Existence and Uniqueness of Mass Conserving Solutions to Safronov-Dubovski Coagulation Equation for Product Kernel

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Abstract. The article presents the existence and mass conservation of solution for the discrete Safronov-Dubovski coagulation equation for the product coalescence coefficients $\phi$ such that $\phi_{i,j} \leq ij \forall i, j \in \mathbb{N}$. Both conservative and non-conservative truncated systems are used to analyse the infinite system of ODEs. In the conservative case, Helly’s selection theorem is used to prove the global existence while for the non-conservative part, we make use of the refined version of De la Vallée-Poussin theorem to establish the existence. Further, it is shown that these solutions conserve density. Finally, the solutions are shown to be unique when the kernel $\phi_{i,j} \leq \min\{i^\eta, j^\eta\}$ where $\eta \in [0, 2]$.

Keywords. Safronov-Dubovski Coagulation Equation; Product Kernel; Existence; Non-Conservative Truncation; Density Conservation

1 Prologue

The Oort-Hulst-Safronov (OHS) equation \[1, 2\] is given by the following expression
\[
\frac{\partial c(t,x)}{\partial t} = -\frac{\partial}{\partial x} \left( c(t,x) \int_0^x y \phi(x,y)c(t,y)dy \right) - c(t,x) \int_0^\infty \phi(x,y)c(t,y)dy,
\]
for $x, t \in \mathbb{R}^+ := [0, +\infty)$. The function $c(t,x)$ is the concentration of particles formed as a result of mutual interaction between particles of different masses (volumes). The term $\phi(x,y)$ is the coalescence coefficient and it represents the frequency of collision between particles of masses $x$ and $y$. The detail explanation of the terms in the equation (1) can be found in \[2, 3\].

The Safronov-Dubovski coagulation equation, which is actually the discrete version of the OHS equation, was introduced by P.B. Dubovski \[4\] in 1999 and then further studied by authors namely Bagland \[5\], Dubovski \[6, 7\] and Davidson \[8, 9\]. The equation is expressed as
\[
\frac{dc_i(t)}{dt} = c_{i-1}(t) \sum_{j=1}^{i-1} j \phi_{i-1,j}c_j(t) - c_i(t) \sum_{j=1}^i j \phi_{i,j}c_j(t) - \sum_{j=i}^\infty \phi_{i,j}c_i(t)c_j(t),
\]
for $t \in [0, \infty)$, $i, j \in \mathbb{N}$ and with the initial condition
\[
c_i(0) = c_i^0 \geq 0.
\]
This equation describes the collision of particles of mass $is_0$ and $js_0$, with $s_0 > 0$ being the mass of the smallest particle in the system. This leads to the formation of monomers from the

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x-mer (a particle of mass $x s_0$) which in turn results in the coalescence of the monomers with the y-mer where $x \leq y$ (here $x = j, y = i$ for the first two terms and $x = i, y = j$ for the last term.). The first summation represents the coagulation of a $(i - 1)$-mer and a monomer, resulting in the addition of an $i$-sized cluster in the system. The second and third terms, followed by a negative sign signifies the death of a $i$-mer. The second expression is present due to the formation of particles of mass larger than $is_0$. Finally, the last term is obtained when a $i$-mer breaks into monomers and coagulate with a $j$-mer. Here, $c_i(t)$ is the concentration of particles of size $i$ at time $t$ and $\phi_{i,j}, i \neq j$ is the rate at which particles of size $i$ and $j$ collide and is called the coagulation kernel. However, by halving the collision rate of a $i$-mer, $\phi_{i,j}$ can be computed. The kernel $\phi_{i,j} \geq 0$ is symmetric, i.e, $\phi_{i,j} = \phi_{j,i}$.

In addition to the concentration of particles, their moments are also of interest in terms of real life applications or in proving some theoretical results. The $r^{th}$ moment is given by

$$\mu_r(t) = \sum_{i=1}^{\infty} i^r c_i(t),$$

in which $r = 0$ leads to the total number of particles (zeroth moment) and the total mass in the system is obtained by taking $r = 1$. Putting $r = 2$ defines the second moment of the solution which can be interpreted as the energy dissipated by the system [10]. The properties of the second moment of the solution of the truncated system (presented in Section 2) are of main focus here in this article, which is defined as

$$\mu_{2,n}(t) = \sum_{i=1}^{n} i^2 c_i^n(t).$$

The equation (2) includes two major events, collision followed by coagulation. There are various examples of the phenomenon explained by the Oort-Hulst-Safronov equation, namely, the collision of asteroids [11], formation of saturn’s rings [12], formation of protoplanetary disc around a newly formed star [13]. There are several real-world problems caused by particle coagulation such as the health hazards of sand and dust storms [14], the analysis of nano-metal dust explosions [15] and the impact on glacier mass as a result of volcanic dust accumulation [16].

1.1 Existing Literature

Dubovski [6], in 1999, developed the model (2) and demonstrated preliminary results for the general kernel $\phi_{i,j}$ including the non-negativity and mass conservation law. The author also discussed the significance of the Oort-Hulst-Safronov model in calculating coagulation front velocity (the rate of displacement of the boundary of the non-zero values of the distribution function). Dubovski also emphasized the relationship between the coagulation front velocity and the mass conservation law, stating that if the velocity escapes to infinity, the equation does not obey this law. Furthermore, Lachowicz et al. [3] pointed out differences between the OHS model [11] and continuous version of Smoluchowski [17] equation and with the help of an $\epsilon$ dependent family of coagulation equations (where $\epsilon \in (0, 1]$), the former is shown to lead as $\epsilon \to 0$ and when $\epsilon = 1$, it is noticed to correspond to the latter.

Laurençot [18] demonstrated the convergence of solutions for equation (1) to the self-similar profiles by applying dynamical systems approach and identifying two Lyapunov functionals with constant kernel $\phi \equiv 1$. Bagland and Laurençot [19] proved the existence of the self-similar solutions and also presented the explicit form of these solutions for the kernel $\phi(x, y) = x^\lambda + y^\lambda, \lambda \in (0, 1)$. They discovered that the self-similar profiles were compactly supported and had the discontinuity at the edge of their support. Laurençot [20] analysed the OHS model and discussed collapse (gelation from astrophysical point of view [6]) and established the self similar profiles for the product kernel $\phi(x, y) = xy$. In 2020, Barik et al. [21] proved the existence of
weak mass conserving solutions to the OHS equation \((1)\) for the singular kernel. Bagland \([5]\) discussed the existence of solution for the equation \((2)\) in case of the kernel of the form,

\[
\lim_{l \to \infty} \phi_{i,l} l = 0.
\]

In 2014, Davidson \([8]\) proved the global existence of the solution for the bounded kernel of the form \(j \phi_{i,j} \leq M\) where \(M\) is a constant and in addition to existence, the density is also shown to be conserved for unbounded kernels \(\phi_{i,j} \leq C \phi_{i} h_{i,j}\) where \(h_{i,j} \to 0\) as \(i \to \infty\). His work includes uniqueness but only for the bounded kernel of the form \(\phi_{i,j} \leq C \phi\). Recently in 2021, the authors \([22]\) established the global existence and density conservation results for the equation \((2)\) when \(\phi_{i,j} \leq C \phi(1 + i + j)^{\alpha}, \alpha \in [0, 1]\) under the assumption that \(\mu_{2,n}(t)\) is bounded when \(t \in [0, T]\).

The article discussed uniqueness of the solution when \(\phi_{i,j} \leq C \phi_{i}\) for \(i \leq j\) where \(\gamma \leq 2\). However, the existence, mass conservation and uniqueness of solutions for the product kernel is not yet analysed in the literature. Also, it would be interesting to study the global existence and mass conservation of the solution through the non-conservative approximation [see \([23]\) for the coagulation-fragmentation equation].

So, this article aims to prove the global existence result and establish the density conservation for the kernel \(\phi_{i,j} \leq ij \forall i, j \in \mathbb{N}\) when \((2)\) is approximated in conservative and non-conservative forms. We also prove the uniqueness of solutions for the restrictive kernel \(\phi_{i,j} \leq \min\{i^n, j^n\} \forall i, j \geq 1\) and \(\eta \in [0, 2]\). The existence of the solution is proved by assuming that \(\mu_{2,n}(0)\) is finite and \(\mu_{2,n}(t)\) is bounded, i.e.,

\[
\sum_{i=1}^{n} i^2 c_i^n(t) \leq \alpha,
\]  

where \(\alpha\) is a function of \(T\) such that \(t \in [0, T]\). To prove the density conservation in non conservative case, an additional physical property of the kernel is required, i.e.,

\[
\phi_{n,j} \to 0 \text{ as } n \to \infty \text{ for every } j \in [1, (n-1)],
\]  

while in the conservative case, the following condition must hold

\[
\mu_2(t) \leq \beta(T), \ t \in [0, T].
\]

Further, to prove the uniqueness, bound on third moment is needed which will be discussed later on.

The novelty of this work is that the existence, density conservation of the solution for the discrete Safronov-Dubovski equation for the product kernel is established by approximating the equation using the mass-conserving as well as non-mass conserving truncated systems. Helly’s theorem is used to establish the existence when the infinite model is approximated using a mass preserving truncation. The density conservation result revolves around obtaining \textit{a priori} estimates on the evolution of the moments and dominated convergence theorem. Moreover, for the non-mass conserving truncation, the refined version of De la Vallé-Poussin theorem is used to prove the global existence and some properties of the convex function ensure the conservation of the first moment, i.e., mass.

The definitions that are essential to lay the groundwork for the results are mentioned. The set of finite mass sequences is defined by

\[
B = \{ z = (z_d) : ||z|| < \infty \},
\]

with

\[
||z|| := \sum_{d=1}^{\infty} d|z_d|,
\]

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\]
where \((B, \| \cdot \|)\) is a Banach space. Since, only the non-negative solutions of (2)-(3) are of interest, we consider the non-negative cone

\[
B^+ = \{ c = (c_i) \in B : c_i \geq 0 \}.
\]  

The concentration \(c_i(t)\) is defined below as

\[\text{Definition 1.1.} \quad \text{Let } T \in (0, \infty]. \text{ A solution } c = (c_i) \text{ of the initial value problem (2)–(3) on } [0, T) \text{ is a function } c : [0, T) \to B^+ \text{ such that}
\]

1. \(\forall i, c_i \in C([0, T)) \text{ and } \sup_{0 \leq t < T} \|c(t)\| < \infty\),
2. \(\int_0^t \sum_{j=1}^\infty \phi_{i,j} c_j(h) dh < \infty\) for every \(i\) and \(\forall t \in [0, T)\),
3. \(\forall i \text{ and } \forall t \in [0, T), \quad c_i(t) = c_i(0) + \int_0^t \left( \delta_i c_{i-1}(h) \sum_{j=1}^{i-1} j \phi_{i-1,j} c_j(h) - c_i(h) \sum_{j=1}^i j \phi_{i,j} c_j(h) - c_i(0) \sum_{j=1}^{n-1} \phi_{i,j} c_j(h) \right) dh, \tag{12}\)

where \(\delta_P = 1 \text{ if } P \text{ is true, and is zero otherwise.}\)

The paper is organized as follows: In Section 2, finite-dimensional conservative and non-conservative systems for the equation (2) are presented. The section also includes the required estimates on the properties of the system which shall be used in Section 3 that presents the global existence theorem in each case. The proof of the conservation of the first moment for equation (2) when it is approximated using conservative and non-conservative truncations is discussed in Section 4. Finally, in Section 5, the global uniqueness theorem is presented.

## 2 A Truncated System

The section addresses a method for dealing with an infinite system of ordinary differential equations comprising of infinite summations and analyses the properties of the said system after its conversion to a finite system of equations. The estimates obtained for the parameters involved in the finite dimensional equation aids in passing the limit \(n \to \infty\) which enables in proving important results for the concerned model (2)-(3).

### 2.1 Conservative Truncation and Required Results

The finite \(n\)-dimensional (F-D) truncated system for the equation (2) that we consider corresponds to the system

\[
\frac{dc_i^n}{dt} = C_i^n(c), \quad \text{for } 1 \leq i \leq n, \tag{13}
\]

where

\[
C_1^n(c) := -\phi_{1,1} c_1^2 - c_1 \sum_{j=1}^{n-1} \phi_{1,j} c_j, \tag{14}
\]

\[
C_i^n(c) := c_{i-1} \sum_{j=1}^{i-1} j \phi_{i-1,j} c_j - c_i \sum_{j=1}^i j \phi_{i,j} c_j - c_i \sum_{j=i}^{n-1} \phi_{i,j} c_j, \quad \text{for } 2 \leq i \leq n - 1, \tag{15}
\]

\[
C_n^n(c) := c_{n-1} \sum_{j=1}^{n-1} j \phi_{n-1,j} c_j. \tag{16}
\]
From what is stated above the initial conditions of interest are

\[ c_i(0) = c_i^0 \geq 0, \quad \text{for } 1 \leq i \leq n. \]  

(17)

The existence and uniqueness of solutions to (13–17) is an easy consequence of the Picard-Lindelöf theorem. The non-negativity of the solution can be proved by the standard procedure of adding a positive \( \varepsilon \) to the right-hand side of all the equations. Denote \( c^\varepsilon = (c^\varepsilon_j) \), the corresponding solution, if \( c^\varepsilon \) satisfy, for some \( t_0 > 0 \), \( c^\varepsilon_i(t_0) > 0 \) and \( c^\varepsilon_j(t_0) = 0 \) for some \( j \in \{1, \ldots, n\} \), then \( \frac{d}{dt}c^\varepsilon_j(t_0) > 0 \), and thus the solution is non-negative for \( t > t_0 \). Passing to the limit \( \varepsilon \to 0 \) proves the result, see [24] Theorem III-4-5.

The properties of the F-D system have a lot of importance in the analysis of the solution for the coupled infinite system. One such property is the evaluation of the moments of the solution of the truncated system, defined as

\[ \mu^n_g(t) := \sum_{i=1}^{n} w_i c_i(t), \]  

where \( w = (w_i) \) is a non-negative sequence. Here in this article, \( w_i \) can take the values 1, \( i \) and \( i^2 \). The following result on \( \mu^n_w(\cdot) \) will be relevant:

**Lemma 2.1.** Let \( c = (c_i)_{i \in \{1, \ldots, n\}} \) be a solution of (13–17) defined in an open interval \( I \) containing 0. Let \( w = (w_i) \) be a real sequence. Then

\[ \frac{d\mu^n_w}{dt} = \sum_{i=1}^{n} \sum_{j=i}^{n-1} (i w_{j+1} - i w_j - w_i) \phi_{i,j} c_i c_j. \]  

(19)

**Proof.** Using the expressions (14–16), one can obtain

\[ \frac{d\mu^n_w}{dt} = \sum_{i=1}^{n} w_i C^n_i(c) = \left( -w_1 \phi_{1,1} c_1^2 - w_1 c_1 \sum_{j=1}^{n-1} \phi_{1,j} c_j \right) + \sum_{i=2}^{n-1} w_i \left( c_{i-1} \sum_{j=1}^{i-1} j \phi_{i-1,j} c_j \right) \]

\[ - \sum_{i=2}^{n-1} w_i \left( c_i \sum_{j=1}^{i} j \phi_{i,j} c_j + c_i \sum_{j=i+1}^{n-1} \phi_{i,j} c_j \right) + w_i c_{n-1} \sum_{j=1}^{n-1} j \phi_{n-1,j} c_j. \]

Rewriting the right-hand side by collecting together the first and third, the second and fifth, and the forth and sixth terms to get

\[ \frac{d\mu^n_w}{dt} = \sum_{i=1}^{n-1} \sum_{j=1}^{i} (w_{i+1} - w_i) j \phi_{i,j} c_i c_j - \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} w_i \phi_{i,j} c_i c_j. \]

Changing the order of summation in the first double series and using the symmetry of the rate coefficients, \( \phi_{i,j} = \phi_{j,i} \), we finally conclude [19]. \( \square \)

For \( w_b := (i^b) \), where \( b = 0, 1, 2 \), consider the simplified notation \( \mu^n_{w_b} := \mu^n_w \). It is clear from (19) that, for all \( t \in I \),

\[ \frac{d\mu^n_{w_b}(t)}{dt} \leq 0, \]  

(20)

and thus \( \mu^n_{w_b}(t) \leq \mu^n_{w_b}(0) \), for all \( t \in I \cap \{t \geq 0\} \). This \textit{a priori} bound implies that non-negative solution of the truncated system (13–16) are globally defined forward in time, i.e., \( I \supset [0, +\infty) \).

Taking \( w_1 = i \) in (19), one can claim that

\[ \frac{d\mu^n_{w_1}(t)}{dt} = 0, \quad \forall t, \]  

(21)
which means that solutions to the truncated system conserve mass. Further, defining a term that is similar to \( \mu^n_1(t) \) (see [24]) as

\[
\kappa^n_m(t) := \sum_{i=m}^{n} ic^n_i(t), \quad m \geq 1.
\]

(22)

Differentiating (22) yields

\[
\frac{d\kappa^n_m(t)}{dt} = \sum_{i=m}^{n-1} \sum_{j=1}^{i} j\phi_{ij}c^n_i c^n_j + mc_{m-1} \sum_{j=1}^{m-1} j\phi_{m-1,j} c^n_j - \sum_{i=m}^{n-1} \sum_{j=i}^{n-1} i\phi_{ij} c^n_i c^n_j.
\]

(23)

Assume now \( 2m < n \) and consider the function \( \chi^n_m(\cdot) \) defined by

\[
\chi^n_m(t) := \sum_{i=m}^{2m} ic^n_i + 2m \sum_{i=2m+1}^{n} c^n_i,
\]

(24)

where, \( c^n = (c^n_1, \ldots, c^n_n) \) is a solution of the \( n \)-dimensional truncated system. Then, after differentiating the above equation, we get

\[
\frac{d\chi^n_m(t)}{dt} = \sum_{i=m}^{2m} iC^n_i + 2m \sum_{i=2m+1}^{n} C^n_i
\]

\[
= mc^n_{m-1} \sum_{j=1}^{m-1} j\phi_{m-1,j} c^n_j(t) + \sum_{i=m}^{2m-1} c^n_i(t) \sum_{j=1}^{i} j\phi_{ij} c^n_j(t) - \sum_{i=m}^{2m} \sum_{j=i}^{n-1} i\phi_{ij} c^n_i(t) c^n_j(t) - 2m \sum_{i=2m+1}^{n} \sum_{j=i}^{n-1} \phi_{ij} c^n_i(t) c^n_j(t).
\]

(25)

Another important result that aids in establishing the existence of a subsequence of the solution \( c^n(t) \) is presented in the following lemma as

**Lemma 2.2.** Take \( c_0 = (c_{0\mu}) \in B^+ \) and, for each \( n \in \mathbb{N} \), consider the point \( c^n_0 \in B^+ \) defined by \( c^n_0 = (c^n_{01}, c^n_{02}, \ldots, c^n_{0n}, 0, 0, \ldots) \) and let it be identified with the point of \( \mathbb{R}^n \) obtained by discarding the \( j \)th components, for \( j > n \). Let \( c^n \) be the solution of the \( n \)-dimensional truncated system \( [13]–[16] \) with initial condition \( c^n(0) = c^n_0 \), both satisfying (9). Assume \( \phi_{ij} \leq ij, \forall i, j \geq 1 \) and let \( \Omega^n_m(t) \) be defined by

\[
\Omega^n_m(t) := e^{-t} \left( \sum_{i=m}^{n} ic^n_i(t) + (m+1)\mu^n_1(0)\alpha \right).
\]

Then,

\[
\frac{d\Omega^n_m(t)}{dt} \leq 0.
\]

(26)

**Proof.** Differentiating \( \Omega^n_m(\cdot) \) and using \( [14]–[16] \) lead to

\[
\frac{d\Omega^n_m(t)}{dt} = e^{-t} \sum_{i=m}^{n} iC^n_i(t) - e^{-t} \left( \sum_{i=m}^{n} ic^n_i(t) + (m+1)\mu^n_1(0)\alpha \right)
\]

\[
\leq e^{-t} \left( \sum_{i=m}^{n} \sum_{j=1}^{i} j(ij)c^n_i c^n_j + mc^n_{m-1} \sum_{j=1}^{m-1} (m-1)j^2 c^n_j - (m+1)\mu^n_1(0)\alpha \right)
\]

\[
\leq e^{-t} (\mu^n_2(0)\alpha + m\mu^n_1(0)\alpha - (m+1)\mu^n_1(0)\alpha) = 0,
\]

which proves the claim of the lemma.
2.2 Non-Conservative Truncation and Required Results

Here, in this subsection, we present an approximating system of equations which do not conserve the first moment. The system is hence called non-conservative and is defined as

\[
\frac{dc_i^n}{dt} = C_i^n(c), \quad \text{for } 1 \leq i \leq n, \tag{27}
\]

where

\[
C_i^n(c) := -\phi_{1,1}c_1^2 - \phi_{1,j}c_j \\text{, for } 1 \leq i \leq n, \tag{28}
\]

\[
C_i^n(c) := c_{i-1} \sum_{j=1}^{i-1} j\phi_{i-1,j}c_j - c_i \sum_{j=1}^{i} j\phi_{i,j}c_j - c_i \sum_{j=i}^{n} \phi_{i,j}c_j, \quad \text{for } 2 \leq i \leq n-1, \tag{29}
\]

\[
C_n^n(c) := c_{n-1} \sum_{j=1}^{n-1} j\phi_{n-1,j}c_j \tag{30}
\]

with the same initial conditions as taken in (17). We compute the \(i\)th moment of the solution of the finite dimensional system which aids in computing its zeroth and first moments.

**Lemma 2.3.** Let \(c = (c_i)_{i \in \{1, \ldots, n\}}\) be a solution of (27)-(30) defined in an open interval \(I\) containing 0. Let \(w = (w_i)\) be a real sequence. Then

\[
\frac{d\mu_i^n}{dt} = \sum_{i=1}^{n} w_i C_i^n(c) = w_1 \left( -\phi_{1,1}c_1^2 - \sum_{j=1}^{n} \phi_{1,j}c_j \right) + w_n \left( c_{n-1} \sum_{j=1}^{n-1} j\phi_{n-1,j}c_j \right) + \sum_{i=2}^{n-1} w_i \left( c_{i-1} \sum_{j=1}^{i-1} j\phi_{i-1,j}c_j - c_i \sum_{j=i}^{n} \phi_{i,j}c_j \right). \tag{31}
\]

**Proof.** Using the expressions (28)-(30), one can obtain

\[
\frac{d\mu_i^n}{dt} = \sum_{i=1}^{n} w_i C_i^n(c) = w_1 \left( -\phi_{1,1}c_1^2 - \sum_{j=1}^{n} \phi_{1,j}c_j \right) + w_n \left( c_{n-1} \sum_{j=1}^{n-1} j\phi_{n-1,j}c_j \right) + \sum_{i=2}^{n-1} w_i \left( c_{i-1} \sum_{j=1}^{i-1} j\phi_{i-1,j}c_j - c_i \sum_{j=i}^{n} \phi_{i,j}c_j \right).
\]

To obtain the desired result, some simple but careful computations are required. One can proceed as follows: expand \(III\) for each \(i\), then combine the first term in \(I\) and the first term in \(III_1\), where \(III_k, k = 1, 2, \cdots, (n-1)\) denotes sub-brackets of \(III\). Further, merge the second term of \(III_1\) with the first term of \(III_2\) and the second term of \(III_2\) with the first term of \(III_3\). Finally, combine \(II\) and the second term of \(III_{(n-1)}\). \(\square\)

The validity of this truncation for a coagulation system of equation can be established by computing the rate of change in the number of particles of the finite dimensional system. To do so, putting \(w_i = 1\) in (31) and the non-negativity of \(c_i, c_j\) and \(\phi_{i,j}\) yield

\[
\frac{d\mu_0^n}{dt} = \sum_{i=1}^{n} C_i^n = -\sum_{i=1}^{n-1} \sum_{j=i}^{n} \phi_{i,j}c_j \leq 0
\]

and thus \(\mu_0^n(t) \leq \mu_0^n(0) \forall t \in I \cap \{t \geq 0\}\) which ensures that the above mentioned truncation is valid.
Taking \( w_1 = i \) in (31), we obtain
\[
\frac{d\mu^n_i(t)}{dt} = \sum_{i=1}^{n} iC^n_i = \sum_{i=1}^{n-1} \sum_{j=1}^{i} j\phi_{i,j}c_{i,j} - \sum_{i=1}^{n-1} \sum_{j=i}^{n} i\phi_{i,j}c_i c_j.
\]
Now, adding and subtracting some terms, changing the order of summation and replacing \( i \leftrightarrow j \) in the second term, it is easy to see that
\[
\frac{d\mu^n_i(t)}{dt} = - \sum_{j=1}^{n-1} j\phi_{n,j}c_n c_j \leq 0, \quad (32)
\]
which means that solutions to the truncated system do not conserve mass, i.e.,
\[
\mu^n_i(t) \leq \mu^n_i(0), \quad \text{for any } t. \quad (33)
\]
When the approximating system does not conserve the first moment, the global existence of the solution is established by the application of Helly’s theorem (see [24, 25]) to \( \{c^n_i(t)\} \). The theorem requires us to prove that the truncated solution is of locally bounded total variation and uniformly bounded at a point. This is achieved by the application of Lemma 2.6, discussed later on. In addition to this, we make use of the refined version of De la Vallée-Poussin theorem (see [27, 28]), which ensures that if \( c^0_i \) is in weighted \( L^1 \) space, there exists a non-negative convex function \( \gamma \in G \) where
\[
G := \{ \gamma \in C^1([0, \infty)) \cap W^{1, \infty}_{\text{loc}}(0, \infty) : \gamma(0) = 0, \gamma'(0) \geq 0, \gamma' \text{ is concave and } \lim_{r \to \infty} \frac{\gamma(r)}{r} = \infty \},
\]
satisfying certain properties given by the following proposition.

**Proposition 2.4.** Let, \( i, j \geq 1 \) and \( \gamma \in G \), then the following holds true
\[
0 \leq \gamma(i+1) - \gamma(i) \leq \frac{(3i + 1)\gamma(1) + 2\gamma(i)}{(i + 1)}.
\]

**Proof.** The proof can be adapted from Lemma A.2 in [29]. \( \square \)

Now, the results which will be useful in proving the existence are presented below.

**Lemma 2.5.** Consider \( T \in (0, \infty) \), \( t \in [0, T] \), and \( \gamma \in G \), then for every \( n \geq 2 \), the following holds
\[
\sum_{i=1}^{n} \gamma(i)c^n_i(t) \leq Q(T) \quad \text{and} \quad (34)
\]
\[
0 \leq \int_{0}^{t} \left( \sum_{i=1}^{n-1} \sum_{j=i}^{n} \gamma(i)\phi_{i,j}c^n_i(h)c^n_j(h) \right)dh \leq Q(T), \quad (35)
\]
where \( Q(T) \) depends only on \( \gamma(1), ||c^0||, \alpha \) (defined in (3)) and \( \mu_0(0) \).

**Proof.** Using \( w_i = \gamma(i) \) in the equation (31) leads to
\[
\sum_{i=1}^{n} \gamma(i)c^n_i(t) = \sum_{i=1}^{n} \gamma(i)c^0_i + \int_{0}^{t} \left( \sum_{i=1}^{n-1} \gamma(i+1) - \gamma(i) \right) \sum_{j=1}^{i} j\phi_{i,j}c^n_i(h)c^n_j(h) - \sum_{i=1}^{n-1} \sum_{j=i}^{n} \gamma(i)\phi_{i,j}c^n_i(h)c^n_j(h) \right)dh. \quad (36)
\]
Since \( \gamma(k) \) and \( c^n_i \) for \( k = i, j \) are non-negative, the above expression simplifies to
\[
\sum_{i=1}^{n} \gamma(i)c^n_i(t) \leq \sum_{i=1}^{n} \gamma(i)c^0_i + \int_{0}^{t} \left( \sum_{i=1}^{n-1} \gamma(i+1) - \gamma(i) \right) \sum_{j=1}^{i} j\phi_{i,j}c^n_i(h)c^n_j(h) \right)dh.
\]
Proof. Using the equations (27) and (29) yield
\[\sum_{i=1}^{n} \gamma(i)c_i^0(t) \leq \sum_{i=1}^{n} \gamma(i)c_i^0(0) + \int_{0}^{t} \left( \sum_{i=1}^{n-1} \sum_{j=1}^{i} (3i+1)\gamma(1) + 2\gamma(i)j^2c_i^0(h)c_j^0(h) \right)dh \]
\[\leq \sum_{i=1}^{n} \gamma(i)c_i^0 + \int_{0}^{t} \left( Q_1(T) + Q_2(T) \sum_{i=1}^{n} \gamma(i)c_i^0(h) \right)dh,\]
where \(Q_1(T) = (3||c_0|| + \mu_0(0))\alpha\gamma(1)\) and \(Q_2(T) = 2\alpha\). Finally, an application of Gronwall’s inequality proves (34) where \(Q(T) = \sum_{i=1}^{n} \gamma(i)c_i^0 + \frac{Q_1(T)}{Q_2(T)}(e^{TQ_2(T)} - 1)\). Combining (31) and (36) leads to
\[\int_{0}^{t} \left( \sum_{i=1}^{n-1} \sum_{j=1}^{i} \gamma(i)\phi_{i,j}c_i^n(h)c_j^n(h) \right)dh \leq Q(T),\]
which establishes (35).

Lemma 2.6. Consider \(T \in (0, \infty), i \in \mathbb{N}\) and \(c_i^n(t)\) satisfies (35). The following result holds true for a constant \(\tilde{Q}(T)\) which depends on \(||c_0||, \alpha\) and \(T\):
\[\sup_{0 \leq t \leq T} \int_{0}^{t} \left| \frac{dc_i^n}{dt} \right|dh = \tilde{Q}(T).\] (37)

Proof. Using the equations (27) and (29) yield
\[\int_{0}^{t} \left| \frac{dc_i^n}{dt} \right|dh \leq \int_{0}^{t} \left( |c_i^{n-1}(h)| \sum_{j=1}^{i-1} j\phi_{i-1,j}c_j^n(h) + |c_i^n(h)| \sum_{j=1}^{i} j\phi_{i,j}c_j^n(h) + |c_i^n(h)| \sum_{j=i}^{\infty} j\phi_{i,j}c_j^n(h) \right)dh \]
\[= \int_{0}^{t} \left( 2|c_i^n(h)| \sum_{j=1}^{i} j\phi_{i,j}c_j^n(h) + |c_i^n(h)| \sum_{j=i}^{\infty} j\phi_{i,j}c_j^n(h) \right)dh \]
\[\leq 2||c_0|| \alpha T + ||c_0||^2T.\]

Also, the absolute values of the equations (28) and (30) can easily be shown bounded by \(2||c_0||^2\) and \(||c_0||^2\), respectively, which completes the proof of the lemma.

Now, we can proceed to study the global existence result for (29)-(31) considering the conservative as well as the non-conservative truncation.

3 Global Existence Theorem

3.1 Case 1: Conservative Truncation

Theorem 1. Let, \(\phi_{i,j}\) be non-negative, symmetric and satisfy \(\phi_{i,j} \leq ij, \forall i,j \in \mathbb{N}\) and let \(c_0 = (c_{0i}) \in B^+.\) Assume that the solution and initial condition of the truncated system (13)-(17) satisfy (9). Then, there exists a non-negative solution \(c(t)\) of (29)-(31) defined in \([0, \infty)\).

Proof. Let \(n\) be an arbitrarily fixed positive integer and \(c_i^n\) is defined as in Lemma 2.2. The initial value problem (13)-(17) has a unique solution, \(c^n = (c_{i}^{n})_{1 \leq i \leq n}\), which is globally defined, non-negative and, by equation (21), also density conserving. By defining \(c_i^n(t) = 0\) when \(i > n, c_i^n(t) \in B^+,\) for all \(t,\) one has
\[||c^n(t)|| = \sum_{i=1}^{\infty} ic_i^n(t) = \sum_{i=1}^{n} ic_i^n(t) \leq \sum_{i=1}^{n} ic_{0i} = \sum_{i=1}^{\infty} ic_{0i} = ||c_0||.\] (38)
By Lemma 2.2, for each $m \in \{1, \ldots, n\}$ the functions $\Omega^m_n(\cdot)$ are of uniformly bounded variation on $[0, \infty)$. Hence, by Helly’s selection theorem, for each fixed $m$ there exists a subsequence of $\Omega^m_n$, not relabelled, and a function of bounded variation $\Omega_m$ such that

$$\Omega^m_n(t) \to \Omega_m(t) \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad t \geq 0.$$ 

By the definition of $\Omega^m_n(t)$, the expression for $c^m_n(t)$ becomes

$$c^m_i(t) = i^{-1}\left(\int_t^{\Omega^m_i(t)} dt - \int_t^{\Omega^m_{i+1}(t)} dt + f^m_i(0)\alpha\right).$$

So, we also have that, for each $i$, there exists a subsequence of $c^m$ (not relabelled) and a function $c_i : [0, \infty) \to \mathbb{R}$, each of bounded variation on every compact subset of $[0, \infty)$ such that $c^m_i(t) \to c_i(t)$ as $n \to \infty$, for each $t \geq 0$. Thus, it is immediately concluded that, for all $t \geq 0$,

$$c_i(t) \geq 0 \quad \text{and} \quad ||c(t)|| \leq ||c_0||.$$  \hspace{1cm} (39)

Our goal is to establish that this limit function $c$ is a weak solution of the initial value problem \textbf{(24–25)}, i.e., fulfills the conditions in Definition \textbf{1.1}. This will be done by passing to the limit $n \to \infty$ in the integrated version of the truncated problem \textbf{(13–17)}, namely

$$c^m_i(t) = c_i(0) + \int_0^t \left(\sum_{j=1}^{i-1} \frac{\partial}{\partial t} c^m_{i-j}(h) - c^m_i(h)\right) dh + \int_0^t \sum_{j=1}^{n-1} \phi_i, c^m_{i-j}(h) c^m_j(h) dh. \hspace{1cm} (40)$$

To do this, and also to satisfy condition 2 in Definition \textbf{1.1} we need to prove that, for every fixed $i \in \mathbb{N}$, $T \geq 0$, and $\varepsilon > 0$, there exists $m$ and $n_0$, with $n_0 > m > i$, such that, for all $n > n_0$,

$$\int_0^T \kappa^m_n(t) dt \leq \varepsilon, \hspace{1cm} (41)$$

where $\kappa^m_n$ is defined in \textbf{(22)}. In order to prove this, integrating \textbf{(23)} with respect to $h \in [0, t]$ and using \textbf{(25)} yield

$$\kappa^m_n(t) = \kappa^m_n(0) + \int_0^t \left(\sum_{i=m}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h) + mc^m_{m-1} - \sum_{i=m}^{n-1} \sum_{j=1}^i \phi_{i,m,j} c^m_i(h) c^m_j(h)\right) dh \hspace{1cm} (42)$$

$$= \kappa^m_n(0) + \chi^m_n(t) - \chi^m_n(0) + \int_0^t \left(2m - \sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h) + \sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h)\right) dh.$$ 

The first double sum above provides

$$\sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h) = \sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h) + \sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h).$$

Changing the order of summation in the second sum and then interchanging $i$ and $j$ yield

$$\sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h) = \sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h) + \sum_{i=2m+1}^{n-1} \sum_{j=1}^i \phi_{i,j} c^m_i(h) c^m_j(h).$$
Substituting this into the above expression for \( \kappa_{m}^{n}(t) \) gives

\[
\kappa_{m}^{n}(t) = \kappa_{m}^{n}(0) + \chi_{m}^{n}(t) - \chi_{m}^{n}(0) + \int_{0}^{t} \left( \sum_{i=2m}^{n-1} \sum_{j=1}^{2m} j \phi_{i,j} c_{i,j}^{n}(h) c_{i,j}^{n}(h) + 2m \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} \phi_{i,j} c_{i,j}^{n}(h) c_{i,j}^{n}(h) \right) dh. \tag{42}
\]

By \( \text{(38), (39), and the pointwise convergence of } c_{i,j}^{n} \text{ to } c_{i,j} \), we conclude that, for all \( t \in [0, T], \varepsilon > 0, p > \frac{4||c_{0}||}{\varepsilon} \), \( \exists n_{0} : \forall n > n_{0} \),

\[
\sum_{i=1}^{\infty} |c_{i}^{n}(t) - c_{i}(t)| = \sum_{i=1}^{p-1} |c_{i}^{n}(t) - c_{i}(t)| + \sum_{i=p}^{\infty} |c_{i}^{n}(t) - c_{i}(t)| < \varepsilon/2 + \frac{2}{p} ||c_{0}|| < \varepsilon,
\]

which allows us to let \( n \to \infty \) in the definition of \( \chi_{m}^{n}(t) \) in \( \text{(24)} \). So, for all \( t \in [0, T], \lim_{m \to \infty} \chi_{m}(t) = \chi_{m}(t) \) and \( |\chi_{m}(t)| \leq ||c_{0}|| \). Therefore, \( \forall \ v > 0, \exists m, n_{0} \) with \( n_{0} > n \), such that, \( \forall m > \tilde{m}, n > n_{0} \) and \( n \geq 2m + 1 \),

\[
\chi_{m}^{n}(t) \leq \frac{\varepsilon}{3}, \tag{43}
\]

and

\[
\kappa_{m}^{n}(0) \leq \frac{\varepsilon}{3}. \tag{44}
\]

By \( \text{(43), (44), and using } \phi_{i,j} \leq ij, \) the right-hand side of \( \text{(42)} \) can be estimated as follows

\[
\kappa_{m}^{n}(t) \leq \frac{2\varepsilon}{3} + \int_{0}^{t} \left( \sum_{i=2m}^{n-1} \sum_{m=1}^{2m} ij^{2} c_{i,j}^{n}(h) c_{i,j}^{n}(h) + 2m \sum_{i=2m+1}^{n-1} \sum_{j=i}^{n-1} ij c_{i,j}^{n}(h) c_{i,j}^{n}(h) \right) dh
\]

\[
\leq \varepsilon + \int_{0}^{t} \left( \sum_{i=m}^{n} ic_{i}(h) \sum_{j=1}^{n} j^{2} c_{i,j}^{n}(h) + 2 \sum_{i=m}^{n} ic_{i}(h) \sum_{j=i}^{n} j^{2} c_{i,j}^{n}(h) \right) dh.
\]

\[
\leq \varepsilon + 3\varepsilon \int_{0}^{t} \kappa_{m}^{n}(h) dh.
\]

Hence, thanks to Gronwall’s lemma, for all \( t \in [0, T] \), one has

\[
\kappa_{m}^{n}(t) \leq k_{1} \varepsilon, \tag{45}
\]

where \( k_{1} = e^{3\varepsilon T} \). This implies that \( \forall \ v > 0, \exists \tilde{m}, n_{0} \) with \( n_{0} > \tilde{m} \) such that \( \forall m > \tilde{m}, n > n_{0} \) and \( n \geq 2m + 1 \),

\[
\int_{0}^{T} \kappa_{m}^{n}(t) dt \leq \varepsilon k_{1} T, \tag{46}
\]

for all \( t \in [0, T] \).

Since, \( c_{i,j}^{n}(t) \) is pointwise convergent to \( c_{i,j}(t) \), the above expression entails that, for all \( \varepsilon > 0 \), there exists \( \tilde{m} \) such that, for all \( m > \tilde{m} \), we have

\[
\int_{0}^{T} \sum_{i=m}^{\infty} ic_{i}(t) dt \leq \varepsilon.
\]

Hence, when \( \phi_{i,j} \leq ij \), for all \( i \geq 1 \),

\[
\int_{0}^{T} \sum_{j=1}^{\infty} \phi_{i,j} c_{j}(t) dt < \infty, \tag{47}
\]

which ensures the validity of condition 2 in Definition 1.1.
Now, for every fixed $i$, take $n > i$ sufficiently large and for any $q$ such that $i < q < n - 1$, write (40) using (23) and (44) as
\[
\left| c^n_i(t) - c_i(0) - \int_0^t \left( c^n_{i-1} \sum_{j=1}^{i-1} j\phi_{i-1,j} c^n_j - c^n_i \sum_{j=1}^i j\phi_{i,j} c^n_j - c^n_{i-1} \sum_{j=1}^{i-1} j\phi_{i-1,j} c^n_j \right) (h) \, dh \right|
\leq |\kappa^n_m(t)| + |\kappa^n_m(0)| + \left| \int_0^t \frac{dc^n_m(t)}{dt} \, dh \right|
\leq k_1 \varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{6k_1 + 1}{3} \varepsilon.
\]
Hence, by the arbitrariness of $\varepsilon$, we can let $q \to \infty$ and conclude that $c = (c_i)$ satisfy (12), which completes the proof.

Next, we establish that the subsequence $c^{n_k}$ of solutions to the truncated system which converges to the solution $c$ of (2)-(3) actually does so in the strong topology of $B$, uniformly for $t$ in compact subsets of $[0, \infty)$.

**Corollary 1.1.** Let $c^{n_k}$ be the corresponding pointwise convergent subsequence of solutions to the finite dimensional system (13)-(17). Then, $c^{n_k} \to c$ in $B$ uniformly on compact subsets of $[0, \infty)$.

**Proof.** To prove this, first we show that $c^{n_k}_i(t) \to c_i(t)$ for each $i$ uniformly on the compact subsets of $[0, \infty)$. For this, let $n_k = n$ and for each $i < m$
\[
\zeta^n_i(t) := e^{-t} \left[ \mu^n_1(t) - \sum_{i=1}^{m-1} ic^n_i(t) + (m + 1)\mu^n_1(0)\alpha \right].
\]
Now, differentiating (48) gives
\[
\frac{d\zeta^n_i(t)}{dt} = e^{-t} \left[ \frac{d\mu^n_1(t)}{dt} - \sum_{i=1}^{m-1} \frac{d\zeta^n_i(t)}{dt} \right] - e^{-t} \left[ \mu^n_1(t) - \sum_{i=1}^{m-1} ic^n_i(t) + (m + 1)\mu^n_1(0)\alpha \right]
\]
where
\[
\frac{d}{dt} \sum_{i=1}^{m-1} ic^n_i(t) = \sum_{i=1}^{m-1} \sum_{j=1}^{i-1} j\phi_{i,j} c^n_j - \sum_{i=1}^{m-1} \sum_{j=1}^{i-1} j\phi_{i,j} c^n_j - (m + 1)\mu^n_{m-1} \sum_{j=1}^{m-1} j\phi_{m-1,j} c^n_j.
\]
Using (21), (22), (50) and $\phi_{i,j} \leq i j \forall i, j$ in (49), one can obtain
\[
\frac{d\zeta^n_i(t)}{dt} \leq e^{-t} \left[ \sum_{i=1}^{m-1} \sum_{j=i}^{m-1} i^2 j\phi_{i,j} c^n_j + (m + 1)\mu^n_{m-1} \sum_{j=1}^{m-1} (m - 1)j^2 e^n_j - \mu^n_1(0)\alpha - m\mu^n_1(0)\alpha \right],
\]
which gives
\[
\frac{d\zeta^n(t)}{dt} \leq 0, \quad n \geq m, \quad t \in [0, T].
\]
Hence, $\zeta^n_i(t) \to \zeta_i(t)$ uniformly on the compact subsets of $[0, T]$ where
\[
\zeta_i(t) := e^{-t} \left[ \mu_i(t) - \sum_{i=1}^{m-1} ic_i(t) + (m + 1)\mu_i(0)\alpha \right].
\]
Let, $K \subset [0, \infty)$ be compact and $t_n \to t$ in $K$, then
\[
\lim_{n \to \infty} ||c^n(t_n)|| = \lim_{n \to \infty} \sum_{i=1}^{\infty} ic^n_i(t_n) = \sum_{i=1}^{\infty} ic_i(t) = ||c(t)||
\]
which ensures that $||c^n|| \to ||c||$ in $C(K, B)$. \qed
3.2 Case 2: Non Conservative Truncation

This section is devoted to proving the global existence of a solution for the non-conservative truncation.

**Theorem 2.** Consider \( c \in B^+ \) and \( \phi_{i,j} \leq ij \ \forall i, j \). Suppose that \( c^n_i(t) \) and \( c^i_0 \) are the solution and initial condition of the non-conservative approximation \((57)-(59)\) satisfying the relation \((6)\), then there exists a solution of the discrete OHS equation \((2)-(3)\) in \( \mathbb{R}^+ \).

**Proof.** Let, \((E, \sigma, M)\) be a measure space with \( E = \mathbb{N}, \sigma = \{ S : S \subset \mathbb{N} \} \) and the measure \( M \) defined by

\[
M(S) = \sum_{i=0}^{\infty} c^i_0.
\]

Since, \( c^0 \in B^+ \), we get, \( z : z \in L^1(E, \sigma, M) \). Then by the refined version of De la Vallée-Poussin theorem, there exists a function \( \gamma_0 \in G \) such that \( z : \gamma_0(z) \in L^1(E, \sigma, M) \) meaning that

\[
\sum_{i=1}^{\infty} \gamma_0(i)c^0_i < \infty. \tag{51}
\]

Using the equation \((53)\) and Lemma \(2.6\) the sequence is locally bounded and absolutely continuous in \((0, T)\) for every \( i \geq 1 \) and \( T \in (0, \infty) \). Thus, by Helly’s theorem, there exists a subsequence of \( \{c^n_i\} \) denoted as \( \{c^i_n\} \) and a sequence \( \{c_i\} \) of functions of locally bounded variation such that

\[
\lim_{n \to \infty} c^n_i = c_i(t), \ \forall i \geq 1, t \geq 0. \tag{52}
\]

Also, if we let \( c_i^n = 0 \) when \( i > n \),

\[
||c^n(t)|| = \sum_{i=1}^{n} ic^n_i(t) \leq \sum_{i=1}^{n} ic^n_0 \leq ||c_0|| \tag{53}
\]

which, followed by the use of \((52)\) gives the non-negativity of the solution \( c_i(t) \). Further, since, \( \gamma_0 \in G \), equation \((51)\) and Lemma \(2.5\) yield the following

\[
\sum_{i=1}^{n} \gamma_0(i)c^n_i(t) \leq Q(T) \quad \& \quad 0 \leq \int_{0}^{T} \sum_{i=1}^{n-1} \sum_{j=1}^{n} \gamma_0(i)\phi_{i,j}c^n_i(h)c^n_j(h)dh \leq Q(T), \tag{54}
\]

for every \( n \geq 2, t \in [0, T] \) where \( T \in [0, \infty) \). If we take \( T \in (0, \infty) \) and \( m \geq 2 \), then using the above equation \((54)\) and the non-negativity of \( \gamma(i) \) give us

\[
\sum_{i=1}^{m} \gamma_0(i)c^n_i(t) \leq Q(T) \quad \& \quad 0 \leq \int_{0}^{T} \sum_{i=1}^{m-1} \sum_{j=1}^{m} \gamma_0(i)\phi_{i,j}c^n_i(h)c^n_j(h)dh \leq Q(T), \tag{55}
\]

Using \((52)\), passing the limit as \( n \to \infty \) in equation \((54)\) and \( m \to \infty \) in \((55)\) yield

\[
\sum_{i=1}^{\infty} \gamma_0(i)c_i(t) \leq Q(T) \quad \& \quad 0 \leq \int_{0}^{T} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \gamma_0(i)\phi_{i,j}c_i(h)c_j(h)dh \leq Q(T). \tag{56}
\]

Following on the lines of \((30)\), the application of \((53), (56)\), the definition of the kernel and the properties of \( \gamma_0 \) confirm the Definition \((11)\) as

\[
\int_{0}^{T} \sum_{j=1}^{\infty} \phi_{i,j}c_j(h)dh < \infty. \tag{57}
\]
Now, by the definition of $\phi_{i,j}$ and equations (6), (53), we have
\[
\sum_{j=i}^{n} \phi_{i,j} c_i^n c_j^n \leq j^2 c_i^n c_j^n \leq \alpha(T) \| c_0 \|.
\]

Further, using (52) followed by the Lebesgue-dominated convergence theorem, the following holds
\[
\lim_{n \to \infty} \int_{0}^{T} \left( \sum_{j=i}^{n} \phi_{i,j} c_i^n c_j^n - \sum_{j=i}^{\infty} \phi_{i,j} c_i c_j \right) (h) dh = 0. \tag{58}
\]

The relations (6), (52), (33), (53) and (58) lead to the fulfilment of Definition 1.1(3). Consequently using (57), the Definition 1.1(1) follows, hence establishing the existence result for the problem (2)-(3) in $\mathbb{R}^+$. \hfill \Box

4 Density Conservation

4.1 Case 1: Conservative Truncation

In this section, we prove that all the solutions of (2)-(3) conserve density using the already known result regarding the conservation of mass of the F-D system.

**Theorem 3.** Let $\phi_{i,j} \leq ij$ for all natural numbers $i$ and $j$. Let $c = (c_i) \in B^+$ be a solution of the Safronov-Dubovski equation (2) satisfying (3), then the density of every solution $c$ is constant.

**Proof.** Let $c = (c_i) \in B^+$ satisfies (12) in $[0, T]$. Multiplying each term in (12) by $w_i$ and adding from $i = 1$ to $n$, we have, after some algebraic manipulations, for all $t \in [0, T]$,
\[
\sum_{i=1}^{n} w_i c_i(t) - \sum_{i=1}^{n} w_i c_{0i} = \int_{0}^{t} \sum_{i=1}^{n} \sum_{j=i}^{n} (j w_{i+1} - j w_i - w_j) \phi_{i,j} c_i(h) c_j(h) dh - \int_{0}^{t} \sum_{j=1}^{n} \sum_{i=n+1}^{\infty} \phi_{i,j} c_i(h) c_j(h) dh - \int_{0}^{t} w_{n+1} c_n(h) \sum_{j=1}^{n} \phi_{n,j} c_j(h) dh \tag{59}
\]
We start by observing that, taking $w_i \equiv i$ in (59) leads to
\[
\sum_{i=1}^{n} i c_i(t) \leq \sum_{i=1}^{n} i c_{0i} \leq \| c_0 \|,
\]
and, as this inequality is valid for all $n$, we can take the limit as $n \to \infty$ and conclude the *a priori* bound $\| c(t) \| \leq \| c_0 \|$. We now use (59) to prove that, under the assumed conditions on $\phi_{i,j}$ all solutions conserve density. Let $A \in \mathbb{N}$ be fixed, and consider the sequence $(w_i^A) \in \ell^\infty$ defined by
\[
w_i^A = \min(i, A). \tag{60}
\]
Then
\[
jw_{i+1}^A - j w_i^A - w_j^A = \begin{cases} -A & \text{on } \{(i, j) : A \leq i \leq j \leq n\}, \\ 0 & \text{on } \{(i, j) : 1 \leq i \leq A - 1 \text{ and } i \leq j \leq n\}, \end{cases}
\]
We shall now estimate the term \( S \) using (8), the above can be estimated as
\[
\sum_{i=1}^{n} w_i^A c_i(t) - \sum_{i=1}^{n} w_i^A c_i(0) = -\int_0^t \left( A \sum_{j=A}^{n} \sum_{i=A}^{j} \phi_{i,j} c_i(h) c_j(h) + \sum_{j=1}^{A} \sum_{i=n+1}^{\infty} j \phi_{i,j} c_i(h) c_j(h) \right) dh
\]
\[
-\int_0^t \left( A \sum_{j=A+1}^{n} \sum_{i=n+1}^{\infty} \phi_{i,j} c_i(h) c_j(h) + A c_n(h) \sum_{j=1}^{n} j \phi_{n,j} c_j(h) \right) dh := -\int_0^t (S_1 + S_2 + S_3 + S_4)(h) dh. \tag{61}
\]
Consider the expression \( S_1 \):
\[
A \sum_{j=A}^{n} \sum_{i=A}^{\infty} \phi_{i,j} c_i c_j \leq A \sum_{j=A}^{n} \sum_{i=A}^{j} i j c_i c_j \leq \sum_{j=A}^{n} j^2 c_j \sum_{i=A}^{\infty} i c_i. \tag{62}
\]
Since, \( c \in B^+ \) implies that (62) converges to zero as \( A \to \infty \). Also, by (8), (62) is bounded above by \( \beta ||c_0|| \), the dominated convergence theorem (DCT) implies that, for all \( \varepsilon > 0 \) there exists \( A_0 \) such that, for all \( n > A \geq A_0 \) the absolute value of \( S_1 \) is smaller than \( \varepsilon / 6 \).

We shall now estimate the term \( S_2 \) for \( n > A \),
\[
\sum_{j=1}^{A} \sum_{i=n+1}^{\infty} j \phi_{i,j} c_i c_j \leq \sum_{j=1}^{A} j^2 c_j \sum_{i=n+1}^{\infty} i c_i. \tag{63}
\]
Using (8), the above can be estimated as
\[
\sum_{j=1}^{A} \sum_{i=n+1}^{\infty} j \phi_{i,j} c_i c_j \leq \beta \sum_{i=n+1}^{\infty} i c_i. \tag{64}
\]
Similarly, \( S_3 \) can be expressed as
\[
A \sum_{j=A+1}^{n} \sum_{i=n+1}^{\infty} \phi_{i,j} c_i c_j \leq \sum_{j=A+1}^{n} j^2 c_j \sum_{i=n+1}^{\infty} i c_i \leq \beta \sum_{i=n+1}^{\infty} i c_i. \tag{65}
\]
Thus, by (63) and (64) we conclude that \( S_2 + S_3 \) is bounded by \( 2 \beta ||c_0|| \) and converge pointwise to zero as \( n \to \infty \). Hence, again by DCT we conclude that the absolute value of the integral \( (S_2) + (S_3) \) is smaller than \( \frac{\varepsilon}{3} \forall n > A \geq A_0 \).

Finally, simplifying \( S_4 \) yields
\[
A c_n \sum_{j=1}^{n} j \phi_{n,j} c_j \leq A c_n \sum_{j=1}^{A} j^2 n c_j \leq A \beta n c_n. \tag{66}
\]
For each fixed \( A \), (66) converges to zero as \( n \to \infty \) and it is bounded above by \( A \beta ||c_0|| \). Thus, DCT implies that, for every \( \varepsilon > 0 \), there exists \( A_0 = A_0(\varepsilon) \) such that, for any fixed \( A > A_0 \), there exists \( n_0 = n_0(\varepsilon, A) \) such that, for all \( n > \max\{n_0, A\} \), absolute value of \( S_4 \) is smaller than \( \varepsilon / 6 \).

To estimate (61), observe that, for every \( n > A \), one can write
\[
\left| \sum_{i=1}^{n} w_i^A c_i(t) - \sum_{i=1}^{n} w_i^A c_i(0) \right| \geq \left| \sum_{i=A}^{n} i c_i(t) - \sum_{i=A}^{n} i c_0_i \right| - A \left| \sum_{i=A+1}^{n} c_i(t) - \sum_{i=A+1}^{n} c_0_i \right|
\]
\[
\geq \left| \sum_{i=A}^{n} i c_i(t) - \sum_{i=A}^{n} i c_0_i \right| - i \left| \sum_{i=A+1}^{n} c_i(t) - \sum_{i=A+1}^{n} c_0_i \right|
\]
\[
\geq \left| \sum_{i=A}^{n} i c_i(t) - \sum_{i=A}^{n} i c_0_i \right| - i \left| \sum_{i=A+1}^{n} c_i(t) - \sum_{i=A+1}^{n} c_0_i \right|
\]
and thus,
\[
\left| \sum_{i=1}^{A} ic_i(t) - \sum_{i=1}^{A} ic_0i \right| \leq \sum_{i=A+1}^{\infty} ic_i(t) + \sum_{i=A+1}^{\infty} ic_0i + \sum_{i=1}^{n} w^A c_i(t) - \sum_{i=1}^{n} w^A c_0i. \tag{66}
\]
Now, for every \( \varepsilon > 0 \) there exists \( A_0 \) such that, for all \( A > A_0 \), each of the first two sums in the right-hand side of (66) can be made smaller than \( \frac{\varepsilon}{6} \), and finally having estimates (62-65) allow us to conclude that
\[
\forall \varepsilon > 0, \exists A_0 : \forall A > A_0, \left| \sum_{i=1}^{A} ic_i(t) - \sum_{i=1}^{A} ic_0i \right| < \varepsilon,
\]
which proves the theorem. \( \square \)

4.2 Case 2: Non-Conservative Truncation

Here, density conservation is demonstrated for the non-mass conserving truncation by the following result.

**Theorem 4.** Let \( \phi_{i,j} \leq ij \) for all natural numbers \( i \) and \( j \). Let \( c = (c_i) \in B^+ \) be a solution of the Safronov-Dubovski equation (2) satisfying all the conditions of Theorem 2. Additionally, if (7) holds, then the solution is density conserving satisfying
\[
\sum_{i=1}^{\infty} ic_i(t) = \sum_{i=1}^{\infty} ic_i(0). \tag{67}
\]

**Proof.** To establish the density conservation, let us consider
\[
\left| \sum_{i=1}^{\infty} ic_i(t) - \sum_{i=1}^{\infty} ic_i(0) \right| = \left| \left( \sum_{i=1}^{m} + \sum_{i=m+1}^{\infty} \right) ic_i(t) - \left( \sum_{i=1}^{m} + \sum_{i=m+1}^{\infty} \right) ic_i(0) \right|. \tag{68}
\]
Now, writing the equation (32) as
\[
\sum_{i=1}^{n} ic_i(0) = \sum_{i=1}^{m} ic_i^n(t) + \sum_{i=m+1}^{n} ic_i^n(t) - \sum_{j=1}^{n-1} j\phi_{n,j}c_n^n(t)c_j^n(t). \tag{69}
\]
Using (69) in (68), one can obtain
\[
\left| \sum_{i=1}^{\infty} ic_i(t) - \sum_{i=1}^{\infty} ic_i(0) \right| \leq \sum_{i=1}^{m} |c_i(t) - c_i^n(t)| + \sum_{i=m+1}^{\infty} |c_i(t) + \sum_{i=m+1}^{n} ic_i^n(t) + \sum_{i=n+1}^{\infty} ic_i(0)
\]
\[
+ \left| c_n^n(t) \sum_{j=1}^{n-1} j\phi_{n,j}c_j^n(t) \right|. \tag{70}
\]
Before passing the limit \( n \to \infty \), the boundedness of the last expression in the above equation needs to be dealt and is given as
\[
|c_n^n(t) \sum_{j=1}^{n} j\phi_{n,j}c_j^n(t)| \leq |c_n^n(t) \sum_{j=1}^{n-1} j\phi_{n,j}c_j^n(t)| \leq \alpha ||c^n||,
\]
where \( \alpha \) is defined in (6). Further, having (7), then passing the limit in (70) and using (52) provide
\[
\lim_{n \to \infty} \left| \sum_{i=1}^{\infty} ic_i(t) - \sum_{i=1}^{\infty} ic_i(0) \right| \leq \sup_{i \geq m} \frac{2ic_i^n(t)\gamma_0(i)}{\gamma_0(i)}.
\]
Hence, the first result given in equation (56) and the properties of \( \gamma_0 \) lead to accomplish our claim. \( \square \)
5 Uniqueness

To prove the uniqueness for \( \phi_{i,j} \leq \min\{i^n, j^n\} \) where \( \eta \in [0, 2] \), the following lemma is required.

**Lemma 5.1.** Let \( c_i(t) \) is the solution of (2) such that the equation (5) holds and \( \mu_k(0), k = 1, 2, 3 \) are finite. Then the third moment \( \mu_3(t) \) is bounded for \( \phi_{i,j} \leq ij \) \( \forall i,j \geq 1 \) such that

\[
\mu_3(t) \leq \xi, \quad t \in [0, T],
\]

(71)

where \( \xi \) is a constant depending on \( \beta, \mu_3(0), \mu_1(0) \) and \( T \).

**Proof.** Multiplying the equation (12) by \( i^3 \) on both the sides and taking summation over \( i \) from 1 to \( n \) lead to

\[
\sum_{i=1}^{n} i^3 c_i(t) = \sum_{i=1}^{n} i^3 c_i(0) + \int_0^t \left( \sum_{i=1}^{n} \sum_{j=1}^{i-1} i^3 c_i c_{i-1,j} \right) j \phi_{i,j} c_j - \sum_{i=1}^{n} \sum_{j=1}^{i} i^3 c_i c_{i,j} c_j - \sum_{i=1}^{n} i^3 c_i \sum_{j=i}^{\infty} \phi_{i,j} c_j (h) \right) \, dh.
\]

Substituting \( i-1 \) by \( i' \), further taking \( i' \to i \) in the first term of the above equation and some simplifications yield

\[
\sum_{i=1}^{n} i^3 c_i(t) = \sum_{i=1}^{n} i^3 c_i(0) + \int_0^t \left( \sum_{i=1}^{n} \sum_{j=1}^{\infty} i^3 \phi_{i,j} c_i c_j (h) - (n+1)^3 c_n(h) \sum_{j=1}^{\infty} j \phi_{n,j} c_j (h) \right) \, dh
\]

\[
- \int_0^t \left( \sum_{i=1}^{n} \sum_{j=1}^{i} i^3 \phi_{i,j} c_i c_j (h) \right) \, dh
\]

\[
\leq \sum_{i=1}^{n} i^3 c_i(0) + \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{\infty} (3i^2 + 3i + 1) j \phi_{i,j} c_i c_j (h) \, dh.
\]

Now, putting \( \phi_{i,j} \leq ij \) and using (5) one can obtain for \( t \in [0, T] \)

\[
\sum_{i=1}^{n} i^3 c_i(t) \leq \sum_{i=1}^{n} i^3 c_i(0) + \int_0^t 3\beta \sum_{i=1}^{n} i^3 c_i(h) \, dh + 3\beta^2 T + \beta \mu_1(0) T.
\]

Thus, Gronwall’s lemma concludes that

\[
\sum_{i=1}^{n} i^3 c_i(t) \leq (\mu_3(0) + 3\beta^2 T + \beta \mu_1(0) T) e^{3\beta t}.
\]

The finiteness of \( \mu_3(0) \), \( \beta \) and \( \mu_1(0) \) allows us to pass the limit into the equation and so letting \( n \to \infty \) gives (71). \( \square \)

**Theorem 5.** Let, \( \phi_{i,j} \leq \min\{i^n, j^n\}, \eta \in [0, 2], \forall i, j \geq 1 \). Further, if \( \mu_3(0) \) is finite and the equations (5) and (71) holds, then the solution of (2) is unique in \( R^+ \).

**Proof.** Let us assume that \( c_i(t) \) and \( d_i(t) \) are two solutions of the equation (2) with the same initial condition, i.e.

\[
c_i(0) = d_i(0).
\]

(72)

Now, consider

\[
\mathcal{u}(t) = \sum_{i=1}^{\infty} |c_i(t) - d_i(t)|,
\]

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where \( c_i(t) \) is given by the equation (12) as

\[
u(t) = \sum_{i=1}^{\infty} c_i(0) + \int_0^t \left( \delta_{i \geq 2} c_{i-1}(h) \sum_{j=1}^{i-1} j \phi_{i-1,j} c_j(h) - c_i(h) \sum_{j=1}^{i} j \phi_{i,j} c_j(h) - \sum_{j=i}^{\infty} \phi_{i,j} c_i(h) c_j(h) \right) dh
\]

\[-d_i(0) - \int_0^t \left( \delta_{i \geq 2} d_{i-1}(h) \sum_{j=1}^{i-1} j \phi_{i-1,j} d_j(h) - d_i(h) \sum_{j=1}^{i} j \phi_{i,j} d_j(h) - \sum_{j=i}^{\infty} \phi_{i,j} d_i(h) d_j(h) \right) dh.\]

Using equation (72) and replacing \( i - 1 \) by \( i' \) in the first and fourth sums, the above equation becomes

\[
u(t) \leq 2 \sum_{i=1}^{\infty} \left| \int_0^t \sum_{j=1}^{i} j \phi_{i,j} (c_i(h) c_j(h) - d_i(h) d_j(h)) dh \right| + \sum_{i=1}^{\infty} \left| \int_0^t \sum_{j=1}^{\infty} \phi_{i,j} (d_i(h) d_j(h) - c_i(h) d_j(h)) dh \right|
\]

\[
\leq \int_0^t \sum_{i=1}^{\infty} \sum_{j=1}^{i} j \phi_{i,j} (|d_i(h)| - c_i(h)|d_j(h)| + c_j(h)|d_i(h)| + d_i(h) + c_i(h)|d_j(h)| - c_j(h)|d_i(h)|) dh
\]

\[
+ \int_0^t \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\phi_{i,j}}{2} (|d_i(h)| - c_i(h)|d_j(h)| + c_j(h)|d_i(h)| + d_i(h) + c_i(h)|d_j(h)| - c_j(h)|d_i(h)|) dh.
\]

Now substituting \( \phi_{i,j} \leq \min\{i^n, j^n\} \) yields

\[
u(t) \leq \int_0^t \left( \sum_{i=1}^{\infty} \sum_{j=1}^{i} j^{n+1} |d_i(h)| - c_i(h)|d_j(h)| + c_j(h)|d_i(h)| + \sum_{i=1}^{\infty} \sum_{j=1}^{i} j^{n+1} |d_i(h)| + c_i(h)|d_j(h)| - c_j(h)|d_i(h)| \right) dh
\]

\[
+ \int_0^t \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{j^n}{2} |d_i(h)| - c_i(h)|d_j(h)| + c_j(h)|d_i(h)| + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{j^n}{2} |d_i(h)| + c_i(h)|d_j(h)| - c_j(h)|d_i(h)| \right) dh.
\]

Using the definition of \( u(t) \), Lemma 5.1 and equation (3), it is easy to see that

\[
u(t) \leq (4\xi + 2\beta) \int_0^t u(h) dh.
\]

Since, \( u(0) = 0 \), an application of Gronwall’s lemma validates \( u(t) \equiv 0 \) when \( 0 \leq t \leq T \). Since, \( c_i(t) \) and \( d_i(t) \) are continuous, so \( c_i(t) = d_i(t) \) \( \forall t \) and \( 0 \leq t \leq T \). Since, \( T \) is arbitrary, this completes the proof.

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