A quantum optical realization of the Ornstein-Uhlenbeck process via simultaneous noise and feedback

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We establish an important connection between coherent quantum feedback and the Ornstein-Uhlenbeck process in quantum optics. We show that an emitter with fluctuating energy levels in front of a mirror results in an Ornstein-Uhlenbeck process for electronic populations, although the fluctuation of the energy levels is assumed to be uncorrelated in time and space. Based on a Heisenberg equation of motion description of the quantum feedback dynamics, we discuss additionally the impact of phase noise on the population dynamics and provide examples in which noise itself is not detrimental but supports and enhances typical features of quantum feedback such as self-stabilization.

The fluctuation-dissipation theorem relates a fluctuating force to its corresponding imposed friction and expresses in this sense the balance between noise and dissipation [1][5]. It is widely used in non-equilibrium statistical mechanics, in which correlation functions replace the partition function as the crucial quantity to characterize properties and the long-time behavior of the system under study [9][8]. Such non-equilibrium systems are often described by the Ornstein-Uhlenbeck (O-U) process [9][11] which formalizes and generalizes Einstein’s description of Brownian motion with

\[
\dot{u} = -\gamma u + F_t, \tag{1}
\]

where \(u\) is the particle velocity, \(\gamma\) the friction and \(F_t\) the corresponding fluctuating force, which enforces thermal equilibrium in the long-time limit [12]. Based on the O-U process which considers a noise correlation \(\langle F_t F_s \rangle = \Gamma \exp[-\gamma|t-s|]\), a mean-squared displacement of a particle can be derived and, using the Einstein-Smoluchowski relation [12][14], reads:

\[
\langle [x(t)]^2 \rangle = \frac{A_0}{\gamma^2} \left[ \gamma t + e^{-\gamma t} - 1 \right], \tag{2}
\]

with \(A_0\) being a constant fulfilling the corresponding self-diffusion coefficient [9][11]. These formulas are applicable to a wide range of physical systems beyond the particular case of Brownian motion and enable the determination of viscosity [15], thermal and electrical conductivity [16][17]. Examples include light scattered by diffusing molecules in spectroscopy measurements [18][20] and the derivation of absorption coefficients such as the Lambert-Beer law determined by the electric-dipole moments of the substance [2][5][14][21].

In this study, we will derive the mean-square displacement formula in Eq. (2) for electronic populations of an atom excited by quantum optical fields [3][22]. Describing the radiative decay of an excited atom in the vicinity of a mirror [23][25], the coherent quantum feedback dynamics lead straightforwardly to the O-U process if fluctuations of the energy levels are accounted for during the emission dynamics. Interestingly, this result can be attributed to a mirror charge-induced dipole-dipole correlation due to the mirror, as has been derived by Zwanzig [2], only here retarded in time and of purely single quantum nature.

The main ingredient for describing spontaneous emission used here beyond the Wigner-Weisskopf limit [29], in which the vacuum field amplitude is considered constant with respect to the frequency of the emitted photon, is the coherent quantum feedback mechanism [30][39]. This kind of feedback has been proven to be a versatile strategy to steer and control systems non-invasively, and is not related to measurement-based, or invasive quantum feedback control [40][41]. The coherent and non-Markovian nature imposes quantum interferences between present and past system states onto the dynamics and allows for interesting two-photon processes [37][42], enhanced entanglement and non-classical photon statistics [43], dimerization [44][45], and a stabilization of quantum coherence due to interference effects between incoming and outgoing probability waves [46]. A typical paradigm for such processes is the formation of dark states and subsequently emerging population trapping [47][48].

In the following, we will establish an important connection between the non-Markovian quantum feedback process and the generation of an Ornstein-Uhlenbeck-type of noise correlations. For this, we derive an equation of motion via the Heisenberg picture for the microscopic coherence operator subjected to white noise fluctuations and quantum feedback. Importantly, we show that white noise-based energy fluctuations are not necessarily detrimental to quantum feedback effects, as it counteracts destructive and unwanted interference effects between the incoming and outgoing photon emission processes, and even supports population trapping for certain quantum feedback phase relations.
Model. In this section, we discuss the impact of white noise on the emission dynamics of a two-level system in front of a mirror. The Hamiltonian of the system reads ($\hbar = 1$):

$$H = (\omega_0 + F_t)P^dP + \int dt \omega \left[ r^a \left( \frac{a^\dagger r}{2} + g_r^a P \right) + \text{h.a.} \right]$$

(3)

where $P = |g\rangle\langle e|$ denotes the microscopic coherence operator from the excited- $|e\rangle$ to the ground-state $|g\rangle$ of the two-level system with a transition energy of $\hbar\omega_0$. The radiative continuum is included via the photon creation and annihilation operators $r^{(\dagger)}_\omega$ for a photon in the mode $\omega = c k$ (c: the speed of light in the waveguide) with bosonic commutation relations: $[r^{(\dagger)}_\omega, r^{(\dagger)}_{\omega'}] = \delta(\omega - \omega')$. The coupling between the emitter and the radiative continuum is denoted by $g_r = g_0 \sin(\omega_\tau/2)$ and includes the mirror imposed boundary condition at a distance $L$ between mirror and atom with a strength of $g_0$. The length defines the feedback round trip time with $\tau = 2L/c$. $F_t$ describes a stochastic force acting upon the excited level of the two-level system and models, e.g. a spectral diffusion process [41, 50, 51]. We assume throughout our analysis a Gaussian white noise with vanishing average $\langle F_t \rangle = 0$ and $\delta$-correlated correlation function $\langle F_t F_s \rangle = \gamma \delta(t - s)$. Next, we solve this model in the Heisenberg picture [52].

The equation of the Heisenberg operator $P^t(t) = U^t(t)P U(t)$ with $U(t) = \exp[-iHt]$ using $\dot{P}^t(t) = i[H, P^t(t)]$ reads in the rotating-frame

$$\dot{P}^t(t) = iF_t P^dP + \int dt \omega g_r e^{i(\omega - \omega_0)t} r^{(\dagger)}_\omega(t)[P(t), P^d(t)].$$

(4)

The coherence operator couples to the inversion and to the quantized light field. Starting with an initial condition at $t = 0$, the goal is to solve for the quantized light-field exactly by integrating out the equation of motion of the photon creation operator:

$$r^{(\dagger)}_\omega(t) = r^{(\dagger)}_\omega(0) + i\omega \int_0^t dt_1 e^{-i(\omega - \omega_0)t_1} P^d(t_1).$$

(5)

This equation allows us to write down the Heisenberg-Langevin equation of motion. Within the one-electron assumption for the two-level system, the inversion operator can be written as $[P(t), P^t(t)] = 1 - 2P^t(t)P(t)$ and for the dynamics of the coherence operator follows

$$\dot{P}^t(t) = -[\Gamma - iF_t]P^d(t) + \Gamma e^{-i\omega_0 \tau} P^t(t - \tau) \theta(t - \tau) - 2\Gamma e^{-i\omega_0 \tau} P^t(t - \tau)P^d(t)P(t)\theta(t - \tau) + i\omega R^t(t)[P(t), P^d(t)],$$

(6)

where $R^t(t) = \int dt_1 r^{(\dagger)}_\omega(0)\sin(\omega_\tau/2)\exp[i(\omega - \omega_0)t]$ includes the quantum noise contribution to conserve the commutation for all times with $\Gamma = g_0^2 \pi/2$. Clearly, the signal $P$ at the feedback delay time $\tau$ occurs in Eq. (6).

In the following, we show that the second and last (third) line vanishes in the case of a reservoir initially in the vacuum state and a system described by the Hamiltonian in Eq. (3).

The solution of Eq. (6) is derived for every $\tau$-interval iteratively [30, 40, 41]. For the time interval, $t \in [0, \tau]$, we evaluate the matrix element of the coherence operator $P^*_\tau(t) = \langle i, vac | P^d(t) | j, vac \rangle$ with $|j, vac \rangle = |j \rangle s | vac \rangle_R$ and $j$ either $e$ or $g$ for the system state and the reservoir in the vacuum state. The dynamics of the polarization reduces to:

$$\dot{P}^*_\tau(t) = -[\Gamma - iF_t]P^*_\tau(t),$$

$$P^*_\tau(t) = e^{-\Gamma \tau + i\phi(t, 0)} P^*_0(0),$$

(7)

(8)

contributing only for $i = e$ and $j = g$ and $\phi(b, a) := \int_0^b F_c dt'$. Note that the matrix element does not represent the expectation value. However, the expectation value can be fully expressed by its corresponding matrix elements, e.g. $\langle P^d(t)P(t) \rangle = |P^*_\tau(t)|^2$. For the second interval, $t \in [\tau, 2\tau]$, the dynamics of the matrix element reads:

$$\dot{P}^*_\tau(t) = -\Gamma P^*_\tau(t) + \Gamma e^{-i\omega_0 \tau} P^*_\tau(t - \tau) - 2\Gamma e^{-i\omega_0 \tau} \langle i, vac | P^d(t - \tau)P^d(t) | j, vac \rangle.$$

(9)

Due to the occurring time delay, we can use $P_{ij}$ for $i = e$ and $j = g$ from Eq. (8) in Eq. (9) to evaluate the second line. For this, we insert now the unity relation $1 = \sum_{i = e, g} |i \rangle_S \langle i | \otimes (|vac \rangle_R \langle vac | + \int d\omega |1_\omega \rangle_R \langle 1_\omega |)$ to evaluate the correlation between the "time-nonlocal" microscopic coherence and the time-local population density, and taking into account that only $\langle i, vac | P^d(t - \tau) | g, vac \rangle$ can contribute non-trivially:

$$\langle i, vac | P^d(t - \tau)P^d(t) | j, vac \rangle = \langle i, vac | P^d(t - \tau) | g, vac \rangle \langle g, vac | P^d(t)P(t) | j, vac \rangle,$$

(10)

having reduced the problem to the matrix element $\langle g, vac | P^d(t)P(t) | j, vac \rangle$. If we now again insert a unity operator between the operators $P^d(t)P(t)$, we reduce this quantity again into further products of matrix elements. Since we know, that only $P^*_\tau(t)$ contributes initially in the first time interval, the quantity vanishes identically in the case of the Hamiltonian dynamics in Eq. (3) due to $\langle g, vac | P^d(t) | \phi \rangle = 0$ for arbitrary $\phi$, and therefore we can conclude that, assuming an initially empty reservoir, the matrix elements of the microscopic coherence operator is governed by the dynamics for all times $t$:

$$\dot{P}^*_\tau(t) = (iF_t - \Gamma) P^*_\tau(t) + \Gamma e^{-i\omega_0 \tau} P^*_\tau(t - \tau) \theta(t - \tau).$$

In the case of $F_t \equiv 0$, this equation can be solved in the Laplace domain [27, 30, 33, 46, 53], yielding the following known dynamics valid for all $t$:

$$P^*_\tau(t) = \sum_{n = 0}^{\infty} \frac{e^{-\Gamma t} t^n}{n!} \left[ \Gamma e^{-i\omega_0 \tau + i\tau} \theta(t - n\tau) \right]^n.$$

(11)
However, in the following we are interested in the case of a non-vanishing noise $F_t \neq 0$, and the equation can only be solved via subsequent integration with respect to time. For example, for $t \in [0, 3\tau]$:

$$P^s_{eg}(t) = e^{-\Gamma t + i\phi(t,0)} \left[ \theta(t) + \theta(t-\tau) \Gamma e^{-i\omega_0\tau + \Gamma \tau} N(t, \tau) 
+ \theta(t-2\tau) (\Gamma e^{-i\omega_0\tau + \Gamma \tau})^2 M(t, 2\tau) \right] \tag{12}$$

with $\phi(b,a) = \int_a^b F_t \, dt'$ and the definitions

$$N(t, \tau) := \int_\tau^t dt_1 e^{-i\phi(t_1,t_1-\tau)}$$
$$M(t, 2\tau) := \int_{2\tau}^t dt_1 e^{-i\phi(t_1,t_1-\tau)} \int_\tau^{t_1} dt_2 e^{-i\phi(t_2,t_2-\tau)} ,$$

which recover in the limit of $\gamma \to 0$, the solution given in Eq. (11), i.e. $N(t, \tau) = (t-\tau)$ and $M(t, 2\tau) = (t-2\tau)^2 / 2$.

In the following, we are interested in the cases (i) $0 \leq t \leq 2\tau$ and (ii) $0 \leq t \leq 3\tau$. For (i), we show that the quantum feedback contribution leads to a Ornstein-Uhlenbeck process for the population dynamics due to the assumed white noise correlation. In (ii), we discuss the impact of phase noise on the coherent quantum feedback dynamics and show that it need not necessarily be detrimental to quantum feedback effects, as disadvantageous destructive interferences, if they occur, are suppressed.

(i) Ornstein-Uhlenbeck. Due to the white noise contribution, we cannot use the solution derived via the Lambert W-function in Eq. (11). We use Eq. (12) to evaluate for the population dynamics. The solution of the population dynamics reads for $\tau \leq t \leq 2\tau$:

$$|P^s_{eg}(t)|^2 = \langle P(t) | P(t) \rangle = e^{-2\Gamma t} \left[ 1 + 2\Gamma e^{-\Gamma(2\tau-t)} \int_0^{t-\tau} ds \text{Re} \left[ e^{i\omega_0 s} e^{i\phi(s',s')} \right] 
+ \left( \Gamma e^{-\Gamma(t-\tau)} \right)^2 \int_0^{t-\tau} ds_1 \int_0^{t-\tau} ds_2 e^{-i\phi(s+s',s)+i\phi(s'+s',s')} \right]$$

where we used the property: $\phi(s+\tau,0) - \phi(s,0) = \phi(s+\tau,\tau)$ [53]. Evaluating the noise correlation, we specify the result to $\omega_0 \tau / (2\pi) = n$ for $n$ integer:

$$\langle |P^s_{eg}(t)|^2 \rangle = e^{-2\Gamma t} \left[ 1 + 2\Gamma e^{\Gamma \tau} \cos(\omega_0 \tau) \langle N(t, \tau) \rangle 
+ \Gamma^2 e^{2\Gamma \tau} \langle N(t, \tau) N^s(t, \tau) \rangle 
+ 2\Gamma^2 e^{2\Gamma \tau} \cos(2\omega_0 \tau) \langle M(t, 2\tau) \rangle \right]$$

$$+ 2\Gamma^3 e^{3\Gamma \tau} \cos(\omega_0 \tau) \langle N^s(t, \tau) M(t, 2\tau) \rangle 
+ \Gamma^4 e^{4\Gamma \tau} \langle M(t, 2\tau) M^s(t, 2\tau) \rangle \right] , \tag{15}$$

with $M(t, 2\tau)$ and $N(t, \tau)$ defined after Eq. (12). Evaluating the noise integrals involves up to four time-ordered
integrals and its corresponding noise-noise correlations. This yields lengthy expressions which are given explicitly in the supplemental material.

Often, the goal of quantum feedback is to stabilize electronic populations due to destructive interference between absorption and re-emission events [37, 40]. This leads in the perfect case of \( \varphi = \omega_0 \tau / (2\pi) = n \) with \( n \) integer to population trapping or a bound state in the continuum [47, 63]. For any phase unequal to a multiple of \( \varphi = \omega_0 \tau / (2\pi) = n \) and without phase noise \( F_1 \equiv 0 \), the population decays inevitably. We show now that the population decay can be slowed down in the presence of phase noise despite a disadvantageous phase choice. In Fig. 1, the dynamics of the population is depicted for the case without feedback (Wigner-Weisskopf case, black line), without noise but with feedback (green line), and with feedback and with noise (orange line) for a phase of \( \varphi = \omega_0 \tau = 3.3 \). As can be seen, phase noise helps to suppress the destructively interfering parts of the solution in Eq. (15) proportional to \( \cos(n \omega_0 \tau) \) with \( n \) integer. These contributions enforce damped oscillations of the population, leading eventually to a complete decay of the electronic excitation into the reservoir with zero excitation left in the emitter (green line). However, these contributions are strongly affected by the phase noise. Here, in the transient regime, noise helps to slow down the decay of the electronic population and prevents it from decaying rapidly to zero (orange line). In Fig. 1, the population in the emitter in case of finite \( F_1 \) is larger or for a short time (\( t/\tau \approx 2.4 \)) equal/slightly less compared to the case with vanishing phase noise. As a comparison, we plot the dynamics imposed by just the Wigner-Weisskopf case (black line). The case with dephasing is always larger than the Wigner-Weisskopf dynamics, whereas the case with feedback and no phase noise oscillates following the decay of the Wigner-Weisskopf solution, indicating an inevitable complete decay for undisturbed feedback. Interestingly, the solution with feedback shows a non-monotonous behavior to the end of the third \( \tau \)-interval where population is gained.

These results, however, depend on the choice of the feedback phase \( \varphi = \omega_0 \tau \). This indicates that the choice of the delay time provides another control parameter to optimize the phase noise action on the population number: In Fig. 2, we plot the difference between the population with and without noise: \( \langle |P_{eg}(t)|^2 \rangle - \langle |\langle P_{eg}(t) \rangle|^2 \rangle \). We clearly see that phase noise is detrimental to population trapping in the vicinity of the phase \( \varphi = 0 \), as this phase choice renders the destructive interference terms already unimportant, i.e. for \( \omega_0 \tau \in (0, \pi/2) \) and \( \omega_0 \tau \in (3\pi/2, 2\pi) \). However, for phases in between \( (\pi/2, 3\pi/2) \), the population is enhanced due to noise. We conclude that for \( \varphi \neq 2\pi \), phase noise still allows for important feedback effects relying on the population trapping mechanism. We like to add that the calculation of the noise contribution up to 3\( \tau \) is straightforward but already lengthy in this approach. So far, the long-time limit is not accessible, as for every \( \tau \)-interval, the noise contributions need to be evaluated separately due to the time-reordering. Here, a fully quantum mechanical model is envisioned to also investigate the long-time behavior without evaluating every time-interval individually.

**Conclusion.** We have studied the impact of white noise on the radiative decay dynamics of an atom in front of a mirror. We have shown that the white noise contribution, here a fluctuation of the excited energy level, leads to an Ornstein-Uhlenbeck process. Herewith, we establish an interesting relation between non-Markovian feedback processes and the fluctuation-dissipation theorem and allow for interpretations of the O-U processes.
in non-equilibrium statistical mechanics in terms of non-Markovianity and feedback acting as a low-pass filtered white noise. Furthermore, we discussed the impact of phase noise on the population trapping dynamics and show advantageous features in a wide range of phase choices.

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