An additive problem over Piatetski–Shapiro primes and almost-primes

Jinjiang Li¹ · Min Zhang² · Fei Xue¹

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Abstract
Let $P_r$ denote an almost-prime with at most $r$ prime factors, counted according to multiplicity. In this paper, we establish a theorem of Bombieri–Vinogradov type for the Piatetski–Shapiro primes $p = \lceil n^{1/\gamma} \rceil$ with $\frac{85}{86} < \gamma < 1$. Moreover, we use this result to prove that, for $0.9989445 < \gamma < 1$, there exist infinitely many Piatetski–Shapiro primes such that $p + 2 = P_3$, which improves the previous results of Lu (Acta Math Sin (Engl Ser) 34(2):255–264, 2018), Wang and Cai (Int J Number Theory 7(5):1359–1378, 2011), and Peneva (Monatsh Math 140(2):119–133, 2003).

Keywords Piatetski–Shapiro prime · Almost-prime · Exponential sum · Bombieri–Vinogradov theorem

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Min Zhang
min.zhang.math@gmail.com

Jinjiang Li
jinjiang.li.math@gmail.com

Fei Xue
fei.xue.math@gmail.com

¹ Department of Mathematics, China University of Mining and Technology, Beijing 100083, People’s Republic of China

² School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, People’s Republic of China
1 Introduction and main result

The ternary Goldbach problem asserts that every odd integer \( n \geq 9 \) can be represented in the form

\[
n = p_1 + p_2 + p_3,
\]

where \( p_1, p_2, p_3 \) are odd prime numbers. In 1937, Vinogradov [36] proved that a representation of the type (1.1) exists for every sufficiently large odd integer. Recently, Helfgott [12–15] completely solved the problem and proved that the ternary Goldbach conjecture is true. The binary Goldbach problem, which states that every even integer \( N \geq 6 \) can be written as the sum of two odd primes, also remains unsettled. Another central problem in the theory of prime distribution, namely the twin prime conjecture, states that there exist infinitely many primes \( p \) such that \( p + 2 \) is also prime. Although the conjecture has resisted all attacks, there have been spectacular partial achievements. One well-known result is due to Chen [4,5], who proved that there exist infinitely many primes \( p \) such that \( p + 2 \) has at most 2 prime factors.

An important approach for studying the binary Goldbach problem is by the use of sieve methods. As usual, we denote by \( \mathcal{P}_r \) an almost-prime with at most \( r \) prime factors, counted according to multiplicity. In 1947, Rényi [31] was the first to prove that there exists an \( r \) such that every sufficiently large even integer \( N \) is representable in the form

\[
N = p + \mathcal{P}_r,
\]

where \( p \) is a prime number. The best result in this direction is due to Chen [4,5] who showed that (1.2) holds for \( r = 2 \).

Let \( \gamma \) be a real number such that \( \frac{1}{2} < \gamma < 1 \). Define

\[
\pi_\gamma (x) := \# \{ p \leq x : p = \lfloor n^{1/\gamma} \rfloor \text{ for some } n \in \mathbb{N} \}.
\]

In 1953, Piatetski–Shapiro [30] showed that

\[
\pi_\gamma (x) \sim \frac{x^\gamma}{\log x} \quad (x \to \infty),
\]

for \( \frac{11}{12} < \gamma < 1 \). The prime numbers of the form \( p = \lfloor n^{1/\gamma} \rfloor \) are called Piatetski–Shapiro primes of type \( \gamma \). Since then, by using the close connection between the lower bound for \( \gamma \) and the estimates of the exponential sums over primes, this range for \( \gamma \) has been enlarged by a number of authors [1,11,16,17,19,20,22,24,26,32]. The best results are given by Rivat and Sargos [33] and Rivat and Wu [34], where it is proved that

\[
\pi_\gamma (x) \sim \frac{x^\gamma}{\log x}
\]

for \( \frac{2426}{2817} < \gamma < 1 \), and

\[
\pi_\gamma (x) \gg \frac{x^\gamma}{\log x}
\]

for \( \frac{205}{243} < \gamma < 1 \), respectively.
In 1992, Balog and Friedlander [2] found an asymptotic formula for the number of solutions of the equation (1.1) with variables restricted to the Piatetski–Shapiro primes. An interesting corollary of their theorem is that every sufficiently large odd integer can be written as the sum of two primes and a Piatetski–Shapiro prime of type $\gamma$, provided that $\frac{8}{9} < \gamma < 1$. Afterwards, their studies in this direction were subsequently continued by Jia [18] and by Kumchev [21], and generalized by Cui [6] and Li and Zhang [25], consecutively and respectively.

Based on the above results, it is interesting to investigate the solvability of the equation (1.2) when $p$ is a Piatetski–Shapiro prime. It is naturally expected that a theorem of Bombieri–Vinogradov type holds for the Piatetski–Shapiro primes. In the early days, the only result in this direction, due to Leitmann [23], gives a very low level of distribution which does not allow us to determine the value of the parameter $r$.

In 2003, Peneva [29] obtained a mean value theorem of Bombieri–Vinigradov’s type for Piatetski–Shapiro primes, by which and sieve methods she showed that, for every sufficiently large even integer $N$, (1.2) is solvable with $p = \lceil n^{1/\gamma} \rceil$ a Piatetski–Shapiro prime, and $r$ is the least positive integer satisfying the inequality

$$r + 1 - \frac{\log \frac{4}{1 + 3^r}}{\log 3} > \frac{1}{\xi(\gamma)} + \varepsilon,$$

where

$$\xi = \xi(\gamma) = \begin{cases} \frac{755}{424} \gamma - \frac{331}{212} - \varepsilon & \text{for } \frac{662}{755} < \gamma \leq \frac{608}{675}, \\ \frac{5}{4} \gamma - \frac{13}{12} - \varepsilon & \text{for } \frac{608}{675} < \gamma < 1. \end{cases}$$

(1.3)

By using the above level $\xi$, Peneva [29] proved that (1.2) is solvable for $r = 7$ with a Piatetski–Shapiro prime $p = \lceil n^{1/\gamma} \rceil$ and $0.9854 < \gamma < 1$. Essentially, from the arguments similar to that in Peneva [29], one can obtain that there exist infinitely many Piatetski–Shapiro primes of type $\gamma$ such that $p + 2 = P_7$ with $0.9854 < \gamma < 1$.

In 2011, by using the same level $\xi$ in (1.3), Wang and Cai [37] improved the result of Peneva [29], and showed that there exist infinitely many Piatetski–Shapiro primes of type $\gamma$ such that $p + 2 = P_5$ with $\frac{29}{30} < \gamma < 1$. Afterwards, Lu [27], in 2018, reestablished a mean value theorem of Bombieri–Vinogradov’s type with level $\xi = \xi(\gamma) = (13\gamma - 12)/4 - \varepsilon$ for $\frac{12}{13} < \gamma < 1$. By using this level, Lu [27] strengthened the result of Wang and Cai [37]. He proved that there exist infinitely many Piatetski–Shapiro primes of type $\gamma$ such that $p + 2 = P_4$ with $0.9993 < \gamma < 1$.

In this paper, we shall continue to improve the result of Lu [27], and establish the two following theorems.

**Theorem 1.1** Suppose that $\gamma$ is a real number satisfying $\frac{85}{86} < \gamma < 1$, $a \neq 0$ is a fixed integer. Then for any given constant $A > 0$ and any sufficiently small $\varepsilon > 0$, there holds

$$\sum_{d \leq x^\xi} \sum_{\substack{p \leq x \mod d \equiv a \Mod d \\{ k^{1/\gamma} \}}} 1 - \frac{1}{\varphi(d)} \pi_{\gamma}(x) \ll \frac{x^\gamma}{(\log x)^A},$$

(1.4)
where
\[ \xi = \xi(\gamma) = \frac{129}{4} \gamma - \frac{255}{8} - \varepsilon; \]
the implied constant in (1.4) depends only on A and \( \varepsilon \).

**Theorem 1.2** Suppose that \( \gamma \) is a real number satisfying \( 0.9989445 < \gamma < 1 \). Then there exist infinitely many Piatetski–Shapiro primes of type \( \gamma \) such that
\[ p + 2 = \mathcal{P}_3. \]

**Remark** The key point of improving the number \( r \) such that \( p + 2 = \mathcal{P}_r \) with Piatetski–Shapiro prime \( p = [n^{1/\gamma}] \) is to enlarge the level \( \xi = \xi(\gamma) \), for \( \gamma \) near to 1, of the mean value theorem of Bombieri–Vinogradov’s type for Piatetski–Shapiro primes. In order to compare our result with the results of Lu [27] and Peneva [29], we list the numerical result as follows:

\[ \xi(\gamma) = \frac{129}{4} \gamma - \frac{255}{8} - \varepsilon \rightarrow \frac{3}{8} = 0.375 \quad \text{for} \quad \gamma \rightarrow 1, \]
\[ \xi(\gamma) = \frac{13}{4} \gamma - \frac{-12}{4} - \varepsilon \rightarrow \frac{1}{4} = 0.25 \quad \text{for} \quad \gamma \rightarrow 1, \]
\[ \xi(\gamma) = \frac{5}{4} \gamma - \frac{13}{12} - \varepsilon \rightarrow \frac{1}{6} = 0.1666\ldots \quad \text{for} \quad \gamma \rightarrow 1. \]

In order to establish Theorem 1.2, we employ the method of Vaughan [35], combined with the weighted sieve of Richert and the method of Chen [5].

**Notation** Throughout this paper, \( x \) is a sufficiently large number; \( \varepsilon \) and \( \eta \) are sufficiently small positive numbers, which may be different in each occurrence. Let \( p \), with or without subscripts, always denote a prime number. We use \([x]\), \( \{x\}\) and \( \|x\| \) to denote the integral part of \( x \), the fractional part of \( x \) and the distance from \( x \) to the nearest integer, respectively. As usual, \( \varphi(n) \), \( \Lambda(n) \), \( \tau(n) \) and \( \mu(n) \) denote Euler’s function, von Mangoldt’s function, the Dirichlet divisor function and Möbius’ function, respectively. Also, we use \( \chi \mod q \) to denote a Dirichlet character modulo \( q \), and \( \chi^0 \mod q \) the principal character. Especially, we use \( \Sigma^* \) to denote sums over all primitive characters. Let \( (m_1, m_2, \ldots, m_k) \) and \( [m_1, m_2, \ldots, m_k] \) be the greatest common divisor and the least common multiple of \( m_1, m_2, \ldots, m_k \), respectively. We write \( L = \log x \); \( e(t) = \exp(2\pi it); \psi(t) = t - [t] - \frac{1}{2}. \) The notation \( n \sim X \) means that \( n \) runs through a subinterval of \( (X, 2X] \), whose endpoints are not necessarily the same in the different occurrences and may depend on the outer summation variables. \( f(x) \ll g(x) \) means that \( f(x) = O(g(x)); f(x) \asymp g(x) \) means that \( f(x) \ll g(x) \ll f(x). \)

### 2 Preliminaries

In this section, we shall reduce the problem of estimating the sum in (1.4) to estimating exponential sums over primes.
For $1/2 < \gamma < 1$, it is easy to see that

$$[-k^\gamma] - [-(k+1)^\gamma] = \begin{cases} 1 & \text{if } k = [n^{1/\gamma}], \\ 0 & \text{otherwise.} \end{cases}$$

For convenience, we put $D = x^{\delta}$. In order to prove (1.4), it is sufficient to prove that

$$\sum_{d \leq D \atop (\alpha, d) = 1} \left| \sum_{n \leq x \atop n \equiv \alpha \pmod{d}} \Lambda(n)((n+1)^\gamma - n^\gamma) - \frac{1}{\varphi(d)} \sum_{n \leq x} \Lambda(n)((n+1)^\gamma - n^\gamma) \right| \ll x^\gamma L^{-A},$$

(2.1)

and

$$\sum_{d \leq D \atop (\alpha, d) = 1} \frac{1}{\varphi(d)} \left| \sum_{n \leq x} \Lambda(n)((\psi(n^\gamma) - \psi((n+1)^\gamma)) \right| \ll x^\gamma L^{-A}$$

(2.2)

The estimate (2.1) can be obtained from the Bombieri–Vinogradov theorem by using partial summation and it holds for every $\gamma \in (1/2, 1)$ and $D = x^{1/2-\varepsilon}$, where $\varepsilon > 0$ is sufficiently small. The estimate (2.3) follows from the arguments of Heath–Brown (see (2) on page 245 in [11]). Thus, we only have to prove (2.2). Obviously, (2.2) will follow, if we can prove that for $X \leq x$, there holds

$$\sum_{d \leq D \atop (\alpha, d) = 1} \left| \sum_{n \sim X \atop n \equiv \alpha \pmod{d}} \Lambda(n)((\psi(n^\gamma) - \psi((n+1)^\gamma)) \right| \ll x^\gamma L^{-A}.$$  

(2.4)

Let $\eta > 0$ be a sufficiently small number. If $X \leq x^{1-\eta}$, then the left-hand side of (2.4) is

$$\ll \sum_{d \leq D \atop (\alpha, d) = 1} \left| \sum_{n \sim X \atop n \equiv \alpha \pmod{d}} \Lambda(n)((n+1)^\gamma - n^\gamma) \right|$$

$$+ \sum_{d \leq D \atop (\alpha, d) = 1} \left| \sum_{n \sim X \atop n \equiv \alpha \pmod{d}} \Lambda(n)([-n^\gamma] - [-(n+1)^\gamma]) \right|$$

$$\ll L \sum_{n \sim X} n^{\gamma-1} \tau(n-a) + L \sum_{n \sim X \atop n = [k^{1/\gamma}]} \tau(n-a) \ll X^{\gamma+\eta/2} \ll x^\gamma L^{-A}.$$
Therefore, we can assume that $x^{1-\eta} \leq X \leq x$. It is easy to see that, for $\xi \leq (1-\eta)/2$, there holds
\[ X^\xi \leq D \leq X^{\xi + \eta/2}. \]

Now, we use the well-known expansions
\[ \psi(t) = -\sum_{0 < |h| \leq H} e(th) + O(g(t, H)), \tag{2.5} \]
where
\[ g(t, H) = \min\left(1, \frac{1}{H ||t||}\right) = \sum_{h=-\infty}^{\infty} b_h e(th) \]
and
\[ b_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\right). \]

Putting (2.5) into the left-hand side of (2.4), the contribution of the error term in (2.5) to the left-hand side of (2.4) is
\[ \sum_{d \leq D} \sum_{(a,d)=1} \sum_{n \sim X} \Lambda(n) \left(g(n^{\gamma}, H) + g((n+1)^{\gamma}, H)\right) = R_1 + R_2, \tag{2.6} \]
say. We only deal with $R_1$, since the estimate of $R_2$ is exactly the same. For $R_1$, we have
\[ R_1 \ll L \sum_{d \leq D} \sum_{(a,d)=1} \sum_{n \sim X} g(n^{\gamma}, H) \ll L \sum_{d \leq D} \sum_{(a,d)=1} \sum_{h=-\infty}^{\infty} |b_h| \left| \sum_{n \sim X} e(hn^{\gamma}) \right|. \tag{2.7} \]

Now, we need the following estimate which is an analogue of Lemma 1 of Heath–Brown [11] for arithmetic progressions.

\textbf{Lemma 2.1} Let $1 \leq d \leq X$, $X < X_1 \leq 2X$. Then
\[ \sum_{X < n \leq X_1, n \equiv a \pmod{d}} e(hn^{\gamma}) \ll \min\left(Xd^{-1}, d^{-1}|h|^{-1}X^{1-\gamma} + d^{\kappa-\ell}|h|^\kappa X^{\kappa \gamma - \kappa + \ell}\right), \]
where $(\kappa, \ell)$ is an exponent pair.

\textbf{Proof} The proof of this lemma is similar to Lemma 2.1 of Lu [27]. We take integer $b$, which satisfies $1 \leq b \leq d$ and $b \equiv a \pmod{d}$. Then we derive that
\[ \sum_{X < n \leq X_1, n \equiv a \pmod{d}} e(hn^{\gamma}) = \sum_{X-b < m \leq X_1-b} e(h(b+md)^{\gamma}). \]
Estimating the sum on the right-hand side of above equation trivially and by any exponent pair \((\kappa, \ell)\), we obtain the desired estimate.

Taking \((\kappa, \ell) = \left(\frac{1}{2}, \frac{1}{2}\right)\) in Lemma 2.1, we obtain

\[
R_1 \ll L \sum_{d \leq D} \left( |b_0| X d^{-1} + \sum_{h \neq 0} |b_h| \left( |h|^{-1} X^{1-\gamma} d^{-1} + |h|^{1/2} X^{\gamma/2} \right) \right)
\]

\[
\ll L^3 H^{-1} X + L X^{1-\gamma} \sum_{d \leq D} d^{-1} \sum_{h \neq 0} |h|^{-2}
\]

\[
+ L X^{\gamma/2} D \left( \sum_{0 < |h| \leq H} |h|^{-1/2} + H \sum_{|h| > H} |h|^{-3/2} \right)
\]

\[
\ll L^3 H^{-1} + L^2 X^{1-\gamma} + L X^{\gamma/2} H^{1/2} D \ll x^{\gamma} L^{-A},
\]

provided that

\[ H = X^{1-\gamma+\eta} \quad \text{and} \quad \gamma > \frac{1}{2} + \xi. \] (2.9)

Therefore, it remains to show that

\[
S := \sum_{d \leq D} \sum_{0 < h \leq H} \frac{1}{h} \left| \sum_{n \sim X \atop n \equiv a (\text{mod } d)} \Lambda(n) \left( e(-hn^{\gamma}) - e(-h(n+1)^{\gamma}) \right) \right| \ll x^{\gamma} L^{-A}.
\] (2.10)

Set

\[
\phi_h(t) = 1 - e(h(n^{\gamma} - (n + 1)^{\gamma})).
\]

By partial summation, the innermost sum on the left-hand side of (2.10) is

\[
\sum_{n \sim X \atop n \equiv a (\text{mod } d)} \Lambda(n) e(-hn^{\gamma}) \phi_h(n) = \int_{X}^{2X} \phi_h(t) d \left( \sum_{n \sim X \atop n \equiv a (\text{mod } d)} \Lambda(n) e(-hn^{\gamma}) \right)
\]

\[
\ll \left| \phi_h(2X) \right| \left| \sum_{n \sim X \atop n \equiv a (\text{mod } d)} \Lambda(n) e(-hn^{\gamma}) \right| + \int_{X}^{2X} \left| \sum_{n \sim X \atop n \equiv a (\text{mod } d)} \Lambda(n) e(-hn^{\gamma}) \right| \left| \frac{\partial \phi_h(t)}{\partial t} \right| dt
\]

\[
\ll h X^{\gamma-1} \cdot \max_{X < t \leq 2X} \left| \sum_{n \sim X \atop n \equiv a (\text{mod } d)} \Lambda(n) e(-hn^{\gamma}) \right|, \quad (2.11)
\]

where we use the estimate

\[
\phi_h(t) \ll h t^{\gamma-1} \quad \text{and} \quad \frac{\partial \phi_h(t)}{\partial t} \ll ht^{\gamma-2}.
\]
Inserting (2.11) into the left-hand side of (2.10), we obtain

\[
S \ll X^{\gamma - 1} \sum_{d \leq D} \sum_{0 < h \leq H} \sum_{n \sim X} \Lambda(n)e(-hn^\gamma) \bigg| \sum_{n \equiv a \pmod{d}} \Lambda(n)e(-hn^\gamma) \bigg|
\]

\[
= X^{\gamma - 1} \sum_{d \leq D} \sum_{0 < h \leq H} c(d, h) \sum_{n \sim X} \Lambda(n)e(-hn^\gamma) \bigg| \sum_{n \equiv a \pmod{d}} \Lambda(n)e(-hn^\gamma) \bigg|
\]

\[
\ll X^{\gamma - 1} \sum_{n \sim X} \Lambda(n) \sum_{0 < h \leq H} e(-hn^\gamma) \sum_{d \leq D} \sum_{d \equiv a \pmod{d}} c(d, h)
\]

\[
= X^{\gamma - 1} \sum_{n \sim X} \Lambda(n)G(n),
\]

where

\[
G(n) = \sum_{0 < h \leq H} \Xi_h(n)e(-hn^\gamma)
\]

and

\[
\Xi_h(n) = \sum_{d \leq D} \sum_{d \equiv a \pmod{d}} c(d, h), \quad |c(d, h)| = 1.
\]

Consequently, in order to establish the estimate (2.10), it is sufficient to show that

\[
\left| \sum_{n \sim X} \Lambda(n)G(n) \right| \ll XL^{-A}.
\] (2.12)

A special case of the identity of Heath–Brown [10] is given by

\[
-\frac{\zeta'}{\zeta} = -\frac{\zeta'}{\zeta} (1 - Z\zeta)^3 - \sum_{j=1}^{3} \binom{3}{j} (-1)^j Z^j \xi j^{-1} (-\zeta'),
\]

where \( Z = Z(s) = \sum_{m \leq X^{1/3}} \mu(m)m^{-s} \). From this we can decompose \( \Lambda(n) \) for \( n \sim X \) as

\[
\Lambda(n) = \sum_{j=1}^{3} \binom{3}{j} (-1)^{j-1} \sum_{m_1 \ldots m_j = n} \mu(m_1) \ldots \mu(m_j) \log m_{2j}.
\]

Thus, for any arithmetic function \( G(n) \), we can express \( \sum_{n \sim X} \Lambda(n)G(n) \) in terms of sums

\[
\sum_{m_1 \ldots m_j \sim X} \mu(m_1) \ldots \mu(m_j) (\log m_{2j}) G(m_1 \ldots m_{2j}),
\]
where \( 1 \leq j \leq 3, \) \( M_1 M_2 \ldots M_{2j} \sim X \) and \( M_1, \ldots, M_j \leq X^{1/3}. \) By dividing the \( M_j \) into two groups, we have

\[
\left| \sum_{n \sim X} \Lambda(n) G(n) \right| \ll \eta X \max \left| \sum_{m \sim X} a(m) b(n) G(mn) \right|, \tag{2.13}
\]

where the maximum is taken over all bilinear forms with coefficients satisfying one of

\[
|a(m)| \leq 1, \quad |b(n)| \leq 1, \tag{2.14}
\]

or

\[
|a(m)| \leq 1, \quad b(n) = 1,
\]

or

\[
|a(m)| \leq 1, \quad b(n) = \log n,
\]

and also satisfying in all cases

\[
M \leq X. \tag{2.15}
\]

We refer to the case (2.14) as being Type II sums and to the other cases as being Type I sums and write for brevity \( \Sigma_{II} \) and \( \Sigma_{I} \), respectively. By dividing the \( M_j \) into two groups in a judicious fashion we are able to reduce the range of \( M \) from (2.15). In Sect. 3, we shall give the estimate of these sums.

In the rest of this section, we shall list several lemmas which is necessary for proving Theorem 1.2.

**Lemma 2.2** If we have real numbers \( 0 < a < 1, \) \( 0 < b < c < 1 \) satisfying

\[
b < \frac{2}{3}, \quad 1 - c < c - b, \quad 1 - a < \frac{c}{2}, \]

then (2.13) still holds when (2.15) is replaced by the conditions

\[
M \leq X^a \quad \text{for Type I sums},
\]

and

\[
X^b \leq M \leq X^c \quad \text{for Type II sums}.
\]

**Proof** See Proposition 1 of Balog and Friedlander [2]. \( \square \)

**Lemma 2.3** For any \( Q \geq 1, \) \( N \geq 1 \) and any sequence \( a(n) \), we have

\[
\sum_{q \leq Q} \frac{\phi(q)}{q} \sum_{\chi \mod q}^{*} \left| \sum_{n=M+1}^{M+N} a(n) \chi(n) \right|^2 \ll (Q^2 + N) \sum_{n=M+1}^{M+N} |a(n)|^2. \tag{2.16}
\]

**Proof** See (9.52) of Friedlander and Iwaniec [8], or Bombieri and Davenport [3]. \( \square \)
Lemma 2.4 For $\frac{1}{2} < \alpha < 1$, $J \geq 1$, $N \geq 1$, $\Delta > 0$, let $\mathcal{N}(\Delta)$ denote the number of solutions of the following inequality:

$$|h_1 n_1^g - h_2 n_2^g| \leq \Delta, \quad h_1, h_2 \sim J, \quad n_1, n_2 \sim N.$$ 

Then we have

$$\mathcal{N}(\Delta) \ll \Delta J N^{2-\alpha} + JN \log(JN).$$

Proof See the arguments on pp. 256–257 of Heath–Brown [11].

Lemma 2.5 For any $A > 0$ and non-principal Dirichlet character $\chi \pmod{q}$ with $q \ll (\log x)^A$, there holds

$$\sum_{p \leq x} \chi(p) \ll x \exp\left(-c(A)\sqrt{\log x}\right),$$

where the implied constant depends only on $A$.

Proof By partial summation and the arguments on p. 132 of Davenport [7], it is easy to derive the desired result.

3 Estimate of exponential sums

In this section, we shall give the estimate of exponential sums which will be used in proving Theorem 1.1.

3.1 The estimate of type II sums

We begin by breaking up the ranges for $n$ and $h$ into intervals $(N, 2N]$ and $(J, 2J]$ so that $MN \asymp X$ and $\frac{1}{2} \leq J \leq H$. Then, for the Type II sums $\Sigma_{II}$, there holds

$$\Sigma_{II} \ll L^2 \sum_{m \sim M} \left| \sum_{n \sim N} \sum_{h \sim J} b(n) \Xi_h(mn)e(h(mn)^\gamma) \right|.$$ 

Then we have

$$0 < hn^\gamma \leq 4JN^\gamma.$$ 

Denote by $T$ a parameter, which will be chosen later. We decompose the collection of available pairs $(h, n)$ into sets $\mathcal{S}_t$ $(1 \leq t \leq T)$, defined by

$$\mathcal{S}_t = \left\{(h, n) : h \sim J, n \sim N, \frac{4JN^\gamma(t-1)}{T} < hn^\gamma \leq \frac{4JN^\gamma t}{T} \right\}.$$
Therefore, we have

$$\Sigma_{II} \ll L^2 \sum_{1 \leq t \leq T} \sum_{m \sim M} \left| \sum_{(h,n) \in \mathscr{I}_t} b(n) \Xi_h(mn) e(h(mn)^\gamma) \right|,$$

which combined with Cauchy’s inequality yields

$$|\Sigma_{II}|^2 \ll L^4 T M \sum_{1 \leq t \leq T} \sum_{m \sim M} \left| \sum_{(h,n) \in \mathscr{I}_t} b(n) \Xi_h(mn) e(h(mn)^\gamma) \right|^2$$

$$\ll L^4 T M \sum_{1 \leq t \leq T} \sum_{(h,n_1) \in \mathscr{I}_t} \sum_{(h,n_2) \in \mathscr{I}_t} \left| \sum_{m \sim M} \sum_{mn_1 \sim X} \sum_{mn_2 \sim X} \Xi_{h_1}(mn_1) \Xi_{h_2}(mn_2) \right|$$

$$\times e\left( (h_1 n_1^\gamma - h_2 n_2^\gamma) m^\gamma \right) \right|$$

$$\ll L^4 T M \sum_{h_1 \sim J} \sum_{h_2 \sim J} \sum_{n_1 \sim N} \sum_{n_2 \sim N} \left| \sum_{\lambda \leq 4 J N^{\gamma} T^{-1}} \sum_{m \sim M} \sum_{mn_1 \sim X} \sum_{mn_2 \sim X} \Xi_{h_1}(mn_1) \Xi_{h_2}(mn_2) e\left( \lambda m^\gamma \right) \right|,$$

where

$$\lambda = h_1 n_1^\gamma - h_2 n_2^\gamma.$$

Denote by $S$ the innermost sum over $m$. First, we use the definition of the quantity $\Xi_h(\cdot)$ and change the order of summation. If the system of the congruences

$$\begin{cases} mn_1 \equiv a \pmod{d_1} \\ mn_2 \equiv a \pmod{d_2} \end{cases}$$

is not solvable, then $S = 0$. If the above system is solvable, then there exists some integer $g = g(n_1, n_2, a, d_1, d_2)$ such that the system is equivalent to $m \equiv g \pmod{[d_1, d_2]}).$ In this case, we have

$$S = \sum_{d_1 \leq D} c\left( d_1, h_1 \right) \sum_{d_2 \leq D} c\left( d_2, h_2 \right) \sum_{m \sim M} e\left( \lambda m^\gamma \right).$$

Therefore, by Lemma 2.1 with $\left( \kappa, \ell \right) = A^2\left( \frac{1}{2}, \frac{1}{2} \right) = \left( \frac{1}{14}, \frac{11}{14} \right)$, we obtain

$$S \ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left| \sum_{m \sim M} e\left( \lambda m^\gamma \right) \right|$$

with $\left( \kappa, \ell \right) = A^2\left( \frac{1}{2}, \frac{1}{2} \right) = \left( \frac{1}{14}, \frac{11}{14} \right)$, we obtain
\[
\ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} \min \left( \frac{M}{[d_1, d_2]} \right) \leq \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left( \frac{(d_1, d_2)}{d_1 d_2} \right)^5 = \sum_{1 \leq r \leq D} \sum_{k_1 \leq \frac{D}{r}} \sum_{k_2 \leq \frac{D}{r}} \frac{1}{r^{5/3} k_1^{5/3} k_2^{5/3}}
\]

From the following estimate

\[
\ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-\gamma} \ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left( \frac{(d_1, d_2)}{d_1 d_2} \right)^5 = \sum_{1 \leq r \leq D} \sum_{k_1 \leq \frac{D}{r}} \sum_{k_2 \leq \frac{D}{r}} \frac{1}{r^{5/3} k_1^{5/3} k_2^{5/3}}
\]

we can see that the total contribution of the term \(|\lambda|^\frac{1}{T} [d_1, d_2]^{-\gamma} M^{\gamma} + \sum_1^5\) to \(|\Sigma_I|^2\) is

\[
\ll |\lambda|^\frac{1}{T} M^{\gamma} + \sum_1^5 \left( \sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-\gamma} \right) \cdot \mathcal{N}(4J N^T T^{-1}) \cdot L^T T M
\]

\[
\ll (J N^T T^{-1}) \frac{1}{T} M^{\gamma} + \sum_1^5 D^7 \cdot \mathcal{N}(4J N^T T^{-1}) \cdot L^T T M
\]

\[
\ll L^T T M \frac{12}{7} D^7 (J M^T N^T T^{-1}) \frac{1}{T} \cdot \mathcal{N}(4J N^T T^{-1}). \tag{3.1}
\]

If \(|\lambda| \leq M^{-\gamma}\), then \(M[d_1, d_2]^{-1} \leq M^{1-\gamma}|\lambda|^{-1}[d_1, d_2]^{-1}\), and thus the contribution of the \(M[d_1, d_2]^{-1}\) term to \(|\Sigma_I|^2\) is

\[
\ll L^T T M \cdot M L^3 \cdot \mathcal{N}(M^{-\gamma}) \ll L^7 T M^2 \cdot \mathcal{N}(M^{-\gamma}), \tag{3.2}
\]

where we use the elementary estimate

\[
\sum_{d_1 \leq D} \sum_{d_2 \leq D} [d_1, d_2]^{-1} \ll (\log D)^3.
\]

If \(|\lambda| > M^{-\gamma}\), then \(M[d_1, d_2]^{-1} > M^{1-\gamma}|\lambda|^{-1}[d_1, d_2]^{-1}\). It follows from the splitting argument that the contribution of the \(M^{1-\gamma}|\lambda|^{-1}[d_1, d_2]^{-1}\) term to \(|\Sigma_I|^2\) is

\[
\ll L^8 T M^{2-\gamma} \cdot \max_{M^{-\gamma} \leq \Delta \leq 4J N^T T^{-1}} \mathcal{N}(2\Delta) \Delta^{-1}, \tag{3.3}
\]

which contains the upper bound estimate (3.2). From Lemma 2.4, we know that

\[
\mathcal{N}(\Delta) \ll \Delta J N^{2-\gamma} + J N L,
\]

which combines (3.1) and (3.3) to derive that

\[
|\Sigma_I|^2 \ll L^T T M^{\frac{12}{7}} D^7 (J X^T T^{-1}) \frac{1}{T} \cdot \mathcal{N}(4J N^T T^{-1})
\]
We take $T$ such that the first term and the fourth term in the above estimate are equal. Consequently, we choose

$$T = \left[ M^{-\frac{6}{5}} X^{\frac{\gamma + 14}{15}} J D^{\frac{4}{15}} \right] + 1. \tag{3.5}$$

Putting (3.5) into (3.4), we obtain

$$|\Sigma_{II}|^2 \ll X^\eta \left( M^{-\frac{2}{5}} X^{\frac{2\gamma + 28}{15}} J^2 D^{\frac{8}{15}} + M^{-\frac{6}{5}} X^{\frac{44 - 14\gamma}{15}} J^2 D^{\frac{4}{15}} + M^{-\frac{1}{5}} X^{\frac{\gamma + 29}{15}} J^2 D^{\frac{4}{15}} + M^{\frac{5}{7}} X^{\frac{\gamma}{15}} J^{\frac{12}{15}} D^{\frac{2}{15}} + X^{2-\gamma} J + J X M \right),$$

which combined with $J \ll H = X^{1-\gamma'^+\eta}$ yields

$$|\Sigma_{II}|^2 \ll X^\eta \left( M^{-\frac{2}{5}} X^{\frac{58 - 28\gamma}{15}} D^{\frac{8}{15}} + M^{-\frac{6}{5}} X^{\frac{74 - 44\gamma}{15}} D^{\frac{4}{15}} + M^{-\frac{1}{5}} X^{\frac{59 - 29\gamma}{15}} D^{\frac{4}{15}} + M^{\frac{5}{7}} X^{\frac{-\gamma}{15}} D^{\frac{2}{15}} + X^{3-2\gamma} + M X^{2-\gamma} \right).$$

According to above arguments, we deduce the following lemma.

**Lemma 3.1** Suppose that $\frac{1}{2} < \gamma < 1$ and $0 < \xi \leq \frac{1}{2}$ satisfy the condition

$$\gamma > \max \left( \frac{29}{32} + \frac{1}{8} \xi + \eta, \frac{1}{4} + \xi + \eta \right). \tag{3.6}$$

If there holds

$$X^{\frac{2\gamma(1-\gamma)+4\xi}{3}+\eta} \ll M \ll X^{\gamma'-\eta},$$

then we have

$$\Sigma_{II} \ll X^{1-\eta}.$$

### 3.2 The estimate of type I sums

As in Sect. 3.1, we begin by breaking up the range for $n$ into intervals $(N, 2N]$, such that $MN \asymp X$. Then according to the definition of the quantity $\Sigma_h(\cdot)$, we change the order of summation and derive that

$$\Sigma_I \ll L^2 \sum_{0 < h \leq H} \mathcal{K}_h. \tag{3.7}$$
where
\[ K_h = \sum_{d \leq D \atop (a, d) = 1} c(d, h) \sum_{m \sim M} a(m) \sum_{n \sim N \atop mn \sim X \atop mn \equiv a \pmod{d}} e(h(mn)^\gamma). \]

By Lemma 2.1 with exponent pair \((\kappa, \ell) = A^6(1, 1) = (\frac{4}{254}, \frac{247}{254})\), we obtain
\[ K_h \ll \sum_{d \leq D \atop (a, d) = 1} \sum_{m \sim M} \left| \sum_{n \sim N \atop mn \equiv a \pmod{d}} e(h(mn)^\gamma) \right| \]
\[ \ll \sum_{d \leq D \atop (a, d) = 1} \sum_{m \sim M} \left( d^{-1} h^{-1} M^{-1} X^{1-\gamma} + d^{-\frac{123}{254}} h^\frac{1}{254} M^{-\frac{123}{254}} X^\frac{1}{254} + \frac{123}{127} \right) \]
\[ \ll h^{-1} X^{1-\gamma} L + h^\frac{1}{254} M^{\frac{4}{127}} X^\frac{1}{254} M^{\frac{123}{127}} D^{\frac{127}{127}}. \]

From (3.7) and (3.8), we have
\[ \Sigma I \ll X^{1-\gamma} + L^2 H^\frac{255}{254} M^{\frac{4}{127}} X^\frac{1}{254} + \frac{123}{127} D^{\frac{127}{127}} \]
\[ \ll L^4 X^{1-\gamma} + X^\frac{501}{254} - \gamma + \frac{4}{127} \xi + \eta M^\frac{4}{127}. \]

According to above estimate, we obtain the following lemma.

**Lemma 3.2** Suppose that \(M\) satisfies the condition
\[ M \ll X^{\frac{127}{124}} - \frac{247}{8} - \xi - \eta. \]

Then we have
\[ \Sigma I \ll X^{1-\eta}. \]

### 4 Proof of Theorem 1.1

In this section, we combine the results of Lemma 2.2, Lemma 3.1 and Lemma 3.2 to complete the proof of Theorem 1.1.

From Lemma 2.2, we take
\[ a = \frac{127}{4} \gamma - \frac{247}{8} - \xi - \eta, \]
\[ b = \frac{29(1 - \gamma) + 4\xi}{3} + \eta, \]
\[ c = \gamma - \eta. \]

It is easy to check the conditions (2.9), (3.6), as well as the inequalities in Lemma 2.2, hold. Hence we obtain (2.12), which suffices to complete the proof of Theorem 1.1.
5 Weighted sieve and Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2 according to the result of Theorem 1.1, weighted sieve of Richert, and the method of Chen [5]. Let

\[ \mathcal{A} = \{ a : a \leq x, \ a = p + 2, \ p = [k^{1/\gamma}] \}. \]

We consider the weighted sum

\[
W(\mathcal{A}, x^{3/32}) := \sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \Omega(a) \leq 3} \left( 1 - \lambda \cdot \sum_{x^{3/32} \leq p < x^{1/\mu} \atop p \mid a} \left( 1 - \frac{u \log p}{\log x} \right) \right),
\]

where \( u = \xi^{-1} + \varepsilon, \ \lambda = (5 - u - \varepsilon)^{-1} \) and

\[
P(z) = \prod_{2 < p < z} p.
\]

For convenience, we write

\[
\mathcal{W}_a = 1 - \lambda \cdot \sum_{x^{3/32} \leq p < x^{1/\mu} \atop p \mid a} \left( 1 - \frac{u \log p}{\log x} \right).
\]

Then we have

\[
W(\mathcal{A}, x^{3/32}) = \sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \Omega(a) \leq 3} \mathcal{W}_a + \sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \Omega(a) = 4, \ \mu(a) \neq 0} \mathcal{W}_a + \sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \Omega(a) \geq 5, \ \mu(a) \neq 0} \mathcal{W}_a + \sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \Omega(a) \geq 4, \ \mu(a) = 0} \mathcal{W}_a.
\]

Obviously, we have

\[
\sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \Omega(a) \geq 4, \ \mu(a) = 0} \mathcal{W}_a \ll \sum_{a \in \mathcal{A}, \ (a, P(x^{3/32})) = 1 \atop \mu(a) = 0} \tau(a) \ll x^{\varepsilon} \sum_{x^{3/32} \leq p_1 \leq x^{1/2}} \sum_{p \equiv -2 \ (\text{mod} \ p_1^2)} 1 \ll x^{\varepsilon} \sum_{x^{3/32} \leq p_1 \leq x^{1/2}} \left( \frac{x}{p_1^2} + 1 \right) \ll x^{\varepsilon} \left( x^{1-3/32} + x^{1/2} \right) \ll x^{29/32 + \varepsilon}.
\]

(5.2)
For given integer $a$ with $a \leq x$, $(a, P(x^{3/32})) = 1$ and $\mu(a) \neq 0$, the weight $W_a$ in the sum $W(A, x^{3/32})$ satisfies

$$
1 - \lambda \sum_{x^{3/32} \leq p < x^{1/3}} \left( 1 - \frac{u \log p}{\log x} \right) \leq \lambda \left( \frac{1}{\lambda} - \sum_{p\mid a} \left( 1 - \frac{u \log p}{\log x} \right) \right)
$$

$$
= \lambda \left( 5 - u - \varepsilon - \Omega(a) + \frac{u \log a}{\log x} \right)
$$

$$
< \lambda (5 - \Omega(a)), \quad (5.3)
$$

and thus $W_a < 0$ for $\Omega(a) \geq 5$. From (5.1)–(5.3), we know that

$$
\sum_{\begin{array}{c} a \in \mathcal{A} \\ (a, P(x^{3/32})) = 1 \\ \Omega(a) \leq 3 \end{array}} W_a = W(A, x^{3/32}) - \sum_{\begin{array}{c} a \in \mathcal{A} \\ (a, P(x^{3/32})) = 1 \\ \Omega(a) = 4 \\ \mu(a) \neq 0 \end{array}} W_a
$$

$$
- \sum_{\begin{array}{c} a \in \mathcal{A} \\ (a, P(x^{3/32})) = 1 \\ \Omega(a) \geq 5 \\ \mu(a) \neq 0 \end{array}} W_a + O(x^{29/32+\varepsilon})
$$

$$
\geq W(A, x^{3/32}) - \sum_{\begin{array}{c} a \in \mathcal{A} \\ (a, P(x^{3/32})) = 1 \\ \Omega(a) = 4 \\ \mu(a) \neq 0 \end{array}} W_a + O(x^{29/32+\varepsilon}). \quad (5.4)
$$

Therefore, if we can show that the contribution of the second term on the right-hand side of (5.4) is strictly less than $(1 - \varepsilon) W(A, x^{3/32})$, then we shall prove Theorem 1.2.

For $W(A, x^{3/32})$, we have

$$
W(A, x^{3/32}) = \sum_{\begin{array}{c} a \in \mathcal{A} \\ (a, P(x^{3/32})) = 1 \end{array}} 1 - \lambda \sum_{x^{3/32} \leq p < x^{1/3}} \left( 1 - \frac{u \log p}{\log x} \right) \sum_{\begin{array}{c} a \in \mathcal{A} \\ (a, P(x^{3/32})) = 1 \end{array}} 1
$$

$$
= S(A, x^{3/32}) - \lambda \sum_{x^{3/32} \leq p < x^{1/3}} \left( 1 - \frac{u \log p}{\log x} \right) S(A, x^{3/32}). \quad (5.5)
$$

Now, we shall use Theorem 8.4 of Halberstam and Richert [9] to give the lower bound of $S(A, x^{3/32})$. Hence in this theorem we take

$$
X = \pi_{\mathcal{A}}(x), \quad \omega(d) = \begin{cases} 
\frac{d}{\varphi(d)} & \text{if } (d, 2) = 1 \text{ and } \mu(d) \neq 0, \\
0 & \text{otherwise}.
\end{cases}
$$

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In this section, as usual, let \( f(s) \) and \( F(s) \) denote the classical functions in the linear sieve theory. Then by (2.8) and (2.9) of Chapter 8 in Halberstam and Richert [9], we have

\[
F(s) = \frac{2e^{C_0}}{s}, \quad 0 < s \leq 3; \quad f(s) = \frac{2e^{C_0} \log(s - 1)}{s}, \quad 2 \leq s \leq 4,
\]

where \( C_0 \) denotes Euler’s constant. Then it is easy to check the conditions \((\Omega_1)\) and \((\Omega_2(1, L))\) hold. Thus, it is sufficient to show that the condition \((R(1, \alpha))\) holds. Set

\[
R(x, d) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{d} \\ p = [m^{1/r}]}} 1 - \frac{1}{\varphi(d)} \pi_{\gamma}(x),
\]

then it follows from Theorem 1.1 that

\[
\sum_{d \leq x^{\frac{3}{32}}} \left| R(x, d) \right| \ll \frac{x^{\gamma}}{(\log x)^A}.
\]

From the trivial estimate \( R(x, d) \ll x^{\gamma}d^{-1} \) and Cauchy’s inequality, we know that

\[
\sum_{d \leq x^{\frac{3}{32}}} \mu^2(d)3^{\nu(d)} \left| R(x, d) \right| \ll x^{\gamma/2} \sum_{d \leq x^{\frac{3}{32}}} \frac{\mu^2(d)3^{\nu(d)}}{d^{1/2}} \left| R(x, d) \right|^{1/2}
\]

\[
\ll x^{\gamma/2} \left( \sum_{d \leq x^{\frac{3}{32}}} \frac{\mu^2(d)9^{\nu(d)}}{d} \right)^{1/2} \left( \sum_{d \leq x^{\frac{3}{32}}} \left| R(x, d) \right| \right)^{1/2}
\]

\[
\ll x^{\gamma/2} \left( \sum_{d_1, d_2, \ldots, d_9 \leq x^{\frac{3}{32}}} \frac{\mu^2(d_1d_2 \ldots d_9)}{d_1d_2 \ldots d_9} \right)^{1/2} \left( \sum_{d \leq x^{\frac{3}{32}}} \left| R(x, d) \right| \right)^{1/2}
\]

\[
\ll x^{\gamma/2} \left( \sum_{n \leq x^{\frac{3}{32}}} \frac{1}{n} \right)^{9/2} \left( \frac{x^{\gamma}}{(\log x)^A} \right)^{1/2} \ll x^{\gamma} \left( \frac{x^{\gamma}}{(\log x)^A} \right)^{1/2}.
\]

from which we know that the condition \((R(1, \alpha))\) holds. By noting the fact that \( 2 < 32\xi/3 < 4 \) holds for \( 171/172 < \gamma < 1 \), then Theorem 8.4 of Halberstam and Richert [9] gives

\[
S(\alpha, x^{3/32}) \geq \pi_{\gamma}(x) V(x^{3/32}) \left( f \left( \frac{32\xi}{3} \right) - o(1) \right)
\]

\[
= \frac{3e^{C_0}}{16} \pi_{\gamma}(x) V(x^{3/32}) \left( \frac{\log \left( \frac{32\xi}{3} \right) - 1}{\xi} - o(1) \right), \quad (5.6)
\]
where \( C_0 \) denotes Euler’s constant, and
\[
V(z) = \prod_{2 < p < z} \left(1 - \frac{\omega(p)}{p}\right).
\]

Moreover, it follows from (1.11) on p. 245 of Halberstam and Richert [9] that
\[
\sum_{x^{3/32} \leq p < x^{1/4}} \left(1 - \frac{u \log p}{\log x}\right) S(\mathcal{A}_p, x^{3/32}) \leq \frac{3eC_0}{16} \pi\gamma(x) V(x^{3/32}) \leq \log \left(\frac{32 \xi - 1}{\xi}\right) \int_u^\frac{32}{\xi} \frac{\beta - u}{\beta(\xi\beta - 1)} d\beta + o(1).
\]

Combining (5.5)–(5.7), we obtain
\[
W(\mathcal{A}, x^{3/32}) \geq \frac{3eC_0}{16} \pi\gamma(x) V(x^{3/32}) \left(\log \left(\frac{32 \xi - 1}{\xi}\right) - \beta \int_u^\frac{32}{\xi} \frac{\beta - u}{\beta(\xi\beta - 1)} d\beta + o(1)\right).
\]

Now, we consider the second term on the right-hand side of (5.4). Set
\[
\mathcal{B} = \left\{ m : m \leq x, m = p_1 p_2 p_3 p_4, x^{3/32} \leq p_1 < p_2 < p_3 < p_4 \right\}
\]
and
\[
\mathcal{E} = \left\{ n : n + 2 \in \mathcal{B}, n = [k^{1/\gamma}] \right\}
\]
From (5.3) we deduce that
\[
\sum_{a \in \mathcal{E}} W_a \leq \lambda \sum_{a \in \mathcal{E}} 1 = \lambda \sum_{p + 2 \leq x, p = [k^{1/\gamma}]} 1 \leq \lambda \cdot S(\mathcal{E}, x^{1/2}) \leq \lambda \cdot S(\mathcal{E}, x^{3/4}).
\]

Let \( \mathcal{E}_d = \{ n \in \mathcal{E} : n \equiv 0 \pmod{d}\} \). Then it is easy to see that
\[
\mathcal{E}_d = \frac{1}{\varphi(d)} \chi + \mathcal{R}_d^{(1)} + \mathcal{R}_d^{(2)} + \mathcal{R}_d^{(3)},
\]
where
\[
\chi = \sum_{n \in \mathcal{B}} (n - 1)^\gamma - (n - 2)^\gamma.
\]
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\[ \mathcal{R}_d^{(1)} = \sum_{\substack{n \in B \\ n \equiv 2 \mod d}} \left( (n - 1)^\nu - (n - 2)^\nu \right) \]
\[ - \frac{1}{\varphi(d)} \sum_{\substack{n \in B \\ (n,d) = 1}} \left( (n - 1)^\nu - (n - 2)^\nu \right), \quad (5.10) \]

\[ \mathcal{R}_d^{(2)} = \sum_{\substack{n \in B \\ n \equiv 2 \mod d}} \left( \psi\left( - (n - 1)^\nu \right) - \psi\left( - (n - 2)^\nu \right) \right), \quad (5.11) \]

\[ \mathcal{R}_d^{(3)} = - \frac{1}{\varphi(d)} \sum_{\substack{n \in B \\ (n,d) > 1}} \left( (n - 1)^\nu - (n - 2)^\nu \right). \quad (5.12) \]

In order to use Theorem 8.4 of Halberstam and Richert [9] to give upper bound for \( S(E, x^{\xi/3}) \), we need to show that

\[ \sum_{d \leq x^\xi \atop (d,2) = 1} \left| \mathcal{R}_d^{(i)} \right| \ll \frac{x^\nu}{(\log x)^{100}}, \quad i = 1, 2, 3. \quad (5.13) \]

We shall prove (5.13) by three following lemmas. For convenience, we put \( D = x^\xi \).

**Lemma 5.1** Let \( \mathcal{R}_d^{(1)} \) be defined as in (5.10). Then we have

\[ \sum_{d \leq D \atop (d,2) = 1} \left| \mathcal{R}_d^{(1)} \right| \ll \frac{x^\nu}{(\log x)^{100}}. \]

**Proof** In order to prove the result of Lemma 5.1, by partial summation with the smooth function \( f(t) = (t - 1)^\nu - (t - 2)^\nu \), we can reduce the matters to considering the case where there is no weight on the \( n \) variable, and only need to show that

\[ \sum_{d \leq D \atop (d,2) = 1} \max_{1 \leq t \leq x} \left| \sum_{n \equiv 2 \mod d \atop n \leq t} 1 - \frac{1}{\varphi(d)} \sum_{n \in B \atop (n,d) = 1 \atop n \leq t} 1 \right| \ll \frac{x}{(\log x)^{100}}. \quad (5.14) \]

By Theorem 9.16 of Friedlander and Iwaniec [8], the Bombieri–Vinogradov theorem for \( B \), i.e. the estimate (5.14), follows immediately, once \( B \) is shown to satisfy the Siegel–Walfisz condition (see (9.68) of [8]). Next, we shall show that the Siegel–Walfisz condition holds for \( B \).

By the orthogonality of Dirichlet characters, we have

\[ \sum_{n \equiv 2 \mod d \atop n \leq t} 1 - \frac{1}{\varphi(d)} \sum_{n \equiv 2 \mod d \atop (n,d) = 1 \atop n \leq t} 1 = \frac{1}{\varphi(d)} \sum_{\chi \mod d} \chi(2) \sum_{\chi \neq \chi^0 \atop \chi \equiv \chi^0 \atop n \leq t} \chi(n). \]
Let $\chi_q^*$ denote the primitive character which induces $\chi_d$, then we have $1 < q | d$ and $\chi_d = \chi_d^0 \chi_q^*$. Consequently, we derive that

$$\max_{1 \leq t \leq x} \left| \sum_{\substack{n \in B \atop n \leq t}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \in B \atop (n,d) = 1 \atop n \leq t}} 1 \right| \leq \frac{1}{\varphi(q)} \sum_{1 \leq q | d \atop n \leq t} \max_{1 \leq t \leq x} \left| \sum_{n \in B \atop n \leq t} (n, d) = 1 \chi^0_d(n) \chi^*_q(n) \right|. \quad (5.15)$$

Now, we consider the innermost sum on the right-hand side of (5.15) in two cases.

**Case 1** For $q \leq (\log x)^{500}$, we have

$$\max_{1 \leq t \leq x} \left| \sum_{n \in B \atop n \leq t} \chi_d^0(n) \chi^*_q(n) \right|$$

$$= \max_{1 \leq t \leq x} \left| \sum_{x^{3/32} \leq p_1 < p_2 < p_3 < (t/(p_1 p_2))^{1/2}} \chi_d^0(p_1 p_2 p_3) \chi^*_q(p_1 p_2 p_3) \times \sum_{p_3 < p_4 < t/(p_1 p_2 p_3)} \chi_d^0(p_4) \chi^*_q(p_4) \right|$$

$$\ll \max_{1 \leq t \leq x} \sum_{x^{3/32} \leq p_1 < p_2 < p_3 < (t/(p_1 p_2))^{1/2}} \left( \left| \sum_{p_3 < p_4 < t/(p_1 p_2 p_3)} \chi^*_q(p_4) \right| + O(1) \right)$$

$$\ll \max_{1 \leq t \leq x} \sum_{x^{3/32} \leq p_1 < p_2 < p_3 < (t/(p_1 p_2 p_3))^{1/2}} \frac{t}{p_1 p_2 p_3} \cdot \exp \left( - c_1 \log^{1/2} t \right)$$

$$\ll \exp \left( - \log^{1/3} x \right).$$

**Case 2** For $q > (\log x)^{500}$, by splitting argument, the innermost sum on the right-hand side of (5.15) can be represented as the sum of at most $O(\log^4 x)$ sums of the form

$$\sum_{p_j \in I_j, j=1,2,3,4} \chi_d^0(p_1 p_2 p_3 p_4) \chi^*_q(p_1 p_2 p_3 p_4),$$

where

$$I_j = [N_j, N'_j], \quad x^{3/32} \leq N_j < N'_j \leq 2N_j, \quad N_1 N_2 N_3 N_4 < N'_1 N'_2 N'_3 N'_4 \leq t.$$
Then we have
\[
\sum_{p \in I_j} \chi_d^0(p) \chi_q^*(p) = g_j(\chi_q^*) + O(1), \quad j = 1, 2, 3, 4,
\]
and thus
\[
\sum_{p_j \in I_j} \chi_d^0(p_1 p_2 p_3 p_4) \chi_q^*(p_1 p_2 p_3 p_4)
\ll \prod_{i=1}^4 \left( g_i(\chi_q^*) + O(1) \right) \ll \sum_{i=1}^4 \sum_{1 \leq j_1 < \cdots < j_i \leq 4} \prod_{k=1}^i |g_{j_k}(\chi_q^*)| + O(1). \quad (5.16)
\]
It is easy to see that one only needs to prove that
\[
\frac{1}{\varphi(d)} \sum_{1 < q | d} \sum_{\chi \mod q} \left| \prod_{k=1}^i g_{j_k}(\chi_q^*) \right| \ll \frac{x}{(\log x)^{500}}
\]
with \(1 \leq i \leq 4, 1 \leq j_1 < \cdots < j_i \leq 4\) and \(q > (\log x)^{500}\). Set
\[
\mathcal{F}_1(\chi_q^*) = g_{j_1}(\chi_q^*) = \sum_{M < m \leq 2M} a(m) \chi_q^*(m)
\]
and
\[
\mathcal{F}_2(\chi_q^*) = \prod_{k=2}^i g_{j_k}(\chi_q^*) = \sum_{N < n \leq 2^{i-1} N} b(n) \chi_q^*(n),
\]
where
\[
x^{3/32} \ll M, N \ll x^{29/32}, \quad MN \asymp x, \quad a(m) \ll 1, \quad b(n) \ll 1.
\]
It follows from Lemma 2.3 with \( Q = 1 \) and Cauchy’s inequality that

\[
\sum_{\chi \mod q}^* \left| \prod_{k=1}^i g_{jk}(\chi_q^*) \right| = \sum_{\chi \mod q}^* \left| \mathcal{F}_1(\chi_q^*) \mathcal{F}_2(\chi_q^*) \right|
\ll \left( \sum_{\chi \mod q}^* \left| \mathcal{F}_1(\chi_q^*) \right|^2 \right)^{1/2} \left( \sum_{\chi \mod q}^* \left| \mathcal{F}_2(\chi_q^*) \right|^2 \right)^{1/2}
\ll \left( M \sum_{m \sim M} |a(m)|^2 \right)^{1/2} \left( N \sum_{n \sim N} |b(n)|^2 \right)^{1/2} \ll MN \ll x,
\tag{5.17}
\]

and thus

\[
\frac{1}{\varphi(d)} \sum_{d \leq D \atop (d,2)=1} \sum_{\chi \mod q}^* \left| \prod_{k=1}^i g_{jk}(\chi_q^*) \right| \ll \frac{x^\gamma}{\varphi(d)} \ll \frac{x}{d^{1-\varepsilon}} \ll \frac{x}{q^{1-\varepsilon}} \ll \frac{x}{(\log x)^{200}}.
\]

Combining the above two cases, we have shown that \( B \) satisfies the Siegel–Walfisz condition. \( \square \)

**Lemma 5.2** Let \( \mathcal{R}_d^{(2)} \) be defined as in (5.11). Then we have

\[
\sum_{d \leq D \atop (d,2)=1} \left| \mathcal{R}_d^{(2)} \right| \ll \frac{x^\gamma}{(\log x)^{100}}.
\]

**Proof** From the definition of \( \mathcal{R}_d^{(2)} \), it suffices to show that, \( X \leq x \), there holds

\[
\sum_{d \leq D \atop (d,2)=1} \left| \sum_{n \sim X \atop n \equiv 2 \pmod{d}} \sum_{n \sim X} \left( \psi(-n-1) - \psi(-n-2) \right) \right| \ll \frac{x^\gamma}{(\log x)^A}.
\tag{5.18}
\]

We can follow the arguments essentially the same as those in Section 2 (i.e. (2.4)–(2.11)), where the weight \( \Lambda(n) \) is instead of \( 1_B(n) \) the characteristic function of \( B \), to derive that

\[
\sum_{d \leq D \atop (d,2)=1} \left| \mathcal{R}_d^{(2)} \right| \ll x^\gamma (\log x)^{-A}
\]

\[
+ \max_{X < u \leq 2X} X^{\gamma - 1} \sum_{d \leq D \atop (d,2)=1} \sum_{0 < h \leq H} \sum_{\ell \equiv 2 \pmod{d} \atop X < \ell \leq u} \alpha(d,h) e(-h \ell^\gamma)
\ll x^\gamma (\log x)^{-A}
\]

\[
+ \max_{X < u \leq 2X} X^{\gamma - 1} \sum_{\ell \sim X} \sum_{0 < h \leq H} \Theta_h(e(-h \ell^\gamma))
\ll x^\gamma (\log x)^{-A}.
\]
where
\[ \Theta_h(\ell) = \sum_{\substack{d \leq D \\ (d, 2) = 1}} \alpha(d, h), \quad |\alpha(d, h)| = 1. \]

Next, we shall illustrate that, for \( \ell = p_1 p_2 p_3 p_4 \in \mathcal{B} \) and \( \ell \sim X > x^{1-\eta} \), there must be some partial product of \( p_1 p_2 p_3 p_4 \) which lies in the interval \([X^{1/2+\eta}, X^{85/86-\eta}]\).

First, since \( p_i \geq x^{3/32} \) and \( p_1 p_2 p_3 p_4 \in [x^{1-\eta}, x] \), we have \( p_i \leq X^{85/86-\eta} \). If there exists some \( p_i \in [X^{1/2+\eta}, X^{85/86-\eta}] \), then the conclusion follows. If this case does not exist, we consider the product \( p_1 p_2 \). At this time, there must be \( p_1 p_2 < X^{1/2+\eta} \). Otherwise, from \( p_1 p_2 \geq X^{1/2+\eta} > (x^{1-\eta})^{1/2+\eta} > x^{1/2} \) we obtain \( p_3 p_4 = n(p_1 p_2)^{-1} < x^{1/2} < p_1 p_2 \), which contradict to \( p_1 < p_2 < p_3 < p_4 \). Now, we consider the product \( p_1 p_2 p_3 \). If \( p_1 p_2 p_3 \in [X^{1/2+\eta}, X^{85/86-\eta}] \), then the conclusion holds. Otherwise, if \( p_1 p_2 p_3 < X^{1/2+\eta} \), then \( p_4 = n(p_1 p_2 p_3)^{-1} > X(X^{1/2+\eta})^{-1} = X^{1/2-\eta} > x^{7/16} \), and thus \( p_1 p_2 p_4 > x^{6/32+7/16} = x^5/8 > X^{1/2+\eta} \). Moreover, \( p_1 p_2 p_4 = n(p_3)^{-1} \leq x^{29/32} \leq X^{85/86-\eta} \). Above all, there must exist some partial product of \( p_1 p_2 p_3 p_4 \) which lies in \([X^{1/2+\eta}, X^{85/86-\eta}]\).

For \( 85/86 < \gamma < 1 \) and the definition of \( \xi \), it is easy to see that
\[ X^{\frac{29(1-\gamma)+4\xi}{3}} + \eta \leq X^{\frac{1}{2}+\eta} < X^{\frac{85}{86}-\eta} \leq X^{\gamma-\eta}, \]
which combined with Lemma 3.1 yields
\[ \sum_{\substack{d \leq D \\ (d, 2) = 1}} \left| \mathcal{R}_d^{(2)}(x) \right| \ll x^\gamma (\log x)^{-A}. \]

This completes the proof of Lemma 5.2. \( \square \)

**Lemma 5.3** Let \( \mathcal{R}_d^{(3)} \) be defined as in (5.12). Then we have
\[ \sum_{\substack{d \leq D \\ (d, 2) = 1}} \left| \mathcal{R}_d^{(3)} \right| \ll \frac{x^\gamma}{(\log x)^{1/3}}. \]

**Proof** We have
\[ \mathcal{R}_d^{(3)} = -\frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{B} \\ (n, d) > 1}} \gamma n^{\gamma-1} - \frac{1}{\varphi(d)} \sum_{\substack{n \in \mathcal{B} \\ (n, d) > 1}} \left( ((n - 1)^\gamma - (n - 2)^\gamma) - \gamma n^{\gamma-1} \right). \]
Hence
\[
\sum_{d \leq D \atop (d, 2) = 1} |\mathcal{E}_d^{(3)}| \ll \sum_{d \leq D} \frac{1}{\varphi(d)} \sum_{n \in B \atop (n, d) > 1} n^{r-1} \\
+ \sum_{d \leq D} \frac{1}{\varphi(d)} \sum_{n \in B \atop (n, d) > 1} \left( ((n - 1)^r - (n - 2)^r) - r(n - 1)^{r-1} \right).
\]
(5.19)

The second term on the right-hand side of (5.19) is
\[
\ll \sum_{d \leq D} \frac{1}{\varphi(d)} \sum_{n \leq x} n^{r-2} \ll x^{r-1} \sum_{d \leq D} \frac{1}{\varphi(d)} \ll x^{r-1+\eta} \ll x^{r} (\log x)^{-A}.
\]
(5.20)

For the first term, which is on the right-hand side of (5.19), we have
\[
\ll \sum_{d \leq D} \frac{1}{\varphi(d)} \sum_{n \in B \atop (n, d) \geq x^{3/32}} n^{r-1} \ll x^{r+\eta} \sum_{d \leq D} \frac{1}{nd} \ll x^{r+\eta} \sum_{d \leq D} \sum_{n \leq x \atop (n, d) = k} \frac{1}{nd} \\
\ll x^{r+\eta} \sum_{d \leq D} \sum_{n \leq x \atop (n, d) = k} \frac{1}{k^2 d_1 n_1} \ll x^{r+2\eta} \sum_{d \leq D} \sum_{n \leq x \atop (n, d) = k} \frac{1}{k^2} \ll x^{r-3/32+\eta}.
\]
(5.21)

Combining (5.19)–(5.21), we derive the desired result of Lemma 5.3. \(\square\)

From Lemma 5.1–5.3, we deduce that
\[
S(\mathcal{E}, x^{\xi/3}) \leq \mathcal{X} V(x^{\xi/3}) \left( F(3) + o(1) \right).
\]
(5.22)

By Theorem 7.11 of Pan and Pan [28], we know that
\[
V(z) = C(\omega) \frac{e^{-C_0}}{\log z} \left( 1 + O \left( \frac{1}{\log x} \right) \right),
\]
(5.23)
where \(C_0\) is Euler’s constant and \(C(\omega)\) is a convergent infinite product defined by
\[
C(\omega) = \prod_p \left( 1 - \frac{\omega(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-1}.
\]

According to (5.23), we get
\[
V(x^{\xi/3}) = \frac{9}{32\xi} V(x^{3/32}) \left( 1 + O(\log x)^{-1} \right),
\]
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from which and (5.22) we deduce that

$$S(\mathcal{E}, x^{2/3}) \leq \frac{3e^{C_0}}{16 \zeta} \mathcal{X}(x^{3/2})(1 + o(1)).$$  \hspace{1cm} (5.24)

Next, we compute the quantity $\mathcal{X}$ definitely. Obviously, we have

$$\mathcal{X} = \sum_{n \in \mathcal{B}} \gamma n^{\gamma - 1} + \sum_{n \in \mathcal{B}} \left( (n - 1)^\gamma - (n - 2)^\gamma - \gamma n^{\gamma - 1}.\right)$$  \hspace{1cm} (5.25)

For the second term in (5.25), we have

$$\sum_{n \in \mathcal{B}} \left( (n - 1)^\gamma - (n - 2)^\gamma - \gamma n^{\gamma - 1} \right) \ll \sum_{n \in \mathcal{B}} n^{\gamma - 2} \ll \left( \sum_{x^{3/32} \leq p \leq x} p^{\gamma - 2} \right)^4 \ll x^{(\gamma - 1)/8} = o(1).$$  \hspace{1cm} (5.26)

For the first term in (5.25), we have

$$\sum_{n \in \mathcal{B}} \gamma n^{\gamma - 1} = \gamma \left( 1 + o(1) \right) \int_{x^{3/32}}^{x^{1/4}} \int_{u_1}^{1/3} \int_{u_2}^{1/2} \int_{u_3} x \sum_{p_1 < p_2 < (x/p_1)^{1/3}} \sum_{p_2 < p_3 < (x/(p_1 p_2))^{1/2}} \sum_{p_3 < p_4 < x/(p_1 p_2 p_3)} (p_1 p_2 p_3 p_4)^{\gamma - 1} \frac{1}{\log u_1} \frac{1}{\log u_2} \frac{1}{\log u_3} \frac{1}{\log u_4}$$

$$= \gamma \left( 1 + o(1) \right) \int_{3/2}^{1} \frac{dt_1}{t_1} \int_{t_1}^{1-t_1} \frac{dt_2}{t_2} \int_{t_2}^{1-t_1-t_2} \frac{dt_3}{t_3} \int_{t_3}^{1-t_1-t_2-t_3} \frac{x(t_1+t_2+t_3+t_4)^\gamma}{4!} dt_4.$$  \hspace{1cm} (5.27)

For the innermost integral in (5.27), we have

$$\int_{t_3}^{1-t_1-t_2-t_3} \frac{x(t_1+t_2+t_3+t_4)^\gamma}{4!} dt_4 = \frac{1}{\gamma \log x} \int_{t_3}^{1-t_1-t_2-t_3} \frac{1}{t_4} dx(t_1+t_2+t_3+t_4)^\gamma \int_{t_4}^{1-t_1-t_2-t_3} \frac{1}{t_4}$$

$$= \frac{1}{\gamma \log x} \left( \frac{x^\gamma}{1 - t_1 - t_2 - t_3} + O\left( \frac{x^\gamma}{\log x} \right) \right) = \frac{1}{1 - t_1 - t_2 - t_3} \cdot \frac{x^\gamma}{\gamma \log x} (1 + o(1)).$$  \hspace{1cm} (5.28)

From (5.26)–(5.28), we deduce that

$$\mathcal{X} = \frac{x^\gamma (1 + o(1))}{\log x} \int_{3/2}^{1} \frac{dt_1}{t_1} \int_{t_1}^{1-t_1} \frac{dt_2}{t_2} \int_{t_2}^{1-t_1-t_2} \frac{dt_3}{t_3} (1 - t_1 - t_2 - t_3).$$  \hspace{1cm} (5.29)
Combining (5.4), (5.8), (5.9), (5.24) and (5.29), we obtain
\[
\sum_{\substack{a \notin f \\ \Omega(a) \leq 3}} \mathcal{W}_a \geq \frac{3e^{C_0}}{16} \frac{x^\gamma}{\log x} V(x^{3/32})(1 + o(1))
\]
\[
\times \left( \log \left( \frac{32}{3} \xi - 1 \right) - \lambda \int_{u}^{32/3} \frac{\beta - u}{\beta(\xi - 1)} d\beta \right)
\]
\[
- \frac{\lambda}{\xi} \int_{1/32}^{1} \frac{dt_1}{t_1} \int_{1/32}^{1-t_1} \frac{dt_2}{t_2} \int_{1-t_2-t_3}^{1-t_1-t_2-t_3} \frac{dt_3}{(1-t_1-t_2-t_3)}
\]
\[
+ O(x^{3/32+\varepsilon}).
\]

By simple numerical calculations, it is easy to see that the number in the above brackets \((-\cdot)\) is \(\geq 0.000060486\), provided that \(0.9989445 < \gamma < 1\). This completes the proof of Theorem 1.2.

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