ASYMPTOTIC BEHAVIOUR OF THE CHRISTOFFEL FUNCTIONS ON THE UNIT BALL IN THE PRESENCE OF A MASS ON THE SPHERE

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Abstract. We present a family of mutually orthogonal polynomials on the unit ball with respect to an inner product which includes a mass uniformly distributed on the sphere. First, connection formulas relating these multivariate orthogonal polynomials and the classical ball polynomials are obtained. Then, using the representation formula for these polynomials in terms of spherical harmonics analytic properties will be deduced. Finally, we analyze the asymptotic behaviour of the Christoffel functions.

1. Introduction

Classical orthogonal polynomials on the unit ball \( B^d \) of \( \mathbb{R}^d \) correspond to the classical inner product

\[
\langle f, g \rangle_{\mu} = \frac{1}{\omega_{\mu}} \int_{B^d} f(x)g(x)W_\mu(x)dx,
\]

where \( W_\mu(x) = (1 - \|x\|^2)^\mu \) on \( B^d \), \( \mu > -1 \), and \( \omega_{\mu} \) is a normalizing constant such that \( \langle 1, 1 \rangle_{\mu} = 1 \).

In the present paper, we consider orthogonal polynomials with respect to the inner product

\[
\langle f, g \rangle_{\mu}^\lambda = \frac{1}{\omega_{\mu}} \int_{B^d} f(x)g(x)W_\mu(x)dx + \frac{\lambda}{\sigma_{d-1}} \int_{S^{d-1}} f(\xi)g(\xi)d\sigma(\xi),
\]

where \( \lambda > 0 \), \( d\sigma \) is the surface measure on \( S^{d-1} \) and \( \sigma_{d-1} \) denotes the sphere area.

Using spherical polar coordinates, we shall construct a family of mutually orthogonal polynomials with respect to \( \langle \cdot, \cdot \rangle_{\mu}^\lambda \), which depends on a sequence of orthogonal polynomials of one variable, namely the Krall polynomials [4]. This sequence of orthogonal polynomials can be expressed in terms of Jacobi polynomials. In a previous paper (see [7]) we have shown that the multivariate polynomials orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle_{\mu}^\lambda \) satisfy a fourth order partial differential equation.

Standard techniques provide us explicit connection formulas relating classical multivariate ball polynomials and our family of orthogonal polynomials. The explicit representations for the norms and the kernels will be obtained.

A very interesting open problem in the theory of multivariate orthogonal polynomials is that of finding asymptotic estimates for the Christoffel functions, because

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these estimates are related to the study of the convergence of the Fourier series. Asymptotics for Christoffel functions associated to the classical orthogonal polynomials on the ball were obtained by Y. Xu in 1996 (see [12]). Recently, more general results on the asymptotic behaviour of the Christoffel functions were established by A. Kroó and D. Lubinsky [5, 6] in the context of universality. Those results include estimates in a quite general case where the orthogonality measure satisfies some regularity conditions. In fact, they provide the ratio asymptotic for the Christoffel functions corresponding to two regular measures supported on the same compact set \( D \in \mathbb{R}^d \), in particular, the ratio of the Christoffel functions converges uniformly on any compact subset of the interior of \( D \).

Since our orthogonal polynomials does not fit into the above mentioned situation, the asymptotic of the Christoffel functions deserves special attention. Not surprisingly, our results show that in any compact subset of the interior of the unit ball Christoffel functions behave exactly as in the classical case, see Theorem 6.3.

On the sphere the situation is quite different and we can perceive the influence of the mass \( \lambda \), see Theorem 6.1.

A similar problem on a Sobolev context where the mass on the sphere was replaced by the normal derivatives has been recently considered in [2].

The paper is organized as follows. In the next section, we state the background materials on orthogonal polynomials on the unit ball and spherical harmonics that we will need later. In Section 3, using spherical polar coordinates we construct explicitly a family of mutually orthogonal polynomials with respect to \( \langle \cdot, \cdot \rangle_{\mu}^\lambda \). Those polynomials are given in terms of spherical harmonics and a family of univariate orthogonal polynomials in the radial part, their properties are studied in Section 4. In Section 5, we deduce explicit connection formulas relating classical multivariate ball polynomials and our family of orthogonal polynomials. Moreover, an explicit representation for the kernels is obtained. The asymptotic behaviour of the corresponding Christoffel functions is studied in Section 6.

2. Classical orthogonal polynomials on the ball

In this section we describe background materials on orthogonal polynomials and spherical harmonics. The first subsection collects some properties on the Jacobi polynomials. The second subsection recalls the basic results on spherical harmonics and classical orthogonal polynomials on the unit ball.

2.1. Classical Jacobi polynomials. For \( \alpha, \beta > -1 \), Jacobi polynomials \( P_n^{(\alpha, \beta)}(t) \) [9] are orthogonal with respect to the Jacobi inner product

\[
(f, g)_{\alpha, \beta} = \int_{-1}^{1} f(t) g(t) (1 - t)^{\alpha} (1 + t)^{\beta} dt.
\]

and satisfy

\[
(2.1) \quad P_n^{(\alpha, \beta)}(1) = \binom{n + \alpha}{n} = \frac{(\alpha + 1)_n}{n!}.
\]

The squares of the \( L^2 \) norms are given by

\[
(2.2) \quad h_n^{(\alpha, \beta)} := \left( P_n^{(\alpha, \beta)}, P_n^{(\alpha, \beta)} \right)_{\alpha, \beta} = \frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) n! \Gamma(n + \alpha + \beta + 1)}.
\]
Furthermore, to Jacobi polynomials we will use the corresponding kernel polynomials defined as

\[ K_n^{(\alpha,\beta)}(t, u) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(t) P_k^{(\alpha,\beta)}(u)}{h_k^{(\alpha,\beta)}}, \]

which are symmetric functions. It is well known (see [9, p. 71]) that

\[ K_n^{(\alpha,\beta)}(1, 1) = \frac{2^{-\alpha-\beta-1} \Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1) \Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1) \Gamma(n+1) \Gamma(\alpha+2)}. \]

2.2. Orthogonal polynomials on the unit ball and spherical harmonics. Let \( \Pi_d \) be the space of polynomials in \( d \) real variables. For a given non-negative integer \( n \), let \( \Pi_n^d \) denote the linear space of polynomials in several variables of (total) degree at most \( n \) and let \( P_n^d \) be the space of homogeneous polynomials of degree \( n \).

The unit ball and the unit sphere in \( \mathbb{R}^d \) are denoted, respectively, by

\[ \mathbb{B}^d = \{ x \in \mathbb{R}^d : \| x \| \leq 1 \} \quad \text{and} \quad S^{d-1} = \{ \xi \in \mathbb{R}^d : \| \xi \| = 1 \}. \]

where \( \| x \| \) denotes as usual the Euclidean norm of \( x \).

For \( \mu \in \mathbb{R} \), the weight function \( W_\mu(x) = (1 - \| x \|^2)^\mu \) is integrable on the unit ball if \( \mu > -1 \). Consider the inner product

\[ \langle f, g \rangle_\mu = \frac{1}{\omega_\mu} \int_{\mathbb{B}^d} f(x) g(x) W_\mu(x) \, dx, \]

where \( \omega_\mu \) is the normalization constant of \( W_\mu \) given by

\[ \omega_\mu := \int_{\mathbb{B}^d} W_\mu(x) \, dx = \frac{\pi^{d/2} \Gamma(\mu+1)}{\Gamma(\mu+1+d/2)}. \]

A polynomial \( P \in \Pi_n^d \) is called orthogonal with respect to \( W_\mu \) on the ball if \( \langle P, Q \rangle_\mu = 0 \) for all \( Q \in \Pi_{n-1}^d \), that is, if it is orthogonal to all polynomials of lower degree. Let \( \mathcal{V}_n^d(W_\mu) \) denote the space of orthogonal polynomials of total degree \( n \) with respect to \( W_\mu \). It is well known that

\[ \dim \Pi_n^d = \binom{n+d}{d} \quad \text{and} \quad \dim \mathcal{V}_n^d(W_\mu) = \binom{n+d-1}{n}. \]

For \( n \geq 0 \), let \( \{ P_n^\nu(x) : |\nu| = n \} \) denote a basis of \( \mathcal{V}_n^d(W_\mu) \). Notice that every element of \( \mathcal{V}_n^d(W_\mu) \) is orthogonal to polynomials of lower degree. If the elements of the basis are also orthogonal to each other, that is, \( \langle P_n^\nu, P_n^\eta \rangle_\mu = 0 \) whenever \( \nu \neq \eta \), we call the basis mutually orthogonal. If, in addition, \( \langle P_n^\nu, P_n^\nu \rangle_\mu = 1 \), we call the basis orthonormal.

Harmonic polynomials of degree \( n \) in \( d \)-variables are polynomials in \( P_n^d \) satisfying the Laplace equation \( \Delta Y = 0 \), where

\[ \Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2} \]

is the usual Laplace operator.
Let $\mathcal{H}_n^d$ be the space of harmonic polynomials of degree $n$. It is well known that

\[
a_n^d := \dim \mathcal{H}_n^d = \binom{n + d - 1}{d - 1} - \binom{n + d - 3}{d - 1}.
\]

Spherical harmonics are the restriction of harmonic polynomials to the unit sphere. If $Y \in \mathcal{H}_n^d$, then in spherical–polar coordinates $x = r\xi$, $r \geq 0$ and $\xi \in S^{d-1}$, we get

\[Y(x) = r^n Y(\xi),\]

so that $Y$ is uniquely determined by its restriction to the sphere. We shall also use $\mathcal{H}_n^d$ to denote the space of spherical harmonics of degree $n$.

If $d\sigma$ denotes the surface measure then the surface area is given by

\[\sigma_{d-1} := \int_{S^{d-1}} d\sigma = \frac{2\pi^{d/2}}{\Gamma(d/2)}\]

Spherical harmonics of different degrees are orthogonal with respect to the inner product

\[\langle f, g \rangle_{S^{d-1}} = \frac{1}{\sigma_{d-1}} \int_{S^{d-1}} f(\xi)g(\xi)d\sigma(\xi).\]

Since the weight function $W_\mu$ is rotationally invariant, in spherical–polar coordinates $x = r\xi$, $r \geq 0$ and $\xi \in S^{d-1}$, a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$ can be shown in terms of Jacobi polynomials and spherical harmonics (see, for instance, [3]).

**Proposition 2.1.** For $n \in \mathbb{N}_0$ and $0 \leq j \leq n/2$, let $\{Y_{\nu}^{n-2j} : 1 \leq \nu \leq a_{n-2j}^d\}$ be an orthonormal basis for $\mathcal{H}_{n-2j}^d$. Let us denote $\beta_{n-2j} = n - 2j + \frac{d-2}{2}$ and define

\[P_{j,\nu}^n(x) := P_{j}^{(\mu,\beta_{n-2j})}(2\|x\|^2 - 1)Y_{\nu}^{n-2j}(x).\]

Then the set $\{P_{j,\nu}^n(x) : 1 \leq j \leq n/2, 1 \leq \nu \leq a_{n-2j}^d\}$ is a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$. More precisely,

\[\langle P_{j,\nu}^n(x), P_{k,\eta}^m(x) \rangle_\mu = H_{j,\nu}^{n,\mu} \delta_{n,m} \delta_{j,k} \delta_{\nu,\eta},\]

where $H_{j,\nu}^{n,\mu}$ is given by

\[H_{j,\nu}^{n,\mu} := \frac{(\mu + 1)j(\frac{d}{2})_{n-j}(n-j+\mu+\frac{d}{2})}{j!(\mu+\frac{d}{2}+1)_{n-j}(n+\mu+\frac{d}{2})} = \frac{c_{n-2j}^\mu}{2^{n-2j}} h_j^{(\mu,\beta_{n-2j})},\]

with

\[c_{\mu}^d = \frac{1}{2^{\mu+\frac{d}{2}+1}} \sigma_{d-1} \omega_\mu.\]
3. An Inner Product on the Unit Ball with an Extra Spherical Term

Let us define the inner product
\[ \langle f, g \rangle^\lambda_{\mu} = \frac{1}{\omega_{\mu}} \int_{B^d} f(x)g(x)W_\mu(x)dx + \frac{\lambda}{\sigma_{d-1}} \int_{S^{d-1}} f(\xi)g(\xi)d\sigma, \]
where \( \lambda > 0 \). As a consequence of the central symmetry of the inner product, we can use a procedure analogous to the construction described in Proposition 2.1 to obtain a basis of \( V_n^d(W_{\mu}, \lambda) \), the linear space of orthogonal polynomials of exact degree \( n \) with respect to \( \langle \cdot, \cdot \rangle^\lambda_{\mu} \). This time, the radial parts constitute a sequence of polynomials in one variable related to Jacobi polynomials.

**Theorem 3.1.** For \( n \in \mathbb{N}_0 \) and \( 0 \leq j \leq n/2 \), let \( \{ Y_n^{\nu-n/2} : 1 \leq \nu \leq a_{n-2j}^d \} \) be an orthonormal basis for \( H_n^{d-2j} \). Let us denote
\[ M_{n-2j} = \frac{\lambda}{c_d^\mu} 2^{n-2j}. \]
Let \( q_j^{(\mu, \beta_{n-2j}, M_{n-2j})}(t) \) be the \( j \)-th orthogonal polynomial with respect to
\[ \langle f, g \rangle^{M_{n-2j}} = \int_{-1}^{1} f(t)g(t)(1-t)^\mu(1+t)^{\beta_{n-2j}}dt + M_{n-2j}f(1)g(1), \]
and having the same leading coefficient as the Jacobi polynomial \( P_j^{(\mu, \beta_{n-2j})}(t) \). Then the polynomials
\[ Q_n^{\nu}(x) = q_j^{(\mu, \beta_{n-2j}, M_{n-2j})}(2\|x\|^2 - 1) Y_n^{\nu-n/2}(x), \]
with \( 1 \leq j \leq n/2 \), \( 1 \leq \nu \leq a_{n-2j}^d \) constitute a mutually orthogonal basis of \( V_n^d(W_{\mu}, \lambda) \). That is,
\[ \langle Q_j^{\nu}(x), Q_k^{\eta}(x) \rangle_{\mu, \beta_{n-2j}}^\lambda = \tilde{H}_{j,\nu,\delta_{n,m}}^{n \mu} \delta_{j,k} \delta_{\nu,\eta}, \]
where \( \tilde{H}_{j,\nu,\delta_{n,m}}^{n \mu} \) is given by
\[ \tilde{H}_{j,\nu,\delta_{n,m}}^{n \mu} := \frac{c_d^\mu}{2^{n-2j}} h_j^{(\mu, \beta_{n-2j}, M_{n-2j})}, \]
with
\[ h_j^{(\mu, \beta_{n-2j}, M_{n-2j})} = (q_j^{(\mu, \beta_{n-2j}, M_{n-2j})}, q_j^{(\mu, \beta_{n-2j}, M_{n-2j})})_{\mu, \beta_{n-2j}}^{M_{n-2j}}. \]

**Proof.** The proof of this theorem uses the following well known identity
\[ \int_{B^d} f(x)dx = \int_0^1 r^{d-1} \int_{S^{d-1}} f(r \xi) d\sigma(\xi) dr \]
that arises from the spherical–polar coordinates \( x = r \xi, r = \|x\|, \xi \in S^{d-1} \).

In order to check the spherical–polar coordinates, we need to compute the product
\[ \langle Q_j^{\nu}(x), Q_k^{\eta}(x) \rangle_{\mu, \beta_{n-2j}}^\lambda = \frac{1}{\omega_{\mu}} \int_{B^d} Q_j^{\nu}(x)Q_k^{\eta}(x)W_\mu(x)dx + \frac{\lambda}{\sigma_{d-1}} \int_{S^{d-1}} Q_j^{\nu}(\xi)Q_k^{\eta}(\xi)d\sigma(\xi). \]
Let us start with the computation of the first integral.
\[ I_1 = \frac{1}{\omega_{\mu}} \int_{B^d} Q_j^{\nu}(x)Q_k^{\eta}(x)W_\mu(x)dx. \]
To simplify our notations, we will write \( q_j^{(\mu, \beta_{n-2j}, M_{n-2j})} = q_j^{(n-2j)} \). Using polar coordinates, relation (4.2), and the orthogonality of the spherical harmonics we obtain
\[
I_1 = \frac{\sigma d - 1}{\omega \mu} \int_0^1 q_j^{(n-2j)}(2r^2 - 1)q_k^{(m-2k)}(2r^2 - 1)(1 - r^2)^{\mu - 1} r^{n-2j + m-2k - 1} \, dr
\times \delta_{n-2j, m-2k} \delta_{\nu \eta}
= \frac{\sigma d - 1}{\omega \mu} \int_0^1 q_j^{(n-2j)}(2r^2 - 1)q_k^{(m-2k)}(2r^2 - 1)(1 - r^2)^{\mu - 1} r^{2(n-2j) + d - 1} \, dr
\times \delta_{n-2j, m-2k} \delta_{\nu \eta}.
\]

Finally, the change of variables \( t = 2r^2 - 1 \) moves the integral to the interval \([-1, 1]\),
\[
I_1 = \frac{c d}{2^{n-2j}} \int_{-1}^1 q_j^{(n-2j)}(t)q_k^{(n-2j)}(t)(1 - t)^{\mu - 1} (1 + t)^{\beta_{n-2j}} dt \delta_{n-2j, m-2k} \delta_{\nu \eta}.
\] (3.4)

For the second integral in (3.3) we get
\[
I_2 = \frac{\lambda}{\sigma d - 1} \int_{S^{d-1}} Q_{\nu, \nu'}^n(\xi)Q_{\mu, \eta}^n(\xi) d\sigma(\xi)
= \frac{\lambda}{\sigma d - 1} q_j^{(n-2j)}(1)q_k^{(m-2k)}(1) \int_{S^{d-1}} Y_{\nu}^{n-2j}(\xi)Y_{\eta}^{m-2k}(\xi) d\sigma(\xi)
= \lambda q_j^{(n-2j)}(1)q_k^{(m-2k)}(1) \delta_{n-2j, m-2k} \delta_{\nu \eta}.
\] (3.5)

To end the proof, we just have to put together (3.4) and (3.5) to get the value of (3.3) in terms of the inner product (3.1) as
\[
\langle Q_{\nu, \nu'}^n, Q_{\mu, \eta}^n \rangle = \frac{c d}{2^{n-2j}} \left( q_j^{(\mu, \beta_{n-2j}, M_{n-2j})}, q_k^{(\mu, \beta_{n-2j}, M_{n-2j})} \right)^{M_{n-2j}}_{\mu, \beta_{n-2j}} \times \delta_{n-2j, m-2k} \delta_{\nu \eta}.
\]
And the result follows from the orthogonality of the polynomial \( q_j^{(\mu, \beta_{n-2j}, M_{n-2j})} \). \( \square \)

4. The Uvarov modification of Jacobi polynomials

In this section we will consider the study of several properties of the univariate orthogonal polynomials involved in (3.1).

Let \( (\cdot, \cdot)_{\alpha, \beta}^M \) be the inner product defined in (3.1)
\[
(f, g)_{\alpha, \beta}^M = \int_{-1}^1 f(t) g(t) (1 - t)^{\alpha} (1 + t)^{\beta} dt + M f(1) g(1)
\] (4.1)
where \( M \) is a positive real number. Let \( \{q_k^{(\alpha, \beta; M)}(t)\}_{k \geq 0} \) be the orthogonal polynomials with respect to (4.1) having the same leading coefficient as the Jacobi polynomial \( P_k^{(\alpha, \beta)} \), and denote by \( K_k^{(\alpha, \beta; M)}(t, u) \) the corresponding kernels.

Following Uvarov (10) these univariate orthogonal polynomials can be expressed in terms of the classical Jacobi polynomials as the first identity in the following lemma shows. Some of these properties are very well known (see [8] p. 131) but we include here a sketch of the proof for the sake of completeness.
Lemma 4.1. For $\alpha, \beta > -1$, it holds

\begin{align}
q^\alpha_k(t) &= P^\alpha_k(t) - MP^\alpha_k(1)K_{k-1}^{(\alpha)}(1, t), \\
\hat{h}^\alpha_k(t) &= (q^\alpha_k, P^\alpha_k)_{\alpha, \beta} = h^\alpha_k(1) + MK^\alpha_k(1, 1), \\
\tilde{K}^\alpha_k(t, u) &= K_k^{(\alpha)}(t, u) - \frac{MK^\alpha_k(1, t)K^\alpha_k(1, u)}{1 + MK_k^{(\alpha)}(1, 1)}, \tag{4.4}
\end{align}

In particular

\begin{align}
\tilde{K}^\alpha_k(1, 1) &= \frac{K^\alpha_k(1, 1)}{1 + MK_k^{(\alpha)}(1, 1)}. \tag{4.5}
\end{align}

Proof. Expand $q^\alpha_k$ in terms of Jacobi polynomials,

\begin{equation}
q^\alpha_k(t) = \sum_{i=0}^k b^\alpha_{k,i}P^\alpha_i(t),
\end{equation}

where $b^\alpha_{k,k} = 1$. For $i = 0, \ldots, k-1$ we have

\begin{equation}
h^\alpha_i b^\alpha_{i,i} = (q^\alpha_k, P^\alpha_i)_{\alpha, \beta} = -Mq^\alpha_k(1)P^\alpha_i(1),
\end{equation}

thus

\begin{equation}
q^\alpha_k(t) = P^\alpha_k(t) - Mq^\alpha_k(1)K_{k-1}^{(\alpha)}(1, t),
\end{equation}

which gives

\begin{equation}
q^\alpha_k(1) = \frac{P^\alpha_k(1)}{1 + MK_{k-1}^{(\alpha)}(1, 1)},
\end{equation}

and therefore (4.2) holds.

Relation (4.3) follows again from (4.6) since

\begin{align}
\hat{h}^\alpha_k &= (q^\alpha_k, P^\alpha_k)_{\alpha, \beta} = q^\alpha_k + MP^\alpha_k(1) = h^\alpha_k(1) + MK^\alpha_k(1, 1), \\
\tilde{K}^\alpha_k(t, u) &= K_k^{(\alpha)}(t, u) - \frac{MK^\alpha_k(1, t)K^\alpha_k(1, u)}{1 + MK_k^{(\alpha)}(1, 1)}.
\end{align}

Now, from (4.2), (4.3) and the identity

\begin{equation}
K_k^{(\alpha)}(t, 1) - K_{k-1}^{(\alpha)}(t, 1) = \frac{P^\alpha_k(t)P^\alpha_k(1)}{h^\alpha_k},
\end{equation}
we can easily prove
\[
q_k^{(α,β,M)}(t) \frac{q_k^{(α,β,M)}(u)}{h_k^{(α,β,M)}} = P_k^{(α,β)}(t) \frac{P_k^{(α,β)}(u)}{h_k^{(α,β)}} - \frac{MK_k^{(α,β)}(1, t)K_k^{(α,β)}(1, u)}{1 + MK_k^{(α,β)}(1, 1)} + \frac{MK_{k-1}^{(α,β)}(1, t)K_{k-1}^{(α,β)}(1, u)}{1 + MK_{k-1}^{(α,β)}(1, 1)}
\]

and a telescopic sum gives (4.4).

5. The kernels

The main purpose of this section is to study the reproducing kernels associated to the orthogonal polynomials \(Q_{n,j,ν}^d(x)\). In particular, we will establish relations with the classical kernels on the unit ball. The \(n\)-th classical kernel on the ball is usually defined as the polynomial

\[
K_n(x, y) := \sum_{m=0}^{n} \left( \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{ν=1}^{a_n^d} \frac{P_m^{(j,ν)}(x)P_m^{(j,ν)}(y)}{H_m^{j,ν}} \right).
\]

Our next result provides a representation of the \(d\)-variable kernels in terms of the univariate Jacobi kernels.

**Theorem 5.1.** Let \(d \geq 3\), for \(x, y \in B^d\) we have

\[
K_n(x, y) = \frac{1}{C_d} \sum_{k=0}^{n} K_{\frac{κ(k+δ)}{2}}^{(μ, k+δ)}(2r^2 - 1, 2s^2 - 1) \times (2rs)^k \frac{k + δ}{δ} C_k^δ(⟨ξ, ̺⟩),
\]

where \(x = rξ, y = s ̺, r = \|x\|, s = \|y\|, ξ, ̺ ∈ S^{d-1}, δ = (d-2)/2\) and \(C_k^δ\) are the Gegenbauer polynomials ([9, (4.7.1) in p. 80]).

For \(d = 2\), (5.1) reduces to

\[
K_n(x, y) = \frac{1}{C_2} \sum_{k=0}^{n} K_{\frac{κ(k+δ)}{2}}^{(μ, k+δ)}(2r^2 - 1, 2s^2 - 1) \times 2k+1 (rs)^k T_k(⟨ξ, ̺⟩),
\]

where \(T_k\) are the first kind Chebyshev polynomials ([9, p. 38]).

**Proof.** For \(n \in \mathbb{N}_0\) and \(0 ≤ j ≤ n/2\), let \(\{Y^{n-2j}_ν : 1 ≤ ν ≤ a_n^d\}\) denote an orthonormal basis for \(H_{n-2j}^d\). In spherical-polar coordinates, \(x = rξ\) and \(y = s ̺\),
since $Y_{\nu}^{m-2j}$ is homogeneous of degree $m-2j$ we get

\[
\mathbb{K}_{n}(x, y) = \sum_{m=0}^{n} \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} a_{m-2j}^{j} \frac{P_{j,\mu}(x)P_{j,\nu}(y)}{H_{j,\mu,\nu}}
\]
\[
= \sum_{m=0}^{n} \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{1}{H_{j,\mu,\nu}} P_{j,(\mu,m-2j+\delta)}(2\|x\|^2 - 1) \, P_{j,(\mu,m-2j+\delta)}(2\|y\|^2 - 1)
\]
\[
\times a_{m-2j}^{j} Y_{\nu}^{m-2j}(x) Y_{\nu}^{m-2j}(y)
\]
\[
= \sum_{m=0}^{n} \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(rs)^{m-2j}}{c_{\nu}^{j} \mu} \frac{1}{H_{j,\mu,\nu}} P_{j,(\mu,m-2j+\delta)}(2\|x\|^2 - 1) \, P_{j,(\mu,m-2j+\delta)}(2\|y\|^2 - 1)
\]
\[
\times (rs)^{m-2j} \sum_{\nu=1}^{\left\lfloor \frac{m}{2} \right\rfloor} Y_{\nu}^{m-2j}(\xi) Y_{\nu}^{m-2j}(\zeta).
\]

Making use of the addition formula of spherical harmonics for $d \geq 3$ (see [1, p. 9])

\[
\sum_{\nu=1}^{\left\lfloor \frac{m}{2} \right\rfloor} Y_{\nu}^{m-k}(\xi) Y_{\nu}^{m-k}(\zeta) = \frac{k + \delta}{\delta} C_{k}^{\delta}(\langle \xi, \zeta \rangle)
\]

we have

\[
\mathbb{K}_{n}(x, y) = \sum_{m=0}^{n} \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(rs)^{m-2j} \sum_{\nu=1}^{\left\lfloor \frac{m}{2} \right\rfloor} Y_{\nu}^{m-k}(\xi) Y_{\nu}^{m-k}(\zeta)}{c_{\nu}^{j} \mu} \frac{1}{H_{j,\mu,\nu}} P_{j,(\mu,m-2j+\delta)}(2\|x\|^2 - 1) \, P_{j,(\mu,m-2j+\delta)}(2\|y\|^2 - 1)
\]
\[
\times (2rs)^{m-2j} \frac{m - 2j + \delta}{\delta} C_{m-2j}^{\delta}(\langle \xi, \zeta \rangle).
\]

Now, make $k = m - 2j$ to change the order in the double sum

\[
\mathbb{K}_{n}(x, y) = \sum_{m=0}^{n} \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(rs)^{k}}{c_{\nu}^{k} \mu} \frac{1}{h_{j,\mu,\nu,\theta}} P_{j,(\mu,k+\delta)}(2\|x\|^2 - 1) \, P_{j,(\mu,k+\delta)}(2\|y\|^2 - 1)
\]
\[
\times (2rs)^{k} \frac{k + \delta}{\delta} C_{k}^{\delta}(\langle \xi, \zeta \rangle),
\]

and therefore

\[
\mathbb{K}_{n}(x, y) = \sum_{k=0}^{n} \frac{(rs)^{k}}{c_{\nu}^{k} \mu} \frac{1}{h_{j,\mu,\nu,\theta}} \frac{1}{h_{j,\mu,\nu,\theta}} P_{j,(\mu,k+\delta)}(2\|x\|^2 - 1) \, P_{j,(\mu,k+\delta)}(2\|y\|^2 - 1)
\]
\[
\times (2rs)^{k} \frac{k + \delta}{\delta} C_{k}^{\delta}(\langle \xi, \zeta \rangle),
\]

The case $d = 2$ follows from the limit relation

\[
\lim_{\delta \to 0} \frac{k + \delta}{\delta} C_{k}^{\delta}(t) = 2 T_{k}(t),
\]

(see [1] (4.7.8) in p. 80).
In a similar way, we define the \( n \)-th kernel associated to the polynomials \( Q_{j,\nu}^m(x) \) as
\[
\tilde{\mathcal{K}}_n(x, y) := \frac{n}{\sum_{m=0}^{\infty} \sum_{j=0}^{d} \sum_{\nu=1}^{\infty} \frac{Q_{j,\nu}^m(x)Q_{j,\nu}^m(y)}{H_{j,\nu}^m}}.
\]
Proceeding as in Theorem 5.1 we can obtain a representation of this kernels in terms of the univariate kernels associated to the Uvarov modifications. Thus, for \( x, y \in \mathbb{B}^d, d \geq 3 \), we have
\[
(5.3) \quad \tilde{\mathcal{K}}_n(x, y) = \frac{1}{c_{\mu}^d} \sum_{k=0}^{n} \tilde{K}^{(\mu, k+\delta, M_k)}(2r^2 - 1, 2s^2 - 1) \times (2r s)^k \frac{k+\delta \epsilon_\lambda}{\delta} C_k^\delta(\langle \xi, \vartheta \rangle).
\]
For \( d = 2 \)
\[
(5.4) \quad \tilde{\mathcal{K}}_n(x, y) = \frac{1}{c_{\mu}^d} \sum_{k=0}^{n} \tilde{K}^{(\mu, k+\delta, M_k)}(2r^2 - 1, 2s^2 - 1) \times 2^{k+1} (r s)^k T_k(\langle \xi, \vartheta \rangle).
\]
Finally, from [4.4] we derive a formula connecting both kernels in terms of the classical Jacobi kernels.

**Proposition 5.2.** Let \( x = r \xi, \ y = s \vartheta, \ r = \|x\|, \ s = \|y\|, \ \xi, \vartheta \in \mathbb{S}^{d-1} \). For \( n \geq 0 \) and \( d \geq 3 \), we get
\[
\mathcal{K}_n(x, y) - \tilde{\mathcal{K}}_n(x, y) = \frac{1}{c_{\mu}^d} \sum_{k=0}^{n} M_k \tilde{K}^{(\mu, k+\delta)}(2r^2 - 1, 1) \times 2^k (r s)^k \frac{k+\delta \epsilon_\lambda}{\delta} C_k^\delta(\langle \xi, \vartheta \rangle).
\]
For \( d = 2 \)
\[
\mathcal{K}_n(x, y) - \tilde{\mathcal{K}}_n(x, y) = \frac{1}{c_{\mu}^d} \sum_{k=0}^{n} M_k \tilde{K}^{(\mu, k+\delta)}(2r^2 - 1, 1) \times 2^{k+1} (r s)^k T_k(\langle \xi, \vartheta \rangle).
\]

### 6. Asymptotics for Christoffel functions

In this section we shall show some asymptotic results for the Christoffel functions. We must restrict ourselves to the case \( \mu \geq -\frac{1}{2} \) because of existing asymptotics for Christoffel functions on the ball have only been established for this range of values. Our results include asymptotics for the interior of the ball as well as for its boundary.

On the boundary of the ball, we recover the value of the mass from the asymptotic of the Christoffel functions.

**Theorem 6.1.** Assume that \( \mu \geq -\frac{1}{2} \). For \( \|x\| = 1 \), we get
\[
(6.1) \quad \lim_{n \to \infty} \frac{\tilde{\mathcal{K}}_n(x, x)}{(n+d-1)/n} = \frac{2}{\lambda}.
\]
Proof. From (5.3) and (4.5), for \( \|x\| = 1 \) we deduce

\[
\tilde{K}_n(x, x) = \frac{1}{c_d^\mu} \sum_{k=0}^{n} \frac{K^{(\mu, k+\delta)}(\frac{n-k}{2}, \frac{n-k}{2})}{1 + M_k K^{(\mu, k+\delta)}(\frac{n-k}{2}, \frac{n-k}{2})} \times 2^k \frac{k + \delta}{\delta} C_k^\delta(1),
\]

then, writing

\[
\delta = \frac{d - 2}{2}, \quad M_k = \frac{\lambda}{c_d^\mu} 2^k \quad \text{and} \quad C_k^\delta(1) = \binom{k + 2\delta - 1}{k}
\]

we get

\[
\tilde{K}_n(x, x) = \frac{1}{\delta} \sum_{k=0}^{n} \frac{2^k K_m^{(\mu, k+\delta)}(1, 1)}{c_d^\mu + \lambda 2^k K_m^{(\mu, k+\delta)}(1, 1)} (k + \delta) \binom{k + d - 3}{k},
\]

where we denote \( m = \left\lfloor \frac{n-k}{2} \right\rfloor \).

Let us split the above sum in two parts

\[
\tilde{K}_n(x, x) = \frac{1}{\delta} (S_{1,n} + S_{2,n}),
\]

with

\[
S_{1,n} = \sum_{k=0}^{n-\lfloor \log n \rfloor} \frac{2^k K_m^{(\mu, k+\delta)}(1, 1)}{c_d^\mu + \lambda 2^k K_m^{(\mu, k+\delta)}(1, 1)} (k + \delta) \binom{k + d - 3}{k},
\]

\[
S_{2,n} = \sum_{k=n-\lfloor \log n \rfloor + 1}^{n} \frac{2^k K_m^{(\mu, k+\delta)}(1, 1)}{c_d^\mu + \lambda 2^k K_m^{(\mu, k+\delta)}(1, 1)} (k + \delta) \binom{k + d - 3}{k},
\]

In order to estimate \( S_{1,n} \) we use relation (2.4) to obtain

\[
2^k K_m^{(\mu, k+\delta)}(1, 1) = \frac{2^{-\mu-\delta-1}}{\Gamma(\mu+1)\Gamma(\mu+2)} \frac{\Gamma(m+k+\mu+\delta+2)\Gamma(m+\mu+2)}{\Gamma(m+k+\delta+1)\Gamma(m+1)}
\]

Now, we can use the following consequence of Stirling’s formula: for fixed \( a, b \), as \( x \to \infty \),

\[
\frac{\Gamma(x+b)}{\Gamma(x+a)} = x^{b-a}(1 + o(1)),
\]

in this way

\[
2^k K_m^{(\mu, k+\delta)}(1, 1) = \frac{2^{-\mu-\delta-1}}{\Gamma(\mu+1)\Gamma(\mu+2)} (m+k)^{\mu+1} m^{\mu+1}(1 + o(1)).
\]

Hence it follows

\[
(6.2) \quad \frac{2^k K_m^{(\mu, k+\delta)}(1, 1)}{c_d^\mu + \lambda 2^k K_m^{(\mu, k+\delta)}(1, 1)} = \frac{1}{\lambda} (1 + o(1))
\]

as \( m \to \infty \).
Next, we estimate \((k + \delta)\binom{k + d - 3}{k}\). For \(0 \leq k \leq n\) we have

\[
\frac{1}{n^{d-2}} (k + \delta) \binom{k + d - 3}{k} = \frac{1}{(d - 3)!} \left( \frac{k + d - 2}{n} \right) \left( \frac{k + d - 3}{n} \right) \ldots \left( \frac{k + 1}{n} \right)
\]

\[
= \frac{1}{(d - 3)!} \left( \frac{k}{n} \right)^{d-2} + a_0 \left( \frac{k}{n} \right)^{d-3} + \ldots + a_{d-3} \left( \frac{k}{n} \right)^{d-2}
\]

(6.3)

\[
\leq \frac{1}{(d - 3)!} \left( \frac{k}{n} \right)^{d-2} + C
\]

where \(C = a_0 + a_1 + \ldots + a_{d-3} > 0\) is independent of \(k\) and \(n\). Therefore

\[
\frac{S_{1,n}}{n^{d-1}} = \frac{1}{\lambda \delta n} \sum_{k=0}^{n-\lfloor \log n \rfloor} \frac{1}{(d - 3)!} \left( \frac{k}{n} \right)^{d-2} + O\left( \frac{1}{n} \right) \left( 1 + o(1) \right)
\]

\[
= \frac{2}{\lambda (d-2)! n} \sum_{k=0}^{n-\lfloor \log n \rfloor} \left( \frac{k}{n} \right)^{d-2} + O\left( \frac{1}{n} \right) \left( 1 + o(1) \right).
\]

Next, since \(0 \leq k \leq n\), we have

\[
0 \leq \frac{1}{n} \sum_{k=0}^{n-\lfloor \log n \rfloor} \left( \frac{k}{n} \right)^{d-2} \leq \frac{1}{n} \sum_{k=0}^{n} 1 \leq \frac{\log n}{n}
\]

which implies

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-\lfloor \log n \rfloor} \left( \frac{k}{n} \right)^{d-2} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \left( \frac{k}{n} \right)^{d-2} = \frac{1}{d-1},
\]

where the last equality follows from Silverman-Toeplitz theorem (see [11, p. 25]). In this way, we conclude

\[
\lim_{n \to \infty} \frac{S_{1,n}}{n^{d-1}} = \frac{2}{\lambda}.
\]

On the other hand

\[
S_{2,n} = \sum_{k=n-\lfloor \log n \rfloor + 1}^{n} \frac{2^k k^{(\mu, k, k+d)_{1, 1}}(1, 1)}{c_{\mu}^d + \lambda 2^k k^{(\mu, k, k+d)_{1, 1}}(1, 1)} (k + \delta) \binom{k + d - 3}{k}
\]

\[
\leq \frac{1}{\lambda} \sum_{k=n-\lfloor \log n \rfloor + 1}^{n} (k + \delta) \binom{k + d - 3}{k}
\]

Using again (6.3), for \(0 \leq k \leq n\) we have

\[
\frac{1}{n^{d-2}} (k + \delta) \binom{k + d - 3}{k} \leq \frac{1}{(d - 3)!} \left( \frac{k}{n} \right)^{d-2} + C < C'
\]

where \(C' > 0\) is a constant independent of \(k\) and \(n\). Hence, we deduce

\[
\frac{S_{2,n}}{n^{d-1}} \leq \frac{C'}{\lambda} \frac{\log n}{n},
\]

this implies

\[
\lim_{n \to \infty} \frac{S_{2,n}}{n^{d-1}} = 0,
\]

and (6.1) follows for \(d \geq 3\).
In the case $d = 2$ and $\|x\| = 1$, from (5.4) we have
\[
\tilde{K}_n(x, x) = 2 \sum_{k=0}^{n} \frac{2^k K_m^{(\mu,k)}(1, 1)}{c_\mu^d + \lambda 2^k K_m^{(n,k)}(1, 1)}
\]
then, proceeding in the same way, from (6.2) we conclude
\[
\lim_{n \to \infty} \frac{1}{n + 1} \tilde{K}_n(x, x) = \frac{2}{\lambda}.
\]

In the following proposition an estimate on the reproducing kernels that is uniform in $k$ is obtained,

**Lemma 6.2.** Fix $\mu > -1, \delta \geq 0$. For $m = \lfloor \frac{n-k}{2} \rfloor \geq 1, k \geq 0$, and $t \in [-1, 1]$,
\[
K_m^{(\mu,k+\delta)}(t, t) \leq C (1 + t)^{-k} \left( \begin{pmatrix} n \cr \frac{k}{2} \end{pmatrix} + 1 \right)^{-\mu - \frac{1}{2}} \left( 1 + t + \frac{1}{(\frac{n}{2} + 1)^2} \right)^{-\delta - \frac{1}{2}}.
\]

Here $C$ depends on $\mu$ and $\delta$ but not on $k, n, t$.

**Proof.** From the extremal properties of Christoffel functions, for $k$ even, say $k = 2\ell$,
we have
\[
K_m^{(\mu,k+\delta)}(t, t) = \sup_{\deg(P) \leq m} \frac{P^2(t)}{\int_{-1}^{1} P^2(s) (1-s)^{\mu} (1+s)^{k+\delta} \, ds} \leq (1 + t)^{-k} \sup_{\deg(P) \leq m} \left( \frac{P(t)(1+t)^{\ell}}{\int_{-1}^{1} \left( P(s)(1+s)^{\ell} \right)^2 (1-s)^{\mu} (1+s)^{\delta} \, ds} \right)^2 \leq (1 + t)^{-k} \sup_{\deg(R) \leq m+\ell} \frac{R(t)^2}{\int_{-1}^{1} R(s)^2 (1-s)^{\mu} (1+s)^{\delta} \, ds} = (1 + t)^{-k} K_m^{(\mu,\delta)}(t, t) \leq (1 + t)^{-k} K_{\lfloor \frac{n}{2} \rfloor}^{(\mu,\delta)}(t, t).
\]

We now use a result from Nevai’s 1979 Memoir [8, p. 108, Lemma 5],
\[
K_{\lfloor \frac{n}{2} \rfloor}^{(\mu,\delta)}(t, t) \leq C \left( \begin{pmatrix} n \cr \frac{n}{2} \end{pmatrix} + 1 \right)^{-\mu - \frac{1}{2}} \left( 1 + t + \frac{1}{(\frac{n}{2} + 1)^2} \right)^{-\delta - \frac{1}{2}},
\]
and the result follows for $k = 2\ell$. 
The case \( k = 2\ell + 1 \) can be deduced from the above reasoning as follows

\[
K_m^{(\mu, 2\ell + 1)}(t, t) \leq (1 + t)^{-2\ell} K_m^{(\mu, \delta + 1)}(t, t)
\]

\[
\leq C(1 + t)^{-2\ell} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)
\times \left( 1 - t + \frac{1}{\left\lceil \frac{n}{2} \right\rceil + 1} \right)^{-\mu - \frac{1}{2}} \left( 1 + t + \frac{1}{\left\lceil \frac{n}{2} \right\rceil + 1} \right)^{-\delta - \frac{1}{2}}
\leq C(1 + t)^{-2\ell-1} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)
\times \left( 1 - t + \frac{1}{\left\lceil \frac{n}{2} \right\rceil + 1} \right)^{-\mu - \frac{1}{2}} \left( 1 + t + \frac{1}{\left\lceil \frac{n}{2} \right\rceil + 1} \right)^{-\delta - \frac{1}{2}},
\]

since for \( t \in [-1, 1] \) we get \( 0 \leq 1 + t < 1 + t + \frac{1}{\left\lceil \frac{n}{2} \right\rceil + 1} \).

\( \Box \)

**Theorem 6.3.** For \( r = \|x\| < 1 \), we have

\[
0 < \| x \| - \tilde{K}_n(x, x) \leq C n^{d-1} \log n \left( 2(1 - r^2) + \frac{4}{n^2} \right)^{-\mu - \frac{1}{2}} \left( 2r^2 + \frac{4}{n^2} \right)^{-\delta - \frac{1}{2}}.
\]

Here \( C \) is independent of \( n \) and \( x \). Consequently if \( \mu \geq -\frac{1}{2} \), uniformly for \( x \) in compact subsets of \( \{ x : 0 < \| x \| < 1 \} \),

\[
\lim_{n \to \infty} \tilde{K}_n(x, x)/\left( \frac{n + d}{d} \right) = \frac{1}{\sqrt{\pi}} \Gamma(\mu + 1) \Gamma\left( \frac{d+1}{2} \right) \left( 1 - \|x\|^2 \right)^{-\frac{d}{2} - \mu}.
\]

This last limit also holds for \( x = 0 \).

**Proof.** Let us consider \( x \in D \), with \( D \) a compact subset of \( \mathbb{R}^d \). With \( t = 2r^2 - 1 \) and \( r = \|x\| \) we can assume that \( t \leq 1 - \eta \) for some \( \eta > 0 \). Then, from Christoffel–Darboux formula and using the convention \( p_j = p_j^{(\mu, k+\delta)} \) for orthonormal Jacobi polynomials, we have

\[
K_m^{(\mu, k+\delta)}(t, 1) = \frac{\gamma_m}{\gamma_{m+1}} \left| \frac{p_{m+1}(t)p_m(1) - p_m(t)p_{m+1}(1)}{t - 1} \right|
\leq C \frac{\sqrt{p_m^2(t) + p_{m+1}^2(t)} \sqrt{p_m^2(1) + p_{m+1}^2(1)}}{\eta}
\leq \frac{C}{2\eta} K_m^{(\mu, k+\delta)}(t, 1)^{1/2} \sqrt{p_m^2(1) + p_{m+1}^2(1)},
\]

where \( \gamma_m \) is the leading coefficient of \( p_m \).

Next, we note that given any real number \( a \), there exists \( C_a > 1 \) such that for all \( x \) with \( \min(x, x+a) \geq 1 \),

\[
C_a^{-1} x^a \leq \frac{\Gamma(x+a)}{\Gamma(x)} \leq C_a x^a.
\]
This follows from Stirling’s formula and the positivity and continuity of $\frac{\Gamma(x+a)}{\Gamma(x)}$ for this range of $x$. Then from (2.4) and (2.2) we get

$$
\left( p_{m,k+\delta}^{(\mu)}(1) \right)^2 \leq C \frac{(m+k)^{\mu+1} m^\mu}{2^k}.
$$

Substituting these bounds into (6.6) for $m \geq 1$ we have

$$
\left( K_m^{(\mu,k+\delta)}(t,1) \right)^2 \leq CK_{m+1}(t,t) \frac{(m+k)^{\mu+1} m^\mu}{2^k}.
$$

In the same way, using (2.4), we deduce

$$
2^k K_m^{(\mu,k+\delta)}(1,1) \geq C' (m+k)^{\mu+1} m^{\mu+1}.
$$

And therefore we conclude

$$
\frac{2^k \left( K_m^{(\mu,k+\delta)}(t,1) \right)^2}{c_\mu + \lambda 2^k K_m^{(\mu,k+\delta)}(1,1)} \leq \frac{CK_{m+1}(t,t)(m+k)^{\mu+1} m^\mu}{c_\mu + \lambda C'(m+k)^{\mu+1} m^{\mu+1}} \leq C K_{m+1}(t,t) \frac{m+1}{m+1}.
$$

(6.7)

This bound holds also for $m = 0$, thought it is obtained in a simpler way since $K_0$ is a constant.

Then, for $d \geq 3$, we have

$$
0 \leq K_n(x,x) - \tilde{K}_n(x,x) \leq C \sum_{k=0}^n 2^k (k+d-3) r^{2k} K_{m+1}(t,t) \frac{m+1}{m+1}.
$$

Using the bound (6.4) and denoting $t = 2r^2 - 1$ and $m = \left\lfloor \frac{n-k}{2} \right\rfloor$, Lemma 6.2 gives

$$
0 \leq K_n(x,x) - \tilde{K}_n(x,x) \leq C n^{d-2} \sum_{k=0}^n \frac{\left\lfloor \frac{n}{2} \right\rfloor + 2}{\left\lfloor \frac{n-k}{2} \right\rfloor + 1} \frac{m+1}{m+1}.
$$

where the last inequality follows from

$$
\lim_{n \to \infty} \log n \sum_{k=0}^n \frac{1}{\left\lfloor \frac{n-k}{2} \right\rfloor + 1} = 2.
$$

Obviously the above inequality holds also in the case $d = 2$. 
Finally, [5, Theorem 1.3] shows
\[
\lim_{n \to \infty} K_n(x, x) / \binom{n + d}{d} = \frac{\omega_\mu W_0(x)}{(1 - \|x\|^2)^\mu} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\mu + 1) \Gamma(d + \frac{1}{2})}{\Gamma(\mu + \frac{d}{2} + 1)} \left(1 - \|x\|^2\right)^{-\frac{d}{2} - \mu},
\]
uniformly for $x$ in compact subsets of the unit ball. Consequently $K_n(x, x)$ grows like $n^d \gg n^{d-1} \log n$, and clearly (6.5) follows.

Finally, in the case $x = 0$, that is $r = 0$, all the terms in $K_n(0, 0) - \tilde{K}_n(0, 0)$ vanish except for $k = 0$. If we write $m = \lfloor \frac{n}{2} \rfloor$ we get
\[
K_n(0, 0) - \tilde{K}_n(0, 0) = \frac{\lambda \left(K_m^{(\mu, \delta)}(-1, 1)\right)^2}{e_\mu^d + \lambda K_m^{(\mu, \delta)}(1, 1)} \leq \frac{CK_m^{(\mu, \delta)}(-1, -1) m^{2\mu + 1}}{e_\mu^d + C' m^{2\mu + 2}} \leq \frac{K_m^{(\mu, \delta)}(-1, -1)}{m + 1}.
\]
Next, Lemma 6.2 implies
\[
K_n(0, 0) - \tilde{K}_n(0, 0) \leq C \left(2 + \frac{1}{\left(\lfloor \frac{n}{2} \rfloor + 2\right)^2}\right)^{-\mu - \frac{1}{2}} \left(1 + \frac{1}{\left(\lfloor \frac{n}{2} \rfloor + 2\right)^2}\right)^{-\delta - \frac{1}{2}} \leq C \left(\left(\frac{n}{2}\right) + 2\right)^{2\delta + 1} \leq C n^{d-1}
\]
and, therefore, (6.5) follows also in this case.

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