THE CONSERVATION LAW APPROACH IN GEOMETRIC PDEs

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Abstract. In this survey paper, we give an overview of the conservation law approach in the study of geometric PDEs that models in particular polyharmonic maps.

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1. Motivation

Interesting geometric partial differential equations are usually of critical or even supercritical nonlinearity in nature. The absence of possible applications of the maximum principle to solutions to non-linear elliptic systems reduces drastically the tools available for the regularity theory of weak solutions. In this aspect, the conservation law approach seems to be an effective strategy to address the regularity issues.

Consider the weakly harmonic map \( u \) from the \( n \)-dimensional unit ball \( B^n \) into the sphere \( S^m \) of \( \mathbb{R}^{m+1} \). Chen [2] and Shatah [23] independently found a conservation law for \( u \). Based on this conservation law, Hélein [10] proved his celebrated regularity theorem for weakly harmonic maps from the two dimensional ball (or more generally surfaces) to sphere. Later, he also succeeded in extending the same result for general manifold targets by introducing the so-called moving frame technique; see [9]. The method of Hélein is beautiful, but it does not apply for critical point of general second order conformally invariant elliptic Lagrangian with quadratic growth.

A major breakthrough towards the regularity issues of general second order conformally invariant elliptic variational problems was made by Rivière [16]. He introduced the following second order linear system

\[
- \Delta u = \Omega \cdot \nabla u
\]

and verified that (1.1) includes the Euler-Lagrange equation of general conformally invariant second order elliptic variational problems with quadratic growth in dimension two. Relying on (a variant of) the gauge theory of Uhlenbeck [24], Rivière succeeded in finding a conservation law for (1.1) and then regularity follows by standard potential theory. It should be noticed that this significant work not only gave a new proof of the regularity

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theorem of Hélein, but also verifies affirmatively two long-standing regularity conjectures of Hildebrandt and Heinz on conformally invariant geometrical problems and the prescribed bounded mean curvature equations respectively; see [16] for details.

The conservation law approach of [16] was soon extended further by Rivière [17] to Willmore surfaces, and by Lamm-Rivière [12] to fourth order linear elliptic system in dimension four, allowing him to give a new proof of the continuity (but not the stronger Hölder continuity) theorems of Chang et al. [1] and Wang [25] for biharmonic maps. Very recently, the conservation law approach was successfully extended by de Longueville and Gastel [3] to general even order linear elliptic systems of Rivière type in the conformal dimension, providing a new proof of the continuity (but not the stronger Hölder continuity) result of polyharmonic maps [4]; see also [11] for a different construction of conservation law. Based on the above mentioned conservation law, Guo, Xiang and Zheng also established the Hölder continuity or indeed even an optimal $L^p$-regularity theory for fourth order system in [5] and for general even order system in [7]; thus yielding a complete recover of the regularity theorems of Chang et al. [1], Wang [25] and Gastel-Scheven [4]. Further more, in a recent work [8], Guo, Xiang and Zheng refined the conservation law of Lamm-Rivière [12] and [3] so that they are fully equivalent to the equation on the whole domain of definition.

In the very recent work [6], Guo and Xiang also partially extended the conservation law of Rivière [16] for (1.1) to supercritical dimensions. In below, we shall give a detailed survey of the conservaiton law approach.

2. THE CONSERVATION LAW OF CHEN AND SHATAH

Recall that a map $u \in W^{1,2}(B^n, S^m)$ is weakly harmonic if it is a weak solution of
\begin{equation}
-\Delta u = |\nabla u|^2 u.
\end{equation}
Because of the conformal invariance of the Dirichlet energy in dimension two, the equation 2.1 is also conformally invariant: if $u \colon B^2 \to S^m$ is a solution of (2.1) and $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a conformal map, then $u \circ \varphi$ is again a solution of (2.1). It is easily observed that the conformal dimension two is critical for (2.1): the right hand side of (2.1) is merely in $L^1$ and so the classical $L^p$ regularity theory does not apply here. The following important conservation law was discovered, independently, by Chen [2] and Shatah [23].

Theorem 2.1 (Chen [2], Shatah [23]). A map $u \in W^{1,2}(B^2, S^m)$ is weakly harmonic if and only if it satisfies the following conservation law
\begin{equation}
\text{div}(u^i \nabla u^j - u^j \nabla u^i) = 0 \quad \text{for all } i, j \in \{1, \ldots, m+1\}.
\end{equation}

The conservation law of Chen and Shatah has a nice interpretation by means of Noether’s theorem on conservation law: the symmetry (rotational invariance) of the target manifold $S^m$ leads to (2.2); see [9] for details.

We have the following celebrated regularity result due to Hélein [10].

Theorem 2.2 (Hélein). If $u \in W^{1,2}(B^2, S^m)$ is weakly harmonic, then it is smooth.

Proof. By the conservation law of Chen and Shatah, we have
\begin{equation}
\text{div}(u^i \nabla u^j - u^j \nabla u^i) = 0 \quad \text{for all } i, j \in \{1, \ldots, n\}.
\end{equation}
By the Hodge decomposition (or Poincaré’s lemma), we know that there exists $B_{ij} \in W^{1,2}(B^2)$ such that $u^i \nabla u^j - u^j \nabla u^i = \nabla^\perp B_{ij}$, where $\nabla^\perp = (-\partial_y, \partial_x)$.

Note that $|u|^2 = 1$ implies that $u^j \nabla u^j = 0$ and so the weakly harmonic map equation (2.1) can be rewritten as

$$\nabla^\perp B_{ij} \cdot \nabla u^j = \nabla^\perp B_{ij} \cdot \nabla u^j.$$

The product curl-grad on the right hand side has a Jacobian structure so that the classical result of Wente [26] allows us to conclude the continuity of $u$. □

**Remark 2.3.** i) Note that $\nabla^\perp B_{ij} \cdot \nabla u^j = -\nabla B_{ij} \cdot \nabla^\perp u^j = -\text{div}(B_{ij} \nabla^\perp u^j)$ and so we have the following conservation law

$$-\text{div}(\nabla u - B_{ij} \nabla^\perp u^j) = 0,$$

or equivalently

$$-\text{div}(\nabla u - B \nabla^\perp u) = 0 \quad \text{in } B^2.$$

ii) Suppose $N \subset \mathbb{R}^m$ is a hypersurface and let $\nu: N \to T^\perp N$ be the smooth unit normal vector field. Given a weakly harmonic map $u: B^2 \to N$ be the composition function $w = \nu \circ u$. Using the fact that $\omega \cdot \partial_j u = 0$, we may rewrite the harmonic map equation as

$$-\Delta u^i = w^i \partial_j w^k \partial_j u^k = (w^i \partial_j w^k - w^k \partial_j w^i) \partial_j u^k.$$

Unlike the case $N = S^m$, the vector fields $v_{ik} := (w^i \partial_j w^k - w^k \partial_j w^i)_{1 \leq j \leq 2}$ is no longer divergence free. Thus the preceding proof fails to apply for general closed target manifold.

As we shall see, following Hélein and Rivière, in spite of this apparent failure of the method, it is possible to once again produce the desired structure after a suitable transformation; in fact, as observed by Rivière [16], this is possible via gauge theory in the spirit of Uhlenbeck [24]. For this, only the anti-symmetry $v_{ik} = -v_{ki}$ is needed.

### 3. The conservation law of Rivière

In the revolutionary work [16], Rivière succeeded in finding a conservation law for the linear system (1.1). More precisely, he proved the following result.

**Theorem 3.1** (Conservation law, [16]). Suppose $\Omega \in L^2(B^2, so_m \otimes \Lambda^1 \mathbb{R}^2)$. If there exist $A \in W^{1,2}(B^2, GL(m))$ and $B \in W^{1,2}(B^2, M(m))$ such that

$$\nabla A + \nabla^\perp B = A \Omega,$$

then $u$ solves (1.1) if and only if the following conservation law holds:

$$\text{div}(A \nabla u - B \nabla^\perp u) = 0.$$

\[1\] Einstein’s summation convention is used throughout this paper
Proof. This follows rather directly from computation:
\[
\text{div}(A\nabla u - B\nabla^\perp u) = \nabla A \cdot \nabla u + A\Delta u - \nabla B \cdot \nabla^\perp u
\]
\[
= (\nabla A + \nabla^\perp B) \cdot \nabla u + A\Delta u
\]
\[
= A\Omega \cdot \nabla u + A\Delta u = A(\Omega \cdot \nabla u + \Delta u).
\]
\[
\square
\]

The main difficulty is thus to find \(A\) and \(B\) as in Theorem 3.1 that satisfies (3.1).

**Theorem 3.2** (Construction of conservation law, [16]). There exists an \(\varepsilon_0 = \varepsilon_0(m) > 0\) such that if \(\Omega \in L^2(B^2, so_m \otimes \Lambda^1 \mathbb{R}^2)\) satisfies
\[
\|\Omega\|_{L^2(B^2)} \leq \varepsilon_0,
\]
then there exist \(A \in W^{1,2} \cap L^\infty(B^2, GL(m))\) and \(B \in W^{1,2}(B^2, M(m))\) such that (3.1) holds. Further more, we have
\[
\|\nabla A\|^2_{L^2(B^2)} + \|\nabla B\|^2_{L^2(B^2)} + \|\text{dist}(A, SO_m)\|_{L^\infty(B^2)} \leq C(m)\|\Omega\|^2_{L^2(B^2)}.
\]

The proof of Theorem 3.2 relies crucially on the following variant of the gauge theory of Uhlenbeck [24].

**Theorem 3.3** (Gauge transform, [16]). Let \(\Omega \in L^2(B^2, so_m \otimes \Lambda^1 \mathbb{R}^2)\). Then there exist \(\xi \in W^{1,2}_0(B^2, GL(m))\) and \(P \in W^{1,2}(B^2, SO_m)\) such that
\[
P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp \xi
\]
and
\[
\|\nabla^\perp \xi\|^2_{L^2(B^2)} + \|\nabla P\|^2_{L^2(B^2)} \leq C(m)\|\Omega\|^2_{L^2(B^2)}.
\]

**Sketch of the proof.** We follow the work of Schikorra [20], who found a very elegant variational proof of this deep theorem. To be more precise, consider the variational problem
\[
\min_{Q \in W^{1,2}(B^2, SO_m)} E(Q) = \min_{Q \in W^{1,2}(B^2, SO_m)} \int_{B^2} |Q^T \nabla Q + Q^T \Omega Q|^2 dx.
\]
One can show that there exists a minimizer \(P \in W^{1,2}(B^2, SO_m)\) for the above functional, which satisfies the following Euler-Lagrange equation:
\[
\text{div}(P^T \nabla P + P^T \Omega P) = 0.
\]
Green’s formula then gives \(\Omega_P \cdot \nu = 0\), where \(\Omega_P := P^T \nabla P + P^T \Omega P\). Standard Hodge decomposition gives the existence of \(\xi \in W^{1,2}_0(B^2, GL(m))\) such that
\[
P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp \xi.
\]
The desired estimates follows easily from the minimizing property of \(P\). For details, see [20, Theorem 2.1].

Using Theorem 3.3, we now give a proof of Theorem 3.2. The idea of the proof is due to Rivière [16], but with some minor technical improvement from [8].
Proof of Theorem 3.2. By Theorem 3.3, there exist $\xi \in W^{1,2}_0(B^2, GL(m))$ and $P \in W^{1,2}_0(B^2, SO_m)$ such that
\[ P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp \xi \]
and
\[ \|\nabla^\perp \xi\|_{L^2(B^2)}^2 + \|\nabla P\|_{L^2(B^2)}^2 + \|\nabla P^{-1}\|_{L^2(B^2)}^2 \leq C(m)\|\Omega\|_{L^2(B^2)}^2. \]

Observe that there exists a pair $(A, B)$ solves (3.1) if and only if for $(\hat{A}, \hat{B})$ with $\hat{A} = AP$, we have
\[ \nabla \hat{A} - \hat{A}\nabla^\perp \xi = \nabla^\perp BP. \]

It is thus sufficient to solve (3.4) in $B^2$.

We now use an extension argument from [8] as follows: extend $\xi$ and $P$ from $B^2$ to $B^2_2$ such that $\xi = 0$ and $P = I$ in $B^2_2 \setminus B^2_{3/2}$ (here we keep the notation $P$ and $\xi$ for the extended functions). Furthermore, we require the norms of $P, \nabla P, \nabla \xi$ in $B^2_2$ is controlled by a constant multiple of the corresponding norms in $B^2$. Our strategy is to solve (3.4) in the enlarged region $B^2_2$.

Claim. There exist $\hat{A} \in W^{1,2}(B^2_2, GL(m))$ and $B \in W^{1,2}_0(B^2_2, M(m))$ such that
\[ \left\{ \begin{array}{l}
\Delta \hat{A} = \nabla \hat{A} \cdot \nabla \perp \xi + \nabla \perp B \cdot \nabla P \\
\Delta B = -\nabla \perp \hat{A} \cdot \nabla P^{-1} - \text{div}((\hat{A} + I)\nabla \xi \cdot P^{-1}) \\
\frac{\partial \hat{A}}{\partial \nu} = 0 \quad \text{on} \ \partial B^2_2 \quad \text{and} \quad \int_{B^2_2} \hat{A} = 0 \\
B = 0 \quad \text{on} \ \partial B^2_2.
\end{array} \right. \]
\]

Moreover,
\[ \|\nabla \hat{A}\|_{L^2} + \|\hat{A}\|_{L^\infty} + \|\nabla B\|_{L^2} \leq C_m\|\Omega\|_{L^2(B^2_2)}. \]

Proof of Claim. Set
\[ X := \{(a, b) \in W^{1,2} \cap L^\infty(B^2_2, GL(m)) \times W^{1,2}_0(B^2_2, M(m)) : \|(a, b)\|_X \leq 1\}, \]
where $\|(a, b)\|_X := \|\nabla a\|_{L^2} + \|\nabla b\|_{L^2} + \|a\|_{L^\infty}$.

Standard elliptic regularity theory implies that for each $(a, b) \in X$, there exists a unique solution $(c, d) \in W^{1,2} \cap L^\infty \times W^{1,2}$ such that
\[ \left\{ \begin{array}{l}
\Delta c = \nabla a \cdot \nabla \perp \xi + \nabla \perp b \cdot \nabla P \\
\Delta d = -\nabla \perp a \cdot \nabla P^{-1} - \text{div}((a + I)\nabla \xi \cdot P^{-1}) \\
\frac{\partial c}{\partial \nu} = 0 \quad \text{on} \ \partial B^2_2 \quad \text{and} \quad \int_{B^2_2} c = 0 \\
d = 0 \quad \text{on} \ \partial B^2_2.
\end{array} \right. \]

Moreover, by Wente’s lemma [26], for some $C = C_m > 0$, we have
\[ \|\nabla c\|_{L^2} + \|c\|_{L^\infty} \leq C\left(\|\nabla a\|_{L^2}\|\nabla \xi\|_{L^2} + \|\nabla b\|_{L^2}\|\nabla P\|_{L^2}\right) \]
\[ \|\nabla d\|_{L^2} \leq C\left(\|\nabla a\|_{L^2}\|\nabla P^{-1}\|_{L^2} + \|a\|_{L^\infty}\|\nabla \xi\|_{L^2} + \|\nabla \xi\|_{L^2}\right). \]
This implies if $\|\Omega\|_{L^2(B^2_2)}$ is sufficiently small (less than $\frac{1}{2C}$), then
\[ \|(c, d)\|_X \leq C\|\Omega\|_{L^2} \cdot \|(a, b)\|_X + C\|\Omega\|_{L^2} \leq 2C\|\Omega\|_{L^2(B^2_2)} < 1, \]
which means \((c, d) \in X\). If we define 
\[
T : X \to X
\]
\[
(a, b) \mapsto T(a, b) := (c, d),
\]
then \(T : X \to X\) is a contraction map. The fixed point theorem then gives a solution 
\((\hat{A}, \hat{B}) \in X\) that solves (3.5).

Set \(\hat{A} := A + I\). Then \((\hat{A}, \hat{B})\) solves
\[
\begin{cases}
\Delta \hat{A} = \nabla \hat{A} \cdot \nabla^\perp \xi + \nabla^\perp B \cdot \nabla P \\
\Delta B = -\nabla^\perp \hat{A} \cdot \nabla P^{-1} - \text{div}(\hat{A} \nabla \xi \cdot P^{-1}) \\
\frac{\partial \hat{A}}{\partial \nu} = 0 \text{ on } \partial B^2 \\
\int_{B^2} \hat{A} = 4\pi I
\end{cases}
\] (3.6)

Moreover, we have
\[
\|\nabla \hat{A}\|_{L^2} + \|\hat{A} - I\|_{L^\infty} + \|\nabla B\|_{L^2} \leq C_m \|\Omega\|_{L^2(B^2)}.
\] (3.7)

Observe that since \(P = I\) and \(\xi = 0\) in \(B^2\setminus B^2_{3/2}\), we have \(\hat{A} = 0 = B\) in \(B^2\setminus B^2_{3/2}\) and so \(\hat{A} = I\) and \(B = 0\) in \(B^2\setminus B^2_{3/2}\).

It remains to show (3.4) holds for the pair \((\hat{A}, \hat{B})\) in \(B^2\) and thus in \(B^2\) as well.

Note first that by (3.6), we have
\[
\text{div}(\nabla \hat{A} - \hat{A} \nabla^\perp \xi - \nabla^\perp BP) = 0 \quad \text{in } B^2.
\]

The Hodge decomposition implies that there exists \(C \in W^{1,2}(B^2_2, M(m))\) such that
\[
\nabla \hat{A} - \hat{A} \nabla^\perp \xi - \nabla^\perp BP = \nabla^\perp C \quad \text{in } B^2_2.
\]

Our aim is to show \(C \equiv 0\) in \(B^2_2\).

By the second equation in (3.6), we have \(\text{div}(\nabla CP^{-1}) = 0\) and so Hodge decomposition implies that there exists \(D \in W^{1,2}(B^2_2, M(m))\) such that
\[
\nabla CP^{-1} = \nabla^\perp D \quad \text{in } B^2_2.
\]

Since \(C = 0\) in \(B^2_2\setminus B^2_{3/2}\), so is \(D\). We may additionally assume \(\int_{B^2_2} D = 0\). Then we have by Wente’s lemma again
\[
\|\nabla D\|_{L^2} \leq C \|\nabla P^{-1}\|_{L^2} \|\nabla C\|_{L^2} \leq C \|\nabla D\|_{L^2} \|\Omega\|_{L^2(B^2)} \leq \frac{1}{2} \|\nabla D\|_{L^2},
\]

if \(\|\Omega\|_{L^2}\) is sufficiently small. This implies that \(\nabla D \equiv 0\) in \(B^2_2\) and so is \(\nabla C\). Since \(C = 0\) in \(B^2_2\setminus B^2_{3/2}\), we thus conclude \(C \equiv 0\) in \(B^2_2\) as desired. The estimate (3.3) follows directly from (3.7). \(\square\)

**Remark 3.4.** It follows from the proof of Theorem 3.2 that there are indeed infinitely many choice of \(A\) and \(B\) as required, as there are infinitely many ways to extend \(P\) and \(\xi\) with the desired properties.
4. The conservation law for higher order system of Rivière type

In an interesting recent work [3], de Longueville and Gastel introduced the following even order linear elliptic system of Rivière type

\[ \Delta^m u = \sum_{l=0}^{m-2} \Delta^l (V_l du) + \sum_{l=0}^{m-1} \Delta^l \delta(w_l du) \quad \text{in } B^{2m}. \]  

System (4.1) includes (both extrinsic and intrinsic) \( m \)-polyharmonic mappings. It reduces to the Lamé-Rivière system [12] when \( m = 2 \), and to (1.1) when \( m = 1 \). The coefficient functions are assumed to satisfy

\[ w_k \in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n \times n}) \quad \text{for } k \in \{0, \ldots, m-2\} \]
\[ V_k \in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^1 \mathbb{R}^{2m}) \quad \text{for } k \in \{0, \ldots, m-1\}. \]

Moreover, the first order potential \( V_0 \) has the decomposition \( V_0 = d\eta + F \) with

\[ \eta \in W^{2-m,2}(B^{2m}, \text{so}(n)), \quad F \in W^{2-m,2}_{m+1}(B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^1 \mathbb{R}^{2m}). \]

To formulate the conservation law, we set

\[ \theta_D := \sum_{k=0}^{m-2} \|w_k\|_{W^{2k+2-m,2}(D)} + \sum_{k=1}^{m-1} \|V_k\|_{W^{2k+1-m,2}(D)} \]
\[ + \|\eta\|_{W^{2-m,2}(D)} + \|F\|_{W^{2-m,2}_{m+1}(D)} \]

for \( D \subset \mathbb{R}^{2m} \).

We have the following conservation law for (4.1).

**Theorem 4.1** (Conservation law, [8]). There exist constants \( \epsilon_m, C_m > 0 \) such that under the smallness assumption \( \theta_{B^{2m}} < \epsilon_m \), there exist \( A \in W^{m,2} \cap L^\infty(B^{2m}, \text{Gl}(n)) \) and \( B \in W^{2-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \Lambda^2 \mathbb{R}^{2m}) \) satisfying

\[ \Delta^{m-1} dA + \sum_{k=0}^{m-1} (\Delta^k A) V_k - \sum_{k=0}^{m-2} (\Delta^k dA) w_k = \delta B \quad \text{in } B^{2m}. \]

Moreover,

\[ \|A\|_{W^{m,2}(B^{2m})} + \|\text{dist}(A, \text{SO}(m))\|_{L^\infty(B^{2m})} + \|B\|_{W^{2-m,2}(B^{2m})} \leq C_m \theta_{B^{2m}}. \]

Consequently, \( u \) solves (4.1) if and only if it satisfies the conservation law

\[ 0 = \delta \left[ \sum_{l=0}^{m-1} \left( \Delta^l A \right) \Delta^{m-l-1} du - \sum_{l=0}^{m-2} \left( d\Delta^l A \right) \Delta^{m-l-1} u \right. \]
\[ - \sum_{k=0}^{m-1} \left( \Delta^k A \right) \Delta^{k-l-1} d(V_k, du) + \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \left( d\Delta^l A \right) \Delta^{k-l-1} (V_k, du) \]
\[ \left. - \sum_{k=0}^{m-2} \sum_{l=0}^{m-2} \left( \Delta^l A \right) d\Delta^{k-l-1} \delta(w_k du) + \sum_{k=0}^{m-2} \sum_{l=0}^{m-2} \left( d\Delta^l A \right) \Delta^{k-l-1} \delta(w_k du) \right) \]

in \( B^{2m} \).
Theorem 4.1 was proved in [12] for the case \( m = 2 \) and in [3] for general \( m \), but with the conservation law (4.6) holding only on \( B_{1/2}^{2m} \). An extra extension argument as was done in the proof of Theorem 3.2 would lead to the current form; see [8] for details.

5. Open problems

In the final section, we shall discuss some natural open problems related to the conservation law.

Rivièrê’s theory about (1.1) is very successful in the critical dimension \( n = 2 \), but it is much less understood in higher dimensions. In [16, Page 7], Rivièrê pointed out that

**Problem 5.1.** Can we establish the conservation law for (1.1) in higher dimensions \( n \geq 3 \)?

In [16], Rivièrê proposed to look for conservation laws in the natural Morrey space \( M^{2,n-2} \), which has direct application to regularity of stationary harmonic maps in higher dimensions. Later, Rivièrê and Struwe [19] constructed an example showing that Wente’s lemma fails with coefficients merely in \( M^{2,n-2} \) and thus it is not easy to find a conservation law in such spaces. In the very recent work [6], we succeeded in finding the conservation law in the smaller Lorentz space \( L^{n,2} \subseteq M^{2,n-2} \). This is, however, far from a satisfied theory in higher dimensions.

The second natural problem is

**Problem 5.2.** What is the essence of Rivièrê’s conservation law?

Unlike Noether’s conservation law, there is no variational characterization of Rivièrê’s conservation law. The existence of \( A \) and \( B \) appearing in (3.1) are obtained through a fixed point type argument and thus are not explicit. Furthermore, as was pointed out in Remark 3.4, there are in fact infinitely many \( A \) and \( B \) such that the conservation law (3.2) holds. Thus it is natural to find a good explanation of this conservation law from a scientific point of view.

Another fundamental problem in Rivièrê’s theory or the theory of harmonic maps is whether one can establish the global regularity in higher dimensions. The following well-known conjecture was asked by Rivièrê in [16, Page 9 Conjecture]:

**Problem 5.3.** Conjecture that for every \( k \leq m \), for every \( n \in \mathbb{N} \), for every \( k \)-dimensional closed submanifold \( N \) of \( \mathbb{R}^m \), and for every \( C > 0 \), there exists \( \delta = \delta(C,n,N) > 0 \) such that if \( u \) is a \( W^{1,2} \) weakly harmonic map from \( B_2^n(0) \) into \( N \) satisfying

\[
\int_{B_2^n(0)} |\nabla u|^2\,dx \leq C,
\]

then

\[
\int_{B_2^n(0)} |\nabla^2 u|\,dx \leq \delta.
\]

Similar type of result has been obtained by Lin [13], Naber and Valtorta [15], for stationary harmonic maps, but with extra topological restrictions on the target manifold \( N \). Without any further requirement on the target manifold, the result was only known for minimizing harmonic maps. The usual small \( \epsilon \)-energy regularity result implies that the conjecture holds if \( C \) is sufficiently small (quantitatively). For large \( C \), there is no
theory available. Note however that a positive answer to Problem 5.3 would imply the corresponding energy identity for stationary harmonic maps into general closed manifolds via the techniques of Lin-Rivière [14]. The classical application of conservation law in regularity issues includes the $L^p$-estimates for inhomogeneous system of Rivière type; see [22, 5, 7]. Towards a possible solution of Problem 5.3, it is suggestive to first establish a conservation law for harmonic maps in higher dimensions.

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