A COMPARISON OF CATEGORICAL AND TOPOLOGICAL ENTROPIES ON WEINSTEIN MANIFOLDS

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ABSTRACT. Let $W$ be a symplectic manifold, and let $\phi : W \to W$ be a symplectic automorphism. Then, $\phi$ induces an auto-equivalence $\Phi$ defined on the Fukaya category of $W$. In this paper, we prove that the categorical entropy of $\Phi$ bounds the topological entropy of $\phi$ from below where $W$ is a Weinstein manifold and $\phi$ is compactly supported. Moreover, being motivated by [CGG21], we propose a conjecture which generalizes the result of [New88, Prz80, Yom87].

1. INTRODUCTION

1.1. Introduction. Let $W$ be a Weinstein manifold equipped with a compactly supported, exact symplectic automorphism $\phi$. The pair $(W, \phi)$ forms a discrete dynamical system. In the current paper, we compare two invariants of the dynamical system.

Let us introduce the invariants we are interested in. The first invariant is called topological entropy. The notion of topological entropy was defined in the '60s for compact spaces and the '70s for noncompact spaces. See [AKM65, Hof72, Hof74]. Let $h_{\text{top}}(\phi)$ denote the classical invariant of $(W, \phi)$.

Recently, [DHKK14] defined the notion of categorical entropy for a pair $(\mathcal{C}, \Phi)$ such that $\mathcal{C}$ is a triangulated category and $\Phi : \mathcal{C} \to \mathcal{C}$ is an auto-equivalence. We point out that our dynamic system $(W, \phi)$ induces a categorical dynamical system $(\mathcal{C}, \Phi)$ in symplectic topology. To be more precise, we recall that

- the (triangulated closure of the) wrapped Fukaya category $\mathcal{W}(W)$ of a Weinstein manifold $W$ is a triangulated category, and
- an exact symplectic automorphism $\phi$ induces an auto-equivalence $\Phi : \mathcal{W}(W) \to \mathcal{W}(W)$.

It induces the second invariant of our symplectic dynamical system $(W, \phi)$, i.e., the categorical entropy of $(W, \phi, \Phi)$. Let $h_{\text{cat}}(\phi)$ denote the second invariant. We call $h_{\text{cat}}(\phi)$ the categorical entropy of $\phi$.

Both entropies are invariants of one dynamical system. Thus, it is natural to compare two invariants. In this paper, we compare them and prove

$$h_{\text{cat}}(\phi) \leq h_{\text{top}}(\phi).$$

Remark 1.1.

(1) We remark that [KO20, Mat21] study the comparison of two entropies in an algebro-geometric setting. Especially, [KO20] considers a pair $(\mathcal{X}, \phi)$ such that $\mathcal{X}$ is a smooth projective variety, and $\phi$ is a surjective endomorphism of $\mathcal{X}$. Then, $\phi$ induces an auto-equivalence $\Phi$ on the derived category of coherent sheaves, and it defines the categorical entropy of $\phi$. For that case, [KO20] proves the equality $h_{\text{cat}}(\Phi) = h_{\text{top}}(\phi)$.

(2) In a symplectic-geometric setting, [BCJ+22] proves the inequality (1.1) for some specific cases.

1.2. Results. One reason for the inequality (1.1) holding is that $h_{\text{cat}}(\phi)$ is an invariant up to compactly supported Hamiltonian isotopy, but $h_{\text{top}}(\phi)$ is not. In other words, if $\phi_1$ and $\phi_2$ are Hamiltonian isotopic to each other, then $h_{\text{cat}}(\phi_1) = h_{\text{cat}}(\phi_2)$. It is because $\phi_1$ and $\phi_2$ induce the same auto-equivalence on $\mathcal{W}(W)$. However, $h_{\text{top}}(\phi_1)$ and $h_{\text{top}}(\phi_2)$ do not need to be the same. Thus, one can expect that the
The topological entropy is more sensitive than the categorical entropy. In other words, one can expect that Theorem 1.2 holds.

**Theorem 1.2** (=Theorem 4.1). The categorical entropy of $\phi$ bounds the topological entropy of $\phi$ from below, i.e.,

$$h_{\text{cat}}(\phi) \leq h_{\text{top}}(\phi).$$

**Sketch of proof.** Let $\mathcal{C}$ be a triangulated category with a generator $G$, and let $\Phi : \mathcal{C} \to \mathcal{C}$ be an auto-equivalence. By [DHKK14, Theorem 2.6], if $\mathcal{C}$ is smooth and proper, we have

$$h_{\text{cat}}(\Phi) = \lim_{n \to \infty} \dim \text{Hom}(G, \Phi^n(G)). \quad (1.2)$$

Let $(W, \phi)$ be a given dynamical system. We note that the wrapped Fukaya category $\mathcal{W}(W)$ of $W$ is smooth, but is not necessarily proper. Thus, we cannot use [DHKK14, Theorem 2.6] directly.

We recall that a fully stopped partially wrapped Fukaya category is proper. Moreover, by Lemma 2.10, the categorical entropies of $\phi$ on $\mathcal{W}(W)$ and on a partially wrapped Fukaya category are the same. Thus, Equation (1.2) holds if $\text{Hom}$ means the morphism space of a fully stopped partially wrapped Fukaya category.

It is left to show that the right-hand side of Equation (1.2) bounds $h_{\text{top}}(\phi)$ from below. A variant of Crofton inequality proves that. See Lemmas 3.2 and 3.4.

We remark that for the categorical entropy of $\phi$, we work on the wrapped Fukaya category $\mathcal{W}(W)$. However, there exists another triangulated category which is also an invariant of $W$. The other is the compact Fukaya category $\mathcal{F}(W)$ (or its triangulated closure). Thus, it would be natural to ask why we work on $\mathcal{W}(W)$ rather than $\mathcal{F}(W)$.

The followings are two reasons why we work on $\mathcal{W}(W)$ rather than $\mathcal{F}(W)$:

- First, it is well-known that there is a Lagrangian generating $\mathcal{W}(W)$. However, for $\mathcal{F}(W)$, the existence of Lagrangian generating $\mathcal{F}(W)$ is not known for a general $W$.
- Second, $\mathcal{W}(W)$ is a smooth category, but $\mathcal{F}(W)$ is not necessarily smooth. Thus, one cannot apply [DHKK14, Theorem 2.6].

However, if one adds some assumptions that resolve the above two difficulties, one can expect the inequality (1.1) holds on $\mathcal{F}(W)$. Based on this, we prove Theorem 1.3.

**Theorem 1.3** (=Theorem 6.3). Let a pair $(W, \phi : W \to W)$ satisfy the assumption in Lemma 6.2. Let $\Phi_{\mathcal{F}(W)}$ denote the functor that $\phi$ induces on the compact Fukaya category $\mathcal{F}(W)$. Then the categorical entropy $h_{\text{cat}}(\Phi_{\mathcal{F}(W)})$ bounds the topological entropy of $\phi$ from below, i.e.,

$$h_{\text{cat}}(\Phi_{\mathcal{F}(W)}) \leq h_{\text{top}}(\phi).$$

1.3. **Further questions.** At the beginning of Section 1.2, we emphasize a reason why we expect the inequality (1.1): categorical entropy cannot distinguish members of a Hamiltonian isotopic class, but topological entropy can. Here, we introduce another philosophical reason for our expectations.

In order to describe the reason, we review a property of topological entropy. By [New88, Prz80, Yom87], it is known that

$$h_{\text{top}}(\phi) = \sup_{\text{compact submanifold } Y \subset W} \left( \text{the exponential growth rate of } \text{Vol}(\phi^n(Y)) \text{ with respect to } n \right).$$

In other words, one can compute $h_{\text{top}}(\phi)$ by taking the supremum over all submanifolds $Y$. On the other hand, the categorical entropy of $\phi$ cares only about the exact Lagrangian submanifolds, and the other submanifolds cannot affect categorical entropy.
As a counterpart of the exponential growth rate of $\text{Vol}(\phi^n(Y))$, we define another entropy which is called barcode entropy. We note that the notion of barcode entropy is a slight modification of relative barcode entropy defined in [CGG21]. By definition, barcode entropy is not an invariant of the dynamical system $(W, \phi)$, but it is an invariant of $(W, \phi, L_1, L_2)$ where $L_i$ is a Lagrangian submanifold of $W$. Let $h_{\text{bar}}(\phi; L_1, L_2)$ denote the barcode entropy for $(W, \phi, L_1, L_2)$. For the details, see Section 7. Then, we prove Proposition 1.4.

**Proposition 1.4** (= Propositions 7.6 and 7.7). For a pair of Lagrangians $(L_1, L_2)$ satisfying conditions in Section 7,

$$h_{\text{cat}}(\phi) \leq h_{\text{bar}}(\phi; L_1, L_2) \leq h_{\text{top}}(\phi).$$

Based on Proposition 1.4 and the above arguments, we ask whether the following equations do hold or do not:

$$h_{\text{cat}}(\phi) = \inf_{L_1, L_2} h_{\text{bar}}(\phi; L_1, L_2),$$

$$h_{\text{top}}(\phi) = \sup_{L_1, L_2} h_{\text{bar}}(\phi; L_1, L_2).$$

1.4. **Structure of the paper.** The paper consists of six sections except Section 1. Section 2 reviews definitions and preliminaries. Sections 3 and 4 prove the main theorem, i.e., Theorem 1.2. Section 5 discusses two examples: the first example shows that the inequality (1.1) can be strict, and the second example shows that the categorical entropy can be larger than the logarithm of spectral radius, which is a well-known lower bound of the topological entropy. Section 6 considers the compact Fukaya category of $W$ under some assumptions. Section 7 is about the further questions described in Section 1.3.

1.5. **Acknowledgment.** In Sections 3 and 7 and the proof of Theorem 4.1, we use the idea given in [CGG21] heavily. Also, Definition 7.5 is originally introduced in [CGG21]. The second named author appreciates Viktor Ginzburg for explaining the key ideas of [CGG21] in a seminar talk and a personal conversation. The first named author is grateful to Otto van Koert for helpful comments.

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2. **Preliminaries**

In this section, we give preliminaries including the definitions of topological entropy, categorical entropy, and some basic stuff of Lagrangian Floer theory.

2.1. **Topological entropy.** Let $X$ be a topological space and let $\phi : X \to X$ be a continuous self-mapping defined on $X$. The notion of topological entropy $h_{\text{top}}(\phi)$ is defined in [AKM65] for compact $X$ and in [Hol72, Hol74] for non-compact $X$.

**Definition 2.1.** Let $X$ be a topological space and let $\phi : X \to X$ be a continuous self-mapping on $X$.

1. Let $\mathcal{O}(X)$ denote the class of all open covers of $X$. Similarly, let $\mathcal{O}_f(X)$ denote the class of all finite open covers of $X$.
2. Let $\alpha_i \in \mathcal{O}(X)$. Then,

$$\bigvee_{i=1}^n \alpha_i := \{U_1 \cap \cdots \cap U_n | U_i \in \alpha_i \} \in \mathcal{O}(X).$$

3. For all $\alpha \in \mathcal{O}(X)$, let $N(\alpha)$ denote the minimal cardinality of a sub-cover of $\alpha$. 


(4) If $X$ is compact, for all $\alpha \in \mathcal{O}(X)$, $h_{top}(\phi, \alpha)$ is a non-negative real number such that

$$h_{top}(\phi, \alpha) := \lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} \phi^{-i}(\alpha) \right).$$

Similarly, if $X$ is not compact, then $h_{top}(\phi, \alpha)$ is a non-negative real number such that

$$h_{top}(\phi) := \sup_{\alpha \in \mathcal{O}(X)} h_{top}(\phi, \alpha).$$

(5) The topological entropy of $\phi$, $h_{top}(\phi)$ is defined as

$$h_{top}(\phi) := \sup_{\alpha \in \mathcal{O}(X)} h_{top}(\phi, \alpha).$$

Remark 2.2. Let $\phi$ be a compactly supported self-mapping defined on a non-compact space $X$. Let $X_0$ be a compact submanifold of $X$ such that $X_0$ contains the support of $\phi$. Then, one can easily show that

$$h_{top}(\phi|_{X_0}) = h_{top}(\phi).$$

In the rest of Section 2.1, let $X$ be a compact manifold (with or without boundary) of dimension $n$, equipped with a Riemannian metric $g$. Then, there is another definition of topological entropy of $\phi : X \to X$. We note that it is known that the new definition gives the same topological entropy with Definition 2.1.

Definition 2.3.

(1) Let $\Gamma^k$ denote the set of strings

$$\Gamma^k := \left\{ (x, \phi(x), \ldots, \phi^{k-1}(x)) \in X^k := X \times \cdots \times X \ (k \text{ factors}) \right\}.$$

(2) An $\epsilon$-cubes in $X^k$ is a product of balls in $X$ of radius $\epsilon$.

(3) For a subset $Y \subset X^k$, $\text{Cap}_\epsilon Y$ is the minimal number of $\epsilon$-cubes needed to cover $Y$.

(4) The topological entropy of $\phi$, denoted by $h_{top}(\phi)$, is given by

$$h_{top}(\phi) := \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{k} \log \text{Cap}_\epsilon \Gamma^k.$$ 

We would like to point out that the $h_{top}(\phi)$ in Definition 2.3 (4) does not depend on a specific choice of a metric $g$. For more details, see [Gro03, Gro87].

We end this subsection by stating a property of topological entropy, that plays a key role in the proof of Lemma 3.4. For a $C^\infty$-submanifold $Y \subset X$ of dimension $m$, let

$$\Gamma^k_{\phi|Y} := \left\{ (y, \phi(y), \ldots, \phi^{k-1}(y)) \in X^k | y \in Y \right\}.$$

We note that the product metric on $X^k$ can induce an $m$-dimensional volume form. Thus, we can measure the volumes of $\Gamma^k_{\phi|Y}$ for all $k$. It is well-known that the exponential growth rate of the volumes is a lower bound of $h_{top}(\phi)$.

Proposition 2.4. The topological entropy of $\phi$ is bounded by the exponential growth rate of the volume of $\Gamma^k_{\phi|Y}$, i.e.,

$$\limsup_{k \to \infty} \frac{1}{k} \log \text{Vol}(\Gamma^k_{\phi|Y}) \leq h_{top}(\phi).$$

See [Yom87, Gro87] for the proof of Proposition 2.4.
2.2. Categorical entropy. We start this subsection by introducing Definition 2.5 which is originally defined in [DHKK14].

**Definition 2.5.** Let \( \mathcal{C} \) be a triangulated category with a split-generator \( G \). Let \( \Phi \) be an auto-equivalence defined on \( \mathcal{C} \).

1. The complexity of \( E_2 \) relative to \( E_1 \) at \( t \) is a number in \([0, \infty)\) given by

\[
\delta_t(E_1; E_2) := \inf \left\{ \sum_{i=1}^{k} e^{n_i t} \mid 0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{k-1} \rightarrow E_2 \oplus E_2' \rightarrow E_1[n_1] \rightarrow E_1[n_2] \rightarrow \cdots \rightarrow E_1[n_k] \right\}.
\]

2. For a given \( t \in \mathbb{R} \), the categorical entropy of \( \Phi \) at \( t \) is defined as

\[
h_{\text{cat}}(\Phi; t) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G; \Phi^n(G)) \in (-\infty) \cup \mathbb{R}.
\]

In the current paper, we only consider the case of \( t = 0 \).

**Definition 2.6.** Let \( \Phi : \mathcal{C} \to \mathcal{C} \) be an auto-equivalence defined on a triangulated category \( \mathcal{C} \) with a generator \( G \). We define the categorical entropy of \( \Phi \) as

\[
h_{\text{cat}}(\Phi) := h_{\text{cat}}(\Phi; 0).
\]

**Remark 2.7.** We note that \( h_{\text{cat}}(\Phi) = h_{\text{cat}}(\Phi; 0) \geq 0 \) by definition.

Let \( \mathcal{D} \) be a fully faithful subcategory of \( \mathcal{C} \) such that

- \( \mathcal{D} \) is a triangulated category, and
- the restriction of \( \Phi \) to \( \mathcal{D} \) defines an auto-equivalence on \( \mathcal{D} \), i.e., \( \Phi(\mathcal{D}) \subset \mathcal{D} \).

It is known that there exists a localisation functor \( l \)

\[
l : \mathcal{C} \to \mathcal{C}/\mathcal{D}.
\]

See [Dri04]. Then, \( \Phi \) induces an auto-equivalence defined on \( \mathcal{C}/\mathcal{D} \) uniquely up to natural transformations.

**Proposition 2.8** (Proposition 3.3 of [BCJ+22]). There exists a unique (up to natural transformation) dg functor

\[
\Phi_{\mathcal{C}/\mathcal{D}} : \mathcal{C}/\mathcal{D} \to \mathcal{C}/\mathcal{D},
\]

satisfying

\[
\Phi_{\mathcal{C}/\mathcal{D}} \circ l = l \circ \Phi.
\]

To be clear, let us use the following notation \( \Phi_{\mathcal{C}}, \Phi_{\mathcal{D}}, \) and \( \Phi_{\mathcal{C}/\mathcal{D}} \),

\[
\Phi_{\mathcal{C}} := \Phi : \mathcal{C} \to \mathcal{C}, \Phi_{\mathcal{D}} := \Phi_{\mathcal{C}/\mathcal{D}} : \mathcal{D} \to \mathcal{D}, \Phi_{\mathcal{C}/\mathcal{D}} : \mathcal{C}/\mathcal{D} \to \mathcal{C}/\mathcal{D}.
\]

Then, [BCJ+22 Theorem 3.8] compares the categorical entropies of \( \Phi_{\mathcal{C}}, \Phi_{\mathcal{D}}, \) and \( \Phi_{\mathcal{C}/\mathcal{D}} \).

**Lemma 2.9** (Theorem 3.8 of [BCJ+22]). The categorical entropies of \( \Phi_{\mathcal{C}}, \Phi_{\mathcal{D}}, \Phi_{\mathcal{C}/\mathcal{D}} \) satisfy

\[
h_{\text{cat}}(\Phi_{\mathcal{C}/\mathcal{D}}) \leq h_{\text{cat}}(\Phi_{\mathcal{C}}) \leq \max\{h_{\text{cat}}(\Phi_{\mathcal{D}}), h_{\text{cat}}(\Phi_{\mathcal{C}/\mathcal{D}})\}.
\]

Let \( W \) be a Weinstein manifold, and let \( \phi : W \to W \) be a compactly supported exact symplectic automorphism. Let \( \Lambda \) be a stop in \( \partial_{\infty} W \). If \( \mathcal{W}(W) \) (resp. \( \mathcal{W}(W, \Lambda) \)) denotes the wrapped Fukaya category of \( W \) (resp. partially wrapped Fukaya category of \( W \) with a stop \( \Lambda \)), \( \phi \) induces functors \( \Phi : \mathcal{W}(W) \to \mathcal{W}(W) \) and \( \Phi_{\Lambda} : \mathcal{W}(W, \Lambda) \to \mathcal{W}(W, \Lambda) \). Thanks to Lemma 2.9, one can compare \( h_{\text{cat}}(\Phi) \) and \( h_{\text{cat}}(\Phi_{\Lambda}) \).
Lemma 2.10 (Theorem 4.2 of [BCJ +22]). The induced functors $\Phi$ and $\Phi_\Lambda$ have the same categorical entropy, i.e.,

$$h_{\text{cat}}(\Phi) = h_{\text{cat}}(\Phi_\Lambda).$$

Proof. We note that

$$\mathcal{W}(W) := \mathcal{W}(W, \Lambda)/\mathcal{D},$$

where $\mathcal{D}$ means the full subcategory of $\mathcal{W}(W, \Lambda)$ generated by all linking disks.

Since $\phi$ is compactly supported, the restriction of $\Phi$ on $\mathcal{D}$ is the identity functor. Thus, the categorical entropy of $\Phi|_{\mathcal{D}}$ is zero.

We note that, as mentioned Remark 2.7,

$$h_{\text{cat}}(\Phi), h_{\text{cat}}(\Phi_\Lambda) \geq 0.$$  

By applying Lemma 2.9, one has

$$0 \leq h_{\text{cat}}(\Phi_\Lambda) \leq h_{\text{cat}}(\Phi) \leq \max\{h_{\text{cat}}(\Phi_\Lambda), 0\} = h_{\text{cat}}(\Phi_\Lambda).$$

This completes the proof. □

Remark 2.11. In Section 1, we used the notation $h_{\text{cat}}(\phi)$ to denote $h_{\text{cat}}(\Phi)$ where $\Phi$ is the induced auto-equivalence on the wrapped Fukaya category of $W$. In above, $\phi$ induced an auto-equivalence on another Fukaya category, partially wrapped Fukaya category. In order to avoid confusion, we let $h_{\text{cat}}(\Phi)$ (resp. $h_{\text{cat}}(\Phi_\Lambda)$) denote the categorical entropy on $W(\hat{\mathcal{W}})$ (resp. $W(\hat{\mathcal{W}}, \Lambda)$).

2.3. Lagrangian Floer theory. Let $\hat{W}$ be a Weinstein manifold with a Liouville one form $\lambda$. Then, there exists a Weinstein domain $W$ whose completion is $\hat{W}$. In other words,

$$\hat{W} := W \cup (\partial W \times [1, \infty]),$$

where $\partial W$ and $\partial W \times \{1\}$ are identified by the natural identification. It is well-known that the Liouville one form $\lambda$ satisfies

$$\lambda|_{\partial W \times [1, \infty]} := r \alpha,$$

where $r$ is the coordinate for $[1, \infty)$ and $\alpha := \lambda|_{\partial W}$. Furthermore, let $J$ be an almost complex structure on $\hat{W}$ that is compatible with the symplectic form $\omega := d\lambda$ and is of contact type at $\infty$. The latter condition is necessary to apply maximum principles to $J$-holomorphic curves in $\hat{W}$ in order to define Lagrangian Floer (co)homology. Then the symplectic structure $\omega$ and the almost complex structure $J$ determine a Riemannian metric $g$ on $\hat{W}$ given by

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot).$$

Definition 2.12. An exact Lagrangian $L$ is cylindrical at $\infty$ if

$$L \cap (\partial W \times [1, \infty)) = \Lambda \times [1, \infty),$$

where $\Lambda := L \times \partial W$.

For convenience, we will use the term “Lagrangian” instead of “exact Lagrangian with Cylindrical end”.

Let us assume that $L_1$ and $L_2$ are a transversal pair of Lagrangians in $\hat{W}$. Since $L_i$ is an exact Lagrangian, there is a primitive function

$$h_i : L_i \to \mathbb{R},$$

such that $\lambda|_{L_i} = df_i$. Let us fix such primitive functions $h_i$ for Lagrangians $L_i$.

Let $\mathcal{P}(L_1, L_2)$ be the space of paths from $L_1$ to $L_2$. We define the action functional

$$\mathcal{A} := \mathcal{A}_{L_1, L_2} : \mathcal{P}(L_1, L_2) \to \mathbb{R}$$
by

$$A(\gamma) := -h_1(\gamma(0)) + h_2(\gamma(1)), \forall \gamma \in \mathcal{P}(L_1, L_2)$$

Let us equip the path space $\mathcal{P}(L_1, L_2)$ with the standard $L^2$-metric induced by $g$. Then it is straightforward to check that

- the critical points of the action functional $A$ are the constant paths from $L_1$ to $L_2$, i.e., the intersection points of $L_1$ and $L_2$, and
- the gradient flows are given by strips

$$u : \mathbb{R} \times [0, 1] \to \hat{W}$$

satisfying the $J$-holomorphic equation

$$\partial_s u + J \partial_t u = 0, \forall (s, t) \in \mathbb{R} \times [0, 1].$$

The Lagrangian Floer complex $CF^*(L_1, L_2)$ is given by the Morse complex for the action functional $A_A(L_1, L_2)$. Indeed, for a given field $k$,

- $CF^*(L_1, L_2)$ is a graded $k$-vector space generated by the intersection points $L_1 \cap L_2$, and
- the differential map is defined by counting the $J$-holomorphic strips between two intersection points of $L_1$ and $L_2$.

We would like to point out that the grading of the Floer cochain complex is not crucial in our remaining arguments. We will just assume that one of the followings holds.

- either the Floer complex $CF^*(L_1, L_2)$ is $\mathbb{Z}/2$-graded, or
- it is $\mathbb{Z}$-graded assuming that

$$2c_1(W) = 0$$

and the Lagrangians $L_1$ and $L_2$ are graded in the sense of [Sei08, Section 12].

We remark that one can extend the action functional $A$ as a function defined on $CF(L_1, L_2)$ as

$$A \left( \sum_{x_i \in L_1 \cap L_2} a_i x_i \right) = \max \{ A(x_i) | a_i \neq 0 \}. \quad (2.2)$$

We need the extended action functional $A$ in Section 7.

3. Crofton’s inequality

The goal of this section is to prove Lemma 3.4 which plays a key role in the proof of Theorem 1.2. In order to prove Lemma 3.4, we construct a family of Lagrangian submanifolds satisfying some conditions in Lemma 3.2. By using the family of Lagrangians, we prove Lemma 3.4 in Section 3.2.

3.1. Lagrangian tomograph. In many place of this paper, we consider pairs of Lagrangians satisfying the following condition.

Definition 3.1. A pair of Lagrangian $(L_1, L_2)$ is good if $L_1$ and $L_2$ are disjoint in the cylindrical part, i.e.,

$$L_1 \cap L_2 \cap (\partial W \times [1, \infty)) = \emptyset.$$
(i) $L_1$ and $L^s$ are Hamiltonian isotopic to each other for all $s \in B_c^d$, 
(ii) $d_H(L_1, L^s) < \frac{\epsilon}{2}$ for all $s \in B_c^d$, and 
(iii) $L^s \cap L_2$ for almost all $s \in B_c^d$.

Before going further, we briefly review the notion of Hofer norm $d_H$ of a Hamiltonian isotopy, which appears in the condition (ii) of Lemma 3.2. Let $\varphi$ be a compactly supported Hamiltonian isotopy. Then, the Hofer norm of $\varphi$ is defined as

$$
\|\varphi\|_{Hofer} := \inf_{H} \int_{S^1} (\max_{M} H - \min_{M} H) \, dt,
$$

where the infimum is taken over all 1-periodic in time Hamiltonian $H$ generating $\varphi$. Moreover, one can define the Hofer distance between two Hamiltonian isotopic Lagrangians $L$ and $L'$ as

$$
d_H(L, L') := \inf\{\|\varphi\|_{Hofer} | \varphi(L) = L'\}.
$$

**Proof of Lemma 3.2.** Since $(L_1, L_2)$ is a good pair, there is a compact set $W_0$ such that 

$$
L_1 \cap L_2 \subset \text{Int}(W_0) \subset W_0 \subset \text{Int}(W),
$$

where $\text{Int}(W_0)$ and $\text{Int}(W)$ denote the interiors of $W_0$ and $W$ respectively. Then, we choose a collection of real-valued functions

$$
\{g_1, \ldots, g_d | g_i : L_1 \rightarrow \mathbb{R}\},
$$

satisfying

(A) $g_i(x) = 0$ if $x \in L_1 \setminus W$, and

(B) for all $x \in L_1 \cap W_0$, the cotangent fiber $T^*_x L_1$ is generated by $\{d g_i(x) | i = 1, \ldots, d\}$.

For any $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$, We set

$$
f_s : L_1 \rightarrow \mathbb{R},
$$

$$
x \mapsto s_1 g_1(x) + \cdots + s_d g_d(x).
$$

We note that there is a small neighborhood of $L_1$ in $\hat{W}$, which is symplectomorphic to a small disk cotangent bundle of $L_1$. Then for $s \in \mathbb{R}^d$ such that $\|s\| < 1$, one can assume that the graph of $d f_s$ is embedded into $\hat{W}$. Let $L^s$ be the embedded image of the graph of $d f_s$ in $\hat{W}$. By the construction of $L^s$, (i) holds obviously.

Let assume that $B_c^d$ is a closed ball in $\mathbb{R}^d$ centered at the origin with a sufficiently small radius. Then, one can observe that (ii) holds for all $s \in B_c^d$. We note that the radius of $B_c^d$ will depend on $\ell$ in Equation (3.2) below.

In order to prove (iii), we would like to show that the following $\Psi$ is a submersion on $L_2$.

$$
\Psi : B_c^d \times L_1 \rightarrow \hat{W},
$$

$$(s, x) \mapsto d f_s(x).
$$

In other words, if $\Psi(s, x) \in L_2$, we would like to show that

$$
D \Psi_{(s, x)} : T_{(s, x)} \left(B_c^d \times L_1\right) \rightarrow T_{\Psi(s, x)} \hat{W}
$$

is surjective.

We note that $\hat{W}$ is equipped with a Riemannian metric $g$ compatible with the symplectic structure. Let

$$
\ell := \min \{d(x, y) | x \in L_1 \cap (W \setminus \text{Int}(W_0)), y \in L_2 \cap (W \setminus \text{Int}(W_0))\},
$$

where $d(x, y)$ is the distance function. Since both $L_1 \cap (W \setminus \text{Int}(W_0))$ and $L_2 \cap (W \setminus \text{Int}(W_0))$ are compact and $L_1 \cap L_2 \subset \text{Int}(W_0)$, $\ell$ is well-defined and positive.
We note that the restriction of \( g \) on \( L_1 \) defines a metric on \( L_1 \). Thus, for all \( x \in L_1 \), one can assume that \( T^*_x L_1 \) is a normed-vector space. If the radius of \( B^d \) is sufficiently small, then for all \((s, x) \in B^d \times W, \| df_s(x)\| < \ell \). It is because \( g \) is compactly supported. Here, \( \| \cdot \| \) means the norm on \( T^*_x L_1 \). We assume that the radius of \( B^d \) is sufficiently small in the rest of the proof.

Let assume that \( \Psi(s, x) \in L_2 \). If \( \Psi(s, x) \in \hat{\hat{W}} \setminus W \), then by (A), \( \Psi(s, x) \in L_1 \). It contradicts to \( L_1 \cap L_2 \subset W_0 \subset W \). If \( \Psi(s, x) \in W \setminus \text{Int}(W_0) \), then \( d(\Psi(s, x), x) = \| df_s(x)\| < \ell \). This is contradict to Equation (3.2).

The above paragraph shows that if \( \Psi(s, x) \in L_2 \), then \( \Psi(s, x) \in W_0 \), i.e., \( L^s \cap L_2 \subset W_0 \). By (B), this proves that \( \Psi \) is a submersion on \( L_2 \). Since \( \text{codim} L_2 = \dim L_1 \), for almost all \( s \in B^d \), \( L^s \cap L_2 \), i.e., (ii) holds.

**Remark 3.3.**

1. We note that the the radius of \( B^d \) is determined by \( \ell, \epsilon \), and the collection \( \{g_1, \ldots, g_d\} \).
2. We would like to point out that \( \frac{\epsilon}{2} \) in Lemma 3.2 (ii) will be used later in the proof of Proposition 3.6.

### 3.2. Crofton's inequality.

In Section 3.2 we prove Lemma 3.4, i.e., a Crofton type inequality, which plays a key role in the proof of Theorem 1.2. We remark that, as mentioned in [CGG21, Section 5.2.2], Lemma 3.4 is well-known to experts. For more details, see [CGG21, Section 5.2.2] and references therein.

In order to state Lemma 3.4 we need some preparation. For \( s \in B^d \) such that \( L^s \cap L_2 \), let

\[
N(s) := |L^s \cap L_2|.
\]

Then, \( N(s) \) is finite for almost all \( s \in B^d \). Moreover, \( N(s) \) is an integrable function on \( B^d \).

Since \( B^d \subset \mathbb{R}^d \), \( B^d \) carries the standard Euclidean metric. Let \( ds \) be the volume form on \( B^d \) induced from the Euclidean metric.

Let

\[
E := \Psi^{-1}(W_0).
\]

Then, let us fix a metric \( g_E \) on \( E \) such that the restriction of \( D\Psi \) to the normals to \( \Psi^{-1}(y), y \in W \) is an isometry. Since \( \Psi \) is a proper submersion, \( \Psi \) is a locally trivial fibration by Ehresmann’s fibration Theorem [Ehr51]. Thus, the existence of such a metric is guaranteed.

Now, we state Lemma 3.4.

**Lemma 3.4.** One has

\[
\int_{B^d} N(s) ds \leq C \cdot \text{Vol}(L_2 \cap W),
\]

where \( C \) is a constant depending only on \( \Psi, ds \), the fixed metric \( g \) on \( \hat{\hat{W}} \), and the fixed metric \( g_E \) on \( E \).

**Proof.** Let \( \Sigma := \Psi^{-1}(L_2 \cap W) \). Then, by definition, for all \( s \in B^d \) such that \( L^s \cap L_2 \), one has

\[
|(s \times L_1) \cap \Sigma| = |L^s \cap L_2| = N(s).
\]

Note that in the proof of Lemma 3.2 we have

\[
L^s \cap L_2 \subset W_0 \subset W,
\]

by choosing a sufficiently small \( B^d \).

We recall that \( B^d \) carries the Euclidean metric and \( L_1 \) also carries a metric \( g_{L_1} \). Thus, \( B^d \times L_1 \) carries a product metric. On \( E \), the restriction of the product metric gives another metric that does not need to be the same as \( g_E \).
Let $\pi : E \to B^d \times L_1 \to B^d_\epsilon$ be the projection to the first factor. Then, if $\text{Vol}_1(\cdot)$ denotes the volume with respect to the product metric on $E$, one has

$$\int_{B^d_\epsilon} N(s) d{s} = \int_{B^d_\epsilon} |(s \times L_1) \cap \Sigma| d{s} = \int_{\Sigma} \pi^* d{s} \leq \text{Vol}_1(\Sigma).$$

Let $\text{Vol}(\cdot)$ (resp. $\text{Vol}_2(\cdot)$) denote the volume with respect to the fixed metric $g$ (resp. $g_E$) on $W$ (resp. $E$). Then, by Fubini theorem, one has

$$\text{Vol}_2(\Sigma) = \int_{L_2 \cap W} Vol_2\left(\Psi^{-1}(y)\right) dy_{L_2} \leq \max_{y \in \Psi(E)} \text{Vol}_2\left(\Psi^{-1}(y)\right) \cdot \text{Vol}(L_2 \cap W).$$

We note that since $E$ is compact,

$$\text{Vol}_1(\Sigma) \leq C_0 \cdot \text{Vol}_2(\Sigma),$$

where $C_0$ is a constant depending only on $g_E$ and the product metric on $E$.

By combining Equations (3.3) – (3.5), one concludes that

$$\int_{B^d_\epsilon} N(s) d{s} \leq C \cdot \text{Vol}(L_2 \cap W),$$

where $C$ is a constant depending only on $\Psi, d{s}, g$, and $g_E$. 

\[\square\]

4. CATEGORICAL VS TOPOLOGICAL ENTROPY

In this Section, we prove our main theorem comparing categorical and topological entropy. To be more precise, let $\phi : \hat{W} \to \hat{W}$ be a compactly supported exact symplectic automorphism of a Weinstein manifold $\hat{W}$. Let $\Phi : \mathcal{W}(\hat{W}) \to \mathcal{W}(\hat{W})$ denote the functor induced from $\phi$, where $\mathcal{W}(\hat{W})$ is the wrapped Fukaya category of $\hat{W}$. Then, we prove Theorem [4.1]

**Theorem 4.1** (=Theorem 1.2). The categorical entropy of $\Phi$ bounds the topological entropy of $\phi$ from below, i.e.,

$$h_{\text{cat}}(\Phi) \leq h_{\text{top}}(\phi).$$

**Proof.** In order to prove Theorem 4.1 we recall that every Weinstein manifold $\hat{W}$ admits a Lefschetz fibration $\pi : \hat{W} \to \mathbb{C}$ by [GP17]. Then, $\pi$ defines a Fukaya-Seidel category. Moreover, it is known by [GPS18] that the corresponding Fukaya-Seidel category is the partially wrapped Fukaya category with the stop $\Lambda = \pi^{-1}(\infty)$. Let $\mathcal{W}(\hat{W}, \Lambda)$ denote the partially wrapped Fukaya category with a stop $\Lambda$. Also, it is known that the Lefschetz thimbles of $\pi$ generate $\mathcal{W}(\hat{W}, \Lambda)$. Let $G$ denote the generating Lagrangian submanifold.

We note that wrapping a Lagrangian $G$ means taking a Hamiltonian isotopy of $G$. Since $\mathcal{W}(\hat{W}, \Lambda)$ is fully stopped, there exists a Hamiltonian isotopy $\varphi_0$ such that

(A) $(\varphi_0(G), \varphi^n(G))$ is a good pair for all $n \in \mathbb{N}$, and
(B) $HW_{\Lambda}\left(\varphi_0(G), \varphi^n(G)\right) = HF\left(\varphi_0(G), \varphi^n(G)\right)$ for all $n \in \mathbb{N}$,

where $HW_{\Lambda}$ means the morphism space of $\mathcal{W}(\hat{W}, \Lambda)$. See Figure 1.

For a given $n \in \mathbb{N}$, we apply Lemma 3.2 for the good pair of Lagrangians $(\varphi_0(G), \varphi^n(G))$. Then, there exists a family of Lagrangian $(L^s)_{s \in B^d_\epsilon}$ such that

(i) $\varphi_0(G)$ and $L^s$ are Hamiltonian isotopic to each other for all $s \in B^d_\epsilon$,
(ii) $d_H(\varphi_0(G), L^s) < \frac{\epsilon}{2}$, and
(iii) $L^s \cap \varphi^n(G)$ for almost all $s \in B^d_\epsilon$. 


Figure 1. The interior of the black dotted circle is the base of a Lefschetz fibration \( \pi \). The star marked points are the singular values and the black dot is \(-\infty\). We note that the stop \( \Lambda \) is give by \( \Lambda = \pi^{-1}(-\infty) \). One can choose \( G \) such that \( \pi(G) \) is the union of all black curves. Similarly, \( \pi(\phi_0(G)) \) is the union of all red curves. Let \( \pi(W) \) be contained in the interior of blue dotted circle. Then, \( (\phi_0(G), \phi^n(G)) \) is a good pair for all \( n \in \mathbb{Z} \).

We note that one can find a family \( \{L^s\}_{s \in B^\epsilon} \) which does not depend on \( n \). To be more precise, we remark that in the proof of Lemma 3.2, the construction of \( \{L^s\}_{s \in B^\epsilon} \) depends only on \( \epsilon \), a collection of functions \( \{g_1, \ldots, g_d\} \), and \( \ell \) in Equation (3.2). Since \( \epsilon \) is a fixed, sufficiently small positive number, \( \epsilon \) is independent of \( n \). Similarly, \( \{g_1, \ldots, g_d\} \) is a collection of functions not depending on \( n \).

We recall that in order to define \( \ell \), we should fix \( W_0 \subset W \) such that \( W_0 \subset \text{Int}(W) \) and \( \phi_0(G) \cap \phi^n(G) \subset \text{Int}(W_0) \).

Without loss of generality, one can assume that \( W_0 \) not only satisfies the above two conditions, but also contains the support of \( \phi \). Then, outside of \( W_0 \), \( \phi^n_1(G) \) and \( \phi^n_2(G) \) agree for all \( n_i \in \mathbb{N} \). Thus, \( \ell \) in Equation (3.2) does not depend on \( n \).

Since we have a family \( \{L^s\}_{s \in B^\epsilon} \) not depending on \( n \), one can define the following function

\[
N_n(s) := |L^s \cap \phi^n(G)|.
\]

We point out that for each \( n \in \mathbb{N} \), \( N_n(s) \) is an integrable function because of (iii).

By applying Lemma 3.4 we have

\[
\int_{B^\epsilon} N_n(s) ds \leq C \cdot \text{Vol}(\phi^n(G) \cap W).
\]

We note that the constant \( C \) in (4.1) is independent of \( n \).

On the other hand, for \( L^s \triangle \phi^n(G) \), we have

\[
\dim \text{HW}_\Lambda(G, \phi^n(G)) = \dim \text{HF}(\phi_0(G), \phi^n(G)) = \dim \text{HF}(L^s, \phi^n(G)) \leq N_n(s).
\]
The first equality holds because of (B), the second equality holds because of (i), and the last inequality holds because of the definition of Lagrangian Floer homology.

By integrating Equation (4.2), one has

\[
\text{Vol}(\mathcal{B}_d) \cdot \dimHW\Lambda(G,\phi^n(G)) = \int_{\mathcal{B}_d} \dimHW\Lambda(G,\phi^n(G)) \, ds \leq \int_{\mathcal{B}_d} N_n(s) \, ds. \tag{4.3}
\]

From two inequalities (4.1) and (4.3), one has

\[
\text{Vol}(\mathcal{B}_d) \cdot \dimHW\Lambda(G,\phi^n(G)) \leq C \cdot \text{Vol}(\phi^n(G) \cap W). \tag{4.4}
\]

By taking \(\limsup_{n \to \infty} \frac{1}{n} \log^+\) for the both hand sides of (4.4), one has

\[
h_{cat}(\Phi_\Lambda : \mathcal{W}(\hat{W},\Lambda) \to \mathcal{W}(\hat{W},\Lambda)) = \limsup_{n \to \infty} \frac{1}{n} \log \dimHW\Lambda(\varphi_0(G),\phi^n(G))
\leq \limsup_{n \to \infty} \frac{1}{n} \log \text{Vol}(\phi^n(G) \cap W). \tag{4.5}
\]

The first equality in Equation (4.5) holds because of [DHKK14, Theorem 2.6] and because \(\mathcal{W}(\hat{W},\Lambda)\) is smooth and proper.

We note that, by Lemma 2.10,

\[h_{cat}(\Phi) = h_{cat}(\Phi_\Lambda).\]

We also note that, by Proposition 2.4

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Vol}(\phi^n(G) \cap W) \leq h_{top}(\phi). \tag{4.6}
\]

Thus, (4.5) and (4.6) complete the proof. \(\square\)

**Remark 4.2.** In the proof of Theorem 4.1, we fix a Lefschetz fibration, and we use the corresponding Fukaya-Seidel category. We note that if one fixes a fully stopped partially wrapped Fukaya category instead of a Fukaya-Seidel category, the same proof still works.

5. **Examples**

In this section, we provide two examples. The first example is a symplectic automorphism for which the inequality (1.1) is strict. The second example shows that categorical entropy can be strictly greater than the spectral radius of its induced map on the homology.

5.1. **The first example.** Let \(W\) be a 2-dimensional Weinstein domain such that \(W \neq \mathbb{D}^2\). It is well-known that its wrapped Fukaya category is generated by the Lagrangian cocores, see [CRGG17, GPS18]. This ensures that the categorical entropy of an endo-functor on \(\mathcal{W}(W)\) is well-defined.

Let \(U\) be a small open ball in \(W\). It is well-known that there is a Hamiltonian diffeomorphism \(\tilde{\phi} : \mathcal{W}(\hat{W},\Lambda) \to \mathcal{W}(\hat{W},\Lambda)\) such that

- \(\tilde{\phi}\) is the identity near the boundary of \(\overline{U}\), and
- \(\tilde{\phi}\) has a positive topological entropy.

Smale’s horseshoe map is an example of such \(\tilde{\phi}\).

Since \(\tilde{\phi}\) is assumed to be the identity near the boundary of the closure of \(U\), it admits a trivial extension to the whole Weinstein domain \(W\), which we will call \(\phi\). Since \(\phi\) is a compactly supported Hamiltonian diffeomorphism, if we let \(\Phi\) be its induced functor on \(\mathcal{W}(W)\) as above, then we have

\[h_{cat}(\Phi) = 0 < h_{top}(\tilde{\phi} = \phi|_U) \leq h_{top}(\phi).\]
Hence $\phi$ is an example showing that the inequality \((1)\) can be strict.

5.2. The second example. Let $\phi_* : H_*(W) \to H_*(W)$ be the linear map on the homology of $W$ which $\phi$ induced. We define the spectral radius of $\phi$ as the maximal absolute value of eigenvalues of $\phi_*$. Let $\operatorname{Rad}(\phi)$ denote the spectral radius of $\phi$.

It is well-known that $\log \operatorname{Rad}(\phi)$ is a lower bound of $h_{\operatorname{top}}(\phi)$. We refer the reader to [Gro87] for more details. Since Theorem 4.1 gives another lower bound of $h_{\operatorname{top}}(\phi)$, i.e., $h_{\operatorname{cat}}(\Phi)$, one can ask the relationship between two lower bounds of $h_{\operatorname{top}}(\phi)$. In this subsection, first, we give an example of $\phi$ such that $\log \operatorname{Rad}(\phi) \neq h_{\operatorname{cat}}(\phi)$, then, we discuss a difficulty of comparing two lower bounds generally.

Let $A$ and $B$ be $n$-dimensional spheres. Then let $W$ be the plumbing of the cotangent bundles $T^*_A$ and $T^*_B$ at a point. In other words, $W$ is the Milnor fiber of $A^2$-type.

Let $\tau_A$ and $\tau_B$ be the Dehn twist defined on $W$ along $A$ and $B$, respectively. Let us consider the symplectic automorphism on $W$ given by
\[
\phi = \tau_A \circ \tau_B^{-1}.
\]
Now observe that the homology of $W$ is given by
\[
H_*(W) = \begin{cases} 
\mathbb{Z} \langle [pt] \rangle & * = 0, \\
\mathbb{Z} \langle [A], [B] \rangle & * = n, \\
0 & \text{otherwise}.
\end{cases}
\]
Since $\phi$ induces the trivial map on the zeroth homology $H_0(W)$, it is enough to consider its induced map on the $n$-dimensional homology $H_n(W)$ to compute $\operatorname{Rad}(\phi_*)$.

For that purpose, we consider the induced map of $\tau_A$ and $\tau_B$ on $H_n(W)$ separately.

1. $(\tau_A)_*([A]) = (-1)^{n-1}[A]$.
2. $(\tau_A)_*([B]) = [A] + [B]$.
3. $(\tau_B)_*([A]) = [A] + (-1)^n[B]$.
4. $(\tau_B)_*([B]) = (-1)^{n-1}[B]$.

In other words, $(\tau_A)_*$ and $(\tau_B)_*$ are represented by the matrices
\[
(\tau_A)_* = \begin{pmatrix} (-1)^{n-1} & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (\tau_B)_* = \begin{pmatrix} 1 & 0 \\ (-1)^n & (-1)^{n-1} \end{pmatrix},
\]
respectively.

Consequently, the map $\phi_* = (\tau_A)_* \circ (\tau_B^{-1})_*$ is represented by
\[
\begin{pmatrix} (-1)^{n-1} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-1)^n & (-1)^{n-1} \end{pmatrix}^{-1} = \begin{pmatrix} (-1)^{n-1} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-1)^n & (-1)^{n-1} \end{pmatrix} = \begin{pmatrix} 1 + (-1)^{n-1} & (-1)^{n-1} \\ 1 & (-1)^{n-1} \end{pmatrix}.
\]

Let us now assume that $n$ is even. Then the above matrix is
\[
\begin{pmatrix} 0 & (-1)^{n-1} \\ 1 & (-1)^{n-1} \end{pmatrix}.
\]
A straightforward computation shows that its eigenvalues are
\[ \frac{-1 + \sqrt{3}i}{2} \quad \text{and} \quad \frac{-1 - \sqrt{3}i}{2}. \]

Hence the spectral radius of \( \phi_* \) is
\[ \left| \frac{-1 + \sqrt{3}i}{2} \right| = \left| \frac{-1 - \sqrt{3}i}{2} \right| = 1. \]

On the other hand, \([BCJ^{22}, \text{Theorem 7.14, Lemma 8.5}]\) says that the categorical entropy of the induced map of \( \phi \) on the wrapped Fukaya category is
\[ \frac{3 + \sqrt{5}}{2}. \]

Finally we have a strict inequality
\[ \logRad(\phi) = 0 < \frac{3 + \sqrt{5}}{2} = h_{cat}(\Phi). \]

So far, we compared \( \logRad(\phi) \) and \( h_{cat}(\phi) \) for a fixed \( \phi \) and showed that
\[ (5.1) \quad \logRad(\phi) \leq h_{cat}(\Phi). \]

Thus, one can ask that inequality \((5.1)\) holds for a general \( \phi \). We note that symplectic automorphisms of a specific type satisfy the inequality \((5.1)\), but we do not know whether it holds or not for a general case. See Remark 5.1 for the special case. In the rest of this subsection, we briefly discuss our strategy for the specific type, and why the strategy does not work for a general symplectomorphism.

Note that \( \phi \) induces a linear map on the Grothendieck group \( K_0(W(W)) \). Let \( \Phi_* \) denote the induced map on \( K_0(W(W)) \). Then, by \([KST^{20}]\), we have
\[ \logRad(\Phi_*) \leq h_{cat}(\Phi), \]
under some conditions.

\([Laz^{19}]\) shows that there is a surjective map from middle-dimensional cohomology of \( W \) to \( K_0(W(W)) \). Thus, one can relate \( \logRad(\Phi_*) \) with the action of \( \phi_* \) on the middle-dimensional homology of \( W \) via duality between homology and cohomology.

On the other hand, \( \logRad(\phi) \) cares about the homology of all dimension. Thus, if \( H_k(W) \) is nontrivial for some \( 1 \leq k \leq n - 1 \), then it is difficult to compare \( h_{cat}(\phi) \) and \( \logRad(\phi) \) directly.

**Remark 5.1.** From the above discussion, one can expect that if \( H_k(W) = 0 \) for all \( 1 \leq k \leq n - 1 \), then
\[ \logRad(\phi) \leq h_{cat}(\phi). \]

We note that the plumbing space of \( T^* S^n \)'s along a tree \( T \) satisfies the condition. Let \( P_n(T) \) denote the plumbing space. Then, on \( P_n(T) \), it is easy to observe that the Grothendieck group \( K_0(W(P_n(T))) \) and \( H_n(P_n(T)) \) are isomorphic.

We note that for a compact core Lagrangian, there is a Dehn twist along it. Then, every Dehn twist induces linear maps on \( K_0(W(P_n(T))) \) and \( H_n(P_n(T)) \). By choosing nice bases for \( K_0(W(P_n(T))) \) and \( H_n(P_n(T)) \), one can easily show that every Dehn twist induces the same matrix on the \( K_0(W(P_n(T))) \) and \( H_n(P_n(T)) \).

By the above argument, if \( \phi : P_n(T) \to P_n(T) \) is a product of positive/negative powers of Dehn twists, then the following equality holds.
\[ \logRad(\phi) \leq h_{cat}(\Phi). \]
6. THE CASE OF COMPACT FUKAYA CATEGORY

As mentioned in the introduction, we prove that a variant of Theorem 4.1 holds for compact Fukaya category under an additional assumption. The assumption we consider is a kind of “duality” between compact and wrapped Fukaya categories of $\hat{W}$. We start Section 6 by giving a specific example satisfying the “duality”.

Let $T$ be a tree and let $P_n(T)$ be the plumbing of the cotangent bundles of $T^*S^n$ along $T$ as in [BCJ^{+}22]. For each vertex $v$ of $T$, let $S_v$ be the Lagrangian sphere in $P_n(T)$ corresponding to $v$, and let $L_v$ be the Lagrangian cocore disk corresponding to $v$. This means that the Lagrangian spheres $S_v$ and the Lagrangian cocore disks $L_v$ intersect transversely and that the intersection numbers between those are given by

$$|S_v \cap L_w| = \begin{cases} 1 & v = w, \\ 0 & \text{otherwise.} \end{cases}$$

We note that [BCJ^{+}22] compares the categorical entropies on compact and wrapped Fukaya categories of $P_n(T)$ by using [AS12, Lemma 2.5] and the above Lagrangians $\{S_v\}$ and $\{L_v\}$. Motivated by this, we will assume the following in this subsection.

Assumption 6.1. There exists a finite collection of exact, closed Lagrangians $\{S_i\}_{i \in I}$ of $W$ indexed by some set $I$ such that

1. the direct sum $S = \bigoplus_{i \in I} S_i$ split-generates the compact Fukaya category $\mathcal{F}(W)$ in such way that every exact, closed Lagrangian $L$ of $W$ is quasi-isomorphic to a twisted complex for $L$ with components $\{S_i\}$, in which none of the arrows are nonzero multiples of the identity morphisms, and

2. there exists another collection of Lagrangian $\{L_i\}_{i \in I}$ of $W$, each of which intersects $S_i \in I$ transversely and satisfies

$$|S_i \cap L_j| = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $S = \bigoplus_{i \in I} S_i$ and $L = \bigoplus_{i \in I} L_i$. Let us denote by $\Phi_{\mathcal{F}(W)}$ the auto-functor on $\mathcal{F}(W)$ induced by $\phi$. Then, since the arguments in [BCJ^{+}22, Lemma 6.5, Theorem 6.6] continue to work under Assumption 6.1, we have Lemma 6.2.

Lemma 6.2. For any exact, compactly-supported symplectic automorphism $\phi$ on $W$, if $W$ satisfies Assumption 6.1, then

$$h_{cat}(\Phi_{\mathcal{F}(W)}) = \lim_{n \to \infty} \frac{1}{n} \log \dim HF^*(\phi^n(S), L).$$

Theorem 6.3. Let a pair $(W, \phi : W \to W)$ be as in Lemma 6.2. Then the categorical entropy $h_{cat}(\Phi_{\mathcal{F}(W)})$ for its induced functor on the compact Fukaya category $\mathcal{F}(W)$ bounds the topological entropy of $\phi$ from below, i.e.

$$h_{cat}(\Phi_{\mathcal{F}(W)}) \leq h_{top}(\phi).$$

Proof. Basically, most arguments in the proof of Theorem 4.1 can be applied to this case. Indeed, for $n \in \mathbb{N}$, we once again apply Lemma 3.2 to the pair $(S, \phi^{-n}(L))$ to get a family of Lagrangian $\{S^s\}_{s \in B^d_c}$ such that

1. $S$ and $S^s$ are Hamiltonian isotopic to each other for all $s \in B^d_c$,

2. $d_H(S, S^s) < \frac{\epsilon}{2}$, and

3. $S^s \cap \phi^{-n}(L)$ for almost all $s \in B^d_c$. 


As mentioned in the proof of 4.1 one can find such a family \( \{S^s\}_{s \in B^d} \), for which the third condition (iii) holds for all \( n \in \mathbb{N} \).

Then we consider the following function.

\[
N_n(s) := |S^s \cap \phi^{-n}(L)|.
\]

By applying Lemma 3.4 once again, we have

\[
\int_{B^d} N_n(s) \, ds \leq C' \cdot \text{Vol}(\phi^{-n}(L) \cap W).
\]

for some constant \( C' \) which does not depend on \( n \).

On the other hand, for \( S^s \pitchfork \phi^{-n}(L) \), we have

\[
\dim \text{HF}(\phi^n(S), L) = \dim \text{HF}(S, \phi^{-n}(L)) \leq N_n(s).
\]

The above inequality holds since \( S^s \) and \( S^s \) are Hamiltonian isotopic.

By integrating Equation (6.2), one has

\[
\text{Vol}(B^d) \cdot \dim \text{HF}(\phi^n(S), L) = \int_{B^d} \dim \text{HF}(\phi^n(S), L) \, ds \leq \int_{B^d} N_n(s) \, ds.
\]

From two inequalities (6.1) and (6.3), one has

\[
\text{Vol}(B^d) \cdot \dim \text{HF}(\phi^n(S), L) \leq C' \cdot \text{Vol}(\phi^{-n}(G) \cap W).
\]

By taking \( \limsup_{n \to \infty} \frac{1}{n} \log^+ \) for the both hand sides of (6.4) and using Lemma 6.2 one has

\[
h_{\text{cat}}(\Phi_{\mathcal{F}(W)}) = \limsup_{n \to \infty} \frac{1}{n} \log \dim \text{HF}(\phi^n(S), L)
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log \text{Vol}(\phi^{-n}(L) \cap W).
\]

Here the latter is again bounded above by \( h_{\text{top}}(\phi) \) due to Proposition 2.4. Therefore, (6.5) proves the assertion.

7. Barcode entropy

In this section, we define another entropy, called barcode entropy. As mentioned in Section 1.2, the notion of barcode entropy is the same as the relative barcode entropy defined in [CGG21]. At the end of Section 7 we give further questions related to categorical, topological, and barcode entropies.

7.1. Preliminaries. In this subsection, we review the theory of persistence module, and we apply the theory to Lagrangian Floer homology. We refer the reader to [PRS20, UZ16] for the theory of Persistence module. Also, we refer the reader to [CGG21] for the details we omitted in the current subsection.

The notion of non-Archimedean norm on a vector space is defined in [UZ16, Definition 2.2]. It is easy to check that \( \mathcal{A} \) in (2.2) is a non-Archimedean norm on the \( k \)-vector space \( CF(L_1, L_2) \). Moreover, \( CF(L_1, L_2) \) is orthogonal with respect to \( \mathcal{A} \).

Now, we are ready to apply [UZ16 Theorem 3.4] for the differential

\[
\delta : CF(L_1, L_2) \to CF(L_1, L_2),
\]

Since \( \delta \) is a linear self-mapping of an orthogonal vector space \( CF(L_1, L_2) \), one obtains a basis \( \Sigma = \{ \alpha_j, \beta_j, \gamma_j \} \) of \( CF(L_1, L_2) \) satisfying

1. \( \partial \alpha_i = 0 \),
(2) $\partial \gamma_j = \beta_j$, and
(3) $\mathcal{A}(\gamma_1) - \mathcal{A}(\beta_1) \leq \mathcal{A}(\gamma_2) - \mathcal{A}(\beta_2) \leq \ldots$.

By using the above, we define the followings.

**Definition 7.1.**

1. A bar of $\text{CF}(L_1, L_2)$ is either $\alpha_i$ or a pair $(\beta_j, \gamma_j)$.
2. The length of a bar $b$ is given by
   \[
   \text{the length of } b = \begin{cases} 
   \infty & \text{if } b = \alpha_i, \\
   \mathcal{A}(\gamma_j) - \mathcal{A}(\beta_j) & \text{otherwise.}
   \end{cases}
   \]
3. Let $b_\epsilon(L_1, L_2)$ be the number of bars of $\text{CF}(L_1, L_2)$ whose lengths are greater than or equal to $\epsilon$.

**Remark 7.2.** We note that $\mathcal{A}$ is unique up to constant. More precisely, for an exact Lagrangian $L_i$, a choice of primitive function $h_i : L_i \to \mathbb{R}$ is not unique, but unique up to constant. Thus, it is easy to show that the length of bars depends only on $L_i$ and independent of the choice of primitive function $h_i : L_i \to \mathbb{R}$.

By Definition 7.1, Lemma 7.3 is obvious.

**Lemma 7.3.** Let $L_1$ and $L_2$ be a transversal pair of Lagrangians. Then, for any $\epsilon \geq 0$,
\[
b_\epsilon(L_1, L_2) \leq b_0(L_1, L_2) \leq |L_1 \cap L_2|.
\]

It is well-known that $b_\epsilon$ is insensitive to small perturbations of the Lagrangians with respect to the Hofer distance. More precisely, Lemma 7.4 holds.

**Lemma 7.4.** Let $L'_1$ be a Lagrangian satisfying
- $L'_1$ and $L_1$ are Hamiltonian isotopic to each other,
- $d_H(L_1, L'_1) < \frac{\epsilon}{2}$ with $\delta < \epsilon$, and
- $L'_1$ and $L_2$ are transversal to each other.

Then,
\[
b_{\epsilon+\delta}(L'_1, L_2) \leq b_{\epsilon}(L_1, L_2) \leq b_{\epsilon-\delta}(L'_1, L_2).
\]

**Proof.** See [CGG21] Equations (3.13) and (3.14)]. We also refer the reader to [KS21][PRSZ20][UZ16].

Now, we extend the barcode counting function $b_\epsilon(L_1, L_2)$ to a good pair $(L_1, L_2)$ defined in Definition 3.1. For a good pair, we set
\[
b_\epsilon(L_1, L_2) := \liminf_{d_H(L_2, L'_2) \to 0} b_\epsilon(L_1, L'_2),
\]
where the limit is taken over Lagrangians $L'_2$ such that $L'_2 \pitchfork L_1$. We also note that $L_2$ and $L'_2$ should be Hamiltonian isotopic so that the Hofer distance between them is defined.

**7.2. Barcode entropy.** In the rest of this paper, we consider the same situation as what we considered in Section 3. For the reader’s convenience, we review the setting.

Let $(\hat{W}, \lambda)$ be a Weinstein manifold and let $\phi : \hat{W} \to \hat{W}$ be a compactly supported exact symplectic automorphism. Then, there is a Weinstein domain $W$ such that
- $\hat{W} = W \cup \partial (W \times [1, \infty))$,
- $\lambda|_{\partial W \times [1, \infty)} = r \alpha$ where $r$ is a coordinate for $[1, \infty)$ and $\alpha := \lambda|_{\partial W}$, and
- the support of $\phi$ is contained in $\text{Int}(W)$.
Let \((L_1, L_2)\) be a good pair of Lagrangians with respect to \(W\), i.e.,
\[
L_1 \cap L_2 \cap (\partial W \times [1, \infty)) = \emptyset.
\]
Since \(\phi\) is the identity outside of \(W\), \((L_1, \phi^n(L_2))\) is also a good pair for any \(n \in \mathbb{Z}\). Then, for a fixed \(\epsilon\), \(b_\epsilon(L_1, \phi^n(L_2))\) is well-defined. We would like to define the barcode entropy of \(\phi\) as the exponential growth rate of \(b_\epsilon(L_1, \phi^n(L_2))\) as \(n \to \infty\).

To be more precise, let \(\log^+ : \mathbb{Z}_{\geq 0} \to \mathbb{R}\) be the function defined as
\[
\log^+(k) = \begin{cases} 0 & \text{if } k = 0, \\ \log(k) & \text{otherwise}, \end{cases}
\]
where the logarithm is taken base \(2\).

**Definition 7.5.**

1. For any \(\epsilon \in \mathbb{R}_{\geq 0}\), the \(\epsilon\)-barcode entropy of \(\phi\) relative to \((L_1, L_2)\) is
   \[
   h_\epsilon(\phi; L_1, L_2) := \lim_{n \to \infty} \frac{1}{n} \log^+ b_\epsilon(L_1, \phi^n(L_2)).
   \]
2. The barcode entropy of \(\phi\) relative to \((L_1, L_2)\) is
   \[
   h_{\text{bar}}(\phi; L_1, L_2) := \lim_{\epsilon \searrow 0} h_\epsilon(\phi; L_1, L_2).
   \]

We note that Definition 7.5 is the same as the notion of relative barcode entropy in [CGG21], except a minor adjustment to our set up.

### 7.3. Barcode vs topological entropy.
In this subsection, we prove that for any good pair \((L_1, L_2)\), the barcode entropy of \(\phi\) bounds the topological entropy of \(\phi\) from below. The proof of Proposition 7.6 is almost same as the [CGG21] Proof of Theorem A).

**Proposition 7.6** (= The second inequality in Proposition 1.4). For any good pair \((L_1, L_2)\),
\[
h_{\text{bar}}(\phi; L_1, L_2) \leq h_{\text{top}}(\phi).
\]

**Proof.** If \(h_{\text{bar}}(\phi; L_1, L_2) = 0\), then there is nothing to prove. Thus, let assume that \(h_{\text{bar}}(\phi; L_1, L_2) > 0\). We would like to show that if \(\alpha \leq h_{\text{bar}}(\phi; L_1, L_2)\), then \(\alpha \leq h_{\text{top}}(\phi)\).

Let \(\delta\) be a positive number. Since
\[
h_{\text{bar}}(\phi; L_1, L_2) := \lim_{\epsilon \searrow 0} h_\epsilon(\phi; L_1, L_2) \geq \alpha,
\]
there is \(\epsilon_0 > 0\) such that if \(\epsilon < \epsilon_0\), then \(h_\epsilon(\phi; L_1, L_2) > \alpha - \delta\). We fix a positive number \(\epsilon\) such that \(2\epsilon < \epsilon_0\).

Since
\[
h_{2\epsilon}(\phi; L_1, L_2) = \limsup_{n \to \infty} \frac{1}{n} \log^+ b_{2\epsilon}(L_1, \phi^n(L_2)) > \alpha - \delta,
\]
there is an increasing sequence of natural numbers \(\{n_i\}_{i \in \mathbb{N}}\) such that
\[
(7.1) \quad b_{2\epsilon}(L_1, \phi^{n_i}(L_2)) > 2^{(\alpha - \delta)n_i}.
\]

Now, we apply Lemma 3.2 to the good pair \((L_1, \phi^{n_i}(L_2))\). Then, one obtains a family of Lagrangians \(\{L^s\}_{s \in B_{\epsilon,n_i}}\) such that

1. \(L_1\) and \(L^s\) are Hamiltonian isotopic for all \(s \in B_{\epsilon,n_i}^d\),
2. \(d_H(L_1, L^s) < \frac{\epsilon}{2}\) for all \(s \in B_{\epsilon,n_i}^d\), and
3. \(L^s \cap \phi^{n_i}(L_2)\) for almost all \(s \in B_{\epsilon,n_i}^d\).
We point out that by the argument in the proof of Theorem 4.1, we can choose a family satisfying (i)–(iii) for all \( n_i \). Let \( \{ L^i \}_{\epsilon \in B^\epsilon} \) denote a fixed Lagrangian tomograph.

Let

\[
N_i(s) := |L^i \cap \phi^{n_i}(L_2)|.
\]

Then, one has

\[
\int_{B^\epsilon^i} N_i(s) ds \leq C \cdot \text{Vol}(\phi^{n_i}(L_2) \cap W),
\]

by applying Lemma 3.4. We note that \( C \) is a constant independent from \( n_i \).

From Lemmas 7.3 and 7.4 and Equation (7.1), we have

\[
2^{(\alpha - \delta)n_i} \leq b_{2\epsilon} \{ L_1, \phi^{n_i}(L_2) \} \leq b_\epsilon \{ L^i, \phi^{n_i}(L_2) \} \leq |L^i \cap \phi^{n_i}(L_2)| = N_i(s).
\]

Then, by taking integration over \( B^\epsilon \), one has

\[
\text{Vol}(B^\epsilon) \cdot 2^{(\alpha - \delta)n_i} \leq C \cdot \text{Vol}(\phi^{n_i}(L_2) \cap W).
\]

Since \( \text{Vol}(B^\epsilon) \) and \( C \) do not depend on \( n_i \),

\[
(\alpha - \delta) \leq \limsup_{i \to \infty} \frac{1}{n_i} \log^+ \text{Vol}(\phi^{n_i}(L_2) \cap W) \leq h_{\text{top}}(\phi|_W).
\]

The last inequality holds because of Proposition 2.4 and because of the fact that

\[
\phi^{n_i}(L_2) \cap W = \phi^{n_i}(L_2 \cap W) = (\phi|_W)^{n_i}(L_2 \cap W).
\]

Finally, we note that \( \phi \) is compactly supported, and that \( \text{supp}(\phi) \subset W \). Thus,

\[
h_{\text{top}}(\phi) = h_{\text{top}}(\phi|_W).
\]

See Remark 2.2.

Thus, one has

\[
\alpha - \delta \leq h_{\text{top}}(\phi).
\]

This completes the proof. \( \square \)

7.4. Barcode vs categorical entropy. In the previous section, for an arbitrary good pair \( (L_1, L_2) \), we compared the barcode entropy of a triple \( (\phi; L_1, L_2) \) and the topological entropy of \( \phi \). As the result, we proved Proposition 7.6. In this subsection, we compare barcode and categorical entropy. However, in order to compare them, we should choose some specific pairs of Lagrangians.

First, we choose a stop \( \Lambda \) giving a fully stopped partially wrapped Fukaya category \( \mathcal{W}(\hat{W}, \Lambda) \). Let \( G \) be an embedded Lagrangian generating \( \mathcal{W}(\hat{W}, \Lambda) \). As we did in the proof of Theorem 4.1, let \( \phi_0 \) be a Hamiltonian isotopy satisfying

(A) \( \{ \phi_0(G), \phi^n(G) \} \) is a good pair for all \( n \in \mathbb{N} \),

(B) \( HW_\Lambda(\phi_0(G), \phi^n(G)) = HF(\phi_0(G), \phi^n(G)) \) for all \( n \in \mathbb{N} \).

As we did before, one has

\[
h_{\text{cat}}(\Phi) = h_{\text{cat}}(\Phi_\Lambda) = \limsup_{n \to \infty} \frac{1}{n} \log^+ \dim HF(\phi_0(G), \phi^n(G)).
\]

Since \( \dim HF(\phi_0(G), \phi^n(G)) \) equals the number of bars having infinite length, one has

\[
\dim HF(\phi_0(G), \phi^n(G)) \leq b_\epsilon(\phi_0(G), \phi^n(G)).
\]

This induces Proposition 7.7.
**Proposition 7.7** (= The first inequality in Proposition 1.4). For $G$ and $\varphi_0$ given above,
\[ h_{\text{cat}}(\Phi) \leq h_{\text{bar}}(\varphi; \varphi_0(G), G). \]

**Remark 7.8.** We note that there always exists a good pair $(L_1, L_2)$ such that $h_{\text{bar}}(\varphi; L_1, L_2) = 0$. By choosing a Lagrangian $L_2$ such that $L_2$ does not intersect the support of $\varphi$, one obtains a such pair. Thus, the choice of $G$ (and $\varphi_0$) in Proposition 7.7 is essential.

### 7.5. Further questions

In this subsection, we discuss the questions given in Section 1.3 in more detail.

We also recall that $h_{\text{top}}(\varphi) \geq \text{the exponential growth rate of } \text{Vol}(\varphi^n(Y)),$
for any compact submanifold $Y$ by [New88, Prz80]. And it is known by [Yom87] that
\[ h_{\text{top}}(\varphi) = \sup_{\text{compact submanifold } Y \subset W} \{ \text{the exponential growth rate of } \text{Vol}(\varphi^n(Y)) \}. \] (7.3)

As one can see in the proof of Proposition 7.6, $h_{\text{bar}}(\varphi; L_1, L_2)$ bounds the exponential volume growth rate of $\varphi^n(L_2)$ from below. Thus,
\[ h_{\text{top}}(\varphi) \geq h_{\text{bar}}(\varphi; L_1, L_2). \]

As a generalization of Equation (7.3), one can ask whether the following equality holds or not:
\[ h_{\text{top}}(\varphi) = \sup_{\text{good pair } (L_1, L_2)} h_{\text{bar}}(\varphi; L_1, L_2). \]

The supremum in the above equation runs over the set of all good pairs. As mentioned in Remark 7.8, it is easy to find a good pair $(L_1, L_2)$ such that $h_{\text{bar}}(\varphi; L_1, L_2) = 0$. Thus, we would like to remove such good pairs from the set where the supremum runs over, for computational convenience.

Finally, we ask whether the following equality holds or not:
\[ h_{\text{top}}(\varphi) = \sup_{G, \varphi_0} h_{\text{bar}}(\varphi; \varphi_0(G), G), \]
where $G$ is a generating Lagrangian and $\varphi_0$ is a Hamiltonian isotopy satisfying the conditions in Section 7.4.

On the other hand, one can ask a similar question for $h_{\text{cat}}(\varphi)$. More precisely, we ask whether the following equality holds or not:
\[ h_{\text{cat}}(\varphi) = \inf_{G, \varphi_0} h_{\text{bar}}(\varphi; \varphi_0(G), G). \]

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