Radiation Reaction in Noncommutative Electrodynamics

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Abstract—We study the radiation reaction acting on an accelerating charge moving in noncommutative spacetime and obtain an expression for it. Radiation reaction, due to a nonrelativistic point charge, is found to receive a small noncommutative correction term. The Abraham–Lorentz equation for a point charge in noncommutative spacetime suffers from the preacceleration and the runaway problems. We explore as an application the radiation reaction experienced by a charge which undergoes harmonic oscillations in a noncommutative plane.

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1. INTRODUCTION

An accelerated charge emits electromagnetic radiation which carries away certain amounts of the energy and momentum of the charge [1–3]. The loss of energy leads to a deceleration of the charge which implies that there must be a force acting on the charge due to the electromagnetic fields it produces. This retarding force is known as the radiation reaction. Thus, the effect caused by the radiation on the dynamics of the charge is to modify the equation of motion of the charge with the radiation reaction.

An accelerated charge produces electromagnetic field, say \( A^\mu(x, y, z, t) \), whose spacetime coordinates commute among themselves, i.e.,

\[
[x_i, x_j] = 0, \quad [x_i, t] = 0; \quad i, j = 1, 2, 3.
\]

However, there are physical instances where the notion of spatial coordinates may not commute

\[
[x_i, x_j] \neq 0; \quad i, j = 1, 2, 3.
\]

could be realized [4].

It is natural to ask: “How the radiation reaction and hence the dynamics of an accelerated charge will change in a spacetime manifold equipped with noncommuting spacetime coordinates?”

Physics at high energy, might alter the continuum nature of spacetime, serves as one of the main motivations behind the study of the spacetime coordinates that do not commute. One of the variants of the granular structure of spacetime could be thought in terms of the uncertainties in the measurements of the spacetime coordinates which give rise to the idea of the noncommutative spacetime.

It was Snyder [5] who first introduced noncommutative space, as a possible cure of ultraviolet divergences which stem from the ill-defined product of the fields at the same space–time point in quantum field theories. Moreover, noncommutative spaces are found to arise in several different contexts. At short distance, the interplay between quantum theory and gravitation suggests a nontrivial structure of space–time and a noncommutative structure of space–time is a possibility. In fact, the concept of space–time as a \( c^\infty \) manifold may break down to the distance-scales of the order of Plank length scale [6]. Noncommutative structure of spacetime naturally appears in string theory [7]. We consider fields defined on noncommutative spacetime manifold which obeys

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu},
\]

where \( \theta^{\mu\nu} \) is a constant real antisymmetric matrix of length dimension two. Electromagnetic field theory in commutative space time could be generalized to a noncommutative space–time via replacing ordinary local product by a Moyal star product of the two functions [8]

\[
(f \star g)(x) = e^{i\theta^{\mu\nu} \partial_x^\mu \partial_y^\nu} f(x)g(y)|_{x=y}.
\]

The question arises: can such spacetime noncommutativity induce any modification to the radiation reaction associated with an accelerated charge in commutative spacetime?

The goal of the present work is to compute the radiation reaction force experienced by an accelerated charge moving in the noncommutative spacetime. We perform the calculation of radiation reaction on an accelerating charge to capture any effect whatsoever arising due to noncommutativity in spacetime with a hope to find the resolutions to the pathologies and
ambiguities associated with the Abraham—Lorentz equation. We find that radiation reaction receives modification in the noncommutative Maxwell theory. The Abraham—Lorentz equation in noncommutative spacetime is found to be plagued with the preacceleration and the runaway solutions. This work serves as a preliminary framework to continue further study of the radiation reaction in the quantum version of noncommutative electrodynamics.

In Section 2, we define and formulate the radiation reaction force in noncommutative spacetime. In Section 3, we derive explicitly the expression for radiation reaction in noncommutative electrodynamics. In Section 4, we study radiation reaction experienced by a charge which undergoes harmonic oscillations in a noncommutative plane.

2. LORENTZ FORCE IN NONCOMMUTATIVE SPACETIME

A charged particle moving in an electromagnetic field experiences Lorentz force. Lorentz four force [1, 3] on a charged particle in electrodynamics in commutative spacetime is defined as

$$G_c^\nu = \int k^\nu d^3x$$

with Lorentz four force density $k^\nu$ given by

$$k^\nu = \frac{1}{c} j_\mu F^{\mu\nu},$$

where $j_\mu$ is four current density vector and $F^{\mu\nu}$ is electromagnetic field tensor constructed using the electromagnetic four vector potential $A^\nu$ and is given by $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$.

For a point charge the four current density reads

$$j_\mu(x) = ec \int_{-\infty}^{+\infty} ds z^\mu \delta[(x - z(s))],$$

where $z(s)$ is the world line of the charged particle parametrized by $s$. In order to calculate self force due to the point charge, electromagnetic field tensor $F^{\mu\nu}$ is replaced by retarded field tensor $F^{\mu\nu}_{\text{ret}}$ and four potential $A^\mu$ is replaced by retarded potential $A^\mu_{\text{ret}}$ [3, 9].

$$F^{\mu\nu}_{\text{ret}} = \frac{1}{2} F^{\mu\nu}_{\text{rad}} + \frac{1}{2} (F^{\mu\nu}_{\text{ret}} + F^{\mu\nu}_{\text{adv}}),$$

where the radiation field defined by

$$F^{\mu\nu}_{\text{rad}} = F^{\mu\nu}_{\text{ret}} - F^{\mu\nu}_{\text{adv}}.$$  

Radiation field $0.5 F^{\mu\nu}_{\text{rad}}$ contributes to radiation reaction where as $0.5 (F^{\mu\nu}_{\text{ret}} + F^{\mu\nu}_{\text{adv}})$ contributes to the electromagnetic mass of point charge.

Therefore the radiation reaction acting on a point charge in commutative electrodynamics is defined by

$$G_c^\nu = \frac{1}{2c} \int j_\mu(x) F^{\mu\nu}_{\text{rad}}(x) d^3x.$$  

Radiation reaction force [3] evaluates to

$$G_c = \frac{2}{3\pi e_c^2} \left( \gamma_i^2 + \frac{3}{4}\gamma_i^2 \gamma^3 (\gamma_i \gamma^3) + \frac{1}{4}\gamma_i^2 (\gamma_i \gamma^3)^2 \gamma^3 \right),$$

where $i = 1, 2, 3$ denote the spatial indices and $\gamma_i$ is the three velocity of the charged particle.

In the nonrelativistic limit, radiation reaction force due to a point charge is given by

$$G_{\text{rad}} = -\frac{2}{3\pi e_c^2} \gamma_i.$$  

Lorentz force in commutative electrodynamics could be generalized to noncommutative electrodynamics as follows:

$$F_{\text{NC}} = \int K^\nu d^3x,$$

with noncommutative Lorentz force density $K^\nu$ [11] given by

$$K^\nu = \frac{1}{c} (j_\mu F^{\mu\nu} + \theta^{0\alpha} \partial_\alpha [A^\beta_j j_\mu F^{\mu\nu}]),$$

where $A^\mu$ and $j_\mu$ [10] are given as:

$$A^\mu = A^\mu_{(0)} + A^\mu_{(0)}(\theta) + O(\theta^2),$$

$$j_\mu = j_\mu^{(0)} + j_\mu^{(1)}(\theta) + O(\theta^2).$$

Radiation reaction four force on a charge in noncommutative electromagnetic theory can therefore be defined by

$$G_{\text{rad}} = \int K_{\text{rad}} d^3x,$$

where

$$K_{\text{rad}} = \frac{1}{2c} j_\mu F^{\mu\nu}_{\text{rad}} + \frac{1}{4c} \theta^{0\alpha} \partial_\alpha [A^\beta_j j_\mu F^{\mu\nu}_{\text{rad}}].$$

3. RADIATION REACTION IN NONCOMMUTATIVE SPACETIME

We shall now turn up to the calculation of the radiation reaction in noncommutative space–time. The radiation reaction in noncommutative spacetime reads:

$$G_{\text{NC}} = \frac{1}{2c} \int j_\mu F^{\mu\nu}_{\text{rad}} d^3x + \frac{\theta^{0\beta}}{4c} \int \partial_\beta (A^\beta_j j_\mu F^{\mu\nu}_{\text{rad}}) d^3x + \frac{\theta^{0\beta}}{4c} \int \partial_\beta (A^\beta_j j_\mu F^{\mu\nu}_{\text{rad}}) d^3x.$$
where \(i,j = 1,2,3\) are spatial indices and index 0 stands for the temporal index. All terms containing \(\theta_{ij}\) involve total spatial derivative and hence must vanish since the vector potential as well as the electromagnetic field tensor vanish at the large spatial boundary. The non-trivial surviving terms are given by

\[
G_{NC}^{\nu} = \frac{1}{2c} \int J_\mu F_{\mu \nu}^{\text{rad}} d^3 x + \frac{\theta_0}{4c} \int \partial_\nu (A_\mu^{\text{rad}} J_\mu F_{\mu \nu}^{\text{rad}}) d^3 x. \tag{17}
\]

The term \(\frac{1}{2c} \int J_\mu F_{\mu \nu}^{\text{rad}} d^3 x\) turns out [3]:

\[
\frac{1}{2c} \int J_\mu F_{\mu \nu}^{\text{rad}} d^3 x = \frac{e}{2} \int J_\mu^{\text{rad}} d^3 x \frac{\partial}{\partial x(t)} (z(s)) - z(s) ds = \frac{e}{2} \frac{\dot{z}_0(s)}{\dot{\zeta}(s)} F_{\mu \nu}^{\text{rad}}(z(s)) = - \frac{2e}{34\pi} v^\nu (\dot{\zeta}^\nu + \dot{\zeta}^\nu)^2. \tag{18}
\]

The second term on the right-hand side of Eq. (17) could be simplified decomposing into two terms (please see Appendix A) as follows:

\[
\int \frac{\partial}{\partial x(t)} (A_\mu^{\text{rad}} J_\mu^{\text{rad}}) d^3 x = \frac{2e}{34\pi} \left( \frac{\dot{\zeta}}{\dot{\zeta}} + \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right) \times \frac{1}{\dot{\zeta}} \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{\dot{\zeta}}{\dot{\zeta}} \left( \frac{1}{2} \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{1}{3} \frac{\dot{\zeta}}{\dot{\zeta}} \right), \tag{19}
\]

and

\[
\int \frac{\partial J_\mu^{\text{rad}}}{\partial x(t)} (A_\mu^{\text{rad}} J_\mu^{\text{rad}}) d^3 x = \frac{2e}{34\pi} \left( \frac{\dot{\zeta}}{\dot{\zeta}} + \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right) \times \frac{1}{\dot{\zeta}} \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{\dot{\zeta}}{\dot{\zeta}} \left( \frac{1}{2} \frac{\dot{\zeta}}{\dot{\zeta}} + \frac{1}{3} \frac{\dot{\zeta}}{\dot{\zeta}} \right). \tag{20}
\]

So that the radiation reaction becomes

\[
G_{NC}^{\nu} = - \frac{2e}{34\pi} \left( \frac{\dot{\zeta}}{\dot{\zeta}} + \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right) + \frac{\theta_0}{16\pi} \frac{e^3}{34\pi} \times \left[ \frac{\dot{\zeta}}{\dot{\zeta}} + \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right] \left( \frac{1}{(\dot{\zeta})^2} - 2 \right) \tag{21}
\]

Moreover, in terms of 3-velocity, radiation reaction takes the following form

\[
G_{NC}^{\nu} = - \frac{2e}{34\pi} \left( \frac{\dot{\zeta}}{\dot{\zeta}} + \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right) + \frac{\theta_0}{16\pi} \frac{e^3}{34\pi} \times \left[ \frac{\dot{\zeta}}{\dot{\zeta}} + \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right] \left( \frac{1}{(\dot{\zeta})^2} - 2 \right) \tag{22}
\]

Thus the radiation reaction force is modified by \(\theta\)-dependent terms and it, in fact, depends on both space–space \(\theta_{ij}\) and space–time \(\theta_{ij}\) noncommutative parameters due to \(\theta_{ij}\) dependence of velocity \(v_j\) and its various order time derivatives.

In the nonrelativistic limit \(\nu/c \to 0\), radiation reaction force turns out to be

\[
G_{NC}^{\nu} = - \frac{2e}{34\pi} \nu\dot{\nu} \left( 1 + \theta_{ij} \frac{e}{4\pi c} \nu \dot{\nu} \right). \tag{23}
\]

The nonrelativistic expression for radiation reaction in noncommutative electrodynamics, unlike its commutative counterpart, receives the \(\theta\)-correction term. Even in the case of zero space–time noncommutativity \(\theta_{ij} = 0\), space–space noncommutativity \(\theta_{ij} \neq 0\) is there through \(\theta_{ij}\) dependence of \(\dot{v}_j\), which could be seen in the example of radiation reaction experienced by charged harmonic oscillator in noncommutative plane which is discussed in Section 4. It is worthy to note at this juncture that, the nonrelativistic limit \(\nu/c \to 0\) does not imply the commutative limit \(\theta_{ij} \to 0\) and vice versa. The Abraham–Lorentz equation in noncommutative spacetime now reads:

\[
ma_j = m\tau a_j - \frac{3m^2 c^2}{2e} \theta_{ij} a_j a_j = F_j, \tag{24}
\]

where \(\dot{v}_j = a_j\), the characteristic time \(\tau = \frac{2e}{34\pi mc^2}\) and \(F_j\) is the external force.

Let us consider one space–one time dimensions and assume \(\theta_{ij} = 0\) then the radiation reaction force becomes

\[
G_{NC}^{\nu} = - \frac{2e}{34\pi c} \left( \dot{\zeta}^\nu (s) \dot{\zeta}^\nu(s) \right) \left( \frac{1}{\dot{\zeta}} - 2 \right) \tag{25}
\]

Now, the equation of motion of a charge particle under a constant external force \(F\) is given by

\[
\frac{3m^3 \theta_0}{2e} a^2 + \tau a - a = \frac{F}{m} = 0. \tag{26}
\]
Equation (30) is quadratic in \( \dot{a} \). The acceptable root which is consistent in the commutative limit \( \theta \rightarrow 0 \) is

\[
\dot{a} = \frac{e}{3m\tau \theta} \left[ -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right].
\]

(27)

Suppose the constant external force \( F \) acts on the charge only during the time interval \( 0 \leq t \leq T \). Therefore, in the region \( \tau \theta \leq t \leq \frac{1}{\tau} \) and Eq. (27) leads to

\[
\dot{a} = \frac{e}{3m\tau \theta} \left[ -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right],
\]

(28)

which, after integration, yields

\[
\left( -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right) \exp \left( -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right) = I_1 \exp \frac{t}{\tau}.
\]

(29)

where \( I_1 \) is the integration constant. In the region \( 0 \leq t \leq \tau \theta \), we have

\[
\left( -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right) \exp \left( -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right) = I_2 \exp \frac{t}{\tau}.
\]

Similarly for the region \( t \geq \tau \theta \), external force \( F \) is zero and the connection between acceleration and time in this region turns out

\[
\left( -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right) \exp \left( -1 + \sqrt{1 + \left( \frac{6m\theta a}{e} \right)^2} \right) = I_3 \exp \frac{t}{\tau}.
\]

(31)

\( I_2 \) and \( I_3 \) are integration constants. The acceleration versus time in the region \( t \geq \tau \theta \) with \( T = 1 \tau \) is plotted as shown in Fig. 1. We notice that the accelerations in regions \( t \geq \tau \theta \) and \( t \leq 0 \) where there are no external forces, are increasing continuously with time \( t \). Therefore, Eqs. (29) and (31) will admit runaway solutions. Thus, Abraham–Lorentz equation in noncommutative spacetime is plagued with preacceleration and runaway solutions. However, Eq. (26) will turn out to be physically applicable leading to the physically reasonable solutions provided \( G^{NC} \ll F \) [12].

4. RADIATION REACTION IN TWO DIMENSIONAL NONCOMMUTATIVE HARMONIC OSCILLATOR

In this section, we explore the radiation reaction force due to a charge undergoing simple harmonic motion in a noncommutative plane \( \{\hat{x}, \hat{y}\} = i\theta \).
experienced by the charged harmonic oscillator in the noncommutative plane using (23) is given by

\[ \mathbf{F}_{\text{Rad}} = \frac{-2 e^2}{34\pi c^3} \left[ \omega_3^3 (C_1 \sin \omega_3 t - C_2 \cos \omega_3 t) \hat{q}_1 + \omega_3^3 (C_4 \sin \omega_4 t - C_1 \cos \omega_1 t) \right] \]

\[ \mathbf{F}_{\text{Rad}} = \frac{-2 e^2}{34\pi c^3} \left[ \omega_3^3 (C_1 \cos \omega_3 t + C_2 \sin \omega_3 t) \hat{q}_1 + \omega_3^3 (C_4 \cos \omega_4 t + C_1 \sin \omega_1 t) \right] \hat{q}_2, \]

where \( \hat{q}_1 \) and \( \hat{q}_2 \) are unit vectors along \( \hat{q}_1 \) and \( \hat{q}_2 \) directions respectively. We shall commit to the specific case when the initial conditions are such that \( C_1 = C_2 = C_3 = C_4 = 1 \). This choice does not alter the generalities of the interpretation of result for radiation reaction force due to 2D noncommutative charged harmonic oscillator. The quantity \( \frac{m\omega^2 t}{2} \) for all practical purposes is rather small and could admit large enough time to observe radiation reaction. Therefore, we can have for our purpose \( \cos \left( \pm \theta + \frac{m\omega^2 t}{2} \right) \sim \cos \theta + \frac{m\omega^2 t}{2} \sin \theta \), so that

\[ q_1 = 2\cos \omega t + \frac{m\omega^2 t}{2} \cos \omega t, \]

\[ q_2 = 2\cos \omega t - \frac{m\omega^2 t}{2} \cos \omega t, \]

\[ q_1 = 2\omega^3 \sin \omega t - m\omega^4 \left( 3 \cos \omega t - \omega t \sin \omega t \right), \]

\[ q_2 = 2\omega^3 \sin \omega t + m\omega^4 \left( 3 \cos \omega t - \omega t \sin \omega t \right). \]

Radiation reaction turns out

\[ \mathbf{F}_{\text{Rad}} = (q_1 + q_2) f(t) + (q_1 - q_2) g(t), \]

where

\[ f(t) = \frac{-2 e^2}{34\pi c^3} (2\omega^3 \sin \omega t), \]

\[ g(t) = \frac{e^2}{34\pi c^3} m\omega^4 (3 \cos \omega t - \omega t \sin \omega t). \]

Let us define

\[ f_s(t) = \frac{f(t)}{2}, \]

\[ g_s(t) = \frac{g(t)}{34\pi c^3} \]

Figure 2 shows a plot between the scaled components \( f_s(t) \) and \( g_s(t) \) of the radiation reaction and time. Radiation reaction in the commutative space vanishes at all times \( t = n\pi/\omega \), where \( n = 0, 1, 2, 3, \ldots \). However, radiation reaction in the noncommutative space does not vanish at any point in time.

5. CONCLUSIONS

The present paper explores whether Maxwell’s electromagnetic theory in the noncommutative spacetime might affect the radiation reaction and could possibly resolve the pathologies and ambiguities associated with the Abraham–Lorentz equation. To this end, we have derived the radiation reaction force acting on an accelerating charge due to the electromagnetic fields that it produces while moving in the noncommutative spacetime. We find that an accelerating point charge in noncommutative Maxwell theory experiences radiation reaction which receives a small \( \theta \)-dependent correction term. Moreover, it turns out that the Abraham–Lorentz equation in noncommutative spacetime suffers from the same long-standing problems of the preacceleration and runaway solutions.

APPENDIX A

CALCULATION OF RADIATION REACTION IN NONCOMMUTATIVE ELECTRODYNAMICS

We have

\[ d^3x \partial_0 \left( A_{\mu}^{\text{rad}}(x)J_{\mu}(x)F_{\text{rad}}^{\mu\nu}(x) \right) \]

\[ = d^3x \frac{\partial}{\partial x_0} \left( A_{\mu}^{\text{rad}}(x)F_{\text{rad}}^{\mu\nu}(x) \right) J_{\mu}(x) \]

\[ + \int d^3x \frac{\partial J_{\mu}(x)}{\partial x_0} A_{\mu}^{\text{rad}}(x)F_{\text{rad}}^{\mu\nu}(x). \]
Four vector current density due to a point charge is given by

\[ J_\mu(x) = e c \int ds \hat{z}_\mu(s) \delta^4(x - z(s)) \]

\[ = e c \frac{\hat{z}_\mu(s)}{\xi^0(s)} \delta^3(\hat{x} - \hat{z}(s)) . \]  

(A.2)

The first term in the above equation after inserting (A.2) gets simplified as

\[ \int d^3x \frac{\partial}{\partial x} (A_\mu^{\text{rad}}(x) F^{\mu\nu}(x)) J_\mu(x) \]

\[ = e c \frac{\hat{z}_\mu(s)}{\xi^0(s)} F^{\mu\nu}(z) \frac{\partial A_\nu^{\text{rad}}}{\partial x} x = z(s) \]

\[ + e c \frac{\hat{z}_\mu(s)}{\xi^0(s)} A_\nu^{\text{rad}}(z) \frac{\partial F^{\mu\nu}}{\partial x} x = z(s). \]  

(A.3)

The second term in Eq. (A.1) becomes

\[ \int d^3x \frac{\partial J_\mu(x)}{\partial x} (A_\mu^{\text{rad}}(x) F^{\mu\nu}(x)) \]

\[ = e c \frac{\hat{z}_\mu(s)}{\xi^0(s)} F^{\mu\nu}(z) \frac{\partial (\hat{z}_\nu(s))}{\partial s} x = z(s) . \]  

(A.4)

Four-vector potential is defined as

\[ A_\mu^{\text{rad}}(x) = e \int ds D(x - z(s)) \hat{z}_\mu(s). \]

At \( x = z(s) \), we have

\[ A_\mu^{\text{rad}}(z(s)) = e \int D(z(s) - z(s)) \hat{z}_\mu(s) ds . \]  

(A.5)

Let \( s = s' + u \) where \( u \) is a small parameter.

\[ \hat{z}_\mu(s) = \hat{z}_\mu(s') + u \frac{\xi^0}{\xi^0(s)} \hat{z}_\mu(s') + \frac{u^2}{2} \frac{\xi^0}{\xi^0(s)} \hat{z}_\mu(s') + \ldots \]

\[ + \ldots \hat{z}_\mu(s) = \hat{z}_\mu(s') + u \frac{\xi^0}{\xi^0(s)} \hat{z}_\mu(s') + \frac{u^2}{2} \frac{\xi^0}{\xi^0(s)} \hat{z}_\mu(s') + \ldots . \]

We know [3]

\[ D(z(s') - z(s)) = \frac{1}{2\pi} \frac{d\delta(u)}{du} . \]

Therefore,

\[ A_\mu^{\text{rad}}(z(s')) = -\frac{e}{2\pi} \int \delta(u) du (\hat{z}_\mu(s') + u \frac{\xi^0}{\xi^0(s)} \hat{z}_\mu(s') + \ldots) . \]

At \( u = 0, s = s' \). Hence

\[ A_\mu^{\text{rad}}(z(s)) = -\frac{e}{2\pi} \hat{z}_\mu(s). \]  

(A.6)

**Calculation of \( \partial_\nu A_\mu^{\text{rad}}(x) \)**

\[ \partial_\nu A_\mu^{\text{rad}}(x) = e \frac{\partial}{\partial x} \int ds D(x - z(s)) \hat{z}_\mu(s) \]

\[ = e \int ds D(x - z(s)) \frac{\hat{z}_\mu(s)(x - z(s))}{(x - z(s))^0} \hat{z}_\nu(s). \]

At \( x = z(s) \)

\[ \partial_\nu A_\mu^{\text{rad}}(z(s)) = e \int ds D((z(s)) \]

\[ - \frac{\hat{z}_\mu(s)(z(s') - z(s)) \hat{z}_\nu(s) d}{ds} (z(s') - z(s))^0 \hat{z}_\nu(s). \]

Then,

\[ (z(s') - z(s))^0 \hat{z}_\nu(s) = -u + o(u^2), \]

\[ \hat{z}_\mu(s)(z(s') - z(s)), \]

\[ = -u \left\{ \hat{z}_\mu(s) \hat{z}_\nu(s') + \frac{u}{2} \hat{z}_\mu(s) \hat{z}_\nu(s') + \frac{u^2}{2} \hat{z}_\mu(s) \hat{z}_\nu(s') + \ldots \right\}, \]

\[ + \frac{u^2}{6} \hat{z}_\mu(s) \hat{z}_\nu(s') + \frac{u^2}{2} \hat{z}_\mu(s) \hat{z}_\nu(s') + \ldots \].

(A.7)

(A.8)

Therefore,

\[ F^{\mu\nu}(z(s)) = (\partial_\nu A_\mu^{\text{rad}} - \partial_\mu A_\nu^{\text{rad}}) x = z(s) \]

\[ = \frac{e}{3\pi} (\hat{z}_\nu(s) \hat{z}_\mu(s') - \hat{z}_\mu(s) \hat{z}_\nu(s')). \]  

(A.9)

**Calculation of the Term**

\[ e c \frac{\hat{z}_\mu(s)}{\xi^0(s)} F^{\mu\nu}(z(s)) \frac{\partial A_\nu^{\text{rad}}}{\partial x} x = z(s) \]

From Eq. (A.9)

\[ (\partial_\nu A_\mu^{\text{rad}}) x = z(s) = -\frac{e}{2\pi} \]

\[ \times \left\{ \frac{1}{3} \hat{z}_\mu(s) \hat{z}_0(s) + \hat{z}_\mu(s) \hat{z}_0(s) + \hat{z}_\mu(s) \hat{z}_0(s) \right\}. \]

Therefore,

\[ ec \frac{\hat{z}_\mu(s)}{\xi^0(s)} F^{\mu\nu}(z(s)) (\partial_\nu A_\mu^{\text{rad}}) x = z(s) \]

\[ = \frac{\hat{z}_\mu(s)}{\xi^0(s)} \frac{e c}{2\pi} \left( \hat{z}_\mu(s) \hat{z}_0(s) - \hat{z}_\mu(s) \hat{z}_0(s) \right) \]

\[ \times \frac{e}{2\pi} \left( \frac{1}{3} \hat{z}_\mu(s) \hat{z}_0(s) + \hat{z}_\mu(s) \hat{z}_0(s) + \hat{z}_\mu(s) \hat{z}_0(s) \right). \]
We know that
\[ \dot{z}^{\mu}(s)z^{\mu}(s) = 1, \quad \dot{z}^{\mu}(s)z^{\mu}(s) = 0, \]
\[ \dot{z}^{\mu}(s)z^{\mu}(s) = 0, \]
\[ \dot{z}^{\mu}(s)\dot{z}^{\mu}(s) = -z^2(s). \]

Then,
\[ ec \frac{\partial \dot{F}^{\mu\nu}_0}{\partial z(\xi)}(z(s))\left(\frac{\partial \dot{F}^{\mu\nu}_0}{\partial A^\alpha}(x)\right)_{x=z(s)} = \frac{2}{3} \left( \dot{z}^\mu(s) + \dot{z}^\nu(s) \right) \frac{1}{\dot{z}(s)} + \frac{1}{3} \left[ \dot{z}^\mu(s)\dot{z}^\nu(s) + \frac{1}{2} \dot{z}^\mu(s)\dot{z}^\nu(s) \right] = \frac{e}{2\pi} \int du d\dot{\delta}(u) \left[ \frac{d}{du} \left( \dot{z}^\mu(s') + \frac{\partial z^\mu(s')}{\partial u} \right) \right] \]
\[ \times \frac{2}{3} \left( \dot{z}^\mu(s') + \dot{z}^\nu(s') \right) \frac{1}{\dot{z}(s')} + \frac{1}{3} \left[ \dot{z}^\mu(s')\dot{z}^\nu(s') + \frac{1}{2} \dot{z}^\mu(s')\dot{z}^\nu(s') \right] \]
\[ = -\frac{e}{2\pi} \int du d\dot{\delta}(u) \left( \frac{d^2 P(u) d^2 Q(u)}{du^2} + \frac{dP(u) d^2 Q(u)}{du^3} \right), \]
\[ \text{where} \]
\[ P(u) = (\dot{z}^\mu(s') + \frac{u}{2} \dot{z}^\mu(s') + \frac{u^2}{6} \dot{z}^\mu(s') \ldots), \]
\[ Q(u) = \dot{z}^\mu(s')\dot{z}^\nu(s') + \frac{u}{2} \dot{z}^\mu(s')\dot{z}^\nu(s') \ldots \]

The first term in Eq. (A.14) can be expressed as
\[ = -\frac{e}{2\pi} \int du d\dot{\delta}(u) \left( \frac{d^2 P(u) d^2 Q(u)}{du^2} + \frac{dP(u) d^2 Q(u)}{du^3} \right), \]

Now, Eq. (A.14) becomes:
\[ = -\frac{e}{2\pi} \int du d\dot{\delta}(u) \left[ \frac{d^2 P(u) d^2 Q(u)}{du^2} + \frac{dP(u) d^2 Q(u)}{du^3} \right] \]
\[ \times \left[ \dot{z}^\mu(s)\dot{z}^\nu(s) + \dot{z}^\mu(s)\dot{z}^\nu(s) \right] + \dot{z}^\mu(s)\dot{z}^\nu(s) \]
\[ = -\frac{e}{2\pi} \int du d\dot{\delta}(u) \left[ \frac{d^2 P(u) d^2 Q(u)}{du^2} + \frac{dP(u) d^2 Q(u)}{du^3} \right] \]
\[ \times \left[ \dot{z}^\mu(s)\dot{z}^\nu(s) + \dot{z}^\mu(s)\dot{z}^\nu(s) \right] + \dot{z}^\mu(s)\dot{z}^\nu(s) \]
\[ = -\frac{e}{2\pi} \int du d\dot{\delta}(u) \left[ \frac{d^2 P(u) d^2 Q(u)}{du^2} + \frac{dP(u) d^2 Q(u)}{du^3} \right] \]
At $x = z(s')$ the second term in Eq. (A.13) is

$$
eq rac{e}{2\pi} \int dz D(z(s') - z(s)) \frac{\partial}{\partial z(s')} \left[ \left( z''(z(s') - z(s)) \right)^2 \right]$$

$$= -\frac{e}{2\pi} \int dz \frac{1}{z''(s')} \delta(u) \frac{d^2}{dz^2}$$

$$\times \left( \frac{z''(s')\tilde{z}'(s') + \frac{2}{3} u z''(s')\tilde{z}'(s')}{2} \right) \quad (A.19)$$

$$+ \frac{3u}{2} z''(s')\tilde{z}'(s') + u z''(s')\tilde{z}'(s')$$

$$+ \frac{2u^2}{3} z''(s')\tilde{z}'(s') + u z''(s')\tilde{z}'(s')$$

$$= -\frac{e}{2\pi} \int dz \frac{1}{z''(s')} \frac{4}{3} u z''(s')\tilde{z}'(s') + 2 z''(s')\tilde{z}'(s').$$

By substituting Eqs. (A.18) and (A.19) in Eq. (A.13), we have,

$$\frac{\partial}{\partial x} \Omega(x)_{z(s')} = -\frac{e}{2\pi}$$

$$\times \left[ \frac{1}{3} \frac{z''(s)}{z''(s)} \left( \frac{1}{2} z''(s)\tilde{z}'(s) + z''(s)\tilde{z}'(s) \right) + \frac{2}{3} u z''(s)\tilde{z}'(s) + 2 z''(s)\tilde{z}'(s) \right]$$

$$+ \frac{4}{3} u z''(s)\tilde{z}'(s) + 2 z''(s)\tilde{z}'(s)$$

Then

$$(\partial_d F^{uv}(x))_{x=z(s')} = \frac{e}{2\pi} \int dz \left[ \frac{z''(s)}{6} \left( z''(s)\tilde{z}'(s) - \tilde{z}'(s)z''(s) \right) + \frac{2}{3} z''(s)\tilde{z}'(s) - \tilde{z}'(s)z''(s) \right]$$

$$+ \frac{1}{2} \frac{z''(s)}{z''(s)} \left( \frac{1}{2} z''(s)\tilde{z}'(s) + \frac{1}{3} \frac{z''(s)}{z''(s)} \right).$$

Therefore,

$$\frac{\partial}{\partial x} \frac{z''(s)}{z''(s)} (A_{rad}(z(s)))_{x=z(s')} = \frac{e^2 c}{4\pi^2} \frac{z''(s)}{z''(s)}$$

$$\times \int \left[ \frac{z''(s)}{6} \left( \frac{1}{2} z''(s)\tilde{z}'(s) - \tilde{z}'(s)z''(s) \right) + \frac{2}{3} z''(s)\tilde{z}'(s) - \tilde{z}'(s)z''(s) \right]$$

$$+ \frac{1}{2} \frac{z''(s)}{z''(s)} \left( \frac{1}{2} z''(s)\tilde{z}'(s) + \frac{1}{3} \frac{z''(s)}{z''(s)} \right).$$

Calculation of the Term

$$ec \cdot A_{rad}^{{rad}}(z(s)) \cdot F^{uv}_{rad}(z(s)) \cdot \frac{\partial}{\partial s} \left( \frac{z''(s)}{z''(s)} \frac{z''(s)}{z''(s)} \right)$$

We have

$$A_{rad}^{{rad}}(z(s)) = \int F^{uv}_{rad}(z(s)) \cdot A_{rad}^{{rad}}(z(s)) - \frac{z''(s)}{z''(s)} \cdot F^{uv}_{rad}(z(s))$$

Then,

$$= -\frac{2 e^2}{3 4\pi^2}$$

$$\times \int \left( \frac{z''(s)}{z''(s)} \right)^2 \left( \frac{z''(s)}{z''(s)} \right)^2$$

and

$$= \frac{2 e^2}{3 4\pi^2} \left( \frac{z''(s)}{z''(s)} \right)^3$$

Therefore,

$$\frac{z''(s)}{z''(s)} \left( \frac{z''(s)}{z''(s)} \right)^2 \left( \frac{z''(s)}{z''(s)} \right)^2$$

From Eqs. (A.11) and (A.21),

$$\int \left( \frac{z''(s)}{z''(s)} \right) \left( \frac{z''(s)}{z''(s)} \right)^2 \left( \frac{z''(s)}{z''(s)} \right)^2$$

$$= \frac{2 e^2}{3 4\pi^2} \left( \frac{z''(s)}{z''(s)} \right)^3$$

$$\times \left( \frac{1}{3} \frac{z''(s)}{z''(s)} + \frac{z''(s)}{z''(s)} \right) \frac{1}{z''(s)}$$

$$+ \frac{2 e^2}{3 4\pi^2} \left( \frac{z''(s)}{z''(s)} \right)^3 \left( \frac{z''(s)}{z''(s)} + \frac{z''(s)}{z''(s)} \right)$$

(A.22)

(A.23)

(A.24)
From Eqs. (A.22) and (A.23), we have
\[ \int \frac{\partial F_{\mu\nu}}{\partial x^\mu} A^{\mu\nu} F_{\mu\nu} \, d^3x = 2e^2 \frac{\dot{z}(s)}{32\pi} \left( \dot{z}^2(s) + \dot{z}^0(s) \dot{z}^0(s) \right) \] \[ - 2e^2 \frac{\dot{z}(s)}{32\pi} \left( \dot{z}^0(s) \dot{z}^0(s) \right) \] \[ (A.26) \]

Radiation reaction becomes
\[ G_{NC}^\nu = -\frac{2e^2}{34\pi \dot{z}^0(s)} \left( \dot{z}^\nu(s) + \dot{z}^\nu(s) \dot{z}^0(s) \right) + \frac{e^2}{16\pi} \] \[ \times \left\{ \frac{1}{\dot{z}^0(s)} \left( \frac{\dot{z}^0(s)}{\dot{z}^0(s)} \right) - 2 \right\} \] \[ - \frac{\dot{z}^0(s) \dot{z}^0(s)}{\dot{z}^0(s)} \left( \frac{1}{2} \dot{z}^0(s) + \frac{1}{3} \dot{z}^0(s) \right) \] \[ - \frac{\dot{z}^0(s) \dot{z}^0(s)}{\dot{z}^0(s)} \left( \frac{1}{2} \dot{z}^0(s) + \frac{1}{3} \dot{z}^0(s) \right) \] \[ \times \frac{1}{\dot{z}^0(s)} \left\{ \frac{1}{3} \dot{z}^0(s) \dot{z}^0(s) + \dot{z}^0(s) \dot{z}^0(s) \right\}, \] \[ (A.27) \]

where
\[ \frac{\dot{z}^0(s)}{\dot{z}^0(s)} \left( \dot{z}^0(s) + \dot{z}^0(s) \dot{z}^0(s) \right) - \frac{1}{\dot{z}^0(s)} \left( \dot{z}^0(s) \right) - 2 \] \[ = -\frac{1}{\gamma c^2} \left( \ddot{v}_j + \frac{2\gamma}{c^2} v_j (\dot{v} \cdot \dot{v}) \right) \] \[ \times \left( \ddot{v}_j + \frac{2\gamma}{c^2} v_j (\dot{v} \cdot \dot{v}) \right) \] \[ - \frac{3\gamma^4}{c^8} (\dot{v} \cdot \dot{v}) v_j \right) \] \[ \times \left( \ddot{v}_j + \frac{2\gamma}{c^2} v_j (\dot{v} \cdot \dot{v}) \right) \] \[ = -\frac{\gamma^4}{3c^6} (\dot{v} \cdot \dot{v}) v_j \] \[ (A.28) \]

and
\[ \frac{\dot{z}^0(s)}{\dot{z}^0(s)} \left( \dot{z}^0(s) + \dot{z}^0(s) \dot{z}^0(s) \right) - \frac{1}{\dot{z}^0(s)} \left( \dot{z}^0(s) \right) - 2 \] \[ = \frac{\gamma^4}{c^8} (\dot{v} \cdot \dot{v}) v_j \] \[ \times \left( \ddot{v}_j + \frac{2\gamma}{c^2} v_j (\dot{v} \cdot \dot{v}) \right) \] \[ - \frac{3\gamma^4}{c^8} (\dot{v} \cdot \dot{v}) v_j \right) \] \[ (A.29) \]

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