Lyapunov-Based Nonlinear Model Predictive Control for Attitude Trajectory Tracking of Unmanned Aerial Vehicles

Duy Nam Bui · Thi Thanh Van Nguyen · Manh Duong Phung

Received: 24 May 2022 / Revised: 6 September 2022 / Accepted: 10 October 2022 / Published online: 27 October 2022
© The Author(s), under exclusive licence to The Korean Society for Aeronautical & Space Sciences 2022

Abstract
This paper presents a new Lyapunov-based nonlinear model predictive controller (LNMPC) for the attitude control problem of unmanned aerial vehicles (UAVs), which is essential for their functioning operation. The controller is designed based on a quadratic cost function integrating UAV dynamics and system constraints. An additional contraction constraint is then introduced to ensure closed-loop system stability. That constraint is fulfilled via a Lyapunov function derived from a sliding mode controller (SMC). The feasibility and stability of the LNMPC are finally proved. Simulation and comparison results show that the proposed controller guarantees the system stability and outperforms other state-of-the-art nonlinear controllers, such as the backstepping controller and SMC. In addition, the proposed controller can be integrated into an existing UAV model in the Gazebo simulator to perform software-in-the-loop tests. The results show that the LNMPC is better than the built-in proportional–integral–derivative controller of the UAV, which confirms the validity and applicability of our proposed approach.

Keywords Unmanned aerial vehicles · Attitude control · Trajectory tracking · Model predictive control

1 Introduction
In recent years, unmanned aerial vehicles (UAVs) have been receiving significant interest due to their applicability in various fields from transportation and agriculture to military and space exploration [1–3]. The key to the success of UAV applications lies in control techniques that drive the UAV to reach its desired trajectory and maintain its stability in conditions subject to disturbances, such as load variation or wind gust. For quadrotor UAVs, it is essential to control their attitude, since it decides their maneuver, i.e., the UAV changes its position and orientation by changing its attitude. Since the dynamic model of quadrotor UAVs is nonlinear with six degrees of freedom but having only four independent inputs, it is an underactuated system [4]. Besides, system constraints are often required to meet the physical limits of the UAV. Therefore, designing controllers for quadrotor UAVs is a challenging problem that requires sufficient investigation.

In the literature, linear control methods such as the proportional–integral–derivative (PID) and linear-quadratic-regulator (LQR) controllers are among the most popular methods used for UAVs [5–7]. In [8], the PID, LQR, and state feedback controllers are used to control the attitude of a UAV with performance sufficient for several tasks. Combining linear controllers with an adaptive method is another approach to improve the tracking performance [9,10]. In [9], a fuzzy approach is used to automatically adjust the control parameters of a PID controller. In [10], the fuzzy PID method is combined with an iterative learning controller to deal with under-actuated dynamics and strong coupling characteristics of the quadrotor. While these approaches can enhance the performance of standard PID controllers, they require linearization, which affects the control performance when the UAV operates in its nonlinear regions.

To overcome that problem, nonlinear control methods are often used. The most popular nonlinear control techniques
include sliding mode control (SMC) and its variations due to the capability to handle model uncertainties [11,12]. In [11], an adaptive twisting SMC algorithm is developed to control the attitudes of a quadrotor with satisfactory performance. In [13], an SMC based on neural networks is designed to minimize the impact of external disturbances on UAV dynamics. The SMC and its variants are also used in [14–16] to regulate the UAV attitude under different conditions. However, the inevitable chattering phenomenon generated by SMC is undesirable to the system and the computational cost is proportional to the order of the system. Hence, another nonlinear control method named the backstepping control (BSC) is often used [17–19]. It can handle nonlinearities in the system dynamics and external disturbances to generate fast and efficient tracking performance. For instance, the adaptive BSC introduced in [20] is capable of controlling the attitude of a quadrotor in harsh conditions with desirable accuracy. However, if the system model is subject to constraints or uncertainties, the control performance is quickly degraded and may lead to divergence. The BSC and SMC can be combined to take advantage of both controllers [21,22], but the undesirable chattering phenomenon then persists. Besides, the system stability is affected if constraints on system states and control signals exist.

Recently, model predictive control (MPC) has been used for UAV control to predict system states and handle constraints. In [23], an adaptive nonlinear MPC is introduced for path tracking of a fixed-wing UAV. The prediction horizon varies according to the path curvature to enhance the tracking result. In [24], MPC is used for the autopilot of a UAV in which a linear parameter-varying model is employed to describe its dynamics and account for estimation errors. In another work, a low complexity MPC algorithm is introduced for real-time trajectory tracking of the UAV with limited computation capacity [25]. MPC is also used in [26–30] for different UAV control tasks, such as guidance, linear tracking, and multitarget–multisensor tracking. However, the stability of those controllers is hardly addressed due to the lack of a direct relationship between control signals and input variables. Since stability is an essential characteristic of control systems, analyzing it is essential for the safe operation of UAVs.

In this work, we address the attitude control problem of UAVs by proposing a nonlinear model predictive controller. First, the dynamic model of the quadcopter UAV is introduced. A cost function relating the current and desired state and having constraints on system states and control signals is then defined. A Newton-type algorithm is finally used to solve the cost function to obtain optimal control signals. Here, our contributions are threefold:

(i) We propose a new Lyapunov-based nonlinear model predictive controller (LNMPC) considering constraints on both attitude states and control signals. Those constraints are essential for the practical use of the controller, since real UAVs are limited in their maneuverability and actuator power.

(ii) We introduce a contraction condition to constrain the system states and based on it, we prove that the stability of the control system is guaranteed. To the best of our knowledge, this is the first time the stability of a nonlinear MPC has been proven for attitude control of the UAV.

(iii) A number of simulations, comparisons, and software-in-the-loop (SIL) tests have been conducted to evaluate the performance of the proposed controller. The results show that our controller outperforms not only the built-in controller of the UAV but also state-of-the-art nonlinear controllers.

The rest of this paper is structured as follows. Section 2 presents the dynamic model of the quadrotor UAV. Section 3 describes the design of the LNMPC. Section 4 analyzes the system stability. Finally, results and conclusions are presented in Sects. 5 and 6.

2 The Dynamic Model of Quadrotor UAVs

The UAV used in this work is a quadrotor drone that has two pairs of propellers rotating in opposite directions. By defining the inertial frame $O = \{X, Y, Z\}$ and the body frame $B = \{B_x, B_y, B_z\}$ as in Fig. 1, a UAV configuration includes its position, $(x, y, z)$, and orientation, $(\phi, \theta, \psi)$, about $x$, $y$, and $z$ axes of the inertial frame, respectively.

According to [31], the dynamic equations of the UAV can be summarized as

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
(g - \cos \phi \cos \theta) \frac{f_t}{m} \\
\dot{x} + \dot{y} I_y - I_z \frac{\tau_y}{m} + I_z \frac{\tau_z}{m} \sin \phi \sin \theta \cos \psi \\
\dot{y} + \dot{z} I_z - I_x \frac{\tau_z}{m} + I_x \frac{\tau_x}{m} \\
\dot{\phi} I_z - I_y \frac{\tau_x}{I_x} + I_y \frac{\tau_y}{I_y} \\
\dot{\theta} I_y - I_x \frac{\tau_y}{I_x} + I_x \frac{\tau_x}{I_x} \\
\dot{\psi} I_x - I_y \frac{\tau_x}{I_y} + I_y \frac{\tau_y}{I_y}
\end{bmatrix},
\]

where $U = [f_t, \tau_x, \tau_y, \tau_z]^T$ is the control input with $f_t$ being the total thrust, and $\tau_x$, $\tau_y$, and $\tau_z$ being the moments about $x$, $y$, and $z$ axes, respectively; $m$ is the mass of the UAV; $g = 9.8\, \text{m/s}^2$ is the gravity; $I_x$, $I_y$, $I_z$ are, respectively, the moments of inertia about $x$, $y$, and $z$ axes of the body frame; and $l_a$ is the UAV arm length.
Since this work focuses on attitude control, only three last equations of (1) are considered. Therefore, the dynamic equations representing UAV’s attitude are expressed as

\[
\begin{bmatrix}
\ddot{\phi} \\
\ddot{\theta} \\
\ddot{\psi}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{I_x} (I_y - I_z) + \frac{l_d}{I_x} \dot{\phi} \\
\frac{1}{I_y} (I_x - I_z) + \frac{l_d}{I_y} \dot{\theta} \\
\frac{1}{I_z} (I_x - I_y) + \frac{l_d}{I_z} \dot{\psi}
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} + \begin{bmatrix}
l_a \\
l_a \\
l_a
\end{bmatrix} \tau
\]

(2)

3 Controller Design

Similar to MPC, LNMPC can handle system constraints to output optimal control signals. Moreover, the stability of LNMPC can be analyzed via the Lyapunov theory. In this section, we present our design of the LNMPC together with the design of a sliding mode controller used to prove the system’s stability.

3.1 Design of the LNMPC

Let \( \xi_1 = [\phi, \theta, \psi]^T \) and \( \xi_2 = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T \) represent the attitude and angular velocities of the UAV, respectively. According to (2), we have

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2, \\
\dot{\xi}_2 &= \begin{bmatrix}
\ddot{\phi} \\
\ddot{\theta} \\
\ddot{\psi}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{I_x} (I_y - I_z) + \frac{l_d}{I_x} \dot{\phi} \\
\frac{1}{I_y} (I_x - I_z) + \frac{l_d}{I_y} \dot{\theta} \\
\frac{1}{I_z} (I_x - I_y) + \frac{l_d}{I_z} \dot{\psi}
\end{bmatrix} \begin{bmatrix}
\dot{\phi} \\
\dot{\theta}
\end{bmatrix} + \begin{bmatrix}
l_a \\
l_a \\
l_a
\end{bmatrix} \tau
\end{align*}
\]

(3)

\[
\begin{bmatrix}
I_y - I_z \\
I_y - I_z \\
I_x - I_y
\end{bmatrix}
\begin{bmatrix}
\ddot{\psi} \\
\ddot{\phi}
\end{bmatrix} + \begin{bmatrix}
\frac{1}{I_x} & 0 & 0 \\
0 & \frac{1}{I_y} & 0 \\
0 & 0 & \frac{1}{I_z}
\end{bmatrix}
\begin{bmatrix}
\tau \phi \\
\tau \theta \\
\tau \psi
\end{bmatrix}
\]

\[
= G_1 \xi_2 + G_2 U,
\]

where \( g(\xi_2) = [\ddot{\phi}, \ddot{\theta}, \ddot{\psi}]^T \).

Let \( \xi = [\xi_1, \xi_2]^T \) be the state vector and \( U = [\tau \phi, \tau \theta, \tau \psi]^T \) be the control signal. The dynamic equations in (3) can be written in a short form as

\[
\dot{\xi} = f(\xi, U).
\]

(4)

Constraints on system states and control signals can be included as

\[
|\xi| \leq \xi_{\text{max}}, \\
|U| \leq U_{\text{max}}.
\]

(5)

Let \( \xi_{1d} = [\phi_d, \theta_d, \psi_d]^T \) be the reference attitude and \( \xi_{2d} = \dot{\xi}_{1d} \). The tracking error is given by

\[
Z = \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
\xi_1 - \xi_{1d} \\
\xi_2 - \xi_{2d}
\end{bmatrix}.
\]

(6)

Denoting the reference state as \( \xi_{\text{ref}} = [\xi_{1d}, \xi_{2d}]^T \), the aim of the LNMPC is to find optimal control signal \( U \) to drive the state variable \( \xi \) to the desired state \( \xi_{\text{ref}} \). This can be carried out by optimizing a cost function with constraints of the form:

\[
J = \min_U \| \xi (t_k + T) - \xi_{\text{ref}} (t_k + T) \|_P^2
\]

\[
+ \int_{t_k}^{t_k+T} \left( \| \xi (t) - \xi_{\text{ref}} (t) \|_Q^2 + \| U (t) \|_R^2 \right) dt,
\]

subject to

\[
\xi (t_k) = \xi (t = t_k),
\]

(8a)

\[
\dot{\xi} (t) = f (\xi (t), U (t)) \quad \forall t \in [t_k; t_k + T],
\]

(8b)

\[
|\xi (t)| \leq \xi_{\text{max}} \forall t \in [t_k; t_k + T],
\]

(8c)

\[
|U (t)| \leq U_{\text{max}} \forall t \in [t_k; t_k + T],
\]

(8d)
\[
\frac{\partial V(\xi)}{\partial \xi} f(\xi(t_k), U(t_k)) \leq \frac{\partial V(\xi)}{\partial \xi} f(\xi(t_k), h(\xi(t_k))) \tag{8e}
\]

where \(t_k\) is the current time instant, \(T\) is the prediction horizon length, \(U(t)\) is the prediction control signal at future time \(t\), \(\|\cdot\|\) denotes the Euclidean norm, \(P\), \(Q\), and \(R\) are the positive definite weight matrices, \(h(\cdot)\) is a Lyapunov-based closed-loop control law, and \(V(\cdot)\) is a Lyapunov function.

The LNMPC problem (7–8) can be solved using a Newton-type algorithm called sequential quadratic programming (SQP) to find a locally optimal solution [32]. Here, we add an additional contraction constraint (8e) to the original nonlinear MPC to maintain the closed-loop stability. Constraint (8e) guarantees that values of the first order derivative of Lyapunov function \(V(\cdot)\) with control input \(U(t_k)\) are smaller than or equal to those values obtained using the Lyapunov-based control law \(h(\xi(t_k))\). This contraction constraint thus allows proving that the LNMPC inherits the stability property of the controller having \(h(\xi)\). According to [33], any Lyapunov-based closed-loop controller can be used to implement \(h(\xi)\). In this work, we use a sliding mode controller for that purpose with its details described below.

### 3.2 Sliding Mode Control Design for the Contraction Constraint

This section presents our design of \(h(\xi)\) in (8e) using the sliding mode control technique. Let us consider the sliding surface of the control system as follows:

\[
s = \dot{z}_1 + \lambda z_1 = z_2 + \lambda z_1, \tag{9}
\]

where \(\lambda\) is a positive gain matrix.

Taking the first derivative of (9), we have

\[
\dot{s} = \dot{z}_2 + \lambda \dot{z}_1 = (\dot{\xi}_2 - \dot{\xi}_2d) + \lambda z_2. \tag{10}
\]

Consider the SMC Lyapunov candidate function

\[
V_{SMC} = \frac{1}{2} s^T s. \tag{11}
\]

Differentiating \(V_{SMC}\) in (11) and noting (3), we get

\[
\dot{V}_{SMC} = s^T \dot{s} = s^T (G_1 g(\xi_2) + G_2 U - \dot{\xi}_2d + \lambda z_2). \tag{12}
\]

To ensure \(V_{SMC}\) negative, we choose the control signal of the form:

\[
U_{SMC} = U_{eq} + U_{sw}, \tag{13}
\]

where \(U_{eq}\) is the equivalent control signal that maintains the system state on the sliding manifold and \(U_{sw}\) is the signal that leads the system to the sliding surface. They are chosen as follows:

\[
U_{eq} = G_2^{-1} (\dot{\xi}_2 - \lambda z_2 - G_1 g(\xi_2)), \tag{14}
\]

\[
U_{sw} = -G_2^{-1} (c_1 \text{sign}(s) + c_2 s), \tag{15}
\]

where \(c_1\) and \(c_2\) are positive diagonal matrices. Substituting (13) and (14) into (12) gives

\[
\dot{V}_{SMC} = -s^T (c_1 \text{sign}(s) + c_2 s) \leq 0. \tag{16}
\]

### 4 Stability Analysis

In this section, we analyze the recursive feasibility and closed-loop stability of the LNMPC. First, we consider the following definition and lemma:

**Definition** If \(x \in \mathbb{R}^n\), the maximum norm of \(x\) is

\[
\|x\|_\infty = \max \{|x_k| : 1 \leq k \leq n\}. \tag{17}
\]

For \(1 \leq p < \infty\), the \(p\)-norm of \(x\) is

\[
\|x\|_p = \left( \sum_{k=1}^n |x_k|^p \right)^{1/p}. \tag{18}
\]

**Lemma 1** For any square matrices \(M, N \in \mathbb{R}^{n \times n}\)

\[
\|MN\|_\infty \leq \|M\|_\infty \|N\|_\infty. \tag{19}
\]

Then, we make the following assumptions.

**Assumption 1** The desired and real configurations of the UAV are smooth and bounded, that is

\[
\|\xi_{1d}(t)\|_\infty \leq \tilde{\xi}_{1d}, \|\dot{\xi}_{1d}(t)\|_\infty \leq \tilde{\xi}_{2d}, \|\ddot{\xi}_{1d}(t)\|_\infty \leq \tilde{\xi}_{3d}, \tag{20}
\]

\[
\|\xi_1(t)\|_\infty \leq \tilde{\xi}_1, \|\dot{\xi}_1(t)\|_\infty \leq \tilde{\xi}_2, \|\ddot{\xi}_1(t)\|_\infty \leq \tilde{\xi}_3. \tag{21}
\]
Assumption 2 The UAV torques are bounded, that is
\[
\tau_\phi \leq \tau_{\max}, \quad \tau_\theta \leq \tau_{\max}, \quad \tau_\psi \leq \tau_{\max}.
\]  
(22)

Those assumptions are reasonable for UAVs in practice due to their physical constraints and limits [34].

Proposition 1 Consider the UAV dynamics in (3), \( G_1, G_2 \) and \( g(\xi_2) \) are then bounded, that is
\[
\|G_1\|_\infty \leq \bar{I}_1, \quad \|G_2\|_\infty \leq \bar{I}_1, \quad \|G_2^{-1}\|_\infty \leq \bar{I}_2, \quad \|g(\xi_2)\|_\infty \leq \bar{\xi}_2^2. \]  
(23)

Proof Since \( \phi, \theta, \psi \) are the elements of \( \xi_1 \), we have \( \|\phi\|_\infty \leq \bar{\xi}_1, \|\theta\|_\infty \leq \bar{\xi}_1, \) and \( \|\psi\|_\infty \leq \bar{\xi}_1, \). Similarly, \( \|\phi\|_\infty \leq \bar{\xi}_2, \|\theta\|_\infty \leq \bar{\xi}_2, \) and \( \|\psi\|_\infty \leq \bar{\xi}_2, \). According to Lemma 1, it follows that
\[
\|g(\xi_2)\|_\infty \leq \bar{\xi}_2 < \infty. \]  
(24)

On the other hand, as the values of UAV parameters are bounded, we have
\[
\|G_1\|_\infty = \max \left\{ \frac{I_y - I_z}{I_x}, \frac{I_z - I_x}{I_y}, \frac{I_x - I_y}{I_z} \right\} \leq \bar{I} < \infty, \]  
(25)

\[
\|G_2\|_\infty = \max \left\{ \frac{I_a}{I_x}, \frac{1}{I_y}, \frac{1}{I_z} \right\} \leq \bar{I}_1 < \infty, \]  
(26)

\[
\|G_2^{-1}\|_\infty = \max \left\{ \frac{I_x}{I_a}, \frac{I_y}{I_a}, \frac{I_z}{I_a} \right\} \leq \bar{I}_2 < \infty. \]  
(27)

Proposition 2 Consider the UAV being controlled by the SMC technique (13–14), the error signal \( z_2 \) and the sliding surface \( s \) are then bounded with \( \|\lambda\|_\infty = \bar{\lambda}. \)

Proof Since \( \dot{V}_{\text{SMC}} \leq 0, \) it follows that
\[
\|z(t)\|_\infty \leq \|z(0)\|_\infty \leq \|z(0)\|_2. \]  
(28)

Besides, we have \( \|z_1\|_\infty \leq \|Z\|_\infty \) and \( \|z_2\|_\infty \leq \|Z\|_\infty \). Let \( \|z_1\|_2 = \bar{Z} \), we have \( \|Z_1\|_\infty \leq \bar{Z} \) and \( \|z_2\|_\infty \leq \bar{Z} \). As the result, the sliding surface is bounded, that is
\[
\|s\|_\infty = \|z_2 + \lambda z_1\|_\infty \leq (1 + \bar{\lambda}) \bar{Z}. \]  
(29)

\[\square\]

Theorem 1 Suppose the control gain matrices are chosen as \( c_1 \) and \( c_2. \) Let \( \|c_1\|_\infty = \bar{c}_1, \) \( \|c_2\|_\infty = \bar{c}_2. \) For \( h(\xi) = U_{\text{SMC}}, \) the LNMPC \((7–8)\) admits recursive feasibility for all \( t \geq 0 \) if the following relation is satisfied:
\[
\bar{I}_2 \left( \xi_{3d} + \bar{\lambda} \bar{Z} + \bar{I}_2 \xi_2 \right) + \bar{I}_2 \left[ \bar{c}_1 + \bar{c}_2 (1 + \bar{\lambda}) \bar{Z} \right] \leq \tau_{\max}. \]  
(30)

Proof Let \( U_{\text{max}} = [\tau_{\phi_{\max}}, \tau_{\theta_{\max}}, \tau_{\psi_{\max}}]^T. \) Given the present system state \( \xi(t) \), \( h(\xi) \) is always feasible for the LNMPC problem \((7–8)\) if \( |h(\xi)| \leq U_{\text{max}}. \)

By taking the norm of both sides of \((13)\) and noting Proposition 1 and 2, we get
\[
\|U_{\text{SMC}}\|_\infty \leq \|U_{\text{eq}}\|_\infty + \|U_{\text{sw}}\|_\infty
\]
\[
= \left\| G_2^{-1} \left( \xi_{2d} - \lambda \bar{Z} - G_1 g(\xi_2) \right) \right\| + \left\| -G_2^{-1} (c_1 \text{sign}(s) + c_2 s) \right\|_\infty
\]
\[
\leq \bar{I}_2 \left( \xi_{3d} + \lambda \bar{Z} + \bar{I}_2 \xi_2 \right) + \bar{I}_2 \left[ \bar{c}_1 + \bar{c}_2 (1 + \lambda) \bar{Z} \right]. \]  
(31)

From \((30)\) and \((31)\), we have \( |h(\xi)| = \|U_{\text{SMC}}\|_\infty \leq \tau_{\max}, \) which completes the proof. \[\square\]

Noting that the SMC acts as an initial guess for the optimization problem \((7–8)\) rather than being used in the optimization process. Therefore, the feasibility is guaranteed by the SMC controller, and the control performance is optimized by the LNMPC controller.

Theorem 2 The closed-loop system under LNMPC is asymptotically stable with respect to the equilibrium \( Z = \begin{bmatrix} z_1^* \\ z_2^* \end{bmatrix} \) and the UAV attitude will converge to the desired reference under the LNMPC.

Proof Since the Lyapunov function \( V_{\text{SMC}} \) in \((11)\) is continuously differentiable and radially unbounded, using the converse Lyapunov theorems [35], there exist functions \( \vartheta_i, i = 1, 2, 3 \) that satisfy the following inequalities:
\[
\vartheta_1 \left( \|\xi\| \right) \leq V_{\text{SMC}}(\xi) \leq \vartheta_2 \left( \|\xi\| \right), \quad \frac{\partial V(\xi)}{\partial \xi} \leq -\vartheta_3 \left( \|\xi\| \right). \]  
(32)

Consider contraction constraint \((8e)\) and note that the optimal solution will be implemented for one sampling period each time, we have
\[
\frac{\partial V(\xi)}{\partial \xi} \leq -\vartheta_3 \left( \|\xi\| \right). \]  
(33)
Based on Theorem 4.16 in [35], we conclude that the closed-loop system is asymptotically stable with a guaranteed region of attraction:

\[
\mathcal{X} = \left\{ \xi \in \mathbb{R}^6 | \bar{l}_2 \left( \xi_{3d} + \bar{\lambda} \dot{Z} + \bar{I}_2 \xi_2^2 \right) + \bar{l}_2 \left[ \bar{c}_1 + \bar{c}_2 \left( 1 + \bar{\lambda} \right) \bar{Z} \right] \leq \tau_{\text{max}} \right\}.
\] (34)

In addition, since there is no other constraints on control gains \( \bar{\lambda}, \bar{c}_1, \bar{c}_2 \), \( \mathcal{X} \) can be enlarged by reducing the magnitude of those control gains as long as (30) is satisfied. The LNMPC is thus recursively feasible, and the closed-loop system is stable.

**Remark 1** Condition (30) can be satisfied by choice of \( \bar{c}_1, \bar{c}_2 \) and \( \bar{\lambda} \) as the remaining parameters are bounded and can be pre-determined. Specifically, UAV parameters represented in (3) are bounded according to Proposition 1 and their values can be determined based on the physical limits of the UAV. Similarly, the reference attitude values, control signals and state limits are all specified. Since \( V_{\text{SMC}} \leq 0 \) is satisfied, the system error is decreased over time leading to the initial error to be considered as the bounded error \( Z \). Therefore, (30) can be fulfilled by varying the values of \( \bar{c}_1, \bar{c}_2, \) and \( \bar{\lambda} \).

**Remark 2** In practice, the proposed controller can be implemented in the embedded computer system of UAVs for real-time control, since the computation time to solve the LNMPC problem (7–8) can be effectively handled by the optimized libraries, such as ACADO [36]. In addition, software and libraries such as CasADi [37], FORCES Pro [38], and VIATOC [39] can also be used to handle the LNMPC problem in different UAV platforms.

## 5 Results

In this section, the performance of the proposed LNMPC controller is evaluated and compared with two state-of-the-art nonlinear controllers, the backstepping controller (BSC) and the sliding mode controller (SMC). The UAV model used is the AscTec Pelican as shown in Fig. 1 with parameters shown in Table 1.

| Parameter   | Value   |
|-------------|---------|
| Mass of the UAV \( m \) | 1.0 kg |
| Gravity \( g \) | 9.8 m/s² |
| Moment of inertia about \( B_x \) axis, respectively | 0.01 kg m² |
| Moment of inertia about \( B_z \) axis | 0.02 kg m² |
| UAV arm length \( l_a \) | 0.21 m |

![Fig. 2](image.png) Step responds of three controllers in Scenario 1

![Fig. 3](image.png) Trajectory tracking results of three controllers for Scenario 2
Constraints on the UAV are defined as

\[-\frac{\pi}{2} \leq \phi, \theta \leq \frac{\pi}{2},\]
\[-\frac{\pi}{2} \leq \dot{\phi}, \dot{\theta}, \dot{\psi} \leq \frac{\pi}{2},\]
\[-0.1 \leq \tau_\phi, \tau_\theta, \tau_\psi \leq 0.1.\] (35)

For the LNMPC, the sampling period is chosen as \(dT = 0.02\) s, the predictive horizon is chosen as \(T = 30dT\), and the weight matrices are chosen as

\[P = Q = \text{diag}(30.0, 30.0, 30.0, 1.0, 1.0, 1.0),\]
\[R = \text{diag}(1.0, 1.0, 1.0).\] (36)

Since the BSC and SMC do not consider the UAV constraints, their control signals are limited to \(u \in \{u_{\text{max}}, u_{\text{min}}\}\) as follows:

\[u = \min\{u_{\text{max}}, \max\{u, u_{\text{min}}\}\}.\] (37)

Comparisons are then conducted in four scenarios with different input references.

### 5.1 Scenario 1: Step Input

This scenario considers a step input. The UAV is initialized at a hovering state, where all attitude angles are zeros. The new reference angles are then set to \(\phi_d = \theta_d = \psi_d = 1\) rad. The system responses are shown in Fig. 2. It can be seen that all controllers can drive the UAV to reach the reference without overshooting. However, the proposed controller smoothly drives the angles to their reference values within 1 s, while other controllers require a longer time, i.e., 2 s with BSC and 2.5 s with SMC. The proposed controller thus introduces the fastest response due to its ability to predict future states to generate more efficient control signals.
5.2 Scenario 2: Time-Varying Reference

In this scenario, the desired attitudes periodically change over time according to the following equations:

\[
\begin{align*}
\phi_d &= \frac{1}{3} \sin(t) + \frac{1}{2} \cos(2t), \\
\theta_d &= \frac{1}{3} \cos(t) + \frac{1}{2} \sin(2t), \\
\psi_d &= \frac{\pi}{10} t + \frac{1}{4} \cos(2t).
\end{align*}
\]

(38)

Figure 3 shows the tracking results. It can be seen that the LNMPC introduces the fastest transient response by first reaching the references. At the steady-state, the LNMPC also has the smallest tracking errors. The BSC, on the other hand, introduces a relatively fast response. However, as shown in Fig. 3b, its performance is affected by the system constraints causing large steady-state errors. The SMC introduces comparable tracking performance to the LNMPC as it is not constrained after reaching the desired reference. However, it has the slowest response and generates chattering control signals as shown in Fig. 4.

5.3 Scenario 3: Saturated Control Signal

To further evaluate the performance of the proposed controller, more challenging reference trajectories are used in Scenario 3, in which control signals become saturated in a certain period. The equations of those trajectories are given by

\[
\begin{align*}
\phi_d &= \frac{1}{3} \sin(2t) + \frac{1}{6} \sin(6t), \\
\theta_d &= \frac{1}{3} \cos(2t) + \frac{1}{6} \cos(4t), \\
\psi_d &= \frac{\pi}{10} t + \frac{1}{5} \sin(3t) + 0.5.
\end{align*}
\]

(39)

Figure 5 shows the tracking results. Similar to previous scenarios, the LNMPC introduces the fastest transient responses and smallest steady-state errors. Its control signals also do not exhibit the chattering phenomenon as with the SMC, as shown in Fig. 6. Especially, the advantage of the LNMPC is clearly shown in period \(6 < t < 7\) s, where the control signals from all controllers are saturated due to the system constraints. The tracking errors of the BSC and SMC for pitch angle \(\theta\) quickly increase, whereas that error of the LNMPC remains the same, as can be seen in Fig. 5a. That advantage comes from the capability of the LNMPC to predict its future states and then provide optimal control signals to better adapt to their changes.

5.4 Scenario 4: Disturbance Rejection

This scenario evaluates the robustness of our controller in the presence of input and output disturbances. The reference signals in this scenario are given by

\[
\begin{align*}
\phi_d &= \frac{1}{3} \cos(2t) + \frac{1}{2} \cos\left(\frac{2}{3} t\right), \\
\theta_d &= \frac{1}{2} \cos(2t) + \frac{1}{6} \sin\left(\frac{2}{3} t\right), \\
\psi_d &= \frac{\pi}{12} t + \cos\left(\frac{3}{5} t\right).
\end{align*}
\]

(40)

The input disturbance is generated from uniformly distributed random noise with magnitude \(d_u = 0.1\) Nm added to the control signals. Note that this magnitude is equal to the maximum input torque mentioned in (35). The tracking results in Fig. 7 show that all controllers are rather insensitive to input disturbances. However, the LNMPC performs the best in terms of transient response and steady state error.

For the case of output disturbance, the system states are added with Gaussian noise of zero mean and variance equal to 0.02 rad. The tracking results are depicted in Fig. 8. It can be seen that the trajectories of BSC and SMC fluctu-
ate about their references implying the influence of noise on those controllers. The LNMPC, on the other hand, copes well with the noise to maintain stable tracking performance. The LNMPC, therefore, is sufficiently robust to deal with disturbances appearing during UAV operation, such as measurement noise and aerodynamic disturbance.

5.5 Validation with Software-in-the-Loop Tests
Based on Gazebo Simulator

To validate the control performance of the LNMPC on flight tasks, we carry out software-in-the-loop (SIL) tests in which the proposed controller is implemented to control a UAV model created in the Gazebo simulating environment, as
shown in Fig. 9. Parameters of the UAV are inherited from the AscTec Pelican drone, as shown in Fig. 1. Since the LNMPC only controls the attitude, we use a standard nonlinear MPC to handle the altitude and position tracking. The results are then compared with the built-in PID controller of the UAV.

The SIL simulation includes two scenarios. One is circular, and the other is a sinusoidal trajectory. The trajectory tracking results are shown in Fig. 10, where the UAV can reach and track the reference trajectories. The attitude tracking results are shown in Fig. 11, where it can be seen that the LNMPC outperforms the PID controller in both the response speed and tracking error. This result can be further verified by Table 2 which shows the root-mean-square error (RMSE). The LNMPC introduces smaller errors in all roll, pitch, and yaw angles, implying its better performance compared to the PID controller.

To evaluate the computational practicality of our proposed controller, we have implemented it on an embedded computer named Jetson Nano, a popular embedded computer used for UAVs [40,41]. Our implementation is based on the ACADO library [36]. The average calculation time for the LNMPC in each processing cycle is 0.016834 s, less than the sampling time, 0.02 s. The result thus confirms the validity of our proposed controller for real UAVs.

6 Conclusion

In this work, we have introduced a new LNMPC designed to track UAVs’ attitude trajectories. Using a contraction condition and the design of a SMC, we proved the feasibility and closed-loop stability of the proposed controller. The results show that our controller outperforms other popular nonlinear controllers in response time, tracking accuracy, and control signal smoothness, especially in challenging situations, where significant changes in control effort are required. In addition, the results of SIL tests with the Gazebo simulator
show the ability to deploy and apply the proposed controller in the embedded control system of UAVs, which then can be used to replace their built-in PID controller. Our future work will extend the LNMPC for the formation control of multiple UAVs.

**Declarations**

**Conflict of Interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

**References**

1. Kim J, Kim S, Ju C, Son HI (2019) Unmanned aerial vehicles in agriculture: a review of perspective of platform, control, and applications. IEEE Access 7:105100–105115. https://doi.org/10.1109/ACCESS.2019.2932119

2. Liu Y, Liu H, Tian Y, Sun C (2020) Reinforcement learning based two-level control framework of UAV swarm for cooperative persistent surveillance in an unknown urban area. Aerosp Sci Technol 98:105671. https://doi.org/10.1016/j.ast.2019.105671

3. Phung MD, Quach CH, Dinh TH, Ha Q (2017) Enhanced discrete particle swarm optimization path planning for UAV vision-based surface inspection. Autom Constr 81:25–33. https://doi.org/10.1016/j.autcon.2017.04.013

4. Hoang VT, Phung MD, Dinh TH, Ha QP (2020) System architecture for real-time surface inspection using multiple UAVs. IEEE Syst J 14(2):2925–2936. https://doi.org/10.1109/JSYST.2019.2922290

5. Najm AA, Ibrahim MK (2019) Nonlinear PID controller design for a 6-DOF UAV quadrotor system. Eng Sci Technol Int J 22(4):1087–1097. https://doi.org/10.1016/j.jestch.2019.02.005

6. Mahmoodabadi MJ, Rezaee Babak N (2020) Robust fuzzy linear quadratic regulator control optimized by multi-objective high exploration particle swarm optimization for a 4 degree-of-freedom quadrotor. Aerosp Sci Technol 97:105598. https://doi.org/10.1016/j.ast.2019.105598

7. Sir Elkhadem A, Naci Engin S (2022) Robust LQR and LQR-PI control strategies based on adaptive weighting matrix selection for a UAV position and attitude tracking control. Alex Eng J 61(8):6275–6292. https://doi.org/10.1016/j.aej.2021.11.057

8. Shehzad MF, Bilal A, Ahmad H (2019) Position and attitude control of an aerial robot (quadrotor) with intelligent PID and state feedback LQR controller: a comparative approach. In: 2019 16th International Bhurban conference on applied sciences and technology (IBCAST), pp 340–346. https://doi.org/10.1109/IBCAST.2019.8667170

9. Fu Y, Zhao Q, Liu Z, Xu Y, Liu X (2018) A self-adaptive cascade fuzzy pid method of attitude control for quadrotor. In: 2018 IEEE 4th international conference on control science and systems engi-

neering (ICCSSSE), pp 269–275. https://doi.org/10.1109/ICCSSSE.2018.8724821

10. Dong J, He B (2019) Novel fuzzy PID-type iterative learning control for quadrotor UAV. Sensors. https://doi.org/10.3390/s19010024

11. Hoang VT, Phung MD, Ha QP (2017) Adaptive twisting sliding mode control for quadrotor unmanned aerial vehicles. In: 2017 11th Asian control conference (ASCC), pp 671–676. https://doi.org/10.1109/ASCC.2017.8287250

12. Alipour K, Robat AB, Tarvirdizadeh B (2019) Dynamics modeling and sliding mode control of tractor-trailer wheeled mobile robots subject to wheels slip. Mech Mach Theory 138:16–37. https://doi.org/10.1016/j.mechmachtheory.2019.03.038

13. Nguyen NP, Mung NX, Thanh HLNN, Huynh TT, Lam NT, Hong SK (2021) Adaptive sliding mode control for attitude and altitude system of a quadcopter UAV via neural network. IEEE Access 9:40076–40085. https://doi.org/10.1109/ACCESS.2021.3064883

14. Yang Y, Yan Y (2016) Attitude regulation for unmanned quadrotors using adaptive fuzzy gain-scheduling sliding mode control. Aerosp Sci Technol 54:208–217. https://doi.org/10.1016/j.ast.2016.04.005

15. Gong W, Li B, Yang Y, Ban H, Xiao B (2019) Fixed-time integral-type sliding mode control for the quadrotor UAV attitude stabilization under actuator failures. Aerosp Sci Technol 95:105444. https://doi.org/10.1016/j.ast.2019.105444

16. Yogi SC, Tripathi VK, Behera L (2021) Adaptive integral sliding mode control using fully connected recurrent neural network for position and attitude control of quadrotor. IEEE Trans Neural Netw Learn Syst 32(12):5955–5969. https://doi.org/10.1109/TNNLS.2021.3071020

17. Das A, Lewis F, Subbarao K (2009) Backstepping approach for controlling a quadrotor using Lagrange form dynamics. J Intell Robot Syst 56(1):127–151

18. Jiang T, Lin D, Song T (2018) Finite-time backstepping control for quadrotors with disturbances and input constraints. IEEE Access 6:62037–62049. https://doi.org/10.1109/ACCESS.2018.2876558

19. Gliota HE, Abdou L, Chelibi A, Sentouh C, Hassene S-EI (2020) Optimal model-free backstepping control for a quadrotor helicopter. Nonlinear Dyn 100(4):3449–3468

20. Fu C, Hong W, Lu H, Zhang L, Guo X, Tian Y (2018) Adaptive robust backstepping attitude control for a multi-rotor unmanned aerial vehicle with time-varying output constraints. Aerosp Sci Technol 78:593–603. https://doi.org/10.1016/j.ast.2018.05.021

21. Tuin LA (2019) Fractional-order fast terminal back-stepping sliding mode control of crawler cranes. Mech Mach Theory 137:297–314. https://doi.org/10.1016/j.mechmachtheory.2019.03.027

22. Chen F, Jiang R, Zhang K, Jiang B, Tao G (2016) Robust backstepping sliding-mode control and observer-based fault estimation for a quadrotor UAV. IEEE Trans Ind Electron 63(8):5044–5056. https://doi.org/10.1109/TIE.2016.2552151

23. Yang K, Kang Y, Sukkarieh S (2013) Adaptive nonlinear model predictive path-following control for a fixed-wing unmanned aerial vehicle. Int J Control Autom Syst 11(1):65–74. https://doi.org/10.1007/s12555-012-0028-y

24. Cavani L, Ippoliti G, Camacho EF (2021) Model predictive control for a linear parameter varying model of an UAV. J Intell Robot Syst 101(3):1–18

25. Bangura M, Mahony R (2014) Real-time model predictive control for quadrotors. IFAC Proc Vol 47(3):11773–11780. https://doi.org/10.3182/20140824-6-ZA-1003.00203 (19th IFAC World Congress)

26. Sarunic P, Evans R (2014) Hierarchical model predictive control of UAVs performing multitarget–multisensor tracking. IEEE Trans Aerosp Electron Syst 50(3):2253–2268. https://doi.org/10.1109/TAES.2014.120780
27. Mammarella M, Capello E, Dabbene F, Guglieri G (2018) Sample-based SMPC for tracking control of fixed-wing UAV. IEEE Control Syst Lett 2(4):611–616. https://doi.org/10.1109/LCSYS.2018.2845546

28. Gavilan F, Vazquez R, Camacho EF (2015) An iterative model predictive control algorithm for UAV payload. IEEE Trans Aeros Electron Syst 51(3):2406–2419. https://doi.org/10.1109/TAES.2015.140153

29. Xu Q, Wang Z, Li Y (2020) Fuzzy adaptive nonlinear information fusion model predictive attitude control of unmanned rotorcrafts. Aerosp Sci Technol 98:105686. https://doi.org/10.1016/j.ast.2020.105686

30. Stastny TJ, Dash A, Siegwart R. (2017) Nonlinear MPC for fixed-wing UAV trajectory tracking: implementation and flight experiments. In: AIAA guidance, navigation, and control conference, p 1512. https://doi.org/10.2514/6.2017-1512

31. Wang C, Song B, Huang P, Tang C (2016) Trajectory tracking control for quadrotor robot subject to payload variation and wind gust disturbance. J Intell Robot Syst 83(2):315–333. https://doi.org/10.1007/s10846-016-0333-4

32. Quirynen R, Vukov M, Zanon M, Diehl M (2015) Autogenerating microsecond solvers for nonlinear MPC: a tutorial using ACADO integrators. Optim Control Appl Methods. https://doi.org/10.1002/oca.2152

33. Christofides PD, Liu J, Muñoz de la Peña D (2011) Lyapunov-based model predictive control. Springer, London, pp 13–45. https://doi.org/10.1007/978-0-85729-582-8_2

34. Salazar S, González-Hernández I, Lopez R, Lozano R (2014) Simulation and robust trajectory-tracking for a quadrotor UAV. In: 2014 International conference on unmanned aircraft systems (ICUAS), pp 1167–1174. https://doi.org/10.1109/ICUAS.2014.6842371

35. Khalil HK (2002) Nonlinear systems, 3rd edn. Prentice-Hall, Upper Saddle River

36. Houska B, Ferreau HJ, Diehl M (2011) ACADO toolkit—an open source framework for automatic control and dynamic optimization. Optim Control Appl Methods 32(3):298–312

37. Andersson JAE, Gillis J, Horn G, Rawlings JB, Diehl M (2019) CasADi—a software framework for nonlinear optimization and optimal control. Math Program Comput 11(1):1–36. https://doi.org/10.1007/s12532-018-0139-4

38. Zanelli A, Domahidi A, Jerez J, Morari M (2020) FORCES NLP: an efficient implementation of interior-point methods for multistage nonlinear nonconvex programs. Int J Control 93(1):13–29. https://doi.org/10.1080/00207179.2017.1316017

39. Kalmar J, Backman J, Visala A (2015) A toolkit for nonlinear model predictive control using gradient projection and code generation. Control Eng Pract. https://doi.org/10.1016/j.conengprac.2015.01.002

40. Jeon J, Jung S, Lee E, Choi D, Myung H (2021) Run your visual-inertial odometry on NVIDIA Jetson: benchmark tests on a micro aerial vehicle. IEEE Robot Autom Lett 6(3):5332–5339. https://doi.org/10.1109/LRA.2021.3075141

41. Menshchikov A, Shadrin D, Prutyanov V, Lopatkin D, Sosnin S, Tsykunov E, Iakovlev E, Somov A (2021) Real-time detection of hogweed: UAV platform empowered by deep learning. IEEE Trans Comput 70(8):1175–1188. https://doi.org/10.1109/TC.2021.3059819

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.