QUANTUM RANDOM WALKER IN PRESENCE OF A MOVING DETECTOR

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In this work, we study the effect of a moving detector on a discrete time one dimensional Quantum Random Walk where the movement is realized in the form of hopping/shifts. The occupation probability \(f(x, t; n, s)\) is estimated as the number of detection \(n\) and number of shift \(s\) vary. It is seen that the occupation probability at the initial position \(x_D\) of the detector is enhanced when \(n\) is small which is a quantum mechanical effect but decreases when \(n\) is large. The ratio of occupation probabilities of our walk to that of an Infinite walk shows a scaling behavior of \(\frac{1}{n}\). It shows a definite scaling behavior with number of shifts \(s\) also. The limiting behaviours of the walk are observed when \(x_D\) is large, \(n\) is large and \(s\) is large and the walker for these cases approach the Infinite Walk, The Semi Infinite Walk and the Quenched Quantum Walk respectively.

I. INTRODUCTION

Discrete time Quantum Random Walk (QRW) on a line caught the attention of scientists at the end of twentieth century. The term quantum walk, the analogue of classical walk was first coined by Aharonov et. al. in 1993 \cite{1}. Several studies have been made in this regard to find out the differences between quantum walk and the classical walk \cite{2–4}. The quantum interference in QRW results in \(\langle x^2 \rangle \propto t^2\) where \(x\) is position of the walker and \(t\) is the time. Therefore the QRW is quadratically faster compared to the Classical Random Walk (CRW). The situation becomes interesting when quenching phenomena is studied for QRW. Slow quenching are usually studied in the aspect of spin glass system where there are many minima in the energy landscape, separated by barriers which may be overcome by quantum tunnelling \cite{3, 4}. Fast quenching is applied for ultracold atoms in an optical lattice by shifting the position of the trap potential and studying its response \cite{5}. In certain systems having a quantum critical point, fast or slow quenching of a few variables are used to study nonequilibrium dynamics \cite{6}. Quenching in QRW is studied earlier in \cite{9} where a detector put initially in the path of the QRW was withdrawn after a certain time. While experimental study of a QRW is becoming more and more important in recent years, the role of a detector in its path is to be studied minutely. The present study is entirely focused to the QRW with a detector in its path.

In this work, we studied a QRW with a detector which is not fixed in its position for long but can move. Here the movement is taken in form of hopping over the sites. In the same section, we present the exact scheme to study QRW in presence of a moving detector. In the same section, we define a few related notations we will use throughout the next sections of this work. Section III consists of the main results of our study. In section IV, a brief summary as obtained from the results is presented.

II. SCHEME TO STUDY QRW IN PRESENCE OF A MOVING DETECTOR

The QRW is drastically different from a Classical Random Walk (CRW). For QRW, there is an additional degree of freedom called “Chirality”. The chirality here can take two states “left” \(|L\rangle\) or “right” \(|R\rangle\) and is coupled to the position. The state of the walker is therefore expressed in \(|x\rangle \otimes |d\rangle\) basis, where \(|x\rangle\) and \(|d\rangle\) are the position and chirality eigenstates respectively. The wavefunction at position \(x\) for time \(t\) is,

\[
\Psi(x, t) = \begin{pmatrix} \psi_L(x, t) \\ \psi_R(x, t) \end{pmatrix}
\]  

(1)
where $L, R$ denote the left and right part of the wavefunction respectively.

The walk may be initialized at the origin which happens to be a boundary condition for the system. In that case, we have $\Psi_L(0,0) = a_0$ and $\Psi_R(0,0) = b_0$; $a_0^2 + b_0^2 = 1$. As we have taken symmetric walk therefore, here, $a_0 = \frac{1}{\sqrt{2}}, b_0 = \frac{1}{\sqrt{2}}$.

The unitary operator we have chosen is Hadamard operator $H$ that acts on the chirality state. $H$ is given as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2)$$

After the action of Hadamard coin the Translation operator $T$ acts in the following way:

$$T |x, L\rangle = |x-1, L\rangle$$
$$T |x, R\rangle = |x+1, R\rangle \quad (3)$$

QRW is usually studied in two specific ways:

- Without a detector: This allows free propagation of the walker on either side of origin. This corresponds to an Infinite Walk (IW).

- With a detector: This is usually studied by placing a detector or absorbing boundary on one side of the path of the walker. This restricts free propagation on one side and therefore leads to Semi Infinite Walk (SIW).

Vis a vis this usual SIW there is another variation called the Quenched Quantum Walk (QQW), where the detector is initially placed at a site and then removed from the path of the walker after a certain time interval. In all the above cases with detector, we already know that presence of a detector at a certain site $x_D$ in the path of the walker affects the occupation probabilities of sites in the following way: If the walker reaches $x_D$ with probability $p_{x_D}$ and the detection probability of the detector is $p_D$, then the absorption probability would obviously be $p_{x_D}p_D$.

However, in our study, we have chosen that, whenever the walker reaches the detector, it is detected with probability unity, i.e., $p_D = 1$. In addition to this, here the detector has a movement which we assume to be a hopping. The detector is initially placed at $x_D$. It then hops to a site $(x_D)_1 = x_D + s$ after making $n$ number of detections. Therefore, if the first detection at $x_D$ is made at time $t$, the first detection at $(x_D)_1$ will be at time $t_1 = t + (2n + s)$. When it completes $n$ detections at $(x_D)_1$ it moves further and detects in the same manner and then again hops to $(x_D)_2 = (x_D)_1 + s$.

From this point of view, we can also say that there must be some kind of velocity associated with the detector. This can also be interpreted from a different point of view in case of experimental arrangements as discussed earlier. A detector at $x_D$ may lose its efficiency to detect after a finite number of detections. In that case, it may be replaced by a new one. In our case the new one is introduced at $(x_D)_1 = x_D + is$, where $i = 1, 2, 3$ and so on for $t_i = t + i(2n + s)$. Note that replacing the old detector by a new one at the same place does not lead to any interesting behavior and therefore, we haven’t shown that here.

The presence of the detector will modify the usual propagation of the walker and therefore occupation probabilities of sites (probability distribution function) will be altered. It should be mentioned here that our usual notation for occupation probabilities $f(x,t)$ should be modified here. As $n$ and $s$ are the two important parameters responsible for the alteration of the occupation probabilities from now on, we will use the notation $f(x,t; n, s)$ or rather a shorter notation $f_{ns}$ to denote the probability distribution function. Also from now on, we will call our walk as Moving Detector Quantum Walk (MDQW). It is to be noted here that according to the measurement scheme proposed in \cite{13}, here $\sum_{x} f(x,t; n, s) = 1 - d$ where $d$ is the probability that it was absorbed earlier. Although we can choose the measurement scheme in two ways, one by normalizing the total probability over sites after each step of absorption/detection, and the other without normalizing, here we are choosing the second scheme, as it is seen in \cite{9} that renormalization does not lead to anything interesting.

### III. RESULTS

To compare our MDQW to the earlier versions of quantum walk, we first present a few snapshots of the occupation probabilities of sites for $t = 1000$ for IW, SIW and MDQW. As we know from the definition of our walk, here we have two parameters, $n$ and $s$. In the subsequent part of this work, we will use a notation to identify the number of detection and amount of hop as $nDsS$. Therefore, it is obvious that 2D6S represents that there will be 2 detections at each position of a detector and amount of hop is 6. If the detector is initially at $x_D$, the walker is not allowed to go beyond $x_D$. The shift/hop of the walker allows the walker to move beyond $x_D$, as then the detector has moved to $x_D + is$ further on that side thereby allowing the walker to cross $x_D$. We expect three limiting cases here as follows:
• For \( x_D >> 0 \), the walker is not at all detected by the detector for long. This will lead to IW behavior.

• With \( n \) large, it is obvious that the detector has to stay at a particular position for long and therefore it can be approximated to SIW.

• For \( s \to \infty \), the situation is as if the detector will detect and then it will be removed from its position. This should resemble QQW.

In the following we will try to see these limiting case along with other important results.

A. MDQW with variation of \( n \)

In the first collage, the occupation probabilities for IW, MDQW and SIW, denoted as \( f_\infty, f_n, \) and \( f \) respectively are shown in form of snapshots. The amount of shift \( s = 1 \) and the number of detections are \( n = 2, 10, 15, 30 \) respectively.

(Fig. 1) with the detector initially at \( x_D = 10 \). As here, the amount of shift is 1, a small number of detection allows more spilling of the probability distribution function compared to a large number of detection. It is clear that as the number of detection \( n \) increases, the occupation probability becomes more and more similar to the SIW occupation probability as expected.

We now wish to see this in more detail. For this purpose, The ratio of occupation probability of a site for a MDQW, i.e. \( f_n \) (a shorter notation for \( f(x,t;n,s) \)) and that of an IW, i.e., \( f_\infty \) is shown against \( t \) for \( nD1S \) where the initial position of the detector \( x_D = 10 \) in Fig. 2. In the same Fig. the corresponding ratio is shown for a SIW where the detector is not replaced or removed (similar to a SIW). As the initial position of the detector is \( x_D = 10 \), it is obvious that upto \( t = x_D = 10 \) (here), the presence of the detector cannot affect the walker. Beyond \( x_D \) the occupation probabilities of sites for the walker starts getting affected by the detector and therefore the ratio starts decreasing below 1. Here the number of detection \( n \) plays a very important role. It is to be mentioned here once again that \( n \) is actually related to the efficiency of the detector. Here \( n \) has shown to vary from 1 through 40. It is clear that as long as the detector is placed at initial position \( x_D \), the plot is same as SIW. If we look closely, the pattern is same as SIW upto \( t = 2n + x_D \). After this time the detector will hop to \( (x_D)_1 \). Thus, when the walker reaches \( x_D \) next, there...
FIG. 2. Ratio of occupation probability of MDQW to that of IW, i.e., $f_{ns}/f_{\infty}$ as a function of $t$ is shown for $nD1S$ for $x_D = 10$. The same is shown for SIW for comparison. The small $n$ behavior of $f_{ns}/f_{\infty}$ is shown in the inset. For MDQW, for each $n$ the ratio approaches a saturation value.

FIG. 3. $\left(\frac{f_{ns}}{f_{\infty}}\right)_{\text{sat}}$ against numbers of detection $n$ for $x_D = 2, 6, 10, 14$ in (a) for $s = 1$, (b) for $s = 10$. A $n^{-2}$ behavior is observed. In the respective insets, data collapses are shown.

is no detection, as the detector is already moved to a new position. The ratio for site 10 will increase therefore. As the walker reaches the detector again the tendency to decrease starts. The hopping of the detector leads to a kind of periodic behavior as seen from the plot. However, after a sufficiently long time the ratio of $f_{ns}/f_{\infty}$ at 10 saturates. In the inset of Fig. 2 the same $f_{ns}/f_{\infty}$ is shown when numbers of detection are small. It is clear from all the plots that the ratio ultimately approaches a saturation value $\left(\frac{f_{ns}}{f_{\infty}}\right)_{\text{sat}}$. For low $n$ values at a large $t$ having a very high $\left(\frac{f_{ns}}{f_{\infty}}\right)_{\text{sat}}$ much greater than 1 suggests that the walker will try to compensate for the sites not reached. In Fig. 3 (a) and (b)
\[
\left( \frac{f_{ns}}{f_{\infty}} \right)_{\text{sat}} \quad \text{is shown as a function of number of detection } n \text{ when the detector is initially placed at } x_D \text{ for } s = 1, 10 \text{ respectively. The plots show a behavior } n^{-\alpha} \text{ for large } n \text{ for all the cases. The exponent } \alpha \text{ here has been observed to be 2 for any fixed } s. \text{ However, for large hopping } s, \text{ small } n \text{ affects the system to a lower degree and therefore the ratio stays close to 1 even for large } n. \text{ This effect is more prominent for higher } s \text{ but we have not shown it here. The collapsed data are shown in the insets of Fig. 3. Therefore, it may be written that for large } n
\]

\[
\left( \frac{f_{ns}}{f_{\infty}} \right)_{\text{sat}} \sim \frac{x_D^2}{n^2}
\]

(4)

It should be mentioned that there is a certain number of detections \( n^* \) for each fixed \( x_D \) and \( s \) beyond which \( \left( \frac{f_{ns}}{f_{\infty}} \right)_{\text{sat}} \) can never attend 1. The plot of \( n^* \) against \( x_D \) shows linear fit as observed in Fig. 4. This supports the nature observed in variation of \( \frac{f_{ns}}{f_{\infty}} \) against \( t \). This is already indicated in Fig. 2 as for small \( n \) if we can wait for sufficiently long time, the system can qualitatively approach IW but not in case of large \( n \).

B. MDQW with variation of \( s \)

Let us now move on to the study of MDQW from the aspect of another parameter which we can control, i.e., the amount of hop/shift \( s \). We now fix the number of detection \( n \) and vary \( s \). In Fig. 5 we have shown the snapshots for \( 2DssS \). It is seen that as we increase the amount of shift \( s \), the tendency of spilling of the probability distribution beyond the initial position of the detector increases as the detector has already moved then. It is to be noted that here the longest hop/shift is taken to be an Infinite Jump, denoted as IJ. This naively suggests that there is a possibility that we may achieve the probability distribution of IW, at least qualitatively, in this case if we can make the amount of shift very high. However, from Fig. 6 it is clear that if the number of detection is more, the spilling is considerably low and even for a longer time snapshot, it is hard to see any significant spilling over the right side where initially the detector was placed. This means the more number of detection affects the nature of the walk significantly.

The ratio \( \frac{f_{ns}}{f_{\infty}} \) for these sets show some interesting facts. When \( n \) is small, for small \( s \), the ratio saturates to a sufficiently high value above 1. This means the walker tries to move more towards the side where it cannot reach before. As we increase \( s \), the saturation first decreases below 1 and then again starts increasing and finally for high \( s \) approaches a value \( F \) below 1. This is shown in Fig. 10 (a). As we increase number of detection, there is more absorption and the ratio saturates to a value which for \( n = 6 \) is maximum. For a very large \( n \) as can be seen from Fig. 4.
FIG. 5. Occupation Probability snapshots for a MDQW (symmetric) for $t = 1000$ for (a) 2D1S, (b) 2D10S, (c) 2D50S, (d) 2DIJ (IJ denotes Infinite Jump). In all the plots the SIW and IW cases are shown for comparison. As $s$ increases, the qualitative approach towards IW behavior is visible. However, this is small $n$ behavior as discussed in the literature. Note that $x_D = 10$ here.

FIG. 6. Occupation Probability snapshots for a MDQW (symmetric) for 30DsS for $t = 1000$ and $t = 2000$ for $x_D = 10$. (b) for $n = 30$, there is more and more absorption and the ratio saturates to a value less than 1. With increase in number of shifts $s$, there is increase in $\left( \frac{f_{ns}}{f_{\infty}} \right)_{sat}$. However, it is checked that beyond $n = 21$ whatever be the value
FIG. 7. Variation of $\frac{f_{\text{ns}}}{f_{\infty}}$ against $t$ for (a) 2D1S, 2DIJ and (b) 30D1S, 30DIJ.

FIG. 8. $\left(\frac{f_{\text{ns}}}{f_{\infty}}\right)_{\text{sat}}$ against number of shift $s$ for $n = 2, 3, 6, 10, 15, 30$. For small $n$, $\left(\frac{f_{\text{ns}}}{f_{\infty}}\right)_{\text{sat}}$ first decreases, then increases and finally approaches $F(n)$. For high $n$ it increases and then decreases slightly to approach $F(n)$. The data shown are for $x_D = 10$. Data collapse for the same is shown in the inset.

The variation ultimately approaches a value $\bar{F}$ which is dependent on $n$ as can be seen from the figure. We found data collapse for $\gamma = 0.6$ and $\delta = 1.2$. The data collapse is shown in the inset of Fig. The approximate behavior of $\left(\frac{f_{\text{ns}}}{f_{\infty}}\right)_{\text{sat}}$ as obtained from the collapsed data is $F(n) + n^{-\delta} \bar{F} ((s - s_{\text{max}}) n^{-\gamma})$. 
C. Probability distribution ratio for $x \neq x_D$

Now let us try to see what happens for other sites $x \neq x_D$. It is clear that when we consider infinite jump the behavior of the walk will be similar to QQW. In Fig. 9 (a) it is shown. The ratio of the occupation probabilities \( \frac{f_{\infty}(x_D + r)}{f_{\infty}(x_D + r)} \) show some interesting behavior as follows:

- For $r < 0$, several peaks are observed with the peak values much greater than 1. For large negative $r$, the ratio is 1 implies that the MDQW and IW behave there in the same way.

- for $r > 0$, $\frac{f_{\infty}(x_D + r)}{f_{\infty}(x_D + r)}$ goes to zero at a finite value of $r$. The decay is smooth for large values of $n$ whatever be the value of $s$. For large $n$ and small $s$ the ratio drops off sharply close to $r = 0$. For small $n$, however, the decay is dependent on $s$. When $s$ is small, its behavior resembles any high $n$ behavior as shown in Fig. 9 (b), but for high $s$, the decay is accompanied by small oscillations. Even when the hop is not infinite but as large as 30S, the behavior is same as IJ as is shown in the 9 (d) plot.

![Graphs showing the ratio of occupation probabilities for MDQW and IW for different values of $n$ and $s$.](image)

**FIG. 9.** Variation of $\frac{f_{\infty}(x_D + r)}{f_{\infty}(x_D + r)}$ against $r$ for MDQW for (a) nDIJ, (b) nD1S, (c) 15DsS and (d) nD30S. The plots corresponding to (a) and (d) are almost similar.

As is evident from Fig. 9 (a), (b), (d) that fixing $s$ and changing $n$ would lead to a shift in the position of the peaks and would also lead to a change in their heights. For small $s$, the effect over the height is not prominent. As we start increasing $n$, at the beginning the effect of shift is strong but when $n$ is large enough the position of peaks become somewhat fixed. Keeping $n$ fixed and changing $s$ affect mostly the peak height of the first peaks although their positions do not change, except from some broadening. The approximate form of the function can be written as:

\[
\frac{f_{\infty}(x_D + r)}{f_{\infty}(x_D + r)} \sim A + Br^\nu \sin(r^\beta) \exp(-r^2/D)
\]

The corresponding parameters are shown in Table II.

As there is variation in $r$, it is important to study the correlation of occupation probabilities in $r$. The correlations for MDQW and IW are defined as:

\[
g_{ns}(x_D + r) = \frac{f_{ns}(x_D + r)}{f_{\infty}(x_D + r)}\frac{f_{ns}(x_D)}{f_{\infty}(x_D)}
g_{\infty}(x_D + r) = \frac{f_{\infty}(x_D + r)}{f_{\infty}(x_D + r)}\frac{f_{\infty}(x_D)}{f_{\infty}(x_D)}
\]
TABLE I. Approximate values of the parameters used in Eq. 5

| n  | s  | A   | B   | ν   | β    | D    |
|----|----|-----|-----|-----|------|------|
| 2  | 1  | 1.05| 0.5 | 0.2 | 0.456| 3 × 10^4|
| 15 | 1  | 1.05| 0.06| 0.2 | 0.4  | 3 × 10^7|
| 30 | 1  | 1.05| 0.06| 0.2 | 0.4  | 3 × 10^8|
| 15 | 1  | 1.7 | 0.6 | 0.19| 0.54 | 5 × 10^5|
| 15 | 1.7| 0.7 | 0.19| 0.54| 7 × 10^4|
| 30 | 1.7| 0.7 | 0.19| 0.54| 6 × 10^4|
| 30 | 1  | 1.7 | 0.75| 0.15| 0.6  | 3 × 10^5|
| 15 | 1.7| 0.86| 0.15| 0.598| 2 × 10^5|
| 30 | 1.7| 0.86| 0.15| 0.598| 2 × 10^5|

respectively. The ratio of these two correlations as in Eq. 6, i.e., \( \frac{g_{ns}}{g_{∞}} \) is an important quantity. For different combinations of \( n \) and \( s \), the behavior of this correlation ratio for different \( r \) is studied as a function of \( t \). The observed specific behavior are as follows:

- For small \( n \):
  
  - In this case if \( s \) is small, the ratio saturates above unity. For sufficiently negative \( r \), it approaches the saturation from well below that value. As \( r \) is increased, the approach towards saturation starts from a value closer to the saturation. However, as \( r > 0 \) again the ratio approaches saturation from well below that. The reason that saturation is much higher than 1 is connected with the fact that after the detection the system has a high tendency to go towards the side where the walker was initially placed. As \( s \) is low the system do not find time to relax and therefore the tendency persists.

  - For large \( s \), the correlation approaches the saturation below 1 for any \( r \) from above. This is because the system now gets enough time to relax and it tries to approach IW.

- For large \( n \):

FIG. 10. Correlation ratio \( \frac{g_{ns}}{g_{∞}} \) as a function of \( t \) for \( r = −20, −10, 10, 20 \) and 40 for (a) 2D1S and (b) 30D30S.
– If $s$ is small the correlation ratio saturates below 1. For $r < 0$ the approach towards saturation is from above and for $r > 0$ from below. The saturation decreases with $n$, as larger $n$ means greater absorption.

– For larger $s$ the correlation ratio saturates close to unity as in that case the system tries to compensate for the occupation probabilities for IW.

IV. SUMMARY

In this work we have studied the detailed effect of a detector in a quantum system from the aspect of Quantum Random Walk. As there are enormous number of experiments on QRW going on in recent years, the role and limitations of detectors is becoming a very important aspect as indicated in several studies. Like QQW, here also it is evident that the occupation probability of sites may be enhanced compared to IW by removing the detector and placing it at a different position. This is a purely quantum mechanical effect. However, this is not true always. If the efficiency of the detector, placed for example at a site towards right, is high, which means it can act longer, then the occupation probability cannot approach the IW picture on the right. It also depends on the initial position of the detector. For a particular initial condition the walker cannot go beyond $x_D$ up to a time $t_D = x_D + 2n$. After that as the detector has hopped to another position, for $t > t_D$, the walker has increased freedom to move beyond $x_D$ but is again restricted by the new position $(x_D)_1$. In this way the walker gets its freedom towards right in steps. From Fig. 1 and Fig. 2 it is obvious that most of the contributions to $x_D$ and beyond come from the density of walkers close to it. This is because the occurrence probability far away from $x_D$ are not much affected by the removal of the detector. At larger times after the replacement of the detector, the occupation probability distribution approaches the IW picture. This effect is prominent for small $n$ and very large $s$. First the local hill like structures closest to $x_D$ smoothed out and at later times slowly the further parts are affected. However, for large $n$ and small $s$, the MDQW approaches SIW. Another important limiting case is when we have infinite jump for the detector, i.e., $s \to \infty$. This case will give results similar to QQW as in [9] and is checked here.

The scaling behavior of \( \left( \frac{f_{\infty}}{f_{\text{sat}}} \right)_{\text{sat}} \) is an important quantity which is found to scale as $\frac{s^2}{n^2}$ here. The $n^{-2}$ behavior is robust against variation of $x_D$ and $s$, although for large $s$ the behavior starts from larger $n$. Also for a fixed $x_D$ and $s$ there is a certain number of detection $n^*$ beyond which the ratio \( \left( \frac{f_{\infty}}{f_{\text{sat}}} \right)_{\text{sat}} \) goes below 1 and cannot increase thereafter. The $n^*$ versus $x_D$ curve shows linear behavior. Another important result is the scaling of \( \left( \frac{f_{\infty}}{f_{\text{sat}}} \right)_{\text{sat}} \) when plotted as a function of $s$ with $n$ as a parameter. The behavior can be approximated as $F(n) + n^{-\gamma} \Delta F((s - s_{\text{max}})n^{-\gamma})$ as is shown in Fig. 3.

The behavior of occupation probability ratio $\frac{f_{\infty}}{f_{\text{sat}}}$ at sites different from the initial position of the detector shows some definite behavior as is shown in Fig. 4. This behavior can be approximated as $A + B r^\nu \sin(\nu^3) \exp(-r^2/D)$ and the variation of the parameters with $n$ and $s$ are studied.

Our work involves a detector with $p_D = 1$ and its replacement is realized in form of a movement as discussed. This work can be extended by studying response of the system when $p_D$ is some definite function of time and thereby comparing it to the experimental data available, if any.

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