LEFT ORDERABILITY AND TAUT FOLIATIONS WITH ONE-SIDED BRANCING

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Abstract. For a closed orientable irreducible 3-manifold $M$ that admits a co-orientable taut foliation with one-sided branching, we show that $\pi_1(M)$ is left orderable.

1. Introduction

The L-space conjecture is proposed by Boyer-Gordon-Watson (cf. [BGW]) and by Juhász (cf. [J]), which states that: for every closed orientable irreducible 3-manifold $M$, $M$ is a non-L-space if and only if $\pi_1(M)$ is left orderable, and if and only if $M$ admits a co-orientable taut foliation.

In [OS], Ozsváth and Szabó proves that $M$ is a non-L-space if $M$ admits a co-orientable taut foliation (see also [B], [KR]). In [G], Gabai proves that $M$ admits taut foliations if $M$ has positive first Betti number. In [BRW], Boyer, Rolfsen and Wiest proves that $\pi_1(M)$ is left orderable if $b_1(M) > 0$. It’s known that the L-space conjecture holds for every graph manifold, by the works of Boyer-Clay ([BC]), Rasmussen ([R]), and Hanselman-Rasmussen-Rasmussen-Watson ([HRRW]).

The main result in this paper is:

Theorem 1.1. Let $M$ be a connected, closed, orientable, irreducible 3-manifold that admits a co-orientable taut foliation with one-sided branching. Then $\pi_1(M)$ is left orderable. In addition, $\pi_1(M)$ is isomorphic to a subgroup of $G_\infty$.

The group $G_\infty$ (cf. Definition 2.5 for the definition) is shown to be left orderable by Navas ([N]) and is shown to have no nontrivial homomorphism to $\text{Homeo}_+(\mathbb{R})$ by Mann ([Ma]).

Furthermore, we show that

Proposition 1.2. Let $L$ be an oriented, connected, simply connected, non-Hausdorff 1-manifold which has one-sided branching. Without loss of generality, we assume that $L$ has branching in the negative side. Let $G$ be a group acting on $L$ via orientation-preserving homeomorphisms.

(a) Given an embedding $e : \mathbb{R} \to L$ such that $e(-\infty), e(+\infty)$ are ends of $L$ which are negative, positive respectively. Then there is a homomorphism $d : G \to G_\infty$ induced by $e$.

(b) Fix an embedding $e : \mathbb{R} \to L$ as in (a). Let $h \in G$. Suppose that for arbitrary $n \in \mathbb{R}$, there is $m \in (n, +\infty)$ with $h(e(m)) \neq e(m)$. Then $d(h) \neq 1$.

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2. Preliminary

2.1. 1-manifolds with one-sided branching. Let $L$ be an oriented, connected, simply connected, non-Hausdorff 1-manifold. We call every element of

$$\{\text{path compactification of } L\} - L$$

an end of $L$. An end $t$ of $L$ is positive (resp. negative) if there is an embedded ray $r : [0, +\infty) \to L$ such that $r(+\infty) = t$ and the increasing orientation on $r$ is consistent with (resp. opposite to) the orientation on $L$. 

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Definition 2.1. $L$ is said to have one-sided branching if either of the following statements holds:

- $L$ has exactly one positive end and infinitely many negative ends. In this case, $L$ is said to have branching in the negative direction.
- $L$ has exactly one negative end and infinitely many positive ends. In this case, $L$ is said to have branching in the positive direction.

2.2. The three types of foliations. As shown in [C2, Definition 4.41], co-orientable taut foliations in closed 3-manifolds have the following three types:

Definition 2.2. Let $F$ be a co-oriented taut foliation of a closed orientable 3-manifold $M$. Let $L(F)$ denote the leaf space of the pull-back foliation of $F$ in the universal cover of $M$ (then $L(F)$ is an orientable, connected, simply connected, possibly non-Hausdorff 1-manifold). We assume that $L(F)$ has an orientation induced from the co-orientation on $F$. Then $F$ has exactly one of the following three types:

(a) $F$ is $\mathbb{R}$-covered if $L(F)$ is homeomorphic to $\mathbb{R}$.
(b) $F$ has one-sided branching if $L(F)$ has one-sided branching.
(c) $F$ has two-sided branching if $L(F)$ has infinitely many positive ends and infinitely many negative ends.

For taut foliations, $\mathbb{R}$-covered is well studied, two-sided branching is the generic case, and one-sided branching has an intermediate role between them.

There are many important examples of taut foliations with one-sided branching. In [Me], Meigniez provides infinitely many taut foliations with one-sided branching in hyperbolic 3-manifolds. See [C1], [F] for more examples and properties of them.

Definition 2.3. Under the assumption of Definition 2.2, the deck transformations of $\tilde{M}$ induces an action of $\pi_1(M)$ on $L(F)$. We call this action the $\pi_1$-action on $L(F)$.

2.3. The blowing-up operation on a foliation.

Definition 2.4. Blowing-up a leaf $\lambda$ of a foliation is to replace $\lambda$ by a product space $\lambda \times I$ foliated with leaves $\{\lambda \times \{t\} \mid t \in I\}$. And blowing-up a point $t$ in an 1-manifold is to replace it by a closed interval $\{t\} \times I$.

2.4. The group $G_\infty$. We define the group $G_\infty$, which follows from [Ma]:

Definition 2.5. Let $G_\infty = \text{Homeo}_+(\mathbb{R})/\sim$, where $\sim$ is the equivalence relation on $\text{Homeo}_+(\mathbb{R})$ defined by $g \sim f$ if there is $n \in \mathbb{R}$ such that the restrictions of $g, f$ to $[n, +\infty)$ are equal. Henceforth, for every $g \in \text{Homeo}_+(\mathbb{R})$, we will always denote by $[g]$ the image of $g$ under the quotient map $\text{Homeo}_+(\mathbb{R}) \to G_\infty$. We assume that the multiplication on $\text{Homeo}_+(\mathbb{R})$ is given by the left group action, i.e. $fg = f \circ g$. Define $[f] : [g] = [fg] = [f \circ g]$ for all $f, g \in \text{Homeo}_+(\mathbb{R})$. This multiplication is well-defined on $G_\infty$ and makes $G_\infty$ a group.

The following theorem is proved by Navas in [N]:

Theorem 2.6 (Navas). $G_\infty$ is left orderable.

Remark 2.7. In [Ma], Mann proves that the cardinality of $G_\infty$ is equal to the cardinality of $\text{Homeo}_+(\mathbb{R})$, but there exists no nontrivial homomorphism $G_\infty \to \text{Homeo}_+(\mathbb{R})$. See [Ma] for more information about $G_\infty$.

3. The proof of the main theorem

3.1. The group action on an 1-manifold with one-sided branching. Let $L$ be an oriented, connected, simply connected, non-Hausdorff 1-manifold which has one-sided branching, and we
assume that $L$ has branching in the negative direction. Let $G$ be a group acting on $L$ via orientation-preserving homeomorphisms. Let $e : \mathbb{R} \to L$ be an embedding with

$$e(+) = \text{the positive end of } L,$$

$$e(-) \in \{\text{negative ends of } L\}.$$

In this subsection, we prove Proposition 1.2:

**Proposition 1.2.** (a) There is a homomorphism $d : G \to G_\infty$.

(b) Let $h \in G$. Suppose that for arbitrary $n \in \mathbb{R}$, there is $m \in (n, +\infty)$ with $h(e(m)) \neq e(m)$. Then $d(h) \neq 1$.

**Notation.** We assume that the action of $G$ on $L$ is a left group action, i.e. for arbitrary $f, g \in G$, we have $fg(x) = f(g(x))$ for every $x \in L$. Henceforth, for arbitrary functions $u : Y \to Z, v : X \to Y$, we will always denote by $uv$ the composition $u \circ v$. For example, given $f, g \in G$, $fge$ will always denote the function $f \circ g \circ e : \mathbb{R} \xrightarrow{e} L \xrightarrow{g} L \xrightarrow{f} L$.

Let $g \in G$. Since $g : L \to L$ is an orientation-preserving homeomorphism, $ge : \mathbb{R} \to L$ is an embedding which also has the properties for $e$ as above. Thus

$$e(+\infty) = ge(+\infty) = \text{the positive end of } L.$$

We can observe that one of the following two possibilities happens:

- $e(\mathbb{R}) = ge(\mathbb{R})$.
- $e(\mathbb{R}) \neq ge(\mathbb{R})$. Then there are $J = (t, +\infty), K = (r, +\infty)$ for some $t, r \in \mathbb{R}$ such that

$$e(J) = ge(K) = e(\mathbb{R}) \cap ge(\mathbb{R}).$$

Thus,

**Fact 3.1.** Let $H = \{g_1, \ldots, g_n\} \subseteq G$ be a finite subset of $G$. Let

$$\mu = \bigcap_{1 \leq i \leq n} g_i e(\mathbb{R}).$$

Then $\mu \neq \emptyset$. Moreover, for each $1 \leq i \leq n$, there is $J_i = [t_i, \infty)$ for some $t_i \in \mathbb{R}$ such that $g_i e(J_i) \subseteq \mu$.

**Definition 3.2.** Let $g \in G$. Let $\mu = e(\mathbb{R}) \cap ge(\mathbb{R})$.

(a) We fix a homeomorphism $g_0 : e(\mathbb{R}) \to ge(\mathbb{R})$ such that the restriction of $g_0$ to $\mu$ is identity.

(b) Let

$$g_* = e^{-1}g_0^{-1}ge : \mathbb{R} \xrightarrow{e} e(\mathbb{R}) \xrightarrow{g} ge(\mathbb{R}) \xrightarrow{g_0^{-1}} e(\mathbb{R}) \xrightarrow{e^{-1}} \mathbb{R}.$$ 

We define $d(g) = [g_*] \in G_\infty$.

**Lemma 3.3.** For each $g \in G$, $d(g)$ is independent of the choice of $g_0$.

*Proof.* Let $\mu = e(\mathbb{R}) \cap ge(\mathbb{R})$. By Fact 3.1, there is $J = [n, +\infty)$ for some $n \in \mathbb{R}$ such that $e(J), ge(J) \subseteq \mu$. Then the restriction of $g_0^{-1}$ to $ge(J)$ is identity. Thus, $$g_* \lvert J = e^{-1}g_0^{-1}(ge \lvert J) = e^{-1} \cdot (ge \lvert J) = (e^{-1}ge) \lvert J.$$ So $g_* \lvert J$ is independent of the choice of $g_0$. \hfill \Box

In the following, we prove that $d : G \to G_\infty$ is a homomorphism.

**Lemma 3.4.** Let $f, g \in G$. Then $d(fg) = d(f)d(g)$.
Proof. We have
\[ f \ast g = (e^{-1} f_0^{-1} f e)(e^{-1} g_0^{-1} g e) \]
\[ = e^{-1} f_0^{-1} f g_0^{-1} g e, \]
and
\[ (f g) \ast = e^{-1} (f g)_0^{-1} f g e. \]

Let \( \mu = e(\mathbb{R}) \cap f e(\mathbb{R}) \cap g e(\mathbb{R}) \cap f g e(\mathbb{R}) \). By Fact 3.1, there is \( J = [t, \infty) \) for some sufficiently large \( t \in \mathbb{R} \) such that
\[ e(J), f e(J), g e(J), f g e(J) \subseteq \mu. \]

Notice that the restriction of \( g_0^{-1} \) to \( g e(J) \) is identity, since
\[ g e(J) \subseteq \mu \subseteq e(\mathbb{R}) \cap g e(\mathbb{R}). \]

And the restriction of \( f_0^{-1} \) to \( f g e(J) \) is identity, since
\[ f g e(J) \subseteq \mu \subseteq e(\mathbb{R}) \cap f e(\mathbb{R}). \]

So
\[ (f \ast g) \mid J = e^{-1} f_0^{-1} g_0^{-1} (g e) \mid J \]
\[ = e^{-1} f_0^{-1} f - 1 \cdot (g e) \mid J \]
\[ = e^{-1} f_0^{-1} (f g e) \mid J \]
\[ = e^{-1} \cdot 1 \cdot (f g e) \mid J \]
\[ = (e^{-1} f g e) \mid J. \]

Also, the restriction of \( (f g)_0^{-1} \) to \( f g e(J) \subseteq \mu \) is identity, since
\[ f g e(J) \subseteq \mu \subseteq e(\mathbb{R}) \cap f g e(\mathbb{R}). \]

So
\[ (f g) \mid J = e^{-1} (f g)_0^{-1} (f g e) \mid J \]
\[ = e^{-1} \cdot 1 \cdot (f g e) \mid J \]
\[ = (e^{-1} f g e) \mid J. \]

It follows that
\[ (f \ast g) \mid J = (e^{-1} f g e) \mid J = (f g) \mid J. \]

By Definition 2.5, we have
\[ d(f g) = [(f g) \ast] = [f \ast] g \ast = d(f) d(g). \]

\( \square \)

It remains to show Proposition 1.2 (b). Let \( h \in G \). Now assume that for arbitrary \( n \in \mathbb{R} \), there is \( m \in (n, +\infty) \) with \( h e(m) \neq e(m) \).

Let \( \mu = e(\mathbb{R}) \cap h e(\mathbb{R}) \). By Fact 3.1, there is \( J = [t, +\infty) \) for some \( t \in \mathbb{R} \) such that \( e(J), h e(J) \subseteq \mu \).

By Definition 3.2, \( d(h) = [e^{-1} h_0^{-1} h e] \). Let \( K = [r, +\infty) \) for some \( r \in (t, +\infty) \). By our assumption, there is \( m \in K \) such that \( h e(m) \neq e(m) \). Since \( m \geq r > t \), \( h e(m) \) is contained in \( \mu \), and thus \( h_0^{-1} h e(m) = h e(m) \). Therefore,
\[ e^{-1} h_0^{-1} h e(m) = e^{-1} h e(m) \neq e^{-1}(e(m)) = m. \]

So \( e^{-1} h_0^{-1} h e \sim id \) (where \( \sim \) denotes the equivalence relation defined in Definition 2.5). Therefore, \( d(h) \neq 1 \). This completes the proof of Proposition 1.2.
3.2. The proof of Theorem 1.1. Let $M$ be a connected, orientable, irreducible 3-manifold. Suppose that $M$ admits a co-oriented taut foliation $\mathcal{F}$ which has one-sided branching. As shown in [C1, Theorem 2.2.7], we can assume that every leaf of $\mathcal{F}$ is dense. In this subsection, we prove that

**Theorem 1.1.** $\pi_1(M)$ is left orderable. Furthermore, $\pi_1(M)$ is isomorphic to a subgroup of $\mathcal{G}_\infty$.

Let $G = \pi_1(M)$. Let $p : \widetilde{M} \to M$ be the universal covering of $M$. Let $\tilde{\mathcal{F}}$ denote the pull-back foliation of $\mathcal{F}$ in $\widetilde{M}$, and let $L(\tilde{\mathcal{F}})$ denote the leaf space of $\tilde{\mathcal{F}}$. Then $L(\tilde{\mathcal{F}})$ is a non-Hausdorff 1-manifold with one-sided branching. We assume that $L(\tilde{\mathcal{F}})$ has an orientation induced from the co-orientation on $\mathcal{F}$, and we assume without loss of generality that $L(\tilde{\mathcal{F}})$ has branching in negative side. In the following, for every $g \in G$ and $t \in L(\tilde{\mathcal{F}})$, $g(t)$ will always denote the image of $t$ under the transformation $g : L(\tilde{\mathcal{F}}) \to L(\tilde{\mathcal{F}})$ given by the $\pi_1$-action on $L(\tilde{\mathcal{F}})$. And we will not distinguish the leaves of $\tilde{\mathcal{F}}$ and the points in $L(\tilde{\mathcal{F}}).

We first give a quick sketch of the proof of Theorem 1.1 in this paragraph. We blow-up some leaf $\lambda$ of $\mathcal{F}$ to obtain a new foliation $\mathcal{F}_0$ with one-sided branching. Let $L(\mathcal{F}_0)$ denote the leaf space of the pull-back foliation of $\mathcal{F}_0$ in $\widetilde{M}$. Then we construct an action $\{\alpha_g : L(\mathcal{F}_0) \to L(\mathcal{F}_0) \mid g \in G\}$ of $G$ on $L(\mathcal{F}_0)$ such that some points in $L(\mathcal{F}_0)$ have trivial stabilizer. Considering the action $\{\alpha_g \mid g \in G\}$ on $L(\mathcal{F}_0)$ and choosing some embedding $e : \mathbb{R} \to L(\mathcal{F}_0)$ as in Subsection 3.1, we can obtain an injective homomorphism $G \to \mathcal{G}_\infty$ by Proposition 1.2.

Now we give the details of the proof. Let $\lambda$ be a leaf of $\mathcal{F}$, and let $\tilde{\lambda}$ be a component of $p^{-1}(\lambda)$.

**Fact 3.5.** Let $\rho : \mathbb{R} \to L$ be an arbitrary embedding such that $\rho(+)\infty)$ is the positive end of $L(\tilde{\mathcal{F}})$ and $\rho(-\infty) \in \{\text{negative ends of } L(\tilde{\mathcal{F}})\}$. Let $J = (n, +\infty)$ for some $n \in \mathbb{R}$. Then there is $g \in G$ such that $g(\tilde{\lambda}) \in \rho(J)$.

**Proof.** Since $\lambda$ is dense in $M$, the closure of $\bigcup_{g \in G} g(\tilde{\lambda})$ is $L(\tilde{\mathcal{F}})$. Notice that $\rho(J)$ is an open set in $L(\tilde{\mathcal{F}})$. So $(\bigcup_{g \in G} g(\tilde{\lambda})) \cap \rho(J) \neq \emptyset$. $\square$

We blow-up the leaf $\lambda$ of $\mathcal{F}$ to obtain a new foliation $\mathcal{F}_0$ of $M$. Let $\tilde{\mathcal{F}}_0$ be the pull-back foliation of $\mathcal{F}_0$ in $\widetilde{M}$, and let $L(\mathcal{F}_0)$ denote the leaf space of $\tilde{\mathcal{F}}_0$. Then $L(\mathcal{F}_0)$ is obtained from blowing-up $\{g(\tilde{\lambda}) \mid g \in G\} \subset L(\tilde{\mathcal{F}})$ in $L(\tilde{\mathcal{F}})$, so $L(\mathcal{F}_0)$ is still a non-Hausdorff 1-manifold with one-sided branching. We assume that $L(\mathcal{F}_0)$ has an orientation induced from the orientation on $L(\tilde{\mathcal{F}})$. Now for every $g \in G$, the point $g(\tilde{\lambda}) \subset L(\tilde{\mathcal{F}})$ is replaced by an interval $\{g(\tilde{\lambda})\} \times I$.

We denote by $K = \{g \in G \mid g(\tilde{\lambda}) = \tilde{\lambda}\}$. Since $\lambda$ is an orientable surface, $K$ is a countable left orderable group or a trivial group (when $\lambda$ is a 2-plane). So there is an action $\phi : K \to \text{Homeo}_+(I)$ of $K$ on $I$ such that: $\phi(g)(\frac{1}{2}) \neq \frac{1}{2}$ for every $g \in K - \{1\}$ (cf. [C2, Lemma 2.43, Remark]). Here, we set $\phi$ to be the trivial homomorphism when $K$ is a trivial group. For each left coset $gK$ ($g \in G$) of $K$ in $G$, we fix an element $x_{gK} \subset gK$. And we set $x_K = 1 \in K$.

**Construction 3.6.** For each $h \in G$, we define a map $\alpha_h : L(\mathcal{F}_0) \to L(\mathcal{F}_0)$ as follows:

- Suppose that $q \in L(\mathcal{F}_0) - \bigcup_{g \in G} \{g(\tilde{\lambda})\} \times I$. Then $q$ is a point of $L(\tilde{\mathcal{F}})$. We define $\alpha_h(q) = h(q)$.
- Suppose that $q \in \bigcup_{g \in G} \{g(\tilde{\lambda})\} \times I$. Then there are $g \in G$, $t \in I$ such that $q = \{g(\tilde{\lambda})\} \times \{t\}$. We define

$$\alpha_h(\{g(\tilde{\lambda})\} \times \{t\}) = \{h(g)(\tilde{\lambda})\} \times \{\phi(x^{-1}_{hgK}h(x_{gK}))(t)\}.$$

Since $x^{-1}_{hgK}h(x_{gK}) \in K$, the map $\alpha_h$ is well-defined. Notice that $\alpha_h$ takes $\{g(\tilde{\lambda})\} \times I$ to $\{h(g)(\tilde{\lambda})\} \times I$ for every $g \in G$. So $\alpha_h$ is an orientation-preserving homeomorphism. Furthermore,

**Lemma 3.7.** $\{\alpha_g : L(\mathcal{F}_0) \to L(\mathcal{F}_0) \mid g \in G\}$ is an action of $G$ on $L(\mathcal{F}_0)$.
Proof. Let $h, r \in G$. It’s clear that $\alpha_{hr}(q) = \alpha_h \alpha_r(q)$ for every $q \in L(\mathcal{F}_0) - \bigcup_{g \in G} \{g(\tilde{\lambda})\} \times I$. Now we choose $q \in \bigcup_{g \in G} \{g(\tilde{\lambda})\} \times I$. Let $g \in G$, $t \in I$ for which $q = \{g(\tilde{\lambda})\} \times \{t\}$. We have

$$\alpha_{hr}(q) = \alpha_{hr}(\{g(\tilde{\lambda})\} \times \{t\}) = \{hr(\tilde{\lambda})\} \times \{\phi(x_{hrghr^1}hhrxgK)(t)\}$$

and

$$\alpha_h \alpha_r(q) = \alpha_h \alpha_r(\{g(\tilde{\lambda})\} \times \{t\}) = \alpha_h(\{rg(\tilde{\lambda})\} \times \{\phi(x_{rg^1}hxrgKx_{rgK^1}hxrgK)(t)\}) = \{hr(\tilde{\lambda})\} \times \{\phi(x_{hrghr^1}hhrxgK)(t)\}.$$ 

Thus $\alpha_{hr}(q) = \alpha_h \alpha_r(q)$. Also, we have $\alpha_1(q) = q$ for every $q \in L(\mathcal{F}_0)$. Therefore, $\{\alpha_g : L(\mathcal{F}_0) \to L(\mathcal{F}_0) \mid g \in G\}$ is an action of $G$ on $L(\mathcal{F}_0)$.

Lemma 3.8. The point $\{\tilde{\lambda}\} \times \{\frac{1}{2}\}$ has trivial stabilizer under $\{\alpha_g : L(\mathcal{F}_0) \to L(\mathcal{F}_0) \mid g \in G\}$.

Proof. Let $g \in G - \{1\}$. We have

$$\alpha_g(\{\tilde{\lambda}\} \times \{\frac{1}{2}\}) = \{g(\tilde{\lambda})\} \times \{\phi(x_{gK^1}hxrgK)(\frac{1}{2})\} = \{g(\tilde{\lambda})\} \times \{\phi(x_{gK^1}g)(\frac{1}{2})\}.$$ 

If $g \notin K$, then $g(\tilde{\lambda}) \neq \tilde{\lambda}$. If $g \in K - \{1\}$, then $\phi(x_{gK^1}g)(\frac{1}{2}) = \phi(g)(\frac{1}{2}) \neq \frac{1}{2}$. So $\alpha_g(\{\tilde{\lambda}\} \times \{\frac{1}{2}\}) \neq \{\tilde{\lambda}\} \times \{\frac{1}{2}\}$ for every $g \in G - \{1\}$.

Let $\tilde{\lambda}_0 = \{\tilde{\lambda}\} \times \{\frac{1}{2}\} \in L(\mathcal{F}_0)$.

Definition 3.9. Let $e : \mathbb{R} \to L(\mathcal{F}_0)$ be an arbitrary embedding such that $e(+\infty)$ is the positive end of $L(\mathcal{F}_0)$ and $e(-\infty) \in \{\text{negative ends of } L(\mathcal{F}_0)\}$.

Lemma 3.10. Let $J = (n, +\infty)$ for some $n \in \mathbb{R}$. Then there is $h \in G$ such that $\alpha_h(\tilde{\lambda}_0) \in e(J)$.

Proof. This follows from Fact 3.5 and the fact that $L(\mathcal{F}_0)$ is obtained from blowing-up $\{g(\tilde{\lambda}) \mid g \in G\}$ in $L(\mathcal{F})$ (and every interval $\{g(\tilde{\lambda})\} \times I (g \in G)$ contains some images of $\tilde{\lambda}_0$ under $\{\alpha_h \mid h \in G\}$).

It follows that

Corollary 3.11. Let $J = [n, +\infty)$ for some $n \in \mathbb{R}$. Then there is $m \in (n, +\infty)$ such that $\alpha_{e}e(m) \neq e(m)$ for every $g \in G - \{1\}$.

Now $(L(\mathcal{F}_0), \{\alpha_g \mid g \in G\}, e)$ can be considered as the triple $(L, G, e)$ as given in Subsection 3.1. By Proposition 1.2 (a), there is a homomorphism $d : G \to \mathcal{G}_\infty$. Combining Corollary 3.11 and Proposition 1.2 (b), we have

Corollary 3.12. For every $g \in G - \{1\}$, $d(g) \neq 1$.

So $d$ is injective, and thus $G$ is isomorphic to a subgroup of $\mathcal{G}_\infty$. By Navas’ Theorem (cf. Theorem 2.6), $\mathcal{G}_\infty$ is a left orderable group. This completes the proof of Theorem 1.1.

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