ON MODULAR CATEGORIES $\mathcal{O}$ FOR QUANTIZED SYMPLECTIC RESOLUTIONS

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Abstract. In this paper we study highest weight and standardly stratified structures on modular analogs of categories $\mathcal{O}$ over quantizations of symplectic resolutions and show how to recover the usual categories $\mathcal{O}$ (reduced mod $p \gg 0$) from our modular categories. More precisely, we consider a conical symplectic resolution that is defined over a finite localization of $\mathbb{Z}$ and is equipped with a Hamiltonian action of a torus $T$ that has finitely many fixed points. We consider algebras $A_\lambda$ of global sections of a quantization in characteristic $p \gg 0$, where $\lambda$ is a parameter. Then we consider a category $\tilde{\mathcal{O}}_\lambda$ consisting of all finite dimensional $T$-equivariant $A_\lambda$-modules. We show that for $\lambda$ lying in a $p$-alcove $^{p}A$, the category $\tilde{\mathcal{O}}_\lambda$ is highest weight (in some generalized sense). Moreover, we show that every face of $^{p}A$ that survives in $^{p}A/p$ when $p \to \infty$ defines a standardly stratified structure on $\tilde{\mathcal{O}}_\lambda$. We identify the associated graded categories for these standardly stratified structures with reductions mod $p$ of the usual categories $\mathcal{O}$ in characteristic 0. Applications of our construction include computations of wall-crossing bijections in characteristic $p$ and the existence of gradings on categories $\mathcal{O}$ in characteristic 0.

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1. Introduction

In this paper we study some aspects of the representation theory of quantizations of symplectic resolutions in large positive characteristic.

1.1. Representations in characteristic 0. We start by recalling some features of the representation theory in characteristic 0, which is more classical and more extensively studied. Let $X$ be a conical symplectic resolution over $\mathbb{C}$ that is defined over a finite localization of $\mathbb{Z}$. We assume that it comes with an action of a torus $T$ that is Hamiltonian, has finitely many fixed points, and is defined over a finite localization of $\mathbb{Z}$. Examples are provided by $X = T^*(G/B)$ (where $G$ is a semisimple algebraic group, $B$ is a Borel subgroup, and $T$ is a maximal torus in $B$) or $X = \text{Hilb}_n(\mathbb{A}^2)$ (and $T$ is a one-dimensional torus acting on $\mathbb{A}^2$ in a Hamiltonian way so that the action lifts to $\text{Hilb}_n(\mathbb{A}^2)$).

Consider the space $\mathfrak{P} := H^2(X, \mathbb{C})$. To $\lambda \in \mathfrak{P}$, we can assign a quantization $A^0_\lambda$ of $X$ that is a microlocal sheaf of filtered algebras on $X$. For example, in the case of $X = T^*(G/B)$ we get the microlocalization of the sheaf of $\lambda - \rho$-twisted differential operators on $G/B$. Taking the global sections of the sheaf $A^0_\lambda$, we get a filtered quantization of $\mathbb{C}[X]$ to be denoted by $A_\lambda$. In the case when $X = T^*(G/B)$ we get the central reductions.
of the universal enveloping algebra $U(g)$, while for the case of $X = \text{Hilb}_n(A^2)$ we recover the spherical subalgebras of the rational Cherednik algebras for $S_n$ (introduced in [EG]). We note that $T$ acts on $A_\lambda$ and the action is Hamiltonian.

Since $T$ has finitely many fixed points in $X$, it makes sense to speak about categories $O$ for $A_\lambda$, see [BLPW Section 5.2]. These categories depend on a choice of a generic one-parameter subgroup $\nu : \mathbb{C}^* \to T$ (where “generic” means that $X^{\nu(\mathbb{C}^*)} = X^T$). The category corresponding to $\nu$ is denoted by $O_\nu(A_\lambda)$. It can be defined as the category of finitely generated $A_\lambda$-modules that admit a $\nu(\mathbb{C}^*)$-equivariant structure so that the weights of $\nu$ are bounded from above. For $X = T^*(G/B)$, we recover an infinitesimal block of the classical BGG category $O$ (and the simples are labelled by the elements of the Weyl group), while for $X = \text{Hilb}_n(A^2)$ we get the category $O$ for the rational Cherednik algebra for $S_n$ (and the simples are labelled by the partitions of $n$). The categories $O$ over quantized symplectic resolutions have been studied very extensively recently from various perspectives, see, e.g., [BLPW, W1, L6, BDGH, BDGHK, L9, W2]—not to mention numerous papers dealing with special cases such as BGG categories $O$, categories $O$ over rational Cherednik algebras and others.

Let us mention some known results about the structure of $O_\nu(A_\lambda)$. We can talk about regular (i.e., non-degenerate) parameters $\lambda$, we will show below that there is a finite union of affine hyperplanes (to be called singular) in $\mathfrak{p}$ such that a parameter outside this finite union is regular. For a regular parameter, the simples in $O_\nu(A_\lambda)$ are labelled by $X^T$, [BLPW Section 5.1]. Also it is known that $O_\nu(A_\lambda)$ is a highest weight category, [BLPW Section 5.2].

In the examples we consider, these finite collections of hyperplanes are as follows. For $X = T^*(G/B)$ we just take the root hyperplanes, $\text{ker} \alpha^\vee$, where $\alpha^\vee$ runs over the set of coroots. For $X = \text{Hilb}_n(A^2)$ we usually consider a shifted parameter, $c = \lambda - 1/2$. Here for the hyperplanes we can take the following collection of points: $\{-\frac{n}{b} + i | 1 \leq a < b \leq n, i \in \mathbb{Z}, |i| \leq \ell\}$ for some non-negative integer $\ell$ (it is expected that we can take $\ell = 0$ but we don’t know how to prove this).

Note that we can also consider the category of $T$-equivariant objects $O^T_\nu(A_\lambda)$ in $O_\nu(A_\lambda)$. This category is the direct sum of several copies of $O_\nu(A_\lambda)$ labelled by the characters of $T$, so we do not get anything new, and the simples are labelled by $X^T \times \mathfrak{x}(T)$, where $\mathfrak{x}(T)$ stands for the character lattice of $T$. In this case, we can define a highest weight order on $X^T \times \mathfrak{x}(T)$ as follows: $(x, \kappa) <_\nu (x', \kappa')$ if $\langle \nu, \kappa \rangle < \langle \nu, \kappa' \rangle$.

### 1.2. Representations in characteristic $p \gg 0$.

Now suppose $\lambda$ is rational, i.e., $\lambda \in \mathfrak{p}_\mathbb{Q} := H^2(X, \mathbb{Q})$. We assume that all the objects from the previous section $(X, \text{the } T$-action, $A_\lambda^0)$ are defined over a finite localization $R$ of $\mathbb{Z}$ depending only on the denominator of $\lambda$. In particular, we have an $R$-algebra $A_{\lambda,R}$ and the base change $A_{\lambda,F} := F \otimes_R A_{\lambda,R}$ for $F := \mathbb{F}_p$ with $p \gg 0$ (we will also impose some congruence conditions on $p$, for example, when $X = \text{Hilb}_n(A^2)$ we will assume that $p+1$ is divisible by $n$!). So we get a quantization $A_{\lambda,F}$ of $F[X]$ for all $\lambda \in \mathfrak{p}_F$. On $A_{\lambda,F}$ we still have a Hamiltonian action of $T_F$.

For $\lambda \in \mathfrak{p}_{F_p} := H^2(X, \mathbb{F}_p)$ (this is the most interesting case), we consider the category $\tilde{O}(A_{\lambda,F})$ consisting of all finite dimensional $T_F$-equivariant $A_{\lambda,F}$-modules. Consider the $p$-singular hyperplanes in $\mathfrak{p}_{F_p}$ that are obtained as the reductions mod $p$ of the singular hyperplanes mentioned in the previous section. We can still talk about the regular parameters in $\mathfrak{p}_{F_p}$: the parameters lying outside these hyperplanes. The $p$-singular hyperplanes split the integral lattice $\mathfrak{p}_\mathbb{Z}$ into the union of regions to be called $p$-alcoves. For
example, for $X = T^*(G/B)$ we get the usual $p$-alcoves (the $p$-singular hyperplanes are of the form $\langle \alpha^\vee, \cdot \rangle = pm$, where $\alpha^\vee$ is a coroot and $m \in \mathbb{Z}$). In particular, we have the so called fundamental $p$-alcove, it is consists from the points $\lambda$ in the weight lattice such that $\langle \alpha^\vee, \lambda \rangle > 0$ (where $\alpha^\vee$ are simple coroots) and $\langle \alpha^\vee, \lambda \rangle > -p$, where $\alpha^\vee_0$ is the minimal coroot. In the case of $X = \text{Hilb}_n(A^2)$, each $p$-alcove consists of integers $z$ satisfying $(p+1)a/b < z < (p+1)a'/b' - \ell$, where $a/b < a'/b'$ are rational numbers with denominators between 2 and $n$ such that the interval $(a/b, a'/b')$ contains no such rational numbers (the nonnegative integer $\ell$ is such as in the previous section).

It turns out that inside of each $p$-alcove categories $\mathcal{O}(A_{\lambda, \xi})$ are naturally equivalent. The categories for two $p$-alcove that are opposite with respect to a common face are perverse equivalent via so called wall-crossing functors.

1.3. Highest weight structures. It turns out that the categories $\mathcal{O}(A_{\lambda, \xi})$ carry highest weight structures (in some generalized sense to be explained below), one per a generic one-parameter subgroup of $T$. Recall that a one-parameter subgroup $\nu : \mathbb{G}_m \to T$ is called generic if $X^{\text{inv}} = X^T$. We fix a one-parameter subgroup $\nu$.

Recall that a highest weight structure on an abelian category is a partial order on the set of irreducible objects subject to certain upper triangularity conditions. Usually, the definition is given in the case when the category of interest is equivalent to that of modules over a finite dimensional algebra. This is not the case with the category $\mathcal{O}(A_{\lambda, \xi})$: the number of simples is infinite (and, moreover, the category does not have projectives, they only exist in a pro-completion). However, we still have a partial order on the set of simples that is defined as follows. The simples in $\mathcal{O}(A_{\lambda, \xi})$ are labelled by pairs $(x, \kappa)$, where $x \in X^T$ and $\kappa \in \mathfrak{X}(T)$. To $x$ we can assign $c_{\nu, \lambda}(x) \in tG$, the highest weight of the corresponding Verma module (over $\mathbb{C}$). We can now define a partial order $\leq_{\nu, \lambda}$ on $X^T \times \mathfrak{X}(T)$ as follows: $(x, \kappa) \leq_{\nu, \lambda} (x', \kappa')$ if $(x, \kappa) = (x', \kappa')$ or

$$\kappa - c_{\nu, \lambda}(x) = \kappa' - c_{\nu, \lambda}(x') \mod p, \langle \nu, \kappa \rangle < \langle \nu, \kappa' \rangle.$$ 

Now pick two integers $z_1 \leq z_2$. We can consider the subquotient $\mathcal{O}(A_{\lambda, \xi})[z_1, z_2]$ corresponding to all labels $(x, \kappa)$ with $z_1 \leq \langle \kappa, \nu \rangle \leq z_2$. Our first principal result, Theorem 8.7, is that this subquotient category is highest weight with respect to the partial order $\leq_{\nu, \lambda}$ (note that when $\dim T = 1$, then the number of simples in $\mathcal{O}(A_{\lambda, \xi})[z_1, z_2]$ is finite, while when $\dim T > 1$, we need to slightly extend the definition of a highest weight category).

It turns out that the equivalences of the categories $\mathcal{O}(A_{\lambda, \xi})$ respect the highest weight structures in a suitable sense.

1.4. Standardly stratified structures. We define the real alcoves (in $\Phi_\mathbb{R}$) as connected components of the complement in $\Phi_\mathbb{R}$ to the hyperplanes that are obtained from the singular ones by translating them by the lattice $\Phi_\mathbb{Z}$. Note that there is a natural bijection between the real alcoves and the $p$-alcoves for $p \gg 0$ (we rescale a $p$-alcove by $1/p$, then there is a unique real alcove whose intersection with a real $p$-alcove is large in a suitable sense).

Now let $A$ be a real alcove, $\partial A$ be the corresponding $p$-alcove. We impose a congruence condition on $p$ requiring that $p+1$ is divisible by a certain number. Then pick a face $\Theta$ of $A$ (of any codimension). To $\Theta$ and a generic one-parameter subgroup $\nu : \mathbb{C}^\times \to T$ (and, in fact, to some additional data) we will assign a standardly stratified structure on
\(\tilde{O}(A_{i,F})\) (that comes from a pre-order on the set of simples and generalizes the notion of a highest weight structure). Such a structure, in particular, gives rise to a filtration on \(\tilde{O}(A_{i,F})\) by Serre subcategories. The main result of this paper, Theorem 9.3,

(i) checks the axioms of a standardly stratified structure,
(ii) relates the associated graded to the category \(O^T_\nu(A_{\bar{\lambda}})\) for a suitable parameter \(\bar{\lambda}\) (informally, the relation is that the associated graded is the reduction to characteristic \(p\) of \(O^T_\nu(A_{\bar{\lambda}})\)),
(iii) and shows that the reductions to characteristic \(p\) of projective and simple objects of \(O^T_\nu(A_{\bar{\lambda}})\) become standard and proper standard objects for the standardly stratified structure.

(ii) basically means that we can recover characteristic 0 categories \(O\) from their characteristic \(p\) analogs, \(\tilde{O}\).

We also relate the wall-crossing functor corresponding to \(\Theta\) to the standardly stratified structures on the categories corresponding to \(A\) (and the alcove opposite to \(A\) with respect to \(\Theta\)). Namely, we show that the wall-crossing functor is a partial Ringel duality functor for the standardly stratified structures, Theorem 9.10.

1.5. Applications. There are two applications of the results outlined in the previous section that we explore in this paper.

First, we use the characterization of the wall-crossing functors as partial Ringel dualities to prove that the wall-crossing bijections (i.e., the bijections between the sets of simples induced by the perverse wall-crossing functors) in characteristic \(p \gg 0\) are the same as in characteristic 0. Computing the wall-crossing bijections is an important ingredient in the Bezrukavnikov-Okounkov program of studying the representations of quantizations of symplectic resolutions in characteristic \(p\).

Second, under some additional assumptions on \(X\) that hold in all examples we know, the category \(\tilde{O}(A_{i,F})\) comes with an additional grading (induced by the contracting torus action on \(X\)). We use (ii) of the previous section to show that the grading carries over to the categories \(O_\nu(A_{\lambda})\). Conjecturally, the grading on \(\tilde{O}(A_{i,F})\) is Koszul and we deduce from here that the resulting grading on \(O_\nu(A_{\lambda})\) is Koszul.

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2. Preliminaries on quantizations

2.1. Symplectic resolutions. Let \(Y\) be a normal Poisson affine variety over \(\mathbb{C}\) with an action of \(\mathbb{C}^X\). The action gives rise to a natural grading on the algebra \(\mathbb{C}[Y]\) of regular functions on \(Y\) with a grading: \(\mathbb{C}[Y] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[Y]_i\). We assume that the grading is positive: \(\mathbb{C}[Y]_i = 0\) when \(i < 0\), and \(\mathbb{C}[Y]_0 = \mathbb{C}\). Further, we assume there is a positive integer \(d\) such that \(\{\mathbb{C}[Y]_i, \mathbb{C}[Y]_j\} \subset \mathbb{C}[Y]_{i+j-d}\) for all \(i, j\). By a symplectic resolution of singularities of \(Y\) one means a pair \((X, \rho)\) of

- a smooth symplectic algebraic variety \(X\) (with form \(\omega\))
- a morphism \(\rho : X \to Y\) of Poisson varieties that is a projective resolution of singularities.
Below we assume that \((X, \rho)\) is a symplectic resolution of singularities. The \(\mathbb{C}^\times\)-action lifts from \(Y\) to \(X\) making \(\rho\) equivariant, see Step 1 of the proof of \([\text{Nam1}],\) Proposition A.7]. Clearly, such a lift is unique.

2.1.1. Structural results. Note that \(\rho^* : \mathbb{C}[Y] \to \mathbb{C}[X]\) is an isomorphism because \(Y\) is normal and \(\rho\) is proper and birational. By the Grauert-Riemenschneider theorem, we have \(H^i(X, O_X) = 0\) for \(i > 0\).

The following is \([\text{BPW}],\) Proposition 2.5.

**Lemma 2.1.** We have \(H^i(X, \mathbb{C}) = 0\) for \(i\) odd.

**Corollary 2.2.** The Chern character map \(\text{Pic}(X) \to H^2(X, \mathbb{Z})\) induces an isomorphism \(\mathbb{Q} \otimes_{\mathbb{Z}} \text{Pic}(X) \xrightarrow{\sim} H^2(X, \mathbb{Q})\).

2.1.2. Deformations. We will be interested in deformations \(\hat{X}/p'\) of \(X\), where \(p'\) is a finite dimensional vector space, and \(\hat{X}\) is a symplectic scheme over \(p'\) with a symplectic form \(\hat{\omega} \in \Omega^2(\hat{X}/p')\) that specializes to the form \(\omega\) on \(X\) and also with a \(\mathbb{C}^\times\)-action on \(\hat{X}\) having the following properties:

- the morphism \(\hat{X} \to p'\) is \(\mathbb{C}^\times\)-equivariant,
- the restriction of the action to \(X\) coincides with the contracting action,
- \(t.\omega := t^d\hat{\omega}\).

It turns out that there is a universal such deformation \(X_p\) over \(p := H^2(X, \mathbb{C})\) (any other deformation is obtained via the pull-back with respect to a linear map \(p' \to p\)). This result for formal deformations is due to Kaledin-Verbitsky, \([\text{KV}],\) but then it carries over to the algebraic setting thanks to the contracting \(\mathbb{C}^\times\)-action on \(X\).

Let \(Y_p\) stand for \(\text{Spec}(\mathbb{C}[X_p])\). Then the natural morphism \(\hat{\rho} : X_p \to Y_p\) is projective and birational and \(Y_p\) is a deformation of \(Y\) over \(p\) meaning, in particular, that \(\mathbb{C}[Y_p]/(p) = \mathbb{C}[Y]\). For \(\lambda \in p\), let \(X_\lambda, Y_\lambda\) denote the fibers of \(X_p, Y_p\) over \(\lambda\). Let \(p^{\text{sing}}\) denote the locus of \(\lambda \in p\) such that \(\rho_\lambda : X_\lambda \to Y_\lambda\) is not an isomorphism. Then, according to Namikawa, \([\text{Nam2}],\) \(p^{\text{sing}}\) is the union of codimension 1 subspaces in \(p\) to be called walls (or classical walls).

2.1.3. Classification of symplectic resolutions. Let us describe the possible symplectic resolutions of \(Y\) following Namikawa.

If \(X^1, X^2\) are two symplectic resolutions of \(Y\), then there are open subvarieties \(\hat{X}^i \subset X^i, i = 1, 2\), with \(\text{codim}_X \hat{X}^i \geq 2\) and \(\hat{X}^1 \sim \hat{X}^2\), see, e.g., \([\text{BPW}],\) Proposition 2.19. This allows to identify the Picard groups \(\text{Pic}(X^1) = \text{Pic}(X^2)\). Let \(p_Z\) be the image of \(\text{Pic}(X)\) in \(H^2(X, \mathbb{C})\).

Set \(p_\mathbb{R} := \mathbb{R} \otimes_{\mathbb{Z}} p_Z\). According to \([\text{Nam2}],\) there is a finite group \(W\) acting on \(p_\mathbb{R}\) as a reflection group, such that the movable cone \(C_{\text{mov}}\) of \(X\) (that does not depend on the choice of a resolution by the previous paragraph) is a fundamental chamber for \(W\). Further, the set of classical walls introduced in Section 2.1.2 is \(W\)-stable and the walls for \(C\) are among the classical walls. So the classical walls further split \(C_{\text{mov}}\) into chambers (to be called classical chambers) and the set of (isomorphism classes of) conical symplectic resolutions of \(Y\) is in one-to-one correspondence with the set of chambers inside \(C_{\text{mov}}\).

For \(\theta \in C_{\text{mov}} \setminus p^{\text{sing}}\) we define \(X^\theta\) to be the symplectic resolutions corresponding to the chamber of \(\theta\). For \(w \in W\), we set \(X^w_\theta := X^\theta\). But we twist the identification of \(p\) with \(H^2(X, \mathbb{C})\) by \(w\) so that the ample cone for \(X\) in \(p\) now contains \(w\theta\).
2.1.4. **Example: cotangent bundles to flag varieties.** Let us proceed to examples.

Take a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let $G$ be the corresponding connected semisimple group of adjoint type and $B \subset G$ be a Borel subgroup. Consider the flag variety $\mathcal{B} = G/B$. For $X$ we can take the cotangent bundle $T^*\mathcal{B}$, then $Y$ is the nilpotent cone $\mathcal{N} \subset \mathfrak{g} \cong \mathfrak{g}^*$ and $\rho : X \rightarrow Y$ is the Springer resolution.

We can identify $p$ with $\mathfrak{h}^* (\cong \mathfrak{h})$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{b}$. Then $W$ is the Weyl group of $\mathfrak{g}$ and $C_{\text{mov}}$ is the positive Weyl chamber. The locus $p^{\text{sing}}$ coincides with the locus of non-regular elements in $\mathfrak{h}^*$ so the classical walls are $\ker \alpha$, where $\alpha$ runs over the set of (positive) roots. The universal deformation $X_p$ is the homogeneous vector bundle $G \times_B \mathfrak{b}$ and $Y_p = \mathfrak{g} \otimes_{\mathfrak{h}/W} \mathfrak{h}$, the morphism $\tilde{\rho} : X_p \rightarrow Y_p$ is Grothendieck’s simultaneous resolution.

This example can be generalized in several ways. For $X$, we can take cotangent bundles to partial flag varieties. More generally, we can also take (parabolic) Slodowy varieties: preimages of transverse slices to nilpotent orbits in the cotangent bundles of (partial) flag varieties.

2.1.5. **Example: Hilb$_n (\mathbb{C}^2)$.** Set $Y = (\mathbb{C}^2)^n / S_n$. This variety admits a symplectic resolution, $X = \text{Hilb}_n (\mathbb{C}^2)$. The space $p$ is one-dimensional and $W = \mathbb{Z}/2\mathbb{Z}$.

There is a classical description of $X$, $Y$ as Nakajima quiver varieties. Consider the vector space $V = \text{End}(\mathbb{C}^n) \oplus \mathbb{C}^{*n}$. It comes equipped with a natural $G := \text{GL}_n (\mathbb{C})$-action. The action extends to a Hamiltonian action on $T^*V = V \oplus V^*$ with moment map $\varphi$, the comoment map $\mathfrak{g} \rightarrow T^*V$ is given by $\varphi^*(x) := x_V$, where $x_V$ is the vector field on $V$ defined by $x \in \mathfrak{g}$.

Now pick a nontrivial character $\theta$ of $G$ so that $\theta = \det^k$, where $k \neq 0$. Consider the semistable locus $(T^*V)^{\theta-ss} \subset T^*V$. Then we can define the Hamiltonian reduction $\varphi^{-1}(0)^{\theta-ss} / G$. It is naturally identified with $\text{Hilb}_n (\mathbb{C}^2)$. Furthermore, $Y = \varphi^{-1}(0)/G$.

This construction of a symplectic resolution can be generalized for $V$ being a framed representation space of any quiver. In this way we get arbitrary Nakajima quiver varieties, all of them are symplectic resolutions.

2.1.6. **Conical slices.** Recall that, by a result of Kaledin, $[K]$, $Y$ has finitely many symplectic leaves. We will impose an additional assumption on $Y$: we want the formal slices to all symplectic leaves come with contracting $\mathbb{C}^*$-actions that rescale the Poisson brackets and lifts to resolutions. We further assume that the slices can be algebrized and the resulting Poisson varieties admit symplectic resolutions (that are obtained from the preimages of slices in $X$).

Let us consider what happens in the examples we consider. In the case of $Y = \mathcal{N}$, the slices are the intersections of the Slodowy slices with $\mathcal{N}$. They are conical (with respect to the so-called Kazhdan action). The formal slices in Nakajima quiver varieties are again quiver varieties, see $[\text{Nak}]$ Section 6 and so are conical.

2.2. **Quantizations.** We will study quantizations of $Y, Y_p, X, X_p$. By a quantization of $Y$, we mean a filtered algebra $\mathcal{A} = \bigcup_i \mathcal{A}_{\leq i}$ such that $[\mathcal{A}_{\leq i}, \mathcal{A}_{\leq j}] \subset \mathcal{A}_{\leq i+j-d}$ for all $i, j$ together with an isomorphism $\text{gr} \mathcal{A} \cong \mathbb{C}[Y]$ of graded Poisson algebras. Similarly, a quantization $\tilde{\mathcal{A}}$ of $Y_p$ is a filtered $\mathbb{C}[[p]]$-algebra (with $p^*$ in degree $d$) together with an isomorphism $\text{gr} \tilde{\mathcal{A}} \cong \mathbb{C}[Y_p]$ of graded Poisson $\mathbb{C}[[p]]$-algebras. For $\lambda \in \mathbb{p}$, we set $\mathcal{A}_\lambda := \mathbb{C}_\lambda \otimes_{\mathbb{C}[[p]]} \tilde{\mathcal{A}}$, this is a filtered quantization of $\mathbb{C}[Y]$. So a quantization of $\mathbb{C}[Y_p]$ can be viewed as a family of quantizations of $\mathbb{C}[Y]$ parameterized by $p$. 

By a quantization of $X = X^\theta$, we mean

- a sheaf $A^\theta$ of filtered algebras in the conical topology on $X$ (in this topology, “open” means Zariski open and $\mathbb{C}^\times$-stable) that is complete and separated with respect to the filtration

- together with an isomorphism $\text{gr}^\theta A \cong O_{X^\theta}$ (of sheaves of graded Poisson algebras).

Similarly, we can talk about quantizations of $X^\rho$.

2.2.1. Classification of quantizations of $X$. ([BPW], Theorem 1.8) (with ramifications given in [L2] Section 2.3) shows that the quantizations $A^\rho$ of $X$ are parameterized (up to an isomorphism) by the points in $\mathfrak{p} = H^2(X, \mathbb{C})$. Below we will use the notation $\mathfrak{p}$ for $\mathfrak{p}$ viewed as a parameter space for quantizations. We $\mathfrak{p}$ view as an affine space, the associated vector space is $\mathfrak{p}$.

More precisely, there is a canonical quantization $A^\rho_{\mathfrak{p}}$ of $X^\rho$ such that the quantization of $X^\rho$ corresponding to $\lambda \in \mathfrak{p}$ is the specialization of $A^\rho_{\mathfrak{p}}$ to $\lambda$. The quantization $A^\rho_{\mathfrak{p}}$ has the following important property: there is an anti-automorphism, $\varsigma$ (to be called the parity anti-automorphism) that preserves the filtration, is the identity on the associated graded, preserves $\mathfrak{p}^* \subset \Gamma(A^\rho_{\mathfrak{p}})$ and induces $-1$ on $\mathfrak{p}^*$. It follows that $A^\rho_{\lambda}$ is isomorphic to $(A^\rho_{\lambda})^{\text{opp}}$.

2.2.2. Algebras of global sections. We set $A^\rho_\mathfrak{p} := \Gamma(A^\rho_{\mathfrak{p}}), A_\lambda = \Gamma(A^\rho_\lambda)$. It follows from [BPW] Section 3.3 that the algebras $A^\rho_\mathfrak{p}, A_\lambda$ are independent of the choice of $\theta$. From $H^i(X^\rho, O_{X^\rho}) = 0$, we deduce that the higher cohomology of both $A^\rho_{\mathfrak{p}}$ and $A^\rho_{\lambda}$ vanish. In particular, $A_\lambda$ is the specialization of $A^\rho_{\mathfrak{p}}$ at $\lambda$. Also we see that $A^\rho_{\mathfrak{p}}$ is a quantization of $\mathbb{C}[X_{\mathfrak{p}}]$ and $A_\lambda$ is a quantization of $\mathbb{C}[X] = \mathbb{C}[Y]$.

From the isomorphism $A^\rho_\lambda \cong A^\rho_{\lambda^{-}}$ we deduce $A_{-\lambda} \cong A^{\text{opp}}_{\lambda}$. Also we have $A_{\lambda} \cong A_{\omega \lambda}$ for all $\lambda \in \mathfrak{p}, w \in W$, see [BPW] Section 3.3.5.

2.2.3. Example: central reductions of universal enveloping algebras. Let us start with the example of $X = T^*B$. Identify the center of the universal enveloping algebra $U(g)$ with $\mathbb{C}[h^*]^W$ by means of the Harish-Chandra isomorphism. Then the quantization $A_\lambda$ of $Y = N$ corresponding to $\lambda \in h^*$ is the central reduction $U(g) \otimes_{\mathbb{C}[h]} W \otimes_{\mathbb{C}[h^*]} \mathbb{C}_\lambda$. The quantization $A^\rho_\lambda$ is (the microlocalization to $T^*B$ of) the sheaf $D_{\lambda^{-}}\rho$ of $\lambda - \rho$-twisted differential operators on $B$. Similarly, the universal quantization $A^\rho_{\mathfrak{p}}$ is $U(g) \otimes_{\mathbb{C}[h]} W \otimes_{\mathbb{C}[h^*]} \mathbb{C}_{\mathfrak{p}}$.

Let us elaborate on the construction of $A^\rho_{\mathfrak{p}}$. Consider the universal sheaf $D_B$ of twisted differential operators. It is constructed as $\varpi_*(D_{G/U})^T$, where $U$ stands for the unipotent radical of $B$, $T = B/U$ is the maximal torus and $\varpi : G/U \to G/B$ is the projection. The microlocalization of $D_B$ is the universal quantization of $T^*B$.

2.2.4. Example: quantizations of Hilb, $(\mathbb{C}^2)^n/S_n$. Now let us describe the quantizations of the Hilbert scheme $X := \text{Hilb}_n(\mathbb{C}^2)$. Consider the algebra $D(V)$ of linear differential operators on the space $V$ from Section 2.1.3. The algebra $D(V)$ carries a Hamiltonian action of $G$. We choose the symmetrized quantum moment map $\Phi(x)$ given by the formula $\Phi(x) = \frac{1}{2}(x_\rho + x_-\rho)$.

We have $\mathfrak{p} = \mathbb{C}$. The quantization $A^\rho_{\mathfrak{p}}$ is given by $\pi_G^*[D_V/D_V\Phi([g, g])/(T^*V)^{\rho-\rho}]^G$, where we write $D_V$ for the microlocal sheaf of algebras on $T^*V$ that is obtained from the algebra $D(V)$ by microlocalization. In our case, the global sections of $A^\rho_{\mathfrak{p}}$ is the quantum Hamiltonian reduction on the level of algebras: $A_\lambda = [D(V)/D(V)\{\Phi(x) - (\lambda, x)|x \in g\}]^G$, it is a quantization of $Y = (\mathbb{C}^2)^n/S_n$. 

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There is an alternative way to construct $A_{\lambda}$ due to Etingof and Ginzburg, \cite{EG}, as the so called \textit{spherical rational Cherednik algebras}. The full rational Cherednik algebra $H_c$, where $c \in \mathbb{C}$ is a parameter, is the quotient of the smash-product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \# S_n$ (where the triangular brackets indicate the free algebra) by the relations

\[
[x_i, x_j] = [y_i, y_j] = 0,
[y_i, x_j] = c(ij), i \neq j,
[y_i, x_i] = 1 - c \sum_{j \neq i} (ij).
\]

Let $e$ be the averaging idempotent in $\mathbb{C}S_n$. We can view $e$ as an element of $H_c$. Then we can consider the unital algebra $eHce$, this is the so called spherical rational Cherednik algebra. It is a quantization of $\mathbb{C}[Y]$.

In fact, by \cite[Theorem 1.3.1]{GG}, $A_{\lambda} \cong eHce$, where $c = \lambda - 1/2$.

Both constructions can be generalized. The quantum Hamiltonian reduction construction produces quantizations of arbitrary Nakajima quiver varieties, while rational Cherednik algebras can be generalized to so called symplectic reflection algebras that give quantizations of the varieties $\mathbb{C}^{2n}/\Gamma$, where $\Gamma$ is a finite subgroup of $\text{Sp}_{2n}(\mathbb{C})$.

2.3. Localization theorems. Here we mostly follow \cite[Sections 5,6]{BPW}. We are interested in the categories $A_{\lambda}$-$\text{mod}$ of all finitely generated $A_{\lambda}$-modules and $\text{Coh}(A_{\lambda}^0)$ of all coherent sheaves of $A_{\lambda}^0$-modules, see \cite[Section 2.3]{BL} for a definition of the latter. These categories are related via the global section functor $\Gamma_{\lambda}^0 : \text{Coh}(A_{\lambda}^0) \to A_{\lambda}$-$\text{mod}$ (mapping a coherent sheaf into its global sections) and its left adjoint, the localization functor $\text{Loc}_{\lambda}^0 := A_{\lambda}^0 \otimes_{A_{\lambda}} \bullet$. These functors have derived versions: $R\Gamma_{\lambda}^0 : D^b(\text{Coh}(A_{\lambda}^0)) \to D^b(A_{\lambda}$-$\text{mod})$ and $L\text{Loc}_{\lambda}^0 : D^{-}(A_{\lambda}$-$\text{mod}) \to D^{-}(\text{Coh}(A_{\lambda}^0))$, the latter restricts to the bounded derived categories whenever $A_{\lambda}$ has finite homological dimension.

We say that \textit{abelian localization holds} for $(\lambda, \theta)$ if the functors $\Gamma_{\lambda}^0, \text{Loc}_{\lambda}^0$ are mutually inverse equivalences between $\text{Coh}(A_{\lambda}^0), A_{\lambda}$-$\text{mod}$. Similarly, we say that \textit{derived localization holds} for $(\lambda, \theta)$ if $R\Gamma_{\lambda}^0, L\text{Loc}_{\lambda}^0$ are mutually inverse equivalences.

2.3.1. Translation bimodules. In order to approach abelian localization, we will need translation bimodules introduced in the present generality in \cite[Section 6.3]{BPW}.

Set $X := X^\theta$. Pick $\chi \in \text{Pic}(X)$ (recall that the Picard groups of different symplectic resolutions of $Y$ are naturally identified). Let $O(\chi)$ denote the corresponding line bundle on $X$. Since $H^i(X, O_X) = 0$ for $i > 0$, the line bundle $O(\chi)$ uniquely deforms to a right $A_{p_{\chi}}^0$-module. It was shown in \cite[Section 5.1]{BPW} that the deformation carries an $A_{p_{\chi}}^0$-bimodule structure, where the adjoint action of $\mu \in p^*$ is by $\langle c_1(\chi), \mu \rangle$. We will denote the resulting bimodule by $A_{p_{\chi}}^0$.

Set $A_{p_{\chi}} := \Gamma(A_{p_{\chi}}^0)$, this is an $A_{p_{\chi}}$-bimodule that is independent of the choice of $\theta$ by \cite[Proposition 6.24]{BPW}. Set $A_{\lambda+\chi} := A_{p_{\chi}} \otimes_{\mathbb{C}[p]} C_{\lambda}$, this is an $A_{\lambda+\chi}$-$A_{\lambda}$-bimodule (here and below we abuse the notation and write $A_{\lambda+\chi}$ instead of $A_{\lambda+c(\chi)}$). We call $A_{\lambda+\chi}$ a \textit{translation bimodule}.

Let us recall some properties of the translation bimodules obtained in \cite{BPW}.
Lemma 2.3 (Proposition 6.31 in [BPW]). Suppose abelian localization holds for \((\lambda + \chi, \theta)\). Then we have a functor isomorphism
\[
\Gamma^\theta_{\lambda + \chi} (\mathcal{A}^\theta_{\lambda, \chi} \otimes \mathcal{A}^\theta_{\lambda} \mathcal{L} \text{Loc}^\theta_{\lambda} (\bullet)) \cong \mathcal{A}_{\lambda, \chi} \otimes L_{\lambda, \chi} \bullet.
\]

Corollary 2.4. Suppose that abelian localization holds for \((\lambda, \theta), (\lambda + \chi, \theta)\). Then the bimodules \(\mathcal{A}_{\lambda, \chi}, \mathcal{A}_{\lambda + \chi, -\chi}\) are mutually inverse Morita equivalences.

Proposition 2.5 (Proposition 5.13 in [BPW]). Let \(\chi\) be in the chamber of \(\theta\). Suppose that the bimodules \(\mathcal{A}_{\lambda + m\chi, \lambda}, \mathcal{A}_{\lambda + (m+1)\chi, -\chi}\) are mutually inverse Morita equivalences for each \(m \geq 0\). Then abelian localization holds for \((\lambda, \theta)\).

The following result was essentially obtained in [BL, Proposition 4.5] (in a special case, a general case can be handled as in the proof of that proposition using [L6, Proposition 2.6]).

Lemma 2.6. The locus of \(\lambda\) such that \(\mathcal{A}_{\lambda, \chi}, \mathcal{A}_{\lambda + \chi, -\chi}\) are mutually inverse Morita equivalences is non-empty and Zariski open.

Let us mention one more important property of translation bimodules.

Lemma 2.7. Let \(\lambda \in \mathfrak{P}\) and \(\chi_1, \chi_2, \chi_3\) be such that abelian localization holds for \((\lambda, \theta), (\lambda + \chi_1, \theta)\) and for \((\lambda' - \chi_3, \theta')\) and \((\lambda', \theta')\), where \(\lambda' = \lambda + \chi_1 + \chi_2 + \chi_3\). Then
\[
\mathcal{A}_{\lambda', -\lambda} = \mathcal{A}_{\lambda + \chi_1 + \chi_2 + \chi_3} \otimes \mathcal{A}_{\lambda + \chi_1 + \chi_2} \otimes \mathcal{A}_{\lambda + \chi_1} \mathcal{A}_{\lambda, \chi_1}.
\]

Proof. We will consider the case when \(\chi_1 = 0\) (the case when \(\chi_3 = 0\) is similar and together they imply the general case). We have a natural bimodule homomorphism
\[
\mathcal{A}_{\lambda', -\chi_2 - \chi_3} \otimes \mathcal{A}_{\lambda, \chi_1} \mathcal{A}_{\lambda, \chi_1} \rightarrow \mathcal{A}_{\lambda', -\lambda}.
\]
But \(\mathcal{A}_{\lambda', -\chi_2 - \chi_3}\) is a Morita equivalence bimodule by Corollary 2.4 and its inverse is \(\mathcal{A}_{\lambda', -\chi_2}\). From here it is easy to see that we get an inverse of (2.1) from the natural homomorphism
\[
\mathcal{A}_{\lambda', -\chi_2} \otimes \mathcal{A}_{\lambda', \chi_2} \mathcal{A}_{\lambda' \chi_1} \mathcal{A}_{\lambda, \chi_1} \rightarrow \mathcal{A}_{\lambda, \chi_1}
\]
by tensoring with \(\mathcal{A}_{\lambda', -\chi_2 - \chi_3}\).

2.3.2. Examples. In the case when \(X = T^*\mathcal{B}\) one can explicitly describe the parameters where abelian and derived localization hold.

Proposition 2.8. Let \(X = T^*\mathcal{B}\). The following is true:

- Derived localization holds for \((\lambda, \theta)\) if and only if \(\lambda\) is regular meaning that \(\langle \lambda, \alpha^\vee \rangle \neq 0\) for every coroot \(\alpha^\vee\).

- Let \(\theta\) be in the positive Weyl chamber. Then abelian localization holds for \((\lambda, \theta)\) if and only if \(\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 0}\) for all positive coroots \(\alpha^\vee\).

These are the classical derived and abelian Beilinson-Bernstein localization theorems, [BB1, BB2].

2.3.3. Example: quantizations of \(\text{Hilb}_n(\mathbb{C}^2)\). Here we have the following result, [GS1, GS2, KR, BE].

Proposition 2.9. Let \(X = \text{Hilb}_n(\mathbb{C}^2)\). The following are true:

1. The homological dimension of \(\mathcal{A}_\lambda\) is infinite, equivalently, derived localization theorem fails for \((\lambda, \theta \neq 0)\), if and only if \(c = \lambda - 1/2\) lies in \((-1, 0)\) and is a rational number with denominator \(\leq n\).
(2) For $\theta > 0$, abelian localization holds for $(\lambda, \theta)$ if and only if
\[
(c + \mathbb{Z}_{\geq 0}) \cap \{-\frac{a}{b} | 1 \leq a < b \leq n\} = \emptyset.
\]

2.4. Harish-Chandra bimodules.

2.4.1. Definition. Let $R$ be a commutative Noetherian ring. Let $\mathcal{A}, \mathcal{A}'$ be two $\mathbb{Z}_{\geq 0}$-filtered $R$-algebras such that $\text{gr} \mathcal{A}, \text{gr} \mathcal{A}'$ are identified finitely generated commutative $R$-algebras, let us denote this algebra by $A$.

By a Harish-Chandra (shortly, HC) bimodule we mean an $\mathcal{A}$-$\mathcal{A}'$-bimodule $\mathcal{B}$ that can be equipped with a good filtration, i.e., an $\mathcal{A}$-$\mathcal{A}'$-bimodule filtration bounded from below subject to the following two properties:

- the induced left and right $A$-actions on $\text{gr} \mathcal{B}$ coincide,
- $\text{gr} \mathcal{B}$ is a finitely generated $A$-module.

By a homomorphism of Harish-Chandra bimodules we mean a bimodule homomorphism. The category of HC $\mathcal{A}$-$\mathcal{A}'$-bimodules is denoted by $\text{HC}(\mathcal{A}$-$\mathcal{A}')$. We also consider the full subcategory $D^b_{\text{HC}}(\mathcal{A}$-$\mathcal{A}')$ of the derived category of $\mathcal{A}$-$\mathcal{A}'$-bimodules with Harish-Chandra homology.

By the associated variety $V(\mathcal{B})$ of $\mathcal{B}$ we mean the support of the finitely generated $A$-module $\text{gr} \mathcal{B}$ in $\text{Spec}(A)$.

2.4.2. Properties. Now let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be three $R$-algebras whose associated graded are identified with the same $R$-algebra $A$ as before. Let $\mathcal{B}_1 \in \text{HC}(\mathcal{A}$-$\mathcal{A}'), \mathcal{B}_2 \in \text{HC}(\mathcal{A}'$-$\mathcal{A}'').$ Then $\text{Tor}_{i}^{\mathcal{A}'}(\mathcal{B}_1, \mathcal{B}_2) \in \text{HC}(\mathcal{A}$-$\mathcal{A}'')$. Indeed, $\text{Tor}_{i}^{\mathcal{A}'}(\mathcal{B}_1, \mathcal{B}_2)$ comes with a natural bounded from below filtration such that $\text{gr} \text{Tor}_{i}^{\mathcal{A}'}(\mathcal{B}_1, \mathcal{B}_2)$ is a subquotient of $\text{Tor}_{i}^{\mathcal{A}}(\text{gr} \mathcal{B}_1, \text{gr} \mathcal{B}_2)$.

Similarly, if $\mathcal{B}_1 \in \text{HC}(\mathcal{A}$-$\mathcal{A}'), \mathcal{B}_2 \in \text{HC}(\mathcal{A}'$-$\mathcal{A}'')$, then $\text{Ext}_{\mathcal{A}}^i(\mathcal{B}_1, \mathcal{B}_2) \in \text{HC}(\mathcal{A}'$-$\mathcal{A}'')$ (and similarly to the Ext’s on the other side).

2.4.3. Examples. We will concentrate on the algebras $\mathcal{A}, \mathcal{A}'$, etc., of the form $\mathcal{A}_\lambda$ (or some other specialization of $\mathcal{A}_p$). For the time being, $R = \mathbb{C}$, while starting Section 6.2 we will also consider the situations when $R$ is a localization of $\mathbb{Z}$ or a positive characteristic field.

For us, the main example of HC bimodules are the translation bimodules $\mathcal{A}_p^{\lambda, \chi}$ and their specializations, see Section 2.3.1. That they are HC was checked in [BPW] Proposition 6.23.

2.4.4. Restriction functors. Let $R = \mathbb{C}$. In [L9] we have constructed the restriction functors for HC bimodules over $\mathcal{A}_\lambda$ (or $\mathcal{A}_p$) under the assumption that $Y$ has conical slices.

Namely, let $y \in Y$ and let $Y, X$ be the algebraizations of the corresponding slices in $Y, X$. We write $\mathcal{A}_\lambda$ for the quantization of $Y$ whose parameter is the image of $\lambda$ in $H^2(\rho^{-1}(y), \mathbb{C})$. Then we have a functor $\bullet_{\lambda, y} : \text{HC}(\mathcal{A}_\lambda$-$\mathcal{A}_\lambda') \to \text{HC}(\mathcal{A}_\lambda$-$\mathcal{A}_\lambda')$. This functor is exact, and intertwines the Tor’s and Ext’s. It annihilates a HC bimodule $\mathcal{B}$ if and only if $y \notin V(\mathcal{B})$. Further, it maps $\mathcal{B}$ to a finite dimensional bimodule if and only if the symplectic leaf of $y$ is open in $V(\mathcal{B})$.

We can also form the algebra $\mathcal{A}_p := \mathbb{C}[\mathbb{P}] \otimes_{\mathbb{C}[\mathcal{P}]} \mathcal{A}_p$. Then we have the restriction functor $\bullet_{\mu, y} : \text{HC}(\mathcal{A}_p \rightarrow \text{HC}(\mathcal{A}_p)$.

2.5. Wall-crossing functors. Here we will recall wall-crossing functors. These functors are a classical tool in the representation theory of semisimple Lie algebras. In the generality we need them they were constructed in [BPW] Section 6.4 and further studied in [L9], where it was shown that some of these functors are perverse.
2.5.1. **Definition.** Let $\lambda \in \mathfrak{p}, \chi \in \text{Pic}(X)$, where $X = X^\theta$. Suppose that abelian localization holds for $(\lambda + \chi, \theta)$, while derived localization holds for $(\lambda, \theta)$. Consider the functor $\mathfrak{MC}_{\lambda + \chi \leftarrow \lambda} := \mathcal{A}_{\lambda, \chi} \otimes_{\mathcal{A}_\lambda} \bullet$, as we have seen in Section 2.3.1 this is a derived equivalence. Note that if abelian equivalence holds for $(\lambda, \theta)$, then $\mathfrak{MC}_{\lambda + \chi \leftarrow \lambda}$ is an abelian equivalence.

2.5.2. **Wall-crossing between simple algebras.** Now let us examine the behavior of $\mathfrak{MC}_{\lambda + \chi \leftarrow \lambda}$, when the algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda + \chi}$ are simple.

**Lemma 2.10.** If the algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda + \chi}$ are simple, then $\mathfrak{MC}_{\lambda + \chi \leftarrow \lambda}$ is an abelian equivalence.

**Proof.** Note that we have natural homomorphisms

$$\mathcal{A}_{\lambda, \chi} \otimes_{\mathcal{A}_\lambda} \mathcal{A}_{\lambda + \chi, -\chi} \to \mathcal{A}_{\lambda + \chi}$$

and $\mathcal{A}_{\lambda, \chi} \otimes_{\mathcal{A}_{\lambda + \chi}} \mathcal{A}_{\lambda + \chi, \lambda} \to \mathcal{A}_\lambda$.

Pick a generic point $y \in Y$ and consider the corresponding restriction functor between the categories of Harish-Chandra bimodules, see Section 2.4.4. The target category for $y$ is $\text{Vect}$ because of the choice of $y$. Moreover, the images of $\mathcal{A}_{\lambda + \chi, -\chi}, \mathcal{A}_{\lambda, \chi}$ are one-dimensional. The functor sends the bimodule homomorphisms above to the identity maps. So their kernels and cokernels vanish, equivalently they have proper associated varieties. This already shows that the homomorphisms are surjective. But the kernels are also Harish-Chandra bimodules. Since the algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda + \chi}$ are simple they have no Harish-Chandra bimodules with proper associated varieties, this follows, for example, from [L5, Lemma 4.4].

2.5.3. **Perversity.** Let us recall the general definition of a perverse equivalence due to Chuang and Rouquier. Let $\mathcal{T}^1, \mathcal{T}^2$ be triangulated categories equipped with $t$-structures that are homologically finite (each object in $\mathcal{T}^i$ has only finitely many nonzero homology groups). Let $\mathcal{C}^1, \mathcal{C}^2$ denote the hearts of $\mathcal{T}^1, \mathcal{T}^2$, respectively.

We are going to define a perverse equivalence with respect to filtrations $\mathcal{C}^i = \mathcal{C}^i_0 \supset \mathcal{C}^i_1 \supset \ldots \supset \mathcal{C}^i_k = \{0\}$ by Serre subcategories. By definition, this is a triangulated equivalence $\mathcal{T}^1 \to \mathcal{T}^2$ subject to the following conditions:

1. **(P1)** For any $j$, the equivalence $\mathcal{F}$ restricts to an equivalence $\mathcal{T}^1_{\mathcal{C}^i_j} \to \mathcal{T}^2_{\mathcal{C}^i_j}$, where we write $\mathcal{T}^i_{\mathcal{C}^i_j}, i = 1, 2$, for the category of all objects in $\mathcal{T}^i$ with homology (computed with respect to the $t$-structures of interest) in $\mathcal{C}^i_j$.
2. **(P2)** For $M \in \mathcal{C}^i_j$, we have $H_j(\mathcal{F}M) = 0$ for $\ell < j$ and $H_\ell(\mathcal{F}M) \in \mathcal{C}^2_{j+1}$ for $\ell > j$.
3. **(P3)** The functor $M \mapsto H_j(\mathcal{F}M)$ induces an equivalence $\mathcal{C}^1_j/\mathcal{C}^1_{j+1} \sim \mathcal{C}^2_j/\mathcal{C}^2_{j+1}$ of abelian categories.

We note that thanks for (P3), $\mathcal{F}$ induces a bijection $\varphi : \text{Irr}(\mathcal{C}^1) \sim \text{Irr}(\mathcal{C}^2)$.

It turns out that the wall-crossing functor $\mathfrak{MC}_{\lambda + \chi \leftarrow \lambda}$ is perverse under certain additional assumptions, [L9, Section 3.1]. Namely, suppose that abelian localization holds for $(\lambda, \theta), (\lambda + \chi, \theta')$ and derived localization holds for $(\lambda, \theta')$, where $\theta, \theta'$ lie in chambers that are opposite with respect to a common face. Then $\mathfrak{MC}_{\lambda + \chi \leftarrow \lambda}$ is perverse, [L9, Theorem 3.1]. Under some more assumptions one can describe the filtrations in terms of annihilators by certain ideals. Namely, let $\Gamma$ be the common face and $\mathfrak{p}_0$ be the subspace of $\mathfrak{p}$ spanned by $\Gamma$. Let us assume that abelian localization holds for $(\lambda, \theta)$ and $(\lambda + \chi, \theta')$ and derived localization holds for $(\hat{\lambda}, \theta')$ for a Weil generic $\hat{\lambda} \in \mathfrak{P}_0 := \mathfrak{p} + \mathfrak{p}_0$. Then it was shown in [L9, Theorem 3.1] that there are chains of ideals $\mathcal{A}_{\mathfrak{p}_0} = \mathcal{I}^k_{\mathfrak{p}_0} \supset \mathcal{I}^{k-1}_{\mathfrak{p}_0} \supset \ldots \supset \mathcal{I}^0_{\mathfrak{p}_0} = \{0\}$...
and \( A_{\mathfrak{p}_0 + \chi} = T_{\mathfrak{p}_0 + \chi}^k \supset T_{\mathfrak{p}_0 + \chi}^{k-1} \supset \cdots \supset T_{\mathfrak{p}_0 + \chi}^0 = \{0\} \), where \( k = \frac{1}{2} \dim X \) with the following properties:

(a) For a Weil generic \( \lambda \in \mathfrak{p}_0 \), the specialization \( I_{\lambda}^j \) is the minimal ideal \( I \subset A_{\chi} \) with \( \text{GK-dim} \ A_{\chi}/I \leq 2(k - j) \). The similar characterization is true for \( I_{\lambda}^i_{\chi + \chi} \subset A_{\chi + \chi} \).

(b) For a Zariski generic \( \lambda \in \mathfrak{p}_0 \) and \( B := A_{\lambda - \chi} \), we have

(b1) For all \( i, j \), we have \( \tau^A_{\lambda}(B, A_{\chi}/I_{\lambda}^j) = 0 \).

(b2) For all \( i, j \), we have \( \tau^B_{A_{\chi}}(A_{\chi}/I_{\lambda}^j, B)I_{\lambda}^j = 0 \).

(b3) We have \( \tau^B_{A_{\chi^j}}(B, A_{\lambda}/I_{\lambda}^i) = 0 \) for \( i < j \).

(b4) We have \( \tau^B_{A_{\chi}}(B, A_{\chi}/I_{\lambda}^j) \tau^B_{A_{\chi}}(A_{\chi}/I_{\chi}, B) \tau^B_{A_{\chi}}(A_{\chi}/I_{\chi}, B)I_{\chi}^{j-1} = 0 \) for \( i > j \).

(b5) Set \( B_j := \tau^B_{A_{\chi}}(B, A_{\chi}/I_{\lambda}^i) \). The kernel and the cokernel of the natural homomorphism

\[ B_j \otimes A_{\chi} \rightarrow \text{Hom}_{A_{\chi}}(B_j, A_{\chi}/I_{\lambda}^j) \rightarrow A_{\chi}/I_{\chi}^j \]

are annihilated by \( I_{\chi}^{j-1} \) on the left and on the right.

(b6) The kernel and the cokernel of the natural homomorphism

\[ \text{Hom}_{A_{\chi}}(B_j, A_{\chi}/I_{\chi}^j) \otimes A_{\chi + \chi} B_j \rightarrow A_{\chi}/I_{\chi}^j \]

are annihilated on the left and on the right by \( I_{\chi}^{j-1} \).

It was shown in [12] Section 3.4] that the functor \( A_{\chi}/I_{\chi} \otimes A_{\lambda} \) is perverse once the conditions of (b) hold. Conversely, it is straightforward to see that if \( A_{\chi}/I_{\chi} \otimes A_{\lambda} \) is perverse, (b1)-(b6) hold.

Remark 2.11. Using techniques of Section [13] below we can show that the locus of \( \lambda \) in (b) is given by removing finitely many hyperplanes from \( \mathfrak{p}_0 \).

Remark 2.12. Note that the filtrations making a derived equivalence perverse are uniquely recovered from that equivalence (provided they exist). In particular, thanks to Lemma 2.7 and (b), the filtrations making \( \mathfrak{mc}_{\lambda + \lambda} \) perverse are always given by annihilation by a suitable chain of two-sided ideals (that are obtained from \( I_{\chi}^j, I_{\lambda + \chi}^j \) via Morita equivalences). In particular, these chains of ideals in \( A_\lambda, A_{\lambda + \chi} \) are determined uniquely.

3. Preliminaries on categories \( O \)

3.1. Highest weight and standardly stratified structures. Let \( \mathbb{K} \) be a field. Let \( C \) be a \( \mathbb{K} \)-linear abelian category equivalent to the category of finite dimensional modules over a split unital associative finite dimensional \( \mathbb{K} \)-algebra. We will write \( T \) for an indexing set for the simple objects of \( C \). Let us write \( L(\tau) \) for the simple object indexed by \( \tau \in T \) and \( P(\tau) \) for the projective cover of \( L(\tau) \).

3.1.1. Highest weight categories. The additional structure of a highest weight category on \( C \) is a partial order \( \leq \) on \( T \) that should satisfy axioms (HW1),(HW2) to be explained below. To state the axioms we need some notation.

The partial order \( \leq \) defines a filtration \( C_{\leq \tau} \) on \( C \) by Serre subcategories indexed by \( T \): the subcategory \( C_{\leq \tau} \) is, by definition, the Serre span of the simples \( L(\tau') \) with \( \tau' \leq \tau \). Define \( C_{< \tau} \) analogously and let \( C_{\tau} \) denote the quotient \( C_{< \tau}/C_{< \tau} \). The first axiom of a highest weight category is as follows:
(HW1) $\mathcal{C}_\tau$ is equivalent to the category of vector spaces for all $\tau$. 

To formulate the second axiom we need some more notation. Let $\pi_\tau$ denote the quotient functor $C_{<\tau} \to C_\tau$. Let us write $\Delta_\tau : C_\tau \to C_{<\tau}$ for the left adjoint functor of $\pi_\tau$. Let $P_\tau$ denote the indecomposable object in $C_\tau$ and set $\Delta(\tau) = \Delta_\tau(P_\tau)$. The object $\Delta(\tau)$ is called \textit{standard}. The second axiom of a highest weight category is as follows:

(HW2) $P(\tau)$ surjects onto $\Delta(\tau)$ in such a way that the kernel is filtered by $\Delta(\tau')$'s with $\tau' > \tau$.

We note that $\mathcal{C}^{\text{opp}}$ is also a highest weight category with respect to the same order. The standard objects in $\mathcal{C}^{\text{opp}}$ are called costandard objects and are denoted by $\nabla(\tau)$. They have the following property: $\dim \text{Ext}^{i}_{\mathcal{C}}(\Delta(\tau), \nabla(\tau')) = \delta_{i,0}\delta_{\tau,\tau'}$.

Below we will need the following lemma.

\textbf{Lemma 3.1.} For $M \in \mathcal{C}$, $\tau \in \mathcal{T}$, the following two conditions are equivalent:

1. $\dim \text{Hom}(\Delta(\tau'), M) = \delta_{\tau,\tau'}$,
2. $M \twoheadrightarrow \nabla(\tau)$.

\textbf{Proof.} It is clear that (2) $\Rightarrow$ (1). Let us prove (1) $\Rightarrow$ (2). Note that $M \in \mathcal{C}_{\leq \tau}$, so we can assume that $\tau$ is maximal. In this case, $\nabla(\tau)$ is injective, $\Delta(\tau)$ is projective. So $L(\tau)$ occurs in $M$ with multiplicity 1 that gives a nonzero homomorphism $M \rightarrow \nabla(\tau)$. We need to show that it is injective. Let $K$ be the kernel. Then $L(\tau)$ is not a composition factor of $K$. It follows that $\text{Hom}(\Delta(\tau'), K) = 0$ for all $\tau'$. Hence $K = 0$. \hfill $\Box$

\textbf{Remark 3.2.} There are various generalizations of the notion of a highest weight category. We will need the situation when $\mathcal{C}$ has finite length property and also enough projectives but not necessarily finitely many simples. Instead of requiring that there are finitely many simples we can impose more general conditions:

- the length of all chains of labels in $(\text{Irr}(\mathcal{C}), \leq)$ is bounded from above by some constant, say $d$,
- and the quotient functors $C_{<\tau} \rightarrow C_\tau$ have left adjoints.

Then we say that $\mathcal{C}$ is a highest weight category if (HW1),(HW2) hold.

3.1.2. \textit{Standardly stratified categories}. Let us define standardly stratified categories following [LW] Section 2.

Let $\mathcal{C}$ be the same as in the first paragraph of this section. The additional structure of a standardly stratified category on $\mathcal{C}$ is a partial \textit{pre-order} $\preceq$ on $\mathcal{T}$ that should satisfy axioms (SS1),(SS2) to be explained below. Let $\Xi$ denote the set of equivalence classes of $\preceq$. Then, for $\xi \in \Xi$, we can define the Serre subcategories $\mathcal{C}_{<\xi} \subset \mathcal{C}_{\leq \xi}$ similarly to Section 3.1.1. their quotient $\mathcal{C}_{\xi}$ together with the quotient functor $\pi_\xi : \mathcal{C}_{<\xi} \rightarrow \mathcal{C}_{\xi}$ and its left adjoint $\Delta_\xi$. Further, for $\tau \in \xi$, we write $L_\xi(\tau)$ for $\pi_\xi(L(\tau))$ and $P_\xi(\tau)$ for the projective cover of $L_\xi(\tau)$ in $\mathcal{C}_{\xi}$. We define the standard (resp. proper standard) objects by $\Delta_{\preceq}(\tau) := \Delta_\xi(P_\xi(\tau))$, resp., $\Delta_{\preceq}(\tau) := \Delta_\xi(L_\xi(\tau))$.

The axioms of a standardly stratified category as defined in [LW] are as follows.

(SS1) The functor $\Delta_\xi$ is exact.
(SS2) The projective object $P(\tau)$ admits an epimorphism onto $\Delta_{\preceq}(\tau)$ with kernel filtered by $\Delta_{\preceq}(\tau')$'s with $\tau' > \tau$.

We will be mostly interested in standardly stratified structures on highest weight categories subject to suitable compatibility conditions. Namely, let $\preceq$ be a partial order on $\mathcal{T}$
defining a highest weight structure on \( \mathcal{C} \). We say that a pre-order \( \preceq \) on \( \mathcal{T} \) is compatible with \( \preceq \) if \( \tau < \tau' \Rightarrow \tau \preceq \tau' \Rightarrow \tau \preceq \tau' \).

**Lemma 3.3.** Let \( \preceq \) be a pre-order compatible with \( \preceq \). Then it defines a standardly stratified structure on \( \mathcal{C} \) if and only if both \( \pi_{\xi}^* \) and \( \pi_{\xi}^* \) (the right adjoint of \( \pi_{\xi} \)) are exact.

**Proof.** \( \mathcal{C} \) satisfies conditions of [L6] Lemma 3.3 because it is highest weight. Our claim follows from [L6] Lemma 3.3. \( \square \)

**Remark 3.4.** Remark 3.2 can be generalized to the setting of standardly stratified categories in a straightforward way.

3.1.3. Partial Ringel dualities. Let \( \mathcal{C} \) be a highest weight category with respect to a partial order \( \preceq \) and let \( \preceq \) be a compatible partial pre-order giving a standardly stratified structure.

Now let \( \mathcal{C}' \) be another category with \( \text{Irr}(\mathcal{C}') \overset{\sim}{\to} \mathcal{T} \). Define a new partial order on \( \mathcal{T} \) by \( \tau \preceq \tau' \) if either \( \tau < \tau' \) or \( \tau \sim \tau' \) and \( \tau \preceq \tau' \). Suppose that \( \mathcal{C}' \) is highest weight with respect to \( \preceq \) and standardly stratified with respect to \( \preceq \).

By a partial Ringel duality functor we mean a derived equivalence \( \psi : D^b(\mathcal{C}) \to D^b(\mathcal{C}') \) that maps \( \Delta(\tau) \) to \( \Delta_{\preceq}(\nabla(\tau)) \), where \( \nabla(\tau) \) is the costandard object in the subquotient category \( \mathcal{C}_{\xi}' \). Note that such a functor is defined uniquely up to pre- or post-composing with an abelian equivalence that is the identity on \( K_0 \). Also \( \mathcal{C}' \) is defined uniquely up to an abelian equivalence which is the identity on the set of simples.

Note that if \( \preceq \) is the trivial pre-order, then we recover the usual notion of Ringel duality. The usual Ringel dual always exists. In general, the existence is unclear, see [L9] Section 4.5.

**Remark 3.5.** We can generalize partial Ringel dualities to categories of the kind considered in Remarks 3.2, 3.4.

3.2. Categories \( \mathcal{O} \): general setup. Let \( \mathcal{A} \) be a Noetherian associative algebra of countable dimension equipped with a rational Hamiltonian \( \mathbb{C}^\times \)-action \( \nu \). Let \( h \in \mathcal{A} \) be the image of 1 under the comoment map and let \( \mathcal{A}_i \) denote the image of \( \mathcal{A}_i := \{ a \mid [h, a] = ia \} \). We set \( \mathcal{A}^{>0} := \bigoplus_{i \geq 0} \mathcal{A}_i, \mathcal{A}^{=0} := \bigoplus_{i > 0} \mathcal{A}_i, \mathcal{C}_{\nu}(\mathcal{A}) := \mathcal{A}^{>0}/(\mathcal{A}^{>0} \cap \mathcal{A}^{>0}). \)

Note that \( \mathcal{A}^{>0} \cap \mathcal{A}^{>0} \) is a two-sided ideal in \( \mathcal{A}^{>0} \). We have a natural isomorphism

\[
\mathcal{C}_{\nu}(\mathcal{A}) \cong \mathcal{A}^{0}/\bigoplus_{i > 0} \mathcal{A}^{-i} \mathcal{A}^i.
\]

Let us note that the definitions of \( \mathcal{A}^{>0}, \mathcal{A}^{>0}, \mathcal{C}_{\nu}(\mathcal{A}) \) make sense even if the action \( \nu \) is not Hamiltonian.

By the category \( \mathcal{O}_{\nu}(\mathcal{A}) \) we mean the full subcategory of the category \( \mathcal{A} \)-mod of the finitely generated \( \mathcal{A} \)-modules consisting of all modules such that \( \mathcal{A}^{>0} \) acts locally nilpotently. We have the Verma module functor \( \Delta_{\nu} : \mathcal{C}_{\nu}(\mathcal{A}) \)-mod \( \to \mathcal{O}_{\nu}(\mathcal{A}) \) given by

\[
\Delta_{\nu}(N) := \mathcal{A} \otimes_{\mathcal{A}^{>0}} N = (\mathcal{A}/\mathcal{A}^{>0}) \otimes_{\mathcal{C}_{\nu}(\mathcal{A})} N.
\]

Note that if \( h \) acts on \( N \) with eigenvalue \( \alpha \) (so that \( N \) is a single generalized \( h \)-eigenspace), then \( h \) acts locally finitely on \( \Delta_{\nu}(N) \) with eigenvalues in \( \alpha + \mathbb{Z}_{\leq 0} \). The module \( \Delta_{\nu}(N) \) has the maximal submodule that does not intersect \( N \) (\( N \) is embedded into \( \Delta_{\nu}(N) \) as the \( \alpha \)-eigenspace for \( h \)), the quotient is denoted by \( L_{\nu}(N) \). The map \( N \leftrightarrow L_{\nu}(N) \) is easily seen to be a bijection \( \text{Irr}(\mathcal{C}_{\nu}(\mathcal{A})) \overset{\sim}{\to} \text{Irr}(\mathcal{O}_{\nu}(\mathcal{A})) \).
Assume now that \( \dim C_\nu(\mathcal{A}) < \infty \). Let \( N_1, \ldots, N_r \) be the full list of the simple \( C_\nu(\mathcal{A}) \)-modules. Let \( \alpha_1, \ldots, \alpha_r \) be the eigenvalues of \( h \) on these modules (note that \( h \) maps into the center of \( C_\nu(\mathcal{A}) \) and so acts by a scalar on every irreducible module). We define a partial order \( \leq \) on the set \( N_1, \ldots, N_r \) by setting \( N_i \leq N_j \) if \( \alpha_i - \alpha_j \in \mathbb{Z}_{>0} \) or \( i = j \). Using this partial order it is easy to show that all the generalized eigenspaces for \( h \) are finite dimensional and all modules in the category \( \mathcal{O} \) have finite length. Moreover, we say that \( N_i, N_j \) (or the corresponding simples \( L_\nu(N_i), L_\nu(N_j) \)) lie in the same \( h \)-block if \( \alpha_i - \alpha_j \in \mathbb{Z} \). Note that the simples of \( \mathcal{O} \) from different \( h \)-blocks lie in different blocks so we can decompose \( \mathcal{O} \) according to \( h \)-blocks: \( \mathcal{O}_\nu(\mathcal{A}) = \sum_{\beta \in \mathbb{C}/\mathbb{Z}} \mathcal{O}_{\nu}^{\beta+\mathbb{Z}}(\mathcal{A}), \) where \( \mathcal{O}_{\nu}^{\beta+\mathbb{Z}}(\mathcal{A}) \) is the Serre span of the simples in the \( h \)-block corresponding to \( \beta \).

We will also need a graded version \( \mathcal{O}_\nu^{gr}(\mathcal{A}) \) of the category \( \mathcal{O} \). By definition, it consists of the modules in \( \mathcal{O}_\nu(\mathcal{A}) \) together with a grading (compatible with the grading on \( \mathcal{A} \)). The homomorphisms are grading preserving. Note that the irreducibles in \( \mathcal{O}_\nu^{gr}(\mathcal{A}) \) are labelled by the graded irreducible \( C_\nu(\mathcal{A}) \)-modules. In the situation when \( \dim C_\nu(\mathcal{A}) < \infty \), the category \( \mathcal{O}_\nu^{gr}(\mathcal{A}) \) splits into the direct sum \( \bigoplus_{\alpha \in \mathbb{C}} \mathcal{O}_\nu^{\alpha}(\mathcal{A}) \), where \( \mathcal{O}_\nu^{\alpha}(\mathcal{A}) \) consists of all modules \( M \) such that \( h \) acts on the graded component \( M(i) \) with single eigenvalue \( \alpha + i \). Note that \( \mathcal{O}_\nu^{\alpha}(\mathcal{A}) \cong \mathcal{O}_\nu^{\alpha+\mathbb{Z}}(\mathcal{A}) \).

3.3. Categories \( \mathcal{O} \): setup of quantized symplectic resolutions. Now we are going to concentrate on the case when \( \mathcal{A} = A_\lambda \) is the algebra of global sections of a quantization \( A_\lambda^q \). Suppose that \( X \) is equipped with a Hamiltonian action of a torus \( T \) such that \( X^T \) is finite. Choose a one-parameter subgroup \( \nu : \mathbb{C}^* \to T \) and assume that \( X^{\nu(\mathbb{C}^*)} \) is a finite set. In this case, \( X^{\nu(\mathbb{C}^*)} = X^T \).

Lemma 3.6. The algebra \( C_\nu(A_\lambda q) \) is finitely generated over \( \mathbb{C}[\mathfrak{p}] \). In particular, \( C_\nu(A_\lambda q) = C_\nu(A_\lambda) \otimes_{\mathbb{C}[\mathfrak{p}]} \mathbb{C}_\lambda \) is finite dimensional.

Proof. The algebra \( C_\nu(A_\lambda q) \) carries a natural filtration and \( C_\nu(\mathbb{C}[Y]) \to \text{gr} C_\nu(A_\lambda q) \). So it is enough to show that \( C_\nu(\mathbb{C}[Y]) \) is finitely generated over \( \mathbb{C}[\mathfrak{p}] \). Note that \( C_\nu(\mathbb{C}[Y])/\mathfrak{p} = C_\nu(\mathbb{C}[Y]) \) is finitely generated and is supported at \( 0 \in Y \); hence is finite dimensional. It follows that \( C_\nu(\mathbb{C}[Y]) \) is finitely generated over \( \mathbb{C}[\mathfrak{p}] \). \( \square \)

For a generic \( \lambda \) one can give a more precise description of \( C_\nu(\mathcal{A}_\lambda) \), see [L6] Proposition 5.3 or [BLPW] Section 5.1 for a somewhat weaker result. Namely, we can define the Cartan subquotient \( C_\nu(A_\lambda^q) \) that will be a sheaf on \( X^{\nu(\mathbb{C}^*)} \), see [L6] Section 5.2. In our case, \( C_\nu(A_\lambda^q) \) is an algebra naturally identified with \( \mathbb{C}[X^{\nu(\mathbb{C}^*)}] \). By the construction, there is a natural homomorphism \( C_\nu(A_\lambda) \to C_\nu(A_\lambda^q) = \mathbb{C}[X^{\nu(\mathbb{C}^*)}] \).

Lemma 3.7 (Proposition 5.3 in [L6]). For a Zariski generic \( \lambda \in \mathfrak{p} \), the homomorphism \( C_\nu(A_\lambda) \to \mathbb{C}[X^T] \) is an isomorphism.

Below we will give a more precise description of this Zariski open subset.

Following [BLPW] Section 3.3 (see also [L6] Section 4.3.), one can define the subcategory \( \mathcal{O}_\nu(\mathcal{A}_\lambda^q) \subset \text{Coh}(A_\lambda^q) \). By definition, it consists of the coherent sheaves of \( A_\lambda^q \)-modules supported on the contracting locus for \( \nu \) that admit a \( \nu(\mathbb{C}^*) \)-equivariant sense (in a weak sense, i.e., without a quantum moment map). It was shown in [BLPW] Corollary 3.19 that the functors \( L_i \text{Loc}_\lambda^q, \text{PRT}_\lambda^q \) map objects from category \( \mathcal{O} \) to objects from category \( \mathcal{O} \). In particular, if abelian localization holds for \((\lambda, \theta)\), then the categories \( \mathcal{O}_\nu(\mathcal{A}_\lambda), \mathcal{O}_\nu(\mathcal{A}_\lambda^q) \) are equivalent (via Loc and \( \Gamma \)).

Note that similarly to Section 3.2 we can consider the graded versions \( \mathcal{O}_\nu^{gr}(\mathcal{A}_\lambda), \mathcal{O}_\nu^{gr}(\mathcal{A}_\lambda^q) \).
Also we can consider the $T$-equivariant versions $\mathcal{O}_p^\nu(A_\lambda), \mathcal{O}_p^\theta(A_\lambda)$ of all $T$-equivariant objects in $\mathcal{O}_p(A_\lambda), \mathcal{O}_p(A_\lambda)$ (when $T$ is one-dimensional, we recover the categories $\mathcal{O}_p^\nu(A_\lambda)$ from Section 3.2).

### 3.4. Families of objects

In this section we are going to recall several families of objects in $\mathcal{O}_p(A_\lambda)$ and $\mathcal{O}_p(A_\lambda)$: various relatives of Verma modules and also dual Verma modules.

#### 3.4.1. Objects $\Delta_{\nu,\lambda}(x)$

Let $x \in X^T$. We can view $C_x$, the simple $\mathbb{C}[X^T]$-module corresponding to $x$, as a module over $C_p(A_\lambda)$ via the homomorphism $C_p(A_\lambda) \to \mathbb{C}[X^T]$. We denote the module $\Delta_p(C_x)$ by $\Delta_{\nu,\lambda}(x)$.

The modules $\Delta_{\nu,\lambda}(x)$ come in a family over $\mathfrak{P}$: we can use the homomorphism $C_p(A_\mathfrak{P}) \to \mathbb{C}[\mathfrak{P}][X^T]$ to form the module $\Delta_{\nu,\mathfrak{P}}(x)$ whose specialization to $\lambda$ is $\Delta_{\nu,\lambda}(x)$. Similarly, we can consider a homogenized version. Namely, form the Rees algebra $A_{p,h}$ of $A_\mathfrak{P}$. We still have a homomorphism $C_p(A_{p,h}) \to \mathbb{C}[p,h][X^T]$ and so can form the $A_{p,h}$-module $\Delta_{\nu,p,h}(x)$ that specializes to $\Delta_{\nu,\mathfrak{P}}(x)$ when $h = 1$.

Note that, for $\lambda \in p \setminus p^{\text{sing}}$, we get $\Delta_{\nu,\lambda,0}(x) = \mathbb{C}[(X_\lambda)_{X_x}^+]$, where $(X_\lambda)_{X_x}^+ := \{x \in X_\lambda | \lim_{t \to 0} \nu(t)x = x\}$. The variety $(X_\lambda)_{X_x}^+$ is a smooth lagrangian subvariety in $X_\lambda$ that is isomorphic to an affine space.

Now let $\kappa \in \mathfrak{X}(T)$. We can form the graded versions $\Delta_{\nu,\lambda}(x,\kappa), \Delta_{\nu,\mathfrak{P}}(x,\kappa)$ etc. by putting $C_x$ in degree $\kappa$.

#### 3.4.2. Objects $\hat{\Delta}_{\nu,\lambda}(x)$

These objects were introduced in [BLPW] Section 5.3 (there they were denoted by $\Theta_\alpha$, where $\alpha \in X^T$). We will start by defining the objects $\hat{\Delta}_{\nu,p,h}(x)$.

Namely, consider the Rees sheaf $R_h(A^{\theta,\wedge}_{p,h})$ and its completion $A^{\theta,\wedge}_{p,h}$ at $x$. This completion is the completed tensor product of $\mathbb{C}[p]$ and the formal Weyl algebra of the symplectic vector space $T_x X$. It is acted on by $T \times \mathbb{C}^\times$. We can form the Verma module $\hat{\Delta}_{\nu,p,h}(\lambda)$ for $A^{\theta,\wedge}_{p,h}$. For $\hat{\Delta}_{\nu,p,h}(x)$ we take the subspace of $T \times \mathbb{C}^\times$-finite vectors in $\hat{\Delta}_{\nu,p,h}(\lambda)$. Then we form various specializations of $\hat{\Delta}_{\nu,p,h}(x)$, such as $\hat{\Delta}_{\nu,\mathfrak{P}}(x)$ and $\hat{\Delta}_{\nu,\lambda}(x)$.

Let us summarize some properties of the modules $\hat{\Delta}_{\nu,p,h}(x)$ that basically appeared in [BLPW] Section 5.3.

**Lemma 3.8.** The following are true:

1. The module $\hat{\Delta}_{\nu,p,h}(x)$ is $[p,h]$-linear and $T \times \mathbb{C}^\times$-equivariantly isomorphic to $\mathbb{C}[p,h][X_x^+]$.
2. Moreover, the specialization $\hat{\Delta}_{\nu,\lambda,0}$ for $\lambda \in p \setminus p^{\text{sing}}$ is $T$-equivariantly and $\mathbb{C}[X_\lambda]$-equivariantly isomorphic to $\mathbb{C}[(X_\lambda)_{X_x}^+]$.

**Proof.** (1) is proved as [BLPW] Proposition 5.20. To prove (2) let us notice that $\hat{\Delta}_{\nu,p,h}(x)/(h)$ coincides with the similarly defined module for $\mathbb{C}[X_\lambda]$. The latter is the $\mathbb{C}[Y_p]$-module $\mathbb{C}[(X_\lambda)_{X_x}^+]$ and its specialization to $\lambda$ is $\mathbb{C}[(X_\lambda)_{X_x}^+]$. □

Now let us investigate a connection between $\Delta_{\nu,\lambda}(x)$ and $\hat{\Delta}_{\nu,\lambda}(x)$. As was observed in [BLPW] Section 5.3, there is a $T$-equivariant homomorphism $\Delta_{\nu,p,h}(x) \to \hat{\Delta}_{\nu,p,h}(x)$ that is the identity on $\mathbb{C}[p,h] \otimes \mathbb{C}_x$ (the highest weight part). The following lemma strengthens [BLPW] Lemma 5.21.

**Lemma 3.9.** The locus of $(\lambda,z)$, where the induced homomorphism $\Delta_{\nu,\lambda,z}(x) \to \hat{\Delta}_{\nu,\lambda,z}(x)$ is an isomorphism, is non-empty and Zariski open.
Proof. Let $K, C$ denote the kernel and the cokernel of $\Delta_{\nu,p,h}(x) \to \hat{\Delta}_{\nu,p,h}(x)$. Let us show that the supports of $K, C$ in $\mathfrak{p} \times \mathbb{C}$ are Zariski closed. Recall that by the support of, say, $K$, we mean the set of all $(\lambda, z) \in \mathfrak{p} \times \mathbb{C}$, such that $\mathbb{C}_{\lambda,z} \otimes_{\mathbb{C}[\mathfrak{p}, \hbar]} K \neq 0$. We note that $K$ carries a rational action of $T$, and that $K$ is generated by $\bigoplus_{\kappa \in T} K_{\kappa}$ for some finite set $\Upsilon$ of characters of $T$. The support of $K$ in $\mathfrak{p} \times \mathbb{C}$ coincides with that of $\bigoplus_{\kappa \in T} K_{\kappa}$. Note that the latter is a finitely generated $\mathbb{C}[\mathfrak{p}, \hbar]$-module so the support of $K$ in $\mathfrak{p} \times \mathbb{C}$ is closed. By the same argument, the support of $C$ is closed.

Now let us show $\Delta_{\nu,\lambda_{\nu}}(x) \to \hat{\Delta}_{\nu,\lambda_{\nu}}(x)$ is an isomorphism if and only if $(\lambda, z)$ does not lie in the union of the supports of $K$ and $C$. The proof repeats that of [BL Proposition 2.7], see Section 3.5 there, because $\hat{\Delta}_{\nu,p,h}(x)$ is flat over $\mathbb{C}[\mathfrak{p}, \hbar]$.

So we see that the locus of $(\lambda, z)$, where $\Delta_{\nu,\lambda_{\nu}}(x) \to \hat{\Delta}_{\nu,\lambda_{\nu}}(x)$ is an isomorphism is Zariski open. To see that it is nonempty we note that it includes $(\mathfrak{p} \setminus \mathfrak{p}^{\text{ring}}, 0)$. Indeed, both modules are $\mathbb{C}[(X_{\lambda_{\nu}})_{\mathfrak{p}}^+]$. \hfill \qed

3.4.3. Contravariant duality. Let us discuss contravariant duality for categories $\mathcal{O}$. Take $M \in \mathcal{O}_{\nu}(A_{\lambda})$. This module decomposes as $\bigoplus_{\alpha \in \mathbb{C}} M_{\alpha}$, where $M_{\alpha}$ is the generalized eigenspace for $h = d_{1}\nu$ with eigenvalue $\alpha$. Recall that all $M_{\alpha}$ are finite dimensional, see, e.g., [BLPW Lemma 3.13]. Then we can consider the restricted dual $M^\vee$, this is a right $A_{\lambda}$-module. One can show that it lies in the category $\mathcal{O}$ for $A_{\lambda}^{\text{opp}}$, more precisely, in $\mathcal{O}_{-\nu}(A_{\lambda}^{\text{opp}})$. Using the isomorphism $A_{\lambda}^{\text{opp}} \cong A_{-\lambda}$, we get $M^\vee \in \mathcal{O}_{-\nu}(A_{-\lambda})$. The quasi-inverse functor of $M \mapsto M^\vee$ is constructed completely analogously and we again denote it by $\bullet^\vee$.

The following result was established in [L3].

Lemma 3.10. For $M_1 \in \mathcal{O}_{\nu}(A_{\lambda})$, $M_2 \in \mathcal{O}_{-\nu}(A_{\lambda}^{\text{opp}})$, we get $\text{Tor}^i_{A_{\lambda}}(M_1, M_2)^* \cong \text{Ext}^i(M_1, M_2^\vee)$.

Note that $\mathbb{C}_{-\nu}(A_{\lambda}^{\text{opp}})$ is naturally identified with $\mathbb{C}_{\nu}(A_{\lambda})^{\text{opp}}$. So for $N \in \mathbb{C}_{\nu}(A_{\lambda})$ let us set $\nabla_{\nu}(N) = A_{\lambda}^{\text{opp}}(N^*)^\vee$, where we write $A_{\lambda}^{\text{opp}}$ for the Verma module functor for the category $\mathcal{O}_{-\nu}(A_{\lambda}^{\text{opp}})$. Alternatively, one can define $\nabla_{\nu}(N)$ as $\text{Hom}_{\mathbb{C}_{\nu}(A_{\lambda})}(A_{\lambda}/A_{\lambda}^{0}A_{\lambda}, N)$, see [L6 Section 4.2] (here we take the restricted Hom, i.e., the direct sum of Hom’s from the graded components).

We can also consider the dual version of $\hat{\Delta}_{\nu}(x)$: the module $\hat{\nabla}_{\nu}(x) := \hat{\Delta}_{-\nu}(x)^\vee$.

3.5. Highest weight structures. Now let us discuss highest weight structures on $\mathcal{O}_{\nu}(A_{\lambda})$, $\mathcal{O}_{\nu}(A_{\lambda}^{0})$.

3.5.1. Ext vanishing. A proof of the following result can be found in [L9 Lemma 4.8].

Lemma 3.11. For a Zariski generic parameter $\lambda \in \mathfrak{P}$ and $x, x' \in X^{\nu(C_{\times})}$, we have $\mathbb{C}_{\nu}(A_{\lambda}) \cong \mathbb{C}[X^{\nu(C_{\times})}]$ and

$$\dim \text{Ext}^i_{A_{\lambda}}(\Delta_{\nu}(x), \nabla_{\nu}(x')) = \delta_{i,0} \delta_{x,x'}.$$

A number of interesting corollaries of this result was deduced in [BLPW Section 5.2].

Corollary 3.12. For $\lambda$ as in Lemma 3.11, the natural morphism $D^b(\mathcal{O}_{\nu}(A_{\lambda})) \to D^b(A_{\lambda} \text{-mod})$ is a fully faithful embedding.

Also Lemma 3.11 implies that the category $\mathcal{O}_{\nu}(A_{\lambda})$ is highest weight, where the order is as in Section 3.2. In more detail, it is as follows. To $\lambda \in \mathfrak{P}$ and $x \in X^{\nu(C_{\times})}$ we can assign the scalar $c_{\nu,\lambda}(x)$ equal to the image of $h \in \mathbb{C}_{\nu}(A_{\lambda})$ under the projection $\mathbb{C}_{\nu}(A_{\lambda}) \to \mathbb{C}$ of
evaluation at \( x \). Then the order \( \leq_\lambda \) on \( X^\nu(C^x) \) is introduced as follows: \( x \leq_\lambda x' \) if \( x = x' \) or \( c_{\nu,\lambda}(x) = c_{\nu,\lambda}(x') \in \mathbb{Z}_{>0} \).

3.5.2. **Highest weight structure on** \( \mathcal{O}_\nu(\mathcal{A}_\lambda^\theta) \). It turns out, \cite[Proposition 6.7]{BLPW}, that the category \( \mathcal{O}_\nu(\mathcal{A}_\lambda^\theta) \) is also highest weight with respect to the contracting order \( \leq_\nu \) on \( X^T \) with respect to \( \nu \). Let us write \( \Delta_\nu^\theta(x) \) for the standard object with respect to that order. This object can be described as follows. To evaluate at \( x \) lying in \( \mathcal{A}_{\nu,\lambda} \), see \cite[Section 6.1]{BLPW}. Then we can define \( \mathcal{O}_\nu(\mathcal{A}_\lambda^\nu) \) as the full subcategory of all objects whose characteristic cycles only include \( X^\nu_+ \) with \( x' \leq_\nu x \). Similarly, we can define \( \mathcal{O}_\nu(\mathcal{A}_\lambda^\nu) \). Then \( \Delta_{\nu,\lambda}(x) \) is the only indecomposable projective in \( \mathcal{O}_\nu(\mathcal{A}_\lambda^\nu) \). This characterization implies the following natural isomorphism

\[
(3.2) \quad \mathcal{A}_{\lambda,\chi}^\theta \otimes \mathcal{A}_\chi^\theta \Delta_{\nu,\lambda}^\theta(x) = \Delta_{\nu,\lambda+\chi}^\theta(x).
\]

Using \((3.2)\), the object \( \Delta_{\nu,\lambda}(x) \) can be described as follows, see \cite[Proposition 5.22]{BLPW}.

**Lemma 3.13.** Let \( \chi \) lie in the interior of \( C \), where \( C \) is the classical chamber containing \( \theta \). Then for \( n \) sufficiently large, we have \( \Delta^{\theta}_{\nu,\lambda+n\chi}(x) = \text{Loc}^{\theta}_{\nu,\lambda+n\chi}(\Delta_{\nu,\lambda+n\chi}(x)) \).

A reason for this lemma is that (the global sections of) the both sides are the standard objects in \( \mathcal{O}_\nu(\mathcal{A}_{\lambda+n\chi}) \) for an order \( \leq_\chi \) refining both \( \leq_{\lambda+n\chi} \) and \( \leq_\nu \). Namely, pick a \( T \)-equivariant structure on \( \mathcal{O}(\chi) \) and let \( \text{wt}_\chi(x) \) be the weight of the fiber \( O(\chi)_x \). We set \( x <_\chi x' \) if \( \langle \text{wt}_\chi(x), \nu \rangle < \langle \text{wt}_\chi(x'), \nu \rangle \). See \cite[Section 6.2]{L6} for a more general situation.

3.6. **\( K_0 \)-groups.** Now let us describe the \( K_0 \) groups of the categories \( \mathcal{O}_\nu(\mathcal{A}_\lambda^\theta) \) and their \( T \)-equivariant versions.

3.6.1. **One-parameter deformations.** Here we will consider one-parameter deformations of the categories of interest. Namely, let \( \chi \) be in the interior of \( C \) and \( \ell = \{ \lambda + z\chi \mid z \in \mathbb{C} \} \). Let \( \ell^\wedge \) denote the formal neighborhood of \( \lambda \) in \( \ell \). Set \( \ell^\wedge_0 = C_{\chi} \) and let \( \ell^\wedge_0 \) denote the formal neighborhood of 0 in \( \ell^\wedge_0 \). We can consider the algebra \( \mathcal{A}_{\ell^\wedge} := \mathbb{C}[\ell^\wedge] \otimes_{\mathbb{C}[\mathbb{P}]} \mathcal{A}_{\mathfrak{p}}, \) the scheme \( X_{\ell^\wedge_0} = \ell^\wedge_0 \times_\mathfrak{p} X_{\mathfrak{p}} \) over \( \ell^\wedge_0 \), and its microlocal quantization \( \mathcal{A}_{\ell^\wedge} \). We still have adjoint functors \( \text{Loc}_{\ell^\wedge}, \Gamma_{\ell^\wedge} \) between the categories \( \mathcal{A}_{\ell^\wedge} \)-mod, \( \text{Coh}(\mathcal{A}_{\ell^\wedge}) \). Note that the kernels and cokernels of the adjunction unit and counit are supported at \( \lambda \) because \( \rho_{\ell^\wedge} \) is an isomorphism outside 0 \( \in \ell^\wedge \). In particular, if abelian localization holds for (\( \lambda, \theta \)), then \( \text{Loc}_{\ell^\wedge}, \Gamma_{\ell^\wedge} \) are mutually inverse equivalences.

We can consider the category \( \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge}) \) consisting of all finitely generated \( \mathcal{A}_{\ell^\wedge} \)-modules that admit a \( T \)-equivariant structure and are supported on the contracting locus of \( \nu \) in \( Y_{\ell^\wedge_0} \). Define the category \( \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge}) \) in a similar way. Then a standard argument, compare with \cite[Section 2.4]{GGOR}, shows that the category \( \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge}) \) contains a projective generator. So \( \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge}) \) is equivalent to the category of modules over an associative \( \mathbb{C}[\ell^\wedge] \)-algebra that is a finitely generated module over \( \mathbb{C}[\ell^\wedge] \).

Suppose, until Lemma 3.15 that Lemma 3.11 holds for \( \lambda \). Then a projective generator of \( \mathcal{O}_\nu(\mathcal{A}_{\lambda}) \) deforms to an object in \( \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge}) \) that is automatically projective. In particular, \( \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge}) \) is equivalent to the category of modules over an associative \( \mathbb{C}[\ell^\wedge] \)-algebra, say \( \mathcal{A}_{\ell^\wedge} \), that is a free finite rank module over \( \mathbb{C}[\ell^\wedge] \).
Note that we have objects $\Delta_{\nu,\ell^\wedge}(x), \hat{\Delta}_{\nu,\ell^\wedge}(x) \in \mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge})$. The former is flat over $\mathbb{C}[\ell^\wedge]$ under our assumptions. Moreover, $\mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge})$ is a highest weight category over $\mathbb{C}[\ell^\wedge]$ and the objects $\Delta_{\nu,\ell^\wedge}(x)$ are standard.

**Lemma 3.14.** The localization of the category $\mathcal{O}_\nu(\mathcal{A}_{\ell^\wedge})$ to the generic point of $\ell^\wedge$ is semisimple.

**Proof.** This localization is highest weight with respect to the order $\leq_{\ell^\wedge}$ defined similarly to $\leq_\lambda$. Namely, let $c_{\nu,\ell^\wedge}(x)$ be the element of $\mathbb{C}[\ell^\wedge]$ by which $h = d_1 \nu$ acts on the highest degree component of $\Delta_{\nu,\ell^\wedge}(x)$; this element specializes to $c_{\nu,\lambda}(x)$. We set $x <_{\ell^\wedge} x'$ if $c_{\nu,\ell^\wedge}(x) < c_{\nu,\ell^\wedge}(x')$.

But the function $c_{\nu,\lambda}(x)$ is affine in $\lambda$ and the restriction of the differential to $\ell_0$ is nonzero, see [L6, Section 6.2]. So the order for the localization to the generic point of $\ell^\wedge$ is trivial. Hence the localization is semisimple. $\square$

Now define the object $\Delta_{\nu,\ell^\wedge}^\theta(x)$ as

$$A_{\ell^\wedge+n\chi,-n\chi}^\theta \otimes \text{Loc}_{\ell^\wedge+n\chi}^\theta(\Delta_{\nu,\ell^\wedge+n\chi}(x)), \text{ for } n \gg 0.$$ 

This is independent of the choice of $n$ for the reasons explained after Lemma 3.13.

Now let us apply the constructions above to check that certain classes in $K_0$ are equal.

**Lemma 3.15.** The following are true:

1. If $\lambda$ satisfies the conditions of Lemma 3.14, then we have $[\Delta_{\nu,\lambda}(x)] = [\hat{\Delta}_{\nu,\lambda}(x)] = [R\Gamma^\lambda \Delta_{\nu,\lambda}^\theta(x)]$.
2. If abelian localization holds for $(\lambda, \theta)$, then $[\Delta_{\nu,\lambda}^\theta(x)] = [\text{Loc}_{\lambda}^\theta \hat{\Delta}_{\nu,\lambda}(x)]$.

**Proof.** Also $\Delta_{\nu,\ell^\wedge}(x), \hat{\Delta}_{\nu,\ell^\wedge}(x), \Delta_{\nu,\ell^\wedge}^\theta(x)$ are flat over $\mathbb{C}[\ell^\wedge]$. It follows that the classes of the specializations of $\Delta_{\nu,\ell^\wedge}(x), \hat{\Delta}_{\nu,\ell^\wedge}(x), R\Gamma \Delta_{\nu,\lambda}^\theta(x)$ to $\lambda$ coincide with the degenerations of the classes of localizations to the generic point. But $\Delta_{\nu,\ell^\wedge}(x), \hat{\Delta}_{\nu,\ell^\wedge}(x), R\Gamma \Delta_{\nu,\lambda}^\theta(x)$ become isomorphic after localizing to the generic point of $\ell^\wedge$. This proves (1). The proof of (2) is similar. $\square$

3.6.2. Identification of $K_0$ with $\mathbb{C}X_T \times \mathcal{X}(T)$. Suppose that $\lambda$ satisfies the conditions of Lemma 3.14 or abelian localization holds for $(\lambda, \theta)$ for some generic $\theta$. Lemma 3.15 implies that the classes $[\Delta_{\nu,\lambda}(x)]$ form a basis in $K_0(\mathcal{O}_\nu(\mathcal{A}_\lambda))$. This gives an identification of $K_0(\mathcal{O}_\nu(\mathcal{A}_\lambda))$ with $\mathbb{C}X_T$. We can also identify $K_0(\mathcal{O}_\nu^T(\mathcal{A}_\lambda))$ with $\mathbb{C}X_T \times \mathcal{X}(T)$ by sending the class of $\Delta_{\nu,\lambda}(x, \kappa)$ to $(x, \kappa)$.

3.6.3. Classes of costandard objects. We assume that $\lambda$ satisfies the conditions of Lemma 3.14. Then $[\Delta_{\nu,\lambda}(x)] = [\nabla_{\nu,\lambda}(x)]$ for all $x$, see [BLPW, Corollary 6.4] for this result under additional assumptions. The reason for this equality is that the localizations of $\Delta_{\nu,\ell^\wedge}(x), \nabla_{\nu,\ell^\wedge}(x)$ to the generic point of $\ell^\wedge$ are isomorphic simples. Similarly, $[\Delta_{\nu,\lambda}(x, \kappa)] = [\nabla_{\nu,\lambda}(x, \kappa)]$. In particular, the classes of standard objects in $\mathcal{O}_\nu(\mathcal{A}_\lambda^T)$ coincide with the classes of costandard objects.

3.7. Examples.
3.7.1. Cotangent bundles to flag varieties. Let us consider the special case when \( X = T^*(G/B) \). Let \( T \) denote a maximal torus in \( B \), it acts on \( X \) in a Hamiltonian fashion. Let \( \nu : \mathbb{C}^* \to T \) be a one-parameter subgroup. The set \( X^{\nu(\mathbb{C}^*)} \) is finite if and only if \( \nu \) is regular, i.e., its centralizer in \( G \) is \( T \). In this case, the fixed points are labelled by \( W \), where \( W = N_G(T)/T \) is the Weyl group of \( g \).

We recover the usual BGG category \( \mathcal{O} \) (in the version, where it consists of finitely generated \( U_\lambda \)-modules with locally finite action of \( b \)). Conditions of Lemma 3.11 hold under the assumption that \( \lambda \) is regular. Here \( \mathcal{C}_\nu(A_\lambda) \) is naturally identified with \( \mathbb{C}[\mathfrak{h}]/\{f \in \mathbb{C}[\mathfrak{h}]| f(w\lambda) = 0, \forall w\} \). Further, we have \( \mathfrak{c}_{\nu,\lambda}(x) = \langle w\lambda, \nu \rangle \) for \( x \in X^{\nu(\mathbb{C}^*)} \) corresponding to \( w \in W \).

3.7.2. Example: rational Cherednik algebras of type A. Consider the case when \( X = \text{Hilb}_n(\mathbb{C}^2) \). Let \( \mathfrak{h} \) denote the span of \( y_1, \ldots, y_n \) and \( \mathfrak{h}^* \) denote the span of \( x_1, \ldots, x_n \). Then \( S(\mathfrak{h}^*), S(\mathfrak{h}) \) are included into \( H_c \) as subalgebras. Moreover, \( \mathbb{C}[Y] = S(\mathfrak{h} \oplus \mathfrak{h}^*)^S_n \).

The category \( \mathcal{O}_c \) was introduced in \([GGOR]\) as the category of all \( H_c \)-modules that are finitely generated over \( S(\mathfrak{h}^*) \) and have locally nilpotent action of \( \mathfrak{h} \). We can introduce the Verma modules \( \Delta_c(\mu) = H_c \otimes S(\mathfrak{h})^\# S_n \mu \), where \( \mu \) runs over the irreducible \( S_n \)-modules (that are naturally labelled by partitions of \( n \)). For \( c \notin \{-n, n\} \), the functor \( M \mapsto eM : H_c \text{-mod} \to eH_c \text{-mod} \) is an equivalence of categories.

On \( X \) we have a one-parameter Hamiltonian torus acting. We choose \( \nu \) so that \( \nu(t).x_i = t^{-1}x_i, \nu(t).y_i = ty_i \). The fixed points in \( X \) are also naturally parameterized by the partitions of \( n \).

The algebras \( eH_c \) and \( H_c \) are Morita equivalent if \( c \) is not a rational number with denominator \( \leq n \) in \((-1, 0)\). Then the simples in \( \mathcal{O}_c(A_\lambda) \) (where \( \lambda = c + \frac{1}{2} \)) are the modules of the form \( eL_c(\mu) \). This labelling is compatible with the standard labelling of \( X^T \) by Young diagrams.

Let us compute the number \( \mathfrak{c}_{\nu,\lambda}(x) \). Let us introduce two important statistics of a Young diagram \( \mu \), namely, \( \text{cont}(\mu) \) and \( n(\mu) \). Recall that by the content of a box in a Young diagram we mean the difference of its horizontal and vertical coordinates. The content of \( \mu \) denoted by \( \text{cont}(\mu) \) is the sum of contents of all boxes, for example, \( \text{cont}((n)) = 0 + 1 + 2 + \ldots + (n - 1) \) (here and below by \( (n) \) me mean a Young diagram with a single row of \( n \) boxes). If \( \mu = (\mu_1, \ldots, \mu_k) \), then we write \( n(\mu) := \sum_{i=1}^k (i - 1)\mu_i \). This is the minimal degree in which \( \mu \) occurs in \( S(\mathfrak{h}) \).

Lemma 3.16. Suppose that \( c \) is Zariski generic. Then, for a suitable choice of \( h \) (it is defined up to adding a constant), we have \( \mathfrak{c}_{\nu,\lambda}(x) = c\text{cont}(\mu) - n(\mu) \) (where \( \mu, x \) correspond to the same Young diagram).

Proof. Note that \( \mathfrak{c}_{\nu,\lambda}(x) \) is an affine function in \( \lambda \), see \([L6]\) Section 6.2. So it is enough to prove the equality assuming \( \lambda \) is Weil generic. Here the simple object corresponding to \( x \) is \( e\Delta_c(\mu) \). It follows that \( \mathfrak{c}_{\nu,\lambda}(x) \) is the highest weight space in \( e\Delta_c(\mu) \). The highest weight in \( \Delta_c(\mu) \) is \( c\text{cont}(\mu) \) and the required equality follows from the fact that \( n(\mu) \) is the minimal degree of \( \mu^* \) in \( S(\mathfrak{h}^*) \).

3.8. Translation and wall-crossing functors. In this section we will study the behavior of the translation and wall-crossing functors on the category \( \mathcal{O} \). Note that \( \mathfrak{M}_\lambda^\chi(\lambda) \) restricts to an equivalence \( D^b_{\lambda}(A_{\lambda-}\text{-mod}) \to D^b_{\lambda}(A_{\lambda-}\text{-mod}) \).
3.8.1. **Behavior on** $K_0(O^\nu_T)$. Pick a $T$-equivariant structure on $O(\chi)$ (defined up to a twist with a character). This gives a $T$-equivariant structure on $A^0_{p,\chi}$ and on $A_{\lambda,\chi}$. So $\mathcal{WC}_{\lambda+\chi-\lambda}$ upgrades to an equivalence $D^b_{O^\nu(T)}(A_{\lambda+\chi}-mod^T) \xrightarrow{\sim} D^b_A(A_{\lambda+\chi}-mod^T)$.

**Proposition 3.17.** We have $[\mathcal{WC}_{\lambda+\chi-\lambda}\Delta_{\nu,\lambda}(\chi, \kappa)] = [\Delta_{\nu,\lambda+\chi}(\chi, \kappa + wt_\lambda(\chi))]$.

**Proof.** We can assume $\kappa = 0$.

We have the degeneration map $K_0(O^\nu_T(A^0_\lambda)) \rightarrow K_0(Coh_{X^\nu}(\chi))$: it is invertible and upper triangular in the basis of the fixed points in the localized $K_0$, this upper triangularity follows from [BLPW] Lemma 6.18. Now consider the objects

$$M^1_{p,h} := A^0_{p,h,\lambda} \otimes_{A_{p,h}} \Delta_{\nu,\lambda}(\chi, \kappa), M^2_{p,h} := A^0_{p,h} \otimes \Delta_{\nu,\lambda}(\chi, \kappa + wt_\lambda(\chi)).$$

These are objects in the category $D^b_{X^\nu}(A^0_{\lambda}-mod^T)$. Note that $\mathcal{WC}_{\lambda+\chi-\lambda}\Delta_{\nu,\lambda}(\chi, \kappa) = C_{\lambda,1} \otimes_{C_{[p,h]}} M^1_{p,h}$, while $\Delta_{\nu,\lambda+\chi}(\chi, \kappa + wt_\lambda(\chi)) = C_{\lambda+\chi,1} \otimes_{C_{[p,h]}} M^2_{p,h}$. Also note that if $M_{p,h} \in D^b_{X^\nu}(A^0_{\lambda}-mod^T)$, then the degenerations of all its specializations $C_{\lambda,1} \otimes_{C_{[p,h]}} M^1_{p,h}$ coincide for $\lambda \in p \setminus p_{sing}$. The former is $O(\chi) \otimes O_{(X_\lambda)^\ast}$ and the latter is $O_{(X_\lambda)^\ast}$ twisted by the character $wt_\lambda(\chi)$. But $(X_\lambda)^+$ is an affine space so the restriction of $O(\chi)$ is a trivial line bundle with the action of $T$ given by $wt_\lambda(\chi)$. The coincidence we need follows. □

3.8.2. **Translations of Verma modules.** Our goal here is to prove the following result.

**Proposition 3.18.** Suppose that $\lambda \in p$ and $\chi \in \text{Pic}(X)$. Assume that the following hold:

- Conditions of Lemma 3.17 hold for $\lambda$, $\lambda + \chi$,
- and $\mathcal{WC}_{\lambda+\chi-\lambda}$ is an abelian equivalence.

Then $A_{\lambda,\chi} \otimes_{A_{\lambda}} \Delta_{\nu,\lambda}(\chi) \cong \Delta_{\nu,\lambda+\chi}(\chi)$.

**Proof.** Note that by Proposition 3.17 (its weaker version that does not take the equivariant structures into account), we have $[A_{\lambda,\chi} \otimes_{A_{\lambda}} \Delta_{\nu,\lambda}(\chi)] \cong [\Delta_{\nu,\lambda+\chi}(\chi)]$. But both $A_{\lambda,\chi} \otimes_{A_{\lambda}} \Delta_{\nu,\lambda}(\chi), \Delta_{\nu,\lambda+\chi}(\chi)$ are standard objects for highest weight structures on $O_\nu(A_{\lambda+\chi})$. As was checked in [GL] Lemma 4.3.2], this implies that the modules are isomorphic. □

3.8.3. **Long wall-crossing as Ringel duality functor.** Let us recall a result from [L6] Section 7.3. Let abelian localization hold for $(\lambda, \theta), (\lambda + \chi, -\theta)$.

**Lemma 3.19.** The wall-crossing functor $\mathcal{WC}_{\lambda+\chi-\lambda} : D^b(O_\nu(A_\lambda)) \xrightarrow{\sim} D^b(O_\nu(A_{\lambda+\chi}))$ is a Ringel duality functor.

4. **Very generic and regular parameters**

4.1. **Generic simplicity.** For a wall $\Gamma$, let $\alpha_\Gamma$ denote an indecomposable element in $p^\ast_Z$ such that $\Gamma = \ker \alpha_\Gamma$ (this element is defined up to multiplication by $\pm 1$). We assume the following condition.

(S) For each of the walls $\Gamma$, there is a finite subset $\Sigma_\Gamma \subset (-1/2, 1/2]$ of rational numbers with the following property: the algebra $A_\lambda$ is simple provided $(\alpha_\Gamma, \lambda) \not\in \Sigma_\Gamma + Z$ for all $\Gamma$.

Fix subsets $\Sigma_\Gamma$ for all $\Gamma$. 
Example 4.1. Let \( X = T^* \mathcal{B} \). Then (S) holds with \( \Sigma_\Gamma = \{0\} \) for all \( \Gamma \): if \( \langle \alpha^\vee, \lambda \rangle \notin \mathbb{Z} \) for all roots \( \alpha \), then the category of Harish-Chandra \( \mathcal{U}_\lambda \)-bimodules is semisimple, hence there are no proper two-sided ideals in \( \mathcal{U}_\lambda \).

In fact, one can show that (S) with \( \Sigma_\Gamma = \{0\} \) holds for all Slodowy varieties. For parabolic Slodowy varieties, \( \Sigma_\Gamma \) consists of one element (that maybe nonzero because the “parabolic \( \rho \)” is not necessary in the weight lattice).

Example 4.2. Let \( X \) be the Hilbert scheme \( \text{Hilb}_n(\mathbb{C}^2) \). Then there is only one hyperplane \( \Gamma \), which is \( \Gamma = \{0\} \). Then for \( \Sigma_\Gamma \) we can take \( \{\frac{a}{b} + \frac{1}{2} | 1 \leq a < b \leq n\} \). This follows from results of [G] (that in this case the algebra \( H_\epsilon \) is simple provided its category \( \mathcal{O} \) is semisimple) and the fact, that follows from [GGOR, Section 6.2], that the category \( \mathcal{O} \) is semisimple for \( c \notin \{-\frac{a}{b} | 1 \leq a < b \leq n\} + \mathbb{Z} \).

The sets \( \Sigma_\Gamma \) can also be computed for quantizations of Nakajima quiver varieties, see [BL, Section 8.4], at least, under some additional assumptions, e.g., that the underlying quiver is of finite or affine type.

Definition 4.3. Parameters \( \lambda \) such that \( \langle \alpha_\Gamma, \lambda \rangle \notin \Sigma_\Gamma + \mathbb{Z} \) for all \( \Gamma \) will be called very generic.

4.2. Category \( \mathcal{O} \) for very generic parameters. We are going to prove a number of results on the category \( \mathcal{O} \) at very generic parameters.

Proposition 4.4. Let \( \lambda \) be a very generic parameter. Then the following is true:

(i) The category \( \mathcal{O}_\nu(\mathcal{A}_\lambda) \) is semisimple.

(ii) The natural homomorphism \( C_\nu(\mathcal{A}_\lambda) \rightarrow \mathbb{C}[X^T] \) is an isomorphism.

(iii) We have \( \dim \text{Tor}_i^A(\Delta_\nu(x), \Delta_{\nu}^\vee(x')) = \delta_{i0}\delta_{x,x'} \).

Proof. Let us prove (i). The algebras \( \mathcal{A}_{\lambda+\chi} \) are simple for all \( \chi \in \text{Pic}(X) \). By Lemma 2.10 all wall-crossing functors \( \mathfrak{WC}_{\lambda+\chi-\lambda} \) are abelian equivalences. By Proposition 2.5 abelian localization holds for \( (\lambda, \theta) \) with any \( \theta \). So the categories \( \mathcal{O}_\nu(\mathcal{A}_\lambda) \) and \( \mathcal{O}_\nu(\mathcal{A}_{\lambda}^\theta) \) are equivalent for any generic \( \theta \).

It follows from Lemma 3.19 that the Ringel duality equivalence is \( t \)-exact. Recall, Section 3.6.3 that the classes of the standard objects in \( K_0(\mathcal{O}_\nu(\mathcal{A}_{\lambda}^\theta)) \) coincide with those of costandard objects. So we can apply [L10] Lemma 4.2 to see that \( \mathcal{O}_\nu(\mathcal{A}_{\lambda}^\theta) \) is indeed semisimple.

Let us prove (ii). First, let us show that the algebra \( C_\nu(\mathcal{A}_\lambda) \) is semisimple. Indeed, otherwise there is a non-split exact sequence \( 0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0 \), where \( N_1, N_2 \) are simple \( C_\nu(\mathcal{A}_\lambda) \)-modules necessarily with the same action of \( h \), say, by scalar \( \alpha \). The object \( \Delta_\nu(N) \) is completely reducible because the category \( \mathcal{O}_\nu(\mathcal{A}_\lambda) \) is semisimple. The \( \alpha \)-eigenspace for \( h \) in \( \Delta_\nu(N) \) is \( N \). Since \( \Delta_\nu(N) \) has no homomorphisms to modules with zero \( \alpha \)-eigenspace, we see that any nonzero direct summand of \( \Delta_\nu(N) \) must have a nonzero intersection with \( N \). This intersection is \( C_\nu(\mathcal{A}_\lambda) \)-stable. Since \( N \) is indecomposable, it follows that \( \Delta_\nu(N) \) is indecomposable, hence simple. But \( \Delta_\nu(N) \rightarrow \Delta_\nu(N_2) \), therefore, this epimorphism is an iso. Since the \( \alpha \)-eigenspace in \( \Delta_\nu(N_2) \) is \( N_2 \) we arrive at a contradiction. We conclude that the algebra \( C_\nu(\mathcal{A}_\lambda) \) is semisimple.

Now let us observe that there is a bijection \( \text{Irr}(C_\nu(\mathcal{A}_\lambda)) \cong X^T \). This follows from the equivalence \( \mathcal{O}_\nu(\mathcal{A}_\lambda) \cong \mathcal{O}_\nu(\mathcal{A}_{\lambda}^\theta) \) and natural bijections \( \text{Irr}(C_\nu(\mathcal{A}_\lambda)) \cong \text{Irr}(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \) and \( X^T \cong \text{Irr}(\mathcal{O}_\nu(\mathcal{A}_{\lambda}^\theta)) \).
Consider the $\mathcal{A}_\lambda$-module $\hat{\Delta}_{\nu,\lambda}(x)$. Since $\mathcal{O}_\nu(\mathcal{A}_\lambda) \cong \mathcal{O}_\nu(\mathcal{A}_\lambda^0)$ is semisimple, thanks to (2) of Lemma 3.15 we have $\hat{\Delta}_{\nu,\lambda}(x) = \Gamma(\Delta_{\nu,\lambda}^0(x))$. Since the objects $\Delta_{\nu,\lambda}^0(x)$ are indecomposable, the objects $\hat{\Delta}_{\nu,\lambda}(x)$ are simple. By Lemma 3.8, the eigenspace in $\hat{\Delta}_{\nu,\lambda}(x)$ corresponding to the largest eigenspace of $h$ is one-dimensional, the corresponding eigenvalue is $c_{\nu,\lambda}(x)$. We have a nonzero homomorphism $\Delta_{\nu,\lambda}(x) \to \Delta_{\nu,\lambda}(x)$. Both sides are simple, so it is an isomorphism. It follows from the previous paragraph that the pull-backs of $\mathbb{C}_x$’s are all the irreducible representations of $\mathcal{C}_\nu(\mathcal{A}_\lambda)$. Since $\mathcal{C}_\nu(\mathcal{A}_\lambda)$ is semisimple, we see that $\mathcal{C}_\nu(\mathcal{A}_\lambda) \to \mathcal{C}_\nu(\mathcal{A}_\lambda^0)$. This finishes the proof of (ii).

Let us prove (iii). Recall, Lemma 3.10, that $\text{Tor}_i^A(\Delta_{\nu}(x), \Delta_{\nu}(x)) = \text{Ext}_A^i(\Delta_{\nu}(x), \nabla_{\nu}(x'))$. Since abelian localization holds for $(\lambda, \theta)$, we see that this Ext is the same as

$$\text{Ext}_{\text{Coh}(\mathcal{A}_\lambda^0)}^i(\text{Loc}_x^0 \Delta_{\nu}(x), \text{Loc}_x^0 \nabla_{\nu}(x')).$$

But the latter Ext is the same as in the category $\mathcal{O}_\nu(\mathcal{A}_\lambda^0)$. This category is semisimple and the objects we consider are simple. So the latter Ext is $\delta_{i,0}\delta_{x,x'}$ we get the claim of (iii). \hfill $\square$

### 4.3. Regular parameters.

#### 4.3.1. Terminology. Let us start by introducing some terminology.

Pick $\lambda \in \mathfrak{P}$. Consider the set of all walls $\Gamma$ with $\langle \alpha_\Gamma, \lambda \rangle \in \Sigma_\Gamma + \mathbb{Z}$. These walls $\Gamma$ will be called integral walls for $\lambda$. They split $p_\mathbb{R}$ into the set of polyhedral chambers to be called integral chambers for $\lambda$.

A subset $\Sigma$ of $\mathbb{C}$ will be called saturated if for any two elements $z, z + n \in \Sigma$ with $n \in \mathbb{Z}_{>0}$, we have $z + 1, z + 2, \ldots, z + n - 1 \in \Sigma$.

Now suppose that we have fixed saturated subsets $\tilde{\Sigma}_\Gamma \subset \Sigma_\Gamma + \mathbb{Z}$ for each wall $\Gamma$. Suppose $\lambda \in \mathfrak{P}$ satisfies $\langle \lambda, \alpha_\Gamma \rangle \not\in \tilde{\Sigma}_\Gamma$ (note that this is automatic if $\Gamma$ is not an integral wall for $\lambda$). Then there is a unique integral chamber $C^\text{int}$ for $\lambda$ with the following property: for $\lambda' \in \lambda + (C^\text{int} \cap p_\mathbb{Z})$ we have $\langle \lambda', \alpha_\Gamma \rangle \not\in \tilde{\Sigma}_\Gamma$ for all $\Gamma$. We will say that $C^\text{int}$ is a positive chamber for $\lambda$.

Now let us define quantum chambers. Let $\lambda \in \mathfrak{P}$ and pick an integral chamber $C^\text{int}$. Quantum chambers will be subsets in $\lambda + \mathfrak{P}_\mathbb{Z}$ defined by linear inequalities. Namely, let $\Gamma_1, \ldots, \Gamma_k$ be all walls such that $\langle \lambda, \alpha_\Gamma \rangle \in \Sigma_\Gamma$. We can assume that all $\alpha_\Gamma$ are non-negative on $C^\text{int}$ (otherwise we replace $\alpha_\Gamma$ with $-\alpha_\Gamma$). Pick minimal integers $m_i, i = 1, \ldots, k$ such that $\tilde{m}_i := \langle \lambda, \alpha_\Gamma \rangle + m_i \not\in \tilde{\Sigma}_\Gamma$. Then we define the quantum chamber $C^q$ as the subset of the set of regular parameters in $\lambda + \mathfrak{P}_\mathbb{Z}$ given by the inequalities $\langle \alpha_\Gamma, \bullet \rangle \geq \tilde{m}_i$. We say that it is shifted from $C^\text{int}$.

Clearly, different quantum chambers are disjoint. Also $C^\text{int}$ is a positive integral chamber for $\lambda' \in \lambda + \mathfrak{P}_\mathbb{Z}$ if and only if $\lambda'$ lies in some shifted (from $C^\text{int}$) quantum chamber. In particular, every regular parameter lies in a unique quantum chamber.

Let us finish this part with an example. Let $X = T^*B$ for $G = \text{SL}_3(\mathbb{C})$. Let $\alpha_1, \alpha_2$ denote the simple roots and let $\alpha_{12} = \alpha_1 + \alpha_2$. Then we have three walls: $\Gamma_1 := \text{ker} \alpha_1^\vee, \Gamma_2 := \text{ker} \alpha_2^\vee, \Gamma_12 := \text{ker} \alpha_{12}^\vee$. Set $\tilde{\Sigma}_{\Gamma_1} = \{0\}, \tilde{\Sigma}_{\Gamma_2} = \{0\}, \tilde{\Sigma}_{\Gamma_12} = \{-2, -1, 0, 1, 2\}$. Let us take an integral weight $\lambda$. One of the integral chambers is the positive (in the usual Lie-theoretic sense) chamber $C^\text{int}$, it is given by $\langle \alpha_1^\vee, \bullet \rangle \geq 0, \langle \alpha_2^\vee, \bullet \rangle \geq 0$. The corresponding quantum chamber is given by $\{x_1, x_2) | x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \geq 3\}$, where $x_1 := \langle \alpha_1^\vee, \lambda \rangle, x_2 := \langle \alpha_2^\vee, \lambda \rangle$. In particular, $C^q$ is not a cone.
4.3.2. **Result.**

**Proposition 4.5.** For each wall $\Gamma$, there is a finite saturated subset $\widetilde{\Sigma}_\Gamma \subset \Sigma_\Gamma + \mathbb{Z}$ such that the following hold:

1. Let $\lambda$ be regular and $C^{\text{int}}$ denote the positive integral chamber for $\lambda$. Then abelian localization holds for $(\lambda', \theta)$ for any generic $\theta \in C^{\text{int}}$ and any $\lambda' \in \lambda + (C^{\text{int}} \cap \mathbb{P}_Z)$.

In particular, the homological dimension of $A_\lambda$ does not exceed $\dim X$.

2. Every regular $\lambda$ satisfies the conditions of Lemma 3.11.

3. Let $\lambda$ be regular. Then $\Delta_{\nu,\lambda}(x) \xrightarrow{\sim} \Delta_{\nu,\lambda}(x)$ and $\nabla_{\nu,\lambda}(x) \xrightarrow{\sim} \nabla_{\nu,\lambda}(x)$.

In what follows the notions of regular parameters, quantum chambers, etc. refer to this collection of $\Sigma_\Gamma$’s. By $\mathfrak{P}^{\text{reg}}$ we will denote the set of regular parameters with respect to this choice of the sets $\Sigma_\Gamma$.

**Proof.** Let us prove (1). Recall, Proposition 2.6, that abelian localization holds for $(\lambda', \theta)$ provided the translation bimodules $A_{\lambda'+k\theta, \theta}, A_{\lambda'+(k+1)\theta, -\theta}$ are mutually inverse Morita equivalences. By Lemma 2.6, the locus $U_\theta \subset \mathfrak{P}$ of such that $A_{\lambda'+k\theta, \theta}, A_{\lambda'+(k+1)\theta, -\theta}$ are mutually inverse Morita equivalences is Zariski open and nonempty. Also this locus contains the locus of very generic elements. It follows that there is a collection of finite subsets $\Sigma^1_\Gamma \subset \Sigma_\Gamma$ such that $\lambda \in U_\theta$ provided $\langle \lambda, \alpha_\Gamma \rangle \not\in \Sigma^1_\Gamma$. Let $\Sigma^1_\Gamma$ denote the saturation of the union $\bigcup_{\theta} \Sigma^1_\Gamma$, where we take one element $\theta$ per classical chamber. Let $U \subset \mathfrak{P}$ be given by $\langle \alpha_\Gamma, \lambda \rangle \not\in \Sigma^1_\Gamma$ for any classical wall $\Gamma$. Now let $\lambda \in U$ so that it makes sense to speak about the positive integral chamber $C^{\text{int}}$ for $\lambda$ (with respect to the saturated subsets $\Sigma^1_\Gamma$). Clearly, $\lambda + (C^{\text{int}} \cap \mathbb{P}_Z) \subset U$. It follows that for any $\theta$ chosen as above (and hence for every generic $\theta \in C^{\text{int}}$) and any $\lambda' \in \lambda + (C^{\text{int}} \cap \mathbb{P}_Z)$, abelian localization holds for $(\lambda', \theta)$. In particular, for every $\lambda \not\in \bigcup_{\Gamma} \Sigma^1_\Gamma$, the homological dimension of $A_\lambda$ does not exceed $\dim X$.

Let us prove (2). First we need to show that the locus of $\lambda$ such that $C_\nu(A_\lambda) \xrightarrow{\sim} C_\nu(A_\lambda^\theta) = \mathbb{C}[X^7]$ is Zariski open. Note that this locus intersects with the locus in $\mathfrak{P}$, where $C_\nu(A_\lambda^\theta) \rightarrow \mathbb{C}[\mathfrak{P}][X^7]$ is an isomorphism. Since both sides are finitely generated over $\mathbb{C}[[\mathfrak{P}]]$, the inner locus is Zariski open. Now let us prove that the locus of $\lambda$, where $\dim \text{Ext}^1_{A_\lambda}(\Delta_{\nu,\lambda}(x), \nabla_{\nu,\lambda}(x')) = \delta_{x',x}^{-1}\delta_{\theta,0}$, is Zariski open. Thanks to Lemma 3.10 we need to check that the locus, where $\dim \text{Tor}^1_{A_\lambda}(\Delta_{\nu,\lambda}(x), \Delta_{\nu,\lambda}^r(x')) = \delta_{x',x}^{-1}\delta_{\theta,0}$, is Zariski open. Now consider the $\mathbb{C}[\mathfrak{P}]$-module $\text{Tor}^1_{A_\lambda}(\Delta_{\nu,\mathbb{P}}(x), \Delta_{\nu,\mathbb{P}}^r(x'))$. It is finitely generated over $\mathbb{C}[[\mathfrak{P}]]$ because the contracting and repelling subvarieties for $\nu$ in $Y$ intersect at a single point, namely 0. So, for all $i, x, x'$, the locus, where the fibers of the coherent sheaf $\text{Tor}^1_{A_\lambda}(\Delta_{\nu,\mathbb{P}}(x), \Delta_{\nu,\mathbb{P}}^r(x'))$ are of minimal rank, is Zariski open.

So the locus of points $\lambda \in \mathfrak{P} \setminus \bigcup_{\Gamma} \Sigma^1_\Gamma$ satisfying the conditions of Lemma 3.11 is Zariski open. From (ii) and (iii) of Proposition 4.4, it follows that, for each wall $\Gamma$, there is a finite subset $\Sigma^2_\Gamma \subset \Sigma_\Gamma + \mathbb{Z}$ such that the conditions of Lemma 3.11 hold for $\lambda$ as long as $\langle \lambda, \alpha_\Gamma \rangle \not\in \Sigma^2_\Gamma$ for all $\Gamma$.

Let us proceed to (3). As we have seen in the proof of Proposition 4.4, if $\lambda$ is very generic, then $\Delta_{\nu,\lambda}(x) \xrightarrow{\sim} \Delta_{\nu,\lambda}(x)$. For a similar reason, we have $\nabla_{\nu,\lambda}(x) \xrightarrow{\sim} \nabla_{\nu,\lambda}(x)$. By Lemma 3.11 the locus of parameters $\lambda$, where the natural homomorphism $\nabla_{\nu,\lambda}(x) \xrightarrow{\sim} \Delta_{\nu,\lambda}(x)$ is an isomorphism, is Zariski open. A similar claim holds for the homomorphism $\nabla_{\nu,\lambda}(x) \xrightarrow{\sim} \nabla_{\nu,\lambda}(x)$. We deduce that there is a collection of finite subsets $\Sigma^3_\Gamma \subset \Sigma_\Gamma + \mathbb{Z}$ such that $\Delta_{\nu,\lambda}(x) \xrightarrow{\sim} \Delta_{\nu,\lambda}(x)$, $\nabla_{\nu,\lambda}(x) \xrightarrow{\sim} \nabla_{\nu,\lambda}(x)$ provided $\langle \alpha_\Gamma, \lambda \rangle \not\in \Sigma^3_\Gamma$.

Now for $\Sigma_\Gamma$ we need to take the saturation of $\Sigma^1_\Gamma \cup \Sigma^2_\Gamma \cup \Sigma^3_\Gamma$. □
Remark 4.6. Note that we can make (1) hold even in the case when there is no torus action. Also it follows from (1) that the homological dimension of $A_{g^+ g}$ does not exceed $\dim X + \dim p$.

4.3.3. Examples. Let us explain what being regular means in the examples we consider: the cotangent bundles $T^*B$ and the Hilbert schemes $\Hilb_n(\mathbb{C}^2)$.

Let us start with the case of $X = T^*B$.

For $X = T^*B$, it is easy to see that we can take $\tilde{\Sigma}_\Gamma = \{0\}$ for all $\Gamma$ so that regular in our sense is the same as regular in the usual sense ($\langle \alpha^\vee, \lambda \rangle \neq 0$ for all roots $\alpha$). Integral chambers for $\lambda$ are given by $\langle \alpha_i^\vee, \bullet \rangle \geq 0$, $i = 1, \ldots, k$, where $\alpha_1, \ldots, \alpha_k$ is a system of simple roots for the integral Weyl group $W$. Quantum chambers are given by $\langle \alpha_i^\vee, \bullet \rangle \geq 1$.

For $X = \Hilb_n(\mathbb{C}^2)$, we have a single wall $\Gamma = \{0\}$ and $\Sigma_{\Gamma} = \{-\frac{a}{b} + 1/2 | 1 \leq a < b \leq n\}$. For $p_1$ we can take $\{\pm 1\}$. We will take $\tilde{\Sigma}_\Gamma$ of the form $\bigcup_{k=\ell}^{\ell+1} (\Sigma_{\Gamma} + k)$, where $\ell$ is a suitable (sufficiently big) positive integer. Clearly, $\tilde{\Sigma}_\Gamma$ is saturated. When $c = \lambda - 1/2$ is irrational, integer, or rational with denominator bigger than $n$, then there is only one integral chamber, while otherwise there are two: $\mathbb{R}_{>0}, \mathbb{R}_{<0}$. The quantum chambers (for $c \in -\frac{a}{b} + \mathbb{Z}$) are of the form $\{-\frac{a}{b} + \ell + m | m \in \mathbb{Z}_{>0}\}$ and $\{-\frac{a}{b} - \ell - m | m \in \mathbb{Z}_{>0}\}$.

5. Alcoves

This is a technical section, our goal here is to discuss various things related to alcoves defined by the walls $\Gamma$ and the subsets $\tilde{\Sigma}_\Gamma$. Here we only deal with questions from elementary combinatorial geometry.

We start with a finitely generated lattice $p_\mathbb{Z}$, a finite collection of codimension one sublattices to be called walls. For each wall $\Gamma$, we have a primitive element $\alpha_\Gamma \in p_\mathbb{Z}$ defined up to a sign with $\Gamma = \ker \alpha_\Gamma$. Further, for each $\Gamma$ we fix a finite saturated subset $\tilde{\Sigma}_\Gamma \subset \mathbb{Q}$. Let $\Sigma_{\Gamma}$ denote the set of classes of elements from $\tilde{\Sigma}_\Gamma$ in $\mathbb{Q}/\mathbb{Z}$.

5.1. Real alcoves and $p$-alcoves. We define two closely related sets of alcoves. One of them, real alcoves, will be in $p_\mathbb{R}$, while the other, $p$-alcoves, will be in $p_\mathbb{Z}$.

Let us start with real alcoves. Consider the hyperplanes of the form $\langle \alpha_\Gamma, \bullet \rangle = m$ for $m + \mathbb{Z} \in \Sigma_{\Gamma}$ for all walls $\Gamma$. By real alcoves we mean the closures of the connected components of $p_\mathbb{R}$ with these hyperplanes removed. Note that the real alcoves do not change if we translate by $p_\mathbb{Z}$.

Example 5.1. Let us consider the very classical case of $X = T^*B$ and the corresponding data of $p_\mathbb{Z}$, walls, and sets $\tilde{\Sigma}_\Gamma = \{0\}$. The hyperplanes we consider are of the form $\langle \alpha_i^\vee, \bullet \rangle = m$, where $\alpha_i^\vee$ is a coroot. Let $\alpha_1^\vee, \ldots, \alpha_r^\vee$ denote the simple coroots and $\alpha_0^\vee$ be the minimal coroot. Then we have the so called fundamental alcove defined by $\langle \alpha_i^\vee, \bullet \rangle \geq 0$ for $i = 1, \ldots, r$, and $\langle \alpha_0^\vee, \bullet \rangle \geq -1$. All other alcoves are obtained from this one by the standard action of the affine Weyl group of $W$ on $p_\mathbb{Z}$.

Example 5.2. Now let us consider the case of $X = \Hilb_n(\mathbb{C}^2)$ (and the parameter $c$).

Here, recall, $p_\mathbb{Z} = \mathbb{Z}, \Gamma = \{0\}$ and $\tilde{\Sigma}_\Gamma = \bigcup_{k=\ell}^{\ell+1} (-\frac{a}{b} + i | 1 \leq a < b \leq n \}$. It follows that the real alcoves are intervals between consecutive rational numbers with denominators from 2 to $n$.

Now fix an integer $p \gg 0$ such that $p + 1$ is divisible by the denominator of any element in $\tilde{\Sigma}_\Gamma$ for all $\Gamma$. Later on, $p$ will assumed to be prime but at this point we do not require
that. Consider the hyperplanes of the form \( \langle \alpha_{\Gamma_i}, \bullet \rangle = (p+1)\sigma + pm \), where \( \sigma \in \hat{\Sigma}_{\Gamma} \) and \( m \in \mathbb{Z} \). We can always choose \( \sigma \) to be maximal in \( \hat{\Sigma}_{\Gamma} \cap (\sigma + \mathbb{Z}) \).

Let \( \hat{A}_{\mathbb{R}} \) denote a connected component of the complement of the union of these hyperplanes in \( \mathcal{P}_{\mathbb{R}} \). By a \( p \)-alcove we mean the intersection of an open subset of the form \( \hat{A}_{\mathbb{R}} \) with \( \mathcal{P}_{\mathbb{Z}} \). So every \( p \)-alcove is the set of integral points \( \lambda \) subject to the inequalities of the form \( \langle \alpha_{\Gamma}, \lambda \rangle > (p+1)\sigma + pm \).

Dividing by \( p \) we get the hyperplanes of the form \( \langle \alpha_{\Gamma_i}, \bullet \rangle = \sigma + m + \frac{z}{p} \). Since \( p \) is large enough, we get a natural bijection between the real chambers and the \( p \)-chambers. Namely, consider a real alcove \( A \) defined by the inequalities \( \langle \alpha_{\Gamma_i}, \bullet \rangle \geq \sigma_i + m_i \), where \( \Gamma_i \) runs over all walls defining codimension 1 faces of \( A \) (where \( \sigma_i \) satisfies the maximality condition as above). Then there is a unique \( p \)-alcove to be denoted by \( pA \), where the points \( \lambda \) satisfy the inequalities \( \langle \alpha_{\Gamma_i}, \lambda \rangle > (p+1)\sigma_i + pm_i \) (there may be other inequalities as well).

So a \( p \)-alcove can have more faces than the corresponding real alcove. However, to every codimension 1 face of a real alcove one can naturally assign a face of the corresponding \( p \)-alcove of codimension 1.

**Example 5.3.** Let \( X = T^*B \). Then the hyperplanes are of the form \( \langle \alpha, \bullet \rangle = pm \) for \( m \in \mathbb{Z} \). We have the fundamental \( p \)-alcove, whose points are all integral \( \lambda \) such that \( \langle \alpha^v, \lambda \rangle \geq 1, \langle \alpha^0, \lambda \rangle \geq 1 - p \). All other \( p \)-alcoves are obtained from the fundamental one by the affine Weyl group action, where an element \( \mu \) of the root lattice acts by the shift by \( p\mu \).

**Example 5.4.** Let \( X = \text{Hilb}_n(\mathbb{C}^2) \). Let us describe the \( p \)-alcoves for the parameter \( c = \lambda - 1/2 \). By our assumption we need to take \( p \) with \( p + 1 \) divisible by \( n! \). The \( p \)-alcoves have form \( [\frac{p+1}{b}a' + \ell + 1, \frac{p+1}{b}a' - \ell - 1] \), where \( \frac{a'}{b} \) are rational numbers with denominators between 2 and \( n \) such that \( (\frac{a'}{b}, \frac{a}{b}) \) has no rational numbers with these denominators.

5.2. **Compatible elements.** In this section we will choose a finite collection of points in \( \mathcal{P}_{\mathbb{Q}} \) with a prescribed behavior mod \( p \).

Namely, pick a real alcove \( A \) and its face \( \Theta \). Let \( \Gamma_1, \ldots, \Gamma_k \) be all classical walls whose translates contain \( \Theta \) and let \( \Gamma'_1, \ldots, \Gamma'_j \) be the remaining hyperplanes whose translates intersect \( A \). We assume that \( A \) is given by \( \langle \alpha_{\Gamma_i}, \bullet \rangle \geq m_i \) and \( \langle \alpha_{\Gamma'_j}, \bullet \rangle \geq m'_j \) for \( m_i \in \Sigma_{\Gamma_i} + \mathbb{Z}, m'_j \in \Sigma_{\Gamma'_j} + \mathbb{Z} \).

Now let \( \lambda \in \mathcal{P}_{\mathbb{Q}} \). Below in this section we consider \( p \) that is large enough and is such that \( (p+1)\lambda \in \mathcal{P}_{\mathbb{Z}} \) and \( p + 1 \) is divisible by the denominators of all elements in \( \hat{\Sigma}_{\Gamma} \) for all \( \Gamma \). By \( pA \) we denote the \( p \)-alcove corresponding to \( A \). Recall that it consists of all \( \lambda' \in \mathbb{Z} \) such that

\[
\langle \alpha_{\Gamma_i}, \bullet \rangle > pm_i + \tilde{m}_i, \quad \langle \alpha_{\Gamma'_j}, \bullet \rangle > pm'_j + \tilde{m}'_j.
\]

Here we write \( \tilde{m}_i \) for the largest element in \( \hat{\Sigma}_{\Gamma_i} \cap (\Sigma_{\Gamma_i} + \mathbb{Z}) \) and \( \tilde{m}'_j \) has the similar meaning. So the right hand sides in the inequalities above are integral.

**Definition 5.5.** We say that \( \lambda \) is compatible with \( (A, \Theta) \) if there is \( p\lambda \in pA \) depending on \( p \) in an affine way such that, for all \( p \gg 0 \) satisfying the congruence conditions in the previous paragraph, we have

(i) \( p\lambda = \lambda + p\mu \), where \( \mu \) is an element \( \lambda + p\mathbb{Z} \),

(ii) the numbers \( \langle \alpha_{\Gamma_i}, p\lambda \rangle - pm_i - \tilde{m}_i \) are independent of \( p \) for all \( i \).
(iii) the numbers $\langle \alpha_{\Gamma_j}, p\lambda \rangle - pm'_j - \tilde{m}'_j$ are affine functions in $p$ with nonzero linear coefficient (that is automatically a positive number).

Lemma 5.6. The following are true:

1. For every $(A, \Theta)$, there is $\lambda$ compatible with $(A, \Theta)$.
2. There is a finite subset $\Lambda \in p\mathbb{Q}$ such that for every $(A, \Theta)$, there is an element $\lambda \in \Lambda$ compatible with $(A, \Theta)$.

Proof. Let us prove (1). We will be looking for $p\lambda$ in the form $\lambda + p\mu$ for $\mu$ such that

\begin{equation}
\lambda - \mu \in p\mathbb{Z}
\end{equation}
so that (i) holds.

Let us see what the condition $p\lambda \in p\mathbb{Z}$ means in terms of $\lambda, \mu$. Recall that we have chosen $p$ so that $(p + 1)\lambda \in p\mathbb{Z}$. So $\lambda + p\mu = (p + 1)\lambda + p(\mu - \lambda)$. This expression is integral for $p$ satisfying the congruence conditions from Section 5.1 provided (5.1) holds.

In (ii) we have

$$\langle \alpha_{\Gamma_i}, p\lambda \rangle - pm_i - \tilde{m}_i = p(\langle \alpha_{\Gamma_i}, \mu \rangle - m_i) + (\langle \alpha_{\Gamma_i}, \lambda \rangle - \tilde{m}_i).$$

So (ii) means

\begin{equation}
\langle \alpha_{\Gamma_i}, \mu \rangle = m_i, \langle \alpha_{\Gamma_i}, \lambda \rangle > \tilde{m}_i.
\end{equation}

Similarly (iii) means that

\begin{equation}
\langle \alpha_{\Gamma'_j}, \mu \rangle > m'_j.
\end{equation}

So for $\mu$ we can take any rational point lying inside $\Theta$, while for $\lambda$ we can take any element in $\mu + p\mathbb{Z}$ such that $\langle \alpha_{\Gamma_i}, \lambda \rangle > \tilde{m}_i$. It is clear by the choice of $\alpha_{\Gamma_1}, \ldots, \alpha_{\Gamma_k}$ that such $\lambda$ exists. This finishes the proof of (1).

Let us prove (2). For this we just need to notice that an element compatible with $(A, \Theta)$ is also compatible with $(A + \chi, \theta)$ for every $\chi \in p\mathbb{Z}$.

Remark 5.7. Let $\lambda$ be compatible with $(A, \Theta)$ and let $\chi \in p\mathbb{Z}$ satisfy $\langle \alpha_{\Gamma_i}, \chi \rangle \geq 0$. Then $\lambda + \chi$ is also compatible with $(A, \Theta)$.

Also let $A^-$ denote the alcove opposite to $A$ with respect to $\Theta$. Let $\lambda, \lambda^-$ be elements compatible with $(A, \Theta), (A^-, \Theta)$. We can assume that $\lambda^- - \lambda \in p\mathbb{Z}$.

Example 5.8. Suppose that we are in the setting of Example 5.4. Pick a real alcove $A = (a/b, a'/b')$. Then, for any $m > \ell$, the element $a/b + m$ is compatible with $(A, \{a/b\})$ and the element $a'/b' - m$ is compatible with $(A, \{a'/b'\})$.

6. R-forms

Below in this section $R$ denotes a finite localization of $\mathbb{Z}$.

6.1. Assumptions.
6.1.1. Assumptions on $R$. In this section we will pick finite sets of elements, $\Lambda \subset \mathfrak{P}_Q$ as well as $\mathcal{P}_1, \mathcal{P}_2 \subset \mathfrak{p}_Z$. Then we will have two finite localizations $R_0 \subset R$ of $\mathbb{Z}$. We will fix $R_0$ in this part and will further replace $R$ with its finite localizations later in this section. In the next two parts we will impose some assumptions on the varieties $X, Y$ and their quantizations.

Let us start by defining finite subsets $\Lambda, \mathcal{P}_1, \mathcal{P}_2$. We choose $\Lambda$ so that for every pair $(A, \Theta)$ of a real alcove and its face, there is $\lambda \in \Lambda$ compatible with $(A, \Theta)$. Let $\Gamma_1, \ldots, \Gamma_k$ be the classical walls whose translates contain $\Theta \subset A$ and let the numbers $m_i$ be given by $\alpha_{\Gamma_i}|_\Theta = m_i, i = 1, \ldots, k$. Thanks to Remark 6.1 and results recalled in Section 2.5.3 we can, in addition, assume that the following hold:

(a) $\lambda$ lies in a quantum chamber $C^\nu$ shifted from an integral chamber $C^{\nu, \text{int}}$ such that the equations $\langle \alpha_{\Gamma_i}, \bullet \rangle = 0, i = 1, \ldots, k$ define a face of $C^{\nu, \text{int}}$.

(b) For each $\lambda \in \Lambda$ compatible with $(A, \Theta)$, there is $\lambda^- \in \Lambda$ compatible with $(A^-, \Theta)$ such that

(b1) $\lambda^- = \lambda + \chi$ for $\chi \in \text{Pic}(X)$,

(b2) $\lambda^-$ lies in the quantum chamber $C^{\nu, \text{int}}$ shifted from $C^{\nu, \text{int}, -}$ (the integral chamber that is opposite to $C^{\nu, \text{int}}$ with respect to the face defined by $\langle \alpha_{\Gamma_i}, \bullet \rangle = 0$, $i = 1, \ldots, k$),

(b3) and $\text{MC}_{\lambda^->\lambda}$ is a perverse derived equivalence $D^b(A_{\lambda^-}\text{-mod}) \sim D^b(A_{\lambda^-}\text{-mod})$.\]

For $\mathcal{P}_1$ we will take the set of all $\chi$ that appear in (b1).

Now let us define a ring $R_0$. By definition, it has the form $\mathbb{Z}[1/N!]$, where $N$ is the maximum of the denominators of elements of all sets $\Sigma_R$ and of all elements in $\Lambda_1$.

For $\mathcal{P}_2$ we take the union of generating sets for all possible monoids $C_2$. Note that $\mathcal{P}_2$ contains a generating set of $C^{\nu, \text{int}}_Z$ for every integral chamber $C^{\nu, \text{int}}$.

We take a finite localization $R$ of $R_0$ as explained below in this section.

6.1.2. Assumptions on $X, Y$. We still assume that the formal slice to any symplectic leaf in $Y$ is conical, see Section 2.1.6 and that $X$ comes with a Hamiltonian action of a torus $T$ with finitely many fixed points.

We further assume that $X, Y, \rho$ and the $\mathbb{C}^\times \times T$-action are defined over some finite localization $R$ of $\mathbb{Z}$. We also assume that all line bundles on $X_R$ are defined over $R$. In particular, $\text{Pic}(X_R) \sim \text{Pic}(X)$. These two assumptions clearly hold in our examples of the cotangent bundles to flag varieties and Hilbert schemes (and, more generally, for Nakajima quiver varieties and Slodowy varieties).

Localizing $R$ further, we may assume that $H^i(X_R, O) = 0$ for $i > 0$ and $H^0(X_R, O) = R[Y]$ is flat over $R$.

Since $H^i(X_Q, O) = 0$, we see that the deformations $X_p, Y_p$ are defined over $\mathbb{Q}$. Replacing $R$ with a finite localization, we achieve that $\mathfrak{p}_R$ is an $R$-lattice in $\mathfrak{p}_Q$. We can pick $R$-forms $X_{pr}, Y_{pr}$ flat over $\mathfrak{p}_R := H^2(X, R)$. They come with contracting $\mathbb{G}_m$-actions and also with Hamiltonian $T_R$-actions (perhaps after a finite localization of $R$). We assume that all $T$-fixed points in $X$ are defined over $\mathbb{Q}$. We also assume that the fixed point subvariety in $X_p$ is defined over $\mathbb{Q}$ (and so are the isomorphisms of connected components of $X_p^T$ with $\mathfrak{p}$).

It is easy to see that these assumptions hold for $X = T^*B$ and for $X = \text{Hilb}_n(\mathbb{A}^2)$ (and, more generally, for all Slodowy varieties associated to principal Levi nilpotent elements and for all quiver varieties of affine type $A$).

In particular, for a generic one-parameter subgroup $\nu$, we see that $\mathcal{C}_\nu(O_{X_Q})$ is the structure sheaf of $X_{Q_p}^T$ and, similarly, $\mathcal{C}_\nu(O_{X_{p, Q}})$ is the structure sheaf of $X_{p, Q}^T$. After a
finite localization of $R$, we achieve that these properties hold over $R$. We can also achieve that $C_\nu(R[Y_\P])$ is a finitely generated module over $R[\P]$.

6.1.3. Assumptions on quantizations. We assume that (S) is satisfied for quantizations of $X$ (over $C$).

We suppose that the canonical quantization $A_{\P,R}^\theta$ of $X_{\P,R}^\theta$ has an $R$-form $A_{\P,R}^\theta$ that is a microlocal quantization of $X_{P,R}$. Let us write $A_{\P,R}$ for the algebra of global sections. Note that $A_\P = C \otimes_R A_{\P,R}$ and $A_\P^\theta = C \hat{\otimes}_R A_{\P,R}^\theta$ (where $\hat{\otimes}$ stands for the completed tensor product with respect to the induced filtration).

This clearly holds in the example of $T^*\mathcal{B}$, where the quantization $A_{\P,R}^\theta$ is obtained as (the microlocalization of) $D_{G/U,R}^{T_\theta}$. This also holds for the Hilbert schemes, where the quantization $A_{\P,R}^\theta$ is obtained as

$$\left(\left[D(V_R)/D(V_R)\Phi([g_R, g_R])\right]|_{T^*X_R^{\theta}}\right)^{G_R}.$$ 

More generally, the conditions hold for the quantizations of Slodowy varieties and of Nakajima quiver varieties.

Now let us discuss the homological dimension for the algebras $A_{\lambda,R}$.

Lemma 6.1. After a finite localization of $R$, for all $\lambda \in \Lambda$, the algebra $A_{\lambda,R}$ has finite homological dimension.

Proof. The algebra $A_\lambda$ has finite homological dimension. To deduce that $A_{\lambda,R}$ has finite homological dimension (after a finite localization of $R$) we can argue as in the proof of [BL, Lemma 6.8].

6.2. Forms of translation bimodules. We will further replace $R$ with its finite localization so that the translation bimodules $A_{R,\chi}$ have good properties as explained below in this section.

Recall the finite sets $\mathcal{P}_1, \mathcal{P}_2 \subset P$ from Section 6.1.1. We lift these sets to Pic($X$). Recall that $-P_1 = \mathcal{P}_1$ by the construction and we require this property to hold for the lifts as well. Abusing the notation, we will view $\mathcal{P}_1, \mathcal{P}_2$ as subsets in Pic($X$) = Pic($X_R$). Set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

As above, we can quantize the line bundles $O(\chi)$ on $X_{P,R}$ getting bimodules $A_{\P,\chi,R}^\theta$. We note that we have natural homomorphisms

$$A_{\P,\chi_2,R}^\theta \otimes_{A_{\P,R}^\theta} A_{\P,\chi_1,R}^\theta \to A_{\P,\chi_1+\chi_2,R}^\theta, \quad \Gamma(A_{\P,\chi_2,R}^\theta) \otimes_{A_{\P,R}^\theta} \Gamma(A_{\P,\chi_1,R}^\theta) \to \Gamma(A_{\P,\chi_1+\chi_2,R}^\theta).$$

Note that the base changes of $\Gamma(A_{\P,\chi,R}^\theta)$ to $Q$ are independent of $\theta$ for the same reason as the $C$-forms are independent of $\theta$. Each of the bimodules $\Gamma(A_{\P,R}^\theta)$ is finitely generated over $A_{\P,R}$. It follows that after a finite localization of $R$, the bimodule $\Gamma(A_{\P,\chi,R}^\theta)$ is independent of $\theta$ for any $\chi \in \mathcal{P}$. We denote this bimodule by $A_{\P,\chi,R}^\theta$. Note that $C \otimes_R A_{\P,\chi,R} = A_{\P,\chi}$.

Note that each bimodule $A_{\P,\chi,R}, \chi \in \mathcal{P}$, is Harish-Chandra. From here one deduces that after a finite localization of $R$ we can achieve that $A_{\P,\chi,R}$ is flat over $R$. This is a consequence of the following more general result that is proved completely analogously to [BL, Lemma 3.5].

Lemma 6.2. Let $M$ be a finitely generated $A_{\P,R}$-module. Then after a finite localization of $R$, the $R$-module $M$ becomes flat.
We will also need some properties of bimodules $A_{\lambda, \chi, R}$ for $\chi \in \mathcal{P}_2$ and $A_{\lambda, \lambda, R}$: where $\lambda \in \Lambda$ and $\chi = \lambda^- - \lambda$.

For $\chi \in \mathcal{P}_2$, let $\mathcal{P}^{\text{reg}[\chi]}$ denote the subset of $\lambda \in \mathfrak{p}$ such that $\lambda, \lambda + \chi$ lie in the same quantum chamber. It is clear that $\mathcal{P}^{\text{reg}[\chi]}$ is the complement to finitely many integral hyperplanes in $\mathcal{P}$. Also note that $\mathcal{P}^{\text{reg}[-\chi]} = \mathcal{P}^{\text{reg}[\chi]} + \chi$. Consider the bimodules $A_{\mathcal{P}^{\text{reg}[\chi]}, \lambda, R} := A_{\mathcal{P}^{\text{reg}[\chi]}, \lambda, R} \otimes_{\mathcal{P}^{\text{reg}[\chi]}, R} \mathcal{P}^{\text{reg}[\chi]}$ (this is a $A_{\mathcal{P}^{\text{reg}[\chi]}, R} - A_{\mathcal{P}^{\text{reg}[\chi]}, R}$-bimodule) and $A_{\mathcal{P}^{\text{reg}[-\chi]}, -\chi, R}$.

**Lemma 6.3.** After a finite localization of $R$, the bimodules $A_{\mathcal{P}^{\text{reg}[\chi]}, \lambda, R}, A_{\mathcal{P}^{\text{reg}[-\chi]}, \lambda, R}$ are mutually inverse Morita equivalence bimodules for all $\chi \in \mathcal{P}_2$.

**Proof.** By Corollary 2.4, $A_{\lambda, \chi}, A_{\lambda + \chi, -\chi}$ are mutually dual Morita equivalence bimodules for $\lambda \in \mathcal{P}^{\text{reg}[\chi]}$. The natural homomorphisms

$$A_{\lambda + \chi, -\chi} \otimes A_{\lambda + \chi} \rightarrow A_{\lambda}, A_{\lambda, \chi} \otimes A_{\chi} A_{\lambda + \chi, -\chi} \rightarrow A_{\lambda + \chi}$$

are obtained by specialization to $\lambda$ from

$$A_{\mathcal{P}^{\text{reg}[-\chi]}, -\chi} \otimes A_{\mathcal{P}^{\text{reg}[\chi]}, \chi} A_{\mathcal{P}^{\text{reg}[-\chi]}, -\chi} \otimes A_{\mathcal{P}^{\text{reg}[\chi]}, -\chi} \rightarrow A_{\mathcal{P}^{\text{reg}[-\chi]}}.$$ (6.1)

It follows that homomorphisms (6.1) are isomorphisms. These homomorphisms are defined over $R$. It follows that the kernels and the cokernels of

$$A_{\mathcal{P}^{\text{reg}[-\chi]}, -\chi, R} \otimes A_{\mathcal{P}^{\text{reg}[\chi]}, R}, A_{\mathcal{P}^{\text{reg}[\chi]}, R} \rightarrow A_{\mathcal{P}^{\text{reg}[\chi]}, R},$$

$$A_{\mathcal{P}^{\text{reg}[-\chi]}, -\chi, R} \otimes A_{\mathcal{P}^{\text{reg}[\chi]}, R} A_{\mathcal{P}^{\text{reg}[-\chi]}, -\chi, R} \rightarrow A_{\mathcal{P}^{\text{reg}[-\chi]}, R}$$

are $R$-torsion. A direct analog of Lemma 6.2 holds for the algebras of the form $A_{\mathcal{P}^{\text{reg}[\chi]}, R} \otimes_R A_{\mathcal{P}^{\text{reg}[-\chi]}, R}$ and so the kernels and the cokernels vanish after a finite localization of $R$. □

Now pick $\lambda \in \Lambda$ and let $\lambda^- \in \Lambda, \chi \in \mathcal{P}_1$ be as before. The bimodule $A_{\lambda, \chi}$ is defined over $Q$. It follows that the filtrations by Serre subcategories on $A_{\lambda} - \text{mod}, A_{\lambda + \chi} - \text{mod}$ that make $\mathcal{W}C_{\lambda^- - \lambda}$ perverse are defined over $Q$. Recall, Section 2.5.3 that these filtrations come from chains of ideals, say $\{0\} = \mathcal{I}_{\lambda}^0 \subset \mathcal{I}_{\lambda}^1 \subset \ldots \subset \mathcal{I}_{\lambda}^k \subset \mathcal{A}_{\lambda},$ where $\lambda^- = \lambda$ or $\lambda^-$. So the ideals are defined over $Q$ as well. We set $\mathcal{I}_{\lambda, R}^j = \mathcal{I}_{\lambda}^j \cap \mathcal{A}_{\lambda, R}$ so that $\mathcal{I}_{\lambda}^j = \mathbb{C} \otimes_R \mathcal{A}_{\lambda, R}$.

**Lemma 6.4.** After a finite localization of $R$ we achieve that for all $\lambda \in \Lambda$ the following hold:

1. $\mathcal{I}_{\lambda, R}^j = (\mathcal{I}_{\lambda}^j)^2_{\lambda, R}$ for all $j$.
2. The functor $A_{\lambda, \chi, R} \otimes_{\lambda, R} : D^b(A_{\lambda, R} - \text{mod}) \rightarrow D^b(A_{\lambda + \chi, R} - \text{mod})$ is a perverse derived equivalence with respect to the filtrations given by the ideals $\mathcal{I}_{\lambda, R}^j, \mathcal{I}_{\lambda + \chi, R}^j$.

**Proof.** We have $\mathcal{I}_{\lambda}^j = (\mathcal{I}_{\lambda}^j)^2$. From here we get $\mathcal{I}_{\lambda, R}^j = (\mathcal{I}_{\lambda, R}^j)^2$ similarly to the proof of Lemma 6.3. Let us proceed to (2).

The functor $A_{\lambda, \chi, R} \otimes_{\lambda, R} :$ is an equivalence if and only if

$$A_{\lambda + \chi, R} \simeq R \text{End}_{A_{\lambda, R}}(A_{\lambda, \chi, R}), \quad A_{\lambda, \chi, R} \otimes_{\lambda, R} R \text{Hom}(A_{\lambda, \chi, R}, A_{\lambda, R}) \simeq A_{\lambda, R}$$

We know that these homomorphisms become iso after base change to $\mathbb{C}$. So they are iso after a finite localization of $R$.

We also know that (b1)-(b6) from Section 2.5.3 hold over $\mathbb{C}$. From here we deduce that (b1)-(b6) hold over $R$, perhaps after a finite localization of $R$. Now that implies that $A_{\lambda, \chi, R} \otimes_{\lambda, R} :$ is perverse as required. □
6.3. Forms of Verma modules. The goal of this and the next sections is to show that most results and constructions that we have for the categories $O$ over $\mathbb{C}$ still carry over to $\mathbb{R}$, perhaps after some finite localization of $\mathbb{R}$.

Recall, Section 6.1.2, that $C_\nu(O_{X_{\mathbb{P}^n}, R}) \cong R[\mathbb{P}][X^T]$. Replacing $\mathbb{R}$ with its finite localization we achieve that the sheaf $C_\nu(A^p_{X, R})$ is a filtered deformation of $C_\nu(O_{X_{\mathbb{P}^n}, R})$, this is proved as the analogous claim over $\mathbb{C}$ in [L6, Section 5.2].

We get a homomorphism $C_\nu(A_{X, R}) \to R[\mathbb{P}][X^T]$. Note that $C_\nu(A_{X, R})$ is a finitely generated $\mathbb{R}$-module. This follows from $C_\nu(R[X_\nu]) \to \text{gr} \ C_\nu(A_{X, R})$.

Since the homomorphism $C_\nu(A_{X_{\mathbb{P}^n}, R}) \to R[\mathbb{P}][X^T]$ is an isomorphism after a base change to $\mathbb{C}$, it is still an isomorphism after a finite localization of $\mathbb{R}$. We assume from now on that this is an isomorphism.

Thanks to the homomorphism $C_\nu(A_{X, R}) \to R[\mathbb{P}][X^T]$ we can define Verma module $\Delta_{\nu, \mathbb{P}^n, R}(x)$ for $x \in X^T$ similarly to the case of $\mathbb{C}$. Clearly, $\mathbb{C} \otimes_{\mathbb{R}} \Delta_{\nu, \mathbb{P}^n, R}(x) = \Delta_{\nu, \mathbb{C}}(x)$. By Lemma 6.5, after a finite localization of $\mathbb{R}$ we can assume that $\Delta_{\nu, \mathbb{P}^n, R}$ is flat over $\mathbb{R}$. It follows that all graded components of $\Delta_{\nu, \mathbb{P}^n, R}(x)$ are finitely generated free $\mathbb{R}$-modules.

**Lemma 6.5.** After a finite localization of $\mathbb{R}$, each $\Delta_{\nu, \mathbb{P}^n, R}(x)$ is projective over $R[\mathbb{P}^{reg}]$.

**Proof.** We can filter the algebra $A_{\mathbb{P}, R}$ (with an ascending $T_{\mathbb{R}}$-stable $\mathbb{Z}_{\geq 0}$-filtration) in such a way that $R[\mathbb{P}]$ is in degree 0 and $\text{gr} A_{\mathbb{P}, R} = R[\mathbb{P}][X]$, compare to [BL, Section 3.5]. Equip $\Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ with a compatible $T_{\mathbb{R}}$-stable good filtration. Then $\text{gr} \Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ is a finitely generated $R[\mathbb{P}^{reg}][X]$-module. Hence, by generic flatness, there is a finite localization $R[\mathbb{P}^{reg}]^0$ of $R[\mathbb{P}^{reg}]$ such that $R[\mathbb{P}^{reg}]^0 \otimes_{R[\mathbb{P}^{reg}]} \text{gr} \Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ is flat over $R[\mathbb{P}^{reg}]^0$. It follows that $R[\mathbb{P}^{reg}]^0 \otimes_{R[\mathbb{P}^{reg}]} \Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ is flat over $R[\mathbb{P}^{reg}]^0$. So every graded component of $\Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ becomes flat (hence projective) after localization to $R[\mathbb{P}^{reg}]^0$. A similar argument works for the specialization of $\Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ to a quotient of $R[\mathbb{P}^{reg}]$ and shows that this specialization becomes flat after a finite localization of that quotient.

We deduce that one can split $\mathbb{P}_{\mathbb{R}}^{reg}$ into a disjoint union locally closed irreducible subschemes in such a way that the restriction of any graded component to any of the subschemes is projective. On the other hand, we know that $\mathbb{Q} \otimes_{\mathbb{R}} \Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ is flat over $\mathbb{Q}[\mathbb{P}^{reg}]$ (because the similar statement holds over $\mathbb{C}$). Combining these two observations, we see that after a finite localization of $\mathbb{R}$ each graded component of $\Delta_{\nu, \mathbb{P}^{reg}, R}(x)$ is flat over $R[\mathbb{P}^{reg}]$. \qed

Now let us study translations of Verma modules. We can consider equivariant Verma modules $\Delta_{\nu, \mathbb{P}, R}(x, \kappa)$ and equivariant lifts $A_{\mathbb{P}, x, R}$, as before, see Section 3.8.1.

**Lemma 6.6.** After a finite localization of $\mathbb{R}$, we have a $T_{\mathbb{R}}$-equivariant isomorphism

\begin{equation}
\Delta_{\nu, \mathbb{P}^{reg}[\lambda], R}(x, \kappa + \text{wt}_\chi(x)) \cong A_{\mathbb{P}^{reg}[\lambda], x, R} \otimes_{A_{\mathbb{P}^{reg}[\lambda], x}} \Delta_{\nu, \mathbb{P}^{reg}[\lambda], R}(x, \kappa).
\end{equation}

**Proof.** By Proposition 3.17 combined with Proposition 3.18 we know that

\begin{equation}
\Delta_{\nu, \chi + \lambda}(x, \kappa + \text{wt}_\chi(x)) \cong A_{\chi + \lambda} \otimes_{A_{\lambda}} \Delta_{\nu, \chi}(x, \kappa).
\end{equation}

Let us first show that this implies

\begin{equation}
\Delta_{\nu, \mathbb{P}^{reg}[\lambda]}(x, \kappa + \text{wt}_\chi(x)) \cong A_{\mathbb{P}^{reg}[\lambda], x} \otimes_{A_{\mathbb{P}^{reg}[\chi], x}} \Delta_{\nu, \mathbb{P}^{reg}[\chi], R}(x, \kappa).
\end{equation}

By (6.3), the highest graded component that appears in the right hand side of (6.4) is $\langle \nu, \kappa + \text{wt}_\chi(x) \rangle$ and the specialization of this component to any point of $\mathbb{P}_{\mathbb{R}}^{reg}[\lambda]$ is one-dimensional. Since $\mathbb{P}_{\mathbb{R}}^{reg}[\chi]$ is a factorial scheme, it follows that the highest graded
component is \( \mathbb{C}[\mathfrak{p}^{\text{reg}}[\lambda]] \), the same as of the left hand side of (6.4). So the \( \mathbb{C}[\mathfrak{p}^{\text{reg}}[\lambda]] \)-module of \( T \)-equivariant homomorphisms from the left hand side to the right hand side is free of rank one. Again, applying (6.3) we see that the invertible elements of \( \mathbb{C}[\mathfrak{p}^{\text{reg}}[\lambda]] \) correspond to isomorphisms.

Now it is enough to show that a version of (6.2) holds over \( \mathbb{Q} \) (then, since our modules are finitely generated, we can take a finite localization of \( R \) to achieve (6.2)). But an isomorphism in (6.2) is rational if and only if its highest degree component is rational. Again, \( \mathbb{Q}[\mathfrak{p}^{\text{reg}}[\lambda]] \) is a factorial algebra, so the highest degree component of the right hand side is a free rank one module. The analog of (6.2) over \( \mathbb{Q} \) follows. \( \square \)

6.3.1. \( \textbf{Tor vanishing.} \) We can also consider the right-handed versions of Verma modules, \( \Delta_{\nu,\mathfrak{p},R}(x) \). Straightforward analogs of Lemmas 6.5 and 6.6 hold for these modules.

**Lemma 6.7.** After a finite localization of \( R \), we have

\[
\text{Tor}^A_{A_{\mathfrak{p}^{\text{reg}},R}}(\Delta_{\nu,\mathfrak{p}^{\text{reg}},R}(x), \Delta_{\nu,\mathfrak{p}^{\text{reg}},R}(x')) = R[\mathfrak{p}^{\text{reg}}]^{-\delta_{\nu,\lambda}}.
\]

**Proof.** Note that, for \( \lambda \in \mathfrak{p}^{\text{reg}} \), we have

\[
\mathbb{C}_\lambda \otimes_{R[\mathfrak{p}^{\text{reg}}]} \left( \Delta^r_{-\nu,\mathfrak{p}^{\text{reg}},R}(x) \otimes_{A_{\mathfrak{p}^{\text{reg}},R}} \Delta_{\nu,\mathfrak{p}^{\text{reg}},R}(x') \right) = \Delta^r_{\nu,\lambda}(x) \otimes_{A_{\lambda}} \Delta_{\nu,\lambda}(x').
\]

The right hand side vanishes if \( x \neq x' \) and is the one-dimensional space in homological degree 0 if \( x = x' \).

Let us show that all homology of the left hand side of (6.5) are finitely generated \( R[\mathfrak{p}^{\text{reg}}] \)-modules. This will follow once we check that \( \text{Tor}^A_{A_{\mathfrak{p}^{\text{reg}},R}}(A_{\mathfrak{p},R}/A_{\mathfrak{p},R}^{<0}, A_{\mathfrak{p},R}/A_{\mathfrak{p},R}^{\geq 0}) \) is a finitely generated \( R[\mathfrak{p}] \)-module. This follows from the observation that

\[
g_{\text{reg}} A_{\mathfrak{p},R}/(g_{\text{reg}} A_{\mathfrak{p},R}^{<0} A_{\mathfrak{p},R} + g_{\text{reg}} A_{\mathfrak{p},R}^{\geq 0})
\]

is a finitely generated graded \( R[\mathfrak{p}] \)-module.

Now let us recall that

\[
\mathbb{Q} \otimes_R \text{Tor}^A_{A_{\mathfrak{p}^{\text{reg}},R}}(\Delta^r_{-\nu,\mathfrak{p}^{\text{reg}},R}(x), \Delta_{\nu,\mathfrak{p}^{\text{reg}},R}(x'))
\]

is zero if \( i \neq 0, x \neq x' \) and is \( \mathbb{Q}[\mathfrak{p}^{\text{reg}}] \) else. Similarly to the proof of Lemma 6.5 (see the last paragraph there, in particular), it follows that

\[
\text{Tor}^A_{A_{\mathfrak{p}^{\text{reg}},R}}(\Delta^r_{-\nu,\mathfrak{p},R}(x), \Delta_{\nu,\mathfrak{p}^{\text{reg}},R}(x'))
\]

becomes flat over \( R[\mathfrak{p}^{\text{reg}}] \) after a finite localization of \( R \) (depending on \( i \)).

It remains to check that the algebra \( A_{\mathfrak{p}^{\text{reg}},R} \) has finite homological dimension after a finite localization of \( R \) (so that we only have finitely many non-vanishing \( \text{Tor}'s \)). Similarly to Lemma 6.1, this will follow if we check that \( A_{\mathfrak{p}^{\text{reg}}} \) has finite homological dimension. By Remark 4.6, the homological dimension of \( A_{\mathfrak{p}^{\text{reg}}} \) does not exceed \( \dim X + \dim \mathfrak{p} \).

So after a finite localization of \( R \), (6.5) holds (for all \( x, x', i \)). \( \square \)

6.4. \( \textbf{Forms of simple, projective, etc. objects in } \mathcal{O}. \) Let us discuss forms of various objects in \( \mathcal{O}_\nu(A_{\lambda}), \lambda \in \Lambda, \) over \( R \). First of all, let us note that, by our assumptions, for \( \lambda \in \mathfrak{p}^{\text{reg}} \), the category \( \mathcal{O}_\nu(A_{\lambda}) \) is defined over \( \mathbb{Q} \). For \( x \in X^T \), let \( L_{\nu,\lambda,Q}(x), P_{\nu,\lambda,Q}(x), I_{\nu,\lambda,Q}(x), \nabla_{\nu,\lambda,Q}(x) \) denote the simple, the projective, the injective and the costandard objects in \( \mathcal{O}_\nu(A_{\lambda,Q}) \) labelled by \( x \).

Let us pick some \( T_R \)-stable \( R \)-forms \( L_{\nu,\lambda,R}(x) \subset L_{\nu,\lambda,Q}(x), \nabla_{\nu,\lambda,R}(x) \subset \nabla_{\nu,\lambda,Q}(x), P_{\nu,\lambda,R}(x) \subset P_{\nu,\lambda,Q}(x), I_{\nu,\lambda,R}(x) \subset I_{\nu,\lambda,Q}(x) \).
Lemma 6.8. After a finite localization of $\mathcal{R}$, for every $\lambda \in \Lambda$, we achieve the following:

1. The objects $\Delta_{\nu,\lambda,R}(x)$, $\nabla_{\nu,\lambda,R}(x)$ are filtered by $L_{\nu,\lambda,R}(x')$’s. Moreover, $\Delta_{\nu,\lambda,R}(x) \to L_{\nu,\lambda,R}(x) \hookrightarrow \nabla_{\nu,\lambda,R}(x)$.

2. The objects $P_{\nu,\lambda,R}(x)$ are filtered by $\Delta_{\nu,\lambda,R}(x')$’s and the objects $I_{\nu,\lambda,R}(x)$ are filtered by $\nabla_{\nu,\lambda,R}(x')$’s (in the increasing order with respect to $\leq_{\nu,\lambda}$).

3. Every object $L_{\nu,\lambda,R}(x)$ has a finite resolution

$$\ldots \to P_{1,R} \to P_{0,R} \to L_{\nu,\lambda,R}(x) \to 0,$$

where each $P_{i,R}$ is the direct sum of $P_{\nu,\lambda,R}(x')$’s. Similarly, $L_{\nu,\lambda,R}(x)$ has a resolution

$$0 \to L_{\nu,\lambda,R}(x) \to I_{0,R} \to I_{1,R} \to \ldots$$

where each $I_{i,R}$ is the direct sum of $I_{\nu,\lambda,R}(x')$’s.

Proof. The proofs of (1)-(3) are similar, let us prove (2). This claim clearly holds over $\mathbb{C}$ and the filtrations are defined over $\mathbb{Q}$. So it holds over $\mathbb{Q}$ as well. After a finite localization, it holds over $\mathcal{R}$ too. 

Lemma 6.9. Let $M_{R}, N_{R}$ be graded $\mathcal{R}$-lattices in $M_{Q}, N_{Q} \in \mathcal{O}_{\nu}^{T}(A_{\lambda,Q})$. Then, for all $i$, $\text{Ext}_{A_{\lambda,R},T}^{i}(M_{R}, N_{R})$ is a finitely generated $\mathcal{R}$-module.

Proof. Let us pick a free graded resolution

$$\ldots \to A_{\lambda,R} \otimes_{\mathcal{R}} V_{1} \to A_{\lambda,R} \otimes_{\mathcal{R}} V_{0} \to M_{R} \to 0,$$

where the $V_{i}$’s are a finite rank graded free $\mathcal{R}$-modules. Then the Ext’s we want to compute are the degree 0 components homology of

$$\ldots \to V_{i} \otimes_{\mathcal{R}} N_{R} \to V_{0} \otimes_{\mathcal{R}} N_{R} \to 0$$

All graded components of $N_{R}$ are finitely generated over $\mathcal{R}$ and our claim follows. 

Pick $m \in \mathbb{Z}$ such that $|c_{\nu,\lambda}(x_{1}) - c_{\nu,\lambda}(x_{2})| \leq m$ for all $x_{1}, x_{2}$ in the same $h$-block.

Corollary 6.10. After a finite localization of $\mathcal{R}$, we have the following. Let $M_{R}, N_{R}$ be modules of the form $L_{\nu,\lambda,R}(x_{i}, \kappa_{i}), \Delta_{\nu,\lambda,R}(x_{i}, \kappa_{i}), \nabla_{\nu,\lambda,R}(x_{i}, \kappa_{i}), P_{\nu,\lambda,R}(x_{i}, \kappa_{i})$ or $I_{\nu,\lambda,R}(x_{i}, \kappa_{i})$, $i = 1, 2$, where $x_{1}, x_{2}$ are in the same $h$-block and $|\kappa_{1} - \kappa_{2}| \leq m$. Then for all $i$, the $\mathcal{R}$-modules $\text{Ext}_{A_{\lambda,R},T}^{i}(M_{R}, N_{R})$ are free and finitely generated over $\mathcal{R}$ (and, automatically, $\mathbb{C} \otimes_{\mathcal{R}} \text{Ext}_{A_{\lambda,R},T}^{i}(M_{C}, N_{C}) \cong \text{Ext}_{A_{\lambda,R},T}^{i}(M_{C}, N_{C})$).

To finish this section let us describe the images of $\Delta_{\nu,\lambda,R}(x, \kappa)$ under the wall-crossing functors $A_{\lambda,R} \otimes_{A_{\lambda,R}} \bullet$. Here we assume that $\lambda \in \Lambda$ and $\chi \in \mathcal{P}_{1}$ is the corresponding element. We choose a $T_{R}$-equivariant structure on $O_{R}(\chi)$ that gives rise to an equivariant structure on $A_{\lambda,R}$.

Lemma 6.11. After a finite localization of $\mathcal{R}$, for all $\lambda \in \Lambda$ and $\chi \in \mathcal{P}_{1}$, we have

$$A_{\lambda,R} \otimes_{A_{\lambda,R}}^{L} \Delta_{\nu,\lambda,R}(x, \kappa) \cong \nabla_{\nu,\lambda,R}(x, \kappa + \text{wt}_{\chi}(x)).$$

Proof. We have such an isomorphism over $\mathbb{C}$, this follows from Lemma 8.19. It follows that the degree $\kappa + \text{wt}_{\chi}(x)$ component in $A_{\lambda,R} \otimes_{A_{\lambda,R}}^{L} \Delta_{\nu,\lambda,Q}(x, \kappa)$ is 1-dimensional. Also it is the irreducible $C_{\nu}(A_{\lambda,Q})$-module corresponding to $x$. So we get a homomorphism

$$A_{\lambda,R} \otimes_{A_{\lambda,Q}}^{L} \Delta_{\nu,\lambda,Q}(x, \kappa) \to \nabla_{\nu,\lambda,Q}(x, \kappa + \text{wt}_{\chi}(x))$$

unique up to proportionality. So it is an isomorphism over $\mathbb{C}$, hence an isomorphism. The claim of the lemma easily follows from here. 

7. Reduction to characteristic $p$

7.1. Assumptions on $p$. Let $p$ be a prime non-invertible in $R$. Let us write $\mathbb{F}$ for $\mathbb{F}_p$. We will be interested in categories of representations of the algebras $A_{\mathbb{F},\lambda} := \mathbb{F}_\lambda \otimes_{R[\mathfrak{P}]} A_{\mathfrak{P},\lambda}$ for $\lambda \in \mathfrak{P}_\mathbb{F}$.

We will impose several additional assumptions on $p$. First of all, we assume that $p+1$ is divisible by the denominators of all elements in $\Sigma_\Gamma$ and $(p+1)\lambda \in \mathfrak{P}_\mathbb{Z}$ for all $\lambda \in \Lambda$. Second, we suppose that $(p+1)c_{\nu,\lambda}(x) \in \mathbb{Z}$ for all $\lambda \in \mathfrak{P}_R$, generic $\nu$ and all $x \in X^T$.

There are infinitely many such primes.

Also we need to assume that $p$ is large enough. In addition to being non-invertible in $R$, this means the following. First, we want the $p$-alcoves to be in bijection with real alcoves as explained before. Second, we want $p$ to be large enough to satisfy the conditions of the next lemma.

Lemma 7.1. For $p$ sufficiently large, for every two points $\lambda_1, \lambda_2$ in the same $p$-alcove, there is a sequence $\chi_1, \ldots, \chi_k \in \Lambda_2$ such that

\[ (*) \quad \text{The elements } \lambda_1 + \sum_{i=1}^k \chi_i \text{ is in the same } p\text{-alcove for all } j = 0, \ldots, k. \]

Proof. Let $\langle \alpha_{\Gamma_i}, \bullet \rangle > \hat{m}_i, i = 1, \ldots, k$ be the equations defining the $p$-alcove of interest. Pick $\epsilon > 0$. Clearly, if $p$ is large enough and $\langle \alpha_{\Gamma_i}, \lambda_\ell \rangle > \hat{m}_i + \epsilon p$, $\ell = 1, 2$, then there are $\chi_1, \ldots, \chi_k$ such that $(*)$ holds.

Now consider $\lambda \in \mathfrak{P}_\mathbb{Z}$ such that $\hat{m}_i < \langle \alpha_{\Gamma_i}, \lambda_\ell \rangle \leq \hat{m}_i + N\epsilon p$ for some $i$, where $N$ is a large enough integer (again, independent of $p$). Let $i = 1, \ldots, k'$ be precisely the indexes $i$, for which this holds. If $p$ is sufficiently large, we can find a classical chamber $C$ whose interior points are positive on all $\alpha_{\Gamma_1}, \ldots, \alpha_{\Gamma_{k'}}$. Pick the generators $\chi_1', \ldots, \chi_s'$ of $C_\mathbb{Z}$. A point of the form $\lambda + (m+1)(\chi_1' + \cdots + \chi_s') + m(\chi_{r+1}' + \cdots + \chi_s')$ lies in the alcove for all $r$ and all $m$ bounded by a function linear in $N$. We can increase $N$ and decrease $\epsilon$ (so that $N\epsilon$ is fixed) in such a way that $\lambda' := \lambda + m(\chi_1' + \cdots + \chi_s')$ now satisfies $\langle \alpha_{\Gamma_i}, \lambda' \rangle > \hat{m}_i + \epsilon p$ for all $i$. Thanks to the previous paragraph, this finishes the proof. $\square$

Below we always assume that the conclusion of Lemma 7.1 holds.

Let us proceed to our next condition. Fix $\chi_1 \in \mathcal{P}_2$. Consider the locus of $\lambda \in \mathfrak{P}_\mathbb{Z}$ such that $\lambda, \lambda + \chi_1$ are in the same quantum chamber. This locus is given by the conditions of the form $\langle \lambda, \alpha_{\Gamma} \rangle \not\in \tilde{\Sigma}_{\Gamma}(\chi)$, where $\Gamma$ runs over all walls $\Gamma$ and $\tilde{\Sigma}_{\Gamma}(\chi)$ is a suitable finite subset such that $\tilde{\Sigma}_{\Gamma} \subset \tilde{\Sigma}_{\Gamma}(\chi) \subset \Sigma_{\Gamma} + \mathbb{Z}$.

We also assume that, for $\lambda \in \mathfrak{P}_\mathbb{Z}$, the following are equivalent:

(i) $\lambda, \lambda + \chi_1$ are in the same $p$-alcove,

(ii) $\langle \lambda, \alpha_{\Gamma} \rangle \not\in (p+1)\sigma + pm$ for all walls $\Gamma$, $\sigma \in \tilde{\Sigma}_{\Gamma}(\chi), m \in \mathbb{Z}$.

This holds for $p \gg 0$.

Finally, we need an assumption on the residues of $c_{\nu,\lambda}(x)$ mod $p$. Let $\tilde{c}_{\nu,\lambda}(x)$ denote this residue, by our previous assumptions on $p$, it is $(p+1)c_{\nu,\lambda}(x) \mod p$. We suppose that the $h$-blocks for $\lambda \in \Lambda$ are ordered in the following sense: if $x_1, x_2$ lie in the same $h$-block for $\lambda$ (which, recall, means that $c_{\nu,\lambda}(x_1) - c_{\nu,\lambda}(x_2) \in \mathbb{Z}$) and $x_1', x_2'$ lie in a different $h$-block, then $\tilde{c}_{\nu,\lambda}(x_1) > \tilde{c}_{\nu,\lambda}(x_2)$ implies $\tilde{c}_{\nu,\lambda}(x_1') > \tilde{c}_{\nu,\lambda}(x_2')$. This is clearly the case for $p$ large enough.

7.2. Equivalences.

7.2.1. Abelian equivalences. For $\lambda \in \mathfrak{P}_\mathbb{F}, \chi \in \mathcal{P}_2$ we define an $A_{\lambda,\mathbb{F}} \otimes A_{\lambda,\mathbb{F}}$-bimodule $A_{\lambda,\mathbb{F}}$ as the specialization of $\mathbb{F} \otimes R A_{\mathfrak{P},\lambda,\mathbb{F}}$ to $\lambda$. 

Proposition 7.2. Let $\lambda, \lambda' = \lambda + \chi$ lie in the same $p$-alcove of $\mathfrak{p}_\mathbb{Z}$. Then the categories $\mathcal{A}_{\lambda, \mathbb{F}}$-mod, $\mathcal{A}_{\lambda', \mathbb{F}}$-mod are equivalent via tensoring with a suitable sequence of bimodules $\mathcal{A}_{\lambda_i, \chi_i, \mathbb{F}}$, where the elements $\lambda_i$ lie in the same $p$-alcove and $\chi_i \in \mathcal{P}_2$.

Proof. Thanks to Lemma 7.1, it is enough to prove that $\mathcal{A}_{\lambda_i, \chi_i, \mathbb{F}}$ is aMorita equivalence $\mathcal{A}_{\lambda_i + \chi_i, \mathbb{F}}$-bimodule as long as $\lambda_i, \lambda_i + \chi_i$ are in the same $p$-alcove. Lemma 6.3 implies that $\mathcal{A}_{\text{pres}[\lambda_i, \chi_i, \mathbb{R}], \mathcal{A}_{\text{pres}[\lambda_i + \chi_i, \mathbb{R}}$ are mutually inverse Morita equivalence bimodules. Thanks to the equivalence of (i) and (ii) in the Section 7.1, we see that $\lambda_i$ (mod $p$) belongs to $\mathfrak{p}_\mathbb{R}^{\text{reg}}$ mod $p$. It follows that $\mathcal{A}_{\lambda_i, \chi_i, \mathbb{F}}$ and $\mathcal{A}_{\lambda_i + \chi_i, \mathbb{F}}$ are mutually inverse Morita equivalence bimodules. \hfill $\square$

7.2.2. Perverse equivalences. Now pick $\lambda \in \Lambda$ (compatible with a real chamber $A$ and its face $\Theta$). Let $\chi$ be the corresponding element in $\mathcal{P}_1$. Define the bimodule $\mathcal{A}_{\lambda, \mathbb{F}}$ as $\mathbb{F} \otimes_{\mathbb{R}} \mathcal{A}_{\lambda, \mathbb{R}}$. Further, consider the chains of ideals $\mathcal{I}^j_{\lambda, \mathbb{F}} := \mathbb{F} \otimes_{\mathbb{R}} \mathcal{I}^j_{\lambda, \mathbb{R}}$ in $\mathcal{A}_{\lambda, \mathbb{F}}$ and the similarly defined chain $\mathcal{I}^j_{\lambda + \chi, \mathbb{F}}$.

Proposition 7.3. The functor $\mathcal{A}_{\lambda, \mathbb{F}} \otimes \mathcal{A}_{\lambda, \mathbb{F}} \mapsto D^b(\mathcal{A}_{\lambda, \mathbb{F}}$-mod) $\sim D^b(\mathcal{A}_{\lambda + \chi, \mathbb{F}}$-mod) is a perverse derived equivalence with respect to the filtrations defined by chains of ideals $\mathcal{I}^j_{\lambda, \mathbb{F}}, \mathcal{I}^j_{\lambda + \chi, \mathbb{F}}$.

Proof. We have $(\mathcal{I}^j_{\lambda, \mathbb{F}})^2 = \mathcal{I}^j_{\lambda, \mathbb{F}}$ (if $\lambda, \lambda + \chi$ because the similar equalities hold over $\mathbb{R}$. Also note that the vanishing conditions (b1)-(b6) from Section 2.5.3 hold over $\mathbb{F}$ because they hold over $\mathbb{R}$, see Lemma 6.4. Note, in particular, that the torsion of the bimodules $\mathcal{B}_{\mathbb{R}, j}$ from (b5) and (b6) is annihilated by the ideals $\mathcal{I}^j_{\mathbb{R}, j}$ (by (b5) and (b6) over $\mathbb{R}$) that insures that (b5) and (b6) hold over $\mathbb{F}$. \hfill $\square$

Corollary 7.4. Let $(A, \Theta)$ be as above and let $A^-$ be the real alcove that is opposite to $A$ with respect to $\Theta$. Let $\lambda_1 \in \mathfrak{p}_\mathbb{Z} A, \lambda_2 \in \mathfrak{p}_\mathbb{Z} A^-$. Then the categories $\mathcal{A}_{\lambda_1, \mathbb{F}}$-mod and $\mathcal{A}_{\lambda_2, \mathbb{F}}$-mod are perverse derived equivalent.

8. Modular categories $\mathcal{O}$

8.1. Definition and basic properties.

8.1.1. Definition. Recall that we have an action of $T_{\mathbb{F}}$ on $X_{\mathbb{F}}, Y_{\mathbb{F}}, \mathcal{A}_{\lambda, \mathbb{F}}$ etc.

For $\lambda \in \mathfrak{p}_\mathbb{Z}$, we consider the category $\bar{\mathcal{O}}(\mathcal{A}_{\lambda, \mathbb{F}})$ consisting of all $T_{\mathbb{F}}$-equivariant finite dimensional $\mathcal{A}_{\lambda, \mathbb{F}}$-modules.

Note that the $p$-central character of a simple module in $\bar{\mathcal{O}}(\mathcal{A}_{\lambda, \mathbb{F}})$ (a point in $Y_{\mathbb{F}}^{(1)}$) is $T_{\mathbb{F}}$-stable. Since $T_{\mathbb{F}}$ has finitely many fixed points in $X_{\mathbb{F}}$, we conclude that the only fixed point in $Y_{\mathbb{F}}^{(1)}$ is 0 and so the $p$-character of any module in $\bar{\mathcal{O}}(\mathcal{A}_{\lambda, \mathbb{F}})$ is zero.

8.1.2. Baby Verma modules. Now suppose that $\lambda \in \mathfrak{p}_\mathbb{Z}$ lies in a $p$-alcove. Let $\nu$ be a generic one-parameter subgroup.

Lemma 8.1. We have $\mathcal{C}_\nu(\mathcal{A}_\lambda) \cong \mathbb{F}[X^T]$.

Proof. Recall that we have $\mathcal{C}_\nu(\mathcal{A}_{\mathfrak{p}_{\mathbb{R}}^{\text{reg}}}) \cong \mathbb{R}[\mathfrak{p}_{\mathbb{R}}^{\text{reg}}][X^T]$. Specializing to $\lambda \in \mathfrak{p}_\mathbb{F}_\text{reg}$, we get the required isomorphism. \hfill $\square$
Let us give an example of an object in \( \hat{\mathcal{O}}(A_{\lambda,F}) \), a baby Verma module. For \( x \in X^T \) and \( \kappa \in X(T) \), we can form the Verma module \( \Delta_{\nu,F}(x,\kappa) \) as in characteristic 0:

\[
\Delta_{\nu,F}(x,\kappa) := A_{\lambda,F}/A_{\lambda,F}A_{\lambda,F}^0 \otimes \mathbb{C}(A_{\lambda,F}) \mathbb{F}_x,
\]

where we put \( \mathbb{F}_x \) in degree \( \kappa \). We define the baby Verma module \( \Delta_{\nu,F}(x,\kappa) \) as the fiber of \( \Delta_{\nu,F}(x,\kappa) \) over \( 0 \in Y_F^{(1)} \).

**Lemma 8.2.** A choice of a generic one-parameter subgroup gives rise to an identification of the set of simples in \( \hat{\mathcal{O}}(A_{\lambda,F}) \) with \( X^T \times X(T) \).

**Proof.** A standard argument shows that every simple is the quotient of a unique baby Verma module. This gives the required classification. \( \square \)

The simple corresponding to \( (x,\kappa) \) will be denoted by \( L_{\nu,\lambda,F}(x,\kappa) \).

### 8.1.3. Equivariant block decomposition

We will also need a direct sum decomposition of the category \( \hat{\mathcal{O}}(A_{\lambda,F}) \). Consider the quantum moment map \( \Phi : t_F \rightarrow A_{\lambda,F} \). Note that \( h \) has degree 0 hence preserves every graded component of \( M \in \hat{\mathcal{O}}(A_{\lambda,F}) \). Then \( M := \bigoplus_{\alpha \in \mathbb{F}_x \otimes \mathbb{C}(T)} M_\alpha \), where \( M_\alpha \) denotes the \( T_F \)-graded subspace of \( M \), where all \( h \in t_F \) acts on the graded component \( M_\kappa \) with the single eigenvalue \( \alpha + \kappa \). Then \( M_\kappa \) is a graded \( A_{\lambda,F} \)-submodule and hence an object in \( \hat{\mathcal{O}}(A_{\lambda,F}) \).

We consider the category \( \hat{\mathcal{O}}^\beta(A_{\lambda,F}), \beta \in t_F^* \), consisting of all \( M \) with \( M = M^\beta \). These categories will be called equivariant blocks. Then \( \hat{\mathcal{O}}(A_{\lambda,F}) = \bigoplus_{\beta \in t_F^*} \hat{\mathcal{O}}^\beta(A_{\lambda,F}) \). Note that all categories \( \hat{\mathcal{O}}^\beta(A_{\lambda,F}) \) are obtained from \( \hat{\mathcal{O}}^0(A_{\lambda,F}) \) by shifting the \( T_F \)-equivariant structure.

For \( x \in X^T \), let us write \( \Phi_x \) for the composition of \( \Phi : t_F \rightarrow A_{\lambda,F}^T \) and the projection \( A_{\lambda,F}^T \rightarrow \mathbb{F} \) corresponding to \( x \). This is an element of \( t_F^* \). We note that \( L_{\nu,\lambda,F}(x,\kappa) \) lies in \( \hat{\mathcal{O}}^\beta(A_{\lambda,F}) \) if and only if \( \Phi_x - \kappa = \beta \) modulo \( p \).

### 8.1.4. Equivalences

Recall that a choice of a \( T_F \)-equivariant structure on \( O_R(\chi) \) gives a \( T_F \)-equivariant structure on \( A_{\phi,\lambda,R} \) and hence on \( A_{\phi,\lambda,F} \).

**Lemma 8.3.** For \( \lambda \in \mathfrak{q}_F \) and \( \chi \in \mathcal{P}_2 \), the functor \( A_{\lambda,F} \otimes_{A_{\lambda,F}} \bullet \) restricts to \( D_{fin}(A_{\lambda,F} \text{-mod}^T) \rightarrow D_{fin}(A_{\lambda+\chi,F} \text{-mod}^T) \).

**Proof.** The claim that \( A_{\lambda,F} \otimes_{A_{\lambda,F}} \bullet \) sends finite dimensional modules to complexes with finite dimensional homology follows from the fact that \( A_{\lambda,F} \) is a HC bimodule. Since \( A_{\lambda,F} \) is also \( T_F \)-equivariant, it follows that \( A_{\lambda,F} \otimes_{A_{\lambda,F}} \bullet \) upgrades to a functor between equivariant categories. \( \square \)

**Proposition 8.4.** Let \( \lambda, \lambda+\chi \) lie in the same alcove. Then \( A_{\lambda,F} \otimes_{A_{\lambda,F}} \bullet \) is an equivalence \( \hat{\mathcal{O}}(A_{\lambda,F}) \) and \( \hat{\mathcal{O}}(A_{\lambda+\chi,F}) \) that sends \( L_{\nu,\lambda,F}(x,\kappa) \) to \( L_{\nu,\lambda+\chi,F}(x,\kappa+\text{wt}_\chi(x)) \).

**Proof.** By Proposition 7.2, \( A_{\lambda,F} \) is a Morita equivalence bimodule. It follows from Lemma 6.6 that \( A_{\lambda,F} \) (with our choice of a \( T_F \)-equivariant structure) sends \( \Delta_{\nu,\lambda,F}(x;\kappa) \) to \( \Delta_{\nu,\lambda+\chi,F}(x;\kappa+\text{wt}_\chi(x)) \). Since \( L_{\nu,\lambda,F}(x;\kappa) \) is the unique graded quotient of \( \Delta_{\nu,\lambda,F}(x;\kappa) \), our claim follows. \( \square \)
8.1.5. **Duality.** Now we want to discuss the contravariant duality between categories \( \hat{\mathcal{O}}(A_{\lambda,\mathbb{F}}) \) and \( \hat{\mathcal{O}}(A_{-\lambda,\mathbb{F}}) \). Recall that we have an isomorphism \( (A_{\lambda,\mathbb{F}})^{\text{opp}} \cong A_{-\lambda,\mathbb{F}} \). Then \( M \mapsto M^* \) defines a contravariant equivalence between \( \hat{\mathcal{O}}(A_{\lambda,\mathbb{F}}) \) and \( \hat{\mathcal{O}}(A_{\lambda,\mathbb{F}}^{\text{opp}}) \). But \( A_{\lambda,\mathbb{F}}^{\text{opp}} \cong A_{-\lambda,\mathbb{F}} \).

**Lemma 8.5.** We have \( L_{\lambda,\mathbb{F}}(x, \kappa) = L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \).

*Proof.* Since \( \bullet^* \) is an equivalence, the object \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \) is simple. First of all, let us show that it corresponds to the point \( x \in X^T \). Recall, see, e.g., [LE Section 4.2], that we have an isomorphism \( C_\nu(A_{\lambda,\mathbb{F}})^{\text{opp}} \cong C_{-\nu}(A_{-\lambda,\mathbb{F}}) \). This isomorphism intertwines the identifications of these algebras with \( \mathbb{F}[X^T] \). The condition that \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^{A_{\lambda,\mathbb{F}}^{<0}} = \mathbb{F}_x \) translates to \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^*/\langle \nu, A_{\lambda,\mathbb{F}}^{<0}, \nu \rangle = \mathbb{F}_x \). Since \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \) is an irreducible module, it follows that \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \cong \mathbb{F}_x \). Hence \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \cong L_{\nu,\lambda,\mathbb{F}}(x, \kappa') \) for some \( \kappa' \). Note also that the highest degree (with respect to \( \nu \)) component of \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \) equals \( (\kappa, \nu) \), while the character of the action of \( T \) on this component is \( \kappa \). So \( L_{-\lambda,\mathbb{F}}(x, -\kappa)^* \cong L_{\nu,\lambda,\mathbb{F}}(x, \kappa) \). \( \square \)

Using the duality we can define analogs of costandard objects of \( \hat{\mathcal{O}}(A_{\lambda,\mathbb{F}}) \) that will be ind-objects to be denoted by \( \nabla^\mathbb{F}_{\nu,\lambda}(x, \kappa) \). Namely, we define \( \nabla^\mathbb{F}_{\nu,\lambda}(x, \kappa) \) as the restricted dual of \( \Delta_{-\lambda,\mathbb{F}}(x, -\kappa) \). We note that \( \nabla^\mathbb{F}_{\nu,\lambda}(x, \kappa) \) is not the same as \( \mathbb{F} \otimes_R \nabla_{\nu,\lambda,R}(x, \kappa) \) (the latter is still a pro-object).

8.2. **Highest weight structure.** Let \( C \) be an abelian Artinian and Noetherian category. Suppose that we have an auto-equivalence \( S \) of \( C \) such that the action of \( \mathbb{Z} \) on \( \text{Irr}(C) \) induced by \( S \) is free. Further, suppose that that \( \text{Irr}(C)/\mathbb{Z} \) is finite.

Further, let \( \leq \) be a partial order on \( \text{Irr}(C) \) subject to the following properties:

- The \( \mathbb{Z} \)-action preserves the order.
- We have \( L < SL \) for every \( L \in \text{Irr}(C) \).
- For any two \( L, L' \) with \( L < L' \), there is \( n \in \mathbb{Z}_{>0} \) such that \( L' < S^nL \).

For \( L \in \text{Irr}(C) \), let \( C_{\leq L} \) denote the Serre span of \( L' \in \text{Irr}(C) \) with \( L' \leq L \). We say that the pair \( (C, \leq) \) is periodic highest weight (shortly, PHW) if the following holds:

(PHW) Each quotient \( C_{\leq L}/C_{\leq L'} \) is a highest weight category with respect to the order \( \leq \) in the usual sense.

Recall that by a poset interval we mean a subset \( \mathcal{I} \) such that if \( L, L' \in \mathcal{I} \) satisfy \( L \leq L' \) and \( L'' \) some other element of the poset with \( L \leq L'' \leq L' \), then \( L'' \in \mathcal{I} \). For an interval \( \mathcal{I} \in \text{Irr}(C) \) we can define a subquotient category \( C_\mathcal{I} \). Namely, consider the set \( \mathcal{F} \) of all simples \( L \) with \( L \leq L' \) for some \( L' \in \mathcal{I} \). Then we can consider the Serre span \( C_\mathcal{F} \) of all simples in \( \mathcal{F} \). By definition, \( C_\mathcal{I} := C_\mathcal{F}/C_{\mathcal{F} \setminus \mathcal{I}} \). We note that (PHW) is equivalent to \( C_\mathcal{I} \) being highest weight (with respect to the restricted order) for any finite interval \( \mathcal{I} \).

8.3. **Main result.** For simplicity, we will assume that \( \dim T = 1 \). We will remark on the general case later.

Let us define a partial order \( \leq_\nu \) on \( \text{Irr}(\hat{\mathcal{O}}(A_{\lambda,\mathbb{F}})) = X^T \times \mathbb{Z} \) as follows: \( (x, \kappa) \leq_\nu (x', \kappa') \) if the following hold:

- \( (x, \kappa) \) and \( (x', \kappa') \) belong to the same equivariant block meaning \( \Phi_x - \kappa = \Phi_{x'} - \kappa' \) in \( \mathbb{F}_p \).
- \( (x, \kappa) = (x', \kappa') \) or \( \kappa < \kappa' \).
We can define a $\mathbb{Z}$-action on this poset as follows: $z.(x, \kappa) := (x, \kappa + zp)$. Clearly, it satisfies the conditions of the previous section.

Now pick a finite interval $\mathcal{I} \subset X^T \times \mathfrak{X}(T)$.

**Lemma 8.6.** Let $(x, \kappa), (x', \kappa') \in \mathcal{I}$. Then there is a graded submodule $M \subset \Delta_{\nu,\lambda,F}(x', \kappa')$ of finite codimension such that $L_{\nu,\lambda,F}(x, \kappa)$ does not occur in $M$ (i.e., does not occur in the quotient $M/M'$ for any graded $M' \subset M$ of finite codimension). The dual statement holds for $\nabla_{\nu,\lambda}^F(x', \kappa')$.

Note that the intersection of graded submodules of finite codimension submodules in $\Delta_{\nu,\lambda,F}(x, \kappa)$.

**Proof.** Note that there is the set of $T_F$-weights in $L_{\nu,\lambda,F}(x, \kappa)$ is finite. All weight spaces in $\Delta_{\nu,\lambda,F}(x', \kappa')$ are finite dimensional. So we can find a $T_F$-stable ideal of finite codimension $m \subset \mathbb{F}[Y^{(1)}_F]$ such that this set of weights doesn’t appear in $M := m\Delta_{\nu,\lambda,F}(x', \kappa')$. The module $\Delta_{\nu,\lambda,F}(x', \kappa')/M$ is finite dimensional.

The claim for $\nabla_{\nu,\lambda}^F(x', \kappa')$ follows by duality. \qed

Thanks to the lemma, we can define the object $\Delta_{\nu,\lambda,F}^3(x', \kappa') \in \tilde{O}(\mathcal{A}_{\lambda,F})_{\mathcal{I}}$ for $(x', \kappa') \in \mathcal{I}$ as the image of $M$ from the previous lemma. Analogously, we can define $\nabla_{\nu,\lambda,F}^3(x', \kappa')$.

The following theorem is the main result of this section.

**Theorem 8.7.** Let $\mathcal{I}$ be an interval in $X^T \times \mathfrak{X}(T)$. The category $\tilde{O}(\mathcal{A}_{\lambda,F})_{\mathcal{I}}$ is a PHW category with respect to the order $\preceq_{\nu}$. The standard and costandard objects in the subquotient $\tilde{O}(\mathcal{A}_{\lambda,F})_{\mathcal{I}}$ corresponding to $(x, \kappa) \in \mathcal{I}$ are $\Delta_{\nu,\lambda,F}^2(x, \kappa)$ and $\nabla_{\nu,\lambda}^3(x, \kappa)$.

**8.3.1. Examples.** Let us consider two examples of the order $\preceq_{\nu}$. First, let $X = T^*(G/B)$. Recall that $X^T$ is identified with $W$ and $c_{\nu,\lambda}(w) = \langle \nu, w\lambda \rangle$. Pick $\nu = \rho^\vee$. So the order on $W \times \mathbb{Z}$ is as follows: $(w, \kappa) \leq (w', \kappa')$ if

- $\langle w\lambda, \rho^\vee \rangle - \kappa = \langle w'\lambda, \rho^\vee \rangle - \kappa'$ in $\mathbb{F}_p$,
- and either $\kappa < \kappa'$ or $(w, \kappa) = (w', \kappa')$.

Now let $X = \text{Hilb}_{\mu}(\mathbb{A}^2)$. Here $X^T$ is in bijection with the set $\mathcal{P}(n)$ of partitions of $n$. So the order on $\mathcal{P}(n) \times \mathbb{Z}$ is as follows (recall that we set $c = \lambda - 1/2$): $(\psi, \kappa) \leq (\psi', \kappa')$ if

- $c \text{ cont}(\psi) - n(\psi) - \kappa = c \text{ cont}(\psi') - n(\psi') - \kappa'$ in $\mathbb{F}_p$,
- and either $\kappa < \kappa'$ or $(\psi, \kappa) = (\psi', \kappa')$.

**8.4. Proof of Theorem [8.7].** The proof is in several steps.

**Step 1.** Pick $(\lambda, \kappa) \in \mathcal{I}$. Set $\mathcal{I}' := \{(x', \kappa') \in \mathcal{I} \mid (x, \kappa) \leq (x', \kappa')\}$. By the construction, the object $\Delta_{\nu,\lambda,F}^2(x, \kappa)$ coincides with $\Delta_{\nu,\lambda,F}^3(x, \kappa)$. By the construction, $\Delta_{\nu,\lambda,F}^3(x, \kappa)$ is the projective cover of $L_{\nu,\lambda,F}(x, \kappa)$ in the Serre subcategory $\tilde{O}(\mathcal{A}_{\lambda,F})_{\mathcal{I}'} \subset \tilde{O}(\mathcal{A}_{\lambda,F})_{\mathcal{I}}$. Dually, $\nabla_{\nu,\lambda}^3(x, \kappa)$ is the injective hull of $L_{\nu,\lambda,F}(x, \kappa)$ in that subcategory.

**Step 2.** What remains to show is that the projectives in $\tilde{O}(\mathcal{A}_{\lambda,F})_{\mathcal{I}}$ are $\Delta$-filtered (Steps 2-4) and that $\Delta$’s appear in a correct order (Step 5).

Note that

\begin{equation}
\text{Ext}_{\mathcal{A}_{\lambda,F}}^i(T(\Delta_{\nu,\lambda,F}(x, \kappa), \nabla_{\nu,\lambda}^F(x', \kappa')) = \text{Tor}_{i}^{\mathcal{A}_{\lambda,F}}(T(\Delta_{\nu,\lambda,F}(x, \kappa), \Delta_{\nu,\lambda,F}^r(x', \kappa'))).
\end{equation}

Using Lemmas [6.7] and [6.5] (as well as the direct analog of the latter for the right-handed Verma modules) we see that the right hand side of (8.1) vanishes. We deduce that

\begin{equation}
\dim \text{Ext}_{\mathcal{A}_{\lambda,F}}^i(T(\Delta_{\nu,\lambda,F}(x, \kappa), \nabla_{\nu,\lambda}^F(x', \kappa')) = 0, \text{ for } i > 0.
\end{equation}
In steps 3 and 4 we will deduce that the projectives are $\Delta$-filtered from (8.2).

**Step 3.** Take an integer $z$ and consider the category $A_{\lambda,F} \text{-mod}^T_{\leq z}$ of all $T_F$-equivariant $A_{\lambda,F}$-modules $M$ such that

- any $T_F$-weight $\kappa$ of $M$ satisfies $\kappa \leq z$,
- all weight spaces are finite dimensional.

This is a Serre subcategory of $A_{\lambda,F} \text{-mod}^T$. Note that $\Delta_{\nu,\lambda,F}(x, \kappa), \nabla_{\nu,\lambda}^F(x, \kappa) \in A_{\lambda,F} \text{-mod}^T_{\leq z}$ if and only if $\kappa \leq z$.

Repeating the proof of Lemma 8.6 we see that every object in $A_{\lambda,F} \text{-mod}^T_{\leq z_2}$ contains a graded submodule $M$ of finite codimension such that $\mathcal{L}_{\nu,\lambda,F}(x, \kappa)$ with $\kappa \in [z_1, z_2]$ does not appear in $M$. It follows that

$$A_{\lambda,F} \text{-mod}^T_{\leq z_2} / A_{\lambda,F} \text{-mod}^T_{\leq z_1} \sim \mathcal{O}(A_{\lambda,F})_{[z_1, z_2]}$$

Thanks to (8.2) and the standard results that the Ext vanishing in an ambient category implies that in any Serre subcategory, we get

$$\dim \text{Ext}^i_{A_{\lambda,F} \text{-mod}^T_{\leq z}}(\Delta_{\nu,\lambda,F}(x, \kappa), \nabla_{\nu,\lambda}^F(x, \kappa')) = 0, \text{ for } i > 0.$$  

**Step 4.** Note that the category $A_{\lambda,F} \text{-mod}^T_{\leq z}$ has enough projectives. Indeed, for $z' \leq z$, the object $P_{z,z'} := A_{\lambda,F}/A_{\lambda,F}A_{\lambda,F}^{z'-z'}$, where the image of 1 has degree $z'$, is projective. This is because $\text{Hom}(P_{z,z'}, M) = M_{z'}$ for $M \in A_{\lambda,F} \text{-mod}^T_{\leq z}$.

Every module in $A_{\lambda,F} \text{-mod}^T_{\leq z}$ is covered by a direct sum of the modules $P_{z,z'}$ (for finitely generated modules we can take a finite sum). A standard argument shows that $P_{z,z'}$ is filtered by $\Delta$’s. For reader’s convenience, let us provide this argument. We prove the existence of filtration by induction on $z - z'$. The case of $z - z' = 0$ is obvious: the object $P_{z,z}$ is the direct sum of Verma modules. Now note that we have a natural surjection $P_{z,z} \twoheadrightarrow P_{z-1,z},$ let $K$ stand for the kernel.

Note that $\text{Ext}^i_{A_{\lambda,F} \text{-mod}^T_{\leq z}}(K, \nabla_{\nu,\lambda,F}(x', \kappa')) = 0$ for $i > 0$. Indeed, for $i > 1$ this follows from $P_{z,z'}$ being projective and $P_{z-1,z'}$ being filtered by $\Delta$’s. We also have an exact sequence

$$0 \rightarrow \text{Hom}(K, \nabla_{\nu,\lambda,F}(x', \kappa')) \rightarrow \text{Hom}(P_{z,z'}, \nabla_{\nu,\lambda,F}(x', \kappa')) \rightarrow \text{Hom}(P_{z-1,z'}, \nabla_{\nu,\lambda,F}(x', \kappa'))$$

$$\rightarrow \text{Ext}^1(K, \nabla_{\nu,\lambda,F}(x', \kappa')) \rightarrow 0.$$  

Now consider two cases. If $\kappa' < z$, then the middle homomorphism is the identity isomorphism $\nabla_{\nu,\lambda,F}(x', \kappa') \sim \nabla_{\nu,\lambda,F}(x', \kappa')$. So

$$\text{Hom}(K, \nabla_{\nu,\lambda,F}(x', \kappa')) = \text{Ext}^1(K, \nabla_{\nu,\lambda,F}(x', \kappa')) = 0$$

If $\kappa' = z$, then $\text{Hom}(P_{z-1,z'}, \nabla_{\nu,\lambda,F}(x', \kappa')) = 0$ and hence

$$\text{Hom}(K, \nabla_{\nu,\lambda,F}(x', \kappa')) = \nabla_{\nu,\lambda,F}(x', \kappa'), \quad \text{Ext}^1(K, \nabla_{\nu,\lambda,F}(x', \kappa')) = 0.$$  

So we see that $\text{Ext}^1(K, \nabla_{\nu,\lambda,F}(x', \kappa')) = 0$ in both cases.

Set $d(x') := \dim \nabla_{\nu,\lambda,F}(x', \kappa')$. We conclude that there is an epimorphism

$$\bigoplus_{x \in X^T} \Delta_{\nu,\lambda,F}(x, z)^{\oplus d(x)} \twoheadrightarrow K.$$  

The kernel has no nonzero homomorphisms to the objects $\nabla_{\nu,\lambda,F}(x', \kappa')$ with $\kappa' \leq z$. On the other hand, if the kernel is nonzero, then it has a finite dimensional graded quotient (compare to the proof of Lemma 8.6) that must have a nonzero Hom to some $\nabla_{\nu,\lambda,F}(x', \kappa')$.  

So we see that $K = 0$. This finishes the proof of the claim that all objects $P_{z,z'}$ admit $\Delta$-filtrations.

Step 5. Note that $\text{Ext}^1_{\mathcal{A}_{\lambda,F}}(\Delta_{\nu,\lambda,F}(x, \kappa), \Delta_{\nu,\lambda,F}(x', \kappa')) \neq 0$, then $\kappa' > \kappa$. So the Verma modules in a $\Delta$-filtration of $P_{z,z'}$ appear in the correct order. Now suppose that $\mathcal{I} = \{(x, \kappa)| z_1 \leq \kappa \leq z_2\}$ for some integers $z_1 < z_2$. The images of $P_{z,z'}$ for $z' \in [z_1, z_2]$ under the quotient functor $\mathcal{A}_{\lambda,F} \rightarrow \mathcal{O}(\mathcal{A}_{\lambda,F})_\mathcal{I}$ are projectives in $\mathcal{O}(\mathcal{A}_{\lambda,F})_\mathcal{I}$ and every simple is covered by one of these projectives. It follows that the indecomposable projectives in $\mathcal{O}(\mathcal{A}_{\lambda,F})_\mathcal{I}$ are filtered by $\Delta_{\nu,\lambda,F}(x, \kappa)$'s in a correct order. So for this choice of the interval $\mathcal{I}$, the category $\mathcal{O}(\mathcal{A}_{\lambda,F})_\mathcal{I}$ is highest weight. For a general interval $\mathcal{J}$, the category $\mathcal{O}(\mathcal{A}_{\lambda,F})_\mathcal{J}$ is a highest weight subquotient in $\mathcal{O}(\mathcal{A}_{\lambda,F})_\mathcal{I}$, where $\mathcal{I} = [z_1, z_2]$ for suitable $z_1, z_2$, and hence a highest weight category itself.

8.4.1. Case of higher dimensional torus. Let us explain what modifications are needed to deal with the case when $\text{dim } T > 1$. We can still define a partial order $\leq_\nu$ on $X^T \times \mathfrak{X}(T)$ (instead of characters themselves we need now to compare their pairings with $\nu$). A technical problem here is that most of intervals for this order are infinite. But we can use a more general definition of a highest weight category in Remark 3.2 and we still get an analog of Theorem 8.7.

9. Standardly stratified structures

In this section we prove the main result of the present paper, Theorem 9.3. Namely, let $\lambda$ lie inside the $p$-alcove $pA$ (corresponding to a real alcove $A$). For each pair $((\Theta, \lambda), \lambda)$, where $\Theta$ is a face of $A$ and $\lambda$ is a parameter compatible with $(A, \Theta)$, see Section 5.2, we will define a standardly stratified structure (in a suitable sense to be explained below) on $\mathcal{O}(\mathcal{A}_{\lambda,F})$. Again, for convenience we assume that $\text{dim } T = 1$, the general case can be treated as in Section 8.4.1.

We will get two results about the standardly stratified structure. First, we will show that the associated graded of $\mathcal{O}(\mathcal{A}_{\lambda,F})$ is essentially a reduction to characteristic $p$ of $\mathcal{O}_\nu(\mathcal{A}_{\lambda,Q})$. We will further show that, roughly speaking, standard and proper standard objects in $\mathcal{O}(\mathcal{A}_{\lambda,F})$ are reductions to characteristic $p$ of projective and simple objects in $\mathcal{O}_\nu(\mathcal{A}_{\lambda,Q})$.

9.1. Main result. In this section, after some preparation we state the main theorem regarding standardly stratified structures on $\mathcal{O}(\mathcal{A}_{\lambda,F})$ determined by faces of the $p$-alcove $A$.

9.1.1. Standardly stratified structures on PHW categories. We use the ramification of the definition of a standardly stratified structure that appeared in [LW]. Let $\mathcal{C}$ be a PHW category with the shift functor $\mathcal{S}$. The additional data of a standardly stratified category is a $\mathbb{Z}$-invariant pre-order on $\text{Irr}(\mathcal{C})$ that is compatible with $\leq$, meaning that

$$L \prec L' \Rightarrow L < L' \Rightarrow L \preceq L'.$$

We also require that $L \prec SL$ for all $L$. Since $|\text{Irr}(\mathcal{C})| < \infty$, we see that the equivalence classes for $\prec$ are finite.

The axiom of a standardly stratified structure in this case is that, for each finite interval $\mathcal{J} \subset \text{Irr}(\mathcal{C})$ that is the union of some equivalence classes for $\prec$, the subquotient category $\mathcal{C}_\mathcal{J}$ is standardly stratified with respect to $\prec$ (in the sense explained in Section 3.1.2).
9.1.2. Pre-order determined by $\Theta$. We will now define a pre-order on $\text{Irr}(\tilde{O}(A_{\lambda, F}))$. Namely, recall that for $\lambda'$ in $\mathcal{P}A$ we have an equivalence $\tilde{O}(A_{\lambda, F}) \cong \tilde{O}(A_{\lambda', F})$ of highest weight categories. We will take $\lambda' := \hat{\nu} \lambda$. We will define a pre-order on $\text{Irr}(\tilde{O}(A_{\lambda', F}))^0$ (then we can transfer this pre-order to $\text{Irr}(\tilde{O}(A_{\lambda, F}))^\beta$ using the natural bijection between these sets, the simples from different equivariant blocks, by definition, are not comparable).

Now note that, for a fixed point $x \in X_T$, the map $p \mapsto c_{\nu, \hat{\nu} \lambda}(x)$ is an affine map in $p$. It follows that the possible values of $\kappa$ such that $(x, \kappa) \in \text{Irr}(\tilde{O}(A_{\lambda', F}))^0$ are affine functions in $p$, they are of the form $\kappa(p) + mp$, where $\kappa$ is one of these functions, and $m$ is an arbitrary integer. Define a pre-order $\preceq_{\nu, \lambda}$ on $\text{Irr}(\tilde{O}(A_{\lambda', F}))^0$ by $(x, \kappa) \preceq_{\nu, \lambda} (x, \kappa')$ if the coefficient of $p$ in the affine function $\kappa' - \kappa$ is $\geq 0$.

Let us describe the equivalence classes of the resulting pre-order on $\text{Irr}(\tilde{O}(A_{\lambda', F}))^0$ (the zero equivariant block). Recall that the labels of simples in that equivariant block are $(x, \kappa)$ such that $x \sim_{\nu, \lambda} x'$ if $c_{\nu, \lambda}(x) - c_{\nu, \lambda}(x') \in \mathbb{Z}$. By the definition of $\sim_{\nu, \lambda}$, the points $x$ and $x'$ are equivalent if and only if the corresponding simples lie in the same $h$-block of $\mathcal{O}_\nu(A_{\lambda})$.

Then we have the following easy elementary lemma that follows from the construction of $\lambda'$ (and that was a starting observation for the present paper).

**Lemma 9.1.** We have $(x, \kappa) \sim (x', \kappa')$ if and only if $x \sim_{\nu, \lambda} x'$ and $c_{\nu, \lambda}(x) - \kappa = c_{\nu, \lambda}(x') - \kappa'$. In particular, the order on the equivalence class for $\preceq$ coincides with $\preceq_{\nu, \lambda}$.

So any equivalence class for $\preceq$ gives rise to a rational number $\beta$ (defined up to adding an integer) as follows: if $(x, \kappa)$ is in the equivalence class, then $\beta - c_{\nu, \lambda}(x) \in \mathbb{Z}$.

**Remark 9.2.** A priori, the pre-order $\preceq_{\nu, \lambda}$ depends on the choice of $\hat{\lambda}$ and not only on $\Theta$. However, in the most interesting case when $\Theta$ is a point, it is easy to see, compare with Lemma 9.9 below, that $\preceq_{\nu, \lambda}$ is independent of the choice of $\hat{\lambda}$.

9.1.3. Reductions to characteristic $p$. Let $M \in \mathcal{O}_\nu(A_{\lambda, \mathbb{Q}})$ and let $M_R \subset M$ be a graded $R$-lattice. Pick $z_2 \in \mathbb{Z}$ such that $M_{z} = 0$ if $z > z_2$. Then $M_{\mathbb{F}} := \mathbb{F} \otimes_R M_R$ lies in $A_{\lambda', F}$-mod$_{\leq z_2}$.

Now choose $z_1 < z$ and set $\mathcal{I} = [z_1, z_2]$. We can view $\mathcal{I}$ as an interval in $\text{Irr}(\tilde{O}(A_{\lambda', F}))^0$.

Set $M_{\mathbb{F}, \mathcal{I}} := \pi_{\mathcal{I}}(M_{\mathbb{F}})$, where we write $\pi_{\mathcal{I}}$ for the quotient functor $\text{A}_{\lambda', F}$-mod$_{\leq z_2} \rightarrow \tilde{O}(A_{\lambda, F})_{\mathcal{I}}$ (that mods out $\text{A}_{\lambda', F}$-mod$_{\leq z_2}$, see (8.3)).

Recall that we have distinguished lattices $L_{\nu, \lambda, R}(x, \kappa) \subset L_{\nu, \lambda, \mathbb{Q}}(x, \kappa)$ and $P_{\nu, \lambda, R}(x, \kappa) \subset P_{\nu, \lambda, \mathbb{Q}}(x, \kappa)$, see Section 6.4. So for $\kappa \in [z_1, z_2]$ we get objects $L_{\nu, \lambda, \mathbb{Q}}(x, \kappa)_\mathcal{I} := \pi_{\mathcal{I}}(L_{\nu, \lambda, \mathbb{F}}(x, \kappa))$.

Also, similarly to Lemma 6.8, we have a unique subobject $M_R \subset P_{\nu, \lambda, R}(x, \kappa)$ that is filtered with $\Delta_{\nu, \lambda, R}(x', \kappa')$, where $\kappa' > \kappa$, while the quotient $P_{\nu, \lambda, R}(x, \kappa)/M_R$ is filtered with $\Delta_{\nu, \lambda, R}(x', \kappa')$ with $\kappa' \leq \kappa$. We set $P_{\nu, \lambda, \mathbb{Q}}(x, \kappa)_\mathcal{I} := \pi_{\mathcal{I}}(P_{\nu, \lambda, \mathbb{F}}(x, \kappa)/M_R)$.

Now for an $h$-block $\mathcal{O}_\nu(A_{\lambda, \mathbb{Q}})_{\beta}$ we define its reduction $\mathcal{O}_\nu(A_{\lambda, F})_{\beta}$. Namely, let $P$ be the direct sum of all indecomposable projectives in $\mathcal{O}_\nu(A_{\lambda, \mathbb{Q}})_{\beta}$ and let $P_R$ be the direct sum of their distinguished $R$-forms. By definition, $\mathcal{O}_\nu(A_{\lambda, F})_{\beta}$ is the category of right modules over $\mathbb{F} \otimes_R \text{End}_{A_{\lambda, R}}(P_R)$.

The following is the main result of this section, and of the entire paper.

**Theorem 9.3.** Pick an interval $\mathcal{I} \subset \text{Irr}(\tilde{O}(A_{\lambda', F}))$ of the form $\{ (x, \kappa) | z - pm \leq \kappa < z \}$, where $z$ is the minimal value of $\kappa$ in some equivalence class for the pre-order $\preceq$. Then the following are true:
(1) The category $\mathcal{O}(A_{\lambda}, \mathbb{F})_3$ is standardly stratified with respect to the pre-order $\preceq$. The objects $P_{\nu, \lambda, \mathbb{F}}(x, \kappa)_3$ are standard and the objects $L_{\nu, \lambda, \mathbb{F}}(x, \kappa)_3$ are proper standard for $(x, \kappa) \in \mathfrak{I}$.

(2) The component of the associated graded category $\text{gr} \mathcal{O}(A_{\lambda}, \mathbb{F})_3$ corresponding to any equivalence class with respect to $\preceq$ is $\mathcal{O}_\nu(A_{\lambda}, \mathbb{F})^\beta$ (as a highest weight category), where $\beta$ is the rational number corresponding to the equivalence class, see the discussion after Lemma 9.4.1 In particular, $\text{gr} \mathcal{O}(A_{\lambda}, \mathbb{F})_3 \cong \mathcal{O}_\nu(A_{\lambda}, \mathbb{F})^{\beta m}$.

9.1.4. Example of $T^*(G/B)$. The most interesting case here is when $\Theta$ is a point.

Consider $X = T^*(G/B)$ and take the fundamental $p$-alcove: $\langle \alpha^\vee_i, \lambda \rangle \geq 1$, $\langle \alpha^\vee_0, \lambda \rangle \geq 1-p$, where we write $\alpha^\vee_0$ for the minimal coroot and $i$ runs over $\{1, 2, \ldots, r\}$. Pick $i \in \{0, \ldots, r\}$ and consider the element $\bar{\lambda}_i$ defined by $\langle \alpha^\vee_i, \bar{\lambda}_i \rangle = 1$ for $j \neq i$ (in particular, $\bar{\lambda}_0 = \rho$). The corresponding point $\lambda'_i = \bar{\rho} \bar{\lambda}_i$ is given by $\lambda'_i = \rho$ if $i = 0$ and by $\langle \alpha^\vee_i, \lambda'_i \rangle = 1$ for $j \neq 0, i$, $\langle \alpha^\vee_0, \lambda'_i \rangle = 1-p$. We have $c_{\nu, \rho, \lambda}(w) = \langle \rho \lambda'_i, \nu \rangle$ is an affine function in $p$.

Two elements $w_1, w_2 \in X^T \cong W$ lie in the same $h$-block if and only if $(w_1 \bar{\lambda} - w_2 \bar{\lambda}, \rho^\vee) \in \mathbb{Z}$. So each of these equivalence classes splits into the union of right cosets for the integral Weyl group $W_{\lambda}(\mathfrak{l})$ (generated by the reflections $s_\alpha$, where $\alpha$ is such that $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$). The categories $\mathcal{O}_\nu(A_{\lambda})$ are the direct sums of the regular blocks of the BGG categories $\mathcal{O}$ for $W_{\lambda}$.

9.1.5. Example of $\text{Hilb}_n(A^2)$. Now consider the case of $X = \text{Hilb}_n(A^2)$ so that the algebra $A_{\lambda}$ is a spherical rational Cherednik algebra for $S_n$. As usual we set $c = \lambda - 1/2$.

Recall that the $p$-alcoves (for the parameter $c$) are the intervals of the form $[\frac{(p+1)a}{b} + \ell, \frac{\ell, (p+1)a' - \ell}{b}]$, where $a/b < a'/b'$ are two rational numbers with denominators between 2 and $n$ such that there are no rational numbers with such denominators between $a/b, a'/b'$.

Pick $\bar{\lambda} = \frac{a}{b} + \ell$. It is again clear that $c_{\nu, \rho, \lambda}(w)$ is an affine function in $p$. We have $\mu \sim_{\nu, \bar{\lambda}} \mu'$ if $\text{cont}(\mu) - \text{cont}(\mu')$ is divisible by $b$. The order within the equivalence class is as follows: $\mu < \mu'$ if $\text{cont}(\mu) > \text{cont}(\mu')$.

We note that, thanks to results of [R, L4], the category $\mathcal{O}_\nu(A_{\lambda})$ is equivalent to the category of modules over the $q$-Schur algebra $S_q(n)$ for the quantum $\mathfrak{gl}_n$ (a labelling preserving equivalence of highest weight categories). It follows that $\mathcal{O}_\nu(A_{\lambda}, \mathbb{F}) \cong S_q, \mathbb{F}(n)$-mod.

9.2. Proof of Theorem 9.3.

9.2.1. Three technical lemmas. In the proof we will use the following three technical lemmas. The first one is a straightforward consequence of Corollary 6.10.

Lemma 9.4. Let $M_R, N_R$ be modules of the form $L_{\nu, \lambda, R}(x, \kappa), P_{\nu, \lambda, R}(x', \kappa')$, where $x, x'$ are in the same $h$-block and $|\kappa - \kappa'| \leq m$, where $m$ has the same meaning as in Corollary 6.11. Then for all $i$, the $R$-modules $\text{Ext}^i_{A_{\lambda}, R}(M_R, N_R)$ are free and finitely generated over $R$. Then we automatically have

$$\mathbb{C} \otimes_R \text{Ext}^i_{A_{\lambda}, R}(M_R, N_R) \cong \text{Ext}^i_{A_{\lambda}, R}(M_C, N_C), \mathbb{F} \otimes_R \text{Ext}^i_{A_{\lambda}, R}(M_R, N_R) \cong \text{Ext}^i_{A_{\lambda}, \mathbb{F}, R}(M_{\mathbb{F}}, N_{\mathbb{F}}).$$

Lemma 9.5. Let $\beta$ be an equivalence class for $\sim_\nu$ and let $M_R$ be an object in $A_{\lambda, R}$-mod$^T$ filtered by $L_{\nu, \lambda, R}(x, \kappa)$, where $(x, \kappa) \in \beta$. Then for any finite interval $\mathfrak{I}$ containing $\beta$ and any $N_{\mathbb{F}} \in \mathcal{O}(A_{\lambda}, \mathbb{F})_{3, < \beta}$ we have

$$\text{Ext}^i_{\mathcal{O}(A_{\lambda}, \mathbb{F})_{3, < \beta}}(M_{\mathbb{F}}, N_{\mathbb{F}}) = 0, \quad \forall i \geq 0.$$
Proof. Of course, it is enough to assume that $M_R = L_{\nu,\lambda,R}(x,\kappa)$. By Lemma 6.8 $M_R$ admits a resolution by objects filtered by $\Delta_{\nu,\lambda,R}(x',\kappa')$ for $(x',\kappa') \in \beta$. So it is enough to prove that

$$\text{Ext}^i_{O(A_{\nu,\beta})}(\Delta_{\nu,\lambda,F}(x',\kappa'), N) = 0, \quad \forall i \geq 0.$$ 

These equalities hold because $\Delta_{\nu,\lambda,F}(x',\kappa')$ is the standard corresponding to $(x',\kappa')$ and $N$ is filtered by simples whose labels are less than the labels in $\beta$. \hfill \square

Corollary 9.6. In the notation of Lemma 9.5, we have $M_F = L\pi^1_\beta \circ \pi_\beta(M_F)$, where we write $\pi_\beta$ for the quotient functor $A_{\lambda,F} - \text{mod}_{\leq \beta} \twoheadrightarrow O(A_{\lambda,F})_\beta$ and $L\pi^1_\beta$ for the left adjoint functor.

Lemma 9.7. Pick a label $(x,\kappa) \in \text{Irr}(O(A_{\lambda,F})^0)$ and let $\beta$ be the $\sim_\nu$ equivalence class of $(x,\kappa)$. Then $P_{\nu,\lambda,F}(x,\kappa)_\beta$ is the projective in $O(A_{\lambda,F})_\beta$ labelled by $(x,\kappa)$, while $L_{\nu,\lambda,F}(x,\kappa)_\beta$ is the simple labelled by $(x,\kappa)$.

Proof. Pick $(x',\kappa') \in \beta$. By Corollary 6.10, we have

$$\text{Hom}_{A_{\lambda,R,T}}(\Delta_{\nu,\lambda,R}(x',\kappa'), L_{\nu,\lambda,R}(x,\kappa)) = \mathbb{R}^{\oplus \delta_{x',x} \delta_{\kappa,\kappa'}}.$$ 

Applying Lemma 9.4 we see that

$$\dim \text{Hom}_{A_{\lambda',R,T}}(\Delta_{\nu,\lambda,F}(x',\kappa'), L_{\nu,\lambda,F}(x,\kappa))) = \delta_{x',x} \delta_{\kappa,\kappa'}.$$ 

By Corollary 9.6 $\Delta_{\nu,\lambda,F}(x',\kappa') = \pi^1_\beta(\Delta_{\nu,\lambda,F}(x',\kappa'))_{\beta}$. From here we deduce that

$$\dim \text{Hom}_{O(A_{\lambda',\beta})}(\Delta_{\nu,\lambda,F}(x',\kappa')_{\beta}, L_{\nu,\lambda,F}(x,\kappa)) = \delta_{x',x} \delta_{\kappa,\kappa'}.$$ 

Recall that the objects $\Delta_{\nu,\lambda,F}(x',\kappa')_{\beta}$ are standard. By Lemma 3.11 $L_{\nu,\lambda,F}(x,\kappa)_\beta \hookrightarrow \nabla_{\nu,\lambda}(x,\kappa)_\beta$. It remains to show that the head of $L_{\nu,\lambda,F}(x,\kappa)_\beta$ coincides with $\pi_\beta(L_{\nu,\lambda,F}(x,\kappa))$.

By Corollary 9.6 $\pi^1_\beta(L_{\nu,\lambda,F}(x,\kappa)) = L_{\nu,\lambda,F}(x,\kappa)$. So using Lemma 9.4 again and arguing as in the first paragraph of the proof of the present lemma, we conclude

$$\dim \text{Hom}_{O(A_{\lambda',\beta})}(L_{\nu,\lambda,F}(x,\kappa)_\beta, L_{\nu,\lambda,F}(x',\kappa')_\beta)) = \delta_{x',x} \delta_{\kappa,\kappa'}.$$ 

Since $L_{\nu,\lambda,F}(x',\kappa')_\beta \hookrightarrow \nabla_{\nu,\lambda}(x,\kappa)_\beta$ we conclude that the head of $L_{\nu,\lambda,F}(x,\kappa)_\beta$ is $\pi_\beta(L_{\nu,\lambda,F}(x,\kappa))$. This finishes the proof that $L_{\nu,\lambda,F}(x,\kappa)_\beta = \pi_\beta(L_{\nu,\lambda,F}(x,\kappa))$.

Let us show that $P_{\nu,\lambda,F}(x,\kappa)_\beta$ is the projective cover of $\pi_\beta(L_{\nu,\lambda,F}(x,\kappa))$. From Lemma 9.4 it follows that

$$\dim \text{Ext}^i_{A_{\lambda',\beta}}(P_{\nu,\lambda,F}(x,\kappa), L_{\nu,\lambda,F}(x',\kappa')) = \delta_{i,0} \delta_{x',x} \delta_{\kappa,\kappa'}.$$ 

By Corollary 9.6 $L\pi^1_\beta P_{\nu,\lambda,F}(x,\kappa)_\beta = P_{\nu,\lambda,F}(x,\kappa)$. We conclude that, for $i = 0,1$, we have

$$\dim \text{Ext}^i_{O(A_{\lambda',\beta})}(P_{\nu,\lambda,F}(x,\kappa)_\beta, L_{\nu,\lambda,F}(x',\kappa')_\beta) = \delta_{i,0} \delta_{x',x} \delta_{\kappa,\kappa'}.$$ 

We already know that $L_{\nu,\lambda,F}(x',\kappa')_\beta$ is the simple object corresponding to $(x',\kappa')$. Hence $P_{\nu,\lambda,F}(x,\kappa)_\beta$ is the projective object corresponding to $(x,\kappa)$. \hfill \square
9.2.2. Completion of the proof.

Proof of Theorem 9.3. Let $\beta$ be as in Lemma 9.7 and let $I$ be an interval containing $\beta$. By Lemma 9.7, for $(x, \kappa) \in I$, the object $L_{\nu, \lambda, F}(x, \kappa)_\beta$ (that coincides with $\pi_\beta(L_{\nu, \lambda, F}(x, \kappa)_\beta)$ by the definitions) is simple. By Corollary 9.5, $L_{\pi, \lambda, F}(x, \kappa)_\beta = L_{\nu, \lambda, F}(x, \kappa)_\beta$. It follows, in particular, that $\pi_\beta$ is exact.

To finish the proof of the claim that $\preceq$ defines a standardly stratified structure, it remains to show that $\pi_\beta : \mathcal{O}(A_{\lambda, F})_\beta \mathcal{O}(A_{\lambda, F})_{I, \preceq}$ is exact. For this we will use the duality functor $\bullet^*$ from Section 8.1.5. Recall that $L_{\nu, \lambda, F}(x, \kappa) = L_{-\nu, -\lambda, F}(x, -\kappa)^*$. It follows that $\bullet^*$ induces an equivalence $\pi_\beta^* : \mathcal{O}(A_{\lambda, F})_\beta \sim \mathcal{O}(A_{-\lambda, F})_{\mathcal{O}^{opr}}$ for any interval $I$. Clearly, the functors $\bullet^{\mathcal{O}^{opr}}$ intertwine $(\pi_\beta^*)$ with $\pi_\beta$. Since we know that $(\pi_\beta^*)$ is exact, we see that $\pi_\beta$ is exact.

It follows that the pre-order $\preceq$ indeed defines a standardly stratified structure. Corollary 9.6 and Lemma 9.7 imply that $L_{\nu, \lambda, F}(x, \kappa)_I, P_{\nu, \lambda, F}(x, \kappa)_I$ are the proper standards and standard objects. This finishes the proof of (1).

Part (2) easily follows from Lemma 9.4 (applied to $M_R = N_R := P_R$) and the projective part of Lemma 9.7.

9.2.3. The case of $\dim T > 1$. The case when $\dim T > 1$ can be handled similarly to the situation of highest weight structures. In particular, we can use a more general notion of a standardly stratified category, see Remark 3.4.

9.3. Wall-crossing for categories $\mathcal{O}$. Now let $A_1, A_2$ be two real alcoves that are opposite with respect to a common face $\Theta$. Let $\lambda_1, \lambda_2$ be elements in $\Lambda$ compatible with $(A_1, \Theta), (A_2, \Theta)$. Let $\chi$ be the corresponding element in $P_1$ so that $\lambda_2 = \lambda_1 + \chi$. Set $\lambda'_i := p^i \lambda_i, i = 1, 2$. Then we have the wall-crossing functor $A_{\lambda'_1, \chi, F} \otimes A_{\lambda'_2, F}^\mathcal{T} \bullet : D^b(A_{\lambda'_1, F}^\mathcal{O}) \sim D^b(A_{\lambda'_2, F}^\mathcal{O})^\mathcal{T}$.

Our goal in this section is to show, that, in a suitable sense, the wall-crossing functor is a partial Ringel duality functor between the categories $\mathcal{O}(A_{\lambda'_1, F})$ and $\mathcal{O}(A_{\lambda'_2, F})$.

9.3.1. Preparation. Let us start with the following elementary lemma.

Lemma 9.8. Let $(x, \kappa)$ and $(x', \kappa')$ lie in the same equivariant block for $\lambda \in P_F$. Then, for any $\chi \in pZ$, $(x, \kappa + \omega_\chi(x)), (x', \kappa + \omega_\chi(x'))$ lie in the same equivariant block.

Proof. Recall, [L6, Section 6.1], $c_\lambda(x) - c_\lambda(x')$ is an affine function in $\lambda$ and $(c_{\lambda + \chi}(x) - c_{\lambda + \chi}(x')) = (c_{\nu, \lambda}(x) - c_{\nu, \lambda}(x')) = \omega_\chi(x) - \omega_\chi(x')$. Our claim easily follows from here.

So by choosing a suitable equivariant structure on $\mathcal{O}(\chi)$, we can assume that $L_{\nu, \lambda, F}(x, \kappa + \omega_\chi(x)) \in \mathcal{O}(A_{\lambda + \chi, F})^0$ if and only if $L_{\nu, \lambda, F}(x, \kappa) \in \mathcal{O}(A_{\lambda, F})^0$.

Arguing as in the proof of Lemma 9.8 and using the condition that $\chi$ is independent of $p$, we get the following.

Lemma 9.9. We have $(x, \kappa) \sim_{\nu, \lambda'_1} (x', \kappa')$ if and only if $(x, \kappa + \omega_\chi(x)) \sim_{\nu, \lambda'_2} (x', \kappa' + \omega_\chi(x'))$.

So we have a bijection between equivalence classes of simples in $\mathcal{O}(A_{\lambda'_1, F})^0$ and in $\mathcal{O}(A_{\lambda'_2, F})^0$. This bijection is compatible with $\preceq$. So to an interval (with respect to $\preceq$) $I \subset \text{Irr}(\mathcal{O}(A_{\lambda'_1, F})^0)$ we can assign an interval $I^x \subset \text{Irr}(\mathcal{O}(A_{\lambda'_2, F})^0)$. 

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9.3.2. Main result. The following is the main result of the section.

**Theorem 9.10.** The functor

\[ \mathcal{WC}_{\lambda' \leftarrow \lambda} : A_{\lambda \times \mathbb{F}} \otimes_{A_{\lambda \times \mathbb{F}}}^L \cdot : D^b(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda' \times \mathbb{F}^T} \text{-mod}) \]

induces a partial Ringel duality functor \( D^b(\mathcal{O}(\mathcal{A}_{\lambda \times \mathbb{F}^T})), D^b(\mathcal{O}(\mathcal{A}_{\lambda' \times \mathbb{F}^T})), \) (with respect to the standardly stratified structures defined by \( \lambda \)).

**Proof.** Recall that for \( z \in \mathbb{Z} \) we have defined the category \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod} \). Inside we can consider the subcategory \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}^{T_{\leq z}} \subset \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}^{T} \) defined similarly to \( \mathcal{O}(\mathcal{A}_{\lambda \times \mathbb{F}^T})^0 \subset \mathcal{O}(\mathcal{A}_{\lambda \times \mathbb{F}^T})^0 \). This subcategory is a direct summand. We have sufficiently many projectives in \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}^{T_{\leq z}} \) and they have no higher self-extensions in \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}^{T} \), this was established in the proof of Theorem 8.7. So the natural functor \( D^b(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}^{T_{\leq z}}) \hookrightarrow D^b(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}^{T}) \) is a full embedding. From here it follows that

\[ \mathcal{WC}_{\lambda' \leftarrow \lambda} \text{ indeed induces a functor } D^b(\mathcal{O}(\mathcal{A}_{\lambda' \times \mathbb{F}^T})), D^b(\mathcal{O}(\mathcal{A}_{\lambda \times \mathbb{F}^T})), \]

It follows from Lemma 6.1 and the R-flatness of \( A_{\lambda \times \mathbb{F}, \Delta^\nu, \lambda, \mathbb{F}}(x, \kappa) \) that

\[ \mathcal{WC}_{\lambda' \leftarrow \lambda} (\Delta^\nu, \lambda, \mathbb{F}) (x, \kappa) = \nabla^\nu, \lambda, \mathbb{F}(x, \kappa + \text{wt}_\lambda(x)). \]

From here it follows that \( \mathcal{WC}_{\lambda' \leftarrow \lambda} \) is compatible with the filtrations on the categories \( D^b(\mathcal{O}(\mathcal{A}_{\lambda \times \mathbb{F}^T})), D^b(\mathcal{O}(\mathcal{A}_{\lambda' \times \mathbb{F}^T})), \) and in particular, the functor induces an equivalence of the associated graded categories. Assume \( \mathcal{I} \) is an equivalence class. For \( (x, \kappa) \in \mathcal{I} \), the costandard object in \( \mathcal{O}(\mathcal{A}_{\lambda' \times \mathbb{F}^T}) \) is \( \pi_\lambda(\nabla^\nu, \lambda, \mathbb{F}(x, \kappa + \text{wt}_\lambda(x))) \). It follows that the equivalence of the associated graded categories induced by \( \mathcal{WC}_{\lambda' \leftarrow \lambda} \) is a Ringel duality functor. So the functor \( D^b(\mathcal{O}(\mathcal{A}_{\lambda' \times \mathbb{F}^T})), D^b(\mathcal{O}(\mathcal{A}_{\lambda \times \mathbb{F}^T})), \) induced by \( \mathcal{WC}_{\lambda' \leftarrow \lambda} \) is a partial Ringel duality. \( \square \)

10. Applications

10.1. Wall-crossing bijections. Let \( \Theta, \lambda', \chi \) be as above. We consider the category \( \mathcal{A}_{\lambda' \times \mathbb{F}^T} \text{-mod}_0 \) of all finite dimensional \( \mathcal{A}_{\lambda' \times \mathbb{F}^T} \text{-modules} \) with generalized zero \( p \)-character. We also consider the category \( D^b_0(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}) \) of all objects in \( D^b_0(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}) \) homology in \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}_0 \). Note that \( \mathcal{WC}_{\lambda' \leftarrow \lambda} = \mathcal{A}_{\lambda' \times \mathbb{F}} \otimes_{\mathcal{A}_{\lambda' \times \mathbb{F}}}^L \cdot \) restricts to a perverse equivalence

\[ D^b_0(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}) \xrightarrow{\sim} D^b_0(\mathcal{A}_{\lambda' \times \mathbb{F}^T} \text{-mod}). \]

Let us classify the irreducible objects in \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}_0 \).

**Lemma 10.1.** Fix a generic one-parameter subgroup \( \nu \). Then we get a bijection between \( \text{Irr}(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}_0) \) and \( X^T \).

**Proof.** Every irreducible from \( \mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}_0 \) has a \( T_\mathbb{F} \)-equivariant structure, unique up to a twist with a character. It is still irreducible as a \( T_\mathbb{F} \)-equivariant module and hence is \( L(x, \kappa) \) for some character \( \kappa \). This implies the claim of the lemma. \( \square \)

As any perverse equivalence, \( \mathcal{WC}_{\lambda' \leftarrow \lambda} \) gives rise to a bijection \( \text{Irr}(\mathcal{A}_{\lambda \times \mathbb{F}^T} \text{-mod}_0) \xrightarrow{\sim} \text{Irr}(\mathcal{A}_{\lambda' \times \mathbb{F}^T} \text{-mod}_0) \), i.e., a self-bijection of \( X^T \). We want to compare this bijection with a similarly defined bijection for categories \( \mathcal{O} \) in characteristic 0.
Proposition 10.2. For $p \gg 0$, the self-bijection of $X^T$ induced by $\mathcal{MC}^{\lambda+\chi_\star-\lambda}$ coincides with the bijection $\text{Irr}(\mathcal{O}_\nu(A_{\lambda})) \xrightarrow{\sim} \text{Irr}(\mathcal{O}_\nu(A_{\lambda+\chi}))$ coming from the wall-crossing functor $\mathcal{MC}^{\lambda+\chi_\star-\lambda}$.

Proof. Let $\varphi$ denote the bijection coming from $\mathcal{MC}^{\lambda+\chi_\star-\lambda}$. What we need to show is that the wall-crossing bijection $\tilde{\varphi}$ for the categories $\tilde{\mathcal{O}}(A_{\lambda,\mathbb{F}})$ has the form $(x, \kappa) \mapsto (\varphi x, \kappa')$. Note that $\mathcal{MC}^{\lambda+\chi_\star-\lambda}$ respects the filtrations on the categories $\tilde{\mathcal{O}}(A_{\lambda,\mathbb{F}})$, $\tilde{\mathcal{O}}(A_{\lambda+\chi,\mathbb{F}})$ coming from the face $\Theta$. So the induced functor between the associated graded categories is perverse and the induced bijection is the same as $\tilde{\varphi}$. On the other hand, the associated graded categories are $\mathcal{O}_\nu^{\mathbb{F}}(A_{\lambda,\mathbb{F}})$ and $\mathcal{O}_\nu^{\mathbb{F}}(A_{\lambda+\chi,\mathbb{F}})$ and the functor induced by $\mathcal{MC}^{\lambda+\chi_\star-\lambda}$ is the Ringel duality. So forgetting the $T$-character component, we see that the bijection $X^T = \text{Irr}(A_{\lambda,\mathbb{F}}-\text{mod}) \xrightarrow{\sim} \text{Irr}(A_{\lambda+\chi,\mathbb{F}}-\text{mod}) = X^T$ coincides with the bijection $\mathcal{O}_\nu(A_{\lambda,\mathbb{F}})$ and $\mathcal{O}_\nu(A_{\lambda+\chi,\mathbb{F}})$ induced by the Ringel duality. But $\mathcal{MC}^{\lambda+\chi_\star-\lambda}$ is the Ringel duality between $\mathcal{O}_\nu(A_{\lambda})$ and $\mathcal{O}_\nu(A_{\lambda+\chi})$ and since $p$ is very large, the bijections induced by the Ringel duality over $\mathbb{F}$ and over $\mathbb{C}$ coincide. This completes the proof. 

Wall-crossing bijections were studied (and sometimes computed combinatorially) in a number of cases. Paper [L7] studied the case of $X = T^*(G/B)$. There we have seen that the wall-crossing bijections through the faces containing 0 define an action of the so-called cactus group $\text{Cact}_W$ on $W$. Using Proposition 10.2 one can show that this action extends to an action of the affine cactus group. We note that combinatorial recipes to compute the action are not known in general.

The case of rational Cherednik algebras of type $A$ (and of more general cyclotomic rational Cherednik algebras) was considered in [L8]. It was shown that the wall-crossing bijections for rational Cherednik algebras of type $A$ are extended Mullineux involutions, see [L8, Corollary 5.7] for a precise statement. Therefore the wall-crossing bijections for rational Cherednik algebras of type $A$ were considered in [L8]. It was shown that the wall-crossing bijections through the faces containing 0 define an action of the so-called cactus group $\text{Cact}_W$ on $W$. Using Proposition 10.2 one can show that this action extends to an action of the affine cactus group. We note that combinatorial recipes to compute the action are not known in general.

10.2. Gradings. In this section, following an idea of Bezrukavnikov, we produce graded lifts of the category $\tilde{\mathcal{O}}(A_{\lambda,\mathbb{F}})$ that come from the contacting torus action on $X$. Then we show that our grading lift induces a grading lift of $\mathcal{O}_\nu(A_{\lambda})$. Finally, we compare Koszulity properties of the grading lifts on $\tilde{\mathcal{O}}(A_{\lambda,\mathbb{F}})$ and on $\mathcal{O}_\nu(A_{\lambda})$.

10.2.1. Grading on $\tilde{\mathcal{O}}(A_{\lambda,\mathbb{F}})$. Let $Y^{\star}_{\mathbb{F}}$ denote the formal neighborhood of 0 in $Y^{\star}_{\mathbb{F}}$ and let $X^{\star}_{\mathbb{F}}$ be its preimage in $X^{\star}_{\mathbb{F}}$.

We assume that the microlocal quantization $A^{\mathbb{F}}_{\lambda,\mathbb{F}}$ of $X_{\mathbb{F}}$ is obtained (by completing with respect to the filtration) from a Frobenius constant quantization (in the sense of [BK2]). By abusing the notation, we denote the corresponding Azumaya algebra on $X^{\star}_{\mathbb{F}}$ also by $A^{\mathbb{F}}_{\lambda,\mathbb{F}}$. This holds in the examples we consider (and in more general examples of Slodowy varieties, [BMR2], and of Nakajima quiver varieties, [BFG]).

Consider the restriction $A^{\mathbb{F}}_{\lambda,\mathbb{F}}$ of $A^{\mathbb{F}}_{\lambda,\mathbb{F}}$ to $X^{\star}_{\mathbb{F}}$. We assume that it splits, let $\mathcal{V}^{\wedge_0}_{\mathbb{F}}$ denote the splitting bundle. Recall that $\mathcal{V}^{\wedge_0}_{\mathbb{F}}$ is defined up to a twist with a line bundle. The splitting bundle again exists in all examples we consider, see [BMR2] for the case of Slodowy varieties and [EL] for the case of Nakajima quiver varieties.

The splitting bundle $\mathcal{V}^{\wedge_0}_{\mathbb{F}}$ has no higher self-extensions. It follows that it admits a $T_{\mathbb{F}} \times \mathbb{F}^\times$-equivariant structure, where we write $\mathbb{F}^\times$ for the contracting torus. We can choose a $T_{\mathbb{F}}$-equivariant structure on $\mathcal{V}^{\wedge_0}_{\mathbb{F}}$ so that the isomorphism $\text{End}(\mathcal{V}^{\wedge_0}_{\mathbb{F}}) \cong A^{\mathbb{F}}_{\lambda,\mathbb{F}}$ is $T_{\mathbb{F}}$-equivariant.
Since the action of $\mathbb{F}^\times$ is contracting we can uniquely extend $\mathcal{V}_F^{\lambda_0}$ to a $T_F \times \mathbb{F}^\times$-equivariant vector bundle $\mathcal{V}_F$. Set $\tilde{A}_F := \text{End}(\mathcal{V}_F)$. We can consider the categories $\tilde{A}_F$-$\text{mod}_{0}^{T}$ of all $T_F$-equivariant finite dimensional $\tilde{A}_F$-modules (automatically supported at $0 \in Y_F^{(1)}$) and $\tilde{A}_F$-$\text{mod}_{0}^{T \times \mathbb{F}^\times}$ of all $T_F \times \mathbb{F}^\times$-equivariant finite dimensional $\tilde{A}_F$-modules. The construction of $\tilde{A}_F$ yields a category equivalence $\tilde{O}(A_{\lambda',F}) \cong \tilde{A}_F$-$\text{mod}_{0}^{T}$. So $\tilde{A}_F$-$\text{mod}_{0}^{T \times \mathbb{F}^\times}$ is a graded lift of $\tilde{O}(A_{\lambda',F})$. Note that this graded lift is independent of the choice of $\lambda'$ in its $p$-alcove.

10.2.2. From graded lift $\tilde{O}(A_{\lambda',F})$ to graded lift of $O_{\nu}(A_{\lambda})$. Now we are going to produce a graded lift of $O_{\nu}(A_{\lambda})$ from a graded lift of $\tilde{O}(A_{\lambda',F})$. Note that a graded lift of $\tilde{O}(A_{\lambda',F})$ induces that of any quotient $\tilde{O}(A_{\lambda',F})_{<z_2}/\tilde{O}(A_{\lambda',F})_{<z_1}$. In particular, $O_{\nu}(A_{\lambda',F})$ is the direct sum of such subquotients so we get graded lifts of $O_{\nu}(A_{\lambda,F})$.

Now we are in the following situation. Let $R$ be a finite localization of $\mathbb{Z}$ and let $B_R$ be an $R$-algebra that is a free finite rank $R$-module (in our case $B_R = \text{End}(R_p)^{opp}$, where $R_p$ is an $R$-form of a pro-generator of $O_{\nu}(A_1)$). We need to check that for $p \gg 0$ there is a natural bijection between graded lifts of $B_R$-$\text{mod}$ and of $B_{\mathbb{C}}$-$\text{mod}$.

For this we need to note that, for a finite dimensional algebra $B$ over an algebraically closed field $\mathbb{K}$, graded lifts of $B$-$\text{mod}$ are parameterized by conjugacy classes of one-parameter subgroups of $H := \text{Aut}(B)/B^\times$.

In our case $H_{\mathbb{K}}$ is the base change of an algebraic group scheme $H_R$ from $R$ to $\mathbb{K}$ assuming that char $\mathbb{K}$ is 0 or is large enough. After taking a finite extension of $R$, we can find a subgroup subscheme $T_R \subset H_R$ that becomes a maximal torus in $H_{\mathbb{K}}$ after a base change to $\mathbb{K}$. The conjugacy classes of one-parameter subgroups in $H_{\mathbb{K}}$ are the orbits of the action of $N_{H_{\mathbb{K}}}(T_{\mathbb{K}})$ on the lattice of one-parameter subgroups in $T_{\mathbb{K}}$. This is clearly independent of $\mathbb{K}$.

This establishes a required bijection between the graded lifts.

10.2.3. Koszulity. Let us now show that if $\tilde{A}_F$-$\text{mod}_{0}^{T \times \mathbb{F}^\times}$ is Koszul for infinitely many $p$, then the category $O_{\nu}(A_{\lambda})$ is Koszul as well.

Recall that being Koszul for the category $\tilde{A}_F$-$\text{mod}_{0}^{T \times \mathbb{F}^\times}$ means that for every simple $L \in \tilde{A}_F$-$\text{mod}_{0}^{T}$ one can find a lift $\tilde{L} \in \tilde{A}_F$-$\text{mod}_{0}^{T \times \mathbb{F}^\times}$ such that $\text{Ext}^i(\tilde{L}, \tilde{L}')$ is concentrated in degree $i$ for all simples $L, L'$ (here the Ext's are taken in $\tilde{A}_F$-$\text{mod}_{0}^{T}$). Let $\tilde{O}(A_{\lambda',F})_{<z}^{F^\times}$ denote the corresponding graded lift of $\tilde{O}(A_{\lambda',F})$.

**Lemma 10.3.** For every interval $\mathcal{I} \subset \mathbb{Z}$ (with respect to $\leq_{\nu}$) the subquotient category the induced graded lift $\tilde{O}(A_{\lambda',F})_{\mathcal{I}}$ is Koszul.

**Proof.** Recall that the natural functor $D^b(\tilde{O}(A_{\lambda',F})_{\leq z}) \hookrightarrow D^b(\tilde{O}(A_{\lambda',F}))$ is a full embedding. So the induced graded lift of $\tilde{O}(A_{\lambda',F})_{\leq z}$ is Koszul. Note that $\tilde{O}(A_{\lambda,F})_{\leq z}$ has enough projectives. Let $P(\tilde{L})$ denote the projective cover of $\tilde{L}$ in $\tilde{O}(A_{\lambda,F})_{\leq z}$. Since $\tilde{O}(A_{\lambda,F})_{\leq z}$ is Koszul, we can find a projective resolution of any $\tilde{L}$ of the form $\ldots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$ such that $P_1$ is the direct sum of $P(\tilde{L}')(i)$’s with some multiplicities, where $(\bullet)$ means a grading shift. Then $\pi_3(P_\bullet)$ is the projective resolution of $\pi_3(\tilde{L})$. Hence $\tilde{O}(A_{\lambda,F})_{\mathcal{I}}$ is Koszul. \qed

In particular, we see that the categories $O_{\nu}(A_{\lambda,F})$ acquire Koszul graded lifts for infinitely many $p$. Let us deduce from here that $O_{\nu}(A_{\lambda})$ does. This follows from the next lemma.
Lemma 10.4. Let $R$ be a ring that is a finite algebraic extension of $\mathbb{Z}$. Let $B_R$ be an $R$-algebra that is a free finite rank $R$-module that has finite homological dimension. Suppose that the fibers of $B_R$ at infinitely many maximal ideals of $R$ carry Koszul gradings. Then $B_{\text{Frac}(R)}$ carries a Koszul grading.

Proof. By replacing $R$ with its finite localization, we can achieve that there are objects $L_i^i, i = 1, \ldots, k$, such that the fibers of these objects at any point of $R$ are the simples over the corresponding algebra. Moreover, we can assume that $\text{Ext}^l_{B_R}(L_i^i, L_j^j)$ are projective $R$-modules for all $i, j, l$ (here we use that $B_R$ has finite homological dimension).

As we have seen above, after replacing $R$ with its finite algebraic extension, we can assume that there is a bijection between gradings on all the fibers of $B_R$, moreover, we can assume that every grading of every fiber comes from a grading on $B_R$. So pick a grading that gives rise to a Koszul grading in some fiber. We claim that it gives a Koszul grading on any fiber. This follows from the observation that all $R$-modules $\text{Ext}^l_{B_R}(L_i^i, L_j^j)$ are graded and all graded components are projective $R$-modules. □

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