INTEGRABILITY VERSUS SEPARABILITY FOR THE MULTI-CENTRE METRICS

Galliano Valent*†

*Laboratoire de Physique Théorique et des Hautes Energies
Unité associée au CNRS UMR 7589
2 Place Jussieu, 75251 Paris Cedex 05, France

†CNRS Luminy Case 907
Centre de Physique Théorique
F-13288 Marseille Cedex 9

Abstract

The multi-centre metrics are a family of euclidean solutions of the empty space Einstein equations with self-dual curvature. For this full class, we determine which metrics do exhibit an extra conserved quantity quadratic in the momenta, induced by a Killing-Stäckel tensor. Our systematic approach brings to light a subclass of metrics which correspond to new classically integrable dynamical systems. Within this subclass we analyze on the one hand the separation of coordinates in the Hamilton-Jacobi equation and on the other hand the construction of some new Killing-Yano tensors.
1 Introduction

The discovery of the generalized Runge-Lenz vector for the Taub-NUT metric [7] has been playing an essential role in the analysis of its classical and quantum dynamics. As shown in [4] this triplet of conserved quantities gives quite elegantly the quantum bound states as well as the scattering states.

The Killing-Stäckel tensors, which are the roots of the generalized Runge-Lenz vector of Taub-NUT, have been derived in [9] using purely geometric tools. As a result the classical integrability of the Taub-NUT metric was established. The classical integrability of the Eguchi-Hanson metric was obtained in [14] where the Hamilton-Jacobi equation was separated. This result was further generalized in [9] to cover the two-centre metric. Despite these successes, a systematic analysis of the full family of the multi-centre metrics was still lacking. It is the aim of this article to fill this gap.

In section 2 we have gathered a summary of known properties of the multi-centre metrics, their geodesic flow and some basic concepts about Killing-Stäckel tensors.

In section 3 we obtain the most general structure of the conserved quantity associated to a Killing-Stäckel tensor: it is a bilinear form in the momenta. Taking this quadratic structure as a starting point, we obtain the system of equations which ensure that such kind of a quantity is preserved by the geodesic flow. This system is analyzed and simplified. Its most important consequence is that the existence of an extra conserved quantity is related to the existence of an extra spatial Killing (besides the tri-holomorphic one), which may be either holomorphic or tri-holomorphic.

In section 3 we first consider the case of an extra spatial Killing which is holomorphic. We find that the extra conserved quantity does exist for the following families, with (minimal) isometry $U(1) \times U(1)$:

1. The most general two-centre metric, with the potential

$$V = v_0 + \frac{m_1}{|\vec{r} + \vec{c}|} + \frac{m_2}{|\vec{r} - \vec{c}|}.$$ 

Our approach explains quite simply why there are three extra conserved quantities for Taub-NUT and only one for Eguchi-Hanson, and their very different nature.

2. A first dipolar breaking of Taub-NUT, with potential

$$V = v_0 + \frac{m}{r} + \frac{\vec{F} \cdot \vec{r}}{r^3}.$$ 

3. A second dipolar breaking of Taub-NUT with potential

$$V = v_0 + \frac{m}{r} + \vec{E} \cdot \vec{r}.$$ 

In the Taub-NUT limit $\mathcal{E} \to 0$ there appears a triplet of extra conserved quantities: the generalized Runge-Lenz vector of [7].

The classical integrability of these three dynamical systems follows from our analysis.

In section 4 we consider the case of an extra spatial Killing which is tri-holomorphic, with (minimal) isometry group still $U(1) \times U(1)$. We find four different families of metrics, which share with the previous ones their classical integrability and, using appropriate coordinates, with potentials:
1. In the first case
\[ V = v_0 + \frac{a\xi\sqrt{\xi^2 - c^2} + b\eta\sqrt{c^2 - \eta^2}}{\xi^2 - \eta^2}. \]

2. In the second case
\[ V = v_0 + m \frac{\cos(2\phi)}{r^2}. \]

3. In the third case
\[ V = \frac{a\xi + b\eta}{\xi^2 + \eta^2}. \]

4. And in the fourth case
\[ V = v_0 + mx. \]

As an application we work out in sections 5 and 6 the separation of variables for the Hamilton-Jacobi equation which gives also a check of the results obtained in the former sections.

Eventually we present in section 7 some new Killing-Yano tensors, and some conclusions in section 8.

2 The Multi-Centre metrics

2.1 Background material

These euclidean metrics on \( M_4 \) have at least one Killing vector \( \tilde{\mathcal{K}} = \partial_t \) and have the local form
\[ g = \frac{1}{V} (dt + \Theta)^2 + V \gamma, \quad V = V(x), \quad \Theta = \Theta_i(x) \, dx^i, \quad (1) \]
where the \( x^i \) are the coordinates on \( \gamma \). They are solutions of the empty space Einstein equations provided that:

1. The three dimensional metric \( \gamma \) is flat. Using cartesian coordinates \( x^i \) we can write
\[ \gamma = d\bar{x} \cdot d\bar{x}. \quad (2) \]

2. Some monopole equation holds
\[ dV = \star_{\gamma} d\Theta. \quad (3) \]

Notice that the integrability condition for the monopole equation is \( \Delta V = 0 \), hence these metrics display an exact linearization of the empty space Einstein equations. They have been derived in many ways [13],[6],[10],[11]. In this last reference the geometric meaning of the cartesian coordinates \( x_i \) was obtained: they are nothing but the momentum maps of the complex structures under the circle action of \( \partial_t \).

Let us summarize some background knowledge on the multi-centre metrics for further use. Taking for canonical vierbein
\[ E_a : \quad E_0 = \frac{1}{\sqrt{V}} (dt + \Theta), \quad E_i = \sqrt{V} \, dx_i \quad (4) \]
and defining as usual the spin connection $\omega_{ab}$ and the curvature $R_{ab}$ by
\[ dE_a + \omega_{ab} \wedge E_b = 0, \quad R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^{c}_{\quad b}, \]
one can check that these metrics have a self-dual spin connection:
\[ \omega_i(-) \equiv \omega_0 - \frac{1}{2} \varepsilon_{ijk} \omega_{jk} = 0, \quad \implies \quad R_i(-) = 0, \]
which implies the self-duality of their curvature. It follows that they are hyperkähler and hence Ricci-flat.

The complex structures are given by the triplet of 2-forms
\[ \Omega_i(-) = E_0 \wedge E_i - \frac{1}{2} \varepsilon_{ijk} E_j \wedge E_k = (dt + \Theta) \wedge dx_i - \frac{1}{2} V \varepsilon_{ijk} dx_j \wedge dx_k, \quad (5) \]
which are closed, in view of the hyperkähler property of these metrics.

Let us note that the self-duality of the complex structures and of the spin connection are opposite and that the Killing vector $\partial_t$ is tri-holomorphic.

It is useful to define the Killing 1-form, dual of the vector $\vec{K} = \partial_t$, which reads
\[ K = \frac{dt + \Theta}{V}, \quad (6) \]
and plays some role in characterizing the multi-centre metrics.

Among these characterizations let us mention:

1. For the multi-centre metrics the differential $dK$ has a self-duality opposite to that of the connection. A proof using spinors may be found in [16] and without spinors in [5].

2. The multi-centre metrics possess at least one tri-holomorphic Killing. For a proof see [9].

### 2.2 Geodesic flow

The geodesic flow is the Hamiltonian flow of the metric considered as a function on the cotangent bundle of $M_4$. Using the coordinates $(t, x_i)$ we will write a cotangent vector as
\[ \Pi_i dx_i + \Pi_0 dt. \]
The symplectic form is then
\[ \omega = dx_i \wedge d\Pi_i + dt \wedge d\Pi_0, \quad (7) \]
and we take for hamiltonian
\[ H = \frac{1}{2} g^{\mu \nu} \Pi_\mu \Pi_\nu = \frac{1}{2} \left( \frac{1}{V} (\Pi_i - \Pi_0 \Theta_i)^2 + V \Pi_0^2 \right). \quad (8) \]
For geodesics orthogonal to the $U(1)$ fibers and affinely parametrized by $\lambda$ the equations for the flow allow on the one hand to express the velocities
\[ \dot{t} \equiv \frac{dt}{d\lambda} = \frac{\partial H}{\partial \Pi_0} = \left( V + \frac{\Theta^2}{V} \right) \Pi_0 - \frac{\Theta_i \Pi_i}{V}, \]
\[ \dot{x}_i \equiv \frac{dx_i}{d\lambda} = \frac{\partial H}{\partial \Pi_i} = \frac{1}{V} p_i, \quad p_i = \Pi_i - \Pi_0 \Theta_i, \quad (9) \]
and on the other hand to get the dynamical evolution equations

\[ \dot{\Pi}_0 = -\frac{\partial H}{\partial t} = 0, \quad q \equiv \Pi_0 = \frac{(\dot{t} + \Theta_i \dot{x}_i)}{V}, \]  

(a) \[ \dot{\Pi}_i = -\frac{\partial H}{\partial x_i} \quad \Rightarrow \quad \dot{p}_i = \left( \frac{H}{V} - q^2 \right) \partial_i V + \frac{q}{V} (\partial_i \Theta_s - \partial_s \Theta_i) p_s. \]  

(b) \hfill (10)

Relation (10a) expresses the conservation of the charge \( q \), a consequence of the \( U(1) \) isometry of the metric. For the multi-centre metrics, use of relation (3) brings the equations of motion (10b) to the nice form

\[ \ddot{p} = \left( \frac{H}{V} - q^2 \right) \nabla V + \frac{q}{V} \dot{p} \land \nabla V. \]  

(11)

The conservation of the energy

\[ H = \frac{1}{2} \left( \frac{p_i^2}{V} + q^2 V \right) = \frac{V}{2} (\dot{x}_i^2 + q^2) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \]  

(12)
is obvious since it expresses the constancy of the length of the tangent vector \( \dot{x}^\mu \) along a geodesic.

### 2.3 Killing-Stäckel tensors and their conserved quantities

A Killing-Stäckel (KS) tensor is a symmetric tensor \( S_{\mu\nu} \) which satisfies

\[ \nabla(\mu S_{\nu\rho}) = 0. \]  

(13)

Let us observe that if \( K \) and \( L \) are two (possibly different) Killing vectors their symmetrized tensor product \( K_{(\mu} L_{\nu)} \) is a KS tensor. So we will define irreducible KS tensors as the ones which cannot be written as linear combinations, with constant coefficients, of symmetrized tensor products of Killing vectors.

For a given KS tensor \( S_{\mu\nu} \) the quadratic form of the velocities:

\[ S = S_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \]  

(14)
is preserved by the geodesic flow.

In all what follows we will look for KS tensors, under the assumptions

**A1** : The KS tensor is preserved by Lie dragging along the tri-holomorphic Killing vector:

\[ \mathcal{L}_{\tilde{K}} S_{\mu\nu} = 0, \quad \tilde{K} = \partial_t \]  

(15)

**A2** : We will consider generic values of \( H \) and \( q \neq 0 \).

Furthermore, instead of focusing ourselves on the KS tensor \( S_{\mu\nu} \), whose usefulness is just to produce the conserved quantity \( S \), let us rather examine more closely the structure of the conserved quantity induced by such a KS tensor. From relation (14) we obtain the following ansatz for the conserved quantity we are looking for:

\[ S = A_{ij}(x_k) p_i p_j + 2q B_i(x_k) p_i + C(x_k), \]  

(16)
where the various unknown functions, as a consequence of A1, are independent of the coordinate on the $U(1)$ fiber.

It is interesting to notice that the knowledge of $S$ is equivalent to the knowledge of the K-S tensor: using (9) one can express $S$ in terms of the velocities and, going backwards, compute the K-S tensor components from relation (14).

Imposing the conservation of $S$ under the geodesic flow gives:

Proposition 1 Under the assumptions A1 and A2 the quantity $S$, given by (16), is conserved iff the following equations are satisfied

$$
\begin{align*}
a) & \quad q \cdot \mathcal{L}_B V = 0 \\
b) & \quad \partial_i A_{ij} = 0 \\
c) & \quad q(\partial_i B_j - A_{si}(\epsilon_j)s u \partial_s V) = 0 \\
d) & \quad \partial_i C + 2(H - q^2 V) A_{is} \partial_s V - 2 q^2 \epsilon_{ist} B_s \partial_t V = 0
\end{align*}
$$

We are now in position to explain why we assumed, in A2, that $q$ should not vanish. Indeed for $q = 0$ the relations (17a) and (17c) are trivially true and we are left with

$$
\begin{align*}
\partial_i A_{ij} = 0, & \quad \partial_i C + 2H A_{is} \partial_s V = 0,
\end{align*}
$$

while the conserved quantity $S$ reduces to

$$
S = A_{ij}(x) p_i p_j + C(x).
$$

It is interesting to notice that, formally, $S$ is preserved by the hamiltonian flow induced by the classical hamiltonian $[15]$

$$
\mathcal{H} = \frac{\vec{p}^2}{2} - H V,
$$

where now $H$ appears as some constant parameter. However the assumption that $q = 0$ leads to a reduced system which has only three degrees of freedom and as such may exhibit integrability. Since we are interested in genuine four dimensional integrability we have to exclude such a possibility.

Let us proceed to the discussion of the system (17). Relation (17a) shows that there are two possible situations:

1. Either the potential $V$ has one (or more) spatial symmetries, with Killing $K$, and then $B$ has to be conformal to this Killing vector.

2. Or the potential has no spatial symmetry, and in this case $B = 0$,

Let us show that this last possibility does not give any new conserved quantity. Indeed relation (17c) can be written

$$
[A, R] = 0, \quad (R)_{ij} = \epsilon_{isj} \partial_s V.
$$

Since $V$ has no Killing the matrix $R$ is a generic matrix in the Lie algebra $so(3)$. By Schur lemma it follows that $A$ has to be proportional to the identity matrix and this does trivialize the corresponding conserved quantity $S$.

So the unique possibility left is the first one. Let us notice that $K$ lifts up to an isometry of the 4 dimensional metric. We have obtained:

\footnote{The assumption A2 implies that $H - q^2 V$ does not vanish identically.}
**Proposition 2** The number of extra conserved quantities, having the structure (16), of a multi-centre metric is at most equal to the number of extra spatial Killing vectors it does possess, besides the tri-holomorphic Killing $\tilde{K} = \partial_t$.

Using this result we can discuss the triaxial generalization of the Eguchi-Hanson metric, with a tri-holomorphic $su(2)$, discovered in [1]. Its potential and cartesian coordinates were given in [8] and the potential has no spatial Killing. From the previous proposition it follows that this metric will not exhibit a conserved quantity of the form (16) for generic values of $H$ and $q \neq 0$.

### 2.4 Transformations of the system

As observed above, the vector $B$ has to be conformal to the Killing $K$. So we define the conformal factor $F$ such that

$$B_i = - F K_i.$$  \hfill (19)

The conserved quantity (16) becomes

$$S = A_{ij}(x) p_i p_j - 2 q F K_i p_i + C(x),$$  \hfill (20)

and equation (17c) transforms into

$$K(i \partial_j) F + A_{s(i} \epsilon_{j)su} \partial_u V = 0.$$  \hfill (21)

Taking its trace we see that $\mathcal{L}_K F = 0$, showing that $V$ and $F$ must have the same Killing.

**Lemma 1** The equation (21) has for consequence:

$$\left(dV \cdot dF\right) K + \star (A[dV] \wedge dV) = 0,$$

$$A[dV] = A_{is} \partial_s V \, dx^i.$$  \hfill (22)

We can proceed to:

**Proposition 3** The relation (21) is equivalent (except possibly at the points where the norm of the Killing $K$ vanishes) to the relations:

$$\begin{cases}
A[K] = a(x) \, K, & a) \\
|K|^2 dF - A[\star (K \wedge dV)] + \star (A[K] \wedge dV) = 0 & b)
\end{cases}$$  \hfill (23)

**Proof:** Contracting relation (21) with $K_j$ gives relation b), while contracting with $K_i K_j$ we have

$$\epsilon_{stu} K_s A[K]_t \partial_u V = 0 \quad \implies \quad A[K]_i = a(x) K_i + b(x) \partial_i V,$$

which is not relation a). To complete the argument we first contract relation (21) with $\epsilon_{iab} K_a$; after some algebra we get

$$K_j \epsilon_{jab} \partial_b F K_a + 2 A[K]_j \partial_b V + A[dV]_b K_j - K_s A[dV]_s \delta_{jb} - A_{ss} K_j \partial_b V = 0,$$

which, upon contraction with $A[K]_b$, gives eventually

$$\left(A[K]_s \partial_s V\right) A[K]_i = \{ - \epsilon_{stu} K_s A[K]_t \partial_u F + A_{ss} A[K]_t \partial_t V - A[K]_s A[dV]_s \} K_i.$$  \hfill (26)
Let us now suppose that \( A[K]_s \partial_s V \neq 0 \). The previous relation shows that in (24) we must have \( b(x) = 0 \), hence \( A[K]_s \partial_s V = 0 \) which is a contradiction.

Let us prove that the converse is true. From (23b) we get

\[
|K|^2 K (j \partial_s) V = (K \partial_s) F + (K \partial_s) \epsilon_{tsu} A[K]_s K \epsilon_{tsu} \partial_u V = 0.
\]

(27)

Use of the identity

\[
A_\ast K \epsilon_{tsu} \partial_u V = (|K|^2 A_\ast \epsilon_{tsu} - A[K]_s K \epsilon_{tsu}) \partial_u V
\]

and of relation (23a) leaves us with (21), up to division by \(|K|^2\). Notice that \(|K|^2\) vanishes at the fixed points under the Killing action, i.e. in subsets of zero measure in \( \mathbb{R}^3 \).

Lemma 2 The relation (23b) is equivalent to

\[
|K|^2 F + (2a - \text{Tr } A) \ast (K \partial_s V) + \ast (K \ast A[dV]) = 0.
\]

(30)

For further use let us prove:

Lemma 3 To the spatial Killing \( K \), leaving the potential \( V \) invariant, there corresponds a quantity \( Q \) invariant under the geodesic flow given by

\[
Q = K_i p_i + qG, \quad \text{with} \quad i(K) = -dG.
\]

(31)

**Proof**: We start from \( \mathcal{L}_K V = 0 \). Since \( K \) is a Killing we have \( \mathcal{L}_K (\ast dV) = \ast d(\mathcal{L}_K V) = 0 \), and (3) implies that \( \mathcal{L}_K d\Theta = 0 \). The closedness of \( d\Theta \) implies \( d(i(K)d\Theta) = 0 \), and since our analysis is purely local in \( \mathbb{R}^3 \), we can define

\[
\eta dG = -i(K) d\Theta, \quad \implies \quad \ast (K \ast dV) = dG.
\]

(32)

Then we multiply (10b) by \( p_i \) and get successively

\[
K_i \dot{p}_i = (K_i \dot{p}_i) - \dot{K}_i p_i = (K_i \dot{p}_i) = \frac{q}{V} K_i (\partial_s \Theta_\ast - \partial_\ast \Theta_i) p_s = -q \dot{x}_s \partial_s G = -q \dot{G},
\]

which concludes the proof.

Let us point out that if we use the coordinate \( \phi \) adapted to the Killing \( \tilde{K} = \partial_\phi \), we can write the connection \( \Theta = G d\phi \), where \( G \) does not depend on \( \phi \).

### 2.5 Integrability equations

We will derive now the integrability conditions for the equations (17c) and (17d). The first one was written using forms in (30) while the second one is

\[
dC + 2(H - q^2 V) A[dV] + 2q^2 F \ast (K \partial_s V) = 0.
\]

(33)

It can now be proved:
**Proposition 4** The integrability condition for (33) is
\[ dA[dV] = 0 \quad \Rightarrow \quad A[dV] = dU \quad \text{and} \quad \mathcal{L}U = 0. \quad (34) \]

**Proof:** The integrability condition is obtained by differentiating (33). We get
\[ 2(H - q^2V) dA[dV] + 2q^2 A[dV] \wedge dV + 2q^2 dF \wedge *(K \wedge dV) + 2q^2 F d*(K \wedge dV) = 0. \quad (35) \]
The last term in this equation vanishes in view of (32). Furthermore we have the identity specific to three dimensional spaces
\[ dF \wedge *\left(K \wedge dV\right) = -(K \cdot dF) \star dV + (dV \cdot dF) \star K = (dV \cdot dF) \star K \]
because \( K \) is a symmetry of \( F \). Relation (35) simplifies to
\[ 2(H - q^2V) dA[dV] + 2q^2 \star \left[(dV \cdot dF) K + \star(A[dV] \wedge dV)\right] = 0, \]
and lemma 1 implies the closedness of \( A[dV] \). Since our analysis is purely local, the existence of \( U \) is a consequence of Poincaré’s lemma.

The relations
\[ \mathcal{L}_K U = i(K) dU = i(K) A[dV] = (A[K] \cdot dV) = a(K \cdot dV) = a \mathcal{L}_K V = 0 \]
show the invariance of \( U \) under the Killing \( K \). □

Let us now turn to equation (30). We will prove:

**Proposition 5** The integrability condition for (30) is
\[ (2a - \text{Tr} A)dV + dU = |K|^2 \star d\tau, \quad \mathcal{L}_K d\tau = 0, \quad (36) \]
for some one form \( \tau \).

**Proof:** Let us define the 1-form
\[ Y = (2a - \text{Tr} A)dV + dU. \quad (37) \]
It allows to write (30) and its integrability condition as
\[ dF = -\star \left(\frac{K \wedge Y}{|K|^2}\right), \quad \delta \left(\frac{K \wedge Y}{|K|^2}\right) = 0, \quad (38) \]
or switching to components
\[ K_i \delta \left(\frac{Y}{|K|^2}\right) + \frac{Y_s \partial_s K_i - K_s \partial_s Y_i}{|K|^2} = 0. \quad (39) \]
Let us examine the last terms. Since \( a \) and \( \text{Tr} A \) are invariant under the Killing \( K \), we obtain
\[ Y_s \partial_s K_i - K_s \partial_s Y_i = -(2a - \text{Tr} A) \partial_i(K_s \partial_s V) - \partial_i(K_s \partial_s U) \quad (40) \]
and both terms vanish because \( V \) and \( U \) are invariant under \( K \). We are left with the vanishing of the divergence of \( Y/|K|^2 \) from which we conclude (local analysis!) that it must have the structure \( \star d\tau \) for some 1-form \( \tau \). From its definition it follows that \( d\tau \) is invariant under \( K \). □

Using this result we can simplify (30) to
\[ dF + \star(K \wedge \star d\tau) = dF - i(K)d\tau = 0. \quad (41) \]
Collecting all these results we have:
Proposition 6 Under the assumptions A1 and A2, the quantity

\[ S = A_{ij}(x)p_i p_j - 2 q F K_i p_i + C(x) \]

is preserved by the geodesic flow of the multi-centre metrics provided that the integrability constraints

\[ \Delta V = 0, \quad A[dV] = dU, \quad (2a - \text{Tr} A) dV + dU = |K|^2 \ast d\tau \] (42)

and the following relations hold:

a) \( \mathcal{L}_K V = 0 \),

b) \( \partial_{(k} A_{ij)} = 0 \), \( A[K] = a K \),

c) \( dF = i(K) d\tau \),

d) \( d(C + 2HU) + 2q^2(-V dU + F dG) = 0 \), \( \ast(K \wedge dV) = dG \).

2.6 Classification of the spatial Killing vectors

An important point, in view of classification, is whether the extra spatial Killing is tri-holomorphic or not. This can be checked thanks to:

Lemma 4 The spatial Killing vector \( K_i \partial_i \) is tri-holomorphic iff

\[ \epsilon_{ist} \partial_s K_t = 0. \]

Otherwise it is holomorphic.

Proof: From [2] we know that, for an hyperkähler geometry, a Killing may be either holomorphic or tri-holomorphic. As shown in [9] such a vector will be tri-holomorphic iff the differential of the dual 1-form \( K = K_i dx_i \) has the self-duality opposite to that of the complex structures. A computation shows that this is equivalent to the vanishing of

\[ dK^{(-)} = -\frac{1}{2} \epsilon_{ijk} \partial_{ij} K_{k]} \left( E_0 \wedge E_i - \frac{1}{2} \epsilon_{ist} E_s \wedge E_t \right), \]

from which the lemma follows. \( \blacksquare \)

Since we are working in a flat three dimensional flat space, there are essentially two different cases to consider:

1. The Killing \( K \) generates a spatial rotation, which we can take, without loss of generality, around the \( z \) axis. In this case we have

\[ K_i p_i = L_z \]

and this Killing vector is holomorphic with respect to the complex structure \( J_3 \), defined in section 2.

2. The Killing \( K \) generates a spatial translation, which we can take, without loss of generality, along the \( z \) axis. In this case we have the

\[ K_i p_i = p_z \]

and this Killing vector is tri-holomorphic.
We will discuss successively these two possibilities, under the simplifying additional assumption:

\textbf{A3 : } the K-S tensor $S_{\mu\nu}$ is also preserved by Lie dragging along the extra spatial Killing vector $K$

\[ \mathcal{L}_K S_{\mu\nu} = 0. \]

### 3 One extra holomorphic spatial Killing vector

The equation (43b) states that $A_{ij}$ is a Killing tensor in flat space. As shown in [12] such a Killing tensor is totally reducible to symmetrized tensor products of Killing vectors and involves 20 free parameters. It is most conveniently written in terms of $A(p,p) \equiv A_{ij} p^i p^j$.

One has:

\[ A(p,p) = \begin{cases} 
\alpha L_x^2 + \beta L_y^2 + \gamma L_z^2 + 2\mu L_y L_z + 2\nu L_z L_x + 2\lambda L_x L_y \\
+ a_1 p_x L_y + a_2 p_x L_z + b_1 p_y L_x + b_2 p_y L_z + c_1 p_z L_x + c_2 p_z L_y \\
+ d_1 p_x L_x + d_2 p_y L_y + a_{ij} p_i p_j.
\end{cases} \]  

(44)

The constraint (A 3) for the rotational Killing, requires $\mathcal{L}_K A_{ij} = 0$, which allows to bring (44) to the form

\[ A(p,p) = \alpha (L_x^2 + L_y^2) + \gamma L_z^2 + b (\vec{p} \wedge \vec{L})_z + a_{33} p_z^2 + a_{11} \vec{p}^2 + \delta p_z L_z. \]  

(45)

We note that the parameter $\gamma$ corresponds to a reducible piece which is just the square of $L_z$. We will take $\gamma = \alpha$ for convenience.

The parameter $a_{11}$ is easily seen, upon integration of the remaining equations in (17), to give rise, in the conserved quantity $S$, to the full piece

\[ a_{11} (\vec{p}^2 - 2HV + q^2V^2) \]  

(46)

which vanishes thanks to the energy conservation (12). So we can take $a_{11} = 0$.

The second relation in (43b) implies the vanishing of $\delta$. Hence, with slight changes in the notation, we end up with

\[ A(p,p) = a \vec{L}^2 + c^2 p_z^2 + b (\vec{p} \wedge \vec{L})_z. \]  

(47)

Let us note that the parameters $a$ and $b$ are real while the parameter $c$ may be either real or pure imaginary.

To take advantage of the rotational symmetry around the z axis we use the coordinates

\[ x = \sqrt{\rho} \cos \phi, \quad y = \sqrt{\rho} \sin \phi, \quad z, \]

and write the connection

\[ \Theta = G d\phi. \]

By lemma 3, this symmetry gives for conserved quantity

\[ J_z = L_z + q G = x \Pi_y - y \Pi_x. \]  

(48)
From the system (43) one can check that the functions $F$ and $U$ are to be determined from
\[
\begin{align*}
F_\rho &= (az + b/2)V_{rz} - a/2 V_z, \\
F_z &= 2(az^2 + bz - c^2)V_\rho - (az + b/2)V_z
\end{align*}
\tag{49}
\]
and
\[
\begin{align*}
U_\rho &= z(az + b)V_\rho - \frac{1}{2}(az + b/2)V_z \\
U_z &= -2\rho(az + b/2)V_\rho + (a\rho + c^2)V_z
\end{align*}
\tag{50}
\]

### 3.1 The two-centre metric

This case corresponds to the choice $a = 1$ and $c \neq 0$. Since $a = 1$, we can get rid of the constant $b$ by a translation of the variable $z$. So, without loss of generality, we can take $b = 0$ and use the new variables $r_{\pm} = \sqrt{x^2 + y^2 + (z \pm c)^2}$. We get the relations
\[
\partial_{r_+} F = -c \partial_{r_+} V, \quad \partial_{r_-} F = +c \partial_{r_-} V
\]
which imply
\[
V = f(r_+) + g(r_-), \quad F = -c(f(r_+) - g(r_-)).
\]
Imposing to the potential $V$ the Laplace equation we have
\[
V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}, \quad F = -c \left( \frac{m_1}{r_+} - \frac{m_2}{r_-} \right) = -c\Delta, \tag{51}
\]
\[i.e., \text{we recover the most general 2-centre metric. Let us recall that only the double Taub-NUT metric, given by real } m_1 = m_2, \text{ is complete. If in addition we take the limit } v_0 \to 0, \text{ we are led to the Eguchi-Hanson [3] metric.} \]

One has then to check the integrability constraint (34) and to determine the functions $U$ and $C$\footnote{We discard constant terms in the function $C$.}
\[
U = -cz\Delta, \quad C = -2(H - q^2 V)U - q^2 r^2 \Delta^2, \quad r^2 = x^2 + y^2 + z^2. \tag{52}
\]
Let us observe that the conserved quantity which we obtain may be real even if $c$ is pure imaginary. In this case $m_1 = m$ may be complex, but if we take $m_2 = m^*$ the functions $V$ and $c\Delta$ are real, as well as $S$. One obtains quite different metrics (as first observed in the particular case of Eguchi-Hanson metric): real $c$ corresponding to type II metric and $c$ pure imaginary for type I metric, in the terminology of [3].

The final form of the conserved quantity for the two-centre metric is therefore
\[
\begin{align*}
\mathcal{S}_I &= \bar{L}^2 + c^2 p_z^2 + 2qc \Delta L_z + 2cz \Delta (H - q^2 V) - q^2 r^2 \Delta^2, \\
V &= v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}, \quad \Delta = \frac{m_1}{r_+} - \frac{m_2}{r_-}, \quad G = m_1 \frac{z + c}{r_+} + m_2 \frac{z - c}{r_-}. \tag{53}
\end{align*}
\]
The relation of our results with the separability of the Hamilton-Jacobi equation for the two-centre metric, obtained in [9], will be discussed in the next section.

From the very definition of the coordinates $r_{\pm}$ it is clear that the previous analysis is only valid for $c \neq 0$. The special case $c = 0$ (it is a singular limit), giving a first dipolar breaking of the Taub-NUT metric, will be examined now.


3.2 First dipolar breaking of Taub-NUT

This case corresponds to the choice $a = 1$ and $c = 0$. Since $a = 1$, we can again get rid of the parameter $b$. Then relation (49) for $F$ implies

$$V = w_0(r) + w_1(r) z, \quad F_r = -r w_1(r), \quad r = \sqrt{x^2 + y^2 + z^2}. \quad (54)$$

Imposing the Laplace equation we obtain

$$V = v_0 + \frac{m}{r} + \mathcal{E} z + \mathcal{F} \frac{z}{r^3}, \quad F = -\frac{\mathcal{E}}{2} r^2 + \frac{\mathcal{F}}{r}. \quad (55)$$

The integrability relations for $U$ require that $\mathcal{E} = 0$ and we have

$$U = \mathcal{F} \frac{z}{r}, \quad C = -2 \mathcal{F} \frac{z}{r} (H - q^2 V) - 2mq^2 \mathcal{F} \frac{z}{r^2} - q^2 \mathcal{F}^2 \frac{(3z^2 - r^2)}{r^4}. \quad (56)$$

The final form of the conserved quantity is therefore

$$S_{II} = L^2 - 2 q \mathcal{F} \frac{z}{r} L_z - 2 \mathcal{F} \frac{z}{r} (H - q^2 v_0) + q^2 \mathcal{F}^2 \frac{(x^2 + y^2)}{r^4},$$

$$V = v_0 + \frac{m}{r} + \mathcal{F} \frac{z}{r^3}, \quad G = m \mathcal{F} \frac{z}{r} - \mathcal{F} \frac{x^2 + y^2}{r^3}. \quad (57)$$

Let us now consider the case $a = 0$, which leads to a second dipolar breaking of Taub-NUT.

3.3 Second dipolar breaking of Taub-NUT

This case corresponds to the choice $a = 0$ and $b = 1$. The relation (49) shows that by a translation of $z$ we can take, without loss of generality, $c = 0$. From the integrability of $F$ we deduce

$$V = \hat{f}(x^2 + y^2 + (z - c)^2) + g(z).$$

Hence by a translation of $z$ we can set $c$ to 0. We are left with

$$V = f(r) + g(z), \quad F = \frac{1}{2} (f(r) - g(z)). \quad (58)$$

Imposing Laplace equation yields

$$V = v_0 + \frac{m}{r} + \mathcal{E} z, \quad F = \frac{1}{2} \left( \frac{m}{r} - \mathcal{E} z \right) \quad (59)$$

Then the integrability conditions for $U$ and $C$ are satisfied and we obtain

$$U = \frac{m z}{2r} - \frac{\mathcal{E}}{4} (x^2 + y^2), \quad C = -2U (H - q^2 v_0) - 2q^2 m \mathcal{E} \frac{(x^2 + y^2)}{r}. \quad (60)$$

The final form of the conserved quantity is therefore

$$S_{III} = (\vec{p} \wedge \vec{L})_z - q \left( \frac{m}{r} - \mathcal{E} z \right) L_z - 2U (H - q^2 v_0) - 2q^2 m \mathcal{E} \frac{(x^2 + y^2)}{r},$$

$$V = v_0 + \frac{m}{r} + \mathcal{E} z, \quad G = m \frac{z}{r} + \frac{\mathcal{E}}{2} (x^2 + y^2). \quad (61)$$
For $E = 0$ we are back to the Taub-NUT metric. In this case the spatial isometries are lifted up from $u(1)$ to $su(2)$. As a result we have now three possible Killings to start with

$$K_i^{(1)} p_i = L_x, \quad K_i^{(2)} p_i = L_y, \quad K_i^{(3)} p_i = L_z$$

and we expect that the conserved quantity found above should be part of a triplet. The two missing conserved quantities can be constructed following the same route which led to $S_{III}$ using the new available spatial Killings given by (62). We recover

$$\vec{S} = \vec{p} \wedge \vec{L} - q \frac{m}{r} \vec{r} + m(q^2 v_0 - H) \frac{\vec{r}}{r}, \quad S_{III}(E = 0) \equiv S_z.$$  

Lemma 3 lifts up $J_z$, given by (48), to a triplet of conserved quantities

$$\vec{J} = \vec{L} + q \frac{m}{r} \vec{r},$$

which allows to write

$$\vec{S} = \vec{p} \wedge \vec{J} + m(q^2 v_0 - H) \frac{\vec{r}}{r},$$

on which we recognize the generalized Runge-Lenz vector discovered by Gibbons and Manton [7].

We have therefore obtained, for the three hamiltonians $H_I$, $H_{II}(\mathcal{F} \neq 0)$ and $H_{III}$, corresponding respectively to the extra conserved quantities $S_I$, $S_{II}$ and $S_{III}$, (the proof of their irreducibility with respect to the Killing vectors is easy) a set of four independent conserved quantities:

$$H, \quad q = \Pi_0, \quad J_z, \quad S,$$

which can be checked to be in involution with respect to the Poisson bracket.

Hence we conclude to:

**Proposition 7** The three hamiltonians $H_I$, $H_{II}(\mathcal{F} \neq 0)$ and $H_{III}$, defined above are integrable in Liouville sense.

### 4 One extra tri-holomorphic spatial Killing vector

This time we have for Killing $K_i p_i = p_z$. Imposing (A 3) for the translational invariance and the constraint $A[K] \propto K$ restricts $A(p, p)$ to have the form

$$A(p, p) = a L_z^2 - 2b p_x L_z + 2c p_y L_z + \sum_{i,j=1}^2 a_{ij} p_i p_j.$$  

We have omitted a term proportional to $p_z^2$ since it is reducible.

The functions $F$ and $U$, which depend only on the coordinates $x$ and $y$, using the system (43), are seen to be determined by

$$\begin{cases} 
F_x = A_{12} V_x - A_{11} V_y \\
F_y = A_{22} V_x - A_{12} V_y \\
U_x = A_{11} V_x + A_{12} V_y \\
U_y = A_{12} V_x + A_{22} V_y
\end{cases}$$

with

$$A_{11} = ay^2 + 2by + a_{11}, \quad A_{22} = ax^2 + 2cx + a_{22}, \quad A_{12} = -axy - bx - cy + a_{12}.$$  

In order to organize the subsequent discussion, let us observe:
1. For \( a \neq 0 \), we may take \( a = 1 \). The spatial translations in the xy-plane allow to take \( b = c = 0 \), and a rotation \( a_{12} = 0 \) as well. Hence we are left with

\[
\mathcal{A}(p, p) = L_z^2 + (a_{11} - a_{22}) p_x^2 + a_{22}(p_x^2 + p_y^2).
\]

Adding the reducible term \( a_{22} p_y^2 \) we recover the piece \( a_{22} \vec{p}^2 \) which can be discarded, as already explained in section 4. So we will take for our first case

\[
\mathcal{A}_1(p, p) = L_z^2 - c^2 p_x^2, \quad c \in \mathbb{R} \cup i\mathbb{R}, \quad c \neq 0.
\]

(69)

2. Our second case, which is the singular limit \( c \to 0 \) of the first case, corresponds to

\[
\mathcal{A}_2(p, p) = L_z^2.
\]

(70)

3. For \( a = 0 \), a first translation allows to take \( a_{12} = 0 \), while the second one allows the choice \( a_{11} = a_{22} \) and the corresponding term \( a_{11}(p_x^2 + p_y^2) \) is disposed of as in the first case. Eventually a rotation will bring \( b \) to zero and \( c \) to 1. Our third case will be

\[
\mathcal{A}_3(p, p) = p_y L_z.
\]

(71)

4. For \( a = b = c = 0 \), we can discard \( p_x^2 + p_y^2 \) and we are left with our fourth case

\[
\mathcal{A}_4(p, p) = \alpha p_y^2 + \beta p_x p_y.
\]

(72)

We will state the results obtained for these four cases without going through the detailed computations, which are greatly simplified by the use of the complex coordinate \( w = x + iy \). In all four cases the metric will have the form

\[
g = \frac{1}{V} (dt + \Theta)^2 + V(dz^2 + d\overline{w} dw), \quad \Theta = G dz.
\]

(73)

### 4.1 First case

Writing the conserved quantity as

\[
S_1 = L_z^2 - c^2 \Pi_x^2 - 2c^2 F \Pi_0 \Pi_z + c^2 (2v_0 U + D) \Pi_0^2 - 2c^2 U H, \quad c \neq 0,
\]

where \( \Pi_z = p_z + G \Pi_0 \) and

- \( V + iG = v_0 + 2m \frac{w}{\sqrt{w^2 + c^2}}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C} \)
- \( U + iF = -m \frac{w + \overline{w}}{\sqrt{w^2 + c^2}} \quad D = -2|m|^2 \frac{(w^2 + \overline{w}^2 + |w|^2 + c^2)}{|\sqrt{w^2 + c^2}|^2} \).

(75)

### 4.2 Second case

Writing the conserved quantity as

\[
S_2 = L_z^2 - 2F \Pi_0 \Pi_z + 2v_0 U \Pi_0^2 - 2U H,
\]

we have:

- \( V + iG = v_0 + \frac{m}{w^2}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C} \)
- \( U + iF = \frac{\overline{w}}{w} \).

(77)
4.3 Third case

Writing the conserved quantity as
\[ S_3 = \Pi_y L_z - 2F \Pi_0 \Pi_z + (2v_0U + D)\Pi_0^2 - 2UH, \]  
we have:
\[ \begin{align*}
&\bullet \quad V + iG = v_0 + 2 \frac{m}{\sqrt{w}}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C}, \\
&\bullet \quad U + iF = -\frac{m}{2} \frac{w - \bar{w}}{\sqrt{w}}, \quad D = |m|^2 \frac{w + \bar{w}}{|\sqrt{w}|^2}.
\end{align*} \]

4.4 Fourth case

In this case we take for the driving term
\[ A_4(p, p) = \alpha p_y^2 + \beta p_x p_y. \]
Using the freedom of rotations in the xy-plane, at the level of the metric, we can take
\[ V = v_0 + mx, \quad G = my. \]
This time there are two conserved quantities
\[ S_4 = \alpha S_4^{(1)} + \beta S_4^{(2)}, \]
given by
\[ \begin{align*}
S_4^{(1)} &= \Pi_y^2 + (\Pi_z - my \Pi_0)^2, \\
S_4^{(2)} &= \Pi_x \Pi_y - V \Pi_0 (\Pi_z - my \Pi_0) - my H.
\end{align*} \]
We added reducible terms of the form \( \Pi_z^2 \) and \( \Pi_z \Pi_0 \) to get a simpler final form.

The metric exhibits one further tri-holomorphic Killing vector and a corresponding conserved quantity
\[ \partial_y - mz \partial_t \quad \Rightarrow \quad \Pi_y - mz \Pi_0. \]
Let us close the algebra of the conserved quantities under Poisson bracket. For the Killing vectors we recover a Bianchi II Lie algebra
\[ \{ \Pi_0, \Pi_z \} = 0, \quad \{ \Pi_z, \Pi_y - mz \Pi_0 \} = m \Pi_0, \quad \{ \Pi_y - mz \Pi_0, \Pi_0 \} = 0. \]
The K-S tensors are invariant under the Killing vectors action, and it may be interesting to note that the Schouten bracket of the two K-S tensors is vanishing. This hamiltonian is therefore super-integrable.

To conclude this section let us notice that, among the four potentials considered, only the second one and the fourth one are uniform functions in the three dimensional flat space.

As was the case when the extra spatial Killing was holomorphic, we have obtained for the four hamiltonians considered in this section, a set of (at least) four conserved quantities
\[ H, \quad q = \Pi_0, \quad \Pi_z, \quad S, \]
and in all the four cases \( S \) is irreducible with respect to the Killing vectors. One can check that these four independent conserved quantities are in involution with respect to the Poisson bracket, hence we have:
Proposition 8 The four hamiltonians determined in this section are integrable in Liouville sense.

As is well known the existence of K-S tensors is related to the separability of the Hamilton-Jacobi (H-J) equation, or equivalently to the separability of the Schrödinger equation. In the next sections we will analyze the separability of the H-J equation according to the nature of the extra Killing vector.

5 H-J separability: extra holomorphic Killing

We write the metric
\[ g = \frac{1}{V} (dt + G \, d\phi)^2 + V (\gamma_1 \, d\xi_1^2 + \gamma_2 \, d\xi_2^2 + \gamma_3 \, d\phi^2), \] (83)
which makes apparent the two commuting Killing vectors \( \tilde{K} = \partial_t \) and \( \tilde{L} = \partial_\phi \), where only the first one is tri-holomorphic.

The hamiltonian is
\[ H = \frac{G^2 + \gamma_3 V^2}{2 \gamma_3 V} \Pi_0^2 - \frac{G}{\gamma_3 V} \Pi_0 \Pi_\phi + \frac{\Pi_\phi^2}{2 \gamma_3 V} + \frac{1}{2 V} \left( \frac{\Pi_1^2}{\gamma_1} + \frac{\Pi_2^2}{\gamma_2} \right). \] (84)

Since the \( \gamma_i \)'s depend only on \( \xi_1 \) and \( \xi_2 \), it follows that \( \Pi_0 \) and \( \Pi_\phi \) are conserved.

5.1 The two-centre case

The H-J equation separability was first used in [9] to get the corresponding K-S tensor. This reference is muddied by so many misprints that we will present its results anew.

Separability relies here on the use of spheroidal coordinates \( \xi_1 = \zeta, \xi_2 = \lambda \), defined by
\[ x = c \sqrt{(\zeta^2 - 1)(1 - \lambda^2)} \cos \phi, \quad y = c \sqrt{(\zeta^2 - 1)(1 - \lambda^2)} \sin \phi, \quad z = c \zeta \lambda. \]
This implies
\[ \gamma_1 = c^2 \frac{\zeta^2 - \lambda^2}{\zeta^2 - 1}, \quad \gamma_2 = c^2 \frac{\zeta^2 - \lambda^2}{1 - \lambda^2}, \quad \gamma_3 = c^2 (\zeta^2 - 1)(1 - \lambda^2). \]

The potential and connection are
\[ V = v_0 + \frac{\sigma \zeta - \delta \lambda}{c(\zeta^2 - \lambda^2)}, \quad G = \frac{\sigma \lambda (\zeta^2 - 1) + \delta \zeta (1 - \lambda^2)}{\zeta^2 - \lambda^2}, \] (85)
with \( \sigma = m_1 + m_2 \) and \( \delta = m_1 - m_2 \).

The hamiltonian is
\[ H = \frac{1}{2c^2 V} \left\{ \left( \frac{(\zeta^2 - 1)}{(\zeta^2 - \lambda^2)} \frac{\Pi_0^2 + (1 - \lambda^2) \Pi_\phi^2}{\Pi_1^2} \right) + \left( \frac{\Pi_0 - G \Pi_0}{\zeta^2 - 1}(1 - \lambda^2) \right)^2 \right\} + \frac{V}{2} \Pi_0^2. \] (86)
The separation constants\footnote{In all what follows each couple of separation constants add up to zero} are
\[ C_\zeta = (\zeta^2 - 1) \Pi_\zeta^2 + \frac{\Pi_\phi^2}{\zeta^2 - 1} - 2 \delta \frac{\zeta}{\zeta^2 - 1} \Pi_0 \Pi_\phi - 2c(v_0 c \zeta^2 + \sigma \zeta) H \]
\[ + \left( \frac{\delta^2}{\zeta^2 - 1} + v_0^2 c^2 \zeta^2 + 2v_0 c \sigma \zeta \right) \Pi_0^2, \tag{87} \]
and
\[ C_\lambda = (1 - \lambda^2) \Pi_\lambda^2 + \frac{\Pi_\phi^2}{1 - \lambda^2} - 2 \sigma \frac{\lambda}{1 - \lambda^2} \Pi_0 \Pi_\phi + 2c(v_0 c \lambda^2 + \delta \lambda) H \]
\[ + \left( \frac{\sigma^2}{1 - \lambda^2} - v_0^2 c^2 \lambda^2 - 2v_0 c \delta \lambda \right) \Pi_0^2. \tag{88} \]

The knowledge of these separation constants is of paramount importance since it reduces the integration of the H-J equation to quadratures. Indeed writing

\[ S = t \Pi_0 + \phi \Pi_\phi + A(\zeta) + B(\lambda), \]

one has just to replace \( \Pi_\zeta \) by \( \frac{dA}{d\zeta} \) in (87) and \( \Pi_\lambda \) by \( \frac{dB}{d\lambda} \) in (88) to get the relevant separated differential equations. In practice the final integrations may be quite tough.

Some algebra allows to relate the conserved quantity obtained in section 3 to the separation constants, with the final simple result

\[ S_I = C_\lambda - (\sigma^2 + \delta^2) \Pi_0^2. \tag{89} \]

In [9] it was conjectured that in the Taub-NUT limit \( c \to 0 \) this separation constant could be related to some component of the generalized Runge-Lenz vector. We can check that this is not true since, using relation (53), we get

\[ \lim_{c \to 0} S_I = \vec{L}^2 - \delta^2 \Pi_0^2. \tag{90} \]

5.2 First dipolar breaking

The H-J equation does separate in spherical coordinates \( \xi_1 = r, \xi_2 = \theta \), for which we have

\[ \gamma_1 = 1, \quad \gamma_2 = r^2, \quad \gamma_3 = r^2 \sin^2 \theta, \]

and

\[ V = v_0 + \frac{m}{r} + \frac{F \cos \theta}{r^2}, \quad G = m \cos \theta - \frac{F \sin^2 \theta}{r}. \tag{91} \]

The separation constants in the H-J equation are

\[ C_r = r^2 \Pi_r^2 + 2 \frac{F}{r} \Pi_0 \Pi_\phi + \left( v_0^2 r^2 + 2v_0 m r + \frac{F^2}{r^2} \right) \Pi_0^2 - 2(v_0 r^2 + m r) H, \tag{92} \]

and

\[ C_\theta = \Pi_0^2 + \frac{\Pi_\phi^2}{\sin^2 \theta} - 2 m \frac{\cos \theta}{\sin^2 \theta} \Pi_0 \Pi_\phi + \left( \frac{m^2}{\sin^2 \theta} + 2v_0 F \cos \theta \right) \Pi_0^2 - 2F \cos \theta H. \tag{93} \]

The relation with the K-S tensor of section 3 is \( S_{II} = C_\theta - m^2 \Pi_0^3 \).
5.3 Second dipolar breaking

The H-J equation does separate in parabolic coordinates \( \xi = \xi, \xi_2 = \eta \), for which we have

\[
\begin{align*}
\gamma_1 &= \frac{(\xi + \eta)}{4\xi}, & \gamma_2 &= \frac{(\xi + \eta)}{4\eta}, & \gamma_3 &= \xi \eta,
\end{align*}
\]

and

\[
V = v_0 + \frac{2m}{\xi + \eta} + \frac{\mathcal{E}}{2}(\xi - \eta), & \quad G = m \frac{\xi - \eta}{\xi + \eta} + \frac{\mathcal{E}}{2} \xi \eta.
\]

The separation constants in the H-J equation are

\[
C_\xi = 4\xi \Pi^2_\xi + \frac{\Pi^2_\phi}{\xi} + 2 \left( \frac{m}{\xi} - \frac{\mathcal{E}}{2} \xi \right) \Pi_0 \Pi_\phi - 2 \left( m + v_0 \xi + \frac{\mathcal{E}^2}{4} \xi^2 \right) H + \left( \frac{m^2}{\xi} + 2v_0m + (v_0^2 + 3m\mathcal{E})\xi + v_0\mathcal{E}\xi^2 + \frac{\mathcal{E}^2}{4}\xi^3 \right) \Pi^2_0,
\]

and

\[
C_\eta = 4\eta \Pi^2_\eta + \frac{\Pi^2_\phi}{\eta} - 2 \left( \frac{m}{\eta} + \frac{\mathcal{E}}{2} \eta \right) \Pi_0 \Pi_\phi - 2 \left( m + v_0 \eta - \frac{\mathcal{E}^2}{4} \eta^3 \right) H + \left( \frac{m^2}{\eta} + 2v_0m + (v_0^2 - 3m\mathcal{E})\eta - v_0\mathcal{E}\eta^2 + \frac{\mathcal{E}^2}{4}\eta^3 \right) \Pi^2_0.
\]

The relation with the K-S tensor of section 3 is \( S_{III} = -\frac{1}{2} C_\xi \).

Having settled the case of an extra holomorphic Killing vector let us now consider the case of an extra tri-holomorphic Killing vector.

6 H-J separability: extra tri-holomorphic Killing

We write the metric in the form

\[
g = \frac{1}{V} (dt + G \, dz)^2 + V \, (dz^2 + \gamma_1 \, d\xi_1^2 + \gamma_2 \, d\xi_2^2),
\]

where the coordinates \( \xi_1 \) and \( \xi_2 \) will be appropriate coordinates in the xy-plane which will ensure separability. The two commuting Killing vectors \( \mathcal{K} = \partial_t \) and \( \mathcal{L} = \partial_z \), both tri-holomorphic, are apparent.

The Hamiltonian is

\[
H = \frac{V^2 + G^2}{2V} \Pi^2_0 - \frac{G}{V} \Pi_0 \Pi_z + \frac{\Pi^2_z}{2V} + \frac{1}{2V} \left( \frac{\Pi^2_1}{\gamma_1} + \frac{\Pi^2_2}{\gamma_2} \right).
\]

It follows that \( \Pi_0 \) and \( \Pi_z \) are conserved.

6.1 First case

We use elliptic coordinates \( \xi_1 = \xi \) and \( \xi_2 = \eta \) in the xy-plane defined by

\[
x = \frac{1}{c} \sqrt{(\xi^2 - c^2)(c^2 - \eta^2)}, \quad y = \frac{1}{c} \xi \eta,
\]
For convenience, we will define
\[ \hat{\xi} = \xi \sqrt{\xi^2 - c^2}, \quad \hat{\eta} = \eta \sqrt{c^2 - \eta^2}. \]

The first case corresponds to
\[ \gamma_1 = \frac{\xi^2 - \eta^2}{\xi^2 - c^2}, \quad \gamma_2 = \frac{\xi^2 - \eta^2}{c^2 - \eta^2}, \quad V = v_0 + \frac{a\hat{\xi} + b\hat{\eta}}{\xi^2 - \eta^2}, \quad G = \frac{-b\hat{\xi} + a\hat{\eta}}{\xi^2 - \eta^2}. \]

The separation constants in the H-J equation are
\[ C_\xi = (\xi^2 - c^2)\Pi_\xi^2 + \left( v_0^2 \xi^2 + 2v_0a\hat{\xi} + (a^2 + b^2)(\xi^2 - c^2/2) \right) \Pi_0^2 \]
\[ + 2b\hat{\xi} \Pi_0 \Pi_z + \xi^2 \Pi_z^2 - 2(v_0 \xi^2 + a\hat{\xi}) H, \]
and
\[ C_\eta = (c^2 - \eta^2)\Pi_\eta^2 + (-v_0^2 \eta^2 + 2v_0b\hat{\eta} + (a^2 + b^2)(\eta^2 - c^2/2)) \Pi_0^2 \]
\[ - 2a\hat{\eta} \Pi_0 \Pi_z - \eta^2 \Pi_z^2 + 2(v_0 \eta^2 - b\hat{\eta}) H. \]

The relation with the K-S tensor obtained in section 4 is
\[ S_1 = -C_\xi + c^2(\Pi_z^2 + v_0^2 \Pi_0^2 - 2v_0 H). \]

### 6.2 Second case

We use polar coordinates \( \xi_1 = r, \xi_2 = \phi \) in the xy-plane. The second case corresponds to
\[ \gamma_1 = 1, \quad \gamma_2 = r^2, \quad V = v_0 + m \frac{\cos(2\phi)}{r^2}, \quad G = -m \frac{\sin(2\phi)}{r^2}. \]

The separation constants in the H-J equation are
\[ \begin{cases} 
C_r = r^2(\Pi_r^2 + \Pi_z^2) + \left( v_0^2 r^2 + \frac{m^2}{r^2} \right) \Pi_0^2 - 2v_0 r^2 H, \\
C_\phi = \Pi_\phi^2 + 2m \sin(2\phi) \Pi_0 \Pi_z + 2m \cos(2\phi) \left( v_0 \Pi_0^2 - H \right). 
\end{cases} \]

The relation with the K-S tensor obtained in section 4 is \( S_2 = C_\phi \).

### 6.3 Third case

We use squared parabolic coordinates \( \xi_1 = \xi, \xi_2 = \eta \) in the xy-plane. The third case corresponds to
\[ \gamma_1 = \gamma_2 = \xi^2 + \eta^2, \quad V = \frac{a\xi + b\eta}{\xi^2 + \eta^2}, \quad G = \frac{b\xi - a\eta}{\xi^2 + \eta^2}. \]

The separation constants in the H-J equation are
\[ \begin{cases} 
C_\xi = \Pi_\xi^2 + (\xi \Pi_z - b \Pi_0)^2 + \frac{1}{2}(a^2 - b^2) \Pi_0^2 - 2a\xi H, \\
C_\eta = \Pi_\eta^2 + (\eta \Pi_z + a\eta \Pi_0)^2 - \frac{1}{2}(a^2 - b^2) \Pi_0^2 - 2b\eta H. 
\end{cases} \]

The relation with the K-S tensor obtained in section 4 is \( S_3 = -\frac{1}{2} C_\xi \).
6.4 Fourth case

We use cartesian coordinates $\xi_1 = x$, $\xi_2 = y$ in the xy-plane. The fourth case corresponds to

$$\gamma_1 = \gamma_2 = 1, \quad V = v_0 + mx, \quad G = my.$$  \hspace{0.5cm} (107)

The separation constants in the H-J equation are

$$\begin{cases}
C_x = \Pi_x^2 + V^2\Pi_0^2 - 2VH, \\
C_y = \Pi_y^2 + (\Pi_z - my\Pi_0)^2.
\end{cases} \hspace{0.5cm} (108)$$

The relation with the K-S tensors obtained in section 4 is merely $S^{(1)}_4 = C_y$.

As a conclusion of these last two sections let us observe that the separable coordinates, known for the various potentials $V$, lift up, without any modification, to separable coordinates for the four dimensional system. Let us turn now to the Killing-Yano tensors.

7 Killing-Yano tensors

An antisymmetric tensor $Y_{\mu\nu}$ is a Killing-Yano (K-Y) tensor iff

$$\nabla_{(\mu} Y_{\nu)\rho} = 0.$$ \hspace{0.5cm} (109)

A complex structure is therefore a K-Y tensor.

The usefulness of such a concept is related to the fact that the symmetrized tensor product of two K-Y tensors does give a K-S tensor, as can be checked by an easy computation. Clearly the triplet of complex structures shared by the multi-centre metrics is not very useful since it gives only trivial K-S tensors so we need extra K-Y tensors. It is the aim of this section to give new examples of these extra K-Y tensors which will give some explicit K-S tensors which do not satisfy assumption (A 3).

We have been able to obtain K-Y tensors for

1. The special case of the second dipolar breaking, corresponding to $V = v_0 + \mathcal{E}z$.

2. The fourth case with an extra tri-holomorphic Killing vector, with potential $V = v_0 + mx$.

Let us consider successively these two cases.

7.1 Special second dipolar breaking

For $m = 0$ the metric simplifies to

$$g = \frac{1}{4V} (2dt - \mathcal{E}y dx + \mathcal{E}x dy)^2 + V(dx^2 + dy^2 + dz^2), \quad V = v_0 + \mathcal{E}z.$$ \hspace{0.5cm} (110)

We have four Killing vectors

$$\partial_t, \quad x \partial_y - y \partial_x, \quad \partial_x + \frac{\mathcal{E}y}{2} \partial_t, \quad \partial_y - \frac{\mathcal{E}x}{2} \partial_t.$$ \hspace{0.5cm} (111)
and the induced conserved quantities have simple Poisson brackets: $\Pi_0$ is central and for the remaining ones

$$\{J_z, p_x\} = p_y, \quad \{J_z, p_y\} = -p_x \quad \{p_x, p_y\} = \mathcal{E} \Pi_0. \quad (112)$$

with

$$J_z = x \Pi_y - y \Pi_x, \quad p_x = \Pi_x + \frac{\mathcal{E}y}{2} \Pi_0, \quad p_y = \Pi_y - \frac{\mathcal{E}x}{2} \Pi_0.$$

Using the canonical vierbein one gets for the K-Y two-form

$$Y = -\mathcal{E}^2 E_0 \wedge (x E_1 + y E_2) + \mathcal{E}^2 (x E_2 \wedge E_3 + y E_3 \wedge E_1) + 2\mathcal{E} V E_1 \wedge E_2. \quad (113)$$

From it and the complex structures we can construct four K-S tensors

$$Y^2, \quad S_i = Y \Omega_i^{(-)} + \Omega_i^{(-)} Y, \quad i = 1, 2, 3$$

We will quote the corresponding conserved quantities instead of the K-S tensors, for the ease of comparison with our earlier results:

$$\frac{Y^2}{\mathcal{E}^2} \rightarrow -4V(\Pi_x^2 + \Pi_y^2) + \mathcal{E}^2(x^2 + y^2)V \Pi_0^2 + 4\mathcal{E} \Pi_0 (x \Pi_x + y \Pi_y) - 2\mathcal{E}^2(x^2 + y^2)H,$$

$$S_1 \rightarrow 4\mathcal{E} V \Pi_0 p_y - 4\mathcal{E} \Pi_0 p_x + 4\mathcal{E}^2 x H,$$

$$S_2 \rightarrow -4\mathcal{E} V \Pi_0 p_x - 4\mathcal{E} \Pi_0 p_y + 4\mathcal{E}^2 y H,$$

$$S_3 \rightarrow 4\mathcal{E}(p_x^2 + p_y^2). \quad (114)$$

Let us observe that $S_3$ is reducible and that $S_1$ and $S_2$ do not satisfy (A3), so we are left with $Y^2$. Some algebra shows how it is related to the conserved quantity obtained in section 3:

$$S_{III}(m = 0) = -\frac{Y^2}{4 \mathcal{E}^3} - \frac{v_0}{4 \mathcal{E}^2} S_3 - v_0 \Pi_0 J_z, \quad (115)$$

so that, up to reducible terms, the two conserved quantities are one and the same. This case is quite similar to the Kerr metric (albeit much simpler) for which the Carter K-S tensor is in fact the square of some K-Y tensor.

### 7.2 The fourth case

Using the canonical vierbein one gets for the K-Y two-form

$$Y = -my \Omega_2^{(-)} - mz \Omega_3^{(-)} + 2V E_2 \wedge E_3. \quad (116)$$

Defining $p_z = \Pi_z - G \Pi_0$, we can write the induced conserved quantities:

$$\frac{Y^2}{4} \rightarrow -V \Pi_y^2 - V p_z^2 + my \Pi_x \Pi_y - my V \Pi_0 p_z + mz \Pi_x p_z + mz V \Pi_0 \Pi_y - \frac{m^2}{2}(y^2 + z^2)H,$$

$$S_1 \rightarrow \Pi_y^2 + p_z^2, \quad p_z = \Pi_z - G \Pi_0, \quad (117)$$

$$S_2 \rightarrow -\Pi_x \Pi_y + V \Pi_0 p_z + my H,$$

$$S_3 \rightarrow -\Pi_x p_z - V \Pi_0 \Pi_y + mz H.$$
We see that $S_1$ and $S_2$ were alredy obtained in section 4. The other two are missing since they don’t satisfy our assumption (A 3). Notice also that the conserved quantity $S^{(2)}_4$ cannot be obtained in that way.

So this example is of some interest since it shows that there do exist K-S tensors which do not satisfy the assumption (A 3). However, since the corresponding conserved quantities do not commute with $\Pi_z$, they are of no use to prove integrability.

8 Conclusion

We have settled the problem of finding all the multi-centre metrics which do exhibit some extra conserved quantity, having the structure (16), under the assumptions (A 1) to (A 3). Since it is induced by a KS tensor, this conserved quantity is quadratic with respect to the momenta, and preserved by the geodesic flow. As we have observed, the existence of this extra conserved quantity is essential to obtain integrability.

However one should keep present that our analysis does not cover all the integrable multi-centre metrics, since integrability could emerge from the existence of more complicated conserved quantities. In fact the concept of Killing-Stäckel tensor can be generalized to symmetric $(n,0)$ tensors with $n \geq 3$ such that

$$\nabla(\lambda S_{\mu_1\cdots\mu_n}) = 0.$$ 

It follows that the geodesic flow preserves the quantity

$$S_{\mu_1\cdots\mu_n} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_n}.$$ 

The corresponding invariants will be cubic, quartic, etc... with respect to the momenta. Little is known about the existence of such conserved quantities, which could produce possibly new integrable multi-centre metrics.

Let us conclude by putting some emphasis on the purely local nature of our analysis: it makes no difference between complete and non-complete metrics. For instance in section 4 we have seen that the most general two-centre metric is integrable, however it is complete only for real $m_1 = m_2$, i. e. for the double Taub-NUT metric.

References

[1] V. A. Belinskii, G. W. Gibbons, D. N. Page and C. N. Pope, Phys. Lett. B 76, 433-435 (1978).
[2] C. P. Boyer and J. D. Finley, J. Math. Phys. 23, 1126-1130 (1982).
[3] T. Eguchi and A.J. Hanson, Phys. Lett. B, 74, 249-251 (1978).
[4] L. G. Feher and P. A. Horváthy, Phys. Lett. B 183, 182-186 (1987).
[5] J. D. Gegenberg and A. Das, Gen. Rel. Grav. 16, 817-829 (1984).
[6] G. Gibbons, S. Hawking, Phys. Lett. B 78, 430-432 (1978).
[7] G. W. Gibbons and N. S. Manton, Nucl. Phys. B 274, 183-224 (1986).
[8] G. W. Gibbons, D. Olivier, P. J. Ruback and G. Valent, *Nucl. Phys.* **B 296**, 679-696 (1988).

[9] G. W. Gibbons and P. J. Ruback, *Commun. Math. Phys.* **115**, 267-300 (1988).

[10] N. Hitchin, *Math. Proc. Camb. Phil. Soc.* **85**, 465-476 (1979).

[11] N. Hitchin, “Monopoles, minimal surfaces and algebraic curves”, in NATO Advanced Study Institute nº 105 Presses Université de Montreal (Québec) Canada (1987).

[12] H. Katzin and J. Levine, *Tensor*, **16** (1965) 97.

[13] S. Kloster, M. Som and A. Das, *J. Math. Phys.* **15**, 1096-1102 (1974).

[14] S. Mignemi, *J. Math. Phys.*, **32**, 3047-3054 (1991).

[15] A. M. Perelomov, “Integrable systems of classical mechanics and Lie algebras”, Birkhäuser Verlag, Basel-Boston-Berlin (1990).

[16] K. P. Tod and R. S. Ward, *Proc. Roy. Soc. London* **A 368**, 411-427 (1979).