Perturbation theory for selfadjoint relations

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Abstract
We study Weyl-type perturbation theorems in the context of closed linear relations. General results on perturbations for dissipative relations are established. In the particular case of selfadjoint relations, we treat finite-rank perturbations and carry out a detailed analysis of the corresponding changes in the spectrum.

Keywords Closed linear relations · Dissipative and Selfadjoint relations · Weyl perturbation theory

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1 Introduction
Closed linear relations in a Hilbert space $\mathcal{H}$ are subspaces (i.e. closed linear sets) of $\mathcal{H} \oplus \mathcal{H}$. A particular realization of a closed linear relation is the graph of a closed linear operator and, since the operator can be identified with its graph, we consider relations as generalizations of operators.

In this work, we study perturbation theory for relations when the essential spectrum is preserved after the relation is submitted to certain types of perturbations. There are various perturbation theorems on the stability of the essential spectrum of operators (cf. [11, Theorem 4.5.35], [4, Theorem 9.1.4]) related to the classical result on perturbations in the selfadjoint case by Weyl [15]. These theorems are known as Weyl-type...
perturbations theorems. Some results of this kind have been obtained for relations being perturbed by relations (see [5, Chapter 7] and [13,14,16,17]).

There are several ways of extending the notions related to Weyl-type perturbation theory from the operator setting to the one of relations. In the first place, the essential spectrum of a relation has to be defined. One can use a general approach to the matter and define the essential spectrum for relations as it is done for closed operators in Banach spaces (see [11, Sect.4.5.6], and [5, Chapter 7] in the relation setting). In connection with this approach, there are various different definitions of the essential spectrum for operators (see [7, Chapter 9]) which can be extended to the case of relations [16]. All these notions reduce to the definition we use here (see Definition 4.2) in the case of selfadjoint relations. Thus, taking into account that the main goal of this paper is the detailed analysis of the spectrum of selfadjoint relations under selfadjoint finite rank perturbations, we restrict ourselves to Definition 4.2. This definition of the essential spectrum is used in [13,14,17], where perturbation theory for relations is treated in a way similar to ours. The results given in this work concerning dissipative relations and the fine-tuning perturbation analysis related to the rank of the perturbation go beyond the results of [13,14,17].

Apart from extending the definition of the essential spectrum, it is necessary to generalize the concepts of relatively bounded and relatively compact perturbations from operators to relations (see [5, Chapter 7] and [16]). In this work, we touch upon additive perturbations of relations only tangentially. Instead, we approach the matter more generally by studying the difference of the resolvents of relations when this difference is a compact operator (for an even more general setting see [2]).

The main goal of this paper is the fine-tuning spectral analysis of selfadjoint relations such that the difference of their resolvents is a finite-rank operator. We develop the theory on the basis of [4, Chapter 9.] and extend some classical Weyl-type perturbation results for operators to selfadjoint relations. To this end, various results on finite-rank perturbations are obtained in Sect. 3 for a setting more general than the selfadjoint one, namely, for dissipative relations. This is done this way for future developments on the spectral theory of dissipative and accretive relations, and, within the latter class, sectorial relations (see in [10] recent results on the matter). In the selfadjoint case, the results of Sect. 3 admit substantial refinements as is shown in Sect. 4. First, we establish Theorem 4.8 which is a general result first proven in [13, Theorem 5.1]. In contrast to [5, Chapter 7] and [16], this result does not require any condition on the multivalued part of the relations. Theorem 4.9 gives bounds on the shift of the spectrum in terms of the rank of the difference of the resolvents. Theorem 4.12 deals with the spectra of selfadjoint extensions of a symmetric relation with finite deficiency indices (cf. [14]). The last Corollaries of Sect. 4 give conditions for spectral interlacing.

Our results are of practical importance in the various theoretical applications that the spectral theory of relations has; for instance in the extension and spectral theories of operators [12] and the theory of canonical systems [9]. The last section provides examples related to the spectral theory of operators.
2 On linear relations

We consider a separable Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) with inner product antilinear in its left argument. Throughout this work, any linear set \(T\) in \(\mathcal{H} \oplus \mathcal{H}\) is called a linear relation. Here, \(\mathcal{H} \oplus \mathcal{H}\) denotes the orthogonal sum of two copies of the Hilbert space \(\mathcal{H}\) (see [4, Sect. 2.3]). Define the sets

\[
\text{dom } T := \left\{ f \in \mathcal{H} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}, \quad \text{ran } T := \left\{ g \in \mathcal{H} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\},
\]

\[
\ker T := \left\{ f \in \mathcal{H} : \begin{pmatrix} f \\ 0 \end{pmatrix} \in T \right\}, \quad \text{mul } T := \left\{ g \in \mathcal{H} : \begin{pmatrix} 0 \\ g \end{pmatrix} \in T \right\},
\]

which turn out to be linear sets in \(\mathcal{H}\). Moreover, if \(T\) is closed, then \(\ker T\) and \(\text{mul } T\) are subspaces (i.e. closed linear sets) of \(\mathcal{H}\).

Given linear relations \(T\) and \(S\), and \(\zeta \in \mathbb{C}\), we consider the linear relations:

\[
T + S := \left\{ \begin{pmatrix} f \\ g + h \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T, \begin{pmatrix} f \\ h \end{pmatrix} \in S \right\}, \quad \zeta T := \left\{ \begin{pmatrix} f \\ \zeta g \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\},
\]

\[
ST := \left\{ \begin{pmatrix} f \\ k \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T, \begin{pmatrix} g \\ k \end{pmatrix} \in S \right\}, \quad T^{-1} := \left\{ \begin{pmatrix} g \\ f \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}.
\]

We assume that the symbols \(\oplus\), \(\ominus\), and \(\oplus\) have their standard meaning, i.e.,

\[
T + S = \left\{ \begin{pmatrix} f \\ g + h \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T, \begin{pmatrix} h \end{pmatrix} \in S, \text{ and } T \cap S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \right\}, \quad T \ominus S = T + S, \text{ with } T \subset S^\perp.
\]

The symbol \(\oplus\) in this context strictly speaking differs from its meaning in the expression \(\mathcal{H} \oplus \mathcal{H}\) given above. It will cause no confusion to use the same symbol.

The adjoint of \(T\) is defined by

\[
T^* := \left\{ \begin{pmatrix} h \\ k \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : \langle k, f \rangle = \langle h, g \rangle, \forall \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\},
\]

which is a closed relation with the properties:

\[
T^* = (-T^{-1})^\perp, \quad S \subset T \Rightarrow T^* \subset S^*, \quad T^{**} = T, \quad (\alpha T)^* = \overline{\alpha} T^*, \text{ with } \alpha \neq 0,
\]

\[
(T^*)^{-1} = (T^{-1})^*, \quad \ker T^* = (\text{ran } T)^\perp. \quad (2.2)
\]

From (2.2), one obtains

\[
\text{dom } T = \overline{\text{ran } T^{-1}} = (\ker(T^{-1})^*)^\perp = (\text{mul } T^*)^\perp. \quad (2.3)
\]
We call a linear relation $T$ bounded if there exists $C > 0$ such that $\|g\| \leq C \|f\|$, for all $\begin{pmatrix} f \\ g \end{pmatrix} \in T$. Note that according to this definition every bounded linear relation is a bounded linear operator. It is worth remarking that, in the context of linear relations, there are other ways of defining boundedness for relations (see [5]) so that a bounded relation is not necessarily an operator.

The quasi-regular set $\hat{\rho}(T)$ of the linear relation $T$ is defined by

$$\hat{\rho}(T) := \{ \zeta \in \mathbb{C} : (T - \zeta I)^{-1} \text{ is bounded} \}.$$ 

It is straightforward to verify that this set is open and for every $\zeta \in \hat{\rho}(T)$ it follows that $\text{ran}(T - \zeta I)$ is closed if and only if $T$ is closed [12, Prop. 2.4]. Furthermore, for any $\zeta \in \hat{\rho}(T)$, the number

$$\eta_\zeta(T) := \dim[\text{ran}(T - \zeta I)]^\perp \quad (2.4)$$

is constant on each connected component of $\hat{\rho}(T)$. We call $\eta_\zeta(T)$ the deficiency index of $T$. We define the deficiency space $N_\zeta(T)$ as follows.

$$N_\zeta(T) := \left\{ \begin{pmatrix} f \\ \xi f \end{pmatrix} \in T \right\}, \quad \zeta \in \mathbb{C}. \quad (2.5)$$

Note that (2.5) is a linear bounded relation which is closed if $T$ is closed. Moreover, by (2.2)

$$\eta_\zeta(T) = \dim \ker (T^* - \overline{\zeta} I) = \dim N_{\overline{\zeta}}(T^*).$$

If $\eta_\zeta(T) = 0$, then $(T - \zeta I)^{-1} \in B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the class of all bounded operators having the whole space $\mathcal{H}$ as their domain.

Define the regular set $\rho(T)$ of the linear relation $T$ by

$$\rho(T) := \{ \zeta \in \mathbb{C} : (T - \zeta I)^{-1} \in B(\mathcal{H}) \}.$$ 

Note that if the linear relation is not closed, then the regular set is empty. Clearly, the regular set is a subset of the quasi-regular set and it is also open. For a relation $T$, we consider the sets

$$\sigma(T) := \mathbb{C} \setminus \rho(T), \quad \hat{\sigma}(T) := \mathbb{C} \setminus \hat{\rho}(T),$$

$s_{\rho}(T) := \{ \zeta \in \mathbb{C} : \ker(T - \zeta I) \neq \{0\} \}$, 

$s_{\rho}^\infty(T) := \{ \zeta \in s_{\rho}(T) : \dim \ker(T - \zeta I) = \infty \}$, 

$s_{e}(T) := \{ \zeta \in \mathbb{C} : \text{ran}(T - \zeta I) \neq \text{ran}(T - \zeta I) \}$. 

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As in the case of operators, one has

\[ \sigma_p(T) \cup \sigma_c(T) = \hat{\sigma}(T). \]  

(2.6)

For any two linear relations \( T \) and \( S \) in \( \mathcal{H} \oplus \mathcal{H} \), define the linear relation \( T_S \) in the Hilbert space \((\text{mul } S)\perp \oplus (\text{mul } S)\perp \) (here \( \oplus \) has the same meaning as in \( \mathcal{H} \oplus \mathcal{H} \)) by

\[ T_S := T \cap (\text{mul } S)\perp \oplus (\text{mul } S)\perp. \]  

(2.7)

If \( T \) is closed, then \( T_S \) is closed and if \( T \) is an operator, then \( T_S \) is an operator. Besides \( (T_S)^{-1} = (T^{-1})_S \).

It is useful to decompose a closed relation \( T \) as follows \( T = T_\oplus \oplus T_\infty \), where

\[ T_\infty = \left\{ \begin{pmatrix} 0 \\ g \end{pmatrix} \in T \right\}, \]

\[ T_\oplus = T \ominus T_\infty \]

are closed linear relations called the multivalued part and the operator part of \( T \), respectively.

**Lemma 2.1** If \( T \) is a closed relation such that \( \text{dom } T \subset (\text{mul } T)\perp \), then

\[ T - \zeta I = (T_T - \zeta I) \oplus T_\infty. \]

**Proof** Since the domain and the range of \( T_\oplus \) belong to \((\text{mul } T)\perp \), it follows from (2.7) that

\[ T_T = T_\oplus. \]  

(2.8)

Moreover,

\[ T - \zeta I = (T_\oplus - \zeta I) \oplus T_\infty \]

from which the assertion follows. \[ \square \]

### 3 Finite-dimensional perturbation of dissipative relations

**Definition 3.1** A relation \( L \) is called dissipative if for every \( \begin{pmatrix} f \\ g \end{pmatrix} \in L \),

\[ \text{Im}(f, g) \geq 0. \]  

(3.1)

If the equality in (3.1) holds, then \( L \) is said to be symmetric. Thus \( L \) is symmetric if and only if \( L \subset L^* \).
As in [12], one can show that
\[ C_\subset \hat{\varrho}(L), \]  
(3.2)
for any closed dissipative relation \( L \). Thus one can consider the deficiency index of \( L \) [see (2.4)] in the connected region \( C_\subset \) and denote it by \( \eta_-(L) \), i.e.
\[ \eta_-(L) := \dim N_{\xi}(L^*), \quad \xi \in C_\subset. \]

If \( L \) is a closed, symmetric relation, then \( \mathbb{C} \setminus \mathbb{R} \subset \hat{\varrho}(L) \) and hence one can consider
\[ \eta_+(L) := \dim N_{\xi}(L^*), \quad \xi \in C_\subset \]
alongside \( \eta_-(L) \). The index \( \eta_- \) is an important characteristic of a dissipative relation, while a symmetric relation is characterized by the pair \( \eta_\pm \).

**Definition 3.2** A dissipative relation \( L \) is maximal when it is closed and \( \eta_-(L) = 0 \). A maximal dissipative relation does not have proper dissipative extensions. Note that maximality of a dissipative relation means that \( C_\subset \) is in the regular set of the relation.

**Remark 3.3** In [3, Lemma 2.1] (see also [12]) it is shown that, for any dissipative relation \( L \),
\[ \text{dom } L \subset (\text{mul } L)^\perp. \]  
(3.3)
Moreover, it is proven in [12, Theorem 2.10] that (3.3) yields
\[ \sigma(L) = \sigma(L_L), \quad \sigma_p(L) = \sigma_p(L_L), \]
\[ \hat{\sigma}(L) = \hat{\sigma}(L_L), \quad \sigma_c(L) = \sigma_c(L_L). \]  
(3.4)
Furthermore, if \( L \) is a closed symmetric relation in \( \mathcal{H} \oplus \mathcal{H} \), then by (2.8) one obtains that \( L_L \) is a closed symmetric operator in \( (\text{mul } L)^\perp \oplus (\text{mul } L)^\perp \).

For any relation \( T \) in \( B(\mathcal{H}) \), we use the notation \( \text{rank } T := \dim(\text{ran } T) \). In [4, Thm 2.6.4] it is shown that rank \( T = m \) if and only if rank \( T^* = m \). Then
\[ \dim(\mathcal{H} \oplus \ker T) = \text{rank } T^* = \text{rank } T. \]  
(3.5)

For maximal dissipative relations \( A \) and \( L \), and \( \xi \in \rho(A) \cap \rho(L) \), we define
\[ F := (L - \xi I)^{-1} - (A - \xi I)^{-1} \in B(\mathcal{H}). \]  
(3.6)
Note that if rank \( F = m < \infty \) for some \( \xi \in \rho(A) \cap \rho(L) \), then the equality holds for every \( \xi \in \rho(A) \cap \rho(L) \). Since we are mostly interested in the rank of the operator \( F \), its dependence on \( \xi \) is not indicated.

The following assertion relies on the fact that, when \( A \) and \( V \) are maximal dissipative relations such that \( \text{dom } V = \mathcal{H} \), the relation \( A + V \) is maximal dissipative [12, Theorem 3.8].
Lemma 3.4 Let $A, V$ be maximal dissipative relations such that $\text{dom } V = \mathcal{H}$. If $L = A + V$ and $F$ is given by (3.6), then $\text{rank } F \leq \text{rank } V$.

**Proof** Take $\zeta \in \rho(L) \cap \rho(A)$ and consider $\left( \begin{array}{c} f \\ h - k \end{array} \right) \in F$, where $\left( \begin{array}{c} f \\ h \end{array} \right) \in (L - \zeta I)^{-1}$ and $\left( \begin{array}{c} f \\ k \end{array} \right) \in (A - \zeta I)^{-1}$. Since $V \in \mathcal{B}(\mathcal{H})$, there is $t \in \mathcal{H}$ such that $\left( \begin{array}{c} k \\ t \end{array} \right) \in V$. Define the set

$$G := \left\{ \left( \begin{array}{c} t \\ h - k \end{array} \right) \in \mathcal{H} \oplus \mathcal{H} : \left( \begin{array}{c} k \\ t \end{array} \right) \in V \text{ and } \left( \begin{array}{c} f \\ h - k \end{array} \right) \in F \right\}.$$ 

A simple computation shows that $G$ is a linear relation. Let $\left( \begin{array}{c} 0 \\ h - k \end{array} \right) \in G$, if $h \neq k$, then $\left( \begin{array}{c} k \\ f + \zeta k \end{array} \right) \in L$ and $\left( \begin{array}{c} h - k \\ \zeta(h - k) \end{array} \right) \in L$, whence $\zeta \in \sigma_p(L) \subset \sigma(L)$, which is impossible since $\zeta \in \rho(L)$. Thus $G$ is a linear operator and therefore

$$\dim \text{ ran } F = \dim \text{ ran } G \leq \dim \text{ dom } G \leq \dim \text{ ran } V.$$

Let us move away from the case when $L$ is obtained from $A$ by an additive perturbation and consider arbitrary maximal dissipative relations.

Lemma 3.5 If $A$ and $L$ are maximal dissipative extensions of a closed dissipative relation $S$ and $F$ is given by (3.6), then $\text{rank } F \leq \eta_{-}(S)$.

**Proof** Let $\zeta \in \mathbb{C}_{-}$. Since $S - \zeta I \subset (A - \zeta I) \cap (L - \zeta I)$, one has $\text{ran}(S - \zeta I)$ is contained in $\ker F$. Hence by (3.5) one obtains

$$\text{rank } F = \dim(\mathcal{H} \ominus \ker F)$$

$$\quad \leq \dim[\mathcal{H} \ominus \text{ran}(S - \zeta I)] = \eta_{-}(S).$$

The following statement is adapted from [12, Proposition 4.10, 4.11].

**Proposition 3.6** Let $S$ be a closed symmetric relation with finite deficiency index $\eta_{-}(S)$. Then for any $\lambda \in \hat{\rho}(S) \cap (\mathbb{C}_{+} \cup \mathbb{R})$ there exists a unique maximal dissipative extension $A$ of $S$ such that $\lambda$ is an eigenvalue of multiplicity at most $\eta_{-}(S)$. Furthermore, $A$ is selfadjoint if $\lambda \in \mathbb{R}$ while for $\lambda \in \mathbb{C}_{+}$ it follows that $A$ is nonselfadjoint.

Now we turn to the analysis of the dimension of eigenspaces of arbitrary maximal dissipative relations such that the difference of their resolvents is a finite rank operator. To simplify the notation, for $\lambda \in \mathbb{C}$ and $A$ being a closed dissipative relation, we put

$$\mu_A(\lambda) := \dim \ker(A - \lambda I).$$
**Proposition 3.7** Let $A$ and $L$ be maximal dissipative relations such that $F$ given by (3.6) is a finite rank operator. If one defines

$$G_\lambda := \ker(A - \lambda I) \cap \ker(L - \lambda I),$$

then

$$\dim[\ker(A - \lambda I) \ominus G_\lambda] \leq \text{rank } F, \quad \dim[\ker(L - \lambda I) \ominus G_\lambda] \leq \text{rank } F, \quad (3.7)$$

$$\mu_A(\lambda) - \text{rank } F \leq \mu_L(\lambda) \leq \mu_A(\lambda) + \text{rank } F. \quad (3.8)$$

**Proof** If we assume, for example, $\dim[\ker(A - \lambda I) \ominus G_\lambda] > \text{rank } F$, then in view of (3.5) there exists a nonzero element $f \in \ker(A - \lambda I) \cap \ker F$ such that $f \perp G_\lambda$. Thus \(\begin{pmatrix} f \\ \lambda f \end{pmatrix} \in A\) and \(\begin{pmatrix} f \\ (\lambda - \xi)^{-1}f \end{pmatrix} \in (A - \xi I)^{-1}\). Since $f \in \ker F$, one has \(\begin{pmatrix} f \\ (\lambda - \xi)^{-1}f \end{pmatrix} \in (L - \xi I)^{-1}\), which implies \(\begin{pmatrix} f \\ \lambda f \end{pmatrix} \in L\). Hence $f \in G_\lambda$ yielding a contradiction. Now we prove the right inequality in (3.8). It follows from (3.7) that

$$\mu_A(\lambda) = \dim[\ker(A - \lambda I) \ominus G_\lambda] + \dim G_\lambda \leq \text{rank } F + \dim G_\lambda \leq \text{rank } F + \mu_L(\lambda). \quad (3.9)$$

To obtain the left inequality in (3.8), interchange the roles of $A$ and $L$ in (3.9). \hfill \Box

Proposition 3.7 shows that the eigenspaces of $A$ and $L$ can differ only by a subspace of dimension at most rank $F$. Besides, it follows from (3.8) that $\sigma_p^\infty(A) = \sigma_p^\infty(L)$.

**Proposition 3.8** If $A$ is a closed dissipative extension of a closed symmetric relation $S$ and $\lambda \in \hat{\rho}(S)$, then $\mu_A(\lambda) \leq \eta_-(S)$.

**Proof** It is clear from (2.6) and (3.2) that $\ker(A - \xi I)$ can only have nontrivial elements when $\xi \in \mathbb{C}_+ \cup \mathbb{R}$. Thus, since $A \subseteq S^*$, one obtains that

$$\mu_A(\lambda) = \dim N_\lambda(A) \leq \dim N_\lambda(S^*) = \eta_-(S)$$

for any $\lambda \in \hat{\rho}(S) \cap (\mathbb{C}_+ \cup \mathbb{R})$. \hfill \Box

### 4 Compact and finite-dimensional perturbation of selfadjoint relations

We begin this section by stating the following characterization of selfadjoint relations which in its operator version is well known. Another characterization can be found in [14, Theorem 2.5].

**Proposition 4.1** For $A$ a closed symmetric relation the following are equivalent:

(i) $A$ is selfadjoint.
(ii) $\eta_{\pm}(A) = 0$.
(iii) $\hat{\rho}(A) = \rho(A)$.
(iv) $\sigma(A) \subset \mathbb{R}$.

**Proof** (i) $\Rightarrow$ (ii): If $\zeta \in \mathbb{C} \setminus \mathbb{R}$, then $(A - \zeta I)^{-1}$ is an operator. Thus

$$\{0\} = \text{mul}(A - \zeta I)^{-1} = \ker(A - \zeta I) = \text{dom} N_{\zeta}(A),$$

whence $\dim N_{\zeta}(A^*) = 0$. (ii) $\Rightarrow$ (iii): If $\zeta \in \hat{\rho}(A) \setminus \mathbb{R}$, then, taking into account (2.4), one concludes that $\text{ran}(A - \zeta I) = \mathcal{H}$ and then $\zeta \in \rho(A)$. Since $\hat{\rho}(A)$ is open and $\eta_{\pm}(A)$ are constants in the connected components of $\hat{\rho}(A)$, if $\zeta \in \hat{\rho}(A) \cap \mathbb{R}$, then $\zeta \in \rho(A)$. Thus we have shown that $\hat{\rho}(A) = \rho(A)$. (iii) $\Rightarrow$ (iv): This is straightforward. (iv) $\Rightarrow$ (i): The hypothesis immediately implies that $\text{dom} N_{\zeta}(A^*) = \{0\}$. If $\begin{pmatrix} f \\ h \end{pmatrix}$ is in $A^*$, then $\begin{pmatrix} g - if \\ h \end{pmatrix} \in (A - iI)^{-1}$. Therefore

$$\begin{pmatrix} h \\ g - i(f - h) \end{pmatrix} \in A \subset A^*.$$  \hspace{1cm} (4.1)

By linearity, $\begin{pmatrix} f - h \\ i(f - h) \end{pmatrix} \in N_{\zeta}(A^*)$. Thus, $f = h$ and (4.1) implies that $\begin{pmatrix} f \\ g \end{pmatrix}$ is in $A$. \hfill $\Box$

**Definition 4.2** The notion of essential spectrum for relations in the case of selfadjoint relations reduces to (see [4, Sect. 9.1.1] for the case of operators and [14] for relations)

$$\sigma_e(A) := \sigma_c(A) \cup \sigma_p^\infty(A).$$

As a consequence of Proposition 4.1-(iv) and Remark 3.3, if $A$ is a selfadjoint relation, then $A$ is a selfadjoint operator. Moreover, one verifies at once, on the basis of (2.8), that $\ker(A - \zeta I) = \ker(A_A - \zeta I)$, which in turn implies that $\sigma_p^\infty(A) = \sigma_p^\infty(A_A)$. Thus,

$$\sigma_e(A) = \sigma_e(A_A).$$  \hspace{1cm} (4.2)

The following definition is a generalization of the notion of singular sequences for selfadjoint operators.

**Definition 4.3** A sequence $\{u_n\}_{n \in \mathbb{N}}$ in the domain of a selfadjoint relation $A$ is said to be singular for $A$ at $\lambda \in \mathbb{R}$, if there exists a sequence $\begin{pmatrix} u_n \\ v_n \end{pmatrix}$ with elements in $A$ such that the following conditions are satisfied:

(i) $\inf_{n \in \mathbb{N}} \|u_n\| > 0,$
(ii) $u_n \rightarrow 0,$
(iii) $(v_n - \lambda u_n) \rightarrow 0,$
where $\rightarrow$ denotes weak convergence.

**Remark 4.4** If in Definition 4.3, one writes $v_n = t_n + s_n$, where $t_n \in \text{ran } A \odot$ and $s_n \in \text{mul } A$, then

$$
\| t_n - \lambda u_n \|^2 \leq \| t_n - \lambda u_n \|^2 + \| s_n \|^2 \\
= \| (t_n + s_n) - \lambda u_n \|^2 \to 0.
$$

Hence, $s_n \to 0$ and the sequence $\left\{ \begin{pmatrix} u_n \\ t_n \\ v_n \end{pmatrix} \right\}_{n \in \mathbb{N}}$, which has elements in $A \odot \subset A$, satisfies (iii). This means that if $A$ is not an operator and $\{u_n\}_{n \in \mathbb{N}}$ is singular, then the sequence $\left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n \in \mathbb{N}}$ in Definition 4.3 is not unique and, moreover, there are sequences $\left\{ \begin{pmatrix} u_n \\ w_n \end{pmatrix} \right\}_{n \in \mathbb{N}}$ in $A$ not satisfying (iii).

**Remark 4.5** Note that Remark 4.4 and (2.8) imply that there is a singular sequence for $A$ at $\lambda$ if and only if there is a singular sequence for $A_A$ at $\lambda$.

The following result is known as the Weyl criterion and it can be found for selfadjoint operators in [4, Th. 9.1.2] (see a more general version for operators in [11, Chapter 4 Sect. 5]).

**Proposition 4.6** Let $A$ be a selfadjoint relation. The real number $\lambda$ belongs to $\sigma_e(A)$ if and only if there exists a singular sequence for $A$ at $\lambda$.

**Proof** The assertion follows from (4.2) and Remark 4.5 since the assertion is true for operators. \hfill $\square$

Denote by $S_\infty(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ the set of compact operators whose domain is $\mathcal{H}$. It is known that $V$ belongs to $S_\infty(\mathcal{H})$ if and only if $V$ maps a weakly convergent sequence into a convergent sequence (see [4, Sect. 2.6]). The following assertion is a Weyl-type perturbation theorem for relations.

**Proposition 4.7** If $A$ and $V$ are selfadjoint relations such that $V \in S_\infty(\mathcal{H})$, then $L = A + V$ is selfadjoint and $\sigma_e(L) = \sigma_e(A)$.

**Proof** The selfadjointness of $L$ follows from the fact that $(A + V)^* = A^* + V^*$ [12, Prop. 2.2]. For any $\left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n \in \mathbb{N}} \subset A$, there is $\left\{ \begin{pmatrix} u_n \\ w_n \end{pmatrix} \right\}_{n \in \mathbb{N}} \subset V$ and

$$
\left\{ \begin{pmatrix} u_n \\ v_n + w_n \end{pmatrix} \right\}_{n \in \mathbb{N}} \subset L.
$$

Then, if $\{u_n\}_{n \in \mathbb{N}}$ is singular for $A$ at $\lambda$, then $w_n \to 0$ and $\left[(v_n + w_n) - \lambda u_n\right] \to 0$, which implies that $\{u_n\}_{n \in \mathbb{N}}$ is singular for $L$ at $\lambda$ and, by Proposition 4.6, one has $\sigma_e(A) \subset \sigma_e(L)$. The other inclusion is obtained by noting that $A = L - V$. \hfill $\square$

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We now extend the previous result using the operator $F$ given in (3.6). An alternative proof of the following theorem is found in [13, Theorem 5.1].

**Theorem 4.8** If $A$ and $L$ are selfadjoint relations and if $F$ belongs to $S_{\infty}(\mathcal{H})$, then $\sigma_e(A) = \sigma_e(L)$.

**Proof** We only need to show that $\sigma_e(A) \subset \sigma_e(L)$. If $\{u_n\}_{n \in \mathbb{N}}$ is singular for $A$ at $\lambda$, then by Remark 4.5 there is a sequence $\left\{ \begin{pmatrix} u_n \\ t_n \end{pmatrix} \right\}_{n \in \mathbb{N}}$ in $A_A$ in which

$$
(t_n - \lambda u_n) \to 0. \tag{4.3}
$$

Note that $\begin{pmatrix} t_n - \xi u_n \\ u_n \end{pmatrix} \in (A - \xi I)^{-1}$ and there exists $\begin{pmatrix} t_n - \xi u_n \\ w_n \end{pmatrix} \in (L - \xi I)^{-1}$ for any $n \in \mathbb{N}$. A short computation shows that

$$
\left\{ \begin{pmatrix} w_n \\ (t_n - \xi (u_n - w_n)) \end{pmatrix} \right\}_{n \in \mathbb{N}} \subset L. \tag{4.4}
$$

In view of Proposition 4.6, it only remains to prove that (4.4) makes $\{w_n\}_{n \in \mathbb{N}}$ be singular for $L$ at $\lambda$. First, one verifies that $\begin{pmatrix} t_n - \xi u_n \\ w_n - u_n \end{pmatrix} \in F$ and $\begin{pmatrix} u_n \\ w_n - u_n \end{pmatrix} \in F(A_A - \xi I)$. Since $A_A$ and $F$ are operators,

$$
F(A_A - \xi I) = F(A_A - \lambda I) + (\lambda - \xi) F.
$$

Then there exist $\begin{pmatrix} u_n \\ s_n \end{pmatrix} \in F(A_A - \lambda I)$ and

$$
\begin{pmatrix} u_n \\ g_n \end{pmatrix} \in (\lambda - \xi) F \tag{4.5}
$$

such that

$$
w_n - u_n = s_n + g_n, \quad n \in \mathbb{N}. \tag{4.6}
$$

The fact that $\begin{pmatrix} u_n \\ t_n - \lambda u_n \end{pmatrix}$ is in $A_A - \lambda I$ implies

$$
\begin{pmatrix} t_n - \lambda u_n \\ s_n \end{pmatrix} \in F. \tag{4.7}
$$

Since $F$ is a compact operator and $u_n \to 0$, it follows from (4.3), (4.5), and (4.7) that $g_n, s_n \to 0$. Thus (4.6) implies

$$
(w_n - u_n) \to 0. \tag{4.8}
$$
The fact that \( \{u_n\}_{n \in \mathbb{N}} \) is singular then yields that \( w_n \to 0 \) and \( \inf_{n \in \mathbb{N}} \|w_n\| > 0 \). To conclude the proof, observe that from (4.3) and (4.8), one has

\[
[t_n - \xi (u_n - w_n)] - \lambda w_n = (t_n - \lambda u_n) - (\lambda - \xi)(w_n - u_n) \to 0.
\]

\[\Box\]

Let us turn to the study of the discrete spectrum of a selfadjoint relation \( A \). In view of Remark 3.3, one can consider the spectral theorem for selfadjoint operators. Let \( E_{AA} \) be the spectral measure of \( A_A \). It follows from [4, Th. 6.1.3] that

1. \( \sigma(A) = \text{supp} \ E_{AA} \).
2. \( \sigma_p(A) = \{ \lambda \in \mathbb{R} : E_{AA}(\lambda) \neq 0 \} \). The eigenspace corresponding to the eigenvalue \( \lambda \) is \( E_{AA}(\lambda)(\text{mul} A) \perp \).
3. \( \sigma_c(A) \) is the set of non-isolated points of \( \sigma(A) \).

Consider a bounded interval \( \Delta \) and define

\[
\mu_A(\Delta) := \dim E_{AA}(\Delta)(\text{mul} A) \perp.
\]

The following assertion does not follow directly from [4, Theorem 9.3.3] and (1)–(3) since the relations \( L \) and \( A \) could be such that \( (\text{mul} A) \perp \) and \( (\text{mul} L) \perp \) do not coincide. This is illustrated by the relations we give as examples in (5.4).

**Theorem 4.9** If \( A \) and \( L \) are two selfadjoint relations and \( \text{rank} \ F \) is finite, then

\[
\mu_A(\Delta) - \text{rank} \ F \leq \mu_L(\Delta) \leq \mu_A(\Delta) + \text{rank} \ F.
\]

**Proof** Only one inequality in (4.9) needs to be proven. If \( \mu_L(\Delta) > \mu_A(\Delta) + \text{rank} \ F \), then there exists a non-zero element \( f \in E_{LL}(\Delta)(\text{mul} L) \perp \cap \ker F \) such that \( f \perp E_{AA}(\Delta)(\text{mul} A) \perp \). This implies that there also exists

\[
\begin{pmatrix} f \\ g \end{pmatrix} \in (A - \xi I)^{-1} \cap (L - \xi I)^{-1}.
\]

Due to (3.3), \( \begin{pmatrix} f \\ g \end{pmatrix} \in (L_{L} - \xi I)^{-1} \). Observe that

\[
\begin{pmatrix} f \\ g \end{pmatrix} \in (L_{L} - \xi I)^{-1} E_{LL}(\Delta) = E_{LL}(\Delta)(L_{L} - \xi I)^{-1}
\]

which implies that \( g \in E_{LL}(\Delta)(\text{mul} L) \perp \). Thereupon, if one defines \( \Delta := (\alpha, \beta) \) and \( \gamma := (\alpha + \beta)/2 \), then

\[
\begin{pmatrix} f + (\xi - \gamma)g \\ g \end{pmatrix} \in (L_{L} - \gamma I).
\]
By the spectral theorem, one concludes
\[
\| f + (\xi - \gamma)g \|^2 = \| (L_L - \gamma I)g \|^2
= \int_{|t - \gamma| < \xi} (t - \gamma)^2 d|g, E_{LL}(t)g| < \xi^2 \|g\|^2,
\]
where \( \xi := (\alpha - \beta)/2 \). Since \( f \perp E_{AA}(\Delta)(\text{mul } A)^{\perp} \), one obtains that
\[
f \in [E_{AA}(\mathbb{R}\setminus \Delta)(\text{mul } A)^{\perp}] \oplus [\text{mul } A].
\]
Thus, one has the decomposition \( f = f_1 + f_2 \), where \( f_1 \in E_{AA}(\mathbb{R}\setminus \Delta)(\text{mul } A)^{\perp} \) and \( f_2 \in \text{mul } A \). On the basis of the fact that \( g \in (\text{mul } A)^{\perp} \), which follows from (4.10), one has \( \left( \begin{array}{c} f_1 \\ g \end{array} \right) \in (A_A - \xi I)^{-1} \). Recurring to the course of reasoning in (4.11), one establishes as before that
\[
\left( \begin{array}{c} f_1 + (\xi - \gamma)g \\ g \end{array} \right) \in (A_A - \gamma I).
\]
Therefore
\[
\| f + (\xi - \gamma)g \|^2 = \| f_1 + f_2 + (\xi - \gamma)g \|^2
\]
\[
= \| f_2 \|^2 + \| f_1 + (\xi - \gamma)g \|^2 
\geq \| f_1 + (\xi - \gamma)g \|^2 
= \| (A_A - \gamma I)g \|^2 
= \int_{|t - \gamma| \geq \xi} (t - \gamma)^2 d\langle g, E_{AA}(t)g \rangle \geq \xi^2 \|g\|^2,
\]
which contradicts (4.12).

\( \square \)

Remark 4.10 As a consequence of Lemma 3.5 and Theorem 4.8, if \( S \) is a closed symmetric relation with finite and equal deficiency indices, then its selfadjoint extensions have the same essential spectrum.

The next definition is based on the analogous notion for operators given in [4, Sect. 9.3].

Definition 4.11 The interval \( \Delta = (\gamma - \xi, \gamma + \xi) \) is a spectral lacuna (or simply lacuna) of a symmetric relation \( S \) when
\[
\| f \| \leq \frac{1}{\xi} \| g - \gamma f \| \quad \text{for all} \quad \left( \begin{array}{c} f \\ g \end{array} \right) \in S.
\]

Notice that a spectral lacuna consists of quasi-regular points of \( S \) and each quasi-regular point of \( S \) belongs to a lacuna of \( S \).

The following result is a generalization of [4, Theorem 9.3.6].
Theorem 4.12 Let $A$ be a selfadjoint extension of a closed symmetric relation $S$. If $\Delta$ is a lacuna of $S$, then $\mu_A(\Delta) \leq \eta_-(S)$.

Proof From (3.3), it follows that $\text{dom } S \subset \text{dom } A \subset (\text{mul } A)\perp$. Thus

$$S_\circ \subset (\text{mul } A)\perp \oplus (\text{mul } A)\perp.$$ 

So $S_A = S_\circ$ and this implies that $S_A$ is a closed symmetric operator and $A_A$ is its selfadjoint extension, which is also an operator. Since $S_A \subset S$, $\Delta$ is also a lacuna of $S_A$. We use (2.1) to obtain that

$$\text{ran}(S - \zeta I) = \text{ran}(S_\circ - \zeta I) \oplus \text{mul } S \subset \text{ran}(S_A - \zeta I) \oplus \text{mul } A,$$

whence

$$\mathcal{H} \ominus \text{ran}(S - \zeta I) \supset \mathcal{H} \ominus (\text{ran}(S_A - \zeta I) \oplus \text{mul } A)$$

$$= (\mathcal{H} \ominus \text{mul } A) \ominus \text{ran}(S_A - \zeta I)$$

$$= (\text{mul } A)\perp \ominus \text{ran}(S_A - \zeta I).$$

Thus

$$\eta_-(S) = \dim \mathcal{H} \ominus \text{ran}(S - \zeta I)$$

$$\leq \dim (\text{mul } A)\perp \ominus \text{ran}(S_A - \zeta I)$$

$$= \eta_-(S_A).$$

Using the fact that the theorem holds for operators, one concludes that

$$\mu_A(\Delta) \leq \eta_-(S_A) \leq \eta_-(S).$$

Remark 4.13 Under the conditions of the previous result, if $\eta_-(S) = n < \infty$, then, for any $\Delta \subset \hat{\rho}(S)$, the spectrum of $A$ in $\Delta$ is discrete and its multiplicity is at most $n$. This is so because every closed bounded subinterval of $\Delta$ can be covered by a finite number of lacunae of $\hat{\rho}(S)$ (see [14, Corollary 5.2]).

Corollary 4.14 Let $S$ be a closed symmetric relation with indices $\eta_-(S) = \eta_+(S) = 1$. Suppose that $A$ and $L$ are distinct selfadjoint extensions of $S$ and $\Delta \subset \hat{\rho}(S)$. Then the spectra of $A$ and $L$ in $\Delta$ are discrete, simple and alternating.

Proof By Remark 4.13, the spectra of $A$ and $L$ in $\Delta$ are discrete and simple. Note that $\text{ran } F = \{0\}$ implies $A = L$, so $\text{rank } F \geq 1$. On the other hand, since $\eta_-(S) = 1$, it follows from Lemma 3.5 that $\text{rank } F \leq 1$. Thus, (4.9) yields

$$\mu_A(\Delta) - 1 \leq \mu_L(\Delta) \leq \mu_A(\Delta) + 1.$$  (4.13)
Moreover, since $A \neq L$, it follows from Proposition 3.6 that the spectra of $A$ and $L$ have empty intersection in $\Delta$. Let $\lambda_1, \lambda_2$ be neighbouring eigenvalues of $A$ in $\Delta$. By (4.13), one has $\mu_L((\lambda_1, \lambda_2)) \leq 1$. If there are no spectral points of $L$ in $[\lambda_1, \lambda_2]$, then the same is true in some open interval $\partial \supset [\lambda_1, \lambda_2]$. Recurring again to (4.13), one arrives at

$$0 = \mu_L(\partial) \geq \mu_A(\partial) - 1 \geq 1,$$

which is a contradiction. Therefore the spectra of $A$ and $L$ are alternating. \hfill \Box

The next result complements Proposition 3.6. It follows directly from Corollary 4.14.

**Corollary 4.15** Suppose that $S$ is closed symmetric relation with indices $\eta_-(S) = \eta_+(S) = 1$. If $\mathbb{R} \subset \hat{\rho}(A)$, then the spectra of the selfadjoint extensions of $S$ are pairwise interlaced and consist solely of isolated eigenvalues of multiplicity one.

### 5 Examples

Let $J$ be a selfadjoint operator in a separable Hilbert space $\mathcal{H}$. For a fixed nonzero $\delta \in \mathcal{H}$, consider the restriction

$$B_\delta := J_{\text{dom } J \ominus \text{span}\{\delta\}}, \quad (5.1)$$

The operator $B_\delta$ is closed, non-densely defined and symmetric. One verifies that

$$B_\delta = J \cap \left( \text{span}\left\{\begin{pmatrix} 0 \\ \delta \end{pmatrix}\right\}^* \right).$$

Observe that $J$ and $\text{span}\left\{\begin{pmatrix} 0 \\ \delta \end{pmatrix}\right\}$ are linearly independent so, in view of [8, Sect. 2], one obtains that $B_\delta$ has indices $\eta_-(B_\delta) = \eta_+(B_\delta) = 1$. Moreover, for any $\tau \in \mathbb{R} \cup \{\infty\}$ there is a unique selfadjoint extension of $B_\delta$ given by

$$J(\tau) = \left\{ \begin{pmatrix} f \\ g + \tau \langle \delta, f \rangle \delta \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in J, \quad \tau \neq \infty \right\}, \quad (5.2)$$

and

$$J(\infty) = B_\delta \oplus \text{span}\left\{\begin{pmatrix} 0 \\ \delta \end{pmatrix}\right\}. \quad (5.3)$$

When $\tau$ runs through the set $\mathbb{R} \cup \{\infty\}$, $J(\tau)$ runs through all selfadjoint extensions of $B_\delta$ [8, Theorem 2.4]. Also, by [8, Eq. 2.2],

$$B_\delta^* = J \oplus \text{span}\left\{\begin{pmatrix} 0 \\ \delta \end{pmatrix}\right\}.$$
A direct consequence of Remark 4.10 is the following assertion.

**Proposition 5.1** For any nonzero \( \delta \in \mathcal{H} \), the essential spectra of all the selfadjoint extensions of the symmetric operator \( B_\delta \) given in (5.1), are equal.

Recall that a selfadjoint operator \( J \) is said to be simple when there exists \( g \in \mathcal{H} \) such that the linear envelope of the vectors \( E_J(\partial)g \), where \( E_J \) is the spectral measure of \( J \) and \( \partial \) runs through all intervals of \( \mathbb{R} \), is dense in \( \mathcal{H} \) (see [1, Sect. 69]). The vector \( g \) is then called a generating element of \( J \).

**Proposition 5.2** Suppose that \( J \) is simple and \( \delta \) is a generating element of it. If \( \partial \) is an interval such that \( \partial \cap \sigma_e(J) = \emptyset \), then \( \partial \subset \hat{\rho}(B_\delta) \).

**Proof** Assume \( \zeta \in \partial \cap \hat{\rho}(B_\delta) \). Then \( \zeta \in \sigma(J) \) and, since \( \partial \cap \sigma_e(J) = \emptyset \), \( \zeta \in \sigma_d(J) \). Moreover \( \zeta \in \sigma_p(B_\delta) \), otherwise \( \zeta \in \sigma_{\infty}(J) \subset \sigma_e(J) \). Therefore

\[
\ker(B_\delta - \zeta I) = E_J(\zeta)\mathcal{H}.
\]

Since \( \delta \) is a generating element of \( J \), one has \( \langle f, \delta \rangle \neq 0 \) for every nonzero \( f \in \ker(B_\delta - \zeta I) \). This contradicts the fact that \( \delta \perp \text{dom } B_\delta \). Therefore \( \partial \cap \hat{\rho}(B_\delta) \) is empty which yields that \( \partial \subset \hat{\rho}(B_\delta) \). \( \square \)

As a consequence of the last result, if the spectrum of \( J \) is purely discrete and \( \delta \) is a cyclic vector of it, then, by Corollary 4.15, the spectra of the extensions (5.2) and (5.3) are pairwise interlaced and consist solely of isolated eigenvalues of multiplicity one. Note that this applies to \( J \) being a selfadjoint Jacobi operator with discrete spectrum and \( \delta = \delta_1 \), where \( \{\delta_k\}_{k \in \mathbb{N}} \) is the canonical basis in \( l_2(\mathbb{N}) \).

Now suppose that \( \{\delta_k\}_{k \in \mathbb{N}} \) is an orthonormal basis of \( \mathcal{H} \) and consider the restriction

\[
S := J|_{\text{dom } J \ominus \text{span}\{\delta_1, \delta_2\}}.
\]

Clearly, by (5.3) one has that

\[
J_{\delta_1} := B_{\delta_1} + \text{span}\left\{ \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\}; \quad J_{\delta_2} := B_{\delta_2} + \text{span}\left\{ \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix} \right\} \tag{5.4}
\]

are selfadjoint extensions of \( S \) and they do not have a common multivalued part. Let us show that the selfadjoint relations (5.4) have the same essential spectrum. One computes that

\[
S = J \cap \left( \text{span}\left\{ \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix} \right\} \right)^*.
\]

According to [6, Lem. 5.1], the indices of \( S \) are \( \eta_-(S) = \eta_+(S) = 2 \). Therefore, by Remark 4.10, the extensions (5.4) have the same essential spectrum.

In the first example, the results of Sect. 4 are applied to selfadjoint extensions of the operator \( B_\delta \). One of these selfadjoint extensions is a relation. In the second example, two selfadjoint extensions of the operator \( S \) are considered. These extensions are relations with different multivalued parts.
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