The Importance of Fourier Phases for the Morphology of Gravitational Clustering

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ABSTRACT

The phases of the Fourier modes appearing in a plane-wave expansion of cosmological density fields play a vital role in determining the morphology of gravitationally-developed clustering. We demonstrate this qualitatively and quantitatively using simulations. In particular, we use cross-correlation and rank-correlation techniques to quantify the agreement between a simulated distribution and phase-only reconstructions. The phase-only reconstructions exhibit a high degree of correlation with the original distributions, showing how meaningful spatial reconstruction of cosmological density fields depends more on phase accuracy than on amplitudes.

Key words: cosmology: theory – large-scale structure of the Universe – methods: statistical

1 INTRODUCTION

The standard theory of the origin of the large-scale structure of the universe involves the assumption that the structure we see today grew by gravitational instability from initial small fluctuations in the density field. In most popular variants of this model, particularly those involving cosmic inflation, the initial fluctuations are of a particularly simple form known as a Gaussian random field. Gaussian random fields are useful because many properties of Gaussian random density fields can be calculated analytically (e.g. Bardeen et al. 1986). Some direct motivation for such an assumption emerges from inflationary models, wherein the fluctuations are generated by quantum oscillations of the scalar field driving inflation (the “inflation”). Even if inflation turns out to be incorrect, however, the Central Limit Theorem tends to produce Gaussian fluctuations in any linear process, so that they are the most generic form for small initial conditions and a natural default assumption.

One particularly interesting property of Gaussian random fields is that the requirement for the density contrast \( \delta(x) = [\rho(x) - \rho_0]/\rho_0 \) to be a Gaussian random field is equivalent to that the real and imaginary parts of its Fourier components \( \tilde{\delta}_k \), where

\[
\delta(x) = \sum_k \tilde{\delta}(k) \exp(ik \cdot x),
\]

are independently distributed. In other words, the Fourier modes \( \tilde{\delta}(k) \),

\[
\tilde{\delta}(k) = |\tilde{\delta}(k)| \exp(i\phi_k),
\]

possess phases \( \phi_k \) which are independently distributed and uniformly random on the interval \([0, 2\pi]\). As the density field is simply a sum over a large number of Fourier modes and if the phases of each of Fourier modes are random, the Central Limit Theorem guarantees that the resulting superposition of the one-point probability distribution \( P(\delta) \) is close to Gaussian and that all of the field’s joint probability distributions are multivariate Gaussians. The statistical properties of an isotropic Gaussian random field are then completely specified by its second-order statistical quantity: the covariance function, or alternatively, its power spectrum \( P(k) = \langle \tilde{\delta}^2(k) \rangle \).

In the framework of gravitational instability, the growth of fluctuations can be understood analytically when the density fluctuation amplitude is small compared to the mean density; the linear perturbation theory tells us that each Fourier mode grows with the same rate independent of wavenumber and the statistical distribution of \( \delta \) remains constant except its variance.

The linear theory breaks down when \( \langle \delta^2 \rangle \) is comparable with unity or beyond, and different Fourier modes start coupling. One way to look at the mode coupling is that \( \delta \) is always constrained to value \( \delta \geq -1 \). When the perturbation is small, the tail of \( P(\delta) \) in the part of \( \delta < -1 \) assigned by Gaussian distribution is negligible, because probability of \( \delta < -1 \) is small. When the density field evolves beyond the linear regime, i.e., \( \sigma^2 \equiv \langle \delta^2 \rangle \sim 1 \), a long tail at high \( \delta \) is generated while lower bound is confined at \( \delta = -1 \). Gaussian condition is therefore invalid, in that mode coupling effect causes the initial condition to skew. Terms in the evolution

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equations for the Fourier modes that represent coupling between different modes are of second (or higher) order in $\delta$ and these are neglected when first-order perturbation theory is considered. Phases of Fourier modes therefore are therefore coupled together in a way which is yet to be fully elucidated but which has been recently investigated by Chiang & Coles (2000) and Coles & Chiang (2000).

One of the reasons for studying Fourier phases in depth is the question of possible primordial non-Gaussianity of the initial density distribution of the Universe. For example, there have been claims of non-Gaussianity from analysis results in the COBE DMR sky maps of cosmic microwave background using a number of different diagnostics: bispectrum analysis (Ferreira et al. 1998); the wavelet transform (Hobson et al. 1999); and Minkowski functionals (Schmalzing & Gorski 1998). The most direct approach, however, is through the distribution of Fourier phases. This diagnostic can also avoid the subtlety of more standard methods that depend on the distribution function $P(\delta)$(Scherrer et al. 1991), in that a density field with Gaussian single-point density distribution $P(\delta)$ is not necessarily a Gaussian field. Phase information is also important as a statistical diagnostic of non-linearity when the evolution of clustering in the non-linear regime which is generally intractable analytically. Indeed, as we shall show, it is also closely related to the morphology of gravitational clustering and through this to the dynamical origin of structure.

The layout of this paper is as follows. In section 2 we present a visual demonstration of the link between Fourier phases and clustering morphology. Section 3 contains a brief discussion of some of the properties of phases and some pitfalls that must be avoided when using them as part of a clustering descriptor. In Section 4 we introduce a cross-correlation parameter $S$ and a rank-correlation $\tau$ as quantitative measures of the agreement between two distributions. In Section 5 we display the results of the correlation tests between sample distributions and phase-based reconstructions, and between simulations evolving from different initial power spectra but the same initial phase set. A brief discussion of the results follows in Section 6.

2 VISUAL DEMONSTRATION

To give a qualitative, visual description of the key ideas in this paper consider Fig. 1, in which we isolate the role of phases in determining clustering morphology. Plates (a), (c) and (e) have the same phase configuration, so do plate (b) and (d). Plate (a), (d) and (f), however, share the same power spectrum, or alternatively, two-point correlation function, so do (b) and (c). See the text for details.

Figure 1. Visual demonstration of the importance of phase information for clustering morphology. Plate (a), (c) and (e) have the same phase configuration, so do plate (b) and (d). Plate (a), (d) and (f), however, share the same power spectrum, or alternatively, two-point correlation function, so do (b) and (c). See the text for details.

and the amplitudes from (b), i.e., $F^{-1}[\hat{\delta}_k^a \exp(i\phi_k^a)]$. Plate (d) is from the phases from (b) but amplitudes from (a). Therefore, plate (a) and (c) share the same phase configuration, so do (b) and (d). It is easy to see resemblance between (a) and (c), and between (b) and (d). Note that plate (a) and (d) have the same power spectrum, or equivalently, two-point correlation function, as do plate (b) and (c). Plate (e) is the inverse Fourier transform from only the phases from (a): $F^{-1}\exp(i\phi_k^a)$; it therefore retains only the phase information from (a). In plate (f), each mode keeps the same amplitude so its power spectrum is unchanged (i.e. the same as plate (a) and (d)) but the phases are redistributed randomly among the modes before inverse Fourier transform. Again, plate (e), the phase-only reconstruction (hereafter phase-only reconstruction), resembles the original distribution plate (a), and plate (c), the same-phase amplitude-swapped reconstruction (hereafter amplitude-swapped reconstruction), but (f) which has the same power spectrum as (a) but random phases (hereafter random-phase reconstruction), is featureless. This experiment suffices to show very clearly how phases determine morphology.

Another interesting property of Fourier phases in clustering morphology is that two realisations evolving from the same initial random phase set, though different power-law power spectra, will have their extrema at the same locations. The fact that Plate (a) and (b) are evolved from different initial phase sets is for demonstration purpose.

3 QUANTIFYING PHASE SHIFTS

Fourier phases reflect the locations of spatial ‘events’ more than Fourier amplitudes do. For a single hypothetical spike, represented by the Dirac $\delta$-function $\delta(x - x_0)$, the amplitudes are constant and phases are $kx_0$. This has a white-noise spectrum, but very strong phase correlation. The phase configuration of a spike is very similar to that of a single density peak evolved from 1D Zel’dovich approximation; see Chiang & Coles (2000) for details. A translation $x$ of the Dirac $\delta$-function density field, $\delta\delta(x - x_0 - x')$, has no effect on the Fourier amplitudes, but the phases now become $k(x_0 + x')$, which suffers a shift by a linear term proportional to wave number $k$. This dependency of phases on the choice of origin means that some care must be taken when trying to extract meaningful information. Some previous studies focused on the evolution of individual phases away from their initial values (Ryden & Gramann 1991; Soda & Suto 1992; Jain & Bertschinger 1998). The mean deviation from the initial phase can be defined as

$$\Delta\phi(k, a) = \langle |\Delta\phi(k, a)| \rangle = \langle |\phi(k, a) - \phi(k, a_0)| \rangle,$$

(3)

where the averages are performed over the different modes within a shell in k-space whose wave numbers lie in the range $k - 0.5 < |k| < k + 0.5$. As long as there are enough modes, the maximal value of $\Delta\phi(k, a)$ is $2\pi\sin 3\phi$ (Jain & Bertschinger 1998), which can be understood as a variable obtained from the change from a random field to a hypothetical spike, i.e., from random phases to a very ordered state, $\phi(k, a) = kx_0 (\mod 2\pi)$, where $x_0$ is the location of the spike and takes on any values except zero. This statistic will change following the translation of ‘events’, e.g., $\Delta\phi(k, a) = \pi$ if the hypothetical spike is shifted to the origin.

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4 CROSS AND RANK CORRELATIONS

Although the examples displayed in Section 2 serve to demonstrate the importance of phases, it is necessary to measure objectively how well the phase-based reconstructions compare not only with the original distribution, but also with reconstruction from, say, random phases. The tool we use here is the cross-correlation statistic introduced by Coles, Melott & Shandarin (1993). A correlation coefficient is defined as

\[ S_{\delta\delta} = \frac{\langle (\delta_{ij} - \overline{\delta})(\delta'_{ij} - \overline{\delta'}) \rangle}{\langle (\delta_{ij} - \overline{\delta})^2 \rangle^{1/2} \langle (\delta_{ij} - \overline{\delta})^2 \rangle^{1/2}}, \]

where \( \delta_{ij} \) and \( \delta'_{ij} \) represent two density distribution and \( \overline{\delta} \) and \( \overline{\delta'} \) are their mean densities, respectively. The indices \( i \) and \( j \) label the pixel positions in the two-dimensional simulations we use here for illustration. The parameter \( S \) compares the density value of each grid point at \((i, j)\) in the original distribution \( \delta \) with the corresponding grid point of the reconstruction \( \delta' \); averages are taken over all grid points. \( S_{\delta\delta} = 1 \) denotes ‘completely positive correlation’, a perfect agreement between \( \delta \) and \( \delta' \), and a value of zero indicates the two distributions are uncorrelated. What is useful about this test is that it compares the morphology between two distributions, point by point, but does not take into account their variances. This is because \( S_{\delta\delta} = 1 \) when \( \delta = C\delta' \), \( C \) being a constant. If the structures of two distributions are different, \( |S| \) is less than unity.

Cross-correlation tests the spatial correspondence of the locations of ‘events’, such as clusters of points or edges, between two distributions. Even the comparison by a small displacement between two identical periodic distributions will result in low value of \( S \). It should be pointed out, therefore, that this grid-by-grid test is severe. As we have explained, phases are related to preservation of locations of ‘events’. Relative magnitudes of the ‘events’, however, are not preserved when the information of Fourier amplitudes is partially or totally lost. To relax the test, comparisons are made not only between density distributions \( \delta \) and \( \delta' \), but also between smoothed distributions \( \delta(x, R) \) and \( \delta'(x, R) \). For this purpose, a Gaussian window function is chosen to smooth the field,

\[ \delta(x, R) = \int d^2x' \delta(x') (\sqrt{2\pi} R)^{-2} \exp(-\frac{|x - x'|^2}{2R^2}). \]

As well as the linear cross-correlation \( S \), we also use a non-parametric rank-correlation, Kendall’s \( \tau \) parameter (Kendall and Gibbons 1990). This parameter uses the relative ordering of ranks to measure the degree of agreement between two compared distributions. For any two grid points \((i, j)\) and \((i', j')\), the given value is +1 if the relative ordering of the density values of \( \delta(i, j) \) and \( \delta(i', j') \) is the same as \( \delta'(i, j) \) and \( \delta'(i', j') \), i.e., either both decreasing or both increasing from \((i, j)\) to \((i', j') \); −1 when one pair is increasing and the other is decreasing. The \( \tau \) parameter compares \( N(N - 1)/2 \) pairs from total \( N^2 \) grid points and is normalized to \([-1, 1]\). Rank correlation measures such as this do not look specifically for linear association between the image and reconstruction, but for one-to-one ordering of the values in one relative to those in the other. If there is strong non-linear association, then \( \tau \) can be close to unity even though \( S \) may be small. Kendall’s \( \tau \) parameter is used here as an auxiliary test, which is again “softened” with a Gaussian window function as in eq. (5).

5 RESULTS

Fig. 2 shows cross-correlation coefficient \( S \) and rank-correlation coefficient \( \tau \) drawn against smoothing scale \( R \) (in computer grid units) for the comparison between sample distributions and its phase-based reconstructions. Phase-only and amplitude-swapped reconstructions retain the morphology of the original structure of the sample realisation, which is shown by a high degree of cross-correlation. Random-phase reconstruction, however, bears no resemblance to the original structure, hence low \( S \). Rank-correlation compares the relative magnitudes of the density peaks. Due to the loss of the information on the Fourier amplitudes in the reconstructions, the relative ordering of the ranks of the density magnitudes is partially disturbed. The phase-only reconstructions keep the ordering better than amplitude-swapped reconstructions.
The n-body simulations in terms of cross-correlation coefficient $S$ between realisations evolving from different initial power-law power spectra but the same initial phase set. The plots are drawn against smoothing scale $R$, as in Fig. [3].

The rank-correlations, on the other hand, show that phase-only reconstructions have higher agreement than amplitude-swapped ones on relative density magnitudes, which is due to the 'twist' on the latter’s power spectra. Surely, the loss of the Fourier amplitude information destroys part of the signal, as shown in the rank-correlation of the right panels. On the morphology of gravitational clustering, however, phases play a much more important role than amplitudes.

It emphasizes this point still further to test the correspondence between simulations evolving from different initial power spectra but the same initial phases. In Fig. [3], we compare the morphology for some relevant examples in terms of cross-correlation coefficient $S$ deployed above. In particular, we show realisations obtained by evolving n-body experiments from different 2D initial power-law power spectra $n = -1$, 0 1 and 2. The left panel is the comparison of stage c in which the scale of non-linearity is $k_{NL} = 64k_f$, and the right panel is that of later stage $f$ with $k_{NL} = 8k_f$ (Chiang & Coles 2000, Beacom et al. 1991). We use the fundamental mode $k_f = 2\pi / L$ as length unit, where $L$ is the length of the side of the simulation square.

Before smoothing, there is little or no correspondence between any of the realisations. This is due to the severe point-by-point test of the cross-correlation. After smoothing, however, there is a dramatic improvement in the correspondence between $n = -1, 0, n = 0, 1,$ and between $n = 1, 2$. We can examine the morphology evolving from the same phase set by the characteristic scale $k_{NL}$. For example, $k_{NL}$ for stage $c$ corresponds to 8 computer grid units, thus the smoothing on scales beyond 8 grid units erases non-linearities, while the larger scale structure remains in the linear regime. The linear growth of the density fluctuations set up by the same initial phases depends only on time. The comparison therefore indicates the intrinsic difference in clustering morphology arising from different initial power spectra. What we see on the scales beyond $k_{NL}$ is that the correspondence is low between $n = -1$ and $n = 1, 2$.

The correspondence deteriorates when the difference between the spectral indices increases, i.e., the difference in intrinsic morphology is significant, or when the evolution goes into highly non-linear regime, where particles move away from their initial Lagrangian position and interact with each other non-linearly. Simulations evolving from the same initial phase configuration tend to develop nonlinear structures at or near the same spatial locations, but these structures appear with different contrast when the initial spectra are different. For example, filaments appear in the nonlinear regime in all cases, but for spectra with large $n$ these tend to be less well defined and broken up into clumps. Our statistic $S$ takes into account both the position and amplitude of structures that form so it indicates a deteriorating agreement for very different spectra. Nevertheless, it is clear that there is strong imprint upon the morphology of the initial phases resulting from the process of gravitational clustering. This can also be seen visually in the pictures shown by Beacom et al. (1991).

The cross-correlation coefficient being a single number, so one cannot infer from it precisely how the phase-based reconstructions perform relative to the morphology of the original distribution. One intriguing question is how the density regions from reconstructions can be compared to those from the original distributions. Is it $\delta \propto \delta'$ that produces high value of $S$? From rank-correlation coefficient $\tau$, this is not the case. In order to get more information between the two distributions, we also made a grid-by-grid comparison between them. A scatter diagram can be drawn in logarithmic scales for the comparison between each cell density from phase-based reconstructions against the corresponding one from the original distribution. If $\delta \propto \delta'$, the points scatter along a straight line of slope equal to unity, and the relative magnitude of ‘events’ is preserved by a linear scale factor. Fig. [4] shows the scatter diagram between original distribution and phase-only, amplitude-swapped reconstructions for $n = 1$ and $n = -1$. Not all the points scatter along the

Figure 3. Comparison of morphology of two different stages in N-body simulations in terms of cross-correlation coefficient $S$ between realisations evolving from different initial power-law power spectra but the same initial phase set. The plots are drawn against smoothing scale $R$, as in Fig. [3].

Figure 4. Scatter diagrams of cell density between smoothed original distributions and reconstructions. The smoothing scale is two grid units to produce a continuous distribution. In each panel only one in four grid points in each spatial direction on the $512^2$ grid are sampled: 16384 points and a straight line with slope equal to unity is added for reference. The top-left panel is a comparison between Fig.1(a) and Fig.1(c), the bottom-left is between Fig.1(a) and (c), and the bottom-right is between (b) and (d).
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Figure 5. Scatter diagrams of cell density between simulations evolving from the same phase set. Here we choose between spectral index $n = -1$ and $n = 0$, and between $n = -1$ and $n = 2$. The smoothing scale is four grid units. As in Fig. 4, 16384 points are sampled and a straight line with slope equal to unity is added for reference.

straight line, as is suggested. Instead, most points scatter horizontally, particularly for phase-only cases. Also notice that, in both reconstruction cases, the density values span only roughly one order of magnitude and the points that do not align well correspond to very small fluctuations. High values of $S$ result from phase configuration preserving the locations of high-density ‘events’ with large magnitudes in the original structure. The relatively flat phase-only reconstructed density distribution is caused by the flat power spectrum, in which the Fourier amplitude for each mode is squashed into unity. The lower two panels are scatter diagrams of the amplitude-swapped reconstructions against the originals, i.e., in Fig. 4 between (c) and (a) and between (d) and (b). The difference is amplitude-swapped reconstructions have the power spectra from the alternative sample distributions. The $n = -1$ amplitude-swapped reconstruction has the power spectrum from the original distribution of $n = 1$, which gives more power on small scales in the reconstruction, the $n = 1$ reconstruction, on the other hand, has more power on large scales. This adverse effect causes the points of the low fluctuations to spread on the scatter diagram, which, however, doesn’t decrease much the cross-correlation coefficient $S$ between phase-only and amplitude-swapped reconstructions. This is in accord with our visual impression that our eyes pick up the maxima between the distributions for comparison.

In Fig. 4 the scatter diagrams for the realisations evolving from the same phase set are also produced. Here only two cases are chosen, i.e., between $n = -1$ and 0, and between $n = -1$ and 2. The left two panels are comparison between $n = -1$ and $n = 0$, and the scattering is expected, which nonetheless follows the straight line. For the right panels, at early stage $c$, the deviation from the straight line is more isotropic, hence the correspondence is low. At late stage $f$, there is even more deviation, which comes from the mapping from the low-density regions of stage $f$, $n = 2$. The reason is as follows. For the realisation of $n = 2$ to reach the same level of non-linearity, the variance is higher than that of $n = -1$, and there are a substantial number of voids. After smoothing to create continuous distribution, the void regions are smoothed as low-density regions, which is seen in the bottom-right panel.

6 CONCLUSION

We have shown the importance of Fourier phases on morphology of gravitational clustering via qualitative and quantitative demonstrations. It is interesting to remark upon the similarity of the results we have obtained here and those presented in Coles et al. (1993). The latter authors were interested in the ability of simple analytic methods to reproduce the clustering displayed by full N-body computations. They showed in particular that the Zel’dovich approximation (Zel’dovich 1970) could reproduce the full numerical results quite well, with cross-correlations similar to those we have found here. The Zel’dovich approximation works as well as it does in this respect because it places the caustic surfaces forming sheets and filaments near to the correct location in the N-body experiment. In other words, the Zel’dovich approximation has a high phase fidelity. It is less good at getting the amplitudes right. As we have shown, however, phases dominate the morphology.

With a new colour representation technique (Coles & Chiang 2000), phase information can be used to distinguish between non-Gaussianity induced by gravitational clustering and that by other mechanisms. For example, in the web page

http://www.nottingham.ac.uk/~ppzpc/phases/cmb.html

the phase configuration of the non-Gaussian temperature fluctuations on the CMB sky induced by cosmic strings can be seen to be intrinsically different from that of hierarchical clustering developed in N-body simulations. A new algorithm based on the analysis of the phase distribution of Fourier components is recently devised to extract noise from point sources from the CMB signal(Naselsky et al. 2000), which is another example practice of the close link between Fourier phases and morphology.

Future large-scale galaxy redshift surveys and microwave sky maps will reveal much morphological information about large-scale structure in the Universe. Existing statistical technology, however, is still dominated by second-order methods that are blind to phase information. Our success in extracting this information will therefore depend on the development of statistical methods sufficiently sensitive to the key ingredient: the distribution of Fourier phases.

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REFERENCES

Bardeen J.M., Bond J.R., Kaiser N., Szalay A.S., 1986, ApJ, 304, 15
Beacom J.F., Dominik K.G., Melott A.L., Perkins S.P., Shandarin S.F., 1991, ApJ, 372, 351
Chiang L.-Y., Coles P., 2000, MNRAS, 311, 809
Coles P., Chiang L.-Y., 2000, Nature, 406, 376
Coles P., Melott A.L., Shandarin S.F., 1993, MNRAS, 260, 765
Ferreira P.G., Magueijo J., Gorski K.M., 1998, ApJ, 503, L1
Hobson M.P., Jones A.W., Lasenby A.N., 1999 MNRAS, 309, 125
Jain B., Bertschinger E., 1996, ApJ, 456, 43
Jain B., Bertschinger E., 1998, ApJ, 509, 517
Kendall M., Gibbons J.D., 1990, Rank Correlation Methods, Oxford University Press, New York
Naselsky P., Novikov D., Silk J., astro-ph/0007133
Ryden B.S., Gramann M., 1991, ApJ, 383, 153
Scherrer R.J., Melott A.L., Shandarin S.F., 1991, ApJ, 377, 29
Schmalzing J., Gorski K.M., 1998, MNRAS, 297, 355
Soda J., Suto Y., 1992, ApJ, 396, 379
Zeldovich Y.B., 1970, A&A, 5, 84

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