Nonlinear multidimensional cosmological models with form fields: stabilization of extra dimensions and the cosmological constant problem

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\textbf{Abstract}

We consider multidimensional gravitational models with a nonlinear scalar curvature term and form fields in the action functional. In our scenario it is assumed that the higher dimensional spacetime undergoes a spontaneous compactification to a warped product manifold. Particular attention is paid to models with quadratic scalar curvature terms and a Freund-Rubin-like ansatz for solitonic form fields. It is shown that for certain parameter ranges the extra dimensions are stabilized. In particular, stabilization is possible for any sign of the internal space curvature, the bulk cosmological constant and of the effective four-dimensional cosmological constant. Moreover, the effective cosmological constant can satisfy the observable limit on the dark energy density. Finally, we discuss the restrictions on the parameters of the considered nonlinear models and how they follow from the connection between the D–dimensional and the four-dimensional fundamental mass scales.

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1 Introduction

Two of the most intriguing problems of modern cosmology are the problem of additional dimensions and the cosmological constant problem (CCP). The first problem follows from theories which unify different fundamental
interactions with gravity, such as M/string theory \[ \text{I} \], and which have their most consistent formulation in spacetimes with more than four dimensions. The problem can be naturally formulated as the following question: if we live in a multidimensional spacetime, why do we not observe the extra dimensions? Within the "old" Kaluza-Klein (KK) framework and the early $E_8 \times E_8$-heterotic string phenomenology the question is answered by assuming the extra dimensions so small (i.e. with a characteristic size $r$ between the Planck and the Fermi scales $10^{-33}\text{cm} \lesssim r \lesssim 10^{-17}\text{cm}$) that they are not accessible by present-day collider experiments. New concepts with the possibility for rich phenomenology opened up with the uncovering of $D$–branes by Polchinski \[ \text{II} \] in 1995. In "brane-world" scenarios of the Universe the usual 4–dimensional physics with its $SU(3) \times SU(2) \times U(1)$ standard model (SM) fields is localized on a 3–dimensional space-like hypersurface (our world-brane) whereas the gravitational field propagates in the whole (bulk) spacetime. Depending on the concrete scenario there are different types of masking of the additional dimensions. Whereas in Arkani-Hamed–Dvali–Dimopoulos (ADD) models \[ \text{III} \] the extra dimensions are curled up to sizes smaller than $10^{-2}\text{cm}$, so that they are in agreement with present table-top Cavendish-type tests of gravity \[ \text{IV} \], they can be infinite in the Randall–Sundrum II (RS II) \[ \text{V} \] and the Dvali–Gabadadze–Porrati (DGP) \[ \text{VI} \] model. In the latter models the appearing four-dimensionality of low-energy physics is achieved by inducing appropriate effective gravitational potentials on the world-brane. Beside their interesting phenomenology, brane-world models provide a possible resolution of the hierarchy problem. In ADD-type models this is due to the connection between the Planck scale $M_{\text{Pl}}$ and the fundamental scale $M_{\text{s}(4+D')} = 3$–dimensional and the $(4+D')$–dimensional spacetime, respectively:

\[
M_{\text{Pl}}(4) \sim V_{D'} M_{\text{s}(4+D')}^{2+D'}. \quad (1.1)
\]

$V_{D'}$ denotes the volume of the compactified $D'$ extra dimensions. It was realized in \[ \text{III} \] \[ \text{IV} \] that localizing the SM fields on a 3–brane allows to lower $M_{\text{s}(4+D')}$ down to the electroweak scale $M_{\text{EW}} \sim 1\text{TeV}$ without contradiction with present observations. Therefore, the compactification scale of the internal space can be of order

\[
r \sim V_{D'}^{-1/3} \sim 10^{22} - 17\text{cm}. \quad (1.2)
\]

With $M_{\text{EW}} \sim 1\text{TeV}$, physically acceptable values correspond to $D' \gtrsim 3$ \[ \text{III} \] (for astrophysical and cosmological bounds see e.g. \[ \text{III} \]); experimental bounds from table-top Cavendish-type experiments are given in \[ \text{IV} \]), and for $D' = 3$ one arrives at a sub-millimeter compactification scale $r \sim 10^{-5}\text{mm}$ of the internal space. If we shift $M_{\text{s}(4+D')}$ to $30\text{TeV}$, as suggested in \[ \text{III} \] (see also \[ \text{IV} \]), then the $D' = 2$ case satisfies all aforementioned bounds and leads to $r \sim 10^{-3} - 10^{-2}\text{mm}$. In order to not exclude this $D' = 2$ with its largest possible compactification scale $r$, we assume that the fundamental scale $M_{\text{s}(4+D')}$ can be of order $30\text{TeV}$. Additionally, the geometry in the ADD approach is to be factorizable as in a standard Kaluza-Klein model. I.e., the topology is the direct product of a non-warped external spacetime manifold and internal space manifolds with warp factors which depend on the external coordinates.

According to observations, the internal space should be static or nearly static at least from the time of primordial nucleosynthesis, (otherwise the fundamental physical constants would vary, see e.g. \[ \text{IV} \]) . This means that at the present evolutionary stage of the Universe the compactification scale of the internal space should either be stabilized and trapped at the minimum of some effective potential, or it should be slowly varying (similar to the slowly varying cosmological constant in the quintessence scenario \[ \text{IV} \]). In both cases, small fluctuations over stabilized or slowly varying compactification scales (conformal scales/geometrical moduli) are possible.

Stabilization of extra dimensions (moduli stabilization) in models with large extra dimensions (ADD-type models) has been considered in a number of papers (see e.g., Refs. \[ \text{V} \] \[ \text{VI} \] \[ \text{VII} \] \[ \text{VIII} \] \[ \text{IX} \] \[ \text{X} \] \[ \text{XI} \] \[ \text{XII} \] \[ \text{XIII} \] \[ \text{XIV} \] \[ \text{XV} \] \[ \text{XVI} \] \[ \text{XVII} \]). In the corresponding approaches, a product topology of the $(4+D')$–dimensional bulk spacetime was constructed from Einstein spaces with scale (warp) factors depending only on the coordinates of the external 4–dimensional component. As a consequence, the conformal excitations had the form of massive scalar fields living in the external spacetime. Within the framework of multidimensional cosmological models (MCM) such excitations were investigated in \[ \text{XVIII} \] \[ \text{XIX} \] \[ \text{XX} \] where they were called gravitational excitons. Later, since the ADD compactification approach these geometrical moduli excitations are known as radions \[ \text{XIII} \].

Most of the aforementioned papers are devoted to the stabilization of large extra dimension in theories with a linear multidimensional gravitational action. String theory suggests that the usual linear Einstein-Hilbert action should be extended with higher order nonlinear curvature terms. In a previous paper \[ \text{XXII} \] we considered

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The M-theory inspired RS-scenarios \[ \text{X} \] \[ \text{XI} \] use a non-factorizable geometry with $D' = 1$. Here, the 4–dimensional spacetime is warped with a factor $\Omega$ which depends on the extra dimension and Eq. \[ \text{XII} \] is modified as: $M_{\text{Pl}}(4) \sim \Omega^{-1} M_{\text{EW}}$. In our paper we concentrate on the factorizable geometry of ADD-type models.

\[ \text{2} \]

In most of these papers, moduli stabilization was considered without regard to the energy-momentum localized on the brane so that the dynamics of the multidimensional universe was mainly defined by the energy-momentum of the bulk matter. A brane matter contribution was taken into account, e.g., in \[ \text{XVIII} \].
a simplified model with multidimensional Lagrangian of the form \( L = f(R) \), where \( f(R) \) is an arbitrary smooth function of the scalar curvature. Without connection to stabilization of the extra-dimensions, such models (4-dimensional as well as multi-dimensional ones) were considered e.g. in Refs. [20, 27, 28]. There, it was shown that the nonlinear models are equivalent to models with linear gravitational action plus a minimally coupled scalar field with self-interaction potential. In [25], we advanced this equivalence towards investigating the stabilization problem for extra dimensions. Particular attention was paid to models with quadratic scalar curvature terms. It was shown that for certain parameter ranges, the extra dimensions are stabilized if the internal spaces have negative constant curvature. In this case, the 4-dimensional effective cosmological constant \( \Lambda_{eff} \) as well as the bulk cosmological constant \( \Lambda_D \) become negative. As a consequence, the homogeneous and isotropic external space is asymptotically AdS\(_4\). Because the considered nonlinear model is a pure geometrical one (only with a bare cosmological constant \( \Lambda_D \) as an exotic matter source included) the equivalent linear model contains only a minimally coupled scalar field as bulk matter. The null energy condition (NEC) \( T_{ab}N^aN^b \geq 0 \) for this field reads \( T_{ab}N^aN^b = (N\partial_a\phi)^2 \geq 0 \) (with \( N \) a future directed null vector) and is satisfied only marginally when the internal spaces are completely stabilized and the scalar field is frozen out. Moreover, the weak energy condition (WEC) \( T_{ab}W^aW^b \geq 0 \) (with \( W \) a future directed time-like vector) is violated in this case because the energy density \( \rho \) of the scalar field is negative definite \( \rho < 0 \). As a result, the aforementioned parameters (the internal space scalar curvatures, \( \Lambda_D \) and \( \Lambda_{eff} \)) are negative in the case of stabilized internal spaces (see also [17, 20, 21]).

However, a negative cosmological constant leads to a deceleration of the Universe instead of an accelerated expansion, as recent observational data indicate. According to these data our Universe is dominated by a dark energy component with negative pressure. For example, from observations of the clusters of galaxies follows that the energy density of the matter components which can clump in various structures is significantly undercritical. But, the position of the first acoustic peak in the angular power spectrum of the cosmic microwave background radiation (CMB) implies that the Universe is, on large scales, nearly flat. In other words, the energy density in the Universe is very close to the critical value. Thus, there must exist a homogeneously distributed exotic (dark) energy component. This observation is in agreement with the conclusion following from the Hubble diagram of type Ia supernovae (SN-Ia) at high redshifts, which also indicate that our Universe currently undergoes an accelerated expansion. Under the assumption of flatness, using the data of the CMB anisotropy measurements, high redshift SN-Ia observations and from local cluster abundances, the authors of Ref. [31] found a constraint on the equation of state parameter \( \omega_Q = P/\rho < -0.85 \) at 68% of confidence level. They concluded that this result is in perfect agreement with the \( \omega_Q = -1 \) cosmological constant case and gives no support to a quintessential field scenario with \( \omega_Q > -1 \). Results obtained in [31] also favor \( \omega_Q \approx -1 \) at the present epoch. In Ref. [29] we already indicated that the effective cosmological constant can be shifted from negative values to positive ones by including into the nonlinear model matter fields which satisfy the NEC. In the present paper, we demonstrate this effect explicitly by endowing the extra dimensions with real-valued solitonic form fields [32]. Such fields naturally arise as Ramond-Ramond (RR) form fields in type II string theory and M-theory. Within a generalized Freund-Rubin setting [33], their influence on the evolutionary dynamics of the Universe has been considered, e.g., in Refs. [34, 35, 36, 37] and due to its simplicity we adopt this ansatz here for the stability analysis of our nonlinear model. From Eqs. (2.7), (2.8) below, it can be easily seen that the real-valued form fields satisfy the NEC as well as the bulk cosmological constant \( \Lambda_D \) become negative. As a consequence, the homogeneous and isotropic external space is asymptotically AdS\(_4\). Because the considered nonlinear model is a pure geometrical one (only with a bare cosmological constant \( \Lambda_D \) as an exotic matter source included) the equivalent linear model contains only a minimally coupled scalar field as bulk matter. The null energy condition (NEC) \( T_{ab}N^aN^b \geq 0 \) for this field reads \( T_{ab}N^aN^b = (N\partial_a\phi)^2 \geq 0 \) (with \( N \) a future directed null vector) and is satisfied only marginally when the internal spaces are completely stabilized and the scalar field is frozen out. Moreover, the weak energy condition (WEC) \( T_{ab}W^aW^b \geq 0 \) (with \( W \) a future directed time-like vector) is violated in this case because the energy density \( \rho \) of the scalar field is negative definite \( \rho < 0 \). As a result, the aforementioned parameters (the internal space scalar curvatures, \( \Lambda_D \) and \( \Lambda_{eff} \)) are negative in the case of stabilized internal spaces (see also [17, 20, 21]).

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\(^3\)For a critical discussion of the different ECs we refer to [38].
generalized Freund-Rubin ansatz for the form fields and perform a dimensional reduction of the action functional to a 4-dimensional effective theory with \((n + 1)\) self-interacting minimally coupled scalar fields (section 3). The stabilization of the extra dimensions is then reduced to the condition that the obtained effective potential for these fields should have a minimum. In section 4 we present a detailed analysis of this problem for a model with one internal space. The main results are summarized and discussed in the concluding section 5.

2 General setup

We consider a \(D = (4 + D')\) - dimensional nonlinear gravitational theory with action

\[
S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|g|} f(\mathcal{R}) - \frac{1}{2} \int_M d^D x \sqrt{|\mathcal{g}|} \sum_{i=1}^n \frac{1}{4} \left( F^{(i)} \right)^2 ,
\]

(2.1)

where \(f(\mathcal{R})\) is an arbitrary smooth function with mass dimension \(O(m^2)\) (\(m\) has the unit of mass) of the scalar curvature \(\mathcal{R} = R[\mathcal{g}]\) constructed from the \(D\) - dimensional metric \(\mathcal{g}_{ab}\) \((a, b = 1, \ldots, D)\).

\[
\kappa_D^2 = 8\pi / M_{(4+D')}^2 \quad (2.2)
\]

denotes the \(D\) - dimensional gravitational constant (subsequently, we assume that \(M_{(4+D')} \sim M_{EW}\)). In action (2.1), \(F^{(i)} = F_{m_i n_i \ldots q_i}, \quad i = 1, \ldots, n\) is an antisymmetric tensor field of rank \(d_i\) (a \(d_i\) - form field strength) with indices from an index set \(s_{(i)} = \{m_i: \max(m_i) - \min(m_i) = d_i\}\), where \(m_i, n_i, \ldots, q_i \in s_{(i)}\). For simplicity, we suppose that the index sets \(s_{(i)}, s_{(j)}\) of tensors \(F^{(i)}, F^{(j)}\) with \(i \neq j\) contain no common elements as well as no indices corresponding to the coordinates of the \(D_0\) - dimensional external spacetime (usually \(D_0 = 4\)). Additionally, we assume that for the sum of the ranks holds \(\sum_{i=1}^n d_i = D - D_0 := D'\). Obviously, this model can be generalized to tensor configurations \(F^{(i)}, F^{(j)}\) with intersecting (overlapping) index sets. In this case explicit field configuration can be obtained, e.g., when the indices satisfy special overlapping rules [36]. Such a generalization is beyond the scope of the present paper. Furthermore, we assume in our subsequent considerations that the index sets \(m_i, n_i, \ldots, q_i \neq 0\) do not contain the coordinates of the external spacetime \(M_0\) and, hence, the field strengths \(F^{(i)}\) can be associated with a magnetic (solitonic) \(p\)-brane system located in the extra dimensions as discussed, e.g., in Refs. [32, 35, 36].

The equation of motion for the gravitational sector of (2.1) reads

\[
f^{\prime} R_{ab} - \frac{1}{2} f \mathcal{G}_{ab} - \nabla_a \nabla_b f' + \mathcal{g}_{ab} \nabla^c f' = \kappa_D^2 T_{ab} [F, \mathcal{g}] ,
\]

(2.3)

where \(a, b = 1, \ldots, D\), \(f' = df / d\mathcal{R}, \quad \mathcal{R}_{ab} = R_{ab}[\mathcal{g}], \quad \mathcal{R} = R[\mathcal{g}], \quad \nabla_a\) and \(\Box\) denote the covariant derivative and the Laplacian with respect to the metric \(\mathcal{g}_{ab}\)

\[
\Box = \Box[\mathcal{g}] = \mathcal{g}^{ab} \nabla_a \nabla_b = \frac{1}{\sqrt{|\mathcal{g}|}} \partial_a \left( \sqrt{|\mathcal{g}|} \mathcal{g}^{ab} \partial_b \right).
\]

(2.4)

Eq. (2.3) can be rewritten in the form

\[
f^{\prime} \mathcal{G}_{ab} + \frac{1}{2} \mathcal{G}_{ab} (\mathcal{R} f' - f) - \nabla_a \nabla_b f' + \mathcal{g}_{ab} \nabla^c f' = \kappa_D^2 T_{ab} [F, \mathcal{g}] ,
\]

(2.5)

where \(\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2} \mathcal{R} \mathcal{g}_{ab}\), and its trace

\[
(D - 1) \Box f' = \frac{D}{2} f - f' \mathcal{R} + \kappa_D^2 T [F, \mathcal{g}]
\]

(2.6)

can be considered as a connection between \(\mathcal{R}\) and \(f\). The energy momentum tensor (EMT) \(T_{ab} [F, \mathcal{g}]\) is defined in the standard way as

\[
T_{ab} [F, \mathcal{g}] \equiv \frac{1}{\sqrt{\mathcal{g}}} \delta \left( \sqrt{|\mathcal{g}|} \sum_{i=1}^n \frac{1}{4} dT (F^{(i)})^2 \right) = \sum_{i=1}^n T_{ab} [F^{(i)}, \mathcal{g}] ,
\]

(2.7)
where

\[ T_{ab}[F^{(i)}, \mathcal{G}] = \frac{1}{d_i!} \left( -\frac{1}{2}g_{ab} F^{(i)}_{m_i n_i \ldots q_i} F^{(i)}_{m_i n_i \ldots q_i} + d_i F^{(i)}_{a m_i \ldots q_i} F^{(i)}_{b n_i \ldots q_i} \right). \]  

(2.8)

For the trace of this tensor we obtain

\[ T[F, \mathcal{G}] = \sum_{i=1}^{n} T[F^{(i)}, \mathcal{G}] \]

(2.9)

with

\[ T[F^{(i)}, \mathcal{G}] = \frac{2d_i - D}{2(d_i!)} F^{(i)}_{m_i n_i \ldots q_i} F^{(i)}_{m_i n_i \ldots q_i}. \]

(2.10)

The field strengths \( F^{(i)} \) satisfy the equations of motion

\[ F^{(i)}_{m_i n_i \ldots q_i} = 0 \iff \frac{1}{\sqrt{|\mathcal{G}|}} \left( \sqrt{|\mathcal{G}|} F^{(i)}_{m_i n_i \ldots q_i} \right)_{q_i} = 0, \quad i = 1, \ldots, n. \]

(2.11)

and the Bianchi identities

\[ F^{(i)}_{[m_i \ldots q_i, a]} = 0, \quad i = 1, \ldots, n. \]

(2.12)

Following Refs. [26, 27, 28], we perform a conformal transformation

\[ g_{ab} = \Omega^{2} \mathcal{G}_{ab} \]

(2.13)

with

\[ \Omega = \left[ f'(\mathcal{R}) \right]^{1/(D-2)} \]

(2.14)

and reduce the nonlinear gravitational theory to a linear one with additional scalar field. This transformation is well defined for \( f'(\mathcal{R}) > 0 \) (concerning the case \( f' \leq 0 \) see footnote [6]). The equivalence of the theories can be easily proven with the help of the auxiliary formulae

\[ \Box = \Omega^{-2} \left[ \Box + (D-2) g^{ab} \Omega^{-1} \partial_{a} \partial_{b} \right] \iff \Box = \Omega^{2} \left[ \Box - (D-2) g^{ab} \Omega^{-1} \partial_{a} \partial_{b} \right], \]

(2.15)

\[ R_{ab} = \mathcal{R}_{ab} + \frac{D - 1}{D - 2} \left( f' - \mathcal{G}^{ab} \partial_{a} \partial_{b} \right) f' - \frac{1}{D - 2} g_{ab} (f')^{-1} \Box f' \]

(2.16)

and

\[ R = (f')^{2/(2-D)} \left( \mathcal{R} + \frac{D - 1}{D - 2} \left( f' - \mathcal{G}^{ab} \partial_{a} \partial_{b} \right) f' - 2 \frac{D - 1}{D - 2} (f')^{-1} \Box f' \right). \]

(2.17)

Defining the scalar \( \phi \) by the relation

\[ f' = \frac{df}{d \mathcal{R}} := e^{A \phi} > 0, \quad A := \sqrt{\frac{D - 2}{D - 1}} \]

(2.18)

and making use of [2.15] - [2.17], Eqs. [2.5] and [2.6] can be rewritten as

\[ G_{ab} = \kappa_{D}^{2} T_{ab} [F, \phi, g] + T_{ab} [\phi, g] \]

(2.19)

and

\[ \Box \phi = \frac{1}{\sqrt{(D - 1)(D - 2)}} e^{-\frac{D}{2} \phi} \left( \frac{D}{2} f - f' \mathcal{R} \right) + \frac{1}{\sqrt{(D - 1)(D - 2)}} \kappa_{D}^{2} T[F, \phi, g]. \]

(2.20)

The EMTs read

\[ T_{ab} [\phi, g] = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} \phi_{,m} \phi_{,n} - \frac{1}{2} g_{ab} e^{-\frac{D}{2} \phi} (\mathcal{R} f' - f), \]

(2.21)

\[ T_{ab} [F, \phi, g] = \sum_{i=1}^{n} e^{-\frac{2d_i - D}{2 \kappa_{D}}} T_{ab} [F^{(i)}, g] \]

(2.22)

and

\[ T[F, \phi, g] = \sum_{i=1}^{n} e^{-\frac{2d_i - D}{2 \kappa_{D}}} T[F^{(i)}, g], \]

(2.23)
where \( T_{ab}[F^{(i)}, g], T[F^{(i)}, g] \) are given by replacing \( \bar{g} \rightarrow g \) in Eqs. (2.8), (2.10). The indices of the field strengths \( F^{(i)} \) are now raised and lowered with the metric \( g \).

The equations of motion (2.11) for \( F^{(i)} \) transform to

\[
\frac{1}{\sqrt{|g|}} \left( \sqrt{|g|} e^{\frac{2d_i - D}{D-1(D-2)}} \phi \right) F^{(i) \, m_{i}, n_{i} \ldots q_{i}} = 0, \quad i = 1, \ldots, n, \tag{2.24}
\]

whereas the Bianchi identities (2.12) do not change.

It can be easily checked that Eqs. (2.19), (2.20) and (2.21) are the equations of motion for the action

\[
S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|g|} \left\{ R[g] - g^{ab} \partial_{(a} \phi \partial_{b)} - 2U(\phi) - \kappa_n^2 \sum_{i=1}^n \frac{1}{d_i!} \sqrt{\frac{2d_i - D}{D-1(D-2)}} \phi \right\} F^{(i)\, m_{i}, n_{i} \ldots q_{i}}, \tag{2.25}
\]

where

\[
U(\phi) := \frac{1}{2} e^{-B \phi} \left[ \phi R(\phi) e^{A \phi} - f(\phi R(\phi)) \right], \quad B := \frac{D}{\sqrt{(D-1)(D-2)}} \tag{2.26}
\]

and Eq. (2.18) is used to express \( \bar{R} \) as a function of \( \phi : R = \bar{R}(\phi) \). The scalar field \( \phi \) is the result and the carrier of the curvature nonlinearity of the original theory. Correspondingly, Eq. (2.20) has a two-fold interpretation. It is the equation of motion for the field \( \phi \) and at the same time it can be considered as constraint equation following from the reduction of the nonlinear theory (2.1) to the linear one (2.25). Furthermore, we note that in the linear theory (2.25) the form fields are non-minimally coupled with the nonlinearity field \( \phi \). (A minimal coupling occurs only for a model with \( n = 1, d_1 = D_0 \), where according to (2.10) the trace of the form field EMT vanishes.) A comparison of the action functional with (2.22) shows that the last term in (2.24) coincides with the expression for the energy density \( -T^D_\theta[F,\phi,g] \) of the solitonic form field (due to \( F^{(i)\, m_{i}, n_{i} \ldots q_{i}} \equiv 0 \) by the definition of \( F^{(i)} \)).

Let us consider what happens if, in some way, the scalar field \( \phi \) tends asymptotically to a constant: \( \phi \rightarrow \phi_0 \) [precisely this situation should hold when the internal space undergoes a (freezing) stabilization]. From Eq. (2.18) we see that in this limit the nonlinearity in (2.1) disappears: \( f(R) \approx c_1 (R - \bar{R}_0) + f(\bar{R}_0) \equiv c_1 R + c_2 \), where \( c_1 := f'(\bar{R}_0) = \exp(\Lambda \phi_0), \quad \bar{R}_0 = \bar{R}(\phi_0), \) and \( -c_2/(2c_1) \) plays the role of a cosmological constant. In the case of homogeneous and isotropic spacetime manifolds, linear purely geometrical theories with constant \( \Lambda \)–term necessarily imply an (A)dS geometry so that the manifolds are Einstein spaces. In our model, the additional form fields destroy this asymptotical behavior. Instead, we obtain from Eqs. (2.6) and (2.5)

\[
\bar{R} \rightarrow \frac{D}{D-2} c_1 \frac{c_2}{D - 2} \frac{1}{2\kappa^2} \sum_{i=1}^n \frac{2d_i - D}{2(d_i!)} \left( F^{(i)} \right)^2 \tag{2.27}
\]

and

\[
\bar{R}_{ab} \rightarrow \left[ -\frac{1}{D-2} \frac{c_2}{c_1} - \frac{2}{c_1 D - 2} \kappa^2 \sum_{i=1}^n \frac{d_i - 1}{2(d_i!)} \left( F^{(i)} \right)^2 \bar{g}_{ab} + \frac{1}{c_1} \kappa^2 \sum_{i=1}^n \frac{2d_i}{2(d_i!)} \left( F^{(i)}_{m_{i}n_{i} \ldots q_{i}} F^{(i)\, m_{i}n_{i} \ldots q_{i}} \right) \right], \tag{2.28}
\]

where the form field product

\[
\left( F^{(i)} \right)^2 := F^{(i)\, m_{i}n_{i} \ldots q_{i}} F^{(i)\, m_{i}n_{i} \ldots q_{i}} \tag{2.29}
\]

is defined with respect to the metric \( \bar{g} \). For a form field, which asymptotically tends to a constant, the scalar curvature and the Ricci tensor also approach constant values. But whereas \( \bar{R} \) and \( R \) are asymptotically connected by the relation [see Eqs. (2.14), (2.10) and (2.21)]

\[
R \rightarrow c_1 \frac{\bar{R}}{\bar{g}}, \tag{2.30}
\]

the Ricci tensor \( \bar{R}_{ab} \) will not be proportional to the metric \( \bar{g}_{ab} \) and, hence, the space will not be Einsteinian. This is in obvious contrast to a nonlinear model of purely geometrical type [25] where the stabilization will result in an asymptotical (A)dS spacetime.

\[^{4}\text{Thus, for brevity, we shall refer to the field } \phi \text{ as nonlinearity scalar field.}\]
In the rest of the paper we consider for simplicity a toy model\(^5\) with quadratic curvature term:
\[
f(R) = R + \alpha R^2 - 2\Lambda_D ,
\]
where the parameter \(\alpha\) has dimension \(\mathcal{O}(m^{-2})\). For this model we obtain
\[
f'(R) = 1 + 2\alpha R = e^{A\phi} \iff R = \frac{1}{2\alpha} (e^{A\phi} - 1)
\]
and
\[
U(\phi) = \frac{1}{2} e^{-B\phi} \left[\frac{1}{4\alpha} (e^{A\phi} - 1)^2 + 2\Lambda_D\right].
\]
The condition\(^6\) \(f' > 0\) implies \(1 + 2\alpha R > 0\) in the limit \(\alpha \to 0\) the nonlinearity is switched off and the linear theory is recovered. Correspondingly, it holds \(f' \to 1\) with implication \(c_1 = 1, \phi_0 \to 0\) so that also \(R \to R\) (in accordance with Eq. (2.31)) and \(U(\phi \to 0) \to \Lambda_D\). The corresponding region of weak nonlinearity is defined by the condition \(\alpha R = e^{A\phi} - 1 \ll 1\).

### 3 Spontaneous compactification and dimensional reduction

The simple block-orthogonal structure of the field strength \(F\) shows that there is a preferable scheme for a spontaneous compactification of the multidimensional spacetime manifold: each of the form fields \(F^{(i)}\) is nested in its own \(d_i\)-dimensional space \(M_i\). Thus, the \(D\)-dimensional spacetime \(M\) can be endowed with the structure of a warped product manifold
\[
M \longrightarrow M = M_0 \times M_1 \times \ldots \times M_n
\]
with metric
\[
\overline{g} = \overline{g}_{ab}(X) dX^a \otimes dX^b = \overline{g}^{(0)} + \sum_{i=1}^{n} \overline{g}^{(i)}.
\]
The coordinates on the \((D_0 = d_0 + 1)\)-dimensional manifold \(M_0\) (usually interpreted as our \((D_0 = 4)\)-dimensional Universe) are denoted by \(x\) and the corresponding metric by
\[
\overline{g}^{(0)} = \overline{g}^{(0)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu.
\]
Let the internal factor manifolds \(M_i\) be \(d_i\)-dimensional warped Einstein spaces with warp factors \(\exp(\overline{f}(x))\) and metrics
\[
\overline{g}^{(i)} = e^{2\overline{f}(x)} \gamma_{m_i,n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i},
\]
i.e.,
\[
R_{m_i,n_i} \left[ \gamma^{(i)} \right] = \lambda r_i^{-1} \gamma_{m_i,n_i}^{(i)}, \quad m_i, n_i = 1, \ldots, d_i
\]
and
\[
R \left[ \gamma^{(i)} \right] = \lambda r_i^{-1} \equiv R_i \sim kr_i^{-2},
\]
where \(k = 0, \pm 1\). The scale \(r_i\) sets the characteristic size of \(M_i\) (modulo the warp factor \(\exp(\overline{f}(x))\)) and can be interpreted as an effective scale factor of the compact Einstein space \(M_i\) with metric \(\gamma^{(i)}\) and corresponding

\(^5\) For considerations on higher order corrections to the gravity sector of M/string theory we refer to [25][33].

\(^6\) Obviously, the conformal transformation \(x \mapsto \lambda x\), \(\phi \mapsto \lambda^2 \phi\) becomes singular when \(f'(R)\) vanishes. The transformation itself can be extended from the \(f' > 0\) branch to the \(f' < 0\) branch with the help of an ansatz \(\Omega = |f'(R)|^{1/(D-2)}\) and a corresponding redefinition of the nonlinearity field \(\phi\): \(e^{A\phi} = |f'|\). As result, one obtains an action functional for the \(f' < 0\) branch which differs from action (2.23) for \(f' > 0\) in its total sign and in the relative signs of the single terms as well as in the potential \(U(\phi)\). Most important, for a fixed sign of the Einstein-Hilbert term the kinetic term of the nonlinearity field has the correct sign, whereas the kinetic terms of additional matter fields (in our case the form fields) have the wrong relative sign. This leads to a set of equations of motions which differ from (2.25). For details we refer to [27]. Unfortunately, the sign switch occurs for all additional matter fields simultaneously and it is not controllable for some selected fields separately. Otherwise, it could have provided a natural mechanism for the generation of a phantom energy component with equation of state parameter \(\omega_Q < -1\) (and a corresponding super-acceleration of the observable Universe) in the sense of [13].
\[ V_{\phi} \equiv \int d^d y \sqrt{|\nabla^i|} \sim r_i^{d_i}, \quad i = 1, \ldots, n, \] 

(3.7)

where \( V_{\phi} \) has dimension \( O(m^{-d_i}). \)

We note that the specific metric ansatz \( \mathbf{32} \) - \( \mathbf{34} \) for the warped product of Einstein spaces results in a scalar curvature \( R \) which depends only on \( x: R|_x = R(x) \). Correspondingly, the nonlinearity field \( \phi \) is also a function only of \( x: \phi = \phi(x) \).

The conformally transformed metric \( \mathbf{34} \) reads

\[ g = \Omega^2 \mathcal{g} = \left( e^{2\phi} \right)^{2/(D-2)} \mathcal{g} := g^{(0)} + \sum_{i=1}^{n} e^{2\beta_i(x)} \gamma^{(i)} \]

(3.8)

with

\[ g^{(0)}_{\mu\nu} = \left( e^{2\phi} \right)^{2/(D-2)} g^{(0)}_{\mu\nu}, \]

(3.9)

\[ \beta^i = \beta + \frac{A}{D-2} \phi. \]

(3.10)

For the field strengths \( F^{(i)} \) we choose a generalized Freund-Rubin ansatz \( \mathbf{33} \) (see also \( \mathbf{34} \) \( \mathbf{35} \) \( \mathbf{36} \) \( \mathbf{37} \)):

\[ F_{m_1 m_2 \ldots q_i}^{(i)} = \sqrt{2} \sqrt{|g^{(i)}|} \varepsilon_{m_1 m_2 \ldots q_i} f^{(i)}(x), \quad F_{m_1 m_2 \ldots q_i}^{(i)} = \left( \sqrt{2} \sqrt{|g^{(i)}|} \right) \varepsilon_{m_1 m_2 \ldots q_i} f^{(i)}(x), \]

(3.11)

where \( g^{(i)} \equiv e^{2\beta^i} \gamma^{(i)} \) and \( \varepsilon_{m_1 m_2 \ldots q_i} \) is the Levi-Civita symbol. We use conventions where for Riemann spaces holds \( \varepsilon_{m_1 m_2 \ldots q_i} = \varepsilon_{m_1 m_2 \ldots q_i} \) and \( \varepsilon_{m_1 m_2 \ldots q_i} \varepsilon_{m_1 m_2 \ldots q_i} = d_i ! \). It can be easily seen that the ansatz \( \mathbf{3.11} \) satisfies Eq. \( \mathbf{2.12} \) (because \( \phi \) and \( f \) depend only on \( x \) and the \( \sqrt{|\gamma^{(0)}|} \) factors cancel). The Bianchi identities \( \mathbf{2.12} \) reduce to the equations

\[ \partial \left( a_i^{d_i}(x)f^{(i)}(x) \right) = 0 \]

(3.12)

with solutions

\[ f^{(i)}(x) = \frac{f_i}{a_i^{d_i}}, \]

(3.13)

where \( a_i := e^{\beta_i} \) are the scale factors of the internal spaces \( M_i \) after conformal transformation \( \mathbf{33} \) and \( f_i \equiv \text{const.} \) We choose the scale factors \( a_i \) dimensionless so that the constants \( f_i^2 \) have dimension \( O(m^{4+D}) \) and \( \kappa_0^2 f_i^2 \sim O(m^2) \). With \( \mathbf{33} \) the energy density of the solitonic form field, and correspondingly the last term in action \( \mathbf{2.23} \), reads

\[ - T_0^0[F, \phi, g] = \sum_{i=1}^{n} \frac{1}{d_i!} e^{4D(1)} \phi F_{m_1 m_2 \ldots q_i}^{(i)} F_{m_1 m_2 \ldots q_i}^{(i)} = \sum_{i=1}^{n} e^{4D(1)} \phi \frac{f_i^2}{a_i^{2d_i}} := \rho(\beta, \phi), \]

(3.14)

where for real form fields \( f_i^2 \geq 0 \). Again we see that for models with \( n = 1 \) and \( d_1 = D_0 \) this energy density decouples from the nonlinearity scalar field \( \phi: \rho(\beta, \phi) \rightarrow \rho(\beta^1) \).

The fact that \( \phi, \beta^i \) and \( \rho \) depend only on \( x \) allows us to perform a dimensional reduction of action \( \mathbf{2.23} \). Without loss of generality we set the compactification scales of the internal spaces \( M_i: i = 1, \ldots, n \) at present time at \( \beta_i = 0 \). This means that at present time the total volume of the internal spaces is completely defined by the characteristic scale factors \( r_i \) (see \( \mathbf{37} \) and footnote \( \mathbf{17} \)):

\[ V_{D_i} \equiv \prod_{i=1}^{n} \int_{M_i} d^{d_i} y \sqrt{|\nabla^i|} \sim \prod_{i=1}^{n} r_i^{d_i}, \]

(3.15)

\(^7\)The volume is well defined for positive curvature spaces \( (k=+1) \). For compact negative and zero curvature spaces \( (k=-1,0) \), i.e. compact hyperbolic spaces (CHSs) \( M_i = H^{d_i}/\Gamma_i \) and tori \( T_i = R^{d_i}/\Gamma_i \), we interpret this volume as scaled volume of the corresponding fundamental domain (“elementary cell”) \( V_{D_i} \sim r_i^{d_i} \times V_{D(1)} \) (see, e.g., \( \mathbf{34} \) and references therein). Here \( H^{d_i}, R^{d_i} \) are hyperbolic and flat universal covering spaces, and \( \Gamma_i, \Gamma_i \) are discrete subgroups of isometries. Furthermore, we assume for the scale factors of the metrics \( \gamma^{(i)} \sim r_i^{d_i} \) scaled such that \( V_{F_D(1)} \sim O(1) \). Thus, the volume \( V_{D_i} \) is mainly defined by \( r_i \). In all three cases \( (k=\pm 1,0) \), the limit \( r_i \rightarrow \infty \) results in an effective decompactification of the internal space with \( V_{D_i} \rightarrow \infty \). In accordance with Eq. \( \mathbf{3.10} \), this means that the positive and negative constant curvature spaces flatten: \( R_{D_i} \rightarrow 0 \). Clearly, for compact Ricci-flat spaces holds \( R_{D_i} \equiv 0 \) by definition and without relation to the compactification scale of the torus.
where $D' = D - D_0 = \sum_{i=1}^{n} d_i$ is the number of extra dimensions and $V_{D'}$ has dimension $O(m^{-D'})$. After dimensional reduction action \cite{2.25} reads

$$
S = \frac{1}{2\kappa_0^2} \int d^{D_0} x \sqrt{|g^{(0)}|} \prod_{i=1}^{n} e^{d_i \beta_i} \left\{ R \left[ g^{(0)} \right] - G_{ij} g^{(0)\mu\nu} \partial_{\mu} \beta_i \partial_{\nu} \beta_j - g^{(0)\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \sum_{i=1}^{n} R \left[ g^{(i)} \right] e^{-2\beta_i} - 2U(\phi) - 2\kappa_D^2 \rho(\beta, \phi) \right\}. \tag{3.16}
$$

where $G_{ij} = d_i \delta_{ij} - d_i d_j$ $(i,j = 1, \ldots, n)$ is the midisuperspace metric \cite{40, 47} and

$$
\kappa_0^2 := \frac{\kappa_D^2}{V_{D'}} \tag{3.17}
$$

denotes the $D_0$-dimensional (4-dimensional) gravitational constant. If we take the electroweak scale $M_{EW}$ and the Planck scale $M_P(4)$ as fundamental ones for $D$-dimensional (see Eq. \cite{2.22}) and 4-dimensional spacetimes $(\kappa_0^2 = 8\pi/M_P^2(4))$ respectively, then we reproduce Eqs. \cite{1.11} and \cite{1.14}.

Action \cite{3.16} is written in the Brans-Dicke frame. Conformal transformation to the Einstein frame \cite{22, 23} yields

$$
\tilde{g}_{\mu\nu}^{(0)} = \left( \prod_{i=1}^{n} e^{d_i \beta_i} \right)^{\frac{2}{n_0 - 2}} g_{\mu\nu}^{(0)} \tag{3.18}
$$

$$
S = \frac{1}{2\kappa_0^2} \int d^{D_0} x \sqrt{\tilde{g}^{(0)}} \left\{ R \left[ \tilde{g}^{(0)} \right] - \tilde{G}_{ij} \tilde{g}^{(0)\mu\nu} \partial_{\mu} \beta_i \partial_{\nu} \beta_j - \tilde{g}^{(0)\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - 2U_{\text{eff}}(\beta, \phi) \right\}. \tag{3.19}
$$

The tensor components of the midisuperspace metric (target space metric on $\mathbb{R}^n_\ell$) $\tilde{G}_{ij}$ $(i,j = 1, \ldots, n)$, its inverse metric $\tilde{G}^{ij}$ and the effective potential are correspondingly

$$
\tilde{G}_{ij} = d_i \delta_{ij} + \frac{1}{D_0 - 2} d_i d_j , \quad \tilde{G}^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D} \tag{3.20}
$$

and

$$
U_{\text{eff}}(\beta, \phi) = \left( \prod_{i=1}^{n} e^{d_i \beta_i} \right)^{-\frac{2}{n_0 - 2}} \left[ -\frac{1}{2} \sum_{i=1}^{n} R_i e^{-2\beta_i} + U(\phi) + \kappa_D^2 \rho(\beta, \phi) \right], \tag{3.21}
$$

where $U(\phi)$ and $\rho(\beta, \phi)$ are defined by Eqs. \cite{2.23} and \cite{3.14}.

A stable compactification of the internal spaces is ensured when the scale factors of the internal spaces $\beta^i$ are frozen at one of the minima of the effective potential $U_{\text{eff}}$. The value of the effective potential at the minimum plays the role of the effective $D_0$-dimensional cosmological constant: $U_{\text{eff}}|_{\text{min}} \equiv \Lambda_{\text{eff}}$. Assuming for the frozen scale factors at present time $\beta^i = 0$, we obtain the non-zero components of the asymptotic Ricci tensor \cite{2.25} as

$$
\overline{R}_{\mu\nu} \to \partial \tilde{g}_{\mu\nu} , \tag{3.22}
$$

$$
\overline{R}_{m,n_i} \to \left( \partial + \frac{2}{c_1} \kappa_D^2 f_i^2 \Omega_0^{2d_j} \right) \tilde{g}_{m,n_i} , \tag{3.23}
$$

where

$$
\partial := -\frac{1}{D - 2} \frac{c_2}{c_1} - \frac{2}{c_1} \frac{\kappa^2_D}{D - 2} \sum_{j=1}^{n} (d_j - 1) f_j^2 \Omega_0^{2d_j} \tag{3.24}
$$

and $\Omega_0 = \left( e^{A\phi_0} \right)^{1/(D-2)}$. Thus, the asymptotic multidimensional spacetime is built up from Einstein-space blocks, but is itself a non-Einsteinian space due to the additional term in \cite{3.23}.
4 Stabilization of the internal space

Without loss of generality\(^8\), we consider in the present section a model with only one \(d_1\)-dimensional internal space. The corresponding action (3.19) reads

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} \left\{ R \left[ g^{(0)} \right] - \tilde{g}^{(0)} \mu \nu \partial_\mu \varphi \partial_\nu \varphi - \tilde{g}^{(0)} \mu \nu \partial_\mu \phi \partial_\nu \phi - 2U_{\text{eff}}(\varphi, \phi) \right\},
\]

where

\[
\varphi := -\sqrt{\frac{d_1(D-2)}{D_0-2}} \beta^1
\]

and

\[
U_{\text{eff}}(\varphi, \phi) = e^{\frac{2\varphi}{\sqrt{(D-2)(D_0-2)}}} \left[ -\frac{1}{2} R_1 e^{\frac{2\phi}{\sqrt{(D-2)(D_0-2)}}} + U(\phi) + \kappa^2_{D} \rho(\phi, \phi) \right].
\]

The potential \(U(\phi)\) of the nonlinearity scalar field is given by Eq. (2.33) and the energy density (3.14) of the solitonic form field reads

\[
\kappa^2_{D} \rho(\phi, \phi) = \kappa^2_{D} f_1^2 e^{\frac{2d_1-D}{10(D-1)})} e^{\frac{2\varphi}{\sqrt{(D-2)(D_0-2)}}}.
\]

For brevity of the notation, we introduce

\[
a := 2 \sqrt{\frac{D_0-2}{d_1(D-2)}}, \quad b := 2 \sqrt{\frac{d_1}{(D-2)(D_0-2)}}, \quad c := \frac{2d_1-D}{(D-1)(D-2)}, \quad h := \kappa^2_{D} f_1^2
\]

so that the effective potential reads

\[
U_{\text{eff}} = e^{b\varphi} \left[ -\frac{1}{2} R_1 e^{a\varphi} + U(\phi) + h e^{c\varphi} e^{ad_1\varphi} \right].
\]

From (4.10) we see that a real-valued form field \(f_1\) implies a non-negative \(h = \kappa^2_{D} f_1^2 \geq 0\). For the rest of the paper, we continue to work with dimensionless scalar fields \(\varphi, \phi\) instead of passing to canonical ones (modulo \(8\pi\)): \(\hat{\phi} = \varphi M_{P(1)}\), \(\hat{\phi} = \phi M_{P(1)}\) and \(U_{\text{eff}} = M^2_{P(1)} U_{\text{eff}}\). The restoration of the correct dimensionality is obvious.

In order to ensure a stabilization and asymptotical freezing of the internal space \(M_1\), the effective potential should have a minimum with respect to the scalar field \(\varphi\)

\[
\partial_\varphi U_{\text{eff}}|_{\text{extr}} = 0,
\]

so that for a minimum position at \(\varphi_0 = 0\) (which corresponds to a compactification scale \(\beta^1 = 0\) at present time) it should hold

\[
\frac{a+b}{2} R_1 = b U(\phi) + (ad_1 + b) h e^{c\phi}.
\]

This formula shows that the potential \(U_{\text{eff}}(\varphi, \phi)\) must also have a minimum with respect to \(\phi\), because without stabilization of \(\phi\) the right hand side remains a dynamical function whereas the left hand side is a constant. This second extremum condition

\[
\partial_\phi U_{\text{eff}}|_{\text{extr}} = 0
\]

yields

\[
\left[ \partial_\phi U + h e^{c\phi} \right]|_{\text{extr}} = 0.
\]

Additionally, the eigenvalues of the mass matrix of the coupled \((\varphi, \phi)\)–field system, i.e. the Hessian of the effective potential at the minimum position,

\[
J := \begin{pmatrix}
    \partial^2_{\varphi\varphi} U_{\text{eff}} & \partial^2_{\varphi\phi} U_{\text{eff}} \\
    \partial^2_{\phi\varphi} U_{\text{eff}} & \partial^2_{\phi\phi} U_{\text{eff}}
\end{pmatrix}
\]

\[^8\text{The difference between a general model with } n > 1 \text{ internal spaces and the particular one with } n = 1 \text{ consists in an additional diagonalization of the geometrical moduli excitations.}\]
should be positive definite
\[ m_{1,2}^2 = \frac{1}{2} \left[ \text{Tr}(J) \pm \sqrt{\text{Tr}^2(J) - 4 \det(J)} \right] > 0. \]  

(4.12)

According to the Sylvester criterion this is equivalent to the conditions
\[ J_{11} > 0, \quad J_{22} > 0, \quad \det(J) > 0. \]  

(4.13)

From (4.11) we see that in the special case of \( \partial^2 \phi U_{\text{eff}} \bigg|_{\text{extr}} = 0 \) the Hessian is diagonal and the excitation modes of the fields decouple. The eigenvalues of \( J \) coincide in this case with the masses squared of the scale factor excitations (gravitational excitons [22]) \( m_1^2 = m_2^2 \) and the excitations of the nonlinearity field \( m_3^2 = m_4^2 \).

Let us now analyze the stability conditions (4.8), (4.10) and (4.13) explicitly. For this purpose we introduce the auxiliary notations
\[ \phi_0 := \phi|_{\text{extr}}, \quad X := e^{A\phi_0} \geq 0, \quad q := 8\alpha \Lambda_D \]  

(4.14)

and rewrite the potentials \( U, U_{\text{eff}} \) and the derivatives of \( U_{\text{eff}} \) at a possible minimum position \((\varphi_0 = 0, \phi_0)\) as
\[ U_0 \equiv U|_{\text{extr}} = \frac{1}{8\alpha} X^{-\frac{D-2}{2}} \left[ (X - 1)^2 + q \right], \]  

(4.15)

\[ U_{\text{eff}}|_{\text{extr}} = -\frac{1}{2} R_1 + U_0(X) + hX^{\frac{2d_1-D}{d_1-1}} \]  

(4.16)

\[ \partial_\phi U_{\text{eff}}|_{\text{extr}} = -\frac{a + b}{2} R_1 + bU_0(X) + (d_1 a + b) hX^{\frac{2d_1-D}{d_1-1}} = 0, \]  

(4.17)

\[ \partial^2 \phi U_{\text{eff}}|_{\text{extr}} = \frac{1}{8\alpha} X^{-\frac{D-2}{2}} \left[ (2A - B)X^2 - 2(A - B)X - (q + 1)B \right] + hcX^{\frac{2d_1-D}{d_1-1}} = 0, \]  

(4.18)

\[ \partial^2 \phi^2 U_{\text{eff}}|_{\text{extr}} = \frac{a^2 - b^2}{2} R_1 - b^2 U_0(X) + \left[ (d_1 a)^2 - b^2 \right] hX^{\frac{2d_1-D}{d_1-1}}, \]  

(4.19)

\[ \partial^2 \phi^2 U_{\text{eff}}|_{\text{extr}} = c d_1 \alpha hX^{\frac{2d_1-D}{d_1-1}}, \]  

(4.20)

\[ \partial^2 \phi^2 U_{\text{eff}}|_{\text{extr}} = \frac{1}{8\alpha} X^{-\frac{D-2}{2}} \left[ (2A - B)^2 X^2 - 2(A - B)^2 X + (q + 1)B^2 \right] + c^2 hX^{\frac{2d_1-D}{d_1-1}}. \]  

(4.21)

(The constants \( A, B \) are defined in Eqs. (2.18) and (2.20), respectively.) We see that, for fixed dimensions \( D_0 \) and \( d_1 \), the two equations (4.17), (4.18) describe a 3-dimensional algebraic variety \( V \subset M \) in the 5-dimensional parameter (moduli) space \( \mathcal{M} = \mathbb{R}^3 \times \mathbb{R}_+^2 \supseteq (\alpha, \Lambda_D, R_1, h, X) \). On the variety, inequalities (4.13) of the Sylvester criterion define then the parameter region \( \mathcal{Y} \subset V \) of stable compactifications. A natural strategy for extracting detailed information about the location of this stability region would consist in solving (4.18) for \( X \) with subsequent back-substitution of the found roots into the inequalities (4.13) and the equation (4.17). In the following consideration we restrict our attention to the three simplest nontrivial cases which are easy to handle analytically.

### 4.1 Zero effective cosmological constant: \( \Lambda_{\text{eff}} \equiv 0 \)

By definition, we have \( \Lambda_{\text{eff}} \equiv U_{\text{eff}}|_{\text{extr}} \) so that in the particular case \( \Lambda_{\text{eff}} \equiv 0 \) Eq. (4.16) yields the additional constraint
\[ U_{\text{eff}}|_{\text{extr}} = -\frac{1}{2} R_1 + U_0(X) + hX^{\frac{2d_1-D}{d_1-1}} = 0. \]  

(4.22)

Combining this constraint with (4.17) we obtain the relation
\[ R_1 = 2 d_1 h X^{\frac{2d_1-D}{d_1-1}} = \frac{2d_1}{d_1-1} U_0(X) \]  

(4.23)

which can be used to eliminate the \( hX^{\frac{2d_1-D}{d_1-1}} \) term from (4.18). As result we arrive at a simple quadratic equation in \( X \) with physically sensible solutions
\[ e^{A\phi_0} \equiv X = \begin{cases} \frac{-1 + \sqrt{1 + (d_1-2)d_1(1+q)}}{d_1-2}, & d_1 > 2, \\ \frac{1+q}{d_1} = 2. & \end{cases} \]  

(4.24)

9The compactification scale (modulus) \( r_1 \) of the internal space \( M_1 \) enters \( V \subset \mathcal{M} \) via curvature scalar \( R_1 \) (see Eq. (6.6)).
With the help of Eqs. (4.23) - (4.25) and repeated use of a substitution-elimination technique, the potential $U_0$ and the second derivatives (4.19) - (4.21) of the effective potential can be rewritten in the simpler form

$$U_0(X) = \frac{d_1 - 1}{4\alpha d_1} X^{-\frac{d_1}{2}} (X - 1)$$

and

$$J_{11} = \left. \frac{\partial^2 U_{\text{eff}}}{\partial \varphi^2} \right|_{\text{extr}} = a^2 d_1 U_0(X),$$

$$J_{22} = \left. \frac{\partial^2 U_{\text{eff}}}{\partial \varphi^2} \right|_{\text{extr}} = \frac{B^2}{4\alpha D^2} X^{-\frac{d_1}{2}} [EX + 4(D_0 - 1)],$$

$$J_{12} = \left. \frac{\partial^2 U_{\text{eff}}}{\partial \varphi \partial \psi} \right|_{\text{extr}} = \frac{cd_1 a}{d_1 - 1} U_0(X),$$

where $E \equiv (D - 4)^2 + 4(d_1 - 2) > 0$ for $d_1 \geq 2$. For the determinant of the Hessian (4.11) we get

$$\det(J) = \frac{1}{\alpha(d_1 - 1)} \frac{D_0 - 2}{D - 1} U_0(X) X^{-\frac{d_1}{2}} [(d_1 - 2)X + 1].$$

With the equations (4.23) - (4.29) at hand, we are well prepared to explicitly describe the location of the stability region $\Upsilon$ in the parameter space $\mathcal{M}$. Let us start with relation (4.23). From the non-negativity conditions $h \geq 0$ and $e^{4\alpha \phi_0} \equiv X \geq 0$ we immediately conclude that for stable spaces $M_1$ it should hold $R_1 \geq 0$ and $U_0(X) \geq 0$. Furthermore, we see from the latter condition and the Sylvester criterion $J_{22} > 0$, $\det(J) > 0$ [applied to (4.23) and (4.25)] that for internal spaces of dimension $d_1 \geq 2$ the parameter $\alpha$ is restricted to positive values $^{10} \alpha > 0$ (the limiting case $\alpha \to 0$ we discuss below). Finally, we note that Eq. (4.25) together with $U_0(X) \geq 0$ and $\alpha > 0$ implies $X \geq 1$ and, hence, we find for $d_1 > 2$ and $d_1 = 2$ from the roots $^{12} q = 8\alpha \Lambda_D \geq 0$ and also $\Lambda_D \geq 0$.

Summarizing the obtained restrictions, we can describe the part $\Theta$ of the parameter space $\mathcal{M}$ where the stability region $\Upsilon$ of the variety $\mathcal{V}$ is located:

$$\Upsilon \subset \mathcal{V} \cap \Theta \subset \Theta = (\alpha \geq 0, \Lambda_D \geq 0, R_1 \geq 0, R_1 \geq 0, X \geq 1) \subset \mathcal{M}. \quad (4.30)$$

It remains to clarify what happens in the various limiting cases when the parameters reach the boundary $\partial \Theta$.

(L.1.1) $q \to +0$: According to (4.24), this limit implies $X \to 1$, $\phi_0 \to 0$. Because of $q = 8\alpha \Lambda_D$ we have to distinguish the two cases $\alpha \to 0$ and $\Lambda_D \to 0$. In these limits we obtain $U_0(X) \to \Lambda_D$ and $U_0(X) \to 0$, respectively.

(L.1.2) $\alpha \to +0, \Lambda_D \neq 0$: The case $\alpha \to 0$ describes the transition to a linear model. Here we have $U(\phi) \to \Lambda_D$, $R_1 \to 2d_1/(d_1 - 1)|\Lambda_D$ and $\Lambda_D \to (d_1 - 1)h$. In this limit, the mass of the $\phi$-field excitations tends to infinity $m_\varphi^2 \to m_\varphi^2 \to J_{12} \to \infty$ and the field itself becomes frozen at the position $\phi_0 \to 0$. The stabilization of the internal space occurs for $R_1, h, \Lambda_D > 0$ with the gravexciton masses $m_1^2 \to m_2^2 \to J_{11} = 4[(D_0 - 2)/(D - 2)]\Lambda_D$. This is in accordance with the results of Ref. 22, where a linear model with monopole terms was considered.

(L.1.3) $\Lambda_D \to 0, \alpha \neq 0$: Due to (4.24) and (L.1.1) this limit implies $X \to 1$, $\phi_0 \to 0$ and $U_0(X) \to 0$ so that according to (4.25) the excitation masses $m_1$, $m_2$ decouple ($J_{12} \to 0$) and the gravexciton mass vanishes $m_1^2 \to m_2^2 \to 0$. Hence, the limit $\Lambda_D \to 0$ is connected with a destabilization of the internal space $M_1$. The mass of the nonlinearity field excitations $m_\varphi$ remains finite $m_\varphi^2 \to m_\varphi^2 \sim 1/\alpha$ for $\alpha \to 0$.

(L.1.4) $h \to +0, \alpha \neq 0$: From (4.23) - (4.25) we see that this limit of a vanishing form field is connected with $R_1 U_0(X), \Lambda_D \to +0$. Thus the excitations of the nonlinearity field $\phi$ decouple from gravexcitons ($J_{12} \to 0$). Simultaneously, because of $R_1 \to +0 \Rightarrow r_1 \to \infty$, the internal space $M_1$ undergoes a decompactification and due to $U_0(X) \to 0$ the effective potential $U_{\text{eff}}$ becomes flat in the $\varphi$-direction ($J_{11} \to 0 \Rightarrow m_\varphi \to 0$). This means that the internal space destabilizes, whereas $U_{\text{eff}}$ remains well behaved with respect to $\phi$. These results completely confirm the conclusions of paper 23 for a nonlinear gravitational model without form fields where a stabilization is only possible for $\Lambda_{\text{eff}} < 0$.

Finally, we note that for a model with $d_1 = D_0$ (and, hence, a vanishing trace of the form field EMT) the excitations of nonlinearity field $\phi$ decouple from the gravexcitons: $J_{12} = 0$ because of $c = 0$ in (1.6), (1.8).
4.2 Traceless EMT of the form field: \( d_1 = D_0 \)

The easy handling of nonlinear models with a traceless form-field EMT as well as of models with a two-dimensional internal space \( M_1 \) is connected with the structure of Eq. (4.11). For \( X > 0, \alpha \neq 0 \) we obtain from Eqs. (4.13), (4.18) and the definitions of \( A \) and \( B \):

\[
\frac{1}{8\alpha} \left[(D-4)X^2 + 4X - (q+1)D\right] + (2d_1 - D)hX^{2d_1/(D-2)} = 0. \tag{4.31}
\]

This algebraic equation reduces to a simple quadratic equation in \( X \) either when the last term vanishes due to \( 2d_1 - D \equiv d_1 - D_0 = 0 \) (the case of a traceless form field EMT) or when its degree \( l(d_1) := 2d_1/(D-2) \) equals 0, 1 or 2. For \( D_0 = 4 \) we have \( l(d_1 = 0) = 0, l(d_1 = 2) = 1 \) and \( l(d_1 \to \infty) \to 2 \) so that as only sensible model remains \( D_0 = 4, d_1 = 2 \). It will be the subject of subsection 4.3.

For \( d_1 = D_0 = 4 \) we find as physically sensible solution of Eq. (4.31)

\[
X = \frac{1}{2} \left(\sqrt{9 + 8q} - 1\right). \tag{4.32}
\]

We use this solution as well as the extremum condition (4.17) to rewrite Eqs. (4.15) - (4.17), (4.19) - (4.21) in the simpler form

\[
U_0(X) = \frac{3}{16\alpha} X^{-1/3}(X - 1), \tag{4.33}
\]

\[
\Lambda_{\text{eff}}(X) = \frac{1}{3} U_0(X) - h, \tag{4.34}
\]

\[
R_1 = 4 \left[\frac{1}{3} U_0(X) + h\right], \tag{4.35}
\]

\[
J_{11} = \partial^2_{\varphi \varphi} U_{\text{eff}}|_{\text{extr}} = \frac{2}{3} \left[9h - U_0(X)\right], \tag{4.36}
\]

\[
J_{22} = \partial^2_{\phi \phi} U_{\text{eff}}|_{\text{extr}} = \frac{1}{14\alpha} X^{-1/3}(2X + 1), \tag{4.37}
\]

\[
J_{12} = \partial^2_{\varphi \phi} U_{\text{eff}}|_{\text{extr}} = 0. \tag{4.38}
\]

Obviously, there is no mixing of the excitations of the nonlinearity field \( \phi \) with gravexcitons \( (J_{12} = 0) \) in this case: \( m^2_{\phi} = J_{11}, m^2_{\varphi} = J_{22} \). Further, we read off from \( J_{22} > 0, X \geq 0 \) that stable internal spaces are again only possible for \( \alpha > 0 \) and from (4.32) and \( X \geq 0 \) that \( q \) is restricted to the half-line \( q \geq -1 \). Additional information can be extracted by combining the condition \( J_{11} > 0 \) with relations (4.34), (4.35), what gives

\[ 16h > R_1 > 16U_0(X)/9 > 8\Lambda_{\text{eff}}. \tag{4.39} \]

For the realistic case of a positive effective cosmological constant we find according to (4.34), (4.35) the conditions

\[ \Lambda_{\text{eff}} > 0: \quad h > R_1/16 > U_0(X)/9 > h/3 > 0, \tag{4.40} \]

and, hence, from (4.32), (4.39) also the implication \( U_0(X) > 0 \implies X > 1 \implies q > 0 \). We therefore conclude that such configurations are only possible for internal spaces with positive scalar curvature \( R_1 > 0 \) and positive bulk cosmological constant \( \Lambda_D > 0 \).

Let us briefly comment on some limiting cases.

(L.2.1) \( h \to +0 \): In this case we recover the result of (4.25) that stable configurations are only possible for \( R_1, \Lambda_D, \Lambda_{\text{eff}} < 0 \) [see the inequality chain (4.39)].

(L.2.2) \( \alpha \to +0, \Lambda_D \neq 0 \): For this transition to the linear model with freezing of the nonlinearity field at \( \phi_0 \to 0 \) and diverging excitation mass \( m^2_{\phi} \to \infty \), the stability sector \( \Theta \subset \mathcal{M} \) can be read off from (4.39) via substitution \( U_0(X) \to \Lambda_D \).

(L.2.3) \( \Lambda_D \to 0, \alpha \neq 0 \): The limit is connected with \( q \to 0, X \to 1, \phi_0 \to 0, U_0(X) \to 0 \) and we have to distinguish two special cases. For a nonvanishing form field strength \( h \neq 0 \) according to (4.30), (4.37) inequalities \( J_{11}, J_{22} > 0 \) hold so that both excitation masses remain finite. For vanishing field strength \( h \to +0 \) we obtain \( R_1 \to 0, \theta_{11} \to 0, m^2_{\phi} = m^2_{\varphi} \to 0 \) and the internal space \( M_1 \) undergoes a destabilization/decompactification with \( r_1 \to \infty \).
4.3 Two-dimensional internal spaces: $d_1 = 2$

According to Eqs. (4.30), (4.31), the extremum condition $\partial_q U_{eff}|_{extr} = 0$ for models with two-dimensional internal space $M_1$ and $D_0 = 4$ can be reduced to a quadratic equation and, hence, allows for an easy analytical handling of the models. Introducing the notation

$$z := 4\alpha h$$

(4.41)

Eq. (4.31) reads

$$X^2 - 2(z - 1)X - 3(q + 1) = 0$$

(4.42)

and has solutions

$$X_{1,2} = z - 1 \pm \sqrt{(z - 1)^2 + 3(q + 1)}.$$  

(4.43)

Furthermore, Eq. (4.31) can be used to simplify the elements of the Hessian $J$. Setting $D_0 = 4$ and $d_1 = 2$ everywhere in (4.35), (4.39) - (4.41) and eliminating $q$ with the help of (4.42) we obtain

$$J_{11} = \frac{\partial^2 U_{eff}}{\partial \phi \partial \phi}|_{extr} = \frac{1}{6\alpha} X^{-1/2} (5z + 1 - X),$$

(4.44)

$$J_{22} = \frac{\partial^2 U_{eff}}{\partial \phi \partial \phi}|_{extr} = \frac{1}{10\alpha} X^{-1/2} (X - z + 1),$$

(4.45)

$$J_{12} = \frac{\partial^2 U_{eff}}{\partial \phi \partial \phi}|_{extr} = -\frac{1}{2\sqrt{3}\alpha} X^{-1/2}z$$

(4.46)

as well as

$$\text{det}(J) = \frac{1}{60\alpha^2 X} (X^2 - 6zX + 8z^2 - 4z - 1).$$

(4.47)

It is now easy to describe the part $\Theta$ of the parameter space $M$ where the stability region $\Upsilon \subset \V$ is located. We start by substituting the solutions (4.43) into $J_{22}$. Taking into account that $X \geq 0$, we get from the condition $J_{22} > 0$ for $X_1, X_2$:

$$\pm \sqrt{(z - 1)^2 + 3(q + 1)}/\alpha > 0.$$  

(4.48)

Thus, the roots $X_1$ and $X_2$ correspond to $\alpha > 0$ and $\alpha < 0$, respectively. Because of $z = 4\alpha h$, $h \geq 0$ this leads to negative values for $X_2$ so that this root is unphysical and stable configurations are restricted to $X = X_1(z, q)$ and $\alpha > 0$. The limiting case $\alpha \rightarrow +0$ will be considered separately below.

Furthermore, we see from the structure of Eqs. (4.30) - (4.35) that the Sylvester criterion selects a region $\Theta(z, X)$ from the $(z, X)$-plane which can be interpreted as the projection of the stability region $\Upsilon$ on this plane. Explicitly we have

$$J_{11} > 0 \implies X < 5z + 1,$$

(4.49)

$$J_{22} > 0 \implies X > z - 1,$$

(4.50)

$$\text{det}(J) > 0 \implies \left\{ \begin{array}{l} X < 3z + \sqrt{z^2 + 4z + 1}, \\ X > 3z - \sqrt{z^2 + 4z + 1}, \end{array} \right.$$  

(4.51)

where the inequalities (4.47) are easily derived from (4.44) by calculating the critical values $X_c(z)$ for which $\text{det}(J[z, X_c(z)]) = 0$. The intersection $\Theta(z, X)$ of the sectors defined by the conditions (4.49) - (4.51) and $X > 0$, $z \geq 0$ is shown on Fig. 1.

In order to obtain information about the values of $q = 8\alpha \Lambda_D$ (and $\Lambda_D$) which allow for a stable internal space $M_1$ it proves convenient to map the region $\Theta(z, X)$ via quadratic equation (4.42) or its solution $X_1(z, q)$ from the $(z, X)$-plane on an equivalent region $\Theta(z, q)$ of the $(z, q)$-plane. For this purpose it is sufficient to transform the inequalities (4.49) - (4.51) and $X \geq 0$, $z \geq 0$ for $X$ and $z$ into an equivalent inequality set for $q$ and $z$. Let us demonstrate the mapping, e.g., for inequality (4.49). Substituting $X = X_1(z, q) = z - 1 + \sqrt{(z - 1)^2 + 3(q + 1)}$ into the equation for the critical line $X = X_c(z) = 5z + 1$ and solving for $q$ we obtain as image of this line $X_c(z)$ a corresponding critical curve $q_c(z) = z(5z + 6)$ on the $(z, q)$-plane. (The same curve can be obtained by substitution of $X_c(z)$ into the quadratic equation (4.49).) With the help of two test points $P_1 = (z_1, q_1 > q_c(z_1))$.

\footnote{The curvature term in $J_{11}$ of Eq. (4.10) cancels because of $a = b = 1$ for $D_0 = 4, d_1 = 2$.}
The resulting intersection region \( \Theta(z,q) \) of Eqs. (4.15) - (4.17) is depicted in Fig. 2.

Let us now turn to the scalar curvature \( R_1 \) and the four-dimensional effective cosmological constant \( \Lambda_{e f f} = U_{e f f}|_{e x t e} \). The structure of Eqs. (4.15), (4.16) and (4.17) suggests to consider \( R_1 \) and \( \Lambda_{e f f} \) as functions of \((z,X,q,\alpha)\). Eliminating \( q \) from Eqs. (4.16), (4.17) (with the help of (4.42)) we obtain

\[
R_1 = \frac{1}{6\alpha} X^{-1/2}(X + 4z - 1),
\]

(4.56)

\[
\Lambda_{e f f} = \frac{1}{12\alpha} X^{-1/2}(X - 2z - 1).
\]

(4.57)

The graphics of the functions

\[
R_1(z,X) = 0 \implies X|_{(R_1=0)} = 1 - 4z
\]

\[
\Lambda_{e f f}(z,X) = 0 \implies X|_{(\Lambda_{e f f}=0)} = 1 + 2z
\]

(4.58)

are included in Fig. 3. For completeness, we map them also on the \((z, q)\)-plane. Following the same scheme as above we obtain

\[
R_1(z,q) = 0 \implies q|_{(R_1=0)} = 2z(4z - 3), \quad 0 \leq z < 1/4
\]

\[
\Lambda_{e f f} = 0 \implies q|_{(\Lambda_{e f f}=0)} = 2z
\]

(4.59)
From Fig. 1 and Fig. 2 we see that the nonlinear model with two-dimensional internal space allows for internal spaces of negative or vanishing scalar curvature $R_1$.

Finally, we comment on some limiting cases.

(L.3.1) $h \rightarrow +0, \ q \neq 0$: According to Figs. 1 and 2 this limit corresponds to a vanishing form field $z \rightarrow +0$, and a stabilization is possible in the case of $R_1 < 0$. Furthermore, for $z \rightarrow 0$ we can approximate $X = X_1(z, q) \approx (v - 1)(1 + z/v)$ with $v := \sqrt{4 + 3q} > 1$ and the masses of the normal excitation modes of the coupled $\varphi - \phi$--field system follow from (4.62) as

$$m_1^2 = \frac{1}{6\alpha} (v - 1)^{-1/2} \left[ 2v + \frac{9}{2}z + O(z^2) \right],$$

$$m_2^2 = \frac{1}{10\alpha} (v - 1)^{-1/2} \left[ v - \frac{2 + v}{2v}z + O(z^2) \right].$$

In the special case $z = 0$ we completely reproduce our earlier results on nonlinear stabilized models without form fields ($d_1 = 2$): $m_\varphi^2 = m_1^2 = -U_0(X)$, $m_\phi^2 = m_2^2 = X^{-1/2}(X + 1)/(10\alpha)$.

(L.3.2) $\alpha \rightarrow +0, \Lambda_D, h \neq 0$: For this transition to a linear model we have as in (L.1.1) $U(X) \rightarrow \Lambda_D$ as well as a freezing of the nonlinearity field at $q_0 \rightarrow 0$, $X \rightarrow 1$. Using the approximation

$$X = X_1(z, q) = 1 + z + (3q - 2z)/4 - (3q - 2z)^2/64 + z^2/4 + O(\alpha^3)$$

we obtain the excitation masses as

$$m_\varphi^2 \rightarrow m_1^2 = \frac{\alpha^{-1} - 2h}{5} + O(\alpha) \rightarrow \infty,$$

$$m_\phi^2 \rightarrow m_2^2 = 3h - \Lambda_D + O(\alpha) > 0.$$
so that the freezing is clearly seen from the diverging mass of the nonlinearity field. Additionally, we find from (L.3.4), (L.3.5) 

\[ R_1 = \Lambda_D + 3h - \frac{\alpha}{6} \left[ 27(h + \Lambda_D)^2 - 8h^2 \right] + \mathcal{O}(\alpha^2) \]  \hspace{1cm} (4.65)  

\[ \Lambda_{\text{eff}} = \frac{\Lambda_D - h}{2} + \frac{3\alpha}{4} \left[ (h + \Lambda_D)^2 - 4\Lambda_D^2 \right] + \mathcal{O}(\alpha^2), \]  \hspace{1cm} (4.66)  

what in the special case of a vanishing effective cosmological constant \( \Lambda_{\text{eff}} = 0 \) reproduces the results of Ref. [24] for a linear model with Freund-Rubin form field: \( h = \Lambda_D = R_1/4, m_\phi^2 = 2h \).

(L.3.3) \( \Lambda_D \to 0, \alpha, h \neq 0 \): In this case we have \( q \to 0 \). A substitution of the approximation \( X = X_1(z, q) = z - 1 + [(z - 1)^2 + 3]^{1/2} + 3(z - 1)^2 + 3^{-1/2} q/2 - \mathcal{O}(q^2) \) into the Hessian shows that there is no special behavior of the excitation masses connected with this limit.

(L.3.4) \( \Lambda_D, h \to 0, \alpha \neq 0 \): From (L.3.4) it follows in this limit \( J_1 \to 0 \), so that beside a decoupling of the gravexciton the graviton mass vanishes \( m_{\phi 2} \to m_{\phi 2}^* \) and the internal space \( M_1 \) destabilizes. This is in full agreement with [24] where a stabilization for \( h = 0 \) requires \( \Lambda_D < 0 \).

(L.3.5) \( R_1 \to 0 \): In the limit \( R_1 \to 0 \) one observes a regular behavior similar to (L.3.3). For parameter points near the line \( X_0(z) := X(R_1=0)(z) = 1 - 4z, 0 \leq z < 1/4 \) we find from Eq. (15.56) \( X = X_0(z) + 6\alpha R_1 X_0^{1/2} + \mathcal{O}(\alpha^2 R_1^2) \) so that the Hessian yields excitation masses of the form \( m_{r,2}^2(R_1 \approx 0) = m_{r,2}^2(R_1 = 0, z) + \sigma_{1,2}(z)\alpha R_1 + \mathcal{O}(\alpha^2 R_1^2) \) with some regular coefficients \( \sigma_{1,2}(z) \) and

\[ m_{r,2}^2(R_1 = 0, z) = \frac{1}{40\alpha} X_0^{-1/2} \left[ 9 - 5X_0 \pm \sqrt{4(3 - 5X_0)^2 + 5(X_0 - 1)^2} \right]. \] \hspace{1cm} (4.67)  

The masses \( m_{r,2}^2(R_1 = 0, z) \) have finite values except at the limiting points \( X_0(z \to 1/4) \to 0 \) (or \( \phi_0 \to -\infty \)) and \( X_0(\alpha \to 0) \to 1/4 \) (or \( \phi_0 \to 0 \)) where both or one of the masses diverge. We see that, with exception of the limiting point \( X_0(\alpha \neq 0) \to 1 \), there occurs no destabilization of the internal space \( M_1 \) for vanishing scalar curvatures \( R_1 \). Due to the smooth behavior of the excitation masses under the transition \( R_1 \to 0 \) we can identify this limit with a stable decompactification \( r_1 \to \infty \) of an internal space \( M_1 \) with fixed topology. Clearly, in our local approach a stable decompactified space with \( r_1 \to \infty \) is indistinguishable from a stabilized internal space which is Ricci-flat from the very beginning.

5 Conclusions and discussion

In the present paper we investigated multidimensional gravitational models with a non-Einsteinian form of the action. In particular, we assumed that the action is an arbitrary smooth function of the scalar curvature \( f(R) \). Additionally, the D-dimensional spacetime was endowed with solitonic form fields of generalized block-orthogonal Freund-Rubin type. This bulk matter ansatz leads to a naturally factorized geometry and a spontaneous compactification can be associated with it. For the considered models, we concentrated on the stabilization problem for the extra dimensions. As technique we used a reduction of the nonlinear gravitational model to a linear one with an additional self-interacting scalar field (nonlinearity scalar field). The factorized geometry as well as the generalized Freund-Rubin ansatz for the solitonic form field allowed for a dimensional reduction of the considered models and a transition to the Einstein frame. As result, we obtained an effective four–dimensional model with nonlinearity scalar field and additional minimally coupled scalar fields which describe conformal excitations of the scale factors of the internal space.

A detailed stability analysis was carried out for the three most simplest configurations of a model with one internal factor space \( M_1 \) and a quadratic curvature term: \( f(R) = R + \alpha R^2 - 2\Lambda_D \), where \( \Lambda_D \) plays the role of a \( D \)–dimensional bare (bulk) cosmological constant. These three configurations are characterized respectively by: 1) a vanishing four-dimensional effective cosmological constant \( \Lambda_{\text{eff}} \), 2) a traceless form-field EMT, or 3) a \( (d_1 = 2) \)–dimensional internal factor space \( M_1 \). For all three configurations, a stabilization of the internal space is only possible in the case of a non-negative nonlinearity parameter \( \alpha \geq 0 \) and a bulk cosmological constant \( \Lambda_D \) restricted by the condition \( q \geq 8\alpha \Lambda_D > -1 \). The transition \( (\Lambda_D \to 0, h \to 0) \) is connected with a decompactification \( (R_1 \to 0, r_1 \to \infty) \) of the internal space \( M_1 \). At the same time, it leads to a flattening of

\footnote{According to (L.6.7), the limit \( X_0(\alpha \neq 0) \to 1 \) gives \( m_{\phi 2}^2 \to m_{\phi 2}^* \to 0, m_\phi^2 \to m_\phi^* \to 1/(5\alpha) \). On the other hand, \( X_0(\alpha \neq 0) \to 1 \) implies \( z = 4\alpha h \to 0 \) and according to (L.3.4) also \( \Lambda_D \to 0 \). Thus, the results of (L.3.1) and (L.3.4) can be used to reproduce the same behavior of the excitation masses via (L.3.5).}
the effective potential in the direction of the scale factor excitations and, hence, to a destabilization of $M_1$ (for a similar limiting behavior see also Ref. 25).

From the three configurations, the model with the two-dimensional internal space shows the richest features. It allows for stable configurations in the cases ($\Lambda_{\text{eff}} \geq 0$, $R_1 > 0$) and ($\Lambda_{\text{eff}} < 0$, any sign of $R_1$) as well as for non-compact internal spaces ($R_1 = 0$). Interestingly, the various stable configurations belong to a connected region in the parameter space $M$ and one can smoothly pass from one type of configuration to another one, including a transition to stable Ricci-flat internal spaces which can be described as “stable decompactifications”:

\[ R_1 \to 0, \quad r_1 \to \infty. \]

As pointed out in the Introduction, such a rich picture became possible due to the presence of the real-valued form fields which satisfy the NEC and the WEC and which compete with the nonlinearity scalar field. The latter satisfies the NEC only marginally and can violate the WEC.

Interestingly, for $(d_1 = 2)$—dimensional internal spaces there exist parameter configurations with $\alpha$, $\Lambda_D$, $h$, $R_1 > 0$ that can provide positive values of the effective four-dimensional cosmological constant $\Lambda_{\text{eff}} > 0$ (see e.g. Eqs. (1.60), (1.61)). Thus, an accelerated expansion of the Universe seems possible in accordance with observational data. Let us assume that the values of the bulk cosmological constant $\Lambda_D$ and the form field strength $h$ are set at some characteristic scale $\Lambda_D \sim h \sim M^2$. Then we find for the parameters $q \sim 8 \alpha M^2$, $z \sim 4 \alpha M^2$ and, hence, $\Lambda \sim 2z$. The latter corresponds to $X \sim 1 + 2z \sim 1 + 8 \alpha M^2$ (see (1.10)) and comparison with (1.60) shows that such configurations should yield an almost vanishing effective cosmological constant $\Lambda_{\text{eff}} \sim 0$. With the help of Eq. (4.50) we can estimate the scalar curvature $R_1$ of the internal space as

\[ R_1 \sim \frac{z}{\alpha} X^{-1/2} \sim \frac{1}{\alpha} \frac{z}{\sqrt{1 + 2z}} \sim \frac{4 M^2}{\sqrt{1 + 8 \alpha M^2}}. \]  

(5.1)

On the other hand, its value is connected with the fundamental scales $M_{s(4+d_1)}$, $M_{Pl(4)}$ by the relations (1.1), (1.2) and (1.6):

\[ R_1 \sim \frac{1}{r_1^2} \sim \left( \frac{M_{s(4+d_1)}}{M_{Pl(4)}} \right)^4 \left( \frac{M_{Pl(4)}}{M_{s(4+d_1)}} \right)^{5/4}. \]  

(5.2)

As mentioned in the discussion after Eq. (2.33), the value of $X$ can be used as a measure of the nonlinearity of the original model: $\alpha R = e^{A_0} - 1 \equiv X - 1$. We see that weakly nonlinear configurations correspond to $X \approx 1$, whereas $X > 1$ leads to a strongly nonlinear regime. With the help of (5.1) and (5.2) we express this dimensionless nonlinearity parameter $X$ in terms of the different scales contained in our model:

\[ X \sim 1 + 8 \alpha M^2 \sim 16 \left( \frac{M}{M_{s(4+d_1)}} \right)^4 \left( \frac{M_{Pl(4)}}{M_{s(4+d_1)}} \right)^{8/d_1}. \]  

(5.3)

From (5.3) we see that setting $M \sim M_{s(4+d_1)}$ we obtain $X \gg 1$ for ADD-type TeV—scale models whereas $X \sim 1$ can only be achieved for standard KK models with $M_{Pl(4)} \sim M_{s(4+d_1)}$. Stably compactified internal spaces in ADD-type models can be obtained within a weakly nonlinear regime $X \sim 1$ if the bulk cosmological constant $\Lambda_D$ and the form field strength $h$ are related with the fundamental scales as

\[ \Lambda_D \sim h \sim M^2, \quad M \sim \frac{1}{2} M_{s(4+d_1)} \left( \frac{M_{s(4+d_1)}}{M_{Pl(4)}} \right)^{2/d_1}. \]  

(5.4)

For $M_{s(4+d_1)} \sim 1 - 30$ TeV and $d_1 = 2$ this implies $M \sim 10^{-4} - 10^{-1}$ eV. It is interesting to note that this mass scale is of the same order as the lowest possible supersymmetry breaking scale $m \sim M^2_{SUSY}/M_{Pl(4)}$ in the minimal supersymmetric extension of the standard model (MSSM) 49 with $M_{SUSY} \sim 1$ TeV.

Above we demonstrated that the assumption $\Lambda_D \sim h$ can result in a small effective cosmological constant $\Lambda_{\text{eff}}$. Let us now estimate the relation between $\Lambda_{\text{eff}}$ and $\Lambda_D$, $h$ in more detail and compare it with the observable value\(^{13}\) of $\Lambda_{\text{eff}} \sim 10^{-123} \Lambda_{Pl(4)}$. For simplicity, we will restrict our consideration to a weak nonlinearity regime with $X \approx 1$, $\alpha \gtrsim 0$ where the approximations (4.65) and (4.66) of (L.3.2) are valid. From (4.66) we see that to ensure a sufficiently small $\Lambda_{\text{eff}}$ the bulk cosmological constant $\Lambda_D$ and the field strength $h$ of the solitonic form field should be connected by

\[ h = (1 + \epsilon) \Lambda_D. \]  

(5.5)

The value of the small $\epsilon$ we will estimate now. With the help of relations (4.65) and (5.1) we find

\[ 10^{-123} \Lambda_{Pl(4)} \sim \Lambda_{\text{eff}} \approx \epsilon \Lambda_D (1 - 6 \alpha \Lambda_D)/2 \]  

\[ \Lambda_D \sim M^2 \sim R_1. \]  

(5.6)

\(^{13}\)In our normalization conventions holds $c = h = 1$ and $\Lambda_{Pl(4)} \sim M^2_{Pl(4)} \sim L^2_{Pl(4)}$. 

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and, hence,

$$10^{-123} \sim \epsilon \left( \frac{M_{*(4+d_1)}}{M_{Pl(4)}} \right)^{4/d_1+2}$$

(5.7)

so that

$$\epsilon \sim 10^{-65}$$

(5.8)

for $d_1 = 2$, $M_{*(4+d_1)} \sim 30\,\text{TeV}$. According to (5.6), this value of $\epsilon$ is not sensitive to changes of the nonlinearity parameter $\alpha$ in a weakly nonlinear curvature regime. Thus, we arrive at the conclusion that the ADD scenario in its simplest extended version can provide a simultaneous stabilization of the extra dimensions together with an adjustment of the effective cosmological constant to its observed value only in the case of a strong fine tuning. Although the solitonic form fields of our model are located in the compactified extra dimensions, the tuning of their effective energy density $h$ to the bulk cosmological constant $\Lambda_D$: $h = (1 + \epsilon)\Lambda_D$ is of a similar type as the four-form-tuning discussed in Weinberg’s no-go theorem for a resolution of the cosmological constant problem (CCP). A shifting of the CCP to a parameter fine tuning is a rather general feature of models with compactified additional dimensions and form fields\(^\text{14}\). In a slightly reshaped form it also appears in the recently proposed brane-world model with two-dimensional “football”-shaped large extra dimensions \(^\text{51}\) (see also \(^\text{52}\)) where the adjustment of the on-brane cosmological constant is shifted to an adjustment of the parameters of the off-brane “football”. A possible resolution of the CCP for similar higher dimensional models with form fields following from an M-theory setup was presented in Ref. \(^\text{53}\). Proposals for a resolution of the CCP within other scenarios comprise various anthropic approaches \(^\text{54}\), \(^\text{55}\), shifting of the CCP to a singularity problem \(^\text{56}\), possible graviton compositeness \(^\text{57}\), a holographic approach \(^\text{57}\) as well as non-local modifications of gravity \(^\text{58}\). However, there is still no satisfactory and comprehensive solution of the CCP. The problem will probably remain challenging the scientific community until a final understanding of quantum gravity will be achieved.

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