \langle L \rangle_S / (\eta_d V / \Sigma) \), which is the ratio between the average length travelled in the actual medium \( V \) and the length that the particle would clape if \( V \) were empty and the paths were straight lines (the meaning of the denominator stems from Cauchy’s formula).

In general, eq. (10) depends on the fine details of the process and of the geometry, since the term \( \langle L \rangle_V \) is not universal. However, when the underlying branching process has \( \nu_1 = 1 \), eq. (10) yields precisely Cauchy’s formula. In this case, the quantity \( \langle L \rangle_S \) would depend only on the geometrical ratio \( V / \Sigma \) and not on the specific details of the random walk. In particular, \( \langle L \rangle_S \) would be independent of the characteristic jump size \( \lambda \).

This simple property unfortunately does not carry over to higher moments of the travelled length. Indeed, for \( m = 2 \) we have \( B_{2,2}[z_1, z_2] = z_1^2 \), and eq. (8) then gives

\[
\langle L^2 \rangle_S = \eta_d \frac{V}{\Sigma} \left[ 2 \langle L \rangle_V + \sigma (\nu_1 - 1) \langle L^2 \rangle_V \right] \\
+ \sigma \nu_2 \langle (L^2)_V \rangle_V \right].
\]

Bell’s polynomials in eq. (8) are the signature of branching, and for \( m \geq 2 \) introduce some extra non-vanishing terms \( (\nu_2 > 0) \) with respect to eq. (9) even when \( \nu_1 = 1 \).

In the absence of branching, \( i.e. \), when random flights can be either scattered or absorbed, with \( p_0 + p_1 = 1 \) and \( p_k = 0 \) for \( k > 2 \), explicit relations for the probability density functions of the travelled length can be also derived. Under these hypotheses, eq. (8) reduces to

\[
\langle L^m \rangle_S = \eta_d \frac{V}{\Sigma} \left[ m \langle L^{m-1} \rangle_V - \sigma_0 \langle L^m \rangle_V \right],
\]

(11)

where \( \sigma_0 = p_0 \sigma \) corresponds to the absorption cross section. In the presence of absorption, then, \( \langle L \rangle_S < \eta_d V / \Sigma \), as expected. We denote by \( f(l) \) and \( g(r) \) the probability density of the total travelled length for a trajectory started on the surface or inside the volume, respectively. Then, eq. (11) can be identically rewritten as

\[
\int_0^{\infty} l^m f(l) dl = \eta_d \frac{V}{\Sigma} \int_0^{\infty} \left[ \frac{m}{r} - \sigma_0 \right] r^m g(r) dr.
\]

(12)

Integrating eq. (12) by parts and using the normalization \( \int_0^{\infty} g(r) dr = 1 \) yields the relation

\[
g(r) = \frac{1}{\eta_d r \Sigma} \left[ 1 + \eta_d \frac{V}{\Sigma} \sigma_0 - \int_0^r f(l) e^{-\sigma_0 l} dl \right] e^{-\sigma_0 r},
\]

(13)

between the two densities \( f(l) \) and \( g(r) \). In the limit of purely diffusive processes \( (p_0 \to 0) \), eq. (13) reduces to

\[
g(r) = \frac{1}{\eta_d V \Sigma} \int_0^{\infty} f(l) dl,
\]

(14)

a relation originally established for straight paths [32] and later extended to Pearson walks [25].

Excursions in sub-domains. — Considerable efforts have been devoted to the study of the occupation statistics of some sub-domain \( V' \) included in \( V \). This issue has been thoroughly investigated, \( e.g. \), in the context of residence times for Brownian motion (with or without branching; see, \( e.g. \), [6–10]) and Pearson walks [26]. Consider a branching exponential flight emitted in \( V \); the particle and its descendants may enter \( V' \), spend some time inside, branch, possibly die in \( V' \) or escape, then re-enter \( V' \), and

2 A \( \mu \)-randomness [23,31]. The term \( \cos \theta = \mathbf{O} \cdot \mathbf{n} \) implies that in polar coordinates trajectories starting on the surface must enter the domain with density \( \theta = \arcsin(2k - 1) \) in two dimensions and \( \theta = 1/2 \arcsin(1 - 2k) \) in three dimensions, \( k \) being uniformly distributed in \( (0, 1) \; \text{see fig. 1.} \)
so on, as illustrated in fig. 2. The total length travelled in \( V' \) can be straightforwardly assessed by resorting to the Feynman-Kac formalism mentioned above. Indeed, its moments \( L^m(r_0, \omega_0) \) satisfy eq. (4), the marker function being restricted to the sub-domain \( V' \). Then, averaging over all angles and positions inside, \( V \) yields

\[
\langle L^m \rangle_{\Sigma} = \frac{V}{\Sigma} \left[ \frac{V'}{V} \langle L^{m-1} \rangle_{V'} + \sigma (\nu_1 - 1) \langle L^m \rangle_{V} + \sigma \sum_{j=2}^{\infty} \nu_j (\langle B_{m,j} \rangle_{\omega}) \right]_{V} \tag{15}
\]

For \( m = 1 \) we have, in particular,

\[
\langle L' \rangle_{\Sigma} = \frac{V}{\Sigma} \left[ \frac{V'}{V} + \sigma (\nu_1 - 1) \langle L' \rangle_{V} \right]. \tag{16}
\]

As a consequence, for trajectories starting on the surface, the ratio between the length travelled in \( V' \) and that travelled in \( V \) reads

\[
\frac{\langle L' \rangle_{\Sigma}}{\langle L \rangle_{\Sigma}} = \frac{V'}{V} \left[ 1 + \frac{\sigma (\nu_1 - 1)}{1 + \sigma (\nu_1 - 1)} \frac{\langle L' \rangle_{V}}{\langle L \rangle_{V}} \right]. \tag{17}
\]

This ratio generally depends on the geometry as well as on the walk features. However, for branching flights with \( \nu_1 = 1 \), we have \( \langle L' \rangle_{\Sigma}/\langle L \rangle_{\Sigma} = V'/V \), i.e., we recover the elegant ergodic-type property that applies to Pearson walks [26].

**Number of collision events.** – The stationary \( m \)-th moment \( N^m \) of the number of collisions in \( V \) performed by a branching exponential flight starting from \( r_0 \) in direction \( \omega_0 \) can be also assessed by resorting to the Feynman-Kac formalism: as shown in [22], \( N^m \) satisfies

\[
\mathcal{L}^* N^m(r_0, \omega_0) + \sigma \sum_{j=2}^{\infty} \nu_j B_{m,j} \langle (N^j)_{\omega} \rangle + \sigma \sum_{k=1}^{m} \left( \sum_{j=0}^{m-k} \nu_j B_{m-k,j} \langle (N^j)_{\omega} \rangle \right) = 0, \tag{18}
\]

which closely resembles eq. (4). At the boundaries, we have \( N^m(r_0 \in \Sigma, \omega_0) = 0 \) when \( \omega_0 \) is directed outward. Then, by averaging eq. (18) over starting positions and directions in \( V \) we get

\[
\langle N^m \rangle_{\Sigma} = \frac{V}{\Sigma} \sigma \sum_{k=1}^{m} \left( \sum_{j=0}^{m-k} \nu_j \langle B_{m-k,j} \rangle_{\omega} \right)_{V} + \nu_1 \langle N^m \rangle_{V} + \sigma \sum_{j=2}^{\infty} \nu_j \langle B_{m,j} \rangle_{\omega} \langle (N^j)_{\omega} \rangle_{V},
\]

which generalizes the results previously found for Pearson walks [33]. For the average collision number we have

\[
\langle N \rangle_{\Sigma} = \frac{V}{\Sigma} \sigma [1 + (\nu_1 - 1) \langle N \rangle_{V}]. \tag{19}
\]

Now, from exponential flights being a Markovian process it follows that \( \langle N \rangle_{V} = \sigma \langle L \rangle_{V} \) [22]. Hence, we have also \( O = \sigma \langle L \rangle_{\Sigma} = \langle N \rangle_{\Sigma} \), which amounts to saying that the opacity can be expressed in terms of the mean number of collisions in \( V \). Similarly as done for the lengths, the number of collisions in a sub-domain \( V' \) can again be computed by using \( \mathcal{L}^* V \), in eq. (18).

For non-branching flights, \( \nu_j = 0 \) for \( j \geq 2 \) and, from \( B_{j,1} [z_i] = z_j \) and \( B_{j,0} [z_i] = 0 \) for \( j \geq 1 \), we have

\[
\langle N^m \rangle_{\Sigma} = \frac{V}{\Sigma} \sigma \left[ p_1 \sum_{k=0}^{m} \langle (N^{m-k})_{V} - (N^m)_{V} + p_0 \right]. \tag{20}
\]

By making use of the binomial formula, we finally get

\[
\langle N^m \rangle_{\Sigma} = \frac{V}{\Sigma} \sigma \left[ p_1 (\langle N + 1 \rangle^m - (N^m)_{V} + p_0 \right]. \tag{21}
\]

In the absence of branching, it is also possible to explicitly derive the collision probabilities. We denote by \( f(i) \) and \( g(j) \) the collision number probability for a trajectory started on the surface or inside the volume, respectively. Then, eq. (20) can be identically rewritten as

\[
\sum_{i=0}^{\infty} t^i f(i) = \frac{V}{\Sigma} \sigma \left[ \sum_{j=0}^{\infty} [p_1 (j + 1)^m - j^m] g(j) + p_0 \right].
\]

By equating the terms of the series, and imposing that the relation holds for arbitrary \( m \geq 1 \), we get then

\[
f(j) = \frac{V}{\Sigma} \sigma [p_1 (j + 1)^m - j^m] g(j) + p_0 \delta_{j,1}, \tag{21}
\]

\( \delta_{i,j} \) being the Kronecker delta. Resumming over \( j \) yields in particular \( f(0) = 1 - \frac{V}{\Sigma} \sigma g(0) \). For Pearson walks, \( p_0 \to 0 \) and we obtain

\[
f(j) = \frac{V}{\Sigma} \sigma [g(j + 1) - g(j)]. \tag{22}
\]
Escape probability. – When the volume $V$ is bounded, the probability $R(r_0, \omega_0)$ that a trajectory starting from $r_0$ in direction $\omega_0$ never visits the exterior of $V$ satisfies [22]
\[-\omega_0 \cdot \nabla_r R(r_0, \omega_0) + \sigma R(r_0, \omega_0) = \sigma \bar{z}_V \cdot G[(R)_\Omega], \tag{23}\]
where $G(z) = \sum_k p_k z^k$ is the generating function associated to the descendant number distribution $p_k$ and $R(r_0 \in \Sigma, \omega_0) = 0$ when $\omega_0$ is directed outward. By definition, the quantity $Q = 1 - R$ represents the escape probability from the volume. Then, averaging eq. (23) over all initial positions and directions yields
\[\langle R \rangle_{\Sigma} = \eta_d \frac{V}{\sum} \sigma [\langle G \rangle \langle R \rangle_{\Omega} - \langle R \rangle_V]. \tag{24}\]
The terms $\langle R \rangle_V$ and $\langle R \rangle_{\Sigma}$ have a simple probabilistic meaning, namely, the probability that a particle born uniformly and isotropically in the volume $V$, or entering the body isotropically from the boundary, respectively, is absorbed with all its descendants in $V$ [2]. Similarly, $\langle R \rangle_{\Omega}(r_0)$ represents the probability that a particle born isotropically at $r_0$ is absorbed (with all its descendants) in $V$. Equation (24) relates therefore the spatial behavior of the probability $R$ to the probabilities $p_k$, via $G(z)$.

When walkers cannot be absorbed in the domain ($p_0 = 0$), trajectories must necessarily escape from the boundaries, and we have $\langle R \rangle_{\Sigma} = 0$. This rather intuitive result can be understood as follows: developing eq. (24) and using the normalization $\sum_{k=0}^{\infty} p_k = 1$ yields
\[\langle R \rangle_{\Sigma} = \eta_d \frac{V}{\sum} \sigma \sum_{k=2}^{\infty} p_k \left[\langle R \rangle_{\Omega}^{k+1} - \langle R \rangle_V \right]. \tag{25}\]
Since $\langle R \rangle_{\Omega}^{k+1} \leq \langle R \rangle_{\Omega}$ ($\langle R \rangle_{\Omega}$ is a probability), we immediately get that $\langle R \rangle_{\Sigma} \leq 0$: then, the probability that a particle entering the body isotropically from the boundary is absorbed with all its descendants in $V$ must vanish, as expected. The same applies to $\langle R \rangle_V$.

In the absence of branching, $G[z] = p_0 + p_1 z + p_0 + p_1 = 1$, and we get
\[\langle R \rangle_{\Sigma} = \eta_d \frac{V}{\sum} \sigma [1 - \langle R \rangle_V], \tag{26}\]
a $d$-dimensional generalization of a theorem originally derived for purely absorbing media and extended to diffusive and absorbing media in three dimensions in [25].

Survival probability. – A fundamental quantity for random walks is the survival probability $S_t(r_0, \omega_0)$, namely, the probability that at time $t$ at least one particle is still in $V$. When $V < V_c$, for long times $S_t(r_0, \omega_0) \to 0$, which physically means that the combined effects of absorptions in $V$ and leakages through $\Sigma$ are sufficient to compensate the population growth due to branching (if any), and trajectories almost surely go to extinction. However, when $\nu_1 > 1$ and $V > V_c$, branching paths have a finite probability of surviving indefinitely in $V$, and there exists a non-trivial limit $S_t(r_0, \omega_0) \to S(r_0, \omega_0) > 0$, depending on the starting position and direction. For branching Brownian motion, for instance, finding the asymptotic survival probability is a long standing issue [11,12]. For branching exponential flights in bounded domains, when $V > V_c$ the probability of ultimate survival $S(r_0, \omega_0)$ satisfies [22]
\[-\omega_0 \cdot \nabla_r S(r_0, \omega_0) + \sigma S(r_0, \omega_0) = \sigma F[(S)_\Omega], \tag{27}\]
where $F[z] = \sum_{k=1}^{\infty} \alpha_k z^k$, with $\alpha_k = (-1)^k \nu_k / k!$. At the boundaries, $S$ must vanish when $\omega_0$ is directed towards the exterior of $V$. Averaging eq. (27) over all initial positions and directions yields then
\[\langle S \rangle_{\Sigma} = \eta_d \frac{V}{\sum} \sigma [\langle F \rangle [(S)_\Omega] - \langle S \rangle_V]. \tag{28}\]
Unfortunately, the complex nature of the alternating series in $F[z]$ seems to prevent from drawing general conclusions based on eq. (28).

Perspectives. – The approach proposed in this letter based on the Feynman-Kac formalism allows the volume-averaged properties of branching exponential flights to be derived by relying upon a minimal number of simplifying hypotheses. The results presented here are fairly general, and as such apply to a broad class of physical and biological systems. Moreover, no assumptions are made concerning the shape of the domain, and our findings remain valid for non-convex bodies, including for instance domains with holes. Further investigations are ongoing in three directions: including the effects of anisotropies and heterogeneities, both in the source terms and at the collision events [34], allowing for other (possibly unbounded) jump distributions, such as Lévy flights [35], and taking into account reflective or mixed boundary conditions [26].

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