Nuclear scissors modes and hidden angular momenta

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The coupled dynamics of low lying modes and various giant resonances are studied with the help of the Wigner Function Moments method generalized to take into account spin degrees of freedom and pair correlations simultaneously. The method is based on Time Dependent Hartree-Fock-Bogoliubov equations. The model of the harmonic oscillator including spin-orbit potential plus quadrupole-quadrupole and spin-spin interactions is considered. New low lying spin dependent modes are analyzed. Special attention is paid to the scissors modes. A new source of nuclear magnetism, connected with counter-rotation of spins up and down around the symmetry axis (hidden angular momenta), is discovered. Its inclusion into the theory allows one to improve substantially the agreement with experimental data in the description of energies and transition probabilities of scissors modes.

I. INTRODUCTION

The idea of the possible existence of the collective motion in deformed nuclei similar to the scissors motion continues to attract the attention of physicists who extend it to various kinds of objects, not necessarily nuclei, (for example, magnetic traps, see the review by Heyde at al [1]) and invent new sorts of scissors, for example, the rotational oscillations of

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neutron skin against a proton-neutron core [2].

The nuclear scissors mode was predicted [3]–[6] as a counter-rotation of protons against neutrons in deformed nuclei. However, its collectivity turned out to be small. From RPA results which were in qualitative agreement with experiment, it was even questioned whether this mode is collective at all [7, 8]. Purely phenomenological models (such as, e.g., the two rotors model [9] and the sum rule approach [10]) did not clear up the situation in this respect. Finally in a very recent review [1] it is concluded that the scissors mode is ”weakly collective, but strong on the single-particle scale” and further: ”The weakly collective scissors mode excitation has become an ideal test of models – especially microscopic models – of nuclear vibrations. Most models are usually calibrated to reproduce properties of strongly collective excitations (e.g. of \( J^{\pi} = 2^+ \) or \( 3^- \) states, giant resonances, …). Weakly-collective phenomena, however, force the models to make genuine predictions and the fact that the transitions in question are strong on the single-particle scale makes it impossible to dismiss failures as a mere detail, especially in the light of the overwhelming experimental evidence for them in many nuclei [11, 12].”

The Wigner Function Moments (WFM) or phase space moments method turns out to be very useful in this situation. On the one hand it is a purely microscopic method, because it is based on the Time Dependent Hartree-Fock (TDHF) equation. On the other hand the method works with average values (moments) of operators which have a direct relation to the considered phenomenon and, thus, make a natural bridge with the macroscopic description. This makes it an ideal instrument to describe the basic characteristics (energies and excitation probabilities) of collective excitations such as, in particular, the scissors mode. Our investigations have shown that already the minimal set of collective variables, i.e. phase space moments up to quadratic order, is sufficient to reproduce the most important property of the scissors mode: its inevitable coexistence with the IsoVector Giant Quadrupole Resonance (IVGQR) implying a deformation of the Fermi surface.

Further developments of the WFM method, namely, the switch from TDHF to Time Dependent Hartree-Fock Bogoliubov (TDHFB) equations, i.e. taking into account pair correlations, allowed us to improve considerably the quantitative description of the scissors mode [13, 14]: for rare earth nuclei the energies were reproduced with \( \sim 10\% \) accuracy and \( B(M1) \) values were reduced by about a factor of two with respect to their non superfluid values. However, they remain about two times too high with respect to experiment. We
have suspected, that the reason of this last discrepancy is hidden in the spin degrees of freedom, which were so far ignored by the WFM method. One cannot exclude, that due to spin dependent interactions some part of the force of M1 transitions is shifted to the energy region of 5-10 MeV, where a 1+ resonance of spin nature is observed [7].

In a recent paper [15] the WFM method was applied for the first time to solve the TDHF equations including spin dynamics. As a first step, only the spin-orbit interaction was included in the consideration, as the most important one among all possible spin dependent interactions because it enters into the mean field. This allows one to understand the structure of necessary modifications of the method avoiding cumbersome calculations. The most remarkable result was the discovery of a new type of nuclear collective motion: rotational oscillations of ”spin-up” nucleons with respect of ”spin-down” nucleons (the spin scissors mode). It turns out that the experimentally observed group of peaks in the energy interval 2-4 MeV corresponds very likely to two different types of motion: the conventional (orbital) scissors mode and this new kind of mode, i.e. the spin scissors mode. The pictorial view of these two intermingled scissors is shown on Fig. 1, which is just the modification (or generalization) of the classical picture for the orbital scissors (see, for example, [1, 9]).

Three low lying excitations of a new nature were found: isovector and isoscalar spin scissors and the excitation generated by the relative motion of the orbital angular momentum and the spin of the nucleus (they can change their absolute values and directions keeping the total spin unchanged). In the frame of the same approach ten high lying excitations were also obtained: well known isoscalar and isovector Giant Quadrupole Resonances (GQR), two resonances of a new nature describing isoscalar and isovector quadrupole vibrations of ”spin-up” nucleons with respect of ”spin-down” nucleons, and six resonances which can be interpreted as spin flip modes of various kinds and multipolarity.

The next step was done in the paper [16], where the influence of the spin-spin interaction on the scissors modes was studied. There was hope that, due to spin dependent interactions, some part of the force of M1 transitions will be shifted to the area of a spin-flip resonance, decreasing in such a way the M1 force of scissors. However, these expectations were not realised. It turned out that the spin-spin interaction does not change the general picture of the positions of excitations described in [15] pushing all levels up proportionally to its strength without changing their order. The most interesting result concerns the $B(M1)$ values of both scissors – the spin-spin interaction strongly redistributes $M1$ strength in
Favour of the spin scissors mode practically without changing their summed strength. One of the main points of this work was, indeed, that we were able to give a tentative explanation of a recent experimental finding [17] where the $B(M1)$ values in $^{232}$Th of the two low lying magnetic states are inverted in strength in favour of the lowest, i.e., the spin scissors mode, when cranking up the spin-spin interaction. Indeed, the explanation with respect to a triaxial deformation given in [17] yields a stronger $B(M1)$ value for the higher lying state, contrary to observation, as remarked by the authors themselves.

In the last work [18] we suggested a generalization of the WFM method which takes into account spin degrees of freedom and pair correlations simultaneously. These two factors, working together, improve considerably the agreement between the theory and experiment in the description of nuclear scissors modes.

The paper is organized as follows. In Sec. 2 the TDHFB equations for the $2 \times 2$ normal and anomalous density matrices are formulated and their Wigner transform is found. In
II. WIGNER TRANSFORMATION OF TIME-DEPENDENT HARTREE-FOCK-BOGOLIUBOV EQUATIONS

The Time-Dependent Hartree–Fock–Bogoliubov (TDHFB) equations in matrix formulation are [19, 20]

\[ i\hbar \dot{\mathcal{R}} = [\mathcal{H}, \mathcal{R}] \tag{1} \]

with

\[ \mathcal{R} = \begin{pmatrix} \hat{\rho} & -\hat{\kappa} \\ -\hat{\kappa}^\dagger & 1 - \hat{\rho}^* \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{h}^* \end{pmatrix} \tag{2} \]

The normal density matrix \( \hat{\rho} \) and Hamiltonian \( \hat{h} \) are hermitian whereas the abnormal density \( \hat{\kappa} \) and the pairing gap \( \hat{\Delta} \) are skew symmetric: \( \hat{\kappa}^\dagger = -\hat{\kappa}^* \), \( \hat{\Delta}^\dagger = -\hat{\Delta}^* \).

The detailed form of the TDHFB equations is

\[ i\hbar \dot{\hat{\rho}} = \hat{h}\hat{\rho} - \hat{\rho}\hat{h} - \hat{\Delta}\hat{\kappa}^\dagger + \hat{\kappa}\hat{\Delta}^\dagger, \]

\[ -i\hbar \dot{\hat{\rho}}^* = \hat{h}^*\hat{\rho}^* - \hat{\rho}^*\hat{h}^* - \hat{\Delta}^\dagger\hat{\kappa} + \hat{\kappa}^\dagger\hat{\Delta}, \]

\[ -i\hbar \dot{\hat{\kappa}} = -\hat{h}\hat{\kappa} - \hat{\kappa}\hat{h}^* + \hat{\Delta} - \hat{\Delta}\hat{\rho}^* - \hat{\rho}\hat{\Delta}, \]

\[ -i\hbar \dot{\hat{\kappa}}^\dagger = \hat{h}^*\hat{\kappa}^\dagger + \hat{\kappa}^\dagger\hat{h} - \hat{\Delta}^\dagger + \hat{\Delta}^\dagger\hat{\rho} + \hat{\rho}^*\hat{\Delta}^\dagger. \tag{3} \]

It is easy to see that the second and fourth equations are complex conjugate to the first and third ones respectively. Let us consider their matrix form in coordinate space keeping all spin indices \( s, s', s'' \) [18]:

\[ i\hbar \langle \mathbf{r}, s | \dot{\hat{\rho}} | \mathbf{r}'', s'' \rangle = \sum_{s'} \int d^3r' \left( \langle \mathbf{r}, s | \hat{h} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\rho} | \mathbf{r}'', s'' \rangle - \langle \mathbf{r}, s | \hat{\rho} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{h} | \mathbf{r}'', s'' \rangle \right) \]
We do not specify the isospin indices in order to make formulae more transparent. They will be re-introduced at the end. Let us introduce the more compact notation \( \langle \mathbf{r}, s | \hat{X} | \mathbf{r}', s' \rangle = X^{ss'}_{rr'} \). Then the set of TDHFB equations (4) with specified spin indices reads

\[
\begin{align*}
\hbar \langle \mathbf{r}, s | \hat{\kappa}^\dagger | \mathbf{r}'', s'' \rangle &= -\langle \mathbf{r}, s | \hat{\Delta} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\kappa} | \mathbf{r}''', s''' \rangle + \langle \mathbf{r}, s | \hat{\kappa} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\Delta} | \mathbf{r}''', s''' \rangle, \\
\hbar \langle \mathbf{r}, s | \hat{\rho}^\dagger | \mathbf{r}'', s'' \rangle &= -\langle \mathbf{r}, s | \hat{\Delta} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\rho} | \mathbf{r}''', s''' \rangle + \langle \mathbf{r}, s | \hat{\rho} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\Delta} | \mathbf{r}''', s''' \rangle, \\
\hbar \langle \mathbf{r}, s | \hat{\kappa}^\dagger | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\rho} | \mathbf{r}''', s''' \rangle + \langle \mathbf{r}, s | \hat{\rho} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\kappa} | \mathbf{r}''', s''' \rangle &= \int d^3r' \left( \langle \mathbf{r}, s | \hat{\rho}^\dagger | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\kappa} | \mathbf{r}''', s''' \rangle - \langle \mathbf{r}, s | \hat{\kappa}^\dagger | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\rho} | \mathbf{r}''', s''' \rangle \right) + \langle \mathbf{r}, s | \hat{\rho} \hat{\kappa} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\rho} \hat{\kappa} | \mathbf{r}''', s''' \rangle + \langle \mathbf{r}, s | \hat{\kappa} \hat{\rho} | \mathbf{r}', s' \rangle \langle \mathbf{r}', s' | \hat{\kappa} \hat{\rho} | \mathbf{r}''', s''' \rangle.
\end{align*}
\]

This set of equations must be complemented by the complex conjugated equations. Writing these equations, we neglected the diagonal matrix elements in spin, \( \kappa^{ss'}_{rr} \) and \( \Delta^{ss'}_{rr} \). It is shown in Appendix A that such approximation works very well in the case of monopole pairing considered here.

We will work with the Wigner transform [20] of equations (5). The relevant mathematical details can be found in [13]. The most essential relations are outlined in Appendix B. Let us remind of some essential details of the Wigner transform of equations (5) on the example of the first of these equations. Its left hand side is transformed with the help of formula (B1) without any approximations, i.e. exactly. The right hand side of this equation contains the
products of two matrices which are transformed with the help of formula (B4), where the
exponent represents an infinite series of terms with increasing powers of \( \hbar \). It was shown
in [13, 14] that after integration of the obtained equation over the phase space with second
order weights \( x_i x_j, x_i p_j, p_i p_j \) only terms proportional to powers in \( \hbar \) less than 2 survive. That
is why we will write out only these terms. From now on, we will not write out the coordinate
dependence \((r, p)\) of all functions in order to make the formulae more transparent. We have

\[
\begin{align*}
\hbar \dot{f}^{\uparrow \uparrow} &= \hbar \{h^{\uparrow \uparrow}, f^{\uparrow \uparrow}\} + h^{\uparrow \uparrow} f^{\uparrow \uparrow} - f^{\uparrow \downarrow} h^{\uparrow \uparrow} + \frac{i \hbar}{2} \{h^{\uparrow \uparrow}, f^{\uparrow \uparrow}\} - \frac{i \hbar}{2} \{f^{\uparrow \uparrow}, h^{\uparrow \uparrow}\} \\
&= -\frac{\hbar^2}{8} \{\{h^{\uparrow \uparrow}, f^{\uparrow \uparrow}\} + \frac{\hbar^2}{8} \{\{f^{\uparrow \downarrow}, h^{\uparrow \uparrow}\} + \kappa \Delta^* - \Delta \kappa^* \\
&+ \frac{i \hbar}{2} \{\kappa, \Delta^*\} - \frac{i \hbar}{2} \{\Delta, \kappa^*\} - \frac{\hbar^2}{8} \{\{\kappa, \Delta^*\} + \frac{\hbar^2}{8} \{\{\Delta, \kappa^*\} + \ldots,
\end{align*}
\]

\[
\begin{align*}
\hbar \dot{f}^{\downarrow \downarrow} &= \hbar \{h^{\downarrow \downarrow}, f^{\downarrow \downarrow}\} + h^{\downarrow \downarrow} f^{\downarrow \downarrow} - f^{\uparrow \downarrow} h^{\downarrow \downarrow} + \frac{i \hbar}{2} \{h^{\downarrow \downarrow}, f^{\downarrow \downarrow}\} - \frac{i \hbar}{2} \{f^{\uparrow \downarrow}, h^{\downarrow \downarrow}\} \\
&= -\frac{\hbar^2}{8} \{\{h^{\downarrow \downarrow}, f^{\downarrow \downarrow}\} + \frac{\hbar^2}{8} \{\{f^{\downarrow \downarrow}, h^{\downarrow \downarrow}\} + \bar{\Delta}^* \bar{\kappa} - \bar{\kappa}^* \bar{\Delta} \\
&+ \frac{i \hbar}{2} \{\bar{\Delta}^*, \bar{\kappa}\} - \frac{i \hbar}{2} \{\bar{\kappa}^*, \bar{\Delta}\} - \frac{\hbar^2}{8} \{\{\bar{\Delta}^*, \bar{\kappa}\} + \frac{\hbar^2}{8} \{\{\bar{\kappa}^*, \bar{\Delta}\} + \ldots,
\end{align*}
\]

\[
\begin{align*}
\hbar \dot{h}^{\uparrow \uparrow} &= \frac{\hbar^2}{8} \{(h^{\uparrow \uparrow} + h^{\downarrow \downarrow}) f^{\uparrow \downarrow} - \hbar^{\uparrow \uparrow} f^{\downarrow \downarrow} - \frac{\hbar^2}{8} \{\{h^{\uparrow \uparrow} + h^{\downarrow \downarrow}\}, f^{\uparrow \downarrow}\} \\
&- h^{\uparrow \downarrow} (f^{\uparrow \downarrow} + f^{\uparrow \uparrow}) + \frac{i \hbar}{2} \{h^{\uparrow \downarrow}, f^{\uparrow \uparrow}\} + \frac{\hbar^2}{8} \{\{h^{\uparrow \downarrow}, f^{\uparrow \uparrow}\} + \ldots,
\end{align*}
\]

\[
\begin{align*}
\hbar \dot{h}^{\downarrow \downarrow} &= \frac{\hbar^2}{8} \{(h^{\downarrow \downarrow} + h^{\uparrow \uparrow}) f^{\downarrow \downarrow} - \hbar^{\downarrow \downarrow} f^{\uparrow \downarrow} - \frac{\hbar^2}{8} \{\{h^{\downarrow \downarrow} + h^{\uparrow \uparrow}\}, f^{\downarrow \downarrow}\} \\
&- h^{\uparrow \downarrow} (f^{\downarrow \downarrow} + f^{\uparrow \downarrow}) + \frac{i \hbar}{2} \{h^{\uparrow \downarrow}, f^{\uparrow \downarrow}\} + \frac{\hbar^2}{8} \{\{h^{\uparrow \downarrow}, f^{\uparrow \downarrow}\} + \ldots,
\end{align*}
\]

\[
\begin{align*}
\hbar \dot{\kappa} &= \kappa (h^{\uparrow \uparrow} + h^{\downarrow \downarrow}) + \frac{i \hbar}{2} \{(h^{\uparrow \uparrow} - h^{\downarrow \downarrow}), \kappa\} - \frac{\hbar^2}{8} \{\{h^{\uparrow \uparrow} + h^{\downarrow \downarrow}\}, \kappa\} \\
&+ \Delta (f^{\uparrow \uparrow} + f^{\downarrow \downarrow}) + \frac{i \hbar}{2} \{(f^{\uparrow \uparrow} - f^{\uparrow \downarrow}), \Delta\} - \frac{\hbar^2}{8} \{\{(f^{\uparrow \uparrow} + f^{\downarrow \downarrow}), \Delta\} - \Delta + \ldots,
\end{align*}
\]

\[
\begin{align*}
\hbar \dot{\kappa}^* &= -\kappa^* (h^{\uparrow \uparrow} + h^{\downarrow \downarrow}) + \frac{i \hbar}{2} \{(h^{\uparrow \uparrow} - h^{\downarrow \downarrow}), \kappa^*\} + \frac{\hbar^2}{8} \{\{(h^{\uparrow \uparrow} + h^{\downarrow \downarrow}), \kappa^*\} \\
&- \Delta^* (f^{\uparrow \uparrow} + f^{\downarrow \downarrow}) + \frac{i \hbar}{2} \{(f^{\uparrow \uparrow} - f^{\uparrow \downarrow}), \Delta^*\} + \frac{\hbar^2}{8} \{\{(f^{\uparrow \uparrow} + f^{\downarrow \downarrow}), \Delta^*\} + \Delta^* + \ldots,
\end{align*}
\]
dynamical equations for $\bar{f}^{\uparrow\downarrow}$, $\bar{f}^{\downarrow\uparrow}$, $\bar{f}^{\downarrow\downarrow}$, $\bar{\kappa}$, $\bar{\kappa}^*$. They are obtained by the change $p \rightarrow -p$ in arguments of functions and Poisson brackets. So, in reality we deal with the set of twelve equations. We introduced the notation $\kappa \equiv \kappa^{\uparrow\downarrow}$ and $\Delta \equiv \Delta^{\uparrow\downarrow}$. Symmetry properties of matrices $\hat{\kappa}$, $\hat{\Delta}$ and the properties of their Wigner transforms (see Appendix B) allow one to replace the functions $\kappa^{\uparrow\downarrow}(r, p)$ and $\Delta^{\uparrow\downarrow}(r, p)$ by the functions $\bar{\kappa}^{\uparrow\downarrow}(r, p)$ and $\bar{\Delta}^{\uparrow\downarrow}(r, p)$.

Following the paper [15] we will write above equations in terms of spin-scalar

$$ f^+ = f^{\uparrow\uparrow} + f^{\downarrow\downarrow} $$

and spin-vector

$$ f^- = f^{\uparrow\downarrow} - f^{\downarrow\uparrow} $$

functions. Furthermore, it is useful to rewrite the obtained equations in terms of even and odd functions $f_e = \frac{1}{2}(f + \bar{f})$ and $f_o = \frac{1}{2}(f - \bar{f})$ and real and imaginary parts of $\kappa$ and $\Delta$: $\kappa^r = \frac{1}{2}(\kappa + \kappa^*)$, $\kappa^i = \frac{1}{2i}(\kappa - \kappa^*)$, $\Delta^r = \frac{1}{2}(\Delta + \Delta^*)$, $\Delta^i = \frac{1}{2i}(\Delta - \Delta^*)$. We have

$$ ih\dot{f}_e^+ = \frac{ih}{2} \left( \{h_o^+, f_e^+\} + \{h_e^+, f_o^+\} + \{h_o^-, f_e^-\} \right) $$

$$ + ih \left( \{h_o^{\uparrow\downarrow}, f_e^{\uparrow\downarrow}\} + \{h_e^{\uparrow\downarrow}, f_o^{\uparrow\downarrow}\} + \{h_e^{\downarrow\uparrow}, f_o^{\downarrow\uparrow}\} \right) $$

$$ + 4i \left( [\kappa_e^r, \Delta_e^r] - [\kappa_e^i, \Delta_e^i] + [\kappa_o^r, \Delta_o^r] - [\kappa_o^i, \Delta_o^i] \right) + \ldots, $$

$$ ih\dot{f}_o^+ = \frac{ih}{2} \left( \{h_o^+, f_e^+\} + \{h_e^+, f_o^+\} + \{h_o^-, f_e^-\} \right) $$

$$ + ih \left( \{h_o^{\uparrow\downarrow}, f_e^{\uparrow\downarrow}\} + \{h_e^{\uparrow\downarrow}, f_o^{\uparrow\downarrow}\} + \{h_e^{\downarrow\uparrow}, f_o^{\downarrow\uparrow}\} \right) $$

$$ + 2ih \left( \{\kappa_e^r, \Delta_e^r\} + \{\kappa_e^i, \Delta_e^i\} + \{\kappa_o^r, \Delta_o^r\} + \{\kappa_o^i, \Delta_o^i\} \right) + \ldots, $$

$$ ih\dot{f}_e^- = \frac{2}{2} \left( \{h_o^+, f_e^+\} + \{h_e^+, f_o^+\} - \{h_o^-, f_e^-\} - \{h_e^-, f_o^-\} \right) $$

$$ + \frac{ih}{2} \left( \{h_o^+, f_e^-\} + \{h_e^+, f_o^-\} + \{h_o^-, f_e^+\} \right) $$

$$ + 2ih \left( \{\kappa_e^r, \Delta_e^r\} + \{\kappa_e^i, \Delta_e^i\} + \{\kappa_o^r, \Delta_o^r\} + \{\kappa_o^i, \Delta_o^i\} \right) + \ldots, $$

$$ ih\dot{f}_o^- = \frac{2}{2} \left( \{h_o^+, f_e^+\} + \{h_e^+, f_o^+\} - \{h_o^-, f_e^-\} - \{h_e^-, f_o^-\} \right) $$

$$ + \frac{ih}{2} \left( \{h_o^+, f_e^-\} + \{h_e^+, f_o^-\} + \{h_o^-, f_e^+\} \right) $$

$$ + 4i \left( [\kappa_e^r, \Delta_e^r] - [\kappa_e^i, \Delta_e^i] + [\kappa_o^r, \Delta_o^r] - [\kappa_o^i, \Delta_o^i] \right) + \ldots, $$

$$ ih\dot{f}_e^{\uparrow\downarrow} = [h_e^-, f_o^{\uparrow\downarrow}] + [h_o^-, f_e^{\uparrow\downarrow}] - [h_e^+, f_o^{-\downarrow}] - [h_o^+, f_e^{-\downarrow}] $$

$$ + \frac{ih}{2} \left( \{h_e^+, f_o^{\uparrow\downarrow}\} + \{h_o^+, f_e^{\uparrow\downarrow}\} + \{h_e^-, f_o^{-\downarrow}\} + \{h_o^-, f_e^{-\downarrow}\} \right) + \ldots, $$

$$ ih\dot{f}_o^{\downarrow\uparrow} = [h_e^-, f_o^{\downarrow\uparrow}] - [h_o^-, f_e^{\downarrow\uparrow}] + [h_e^+, f_o^{-\uparrow}] + [h_o^+, f_e^{-\uparrow}] $$

$$ + \frac{ih}{2} \left( \{h_e^+, f_o^{\downarrow\uparrow}\} + \{h_o^+, f_e^{\downarrow\uparrow}\} + \{h_e^-, f_o^{-\uparrow}\} + \{h_o^-, f_e^{-\uparrow}\} \right) + \ldots, $$
\[ i \hbar f_\alpha^{\uparrow \downarrow} = [h_\alpha^{\uparrow \downarrow} f_\alpha^{\uparrow \downarrow}] + [h_\alpha^{\downarrow \uparrow} f_\alpha^{\downarrow \uparrow}] - [h_\alpha^{\uparrow \downarrow} f_\alpha^{\downarrow \uparrow}] - [h_\alpha^{\downarrow \uparrow} f_\alpha^{\downarrow \uparrow}] + \frac{i \hbar}{2} \left( \{h_\alpha^{\uparrow \downarrow}, f_\alpha^{\uparrow \downarrow}\} + \{h_\alpha^{\downarrow \uparrow}, f_\alpha^{\downarrow \uparrow}\} + \{h_\alpha^{\uparrow \downarrow}, f_\alpha^{\downarrow \uparrow}\} + \{h_\alpha^{\downarrow \uparrow}, f_\alpha^{\downarrow \uparrow}\} \right) + \ldots, \]

\[ i \hbar f_\alpha^{\downarrow \uparrow} = -[h_\alpha^{\downarrow \uparrow} f_\alpha^{\downarrow \uparrow}] - [h_\alpha^{\uparrow \downarrow} f_\alpha^{\uparrow \downarrow}] + [h_\alpha^{\uparrow \downarrow} f_\alpha^{\downarrow \uparrow}] + [h_\alpha^{\downarrow \uparrow} f_\alpha^{\downarrow \uparrow}] + \frac{i \hbar}{2} \left( \{h_\alpha^{\uparrow \downarrow}, f_\alpha^{\uparrow \downarrow}\} + \{h_\alpha^{\downarrow \uparrow}, f_\alpha^{\downarrow \uparrow}\} + \{h_\alpha^{\uparrow \downarrow}, f_\alpha^{\downarrow \uparrow}\} + \{h_\alpha^{\downarrow \uparrow}, f_\alpha^{\downarrow \uparrow}\} \right) + \ldots, \]

The following notation is introduced here: \( h^\pm = h^{\uparrow \downarrow} \pm h^{\downarrow \uparrow}, [ab] = ab - \frac{h^2}{8} \{\{a, b\}\}. \)

These twelve equations will be solved by the method of moments in a small amplitude approximation. To this end all functions \( f(\mathbf{r}, \mathbf{p}, t) \) and \( \kappa(\mathbf{r}, \mathbf{p}, t) \) are divided into an equilibrium part and a deviation (variation): \( f(\mathbf{r}, \mathbf{p}, t) = f(\mathbf{r}, \mathbf{p})_{\text{eq}} + \delta f(\mathbf{r}, \mathbf{p}, t), \kappa(\mathbf{r}, \mathbf{p}, t) = \kappa(\mathbf{r}, \mathbf{p})_{\text{eq}} + \delta \kappa(\mathbf{r}, \mathbf{p}, t). \) Then equations are linearized neglecting quadratic in variations terms.

From general arguments one can expect that the phase of \( \Delta \) (and of \( \kappa \), since both are linked, according to equation (22)) is much more flexible than its magnitude, since the former determines the superfluid velocity. After linearization, the phase of \( \Delta \) (and of \( \kappa \)) is expressed by \( \delta \Delta^i \) (and \( \delta \kappa^i \)), while \( \delta \Delta^r \) (and \( \delta \kappa^r \)) describes oscillations of the magnitude of \( \Delta \) (and of \( \kappa \)). Let us therefore assume that

\[ \delta \kappa^r(\mathbf{r}, \mathbf{p}) \ll \delta \kappa^i(\mathbf{r}, \mathbf{p}). \] (8)

This assumption was explicitly confirmed in [21] for the case of superfluid trapped fermionic atoms, where it was shown that \( \delta \Delta^r \) is suppressed with respect to \( \delta \Delta^i \) by one order of \( \Delta/E_F \), where \( E_F \) denotes the Fermi energy.

The assumption (8) allows one to neglect all terms containing the variations \( \delta \kappa^r \) and \( \delta \Delta^r \) in the equations (7) after their linearization. In this case the "small" variations \( \delta \kappa^r \) and \( \delta \Delta^r \) will not affect the dynamics of the "big" variations \( \delta \kappa^i \) and \( \delta \Delta^i \). This means that the
dynamical equations for the ”big” variations can be considered independently from that of the ”small” variations, and we will finally deal with a set of only ten equations.

III. MODEL HAMILTONIAN

The microscopic Hamiltonian of the model, harmonic oscillator with spin orbit potential plus separable quadrupole-quadrupole and spin-spin interactions is given by

\[ H = \sum_{i=1}^{A} \left[ \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega_i^2 \mathbf{r}_i^2 - \eta \hat{l}_i \hat{S}_i \right] + H_{qq} + H_{ss} \]  

(9)

with

\[ H_{qq} = \sum_{\mu=-2}^{2} (-1)^\mu \left\{ i \sum_{i,j} Z_i N_j + \frac{\kappa}{2} \left[ \sum_{i,j(i\neq j)} Z_i + \sum_{i,j(i\neq j)} N_i \right] \right\} q_{2-\mu}(r_i)q_{2\mu}(r_j), \]  

(10)

\[ H_{ss} = \sum_{\mu=-1}^{1} (-1)^\mu \left\{ \chi \sum_{i,j} Z_i N_j + \frac{\chi}{2} \left[ \sum_{i,j(i\neq j)} Z_i + \sum_{i,j(i\neq j)} N_i \right] \right\} \hat{S}_{-\mu}(i) \hat{S}_{\mu}(j) \delta(r_i - r_j), \]  

(11)

where \( N \) and \( Z \) are the numbers of neutrons and protons and \( \hat{S}_\mu \) are spin matrices [22]:

\[ \hat{S}_1 = -\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{S}_0 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{S}_{-1} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]  

(12)

A. Mean Field

Let us analyze the mean field generated by this Hamiltonian.

1. Spin-orbit Potential

Written in cyclic coordinates, the spin orbit part of the Hamiltonian reads

\[ \hat{h}_{ls} = -\eta \sum_{\mu=-1}^{1} (-)^{\mu} \hat{l}_\mu \hat{S}_{-\mu} = -\eta \left( \hat{l}_{\frac{\mu}{2}} \hat{S}_{-\frac{\mu}{2}} - \hat{l}_{-\frac{\mu}{2}} \hat{S}_{\frac{\mu}{2}} \right), \]

where [22]

\[ \hat{l}_\mu = -\hbar \sqrt{2} \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{\lambda\mu} r_\nu \nabla_\alpha, \]  

(13)

cyclic coordinates \( r_{-1}, r_0, r_1 \) are also defined in [22], \( C_{1\sigma,1\nu}^{\lambda\mu} \) is a Clebsch-Gordan coefficient, and

\[ \hat{l}_1 = \hbar (r_0 \nabla_1 - r_1 \nabla_0) = -\frac{1}{\sqrt{2}} (\hat{l}_x + i\hat{l}_y), \]
\[ \hat{t}_0 = \hbar (r_{-1} \nabla_1 - r_1 \nabla_{-1}) = \hat{t}_z, \]
\[ \hat{t}_{-1} = \hbar (r_{-1} \nabla_0 - r_0 \nabla_{-1}) = \frac{1}{\sqrt{2}} (\hat{t}_x - i \hat{t}_y), \]
\[ \hat{t}_x = -i \hbar (y \nabla_z - z \nabla_y), \quad \hat{t}_y = -i \hbar (z \nabla_x - x \nabla_z), \]
\[ \hat{t}_z = -i \hbar (x \nabla_y - y \nabla_x). \] (14)

Matrix elements of \( \hat{h}_{ls} \) in coordinate space can obviously be written [15] as
\[
\langle r_1, s_1 | \hat{h}_{ls} | r_2, s_2 \rangle = -\frac{\hbar}{2} \eta \left[ \hat{t}_0(r_1) (\delta_{s_1 \uparrow \delta_{s_2 \uparrow}} - \delta_{s_1 \downarrow \delta_{s_2 \downarrow}}) + \sqrt{2} \hat{t}_{-1}(r_1) \delta_{s_1 \uparrow \delta_{s_2 \downarrow}} - \sqrt{2} \hat{t}_1(r_1) \delta_{s_1 \downarrow \delta_{s_2 \uparrow}} \right] \delta(r_1 - r_2). \] (15)

Their Wigner transform reads [15]:
\[
\hat{h}_{ls}^{s_1 s_2}(r, p) = -\frac{\hbar}{2} \eta \left[ \hat{t}_0(r, p) (\delta_{s_1 \uparrow \delta_{s_2 \uparrow}} - \delta_{s_1 \downarrow \delta_{s_2 \downarrow}}) + \sqrt{2} \hat{t}_{-1}(r, p) \delta_{s_1 \uparrow \delta_{s_2 \downarrow}} - \sqrt{2} \hat{t}_1(r, p) \delta_{s_1 \downarrow \delta_{s_2 \uparrow}} \right], \] (16)

where \( l_\mu = -i \sqrt{2} \sum_{\nu, \alpha} C_{l \nu, 1 \alpha} r_{\nu} p_{\alpha} \).

### 2. Quadrupole-quadrupole interaction

The contribution of \( H_{qq} \) to the mean field potential is easily found by replacing one of the \( q_{2\mu} \) operators by the average value. We have
\[
V^{\tau}_{qq} = \sqrt{6} \sum_\mu (-1)^\mu Z^{\tau+}_{2-\mu} q_{2\mu}. \] (17)

Here
\[
Z^{\tau+}_{2\mu} = \kappa R^{\tau+}_{2\mu} + \bar{\kappa} R^{\tau+}_{2\mu}, \quad Z^{\tau+}_{2\mu} = \kappa R^{\tau+}_{2\mu} + \bar{\kappa} R^{\tau+}_{2\mu},
\]
\[
R^{\tau+}_{2\mu}(t) = \frac{1}{\sqrt{6}} \int d(p, r) q_{2\mu}(r) f^{\tau+}(r, p, t) \] (18)

with \( \int d(p, r) \equiv (2\pi \hbar)^{-3} \int d^3 p \int d^3 r \) and \( \tau \) being the isospin index.

### 3. Spin-spin interaction

The analogous expression for \( H_{ss} \) is found in the standard way, with the Hartree-Fock contribution given [20] by:
\[
\Gamma_{kk'}(t) = \sum_{l'} \bar{V}_{kk' l'} \rho_{l'}(t), \] (19)
where $\tilde{v}_{kk'}$ is the antisymmetrized matrix element of the two body interaction $v(1,2)$. Identifying the indices $k, k', l, l'$ with the set of coordinates $(r, s, \tau)$, i.e. (position, spin, isospin), one rewrites (19) as

$$V_{HF}(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1; t) =$$

$$\int dr_2 \int dr'_2 \sum_{s_2, s'_2} \sum_{\tau_2, \tau'_2} \langle r_1, s_1, \tau_1; r_2, s_2, \tau_2 | \tilde{v} | r'_1, s'_1, \tau'_1; r'_2, s'_2, \tau'_2; r_2, s_2, \tau_2; t \rangle \rho(r'_2, s'_2, \tau'_2; r_2, s_2, \tau_2; t).$$

Let us consider the neutron-proton part of the spin-spin interaction. In this case

$$\hat{v} = v(r_1 - r_2) \sum_{\mu=-1}^{1} (-1)^\mu \hat{S}_{-\mu}(1) \hat{S}_{\mu}(2) \delta_{\tau_1 p} \delta_{\tau_2 n},$$

where $\hat{r}_1$ is the position operator: $\hat{r}_1 | r_1 \rangle = r_1 | r_1 \rangle$, $\langle r_1 | \hat{r}_1 | r'_1 \rangle = \langle r_1 | r'_1 \rangle = \delta(r_1 - r'_1) r'_1$.

For the Hartree term one finds:

$$\langle r_1, s_1, \tau_1; r_2, s_2, \tau_2 | \hat{v} | r'_1, s'_1, \tau'_1; r'_2, s'_2, \tau'_2 \rangle =$$

$$\delta(r_1 - r'_1) \delta(r_2 - r'_2) v(r_1 - r'_1) \sum_{\mu=-1}^{1} (-1)^\mu \langle s_1, \tau_1; s_2, \tau_2 | \hat{S}_{-\mu}(1) \hat{S}_{\mu}(2) \delta_{\tau_1 p} \delta_{\tau_2 n} | s'_1, \tau'_1; s'_2, \tau'_2 \rangle,$n

$$V_{HF}^{H}(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1; t) =$$

$$\int dr_2 \int dr'_2 \sum_{s_2, s'_2} \sum_{\tau_2, \tau'_2} \langle r_1, s_1, \tau_1; r_2, s_2, \tau_2 | \hat{v} | r'_1, s'_1, \tau'_1; r'_2, s'_2, \tau'_2; r_2, s_2, \tau_2; t \rangle =$$

$$\delta_{\tau_1 p} \delta_{\tau_2 n} \sum_{s_2, s'_2} \sum_{\mu=-1}^{1} (-1)^\mu \langle s_1 | \hat{S}_{-\mu}(1) | s'_1 \rangle \langle s_2 | \hat{S}_{\mu}(2) | s'_2 \rangle$$

$$\times \delta(r_1 - r'_1) \int dr_2 v(r_1 - r_2) \rho(r_2, s'_2, n; r_2, s_2, n; t).$$

The Fock term reads:

$$\langle r_1, s_1, \tau_1; r_2, s_2, \tau_2 | \hat{v} | r'_2, s'_2, \tau'_2; r'_1, s'_1, \tau'_1 \rangle =$$

$$\delta(r_1 - r'_2) \delta(r_2 - r'_1) v(r'_2 - r'_1) \sum_{\mu=-1}^{1} (-1)^\mu \langle s_1, \tau_1; s_2, \tau_2 | \hat{S}_{-\mu}(1) \hat{S}_{\mu}(2) \delta_{\tau_1 p} \delta_{\tau_2 n} | s'_2, \tau'_2; s'_1, \tau'_1 \rangle,$n

$$V_{HF}^{F}(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1; t) =$$

$$- \int dr_2 \int dr'_2 \sum_{s_2, s'_2} \sum_{\tau_2, \tau'_2} \langle r_1, s_1, \tau_1; r_2, s_2, \tau_2 | \hat{v} | r'_2, s'_2, \tau'_2; r'_1, s'_1, \tau'_1; r_2, s_2, \tau_2; t \rangle =$$

$$- \delta_{\tau_1 p} \delta_{\tau_2 n} \sum_{s_2} \sum_{\mu=-1}^{1} (-1)^\mu \langle s_1 | \hat{S}_{-\mu}(1) | s'_2 \rangle \langle s_2 | \hat{S}_{\mu}(2) | s'_1 \rangle v(r_1 - r'_1) \rho(r_1, s'_1, p; r'_1, s_2, n; t).$$
The proton mean field is defined as the sum of these two terms:

\[ \langle s| \hat{S}_\tau |s'\rangle = \frac{\hbar}{\sqrt{2}} \delta_{s1} \delta_{s'\uparrow}, \quad \langle s| \hat{S}_0 |s'\rangle = \frac{\hbar}{2} \delta_{s,s'} (\delta_{s\uparrow} - \delta_{s\downarrow}), \quad \langle s| \hat{S}_1 |s'\rangle = -\frac{\hbar}{\sqrt{2}} \delta_{s\uparrow} \delta_{s'\downarrow} \]

and \( v(r - r') = \bar{\chi} \delta(r - r') \) one finds for the mean field generated by the proton-neutron part of \( H_{ss} \):

\[
\Gamma_{pn}(r, s, \tau; r', s', \tau'; t) = \frac{\hbar^2}{2\bar{\chi}} \left\{ \delta_{\tau p} \delta_{\tau' p} \left[ \delta_{s1} \delta_{s'\uparrow} \rho(r, \uparrow, n; r', \uparrow, n; t) + \delta_{s\uparrow} \delta_{s'\downarrow} \rho(r, \uparrow, n; r', \downarrow, n; t) \right] \\
- \delta_{\tau p} \delta_{\tau' n} \left[ \delta_{s1} \delta_{s'\uparrow} - \delta_{s\uparrow} \delta_{s'\downarrow} \right] \left[ \rho(r, \uparrow, n; r', \uparrow, n; t) - \rho(r, \downarrow, n; r', \downarrow, n; t) \right] \\
+ \frac{1}{2} \delta_{\tau p} \delta_{\tau' n} \left[ \delta_{s1} \delta_{s'\uparrow} \rho(r, \uparrow, p; r', \downarrow, n; t) + \delta_{s\uparrow} \delta_{s'\downarrow} \rho(r, \downarrow, p; r', \uparrow, n; t) \right] \\
- \delta_{s\uparrow} \delta_{s'\rho} \left[ \rho(r, \uparrow, p; r', \uparrow, n; t) - \delta_{s\downarrow} \delta_{s'\downarrow} \rho(r, \downarrow, p; r', \downarrow, n; t) \right] \left[ \rho(r, \uparrow, n; r', \uparrow, n; t) - \rho(r, \downarrow, n; r', \downarrow, n; t) \right] \right\} \delta(r - r'). \tag{20} \]

The expression for the mean field \( \Gamma_{pp}(r, s, \tau; r', s', \tau'; t) \) generated by the proton-proton part of \( H_{ss} \) can be obtained from (20) by replacing index \( n \) by \( p \) and the strength constant \( \bar{\chi} \) by \( \chi \).

The proton mean field is defined as the sum of these two terms \( \Gamma_{pp}(r, s, p; r', s', p; t) + \Gamma_{pn}(r, s, p; r', s', t; p; t) \). Its Wigner transform can be written as

\[
V_{sp}^{ss'}(r, t) = 3\chi \frac{\hbar^2}{8} \left\{ \delta_{s1} \delta_{s'\uparrow} n_p^{\uparrow} + \delta_{s\uparrow} \delta_{s'\downarrow} n_p^{\downarrow} - \delta_{s1} \delta_{s'\downarrow} n_p^{\uparrow} - \delta_{s\uparrow} \delta_{s'\uparrow} n_p^{\downarrow} \right\} \\
+ \bar{\chi} \frac{\hbar^2}{8} \left\{ 2\delta_{s1} \delta_{s'\uparrow} n_p^{\uparrow} + 2\delta_{s\uparrow} \delta_{s'\downarrow} n_p^{\downarrow} + (\delta_{s\uparrow} \delta_{s'\uparrow} - \delta_{s\downarrow} \delta_{s'\downarrow}) (n_p^{\uparrow} - n_p^{\downarrow}) \right\}, \tag{21} \]

where \( n_{sp}^{ss'}(r, t) = \int \frac{dp}{(2\pi \hbar)^3} f_{sp}^{ss'}(r, p, t) \). The Wigner transform of the neutron mean field \( V_n^{ss'} \) is obtained from (21) by the obvious change of indices \( p \leftrightarrow n \).

The Wigner function \( f \) and density matrix \( \rho \) are connected by the relation

\[
f_{sp}^{ss'}(r, p, t) = \int dq e^{-ipq/h} \rho(q, s, \tau; r, s', \tau'; t), \quad \text{with} \quad q = r_1 - r_2 \quad \text{and} \quad r = \frac{1}{2}(r_1 + r_2). \]

Integrating this relation over \( p \) with \( \tau' = \tau \) one finds:

\[
n_{sp}^{ss'}(r, t) = \rho(r, s, \tau; r, s', \tau; t). \]

By definition the diagonal elements of the density matrix describe the proper densities. Therefore \( n_{s}^{ss}(r, t) \) is the density of spin-up nucleons (if \( s = \uparrow \)) or spin-down nucleons (if \( s = \downarrow \)).

Off diagonal in spin elements of the density matrix \( n_{sp}^{ss'}(r, t) \) are spin-flip characteristics and can be called spin-flip densities.
B. Pair potential

The Wigner transform of the pair potential (pairing gap) $\Delta(r,p)$ is related to the Wigner transform of the anomalous density by [20]

$$\Delta(r,p) = -\int \frac{dp'}{(2\pi\hbar)^3}v(|p-p'|)\kappa(r,p'),$$  \hspace{1cm} (22)

where $v(p)$ is a Fourier transform of the two-body interaction. We take for the pairing interaction a simple Gaussian of strength $V_0$ and range $r_p$ [20]

$$v(p) = \beta e^{-\alpha p^2},$$  \hspace{1cm} (23)

with $\beta = -|V_0|(r_p\sqrt{\pi})^3$ and $\alpha = r_p^2/4\hbar^2$. For the values of the parameters, see section VA.

IV. EQUATIONS OF MOTION

Integrating the set of equations (7) over phase space with the weights

$$W = \{r \otimes p\}_{\lambda\mu}, \{r \otimes r\}_{\lambda\mu}, \{p \otimes p\}_{\lambda\mu}, \text{ and } 1$$  \hspace{1cm} (24)

one gets dynamic equations for the following collective variables:

$$L_{\lambda\mu}^{\tau\varsigma}(t) = \int d(p,r)\{r \otimes p\}_{\lambda\mu}f_{\alpha}^{\tau\varsigma}(r,p,t),$$

$$R_{\lambda\mu}^{\tau\varsigma}(t) = \int d(p,r)\{r \otimes r\}_{\lambda\mu}f_{\epsilon}^{\tau\varsigma}(r,p,t),$$

$$P_{\lambda\mu}^{\tau\varsigma}(t) = \int d(p,r)\{p \otimes p\}_{\lambda\mu}f_{\epsilon}^{\tau\varsigma}(r,p,t),$$

$$F^{\tau\varsigma}(t) = \int d(p,r)f_{\epsilon}^{\tau\varsigma}(r,p,t),$$

$$\tilde{L}_{\lambda\mu}^{\tau}(t) = \int d(p,r)\{r \otimes p\}_{\lambda\mu}\kappa_{\sigma}^{\tau\varsigma}(r,p,t),$$

$$\tilde{R}_{\lambda\mu}^{\tau}(t) = \int d(p,r)\{r \otimes r\}_{\lambda\mu}\kappa_{\sigma}^{\tau\varsigma}(r,p,t),$$

$$\tilde{P}_{\lambda\mu}^{\tau}(t) = \int d(p,r)\{p \otimes p\}_{\lambda\mu}\kappa_{\sigma}^{\tau\varsigma}(r,p,t),$$  \hspace{1cm} (25)

where $\varsigma = +, -, \uparrow\downarrow$, and $\{r \otimes r\}_{\lambda\mu} = \sum_{\sigma,\nu}C_{1\sigma,1\nu}^{\lambda\mu} r_{\sigma} r_{\nu}$. We already called the functions $f^+ = f^{\uparrow\uparrow} + f^{\downarrow\downarrow}$ and $f^- = f^{\uparrow\downarrow} - f^{\downarrow\uparrow}$ spin-scalar and spin-vector ones, respectively. It is, therefore, natural to call the corresponding collective variables $X_{\lambda\mu}^{+}(t)$ and $X_{\lambda\mu}^{-}(t)$ spin-scalar and spin-vector variables.
The required expressions for \( h^\pm, h^{\uparrow\downarrow} \) and \( h^{\uparrow\uparrow} \) are

\[
h^\pm_\tau = \frac{p^2}{m} + m \omega^2 r^2 + 12 \sum_\nu (-1)^\nu Z^\pm_{2\nu}(t)\{r \otimes r\}_{2-\nu} + V^\pm_\tau(\mathbf{r}, t) - \mu^\tau,
\]

\( \mu^\tau \) being the chemical potential of protons (\( \tau = p \)) or neutrons (\( \tau = n \)),

\[
h^-_\tau = -\eta \eta l_0 + V^-_\tau(\mathbf{r}, t), \quad h^{\uparrow\downarrow}_\tau = -\frac{\hbar}{\sqrt{2}} \eta l_{-1} + V^{\uparrow\downarrow}_\tau(\mathbf{r}, t), \quad h^{\uparrow\uparrow}_\tau = \frac{\hbar}{\sqrt{2}} \eta l_1 + V^{\uparrow\uparrow}_\tau(\mathbf{r}, t),
\]

where according to (21)

\[
V^+_p(\mathbf{r}, t) = -3 \frac{\hbar^2}{8} \chi n^+_p(\mathbf{r}, t),
\]

\[
V^-_p(\mathbf{r}, t) = 3 \frac{\hbar^2}{8} \chi n^-_p(\mathbf{r}, t) + \frac{\hbar^2}{4} \chi n^-_n(\mathbf{r}, t),
\]

\[
V^{\uparrow\downarrow}_p(\mathbf{r}, t) = 3 \frac{\hbar^2}{8} \chi n^{\uparrow\downarrow}_p(\mathbf{r}, t) + \frac{\hbar^2}{4} \chi n^{\uparrow\downarrow}_n(\mathbf{r}, t),
\]

\[
V^{\uparrow\uparrow}_p(\mathbf{r}, t) = 3 \frac{\hbar^2}{8} \chi n^{\uparrow\uparrow}_p(\mathbf{r}, t) + \frac{\hbar^2}{4} \chi n^{\uparrow\uparrow}_n(\mathbf{r}, t)
\]

and the neutron potentials \( V^+_n \) are obtained by the obvious change of indices \( p \leftrightarrow n \).

The integration of equations (7) with the weights (24) yields:

\[
\dot{L}^+_\lambda^\mu = \frac{1}{m} P^+_\lambda^\mu - m \omega^2 R^+_\lambda^\mu + 12 \sqrt{5} \sum_{j=0}^2 \sqrt{2j + 1} \{ \frac{11j}{2}\lambda_1 \} \{ Z^+_2 \otimes R^j \} \lambda^\mu
\]

\[-i \hbar \frac{\eta}{2} \left[ L^\mu L^-_\lambda + \sqrt{(\lambda - \mu)(\lambda + \mu + 1)} L^{\uparrow\downarrow}_\lambda + \sqrt{(\lambda + \mu)(\lambda - \mu + 1)} L^{\uparrow\downarrow}_{\lambda-1} \right] \]

\[-i \hbar \frac{\eta}{2} \delta_{\lambda,1} \left[ \delta_{\mu-1} F^{\uparrow\downarrow} + \delta_{\mu,1} F^{\uparrow\downarrow} \right] - \frac{1}{2} \int d^3r \left[ n^- \{ r \otimes \nabla \} \lambda^\mu V^+ + n^+ \{ r \otimes \nabla \} \lambda^\mu V^- \right] \]

\[-2 \frac{i}{\hbar} \int d(\mathbf{p}, \mathbf{r}) \{ r \otimes p \} \lambda^\mu \left[ h^{\uparrow\downarrow} f^{\uparrow\downarrow} - h^{\uparrow\downarrow} f^{\uparrow\downarrow} \right],
\]

\[
\dot{L}^{\uparrow\downarrow}_\lambda^\mu = \frac{1}{m} P^{\uparrow\downarrow}_\lambda^\mu - m \omega^2 R^{\uparrow\downarrow}_\lambda^\mu + 12 \sqrt{5} \sum_{j=0}^2 \sqrt{2j + 1} \{ \frac{11j}{2}\lambda_1 \} \{ Z^{\uparrow\downarrow}_2 \otimes R^j \} \lambda^\mu
\]

\[-i \hbar \frac{\eta}{4} \left[ \delta_{\mu,0} \delta_{\lambda,1} \right] - \frac{1}{2} \int d^3r \left[ n^{\uparrow\downarrow} \{ r \otimes \nabla \} \lambda^\mu V^+ + n^+ \{ r \otimes \nabla \} \lambda^\mu V^{\uparrow\downarrow} \right] \]

\[-2 \frac{i}{\hbar} \int d(\mathbf{p}, \mathbf{r}) \{ r \otimes p \} \lambda^{\uparrow\downarrow} \left[ h^{\uparrow\downarrow} f^{\uparrow\downarrow} - h^{\uparrow\downarrow} f^{\uparrow\downarrow} \right],
\]
\[ \hat{L}_{\lambda_{\mu}-1} = \frac{1}{m} P^{\dagger}_{\lambda_{\mu}1} - m \omega^2 R^{\dagger}_{\lambda_{\mu}1} + 12 \sqrt{5} \sum_{j=0}^{2} \sqrt{2j + 1} \left\{ \frac{11j}{2j+1} \right\} \{ Z_{2}^{\dagger} \otimes R_{j}^{\dagger} \}_{\lambda_{\mu}-1} \]
\[ - \frac{\hbar \eta}{4} \sqrt{(\lambda + \mu)(\lambda - \mu + 1)}L_{\lambda_{\mu}}^{\dagger} + \frac{\hbar^2}{4} \delta_{\lambda,1} \left\{ \delta_{\mu,0} F^+ - \sqrt{2} \delta_{\mu,1} F^{1\uparrow} \right\} \]
\[ - \frac{1}{2} \int d^3r \left[ n^{\dagger} \{ r \otimes \nabla \} \lambda_{\mu}-1 V^+ + n^{+} \{ r \otimes \nabla \} \lambda_{\mu}-1 V^{1\uparrow} \right] \]
\[ - \frac{i}{\hbar} \int d(p, r) \{ r \otimes p \} \lambda_{\mu}-1 \left[ h^{\dagger} f^- - h^- f^{1\uparrow} \right], \]
\[ \hat{F}^- = 2 \eta \left[ L_{1\uparrow}^{\dagger} + L_{1\uparrow}^{1\uparrow} \right], \]
\[ \hat{F}^{1\uparrow} = - \eta [L_{1\uparrow} - \sqrt{2} L_{1\downarrow}], \]
\[ \hat{F}^{\dagger} = - \eta [L_{1\downarrow} + \sqrt{2} L_{1\uparrow}]; \]
\[ \hat{R}_{\lambda_{\mu}}^{\dagger} = \frac{2}{m} \sqrt{(\lambda - \mu)(\lambda + \mu + 1)} R^{\dagger}_{\lambda_{\mu}1} + \sqrt{(\lambda + \mu)(\lambda - \mu + 1)} R_{\lambda_{\mu}-1}^{\dagger}, \]
\[ \hat{R}_{\lambda_{\mu}} = \frac{2}{m} \sqrt{(\lambda - \mu)(\lambda + \mu + 1)} R_{\lambda_{\mu}}^{\dagger} - \frac{i}{\hbar} \int d(p, r) \{ r \otimes r \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ \hat{R}_{\lambda_{\mu}+1}^{\dagger} = \frac{2}{m} L^{\dagger}_{\lambda_{\mu}1} - \frac{i}{\hbar} \int d(p, r) \{ r \otimes r \} \lambda_{\mu+1} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ \hat{R}_{\lambda_{\mu}-1}^{\dagger} = \frac{2}{m} L^{\dagger}_{\lambda_{\mu}1} - \frac{i}{\hbar} \int d(p, r) \{ r \otimes r \} \lambda_{\mu-1} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ \hat{P}_{\lambda_{\mu}}^{\dagger} = - 2m \omega^2 L^{\dagger}_{\lambda_{\mu}1} + 24 \sqrt{5} \sum_{j=0}^{2} \sqrt{2j + 1} \left\{ \frac{11j}{2j+1} \right\} \{ Z_{2}^{\dagger} \otimes L_{j}^{\dagger} \}_{\lambda_{\mu}} \]
\[ - \frac{i}{2} \int d^3r \left[ \{ J^{\dagger} \otimes \nabla \} \lambda_{\mu} V^+ + \{ J^{\dagger} \otimes \nabla \} \lambda_{\mu} V^{1\uparrow} \right] \]
\[ = - \frac{\hbar \eta}{2} \left[ \mu P_{\lambda_{\mu}}^{\dagger} + \sqrt{(\lambda + \mu)(\lambda - \mu + 1) P_{\lambda_{\mu}+1}^{\dagger} + \sqrt{(\lambda + \mu)(\lambda - \mu + 1) P_{\lambda_{\mu}-1}^{\dagger}} \right] \]
\[ - \frac{i}{\hbar} \int d(p, r) \{ p \otimes p \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ \hat{P}_{\lambda_{\mu}+1}^{\dagger} = - 2m \omega^2 L^{\dagger}_{\lambda_{\mu}1} + 24 \sqrt{5} \sum_{j=0}^{2} \sqrt{2j + 1} \left\{ \frac{11j}{2j+1} \right\} \{ Z_{2}^{\dagger} \otimes L_{j}^{\dagger} \}_{\lambda_{\mu}+1} - \frac{i}{\hbar} \int d(p, r) \{ p \otimes p \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ \hat{P}_{\lambda_{\mu}-1}^{\dagger} = - 2m \omega^2 L^{\dagger}_{\lambda_{\mu}1} + 24 \sqrt{5} \sum_{j=0}^{2} \sqrt{2j + 1} \left\{ \frac{11j}{2j+1} \right\} \{ Z_{2}^{\dagger} \otimes L_{j}^{\dagger} \}_{\lambda_{\mu}-1} - \frac{i}{\hbar} \int d(p, r) \{ p \otimes p \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ - \frac{i}{\hbar} \int d(p, r) \{ p \otimes p \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ - \frac{i}{\hbar} \int d(p, r) \{ p \otimes p \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ - \frac{i}{\hbar} \int d(p, r) \{ p \otimes p \} \lambda_{\mu} \left[ h^{\dagger} f^{1\uparrow} - h^{1\uparrow} f^{\dagger} \right], \]
\[ (29) \]
where $\{^{11j}_{2\lambda j}\}$ is the Wigner 6-$j$-symbol and $J_\nu^\lambda(r, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} p_\nu f^\lambda(r, \mathbf{p}, t)$ is the current. For simplicity the isospin and the time dependence of tensors is not written out. It is easy to see that equations (29) are nonlinear due to quadrupole-quadrupole and spin-spin interactions. We will solve them in the small amplitude approximation, by linearizing the equations. This procedure helps also to solve another problem: to represent the integral terms in (29) as the linear combination of collective variables (25), that allows to close the whole set of equations (29). The detailed analysis of the integral terms is given in the appendix C.

We are interested in the scissors mode with quantum number $K^\pi = 1^\pm$. Therefore, we only need the part of dynamic equations with $\mu = 1$.

### A. Linearized equations ($\mu = 1$), isovector, isoscalar

Writing all variables as a sum of their equilibrium value plus a small deviation

\[
R_{\lambda\mu}(t) = R_{\lambda\mu}(\text{eq}) + \mathcal{R}_{\lambda\mu}(t), \quad P_{\lambda\mu}(t) = P_{\lambda\mu}(\text{eq}) + \mathcal{P}_{\lambda\mu}(t),
\]

\[
L_{\lambda\mu}(t) = L_{\lambda\mu}(\text{eq}) + \mathcal{L}_{\lambda\mu}(t), \quad F(t) = F(\text{eq}) + \mathcal{F}(t),
\]

one gets dynamic equations for variations of the collective variables (25):

\[
\mathcal{L}_{\lambda\mu}^{TS}(t) = \int d(\mathbf{p}, r) \{r \otimes p\} \lambda_\mu \delta f_{\sigma}^{TS}(r, \mathbf{p}, t),
\]

\[
\mathcal{R}_{\lambda\mu}^{TS}(t) = \int d(\mathbf{p}, r) \{r \otimes r\} \lambda_\mu \delta f_{e}^{TS}(r, \mathbf{p}, t),
\]

\[
\mathcal{P}_{\lambda\mu}^{TS}(t) = \int d(\mathbf{p}, r) \{p \otimes p\} \lambda_\mu \delta f_{e}^{TS}(r, \mathbf{p}, t),
\]

\[
\mathcal{F}^{TS}(t) = \int d(\mathbf{p}, r) \delta f_{e}^{TS}(r, \mathbf{p}, t),
\]

\[
\tilde{\mathcal{L}}_{\lambda\mu}(t) = \int d(\mathbf{p}, r) \{r \otimes p\} \lambda_\mu \delta \kappa_{o}^{T\dagger}(r, \mathbf{p}, t),
\]

\[
\tilde{\mathcal{R}}_{\lambda\mu}(t) = \int d(\mathbf{p}, r) \{r \otimes r\} \lambda_\mu \delta \kappa_{e}^{T\dagger}(r, \mathbf{p}, t),
\]

\[
\tilde{\mathcal{P}}_{\lambda\mu}(t) = \int d(\mathbf{p}, r) \{p \otimes p\} \lambda_\mu \delta \kappa_{e}^{T\dagger}(r, \mathbf{p}, t).
\]

Neglecting quadratic deviations, one obtains the linearized equations. Naturally one needs to know the mean fields variations and the equilibrium values of all variables.

Variations of mean fields read:

\[
\delta h_{\tau}^\pm = 12 \sum_{\mu} (-1)^\mu \delta Z_{2\mu}^{\tau\dagger}(t) \{r \otimes r\}_{2-\mu}^\pm + \delta V_{\tau}^+(r, t),
\]
where $\delta Z_{2\mu}^+ = \kappa \delta R_{2\mu}^p + \bar{\kappa} \delta R_{2\mu}^n$, $\delta R_{\lambda\mu}^+(t) \equiv \mathcal{R}_{\lambda\mu}^+(t)$ and

\[
\delta V_p^+(r, t) = -\frac{3h^2}{8} \chi \delta n_p^+(r, t),
\]

\[
\delta n_p^+(r, t) = \int \frac{d^3 p}{(2\pi \hbar)^3} \delta f_p^+(r, p, t).
\]

Variations of $h^-$, $h^{\uparrow\downarrow}$ and $h^{\downarrow\uparrow}$ are obtained in a similar way. Variation of the pair potential is

\[
\delta \Delta(r, p, t) = -\int \frac{d^3 p'}{(2\pi \hbar)^3} v(|p - p'|) \delta \kappa(r, p', t).
\]  

(31)

Evident equilibrium conditions for an axially symmetric nucleus are:

\[
R^+_{2\pm 1}(eq) = R^+_{2\pm 2}(eq) = 0, \quad R^+_{20}(eq) \neq 0, \quad R^+_{00}(eq) \neq 0.
\]  

(32)

It is obvious that all ground state properties of the system of spin up nucleons are identical to the ones of the system of nucleons with spin down. Therefore

\[
R^-_{\lambda\mu}(eq) = P^-_{\lambda\mu}(eq) = L^-_{\lambda\mu}(eq) = 0.
\]  

(33)

We also will suppose

\[
L^+_{\lambda\mu}(eq) = L^+_{\lambda\mu}(eq) = L^+_{\lambda\mu}(eq) = 0 \quad \text{and} \quad R^+_{\lambda\mu}(eq) = R^+_{\lambda\mu}(eq) = 0.
\]  

(34)

It also is natural to define isovector and isoscalar strength constants $\kappa_1 = \frac{1}{2}(\kappa - \bar{\kappa})$ and $\kappa_0 = \frac{1}{2}(\kappa + \bar{\kappa})$ connected by the relation $\kappa_1 = \alpha \kappa_0$ [23]. Then the equations for the neutron and proton systems are transformed into isovector and isoscalar ones. Supposing that all
equilibrium characteristics of the proton system are equal to that of the neutron system one
decouples isovector and isoscalar equations. This approximations looks rather crude. In the
paper [14] we have tried to improve it by employing more accurate approximation which
works very well in the case of collective motion:

\[ Q^n / N = \pm Q^n / Z, \]

where \( Q \) is any of collective variables (30) and the sign (+,−) is utilized for the isoscalar
(isovector) motion. The corrections to the more simple approximatin turned out of the
order \( (\frac{N-Z}{A})^2 \). For rare earth nuclei this gives an error about 4%, that is admissible for us,
because the main goal of this paper is to understand the influence of the simultaneous action
of pairing and spin degrees of freedom on the scissors mode. So, to keep final formulae more
transparent, we prefer to use the more simple approximations.

With the help of the above equilibrium relations one arrives at the following final set of
equations for isovector variables:

\[
\mathcal{L}_{21}^+ = \frac{1}{m} \mathcal{P}_{21}^+ - \left[ m \omega^2 - 4/3 \kappa_0 R_{00}^{	ext{eq}} + \sqrt{6}(1+\alpha)\kappa_0 R_{20}^{	ext{eq}} \right] \mathcal{R}_{21}^+ - i h \frac{\eta}{2} \left[ \mathcal{L}_{21}^- + 2\mathcal{L}_{22}^{\uparrow \downarrow} + \sqrt{6}\mathcal{L}_{20}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{L}_{21}^- = \frac{1}{m} \mathcal{P}_{21}^- - \left[ m \omega^2 + \sqrt{6}\kappa_0 R_{20}^{	ext{eq}} - \frac{\sqrt{3}}{20} h^2 \left( \chi - \frac{\chi}{3} \right) \left( \frac{I_1}{a_0^2} + \frac{I_1}{a_1^2} \right) \left( \frac{a_0^2}{A_2} - \frac{a_1^2}{A_1} \right) \right] \mathcal{R}_{21}^- - i h \frac{\eta}{2} \mathcal{L}_{21}^+ + \frac{4}{h} |V_0| I_{rp}^\Delta (r') \mathcal{L}_{21},
\]

\[
\mathcal{L}_{22}^{\uparrow \downarrow} = \frac{1}{m} \mathcal{P}_{22}^{\uparrow \downarrow} - \left[ m \omega^2 - 2/3 \kappa_0 R_{20}^{	ext{eq}} \right] \mathcal{R}_{22}^{\uparrow \downarrow} + \frac{2}{3} \kappa_0 R_{20}^{	ext{eq}} \mathcal{R}_{00}^{\uparrow \downarrow} - i h \frac{\eta}{2} \sqrt{2/2} \mathcal{L}_{21}^{\uparrow \downarrow},
\]

\[
\mathcal{L}_{20}^{\uparrow \downarrow} = \frac{1}{m} \mathcal{P}_{20}^{\uparrow \downarrow} - \left[ m \omega^2 + 2/3 \kappa_0 R_{20}^{	ext{eq}} \right] \mathcal{R}_{20}^{\uparrow \downarrow} - i h \frac{\eta}{2} \left[ \mathcal{L}_{21}^{\uparrow \downarrow} + \sqrt{2} \mathcal{L}_{20}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{L}_{11}^{\uparrow \downarrow} = -3\sqrt{6}(1-\alpha)\kappa_0 R_{20}^{	ext{eq}} \mathcal{R}_{11}^{\uparrow \downarrow} - i h \frac{\eta}{2} \left[ \mathcal{L}_{11}^- + \sqrt{2} \mathcal{L}_{10}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{L}_{11}^+ = -3\sqrt{6}(1-\alpha)\kappa_0 R_{20}^{	ext{eq}} \mathcal{R}_{11}^+ - i h \frac{\eta}{2} \left[ \mathcal{L}_{11}^- + \sqrt{2} \mathcal{L}_{10}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{L}_{10}^{\uparrow \downarrow} = -i h \frac{\eta}{2} \left[ i \mathcal{L}_{11}^+ + h \mathcal{F}_{10}^{\uparrow \downarrow} \right] + \frac{4}{h} |V_0| I_{rp}^\Delta (r') \mathcal{L}_{11},
\]

\[
\mathcal{F}_{10}^{\uparrow \downarrow} = -\eta \left[ \mathcal{L}_{11}^- + \sqrt{2} \mathcal{L}_{10}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{L}_{21}^- = \frac{2}{m} \mathcal{L}_{21}^- - i h \frac{\eta}{2} \left[ \mathcal{R}_{21}^- + 2\mathcal{R}_{22}^{\uparrow \downarrow} + \sqrt{6}\mathcal{R}_{20}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{L}_{21}^- = \frac{2}{m} \mathcal{L}_{21}^- - i h \frac{\eta}{2} \mathcal{R}_{21}^-,
\]

\[
\mathcal{F}_{21}^+ = \frac{2}{m} \mathcal{L}_{21}^+ - i h \frac{\eta}{2} \left[ \mathcal{R}_{21}^- + 2\mathcal{R}_{22}^{\uparrow \downarrow} + \sqrt{6}\mathcal{R}_{20}^{\uparrow \downarrow} \right],
\]

\[
\mathcal{R}_{21}^- = \frac{2}{m} \mathcal{L}_{21}^- - i h \frac{\eta}{2} \mathcal{R}_{21}^-,
\]
\[ \dot{R}_{22}^{\perp} = \frac{2}{m} \mathcal{L}_{22}^{\perp} - i\hbar \frac{\eta}{2} R_{21}^{+}, \]
\[ \dot{R}_{20}^{\perp} = \frac{2}{m} \mathcal{L}_{20}^{\perp} - i\hbar \frac{3}{2} \sqrt{\frac{3}{2}} R_{21}^{+}, \]
\[ \dot{P}_{21}^{+} = -2 \left[ m \omega^2 + \sqrt{6} \kappa_0 R_{20}^{eq} \right] \mathcal{L}_{21}^{\perp} + 6 \sqrt{6} \kappa_0 R_{20}^{eq} \mathcal{L}_{11}^{\perp} - i \hbar \frac{\eta}{2} \left[ P_{21}^{\perp} + 2 P_{22}^{\perp} + \sqrt{6} P_{20}^{+} \right] + \frac{3 \sqrt{3}}{4} \hbar^2 \chi \frac{I_2}{A_1 A_2} \left[ (A_1 - A_2) \mathcal{L}_{21}^{\perp} + (A_1 + A_2) \mathcal{L}_{11}^{\perp} \right] + \frac{4}{\hbar} V_0 I_{pp}^{\Delta}(r') \dot{P}_{21}, \]
\[ \dot{P}_{22}^{\perp} = -2 \left[ m \omega^2 + \sqrt{6} \kappa_0 R_{20}^{eq} \right] \mathcal{L}_{21}^{\perp} + 6 \sqrt{6} \kappa_0 R_{20}^{eq} \mathcal{L}_{11}^{\perp} - 6 \sqrt{2} \alpha \kappa_0 L_{10}^{\perp}(eq) R_{21}^{+} - i \hbar \frac{\eta}{2} P_{21}^{+} + \frac{3 \sqrt{3}}{4} \hbar^2 \chi \frac{I_2}{A_1 A_2} \left[ (A_1 - A_2) \mathcal{L}_{21}^{\perp} + (A_1 + A_2) \mathcal{L}_{11}^{\perp} \right], \]
\[ \dot{P}_{20}^{\perp} = -\left[ 2 m \omega^2 - 4 \sqrt{6} \kappa_0 R_{20}^{eq} \mathcal{L}_{20}^{\perp} + 3 \sqrt{3} \hbar^2 \frac{I_2}{A_2} \right] \mathcal{L}_{22}^{\perp} - i \hbar \frac{\eta}{2} P_{21}^{+}, \]
\[ \dot{P}_{00}^{+} = \frac{1}{m} \mathcal{P}_{00}^{+} - m \omega^2 \mathcal{R}_{00}^{+} + 4 \sqrt{3} \kappa_0 R_{20}^{eq} \mathcal{R}_{20}^{+} \]
\[ + \frac{\hbar^2}{2 \sqrt{3} A_1 A_2} \left[ \left( \chi - \frac{\bar{\lambda}}{3} \right) I_1 - \frac{9}{4} \chi I_2 \right] \left[ (2A_1 - A_2) \mathcal{R}_{00}^{+} + \sqrt{2} (A_1 + A_2) \mathcal{R}_{20}^{\perp} \right], \]
\[ \dot{R}_{00}^{\perp} = \frac{2}{m} \mathcal{L}_{00}^{\perp}, \]
\[ \dot{R}_{00}^{+} = -2 \left[ m \omega^2 \mathcal{L}_{00}^{\perp} + 8 \sqrt{3} \kappa_0 R_{20}^{eq} \mathcal{L}_{20}^{\perp} + \frac{\sqrt{3}}{2} \hbar^2 \chi \frac{I_2}{A_1 A_2} \right] \left[ (2A_1 - A_2) \mathcal{R}_{00}^{\perp} + \sqrt{2} (A_1 + A_2) \mathcal{R}_{20}^{\perp} \right], \]
\[ \dot{R}_{21} = -\left( \frac{16}{5} \alpha \kappa_0 K_4 + \Delta_0(r') + \frac{3}{8} h^2 \kappa_0(r') \right) \mathcal{R}_{21}^{+}, \]
\[ \dot{P}_{21} = -\frac{1}{h} \Delta_0(r') \mathcal{P}_{21}^{+} + 6 h \alpha \kappa_0 K_0 \mathcal{R}_{21}^{+}, \]
\[ \dot{L}_{21} = -\frac{1}{h} \Delta_0(r') \mathcal{L}_{21}^{\perp}, \]
\[ \dot{L}_{11} = -\frac{1}{h} \Delta_0(r') \mathcal{L}_{11}^{\perp}, \]
\[ \mathcal{A} - \mathcal{A} = \mathcal{R}_0 \left( \frac{1 - (2/3)\delta}{1 + (2/3)\delta} \right)^{1/6} \text{ and } \mathcal{A} = \mathcal{R}_0 \left( \frac{1 - (2/3)\delta}{1 + (2/3)\delta} \right)^{-1/3} \text{ are semiaxes of ellipsoid by which the shape of nucleus is approximated, } \delta \text{ – deformation parameter, } R_0 = 1.2A^{1/3} \text{ fm – radius of} \]

\[ A_1 = \sqrt{2} R_{20}^{eq} - R_{00}^{eq} = \frac{Q_{00}}{\sqrt{3}} \left( 1 + \frac{4}{3} \delta \right), \]
\[ A_2 = R_{20}^{eq}/\sqrt{2} + R_{00}^{eq} = -\frac{Q_{00}}{\sqrt{3}} \left( 1 - \frac{2}{3} \delta \right). \]
nucleus, $R_{\lambda\mu}^{eq} \equiv R_{\lambda\mu}(eq)$.

$$I_1 = \frac{\pi}{4} \int_0^\infty dr r^4 \left( \frac{\partial n(r)}{\partial r} \right)^2, \quad I_2 = \frac{\pi}{4} \int_0^\infty dr r^2 n(r)^2,$$

$$n(r) = n_0 \left(1 + e^{-\frac{r-R_0}{a}}\right)^{-1} - \text{nuclear density},$$

$$K_0 = \int d(r, p) \kappa(r, p), \quad K_4 = \int d(r, p) r^4 \kappa(r, p).$$

The functions $\kappa_0(r'), \Delta_0(r'), I_{pp}^{\kappa\Delta}(r')$ and $I_{pp}^{\kappa}(r')$ are discussed in the next section and are outlined in Appendix E. Deriving these equations we neglected double Poisson brackets containing $\kappa$ or $\Delta$, which are the quantum corrections to pair correlations. The isoscalar set of equations is easily obtained from (36) by taking $\alpha = 1$, replacing $\bar{\chi} \rightarrow -\bar{\chi}$ and putting the marks "bar" above all variables.

### B. Integrals of motion and the angular momentum conservation

Imposing the time evolution via $e^{iEt/\hbar}$ for all variables one transforms (36) into a set of algebraic equations. It contains 23 equations. To find the eigenvalues we construct the $23 \times 23$ determinant and seek (numerically) for its zeros. We find seven roots with exactly $E = 0$ and 16 roots which are non zero: eight positive ones (shown in the tables) and eight negative ones (not shown, situation exactly same as with RPA; see [24] for connection of WFM and RPA).

The integrals of motion corresponding to Goldstone modes (zero roots) can be found analytically. In the isovector case we have

$$i\hbar \frac{\eta}{2} \left[ L^+_{11} - i \frac{\hbar}{2} \mathcal{F}^{\uparrow\downarrow} \right] - 3\sqrt{6}(1 - \alpha)\kappa_0 R_{20}^{eq} \left[ \frac{2}{\sqrt{3}c_2 m} \mathcal{P}^{\uparrow\downarrow}_{00} + \frac{c_1}{\sqrt{3}c_2} \mathcal{R}^{\perp\perp}_{00} - \sqrt{\frac{2}{3}} \mathcal{R}^{\uparrow\downarrow}_{20} \right] = \text{const},$$

$$\frac{\mathcal{P}^{\uparrow\downarrow}_{22}}{2} - \sqrt{\frac{2}{3}} \left( \mathcal{P}^{\uparrow\downarrow}_{20} + \sqrt{2}\mathcal{P}^{\uparrow\downarrow}_{00} \right) + \frac{m}{2}(c_1 - c_2) \left[ \mathcal{R}^{\perp\perp}_{22} - \sqrt{\frac{2}{3}} \left( \mathcal{R}^{\perp\perp}_{20} + \sqrt{2}\mathcal{R}^{\perp\perp}_{00} \right) \right] = \text{const},$$

$$i\hbar \frac{\eta}{2} L^-_{21} - \hbar^2 \eta^2 \frac{m}{8} \left[ \mathcal{R}^{\perp\perp}_{21} + 2\mathcal{R}^{\perp\perp}_{22} \right] + \sqrt{\frac{2}{3}} \left( \frac{3}{8} \hbar^2 \eta^2 m - c_3 \right) \mathcal{R}^{\perp\perp}_{26} + \frac{\sqrt{2}}{3 m} \mathcal{P}^{\uparrow\downarrow}_{20}$$

$$+ \frac{1}{2\sqrt{3}c_2} \left( c_1 - c_2 - 2c_1c_2 - 2c_1c_3 - \frac{3}{2} \hbar^2 \eta^2 m \right) \mathcal{R}^{\perp\perp}_{00}$$

$$+ \frac{1}{\sqrt{3}c_2 m} \left( c_1 + c_2 + 2c_3 - \frac{3}{2} \hbar^2 \eta^2 m \right) \mathcal{P}^{\uparrow\downarrow}_{00} = \text{const},$$
Isoscalar integrals of motion are easily obtained from isovector ones by taking \( \alpha = 1 \) and putting bars above all variables. In the case of harmonic oscillations all constants "const" are obviously equal to zero.

The physical sense of variables entering into above integrals of motion can be understood with the help of their definitions (30). The variables (or matrix elements) \( R_{\lambda\mu}^{ss'}(t) \) describe the quadrupole \( (\lambda = 2) \) and monopole \( (\lambda = 0) \) deformation of the density of nucleons with spin \( s \), if \( s = s' \), otherwise they describe the simultaneous deformation and spin flip. The variables \( P_{\lambda\mu}^{ss'}(t) \) describe the analogous situation in the momentum space, i.e. the Fermi surface deformation, if \( s = s' \), or the deformation accompanied by spin flip, if \( s \neq s' \).

The variables \( L_{\lambda\mu}^{ss'}(t) \) with \( \lambda = 2, 0 \) describe the similar situation in the phase space \((r,p)\). Looking on the equations of motion for \( R_{\lambda\mu}^{ss'}(t) \) and \( P_{\lambda\mu}^{ss'}(t) \) in the case without spin orbital field and spin-spin forces one can see that, roughly speaking, these variables \( (L_{\lambda\mu}^{ss'}) \) determine the velocities \( \frac{d}{dt} R_{\lambda\mu}^{ss'} \) and \( \frac{d}{dt} P_{\lambda\mu}^{ss'} \) of the density deformation and the Fermi surface deformation respectively.
The variables $\mathcal{L}_{tt'}^{ss'}(t)$ describe the dynamics of the orbital angular momentum of nucleons with spin $s$, if $s = s'$, otherwise they describe the dynamics of the orbital angular momentum together with spin flip. The variables $\mathcal{F}_{tt'}^{ss'}(t)$ describe the dynamics of the number of nucleons with spin $s$, if $s = s'$, or dynamics of spin, i.e. the spin flip, if $s \neq s'$.

Having this information we can give the physical interpretation of some integrals of motion. The first isoscalar integral is the most simple one:

$$2i\mathcal{L}_{11}^+(t) + \hbar \mathcal{F}_\uparrow^+(t) = \text{const}$$

and has a clear physical interpretation – the conservation of the total angular momentum $\langle \hat{J}_1 \rangle = \langle \hat{l}_1 \rangle + \langle \hat{S}_1 \rangle$. Really, by definition

$$\langle \hat{l}_1 \rangle = Tr(\hat{l}_1 \hat{\rho}) = \sum_{\tau, s} \int d^3r \int d^3r' \langle \hat{l}_1(r) | \hat{l}_1(r') \rangle \langle r' | \hat{\rho}^\tau \hat{r} | r, s \rangle$$

$$= \sum_{\tau, s} \int d^3r \int d^3r' \hat{l}_1(r) \delta(r - r') \langle r' | \hat{\rho}^\tau | r, s \rangle = \sum_{\tau} \int d^3r \hat{l}_1(r) \left( [\langle r | \hat{\rho}^\tau | r' \rangle \rangle^\tau + \langle r | \hat{\rho}^\tau | r' \rangle \rangle^\tau \right)$$

$$= \int d(p, r) l_1(r, p) f_\uparrow^+ (r, p, t) = -i\sqrt{2} \int d(p, r) \{ r \otimes p \}_{11} f_\uparrow^+ (r, p, t) = -i\sqrt{2} \mathcal{L}_{11}^+(t). (38)$$

The average value of the spin operator $\hat{S}_1$ reads:

$$\langle \hat{S}_1 \rangle = Tr(\hat{S}_1 \hat{\rho}) = \sum_{\tau, s, s'} \int d^3r \langle \hat{S}_1 | s' \rangle \langle r, s' | \hat{\rho}^\tau | r, s \rangle$$

$$= \sum_{s, s'} \langle s | \hat{S}_1 | s' \rangle \int d(p, r) \hat{f}_\uparrow^s (r, p, t) = -\frac{\hbar}{\sqrt{2}} \sum_{s, s'} \delta_{s^+} \delta_{s'^+} \hat{F}^s(t) = -\frac{\hbar}{\sqrt{2}} \hat{F}_\uparrow^+(t). (39)$$

As a result $\langle \hat{J}_1 \rangle = -\frac{1}{\sqrt{2}} \left( 2i\mathcal{L}_{11}^+ + \hbar \mathcal{F}_\uparrow^+ \right)$. It is easy to see that such a combination of the respective equations of motion in (36) is equal to zero in the isoscalar case ($\alpha = 1$), i.e. the total angular momentum is conserved. The isovector counterpart of this integral of motion implies that the relative (neutrons with respect of protons) total angular momentum oscillates in phase (we recall that $\kappa_0 < 0$) with the linear combination of three variables $\mathcal{R}_{20}^\uparrow, \mathcal{R}_{00}^\uparrow$ and $\mathcal{P}_{00}^\uparrow$.

The second integral of motion can be interpreted saying that the definite combination of variables $\left( \sqrt{\frac{3}{2}} \mathcal{P}_{22}^\uparrow - \mathcal{P}_{20}^\uparrow - \sqrt{2} \mathcal{P}_{00}^\uparrow \right)$, describing the quadrupole and monopole deformations of the Fermi surface together with the spin flip, oscillates out of phase with the exactly the same combination of variables $\left( \sqrt{\frac{3}{2}} \mathcal{R}_{22}^\uparrow - \mathcal{R}_{20}^\uparrow - \sqrt{2} \mathcal{R}_{00}^\uparrow \right)$, describing the quadrupole and monopole deformations of the density distribution together with the spin flip. It is interesting to note that in the analogous problem without spin [23] there is the similar integral, saying that the
nuclear density and the Fermi surface oscillate out of phase. The physical interpretation of the third integral and the integrals 4, 5, 6, 7 appearing due to pairing, seems not to be obvious.

Let us to prove that the conservation of the total angular momentum follows from the set of equations (29), which describe the motion without any restrictions on the values (small or large) of amplitudes. It is necessary to consider the first equation of (29) in the isoscalar case for \( \lambda = \mu = 1 \). Having in mind that \( R_{11}^+ = P_{11}^+ = 0 \) we find

\[
\dot{L}_{11}^+ = 60 \left\{ \frac{1}{211} \left\{ \{ \lambda \mu \}^+_{11} \right\} Z_{211}^+ \otimes R_{211}^+ \right\}_{11} - i \hbar \eta \left( \frac{1}{2} \sum_{\nu, \sigma} C_{22,22}^1 \left( \kappa R_{22}^{\nu+} + \bar{\kappa} R_{22}^{\nu-} \right) R_{22}^{\nu+} \right)
\]

\[
- \int d^3r \left[ \frac{1}{2} n_{11}^+ \{ r \otimes \nabla \} _{11} V_{11}^+ + \frac{1}{2} n_{11}^- \{ r \otimes \nabla \} _{11} V_{11}^- + n_{11}^{+\dagger} \{ r \otimes \nabla \} _{11} V_{11}^{+\dagger} + n_{11}^{-\dagger} \{ r \otimes \nabla \} _{11} V_{11}^{-\dagger} \right].
\]

(40)

Let us analyze the first term. We have for protons:

\[
\{ \{ \lambda \mu \}^{p+}_{22} \otimes R_{22}^{p+} \}_{11} = \sum_{\nu, \sigma} C_{22,22}^1 \left( \kappa R_{22}^{\nu+} + \bar{\kappa} R_{22}^{\nu-} \right) R_{22}^{\nu+}
\]

(41)

We have used here the definition (18) of \( Z_{22}^{\nu+} \) and the equality \( C_{22,22}^1 = -C_{22,22}^1 \). Analogously one finds for neutrons:

\[
\{ \{ \lambda \mu \}^{n+}_{22} \otimes R_{22}^{n+} \}_{11} = \sum_{\nu, \sigma} C_{22,22}^1 \left( \kappa R_{22}^{\nu+} + \bar{\kappa} R_{22}^{\nu-} \right) R_{22}^{\nu-}
\]

(42)

The sum of (41) and (42) is obviously equal to zero.

The integral in (40) consists of four terms. The first one is (see the definition of \( V_{11}^+ \) in (28)):

\[
-\frac{3}{16} \hbar^2 \chi \int d^3r n_{11}^+ C_{11,10}^{11} \left[ r_1 \nabla_0 - r_0 \nabla_1 \right] n_{11}^+ = -\frac{3}{32} \hbar^2 \chi \int d^3r C_{11,10}^{11} \left[ r_1 \nabla_0 - r_0 \nabla_1 \right] (n_{11}^+)^2.
\]

(43)

Integrating by parts we find that this integral is equal to zero because \( \nabla_1 r_0 = \nabla_0 r_1 = 0 \).

The second term of integral in (40) can be written (for protons) as

\[
\frac{\hbar^2}{8} \int d^3r n_p C_{11,10}^{11} \left[ r_1 \nabla_0 - r_0 \nabla_1 \right] \left( \frac{3}{2} \chi n_p^- + \bar{\chi} n_n^- \right) = \frac{\hbar^2}{8} \chi \int d^3r C_{11,10}^{11} n_p^- \left[ r_1 \nabla_0 - r_0 \nabla_1 \right] n_n^-.
\]

(44)

Changing here the indices \( p \leftrightarrow n \) we obtain the analogous integral for neutrons. Their sum is obviously equal to zero.
The third and fourth terms of integral in (40) must be analyzed together. We have for protons:

\[
\frac{\hbar^2}{4} C_{11}^{11,10} \int d^3r \left[ n_p^{\uparrow \downarrow} (r_1 \nabla_0 - r_0 \nabla_1) \left( \frac{3}{2} \chi n_p^{\uparrow \uparrow} + \bar{\chi} n_p^{\downarrow \downarrow} \right) + n_p^{\downarrow \uparrow} (r_1 \nabla_0 - r_0 \nabla_1) \left( \frac{3}{2} \chi n_p^{\downarrow \uparrow} + \bar{\chi} n_p^{\uparrow \downarrow} \right) \right] = \frac{\hbar^2}{4} C_{11}^{11,10} \bar{\chi} \int d^3r \left[ n_p^{\uparrow \downarrow} (r_1 \nabla_0 - r_0 \nabla_1) n_n^{\downarrow \downarrow} + n_p^{\downarrow \uparrow} (r_1 \nabla_0 - r_0 \nabla_1) n_n^{\uparrow \downarrow} \right].
\]

(45)

The sum of this integral with the analogous one for neutrons (which is obtained by changing indices \( p \leftrightarrow n \)) is obviously equal to zero.

So, finally we have found that the isoscalar variant of equation (40) can be written (in variables defined in (35)) as

\[
\dot{L}_{11}^p(t) + \dot{L}_{11}^n(t) \equiv \hat{L}_{11}(t) = -i\hbar \frac{\gamma}{2} \left[ L_{11}^-(t) + \sqrt{2} L_{10}^\uparrow(t) \right].
\]

(46)

It is easy to see that the proper combination of this equation with the seventh equation in (29) gives the required result:

\[
-\frac{1}{\sqrt{2}} \left( 2i \dot{L}_{11}^\uparrow + \hbar \dot{F}_{11}^{\uparrow \downarrow} \right) = \frac{d}{dt} \langle \hat{J}_1 \rangle = 0,
\]

i.e. the total angular momentum is conserved for arbitrary amplitudes, not only in a small amplitude approximation. One must note that this result is not influenced by the approximate treatment of integral terms in (29).

V. RESULTS OF CALCULATIONS

A. Choice of parameters

- Following our previous publications [23, 24] we take for the isoscalar strength constant of the quadrupole-quadrupole residual interaction \( \kappa_0 \) the self consistent value [25] \( \kappa_0 = -m\bar{\omega}^2/(4Q_{00}) \) with \( Q_{00} = \frac{3}{5} AR_0^2 \), \( \bar{\omega}^2 = \omega_0^2/[1 + (\frac{4}{3}\delta)^2/3(1 - \frac{2}{3}\delta)^1/3] \), \( \hbar\omega_0 = 41/A^{1/3} \) MeV.

- The equations (36) contain the functions \( \Delta_0(r') \equiv \Delta_{eq}(r', p_F(r')) \), \( I_{r \rho}^{n\Delta}(r') \equiv I_{r \rho}^{n\Delta}(r', p_F(r')) \), \( I_{r \rho}^{n\Delta}(r') \equiv I_{r \rho}^{n\Delta}(r', p_F(r')) \) and \( \kappa_0(r') \equiv \kappa(r', r') \) depending on the radius \( r' \) and the local Fermi momentum \( p_F(r') \) (see Fig. 2). The value of \( r' \) is not fixed by the theory and can be used as the fitting parameter. We have found in our previous paper [14] that the best agreement of calculated results with experimental data is achieved at the point \( r' \) where the function \( I_{r \rho}^{n\Delta}(r', p_F(r')) \) has its maximum. Nevertheless, to get rid off the fitting
FIG. 2: The pair field (gap) $\Delta_0(r)$, the function $\Delta(r) = |V_0| I_{pp}^\Delta(r)$ and the nuclear density $n(r)$ as the functions of radius $r$. The solid lines – calculations without the spin-spin interaction $H_{ss}$, the dashed lines – $H_{ss}$ is included.

parameter, we use the averaged values of these functions: $\bar{\Delta}_0 = \int dr \, n(r) \Delta_0(r, p_F(r))/A$, etc. The gap $\Delta(r, p_F(r))$, as well as the integrals $I_{pp}^\Delta(r, p_F(r))$, $K_4$ and $K_0$, were calculated with the help of the semiclassical formulae for $\kappa(r, p)$ and $\Delta(r, p)$ (see Appendix E), a Gaussian being used for the pairing interaction with $r_p = 1.9$ fm and $V_0 = 25$ MeV [20]. Those values reproduce usual nuclear pairing gaps.

• The used spin-spin interaction is repulsive, the values of its strength constants being taken from the paper [26], where the notation $\chi = K_s/A$, $\bar{\chi} = q\chi$ was introduced. The constants were extracted by the authors of [26] from Skyrme forces following the standard procedure, the residual interaction being defined in terms of second derivatives of the Hamiltonian density $H(\rho)$ with respect to the one-body densities $\rho$. Different variants of Skyrme forces produce different strength constants of the spin-spin interaction. The most consistent results are obtained with SG1, SG2 [27] and Sk3 [28] forces. To compare theoretical results with experiment the authors of [26] preferred to use the force SG2. Nevertheless they have noticed that “As is well known, the energy splitting of the HF states around the Fermi level is too large. This has an effect on the spin $M1$ distributions that can be roughly compensated by reducing the $K_s$ value”. According to this remark they changed the original self-consistent SG2 parameters from $K_s = 88$ MeV, $q = -0.95$ to $K_s = 50$ MeV, $q = -1$. It was found that this modified set of parameters gives better agreement with experiment for some nuclei in the description of spin-flip resonance. So we will use $K_s = 50$ MeV and $q = -1$. 

Our calculations without pairing [16] have shown that the results for scissors modes are strongly dependent on the values of the strength constants of the spin-spin interaction. The natural question arises: how sensitive are they to the strength of the spin-orbital potential?

The results of the demonstrative calculations are shown in Fig. 3. The $M1$ strengths were computed using effective spin gyromagnetic factors $g_s^{\text{eff}} = 0.7g_s^{\text{free}}$. One observes a rather strong dependence of the results on the value of $\eta$: the splitting $\Delta E$ and the $M1$ strength of the spin scissors grow with increasing $\eta$, the $B(M1)$ of the orbital scissors being decreased. At some critical point $\eta_c$ the $M1$ strength of the spin scissors becomes bigger than that of the orbital scissors.

The inclusion of the spin-spin interaction does not change the qualitative picture, as well as the inclusion of pair correlations (see Fig. 3). Nevertheless, it is necessary to note, that the
spin-spin interaction decreases remarkably the value of $\eta_c$, whereas pair correlations slightly increase it.

What value of $\eta$ to use? Accidentally, the choice of $\eta$ in our papers [15, 16] was not very realistic. The main purpose of the first paper was the introduction of spin degrees of freedom into the WFM method, and the aim of the second paper was to study the influence of spin-spin forces on both scissors – we did not worry much about the comparison with experiment. Now, both preliminary aims being achieved, one can think about the agreement with experimental data, therefore the precise choice of the model parameters becomes important. Of course, we could try to choose $\eta$ according to the standard requirement of the best agreement with experiment. However, in reality we are not absolutely free in our choice. It turns out that we are already restricted by the other constraints. As a matter of fact we work with the Nilsson potential, parameters of which are very well known. Really, the mean field of our model (9) is the deformed harmonic oscillator with the spin-orbit potential, the Nilsson $\ell^2$ term being neglected because it generates the fourth order moments and, anyway, they are probably not of great importance. In the original paper [29] Nilsson took the spin-orbit strength constant $\kappa_{\text{Nils}} = 0.05$ for rare earth nuclei. Later the best value of $\kappa_{\text{Nils}}$ for rare earth nuclei was established [20] to be 0.0637. For actinides there were established different values of $\kappa_{\text{Nils}}$ for neutrons (0.0635) and protons (0.0577). The numbers $\kappa_{\text{Nils}} = 0.0637$, $\kappa_{\text{Nils}} = 0.05$ and $\kappa_{\text{Nils}} = 0.024$ (corresponding to $\eta = 0.36$ used in our calculations [15, 16]) are marked on Figs 3, 5 by the dotted vertical lines. Of course we will use the conventional [20] parameters of the Nilsson potential and from now on we will speak only about the Nilsson [29] spin-orbital strength parameter $\kappa_{\text{Nils}}$, which is connected with $\eta$ by the relation $\kappa_{\text{Nils}} = \eta/2\hbar\omega$.

B. Discussion and interpretation of the results without pairing

The dependence of energies and excitation probabilities on the spin-spin interaction is demonstrated in Table I (isovector) and in Table II (isoscalar). These results are obtained by the solution of the set of equations (36) without pairing.

To avoid misunderstanding we want to recall here that quantum numbers of all levels are $K^\pi = 1^+$ (the projection of the total angular momentum and parity). The first columns of tables I and II demonstrate the labels ($\lambda, \mu$ and spin projections $\uparrow, \downarrow$) of variables which
TABLE I: Isovector energies and excitation probabilities of $^{164}$Dy. Deformation parameter $\delta = 0.26$, spin-orbit constant $\kappa_{N\hbar} = 0.0637/\hbar^2$. Spin-spin interaction constants are: I $- K_s = 0$ MeV/$\hbar^2$; II $- K_s = 50$ MeV/$\hbar^2$, $q = -0.5$; III $- K_s = 92$ MeV/$\hbar^2$, $q = -0.8$. Quantum numbers (including indices $\varsigma = +, -, \uparrow\downarrow, \downarrow\uparrow$) of variables responsible for the generation of the present level are shown in the first column. For example: $(1,1)^- –$ spin scissors, $(1,1)^+ –$ conventional scissors, etc.. The numbers in the last line are imaginary, so they are marked by the letter i.

| $(\lambda, \mu)^\varsigma$ | $E_{iv}$, MeV | $B(M1)^\uparrow$, $\mu^2_N$ | $B(E2)$, W.u. |
|-----------------------------|---------------|-----------------|----------------|
|                            | I  | II  | III | I  | II  | III | I  | II  | III |
| $(1,1)^-$                   | 1.47 | 1.67 | 1.80 | 5.58 | 6.45 | 7.02 | 0.14 | 0.21 | 0.28 |
| $(1,1)^+$                   | 2.64 | 2.76 | 2.86 | 1.99 | 1.82 | 1.73 | 0.90 | 1.01 | 1.11 |
| $(0,0)^{\uparrow\downarrow}$ | 12.66 | 14.60 | 15.87 | 0.05 | 0.07 | 0.09 | 0.27 | 0.50 | 0.72 |
| $(2,1)^-$                   | 14.45 | 16.34 | 17.66 | 0.06 | 0.08 | 0.10 | 0.22 | 0.37 | 0.58 |
| $(2,0)^{\uparrow\downarrow}$ | 16.10 | 18.22 | 19.62 | 0.12 | 0.18 | 0.36 | 1.06 | 1.88 | 4.29 |
| $(2,2)^{\uparrow\downarrow}$ | 16.25 | 19.32 | 21.21 | 0 | 0.09 | 0.82 | 0 | 1.00 | 10.55 |
| $(2,1)^+$                   | 20.78 | 21.26 | 21.88 | 2.83 | 2.49 | 1.40 | 34.33 | 31.49 | 18.58 |
| $(1,0)^{\uparrow\downarrow}$ | 0.70 | 1.69 | 0.69 | -9.8 | -19.7 | -19.7 | i0.04 | i0.04 | i0.04 |

are responsible (approximately, because all equations are coupled) for the generation of the corresponding eigenvalue.

One can see from Table I that the spin-spin interaction does not change the qualitative picture of the positions of excitations described in [15]. It pushes all levels up proportionally to its strength (2-20% in the case II and 5-30% in the case III) without changing their order. The most interesting result concerns the relative $B(M1)$ values of the two low lying scissors modes, namely the spin scissors $(1,1)^-$ and the conventional (orbital) scissors $(1,1)^+$ mode. As can be noticed, the spin-spin interaction strongly redistributes M1 strength in the favour of the spin scissors mode. We tentatively want to link this fact to the recent experimental finding in isotopes of Th and Pa [17]. The authors have studied deuteron and $^3$He-induced reactions on $^{232}$Th and found in the residual nuclei $^{231,232,233}$Th and $^{232,233}$Pa ”an unexpectedly strong integrated strength of $B(M1) = 11 – 15 \mu^2_N$ in the $E_\gamma = 1.0 – 3.5$ MeV region”. The $B(M1)$ force in most nuclei shows evident splitting into two Lorentzians. "Typically, the experimental splitting is $\Delta \omega_{M1} \sim 0.7$ MeV, and the ratio of the strengths
between the lower and upper resonance components is $B_L/B_U \sim 2^\circ$. (Note a misprint in that paper: it is written erroneously $B_2/B_1 \sim 2$ whereas it should be $B_1/B_2 \sim 2$. To avoid misunderstanding, we write here $B_L$ instead of $B_1$ and $B_U$ instead of $B_2$.) The authors have tried to explain the splitting by a $\gamma$-deformation. To describe the observed value of $\Delta \omega_{M1}$ the deformation $\gamma \sim 15^\circ$ is required, that leads to the ratio $B_L/B_U \sim 0.7$ in an obvious contradiction with experiment. The authors conclude that "the splitting may be due to other mechanisms". In this sense, we tentatively may argue as follows. On one side, theory [30] and experiment [31] give zero value of $\gamma$-deformation for $^{232}$Th. On the other side, it is easy to see that our theory suggests the required mechanism. The calculations performed for $^{232}$Th give $\Delta \omega_{M1} \sim 0.32$ MeV and $B_L/B_U \sim 1.6$ for the first variant of the spin-spin interaction and $\Delta \omega_{M1} \sim 0.28$ MeV and $B_L/B_U \sim 4.1$ for second one in reasonable agreement with experimental values.

The general picture of the influence of the spin-spin interaction on isoscalar energies and excitation probabilities (Table II) is quite close to that observed in the isovector case. The only difference is the low lying mode marked by $(1, 1)^+$ which is practically insensitive to the spin-spin interaction. In ref [17] the assignment of the resonances to be of isovector type is only tentative based on the assumption that at such low energies there is no collective mode other than the isovector scissors mode. However, from [17] one cannot exclude that also an isoscalar spin scissors mode is mixed in. From our analysis we see that the isoscalar spin scissors (the level $(1, 1)^-$) where all nucleons with spin up counter-rotate with respect the ones of spin down comes more or less at the same energy as the isovector scissors. So it would be very important for the future to pin down precisely the quantum numbers of the resonances.

Let us discuss in more detail the nature of the predicted excitations. As one sees, the generalization of the WFM method by including spin dynamics allowed one to reveal a variety of new types of nuclear collective motion involving spin degrees of freedom. Two isovector and two isoscalar low lying eigenfrequencies and five isovector and five isoscalar high lying eigenfrequencies have been found.

Three low lying levels correspond to the excitation of new types of modes. For example the isovector level marked by $(1, 1)^-$ describes rotational oscillations of nucleons with the spin projection "up" with respect of nucleons with the spin projection "down", i.e. one can talk of a nuclear spin scissors mode. Having in mind that this excitation is an isovector one,
TABLE II: The same as in Table I, but for isoscalar excitations.

| (λ, µ) | $E_{is}$, MeV | $B(M1)\uparrow$, $\mu_N^2$ | $B(E2)$, W.u. |
|--------|---------------|-----------------|----------------|
|        | I  | II | III | I  | II | III | I  | II | III | I  | II | III | I  | II | III |
| (1,1)$^+$ | 0.90 | 0.91 | 0.91 | -0.22 | -0.22 | -0.22 | 37.51 | 39.76 | 39.76 |
| (1,1)$^-$ | 1.87 | 2.00 | 2.00 | 0.13 | 0.12 | 0.12 | 9.21 | 7.11 | 7.11 |
| (2,1)$^+$ | 9.99 | 10.75 | 10.75 | 0 | 0 | 0 | 62.09 | 58.96 | 58.96 |
| (0,0)$^\uparrow\downarrow$ | 12.89 | 14.04 | 14.04 | 0 | 0 | 0 | 3.40 | 2.61 | 2.61 |
| (2,1)$^-$ | 14.54 | 15.77 | 15.77 | 0 | 0 | 0 | 0.77 | 0.60 | 0.60 |
| (2,2)$^\uparrow\downarrow$ | 16.25 | 17.69 | 17.69 | 0 | 0 | 0 | 0.44 | 0.44 |
| (2,0)$^\uparrow\downarrow$ | 16.37 | 17.90 | 17.90 | 0 | 0 | 0 | 1.31 | 0.49 | 0.49 |
| (1,0)$^\uparrow\downarrow$ | i0.54 | i0.54 | i0.54 | i0.10 | i0.10 | i0.10 | i14.3 | i14.0 | i14.0 |

we can see that the resulting motion looks rather complex – proton spin scissors counter-rotates with respect to the neutron spin scissors (see Fig. 1). Thus the experimentally observed group of $1^+$ peaks in the interval 2-4 MeV, associated usually with the nuclear scissors mode, in reality consists of the excitations of the "spin" scissors mode together with the conventional [1] scissors mode (the level $(1,1)^+$ in our case). The isoscalar level $(1,1)^-$ describes the real spin scissors mode: all spin up nucleons (protons together with neutrons) oscillate rotationally out of phase with all spin down nucleons.

Such excitations were, undoubtedly, produced implicitly by other methods (e.g. RPA [1, 2, 26, 32]), but they never were analyzed in such terms. It is interesting to note, for example, that in [2] the scissors mode was analyzed in so-called spin and orbital components. Roughly speaking there are two groups of states corresponding to these two types of components, not completely dissimilar to our finding. Whereas the nature of the orbital, i.e. conventional scissors is quite clear, the authors did not analyze the character of their states which consist of the spin component. It can be speculated that those spin components just correspond to the isovector spin scissors mode discussed in our work here. It would be interesting to study whether our suggestion is correct or not. This could for example be done in analyzing the current patterns.

One more new low lying mode (isoscalar at 0.90 MeV, marked by $(1,1)^+$) is generated by the relative motion of the orbital angular momentum and spin of the nucleus. They can
change their absolute values and directions keeping the total spin unchanged. If there was not
the spin-orbit coupling, orbital angular momentum and spin would be constants of motion
separately, see dynamical equations for $L_{i1}$ (the orbital angular momentum variable) and $F^{\uparrow\downarrow}$
(the spin variable) in the isoscalar variant of the set of equations (36). Apparently spin-orbit
force is too weak to lift up this zero mode strongly. Physically it is quite understandable that
such a mode can exist. We want to call it ’collective spin-orbit mode’. Another question is
whether such a collective spin-orbit mode can be excited experimentally. In any case, to our
knowledge a low lying mode of this type with a strong $B(E2)$ has so far not been identified
experimentally. On the other hand the negligibly small negative $B(M1)$ value probably has
to do with the approximate treatment of integrals in the equations of motion (29) (especially
the neglect by the terms generating fourth order moments, see appendix A).

In order to complete the picture of the low-lying states, it is important to discuss the
state which is slightly imaginary. Let us first state that the nature of this state has nothing
to do with neither spin scissors nor with conventional scissors. It can namely be seen from
the structure of our equations that this state corresponds to a spin flip induced by the spin-
orbit potential. Such a state is of purely quantal character and it cannot be hoped that we
can accurately describe it with our WFM approach restricting the consideration by second
order moments only. For its correct treatment, we certainly should consider higher moments
like fourth order moments, for instance. The spin-orbit potential is the only term in our
theory which couples the second order moments to the fourth order ones. As mentioned, we
decoupled the system in neglecting the fourth order moments. Therefore, it is no surprise
that this particular spin flip mode is not well described. Nevertheless, one may try to better
understand the origin of this mode almost at zero energy. For this, we make the following
approximation of our diagonalisation procedure to get the eight eigenvalues listed in Table I.
We neglect in (36) all couplings between the set of variables $X_{\lambda\mu}^+, \lambda\mu$ and the set of variables
$X_{\lambda\mu}^{\uparrow\downarrow}, X_{\lambda\mu}^\downarrow\uparrow$ ($X \equiv L, R, P, F$). To this end in the dynamical equations for $X_{\lambda\mu}^+, \lambda\mu$ we omit all
terms containing $X_{\lambda\mu}^{\uparrow\downarrow}, X_{\lambda\mu}^\downarrow\uparrow$ and in the dynamical equations for $X_{\lambda\mu}^{\uparrow\downarrow}, X_{\lambda\mu}^\downarrow\uparrow$ we omit all terms
containing $X_{\lambda\mu}^+, X_{\lambda\mu}^-$. In such a way we get two independent sets of dynamical equations.
The first one (for $X_{\lambda\mu}^+, X_{\lambda\mu}^-$) was already studied in [15], where we have found that such
approximation gives satisfactory (in comparison with the exact solution) results but must
be used cautiously because of the problems with the angular momentum conservation. The
second set of equations (for $X_{\lambda\mu}^{\uparrow\downarrow}, X_{\lambda\mu}^\downarrow\uparrow$) splits into three independent subsets. Two of them
were already analyzed in [15] (it turns out that these subsets can be obtained also in the limit \( \eta \to 0 \), which was studied there), where it was shown that the results of approximate calculations are very close to that of exact calculations, i.e. the coupling between the respective variables \( X_{\lambda\mu}^{\uparrow\downarrow}, X_{\lambda\mu}^{\downarrow\uparrow} \) and \( X_{\lambda\mu}^{\uparrow +}, X_{\lambda\mu}^{\downarrow -} \) is very weak. The only new subset of equations reads:

\[
\dot{\mathcal{L}}_{10}^{\uparrow\downarrow} = -\hbar^2 \frac{\eta}{2\sqrt{2}} \mathcal{F}_{\downarrow\uparrow}^{\uparrow\downarrow}, \\
\dot{\mathcal{F}}_{\downarrow\downarrow}^{\uparrow\downarrow} = -\eta \sqrt{2}\mathcal{L}_{10}^{\uparrow\downarrow}.
\]  

(47)

The solution of these equations is \( E = \frac{i\hbar}{2\sqrt{2}} \eta = 0.676 i \) what practically coincides with the number of the full diagonalisation. So the non-zero (purely imaginary) value of this root only comes from the fact that z-component of orbital angular momentum is not conserved (only total spin \( J \) is conserved). However, the violation of the conservation of orbital angular momentum is very small as can be seen from the numbers. In any case, we see that this spin flip state has nothing to do with neither the spin scissors nor with the conventional scissors.

Two high lying excitations of a new nature are found. They are marked by \((2, 1)^-\) and following the paper [32] can be called spin-vector giant quadrupole resonances. The isovector one corresponds to the following quadrupole motion: the proton system oscillates out of phase with the neutron system, whereas inside of each system spin up nucleons oscillate out of phase with spin down nucleons. The respective isoscalar resonance describes out of phase oscillations of all spin up nucleons (protons together with neutrons) with respect of all spin down nucleons.

Six high lying modes can be interpreted as spin-flip giant monopole (marked by \((0, 0)^{-}\)) and quadrupole (marked by \((2, 0)^{\downarrow\uparrow}\) and \((2, 2)^{\uparrow\downarrow}\)) resonances.

C. Discussion of results with pairing

The results of calculations with pairing taken into account are compared with that of without pairing in Tables III (isovector) and IV (isoscalar). As it was expected the energies of all four low lying modes increased approximately by 1 Mev after inclusion of pairing. The behaviour of transition probabilities turned out less predictable. The \( B(M1) \) value of the spin scissors decreased approximately by 1.5 \( \mu^2 \), whereas \( B(M1) \) value of the orbital scissors turned out practically insensitive to the inclusion of pair correlations, \( B(E2) \) values
TABLE III: Isovector energies and excitation probabilities of $^{164}$Dy. Deformation parameter $\delta = 0.26$, spin-orbit constant $\kappa_{\text{Nil}} = 0.0637$, spin-spin interaction constants are $K_s = 50$ MeV, $q = -0.5$. Results: a – without pair correlations, b – with pair correlations. The notation of the first column is the same as in Table I.

| $(\lambda, \mu)$ | $E_{iv}$, MeV | $B(M1)^\dagger$, $\mu_N^2$ | $B(E2)$, W.u. |
|-----------------|----------------|------------------|-----------------|
|                 | a   | b   | a   | b   | a   | b   |
| (1,1)$^-$       | 1.67| 2.80| 6.45| 4.98| 0.21| 0.52|
| (1,1)$^+$       | 2.76| 3.57| 1.82| 1.71| 1.01| 1.94|
| (0,0)$^\dagger$ | 14.60| 14.60| 0.07| 0.07| 0.50| 0.47|
| (2,1)$^-$       | 16.34| 16.50| 0.08| 0.08| 0.37| 0.38|
| (2,0)$^\dagger$ | 18.22| 18.23| 0.18| 0.17| 1.88| 1.76|
| (2,2)$^\dagger$ | 19.32| 19.32| 0.09| 0.08| 1.00| 0.93|
| (2,1)$^+$       | 21.26| 21.33| 2.49| 2.47| 31.49| 31.32|
| (1,0)$^\dagger$ | i0.69| i0.59| -i9.7| -i9.7| i0.04| -i0.02|

TABLE IV: The same as in Table III, but for isoscalar excitations.

| $(\lambda, \mu)$ | $E_{is}$, MeV | $B(M1)^\dagger$, $\mu_N^2$ | $B(E2)$, W.u. |
|-----------------|----------------|------------------|-----------------|
|                 | a   | b   | a   | b   | a   | b   |
| (1,1)$^+$       | 0.91| 1.55| -0.22| -0.08| 39.76| 50.97|
| (1,1)$^-$       | 2.00| 2.97| 0.04| 0.04| 7.11| 2.14|
| (2,1)$^+$       | 10.75| 11.05| 0  | 0  | 58.96| 53.47|
| (0,0)$^\dagger$ | 14.04| 14.05| 0  | 0  | 2.61| 2.90|
| (2,1)$^-$       | 15.77| 15.93| 0  | 0  | 0.60| 0.58|
| (2,2)$^\dagger$ | 17.69| 17.69| 0  | 0  | 0.44| 0.45|
| (2,0)$^\dagger$ | 17.90| 17.90| 0  | 0  | 0.49| 0.52|
| (1,0)$^\dagger$ | i0.54| i0.49| i0.10| i0.02| i14.0| i6.93|

of both isovector scissors increased two times. The $B(M1)$ value of the isoscalar scissors (excitation $(1,1)^-$) does not feel pairing, whereas its $B(E2)$ value decreased three times. Rather interesting is the situation with $B(M1)$ value of the isoscalar excitation $(1,1)^+$. It has small negative value because the
appropriate linear response is calculated not enough accurately, that in its turn is explained by the neglect of the fourth order moments. It is seen that taking into account pairing makes the role of this error less important, reducing this (ridiculous) negative value three times. It is necessary to note also the remarkable (∼30%) increase of $B(E2)$ value of this mode.

The influence of pairing on energies and excitation probabilities of all high lying modes is negligible. The more or less remarkable change happens with isoscalar GQR $(2,1)^+$ only – its energy increased by 0.3 MeV (∼3%), whereas its $B(E2)$ value decreased about 9%.

We are interested mostly in the scissors modes. Let us compare the summed $B(M1)_\Sigma = B(M1)_{\text{or}} + B(M1)_{\text{sp}}$ values and the centroid of both scissors energies

$$E_{\text{cen}} = [E_{\text{or}}B(M1)_{\text{or}} + E_{\text{sp}}B(M1)_{\text{sp}}]/B(M1)_{\Sigma}$$

with the results of the paper [14] where no spin degrees of freedom had been considered and with the experimental data. The respective results are shown in the Table V. It is seen

| $^{164}$Dy | $E_{\text{cen}}$, MeV | $B(M1)_{\Sigma}$, $\mu^2_N$ |
|---------|-----------------|-----------------|
| $K_s = 50$ | Ref. [14] | Exp. | $K_s = 50$ | Ref. [14] | Exp. |
| $\bar{\Delta}_0 = 0$ | 1.91 | 2.17 | 8.27 | 9.59 | 3.14 | 3.18 |
| $\bar{\Delta}_0 \neq 0$ | 2.99 | 3.60 | 6.69 | 5.95 |

that the inclusion of spin degrees of freedom in the WFM method does not change markedly our results (in comparison with previous ones [14]). Of course, the energy changed in the desired direction and now practically coincides with the experimental value. However, the situation with the $B(M1)$ values did not change (and even becomes worse). Our hope, that spin degrees of freedom can improve the situation with the $B(M1)$ values, did not become true: the theory so far gives two-times-bigger values of $B(M1)$ than the experimental ones, exactly as it was the case in the paper [14].

The result look discouraging. However, a phenomenon, which was missed in our previous papers and described in the next section will save the situation.
VI. COUNTER-ROTATING ANGULAR MOMENTA OF SPINS UP/DOWN
(HIDDEN ANGULAR MOMENTA)

The equilibrium (ground state) orbital angular momentum of any nucleus is composed of two equal parts: half of nucleons (protons + neutrons) having spin projection up and other half having spin projection down. It is known that the huge majority of nuclei have zero angular momentum in the ground state. We will show below that as a rule this zero is just the sum of two rather big counter directed angular momenta (hidden angular momenta, because they are not manifest in the ground state) of the above mentioned two parts of any nucleus. Being connected with the spins of nucleons this phenomenon naturally has great influence on all nuclear properties connected with the spin, in particular, the spin scissors mode.

Let us analyze the procedure of linearization of the equations of motion for collective variables (25). We consider small deviations of the system from equilibrium, so all variables are written as a sum of their equilibrium value plus a small deviation:

\[ L(t) = L(eq) + L(t), \]

et al.

Neglecting quadratic deviations one obtains the set of linearized equations for deviations depending on the equilibrium values \( R_{\tau \varsigma}^{\rho \mu}(eq) \) and \( L_{\tau \varsigma}^{\rho \mu}(eq) \), which are the input data of the problem. In the paper [16] we made the choice shown in equations (32)-(34). For the sake of convenience we write it again:

\[ R_{2 \pm 1}^{\tau \varsigma}(eq) = R_{2 \pm 2}^{\tau \varsigma}(eq) = 0, \]

\[ R_{20}^{\tau \varsigma}(eq) \neq 0, \quad R_{00}^{\tau \varsigma}(eq) \neq 0, \quad (48) \]

\[ R_{1 \mu}^{\tau \varsigma}(eq) = R_{1 \mu}^{\tau \varsigma}(eq) = 0, \quad (49) \]

\[ L_{\lambda \mu}^{\tau \varsigma}(eq) = 0, \quad R_{\lambda \mu}^{-}(eq) = 0. \quad (50) \]

At first glance, this choice looks quite natural. Really, relations (48) follow from the axial symmetry of nucleus. Relations (49) are justified by the fact that these quantities should be diagonal in spin at equilibrium. The variables \( L_{\lambda \mu}^{\tau \varsigma}(t) \) contain the momentum \( p \) in their definition which incited us to suppose zero equilibrium values as well (we will show below that it is not true for \( L_{10}^{-} \) because of quantum effects connected with spin).

The relation \( R_{\lambda \mu}^{-}(eq) = 0 \) follows from the shell model considerations: the nucleons with spin projection "up" and "down" are sitting in pairs on the same levels, therefore all
average properties of the "spin up" part of the nucleus must be identical to that of the "spin down" part. However, a careful analysis shows that being undoubtedly true for variables $R_{\lambda\mu}^\dagger$, $R_{\lambda\mu}^\dagger$, this statement turns out erroneous for variables $L_{10}^\dagger$, $L_{10}^\dagger$. Let us demonstrate it. By definition

$$L_{\lambda\mu}^{ss'}(t) = \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} \{ r \otimes p \} \lambda_{\mu} f^{ss'}(r, p, t) = \int d^3r \{ r \otimes J^{ss'} \} \lambda_{\mu}, \quad (51)$$

where

$$J_i^{ss'}(r, t) = \int \frac{d^3p}{(2\pi\hbar)^3} p_i f^{ss'}(r, p, t) = \int \frac{d^3pp_i}{(2\pi\hbar)^3} \int d^3qe^{-\frac{i}{\hbar}q \cdot r} \rho \left( r + \frac{q}{2}, s; r - \frac{q}{2}, s'; t \right) \quad (52)$$

is the $i$-th component of the nuclear current. In the last relation the definition (B1) of Wigner function is used. Performing the integration over $p$ one finds:

$$J_i^{ss'}(r, t) = i\hbar \int d^3q [\frac{\partial}{\partial q_i} \delta(q)] \rho(r + \frac{q}{2}, s; r - \frac{q}{2}, s'; t)$$

$$= -i\hbar \int d^3q \frac{\partial}{\partial q_i} \rho(r + \frac{q}{2}, s; r - \frac{q}{2}, s'; t) = -\frac{i\hbar}{2} \{ (\nabla_{1i} - \nabla_{2i}) \rho(r_1, s; r_2, s'; t) \}_{r_1 = r_2 = r}, \quad (53)$$

where $r_1 = r + \frac{q}{2}$, $r_2 = r - \frac{q}{2}$. The density matrix of the ground state nucleus is defined [20] as

$$\rho(r_1, s; r_2, s'; t) = \sum_{\nu} v_{\nu}^2 \psi_\nu(r_1 s) \psi_\nu^*(r_2 s'), \quad (54)$$

where $v_{\nu}^2$ are occupation numbers and $\psi_\nu$ are single particle wave functions. For the sake of simplicity we will consider the case of spherical symmetry. Then $\nu = nljm$ and

$$\psi_{nljm}(r, s) = \mathcal{R}_{nlj}(r) \sum_{\lambda, \sigma} C_{l\lambda, 2\sigma}^{jm} Y_{l\lambda}(\theta, \phi) \chi_{2\sigma}(s), \quad (55)$$

$$J_i^{ss'}(r) = -\frac{i\hbar}{2} \sum_{\nu} v_{\nu}^2 [\nabla_i \psi_\nu(r, s) \cdot \psi_\nu^*(r, s') - \psi_\nu(r, s) \cdot \nabla_i \psi_\nu^*(r, s')] \quad (56)$$

$$= -\frac{i\hbar}{2} \sum_{nljm} v_{nljm}^2 \mathcal{R}_{nlj}^2 \sum_{\lambda, \sigma, \lambda', \sigma'} C_{l\lambda, 2\sigma}^{jm} C_{l\lambda', 2\sigma'}^{jm} [Y_{l\lambda}^* \nabla_i Y_{l\lambda} - Y_{l\lambda} \nabla_i Y_{l\lambda}^*] \chi_{2\sigma}(s) \chi_{2\sigma'}^*(s'). \quad (57)$$

Inserting this expression into (51) one finds:

$$L_{10}^{ss'}(eq) = \frac{i}{2} \sum_{nljm} v_{nljm}^2 \sum_{\lambda, \sigma, \lambda', \sigma'} C_{l\lambda, 2\sigma}^{jm} C_{l\lambda', 2\sigma'}^{jm} \chi_{2\sigma}(s) \chi_{2\sigma'}^*(s') \int d^3r \mathcal{R}_{nlj}^2 \{ Y_{l\lambda}^* \{ r \otimes \nabla \}_{10} Y_{l\lambda} - Y_{l\lambda} \{ r \otimes \nabla \}_{10} Y_{l\lambda}^* \}$$

$$= \frac{i}{2\sqrt{2}} \sum_{nljm} v_{nljm}^2 \sum_{\lambda, \sigma, \lambda', \sigma'} C_{l\lambda, 2\sigma}^{jm} C_{l\lambda', 2\sigma'}^{jm} \chi_{2\sigma}(s) \chi_{2\sigma'}^*(s') \int d^3r \mathcal{R}_{nlj}^2 \{ Y_{l\lambda}^* \tilde{t}_0 Y_{l\lambda} - Y_{l\lambda} \tilde{t}_0 Y_{l\lambda}^* \}$$

$$= \frac{i}{2\sqrt{2}} \sum_{nljm} v_{nljm}^2 \sum_{\lambda, \sigma, \lambda', \sigma'} C_{l\lambda, 2\sigma}^{jm} C_{l\lambda', 2\sigma'}^{jm} \chi_{2\sigma}(s) \chi_{2\sigma'}^*(s') \int d^3r \mathcal{R}_{nlj}^2 \{ Y_{l\lambda}^* \tilde{t}_0 Y_{l\lambda} - Y_{l\lambda} \tilde{t}_0 Y_{l\lambda}^* \}$$
By definition (25)

\[ \hat{\mu} = -\hbar \sqrt{2} \{ r \otimes \nabla \} \cdot \mu, \]

formula \( \hat{\mu}_0 Y_{l \Lambda} = \Lambda Y_{l \Lambda} \) and normalization of functions \( R_{nlj} \) were used. Remembering the definition of the spin function \( \chi_{\frac{s}{2}}(s) = \delta_{ss} \) we get finally:

\[ L_{10}^{ss'}(eq) = \frac{i}{\sqrt{2}} \sum_{nljm} v_{nljm}^2 \sum_{\Lambda} \left( C_{l \Lambda \frac{s}{2}}^{jm}(C_{l \Lambda \frac{s}{2}}^{jm})' \right)^2 \delta_{ss'} \delta_{ss'} \frac{i}{\sqrt{2}} \sum_{nljm} v_{nljm}^2 \left( C_{l \Lambda \frac{s}{2}}^{jm}(C_{l \Lambda \frac{s}{2}}^{jm})' \right)^2 (m - s). \]  

Now, with the help of analytic expressions for Clebsh-Gordan coefficients one obtains the final expressions

\[ L_{10}^{+}(eq) = \frac{i}{\sqrt{2}} \sum_{nl} \left[ \sum_{m = -(l + \frac{1}{2})}^{l + \frac{1}{2}} v_{nlj}^2 \frac{l + \frac{1}{2} + m}{2l + 1} + \sum_{m = -(l - \frac{1}{2})}^{-l - \frac{1}{2}} v_{nlj}^2 \frac{l + \frac{1}{2} - m}{2l + 1} \right] \left( m - \frac{1}{2} \right), \]  

\[ L_{10}^{-}(eq) = \frac{i}{\sqrt{2}} \sum_{nl} \left[ \sum_{m = -(l + \frac{1}{2})}^{l + \frac{1}{2}} v_{nlj}^2 \frac{l + \frac{1}{2} - m}{2l + 1} + \sum_{m = -(l - \frac{1}{2})}^{-l - \frac{1}{2}} v_{nlj}^2 \frac{l + \frac{1}{2} + m}{2l + 1} \right] \left( m + \frac{1}{2} \right), \]  

where the notation \( j^\pm = l \pm \frac{1}{2} \) is introduced. Replacing in (60) \( m \) by \( -m \) we find that

\[ L_{10}^{+}(eq) = -L_{10}^{-}(eq). \]

By definition (25) \( L_{10}^{+}(eq) = L_{10}^{+}(eq) \pm L_{10}^{+}(eq) \). Combining linearly (60) and (61) one finds:

\[ L_{10}^{+}(eq) = \frac{i}{\sqrt{2}} \sum_{nl} \left[ \sum_{m = -(l + \frac{1}{2})}^{l + \frac{1}{2}} v_{nlj}^2 \frac{2l + 2m}{2l + 1} m + \sum_{m = -(l - \frac{1}{2})}^{-l - \frac{1}{2}} v_{nlj}^2 \frac{2l - 2m}{2l + 1} m \right], \]  

\[ L_{10}^{-}(eq) = \frac{i}{\sqrt{2}} \sum_{nl} \left[ \sum_{m = -(l + \frac{1}{2})}^{l + \frac{1}{2}} v_{nlj}^2 \frac{2m^2 - l - \frac{1}{2}}{2l + 1} - \sum_{m = -(l - \frac{1}{2})}^{-l - \frac{1}{2}} v_{nlj}^2 \frac{2m^2 + l + \frac{1}{2}}{2l + 1} \right]. \]  

These formulas are valid for spherical nuclei. However, with the scissors and spin-scissors modes, we are considering deformed nuclei. For the sake of the discussion, let us consider the case of infinitesimally small deformation, when one can continue to use formulae (63, 64). Now only levels with quantum numbers \( \pm m \) are degenerate. According to, for example, the Nilsson scheme [29] nucleons will occupy pairwise precisely those levels which leads to the zero value of \( L_{10}^{+}(eq) \).

What about \( L_{10}^{-}(eq) \)? It only enters (36) in the equation for \( \hat{P}_{21} \). Let us analyze the structure of formula (64) considering for the sake of simplicity the case without pairing.
Two sums over \( m \) (let us note them \( \Sigma_1 \) and \( \Sigma_2 \)) represent the two spin-orbital partners: in the first sum the summation goes over the levels of the lower partner \((j = l + \frac{1}{2})\) and in the second sum – over the levels of the higher partner \((j = l - \frac{1}{2})\). The values of both sums depend naturally on the values of occupation numbers \( n_{nljm} = 0, 1 \). There are three possibilities. The first one is trivial: if all levels of both spin-orbital partners are disposed above the Fermi surface, then the respective occupation numbers \( n_{nljm} = 0 \) and both sums are equal to zero identically. The second possibility: all levels of both spin-orbital partners are disposed below the Fermi surface. Then all respective occupation numbers \( n_{nlj+m} = n_{nlj-m} = 1 \). The elementary analytical calculation (for arbitrary \( l \)) shows that in this case the two sums in (64) exactly compensate each other, i.e. \( \Sigma_1 + \Sigma_2 = 0 \). The most interesting is the third possibility, when one part of levels of two spin-orbital partners is disposed below the Fermi surface and another part is disposed above it. In this case the compensation does not happen and one gets \( \Sigma_1 + \Sigma_2 \neq 0 \) what leads to \( L_{10}^{(eq)} \neq 0 \). In the case of pairing, things are not so sharply separated and \( L_{10}^{(eq)} \) has always a finite value. However, the modifications with respect to mean field are very small.

Let us illustrate the above analysis by the example of \(^{164}\text{Dy}\) (protons). Its deformation is \( \delta = 0.26 \) (\( \epsilon = 0.28 \)) and \( Z=66 \). Looking on the Nilsson scheme (for example, Fig. 1.5 of [19] or Fig. 2.21c of [20]) one easily finds, that only three pairs of spin-orbital partners give a nonzero contribution to \( L_{10}^{(eq)} \). They are: \( N = 4, d_{5/2} - d_{3/2} \) (two levels of \( d_{5/2} \) are below the Fermi surface, all the rest – above); \( N = 4, g_{9/2} - g_{7/2} \) (one level of \( g_{7/2} \) is above the Fermi surface, all the rest – below); \( N = 5, h_{11/2} - h_{9/2} \) (four levels of \( h_{11/2} \) are below the Fermi surface, all the rest – above). It is possible to make the crude evaluation of \( L_{10}^{(eq)} \) using the quantum numbers indicated on Fig. 1.5 of [19] or Fig. 2.21c of [20]. The result turns out rather close to the exact one, computed with the help of formulas (51,56) and Nilsson wave functions. The influence of pair correlations is very small.

Indeed, from the definitions (51) and (58) one can see that \( L_{10}^{(eq)} \) is just the average value of the z-component of the orbital angular momentum of nucleons with the spin projection \( s \) (\( \frac{1}{2} \) or \( -\frac{1}{2} \)). So, the ground state nucleus consists of two equal parts having nonzero angular momenta with opposite directions, which compensate each other resulting in the zero total angular momentum. This is graphically depicted in Fig. 4(a).

On the other hand, when the opposite angular momenta become tilted, one excites the system and the opposite angular momenta are vibrating with a tilting angle, see Fig. 4(b).
FIG. 4: (a) Protons with spins $\uparrow$ (up) and $\downarrow$ (down) having nonzero orbital angular momenta at equilibrium. (b) Protons from Fig.(a) vibrating against one-another.

Actually the two opposite angular momenta are oscillating, one in the opposite sense of the other. It is rather obvious from Fig. 1 that these tilted vibrations happen separately in each of the neutron and proton lobes. These spin-up against spin-down motions certainly influence the excitation of the spin scissors mode. So, classically speaking the proton and neutron parts of the ground state nucleus consist each of two identical gyroscopes rotating in opposite directions. One knows that it is very difficult to deviate gyroscope from an equilibrium. So one can expect, that the probability to force two gyroscopes to oscillate as scissors (spin scissors) should be small. This picture is confirmed in the next section.

VII. RESULTS OF CALCULATIONS CONTINUED

We made the calculations taking into account the non zero value of $L_{10}^{(eq)}$ (which was computed according to formulas (51,56) and Nilsson wave functions). The results are presented in Fig. 5 and Table VI.

Figure 5 demonstrates the dependence of the scissors modes energies and $B(M1)$ values on the spin orbital strength constant $\kappa_{\text{Nils}}$. One can observe the strong influence of the hidden angular momenta on the spin scissors mode, whose $B(M1)$ value is strongly decreasing with increasing $\kappa_{\text{Nils}}$. The $B(M1)$ value of the orbital scissors also is reduced, but not so much,
FIG. 5: The energies $E$ and $B(M1)$-factors as a functions of the spin-orbital strength constant $\kappa_{\text{Nils}}$. The dashed lines – calculations without $L_{10}^{-}\,(\text{eq})$, the solid lines – $L_{10}^{-}\,(\text{eq})$ are taken into account. $H_{ss}$ and pairing are included.

the value of the reduction being practically independent on $\kappa_{\text{Nils}}$. The influence of $L_{10}^{-}\,(\text{eq})$ on the energies of both scissors is negligible, leading to a small increase of their splitting. Now the energy centroid of both scissors and their summed $B(M1)$ value at $\kappa_{\text{Nils}} = 0.0637$ are $E_{\text{cen}} = 3.07 \text{ MeV}$ and $B(M1)_{\Sigma} = 3.78 \mu_{N}^{2}$. The general agreement with experiment becomes considerably better (compare with Table V).

Table VI demonstrates the energies and transition probabilities of all (low and high lying) isovector modes of $^{164}\text{Dy}$ obtained by the solution of equations (36) with hidden angular momenta taken into account. The results for both scissors coincide with that of Fig. 5 for $\kappa_{\text{Nils}} = 0.0637$. It is seen also that all high lying modes are completely insensitive to the value of $L_{10}^{-}\,(\text{eq})$. The same is also true for all isoscalar modes. It is worth noting, nevertheless, that the small negative $B(M1)$ value of the isoscalar excitation $(1,1)^{+}$ (see Tab. IV) goes at last to zero!

The results of systematic calculations for the rare-earth nuclei are presented in Tables VII
TABLE VI: Isovector energies and excitation probabilities of \( ^{164}\text{Dy} \) with pair correlations taken into account. Deformation parameter \( \delta = 0.26 \), spin-orbit constant \( \kappa_{\text{Nils}} = 0.0637 \), spin-spin interaction constants are \( K_s = 50 \text{ MeV}, q = -0.5 \). Results: a – without \( L_{10}^- \) (eq), b – with \( L_{10}^- \) (eq).

| \((\lambda, \mu)^\dagger\) | \( E_{iv}, \text{MeV} \) | \( B(M1)^\dagger, \mu_N^2 \) | \( B(E2), \text{W.u.} \) |
|-----------------|-----------------|-----------------|-----------------|
|                 | \( a \)         | \( b \)         | \( a \)         | \( b \)         | \( a \)         | \( b \)         |
| (1,1)           | 2.80            | 2.77            | 4.98            | 2.44            | 0.52            | 0.49            |
| (1,1)           | 3.57            | 3.60            | 1.71            | 1.36            | 1.94            | 2.00            |
| (0,0)           | 14.60           | 14.60           | 0.07            | 0.07            | 0.47            | 0.47            |
| (2,1)           | 16.50           | 16.51           | 0.08            | 0.07            | 0.38            | 0.36            |
| (2,0)           | 18.23           | 18.22           | 0.17            | 0.17            | 1.76            | 1.80            |
| (2,1)           | 19.32           | 19.32           | 0.08            | 0.07            | 0.93            | 0.94            |
| (1,0)           | 21.33           | 21.32           | 2.47            | 2.48            | 31.32           | 31.30           |
|                 | \( i0.59 \)     | \( i0.58 \)     | -i5.4           | -i65            | -i0.02          | i0.0            |

and VIII and displayed in Fig. 6. Table VII contains the results for well deformed nuclei with \( \delta \geq 0.18 \). It is easy to see that the overall (general) agreement of theoretical results with experimental data is substantially improved (in comparison with our previous calculations [14]).

The results of calculations for two groups (“light” and “heavy”) of weakly deformed nuclei with deformations \( 0.14 \leq \delta \leq 0.17 \) are shown in the Table VIII. They require some discussion, because of the self-consistency problem. These two groups of nuclei are transitional between well deformed and spherical nuclei. Systematic calculations of equilibrium deformations [19] predict \( \delta_{\text{eq}}^{th} = 0.0 \) for \( ^{134}\text{Ba} \), \( \pm 0.1 \) for \( ^{148}\text{Nd} \), 0.15 or -0.12 for \( ^{150}\text{Sm} \), 0.1 or -0.14 for \( ^{190}\text{Os} \) and -0.1 for \( ^{192}\text{Os} \), whereas their experimental values are \( \delta_{\text{eq}} = 0.14, 0.17, 0.16, 0.15 \) and 0.14 respectively. As one sees, the discrepancy between theoretical and experimental \( \delta_{\text{eq}} \) is large. Uncertain signs of theoretical equilibrium deformations are connected with very small (\( \sim 0.1-0.2 \text{ MeV} \)) difference between the values of deformation energies \( E_{\text{def}} \) at positive and negative \( \delta_{\text{eq}} \). Even more so, the values of deformation energies of these nuclei are very small: \( E_{\text{def}} = 0.20, 0.50, 0.80 \) and 0.70 MeV for \( ^{148}\text{Nd} \), \( ^{150}\text{Sm} \), \( ^{190}\text{Os} \) and \( ^{192}\text{Os} \) respectively. This means that these nuclei are very ”soft” with respect of \( \beta \)- or \( \gamma \)-vibrations and probably they have more complicated equilibrium shapes, for example, hexadecapole or octupole
TABLE VII: Scissors modes energy centroids $E_{cen}$ and summarized transition probabilities $B(M1)\Sigma$. Parameters: $\kappa_{Nils} = 0.0637$, pairing strength constant $V_0 = 25$ ($V_0 = 27$ for $^{182,184,186}W$).

The experimental values of $E_{cen}$, $\delta$, and $B(M1)\Sigma$ are from [33, 34] and references therein.

| Nuclei | $\delta$ | $E_{cen}$, MeV | $B(M1)\Sigma$, $\mu^2_N$ |
|--------|---------|----------------|----------------------|
|        |         | Exp. WFM Ref. [14] $\Delta = 0$ | Exp. WFM Ref. [14] $\Delta = 0$ |
| $^{150}$Nd | 0.22  | 3.04 2.88 3.44 | 1.92 1.61 1.64 4.17 | 7.26 |
| $^{152}$Sm | 0.24  | 2.99 2.99 3.46 | 2.02 2.26 2.50 4.68 | 7.81 |
| $^{154}$Sm | 0.26  | 3.20 3.10 3.57 | 2.17 2.18 3.34 5.42 | 8.65 |
| $^{156}$Gd | 0.26  | 3.06 3.09 3.60 | 2.16 2.73 3.44 5.42 | 8.76 |
| $^{158}$Gd | 0.26  | 3.14 3.09 3.60 | 2.19 3.39 3.52 5.72 | 9.12 |
| $^{160}$Gd | 0.27  | 3.18 3.14 3.61 | 2.21 2.97 4.02 5.90 | 9.38 |
| $^{160}$Dy | 0.26  | 2.87 3.08 3.59 | 2.13 2.42 3.60 5.53 | 9.03 |
| $^{162}$Dy | 0.26  | 2.96 3.07 3.61 | 2.14 2.49 3.69 5.66 | 9.25 |
| $^{164}$Dy | 0.26  | 3.14 3.07 3.60 | 2.17 3.18 3.78 5.95 | 9.59 |
| $^{164}$Er | 0.25  | 2.90 3.01 3.57 | 2.10 1.45 3.39 5.62 | 9.26 |
| $^{166}$Er | 0.26  | 2.96 3.06 3.53 | 2.13 2.67 3.86 5.96 | 9.59 |
| $^{168}$Er | 0.26  | 3.21 3.06 3.53 | 2.10 2.82 3.95 5.95 | 9.67 |
| $^{170}$Er | 0.26  | 3.22 3.05 3.57 | 2.09 2.63 4.03 5.91 | 9.79 |
| $^{172}$Yb | 0.25  | 3.03 2.99 3.55 | 2.05 1.94 3.72 5.84 | 9.79 |
| $^{174}$Yb | 0.25  | 3.15 2.98 3.47 | 2.02 2.70 3.80 5.89 | 9.82 |
| $^{176}$Yb | 0.24  | 2.96 2.92 3.45 | 1.94 2.66 3.46 5.54 | 9.58 |
| $^{178}$Hf | 0.22  | 3.11 2.81 3.43 | 1.79 2.04 2.67 4.86 | 9.00 |
| $^{180}$Hf | 0.22  | 2.95 2.81 3.36 | 1.76 1.61 2.69 4.85 | 8.97 |
| $^{182}$W | 0.20  | 3.10 3.28 3.30 | 1.63 1.65 2.05 4.31 | 8.43 |
| $^{184}$W | 0.19  | 3.31 3.24 3.28 | 1.55 1.12 1.72 3.97 | 8.14 |
| $^{186}$W | 0.18  | 3.20 3.19 3.26 | 1.49 0.82 1.40 3.76 | 7.95 |

Deformations in addition to the quadrupole one. This means that for the correct description of their dynamical and equilibrium properties it is necessary to include higher order Wigner function moments (at least fourth order) in addition to the second order ones. In this case it would be natural also to use more complicate mean field potentials (for example, the
TABLE VIII: Scissors modes energy centroids $E_{cen}$ and summarized transition probabilities $B(M1)_{\Sigma}$. Parameters: $\kappa_{\text{Nils}} = 0.05$ ($\kappa_{\text{Nils}} = 0.0637$ for $^{182,184,186}$W), pairing strength constant $V_0 = 27$.

| Nuclei | $\delta$ | $E_{cen}$, MeV | $B(M1)_{\Sigma}$, $\mu^2_N$ |
|--------|---------|----------------|------------------|
|        |         | Exp. | WFM | Ref. [14] | $\Delta = 0$ | Exp. | WFM | Ref. [14] | $\Delta = 0$ |
| $^{134}$Ba | 0.14    | 2.99 | 3.04 | 3.09 | 1.28 | 0.56 | 0.68 | 1.67 | 3.90 |
| $^{148}$Nd | 0.17    | 3.37 | 3.22 | 3.18 | 1.48 | 0.78 | 1.28 | 2.58 | 5.39 |
| $^{150}$Sm | 0.16    | 3.13 | 3.17 | 3.13 | 1.42 | 0.92 | 1.12 | 2.45 | 5.26 |
| $^{182}$W   | 0.20    | 3.10 | 3.28 | 3.30 | 1.63 | 1.65 | 2.05 | 4.31 | 8.43 |
| $^{184}$W   | 0.19    | 3.31 | 3.24 | 3.28 | 1.55 | 1.12 | 1.72 | 3.97 | 8.14 |
| $^{186}$W   | 0.18    | 3.20 | 3.19 | 3.26 | 1.49 | 0.82 | 1.40 | 3.76 | 7.95 |
| $^{190}$Os  | 0.15    | 2.90 | 3.14 | 3.12 | 1.21 | 0.98 | 1.38 | 2.67 | 6.64 |
| $^{192}$Os  | 0.14    | 3.01 | 3.11 | 3.12 | 1.15 | 1.04 | 1.00 | 2.42 | 6.37 |

FIG. 6: The energies $E$ and $B(M1)$-factors as a function of the mass number $A$ for nuclei listed in the Tables VII, VIII.
Woods-Saxon one or the potential extracted from some of the numerous variants of Skyrme forces) instead of the too simple Nilsson potential. Naturally, this will be the subject of further investigations. However, to be sure that the situation with these nuclei is not absolutely hopeless, one can try to imitate the properties of the more perfect potential by fitting parameters of the Nilsson potential. As a matter of fact this potential contains one single but essential parameter – the spin-orbital strength $\kappa_{\text{Nils}}$. It turns out that changing its value from 0.0637 to 0.05 (the value used by Nilsson in his original paper [29]) is enough to obtain a reasonable description of $B(M1)$ factors (see Table VIII). To obtain a reasonable description of the scissors energies we use the "freedom" of choosing the value of the pairing interaction constant $V_0$ in (23). It turns out that changing its value from 25 MeV to 27 MeV is enough to obtain satisfactory agreement between the theoretical and experimental values of $E_{\text{sc}}$ (Table VIII).

The isotopes $^{182-186}\text{W}$ turn out intermediate between weakly deformed and well deformed nuclei: reasonable results are obtained with $\kappa_{\text{Nils}} = 0.0637$ (as for well deformed) and $V_0 = 27$ MeV (as for weakly deformed). That is why they appear in both Tables.

Returning to the group of well deformed nuclei with $\delta \geq 0.18$ (Table VII) it is necessary to emphasize that all presented results for these nuclei were obtained without any fitting. In spite of it the agreement between the theory and experiment looks more or less satisfactory for all nuclei of this group except two: $^{164}\text{Er}$ and $^{172}\text{Yb}$, where the theory overestimates $B(M1)$ values approximately two times. However, these two nuclei fall out of the systematics and one can suspect, that there the experimental $B(M1)$ values are underestimated. Therefore one can hope, that new experiments will correct the situation with these nuclei, as it happened, for example, with $^{232}\text{Th}$ [35].

It is interesting to compare our results with that of RPA calculations. The only systematic calculations for rare earth nuclei was done in the frame of the extended RPA formalism (Quasiparticle-Phonon Nuclear Model (QPNM)) [36]. We took the Table IX from this paper adding there, for the sake of comparison, the column with our results (WFM). It is easy to see that QPNM results practically coincide with experimental ones, whereas deviations of our results from experimental data reach sometimes 50%. However, it is necessary to emphasize here, that such naive comparison is not fully legitimate, because the objects of comparison are slightly different. The numbers presented in third column of Table IX are just the sums of all M1 strength found experimentally in the energy interval shown in second
TABLE IX: Scissors modes summarized transition probabilities $\sum B(M1)$. The experimental values $\sum B(M1)$ are from [33].

| Nuclei | $E$, MeV | $\sum B(M1)$, $\mu_N^2$ |
|--------|----------|-------------------------|
|        | Exp. [33]| QPNM [36] | WFM |
| $^{156}$Gd | 2.7 – 3.7 | 2.73 | 2.95 | 3.44 |
| $^{158}$Gd | 2.7 – 3.7 | 3.39 | 3.41 | 3.52 |
| $^{160}$Gd | 2.7 – 3.7 | 2.97 | 2.86 | 4.02 |
| $^{160}$Dy | 2.7 – 3.7 | 2.42 | 2.46 | 3.60 |
| $^{162}$Dy | 2.7 – 3.7 | 2.49 | 2.60 | 3.69 |
| $^{164}$Dy | 2.7 – 3.7 | 3.18 | 2.92 | 3.78 |
| $^{166}$Er | 2.4 – 3.7 | 2.67 | 2.51 | 3.86 |
| $^{168}$Er | 2.4 – 3.7 | 2.82 | 2.87 | 3.95 |
| $^{172}$Yb | 2.4 – 3.7 | 1.94 | 2.27 | 3.72 |
| $^{174}$Yb | 2.4 – 3.7 | 2.70 | 2.84 | 3.80 |
| $^{178}$Hf | 2.4 – 3.7 | 2.04 | 2.30 | 2.67 |

Theorists, working in RPA, represent their results exactly in the same manner – the sum of $B(M1)$ values of all pikes in the respective energy interval.

In principle, RPA calculations [7, 36] predict some $M1$ strength at energies higher than 3.7 MeV (up to 10 MeV). "Because of the dominance of spin-flip and the high level density in this region there is little hope that reliable measurements of this strength will ever be possible" [7]. This just the point: the WFM approach implicitly takes in account the whole configuration space. Then, the two scissors modes (spin and orbital), found by the WFM method, include this part of the $M1$ strength which is inaccessible, even for the modern experiments.

In the light of the aforesaid it becomes clear that the summarized $M1$ strength of spin and orbital scissors is to become somewhat bigger than the number presented as the experimental $B(M1)$ value of the scissors mode. So, in evaluating the quality of agreement between theoretical and experimental results, one has to have in mind this element of uncertainty.
VIII. CONCLUDING REMARKS

In this work, we continued the investigation of spin modes [15] with the Wigner Function Method in studying the influence of spin-spin forces. This method, when pushed to high order moments, is equivalent to the exact solution of TDHFB equation. For lower rank moments, it yields a coarse grained spectrum. It has the advantage that the moments allow for a direct physical interpretation and, thus, the spin or orbital structure of the found states comes directly at hand.

The inclusion of spin-spin interaction does not change qualitatively the picture concerning the spectrum of the spin modes found in [15]. It pushes all levels up without changing their order. However, it strongly redistributes M1 strength between the conventional and spin scissors mode in the favour of the last one.

We mentioned the recent experimental work [17], where for the two low lying magnetic states a stronger B(M1) transition for the lower state with respect to the higher one was found. A tentative explanation in terms of a slight triaxial deformation in [17] failed. However, our theory can naturally predict such a scenario with a non vanishing spin-spin force. It would indeed be very exciting, if the results of [17] had already discovered the isovector spin scissors mode. However, much deeper experimental and theoretical results must be obtained before a firm conclusion on this point is possible.

The method of Wigner function moments is generalized to take into account spin degrees of freedom and pair correlations simultaneously [18]. The inclusion of the spin into the theory allows one to discover several new phenomena. One of them, the nuclear spin scissors, was described and studied in [15, 16]. Another phenomenon, the opposite rotation of spin up/down nucleons, or in other words, the phenomenon of hidden angular momenta, is described in paper [18]. Being determined by the spin degrees of freedom this phenomenon has great influence on the excitation probability of the spin scissors mode. On the other hand the spin scissors $B(M1)$ values and the energies of both, spin and orbital, scissors are very sensitive to the action of pair correlations. As a result, these two factors, the spin up/down counter-rotation and pairing, working together, improve substantially the agreement between the theory and experiment in the description of the energy centroid of two nuclear scissors and their summed excitation probability. More precisely, a satisfactory agreement is achieved for well deformed nuclei of the rare earth region with standard values of all possible
parameters. The accuracy of the description of the scissors mode by the WFM method is comparable with that of RPA, if to take into account the principal difference in definitions of scissors in WFM method and RPA and experiment. A satisfactory agreement is also achieved for weakly deformed (transitional) nuclei of the same region by a very modest re-fit of the spin-orbit and pairing strength. We suppose that fourth order moments and more realistic interactions are required for the adequate description of transitional nuclei. This shall be the object of future work.

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Appendix A:

Abnormal density

According to formula (D.47) of [20] the abnormal density in coordinate representation \( \kappa(r, s;r', s') \) is connected with the abnormal density in the representation of the harmonic oscillator quantum numbers \( \kappa_{\nu,\nu'} = \langle \Phi | a_\nu a_{\nu'} | \Phi \rangle \) by the relation

\[
\kappa(r, s;r', s') = \langle \Phi | a(r, s)a(r', s') | \Phi \rangle = \sum_{\nu,\nu'} \psi_\nu(r, s)\psi_{\nu'}(r', s') \langle \Phi | a_{\nu} a_{\nu'} | \Phi \rangle,
\]

where \( \nu \equiv k, \varsigma \) (with \( k \equiv n, l, j, |m| \) and \( \varsigma \equiv \text{sign}(m) = \pm \)), \( k, + \equiv \nu \); \( k, - \equiv \nu' \), \( \psi_\nu(r, s) = T\psi_\nu(r, s) \). \( T \) - time reversal operator defined by formula (XV.85) of [37]: \( T = -i\sigma_y K_0 \),

where \( \sigma_y \) is the Pauli matrix and \( K_0 \) is the complex-conjugation operator.

According to formula (7.12) of [20]

\[
a_{k,\varsigma} = u_k \alpha_{k,\varsigma} - \varsigma v_k \alpha_{k,-\varsigma}, \quad \alpha_\nu | \Phi \rangle = 0,
\]

\[
\langle \Phi | a_\nu a_{\nu'} | \Phi \rangle \equiv \kappa_{\nu,\nu'} = -\varsigma' u_k v_k' \langle \Phi | \alpha_{k,\varsigma} \alpha_{k',-\varsigma'}^\dagger | \Phi \rangle = -\varsigma' u_k v_k' \delta_{k,k'} \delta_{\varsigma,-\varsigma'}.
\]

This result means that in accordance with the theorem of Bloch and Messiah we have found the basis \( |\nu\rangle \) in which the abnormal density \( \kappa_{\nu,\nu'} \) has the canonical form. Therefore the spin structure of \( \kappa_{\nu,\nu'} \) is

\[
\kappa_{\nu,\nu'} = \begin{pmatrix} 0 & u_k v_k' \\ -u_k v_k' & 0 \end{pmatrix},
\]
or \( \kappa_{\nu,\nu} = -\kappa_{\nu,\nu} \) and \( \kappa_{\nu,\nu} = \kappa_{\nu,\nu} = 0 \).

With the help of (A2) formula (A1) can be transformed into

\[
\kappa(\mathbf{r}, s; \mathbf{r}', s') = \sum_{k,s} \xi u_k v_k \psi_{k,\xi}(\mathbf{r}, s) \psi_{k,-\xi}(\mathbf{r}', s')
= \sum_{\nu > 0} u_\nu v_\nu [\psi_\nu(\mathbf{r}, s) \psi_\nu(\mathbf{r}', s') - \psi_\nu(\mathbf{r}, s) \psi_\nu(\mathbf{r}', s')].
\]

(A4)

that reproduces formula (D.48) of [20].

What is the spin structure of \( \kappa(\mathbf{r}, s; \mathbf{r}', s') \)?

Let us consider the spherical case:

\[
\psi_\nu(\mathbf{r}, s) = \mathcal{R}_{nlj}(r) \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} \sigma}^{jm} Y_{l\Lambda}(\theta, \phi) \chi_{\frac{1}{2} \sigma}(s) \equiv \mathcal{R}_{nlj}(r) \phi_{ljm}(\Omega, s),
\]

(A5)

where \( \phi_{ljm}(\Omega, s) = \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} \sigma}^{jm} Y_{l\Lambda}(\theta, \phi) \chi_{\frac{1}{2} \sigma}(s), \) spin function \( \chi_{\frac{1}{2} \sigma}(s) = \delta_{\sigma s} \) and angular variables are denoted by \( \Omega \).

Time reversal:

\[
TY_{l\Lambda} = Y_{l\Lambda}^* = (-1)^l Y_{l-\Lambda},
\]

\[
T \chi_{l, \frac{1}{2} \sigma} = \chi_{l, -\frac{1}{2} \sigma}, \quad T \chi_{l, -\frac{1}{2} \sigma} = -\chi_{l, \frac{1}{2} \sigma} \quad \rightarrow \quad T \chi_{l, \sigma} = (-1)^{l+\frac{1}{2}} \chi_{l, -\sigma};
\]

\[
T \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} \sigma}^{jm} Y_{l\Lambda} \chi_{\frac{1}{2} \sigma} = \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} \sigma}^{jm} Y_{l-\Lambda} \chi_{\frac{1}{2} -\sigma} = \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} -\sigma}^{jm} Y_{l\Lambda} \chi_{\frac{1}{2} \sigma} = \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} \sigma}^{jm} Y_{l\Lambda} \chi_{\frac{1}{2} \sigma} (-1)^{\Lambda -\sigma};
\]

(A6)

As a result

\[
\psi_\nu(\mathbf{r}, s) = (-1)^{l-\nu+m} \mathcal{R}_{nlj}(r) \sum_{\Lambda, \sigma} C_{l,\Lambda, \frac{1}{2} \sigma}^{jm} Y_{l\Lambda}(\theta, \phi) \chi_{\frac{1}{2} \sigma}(s) = (-1)^{l-\nu+m} \mathcal{R}_{nlj}(r) \phi_{ljm}(\Omega, s),
\]

that coincides with formula (2.45) of [20]. Formula (A4) can be rewritten now as

\[
\kappa(\mathbf{r}_1, s_1; \mathbf{r}_2, s_2) = \sum_{nljm>0} (uv)_{nljm} \mathcal{R}_{nlj}(r_1) \mathcal{R}_{nlj}(r_2) (-1)^{l-\nu+m} \left[ \phi_{ljm}(\Omega_1, s_1) \phi_{ljm}(\Omega_2, s_2) - \phi_{ljm}(\Omega_2, s_2) \phi_{ljm}(\Omega_1, s_1) \right]
= \sum_{nljm>0} (uv)_{nljm} \mathcal{R}_{nlj}(r_1) \mathcal{R}_{nlj}(r_2) (-1)^{l-\nu+m}
\]

\[
\times \sum_{\Lambda, \Lambda'} C_{l\Lambda, \frac{1}{2} s_1}^{jm} C_{l\Lambda', \frac{1}{2} s_2}^{jm} Y_{l\Lambda}(\Omega_1) Y_{l\Lambda'}(\Omega_2) - C_{l\Lambda, \frac{1}{2} s_2}^{jm} C_{l\Lambda', \frac{1}{2} s_1}^{jm} Y_{l\Lambda}(\Omega_2) Y_{l\Lambda'}(\Omega_1)
\]

(A7)

\[
= \sum_{nljm>0} (uv)_{nljm} \mathcal{R}_{nlj}(r_1) \mathcal{R}_{nlj}(r_2) (-1)^{l-\nu+m}
\]

\[
\times \sum_{\Lambda, \Lambda'} Y_{l\Lambda}(\Omega_1) Y_{l\Lambda'}(\Omega_2) \left[ C_{l\Lambda, \frac{1}{2} s_1}^{jm} C_{l\Lambda', \frac{1}{2} s_2}^{jm} - C_{l\Lambda, \frac{1}{2} s_2}^{jm} C_{l\Lambda', \frac{1}{2} s_1}^{jm} \right].
\]
It is obvious that $\kappa(r, \uparrow; r', \downarrow) \neq -\kappa(r, \downarrow; r', \uparrow)$, i.e. in the coordinate representation the spin structure of $\kappa$ has nothing common with (A3).

The anomalous density defined by (A7) has not definite angular momentum $J$ and spin $S$. It can be represented as the sum of several terms with definite $J, S$. We have:

$$
\phi_{lm}(1)\phi_{l-m}(2) = \sum_{0 \leq J \leq 2j} C_{jm,j-m}^{J0} \{\phi_j(1) \otimes \phi_j(2)\}_{J0}
$$

$$
= C_{jm,j-m}^{00} \{\phi_j(1) \otimes \phi_j(2)\}_{00} + \sum_{1 \leq J \leq 2j} C_{jm,j-m}^{J0} \{\phi_j(1) \otimes \phi_j(2)\}_{J0}. \tag{A8}
$$

We are interested in the monopole pairing only, so we omit all terms except the first one:

$$
[\phi_{lm}(1)\phi_{l-m}(2)]_{J=0} = C_{jm,j-m}^{00} \{\phi_j(1) \otimes \phi_j(2)\}_{00}
$$

$$
= (-1)^{j-m} \frac{1}{\sqrt{2j+1}} \sum_{\nu,\sigma} C_{j\nu,j\sigma}^{00} \phi_{j\nu}(1)\phi_{j\sigma}(2) = \frac{1}{2j+1} \sum_{\nu} (-1)^{\nu-m} \phi_{j\nu}(1)\phi_{j-\nu}(2). \tag{A9}
$$

Remembering the definition of $\phi$ function we find

$$
(-1)^m [\phi_{lm}(\Omega_1, s_1)\phi_{l-m}(\Omega_2, s_2)]_{J=0} = \frac{1}{2j+1} \sum_{\nu} \sum_{\Lambda,\sigma,\Lambda',\sigma'} C_{\nu,\lambda,\sigma}^{\Lambda,\sigma,\Lambda',\sigma'} \chi_{\nu}^{(1)}(\Omega_1)Y_{\Lambda\lambda}(\Omega_1)Y_{\Lambda'\sigma}(\Omega_2)\chi_{\sigma}^{(2)}(s_1)\chi_{\sigma'}^{(2)}(s_2). \tag{A10}
$$

The direct product of spin functions in this formula can be written as

$$
\chi_{\frac{1}{2},\sigma}^{(s_1)}\chi_{\frac{1}{2},\sigma'}^{(s_2)} = \sum_{s,\Sigma} C_{\frac{1}{2},\sigma,\frac{1}{2},\sigma'}^{s,\Sigma}\{\chi_{\frac{1}{2}}^{(s_1)} \otimes \chi_{\frac{1}{2}}^{(s_2)}\}_{s\Sigma}
$$

$$
= C_{\frac{1}{2},\sigma,\frac{1}{2},\sigma'}^{00} \{\chi_{\frac{1}{2}}^{(s_1)} \otimes \chi_{\frac{1}{2}}^{(s_2)}\}_{00} + \sum_{\Sigma} C_{\frac{1}{2},\sigma,\frac{1}{2},\sigma'}^{1,\Sigma} \{\chi_{\frac{1}{2}}^{(s_1)} \otimes \chi_{\frac{1}{2}}^{(s_2)}\}_{1\Sigma}. \tag{A11}
$$

According to this result the formula for $\kappa$ consists of two terms: the one with $S = 0$ and another one with $S = 1$. It was shown in the paper [38] that the term with $S = 1$ is an order of magnitude less than the term with $S = 0$, so we can neglect by it. Then

$$
\chi_{\frac{1}{2},\sigma}^{(s_1)}\chi_{\frac{1}{2},\sigma'}^{(s_2)} = (-1)^{\frac{1}{2}-\sigma} \frac{1}{\sqrt{2}} \delta_{\sigma,-\sigma'} \{\chi_{\frac{1}{2}}^{(s_1)} \otimes \chi_{\frac{1}{2}}^{(s_2)}\}_{00}
$$

$$
= (-1)^{\frac{1}{2}-\sigma} \frac{1}{\sqrt{2}} \delta_{\sigma,-\sigma'} \sum_{\nu,\nu'} C_{\nu,\nu,\frac{1}{2},\sigma}^{00} \chi_{\frac{1}{2}}^{(s_1)}\chi_{\nu}^{(s_2)}
$$

$$
= (-1)^{\frac{1}{2}-\sigma} \frac{1}{\sqrt{2}} \delta_{\sigma,-\sigma'} \sum_{\nu=-1/2}^{1/2} (-1)^{\nu} \frac{1}{\sqrt{2}} \chi_{\frac{1}{2}}^{(s_1)}\chi_{\frac{1}{2}}^{(s_2)}
$$

$$
= (-1)^{\frac{1}{2}-\sigma} \frac{1}{2} \delta_{\sigma,-\sigma'} \left[\chi_{\frac{1}{2}}^{(s_1)}\chi_{\frac{1}{2}}^{(s_2)} - \chi_{\frac{1}{2}}^{(s_1)}\chi_{-\frac{1}{2}}^{(s_2)} - \chi_{\frac{1}{2}}^{(s_1)}\chi_{\frac{1}{2}}^{(s_2)}\right]
$$

$$
= \frac{1}{2} \delta_{\sigma,-\sigma'} (-1)^{\frac{1}{2}-\sigma} \left[\delta_{s_1,1}\delta_{s_2,-\frac{1}{2}} - \delta_{s_1,-\frac{1}{2}}\delta_{s_2,1}\right]. \tag{A12}
$$
Inserting this result into (A10) we find
\[
(-1)^m [\phi_{ljm}(\Omega_1, s_1) \phi_{lj-m}(\Omega_2, s_2)]_{J=0} = \\
= \frac{1}{2} \left[ \delta_{s_1 \frac{1}{2}} \delta_{s_2 - \frac{1}{2}} - \delta_{s_1 - \frac{1}{2}} \delta_{s_2 \frac{1}{2}} \right] \frac{1}{2j + 1} \sum_{\Lambda, \Lambda'} Y_{\Lambda}(\Omega_1) Y_{\Lambda'}(\Omega_2) \sum_{\nu, \sigma} (-1)^{\nu + \frac{1}{2} - \sigma} C_{\lambda \frac{1}{2} \sigma}^{\nu} C_{\lambda' \frac{1}{2} - \sigma}^{\nu} \\
= \frac{1}{2} \left[ \delta_{s_1 \frac{1}{2}} \delta_{s_2 - \frac{1}{2}} - \delta_{s_1 - \frac{1}{2}} \delta_{s_2 \frac{1}{2}} \right] \frac{1}{2j + 1} \sum_{\Lambda, \Lambda'} Y_{\Lambda}(\Omega_1) Y_{\Lambda'}(\Omega_2) \\
\times (1 - 1)^{\frac{1}{2} + \Lambda \frac{2j + 1}{2l + 1}} (-1)^{1 + j + \frac{1}{2} - \sigma} C_{\lambda \frac{1}{2} \sigma}^{\nu} C_{\lambda' \frac{1}{2} - \sigma}^{\nu} \\
= \frac{1}{2} \left[ \delta_{s_1 \frac{1}{2}} \delta_{s_2 - \frac{1}{2}} - \delta_{s_1 - \frac{1}{2}} \delta_{s_2 \frac{1}{2}} \right] \frac{1}{2l + 1} \sum_{\Lambda, \Lambda'} Y_{\Lambda}(\Omega_1) Y_{\Lambda'}(\Omega_2) (-1)^{\Lambda \delta_{\Lambda, -\Lambda'}} \\
= \frac{1}{2} \left[ \delta_{s_1 \frac{1}{2}} \delta_{s_2 - \frac{1}{2}} - \delta_{s_1 - \frac{1}{2}} \delta_{s_2 \frac{1}{2}} \right] (-1)^{-j - l} \frac{1}{4\pi} P_l(\cos \Omega_{12}), \tag{A13}
\]
where \( P_l(\cos \Omega_{12}) \) is Legendre polynomial and \( \Omega_{12} \) is the angle between vectors \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \).

With the help of this result formula (A7) is transformed into
\[
\kappa(\mathbf{r}_1, s_1; \mathbf{r}_2, s_2)_{J=0} = \left[ \delta_{s_1 \frac{1}{2}} \delta_{s_2 - \frac{1}{2}} - \delta_{s_1 - \frac{1}{2}} \delta_{s_2 \frac{1}{2}} \right] \frac{1}{4\pi} \sum_{nljm > 0} (uv)_{nljm} \mathcal{R}_{nlj}(r_1) \mathcal{R}_{nlj}(r_2) P_l(\cos \Omega_{12}). \tag{A14}
\]

Now it is obvious that in the coordinate representation \( \kappa \) with \( J = 0, S = 0 \) has the spin structure similar to the one demonstrated by formula (A3):
\[
\kappa(\mathbf{r}_1, s_1; \mathbf{r}_2, s_2)_{J=0} = \left( \begin{array}{cc} 0 & \kappa(\mathbf{r}_1, \mathbf{r}_2) \\ -\kappa(\mathbf{r}_1, \mathbf{r}_2) & 0 \end{array} \right) \tag{A15}
\]
with
\[
\kappa(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi} \sum_{nljm > 0} (uv)_{nljm} \mathcal{R}_{nlj}(r_1) \mathcal{R}_{nlj}(r_2) P_l(\cos \Omega_{12}). \tag{A16}
\]

**Appendix B:**

**Wigner transformation**

The Wigner Transform (WT) of the single-particle operator matrix \( \hat{F}_{r_1,\sigma;r_2,\sigma'} \) is defined as
\[
[\hat{F}_{r_1,\sigma;r_2,\sigma'}]_{\text{WT}} = F_{\sigma,\sigma'}(\mathbf{r}, \mathbf{p}) = \int d\mathbf{s} e^{-i\mathbf{p} \cdot \mathbf{s}/\hbar} \hat{F}_{r+s/2,\sigma';r-s/2,\sigma} \tag{B1}
\]
with \( \mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2 \) and \( \mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2 \). It is easy to derive a pair of useful relations. The first one is
\[
F_{\sigma,\sigma'}^{*}(\mathbf{r}, \mathbf{p}) = \int d\mathbf{s} e^{i\mathbf{p} \cdot \mathbf{s}/\hbar} \hat{F}_{r+s/2,\sigma;r-s/2,\sigma'} = \int d\mathbf{s} e^{-i\mathbf{p} \cdot \mathbf{s}/\hbar} \hat{F}_{r-s/2,\sigma;r+s/2,\sigma'} = \int d\mathbf{s} e^{-i\mathbf{p} \cdot \mathbf{s}/\hbar} \hat{F}_{r+s/2,\sigma';r-s/2,\sigma} = \left[ \hat{F}_{r_1,\sigma';r_2,\sigma} \right]_{\text{WT}}, \tag{B2}
\]
i.e., \([\hat{F}^\dagger_{r_1,\sigma;r_2,\sigma'}]_{\text{WT}} = [\hat{F}_{r_1,\sigma';r_2,\sigma}]_{\text{WT}} = F^*_\sigma\sigma(r,p)\). The second relation is

\[
\hat{F}_{\sigma\sigma'}(r,p) \equiv F_{\sigma\sigma'}(r,-p) = \int ds \ e^{ip\cdot s/\hbar} \hat{F}_{r+s/2,\sigma;r-s/2,\sigma'} = \int ds \ e^{-ip\cdot s/\hbar} [\hat{F}^\dagger_{r+s/2,\sigma';r-s/2,\sigma}].
\]

(B3)

For the hermitian operators \(\hat{\rho}\) and \(\hat{h}\) this latter relation gives

\[
[\hat{\rho}^*_r,\sigma;r_2,\sigma]_{\text{WT}} = \rho_{\sigma\sigma}(r,-p) \quad \text{and} \quad [\hat{h}^*_r,\sigma;r_2,\sigma]_{\text{WT}} = h_{\sigma\sigma}(r,-p).
\]

The Wigner transform of the product of two matrices \(F\) and \(G\) is

\[
[\hat{F}\hat{G}]_{\text{WT}} = F(r,p) \exp\left(\frac{i\hat{\mathcal{A}}}{2}\right) G(r,p),
\]

where the symbol \(\hat{\mathcal{A}}\) stands for the Poisson bracket operator

\[
\hat{\mathcal{A}} = \sum_{i=1}^3 \left( \frac{\partial}{\partial r_i} \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial}{\partial r_i} \right).
\]

(B5)

**Appendix C:**

All derivations of this section will be done in the approximation of spherical symmetry. The inclusion of deformation makes the calculations more cumbersome without changing the final conclusions. Let us consider, as an example, the integral

\[
I_h = \int d(p,r) \{r \otimes p\} \chi_{\sigma}(h^{\uparrow\downarrow} f^{\downarrow\uparrow} - h^{\downarrow\uparrow} f^{\uparrow\downarrow}).
\]

It can be divided in two parts corresponding to the contributions of spin-orbital and spin-spin potentials: \(I_h = I_{so} + I_{ss}\), where

\[
I_{so} = -\frac{\hbar}{\sqrt{2}} \eta \int d(p,r) \{r \otimes p\} \chi_{\sigma}[l_{-1} f^{\uparrow\downarrow} + l_{1} f^{\downarrow\uparrow}],
\]

\[
I_{ss} = \int d(p,r) \{r \otimes p\} \chi_{\sigma}[V^{\uparrow\downarrow}_\tau f^{\downarrow\uparrow} - V^{\downarrow\uparrow}_\tau f^{\uparrow\downarrow}],
\]

\(V^{ss'}\) being defined in (28). It is easy to see that the integral \(I_{so}\) generate moments of fourth order. According to the rules of the WFM method \([39]\) this integral is neglected.

Let us analyze the integral \(I_{ss}\) (to be definite, for protons). In this case

\[
V_p^{\uparrow\downarrow}(r) = \frac{\hbar^2}{8} \chi_{n_p^{\uparrow\downarrow}}(r) + \frac{\hbar^2}{4} \chi_{n_p^{\downarrow\uparrow}}(r),
\]
We are interested in the value of $\mu$ the density and current variations as a series (see appendix D). It can not be solved exactly, so we will use the approximation suggested in [39] and expand it is necessary to represent this integral in terms of the collective variables (30). This problem can not be solved exactly, so we will use the approximation suggested in [39] and expand the density and current variations as a series (see appendix D).

Let us consider the second part of integral (C2). With the help of formula (D4) we find

$$I_2 \equiv \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{\lambda\mu} \int d^3r r_\nu J_\alpha^{\uparrow \downarrow} \delta n^{\uparrow \downarrow}$$

$$= - \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{\lambda\mu} \int d^3r r_\nu J_\alpha^{\uparrow \downarrow} \delta n^{\uparrow \downarrow} \left\{N^{\uparrow \downarrow}_{\beta,-\beta}(t) n^+ + \sum_\gamma (-1)^\gamma N^{\uparrow \downarrow}_{\beta,\gamma}(t) \frac{1}{r} \frac{\partial n^+}{\partial r} r_{-\beta r_{-\gamma}} \right\}.$$ (C3)

Let us analyze at first the more simple part of this expression:

$$I_{2,1} \equiv - \sum_{\beta} (-1)^\beta N^{\uparrow \downarrow}_{\beta,-\beta}(t) \int d^3r \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{\lambda\mu} r_\nu J_\alpha^{\uparrow \downarrow} \delta n^+ = - \sum_{\beta} (-1)^\beta N^{\uparrow \downarrow}_{\beta,-\beta} X_{\lambda \mu}. \quad (C4)$$

We are interested in the value of $\mu = 1$, therefore it is necessary to analyze two possibilities: $\lambda = 1$ and $\lambda = 2$.

In the case $\lambda = 1$, $\mu = 1$ we have

$$X_{11} \equiv \int d^3r n^+ \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{11} r_\nu J_\alpha^{\uparrow \downarrow} \delta n^+ = \int d^3r n^+ \frac{1}{\sqrt{2}} \left[ r_1 J_0^{\uparrow \downarrow} \delta n^+ - r_0 J_1^{\uparrow \downarrow} \delta n^+ \right]. \quad (C5)$$

Inserting the definition (57) into (C5) one finds

$$X_{11} = \frac{i\hbar}{2 \sqrt{2}} \sum_{n,ljm} v_{nljm}^2 \int d^3r n^+(r) R_{nljm}^2(r) C_{l\lambda,1\frac{3}{2}}^{jm} C_{l\lambda,1\frac{1}{2}}^{jm} \left[ Y_{l\lambda}(r_1 \nabla_0 - r_0 \nabla_1) Y_{l\lambda}^* - Y_{l\lambda}^*(r_1 \nabla_0 - r_0 \nabla_1) Y_{l\lambda} \right]$$

with $\Lambda = m - \frac{1}{2}$ and $\Lambda' = m + \frac{1}{2}$. Remembering the definition (14) of the angular momentum $\hat{l}_1 = \hbar(r_0 \nabla_1 - r_1 \nabla_0)$ and using the relation [22] $\hat{l}_{\pm 1} Y_{l\lambda} = \pm \frac{1}{\sqrt{2}} \sqrt{(l \pm \Lambda)(l \pm \Lambda + 1)} Y_{l\lambda \pm 1}$ one
transforms (C6) into

\[
X_{11} = -i\hbar \frac{1}{2} \sum_{nljm} v_{nljm} \int dr n^+(r) r^2 R_{nlj}^2(r) C_{lM,\frac{1}{2} \pm \frac{1}{2}}^{jm} C_{lM,\frac{1}{2} \mp \frac{1}{2}}^{jm} \frac{2}{\sqrt{2}} \sqrt{(l - \Lambda)(l + \Lambda + 1)}
\]

\[
= -i\hbar \sum_{nl} \sum_{m=\pm \frac{1}{2}} \frac{[(l + \frac{1}{2})^2 - m^2]}{2l + 1} \int dr n^+(r) r^2 \left[ v_{nl,+\frac{1}{2}m}^2 R_{nl,+\frac{1}{2}m}^2(r) - v_{nl,-\frac{1}{2}m}^2 R_{nl,-\frac{1}{2}m}^2(r) \right].
\]  

(C7)

As it is seen, the value of this integral is determined by the difference of the wave functions of spin-orbital partners \((vR)^2_{nl,+\frac{1}{2}m} - (vR)^2_{nl,-\frac{1}{2}m}\), which is usually very small, so we will neglect it. The only remarkable contribution can appear in the vicinity of the Fermi surface, where some spin-orbital partners with \(j = l + \frac{1}{2}\) and \(j = |l - \frac{1}{2}|\) can be disposed on different sides of the Fermi surface. In reality such situation happens very frequently, nevertheless we will not take into account this effect, because the values of the corresponding integrals are considerably smaller than \(R_{20}(eq)\), the typical input parameter of our model.

Let us consider now the integral \(I_{2,1}\) (formula (C4)) for the case \(\lambda = 2, \mu = 1\). We have

\[
X_{21} \equiv \int d^3 r n^+ \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{21} r_\nu J_{\alpha}^{14}(eq) = \int d^3 r n^+ C_{11,10}^{21} \left[ r_1 J_{0}^{14}(eq) + r_0 J_{1}^{14}(eq) \right].
\]  

(C8)

With the help of formulae (57) one can show by simple algebraic transformations that

\[
\int d\Omega r_1 J_{0}^{14}(eq) = - \int d\Omega r_0 J_{1}^{14}(eq),
\]

where \(d\Omega\) means the integration over angles. As a result \(X_{21} = 0\).

Let us consider the second, more complicated, part of integral \(I_2\):

\[
I_{2,2} = - \sum_{\beta,\gamma} (-1)^{\beta + \gamma} N_{-\beta, -\gamma}(t) \sum_{\nu,\alpha} C_{1\nu,1\alpha}^{\lambda\mu} \int d^3 r r_\nu J_{\alpha}^{14}(eq) \frac{1}{r} \frac{\partial n^+}{\partial r} r_\beta r_\gamma
\]

\[
= - \sum_{\beta,\gamma} (-1)^{\beta + \gamma} N_{-\beta, -\gamma}(t) X_{\lambda\mu}^{\lambda\mu}(\beta, \gamma).
\]  

(C10)

The case \(\lambda = 1, \mu = 1\):

\[
X_{11}'(\beta, \gamma) = \frac{1}{\sqrt{2}} \int d^3 r \frac{1}{r} \frac{\partial n^+}{\partial r} \left[ r_1 J_{0}^{14}(eq) - r_0 J_{1}^{14}(eq) \right] r_\beta r_\gamma
\]

\[
= - \frac{i\hbar}{4} \sum_{nljm} v_{nljm} \int d^3 r \frac{1}{r} \frac{\partial n^+}{\partial r} R_{nj}^2(r) C_{lM,\frac{1}{2} \pm \frac{1}{2}}^{jm} C_{lM,\frac{1}{2} \mp \frac{1}{2}}^{jm} \times \sqrt{(l - \Lambda)(l + \Lambda + 1)} \left[ Y_{lM}^* Y_{lM} + Y_{lM}^* Y_{lM} \right] r_\beta r_\gamma.
\]

(C11)

The angular part of this integral is

\[
\int d\Omega \left[ Y_{lM}^* Y_{lM} + Y_{lM}^* Y_{lM} \right] r_\beta r_\gamma = \sum_{L, M} C_{13,1,1}^{LM} \int d\Omega \left[ Y_{lM}^* Y_{lM} + Y_{lM}^* Y_{lM} \right] \{ r \otimes r \}_{LM}
\]
\[ \begin{align*}
&= -\frac{2}{\sqrt{3}} r^2 C^{00}_{1,1,\gamma} + \sqrt{\frac{8\pi}{15}} r^2 \sum_{M} C^{2M}_{1,1,\gamma} \int d\Omega [Y_{1\lambda} Y^*_{1\lambda} + Y^*_{1\lambda} Y_{1\lambda}] Y_{2M} \\
&= \frac{2}{3} r^2 \delta_{\gamma, -\beta} \left\{ 1 - \sqrt{\frac{5}{2}} C^{d0}_{1,0,20} C^{d\beta}_{1,3,20} \left[ C^{d\lambda}_{1,\lambda,2M} + C^{d\lambda'}_{1,\lambda',2M} \right] \right\}. \quad \text{(C12)}
\end{align*} \]

Therefore
\[ \begin{align*}
X_{11}'(\beta, \gamma) &= -\frac{i\hbar}{6} \delta_{\gamma, -\beta} \int dr \frac{\partial n^+(r)}{\partial r} r^3 \sum_{nljm} \left\{ 1 - \sqrt{\frac{5}{2}} C^{d\lambda}_{1,\lambda,20} C^{d\lambda'}_{1,\lambda',20} \right\} \\
&\times v_{nljm} R^2_{nlj}(r) C^{00}_{1,0,20} C^{d0}_{1,0,20} \sqrt{(l - \Lambda)(l + \Lambda + 1)} \\
&= -\frac{i\hbar}{3} \delta_{\gamma, -\beta} \sum_{nl} \left\{ 1 - \sqrt{\frac{5}{2}} C^{d\lambda}_{1,\lambda,20} C^{d\lambda'}_{1,\lambda',20} \right\} \\
&\times \sum_{m=\frac{1}{2}}^{|l - \frac{1}{2}|} \frac{(l + \frac{1}{2})^2 - m^2}{2l + 1} \int dr \frac{\partial n^+(r)}{\partial r} r^3 \left[ v_{nll+\frac{1}{2}m} R^2_{nll+\frac{1}{2}}(r) - v_{nll-\frac{1}{2}m} R^2_{nll-\frac{1}{2}}(r) \right]. \quad \text{(C13)}
\end{align*} \]

One sees that, exactly as in formula (C7), the value of this integral is determined by the difference of the wave functions of spin-orbital partners \( (vR)^2_{nl\ell, \frac{1}{2}m} - (vR)^2_{nl\ell, -\frac{1}{2}m} \) near the Fermi surface, so it can be omitted together with \( X_{11} \) following the same arguments.

The case \( \lambda = 2, \mu = 1 \) can be analyzed in full analogy with formulae (C8,C9) that allows us to take \( X_{21}' = 0 \).

So, we have shown that the integral \( I_2 \) can be approximated by zero. Let us consider now the first part of the integral (C2):
\[ \begin{align*}
I_1 &= \sum_{\nu, \alpha} C^{\lambda\mu}_{\nu,\alpha 1,\gamma} \int d^3r_r n^+(eq) \delta J^\dagger_\alpha = \sum_{\nu, \alpha} C^{\lambda\mu}_{\nu,\alpha 1,\gamma} \int d^3r_r n^+(eq) n^+(r) \sum_\gamma (-1)^\gamma K^\dagger_\alpha, -\gamma(t) r_\gamma \\
&= \sum_{\nu, \alpha} C^{\lambda\mu}_{\nu,\alpha 1,\gamma} \sum_\gamma (-1)^\gamma K^\dagger_\alpha, -\gamma(t) \int d^3r n^+(eq) n^+(r) \sum_{L,M} C^{LM}_{1,\nu,\gamma} \{ r \otimes r \}_LM. \quad \text{(C14)}
\end{align*} \]

This integral can be estimated in the approximation of constant density \( n^+(r) = n_0 \). Then
\[ \begin{align*}
I_1 &= n_0 \sum_{\nu, \alpha} C^{\lambda\mu}_{\nu,\alpha 1,\gamma} \sum_\gamma (-1)^\gamma K^\dagger_\alpha, -\gamma(t) \sum_{L,M} C^{LM}_{1,\nu,\gamma} R^\dagger_{LM}(eq) = 0. \quad \text{(C15)}
\end{align*} \]

It is easy to show, that \( R^\dagger_{LM}(eq) = 0 \). Let us consider, for example, the case with \( L = 2 \):
\[ R^\dagger_{2M} = \int d\{p, r\} \{ r \otimes r \}_{2M} f^\dagger_{ss'}(r, p) = \int d^3r \{ r \otimes r \}_{2M} n^+(r) = \sqrt{\frac{8\pi}{15}} \int d^3r Y_{2M} n^+(r). \quad \text{(C16)} \]

By definition
\[ \begin{align*}
n^{ss'}(r) &= \int \frac{d^3p}{(2\pi\hbar)^3} f^{ss'}(r, p) = \sum_k v^2_k \psi_k (r, s) \psi_k ^*(r, s'). \quad \text{(C17)}
\end{align*} \]
with $\psi_k$ defined in (55). Therefore

$$R_{2M}^{\uparrow\uparrow} = \sqrt{\frac{8\pi}{15}} \int d^3 r Y_{2M} \sum_{nljm} v_{nljm}^2 R_{nljm}(r) C_{l\Lambda,\frac{1}{2}}^{lm} C_{l\Lambda,\frac{1}{2}}^{jm} Y_l Y^*_l$$

$$= \sqrt{\frac{2}{3}} \sum_{nljm} v_{nljm}^2 \int dr^4 R_{nljm}(r) C_{l\Lambda,\frac{1}{2}}^{lm} C_{l\Lambda,\frac{1}{2}}^{jm} C_{\Lambda,0,0}^{lm} C_{2M,\Lambda}^{lm} = 0,$$  \hspace{1cm} (C18)

where $\Lambda = m - \frac{1}{2}$ and $\Lambda' = m + \frac{1}{2}$. The zero is obtained due to summation over $m$. Really, the product $C_{l\Lambda,\frac{1}{2}}^{lm} C_{l\Lambda',\frac{1}{2}}^{jm} = \pm \sqrt{(l+\frac{1}{2})^2 - m^2} (\text{for } j = l \pm \frac{1}{2})$ does not depend on the sign of $m$, whereas the Clebsh-Gordan coefficient $C_{2M,\Lambda}^{lm} = C_{2-1,lm+1}^{lm-\frac{1}{2}}$ changes its sign together with $m$.

Summarizing, we have demonstrated that $I_1 + I_2 \simeq 0$, hence one can neglect the contribution of the integrals $I_h$ in the equations of motion.

- It is necessary to analyze also the integrals with the weight $\{p \otimes p\}_{\lambda\mu}$:

$$I'_h = \int d(p, r) \{p \otimes p\}_{\lambda\mu} [h^{\uparrow\downarrow} f^{\uparrow\downarrow} - h^{\downarrow\uparrow} f^{\downarrow\uparrow}] = I'_{so} + I'_{ss}.$$

Again we neglect the contribution of the spin-orbital part $I'_{so}$, which generates fourth order moments. For the spin-spin contribution, we have

$$I'_{ss4} = \int d(p, r) \{p \otimes p\}_{\lambda\mu} n^{\uparrow\downarrow}(r, t) f^{\uparrow\downarrow}(r, p, t) = \int d^3 r \Pi^{\uparrow\downarrow}_{\lambda\mu}(r, t) n^{\uparrow\downarrow}(r, t),$$ \hspace{1cm} (C19)

where $\Pi^{\uparrow\downarrow}_{\lambda\mu}(r, t) = \int \frac{d^3 p}{(2\pi \hbar)^3} \{p \otimes p\}_{\lambda\mu} f^{\uparrow\downarrow}(r, p, t)$ is the pressure tensor. The variation of this integral reads:

$$\delta I'_{ss4} = \int d^3 r [n^{\uparrow\downarrow}(eq) \delta \Pi^{\uparrow\downarrow}_{\lambda\mu}(r, t) + \Pi^{\uparrow\downarrow}_{\lambda\mu}(eq) \delta n^{\uparrow\downarrow}(r, t)].$$ \hspace{1cm} (C20)

The pressure tensor variation is defined in appendix D. With formula (D6) one finds for the first part of (C20):

$$I'_1 = \int d^3 r n^{\uparrow\downarrow}(eq) \delta \Pi^{\uparrow\downarrow}_{\lambda\mu}(r, t) \simeq T^{\uparrow\downarrow}_{\lambda\mu}(t) \int d^3 r n^{\uparrow\downarrow}(eq) n^+(r) \simeq T^{\uparrow\downarrow}_{\lambda\mu}(t) n_0 \int d^3 r n^{\uparrow\downarrow}(eq) = 0.$$ \hspace{1cm} (C21)

The last equality follows obviously from the definition of $n^{\uparrow\downarrow}$ (C17).

The second part of (C20) reads:

$$I'_2 = \int d^3 r \Pi^{\uparrow\downarrow}_{\lambda\mu}(eq) \delta n^{\uparrow\downarrow}(r, t)$$

$$= - \sum_{\beta} (-1)^{\beta} \int d^3 r \Pi^{\uparrow\downarrow}_{\lambda\mu}(eq) \left\{ N^{\uparrow\downarrow}_{\beta} n^+(r) + \sum_{\gamma} (-1)^{\gamma} N^{\uparrow\downarrow}_{\beta\gamma} \frac{1}{r} \frac{\partial n^+}{\partial r} r_\beta r^-_{\gamma} \right\}.$$ \hspace{1cm} (C22)
Let us consider at first the simpler part of this integral
\[
- \sum_{\beta} (-1)^{\beta} N_{\beta,-\beta} (t) \int d^3 r \Pi_{\lambda \mu}^{\uparrow \downarrow} (\text{eq}) n^+ (r).
\]
(C23)

The value of the last integral is determined by the angular structure of the function \( \Pi_{\lambda \mu}^{\uparrow \downarrow} (r) \). We are interested in \( \lambda = 2, \mu = 1 \). By definition
\[
\Pi_{21}^{\uparrow \downarrow} (r) = \int \frac{d^3 p}{(2\pi \hbar)^3} \{ p \otimes p \} 21 f^{\uparrow \downarrow} (r, p) = \sum_{\nu, \sigma} C_{1\nu,1\sigma}^{21} \int \frac{d^3 p}{(2\pi \hbar)^3} p_{\nu} p_{\sigma} f^{\uparrow \downarrow} (r, p)
\]
\[
= 2 C_{11,10}^{21} \int \frac{d^3 p}{(2\pi \hbar)^3} p_{1} p_{0} f^{\uparrow \downarrow} (r, p) = - \frac{\hbar^2}{2\sqrt{2}} \left[ (\nabla_1 - \nabla_1)(\nabla_0 - \nabla_0) \rho (r' \uparrow, r \downarrow) \right]_{r' = r}
\]
\[
= - \frac{\hbar^2}{2\sqrt{2}} \sum_k v_k^2 \left\{ \left[ \nabla_1 \nabla_0 \psi_k (r, \uparrow) \right] \psi_k^* (r, \downarrow) - \left[ \nabla_1 \psi_k (r, \uparrow) \right] \left[ \nabla_0 \psi_k^* (r, \downarrow) \right] - \left[ \nabla_0 \psi_k (r, \uparrow) \right] \left[ \nabla_1 \psi_k^* (r, \downarrow) \right] + \psi_k (r, \uparrow) \left[ \nabla_1 \nabla_0 \psi_k (r, \downarrow) \right] \right\}
\]
(C24)

with \( \psi_k \) being defined by (55). Taking into account formulae [22]
\[
\nabla_{\pm 1} Y_{\lambda} = - \sqrt{\frac{(l + \Lambda + 1)(l + \Lambda + 2)}{2(l + 1)(2l + 3)}} Y_{l+1, \Lambda \pm 1} - \sqrt{\frac{(l + \Lambda - 1)(l + \Lambda)}{2(l - 1)(2l + 1)}} Y_{l-1, \Lambda \pm 1},
\]
\[
\nabla_0 Y_{\lambda} = - \sqrt{\frac{(l + 1)^2 - \Lambda^2}{2(l + 1)(2l + 3)}} Y_{l+1, \Lambda} + \sqrt{\frac{l^2 - \Lambda^2}{2(l - 1)(2l + 1)}} Y_{l-1, \Lambda}
\]
one finds that
\[
\int d^3 r \Pi_{\lambda \mu}^{\uparrow \downarrow} (\text{eq}) n^+ (r) = \hbar^2 \sum_{nljm} v_{nljm}^2 \int d r n^+ (r) R_{nljm}^2 (r) \left( \delta_{j, l+\frac{1}{2}} - \delta_{j, l-\frac{1}{2}} \right) \frac{l(l + 1)}{(2l + 3)(2l + 1)(2l - 1)} m = 0
\]
(C25)
due to summation over \( m \). The more complicated part of the integral (C22) is calculated in a similar way with the same result, hence \( I_2 = 0 \).

So, we have shown that \( I_1 + I_2 \simeq 0 \), therefore one can neglect by the contribution of integrals \( I_h' \) (together with \( I_h \)) into equations of motion.

- And finally, just a few words about the integrals with the weight \( \{ r \otimes r \}_\lambda \mu \):
\[
I''_h = \int d(p, r) \{ r \otimes r \}_\lambda \mu \left[ h^{\uparrow \downarrow} f^{\uparrow \downarrow} - h^{\uparrow \downarrow} f^{\downarrow \downarrow} \right] = I''_{so} + I''_{ss}.
\]
The spin-orbital part \( I''_{so} \) is neglected and for the spin-spin part we have
\[
I''_{ss4} = \int d(p, r) \{ r \otimes r \}_\lambda \mu n^{\uparrow \downarrow} (r, t) f^{\uparrow \downarrow} (r, p, t) = \int d^3 r \{ r \otimes r \}_\lambda \mu n^{\uparrow \downarrow} (r, t) n^{\downarrow \downarrow} (r, t).
\]
(C26)
The variation of this integral reads:

\[ \delta I''_{ss4} = \int d^3r \{r \otimes r\} \lambda \mu \left[ n^{\uparrow\downarrow}(r, t) + n^{\downarrow\uparrow}(r, t) \right]. \]  

(C27)

With the help of formulae (C17) and (D4) the subsequent analysis becomes quite similar to that of the integral (C14) with the same result, i.e. \( I''_h \simeq 0 \).

- The integrals \( \int d(p, r)W_{\lambda \mu} [h, f^{\downarrow\uparrow} - h^{\downarrow\uparrow} f] \) and \( \int d(p, r)W_{\lambda \mu} [h, f^{\uparrow\downarrow} - h^{\uparrow\downarrow} f] \), where \( W_{\lambda \mu} \) is any of the above mentioned weights, can be analyzed in an analogous way with the same result.

Appendix D:

According to the approximation suggested in [39], the variations of density, current, and pressure tensor are expanded as the series

\[ \delta n^\varsigma(r, t) = - \sum_\beta (-1)^\beta \nabla - \beta \left\{ n^+(r) \left[ N^\varsigma_\beta(t) + \sum_\gamma (-1)^\gamma N^\varsigma_{\beta, \gamma}(t)r_{-\gamma} \right. \right. \]

\[ + \sum_{\lambda', \mu'} (-1)^{\mu'} N^\varsigma_{\beta, \lambda', \mu'}(t) \{r \otimes r\} \lambda' - \mu' + \ldots \right\}, \]  

(D1)

\[ \delta J^\beta_\gamma(r, t) = n^+(r) \left[ K^\beta_\gamma(t) + \sum_\gamma (-1)^\gamma K^\beta_{\gamma,-\gamma}(t)r_{\gamma} \right. \]

\[ + \sum_{\lambda', \mu'} (-1)^{\mu'} K^\beta_{\lambda', \lambda', \mu'}(t) \{r \otimes r\} \lambda' - \mu' + \ldots \right\}, \]  

(D2)

\[ \delta \Pi_{\lambda \mu}^\varsigma(r, t) = n^+(r) \left[ T^\varsigma_{\lambda \mu}(t) + \sum_\gamma (-1)^\gamma T^\varsigma_{\gamma, \gamma,-\gamma}(t)r_{\gamma} \right. \]

\[ + \sum_{\lambda', \mu'} (-1)^{\mu'} T^\varsigma_{\lambda', \lambda', \mu'}(t) \{r \otimes r\} \lambda' - \mu' + \ldots \right\}. \]  

(D3)

Putting these series into the integrals (C2, C20), one discovers immediately that all terms containing expansion coefficients \( N, K, T \) with odd numbers of indices disappear due to axial symmetry. Furthermore, we truncate these series omitting all terms generating higher (than second) order moments. So, finally the following expressions are used:

\[ \delta n^\varsigma(r, t) \simeq - \sum_\beta (-1)^\beta \nabla - \beta \left\{ n^+(r) \sum_\gamma (-1)^\gamma N^\varsigma_{\beta, \gamma}(t)r_{-\gamma} \right\} \]

\[ = - \sum_\beta (-1)^\beta \left\{ N^\varsigma_{\beta,-\gamma}(t)n^+ + \sum_\gamma (-1)^\gamma N^\varsigma_{\beta, \gamma}(t)\frac{1}{r} \frac{\partial n^+}{\partial r} r_{-\beta r_{-\gamma}} \right\}, \]  

(D4)
\(\delta J_\beta^c(r, t) \simeq n^+(r) \sum_\gamma (-1)^\gamma K_{\beta, -\gamma}^c(t) r_\gamma \) \hspace{1cm} (D5)

and

\(\delta \Pi_{\lambda \mu}^c(r, t) \simeq n^+(r) T_{\lambda \mu}^c(t)\). \hspace{1cm} (D6)

The coefficients \(N_{\beta, \gamma}^c(t)\) and \(K_{\beta, -\gamma}^c(t)\) are connected by the linear relations with collective variables \(R_{\lambda \mu}^c(t)\) and \(L_{\lambda \mu}^c(t)\) respectively.

\[
R_{\lambda \mu}^c = \int d^3r \{r \otimes r\}_{\lambda \mu} \delta n^c(r)
\]

\[
= \frac{2}{\sqrt{3}} \left[ A_1 C_{1,10,0}^\lambda N_{\mu,0}^c - A_2 \left( C_{1,11,1}^\lambda N_{\mu,1,1}^c + C_{1,11,11,1}^\lambda N_{\mu,1,1}^c \right) \right],
\]

where

\[
A_1 = \sqrt{2} \frac{R_{00}^c - R_{02}^c}{R_{00}^c \sqrt{3}} \left( 1 + \frac{4}{3} \delta \right), \quad A_2 = \frac{R_{00}^c}{\sqrt{2}} + \frac{R_{02}^c}{R_{00}^c} = - \frac{Q_{00}^c}{\sqrt{3}} \left( 1 - \frac{2}{3} \delta \right), \hspace{1cm} (D7)
\]

\[
R_{20}^c = Q_{20}/\sqrt{6}, \quad R_{00}^c = -Q_{00}/\sqrt{3}, \quad Q_{20} = \frac{4}{3} \delta Q_{00}, \quad Q_{00} = A(r^2) = \frac{3}{5} A R_0^2.
\]

\[
N_{-1,1}^c = -\frac{3}{2} R_{2}^c, \quad N_{-1,0}^c = \frac{6}{4 A_1} R_{1}^c, \quad N_{-1,1}^c = -\frac{R_{30}^c + R_{20}^c/\sqrt{2}}{2 A_2},
\]

\[
N_{0,-1}^c = -\frac{6}{4 A_2} R_{-1}^c, \quad N_{0,0}^c = \frac{2 R_{0}^c - R_{0,0}^c}{2 A_1}, \quad N_{0,1}^c = -\frac{\sqrt{6} R_{-1}^c}{4 A_2},
\]

\[
N_{1,-1}^c = N_{-1,1}^c, \quad N_{1,0}^c = \frac{\sqrt{6} R_{1}^c}{4 A_1}, \quad N_{1,1}^c = -\frac{3}{2 A_2} R_{2}^c.
\]

\[
L_{\lambda \mu}^c = \int d^3r \{r \otimes \delta J^c\}_{\lambda \mu}
\]

\[
= \frac{1}{\sqrt{3}} (-1)^\lambda \left[ A_1 C_{1,10,0}^\lambda K_{\mu,0}^c - A_2 \left( C_{1,11,1}^\lambda K_{\mu,1,1}^c + C_{1,11,11,1}^\lambda K_{\mu,1,1}^c \right) \right].
\]

\[
K_{-1,1}^c = -\frac{3}{2 A_2} L_{-2}^c, \quad K_{-1,0}^c = \frac{3}{\sqrt{2} A_1} (L_{-1,1}^c + L_{-1,-1}^c), \quad K_{-1,1}^c = -\frac{3}{\sqrt{2} A_2} L_{0}^c + L_{0,0}^c + \frac{\sqrt{2}}{\sqrt{2} A_2} L_{20}^c
\]

\[
K_{0,-1}^c = \frac{3}{\sqrt{2} A_2} (L_{-1}^c - L_{-1,-1}^c), \quad K_{0,0}^c = \frac{2}{A_1} L_{2,0}^c - L_{0,0}^c, \quad K_{0,1}^c = -\frac{3}{\sqrt{2} A_2} (L_{1,1}^c + L_{21}^c)
\]

\[
K_{1,-1}^c = \frac{3}{\sqrt{2} A_2} (L_{10}^c - L_{20}^c - \sqrt{2} L_{00}^c), \quad K_{1,0}^c = \frac{3}{\sqrt{2} A_1} (L_{20}^c - L_{11}^c), \quad K_{1,1}^c = -\frac{3}{A_2} L_{22}^c.
\]

The coefficient \(T_{\lambda \mu}^c(t)\) is connected with \(P_{\lambda \mu}^c(t)\) by the relation \(P_{\lambda \mu}^c(t) = AT_{\lambda \mu}^c(t)\), \(A\) being the number of nucleons.
Appendix E:

\begin{equation}
I_{pp}^{\kappa \Delta}(r, p) = \frac{r^3}{\sqrt{\pi \hbar^3}} e^{-\alpha p^2} \int \kappa^r(r, p') \left[ \phi_0(x) - 4\alpha^2 p'^4 \phi_2(x) \right] e^{-\alpha p'^2} p'^2 dp',
\end{equation}

\begin{equation}
I_{rp}^{\kappa \Delta}(r, p) = \frac{r^3}{\sqrt{\pi \hbar^3}} e^{-\alpha p^2} \int \kappa^r(r, p') \left[ \phi_0(x) - 2\alpha p'^2 \phi_1(x) \right] e^{-\alpha p'^2} p'^2 dp',
\end{equation}

where \( x = 2\alpha p p' \),

\begin{align*}
\phi_0(x) &= \frac{1}{x} \sinh(x), \\
\phi_1(x) &= \frac{1}{x^2} \left[ \cosh(x) - \frac{1}{x} \sinh(x) \right], \\
\phi_2(x) &= \frac{1}{x^3} \left[ \left( 1 + \frac{3}{x^2} \right) \sinh(x) - \frac{3}{x} \cosh(x) \right].
\end{align*}

The detailed derivation of these formulae can be found in [14].

Anomalous density and semiclassical gap equation [20]:

\begin{equation}
\kappa(r, p) = \frac{1}{2} \frac{\Delta(r, p)}{\sqrt{\hbar^2(r, p) + \Delta^2(r, p)}},
\end{equation}

\begin{equation}
\Delta(r, p) = -\frac{1}{2} \int \frac{dp'}{(2\pi \hbar)^3} v(|p - p'|) \frac{\Delta(r, p')}{\sqrt{\hbar^2(r, p') + \Delta^2(r, p')}}.
\end{equation}

where \( v(|p - p'|) = \beta e^{-\alpha |p - p'|^2} \) with \( \beta = -|V_0|(r_p \sqrt{\pi})^3 \) and \( \alpha = r_p^2 / 4\hbar^2 \).

\[\text{References}\]

[1] K. Heyde, P. von Neuman-Cosel, and A. Richter, Rev. Mod. Phys. 82, 2365 (2010).
[2] D. Pena Arteaga and P. Ring, arXiv:0912.0908v1 [nucl-th], (2009).
[3] R. R. Hilton, Talk presented at the International Conference on Nuclear Structure (Joint Institute for Nuclear Research, Dubna, Russia, 1976) (unpublished).
[4] R. R. Hilton, Ann. Phys. (NY) 214, 258 (1992).
[5] T. Suzuki and D. J. Rowe, Nucl. Phys. A 289, 461 (1977).
[6] N. Lo Iudice and F. Palumbo, Phys. Rev. Lett. 41, 1532 (1978).
[7] D. Zawischa, J. Phys. G: Nucl. Part. Phys. 24, 683 (1998).
[8] V. G. Soloviev, A. V. Sushkov, N. Yu. Shirikova, and N. Lo Iudice, Nucl. Phys. A 600, 155 (1996).
[9] N. Lo Iudice, La Rivista del Nuovo Cimento 23(9), 1 (2000).
[10] E. Lipparini and S. Stringari, Phys. Rep. 175, 103 (1989).
[11] U. Kneissl, H. H. Pitz, and A. Zilges, Prog. Part. Nucl. Phys. 37, 349 (1996).
[12] A. Richter, Prog. Part. Nucl. 34, 261 (1995).
[13] E. B. Balbutsev, L. A. Malov, P. Schuck, M. Urban, and X. Viñas, Phys. At. Nucl. 71, 1012 (2008).
[14] E. B. Balbutsev, L. A. Malov, P. Schuck, and M. Urban, Phys. At. Nucl. 72, 1305 (2009).
[15] E. B. Balbutsev, I. V. Molodtsova, and P. Schuck, Nucl. Phys. A 872, 42 (2011).
[16] E. B. Balbutsev, I. V. Molodtsova, and P. Schuck, Phys. Rev. C 88, 014306 (2013).
[17] M. Guttormsen, L. A. Bernstein, A. Bürger, A. Görgen, F. Gunsing, T. W. Hagen, A. C. Larsen, T. Renstrøm, S. Siem, M. Wiedeking, and J. N. Wilson, Phys. Rev. Lett. 109, 162503 (2012).
[18] E. B. Balbutsev, I. V. Molodtsova, and P. Schuck, Phys. Rev. C 91, 064312 (2015).
[19] V. G. Soloviev, Theory of complex nuclei (Pergamon Press, Oxford, 1976).
[20] P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer, Berlin, 1980).
[21] M. Urban, Phys. Rev. A 75, 053607 (2007).
[22] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonski, Quantum Theory of Angular Momentum (World Scientific, Singapore, 1988).
[23] E. B. Balbutsev and P. Schuck, Nucl. Phys. A 720, 293 (2003); 728, 471 (2003).
[24] E. B. Balbutsev and P. Schuck, Ann. Phys. 322, 489 (2007).
[25] A. Bohr and B. Mottelson, Nuclear Structure, Vol. 2 (Benjamin, New York, 1975).
[26] P. Sarriguren, E. Moya de Guerra, and R. Nojarov, Phys. Rev. C 54, 690 (1996); P. Sarriguren, E. Moya de Guerra, and R. Nojarov, Z. Phys. A 357, 143 (1997).
[27] N. Van Giai and H. Sagawa, Phys. Lett. B 106, 379 (1981).
[28] M. Beiner, H. Flocard, N. Van Giai, and P. Quentin, Nucl. Phys. A 238, 29 (1975).
[29] S. G. Nilsson, Mat.-Fys. Medd.-K. Dan. Vidensk. Selsk. 29(16), 1 (1955).
[30] Gogny D1S web mass table, http://www-phynu.cea.fr/HFB-Gogny.htm.
[31] W. Korten, T. Härtlein, J. Gerl, D. Habs, and D. Schwalm, Phys. Lett. B 317, 19 (1993).
[32] F. Osterfeld, Rev. Mod. Phys. 64, 491 (1992).
[33] N. Pietralla, P. von Brentano, R.-D. Herzberg, U. Kneissl, J. Margraf, H. Maser, H. H. Pitz, and A. Zilges, Phys. Rev. C 52, R2317 (1995).
[34] N. Pietralla, P. von Brentano, R.-D. Herzberg, U. Kneissl, N. Lo Iudice, H. Maser, H. H. Pitz,
and A. Zilges, Phys. Rev. C 58, 184 (1998).

[35] A. S. Adekola, C. T. Angell, S. L. Hammond, A. Hill, C. R. Howell, H. J. Karwowski, J. H. Kelley, and E. Kwan, Phys. Rev. C 83, 034615 (2011).

[36] V. G. Soloviev, A. V. Sushkov, and N. Yu. Shirikova, Phys. Part. Nucl. 31, 385 (2000)

[37] A. Messiah, Quantum Mechanics, Vol. 2 (North-Holland, Amsterdam, 1961).

[38] N. Pillet, N. Sandulescu, P. Schuck, and J.-F. Berger, Phys. Rev. C 81, 034307 (2010).

[39] E. B. Balbutsev, Sov. J. Part. Nucl. 22, 159 (1991).