ON THE BRUHAT-TITS STRATIFICATION OF A QUATERNIONIC UNITARY RAPOPORT-ZINK SPACE

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Abstract. In this note we study the special fiber of the Rapoport-Zink space attached to a quaternionic unitary group. The special fiber is described using the so called Bruhat-Tits stratification and is intimately related to the Bruhat-Tits building of a split symplectic group. As an application we describe the supersingular locus of the related Shimura variety.

1. Introduction

1.1. Motivations. In this note we study the basic locus of a specific Shimura variety. Namely the Shimura variety associated to a quaternionic unitary group. This Shimura variety is particular interesting as it is of PEL-type and is intimately related to Siegel and Orthogonal type Shimura varieties. This note is inspired by the paper [KR00] where the basic locus of this Shimura variety is studied at a prime $p$ where the quaternion algebra is split. Using this description, special cycles on this Shimura variety is defined and their intersection numbers are related to Eisenstein series. In this case the Shimura variety has good reduction and the quaternionic unitary group at $p$ agrees with the symplectic group of degree 4. Therefore the supersingular locus can be studied essentially in the same way as in the case of a Siegel threefold of hyperspecial level. An explicit description of this supersingular locus is available in [KO87]. However in [KR00] the authors are able to relate the description to the Bruhat-Tits building of an inner form of the symplectic group. In this note we treat the case where the quaternion algebra is ramified at $p$ and we describe the supersingular locus with its relation to Bruhat-Tits building. Note in this case the Shimura variety has bad reduction at $p$ and the local model is discussed in this note. In a subsequent work we will define and study special cycles on this Shimura variety.

In general, to describe the supersingular locus of a PEL-type Shimura variety, one can pass to the associated moduli space of $p$-divisible groups then use the uniformization theorem of Rapoport-Zink [RZ96]. These moduli spaces of $p$-divisible groups are known as the basic Rapoport-Zink spaces and we are interested in calculating their special fibers. By passing from $p$-divisible groups to their associated Dieudonné modules, one obtain lattices in an isocrystal. By comparing the relative position between these lattices and the lattices representing the faces of the base alcove of an inner form of

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the underlying group of the PEL-problem, once can parted the Rapoport-Zink space into pieces which are related to classical Deligne-Lusztig varieties. This decomposition is known as the Bruhat-Tits stratification. The terminology come from the pioneering work of [Vol10] where the case of the Rapoport-Zink space of $GU(1, n-1)$ at an inert prime is analyzed. The analysis is complete in [VW10]. This program has been extended to cover other PEL-type problems by many authors. We provide a list of works of this type and acknowledge their influences of this note.

- For $GU(1, n-1)$, $p$-inert and hyperspecial level, this is the work of [Vol10] and [VW10]. In a recent preprint [Cho18], Sungyoon Cho is able to treat many cases of parahoric level structures.
- For $GU(1, n-1)$, $p$-ramified and with level structure related to self-dual lattices, this is the work of [RTW14]. When the level structure is related to a special parahoric and the Rapoport-Zink space admits the so-called exotic good reduction, this is the work of [Wu16].
- For $GU(2, 2)$, $p$-split, this is considered by the author in [Wang18]. For $GU(2, 2)$, $p$-inert, this is done in [HP14] by transferring the problem to $GSpin(4, 2)$. In the process of preparing of this note, we find the methods in this note can be also used to deal with $GU(2, 2)$, $p$-inert directly and this is documented in [Wang18].
- For $GSpin(n-2, 2)$ with hyperspecial level, this is the work of [HP17]. In this case, the Shimura variety is not of PEL-type in general except for small ranks case where exceptional isomorphism happens. For example, $GU(2, 2)$ is essentially $GSpin(4, 2)$ and $GSp(4)$ is essentially $GSpin(3, 2)$. The two Rapoport-Zink space studied in this note are both of non-hypersepcial level and therefore is not covered in [HP17], however they are all $GSpin(3, 2)$-type Shimura varieties.

A general understanding of the Bruhat-Tits stratification is achieved in the powerful work of [GH15] and the subsequent work of [GHN16]. There the problem is studied in the setting of affine Deligne-Lusztig varieties and a general group theoretic method is employed. Their work not only classifies which types of Shimura varieties admit Bruhat-Tits stratification but also gives an algorithm to compute the Deligne-Lusztig varieties occurring in the strataums. The types of Shimura varieties in the above list as well as the two Shimura varieties in this note are covered by their work. A comparison between our results with [GH15] is contained in the very last section of this note.

Besides the Bruhat-Tits stratification, there are many other strategies to study the supersingular locus. We will only mention the works [Hel], [HTX17] and [CV18]. The method used in [Hel], [HTX17] is the so-called isogeny trick and the isogeny is referred to the isogeny between the universal abelian variety on different Shimura varieties. Notice that the isogeny trick used in section 4 is of same nature but in a local setup. The work of [CV18] introduced the so called $J$-stratification for the affine Deligne-Lusztig varieties and is closely related to Bruhat-Tits stratification. In fact it can be shown that the $J$-stratification agrees with the Bruhat-Tits stratification for those classified by [GH15], see [Gor18].
1.2. Results on Rapoport-Zink spaces. We now introduce some notations and state the local results proved in this note. Let $p$ be a prime and let $F$ be an algebraically closed field containing $\mathbb{F}_p$. Let $W(F)$ be the Witt ring of $F$ and $W(F)_Q$ its fraction field. Let $B$ be a quaternion division algebra over $Q$ which ramifies at $p$ and splits at $\infty$. Denote by $\ast$ the main involution of $B$. We fix a maximal order $O_B$ that is stable under $\ast$. Let $N$ be a height 8 isocrystal with slope $\frac{1}{2}$ with $\iota: B_p \to \text{End}(N)$ and an alternating form $(\cdot, \cdot): N \times N \to W(F)_Q$. We are concerned with the following moduli space $p$-divisible groups with additional structure. Let $(\text{Nilp})$ be the category of $N$-valued functor and a polarization $N$-schemes that $p$ is locally nilpotent. We fix a $p$-divisible group $X$ whose associated isocrystal is $N$ and a polarization $\lambda: X \to X^\vee$ corresponding to $(\cdot, \cdot)$ on $N$. We consider the set valued functor $\mathcal{N}$ that sends $S \in (\text{Nilp})$ to the isomorphism classes of the collection $(X, \iota_X, \lambda_X, \rho_X)$ where

- $X$ is a $p$-divisible group of height 8 over $S$;
- $\lambda_X: O_{B_p} \to \text{End}_S(X)$ is an action of $O_{B_p}$ defined over $S$ satisfying certain Kottwitz condition;
- $\lambda_X: X \to X^\vee$ is a principal polarization;
- $\rho_X: X \times S S_0 \to X \times S S_0$ is an $O_{B_p}$-linear quasi-isogeny where $S_0$ is the special fiber of $S$ at $p$.

The moduli space is representable by a separated formal scheme $\mathcal{N}$. The reduced scheme $\mathcal{N}_{\text{red}}$ of the special fiber of $\mathcal{N}$ can be decomposed into $\mathcal{N}_{\text{red}} = \bigcup_{i \in \mathbb{Z}} \mathcal{N}_{\text{red}}(i)$ where each $\mathcal{N}_{\text{red}}(i)$ is isomorphic to $\mathcal{N}_{\text{red}}(0)$. Denote by $\mathcal{M} = \mathcal{N}_{\text{red}}(0)_{\mathbb{F}_p[2]}$. Then our main result concerns the structure of $\mathcal{M}$.

**Theorem 1.** The scheme $\mathcal{M}$ can be decomposed into $\mathcal{M} = \mathcal{M}_{\{0\}} \cup \mathcal{M}_{\{2\}}$. The irreducible components of $\mathcal{M}_{\{0\}}$ and $\mathcal{M}_{\{2\}}$ are all isomorphic to the surface defined by the equation $x_3^3x_0 - x_0^2x_3 + x_2^2x_1 - x_1^2x_2 = 0$ with $x_i$ the projective coordinate on $\mathbb{P}^3$. If an irreducible component in $\mathcal{M}_{\{0\}}$ intersect with an irreducible component of $\mathcal{M}_{\{2\}}$ non-trivially, then the intersection is isomorphic to $\mathbb{P}^1$. If an irreducible component in $\mathcal{M}_{\{0\}}$ intersect with an irreducible component of $\mathcal{M}_{\{0\}}$ non-trivially, then the intersection is a point which is superspecial. If an irreducible component in $\mathcal{M}_{\{2\}}$ intersect with an irreducible component of $\mathcal{M}_{\{2\}}$ non-trivially, then the intersection is a point which is superspecial.

The decomposition in Theorem 1 can be made finer and we have the Bruhat-Tits stratification of $\mathcal{M}$.

**Theorem 2.** We have the Bruhat-Tits stratification

$$\mathcal{M} = \mathcal{M}_{\{0\}}^0 \cup \mathcal{M}_{\{2\}}^0 \cup \mathcal{M}_{\{0,2\}}^0 \cup \mathcal{M}_{\{1\}}.$$ 

The stratum $\mathcal{M}_{\{0\}}^0$ is open and dense in $\mathcal{M}_{\{0\}}$ and the complement is precisely $\mathcal{M}_{\{0,2\}}^0 \cup \mathcal{M}_{\{1\}}$. Similarly the stratum $\mathcal{M}_{\{2\}}^0$ is open and dense in $\mathcal{M}_{\{2\}}$ and the complement is precisely $\mathcal{M}_{\{0,2\}}^0 \cup \mathcal{M}_{\{1\}}$. The irreducible components of $\mathcal{M}_{\{0\}}^0$, $\mathcal{M}_{\{2\}}^0$ and $\mathcal{M}_{\{0,2\}}^0$ can be identified with classical Deligne-Lusztig varieties.

We explain the connection of this stratification with Bruhat-Tits building. The quaternionic unitary Rapoport-Zink space is acted upon by the split symplectic group.
of 4-variables. If we label the vertices of a base alcove of the Bruhat-Tits building of \( \text{Sp}(4) \) by \( \{0, 1, 2\} \), then the strata:\[ M_0 \{0\}, M_2 \{2\}, M_{\{1\}} \text{ and } M_{\{0, 2\}} \] correspond to precisely the vertices 0, 2, 1 and the face \( \{0, 2\} \). The intersection behaviour is also determined by the incidence relation among faces of the Bruhat-Tits building. We refer the reader to section 5.1 for details.

The quaternionic unitary Rapoport-Zink space in this note is closely related to the Rapoport-Zink space for \( \text{GSp}(4) \). Their relation is formally similar to the relation between the Lubin-Tate space and the Drinfeld upper-half plane. But from a group theoretic point of view or from the point of view of local models, the quaternionic unitary Rapoport-Zink space is closer to the Rapoport-Zink space for \( \text{GSp}(4) \) with the so-called paramodular level structure instead of with the hyperspecial level. Therefore we also include the calculation for the Bruhat-Tits stratification of the Rapoport-Zink space for \( \text{GSp}(4) \) with the paramodular level. However we should point out the description of the supersingular locus for the corresponding Shimura variety in the paramodular case is already known, we refer the reader to [Yu06] and [Yu11] for the statement and proof of this result. The calculations in [Yu06] and [Yu11] do not involve Rapoport-Zink space or Bruhat-Tits building and are indirect in the sense that they involve Shimura varieties with different level structures. Our calculation is direct and it is interesting to compare the two cases.

1.3. Applications to the supersingular locus. We consider the following moduli problem \( S_{h_{U'}p} \) in [KR00]: to a scheme \( S \) over \( \mathbb{Z} \), we classify the set of isomorphism classes of the collection \( \{(A, \iota, \lambda, \eta)\} \) where:

- \( A \) is an abelian scheme of relative dimension 4 over \( S \);
- \( \iota : \mathcal{O}_B \to \text{End}_S(A) \otimes \mathbb{Z} \) is a morphism such that \( \lambda \circ \iota(a^*) = \iota(a)^\vee \circ \lambda \) and satisfies the Kottwitz condition

\[
\det(a, \text{Lie}(A)) = N^0(a)^2
\]

for all \( a \in \mathcal{O}_B \) and \( N^0 \) is the reduced norm of \( B \). This should be understood as a polynomial identity.

- \( \lambda : A \to A^\vee \) is a prime to \( p \) polarization which is \( \mathcal{O}_B \)-linear.

- \( \eta \) is a suitable level structure.

We are interested in the closed subscheme \( S_{h_{U'}p}^{ss} \) in \( S_{h_{U'}p} \otimes \mathbb{F} \) known as the supersingular locus. The uniformization theorem of Rapoport-Zink [RZ96] Theorem 6.1 transfers the problem of describing the supersingular locus to that of describing the Rapoport-Zink space. More precisely there is an isomorphism of \( \mathbb{F} \) schemes

\[
S_{h_{U'}p}^{ss} \cong I(\mathbb{Q}) \backslash N_{\text{red}, \mathbb{F}} \times G(k_f^p)/U^p.
\]

where \( I \) is an inner form of \( G = \text{GU}_B(2) \).

**Theorem 3.** The scheme \( S_{h_{U'}p}^{ss} \) is pure of dimension 2. For \( U^p \) sufficiently small, the irreducible components are isomorphic to the surface \( x_0^3x_0 - x_0^2x_3 + x_2^2x_1 - x_1^2x_2 = 0 \). The intersection of two irreducible components is either empty or is isomorphic to a \( \mathbb{P}^1 \) or is a point.
As mentioned previously, we also treat the case of a Siegel threefold with paramodular level structures. This is the content of section 7.

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1.5. Notations. Let p be a prime and let $\mathbb{F}$ be an algebraically closed field containing $\mathbb{F}_p$. Let $W(\mathbb{F})$ be the Witt ring of $\mathbb{F}$ and $W(\mathbb{F})_\mathbb{Q}$ its fraction field. Denote by $\psi_0$ and $\psi_1$ the two embeddings of $\mathbb{F}_p^2$ in $\mathbb{F}$. Let $\sigma$ be the Frobenius on $\mathbb{F}$. Let $M_1 \subset M_2$ be two $W(\mathbb{F})$-modules we write $M_1 \subset^d M_2$ if the colength of the inclusion is $d$. If $L$ is a $\mathbb{Z}_p$-module, then we define $L_\mathbb{F} = L \otimes W(\mathbb{F})$. Let $B$ be a quaternion division algebra over $\mathbb{Q}$ which ramifies at $p$ and splits at $\infty$. Denote by $\ast$ the main involution of $B$. We fix a maximal order $\mathcal{O}_B$ that is stable under $\ast$.

2. Quaternionic unitary Rapoport-Zink space

2.1. Isocrystal with quaternionic multiplication. Let $N$ be a height 8 isocrystal of slope $\frac{1}{2}$ with $\iota : B_p \rightarrow \text{End}(N)$. We can write $B_p$ as $\mathbb{Q}_p^2 + \mathbb{Q}_p^2 \Pi$ with $\Pi^2 = p$. Suppose $N$ is equipped with an alternating form $(\cdot, \cdot) : N \times N \rightarrow W(\mathbb{F})$ with the property that $(Fx, y) = (x, V y)^{\sigma}$ and $(bx, y) = (x, b^* y)$ where $b^*$ is the main involution on the quaternion algebra. The action of $\text{End}(B_p)$ decomposes $N$ into $N = N_0 \oplus N_1$. Restricting the symplectic form $(x, y)_0 := (x, \Pi y)$ to $N_0$ gives an identification between the group $\text{GU}_{B_p}(N)$ and $\text{GSp}(N_0)$ over $W(\mathbb{F})_\mathbb{Q}$ cf. [RZ96] 1.42.

We will use covariant Dieudonné theory throughout this note. A Dieudonné lattice $M$ is a lattice in $N$ with the property that $pM \subset FM \subset M$. A Dieudonné lattice is superspecial if $F^2 M = pM$ and in this case $F = V$. We are concerned with Dieudonné lattice with additional endomorphism $\iota : \mathcal{O}_{B_p} \rightarrow \text{End}(M)$. Here we can present $\mathcal{O}_{B_p}$ as $\mathbb{Z}_p^2 + \mathbb{Z}_p^2 \Pi$ and hence we can decompose $M$ as $M_0 \oplus M_1$ with an additional operator $\Pi$ that swaps the two components. The alternating form $(\cdot, \cdot)$ restricts to $M$ and induces a pairing $(\cdot, \cdot) : M_0 \times M_1 \rightarrow W(\mathbb{F})$.

2.2. Rapoport-Zink space for quaternionic unitary group. We consider the following PEL-type Rapoport-Zink space. Let $(\text{Nilp})$ be the category of $\mathbb{Z}_p$-schemes that $p$ is locally nilpotent. We fix a $p$-divisible group $X$ whose associated isocrystal is $N$ and a polarization $\lambda : X \rightarrow X^\vee$ corresponding to $(\cdot, \cdot)$ on $N$. We consider the set valued functor $\mathcal{N}$ that sends $S \in (\text{Nilp})$ to the isomorphism classes of the collection $(X, \iota_X, \lambda_X, \rho_X)$ where

- $X$ is a $p$-divisible group of height 8;
- $\iota_X : \mathcal{O}_{B_p} \rightarrow \text{End}_S(X)$ is an action of $\mathcal{O}_{B_p}$;
- $\lambda_X : X \rightarrow X^\vee$ is a $p$-principal polarization;
\(- \rho_X : X \times_S S_0 \to \mathbb{X} \times_S S_0\) is an \(\mathcal{O}_{B_p}\)-linear quasi-isogeny.

We require that \(\iota_X\) satisfies the Kottwitz condition
\[
\det(\iota(c); \text{Lie}(X)) = N^0(c)^2
\]
for \(c \in \mathcal{O}_{B_p}\) here the equality should be understood as an identity of polynomial functions and \(N^0(c)\) is the reduced norm of \(c\). For \(\rho_X : X \times_S S_0 \to \mathbb{X} \times_S S_0\), we require that
\[
\rho^*_X \lambda_X = c(\rho) \lambda_X
\]
for a \(\mathbb{Q}_p\)-multiple \(c(\rho)\).

**Lemma 2.3.** The functor \(\mathcal{N}\) is representable by a separated formal scheme locally formally of finite type over \(\text{Spf}(\mathbb{Z}_p)\). Moreover \(\mathcal{N}\) is flat and formally smooth away from the superspecial point.

**Proof.** The representability follows from [RZ96] and second assertion on flatness and the singularity relies on the analysis of the local model below. We will see after unramified base change it is equivalent the local model for the split symplectic group of dimension 4 with paramodular level. Then the result we need here is proved in [Yu11]. \(\square\)

### 2.3. Local model

The local model for this PEL-type Rapoport-Zink space is defined as in [RZ96] definition 3.27. Let \(M^{\text{loc}}\) be the functor that to each \(\mathbb{Z}_p\)-scheme \(S\) the \(S\)-point \(M^{\text{loc}}(S)\) is the set of \((\Lambda, \mathcal{F}_\Lambda)\) where \(\Lambda\) is a \(\mathcal{O}_{B_p}\)-module of \(\mathbb{Z}_p\)-rank 8 and \(\mathcal{F}_\Lambda \subset \Lambda \otimes \mathcal{O}_{S}\) is a locally free \(\mathcal{O}_{B_p} \otimes \mathbb{Z}_p \otimes \mathcal{O}_{S}\) submodule of rank 4. The module \(\Lambda\) is equipped with a alternating form \((\cdot, \cdot)\) that satisfies \((bx, y) = (x, b^*y)\) and we require that \(\Lambda \cong \Lambda^\perp\) where \(\Lambda^\perp\) is the dual of \(\Lambda\) with respect to the form \((\cdot, \cdot)\) and that \(\mathcal{F}_\Lambda\) is isotropic. As we have seen in 2.1 for each \((\Lambda, \mathcal{F}_\Lambda)\) we have a decomposition \(
\Lambda \otimes \mathbb{Z}_p^2 = \Lambda_0 \oplus \Lambda_1 \supset \mathcal{F}_\Lambda \otimes \mathbb{Z}_p^2 = \mathcal{F}_0 \oplus \mathcal{F}_1\)\end{equation}
. The conditions we put on \((\Lambda, \mathcal{F}_\Lambda)\) translates to the conditions that \(\Lambda_0^\perp = \Lambda_1\), \(\Pi \Lambda_0 \subset \Lambda_1\) and \(\Pi \mathcal{F}_0 \subset \mathcal{F}_1\), \(\Pi \Lambda_1 \subset \Lambda_0\) and \(\Pi \mathcal{F}_1 \subset \mathcal{F}_0\). We define a new alternating form on \(\Lambda\) by \((x, y)_0 = (x, \Pi y)\). With respect to this new form, we see that the pair \((\Lambda, \mathcal{F}_\Lambda)\) is equivalent to the pair \((\Lambda_0, \mathcal{F}_0)\) that satisfies the condition \(p\Lambda_0^\perp \subset \Lambda_0 \subset \Lambda_0^\perp\), \(\mathcal{F}_0\) is isotropic with respect to \((\cdot, \cdot)_0\). For example the first condition can be seen from the following computation,

\[
\Pi \Lambda_1 = \Pi \Lambda_0^\perp = \{ x \in \Lambda_0 \otimes \mathbb{Q}_p : (\Lambda_0, \Pi x) \subset \mathbb{Z}_p^2 \} = \{ x : (\Lambda_0, \frac{1}{p} \Pi x) \subset \mathbb{Z}_p^2 \} = p \Lambda_0^\perp.
\]

Hence after base change to \(\mathbb{Z}_p^2\) we see that \(M^{\text{loc}}_{\mathbb{Z}_p^2}\) is equivalent to the functor \(M^{\text{loc}}_{\{1\}, \mathbb{Z}_p^2}\) such that for a \(\mathbb{Z}_p^2\)-scheme \(S\), \(M^{\text{loc}}_{\{1\}, \mathbb{Z}_p^2}(S) = \{(\Lambda, \mathcal{F}_\Lambda)\}\) and where \(\Lambda\) is a \(\mathbb{Z}_p\)-module of rank 4 and \(\mathcal{F}_\Lambda \subset \Lambda \otimes \mathcal{O}_{S}\) is locally free of rank 2. And we require that \(p\Lambda^\perp \subset \Lambda \subset \Lambda^\perp\) for an alternating form \((\cdot, \cdot)_0\) on \(\Lambda\) and \(\mathcal{F}_\Lambda\) is isotropic with respect to \((\cdot, \cdot)_0\). The latter local model \(M^{\text{loc}}_{\{1\}}\) arises when one studies the Siegel threefold with paramodular level structure and we refer the reader to the article [Yu11] for the study of this local model. The following theorem is [Yu11] Theorem 1.3.
Theorem 2.4. The special fiber of the local model $M_{\{1\}, W(F)}^{\text{loc}}$ is singular in a discrete set of points and the formal completion of the local ring at a singular point in the special fiber is given by

$$\mathbb{F}[\![x_{11}, x_{12}, x_{21}, x_{22}]!/ (x_{11}x_{22} - x_{12}x_{21})].$$

2.4. Points of the Rapoport-Zink space. By passing to Dieudonné modules, the $\mathbb{F}$-points of the Rapoport-Zink space $N(F)$ can be identified with the following set:

$$N(F) = \{ M \subset N : M^\perp = cM, pM \subset VM \subset M, \Pi M_0 \subset^2 M_1, \Pi M_1 \subset^2 M_0, VM_1 \subset^2 M_0, VM_2 \subset^2 M_1 \}.$$  

(2.5)

The last two conditions of the definition follows from the Kottwitz condition and similar condition for $\Pi$ follows from the following commutative diagram:

$$\begin{array}{ccc}
M_0 & \overset{V}{\rightarrow} & M_1 \\
\downarrow^{\Pi} & & \downarrow^{\Pi} \\
M_1 & \overset{V}{\rightarrow} & M_0.
\end{array}$$

We claim the following Pappas condition on the special fiber of the Rapoport-Zink space is automatic in our case:

$$\wedge^2 (\iota(\Pi); \text{Lie}(X)_i) = 0 \text{ for } i = 0, 1.$$  

(2.6)

Indeed, since we have the composite of $\Pi : M_0/VM_1 \rightarrow M_1/VM_0$ and $\Pi : M_1/VM_0 \rightarrow M_0/VM_1$ being multiplication by $p$, at least one of them is not invertible. Note also that we have $(\Pi x, y) = (x, \Pi y)$, the other is also not invertible.

On the level of $\mathbb{F}$-points, this translates to

$$\text{dim}_\mathbb{F} VM_0 + \Pi M_0/VM_0 \leq 1 \text{ and dim}_\mathbb{F} VM_1 + \Pi M_1/VM_1 \leq 1.$$  

(2.7)

We can restrict to the component of $N(0)$ that classifies degree 0 quasi-isogenies in the definition of the Rapoport-Zink space i.e $c = 1$ and $M = M^\perp$ in $N$ . In light of what we discussed above that the identification between $\text{GU}_{B_p}(2)$ and $\text{GSp}(4)$ over $W(F)_Q$ is via restricting to the 0-th component $N_0$, we will similarly restrict Dieudonné lattices to their 0-th component. We put $\tau := \Pi V^{-1}$ and for $M \subset N_0$ we denote by $M^\vee$ the dual of $M$ in $N_0$ with respect to $(\cdot, \cdot)_0$. The set theoretic description (2.5) of the Rapoport-Zink space then can be reformulated in terms of lattices in $N_0$ with the symplectic form $(\cdot, \cdot)_0$. We are interested in the reduced scheme structure of $N_{\text{red}}(0)$ modulo $p$ and we denote this reduced scheme by $N_{\text{red}}(0)$. Denote by $M$ the reduced scheme $N_{\text{red}}(0) \otimes \mathbb{F}_{p^2}$. The $\mathbb{F}$-points of $M$ are given as follows.

Proposition 2.8. There is a bijection between $M(\mathbb{F})$ and the following set

$$M(\mathbb{F}) = \{ D \subset N_0 : pD^\vee \subset^2 D \subset^2 D^\vee, pD^\vee \subset^2 \tau(D) \subset^2 D^\vee \}.$$  

The map is sending $M = M_0 \oplus M_1$ to $D = M_0$. Moreover dim$_\mathbb{F} D + \tau(D)/D \leq 1$.

Proof. The first condition corresponds to the condition $\Pi M_0 \subset^2 M_1$. The second condition corresponds to the Kottwitz condition. We only spell out the details for the
second condition as similar computation is used throughout the notes. Since \( pM_2 \subset FM_1 \subset M_2 \), we have \( \Pi M_2 \subset \Pi V^{-1}M_1 = \tau(M_1) \subset \Pi^{-1}M_2 \). Notice we have:
\[
\Pi M_2 = \Pi M_1^\perp = \{ x \in N_2 : (M_1, \Pi x) \subset W(F) \} = \{ x : (M_1, 1_p \Pi x) \subset W(F) \} = pM_1^\perp.
\]
The second condition follows. The moreover part comes from the Pappas condition (2.7).

3. Bruhat-Tits Stratification for quaternionic unitary group

3.1. Deligne-Lusztig varieties. Let \( G_0 \) be a connected reductive group over \( \mathbb{F}_p \) and let \( G = G_0 \otimes \mathbb{F} \) be the base change to \( \mathbb{F} \). Let \( T \) be a maximal torus contained in \( G \) and \( B \) a Borel subgroup containing \( T \). We can assume \( T \) is defined over \( \mathbb{F}_p \). Let \( W \) be the Weyl group corresponding to \( (T, B) \). Then \( W \) affords an action by \( \sigma \) induced by the Frobenius action on \( G \).

Let \( \Delta^* = \{ \alpha_1, \cdots, \alpha_n \} \) be the simple roots corresponding to \( (T, B) \) and let \( s_i \) be the simple reflection corresponding to the root \( \alpha_i \). For \( I \subset \Delta^* \), let \( W_I \) be the subgroup of \( W \) generated by \( \{ s_i, i \in I \} \). Consider \( P_I \) the corresponding parabolic subgroup of \( G \).

For another set \( J \subset \Delta^* \), we have a decomposition
\[
G = \bigcup_{w \in W_I \backslash W/W_J} P_I w P_J.
\]
and hence a bijection
\[
P_I \backslash G/P_J \cong W_I \backslash W/W_J.
\]
We define the relative position map
\[
\text{inv} : G/P_I \times G/P_J \to W_I \backslash W/W_J
\]
by sending \((g_1, g_2)\) to \( g_1^{-1}g_2 \in P_I \backslash G/P_J \).

**Definition 3.1.** For a given \( w \in W_I \backslash W/W_{\sigma(I)} \), the Deligne-Lusztig variety \( X_{P_I}(w) \) is the locally closed reduced subscheme of \( G/P_I \) whose \( \mathbb{F} \)-points is described by
\[
X_{P_I}(w) = \{ gP_I \in G/P_I ; \text{inv}(g, \sigma(g)) = w \}.
\]

3.2. The symplectic group. Let \( G_0 \) be the symplectic group over \( \mathbb{F}_p \) defined by a symplectic space \((V, \langle \cdot, \cdot \rangle)\) of dimension 4 over \( \mathbb{F}_p \). We choose a basis \((e_1 \cdots e_4)\) of \( V \) that \((e_i, e_5-j) = \pm \delta_{i,j} \) for \( i = 1, \cdots, 4 \). The simple reflections in \( W \) can be understood as follows, as elements in \( S_4 \),
- \( s_1 \) is \((12)(34)\);
- \( s_2 \) is \((23)\);
- \( s_1 s_2 \) is \((1243)\).

We define the elements \( w_1, w_2 \in W \) by \( w_1 = s_1 \) and \( w_2 = s_1 s_2 \).

Denote by \( G \) the group \( G = G_0 \otimes \mathbb{F} \). We are concerned with the Deligne-Lusztig variety \( X_{P_I}(w) \) for \( I = \{ 2 \} \) where \( P_{\{2\}} \) is the Klingen parabolic. Consider the following closed subvarieties of \( G/P_{\{2\}} \).
(3.2) \( Y_V = \{ U \subset V_\mathbb{F}; \dim_\mathbb{F} U = 1, U \subset U_\perp, \dim_\mathbb{F} U + \sigma(U)/U \leq 1, U \subset U_\perp \cap \sigma(U_\perp) \}; \)
and
(3.3) \( Y_V = \{ U \subset V_\mathbb{F}; \dim_\mathbb{F} U = 3, U_\perp \subset U, \dim_\mathbb{F} U/\sigma(U) \leq 1, U_\perp \subset U \cap \sigma(U) \} \).

They are obviously isomorphic. We would like to define a stratification on \( Y_V \).

**Theorem 3.4.** There is a decomposition of \( Y_V \) into disjoint union of locally closed subvarieties

\[
Y_V = X_{P_{(2)}}(1) \cup X_B(w_1) \cup X_B(w_2)
\]

where \( w_1 \) and \( w_2 \) are as in (3.2). Here \( X_{P_{(2)}}(1) \) is closed and of dimension 0, \( X_B(w_1) \) is of dimension 1 and \( X_B(w_2) \) is two dimensional. Moreover \( Y_V \) is the closure of \( X_B(w_2) \).

**Proof.** We use the description of \( Y_V \) as in (3.3). One has \( Y_V = X_{P_{(2)}}(1) \cup X_{P_{(2)}}(w_1) \) and \( X_{P_1}(w_1) = X_B(w_1) \cup X_B(w_2) \). In fact using the description as (3.3) one has the following descriptions

- \( X_{P_{(2)}}(1) \) consists of \( U \) that is \( \sigma \)-stable;
- \( X_B(w_1) \) consists of \( U \) that is not \( \sigma \)-stable but \( U \cap \sigma(U) \) is \( \sigma \)-stable and is a totally isotropic plane;
- \( X_B(w_2) \) consists of \( U \) that is not \( \sigma \)-stable and \( U \cap \sigma(U) \) is not \( \sigma \)-stable but is a totally isotropic plane.

There is a more concrete description of the each stratum. The closed stratum \( X_{P_{(2)}}(1) \) consists of \( \mathbb{F}_{p^2} \)-points of \( Y_V(\mathbb{F}) \).

The irreducible components of \( X_B(w_1) \) are the complement of \( \mathbb{F}_{p^2} \)-points in \( \mathbb{P}^1 \). Indeed, since \( U \) is not \( \tau \)-stable, \( U \cap \tau(U) \) is of dimension 2. Moreover \( \bar{U} \cap \tau(U) \) has the property that \( \bar{U} \cap \tau(U) = (U \cap \tau(U))^\perp = (U \cap \tau(U)) \perp \). The line \( U_\perp \subset U \cap \tau(U) \) gives a point in \( \mathbb{P}^1 \). This point is not \( \mathbb{F}_{p^2} \)-rational as \( U \) is not \( \tau \)-stable. Conversely, given any \( \tau \)-stable plane \( T \) in \( V \) with the property that \( T = T^\perp \) and a line \( U \subset T \), \( U \) gives a point in \( X_B(w_1) \).

The open stratum \( X_B(w_2) \) is the surface defined as in [DL76] 2.4. It is the surface defined by the set of \( x \in \mathbb{P}(V) \) such that \( (x, \sigma(x)) = 0 \) and \( (x, \sigma^2(x)) \neq 0 \). By choosing a coordinate system of \( \mathbb{P}(V) \), say \( x_0, x_1, x_2, x_3 \), we see it is defined by removing from the surface \( x_0^p x_0 - x_0^p x_3 + x_0^p x_1 - x_0^p x_2 = 0 \) a collection of \( \mathbb{P}^1 \), one for each isotropic plane. The closure of \( X_B(w_2) \) is \( Y_V \) and it is the surface defined by \( x_0^p x_0 - x_0^p x_3 + x_0^p x_1 - x_0^p x_2 = 0 \). □

### 3.3. Vertex lattices in \( \text{GSp}(4) \).

The local Dynkin diagram of type \( \tilde{C}_2 \) is

```
0 ———— 1 ———— 2
```

where the nodes 0 and 2 are special and 1 is not special in the sense that the parahoric subgroup corresponding to 0 and 2 are special parahoric subgroups.

The vertices of a base alcove of the building for \( \text{Sp}(4) \) is identified with the nodes in the Dynkin diagram. We label the vertices by the set \( \{0, 1, 2\} \). The vertex 0 corresponds
to a lattice $L$ with $pL^\vee \subset 4 \subset L \subset 0 \subset L^\vee$, the vertex 1 corresponds to a lattice $L$ with $pL^\vee \subset 2 \subset L \subset 2 \subset L^\vee$ and the vertex 2 corresponds to a lattice $L$ with $pL^\vee \subset 0 \subset L \subset 4 \subset L^\vee$.

Let $b$ be an element in its $\sigma$-conjugacy class which gives the isocrystal $N$ and let $J_b$ be the group of isomorphisms of $N$ that respect the action of $B_p$ and the polarization $(\cdot, \cdot)$. Consider the slope zero isocrystal $(N_0, \tau)$, by the theorem of Dieudonné it gives rise to a factorization $N_0 = C \otimes_{\mathbb{Q}_b} W(\mathbb{F})_{\mathbb{Q}}$.

**Lemma 3.6.** There is an isomorphism between the group $J_b$ the $\sigma$-conjugate centralizer of $b$ which gives the isocrystal $N$ and the group $\text{GSp}(C, (\cdot, \cdot)_0)$.

**Proof.** By definition, we have $J_b = \{ g \in \text{GU}_{B_p}(N); gF = Fg \}$. Since $g$ commutes with the action of $B_p$, it respects the decomposition of $N = N_0 \oplus N_1$. Since it also commutes with the Frobenius $F$, the action is uniquely determined on its restriction to $N_0$. The result then follows from the fact that $g$ also commutes with $\tau$. This claim is also stated in [RZ96] 1.42. \qed

We consider lattices in $C$. We call a lattice $L \subseteq C$ a vertex lattice if it corresponds to a vertex in the base alcove as above and we call $L$ a $i$-vertex if it corresponds to the $i$-th vertex as in the description above for $i \in \{0, 1, 2\}$.

**Definition 3.7.** For each vertex lattice, we define the associated set theoretic lattice stratum as follows

1. For a 0-vertex $L_0$, the stratum $\mathcal{M}_{L_0}(\mathbb{F})$ is the set $\{ D \in \mathcal{M}(\mathbb{F}) : D \subseteq L_0, F \}$;
2. For a 2-vertex $L_2$, the stratum $\mathcal{M}_{L_2}(\mathbb{F})$ is the set $\{ D \in \mathcal{M}(\mathbb{F}) : L_2 \subseteq D \}$;
3. For a 1-vertex $L_1$, the stratum $\mathcal{M}_{L_1}(\mathbb{F})$ is the set $\{ D \in \mathcal{M}(\mathbb{F}) : L_1 \subseteq D \}$.

To show these lattice strata give reasonable subsets of $\mathcal{M}(\mathbb{F})$, we will rely on an analogue of the crucial lemma as in [Vol10] Lemma 2.1.

**Proposition 3.8.** Given $D \in \mathcal{M}(\mathbb{F})$, we can find a $\tau$ stable lattice $L(D)$ in $N_0$ that either fits in the following chain of lattices:

\[(3.9) \quad pL(D)^\vee \subset pD^\vee \subset D \subset L(D) \subset L(D)^\vee \subset D^\vee \]
or it fits in

\[(3.10) \quad pD^\vee \subset pL(D)^\vee \subset L(D) \subset D \subset D^\vee \subset L(D)^\vee .\]

**Proof.** The proof of this is similar to [Vol10] Lemma 2.1 but there the Kottwitz condition is different.

Suppose $D$ is $\tau$-stable, then there is nothing to prove and we simply let $L(D) = D$. Otherwise, the Pappas condition shows that $D \subseteq D + \tau(D)$. Suppose that $D + \tau(D)$ is $\tau$-stable, then we set $L(D) = D + \tau(D)$. Notice that Since $D \in \mathcal{D}(\mathbb{F})$, we have

\[(3.11) \quad pD^\vee \subset D \subset D^\vee \quad \text{and} \quad pD^\vee \subset \tau(D) \subset D^\vee .\]

Therefore $L(D) = D + \tau(D) \subseteq L(D)^\vee = D^\vee \cap \tau(D)^\vee$ and $L(D)$ fits in (3.9).

Now suppose $D + \tau(D)$ is not $\tau$-stable and we claim that $D \cap \tau(D)$ is $\tau$-stable. Indeed, suppose otherwise $D \cap \tau(D)$ is not $\tau$-stable, by (3.11), we have $p\tau(D)^\vee \subset 1 \subset D \cap \tau(D)$.
and \( p\tau(D)^\vee \subset \tau(D) \cap \tau^2(D) \). Hence

\[
D \cap \tau(D) \cap \tau^2(D) = p\tau(D)^\vee.
\]

(3.12)

Now, we consider the family of lattices \( L_j(D) = D + \tau(D) + \cdots + \tau^j(D) \). There is a minimal \( d \) such that \( L_d(D) = \tau(L_d(D)) \) by [RZ96] Proposition 2.7 and we set \( L(D) = L_d(D) \). From the Pappas condition in 2.8 we deduce that

\[
\tau(L_{j-2}(D)) \subset L_{j-1}(D) \subset L_j(D)
\]

and

\[
\tau(L_{j-2}(D)) \subset \tau(L_{j-1}(D)) \subset L_j(D).
\]

It follows that

(3.13)

\[
\tau(L_{j-2}(D)) = L_{j-1}(D) \cap \tau(L_{j-1}(D)).
\]

By (3.11) \( D + \tau(D) \subset D^\vee \subset p^{-1}\tau(D) \) and \( \tau(D) + \tau^2(D) \subset \tau(D)^\vee \subset p^{-1}\tau(D) \). Then we have \( D + \tau(D) + \tau^2(D) \subset p^{-1}\tau(D) \). Thus

\[
L_d(D) = D + \tau(D) + \cdots + \tau^d(D) \subset p^{-1}\tau(D) + p^{-1}\tau^2(D) + \cdots p^{-1}\tau^{d-1}(D) \subset p^{-1}L_{d-1}(D).
\]

Applying \( \tau^l \) for any integer \( l \) on both sides, we get \( L_d(D) \subset p^{-1}\bigcap_{l \in \mathbb{Z}} \tau^l(L_{d-1}(D)) \). It follows from (3.13), \( \bigcap_{l \in \mathbb{Z}} L_{d-1}(D) = \bigcap_{l \in \mathbb{Z}} L_{d-2}(D) = \cdots = \bigcap_{l \in \mathbb{Z}} D \). Hence

\[
L_d(D) \subset p^{-1}\bigcap_{l \in \mathbb{Z}} \tau^l(L_{d-1}(D)) = p^{-1}\bigcap_{l \in \mathbb{Z}} \tau^l(D).
\]

Now we apply (3.12) and we find

\[
L_d(D) \subset p^{-1}\bigcap_{l \in \mathbb{Z}} \tau^l(L_{d-1}(D)) = p^{-1}\bigcap_{l \in \mathbb{Z}} \tau^l(D) = \bigcap_{l \in \mathbb{Z}} \tau^l(D)^\vee = L_d(D)^\vee.
\]

Then \( L_d(D) \) satisfies (3.9) and it follows that the indices among the lattices has to be \( pL_d(D)^\vee \subset \tau(D) \subset pD^\vee \subset D \subset L(D) \subset \tau(D) \subset D^\vee \). This implies \( L_d(D) = D + \tau(D) \) and this is a contradiction. Hence \( D + \tau(D) \) is not \( \tau \)-stable. This proves in this case \( D \cap \tau(D) \) is \( \tau \)-stable and we set \( L(D) = D \cap \tau(D) \). Then one checks easily as before that \( L(D) \) satisfies (3.10).

\[\square\]

**Definition 3.14.** For \( i = 0, 1, 2 \), we define

\[
\mathcal{M}_{\{i\}}(\mathbb{F}) = \bigcup_{L_i} \mathcal{M}_{L_i}(\mathbb{F})
\]

where the union is taken over all the vertex lattices of type \( i \). We call \( \mathcal{M}_{\{i\}} \) the \( i \)-vertex stratum.

**Corollary 3.15.** We have a decomposition of the set \( \mathcal{M}(\mathbb{F}) \) by

\[
\mathcal{M}(\mathbb{F}) = \mathcal{M}_{\{0\}}(\mathbb{F}) \cup \mathcal{M}_{\{1\}}(\mathbb{F}) \cup \mathcal{M}_{\{2\}}(\mathbb{F}).
\]
Definition 3.16. We define the following subsets of the lattice stratum \( \mathcal{M}_{L_0}(\mathbb{F}) \).

1. \( \mathcal{M}_{L_0}^0(\mathbb{F}) = \mathcal{M}_{L_0}(\mathbb{F}) \setminus (\mathcal{M}_{\{2\}} \cup \mathcal{M}_{\{1\}}) \);
2. \( \mathcal{M}_{L_0,\{2\}}(\mathbb{F}) = \mathcal{M}_{L_0}(\mathbb{F}) \cap \mathcal{M}_{\{2\}}(\mathbb{F}) \);
3. \( \mathcal{M}_{L_0,\{2\}}^0(\mathbb{F}) = (\mathcal{M}_{L_0}(\mathbb{F}) \cap \mathcal{M}_{\{2\}}(\mathbb{F})) \setminus \mathcal{M}_{\{1\}}(\mathbb{F}) \);
4. \( \mathcal{M}_{L_0,\{1\}}(\mathbb{F}) = \mathcal{M}_{L_0}(\mathbb{F}) \cap \mathcal{M}_{\{1\}}(\mathbb{F}). \)

Then \( \mathcal{M}_{L_0}(\mathbb{F}) \) admits the set theoretic Bruhat-Tits stratification
\[
\mathcal{M}_{L_0}(\mathbb{F}) = \mathcal{M}_{L_0}^0(\mathbb{F}) \cup \mathcal{M}_{L_0,\{2\}}(\mathbb{F}) \cup \mathcal{M}_{L_0,\{1\}}(\mathbb{F}).
\]

We have the following descriptions of them in terms of a Deligne-Lusztig variety. We define the symplectic space \( V(L_0) \) of dimension 4 by \( L_0/pL_0^\vee \) and we put \( Y_{L_0} = Y_{V(L_0)}. \)

Proposition 3.17. There is a bijection between \( \mathcal{M}_{L_0}(\mathbb{F}) \) and \( Y_{L_0}(\mathbb{F}). \) This bijection is compatible with the stratification on both sides in the sense that the stratum \( \mathcal{M}_{L_0}^0(\mathbb{F}) \) is in bijection with \( X_B(w_2)(\mathbb{F}), \mathcal{M}_{L_0,\{2\}}^0(\mathbb{F}) \) is in bijection with \( X_B(w_1)(\mathbb{F}) \) and \( \mathcal{M}_{L_0,\{1\}}(\mathbb{F}) \) is in bijection with \( X_{F(2)}(1)(\mathbb{F}). \)

Proof. Given a point \( D \in \mathcal{M}_{L_0} \), we have the following chain of inclusions:
\[
pL_0^\vee \subset pD^\vee \subset D \subset L_0, \mathbb{F}.
\]
Let \( V(L_0) = L_{0,\mathbb{F}}/pL_{0,\mathbb{F}}^\vee \), then sending \( D \) to the flag
\[
0 \subset U^1 = pD^\vee /pL_{0,\mathbb{F}}^\vee \subset U = D/pL_{0,\mathbb{F}}^\vee \subset L_{0,\mathbb{F}}/pL_{0,\mathbb{F}}^\vee
\]
gives a map from \( \mathcal{M}_{L_0}(\mathbb{F}) \) to \( Y_{L_0}(\mathbb{F}). \)

Given \( D \in \mathcal{M}_{L_0}^0(\mathbb{F}), D \cap \tau(D) \) is not \( \tau \)-stable by definition of \( \mathcal{M}_{L_0}^0 \) and hence \( U \cap \sigma(U) \) is not \( \sigma \)-stable. This gives a point in \( X_B(w_2). \)

Given \( D \in \mathcal{M}_{L_0,\{2\}}(\mathbb{F}), \) then
\[
pL_0^\vee \subset pD^\vee \subset L_{2,\mathbb{F}} \subset D \subset L_{0,\mathbb{F}}.
\]
Hence comparing \( D \cap \tau(D) \) and \( L_{2,\mathbb{F}} \) shows that \( L_{2,\mathbb{F}} = D \cap \sigma(D). \) Then \( U^1 = pD^\perp /pL_{0,\mathbb{F}}^\perp \) is not \( \sigma \)-stable but \( U \cap \sigma(U) \) is \( \sigma \)-stable and this is precisely in \( X_B(w_1). \)
Given $D \in \mathcal{M}_{L_0, L_1}^\circ (\mathbb{F})$, then
\[ pL_{0, \mathbb{F}}^\vee \subset pD^\vee \subset pL_{1, \mathbb{F}}^\vee \subset D \subset L_{0, \mathbb{F}}. \]
This forces $D = L_1$ and $U^\perp = pD^\vee / pL_{0, \mathbb{F}}^\vee$ is $\sigma$-stable and this is precisely in $X_{P_{(2)}}(1)$.

Conversely, if $0 \subset U^\perp \subset U \subset V(L_0)_{\mathbb{F}} = L_{0, \mathbb{F}} / pL_{0, \mathbb{F}}^\vee$, then define $D$ as the preimage of $U$ in $L_{0, \mathbb{F}}$. One check easily this gives the desired inverse map. \hfill $\square$

**Definition 3.18.** We consider the following subsets of the stratum $\mathcal{M}_{L_2}(\mathbb{F})$.

1. $\mathcal{M}_{L_2,0}(\mathbb{F}) := \mathcal{M}_{L_2}(\mathbb{F}) \setminus (\mathcal{M}_{1}(\mathbb{F}) \cup \mathcal{M}_{(0)}(\mathbb{F})$);
2. $\mathcal{M}_{L_2,(0)}(\mathbb{F}) = \mathcal{M}_{L_2}(\mathbb{F}) \cap \mathcal{M}_{(0)}(\mathbb{F})$;
3. $\mathcal{M}_{L_2,(1)}(\mathbb{F}) = (\mathcal{M}_{L_2}(\mathbb{F}) \cap \mathcal{M}_{(0)}(\mathbb{F})) \setminus \mathcal{M}_{1}(\mathbb{F})$;
4. $\mathcal{M}_{L_2,1}(\mathbb{F}) = \mathcal{M}_{L_2}(\mathbb{F}) \cap \mathcal{M}_{1}(\mathbb{F})$.

Then $\mathcal{M}_{L_2}(\mathbb{F})$ admits the following Bruhat-Tits stratification
\[ \mathcal{M}_{L_2}(\mathbb{F}) = \mathcal{M}_{L_2,0}(\mathbb{F}) \cup \mathcal{M}_{L_2,1}(\mathbb{F}) \cup \mathcal{M}_{L_2,1}(\mathbb{F}). \]

We define the symplectic space $V(L_2)$ of dimension 4 as $L_2^\vee / L_2$ and we put $Y_{L_2} = Y_{V(L_2)}$.

**Proposition 3.19.** There is a bijection between $\mathcal{M}_{L_2}(\mathbb{F})$ and $Y_{L_2}(\mathbb{F})$. This bijection is compatible with the stratification on both sides in the sense that $\mathcal{M}_{L_2,0}(\mathbb{F})$ is in bijection with $X_B(w_2)(\mathbb{F})$, $\mathcal{M}_{L_2,(0)}(\mathbb{F})$ is in bijection with $X_B(w_1)(\mathbb{F})$ and $\mathcal{M}_{L_2,1}(\mathbb{F})$ is in bijection with $X_{P_{(2)}}(1)(\mathbb{F})$.

**Proof.** Given $D \in \mathcal{M}_{L_2}$, we have the following chain of inclusions:
\[ L_{2, \mathbb{F}} \subset D \subset L_{1, \mathbb{F}}. \]
Let $V = \frac{L_{2, \mathbb{F}}}{L_{2, \mathbb{F}}}$, then sending $D$ to the flag
\[ 0 \subset D / L_{2, \mathbb{F}} \subset D^\vee / L_{2, \mathbb{F}} \subset L_{2, \mathbb{F}} / L_{2, \mathbb{F}} \]
gives the desired flag. The rest is the same as in the previous case. \hfill $\square$

For a 1-vertex $L_1$, the stratum $\mathcal{M}_{L_1}(\mathbb{F})$ is the set $\{ M \in \mathcal{M}(\mathbb{F}) : L_{1, \mathbb{F}} \subset M \}$. This stratum consists of a superspecial point.

**Proposition 3.20.** The stratum $\mathcal{M}_{L_1}(\mathbb{F})$ consists of a superspecial point. Therefore $\mathcal{M}_{(1)}$ consists of all the superspecial points.

**Proof.** Given $D \in \mathcal{M}_{L_1}(\mathbb{F})$, we have the following chain:
\[ L_{1, \mathbb{F}} \subset D \subset D^\vee \subset L_{1, \mathbb{F}}. \]
This forces $L_{1, \mathbb{F}} = D$. Since $\tau(L_{1, \mathbb{F}}) = L_{1, \mathbb{F}}$, it follows that $VD = \Pi D$. As $\Pi^2 = p$, $D$ is superspecial. Note that $M$ is superspecial if and only if $M_0 = D$ is superspecial. The result follows. It is also clear that all the superspecial points arise this way. \hfill $\square$

**Lemma 3.21.** Let $L_1$ be a 1-vertex. Then we can find a 0-vertex $L_0$ containing $L_1$ and a 2-vertex $L_2$ contained in $L_1$. 

Moreover there is a commutative diagram:

This lemma shows that any superspecial point is contained in the intersection $\mathcal{M}_{L_0}(\mathbb{F}) \cap \mathcal{M}_{L_2}(\mathbb{F})$ for a 0-vertex $L_0$ and a 2-vertex $L_2$.

Finally we study the intersection of any two 2-dimensional stratum. We have already seen that for $L_0$ and $L_2$, the intersection is $\mathcal{M}_{L_0}(\mathbb{F}) \cap \mathcal{M}_{L_2}(\mathbb{F})$ is a $\mathbb{P}^1(\mathbb{F})$. Let $L_0$ and $L_0'$ be two vertex lattices of type 0. Similarly let $L_2$ and $L_2'$ be two vertex lattices of type 2.

**Lemma 3.22.** Suppose the intersection of $\mathcal{M}_{L_0}(\mathbb{F})$ and $\mathcal{M}_{L_0'}(\mathbb{F})$ is nonempty, then it consists of a superspecial point corresponding to the 1-vertex $L_0 \cap L_0'$. Similarly if the intersection of $\mathcal{M}_{L_2}(\mathbb{F})$ and $\mathcal{M}_{L_2'}(\mathbb{F})$ is nonempty, then it consists of a superspecial point corresponding to the 1-vertex $L_2 + L_2'$.

**Proof.** Suppose $M \in \mathcal{M}_{L_0} \cap \mathcal{M}_{L_0'}$. Then $pL_{0,F}^\vee C^1 \ pM^\vee C^2 M \subset L_{0,F}$ and $p(L_0')^\vee C^1 \ pM^\vee C^2 M \subset L_{0,F}'$, this forces $M = L_{0,F} \cap L_{0,F}'$. Moreover $L_0 \cap L_0'$ is a 1-vertex. The statement for $L_2, L_2'$ is proved in the same way.

4. The isogeny trick

For each vertex lattice $L$, we now would like to equip the set $\mathcal{M}_L(\mathbb{F})$ with a scheme theoretic structure.

If $L_0$ is a 0-vertex lattice, we define $L_0^+ = L_{0,F} \oplus \Pi^{-1}L_{0,F}$ and $L_0^- = (L_0^+)^\perp$ where $\perp$ is taken with respect to the form $(\cdot, \cdot)$ inside of $N_0$. Notice by an easy computation $L_{0,F}^\perp = \Pi L_{0,F}^\vee = \Pi L_{0,F}$ and $(\Pi^{-1}L_{0,F})^\perp = pL_{0,F}^\vee = pL_{0,F}$.

**Lemma 4.1.** The lattices $L_0^+$ and $L_0^-$ give rise to p-divisible groups $\mathbb{X}_{L_0^+}$ and $\mathbb{X}_{L_0^-}$. Moreover there is a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{X}_{L_0^+} & \xrightarrow{\sim} & \mathbb{X}_{L_0^-} \\
\downarrow \rho_{L_0^+} & & \downarrow \rho_{L_0^-} \\
\mathbb{X} & \xrightarrow{\lambda_{L_0^-}} & \mathbb{X}^\vee.
\end{array}
\]

**Proof.** To see $L_0^+$ defines a p-divisible groups, we need to verify that $pL_{0,F}^+ \subset VL_{0,F}^+ \subset L_{0,F}^+$. This is equivalent to $p\Pi^{-1}L_{0,F} \subset VL_{0,F} \subset \Pi^{-1}L_{0,F}$ and $pL_{0,F} \subset \Pi^{-1}L_{0,F} \subset L_{0,F}$. Notice that $\Pi = V = F$ on $L_{0,F}$ and the two conditions above are easily verified. Therefore $L_{0,F}^+$ gives a p-divisible group $\mathbb{X}_{L_0^+}$. The dual $L_0^-$ of $L_0^+$ gives the dual p-divisible group $\mathbb{X}_{L_0^-} \cong \mathbb{X}_{L_0^+}^\vee$. The maps $\rho_{L_0^+}$ and $\rho_{L_0^-}$ are given by the inclusions of $L_{0,F}^+ \subset N$ and $L_{0,F}^- \subset N$.\[\square\]
Let \((X, \iota_X, \lambda_X, \rho_X) \in \mathcal{M}(R)\) for a \(\mathbb{F}_{p^2}\)-algebra \(R\), consider the quasi-isogenies defined by

\[(4.2) \quad \rho_{X,L_0^+} : X \xrightarrow{\rho_X} \mathbb{X}_R \xrightarrow{\rho_{L_0^+}^{-1}} \mathbb{X}_{L_0^+,R}\]

and

\[(4.3) \quad \rho_{X,L_0^-} : \mathbb{X}_{L_0^-,R} \xrightarrow{\rho_{L_0^-}} \mathbb{X}_R \xrightarrow{\rho_X^{-1}} X.\]

We define the lattice stratum \(\mathcal{M}_{L_0}\) associated to \(L_0\) as the subscheme of \(\mathcal{M}\) consisting of those points that \(\rho_{X,L_0^+}\) is an isogeny. This is equivalent to \(\rho_{X,L_0^-}\) is an isogeny.

**Proposition 4.4.** The subscheme \(\mathcal{M}_{L_0}\) is a closed subscheme of \(\mathcal{M}\). Moreover \(\mathcal{M}_{L_0}\) is projective.

**Proof.** First of all, \(\mathcal{M}_{L_0}\) is a closed subscheme by [RZ96] 2.9. Moreover \(\mathcal{M}_{L_0}\) is bounded in the sense of [RZ96] 2.30. This follows from the fact that we have isogenies

\[\mathbb{X}_{L_0^-,R} \xrightarrow{\rho_{X,L_0^-}} X \xrightarrow{\rho_X^{-1}} \mathbb{X}_{L_0^+,R}.\]

Hence \(\mathcal{M}_{L_0}\) is also quasi-projective [RZ96] 2.31. By sending

\[(X, \iota_X, \lambda_X, \rho_X, \rho_{X,L_0^+}, \rho_{X,L_0^-}) \in \mathcal{M}_{L_0}\]

to the kernel \(H = \ker(\rho_{L_0^-})\) gives a morphism from \(\mathcal{M}_{L_0}\) to the functor \(\mathcal{T}\) on \(\mathbb{F}_{p^2}\)-algebras \(R\) such that \(\mathcal{T}(R)\) is the set of \(\mathcal{O}_{B_0}\)-subgroup schemes \(H\) of \(\mathbb{X}_{L_0^-}\) such that the polarization on \(\mathbb{X}_{L_0^-}\) induces an isomorphism on \(\mathbb{X}_{L_0^-}/H\). It is easy to this morphism is an isomorphism as we can define an inverse functor by sending \(H \subset \mathbb{X}_{L_0^-}\) to \(X = \mathbb{X}_{L_0^-}/H\). The functor \(\mathcal{T}\) is a closed subscheme of the projective scheme classifying subgroup schemes in \(\mathbb{X}_{L_0^-}\) of height 2. Therefore \(\mathcal{M}_{L_0}\) is projective. \(\square\)

**Lemma 4.5.** The set \(\mathcal{M}_{L_0}(\mathbb{F})\) as in Definition 3.7 is precisely the set of \(\mathbb{F}\)-points of \(\mathcal{M}_{L_0}\).

**Proof.** Giving a \(\mathbb{F}\)-point of \(\mathcal{M}_{L_0}\) is equivalent to giving a Dieudonné module \(M \subset L_0^+\) but this is equivalent to \(M_0 \subset L_{0,\mathbb{F}}\). Indeed, if \(M \subset L_0^+\), then \(M_0 \subset L_{0,\mathbb{F}}\) is obvious. Conversely if \(M_0 \subset L_{0,\mathbb{F}}\), then \(M_1 = \Pi M_0^\vee \subset \Pi^{-1}M_0 \subset \Pi^{-1}L_{0,\mathbb{F}}\). \(\square\)

If \(L_{2,\mathbb{F}}\) is a 2-vertex lattice, we define \(L_2^+ = L_{2,\mathbb{F}} \oplus \Pi L_{2,\mathbb{F}}\) and \(L_2^- = (L_{2,\mathbb{F}}^+)\perp\) where \(\perp\) is taken with respect to the form \((\cdot, \cdot)\) inside of \(N_0\). The proof of the following statements are proved exactly the same way as in the \(L_0\) case.

**Lemma 4.6.** The lattices \(L_2^+\) and \(L_2^-\) gives rise to \(p\)-divisible groups \(\mathbb{X}_{L_2^+}\) and \(\mathbb{X}_{L_2^-}\). Moreover there is a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{X}_{L_2^+} & \xrightarrow{\rho_{L_2^+}} & \mathbb{X}_{L_2^-} \\
\downarrow{\rho_{L_2^+}} & & \downarrow{\rho_{L_2^-}} \\
\mathbb{X} & \xrightarrow{\lambda_X} & \mathbb{X}^\vee.
\end{array}
\]
Let \((X, \iota_X, \lambda_X, \rho_X)\) for a \(\mathbb{F}_{p^2}\)-algebra \(R\), consider the quasi-isogeny defined by

\begin{equation}
\rho_{X,L_2^+} : \mathbb{X}_{L_2^+} \xrightarrow{\rho_{L_2^+}} X, \quad \rho_{X,L_2^-} : \mathbb{X}_{L_2^-} \xrightarrow{\rho_{L_2^-}} X,
\end{equation}

and

\begin{equation}
\rho_{X,L_2^+} : X \xrightarrow{\rho_X} \mathbb{X}_R \xrightarrow{\rho_{L_2}} \mathbb{X}_{L_2^+.R}.
\end{equation}

We define the lattice stratum \(\mathcal{M}_{L_2}\) associated to \(L_2\) as the subscheme of \(\mathcal{M}\) consisting of those points that \(\rho_{X,L_2^+}\) is an isogeny or equivalently \(\rho_{X,L_2^-}\) is an isogeny.

**Proposition 4.9.** The subscheme \(\mathcal{M}_{L_2}\) is a closed subscheme of \(\mathcal{M}\). Moreover \(\mathcal{M}_{L_2}\) is projective. The set \(\mathcal{M}_{L_2}(\mathbb{F})\) as in Definition [3.7] is precisely set of \(\mathbb{F}\)-points of \(\mathcal{M}_{L_2}\).

**4.1. The isomorphism between \(\mathcal{M}_{L_i}\) and \(Y_{L_i}\).** Let \(L_0\) be a 0-vertex and \(L_2\) a 2-vertex. The goal of this subsection is to define a map \(f_i : \mathcal{M}_i \rightarrow Y_{L_i}\) for \(i = 0, 2\) and show that it is an isomorphism.

We begin with \(L_0\). Given a point \((X, \iota_X, \lambda_X, \rho_X) \in \mathcal{M}_{L_0}(R)\) for a \(\mathbb{F}_{p^2}\)-algebra \(R\). Consider the isogenies

\[
\rho_{L_0} : \mathbb{X}_{L_0^+} \xrightarrow{\rho_{X,L_0^+}} X \quad \rho_{X,L_0^-} \xrightarrow{\rho_{L_0^-}} \mathbb{X}_{L_0^-}.\]

The kernel of \(\rho_{L_0}\) is the quotient \(L^+_{0,R}/L^-_{0,R}\) which we compute to be \(L^+_{0,R}/pL^+_{0,R} \oplus \Pi^{-1}L^+_{0,R}/\Pi\). The kernel of \(\rho_{X,L_0}\) is a direct summand of \(L^0_{0,R}/pL^0_{0,R} \oplus \Pi^{-1}L^0_{0,R}/\Pi\) which we compute to be \(M_0/pL^0_{0,R} \oplus \Pi M_0^\vee/\Pi L^0_{0,R}\) where \(M = M_0 \oplus M_1\) is the covariant Dieudonné crystal \(\mathfrak{D}(X)(R)\) of \(X\). Sending \((X, \iota_X, \lambda_X, \rho_X) \in \mathcal{M}_{L_0}(R)\) to \(M_0/pL^0_{0,R}\) gives a map from \(\mathcal{M}_{L_0}\) to \(Y_{L_0}\). We denote this map by \(f_0\).

We consider a 2-vertex \(L_2\). Given a point \((X, \iota_X, \lambda_X, \rho_X) \in \mathcal{M}_{L_2}(R)\) for a \(\mathbb{F}_{p^2}\)-algebra \(R\). Consider the isogenies

\[
\rho_{L_2} : \mathbb{X}_{L_2^+} \xrightarrow{\rho_{X,L_2^+}} X \quad \rho_{X,L_2^-} \xrightarrow{\rho_{L_2^-}} \mathbb{X}_{L_2^-}.\]

The kernel of \(\rho_{L_2}\) is the quotient \(L^+_{2,R}/L^-_{2,R}\) which we compute to be \(L^+_{2,R}/L^0_{2,R} \oplus \Pi L^0_{2,R}/\Pi\). The kernel of \(\rho_{X,L_2}\) is a direct summand of \(L^0_{2,R}/L^+_{2,R} \oplus \Pi L^0_{2,R}/\Pi\) which we compute to be \(M_0/L^0_{2,R} \oplus \Pi M_0^\vee/\Pi L_{2,R}\) where \(M = M_0 \oplus M_1\) is the covariant Dieudonné crystal \(\mathfrak{D}(X)(R)\) of \(X\). Sending \((X, \iota_X, \lambda_X, \rho_X) \in \mathcal{M}_{L_2}(R)\) to \(M_0/L^0_{2,R}\) gives a map from \(\mathcal{M}_{L_2}\) to \(Y_{L_2}\). We denote this map by \(f_2\).

**Proposition 4.10.** The maps \(f_i : \mathcal{M}_{L_i} \rightarrow Y_{L_i}\) are isomorphisms for \(i = 0, 2\).

**Proof.** Using the theory of windows of display [Zin99] in place of Dieudonné modules, one can show \(f\) is bijective on \(k\)-points for any field extension \(k\) over \(\mathbb{F}_{p^2}\) following the same proof as in [3.17]. Since \(Y_{L_0}\) is a closed subscheme of a projective scheme, it is projective. Moreover it is smooth and hence normal. Note that \(f_0\) is proper since \(\mathcal{M}_{L_0}\) is proper and \(Y_{L_0}\) is separated. Therefore we can apply Zariski’s main theorem, \(f_0\) is an isomorphism. The proof for \(f_2\) is exactly the same. \(\square\)
4.2. The Bruhat-Tits stratification. We can now transfer the results on the Bruhat-Tits stratification from the set theoretic level in 3.17 and 3.19 to the scheme theoretic level. For $i = 0, 1, 2$, the $i$-vertex stratum is defined to be $M_{(i)} = \bigcup L_i M_{L_i}$ where $L_i$ ranges over all the $i$-vertex.

Definition 4.11. We define the Bruhat-Tits stratums of $M_{L_0}$ by the following schemes.

1. The open stratum $M_{L_0}^0 = M_{L_0} \setminus (M_{(2)} \cup M_{(1)})$;
2. The closed stratum $M_{L_0,\{2\}} = M_{L_0} \cap M_{\{2\}}$;
3. The stratum $M_{L_0,\{2\}} = (M_{L_0} \cap M_{\{2\}}) \setminus M_{(1)}$;
4. The closed stratum $M_{L_0,\{1\}} = M_{L_0} \cap M_{\{1\}}$.

We call the natural decomposition $M_{L_0} = M_{L_0}^0 \cup M_{L_0,\{2\}} \cup M_{L_0,\{1\}}$ the Bruhat-Tits stratification of $M_{L_0}$.

Similarly, we define the Bruhat-Tits stratums of $M_{L_2}$ by the following schemes.

1. The open stratum $M_{L_2}^0 = M_{L_2} \setminus (M_{\{0\}} \cup M_{\{1\}})$;
2. The closed stratum $M_{L_2,\{0\}} = M_{L_2} \cap M_{\{0\}}$;
3. The stratum $M_{L_2,\{0\}} = (M_{L_2} \cap M_{\{0\}}) \setminus M_{\{1\}}$;
4. The closed stratum $M_{L_2,\{1\}} = M_{L_2} \cap M_{\{1\}}$.

We call the natural decomposition $M_{L_2} = M_{L_2}^0 \cup M_{L_2,\{0\}} \cup M_{L_2,\{1\}}$ the Bruhat-Tits stratification of $M_{L_2}$.

Theorem 4.12. Let $L_i$ be $i$-vertex. The isomorphism $f_i : M_{L_i} \to Y_{L_i}$ is compatible with the stratification on both sides: for $M_{L_i}$, we consider the Bruhat-Tits stratification and for $Y_{L_i}$, we consider the stratification in Theorem 3.4.

Proof. Once we know that $f$ is an isomorphism, the statements about the stratification can be checked on $\mathbb{F}$-points. These are already done in Proposition 3.17 and Proposition 3.19. More explicitly, for $L_0$

1. $M_{L_0}^0$ is isomorphic to $X_B(w_2)$;
2. $M_{L_0,\{2\}}$ is isomorphic to $X_B(w_1)$;
3. $M_{L_0,\{0\}}$ is isomorphic to $X_{P(2)}(1)$.

And for $L_2$, we have

1. $M_{L_2}^0$ is isomorphic to $X_B(w_2)$;
2. $M_{L_2,\{0\}}$ is isomorphic to $X_B(w_1)$;
3. $M_{L_2,\{1\}}$ is isomorphic to $X_{P(2)}(1)$.

\[ \square \]

We can also define the Bruhat-Tits stratification of $M$. The stratums are given by the union of the Bruhat-Tits stratums of $M_{L_0}$ and $M_{L_2}$. We define $M_{(0)} = \bigcup L_0 M_{L_0}^0$, $M_{(2)} = \bigcup L_2 M_{(2)}$, and $M_{(0,2)} = \bigcup L_0 M_{(0,2)}$. Recall also we have $M_{(1)}$ the set of superspecial points on $M$. We refer to the following natural decomposition the Bruhat-Tits stratification of $M$

\[ M = M_{(0)} \cup M_{(2)} \cup M_{(0,2)} \cup M_{(1)}. \]
5. The supersingular locus of the quaternionic unitary Shimura variety

5.1. The main result on the Rapoport-Zink space. We summarize the results from previous sections and finish the description of the Rapoport-Zink space in the following theorem.

Theorem 5.1. The reduced scheme \( \mathcal{N}_{\text{red}} \) can be written as \( \mathcal{N}_{\text{red}} = \bigcup_{l \in \mathbb{Z}} \mathcal{N}_{\text{red}}(l) \). The connected components \( \mathcal{N}_{\text{red}}(l) \) are all isomorphic to \( \mathcal{N}_{\text{red}}(0) \). Let \( \mathcal{M} = \mathcal{N}_{\text{red}}(0) \otimes \mathbb{F}_p^2 \), then \( \mathcal{M} \) is pure of dimension 2.

1. \( \mathcal{M} \) can be decomposed into

\[
\mathcal{M} = \mathcal{M}^0_{(0)} \cup \mathcal{M}^0_{(2)} \cup \mathcal{M}^0_{(0,2)} \cup \mathcal{M}_{(1)}
\]

This is called the Bruhat-Tits stratification of \( \mathcal{M} \).

2. \( \mathcal{M}^0_{(0)} = \bigcup_{L_0} \mathcal{M}^0_{L_0} \) where \( L_0 \) runs through all the 0-vertices. For each \( L_0 \), the closure of \( \mathcal{M}^0_{L_0} \) is \( \mathcal{M}_{L_0} \) and is isomorphic to the hypersurface \( x_0^2x_0 - x_0x_3 + x_2^2x_1 - x_1^2x_2 = 0 \). Moreover it admits a stratification

\[
\mathcal{M}_{L_0} = \mathcal{M}^0_{L_0} \cup \mathcal{M}^0_{L_0,(2)} \cup \mathcal{M}_{(1)}
\]

called the Bruhat-Tits stratification of \( \mathcal{M}_{L_0} \). Here the closure of \( \mathcal{M}^0_{L_0,(2)} \) is \( \mathcal{M}_{L_0,(2)} \) and its irreducible component is isomorphic to \( \mathbb{P}^1 \). The complement of \( \mathcal{M}^0_{L_0,(2)} \) in \( \mathcal{M}_{L_0,2} \) is precisely \( \mathcal{M}_{L_0,(1)} \) which consists of superspecial points. The same statements hold true for \( \mathcal{M}^0_{(2)} \) with obvious modification.

3. The intersection between \( \mathcal{M}_{L_0} \) and \( \mathcal{M}_{L_2} \) for a 0-vertex \( L_0 \) and a 2-vertex \( L_2 \) if nonempty is isomorphic to a \( \mathbb{P}^1 \). The intersection between \( \mathcal{M}_{L_0} \) and \( \mathcal{M}_{L_0'} \) for a 0-vertex \( L_0 \) and a different 0-vertex \( L_0' \) if nonempty is a point which is superspecial. The intersection between \( \mathcal{M}_{L_2} \) and \( \mathcal{M}_{L_2'} \) for a 2-vertex \( L_2 \) and a 2-vertex \( L_2' \) if nonempty is a point which is superspecial.

5.2. Integral model of the Shimura variety. Let \( V \) be vector space of dimension 8 over \( \mathbb{Q} \) with an action \( \iota : B \to \text{End}(V) \). We assume that \( V \) is equipped with an alternating form \( (\cdot, \cdot) \) such that \( (x, by) = (b^*x, y) \). Then we define \( G(\mathbb{Q}) = \{ g \in \text{GL}_B(V) ; (gx, gy) = c(g)(x,y) \} \). Since \( B \) is split at \( \infty \), \( G(\mathbb{R}) = \text{GSp}(W) \) for some \( W \) of dimension 4. Let \( h : \mathbb{G}_m \to G(\mathbb{R}) \) be the cocharacter sending \( z \) to diag(\( z, 1, z, 1 \)). Moreover \( h \) defines a decomposition \( V_\mathbb{C} = V_1 \oplus V_2 \) where \( h(z) \) acts on \( V_1 \) by \( z \) and on \( V_2 \) by \( \bar{z} \). We fix an open compact subgroup \( U \) of \( G(h) \) and we assume \( U_p \) is sufficiently small and \( U_p = G(\mathbb{Z}_p) \). We also assume that there is an \( \mathcal{O}_B \) lattice \( \Lambda \) in \( V \) such that \( G(\mathbb{Z}_p) \) stabilizes \( \Lambda \otimes \mathbb{Z}_p \).

To \( (B, *, V, (\cdot, \cdot), h, \Lambda) \), we associate the moduli problem following moduli problem \( \mathcal{G}h_{U} \) over \( \mathbb{Z}_p \): for a scheme \( S \) over \( \mathbb{Z}_p \) we associate set of isomorphism classes of the collection \( \{ (\Lambda, \iota, \lambda, \eta) \} \) where:

- \( \Lambda \) is an abelian scheme of relative dimension 4 over \( S \);
- \( \iota : \mathcal{O}_B \to \text{End}_S(A) \otimes \mathbb{Z}_p \) is a morphism such that \( \lambda \circ i(a^*) = i(a) \circ \lambda \) and satisfies the Kottwitz condition

\[
\text{det}(a, \text{Lie}(A)) = \text{det}(a, V_1) = N^0(a)^2
\]
for all \( a \in \mathcal{O}_B \) and \( N^0 \) is the reduced norm of \( B \). This should be understood as a polynomial identity.

- \( \lambda : A \to A^\vee \) is a prime to \( p \) polarization which is \( \mathcal{O}_B \)-linear.
- \( \eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}_f^{(p)} \to V^{(p)}(A) \) is a \( U^p \)-orbit of \( B \otimes_{\mathbb{Q}} \mathbb{A}_f^{(p)} \)-linear isomorphisms which is compatible with the Weil-pairing on the righthand side and the form \((\cdot, \cdot)\) on the lefthand side.

This moduli problem is representable by a quasi-projective variety \( \mathcal{S}_h \) over \( \mathbb{Z}_p \). Note that in \( \text{[KR00]} \) Proposition 1.3. Note that in \( \text{[KR00]} \), the moduli space is presented in terms of a GSpin Shimura variety but the equivalence between the two moduli problem is clear.

We are interested in the closed subscheme of the supersingular locus \( \mathcal{S}_{hU^p} \) in \( \mathcal{S}_h \otimes \mathbb{F} \) considered as a closed reduced subscheme. The uniformization theorem of Rapoport-Zink \( \text{[RZ96]} \) Theorem 6.1 transfers this problem to the descriptions of the Rapoport-Zink space.

**Theorem 5.3.** There is an isomorphism of \( \mathbb{F} \)-schemes
\[
\mathcal{S}_{hU^p} \cong I(\mathbb{Q}) \setminus \mathcal{N}_{\text{red}, \mathbb{F}} \times G(\mathbb{A}_f^{(p)})/U^p.
\]

Here \( I \) is an inner form of \( G \) with \( I(\mathbb{Q}_p) = J(\mathbb{Q}_p) \). It is defined in the following way. Let \((A, \iota, \lambda, \eta)\) be a fixed supersingular point in \( \mathcal{S}_{hU^p} \). Then \( I = \text{End}_{\mathcal{O}_B}(A, \lambda) \otimes \mathbb{Q} \).

**5.3. Main theorem on the supersingular locus.** We apply the results obtained previously to describe the supersingular locus of the quaternionic unitary Shimura variety.

**Theorem 5.4.** The \( \mathbb{F} \)-scheme \( \mathcal{S}_{hU^p} \) is pure of dimension 2. For \( U^p \) sufficiently small, the irreducible components are isomorphic to the surface 
\[
x_3x_0 - x_0x_3 + x_2x_1 - x_1x_2 = 0.
\]
The intersection of two irreducible components is either empty or is isomorphic to a \( \mathbb{P}^1 \) or is a superspecial point.

**Proof.** Using the uniformization theorem 5.3 the result follows from the main theorem for the Rapoport-Zink space 5.1. \( \square \)

6. **Rapoport-Zink space for \( \text{GSp}(4) \) with paramodular level**

In this section we treat the case when \( G \) corresponds to the split \( \text{GSp}(4) \) but where the level structure is parahoric corresponding to the node 1 in the local Dynkin diagram. This level structure is known as the paramodular level and is well studied. In particular the structure of the supersingular locus is known. See for example \( \text{[Yu06]} \) Theorem 4.7. The purpose of this section is to revisit the description of the supersingular locus from a Bruhat-Tits perspective. First we define the Rapoport-Zink space we will study.

6.1. **Rapoport-Zink space with paramodular level.** Let \( N \) be an isocrystal of slope \( \frac{1}{2} \) and height 4 equipped with an alternating form \((\cdot, \cdot)\). We fix a \( p \)-divisible group \( X \) whose associated isocrystal agrees with \( N \) and a polarization \( \lambda_X : X \to X^\vee \) which corresponds to the form \((\cdot, \cdot)\). We assume that the height of \( \lambda_X \) is 2. Let \( (\text{Nilp}) \) be the category of \( \mathbb{Z}_p \)-schemes that \( p \) is locally nilpotent. We consider the set valued functor \( \mathcal{N} \) that sends \( S \in (\text{Nilp}) \) to the isomorphism classes of the collection \((X, \lambda_X, \rho_X)\) where
- $X$ is a $p$-divisible group of dimension 2 and height 4;
- $\lambda_X : X \to X^\vee$ is a quasi-polarization of height 2;
- $\rho_X : X \times_S S_0 \to \mathbb{X} \times_S S_0$ is a quasi-isogeny.

For $\rho_X : X \times_S S_0 \to \mathbb{X} \times_S S_0$, we require that $\rho_X^* \lambda_X = c(\rho) \lambda_X$ for a $\mathbb{Q}_p$-multiplier $c(\rho)$. The moduli problem $\mathcal{N}$ is representable.

**Lemma 6.1.** The functor $\mathcal{N}$ is representable by a separated formal scheme locally formal of finite type over $\text{Spf}(\mathbb{Z}_p)$. Moreover $\mathcal{N}$ is flat and formally smooth away from the superspecial points.

**Proof.** The representability follows from [RZ96] and second assertion on flatness and the singularity is proved in [Yu11].

By passing to Dieudonné modules, the $\mathbb{F}$-points of the Rapoport-Zink space $\mathcal{N}(\mathbb{F})$ can be identified with the following set:

\[(6.2) \quad \mathcal{N}(\mathbb{F}) = \{ M \subset N : c p M^\perp \subset^2 M \subset c M^\perp, p M \subset^2 V M \subset^2 M, \} \]

The first description comes from the fact that $\lambda_X$ is a quasi-polarization of height 2 and second one comes from the fact that $X$ is of dimension 2.

We write $\mathcal{N}_{\text{red}}$ the reduced scheme of the special fiber $\mathcal{N}$ modulo $p$. To describe $\mathcal{N}_{\text{red}}$, we consider the subscheme $\mathcal{N}(i)$ the open and closed formal subscheme of $\mathcal{N}_{\text{red}}$ defined by the condition that the quasi-isogeny $\rho$ has height $ni$. Moreover $\mathcal{N}(i)$ is isomorphic to $\mathcal{N}(0)$ and hence it suffices to only consider $\mathcal{N}(0)$. Therefore we set $\mathcal{M} = \mathcal{N}(0)$. Then by the previous description, we arrive at the following lemma.

**Lemma 6.3.** The $\mathbb{F}$-points of $\mathcal{M}$ can be described by

\[ \mathcal{M}(\mathbb{F}) = \{ M \subset N; p M^\perp \subset^2 M \subset c M^\perp, p M \subset^2 V M \subset^2 M \} \]

6.2. **The group $J_b$.** This Rapoport-Zink space is associated to $G = \text{GSp}(4)$ and a parahoric group of $G(\mathbb{Q}_p)$ known as the paramodular group. We define the $\sigma^2$-linear operator $\tau = V^{-1}F$. Then the slope 0 isocrystal $(N, \tau)$ admits a factorization $N = C \otimes_{\mathbb{Q}_p} W(\mathbb{F})_0$ for a $\mathbb{Q}_p$-vector space $C$. By letting $\Pi$ act on $C$ by $F$, we can consider the vector space $C$ as a $B_{\text{red}}$-module of rank 2. The isocrystal $N$ gives an element $b \in B(G)$ where $B(G)$ is the set of $\sigma$-conjugate classes of $G$. Recall the group $J_b$ is defined to be the $\sigma$-conjugate centralizer of $b$. This group can be computed as follows.

**Lemma 6.4.** The group $J_b(\mathbb{Q}_p)$ can be identified with $\text{GU}_{B_p}(C)$.

**Proof.** The group $J_b(\mathbb{Q}_p)$ can be computed by the group

\[ \{ g \in \text{End}^0(N); g F = F g, (gx, gy) = c(g)(x, y) \} \]

for some $c(g) \in \mathbb{Q}_p$. Since $g$ commutes with $\tau = V^{-1}F$, it restricts to an action on $C$ and is determined by this restriction. Since $F$ acts on $C$ by $\Pi$,

\[ J_b(\mathbb{Q}_p) = \{ g \in \text{End}^0_{B_p}(C); (gx, gy) = c(g)(x, y) \} = \text{GU}_{B_p}(C). \]
7. Bruhat-Tits stratification for Rapoport-Zink space with paramodular level

7.1. Vertex lattices in \( GU_B(2) \). The group \( GU_B(2) \) splits over \( W(\mathbb{F})_Q \) and can be identified with \( GSp(4) \) over \( W(\mathbb{F})_Q \). We consider the same vertex lattices as in Section 3.3. We use the same terminology as in 3.3 and recall a 0-vertex corresponds to a lattice \( L \) with \( pL\perp \subset 4L \subset 0L \perp \), the 1-vertex corresponds to a lattice \( L \) with \( pL\perp \subset 2L \subset 2L \perp \) and the 2-vertex corresponds to a lattice \( L \) with \( pL\perp \subset 0L \subset 4L \perp \). Here the dual \( \perp \) is taken with respect to the

Remark 7.1. The relation between the group \( GSp(4) \) and \( GU_B(2) \) can be also demonstrated on the local Dynkin diagram. For \( GSp(4) \), it is given by

\[
\bullet \rightarrow \bullet \rightarrow \bullet
\]

with Frobenius acting trivially on the diagram. For \( GU_B(N) \), it is given by the same local Dynkin diagram

\[
\bullet \rightarrow \bullet \rightarrow \bullet \rightleftarrows \bullet
\]

with Frobenius acting on the diagram by permuting the nodes 0 and 2. Therefore the 0-vertex and 2-vertex should be identified when we consider the vertex lattices for \( GU_B \).

We define a \( \{0,2\} \)-vertex as a pair of lattices \((L_0, L_2)\) where \( L_0 \) is a 0-vertex and \( L_2 \) is a 2-vertex.

Let \( L_1 \) be a 1-vertex and \((L_0, L_2)\) be a \( \{0,2\} \)-vertex.

Definition 7.2. We define the following set theoretic lattice stratum.

1. For a \( \{0,2\} \)-vertex \((L_0, L_2)\), the stratum \( M_{L_0,L_2}(\mathbb{F}) \) is the set \( \{ M \in M(\mathbb{F}) : L_2, F \subset M \subset L_0, F \} \);
2. For a 1-vertex \( L_1 \), the stratum \( M_{L_1}(\mathbb{F}) \) is the set \( \{ M \in M(\mathbb{F}) : L_1, F \subset M \} \).

7.2. Vertex stratums. To see that the set theoretic lattice stratums form a covering of the set \( M(\mathbb{F}) \), we need the following result.

Proposition 7.3. Given \( M \in M(\mathbb{F}) \), there are \( \tau \)-stable lattices \( L^-(M) \) and \( L^+(M) \) of \( N \) such that we have

\[
M \subset L^+(M) \subset L^+(M)\perp \subset M\perp
\]

and

\[
pM\perp \subset pL^-(M)\perp \subset L^-(M) \subset M.
\]

Before we prove this proposition, we need to recall a few basic facts about Dieudonné modules that are stated in [NO80](11a)-(11d). Define the \( a \)-number of the Dieudonné module \( M \) by

\[
a(M) = \dim_{\mathbb{F}} M/(FM + VM).
\]

Notice that \( a(M) = 2 \) if and only if \( M \) is superspecial.

Lemma 7.6. Suppose \( M \) has \( a(M) = 1 \), then we have
(1) \(F^2(M)/pM\) is the unique sub-\(\mathbb{F}[F,V]\)-module of \(F(M)/pM\) of rank 1;
(2) \(F^2(M) + pM = FM \cap VM = V^2(M) + pM\).

**Proof.** This is precisely the statement of 11\((b)\), 11\((c)\) and 11\((d)\) in [NO80].

\(\square\)

**Proof of Proposition 7.3.** Given \(M \in \mathcal{M}(\mathbb{F})\), then we set \(L^+(M) = M + \tau(M)\) and \(L^-(M) = M \cap \tau(M)\). First suppose \(M\) is \(\tau\)-stable, then there is nothing to prove as \(M\) and \(L^+(M) = L^-(M) = M\). Note in this case \(M\) is \(\tau\)-stable and hence \(M\) is superspecial with \(a(M) = 2\).

Now suppose \(M\) is not \(\tau\)-stable. Since \(M\) corresponds to a supersingular \(p\)-divisible group, \(a(M)\) is at least 1 and is equal to 1 in this case. By Lemma 7.6 we have \(FM \cap VM = V^2M + pM\). Therefore \(L^+(M) = M \cap \tau(M) = FM + V(M)\) and \(L^-(M) = M + \tau(M) = V^{-1}(M \cap \tau(M))\). Since \(FL^-(M) = F^2M + pM\) and \(VL^-(M) = V^2M + pM\), \(FL^-(M) = VL^-(M) = VM\) is superspecial and \(\tau\)-stable. An easy computation shows that \(M\) is \(\tau\)-stable if and only if \(M^\perp\) is \(\tau\)-stable. Therefore \(a(M^\perp) = 1\) and we can apply Lemma 7.6 to \(M^\perp\) to get \(F^2(M^\perp) + pM^\perp\) is the unique colength 1 sub-Dieudonné-module of \(F(M^\perp)\). But \(M \cap F(M^\perp)\) is also colength 1 in \(F(M^\perp)\) as otherwise \(M = F(M^\perp)\) will imply \(a(M^\perp) = 2\). Indeed, take the dual of this equation give \(M = V(M^\perp)\). This implies that \(M \cap F(M^\perp) = F^2M^\perp + pM^\perp\) and therefore \(F^2(M^\perp) \subset M\). This is the same as \(p\tau(M^\perp) \subset M\) and hence \(pM^\perp + p\tau(M^\perp) \subset M \cap \tau(M)\). But \(pL^-(M)^\perp = pM^\perp + p\tau(M^\perp)\), thus \(pL^-(M)^\perp \subset L^-(M)\). This finishes the proof for (7.4).

Since \(L^+(M) = M + \tau(M) = V^{-1}(L^-(M))\),

\[
FL^+(M) = FV^{-1}(L^-(M)) = \tau^{-1}L^-(M) = L^-(M)
\]

and \(VL^+(M) = VV^{-1}(L^-(M)) = L^-(M)\). This implies that \(a(L^+(M)) = 2\). By the same argument as above using Lemma 7.6 \(F^2(M) + pM = pM^\perp \cap F(M)\). Therefore \(\tau(M) \subset M^\perp\) and \(L^+(M) = M + \tau(M) \subset M^\perp \cap \tau(M)^\perp = L^+(M)^\perp\). This shows (7.4).

\(\square\)

**Remark 7.7.** It is clear by considering the indices in (7.4) and (7.5) that if \(M\) is \(\tau\)-stable, then \(L^+(M) = L^-(M) = M\) is a 1-vertex. If \(M\) is not \(\tau\)-stable, then \(L^+(M)\) is a 0-vertex and \(L^-(M)\) is a 2-vertex. Hence \((L^-(M), L^+(M))\) gives rise to a \(\{0, 2\}\)-vertex.

We define similarly as in the quaternionic case \(\mathcal{M}_{0,2}(\mathbb{F}) = \bigcup_{L_0,L_2} \mathcal{M}_{L_0,L_2}(\mathbb{F})\) and \(\mathcal{M}_{\{1\}}(\mathbb{F}) = \bigcup_{L_1} \mathcal{M}_{L_1}(\mathbb{F})\). By the above remark we have the following decomposition of the set \(\mathcal{M}(\mathbb{F})\).

**Lemma 7.8.** We have \(\mathcal{M}(\mathbb{F}) = \mathcal{M}_{0,2}(\mathbb{F}) \cup \mathcal{M}_{\{1\}}(\mathbb{F})\) and \(\mathcal{M}_{\{1\}}\) consists of superspecial points.

**Proof.** The first statement is a translation of proposition 3.8 and the second statement is clear as \(M\) is superspecial if and only if \(M\) is \(\tau\)-stable. \(\square\)
7.3. Bruhat-Tits stratification. Our next goal is to relate the set theoretic lattice stratum \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) to a Deligne-Lusztig variety which in our case is simply \( \mathbb{P}^1(\mathbb{F}) \). Suppose \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) is non-empty, we define a two dimensional \( \mathbb{F} \)-vector space \( V_{L_0,2} = L_{0,2}/L_{2,2} \).

**Proposition 7.9.** Let \( L_0 \) be 0-vertex and \( L_2 \) be 2-vertex. Suppose \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) is non-empty. There is a bijection between \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) and \( \mathbb{P}^1(V_{L_0,2})(\mathbb{F}) \). Moreover the superspecial points on \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) correspond to the \( \mathbb{F}_{p^2} \)-points on \( \mathbb{P}^1(V_{L_0,2})(\mathbb{F}) \).

**Proof.** We define a map \( f: \mathcal{M}_{L_0,L_2}(\mathbb{F}) \to \mathbb{P}^1(V_{L_0,2})(\mathbb{F}) \) by sending \( M \in \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) to \( M/L_{2,2} \subset L_0/L_{2,2} \). Conversely let \( m \in V_{L_0,2} = L_{0,2}/L_{2,2} \) be a point in \( \mathbb{P}^1(V_{L_0,2}) \) and denote by \( M \subset L_{0,2} \) the preimage of \( m \) under the natural reduction map. By the proof of proposition 3.8 we have \( L_{2,2} = VL_{0,2} \). Then we have \( VM \subset VL_{0,2} = L_{2,2} \subset M \) and \( pM \subset pL_{2,2} = V^2L_{2,2} \subset V^2M \subset V(M) \). Hence \( M \) is a Dieudonné module. It is clear these two maps are inverse to each other and we have a bijection. It is also clear from the construction that the superspecial points are the \( \mathbb{F}_{p^2} \)-points.

**Lemma 7.10.** Let \( L_1 \) be a 1-vertex. Then we can find a 0-vertex \( L_0 \) containing \( L_1 \) and a 2-vertex \( L_2 \) contained in \( L_1 \).

**Proof.** This is lemma 3.21 and we state it again for the reader’s convenience.

The lemma and its proof shows that any superspecial point is contained in the \( \{0,2\} \)-vertex stratum. Let \( (L_0, L_2) \) and \( (L'_0, L'_2) \) be two \( \{0,2\} \)-vertices. We study the set theoretic relation between \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) and \( \mathcal{M}_{L'_0,L'_2}(\mathbb{F}) \).

**Lemma 7.11.** Suppose that the intersection \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \cap \mathcal{M}_{L'_0,L'_2}(\mathbb{F}) \) is non-empty. Then \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \cap \mathcal{M}_{L'_0,L'_2}(\mathbb{F}) \) consists of a superspecial point.

**Proof.** Given \( M \in \mathcal{M}_{L_0,L_2}(\mathbb{F}) \cap \mathcal{M}_{L'_0,L'_2}(\mathbb{F}) \), we have \( L_{0,2} \subset M \subset L_{2,2} \) and \( L'_{0,2} \subset M \subset L'_{2,2} \). By considering the index, \( L_{0,2} + L'_{0,2} = M \) and \( L'_{2,2} \cap L_{2,2} = M \). This forces \( M \) to be \( \tau \)-stable and hence superspecial.

7.4. The isogeny trick. We are now ready to translate the set theoretic results in the previous sections to the scheme theoretic setting. We will again rely on the isogeny trick used in section 5. Let \( (L_0, L_2) \) be a \( \{0,2\} \)-vertex such that \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) is non-empty.

**Lemma 7.12.** The lattices \( L_0 \) and \( L_2 \) gives rise to a \( p \)-divisible groups \( \mathbb{X}_{L_0} \) and \( \mathbb{X}_{L_2} \). Moreover there is a commutative diagram:

\[
\begin{array}{ccc}
X_{L_2} & \longrightarrow & X_{L_0}^Y \\
\downarrow_{\rho_{L_2}} & & \uparrow_{\rho_{L_0}^\vee} \\
X & \longrightarrow & X^\vee \\
\end{array}
\]

**Proof.** Since \( \mathcal{M}_{L_0,L_2}(\mathbb{F}) \) is non-empty, we have \( VL_0 = L_2 \) and a chain

\[(7.13)\quad pM^+ \subset pL_{2,2}^+ \subset 0 L_{2,2} \subset M \subset L_{0,2} \subset 0 L_{0,2}^+ \subset M^+.
\]
Therefore
\[ pL_{0,F} = pL_{0,F}^\perp \subset pL_{2,F} = L_{2,F} = VL_{0,F} \subset L_{0,F} \]
and \( L_0 \) is a Dieudonné module. Since \( L_0 \) is \( \tau \)-stable, \( V^2L_{0,F} = pL_{0,F} \). Then \( pL_{2,F} \subset pL_{0,F} = VL_{2,F} = VL_{0,F} = L_{2,F} \). The rest of the diagram follows from the chain \( \tau \). □

Given \((X, \lambda_X, \rho_X) \in \mathcal{M}(R)\) for a \( \mathbb{F}_p \)-algebra \( R \), we define two quasi-isogenies by the following composite

\[ (7.14) \]
\[ \rho_{X,L_0} : X \xrightarrow{\rho_X} X_R \xrightarrow{\rho_{L_0}^{-1}} X_{L_0,R}, \]

and

\[ (7.15) \]
\[ \rho_{X,L_2} : X_{L_2,R} \xrightarrow{\rho_{L_2}} X_R \xrightarrow{\rho_{X,L_0}^\perp} X. \]

We define the subfunctor \( \mathcal{M}_{L_0,L_2} \) of \( \mathcal{M} \) over \( \mathbb{F}_p \) by classifying those \((X, \lambda_X, \rho_X)\) such that \( \rho_{X,L_0} \) and \( \rho_{X,L_2} \) are both actual isogenies.

**Proposition 7.16.** The functor \( \mathcal{M}_{L_0,L_2} \) is representable by a closed subscheme of \( \mathcal{M} \) and \( \mathcal{M}_{L_0,L_2} \) is projective.

**Proof.** First of all, \( \mathcal{M}_{L_0,L_2} \) is a closed subscheme by \( \text{[RZ96] 2.9} \). Moreover \( \mathcal{M}_{L_0,L_2} \) is bounded in the sense of \( \text{[RZ96] 2.30} \). This follows from the fact that we have isogenies

\[ X_{L_2,R} \xrightarrow{\rho_{X,L_2}} X \xrightarrow{\rho_{X,L_0}^\perp} X_{L_0,R}. \]

Hence \( \mathcal{M}_{L_0,L_2} \) is also quasi-projective \( \text{[RZ96] 2.31} \).

Consider the map sending \((X, \lambda_X, \rho_X)\) to \((\ker(\rho_{X,L_2}), \ker(\rho_{X,L_0}^\perp))\) gives a map from \( \mathcal{M}_{L_0,L_2} \) to a product of projective schemes. It is clear that this map is a closed immersion since we have a commutative diagram

\[ \begin{array}{ccc}
X_{L_2,R} & \xrightarrow{\rho_{X,L_2}} & X_{L_0,R}^\perp \\
| & & | \\
\rho_{X,L_2} & & \rho_{X,L_0}^\perp \\
\downarrow & & \downarrow \\
X & \xrightarrow{\lambda_X} & X^\perp
\end{array} \]

where the map on the top is the natural map induced by the inclusion of \( L_2 \subset L_0 = L_0^\perp \).

It follows that \( \mathcal{M}_{L_0,L_2} \) is projective. □

We define a map \( f : \mathcal{M}_{L_0,L_2} \to \mathbb{P}^1(V_{L_0,2}) \) by sending \((X, \rho_X, \lambda_X)\) to

\[ \mathbb{D}(X)(R)/L_{2,R} \subset L_{0,R}/L_{2,R} \]

for any \( \mathbb{F}_p \)-algebra \( R \).

**Proposition 7.17.** The map \( f : \mathcal{M}_{L_0,L_2} \to \mathbb{P}^1(V_{L_0,2}) \) is an isomorphism.
Proof. Using the theory of windows of display\footnote{Zin99} in place of Dieudonné modules, one can show that $f$ is bijective on $k$-points for any field extension $k$ over $\mathbb{F}_{p^2}$ following the same proof as in proposition 7.9. Since $\mathbb{P}^1(V_{L_0,L_2})$ is normal and $\mathcal{M}_{L_0,L_2}$ is a projective scheme, we can apply Zariski’s main theorem to show $f$ is an isomorphism. $\square$

The scheme $\mathcal{M}_{L_0,L_2}$ admits the following stratification called the Bruhat-Tits stratification. Denote by

\begin{equation}
\mathcal{M}^0_{L_0,L_2} = \mathcal{M}_{L_0,L_2} \setminus \mathcal{M}\{1\}
\end{equation}

and

\begin{equation}
\mathcal{M}_{L_0,L_2,\{1\}} = \mathcal{M}_{L_0,L_2} \cap \mathcal{M}\{1\}.
\end{equation}

Corollary 7.20. The superspecial points on $\mathcal{M}_{L_0,L_2}$ are precisely $\mathcal{M}_{L_0,L_2,\{1\}}$. The isomorphism respects the Bruhat-Tits stratification in the sense that $\mathcal{M}_{L_0,L_2,\{1\}}$ corresponds to the $\mathbb{F}_{p^2}$ points on $\mathbb{P}^1$.

Proof. The statements can be checked on $\mathbb{F}$-points and they follow from proposition 7.9. $\square$

7.5. The main result in the paramodular Rapoport-Zink space. We summarize the results obtained from previous sections. We introduce the following Bruhat-Tits strata for $\mathcal{M}$. Define $\mathcal{M}^0_{\{0,2\}} = \bigcup_L \mathcal{M}^0_L$ where $L$ runs through all the vertices of type $\{0,2\}$ and with $\mathcal{M}^0_L$ defined in (7.18).

Theorem 7.21. The reduced scheme $\mathcal{N}_{\text{red}}$ can be written as $\mathcal{N}_{\text{red}} = \bigcup_{l \in \mathbb{Z}} \mathcal{N}_{\text{red}}(l)$. The connected components $\mathcal{N}_{\text{red}}(l)$ are all isomorphic to $\mathcal{N}_{\text{red}}(0)$. We denote $\mathcal{M} = \mathcal{N}_{\text{red},\mathbb{F}_{p^2}}$. Then $\mathcal{M}$ is pure of dimension 2.

1. $\mathcal{M}$ can be decomposed into

$$\mathcal{M} = \mathcal{M}^0_{\{0,2\}} \cup \mathcal{M}\{1\}.$$ $\,$

We call this the Bruhat-Tits stratification of $\mathcal{M}$.

2. $\mathcal{M}_L$ for $L$ a $\{0,2\}$-vertex is the closure of $\mathcal{M}^0_L$ and is isomorphic to $\mathbb{P}^1$. Moreover it admits a stratification

$$\mathcal{M}_L = \mathcal{M}^0_L \cup \mathcal{M}_L,\{1\}$$

called the Bruhat-Tits stratification for $\mathcal{M}_L$. The complement $\mathcal{M}_L,\{1\}$ of $\mathcal{M}^0_L$ in $\mathcal{M}_L$ corresponds to the $\mathbb{F}_{p^2}$-points of $\mathbb{P}^1$.

3. The intersection between $\mathcal{M}_L$ and $\mathcal{M}_{L'}$ for a $\{0,2\}$-vertex $L$ and another $\{0,2\}$-vertex $L'$ if nonempty is a superspecial point.

Proof. The first statement (1) is seen from lemma 3.21. The second statement (2) is proved in proposition 7.20. The third statement (3) is proved in 7.11 $\square$
7.6. **Application to the supersingular locus.** Let $V$ be vector space of dimension 4 over $\mathbb{Q}$. We assume that $V$ is equipped with a symplectic form $(\cdot, \cdot)$. Then we define $G(\mathbb{Q}) = \{ g \in \text{GL}(V); (gx, gy) = c(g)(x, y) \}$. Then $G = \text{GSp}(4)$. Let $h : \mathbb{G}_m \to G(\mathbb{R})$ be the cocharacter sending $z$ to $\text{diag}(z, 1, z, 1)$. Moreover $h$ defines a decomposition $V_C = V_1 \oplus V_2$ where $h(z)$ acts on $V_1$ by $z$ and on $V_2$ by $\bar{z}$. We fix an open compact subgroup $U$ of $G(\mathbb{A}_f)$ and we assume $U_p$ is sufficiently small. We also assume that there is a lattice $\Lambda$ in $V$ such that $\Lambda$ is paramodular in the sense that $p\Lambda^\perp \subset 2\Lambda \subset 2\Lambda^\perp$. We choose $U_p$ that stabilize $\Lambda \otimes \mathbb{Z}_p$.

To $(V, (\cdot, \cdot), h, \Lambda)$, we consider the following moduli problem $\mathcal{S}_h$ over $\mathbb{Z}(p)$: for a scheme $S$ over $\mathbb{Z}(p)$ we associate set of isomorphism classes of the collection $\{(A, \iota, \lambda, \eta)\}$ where:

- $A$ is an abelian scheme of relative dimension 2 over $S$;
- $\lambda : A \to A^\vee$ is a polarization which is degree $p^2$.
- $\eta_p : V \otimes \mathbb{Q} A_f(p) \to V(p)(A)$ is a $U_p$-orbit of isomorphisms which is compatible with the Weil-pairing on the righthand side and the form $(\cdot, \cdot)$ on the lefthand side.

This moduli problem is representable by a quasi-projective variety $\mathcal{S}_h$ which is well-known. We are interested in the closed subscheme of the supersingular locus $\mathcal{S}_h^{ss}$ in $\mathcal{S}_h$ considered as a closed reduced subscheme. The uniformization theorem of Rapoport-Zink [RZ96] Theorem 6.1 transfers this problem to the descriptions of the Rapoport-Zink space.

**Theorem 7.22.** There is an isomorphism of $\mathbb{F}$ schemes

$$\mathcal{S}_h^{ss} \cong I(\mathbb{Q}) \backslash \mathcal{N}_{\text{red}, \mathbb{F}} \times G(\mathbb{A}_f^p)/U_p.$$ 

Here $I$ is an inner form of $G$ with $I(\mathbb{Q}_p) = J(\mathbb{Q}_p)$. It is defined in the following way. Let $(A, \iota, \eta)$ be a fixed supersingular point in $\mathcal{S}_h^{ss}$. Then $I = \text{End}(A, \lambda) \otimes \mathbb{Q}$.

7.7. **Main theorem on the supersingular locus.** We apply the results obtained previously to describe the supersingular locus of the Siegel modular variety with paramodular level structure

**Theorem 7.23.** The $\mathbb{F}$ scheme $\mathcal{S}_h^{ss}$ is pure of dimension 1. For $U_p$ sufficiently small, the irreducible components are isomorphic to the projective line $\mathbb{P}^1$. The intersection of two irreducible components is a superspecial point.

**Proof.** Using the uniformization theorem 7.22 the result follows from the main theorem for the Rapoport-Zink space [RZ96] Theorem 6.1.

8. **Comparison with Affine Deligne-Lusztig varieties**

In this final section we would like point out how our results fit in the results proved in Götzt and He [GH15] in terms of the affine Deligne-Lusztig varieties. In the following we will abbreviate affine Deligne-Lusztig variety to ADLV.
8.1. **Affine Deligne-Lusztig variety.** Let $F$ be a finite extension of $\mathbb{Q}_p$ and $\tilde{F}$ be the completion of the maximal unramified extension of $F$. Let $G$ be a connected reductive group over $F$ and we write $\hat{G}$ its base change to $\tilde{F}$. Then $\hat{G}$ is quasi split and we choose a maximal split torus $S$ and denote by $T$ its centralizer. We know $T$ is a maximal torus and we denote by $N$ its normalizer. The relative Weyl group is defined to be $W_0 = N(\tilde{F})/T(\tilde{F})$. Let $I$ be the Galois group of $\tilde{F}$ and we have the following Kottwitz homomorphism:

$$\kappa_G : G(\tilde{F}) \to X_*(\hat{G})_I.$$  

Denote by $\tilde{W}$ the Iwahori Weyl group of $\hat{G}$ which is by definition $\tilde{W} = N(\tilde{F})/T(\tilde{F})_1$ where $T(\tilde{F})_1$ is the kernel of the Kottwitz homomorphism for $T(\tilde{F})$. Inside the Iwahori Weyl group $\tilde{W}$, there is a copy of the affine Weyl group $W_a$ which can be identified with $N(\tilde{F}) \cap G(\tilde{F})_1/T(\tilde{F})_1$ where $G(\tilde{F})_1$ is the kernel of the Kottwitz morphism for $G$. The group $\tilde{W}$ is not quite a Coxeter group while $W_a$ is generated by the affine reflections denoted by $\hat{S}$ and $(\tilde{W}, \hat{S})$ form a Coxeter system. We in fact have $\tilde{W} = W_a \rtimes \Omega$ where $\Omega$ is the normalizer of a fixed base alcove and more canonically $\Omega = X_*(T)_I/X_*(T_{sc})_I$ where $T_{sc}$ is the preimage of $T \cap G_{der}$ in the simply connected cover $G_{sc}$.

Let $\mu \in X_*(T)$ be a minuscule cocharacter of $G$ over $\tilde{F}$ and $\lambda$ its image in $X_*(T)_I$. We denote by $\tau$ the projection of $\lambda$ in $\Omega$. The *admissible subset* of $\tilde{W}$ is defined to be

$$\text{Adm}(\mu) = \{ w \in \tilde{W}; w \leq x(\lambda) \text{ for some } x \in W_0 \}.$$  

Here $\lambda$ is considered as a translation element in $\tilde{W}$. Let $K \subset \hat{S}$ and $\tilde{K}$ its corresponding parahoric subgroup. Let $\tilde{W}_K$ be the subgroup defined by $N(\tilde{F}) \cap \tilde{K}/T(\tilde{F})_1$. We have the following decomposition $\tilde{K}\backslash G(\tilde{F})/\tilde{K} = \tilde{W}_K \backslash \tilde{W}/\tilde{W}_K$. Therefore we can define a relative position map

$$\text{inv} : G(\tilde{F})/\tilde{K} \times G(\tilde{F})/\tilde{K} \to \tilde{W}_K \backslash \tilde{W}/\tilde{W}_K.$$  

For $w \in \tilde{W}_K \backslash \tilde{W}/\tilde{W}_K$ and $b \in G(\tilde{F})$, we define the *affine Deligne-Lusztig variety* to be the set

$$X_w(b) = \{ g \in G(\tilde{F})/\tilde{K}; \text{inv}(g, b \sigma(g)) = w \}.$$  

Thanks to the work of [BS17] and [Zhu17], this set can be viewed as an ind-closed-subscheme in the affine flag variety $\hat{G}/\tilde{K}$. In this note, we only consider it as a set. The Rapoport-Zink space is not directly related to the affine Deligne-Lusztig variety but rather to the following union of affine Deligne-Lusztig varieties

$$X(\mu, b)_K = \{ g \in G(\tilde{F})/\tilde{K}; g^{-1} b \sigma(g) \in \tilde{K} w \tilde{K}, w \in \text{Adm}(\mu) \}.$$  

We recall the group $J_b$ is defined by the $\sigma$-centralizer of $b$ that is

$$J_b(R) = \{ g \in G(R \otimes_F \tilde{F}); g^{-1} b \sigma(g) = b \}$$

for any $F$-algebra $R$. In the following we will assume that $b$ is basic and in this case $J_b$ is an inner form of $G$. 


8.2. Coxeter type ADLV. We define $\text{Adm}^K(\mu)$ to be the image of $\text{Adm}(\mu)$ in $\widetilde{W}_K \backslash \widetilde{W} / \widetilde{W}_K$ and $K\widetilde{W}$ to be the set of elements of minimal length in $\widetilde{W}_K \backslash \widetilde{W}$. We define the set $EO^K(\mu) = \text{Adm}^K(\mu) \cap K\widetilde{W}$. For $w \in W$, we set

$$\text{supp}_\sigma(w\tau) = \bigcup_{n \in \mathbb{Z}} (\tau \sigma)^n(\text{supp}(w)).$$

If the length $l(w)$ of $w$ agrees with the cardinality of $\text{supp}_\sigma(w\tau) / \langle \tau \sigma \rangle$, we say $w\tau$ is a $\sigma$-Coxeter element. We denote by $EO^K_{\sigma, \text{cox}}(\mu)$ the subset of $EO^K(\mu)$ such that $w$ is a $\sigma$-Coxeter element and $\text{supp}_\sigma(w)$ is not $\widetilde{S}$. A $K$-stable piece is a subset of $G(\overline{F})$ of the form $K \cdot \sigma I_w I$ where $\cdot \sigma$ means $\sigma$-conjugation and $I$ is an Iwahori subgroup and $w \in K\widetilde{W}$. Then we define the Ekedahl-Oort stratum attached to $w \in EO^K(\mu)$ of $X(\mu, b)_{\mathbb{K}}$ by the set $X_{K, w}(b) = \{g \in G(\overline{F}) / \widetilde{K}; g^{-1}b\sigma(g) \in K \cdot \sigma I_w I\}$. Then by [GH15] we have the following EO-stratification

$$X(\mu, b)_{\mathbb{K}} = \bigcup_{w \in EO^K(\mu)} X_{K, w}(b).$$

The case when

$$X(\mu, b)_{\mathbb{K}} = \bigcup_{w \in EO^K_{\sigma, \text{cox}}(\mu)} X_{K, w}(b)$$

is particular interesting and when this happens we say the datum $(G, \mu, K)$ is of Coxeter type. The datum $(G, \mu, K)$ being Coxeter type or not depends only on the associated datum $(\widetilde{W}, \lambda, K, \sigma)$ where $\lambda$ is the image of $\mu \in X_*(T)_I$ and $\sigma$ is the induced automorphism of the Frobenius $\sigma$ on the local Dynkin diagram. The set of $(G, \mu, K)$ is classified in [GH15] Theorem 5.11. This includes the two cases we studied in the previous sections.

- The quaternionic unitary case corresponds to $G = GU_{B_{1,1}}$ and the datum

$$(\widetilde{C}_2, \omega_2, \widetilde{S} - \{1\}, \tau_2)$$

where $\sigma$ acts on the local Dynkin diagram by the image $\tau_2$ of $\omega_2$ in $\Omega$.

- The paramodular case corresponds to $G = GSp(4)$ and the datum

$$(\widetilde{C}_2, \omega_2, \widetilde{S} - \{1\}, \text{id})$$

where $\sigma$ acts on the local Dynkin diagram by the identity.

8.3. Bruhat-Tits stratification of ADLV. Now we assume that $K$ is a maximal proper subset of $\widetilde{S}$ such that $\sigma(K) = K$. Consider the following set

$$\mathcal{J} = \{\Sigma \subset \widetilde{S}; \emptyset \neq \Sigma \text{ is } \tau\sigma\text{-stable and } d(v) = d(v') \text{ for every } v, v' \in \Sigma\},$$

where $d(v)$ is the distance between $v$ and the unique vertex not in $K$. In fact every $w \in EO^K_{\sigma, \text{cox}}(\mu)$ corresponds to a $\Sigma \in \mathcal{J}$ and we write $w$ as $w_\Sigma$. If $(G, \mu, K)$ is of
Coxeter type, for any \( w_\Sigma \in EO^K_{\sigma, \text{cox}}(\mu) \),

\[
(8.3) \quad X_{K, w_\Sigma}(b) = \bigcup_{i \in J_b / J_b \cap \tilde{K}_{\tilde{\Sigma} - \Sigma}} i \cdot X(w_\Sigma).
\]

Here \( \tilde{K}_{\tilde{\Sigma} - \Sigma} \) is the parahoric subgroup associated to the set \( \tilde{S} - \Sigma \) and \( X(w_\Sigma) \) is a classical Deligne-Lusztig variety defined by

\[
X(w_\Sigma) = \{ g \in \tilde{K}_{\text{supp}_\sigma(w_\Sigma)}/\tilde{I}; g^{-1} \tau \sigma(g) \in \tilde{I}w\tilde{I} \}
\]

which is a Deligne-Lusztig variety attached to the maximal reductive quotient \( G_w \) of the special fiber of \( \tilde{K}_{\tilde{\Sigma} - \Sigma} \). Combine (8.2) and (8.3) we arrive at the following Bruhat-Tits stratification of \( X(\mu, b)_K \):

\[
(8.4) \quad X(\mu, b)_K = \bigcup_{J_b / J_b \cap \ker(\kappa_G)} \bigcup_{w_\Sigma \in EO^K_{\sigma, \text{cox}}} \Delta_{\Sigma}^w \cup \bigcup_{i \in J_b / J_b \cap \tilde{K}_{\tilde{\Sigma} - \Sigma}} i \cdot X(w_\Sigma).
\]

where

\[
(8.5) \quad \Delta_{\Sigma}^w = \bigcup_{i \in J_b / J_b \cap \ker(\kappa_G) / J_b \cap \tilde{K}_{\tilde{\Sigma} - \Sigma}} i \cdot X(w_\Sigma).
\]

8.3.1. Quaternionic unitary case. In the quaternionic unitary case, we can compute

| \( w_\Sigma \) | \( \tau \) | \( s_1 \tau \) | \( s_1 s_2 \tau \) | \( s_1 s_0 \tau \) |
| --- | --- | --- | --- | --- |
| \( \tilde{S} - \Sigma \) | \( \{0,2\} \) | \( \{1\} \) | \( \{1,2\} \) | \( \{0,1\} \) |
| \( \text{supp}_\sigma(w_\Sigma) \) | \( \emptyset \) | \( \{1\} \) | \( \{1,2\} \) | \( \{0,1\} \). |

In this case the Deligne-Lusztig varieties \( X(w_{\{0\}}) \) and \( X(w_{\{2\}}) \) agrees with \( X_B(w_2) \) in Theorem 3.4. The Deligne-Lusztig variety \( X(w_{\{0,2\}}) \) is isomorphic to \( X_B(w_1) \) and \( X(w_{\{1\}}) \) is 0-dimensional and agrees with \( X_P(2)^{\{1\}} \). Therefore we have the following comparison between the Bruhat-Tits stratification for ADLV and Bruhat-Tits stratification for the Rapoport-Zink space studied in the quaternionic unitary case. First recall that for \( \mathcal{M} \) the Brhat-Tits stratification in Theorem 5.1 reads

\[
(8.6) \quad \mathcal{M} = \mathcal{M}_{\{0\}}^{\{0\}} \cup \mathcal{M}_{\{2\}}^{\{2\}} \cup \mathcal{M}_{\{0,2\}}^{\{0,2\}} \cup \mathcal{M}_{\{1\}}^{\{1\}}.
\]

- \( \mathcal{M}_{\{0\}}^{\{0\}} \) in (8.5) is identified with \( \mathcal{M}_{\{0\}}^{\{0\}} \) in (8.6);
- \( \mathcal{M}_{\{2\}}^{\{2\}} \) in (8.5) is identified with \( \mathcal{M}_{\{2\}}^{\{2\}} \) in (8.6);
- \( \mathcal{M}_{\{0,2\}}^{\{0,2\}} \) in (8.5) is identified with \( \mathcal{M}_{\{0,2\}}^{\{0,2\}} \) in (8.6);
- \( \mathcal{M}_{\{1\}}^{\{1\}} \) in (8.5) is identified with \( \mathcal{M}_{\{1\}}^{\{1\}} \) in (8.6) and is the set of superspecial points.
8.3.2. Paramodular Siegel case. In the paramodular Siegel case, we can compute

\[
\begin{align*}
\Sigma & \quad \{1\} \quad \{0, 2\} \\
\omega & \quad \tau \quad s_1 \tau \\
\bar{S} - \Sigma & \quad \{0, 2\} \quad \{1\} \\
supp_{\sigma}(w_{\Sigma}) & \quad \emptyset \quad \{1\}.
\end{align*}
\]

In this case the Deligne-Lusztig varieties \( X(w_{\{1\}}) \) is 0-dimensional and \( X(w_{\{0, 2\}}) \) is isomorphic the complement of \( \mathbb{F}_p \)-points in \( \mathbb{P}^1(\mathbb{F}) \). Therefore we have the following comparison between the Bruhat-Tits stratification for ADLV and Bruhat-Tits stratification for the Rapoport-Zink space studied in the paramodular Siegel case. Recall that for \( \mathcal{M} \) the Bruhat-Tits stratification in Theorem 7.21 reads

(8.7) \[
\mathcal{M} = \mathcal{M}^0_{\{0, 2\}} \cup \mathcal{M}^0_{\{1\}}.
\]

- \( \lambda^0_{\{0, 2\}} \) in (8.5) can be identified with \( \mathcal{M}^0_{\{0, 2\}} \) in (8.7).
- \( \lambda^0_{\{1\}} \) in (8.5) can be identified with \( \mathcal{M}^0_{\{1\}} \) in (8.7) and is the set of superspecial points.

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