Spherical harmonic polynomials for higher bundles

Yasushi Homma*

Abstract

We give a method of decomposing bundle-valued polynomials compatible with the action of the Lie group $Spin(n)$, where important tools are $Spin(n)$-equivariant operators and their spectral decompositions. In particular, the top irreducible component is realized as an intersection of kernels of these operators.

0 Introduction

Spherical harmonic polynomials or spherical harmonics are polynomial solutions of the Laplace equation $\Box \phi(x) = \sum \partial^2 \phi / \partial x_i^2 = 0$ on $\mathbb{R}^n$. These are fundamental and classical objects in mathematics and physics. It is natural that we consider vector-valued spherical harmonic polynomials. For example, the polynomial solutions of the Dirac equation $D\phi(x) = 0$ on $\mathbb{R}^n$ are studied in Clifford analysis (see [3], [8], and [14]). They are spinor-valued polynomials and called spherical monogenics. We also have other examples in [5], [7], [9], and [12], where we can give spectral information of some basic operators on sphere. Recently, the first-order $Spin(n)$-equivariant differential operators have been studied like Dirac operator and Rarita-Schwinger operator (see [1]-[5], [10], and [11]). These operators are called higher spin Dirac operators or Stein-Weiss operators. In this paper, we give a method to analyze polynomial sections for natural bundles on $\mathbb{R}^n$ by using higher spin Dirac operators and Clifford homomorphisms. Here, Clifford homomorphism is a natural generalization of Clifford algebra given in [10] and [11].

Let $S^q$ (resp. $H^q$) be the spaces of polynomials (resp. harmonic polynomials) with degree $q$ on the $n$-dimensional Euclidean space $\mathbb{R}^n$. We know

*Department of Mathematical Sciences, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, JAPAN.

*e-mail address: homma@gm.math.waseda.ac.jp
that $H^q$ is an irreducible representation space for $Spin(n)$, and $S^q$ has irreducible decomposition, $\oplus_{0\leq k\leq [q/2]} r^{2k} H^{q-2k}$, where $r$ is $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

To give such a decomposition, we use the invariant operator $-r^2\Box$ and its spectral decomposition. In particular, the top component $H^q$ is the kernel of the operator $-r^2\Box$. Now, we consider natural irreducible bundle $\mathbb{R}^n \times V_\rho$ on $\mathbb{R}^n$, where $V_\rho$ is an irreducible representation space with highest weight $\rho$ for $Spin(n)$. Our interest is to analyze the space of $V_\rho$-valued polynomials, $S^q \otimes V_\rho$. For that purpose, we use higher spin Dirac operators $\{D_{\lambda_k}\}_k$ and algebraic operators $\{x_{\lambda_k}\}_k$. Then we have an invariant operator $E$ whose spectral decomposition gives the irreducible decomposition of $S^q \otimes V_\rho$ like the operator $-r^2\Box$. In particular, the top irreducible component is the kernel of $E$ and realized as an intersection of kernels of higher spin Dirac operators.

### 1 Clifford Homomorphisms

In this section, we review Clifford homomorphisms given in [11]. Let $\mathfrak{spin}(n) \simeq \mathfrak{so}(n)$ be the Lie algebra of the spin group $Spin(n)$ or orthogonal group $SO(n)$. The Lie algebra $\mathfrak{spin}(n)$ is realized by using the Clifford algebra $Cl_n$ associated to $\mathbb{R}^n$: we choose the standard basis $\{e_i\}_i$ of $\mathbb{R}^n$ and put $[e_i, e_j] := e_i e_j - e_j e_i$ in $Cl_n$. Then $\{[e_i, e_j]\}_{i,j}$ span the Lie algebra $\mathfrak{spin}(n)$ in $Cl_n$.

The irreducible finite dimensional unitary representations of $\mathfrak{spin}(n)$ or $Spin(n)$ are parametrized by dominant weights $\rho = (\rho^1, \ldots, \rho^m) \in \mathbb{Z}^m \cup (\mathbb{Z} + 1/2)^m$ satisfying that

$$\rho^1 \geq \cdots \geq \rho^{m-1} \geq |\rho^m|, \quad \text{for } n = 2m, \quad (1.1)$$

$$\rho^1 \geq \cdots \geq \rho^{m-1} \geq \rho^m \geq 0, \quad \text{for } n = 2m + 1. \quad (1.2)$$

We denote by $(\pi_\rho, V_\rho)$ not only the representation of $Spin(n)$ but also its infinitesimal one of $\mathfrak{spin}(n)$ with highest weight $\rho$. When writing dominant weights, we denote a string of $j$ k’s for $k$ in $\mathbb{Z} \cup (\mathbb{Z} + 1/2)$ by $k_j$. For example, the adjoint representation $(\text{Ad}, \mathbb{R}^n \otimes \mathbb{C})$ of $Spin(n)$ (resp. $\mathfrak{spin}(n)$) has the highest weight $(1, 0_{m-1})$, where the action is $\pi_{\text{Ad}}(g)u = gug^{-1}$ for $g$ in $Spin(n)$ (resp. $\pi_{\text{Ad}}([e_i, e_j])u := [e_i, e_j], u)$).

We consider an irreducible representation $(\pi_\rho, V_\rho)$ and the tensor representation $(\pi_\rho \otimes \pi_{\text{Ad}}, V_\rho \otimes \mathbb{C} \mathbb{R}^n)$. We decompose it to irreducible components, $V_\rho \otimes \mathbb{C} \mathbb{R}^n = \sum_{0 \leq k \leq N} V_{\lambda_k}$. For $u$ in $\mathbb{R}^n$, we have the following bilinear mapping for each $k$:

$$\mathbb{R}^n \times V_\rho \ni (u, \phi) \mapsto p_{\lambda_k}^\rho(u)\phi := \Pi_{\lambda_k}^\rho(\phi \otimes u) \in V_{\lambda_k}, \quad (1.3)$$
where $\Pi^\rho_{\lambda_k}$ is the orthogonal projection from $V_\rho \otimes_\mathbb{C} \mathbb{R}^n$ onto $V_{\lambda_k}$. We call the linear mapping $p^\rho_{\lambda_k}(u) : V_\rho \to V_{\lambda_k}$ the Clifford homomorphism from $V_\rho$ to $V_{\lambda_k}$, and denote by $(p^\rho_{\lambda_k}(u))^*$ the adjoint operator of $p^\rho_{\lambda_k}(u)$ with respect to the inner products on $V_\rho$ and $V_{\lambda_k}$. If we consider the spinor representation $(\pi_\Delta, V_\Delta)$, then the Clifford homomorphism from $V_\Delta$ to itself is the usual Clifford action of $\mathbb{R}^n$ on $V_\Delta$, which satisfy the relation $e_i e_j + e_j e_i = -\delta_{ij}$. In general cases, we have a lot of relations among these homomorphisms.

**Theorem 1.1 ([11]).** For any non-negative integer $q$, we define the bilinear mapping $r^q_\rho$ as follows:

$$r^q_\rho : \mathbb{R}^n \times \mathbb{R}^n \ni (u, v) \mapsto \left(-\frac{1}{4}\right)^q \sum_{l_1, \ldots, l_{q-1}} \pi_\rho([u, e_{l_1}]) \pi_\rho([e_{l_1}, e_{l_2}]) \cdots \pi_\rho([e_{l_{q-1}}, v]) \in \text{End}(V_\rho),$$

(1.4)

and $r^0_\rho(u, v) := \langle u, v \rangle$. Then we have

$$\sum_{0 \leq k \leq N} m(\lambda_k)^q (p^\rho_{\lambda_k}(u))^* p^\rho_{\lambda_k}(v) = r^q_\rho(u, v),$$

(1.5)

where $m(\lambda_k)$ is the conformal weight assigned from $V_\rho$ to $V_{\lambda_k}$.

In this paper, we will use the case of $q = 0$ and $q = 1$:

$$\sum_{0 \leq k \leq N} (p^\rho_{\lambda_k}(e_j))^* p^\rho_{\lambda_k}(e_i) = \delta_{ij},$$

(1.6)

$$\sum_{0 \leq k \leq N} m(\lambda_k)(p^\rho_{\lambda_k}(e_j))^* p^\rho_{\lambda_k}(e_i) = -\frac{1}{4} \pi_\rho([e_j, e_i]).$$

(1.7)

**Remark 1.1.** The endomorphisms $\{r^q_\rho(e_i, e_j)\}_{i,j}$ are useful to compute the eigenvalues of the higher Casimir operators (see [13] and [14]).

The Clifford homomorphisms also satisfy the following properties.

**Proposition 1.2 ([11]).** Let $u$ be in $\mathbb{R}^n$, $g$ in $\text{Spin}(n)$, and $[e_i, e_j]$ in $\text{spin}(n)$. Then we have

$$p^\rho_{\lambda_k}(g u g^{-1}) = \pi_{\lambda_k}(g) p^\rho_{\lambda_k}(u) \pi_{\rho}(g^{-1}),$$

(1.8)

and

$$p^\rho_{\lambda_k}([e_i, e_j], u) = \pi_{\lambda_k}([e_i, e_j]) p^\rho_{\lambda_k}(u) - p^\rho_{\lambda_k}(u) \pi_{\rho}([e_i, e_j]).$$

(1.9)
2 Invariant operators on polynomials for higher bundles

In the first part of this section, we give a well-known method to decompose the space of complex-valued polynomials on \( \mathbb{R}^n \). We denote the canonical coordinate on \( \mathbb{R}^n \) by \((x_1, \cdots, x_n)\), and the space of complex-valued polynomials with degree \( q \) on \( \mathbb{R}^n \) by \( S^q \). The vector space \( \sum q S^q \) has the Hermitian inner product satisfying \((\partial/\partial x_i f(x), g(x)) = (f(x), x_i g(x))\). The polynomial representation \((\pi_s, \sum S^q)\) of \( \text{spin}(n) \) is defined by

\[
(\pi_s([e_k, e_l])f)(x) := 4(-x_k \frac{\partial}{\partial x_l} + x_l \frac{\partial}{\partial x_k})f(x).
\] (2.1)

To decompose the space \( \sum S^q \), we use invariant operators compatible with the action of \( \text{spin}(n) \). When the operator on \( S^q \) maps to \( S^{q-k} \), the order of the operator is said to be \( k \). On \( \sum S^q \), we have the following invariant operators: the Laplacian operator \( \Box := -\sum \frac{\partial^2}{\partial x_i^2} \), and the 0-th order operator \( r\partial/\partial r = \sum x_i\partial/\partial x_i \) called the Euler operator, where \( r^2 = \sum x_i^2 \). The Euler operator measures the degree of polynomials. In other words, the vector space \( S^q \) is the eigenspace with eigenvalue \( q \) for the operator \( r\partial/\partial r \). To decompose \( S^q \) further, we use the 0-th order invariant operator \(-r^2\Box\). This operator has the spectral decomposition corresponding to the irreducible decomposition. In fact, we show that \( S^q \) is isomorphic to \( \bigoplus_{0 \leq k \leq [q/2]} r^{2k}H^{q-2k} \) and the eigenvalue of \(-r^2\Box\) on \( r^{2k}H^{q-2k} \) is \( k(2q - 2k + n - 2) \), where \( H^q \) is the space of harmonic polynomials with degree \( q \). In particular, the top component \( H^q \) is the kernel of \(-r^2\Box\) and has the highest weight \( h^q := (q, 0, \cdots, 0) \). Thus, to decompose a representation space into irreducible components, we should investigate the spectral decompositions of invariant operators.

Now, we shall consider the space of polynomials for higher bundles on \( \mathbb{R}^n \). Let \((\pi_\rho, V_\rho)\) be an irreducible unitary representation of \( \text{spin}(n) \). Then we have the (trivial) higher bundle \( S_\rho := \mathbb{R}^n \times V_\rho \), and consider the polynomial sections of \( S_\rho \), that is, the \( V_\rho \)-valued polynomials \( \sum S^q \otimes V_\rho \). This vector space is a representation space on where more invariant operators exist in addition to \(-r^2\Box\) and \( r\partial/\partial r \). Here, the action of \( \text{spin}(n) \) on \( \sum S^q \otimes V_\rho \) is given as the tensor representation:

\[
\text{spin}(n) \times S^q \otimes V_\rho \ni ([e_k, e_l], f \otimes \phi) \rightarrow
4(-x_k \frac{\partial}{\partial x_l} + x_l \frac{\partial}{\partial x_k})f \otimes \phi + f \otimes \pi_\rho([e_k, e_l])\phi \in S^q \otimes V_\rho.
\] (2.2)

We recall the Clifford homomorphism from \( V_\rho \) to \( V_\lambda \) given in Section 1.
By using the Clifford homomorphism, we introduce the following operators:

\[ x^\rho_{\lambda_k} := \sum x_i D^\rho_{\lambda_k}(e_i) : S^q \otimes V_\rho \rightarrow S^{q+1} \otimes V_{\lambda_k}, \quad (2.3) \]

\[ (x^\rho_{\lambda_k})^* := \sum x_i (p^\rho_{\lambda_k}(e_i))^* : S^q \otimes V_{\lambda_k} \rightarrow S^{q+1} \otimes V_\rho, \quad (2.4) \]

\[ D^\rho_{\lambda_k} := \sum p^\rho_{\lambda_k}(e_i) \frac{\partial}{\partial x_i} : S^q \otimes V_\rho \rightarrow S^{q-1} \otimes V_{\lambda_k}, \quad (2.5) \]

\[ (D^\rho_{\lambda_k})^* := -\sum (p^\rho_{\lambda_k}(e_i))^* \frac{\partial}{\partial x_i} : S^q \otimes V_{\lambda_k} \rightarrow S^{q-1} \otimes V_\rho. \quad (2.6) \]

The differential operators \( D^\rho_{\lambda_k} \) and \((D^\rho_{\lambda_k})^*\) are called the higher spin Dirac operators, which are generalization of the Dirac operator for higher bundles. If we define the inner product on \( S^q \otimes V_\rho \) by the tensor inner product, then we show that the adjoint operators of \( x^\rho_{\lambda_k} \) and \((x^\rho_{\lambda_k})^*\) are \(-(D^\rho_{\lambda_k})^*\) and \(D^\rho_{\lambda_k}\), respectively.

We can show that the above operators are invariant operators on the \( \text{spin}(n) \)-module \( \sum_q S^q \otimes V_\rho \).

**Proposition 2.1.** The operators \((2.3)-(2.6)\) are invariant operators.

**Proof.** We prove only the invariance of \( x^\rho_{\lambda_k} \). It follows from the equation \((1.3)\) that we have

\[
(-4x_k \frac{\partial}{\partial x_l} + 4x_l \frac{\partial}{\partial x_k} + \pi_{\lambda_k}([e_k, e_l]))x^\rho_{\lambda_k}
\]

\[
= \sum_i (-4x_k \frac{\partial}{\partial x_l} + 4x_l \frac{\partial}{\partial x_k} + \pi_{\lambda_k}([e_k, e_l]))x_i p^\rho_{\lambda_k}(e_i)
\]

\[
= \sum_i 4p^\rho_{\lambda_k}(e_i)(-\delta_{li}x_k - x_kx_l \frac{\partial}{\partial x_l} + \delta_{ki}x_l + x_kx_l \frac{\partial}{\partial x_k})
\]

\[
+ x_i(p^\rho_{\lambda_k}(e_i)\pi_\rho([e_k, e_l]) + p^\rho_{\lambda_k}([[e_k, e_l], e_i]) \}
\]

\[
=4(-p^\rho_{\lambda_k}(e_l)x_k + p^\rho_{\lambda_k}(e_k)x_l) + x^\rho_{\lambda_k} 4(-x_k \frac{\partial}{\partial x_l} + x_l \frac{\partial}{\partial x_k})
\]

\[
+ x^\rho_{\lambda_k} \pi_\rho([e_k, e_l]) + \sum x_i(4\delta_{ki}p^\rho_{\lambda_k}(e_l) - 4\delta_{li}p^\rho_{\lambda_k}(e_k))
\]

\[
= x^\rho_{\lambda_k} (-4x_k \frac{\partial}{\partial x_l} + 4x_l \frac{\partial}{\partial x_k} + \pi_\rho([e_k, e_l])).
\]

\[\square\]

We shall investigate relations among these invariant operators, and re-construct the Laplacian operator and the Euler operator. First, the formula \((1.6)\) induces the following lemma.
Lemma 2.2. The invariant operators (2.3)-(2.6) satisfy that
\[
\sum_{0 \leq k \leq N} (x^o_{\lambda_k})^* x^o_{\lambda_k} = \sum_i (x_i)^2 = r^2, \quad \sum_{0 \leq k \leq N} (D^o_{\lambda_k})^* D^o_{\lambda_k} = \Box, \tag{2.8}
\]
\[
\sum_{0 \leq k \leq N} (D^o_{\lambda_k})^* x^o_{\lambda_k} = -n - r \frac{\partial}{\partial r}, \quad \sum_{0 \leq k \leq N} (x^o_{\lambda_k})^* D^o_{\lambda_k} = r \frac{\partial}{\partial r}. \tag{2.9}
\]

In similar way, the formula (1.7) gives the following lemma.

Lemma 2.3. The invariant operators (2.3)-(2.6) satisfy that
\[
\sum_{0 \leq k \leq N} m(\lambda_k) (x^o_{\lambda_k})^* x^o_{\lambda_k} = 0, \quad \sum_{0 \leq k \leq N} m(\lambda_k) (D^o_{\lambda_k})^* D^o_{\lambda_k} = 0. \tag{2.10}
\]

Remark 2.1. The second equation in (2.10) means that \( \mathbf{R}^n \) is a flat space (see [11]).

Since we have already given the decomposition of \( S^n \), we shall decompose the \( V_\rho \)-valued harmonic polynomials \( H^n \otimes V_\rho \). So we need relations among the Laplacian and the operators (2.3)-(2.6).

Lemma 2.4. The Laplace operator \( \Box \) and the operators (2.3)-(2.6) satisfy that
\[
[\Box, (D^o_{\lambda_k})^* ] = 0, \quad [\Box, D^o_{\lambda_k} ] = 0, \tag{2.11}
\]
\[
[\Box, x^o_{\lambda_k} ] = -2 D^o_{\lambda_k}, \quad [\Box, (x^o_{\lambda_k})^* ] = 2(D^o_{\lambda_k})^*. \tag{2.12}
\]

From Lemma 2.3 and 2.4, we have 0-th order invariant operators compatible with the Laplacian \( \Box \).

Corollary 2.5. We consider the 0-th order operators \( \sum_k m(\lambda_k) (D^o_{\lambda_k})^* x^o_{\lambda_k} \) and \( \sum_k m(\lambda_k) (x^o_{\lambda_k})^* D^o_{\lambda_k} \). These operators commute with the Laplace operator:
\[
[\Box, \sum_k m(\lambda_k) (D^o_{\lambda_k})^* x^o_{\lambda_k} ] = [\Box, \sum_k m(\lambda_k) (x^o_{\lambda_k})^* D^o_{\lambda_k} ] = 0. \tag{2.13}
\]

Furthermore, these two operators coincide with each other.

Proof. We can easily show that
\[
\sum_k m(\lambda_k) (-(D^o_{\lambda_k})^* x^o_{\lambda_k} + (x^o_{\lambda_k})^* D^o_{\lambda_k})
\]
\[
= - \sum_{i,j} (x_j \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_j})(\frac{1}{4} \pi_\rho ([e_j, e_i]))
\]
\[
= 0.
\]
So we have proved the lemma.
This corollary means that the operator \( \sum_k m(\lambda_k)(x_{\lambda_k}^\rho)^*D_{\lambda_k}^\rho \) acts on \( H^q \otimes V_\rho \) and has a spectral decomposition.

**Proposition 2.6.** Let \( (\sum_\mu \pi_\mu, \sum_\mu V_\mu) \) be the irreducible decomposition of \( (\pi_{h^q} \otimes \pi_\rho, H^q \otimes V_\rho) \). The 0-th order invariant operator \( \sum_k m(\lambda_k)(x_{\lambda_k}^\rho)^*D_{\lambda_k}^\rho \) has the following spectral decomposition on \( H^q \otimes V_\rho \):

\[
\sum_k m(\lambda_k)(x_{\lambda_k}^\rho)^*D_{\lambda_k}^\rho = m(\mu, q) \quad \text{on } V_\mu.
\]

The constant \( m(\mu, q) \) is given by

\[
m(\mu, q) := \frac{1}{2}(q^2 + (n - 2)q + \| \rho + \delta \|^2 - \| \mu + \delta \|^2),
\]

where \( \delta \) is half the sum of positive roots, and \( \| \cdot \| \) is the canonical norm on the weight space, that is, \( \| \nu \|^2 = \sum_{1 \leq i \leq m} (\nu^i)^2 \).

**Proof.** We can show that

\[
\sum_k m(\lambda_k)(-(D_{\lambda_k}^\rho)^*x_{\lambda_k}^\rho - (x_{\lambda_k}^\rho)^*D_{\lambda_k}^\rho)
= -\sum_{ij}(x_j^i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_j})(\frac{1}{4} \pi_\rho([e_j, e_i]))
= -2\sum_{ij}\frac{1}{32} \pi_{h^q}\([e_i, e_j]\) \otimes \pi_\rho([e_i, e_j])
\]

The last equation is realized by using the Casimir operators. In fact, we can show that

\[
\sum_{ij} \frac{1}{32} \pi_{h^q}\([e_i, e_j]\) \otimes \pi_\rho([e_i, e_j]) = C_{h^q \otimes \rho} - C_{h^q} \otimes \text{id} - \text{id} \otimes C_\rho.
\]

Here, the Casimir operator \( C_\nu \) on the irreducible representation space \( V_\nu \) is defined by

\[
C_\nu := \frac{1}{64} \sum_{ij} \pi_\nu([e_i, e_j])\pi_\nu([e_i, e_j]),
\]

and acts as the constant \( -(|\delta + \nu|^2 - |\delta|^2)/2 \) on \( V_\nu \). Thus we have proved the proposition. \( \blacksquare \)
Instead of the 0-th order operator in the above proposition, we consider the following operator corresponding to the Bochner type Laplacian on the bundle $S_{\rho}$ (see [11]):

$$E := \sum_{1 \leq k \leq N} (1 - \frac{m(\lambda_k)}{m(\lambda_0)}) (x_{\lambda_k}^\rho)^* D_{\lambda_k}^\rho,$$

(2.19)

where the weights $\{\lambda_k\}_k$ satisfy that $\lambda_0 > \lambda_1 > \cdots > \lambda_N$ with respect to the lexicographical order on the weight space. This operator $E$ is obtained by eliminating the top operator $(x_{\lambda_0}^\rho)^* D_{\lambda_0}^\rho$ from the equations $\sum_k (x_{\lambda_k}^\rho)^* D_{\lambda_k}^\rho$ and $\sum_k m(\lambda_k)(x_{\lambda_k}^\rho)^* D_{\lambda_k}^\rho$. Then we have the following theorem.

**Theorem 2.7.** Let $(\sum_\mu \pi_\mu, \sum_\mu V_\mu)$ be the irreducible decomposition of $(\pi_{h^q} \otimes \pi_{\rho}, H^q \otimes V_\rho)$, where $h^q = (q, 0_{m-1})$ and $\rho = (\rho^1, \cdots, \rho^m)$. The 0-th order invariant operator $E$ is a non-negative operator and has the spectral decomposition on $H^q \otimes V_\rho$ as follows:

$$E = q + \frac{m(\mu, q)}{\rho^1} \text{ on } V_\mu,$$

(2.20)

where the constant $m(\mu, q)$ is given in (2.15). In particular, the 0-eigenspace is the irreducible representation space with highest weight $\mu_0 := h^q + \rho$.

In this theorem, we remark that the eigenvalues $\{e(\mu)\}$ of $E$ order as $0 = e(\mu_0) < e(\mu_1) \leq e(\mu_2) \leq \cdots$ for $\mu_0 > \mu_1 \geq \mu_2 \geq \cdots$. Here, the top component $(\pi_{\mu_0}, V_{\mu_0})$ certainly exists with multiplication one.

**Corollary 2.8.** The irreducible representation with highest weight $\mu_0$ in $H^q \otimes V_\rho$ is realized as follows:

$$V_{\mu_0} = \bigcap_{1 \leq k \leq N} \ker D_{\lambda_k}^\rho,$$

(2.21)

where $\ker D_{\lambda_k}^\rho$ is the kernel of $D_{\lambda_k}^\rho$ on $H^q \otimes V_\rho$.

### 3 Examples

In this section, we give some examples: spinor-valued harmonic polynomials and $p$-form-valued harmonic polynomials (see [8], [7]-[12], [14]).

**Example 3.1 (spinor-valued harmonic polynomials).** We shall investigate only the odd dimensional case, that is, the case of $n = 2m+1$. Let $V_\Delta$ be the spinor
space with highest weight $\Delta = ((1/2)_m)$. We consider the spinor-valued harmonic polynomials $H^q \otimes V_\Delta$, and have invariant operators: the Clifford multiplication $x = -x^* = \sum x_i e_i$ and the Dirac operator $D = D^* = \sum e_i \partial / \partial x_i$, twistor operator $T$ and so on. Then the 0-th order invariant operator $E$ in Theorem 2.7 is $-xD = x^*D$.

Now, we show that $H^q \otimes V_\Delta$ has the irreducible decomposition $V_{\mu_0} \oplus V_{\mu_1}$, where $\mu_0 = h^q + \Delta = (q+1/2, (1/2)m^{-1})$ and $\mu_1 = (q-1/2, (1/2)m^{-1})$. Then we have the spectral decomposition of $-xD$:

$$-xD = \begin{cases} 0 & \text{on } V_{\mu_0} \\ n+2q-2 & \text{on } V_{\mu_1}. \end{cases}$$  \hspace{1cm} (3.1)

In particular, we have

$$V_{\mu_0} = \ker D, \quad V_{\mu_1} = H^q \otimes V_\Delta / \ker D.$$  \hspace{1cm} (3.2)

**Example 3.2 (p-form-valued harmonic polynomials).** Let $\Lambda^p$ be the exterior tensor product space of $\mathbb{R}^n$ with degree $p$, which is the irreducible representation space with highest weight $(1_p, 0_{m-p})$. We consider the $p$-form-valued harmonic polynomials $H^q \otimes \Lambda^p$, and have invariant operators: the exterior derivative $d = \sum e_i \partial / \partial x_i$, its adjoint $d^* = -\sum i(e_i) \partial / \partial x_i$, the conformal killing operator $C$, $x_\Lambda = \sum x_i e_i \Lambda$, and $i(x) = \sum x_i i(e_i)$ and so on. Here, $i(e_i)$ denotes the interior product of $e_i$. Then we have the spectral decomposition of $E = i(x)d - x_\Lambda d^*$ on $H^q \otimes \Lambda^p$:

$$i(x)d - x_\Lambda d^* = \begin{cases} 0 & \text{on } V_{\mu_0} \\ q+p & \text{on } V_{\mu_1} \\ n+q-p & \text{on } V_{\mu_2} \\ n+2q-2 & \text{on } V_{\mu_3} \text{ (for } q \geq 2), \end{cases}$$  \hspace{1cm} (3.3)

where $\mu_0 = (q+1, 1_{p-1}, 0_{m-p})$, $\mu_1 = (q, 1_p, 0_{m-p-1})$, $\mu_2 = (q, 1_{p-2}, 0_{m-p+1})$, and $\mu_3 = (q-1, 1_{p-1}, 0_{m-p})$. In particular, we have $V_{\mu_0} = \ker d \cap \ker d^*$.}

**4 Discussion**

In the case of $p$-form-valued harmonic polynomials, we can show that

$$V_{\mu_1} = \ker d / \ker d \cap \ker d^*, \hspace{1cm} (4.1)$$

$$V_{\mu_2} = \ker d^* / \ker d \cap \ker d^*, \hspace{1cm} (4.2)$$

$$V_{\mu_3} = H^q \otimes \Lambda^p / (\ker d + \ker d^*). \hspace{1cm} (4.3)$$
Thus, we can realize the irreducible components by using kernels of $d$ and $d^*$. In general case, we may realize any irreducible component of $H^q \otimes V_\rho$ by using kernels of higher spin Dirac operators (for the case of Rarita-Schwinger operator, see [3]).

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**References**

[1] T. Branson, *Harmonic Analysis in Vector Bundles Associated to the Rotation and Spin Groups*, J. Funct. Anal. **106** (1992), 314-328.

[2] T. Branson, *Stein-Weiss Operators and Ellipticity*, J. Funct. Anal. **151**, (1997), 334-383.

[3] T. Branson, *Spectra of self-gradients on spheres*, J. Lie Theory. **9**, (1999), 491-506.

[4] J. Bureš and V. Souček, *Eigenvalues of conformally invariant operators on spheres*, in ‘The 18th Winter School “Geometry and Physics”’, Rend. Circ. Mat. Palermo (2) Suppl. No. 59, (1999), 109–122. (math. DG/9807050).

[5] J. Bureš, *The higher spin Dirac operators*, in ‘Differential geometry and applications’, Masaryk Univ., Brno, (1999), 319-334. (math. DG/9901039).

[6] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor-Valued Functions*, Kluwer Ac. Publishers, 1992.

[7] G. B. Folland, *Harmonic analysis of the de Rham complex on the sphere*, J. reine angew. Math. **398** (1989), 130-143.

[8] Y. Homma, *A representation of Spin(4) on the eigenspinors of the Dirac operator on $S^3$*, to appear in Tokyo J. Math.

[9] Y. Homma, *Spinor-valued and Clifford Algebra-valued Harmonic Polynomials* to appear in J. Geom. Phys.
[10] Y. Homma, *The Higher Spin Dirac Operator on 3-Dimensional Manifolds* preprint, 1999. [math.DG/0006210].

[11] Y. Homma, *Clifford Homomorphisms and Higher Spin Dirac Operators* preprint, 2000. [math.DG/0007052].

[12] A. Ikeda and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on $S^n$ and $P^n(\mathbb{C})$*, Osaka J. Math. 15 (1978), no. 3, 515-546.

[13] C. O. Nwachuku and M. A. Rashid, *Eigenvalues of the Casimir operators of the orthogonal and symplectic groups*, Journal of Mathematical Physics, 17 (1976), no. 8, 1611-1616.

[14] A. Trautman, *The Dirac operator on hypersurfaces*, Acta Physica Polonica B26 (1995), no. 7, 1283-1310.

[15] D. P. Želobenko, *Compact Lie Groups and Their Representations*, Trans. Math. Monographs. vol 40, A.M.S. 1973.