On strong extension groups of Cuntz–Krieger algebras

Kengo Matsumoto
Department of Mathematics
Joetsu University of Education
Joetsu, 943-8512, Japan

May 17, 2023

Abstract

In this paper, we study the strong extension groups of Cuntz–Krieger algebras, and present a formula to compute the groups. We also detect the position of the Toeplitz extension of a Cuntz–Krieger algebra in the strong extension group and in the weak extension group to see that the weak extension group with the position of the Toeplitz extension is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra associated with its transposed matrix.

Mathematics Subject Classification: Primary 46L80; Secondary 19K33.

Keywords and phrases: Extension group, $C^*$-algebra, extension, Cuntz–Krieger algebra, strong extension group, Toeplitz extension, Fredholm index

1 Preliminary

There are several kinds of extension groups $\text{Ext}_s(A)$ for a $C^*$-algebra $A$. Among them two extension groups $\text{Ext}_w(A)$ and $\text{Ext}_s(A)$ for a unital nuclear separable $C^*$-algebra $A$ have been studying in many papers (see [2], [4], [7], [8], [9], [13], [14], [16], [17], [18], etc.). In this paper, we study the strong extension groups $\text{Ext}_s(O_A)$ of Cuntz–Krieger algebras $O_A$, and present a formula to compute the groups. We also detect the position of the Toeplitz extension $T_A$ of a Cuntz–Krieger algebra $O_A$ in the weak extension group $\text{Ext}_w(O_A)$ to show that it is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra $O_A$ for the transposed matrix $A^t$ of $A$ by using Rørdam’s classification result.

In what follows, $H$ stands for a separable infinite dimensional Hilbert space. Let us denote by $K(H)$ the $C^*$-algebra of compact operators on $H$. It is a closed two-sided ideal of the $C^*$-algebra $B(H)$ of bounded linear operators on $H$. The quotient $C^*$-algebra $B(H)/K(H)$ is called the Calkin algebra, denoted by $Q(H)$. The quotient map $B(H) \to Q(H)$ is denoted by $\pi$.

Let $A$ be a unital separable $C^*$-algebra. Throughout the paper, a unital $*$-monomorphism $\tau : A \to Q(H)$ is called an extension. Two extensions $\tau_1, \tau_2 : A \to Q(H)$ are said to be strongly equivalent, written $\tau_1 \sim_s \tau_2$, if there exists a unitary $U \in B(H)$ on $H$ such that $\tau_1(a) = \pi(U)\tau_2(a)\pi(U^*)$ in $Q(H)$ for all $a \in A$. They are said to be weakly equivalent, written $\tau_1 \sim_w \tau_2$, if there exists a unitary $u \in Q(H)$ such that $\tau_1(a) = u\tau_2(a)u^*$ in $Q(H)$ for all $a \in A$. The strong equivalence class of an extension $\tau : A \to Q(H)$ is denoted by $[\tau]_s$, and
and similarly the weak equivalence class is denoted by $[\tau]_w$. We note that weakly equivalent extensions are strongly equivalent if one may take a unitary $u \in Q(H)$ of Fredholm index zero such that $\tau_1(a) = u \tau_2(a) u^*$ in $Q(H)$ for all $a \in A$. An extension $\tau : A \to Q(H)$ is said to be trivial if there exists a unital $*$-monomorphism $\rho : A \to B(H)$ such that $\tau = \pi \circ \rho$. We regard $Q(H) \oplus Q(H) \subset Q(H \oplus H)$ in a natural way and identify $H \oplus H$ with $H$, so that $Q(H) \oplus Q(H) \subset Q(H)$. The sum of extensions $\tau_1, \tau_2 : A \to Q(H)$ are defined by

$$(\tau_1 + \tau_2)(a) = \tau_1(a) \oplus \tau_2(a) \in Q(H) \oplus Q(H) \subset Q(H), \quad a \in A$$

gives rise to an extension $\tau_1 \oplus \tau_2 : A \to Q(H)$. Let us denote by $\text{Ext}_s(A)$ the set of strong equivalence classes of extensions. Similarly the set of weak equivalence classes is denoted by $\text{Ext}_w(A)$. Both $\text{Ext}_s(A)$ and $\text{Ext}_w(A)$ have commutative semigroup structure by the above sums. There is a canonical surjective homomorphism $q_A : \text{Ext}_s(A) \to \text{Ext}_w(A)$ of commutative semigroups defined by $q_A([\tau]) = [\tau]_w$.

By virtue of Voiculescu’s theorem in [20], the following basic lemma holds:

**Lemma 1.1** ([20]). Let $A$ be a unital separable $C^*$-algebra. For any two trivial extensions $\tau_1, \tau_2 : A \to Q(H)$, there exists a unitary $U \in B(H)$ such that $\tau_2 = \text{Ad}(\pi(U)) \circ \tau_1$, that is, $\tau_1 \sim \tau_2$. The strong (resp. weak) equivalence class of a trivial extension is the neutral element of $\text{Ext}_s(A)$ (resp. $\text{Ext}_w(A)$).

Choi–Effros in [5] (cf. [11]) proved that if $A$ is nuclear, the semigoups $\text{Ext}_s(A), \text{Ext}_w(A)$ become groups, that is, any element has its inverse. The following lemma is seen in [17].

**Lemma 1.2.** Let $A$ be a unital separable nuclear $C^*$-algebra. For $m \in \mathbb{Z}$, take a unitary $u_m \in Q(H)$ of Fredholm index $m$. Take a trivial extension $\tau : A \to Q(H)$ and consider the extension $\sigma_m = \text{Ad}(u_m) \circ \tau : A \to Q(H)$. Then the map $\iota_A : m \in \mathbb{Z} \to [\sigma_m] \in \text{Ext}_s(A)$ gives rise to a homomorphism of groups such that the sequence

$$\mathbb{Z} \xrightarrow{\iota_A} \text{Ext}_s(A) \xrightarrow{q_A} \text{Ext}_w(A).$$

is exact at the middle, that is, $\iota_A(\mathbb{Z}) = \text{Ker}(q_A)$, so that

$$\text{Ext}_s(A)/\iota_A(\mathbb{Z}) \cong \text{Ext}_w(A).$$

The groups $\text{Ext}_s(A)$ and $\text{Ext}_w(A)$ for a unital separable nuclear $C^*$-algebra $A$ are called the strong extension group for $A$ and the weak extension group for $A$, respectively.

Let $e \in Q(H), E \in B(H)$ be projections such that $e = \pi(E)$. For an element $x \in Q(H)$ such that $exe \in eQ(H)e$ is invertible in $eQ(H)e$, one may denote by $\text{ind}_e x$ the Fredholm index $\text{ind}_E X$ for $X \in B(EH)$ satisfying $x = \pi(X)$. As the Fredholm index is invariant under compact perturbations, the integer $\text{ind}_e x$ does not depend on the choice of $E$ and $X$ as long as $e = \pi(E), x = \pi(X)$. The following lemma is well-known (cf. [3] Lemma 5.1).

**Lemma 1.3.** Let $e, f \in Q(H)$ be projections. Suppose that $x \in Q(H)$ commutes with $e$ and $f$, and $exe, fxf$ are invertible in $eQ(H)e$ and $fQ(H)f$, respectively.

(i) If $ef = 0$, then $\text{ind}_{e+f} x = \text{ind}_e x + \text{ind}_f x$.

(ii) If $x, y \in eQ(H)e$ are both invertible in $eQ(H)e$, then $\text{ind}_e xy = \text{ind}_e x + \text{ind}_e y$. 

2
2 Ext-groups for Cuntz–Krieger algebras

Let $A = [A(i,j)]_{i,j=1}^{N}$ be an irreducible non permutation matrix with entries in $\{0,1\}$ with $N > 1$. The Cuntz–Krieger algebra $\mathcal{O}_A$ is defined to be the universal $C^*$-algebra generated by $N$ partial isometries $S_1, \ldots, S_N$ subject to the operator relations:

$$\sum_{j=1}^{N} S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*, \quad i = 1, \ldots, N \quad (2.1)$$

It is a nuclear $C^*$-algebra uniquely determined by the operator relations (2.1) (see [8]). For the matrix $A$ with all of the entries are one’s, the $C^*$-algebra $\mathcal{O}_A$ is called the Cuntz algebra written $\mathcal{O}_N$ ([6]).

In [8], Cuntz–Krieger pointed out the $C^*$-algebras $\mathcal{O}_A$ are closely related to dynamical property of the underlying topological Markov shifts. Among other things, they proved that the weak extension group $\text{Ext}_w(\mathcal{O}_A)$ is isomorphic to the abelian group $\mathbb{Z}^N/(1 - A)\mathbb{Z}^N$.

We note that the group $\text{Ext}_w(\mathcal{O}_A)$ was written as $\text{Ext}(\mathcal{O}_A)$ in the Cuntz–Krieger’s paper [8]. For the Cuntz algebra $\mathcal{O}_N$, both of the groups $\text{Ext}_s(\mathcal{O}_N)$ and $\text{Ext}_w(\mathcal{O}_N)$ had been computed as $\mathbb{Z}$ and $\mathbb{Z}/(1 - N)\mathbb{Z}$, respectively by Pimsner–Popa [17] and Paschke–Salinas [16].

In this paper, we will compute the strong extension group $\text{Ext}_s(\mathcal{O}_A)$ for $\mathcal{O}_A$ and present a formula (2.3) stated in the theorem below. For $n = 1, \ldots, N$, let $R_n = [R_n(i,j)]_{i,j=1}^{N}$ be the $N \times N$ matrix defined by

$$R_n(i,j) = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

meaning that the only $n$th row is the vector $[1, \ldots, 1]$ but the other rows are zero vectors. The homomorphisms $\iota_A : \mathbb{Z} \longrightarrow \text{Ext}_s(\mathcal{O}_A)$ and $q_A : \text{Ext}_s(\mathcal{O}_A) \longrightarrow \text{Ext}_w(\mathcal{O}_A)$ in (1.1) for $A = \mathcal{O}_A$ are denoted by $\iota_A : \mathbb{Z} \longrightarrow \text{Ext}_s(\mathcal{O}_A)$ and $q_A : \text{Ext}_s(\mathcal{O}_A) \longrightarrow \text{Ext}_w(\mathcal{O}_A)$, respectively.

Theorem 2.1 (Theorem 2.6 and Theorem 3.3).

(i) The strong extension group $\text{Ext}_s(\mathcal{O}_A)$ for the Cuntz–Krieger algebra $\mathcal{O}_A$ is

$$\text{Ext}_s(\mathcal{O}_A) = \mathbb{Z}^N/(1 - \widehat{A})\mathbb{Z}^N \quad (2.3)$$

where the matrix $\widehat{A}$ is $\widehat{A} = A + R_1 - AR_1$.

(ii) The homomorphism $\iota_A : \mathbb{Z} \longrightarrow \text{Ext}_s(\mathcal{O}_A)$ in (1.1) is injective if and only if $\det(I - A) \neq 0$. Hence the short exact sequence

$$0 \longrightarrow \mathbb{Z} \overset{\iota_A}{\longrightarrow} \text{Ext}_s(\mathcal{O}_A) \overset{q_A}{\longrightarrow} \text{Ext}_w(\mathcal{O}_A) \longrightarrow 0 \quad (2.4)$$

holds if and only if $\det(I - A) \neq 0$. 

3
The given proof in this paper for the formula \([2,3]\) presented as Theorem [2.6] is basically follows the proof of [8, Theorem 5.3] that showed the formula \(\text{Ext}_w(\mathcal{O}_A) = \mathbb{Z}^N/(I - A)\mathbb{Z}^N\).

Among various extensions of a Cuntz–Krieger algebra \(\mathcal{O}_A\), there is one specific extension called the Toeplitz extension \(\sigma_{T_A}\) of \(\mathcal{O}_A\). It arises from the short exact sequence:

\[
0 \longrightarrow K(H_A) \xrightarrow{k} \mathcal{T}_A \xrightarrow{q} \mathcal{O}_A \longrightarrow 0
\]

of the Toeplitz algebra \(\mathcal{T}_A\) on the sub Fock space \(H_A\) (cf. \([10, 12]\)). We will detect the positions of the Toeplitz extension \(\sigma_{T_A}\) of \(\mathcal{O}_A\) in the strong extension group \(\text{Ext}_s(\mathcal{O}_A)\) and in the weak extension group \(\text{Ext}_w(\mathcal{O}_A)\) \((\text{Theorem } 1.4)\). As a result, we will know that the group \(\text{Ext}_w(\mathcal{O}_A)\) with the position \([\mathcal{T}_A]_w\) of the Toeplitz extension \(\sigma_{T_A}\) in \(\text{Ext}_w(\mathcal{O}_A)\) is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra \(\mathcal{O}_{A'}\) for its transposed matrix \(A'\) of \(A\) by using Rørdam’s classification result \((\text{Corollary } 4.5)\).

Let us denote by \(P_i\) the projection \(S_iS_i^*\). Let \(\sigma : \mathcal{O}_A \longrightarrow Q(H)\) be an extension. Put \(e_i = \sigma(P_i)\). There exists a trivial extension \(\tau : \mathcal{O}_A \longrightarrow Q(H)\) such that \(\tau(P_i) = \sigma(P_i), i = 1, \ldots, N\). As the partial isometry \(\sigma(S_i)\tau(S_i^*)\) commutes with \(e_i, e_i\sigma(S_i)\tau(S_i^*)e_i\) becomes a unitary in \(e_iQ(H)e_i\). One may define \(\text{ind}_{e_i}\sigma(S_i)\tau(S_i^*)\) denoted by \(d_i(\sigma, \tau)\), that is

\[
d_i(\sigma, \tau) = \text{ind}_{e_i}\sigma(S_i)\tau(S_i^*), \quad i = 1, \ldots, N.
\]

The proof of \([8, \text{Proposition } 5.2]\) describes the following lemma. We give its proof for the sake of completeness.

**Lemma 2.2** (cf. \([8, \text{Proposition } 5.2]\)). Let \(\sigma : \mathcal{O}_A \longrightarrow Q(H)\) be an extension. Put \(e_i = \sigma(P_i)\). Let \(\tau_1, \tau_2 : \mathcal{O}_A \longrightarrow Q(H)\) be trivial extensions such that \(\tau_j(P_i) = \sigma(P_i), j = 1, 2, i = 1, \ldots, N\). Then there exists a vector \([k_i]_i = 1^N \in \mathbb{Z}^N\) such that

1. \(d_i(\sigma, \tau_2) = d_i(\sigma, \tau_1) - k_i + \sum_{j=1}^{N} A(i, j)k_j\),
2. \(\sum_{i=1}^{N} k_i = 0\).

**Proof.** By Lemma 1.1 one may find a unitary \(U \in B(H)\) such that \(\tau_2(x) = \pi(U)\tau_1(x)\pi(U^*), x \in \mathcal{O}_A\). Put \(u = \pi(U) \in Q(H)\). Since

\[
(e_iue_i)(e_iue_i)^* = (\tau_2(P_i)\pi(U)\tau_1(P_i)\pi(U^*)\tau_2(P_i)\pi(U)\tau_1(P_i)\pi(U^*)\tau_2(P_i)) = \tau_2(P_i)\tau_2(P_i) = e_i
\]

and similarly \((e_iue_i)^*(e_iue_i) = e_i\), we see that \(e_iue_i\) is a unitary in \(e_iQ(H)e_i\). By putting \(k_i = \text{ind}_{e_i}u\), the equality

\[
d_i(\sigma, \tau_2) = d_i(\sigma, \tau_1) - k_i + \sum_{j=1}^{N} A(i, j)k_j
\]
holds, following the proof of [8, Proposition 5.2]. We in fact see that

\[ d_i(\sigma, \tau) = \text{ind}_{e_i} \sigma(S_i) \tau_1(S^*_i) \]

\[ = \text{ind}_{e_i} \sigma(S_i) \sigma(S^*_i S_i) u \tau_1(S^*_i S_i) \tau_1(S^*_i) u^* \]

\[ = \text{ind}_{e_i} \sigma(S_i) \tau_1(S^*_i S_i) u \tau_1(S^*_i S_i) \tau_1(S^*_i) \tau_1(S^*_i) u^* \]

\[ = \text{ind}_{e_i} \sigma(S_i) \tau_1(S^*_i) \left( \tau_1(S_i) \sum_{j=1}^{N} A(i, j) u \tau_1(S_j S^*_j) \tau_1(S^*_i) \right) e_i u^* \]

\[ = \text{ind}_{e_i} \sigma(S_i) \tau_1(S^*_i) \left( \tau_1(S_i) \sum_{j=1}^{N} A(i, j) e_j u e_j \tau_1(S^*_i) \right) e_i u^* e_i \]

\[ = \text{ind}_{e_i} \sigma(S_i) \tau_1(S^*_i) + \text{ind}_{e_i} \left( \tau_1(S_i) \sum_{j=1}^{N} A(i, j) e_j u e_j \tau_1(S^*_i) \right) + \text{ind}_{e_i} u^* \]

\[ = d_i(\sigma, \tau_1) + \sum_{j=1}^{N} A(i, j) \text{ind}_{e_i} \tau_1(S_i) e_j u e_j \tau_1(S^*_i) - k_i. \]

As \( \text{ind}_{e_i} \tau_1(S_i) e_j u e_j \tau_1(S^*_i) = \text{ind}_{e_j} u = k_j \) whenever \( A(i, j) = 1 \), we obtain the equality (2.6).

Since \( u = \pi(U) \) for some unitary \( U \) on \( H \), Lemma [1.3] tells us

\[ \sum_{i=1}^{N} k_i = \sum_{i=1}^{N} \text{ind}_{e_i} u = \text{ind}_{\sum_{i=1}^{N} e_i} u = \text{ind} U = 0. \]

Define a subgroup \( \text{Im}(1 - A)_0 \subset \mathbb{Z}^N \) by setting

\[ \text{Im}(1 - A)_0 = \{(1 - A)[k_i]_{i=1}^{N} \in \mathbb{Z}^N \mid [k_i]_{i=1}^{N} \in \mathbb{Z}^N \text{ with } \sum_{i=1}^{N} k_i = 0\}. \]

We thus see that an extension \( \sigma : \mathcal{O}_A \rightarrow Q(H) \) defines an element of \( \mathbb{Z}^N / \text{Im}(1 - A)_0 \) in a unique way by

\[ d_{\sigma} := [d_i(\sigma, \tau)]_{i=1}^{N} \in \mathbb{Z}^N / \text{Im}(1 - A)_0 \]

for a trivial extension \( \tau : \mathcal{O}_A \rightarrow Q(H) \) satisfying \( \tau(P_i) = \sigma(P_i), i = 1, \ldots, N \).

**Lemma 2.3.** Let \( \sigma_1, \sigma_2 : \mathcal{O}_A \rightarrow Q(H) \) be extensions. If \( \sigma_1 \sim \sigma_2 \), then \( d_{\sigma_1} = d_{\sigma_2} \) in \( \mathbb{Z}^N / \text{Im}(1 - A)_0 \).

**Proof.** Assume that \( \sigma_1 \sim \sigma_2 \) so that, by Lemma [1.1] one may find a unitary \( V \) in \( B(H) \) such that \( \sigma_2 = \text{Ad}(\pi(V)) \circ \sigma_1 \). Put \( e_1^{i} = \sigma_1(P_i), e_2^{i} = \sigma_2(P_i), i = 1, \ldots, N \) and \( v = \pi(V) \in Q(H) \), and hence \( v e_1^{i} v^* = e_2^{i} \). Take a trivial extension \( \tau_1 : \mathcal{O}_A \rightarrow Q(H) \) such that \( \tau_1(P_i) = e_1^{i}, i = 1, \ldots, N \). We set \( \tau_2 = \text{Ad}(v) \circ \tau_1 \) so that \( \tau_2(P_i) = v \tau_1(P_i) v^* = e_2^{i} \). We then have

\[ d_i(\sigma_2, \tau_2) = \text{ind}_{e_1^{i}} \sigma_2(S_i) \tau_2(S^*_i) = \text{ind}_{v e_1^{i} v^*} v \sigma_1(S_i) \tau_1(S^*_i) v^* = d_i(\sigma_1, \tau_1). \]

\[ \square \]
Let us define $d_s : \text{Ext}_s(O_A) \to \mathbb{Z}^N / \text{Im}(I - A)_0$ by setting

$$d_s([\sigma], \tau) = [[d_i(\sigma, \tau)]_{i=1}^N] \in \mathbb{Z}^N / \text{Im}(I - A)_0$$

for a trivial extension $\tau : O_A \to Q(H)$ satisfying $\tau(P_i) = \sigma(P_i), i = 1, \ldots, N$.

**Proposition 2.4.** $d_s : \text{Ext}_s(O_A) \to \mathbb{Z}^N / \text{Im}(I - A)_0$ is an isomorphism of groups.

**Proof.** It is obvious that $d_s : \text{Ext}_s(O_A) \to \mathbb{Z}^N / \text{Im}(I - A)_0$ is a homomorphism of groups. It remains to show that $d_s$ is injective. We will first show that $d_s$ is bijective. We will first show that $d_s$ is bijective. Let $\sigma : O_A \to Q(H)$ be an extension such that $d_s([\sigma]) = 0$ in $\mathbb{Z}^N / \text{Im}(I - A)_0$. Take a trivial extension $\tau$ such that $\tau(P_i) = \sigma(P_i), i = 1, \ldots, N$. Put $d_i = d_i(\sigma, \tau) \in \mathbb{Z}$. Let $\rho_\tau : O_A \to B(H)$ be a unital $*$-monomorphism such that $\tau = \pi \circ \rho_\tau$. By the assumption, there exists $[k_i]_{i=1}^N \in \mathbb{Z}^N$ such that

$$[d_i]_{i=1}^N = (I - A)[k_i]_{i=1}^N, \quad \sum_{i=1}^N k_i = 0.$$

Put $e_i = \tau(P_i)$ and $E_i = \rho_\tau(P_i)$ so that $\pi(E_i) = e_i$. Take an isometry or coisometry $V_i \in B(E_i H)$ such that $\text{ind}(V_i) = -k_i$. Put $V = \sum_{i=1}^N V_i \in B(H)$ and $v = \pi(V)$. Since $v$ is a unitary in $Q(H)$ such that $\text{ind}(v) = \sum_{i=1}^N \text{ind}(V_i) = -\sum_{i=1}^N k_i = 0$, one may take a unitary $U$ in $B(H)$ such that $v = \pi(U)$. By following the proof of [8] Theorem 5.3, we have

$$\text{ind}_{e_i} \pi(U) \sigma(S_i) \pi(U^*) \tau(S_i^*)$$

$$= \text{ind}_{e_i} \pi(V_i) \sigma(S_i) (S_i^* S_i) \pi \left( \sum_{n=1}^N V_n^* \right) \tau(S_i^*)$$

$$= \text{ind}_{e_i} \pi(V_i) \sigma(S_i) \left( \sum_{j=1}^N A(i, j) \pi(E_j) \right) \pi \left( \sum_{n=1}^N E_n V_n^* \right) \tau(S_i^*)$$

$$= \text{ind}_{e_i} \pi(V_i) \sigma(S_i) \sigma(S_i^* S_i) \pi \left( \sum_{j=1}^N A(i, j) V_j^* \right) \tau(S_i^*)$$

$$= \text{ind}_{e_i} \pi(V_i) \sigma(S_i) \tau(S_i^*) \left( \tau(S_i) \pi \left( \sum_{j=1}^N A(i, j) V_j^* \right) \tau(S_i^*) \right)$$

$$= \text{ind}_{e_i} \pi(V_i) + \text{ind}_{e_i} \sigma(S_i) \tau(S_i^*) + \text{ind}_{e_i} \tau(S_i) \pi \left( \sum_{j=1}^N A(i, j) V_j^* \right) \tau(S_i^*)$$

$$= -k_i + d_i + \sum_{j=1}^N A(i, j) \text{ind}_{e_i} \tau(S_i) \pi(V_j^*) \tau(S_i^*).$$

Since $\text{ind}_{e_i} \tau(S_i) \pi(V_j^*) \tau(S_i^*) = \text{ind}_{e_j} \pi(V_j^*) = k_j$ whenever $A(i, j) = 1$, we have

$$\text{ind}_{e_i} \pi(U) \sigma(S_i) \pi(U^*) \tau(S_i^*) = -k_i + d_i + \sum_{j=1}^N A(i, j) k_j = 0$$

6
so that there exists a unitary $W_i \in B(E_iH)$ on $E_iH$ such that

$$\pi(U)\sigma(S_i)\pi(U^*)\tau(S_i^*) = \pi(W_i), \quad i = 1, \ldots, N.$$ 

By putting $T_i = W_i\rho_r(S_i)$, $i = 1, \ldots, N$, we have

$$\sum_{j=1}^NT_jT_j^* = \sum_{j=1}^NW_j\rho_r(S_j)\rho_r(S_j^*)W_j^* = \sum_{j=1}^NW_jW_j^* = \sum_{j=1}^NE_j = 1,$$

and

$$T_i^*T_i = \rho_r(S_i^*)W_i^*W_i\rho_r(S_i) = \rho_r(S_i^*)\rho_r(S_iS_i^*)\rho_r(S_i) = \sum_{j=1}^NA(i,j)\rho_r(S_jS_j^*).$$

As $\rho_r(S_jS_j^*) = T_jT_j^*$, we see that $T_i^*T_i = \sum_{j=1}^NA(i,j)T_jT_j^*$. Define $\rho_\sigma(S_i) = T_i \in B(H), i = 1, \ldots, N$ so that $\rho_\sigma : O_A \rightarrow Q(H)$ is a unital *-monomorphism such that

$$\pi \circ \rho_\sigma(S_i) = \pi(W_i\rho_r(S_i)) = \pi(U)\sigma(S_i)\pi(U^*)\tau(S_i)\tau(S_i^*) = \pi(U)\sigma(S_i)\tau(S_iS_i^*)\pi(U^*) = \pi(U)\sigma(S_i)\pi(U^*).$$

Hence we have $\text{Ad}(\pi(U)) \circ \sigma = \pi \circ \rho_\sigma$. This shows that $\sigma$ is strongly equivalent to the trivial extension $\pi \circ \rho_\sigma$ proving $[\sigma]_s = 0$ in $\text{Ext}_s(O_A)$.

We will next show that $d_s$ is surjective. We will show that there exist an extension $\sigma : O_A \rightarrow Q(H)$ and a trivial extension $\tau : O_A \rightarrow Q(H)$ such that $\tau(P_i) = \sigma(P_i)$ denoted by $e_i$ and

$$\text{ind}_{e_i}\sigma(S_i)\tau(S_i^*) = \begin{cases} -1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

Decompose the Hilbert space $H$ as $H = H_1 \oplus \cdots \oplus H_N$ such that $\dim H_i = \dim H, i = 1, \ldots, N$. Take a nonzero vector $v_1 \in H_1$ and put its orthogonal complement $H_1^0 = \{\mathbb{C}v_1\}^\perp \cap H_1$ in $H_1$. Let $E_i$ be the orthogonal projection onto $H_i, i = 1, \ldots, N$. The orthogonal projection onto $H_1^0$ is denoted by $E_1^0$, so that $\sum_{i=1}^NE_i = 1$ and $E_1 - E_1^0$ is the projection onto $\mathbb{C}v_1$. Take partial isometries $T_1, \ldots, T_N$ and $V_1, \ldots, V_N$ on $H$ such that

$$T_1T_1^* = E_1^0, \quad T_iT_i^* = E_i, \quad i = 2, \ldots, N, \quad V_iV_i^* = E_i, \quad i = 1, \ldots, N$$

and

$$T_i^*T_i = V_i^*V_i = \sum_{j=1}^NA(i,j)E_j, \quad i = 1, \ldots, N.$$

We know that

$$\pi(T_i)\pi(T_i)^* = \pi(V_i)\pi(V_i)^* = \pi(E_i), \quad \sum_{i=1}^N\pi(E_i) = 1$$

and

$$\pi(T_i)^*\pi(T_i) = \pi(V_i)^*\pi(V_i) = \sum_{j=1}^NA(i,j)\pi(E_j), \quad i = 1, \ldots, N.$$
By setting $\sigma(S_i) = \pi(T_i), \tau(S_i) = \pi(V_i), i = 1, \ldots, N$, we have extensions $\sigma, \tau : \mathcal{O}_A \to Q(H)$ such that $\tau$ is a trivial extension. Put $e_i = \pi(E_i), i = 1, \ldots, N$. Since $\sigma(S_i)\tau(S_i^*) = \pi(T_iV_i^*), i = 1, \ldots, N$, we have $\text{ind}_{e_i}\sigma(S_i)\tau(S_i^*) = \text{ind}_{E_i}T_iV_i^*$ so that the equality (2.7) holds. Therefore we have $d_s(\sigma) = [(i-1, 0, \ldots, 0)]$ in $\mathbb{Z}^N/\text{Im}(I - A)_0$. One may show that $d_s : \text{Ext}_s(\mathcal{O}_A) \to \mathbb{Z}^N/\text{Im}(I - A)_0$ is surjective by a similar fashion. 

Recall that the $N \times N$ matrix $R_n$ for $n = 1, \ldots, N$ is defined in (2.2).

**Lemma 2.5.** For $n = 1, 2, \ldots, N$, put $\tilde{A}_n = A + R_n - AR_n$. Then we have

$$\text{Im}(I - A)_0 = (I - \tilde{A}_n)\mathbb{Z}^N. \tag{2.8}$$

In particular for $n = 1$, we put $\tilde{A} = \tilde{A}_1$ so that we have $\text{Im}(I - A)_0 = (I - \tilde{A})\mathbb{Z}^N$.

**Proof.** As $\text{Im}(I - A)_0 = \{(I - A)[k_i]_{i=1}^N \mid \sum_{i=1}^N k_i = 0\}$, a vector $[k_i]_{i=1}^N \in \mathbb{Z}^N$ satisfies $\sum_{i=1}^N k_i = 0$ if and only if $[k_i]_{i=1}^N = (I - R_n)[k_i]_{i=1}^N$. Hence we have

$$\text{Im}(I - A)_0 = (I - A)(I - R_n)\mathbb{Z}^N.$$ 

Since $(I - A)(I - R_n) = I - \tilde{A}_n$, we have $\text{Im}(I - A)_0 = \text{Im}(I - \tilde{A}_n)\mathbb{Z}^N$. 

Therefore we reach the following theorem.

**Theorem 2.6.** $\text{Ext}_s(\mathcal{O}_A) \cong \mathbb{Z}^N/(I - \tilde{A})\mathbb{Z}^N$, where $\tilde{A} = A + R_1 - AR_1$.

### 3 The homomorphism $\iota_A : \mathbb{Z} \to \text{Ext}_s(\mathcal{O}_A)$

For $m \in \mathbb{Z}$, take $k_1, \ldots, k_N \in \mathbb{Z}$ such that $m = \sum_{i=1}^N k_i$. Take trivial extensions $\tau, \tau' : \mathcal{O}_A \to Q(H)$ such that $\tau(P_i) = \tau'(P_i)$ denoted by $e_i, i = 1, \ldots, N$. Let $\rho, \rho' : \mathcal{O}_A \to B(H)$ be unital *-isomorphisms such that $\tau = \pi \circ \rho$, $\tau' = \pi \circ \rho'$, respectively. Put $E_i = \rho(P_i)$ so that $\pi(E_i) = e_i$. Take an isometry or coisometry $V_i \in B(E_iH)$ such that $\text{ind}_{E_i}V_i = k_i$ and put $V = \sum_{i=1}^N V_i \in B(H)$. Hence we see that

$$\text{ind}_{e_i}\pi(V) = k_i, \quad i = 1, \ldots, N.$$ 

Recall that the extension $\sigma_m : \mathcal{O}_A \to Q(H)$ is defined by setting $\sigma_m = \text{Ad}(\pi(V)) \circ \tau : \mathcal{O}_A \to Q(H)$. Put $d_i = d_i(\sigma_m, \tau') = \text{ind}_{e_i}\sigma_m(S_i)\tau'(S_i^*)$. Then $d_s(\sigma_m) = [(d_1, \ldots, d_N)] \in \mathbb{Z}^N/(I - \tilde{A})\mathbb{Z}^N$ does not depend on the choice of trivial extensions $\tau, \tau'$, because of Lemma 2.2 and Lemma 2.3.

**Proposition 3.1.** Define $\iota_A : \mathbb{Z} \to \mathbb{Z}^N/(I - \tilde{A})\mathbb{Z}^N$ by setting $\iota_A(m) = [(I - A)[k_i]_{i=1}^N]$ for $m = \sum_{i=1}^N k_i$. Then we have

(i) $\iota_A(m) = [(I - A)[k_i]_{i=1}^N]$ does not depend on the choice of $[k_i]_{i=1}^N$ as long as $m = \sum_{i=1}^N k_i$.

(ii) The diagram

$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\iota_A} & \text{Ext}_s(\mathcal{O}_A) \\
\downarrow & & \downarrow d_s \\
\mathbb{Z} & \xrightarrow{i_A} & \mathbb{Z}^N/(I - \tilde{A})\mathbb{Z}^N
\end{array}$$

is commutative, that is $d_s(\iota_A(m)) = i_A(m)$, where $\iota_A(m) = [\sigma_m]$. 

8
(iii) The position \(i_A(1)\) in \(\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N\) is invariant under the isomorphism class of \(\mathcal{O}_A\).

(iv) If \(\det(I - A) \neq 0\), then we have a short exact sequence

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot \iota_A} \text{Ext}_s(\mathcal{O}_A) \xrightarrow{\cdot q_A} \text{Ext}_w(\mathcal{O}_A) \longrightarrow 0.
\]

**Proof.** (i) Suppose that \(m = \sum_{i=1}^N k_i = \sum_{i=1}^N k_i'\) for some \(k_i, k_i' \in \mathbb{Z}\). Put \(l_i = k_i - k_i'\) so that \(\sum_{i=1}^N l_i = 0\) and \((I - A)[k_i]_{i=1}^N - (I - A)[k_i']_{i=1}^N = (I - A)[l_i]_{i=1}^N \in (I - \hat{A})\mathbb{Z}^N\) by Lemma 2.5. This shows that \([ (I - A)[k_i]_{i=1}^N \] = \([ (I - A)[k_i']_{i=1}^N \] in \(\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N\).

(ii) Keep the notation stated before Proposition 3.1. Since \(I - A\), the assertion (ii) says that \(d_i = \text{ind}_{e_i} \sigma_m(S_i) \tau(S_i^*)\) does not depend on the choice of a trivial extension \(\tau' : \mathcal{O}_A \longrightarrow Q(H)\) as long as \(\tau(P_i) = \tau'(P_i)\), we may take \(\tau'\) as \(\tau\). We then have

\[
d_i = \text{ind}_{e_i} \sigma_m(S_i) \tau(S_i^*)
\]

\[
= \text{ind}_{e_i} \pi(V) \tau(S_i) \pi(V^*) \tau(S_i^*)
\]

\[
= \text{ind}_{e_i} \pi(V) + \text{ind}_{e_i} \tau(S_i) \pi(V^*) \tau(S_i^*)
\]

\[
= k_i + \text{ind}_{\tau(S_i^*P_iS_i)} \pi(V^*)
\]

\[
= k_i + \sum_{j=1}^N A(i,j) \text{ind}_{\tau(P_j)} \pi(V^*)
\]

\[
= k_i - \sum_{j=1}^N A(i,j) k_j
\]

so that we obtain

\[
d_s(\iota_A(m)) = d_s([\sigma_m]_s) = [d_i]_{i=1}^N = [(I - A)[k_i]_{i=1}^N] = i_A(m).
\]

(iii) By the construction, the map \(\iota_A : m \in \mathbb{Z} \longrightarrow [\sigma_m]_s \in \text{Ext}_s(A)\) as well as the position \(\iota_A(1)\) in \(\text{Ext}_s(A)\) is invariant under the isomorphism class of a \(C^*\)-algebra \(A\). For \(\mathcal{A} = \mathcal{O}_A\), the assertion (ii) says that

\[
(\text{Ext}_s(\mathcal{O}_A), \iota_A(1)) \cong (\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N, i_A(1))
\]

so that the position of \(i_A(1)\) in the group \(\mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N\) is invariant under the isomorphism class of \(\mathcal{O}_A\).

(iv) Assume that \(\det(I - A) \neq 0\). Let \(m \in \mathbb{Z}\) satisfy \(\iota_A(m) = 0\). Take \(k_1, \ldots, k_N \in \mathbb{Z}\) such that \(m = \sum_{i=1}^N k_i\) and hence \(i_A(m) = [(I - A)[k_i]_{i=1}^N]\). As \(i_A(m) = d_s(\iota_A(m)) = 0\), there exists \([n_i]_{i=1}^N \in \mathbb{Z}^N\) such that \(\sum_{i=1}^N n_i = 0\) and \(i_A(m) = (I - A)[n_i]_{i=1}^N\). We then have \((I - A)[k_i]_{i=1}^N = (I - A)[n_i]_{i=1}^N\). By the assumption \(\det(I - A) \neq 0\), we have \([n_i]_{i=1}^N = [k_i]_{i=1}^N\) so that \(m = \sum_{i=1}^N n_i = 0\).

Since \(I - \hat{A} = (I - A)(I - R_1)\), the inclusion relation \((I - \hat{A})\mathbb{Z}^N \subset (I - A)\mathbb{Z}^N\) holds.

There exists a natural quotient map \(\hat{q}_A : \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \longrightarrow \mathbb{Z}^N/(I - A)\mathbb{Z}^N\). In [8], Cuntz–Krieger proved that the map \(d_w : \text{Ext}_s(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(I - A)\mathbb{Z}^N\) defined by \(d_w([\sigma]_w) = [(d_1, \ldots, d_N)] \in \mathbb{Z}^N/(I - A)\mathbb{Z}^N\) yields an isomorphism of groups.
Let us denote by \( \text{Ker}(I - A), \text{Ker}(I - \widehat{A}) \) the subgroups of \( \mathbb{Z}^N \) defined by the kernels in \( \mathbb{Z}^N \) of the matrices \( I - A \) and of \( I - \widehat{A} \), respectively. Define homomorphisms of groups

\[
i_1 : \mathbb{Z} \longrightarrow \text{Ker}(I - \widehat{A}), \quad j_A : \text{Ker}(I - \widehat{A}) \longrightarrow \text{Ker}(I - A), \quad s_A : \text{Ker}(I - A) \longrightarrow \mathbb{Z}
\]

by setting

\[
i_1 : n \mapsto \begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad j_A : [l_i]_{i=1}^N \mapsto \begin{bmatrix} -\sum_{i=2}^N l_i \\ l_2 \\ \vdots \\ l_N \end{bmatrix}, \quad s_A : [l_i]_{i=1}^N \mapsto \sum_{i=1}^N l_i.
\]

**Lemma 3.2.** We have the following long exact sequence.

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{i_1} \text{Ker}(I - \widehat{A}) \xrightarrow{j_A} \text{Ker}(I - A) \xrightarrow{s_A} \mathbb{Z}
\]

Proof. It suffices to show the exactness at the lower right corner

\[
\text{Ker}(I - A) \xrightarrow{s_A} \mathbb{Z} \xrightarrow{i_A} \mathbb{Z}^N/(I - A)\mathbb{Z}^N.
\]

Suppose that \( m \in \mathbb{Z} \) satisfies \( \iota_A(m) = 0 \). Take \( k_1, \ldots, k_N \in \mathbb{Z} \) such that \( m = \sum_{i=1}^N k_i \) and hence \( (I - A)[k_i]_{i=1}^N \) belongs to \( \text{Im}(I - A) \). There exists \( [n_i]_{i=1}^N \in \mathbb{Z}^N \) such that \( (I - A)[k_i]_{i=1}^N = (I - A)[n_i]_{i=1}^N \) and \( \sum_{i=1}^N n_i = 0 \). Put \( l_i = k_i - n_i \). Hence \( [l_i]_{i=1}^N \in \text{Ker}(I - A) \) and \( \sum_{i=1}^N l_i = \sum_{i=1}^N k_i = m \) so that \( s_A([l_i]_{i=1}^N) = m \), proving \( \text{Ker}(\iota_A) \subset s_A(\text{Ker}(I - A)) \).

Conversely, for \( [l_i]_{i=1}^N \in \text{Ker}(I - A) \), we have \( \iota_A(s_A([l_i]_{i=1}^N)) = \iota_A(\sum_{i=1}^N l_i) = [(I - A)]_{i=1}^N \iota_A(1) = 0 \), so that \( s_A(\text{Ker}(I - A)) \subset \text{Ker}(\iota_A) \). Hence the sequence (3.2) is exact at the middle. Exactness at the other places are easily seen.

We thus have the following theorem.

**Theorem 3.3.** (i) The isomorphisms

\[
d_w : \text{Ext}_w(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(I - A)\mathbb{Z}^N, \quad d_s : \text{Ext}_s(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N
\]

of groups and a homomorphism \( \iota_A : \mathbb{Z} \longrightarrow \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N \) defined by \( \iota_A(m) = (I - A)[k_i]_{i=1}^N \) with \( m = \sum_{i=1}^N k_i \) yield the commutative diagrams:

\[
\begin{array}{cccc}
\mathbb{Z} & \xrightarrow{\iota_A} & \text{Ext}_s(\mathcal{O}_A) & \xrightarrow{q_A} & \text{Ext}_w(\mathcal{O}_A) \\
\downarrow & & \downarrow d_s & & \downarrow d_w \\
\mathbb{Z} & \xrightarrow{i_A} & \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N & \xrightarrow{\widehat{q}_A} & \mathbb{Z}^N/(I - A)\mathbb{Z}^N.
\end{array}
\]

(ii) The pair \( (\mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N, \iota_A(1)) \) showing the position \( \iota_A(1) = [(I - A)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}] \) in the

group \( \mathbb{Z}^N/(I - \widehat{A})\mathbb{Z}^N \) is invariant under the isomorphism class of \( \mathcal{O}_A \).
(iii) The homomorphism \( \iota_A : \mathbb{Z} \longrightarrow \text{Ext}_s(O_A) \) is injective if and only if \( \det(I - A) \neq 0 \).

In this case, we have a short exact sequence

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{\iota_A} \text{Ext}_s(O_A) \xrightarrow{\sigma_A} \text{Ext}_w(O_A) \longrightarrow 0. \tag{3.3}
\]

\section{Toeplitz extension}

Among various extensions of \( O_A \), there is a specific extension \( \sigma_A \) of \( O_A \) called the Toeplitz extension (cf. \cite{10, 12}). We fix an irreducible non permutation matrix \( A = [A(i,j)]_{i,j=1}^N \) with entries in \( \{0,1\} \). Let \( \mathbb{C}^N \) be an \( N \)-dimensional Hilbert space with orthonormal basis \( \{\xi_1, \ldots, \xi_N\} \). Let \( H_0 \) be a one-dimensional Hilbert space with unit vector \( v_0 \). Let \( H^\otimes n \) be the \( n \)-fold tensor product \( \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N \). Consider the full Fock space \( F_N = H_0 \oplus (\oplus_{n=1}^\infty H^\otimes n) \). Define a sub Fock space \( H_A \) to be the closed linear span of vectors

\[
\{v_0\} \cup \{\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \mid A(i_j, i_{j+1}) = 1 \text{ for } j = 1, \ldots, n-1, n = 1, 2, \ldots\}.
\]

Define creation operators \( T_i \) for \( i = 1, \ldots, N \) on \( H_A \) by

\[
T_i v_0 = \xi_i,
\]

\[
T_i (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \begin{cases} 
\xi_i \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} & \text{if } A(i, i_1) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Let us denote by \( E_0 \) the rank one projection onto the subspace \( H_0 \) on \( H_A \). The operators \( T_i, i = 1, \ldots, N \) on \( H_A \) are partial isometries satisfying the relations

\[
\sum_{j=1}^N T_j T^*_j = 1 - E_0, \quad T^*_iT_i = \sum_{j=1}^N A(i, j)T_j T^*_j + E_0, \quad i = 1, \ldots, N \quad \text{(10, 12).} \tag{4.1}
\]

The Toeplitz algebra for the matrix \( A \) is defined to be the \( C^* \)-algebra \( C^*(T_1, \ldots, T_N) \) on \( H_A \) generated by the partial isometries \( T_i, i = 1, \ldots, N \). By \cite{11}, we know that the correspondence \( S_i \in O_A \longrightarrow \pi(T_i) \in \pi(H_A) = B(H_A)/K(H_A) \) gives rise to a unital *-homomorphism, that is called the Toeplitz extension denoted by \( \sigma_{T_A} \). In this section, we will detect the positions \( d_s([\sigma_{T_A}]_s) \) in \( \text{Ext}_s(O_A) \) and \( d_w([\sigma_{T_A}]_w) \) in \( \text{Ext}_w(O_A) \), respectively. The classes \( [\sigma_{T_A}]_s \) and \( [\sigma_{T_A}]_w \) are simply denoted by \( [T_A]_s \) and \( [T_A]_w \), respectively.

For \( j = 1, \ldots, N \), let \( H_{A,j} \) be the closed linear subspace of \( H_A \) spanned by the vectors \( \{\xi_j \otimes \eta \in H_A \mid \eta \in H_A\} \), so that \( H_A = H_0 \oplus H_{A,1} \oplus \cdots \oplus H_{A,N} \). Let us denote by \( E_{A,i} \) the projection on \( H_A \) onto the subspace \( H_{A,i} \). We then see that \( E_0 + \sum_{j=1}^N E_{A,j} = 1 \) and

\[
T_i T^*_i = E_{A,i}, \quad T^*_iT_i = E_0 + \sum_{j=1}^N A(i, j)E_{A,j}, \quad i = 1, \ldots, N. \tag{4.2}
\]

We fix \( m \in \{1, \ldots, N\} \) for a while. By setting

\[
H_j := \begin{cases} 
H_{A,j} \oplus H_0 & \text{if } j = m, \\
H_{A,j} & \text{if } j \neq m.
\end{cases}
\]
we have a decomposition $H_A = H_1 \oplus \cdots \oplus H_N$ of $H_A$ depending on $m$. Let us denote by $E_i$ the orthogonal projection on $H_A$ onto the subspace $H_i$, so that we have $\sum_{j=1}^N E_j = 1$. Take a family of partial isometries $V_1, \ldots, V_N$ on $H_A$ satisfying the relations

$$V_i V_i^* = E_i, \quad V_i^* V_i = \sum_{j=1}^N A(i,j) E_j, \quad i = 1, \ldots, N. \quad (4.3)$$

**Lemma 4.1.** For a fixed $m \in \{1, \ldots, N\}$, we have for $i = 1, \ldots, N$,

$$E_i = \begin{cases} E_{A,i} + E_0 & \text{if } i = m, \\ E_{A,i} & \text{if } i \neq m, \end{cases} \quad V_i^* V_i = \begin{cases} T_i^* T_i & \text{if } A(i,m) = 1, \\ T_i^* T_i - E_0 & \text{if } A(i,m) = 0. \end{cases}$$

For $i = 1, \ldots, N$, the operator $T_i E_0 T_i^*$ on $H_A$ is a rank one projection on $H_A$ onto the one-dimensional subspace spanned by the vector $\xi_i$. We note that the operator $T_i V_i^* : H_i \to H_i$ is a (not necessarily onto) partial isometry. We then have

**Lemma 4.2.** For $i = 1, \ldots, N$, we have $(T_i V_i^*)^* T_i V_i^* = V_i V_i^* = E_i$ and

$$T_i V_i^* (T_i V_i^*)^* = \begin{cases} E_i - E_0 & \text{if } i = m, \ A(i,m) = 1, \\ E_i - E_0 - T_i E_0 T_i^* & \text{if } i = m, \ A(i,m) = 0, \\ E_i & \text{if } i \neq m, \ A(i,m) = 1, \\ E_i - T_i E_0 T_i^* & \text{if } i \neq m, \ A(i,m) = 0. \end{cases} \quad (4.4)$$

Since the partial isometries $V_i, i = 1, \ldots, N$ on $H_A$ satisfy (2.1), there exists a unital $*$-monomorphism $\tau_m : \mathcal{O}_A \to B(H_A)$ satisfying $\tau_m(S_i) = V_i, i = 1, \ldots, N$, so that $\pi \circ \tau_m : \mathcal{O}_A \to Q(H_A)$ is a trivial extension. The above lemma says the following proposition.

**Proposition 4.3.** For a fixed $m \in \{1, \ldots, N\}$, we have

$$d_i(\sigma_{\tau_A}, \tau_m) = \begin{cases} -1 & \text{if } i = m, \ A(i,m) = 1, \\ -2 & \text{if } i = m, \ A(i,m) = 0, \\ 0 & \text{if } i \neq m, \ A(i,m) = 1, \\ -1 & \text{if } i \neq m, \ A(i,m) = 0. \end{cases} \quad (4.5)$$

**Proof.** As $H_i = E_i H_A$ and

$$d_i(\sigma_{\tau_A}, \tau_m) = \text{ind}_{E_i} (T_i V_i^* \tau_m)$$

$$= \dim(\text{Ker}(T_i V_i^*) \text{ in } H_i) - \dim(\text{Coker}(T_i V_i^*) \text{ in } H_i)$$

$$= - \dim(H_i/T_i V_i^* (T_i V_i^*)^* H_i),$$

we get the formula (4.3) by (4.4).

Therefore we have

**Theorem 4.4.** Let us denote by $[\tau_A]$ the class in $\text{Ext}_*(\mathcal{O}_A)$ of the Toeplitz extension $\sigma_{\tau_A}$ of $\mathcal{O}_A$. We then have

$$\Box$$

12
(i) \( d_s([T_A]_s) = -i_A(1) - [1_N] \) in \( \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \),
(ii) \( d_w([T_A]_w) = -[1_N] \) in \( \mathbb{Z}^N/(I - A)\mathbb{Z}^N \),

where \([1_N] = [(1, \ldots , 1)]\) means the class of the vector \((1, \ldots , 1) \in \mathbb{Z}^N\).

**Proof.** Let us denote by \( v(m) \in \mathbb{Z}^N \) the column vector in \( \mathbb{Z}^N \) whose \( m \)th component is one and the other components are zero's. Denote by \((1, \ldots , 1)^t\) the column vector defined by the transpose of the row vector whose components are all one's. By \((4.5)\), we have
\[
[d_i(\sigma_{T_A}, r_m)]_{i=1}^N = -(1, \ldots , 1)^t - v(m) + [A(i, m)]_{i=1}^N
= - (I - A)v(m) - (1, \ldots , 1)^t.
\]
Since \([(I - A)v(m)] = i_A(1) in \( \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \), we have \( d_s([T_A]_s) = -i_A(1) - [1_N] \) in \( \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \). As \( i_A(1) = 0 in \( \mathbb{Z}^N/(I - A)\mathbb{Z}^N \), we have \( d_w([T_A]_w) = -[1_N] \) in \( \mathbb{Z}^N/(I - A)\mathbb{Z}^N \). \( \Box \)

By virtue of the Rørdam's classification theorem for Cuntz–Krieger algebras [19] (7), cf. [11]) showing that the \( K_0 \)-group \( K_0(O_A) \) with the position of the class \([1]\) of the unit 1 of \( O_A \) in \( K_0(O_A) \) is a complete invariant of the isomorphism class of the algebra \( O_A \), we obtain the following corollary.

**Corollary 4.5.** The pair \((\text{Ext}_w(O_A), [T_A]_w)\) of the weak extension group \( \text{Ext}_w(O_A) \) and the weak equivalence class \([T_A]_w\) of the Toeplitz extension \( \sigma_{T_A} \) of the Cuntz–Krieger algebra \( O_A \) is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra \( O_A \), for the transposed matrix \( A^t \) of the matrix \( A \). This shows that two Cuntz–Krieger algebras \( O_A \) and \( O_B \) are isomorphic if and only if there exists an isomorphism \( \varphi : \text{Ext}_w(O_{A^t}) \rightarrow \text{Ext}_w(O_{B^t}) \) of groups such that \( \varphi([T_A]_w) = [T_B]_w \).

**Proof.** As \( K_0(O_{A^t}) \cong \mathbb{Z}^N/(I - A)\mathbb{Z}^N \) and \((\mathbb{Z}^N/(I - A)\mathbb{Z}^N, [1_N]) \cong (\mathbb{Z}^N/(I - A)\mathbb{Z}^N, [1_N])\), we have
\[(\text{Ext}_w(O_A), [T_A]_w) \cong (\mathbb{Z}^N/(I - A)\mathbb{Z}^N, [1_N]) \cong (K_0(O_{A^t}), [1]).\]

By virtue of the Rørdam’s classification result for Cuntz–Krieger algebras [19] (7), cf. [11]), we obtain the desired assertion. \( \Box \)

**Remark 4.6.**
(i) The position \([T_A]_s \) in \( \text{Ext}_s(O_A) \) is not necessarily invariant under the isomorphism class of \( O_A \) (see Example 2 in the next section).

(ii) The abelian groups \( \text{Ext}_w(O_A) \) and \( K_0(O_A) \) are isomorphic, and two \( C^* \)-algebras \( O_A \otimes K(H) \) and \( O_{A^t} \otimes K(H) \) are always isomorphic for every matrix \( A \). There is however an example of an irreducible non permutation matrix \( A \) such that \( O_A \) is not isomorphic to \( O_{A^t} \) as in the classification table in [11] of the Cuntz–Krieger algebras for \( 3 \times 3 \) matrices (see also [11] Example 2.1, or Example 4 in the next section).

## 5 Examples

1. Let \( A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \) be the \( N \times N \) matrix with all entries are one’s with \( N > 1 \). The Cuntz–Krieger algebra \( O_A \) is nothing but the Cuntz algebra \( O_N \) (see [6]). The element
\( \iota_A(1) \) in \( \text{Ext}_s(\mathcal{O}_N) \) is denoted by \( \iota_N(1) \). The Toeplitz algebra \( \mathcal{T}_A \) is also denoted by \( \mathcal{T}_N \). As \( AR_1 = A \), we have \( \hat{A} = A + R_1 - AR_1 = R_1 \), so that

\[
I - \hat{A} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.
\]

Define

\[
L_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}
\]

so that

\[
L_N(I - \hat{A}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.
\]

Hence \( L_N \) induces an isomorphism from \( \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \) to \( L_N\mathbb{Z}^N/L_N(I - \hat{A})\mathbb{Z}^N \cong \mathbb{Z} \) such that

\[
[v] \in \mathbb{Z}^N/(I - \hat{A})\mathbb{Z}^N \longrightarrow [L_Nv] \in L_N\mathbb{Z}^N/L_N(I - \hat{A})\mathbb{Z}^N \longrightarrow (L_Nv)_1 \in \mathbb{Z}.
\]

For \( [v] = \iota_N(1) = [(I - A) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}] \), we see that

\[
L_Nv = L_N(I - A) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - N \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

so that \((L_Nv)_1 = 1 - N\). Therefore we have \((\text{Ext}_s(\mathcal{O}_N), \iota_N(1)) \cong (\mathbb{Z}, 1 - N)\) and hence the exact sequence (3.3) goes to

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{x(1-N)} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/(1-N)\mathbb{Z} \longrightarrow 0. \tag{5.1}
\]

By using Theorem 4.4 one may easily compute that

\[
(\text{Ext}_w(\mathcal{O}_N), [\mathcal{T}_N]_w) \cong (\mathbb{Z}/(1-N)\mathbb{Z}, -1), \quad (\text{Ext}_s(\mathcal{O}_N), [\mathcal{T}_N]_s, \iota_N(1)) \cong (\mathbb{Z}, -1, 1 - N).
\]

2. Let us denote by \( F \) the Fibonacci matrix \( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). It is well-known that the Cuntz-Krieger algebra \( \mathcal{O}_F \) is isomorphic to the Cuntz algebra \( \mathcal{O}_2 \). Hence we have \( \text{Ext}_w(\mathcal{O}_F) \cong \text{Ext}_w(\mathcal{O}_2) \cong \{0\} \), and \( \text{Ext}_s(\mathcal{O}_F) \cong \text{Ext}_s(\mathcal{O}_2) \cong \mathbb{Z} \). By the formula in Theorem 4.4 together with the above Example 1, we see

\[
(\text{Ext}_s(\mathcal{O}_F), [\mathcal{T}_F]_s, \iota_F(1)) = (\mathbb{Z}, -2, -1), \quad (\text{Ext}_s(\mathcal{O}_2), [\mathcal{T}_2]_s, \iota_2(1)) = (\mathbb{Z}, -1, -1).
\]

14
Hence the position \([T_F]_s\) in \(\text{Ext}_s(\mathcal{O}_F)\) is different from the position \([T_2]_s\) in \(\text{Ext}_s(\mathcal{O}_2)\).

3. The weak extension groups \(\text{Ext}_w(\mathcal{O}_{A_i}), i = 1, 2, 3, 4\) of \(\mathcal{O}_{A_i}, i = 1, 2, 3, 4\) for the following list of matrices \(A_i, i = 1, 2, 3, 4\) have been presented in [8, Remark 3.4]. Their strong extension groups \(\text{Ext}_s(\mathcal{O}_{A_i})\) with the positions of the element \(\iota_{A_i}(1), i = 1, 2, 3, 4\) are easily computed by using Theorem 3.3. We also easily know the positions \([T_{A_i}]_s\) in \(\text{Ext}_s(\mathcal{O}_{A_i})\) by Theorem 4.4. We present the list in the following, computed without difficulty by hand.

- \(A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\), \((\text{Ext}_w(\mathcal{O}_{A_1}), [T_{A_1}]_w) \cong (\mathbb{Z}/3\mathbb{Z}, 2)\),
  \((\text{Ext}_s(\mathcal{O}_{A_1}), [T_{A_1}]_s, \iota_{A_1}(1)) \cong (\mathbb{Z}, 4, 3)\).

- \(A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\), \((\text{Ext}_w(\mathcal{O}_{A_2}), [T_{A_2}]_w) \cong (\mathbb{Z}/4\mathbb{Z}, 2)\),
  \((\text{Ext}_s(\mathcal{O}_{A_2}), [T_{A_2}]_s, \iota_{A_2}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, -2 \oplus 0, 2 \oplus 1)\).

- \(A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\), \((\text{Ext}_w(\mathcal{O}_{A_3}), [T_{A_3}]_w) \cong (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, 0 \oplus 0)\),
  \((\text{Ext}_s(\mathcal{O}_{A_3}), [T_{A_3}]_s, \iota_{A_3}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, -2 \oplus 0\oplus 0, 1 \oplus 1\oplus 1)\).

- \(A_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\), \((\text{Ext}_w(\mathcal{O}_{A_4}), [T_{A_4}]_w) \cong (\mathbb{Z}, -1)\),
  \((\text{Ext}_s(\mathcal{O}_{A_4}), [T_{A_4}]_s, \iota_{A_4}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}, -2 \oplus (-1), 1 \oplus 0)\).

4. The matrices

\[
A_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_6 = A_6^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\]

are examples presented in [11, Example 2.1] such that \((K_0(\mathcal{O}_{A_5}), [1]) \cong (\mathbb{Z}/2\mathbb{Z}, 1)\) and \((K_0(\mathcal{O}_{A_6}), [1]) \cong (\mathbb{Z}/2\mathbb{Z}, 0)\), so that \(\mathcal{O}_{A_5}\) is not isomorphic to \(\mathcal{O}_{A_6}\). We then see that

\[(\text{Ext}_w(\mathcal{O}_{A_5}), [T_{A_5}]_w) \cong (\mathbb{Z}/2\mathbb{Z}, 0), \quad (\text{Ext}_w(\mathcal{O}_{A_6}), [T_{A_6}]_w) \cong (\mathbb{Z}/2\mathbb{Z}, 1)\).

We also easily see that

\[(\text{Ext}_s(\mathcal{O}_{A_5}), [T_{A_5}]_s, \iota_{A_5}(1)) \cong (\mathbb{Z}, -2, -2), \quad (\text{Ext}_s(\mathcal{O}_{A_6}), [T_{A_6}]_s, \iota_{A_6}(1)) \cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, -1 \oplus 0, -1 \oplus (-1)),\]

and hence \(\text{Ext}_s(\mathcal{O}_{A_5})\) is not isomorphic to \(\text{Ext}_s(\mathcal{O}_{A_6})\).

Some of the results in this paper will be generalized to more general setting in a class of \(C^*\)-algebras associated with symbolic dynamical systems in [15].

Acknowledgment: The author would like to thank Joachim Cuntz for his useful comments and suggestions on a preliminary version of this paper. This work was supported by JSPS KAKENHI Grant Number 19K03537.
References

[1] W. B. Arveson, *Notes on extensions of C*-algebras*, Duke Math. J. 44(1977), pp. 329–335.

[2] B. E. Blackadar, *K-theory for operator algebras*, MSRI Publications 5, Springer-Verlag, Berlin, Heidelberg and New York, 1986.

[3] R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets*, Ann. Math. 106(1977), pp. 73–92.

[4] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of C*-algebras and K-homology*, Ann. Math. 105(1977), pp. 265–324.

[5] M. D. Choi and E. G. Effros, *The completely positive lifting problem for C*-algebras*, Ann. Math. 104(1976), 585–609.

[6] J. Cuntz, *Simple C*-algebras generated by isometries*, Comm. Math. Phys. 57(1977), pp. 173–185.

[7] J. Cuntz, *A class of C*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C*-algebras*, Invent. Math. 63(1980), pp. 25–40.

[8] J. Cuntz and W. Krieger, *A class of C*-algebras and topological Markov chains*, Invent. Math. 56(1980), pp. 251–268.

[9] R. G. Douglas, *C*-algebra extensions and K-homology*, Princeton University Press, Princeton, New Jersey, 1980.

[10] M. Enomoto, M. Fujii and Y. Watatani, *Tensor algebras on the sub Fock space associated with OA*, Math. Japon. 24(1979), pp. 463–468.

[11] M. Enomoto, M. Fujii and Y. Watatani, *K0-groups and classifications of Cuntz-Krieger algebras*, Math. Japon. 26(1981), pp. 443–460.

[12] D. Evans, *Gauge actions on OA*, J. Operator theory 7(1982), pp. 79–100.

[13] N. Higson and J. Roe, *Analytic K-HOMOLOGY*, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, Oxford, 2000

[14] G. G. Kasparov, *The operator K-functor and extensions of C*-algebras*, Math. USSR Izvestija 16(1981), pp. 513–572.

[15] K. Matsumoto *On extension groups of C*-algebras associated with symbolic dynamical systems, (tentative title)*, in preparation.

[16] W. L. Paschke and N. Salinas, *Matrix algebras over ON*, Michigan Math. J. 26(1979), pp. 3–12.

[17] M. Pimsner and S. Popa, *The Ext-groups of some C*-algebras considered by J. Cuntz*, Rev. Roum. Math. Pures et Appl. 23(1978), pp. 1069–1076.
[18] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-products C*-algebras*, J. Operator Theory 4(1980), pp. 93–118.

[19] M. Rørdam, *Classification of Cuntz-Krieger algebras*, K-theory 9(1995), pp. 31–58.

[20] D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roum. Math. Pures et Appl. 21(1976), pp. 97–113.