A NOTE ON LOG CANONICAL THRESHOLDS

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Abstract. We prove that the largest accumulation point of the set $T_3$ of all three-dimensional log canonical thresholds $c(X, F)$ is $5/6$.

1. Introduction

Let $(X, \Omega)$ be a log variety and let $F$ be an effective non-zero Weil $\mathbb{Q}$-Cartier divisor on $X$. Assume that $(X, \Omega)$ has at worst log canonical singularities. The log canonical threshold of $F$ with respect to $(X, \Omega)$ is defined by

$$c(X, \Omega, F) = \sup \{ c \mid (X, \Omega + cF) \text{ is log canonical} \}.$$ 

It is known that $c(X, \Omega, F)$ is a rational number from the interval $[0, 1]$ (see [3]). We frequently write $c(X, F)$ instead of $c(X, 0, F)$.

For each $d \in \mathbb{N}$ define the set $T_d \subset [0, 1]$ by

$$T_d := \left\{ c(X, F) \mid \dim X = d, X \text{ has only log canonical singularities and } F \text{ is an effective non-zero Weil } \mathbb{Q} \text{-Cartier divisor} \right\}.$$ 

The structure of $T_d$ is interesting for applications to the problem of termination some inductive procedures appearing in the Minimal Model Program [11], [4]. The interest in log canonical thresholds was also inspired in connection with the complex singular index and Bernstein-Sato polynomials (see [3]).

Conjecture 1.1 ([10]). $T_d$ satisfies the ascending chain condition, i.e. any increasing chain of elements terminates.

The set $T_2$ is completely described (see [7]). Concerning $T_3$ it is known the following:

(i) Conjecture [11] holds true for $T_3$ [11], [3], Ch. 18];
(ii) $T_3 \cap (41/42, 1) = \emptyset$;
(iii) $T_3 \cap [6/7, 1]$ is finite [3].

Actually, the structure of $T_d$ is rather complicated: it has a lot of accumulation points [3, 8.21]. However adopting Conjecture [11] we see that $T_d$ is discrete near 1.

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Our main result is the following theorem which generalizes the result of [9].

**Theorem 1.2.** The largest accumulation value of $T_3$ is $5/6$.

**Remark 1.3.** (i) The two-dimensional analog of our theorem easily follows from the description of $T_2$ ([7]): the largest accumulation value of $T_2$ is $1/2$.

(ii) T. Kuwata described the set of all values $c(C^3, F)$ in the interval $[5/6, 1]$, where $F$ is a hypersurface in $C^3$. His proof is done by studying the local equation of $F$. Our proof uses quite different method and based on Alexeev’s result [2].

The essential part of the proof is to show the finitedness of $T_3 \cap [5/6 + \epsilon, 1]$ for any $\epsilon > 0$. The easy example below shows that $5/6$ is an accumulation point of $T_3$.

**Example 1.4.** Let $X = C^3$ and let $F_r$ be the hypersurface given by $x^2 + y^3 + z^r$, $r \geq 7$. This singularity is quasihomogeneous. By [3, 8.14] we have $c(C^3, F_r) = 5/6 + 1/r$. Thus $\lim_{r \to \infty} c(C^3, F_r) = 5/6$.

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2. Preliminary results

All varieties are assumed to be algebraic varieties defined over the field $C$. A log variety (or a log pair) $(X, D)$ is a normal quasiprojective variety $X$ equipped with a boundary, a $Q$-divisor $D = \sum d_i D_i$ such that $0 \leq d_i \leq 1$ for all $i$. We use terminology, definitions and abbreviations of the Minimal Model Program [5].

**Proposition-Definition 2.1** ([10, §3], [4, Ch. 16]). Let $(X, S + B)$ be a log variety, where $S = [S + B] \neq 0$ and divisors $S, B$ have no common components. Assume that $K_X + S$ is lc in codimension two. Then there is a naturally defined effective $Q$-divisor $\text{Diff}_S(B)$ on $S$ called the different of $B$ such that

$$K_S + \text{Diff}_S(B) \sim_Q (K_X + S + B)|_S.$$ 

2.2. Let $\Phi$ be a subset of $Q$. For a $Q$-divisor $D = \sum d_i D_i$, we write $D \in \Phi$ if $d_i \in \Phi$ for all $i$. Define the following sets

$$\Phi_{sm} := \left\{ 1 - 1/m \mid m \in \mathbb{N} \cup \{\infty\} \right\},$$

$$\Phi_{sm}^\alpha := \Phi_{sm} \cup [\alpha, 1], \text{ for } \alpha \in [0, 1].$$
We distinguish them because they are closed under some important operations (see e.g. Corollary 2.5 below). Usually the numbers from $\Phi_{\text{sm}}$ are called standard.

**Proposition 2.3** ([10, Prop. 3.9]). Let $(X, S)$ be a $d$-dimensional plt log variety, where $S$ is integral. Let $W \subset S$ be an irreducible subvariety of codimension 1. Then near the general point $P \in W$ there is an analytic isomorphism

$$(X, S, W) \simeq \left( (\mathbb{C}^d, \{x_1 = 0\}, \{x_1 = x_2 = 0\})/\mathbb{Z}_m(1, q, 0 \ldots, 0) \right),$$

where $m, q \in \mathbb{N}$, $\gcd(m, q) = 1$.

**Corollary 2.4** ([10, 3.10, 3.11]). Let $(X, S + B)$ be a log variety, where $S := \lfloor S + B \rfloor$ and divisors $S, B$ have no common components. Assume that $(X, S)$ is plt. Let $W \subset S$ be an irreducible subvariety of codimension 1. If $B = \sum b_iB_i$, then the coefficient of $\text{Diff}_S(B)$ along $W$ is equal to

$$1 - \frac{1}{m} + \sum_{B_i \supset W} \frac{n_ib_i}{m},$$

where $m$ is such as in (2.1) and $n_i \in \mathbb{N}$. Moreover, if $(X, S + B)$ is plt and $B \in [1/2, 1]$, then there is at most one component $B_i$ of $B$ containing $W$ and $n_i = 1$.

**Corollary 2.5** ([10, 3.11, 4.2]). Let $(X, S + B)$ be a log variety, where $S := \lfloor S + B \rfloor$ and divisors $S, B$ have no common components. Assume that $(X, S)$ is plt and $(X, S + B)$ is plt. Take $\alpha \in [0, 1]$. If $B \in \Phi_{\text{sm}}^\alpha$, then $\text{Diff}_S(B) \in \Phi_{\text{sm}}^\alpha$.

**Proposition-Definition 2.6** ([8]). Let $(X, D)$ be a log variety such that $(X, D)$ is lc but not plt, $X$ is klt and $\mathbb{Q}$-factorial. Assume the log MMP in dimension $\dim(X)$. Then there exists a blow-up $f : Y \to X$ such that

(i) the exceptional set of $f$ contains an unique prime divisor $S$;
(ii) $K_Y + D_Y = f^*(K_X + D)$ is lc, where $D_Y$ is the proper transform of $D$;
(iii) $K_Y + S + (1 - \varepsilon)D_Y$ is plt and anti-ample over $X$ for any $\varepsilon > 0$;
(iv) $Y$ is $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$.

Such a blow-up we call an inductive blow-up of $(X, D)$.  

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3. Lemmas

**Lemma 3.1.** Let $\Lambda$ be a boundary on $\mathbb{P}^1$ such that $\Lambda \in \Phi_{5/6}^{5/6}$ and $K_\mathbb{P}^1 + \Lambda \equiv 0$. Then $\Lambda \in \Phi_{\text{sm}} \cap [0, 5/6] \cup \{1\}$.

*Proof.* Write $\Lambda = \sum \lambda_i \Lambda_i$. Then $\lambda_i \in \Phi_{\text{sm}}^{5/6}$ and $\sum \lambda_i = 2$. If $\lfloor \lambda \rfloor \neq 0$, then there are only two possibilities: $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = 2\lambda_2 = 2\lambda_3 = 1$. Otherwise $\lambda_i < 1$ and easy computations give us $\lambda_i \leq 5/6$, so $\lambda \in \Phi_{\text{sm}}$.

**Lemma 3.2.** Let $(S, \Delta = \sum \delta_i \Delta_i)$ be a lc log surface such that $\delta_i \in \Phi_{\text{sm}}^{5/6}$ and let $C$ be an effective Weil divisor on $S$. Then either $c(S, \Delta, C) \leq 5/6$ or $c(S, \Delta, C) = 1$.

*Proof.* Put $c := c(S, \Delta, C)$. Assume that $5/6 < c < 1$. By [3, 8.5] there is an exceptional divisor $E$ such that $a(E, \Delta + cC) = -1$ and $a(E, \Delta) > -1$. Put $P := \text{Center}(E)$. Regard $S$ as a germ near $P$.

Let $\varphi : \tilde{S} \to S$ be an inductive blowup of $(S, \Delta + cC)$. Write

$$K_{\tilde{S}} + \tilde{\Delta} + c\tilde{C} + \tilde{E} = \varphi^*(K_S + \Delta + cC),$$

where $\tilde{E}$ is the exceptional divisor, $\tilde{C}$ and $\tilde{\Delta}$ are proper transforms of $C$ and $\Delta$, respectively. By Corollary 2.3, $\text{Diff}_E(\tilde{\Delta} + c\tilde{C}) \in \Phi_{\text{sm}}^{5/6}$. On the other hand, $K_E + \text{Diff}_E(\tilde{\Delta} + c\tilde{C}) \equiv 0$. By Lemma 3.1, $\text{Diff}_E(\tilde{\Delta} + c\tilde{C}) \in [0, 5/6]$. Clearly, $\tilde{E} \cap C \neq \emptyset$. Applying Corollary 2.2 to our situation we obtain $1 - 1/m + c/m \leq 5/6$ for some $m \in \mathbb{N}$. This yields $c \leq 5/6$, a contradiction.

**Lemma 3.3** (cf. [11]). Let $(S \ni o, \Lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2)$ be a log surface germ such that $\lambda_1, \lambda_2 \geq 5/6$. Assume that $\text{discr}(S, \Lambda) \geq -5/6$ at $o$. Then $\lambda_1 + \lambda_2 \leq 11/6$.

*Proof.* By Lemma 3.2, $K_S + \Lambda_1 + \Lambda_2$ is lc at $o$. In this situation there is an analytic isomorphism (cf. Proposition 2.3)

$$(S, \Lambda, o) \simeq (\mathbb{C}^2, \{xy = 0\}, 0)/\mathbb{Z}_m(1, q),$$

where $m \in \mathbb{N}$ and $\text{gcd}(m, q) = 1$. Take $q$ so that $1 \leq q < m$ and consider the weighted blow up with weights $\frac{1}{m}(1, q)$. We get the exceptional divisor $E$ with discrepancy

$$-\frac{5}{6} \leq a(E, \Lambda) = -1 + \frac{1 + q}{m} - \frac{\lambda_1}{m} - \frac{q\lambda_2}{m}.$$  

Thus

$$0 \leq 1 + q - \lambda_1 - q\lambda_2 - \frac{m}{6} \leq 1 + q - \frac{5}{6}(1 + q) - \frac{m}{6} = \frac{1 + q - m}{6}.$$
If $m \geq 2$, this gives as $q = m - 1$ and equalities $\lambda_1 = \lambda_2 = 5/6$. In the case $m = 1$, $q = 1$ we have $0 \leq 2 - \lambda_1 - \lambda_2 - 1/6$, i.e. $\lambda_1 + \lambda_2 \leq 2 - 1/6$.

4. Proof of the main theorem

In this section we prove Theorem 1.2. First we reduce the problem to the case when $X$ is $\mathbb{Q}$-factorial and has only log terminal singularities. These arguments are quite standard, so the reader can skip them.

Lemma 4.1. Let $(X, \Omega)$ be a $d$-dimensional lc log variety such that $\Omega \in \Phi_{\text{sm}}$ and let $F$ be an effective Weil $\mathbb{Q}$-Cartier divisor on $X$. Assume that the log MMP in dimension $d$ holds. Then there is a $\mathbb{Q}$-factorial $d$-dimensional klt variety $X'$ and an effective Weil $\mathbb{Q}$-Cartier divisor $F'$ on $X'$ such that $c(X, \Omega, F) = c(X', F')$.

Proof. We prove our lemma by induction on $d$. Put $c := c(X, \Omega, F)$. Clearly, we may assume that $0 < c < 1$. Consider minimal dlt $\mathbb{Q}$-factorial modification $g: (\tilde{X}, \tilde{\Omega}) \to (X, \Omega)$ (see [5, 17.10]). By definition, this is a birational morphism $g: \tilde{X} \to X$ such that $\tilde{X}$ is $\mathbb{Q}$-factorial and

$$K_{\tilde{X}} + \tilde{\Omega} + \sum E_i = g^*(K_X + \Omega)$$

is dlt, where $\tilde{\Omega}$ is the proper transform of $\Omega$ and the $E_i$ are prime exceptional divisors (if $(X, \Omega)$ is dlt, one can take $\sum E_i = 0$). Since $c > 0$ and because $a(E_i, \Omega) = -1$, $F$ cannot contain $g(E_i)$. Therefore the proper transform of $F$ coincides with its pull-back $g^*F$. Replace $(X, \Omega, F)$ with $(\tilde{X}, \tilde{\Omega}, g^*F)$. From now on we may assume that $(X, \Omega)$ is dlt and $X$ is $\mathbb{Q}$-factorial. There is an exceptional divisor $E$ such that $a(E, \Omega + cF) = -1$ and $a(E, \Omega) > -1$. Regard $X$ as a germ near a point $P \in \text{Center}(E)$.

Assume that $[\Omega] \neq 0$. Let $S$ be a component of $[\Omega]$ (passing through $P$). Then $(S, \text{Diff}_S(\Omega - S))$ is lc [3, 17.7] and $\text{Diff}_S(\Omega - S) \in \Phi_{\text{sm}}$ (see Corollary 2.3). Then it is easy to see that $c(X, \Omega, F) = c(S, \text{Diff}_S(\Omega - S), F|_S)$. Taking into account $\mathcal{T}_{d-1} \subset \mathcal{T}_d$ (see [3, 8.21]), we get our assertion.

Now consider the case $[\Omega] = 0$. Then $(X, \Omega)$ is klt. Since $X$ is a germ near $P$, $n(K_X + \Omega) \sim 0$ for some $n \in \mathbb{N}$. Take $n$ to be minimal with this property. Then the isomorphism $\mathcal{O}_X(n(K_X + \Omega)) \simeq \mathcal{O}_X$ defines an $\mathcal{O}_X$-algebra structure on $\sum_{i=0}^{n-1} \mathcal{O}_X([iK_X - i\Omega])$ this gives us a cyclic $\mathbb{Z}_n$-cover

$$\varphi: X' := \text{Spec} \left( \sum_{i=0}^{n-1} \mathcal{O}_X([-iK_X - i\Omega]) \right) \to X.$$
The ramification divisor of $\varphi$ is $\Omega$. Hence $\varphi^*(K_X + \Omega) = K_{X'}$ and $X'$ has only log terminal singularities [3, 20.3]. Put $F' := \varphi^*F$. Then $c(X, \Omega, F) = c(X', F')$ (see [3, 8.12]). Replacing $X'$ with its $\mathbb{Q}$-factorialization we get the desired log pair.

4.2. **Notation.** Let $X$ be a three-dimensional $\mathbb{Q}$-factorial normal variety with only log terminal singularities and let $F$ be an effective Weil $\mathbb{Q}$-Cartier divisor on $X$. Put $F' := \varphi^*F$.

Then $c(X, \Omega, F) = c(X', F')$ (see [3, 8.12]). Replacing $X'$ with its $\mathbb{Q}$-factorialization we get the desired log pair.

4.3. **Main assumption.** Fix $\epsilon > 0$ and assume that $1 > c > 5/6 + \epsilon$. We prove that there are only a finite number of possibilities for such $c$.

**Lemma 4.4.** $f(S)$ is a point.

**Proof.** Otherwise $f(S)$ is a curve and the pair $(X, cF)$ is lc but not klt along $f(S)$. Taking a general hyperplane section we derive a contradiction with Lemma 3.2.

**Lemma 4.5.** $(Y, S + cF_Y)$ is plt.

**Proof.** Assume the converse. Then there is an exceptional divisor $E$ such that $a(E, S + cF_Y) = -1$. Since $(Y, S)$ is plt, $\text{Center}(E) \subset E \cap F_Y$.

If $\text{Center}(E)$ is a curve, then $(Y, S + cF_Y)$ is lc but not klt along $\text{Center}(E)$. As in the proof of Lemma 4.4 we derive a contradiction. Thus we may assume that $(Y, S + cF_Y)$ is plt in codimension two. By Adjunction [3, Th. 17.6] this implies that $[\Theta] = 0$.

Hence $\text{Center}(E)$ is a point. Again by Adjunction $(S, \Theta)$ is lc but not klt near $\text{Center}(E)$. As above, we have a contradiction with Lemma 3.2.

**Corollary 4.6.** $(S, \Theta)$ is klt.

4.7. Now we are going to construct a “good” birational model $(\tilde{S}, \tilde{\Theta})$ of $(S, \Theta)$. The construction is similar to that in [11]. Assumption 4.3 gives us that $\Theta \in \Phi^{5/6}_{\text{sm}}$. If $\text{discr}(S, \Theta) \geq -5/6$ and $\rho(S) = 1$, we put $(\tilde{S}, \tilde{\Theta}) = (S, \Theta)$.

From now on we assume either $\text{discr}(S, \Theta) < -5/6$ or $\rho(S) > 1$. Since $(S, \Theta)$ is klt, there is only a finite set $\mathcal{E}$ of divisors $E$ with $a(E, \Theta) < -5/6$ [3, 2.12.2]. Let $\mu : \tilde{S} \to S$ be the blow-up of all divisors $E \in \mathcal{E}$ (see [3, Th. 17.10]) and let $\tilde{\Theta}$ be the crepant pull-back:

$$K_{\tilde{S}} + \tilde{\Theta} = \mu^*(K_S + \Theta), \quad \mu_*\tilde{\Theta} = \Theta.$$
Then $\text{discr}(\tilde{S}, \tilde{\Theta}) \geq -5/6$ and again we have $\tilde{\Theta} \in \Phi_{\text{sm}}^{5/6}$. Write $\tilde{\Theta} = \sum \vartheta_i \tilde{\Theta}_i$ and consider the boundary $\tilde{\Xi}$ with $\text{Supp}(\tilde{\Xi}) = \text{Supp}(\tilde{\Theta})$:

$$
\tilde{\Xi} := \sum \xi_i \tilde{\Theta}_i, \quad \xi_i = \begin{cases} 1 & \text{if } \vartheta_i > 5/6, \\ \vartheta_i & \text{otherwise.} \end{cases}
$$

For sufficiently small positive $\alpha$, the $\mathbb{Q}$-divisor $\tilde{\Theta} - \alpha(\tilde{\Xi} - \tilde{\Theta})$ is a boundary. It is clear that

$$
K_{\tilde{S}} + \tilde{\Theta} - \alpha(\tilde{\Xi} - \tilde{\Theta}) \equiv -\alpha(\tilde{\Xi} - \tilde{\Theta})
$$

cannot be nef. By our assumption, $\rho(\tilde{S}) > 1$. Note also that $(\tilde{S}, \tilde{\Xi})$ is lc (see Lemma 3.2). Run $K_{\tilde{S}} + \tilde{\Theta} - \alpha(\tilde{\Xi} - \tilde{\Theta})$-MMP. On each step we contract an extremal ray $R$ such that

$$(K_{\tilde{S}} + \tilde{\Xi}) \cdot R = (\tilde{\Xi} - \tilde{\Theta}) \cdot R > 0.$$ 

Consider such a contraction $\varphi: \tilde{S} \rightarrow S^\sharp$.

4.8. Assume that $\dim S^\sharp = 1$ and let $C$ be a general fiber. Since $(\tilde{\Xi} - \tilde{\Theta}) \cdot C > 0$, there is a component $\tilde{\Theta}_i$ of $\tilde{\Theta}$ with coefficient $\vartheta_i > 5/6$ meeting $C$. Hence $\text{Diff}_C(\tilde{\Theta})$ also has a component with coefficient $> 5/6$. By Adjunction $K_C + \text{Diff}_C(\tilde{\Theta})$ is klt. On the other hand,

$$
K_C + \text{Diff}_C(\tilde{\Theta}) \equiv 0 \quad \text{and} \quad \text{Diff}_C(\tilde{\Theta}) \in \Phi_{\text{sm}}^{5/6}
$$

(see Corollary 2.5). This contradicts Lemma 3.1. Thus, $\varphi$ is birational.

4.9. We claim that $\varphi$ cannot contract a component of $[\tilde{\Xi}]$. Indeed, assume that $\varphi$ contracts a curve $C \subset [\tilde{\Xi}]$. Take $\tilde{\Theta}' := \tilde{\Theta} + \alpha C$ so that $[\tilde{\Theta}'] = C$ and $\tilde{\Theta}' \leq \tilde{\Xi}$. Since $C^2 < 0$, we have $(K_{\tilde{S}} + \tilde{\Theta}') \cdot C < 0$. Therefore

$$
(K_{\tilde{S}} + \tilde{\Theta}' + \beta(\tilde{\Xi} - \tilde{\Theta}')) \cdot C = 0
$$

for some $0 < \beta < 1$. Put $\tilde{\Theta}'' := \tilde{\Theta}' + \beta(\tilde{\Xi} - \tilde{\Theta}')$. Then $\tilde{\Theta}'' \leq \tilde{\Xi}$, so $(\tilde{S}, \tilde{\Theta}'')$ is lc. Moreover $\tilde{\Theta}'' \in \Phi_{\text{sm}}^{5/6}$. Since $(\tilde{\Xi} - \tilde{\Theta}'') \cdot C > 0$, there is a component of $\tilde{\Xi} - \tilde{\Theta}''$ meeting $C$. By Lemma 3.2, $(\tilde{S}, \tilde{\Theta}'')$ is plt near $C \cap \text{Supp}(\tilde{\Xi} - \tilde{\Theta}'')$. As in 4.8 we derive a contradiction by Lemma 3.1.

Put $\Xi^\sharp := \varphi_* \tilde{\Xi}$ and $\Theta^\sharp := \varphi_* \tilde{\Theta}$. By [1, 2.28],

$$
\text{discr}(S^\sharp, \Theta^\sharp) = \text{discr}(\tilde{S}, \tilde{\Theta}) \geq -5/6.
$$
Thus all the assumptions hold for \((S^\sharp, \Theta^\sharp)\). Again

\[
K_{S^\sharp} + \Theta^\sharp - \alpha(\Xi^\sharp - \Theta^\sharp) \equiv -\alpha(\Xi^\sharp - \Theta^\sharp)
\]
cannot be nef.

Continuing the process we get a new pair \((\bar{S}, \bar{\Theta})\) such that

\[
\rho(\bar{S}) = 1, \quad \bar{\Theta} \in \Phi_{\text{sm}}^{5/6}, \quad (\bar{S}, \bar{\Theta}) \text{ is klt, } K_{\bar{S}} + \bar{\Theta} \equiv 0, \quad \text{and } \text{discr}(\bar{S}, \bar{\Theta}) \geq -5/6.
\]

Note that all our birational modifications are \((K + \Theta)\)-crepant. Hence

\[
\text{totaldiscr}(S, \Theta) = \text{totaldiscr}(\bar{S}, \bar{\Theta}) = \text{totaldiscr}(\tilde{S}, \tilde{\Theta})
\]
(see [3, 3.10]). Consider the decomposition \(\Theta = \Theta^a + \Theta^b\), where

\[
\Theta^a = \sum_{\Theta_i \subset F_Y} \vartheta_i \Theta_i, \quad \Theta^b = \sum_{\Theta_i \subset F_Y} \vartheta_i \Theta_i.
\]

Similarly, \(\bar{\Theta} = \bar{\Theta}^a + \bar{\Theta}^b + \bar{\Theta}^c\), where \(\bar{\Theta}^a\) and \(\bar{\Theta}^b\) are proper transforms of \(\Theta^a\) and \(\Theta^b\), respectively, and components of \(\bar{\Theta}^a = \bar{\Theta} - \bar{\Theta}^a - \bar{\Theta}^b\) are proper transforms of exceptional divisors of \(\mu\).

It is clear that \(\Theta^b, \bar{\Theta}^b \in \Phi_{\text{sm}}\) and \(\bar{\Theta}^c \in (5/6, 1)\). Since the coefficients of \(\Theta^a\) (as well as \(\bar{\Theta}^a\)) are of the form

\[
\vartheta_i = 1 - 1/m_i + c/m_i \geq c > 5/6 + \epsilon,
\]
we have \(\Theta^a \in (5/6 + \epsilon, 1)\). By our assumptions \(\Theta^a \neq 0\).

We need the following result of Alexeev [2]:

**Theorem 4.10.** Fix \(\epsilon > 0\). Consider the class of all projective log surfaces \((S, \Theta)\) such that \(- (K_S + \Theta)\) is nef and \(\text{totaldiscr}(S, \Theta) > -1 + \epsilon\) excluding only the case

- \(\Theta = 0, \quad K_S \equiv 0\) and the singularities of \(S\) are at worst Du Val.

Then the class \(\{S\}\) is bounded, i.e. \(S\) belongs to a finite number of algebraic families.

4.10.1. Let \(\bar{\Theta}_1\) be a component of \(\bar{\Theta}^a\). Then \(\vartheta_1 > 5/6 + \epsilon\). Since \(\rho(\bar{S}) = 1\), every two components of \(\bar{\Theta}\) intersects each other. Applying Lemma [9] we obtain

\[
\vartheta_j \leq 11/6 - \vartheta_1 < 11/6 - 5/6 - \epsilon = 1 - \epsilon
\]
for all \(j \neq 1\). Since \(\bar{\Theta}^b \in \Phi_{\text{sm}}\), there is only a finite number of possibilities for the coefficients of \(\bar{\Theta}^b\) (and \(\Theta^b\)).
4.10.2. If $\bar{\Theta}^a$ has at least two components, say $\bar{\Theta}_1$ and $\bar{\Theta}_2$, then by Lemma 3.3 the inequality $\vartheta_k < 1 - \epsilon$ holds for all $\vartheta_k$. Thus

$$\text{totaldiscr}(S, \Theta) = \text{totaldiscr}(\bar{S}, \bar{\Theta}) > -1 + \epsilon.$$  

Apply 4.10 to $(S, \Theta)$.

For all coefficients of $\Theta$ we have $\vartheta_i \geq 1/2$. Fix a very ample divisor $H$ on $S$. Then $H \cdot \sum \Theta_i \leq 2H \cdot K_S \leq \text{Const}$. This shows that the pair $(S, \text{Supp}(\Theta))$ is also bounded.

As above, $(S, \text{Supp}(\Theta))$ is bounded. From the equality $0 = K_S^2 + K_S \cdot \Theta^a + K_S \cdot \Theta^b$ we obtain

$$\sum_{a_i \not\in F_Y} (1 - 1/m_i + c/m_i)(K_S \cdot \Theta_i) = -K_S^2 - K_S \cdot \Theta^b,$$

where $1 - 1/m_i + c/m_i < 1 - \epsilon$. This gives us a finite number of possibilities for $c$.

4.10.3. Assume that $\bar{\Theta}^a = \vartheta \bar{\Theta}_1$, where $\vartheta = 1 - 1/m_1 + c/m_1$. If $\vartheta < 1 - \epsilon$, then we can argue as above. Let $\vartheta_1 \geq 1 - \epsilon$. Then $\Theta_1$ is the only divisor with discrepancy $a(\Theta_1, \Theta) \leq -1 + \epsilon$. Put $\Lambda := \Theta - \vartheta \bar{\Theta}_1$. Then $a(\Theta_1, \Lambda) = 0$, so totaldiscr$(S, \Lambda) > -1 + \epsilon$. Note that $\Theta_1$ is ample (because $\Theta_1 = (F_Y|_S)_{\text{red}}$ and $F_Y$ is $f$-ample, see 2.6. (iii)). Hence $-(K_S + \Lambda)$ is also ample. By 4.10 (S, Supp($\Lambda$)) is bounded and so is $(S, \text{Supp}(\Theta))$. As in 4.10.2 there is only a finite number of possibilities for $c$.

The following example illustrates our proof:

**Example 4.11.** Notation as in Example 1.4. Assume that $\gcd(6, r) = 1$. Let $f : Y \to X$ be the weighted blowup with weights $(3r, 2r, 6)$. Then $f$ is an inductive blowup of $(X, cF)$ and the exceptional divisor $S$ is isomorphic to $\mathbb{P}(3r, 2r, 6) \simeq \mathbb{P}^2$. It is easy to compute that $\Theta = \text{Diff}_S(cF_Y) = \frac{1}{2}L_1 + \frac{2}{3}L_2 + \frac{r}{6}L_3 + cL_0$, where $c = 5/6 + 1/r$ and $L_1, L_2, L_3, L_0$ are lines on $S \simeq \mathbb{P}^2$ given by equations $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$, respectively. Thus $\text{discr}(S, \Theta) \geq -5/6$ and $\bar{S} = \bar{S} = S \simeq \mathbb{P}^2$.

**Concluding remark.** (i) Using the same arguments one can see that see that the set $\mathcal{T}_3$ in Theorem 1.2 can be replaced with $\mathcal{T}_3(\Phi_{\text{sm}})$, the set of all values $c(X, \Omega, F)$ with $\Omega \in \Phi_{\text{sm}}$.

(ii) We expect that our proof of Theorem 1.2 can be generalized in higher dimensions modulo the following facts: the log MMP, boundedness result 4.10 and lemmas 3.1 and 3.2. Also we hope that our method allow us to get the complete description of $\mathcal{T}_3 \cap [5/6, 1]$. 
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