Testing randomness of spatial point patterns with the Ripley statistic

Gabriel Lang*

UMR 518 Mathématique et Informatique appliquées,
AgroParisTech,
19 avenue du Maine,
75732 PARIS CEDEX 15, France.
e-mail: gabriel.lang@agroparistech.fr

Eric Marcon

UMR 745 Ecologie des Forêts de Guyane,
AgroParisTech,
Campus agronomique BP 316,
97379 KOUROU CEDEX, France.
e-mail: eric.marcon@ecofog.gf

Abstract: Aggregation patterns are often visually detected in sets of location data. These clusters may be the result of interesting dynamics or the effect of pure randomness. We build an asymptotically Gaussian test for the hypothesis of randomness corresponding to a Poisson point process. We first compute the exact first and second moment of the Ripley K-statistic under the homogeneous Poisson point process model. Then we prove the asymptotic normality of a vector of such statistics for different scales and compute its covariance matrix. From these results, we derive a test statistic that is chi-square distributed. By a Monte-Carlo study, we check that the test is numerically tractable even for large data sets and also correct when only a hundred of points are observed.

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1. Introduction

Analysis of point patterns is relevant in many sciences: cell biology, ecology or spatial economics. The observation of clusters in point locations is considered as a hint for non observable dynamics. For example the clustering of tree locations in a forest may come from better soil conditions or from spreading of seeds of a same mature individual; but clusters are also observed in random distribution as a Poisson point process sample. It is therefore essential to distinguish between clusters resulting from relevant interactions or from complete randomness. Ripley (1976, 1977)— is a widely used tool to quantify the structure of point patterns, especially in ecology, and is well referenced in handbooks (Ripley,
Up to a renormalization by the intensity of the process, this statistic denoted here $\hat{K}(r)$ estimates the expectation $K(r)$ of the number of neighbors at distance less than $r$ of a point in the sample. The observed $\hat{K}(r)$ is compared to the value of $K(r)$ for a homogeneous Poisson point process with the same intensity as the data, chosen as a null hypothesis: the Poisson point process is characterized by an independence of point locations, modelling an absence of interactions between individuals in ecosystems. In this case $K(r)$ is simply the mean number of points in a ball of radius $r$ divided by the intensity, that is $\pi r^2$. If $\hat{K}(r)$ is significantly larger than $\pi r^2$ (respectively smaller), the process is considered as aggregated (respectively over-dispersed) at distance $r$.

To decide if the difference is statistically significant, we build a test of the Poisson process hypothesis; we need to know the distribution of $\hat{K}(r)$ for this process. But even the variance is not known and statistical methods generally rely on Monte-Carlo simulations. Ripley (1979) used them to get confidence intervals. Starting from previous results (Saunders & Funk, 1977), he also gave critical values for the $L$ function, a normalized version of $K$ introduced by Besag (1977). These critical values are valid asymptotically, for a large number of points but low intensity, so that both edge effects and point-pair dependence can be neglected. Further computations of confidence interval bands based on simulation have been proposed in Koen (1991) and corrected in Chiu (2007).

But the simulation is a practical issue for large point patterns, because computation time is roughly proportional to the square of the number of points (one has to calculate the distances between all pairs of points) multiplied by the number of simulations.

We propose here to compute the exact variance of the Ripley statistic. Ward & Ferrandino (1999) studied this variance. But they ignored that point pairs are not independent even though points are (eq. A8, p. 235), thus their derivation of the variance of $\hat{K}(r)$ was erroneous. The right way to compute the covariance is to consider that it is a $U$-statistic as remarked in Ripley (1979), then to use the Höffding decomposition. As the variance is not enough to build a test, we study the distribution of the statistic. We prove its asymptotic normality as the size of the observation window grows. It is then easy to build an asymptotically Gaussian test.

Another concern is to test simultaneously the aggregation/dispersion at different scales. This is rarely correctly achieved in practical computations with Monte-Carlo simulations. The confidence bands or test rejection zone are often determined without taking the dependence between the numbers of neighbors at different scales into account. As an exception Duranton & Overman (2005) provide a heuristic multiscale test. In our main theorem, we consider a set of scales $(r_1, \ldots, r_d)$, compute the covariance matrix of the $K(r_i)$ and prove the asymptotic normality for the vector $(\hat{K}(r_1), \ldots, \hat{K}(r_d))$. From this we propose the first rigorous multiscale test of randomness for point patterns.

The paper is built as follows: Section 2 introduces the precise definition of $K(r)$ and the current definition of $\hat{K}(r)$. In Section 3, after the definition of our statistics (no edge-effects correction, known or unknown intensity), we list
the main results of the paper: exact bias due to the edge effects and exact variance of $\hat{K}(r)$ for a homogeneous Poisson process with known or unknown intensity; covariance between $\hat{K}(r)$ and $\hat{K}(r')$ for two different distances $r$ and $r'$. The main theorem contains the convergence of the vector $(\hat{K}(r_1), \ldots, \hat{K}(r_d))$ to a Gaussian distribution with explicit covariance in the following asymptotic framework: data from the same process are collected on growing squares of observation. These results allow a simple, multiscale and efficient test procedure of the Poisson process hypothesis. Section 4 provides a Monte-Carlo study of the test and Section 5 gives our conclusions. The last section contains the proofs. Technical integration lemmas are postponed in the appendix.

2. Definition of the Ripley $K$-function

We recall the characterizations of the dependence of the locations for a general point process $X$ over $\mathbb{R}^2$. We refer to the presentation of Møller & Waagepetersen (2004).

2.1. Definitions

For a point process $X$, define the point process $X^{(2)}$ on $\mathbb{R}^2 \times \mathbb{R}^2$ of all the couples of two different points of the original process. The intensity of this new process gives information on the simultaneous presence of points in the original process. Denote $\rho^{(2)}(x, y)$ its density (called the second-order product density). The Poisson process of density $\rho(x)$ is such that $\rho^{(2)}(x, y) = \rho(x)\rho(y)$.

The Ripley statistic is a way to estimate the density $\rho^{(2)}(x, y)$. Precisely it is an estimate of the integral on test sets of the ratio $g(x, y) = \rho^{(2)}(x, y)/\rho(x)\rho(y)$. The function $g(x, y)$ characterizes the fact that the points $x$ and $y$ appear simultaneously in the samples of $X$. If $g(x, y) = 1$, the points appear independently. If $g(x, y) < 1$, they tend to exclude each other; if $g(x, y) > 1$, they appear more frequently together.

We assume the translation invariance of the point process: $g(x, y) = g(x - y)$. In order to estimate the function $g$, we define its integral as the set function $\mathcal{K}$.

Let $A$ be a Borel set:

$$\mathcal{K}(A) = \int_A g(x) dx.$$  

If we also assume that the point process is isotropic, we define the Ripley $K$-function as

$$K(r) = \mathcal{K}(B(x, r)),$$

where $B(x, r)$ is the closed ball with center $x$ and radius $r$. The translation invariance implies that $\mathcal{K}(B(x, r))$ does not depend on $x$. For example, if the process is a Poisson process then $g(x) = 1$ and $K(r) = \pi r^2$. We define the Ripley statistic that estimates the $K$-function. Let $A$ be a bounded Borel set of the plane $\mathbb{R}^2$, $m$ the Lebesgue measure and $\hat{\rho}$ an estimator of the local intensity.
of the process; for a realization $S$ of the point process $X$, $S = \{X_1, \ldots, X_N\}$, the Ripley statistic is defined by

$$\hat{K}_A(r) = \frac{1}{m(A)} \sum_{X_i \neq X_j \in S} \frac{I\{d(X_i, X_j) \leq r\}}{\hat{\rho}(X_i) \hat{\rho}(X_j)}.$$

### 3. Main results

This section presents the theoretical results on the Ripley statistic and the resulting test.

#### 3.1. Definitions

Throughout the paper, we refer to the indicator function $I$, the expectation $e_{r,n}$, the centred indicator function $h$ and its conditional expectation $h_1$. We gather here these definitions.

Let $n$ be an integer; $A_n$ denotes the square $[0,n]^2$; $U$ is a random location in $A_n$ with an uniform random distribution; its density is $1/n^2$ with respect to the Lebesgue measure $d\xi_1 d\xi_2$ over $A_n$. $V$ is a random location with the same distribution as $U$ and independent of $U$. We denote $d(x,y)$ the Euclidean distance between $x$ and $y$ in the plane, and $I\{A\}$ the indicator function of set $A$. We define $e_{r,n} = \mathbb{E}( I\{d(U,V) \leq r\} )$, $h(x,y,r) = I\{d(x,y) \leq r\} - e_{r,n}$ and $h_1(x,r) = \mathbb{E}(h(U,V,r) | V = x)$.

#### 3.2. Assumptions

We assume that $X$ is a homogeneous Poisson process on $\mathbb{R}^2$ with intensity $\rho$. We consider that the data are available on the square $A_n$. $S = \{X_1, \ldots, X_N\}$ is the sample of observed points. We consider two cases:

1. If the intensity $\rho$ is known, the Ripley statistic is expressed as

$$\hat{K}_{1,n}(r) = \frac{1}{n^2 \rho^2} \sum_{X_i \neq X_j \in S} I\{d(X_i, X_j) \leq r\}.$$

2. If the intensity $\rho$ is unknown, we choose to estimate $\rho^2$ by the unbiased estimator $\hat{\rho}^2 = N(N-1)/n^4$ (Stoyan & Stoyan, 2000) and define

$$\hat{K}_{2,n}(r) = \frac{n^2}{N(N-1)} \sum_{X_i \neq X_j \in S} I\{d(X_i, X_j) \leq r\}.$$

#### 3.3. Bias

It is known that a large number of neighbors of the points located near the edges of $A_n$ may lie outside $A_n$ causing a bias in the estimation. We compute the bias due to this edge effect.
Proposition 1. Assume that \( r/n < 1/2 \).

\[
\E \hat{K}_{1,n}(r) - K(r) = r^2 \left(- \frac{8r}{3n} + \frac{r^2}{2n^2} \right).
\]

\[
\E \hat{K}_{2,n}(r) - K(r) = r^2 \left(- \frac{8r}{3n} + \frac{r^2}{2n^2} \right) - r^2 e^{-\rho n^2} \left( \pi - \frac{8r}{3n} + \frac{r^2}{2n^2} \right) \left( 1 + \rho n^2 e^{-\rho n^2} \right).
\]

Notes:

- The assumption that \( r/n \) is less than \( 1/2 \) means that at least some balls of radius \( r \) are included in the square \( A_n \).
- The additional term for \( K_{2,n} \) corresponds to the probability to draw a sample with zero or one point in the square. This probability is so low that the term gives a zero contribution as soon as the mean number of points \( \rho n^2 \) is larger than 20.
- The proof may be adapted for a convex polygon of perimeter \( L_n \) to compute the first order term of the bias; for \( u = 1 \) or 2:

\[
\E \hat{K}_{u,n}(r) - K(r) = - \frac{2Lr^2}{3} + O \left( \frac{r^2}{n^2} \right).
\]

3.4. Variance

We compute the covariance matrix of \( \hat{K}_{u,n}(r) \) for \( u = 1 \) or 2. We get an exact computation for the variance, that can be used for any value of \( n \).

Proposition 2. For \( 0 < r < r' \),

\[
\text{var}(\hat{K}_{1,n}(r)) = \frac{2e_{r,n}}{\rho^2} + \frac{4n^2e_{r,n}^2}{\rho} + 4n^2 \E h_1^2(U, r),
\]

\[
\text{cov}(\hat{K}_{1,n}(r), \hat{K}_{1,n}(r')) = \frac{2e_{r,n}}{\rho^2} + \frac{4n^2e_{r,n}e_{r',n}}{\rho} + 4n^2 \rho \text{cov}(h_1(U, r'), h_1(U, r)),
\]

\[
\text{var}(\hat{K}_{2,n}(r)) = 2n^4 \E \left( \frac{\I{N > 1}}{N(N-1)} \right) (e_{r,n} - e_{r,n}^2)
+ 4n^4 \E \left( \frac{\I{N > 1}(N-2)}{N(N-1)} \right) \E h_1^2(U, r)
+ n^4 e^{-\rho n^2} \left( 1 + \rho n^2 \right) \left( 1 - e^{-\rho n^2} - \rho n^2 e^{-\rho n^2} \right) e_{r,n}^2,
\]

\[
\text{cov}(\hat{K}_{2,n}(r), \hat{K}_{2,n}(r')) = 2n^4 \E \left( \frac{\I{N > 1}}{N(N-1)} \right) (e_{r,n} - e_{r',n} e_{r,n})
+ 4n^4 \E \left( \frac{\I{N > 1}(N-2)}{N(N-1)} \right) \text{cov}(h_1(U, r'), h_1(U, r))
+ n^4 e^{-\rho n^2} \left( 1 + \rho n^2 \right) \left( 1 - e^{-\rho n^2} - \rho n^2 e^{-\rho n^2} \right) e_{r,n} e_{r,n},
\]
where
\[ e_{r,n} = \frac{\pi r^2}{n^2} - \frac{8r^3}{3n^3} + \frac{r^4}{2n^4}, \]
\[ \mathbb{E} h^2_1(U, r) = \frac{r^5}{n^5} \left( \frac{8}{3\pi} - \frac{256}{45} \right) + \frac{r^6}{n^6} \left( \frac{11}{48\pi} - \frac{56}{9} \right) + \frac{8r^7}{3n^7} - \frac{1}{4} \frac{r^8}{n^8}. \]

Notes:
- The variances of both estimators are exact and can be computed at any precision, as inverse moments of the Poisson variable correspond to fast converging series. But these series may be difficult to evaluate with mathematical softwares, because of the large value of the Poisson parameter.
- The covariances are not explicit because the terms \( \text{cov}(h^2_1(U, r'), h^2_1(U, r)) \) involve terms that have to be numerically integrated.
- The leading terms of the variances of \( K_{1,n}(r) \) and \( K_{2,n}(r) \) as \( n \) tends to infinity are \( 2\pi r^2/n^2\rho^2 + 4\pi r^4/n^2\rho \) and \( 2\pi r^2/n^2\rho^2 \).

3.5. Central Limit Theorem

We show that a normalized vector of Ripley statistics for different \( r \) converges in distribution to a normal vector. Let \( \mathcal{N}(0, \Sigma) \) denote the Gaussian multivariate centred distribution with covariance matrix \( \Sigma \).

**Theorem 1.** Let \( d \) be an integer, \( 0 < r_1 < \ldots < r_d \) a set of reals and define \( K_{u,n} = (\hat{K}_{u,n}(r_1), \ldots, \hat{K}_{u,n}(r_d)) \). Then \( n\sqrt{\rho}(K_{u,n} - \pi(r_1^2, \ldots, r_d^2)) \) converges in distribution to \( \mathcal{N}(0, \Sigma) \) as \( n \) tends to infinity, where for \( s \) and \( t \) in \( \{1, \ldots, d\} \)

- if \( u = 1 \), \( \Sigma_{s,t} = \frac{2\pi(r_s^2 \land r_t^2)}{\rho} + 4\pi^2 r_s^2 r_t^2 \).
- if \( u = 2 \), \( \Sigma_{s,t} = \frac{2\pi(r_s^2 \land r_t^2)}{\rho} \).

Note: The first term of the variance corresponds to a situation where the couples of points are independent from each others; this was used as an approximation without proof in Ward & Ferrandino (1999); our work proves that the actual variance and limit process are different in the first case and that the approximation holds only in the second case.

3.6. Applications to test statistics

From Theorem 1, we deduce that \( T_u = \Sigma^{-1/2}K_{u,n} \) is asymptotically \( \mathcal{N}(0, I_d) \) distributed. For the hypothesis

- \( H_0: X \) is a homogeneous Poisson process of intensity \( \rho \)
- we use \( T^2 = \|T_u\|^2_2 \) as a test statistic with rejection zone for the level \( \alpha \):

\[ T^2 > \chi^2_\alpha(d). \]
where $\chi^2_\alpha(d)$ is the $(1-\alpha)$-quantile of the $\chi^2(d)$ distribution.

Note: the covariance matrix $\Sigma$ depends on the intensity parameter $\rho$, so that in the case of the unknown parameter we have to use an estimate of $\rho$ in the formula defining $\Sigma$.

4. Simulations

We study the empirical variance of the proposed statistics by a Monte-Carlo simulation. Then we apply the test procedure to simulated data sets, observe the number of rejections and compare it to the level of the test.

4.1. Variance

We simulate a sample of 1000 repetitions with $\rho = 5$ and compare (after renormalization by $n\sqrt{\rho}$) the empirical variance and the exact computed variance with the limit variance for different value of $n$ (figure 1). With 1000 repetitions, the oscillations of the empirical variance are still large; we will use a larger number of repetitions in the following study of the test.

The convergence of the computed variance to the limit value is not so fast and for applications with hundreds of points (corresponding in figure 1 to $n < 15$) the distance between the variances is still large. A preliminary study, not presented here, showed that the test procedure is perturbed by a small error in the covariance matrix, as we tried simplified versions of the covariance by bounding.
or ignoring the corner contribution \( C(A_n^{3,3}) \) (see in the proof section). It is crucial to use an accurate computation of the covariance matrix to have a correct approximation of the square root inverse matrix \( \Sigma^{-1/2} \). Therefore we will use the exact formula instead of the asymptotic formula in the test procedure.

### 4.2. Test

In the known parameter case, the computation of the test statistic \( T_1 \) is straightforward; we also build a statistic \( T_1^* \) using the empirical covariance matrix of the sample. The advantage of \( T_1^* \) is that it is orthogonal by construction and should lead to better results. But the covariance matrix is not observable when we dispose of one sample, so that the test procedure based on \( T_1^* \) is unfeasible. It is an idealized version, used to compare the corresponding number of rejections. To avoid the statistical dependence between the sample and the estimator of the covariance matrix, we also build a statistic \( T_1' \) where we generate an additional independent sample of the Poisson process with intensity \( \rho \) to compute the empirical covariance matrix.

In the unknown parameter case, the computation of the test statistic \( T_2 \) is similar. In the variance formula the unknown parameter \( \rho \) is replaced by the estimator \( N/n^2 \). We also choose to replace the expectation \( E\{I\{N > 1\}/(N(N-1))\} \) by the observed value \( 1/(N(N-1)) \) and \( E\{I\{N > 1\}(N-2)/(N(N-1))\} \) by \( (N-2)/(N(N-1)) \), because the dispersion of a Poisson variable is low with respect to the expectation when its intensity is large. The construction of \( T_2^* \) is the same as for \( T_1^* \). The case of \( T_2' \) is not studied because, as \( \rho \) is unknown, one would have to generate an additional sample for each estimated value of \( \rho \).

The test output is a Bernoulli random variable with parameter \( \alpha \). With a sufficient index of repetition \( m \), the mean number of rejection is close to a normal variable with expectation \( \alpha(1-\alpha)/m \). We consider that the test works when the observed frequency of rejection is in the 95% Gaussian confidence interval \([\alpha - 1.96\sqrt{\alpha(1-\alpha)/m}, \alpha + 1.96\sqrt{\alpha(1-\alpha)/m}]\). With \( m = 10000 \) and \( \alpha = 0.05 \), the interval is \([0.0457; 0.0543]\) so that the percentile of rejection in table 1 should lie in \([4.57; 5.43]\). Stars indicate the values outside the confidence interval.

The performances in the case of a known parameter \( (T_1, T_1^* \text{ and } T_1') \) are good except when the number of points is small. The unfeasible tests \( T_1^* \) and \( T_1' \) based

| Poisson | \( T_1^* \) | \( T_1' \) | \( T_1 \) | \( T_2 \) | \( T_2^* \) |
|---------|-------------|-------------|-------------|-------------|-------------|
| n = 30  | \( \rho = 1 \) | r = (1, 2, 5) | 5.40 | 5.04 | 5.20 | 5.01 | 5.10 |
| n = 10  | \( \rho = 5 \) | r = (1, 2, 5) | 5.61* | 5.40 | 5.19 | 5.38 | 5.37 |
| n = 10  | \( \rho = 5 \) | r = (1, 2, ..., 10) | 5.13 | 5.32 | 5.76* | 6.67* | 6.01* |
| n = 10  | \( \rho = 1 \) | r = (1, 2, 5) | 5.67* | 5.86* | 5.81* | 5.30 | 5.25 |
| n = 10  | \( \rho = .5 \) | r = (1, 2, 5) | 5.52* | 5.73* | 5.52* | 5.60* | 4.91 |
| n = 10  | \( \rho = .2 \) | r = (1, 2, 5) | 6.40* | 6.84* | 6.59* | 6.59* | 5.22 |
on the empirical covariance have no better performance than the test $T_1$. The error of the empirical covariance is probably still too large. The only exception is the third line where a large number of values of $r$ are considered simultaneously. The test $T_2$ performs better than $T_1$ for small data sets. The only exception is the case of a large number of scales. The poor performance of $T_1$ and $T_2$ in this case may result from numerical instabilities in the covariance matrix inversion as its dimension is larger. The departure from normality may also be larger in this case (some classes of inter-point distances being weakly represented in the sample). With this exception, the test based on $T_2$ works perfectly.

In table 2, we investigate the power of the test $T_2$ by simulating two Thomas cluster processes (Thomas, 1949). A Thomas process is a Neyman-Scott process; the germs of the clusters are drawn as a sample of a homogeneous Poisson process of intensity $\kappa$. For each germ, an inhomogeneous Poisson process is drawn with intensity measure $\mu f$, where $f$ is the density of the Gaussian two-dimensional vector centered on the germ and with independent coordinates of variance $\sigma$. The Thomas process results from the superposition of these Poisson processes. The germs are not conserved. The parameters of the two processes are such that clusters are not visually detectable in the first process and evident in the second one. The test rejects 71% of the first sample and systematically the second one. The test is more powerful than a visual observation of the data, detecting invisible clusters. A rigorous analysis of the distribution of the statistic for dependent point process models should allow to conclude on the power of our test but such a study is beyond the scope of this paper.

5. Conclusion

We provide an efficient test of the null hypothesis of a homogeneous Poisson process for point patterns in a square domain. This is a theoretical and practical improvement on preexisting methods: Monte-Carlo simulations are untractable when the number of points increases. With a personal computer, calculating $K$ for 10,000 simulations of a 10,000-point set is not feasible (or it will take months). Marcon & Puech (2003) applied $K$ to a 36,000-point data set (the largest ever published as far as we know), but had to limit the number of simulations to 20. We suggest to change the treatment of edge effects. Instead of correcting edge effect on each sample to reduce the bias, we compute the exact bias. The use of sample correction (for each point of the data) has not been questioned since Ripley’s original paper, except by Ward & Ferrandino (1999).

We also point out that the test can be used on samples with a few dozens of points as encountered in actual data sets. It works correctly with such small
data sets, even if it is based on asymptotic normality. This is due to the fact that the bias and variance are known exactly and not asymptotically; the non-normality of the statistics for small data sets seems to have lesser effects than approximating the variance.

Our work should be extended in two directions: to other domain shapes that are of interest for the practitioners and to 3-dimensional data for high resolution medical imagery. A further study of the asymptotics of the distribution of \( \hat{K}(r) \) for dependent point process models such as Markov or Cox processes should also be achieved to inform on the power of our test.

6. Proofs

6.1. Proof of proposition 1

Recall that \( U \) and \( V \) are two independent uniform variables on \( A_n \). The expectations of the Ripley statistics are

\[
\mathbb{E}\hat{K}_{1,n}(r) = \frac{1}{n^2 \rho^2} \mathbb{E} \left( \sum_{X_i \neq X_j \in S} \mathbb{I}\{d(X_i, X_j) \leq r\} \right)
= \frac{\mathbb{E}(N(N-1))}{n^2 \rho^2} \mathbb{E}\left( \mathbb{I}\{d(U, V) \leq r\} \right)
= n^2 e_{r,n}.
\]

\[
\mathbb{E}\hat{K}_{2,n}(r) = n^2 \mathbb{E} \left( \frac{1}{N(N-1)} \sum_{X_i \neq X_j \in S} \mathbb{I}\{d(X_i, X_j) \leq r\} \right)
= n^2 \mathbb{P}(N > 1) \mathbb{E}\left( \mathbb{I}\{d(U, V) \leq r\} \right)
= n^2 \left(1 - e^{-\rho n^2} - \rho n^2 e^{-\rho n^2}\right) e_{r,n}.
\]

The following lemma allows to conclude:

Lemma 1.

\[
e_{r,n} = \frac{\pi r^2}{n^2} - \frac{8r^3}{3n^3} + \frac{r^4}{2n^4}.
\]

Proof: We split \( A_n \) into four parts to compute \( e_{r,n} \):

\[
e_{r,n} = \int_{\xi \in A_n^1} \int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \frac{1}{n^4} d\xi d\eta + \int_{\xi \in A_n^2} \int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \frac{1}{n^4} d\xi d\eta + \int_{\xi \in A_n^3} \int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \frac{1}{n^4} d\xi d\eta + \int_{\xi \in A_n^4} \int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \frac{1}{n^4} d\xi d\eta
\]
where (see figure 2)

- (interior) \( A^1_n = \{ \xi, \xi \text{ is at distance larger than } r \text{ from the boundary} \} \)
- (edge) \( A^2_n = \{ \xi, \xi \text{ is at distance less than } r \text{ from an edge, larger than } r \text{ from the others} \} \)
- (two edges) \( A^3_n = \{ \xi, \xi \text{ is at distance less than } r \text{ from two edges and larger than } r \text{ from the corner} \} \)
- (corner) \( A^4_n = \{ \xi, \xi \text{ is at distance less than } r \text{ from the corner} \} \)

Note that \( A^2_n, A^3_n \) and \( A^4_n \) are composed of four parts that contribute identically. We establish formulas only for one of these parts.

**Lemma 2.** Define function \( g(x) = \arccos(x) - x\sqrt{1 - x^2} \). If \( \xi \in A^1_n \),
\[
\int_{\eta \in A_n} I\{d(\xi, \eta) \leq r\} d\eta = \pi r^2.
\]
If \( \xi \in A^2_n \), with \( n - r < \xi_1 < n \), \( x_1 = \frac{1}{r}(n - \xi_1) \),
\[
\int_{\eta \in A_n} I\{d(\xi, \eta) \leq r\} d\eta = r^2(\pi - g(x_1))
\]
If \( \xi \in A^3_n \), with \( n - r < \xi_1 < n, n - r < \xi_2 < n \) and \( (x_1, x_2) = \frac{1}{r}(n - \xi_1, n - \xi_2) \),
\[
\int_{\eta \in A_n} I\{d(\xi, \eta) \leq r\} d\eta = r^2(\pi - g(x_1) - g(x_2))
\]
If \( \xi \in A^4_n \), with \( n - r < \xi_1 < n, n - r < \xi_2 < n \) and \( (x_1, x_2) = \frac{1}{r}(n - \xi_1, n - \xi_2) \),
\[
\int_{\eta \in A_n} I\{d(\xi, \eta) \leq r\} d\eta = r^2 \left( \frac{3\pi}{4} + x_1 x_2 - \frac{g(x_1) + g(x_2)}{2} \right).
\]
Figure 3. Geometrical interpretation of \( g \)

Note: Function \( g(x) \) is the area of the part of a ball of radius 1 that lies outside the square when the ball intersects one of its edges (see figure 3).

Proof. For the interior points \( \xi \in A^1_n, B(\xi, r) \subset A_n \). Let \( \xi \in A^2_n \). We compute the area of \( B(\xi, r) \cap A_n \).

\[
\int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \, d\eta = \frac{\pi r^2}{2} + 2r^2 \int_0^{x_1} \sqrt{1-t^2} \, dt
\]

\[
= r^2 \left( \pi - \arccos(x_1) + x_1 \sqrt{1-x_1^2} \right)
\]

\[
= r^2 \left( \pi - g(x_1) \right).
\]

Note that \( r^2 g(x) \) is the part of the ball that lies out of the square \( A_n \) if the center is at distance \( xr \) from the edge of the square.

Let \( \xi \in A^3_n \). Here the ball intersects two edges of the square and the area of \( B(\xi, r) \cap A_n \) is

\[
\int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \, d\eta = r^2 \left( \pi - g(x_1) - g(x_2) \right).
\]

Let \( \xi \in A^4_n \). Divide the ball into four quarters along axes parallel to the coordinate axes. One of the quarter is inside the square, two intersect the edges, leaving outside an area equal to \( (g(x_1) + g(x_2))/2 \). The area of the intersection of the last quarter with the square is \( x_1 x_2 \) so that the area of \( B(\xi, r) \cap A_n \) is

\[
\int_{\eta \in A_n} \mathbb{I}\{d(\xi, \eta) \leq r\} \, d\eta = r^2 \left( \frac{3\pi}{4} + x_1 x_2 - \frac{g(x_1) + g(x_2)}{2} \right). \quad \square
\]

Proof of lemma 1 (continued). The left-hand side of (1) is \( m(A^1_n) \pi r^2 = \pi(n - 2r)^2 \). Recall that \( A^2_n \) is composed of four parts that contribute identically. We integrate function \( g \).
Lemma 3. \[ G(x) = \int_0^x g(u) du = x \arccos(x) - \sqrt{1 - x^2} + \frac{1}{3}(1 - x^2)\frac{3}{2} + \frac{2}{3}. \]

Proof. Changing variables and integrating by parts
\[
\int_0^x \arccos(u) du = -\int_{\arccos(x)}^{\pi/2} t \sin(t) dt
= [t \cos(t)]_{\arccos(x)}^{\pi/2} + \int_{\arccos(x)}^{\pi/2} \cos(t) dt
= x \arccos(x) - \sqrt{1 - x^2} + 1.
\]
Changing the variable \( v = \sqrt{1 - u^2} \), we get
\[
- \int_0^x u \sqrt{1 - u^2} du = \int_1^{\sqrt{1-x^2}} v^2 dv = \frac{1}{3} \left( (1 - x^2)^{3/2} - 1 \right). \]

Then the contribution (2) is equal to
\[
4r \int_0^{n-r} d\xi_2 \int_0^1 r^2 (\pi - g(x)) dx = 4r^3 (n-2r) (\pi - G(1)) = \left( 4\pi - \frac{8}{3} \right) r^3 (n-2r).
\]
We consider \( A_3 \); the domain of integration is symmetric in \((x_1, x_2)\) so that the contribution (3) is equal to
\[
4r^4 \int_0^1 dx_1 \int_{\sqrt{1-x_1^2}}^{1-x_1^2} (\pi - 2g(x_1)) dx_2 = 4r^4 \left( \pi \left( 1 - \frac{\pi}{4} \right) - 2 \int_0^1 g(x_1) dx_1 \int_{\sqrt{1-x_1^2}}^{1-x_1^2} dx_2 \right).
\]
From Lemma 6,
\[
\int_0^1 g(x_1) dx_1 \int_{\sqrt{1-x_1^2}}^{1-x_1^2} dx_2 = G(1) - \int_0^1 g(x_1) \sqrt{1 - x_1^2} dx_1 = \frac{2}{3} - \frac{\pi^2}{16},
\]
so that contribution (3) is equal to \( r^4 \left( 4\pi - \frac{\pi^2}{2} - \frac{16}{3} \right) \).

We consider \( A_4 \); the contribution (4) is equal to
\[
4r^4 \int_0^1 dx_1 \int_0^{\sqrt{1-x_1^2}} \left( \frac{3\pi}{4} + x_1 x_2 - g(x_1) \right) dx_2 = r^4 \left( \frac{3\pi^2}{4} + \frac{1}{2} - 4 \int_0^1 g(x_1) \sqrt{1 - x_1^2} dx_1 \right)
= r^4 \left( \frac{\pi^2}{2} + \frac{1}{2} \right).
\]
Gathering the four contributions, we get
\[
e_{r,n} = \frac{r^2}{n^2} \left( \pi \left( 1 - \frac{2r}{n} \right)^2 + \left( 4\pi - \frac{8}{3} \right) \frac{r}{n} \left( 1 - \frac{2r}{n} \right) + \left( 4\pi - \frac{29}{6} \right) \frac{r^2}{n^2} \right)
= \frac{r^2}{n^2} \left( \pi - \frac{8}{3} \frac{r}{n} + \frac{1}{2} \frac{r^2}{n^2} \right). \]
6.2. Proof of proposition 2

We decompose the variance of $K_{s,A_n}(r)$ by conditioning the variable with respect to the number $N$ of points in the sample. Conditionally to $N$, $K_{s,A_n}(r)$ has the form of a $U$-statistic. Then we apply the Höffding decomposition to this $U$-statistic.

For $s = 1, 2$, we use the relation
\[
\text{var}(\hat{K}_{s,A_n}(r)) = \text{var}(\hat{K}_{s,A_n}(r)|N) + \mathbb{E}\text{var}(\hat{K}_{s,A_n}(r)|N).
\]

We first consider the conditional expectation of $\hat{K}_{s,A_n}(r)$.

\[
\mathbb{E}(\hat{K}_{1,n}(r)|N) = \frac{1}{n^2\rho^2} \left( \sum_{i \neq j=1}^{N} \mathbb{E} I\{d(X_i, X_j) \leq r\} \right) = \frac{N(N-1)e_{r,n}}{n^2\rho^2},
\]

\[
\mathbb{E}(\hat{K}_{2,n}(r)|N) = \frac{n^2}{N(N-1)} \sum_{i \neq j=1}^{N} \mathbb{E} I\{d(U_i, U_j) \leq r\} = n^2 e_{r,n} I\{N > 1\}.
\]

Because $N$ is a Poisson variable with intensity $\rho n^2$
\[
\mathbb{E}N^2(N-1)^2 = \mathbb{E}N(N-1)(N-2)(N-3) + 4\mathbb{E}N(N-1)(N-2) + 2\mathbb{E}N(N-1) = \rho^4 n^8 + 4\rho^3 n^6 + 2\rho^2 n^4.
\]

\[
\text{var} N(N-1) = 4\rho^3 n^6 + 2\rho^2 n^4.
\] (5)

Then
\[
\text{var}\mathbb{E}(\hat{K}_{1,n}(r)|N) = \frac{(4\rho n^2 + 2)e_{r,n}}{\rho^2},
\] (6)

\[
\text{var}\mathbb{E}(\hat{K}_{2,n}(r)|N) = n^2 \mathbb{E}\{N > 1\}(1 - \mathbb{P}\{N > 1\})e_{r,n}^2 = n^4 e^{-\rho n^2} (1 + \rho n^2) \left( 1 - e^{-\rho n^2} (1 + \rho n^2) \right) e_{r,n}^2.
\] (7)

We compute the conditional variances.

\[
\text{var}(\hat{K}_{1,n}(r)|N) = \frac{1}{n^4\rho^4} \text{var}\left( \sum_{i \neq j=1}^{N} h(X_i, X_j, r) \right),
\]

\[
\text{var}(\hat{K}_{2,n}(r)|N) = \frac{n^4}{N^2(N-1)^2} \text{var}\left( \sum_{i \neq j=1}^{N} h(X_i, X_j, r) \right).
\]

Conditionally to $N$, the locations of the points are independent and uniformly distributed variables $U_i$ over $A_n$. We introduce the Höffding decomposition of the $U$-statistic kernel $h$:

\[
h(x, y, r) = h_1(x, r) + h_1(y, r) + h_2(x, y, r),
\]
where $h_1(x) = \mathbb{E}(h(U, V, r)|V = x)$, $(U, V)$ being two independent uniform random variables on $A_n$.
Then $\mathbb{E}h_1(U, r) = 0$ and $\mathbb{E}(h_2(U, V, r)|U) = \mathbb{E}(h_2(U, V, r)|V) = 0$, so that

$$\text{var } h(U, V, r) = \text{var } h_1(U, r) + \text{var } h_1(V, r) + \text{var } h_2(U, V, r) = 2\mathbb{E}h_1^2(U, r) + \text{var } h_2(U, V, r).$$

From

$$\sum_{i \neq j=1}^{N} h(U_i, U_j, r) = 2(N - 1) \sum_{i=1}^{N} h_1(U_i, r) + \sum_{i \neq j=1}^{N} h_2(U_i, U_j, r).$$

we get

$$\text{var}(\hat{K}_{1,n}(r)|N) = \frac{4(N - 1)^2}{n^4 \rho^4} \text{var} \left( \sum_{i=1}^{N} h_1(U_i, r) \right) + \frac{1}{n^4 \rho^4} \text{var} \left( \sum_{i \neq j=1}^{N} h_2(U_i, U_j, r) \right)$$

$$= \frac{4N(N - 1)^2}{n^4 \rho^4} \mathbb{E}h_1^2(U, r) + \frac{2}{n^4 \rho^4} \sum_{i \neq j=1}^{N} \text{var } h_2(U_i, U_j, r)$$

$$= \frac{4N(N - 1)^2}{n^4 \rho^4} \mathbb{E}h_1^2(U, r) + \frac{2N(N - 1)}{n^4 \rho^4} (\text{var } h(U, V, r) - 2\mathbb{E}h_1^2(U, r))$$

$$= \frac{4N(N - 1)(N - 2)}{n^4 \rho^4} \mathbb{E}h_1^2(U, r) + \frac{2N(N - 1)}{n^4 \rho^4} \text{var } h(U, V, r),$$

Now $\text{var } h(U, V, r) = e_{r, n} - e_{r, n}^2$ and using factorial moments of the Poisson distribution

$$\mathbb{E} \text{var}(\hat{K}_{1,n}(r)|N) = \frac{4n^2}{\rho} \mathbb{E}h_1^2(U, r) + \frac{2}{\rho^2} (e_{r, n} - e_{r, n}^2). \quad (8)$$

Lemma 4 gives the exact value of $\mathbb{E}h_1^2(U, r)$. With relations (6) and (8), we get

$$\text{var}(\hat{K}_{1,n}(r)) = \frac{2e_{r, n}}{\rho^2} + \frac{4n^2 e_{r, n}}{\rho} + \frac{4n^2}{\rho} \mathbb{E}h_1^2(U, r)$$

$$= \frac{1}{n^2} \left( \frac{2\pi^2}{\rho^2} + \frac{4\pi^2 r^4}{\rho} \right)$$

$$- \frac{1}{n^3} \left( \frac{16 r^3}{3} + \left( \frac{32\pi}{3} + \frac{1024}{45} \right) \frac{r^5}{\rho} \right)$$

$$+ \frac{1}{n^4} \left( \frac{r^4}{\rho^2} + \left( \frac{59\pi}{12} + \frac{32}{9} \right) \frac{r^6}{\rho} \right).$$
Similarly

\[ \text{var}(\hat{K}_{2.1}(r)|N) = \frac{\sum_{i=1}^{N} h_1(U_i, r')}{n^4 \rho^4} \text{cov} \left( \sum_{i=1}^{N} h_1(U_i, r'), \sum_{i=1}^{N} h_1(U_i, r) \right) \]

\[+ \frac{1}{n^4 \rho^2} \text{cov} \left( \sum_{i \neq j=1}^{N} h_2(U_i, U_j, r'), \sum_{i \neq j=1}^{N} h_2(U_i, U_j, r) \right) \]

\[= \frac{4N(N-1)(N-2)}{n^4 \rho^4} \text{cov} (h_1(U, r'), h_1(U, r)) \]

\[+ \frac{2N(N-1)}{n^4 \rho^2} \text{cov} (h(U, V, r'), h(U, V, r)). \]

\[ \mathbb{E} \text{cov}(\hat{K}_{1.0}(r'), \hat{K}_{1.0}(r)|N) = \frac{4n^2}{\rho} \text{cov} (h_1(U, r'), h_1(U, r)) + \frac{2}{\rho^2} (e_{r,n} - e_{r',n} e_{r,n}). \]

To compute \( \text{cov}(h_1(U, r'), h_1(U, r)) \), the square \( A_n \) should now be split into 16 different zones according to the 4 zones of the preceding section with respect to \( r \) and the 4 zones with respect to \( r' \). Because of inclusions, the actual number of zones to consider is reduced to 9. The corresponding computation is easy in the center zone, but cannot be achieved in a close form in the edge bands and in the corner. We consider the following zones:

- (interior) \( A_n^{1.1} = \{\xi, \xi \text{ is at distance larger than } r' \text{ from the boundary}\}, \)
• (interior-edge) $A_{n}^{1.2} = \{ \xi, \xi \text{ is at distance between } r \text{ and } r' \text{ from an edge, larger than } r' \text{ from the others} \}$,
• (edge) $A_{n}^{2.2} = \{ \xi, \xi \text{ is at distance less than } r \text{ from an edge, larger than } r' \text{ from the others} \}$,
• (corner) $A_{n}^{3.3} = \{ \xi, \xi \text{ is at distance less than } r' \text{ from two edges} \}$.

Denoting $x_1 = \frac{1}{r}(n - \xi_1)$ and $x'_1 = \frac{1}{r'}(n - \xi_1)$ we get

\[
h_1(X_j, r')h_1(X_j, r) = \left( \frac{\pi r'^2 - e_{r', n}}{n^2} \right) \left( \frac{\pi r^2 - e_{r, n}}{n^2} \right)
\text{ on } A_{n}^{1.1},

\[
= \left( \frac{\pi r'^2}{n^2} - e_{r', n} - \frac{r'^2}{n^2} g(x'_1) \right) \left( \frac{\pi r^2}{n^2} - e_{r, n} \right)
\text{ on } A_{n}^{1.2},

\[
= \left( \frac{\pi r'^2}{n^2} - e_{r', n} - \frac{r'^2}{n^2} g(x'_1) \right) \left( \frac{\pi r^2}{n^2} - e_{r, n} - \frac{r^2}{n^2} g(x_1) \right)
\text{ on } A_{n}^{2.2}.
\]

Denote $b_{r, n} = \left( \pi - \frac{n^2}{r^2}e_{r, n} \right) = \frac{8r}{2n} - \frac{r^2}{2n^2}$.

\[
\text{cov} (h_1(X_j, r'), h_1(X_j, r)) = C(A_{n}^{1.1}) + C(A_{n}^{1.2}) + C(A_{n}^{2.2}) + C(A_{n}^{3.3})
\]

\[
C(A_{n}^{1.1}) = \frac{r'^2 r^2}{n^4} \left( 1 - \frac{2r'}{n} \right)^2 b_{r', n} b_{r, n}
\]

\[
C(A_{n}^{1.2}) = 4 \left( 1 - \frac{2r'}{n} \right) \frac{r'^3 r^2}{n^5} b_{r, n} \int_{r/r'}^{1} (b_{r', n} - g(x_1')) dx'_1
\]

\[
C(A_{n}^{2.2}) = 4 \left( 1 - \frac{2r'}{n} \right) \frac{r'^3 r^2}{n^5} \int_{0}^{1} (b_{r', n} - g(rx_1/r'))(b_{r, n} - g(x_1))dx_1.
\]

The first integral may be expressed in terms of function $G$, the second integral is elliptic and has to be numerically evaluated; as the integrand is bounded and very smooth this can be achieved without difficulties. To compute the term $C(A_{n}^{3.3})$, we rewrite the different values of function $h_1$ with the help of indicator functions:

\[
h_{A_1}(x, r) = b_{r, n} \mathbb{I}\{x_1 \geq 1; x_2 \geq 1\}
\]

\[
h_{A_2}(x, r) = (b_{r, n} - g(x_2)) \mathbb{I}\{x_1 \geq 1; x_2 < 1\} + (b_{r, n} - g(x_1)) \mathbb{I}\{x_2 \geq 1; x_1 < 1\}
\]

\[
h_{A_3}(x, r) = (b_{r, n} - g(x_1) - g(x_2)) \mathbb{I}\{x_1 < 1; x_2 < 1; x_1^2 + x_2^2 \geq 1\}
\]

\[
h_{A_4}(x, r) = (b_{r, n} - \pi/4 + x_1 x_2 - (g(x_1) + g(x_2))/2) \mathbb{I}\{x_1^2 + x_2^2 < 1\}
\]

For $x' = \frac{1}{r}(n - \xi_1, n - \xi_2)$

\[
C(A_{n}^{3.3}) = 4 \frac{r'^2 r^4}{n^6} \int_{0}^{1} \int_{0}^{1} \sum_{i=1}^{4} h_{A_i}(r' x'/r, r) \times \sum_{i=3}^{4} h_{A_i}(x', r')dx'_1dx'_2
\]

and this integral also can be numerically evaluated.

Note: the whole computation of this term of the covariance could be numerically
achieved, but it is preferable to use an exact computation whenever it is possible. The case of the covariance of $K_{2,n}(r)$ is analogous:
\[
\text{cov} \left( \mathbb{E}(\hat{K}_{2,n}(r')|N), \mathbb{E}(\hat{K}_{2,n}(r)|N) \right) = n^4 e^{-\rho n^2}(1 + \rho n^2)(1 - e^{-\rho n^2}(1 + \rho n^2))e_{r',n}e_{r,n}.
\]
\[
\mathbb{E} \text{ cov} \left( \hat{K}_{2,n}(r'), \hat{K}_{2,n}(r)|N \right) = 4n^4 \mathbb{E} \left( \frac{\mathbb{I}\{N > 1\}(N-2)}{N(N-1)} \right) \text{ cov} \left( h_1(U, r'), h_1(U, r) \right) + 2n^4 \mathbb{E} \left( \frac{\mathbb{I}\{N > 1\}}{N(N-1)} \right) (e_{r,n} - e_{r',n}e_{r,n}).
\]

6.3. **Proof of Theorem 1.**

We show that any linear combination of the $K_{1,n}(r_t)$ is asymptotically normal. Let $\Lambda = (\lambda_1, \ldots, \lambda_d)$ be a vector of real coefficients. Define $Z_1 = \sum_{t=1}^{d} \lambda_t K_{1,n}(r_t)$. We use the Bernstein blocks technique (Bernstein, 1939): we divide the square $A_n$ into squares of side $p$ with $p = o(n)$. These squares are separated by gaps of width $2r_d$ so that the sums over couples of points in each square are independent. The couples of points with at least one point in the gaps give a negligible contribution, so that the statistic $Z_1$ is equivalent to a sum of independent variables and asymptotically normal.

Set $p = n^{1/4}$. Assume that the Euclidean division of $n$ by $(p + 2r_d)$ gives a quotient $a$ and a remainder $q$. For $l = 0, \ldots, a$, we define the segment $I_l = [(p + 2r_d)l, (p + 2r_d)l + p - 1]$. We order the set $\{0, \ldots, a\}^2$ by the lexicographic order. To any integer $i$ such that $1 \leq i \leq k = (a + 1)^2$, corresponds an element $(j_1, j_2)$ of this set; we define the block $P_{i,n} = I_{j_1} \times I_{j_2}$ and $Q = A_n \setminus \bigcup_i P_{i,n}$ the set of points that are in none of the $P_{i,n}$'s. For each block $P_{i,n}$ and $Q$, we define the partial sums:

\[
u_i,n = \frac{1}{np^{3/2}} \sum_{X_i \neq X_m \in P_{i,n}} \sum_{t=1}^{d} \lambda_t \mathbb{I}\{d(X_i, X_m) \leq r_t\},
\]

\[
u_{i,n} = \frac{1}{np^{3/2}} \sum_{X_i \in P_{i,n}, X_m \in Q} \sum_{t=1}^{d} \lambda_t \mathbb{I}\{d(X_i, X_m) \leq r_t\}
\]

\[
u_{w,n} = \frac{1}{np^{3/2}} \sum_{X_i \neq X_m \in Q} \sum_{t=1}^{d} \lambda_t \mathbb{I}\{d(X_i, X_m) \leq r_t\}.
\]

then

\[
\sqrt{n}(Z_1 - \mathbb{E}Z_1) = \sum_{i=1}^{k} \left( \nu_i,n - \mathbb{E}\nu_i,n \right) + \sum_{i=1}^{k} \left( \nu_{i,n} - \mathbb{E}\nu_{i,n} \right) + \nu_{w,n} - \mathbb{E}\nu_{w,n}.
\]

We show that the sum of the $u_{i,n}$ converges in distribution to a Gaussian variable and that the other term are negligible in $\sqrt{n}$. We check the conditions of the following CLT adapted from Bardet et al. (2008).
Theorem 2. Let \((z_{i,n})_{0 \leq i \leq k(n)}\) be an array of random variables satisfying

1. There exists \(\delta > 0\) such that \(\sum_{i=0}^{k(n)} \mathbb{E}|z_{i,n}|^{2+\delta}\) tends to 0 as \(n\) tends to infinity,

2. \(\sum_{i=0}^{k(n)} \text{var} z_{i,n}\) tends to \(\sigma^2\) as \(n\) tends to infinity,

then \(\sum_{i=0}^{k(n)} z_{i,n}\) tends in distribution to \(N(0, \sigma^2)\) as \(n\) tends to infinity.

To check Condition 1, we compute the fourth order moment of \(u_{i,n} - \mathbb{E}u_{i,n}\). Let \(N_i\) be the number of points of \(S\) that fall in \(P_{i,n}\). Define

\[
f(x, y) = \sum_{t=1}^{d} \lambda_t \left( \mathbb{I}\{d(x, y) \leq r_t\} - e_{r,p}\right) = \sum_{t=1}^{d} \lambda_t h(x, y, r_t)
\]

\[
\mathbb{E}((u_{i,n} - \mathbb{E}u_{i,n})^4 | N_i) = \frac{1}{n^4 \rho^4} \mathbb{E} \left( \sum_{l \neq m=1}^{N_i} f(U_l, U_m) \right)^4
\]

Denote \(f_1\) and \(f_2\) the decomposing functions of \(f\):

\[
\mathbb{E}(f_1(U_i)) = 0, \mathbb{E}(f_1(U_i)f_2(U_i, U_m)) = \mathbb{E}(f_1(U_m)f_2(U_i, U_m)) = 0, \text{ for } U_i \text{ and } U_m
\]

two independent uniform variables on \(P_{i,n}\).

\[
\sum_{l \neq m=1}^{N_i} f(U_l, U_m) = 2(N_i - 1) \sum_{l=1}^{N_i} f_1(U_l) + \sum_{l \neq m=1}^{N_i} f_2(U_l, U_m).
\]

Note that \(|h_1(x, r)| \leq \pi r^2 p^{-2}\) so that \(f_1\) is bounded by \(Cp^{-2}\).

Define \(M_1 = \mathbb{E} \left( \sum_{l=1}^{N_i} f_1(U_l) \right)^4\). Then \(M_1 = N_i \mathbb{E}(f_1^4(U)) + 6N_i(N_i - 1) \mathbb{E}(f_1^2(U))^2\) and

\[
\mathbb{E}(N_i - 1)^4 M_1 = O(1).
\]

Define \(M_2 = \mathbb{E} \left( \sum_{l \neq m=1}^{N_i} f_2(U_l, U_m) \right)^4\). Because \(f_2\) is zero mean with respect to one coordinate, only the products where variables appear at least two times contribute.

\[
M_2 = 8 \sum_{l \neq m=1}^{N_i} \mathbb{E} f_2^4(U_l, U_m) + 16 \sum_{l \neq m \neq u=1}^{N_i} \mathbb{E} f_2^2(U_l, U_u) f_2^2(U_m, U_u)
\]

\[
+ 32 \sum_{l \neq m \neq u=1}^{N_i} \mathbb{E} f_2^2(U_l, U_m) f_2(U_m, U_u) f_2(U_l, U_u)
\]

\[
+ 4 \sum_{l \neq m \neq u \neq v=1}^{N_i} \mathbb{E} f_2^2(U_l, U_m) f_2^2(U_u, U_v)
\]

\[
+ 16 \sum_{l \neq m \neq u \neq v=1}^{N_i} \mathbb{E} f_2(U_l, U_m) f_2(U_m, U_u) f_2(U_u, U_v) f_2(U_v, U_l).
\]
Because $f_2$ is bounded, $E M_2 = O(E N_i (N_i - 1)(N_i - 2)(N_i - 3)) = O(p^8)$, so that
\[ \sum_{i=0}^{k} E(u_{i,n} - E u_{i,n})^4 = O(p^6 n^{-2}). \]

As $p = n^{1/4}$, we get condition 1.

To check condition 2, note that the vector $(K_{1,p_i}(r_1), \ldots, K_{1,p_i}(r_d))$ has a co-variance matrix $\Sigma_p$ defined by Proposition 2 by substituting $p$ to $n$ in the expressions. The $u_{i,n} = \frac{p_n}{n^2} \sum_{i=1}^d \lambda_i (K_{1,p_i}(r_i) - E K_{1,p_i}(r_i))$ are i.i.d variables with variance equal to $\frac{p^2 \rho \Sigma_p}{n^2} = \Lambda^t \Sigma_p \Lambda$. But $p^2 \rho \Sigma_p$ tends to $\Sigma$ as $p$ tends to infinity and
\[ \sum_{i=0}^{k} \text{var} u_{i,n} = \frac{k p^2 \rho}{n^2} \Lambda^t \Sigma_p \Lambda \rightarrow \Lambda^t \Sigma \Lambda \]
so that $\sum_{i=1}^{k} u_{i,n}$ tends in distribution to $\mathcal{N}(0, \Lambda^t \Sigma \Lambda)$.

Note that the $v_{i,n}$ are $k$ independent variables. Denote $N_{i,r_d}$ the number of points $X_i$ in the boundary region $P_{i,r_d}$ of $P_{i,n}$ such that the ball $B(X_i, r_d)$ intersects $Q$ and let $D(X_i)$ denote this intersection. Note that
\[ EN_{i,r_d} = \rho m(P_{i,r_d}) \leq C p r_d. \]

\[ \text{var} v_{i,n} \leq \frac{C}{n^2} E \left( \sum_{l=1}^{N_{i,r_d}} \sum_{m=1}^{N_Q} I\{X_m \in D(X_l)\} \right)^2 \leq \frac{C}{n^2} (T_1 + T_2), \]
where
\[
T_1 = \sum_{l=1}^{N_{i,r_d}} \sum_{m=1}^{N_Q} \sum_{u=1}^{N_{i,r_d}} I\{X_m \in D(X_l)\} I\{X_u \in D(X_l)\} I\{X_u \in D(X_m)\},
\]
\[
T_2 = \sum_{l=1}^{N_{i,r_d}} \sum_{m=1}^{N_Q} \sum_{u=1}^{N_{i,r_d}} I\{X_m \in B(X_l, 2r_d)\} I\{X_u \in D(X_l) \cap D(X_m)\}.
\]

\[ T_1 \leq EN_{i,r_d} EN_{Q}^2 \mathbb{P}\{X_m \in D(X_l)|X_m \in Q\} \]
\[ \leq \rho^3 m(P_{i,r_d}) (m^2(Q) + m(Q)) \left( \frac{m^2 r_d}{2m(Q)} \right) = O(p). \]
\[ T_2 \leq EN_{i,r_d} EN_{Q} \mathbb{P}\{X_m \in B(X_l, r_d)|X_m \in P_{i,r_d}\} \mathbb{P}\{X_u \in D(X_l)|X_u \in Q\} \]
\[ \leq \rho^3 (m^2(P_{i,r_d}) + m(P_{i,r_d})) \left( \frac{m^2 r_d}{m(P_{i,r_d})} \right) m(Q) \left( \frac{m^2 r_d}{2m(Q)} \right) = O(p). \]
and \( \text{var} \left( \sum_{i=1}^{k} v_{i,n} \right) = O \left( kp/n^2 \right) = O \left( p^{-1} \right) \), so that this sum is negligible in \( n^2 \). Similarly

\[
\text{var} \left( w_n \right) \leq \frac{C}{n^2} \mathbb{E} \left( \sum_{i \neq m=1}^{N_Q} I\{X_m \in B(X_i, r_d)\} \right)^2 \leq \frac{C}{n^2} (T_1 + T_2).
\]

where

\[
T_1 = \mathbb{E} \sum_{l=1}^{N_Q} \sum_{m=1}^{N_Q} I\{X_m \in B(X_l, r_d)\}
\]

\[
\leq \mathbb{E} N_Q (N_Q - 1) \mathbb{P} \{X_m \in B(X_l, r_d) \mid X_m \in Q\} \leq m^2(Q) \frac{\pi r_d^2}{m(Q)}.
\]

\[
T_2 = \mathbb{E} \sum_{l=1}^{N_Q} \sum_{m=1}^{N_Q} \sum_{u=1}^{N_Q} I\{X_m \in B(X_l, r_d)\} I\{X_u \in B(X_l, r_d)\} \leq \mathbb{E} N_Q^2 (N_Q - 1) \mathbb{P}^2 \{X_m \in B(X_l, r_d) \mid X_m \in Q\}
\]

\[
\leq \left( m^3(Q) + 2m^2(Q) \right) \left( \frac{\pi r_d^2}{m(Q)} \right)^2.
\]

Then \( \text{var} \left( w_n \right) = O \left( m(Q)/n^2 \right) = O \left( p^{-1} \right) \) and \( w_n \) is negligible in \( n^2 \).

Consider now \( K_{2,n}(r) \). Define \( Z_2 = \sum_{l=1}^{d} \lambda_l K_{2,n}(r_l) = A_{N,n} Z_1 \) where \( A_{N,n} = \frac{n^3}{N(N-1)} \). We have \( \mathbb{E}(A_{N,n}^{-1}) = 1 \) and from (5), \( \text{var}(A_{N,n}^{-1}) = \frac{4}{n^3 \rho^2} + \frac{2}{n^4 \rho^3} \).

For \( \delta > 0 \), the Markov inequality gives

\[
\mathbb{P}(|A_{N,n}^{-1} - 1| > \delta) \leq \frac{\text{var}(A_{N,n}^{-1})}{\delta^2}.
\]

Then, with \( \delta = n^{-1/4} \)

\[
\sum_{n=1}^{\infty} \mathbb{P}(|A_{N,n}^{-1} - 1| > n^{-1/4}) < \sum_{n=1}^{\infty} \frac{4}{n^{3/2} \rho^2} + \frac{2}{n^{7/2} \rho^2} < \infty.
\]

From the Borel-Cantelli lemma, we get that \( A_{N,n}^{-1} \) converges a.s. to 1. By the Slutsky lemma, \( A_{N,n} Z_1 \) converges in distribution to \( \mathcal{N}(0, \Lambda^t \Sigma \Lambda) \). \( \square \)

### 6.4. Computation of \( \mathbb{E}h^2_1(U, r) \)

**Lemma 4.**

\[
\mathbb{E}h^2_1(U, r) = \frac{r^5}{n^5} \left( \frac{8}{3} \pi - \frac{256}{45} \right) + \frac{r^6}{n^6} \left( \frac{11}{48} \pi - \frac{56}{9} \right) + \frac{8}{3 n^7} - \frac{1}{4 n^8}.
\]
Proof: From the computation of the bias, denoting \( x_i = \frac{1}{n}(n - \xi_i) \), we get

\[
h_1(\xi, r) = \frac{\pi r^2}{n^2} - e_{r,n} \text{ on } A^1_n
\]

\[
= \frac{r^2}{n^2}(\pi - g(x_1)) - e_{r,n} \text{ on } A^2_n
\]

\[
= \frac{r^2}{n^2}(\pi - g(x_1) - g(x_2)) - e_{r,n} \text{ on } A^3_n
\]

\[
= \frac{r^2}{n^2} \left( \frac{3\pi}{4} + x_1 x_2 - \frac{g(x_1) + g(x_2)}{2} \right) - e_{r,n} \text{ on } A^4_n
\]

\[
E(h_1(X,j))^2 = \pi^2 \left( 1 - \frac{2r}{n} \right)^2 \frac{r^4}{n^4} - e_{r,n}^2 + 4 \left( 1 - \frac{2r}{n} \right) \frac{r^5}{n^5} T_1 + 4 \frac{r^6}{n^6} (T_2 + T_3)
\]

\[
T_1 = \int_0^1 (\pi - g(x_1))^2 dx_1
\]

\[
T_2 = \int_0^1 dx_1 \int_0^1 (\pi - g(x_1) - g(x_2))^2 dx_2
\]

\[
T_3 = \int_0^1 dx_1 \int_0^1 \left( \frac{3\pi}{4} + x_1 x_2 - \frac{g(x_1) + g(x_2)}{2} \right)^2 dx_2.
\]

To compute these three terms, we need integral computations on function \( g \).

**Lemma 5.** For \( n \geq 1 \),

\[
I_n = \int_0^1 u^{2n-1} \arccos(u) du = \frac{\pi (2n)!}{n^{2n+2}(n!)^2}
\]

\[
J_n = \int_0^1 u^n \sqrt{1 - u^2} du = -(2n+2) I_{n+1} + 2n I_n.
\]

\[
\int_0^1 \sqrt{1 - u^2} \arccos(u) du = \frac{\pi^2}{16} + \frac{1}{4}
\]

\[
\int_0^1 \sqrt{1 - u^2} \arccos^2(u) du = \frac{\pi^3}{48} + \frac{\pi}{4}
\]

**Note:** in the following, we use \( I_1 = \pi/8, I_2 = 3\pi/64, J_1 = \pi/16 \) and \( J_2 = \pi/32 \).

**Lemma 6.**

\[
\int_0^1 g(u) \sqrt{1 - u^2} du = \frac{\pi^2}{16}
\]

\[
\int_0^1 g^2(u) du = \frac{2\pi}{3} - \frac{64}{45}
\]

\[
\int_0^1 g^2(u) \sqrt{1 - u^2} du = \frac{\pi^3}{48}
\]

\[
\int_0^1 g(u) G \left( \sqrt{1 - u^2} \right) du = \frac{\pi^3}{96} - \frac{5\pi}{48} + \frac{4}{9}
\]
Proofs are postponed in the appendix. Using these lemmas, we get

\[ T_1 = \pi^2 - 2\pi G(1) + \int_0^1 g^2(x_1)dx_1 = \pi^2 - \frac{64}{45} \frac{2\pi}{3}, \] (19)

\[ T_2 = \pi^2 \left( 1 - \frac{\pi}{4} \right) - 4\pi \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 + 2 \int_0^1 g^2(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 
+ 2 \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 g(x_2)dx_2. \]

From the computation of the bias, 
\[-4\pi \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 = \frac{8\pi}{3} + \frac{\pi^3}{4}.\]

From (16), (17) and (18), we get

\[ 2 \int_0^1 g^2(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 = 2 \int_0^1 g^2(x_1)dx_1 - 2 \int_0^1 \sqrt{1-x_1^2} g^2(x_1)dx_1 = \frac{4\pi}{3} - \frac{128}{45} \frac{\pi^3}{24}. \]

\[ 2 \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 g(x_2)dx_2 = 2G^2(1) - 2 \int_0^1 g(x_1)G(\sqrt{1-x_1^2}) dx_1 = -\frac{\pi^3}{48} + \frac{5\pi}{24}. \]

Adding these results, we obtain

\[ T_2 = -\frac{\pi^3}{16} + \pi^2 - \frac{9\pi}{8} - \frac{128}{45}. \] (20)

To compute \( T_3 \), we write

\[ T_3 = \frac{9\pi^3}{64} + \int_{x_1^2}^1 x_2 dx_1 \int_{\sqrt{1-x_1^2}}^1 x_2^2 dx_2 - \frac{3\pi}{2} \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 
+ \frac{1}{2} \int_0^1 g^2(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 + \frac{3\pi}{2} \int_0^1 x_1 dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 
+ \frac{1}{2} \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 g(x_2)dx_2 - 2 \int_0^1 x_1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2. \]

\[ \int_0^1 x_1^2 dx_1 \int_{\sqrt{1-x_1^2}}^1 x_2^2 dx_2 = \frac{1}{3} \int_0^1 x_1^2 (1-x_1^2) \sqrt{1-x_1^2} dx_1 = \frac{1}{3} (J_1 - J_2) = \frac{\pi}{96}. \]

From (15), \[-\frac{3\pi}{2} \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 = -\frac{3\pi^3}{32}. \]

From (17), \[\frac{1}{2} \int_0^1 g^2(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 dx_2 = -\frac{\pi^3}{96}. \]

\[ \frac{3\pi}{2} \int_0^1 x_1 dx_1 \int_{\sqrt{1-x_1^2}}^1 x_2 dx_2 = \frac{3\pi}{4} \int_0^1 x_1 (1-x_1^2) dx_1 = \frac{3\pi}{16}. \]

From (18), \[\frac{1}{2} \int_0^1 g(x_1)dx_1 \int_{\sqrt{1-x_1^2}}^1 g(x_2)dx_2 = \frac{\pi^3}{192} + \frac{5\pi}{96} + \frac{2}{9}. \]
\[ -2 \int_0^1 x_1 g(x_1) dx_1 \int_0^{\sqrt{1-x_1^2}} x_2 dx_2 = \int_0^1 (x_1^3 - x_1) g(x_1) dx_1 = -\frac{3\pi}{64}. \]

Adding these results, we get

\[ T_3 = \frac{\pi^3}{16} + \frac{19\pi}{192} + \frac{2}{9}. \]  

(21)

Gathering (19), (20) and (21) gives the result. □

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**Appendix A: Integration lemmas**

**A.1. Proof of Lemma 5**

Integrating by parts

\[
\int_0^1 u^{2n-1} \arccos(u) du = \int_0^{\pi/2} t \cos^{2n-1}(t) \sin(t) dt = \frac{1}{2n} \int_0^{\pi/2} \cos^{2n}(t) dt.
\]

Using De Moivre formula

\[
\cos^{2n}(t) = \frac{1}{2^{2n}} \left( 2 \cos(2nt) + 2 \left( \frac{2n}{1} \right) \cos(2(n-1)t) + \cdots + \left( \frac{2n}{n} \right) \right).
\]

Only the last term gives a non zero integral, giving the result for \( I_n \).

\[
J_n = \int_0^1 (u^{2n+2} - u^{2n}) (-(1-u^2)^{-1/2}) du
\]

\[
= \left[ (u^{2n+2} - u^{2n}) \arccos(u) \right]_0^1 - \int_0^1 ((n+2)u^{2n+1} - nu^{2n-1}) \arccos(u) du
\]

and the term under brackets is zero, giving the result.

\[
\int_0^1 \sqrt{1-u^2} \arccos(u) du = \int_0^{\pi/2} t \sin^2(t) dt = \int_0^{\pi/2} t \frac{2}{2} - \frac{t \cos(2t)}{2} dt
\]

\[
= \frac{\pi^2}{16} - \left[ \frac{t \sin(2t)}{4} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{t \sin(2t)}{4} dt = \frac{\pi^2}{16} + \frac{\pi}{4}.
\]

\[
\int_0^1 \sqrt{1-u^2} \arccos^2(u) du = \int_0^{\pi/2} t^2 \sin^2(t) dt = \int_0^{\pi/2} t^2 \frac{2}{2} - \frac{t^2 \cos(2t)}{2} dt
\]

\[
= \frac{\pi^3}{48} - \left[ \frac{t^2 \sin(2t)}{4} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{t \sin(2t)}{4} dt
\]

\[
= \frac{\pi^3}{48} - \left[ \frac{t \cos(2t)}{4} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{t \cos(2t)}{4} dt = \frac{\pi^3}{48} + \frac{\pi}{8}. \quad \square
\]
A.2. Proof of lemma 6

Equation (15) follows from equation (13). Write \( g^2(u) = \arccos^2(u) + u^2 - u^4 - 2u\sqrt{1-u^2}\arccos(u) \) and

\[
\int_0^1 \arccos^2(u) du = \int_0^\pi/2 t^2 \sin(t) dt = \left[ t^2 \cos(t) \right]_0^{\pi/2} + 2 \int_0^\pi/2 t \cos(t) dt = 2 \left[ t \sin(t) \right]_0^{\pi/2} + 2 \int_0^\pi/2 \sin(t) dt = \pi - 2,
\]

\[
\int_0^1 (u^2 - u^4) du = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.
\]

\[
\int_0^1 u \sqrt{1 - u^2} \arccos(u) du = \int_0^\pi/2 t \cos(t) \sin^2(t) dt
\]

\[
= \left[ \frac{t}{3} \sin^3(t) \right]_0^{\pi/2} - \frac{1}{3} \int_0^{\pi/2} \sin^3(t) dt
\]

\[
= \frac{\pi}{6} - \frac{1}{3} \int_0^{\pi/2} \sin(t) dt + \frac{1}{3} \int_0^{\pi/2} \cos^2(t) \sin(t) dt
\]

\[
= \frac{\pi}{6} - \frac{1}{3} - \frac{1}{9} \left[ \cos^3(t) \right]_0^{\pi/2} = \frac{\pi}{6} - \frac{2}{9}.
\]

Collecting the three parts yields to \((16)\).

\[
\int_0^1 g^2(u) \sqrt{1 - u^2} du = \int_0^1 \sqrt{1 - u^2} \arccos^2(u) du
\]

\[
-2 \int_0^1 (u - u^3) \arccos(u) du + \int_0^1 \sqrt{1 - u^2} (u^2 - u^4) du
\]

\[
= \frac{\pi^3}{48} + \frac{\pi}{8} - 2 \left( \frac{\pi}{8} - \frac{3\pi}{64} \right) + \frac{\pi}{16} - \frac{\pi}{32} = \frac{\pi^3}{48}.
\]

Write \( G \left( \sqrt{1 - x^2} \right) = \sqrt{1 - x^2} \left( \frac{\pi}{2} - \arccos(x) \right) + \frac{x^3}{3} - x + \frac{2}{3} \)

\[
\int_0^1 g(x) \sqrt{1 - x^2} dx = \int_0^1 \sqrt{1 - x^2} \left( \frac{\pi}{2} - \arccos(x) \right) \arccos(x) dx
\]

\[
- \int_0^1 (x - x^3) \left( \frac{\pi}{2} - \arccos(x) \right) dx
\]

\[
+ \int_0^1 \left( \frac{x^3}{3} - x + \frac{2}{3} \right) \arccos(x) dx
\]

\[
+ \int_0^1 \left( - \frac{x^4}{3} + x^2 - \frac{2x}{3} \right) \sqrt{1 - x^2} dx
\]

\[
= \frac{\pi^3}{96} - \frac{5\pi}{48} + \frac{4}{9}. \quad \square
\]