Alternative description of particle shower longitudinal profile

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I. INTRODUCTION

The parameterization of longitudinal profiles for particle showers produced by the primary nuclei in atmosphere, is an essential tool for the primary nuclei identification and the evaluation of primary energy. The experiments sampling the longitudinal development by the Cherenkov light images [1] or air fluorescence [2, 3] from different traversed atmosphere depths, extract the position of shower maximum, which is sensitive to the incident primary nucleus. The integral of shower profile strongly correlates with primary energy [4].

The shower longitudinal profile is a dependence of shower particle number \( N \) on a given traversed atmospheric depth, \( T \). The parameterization of shower profile commonly used in cosmic-ray experiments is Gaisser-Hillas formula [5]:

\[
N(X) = N_{\text{max}} \left( \frac{X}{X_{\text{max}}} \right)^{X_{\text{max}}} \exp \left( X_{\text{max}} - X \right),
\]

where \( X = (T - T_0)/\lambda \) and \( X_{\text{max}} = (T_{\text{max}} - T_0)/\lambda \).

The maximum number of shower particles \( N_{\text{max}} \) at the traversed atmosphere depth \( T_{\text{max}} \) along with \( T_0 \) and \( \lambda \) in expression (1) are free parameters depending on primary nucleus and energy.

Standard primary nuclei composition consists of the first 28 nuclei of periodic table with mass (nucleon) numbers \( A = 1, \ldots, 56 \) usually divided into 4-6 groups (species) H, He, CNO-like, Si-like, Fe-like. The large number of nuclei species (more than 4) increases the uncertainties of inverse problem \((E \) and \( A \) reconstruction) falsely improving the agreement of experiment with the theory [6].

The primary energy region responsible for particle shower detection at the observation level begins at about \( E > 1 \) PeV and ends at GZK cutoff energies [2].

The efficiency of 4-parametric parameterization (1) is in its applicability to wide range of energies and primary nuclei. However, the observed correlations between parameters result in a loss of the physical meaning of \( X_0 \) and \( \lambda \) and reduce the range of effective atmosphere depths for (1).

II. PARAMETERIZATION

Here, an alternative parameterization \( N(T,E,\varepsilon) \) for particle shower longitudinal profile is proposed using three non-correlating parameters depending on primary particle energy and nucleon energy, \( \varepsilon \):

\[
N(x) = N_{\text{max}} \exp \left( -\frac{1}{2} \left( \frac{\ln x}{\delta(x)} \right)^2 \right),
\]

where

\[
\delta(x) = \alpha - \beta (\tanh x)^{\frac{1}{4}}.
\]

is the profile shape function of variable

\[
x = \frac{T}{T_{\text{max}}}.\]

The shower maximum position, \( T_{\text{max}}(\varepsilon) \), and shape function, \( \delta(x,\varepsilon) \), turned out to be dependent on primary particle energy per nucleon,

\[
\varepsilon = \frac{E}{A}, \ (\text{PeV/}\text{n}).
\]

The maximum number of shower particles, \( N_{\text{max}}(E,\varepsilon) \) is factored into the primary energy and a function of nucleon energy only. The corresponding approximations for the parameters of shower longitudinal profile (2,3) are:

\[
\alpha = 0.707 + 0.209 \varepsilon^{-0.084}, \quad \beta = \sqrt{\alpha/2.59}, \quad (4a)
\]
\[
T_{\text{max}} = 433.5 + 38.9 \ln(\varepsilon A_{Fe})^{0.857}, \ (\text{g/cm}^2), \quad (4b)
\]
\[
N_{\text{max}} = 0.653(E/1\text{GeV})(1 - e^{-2.5\varepsilon^{0.12}}), \quad (4c)
\]

where \( A_{Fe} = 56 \) and \( \varepsilon \) in the units of PeV/n. The goodness-of-fit tests for (4a-c) were \( \chi^2 < 1 \) at negligible correlations between \( \alpha, T_{\text{max}} \) and \( N_{\text{max}} \) parameters.

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III. SHOWER PROFILES

The values of free parameters in expressions (1) and (2) were obtained from simulated shower profiles (training sample) using CORSIKA [8] (SIBYLL [9]) code for four primary nuclei $A \equiv 1, 4, 16, 56$ at six energies $E \equiv 1, 10, 100, 500, 2500, 10^4$ PeV. Shower profiles were studied for 10 atmosphere depths $T \equiv 100, 200, \ldots, 1000$ g/cm$^2$ at two zenith angles, $\cos \theta = 0.7$ and 1. The shower particle energy threshold was $E_e > 1$ MeV. Simulation statistics were provided for less than 2-3% statistical errors in the whole measurement range.

The averaged shower profiles were approximated by expressions (1) and (2) using 13 reference depths. Results are presented in Fig. 1. It is seen that the parameterization (1) (dashed lines) underestimates the shower sizes at large atmosphere depths.

The parameterization errors of expressions (1) and (2) and corresponding $\chi^2_{\text{fit}}$ are presented in Fig. 2 for different primary energies and nuclei. Upper and middle panels show the errors of the 4-parametric approximations of CORSIKA simulated shower profiles using the $N_{\text{max}}, T_{\text{max}}, X_0$ and $\lambda$ parameters of expression (1) and the $N_{\text{max}}, T_{\text{max}}, \alpha$ and $\beta$ of expressions (2,3). The lower panel of Fig. 2 shows the errors of shower profiles $N(T, E, A)$ from expressions (2-4).

The normalized simulated (symbols) and parameterized (lines) shower profiles are presented in Fig. 3. It is seen that parameterization (2) effectively describe the shower profiles in the regions of both the maximum ($x \approx 1$, inset figure, solid line) and asymptotic depths ($x \approx 3$). The expression (1) is systematically biased about $-2\%$ (inset figure, dashed line) at $x \approx 1$.

IV. PARAMETERS

The study of $T_{\text{max}}(\varepsilon)$ and $N_{\text{max}}(\varepsilon)$ dependence on nucleon energy ($\varepsilon$) are presented in the upper and lower panel of Fig. 4 respectively. The approximations of...
shower profiles using parameterizations (1) and (2) were trailed for different lower ($T_{\text{low}}$) and upper ($T_{\text{up}}$) limits of traversed atmosphere depth.

The estimated values for $T_{\text{max}}$ (Fig. 4, upper panel) were unbiased for all trails. The line in Fig. 4 corresponds to expression (4b). The asterisk and cross symbols in Fig. 4 are correspondingly renormalized CORSIKA simulated data from [10].

Estimations of $N_{\text{max}}$ (Fig. 4, lower panel) using expression (1) for approximations of shower profile turned out to be depending on boundary conditions for atmosphere depth (hollow and bold star symbols), whereas the expression (2) remained practically unbiased (hollow and bold circle symbols) for different boundaries.

The shower profile shape functions $\delta(\varepsilon | T)$ and $\delta(x | \varepsilon)$ are presented in Fig. 5 where symbols (left panel) are the data extracted from CORSIKA simulated training sample. Solid lines in both panels correspond to expressions (3,4a). Dashed lines in Fig. 5 (right panel) are the 0.5% accuracy logarithmic simplifications of shape function according to (5,6).

\[ \delta(x) \approx \begin{cases} a - b \ln x, & \text{if } 0.07 \lesssim x \lesssim 1; \\ a - b \ln x / (1 + 1.59 \ln x), & \text{if } x \geq 1, \end{cases} \]

where

\[ a = 0.215 + 0.145 \varepsilon^{-0.084}, \]
\[ b = 0.086 + 0.011 \varepsilon^{-0.084}, \]

at $\chi^2/450 = 0.7$. Approximation (5) provides for the analytic solution for inverse function $f_1^{-1}(x)$ for $x > 1$.

V. VERIFICATION

Verification of universality of approximations (1) and (2) was performed extrapolating the shower profiles from $100 - 1428 \text{ g/cm}^2$ interval to the $T = 10 \text{ g/cm}^2$ observation level corresponding to the earlier stage of shower development. The results are presented in Fig. 6. The symbols at $T = 10 \text{ g/cm}^2$ in Fig. 6 are the corresponding data from CORSIKA simulated control sample, whereas symbols at $T = 100 \text{ g/cm}^2$ are the representatives of the training sample (Section III).

It is seen that parameterization (1) being trained in the $100 - 1428 \text{ g/cm}^2$ depth interval can not be extrapolated to the region less than about 50 $\text{ g/cm}^2$ (lines, left panel), whereas parameterization (2) works correctly up to the beginning of atmosphere (lines, right panel).

Verification of shower profiles (2-4) by the control samples of different nuclei and energies are shown in Fig. 7. The shower profile for primary Fe nucleus with energy $E = 500 \text{ PeV}$ and corresponding $\varepsilon \simeq 8.93 \text{ PeV/n}$ from training sample (Section III) are compared to the control sample of shower profiles produced by primary H, He, C, O and Si nuclei with the same energy per nucleon (symbols). Lines in Fig. 7 are the corresponding congruent predictions from the parameterizations (2-4).

The results in Fig. 7 confirm the $\varepsilon$-dependence of shower longitudinal profile shape (expressions 4a,b). The shower profile amplitude ($N_{\text{max}}$) also depends linearly on $f_1^{-1}(x)$ for $x > 1$.
FIG. 6: Extrapolations of the parameterization (1), left panel, and (2), right panel, to the earliest stage of shower development for different primary nuclei and energies (lines). Symbols are CORSIKA simulated data.

FIG. 7: Control samples of shower profiles (symbols) produced by the different primary nuclei with the same nucleon energy $\varepsilon \approx 8.93$ PeV/n. Lines are the predictions from (2-4).

VI. INTEGRAL

The right hand side of parameterization (2) at corresponding normalization can be considered as a probability density function and be used further for primary energy evaluation [2, 4]. Unfortunately this function was missed by mathematicians and using numerical technique, the required normalization

$$
\int_0^\infty f(x, \varepsilon)dx \approx 1 \pm 10^{-4}
$$

was provided for probability density function

$$
f(x) = \frac{1}{\sqrt{2\pi\delta_0}} \exp\left(-\frac{1}{2} \left(\frac{\ln x}{\delta(x)}\right)^2\right)$$

with additional parameter

$$\delta_0 = 0.226 + 0.148\varepsilon^{-0.092}.$$  

The goodness-of-fit test for $\delta_0(\varepsilon)$ was $\chi^2 = 0.01$ in the $10^{-2} \leq \varepsilon \leq 10^4$ (PeV/nucleon) interval and the upper limit of integral (7), $x_{max} = 3$.

It is interesting to note the relation between parameters $\delta_0$ and shape function $\delta(x)$ from expression (3):

$$\delta_0(\varepsilon) \approx \frac{1}{x_{max}} \int_0^{x_{max}} \delta(x)dx \pm 1\%.$$  

The statistical parameters, the average ($\bar{x}$) and standard deviation ($\sigma_x$) of distribution (8) are well approximated (0.1% errors) by the following expressions depending on nucleon energy:

$$\bar{x} = 1.036 + 0.094\varepsilon^{-0.12}$$

at $\chi^2 = 0.1$, and

$$\sigma_x = 0.226 + 0.176\varepsilon^{-0.092}$$

at $\chi^2 = 1.1$.

VII. FLUCTUATIONS

The main source of shower profile fluctuations is the depth of the first interaction of primary particles in the atmosphere [11]. The exponentially distributed uncertainty of the first interaction point results in the corresponding fluctuations of shower profile (2) depending on the rate of change $(dN/dx)$ of profile with respect to the depth, $x$. Thus, the fluctuations should be maximal at the beginning of shower development ($x \approx 0$, Fig. 3), and minimal in the region of shower maximum, $x = 1$.

The dependence of interaction length, $\lambda(A, E)$, on primary particle mass, $A$, and energy $(E)$ dependences of shower profile fluctuations.

The statistical measure of fluctuations is the standard deviation of shower particle number, $\sigma_N$. The corresponding values of $\sigma_N(x, A, E)/N(x)$ obtained from the

primary energy, $E$ (expression 4c).

The good agreement in Fig. 7 between predictions (lines) and simulated data indicates the correctness of expressions (2-4) for shower profile description at least with accuracies of about $2 - 3\%$ in the whole measurement range.
FIG. 8: Normalized standard deviations \((\sigma_N/N)\) of shower particles for different primary nuclei and primary energies (symbols). Lines are the parameterization (10) for energies 1 PeV (dotted lines) and 10 EeV (solid lines). Dashed lines describe the fluctuations for intermediate energies. Inset panel zooms in on the region of minimal fluctuations at \(x \approx 1\).

shower simulated dataset (Section III) are presented in Fig. 8 (symbols). Inset panel shows the region of minimal fluctuations in detail. Lines in Fig. 8 correspond to the parameterizations

\[
\frac{\sigma_N}{N} \approx \begin{cases} 
  a_1 - a_2 \ln x, & \text{if } x \leq 1; \\
  a_1 + a_3 (\ln^9 x)/x, & \text{if } x \geq 1,
\end{cases}
\]

(10)

where

\[
\begin{align*}
  a_1 &= 0.165 A^{-0.32} E^{-0.13}, \\
  a_2 &= 0.68 A^{-0.185} E^{-0.009}, \\
  a_3 &= 3.77 A^{-0.386} E^{-0.035}, \\
  \eta &= 2.67 A^{-0.080} E^{-0.027}
\end{align*}
\]

at \(\chi^2/470 \approx 1.7\).

VIII. CONCLUSION

Parameterizations (2-4) describe the longitudinal profile of particle showers \(N(T, E, \varepsilon)\) with accuracy of about 2 – 3%.

The position of shower maximum, \(T_{\text{max}}(\varepsilon)\), and profile shape function, \(\delta(x, \varepsilon)\), depend only on primary nucleon energy \(\varepsilon\), which is in agreement with prediction of superposition model \([12]\).

Amplitude of profile \(N_{\text{max}}(E, \varepsilon)\) depends on both primary energy \((E)\) and nucleon energy \(\varepsilon\).

The inter-correlations between \(N_{\text{max}}(E, \varepsilon)\), \(T_{\text{max}}(\varepsilon)\) and \(\delta(x, \varepsilon)\) shower profile parameters are negligible.

The applicability of parameterization (2) is in the energy range of 1 PeV to at least 10 EeV for all primary nuclei \((A \in 1, \ldots, 56)\) and atmosphere slant depths \(T \in 0 - 1500 \text{ g/cm}^2\).

The profile shape function, \(\delta(x, \varepsilon)\), from (3) has simple logarithmic representation (5) providing for an analytic solution for corresponding inverse function.

The fluctuations of particle shower longitudinal profile (parameterization (10) and Fig. 8) depend on the energy and mass number of primary nuclei.

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