A Control Performance Index for Multicopters Under Off-nominal Conditions

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Abstract—In order to prevent loss of control (LOC) accidents, the real-time control performance monitoring (CPM) problem is studied for multicopters. Different from the existing literature, this paper does not try to monitor the performance of the controllers directly. Conversely, the unknown disturbances of the multicopter under off-nominal conditions are modeled and assessed. The monitoring results will tell the user whether a multicopter will be LOC or not. Firstly, a new degree of controllability (DoC) will be proposed for multicopters subject to control constrains and off-nominal conditions. Then a control performance index (CPI) will be defined based on the new DoC to reflect the control performance of the multicopters. Besides, the proposed CPI is applied to a new switching control framework to guide the control decision of multicopter under off-nominal conditions. Finally, simulation and experimental results will show the effectiveness of the CPI and the switching control framework proposed in this paper.

Index Terms—Multicopters, loss of control, control performance monitoring, degree of controllability.

I. INTRODUCTION

MULTICOPTERS are attracting increasing attention in recent years. The growing interest is partly due to the fact that multicopters can be used in numerous applications such as surveillance, inspection, and mapping. Besides, applications of large multicopters are becoming eye-catching, whilst there exists potential risk in civil safety if they crash [1], especially in urban areas. Therefore, it is of great importance to consider the flight safety problem and prevent loss of control (LOC) [2], [3] accidents of multicopters.

Current multicopter autopilots are primarily designed for operation under nominal conditions (e.g., predefined vehicle weight distribution, good vehicle health, and acceptable wind disturbances) by the makers. However, it is unavoidable to use the multicopter which may work under off-nominal conditions [2] (e.g., additional payloads, propulsor degradation, unacceptable wind disturbances). As mentioned in [4], it is necessary to assess the performance of the vehicle that can be achieved in the presence of off-nominal conditions. Based on the performance assessment results, proper failsafe control actions can be performed which do not worsen the situation any further. Therefore, a performance index, which will warn the users or guide the autopilots if the multicopter is working under off-nominal conditions, is required to monitor the control performance of the multicopter autopilot.

Control performance monitoring (CPM) is an interesting and important topic in the literature. This can be evidenced by the reviews in [5]–[7] and the references therein. In [5], an overview of the status of control performance monitoring and assessment using minimum variance principles is provided. The overview of MIMO control performance monitoring is given by [6] while the review [7] has reported the most recent results in CPM research and their use in industry. From the literature, one can see that most of the CPM methods focus on the industrial production processes and most of the CPM methods try to monitor the output variance, step changes, settling time, decay ratio, or stability margin of the control systems online. For aircraft, the authors in [8] have provided a software tool for monitoring control law stability margins on-line in quasi-real-time. Robust tracking performance is proposed as a metric for the quantification of the permissible flight envelope in [9]. In this paper, the CPM problem of multicopter systems is considered. Here, we do not try to monitor the performance of the controllers. Conversely, the unknown disturbances of the multicopter under off-nominal conditions are modeled and monitored. The monitoring results could tell the user whether a multicopter will be LOC or not. The advantages of this idea lies in that only the disturbances are monitored and compared with a specific threshold which can be determined in the development process of a multicopter.

In this paper, a new Control Performance Index (CPI) will be proposed based on Degree of Controllability (DoC) [10]–[13]. DoC is a reasonable choice to reflect the performance that can be achieved in the presence of off-nominal conditions. However, the existing definitions of DoC suffer limited feasibility in practice because of the following reasons: i) Most of these definitions, such as the Grammian matrix based DoC [10], [13], do not consider the control constraints. ii) The state norm based DoC [11], [12] only considers symmetrical control constraints and is recovery time related. Besides, consider the system expressed by $\dot{x} = Ax + Bu - d$, where $x \in \mathbb{R}^n$, and $u \in U \subset \mathbb{R}^m$, the control constraint set $U$ shrinks in presence of the external disturbance $d$. But as far as the authors understand, this scenario is not considered in the existing results. This paper will define a new kind of DoC for multicopters based on the Available Control Authority Index (ACAI) as in [14]. Comparing with existing DoCs, the new DoC has the following advantages: i) It is independent of recovery time. ii) It can reflect the effect of the disturbance $d$. iii) It considers the control constraints. A new CPI is defined based on the new DoC to demonstrate the control performance of multicopters,

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1 A propulsor is composed of a propeller, a motor, and an electronic speed control module, and is powered by a battery. In the literature, the term “rotor”, “thruster” are also used.
because the effect of off-nominal conditions can be lumped into $d$. The CPI can tell the user whether the multicopter is stable or not. Besides, the CPI proposed in this paper can be integrated into the open source autopilot, such as the Ardupilot project \cite{15}, without the need for extra sensors.

In Section II, the dynamic models of multicopter systems are introduced and the objective of the paper is provided. In Section III, a preliminary on the ACAA is given, based on which a new DoC and a new CPI are defined for multicopters. Besides, a step-by-step procedure is provided to obtain the proposed CPI. In Section IV, the proposed CPI is applied to a switching control framework. Finally, the effectiveness of the new CPI is demonstrated by numerical and experimental results in Section V. Section VI concludes the paper. The major contributions of this paper lie in: 1) the definition of the new DoC for multicopter systems, based on which a new CPI is proposed and used to monitor the real-time control performance of multicopters working under off-nominal conditions, 2) the proposed CPI is used in a switching control framework to guide the control decision of multicopters under off-nominal conditions.

II. PROBLEM FORMULATION

A. Mathematical Model of Multicopters

As shown in \cite{14}, there are various types of multicopters with different number of propulsors and different configurations. Let $m_a$ denotes the mass of the multicopter, $h, v_h$ denote the altitude and vertical velocity, $\phi, \theta, \psi$ denote the roll, pitch and yaw angles, and $p, q, r$ denote the roll, pitch and yaw angular velocities of multicopter, respectively. Then, the mathematical model of multicopters is nonlinear and the rigid body equations of motion of the airframe are \cite{16}

$$
\begin{align*}
\dot{h} &= v_h \\
\dot{\omega}_R &= R \omega_x \\
J \ddot{\omega} &= -\omega \times J + u_\tau,
\end{align*}
$$

where $u_\tau$ is the total thrust, $g$ is the acceleration of gravity, $\omega = [p \ q \ r]^T \in \mathbb{R}^3$, $u_\tau = [\tau_r \ \tau_p \ \tau_y]^T \in \mathbb{R}^3$ (where $\tau_r, \tau_p, \tau_y$ are the airframe roll, pitch and yaw torque of multicopter, respectively) and the notation $\omega_x$ denotes the skew-symmetric matrix, such that $\omega_x v = \omega \times v$ for the vector cross product $\times$ and any vector $v$. The matrix $J = \text{diag}(J_x, J_y, J_z) \in \mathbb{R}^{3 \times 3}$ is the constant inertia matrix where $J_x, J_y, J_z$ are the moment of inertia around the roll, pitch and yaw axes of the multicopter frame, respectively. Here, the horizontal position channel model is omitted for simplification. The orientation of the multicopter is given by the rotation matrix $R$ (see \cite{16}).

However, if the multicopter is hovering, the aerodynamic damping and stiffness is ignorable, and the linear dynamical model of a multicopter in the neighborhood of the equilibrium is given as \cite{14}

$$
\dot{x} = Ax + B(u_f - g)
$$

where $x \in \mathbb{R}^8$ is the state vector, $A \in \mathbb{R}^{8 \times 8}, B \in \mathbb{R}^{8 \times 4}$ are real matrices, $g \in \mathbb{R}^4$ is the gravity vector of the multicopter, $u_f \in \mathbb{R}^4$ is the total thrust and torques vector of the system and

$$
x = \begin{bmatrix} h & \phi & \theta & \psi & v_h & p & q & r \end{bmatrix}^T \in \mathbb{R}^8
$$

$$
u_f = \begin{bmatrix} u_t & \tau_r & \tau_p & \tau_y \end{bmatrix}^T \in \mathbb{R}^4
$$

$$
A = \begin{bmatrix} 0_{4 \times 4} & I_4 \\
0_{4 \times 4} & 0_{4 \times 4} \end{bmatrix} \in \mathbb{R}^{8 \times 8}
$$

$$
B = \begin{bmatrix} 0_{4 \times 4} \\
J_f \end{bmatrix} \in \mathbb{R}^{8 \times 4}
$$

$$
J_f = \text{diag}(-m_a J_x, J_y, J_z) \in \mathbb{R}^{4 \times 4}
$$

According to \cite{13} and \cite{17}, the mapping from the propulsor thrust $f_i, i = 1, \ldots, n_p$ to the thrust and torques vector $u_f$ is

$$
u_f = B f
$$

where $Bf \in \mathbb{R}^{4 \times n_p}$ is the control effectiveness matrix.

B. Multicopter Modeling Under Off-nominal Conditions

The off-nominal conditions of interest in this paper cover additional payloads, propulsor degradation, and unacceptable wind disturbances. Additional payloads will change $Bf$ of the multicopter. The propulsor degradation will change the control effectiveness matrix $B_f$ to $E = B_f (I_{n_p} - \Gamma) \in \mathbb{R}^{4 \times n_p}$, where $\Gamma = \text{diag}(\eta_1, \ldots, \eta_{n_p}) \in \mathbb{R}^{n_p \times n_p}$ and $\eta_i \in [0, 1], i = 1, \ldots, n_p$ is used to account for propulsor efficiency degradation which means that the effectiveness of propulsor $i$ is reduced by $100\eta_i \%$. If the $i$th propulsor completely fails, then $\eta_i = 1$. As the wind disturbance only affect the dynamics of multicopters but not the multicopter kinematics, the wind disturbances can be denoted as $d_w \in \mathbb{R}^4$. This means that the wind effect is formulated as a thrust/torque disturbance added to the multicopter. Then, system (2) is modified as

$$
\dot{x} = Ax + \begin{bmatrix} 0_{4 \times 4} \end{bmatrix} \begin{bmatrix} I_4 \\
J_f - \Delta J_f^{-1} \end{bmatrix} (u_f - B_f \Gamma f - g - \Delta g - d_w)
$$

where $\Delta J_f^{-1}$ and $\Delta g$ are the variations (introduced by the additional payloads) of $J_f^{-1}$ and $g$, respectively. From the above, (5) can be rewritten as

$$
\dot{x} = Ax + B (u_f - d)
$$

Here, the lumped external disturbance $d$ is used to reflect the effect of the off-nominal conditions and $d = B_f \Gamma f + g + \Delta g + d_w - J_f \Delta J_f^{-1} (u_f - B_f \Gamma f - g - \Delta g - d_w) \in \mathbb{R}^4$. For a multicopter system without off-nominal conditions, one has $d = g$.

In practice, $f_i \in [0, K_i], i = 1, \ldots, n_p$ (where $K_i$ is the maximum thrust of the $i$-th propulsor) because the propulsors can only provide unidirectional thrust (upward or downward). As a result, the propulsor thrust $f$ is constrained by

$$
f \in U_f = \{ f | f_i \in [0, K_i], i = 1, \ldots, n_p \}.
$$

Let $u = u_f - d \in \mathbb{R}^4$ denote the control vector of (6), then one has

$$
u_f \in \Omega = \{ u_f | u_f = B f, f \in U_f \} \subset \mathbb{R}^4
$$

$$
u \in U = \{ u | u = u_f - d, u_f \in \Omega \} \subset \mathbb{R}^4
$$

according to (4) and (7). Obviously, the system (6) is a nonlinear one with constrained control inputs.
C. Objective of the Paper

The objective of this paper is to solve the following problems: (i) How to monitor the control performance of the multicopters that is controlled by autopilots designed under nominal conditions, and (ii) how to guide the users or the autopilots based on the monitoring results.

III. A CONTROL PERFORMANCE INDEX FOR MULTICOPTERS

A. Preliminaries

In practice, if the unknown disturbance \( d \) introduced by the off-nominal conditions makes the system in (6) uncontrollable, then the multicopter will be LOC. In order to test the controllability of the disturbance driven system in (6), the ACAI based controllability analysis method in [14] is used. The ACAI, which is spurred by the research in [17], was first proposed in [14] and was originally used to check on the positive controllability of multicopters. According to [14], a new measurement on the available control authority is defined as

\[
\rho(d, \partial) \triangleq \left\{ \begin{array}{cl} \min \{ \| \alpha - \beta \| : \alpha \in \Omega, \beta \in \partial \Omega \} & \text{if} \ |\| \rho(d, \partial)\| < 0. \\
\min \{ \| \alpha - \beta \| : \alpha \in \Omega^C, \beta \in \partial \Omega \} & \text{if} \ |\| \rho(d, \partial)\| \geq 0. \end{array} \right.
\]

(9)

where \( \partial \Omega \) is the boundary of \( \Omega \), \( \Omega^C \) is the complementary set of \( \Omega \), and \( \rho(d, \partial) \) represents the distance from \( \partial \Omega \) to \( \partial \Omega \). Then the ACAI of system (6) is defined as \( \rho(d, \partial) \in \mathbb{R} \) and the following theorem is obtained directly according to the results in [14].

Theorem 1. The system in (6) is controllable if and only if \( \rho(d, \partial) > 0 \).

Obviously, \( \rho(d, \partial) \) is the radius of the biggest enclosed sphere centered at \( d \) in the attainable control set \( \partial \Omega \). It is obvious that the larger the value of \( \rho(d, \partial) \) is, the larger is the attainable control set. Then the system has more control authority and can be more resilient to disturbances. On the contrary, the smaller the value of \( \rho(d, \partial) \) is, the smaller is the attainable control set. Then the system has less control authority, and may be less resilient to disturbances. In particular, if \( \rho(d, \partial) = 0 \), then the system is therefore LOC. In order to compute the value of \( \rho(d, \partial) \), a step-by-step ACAI computing procedure is given by [14], and the readers are referred to the toolbox provided in [18].

B. A Control Performance Index

It is easy to see that, the ACAI \( \rho(d, \partial) \) can be used to indicate the disturbance resilience and control performance of multicopters. However, it is not a control performance index intuitively. To account for this, a new DoC is defined first, based on which a CPI is proposed.

1) A New Definition of the Degree of Controllability: Based on the results in [14], a virtual ACAI \( \rho(u_{fc}, \partial) \) is used to normalize the ACAI, where \( u_{fc} = B \beta \) is the center of \( \Omega \) and

\[
\beta = \frac{1}{2} \begin{bmatrix} K_1 & K_2 & \cdots & K_{n_p} \end{bmatrix}^T.
\]

(10)

According to (2), the following lemma is obtained.

Lemma 1. \( \rho(u_{fc}, \partial) \) is the maximum ACAI of system (6).

Proof. See Appendix for details. \( \Box \)

In the following, \( \rho(u_{fc}, \partial) \) will be used to define the DoC of the multicopter system:

Definition 1 (Degree of Controllability for Multicopters). The degree of controllability for the multicopter system in (6) is defined as

\[
\sigma = \frac{\rho(d, \partial)}{\rho(u_{fc}, \partial)} \in \mathbb{R}
\]

(11)

where \( \rho(d, \partial) \) is the ACAI of the multicopter system. According to Definition 1, it is obvious that

\[
\sigma = 0 \quad \text{if} \ \rho(d, \partial) \leq 0.
\]

and \( \sigma \) shows the impact of the disturbance \( d \) on the virtual multicopter system where the disturbance in the system (6) satisfies \( d = u_{fc} \). Consider an extreme case where \( d = 0 \) when the multicopter is hovering, then there is no control margin to land the multicopter as the propulsors can only provide upwards thrust. Similarly, if \( d = 2u_{fc} \), namely all the propulsors are providing the maximum thrust, then there is no control margin to lift the multicopter. In order to ensure a sufficient control margin and obtain a more controllable multicopter, the designer often designs \( u_{fc} = g \) by choosing proper propulsion system in practice. If \( g = u_{fc} \) and other disturbance is ignored, the system (6) will obtain the maximum ACAI because \( d = g = u_{fc} \) and \( \rho(d, \partial) = \rho(u_{fc}, \partial) \).

According to (2), \( \rho(d, \partial) \leq 0 \) if the multicopter system is uncontrollable. For the sake of simplicity, \( \sigma = 0 \), if \( \rho(d, \partial) \leq 0 \).

Then it is obvious that \( \sigma = 0 \) when the multicopter system in (6) is uncontrollable. Then the following theorem holds:

Theorem 2. For the system in (6), the DoC satisfies \( \sigma \in [0, 1] \).

Proof. As \( \sigma \geq 0 \), it is obviously that \( \sigma \in [0, 1] \) according to Lemma 1 and Definition 1. \( \Box \)

2) Definition of the Control Performance Index: It is known that the DoC \( \sigma \) shows how controllable the system is. However, controllability does not imply stability. In practice, people concern stability more than controllability of a multicopter hovering in the air. In order to show the stability performance of the flying vehicle, this paper will define a new control performance index based on the DoC \( \sigma \).

Suppose that \( d \in U_d \). Then a new constraint set \( U_{\alpha_0} \) is defined as

\[
U_{\alpha_0} = \left\{ \frac{d}{\sigma} | \rho(d, \partial) \geq \sigma_0, d \in U_d \right\}.
\]

(14)

where \( U_{\alpha_0} \) contains all the disturbances satisfying \( \rho(d, \partial) \geq \sigma_0 \rho(u_{fc}, \partial) \). Given a controller \( u = u(x, t) \), a large enough disturbance will make system (6) unstable. Then a definition of control performance threshold (CPT) \( \sigma_h \) is given as follows:

Definition 2 (Control Performance Threshold). The CPT of the system in (6) is defined as

\[
\sigma_h = \inf_{\sigma \in [0, 1]} \{ \sigma \}
\]

(15)
where the multicopter system (6) subject to disturbance \( \forall d \in U_{\sigma_0} \sigma_i \) (i.e., any \( d \) satisfying \( \rho (d, \partial \Omega) \geq \sigma \rho (u, \partial \Omega) \)) is stable given a control strategy \( u = u(x,t) \).

Without loss of generality, the CPT \( \sigma_{th} < 1 \) for the system (6) controlled by a reasonable and robust strategy \( u = u(x,t) \).

If \( \sigma_{th} \) is determined for the control strategy \( u = u(x,t) \), then the system is stable if \( \sigma \geq \sigma_{th} \) and the system is unstable if \( \sigma < \sigma_{th} \). In order to show how stable the system is, a control performance index is defined based on Definition 1 and Definition 2:

**Definition 3 (Control Performance Index).** The CPI of the multicopter system in (6) is defined as

\[
S = \frac{\sigma - \sigma_{th}}{1 - \sigma_{th}} \tag{16}
\]

where \( \sigma_{th} < 1 \) is CPT of the system in (6).

From Definition 3, it is obvious that the multicopter is stable if \( S \geq 0 \) and unstable if \( S < 0 \). As \( \sigma \in [0,1] \), then one has

\[
S \in \left[ -\frac{\sigma_{th}}{1 - \sigma_{th}}, 1 \right]. \tag{17}
\]

Now the stability of the multicopter system can be indicated by the CPI \( S \).

**C. Threshold Value Determination**

As mentioned above, there is a CPT \( \sigma_{th} \) for the specified control strategy \( u = u(x,t) \), and the control strategy is stable if \( \sigma \geq \sigma_{th} \). In practice, it is hard to compute \( \sigma_{th} \) theoretically due to the complexity of the control strategies. In this paper, the CPT \( \sigma_{th} \) is obtained through repetitive simulations and real flight experiments. It should be pointed out that the nonlinear dynamics shown in equation (1) is applied in the simulations whilst the linear model (6) is only used to compute the DoC of the multicopter. The lumped disturbance \( d \) is considered and the computing procedure is as follows:

**Step 1:** Generate the disturbance grid set \( \Theta \subset \mathbb{R}^4 \) of the disturbance \( d \in U_d \). Denote \( d = [d_1 \ d_2 \ d_3 \ d_4]^T \). As \( d \in U_d \), the constraint of \( d_i \) can be obtained as \( d_i \in [d_{i,\min}, d_{i,\max}] \) where \( i = 1, 2, 3, 4 \) and \( d_{i,\min}, d_{i,\max} \) are the minimum and maximum value of \( d_i \) respectively. Suppose that \([d_{i,\min}, d_{i,\max}] \) is divided into \( n_d \) grid points, then \( U_d \) changes to \( \Theta \subset \mathbb{R}^4 \) with \( n_d \) points.

**Step 2:** Compute the ACAI of the multicopter system (6) corresponding to each disturbance grid points in \( \Theta \).

**Step 3:** Compute the DoC \( \sigma \) of each disturbance grid points in \( \Theta \), and the results are denoted by set \( \Lambda \).

**Step 4:** Let \( k = 0 \) and \( \Delta \sigma \in (0,1] \).

**Step 5:** Let \( k = k + 1 \). If \( k \Delta \sigma > 1 \), go to Step 8.

**Step 6:** Check the stability of the specified control strategy \( u = u(x,t) \) for all the disturbance grid points satisfying \( 1 - k \Delta \sigma \leq \sigma < 1 - (k - 1) \Delta \sigma \), which is denoted by \( \Theta_k \).

**Step 7:** If the control strategy \( u = u(x,t) \) is stable under all the specified disturbance grid points in \( \Theta_k \), go to Step 5. If the control strategy is unstable under any of the specified disturbance grid points in \( \Theta_k \), go to Step 8.

**Step 8:** The CPT is obtained as \( \sigma_{th} = 1 - (k - 1) \Delta \sigma \).

It is obvious that the larger the value of \( n_d \) is and the smaller the value of \( \Delta \sigma \) is, the more accurate is the threshold \( \sigma_{th} \). In practice, the CPT needs to be checked by real flight experiments. Fortunately, there is no need to check all the disturbance grid points that satisfy \( \sigma \geq \sigma_{th} \). If the control strategy \( u = u(x,t) \) is stable with all the disturbance grid points that satisfy \( \sigma_{th} \leq \sigma \leq \sigma_{th} + C_{\sigma} \), then the system is always stable if \( \sigma > \sigma_{th} + C_{\sigma} \), where \( C_{\sigma} \) is a specified confidence value. Therefore, only the disturbance grid points that satisfy \( \sigma_{th} \leq \sigma \leq \sigma_{th} + C_{\sigma} \) need to be tested by experiments.

**D. Confidence Analysis of the Real-time Control Performance Index**

In order to get the CPI \( S \), one needs to compute the DoC \( \sigma \) and the disturbance \( d \) needs to be estimated online where a Kalman filter is usually used. Denote the estimation value of \( d \) as \( \hat{d} \). However, the disturbance estimation \( \hat{d} \) is not equal to the real disturbance \( d \). For a given disturbance \( d \), the estimation \( \hat{d} \) is subject to noise in nature. In order to avoid false alerts, confidence analysis needs to be performed.

Due to the stochastic nature of \( \hat{d} \), the stochastic distribution of \( S \) is assumed to be \( S = \mu_S + \varepsilon_S \) where \( \mu_S \) is the mean value of \( S \), and \( \varepsilon_S \sim \mathcal{N}(0, \Sigma_S) \) is the estimation noise of \( S \) which is assumed to be white Gaussian noise and the variance of \( \varepsilon_S \) is \( \Sigma_S \). Then the stochastic distribution of \( S \) is obtained as

\[
E(S) = \mu_S, \text{Cov}(S) = \text{Cov}(\varepsilon_S) = \Sigma_S. \tag{18}
\]

From (18), the CPI \( S \sim \mathcal{N}(\mu_S, \Sigma_S) \). In this paper, it is assumed that if the probability that \( S \) is nonnegative is 95%, i.e. \( P(S \geq 0) \geq 95\% \), the system (6) is considered to be stable. Then the following proposition holds:

**Proposition 1.** \( P(S \geq 0) \geq 95\% \) if the following condition is satisfied

\[
\frac{\mu_S}{\sqrt{\Sigma_S}} \geq 1.65 \tag{19}
\]

where \( \mu_S \) is the mean value of \( S \), and \( \Sigma_S \) is the variance of \( S \).

**Proof.** According to [19], one has \( P(S \geq 0) = 1 - P(S < 0) \).

Besides, \( S \) has a \( \mathcal{N}(\mu_S, \Sigma_S) \) distribution if and only if \( Z_S = S - \mu_S \sqrt{\Sigma_S} \) has a \( \mathcal{N}(0, 1) \) distribution. Then, \( P(S < 0) \) is given by the following expression

\[
P(S < 0) = P(Z_S < -\frac{\mu_S}{\sqrt{\Sigma_S}}) = 1 - P(Z_S < \frac{\mu_S}{\sqrt{\Sigma_S}}) \leq 5\%. \tag{20}
\]

By looking up the table of normal distribution in [19], \( P(S < 0) \leq 5\% \) if and only if \( \mu_S/\sqrt{\Sigma_S} \geq 1.65 \).

**IV. Application of the Control Performance Index**

In this section, the proposed CPI is used for a switching control framework for multicopters, which is composed by a CPI estimator, two control strategies, and a switching control allocation module. This framework can switch between the two control strategies based on the real-time CPI value.
A. Switching Control Framework

The diagram of the switching control framework is shown in Fig. 1. Practically, the reference signal \((h_c, v_c, \phi_c, p_c, \theta_c, q_c, \psi_c, r_c)\)^T is given by top level guidance module or the pilots on the ground. This framework has four major assemblies: i) Two control strategies: the nominal control strategy and the degraded control strategy; ii) a real-time CPI estimator, iii) a disturbance observer, and iv) a switching control allocation module based on the CPI. The aim of this setup is to update the CPI online in case of off-nominal conditions, and compensate for the effect of the lumped disturbance by using the control strategies and the switching control allocation system.

There are many research on the nominal control of multicopters, see [16], [20], [21] and the references therein. In order to make this paper more extensible, the nominal control strategy in the framework is not specified. For the case that the multicopter under severe off-nominal conditions is uncontrollable, a degraded control strategy will be adopted. The papers [22]–[26] studied a relaxed hover solution for multicopters where the vehicle may rotate at a constant velocity in hover, which gives up the control of the yaw angle, i.e. the yaw states are ignored. In the following, the lumped disturbance, DoC, CPT and CPI of the degraded system without yaw states are denoted by \(\bar{d}, \bar{\sigma}, \bar{\sigma}_{th}\) and \(\bar{S}\), respectively. Then, these strategies are now integrated with an online estimator for the CPI, resulting in a switching control allocation system that is robust against off-nominal conditions. Besides, this framework can deliver the real-time control performance state of the multicopters to the ground station so that the on-ground pilots can decide whether to continue or abort the mission.

In practice, a Kalman filter based on the dynamic model shown in (6) is a reasonable choice to estimate the disturbances. The ACAI can be obtained according to the computation procedure given in [14] and the toolbox in [18]. Based on the ACAI, the DoC and the CPI of the system are obtained according to the results in Section III. In the following, a switching control allocation module based on the CPI is proposed.

B. Switching Control Allocation Based on the Control Performance Index

According to Definition 1, Definition 2, and Definition 3, the following observations are obtained:

**Observation 1.** If \(\sigma \geq \sigma_{th}\), then \(S \geq 0\). The multicopter system is controllable and the nominal control strategy is stable, then it is safe to continue the flight.

**Observation 2.** If \(0 < \sigma < \sigma_{th}\) and \(\bar{\sigma} \geq \bar{\sigma}_{th}\), then \(S < 0, \bar{S} \geq 0\). The multicopter system \(S\) is controllable but the nominal control strategy is unstable, then it is unsafe to continue the mission but the multicopter can be landed with a degraded performance.

**Observation 3.** If \(\sigma \leq 0\) and \(\bar{\sigma} \geq \bar{\sigma}_{th}\), then \(S < 0, \bar{S} < 0\). The multicopter system \(S\) is uncontrollable, and it is unsafe to continue the mission but the multicopter can be landed with a degraded performance.

**Observation 4.** If \(\sigma \leq 0\) and \(\bar{\sigma} < \bar{\sigma}_{th}\), then \(S < 0, \bar{S} < 0\). The multicopter system \(S\) will be in LOC status.

Then the idea behind the switching control allocation is to give up the control of the yaw angle if \(P(S \geq 0) < 95\%\) and \(P(\bar{S} \geq 0) \geq 95\%\), which is shown in Fig. 2.

As shown in Fig. 2, there are two safe control allocation modes and one LOC mode:

i) **Safe Flight mode**, where the total thrust \(u_t\) and all three control torques \(\tau_c, r_c, \tau_t\) are allocated to all the propulsors.

ii) **Safe Landing mode**, where the total thrust \(u_t\) and roll-pitch control torques \(\tau_c, \tau_t\) are allocated to all the propulsors.

iii) **Loss-of-control** is an unsafe state that the multicopter will crash onto the ground regardless of the control strategies used.

As the details of the degraded control strategy are beyond the scope of this paper, the readers are referred to [25], [26] for more information.

V. SIMULATION AND EXPERIMENTS

In order to show the effectiveness of the proposed CPI, both numerical and experimental results are given in this section. As the CPI proposed by this paper can be applied to most kinds of multicopters (e.g., quadcopters, hexacopters), a hexacopter subject to propulsor faults is simulated to show the effectiveness of the proposed CPI and switching control framework. Moreover, a number of real flight experiments are carried out to show the effectiveness of the CPI based on a quadcopter platform. As mentioned before, off-nominal conditions (i.e., additional payloads, propulsor degradation,
and unacceptable wind disturbances) are considered in this paper. However, the wind disturbance is not easy to simulate for multicopters. Therefore, only the propulsor degradation (in the simulations) and additional payloads (in the experiments) are considered in this section.

A. Simulations and Results

Here, a traditional hexacopter with symmetric configuration (see [14] for the detailed parameters of the hexacopter) is considered to show the effectiveness of the proposed CPI and the switching control framework. The simulation model of the hexacopter is constructed which consists of three main modules: i) two control strategies: the nominal control strategy and the degraded control strategy, ii) a real-time estimator for the CPI $\bar{S}$ and $\bar{S}$, iii) a switching control allocation method based on the CPI, which is consistent with the switching control allocation system shown in Fig. 2. In the simulation, the hexacopter is controlled to 1 meter above the ground ($h_c = 1$), and maintains the level state ($\psi_c = \theta_c = \phi_c = 0$).

In order to compute the ACAI $\rho(\bar{d}, \bar{\Omega})$, a Kalman filter is used to estimate the lumped disturbance $\bar{d}$. Based on the estimated disturbance $\bar{d}$, the value of the ACAI can be computed according to the procedure presented in [14]. Then, the DoC $\sigma$ and the CPI $S$ can be computed based on the ACAI. Similarly, the lumped disturbance of the degraded system can be estimated and the DoC $\bar{\sigma}$ and the CPI $\bar{S}$ can be computed. The details are omitted here.

1) Threshold Value Determination: According to the computing procedure for the threshold value in Section III.C, we set $n_p = 21$, $\Delta \sigma = 0.1$, and simulations are performed at 194481 points and the results are shown in Table I. From Table I it can be seen that the system $\sigma_h$ controlled by the nominal control strategy is always stable if $\sigma \geq 0.4$. Similarly, the degraded system controlled by a degraded control strategy is simulated and is always stable if $\bar{\sigma} \geq 0.4$ (the details are omitted here). Then we get the threshold value of the considered hexacopter as follows

$$\sigma_{th} = 0.4, \quad \bar{\sigma}_{th} = 0.4.$$  \hspace{1cm} (21)

2) Simulation Results: In the simulation, the hexacopter is hovering with full attitude control initially, i.e., the altitude and the roll-pitch-yaw angles are all under control. At time $t = 5s$, propulsor 2 fails and the reduced attitude control is switched to based on the switching control methodology. The simulation results are shown in Fig. 3 and Fig. 4. In Fig. 3, the real-time altitude and attitude information are shown, where the multicopter is in Safe Flight mode when no faults occurred and then switched to the Safe Landing mode after propulsor 2 fails. The real-time CPIs $S, \bar{S}$ are shown in Fig. 4 from which it is seen that $S < 0$ and $\bar{S} > 0$ after the failure of propulsor 2.

According to the simulations results, it is shown that the proposed switching control framework based on the proposed CPI is effective. In the following, experiments are carried out to show the effectiveness of the real-time CPI estimator in the switching control framework.

B. Experimental Results

A quadcopter platform named Qball-X4 [27] (a quadcopter developed by Quanser) is used in the experiments to show that the CPI can be used to monitor the performance of the vehicle. A group of PID controllers are offered by the manufacturer of the Qball-X4 for altitude and attitude control purpose. In order to get the CPT of Qball-X4, the Qball-X4 simulation model offered by the Quanser Company is modified slightly and the threshold value determination procedures are used. Here, the details are omitted and the CPT of the Qball-X4 is $\sigma_{th} = 0.3993$. 

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### Table I

| $\sigma$   | $N_{total}$ | $N_{stable}$ | percentage |
|-----------|-------------|--------------|------------|
| $[0,9]$   | 3           | 3            | 100%       |
| $[0.9,0,9]$ | 9           | 9            | 100%       |
| $[0.7,0.8]$ | 242         | 242          | 100%       |
| $[0.6,0.7]$ | 478         | 478          | 100%       |
| $[0.5,0.6]$ | 843         | 843          | 100%       |
| $[0.4,0.5]$ | 1329        | 1329         | 100%       |
| $[0.3,0.4]$ | 1865        | 1848         | 99%        |
| $[0.2,0.3]$ | 2705        | 2380         | 88%        |
| $[0.1,0.2]$ | 3190        | 2245         | 70%        |
| $[0.0,1]$  | 183727      | 1544         | 0.1%       |

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![Fig. 3. The real-time state of the considered hexacopter](image-url)

![Fig. 4. The real-time CPIs $S, \bar{S}$](image-url)
In the experiments, different weights are attached to the same specified place as shown in Fig. 5 and the real-time CPI will show the control performance of the quadcopter. In order to verify the experimental results, the maximum weight allowed by the quadcopter, which is $m_{\text{max}} = 126g$, is obtained by plenty of simulations. The main purpose of the experiments here is to verify the effectiveness of the CPI which are used to show the real-time control performance of the quadcopter and guide the autopilot to make safety decisions. The safety decision (1 for stable and 0 for unstable) is made based on (19). These experiments are recorded in the online video [28] or [29].

1) Case 1: A 100g weight was attached to the Qball-X4: Fig. 6 shows the experimental results where 100g weight is attached to specified place as shown in Fig. 5. The 100g weight was attached at time $t = 81s$, and the aircraft is still stable with the PID controllers. Then the 100g weight was removed at time $t = 96s$. The CPI results in Fig. 6 show that the Qball-X4 is always stable during the flight.

2) Case 2: Totally 150g weight was attached to the Qball-X4: However, in the second flight, totally 150g weight was attached to the same place and the results are shown in Fig. 7. Firstly, 100g weight was attached at time $t = 33s$, and the aircraft is still stable. Then another 50g weight was attached to the same place as the 100g weight at time $t = 63s$, and the CPI results in Fig. 7 show that the Qball-X4 is unstable after the second weight is attached. From the video recording of this experiment, it is seen that the Qball-X4 crashed eventually.

3) Case 3: Totally 200g weight was attached to the Qball-X4: In the last flight, totally 200g weights was attached to the same place and the results are shown in Fig. 8. Firstly, 100g weight was attached at time $t = 57s$, the aircraft is stable. Then another 100g weight was attached to the same position at time $t = 79s$, and the CPI results in Fig. 8 show that the Qball-X4 is unstable after the second weight is attached. From the video recording of this experiment, it is seen that the Qball-X4 crashed eventually.

From the above experiments, it can be seen that the CPI proposed by this paper is effective in practice. The CPI can be used to monitor the real-time control performance of the multicopters and tell the users whether the vehicle is safe to operate or not.
based on the new DoC to reflect the control performance of the multicopters. A step-by-step procedure was also provided to obtain the control performance threshold (CPT) which would be used to compute the CPI. Besides, the proposed CPI is used to guide the switching control of multicopters in a new switching control framework. Finally, simulation and experimental results showed the effectiveness of the switching control framework and the CPI proposed in this paper.

APPENDIX: PROOF OF LEMMA 1

In the following, it is assumed that \( \rho(\mathbf{d}, \partial \Omega) > 0 \). According to Theorem 3 in [14], if rank \( \mathbf{B}_f \) = 4, then the ACAI \( \rho(\mathbf{d}, \partial \Omega) \) is given by

\[
\rho(\mathbf{d}, \partial \Omega) = \min\{d_1, d_2, \cdots, d_m\} \tag{22}\]

where \( d_j = +\infty \) if rank \( \mathbf{B}_{1j} \) < 3 and

\[
d_j = \frac{1}{2} \text{sign} \left( \xi_j^T \mathbf{B}_{2j} \right) \mathbf{A}_j \left( \xi_j^T \mathbf{B}_{2j} \right)^T - \left| \xi_j^T (\mathbf{u}_f - \mathbf{d}) \right| \tag{23}
\]

if rank \( \mathbf{B}_{1j} \) = 3. The matrices \( \mathbf{B}_{1j} \in \mathbb{R}^{4 \times 3} \) and \( \mathbf{B}_{2j} \in \mathbb{R}^{4 \times (n_{p} - 3)} \) are composed of arbitrary three columns and the remaining \( n_{p} - 3 \) columns, respectively. There are totally \( s_m \) cases of \( \mathbf{B}_{1j} \in \mathbb{R}^{4 \times 3} \) and \( \mathbf{B}_{2j} \in \mathbb{R}^{4 \times (n_{p} - 3)} \) where \( s_m = \frac{n_{p}!}{(n_{p} - 3)!} \). The vector \( \xi_j \in \mathbb{R}^4 \) satisfies \( \xi_j^T \mathbf{B}_{1j} = 0, \left| \xi_j \right| = 1 \).

Similarly, if rank \( \mathbf{B}_f \) = 4, then the virtual ACAI \( \rho(\mathbf{u}_f, \partial \Omega) \) is given by

\[
\rho(\mathbf{u}_f, \partial \Omega) = \min\{d'_1, d'_2, \cdots, d'_m\} \tag{24}\]

where \( d'_j = +\infty \) if rank \( \mathbf{B}_{1j} \) < 3 and

\[
d'_j = \frac{1}{2} \text{sign} \left( \xi_j^T \mathbf{B}_{2j} \right) \mathbf{A}_j \left( \xi_j^T \mathbf{B}_{2j} \right)^T. \tag{25}\]

According to (23) and (25), one has \( d_j \leq d'_j \). Then \( \rho(\mathbf{d}, \partial \Omega) \leq \rho(\mathbf{u}_f, \partial \Omega) \) according to (22) and (24). Then, \( \rho(\mathbf{u}_f, \partial \Omega) \) is the maximum ACAI of system (6).

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REFERENCES

[1] C. M. Belcastro, J. V. Foster, R. L. Newman, L. Groff, D. A. Crider, and D. H. Khlyde, “Aircraft loss of control: problem analysis for the development and validation of technology solutions,” in AIAA Guidance, Navigation, and Control Conference, no. AIAA 2016-0092, 2016.

[2] C. M. Belcastro and S. R. Jacobson, “Future integrated system concepts for preventing aircraft loss-of-control accidents,” in AIAA Guidance, Navigation, and Control Conference, no. AIAA 2010-8142, 2010.

[3] C. M. Belcastro, “Validation and verification of future integrated safety-critical systems operating under off-nominal conditions,” in AIAA Guidance, Navigation, and Control Conference, no. AIAA 2010-8143, 2010.

[4] K. Krishnakumar, S. Viken, and N. Nguyen, “Stability, maneuverability, and safe landing in the presence of adverse conditions,” http://www.aeronautics.nasa.gov/terra_pdf/foree_tech_plan_c1.pdf, 2009.

[5] S. J. Qin, “Control performance monitoring—a review and assessment,” Computers & Chemical Engineering, vol. 23, no. 2, pp. 173–186, 1999.

[6] S. J. Qin and J. Yu, “Recent developments in multivariable controller performance monitoring,” Journal of Process Control, vol. 17, no. 3, pp. 221–227, 2007.

[7] M. Bauer, A. Horch, L. Xie, M. Jelali, and N. Thornhill, “The current state of control loop performance monitoring—a survey of application in industry,” Journal of Process Control, vol. 38, pp. 1–10, 2016.

[8] M. Lichter, A. Bateman, and G. Balas, “Flight test evaluation of a run-time stability margin estimation tool,” in AIAA Guidance, Navigation, and Control Conference, no. AIAA 2009-6257, 2009.

[9] H. Pfifer, R. Venkataraman, and P. Seller, “Quantifying loss-of-control envelopes via robust tracking analysis,” Journal of Guidance, Control, and Dynamics, published online, December 28, 2016.

[10] P. Müller and H. Weber, “Analysis and optimization of certain qualities of controllability and observability for linear dynamical systems,” Automatica, vol. 8, no. 3, pp. 237–246, 1972.

[11] C. Viswanathan, R. Longman, and P. Likins, “A degree of controllability definition—fundamental concepts and application to modal systems,” Journal of Guidance, Control, and Dynamics, vol. 7, no. 2, pp. 222–230, 1984.

[12] G. Klein, R. Ongman, and R. Indberg, “Computation of a degree of controllability via system discretization,” Journal of Guidance, Control, and Dynamics, vol. 32, no. 5, pp. 1658–1661, 2009.

[13] O. Kang, Y. Park, Y. Park, and M. Suh, “New measure representing a degree of controllability for disturbance rejection,” Journal of guidance, control, and dynamics, vol. 32, no. 5, pp. 1658–1661, 2009.

[14] G.-X. Du, Q. Quan, B. Yang, and K.-Y. Cai, “Controllability analysis for multicopter helicopter rotor degradation and failure,” Journal of Guidance, Control, and Dynamics, vol. 38, no. 5, pp. 978–985, 2015.

[15] C. Anderson, “Ardupilot project,” http://copter.ardupilot.com, 2016.

[16] R. Mahony, V. Kumar, and P. Corke, “Multicopter aerial vehicles: Modeling, estimation, and control of a quadrotor,” IEEE robotics & automation magazine, vol. 19, no. 3, pp. 20–32, 2012.

[17] T. Schneider, “Fault-tolerant multicopter systems,” ETH Master Thesis, 2011.

[18] G.-X. Du and Q. Quan, “A matlab toolbox for calculating an available control authority index of multicopters,” http://rfc.buaa.edu.cn/resources, 2016.

[19] R. V. Hogg, J. W. McKean, and A. T. Craig, Introduction to Mathematical Statistics (Sixth Edition). Upper Saddle River, New Jersey, USA: Prentice Hall, 2005.

[20] A. Tayebi and S. McGilvray, “Attitude stabilization of a vtol quadrotor aircraft,” IEEE Transactions on control systems technology, vol. 14, no. 3, pp. 562–571, 2006.

[21] D. Lee, H. J. Kim, and S. Sastry, “Feedback linearization vs. adaptive sliding mode control for a quadrotor helicopter,” International Journal of control, Automation and systems, vol. 7, no. 3, pp. 419–428, 2009.

[22] A. Freddi, A. Lanzon, and S. Longhi, “A feedback linearization approach to fault tolerance in quadrotor vehicles,” IFAC Proceedings Volumes, vol. 44, no. 1, pp. 5413–5418, 2011.

[23] M. W. Mueller and R. D’Andrea, “Relaxed hover solutions for multicopters: Application to algorithmic redundancy and novel vehicles,” The International Journal of Robotics Research, p. 0278349515596233, 2015.

[24] Z. Zhang, A. Chamseddine, C. Rabbath, B. Gordon, C.-Y. Su, S. Rakheja, C. Fulford, J. Apkarian, and P. Gosselin, “Development of advanced fdd and ftc techniques with application to an unmanned quadrotor helicopter testbed,” Journal of Guidance, Control, and Dynamics, vol. 37, no. 2, pp. 580–591, 2014.

[25] G.-X. Du, Q. Quan, and K.-Y. Cai, “Controllability analysis and degraded control for a class of hexacopters subject to rotor failures,” Journal of Intelligent & Robotic Systems, vol. 78, no. 1, pp. 143–157, 2015.