Modified Relativity from the $\kappa$-deformed Poincaré Algebra

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Abstract

The theory of the $\kappa$-deformed Poincaré algebra is applied to the analysis of various phenomena in special relativity, quantum mechanics and field theory. The method relies on the development of series expansions in $\kappa^{-1}$ of the generalised Lorentz transformations, about the special-relativistic limit. Emphasis is placed on the underlying assumptions needed in each part of the discussion, and on in principle limits for the deformation parameter, rather than on rigorous numerical bounds. In the case of the relativistic Doppler effect, and the Michelson-Morley experiment, comparisons with recent experimental tests yield the relatively weak lower bounds on $\kappa c$ of 90 $eV$ and 250 $keV$, respectively. Corrections to the Casimir effect and the Thomas precession are also discussed.
1 Introduction and Main Results

In recent years there has been much interest in the implications for physical theories of the notion of ‘deformed’ symmetries. First introduced in the context of the inverse scattering transform and exactly solvable models in statistical mechanics, the study of ‘quantum groups’ [1] in particular entrains the mathematical structures of Hopf algebras, which provide a more powerful and general language for the discussion of symmetry principles than the traditional one of Lie algebras and Lie groups. The deformations also typically involve one or more scalar parameters; these may be coupling constants as in the case of lattice systems, but in the context of deformed theories generally, may be new types of fundamental dimensionful constants. It is natural to speculate on their role as, say, cutoff parameters in the renormalization of quantum field theory, or unification scales in applications to symmetry breaking phenomena.

The present work addresses the question of deformed space-time symmetries. In particular we take up the case of the so-called $\kappa$-deformed Poincaré algebra introduced recently and studied in several papers [2]. In this case the deformation parameter gives a fundamental inverse length or energy scale beyond which deviations from special relativity are expected. The paper concerns the detailed working-out of the implications of this class of deformed space-time algebra for a variety of physical, experimentally tested effects. Specifically, we develop a series expansion in inverse powers of $\kappa$ for the generalised ‘Lorentz transformations’ about the special relativistic limit, and we consider the lowest order corrections to the experimental predictions of special relativity for the various effects, in order to deduce lower limits on $\kappa$. Our main results (see §2 below) are that the existing data on the transverse Doppler shift and the Michelson-Morley experiment yield limits of $90 \text{ eV}$ and $250 \text{ keV}$ respectively for $\kappa_c$.

The $\kappa$-deformed space-time symmetries [3] were first derived from a systematic application of the formalism of $q$-deformations to real forms of simple Lie algebras related to compactified Minkowski space, followed by appropriate contraction [3,4]. Since their introduction further developments have been considered, including higher dimensional [5,6] and supersymmetric [7,8,9] versions, the formalism of $\kappa$-relativistic fields [10], the integration of the infinitesimal transformations to finite transformations [11], $\kappa$-Dirac and higher spin equations [11,12], as well as the elements of induced representation theory [13]. Formal developments have included consideration of the coproduct structure and tensor operators, [14] for the $\kappa$-deformed algebra. Analogously to the standard theory of quantum groups, theorems have been proved [14] which ensure that for generic values of $\kappa$, representations of the undeformed algebra also admit actions under the deformed algebra.

Deformations of symmetries have also arisen in a more geometrical arena in connection with ‘non commutative geometry’ [15]. In this dual approach, abstract generalisations of functions on manifolds would imply the existence of coordinates which are no longer point like, but belong to a non commutative algebra. Applied to space-time symmetries this leads to consideration of the quantum group $SL_q(2,\mathbb{C})$ [16]. This framework seems a natural one for the introduction of discrete substructure, as might be expected in quantum theories of gravity at the highest Planck energies. Indeed one proposal involving applications of Hopf algebras in geometric quantisation leads to a natural ‘bicrossproduct’ structure [17,18] in which the role of Planck’s constant in quantisation of the Weyl algebra of quantum mechanics is dually matched by a second fundamental deformation parameter in the quantisation and coproduct structure of accompanying symmetries. In particular for model systems [17,18] this parameter can be plausibly related to Newton’s gravitational constant. The further ramifications of this line of thinking lead to a fertile concept of ‘braided’ generalisations of Lie algebras [19].

It has recently been shown that the $\kappa$-deformed algebra itself admits a bicrossproduct struc-
ture in the above sense \[20\], giving some indication that the numerical value of the deformation parameter $\kappa$ might be comparable with the Planck energy. At the same time, this suggests that the correct context for detailed study of the physical consequences of the $\kappa$-relativity should be that of noncommutative geometry \[20\].

In the present work we eschew such theoretical sophistication, preferring to work with the deformed relativity as a theory which induces small but testable corrections to experimental results: we claim that, even with the full apparatus of non-commutative geometry, if indeed it is required to make a consistent theory, actual deviations from relativity will involve much the same type of algebraic consequences as given here. Here we treat some of the main predictions of special relativity; a more comprehensive analysis would consider the whole gamut of Post-Newtonian parametrisations \[21\] (see below).

For the purposes of this work the detailed Hopf structure of the $\kappa$-Poincaré algebra is not essential (the salient definitions are given in the appendix for completeness). Amongst the Lorentz generators, deformed commutation relations arise only for the boosts $L_i$ for $i = 1, 2, 3$, and between the boosts and the momentum generators $P_0$, and $P_i$, with the standard expressions modified by rational functions of $\exp P_0/\kappa$; commutation relations for the angular momentum generators $M_i$ are undeformed. The quadratic Casimir invariant becomes

$$C_1 = P^2 - (2\kappa \sinh P_0/2\kappa)^2$$

with an expression for the spin invariant corresponding to a Pauli-Lubanski vector which has been deformed in an analogous way.

The modified dispersion relation \(\Pi\) leads directly to experimental determinations of (limits on) $\kappa$, and high energy astrophysical processes have been analysed by assuming that photons are ‘massless’, and that $\Pi$ prescribes an effective energy dependence for the photon phase velocity $\omega/k = E/p$. Thus from the coincidence measurements on the arrival times of sharp bursts from distant events over widely different energies \[22\] the lower limit on $\kappa$ of $10^{12}$ GeV was derived.

Although $\Pi$ entails a mild deviation from special relativity at laboratory energies, in the context of quantum field theory it is possible that the entire ultraviolet structure is changed. The simplest manifestation of this is the Casimir effect, which depends only on the zero point energy of virtual modes. Following the standard textbook derivations \[23,24\] for plane square parallel plates of side $L$ and separation $d$, the $\kappa$–relativistic dispersion relation leads to the usual Casimir interaction energy plus a series of $\kappa$–dependent corrections

$$U(d) = -\frac{\pi^2 \hbar c L^2}{720 d^3} \left(1 + \frac{1}{84} \left(\frac{\pi \hbar}{\kappa d}\right)^2 + \frac{9}{896} \left(\frac{\pi \hbar}{\kappa d}\right)^4 + \cdots\right)$$

with the general $n$th coefficient (for $n > 1$) being given in terms of Bernoulli numbers, \[25\]

$$U(d) \frac{2d^3}{\pi^2 \hbar c L^2} = (-1)^{(n+1)} \frac{B_{2n+2}}{(2n+1)(2n+2)(2n-1)} \frac{\prod_{m=1}^{n-1} (2m-1)}{\prod_{m=1}^{n-1} 2m} \left(\frac{\pi \hbar}{2\kappa d}\right)^{2(n-1)}.$$  \(3\)

Due to the $n!$ behaviour of the Bernoulli numbers for large $n$, \(\Pi\) is in fact divergent for all finite $\kappa$ and $d$, which is of course very different from the case of the usual Casimir effect. On the other hand, if we assume that \(\Pi\) is an asymptotic series, as often arises in field theory, it is found that the Casimir force will be $\kappa$-affected at small separations only. In practice this is a very difficult experimental regime (see for example \[25\]) due to the skin depth effect in real conductors, so that any bound on $\kappa c$ from existing data is in the eV range at best. However,
the effect of the deformation has potentially drastic consequences for a variety of phenomena which involve the Casimir effect, such as black hole evaporation.

Before turning to the analysis of special relativistic tests, we present a further example, that of the Thomas precession, to illustrate our general approach. Whereas the above discussion of the Casimir effect relies essentially on the modifications to the energy-momentum relationship, quantum mechanical wave equations in $\kappa$-relativity, especially for spinning particles (see for example [11], [12]) necessarily must address issues in the representation theory of the full deformed algebra. On the other hand, the original arguments for the relativistic origins of the atomic spin-orbit coupling may be discussed using purely kinematical arguments involving, for example, the commutator of two boosts in perpendicular directions. The same textbook argument goes through in the $\kappa$-deformed algebra, and would lead to a modified spin-orbit coupling of the type

$$\Delta E = \frac{1}{r} \frac{dV(r)}{dr} \left( \frac{\Sigma_\kappa}{2m^2c^2} \right) \mathbf{S} \cdot \mathbf{L}, \quad (4)$$

where $\Sigma_\kappa = 1 + O(m^2c^2/\kappa^2)$ (see below for comments about expansions in powers of $\kappa$). This semi-classical derivation is obviously incomplete, in that it has not required $\kappa$–deformed Dirac or Maxwell equations. However, it still gives some insight into how the deformation will alter the energy levels of the hydrogen atom, and moreover because of the paucity of underlying assumptions, we would claim that any consistent theory must contain terms of the same general form.

In §2 below we return to the question of special relativistic tests of $\kappa$-relativity. The derivations use the results of the appendix, where we present the evaluation of the series expansion of the ‘$\kappa$-deformed Lorentz transformations’ about the special relativistic limit in inverse powers of $\kappa$. This involves various manipulations of the Jacobi elliptic functions which arise from integrating the infinitesimal transformations generated by the $\kappa$-deformed boosts. The result is that the usual linear momentum transformation $p'_\mu = \Lambda_{\mu \nu} p_\nu$ is now the first term in the expansion, the general relation being of the form $p'_\mu = L_\mu(p)$. In §2 these expansions are used in an analysis of two fundamental experimental tests of relativity: the transverse Doppler effect, and the Michelson-Morley experiment. The former is a direct application of the formulae of the appendix. The latter is analysed by modelling radiation between the etalon plates as a scalar field. However, in order to complete the discussion, additional assumptions about coordinate space aspects of $\kappa$-relativity, and also about the superposition principle, are required. These are discussed as the analysis is developed.

In our conclusions (§3), we summarize our results and give some final remarks on the prospects for a more comprehensive, systematic analysis of experimental data which may provide tests of deformed relativity. Our work is based on [26] (unpublished), but the presentation of the material given in this paper is in principle self-contained.

## 2 Special relativistic tests

### Transverse Doppler Effect

A fundamental relativistic phenomenon is the relativistic Doppler effect, which has in the past been used as an important experimental test of special relativity. We can obtain the $\kappa$-generalised relativistic Doppler effect from the $\kappa$ deformed four-momentum transformations (A.7)-(A.8) by simply replacing the four-momentum parameters by their corresponding four wave-numbers, using the deBroglie relation. One of the main problems associated with experimental tests of the relativistic Doppler effect is the presence of two distinct contributions to the total Doppler
effect observed. These contributions are best described as the ‘relativistic component’ which we
are primarily interested in, and the ‘non-relativistic contribution’, i.e \( k_0' = k_0(1 + v/c) \). In the
transverse Doppler effect, this relativistic component is the sole contributor, i.e
\[
\frac{k_0'}{k_0} = \cosh \alpha \equiv \gamma,
\]
which is found to \( \kappa \) generalise to,
\[
\frac{k_0'}{k_0} = \cosh \alpha + \frac{\hbar^2 k_0^2}{8\kappa^2} \left[ \frac{\cosh \alpha}{3} - \frac{\cosh^3 \alpha}{3} + \frac{\alpha \sinh \alpha}{2} + \frac{\sinh^2 \alpha \cosh \alpha}{2} \right] + 0(\kappa^{-4})
\]  
(6)
(compare with (A.7)-(A.8)). In the past experimental tests of the relativistic Doppler effect have
concentrated on the measurement of this transverse component. There are however practical
problems associated with this measurement, due to the fact that any small aberrant longitudinal
component will swamp this weak transverse effect.

Recent measurements of the relativistic Doppler effect have however been improved (see for
example [27]) by using a technique in which the relativistic component is isolated using a two-
photon absorption process (TPA). In two-photon spectroscopy, the first order Doppler shift is
effectively absent, and the second order term becomes dominant. By measuring the frequency
difference between a two-photon transition in a fast neon-atom beam and a stationary neon
sample, using two lasers, the second order Doppler effect was confirmed [27] to accuracy \( 4 \times 10^{-5} \).

The analysis of this experiment can be performed using our \( \kappa \)-modified expressions for the
relativistic Doppler effect, obtained explicitly from (A.7)-(A.8). For the kinematics of [27],
inferring the boost parameter from the beam velocity (which is measured accurately in the TPA
resonance absorption technique via the shifted laser frequencies), a lower limit on \( \kappa c \) of 91 eV
is obtained directly by attributing the experimental uncertainty quoted above, to the presence
of the new \( \kappa \)-dependent terms modifying the special relativistic prediction. The main factors
limiting the magnitude of this lower bound include the low energy (optical) transitions which
were used, and the low velocity of the atom beam employed (\( v/c \sim 4 \times 10^{-3} \)).

Michelson-Morley Experiment

Another fundamental special relativistic test is the Michelson-Morley experiment. As we shall
see below, in order to \( \kappa \)-generalise the analysis of this experiment, we need some consideration
as to the likely forms of the \( \kappa \)-deformed space-time transformations. Due to the semigroup
structure of the \( \kappa \)-Lorentz transformations [11] in momentum space, the extension to coordinate
space is most naturally introduced in the context of relativistic mechanics using the full eight-
dimensional phase space. If it is assumed that the canonical phase space Poisson bracket
\[
\{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{p_\mu, p_\nu\} = 0 = \{x^\mu, x^\nu\}
\]  
(7)
be preserved under the \( \kappa \)-Lorentz transformations [11], then the space-time coordinates \( x^\mu \)
behave as a four-vector dual to the differentials \( dp_\nu \), so that the complete phase-space transfor-
mation rules are
\[
p'_\mu = L_\mu(\alpha; p), \quad x'^\mu = (L^{-1})_\rho^\mu (x^\rho - a^\rho)
\]  
(8)
where \( \alpha \) is the rapidity parameter of the boost, \( a^\mu \) describes a four-translation, and the matrix
\( L^{-1} \) is the inverse of the matrix of partial derivatives,
\[
L_\mu^\nu = \frac{\partial L_\mu(\alpha; p)}{\partial p_\nu}.
\]  
(9)
From (8) we can deduce the transformation properties of classical fields in space-time from that of their realisations in momentum space by a standard Fourier transform. Hence for the simple case of scalar fields $\Phi(x)$, we obtain the following expression for the $\kappa$–transformed field:

$$
\Phi'(x') = \int d^4x K(\alpha; x' - a, x) \Phi(x),
$$

$$
K(\alpha; x', x) = \frac{1}{(2\pi)^4} \exp(i(p_\mu x'^\mu - L_\mu(\alpha; p)x^\mu)).
$$

In the special case of a plane wave, $\Phi(x) = A \exp(ik_\mu x^\mu)$, we have

$$
\Phi'(x') = A |L| p = \bar{h} k \exp i(L_\mu(\alpha; \bar{h} k)x'^\mu/\bar{h})
$$

where the determinant $|L|$ arises as a Jacobian factor.

The re-analysis of the Michelson-Morley experiment is accomplished using a classical scalar field model. Assuming that the $\kappa$–deformed approach is compatible with a preferred-frame theory, we wish to set up a formalism in which monochromatic radiation in each of two orthogonal arms of the interferometer in the laboratory frame (described by plane waves) is written in terms of the $\kappa$–transformed expression for the scalar field as seen by a moving observer (the preferred frame). Ultimately, we need to evaluate the $\kappa$–transformed fields at the coordinates of the recombination point (see figure 1). Unfortunately however, the expression (10) only gives these fields in the coordinates of the moving observer. Thus we need to use the $\kappa$–deformed space-time transformations, (8). We make the additional assumption that the four-momentum dependence inherent in these space time transformations (c.f. (9)) is simply the four-momentum of the radiation used. The $\kappa$–Lorentz transformed plane wave scalar field, in terms of the laboratory coordinates therefore takes the form

$$
\Phi'(x')(x) = A |L| p = \bar{h} k \exp i(L_\mu(\alpha; \bar{h} k)x'^\mu/\bar{h})
$$

It is the ‘superposition’ of two such quantities, corresponding to the appropriate field amplitudes from each of the interferometer arms, which will determine the observed phase shift at the recombination point (see (12) below).

The exponent of (11) will be written as $i(L_\mu \frac{\partial}{\partial L_\mu} p_\rho x^\rho/\bar{h})$. This shorthand notation for the phase indicates a possible procedure for its evaluation, using the implicitly defined function $p_\mu(L)$, via inversion of the relevant series expansions. From this point of view it is also evident how the frame independence of the phase, and hence the usual null result for the Michelson-Morley experiment, arises from the linearity of the standard Lorentz transformation. In practice, the exponent will be evaluated using the series expansions of the appendix, and explicitly inverting the matrix $L$ to the appropriate order in $\kappa$.

Using the geometry of figure 1, where the $x$ axis is vertical and the $z$ axis horizontal, the laboratory wave 4-vectors of the radiation in the emitted, or outward $(+)$ and reflected, or inward $(-)$, moving beams are

$$
(k^\parallel)^\pm = (k_0, 0, 0, \pm k_3)
$$

$$
(k^\perp)^\pm = (k_0, \pm k_1, 0, 0)
$$

respectively. Ignoring an inessential overall phase, and phase change on reflection, and matching the phase of the emitted and reflected beams at the mirror positions

$$
(m^\parallel) = (ct, 0, 0, L),
$$

$$
(m^\perp) = (ct, L, 0, 0),
$$

respectively.
the total field amplitude at the recombination point \((r) = (ct, 0, 0, 0)\) (taken to be the origin in the laboratory frame) is finally

\[
\Phi'(r) = A'_\parallel \exp(L \frac{\partial}{\partial L} k^- \cdot r + L \frac{\partial}{\partial L} (k^- - k^+) \cdot m_\parallel) + A'_\perp \exp(L \frac{\partial}{\partial L} k_\perp \cdot r + L \frac{\partial}{\partial L} (k_\perp - k^+ \cdot m_\perp), \tag{12}
\]

where \(A'_\parallel, A'_\perp\) are the modified plane wave amplitudes, as in (11).

Working to order \(\kappa^{-2}\), the effect of the (spacetime independent) difference between the \(A'\) factors on any resultant fringe pattern can be ignored (since it contributes only to the overall amplitude of the intensity modulation). The modulation itself is determined by the phase difference \(\delta \phi\) between the two terms in (12). Using the series expansions from the appendix, we find

\[
\delta \phi = \frac{\hbar^2 k^3}{12\kappa^2} \left[ L \sinh^2 \alpha (6 + 10 \sinh^2 \alpha) - ct \sinh \alpha (6 \cosh \alpha + 10 \cosh \alpha \sinh \alpha + 9 \sinh^3 \alpha) \right]. \tag{13}
\]

As mentioned above, this non-null prediction for the Michelson-Morley experiment appears to imply the existence of a preferred frame. It is only when the interferometer is in this preferred frame that there will be found to be no phase shift for all orientations. If we \(\kappa\)-Lorentz boost to the frame of the Earth for example, a phase shift will be observed depending upon the orientation of the interferometer apparatus with respect to the preferred frame. An important feature of (13) is its time dependence. Of course, if the phase variation is rapid compared with the time over which intensity observations are made, then the phase difference washes out and the experiment apparently is consistent with a null result (leading in principle to an upper limit for \(\kappa c\)). Moreover, it can be seen from the \(\sinh \alpha\) dependence of the terms in (13) that the time variation of the phase shift is in fact more sensitive to \(\kappa\) than the dependence on the interferometer arm length \(L\).

To obtain limits on \(\kappa\) from the above, consider for example the experiment of [28]. By considering the fractional change in an etalon of length, the anisotropy of space was measured to be less than \(\delta L/L = \varepsilon = 1.5 \times 10^{-15}\), by means of a frequency servo-stabilised laser system which tracked a particular etalon cavity mode to within \(\delta \omega/\omega = \varepsilon\) as the (single) arm rotated on a table at angular frequency \(\delta \omega\). If it is assumed for this case that the phase can be analysed along the above lines, then no signature of the time dependent term \(\omega_\kappa t\) in (13) would appear provided \(\omega_\kappa < \delta \omega\) (a frequency variation found at \(\delta \omega\) in the experiment [28] was attributed to gravitational flexure of the apparatus). Thus for small \(\alpha\) we have from (13)

\[
\frac{\hbar^2 k^3 / 2\kappa^2 \alpha c}{kc} < \varepsilon
\]

leading, for the laser frequency used, and taking the cosmic background radiation frame \((\beta = 1.3 \times 10^{-3})\) as the preferred frame, to a lower limit on \(\kappa c\) of 250 keV. Clearly, a true Michelson-Morley experiment explicitly designed to confirm the stability of the phase to the level of \(\varepsilon\) over an interval of several months or years would potentially provide a very stringent test of \(\kappa\).

On the other hand, the instantaneous phase shift (the length-dependent piece of (13)) is difficult to isolate experimentally without additional means to null the time variation (perhaps by means of a multiple-arm apparatus). Nevertheless, for the purposes of illustration, we can interpret the limit \(\delta L/L = \varepsilon = 1.5 \times 10^{-15}\) of [28] as providing a null result for an experiment of this type. Again, for small \(\alpha\) we have from (13) in this case

\[
\frac{\hbar^2 k^3 / 4\kappa^2 \alpha^2}{kL} < \varepsilon
\]

leading to the lower limit on \(\kappa c\) of 6.2 keV.
3 Conclusions

In this paper we have investigated some of the important consequences associated with the \( \kappa \)-deformed Poincaré group based on and extending from the work done by [11]. In the two major relativistic tests studied, we have obtained lower limits on \( \kappa \), by comparing the modified theory with experiment. The lower limits on \( \kappa \) thus obtained were however found to be small, particularly when compared with limits obtained from particle accelerator [2] and astrophysical [22] tests. It is perhaps surprising that two of the standard tests, normally regarded as strong confirmation of special relativity, turn out to be relatively insensitive to \( \kappa \). The main reason is the low energy nature of the precision tests. In fact, any modifications to special relativity, including the \( \kappa \)-deformation, are most likely to be significant for high energy processes only. One could for example substantially improve the lower limits obtained on \( \kappa \) by performing the Michelson-Morley experiment using particles with a non-zero rest mass, say neutrons. The four-momentum component \( p_0 c \) will now have a very large rest mass contribution (\( \sim 1 \text{ GeV} \)). Assuming that an experimental accuracy of \( 1.5 \times 10^{-15} \) is again feasible, much larger lower limits on \( \kappa \) in the \( 10^3 \text{ GeV} \) range, comparable with the limits [2] from particle accelerator experiments, are obtainable.

Beyond relativistic tests, our approach in the paper has been to underline in principle various physical implications of the modified relativity, pointing out the essential assumptions. The transverse Doppler analysis and the Thomas precession use only the \( \kappa \)-deformed Poincaré algebra, whereas both the Casimir effect and the Michelson-Morley analysis involve additional assumptions. In the former, additivity of the energy of (virtual) multiphoton states has been used implicitly in addition to the modified dispersion relation. While this seems justified given the fact that the \( P_0 \) coproduct is undeformed, nevertheless other more subtle effects to do with composite systems in a fully developed \( \kappa \)-deformed Maxwell gauge theory cannot be ruled out—perhaps the divergence of the result (3) is an indication of the incompleteness of the analysis. In the case of the Michelson-Morley experiment, while our procedure involving momentum-dependent spacetime transformations appears to have been invoked merely to provide a recipe for obtaining a concrete answer for this particular case, it is certainly to be expected that a complete theory would involve some kind of modification to the ‘superposition’ principle. Thus at the level of quantum fields, where intensity measurements correspond to appropriate correlation functions, it is well possible that the necessity for nontrivial coproducts force the intervention of certain grouplike operators (such as \( \exp(P_0/\kappa) \)). These may have the same kind of effect on the amplitudes as has been invoked here as an extra assumption to do with the classical fields.

In conclusion, the analysis of experimental effects presented here can be regarded as a preliminary to a more far-reaching study. In the absence of a definitive and consistent deformed field theory, one more modest goal in this context would be to develop further a \( \kappa \)-relativistic particle mechanics, especially for spin. Certainly, if the formalism can be re-cast as an effective metric theory, then a host of experimental results, intended as tests of general relativity, could be re-interpreted in terms of limits on the deformation parameter \( \kappa \) in the spirit of this work.

Acknowledgements

The authors would like to thank Prof Henri Ruegg for helpful suggestions in the early stages of the work, and for drawing our attention to ref [30]; Dr Lindsay Dodd for correspondence on the deformed Maxwell equations; Prof Angas Hurst for critical comments; Prof Jerzy Lukierski for supplying us with a copy of the Domokos preprint (ref [22]), and for continuous support and encouragement, and Prof Geoff Stedman for useful discussions. We also thank the referees for
enforcing a more focussed presentation.

Appendix

The $\kappa$–deformed Poincaré algebra [11] has the following structure.

\[
\begin{align*}
[M_i, M_j] &= i\hbar\epsilon_{ijk}M_k & [P_\mu, P_\nu] &= 0 \\
[L_i, M_j] &= i\hbar\epsilon_{ijk}L_k & [M_i, P_0] &= 0 \\
[M_i, P_j] &= i\hbar\epsilon_{ijk}P_k & [L_i, P_j] &= 0 \\
[L_i, P_j] &= i\hbar\kappa\delta_{ij} \sinh \frac{P_0}{\kappa} & [L_i, P_0] &= i\hbar P_i \\
[L_i, L_j] &= -i\hbar\epsilon_{ijk} \left( M_k \cosh \frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k (P.M) \right),
\end{align*}
\]

where $M_i$ is the rotation operator, $L_i$ is the Lorentz boost generator, and $P_\mu$ is the momentum operator, all of which are Hermitian. The deformed elements in (A1) revert to their normal Poincaré algebraic forms in the limit of $\kappa \to \infty$.

The coalgebra, coproduct, and the associated counit and antipodes, of the Hopf algebra are given [11] by:

\[
\begin{align*}
\triangle (M_i) &= M_i \otimes 1 + 1 \otimes M_i \\
\triangle (L_i) &= L_i \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes L_i + \frac{1}{2\kappa} \epsilon_{ijk} \left( P_j \otimes M_k e^{P_0/2\kappa} + e^{-P_0/2\kappa} M_j \otimes P_k \right) \\
\triangle (P_i) &= P_i \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes P_i, \quad \triangle (P_0) = P_0 \otimes 1 + 1 \otimes P_0
\end{align*}
\]

where the counits $\epsilon$ of all generators are zero. The antipode is

\[
S(M_i) = -M_i, \quad S(P_\mu) = -P_\mu, \quad S(L_i) = -L_i + \frac{3}{2\kappa} P_i.
\]

It should be noted that the coproduct for the space translations, and boosts are deformed, as is the antipode for the boosts.

From the $\kappa$–deformed Poincare algebra (A1) we can obtain the corresponding generalised Lorentz transformations in four-momentum space. This can be achieved by considering the orbits in $(P_0, P_3)$ space of the ‘one parameter subgroups’ generated by exponentiation, i.e.

\[
P_\mu(\eta) = e^{i\eta L_3} P_\mu e^{-i\eta L_3}.
\]

By differentiating (A4) with respect to $\eta$, and using the commutation relations (A1) we can express the deformed algebra (A3) as the following differential equation,

\[
\ddot{p}_0 - \kappa \sinh \frac{P_0}{\kappa} = 0.
\]

This equation has the same form as the differential equation of the hyperbolic pendulum, with the ‘rapidity’ $\eta$ playing the role of time, and with the first integral of motion describing the constant energy surfaces. By integrating (A5) with respect to $p_0$ we obtain the set of $\kappa$–generalised four-momentum transformations, (see [11]). In these transformations the hyperbolic functions $\cosh$ and $\sinh$ associated with the Lorentz transformation are replaced by the Jacobi elliptic functions $nc$, $sc$ and $dc$. For example for the usual boost with rapidity $\alpha$ in the $x$ direction, the role of $\cosh\alpha$ in the Lorentz transformation is played by,

\[
nc \left( \left(1 + \epsilon^2 \right)^{1/2} |(1 + \epsilon^2)^{-1} \right) \quad \text{where} \quad \epsilon = \frac{\sqrt{c^2 M_0^2 + p_1^2 + p_2^2}}{2\kappa}.
\]
We apply these $\kappa$-generalised four-momentum transformations to relativistic problems in a series expanded form. To evaluate these it is first necessary to find suitable expansions of the Jacobi elliptical functions. These Jacobi elliptic functions all have the same parameter $m = 1/(1 + \epsilon^2)$ which is very close to unity for large $\kappa$, (i.e. small $\epsilon$). Alternatively as the complementary parameter $m_1 = 1 - m$ is very small, it is useful to express these elliptical functions in terms of a power series in $m_1$. This involves the use [29] of the Jacobi imaginary transformations $nc(u|m) = cn(iu|m_1)$, $sc(u|m) = -isn(iu|m_1)$, and $dc(u|m) = dn(iu|m_1)$, and the following Fourier-like expansions of the Jacobi elliptical functions in powers of the parameter $m$, 

\[\begin{align*}
    cn(u|m) & = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \cos(2n + 1)\nu, \\
    sn(u|m) & = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 - q^{2n+1}} \sin(2n + 1)\nu, \\
    dn(u|m) & = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos 2n\nu, 
\end{align*}\]  

(A.6)  

where the nome $q = \exp(-\pi K'/K)$ and the argument $\nu = \pi u/2K$ are defined in terms of the real and imaginary quarter periods $K, K'$, both of which are expressible in series expansion form. Finally, the series expansions of the generalised transformations are found to take the following forms,

\[\begin{align*}
    p'_0 & = p_0 \cosh \alpha + p_3 \sinh \alpha + \frac{1}{4\kappa^2} \left( - \frac{(p_3 \sinh \alpha + p_0 \cosh \alpha)^3}{6} + \frac{p_0^3 \cosh \alpha}{6} \right) \\
    & + \frac{p_3 \mu^2}{4\kappa^2} \left( \frac{\alpha \cosh \alpha}{2} - \frac{\sinh \alpha}{2} + \frac{p_0^2 \sinh \alpha}{2} + \frac{\sinh^3 \alpha}{\mu^2} + \frac{p_3 \sinh^3 \alpha}{8} + \frac{\cosh (\sinh 2\alpha - 2\alpha)}{8} \right) \\
    & + \frac{p_0 \mu^2}{4\kappa^2} \left( \frac{\alpha \sinh \alpha}{2} + \frac{p_0^2 \cosh \alpha \sinh^2 \alpha}{\mu^2} + \frac{\sinh \alpha (\sinh 2\alpha - 2\alpha)}{8} \right) + 0(\kappa^{-4}) 
\end{align*}\]  

(A.7)  

\[\begin{align*}
    p'_3 & = p_3 \cosh \alpha + p_0 \sinh \alpha \\
    & + \frac{1}{4\kappa^2} \left( \frac{p_3^2 \sinh \alpha}{6} + \frac{p_3 \sinh \alpha (2\alpha \mu^2 + 3\mu^2 \sinh 2\alpha + 4p_3^3 \sinh 2\alpha)}{8} \right) \\
    & + \frac{p_0^2 \mu^2}{4\kappa^2} \left( \frac{\alpha \cosh \alpha}{2} - \frac{\sinh \alpha}{2} + \frac{p_0^2 \sinh \alpha}{2\mu^2} + \frac{p_3 \sinh^3 \alpha}{\mu^2} + \frac{\cosh (\sinh 2\alpha - 2\alpha)}{8} \right) \\
    & + 0(\kappa^{-4}) 
\end{align*}\]  

(A.8)  

where $\mu = \sqrt{c^2 M_0^2 + \pi_1^2 + \pi_2^2}$, and $\pi_i \equiv p_i$ for the transverse components.

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Figure 1: Diagram of the Michelson Morley setup. The parallel direction is taken along the \( z \) axis, and the transverse direction is taken to be along the \( x \) axis. The length of each arm of the interferometer is \( L \), and the preferred frame moves with rapidity \( \alpha \) with respect to the laboratory frame.