UNIFORM DISTRIBUTION OF KAKUTANI PARTITIONS GENERATED BY SUBSTITUTION SCHEMES

BY

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ABSTRACT

Substitution schemes provide a classical method for constructing tilings of Euclidean space. Allowing multiple scales in the scheme, we introduce a rich family of sequences of tile partitions generated by the substitution rule, which include the sequence of partitions of the unit interval considered by Kakutani as a special case. Using our recent path counting results for directed weighted graphs, we show that such sequences of partitions are uniformly distributed, thus extending Kakutani's original result. Furthermore, we describe certain limiting frequencies associated with sequences of partitions, which relate to the distribution of tiles of a given type and the volume they occupy.

1. Introduction and main results

1.1. The Kakutani splitting procedure. The splitting procedure introduced by Kakutani in [Ka] and the associated sequences of partitions \( \{ \pi_m \} \) of the unit interval \( I = [0, 1] \) are defined as follows: Fix a constant \( \alpha \in (0, 1) \) and define the first element of the sequence to be the trivial partition \( \pi_0 = I \). Next, define \( \pi_1 \) to be the partition of \( I \) into two subintervals \( [0, \alpha] \) and \( [\alpha, 1] \), that is, into two intervals of disjoint interior, one of length \( \alpha \) and one of length \( 1 - \alpha \). Assuming \( \pi_{m-1} \) is defined, define the partition \( \pi_m \) by splitting every interval of

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maximal length in $\pi_{m-1}$ into two subintervals with disjoint interiors, proportional to the two subintervals that constitute $\pi_1$. This sequence is known as the $\alpha$-Kakutani sequence of partitions, and the procedure that generates it is called the Kakutani splitting procedure.

For example, the trivial partition $\pi_0$ and the first four non-trivial partitions $\pi_1, \pi_2, \pi_3$ and $\pi_4$ of the $\frac{1}{3}$-Kakutani sequence of partitions of the unit interval $I$ are illustrated in Figure 1.1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.1.png}
\caption{From left to right, the first few partitions in the $\frac{1}{3}$-Kakutani sequence of partitions of the unit interval.}
\end{figure}

A sequence $\{\pi_m\}$ of partitions of $I$ is said to be uniformly distributed if for any continuous function $f$ on $I$

$$\lim_{m \to \infty} \frac{1}{k(m)} \sum_{i=1}^{k(m)} f(t_i^m) = \int_I f(t) dt,$$

where $k(m)$ is the number of intervals in the partition $\pi_m$ and $t_i^m$ is the right endpoint of the interval $i$ in the partition $\pi_m$. We remark that the choice of sampling points as the right endpoints of the intervals is arbitrary.

**Theorem (Kakutani):** For any $\alpha \in (0, 1)$, the $\alpha$-Kakutani sequence of partitions is uniformly distributed.

Our main result is a proof of uniform distribution for sequences of partitions generated by multiscale substitution schemes in $\mathbb{R}^d$, constituting a generalization of the $\alpha$-Kakutani sequences considered above.

### 1.2. Tiles, partitions and multiscale substitution schemes.

**Definition 1.1:** A **tile** $T \subset \mathbb{R}^d$ is a Lebesgue measurable bounded set with positive Lebesgue measure and boundary of measure zero. The measure of $T$ is denoted by $\text{vol} T$ and referred to as the volume of $T$.

**Definition 1.2:** Let $U \subset \mathbb{R}^d$ be a bounded set. A **partition** of $U$ is a set of subsets of $U$ with pairwise disjoint interiors, the union of which is equal to $U$. Given a list of tiles $L$, we say that $L$ **tiles** $U$, and that $U$ is **partitioned** into
elements of $L$, if there exists a partition of $U$ such that each element of the partition is isometric to exactly one tile in $L$.

**Definition 1.3:** A **multiscale substitution scheme** $\sigma = (\tau_\sigma, \omega_\sigma, \Sigma_\sigma)$ in $\mathbb{R}^d$ consists of a finite list $\tau_\sigma = (T_1, \ldots, T_n)$ of labeled tiles in $\mathbb{R}^d$ called **prototiles**, each equipped with a finite list of **substitution tiles**

$$\omega_\sigma(T_i) = (\alpha_{ij}^{(k)} T_j : j = 1, \ldots, n \text{, } k = 1, \ldots, k_{ij})$$

so that $\omega_\sigma(T_i)$ tiles $T_i$, and a family $\Sigma_\sigma$ of **substitution rules**, where each $\varrho_\sigma \in \Sigma_\sigma$ assigns a partition $\varrho_\sigma(T_i)$ of $T_i$ into elements of $\omega_\sigma(T_i)$. By assumption $\Sigma_\sigma$ is non-empty.

For all $i, j$ and $k$ the scaling constants $\alpha_{ij}^{(k)}$ are positive, and the **constants of substitution**

$$(1.1) \quad \beta_{ij}^{(k)} = \left(\frac{\text{vol} T_j}{\text{vol} T_i}\right)^{1/d} \alpha_{ij}^{(k)}$$

satisfy $0 < \beta_{ij}^{(k)} < 1$.

It is convenient to think of the analogy to jigsaw puzzles, where the prototiles $\tau_\sigma$ are puzzles to be solved using the pieces in $\omega_\sigma$, and every $\varrho_\sigma \in \Sigma_\sigma$ gives a solution $\varrho_\sigma(T_i)$ to each of the puzzles. (See Figure 1.2.)

![Figure 1.2](image)

**Figure 1.2.** A substitution scheme $\sigma$ on a rectangle $\mathcal{R}$ and a square $\mathcal{S}$ in $\mathbb{R}^2$. From left to right, the prototiles $\tau_\sigma = (\mathcal{R}, \mathcal{S})$, the substitution tiles $\omega_\sigma(\mathcal{R})$ (top) and $\omega_\sigma(\mathcal{S})$, and a substitution rule $\varrho_\sigma$.

A set of prototiles $\tau_\sigma$ may contain two or more labeled tiles that are identical as sets in $\mathbb{R}^d$, or more generally are the images of each other under conformal maps of $\mathbb{R}^d$, but carry distinct labels. When applying a rescaling or an isometry to a labeled tile, we assume that the label is preserved. This allows us to naturally extend substitution rules to the image of a tile under such mappings.
More precisely, given a tile $T = \alpha \cdot \varphi(T_i)$, where $\alpha > 0$ and $\varphi$ is an isometry of $\mathbb{R}^d$, for any $\varrho_\sigma \in \Sigma_\sigma$ we define

$$\varrho_\sigma(T) = \varrho_\sigma(\alpha \cdot \varphi(T_i)) = \alpha \cdot \varphi(\varrho_\sigma(T_i))$$

and

$$\omega_\sigma(T) = \omega_\sigma(\alpha \cdot \varphi(T_i)) = (\alpha T' : T' \in \omega_\sigma(T_i)).$$

For every $k \in \mathbb{N}$ we can thus define inductively

$$\varrho^{k+1}_\sigma(T) = \bigsqcup_{T' \in \varrho^k_\sigma(T)} \varrho_\sigma(T') \quad \text{and} \quad \omega^{k+1}_\sigma(T) = (T'' \in \omega_\sigma(T') : T' \in \omega^k_\sigma(T)).$$

**Definition 1.4:** Let $\sigma$ be a multiscale substitution scheme. Denote by

$$\mathcal{O}^\sigma_i = (T \in \omega^k_\sigma(T_i) : k \in \mathbb{N})$$

the list of substitution tiles defined by finitely many applications of elements of $\Sigma_\sigma$ to $T_i$. A tile $T \in \mathcal{O}^\sigma_i$ is of type $j$ if it is labeled $j$, and $\sigma$ is **irreducible** if $\mathcal{O}^\sigma_i$ contains a tile of type $j$ for any $1 \leq i, j \leq n$.

**1.3. Sequences of partitions generated by multiscale substitution schemes.**

**Definition 1.5:** Let $\sigma$ be a multiscale substitution scheme.

(1) A **Kakutani sequence of partitions** $\{\pi_m\}$ of $T_i \in \tau_\sigma$ is defined as follows: The trivial partition is $\pi_0 = T_i$. For any $m \in \mathbb{N}$, if $\pi_{m-1}$ is a partition of $T_i$ consisting of labeled tiles, then the partition $\pi_m$ is defined by substituting tiles of maximal volume in $\pi_{m-1}$, each tile substituted according to an arbitrarily and independently chosen substitution rule $\varrho_\sigma \in \Sigma_\sigma$.

(2) A **generation sequence of partitions** $\{\delta_k\}$ of $T_i \in \tau_\sigma$ is defined as follows: The trivial partition is $\delta_0 = T_i$. For any $k \in \mathbb{N}$, if $\delta_{k-1}$ is a partition of $T_i$ consisting of labeled tiles, then the partition $\delta_k$ is defined by substituting all tiles in $\delta_{k-1}$, each tile substituted according to an arbitrarily and independently chosen substitution rule $\varrho_\sigma \in \Sigma_\sigma$.

For illustrated examples of Kakutani and generation sequences of partitions, see Section 2. Note that for any Kakutani sequence $\{\pi_m\}$ and generation sequence $\{\delta_k\}$ of $T_i$, every tile $T \in \mathcal{O}^\sigma_i$ corresponds to at least one partition $\pi_m$ and to exactly one partition $\delta_k$, and that $\mathcal{O}^\sigma_i$ is the union of all substitution tiles used to tile the elements of $\{\pi_m\}$ and $\{\delta_k\}$. 
Remark 1.6: In the terminology of [Ka], the splitting procedures that define \( \{ \pi_m \} \) and \( \{ \delta_k \} \) correspond to “maximal refinement” and “refinement”, respectively, and the generation of a tile is its “rank”.

1.4. Uniform distribution.

Definition 1.7: Let \( U \subset \mathbb{R}^d \) be a bounded measurable set of positive measure, and for every \( m \in \mathbb{N} \) let \( x_m \) be a finite set of points in \( U \) of cardinality \( |x_m| \) tending to infinity with \( m \). The sequence \( \{ x_m \} \) is uniformly distributed in \( U \) if for any continuous function \( f \) on \( U \)

\[
\lim_{n \to \infty} \frac{1}{|x_m|} \sum_{x \in x_m} f(x) = \frac{1}{\text{vol} U} \int_U f(t) dt,
\]

where the integration is with respect to Lebesgue measure.

Definition 1.8: Let \( \{ \gamma_m \} \) be a sequence of partitions of \( U \). A marking sequence \( \{ x_m \} \) of \( \{ \gamma_m \} \) is a sequence of sets of points in \( U \), such that every set in the partition \( \gamma_m \) contains a single point of \( x_m \), and all points in \( x_m \) are distinct. The sequence of partitions \( \{ \gamma_m \} \) is uniformly distributed if there exists a marking sequence \( \{ x_m \} \) of \( \{ \gamma_m \} \) that is uniformly distributed in \( U \).

Definition 1.7 is equivalent to the weak-* convergence of the normalized sampling measures

\[
\frac{1}{|x_m|} \sum_{x \in x_m} \delta_x
\]

to the normalized Lebesgue measure on \( U \), where \( \delta_x \) is the Dirac measure concentrated at \( x \). The marking introduced in [Ka] consists of the boundary points of the intervals constituting each partition of \( \mathcal{I} \).

The following is our main result.

Theorem 1.9: Let \( \sigma \) be an irreducible multiscale substitution scheme, and let \( \{ \pi_m \} \) be a Kakutani sequence of partitions of \( T_i \in \tau_\sigma \). Then \( \{ \pi_m \} \) is uniformly distributed in \( T_i \).

The irreducibility condition is crucial. As shown in Section 2 of [Vo], there are simple examples of sequences of partitions generated by non-irreducible schemes which are not uniformly distributed.
Remark 1.10: In our proof of Theorem 1.9 as well as in the proofs of the results stated below, the choice of marking sequences is arbitrary. In fact, we show that every marking sequence of the sequences of partitions is uniformly distributed.

1.5. INCOMMENSURABLE SCHEMES AND FREQUENCIES OF TYPES.

Definition 1.11: An irreducible multiscale substitution scheme \( \sigma \) is incommensurable if there exist \( 1 \leq i, j \leq n \) and two tiles \( T_1 \in \mathcal{O}_i^\sigma \) of type \( i \) and \( T_2 \in \mathcal{O}_j^\sigma \) of type \( j \) so that

\[
\frac{\log \text{vol} T_1}{\text{vol} T_i} \notin \mathbb{Q} \frac{\log \text{vol} T_2}{\text{vol} T_j}.
\]

Otherwise the scheme is commensurable.

Easy non-trivial examples arise from \( \alpha \)-Kakutani sequences, which can be formulated in the language of multiscale substitution schemes with a prototile set consisting only of the unit interval, and a substitution rule partitioning the unit interval into the union of the intervals \([0, \alpha]\) and \([\alpha, 1]\); see also Example 2.1. If \( \alpha = \frac{1}{3} \), we can simply pick \( T_1 = [0, \frac{1}{3}] \) and \( T_2 = [\frac{1}{3}, 1] \), and since \( \frac{\log 3}{\log 2} \) is irrational, incommensurability follows. On the other hand, since all intervals that appear in an \( \alpha \)-Kakutani sequence are of length \( \alpha^k(1 - \alpha)^\ell \), if \( \varphi \) is the golden ratio and \( \alpha = \frac{1}{\varphi} \), then since \( 1 - \frac{1}{\varphi} = \frac{1}{\varphi^2} \) all interval lengths are an integer power of \( \varphi \), and commensurability follows.

Admittedly, the definition of incommensurability strictly in terms of the substitution scheme \( \sigma \) seems rather mysterious. As we will see, it is more natural and easier to verify when considered in the context of the graph \( G_{\sigma} \) associated with the substitution scheme \( \sigma \), introduced in Section 4. For now we only note that incommensurable schemes are generic in the sense that for almost any choice of constants of substitution, the resulting scheme is incommensurable. We also refer to Section 5 for equivalent definitions and additional examples of commensurable and incommensurable substitution schemes.

Definition 1.12: Let \( \sigma \) be a multiscale substitution scheme and let \( \{ \gamma_m \} \) be either a Kakutani or a generation sequence of partitions of \( T_i \in \tau_\sigma \). For \( 1 \leq r \leq n \), an \( r \)-marking sequence \( \{ x_m^{(r)} \} \) of \( \{ \gamma_m \} \) is a sequence of sets of points in \( T_i \), such that every tile of type \( r \) in the partition \( \gamma_m \) contains a single point of \( x_m^{(r)} \), and all points in \( x_m^{(r)} \) are distinct.
THEOREM 1.13: Let $\sigma$ be an irreducible incommensurable multiscale substitution scheme and let $\{\pi_m\}$ be a Kakutani sequence of partitions of $T_i \in \tau_\sigma$. Let $1 \leq r \leq n$.

1. Any $r$-marking sequence $\{x_m^{(r)}\}$ of $\{\pi_m\}$ is uniformly distributed in $T_i$.

2. The ratio between the number of tiles of type $r$ in $\pi_m$ and the total number of tiles is

$\frac{\sum_{h=1}^{n} q_h \sum_{k=1}^{k_{hr}} (1 - (\beta_{hr}^{(k)})^d)}{\sum_{h,j=1}^{n} q_h \sum_{k=1}^{k_{hj}} (1 - (\beta_{hj}^{(k)})^d)} + o(1), \ m \to \infty.$

3. The volume of the region covered by tiles of type $r$ in $\pi_m$ is

$\text{vol} T_i \cdot \sum_{h=1}^{n} q_h \sum_{k=1}^{k_{hr}} (\beta_{hr}^{(k)})^d \log \frac{1}{\beta_{hr}^{(k)}} + o(1), \ m \to \infty.$

In the equations above, $q_h$ is any entry of column $h$ of a matrix $Q_\sigma \in M_n(\mathbb{R})$ of equal rows, given by

$Q_\sigma = \frac{\text{adj}(I - M_\sigma)}{-\text{tr}(\text{adj}(I - M_\sigma) \cdot M_\sigma)}$

with

$(M_\sigma)_{ij} = \sum_{k=1}^{k_{ij}} (\beta_{ij}^{(k)})^d \quad \text{and} \quad (M'_\sigma)_{ij} = \sum_{k=1}^{k_{ij}} (\beta_{ij}^{(k)})^d \log \beta_{ij}^{(k)}.$

For Kakutani sequences generated by commensurable schemes, which are described in Section 5 and include schemes of fixed scale, Theorem 1.13 does not necessarily hold, and the limits appearing in it may not even exist; see Example 6.14.

1.6. Generation sequences of partitions. Generation sequences of partitions are generally not uniformly distributed. For example, the generation sequence generated by the $\alpha$-Kakutani multiscale substitution scheme is not uniformly distributed for any $\alpha \neq \frac{1}{2}$; see also Example 2.1.

Definition 1.14: A multiscale substitution scheme is fixed scale if there exists $\alpha \in (0, 1)$ so that

$\alpha_{ij}^{(k)} = \alpha$

for all $i, j$ and $k$. In this case $\alpha$ is called the contraction constant.
Clearly, fixed scale substitution schemes are commensurable, since for every 1 ≤ i ≤ n and every tile T of type i in $\sigma^T$, the ratio $\frac{\text{vol} T}{\text{vol} T_i}$ is an integer power of $\alpha$.

Remark 1.15: Fixed scale substitution schemes are the classical setup of substitution tilings.

Theorem 1.16: Let $\sigma$ be an irreducible fixed scale substitution scheme and let $\{\delta_k\}$ be a generation sequence of partitions of $T_i \in \tau_\sigma$. Then $\{\delta_k\}$ is uniformly distributed in $T_i$.

The counterpart of Theorem 1.13 for generation sequences generated by fixed scale schemes do not necessarily hold without an additional assumption of primitivity, as demonstrated by Corollary 6.10. In Section 6 we prove the special case concerning primitive schemes, see Theorem 6.12.

1.7. ADDITIONAL BACKGROUND AND RELATED TOPICS. Kakutani’s original proof of uniform distribution of the $\alpha$-Kakutani sequences given in [Ka], as well as Adler and Flatto’s proof given in [AF], both use ergodic and measure theoretical tools and are of a different nature than the proof presented here. Various generalizations of the Kakutani splitting procedure have been studied, and in recent years the procedure has been generalized in two directions. In [CV], Carbone and Volčič define a generalization of the splitting procedure which generates sequences of partitions of higher dimensional sets. Volčič also studies in [Vo] an extended Kakutani one-dimensional splitting procedure in which $\pi_1$ consists of any finite number of subintervals of $I$, with subsequent partitions defined by composing the same splitting rule with a dilation. In both cases the resulting sequences of partitions are shown to be uniformly distributed, and while the higher dimensional construction introduced in [CV] is different from the one presented here, the extended one-dimensional construction can be interpreted as a multiscale substitution scheme on a single prototile in $\mathbb{R}^1$. This construction is further studied in [AH], [DI] and in [In], which also contains a survey of results on uniform distribution of points and partitions.

Multiscale substitution schemes and their associated graphs were originally considered by the author in a joint work with Yaar Solomon [SyS] concerning the study of a new family of tilings of the Euclidean space. Denote by $\mathcal{X}$ the collection of closed sets in $\mathbb{R}^d$; then equipped with an appropriate topology $\mathcal{X}$ is compact. The multiscale scheme is used to define tilings of bounded sets of
growing diameter in \( \mathbb{R}^d \), which constitute a sequence of closed sets in \( \mathcal{X} \). Tilings of the entire space are defined as partial limits of such sequences, and the space of all tilings of \( \mathbb{R}^d \) constructed this way is denoted by \( \mathcal{X}_\sigma \), which is a compact subspace of \( \mathcal{X} \). Our paper contains more information about the tilings \( \tau \in \mathcal{X}_\sigma \) themselves as well as the tiling space \( \mathcal{X}_\sigma \). Examples include Sadun’s generalizations of the pinwheel tiling presented in [Sa]; see also Example 5.8 and Remark 5.9 below. In addition, the construction described in Section A.5 of [FS] is closely related to the \( \frac{1}{3} \)-Kakutani scheme given in Example 2.1 below, and is studied there in the context of fusion tilings of infinite local complexity.

This work is inspired by the theory of aperiodic tilings of Euclidean space, and more specifically by the hierarchical construction known as substitution, which is used to generate some well known aperiodic tilings, most famously the Penrose tiling. We visit in examples some well known substitutions, including the generalized pinwheel [Sa], the Rauzy fractal [Rau] and the Penrose–Robinson substitution; see [BG] for more about Penrose-type constructions, substitutions and other aperiodic tilings. This work is also related to the construction and study of self-similar fractal strings [LV] and of graph-directed fractal sprays [CKOU].

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2. Examples

It is convenient to represent substitution schemes visually as a set of partitions of the elements of \( \tau_\sigma \), where labels are visualized using colors, as is done throughout this section. Every such representation involves a choice of a substitution rule \( \varrho_\sigma \in \Sigma_\sigma \), which in general may consist of various distinct substitution rules. A substitution scheme, as well as any sequence of partitions it generates, is said to be of constant configuration if \( \Sigma_\sigma \) consists of a single element. Although such examples are relevant to the study of hierarchical structures such as the tilings studied in [SyS], in this paper constant configuration is nowhere assumed.
Nevertheless, unless otherwise stated, all illustrations of sequences of partitions below are of constant configuration, as these tend to be aesthetically pleasing.

Example 2.1 ($\alpha$-Kakutani): The $\alpha$-Kakutani sequence of partitions can be represented as a Kakutani sequence of partitions generated by a multiscale substitution scheme of constant configuration in $\mathbb{R}^1$. This $\alpha$-Kakutani scheme is defined on a single prototile $\mathcal{I}$, the unit interval, with $\omega_\alpha(\mathcal{I}) = (\alpha \mathcal{I}, (1 - \alpha)\mathcal{I})$ and $\Sigma_\alpha = \{\varrho_\alpha\}$, where $\varrho_\alpha(\mathcal{I}) = \alpha \mathcal{I} \cup (\alpha + (1 - \alpha)\mathcal{I})$. For example, the $\frac{1}{3}$-Kakutani scheme of constant configuration and substitution rule as illustrated in Figure 2.1, generates the $\frac{1}{3}$-Kakutani sequence described in Figure 1.1.

Figure 2.1. The $\frac{1}{3}$-Kakutani scheme on $\mathcal{I}$.

A $\frac{1}{3}$-Kakutani sequence of partitions of non-constant configuration generated by a similar scheme but with a second substitution rule

$$\varrho'_{1/3}(\mathcal{I}) = \frac{2}{3} \mathcal{I} \sqcup \left(\frac{2}{3} + \frac{1}{3} \mathcal{I}\right)$$

added to $\Sigma_{1/3}$, is shown in Figure 2.2. Note that indeed this sequence $\{\pi_m\}$ is not of constant configuration, because when passing from $\pi_0$ to $\pi_1$ the substitution rule $\varrho'_{1/3}$ is applied, but when passing from $\pi_1$ to $\pi_2$ the substitution is done via $\varrho_{1/3}$.

Figure 2.2. A $\frac{1}{3}$-Kakutani sequence of partitions of non-constant configuration.

A generation sequence of partition generated by the $\frac{1}{3}$-Kakutani scheme is illustrated in Figure 2.3.

Figure 2.3. A generation sequence of partitions generated by the $\frac{1}{3}$-Kakutani scheme on $\mathcal{I}$. 
Example 2.2 (Penrose–Robinson): The Penrose-type fixed scale substitution scheme, also known as the Penrose–Robinson substitution scheme, is described in Figure 2.4. The prototiles are a tall triangle $T$ and a short triangle $S$ in $\mathbb{R}^2$, and the contraction constant is $\alpha = \frac{1}{\varphi}$, where $\varphi$ is the golden ratio. This substitution scheme is well known for generating aperiodic tilings of the Euclidean plane, and has been studied extensively; see Chapter 6 in [BG] and references within.

![Figure 2.4. The Penrose–Robinson fixed scale substitution scheme.]

Figure 2.4. The Penrose–Robinson fixed scale substitution scheme.

Figure 2.5 illustrates the first few partitions in a Kakutani sequence $\{\pi_m\}$ of $T$.

![Figure 2.5. A Kakutani sequence of partitions generated by the Penrose–Robinson scheme.]

The first few partitions in a generation sequence of partitions $\{\delta_k\}$ of $T$ are shown in Figure 2.6.

![Figure 2.6. A generation sequence of partitions generated by the Penrose–Robinson scheme.]

By Theorem 1.16 the sequence \(\{\delta_k\}\) is uniformly distributed in \(\mathcal{T}\). The two frequencies of types, as defined in Theorem 1.13 for the incommensurable case, can be also calculated using the formulas given in Theorem 6.12:

\[
\lim_{k \to \infty} \frac{|\{\text{Short triangles } \in \delta_k\}|}{|\{\text{Tiles } \in \delta_k\}|} = \frac{1}{\varphi + 1}
\]

\[
\lim_{k \to \infty} \frac{\text{vol}(\bigcup \{\text{Short triangles } \in \delta_k\})}{\text{vol}(\bigcup \{\text{Tiles } \in \delta_k\})} = \frac{1}{\varphi + 2}.
\]

Further details are given in Example 6.14, where it is shown that in the case of the the Kakutani sequence \(\{\pi_m\}\) illustrated in Figure 2.5, the corresponding limits do not exist.

Remark 2.3: We note that the fact that the generation sequence of partitions \(\{\delta_k\}\) in Figure 2.6 is a subsequence of the Kakutani sequence of partitions \(\{\pi_m\}\) in Figure 2.5 is coincidental. In general this is not the case, as can be easily demonstrated by examples in which \(\tau_\sigma\) contains two prototiles \(T_i\) and \(T_j\) for which \(\text{vol}(\alpha T_i) > \text{vol} T_j\), where \(\alpha\) is the contraction constant.

Example 2.4 (The rectangle and the square): The construction shown in Figure 2.7 is an example of a multiscale substitution scheme \(\sigma\) in \(\mathbb{R}^2\) with two substitution rules \(\Sigma_\sigma = \{\varrho_\sigma^{(1)}, \varrho_\sigma^{(2)}\}\), where \(\tau_\sigma = (R, S)\) and \(\omega_\sigma\) are as illustrated in Figure 1.2.

![Figure 2.7. Two distinct substitution rules \(\varrho_\sigma^{(1)}\) and \(\varrho_\sigma^{(2)}\) in \(\Sigma_\sigma\).](image)

The substitution tiles are

\[
\omega_\sigma(R) = \left(\frac{1}{2}R, \frac{1}{2}S, \frac{1}{2}S\right) \quad \text{and} \quad \omega_\sigma(S) = \left(\frac{1}{3}R, \frac{1}{3}R, \frac{1}{3}S, \frac{2}{3}S\right),
\]
and the scheme is clearly incommensurable because $\frac{1}{3} S, \frac{2}{3} S \in \omega_\sigma(S)$. The constants of substitution are

$$
\beta_{11}^{(1)} = \frac{1}{2}, \quad \beta_{12}^{(1)} = \frac{1}{\sqrt{2}}, \quad \beta_{12}^{(2)} = \beta_{12}^{(3)} = \frac{1}{2\sqrt{2}}, \\
\beta_{21}^{(1)} = \beta_{21}^{(2)} = \frac{\sqrt{2}}{3}; \quad \beta_{22}^{(1)} = \frac{1}{3}, \quad \beta_{22}^{(2)} = \frac{2}{3}.
$$

The first few elements of the constant configuration Kakutani sequence of partitions of the rectangle $\mathcal{R}$, with $\sigma = (\tau_\sigma, \omega_\sigma, \varrho_\sigma^{(1)})$ and $\varrho^{(1)}_\sigma$ as in the left-hand side of Figure 2.7, are illustrated in Figure 2.8.

Figure 2.8. A Kakutani sequence of partitions of the rectangle $\mathcal{R}$.

The matrices $M_\sigma$ and $Q_\sigma$ defined in Theorem 1.13 are given by

$$
M_\sigma = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} \\
\frac{4}{9} & \frac{5}{9}
\end{pmatrix}, \\
Q_\sigma = \frac{\text{adj}(I - M_\sigma)}{-\text{tr}(\text{adj}(I - M_\sigma) \cdot M'_\sigma)} = \frac{1}{3} \log 3 - \frac{1}{9} \log 2 \left( \frac{4}{9} \frac{3}{4} \frac{5}{9} \right).
$$

In any Kakutani sequence of partitions generated by this scheme, either of the rectangle $\mathcal{R}$ or of the square $S$, we get

$$
\lim_{m \to \infty} \frac{|\{\text{Squares} \in \pi_m\}|}{|\{\text{Tiles} \in \pi_m\}|} = \frac{25}{43}, \\
\lim_{m \to \infty} \frac{\text{vol}(\bigcup \{\text{Squares} \in \pi_m\})}{\text{vol}(\bigcup \{\text{Tiles} \in \pi_m\})} = \frac{5}{12} \log 3 - \frac{1}{18} \log 2 \left( \frac{4}{9} \frac{3}{4} \frac{5}{9} \right).$$

Thus for large $m$ roughly 58% of the tiles in $\pi_m$ are squares, covering approximately 56% of the area.
3. Preliminaries and reduction to counting

3.1. Constants of substitution and equivalence of multiscale substitution schemes.

**Lemma 3.1:** Let \( \sigma \) be a multiscale substitution scheme and let \( \alpha_{ij}^{(k)} \) and \( \beta_{ij}^{(k)} \) be the constants appearing in Definition 1.3. Then

1. \( \text{vol}(\alpha_{ij}^{(k)} T_j) = \text{vol}(\beta_{ij}^{(k)} T_i) \),
2. \( \sum_{j=1}^n \sum_{k=1}^{k_{ij}} (\beta_{ij}^{(k)})^d = 1 \).

**Proof.** This follows from Equation (1.1) and the fact that the volume of \( T_i \) is equal to the sum of volumes of the tiles in \( \omega_\sigma(T_i) \).

**Definition 3.2:** A multiscale substitution scheme \( \sigma \) is called **normalized** if all prototiles in \( \tau_\sigma \) are of volume 1. In a normalized scheme

\[ \alpha_{ij}^{(k)} = \beta_{ij}^{(k)} \]

for all \( i, j \) and \( k \). Two multiscale substitution schemes \( \sigma_1 \) and \( \sigma_2 \) are **equivalent** if \( \tau_{\sigma_1} \) and \( \tau_{\sigma_2} \) consist of the same tiles up to scale changes, that is if

\[ \tau_{\sigma_1} = (T_1, \ldots, T_n) \iff \tau_{\sigma_2} = (\lambda_1 T_1, \ldots, \lambda_n T_n) \]

and the tilings of the prototiles given by the substitution rules are the same up to a change of scales defined accordingly. Every equivalence class includes a single normalized scheme.

For example, the non-normalized substitution scheme illustrated in Figure 1.2 is equivalent to the normalized one described in Figure 4.2. Clearly the constants \( \alpha_{ij}^{(k)} \) may vary between equivalent schemes, as they depend on the volumes of the prototiles and their ratios, but sequences of partitions generated by equivalent schemes are identical up to a uniform rescaling of all tiles in all partitions. The next lemma follows from a straightforward computation using Equation (1.1).

**Lemma 3.3:** Equivalent multiscale substitution schemes have the same constants of substitution \( \beta_{ij}^{(k)} \).

The assumption that the constants of substitution \( \beta_{ij}^{(k)} \) are a finite set of constants which are all strictly smaller than 1 yields the following statement.
Lemma 3.4: Let $\sigma$ be a multiscale substitution scheme and let $\{\gamma_m\}$ be either a Kakutani or a generation sequence of partitions of $T_i \in \tau_\sigma$. Then for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that all tiles in $\gamma_m$ are of diameter less than $\varepsilon$ for all $m \geq m_0$.

3.2. Counting and uniform distribution. A key step in the proof of Theorem 1.9 is the following Lemma. It implies that in order to prove uniform distribution of sequences of partitions generated by multiscale substitution schemes, it is enough to apply a counting argument.

Definition 3.5: Let $\sigma$ be a multiscale substitution scheme and let $\{\gamma_m\}$ be a sequence of partitions of $T_i \in \tau_\sigma$ generated by $\sigma$. Denote by

$$T \in \mathcal{T}_i^\sigma(\{\gamma_m\}) = \{T' \in \gamma_m : m \in \mathbb{N}\}$$

the set of all tiles that appear in elements of the sequence $\{\gamma_m\}$.

Note that tiles of $\mathcal{T}_i^\sigma(\{\gamma_m\})$ are actual subsets of $T_i$, while those of $O_i^\sigma$ from Definition 1.4 are the substitution tiles copies of which appear in elements of $\{\gamma_m\}$. Clearly, if $\{\gamma_m\}$ is a Kakutani or a generation sequence then there is a one-to-one correspondence between $\mathcal{T}_i^\sigma(\{\gamma_m\})$ and $O_i^\sigma$. We write $\mathcal{T}_i^\sigma = \mathcal{T}_i^\sigma(\{\gamma_m\})$ if the sequence of partitions $\{\gamma_m\}$ it refers to is clear from the context.

Lemma 3.6: Let $\sigma$ be a multiscale substitution scheme and let $\{\gamma_m\}$ be a sequence of partitions of $T_i \in \tau_\sigma$ generated by $\sigma$. Assume that for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ so that all tiles in $\gamma_m$ are of diameter less than $\varepsilon$ for any $m \geq m_0$. If there exists a marking sequence $\{x_m\}$ of $\{\gamma_m\}$ so that

$$\lim_{m \to \infty} \frac{|x_m \cap T|}{|x_m|} = \frac{\text{vol} T}{\text{vol} T_i}$$

holds for any tile $T \in \mathcal{T}_i^\sigma(\{\gamma_m\})$, then $\{\gamma_m\}$ is uniformly distributed in $T_i$.

Proof. Assume for simplicity $\text{vol} T_i = 1$. We want to show that

$$\mu_m := \frac{1}{|x_m|} \sum_{x \in x_m} \delta_x \rightarrow \text{vol}$$

in the weak-* topology, where $\text{vol}$ is the Lebesgue measure in $\mathbb{R}^d$, restricted to $T_i$. 


Let $\mu$ be a partial limit of $\mu_m$. It is enough to show that $\mu(R) = \text{vol} R$ for any axis-parallel box $R \subset T_i$ in order to prove $\mu = \text{vol}$. By assumption, for any $T \in \mathcal{T}^\sigma_i$

$$\mu(T) = \text{vol} T.$$ 

Let $\varepsilon > 0$. Since $\mathcal{T}^\sigma_i$ contains tiles of arbitrarily small diameter, there exist subsets $I_\varepsilon, C_\varepsilon \subset \mathcal{T}^\sigma_i$ such that

$$\bigcup_{T \in I_\varepsilon} T \subset R \subset \bigcup_{T \in C_\varepsilon} T$$

and

$$\text{vol} R - \varepsilon < \sum_{T \in I_\varepsilon} \text{vol} T \leq \text{vol} R \leq \sum_{T \in C_\varepsilon} \text{vol} T < \text{vol} R + \varepsilon.$$ 

It is enough to prove that for all $T \in \mathcal{T}^\sigma_i$

$$\mu(\partial T) = 0,$$ 

(3.1) 

since then

$$\text{vol} R - \varepsilon < \sum_{T \in I_\varepsilon} \text{vol} T = \sum_{T \in I_\varepsilon} \mu(t) = \mu\left(\bigcup_{T \in I_\varepsilon} T\right) \leq \mu(R)$$

and

$$\mu(R) \leq \mu\left(\bigcup_{T \in C_\varepsilon} T\right) = \sum_{T \in C_\varepsilon} \mu(t) = \sum_{T \in C_\varepsilon} \text{vol} T < \text{vol} R + \varepsilon.$$ 

This holds for any $\varepsilon > 0$ and the lemma follows.

We now prove Equation (3.1) for all $T \in \mathcal{T}^\sigma_i$. Let $T \in \mathcal{T}^\sigma_i$ and let $\eta > 0$. Since $\text{vol} \partial T = 0$, there exists a countable sequence of cubes $Q_n$ such that

$$\partial T \subset \bigcup Q_n$$

and

$$\sum_n \text{vol} Q_n < \eta.$$ 

For any $\varepsilon_n > 0$ there exists a covering of $Q_n$ by a countable set of tiles $T_n \subset \mathcal{T}^\sigma_i$ such that

$$\mu(Q_n) \leq \mu\left(\bigcup_{T \in T_n} T\right) \leq \sum_{T \in T_n} \mu(t) = \sum_{T \in T_n} \text{vol} T \leq \text{vol} Q_n + \varepsilon_n.$$
Choose $\varepsilon_n = \text{vol} Q_n$. The boundary $\partial T$ is thus covered by a countable union of tiles in $\mathcal{J}_i^\sigma$, and

$$\mu(\partial T) \leq \mu \left( \bigcup_{n} \bigcup_{T \in T_n} T \right) \leq \sum_{n} \sum_{T \in T_n} \mu(t) \leq \sum_{n} (\text{vol} Q_n + \varepsilon_n) = 2 \sum_{n} \text{vol} Q_n < 2\eta.$$ 

This holds for any $\eta > 0$, finishing the proof of Equation (3.1) for all $T \in \mathcal{J}_i^\sigma$, thus proving the lemma.

Remark 3.7: A counterpart of Lemma 3.6 appears as Lemma 2.5 in [CV].

4. Graphs associated with multiscale substitution schemes

A key element in our study of Kakutani sequences of partitions is the directed weighted graph, which we regard as a geometric object, not only combinatorial. As in [KSS], denote by $G = (\mathcal{V}, \mathcal{E}, l)$ a directed weighted metric multigraph with a set of vertices $\mathcal{V}$ and a set of weighted edges $\mathcal{E}$, with positive weights which are regarded as lengths. A path in $G$ is a directed walk on the edges of $G$ that originates and terminates at vertices of $G$. More generally, a metric path in $G$ is a directed walk on edges of $G$, which does not necessarily originate or terminate at vertices of $G$. An edge of weight $a$ is equipped with a parameterization by the interval $[0, a]$, and the parameterization is used to define the path metric $l$ on edges, paths and metric paths in $G$. We assume throughout that as a subset of the metric graph, an edge contains its terminal vertex but not its initial one.

4.1. Graphs associated with multiscale substitution schemes.

Definition 4.1: Let $\sigma$ be a multiscale substitution scheme. The graph associated with $\sigma$ is a directed weighted graph $G_\sigma$, the vertices of which model the prototiles in $\tau_\sigma$, and the weighted edges model the substitution tiles in $\omega_\sigma$ and their scales. More precisely, $G_\sigma$ has a set of vertices $\mathcal{V}_\sigma = \{1, \ldots, n\}$, where the prototile $T_i \in \tau_\sigma$ is associated with the vertex $i \in \mathcal{V}_\sigma$, and every substitution tile in $\omega_\sigma(T_i)$ is associated with a distinct edge in $\mathcal{E}_\sigma$ with initial vertex $i$. In addition, if $\varepsilon \in \mathcal{E}_\sigma$ is an edge in $G_\sigma$ associated with the substitution tile $\alpha T_j \in \omega_\sigma(T_i)$, then $\varepsilon$ terminates at vertex $j$ and is of length

$$l(\varepsilon) = \log \frac{1}{\alpha} = \log \frac{1}{\beta} + \frac{1}{d} (\log \text{vol} T_j - \log \text{vol} T_i),$$

where $\beta$ is the corresponding constant of substitution, as defined in Definition 1.3.
Note that \( G_\sigma \) does not depend on the substitution rules in \( \Sigma_\sigma \), but only on the prototiles \( \tau_\sigma \) and the substitution tiles \( \omega_\sigma \). A priori, since in the definition of the multiscale substitution scheme we do not assume \( \alpha < 1 \), under the definition above it may be the case that an edge in the associated graph carries a non-positive weight. For example, in the non-normalized scheme defined in Figure 2.7, the edge associated with \( S \in \omega_\sigma(\mathcal{R}) \) is of “length” \( 0 = \log 1 \). However, since we assume \( 0 < \beta < 1 \) for all constants of substitution, all edges in graphs associated with normalized substitution schemes carry positive weights, and these are the graphs involved in the proof of the main results of this paper.

**Example 4.2:** The graph associated with the \( \alpha \)-Kakutani scheme is illustrated in Figure 4.1. It consists of a single vertex corresponding to the single prototile \( I \), and two loops of lengths \( \log \frac{1}{\alpha} \) and \( \log \frac{1}{1-\alpha} \).

![Figure 4.1](image1.png)

**Figure 4.1.** The graph associated with the \( \alpha \)-Kakutani scheme on \( I \).

**Example 4.3:** The graph associated with a normalized multiscale substitution scheme, equivalent to the scheme represented in Figure 1.2, is illustrated in Figure 4.2. The left vertex is associated with the rectangle \( \mathcal{R} \) and every one of its outgoing edges is associated with a distinct tile of matching color in the substitution rule on \( \mathcal{R} \), where multiple arrow heads represent multiple distinct edges of the same length, origin, termination and direction. Similarly, the right vertex and its outgoing edges are associated with the square \( S \) and the tiles which appear in its substitution rule.

![Figure 4.2](image2.png)

**Figure 4.2.** The graph associated with a normalized scheme on \( \mathcal{R} \) and \( S \).
The next lemma describes the relation between associated graphs of equivalent schemes, and follows from the definition of the associated graph.

**Lemma 4.4:** Let \( \sigma_1 \) and \( \sigma_2 \) be two equivalent multiscale substitution schemes with \( \tau_{\sigma_1} = (T_1, \ldots, T_n) \), and \( \tau_{\sigma_2} = (T_1, \ldots, \alpha T_j, \ldots, T_n) \), with \( \alpha > 0 \), and let \( G_{\sigma_1} \) and \( G_{\sigma_2} \) be the associated graphs.

1. The graphs \( G_{\sigma_1} \) and \( G_{\sigma_2} \) have the same sets of vertices and edges.
2. The weights of the edges are not changed, except for edges \( \varepsilon \) with terminal vertex \( j \), for which
   \[ l_{G_{\sigma_2}}(\varepsilon) = l_{G_{\sigma_1}}(\varepsilon) + \log \alpha, \]
   and edges \( \varepsilon \) with initial vertex \( j \), for which
   \[ l_{G_{\sigma_2}}(\varepsilon) = l_{G_{\sigma_1}}(\varepsilon) - \log \alpha. \]

In view of Lemma 4.4, in order to define the graph \( G_{\sigma_2} \), one simply “slides” the vertex \( j \) in \( G_{\sigma_1} \) along the edges, in a way that does not change the length of any closed path in the associated graph. See also Example 7.3 and Figures 7.1, 7.2 and 7.3 within.

### 4.2. Paths in \( G_{\sigma} \) and Tiles in Partitions

Let \( \sigma \) be a multiscale substitution scheme and let \( G_{\sigma} \) be the associated graph. There is a natural one-to-one correspondence between tiles of type \( j \) in \( \mathcal{O}_i^\sigma \) and finite paths in \( G_{\sigma} \) with initial vertex \( i \in \mathcal{V}_\sigma \) and terminal vertex \( j \in \mathcal{V}_\sigma \), where a tile of generation \( m \) corresponds to a path consisting of \( m \) edges. Denote by \( \gamma_T \) the unique path in \( G_{\sigma} \) that corresponds to the tile \( T \in \mathcal{O}_i^\sigma \).

A partition of \( T_i \) that is an element either of a Kakutani or of a generation sequence of partitions generated by the scheme, corresponds to a finite collection of paths with initial vertex \( i \in \mathcal{V}_\sigma \). Every infinite path in \( G_{\sigma} \) with initial vertex \( i \) is the continuation of a unique finite path in this finite collection.

**Lemma 4.5:** Let \( T \in \mathcal{O}_i^\sigma \) be a tile of type \( j \), and assume \( T \) corresponds to the path \( \gamma_T \) in \( G_{\sigma} \). Then
\[
\text{vol} \, T = e^{-l(\gamma_T)d} \cdot \text{vol} \, T_j.
\]

**Proof.** This follows directly from the definition of \( G_{\sigma} \). 

**Corollary 4.6:** Graphs associated with equivalent multiscale substitution schemes have the same set of lengths of closed paths.
Proof. If $\gamma$ is a closed path in $G_{\sigma}$, that is if $i = j$, then $l(\gamma)$ does not depend on the volumes of the tiles in $\tau_{\sigma}$, but only on the constants of substitution $\beta_{ij}^{(k)}$. The corollary now follows from Lemma 3.3

**Corollary 4.7:** Let $G_{\sigma}$ be the graph associated with a normalized multiscale substitution scheme $\sigma$. Let $T_1, T_2 \in \mathcal{O}_i{\sigma}$ and let $\gamma_{T_1}$ and $\gamma_{T_2}$ be the corresponding paths in $G_{\sigma}$. Then

$$\text{vol} T_1 < \text{vol} T_2 \quad \text{if and only if} \quad l(\gamma_{T_1}) > l(\gamma_{T_2}).$$

**Proof.** This is clear from Lemma 4.5.

It follows that the set of lengths of paths in a graph associated with a normalized scheme may be used as indices of a Kakutani sequences of partitions, in the following sense:

**Lemma 4.8:** Let $\sigma$ be a multiscale substitution scheme, let $G_{\sigma}$ be the graph associated with an equivalent normalized scheme, and let $\{\pi_m\}$ be a Kakutani sequence of partitions of $T_i \in \tau_{\sigma}$. Let $\{l_m\}$ be the increasing sequence of lengths of paths in $G_{\sigma}$ that originate at $i \in V_{\sigma}$. Let $\varepsilon \in \mathcal{E}_{\sigma}$ be an edge associated with an element $\alpha T_j \in \omega_{\sigma}(T_h)$ for some $T_j, T_h \in \tau_{\sigma}$. There is a one-to-one correspondence between tiles of type $j$ in the partition $\pi_m$ that appear as a result of a substitution of a tile of type $h$ and are associated with the substitution tile $\alpha T_j \in \omega_{\sigma}(T_h)$, and metric paths of length $l_m$ that originate at $i \in V_{\sigma}$ and terminate at a point on the edge $\varepsilon$.

**Proof.** Let $T \in \pi_m$ be a tile associated with the substitution tile $\alpha T_j \in \omega_{\sigma}(T_h)$ as in the lemma. Then $T$ corresponds to a path $\gamma_T$ with initial vertex $i$, terminal vertex $j$ and final edge $\varepsilon$. If $T$ is of maximal volume in $\pi_m$, then $l(\gamma_T) = l_m$ and $\gamma_T$ is also the metric path corresponding to $T$ as an element of $\pi_m$. Otherwise $l(\gamma_T) > l_m$, and the unique metric path of length exactly $l_m$ that is defined by truncating $\gamma_T$ terminates at a point on $\varepsilon$, and is the metric path corresponding to $T$ as an element of $\pi_m$.

Consider a metric path with initial vertex $i$ and length $l_m$ that terminates at a point on $\varepsilon$. Then it can be uniquely extended to a path $\gamma$ in $G_{\sigma}$ with final edge $\varepsilon$. Let $T \in \mathcal{F}_i{\sigma}$ be the tile corresponding to $\gamma$. Then there exist $m_0 \leq m \leq m_1$ such that $T$ first appears in partition $\pi_{m_0}$ and is of maximal volume among tiles in partition $\pi_{m_1}$. It follows that $T \in \pi_m$, and $T$ is the tile corresponding to the metric path at hand.
Example 4.9: Consider the $\frac{1}{3}$-Kakutani multiscale substitution scheme and its associated graph, which consists of a single vertex and two loops, as illustrated in Figure 4.1. The two loops are of lengths $\log 3$ and $\log \frac{3}{2}$, and so all paths are of lengths $a \log \frac{3}{2} + b \log 3$ for integers $a, b \geq 0$. The first few elements of the sequence $\{l_m\}$ are therefore $l_0 = 0, l_1 = \log \frac{3}{2}, l_2 = 2 \log \frac{3}{2}$ and $l_3 = \log 3$.

Figure 4.3 illustrates all metric paths of lengths $l_0, l_1$ and $l_2$ together with their corresponding intervals in $\pi_0, \pi_1$ and $\pi_2$. As can be clearly seen from the illustration, the interval of maximal length in the next partition $\pi_3$ would be the interval $[0, \frac{1}{3}]$, which corresponds to the loop of length $l_3 = \log 3$.

![Figure 4.3. Elements of the $\frac{1}{3}$-Kakutani sequence and the corresponding metric paths.](image)

4.3. The graph matrix function of graphs associated with multiscale substitutions.

Definition 4.10: Let $G$ be a directed weighted graph with a set of vertices $\mathcal{V} = \{1, \ldots, n\}$. Let $i, j \in \mathcal{V}$ be a pair of vertices in $G$, and assume that there are $k_{ij} \geq 0$ edges $\varepsilon_1, \ldots, \varepsilon_{k_{ij}}$ with initial vertex $i$ and terminal vertex $j$. The graph matrix function of $G$ is the matrix valued function $M : \mathbb{C} \to M_n(\mathbb{C})$ defined by

$$M_{ij}(s) = e^{-s \cdot l(\varepsilon_1)} + \cdots + e^{-s \cdot l(\varepsilon_{k_{ij}})}.$$  

If $i$ is not connected to $j$ by an edge, put $M_{ij}(s) = 0$. Note that the restriction of $M$ to $\mathbb{R}$ is real valued.
Lemma 4.11: Let $\sigma$ be a multiscale substitution scheme in $\mathbb{R}^d$ and let $G_\sigma$ be the associated graph. Then $M(d)$ is a non-negative real valued matrix with a positive right eigenvector

$$v_{\text{vol}} = (\text{vol } T_1, \ldots, \text{vol } T_n)^T \in \mathbb{R}^n$$

and associated eigenvalue $\mu = 1$.

Proof. The non-zero entries of the graph matrix function of $G_\sigma$ are given by

$$M_{ij}(s) = \sum_{k=1}^{k_{ij}} (\alpha_{ij}^{(k)})^s.$$ 

By Lemma 3.1 we have

$$(M(d) \cdot v_{\text{vol}})_i = \sum_{j=1}^{n} M_{ij}(d) \text{vol } T_j$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{k_{ij}} \frac{\text{vol } T_i}{\text{vol } T_j} (\beta_{ij}^{(k)})^d \text{vol } T_j$$

$$= \text{vol } T_i,$$

and so $M(d) \cdot v_{\text{vol}} = 1 \cdot v_{\text{vol}}$. 

4.4. Irreducibility of multiscale substitution schemes.

Definition 4.12: A graph $G$ is called strongly connected if for every pair of vertices $i, j \in \mathcal{V}$ there exists a path in $G$ with initial vertex $i$ and terminal vertex $j$. A non-negative real valued matrix $A \in M_n(\mathbb{R})$, that is, a matrix with non-negative entries, is called irreducible if for every pair of indices $i, j$ there exists $k \in \mathbb{N}$ for which $(A^k)_{ij} > 0$.

Recall the definition of irreducible multiscale substitution schemes given in Definition 1.4.

Lemma 4.13: A multiscale substitution scheme is irreducible if and only if the associated graph $G_\sigma$ is strongly connected if and only if $M(d)$ is an irreducible matrix.

Proof. This follows directly from the definitions of $G_\sigma$ and $M(d)$. 

The following result concerns irreducible matrices and is due to Perron and Frobenius (full statements and proofs can be found in Chapter XIII of [Ga]).

**Theorem (Perron–Frobenius theorem for irreducible matrices):** Let \( A \in M_n(\mathbb{R}) \) be a non-negative irreducible matrix.

1. There exists \( \mu > 0 \) which is a simple eigenvalue of \( A \), and \( |\mu_j| \leq \mu \) for any other eigenvalue \( \mu_j \).
2. There exists \( v \in \mathbb{R}^n \) with positive entries such that \( Av = \mu v \). Moreover, every right eigenvector with non-negative entries is a positive multiple of \( v \).

**Definition 4.14:** The eigenvalue \( \mu \) is called the **Perron–Frobenius eigenvalue**, and an associated positive right eigenvector is called a **right Perron–Frobenius eigenvector**.

**Corollary 4.15:** Let \( M \) be the graph matrix function of a graph associated with an irreducible multiscale substitution scheme in \( \mathbb{R}^d \). The Perron–Frobenius eigenvalue of \( M(d) \) is \( \mu = 1 \) and \( v_{\text{vol}} \) is a right Perron–Frobenius eigenvector of \( M(d) \).

**Proof.** This is a straightforward application of the Perron–Frobenius theorem for irreducible matrices, combined with Lemma 4.11. ■

### 5. Incommensurable and commensurable schemes

**Definition 5.1:** A strongly connected graph \( G \) is **incommensurable** if there exist two closed paths \( \gamma_1, \gamma_2 \) in \( G \) which are of incommensurable lengths, that is

\[
\frac{l(\gamma_1)}{l(\gamma_2)} \notin \mathbb{Q}.
\]

Otherwise, the graph is called **commensurable**.

**Lemma 5.2:** Let \( \sigma \) be a multiscale substitution scheme. Then \( \sigma \) is incommensurable if and only if the associated graph \( G_\sigma \) is incommensurable. In addition, the incommensurability of a scheme depends only on its equivalence class.

**Proof.** By definition, there is a one-to-one correspondence between tiles in \( \Theta_i^{\sigma} \) of type \( i \), and closed paths in \( G_\sigma \) with initial and terminal vertex \( i \in V_\sigma \). If \( T \)
is such a tile, then by Lemma 4.5 the associated path $\gamma_T$ is of length
\[
l(\gamma_T) = -\frac{1}{d} \log \frac{\text{vol } T}{\text{vol } T_j}.
\]
The equivalence is now clear from a comparison of Definitions 1.11 and 5.1 for incommensurable schemes and graphs, respectively. The second statement is immediate from Corollary 4.6, which states that graphs associated with equivalent schemes have the same set of lengths of closed paths.

**Lemma 5.3:** Let $G$ be a strongly connected directed weighted graph. Then $G$ is incommensurable if and only if the set of lengths of all closed paths in $G$ is not a uniformly discrete subset of $\mathbb{R}$.

**Proof.** The proof appears in the introduction of [KSS]; it is included here for the sake of completeness. First, assume $G$ is incommensurable. By Dirichlet’s approximation theorem, for every $\varepsilon > 0$ there exist $p, q \in \mathbb{N}$ such that
\[
|l(\gamma_1)q - l(\gamma_2)p| < \varepsilon,
\]
and so the set of lengths of closed paths in $G$ is not uniformly discrete. Conversely, if the set of lengths of closed paths is rationally dependent, then the finiteness of the graph implies that there is a finite set $L$ of lengths for which the length of any closed path in $G$ is a linear combination with integer coefficients of elements in $L$. It follows that the set of lengths of closed paths in $G$ is uniformly discrete.

**5.1. Some examples.**

**Example 5.4:** As we have seen, the Penrose–Robinson scheme illustrated in Figure 2.4, as well as any other fixed scale scheme, is commensurable. If the contracting constant is $\alpha$, then all edges of the associated graph are of equal length $\log \frac{1}{\alpha}$, and so the associated graph is clearly commensurable.

**Example 5.5:** The graph $G_\sigma$ associated with the multiscale substitution scheme on the rectangle $R$ and the square $S$, both illustrated in Figure 4.2, is incommensurable. The graph $G_\sigma$ contains two loops of lengths $\log 3$ and $\log 2$ associated with the substitution tiles $\frac{1}{3}S, \frac{2}{3}S \in \omega_\sigma(S)$, respectively. Since $\frac{\log 2}{\log 3} \notin \mathbb{Q}$, the associated graph $G_\sigma$ is incommensurable.
Example 5.6: The Rauzy fractal introduced by Rauzy in [Rau] is an example of a set with non-polygonal boundary that admits a multiscale substitution scheme, as illustrated in Figure 5.1. The substitution tiles are

$$\omega_\sigma(\mathcal{R}) = \left( \frac{1}{\tau} \mathcal{R}, \frac{1}{\tau^2} \mathcal{R}, \frac{1}{\tau^3} \mathcal{R} \right)$$

where $\tau$ is the tribonacci constant satisfying

$$\tau^3 + \tau^2 + \tau = 1.$$

The associated graph has a single vertex corresponding to the single prototile $\mathcal{R}$, and three loops of lengths $\log \tau, 2 \log \tau$ and $3 \log \tau$, and so the scheme is not fixed scale but is nevertheless commensurable.

![Figure 5.1. The commensurable scheme on the Rauzy fractal $\mathcal{R}$, courtesy of [Wi].](image)

Example 5.7: The $\alpha$-Kakutani scheme on the unit interval $I$ is incommensurable for all but countably many values of $\alpha \in (0, 1)$. As illustrated in Figure 4.1, the associated graph consists of a single vertex and two loops of lengths $\log \frac{1}{\alpha}$ and $\log \frac{1}{1-\alpha}$, and so the $\alpha$-Kakutani scheme is commensurable if and only if

$$\frac{\log \alpha}{\log(1 - \alpha)} \in \mathbb{Q}.$$  

Indeed as we have seen, the $\frac{1}{3}$-Kakutani scheme is incommensurable, while the $\frac{1}{\varphi}$-Kakutani scheme is commensurable, where $\varphi$ is the golden ratio.
Example 5.8: The generalized pinwheel construction introduced by Sadun and used in [Sa] to define and study tilings of the Euclidean plane, can be viewed as a one parameter family of multiscale substitution schemes. Each scheme is defined on a single prototile which is a right triangle with an angle \( \theta \), and the substitution rule defines a decomposition of the triangle into five rescaled and rotated similar triangles, four of which of the same volume, as illustrated in Figure 5.2. The generalized pinwheel scheme is incommensurable for all but countably many values of \( \theta \in (0, \frac{\pi}{2}) \). Indeed, the two constants of substitution depend on \( \theta \) and the scheme is commensurable if and only if

\[
\frac{\log \sin \theta}{\log \cos \frac{\theta}{2}} \in \mathbb{Q}.
\]

Remark 5.9: The tilings of the plane studied in [Sa] are generated by a succession of applications of the substitution rule on triangles, each followed by a suitable rescaling and repositioning. The substitution rule is always applied to triangles of maximal volume, as is done in the Kakutani splitting procedure. This defines a sequence of nested partitions of finite regions, the union of which is a tiling of the entire plane. The fixed scale scheme defined by putting \( \theta = \arctan \frac{1}{2} \) generates the Conway and Radin’s classical pinwheel tiling; see [Rad]. Measures associated with sequences of partitions generated by this construction are studied in [Ol], where uniform distribution for the corresponding Kakutani sequences is established.
6. Generation sequences generated by fixed scale substitution schemes

6.1. The substitution matrix and the associated graph.

**Definition 6.1:** Let $\sigma$ be a fixed scale scheme in $\mathbb{R}^d$ with contraction constant $\alpha$. The **substitution matrix** of $\sigma$ is an integer valued matrix $S_{\sigma}$ with entries

$$(S_{\sigma})_{ij} = k_{ij} = \# \{ \text{copies of } \alpha T_j \text{ in } \omega_{\sigma}(T_i) \}.$$

All edges of the graph $G_{\sigma}$ associated with a fixed scale substitution scheme with contraction constant $\alpha$ are of length $\log \frac{1}{\alpha}$. The entries of the graph matrix function as defined in Section 4 are thus given by

$$M_{ij}(s) = k_{ij} \alpha^s,$$

and so $S_{\sigma}$, which is also the adjacency matrix of $G_{\sigma}$, can be written as

$$S_{\sigma} = \frac{1}{\alpha^d} M(d).$$

By Lemma 4.11, $\mu = 1$ is a Perron–Frobenius eigenvalue and $v_{\text{vol}}$ is a right Perron–Frobenius eigenvector of $M(d)$, and so

$$S_{\sigma} \cdot v_{\text{vol}} = \frac{1}{\alpha^d} \cdot v_{\text{vol}}.$$

Therefore $\mu = \frac{1}{\alpha^d}$ is a Perron–Frobenius eigenvalue and $v_{\text{vol}}$ is a right Perron–Frobenius eigenvector of $S_{\sigma}$.

**Example 6.2:** The substitution matrix of the Penrose–Robinson fixed scale substitution scheme on a tall triangle $T$ and a short triangle $S$ in $\mathbb{R}^2$, as described in Figure 2.4, is given by

$$S_{\sigma} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$ 

The Perron–Frobenius eigenvalue of $S_{\sigma}$ is $\mu = \varphi^2$, and a right Perron–Frobenius eigenvector can be chosen to be $v = (\varphi, 1)$, where $\varphi$ is the golden ratio. Indeed, the contracting constant of the scheme is $\alpha = \frac{1}{\varphi}$, and the scale of the prototiles can be chosen so that

$$\text{vol } T = \varphi \quad \text{and} \quad \text{vol } S = 1,$$

and so $v = v_{\text{vol}}$. 
Lemma 6.3: Let $\sigma$ be a fixed scale scheme, and let $\{\delta_k\}$ be a generation sequence of partitions of $T_i \in \tau_\sigma$. Then

$$(S^k_\sigma)_{ij} = |\{\text{Tiles of type } j \text{ and generation } k \text{ in a tiling of } T_i\}|.$$ 

Proof. This is a standard result about adjacency matrices of graphs, and is proved by induction on $k$.  

6.2. Primitive fixed scale substitution schemes.

Definition 6.4: An irreducible real valued matrix $A \in M_n(\mathbb{R})$ is called primitive if there exists $k \in \mathbb{N}$ for which $A^k$ is a matrix with all entries strictly positive, and non-primitive otherwise. An irreducible fixed scale substitution scheme is called primitive if the associated substitution matrix is primitive, and non-primitive otherwise.

Remark 6.5: In the case of a fixed scale substitution scheme $\sigma$ with contacting constant $\alpha$, the substitution matrix $S_\sigma$ is primitive if and only if $G_\sigma$ is aperiodic, which in this case means that the greatest common divisor of the set of lengths of all closed paths is $\log \frac{1}{\alpha}$.

The following is the Perron–Frobenius theorem for primitive matrices, which gives a stronger result than its counterpart for general irreducible matrices (see Chapter XIII of [Ga] for more details).

Theorem (Perron–Frobenius theorem for primitive matrices): Let $A \in M_n(\mathbb{R})$ be a primitive matrix.

1. There exists $\mu > 0$ which is a simple eigenvalue of $A$, and $|\mu_j| < \mu$ for any other eigenvalue $\mu_j$.

2. There exist $v, u \in \mathbb{R}^n$ with positive entries such that $Av = \mu v$ and $u^T A = \mu u^T$. Moreover, every right eigenvector with non-negative entries is a positive multiple of $v$, and every left eigenvector with non-negative entries is a positive multiple of $u$.

3. The following holds:

$$\lim_{k \to \infty} \left( \frac{1}{\mu} A \right)^k = \frac{vu^T}{u^Tv}.$$ 

Definition 6.6: The limit matrix

$$P = \frac{vu^T}{u^Tv} \in M_n(\mathbb{R})$$

is called the Perron projection of $A$. 
6.3. Irreducible non-primitive fixed scale substitution schemes.

Definition 6.7: A matrix $A \in M_n(\mathbb{R})$ is a **cyclic matrix of period** $p$ if up to a permutation of the indices, $A$ is of the block form

$$A = \begin{pmatrix} 0_0 & A_0 \\ 0_1 & \ddots \\ \vdots & \ddots & A_{p-2} \\ A_{p-1} & \cdots & 0_{p-1} \end{pmatrix},$$

where $n_0 + \cdots + n_{p-1} = n$, and $0_r \in M_{n_r}(\mathbb{R})$ is the square zero matrix of order $n_r$.

Proofs of the following result can be found in the discussion on irreducible matrices in Part 1 of [Se].

Lemma 6.8: Let $A \in M_n(\mathbb{R})$ be an irreducible non-primitive matrix with Perron–Frobenius eigenvalue $\mu$. Then there exists $p \geq 2$ such that $A$ is a cyclic matrix of period $p$. In addition

$$A^p = \begin{pmatrix} B_0 \\ \vdots \\ B_{p-1} \end{pmatrix},$$

where $B_r \in M_{n_r}(\mathbb{R})$ are primitive square matrices with Perron–Frobenius eigenvalue $\mu^p$.

Corollary 6.9: Let $\sigma$ be an irreducible non-primitive fixed scale substitution scheme. Then the prototiles in $\tau_\sigma$ can be divided into $p$ classes $C_0, \ldots, C_{p-1}$, so that the substitution tiles of prototile in $C_r$ contain only tiles of types in $C_{r+1(\text{mod } p)}$.

In addition, the associated generation sequences of partition have the following useful property.

Corollary 6.10: Let $\sigma$ be an irreducible non-primitive scheme on, and let $\{\delta_k\}$ be a generation sequence of partitions of, $T_i \in \tau_\sigma$. Assume without loss of generality that $T_i \in C_0$.

1. The sequence $\{\delta_k\}$ consists of $p$ subsequences $\{\delta_{pk+r}\}$, and partition $\delta_{pk+r}$ contains only tiles of class $C_r$. 

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(2) The subsequence \(\{\delta_{pk+r}\}\) is geometrically identical to the union of generation sequences of partitions of tiles which appear in the partition \(\delta_r\), generated by an irreducible primitive fixed scale substitution scheme on \(C_r\). The substitution matrix of this scheme is the primitive matrix \(B_r\) described in Lemma 6.8, and the contraction constant is \(\alpha^p\).

Example 6.11: The substitution defined in Figure 6.1 is an irreducible non-primitive fixed scale substitution scheme on a square and two rectangles, with period \(p = 2\) and contraction constant \(\alpha = \frac{1}{\sqrt{3}}\).

![Figure 6.1. A non-primitive fixed scale substitution scheme with period \(p = 2\).](image_url)

The first class of prototiles \(C_0\) consists of the square, and the second class \(C_1\) consists of the two rectangles, therefore \(n_0 = 1\) and \(n_1 = 2\). The substitution matrix \(S_\sigma\) and the block diagonal \(S_\sigma^2\), with two primitive blocks \(B_0\) and \(B_1\), are given by

\[
S_\sigma = \begin{pmatrix}
0 & 1 & 1 \\
3 & 0 & 0 \\
6 & 0 & 0
\end{pmatrix}, \quad S_\sigma^2 = \begin{pmatrix}
9 & 0 & 0 \\
0 & 3 & 3 \\
0 & 6 & 6
\end{pmatrix}.
\]

6.4. Proof of uniform distribution of generation sequences generated by fixed scale schemes.

Proof of Theorem 1.16. Let \(T \in \mathcal{T}_i\) and assume that \(T\) is a tile of type \(j\) that appears at partition \(\delta_{k_0}\).

First, assume that scheme \(\sigma\) is primitive. By Lemma 6.3, for any \(k > k_0\) we have

\[
\frac{|x_k \cap T|}{|x_k|} = \frac{\sum_{h=1}^n (S^{k-k_0}_{\sigma})_{jh} \cdot (S^k_{\sigma} \cdot 1)_j}{\sum_{h=1}^n (S^k_{\sigma})_{jh}} = \frac{(S^{k-k_0}_{\sigma} \cdot 1)_j}{(S^k_{\sigma} \cdot 1)_i}.
\]
where \( \mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n \). The Perron–Frobenius eigenvalue of \( S_\sigma \) is \( \mu = \frac{1}{\alpha^d} \) and \( \nu_{\text{vol}} \) is a Perron–Frobenius eigenvector of \( S_\sigma \). The matrix \( S_\sigma \) is primitive, the Perron projection of \( S_\sigma \) satisfies
\[
\lim_{k \to \infty} \left( \frac{1}{\mu} S_\sigma \right)^k = \lim_{k \to \infty} (\alpha^d S_\sigma)^k = P,
\]
and the columns of \( P \) are spanned by \( \nu_{\text{vol}} \). From this we get
\[
\lim_{k \to \infty} \frac{|x_k \cap T|}{|x_k|} = \lim_{k \to \infty} \frac{(S^{k-k_0} \cdot \mathbf{1})_j}{(S^k \cdot \mathbf{1})_i} = \lim_{k \to \infty} (\alpha^d)^{k-k_0} \frac{(\alpha^d)^k (S^{k-k_0} \cdot \mathbf{1})_j}{(\alpha^d)^k (S^k \cdot \mathbf{1})_i}
\]
\[
= (\alpha^d)^{k_0} \lim_{k \to \infty} \frac{((\alpha^d S_\sigma)^{k-k_0} \cdot \mathbf{1})_j}{((\alpha^d S_\sigma)^k \cdot \mathbf{1})_i} = (\alpha^d)^{k_0} \frac{(P \cdot \mathbf{1})_j}{(P \cdot \mathbf{1})_i}
\]
\[
= (\alpha^d)^{k_0} \frac{\text{vol} T_j}{\text{vol} T_i} \frac{\text{vol}(\alpha^{k_0} T_j)}{\text{vol}(\alpha^{k_0} T_i)} = \frac{\text{vol} T}{\text{vol} T_i}.
\]
By Lemma 3.6, uniform distribution of \( \{\delta_k\} \) is established for the primitive case.

Next, assume the scheme is non-primitive with period \( p \), and assume without loss of generality that \( T_i \in C_0 \). Let \( T \in \mathcal{S}_i^\sigma \) and assume that \( T \in \delta_{k_0} \) and \( k_0 \equiv r_0 \mod p \), so \( T \) is a rescaled copy of a tile in \( C_{r_0} \). For every \( 0 \leq c \leq p-1 \), let \( T_1^{(c)}, \ldots, T_{s_c}^{(c)} \) be the tiles in the partition \( \delta_{k_0+c} \) such that
\[
T = \bigsqcup_{i=1}^{s_c} T_i^{(c)}.
\]
Let \( 0 \leq r \leq p-1 \) and let \( c \equiv r - r_0 \mod p \). By Corollary 6.10 and what we have just proved for primitive schemes
\[
\lim_{k \to \infty} \frac{|x_{pk+r} \cap T|}{|x_{pk+r}|} = \sum_{i=1}^{s_c} \lim_{k \to \infty} \frac{|x_{pk+r} \cap T_i|}{|x_{pk+r}|} = \sum_{i=1}^{s_c} \frac{\text{vol} T_i}{\text{vol} T_i} = \frac{\text{vol} T}{\text{vol} T_i}.
\]
The limit does not depend on \( r \), and so
\[
\lim_{k \to \infty} \frac{|x_k \cap T|}{|x_k|} = \frac{\text{vol} T}{\text{vol} T_i}.
\]
By Lemma 3.6 this finishes the proof of Theorem 1.16.

6.5. TYPE FREQUENCIES IN GENERATION SEQUENCES GENERATED BY PRIMITIVE FIXED SCALE SCHEMES. The following is the counterpart of Theorem 1.13 for generation sequences generated by primitive fixed scale substitution schemes. These results follow from the theory of Perron–Frobenius, and are included here to allow comparison with their incommensurable counterparts.
THEOREM 6.12: Let $\sigma$ be an irreducible primitive fixed scale scheme in $\mathbb{R}^d$ with contraction constant $\alpha$. Let $\{\delta_k\}$ be a generation sequence of partitions of $T_i \in \tau_\sigma$, and let $1 \leq r \leq n$.

(1) Any $r$-marking sequence $\{x^{(r)}_k\}$ of $\{\delta_k\}$ is uniformly distributed in $T_i$.

(2) The ratio between the number of tiles of type $r$ in $\delta_k$ and the total number of tiles is

$$u_r + o(1), \quad k \to \infty,$$

where $u = (u_1, \ldots, u_n)^T$ is a left Perron–Frobenius eigenvector of $S$ normalized so that $\sum u_j = 1$.

(3) The volume of the region covered by tiles of type $r$ in $\delta_k$ is

$$\text{vol}(T_i)w_r + o(1), \quad k \to \infty,$$

where $w = (w_1, \ldots, w_n)^T$ is a left Perron–Frobenius eigenvector of $M(d)$ normalized such that $\sum w_j = 1$, and $M$ is the graph matrix function of the graph associated with the equivalent normalized scheme.

Remark 6.13: Estimates for the error terms can be found in the literature; see for example [So].

Proof. The uniform distribution of any $r$-marking sequence $\{x^{(r)}_k\}$ of $\{\delta_k\}$ follows from the same arguments as those described above, by replacing the vector $1 \in \mathbb{R}^n$ with $e_r \in \mathbb{R}^n$, the vector $r$ of the standard basis of $\mathbb{R}^n$.

The second part of the theorem follows from the Perron–Frobenius theorem applied to the substitution matrix $S_\sigma$ and a direct calculation:

$$\lim_{k \to \infty} \frac{|x^{(r)}_k|}{|x_k|} = \lim_{k \to \infty} \frac{(S^k_\sigma \cdot e_r)_i}{(S^k_\sigma \cdot 1)_i} = \frac{(vu^T \cdot e_r)_i}{(vu^T \cdot 1)_i} = \frac{u_r v_i}{v_i} = u_r.$$

To prove the third part of the theorem, define the weighted substitution matrix by

$$(W_\sigma)_{ij} = \frac{\text{vol}(\alpha T_j)}{\text{vol}T_i} k_{ij}.$$ 

Similarly to Lemma 6.3, the matrix $W_\sigma$ satisfies

$$(W^k_\sigma)_{ir} = \frac{\text{vol}(\bigcup\{\text{Tiles of type } r \in \delta_k\})}{\text{vol}T_i},$$

and is primitive if and only if $S_\sigma$ is primitive. Observe that

$$\frac{\text{vol}(\alpha T_j)}{\text{vol}T_i} = \beta^{(k)}_{ij}.$$
for all $k = 1, \ldots, k_{ij}$, and so $W_{\sigma} = M(d)$, where $M$ is the graph matrix function of the graph associated with the equivalent normalized scheme. It follows by Lemma 4.11 that $W_{\sigma}$ has Perron–Frobenius eigenvalue $\mu = 1$ and a right positive Perron–Frobenius eigenvector $v = 1 \in \mathbb{R}^n$.

Let $w \in \mathbb{R}^n$ be a left positive eigenvector of $W_{\sigma}$ chosen such that $\sum w_j = 1$. Then

$$\lim_{k \to \infty} \frac{\text{vol}(\bigcup \{\text{Tiles of type } r \in \delta_k\})}{\text{vol} T_i} = \lim_{k \to \infty} (W_k^T \cdot e_r)_i = \left(\frac{vw^T}{w^Tv} \cdot e_r\right)_i = \left(\frac{1}{w^T} \cdot e_r\right)_i = w_r$$

finishing the proof.

**Example 6.14:** Consider the Penrose–Robinson fixed scale substitution scheme on a tall triangle $T$ and a short triangle $S$, as defined in Figure 2.4. The substitution matrix and the weighted substitution matrices are given by

$$S_{\sigma} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad W_{\sigma} = \frac{1}{\varphi^2} \begin{pmatrix} 2 & \varphi^{-1} \\ \varphi & 1 \end{pmatrix},$$

and so $u^T = \left(\frac{\varphi}{\varphi+1}, \frac{1}{\varphi+1}\right)$ and $w^T = \left(\frac{\varphi+1}{\varphi+2}, \frac{1}{\varphi+2}\right)$, from which the frequencies of the generation sequence of partitions $\{\delta_k\}$ of $T$ illustrated in Figure 2.6 are deduced; see Example 2.2.

Consider now the Kakutani sequence of partitions $\{\pi_m\}$ of $T$ generated by the Penrose–Robinson scheme, illustrated in Figure 2.5. Here $\delta_k = \pi_{2k-1}$ for all $k \in \mathbb{N}$, and so the number of tall and short triangles in $\pi_{2k-1}$ is given by the first row of $S_{\sigma}^k$. Since $\pi_{2k}$ is the result of substituting all tall triangles in $\pi_{2k-1}$ according to the substitution rule, the number of tall and short triangles in $\pi_{2k}$ is given by the first row of

$$K_k = S_{\sigma}^k \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$
and so the limit of the ratio between the number of short triangles in $\pi_m$ and the total number of tiles does not exist. It can be shown in a similar way that the sequence of volumes of the regions covered by short triangles in $\pi_m$ does not converge.

7. Kakutani sequences generated by commensurable substitution schemes

The proof of Theorem 1.9 will be given in two stages. In this section we prove the theorem for the commensurable case.

Let $\sigma$ be an irreducible commensurable multiscale substitution scheme and let $G_\sigma$ be the graph associated with an equivalent normalized scheme. Then $G_\sigma$ is commensurable, and by Lemma 5.3 the set of lengths of paths in $G_\sigma$ is uniformly discrete, that is, there exists $\rho > 0$ so that for any two paths $\gamma_1, \gamma_2$ of different lengths in $G_\sigma$

$$|l(\gamma_1) - l(\gamma_2)| \geq \rho.$$ 

It follows that it is possible to slightly rescale the prototiles of the equivalent normalized scheme while remaining in the same equivalent class of schemes. By Lemma 4.4 this amounts to slightly sliding the vertices in $G_\sigma$ along the edges, without losing the monotonicity of the map from the lengths of paths originating at vertex $i \in V_\sigma$ in the associated graph to the volumes of tiles in $O_i^{\sigma'}$, described in Corollary 4.7. This proves the following Lemma 7.1.

**Lemma 7.1:** Let $\sigma$ be an irreducible commensurable multiscale substitution scheme. Then $\sigma$ is equivalent to a scheme $\sigma'$ for which

1. Any two edges $\varepsilon_1$ and $\varepsilon_2$ in the associated graph $G_{\sigma'}$ satisfy

$$\frac{l(\varepsilon_1)}{l(\varepsilon_2)} \in \mathbb{Q}.$$ 

2. The correspondence between tiles in $O_i^{\sigma'}$ and finite paths in $G_{\sigma'}$ with original vertex $i \in V_{\sigma'}$, is monotone in the sense of Corollary 4.7, that is, if $\gamma_{T_1}$ and $\gamma_{T_2}$ are the paths in $G_{\sigma'}$ corresponding to the tiles $T_1, T_2 \in O_i^{\sigma'}$, then

$$\text{vol} T_1 < \text{vol} T_2 \quad \text{if and only if} \quad l(\gamma_{T_1}) > l(\gamma_{T_2}).$$
Theorem 7.2: Let $\sigma$ be an irreducible commensurable multiscale substitution scheme and let $\{\pi_m\}$ be a Kakutani sequence of partitions of $T_i \in \tau_{\sigma}$. Then there exist a set of prototiles $\tau_{\sigma}$ that contains $\tau_{\sigma}$ as a subset, and a fixed scale substitution scheme $\tilde{\sigma}$ with prototile set $\tau_{\tilde{\sigma}}$, so that $\{\pi_m\}$ is a subsequence of a generation sequence of partitions $\{\delta_k\}$ of $T_i \in \tau_{\tilde{\sigma}}$, generated by the fixed scale scheme $\tilde{\sigma}$.

Proof. The stages of the process described in the proof are illustrated in Example 7.3 below.

Put $\sigma_1 = \sigma$, and let $G_{\sigma_1}$ be the associated graph. By Lemma 4.8, there is a one-to-one correspondence between tiles in the partition $\pi_m$ and metric paths of length $l_m$ in $G_{\sigma_2}$, which is the graph associated with the equivalent normalized scheme $\sigma_2$, and has the same set of closed path lengths as $G_{\sigma_1}$. By Lemma 7.1, the normalized scheme is equivalent to a scheme $\sigma_3$ with an associated graph $G_{\sigma_3}$ with rationally dependent edge lengths and the same set of closed path lengths as $G_{\sigma_1}$ and $G_{\sigma_2}$, and for which the statement of Lemma 4.8 still holds. Let

$$\tau_{\sigma_3} = (T_1, \ldots, T_n)$$

be the list of prototiles on which the “rationalized” scheme $\sigma_3$ is defined.

Assume there is an edge of length $\log \frac{1}{a}$ in $G_{\sigma_3}$, and let $a \in \mathbb{N}$ be the minimal positive integer such that for every $\varepsilon \in G_{\sigma_3}$ there exists $b_\varepsilon \in \mathbb{N}$ coprime to $a$ with

$$l(\varepsilon) = \frac{b_\varepsilon}{a} \log \frac{1}{\alpha}.$$

It follows that there exists an increasing sequence $\{k_m\} \subset \mathbb{N}$ such that

$$l_m = \frac{1}{a} \log \frac{1}{\alpha} \cdot k_m.$$

By appropriately adding vertices to $G_{\sigma_3}$, define a new graph $G_{\sigma_4}$ with edges all of length $\frac{1}{a} \log \alpha$. The graph $G_{\sigma_4}$ is associated with a fixed scale substitution scheme $\sigma_4$ with contracting constant $\alpha^{\frac{1}{a}}$ defined on a prototile set $\tau_{\sigma_4}$, that contains rescaled copies of elements of $\tau_{\sigma}$ and new prototiles associated with the new vertices.

Let $\varepsilon$ be an edge in $G_{\sigma_3}$ with initial and terminal vertices $i$ and $j$, and let $v_1, \ldots, v_t$ be the new vertices in $G_{\sigma_4}$ added on $\varepsilon$; then they each have a single outgoing edge in $G_{\sigma_4}$. It follows that the substitution tiles of the associated new prototiles in $\tau_{\sigma_4}$ are geometrically trivial, that is

$$\omega_{\sigma_4}(T_{v_s}) = \alpha^{\frac{1}{a}} T_{v_{s+1}}$$ and $$\omega_{\sigma_4}(T_{v_t}) = \alpha^{\frac{1}{a}} T_j,$$
and so they are all rescaled copies of $T_j$. The rescaled copy of $T_j$ in the substitution rule on $T_i$ that corresponds to the edge $\varepsilon$ in the original scheme, is replaced in the fixed scale scheme by a copy of $\alpha^{\frac{1}{a}} T_{v_1}$.

The partition $\delta_k$ in the generation sequence generated by the fixed scale scheme $\sigma_4$ on $\tau_{\sigma_4}$ defined above corresponds to paths in $G_{\sigma_4}$ of length $\frac{1}{a} \log \alpha \cdot k$, which can be regarded as metric paths of the same length in $G_{\sigma_3}$. Since $\pi_m$ corresponds to metric paths of length $l_m = \frac{1}{a} \log \frac{1}{\alpha} k_m$, by the construction above $\pi_m$ is geometrically identical to $\delta_{k_m}$. Putting $\tilde{\sigma} = \sigma_4$ finishes the proof.

**Proof of Theorem 1.9 for commensurable schemes.** By Theorem 7.2 a Kakutani sequence of partitions generated by a commensurable scheme is a subsequence of a generation sequence of partitions generated by a fixed scale scheme, which by Theorem 1.16 is uniformly distributed, finishing the proof.

**Example 7.3:** Consider the fixed scale substitution scheme on the triangle $T$ and the rhombus $R$ in $\mathbb{R}^2$, as defined in Figure 7.1 where it is shown together with its associated graph $G_{\sigma_1}$.

Figure 7.1. A fixed scale substitution scheme $\sigma_1$ on $T$ and $R$ and its associated graph $G_{\sigma_1}$. The vertex on the left corresponds to $T$ and the vertex on the right corresponds to $R$. Multiple arrows stand for multiple edges of the same length.

This substitution scheme is studied in Chapter 6 of [BG]. The substitution matrix of this scheme is given by

$$S_{\sigma} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$  

The matrix $S$ has Perron–Frobenius eigenvalue $\mu = (1 + \sqrt{2})^2$ and $v = (1, \sqrt{2})$ is a right Perron–Frobenius eigenvector. The contraction constant of the scheme is thus $\alpha = \frac{1}{1 + \sqrt{2}}$, the volumes of the prototiles are $\text{vol} \ T = 1$ and $\text{vol} \ R = \sqrt{2}$ and the edges of the associated graph $G_{\sigma_1}$ are all of length $\log \frac{1}{\alpha}$. 


An equivalent normalized scheme, in which \( \text{vol } T = \text{vol } R = 1 \), is illustrated next to its associated graph \( G_{\sigma_2} \) in Figure 7.2. Note that the contraction of the rhombus \( R \) corresponds to the sliding of its associated vertex backwards along the edges, as described in Lemma 4.4.

By contracting the rhombus \( R \) some more, and continuing to slide the associated vertex along the edges accordingly, we obtain an equivalent scheme \( \sigma_3 \) on a set of prototiles \( \tau_{\sigma_3} \) with an associated graph \( G_{\sigma_3} \), which has the properties described in Lemma 7.1. This is illustrated in Figure 7.3.

The lengths of the edges in the graph \( G_{\sigma_3} \) are all integer multiples of \( \frac{1}{2} \log \alpha \). By adding vertices to the graph we define a new graph \( G_{\sigma_4} \) in which all edges are of equal length \( \frac{1}{2} \log \alpha \). Note that in this example there are \( k \geq 2 \) edges of the same length and initial and terminal vertices and so we can choose one of these edges, define new vertices as described in the proof of Theorem 7.2, and define \( k \) new edges between the initial vertex and the first new one. This is illustrated in Figure 7.4, and may be useful for computations.
The procedure by which the graph $G_{\sigma_4}$ is derived from the graph $G_{\sigma_3}$ corresponds to the addition of four rescaled copies of the original prototiles to the set of prototiles $\tau_{\sigma_3}$ illustrated in Figure 7.3. The graph $G_{\sigma_4}$ is the graph associated with the fixed scale substitution scheme $\sigma_4$ defined on the extended set of prototiles $\tau_{\sigma_4}$, with substitution rule as shown in Figure 7.5 and contraction constant $\sqrt{\alpha}$. The generation sequence generated by this scheme on $\tau_{\sigma_4}$ is geometrically identical to the Kakutani sequence generated by the original scheme.

8. Kakutani sequences generated by incommensurable schemes

In this section we prove Theorem 1.9 on uniform distribution of Kakutani sequences of partitions generated by incommensurable multiscale substitution schemes, and present a proof for Theorem 1.13.
8.1. **Counting metric paths in incommensurable graphs.** In the previous sections, we have seen how the Perron–Frobenius theorem is used to imply uniform distribution of sequences of partitions generated by commensurable schemes. These are no longer sufficient in the incommensurable case, and a new tool for counting tiles is required. As we will see below, the role of the Perron–Frobenius theorem is taken by the path counting results on incommensurable graphs that appear as parts (ii) of Theorems 1 and 2 in [KSS].

**Theorem** (Part (ii) of Theorem 1 in [KSS]): Let $G$ be a strongly connected incommensurable graph with a set of vertices $V = \{1, \ldots, n\}$. There exist a positive constant $\lambda$ and a matrix $Q(M) \in M_n(\mathbb{R})$ with positive entries, so that if $\varepsilon \in \mathcal{E}$ is an edge in $G$ with initial vertex $h \in V$, then the number of metric paths of length exactly $x$ from vertex $i \in V$ to a point on the edge $\varepsilon$ grows as

$$1 - \frac{e^{-l(\varepsilon)\lambda}}{\lambda} Q_{ih} e^{\lambda x} + o(e^{\lambda x}), \quad x \to \infty.$$ 

The constant $\lambda$ is the maximal real value for which the spectral radius of $M(\lambda)$ is equal to 1, and

$$Q(M) = \frac{\text{adj}(I - M(\lambda))}{-\text{tr}(\text{adj}(I - M(\lambda)) \cdot M'(\lambda))},$$

where $M$ is the graph matrix function defined in Section 4, $M'$ is the entry-wise derivative of $M$, and $\text{adj} A$ is the adjugate or classical adjoint matrix of $A$, that is, the transpose of its cofactor matrix.

The proof involves the analysis of the Laplace transform of the corresponding counting function and the application of the Wiener–Ikehara Tauberian theorem. For a complete proof see [KSS], where further details regarding walks on directed weighted graphs may be found.

Although for general graphs the value of the constant $\lambda$ and therefore also of the entries of the matrix $Q(M)$ appearing in Theorem 1 in [KSS] may be difficult to compute, they can be given explicitly in the case of graphs associated with multiscale substitution schemes, as follows from Lemma 8.1

**Lemma 8.1:** Let $\sigma$ be an irreducible incommensurable substitution scheme in $\mathbb{R}^d$, and let $G_\sigma$ be the associated graph. Then $\lambda = d$, and the columns of $Q(M)$ are spanned by $v_{\text{vol}}$, that is, there exist $q_1, \ldots, q_n > 0$ such that $q_h \cdot v_{\text{vol}}$ is column $h$ of the matrix $Q(M)$. 
Proof. By Corollary 4.15, the spectral radius of $M(d)$ is exactly 1. For any $x > d$ the sums of all rows of the matrix $M(x)$ are strictly smaller than 1, and so the Perron–Frobenius eigenvalue of $M(x)$ is strictly smaller than 1 (see page 63 of [Ga]). It follows that the spectral radius of $M(x)$ is strictly smaller than 1 for all $x > d$, and so $\lambda = d$.

As proved in [KSS], the columns of $Q(M)$ are positive multiples of a right Perron–Frobenius eigenvector of $M(d)$. By Corollary 4.15 this vector can be chosen to be $v_{vol}$.

Given an irreducible incommensurable scheme $\sigma$ in $\mathbb{R}^d$, consider the graph associated with an equivalent normalized scheme, and let $M$ be the corresponding graph matrix function. We denote $M_\sigma = M(d)$, $M'_\sigma = M'(d)$ and $Q_\sigma = Q(M)$.

**Corollary 8.2:** Let $G_\sigma$ be a graph associated with a normalized irreducible incommensurable multiscale substitution scheme in $\mathbb{R}^d$. Let $\varepsilon \in \mathcal{E}_\sigma$ be an edge associated with the tile $\alpha T_j \in \omega_\sigma(T_h)$, that is $\varepsilon$ has initial vertex $h$, terminal edge $j$ and is of length $\log \frac{1}{\alpha}$. Then the number of metric paths of length exactly $x$ from vertex $i$ to a point on the edge $\varepsilon$ grows as

$$
\frac{1 - \beta^d}{d} q_h e^{dx} + o(e^{dx}), \quad x \to \infty,
$$

where $\beta$ is the constant of substitution associated with $\alpha$, and $q_h \cdot v_{vol} = q_h \cdot 1$ is column $h$ of the matrix $Q_\sigma$.

**8.2. Tiles in partitions generated by incommensurable multiscale schemes.** In the language of tiles and Kakutani sequences of partitions generated by multiscale substitution schemes, as a result of Lemma 4.8, Corollary 8.2 amounts to the following result.

**Theorem 8.3:** Let $\sigma$ be an irreducible incommensurable multiscale substitution scheme in $\mathbb{R}^d$ and let $\{\pi_m\}$ be a Kakutani sequence of partitions of $T_i \in \tau_\sigma$. Denote

$$
b_{hj} := \sum_{k=1}^{k_{hj}} \frac{1 - (\beta_{hj}^{(k)})^d}{d}.
$$

Then the number of tiles of type $j$ appearing in the partition $\pi_m$ of $T_i$ grows as

$$
|j_m| = \sum_{h=1}^{n} b_{hj} q_h e^{dl_m} + o(e^{dl_m}), \quad m \to \infty,
$$

independent of $i$, where $q_h \cdot 1$ is column $h$ of the matrix $Q_\sigma$. 

Proof of Theorem 1.9 for incommensurable schemes. Let \( T \in \mathcal{T}_i^\sigma \) and assume \( T \) is a tile of type \( i_0 \) that appears in partition \( \pi_{m_0} \). The path \( \gamma_T \) corresponding to \( T \) in the graph \( G^\sigma \) associated with an equivalent normalized scheme terminates at vertex \( i_0 \) and is of length \( l(\gamma) = l_{m_0} \), and therefore

\[
\operatorname{vol} T = e^{-l_{m_0}d} \cdot \operatorname{vol} T_{i_0}.
\]

Let \( m > m_0 \). Then by Theorem 8.3

\[
\frac{|x_m \cap T|}{|x_m|} = \frac{\sum_{h,j=1}^{n} b_{hj} q_h e^{dl_{m_0}}}{\sum_{h,j=1}^{n} b_{hj} q_h e^{dl_m}} + o(1),
\]

where \( q_h \cdot 1 \) is column \( h \) of \( Q^\sigma \). Since the scheme is normalized, \( \frac{\operatorname{vol} T_{i_0}}{\operatorname{vol} T_i} = 1 \), and so

\[
\sum_{h,j=1}^{n} b_{hj} q_h e^{dl_{m_0}} = e^{dl_{m_0}} \cdot \frac{\sum_{h,j=1}^{n} b_{hj} q_h}{\sum_{h,j=1}^{n} b_{hj} q_h} = e^{-l_{m_0}d} = e^{-l_{m_0}d} \frac{\operatorname{vol} T_{i_0}}{\operatorname{vol} T_i} = \frac{\operatorname{vol} T}{\operatorname{vol} T_i}.
\]

Therefore

\[
\lim_{m \to \infty} \frac{|x_m \cap T|}{|x_m|} = \frac{\operatorname{vol} T}{\operatorname{vol} T_i},
\]

and combined with Lemma 3.6, this completes the proof of Theorem 1.9.  

8.3. Frequencies of types.

Proof of parts (1) and (2) of Theorem 1.13. For the first part, simply follow the proof of Theorem 1.9 and replace \( \sum_{h,j=1}^{n} b_{hj} \) with \( \sum_{h=1}^{n} b_{hr} \).

By Theorem 8.3, the ratio between the number of tiles of type \( r \) in \( \pi_m \) and the total number of tiles is given by

\[
\frac{|x_m^{(r)}|}{|x_m|} = \frac{\sum_{h=1}^{n} b_{hr} q_h e^{dl_m} + o(e^{dl_m})}{\sum_{h,j=1}^{n} b_{hj} q_h e^{dl_m} + o(e^{dl_m})} = \frac{\sum_{h=1}^{n} b_{hr} q_h}{\sum_{h,j=1}^{n} b_{hj} q_h} + o(1), \quad m \to \infty.
\]

Plugging in the definition of \( b_{hj} \) finishes the proof.



For the proof of the third part of Theorem 1.13, an additional result on random walk on graphs is needed. Let \( G \) be a directed weighted graph. For any \( i \in \mathcal{V} \) and \( \varepsilon \in \mathcal{E} \) with initial vertex \( i \), denote by \( p_{i\varepsilon} > 0 \) the probability that a walker who is passing through vertex \( i \) chooses to continue his walk through edge \( \varepsilon \), and assume that for any vertex in \( G \), the sum of the probabilities over all
edges originating at that vertex is equal to 1. Let \( \varepsilon_1, \ldots, \varepsilon_{k_{ij}} \) be the edges in \( G \) with initial vertex \( i \) and terminal vertex \( j \). The **graph probability matrix function** \( N : \mathbb{C} \to M_n(\mathbb{C}) \) is defined by

\[
N_{ij}(s) = p_i \varepsilon_1 e^{-s \cdot l(\varepsilon_1)} + \cdots + p_i \varepsilon_{k_{ij}} e^{-s \cdot l(\varepsilon_{k_{ij}})},
\]

and if \( i \) is not connected to \( j \) by an edge, put \( N_{ij}(s) = 0 \).

**Theorem (Special case of part (ii) of Theorem 2 in [KSS]):** Let \( G \) be a strongly connected incommensurable graph with a set of vertices \( \mathcal{V} = \{1, \ldots, n\} \), and consider a walker on \( G \) advancing at constant speed 1, obeying the probabilities \( p_i \varepsilon \) attached to the edges of \( G \). There exists a matrix \( Q \in M_n(\mathbb{R}) \) with positive entries such that for any \( \varepsilon \in \mathcal{E} \) with initial vertex \( h \in \mathcal{V} \), the probability that a walker who has left vertex \( i \in \mathcal{V} \) at time \( t = 0 \) is on the edge \( \varepsilon \in \mathcal{E} \) at time \( t = T \) is

\[
p_h \varepsilon l(\varepsilon) Q_{ih} + o(1), \quad T \to \infty.
\]

Here

\[
Q^{(N)} = \frac{\text{adj}(I - N(0))}{-\text{tr}(\text{adj}(I - N(0)) \cdot N'(0))}.
\]

**Proof of part (3) of Theorem 1.13.** The probability that a point in \( T_i \) is in a tile \( \alpha \cdot \varphi(T_j) \in \varrho_{\sigma}(T_i) \) after the application of the substitution rule once, is given by

\[
\frac{\text{vol}(\alpha \cdot \varphi(T_j))}{\text{vol} T_i} = \beta^d.
\]

Therefore, if \( \varepsilon \) is the edge associated with \( \alpha T_j \in \omega_{\sigma}(T_i) \) in the graph \( G_{\sigma} \) associated with an equivalent normalized scheme, then we set

\[
p_i \varepsilon = \alpha^d = \beta^d.
\]

By Lemma 3.1, the sum of the probabilities on all edges with a common initial vertex equals 1. Observe that in the case studied here \( N(0) = M(d) = M_{\sigma} \) and \( N'(0) = M'(d) = M'_{\sigma} \), where \( M \) and \( N \) are the graph and the graph probability matrix functions of \( G_{\sigma} \), and so

\[
Q^{(N)} = Q^{(M)} = Q_{\sigma}.
\]

The result now follows from the special case of part (ii) of Theorem 2 in [KSS] and from the previous discussion.
Since the formulas for the path counting functions on incommensurable graphs that are given in [KSS] are stated separately for every edge in the graph, they imply results which are more refined than those stated in Theorem 1.13. A nice consequence of this more refined version of Theorem 1.13 is given in Example 8.4.

Example 8.4: Let \( \{\pi_m\} \) be the \( \frac{1}{3}\)-Kakutani sequence of partitions of \( I \) as illustrated in Figure 1.1. Whenever a partition is made, color the shorter of the two new intervals light pink and the longer one dark blue. The first non-trivial elements of this colored sequence of partitions are illustrated in Figure 8.1.

![Figure 8.1](image)

Figure 8.1. After each substitution, the shorter of the new intervals is light pink, and the longer one dark blue.

Then by the arguments used in the proof of parts (2) and (3) of Theorem 1.13, we have

\[
\lim_{m \to \infty} \frac{|\{\text{Pink intervals } \in \pi_m\}|}{|\{\text{Intervals } \in \pi_m\}|} = \frac{2}{3}
\]

and

\[
\lim_{m \to \infty} \text{vol} \left( \bigcup \{\text{Pink intervals } \in \pi_m\} \right) = \frac{1}{3} \log \frac{1}{3} + \frac{2}{3} \log \frac{2}{3}.
\]

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