Holomorphic Parafermions in the Potts model and SLE

V. Riva$^{a,b}$ and J. Cardy$^{a,c}$

$^a$Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK
$^b$Wolfson College, Oxford
$^c$All Souls College, Oxford

Abstract

We analyse parafermionic operators in the $Q$–state Potts model from three different perspectives. First, we explicitly construct lattice holomorphic observables in the Fortuin-Kasteleyn representation, and point out some special simplifying features of the particular case $Q = 2$ (Ising model). In particular, away from criticality, we find a lattice generalisation of the massive Majorana fermion equation. We also compare the parafermionic scaling dimensions with known results from CFT and Coulomb gas methods in the continuum. Finally, we show that expectation values of these parafermions correspond to local observables of the SLE process which is conjectured to describe the scaling limit of the $Q$–state Potts model.

E-mail addresses: j.cardy1, v.riva1@physics.ox.ac.uk
1 Introduction

Among the special features of two-dimensional physics, one of the most intriguing is semi-locality, which is marked by the appearance of a complex phase in the correlation function of two or more excitations once their positions have been exchanged. The study of this type of excitations, called parafermions and characterized by fractional spin, has found fruitful applications in diverse areas of theoretical physics. Parafermions are rather well understood in critical systems in the continuum limit, since the discrete symmetries they are associated to combine with conformal invariance and permit an extensive classification of the so-called parafermionic Conformal Field Theories. The first of such theories, constructed by Fateev and Zamolodchikov [1], describes the critical point of certain lattice models with $\mathbb{Z}_N$ symmetry, which reduce to the Ising and 3-state Potts models for $N = 2$ and $N = 3$, respectively. The parafermionic fields represent the chiral conserved currents associated to the extra symmetry, the conservation laws being expressed in complex coordinates as a holomorphicity condition on the parafermions. In the Ising case, the parafermionic current is precisely the chiral component of the Majorana fermion that is known to describe the model at criticality and in the neighbouring scaling regime, and the holomorphicity condition is deformed, away from criticality, by terms proportional to the fermion mass.

In this paper, we analyse parafermionic observables in the $Q$–state Potts model, both on the lattice and in the continuum limit. With respect to the literature on the subject, our approach provides a new explicit proof of holomorphicity on the lattice, and the first analysis of its relation to Stochastic Loewner Evolution (SLE) in the continuum limit.

On the lattice, we adopt the Fortuin–Kasteleyen (FK) representation and define the parafermions in terms of the corresponding gas of loops. This allows us to show that, for every value of $Q$, there is a particular value of the spin at which the parafermionic correlation functions are purely holomorphic (in a lattice sense) at criticality. The resulting scaling dimensions agree with CFT and Coulomb gas results in the continuum.

It is conjectured that the continuum limit of the $Q$–state Potts model in the FK representation is described by SLE$_\kappa$ with $\sqrt{Q} = -2 \cos (4\pi/\kappa)$, $4 < \kappa < 8$. We provide further evidence for this correspondence by constructing SLE observables which are holomorphic, have the appropriate scaling dimension and satisfy the same boundary conditions as the lattice parafermions mentioned above. Finally, we explicitly discuss the relation between parafermions and SLE on the lattice in a particular example.

Our work, although carried out independently, has some overlap with that of the programme of Smirnov [2] for proving that the scaling limit of certain curves in lattice $O(n)$ and $Q$-state Potts models is SLE with the appropriate value of $\kappa$ (which has been pushed to completion by Smirnov for the Ising case ($n = 1, Q = 2$)). Our emphasis, however, is on the lattice holomorphicity of the multi-point correlations of the parafermions, and their relations to earlier physics CFT literature, rather than on detailed considerations of the one-point function in an arbitrary domain which are required to prove convergence to SLE. That fact that analogous holomorphic observables
exist in SLE was already pointed out by B. Doyon and the present authors in an earlier paper \[3\].

The paper is organized as follows. In Section \[2\] we briefly review the FK representation of the Potts model. In Section \[3\] we define the parafermionic operators of spin \( p \), and we show that their correlation functions are purely holomorphic at a particular value of \( p \). Special features of the Ising model are discussed in Section \[4\]. Section \[5\] presents the continuum results, first the ones already known from Coulomb gas and CFT methods, then the new prediction obtained by SLE. Finally, in Section \[6\] we discuss the relation between the SLE and lattice results, and we conclude the paper in Section \[7\].

## 2 Q-state Potts model

Let us consider the \( Q \)-state Potts model on the square lattice, at each site of which is associated a spin variable \( S_i \in \{0, 1, \ldots, Q-1\} \). The partition function is

\[
Z = \sum_{\{S\}} e^{P_{i,j}} J_{\delta(S_i, S_j)} = \sum_{\{S\}} \prod_{<i,j>} \left[ 1 + u \delta(S_i, S_j) \right], \quad u = e^J - 1. \tag{2.1}
\]

By expanding the product in (2.1), one obtains the Fortuin–Kasteleyn (FK) representation \[4\]

\[
Z = \sum_G u^b Q^c \tag{2.2}
\]

where \( G \) is any subgraph of the original domain, consisting of all the sites and some bonds placed arbitrarily on the lattice edges, \( b \) is the number of bonds in \( G \), and \( c \) is the number of clusters of connected sites into which the bonds partition the lattice (FK clusters, which also include single sites). Fig. 1 shows a configuration in this expansion, where wired boundary conditions have been assigned (i.e. \( u = \infty \) on the boundary edges). Alternatively, one could assign \( u = 0 \) to the boundary edges, which corresponds to free boundary conditions on the Potts spins.

Although the original formulation of the model requires \( Q \) to be a positive integer, the FK representation (2.2) allows one to interpret \( Q \) as taking arbitrary complex values. The model is known to have a critical point with a diverging correlation length at the self-dual value of \( u \) for \( Q \) real and \( 0 < Q \leq 4 \).

The partition function (2.2) can be also expressed as that of a gas of fully packed loops on the medial lattice (the dashed lines in Fig. 1), whose number is \( d = l + c \) for each given \( G \), where \( l \) is the number of independent loops in \( G \).

By using Euler’s relation

\[
c - l = N - b, \tag{2.3}
\]

where \( N \) is the number of vertices in \( G \) (i.e. the total number of vertices in the lattice), the partition function can be rewritten as

\[
Z = Q^{\frac{N}{2}} \sum_G \left( \frac{u}{\sqrt{Q}} \right)^b \sqrt{Q}^d. \tag{2.4}
\]
By reexpressing (2.4) in terms of the dual lattice and imposing self duality at the critical point, one extracts the critical value

$$u_c = \sqrt{Q},$$

(2.5)

which implies that the weights for the FK clusters at criticality are equivalent to counting each loop on the medial lattice with a fugacity $\sqrt{Q}$.

3 Holomorphic observables on the lattice

In this Section, we will recall the definition of parafermion (or spinor) operators, and we will give it an explicit geometrical meaning in the context of the FK representation. We will then show that, at a particular value of the spin, the parafermions are purely holomorphic observables in a lattice sense.

Let us first briefly recall the definitions of order and disorder operators, which are the basic ingredients in the construction of lattice parafermions (for more details, see \cite{5, 6}). An order (or spin) operator $\sigma_i$ is associated to every site of the original lattice (dots in Fig. 1), taking values in \( \{ e^{2\pi i S_i / Q} \} \), where \( S_i \in \{ 0, 1, ..., Q - 1 \} \). The spin–spin correlation function \( \langle \sigma_i \sigma_j^* \rangle = \langle e^{2\pi i (S_i - S_j) / Q} \rangle \) is proportional to a sum like (2.2) where only those graphs \( \tilde{G} \subset \{ G \} \) appear on which the sites \( i \) and \( j \) belong to the same FK cluster. The contributions of all other terms vanish, since \( S_i \) and \( S_j \) belong to different clusters and are freely summed over. Disorder operators $\mu_k$ can be associated to the sites of the dual lattice (crosses in Fig. 1) by defining their correlation function as \( \langle \mu_k \mu_l^* \rangle = Z'/Z' \), where $Z'$ is obtained from (2.1) by modifying $\delta(S_k, S_l) \rightarrow \delta(S_k + 1 \mod Q, S_l)$ along a path on the dual lattice that connects sites \( k \) and \( l \). It has been shown \cite{5, 6} that this definition of the correlation function does not depend on the particular path chosen, and that the resulting operator $\mu$ is dual to the spin operator $\sigma$. Hence the disorder correlation function only gets contributions from those graphs where the
sites belong to the same cluster on the dual lattice.

If we now look at the covering lattice, which consists of the sites of both the original and the dual lattice, we can define parafermion (or spinor) operators \( \psi_p(e) \) on each of its edges \( e \) as the product of the spin and the disorder operator at the sites connected by \( e \), times a phase factor which encodes \( p \):

\[
\psi_p(e) = \sigma(e) \mu(e) e^{-ip \theta(\gamma,e)}.
\] (3.1)

The meaning of the angle \( \theta(\gamma,e) \) will be clarified below. The existence of several spinor operators, labelled by \( p \), at each value of \( Q \), is due to the fact that the disorder operator can be modified by replacing \( \delta(S_k,S_l) \) with \( \delta(S_k + n \text{ (mod } Q), S_l) \), with \( n = 1, 2, ..., Q \). It was shown in [6] that the different parafermions can be labelled by a fractional index \( p \) which has the physical meaning of spin, since the spinor correlation function acquires a phase factor \( e^{i4\pi p} \) when the spinors circle one another.

In order to give a precise meaning to the angle \( \theta(\gamma,e) \) in (3.1), we now geometrically identify the spinor operators in terms of the fully packed loops in the FK representation of the Potts model (dashed lines in Fig. 1). It follows from (3.1) that spinor correlation functions are observables of such loops, since they only get contributions from terms in (2.2) where all order and all disorder operators belong to the same FK cluster on the original and dual lattice, respectively, hence all edges \( e \) belong to the same loop on the medial lattice. For each term in (2.2) which contributes to a spinor correlation function, we can define the parafermions’ phases as

\[
\arg [\psi_p(e)] = -p \theta(\gamma,e),
\] (3.2)

where \( \gamma \) indicates the loop which crosses all edges involved in the given correlation function, and the angle \( \theta(\gamma,e) \) is the inclination at which \( \gamma \) intersects the edge \( e \). In the cases of interest \( p \) is fractional and (3.2) defines the left hand side only up to a multiple of \( 2\pi p \). However we can give it a unique meaning as follows: for a curve \( \gamma \) which begins and ends on the boundary we choose \( \theta(\gamma,e) \) for \( e \) on the boundary to be in the range \([0, 2\pi]\), and then define it recursively at other points by requiring that between neighbouring edges on the medial lattice it can change only by \( \pm\pi/2 \). With this definition we see that \( \theta(\gamma,e) \) can be arbitrarily large, depending on the degree of winding of \( \gamma \) around the edge \( e \). If \( \gamma \) is a closed loop, we choose a given edge \( e \) at random, require again that \( \theta(\gamma,e) \in [0, 2\pi] \), and define the angles of the other edges recursively as before. Since all observables will turn out to depend only on the differences of the \( \theta(\gamma,e) \) between different edges on the same curve, this arbitrariness is not important.

### 3.1 Lattice Holomorphicity

We now show that, at the critical point, \( \langle \psi_p(e_1)\psi_p(e_2)\cdots\psi_p(e_n) \rangle \) is a “lattice–holomorphic” function of each of its \( n \) variables, i.e. it satisfies a discrete version of the Cauchy–Riemann equations, if we properly choose the value of \( p \). To this purpose, we will prove that, if

\[
\sqrt{Q} = 2 \sin \left(p \frac{\pi}{2}\right),
\] (3.3)
then
\[ \sum_{e \in C} \langle \psi_p(e_1) \cdots \psi_p(e_n) \psi_p(e) \rangle \delta z_e = 0, \quad (3.4) \]

for every closed contour \( C \) of the covering lattice which does not include \( e_1, \ldots, e_n \). This equation has a direct meaning as lattice version of a vanishing contour integral. Furthermore, if we consider \( C \) as the boundary of an elementary plaquette of the covering lattice, (3.4) is equivalent to the discrete Cauchy–Riemann equations, which constitute its real and imaginary part (for a systematic study of discrete Cauchy–Riemann equations see [7]).

**Proof of (3.4).** Let us first consider the case when \( C \) is the boundary of an elementary plaquette of the covering lattice. A loop \( \gamma \), which intersects all edges \( e_1, \ldots, e_n \) and \( e \), can visit the plaquette in eight inequivalent ways, which are depicted in Fig. 2, where \( \gamma \) is represented by the continuous line. The dashed lines represent other loops in the same configuration; their paths outside the plaquette can be different from the picture, but their particular shape does not matter, as we shall see below. To each edge it is associated the value of the inclination at which \( \gamma \) intersects the edge in the considered configuration, up to a multiple of \( 2\pi \), which is, however, the same for every edge in the plaquette which the curve visits.

![Figure 2: Inequivalent configurations on a single plaquette](image)

We will now show that configurations (1a) and (1b) cancel each other in the sum (3.4), and the same can be verified in the other cases. The result automatically holds in all winding classes, since winding simply implies a global rotation of all angles.

---

1. Singularities occur in the correlation function when two of its arguments coincide.
2. For definiteness, we have oriented the loop so the sites of the original lattice, carrying the Potts spins, lie immediately to its left, and the dual sites to its immediate right.
The sum over (1a) and (1b) gives

\[
\sum_{e \in C} \langle \psi_p(e_1) \cdots \psi_p(e_n) \psi_p(e) \rangle \delta z_e \bigg|_{1a,1b} = (3.5)
\]

\[
= \left(1 + i e^{ip \frac{\pi}{2}}\right) P(1a) + \left(1 + i e^{ip \frac{\pi}{2}} + (-1) e^{ip \pi} + (-i) e^{-ip \frac{\pi}{2}}\right) P(1b) ,
\]

where \( P(1a) \) and \( P(1b) \) are the weights of all graphs in the class (1a) or (1b), respectively. From (2.4, 2.5) we know that, at criticality,

\[
P(1a) = \sqrt{Q} P(1b) ,
\]

(3.6)

since the two types of configuration differ from the presence/absence of a closed loop, weighted with a factor \( \sqrt{Q} \) (independently of its particular shape). Eq. (3.6) can be interpreted as an "equation of motion", since it expresses the variation of the partition function under a local rearrangement of configuration. Therefore, (3.5) vanishes for \( \sqrt{Q} = 2 \sin \left( \frac{p \pi}{2} \right) \), i.e. when (3.3) holds. It is easy to check that the same cancellation mechanism takes place for the other pairs of configurations, which proves (3.4) for the single plaquette case.

The validity of (3.5) for an arbitrary closed lattice contour \( C \) follows directly since it is equal to the sum of the contour integrals around each elementary plaquette contained inside \( C \).

\[\Box\]

### 3.2 One–point function

By choosing suitable boundary conditions, it is possible to obtain a non–vanishing expectation value of \( \psi_p \). We will now focus on this situation, in view of its simplicity and its relation to SLE, which will be discussed in Section 6. If we impose wired boundary conditions on one connected component of the boundary and free boundary conditions on its complement (see Fig. 3), a domain wall \( \gamma \) will be generated on the medial lattice (continuous line), which is simultaneously the boundary of the cluster attached to the wired part of the boundary, and of the dual cluster attached to the rest of the boundary. In addition, of course, there will in general be closed loops (dashed lines).

The expectation value \( \langle \psi_p(e) \rangle \) vanishes every time the values of \( \sigma(e) \) and/or \( \mu(e) \) can be summed over freely in the partition function. As a consequence, \( \langle \psi_p(e) \rangle \) is different from zero only on the edges crossed by the domain wall, since in this case both \( \sigma(e) \) and \( \mu(e) \) belong to a cluster connected to the boundary and therefore have a fixed value (wired boundary conditions are equivalent to fixing the value of \( \sigma \) and free boundary conditions to fixing \( \mu \)). Therefore, we can define the phase \( \theta(\gamma,e) \) in (3.2) as the inclination at which the domain wall \( \gamma \) intersects the edge \( e \). It is easy to check that the proof of (3.4) also holds for

\[
\sum_{e \in C} \langle \psi_p(e) \rangle \delta z_e = 0 ,
\]

(3.7)
by simply considering \( \gamma \) as the domain wall described above. Therefore, \( \langle \psi_p(e) \rangle \) is a lattice holomorphic function.

Moreover, we can determine which are the boundary conditions satisfied by \( \langle \psi_p(e) \rangle \). To this purpose, we have to look at the plaquettes of the covering lattice which are cut in half by the boundary edges of the original or dual lattice, and therefore have a triangular shape. The presence of the boundary implies that \( \langle \psi_p(e) \rangle \) only differs by a phase on the two edges of any of these plaquettes, since the domain wall automatically intersects the second edge of a plaquette once it entered the first. Therefore, we can limit our analysis to half of these edges, and we choose for convenience the entering edges \( e_L \) of the plaquettes which lie on the left side of the boundary (with wired boundary conditions), and the exiting edges \( e_R \) of the plaquettes which lie on the right side (with free boundary conditions). Since winding is not possible around these plaquettes, the phase \( \theta(\gamma,e) \) is uniquely determined by the boundary edge under consideration:

\[
\theta(e_L) = \theta_b - \pi , \quad \theta(e_R) = \theta_b ,
\]

where \( \theta_b \) is the inclination of the boundary itself (measured counterclockwise from the starting point of the curve) and we set to zero the value of \( \theta(\gamma,e) \) at the starting edge in Fig. 3 (these arguments can be easily generalized to more general domains than the one in Fig. 3). Therefore, the phase of \( \langle \psi_p(e) \rangle \) on the boundary is fixed as

\[
\arg[\langle \psi_p(e_L) \rangle] = -p(\theta_b - \pi) , \quad \arg[\langle \psi_p(e_R) \rangle] = -p\theta_b .
\]  

This is the generalization to an arbitrary domain of the boundary conditions satisfied by the function \( z^{-p} \) on the upper half plane when the domain wall starts from \( x = 0 \) on the real axis, since in that geometry our convention implies \( \theta_b = 0 \) for \( x > 0 \) and \( \theta_b = 2\pi \) for \( x < 0 \).
4 The Ising model

In this Section, we will analyse in further detail the properties of $\langle \psi_p(e) \rangle$, enlightening some special features of the case $Q = 2$, i.e. the Ising model (for previous studies of the fermion operator in this particular case, see [3, 7, 9]).

4.1 Additional equation

Let us start by noticing that, in general, eq. (3.7) is not sufficient to determine the function $\langle \psi_p(e) \rangle$. The reason is that, on a covering lattice with $N$ sites, we have to determine the complex function $\langle \psi_p(e) \rangle$ on $2^N$ edges, but we only have the single complex equation (3.7) for each of the $O(N)$ plaquettes.

However, in the particular case $Q = 2$ and $p = \frac{1}{2}$ it is possible to see that another complex relation holds for every elementary plaquette. This has the form

$$\langle \psi_{1/2}(e_1) \rangle - \langle \psi_{1/2}(e_2) \rangle + \langle \psi_{1/2}(e_3) \rangle - \langle \psi_{1/2}(e_4) \rangle = 0,$$

where $e_1, ..., e_4$ are the four consecutive edges of the plaquette.

In order to formulate this in a more systematic fashion, let us consider again the eight classes of configuration in Fig. 2. Let us define

$$g_i = \sum_{w=-\infty}^{\infty} \sum_{G \in C_{ib}^{(w)}} G \cdot e^{-ip2\pi w} P(G),$$

(4.1)

where $C_{ib}^{(w)}$ is the set of all graphs in the FK expansion which produce configurations of type $(ib)$ $(i = 1, ..., 4)$, with winding number $w$, and $P(G)$ is the weight of a graph $G$. It follows from (2.4) that for every $G \in C_{ib}$ there is a corresponding $G \in C_{ia}$ such that

$$P(G_{1a}) = uP(G_{1b}), \quad P(G_{2a}) = \frac{Q}{u}P(G_{2b}), \quad P(G_{3a}) = \frac{Q}{u}P(G_{3b}), \quad P(G_{4a}) = uP(G_{4b}).$$

(at criticality all factors reduce to $\sqrt{Q}$). We therefore have

$$\langle \psi_p(e_1) \rangle = \left\{ (u+1)g_1 + \left( \frac{Q}{u} + 1 \right) g_2 + g_3 + g_4 \right\},$$

(4.2)

$$\langle \psi_p(e_2) \rangle = \left\{ (u+1)e^{ip\frac{\pi}{2}} g_1 + e^{ip\frac{\pi}{2}} g_2 + \left( \frac{Q}{u} + 1 \right) e^{ip\frac{\pi}{2}} g_3 + e^{-ip\frac{\pi}{2}} g_4 \right\},$$

$$\langle \psi_p(e_3) \rangle = \left\{ e^{ip\pi} g_1 + e^{-ip\pi} g_2 + \left( \frac{Q}{u} + 1 \right) e^{ip\pi} g_3 + (u+1)e^{-ip\pi} g_4 \right\},$$

$$\langle \psi_p(e_4) \rangle = \left\{ e^{-ip\frac{\pi}{2}} g_1 + \left( \frac{Q}{u} + 1 \right) e^{-ip\frac{\pi}{2}} g_2 + e^{ip\frac{\pi}{2}} g_3 + (u+1)e^{-ip\frac{\pi}{2}} g_4 \right\},$$

which can be written in the matrix notation

$$\langle \psi \rangle = M \cdot g.$$  

(4.3)
It is now easy to check that

$$\operatorname{rank}(M) = \begin{cases} 
4 & \text{for generic } \sqrt{Q} \\
3 & \text{for } \sqrt{Q} = 2 \sin \frac{p\pi}{2} \\
2 & \text{for } Q = 2, \quad p = \frac{1}{2} 
\end{cases} \quad \forall u \, . \quad (4.4)$$

Thus in general, if we choose $p$ correctly, there is one linear relation between the parafermions around the 4 edges of each plaquette, while for $Q = 2$ there are two.

### 4.2 Continuum limit of the lattice equations

In this section we make some remarks on the question of the extent to which our results on lattice holomorphicity imply that in the continuum limit the parafermion correlators are truly holomorphic functions of their arguments. In general this does not follow without further assumptions. Recall Morera’s theorem\cite{8} which states that if $\oint_C f(z)\,dz$ vanishes for every closed contour $C$ in some domain $D$, and $f(z)$ is continuous, then $f(z)$ is a complex analytic function in $D$. Although our lattice result shows that there is a subsequence of approximations to $\oint_C f(z)\,dz$ which do vanish identically, this does not imply that the limit vanishes without further smoothness assumptions. Although we expect these to be valid, it would require further work to show that they follow from the lattice model. Nevertheless, if true, this implies that in, for example, the upper half plane when the curve runs from 0 to infinity, $\langle \psi_p(z) \rangle \to \text{const} \cdot z^{-p}$.

In the case of the Ising model, $Q = 2$, the situation is better. In that case there are sufficient equations to determine the lattice function $\langle \psi(e) \rangle$. Since they are linear and translationally invariant, it should be a simple matter to solve then by Fourier analysis in a sufficiently simple domain, and then to show that, in the limit when the size of the domain is much larger than the lattice spacing, the Fourier components near $k = 0$ dominate, and in the limit, are given by the solution of the continuum problem. However, this approach is not adequate for more irregular domains.

### 4.3 Off–criticality

As soon as we move away from the critical point, eq. (3.7) does not hold anymore, since the necessary cancellation mechanism relies on the particular value $u_c = \sqrt{Q}$. However, linear relations between the values of $\langle \psi_p \rangle$ at different edges are still present if $\sqrt{Q} = 2 \sin \frac{p\pi}{2}$, since (4.4) holds for any value of $u$. For instance, if we define $v = u - \sqrt{Q}$ we have

$$\langle \psi_p(e_1) \rangle + i\langle \psi_p(e_2) \rangle - \langle \psi_p(e_3) \rangle - i\langle \psi_p(e_4) \rangle = X \left( \langle \psi_p(e_3) \rangle - \langle \psi_p(e_1) \rangle \right), \quad (4.5)$$

where

$$X = v \frac{1 + i e^{ip\pi/2}}{e^{ip\pi/2} - 1 - 2 \sin \frac{p\pi}{2} - v}.$$ 

We will now give a precise meaning to this off–critical relation in the case $Q = 2$. It is well known that the Ising model can be described, in the scaling regime near its phase transition, by
a field theory of a free Majorana fermion of chiral components $\chi, \bar{\chi}$, with the action

$$S = \frac{1}{2\pi} \int d^2z \left( \chi \partial_z \chi + \bar{\chi} \partial\bar{z} \bar{\chi} + im\bar{\chi}\chi \right), \quad (4.6)$$

where the parameter $m$ measures the deviation from the critical temperature. In the above notation, $m$ is positive for $T < T_c$ and negative for $T > T_c$. The absolute value $|m|$ is the mass of the elementary excitation in both regimes. The action (4.6) implies the equations of motion

$$\partial_z \chi = \frac{m}{2} \bar{\chi}, \quad \partial_{\bar{z}} \bar{\chi} = -\frac{m}{2} \chi. \quad (4.7)$$

The left hand side of (4.5) is the discrete version of $2i\partial_z \langle \psi_p \rangle$, therefore we have to reexpress the right hand side in terms of $\langle \bar{\psi}_p \rangle$ in order to reproduce a discrete version of (4.7). This is possible only in the case $Q = 2$, because at the value $p = \frac{1}{2}$ the functions $g_i$ defined in (4.1) coincide with their complex conjugates $\bar{g}_i$. In more general cases, the right hand side of (4.5) cannot be just written in terms of $\langle \bar{\psi}_p \rangle$, and this is consistent with the fact that no simple equation like (4.7) holds in the corresponding field theory descriptions. Going back to the Ising case, we have

$$\langle \psi_{1/2}(e_1) \rangle + i \langle \psi_{1/2}(e_2) \rangle - \langle \psi_{1/2}(e_3) \rangle - i \langle \psi_{1/2}(e_4) \rangle =$$

$$= \frac{v}{(1 + \sqrt{2})(2\sqrt{2} + v)} \left( \langle \bar{\psi}_{1/2}(e_1) \rangle + \langle \bar{\psi}_{1/2}(e_2) \rangle + \langle \bar{\psi}_{1/2}(e_3) \rangle + \langle \bar{\psi}_{1/2}(e_4) \rangle \right),$$

which is the discrete version of

$$2i\partial_z \langle \psi_{1/2} \rangle = \frac{v}{(1 + \sqrt{2})(2\sqrt{2} + v)} 4\langle \bar{\psi}_{1/2} \rangle$$

where the antiholomorphic operator in the right hand side is uniformly distributed on the plaquette. This relation and its complex conjugate are equivalent to (4.7) if the fields are related by

$$\chi = \alpha \psi_{1/2}, \quad \bar{\chi} = \beta \bar{\psi}_{1/2}, \quad \text{with} \quad \frac{\alpha}{\beta} = i$$

and the parameter $m$ is given by

$$m = \frac{4v}{(1 + \sqrt{2})(2\sqrt{2} + v)} = \frac{e^J - 1 - \sqrt{2}}{(1 + \sqrt{2})(\sqrt{2} - 1 + e^J)} \simeq \sqrt{2} (J - J_c) + O(J - J_c)^2. \quad (4.8)$$

We can now compare this result with the known value of the correlation length $\xi$ on the square lattice, expressed in terms of the modular parameter $k = (\sinh J)^{-2}$ [10] and evaluated in the scaling limit:

$$\frac{1}{\xi} = \begin{cases} -\log k & \simeq 2\sqrt{2} (J - J_c) + O(J - J_c)^2 \quad \text{for} \quad T < T_c \\ \frac{1}{2} \log k & \simeq \sqrt{2} (J_c - J) + O(J_c - J)^2 \quad \text{for} \quad T > T_c \end{cases} \quad (4.9)$$

From the definition of the correlation length in terms of the connected two–point function of the order operator $\sigma$

$$\langle \sigma(x)\sigma(0) \rangle \sim x^{-\gamma} e^{-x/\xi} \quad \text{as} \quad x \to \infty,$$
it follows that $1/\xi$ must be equal to the mass of the lightest excitation which couples to $\sigma$. The $Z_2$ symmetry of the Ising model implies that $\sigma$ only couples to states with odd numbers of particles at high temperature and with even number of particles at low temperature. Hence (4.8) and (4.9) are in perfect agreement.

5 Continuum predictions

In this Section, we will show that, despite difficulties in proving convergence to the continuum limit, the lattice holomorphic objects defined above nevertheless have their counterparts in the continuum formulations of these models. In particular, we will review results obtained by CFT and Coulomb gas methods, and we will describe a new prediction from SLE.

5.1 CFT and Coulomb gas results

From the CFT point of view, the holomorphicity condition (3.3) has a clear meaning both for $Q = 2$ and $Q = 3$, where the values $p = \frac{1}{2}$ and $p = \frac{2}{3}$ coincide with the spins of the parafermionic currents predicted by Fateev and Zamolodchikov [1] for CFT with extended $Z_2$ and $Z_3$ symmetry, respectively.

A more general result was obtained in [6] for the Potts model by using Coulomb gas methods. Nienhuis and Knops showed that for every $Q$, there is a series of spinor operators parameterized by a rational number $p < 1$, whose denominator is the order of one of the cyclic permutations into which the permutations of $\{0, 1, ..., Q - 1\}$ can be decomposed. For instance, at $Q = 2$ we only have $p = 1/2$, while at $Q = 3$ we have $p = 1/2, 1/3$ or $2/3$. The scaling dimension $x_p$ and spin $s_p$ of the spinors are given by

$$x_p = 1 + \frac{1}{2 - y} (p^2 - 1), \quad s_p = p,$$

where $y$ is defined through

$$\sqrt{Q} = 2 \cos \left( \frac{\pi y}{2} \right).$$

For every $Q$, there is a single holomorphic operator among these, characterized by $x_{p^*} = s_{p^*}$, i.e.

$$p^* = 1 - y.$$

This result is in perfect agreement with our requirement (5.3) for the holomorphicity of the parafermionic correlation functions on the lattice.

5.2 The SLE prediction

In this Section, we will show that some local observables associated to an SLE curve are characterized by the scaling dimension of the spinor operators discussed above. To this purpose, we will recall and generalize results obtained in [3] for the analysis of another operator, the stress–energy tensor.
Let us consider a chordal SLE$_\kappa$ process (for $0 < \kappa < 8$) on the upper half plane described by complex coordinates $w, \bar{w}$ \[\text{[11]}. The probability $P(w_1, w_2, \bar{w}_1, \bar{w}_2)$ that the curve passes between two points $w_1, w_2$ (or to the left or right of both) satisfies the equation
\[
\left\{ \frac{\kappa}{2} (\frac{\partial w_1 + \partial \bar{w}_1 + \partial w_2 + \partial \bar{w}_2}{\bar{w}_1} + \frac{\partial w_1}{\bar{w}_1} + \frac{\partial \bar{w}_1}{\bar{w}_1} + \frac{\partial w_2}{\bar{w}_2} + \frac{\partial \bar{w}_2}{\bar{w}_2}) \right\} P(w_1, w_2, w_1, \bar{w}_2) = 0 .
\] (5.4)

If we parameterize the event by the middle point $w$ of a straight segment, by its length $\epsilon$ and by the angle $\theta$ that it makes with the positive imaginary direction:
\[
w_1 = w - \frac{\epsilon}{2} e^{i\theta} , \quad w_2 = w + \frac{\epsilon}{2} e^{i\theta} ,
\]
we obtain, at leading order in $\epsilon$, the following equation
\[
\left\{ \frac{\kappa}{2} \left( \frac{\partial w + \partial \bar{w}}{w} \right)^2 + \frac{2}{w} \partial w + \frac{2}{\bar{w}} \partial \bar{w} - \left( \frac{1}{w^2} + \frac{1}{\bar{w}^2} \right) \epsilon \partial \epsilon + \left( \frac{1}{w^2} - \frac{1}{\bar{w}^2} \right) i \partial \theta + O(\epsilon^2) \right\} P(w, \bar{w}, \epsilon, \theta) = 0 .
\] (5.5)

In the above expansion we have assumed that each of the Fourier modes of the probability, defined as
\[
Q_p(w, \bar{w}, \epsilon) = \int_0^\infty d\theta e^{-ip\theta} P(w, \bar{w}, \epsilon, \theta) ,
\] (5.6)
vanishes with a power law $\epsilon^{-p}$ as $\epsilon \to 0$, hence $\partial_\epsilon = O(\epsilon^{-1})$. The parameter $p$ has the physical meaning of "spin" and took the value $p = 2$ in the case of the stress-energy tensor studied in \[\text{[3]}\].

Here we are interested in more general values of $p$, in particular non-integer values. Therefore, the domain of integration for the variable $\theta$ in (5.6) is not restricted between 0 and $2\pi$, since the winding of the SLE curve around the segment is associated to a shift $\theta + 2\pi n$, where $n \in \mathbb{Z}$ is the winding number. If $p$ is integer this phenomenon is irrelevant, but if $p < 1$ winding has to be taken into account. In particular, there are $d$ inequivalent winding classes, where $d$ is the denominator of $p$. We can define the probability that the SLE curve passes between the ending points of the segment with a winding number $n = k \pmod{d}$ as
\[
P_k(w, \bar{w}, \epsilon, \theta) = \sum_{m=-\infty}^{\infty} P(w, \bar{w}, \epsilon, \theta + 2\pi k + 2\pi md) .
\]

In this notation, the Fourier modes can be expressed as
\[
Q_p(w, \bar{w}, \epsilon) = \sum_{k=0}^{d-1} e^{-ip2\pi k} \int_0^{2\pi} d\theta e^{-ip\theta} P_k(w, \bar{w}, \epsilon, \theta) .
\] (5.7)

By integrating eq. (5.5) over $\int_0^{2\pi} d\theta e^{-ip\theta}$ we obtain, to leading order in $\epsilon$,
\[
\left\{ \frac{\kappa}{2} \left( \frac{\partial w + \partial \bar{w}}{w} \right)^2 + \frac{2}{w} \partial w + \frac{2}{\bar{w}} \partial \bar{w} - \left( \frac{1}{w^2} + \frac{1}{\bar{w}^2} \right) \epsilon \partial \epsilon - p \left( \frac{1}{w^2} - \frac{1}{\bar{w}^2} \right) \right\} Q_p(w, \bar{w}, \epsilon) + \text{corrections} = 0 ,
\] (5.8)
Among the several solutions of \( (5.8) \), we are now interested in the one which satisfies appropriate boundary conditions for the event of passing in between the two initial points. This particular solution was obtained in \[3\] by mapping the problem into the disk geometry; for completeness, we report the detailed discussion in Appendix A. The solution is

\[
Q_p(w, \bar{w}, \epsilon) = c_p \epsilon^{x_p} w^{\alpha_p} \bar{w}^{\beta_p} (w - \bar{w})^{\gamma_p}, \tag{5.9}
\]

with

\[
\alpha_p = \frac{\kappa - 8}{2\kappa} - \frac{p}{2}, \quad \beta_p = \frac{\kappa - 8}{2\kappa} + \frac{p}{2}, \quad \gamma_p = \frac{(8 - \kappa)^2 - \kappa^2 p^2}{8\kappa}, \quad x_p = 1 - \frac{\kappa}{8} + \frac{\kappa}{8} p^2. \tag{5.10}
\]

The function \( (5.9) \) gives the correct leading behaviour for small \( \epsilon \) of \( Q_p(w, \bar{w}, \epsilon) \) only when the higher order terms in \( (5.5) \) do not mix the Fourier components at leading order. This is guaranteed if

\[
x_p < 2m + x_{p-2m} \quad \forall \ m = 1, 2, 3, \ldots \text{ such that } c_{p-2m} \neq 0, \tag{5.11}
\]

where \( m \) labels the powers of \( \epsilon^2 \) in the expansion of the differential equation \( (5.4) \). This automatically holds for any \( 0 \leq p \leq 1 \), which is the case of interest in this paper.

At the particular value

\[
\kappa_p = \frac{8}{p + 1}, \tag{5.12}
\]

\( (5.9) \) simplifies to the purely holomorphic function

\[
Q_p(w, \bar{w}, \epsilon) = \text{const} \times \left( \frac{\epsilon}{\bar{w}} \right)^p, \tag{5.13}
\]

where spin and scaling dimension are equal.

In order to compare the SLE prediction with the other results described in this paper, let us recall that the \( Q \)-states Potts model in the FK representation is conjectured \[12\] to be described, in the continuum limit, by \( \text{SLE}_\kappa \) where

\[
\sqrt{Q} = -2 \cos \left( \frac{\pi}{4} \right), \quad 4 < \kappa < 8. \tag{5.14}
\]

It is then easy to check that \( (5.10) \) perfectly agrees with the Coulomb gas results \( (5.1) \) for every \( 0 < p < 1 \), and in particular at the holomorphic value \( (5.12) \).

The holomorphic solution \( (5.13) \) has a natural CFT interpretation as the expectation value of a purely holomorphic operator of spin \( p \), given the appropriate boundary conditions which generate the domain wall. By recalling the relation between \( \kappa \) and the central charge \( c \) of CFT \[13\]

\[
c(\kappa) = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}, \tag{5.15}
\]

one can see that \( (5.12) \) implies

\[
p = \begin{cases} 
  h_{3,1} & \text{for } 0 < \kappa < 4 \\
  h_{1,3} & \text{for } 4 < \kappa < 8 
\end{cases}, \tag{5.16}
\]
where \( h_{i,j} \) are the conformal weights of primary operators in the Kac table of CFT. For Ising and 3–state Potts models, this is again in agreement with the other results presented in this paper. Relation (5.16), however, is more general and deserves further study in connection with other statistical models.

6 Relation between SLE and lattice results

In the last Section, we have checked that the SLE prediction agrees with other continuum limit results. Here, we will discuss the explicit relation between the purely holomorphic function (5.13) obtained from SLE and the lattice spinor \( \psi_p \) defined in (3.2).

To this purpose, we have to look at the one–point function \( \langle \psi_p(e) \rangle \), which is an observable of the domain wall that is conjectured to converge to SLE in the continuum limit. We have shown in Sect.3.2 that \( \langle \psi_p(e) \rangle \) is purely holomorphic when \( \sqrt{Q} = 2 \sin \frac{p \pi}{2} \), i.e. when \( \kappa = \frac{8}{p+1} \) as in (5.12). Moreover, we know that \( \langle \psi_p(e) \rangle \) satisfies the same boundary conditions obeyed by \( z^{-p} \) in the continuum limit, hence we expect

\[
Q_p(w, \bar{w}, \epsilon) = \text{const} \times \epsilon^p \langle \psi_p(w) \rangle^\text{cont},
\]

where \( \langle \psi_p(w) \rangle^\text{cont} \) is the function to which \( \langle \psi_p(e) \rangle \) is supposed to converge in the continuum limit.\(^3\)

The advantage of SLE, with respect to other continuum limit approaches, is to have a direct and explicit meaning in terms of the lattice formulation of the problem. This allows us to analyse the correspondence between \( Q_p(w, \bar{w}, \epsilon) \) and \( \langle \psi_p(e) \rangle \) on the lattice itself, by defining a discretized version \( Q_p^\text{lat} \) of \( Q_p \) where the SLE probability distributions is replaced by the lattice weights in the FK representation of the Potts model. In principle, one should check the relation for segments connecting two arbitrary points \( w_1, w_2 \) on the lattice. As a matter of fact, the computation becomes very complicated as soon as several plaquettes are involved, and we have been able to carry it out explicitly only for the case of a single plaquette. This corresponds to the rotating segment shown in Fig.4.

\(^3\)For \( p \neq 1/2 \), we need a continuity assumption to justify this statement, as discussed in Sect 4.
Let us first obtain the lattice version of $Q_p$. The range of integration in (5.7) can be further reduced to $\theta \in [0, \pi]$ by noticing that, in every winding sector, the curve can pass between the end points of the segment by leaving point $w_1$ to its left and $w_2$ to its right, or vice versa, and the two events are related by the transformation $\theta \to \theta + \pi$:

$$Q_p(w, \bar{w}, \epsilon) = \sum_k e^{-ip2\pi k} \int_0^\pi d\theta e^{-ip\theta} \left[ P_{k,1L,2R}(w, \bar{w}, \epsilon, \theta) + e^{-ip\pi} P_{k,1R,2L}(w, \bar{w}, \epsilon, \theta + \pi) \right].$$

This implies that we only have to analyse cases (1) and (2) in Fig. 4. The discretisation of the integral in $\theta$ gives

$$Q_p^{\text{lat}} = \frac{\pi}{2} \sum_k e^{-ip2\pi k} \left\{ e^{-ip\frac{\pi}{4}} \left[ P_{k,1L,2R}^{(1)} + e^{-ip\pi} P_{k,1R,2L}^{(1)} \right] + e^{-ip\frac{3\pi}{4}} \left[ P_{k,1L,2R}^{(2)} + e^{-ip\pi} P_{k,1R,2L}^{(2)} \right] \right\} =$$

$$= \frac{\pi}{2} \left\{ e^{-ip\frac{\pi}{4}} \left[ \sqrt{Q} g_2 + e^{-ip\pi} e^{ip2\pi} \sqrt{Q} g_3 \right] + e^{-ip\frac{3\pi}{4}} \left[ \sqrt{Q} g_4 + e^{-ip\pi} e^{ip2\pi} \sqrt{Q} g_1 \right] \right\}$$

(6.2)

where the functions $g_i$ are defined in (4.1), and their relation to the probabilities of passing between the end points of the segment can be inferred from Fig. 2. The factors $e^{ip2\pi}$ in the last line are due to the fact that the domain wall in configurations (1a) and (3a) of Fig. 2 passes between the end points of the segment with winding $-1$.

We expect from (6.1) that (6.2) is proportional to $\langle \psi_p(w) \rangle$ evaluated at the middle point $w$ of the segment, i.e. at the center of the plaquette. Assuming the spinor operator to be uniformly distributed on the plaquette, we have

$$\langle \psi_p(w) \rangle = \frac{1}{4} \left( \langle \psi_p(e_1) \rangle + \langle \psi_p(e_2) \rangle + \langle \psi_p(e_3) \rangle + \langle \psi_p(e_4) \rangle \right) = \frac{1}{4} \sum_{i=1}^4 S_i g_i,$$

where the coefficients $S_i$ can be easily obtained from (4.12) at $u = \sqrt{Q}$. It is now easy to check that

$$Q_p^{\text{lat}} = 2\pi \frac{\sqrt{Q} \ \text{sec} \ p \frac{\pi}{4}}{2 \left( \sqrt{Q} + 2 \cos p \frac{\pi}{2} \right)} \langle \psi_p(w) \rangle,$$

(6.3)

which confirms (6.1).

It is worth noting that (6.3) is valid for any value of $Q$, not only for the purely holomorphic case $\sqrt{Q} = 2 \sin p \frac{\pi}{2}$. This agrees with the general SLE prediction that, for every $Q$, the several spinor operators obtained by Coulomb gas methods in [6] are local observables of the domain wall.

Let us conclude this Section by analysing the slightly different situation in which $Q_p$ is defined as the Fourier component of the probability that the domain wall *intersects* the segment without necessarily passing through its end points. Additional configurations have to be taken into account (see Fig. 2), and (6.2) is modified as

$$Q_p^{\text{lat}} = Q_p^{\text{lat}} + \frac{\pi}{2} \left\{ e^{-ip\frac{\pi}{4}} \left[ g_1 + e^{-ip\pi} g_4 \right] + e^{-ip\frac{3\pi}{4}} \left[ g_2 + e^{-ip\pi} e^{ip2\pi} g_3 \right] \right\},$$

(6.4)

which still implies a proportionality with $\langle \psi_p(w) \rangle$:

$$Q_p^{\text{lat}} = 2\pi - \frac{\left( \sqrt{Q} + e^{-ip\frac{\pi}{4}} \right) \ \text{sec} \ p \frac{\pi}{4}}{2 \left( \sqrt{Q} + 2 \cos p \frac{\pi}{2} \right)} \langle \psi_p(w) \rangle.$$

(6.5)
This is consistent with the SLE result, because the boundary conditions that select solution (5.9) are the same if we consider the events of intersecting the segment or passing between its end points (in the disk geometry defined in Appendix A, both events require the curve to pass by the center of the disk). Therefore, if we assume that the distortion of the segment under the Loewner conformal map only introduces terms of higher order in $\epsilon$ in (5.5) and (5.8), $Q_p$ and $Q_p'$ are expected to only differ by a constant in the continuum limit, and this is nicely confirmed by the lattice example analysed here.

7 Conclusions

In this paper, we have carried out a detailed study of parafermionic operators in the $Q$-state Potts model, both on the lattice and in the continuum limit.

Our discrete setting was a planar square lattice. By geometrically defining the parafermions in terms of the FK representation of the model, we obtained an explicit proof of lattice holomorphicity for certain values of the spin. We believe that this is first time that these parafermions have been identified within the FK representation of the Ising and Potts models, and that their holomorphicity has been directly demonstrated at this level. It would be interesting to extend our analysis to different lattices and geometries.

After reviewing known results on the scaling dimensions of parafermions from CFT and Coulomb gas methods in the continuum limit, we have shown that the same scaling dimensions are associated to local observables of SLE$_\kappa$ when (5.14) holds, i.e. at the value of $\kappa$ which is conjectured to describe the $Q$-state Potts model in the FK representation. Moreover, we have explicitly discussed the relation between the SLE observable and the lattice parafermions, both in the discrete and continuum settings.

Natural extensions of this work are the full analysis of the situation when the parafermionic operators are not purely holomorphic, and the study of the SLE prediction in the context of other statistical systems. In particular, it will be interesting to exploit the information provided by SLE for models associated with non–unitary or non–minimal CFT.

Acknowledgments

After we had established the lattice holomorphicity equations of this paper we learned that they had also been derived (together with similar equations for the O($n$) model on a hexagonal lattice) by S. Smirnov in the course of his programme to show convergence of lattice curves to SLE [2]. One of us (JLC) thanks him for enlightening discussions on this subject.

We also thank B. Doyon and A. Sportiello for useful discussions. This work was supported by EPSRC under the grant GR/R83712/01. It was completed while JLC was a visitor to the Kavli Institute for Theoretical Physics, Santa Barbara, supported by the NSF under Grant No. PHY99-07949.
A SLE probabilities in the disk geometry

The ansatz

\[ Q_\rho(w, \bar{w}, \epsilon) = c_\rho \epsilon^{2p} w^{\alpha_p} \bar{w}^{\beta_p} (w - \bar{w})^{\gamma_p} \]  

(A.1)
solves eq. (5.8) for two different choices of the parameters:

\[ \alpha_p = -\frac{2p}{\kappa - 4}, \quad \beta_p = \frac{2p}{\kappa - 4}, \quad \gamma_p = -\frac{2\kappa p^2}{(\kappa - 4)^2}, \quad x_p = \frac{2\kappa p^2}{(\kappa - 4)^2} \]  

(A.2)

In order to select the correct set of parameters, it is convenient to map our problem onto the unit disk \( \mathbb{D} \), through the transformation \( z' = \frac{z - \bar{w}}{w - \bar{w}} \) for \( z \in \mathbb{H} \) and \( z' \in \mathbb{D} \). This transformation maps the point \( w \) to the center of the disk, the length \( \epsilon \) to \( \epsilon/|w - \bar{w}| \), and it shifts the angle \( \theta \) by an angle of \( \pi/2 \). Also, the point 0 is mapped to \( w/\bar{w} \) on the boundary of the disk, and the point \( \infty \) to 1. We are then describing an SLE curve on the unit disk started at \( w/\bar{w} \) and required to end at 1. Fixing the power of \( \epsilon/|w - \bar{w}| \) to be some number \( x_p \) (the “scaling dimension”), we are left, after integration over \( \theta \) as in (5.7), with a second order ordinary differential equation in the angle \( \alpha = \arg(w/\bar{w}) \in [0, 2\pi) \). This equation is the eigenvalue equation for an eigenfunction of the two-particle Calogero-Sutherland Hamiltonian with eigenvalue (energy) \( 2x_p/\kappa \) and with total momentum \( p \) [14]. For generic \( \kappa \), the Calogero-Sutherland Hamiltonian admits only two types of series expansions \( C^\omega [[\alpha^2]] \) (with \( C \neq 0 \)) as \( \alpha \to 0^\pm \) for its eigenfunctions: one with a leading power \( \omega = 8/\kappa - 1 \), the other with a leading power \( \omega = 0 \). It admits the same two types of series expansions \( C'(2\pi - \alpha)^\omega [[(2\pi - \alpha)^2]] \) (with \( C' \neq 0 \)) as \( \alpha \to 2\pi^- \). Allowing only one type of series expansion at 0 and only one at \( 2\pi \) (the possibilities give the Calogero-Sutherland system in the fermionic sector \( \omega = \omega' = 8/\kappa - 1 \), bosonic sector \( \omega = \omega' = 0 \) or mixed sector, \( \omega \neq \omega' \)), the Calogero-Sutherland Hamiltonian has a discrete set of eigenfunctions, with eigenvalues bounded from below (since it is a self-adjoint operator on the space of functions with these asymptotic conditions). The lowest eigenvalue is obtained for the eigenfunction (the ground state) with the least number of nodes (zeros of the eigenfunction). If the leading powers \( \omega \) and \( \omega' \) are chosen equal to each other, then the ground state (in the sector with total momentum \( n \)) is described by the solutions (A.1) with (A.2) (for \( \omega = 0 \)) or (A.3) (for \( \omega = 8/\kappa - 1 \)), which, in the coordinates of the disk, take the form

\[ Q_\rho(|w - \bar{w}|, \alpha, \epsilon) = C_\rho \epsilon^{\frac{\alpha}{|w - \bar{w}|}} e^{i\frac{\alpha p - \beta_p}{2} \alpha} \left( \sin \frac{\alpha}{2} \right)^{\gamma_p + x_p} \]  

(A.4)

If we consider the probability that the curve passes in between the two points in the original formulation on the half plane, then the curve is required here to pass by the center of the disk. Hence, the probability vanishes when the starting point of the SLE curve is brought toward its ending point on the disk, from any direction; this fixes the power to be \( 8/\kappa - 1 \) (for \( \kappa < 8 \)) at both values \( \alpha = 0, 2\pi \) and therefore selects the solution in the fermionic sector (A.3). Note
that since the probability could be given by an excited state in the fermionic sector (which corresponds to a higher value in place of the exponent \( x_p \)), we do not have the condition that \( \tilde{c}_p \) is nonzero.

**References**

[1] A. Zamolodchikov and V. Fateev, Sov. Phys. JETP 62(2) (1985) 215.

[2] S. Smirnov, *Towards conformal invariance of 2D lattice models*, in Proceedings of the International Congress of Mathematicians (Madrid, August 22-30, 2006), European Mathematical Society, 2006, Volume II, 1421–1451.

[3] B. Doyon, V. Riva and J. Cardy, *math-ph/0511054*, to appear in Commun. Math. Phys.

[4] E. Fortuin and P. Kasteleyen, Physica 57 (1972) 536.

[5] L. Kadanoff and H. Ceva, Phys. Rev. B 3(11) (1971) 3918; E. Fradkin and L. Kadanoff, Nucl. Phys. B 179 (1980) 1.

[6] B. Nienhuis and H. Knops, Phys. Rev. B 32 (1985) 1872.

[7] C. Mercat, Commun. Math. Phys. 218 (2001) 177.

[8] [http://mathworld.wolfram.com/MorerasTheorem.html](http://mathworld.wolfram.com/MorerasTheorem.html)

[9] V. Dotsenko and V. Dotsenko, Adv. in Phys. 32(2) (1983) 129.

[10] R. Baxter, *Exactly solved models in statistical mechanics*, London, Academic Press, 1982.

[11] O. Schramm, *math.PR/9904022*, Israel J. Math. 118 (2000) 221.

[12] S. Rhode and O. Schramm, *math.PR/0106036*, Ann. Math. 161(2) (2005) 879.

[13] M. Bauer, D. Bernard, *math-ph/0206028*, Phys. Lett. B 543 (2002) 135; *hep-th/0210015*, Commun. Math. Phys. 239 (2003) 493.

[14] Cardy, J., Phys. Lett. B 582 (2004) 121.