GRADIENT ESTIMATE OF SUBELLIPTIC HARMONIC MAPS WITH POTENTIAL

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Abstract. In this paper, we investigate subelliptic harmonic maps with potential from noncompact complete sub-Riemannian manifolds corresponding to totally geodesic Riemannian foliations. Under some suitable conditions, we give the gradient estimates of these maps and establish a Liouville type result.

1. Introduction

Subelliptic harmonic maps were first introduced by Jost and Xu in [JX98], which can be regarded as a natural counterpart of harmonic maps in the realm of sub-Riemannian geometry. Barletta et al. [BDU01] studied pseudo-harmonic maps from pseudoconvex CR manifolds, which are actually subelliptic harmonic maps defined with respect to the Webster metrics. Dong [Don21] obtained Eells-Sampson type results for subelliptic harmonic maps in some general cases. On the other hand, to study some important physics phenomena, the harmonic maps with potential were introduced in [FR97]. Some existence results for harmonic maps with potential can be found in [FRR00] and [Che99].

Suppose that \((M, H, g_H)\) is a sub-Riemannian manifold with a smooth measure \(d\mu\) and \((N, h)\) is a Riemannian manifold. Given a function \(G \in C^\infty(N)\), we consider the following energy functional

\[
E_{G}(u) = \frac{1}{2} \int_{M} [ |du_H|^2 - 2G(u) ] d\mu
\]

where \(u : M \to N\) is a smooth map and \(du_H = du|_H\). A smooth map \(u : (M, H, g_H) \to (N, h)\) is referred to as a subelliptic harmonic map with potential \(G\) if it is a critical point of \((1.1)\). The subelliptic harmonic maps with potential can be viewed as a generalization of both harmonic maps with potential and subelliptic harmonic maps. Some Eells-Sampson type existence results of such maps have been achieved in [DLY22].

Many Liouville type theorems have been established in both Riemannian geometry and sub-Riemannian geometry. One may often derive Liouville type results directly from gradient estimates. At first, Yau [Yau75] showed that on a complete Riemannian manifold with nonnegative Ricci curvature, there doesn’t exist any nontrivial harmonic function bounded from one side. Later, Cheng [Che80] proved the Liouville theorem for harmonic maps whose images satisfy the sublinear growth condition when the source manifolds
have nonnegative Ricci curvature and the target manifolds have nonpositive sectional curvature. Chen [Che98] gave Liouville theorems for harmonic maps with potential whose images are contained in a regular ball on a manifold with sectional curvature bounded from above or a horoball on a Cartan-Hadamard manifold. Ren [Ren20] achieved a Liouville type results for positive pseudo-harmonic functions on complete pseudo-Hermitian manifolds. Chong et al. [CDRZ20] established a Liouville theorem for pseudo-harmonic maps which generalizes the one for harmonic maps by Choi [Cho82]. Zou [Zou21] got a gradient estimate of the subelliptic harmonic maps when the source manifolds are step-2 complete totally geodesic Riemannian foliations. In this paper, we aim to give gradient estimates of subelliptic harmonic maps with potential, and establish Liouville type results for them.

For a Riemannian foliation \((M, g; \mathcal{F})\) with a bundle-like metric \(g\), we define \(H = (T\mathcal{F})^\perp\) as the horizontal subbundle of the foliation \(\mathcal{F}\) with respect to \(g\) and denote \(g_H = g|_H\). It is easy to check that if \(H\) is bracket generating for \(TM\), then we have a sub-Riemannian manifold \((M, H, g_H; g)\) corresponding to \((M, g; \mathcal{F})\). In this paper, the subbundle \(H\) is always required to have the bracket generating property for \(TM\). We call \(M\) a step-\(r\) sub-Riemannian manifold if sections of \(H\) together with their Lie brackets up to order \(r\) spans \(T_xM\) at each point \(x\). Henceforth, we always assume that the source manifold \((M^{m+d}, H, g_H; g)\) is a complete step-\(r\) sub-Riemannian manifold whose sub-Riemannian structure comes from a totally geodesic Riemannian foliation and the target manifold \((N, h)\) is a complete Riemannian manifold.

Suppose the sectional curvature of \((N, h)\) is bounded above by \(\bar{K}\), where \(\bar{K} \geq 0\) is a constant. Let \(B_q(\tau)\) denote a geodesic ball of radius \(\tau < \pi/2\sqrt{\bar{K}}\) centered at \(q \in N\), which lies inside the cut locus of \(q\). Set

\[
\phi(t) = \begin{cases} 
(1 - \cos \sqrt{\bar{K}}t)/\bar{K}, & \bar{K} > 0 \\
\bar{K} = 0 
\end{cases}
\]

and

\[
\psi(\cdot) = \phi \circ \rho(\cdot)
\]

where \(\rho(\cdot) = d_N(\cdot, q)\) denotes the Riemannian distance from \(q\) in \(N\). Let \(M_H\) and \(m_H\) be constants such that

\[
\text{Hess} G(Y, Y) \leq M_H G(Y, Y)_h, \quad \forall Y \in T_yN, y \in B_q(\tau)
\]

and

\[
m_G = \sup_{y \in B_q(\tau)} \langle \nabla\psi, \nabla G \rangle_h(y)
\]

where \(\nabla\) is the Riemannian connection on \((N, h)\), \(\text{Hess} G\) is the Hessian matrix of \(G\) with respect to \(\nabla\). Choosing a constant \(\delta > 0\) such that \(\tau < \delta < \pi/2\sqrt{\bar{K}}\), we set

\[
\beta_1 = \cos \sqrt{\bar{K}} \tau - \cos \sqrt{\bar{K}} \delta, \quad \beta_2 = 1 - \cos \sqrt{\bar{K}} \delta
\]
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and

(1.2) \quad f = \beta_2 - \bar{K}\psi.

It is easy to see that

(1.3) \quad 0 < \beta_1 < f < \beta_2 < 1.

Let $B_p(R)$ denote the Riemannian geodesic ball of radius $R$ centered at $p \in M$, we have the following

**Theorem 1.1.** Let $(M^{m+d}, H, g_H; g)$ be a noncompact complete sub-Riemannian manifold corresponding to a totally geodesic Riemannian foliation with

\[ \text{Ric}_H \geq -k_1, \quad \text{and} \quad |T|, |\text{div}_HT| \leq k_2 \]

where $\text{Ric}_H$ and $T$ are the horizontal Ricci curvature and the torsion of the generalized Bott connection $\nabla_B^H$ respectively (see Sect. 2 for the detailed definitions), $\text{div}_HT$ is the horizontal divergence of $T$, $k_1, k_2 \geq 0$ are constants. Let $N, G$ and $\bar{B}_q(\tau)$ be as above. Suppose $u : M \to \bar{B}_q(\tau) \subset N$ is a subelliptic harmonic map with potential $G$. If

\[ k_1 + M_G + \frac{2k_2^2}{(1 - \beta_2)\beta_1} + k_2 + \bar{K}\max \left( \frac{m_G}{\beta_1}, \frac{m_G}{\beta_2} \right) < 0, \]

then $u$ must be a constant map and $u(M) \subset \Sigma_G := \{ y \in N | \tilde{\nabla}_G(y) = 0 \}$. Otherwise, we have the following estimate

\[ \sup_{B_p(R)} e_H(u) \leq C \left( k_1 + M_G + \frac{4k_2^2}{(1 - \beta_2)\beta_1} + k_2 + \bar{K}\max \left( \frac{m_G}{\beta_1}, \frac{m_G}{\beta_2} \right) + \frac{1 + R}{R^2} \right) \]

where $C(m, k_1, k_2, \bar{K}, \tau)$ is a positive constant.

If $H$ is 2-step bracket generating for $TM$, adding some suitable condition on $T$, we may get a better estimate as follows.

**Theorem 1.2.** Let $(M^{m+d}, H, g_H; g)$ be a step-2 noncompact complete sub-Riemannian manifold corresponding to a totally geodesic Riemannian foliation with

\[ \text{Ric}_H \geq -k_1, \quad |T| \leq k_2, \quad \text{and} \quad \text{div}_HT = 0 \]

where $k_1, k_2 \geq 0$ are constants. Let $N, G$ and $\bar{B}_q(\tau)$ be as above. Suppose $u : M \to \bar{B}_q(\tau) \subset N$ is a subelliptic harmonic map with potential $G$. We have the following estimate

\[ \sup_{B_p(R)} e_H(u) \leq C(m, k_1, k_2, \bar{K}, \tau) \left( k_1 + M_G + \frac{1 + R}{R^2} \right) \]

where $C(m, k_1, k_2, \bar{K}, \tau)$ is a positive constant.

A direct application of Theorem 1.2 is the following Liouville type result.
Corollary 1.3. Let \((M^{m+d}, H, g_H; g)\) be a step-2 noncompact complete sub-Riemannian manifold corresponding to a totally geodesic Riemannian foliation with
\[
\text{Ric}_H \geq 0, \quad |T| \leq k_2, \quad \text{and} \quad \text{div}_H T = 0
\]
where \(k_2 \geq 0\) is a constant. Let \(N\) and \(\overline{B}_q(\tau)\) be as above, and let \(G\) be a smooth function with \(\text{Hess}G \leq 0\) on \(N\). If \(u : M \to \overline{B}_q(\tau) \subset N\) is a subelliptic harmonic map with potential \(G\), then \(u\) must be a constant map and \(u(M) \subset \Sigma_G\).

Note that Theorem 1.2 and Corollary 1.3 hold true in the case that the source manifolds are Sasakian manifolds. Since Sasakian manifolds can be seen as step-2 sub-Riemannian manifolds and the Reeb foliation on any Sasakian manifold is actually totally geodesic with the bundle-like metric and of Yang-Mills type, that is, \(\text{div}_H T = 0\) (cf.
\[\text{Bau16}\]).

2. Preliminaries

A sub-Riemannian manifold is defined as a triple \((M, H, g_H)\), where \(M\) is a connected smooth manifold, \(H\) is a subbundle bracket generating for \(TM\), and \(g_H\) is a smooth fiberwise metric on \(H\). According to \[\text{Str86}\], there always exists a Riemannian metric \(g\) on \(M\) such that \(g|_H = g_H\), where \(g\) is referred to as a Riemannian extension of \(g_H\). From now on, we always fix a Riemannian extension \(g\) on the sub-Riemannian manifold \((M, H, g_H)\), and consider the quadruple \((M, H, g_H, g)\). According to \(g\), the tangent bundle \(TM\) has the following orthogonal decomposition:
\[
TM = H \oplus V.
\]
which induces the projections \(\pi_H : TM \to H\) and \(\pi_V : TM \to V\). Such \(H\) and \(V\) are called the horizontal distribution and the vertical distribution, respectively.

On a sub-Riemannian manifold \((M^{m+d}, H, g_H; g)\), there are two canonical distances. One is the Carnot-Carathéodory distance \(d_{CC}\) of the sub-Riemannian structure \((H, g_H)\) (cf.
\[\text{Str86}\]), and the other is the Riemannian distance \(d_{Rm}\) of \(g\). It was proved in \[\text{NSW85}\] that \(d_{CC}\) and \(d_{Rm}\) induce the same topology, if \(H\) is bracket generating for \(TM\). Note that the Riemannian distance has better regularity and its variational theory is well studied in Riemannian geometry. In this paper, we restrict our discussion to the Riemannian distance \(d_{Rm}\) on \(M\) and \(B_p(R)\) which is the Riemannian geodesic ball of radius \(R\) centered at \(p \in M\).

The generalized Bott connection on sub-Riemannian manifold is given by (cf.
\[\text{Bau16}, \text{BF13}, \text{Don21}\])
\[
\nabla^B_X Y = \begin{cases} 
\pi_H(\nabla^R_X Y), & X, Y \in \Gamma(H) \\
\pi_H([X, Y]), & X \in \Gamma(V), Y \in \Gamma(H) \\
\pi_V([X, Y]), & X \in \Gamma(H), Y \in \Gamma(V) \\
\pi_V(\nabla^R_X Y), & X, Y \in \Gamma(V)
\end{cases}
\]
where $\nabla^R$ denotes the Riemannian connection of $g$. It is convenient for computations on sub-Riemannian manifold by using the above connection, since $\nabla^g$ preserves the decomposition (2.1). However, in general, $\nabla^g$ does not preserve the Riemannian metric $g$. The torsion and the curvature of $\nabla^g$ can be expressed as (c.f. [Don21])

$$T(X,Y) = \nabla^g_X Y - \nabla^g_Y X - [X,Y]$$

and

$$R(X,Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z.$$

respectively, for $X, Y \in \Gamma(TM)$. Choose a local orthonormal frame field $\{e_A\}_{A=1}^{m+d}$ on an open domain $\Omega$ of $(M,g)$ such that $\text{span}\{e_i\}_{i=1}^m = H$, and thus $\text{span}\{e_{\alpha}\}_{\alpha=m+1}^{m+d} = V$. We call such a frame field an adapted frame field for $(M,H,g_H)$. Denote its dual frame field by $\{\omega^A\}_{A=1}^{m+d}$. From now on, we will make use of the following convention on the ranges of induces in $M$:

$$1 \leq A, B, C, \ldots, \leq m + d; \quad 1 \leq i, j, k, \ldots, \leq m;$$

$$m + 1 \leq \alpha, \beta, \gamma, \ldots, \leq m + d,$$

and the Einstein summation convention. Using the frame field, the torsion components and the curvature components are given by

$$T^A_{BC} = \langle T(e_B,e_C),e_A \rangle$$

and

$$R^A_{BCD} = \langle R(e_C,e_D)e_B,e_A \rangle$$

respectively. Consequently, the horizontal divergence of $T$ is

$$\text{div}_HT(X) = \text{trace}_H(\nabla^g_{e_i}T)(X,e_i) \quad \text{for} \quad X \in \Gamma(TM),$$

and horizontal Ricci tensor is

$$Ric_H(X) = \sum_{i=1}^m R(X,e_i)e_i \quad \text{for} \quad X \in \Gamma(TM).$$

The horizontal gradient of a smooth function $f$ is defined by

$$\nabla^H f = \pi_H(\nabla^g f).$$

As we know, the divergence of a vector field $X$ on $M$ is given by

$$\text{div}_g X = \sum_{A=1}^{m+d} \{e_A \langle X,e_A \rangle - \langle X,\nabla^R_{e_A}e_A \rangle \}.$$

Then one can define sub-Laplacian of a function $f$ on $(M,H,g_H;g)$ as

(2.3) \hspace{1cm} \Delta_H f = \text{div}_g(\nabla^H f) = \text{trace}(\nabla^g \text{df}|_{H \times H}) - \zeta f$$

where $\zeta = \pi_H(\sum_{\alpha} \nabla^R_{e_\alpha}e_\alpha)$ is referred to as the mean curvature vector field of the vertical distribution $V$. 

We call $M$ a step-$r$ sub-Riemannian manifold if sections of $H$ together with their Lie brackets up to order $r$ spans $T_xM$ at each point $x$. Let $S(V) = \{v \in V : \|v\|_g = 1\}$ denote the unit sphere bundle of the vertical bundle $V$. For any $v \in S(V)$, the $v$-component of $T(\cdot, \cdot)$ is given by $T^v(\cdot, \cdot) = \langle T(\cdot, \cdot), v \rangle$. Then we have a smooth function $\eta(v) = \frac{1}{2} \| T^v \|_g^2 : S(V) \to \mathbb{R}$. If $(M^{m+d}, H, g_H; g)$ is a step-2 sub-Riemannian manifold, we have the following

**Lemma 2.1.** ([Don21, Lemma 6.6]) $H$ is 2-step bracket generating if and only if $\eta(v) > 0$ for each $v \in S(V)$.

The above lemma suggests that $\eta(v)$ achieves a positive minimal value $\eta_{\text{min}}$ on $S(V)$.

For a Riemannian foliation $(M, g; \mathcal{F})$, let $H$ be the orthogonal component of $V = T\mathcal{F}$ and $g_H$ be the restriction of $g$ to $H$. Then we have a sub-Riemannian manifold $(M, H, g_H; g)$ corresponding to $(M, g; \mathcal{F})$. From [GW09, Lemma 1.4.1], one may find that $R^j_{\alpha k} = 0$ and $\text{Ric}_H$ is symmetric. Furthermore, if $\mathcal{F}$ is totally geodesic, then the vector field $\zeta = 0$ and $\nabla g = 0$. Readers may refer to [Bau16], [Don21] for details.

We also need the estimate of sub-Laplacian of Riemannian distance on $M$. Although sub-Laplacian comparison theorems have been investigated for some special cases in [AL15], [BGKT18], [CKLT19], [LL18], [CDRZ20], there is no satisfactory comparison theorem for a sub-Riemannian manifold in general up to now. Fortunately, when $M$ is a complete totally geodesic Riemannian foliation, according to [HY22, Theorem 3.1], one may get the following result:

$$\Delta_H r \leq C (1 + \frac{1}{r}), \quad \text{on} \quad B_p(R) \setminus \text{Cut}(p)$$

where $C(m, k_1, k_2)$ is a positive constant, $r$ is the Riemannian distance from the fixed point $p$ and $\text{Cut}(p)$ is the cut locus of $p$.

Let $(N, h)$ be a Riemannian manifold with the Riemannian connection $\hat{\nabla}$ and the Riemannian curvature $\hat{R}$. We choose an orthonormal frame field $\{\tilde{e}_I\}_{I=1, \ldots, n}$ in $(N, h)$ and make use of the following convention on the ranges of indices in $N$:

$$I, J, K = 1, \ldots, n.$$

For a smooth map $u : M \to N$, in terms of the frame fields in $M$ and $N$, the differential $du$ and the second fundamental form $\beta$ can be written as

$$du = u^I_A \omega^A \otimes \tilde{e}_I,$$

and

$$\beta = u^I_{AB} \omega^A \otimes \omega^B \otimes \tilde{e}_I$$

respectively. Apart from the differential $du$, we also introduce two partial differentials $du_H = du|_H \in \Gamma(H^* \otimes u^{-1}TN)$ and $du_V = du|_V \in \Gamma(V^* \otimes u^{-1}TN)$. Then, we get

$$|du_H|^2 = (u^I_H)^2, \quad |du_V|^2 = (u^I_V)^2, \quad |du|^2 = (u^I_A)^2.$$
Define
\[ e_H(u) = \frac{1}{2} |du_H|^2, \quad e_V(u) = \frac{1}{2} |du_V|^2, \quad e(u) = \frac{1}{2} |du|^2. \]

For any potential function \( G \in C^\infty(N) \), we introduce the following energy:
\[ (2.5) \quad E_G(u) = \int_M [e_H(u) - G(u)] \, dv_g. \]

The energy \( E_G(u) \) is called horizontal energy with potential \( G \).

**Definition 2.2.** \([DLY22]\) A map \( u : (M, H, g_H) \to (N, h) \) is called a subelliptic harmonic map with potential \( G \) if it is a critical point of the energy \( E_G(u) \).

The Euler-Lagrange equation of (2.5) is
\[ (2.6) \quad \tau_G(u) = \tau_H(u) + (\tilde{\nabla} G)(u) = \beta(e_i, e_i) - du(\zeta) + (\tilde{\nabla} G)(u) = 0 \]
where \( \tau_H(u) \) is the subelliptic tension field associated with the horizontal energy (c.f. [Don21]). Therefore, we have the following equivalent characterization of subelliptic harmonic maps with potential \( G \).

**Proposition 2.3.** \([DLY22]\) A map \( u : (M, H, g_H; G) \to (N, h) \) is a subelliptic harmonic map with potential \( G \) if and only if it satisfies the Euler-Lagrange equation
\[ \tau_G(u) = 0. \]
We call \( \tau_G(u) \) the subelliptic tension field of \( u \) with potential \( G \).

For our purpose, we need the following Bochner type inequality for \( e(u) \).

**Lemma 2.4.** Let \((M^{m+d}, H, g_H; g)\) be a complete sub-Riemannian manifold which is corresponding to a totally geodesic Riemannian foliation with
\[ \text{Ric}_H \geq -k_1, \quad \text{and} \quad |T|, |\text{div}_H T| \leq k_2. \]
Suppose \((N, h)\) is a complete Riemannian manifold with \( K_N \leq \bar{K} \) and \( u : M \to N \) is a subelliptic harmonic map with potential \( G \), then one has
\[ (2.7) \quad \Delta_H e(u) \geq (1 - k_2 \epsilon)(u_{Ak}^I)^2 - \left( 2M_G + 2k_1 + \frac{(2 + \epsilon)k_2}{\epsilon} \right) e(u) - 4\bar{K}e(u) \cdot e_H(u) \]
for any given \( \epsilon > 0 \). In particular, if \( H \) is 2-step bracket generating for \( TM \) and \( \text{div}_H T = 0 \), we have
\[ (2.8) \quad \Delta_H e_H \geq (1 - \epsilon_1)(u_{ik}^I)^2 - \left( 2k_1 + \frac{C}{\epsilon_2} + 2M_G \right) e_H(u) \]
\[ + \frac{1}{2} \epsilon_1 \eta_{\text{min}} e_V(u) - C\epsilon_2(u_{Ak}^I)^2 - 4\bar{K}e_H^2(u) \]
and
\[ (2.9) \quad \Delta_H e_V \geq (u_{Ak}^I)^2 - 2M_G e_V(u) - 4\bar{K}e_H(u) \cdot e_V(u) \]
for any given \( 0 < \epsilon_1 < 1 \) and \( \epsilon_2 > 0 \), where \( C(k_2) \) is a positive constant only depending on \( T \).
Proof of Lemma 2.4. Denote covariant derivatives of \( \zeta^k \) and \( T_{BC}^A \) by \( \zeta_{,A}^k \) and \( T_{BC;D}^A \) respectively. From [Don21], we know

\[
\Delta_H e_H(u) = (u^I_{ik})^2 + u^I_{i} \tau_{H,i} + u^I_{ij} \zeta_{,i}^k u_k + \zeta^k u^I_{i} u^I_{a} T_{a}^a \\
+ u^I_{i} u^I_{j} R_{ki}^j + 2 u^I_{i} u^I_{a} T_{ik}^a - u^I_{i} u^I_{a} K R_{KJL}^K u^I_{k} u^I_{L} + u^I_{a} u^I_{i} T_{ik,j}
\]

and

\[
\Delta_H e_V(u) = (u^I_{ak})^2 + u^I_{a} \tau_{H,a} + u^I_{i} \zeta_{,a}^k u_k + \zeta^k u^I_{a} R_{kak}^j - u^I_{a} u^I_{i} K R_{KJL}^K u^I_{a} u^I_{k}.
\]

Since \( \tau^I = \tau_{H}^I + [(\nabla G)(u)]^I \), then

\[
\Delta_H e_H(u) = (u^I_{ik})^2 + u^I_{i} \tau_{H,i} + u^I_{ij} \zeta_{,i}^k u_k + \zeta^k u^I_{i} u^I_{a} T_{a}^a \\
+ u^I_{i} u^I_{j} R_{ki}^j + 2 u^I_{i} u^I_{a} T_{ik}^a - u^I_{i} u^I_{a} K R_{KJL}^K u^I_{k} u^I_{L} + u^I_{a} u^I_{i} T_{ik,j}
\]

and

\[
\Delta_H e_V(u) = (u^I_{ak})^2 + u^I_{a} \tau_{H,a} - u^I_{i} G_{IJ} u^I_{a} \\
+ u^I_{a} \zeta_{,a}^k u_k + \zeta^k u^I_{a} R_{kak}^j - u^I_{a} u^I_{i} K R_{KJL}^K u^I_{a} u^I_{k}.
\]

where \( \text{Hess } G = (G_{IJ}) \). Since \( \zeta = 0 \) and \( R^j_{kak} = 0 \), we get

\[
u^I_{i} \zeta_{,i}^k u_k + \zeta^k u^I_{i} u^I_{a} T_{a}^a = 0
\]

and

\[
u^I_{a} \zeta_{,a}^k u_k + u^I_{a} u^I_{i} R_{kak}^j = 0.
\]

For any given \( \epsilon > 0 \), by Schwarz inequality, we obtain

\[
u^I_{i} G_{IJ} u^I_{i} + u^I_{a} G_{IJ} u^I_{a} \leq 2 M_G \cdot e(u),
\]

\[
u^I_{i} u^I_{j} R^j_{kik} \geq -2 k_1 e_H(u) \geq -2 k_1 e(u),
\]

\[
2 u^I_{i} u^I_{j} T^a_{ik} + u^I_{a} u_{i} T^a_{ik,a} \geq - \frac{(2 + \epsilon) k_2}{\epsilon} e_H(u) - k_2 e_V(u) - k_2 e(u^I_{ak})^2 \\
\geq - \frac{(2 + \epsilon) k_2}{\epsilon} e(u) - k_2 e(u^I_{ak})^2,
\]

\[
u^I_{i} u^I_{a} K R^j_{KJL} u^I_{k} u^I_{L} + u^I_{a} u^I_{i} K R^j_{KJL} u^I_{a} u^I_{k} \leq 4 K e(u) \cdot e_H(u).
\]

The above estimates give (2.7). Due to [Don21] Section 4, we have the following equality

\[
(2.10) \quad u^I_{ij} - u^I_{ji} = u^I_{a} T_{ij,}^a.
\]
When $H$ is 2-step bracket generating for $TM$, in terms of Lemma 2.1 and (2.10), we know that
\[
(u^I_{ik})^2 \geq \frac{1}{2} \sum_I \sum_{i<j} ((u^I_{ij} + u^I_{ji})^2 + (u^I_{ij} - u^I_{ji})^2)
\]
\[
\geq \frac{1}{2} \sum_I \sum_{i<j} \sum_{\alpha} (u^I_{\alpha})^2 (T^\alpha_{ij})^2
\]
\[
= \frac{1}{2} \sum_I \sum_{\alpha} (u^I_{\alpha})^2 \eta(e_{\alpha})
\]
\[
\geq \eta_{\text{min}} e_V(u).
\]
Using Schwarz inequality and $\text{div}_H T = 0$, we get (2.8) and (2.9). \hfill \Box

3. Proof of the theorem

Proof of Theorem 1.1. Set
\[
A(x) = \frac{e(u)(x)}{f^2(u)(x)}
\]
where $f$ is defined in (1.2). To simplify the notations, we write $e = e(u), e_H = e_H(u), f = f(u)$. By a direct computation, we have
\[
\Delta_H A = \frac{\Delta_H e}{f^2} \frac{4\nabla H e \cdot \nabla H f}{f^3} - \frac{2e \Delta_H f}{f^3} + \frac{6e |\nabla H f|^2}{f^4}.
\]

In terms of (1.2) and Riemannian comparison theorem, we get
\[
\Delta_H f = \text{Hess}(f)(du_H, du_H) + ((\tilde{\nabla} f)(u), \tau_H(u))
\]
\[
= \text{Hess}(f)(du_H, du_H) - ((\tilde{\nabla} f)(u), (\tilde{\nabla} G)(u))
\]
\[
\leq -\bar{K} \text{Hess}(\psi)(du_H, du_H) + \bar{K} (|\tilde{\nabla} \psi|(u), (\tilde{\nabla} G)(u))
\]
\[
\leq -2\bar{K}(1 - \beta^2 + f) e_H + \bar{K} m_G
\]

Applying (2.7) and (3.2) to (3.1), we have
\[
\Delta_H A \geq - \frac{2k_1 e}{f^2} - \frac{4\bar{K} e \cdot e_H}{f^2} - \frac{2e}{f^2} M_G - \frac{(2+\epsilon) k_2 e}{f^3} + \frac{4\bar{K}(1 - \beta^2 + f) e \cdot e_H}{f^3}
\]
\[
- \frac{2\bar{K} e}{f^3} m_G + \left( \frac{1 - k_2 e}{f^2} (u_A^I u_{Ak})^2 \right) - \frac{4\nabla H e \cdot \nabla H f}{f^3} + \frac{6e |\nabla H f|^2}{f^4}
\]
where $\epsilon$ is a positive constant to be decided. Note that
\[
\nabla H e = u_A^I u_{Ak} \leq \sqrt{(u_A^I u_{Ak})^2} = |u_A^I| \sqrt{2e}
\]
and
\[
|\nabla H f|^2 \leq 2|\nabla f|^2 e_H \leq 2\bar{K} e_H.
\]
By using \([3.4]\) and \([3.5]\), the last term on the right-hand side of \([3.3]\) becomes

\[
\cdots = \frac{(1 - k_2 \epsilon)(u_{Ak}'^2)}{f^2} - \frac{2(1 - k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^3} + \frac{2(1 - k_2 \epsilon)|\nabla^H f|^2}{f^4}
\]

\[
- \frac{2(1 + k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^3} + \frac{(4 + 2k_2 \epsilon)|\nabla^H f|^2}{f^4}
\]

\[
\geq \frac{(1 - k_2 \epsilon)}{f^2} \left[ (u_{Ak}')^2 - 2|u_{Ak}'|\sqrt{2\epsilon} \cdot \frac{|\nabla^H f|}{f^3} + 2\epsilon \frac{|\nabla^H f|^2}{f^4} \right]
\]

\[
- \frac{2(1 + k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^3} - \frac{2k_2 \epsilon e|\nabla^H f|^2}{f^4}
\]

\[
\geq - \frac{2(1 + k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^3} - \frac{4k_2 \epsilon e \cdot e_H}{f^4}.
\]

Therefore, we get

\[
\Delta_H A \geq \frac{4\bar{K}(1 - \beta_2)\epsilon \cdot e_H}{f^3} - \frac{4k_2 \epsilon e \cdot e_H}{f^4} - \frac{2(1 + k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^4} - \frac{2(1 + k_2 \epsilon)|\nabla^H e|^2}{f^4}
\]

\[
- 2 \left( k_1 + M_G + \frac{(2 + \epsilon)k_2}{\epsilon} + \frac{\bar{K}m_G}{f} \right) e \frac{1}{f^2}
\]

\[
\geq 4\bar{K}[ (1 - \beta_2)f - k_2 \epsilon ] \frac{Ae_H}{f^2} - \frac{2(1 + k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^4} - \frac{2(1 + k_2 \epsilon)|\nabla^H e|^2}{f^4}
\]

\[
- 2 \left( k_1 + M_G + \frac{(2 + \epsilon)k_2}{\epsilon} + \frac{\bar{K}m_G}{f} \right) A.
\]

Choosing \(0 < \epsilon < \frac{(1 - \beta_2)\beta_1}{k_2}\), we have

\[
\Delta_H A \geq C_0 \frac{Ae_H}{f^2} - 2s_G A - \frac{2(1 + k_2 \epsilon)\nabla^H e \cdot \nabla^H f}{f^4}
\]

where \(C_0 = 4\bar{K}[ (1 - \beta_2)f - k_2 \epsilon ] > 0 \) and \(s_G(\epsilon) = k_1 + M_G + \frac{(2 + \epsilon)k_2}{\epsilon} + \bar{K}\max \left( \frac{m_G}{\beta_1^2}, \frac{m_G}{\beta_2^2} \right) \). It is easy to see that \(s_G(\epsilon)\) is a decreasing function of \(\epsilon\) for \(\epsilon > 0\).

Choose a cut-off function

\[
\varphi |_{[0,1]} = 1, \varphi |_{[2,\infty)} = 0, -C'_1|\varphi|^{\frac{3}{2}} \leq \varphi' \leq 0
\]

where \(C'_1\) is a positive constant. Let \(\chi(r) = \varphi\left(\frac{r}{R}\right)\), owing to \([2.4]\), we find that

\[
\frac{|\nabla^H \chi|^2}{\chi} \leq \frac{C_1}{R^2}
\]

and

\[
\Delta_H \chi \geq -\frac{C_1}{R^2} \text{ in } B(2R) \setminus \text{Cut}(p)
\]

where \(C_1(m,k_1,k_2)\) is a positive constant.
Let $x_R$ be a maximum point of $\chi A(x) = \chi(r(x))A(x)$ in $B_R(p)$, then at $x_R$, we have
\begin{equation}
\nabla^H(\chi A)(x_R) = \chi \nabla^H A + A \nabla^H \chi = 0
\end{equation}
and
\begin{equation}
\Delta_H(\chi A)(x_R) \leq 0.
\end{equation}
Substituting (3.6), (3.7), (3.8) and (3.9) into (3.10) yields, at point $x_R$
\begin{equation}
0 \geq A \Delta_H \chi + \chi \Delta_H A + 2 \nabla^H \chi \cdot \nabla^H A
= A \Delta_H \chi + \chi \Delta_H A - 2 \left( \frac{\nabla^H \chi}{\chi} \right) A
\geq \left( C_0 \frac{e_H}{f^2} A - 2s_G A - 2(1 + k_2^2) \frac{\nabla^H A \cdot \nabla^H f}{f} \right) \chi
- C_1 \frac{R}{A} - 2 \frac{C_1}{R^2} A.
\end{equation}
Multiplying both sides with $\chi$, we obtain
\begin{equation}
0 \geq C_0 \frac{e_H}{f^2} A \chi^2 - 2s_G A \chi^2 - C_2 \frac{R}{A} (1 + R) A \chi - 2(1 + k_2^2) \frac{\nabla^H A \cdot \nabla^H f}{f} \chi^2
\end{equation}
where $C_2(m, k_1, k_2)$ is a positive constant. From (3.10) and (3.9), we know
\begin{equation}
- \chi \frac{\nabla^H A \cdot \nabla^H f}{f} = \frac{A \nabla^H \chi \cdot \nabla^H f}{f}
\geq - A |\nabla^H \chi| \frac{|\nabla^H f|}{f}
\geq - A |\nabla^H \chi| \frac{\sqrt{2K e_H}}{f}
\geq - \frac{C_3}{R} \sqrt{\frac{e_H}{f^2}} A
\end{equation}
where $C_3(m, k_1, k_2, \bar{K})$ is a positive constant. By (3.11) and (3.12), we have
\begin{equation}
0 \geq C_0 \frac{e_H}{f^2} A \chi - 2s_G \chi \cdot A \chi - C_2 \frac{R}{A} (1 + R) \chi - C_4 \frac{R}{f} \sqrt{\frac{e_H}{f^2}} \chi
\end{equation}
where $C_4(m, k_1, k_2, \bar{K}, \epsilon)$ is a positive constant. We assume that $A \chi(x_R) > 0$, otherwise $\chi$ is a constant. Multiplying both sides of (3.13) with $\frac{1}{A \chi}$, we get
\begin{equation}
0 \geq C_0 \frac{e_H \chi}{f^2} - 2s_G \chi - C_2 \frac{R}{f^2} (1 + R) - C_4 \frac{R}{f} \sqrt{\frac{e_H \chi}{f^2}}.
\end{equation}
If $s_G\left(\frac{(1 - \beta_2)\beta_1}{k_2}\right) < 0$, that is,
\begin{equation}
k_1 + M_G + \frac{2k_2^2}{(1 - \beta_2)\beta_1} + k_2 + \bar{K} \max\left(\frac{m_G}{\beta_1}, \frac{m_G}{\beta_2}\right) < 0,
\end{equation}
by the monotonicity of \( s_G(\epsilon) \), we can choose a fixed number \( 0 < \epsilon_0 < \frac{(1-\beta_2)\beta_1}{k_2} \), such that \( s_G(\epsilon_0) \leq 0 \). Then, we obtain

\[
0 \geq C_0 \frac{e^{H \chi}}{f^2} - \frac{C_4}{R} \sqrt{\frac{e^{H \chi}}{f^2}} - \frac{C_2}{R^2} (1 + R).
\]

It follows easily that

\[
e^{H \chi} \leq \frac{C_5}{R^2} (1 + R)
\]

where \( C_5(m, k_1, k_2, \bar{K}, \epsilon, \tau) \) is a positive constant. Since \( f < \beta_2 \), we deduce that

\[
\frac{1}{\beta_2^2} \sup_{B_p(R)} e_H(u) \leq \sup_{B_p(2R)} \frac{e_H \chi(u)}{f^2(u)} \leq \frac{C_5}{R^2} (1 + R).
\]

Letting \( R \to \infty \), we conclude that \( u \) must be a constant map and therefore \( u(M) \subset \Sigma_G \).

If \( s_G(\frac{(1-\beta_2)\beta_1}{k_2}) \geq 0 \), then \( s_G(\epsilon) > 0 \) for \( 0 < \epsilon < \frac{(1-\beta_2)\beta_1}{k_2} \). It follows from (3.14) that

\[
0 \geq C_0 \frac{e^{H \chi}}{f^2} - \frac{C_4}{R} \sqrt{\frac{e^{H \chi}}{f^2}} - \left( 2s_G + \frac{C_2}{R^2} (1 + R) \right).
\]

Consequently,

\[
e^{H \chi} \leq C_6(2s_G + \frac{1 + R}{R^2})
\]

where \( C_6(m, k_1, k_2, \bar{K}, \epsilon, \tau) \) is a positive constant. Setting \( \epsilon = \frac{(1-\beta_2)\beta_1}{2k_2} \), we have the following estimate

\[
\sup_{B_p(R)} e_H(u) \leq C \left( k_1 + M_G + \frac{4k_2^2}{(1 - \beta_2)\beta_1} + k_2 + \bar{K} \max \left( \frac{m_G}{\beta_1}, \frac{m_G}{\beta_2} \right) + \frac{1 + R}{R^2} \right)
\]

where \( C(m, k_1, k_2, \bar{K}, \epsilon) \) is a positive constant. \( \square \)

In order to prove Theorem 1.2, we also need the following lemma.

**Lemma 3.1.** \([\text{CDRZ20, Zou21}]\) For any \( 0 < \tau < \frac{\tau}{2\sqrt{K}} \), there exist \( \nu \in [1, 2) \), \( b > \phi(\tau) \) and \( \delta > 0 \) only depending on \( \tau \) such that

\[
\nu \frac{\cos(\sqrt{K} t)}{b - \phi(t)} - 2\bar{K} > \delta, \quad \forall t \in [0, \tau].
\]

Furthermore, if \( u : M \to \bar{B}_q(\tau) \subset N \) is a subelliptic harmonic map with potential \( G \), then we have

\[
\nu \frac{\Delta_H(\psi \circ u)}{b - \psi \circ u} - 4\bar{K} e_H(u) > 2\delta e_H(u).
\]

**Proof of Theorem 1.2.** Define the following auxiliary function

\[
\Phi_{\mu \chi} = e_H(u) + \mu \chi e_V(u)
\]
where $\chi$ is the same cut-off function we use in the proof of Theorem 1.1. $\mu$ is a constant number to be decided. Applying $\epsilon_2 = \frac{\epsilon_1}{C}$ to (3.8) and using (2.9), we deduce that

\begin{align}
(3.15) \\
\Delta H \Phi_{\mu\chi} &= \Delta H (e_H(u) + \mu \chi \phi_V(u)) \\
&\geq (1 - \epsilon_1) (u_{ik}^I)^2 - \left( 2k_1 + \frac{C}{\epsilon_2} + 2MG \right) e_H(u) \\
&\quad + \frac{1}{2} \epsilon_1 \eta_{\text{min}} \phi_V(u) - C \epsilon_2 (u_{\alpha k}^I)^2 - 4\bar{K} e_H(u) \phi_V(u) + \mu \Delta H \chi \phi_V(u) \\
&\quad + 2\mu \chi_k u_{\alpha}^I u_{\alpha k}^I + \mu \chi (u_{\alpha k}^I)^2 - 2\mu \chi M_G \phi_V(u) - 4\bar{K} \mu \chi H e_H(u) \phi_V(u) \\
&\quad = (1 - \epsilon_1) \left( (u_{ik}^I)^2 + \mu \chi (u_{\alpha k}^I)^2 \right) + 2\mu \chi_k u_{\alpha}^I u_{\alpha k}^I - 4\bar{K} \Phi_{\mu\chi} e_H(u) \\
&\quad + \left( \frac{1}{2} \epsilon_1 \eta_{\text{min}} + \mu \Delta H \chi - 2\mu \chi M_G \right) e_V(u) \\
&\quad - \left( 2k_1 + \frac{C}{\epsilon_2} + 2MG \right) e_H(u)
\end{align}

where $\chi_k$ is $k$-component of $\nabla^H \chi$ given by $(\nabla^H \chi, e_k)$. By Schwarz inequality, we have

\begin{align}
(3.16) \\
|\nabla^H \Phi_{\mu\chi}|^2 &= |\nabla^H (e_H(u) + \mu \chi \phi_V(u))|^2 \\
&\leq |u_{ik}^I + \sqrt{\mu \chi} u_{\alpha}^I| \cdot |u_{ik}^I + \sqrt{\mu \chi} u_{\alpha k}^I + \sqrt{\mu \nabla^H \chi \phi_V(u)}| \\
&= 2\Phi_{\mu\chi} \left( (u_{ik}^I)^2 + \mu \chi (u_{\alpha k}^I)^2 + \mu \phi_V(u) + \mu \chi_k u_{\alpha}^I u_{\alpha k}^I \right).
\end{align}

Choosing $0 < \epsilon_1 < \frac{1}{2}$, from (3.16), we have the following estimate

\begin{align}
(3.17) \\
&\geq (1 - 2\epsilon_1) \left( (u_{ik}^I)^2 + \mu \chi (u_{\alpha k}^I)^2 \right) + 2\mu \chi_k u_{\alpha}^I u_{\alpha k}^I \\
&\geq \left( \frac{1}{2} - \epsilon_1 \right) \frac{|\nabla^H \Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - \left( \frac{1}{2} - \epsilon_1 \right) \frac{\mu |\nabla^H \chi|^2}{\chi} e_V(u) \\
&\quad + (1 + 2\epsilon_1) \mu \chi_k u_{\alpha}^I u_{\alpha k}^I + \epsilon_1 \mu \chi (u_{\alpha k}^I)^2 \\
&\geq \left( \frac{1}{2} - \epsilon_1 \right) \frac{|\nabla^H \Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - \left( \frac{1}{2} - \epsilon_1 \right) \frac{\mu |\nabla^H \chi|^2}{\chi} e_V(u) \\
&\quad - \frac{1}{4\epsilon_1} \frac{\mu |\nabla^H \chi|^2}{\chi} e_V(u) \\
&\geq \left( \frac{1}{2} - \epsilon_1 \right) \frac{|\nabla^H \Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - 4\epsilon_1 \frac{\mu |\nabla^H \chi|^2}{\chi} e_V(u),
\end{align}
\[ \frac{1}{2} - \epsilon_1 + \frac{(1 + 2\epsilon_1)^2}{4\epsilon_1} \leq \frac{3}{2} + \frac{1}{4}\epsilon_1^{-1} \leq 2\epsilon_1^{-1}. \]

Combining (3.15) with (3.17), we get

\[ \Delta \Phi \frac{\chi}{\mu} = \left( \frac{1}{2} - \epsilon_1 \right) \frac{|\nabla H\Phi|}{\Phi} - 4\bar{K}\Phi e_H(u) \]

\[ + \left( \frac{1}{2}\epsilon_1 \eta_{\text{min}} + \mu \Delta \chi - 2\epsilon_1^{-1}\frac{\mu|\nabla H\chi|^2}{\chi} - 2\mu \chi M_G \right) e_V(u) \]

\[ - \left( 2k_1 + \frac{C}{\epsilon_2} + 2M_G \right) e_H(u). \]

We consider the following function

\[ F_{\mu \chi} = \frac{\Phi_{\mu \chi}}{(b - \psi \circ u)^\nu}. \]

Suppose \( \chi F_{\mu \chi} \) attains its maximum in \( B_R(p) \) at \( x_R \), then

\[ 0 = \nabla^H \ln(\chi F_{\mu \chi})(x_R) = \frac{\nabla^H \chi}{\chi} + \frac{\nabla^H \Phi_{\mu \chi}}{\Phi_{\mu \chi}} + \nu \frac{\nabla^H (\psi \circ u)}{b - \psi \circ u} \]

and

\[ 0 \geq \Delta \ln(\chi F_{\mu \chi})(x_R) = \frac{\Delta \chi}{\chi} - \frac{|\nabla^H \chi|^2}{\chi^2} + \frac{\Delta \Phi_{\mu \chi}}{\Phi_{\mu \chi}} \]

\[ - \frac{|\nabla^H \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}^2} + \frac{\Delta (\psi \circ u)}{b - \psi \circ u} - \nu \frac{|\nabla^H (\psi \circ u)|^2}{(b - \psi \circ u)^2}. \]

Substituting (3.18) into (3.20), we get

\[ 0 \geq \frac{\Delta \chi}{\chi} - \frac{|\nabla^H \chi|^2}{\chi^2} - \left( \frac{1}{2} + \epsilon_1 \right) \frac{|\nabla^H \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}^2} - 4\bar{K} e_H(u) \]

\[ + \nu \frac{\Delta (\psi \circ u)}{b - \psi \circ u} - \nu \frac{|\nabla^H (\psi \circ u)|^2}{(b - \psi \circ u)^2} \]

\[ + \left( \frac{1}{2}\epsilon_1 \eta_{\text{min}} + \mu \Delta \chi - 2\epsilon_1^{-1}\frac{\mu|\nabla^H \chi|^2}{\chi} - 2\mu \chi M_G \right) e_V(u) \]

\[ - \left( 2k_1 + \frac{C}{\epsilon_2} + 2M_G \right) e_H(u). \]
Choosing \( \epsilon_1 = \frac{1}{2\nu} - \frac{1}{4} \), by (3.19) and Schwarz inequality, we derive that

\[
\left( \frac{1}{2} + \epsilon_1 \right) \frac{|\nabla H \Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}^2} = \left( \frac{1}{2} + \epsilon_1 \right) \left( \frac{1}{\chi} \nabla H \chi + \nu \frac{\nabla H (\psi \circ u)}{b - \psi \circ u} \right)^2 \leq \left( \frac{1}{2} + \epsilon_1 \right) \left( 1 + \frac{2 + \nu}{2 - \nu} \right) \left( \frac{1}{\chi} \nabla H \chi \right)^2 + \left( \frac{1}{2} + \epsilon_1 \right) \left( 1 + \frac{2 - \nu}{2 + \nu} \right) \nu^2 \frac{|\nabla H (\psi \circ u)|^2}{(b - \psi \circ u)^2}.
\]

Therefore, (3.21) becomes

\[
0 \geq \frac{\Delta H \chi}{\chi} - \left( 1 + \frac{2 + \nu}{\nu(2 - \nu)} \right) \frac{|\nabla H \chi|^2}{\chi^2} - 4\bar{K} e_H (u) + \nu \frac{\Delta H (\psi \circ u)}{b - \psi \circ u} + \left( \frac{1}{2} \epsilon_1 \eta_{\text{min}} + \mu \Delta H \chi - 2\epsilon_1^{-1} \mu |\nabla H \chi|^2 - 2\mu \chi M_G \right) \frac{e_V (u)}{\Phi_{\mu\chi}} - \left( 2k_1 + \frac{C}{\epsilon_2} + 2M_G \right) \frac{e_H (u)}{\Phi_{\mu\chi}}.
\]

By (3.7), (3.8), Lemma 3.1 and (3.22),

\[
0 \geq - \frac{C_1}{\chi R} - \left( 1 + \frac{2 + \nu}{\nu(2 - \nu)} \right) \frac{C_1}{\chi R^2} + 2\delta e_H (u) + \left( \frac{1}{2} \epsilon_1 \eta_{\text{min}} - \mu \frac{C_1}{R} - 2\epsilon_1^{-1} \mu \frac{C_1}{R^2} - 2\mu \chi M_G \right) \frac{e_V (u)}{\Phi_{\mu\chi}} - \left( 2k_1 + \frac{C}{\epsilon_2} + 2M_G \right) \frac{e_H (u)}{\Phi_{\mu\chi}}.
\]

where \( C_7 (m, k_1, k_2, \nu) \) is a constant and \( \delta \) is given by Lemma 3.1. Since

\[
e_V (u) = \mu^{-1} \chi^{-1} (\Phi_{\mu\chi} - e_H (u)),
\]
from (3.23), we obtain

\begin{align}
0 & \geq -C_7 \left(1 + \frac{R}{R^2} \right) + 2\epsilon \frac{e_H(u)}{R^2} + \left(\frac{1}{2} \epsilon \eta_{\min} - \mu \frac{C_7 (1 + R)}{R^2} - 2\mu \chi M_G \right) \mu^{-1} \chi^{-1} \\
& + \left(\frac{1}{2} \epsilon \eta_{\min} - \frac{C_7 (1 + R)}{R^2} - 2k_1 - \frac{C}{\epsilon^2} \right) \frac{e_H(u)}{\Phi_{\mu} \chi} \\
& \geq \frac{1}{\chi} \left(\frac{1}{2} \epsilon \eta_{\min} - \frac{2C_7 (1 + R)}{R^2} - 2\chi M_G \right) \\
& + \left[2\epsilon \Phi_{\mu} \chi - \left(\frac{1}{2} \epsilon \eta_{\min} + 2k_1 + \frac{C}{\epsilon^2} \right) \frac{e_H(u)}{\chi^{\Phi_{\mu} \chi}} \right]
\end{align}

Setting

\begin{align}
\mu^{-1} = \frac{4C_7 (1 + R)}{\epsilon \eta_{\min} R^2} + \frac{4\chi M_G}{\epsilon \eta_{\min}}
\end{align}

such that

\begin{align}
\frac{1}{2} \epsilon \eta_{\min} - \frac{2C_7 (1 + R)}{R^2} - 2\chi M_G > 0,
\end{align}

then we have

\begin{align}
\Phi_{\mu \chi} \leq \delta^{-1} \left( \frac{C_7 (1 + R)}{R^2} + k_1 + \chi M_G + \frac{C^2}{2\epsilon \mu} \right) \\
\leq \delta^{-1} C_8
\end{align}

where

\begin{align}
C_8 = \frac{C_7 (1 + R)}{R^2} + k_1 + M_G + \frac{C^2}{2\epsilon \mu}.
\end{align}

Therefore, we have

\begin{align}
\max_{B_{p(R)}} \chi F_{\mu \chi} \leq \frac{\chi \Phi_{\mu \chi}}{(b - \psi \circ u)^{\nu}}(x) \leq \frac{C_8}{\delta (b - \phi(\tau))^{\nu}}.
\end{align}

Furthermore, we get

\begin{align}
\max_{B_{p(R)}} e_H(u) \leq b^{\nu} \max_{B_{p(R)}} \chi F_{\mu \chi} \leq \frac{C_8 b^{\nu}}{\delta (b - \phi(\tau))^{\nu}}.
\end{align}

In terms of Lemma 3.1, we know \( \delta, b \) and \( \mu \) all depend on \( \bar{K} \). By (3.25), we obtain the following estimate

\begin{align}
\max_{B_{p(R)}} e_H(u) \leq C_9 \left( \frac{C_7 (1 + R)}{R^2} + k_1 + M_G + \frac{C^2}{2\epsilon \mu} \right) \\
\leq C_{10} \left( k_1 + M_G + \frac{1 + R}{R^2} \right)
\end{align}

where \( C_9(\bar{K}), C_{10}(m, k_1, k_2, \bar{K}) \) are constants. \( \square \)
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