Bayesian Transfer Reinforcement Learning with Prior Knowledge Rules

Michalis K. Titsias\(^1\) and Sotirios Nikoloutsopoulos\(^1\)

\(^1\)Athens University of Economics and Business

Abstract

We propose a probabilistic framework to directly insert prior knowledge in reinforcement learning (RL) algorithms by defining the behaviour policy as a Bayesian posterior distribution. Such a posterior combines task-specific information with prior knowledge, thus allowing to achieve transfer learning across tasks. The resulting method is flexible and it can be easily incorporated to any standard off-policy and on-policy algorithms, such as those based on temporal differences and policy gradients. We develop a specific instance of this Bayesian transfer RL framework by expressing prior knowledge as general deterministic rules that can be useful in a large variety of tasks, such as navigation tasks. Also, we elaborate more on recent probabilistic and entropy-regularised RL by developing a novel temporal learning algorithm and show how to combine it with Bayesian transfer RL. Finally, we demonstrate our method for solving mazes and show that significant speed ups can be obtained.

1 Introduction

Humans use prior knowledge to act and learn from sequential decision making tasks. Prior knowledge is important since it allows for efficient solving new tasks with minimal additional learning and computational effort. Such knowledge is gradually built through interaction with the world through a process that can be credited to the ability of the human brain to understand about how the world works and efficiently organise such knowledge for future use. A simple, but highly interpretable, way to describe knowledge is by using a set of rules that have been induced from past experience. These rules can be used to guide human decision making in new tasks by essentially shaping future behaviour so that sample efficient learning is achieved. Such rules could be inter-related, e.g. hierarchically ordered, and probabilistic. For instance, from early childhood someone can pick up the rule "things fall" and also "certain things made of glass when they fall they can break". By knowing already that glass can break, a child playing with a toy made of glass will probably avoid actions that can result in breaking the toy, until the game becomes less interesting and breaking or not breaking the toy are not that different in terms of enjoying the game. The main aspect of this and many similar examples is that past knowledge is used (together with task-specific information such as reward) to shape current behaviour.

Motivated by the above we propose a probabilistic framework, referred to as Bayesian transfer reinforcement learning (RL), to directly insert prior knowledge in RL algorithms by defining the behaviour policy as a Bayesian posterior distribution. Such a posterior combines task-specific information with prior knowledge, thus allowing to achieve transfer learning across tasks. In other words, we perform transfer or meta-learning RL \cite{Duan2016, Wang2016, Stadie2018, Gupta2018, Teh2017} through a Bayesian behaviour policy.
while other aspects of the RL learning algorithm, such as whether we are based on value function estimation or direct policy estimation, can remain largely unchanged. Further, the incorporation of prior knowledge can be arbitrarily complex in the sense that the behaviour policy can be constructed using high-order non-Markovian relationships associated with the history of observed states and actions, memory, deterministic rules etc. In the current implementation we are based on simple deterministic rules that are incorporated into a given task using a procedure called prior realisation, while more advanced methods for more drastically learning such rules is left for future research.

Furthermore, we illustrate our method using a probabilistic formulation of RL [Todorov, 2009 Kappen et al., 2012; Toussaint, 2009; Azar et al., 2011; Rawlik et al., 2013] since it can lead to algorithms, such as entropy-regularised methods [Williams and Peng, 1991; Peters et al., 2010; Schulman et al., 2015; Mnih et al., 2016], that can naturally capture uncertainty. For probabilistic RL we also develop a novel temporal learning algorithm, similar to those in [Haarnoja et al., 2017; Fox et al., 2016; Asadi and Littman, 2017], that for the tabular case approximates a set of linear equations by sequentially updating a certain state-action value function. In the experiments we apply this temporal learning algorithm jointly with Bayesian transfer RL for solving mazes and show that significant speed ups can be obtained.

2 Bayesian transfer reinforcement learning

Consider a RL problem where at each time step $t$ an agent performs an action $\alpha_t \in \mathcal{A}$, observes a new state $s_{t+1} \in \mathcal{S}$ drawn from $p(s_{t+1}|s_t, \alpha_t)$ and receives reward $r_t = r(s_t, \alpha_t)$. The transition density $p(s_{t+1}|s_t, \alpha_t)$ and the reward function $r(s_t, \alpha_t)$ can be unknown or partially known to the agent. By taking actions and observing states the agent wishes to learn a policy that maximises a measure that depends on future reward, such as expected discounted reward or average reward; see [Sutton and Barto, 1998] for full details. The policy according to which actions are taken is the behaviour policy. In off-policy RL the behaviour policy is different from the actual policy that the agent eventually learns, while in on-policy RL the two policies coincide. Next we introduce a probabilistic Bayesian procedure that introduces prior knowledge into the structure of the behaviour policy. We discuss separately the two distinct cases, i.e. off-policy and on-policy learning.

Off-policy RL is usually based on temporal difference methods, such as Q-learning [Watkins, 1989]. Let us assume that the action space $\mathcal{A}$ is discrete and the agent has an estimate of the state-action value function $Q(s_t, \alpha_t)$ and it can update such estimate at each iteration. A behaviour policy based on the current estimate of the $Q$ function typically takes the softmax or Boltzmann form,

$$ \pi(\alpha_t|s_t) = \frac{\exp\{\beta Q(s_t, \alpha_t)\}}{\sum_{\alpha \in \mathcal{A}} \exp\{\beta Q(s_t, \alpha)\}}, $$

where $\beta \geq 0$ is a hyperparameter that specifies the amount of uncertainty; when $\beta = 0$ the above becomes the uniform distribution while for large $\beta \gg 0$ actions will be chosen based on the maximum $Q$ value. The above policy uses only task-specific information, i.e. the state-action value function for the task at hand. To combine this with prior knowledge we can express the following Bayesian behaviour policy,

$$ q(\alpha_t|s_t, \mathcal{M}_t) \propto \exp\{\beta Q(s_t, \alpha_t)\} \times f(\alpha_t; \mathcal{M}_t). \tag{1} $$

Here, the non-negative function $f(\alpha_t; \mathcal{M}_t) \geq 0$ is an unnormalised probability distribution that takes high values for actions $\alpha_t$ that conform well with the current prior knowledge and small
values otherwise. $\mathcal{M}_t$ represents all stored information needed to express prior knowledge and compute values of $f(\cdot)$. For instance, $\mathcal{M}_t$ can include tunable parameters, histories of previous states and actions, reward values etc. In section 2.1 we give an example of how to construct $f(\alpha_t; \mathcal{M}_t)$ using indicator functions and simple deterministic rules. The $\beta$ hyperparameter allows to balance between task-specific information and prior knowledge.

The above Bayesian behaviour policy offers a prior-informed exploration that can be used for off-policy RL, i.e. to learn the state-action value function. However, unlike standard exploration mechanisms such as e-greedy procedures, the policy in (1) may completely ignore uninteresting parts of the state space, i.e. the ones unfavored by the prior $f(\alpha_t; \mathcal{M}_t)$. Clearly, this can be highly desirable (given that these uninteresting parts truly do not contain information about the optimal policy) and arguably it is the only way to achieve sample efficient learning in very large or continuous state spaces.

A second way to use Bayesian behaviour policies is as part of on-policy algorithms. Such schemes are typically used with direct policy optimisation based on algorithms such as REINFORCE [Williams, 1992] and actor-critic methods. To extend our method to cover such cases we can modify the policy from (1) according to

$$q(\alpha_t|s_t, \mathcal{M}_t) \propto p(\alpha_t|s_t) \times f(\alpha_t; \mathcal{M}_t).$$

where $p(\alpha_t|s_t)$ is a policy that aims at capturing task-specific information while $f(\alpha_t; \mathcal{M}_t)$ is the same prior that appears in (1). $q(\alpha_t|s_t, \mathcal{M}_t)$ consists of the overall policy that can be optimised with respect to the parameters of $p(\alpha_t|s_t)$ and possibly of any parameters of the prior $f(\alpha_t; \mathcal{M}_t)$.

In the next section we present a simple example of how to construct $f(\alpha_t; \mathcal{M}_t)$ using indicator functions and simple deterministic rules.

### 2.1 Incorporate prior knowledge using deterministic rules

Prior knowledge in RL means that we know something about how the world works. Specifically, this translates to knowing something about the environmental transition densities $p(s_{t+1}|s_t, \alpha_t)$ and possibly the reward function $r(s_t, \alpha_t)$. We shall focus on general purpose prior knowledge, phrased as simple intuitive rules, that can be useful to a large number of tasks. An example of general purpose prior knowledge is that of knowing that the environmental transition densities $p(s_{t+1}|s_t, \alpha_t)$ are deterministic and/or stationary, which can already be a truly powerful prior information that can lead to practical knowledge rules as the following one:

*In a roughly deterministic and stationary world, past plans that resulted in no progress for solving a task need to be tried out less frequently in the future*

Humans possibly pick up such rule from past experience and particularly by observing that the world is largely deterministic and the rules about how the world works do not unpredictably change

To see an example of how a human applies this rule, suppose a car driver tries to reach a certain destination. In case the driver starts at state/location $s_0$, follows a certain route (consisted of several locations and actions) and returns to the same location $s_0$, he knows that trying again the same route is largely pointless. In the remaining of this section we present a way to implement simple versions of the above rule and incorporate them into the prior $f(\alpha_t; \mathcal{M}_t)$.

Given that a plan has certain length, corresponding to the number of actions comprising the plan, we can define M-order rules that conform with the above general rule and where $M$ corresponds to length. We shall focus on the following 1-order and 2-order rules since they are the simplest ones:

---

1. I.e. the world at the time scale where one solves a certain task is often stationary.
• 1-order rule: Suppose we are at state $s_t = s$, apply action $\alpha_t = \alpha$ and the state remains unchanged, i.e. the next state is $s_{t+1} = s$. Then the plan of taking action $\alpha$ whenever we are at state $s$ should never be tried, unless there is reward for staying at $s$.

• 2-order rule: Suppose we are at state $s_t = s$, apply action $\alpha_t = \alpha$, move to the state $s_{t+1} = s' \neq s$ and by applying a second action $\alpha_{t+1} = \alpha'$ we return to the initial state $s_{t+2} = s$. Then the plan of taking the sequences of actions $(\alpha, \alpha')$ whenever we are at state $s$ should never be tried, unless there is reward for staying at $s$ or $s'$.

Intuitively, the first case above describes situations like trying to walk through a wall or a fly trying to go through a window. The second case describes pairs of undoing or opposite actions such as (left,right) and (up,down). An undoing pair is universal when this holds for any state $s$, which is often the case for many tasks such as navigation tasks. Notice, that two actions in order to really be undoing pairs they must result in returning to the same state after consecutively applying both actions. For instance, the (left,right) pair in a navigation task, such as escaping from a maze, is truly an undoing pair, but this might not hold for other tasks such as when playing an Atari game (where the state of the game such as the location of an object might not be the same after applying a left and then a right move) or in continuous control problems. For these latter cases the above rules might hold in an approximate or probabilistic manner; see Section 5 for further discussion.

We can incorporate the above 1-order and 2-order rules in the prior $f(\alpha_t; M_t)$ using indicator functions (essentially simple if-then-else rules). For any given task, we need to store in memory $M_t$ all cases where the 1-order rule applies and all pairs of undoing actions (assuming that such pairs are universal). Notice that regarding the latter case we can express all indicator functions as a binary matrix $g(\alpha_t; \alpha_{t-1})$ that takes the value zero in any entry where the current action $\alpha_t$ and the previous action $\alpha_{t-1}$ are undoing pairs. The process of gradually building this memory $M_t$ with all these cases is referred to as prior realisation (since it realises our prior knowledge to the specifics of a given task) and it is carried out through actual experience, i.e. as the agent interacts with the environment.

3 Probabilistic reinforcement learning

Bayesian transfer RL could work in conjunction with standard off-policy and on-policy algorithms. However, it can more naturally be used together with algorithms that represent uncertainty when estimate task-specific policies, such soft-$Q$ learning [Haarnoja et al., 2017] and related methods [Fox et al., 2016] [Asadi and Littman, 2017] [Azar et al., 2011] [Rawlik et al., 2013] as well as policy gradient methods with entropy regularization [Williams and Peng, 1991] [Peters et al., 2010] [Schulman et al., 2015] [Mnih et al., 2016]. Therefore here we re-visit the probabilistic RL framework [Todorov, 2009] [Kappen et al., 2012] [Toussaint, 2009] [Azar et al., 2011] [Rawlik et al., 2013] [Levine, 2018] and introduce also a novel temporal learning algorithm. In the experiments we apply this algorithm together with Bayesian transfer RL for solving mazes.

Consider an episodic RL setting, where we start at state $s_0$ and we generate a sequence of states and actions according to the joint distribution

$$p(\alpha_{0:h-1}, s_{1:h}|s_0) = \prod_{t=0}^{h-1} \pi_0(\alpha_t|s_t)p(s_{t+1}|s_t, \alpha_t).$$  \hspace{1cm} (3)$$

The episode ends when we reach a terminal state $s \in T \subset S$. $\pi_0(\alpha_t|s_t)$ is a baseline stochastic policy which could be a very broad distribution, e.g. for discrete action spaces it could be
uniform. We introduce rewards \( r_t = r(s_t, \alpha_t) \) associated with a certain task and assume that \( r_t \) is a deterministic function of the state-action pair \((s_t, \alpha_t)\) (extending to random rewards where \( r_t \sim p(r_t|s_t, \alpha_t) \) is straightforward). The rewards aim at constraining the above joint distribution towards state-action sequences leading to high values of accumulated reward \( \sum_{t=0}^{h-1} r_t \).

To incorporate such constraint into the joint distribution we introduce the exponentiated reward factors \( \exp(\beta r_t) \), where \( \beta > 0 \) is a hyperparameter, and consider the factorisation

\[
f(r_0:h-1, \alpha_0:h-1, s_{1:h}|s_0) = \prod_{t=0}^{h-1} \exp(\beta r_t) \pi_0(\alpha_t|s_t)p(s_{t+1}|s_t, \alpha_t). \tag{4}
\]

This factorization does not define a joint probability distribution since the factors \( \exp(\beta r_t) \) are not distributions but soft constraints that favour high reward values. Several authors [Rawlik et al., 2013; Levine, 2018] interpret each term \( \exp(\beta r_t) \) as the probability \( p(O_t = 1|s_t, \alpha_t) \) of an auxiliary binary variable \( O_t \), so that \( \exp(\beta \sum_{t=0}^{h-1} r_t) \) is precisely the likelihood \( \prod_{t} p(O_t = 1|s_t, \alpha_t) \) of all of these binary variables taking the value one. However, this interpretation is rather artificial and also restrictive since it is valid only when \( r_t \leq 0 \) so that \( \exp(\beta r_t) \in [0, 1] \). Instead, here we assume that \( r_t \) takes arbitrary finite values and view the overall factorization in (4) as a potential function (similarly to undirected graphical models) that allows us to define the following posterior distribution,

\[
p(a_0:h-1, s_{1:h}|s_0, r_{0:h-1}) = \frac{1}{Z} \exp \left( \beta \sum_{t=0}^{h-1} r_t \right) \prod_{t=0}^{h-1} \pi_0(\alpha_t|s_t)p(s_{t+1}|s_t, \alpha_t), \tag{5}
\]

where \( Z \) denotes the normalizing constant. The hyperparameter \( \beta > 0 \) determines the relative strength of the rewards factor \( \exp(\beta \sum_{t=0}^{h-1} r_t) \) versus \( p(a_0:h-1, s_{1:h}|s_0) \).

In order to utilise the posterior distribution in (5) for RL we need to compute the optimal policy that is consistent with the full posterior. At any given time all past states, actions and rewards have been observed and the agent needs to take the current action by marginalising out all possible future sequences that could be possibly realised after taking this action. Given that we are at state \( s_t \) we are interested in computing the marginal posterior distribution over action \( \alpha_t \) conditioning on all rewards \( r_{0:h-1} \) but also on the full history of all past states and actions \((s_{0:t-1}, a_{0:t-1})\),

\[
p(\alpha_t|s_{0:t-1}, a_{0:t-1}, r_{0:h-1}) = p(\alpha_t|s_{t}, r_{t:h-1}). \tag{6}
\]

The simplification in the r.h.s. is because when conditioning on the current state \( s_t \) the action \( \alpha_t \) becomes independent from all previous states, actions and rewards, and it depends only on the future rewards. Of course, this simplification is due to the Markovian nature of the model. From probabilistic inference perspective \( p(\alpha_t|s_t, r_{t:h-1}) \) is the optimal policy. Such policy satisfies a Bellman-type of recursive equation as stated next.

**Proposition 1.** For the posterior in (5) the optimal policy \( p(\alpha_t|s_t, r_{t:h-1}) \) is computed as

\[
p(\alpha_t|s_t, r_{t:h-1}) = \frac{B(s_t, \alpha_t)}{\sum_{\alpha_t} B(s_t, \alpha_t)} = \frac{B(s_t, \alpha_t)}{A(s_t)} \tag{7}
\]

\[
B(s_t, \alpha_t) = e^{\beta r_t} \pi_0(\alpha_t|s_t) \sum_{s_{t+1}} p(s_{t+1}|s_t, \alpha_t) A(s_{t+1}), \quad A(s_{t+1}) = \sum_{\alpha_{t+1}} B(s_{t+1}, \alpha_{t+1}), \quad A(s) = 1 \forall s \in \mathcal{T}.
\]

The proof is given in the Appendix. From the above we can also conclude that \( A(s_t) \) satisfies the recursion \( A(s_t) = \sum_{\alpha_t} e^{\beta r_t} \pi_0(\alpha_t|s_t) \sum_{s_{t+1}} p(s_{t+1}|s_t, \alpha_t) A(s_{t+1}) \). \( B(s_t, \alpha_t) \) can be considered
as a state-action value function while $A(s_t)$ as a state value function. When the environmental dynamics $p(s_{t+1}|s_t, \alpha_t)$ are known (and each $s_t$ and $\alpha_t$ take discrete values) we can compute the $B$ function, and subsequently the optimal policy, by unfolding the recursion backwards or by applying a linear system solver. However for RL, where $p(s_{t+1}|s_t, \alpha_t)$ are unknown, we will need to apply stochastic approximation to learn from actual experience as discussed shortly.

The state-action value function $B(s_t, \alpha_t)$ connects with the optimal $Q$ function in the regular reinforcement learning as shown in the following statement.

**Proposition 2.** Suppose discrete state and action spaces, deterministic transitions such that $p(s'|s, \alpha) = \delta(s' - d(s, \alpha))$ and $\pi_0(\alpha|s) > 0$ for any $s, \alpha$. Then as $\beta \to \infty$, $\frac{1}{\beta} \log B(s, \alpha)$ converges to the state-action value $Q_*(s, \alpha) = r + \max_\alpha \{Q_*(d(s, \alpha), \alpha)\}$ associated with the optimal deterministic policy in regular reinforcement learning.

The proof is given in the Appendix. Clearly also for the deterministic environmental transitions $(1/\beta) \log A(s_t)$ converges to the value function $V(s_t)$. For stochastic environmental transitions the state-action function $B(s_t, \alpha_t)$ will be generally different from $Q(s_t, \alpha_t)$ in regular RL. For instance, observe that

$$(1/\beta) \log B(s_t, \alpha_t) = r_t + (1/\beta) \log \pi_0(\alpha_t|s_t) + (1/\beta) \log \sum_{s_{t+1}} p(s_{t+1}|s_t, \alpha_t) A(s_{t+1}),$$

where the expectation under $p(s_{t+1}|s_t, \alpha_t)$ is inside the log while in the $Q$ function is outside. As discussed in Levine, 2018 this can result in an optimistic policy that is unrealistic in most control problems. To overcome this, we could replace the optimal policy with a variational approximation obtained by imposing the actual transition densities $p(s_{t+1}|s_t, \alpha_t)$ as part of the approximation; see Levine, 2018 and Rawlik et al., 2013 for full details.

Given that the logarithm of $B(s_t, \alpha_t)$ connects with the $Q$ function in regular RL led many authors to derive temporal difference stochastic approximation algorithms, such soft $Q$-learning and $Q$-learning that operate in the log space Haarnoja et al., 2017 Fox et al., 2016 Asadi and Littman, 2017 Azar et al., 2011 Rawlik et al., 2013. However, from probabilistic inference viewpoint another direct way to apply stochastic approximation is to be based on the initial linear recursions of Proposition 1.

More precisely, for discrete states and actions we wish to directly approximate the Bellman equation in Proposition 1 so that to stochastically approximate the state-action value $B(s_t, \alpha_t)$. Notice that in this discrete setting the whole function $B(s, \alpha)$ reduces to a table of size $|S| \times |A|$. For any terminal state $s \in T$ we set $B(s, \alpha) = 1/|A|$ and the remaining values to arbitrary strictly positive values. Then, at each time step $t$ we perform an action based on some behaviour policy and we obtain the following stochastic estimate for the entry $B(s_t, \alpha_t)$,

$$\tilde{B}(s_t, \alpha_t) = e^{\beta r(s_t, \alpha_t)} \pi_0(\alpha_t|s_t) \sum_{\alpha_{t+1}} B(s_{t+1}, \alpha_{t+1}),$$

and then we do a stochastic optimization update

$$B(s_t, \alpha_t) = (1 - \rho_t) B(s_t, \alpha_t) + \rho_t \tilde{B}(s_t, \alpha_t),$$

where $\{\rho_t\}$ is the learning rate sequence satisfying the standard Robins-Monroe conditions and where for each terminal state $s \in T$ the values are fixed to $B(s, \alpha) = 1/|A|$ which ensures that

$^2$For numerical stability the update is performed based on the logsumexp trick by keeping track of the logarithm of $B(s_t, \alpha_t)$. 

6
\[ \sum_{\alpha} B(s, \alpha) = A(s) = 1. \] Repeated application of the update in (8) stochastically approximates a set of linear equations.

If we wish to combine the above temporal learning algorithm with Bayesian transfer RL from the previous section we simply need to consider an off-policy algorithm where the actual experience of the RL agent is collected based on the following behaviour policy

\[ q(\alpha_t | s_t, \mathcal{M}_t) \propto B(s_t, \alpha_t) \times f(\alpha_t; \mathcal{M}_t), \]

where \( B(s_t, \alpha_t) \) is the current estimate of the state-action value, updated according to (8), and \( f(\alpha_t; \mathcal{M}_t) \) is the prior that allows to transfer past knowledge. In the next section we consider the prior that represents the deterministic rules from Section 2.1 and apply the overall scheme for solving mazes.

4 Experiments

Here, we demonstrate the Bayesian transfer RL algorithm for solving mazes implemented using openai gym. We generated 100 random mazes which consist of \( 10 \times 10 \) grids, such as those shown in Figure 1 where the agent starts at blue top-left corner and wishes to reach the red bottom-right corner. We assume that at each state there are four possible actions (up, down, right, left), the environmental dynamics are deterministic and the semantics of the four actions are the same across all tasks. A reward of 1 is given when the agent reaches the goal, while for every step in the maze the agent receives a reward of value \(-0.001\). For all experiments below we fix \( \beta = 1000 \) based on the simple heuristic that a good value is such that \( \beta \approx O(\frac{1}{|r|}) \) where \( r \) is a typical reward value.

We implemented and compared three different methods: (i) The temporal learning algorithm introduced in Section 3 using as behaviour policy \( p(\alpha_t | s_t) \propto B(s_t, \alpha_t) \) which corresponds to an on-policy procedure. We refer to this method as NO-PRIOR since no prior knowledge is used. (ii) The off-policy scheme where the behaviour policy is \( q(\alpha_t | s_t, \mathcal{M}_t) \propto B(s_t, \alpha_t) \times f(\alpha_t; \mathcal{M}_t) \) and where the prior accounts only for the 1-order rule from Section 2.1. We refer to this method as 1-prior. (iii) A scheme similar to (ii) but where the prior accounts for both the 1-order and the 2-order rule from Section 2.1. This third method is referred to as 1&2-prior. For the 1-prior case the prior realisation process is such that the memory \( \mathcal{M}_t \) starts from the empty set and is updated on the fly by inserting state-action pairs \((s, \alpha)\) that result in no change in the state. Subsequently, the actions in \( \mathcal{M}_t \) given that we are in the corresponding state are never taken. For the 1&2-prior case \( \mathcal{M}_t \) also includes the two pairs of opposite actions which are assumed to be universal (see Section 2.1) and they are quickly discovered in the first few moves when solving the first maze, so that this knowledge is transferred through the behaviour policy to all subsequent iterations and different mazes.

Figure 2 shows average performance for all three methods. Clearly, by adding prior knowledge in the behaviour policy learning is speeded up in a systematic way so that 1-prior is better than NO-PRIOR and 1&2-PRIOR is better than 1-PRIOR.

5 Discussion

We proposed a framework to carry out transfer learning in RL through Bayesian behaviour policies that can combine task-specific information with prior knowledge. The resulting method is very general and it can work together with standard off-policy and on-policy RL algorithms, although probabilistic versions of such algorithms are the most suitable due to their natural ability to capture uncertainty.
We showed how to represent prior knowledge using intuitive deterministic rules and demonstrated this for solving mazes. However, for more realistic applications these rules will not hold in a fully deterministic manner because of the uncertainty and high complexity of the real-world environments. Thus, one main direction for future work is to define soft or probabilistic relaxations of the initial deterministic rules so that to deal with real-world applications. For instance, given that the constraints associated with undoing pairs of actions can be represented using a binary matrix \( g(\alpha_t; \alpha_{t-1}) \) (see Section 2.1), by relaxing this and parametrising each entry of this matrix with the sigmoid function we could potentially learn arbitrary transition relationships between consecutive actions. Similarly, it would be useful to investigate whether it is possible to more drastically learn prior rules (or the full structure of the prior \( f(\alpha_t; M_t) \)) based on gradient-based optimisation and deep learning.
A Proofs

We first prove Proposition 1 by showing how to compute $B(s_0, \alpha_0)$ while the general case is similar. Given that we start at state $s_0$ we wish to compute the optimal policy $p(\alpha_0|s_0, r_{0:h-1})$:

$$p(\alpha_0|s_0, r_{0:h-1}) = \frac{B(s_0, \alpha_0)}{\sum_{\alpha_0} B(s_0, \alpha_0)} = \frac{B(s_0, \alpha_0)}{A(s_0)}$$

$B(s_0, \alpha_0)$ is written as

$$B(s_0, \alpha_0) = \sum_{s_1, s_0, \alpha_1} e^{\beta \sum_{t=0}^{h-1} r_t} p(\alpha_0|s_0, s_1, \alpha_1)$$

$$= e^{\beta r_0} \pi_0(\alpha_0|s_0) \sum_{s_1} p(s_1|s_0, \alpha_0) \sum_{\alpha_1} e^{\beta \sum_{t=1}^{h-1} r_t} p(\alpha_1|s_1, \alpha_1)$$

$$= e^{\beta r_0} \pi_0(\alpha_0|s_0) \sum_{s_1} p(s_1|s_0, \alpha_0) \sum_{\alpha_1} B(s_1, \alpha_1)$$

(9)

More generally, we have the recursion

$$B(s_t, \alpha_t) = e^{\beta r_t} \pi_0(\alpha_t|s_t) \left( \sum_{s_{t+1}} p(s_{t+1}|s_t, \alpha_t) \sum_{\alpha_{t+1}} B(s_{t+1}, \alpha_{t+1}) \right)$$

(10)

$$= e^{\beta r_t} \pi_0(\alpha_t|s_t) \sum_{s_{t+1}} p(s_{t+1}|s_t, \alpha_t) A(s_{t+1})$$

(11)

where for any terminal state $s \in \mathcal{T}$ (for which we take no further actions) $B(s, \alpha)$ is such that $\sum_{\alpha} B(s, \alpha) = 1$, which is consistent with the recursion.

We now prove Proposition 2. From (10) by taking logarithms of both sides, dividing by $\beta$ and by using the fact that $p(s'|s, \alpha) = \delta(s' - d(s, \alpha))$ we have that

$$\frac{1}{\beta} \log B(s, \alpha) = r + \frac{1}{\beta} \log \pi_0(\alpha|s) + \frac{1}{\beta} \log \sum_{\alpha} B(d(s, \alpha), \alpha)$$

Now set $M = \max_{\alpha} \{ \log B(d(s, \alpha), \alpha) \}$ and let $\alpha_*$ be the action for which this maximum is attained. The above is written as

$$\frac{1}{\beta} \log B(s, \alpha) = r + \frac{1}{\beta} \log \pi_0(\alpha|s) + \max_{\alpha} \left\{ \frac{1}{\beta} \log B(d(s, \alpha), \alpha) \right\} + \frac{1}{\beta} \log (1 + \sum_{\alpha \neq \alpha_*} e^{\log B(d(s, \alpha), \alpha) - M})$$

It holds that $0 \leq \log (1 + \sum_{\alpha \neq \alpha_*} e^{\log B(d(s, \alpha), \alpha) - M}) \leq \log K$. Thus, by taking the limit $\beta \to \infty$ the terms $\frac{1}{\beta} \log \pi_0(\alpha|s)$ and $\frac{1}{\beta} \log (1 + \sum_{\alpha \neq \alpha_*} e^{\log B(d(s, \alpha), \alpha) - M})$ tend to zero from which we conclude that $\frac{1}{\beta} \log B(s, \alpha) \to Q_*(s, \alpha)$.

References

[Asadi and Littman, 2017] Asadi, K. and Littman, M. L. (2017). An alternative softmax operator for reinforcement learning. In Precup, D. and Teh, Y. W., editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 243–252, International Convention Centre, Sydney, Australia. PMLR.
[Azar et al., 2011] Azar, M. G., Gomez, V., and Kappen, B. (2011). Dynamic policy programming with function approximation. In Gordon, G., Dunson, D., and Dudík, M., editors, Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics, volume 15 of Proceedings of Machine Learning Research, pages 119–127, Fort Lauderdale, FL, USA. PMLR.

[Duan et al., 2016] Duan, Y., Schulman, J., Chen, X., Bartlett, P. L., Sutskever, I., and Abbeel, P. (2016). RL$^2$: Fast reinforcement learning via slow reinforcement learning. CoRR, abs/1611.02779.

[Fox et al., 2016] Fox, R., Pakman, A., and Tishby, N. (2016). Taming the noise in reinforcement learning via soft updates. In Proceedings of the Thirty-Second Conference on Uncertainty in Artificial Intelligence, UAI’16, pages 202–211, Arlington, Virginia, United States. AUAI Press.

[Gupta et al., 2018] Gupta, A., Mendonca, R., Liu, Y., Abbeel, P., and Levine, S. (2018). Meta-reinforcement learning of structured exploration strategies. CoRR, abs/1802.07245.

[Haarnoja et al., 2017] Haarnoja, T., Tang, H., Abbeel, P., and Levine, S. (2017). Reinforcement learning with deep energy-based policies. In Precup, D. and Teh, Y. W., editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 1352–1361, International Convention Centre, Sydney, Australia. PMLR.

[Kappen et al., 2012] Kappen, H. J., Gómez, V., and Oppermann, M. (2012). Optimal control as a graphical model inference problem. Machine Learning, 87(2):159–182.

[Levine, 2018] Levine, S. (2018). Reinforcement learning and control as probabilistic inference: Tutorial and review. CoRR, abs/1805.00909.

[Mnih et al., 2016] Mnih, V., Badia, A. P., Mirza, M., Graves, A., Lillicrap, T., Harley, T., Silver, D., and Kavukcuoglu, K. (2016). Asynchronous methods for deep reinforcement learning. In Balcan, M. F. and Weinberger, K. Q., editors, Proceedings of The 33rd International Conference on Machine Learning, volume 48 of Proceedings of Machine Learning Research, pages 1928–1937, New York, New York, USA. PMLR.

[Peters et al., 2010] Peters, J., Mülling, K., and Altün, Y. (2010). Relative entropy policy search. In Fox, M. and Poole, D., editors, Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence (AAAI 2010), pages 1607–1612. AAAI Press.

[Rawlik et al., 2013] Rawlik, K., Toussaint, M., and Vijayakumar, S. (2013). On stochastic optimal control and reinforcement learning by approximate inference (extended abstract). In Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence, IJCAI ’13, pages 3052–3056. AAAI Press.

[Schulman et al., 2015] Schulman, J., Levine, S., Moritz, P., Jordan, M., and Abbeel, P. (2015). Trust region policy optimization. In Proceedings of the 32Nd International Conference on International Conference on Machine Learning - Volume 37, ICML’15, pages 1889–1897. JMLR.org.

[Stadie et al., 2018] Stadie, B. C., Yang, G., Houthooft, R., Chen, X., Duan, Y., Wu, Y., Abbeel, P., and Sutskever, I. (2018). Some considerations on learning to explore via meta-reinforcement learning. CoRR, abs/1803.01118.

[Sutton and Barto, 1998] Sutton, R. S. and Barto, A. G. (1998). Introduction to Reinforcement Learning. MIT Press, Cambridge, MA, USA, 1st edition.
[Teh et al., 2017] Teh, Y. W., Bapst, V., Czarnecki, W. M., Quan, J., Kirkpatrick, J., Hadsell, R., Heess, N., and Pascanu, R. (2017). Distral: Robust multitask reinforcement learning. CoRR, abs/1707.04175.

[Todorov, 2009] Todorov, E. (2009). Efficient computation of optimal actions. Proceedings of the National Academy of Sciences, 106(28):11478–11483.

[Toussaint, 2009] Toussaint, M. (2009). Robot trajectory optimization using approximate inference. In Proceedings of the 26th Annual International Conference on Machine Learning, ICML ’09, pages 1049–1056, New York, NY, USA. ACM.

[Wang et al., 2016] Wang, J. X., Kurth-Nelson, Z., Tirumala, D., Soyer, H., Leibo, J. Z., Munos, R., Blundell, C., Kumaran, D., and Botvinick, M. (2016). Learning to reinforcement learn. CoRR, abs/1611.05763.

[Watkins, 1989] Watkins, C. J. C. H. (1989). Learning from Delayed Rewards. PhD thesis, King’s College, Oxford.

[Williams, 1992] Williams, R. J. (1992). Simple statistical gradient-following algorithms for connectionist reinforcement learning. Mach. Learn., 8(3-4):229–256.

[Williams and Peng, 1991] Williams, R. J. and Peng, J. (1991). Function optimization using connectionist reinforcement learning algorithms. Connection Science, 3(3):241–268.