On the Transfer of Adiabatic Fluctuations through a Nonsingular Cosmological Bounce

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We study the transfer of cosmological perturbations through a nonsingular cosmological bounce in a special model in which the parameters of the bounce and the equation of state of matter are chosen such as to allow for an exact calculation of the evolution of the fluctuations. We find that the growing mode of the metric fluctuations in the contracting phase goes over into the growing mode in the expanding phase, a result which is different from what is obtained in analyses in which fluctuations are matched at a singular hypersurface. Consequences for Ekpyrotic cosmology are discussed in a limit when the equation of state of a fluid becomes large.

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I. INTRODUCTION

It has long been appreciated that Standard Big Bang cosmology cannot be the complete theory of the early universe since it is plagued by an initial singularity \cite{1}. The singularity theorems have also been extended to apply to scalar field-driven inflationary cosmology \cite{2}. It is possible that the initial singularity is resolved by quantum gravity. Another possibility, however, is that correction terms to the gravitational Lagrangian which describe the dynamics of space-time leads to a bouncing universe. In a bouncing cosmology, the Hubble radius \( H^{-1}(t) \) (where \( H \) is the Hubble expansion rate and \( t \) is physical time) decreases faster than the physical wavelength of fluctuations (which have a constant wavelength in comoving coordinates). Hence, it is conceivable that processes acting in the contracting phase can lead to a non-inflationary mechanism for the origin of structure in the universe. Recent proposals which involve bouncing cosmologies are the Pre-Big-Bang \cite{3}, Ekpyrotic \cite{4}, and higher derivative modification of an Einstein gravity \cite{5} scenarios.

The transfer of metric fluctuations through the bounce from the contracting to the expanding phase has been an outstanding problem in the discussions of Pre-Big-Bang and Ekpyrotic cosmology. Often, the evolution of the background is modeled by a contracting phase modeled by a solution to the Einstein equations matched to an expanding phase of Standard Big Bang cosmology through instantaneous and often singular transition along a space-like hypersurface. One proposal has been to use the analog of the Israel matching conditions (matching conditions \cite{6} which describe the merger of two solutions of the Einstein equations along a time-like hypersurface). These equations were discussed in \cite{7,8}.

If we consider Einstein gravity, then in Fourier space the space of solutions for fixed comoving wavenumber \( k \) is two-dimensional (see e.g. \cite{9} for a comprehensive review of the theory of cosmological perturbations and \cite{10} for a pedagogical overview). We eliminate the coordinate ambiguities in the description of the fluctuations by working in a specific coordinate system, namely longitudinal gauge, in which the metric is given by (in the absence of anisotropic stress)

\[
ds^2 = a(\tau)^2 \left[(1 + 2\Phi)dt^2 - (1 - 2\Phi)dx^2\right],
\]

where \( \tau \) is conformal time related to physical time \( t \) via \( dt = a d\tau \), and the relativistic potential \( \Phi(x, \tau) \) is the field describing the fluctuations.

On super-Hubble scales and in an expanding universe, the dominant mode of \( \Phi \) is constant in time if the equation of state of the cosmological background is constant. We will call this mode the D-mode. The second fundamental solution, the S-mode, is decreasing in time. In a contracting phase, the D-mode is sub-dominant, and the S-mode is growing and hence dominant.

Both in the context of the Pre-Big-Bang scenario \cite{11} and in the original effective field theory description of the Ekpyrotic cosmology \cite{12} it was found that if the fluctuations are matched across a hypersurface of fixed matter field value (the hypersurfaces singled out in the scenarios of \cite{3,4}), then the growing mode in the contracting universe couples almost exclusively to the subdominant mode in the expanding phase (the coupling to the dominant mode is suppressed by \( k^2 \)). In the Ekpyrotic scenario \cite{4} a scale-invariant spectrum of the dominant mode of cosmological fluctuations is generated in the contracting phase \cite{12}, whereas the decaying mode has a \( n = 3 \) spectrum \cite{11}. Due to the suppression of the coupling, only a \( n = 3 \) spectrum in the expanding phase is induced \cite{11,12}.

\textsuperscript{1} We are employing the standard notation in which the index of the spectrum of scalar metric fluctuations is denoted by \( n - 1 \), with \( n = 1 \) standing for a scale-invariant spectrum.

\textsuperscript{2} In the case of the Ekpyrotic scenario, the underlying physics is higher-dimensional, and the reduction of the analysis to a four space-time dimensional effective field theory without adding entropy modes does not correctly model the full physics of the fluctuations. An analysis of the transfer of metric fluctuations in five space-time dimensions \cite{12} (see also \cite{12} for a similar anal-
As initially pointed out in [16, 17], the result for the spectrum of fluctuations in the expanding phase depends quite sensitively on the details of the matching, and it is in fact not clear whether the matching prescription of [7, 8] can be applied at all due to the inconsistency of the matching of the background. It is thus of great interest to study how the cosmological fluctuations propagate through a nonsingular cosmological bounce. A first analysis of the evolution of fluctuations through a specific nonsingular bounce obtained by using a higher derivative gravity action was made in [18] (see also [19]), showing that initial scale-invariant fluctuations do not pass through the bounce, thus confirming the results of [12]. On the other hand, bounces obtained by adding spatial curvature and matter with wrong-sign kinetic terms, but with the standard gravitational action, have been studied for a recent review). The action of [5] has the property that, although the cosmological background is changed, the correction terms in the equation of motion for long-wavelength (compared to the scale of new physics) contribute to the Hubble equation, it is completely ignorant of local fluctuations of the volume (metric) and therefore does not alter the perturbation equations. This situation is very similar to the usual vacuum energy (cosmological constant) that gravitates (and therefore contributes to the Hubble equation) but does not contribute to the perturbation equation. Another example of this kind is the Casimir energy, that represents only a global shift in the vacuum energy, which can be negative and therefore resolve the singularity without effecting perturbations. Yet another interesting scenario is to consider higher derivative gravity correction terms to the Einstein action which we expect will become dominant near the bounce, as long as the curvature at the bounce point becomes comparable to Planck-scale curvature. Recently, a ghost-free and asymptotically free higher derivative gravity model leading to a cosmological bounce was proposed in [2], and in a follow-up paper [30] it was suggested that a long bouncing phase may lead to the correct thermal string gas initial conditions for the new structure formation scenario [31, 32].

Although such a bouncing cosmology can provide us with a non-singular description of the universe and possibly a new mechanism for the origin of cosmological perturbations, the models have not yet addressed the question of why after the bounce, in the expanding branch, we are mostly left with Standard Model particles. One has to perhaps find a mechanism similar to “reheating” where after the bounce most of the energy is converted back into a scale-invariant spectrum in the expanding phase.

4 Note that there is a technical problem with some of the analyses in these references, as discussed in [22].

5 See also [24] for analysis of how fluctuations arise in a bouncing cosmology via quantum cosmological methods.
nario is resolved by a nonsingular bounce, it will not be necessary to invoke a scale-invariant spectrum of initial entropy fluctuations in addition to the existing scale-invariant spectrum of adiabatic modes in order to obtain a scale-invariant spectrum in the expanding phase.

II. THE MODEL

The starting point of our analysis is the generalization of Einstein’s equations to

\[ G_{\mu}^{\nu} = T_{\mu}^{\nu} + Q_{\mu}^{\nu} \]  

(2)

where \( Q_{\mu\nu} \) arises from some unknown new physics responsible for resolving the Big Bang singularity. A priori, \( Q_{\mu\nu} \) may arise either in the gravity sector (see for instance [3]) or in the matter sector [26, 27, 28, 29] but we purposely do not specify it. As usual the above equation can be broken down into the background (which only depends on time) and perturbed (depends on both space and time) quantities such as

\[ G_{\mu}^{\nu} \equiv <G_{\mu}^{\nu}>(\tau) + \delta G_{\mu}^{\nu}(x) \]  

(3)

and similarly for \( T_{\mu}^{\nu} \) and \( Q_{\mu}^{\nu} \). For homogeneous isotropic cosmology, the background quantities generalize the Hubble equation to

\[ 3H^2 = <G_{0}^{0}> = <T_{0}^{0}> + <Q_{0}^{0}> = <\rho> + <Q_{0}^{0}> \]  

(4)

To keep things as general as possible let us not specify the new physics, i.e., \( <Q_{0}^{0}> \) that causes the universe to bounce, but simply make an ansatz\(^6\) for the time evolution of the Hubble rate. Near the bounce, the most generic behavior of the scale factor is given by

\[ a(\tau) = 1 + \left(\frac{\tau}{\tau_0}\right)^2 \]  

(5)

where \( \tau_0 \) corresponds to the bounce time scale. \(^5\) leads to a Hubble evolution of the form

\[ H \sim \frac{\tau}{\tau^2 + \tau_0^2}, \]  

(6)

where \( H \) is the Hubble rate in conformal time. Now, we also know that at late times \( H \) must have the right asymptotic property, \( H \to \frac{2}{\tau} \), with \( q < 1 \). Of course, in practice the precise transition from the accelerating bouncing region \(^6\) to the decelerating late time regime will depend on the new physics that is introduced. However, at least in the case where there is really only one new fundamental scale we expect only the details to depend on the precise functional form of the transition.

Thus, for the purpose of illustration and technical simplicity we choose to work with a specific ansatz\(^7\) to capture the essential effects on the perturbations as they evolve across the bounce:

\[ H = \frac{q\tau}{\tau^2 + \tau_0^2}, \]  

(7)

We note that this has the right asymptotic property as \( \tau \to \pm \infty \), while during the bounce \( H \) goes linearly with \( \tau \) as expected.

As matter we consider a fluid with an equation of state

\[ p = \omega \rho, \]  

(8)

where \( \rho \) and \( p \) stand for the energy density and the pressure of the fluid, respectively. Inserting this equation of state into the general relativistic Friedmann equations yields the following time evolution of the scale factor:

\[ a(\tau) = a_0|\tau|^q \]  

(9)

where \( a_0 \) is a constant.

III. PERTURBATIONS ACROSS THE BOUNCE

Starting from (2) and using perturbative expansions such as \(^8\) we obtain the generalized perturbation equation:

\[ \delta G_{\mu}^{\nu} = \delta T_{\mu}^{\nu} + \delta Q_{\mu}^{\nu} \]  

(10)

We now make the crucial assumption that \( \delta Q_{\mu}^{\nu} \) vanishes (or remains negligible during the bounce). As mentioned in the Introduction, when the resolution of the Big Bang singularity involves new non-local physics such as \([5, 26, 27, 28, 30]\), then at least for fluctuations whose length scales during the bounce are much larger than \( \tau_0 \), this can be justified.

Returning to (11), one can now check that just by using the usual expressions of \( \delta G_{\mu}^{\nu} \) and \( \delta T_{\mu}^{\nu} \) in terms of \( \Phi \) and \( \delta \rho \) for an ideal gas, one can derive the usual General Relativistic perturbation equation for the Bardeen

\(^7\) If we believe that the singularity at \( \tau = 0 \) will be resolved, then it is clear that the pole at \( \tau = 0 \) will have to shift from the real axis to the complex plane. Since we want \( H \) to be finite and real in the entire real axis, it is easy to see that in fact the poles must lie along the imaginary axis and come in conjugate pairs. Therefore, the simplest ansatz that one can make for \( H \) is to consider only a pair of simple poles as in (4). However, the general algorithm that will be advocated here can be carried forward for any non-singular bounce ansatz that one may get for a given new physics. Also, we believe that this ansatz for the Hubble parameter may be realized as non perturbative solutions of non metric theories of gravity as well as from four dimensional, higher order corrected Heterotic M theory actions. These constructions will be pursued in future work [29].

\(^6\) If one wants, one can deduce \( <Q_{0}^{0}> \) from the ansatz.
potential $\Phi$

$$\Phi'' + 3(1 + \omega)H\Phi' + \omega k^2 \Phi + [2H' + (1 + 3\omega)H^2] \Phi = 0.$$  \hspace{1cm} (11)

Crucially we note that this derivation does not "re-use" the background Hubble equation and therefore goes through even though the background equation is modified. As we have explained in the appendix, this is not the case for the standard evolution equation for the Mukhanov variable (also see section ??) and in fact this is the reason why one cannot trust the General Relativistic equation for the Mukhanov variable to track perturbations around the bounce.

In passing we note that for GR, in case of an ideal fluid the last term drops out. However, when one modifies gravity, this is no longer true, and this is what makes the analysis a lot more interesting.

Substituting our ansatz \( \Phi \) into (11), we obtain the following equation for the metric fluctuations

$$\Phi'' + \frac{3(1 + \omega)q\tau}{\tau^2 + \tau_0^2} \Phi' + \left[ \omega k^2 + \frac{2q^2\tau_0^2}{(\tau^2 + \tau_0^2)^2} \right] \Phi = 0.$$  \hspace{1cm} (12)

As one can see, the coefficients in the above differential equation are all non-singular and completely well defined. Therefore one can in principle solve for $\Phi_k$ in the interval $(-\infty, \infty)$ and understand how an initial perturbation evolves from the contracting to the expanding phase, without having to implement any matching conditions anywhere. Although this is in general only possible to achieve numerically, as we will argue now, one can make significant progress analytically.

Let us first identify some of the regimes where the differential equation may simplify. Firstly, one has the usual division between the super- and sub-Hubble regimes separated at points where

$$k = |H| \Rightarrow k|\tau|^2 - q|\tau| + \tau_0^2 k = 0.$$  \hspace{1cm} (13)

As is typical in bouncing cosmologies, there are four solutions to the above equation:

$$\tau = \pm \left( \frac{q \pm \sqrt{q^2 - 4k\tau_0}}{2k} \right) = \pm \tau_{\pm}.$$  \hspace{1cm} (14)

Two occur during contraction ($-\tau_{\pm}$), and two during expansion ($\tau_{\pm}$). The initial and final crossings are the usual ones for a contracting and an expanding universe, respectively, whereas the two middle ones occur near the bounce point and are a special feature of nonsingular bouncing cosmologies. The first crossing occurs at $-\tau_{\pm}$, when GR is a valid description of the background and modes exit the Hubble radius (note the universe starts from a cold $H \rightarrow 0$ phase, so that all the modes are sub-Hubble to begin with). Around $\tau \sim \tau_0$, the new physics kicks in and $H$ starts to decrease again. Thus, all the modes which are super-Hubble start to come back inside the Hubble radius, and for a given mode this happens at $-\tau_{\pm}$. By the time the bounce occurs all the modes are again sub-Hubble, but after the bounce as the Hubble radius reaches a finite value, modes again exit the Hubble radius. This happens at $\tau_+$. Finally, as the expansion continues according to the equations of standard cosmology, modes eventually enter our horizon at $\tau_+$. A second way to divide the evolution is to consider the range $|\tau| > \tau_0$ and $|\tau| < \tau_0$ separately, where the former corresponds to considering the intervals when GR is a good description of the background, while for the latter one has to include the effects of the new physics. So, how should one proceed?

First let us look at (12) as $|\tau| \gg \tau_0$, where we recover the general relativistic limit

$$\Phi'' + \frac{6(1 + \omega)}{\tau(1 + 3\omega)} \Phi' + \omega k^2 \Phi = 0,$$  \hspace{1cm} (15)

which has the following analytical solution

$$\Phi_k = \tau^{-\nu}[k^{-\nu} D_\pm(k)J_\nu(\sqrt{\omega k}\tau) + k^{\nu} S_\pm(k)J_{-\nu}(\sqrt{\omega k}\tau)],$$  \hspace{1cm} (16)

where

$$\nu = \frac{1}{2} \left( \frac{5 + 3\omega}{1 + 3\omega} \right).$$  \hspace{1cm} (17)

and $+\,$ or $-\,$ labels the coefficients corresponding to the expanding ($\tau_0, \infty$) and contracting ($-\infty, -\tau_0$) phases respectively where the solution is a good approximation. In particular, it incorporates the sub-Hubble phases at early and late times, i.e. between ($-\infty, -\tau_+$) and ($\tau_+, \infty$). Also, we have included the $k$-dependent factors in front of the coefficients $D_\pm$ and $S_\pm$ to ensure that in the super-Hubble phase the $k$ dependence is completely contained in $D(k)$ and $S(k)$. By using the asymptotic form of Bessel functions:

$$\lim_{\tau \rightarrow 0} J_\nu(x) = x^\nu,$$  \hspace{1cm} (18)

and taking the appropriate $\tau \rightarrow 0$ limit of (11), this can easily be seen:

$$\lim_{\tau \rightarrow 0} \Phi_k = D_\pm(k)\omega^{\nu/2} + S_\pm(k)\omega^{-\nu/2}(\pm \tau)^{-2\nu}.$$  \hspace{1cm} (19)

Now of course, if we know the solution to (12) in the entire range ($-\infty, \infty$), then we will know how $\{D_+, S_+\}$ is related to $\{D_-, S_-\}$, but as we will now explain, one requires less. Finding this "transfer matrix" without any ambiguities is in fact the main endeavor of this paper.

In order to achieve this let us next look at the super-Hubble phases. As usual, in these phases, the $k$-terms can be ignored, as $H > k$ and the second term dominates. Finally, we are left with the sub-Hubble phase around the bounce and now comes a crucial observation. Unlike in the usual GR scenario, in this phase which occurs once the new physics has kicked in, i.e. $|\tau| < \tau_0$, the $k$-term can again be ignored in favor of the fourth term. This is because for the modes that we observe today, typically\(^8\)

\(^8\) If there is no prolonged inflationary phase.
\( k \ll \tau_0^{-1} \), the latter being the scale of new physics which is expected to be close to the string or the Planck scale. As a result, we find that for the entire range \((-\tau_+, \tau_+)\) we can ignore the \( k \)-term and need only solve

\[
\Phi_k'' + \frac{3(1 + \omega)q\tau}{\tau^2 + \tau_0^2} \Phi_k' + \frac{2q\tau_0^2}{(\tau^2 + \tau_0^2)^2} \Phi_k = 0 .
\] (20)

A few comments are now in order. Firstly, observe that we have completely bypassed having to do any matching at \( \pm \tau_0 \) or at \( \tau = 0 \), as is often done in literature. Secondly, the formalism described here divides the entire evolution into three overlapping regions: \((-\infty, -\tau_0)\), \((-\tau_+, \tau_+)\) and \((\tau_0, \infty)\). As a result, the matching of the solutions to (15) and (20) can be done unambiguously by looking at the asymptotic properties of the solutions. This is to be contrasted with matching conditions which are imposed at specific points in time (like between sub-Hubble phases). A final remark, as is clear the sub-Hubble phase around the bounce is very different from the usual sub-Hubble phase (for instance in inflationary cosmology) because the \( k \)-term is unimportant, and thus any analysis based on the usual framework of solving the equation keeping only the \( k \)-term is bound to give incorrect results.

Above, we already obtained exact solutions (16) for the 1st and 3rd region, \( i.e. \) in the intervals \((-\infty, -\tau_0)\) and \((\tau_0, \infty)\). In order to avoid any ambiguity involving mode matching we need to find exact solutions to (20). To do this it is convenient to define

\[
y_k \equiv \Phi_k(\tau^2 + \tau_0^2)^{-n} .
\] (21)

The differential equation then becomes

\[
(\tau^2 + \tau_0^2)y'' + (\tau^2 + \tau_0^2)\tau(4n + 3(1 + \omega)q)y' + 2n(\tau^2 + \tau_0^2) + (4n(n - 1)) + 6nq(1 + \omega)\tau^2 + 2q\tau_0^2)y = 0 .
\] (22)

If we now choose \( n \) to satisfy

\[
4n(n - 1) + 6nq(1 + \omega) = 2q ,
\] (23)

which leads to

\[
n_\pm = \frac{-1 \pm \sqrt{2 + 3\omega}}{1 + 3\omega} ,
\] (24)

and in particular choose to work with \( n = n_- \), then the ordinary differential equation greatly simplifies and one has

\[
(\tau^2 + \tau_0^2)y'' + \tau\beta y' - 2n_+ y = 0 ,
\] (25)

where

\[
\beta \equiv \frac{2(1 + 3\omega - \sqrt{2 + 3\omega})}{1 + 3\omega} .
\] (26)

**IV. MODE SWITCH: AN EXACT SPECIAL CASE**

In order to have a clear understanding of the physics we will focus on a special case where the solution can be written in terms of familiar functions. This is the case when the coefficient of the second term, \( \beta \), equals one, which happens when \( \omega \approx 5.314 \).

We note in passing that such a large \( \omega \) can actually be physically interesting as it is known that as \( \omega \rightarrow \infty \) we produce a scale-invariant spectrum in one of the modes (the Ekpyrotic scenario). In any case, the reason we want to consider the above \( \omega \) is because it lends technical simplicity to extract the physics which will be useful when we study the more general case.

For the above special case the differential equation simplifies to

\[
(\tau^2 + \tau_0^2)y'' + \tau y' - \alpha^2 y = 0
\] (27)

with

\[
\alpha^2 \equiv 2n_+ > 0 .
\] (28)

Its solution reads

\[
y = B_1 \exp\left(\frac{\alpha}{\tau_0} \sinh \frac{\tau}{\tau_0} \right) + B_2 \exp\left(-\alpha \sinh \frac{\tau}{\tau_0} \right) .
\] (29)

In order to understand how the perturbations propagate through the bounce we have to look at the behavior of \( y \) as \( \tau \rightarrow \pm \infty \). Now,

\[
\lim_{\tau \rightarrow \pm \infty} \sinh \frac{\tau}{\tau_0} = \pm \ln \left( \frac{\pm 2\tau}{\tau_0} \right) .
\] (30)

As a result, the asymptotic values of \( y \) reads

\[
\lim_{\tau \rightarrow \pm \infty} y = B_1 \left( \frac{\pm 2\tau}{\tau_0} \right)^{\pm \alpha} + B_2 \left( \frac{\pm 2\tau}{\tau_0} \right)^{\mp \alpha} .
\] (31)

Something remarkable has happened: in going from contraction to expansion, the coefficients corresponding to the two modes \( \tau^\alpha \) and \( \tau^{-\alpha} \) have completely switched. The dominant mode in the contracting phase goes over into the dominant mode in the expanding phase, unlike what happens in a singular bounce making use of the Hwang-Vishniac and Deruelle-Mukhanov matching conditions. To see this more precisely, by matching the bouncing solution (22) to the late time solutions given by (16) in the overlapping regions one finds the exact relation:

\[
D_+ = S_+ \left( \frac{\omega \tau_0^2}{2} \right)^{-\nu} \quad \text{and} \quad S_+ = D_- \left( \frac{\omega \tau_0^2}{2} \right)^{\nu} .
\] (32)

For a general equation of state, although we do not expect such a complete switch, the above result certainly suggests that there would be mode mixing.
V. MODE MIXING AS $\omega \gg 1$

It is known that the Ekpyrotic scalar which mimics an ideal fluid with a large equation of state parameter produces a (nearly) scale-invariant spectrum of perturbations during the phase of contraction, but in the growing mode. It was argued that this mode matches the decaying mode exclusively during the expansion phase and therefore cannot explain the near scale-invariance observed in the CMB today. The previous exact analysis already suggests that this may not be true if the Ekpyrotic bounce is smoothed out.

Quite remarkably, in the $\omega \to \infty$ limit the differential equation (22) simplifies to

$$ (\tau^2 + \tau_0^2) y'' + 2\tau y' = 0 , $$  

and again becomes amenable to an exact treatment. The equation has the following solution

$$ y = B_1 + B_2 \tan^{-1}(\tau/\tau_0) . $$  

Now, in the case $\omega \gg 1$ we obtain from (34)

$$ y_k \approx \Phi_k = D_\pm(k)\omega^{1/4} + S_\pm(k)\omega^{-1/4}(\pm \tau)^{-1} , $$

where we have used that $n \to 0$ as $\omega \to \infty$. We can now relate the $+$ coefficients with $-$ via the bounce solution (35). By considering the $\tau \to \pm \infty$ limit of (34) one easily finds

$$ \omega^{1/4} D_\pm = B_1 \pm \frac{\pi B_2 \tau_0}{2} \text{ and } \pm \omega^{-1/4} S_\pm = -B_2 \tau_0 . $$

This leads us to

$$ D_+(k) = D-(k) - \frac{\pi}{\omega^{1/4} \tau_0} S_+(k) \text{ and } S_+(k) = -S_-(k) . \quad (37) $$

We notice that the constant mode $D_+$ gets contributions from both the modes in the contracting phase. In particular, the contribution from $S_-$ gives a scale invariant spectrum $\gamma$!

VI. DISCUSSION

In this paper, we have followed the evolution of cosmological fluctuations across the bounce using the relativistic potential $\Phi$, the metric fluctuation in longitudinal gauge. From experience built up in investigations of inflationary universe models (see e.g. [45, 46]), Pre-Big-Bang cosmologies and the Ekpyrotic scenario, working in terms of $\Phi$ can be dangerous because of very sensitive dependence of the evolution on matching conditions at times when the equation of state of the cosmological background changes, e.g. during the period of reheating in the context of inflationary cosmology. In our example we are safe from this danger because we have an exact solution and hence do not need to invoke any matching conditions.

On the other hand, from the point of view of the quantum theory of cosmological perturbations (see [17, 18] for pioneering works), a variable different from $\Phi$ carries more physical meaning, namely the variable $v$ in terms of which the action for cosmological fluctuations has canonical kinetic term. In the case of hydrodynamical matter, the variable $v$, which determines the curvature perturbation in comoving gauge, is related to $\Phi$ via

$$ c_s v \equiv u' - \frac{\theta}{\rho} u , $$

where $c_s$ is the speed of sound,

$$ \theta \equiv \frac{1}{a} \left(1 + \frac{\rho}{\rho}ight)^{-1/2} , $$

and

$$ u \equiv \frac{\Phi}{\sqrt{\rho + p}} . \quad (40) $$

The variable $v$ obeys the following field equation:

$$ v'' - c_s^2 \nabla^2 v - \frac{\alpha}{z} v = 0 , $$

where

$$ z \equiv \frac{a \sqrt{\rho}}{-\mathcal{H} c_s} \text{ and } \beta \equiv \mathcal{H}^2 - \mathcal{H}' . \quad (42) $$

In inflationary cosmology, the variable $v$ remains constant during the transition of the equation of state which takes place at the time of reheating, whereas $\Phi$ jumps by a large factor. It is thus the variable $v$ which is a more robust one to follow. In fact, in the absence of entropy fluctuations, one can show that on super-Hubble scales, the variable

$$ \mathcal{R} = \frac{v}{z} $$

is conserved [45, 46, 49]:

$$ (1 + \omega) \mathcal{R} = 0 . $$

In the contracting phase of an Einstein universe it can be shown [12] that the dominant mode of $\Phi$ does not couple to the variable $v$, and hence the curvature fluctuation is only sensitive to the decaying mode of $\Phi$, a mode which does not lead to a scale-invariant spectrum.

In a bouncing cosmology, it becomes problematic to use the equation of motion (11) for $v$. For instance, at

9 To see this note that according to conventional quantization of the Mukhanov variable, as $\tau \to -\infty$, $v \sim (V_0/\sqrt{\mathcal{R}}) [\cos(\sqrt{\mathcal{R}} \tau) + \sin(\sqrt{\mathcal{R}} \tau)]$. Using the relation $v \sim \tau^{(1+\omega)/(1+2\omega)} [\Phi' + 2\nu(\Phi/\tau)]$ valid for ideal fluids in the General Relativistic limit ($\approx -\tau_\gamma$), one then finds $D_- \sim k^{-1/2}$ and $S_- \sim k^{-3/2}$, the latter giving rise to a scale-invariant contribution to the power spectrum.
the bounce point the variable $z$ blows up and hence the
equation is singular. Essentially the new physics which
solves the Big Bang singularity necessarily has to modify
\[41\]. As we discuss in details in the appendix, in general
it is not even possible to come up with an equation for $v$
(see \[50\] for an attempt in this direction) while at least
in our scenario the perturbation equation for $\Phi$ is still
correct. Thus it is not justified to use the large scale
analysis and conclude that $\mathcal{R}$ is constant. These are our
reasons for focusing on the evolution equation for $\Phi$ and
not for $v$.

VII. CONCLUSIONS

In this paper we have presented a particular nonsingu-
lar bouncing cosmological background, in which the usual
general relativistic equations for cosmological perturba-
tions can be solved exactly. We find that the growing
mode in the contracting phase goes over into the domi-
nant mode in the expanding phase. This is unlike what
happens in four dimensional effective field theories of Pre-
Big-Bang or Ekpyrotic type, in which the fluctuations are
matched at a distinguished but singular hypersurface us-
ing the analog of the Israel matching conditions. Our
result supports the conclusions of \[16, 17\] that the trans-
fers of fluctuations is very sensitive to the details of the
bounce.

Our analysis assumes that terms in the Lagrangian dif-
ferent from the Einstein-Hilbert terms generate the non-
singular bounce. At the same time, it is crucial that these
new terms not effect the equations for the IR fluctuation
modes. An example where both conditions are satisfied
is given in \[3, 30\].

Applied to the Ekpyrotic scenario, our result implies
that one may not be required to invoke entropy fluctua-
tions with a scale-invariant spectrum \[38, 39, 40, 41, 42\]
in order to obtain a scale-invariant spectrum of curvature
fluctuations in the expanding phase.

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VIII. APPENDIX

In this appendix we explain why in our bouncing uni-
verse scenario we used the Bardeen potential $\Phi$ to track
the perturbations across the bounce rather than the more
conventional Mukhanov variable $v$. The discussion will
also clarify how we can evade the usual conservation ar-
guments involving the curvature perturbation \[43\] to
get mode-mixing. Let us first define the intermediate $u$
variable:

$$u \equiv \exp \left[ \frac{3}{2} \int (1 + c_s^2) \mathcal{H} \; d\tau \right] \Phi$$

(45)

For an ideal gas using the continuity equation

$$\rho' + 3\mathcal{H}(\rho + p) = 0$$

(46)

$u$ simplifies to

$$u = \frac{\Phi}{\sqrt{\rho}}$$

(47)

If one now uses the relation

$$2\mathcal{H}' + (1 + 3\omega)\mathcal{H}^2 = 0$$

(48)

which is valid only for a General Relativistic background
one can derive a relatively simple differential equation for
$u$ from \[41\]:

$$u'' - \omega \nabla^2 u - \frac{\theta''}{\theta} u = 0$$

(49)

where

$$\theta \equiv \frac{1}{a \sqrt{1 + \omega}}$$

Since the new physics that resolves the singularity al-
ters the background equations, \[43\] can no longer be
valid (in fact one can check that \[48\] always lead to a
singular universe), and therefore one cannot trust \[49\].
Nevertheless, one can actually derive a similar equation
for $u$ which does not use \[48\] and therefore will is valid
irrespective of the new physics. The equation reads:

$$u'' - \omega \nabla^2 u - (\alpha \mathcal{H}' + \beta \mathcal{H}^2) u = 0$$

(50)

where

$$\alpha \equiv \frac{3\omega - 1}{2}$$

and

$$\beta \equiv \frac{9\omega^2 + 6\omega - 5}{4}$$

(51)

One can easily verify that provided \[48\] is valid, \[50\]
reduces to \[49\].

In the usual General Relativistic analysis one typically
defines the Mukhanov variable via \[38\] which for an ideal
gas simplifies to

$$v = u' + \mathcal{H} u$$

(52)

and turns out to be the right canonical variable for the
effective quantum action \[3, 10\] (again in the General Rel-
avitistic limit) and therefore let’s one impose the “sub-
hubble quantum fluctuations” as initial conditions. In
this context we note that using the relations between
$\Phi$, $u$ and $v$ \[40, 48\], one can also read off the appropriate
initial conditions for $\Phi$ or $u$, from that of $v$, and then
choose to analyse the propagation of the fluctuation in
any of the variables that one chooses to (see footnote
9 for instance). We should also point out that all the
definitions \[40, 48\] are non-singular, and one expects the
solutions for all these variables to be regular when one uses the correct evolution equations for them. Actually, this is where the Mukhanov variable as defined in (38) loses its usefulness. Unlike $\phi$ and $u$, one cannot even find an evolution equation for $v$ like (41) in the general case when one has introduced some new physics to resolve the singularity. Instead one finds the following equation

$$v'' - \omega \Delta v - [(\alpha + 2)\dot{H} + \beta \dot{H}^2](u' + fu) = 0 \quad (53)$$

where we have defined a function which depends on the background:

$$f \equiv \frac{\beta \dot{H}^2 + (\alpha + 1)\dot{H}'' + (2\beta + \alpha)\dot{H}'}{(\alpha + 2)\dot{H}' + \beta \dot{H}^2} \quad (54)$$

10 It may be an interesting exercise to find whether it is possible to generalize the definition of $v$ such that it reduces to (41) in the GR limit, but differs from it near the bounce in such a way that one can obtain an evolution equation for $v$ like its GR cousin.

Clearly, in order to have an equation only in terms of $v$, the terms involving $u$ must combine to give us $v$ which can happen iff one can satisfy

$$3(1 + \omega)\dot{H}'' + (1 + 3\omega)^2 \dot{H} = 0 \quad (55)$$

Indeed for GR this condition is satisfied. However, it is clear that when one introduces new physics, in general there is no simple evolution equation for $v$ and (11) is certainly not valid. It also explains why one can’t use the conservation law (11) which is based on (11). This is why a mode-mixing is actually possible in our scenario.

For such an attempt in the context of quantum cosmological models see [54].

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