LOCAL-GLOBAL PRINCIPLE AND INTEGRAL TATE CONJECTURE FOR CERTAIN VARIETIES

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Abstract. We give a geometric criterion to check the validity of integral Tate conjecture for one cycles on separably rationally connected fibrations over a curve, and to check that the Brauer-Manin obstruction is the only obstruction to local-global principle for zero cycles on a separably rationally connected variety defined over a global function field.

We prove that the Brauer-Manin obstruction is the only obstruction to local-global principle for zero cycles on all geometrically rational surfaces defined over a global function field, and to Hasse principle for rational points on del Pezzo surfaces of degree four defined over a global function field of odd characteristic.

Along the way, we also prove some results about the space of one cycles on a separably rationally connected fibration over a curve, which leads to the equality of the coniveau filtration and the strong coniveau filtration (introduced by Benoist-Ottem and Voisin) on degree 3 homology of such varieties.

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1. Introduction

1.1. Local-global principle for zero cycles. Given a smooth projective variety defined over a global field, a natural and important problem is to find criteria for the existence of rational points and a description of the set of all rational points. Hasse principle and weak approximation problem, or local-global principle, gives a characterization of this set in terms of the adelic points. There are various obstructions...
for local-global principle to hold, notably the so called Brauer-Manin obstruction. A conjecture due to Colliot-Thélène states that for rationally connected varieties defined over a global field, this is the only obstruction. The study of zero cycles, as natural generalizations of rational points, has also drawn lots of attentions in recent years. Motivated by the case of rational points, Colliot-Thélène has formulated the following conjectures on the local-global principle for zero cycles.

**Conjecture 1.1.** [CT99, Conjecture 2.2] Let \( X \) be a smooth projective variety defined over the function field \( \mathbb{F}_q(B) \) of a smooth curve \( B \) defined over a finite field \( \mathbb{F}_q \). For every place \( \nu \) of \( \mathbb{F}_q(B) \), let \( z_\nu \in CH_0(X_\nu) \). Suppose that for all element \( A \in Br(X)\{\ell\} \), we have \( \sum_\nu Inv(A(z_\nu)) = 0 \). Then for all \( n > 0 \), there is a cycle \( z_n \in CH_0(X) \) such that for all \( \nu \) we have that 

\[
\text{cl}(z_n) = \text{cl}(z_\nu) \in H^{2d}_{\text{ét}}(X_\nu, \mu_{\ell^n}).
\]

Here \( Inv(A(z_\nu)) \) means the value of \((A, z_\nu)\) under the pairing \( Br(X_\nu)\{\ell\} \times CH_0(X_\nu) \rightarrow \mathbb{Q}/\mathbb{Z} \).

A particular case of the above conjecture is the following.

**Conjecture 1.2.** Let \( X \) be a smooth projective variety defined over the function field \( \mathbb{F}_q(B) \) of a smooth curve \( B \) defined over a finite field \( \mathbb{F}_q \). Suppose that for every place \( \nu \) of \( \mathbb{F}_q(B) \), there is a zero cycle \( z_\nu \in CH_0(X_\nu) \) of degree prime to \( \ell \). Suppose that for all element \( A \in Br(X)\{\ell\} \), we have \( \sum_\nu Inv(A(z_\nu)) = 0 \). Then there is a cycle \( z \in CH_0(X) \) of degree prime to \( \ell \).

In this paper, for an abelian group \( A \), we use \( A \hat{\otimes} \mathbb{Z}_\ell \) to denote the inverse limit \( \lim_{\leftarrow} A/\ell^n A \). The following stronger form of the above conjecture is also well-known.

**Conjecture 1.3.** Let \( X \) be a smooth projective variety defined over a global field \( K \). Let \( l \) be a prime number invertible in \( K \). There is an exact sequence:

\[
CH_0(X) \hat{\otimes} \mathbb{Z}_\ell \rightarrow \Pi_{\nu \in \Omega(K)} CH_0(X_\nu) \hat{\otimes} \mathbb{Z}_\ell \rightarrow Hom(Br(X)\{\ell\}, \mathbb{Q}/\mathbb{Z}).
\]

Conjectures 1.1 and 1.2 are consequences of this via considering the commutative diagram of various cycle class maps. On the other hand, if the cycle class map \( CH_0(X_\nu) \hat{\otimes} \mathbb{Z}_\ell \rightarrow H^{2d}_{\text{ét}}(X_\nu, \mu_{\ell^n}(d)) \) is injective, Conjecture 1.1 and this stronger conjecture 1.3 are equivalent. In general the injectivity fails. But we will see that in many (and conjecturally all) cases of interest to us, the injectivity holds.

One of the main theorems of this article is the following.

**Theorem 1.4.** Let \( X \) be a smooth projective geometrically rational surface defined over the function field \( \mathbb{F}_q(B) \) of a smooth projective curve \( B \). Then Conjectures 1.3, and hence 1.1 and 1.2 hold true for \( X \).

By a happy coincidence, we deduce a corollary for rational points.

**Theorem 1.5.** Let \( X \) be a del Pezzo surface of degree 4 defined over a global function field of odd characteristic. Then Brauer-Manin obstruction is the only obstruction for Hasse principle for rational points on \( X \) to hold.
Proof. If there is a rational point everywhere locally and satisfies the Brauer-Manin constraint, there is a zero cycle of degree 1 over the function field by Theorem 1.4. A del Pezzo surface of degree 4 is a complete intersection of 2 quadrics in $\mathbb{P}^4$. Such a complete intersection has a rational point if and only if it has an odd degree 0-cycle $\text{Br}_1$. Hence we have the result. 

Remark 1.6. One is also interested to study weak approximation for rational points on a del Pezzo surface of degree 4. For a del Pezzo surface of degree 4 over a number field, assuming that there is a rational point, Salberger and Skorobogatov [SS91] prove that the Brauer-Manin obstruction is the only obstruction to weak approximation. As the author has been informed by Colliot-Thélène, essentially the same argument also proves that over a global function field of odd characteristic, Brauer-Manin obstruction is the only obstruction to weak approximation once there is a rational point. In characteristic 2, some partial results are contained in the joint work of the author with Letao Zhang [TZ18].

We finish this section with some previously known results. There is a vast literature on the local-global principles for zero-cycles/rational points on geometrically rational surfaces. Let us only mention a few relevant results and refer the readers to survey articles such as [Wit18] etc. for a more comprehensive list.

Colliot-Thélène proved Conjecture 1.3 holds for ruled surfaces defined over number fields [CT00]. The global function field version for ruled surfaces is proved by Parimala-Suresh [PS16], whose proof depends on the computation of degree 3 unramified cohomology and also establishes the integral Tate conjecture for conic bundle over surfaces defined over finite fields. An interesting example of cubic surfaces of the form $(f + tg = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1$ is studied by Colliot-Thélène-Swinnerton-Dyer [CTSD12]. In addition to proving that Hasse-principle for zero cycles holds for cubic surfaces of this form, they also prove that the existence of rational points is equivalent to the existence of a zero cycle of degree 1 for such surfaces.

The study of complete intersection of two quadrics has also drawn lots of attention. It starts with the work of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [CTSSD87]. Heath-Brown proved that Hasse principle for rational points holds for smooth complete intersections of two quadrics in $\mathbb{P}^7$ over number fields [HB18], see also a different proof by Colliot-Thélène [CT22]. Under the assumption on finiteness of Tate-Shafarevich groups of elliptic curves and the validity of Schinzel’s hypothesis, Wittenberg proved Hasse principle holds for such complete intersections in $\mathbb{P}^5$ and some case in $\mathbb{P}^4$ over number fields [Wit07]. The author has shown in a previous paper [Tia17] that Hasse principle for rational points holds for smooth complete intersections of two quadrics in $\mathbb{P}^n, n \geq 5$ defined over a global function field of odd characteristic.

1.2. Integral Tate conjecture. Our approach to Theorem 1.4 is based on the close relation between an integral version of Tate conjecture and Colliot-Thélène’s conjectures, first studied by Saito [Sai89] and Colliot-Thélène [CT99].

Let $X$ be a smooth projective geometrically irreducible variety of dimension $d$ defined over a finite field $\mathbb{F}$. We have the cycle class maps:

\[ CH_1(X) \otimes \mathbb{Z}_d \to H^{2d-2}(X, \mathbb{Z}_d(d-1)), \]

\[ CH_1(X) \otimes \mathbb{Z}_d \to H^{2d-2}(X, \mathbb{Z}_d(d-1)) \to H^{2d-2}(\overline{X}, \mathbb{Z}_d(d-1))^G, \]

where $G$ is the Galois group.
We also have the corresponding cycle class maps after tensoring with $\mathbb{Q}_\ell$. Tate conjecture predicts that the cycle class map on codimension $r$ cycles

$$\text{CH}^r(X) \otimes \mathbb{Q}_\ell \to H^{2r}_{\text{et}}(X, \mathbb{Q}_\ell(r))$$

is surjective for any smooth projective varieties defined over a finite field. While the cycle class map is in general not surjective for $\mathbb{Z}_\ell$ coefficients, one is still interested in knowing in which cases surjectivity still holds. This is usually called the integral Tate conjecture (even though it is not true in general).

**Remark 1.8.** The term SRC is introduced by Kollár-Miyaoka-Mori [KMM92]. The term SRC in codimension 1 is introduced in [KT23]. Main examples include SRC varieties and fibrations over a curve with smooth proper SRC general fibers. In characteristic 0, these are all the examples. In positive characteristic there are more examples. In any case, one can take the quotient by free rational curves on a variety that is SRC in codimension 1. The quotient is either a curve or a point. In particular, the Chow group of 0-cycles on such varieties is supported in a curve.

The results of this paper, together with some results proved by the author in [Tia20], strongly suggest that the following is true.

**Conjecture 1.9.** Let $X$ be a smooth projective variety defined over a finite field. Assume that $X$ is separably rationally connected in codimension 1. Then the cycle class map

$$\text{CH}^1(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}_{\text{et}}(X, \mathbb{Z}_\ell(d-1))$$

is surjective, where $d = \dim X$.

We refer the readers to Theorem 1.13 and Remark 1.15 for evidences of this conjecture.

The connection between integral Tate conjecture and Conjecture 1.1, 1.2 is the following.

**Theorem 1.10** ([CT99] Proposition 3.2, [Sai89] Corollary (8-6)). Let $\mathbb{F}$ be a finite field, $C$ a smooth projective geometrically connected curve over $\mathbb{F}$, and $K$ the function field of $C$. Let $X$ be a smooth projective geometrically connected variety of dimension $d + 1$ defined over $\mathbb{F}$, equipped with a morphism $p : X \to C$, whose generic fiber is smooth and geometrically irreducible. Let $l$ be a prime different from the characteristic.

1. If the cycle class map

$$\text{CH}^d(X) \otimes \mathbb{Z}_l \to H^{2d}_{\text{et}}(X, \mathbb{Z}_l(d))$$

is surjective, Conjectures 1.1 and 1.2 are true.
If the cycle class map
\[ CH^d(X) \otimes \mathbb{Z}_\ell \to H^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d)) \to H^{2d-2}_{\text{ét}}(X, \mathbb{Z}_\ell(d-1))^G \]
is surjective, or if
\[ CH^d(X) \otimes \mathbb{Z}_\ell \to H^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d)) \]
is surjective modulo torsion, Conjecture 1.2 is true.

Remark 1.11. The cited references only contain a proof of the first statement. But the second statement follows from the same proof. The general result of Saito produces a cohomology class \( \xi \in H^{2d}(X, \mathbb{Z}_\ell(d)) \) whose restriction to each local place coincide with the class of \( z_\mu \) ([CT99, Proposition 3.1]). The various types of integral Tate conjecture are simply used to find a global cycle whose class agrees with \( \xi \) in various cohomology groups. See also Page 19 of the slide of Colliot-Thélène’s lecture at Cambridge in 2008 (available at https://www.imo.universite-paris-saclay.fr/~jean-louis.colliot-thelene/expocambridge240809.pdf).

A result of Schoen [Sch98] says that if the Tate conjecture is true for divisors on all smooth projective surfaces defined over finite fields, then for any smooth projective variety \( V \) defined over a finite field \( \overline{F} \), the cycle class map
\[ CH_1(\overline{V}) \otimes \mathbb{Z}_\ell \to \cup_{K'/\overline{F}} H^{2d-2}(V, \mathbb{Z}_\ell(d-1))^\text{Gal}(\overline{F}/K) \subset H^{2d-2}(\overline{V}, \mathbb{Z}_\ell(d-1)) \]
is surjective, where \( \overline{V} \) is the base change of \( V \) to an algebraic closure of \( \overline{F} \).

If furthermore \( V \) is SRC in codimension 1, since its Chow group of zero cycles is supported in a curve, it is easy to see that every class in \( H^{2d-2}(V, \mathbb{Z}_\ell(d-1)) \) is algebraic. Thus every class in \( H^{2d-2}(\overline{V}, \mathbb{Z}_\ell(d-1)) \) is fixed by some open subgroup of the Galois group. So in this case, Schoen’s theorem implies that we always have a surjection
\[ CH_1(V) \otimes \mathbb{Z}_\ell \to H^{2d-2}(V, \mathbb{Z}_\ell(d-1)), \]
provided that Tate conjecture holds for all surfaces.

The paper [CTS10] discussed the implication of Schoen’s result for varieties defined over \( \overline{F}(C) \), the function field of a curve defined over \( \overline{F} \). Colliot-Thélène and Kahn analyzed the surjectivity of the \( \mathbb{Z}_\ell \)-coefficient cycle class map of codimension 2 cycles and its relation with degree 3 unramified cohomology \( H^3_{\text{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) in [CTK13] (over the complex numbers, such a relation is studied in [CTV12]). For the sake of brevity, and since we will not need these notions for the other parts of this paper, we will not define this invariant. Instead, we refer the interested reader to these papers and the references therein for definitions and properties of unramified cohomology. In particular, if the unramified cohomology \( H^3_{\text{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) vanishes, the cokernel of \( CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)) \) is torsion free [CTK13 Théorème 2.2]. Thus if, in addition, we know that the cokernel is torsion (for instance, if the Chow group of zero cycles with rational coefficients is universally supported in a surface [CTK13 Proposition 3.2]), we know the cycle class map is surjective. One should also note that by the Tate conjecture, one expects the cokernel to be torsion. In general, they deduced a short exact sequence relating various Chow groups of codimension 2 cycles and degree 3 unramified cohomology. Their short exact sequence for varieties over finite fields reads the following ([CTK13]):
Theorem 6.8]:

\[ 0 \to \ker(CH^2(X) \to CH^2(\breve{X})) \to H^1(\mathbb{F}, \oplus \ell H^3_{\text{ét}}(X, \mathbb{Z}/\ell \mathbb{Z}(2))) \to \ker(H^3_{\text{nr}}(\mathbb{X}, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\text{nr}}(\breve{\mathbb{X}}, \mathbb{Q}/\mathbb{Z}(2))) \to \coker(CH^2(X) \to CH^2(\breve{X})) \to 0 \]

Of course, one can deduce from this a similar exact sequence for $\ell$-primary torsions.

In particular, we can apply their results to 3-folds, which then relates the integral Tate conjecture to the vanishing of degree 3 unramified cohomology. Note that by the Lefschetz hyperplane theorem, if we can prove integral Tate conjecture for one cycles on all 3-folds, we prove integral Tate conjecture for one cycles on all smooth projective varieties.

Several groups of authors proved the vanishing of the degree 3 unramified cohomology on certain threefolds and deduce the integral Tate conjecture for one cycles, and thus proving Conjecture 1.1 and 1.2 for some surfaces defined over a global function field. See, for example, [PS16] for the case of conic bundles over a surface, [CTS21] and [Sca22] for the case of a product of a curve with a $CH_0$-trivial surface.

We prove Theorem 1.4 as a consequence of the following case of the integral Tate conjecture for one cycles.

**Theorem 1.12.** Let $\pi : X \to B$ be a projective flat family of surfaces over a smooth projective curve $B$ defined over a finite field $\mathbb{F}_q$. Assume that $X$ is smooth and that the geometric generic fiber is a smooth rational surface. Then integral Tate conjecture holds for one cycles. More concretely, the cycle class map

\[ CH^1(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(\breve{X}, \mathbb{Z}/\ell \mathbb{Z}(2)) \]

is surjective.

In general, one can deduce the following geometric criterion for the validity of the integral Tate conjecture and the local-global principles for separably rationally connected varieties defined over global function fields.

Given a variety $V$ defined over a field $k$, we denote by $A_1(V)$ the group of one cycles in $V$ modulo algebraic equivalence. We also use $\overline{V}$ to denote the base change of $V$ to an algebraic closure of $k$.

**Theorem 1.13.** Let $\pi : X \to B$ be a projective flat family of varieties over a smooth projective curve $B$ defined over a finite field $\mathbb{F}_q$. Assume that $X$ is smooth and that the generic fiber is smooth, separably rationally connected, and of dimension $d$. Consider the following hypothesis:

(A) The cycle class map $A_1(X) \otimes \mathbb{Z}_\ell \to H^{2d}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(d))$ is surjective.

(B) The cycle class map $A_1(X) \otimes \mathbb{Z}_\ell \to H^{2d}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(d))$ is injective.

(C) The cycle class map from higher Chow groups

\[ \varprojlim_{n} CH^1(X, 1, \mathbb{Z}/\ell^n \mathbb{Z}) \to H^{2d-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(d)) \]

is surjective.

(D) The coniveau filtration $N^1 H^{2d-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ is the whole cohomology group $H^{2d-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$.

If $\overline{X}$ satisfies hypothesis (A) and (B), then the cycle maps

\[ CH^1(X) \otimes \mathbb{Z}_\ell \to H^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d)) \to H^{2d}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(d))^{\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)} \]
is surjective, and Conjecture 1.2 holds for the generic fiber $X$ over $\mathbb{F}_q(B)$.

If $\overline{X}$ satisfies hypothesis (C) or (D), then the cycle maps

$$CH_1(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \to H^1(\mathbb{F}_q, H^{2d-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(d)))$$

is surjective.

Thus Conjecture 1.1 holds for the generic fiber $X$ over $\mathbb{F}_q(B)$ if either hypothesis (A), (B), (C), or (A), (B), (D) hold.

Remark 1.14. The statements in Hypothesis (A), (B), (C), (D) only depend on the stable birational class of the generic fiber of $X \to \overline{B}$. In particular, they only depend on the stable birational class of the generic fiber $X$ over the field $\mathbb{F}_q(B)$ (assuming that there is a smooth projective model for every stable birational class of $X$). Also note that Conjectures 1.1, 1.2, 1.3, 1.9 only depend on the stable birational class of the variety over $\mathbb{F}_q(B)$ (or $\mathcal{F}$ for Conjecture 1.9).

Remark 1.15. We make a few simple remarks about the validity of the hypothesis above. First of all, it is a simple exercise to prove that all these hypothesis hold if we use $\mathbb{Q}_\ell$-coefficient, and that they hold for all but finitely many $\ell$.

As discussed above, Hypothesis (A) follows from Tate’s conjecture on surfaces. The author has made conjectures on the Kato homology of rationally connected fibrations over an algebraically closed field of characteristic 0 in [Tia20]. A special case of the conjecture predicts that for a rationally connected fibration over a curve defined over an algebraically closed field of characteristic 0, hypothesis (A), (B), (C), and (D) hold. It is quite reasonable to believe that the same is true for separably rationally connected fibrations in characteristic $p > 0$. We discuss some examples in Section 6.

Now we explain a corollary of Theorem 1.12, which confirm a conjecture of Colliot-Thélène and Kahn ([CTK13, Conjecture 5.8]) up to torsion.

Corollary 1.16. Let $X$ be a smooth projective threefolds defined over a finite field $\mathcal{F}$. Assume that $X$ admits a fibration structure over a smooth projective curve with smooth projective geometrically rational generic fiber. Then we have

$$H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0,$$

for any $\ell$ invertible in $\mathcal{F}$, and a short exact sequence

$$0 \to H^1(\mathcal{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\}) \to CH_1(X) \otimes \mathbb{Z}_\ell \to CH_1(\overline{X})^G \otimes \mathbb{Z}_\ell \to 0.$$

Proof. The vanishing of $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ follows from Theorem 1.12, [CTK13, Theorem 2.2, Proposition 3.2], and the fact that $CH_0(\overline{X})$ is supported in a curve.

It then follows from the exact sequence (4) that we have the above description of the Chow groups of $X$ and $\overline{X}$. $\square$

Remark 1.17. Theorem 1.13 holds for smooth projective separably rationally connected varieties. We have made the proof works in both cases. A cheaper way to get this result is to note that the validity of the integral Tate conjecture is a stable birational invariant and apply the above theorems to the product $\mathbb{P}^1 \times X$ ($X$ separably rationally connected) as a fibration over $\mathbb{P}^1$. Unfortunately, for a separably rationally connected threefold $V$ defined over $\mathbb{F}_q$, we do not know if the cycle class map

$$CH_1(\overline{V}) \otimes \mathbb{Z}_\ell \to H^4(\overline{V}, \mathbb{Z}_\ell(2))$$
is surjective. Once we know this (e.g. if we are willing to assume the Tate conjecture for surfaces), the same argument as above shows that
\[ CH_1(V) \otimes \mathbb{Z}_\ell \to H^4(V, \mathbb{Z}_\ell(2)) \]
is surjective. One can also deduce the vanishing of degree 3 unramified cohomology and the short exact sequence of Chow groups as above.

1.3. Coniveau and strong coniveau. This section is a digression into several a priori different notions of coniveau filtrations on the cohomology of a variety introduced by Benoist-Ottom and Voisin in [BO21, Voi22]. These notions will play an important role when we return to discuss the integral Tate conjecture in the next section.

Let us first review the definitions.

**Definition 1.18.** Let \( X \) be a smooth projective variety of dimension \( n \) defined over an algebraically closed field. Given an abelian group \( A \) that is one of \( \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Z}, \mathbb{Q}, \) or \( \mathbb{Q}_\ell \), we simply write \( H^k(X, A) \) as either the étale cohomology with coefficient \( A \) or the singular cohomology with coefficient \( A \) (if \( X \) is a complex variety), we have the following closely related filtrations on the cohomology \( H^k(X, A) \).

1. The coniveau filtration:
\[
N^c H^k(X, A) := \sum_{f: Y \to X} f_* (H_{2n-k}(Y, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A),
\]
where the sum is taken over all morphisms from projective algebraic sets \( f: Y \to X, \dim Y \leq n - c \);

2. The strong coniveau filtration:
\[
\tilde{N}^c H^k(X, A) := \sum_{f: Y \to X} f_* (H_{2n-k}(Y, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A),
\]
where the sum is taken over all morphisms from smooth projective varieties \( f: Y \to X, \dim Y \leq n - c \).

3. The strong cylindrical filtration:
\[
\tilde{N}_{c,cyl} H^k(X, A) := \sum_{f: Y \to X} \Gamma_*(H_{2n-k-2c}(Z, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A)
\]
where the sum is taken over all smooth projective varieties \( Z \) and correspondences \( \Gamma \subset Z \times X \) of relative dimension \( c \) over \( Z \).

4. The strong equidimensional cylindrical filtration:
\[
\tilde{N}_{c,cyl,eq} H^k(X, A) := \sum_{f: Y \to X} \Gamma_*(H_{2n-k-2c}(Z, A)) \subset H_{2n-k}(X, A) \cong H^k(X, A)
\]
where the sum is taken over all smooth projective varieties \( Z \) and correspondences \( \Gamma \subset Z \times X \) that is equidimensional of relative dimension \( c \) over \( Z \).

5. The semi-stable filtration: \( N_{c, st, cyl} H^k(X, Z) \) as the group generated by the cylinder homomorphisms
\[
f_* \circ p^*: H_{2n-k-2c}(Z, \mathbb{Z}) \to H_{2n-k}(X, \mathbb{Z}) \cong H^k(X, \mathbb{Z}),
\]
for all morphisms \( f: Y \to X, \) and flat projective morphisms \( p: Y \to Z \) of relative dimension \( c \) with simple normal crossing fibers, where \( \dim Z \leq 2n - k - 2c \).
(6) We use the notations $N^c H_k$ etc. to denote the filtrations on Borel-Moore or singular homology $H_k$. Since $X$ is smooth, this is the same as the filtrations $N^c H^{2d-k}$.

A general relation between these filtrations is the following.

**Lemma 1.19.** [Voi22, Proposition 1.3] We have the following inclusions:

$\tilde{N}_n-cyl,eq H^{2c-1}(X, A) \subset \tilde{N}_{n-c, cyl} H^{2c-1}(X, A) = \tilde{N}^c H^{2c-1}(X, A) \subset N^c H^{2c-1}(X, A)$. 

The only non-obvious part, the equality in the middle, is proved by Voisin [Voi22, Proposition 1.3].

A natural question is whether or not these filtrations agree with each other. If we use $\mathbb{Q}$ or $\mathbb{Q}_\ell$ coefficients, the theory of weights shows that the strong coniveau and coniveau filtrations are equivalent. Since the difference between some of these filtrations also gives stable birational invariants, one wonders if this could be used to prove non-stable-rationality for some rationally connected varieties.

Examples with $\mathbb{Z}$-coefficients where the strong coniveau filtration and coniveau filtration differ are constructed in [BO21]. More precisely, they prove the following.

**Theorem 1.20.** [BO21, Theorem 1.1] For all $c \geq 1$ and $k \geq 2c + 1$, there is a smooth projective complex variety $X$ such that the inclusion

$\tilde{N}^c H^k(X, \mathbb{Z}) \subset N^c H^k(X, \mathbb{Z})$

is strict. One may choose $X$ to have torsion canonical bundle. If $c \geq 2$, one may choose $X$ to be rational.

The examples as above usually have large dimension especially when $c$ is large. For lower dimensional examples, Benoist-Ottem proved the following.

**Theorem 1.21.** [BO21, Theorem 1.2] For $k \in \{3, 4\}$, there is a smooth projective complex variety $X$ of dimension $k + 1$ with torsion canonical bundle such that the inclusion

$\tilde{N}^1 H^k(X, \mathbb{Z}) \subset N^1 H^k(X, \mathbb{Z})$

is strict.

These examples leave the case of $c = 1$ open for threefolds and for rationally connected varieties. Voisin studied the strong coniveau filtrations on $H^{2d-3}$ [Voi22].

**Theorem 1.22.** [Voi22, Theorem 2.6, Corollary 2.7, Theorem 2.17] Let $X$ be a smooth projective variety of dimension $d$ defined over $\mathbb{C}$.

1. Assume the Walker Abel-Jacobi map ([Wal07])

$\phi : CH_1(X)_{alg} \to J(N^1 H^{2d-3}(X, \mathbb{Z}))$

is injective on torsions. Then we have

$N_{1, st, cyl} H^{2d-3}(X, \mathbb{Z})/\text{Tor} = N^1 H^{2d-3}(X, \mathbb{Z})/\text{Tor}$.

2. If $\dim X$ is 3, we have

$N_{1, cyl, st} H^3(X, \mathbb{Z})/\text{Tor} = N^1 H^3(X, \mathbb{Z})/\text{Tor}$.

3. If $X$ is rationally connected, we have

$N_{1, cyl, st} H^{2d-3}(X, \mathbb{Z}) = \tilde{N}_1 H^{2d-3}(X, \mathbb{Z})$

As a consequence,

$N_{1, cyl, st} H^{2d-3}(X, \mathbb{Z})/\text{Tor} = \tilde{N}^1 H^{2d-3}(X, \mathbb{Z})/\text{Tor} = N^1 H^{2d-3}(X, \mathbb{Z})/\text{Tor}$. 

We prove an improvement of Voisin’s results.

**Theorem 1.23.** Let $X$ be a complex smooth projective variety of dimension $d$. Then the following two filtrations agree with each other:

$$N_{1, st, eq} H_3(X, \mathbb{Z}) = N^1 H^{2d-3}(X, \mathbb{Z}).$$

Assume furthermore that $X$ is SRC in codimension 1. There is a smooth projective curve $C$ with a family of 1-dimensional cycles $\Gamma \subset C \times X$ such that

$$\Gamma_* : H_1(C, \mathbb{Z}) \to H_3(X, \mathbb{Z})$$

has the same image as the $s$-map $s : L_1 H_3(X) \to H_3(X)$, which is the same as the $N^1 H^3(X, \mathbb{Z})$. In particular, the following filtrations on $H^{2d-3}(X, \mathbb{Z})$ introduced in Definition 1.18 are the same:

$$\tilde{N}_{1, eq, eq} H^{2d-3}(X, \mathbb{Z}) = \tilde{N}_{1, eq} H^3(X, \mathbb{Z}) = \tilde{N}^1 H^3(X, \mathbb{Z}) = N^1 H^3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}).$$

For the definition of Lawson homology and $s$-map, see Definition 3.5 and 3.10 in Section 3.

An immediate corollary is the following.

**Theorem 1.24.** Let $X$ be a complex smooth projective variety, that is SRC in codimension 1. Then the following filtrations on $H^3(X, \mathbb{Z})$ introduced in Definition 1.18 coincide with the whole cohomology group:

$$\tilde{N}_{1, cyl, eq} H^3(X, \mathbb{Z}) = \tilde{N}_{1, cyl} H^3(X, \mathbb{Z}) = \tilde{N}^1 H^3(X, \mathbb{Z}) = N^1 H^3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}).$$

**Remark 1.25.** Using the decomposition of the diagonal argument, one can show that when $X$ is SRC in codimension 1, for each $i$, there is a smooth projective variety $Y_i$ and a family of cycles $\Gamma_i \subset Y_i \times X$ such that the cokernel of $\Gamma_* : H_i(Y_i) \to H_{i+2}(X)$ is $N$-torsion for a fixed $N$. So we may consider the $s$-map from $\mathbb{Z}/N$ Lawson homology (defined as the homotopy group of the topological group $Z_r(X) \otimes \mathbb{Z}/N$) to $H_3(X, \mathbb{Z}/N)$. We have long exact sequences

$$L_1 H_i(X, \mathbb{Z}) \xrightarrow{N} L_1 H_i(X, \mathbb{Z}) \xrightarrow{} L_1 H_i(X, \mathbb{Z}/N) \xrightarrow{} L_1 H_{i-1}(X, \mathbb{Z})$$

By results of Suslin-Voevodsky [SV90] and the Bloch-Kato conjecture proved by Voevodsky, there is an isomorphism

$$L_1 H_{2+i}(X, \mathbb{Z}/N) \cong CH_1(X, i, \mathbb{Z}/N) \cong \mathbb{H}^i(X, \tau \leq \dim X - 1 R\pi_* (\mathbb{Z}/N))$$

between torsion Lawson homology, Bloch’s higher Chow group, and certain Zariski cohomology group, where $\pi : X_{\text{cl}} \to X_{\text{zar}}$ is the continuous map from $X(\mathbb{C})$ with the analytic topology to $X$ with the Zariski topology.

We also have a long exact sequence:

$$\ldots \to L_1 H_k(X, \mathbb{Z}/N) \to H_k(X, \mathbb{Z}/N) \to KH_k(X, \mathbb{Z}/N) \to L_1 H_{k-1}(X, \mathbb{Z}/N) \to \ldots$$

where $KH_k(X, \mathbb{Z}/N) = H^{\dim X - k}(X, R^{\dim X - k} \pi_* \mathbb{Z}/N)$ is the so-called Kato homology. The author has made a number of conjectures about Kato homologies of a rationally connected fibration in [Tia20]. Special cases of these conjectures imply that there is an isomorphism $L_1 H_i(X, \mathbb{Z}/N) \cong H_i(X, \mathbb{Z}/N)$ for all $k$ and all rationally connected varieties and rationally connected fibrations over a curve defined over the
complex numbers. This in turn would imply the s-maps $L_1H_s(X, \mathbb{Z}) \to H_s(X, \mathbb{Z})$ are isomorphisms.

We have a similar result that applies to fields of positive characteristic.

**Theorem 1.26** (=Theorem 1.13). Let $X$ be a $d$-dimensional smooth projective variety defined over an algebraically closed field, which is separably rationally connected in codimension 1. There is a smooth projective curve $C$ with a family of 1-dimensional cycles $\Gamma \subset C \times X$ such that

$$\Gamma_* : H^1_{BM}(C, \mathbb{Z}_\ell) \to H^3_{BM}(X, \mathbb{Z}_\ell) \cong H^{2d-3}_\ell(X, \mathbb{Z}_\ell)$$

surjects onto $N^1H^{2d-3}_\ell(X, \mathbb{Z}_\ell)$.

**Theorem 1.27** (=Theorem 1.16). Let $X$ be a smooth projective defined over an algebraically closed field, which is separably rationally connected in codimension 1. Assume $X$ is a 3-fold. Then the following filtrations on $H^3(X, \mathbb{Z}_\ell)$ introduced in Definition 1.18 equal the whole cohomology group:

$$\widetilde{N}_{1,\text{cyl},\text{eq}}H^3(X, \mathbb{Z}_\ell) = \widetilde{N}_1,\text{cyl}H^3(X, \mathbb{Z}_\ell) = N^1H^3(X, \mathbb{Z}_\ell) = N^3H^3(X, \mathbb{Z}_\ell) = H^3(X, \mathbb{Z}_\ell).$$

1.4. **Integral Tate conjecture for one cycles: arithmetic part.** We continue the discussion on integral Tate conjecture in this section.

The Serre-Hochschild spectral sequence gives an exact sequence:

$$0 \to H^1(\mathbb{F}_q, H^{2d-2r-1}_\text{et}(\tilde{X}, \mathbb{Z}_\ell(d-r))) \to H^{2d-2r}_\text{et}(X, \mathbb{Z}_\ell(d-r)) \to H^{2d-2r}_\text{et}(\tilde{X}, \mathbb{Z}_\ell(d-r))^G \to 0.$$ 

Thus the integral Tate conjecture consists of a geometric part, i.e. surjectivity of

$$CH_r(X) \otimes \mathbb{Z}_\ell \to H^{2d-2r}_\text{et}(\tilde{X}, \mathbb{Z}_\ell(d-r))^G,$$

and an arithmetic part, i.e. surjectivity of

$$CH_r(X)_{\text{hom}} \otimes \mathbb{Z}_\ell \to H^1(\mathbb{F}_q, H^{2d-2r-1}_\text{et}(\tilde{X}, \mathbb{Z}_\ell(d-r))),$$

where $CH_r(X)_{\text{hom}} \otimes \mathbb{Z}_\ell$ is the “geometrically homologically trivial” part, i.e. the kernel of

$$CH_r(X) \otimes \mathbb{Z}_\ell \to H^{2d-2r}_\text{et}(\tilde{X}, \mathbb{Z}_\ell(d-r))^G.$$

In a recent preprint [SS22], Scavia and Suzuki systematically investigated the question of the surjectivity in the arithmetic part and relate them to the strong coniveau filtration. For codimension 2 cycles, they obtain the following results.

**Theorem 1.28.** [SS22] Theorem 1.3] Let $\mathbb{F}$ be a finite field and $\ell$ be a prime number invertible in $\mathbb{F}$, and suppose that $\mathbb{F}$ contains a primitive $\ell^2$-th root of unity. There exists a smooth projective geometrically connected $\mathbb{F}$-variety $X$ of dimension $2\ell + 2$ such that the map

$$CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{\text{et}}(X, \mathbb{Z}_\ell(4))^G$$

is surjective whereas the map

$$CH^2(X)_{\text{hom}} \otimes \mathbb{Z}_\ell \to H^1(\mathbb{F}_q, H^3_{\text{et}}(\tilde{X}, \mathbb{Z}_\ell(2)))$$

is not.

**Theorem 1.29.** [SS22] Theorem 1.4] Let $p$ be an odd prime. There exist a finite field $\mathbb{F}$ of characteristic $p$ and a smooth projective geometrically connected fourfold $X$ over $\mathbb{F}$ for which the image of the composition

$$H^1(\mathbb{F}, H^3_{\text{et}}(X, \mathbb{Z}_2(2))) \to H^1(\mathbb{F}, H^3_{\text{et}}(X, \mathbb{Z}_2(2))) \to H^4(X, \mathbb{Z}_2(2))$$

contains a non-algebraic torsion class.
Shortly after Scavia-Suzuki’s preprint appeared, Benoist studied steenrod operations on Chow groups and cohomologies [Ben22]. He obtained new examples of non-algebraic cohomology classes over many fields (\(\mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{F}_q\)) and for cohomology classes on algebraizable smooth manifolds. In the case of finite fields, his results removed the assumptions on \(\ell^2\)-th roots of unity in Scavia-Suzuki’s results.

**Theorem 1.30.** [Ben22 Theorem 4.12] Let \(p \neq \ell\) be prime numbers, and let \(\mathbb{F}\) be a finite subfield of \(\mathbb{F}_p/\mathbb{F}_p\). There exist a smooth projective geometrically connected variety \(X\) of dimension \(2\ell + 3\) over \(\mathbb{F}\) and a non-algebraic class \(x \in \text{Ker}(H^{4}_{\text{et}}(X, \mathbb{Z}_{\ell}(2)) \to H^{4}_{\text{et}}(\bar{X}, \mathbb{Z}_{\ell}(2)))\).

The failure of the surjectivity is related to the discrepancy of the strong coniveau and coniveau filtration by the following.

**Theorem 1.31.** [SS22 Theorem 1.5, Proposition 7.6] Let \(X\) be a smooth projective geometrically connected variety over a finite field \(\mathbb{F}\) and \(\ell\) be a prime number invertible in \(\mathbb{F}\). Suppose that the coniveau and strong coniveau on \(H^{3}_{\text{et}}(X, \mathbb{Z}_{\ell}(2))\) coincide: \(N^{1}H^{3}(X, \mathbb{Z}_{\ell}(2)) = \tilde{N}^{1}H^{3}_{\text{et}}(X, \mathbb{Z}_{\ell}(2))\). Then the \(\ell\)-adic algebraic Abel-Jacobi map is an isomorphism:

\[
CH^{3}(X)_{\text{alg}} \otimes \mathbb{Z}_{\ell} \to H^{1}(\mathbb{F}, N^{1}H^{3}(X, \mathbb{Z}_{\ell}(2))).
\]

In general, we have a surjection

\[
CH^{r}(X)_{\mathbb{F}, \text{alg}} \otimes \mathbb{Z}_{\ell} \to \text{Image}(H^{1}(\mathbb{F}, \tilde{N}^{r-1}H^{2r-1}(X, \mathbb{Z}_{\ell}(r)))) \to H^{1}(\mathbb{F}, N^{r-1}H^{2r-1}(X, \mathbb{Z}_{\ell}(r)))).
\]

In the paper of Scavia-Suzuki [SS22], they use \(CH^{r}(X)_{\mathbb{F}, \text{alg}}\) to denote cycles algebraically equivalent to zero over \(\mathbb{F}\), and \(CH^{r}(X)_{\text{alg}}\) to denote cycles defined over \(\mathbb{F}\) that are algebraically equivalent to zero over \(\bar{\mathbb{F}}\). However, for codimension 2 cycles on varieties defined over \(\mathbb{F}\), there is no known example where \(CH^{2}(X)_{\mathbb{F}, \text{alg}}\) and \(CH^{2}(X)_{\text{alg}}\) differ (Question 8.2, [SS22]), and \(CH^{2}(X)_{\mathbb{F}, \text{alg}} \otimes \mathbb{Z}_{\ell}\) and \(CH^{2}(X)_{\text{alg}} \otimes \mathbb{Z}_{\ell}\) are isomorphic if the strong coniveau coincides with the coniveau filtration on \(H^{3}\) ([SS22 Proposition 7.10, 7.11]). Moreover, for one cycles on a separably rationally connected variety or a separably rationally connected fibration over a curve defined over a finite field, we know \(CH_{1}(X)_{\mathbb{F}, \text{alg}}\) and \(CH_{1}(X)_{\text{alg}}\) are the same [KT23 Theorem 6].

While the surjectivity in the arithmetic part is not true in general, we do expect this to be true for one cycles on varieties that is separably rationally connected in codimension 1.

As a corollary of Theorem 1.7 and the work of Scavia-Suzuki, we get the following results regarding the arithmetic part of the cycle class map.

**Corollary 1.32** (=Corollary 4.17). Let \(X\) be a smooth projective variety of dimension \(d\) defined over a finite field \(\mathbb{F}_q\), which is separably rationally connected in codimension 1. Then we have a surjection

\[
CH_{1}(X)_{\text{alg}} \to H^{1}(\mathbb{F}_q, N^{1}H^{2d-3}_{\text{et}}(\bar{X}, \mathbb{Z}_{\ell}(d - 1))).
\]

Furthermore, assume one of the followings

1. \(N^{1}H^{2d-3}_{\text{et}}(\bar{X}, \mathbb{Z}_{\ell}(d - 1)) = H^{2d-3}_{\text{et}}(\bar{X}, \mathbb{Z}_{\ell}(d - 1))\).
2. The cycle class map

\[
cl : \lim_{\ell \to \infty} CH_{1}(\bar{X}, 1, \mathbb{Z}/\ell^{n}) \to H^{2d-3}_{\text{et}}(\bar{X}, \mathbb{Z}_{\ell}(d - 1))
\]

is surjective.
Then every class in $H^1(\mathbb{F}_q, H^3(\bar{X}, \mathbb{Z}/d-1))$ is the class of an algebraic cycle. In particular, this holds if $X$ has dimension 3.

1.5. Algebraic equivalence. The key technical result in proving the main theorems is the joint work of the author with János Kollár [KT23] studying algebraic equivalences of one cycles on smooth projective varieties.

Algebraically equivalence between two cycles means that one has to add complicated cycles to both of them and then get a family of cycles over a curve. Given two stable maps, supposing that they give algebraically equivalent cycles, it is not clear that this algebraic equivalence of cycles can be realized as a deformation of stable maps. In the joint work with Kollár, we prove that this is always possible for curves on smooth projective varieties.

**Theorem 1.33.** [KT23] Theorem 1] Let $X$ be a smooth, projective variety over an algebraically closed field $K$. Let $\pi_i : C_i \to X$ (for $i \in I$) be finitely many morphisms of nodal curves to $X$ such that the $(\pi_i)_* [C_i]$ are algebraically equivalent to each other. Then there is a morphism from a single nodal curve $\pi_R : R \to X$ and a family of connected nodal curves over a connected curve $B$ with a morphism to $X$:

$B \leftarrow S \xrightarrow{\pi} X$ such that for any $i \in I$, there is a point $b_i \in B$ with fiber $S_i \cong C_i \cup R$ and $(\pi|_{S_i} : C_i \to X) \cong (\pi_i : C_i \to X), (\pi|_{S_i} : R \to X) \cong (\pi_R : R \to X)$.

That is, the deformation of cycles is visible as deformation of maps. This result yields many interesting applications. For example, the study leads to the following theorem.

**Theorem 1.34.** [KT23] Theorem 6] Let $X_k$ be a smooth, projective variety over a perfect field $k$, which is separably rationally connected in codimension 1. Then the kernel of the natural map

$$A_1(X_k) \to A_1(\bar{X}_k)$$

is either trivial or $\mathbb{Z}/2\mathbb{Z}$. More precisely,

1. The kernel is trivial if $X_k$ contains an odd degree 0-cycle, and
2. if $Z = \sum d_i C_i$ and $Z_k$ is algebraically equivalent to 0 over $\bar{k}$, then $Z$ is algebraically equivalent to 0 over $k$ if and only if the index of $X$ divides $\chi(Z) := \sum d_i \chi(C_i, \mathcal{O}_{C_i})$.

Recall that a pseudo algebraically closed field (or a PAC field for short) is a field where every geometrically integral variety has a rational point.

**Theorem 1.35.** [KT23] Theorem 7] Let $X_k$ be a smooth projective variety defined over a perfect field $k$. Assume that every geometrically irreducible $k$-variety has a 0-cycle of degree 1 (e.g. $k$ is a finite field or a PAC field). Assume that $X$ is separably rationally connected in codimension 1. We have an isomorphism

$$A_1(X_k) \cong A_1(X_k)^G,$$

where $G$ is the absolute Galois group of $k$.

If $k$ is not perfect (and every geometrically irreducible $k$-variety has a 0-cycle of degree 1), then we have an isomorphism after inverting the characteristic $p$. 
1.6. Structure of the paper. This paper consists of applications of the results in [KT23].

The first application is in Section 2 where we describe some structural results of the space of one cycles. These structural results are then used in Sections 3 and 4 to study the coniveau filtration on $H^3$ of a separably rationally connected fibration over a curve. The case of complex varieties uses Lawson homology and is topological. Moreover, it gives “integral” results. We present this first to give the readers some flavor of the argument. The general case has to use the Chow sheaves introduced by Suslin and Voevodsky, hence more abstract. Unfortunately, in this case we only have results for torsion coefficients and have to pass to the inverse limit from time to time. The criterion for the surjection of the cycle class map onto the arithmetic part is proved in Corollary 4.17.

Finally, the applications to local-global principles are discussed in Section 5. In Section 6 we give some examples where the criterion in Theorem 1.13 can be effectively checked.

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2. Space of one cycles

In this section, we fix an algebraically closed field $k$ of any characteristic. We remind the readers that a variety is always assumed to be irreducible throughout the paper, and thus connected. Sometimes we add the word irreducible just to emphasize this.

**Definition 2.1.** Let $X, Y$ be finite type reduced separated $k$-schemes. A family of relative cycles of equi-dimension $r$ over $Y$ is a formal linear combination of integral subschemes $Z = \sum m_i Z_i, Z_i \subset Y \times X, m_i \in \mathbb{Z}$ such that

1. Each $Z_i$ dominates one irreducible component of $Y$
2. Each fiber of $Z_i \to Y$ has dimension $r$.
3. A fat point condition is satisfied (see [Kol96] Chapter I, 3.10.4 or [SV00] Definition 3.1.3).
4. A field of definition condition is satisfied. Namely, for any point $y \in Y$, there are integral subschemes $\gamma_i$ of $X$ defined over the residue field $\kappa(y)$ such that the cycle theoretic fiber ([Kol96] Chapter I, 3.10.4) of $Z$ over $y$ is $\sum m_i \gamma_i$.

We write $Z^+$ and $Z^-$ as the positive and negative parts, i.e. the sum of $Z_i$’s with positive and negative coefficients.

We say this family has proper support if $Z_i \to Y$ are proper for every $i$.

It is often convenient to allow some redundancy in the linear combination, especially when we consider pull-back and restrictions. For example, we allow an expression of the form $(Z_1 + Z_3) - (Z_2 + Z_3)$. When this happens, we think of $(Z_1 + Z_3)$ (resp. $(Z_2 + Z_3)$) as the positive (resp. negative) part.
We refer the interested readers to [Kol96] Section I.3, I.4 and [SV00] Section 3 for details about the last two conditions and the subtle points about these definitions. We only remark that with this definition, one can pull-back families of cycles.

We adopt Kollár’s convention of only considering families of cycles over a reduced base. Suslin and Voevodsky [SV00] consider more general base.

For our purpose, it is perfectly fine to always work over a reduced base. We call a separated, finite type, reduced $k$-scheme an algebraic set.

**Remark 2.2.** We would also like to mention that for a normal variety, condition 3 is automatic. Condition 4 is always satisfied in characteristic 0, or if the base is normal/smooth/projective, we may choose $S$ to be normal/smooth/projective.

**Definition 2.3.** Given a family of projective curves $S \to B$ and a $B$-morphisms $F : S \to B \times X$, we can associate to a family of cycles $\Gamma_S \subset B \times X$. So given a family of cycles $\Gamma \subset B \times X$ over $B$, we say it is nodal, if it is induced from a family of nodal curves as above.

**Proposition 2.4.** Let $S_1, S_2$ be two connected algebraic sets and $\Gamma_1 \subset S_1 \times X, \Gamma_2 \subset S_2 \times X$ be two equidimensional families of $r$-cycles in $X$. Assume that the cycles in the two families are algebraically equivalent. There is a connected algebraic set $S$ and an equidimensional family of $r$-cycles $\Gamma \subset S \times X$, and morphisms $f_1 : S_1 \to S, f_2 : S_2 \to S$ such that $\Gamma_1 = f_1^*\Gamma, \Gamma_2 = f_2^*\Gamma$. Moreover, if both $S_1$ and $S_2$ are normal/smooth/projective, we may choose $S$ to be normal/smooth/projective.

**Proof.** Take two points $s_1 \in S_1, s_2 \in S_2$. Denote by $\gamma_1, \gamma_2$ the cycle over $s_1, s_2$. By assumption, $\gamma_1, \gamma_2$ are algebraically equivalent. Thus there is a smooth projective curve $S_3$, two points $a, b \in S_3$ and a family of $r$-cycles $\Gamma_3 \subset S_3 \times X$ such that the cycle over $a, b$ are $\gamma_1, \gamma_2$. This follows from the definition of algebraic equivalence.

Indeed, by definition of algebraic equivalence, we may find a cycle $\Delta$ and a family of cycles $\Gamma_T$ over a smooth projective curve $T$ and two points $t_1, t_2 \in T$ such that the cycle over $t_1$ (resp. $t_2$) is $\gamma_1 + \Delta$ (resp. $\gamma_2 + \Delta$). Then we take $S_3$ to be $T$ and the family of cycles $\Gamma_3$ to be $\Gamma_T - T \times \Delta$.

Define $S = S_1 \times S_2 \times S_3$, with $p_i : S \to S_i, i = 1, 2, 3$, the projections. Define $\Gamma = p_1^*\Gamma_1 + p_2^*\Gamma_2 - p_3^*\Gamma_3$. Finally, define

$$f_1 : S_1 \to S, x \mapsto (x, s_2, b)$$

and

$$f_2 : S_2 \to S, y \mapsto (s_1, y, a).$$

One easily checks these satisfy the conditions.

Since $S_3$ is a smooth projective curve, if both $S_1, S_2$ are normal, or smooth, or projective, so is $S$. \qed

Now we can state the main technical result of this section.

**Theorem 2.5.** Let $X$ be a smooth projective variety defined over an algebraically closed field $k$. Let $(U, \Gamma_U)$ be an equi-dimensional family of one dimensional cycles over an irreducible variety $U$ and let $u_0, u_1 \in U$ be two points in $U$ such that

$$\gamma_0 := \Gamma_U|_{u_0} = \gamma_1 := \Gamma_U|_{u_1} = \gamma$$
as cycles. Then there is a family of one-dimensional cycles \((V, \Gamma_V)\) over a smooth quasi-projective variety \(V\) with a morphism \(f : V \to U\) such that \(f^*\Gamma_U = \Gamma_V\), and a lifting \(v_0, v_1\) of \(u_0, u_1\), such that

1. The morphism \(f : V \to U\) is projective and surjective.
2. We may take the family over \(V\) to be nodal as in Definition 2.3. We still use \(\Gamma_V\) to denote the family of nodal curves in the following.
3. There is a nodal deformation equivalence \(T \leftarrow \Gamma_T \to X\) over a two pointed connected nodal curve \((T, t_0, t_1)\) between \(\Gamma_V|_{v_0}\) and \(\Gamma_V|_{v_1}\).
4. For each \(k\)-point \(t\) of \(T\), the cycle over \(t\) is \(\gamma\).
5. For a general point \(u\) in \(U\), and for any pair of points \(v, v'\) in the inverse image of \(u\) in \(V\), there is a nodal deformation equivalence \((C_{v,v'}, c, c') \leftarrow \Gamma_C \to X, \Gamma_C|_c \cong \Gamma_V|_{v}, \Gamma_C|_{c'} \cong \Gamma_V|_{v'}\) over a connected two pointed nodal curve \((C_{v,v'}, c, c')\).
6. For any point \(c\) of \(C\), the cycle of \(\Gamma_C\) at \(c\) is the same as that of \(\Gamma_U\) at the point \(u\).

**Theorem 2.6.** Keep the same assumptions as in Theorem 2.5. Assume furthermore that \(X\) is separably rationally connected in codimension 1. Then \(V, T, C_{v,v'}\) admit a morphism to \((W, \Gamma_W)\), where \(W\) is a normal projective variety and \(\Gamma_W\) is a family cycles over \(W\). In characteristic 0, we may choose \(W\) to be irreducible, smooth and projective. In general, we may take \(W\) to be irreducible, normal, projective and smooth near the nodal points of \(T, C\), and \(v_0, v_1\).

**Remark 2.7.** This theorem is special to one cycles on varieties that is SRC in codimension 1.

Indeed, if the statements were true for a variety \(X\) and families of \(r\)-dimensional cycles, then the same argument as in Sections 3 and 4 would prove that the strong coniveau filtration \(N^r H^{2r+1}(X)\) coincide with the coniveau filtration \(N^r H^{2r+1}\). But the examples of Benoist-Ottem [BO21], Scavia-Suzuki [SS22] (cited in Section 1.3 and 1.4 as Theorems 1.20, 1.21, 1.28, 1.29) shows that this is not true in general.

**Remark 2.8.** We remark that the positive and negative part of the families of cycles may vary along \(T\). The theorem only states that the difference remains constant.

**Remark 2.9.** Even if we start with a family of effective cycles, for the statements to be true, we have to use non-effective cycles.

Moreover, the statement is not a simple corollary of the existence of the universal family over the Chow variety (which is true only in characteristic 0). This is because we require that the family is parameterized by a normal variety, while the Chow variety is only semi-normal in [Kol96] by definition or satisfies no such normality condition at all in some other references such as [Fri91].

It is possible that the morphism from the normalization of the Chow variety to the Chow variety maps two points to the same point. If this happens, we take \(U\) to be the normalization of the Chow variety, \(u, u'\) to be the two points mapping to the same point in the Chow variety, the existence of \(V, T, W\) in this case cannot be deduced from the existence of the universal family over the Chow variety.

**Remark 2.10.** Finally we remark that \(U\) being irreducible is not essential in the proof. But it simplifies the argument. If \(U\) is reducible and connected, one can use
similar argument as in [KT23, Section 8] to find a connected algebraic set $V$. But 
in this case, we cannot choose $V$ to admit a morphism to $U$. The best one can have 
is that for each irreducible component of $U$, there is an irreducible component of $V$ with a projective, surjective morphism to this component.

**Proof of Theorem 2.3.** First, using Nagata’s compactification and Chow lemma, we 
can make a base change that is a birational projective morphism and replace $U$ with 
a quasi-projective variety. So in the following, we assume $U$ is quasi-projective.

Up to a purely inseparable base change, we may assume that the generic fiber 
of the family is comes from nodal curves.

We write $\Gamma_U = \Gamma_U^+ - \Gamma_U^-$ as its positive and negative part. We write 
$$
\gamma_0^+ = \Gamma_U^{+|\omega_0}, \quad \gamma_0^- = \Gamma_U^{-|\omega_0}, \quad \gamma_1^+ = \Gamma_U^{+|u_1}, \quad \gamma_1^- = \Gamma_U^{-|u_1}.
$$

By assumption,
$$
\gamma_0^+ - \gamma_0^- = \gamma_1^+ - \gamma_1^-.
$$

We take a general complete intersection $V'$ (of the same dimension as $U$) contain-
ing $v_0 = (u_0, u_1)$ and $v_1 = (u_1, u_0)$ in the product $U \times U$. There are two families 
of nodal curves $\Gamma_p = \Gamma_p^{+} - \Gamma_p^{-}, \Gamma_q = \Gamma_q^{+} - \Gamma_q^{-}$ over $V'$ induced by the two projections 
$p, q : V' \to U$. We have the family of cycles over $V'$ 
$$
\Gamma_{V'} = (\Gamma_p^{+} + \Gamma_q^{-}) - (\Gamma_p^{-} + \Gamma_q^{+}).
$$

Then the positive part over $v_0$ is $\gamma_0^+ + \gamma_1^-$, and the positive part over $v_1$ is $\gamma_1^+ + \gamma_0^-$. 
Thus they are the same as cycles. Moreover, the restriction of the negative parts 
of $\Gamma_{V'}$ to $v_0, v_1$ are the same.

So now we have two families of cycles, and the restriction of each family to $v_0, v_1$ 
gives the same cycle. We first prove the statement for each family. Then we take 
base changes for both families such that they are over the same base and subtract them. Here we use the fact that the fiber of the family over the singular points 
of $T, C$ and $v_0, v_1, v, v'$ are all nodal curves, and thus the base change will change nothing in their neighborhood.

From now on we assume there is a nodal family of effective cycles.

The existence of a generically finite base change $V'' \to U$, the lifting $v_0, v_1$ and the curve $T$ follows from [KT23, Theorem 58]. Unwrapping the content of this theorem, one gets the following:

1. There is a nodal curve $Z$ and a family of $r$-tuple pf complete intersection 
curves $r|H^2| \to V''$.
2. One can glue them together to get a family of nodal curves $C_{V''} \to V''$.
3. There is a family of nodal curves over a two pointed curve $X \leftarrow C_T \to (T, t_0, t_1)$, such that the restriction of the family $C_T$ to $t_0, t_1$ coincide with 
the restriction of of $C_{V''}$ to $v_0, v_1$.
4. For each $t \in T$, the cycle over $t$ is $Z + \sum_{i=1}^r L_i(t)$, where $\cup L_i \to T$ is a family of $r$-tuple complete intersection curves.
5. The restriction of $\cup L_i$ to $t_0, t_1$ coincide with the restriction of $r|H^2|$ to 
$v_0, v_1$.
6. The families of curves induces a morphism $V'' \cup T \to \text{Hilb}_1(X \times \mathbb{P}^3)$.

This is almost what we want, except that the cycles over $V''$ and $T$ changes by a 
constant cycle $Z$ and a family of $r$-tuples of complete intersection curves in $r|H^2|$. 
So we subtract the corresponding family.
To get a finite base change \( V \to U \), one can take the graph closure of \( V' \) with respect to the morphism to \( \text{Hilb}_1(X \times \mathbb{P}^3) \) and then apply semi-stable reduction.

We may assume that \( V \) is smooth using de Jong’s alteration (or resolution of singularities in characteristic zero).

We note that the base change \( V \to U \) consists of two steps: first a purely inseparable base change \( V' \to U \) such that a general fiber becomes nodal, then a further base change \( V \to V' \) such that all fibers become semi-stable. The second step does not change general fibers. Therefore, for a general point \( u \in U \), denote by \( v^1 \in V' \) its inverse image in \( V' \), and two of its inverse image by \( v, v' \in V \). The fiber of the family of stable maps over \( v, v' \) consists of complete intersection curves and the nodal curve over \( v^1 \). They only differ by the complete intersection curves. So we can construct a deformation over a curve \( C \) from the fiber over \( v \) to the fiber over \( v' \) by deforming the complete intersection curves. □

Proof of Theorem 2.6. This is essentially [KT23, Corollary 59]. As in the proof of Theorem 2.5 we reduces the theorem to the case of an effective family. The point is, we can attach families of curves constructed in [KT23, Theorem 43] to the family (after a base change), so that the fiber (of the new family of curves) over the nodes of \( T, C \) and \( v_0, v_1 \) has unobstructed deformation. Thus the nodes in \( T, C \) and \( v_0, v_1 \) map to smooth points in the Hilbert scheme of \( X \times \mathbb{P}^3 \). We take \( W \) to be the normalization of the unique geometrically irreducible component containing the image of \( V, T, C \). In characteristic 0, we may even take a resolution of singularities of \( W \) that is isomorphic over the smooth locus. □

3. Lawson homology

Let \( X \) be a complex projective variety and we fix a very ample line bundle \( \mathcal{O}(1) \).

All the degree’s are taken with respect to this line bundle. Let \( \text{Chow}_{r,d}(X) \) be the Chow variety parameterizing degree \( d \), \( r \)-dimensional cycles of \( X \) and

\[
\text{Chow}_r(X) = \coprod_{d \geq 0} \text{Chow}_{r,d}(X),
\]

where \( \text{Chow}_{r,d}(X) \) is defined to be a single point corresponding to the zero cycle. We give the set \( \text{Chow}_r(X) \) the structure of a topological monoid, where the topological structure comes from the analytic topology on \( \text{Chow}_{r,d}(X) \) and the monoid structure is the sum of cycles. Define \( Z_r(X) \) to be the group completion of \( \text{Chow}_r(X) \). It has a topological group structure. The topology can be defined in several equivalent ways. These are studied by Lima-Filho [LF94].

Definition 3.1. We first define the category \( I^\text{eq} \). The objects are pairs \((S, \Gamma)\) consisting of a normal variety \( S \) and a family of equi-dimensional \( r \)-dimensional cycles \( \Gamma \), and whose morphisms between \((S, \Gamma)\) and \((S', \Gamma')\) are all the morphisms \( f : S \to S' \) such that \( \Gamma = f^* \Gamma' \).

Define the topological space \( Z_r(X)^\text{eq} \) as the colimit of all the topological spaces \( Z_r(X) \) over the category \( I^\text{eq} \).

More precisely, each \((S, \Gamma)\) in \( I^\text{eq} \) gives a map of sets \( \phi_S : S(\mathbb{C}) \to Z_r(X) \). The topology of \( Z_r(X)^\text{eq} \) is defined in such a way that a subset \( T \subset Z_r(X) \) is closed if and only if \( \phi^{-1}_S(T) \) is closed for all \((S, \Gamma)\).
Lemma 3.2. In the definition of $Z_r(X)^{eq}$, we may take a subset consisting of family of equidimensional cycles over normal projective varieties (or smooth projective varieties).

Proof. Given any family of equidimensional cycles $\Gamma \to S$, we may find a normal projective variety (resp. smooth projective variety) $T$, a family $\Gamma_T \to T$, and an open subset $T^0$ of $T$ such that there is a surjective proper map $p : T^0 \to S$ and $\Gamma_T|_{T^0}$ is $\Gamma \times_S T^0$.

Note that we have a factorization $T^0(C) \to S(C) \to Z_r(X)$. A set in $S(C)$ is closed if and only if its inverse image under $p^{-1}$ in $T^0(C)$ is closed. That is, the topology of $S(C)$ is the quotient topology coming from $T^0(C) \to S(C)$.

Thus the topology on $Z_r(X)^{eq}$ is determined by families over normal varieties (resp. smooth varieties) such that the family has an extension over a normal (resp. smooth) projective compactification.

Therefore, when defining $Z_r(X)^{eq}$ as a colimit, we may take only normal (resp. smooth) projective varieties. □

Definition 3.3. Define the topological space $Z_r(X)^{Chow}$ as the quotient of $\operatorname{Chow}_r(X)(\mathbb{C}) \times \operatorname{Chow}_r(X)(\mathbb{C})$ by $\operatorname{Chow}_r(X)(\mathbb{C})$, where the action is $(a, b) \mapsto (a + c, b + c)$ for $c \in \operatorname{Chow}_r(X)(\mathbb{C})$.

Theorem 3.4 ([LF94], Theorem 3.1, Theorem 5.2, Corollary 5.4). The identity map induces homeomorphisms $Z_r(X)^{eq} \cong Z_r(X)^{Chow}$.

Here is the definition of Lawson homology, first studied in [Law89].

Definition 3.5. Let $X$ be a complex projective variety. Define the Lawson homology $L_r H_{n+2r}(X)$ as the homotopy group $\pi_{n+2r}(Z_r(X))$.

Example 3.6 (Dold-Thom isomorphism). Consider $Z_0(X)$, the group of zero cycles on $X$. The classical Dold-Thom theorem implies that there is an isomorphism

$L_0 H_n(X) \cong H_n(X,\mathbb{Z})$.

Example 3.7 (Hurewitz map). The Hurewitz map is induced by the inclusion $X \to Z_0(X)$:

$\pi_k(X) \to \pi_k(Z_0(X)) \cong H_k(X,\mathbb{Z})$.

Theorem 3.8. Let $X$ be a complex smooth projective variety. Then for any loop $L$ in $Z_1(X)$, there is a projective algebraic set $Y$ with a family of nodal curves $\Gamma \to Y \times X$ over $Y$ such that the map

$\Phi : L_0 H_1(Y) = \pi_1(Z_0(Y)) \to L_1 H_3(X) = \pi_1(Z_1(X))$

induced by the family $\Gamma$ contains the class $[L]$ in $L_1 H_3(X)$. Assume furthermore that either $X$ is rationally connected or $X$ is a rationally connected fibration over a curve, we may take $Y$ to be smooth.

We first introduce some notations. Given a projective algebraic set $S$ parameterizing a family of one dimensional cycles of $X$, there is an induced continuous map between topological groups:

$Z_0(S) \to Z_1(X)$. 
We denote by $I(S)$ the image of this map, i.e. the closed subgroup of $Z_1(X)$ generated by the cycles over $S$, and $K(S)$ the kernal of this map.

The first observation in the proof of Theorem 3.8 is the following.

Lemma 3.9. Let $X$ be a complex smooth projective variety. For any class $[L]$ in $L_1H_3(X) = \pi_1(Z_1(X))$, there is a normal projective variety $U$ and a family of equidimensional one cycles $\gamma_U$ over $U$ such that $[L]$ is represented by a continuous map

$$I = [0,1] \to U \to Z_1(X).$$

Note that $I \to U$ is not a loop in general.

Proof. Denote by $Z_1(X)^0$ the neutral component of the topological group $Z_1(X)$, i.e., the connected component containing the identity. We may assume $L$ lies in $Z_1(X)^0$. Cycles in $Z_1(X)^0$ are precisely the cycles algebraically equivalent to 0. By Proposition 2.4 the topological group $Z_1(X)^0$ is a filtered colimit over closed subgroups generated by one-dimensional cycles parameterized by normal projective varieties.

Homotopy groups commute with filtered colimits. Thus there is an irreducible normal projective variety $S$ with a family of one dimensional cycles over $S$ such that the induced map

$$\pi_1(I(S)) \to \pi_1(Z_1(X)) \cong L_1H_3(X)$$

contains the class $[L]$ in $\pi_1(Z_1(X))$.

The fibration $K(S) \to Z_0(S) \to I(S)$ gives a long exact sequence of homotopy groups:

$$\ldots \to \pi_1(Z_0(S)) \to \pi_1(I(S)) \to \pi_0(K(S)) \to \ldots$$

A loop in $I(S)$ lifts to a continuous map from the unit interval $I = [0,1]$ to $Z_0(S)$, such that 0, 1 map to two points in $Z_0(S)$ that parameterize the same cycle in $X$.

We may assume that the family over $I$ is the difference of two families of effective 0-cycles of degree $d+ / d-$ in $S$. That is, it corresponds to the difference of two continuous maps $f^+ : I \to S^{(d+)}$, $f^- : I \to S^{(d-)}$, which is the same as a continuous map $f : I \to S^{(d+)} \times S^{(d-)}$ with 0 mapping to a point $x = (x^+, x^-)$ and 1 mapping to a point $y = (y^+, y^-)$.

A family of one cycles over $S$ induces a family of cycles over $S^{(d+)}$ and $S^{(d-)}$. Let us denote them by $\Gamma_{d+}, \Gamma_{d-}$.

The loop is the composition $I \to S^{(d+)} \times S^{(d-)} \to Z_0(S) \to Z_1(X)$, where the middle map is taking the difference.

Let us use a different family of cycles $\pi_{+}^{*}\Gamma_{d+} - \pi_{-}^{*}\Gamma_{d-}$ on the product $S^{(d+)} \times S^{(d-)}$, where $\pi_{+/-}$ is the projection to $S^{(d+)/(d-)}$. This family of cycles induces a continuous map $S^{(d+)} \times S^{(d-)} \to Z_1(X)$ such that the composition $I \to S^{(d+)} \times S^{(d-)} \to Z_1(X)$ is the loop $L$.

We take $U$ to be $S^{(d+)} \times S^{(d-)}$ and $\gamma_U$ to be $\pi_{+}^{*}\Gamma_{d+} - \pi_{-}^{*}\Gamma_{d-}$.

Proof of Theorem 3.8. By Lemma 3.9 there is a normal projective variety $U$ and a family of equidimensional one cycles $\gamma_U$ over $U$ such that $[L]$ is represented by a continuous map

$$f : I = [0,1] \to U \to Z_1(X).$$

Denote by $x, y \in U$ the image of 0, 1 by $f$. The cycle over $x, y$ are the same by assumption. Now we are in the set-up of Theorem 2.9. Thus there is a smooth
projective variety $V$ with a generically finite surjective morphism $V \to U$, a lifting $x_V, y_V$ of $x, y$ to $V$, and families of constant cycles parameterized by curves $T, C$ connecting $x_V, y_V$ and respectively inverse images of a general point in $S$. We take $Y$ to be $V \cup T \cup C$ in this case.

The morphism $V \to U$ induces a continuous map between topological groups $Z_0(V) \to Z_0(U)$. Denote by $K$ the kernel topological group. As a group, $K$ is generated by cycles of the form $a - b$, where $a, b$ are points in a fiber of $V \to U$. Note that $I(V) = I(U)$. Thus we have a fibration sequence of topological groups:

$$0 \to K \to K(V) \to K(U) \to 0.$$ 

We have commutative diagrams:

$$\begin{array}{ccc}
\pi_1(Z_0(Y)) & \longrightarrow & \pi_1(I(Y)) \\
\uparrow & & \uparrow \\
\pi_1(Z_0(V)) & \longrightarrow & \pi_0(K(V)) \longleftrightarrow \pi_0(K) \\
\downarrow & & \downarrow \\
\pi_1(Z_0(U)) & \longrightarrow & \pi_1(I(U)) \longrightarrow \pi_0(K(U))
\end{array}$$

The obstruction of lifting the class $[L]$ in $\pi_1(I(V))$ is in $\pi_0(K(V))$ and maps to $[x - y]$ in $\pi_1(K(U))$. The class $[x_V - y_V]$ differs from the obstruction class by an element in $\pi_0(K)$.

We take the stein factorization $V \to V' \to U$, where $V \to V'$ has connected fibers (hence birational) and $V' \to U$ is finite. Therefore $\pi_0(K)$ is finitely generated by classes of the form $[a - b]$, where $a, b$ are points in the fiber over a general point in $U$.

The class $[L]$ maps to $\pi_1(I(Y))$, with obstruction class the push-forward of $[x_V - y_V]$ modulo classes in $\pi_0(K)$.

By the existence of the families of constant cycles over the curves $T, C$ in Theorem 2.6, we have

1. The composition

$$\pi_0(K) \to \pi_0(K(V)) \to \pi_0(K(Y))$$

is the zero map.

2. The push-forward of the class $[x_V - y_V]$ vanishes in $\pi_0(K(Y))$.

Thus the class of the loop $L$ in $\pi_1(I(Y))$ is contained in $\pi_1(Z_0(Y))$.

Finally, if $X$ is rationally connected in codimension 1, by Theorem 2.6 all these families over $V, T, C$ come from pulling back from a family of cycles over a smooth projective variety. In this case, we take $Y$ to be this smooth projective variety. □

Now we introduce another ingredient.

**Lemma 3.10.** [FM04] Page 709, 1.2.1] There is a continuous map, the s-map: $Z_r(X) \wedge \mathbb{P}^1 \to Z_{r-1}(X)$ inducing the s-map on Lawson homologies $s : L_rH_k(X) \to L_{r-1}H_k(X)$. 

**Remark 3.11.** The construction of the s-map depends on a deep result: Lawson’s algebraic suspension theorem. A geometric way of describing the s-map is the following. Given a cycle $\Gamma$, take a general pencil of divisors $D_t(t \in \mathbb{P}^1)$ that intersect $\Gamma$ properly, and the s-map sends $([\Gamma], t)$ to the cycle $\Gamma \cdot D_t - \Gamma \cdot D_0$. 
**Definition 3.12.** Let $Y$ be a semi-normal variety. Let $Z \subset Y \times X$ be a family of $r$-dimensional cycle over $Y$ corresponding to a morphism $f : Y \to Z_r(X)$. We define the correspondence homomorphism

$$\Phi_f : H_k(Y, \mathbb{Z}) \to H_{k+2r}(X, \mathbb{Z})$$

as the composition

$$H_k(Y, \mathbb{Z}) \cong \pi_k(Z_0(Y)) \to \pi_k(Z_r(X)) \xrightarrow{\text{th}} \pi_{k+2r}(Z_0(X)) \cong H_{k+2r}(X, \mathbb{Z}),$$

where the map $\pi_k(Z_r(X)) \to \pi_{k+2r}(Z_0(X))$ is induced by $k$-th iterations of the $s$-map.

**Theorem 3.13 ([FM94] Theorem 3.4).** Let $Y$ be a smooth projective variety and $\Gamma \subset Y \times X$ be a family of $r$-dimensional cycle over $Y$ corresponding to a morphism $f : Y \to Z_r(X)$. We have

$$\Phi_f = \Gamma_* : H_k(Y, \mathbb{Z}) \to H_{k+2r}(X, \mathbb{Z}),$$

where $\Gamma_*$ is the map defined using $Z$ as a correspondence.

With this result, we can prove the main results over complex numbers.

**Theorem 3.14.** Let $X$ be a smooth projective variety. There is a projective curve $C$ with a nodal family of cycles $\Gamma \subset C \times X$ inducing a map $f : C \to Z_1(X)$ such that

$$\Phi_f : H_1(C, \mathbb{Z}) \to H_3(X, \mathbb{Z})$$

has the same image as the $s$-map $s : L_1H_3(X) \to H_3(X)$, which is the same as the coniveau filtration $N^1H_3(X, \mathbb{Z})$.

Furthermore, if $X$ is rationally connected in codimension 1, we may take $C$ to be a smooth projective curve, and $\Phi_f = \Gamma_*$. 

**Proof.** Recall that there is an isomorphism $L_0H_3(S) \cong H_k(S)$ for any projective algebraic set $S$ by the Dold-Thom theorem. We have a commutative diagram:

\[
\begin{array}{ccc}
L_0H_1(Y) & \xrightarrow{f_*} & L_1H_3(X) \\
\cong & \downarrow & \downarrow s \\
H_1(Y) & \xrightarrow{\Phi_f} & H_3(X)
\end{array}
\]

The image of the $s$-map is finitely generated. Thus we may find finitely many projective algebraic sets $Y_i$ and families of semistable curves $\Gamma_i \to Y_i \times X$ such that the generators of the image of $s$-map are contained in the image of the correspondence homomorphisms $\Phi_{i*} : H_1(Y_i) \to H_3(X)$. Then we take $Y$ to be the product $\Pi Y_i$ and $\Gamma = \sum \pi_i^*\Gamma_i$. Clearly $\Phi$ contains the image of all the $\Phi_{i*}$.

By taking general hyperplane sections containing all the singularities and all the one dimensional irreducible components, we may find a projective curve $C \subset Y$ such that $\pi_1(C_1) \to \pi_1(Y)$ is surjective. Then we simply restrict the family of cycles to $C$.

If $X$ is rationally connected in codimension 1, we may take all the $Y_i$‘s to be smooth by Theorem 3.5, and thus $C$ to be a general complete intersection of very ample divisors, which is a smooth projective curve.

Finally, we note that the image of the $s$-map

$$s : L_1H_3(X) \to H_3(X)$$
is $N^{1}H^{3}(X, \mathbb{Z})$ by [Wal07] Proposition 2.8]. □

The immediate consequence is the following.

**Theorem 3.15.** Let $X$ be a smooth projective rationally connected variety or a rationally connected fibration over a curve. Assume $X$ is a 3-fold. Then all the filtrations on $H^{3}(X, \mathbb{Z})$ introduced in Definition 1.18 equal the whole cohomology group:

$$\tilde{N}_{1, cpl, eq}H^{3}(X, \mathbb{Z}) = \tilde{N}_{1, cpl}H^{3}(X, \mathbb{Z}) = N^{1}H^{3}(X, \mathbb{Z}) = H^{3}(X, \mathbb{Z}).$$

**Proof.** By the decomposition of the diagonal argument,

$$L_{1}H_{k}(X) \otimes \mathbb{Q} \cong H_{k}(X, \mathbb{Q}) \cong H^{k}(X, \mathbb{Q}).$$

Thus we know that

$$s : L_{1}H_{3}(X) \rightarrow H_{3}(X, \mathbb{Z}) \cong H^{3}(X, \mathbb{Z})$$

is surjective by [Voi08, Corollary 3.1]. □

4. **Chow sheaves**

In this section we discuss the general case over an algebraically closed field $k$.

Sometimes we may “invert $p$”, by taking the tensor product of a sheaf with $\mathbb{Z}[\frac{1}{p}]$. In this scenario, we understand that $p$ is 1 if the base field has characteristic 0 and equal the characteristic otherwise. We use $\mathbb{Z}_{l}$ coefficient étale cohomology for $\ell$ a prime number, non-zero in the field $k$.

One can also define Lawson homology with $\mathbb{Z}_{l}$ coefficients in this context [Fri91]. But the construction of the analogue of $Z_{r}(X)$ is more complicated. Also Lawson homology in this context is much less studied. Many of the known results for complex varieties have not been explicitly stated to hold, even though one can imagine that they are still true. For example, the author do not know a reference for the construction of the s-map, Neither could the author find the analogue of Friedlander-Mazur’s result (Theorem 3.13) explicitly stated. If one had developed all the necessary results in this general context, presumably the argument in last section works the same way.

So we decided to use another approach for the general case.

**Definition 4.1.** A finite correspondence from $Y$ to $X$ is a family of relative cycles of dimension 0 with proper support over $Y$.

**Definition 4.2.** Let $\text{Sch}_{k}$ be the category of finite type separated $k$-schemes. Let $\text{Cor}_{k}$ be category whose objects are separated finite type $k$-schemes and morphisms finite correspondences. Let $\text{SmCor}_{k}$ be the full subcategory whose objects are smooth $k$-varieties. In this subcategory a finite correspondence between from $X$ to $Y$ is a linear combination of closed subvarieties of $X \rightarrow Y$ that are finite surjective onto one of the irreducible components of $X$.

Recall that the h-topology is generated by covers that are universal topological epimorphisms. Since we will only deal with noetherian schemes, this is the same as the topology generated by Zariski covers and covers that are proper surjective morphisms.

The qfh-topology is generated by covers in the h-topology that are also quasi-finite.

Later we will only use the fact that a surjective proper morphism is an h-cover.
Definition 4.3. We define the presheaf $Z_{\text{fl}}(X, r)$ on the category $\text{Sch}_k$ whose value on a scheme $S$ is a formal linear combination of integral subschemes $Z \subset S \times X$ that is flat, of equidimension $r$ over $S$.

We also define $Z_{\text{eq}}(X, r)$ on the category $\text{Sch}_k$ whose value on a scheme $S$ is the group of families of cycles in $X$ of equidimension $r$ over $S$.

We also define $Z(X, r)$ on the category $\text{Sch}_k$ whose value on a scheme $S$ is the group of families of cycles of dimension $r$ in $X$ over $S$.

We define $Z_{\text{fl}}^\text{eff}(X, r)$ (resp. $Z_{\text{eq}}^\text{eff}(X, r), Z^\text{eff}(X, r)$) on the category $\text{Sch}_k$ whose value on a scheme $S$ is the monoid of families of effective cycles in $X$ of equidimension $r$ over $S$.

Similarly, we define $C_{\text{fl}}(X, r), C_{\text{eq}}(X, r), C(X, r), C_{\text{fl}}^\text{eff}(X, r), C_{\text{eq}}^\text{eff}(X, r), C^\text{eff}(X, r)$ as the counterpart of the above presheaves for families of cycles with proper support over $S$.

The sheaf $C_{\text{fl}}(X, r)$ is denoted by $\mathbb{Z}\text{PropHilb}$ in [SV00].

Since later we will consider cycles on proper schemes, with the purpose of keeping the names consistent with previous section, we will use $Z_{\text{fl}}(X, r)$ etc. notations.

Note that we do not require the subschemes to be equidimensional over $S$ in the definition of $Z(X, r)$ and $C(X, r)$. It is possible to have higher dimensional fibers ([SV00] Example 3.1.9]). However, $Z^\text{eff}(X, r)$ is the same as $Z_{\text{fl}}^\text{eff}(X, r)$. Similarly for the properly supported version.

We have the following.

Proposition 4.4. [SV00] Proposition 4.2.7, 4.2.6, Lemma 4.2.13] The presheaf $Z_{\text{eq}}(X, r) \otimes \mathbb{Z}[\frac{1}{p}]$ is a qfh-sheaf and the presheaf $Z(X, r) \otimes \mathbb{Z}[\frac{1}{p}]$ is an $h$-sheaf. Moreover, the sheafification in the $h$ topology of $Z_{\text{eq}}(X, r)$ is the same as that of $Z(X, r)$.

In the following, we write $Z_{\text{fl}}^h(X)$ as the $h$-sheaf associated to $Z_{\text{fl}}(X, r)$ (which is the same as that of $Z_{\text{eq}}(X, r)$ or $Z(X, r)$).

In the following, given a presheaf $\mathcal{F}$, we use $C^\ast(\mathcal{F})$ to denote the Suslin complex of presheaves (with non-positive degrees). That is, $C^{-i}(\mathcal{F})(S) = \mathcal{F}(S \times \Delta^i)$, where $\Delta^i = \text{Spec} k[t_0, \ldots, t_i]/\langle \sum t_j = 1 \rangle$ is the algebraic $i$-simplex.

If $\mathcal{F}$ is a torsion, homotopy invariant étale sheaf with transfers, or a qfh, or $h$ sheaf on the category of schemes over $X$, with torsion order prime to $p$, the Suslin rigidity theorem [SV96] Theorem 4.5] implies that $\mathcal{F}$ is a locally constant sheaf. Since we work over an algebraically closed field, locally constant is the same as constant. Thus for any torsion sheaf $\mathcal{G}$ with torsion order prime to $p$, its Suslin complex $C^\ast(\mathcal{G})$ is isomorphic to the pull-back of a complex of locally constant sheaves. Moreover if $\mathcal{F}$ is a constant étale sheaf, we have isomorphisms ([SV96 Theorem 10.2, 10.7])

$$H^i_{\text{et}}(X, \mathcal{F}) \cong H^i_{\text{qfh}}(X, \mathcal{F}^{\text{qfh}}) \cong H^i_h(X, \mathcal{F}^h).$$

Since we assume that $k$ is algebraically closed, Spec $k$ has no higher cohomology for any sheaf in any of these three topologies. Therefore for any complex of constant sheaves, we also have the isomorphism of cohomologies for $X = \text{Spec} k$. In particular, above discussions apply to the complex $C^\ast(Z_{\text{eq}}(X, r)) \otimes \mathbb{Z}/NZ$. We may identify the cohomology of this complex.

Theorem 4.5. Let $X$ be a quasi-projective variety defined over an algebraically closed field $k$. Let $N$ be an integer, non-zero in the field. We have the following
isomorphisms.

\[ H^i_h(\text{Spec } k, C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{qfh}(\text{Spec } k, C^*(Z_{eq}^h(X, r) \otimes \mathbb{Z}/N\mathbb{Z})) \]

\[ \cong H^i_{\text{et}}(\text{Spec } k, C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{\text{Ab}}(C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z}))(\text{Spec } k) \]

\[ \cong \text{CH}_i(X, -i, \mathbb{Z}/N\mathbb{Z}). \]

**Proof.** The first three cohomology groups are isomorphic as discussed above. They are all equal to the cohomology of the complex \( C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z})(\text{Spec } k) \), since this complex computes the cohomology group in the qfh and étale topology.

Finally, under the hypothesis that resolution of singularities holds, Suslin [Sus00, Theorem 3.2] proves that for any quasi-projective variety \( X \), we have an isomorphism

\[ H^i_{\text{et}}(C^*(Z_{eq}(X, r) \otimes \mathbb{Z}/N\mathbb{Z}))(\text{Spec } k) \cong \text{CH}_r(X, -i, \mathbb{Z}/N\mathbb{Z}). \]

Using Gabber’s refinement of de Jong’s alteration theorem, Kelly [Kel17, Theorem 5.6.4] removed the resolution of singularities hypothesis. \( \square \)

**Corollary 4.6.** Let \( X \) be a quasi-projective variety defined over an algebraically closed field \( k \). Let \( N \) be an integer, non-zero in the field. We have the following isomorphisms.

\[ H^i_{\text{et}}(\text{Spec } k, C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{qfh}(\text{Spec } k, C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z})) \]

\[ \cong H^i_h(\text{Spec } k, C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z})) \cong H^i_{\text{Ab}}(C^*(Z_{eq}(X, 0) \otimes \mathbb{Z}/N\mathbb{Z}))(\text{Spec } k) \]

\[ \cong \text{CH}_0(X, -i, \mathbb{Z}/N\mathbb{Z}) \cong H^{2d-i}_{\text{et}}(X, \mathbb{Z}/N\mathbb{Z}), \]

where \( d \) is the dimension of \( X \). In particular, all the groups are finite.

**Proof.** The last equality follows from [SV96, Corollary 7.8] and [Gei00, Theorem 3.5], [Kel17, Theorem 5.6.1]. Clearly the étale cohomology group is finite. \( \square \)

We use \( A_1(X) \) to denote the group of one cycles in \( X \) modulo algebraic equivalence. For any abelian group \( A \) and any integer \( m \), we use \( A/m \) to denote the group of \( m \)-torsions in \( A \), and \( A/mA \) to denote the quotient \( A/mA \).

For any integer \( N \) invertible in the field \( k \), we have a homomorphism

\[ \text{CH}_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to \text{CH}_1(X, 0, \mathbb{Z})[N] \]

that comes from the long exact sequence of higher Chow groups with \( \mathbb{Z} \) and \( \mathbb{Z}/N \) coefficients. Composing with the surjective map

\[ \text{CH}_1(X, 0, \mathbb{Z})[N] \to A_1(X)[N], \]

we have a homomorphism

\[ \text{CH}_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to A_1(X)[N]. \]

Now we can state the counterpart of Theorem 3.8

**Theorem 4.7.** Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) of characteristic \( p \). Fix an integer \( N \) that is invertible in \( k \). For any class \( |L| \) in the kernel of the map

\[ H^1_h(\text{Spec } k, C^*(Z_{eq}^h(X)) \otimes \mathbb{Z}/N\mathbb{Z}) \cong \text{CH}_1(X, 1, \mathbb{Z}/N\mathbb{Z}) \to A_1(X)[N], \]
there is a projective algebraic set \( Z \) and a family of one-dimensional nodal curves over \( Z \) such that the class \([L]\) is in the image of
\[
H^{-1}_{\text{cl}}(\text{Spec } k, C^*(Z_0^1(Z)) \otimes \mathbb{Z}/NZ) \to H^{-1}_{\text{cl}}(\text{Spec } k, C^*(Z_1^1(X)) \otimes \mathbb{Z}/NZ)
\]
induced by this family of cycles. Assume furthermore that \( X \) is separably rationally connected in codimension 1. We may take \( Z \) to be normal.

**Remark 4.8.** This result is a priori weaker than Theorem 3.8 over complex numbers. We have a short exact sequence
\[
0 \to L_1H_3(X)/N \to L_1H_3(X, \mathbb{Z}/N) \to L_1H_2(X)[N] \to 0.
\]
Since \( L_1H_3(X, \mathbb{Z}/N) \cong CH_1(X, 1, \mathbb{Z}/N) \) and \( L_1H_2(X)[N] \cong A_1(X)[N] \), Theorem 4.7 only says that classes in \( L_1H_3(X)/N \) comes from a smooth projective variety.

But if we know that \( L_1H_3(X) \) is finitely generated, then we can find the lift. Conjecturally, this group is isomorphism to \( H_3(X, \mathbb{Z}) \), thus finitely generated.

The proof of Theorem 4.7 is analogous to that of Theorem 3.8. We first prove the analogue of Lemma 3.9.

**Lemma 4.9.** Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) of characteristic \( p \). Assume that \( X \) is either a separably rationally connected variety or a separably rationally connected fibration over a curve. Fix an integer \( N \) that is invertible in \( k \). For any class \([L]\) in
\[
H^{-1}_{\text{cl}}(\text{Spec } k, C^*(Z_1^1(X)) \otimes \mathbb{Z}/NZ) \cong CH_1(X, 1, \mathbb{Z}/NZ),
\]
there is a normal projective variety \( U \), a family of equidimensional one cycles \( \gamma_U \) over \( U \) and a morphism \( f : \Delta^1 \to U \) such that \([L]\) is represented by \( f^*\gamma_U \) over \( \Delta^1 \).

**Proof.** We could translate the proof of Theorem 3.8 in the context of h-sheaves. But here is an easier argument using Hilbert schemes.

The class \([L]\) is represented by a family of cycles \( \sum_i m_i \Gamma_i, m_i \in \mathbb{Z}/N \) over \( \Delta^1 \), where \( \Gamma_i \subset \Delta^1 \times X \) is an integral subvariety. Since \( \Delta^1 \) is one dimensional, the projection \( \Gamma_i \to \Delta^1 \) is flat. Thus we get a morphism \( f_i \) from \( \Delta^1 \) to the Hilbert scheme. The universal subscheme over the Hilbert scheme gives a family of cycles over the Hilbert scheme. Therefore, we may take \( U \) to be the normalization of of products of irreducible components of the Hilbert scheme and \( \gamma_U \) to be the family of cycles (with appropriate multiplicity) coming from universal subschemes. \( \square \)

We will need the following observation later in the proof.

**Lemma 4.10.** Let \( T \) be a connected projective algebraic set over an algebraically closed field \( k \), and let \( x, y \) be two points in \( T \). Let \( \mathcal{F} \) be a sheaf of abelian groups in the qfh or h topology, or an étale sheaf with transfers. Fix an integer \( N \) invertible in \( k \). Write \( F_x \) (resp. \( F_y \)) the restriction of \( F \in \mathcal{F} \otimes \mathbb{Z}/NZ(T) \) to \( x \) (resp. \( y \)). Then \( F_x = F_y \) in \( H^0(\text{Spec } k, C^*(\mathcal{F}) \otimes \mathbb{Z}/NZ) \), where the cohomology in taken in the étale topology, qfh topology or h topology.

**Proof.** Elements in \( \mathcal{F} \otimes \mathbb{Z}/NZ(T) \) induces a unique morphism
\[
Z_0(T) \otimes \mathbb{Z}/N \to \mathcal{F} \otimes \mathbb{Z}/NZ.
\]
If \( \mathcal{F} \) is a sheaf with transfers, this is the Yoneda Lemma. If \( \mathcal{F} \) is a qfh sheaf or h sheaf, this follows from the fact that the qfh sheafification of \( Z_0(T)_{\frac{1}{p}} \) is the free
sheaf $\mathcal{Z}[\gamma](T)$ generated by the presheaf of sets $\text{Hom}(\gamma, T)$ (SV96 Theorem 6.7). Thus the class $[\mathcal{F}_x]$ (resp. $[\mathcal{F}_y]$) is the image of $[x]$ (resp. $[y]$) under the map 

$$H^0(\text{Spec } k, C^*(Z_0(T)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H^0(\text{Spec } k, C^*(\mathcal{F}) \otimes \mathbb{Z}/N\mathbb{Z}).$$

So it suffices to show that $[x] = [y]$ in $H^0(\text{Spec } k, Z_0(T) \otimes \mathbb{Z}/N\mathbb{Z})$. But the latter cohomology group is $CH_0(T) \otimes \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/N\mathbb{Z}$ by Lemma 4.6 and the isomorphism is given by the degree map. Any two points $x, y$ give the same class in $H^0(\text{Spec } k, Z_0(T) \otimes \mathbb{Z}/N\mathbb{Z})$. \hfill $\blacksquare$

Now we begin the proof Theorem 4.7

**Proof of Theorem 4.7.** Given a normal projective variety $S$ parameterizing a family of one dimensional cycles of $X$, there is an induced morphism of h-sheaves:

$$Z^h_0(S) \to Z^h(X).$$

We denote by $I(S)$ (resp. $K(S)$) the image h-sheaf (resp. kernel h-sheaf) of this map.

By Lemma 4.3, there is a normal projective variety $U$, a family of equidimensional one cycles $\gamma_U$ over $U$ and a morphism $f: \Delta^1 \to U$ such that $[L]$ is represented by $\pi^*\gamma_U$ over $\Delta^1$.

Denote by $\gamma_0, \gamma_1$ the restriction of the family of cycles $\gamma_U$ over $U$ to $0, 1 \in \Delta^1$. Then $\gamma_0 - \gamma_1 = N(\gamma_{0,1})$ for some cycle $\gamma_{0,1}$. The image of $[L]$ in $CH_1(X, 0, \mathbb{Z})[N]$ and $A_1(X)[N]$ is the class of $\gamma_{0,1}$.

If $\gamma_{0,1}$ is zero in $A_1(X)[N]$, that is, if $\gamma_{0,1}$ is algebraically equivalent to 0, then by Proposition 2.4, there is a smooth projective curve $D$ with a family of cycles $\gamma_D$ and two points $d, d'$ such that $\Gamma_d$ is 0 and $\Gamma_{d'}$ is $\gamma_{0,1}$.

Consider the product $U \times D$. We have a family of cycles $\gamma = \pi_U^*\gamma_U + N\pi_D^*\gamma_D$.

There are three points in $S = U \times D$,

$$x = (f(0), d), y = (f(1), d), z = (f(1), d')$$

such that

1. $\gamma_x = \gamma_z = \gamma_0$.
2. There is a curve $C$ containing $y, z$ such that for every point $c \in C$, the cycle $\gamma_c$ equals $\gamma_1$ in $Z_1(X) \otimes \mathbb{Z}/N(\text{Spec } k)$.

As in the proof of Theorem 3.8, we apply the second part of Theorem 2.6 to find a normal projective variety $V$, a projective algebraic set $Y$ with a surjective projective morphism $V \to S$ and an embedding $V \to Y$, and liftings $x_V, y_V, z_V$ of the points $x, y, z$ such that

1. The two points $x_V$ and $z_V$ are connected by a chain of curves in $Y$ parameterizing constant cycles.
2. The two points $y_V$ and $z_V$ are connected by a curve $D_V$ in $V$ parameterizing cycles divisible by $N$.

Denote by $K$ the kernal of the morphism between h sheaves

$$Z_0(V) \to Z_0(S).$$

Here $V \to S$ is proper and surjective. So the above morphism of h sheaves is surjective. Then we have an isomorphism of h sub-sheaves of $Z_1(X)$

$$I(V) \cong I(S).$$
It follows that we have a short exact sequence of sheaves:

$$0 \to K \to K(V) \to K(S) \to 0.$$ 

We have commutative diagrams:

$$\begin{array}{c}
H^{-1}_h(C^*(Z_0(Y))/N) \longrightarrow H^{-1}_h(C^*(I(Y))/N) \longrightarrow H^0_h(C^*(K(Y))/N) \\
\uparrow \quad \uparrow \quad \uparrow \\
H^{-1}_h(C^*(Z_0(V))/N) \longrightarrow H^{-1}_h(C^*(I(V))/N) \longrightarrow H^0_h(C^*(K(V))/N) \\
\downarrow \quad \downarrow \quad \downarrow \\
H^{-1}_h(C^*(Z_0(S))/N) \longrightarrow H^{-1}_h(C^*(I(S))/N) \longrightarrow H^0_h(C^*(K(S))/N)
\end{array}$$

The obstruction of lifting the class \([L]\) in \(H^{-1}_h(\text{Spec } k, C^*(I(V)) \otimes \mathbb{Z}/N\mathbb{Z})\) is in \(H^0_h(\text{Spec } k, C^*(K(V)))\) and maps to \([x - y]\) in \(H^0_h(\text{Spec } k, C^*(K(S)) \otimes \mathbb{Z}/N\mathbb{Z})\). The class \([x_T - y_T]\) differs from the obstruction class by an element in \(H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})\).

Given the morphism \(V \to S\), we have a long exact sequence

$$\ldots \to H^{-1}_h(\text{Spec } k, C^*(Z_0(V)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H^{-1}_h(\text{Spec } k, C^*(Z_0(S)) \otimes \mathbb{Z}/N\mathbb{Z}) \to H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z}) \to \ldots$$

Therefore \(H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})\) is finitely generated by Corollary \(4.12\). By Lemma \(4.12\) any class in \(H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})\) is equivalent to the class of the form \([a - b]\), where \(a, b\) are points in the fiber over a general point in \(S\).

The class \([L]\) maps to \(H^{-1}_h(\text{Spec } k, C^*(I(V)) \otimes \mathbb{Z}/N\mathbb{Z})\), with obstruction class the push-forward of \([x_V - y_V]\) modulo classes in \(H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/N\mathbb{Z})\).

By the existence of the families of constant cycles in the second part of Theorem \(2.6\) and by Lemma \(4.10\) we have

1. The composition

\(H^0_h(\text{Spec } k, C^*(K))/N) \to H^0_h(\text{Spec } k, C^*(K(V))/N) \to H^0_h(\text{Spec } k, C^*(K(Y))/N)\)

is the zero map.

2. The push-forward of the class \([x_V - y_V]\) vanishes in \(H^0_h(\text{Spec } k, C^*(K(Y)) \otimes \mathbb{Z}/N\mathbb{Z})\) (by applying Lemma \(4.10\) to \([x_V - y_V]\) and \([x_V - x_V] = 0\)).

Thus the class \([L]\) in \(H^{-1}_h(\text{Spec } k, C^*(I(V)))/N)\) comes from \(H^{-1}_h(\text{Spec } k, C^*(Z_0(Y))/N)\).

Finally, if \(X\) is separably rationally connected in codimension 1, we may take \(Y\) to be normal by Theorem \(2.6\). We use Gabber’s refinement of de Jong’s alteration to find a smooth projective variety \(Z\) and a projective alteration \(Z \to Y\) whose degree is relatively prime to \(N\). Then

\(\text{CH}_0(\mathbb{Z}, 1, \mathbb{Z}/N\mathbb{Z}) \to \text{CH}_0(\mathbb{Y}, 1, \mathbb{Z}/N\mathbb{Z})\)

is surjective by Lemma \(4.13\) Pulling back the families of cycles over \(Y\) gives a family of cycles over \(Z\). Then the theorem follows from the following commutative diagram

$$\begin{array}{c}
\text{CH}_0(Z, 1, \mathbb{Z}/N) \longrightarrow \text{CH}_0(Y, 1, \mathbb{Z}/N) \longrightarrow \text{CH}_1(X, 1, \mathbb{Z}/N) \\
\hat{=} \downarrow \quad \hat{=} \downarrow \quad \hat{=} \downarrow \\
H_1(Z, \mathbb{Z}/N) \longrightarrow H_1(Y, \mathbb{Z}/N) \longrightarrow H_3(X, \mathbb{Z}/N).
\end{array}$$
Let $X \to Y$ be a flat and finite morphism defined over an algebraically closed field $k$, where $Y$ is a normal variety (but $X$ is not necessarily normal). Let $N$ be an integer invertible over $k$. Denote by $K$ the kernel sheaf of $Z_0(X) \to Z_0(Y)$. Then for any chosen general point in $Y$, $H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/NZ)$ is generated by class of the form $[t_1 - t_2]$, for $t_1, t_2$ in the fiber of this chosen general point in $Y$.

Proof. Let $x_1, x_2$ be two points in the fiber over $y \in Y$. Clearly $H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/NZ)$ is generated by class of the form $[x_1 - x_2]$ for all the pairs of points with the same image in $Y$. We will show that for any chosen general point $t \in Y$, the class $[x_1 - x_2]$ is equivalent to a class $[t_1 - t_2]$ for some points $t_1, t_2$ in the fiber over $t$. Consider the correspondence $X \times_Y X \subset X \times X$. Since $X \to Y$ is assumed to be flat, $X \times_Y X \to X$ is flat. We take an irreducible component $D$ containing $(x_1, x_2)$, which dominates (and thus surjects onto) $X$. There are two points $x_D, t_D$ in $D$ such that the following conditions are satisfied.

1. There is a surjective morphism $f : D \to Y$ that maps $x_D$ (resp. $t_D$) to $y$ (resp. $t$).
2. There are two morphisms $f_1, f_2 : D \to X$ such that $f_1(x_D) = x_1, f_2(x_D) = x_2$.
3. The composition of $f_1, f_2$ with the morphism $g : X \to Y$ gives the morphism $f : D \to Y$.

Then by Lemma 4.10, the class $[x_1 - x_2]$ is the same as $[f_1(t_D) - f_2(t_D)]$.

Lemma 4.12. Let $p : X \to Y$ be a generically finite surjective morphism between normal projective varieties over an algebraically closed field $k$. Let $N$ be an integer invertible over $k$. Denote by $K$ the kernel sheaf of $Z_0(X) \to Z_0(Y)$. Then the cohomology group $H^0_h(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/NZ)$ is generated by class of the form $[t_1 - t_2]$, for $t_1, t_2$ in the fiber of any chosen general point in $Y$.

Proof. There is a birational projective variety morphism $Y' \cong Y$ such that the strict transform $X'$ of $X$ is flat over $Y'$. That is, we have a commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{q'} & X \\
p' \downarrow & & \downarrow p \\
Y' & \xrightarrow{q} & Y
\end{array}
$$

We denote by $K(p)$ etc. to denote the kernel sheaf of $Z_0(X) \to Z_0(Y)$ etc.. There is a commutative diagram of short exact sequences of sheaves:

$$
\begin{array}{cccccc}
0 & \longrightarrow & K(q') & \longrightarrow & Z_0(X') & \xrightarrow{q'} & Z_0(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K(q) & \longrightarrow & Z_0(Y') & \xrightarrow{q} & Z_0(Y) & \longrightarrow & 0
\end{array}
$$

which also gives commutative diagrams after tensoring with $\mathbb{Z}/NZ$. Then we have long exact sequences:

$$
CH_1(X, 1, \mathbb{Z}/NZ) \to CH_1(Y, 1, \mathbb{Z}/NZ) \to H^0_h(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/NZ)\ldots
$$
The cohomology group $H^0_k(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/\mathbb{Z})$ (resp. $H^0_k(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/\mathbb{Z})$) is generated by cycles of the form $y_1 - y_2$ for $y_1, y_2$ in the same fiber. So it suffices to show that such cycles are zero.

We first show that $\text{CH}_0(Y', 1, \mathbb{Z}/\mathbb{Z}) \to \text{CH}_0(Y, 1, \mathbb{Z}/\mathbb{Z})$ is surjective. This is because $Y' \to Y$ has connected fibers. So for any two points in the same fiber, by Lemma 4.10, the class of the difference is zero. Since

$\text{CH}_0(Y', 0, \mathbb{Z}/\mathbb{Z}) \to \text{CH}_0(Y, 0, \mathbb{Z}/\mathbb{Z})$

is an isomorphism, we know that $H^0_k(\text{Spec } k, C^*(K(q)) \otimes \mathbb{Z}/\mathbb{Z})$ vanishes.

By the same argument, $H^0_k(\text{Spec } k, C^*(K(q')) \otimes \mathbb{Z}/\mathbb{Z})$ vanishes. Then a simple diagram chasing shows that $H^0_k(\text{Spec } k, C^*(K(p')) \otimes \mathbb{Z}/\mathbb{Z}) \to H^0_k(\text{Spec } k, C^*(K(p)) \otimes \mathbb{Z}/\mathbb{Z})$ is surjective.

Thus the statements follow form Lemma 4.11.

Lemma 4.13. Let $p : X \to Y$ be a generically finite morphism between normal projective varieties over an algebraically closed field $k$. Let $N$ be an integer invertible over $k$. Then we have a surjection

$\text{CH}_0(X, 1, \mathbb{Z}/\mathbb{Z}) \to \text{CH}_0(Y, 1, \mathbb{Z}/\mathbb{Z}).$

Proof. By Lemma 4.12 and the long exact sequence

$\text{CH}_0(X, 1, \mathbb{Z}/\mathbb{Z}) \to \text{CH}_0(Y, 1, \mathbb{Z}/\mathbb{Z}) \to H^0_k(\text{Spec } k, C^*(K)/N)$

it suffices to show that for a general point $y \in Y$ and any two points $x_1, x_2$ in the fiber of $y$, the class $[x_1 - x_2]$ is zero in $H^0_k(\text{Spec } k, C^*(K)/N)$.

By the Bertini theorem for étale fundamental groups, there is a general complete intersection curve $H$ such that the inverse image $H'$ in $Y'$ is irreducible. For $H$ general, the morphism $H' \to H$ is flat and finite of degree prime to $N$. Thus

$\text{CH}_0(H', 1, \mathbb{Z}/\mathbb{Z}) \to \text{CH}_0(H, 1, \mathbb{Z}/\mathbb{Z}) \to H^0_k(\text{Spec } k, C^*(K_H) \otimes \mathbb{Z}/\mathbb{Z})$

is an isomorphism. On the other hand, since $p : H' \to H$ is flat and finite, we have pull-back and push-forward on all the higher Chow groups. The composition of pull-back and push-forward

$\text{CH}_0(H, 1, \mathbb{Z}/\mathbb{Z}) \xrightarrow{p^*} \text{CH}_0(H', 1, \mathbb{Z}/\mathbb{Z}) \xrightarrow{p_*} \text{CH}_0(H, 1, \mathbb{Z}/\mathbb{Z})$ is multiplication by $\deg p$. Since the degree of the map is relatively prime to $N$,

$\text{CH}_0(H', 1, \mathbb{Z}/\mathbb{Z}) \xrightarrow{\partial} \text{CH}_0(H, 1, \mathbb{Z}/\mathbb{Z})$

is surjective. Thus for any two points $t_1, t_2$ over a general point $t \in Y$, the class $[t_1 - t_2]$ vanishes in $H^0_k(\text{Spec } k, C^*(K_H) \otimes \mathbb{Z}/\mathbb{Z})$. So does its push-forward in $H^0_k(\text{Spec } k, C^*(K) \otimes \mathbb{Z}/\mathbb{Z})$. 

\qed
Fix a prime number $\ell$ different from the characteristic of $k$. In the following theorem, we omit all the Tate twists for simplicity of notations.

**Theorem 4.14.** Let $X$ be a smooth projective variety defined over an algebraically closed field, which is separably rationally connected in codimension 1. There is a smooth projective curve $C$ with a family of 1-dimensional cycles $\Gamma \subset C \times X$ such that

$$\Gamma_* : H^1_{BM}(C, \mathbb{Z}_\ell) \to H^3_{BM}(X, \mathbb{Z}_\ell)$$

surjects onto $N^1H_3(X, \mathbb{Z}_\ell)$.

**Proof.** In the following, we use Borel-Moore homology. For simplicity of notations, we only write them as $H_1, H_3$. Let $NH_3(X, \mathbb{Z}/\ell^n)$ be the coniveau filtration on the homology $N^1H_3(X, \mathbb{Z}/\ell^n)$. Denote by $\tilde{NH}_3(X, \mathbb{Z}/\ell^n)$ the strong coniveau filtration $N^1H_3(X, \mathbb{Z}/\ell^n)$.

For a smooth projective variety $Y$, we have $H_1(Y, \mathbb{Z}_\ell)/\ell^n \cong H_1(Y, \mathbb{Z}/\ell^n)$, since $H_0(Y, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ is torsion free. Therefore, $\tilde{NH}_3(X, \mathbb{Z}_\ell)/\ell^n \to \tilde{NH}_3(X, \mathbb{Z}/\ell^n)$ is surjective.

We have a commutative diagram

$$
\begin{array}{cccc}
\oplus_{(S, \gamma_S)} CH_0(S, 1, \mathbb{Z}/\ell^n) & \xrightarrow{\oplus \Gamma_S*} & CH_1(X, 1, \mathbb{Z}/\ell^n) & \xrightarrow{A_1(X)[\ell^n]} 0 \\
0 & \rightarrow & H_3(X, \mathbb{Z}_\ell)/\ell^n \cap NH_3(X, \mathbb{Z}/\ell^n) & \rightarrow \tilde{NH}_3(X, \mathbb{Z}/\ell^n) & \rightarrow H_2(X, \mathbb{Z}_\ell)[\ell^n]
\end{array}
$$

where the direct sum is taken over families of equidimensional one cycles over smooth projective varieties.

By Theorem 4.13, the upper row is exact. The lower row is also exact, since it comes from

$$0 \rightarrow H_3(X, \mathbb{Z}_\ell)/\ell^n \rightarrow H_3(X, \mathbb{Z}/\ell^n) \rightarrow H_2(X, \mathbb{Z}_\ell)[\ell^n] \rightarrow 0.$$ 

The vertical maps are cycle class maps.

The middle vertical map

$$CH_1(X, 1, \mathbb{Z}/\ell^n) \rightarrow NH_3(X, \mathbb{Z}/\ell^n)$$

is surjective, since for any surface $\Sigma$, not necessarily smooth, we have a surjection

$$CH_1(\Sigma, 1, \mathbb{Z}/\ell^n) \rightarrow H_3(\Sigma, \mathbb{Z}/\ell^n).$$

When this surface is smooth, it is a consequence of the Bloch-Kato conjecture. The general case can be proven using the localization sequence for higher Chow groups and Borel-Moore homology.

The left vertical arrow is the direct sum of the composition

$$CH_0(S, 1, \mathbb{Z}/\ell^n) \rightarrow H_1(S, \mathbb{Z}/\ell^n) \cong H_1(S, \mathbb{Z}_\ell)/\ell^n \xrightarrow{\Gamma_S*} H_3(X, \mathbb{Z}_\ell)/\ell^n,$$

Since the cycle class map induces an isomorphism $CH_0(S, 1, \mathbb{Z}/\ell^n) \cong H_1(S, \mathbb{Z}/\ell^n) \cong H_1(S, \mathbb{Z}_\ell)/\ell^n$, the left vertical arrow has the same cokernel as

$$\tilde{NH}_3(X, \mathbb{Z}_\ell)/\ell^n \rightarrow H_3(X, \mathbb{Z}_\ell)/\ell^n \cap NH_3(X, \mathbb{Z}/\ell^n).$$

By the snake lemma, this cokernel is isomorphic to the cokernel $C_n$ of

$$\text{Ker}(CH_1(X, 1, \mathbb{Z}/\ell^n) \rightarrow H_3(X, \mathbb{Z}/\ell^n)) \rightarrow \text{Ker}(A_1[\ell^n] \rightarrow H_2(X, \mathbb{Z}_\ell)[\ell^n]).$$
The connecting maps $C_{n+m} \to C_n$ are multiplication by $\ell^m$. Therefore the inverse limit $\lim \nolimits_{\ell} C_n$ is torsion free.

We have a factorization
\[
\tilde{N} H_3(X, \mathbb{Z}_\ell)/\ell^n \to N H_3(X, \mathbb{Z}_\ell)/\ell^n \to H_3(X, \mathbb{Z}_\ell)/\ell^n \cap N H_3(X, \mathbb{Z}/\ell^n) \subset H_3(X, \mathbb{Z}_\ell)/\ell^n.
\]
Taking inverse limit, we get
\[
\tilde{N} H_3(X, \mathbb{Z}_\ell) \to N H_3(X, \mathbb{Z}_\ell) \to \lim \nolimits_{\ell^n} H_3(X, \mathbb{Z}_\ell)/\ell^n \cap N H_3(X, \mathbb{Z}/\ell^n) \subset H_3(X, \mathbb{Z}_\ell).
\]
Therefore the map
\[
N H_3(X, \mathbb{Z}_\ell)/\ell^n \cap N H_3(X, \mathbb{Z}/\ell^n)
\]
is injective. Since the cokernal of $N H_3(X, \mathbb{Z}_\ell)/\ell^n \to \lim \nolimits_{\ell^n} H_3(X, \mathbb{Z}_\ell)/\ell^n \cap N H_3(X, \mathbb{Z}/\ell^n)$
is torsion free, so is the cokernal of $\tilde{N} H_3 \to N H_3$. On the other hand, we know that the cokernal is torsion. So it has to be zero. That is, the strong coniveau filtration coincide with the coniveau filtration.

Therefore there is a smooth projective variety $Y$ and a family of cycles $\Gamma$ such that the induced map
\[
\Gamma_{Y*} : H_3(Y, \mathbb{Z}_\ell) \to N^1 H_3(X, \mathbb{Z}_\ell)
\]
is surjective. By taking hyperplane sections in $Y$, we may find a smooth projective curve $C$ with a family of cycles $\Gamma$ such that
\[
\Gamma_* : H_3(C, \mathbb{Z}_\ell) \to N^1 H_3(X, \mathbb{Z}_\ell)
\]
is surjective.

Finally, for later use, we note that in the proof, we also prove that for $X$ SRC in codimension $1$, the maps
\[
(5) \quad \tilde{N}^1 H_3(X, \mathbb{Z}_\ell) \to N^1 H_3(X, \mathbb{Z}_\ell) \to \lim \nolimits_{\ell^n} H_3(X, \mathbb{Z}_\ell)/\ell^n \cap N H_3(X, \mathbb{Z}/\ell^n)
\]
are isomorphisms. The first isomorphism is already shown above. We have already shown that the second map is injective. The cokernal is torsion since the cokernal of $N H_3 \to H_3$ is torsion. By the first isomorphism and the fact that the composition has torsion free cokernal, the cokernal of the second map is also torsion free, and thus zero. \(\square\)

**Remark 4.15.** When $X$ is only smooth projective, one can prove that the filtration $N_{1,cyl,et} H_3(X, \mathbb{Z}_\ell)$ is the same as $N^1 H_3(X, \mathbb{Z}_\ell)$ by the same argument.

**Theorem 4.16.** Let $X$ be a smooth projective 3-fold over an algebraically closed field. Assume that $X$ is separably rationally connected in codimension 1. Then all the filtrations on $H^3(X, \mathbb{Z}_\ell)$ introduced in Definitions 4.15 equal the whole cohomology group:
\[
\tilde{N}_{1,cyl,eq} H^3(X, \mathbb{Z}_\ell) = \tilde{N}_{1,eq} H^3(X, \mathbb{Z}_\ell) = \tilde{N}^1 H^3(X, \mathbb{Z}_\ell) = N^1 H^3(X, \mathbb{Z}_\ell) = H^3(X, \mathbb{Z}_\ell).
\]

**Corollary 4.17.** Let $X$ be a smooth projective variety of dimension $d$ defined over a finite field $\mathbb{F}_q$, that is separably rationally connected in codimension 1. Assume one of the followings
\[
(1) \quad N^1 H^{2d-3}_{\ell et}(X, \mathbb{Z}_\ell(d-1)) = H^{2d-3}_{\ell et}(X, \mathbb{Z}_\ell(d-1)).
\]
(2) The cycle class map

\[ cl : \lim_{n} CH_1(\bar{X}, 1, Z/\ell^n) \to H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \]

is surjective.

Then every class in \( H^1(\mathbb{F}_q, H^3(\bar{X}, Z_\ell(d-1))) \) is the class of an algebraic cycle defined over \( \mathbb{F}_q \). In particular, this holds if \( X \) has dimension 3.

Proof. We first show that the surjectivity of the cycle class map

\[ cl : \lim_{n} CH_1(\bar{X}, 1, Z/\ell^n) \to H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \]

implies that \( N^1 H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) = H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \). In fact, we have

\[
\lim_{n} CH_1(\bar{X}, 1, Z/\ell^n) \to \lim_{n} N^1 H^{2d-3}_\text{ét}(\bar{X}, Z/\ell^n(d-1))
\]

\[ \to \lim_{n} H^{2d-3}_\text{ét}(\bar{X}, Z/\ell^n(d-1)) = H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)). \]

Therefore

\[
\lim_{n} N^1 H^{2d-3}_\text{ét}(\bar{X}, Z/\ell^n(d-1)) \to \lim_{n} H^{2d-3}_\text{ét}(\bar{X}, Z/\ell^n(d-1)) = H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1))
\]

is surjective. On the other hand, since \( N^1 H_3(X, Z/\ell^n) \) is a subgroup of \( H_3(X, Z/\ell^n) \), the inverse limit is injective, hence an isomorphism. We have an exact sequence

\[ 0 \to \lim_{n} H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1))/\ell^n \cap N^1 H^{2d-3}_\text{ét}(\bar{X}, Z/\ell^n(d-1)) \]

\[ \to \lim_{n} N^1 H^{2d-3}_\text{ét}(\bar{X}, Z/\ell^n(d-1)) \to \lim_{n} H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \]

where the first inverse limit is \( N^1 H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \) by (5), and the last inverse limit is torsion free. Since the quotient of \( H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1))/N^1 H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \) is torsion for separably rationally connected varieties or separably rationally connected fibrations over a curve, we know that \( \phi \) is an isomorphism and thus \( N^1 H^{2d-3}_\text{ét}(X, Z_\ell) \to H^{2d-3}_\text{ét}(X, Z_\ell) \) is an isomorphism.

Therefore, by Theorem 4.14 there is a smooth projective curve \( C \) defined over \( \mathbb{F}_q \) with a family of one-dimensional cycles \( \Gamma \subset C \times \bar{X} \) such that

\[ \Gamma_* : H^1_{\text{ét}}(C, Z_\ell(1)) \to H^{2d-3}_\text{ét}(\bar{X}, Z_\ell(d-1)) \]

is surjective. Then this corollary follows from [SS22 Proposition 7.6], the statement of which is recalled in Theorem 1.31. \( \square \)

5. Integral Tate conjecture and local-global principle for zero cycles

Let \( X \) be a smooth projective geometrically irreducible variety of dimension \( d \) defined over a finite field \( \mathbb{F} \). We have the cycle class maps:

\[ CH_1(X) \otimes Z_\ell \to H^{2d-2}(X, Z_\ell(d-1)). \]

Recall the integral Tate conjecture asks the following question.

**Question 5.1.** For which smooth projective variety \( X \) defined over \( \mathbb{F} \), and which \( r \), is the cycle class map \( [6] \) surjective?
We mention another closely related question.

**Question 5.2.** Let $X$ be a smooth projective variety defined over a henselian local field with finite residue field. Is the cycle class map 

$$CH_0(X) \hat{\otimes} \mathbb{Z}_\ell \to H^{2d}(X, \mathbb{Z}_\ell(d))$$

injective? Here $\ell$ is invertible in the residue field.

**Remark 5.3.** Question 5.2 has a positive answer if $X$ is a geometrically rational surface, and has a regular model with SNC central fiber ([EW16, Theorem 3.1] in general and [Sai91, Theorem A] for the case of $p$-adic fields). In this case, the proof in [EW16] also shows that the closed fiber also satisfies a version of the integral Tate conjecture. For $X$ defined over a Laurent field $\mathbb{F}_q((t))$, a regular model with SNC central fiber always exists since we have resolution of singularities for 3-folds.

If Question 5.2 has a positive answer for the generic fiber $X$, then Conjectures 1.1 and 1.3 are equivalent for $X$.

**Remark 5.4.** We also note that the results in [Tia20] suggest that Question (5.2) should be true for separably rationally connected varieties, provided that the characteristic $p$ analogues of the conjecture $R(n, 3)$ about Kato homology in loc. cit. is true, and that we have the minimal model program established in positive and mixed characteristic.

As discussed in Theorem 1.10 and the remark that follows this theorem in the introduction, various types of integral Tate conjectures would imply various versions of Colliot-Thélène’s conjectures 1.1, 1.2.

We can deduce Theorem 1.13 from Theorem 1.35 and Corollary 4.17.

**Proof of Theorem 1.13.** Recall that $G = \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is the absolute Galois group. By Theorem 1.35 we have an isomorphism

$$A_1(X) \cong A_1(\overline{X})^G.$$

Under the assumptions (A), (B) of Theorem 1.13 we know that there is an isomorphism of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules:

$$A_1(\overline{X}) \otimes \mathbb{Z}_\ell \cong H^{2d}(\overline{X}, \mathbb{Z}_\ell(d)).$$

Note that $G$ is generated by the Frobenius $F$ and $A_1(\overline{X})^G$ is the kernel of $F^* - \text{id}$. Since $\mathbb{Z}_\ell$ is a flat $\mathbb{Z}$-module, we have $A_1(\overline{X})^G \otimes \mathbb{Z}_\ell$ is the kernel of

$$(F^* - \text{id}) \otimes \text{id}_{\mathbb{Z}_\ell} : A_1(\overline{X}) \otimes \mathbb{Z}_\ell \to A_1(\overline{X}) \otimes \mathbb{Z}_\ell.$$

That is,

$$A_1(\overline{X})^G \otimes \mathbb{Z}_\ell \cong (A_1(\overline{X}) \otimes \mathbb{Z}_\ell)^G \cong H^{2d}(\overline{X}, \mathbb{Z}_\ell)^G,$$

where $\mathbb{Z}_\ell$ is equipped with the trivial action of $G$.

By part 2 of Theorem 1.10 this proves the first part of the theorem. The second part of the theorem is just Corollary 4.17.

**Proof of Theorem 1.12.** First note that the surjectivity of the cycle class map is a birational invariant. So using resolution of singularities for 3-folds [CP05, CP09, Abh98], we may assume that the singular fibers are SNC divisors. The result of Bloch-Srinivas [BS83] shows that the Griffiths group of 1 cycles on $\overline{X}$ is $p$-torsion.
Thus the hypothesis (B) in Theorem 1.13 is satisfied. As for hypothesis (A), we have a commutative diagram of localization exact sequences:

\[
\begin{array}{cccccc}
\oplus CH_1(\mathcal{X}_i) \otimes \mathbb{Z}_\ell & \longrightarrow & CH_1(\mathcal{X}) \otimes \mathbb{Z}_\ell & \longrightarrow & CH_1(\mathcal{X}^0) \otimes \mathbb{Z}_\ell & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\oplus H^4_{\acute{e}t}(\mathcal{X}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4(\mathcal{X}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4(\mathcal{X}^0, \mathbb{Z}_\ell(2)) & \\
\end{array}
\]

Here \( \mathcal{X}_i \) is a singular fiber of the fibration \( \mathcal{X} \to B \) and \( \mathcal{X}^0 \) is the complement of all the singular fibers \( \mathcal{X}_i \). By Section 4.3 in \[EW16\], the first vertical map is surjective. We may assume that \( \mathcal{X}^0 \) is over an affine curve (i.e. the direct sum on the left is non-trivial). A simple calculation then shows that \( H^4(\mathcal{X}^0) \) is one dimensional and spanned by the class of a section. Thus the third cycle class map is also surjective by \[dJS03\]. So the middle one is surjective.

Hypothesis (C) and (D) are also satisfied in this case. For simplicity, we only explain how to prove hypothesis (D). On the one hand, the cokernel is torsion for separably rationally connected fibrations by a decomposition of diagonal argument. On the other hand, Bloch-Kato conjecture proved by Voevodsky (in dimension 3, we can also use the Merkurjev-Suslin theorem) implies that it is torsion free for all smooth projective 3-fold. Hence the cokernel has to vanish.

Thus Theorem 1.13 implies that \( CH_1(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^4(\mathcal{X}, \mathbb{Z}_\ell(2)) \) is surjective. \( \square \)

**Proof of Theorem 1.4** This follows from combining Theorems 1.10, 1.12, and Remark 5.3. \( \square \)

### 6. Examples

We conclude this article with some examples where one can check the conditions in Theorem 1.13.

**Proposition 6.1.** Let \( \mathcal{X} \subset \mathbb{P}B(E) \) be a family of complete intersections of degree \( d_1, \ldots, d_c \) in \( \mathbb{P}^n \) over a smooth projective curve \( B \) over \( \mathbb{F}_q \). Assume that the generic fiber \( X \) is smooth separably rationally connected of dimension \( d, d \geq 5 \) and that \( \sum d_i^2 \leq n \). Also assume that \( \mathcal{X} \) is smooth. Then the cycle class map

\[
CH^d(\mathcal{X}) \otimes \mathbb{Z}_\ell \to H^d_{\acute{e}t}(\mathcal{X}, \mathbb{Z}_\ell(d))
\]

is surjective and Conjectures 1.1 and 1.2 hold for the generic fiber \( X \) over \( \mathbb{F}_q(B) \).

**Proof.** By the dimension assumption and the affine vanishing for étale cohomology, there is an isomorphism

\[
H^d_{\acute{e}t}(\mathcal{X}, \mathbb{Z}_\ell(d)) \cong H^{2d-2}(\mathcal{X}, \mathbb{Z}_\ell(d-1))^G,
\]

which is spanned by the class of a section and a line in a fiber. So it suffices to show that over the algebraic closure \( \overline{\mathbb{F}}_q \), every multisection is rationally equivalent to a multiple of any fixed section modulo lines in fibers, and that every line in a fiber is algebraically equivalent to any line in any smooth fiber.

Both statements follows from some well-known argument. More precisely, the space of a chain of two lines passing through two points in a complete intersection of degree \( d_1, \ldots, d_c \) is defined by equations of degree \( 1, 1, 2, 2, \ldots, d_1 - 1, d_1 - 1, d_2 - 1, d_2 - 1, \ldots, d_c - 1, d_c - 1, d_c \) in \( \mathbb{P}^n \) (see, for example, Lemma 3.4 in \[Pan18\]). Thus by the classical Tsen-Lang theorem, for any family of...
complete intersections of degree \((d_1, \ldots, d_c)\) over a smooth curve \(T/\bar{\mathbb{F}}_q\), and for any two sections of this family, there is a family of chain of two lines in the complete intersections over \(T\) such that the two sections lie in this family of chain of two lines. Any two sections in a \(\mathbb{P}^1\)-bundle over a curve are rationally equivalent modulo general fibers. Thus any two sections are rationally equivalent up to lines in general fibers. This in turn implies that any two multi-sections of the same degree are rationally equivalent up to lines in general fibers. Since any curve in a fiber is rationally equivalent to the difference of two multi-sections of the same degree, it is also rationally equivalent to lines in general fibers. Finally, since the Fano scheme of lines of a complete intersection is connected as long as it has positive dimension \cite{DM98} Théorème 2.1, all lines in a fiber are algebraically equivalent. □

Remark 6.2. The dimension assumption is not restrictive. The only low dimensional examples satisfying the numerical conditions are quadrics and linear spaces. One can check by hands that integral Tate conjecture holds for them.

Remark 6.3. In general it is still an open question if a smooth Fano complete intersection is separably rationally connected. However, one can show that if the characteristic \(p\) is larger than all the \(d_i\), then every smooth Fano complete intersection of degree \(d_1, \ldots, d_c\) is separably rationally connected \cite{STZ22}.

Proposition 6.4. Let \(X\) be a smooth proper variety that is also a homogeneous variety under an integral linear algebraic group \(G\) over \(\mathbb{F}_q(B)\). Assume that \(X\) admits a regular projective model \(X \to B\). Then the cycle class map \(\text{CH}^d(X) \otimes \mathbb{Z}_\ell \to H^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d))\) is surjective and Conjecture \(\bar{1}\) holds for \(X\).

Proof. It is well-known that \(\bar{G}\) over \(\bar{\mathbb{F}}_q(B)\) is rational. Thus \(\bar{X} \to \bar{B}\) is birational to \(\bar{B} \times_{\bar{\mathbb{F}}_q} \mathbb{P}^n \to \bar{B}\). So the conditions in Theorem \(\bar{1}\) are satisfied. □

Remark 6.5. Liang \cite{Lia13} proved that if the Brauer-Manin obstruction is the only obstruction to weak approximation of rational points in a rationally connected variety over a number field \(K\) and all of its finite field extensions, the number field analogue of Conjecture \(\bar{1}\) is true. As a corollary, he proved Conjecture \(\bar{1}\) for all smooth proper varieties birational to a homogeneous space under a linear algebraic groups with connected stabilizer. Harpaz and Wittenberg \cite{HW20} proved Conjecture \(\bar{1}\) for all smooth proper varieties birational to a homogeneous space under a linear algebraic group. One could expect that this also holds in the global function field case by essentially the same proof (modulo some characteristic \(p\) issues).

Theorem 6.6. Let \(C\) be a smooth projective geometrically integral curve defined over a global function field \(\mathbb{F}(B)\). Assume that \(C\) has a zero cycle of degree 1 over \(\mathbb{F}(B)\). Let \(X(r, L)\) be the moduli space of stable vector bundles of rank \(r\) and fixed determinant \(L \in \text{Pic}^d(X)\), with \(r, d\) relatively prime to each other. Assume that \(X(r, d)\) has a smooth projective model over \(B\). Then the local-global principle for zero cycles (i.e. Conjecture \(\bar{1}\)) holds for \(X(r, L)\).

Proof. It is well-known that \(X(r, L)\) is geometrically rational \cite{Hof07, New75, New80, KS99} and has geometric Picard group isomorphic to \(\mathbb{Z}\). In fact, as long as the curve, defined over any field \(k\), has a \(k\)-rational point, \(X(r, L)\) is rational over \(k\) \cite{Hof07}.
Using a norm argument, one can prove that the hypothesis (A)-(D) in Theorem 1.13 holds for $X(r, L) \otimes \overline{\mathbb{F}}$ under our assumptions. Hence Theorem 1.13 implies the statements.

\[\square\]

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