Comment on “Remarks on flavor-neutrino propagators and oscillation formulae”

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We comment on the paper “Remarks on flavor-neutrino propagators and oscillation formulae” [Phys. Rev. D \textbf{64}, 013011 (2001)]. We show that the conclusions presented in that paper do not apply to the exact field theoretical oscillation formulae obtained in the BV formalism (for three flavors) which are free from the dependence on arbitrary mass parameters, account for CP violation and reduce to the usual quantum mechanical (Pontecorvo) three flavor oscillation formulae in the relativistic limit.

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I. INTRODUCTION

In recent years exact formulae for neutrino oscillations have been obtained in the quantum field theory (QFT) framework \cite{1,2,3,4} (hereafter referred to as the BV formalism by following Ref. \cite{6}). In the paper “Remarks on flavor-neutrino propagators and oscillation formulae” [Phys. Rev. D \textbf{64}, 013011 (2001)] \cite{7} it has been remarked that by the use of the retarded propagators different formulae can be obtained which are free from the dependence on arbitrary mass parameters and in the case of three flavor oscillations they reduce to the formulae obtained in the BV formalism only when the mixing matrix is real. No problem arises in the two flavor case, in the sense that the retarded propagator derivation gives same oscillation formulae as obtained in \cite{2}.

In the following we observe that the formulae obtained in Ref. \cite{7} are not physically acceptable since they do not allow CP violation and do not reduce to the usual Pontecorvo three flavor formulae in the relativistic limit. We note that, on the contrary, our formulae do account for CP violation, are independent of arbitrary mass parameters and do reduce to the usual Pontecorvo three flavor formulae in the relativistic limit.

Apart from the latter property, which by itself is not at all a negligible requirement to be satisfied, the consistency with CP violation discriminates with a clear cut between our formulae and those derived by the use of the retarded propagator. Moreover, it also points to the necessity of using the flavor Hilbert space, as indeed emerges in our treatment. In this respect, we observe that the authors of Ref. \cite{7} actually do not exclude the possibility of using such a flavor state space. On the contrary, they analyze the arguments presented in Ref. \cite{8} and conclude that the assertion there presented against the flavor space is not appropriate.

The paper is organized as follows. We first introduce the main lines of our derivation for the three flavor case (see also \cite{5}). Subsequently, we discuss the results of Ref. \cite{7} and see that they are physically not acceptable for the reasons said above.

The derivation of the three flavor oscillation formulae proceeds along the same line of the derivation for the two flavor case. We use standard QFT for the neutrino fields and the familiar parameterization of the CKM matrix \cite{9}. We write down the generator for the mixing transformations in terms of the Dirac neutrino fields \( \nu_i \) with masses \( m_i \), \( i = 1, 2, 3 \), then we consider the charges for \( \nu_i \) and for the flavor neutrinos \( \nu_\sigma, \sigma = e, \mu, \tau \). We construct the flavor state space, which is found to be unitarily inequivalent to the state space for the mass eigenstate neutrinos, and we compute the expectation values of the flavor charges in the flavor states, thereby obtaining the exact oscillation formulae. We finally discuss the CP violation. Besides the obvious higher level of computational complexity, the essential, non-trivial difference with respect to the two flavor case is indeed in the physically relevant fact that the three flavor oscillation formulae must account for CP violation.

We then analyze the derivation of the formulae given in \cite{7} and discuss their independence of the CP violating phase and their failure in reducing to the usual Pontecorvo formulae in the relativistic case.

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We observe that the conclusions reached in the present paper can be extended as well to the case of boson mixing with number of flavors larger than two.

We consider the CKM matrix for neutrinos:

\[
\Psi_f(x) = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{13}e^{i\delta} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{13}c_{13}
\end{pmatrix} \Psi_m(x),
\]

(1)

with \(c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}\) being \(\theta_{ij}\) the mixing angles, \(\Psi_T^m = (\nu_1, \nu_2, \nu_3)\) and \(\Psi_T^f = (\nu_e, \nu_\mu, \nu_\tau)\). The CKM matrix is generated as

\[
\nu_\sigma^x(\mathbf{x}) = G_{\mathbf{\theta}}^{-1}(\mathbf{x}) \nu_\sigma(\mathbf{x}) G_{\mathbf{\theta}}(\mathbf{x})
\]

(2)

where \((\sigma, j) = (e, 1), (\mu, 2), (\tau, 3)\) and

\[
G_{\mathbf{\theta}}(\mathbf{x}) = G_{23}(\mathbf{x})G_{13}(\mathbf{x})G_{12}(\mathbf{x}),
\]

(3)

\[
G_{12}(\mathbf{x}) = \exp \left[ \theta_{12} \int d^3x (\nu_1^\dagger(\mathbf{x})\nu_2(\mathbf{x}) - \nu_2^\dagger(\mathbf{x})\nu_1(\mathbf{x})) \right]
\]

(4)

\[
G_{23}(\mathbf{x}) = \exp \left[ \theta_{23} \int d^3x (\nu_2^\dagger(\mathbf{x})\nu_3(\mathbf{x}) - \nu_3^\dagger(\mathbf{x})\nu_2(\mathbf{x})) \right]
\]

(5)

\[
G_{13}(\mathbf{x}) = \exp \left[ \theta_{13} \int d^3x (\nu_1^\dagger(\mathbf{x})\nu_3(\mathbf{x})e^{-i\delta} - \nu_3^\dagger(\mathbf{x})\nu_1(\mathbf{x})e^{i\delta}) \right]
\]

(6)

The free fields \(\nu_i\) (i=1,2,3) can be quantized in the usual way and expressed as (we use \(t \equiv x_0\)):

\[

\nu_i(\mathbf{x}) = \sum_r \int d^3k \left[ u_{k,i}^r \alpha_{k,i}^r(t) + v_{-k,i}^r \beta_{-k,i}^r(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}},
\]

(7)

with \(\alpha_{k,i}^r(t) = e^{-i\omega_{k,i}t} \alpha_{k,i}^0(0), \beta_{k,i}^r(t) = e^{-i\omega_{k,i}t} \beta_{k,i}^0(0)\) and \(\omega_{k,i} = \sqrt{k^2 + m_i^2}\). The vacuum for the mass eigenstates is denoted by \(|0\rangle_m\): \(\alpha_{k,i}^0|0\rangle_m = \beta_{k,i}^0|0\rangle_m = 0\). The anticommutation relations are the usual ones; the wave function orthonormality and completeness relations are those of Ref. [4].

The flavor vacuum is defined as

\[
|0(\mathbf{t})\rangle_f = G_{\mathbf{\theta}}^{-1}(\mathbf{t})|0\rangle_m,
\]

(8)

and its orthogonality (unitary inequivalence) in the infinite volume limit to the vacuum \(|0\rangle_m\) is obtained as shown in Ref. [4]. We observe that the unitary inequivalence of the flavor vacuum to the mass eigenstate vacuum has been rigorously proved for the general case of any number of flavors in Ref. [4].

The flavor fields are expanded as:

\[

\nu_{\sigma}(\mathbf{x}) = \sum_r \int d^3k \left[ u_{k,j}^{\sigma r} \alpha_{k,j}^{\sigma r}(t) + v_{-k,j}^{\sigma r} \beta_{-k,j}^{\sigma r}(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}},
\]

(9)

where \(\alpha_{k,j}^{\sigma r}(t) \equiv G_{\mathbf{\theta}}^{-1}(t) \alpha_{k,j}^r(t) G_{\mathbf{\theta}}(t)\) and \(\beta_{-k,j}^{\sigma r}(t) \equiv G_{\mathbf{\theta}}^{-1}(t) \beta_{-k,j}^r(t) G_{\mathbf{\theta}}(t)\) with \((\sigma, j) = (e, 1), (\mu, 2), (\tau, 3)\).

In order to derive oscillation formulae, we define the flavor charges \(Q_{\sigma}\) (\(\sigma = e, \mu, \tau\)) as

\[
Q_{\sigma}(t) = \sum_r \int d^3k \left( \alpha_{k,j}^{\sigma r}(t) \alpha_{k,j}^{\sigma r}(t) - \beta_{-k,j}^{\sigma r}(t) \beta_{-k,j}^{\sigma r}(t) \right),
\]

(10)

which are connected to the (conserved) Noether charges \(Q_i\) of the free fields via the mixing generator: \(Q_{\sigma}(t) = G_{\mathbf{\theta}}^{-1}(t)Q_i G_{\mathbf{\theta}}(t)\), \((\sigma, j) = (e, 1), (\mu, 2), (\tau, 3)\). As usual in QFT, one must perform subtraction of the vacuum contributions, or, in other words, use normal ordering with respect to the vacuum where one operates with charges and currents.

We define the \(\rho\)-flavor neutrino state with a given momentum and helicity as \(|\nu_\rho\rangle \equiv |\alpha_{k,\rho}^0|0\rangle_f\) and similarly for antiparticles. In the following for simplicity we use \(\alpha_{\rho} \equiv \alpha_{k,\rho}^0\) and \(\beta_{\rho} \equiv \beta_{-k,\rho}^0\). We then obtain the oscillation formulae for neutrinos and antineutrinos as
with \( \sum_i \mathcal{Q}^\rho_i(t) = - \sum_i \mathcal{Q}^\rho_i(t) = 1 \).

The above formulae coincide with the usual quantum-mechanical ones in the relativistic limit [8]. Indeed, in this limit, the (anti-)neutrino state reduces to the usual quantum-mechanical one, defined on the vacuum \(|0\rangle_m\) and one has [6] (considering neutrinos for example):

\[
\mathcal{Q}^\rho_{\sigma}(t) \rightarrow \langle \{\alpha^\rho_{\sigma}(t), \alpha^\rho_{\bar{\sigma}}(0)\} \rangle^2 = \langle \nu_{\rho}| \bar{N}_{\sigma}(t)|\nu_{\rho}\rangle = |\langle \nu_{\rho}(t)|\nu_{\rho}(0)\rangle|^2, \quad \text{for} \quad |k| \gg m_i, \quad i = 1, 2, 3,
\]

which is interpretable as a transition probability.

The \( CP \) and \( T \) violations are calculated as [8]:

\[
\Delta^{\rho}_{CP}(t) \equiv \mathcal{Q}^\rho_{\sigma}(t) + \mathcal{Q}^\rho_{\bar{\sigma}}(t)
\]

\[
\Delta^{\rho}_{T}(t) \equiv \mathcal{Q}^\rho_{\sigma}(t) - \mathcal{Q}^\rho_{\bar{\sigma}}(-t)
\]

with \( \rho, \sigma = e, \mu, \tau \) and \( \Delta^{\sigma}_{CP}(t) = \Delta^{\rho}_{CP}(t) \neq 0 \) when \( \delta \neq 0 \) and \( \rho \neq \sigma \).

In Ref. [4], it was noticed that expanding the flavor fields in the same basis as the (free) fields with definite masses (cf. [3]) is actually a special choice, and that a more general possibility exists. In other words, in the expansion Eq. (3) one could use eigenfunctions with arbitrary masses \( \mu_{\sigma} \), and therefore not necessarily the same as the masses which appear in the Lagrangian. On this basis, the authors of Ref. [6] have generalized the BV formalism by writing the flavor fields as

\[
\nu_{\sigma}(x) = \sum_r \int d^4 k \left[ u^\dagger_{k,\sigma} \bar{\alpha}_{k,\sigma}(t) + \bar{c}_{k,\sigma} \bar{\beta}_{k,\sigma}(t) \right] e^{ik \cdot x},
\]

where \( u_{\sigma} \) and \( v_{\sigma} \) are the helicity eigenfunctions with mass \( \mu_{\sigma} \). We denote by a tilde the generalized flavor operators introduced in Ref. [6], in order to distinguish them from the ones in Eq. (3). The expansion Eq. (3) is more general than the one in Eq. (3) since the latter corresponds to the particular choice \( \mu_{\mu} \equiv m_1, \mu_{\mu} \equiv m_2 \). Of course, the flavor fields in Eq. (3) and Eq. (3) are the same fields. The relation, given in Ref. [8], between the general flavor operators and the BJ ones is

\[
\left( \begin{array}{c} \bar{\alpha}_{k,\sigma}(t) \\ \bar{\beta}_{k,\sigma}(t) \end{array} \right) = J^{-1}(t) \left( \begin{array}{c} \alpha_{k,\sigma}(t) \\ \beta_{k,\sigma}(t) \end{array} \right) J(t),
\]

\[
J(t) = \prod_{(\sigma,j)} \exp \left\{ i \sum_{(\sigma,j)} \xi_{\sigma,j} \left[ \alpha_{k,\sigma j}^\dagger(t) \beta_{k,\sigma j}^\dagger(t) + \beta_{k,\sigma j}^\dagger(t) \alpha_{k,\sigma j}(t) \right] \right\}.
\]

with \( \xi_{\sigma,j} = (\sigma, 1), (\mu, 2), (\tau, 3) \), \( \xi_{\sigma,j} = (\chi_k - \chi_k)/2 \) and \( \cot \chi_k = |k|/\mu_{\sigma}, \cot \chi_j = |k|/m_j \). For \( \mu_{\sigma} \equiv m_j \), one has \( J(t) = 1 \).

As already noticed in Ref. [8], the flavor charge operators are the Casimir operators for the Bogoliubov transformation [7], i.e. they are free from arbitrary mass parameters: \( \tilde{Q}_{\sigma}(t) = Q_{\sigma}(t) \). This is obvious also from the fact that they can be expressed in terms of flavor fields (see Ref. [8]).

Physical quantities should not carry any dependence on the \( \mu_{\sigma} \): in the two–flavor case, it has been shown [8] that the expectation values of the flavor charges on the neutrino states are free from the arbitrariness. For three generations, the question is more subtle due to the presence of the CP violating phase. Indeed, in Ref. [8] it has been found that the corresponding generalized quantities depend on arbitrary mass parameters. However, we find that:

\[
\tilde{Q}_{\rho}(t) = \left| \langle \bar{\alpha}_{\sigma}(t), \bar{\alpha}_{\rho}^\dagger(0) \rangle \right|^2 + \left| \langle \bar{\beta}_{\rho}(t), \bar{\alpha}_{\rho}^\dagger(0) \rangle \right|^2 = Q_{\rho}(t) + F(\mu_{\rho}, t)
\]

\[
\tilde{Q}_{\rho}(t) = - \left| \langle \beta_{\rho}(t), \alpha_{\rho}^\dagger(0) \rangle \right|^2 - \left| \langle \alpha_{\rho}(t), \beta_{\rho}^\dagger(0) \rangle \right|^2 = Q_{\rho}(t) + F(\mu_{\rho}, t)
\]

where \( F(\mu_{\rho}, t) \), whose explicit form we do not report here for sake of shortness, goes to zero at \( t = 0 \). It can be shown that \( F(\mu_{\rho}, t) \) vanishes for \( \delta = 0 \) and/or \( \mu_{\rho} = m_j \), \( \rho, j = (e, 1), (2, \mu), (3, \tau) \). This proves that the invariant (physical) quantities in the generalized theory are \( \tilde{Q}_{\rho}(t) - F(\mu_{\rho}, t) \) and \( \tilde{Q}_{\rho}(t) - F(\mu_{\rho}, t) \) which in fact coincide with
the oscillation formulae of Eqs.(11) and (12). In order to understand the origin of such an invariance, it would be useful to obtain \( F(\mu, t) \) as expectation value of some operator. But this goes beyond the task of the present paper.

We remark that the quantities proposed in Ref. [7] as probabilities for flavor oscillations are ruled out by the present analysis. Indeed, it was there shown that by using the retarded propagators one could arrive at the same oscillation formulae of the BV formalism. However, this coincidence holds only for two flavors since in the three flavor case, the quantity (Eq.(3.7) in Ref [7]):

\[
\rho_{\mu \sigma}(k; t) = \frac{1}{4} \text{Tr}[\rho_{\mu \sigma}(k; t)]
\]

is manifestly CP invariant and thus cannot be used as a correct QFT generalization of the QM oscillation formulae. This is, on the other hand, also confirmed by the fact that the relativistic limit of the above expression does not give the three flavor Pontecorvo formulae. On the contrary, we have shown that our formulae do exhibit all the expected features in the presence of a CP violating phase and Pontecorvo formulae are recovered in the relativistic limit.

Finally, we remark that the formulae Eqs.(11),(12) can be also obtained by use of the unordered Green’s functions [2] as follows (Eq.(2.37) in Ref [7]):

\[
P_{\nu_{\mu} \to \nu_{\mu}}(k; t) = \frac{1}{4} \text{Tr}[g^{(ret)}_{\mu \sigma}(k; t)g^{(ret)*}_{\sigma \mu}(k; t)]
\]

is manifestly CP invariant and thus cannot be used as a correct QFT generalization of the QM oscillation formulae. This is, on the other hand, also confirmed by the fact that the relativistic limit of the above expression does not give the three flavor Pontecorvo formulae. On the contrary, we have shown that our formulae do exhibit all the expected features in the presence of a CP violating phase and Pontecorvo formulae are recovered in the relativistic limit.

Finally, we remark that the formulae Eqs.(11),(12) can be also obtained by use of the unordered Green’s functions [2] as follows (Eq.(2.37) in Ref [7]):

\[
\left| \{\alpha_{\rho}(t), \alpha_{\sigma}^{\dagger}(0)\} \right|^{2} + \left| \{\beta_{\rho}(t), \beta_{\sigma}^{\dagger}(0)\} \right|^{2} = \frac{1}{2} \text{Tr}[G^{\rho \sigma}(k; t)G^{\rho \sigma\dagger}(k; t)]
\]

(22)

In Ref. [7], it is observed that these quantities are generally dependent on the arbitrary mass parameters (in the case one calculate the propagators within the generalized framework) and also that they can be interpreted as (oscillation) probabilities since they satisfy the required boundary conditions. We have already shown how to renormalize the above quantities with respect to the arbitrary mass parameters. As for the probabilistic interpretation, we need to stress that flavor states are essentially multiparticle ones and thus one cannot really talk of probabilities for the above quantities with respect to the arbitrary mass parameters. As for the probabilistic interpretation, we need to stress that flavor states are essentially multiparticle ones and thus one cannot really talk of probabilities for the evolution of such states. Rather, the correct interpretation of our exact oscillation formulae is the one of Eqs.(11),(12), i.e. as expectation values of the flavor charges on states defined on the flavor Hilbert space. This is evident in the bosonic case, where the corresponding quantities can assume values larger than one as well as negative values [11, 14].

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