P-CONNECTION ON RIEMANNIAN ALMOST PRODUCT MANIFOLDS

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Abstract
In the present work, we introduce a linear connection (preserving the almost product structure and the Riemannian metric) on Riemannian almost product manifolds. This connection, called P-connection, is an analogue of the first canonical connection of Lichnerowicz in the Hermitian geometry and the B-connection in the geometry of the almost complex manifolds with Norden metric. Particularly, we consider the P-connection on a class of manifolds with nonintegrable almost product structure.

Key words: Riemannian manifold, Riemannian metric, almost product structure, linear connection, parallel torsion.

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1. INTRODUCTION

In [1] a linear connection, called B-connection, is introduced on almost complex manifolds with Norden (or anti-Hermitian) metric. This connection (preserving the almost complex structure and the Norden metric) is an analogue of the first canonical connection of Lichnerowicz in the Hermitian geometry ([2], [3], [4]). In [5] the B-connection is considered on a class of almost complex manifolds with Norden metric and nonintegrable almost complex structure. This is the class $W_3$ of the quasi-Kähler manifolds with Norden metric.

In the present work, we introduce a linear connection (preserving the almost product structure and the Riemannian metric) on Riemannian almost product manifolds. This connection, called P-connection, is an analogue of the first canonical connection of Lichnerowicz in the Hermitian geometry and the B-connection in the geometry of the almost complex manifolds with Norden metric. Particularly, we consider the P-connection on the manifolds of the class $W_3$ from the classification in [6].

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano [4]. In [7] A. M. Naveira gives a classification of these manifolds with respect to the covariant differentiation of the almost product structure. Having in mind the results in [7], M. Staikova and

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K. Gribachev give in [6] a classification of the Riemannian almost product manifolds with zero trace of the almost product structure.

2. Preliminaries

Let \((M, P, g)\) be a Riemannian almost product manifold, i.e. a differentiable manifold \(M\) with a tensor field \(P\) of type \((1, 1)\) and a Riemannian metric \(g\) such that

\[
P^2 x = x, \quad g(Px, Py) = g(x, y)
\]

for arbitrary \(x, y\) of the algebra \(\mathfrak{X}(M)\) of the smooth vector fields on \(M\).

Obviously \(g(Px, y) = g(x, Py)\).

Further \(x, y, z, w\) will stand for arbitrary elements of \(\mathfrak{X}(M)\).

In this work we consider Riemannian almost product manifolds with \(\text{tr}P = 0\). In this case \((M, P, g)\) is an even-dimensional manifold.

The classification in [6] of Riemannian almost product manifolds is made with respect to the tensor field \(F\) of type \((0,3)\), defined by

\[
F(x, y, z) = g((\nabla_x P)y, z),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). The tensor \(F\) has the following properties:

\[
F(x, y, z) = F(x, z, y) = -F(x, Py, Pz),
\]

\[
F(x, y, Pz) = -F(x, Py, z).
\]

The basic classes of the classification in [6] are \(W_1\), \(W_2\) and \(W_3\). Their intersection is the class \(W_0\) of the Riemannian \(P\)-manifolds, determined by the condition \(F(x, y, z) = 0\) or equivalently \(\nabla P = 0\). In the classification there are include the classes \(W_1 \oplus W_2\), \(W_1 \oplus W_3\), \(W_2 \oplus W_3\) and the class \(W_1 \oplus W_2 \oplus W_3\) of all Riemannian almost product manifolds.

In the present work we consider manifolds from the class \(W_3\). This class is determined by the condition

\[
\mathcal{S}_{x,y,z} F(x, y, z) = 0,
\]

where \(\mathcal{S}_{x,y,z}\) is the cyclic sum by \(x, y, z\). This is the only class of the basic classes \(W_1\), \(W_2\) and \(W_3\), where each manifold (which is not Riemannian \(P\)-manifold) has a nonintegrable almost product structure \(P\). This means that in \(W_3\) the Nijenhuis tensor \(N\), determined by

\[
N(x, y) = (\nabla_x P)y - (\nabla_{Py} P)x + (\nabla_y P)Px - (\nabla_{Py} P)x,
\]

is non-zero.

Further, manifolds of the class \(W_3\) we call Riemannian \(W_3\)-manifolds.

As it is known the curvature tensor field \(R\) of a Riemannian manifold with metric \(g\) is determined by \(R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z\) and the
corresponding tensor field of type $(0, 4)$ is defined as follows $R(x, y, z, w) = g(R(x, y)z, w)$.

Let $(M, P, g)$ be a Riemannian almost product manifold and $\{e_i\}$ be a basis of the tangent space $T_pM$ at a point $p \in M$. Let the components of the inverse matrix of $g$ with respect to $\{e_i\}$ be $g^{ij}$. If $\rho$ and $\tau$ are the Ricci tensor and the scalar curvature, then $\rho^*$ and $\tau^*$, defined by $\rho^*(y, z) = g^{ij}R(e_i, y, z, Pe_j)$ and $\tau^* = g^{ij}\rho^*(e_i, e_j)$, are called an associated Ricci tensor and an associated scalar curvature, respectively. We will use also the trace $\tau^{**} = g^{ij}g^{ks}R(e_i, e_k, P e_s, P e_j)$.

The square norm of $\nabla P$ is defined by

\[ \|\nabla P\|^2 = g^{ij}g^{ks}g((\nabla e_i P) e_k, (\nabla e_j P) e_s). \]

Obviously $\|\nabla P\|^2 = 0$ iff $(M, P, g)$ is a Riemannian $P$-manifold. In [8] it is proved that if $(M, P, g)$ is a Riemannian $W_3$-manifold then

\[ \|\nabla P\|^2 = -2g^{ij}g^{ks}g((\nabla e_i P) e_k, (\nabla e_j P) e_j) = 2(\tau - \tau^{**}). \]

A tensor $L$ of type $(0, 4)$ with the properties

\[ L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \]
\[ \bigotimes_{x,y,z} L(x, y, z, w) = 0 \quad \text{(the first Bianchi identity)} \]

is called a curvature-like tensor. Moreover, if the curvature-like tensor $L$ has the property

\[ L(x, y, Pz, Pw) = L(x, y, z, w), \]

we call it a Riemannian $P$-tensor.

If the curvature tensor $R$ on a Riemannian $W_3$-manifold $(M, P, g)$ is a Riemannian $P$-tensor, i.e. $R(x, y, Pz, Pw) = R(x, y, z, w)$, then $\tau^{**} = \tau$. Therefore $\|\nabla P\|^2 = 0$, i.e. $(M, P, g)$ is a Riemannian $P$-manifold.

3. $P$-connection

A linear connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ preserving $P$ and $g$, i.e. $\nabla' P = \nabla' g = 0$, is called a natural connection [9].

**Definition 3.1.** The natural connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ determined by

\[ \nabla'_x y = \nabla_x y - \frac{1}{2}(\nabla x P)Py, \]

is called a $P$-connection.
Let $T$ be a torsion tensor of the $P$-connection $\nabla'$ determined on $(M, P, g)$ by (3.1). Because of the symmetry of $\nabla$, from (3.1) we have $T(x, y) = -\frac{1}{2}\{\nabla_x P y - \nabla_y P x\}$. Then, having in mind (3.1), we obtain

$$T(x, y, z) = g(T(x, y), z) = -\frac{1}{2}\{F(x, P y, z) - F(y, P x, z)\}.$$ 

Hence and (2.3) we have

$$S_{x,y,z} T(x, y, P z) = 0.$$ 

Let $Q$ be the tensor field determined by

$$Q(y, z) = -\frac{1}{2}(\nabla_y P) P z.$$ 

Having in mind (2.2), for the corresponding $(0,3)$-tensor field we have

$$Q(y, z, w) = -\frac{1}{2} F(y, P z, w).$$ 

Because of the properties (2.3), (3.4) implies

$$Q(y, z, w) = -Q(y, w, z).$$ 

Let $R'$ be the curvature tensor of the $P$-connection $\nabla'$. Then, according to (3.1) and (2.5) we have

$$R'(x, y, z, w) = R(x, y, z, w) - (\nabla_x F)(y, z, w) + (\nabla_y F)(x, z, w) + Q(x, Q(y, z), w) - Q(y, Q(x, z), w).$$ 

After a covariant differentiation of (3.4), a substitution in (3.5), a use of (2.2), (2.3), (2.4) and some calculations, from (3.5) we obtain

$$R'(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} \left[ g((\nabla y P) z, (\nabla x P) w) - g((\nabla x P) z, (\nabla y P) w) \right].$$ 

The last equality, having in mind the Ricci identity for Riemannian almost product manifolds

$$(\nabla_x F)(y, z, w) - (\nabla_y F)(x, z, w) = R(x, y, P z, w) - R(x, y, z, P w),$$ 

implies

$$R'(x, y, z, w) = \frac{1}{4} \left\{ 2R(x, y, z, w) + 2R(x, y, P z, P w) + K(x, y, z, w) \right\},$$

where $K$ is the tensor determined by

$$K(x, y, z, w) = -g((\nabla x P) z, (\nabla y P) w) + g((\nabla y P) z, (\nabla x P) w).$$ 

In this way, the following theorem is valid.

**Theorem 3.1.** The curvature tensor $R'$ of the $P$-connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ has the form (3.6). \hfill \Box
From (3.6) follows immediately that the property (2.7) and (2.9) are valid for \( R' \). Therefore, the property (2.8) for \( R' \) is a necessary and sufficient condition \( R' \) to be a Riemannian \( P \)-tensor. Since \( R \) satisfies (2.8), then from (3.6) we obtain immediately the following

**Theorem 3.2.** The curvature tensor \( R' \) of the \( P \)-connection \( \nabla' \) on a Riemannian \( W_3 \)-manifold \((M, P, g)\) is a Riemannian \( P \)-tensor iff

\[
2 \sum_{x, y, z} R(x, y, Pz, Pw) = - \sum_{x, y, z} K(x, y, z, w).
\]

□

Let the following condition be valid for the Riemannian almost product manifold \((M, P, g)\):

\[
(3.9) \quad \sum_{x, y, z} R(x, y, Pz, Pw) = 0.
\]

We say that the condition (3.9) characterizes a class \( L_2 \) of the Riemannian almost product manifolds.

The equality (3.7) implies immediately the properties (2.7) and (2.9) for \( P \). Then, according to (3.8) and (3.9), we obtain the following

**Theorem 3.3.** Let \((M, P, g)\) belongs to the class \( L_2 \). Then the curvature tensor \( R' \) of the \( P \)-connection \( \nabla' \) is a Riemannian \( P \)-tensor iff the tensor \( P \) determined by (3.7) is a Riemannian \( P \)-tensor, too. □

Having in mind (3.6), the last theorem implies the following

**Corollary 3.4.** Let the curvature tensor \( R' \) of the \( P \)-connection \( \nabla' \) be a Riemannian \( P \)-tensor on \((M, P, g) \in L_2 \). Then the tensor \( H \), determined by

\[
(3.10) \quad H(x, y, z, w) = R(x, y, z, w) + R(x, y, Pz, Pw)
\]

is a Riemannian \( P \)-tensor, too. □

## 4. Curvature properties of the \( P \)-connection in \( W_3 \cap L_2 \)

Let us consider the manifold \((M, P, g) \in W_3 \cap L_2 \) with a Riemannian \( P \)-tensor of curvature \( R' \) of the \( P \)-connection \( \nabla' \). Then, according to Theorem 3.3 and Corollary 3.4, the tensors \( K \) and \( H \), determined by (3.7) and (3.10), respectively, are also Riemannian \( P \)-tensors.

Let \( \rho' \) and \( \rho(K) \) be the Ricci tensors for \( R' \) and \( K \), respectively. Then we obtain immediately from (3.6)

\[
(4.1) \quad \rho(y, z) + \rho^*(y, Pz) = 2\rho'(y, z) - \frac{1}{2} \rho(K)(y, z).
\]

From (4.1) we have

\[
(4.2) \quad \tau + \tau^{**} = 2\tau' - \frac{1}{2} \tau(K),
\]
where $\tau'$ and $\tau(K)$ are the scalar curvatures for $R'$ and $K$, respectively. It is known from [8], that $\|\nabla P\|^2 = 2(\tau - \tau^{**})$. Then (4.2) implies
\[(4.3) \quad \tau = \tau' - \frac{1}{4}(\tau(K) - \|\nabla P\|^2).\]
From (3.7) we obtain
\[\rho(K)(y, z) = -g^{ij}g((\nabla e_i P) z, (\nabla y P) e_j),\]
from where
\[\tau(K) = g^{ij}g^{ks}g((\nabla e_i P) e_s, (\nabla e_k P) e_j).\]
Hence, applying (2.6), we get
\[(4.4) \quad \tau(K) = \frac{1}{2} \|\nabla P\|^2.\]
From (4.3) and (4.4) it follows
\[(4.5) \quad \tau = \tau' + \frac{1}{8} \|\nabla P\|^2.\]
The equalities (4.2), (4.3), (4.4) and (4.5) implies the following
\[\text{Proposition 4.1. Let the curvature tensor } R' \text{ of the } P\text{-connection } \nabla' \text{ be a Riemannian } P\text{-tensor on } (M, P, g) \in W_3 \cap L_2. \text{ Then}\]
\[(4.6) \quad \|\nabla P\|^2 = -8(\tau' - \tau) = \frac{8}{3} (\tau' - \tau^{**}) = 2\tau(K).\]
\[\square\]
\[\text{Corollary 4.2. Let the curvature tensor } R' \text{ of the } P\text{-connection } \nabla' \text{ be a Riemannian } P\text{-tensor on } (M, P, g) \in W_3 \cap L_2. \text{ Then the following assertions are equivalent:}\]
1) $(M, P, g)$ a Riemannian $P$-manifold;
2) $\tau' = \tau$;
3) $\tau' = \tau^{**}$;
4) $\tau(K) = 0$.\]
From (3.6) and (3.10) we obtain

\[
\tau(H) = \frac{4\tau' - \tau(K)}{2}, \quad \tau^*(H) = \frac{4\tau^{*s} - \tau^*(K)}{2},
\]

where \(\tau^{*s}\) and \(\tau^*(K)\) are the associated scalar curvature to \(\tau'\) and \(\tau(K)\), respectively.

We apply (4.8) to (4.7) and thus we obtain the following

**Proposition 4.3.** Let the curvature tensor \(R'\) of the \(P\)-connection \(\nabla'\) be a Riemannian \(P\)-tensor on a 4-dimensional manifold \((M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_2\). Then

\[
H = \frac{4\tau' - \tau(K)}{16}(\pi_1 - \pi_2) + \frac{4\tau^{*s} - \tau^*(K)}{16}\pi_3.
\]

\[\square\]

Let \(\mathcal{L}_1\) is the subclass of \(\mathcal{L}_2\) determined by

\[
R(x, y, Pz, Pw) = R(x, y, z, w).
\]

The equalities (4.9) and (3.6) imply the following

**Proposition 4.4.** Let \((M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_1\). Then

\[
R = R' - \frac{1}{4}K.
\]

\[\square\]

**Corollary 4.5.** Let \((M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_1\). Then

\[
\tau = \tau' - \frac{1}{4}\tau(K), \quad \tau^* = \tau^{*s} - \frac{1}{4}\tau^*(K).
\]

\[\square\]

**Corollary 4.6.** Let \((M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_1\) and \(\dim M = 4\). Then

\[
\tau = \frac{1}{2}\tau(H), \quad \tau^* = \frac{1}{2}\tau^*(H).
\]

\[\square\]

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