The \((N, M)\)-th KdV hierarchy
and the associated \(W\) algebra

L. Bonora
International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2, 34014 Trieste, Italy
INFN, Sezione di Trieste

C. S. Xiong
Physikalisches Institut, der Universität Bonn
Nussallee 12, 53115 Bonn, Germany
Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, China

Abstract
We discuss a differential integrable hierarchy, which we call the \((N, M)\)-th KdV hierarchy, whose Lax operator is obtained by properly adding \(M\) pseudo-differential terms to the Lax operator of the \(N\)-th KdV hierarchy. This new hierarchy contains both the higher KdV hierarchy and multi-field representation of KP hierarchy as sub-systems and naturally appears in multi-matrix models. The \(N + 2M - 1\) coordinates or fields of this hierarchy satisfy two algebras of compatible Poisson brackets which are local and polynomial. Each Poisson structure generate an extended \(W_{1+\infty}\) and \(W_\infty\) algebra, respectively. We call \(W(N, M)\) the generating algebra of the extended \(W_\infty\) algebra. This algebra, which corresponds with the second Poisson structure, shares many features of the usual \(W_N\) algebra. We show that there exist \(M\) distinct reductions of the \((N, M)\)-th KdV hierarchy, which are obtained by imposing suitable second class constraints. The most drastic reduction corresponds to the \((N + M)\)-th KdV hierarchy. Correspondingly the \(W(N, M)\) algebra is reduced to the \(W_{N+M}\) algebra. We study in detail the dispersionless limit of this hierarchy and the relevant reductions.
1 Introduction

Integrable hierarchies are a topic of increasing importance in theoretical physics. It has been realized recently that they play an essential role in the study of 2D quantum gravity and topological field theories, as well as in matrix models. In particular matrix models exhibit an extremely rich integrable structure. The origin of this interest in integrable hierarchies goes back to 1989 when three different groups, through the so-called double scaling limit technique, obtained remarkable non-perturbative results in 2D quantum gravity. In particular they found that the partition function of one-hermitean matrix model with even potential satisfies the KdV hierarchy equations [1]. Later on it has been shown that topological field theories coupled to topological gravity and the Kontsevich model also possess this integrable structure [2][3]. After these successes several authors have conjectured that multi-matrix models should be governed by higher KdV hierarchies [4]. Now, it is rather straightforward to extract discrete hierarchies from matrix models. The difficulties start when we try to pass to differential hierarchies. In particular the double scaling limit technique does not prove as powerful and manageable in multi-matrix models as in one-matrix models.

In [5] we proposed an alternative approach to investigate one-matrix models, by which we could extract a differential integrable hierarchy from the discrete one without reference to any continuum limit. Our basic observation is the following one: in one-hermitean matrix model there naturally exists a discrete (or lattice) integrable hierarchy – the Toda lattice hierarchy; if we treat the first flow parameter as space coordinate, this discrete integrable hierarchy can be re-expressed as a continuum (differential) integrable hierarchy, which admits the following Lax pair representation

\[ L_2 = \partial + R \frac{1}{\partial - S} \]

\[ \frac{\partial}{\partial t_1} L_2 = [(L_2^t)_+, L_2] \]

where \( \partial \) = \( \frac{\partial}{\partial x} = \frac{\partial}{\partial t_1} \), \( t_1 \) is the space coordinate, while \( t_r (r \geq 2) \) are real ‘time’ (coupling) parameters. The subscript “+” means that we keep only the terms containing non-negative powers of \( \partial \). \( R \) and \( S \) are the independent fields of the system. This differential integrable hierarchy is referred to as two-boson representation of the KP hierarchy. This hierarchy can be reduced to the KdV hierarchy by imposing the second class constraint \( S = 0 \).

In [8] we showed that this approach is also applicable to multi-matrix models (with bilinear couplings among different matrices) and permits a systematic analysis of these models. Such multi-matrix models are characterized by generalized Toda lattice integrable hierarchies which, via the same procedure applied to one-matrix models, can be rewritten as the following differential integrable hierarchies

\[ L_{2M} = \partial + \sum_{l=1}^{M} a_l \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1} \]

\[ \frac{\partial}{\partial t_r} L_{2M} = [(L_{2M}^r)_+, L_{2M}] \]

where \( a_1, ..., a_M, S_1, ..., S_M \) are independent coordinates (fundamental fields or dynamical variables) of the system. These integrable hierarchies are called 2M-field representations of the KP hierarchy. It is just these differential integrable hierarchies (together with the string equations) that contain

*This integrable structure also shows up in WZW model and Conformal Affine Toda field theories (CAT models). For a more mathematical approach see [7] and references therein.
the information of multi–matrix models. In [9] we showed that they are related to the higher KdV hierarchies via Hamiltonian reduction.

In particular in [9] we examined in detail the 4–field representation of the KP hierarchy and we obtained two distinct integrable hierarchies by suppressing successively the fields \( S_1, S_2 \). The corresponding Lax operators are

\[
L_{21} = \partial^2 + a_1 + a_2 \frac{1}{\partial - S_2}, \quad L_3 = \partial^3 + a_1 \partial + a_2.
\]

In the present paper we carry our analysis further along the same line. Precisely we will show that:

\( i \). There exists a general integrable differential hierarchy

\[
\frac{\partial}{\partial t_r} L = \left[(L^N)_{+}, L \right]
\]

with the following Lax operator

\[
L = \partial^N + \sum_{l=1}^{N-1} a_l \partial^{N-l-1} + \sum_{l=1}^{M} a_{N+l-1} \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \ldots \frac{1}{\partial - S_1}, \quad N \geq 1, \quad M \geq 0;
\]

which involves \((N+2M-1)\) independent fields. We call them the coordinates of the hierarchy. Since the Lax operator \((1.4)\) is obtained by adding \(M\) pseudo–differential terms to the Lax operator of the usual \(N\–th\ KdV\ hierarchy, and in the lack of a preexisting terminology, we refer to this integrable hierarchy as the \((N,M)\–th\ KdV\ hierarchy\ (or the \(M\–extended\ \(N\–th\ KdV\ hierarchy). We refer at times to the \(N + 2M - 1\) fields together with all the properties implied by integrability as ‘model’. So, for example, we will be talking about the \((N,M)\ model\ in connection with the hierarchy just introduced.

\( ii \). Secondly we will show that the coordinates of the \((N,M)\–th\ KdV\ hierarchy\ form two closed algebras with respect to the two Poisson brackets which constitute the bi–Hamiltonian structure. They generate the extended \(W_{1+\infty}\) and \(W_{\infty}\) algebras, respectively: in other words, there exist combinations of the fields and their derivatives which satisfy the extended \(W_{1+\infty}\) and \(W_{\infty}\) algebras.

\( iii \). Finally we will prove that there exist various Hamiltonian reductions. Precisely we show that it is possible to suppress the \(S\) fields one by one and still obtain integrable hierarchies; at the end of this cascade reduction one gets a KdV hierarchy of order \(N + M\).

The motivation for this work stems from the fact that the pseudodifferential operator \((1.4)\) is, after reduction to the standard form, the most general operator which appears in multi–matrix models. In a companion paper [10] we use the hierarchies of this paper to calculate the correlation functions of the corresponding matrix models.

We remark that there is not only one possible viewpoint to study the system specified by \((1.3)\) and \((1.4)\). In fact, a posteriori, one will remark that the Lax operator \((1.4)\) can be envisaged as a reduction of Lax operator of the type \((1.2a)\) with \(M\) replaced by \(N + M - 1\) and in which \(N - 1\) of the \(S\) fields have been suppressed. The results of the analysis are of course invariant if we change the point of view.

This paper is organized as follows. In section 2 we present a brief review on the pseudo–differential analysis of integrable systems with reference to the general pseudo–differential operator \((2.1)\) and write down the corresponding integrable hierarchy \((2.10)\). In section 3 we will show that this general integrable hierarchy \((2.10)\) admits a particular restriction which leads to the \((N,M)\–th\ hierarchy \((1.3)\). As stated above, the independent fields of \((1.4)\) satisfy two \((N + 2M - 1)\–dimensional\ algebras. We
refer in particular to the one corresponding to the second Hamiltonian structure as the \( W(N, M) \)–algebra. Its properties will be illustrated in section 4. In section 5 we will discuss the reduction procedures. We will show that the \( (N, M) \)–th KdV hierarchy possesses \( M \) different reductions which are characterized by \( M \) second class constraints \( S_l = 0 (l = 1, 2, \ldots, M) \). In particular, when we impose the constraint \( S_1 = 0 \), the \( (N, M) \)–th KdV hierarchy and \( W(N, M) \) algebra are reduced to \( (N + 1, M - 1) \)–th KdV hierarchy and \( (N + 1, M - 1) \) algebra, respectively. If we suppress all the \( S_l \) fields in succession, we will obtain the following two sequences

\[
(N, M) \rightarrow \text{thKdV} \xrightarrow{S_1=0} (N + 1, M - 1) \rightarrow \text{thKdV} \xrightarrow{S_2=0} (N + 2, M - 2) \rightarrow \text{thKdV} \rightarrow \ldots.
\]

\[
S_{M-1}=0 \rightarrow (N + M - 1, 1) \rightarrow \text{thKdV} \xrightarrow{S_M=0} (N + M, 0) = (N + M) - \text{thKdV}. \tag{1.5}
\]

for the hierarchies, and

\[
W(N, M) \xrightarrow{S_1=0} W(N + 1, M - 1) \xrightarrow{S_2=0} W(N + 2, M - 2) \rightarrow \ldots.
\]

\[
S_{M-1}=0 \rightarrow W(N + M - 1, 1) \xrightarrow{S_M=0} W(N + M, 0) = W_{N+M}. \tag{1.6}
\]

for the \( W(N, M) \) algebras, respectively. The double arrow \( \rightarrow \rightarrow \) means reduces to throughout the paper.

Our results on reductions can be nicely summarized by means of a Drinfeld–Sokholov representation, section 6.

We also consider the dispersionless version of the \( (N, M) \)–th KdV hierarchy in section 7 and 8. They constitutes the genus 0 part of the hierarchy in the context of matrix models, see [10]. We find that the dispersionless \( (N, M) \)–th KdV hierarchy admits, in addition to the ones already considered, another type of reductions, in which some \( S_l \) fields are identified with one another instead of being suppressed. In this way we get more possible reductions. All of them can be seen as restrictions, i.e. we can obtain the reduced hierarchies by imposing the constraints on the Lax pair. Section 9 contains some final remarks.

## 2 The pseudo–differential analysis

In this section we give a short review of integrable differential hierarchies, in particular we collect some results we will need later on concerning pseudo–differential analysis, coadjoint orbit method and bi–Hamiltonian structure. We refer to existing books and reviews (see [11] and references therein) for more detailed explanations.

### 2.1 Pseudo-differential analysis

Let us begin with the most general pseudo–differential operator (denoted PDO)

\[
A = \sum_{-\infty}^{N} u_i(x) \partial_i. \tag{2.1}
\]
where $u_i(x)$’s are ordinary functions. As usual $\partial = \frac{\partial}{\partial x}$ is the derivative with respect to the space coordinate $x$, while $\partial^{-1}$ is a formal integration operator, which has the following properties

$$\partial^{-1} \partial = \partial \partial^{-1} = 1,$$

$$\partial^{-j-1} u = \sum_{v=0}^{\infty} (-1)^v \binom{j+v}{v} u^{(v)} \partial^{-j-v-1}, \quad (2.2)$$

where $u' = \frac{\partial u}{\partial x}, u'' = \frac{\partial^2 u}{\partial x^2}, \ldots, u^{(v)} = \frac{\partial^v u}{\partial x^v}$. The above formula together with the usual Leibnitz rule,

$$\partial u(x) = u(x) \partial + u'(x), \quad [\partial, x] = 1$$

provides an algebraic structure on the whole operator space formed by the general PDO’s (2.1). We call this algebra \textit{pseudo–differential algebra} $\varphi$.

For any pseudo–differential operator $A$ of type (2.1), we call order the number $N$ (the highest power of $\partial$). We will use the conventions

$$\begin{cases} 
A_+ : \text{pure differential part of } A; \\
A_- : \text{pure integration part of } A; \\
A_{(i,j)} \equiv \sum_{l=i}^{j} u_l(x) \partial^l, \quad i > j; \\
A_{(i)} \equiv u_i \partial^i, \quad -\infty < i \leq N.
\end{cases}$$

Therefore any pseudo–differential operator has a natural decomposition

$$A = A_+ + A_-.$$ 

which leads to an analogous decomposition in $\varphi$

$$\varphi = \varphi_+ + \varphi_-$$

In terms of this truncation, we can introduce the useful mapping $[15]$

$$\mathcal{R} A \equiv A_+ - A_- \quad (2.3)$$

This mapping defines another algebraic structure on $\varphi$

$$[X,Y]_{\mathcal{R}} \equiv \frac{1}{2} \left( [\mathcal{R}X,Y] + [X,\mathcal{R}Y] \right) = [X_+,Y_+] - [X_-,Y_-], \quad \forall X,Y \in \varphi. \quad (2.4)$$

This $\mathcal{R}$–commutator will be very important in our discussion on Hamiltonian structures. But before that, we need more notations. We call $u_{-1}(x)$ the residue of pseudo–differential operator $A$ of type (2.1) and denote it by

$$\text{res}_\partial A = u_{-1}(x) \quad \text{or} \quad A_{(-1)}.$$

The following functional integral

$$\langle A \rangle = \text{Tr}(A) = \int (\text{res}A) = \int u_{-1}(x)dx \quad (2.5)$$
naturally defines an inner product on the algebra $\wp$, which is nondegenerate, symmetric and invariant $^\dag$. Using this inner product, we may express a functional of the fields $u_i$’s as follows

$$f_X(A) = \langle AX \rangle = A(X), \quad X = \sum_i \partial^i \chi_i(x)$$

where $\chi_i(x)$’s are testing functions, and $X$ may be thought of as a cotangent vector at the point $A$. Generally, the cotangent vector $df$ is defined as

$$\delta f(A) = f(A + \delta A) - f(A) = \langle df, \delta A \rangle.$$ 

We denote by $\mathcal{F}(\wp)$ the functional space formed by all functionals $f_X$ defined as above.

### 2.2 Bi–Hamiltonian structure

By means of the $\mathcal{R}$–commutator introduced in the previous subsection, we may define the adjoint action of the algebra $\wp$ on itself $^\ddagger$

$$Ad_Y X \overset{\text{def}}{=} [X, Y]_\mathcal{R}.$$ 

The coadjoint action of $\wp$ on the functional space $\mathcal{F}(\wp)$ is specified by

$$Ad^*_Y f_X(A) \overset{\text{def}}{=} A(Ad_Y X) = A([X, Y]_\mathcal{R}),$$

For a fixed $f_X$, as $Y$ varies in $\wp$, $Ad^*_Y f_X$ spans an orbit in the functional space $\mathcal{F}(\wp)$ which is called the co–adjoint orbit. Since we may view $Y$ as a co–tangent vector, this co–adjoint action naturally defines a Poisson structure on $\mathcal{F}(\wp)$

$$\{f_X, f_Y\}_1(A) = A([X, Y]_\mathcal{R}). \tag{2.6}$$

With respect to this Poisson bracket, the conserved quantities (Hamiltonians) have very simple form

$$H_r = \frac{N}{r} \langle A^\frac{r}{N} \rangle, \quad \forall r \geq 1; \tag{2.7}$$

The corresponding cotangent vector at the point $A$ reads

$$dH_r = A^{\frac{N}{N(1-N,\infty)}}, \quad \forall r \geq 1. \tag{2.8}$$

The time evolution of a function $f_X(A)$ is given by

$$\frac{\partial}{\partial t_r} f_X(A) = \{f_X(A), H_{r+N}\}_1 = \langle A, [X, dH_{r+N}]_\mathcal{R} \rangle,$$

$^\ddagger$ The product is symmetric since $\langle AB \rangle = \langle BA \rangle$, while invariance means $\langle A[B, C] \rangle = \langle [A, B]C \rangle$. Occasionally we will also denote the product in other ways: $\langle AB \rangle = A(B) = Tr(AB)$.

$^\dag$ The usual adjoint action is $Ad_Y X = [X, Y]$, here we use the same notation to denote the adjoint action generated by the $\mathcal{R}$–commutator.
From eqs. (2.4, 2.8) we see that
\[
\langle A, [X, dH_{r+N}] \rangle_R = \langle A, [X_+, (A^\diamond_+)_{+}] - [X_-, (A^\diamond_-)_{(1-N,-1)}] \rangle_R
\]
\[
= - \langle [A, (A^\diamond_+)_{+}], X_+ \rangle + \langle [A, (A^\diamond_-)_{(1-N,-1)}], X_- \rangle
\]
\[
= - \langle [A, (A^\diamond_+)_{+}], X_+ \rangle + \langle [A, (A^\diamond_-)_{+}], X_- \rangle
\]
\[
= \langle X, [(A^\diamond_+)_{+}, A] \rangle.
\]
In the second and the last equalities we have used the invariance of the inner product; in the third step we have added a vanishing term \( \langle [A, (A^\diamond_+)_{+}], X_- \rangle \), since
\[
[A, A^\diamond_{(\infty,-N)}] \in \varphi_+\quad \text{and} \quad \langle Y, Z \rangle = 0, \quad \text{if} \quad Y, Z \in \varphi_+.
\]
Therefore we have
\[
\frac{\partial}{\partial t_r} f_X(A) = \langle X, [(A^\diamond_+)_{+}, A] \rangle, \quad (2.9)
\]
Suppose that \( X \) is time independent; then we obtain the time evolution equations of the pseudo–differential operator (2.1)
\[
\frac{\partial}{\partial t_r} A = [(A^\diamond_+)_{+}, A], \quad \forall r \geq 1. \quad (2.10)
\]
This is the general integrable differential hierarchy we have obtained from pseudo–differential analysis
\[\text{‡}\]. In order to prove its integrability we derive the second Hamiltonian structure, [13], compatible with the first, i.e. a second Poisson bracket such that
\[
\{ f_X, H_{r+N} \}_1 = \{ f_X, H_r \}_2. \quad (2.11)
\]
The LHS is just (2.9). Taking the residue of the following equality
\[
[(A^\diamond_+)_{(1-N, \infty)}, A] = [A, (A^\diamond_+)_{(\infty, -N)}],
\]
we get
\[
\mathcal{N} (A^\diamond_{(-N)}) = [A^\diamond_{(1-N, \infty)}, A]_{(-1)}.
\]
On the other hand,
\[
(A^\diamond_+) = (A \cdot A^\diamond)_{(1-N, \infty)} \cdot (A^\diamond_{(1-N, \infty)})_{+} + (A^\diamond_{(-N)})_{(-N)}
\]
\[
= [(A^\diamond_{(1-N, \infty)}) + (A^\diamond_{(-N)})]_{+}.
\]
\[\text{‡}\]If we define another kind of \( \mathcal{R} \)–operator such as
\[
\mathcal{R}(X) = X_{1, \infty} - X_{-\infty, t-1},
\]
we obtain non–standard integrable hierarchies for some values of \( l \) [12].
Therefore, we have

$$\text{LHS of eq.}(2.11) = <AX(A_{\infty}^{N})> - <XA(A_{\infty}^{N})>$$

$$= <AX\left(A_{(1-N,\infty)}^{N}\right)> - <XA\left(A_{(1-N,\infty)}^{N}\right)> + <AX - AX\left(A_{(1-N,\infty)}^{N}\right)>$$

$$= <AX\left(AdH_{r}\right)> - <XA\left(dH_{r}\right)> + \frac{1}{N} \int [A, dH_{r}]_{(-1)} \left(\partial^{-1}[dH_{r}, A]_{(-1)}\right)$$

Now we can see that all the terms in the last equality are linear in the cotangent vector $dH_{r}$ and in $X$. This is therefore a candidate for a second Poisson bracket. If we replace $dH_{r}$ by an arbitrary cotangent vector $Y$, we have the general expression of it (2.10)

$$\{f_{X}, f_{Y}\}_{2}(A) = <(AX)_{+}Y A> - <(AX)_{+} AY>$$

$$+ \frac{1}{N} \int [A, Y]_{-1} \left(\partial^{-1}[A, X]_{-1}\right). \tag{2.12}$$

One can show that this Poisson bracket satisfies the Jacobi identity [14], so it is indeed a well–defined Poisson structure. With respect to this bracket, the conserved quantities are the same as in eq. (2.7), and generate all the flows in the hierarchy (2.10). Therefore we proved that the system (2.10) possesses a bi–Hamiltonian structure and is therefore integrable.

3 The $(N, M)$–th KdV hierarchy

In the previous section we have constructed the bi–Hamiltonian structure for a general pseudo–differential operator and the corresponding integrable hierarchy. In this section we will prove that the general integrable hierarchy (2.10) admits a particular restriction which leads to the $(N, M)$–th KdV hierarchy (1.3) with the Lax operator (1.4).

3.1 The consistency of the general flows

In order to show that the restricted Lax operator (1.4) preserves the hierarchy (2.10), and gives in fact the $(N, M)$–th KdV hierarchy, we should first prove the consistency of the flows defined by (1.3), and then check that they commute.

For the usual $N$–th KdV hierarchy ( KP hierarchy), the scalar Lax operator is a pure differential operator ( pseudo–differential operator), so it is very easy to see its form invariance under the time evolutions. For instance, the scalar Lax operator involved in the $N$–th KdV hierarchy is

$$A_{N_{k}N_{v}} = \partial^{N} + \sum_{l=1}^{N-1} a_{l} \partial^{N-l-1},$$

Obviously we have

$$[\left(\frac{A_{N_{k}N_{v}}}{\infty}\right), A_{N_{k}N_{v}}] = [A_{N_{k}N_{v}}\left(A_{N_{k}N_{v}}\right)^{\infty}].$$
The LHS is a purely differential operator, while the RHS indicates that this operator is of order $N - 2$. This enables us to define the flows as follows

$$\frac{\partial}{\partial t} A_{NKdV} = \left[A_{NKdV}, A_{NKdV}\right].$$

Therefore, the consistency of the flows in the $N$–th KdV hierarchy (or KP hierarchy) case is very simple.

However the present case is much more complicated since the scalar Lax operator (1.4) contains both differential and pseudo–differential parts. It is a nontrivial result that the time evolutions of the Lax operator (1.4) are form invariant. This subsection is devoted to proving exactly this.

Proposition 3.1 The LHS and the RHS of eq.(1.3) are compatible.

Proof. To start with we write down the $r$–th time evolution of the Lax operator (1.4)

$$\frac{\partial}{\partial t} L = \sum_{l=1}^{N-1} \left(\frac{\partial a_l}{\partial t}\right) \frac{\partial^{N-l-1}}{\partial s_{l-1}} \frac{1}{\partial s_{l-1}} \cdots \frac{1}{\partial s_1}$$

$$+ \sum_{l=1}^{M} \sum_{k=1}^{l} a_{N+l-1} \frac{1}{\partial s_{l}} \cdots \frac{1}{\partial s_{k}} \left(\frac{\partial s_{k}}{\partial \tau}\right) \frac{1}{\partial s_{k}} \cdots \frac{1}{\partial s_1}. \tag{3.1}$$

The problem is to prove that the LHS of eq.(1.3) contains the same type of terms. Our first task will thus be to classify the terms in RHS of this equation. To this end we introduce a new basis set for the pseudo–differential operator algebra $\mathcal{P}$. We split the basis set into several classes:

- **the differential operator class**: we adopt the usual basis
  $$\{f_i \partial^i, i \in \mathbb{Z}_+\},$$
  with arbitrary functions $f_i$’s. By definition this basis spans the operator space $F^0$.

- **the first class of integration operators**:
  $$f_l \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1}, \quad 1 \leq l \leq M;$$
  where $f_l$’s are arbitrary functions. For a fixed $l$, the above operators span by definition the space $F^1_l$. We also define the direct sum space $F^1 = \oplus_l F^1_l$.

- **the second class of integration operators**:
  $$f_{lk} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} \frac{1}{\partial - S_{k-1}} \cdots \frac{1}{\partial - S_1}, \quad 1 \leq l \leq M; \quad 1 \leq k \leq l.$$ 
  Once again $f_{lk}$’s are arbitrary functions. We denote by $F^2_{l,k}$ the space spanned by the above operators with fixed $l, k$, and

$$F^2 = \{\oplus_{l,k} F^2_{l,k}, \ 1 \leq l \leq M; \ 1 \leq k \leq l\}.$$ 

All these operators are linearly independent. Actually they do not constitute a complete basis of the algebra $\mathcal{P}$. However, since the residual basis subset will not show up in our later discussion, we do not need to write them out explicitly. Going back to eq.(3.1), we immediately see that its RHS contains all these terms. We will say that a pseudodifferential operator takes the standard form $F^0, F^1$ or $F^2$ if it belongs to the corresponding vector subspace.
What we should do next is to prove that the RHS of eq.(1.3) can be recast into these standard forms.

In order to simplify the calculation of the commutator in eq. (1.3), we adopt the notation

\[ O_r^{(r)} = \left( L^N \right)_+ = \partial^r + \sum_{l=1}^{r-2} \alpha_l \partial^l, \quad \forall r \geq 1. \]

where the \( \alpha_l \)'s are certain functions of the \( \{a_l\} \)'s and the \( \{S_l\} \)'s. The commutator in eq.(1.3) can be decomposed into two parts

\[ [O_r, L] = [O_r, L+] + [O_r, L^-]. \]

Obviously the first part is a purely differential operator of order \( N-1 \), while the second part can be further simplified

\[
[O_r, L^-] = [O_r, \sum_{l=1}^{M} \frac{a_{N+l-1}}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1}]
- \sum_{l=1}^{M} \sum_{k=1}^{l} a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} \left[ \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \right] [O_r, \partial - S_l] \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1}, \tag{3.2}
\]

Here we have used

\[ [O_r, \frac{1}{\partial - S_k}] = -\frac{1}{\partial - S_k} [O_r, \partial - S_k] \frac{1}{\partial - S_k}. \]

The remaining two commutators in the expression (3.2) only involve purely differential operators, both of them are of order \( r-1 \). Therefore we find that the RHS of eq.(1.3) contains the following three kinds of terms

\[
\begin{align*}
G_0 & : \text{purely differential operator part } [O_r, L^+] \\
I^{(r)}_l & : O_{(r-1)} \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1}, \quad 1 \leq l \leq M; \\
J^{(r-1)}_{l,k} & : O_{(0)} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} O_{(r-1)} \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1}, \quad 1 \leq k \leq l \leq M.
\end{align*}
\]

where \( O_r \) is an arbitrary purely differential operator of \( r \)-th order, while \( O_{(0)} \) is an ordinary function. We will denote

\[
G_1 = \{ \sum_l I_l, \quad 1 \leq l \leq M \},
\]

\[
G_2 = \{ \sum_{l,k} J^{(r-1)}_{l,k}, \quad 1 \leq k \leq l \leq M \}.
\]

We will need one more operator space, \( G^3 \), which contains the following type of terms

\[
\nabla^{(r-1)}_{l,k} : O_{(0)} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} O_{(r-1)} \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1}, \quad 1 \leq k \leq l \leq M.
\]
Here $\mathcal{O}_{(r-1)}$ is another purely differential operator. In the case $k = l$, this is the same as $G_1$, i.e.

$$
V_{l,l}^{(r)} = I_l^{(r)}.
$$

(3.3)

In the remaining part of this subsection we will show how to recast the above terms into the standard forms $F^0, F^1, F^2$.

$G_0$–terms:

We notice that

$$
[(L^{\pm})_+, L] = [L, (L^{\pm})_-],
$$

so the power expansion of the above commutator has order $< N - 1$, i.e. $G_0$–terms are pure differential operators of $(N - 2)$–th order, i.e. they have exactly the same form as $F^0$–terms. Therefore $G_0$–terms contribute to the time evolutions of the fields $a_l$ ($1 \leq l \leq N$).

$G_1$–terms:

In order to simplify $G_1$–terms, we recall that for any purely differential operator $O_{(r)}$ of $r$–th order, and any ordinary function $f$, we can find two $(r-1)$–th order purely differential operators $\tilde{O}_{(r-1)}$ and $\mathcal{O}_{(r-1)}$ such that

$$
O_{(r)} = (\partial - f)O_{(r-1)} + g, \quad \text{and} \quad O_{(r)} = \tilde{O}_{(r-1)}(\partial - f) + \tilde{g}.
$$

(3.4)

where $g$ and $\tilde{g}$ are two ordinary functions specified by the above equalities. The second equality immediately leads to the following results

$$
I_l^{(r-1)} \rightarrow I_l^{(r-2)} \oplus F^1_l.
$$

repeating this procedure we can reduce all the $G_1$–terms to $F^1$ (possibly $F^0$) terms. For instance

$$
[(L^{\pm})_+, a_{N+l-1}] = O_{(r-1)} = f_0 + O_{(r-2)}(\partial - S_l)
$$

$$
= f_0 + \left( f_1 + O_{(r-3)}(\partial - S_{l-1}) \right)(\partial - S_l)
$$

$$
= \ldots \ldots
$$

$$
= f_0 + \sum_{i=1}^{l-1} f_i(\partial - S_{l-i+1})(\partial - S_{l-i}) \ldots (\partial - S_l)
$$

$$
+ O_{(r-l-1)}(\partial - S_1)(\partial - S_2) \ldots (\partial - S_l),
$$

where all the functions $f_i$'s and the operator $O_{(r-l-1)}$ are completely determined by the commutator. Therefore the $G_1$–term

$$
[(L^{\pm})_+, a_{N+l-1}] = \frac{1}{\partial - S_1} \frac{1}{\partial - S_{l-1}} \ldots \frac{1}{\partial - S_l}
$$

becomes an $F^1$–term. If $r \geq l + 1$, the last term in eq.(3.5) contributes an $F^0$–term $O_{(r-l-1)}$. We may summarize the procedure with the following diagram

$$
I_l^{(r)} \rightarrow I_{l-1}^{(r-1)} \rightarrow I_{l-2}^{(r-2)} \rightarrow \ldots \rightarrow I_{l}^{(r-l-1)} \rightarrow I_{0}^{(r-l)}
$$

(3.6)
$G_2$–terms:

Now let us consider the $G_2$–terms. Their form is

\[
J_{l,k}^{(r-1)} = a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} \left[ (L^2)_{+,} \right. \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \\
= a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} O^{(r-1)} \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \\
= a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} O^{(0)} \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \\
+ a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_{k+1}} O^{(r-2)} \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1},
\]

in the last step we have used the first equality in eqs. (3.4). This decomposition simply means

\[
J_{l,k}^{(r-1)} \implies V_{l,k}^{(r-2)} \oplus F_{l,k}^2.
\] (3.7)

Here and in the following $\implies$ means decomposition. In other words, $G_2$–terms can be decomposed into $F^2$ and $G_3$ terms. Therefore it remains for us to treat $G_3$–terms.

$G_3$–terms:

As we know, a $G_3$–term is of the form

\[
V_{l,k}^{(r-1)} = a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_k} O^{(r-2)} \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1}.
\]

With a simple calculation we get

\[
\frac{1}{\partial - S_{k+1}} O^{(r-2)} = O^{(r-2)} \frac{1}{\partial - S_{k+1}} + \left[ \frac{1}{\partial - S_{k+1}}, O^{(r-2)} \right]
\]

\[
= O^{(r-2)} \frac{1}{\partial - S_{k+1}} - \frac{1}{\partial - S_{k+1}} [\partial - S_{k+1}, O^{(r-2)}] \frac{1}{\partial - S_{k+1}}
\]

\[
= O^{(r-2)} \frac{1}{\partial - S_{k+1}} - \frac{1}{\partial - S_{k+1}} - \frac{1}{\partial - S_{k+1}} O^{(r-3)} \frac{1}{\partial - S_{k+1}}.
\]

This result shows

\[
V_{l,k}^{(r-1)} \implies V_{l,k+1}^{(r-2)} \oplus J_{l,k+1}^{(r-1)}.
\] (3.8)

The crucial feature of this step is that we have moved the operators in $G_3$ one step to the right.

Combining the procedures (3.7) and (3.8), and remembering the fact (3.3), we can recast the $G_2, G_3$–terms into the standard forms. Diagrammatically

\[
\begin{align*}
V_{l,k}^{(r-1)} & \implies V_{l,k+1}^{(r-2)} \implies V_{l,k+1}^{(r-2)} \implies \cdots \implies V_{l,l}^{(r-l+k-1)} = I_{l}^{(r-l+k-1)} \\
J_{l,k}^{(r-1)} & \implies J_{l,k+1}^{(r-2)} \implies J_{l,k+2}^{(r-2)} \implies \cdots \implies J_{l,l}^{(r-l+k)} \\
F_{l,k}^2 & \implies F_{l,k+1}^2 \implies F_{l,k+2}^2 \implies \cdots \implies F_{l,l}^2
\end{align*}
\] (3.9)
The last term in the first line can be treated through the procedure (3.6). So we finally proved that the RHS of eq. (1.3) has exactly the same form as its LHS. Comparing their explicit expressions, we can obtain the equations of motion of the fundamental fields.

3.2 Commutativity of the flows

Our next problem is to prove the following

**Proposition 3.2** All the flows defined in (1.3) commute with one another, i.e.

\[
\frac{\partial}{\partial t_l} \left( \frac{\partial}{\partial t_r} L \right) = \frac{\partial}{\partial t_r} \left( \frac{\partial}{\partial t_l} L \right).
\]  

**Proof.** This is elementary to prove since we have explicitly

\[
\text{RHS} = \left[ [L_N^+, L^+_{N-1}, L] + [L_N^+, [L_N^+, L]] \right] + \left[ [L_{N-1}^+, L] - [L_N^+, [L_{N-1}^+, L]] \right] = \left[ [L_N^+, L^+_{N-1}, L] + [L_N^+, [L_N^+, L]] \right] = \text{LHS}.
\]

In the second equality we have used the Jacobi identity.

3.3 The non–linear evolution equations

Now let us give here the first non–trivial flow of the hierarchy (3.11a) (see Appendix A for the derivation),

\[
\frac{\partial}{\partial t_2} a_l = a_l'' + 2a_{l+1}' - \frac{2}{N} \left( \begin{array}{c} N \\ l + 1 \end{array} \right) a_1^{(l+1)}, \quad 1 \leq l \leq N - 1; \quad (3.11a)
\]

\[
\frac{\partial}{\partial t_2} a_{N+l} = a_{N+l}'' + 2a_{N+l+1}' + 2a_{N+l}'S_{l+1} + 2a_{N+l} \left( \sum_{k=1}^{l+1} S_k \right)', \quad 0 \leq l \leq M - 1; \quad (3.11b)
\]

\[
\frac{\partial}{\partial t_2} S_l = \frac{2}{N} a'_1 + 2S_lS'_l - S_l'' - 2\left( \sum_{k=1}^{l-1} S_k \right)'', \quad 1 \leq l \leq M. \quad (3.11c)
\]

This set of non–linear evolution equations are crucial objects, all the important ingredients like the bi–Hamiltonian structure and the algebraic structures, as well as the conserved quantities, are rooted in these equations; the pseudo–differential analysis is only a powerful tool to make them explicit. In other words, we can say that the above equations define a bi–Hamiltonian system with two Poisson structures (2.6) and (2.12), the hierarchical equations (1.3) can be thought of as symmetries of these non–linear equations.

We end this section with a few remarks.

(i). Comparing the first \(N - 1\) equations (3.11a) with the \(N\)–th KdV equations, we see that they are the same, except that the \(a_N\) field is involved in the time evolution of the field \(a_{N-1}\) in eq. (3.11a). Therefore, if we set \(a_{N+l-1} = S_l = 0, l = 1, 2, \ldots, M\); we recover the \(N\)–th KdV hierarchy. So we may
view this set of equations (3.11a)—(3.11d) as a generalization or a perturbation of the \( N \)--th KdV equations by means of the fields \( \{ a_{N+l-1}, S_l; 1 \leq l \leq M \} \).

(ii). In the \( N = 1 \) case, the integrable hierarchy (1.3) is nothing but the \( 2M \)--field representation of KP hierarchy.

In this sense we can say that the \( (N,M) \)--th KdV hierarchy (1.3) contains both \( N \)--th KdV hierarchy and the multi--boson representations of KP hierarchy.

4 \( W(N,M) \)--algebra (or the extended \( W \)--algebra)

As we know, the second Hamiltonian structure of the \( N \)--th KdV hierarchy gives rise to a \( W_N \) algebra, and the second Hamiltonian structure of the KP hierarchy leads to a \( W_\infty \) algebra. We will see that the bi--Hamiltonian structure of the integrable hierarchy (1.3) results in two finite dimensional algebras, which generate the extended \( W_{1+\infty} \) and \( W_\infty \) algebras, respectively.

Before we proceed we wish to clarify the meaning of our choice of coordinates in (1.4). Up to now in fact we have only used \( \{ a_l, l = 1,2,\ldots,N-1; a_{N+l-1}, S_l, l = 1, 2, \ldots,M \} \) as our dynamical fields. However, one might choose any other system of coordinates. For instance, we can introduce a new set of coordinates in the following way

\[
\begin{align*}
(\ln r_l)' &= -S_l, \\
q_l &= \frac{a_{N+l-1}}{r_l}, \quad 1 \leq l \leq M;
\end{align*}
\]

Noting

\[
\frac{1}{\partial T_l} = r_l \frac{1}{\partial + (\ln r_l)'};
\]

we can immediately rewrite (1.4) as follows

\[
L = \partial^N + \sum_{l=1}^{N-1} a_l \partial^{N-l-1} + \sum_{l=1}^{M} q_l \partial^{-1} \left( \frac{T_l}{r_l-1} \right) \partial^{-1} \ldots \left( \frac{T_2}{r_1} \right) \partial^{-1} r_1. 
\]

Each set of coordinates have their own advantages (and in general different physical meaning). For example, the first choice (1.4) leads to simple and local Poisson algebras, while the Poisson algebras in the second choice (4.2) will contain non--local terms. We can think of the passage from one set of coordinates to another set as a field--dependent gauge transformation of the corresponding linear system (see the representations of the Lax operators in terms of linear systems of section 6). For this reason we will refer to different choices of the coordinates as “different gauges”.

We notice however that we can write

\[
L \overset{\text{def}}{=} \partial^N + \sum_{l=1}^{N-1} a_l \partial^{N-l-1} + \sum_{n=0}^{\infty} u_n \partial^{-n-1},
\]

where

\[
u_n = \sum_{l=1}^{n} \sum_{\mu_i=n-l+1} a_{N+l-1}(-\partial + S_l)^{\mu_l}(-\partial + S_{l-1})^{\mu_{l-1}} \ldots (-\partial + S_1)^{\mu_1} \cdot 1,
\]

\[0 \leq n \leq M - 1;\]
The form of the operator \( L \) is gauge invariant and by (a suggestive) abuse of language we call the coordinates \( \{ a_l, 1 \leq l \leq N - 1; u_l, l \geq 0 \} \) ‘gauge invariant’.

The expressions (4.4) have the following remarkable features:

- the \( u_l \)'s are linear in the \( a_l \) \((1 \leq l \leq N + M - 1)\) fields, but highly non–linear in the \( S_l \) fields;
- the \( u_l \)'s do not contain any derivative of the \( a_l \) \((1 \leq l \leq N + M - 1)\) fields while they contain derivatives of the \( S_l \) fields.
- these formulas show that the subsets of fields \( \{ a_l, 1 \leq l \leq N - 1 \} \) and \( \{ a_{N+l}, 0 \leq l \leq M - 1 \} \) introduced in (1.4) are of quite different nature, since the former set is canonical the other is not. However for later convenience we will keep the notation \( \{ a_l \} \) for both subsets.
- in the gauge (4.2), the \( u_l \)'s will be linear in the \( q_l \) fields, but non–linear and non–polynomial in the \( r_l \) fields.

In the following we will mostly work with the gauge (1.4).

### 4.1 The extended \( W_\infty \) algebras

If we substitute the Lax operator (4.3) into (2.6) and (2.12), we will get two infinite dimensional algebras, which we will call extended \( W_\infty \) algebras. The calculation is in principle straightforward, but it requires some skillful use of suitable techniques to simplify the results. However we will omit any detail and only give the results.

**The extended \( W_{1+\infty} \) algebra:**

**Proposition 4.1** The first Poisson structure (2.6) leads to the following explicit algebra

\[
\begin{align*}
\{ a_i, a_j \} & = \left[ \left( \begin{array}{c} j \\ N - i - 1 \end{array} \right) + (-1)^{i+j-N} \left( \begin{array}{c} i \\ N - j - 1 \end{array} \right) \right] \partial^{i+j-N+1} \\
& + \sum_{l=i}^{N-1} (-1)^{j+l-N-1} \left( \begin{array}{c} i-l-1 \\ N-j-1 \end{array} \right) \partial^{i+l-N} a_l \\
& + \sum_{l=j}^{N-1} \left( \begin{array}{c} j-l-1 \\ N-i-1 \end{array} \right) a_l \partial^{i+l-N} \end{align*}
\]  
(4.5a)

\[
\begin{align*}
\{ a_i, u_j \} & = 0, \quad \forall i,j \\
\{ u_i, u_j \} & = \sum_{l=1}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^l u_{i+j-l} + \sum_{l=1}^{i} (-1)^{l+1} \left( \begin{array}{c} i \\ l \end{array} \right) u_{i+j-l} \partial^l \end{align*}
\]  
(4.5b)

In the above Poisson brackets (as well as in subsequent ones) we use a shorthand notation. They must be understood in the following way

\[ \{ f(x), g(y) \} = \hat{O}(x) \delta(x-y), \quad \forall f,g; \]
Moreover either the indices of \( a_l \) \( (u_l) \) fields are in the region \( 1 \leq l \leq N - 1 \) \( (l \geq 0) \) or the corresponding terms are understood to be absent. Similarly, either the powers of \( \partial \) are non–negative or the corresponding terms are absent. We remark that a central term is contained only in eq.(4.5a).

The above Poisson algebra is a direct sum of two sub–algebras, the sub–algebra \((4.5a)\) coincides exactly with the first Poisson algebra in the \( N \)–th KdV hierarchy. From eq.(4.5a), we see that

\[
\{ u_1, u_1 \} = (u_1 \partial + \partial u_1) \delta(x - y),
\]

Therefore \( u_1 \) can be viewed as conformal tensor of weight 2. The Poisson bracket \( \{ u_1, u_0 \} \) indicates that \( u_0 \) has conformal spin 1. Such sub–algebra \((4.5c)\) is commonly referred to as a \( W_{1+\infty} \) algebra.

**The extended \( W_{\infty} \) algebra:**

The second Poisson algebra is much more complicated than the former one.

**Proposition 4.2** The explicit form of the second Poisson brackets is

\[
\{ a_i, a_j \} = \left[ c_{ij} \partial^{i+j+1} + \left( \sum_{l=i+j-N+1}^{i+j-1} (-1)^{l+1} \binom{i}{l} \partial^l \right) a_{i+j-l} \partial^l \right. \\
+ \sum_{l=1}^{i+j-N} \left( \binom{j}{l} \partial^l u_{i+j-l-N} + (-1)^{l+1} \binom{j}{l} u_{i+j-l-N} \partial^l \right) \\
+ \left( \frac{1}{N} \sum_{l=i+1}^{i+j-1} (-1)^{l+1} \binom{N}{i+1} \left( N + l - i - j - 1 \right) \right) \\
+ \sum_{k=i+j-l-N+1}^{i+j-1} b_{ijl}^2 \partial^l a_{i+j-l} \right.
\]

\[
+ \sum_{l=1}^{i-j-1} \sum_{k=1}^{i-j-l} \left( -1 \right)^{k+1} \binom{i-j-l-1}{k} a_{i+j-l-k-1} \partial^k a_l + \sum_{k=i+j-l-N}^{i+j-l-2} b_{ijlk}^2 \partial^l a_{i+j-l-k-1} \right]
\]

\[
+ \sum_{l=1}^{i-j-l-N-1} \sum_{k=1}^{i-j-l-N-1} \left( -1 \right)^{k+1} \binom{i-j-l-1}{k} u_{i+j-l-k-N-1} \partial^k a_l + \left( j - l - 1 \right) a_l \partial^k u_{i+j-l-k-N-1} \right)
\]

\[
+ \frac{1}{N} \sum_{l=1}^{i-j-l-k-N-1} \sum_{k=1}^{i-j-l-k-N-1} \left( -1 \right)^{k+1} \binom{i-j-l-k-1}{k} a_l \partial^{i+j-l-k-1} a_k \right] \delta(x - y);
\]

\[
\{ a_i, u_j \} = \left[ \sum_{l=1}^{i+j} \binom{N+j}{l} \partial^l u_{i+j-l} + \sum_{l=1}^{i+j-N+1} (-1)^{l+1} \binom{i}{l} u_{i+j-l} \right]
\]

\[
+ \sum_{l=1}^{N-j-l-1} \sum_{k=1}^{N-j-l-1} \binom{N-j-l-1}{k} a_l \partial^k u_{i+j-l-k-1} - \frac{1}{N} \sum_{l=0}^{N-j-1} \binom{N}{i+1} \binom{j}{l} \partial^{i+j-l} \right]
\]

\[
+ \sum_{l=1}^{i-j-l-1} \sum_{k=1}^{i-j-l-1} \left( -1 \right)^{k+1} \binom{i-j-l-1}{k} u_{i+j-l-k-1} \partial^k a_l \right]
\]

\[
- \frac{1}{N} \sum_{l=1}^{i-j-l-1} \sum_{k=0}^{N-j-l-1} \binom{N-j-l-1}{k} \binom{j}{k} a_l \partial^{i+j-l-k-1} u_k \right] \delta(x - y);
\]

\[\text{(4.6b)}\]

16
\[
\{u_i, u_j\}_2 = \left[ \sum_{l=1}^{N+j} \binom{N+j}{l} \partial^l u_{i+j+N-l} + \sum_{l=1}^{N+i} (-1)^{l+1} \binom{N+i}{l} u_{i+j+N-l} \partial^l \right] \\
+ \sum_{l=0}^{i-1} \left[ \sum_{k=1}^{j-l-1} \binom{j-l-1}{k} u_l \partial^k u_{i+j-l-k-1} \right] + \sum_{l=1}^{N-1} \sum_{k=1}^{N-j-l-1} \binom{N+j-l-1}{k} a_l \partial^k u_{i+j+N-l-k-1} \\
+ \sum_{l=1}^{N-1} \sum_{k=1}^{N+i-l-1} (-1)^{k+1} \binom{N+i-l-1}{k} u_{i+j+N-l-k-1} \partial^k a_l \\
+ \frac{1}{N} \sum_{l=1}^{i} \sum_{k=1}^{j} (-1)^{l+1} \binom{i}{l} \binom{j}{k} u_{i-l} \partial^{l+k-1} u_{j-k} \delta(x-y)
\]

where

\[
c_{ij} = \frac{(-1)^j}{N} \left( \binom{N}{i+1} + \sum_{l=0}^{i} (-1)^{i+l+1} \binom{N}{l} \binom{N+i-l}{N-j} \right)
\]

\[
b_{ijl} = \frac{(-1)^j}{N} \left( \binom{N}{j+1} \binom{N+l-i-j-1}{l-j} \right) + \sum_{k=0}^{l-j-1} (-1)^{k+l} \binom{N+l-i-j-1}{k} \binom{N+l-j-k-1}{N-j-1}
\]

\[
b_{ijl}^2 = \sum_{k=0}^{i} (-1)^{l+k} \binom{N}{k} \binom{N+l-j-k-1}{N-j-1}
\]

\[
b_{ijlk} = \sum_{r=0}^{i-l-1} (-1)^{r+k} \binom{N-l-1}{r} \binom{N+k-i-r-1}{N-j-1}
\]

From eq. (4.6c) can immediately extract a Virasoro subalgebra

\[
\{a_1, a_1\}_2 = \left( \frac{1}{2} \binom{N+1}{3} \partial^3 + a_1 \partial + \partial a_1 \right) \delta(x-y),
\]

i.e. \(a_1\) can be interpreted as a semi-classical energy momentum tensor. The Poisson brackets between \(a_1\) and the other fields tell us the conformal dimensions (or spin contents) \([ \cdot \cdot \cdot \] of our coordinates

\[
[a_l]_{\text{conf}} = l + 1, \quad l = 1, 2, \ldots, N + M - 1; \\
[S_l]_{\text{conf}} = 1, \quad l = 1, 2, \ldots, M; \\
[u_l]_{\text{conf}} = N + l + 1, \quad l = 0, 1, \ldots, \infty.
\]

If we assign to \(\delta\)–function a conformal weight 1, then we have

\[
[\{ \cdot, \cdot \}]_{\text{conf}} = 0.
\]
The algebra (4.6a–4.6d) contains fields with spins from 2 to infinity, and is linear or bilinear in the gauge invariant functions. There exist central extensions represented by the coefficients $c_{ij}$. For this reason we call this algebra the extended $W_{\infty}$ algebra.

### 4.2 The finite dimensional algebras associated to the $(N,1)$–th hierarchy

The above algebras are independent of the particular coordinatization we choose for the Lax operator. However, the physical meaning of a hierarchy may essentially depend on the gauge, in particular on the number of fields. Therefore we are very much interested in the algebras formed by the coordinates we choose. In particular the independent fundamental coordinate fields in the integrable hierarchy (1.3) are finite. Therefore we expect two finite algebras corresponding to the two compatible Poisson structures. They in turn generate the infinite dimensional algebras of the previous subsection. We can derive these algebras from the infinite dimensional algebras (4.5a–4.5c) and (4.6a–4.6c) by making use of the expressions (4.4). As an example, we consider the $(N,1)$–th hierarchy, in which the scalar Lax operator is

$$L = \partial^N + \sum_{l=1}^{N-1} a_l \partial^{N-l-1} + a_N \frac{1}{\partial - S_1}. \quad (4.8)$$

The gauge invariant functions are $\{a_1, a_2, \ldots, a_{N-1}\}$ and a set of infinite many functions (they are generated by only two fields)

$$u_l = a_N \alpha_l, \quad \alpha_l \equiv (-\partial + S_1)^l \cdot 1, \quad l \geq 0. \quad (4.9)$$

**Proposition 4.3** The first Hamiltonian structure leads to the following Poisson algebra

\[
\begin{align*}
\{a_i, a_j\}_1 &= (4.5a), \quad 1 \leq i, j \leq N - 1; \\
\{a_i, a_N\}_1 &= 0, \quad i = 1, 2, \ldots, N; \\
\{a_i, S_1\}_1 &= \delta_{i,N} \delta'(x - y), \quad i = 1, 2, \ldots, N; \\
\{S_1, S_1\}_1 &= 0, \\
\end{align*}
\]

This $(N+1)$ dimensional algebra generates the $W_{1+\infty}$ algebra (4.5a 4.5c) through the transformation (4.4).

**Proposition 4.4** The second Hamiltonian structure leads to the following Poisson algebra

\[
\begin{align*}
\{a_i, a_j\}_2 &= (4.6a), \quad i, j = 1, 2, \ldots, N - 1; \\
\{a_N, a_N\}_2 &= \{u_0, u_0\}_2; \\
\{a_i, a_N\}_2 &= \{a_i, u_0\}_2, \quad 1 \leq i \leq N - 1; \\
\{S_1, S_1\}_2 &= \frac{N+1}{N} \delta'(x - y). \\
\{a_j, S_1\}_2 &= \frac{j}{N} \left( \begin{array}{c} N + 1 \\ j + 1 \end{array} \right) \partial^{j+1} + \sum_{l=1}^{j-1} \frac{(N+1)j - N(l+1)}{N(N-l)} \left( \begin{array}{c} N - l \\ N - j \end{array} \right) a_l \partial^{j-l} \\
&+ \sum_{l=0}^{j-1} \partial^{j-l} \left( \begin{array}{c} N \\ j - l - 1 \end{array} \right) \alpha_l S_1 - \left( \begin{array}{c} N + 1 \\ j - l \end{array} \right) \alpha'_l + \sum_{l=0}^{j-1} (-1)^{j-l} \left( \begin{array}{c} j \\ l \end{array} \right) \alpha_l S_1^{(j-l)} \quad (4.11e)
\end{align*}
\]
\[ + \sum_{l=1}^{j-2} \sum_{k=0}^{l-2} a_l \partial^{j-l-k-1} \left( \binom{N-l}{j-l-k-1} \binom{N-l-1}{j-l-k-2} \right) \alpha_k S_1 \]

\[ + \sum_{l=1}^{j-2} \sum_{k=0}^{l-2} (-1)^{j-l-k} \binom{j-l-1}{k} a_l \alpha_k S_1^{(j-l-k-1)} \delta(x-y), \quad 1 \leq j \leq N-1; \]

\[ \{ a_N, S_1 \}_2 = (\partial + S_1)^N + \sum_{l=1}^{N-1} a_l (\partial + S_1)^{N-l-1} \delta'(x-y). \quad \text{(4.11f)} \]

The proofs of the above two propositions are not very difficult, so we skip them.

4.3 The general $W(N, M)$–algebra

The Poisson algebras defined by the first Hamiltonian structure is relatively simple. On the other hand the second Hamiltonian structure certainly plays a more important role (see the Hamiltonian reduction below). Hereafter we only pay attention to the second Hamiltonian structure, and as we mentioned before, we denote by $W(N, M)$ the finite algebra represented by the second Poisson brackets of the fundamental fields of the $(N, M)$ model which is encoded in (4.6a–4.6c). Contrary to the latter the $W(N, M)$ algebras are gauge–dependent, i.e. they depend on the coordinates we choose for the model. In Appendix B we give several explicit examples of the simplest $W(N, M)$ algebras. In principle we can calculate any $W(N, M)$ algebra. Unfortunately we cannot exhibit a compact explicit form of $W(N, M)$ with arbitrary $N$ and $M$. However it is not difficult to extract some general properties of these algebras. This is the aim of the present subsection.

(i). $W(N, M)$ in the gauge (1.4):

In this gauge

- $W(N, M)$ algebras are local and polynomial, i.e. the Poisson brackets contain neither integration operators $\partial^{-1}$ nor fractional or negative powers of the coordinates.
- $W(N, M)$–algebras are linear or bilinear in the $a_l$ fields but, in general, highly non–linear in the $S_l$ fields.
- The fields are characterized by their conformal spin; there are $(N + M - 1)$ fields with spin ranging from 2 to $(N + M)$; in addition, there are $M$ spin one fields.
- For any Poisson bracket, the $\partial^0$–terms in its RHS are either absent or contain derivatives of the fields. In other words, for any two coordinates $f$ and $g$, if

\[ \{ f, g \}_2 = \hat{B} \delta(x-y), \quad \hat{B} = \sum_{l \geq 0} b_l \partial^l \]

then either $b_0 = 0$ or $b_0$ contains derivatives of the fundamental coordinates.

The first and the third assertions can be checked case by case, and in fact they are true for all our examples in Appendix B and the previous subsection. The second property can be obtained from the transformation (1.4) and the algebra (1.6a 1.6c). Since the algebra (1.6a 1.6c) is at most bilinear in the gauge invariant functions, while the $u_l$ fields are linear in the $a_l$ fields and non–linear in the $S_l$ fields, then $W(N, M)$ must be at most bilinear in the $a_l$ fields but, in general highly non–linear in the $S_l$ fields.
(ii). \( W(N, M) \) in the gauge (4.2):

- \( W(N, M) \) algebras are, in general, non-local and non-polynomial.
- Spin content: there are \((N-1)\) fields with integer spin running from 2 to \(N\), and \(2M\) fields with spins taking value
  \[
  \begin{align*}
  \l_{\text{conf}} &= \frac{N+1}{2}, \\
  \l_{\text{conf}} &= l + \frac{N-1}{2}, \quad 1 \leq l \leq M.
  \end{align*}
  \]

In Appendix C we will give some explicit examples of algebras in this gauge.

(iii). \( W(N, M) \) algebras and \( W_N \)-algebras

\( W(N, M) \)-algebras are intimately related to \( W_N \)-algebras in the following sense:

- They share the same Virasoro sub-algebra (4.7).
- The fields in the \( W_N \)-algebras are gauge invariant, while the \( W(N, M) \) algebras contain all these gauge invariant fields plus some gauge dependent fields, whose spins are integers or half-integers (depending on the gauge choice).
- (4.6a) is exactly a \( W_N \)-algebra modulo \( u_l \)-dependent terms.
- In the next section we will show that the \( W(N, M) \) algebra can be reduced to the usual \( W_{N+M} \) algebra.

(iv). Recurrences among \( W(N, M) \) algebras

From Appendix B, we find that \( W(1, 2) \) is almost a sub-algebra of \( W(1, 3) \), the only discrepancy being that \( \{a_2, a_2\}_2 \) in the latter case contains \( a_3 \)-linearly-dependent terms. This property holds for two more pairs: \( W(2, 2) \) and \( W(2, 1) \), \( W(3, 1) \) and \( W_3 \). In fact, in general, we can write

\[
W_N \subset W(N, 1) \subset \ldots \subset W(N, \tilde{M}) \subset W(N, M).
\]

This is to be interpreted in the following way: for any \( \tilde{M} < M \), let us pick out from \( W(N, \tilde{M}) \) the Poisson brackets among \( (a_1, \ldots, a_{N+\tilde{M}-1}; S_1, S_2, \ldots, S_{\tilde{M}}) \), and mode out \( (a_{N+\tilde{M}}, \ldots, a_{N+M-1}; S_{\tilde{M}+1}, S_{\tilde{M}+2}, \ldots, S_M) \)-dependent terms; then we get the \( W(N, \tilde{M}) \)-algebra.

(v). M dimensional subalgebras

Beside the Virasoro algebra (4.7), \( W(N, M) \) has another very simple subalgebra, the Poisson algebra formed by the \( S_i \) fields

\[
\{S_i, S_j\}_2 = (\delta_{ij} + \frac{1}{N}) \delta'(x - y). \tag{4.12}
\]

To derive it we notice that on a simple basis of dimensional counting the only possible brackets are

\[
\{S_i, S_j\}_2 = \text{const.}\delta'(x - y) \tag{4.13}
\]

Then such form must remain unchanged when we take the dispersionless limit (see below). In section 7, when we analyse the dispersionless \((N, M)\)-th KdV hierarchy, we will prove that the constant is just \((\delta_{ij} + \frac{1}{N})\). From this (4.12) follows in general.

Concerning this subalgebra let us make a side remark. Let us define a new set of fields

\[
\tilde{S}_1 = S_1, \quad \tilde{S}_j = S_j - S_{j-1}, \quad 2 \leq j \leq M.
\]
Then the Poisson brackets among these new fields can be expressed in the following matrix form

\[
\begin{pmatrix}
\frac{N+1}{N} & -1 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & -1 & 2 & -1 \\
0 & 0 & 0 & \ldots & 0 & -1 & 2 \\
\end{pmatrix}.
\]  
(4.14)

This is almost the Cartan matrix of the \( sl(M+1) \) Lie algebra, except for the element in the upper left corner, which is the Poisson bracket \( \{ \tilde{S}_1, \tilde{S}_1 \} \). This may hide some still unknown relation between \( W(N, M) \) algebras and ordinary finite dimensional Lie algebras.

**(vi). Some useful Poisson brackets**

**Proposition 4.5** In the \( W(N, M) \) algebras, we find in particular the following brackets:

\[
\{ a_j, S_1 \}_2 = \left[ \frac{j}{N} \right] \left( \frac{N+1}{j+1} \right) \partial^{j+1} + \sum_{l=1}^{j-1} \frac{(N+1)j - N(l+1)}{N(N-l)} \left( \frac{N-l}{N-j} \right) a_l \partial^{j-l} + S_1 \text{ – dependent terms} \quad j \leq N-1;
\]  
(4.15a)

\[
\{ a_N, S_1 \}_2 = \left( \partial S_1 \right)^N + \sum_{l=1}^{N-1} a_l (\partial + S_1)^{N-l-1} \delta'(x-y);
\]  
(4.15b)

\[
\{ a_{N+i}, S_j \}_2 = 0, \quad i \geq j;
\]  
(4.15c)

\[
\{ S_i, S_j \}_2 = \left( \delta_{ij} + \frac{1}{N} \right) \delta'(x-y).
\]  
(4.15d)

These brackets are particularly important since they are crucial for Hamiltonian reductions.

**Proof**: Let us start from eq. (4.15d); we see that it is true for all the algebras given in Appendix B. Now let us consider the case of arbitrary \( N \) and \( M = 2 \) in which the gauge invariant functions have the following realization

\[
\tilde{u}_l = a_N(-\partial + S_1)^l \cdot 1 + \sum_{l_1+l_2=l-1} a_{N+1}(-\partial + S_2)^{l_2}(-\partial + S_1)^{l_1} \cdot 1.
\]

Comparing with eq. (4.14), we see that

\[
\tilde{u}_l = u_l + \delta u_l, \quad \delta u_l = a_{N+1} \text{ – dependent terms}
\]

Since \( \tilde{u}_l \)'s and \( u_l \)'s (together with \( \{ a_l \mid 1 \leq l \leq N-1 \} \)) form the same algebra (4.6a – 4.6c), we may use this invariance to derive the additional Poisson brackets. For example

\[
\{ \tilde{u}_0, \tilde{u}_1 \}_2 = \{ u_0, u_1 + a_{N+1} \}_2
\]

The \( a_{N+1} \)-dependent terms on the LHS determine \( \{ u_0, a_{N+1} \}_2 = \{ a_N, a_{N+1} \}_2 \). Similarly

\[
\{ \tilde{u}_1, \tilde{u}_1 \}_2 - \{ u_1, u_1 \}_2 = \{ a_{N+1}, u_1 \}_2 + \{ u_1, a_{N+1} \}_2 + \{ a_{N+1}, a_{N+1} \}_2,
\]

The RHS must be \( a_{N+1} \)-dependent, since all the \( a_{N+1} \)-independent terms are excluded. This requirement implies that

\[
\{ a_{N+1}, S_1 \}_2 \text{ must be } a_{N+1} \text{ – dependent}
\]

21
However, by dimension counting, it is easy to see that \( a_{N+1} \)-dependent terms are not allowed in local polynomial Poisson brackets. Therefore we must have
\[
\{a_{N+1}, S_1\}_2 = 0.
\]

For the same reason, when we consider \( M \geq 3 \), i.e. we add more pseudo-differential terms to the Lax operator, we will obtain (4.15a). Therefore, when we add more pseudo-differential operator terms to the Lax operator, they will not change the Poisson brackets involving the \( S_l \) fields already considered, thus we must have in general (4.15a).

(vii). The generating algebras of the usual W–infinity algebras

Finally let us consider the \( N = 1 \) case. As we have pointed out several times the gauge invariant functions \( \{u_l\} \) form a \( W_{1+\infty} \) and a \( W_\infty \) algebra. However these two algebras can be realized also by means of \( 2M \) fields only, via the combinations (4.4) and the finite dimensional algebras determined by the first and second Poisson brackets of the fields. We could therefore phrase this situation by saying that the \( W_{1+\infty} \) and the \( W_\infty \) algebras reduce to the latter finite algebras.

Since \( M \) can be any positive integer we see that there are infinite many different realizations of the \( W_{1+\infty} \) and the \( W_\infty \) algebras. For instance, \( W(1, 2) \) and \( W(1, 3) \) give the four– and six–field realizations of the \( W_\infty \) algebra. Alternatively we can say that \( W_{1+\infty} \) and \( W_\infty \) algebras are highly reducible, and multi–field representations of the KP hierarchy provide a way to classify the reduced algebras.

5 Reductions of the \((N, M)\)–th KdV hierarchy

We have claimed above that \( W(N, M) \) algebra can be reduced to the usual \( W_{N+M} \) algebra. In this section we will show this and the more general relations (1.5) and (1.6). Let us first summarize the two schemes for Hamiltonian reduction with second class constraints we will be using.

5.1 Two reduction schemes for Hamiltonian systems

Let us consider a Hamiltonian system, no matter whether it is integrable or non–integrable. It will have fields generically denoted by \( f_i \), a Hamiltonian \( H \) and a Poisson bracket. The equations of motion are
\[
\frac{\partial}{\partial t} f_i = \{f_i, H\}, \quad i = 1, 2, \ldots, n.
\]

Now we want to impose, for example, \( C = 0 \) where \( C \) is a particular combination of the fields and their derivatives, and suppose that it is second class, i.e.
\[
\{C, C\} = \Delta \delta(x - y), \quad \Delta|_{C=0} \neq 0,
\]
then, in order to avoid inconsistencies, we can proceed in two ways.

The first scheme

If we explicitly know all the Poisson brackets among the independent fields, we can first improve the Poisson bracket, then derive the reduced equations of motion.

step 1: Introduce in the reduced phase space the Dirac–Poisson bracket
\[
\{f_i, f_j\}_D \overset{\text{def}}{=} \left( \{f_i, f_j\} - \frac{1}{\Delta} \{f_i, C\} \right)_{C=0},
\]
The second scheme

If it so happens that we do not know the full Poisson algebra, but we know the Poisson brackets between the constraint and all the fields, then we can improve the Hamiltonian with the addition of a suitable Lagrange multiplier term, and make use of the known Poisson brackets to derive the equations of motion of the reduced system.

**step 1**: Add a Lagrange multiplier term to $H$

$$H = \tilde{H} = H + \int \alpha f_1,$$

where $\alpha$ is an expression to be determined.

**step 2**: Using this new Hamiltonian and the original Poisson brackets derive the equation of motion for $C$,

$$\frac{\partial}{\partial t} C = \{C, \tilde{H}\}.$$

**step 3**: Determine $\alpha$ so that the second class constraint is preserved by the time evolution generated by the new Hamiltonian, i.e. turn $C = 0$ into a first class constraint.

**step 4**: Using this new Hamiltonian and the original Poisson brackets derive the equations of motion for the reduced system

$$\frac{\partial}{\partial t} f_i = \{f_i, \tilde{H}\}_{C=0}, \quad i = 2, 3, \ldots, n.$$

Comment concerning integrable Hamiltonian systems

What we said so far is valid for any Hamiltonian system. Now let us turn our attention to the reduction of an integrable Hamiltonian system. After implementing either of the above two schemes, we have to make sure that the reduced system is still integrable. In other words we should further construct its bi–Hamiltonian structure, which is a recursive but lengthy procedure. A shortcut in this sense consists in finding its Lax pair representation.

Let us finally remark that the two schemes presented above lead to the same results, to the extent we have been able to produce explicit results for both of them.

### 5.2 Reduction of $W(N, M)$ algebras

We want to impose the constraint $S = 0$ for some of the $S$ fields of the algebra, and this is second class.

Applying the first reduction scheme to the $W(1, 2)$–algebra, one can prove the following sequence of reductions

$$W(1, 2) \xrightarrow{S_1=0} W(2, 1) \xrightarrow{S_2=0} W(3, 0) = W_3.$$

For the algebras given in Appendix B, we can find another sequence

$$W(1, 3) \xrightarrow{S_1=0} W(2, 2) \xrightarrow{S_2=0} W(3, 1) \xrightarrow{S_3=0} W(4, 0) = W_4.$$
These two sequences confirm the relation (1.6). However, since we do not have at hand a compact explicit form of the algebra \( W(N, M) \) we can verify that with \( S_1 = 0 \), \( W(N, M) \) reduces to \( W(N + 1, M - 1) \) only case by case. To overcome this difficulty we have to resort to the second reduction scheme.

5.3 Reduction of the hierarchy

Proposition 5.1: When we impose the constraint \( S_1 = 0 \), the \( W(N, M) \)-algebra reduces to the \( W(N + 1, M - 1) \)-algebra

\[
W(N, M) \xrightarrow{S_1=0} W(N + 1, M - 1) \quad (5.2)
\]

and, simultaneously, the \((N, M)\) hierarchy reduces to the \((N + 1, M - 1)\) hierarchy.

Proof: As anticipated above, we adopt the second reduction scheme. Our starting point is eqs.(3.11a–3.11c), which are generated by the second Hamiltonian \( H_2 = \frac{N}{2} < \hat{L}^2 > \) through the second Poisson structure. Obviously the constraint \( S_1 = 0 \) is not preserved by the time evolutions, so we modify \( H_2 \) as follows

\[
H_2 \Rightarrow \tilde{H}_2 = H_2 + \int \alpha S_1,
\]

with \( \alpha \) to be determined. This new Hamiltonian generates the following equation of motion for the \( S_1 \) field (with respect to the original second Poisson bracket)

\[
\left( \frac{\partial}{\partial t_2} S_1 \right)_{\text{improved}} = \frac{2}{N} a'_l + 2S_1S'_l - S''_l + \frac{N + 1}{N} \alpha + \{S_1, \alpha\}_2 S_1.
\]

Setting \( S_1 = 0 \), we obtain

\[
\alpha = -\frac{2}{N + 1} a_1
\]

therefore our improved Hamiltonian is

\[
\tilde{H}_2 = 2 \int \left( a_2(x) + \left( \frac{\delta_{N,1}}{\delta_{N,1}} - \frac{1}{N + 1} \right) a_1(x) S_1(x) \right) dx. \quad (5.3)
\]

Using eq.(4.15a) and (4.15c), we can derive the explicit form of the equations of motion of the reduced system

\[
\frac{\partial}{\partial t_2} a_l = a''_l + 2a'_{l+1} - \frac{2}{N + 1} \binom{N + 1}{l + 1} a^{(l+1)}_1,
\]

\[
-\frac{2}{N + 1} \sum_{k=1}^{l-1} \binom{N - k}{l - k} a_k a^{(l-k)}_1, \quad 1 \leq l \leq N; \quad (5.4a)
\]

\[
\frac{\partial}{\partial t_2} a_{N+l} = \frac{\partial}{\partial t_2} a'_{N+l} + 2a'_{N+l+1} + 2a'_{N+l} S_{l+1} + 2a_{N+l} \left( \sum_{k=1}^{l+1} S_k \right)' , \quad 1 \leq l \leq M - 1; \quad (5.4b)
\]

\[
\frac{\partial}{\partial t_2} S_l = \frac{2}{N + 1} a'_l + 2S_lS'_l - S''_l - 2 \left( \sum_{k=1}^{l-1} S_k \right)' , \quad 2 \leq l \leq M. \quad (5.4c)
\]
We immediately recognize that this is nothing but the first non–trivial flow of the \((N + 1, M - 1)\)–th KdV hierarchy. The only effect of our reduction is the shift
\[
N \rightarrow N + 1, \quad M \rightarrow M - 1.
\]
The algebra associated to this hierarchy is \(W(N + 1, M - 1)\), thus we are led to the relation (5.2).

But now we remark that our proof is valid for any value of \(M\). Therefore we can adapt the above proof to the case when we set \(S_2 = 0, S_3 = 0, \ldots, S_M = 0\). Finally we get the \((N + M)\)–th KdV hierarchy and the \(W_{N+M}\) algebra.

6 Drinfeld–Sokholov representation

The integrable models and reductions we have been considering so far can be synthesized in a very compact and useful form via suitable Drinfeld–Sokholov (DS) linear systems. From them one can easily extract the corresponding Lax pair.

6.1 The linear system associated to the \((N, M)\)–th KdV hierarchy

The \((N, M)\)–th KdV hierarchy (1.3) can be viewed as the consistency condition of the following linear system
\[
\begin{align*}
L \psi_0(\lambda, t) &= \lambda \psi_0(\lambda, t), \\
\frac{\partial}{\partial t_r} \psi_0(\lambda, t) &= \left( L_\lambda^{*} \right) \psi_0(\lambda, t)
\end{align*}
\]  
(6.1)

where \(\lambda\) is the spectral parameter, and \(\psi_0\) is referred to as Baker–Akhiezer function. In [8] we have shown that this linear system naturally appears in multi–matrix models.

In terms of Baker–Akhiezer function, we can introduce an important ingredient – the \(\tau\)–function,
\[
\psi_0(\lambda, t) = \frac{V(\lambda, t) \tau(t)}{\tau(t)},
\]
(6.2)

where
\[
V(\lambda, t) = \exp \left( \sum_{r=1}^{\infty} t_r \lambda^r \right) \exp \left( - \sum_{r=1}^{\infty} \frac{1}{r \lambda^r} \frac{\partial}{\partial t_r} \right)
\]
is a vertex operator. One can show that the \(\tau\)–function satisfies the following relations [11]
\[
\frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau(t) = \text{res}_\lambda L_\lambda^{*}.
\]
(6.3)

6.2 The generalized DS representation

Now let us return to the spectral problem (6.1). The first equation of (6.1) is highly non–linear, our aim is to linearize it. In order to do so, we introduce some more notations,
\[
(E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad 1 \leq i, j, k, l \leq N + M,
\]
which is the \((N + M) \times (N + M)\) matrix with only one non–zero element at the position \((i, j)\). Define

\[
I_+ \equiv \sum_{i=1}^{N+M-1} E_{i,i+1},
\]

\[
\Psi \equiv (\Psi_{-M}, \Psi_{1-M}, \ldots, \Psi_{-1}, \Psi_1, \ldots, \Psi_N)^T,
\]

\[
S \equiv \sum_{i=1}^M S_{M-i+1}E_{i,i}, \quad A \equiv \sum_{i=1}^{N+M-1} a_i E_{N+M,N+M-i}.
\]

So \(\Psi \equiv \Psi(\lambda, t)\) is a column vector, the other are matrices. \(\Psi_0\) is the Baker–Akhiezer function of eq.(6.1). Now we can express the spectral equation in linear form as follows

\[
\left( \partial + S + A - \lambda E_{N+M,M+1} - I_+ \right) \Psi = 0.
\] (6.5)

If we eliminate all the \(\Psi_l\) in favor of \(\Psi_0\), we recover the spectral equation in (6.1).

Now let us consider some particular cases

(i). \(M = 0\) case, this is the \(N\)–th KdV hierarchy, the above linear system is just the original Drinfeld–Sokholov representation, in which the spectral parameter \(\lambda\) is put on the lower left corner [16].

(ii). \(N = 1\) case, this is 2\(M\)–field representation of KP hierarchy. The spectral parameter in this case is located on the main diagonal line (on the lower right corner).

For general ‘\(N\)’ and ‘\(M\)’, the spectral parameter appears in the last row. When we make the reductions discussed in section 5, Imposing \(S_1 = 0\) is equivalent to replacing

\[
N \rightarrow N + 1, \quad M \rightarrow M - 1.
\]

In (6.5), this change corresponds to moving spectral parameter \(\lambda\) from the right to the left by one step. Repeating the procedure, we finally can move \(\lambda\) to the lower left corner, that is, we get a KdV hierarchy.

7 The dispersionless \((N, M)\)–th KdV hierarchy

In this section we will consider the dispersionless limit of the \((N, M)\)–th KdV hierarchy (1.3). The usual procedure to get this limit is the following: we simply ignore the higher than first derivatives in the equations of motion as well as in the Poisson brackets. This is equivalent to substituting the commutators by the basic canonical Poisson relation, i.e.

\[
[\partial, x] \implies \{p, x\} = 1,
\] (7.1)

where \(p\) denotes the canonical conjugate momentum. In terms of this elementary Poisson structure, the dispersionless \((N, M)\)–th KdV hierarchy can be written as

\[
\frac{\partial}{\partial t_r} \mathcal{L} = \{\mathcal{L}'_r, \mathcal{L}\},
\] (7.2)

with the dispersionless Lax operator

\[
\mathcal{L} = p^N + \sum_{l=1}^N a_l p^{N-l} + \sum_{l=1}^M a_{N+l-1} \frac{1}{(p-S_1)(p-S_2)\ldots(p-S_l)} = p^N + \sum_{l=1}^N a_l p^{N-l} + \sum_{l=1}^\infty u_l p^{-l-1}.
\] (7.3)
The subscript ‘+’ in (7.2) means that we keep only non-negative powers of \( p \). As an example we give the second dispersionless flow equations

\[
\frac{\partial}{\partial t_2} a_l = 2a_{l+1}' - \frac{2(N-l)}{N} a_{l-1} a_l', \quad 1 \leq l \leq N - 1; \quad (7.4a)
\]

\[
\frac{\partial}{\partial t_2} a_{N+l} = 2a_{N+l+1}' + 2a_{N+l} a_{N+l+1}' + 2a_{N+l} \left( \sum_{k=1}^{l+1} S_k \right)', \quad 0 \leq l \leq M - 1; \quad (7.4b)
\]

\[
\frac{\partial}{\partial t_2} S_l = \frac{2}{N} a_l' + 2S_l S_l', \quad 1 \leq l \leq M. \quad (7.4c)
\]

Comparing with eqs (3.11a–3.11c), we see that only the first order derivatives survive. In the dispersionless limit, the gauge invariant functions \( u_i \)’s have very simple expressions. The first few are

\[
u_0 = a_N, \quad u_1 = a_{N+2} + a_N S_1, \quad u_2 = a_{N+3} + a_{N+2}(S_1 + S_2) + a_N S_1^2.
\]

In order to find general compact formulas we introduce a set of completely symmetric and homogeneous polynomials \( e_k \)

\[
e_k(S_1, S_2, \ldots, S_M) \overset{\text{def}}{=} \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq M} S_{i_1} S_{i_2} \ldots S_{i_k}, \quad k \in \mathbb{Z}_+ \quad (7.5)
\]

In particular we have

\[
e_k(S_1, S_2, \ldots, S_M) = \sum_{l=0}^{k} S_l' e_{k-l}(S_2, S_3, \ldots, S_M). \quad (7.6)
\]

Without further specification hereafter we understand \( e_k = e_k(S_1, S_2, \ldots, S_M) \). It is not difficult to show that these symmetric functions satisfy the following identities (see Appendix D for the proofs)

\[
\frac{\partial}{\partial S_l} e_{i+1} = \sum_{\mu=0}^{i} S_l^\mu e_{i-\mu}; \quad (7.7a)
\]

\[
\sum_{l=1}^{M} \sum_{\mu=0}^{i} S_l^\mu e_{i-\mu} = (i + M) e_i; \quad (7.7b)
\]

\[
e_{i+1}' \overset{\text{def}}{=} \sum_{l=1}^{M} S_l' \frac{\partial}{\partial S_l} e_{i+1} = \sum_{l=1}^{M} \sum_{\mu=0}^{i} S_l^\mu e_{i-\mu}; \quad (7.7c)
\]

\[
\sum_{l=1}^{M} \sum_{\alpha=\beta+1}^{i} S_l^\beta e_{i-\alpha} e_{j+\alpha-\beta} = \sum_{\alpha=1}^{i} (i - \alpha + M) e_{i-\alpha} e_{j+\alpha}; \quad (7.7d)
\]

\[
\sum_{l=1}^{M} \sum_{\alpha=0}^{\beta-1} (\alpha - \beta + 1) S_l^\beta e_{i-\alpha} e_{j+\alpha-\beta-1} = \sum_{\alpha=1}^{i} \alpha e_{i-\alpha} e_{j+\alpha}; \quad (7.7e)
\]

\[
\sum_{l=1}^{M} \left( \frac{\partial}{\partial S_l} e_{i+1} \right) \frac{\partial}{\partial S_l} e_{j+1} = (j + M) e_i \partial e_j + \sum_{l=1}^{i} (l e_{j+l} \partial e_{i-l} + (j - i + l) e_{i-l} \partial e_{j+l}). \quad (7.7f)
\]
The last one is an operatorial equation. Using these symmetric polynomials we obtain

\[ u_l = \sum_{k=1}^{l+1} a_{N+k-1} e_{l-k+1}(S_1, S_2, \ldots, S_k), \quad 0 \leq l \leq M - 1; \]

\[ u_l = \sum_{k=1}^{M} a_{N+k-1} e_{l-k+1}(S_1, S_2, \ldots, S_k), \quad l \geq M. \quad (7.8) \]

In the dispersionless limit the two Hamiltonian structures are extremely simplified. With a little exercise, we get the first Poisson algebra

\[ \{a_i, a_j\}_1 = \left( N \delta_{i+j,N} \partial + (N - i)a_{i+j-N-1} \partial + (N - j)\partial a_{i+j-N-1} \right) \delta(x - y), \quad (7.9a) \]

\[ \{u_i, u_j\}_1 = (iu_{i+j-1} \partial + j\partial u_{i+j-1}) \delta(x - y), \quad (7.9b) \]

\[ \{a_i, u_j\}_1 = 0. \quad (7.9c) \]

The second Poisson algebra is

\[ \{a_i, a_j\}_2 = [ia_{i+j-1} \partial + j\partial a_{i+j-1} + iu_{i+j-N-1} \partial + j\partial u_{i+j-N-1} \]
\[ + \sum_{l=1}^{i-1} ((i - l - 1)a_{i+j-l-2} \partial a_l + (j - l - 1)a_l \partial a_{i+j-l-2}) \]
\[ + \sum_{l=1}^{i-2} ((j - l - 1)a_l \partial u_{i+j-l-N-2} + (i - l - 1)u_{i+j-l-N-2} \partial a_l) \]
\[ + \frac{(N - i)(N - j)}{N} a_{i-1} \partial a_{j-1} \delta(x - y), \quad (7.10a) \]

\[ \{u_i, u_j\}_2 = [(N + i)u_{i+j+N-1} \partial + (N + j)\partial u_{i+j+N-1} + \frac{ij}{N} u_{i-1} \partial u_{j-1} \]
\[ + \sum_{l=1}^{N-1} ((N + j - l - 1)a_l \partial u_{i+j+N-l-2} + (N + i - l - 1)u_{i+j+N-l-2} \partial a_l) \]
\[ + \sum_{l=0}^{i-1} ((i - l - 1)u_{i+j+N-l-2} \partial u_l + (j - l - 1)u_l \partial u_{i+j+N-l-2}) \delta(x - y), \quad (7.10b) \]

\[ \{a_i, u_j\}_2 = [iu_{i+j-1} \partial + (N + j)\partial u_{i+j-1} - \frac{(N - i)j}{N} a_{i-1} \partial u_{j-1} \]
\[ + \sum_{l=1}^{i-2} ((i - l - 1)u_{i+j-l-2} \partial a_l + (N + j - l - 1)a_l \partial u_{i+j-l-2}) \] \delta(x - y). \quad (7.10c) \]

The conserved quantities are

\[ H_r = \frac{N}{r} \int \text{Resp} \left( \mathcal{L}^r \right), \quad r \geq 1. \quad (7.11) \]

One can check that the two Poisson brackets are compatible w.r.t. these quantities, and they indeed generate the flows \((7.2)\). In the remaining part of this section we will try to derive the Poisson brackets among different \(S_i\)'s, and the Poisson relations between \(S_1\) and other fields.

*Here the residue means simply the coefficient of the \(p^{-1}\) term.*
Proposition 7.1 The symmetric polynomials \( e_i \) satisfy the following Poisson brackets

\[
\{ e_{i+1}, e_{j+1} \}_2 = \left[ \frac{(i + M/N + 1)(j + M)e_i e_j}{N} + \sum_{l=1}^{i} \left( l e_{j+l} \partial e_{i-l} + (j - i + l)e_{i-l} \partial e_{j+l} \right) \right] \delta(x - y)
\] (7.12)

Proof: As we discussed in subsection 4.3 the possible Poisson brackets among the \( S_l \) fields are of the form (4.13). Taking the Poisson bracket between two monomials of the \( S_l \) fields reduces the total number of \( S_l \) fields by 2. Eq.(4.4) shows that in \( u_{i+M}(i \geq 0) \) the terms with lowest powers of \( S_l \) fields are \( a^{N+M-1}_N e_{i+1} \). Now let us consider the Poisson bracket \( \{ u_{i+M}, u_{j+M} \}_2 \) with \( i, j \geq 0 \). In this expression the terms with lowest powers of \( S_l \) fields will come from \( a^{N+M-1}_N \{ e_{i+1}, e_{j+1} \}_2 a^{N+M-1}_N \)

On the other hand, the comparable terms in the RHS of the Poisson bracket (7.10b) are

\[
a^{N+M-1}_N \left[ \left( \frac{i + M/N + 1}{N} \right)(j + M)e_i \partial e_j + \sum_{l=1}^{i} \left( l e_{j+l} \partial e_{i-l} + (j - i + l)e_{i-l} \partial e_{j+l} \right) \right] a^{N+M-1}_N \delta(x - y)
\]

Comparing these two expressions, we obtain (7.12).

Proposition 7.2 The Poisson brackets (7.12) imply the Poisson algebra (4.15d).

Proof. This is not difficult to prove. We have already noticed that the bracket \( \{ S_1, S_j \} \) must have the form (4.13). Now making use of the relations (7.6) and (7.7a–7.7f), we can prove that (4.15d) indeed leads to eqs.(7.12). This uniquely fixes the undetermined constant of (4.13).

The Poisson brackets between \( S_1 \) and \( a_l(1 \leq l \leq N + M - 1) \) are easier to derive. In fact we can get them directly from eq.(4.15a–4.15d) by suppressing all higher order derivatives, the result is

\[
\{ a_i, S_1 \}_2 = [S_1^i + \sum_{l=1}^{i-2} a_l S_1^{i-l-1} + \frac{i}{N} a_{i-1} ] \delta'(x - y), \quad 1 \leq i \leq N - 1; \quad (7.13a)
\]

\[
\{ a_N, S_1 \}_2 = \left( S_1^N + \sum_{l=1}^{N} a_l S_1^{N-l} \right) \delta'(x - y); \quad (7.13b)
\]

\[
\{ a_{N+l-1}, S_1 \}_2 = 0, \quad l = 2, 3, \ldots, M. \quad (7.13c)
\]

8 Reductions of the dispersionless \((N, M)\)-th KdV hierarchy

Starting from the previous results we will examine in this section the reduction of the dispersionless \((N, M)\)-th KdV hierarchy. To find reductions we may suppress the fields \( S_l \) one by one. This was already done in section 5 and in [9] for the dispersive (general) case and will not be repeated here. In this section we are interested in the possible existence of other reductions. Let us take the dispersionless four–boson representation of KP hierarchy as an example. The Lax operator is

\[
L_1 = p + \frac{a_1}{p - S_1} + \frac{a_2}{(p - S_1)(p - S_2)}. \quad (8.1)
\]
The second Poisson algebra can be obtained from (B.1) by killing all the higher derivatives
\[
\{a_1, a_1\} = (a_1 \partial + \partial a_1) \delta(x - y), \quad \{a_1, a_2\} = (a_2 \partial + 2 \partial a_2) \delta(x - y),
\]
\[
\{a_1, S_1\} = S_1 \delta'(x - y), \quad \{a_1, S_2\} = S_2 \delta'(x - y),
\]
\[
\{a_2, a_2\} = [a_2(2S_2 - S_1) \partial + \partial a_2(2S_2 - S_1)] \delta(x - y), \quad \{a_2, S_2\} = (a_1 + S_2(S_2 - S_1)) \delta'(x - y),
\]
\[
\{a_2, S_1\} = 0, \quad \{S_i, S_j\} = (\delta_{ij} + 1) \delta'(x - y), \quad i, j = 1, 2.
\]
We call it the classical \(u(1, 2)\) algebra. With respect to this Poisson algebra, the second Hamiltonian
\[
H_2 = \int (a_2 + a_1 S_1),
\]
will generate the first non–trivial flow equations
\[
\frac{\partial}{\partial t_2} a_1 = 2(a_2 + a_1 S_1)',
\]
\[
\frac{\partial}{\partial t_2} a_2 = 2a'_2 S_2 + 2a_2(S_1 + S_2)',
\]
\[
\frac{\partial}{\partial t_2} S_1 = (2a_1 + S_2')',
\]
\[
\frac{\partial}{\partial t_2} S_2 = (2a_1 + S_2')'.
\]

Now, instead of imposing \(S_1 = 0\) as we have done previously, we let \(S_1 = S_2\). Starting from the algebra (8.2) we get an improved algebra
\[
\{a_1, a_1\} = (a_1 \partial + \partial a_1) \delta(x - y), \quad \{a_1, a_2\} = (a_2 \partial + 2 \partial a_2) \delta(x - y),
\]
\[
\{a_2, a_2\} = (a_2 S \partial + \partial a_2 S - \frac{1}{2} a_1 \partial a_1) \delta(x - y),
\]
\[
\{a_1, S\} = S \delta'(x - y), \quad \{a_2, S\} = \frac{1}{2} a_1 \delta'(x - y), \quad \{S, S\} = \frac{3}{2} \delta'(x - y).
\]
With respect to this reduced algebra, the original Hamiltonian (8.3) generates the following flow equations
\[
\frac{\partial}{\partial t_2} a_1 = 2(a_2 + a_1 S)',
\]
\[
\frac{\partial}{\partial t_2} a_2 = 2a'_2 S + 4a_2 S',
\]
\[
\frac{\partial}{\partial t_2} S_2 = (2a_1 + S^2)'.
\]

It turns out that these equations admit the following Lax representation [1]
\[
\frac{\partial}{\partial t} \mathcal{L}_4 = \{\mathcal{L}_4', \mathcal{L}_4\}.
\]

*This implies integrability for the reduced system. The \(S_1 = S_2\) reduction for the corresponding dispersive hierarchy was studied in [3]; the reduced system however is not integrable, at least as long as we stick to locality and polynomiality.
with
\[ L_2 = p + \frac{a_1}{p - S} + \frac{a_2}{(p - S)^2}. \] (8.5)

As we explained in [3], imposing \( S = 0 \) will reduce the above hierarchy to the dispersionless Boussinesq hierarchy.

The obvious generalization of the \( S_1 = S_2 \) reduction to other hierarchies consists in picking out \( i \) (\( i = 1, ..., M \)) of the \( S_l \) fields and setting them equal. So altogether we expect to find \( k \) distinct reductions with
\[
k = \binom{M}{2} + \binom{M}{3} + \ldots + \binom{M}{M} = 2^M - M - 1.
\]

We have checked this for the 6–boson field representation of KP hierarchy.

### 9 Conclusions

The content of our paper can be summarized as follows. We have systematically discussed the \((N, M)\)-th KdV hierarchy. The large integrable differential hierarchy (1.3) contains both the higher KdV hierarchy and the multi–field representations of KP hierarchy. The second Hamiltonian structure of this hierarchy leads to the extended \( W \)-algebra, \( W(N, M) \). By suppressing the \( S_l \) fields in succession we can reduce the \( W(N, M) \) algebra to the usual \( W_{N+M} \)-algebra. Simultaneously the corresponding \((N, M)\)-th KdV hierarchy reduces to \((N + M)\)-th KdV hierarchy. There do not seem to exist other integrable reductions as long as we put restrictions only on the fields \( S_l \). However in the dispersionless limit there do exist another kind of reduction in which we identify some of the \( S_l \) fields. The resulting integrable hierarchy can be obtained by directly imposing this constraint in the Lax operator.

In [10] we use the results of this papers to compute correlation functions of the two–matrix models. As far as the study of hierarchies and algebras is concerned, there remain some open questions. One is whether there are other reductions of the \((N, M)\) hierarchy of different nature than the ones we have considered. Another problem concerns the quantum versions of the algebras \( W(N, M) \). We have seen in section 4 that each algebra \( W(N, M) \) contains a Virasoro sub–algebra (4.7); this may suggest the quantum version of \( W(N, M) \) algebra to correspond to some new conformal field theory. [17] contains some hints on this direction: the \( W(2, 1) \) algebra is examined and its several free field realizations and quantum version are derived.

A third interesting question is whether our \( W(N, M) \) algebra has any relations with the \( W_N^{(l)} \) introduced in [18]. The latter is deduced from a WZW model via Hamiltonian reduction. Finally it would be quite interesting to understand the relation between the extended KdV hierarchy (1.3) and the conformal affine Toda theories (CAT models), for we know the two boson representation of KP hierarchynaturally appears in the \( sl(2) \) CAT model.

### Appendices
A Derivation of the second flow equations

In this Appendix we will derive the second flow equations eqs. (3.11a–3.11c). First we see that

\[
\frac{\partial}{\partial t_2} L = \sum_{l=1}^{N-1} \left( \frac{\partial a_l}{\partial t_2} \right) \partial^{N-l-1} + \sum_{l=1}^{M} \left( \frac{\partial a_{N+l-1}}{\partial t_2} \right) \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \ldots \frac{1}{\partial - S_1} \\
+ \sum_{l=1}^{M} \sum_{k=1}^{l} a_{N+l-1} \frac{1}{\partial - S_l} \frac{1}{\partial - S_k} \left( \frac{\partial S_k}{\partial t_2} \right) \frac{1}{\partial - S_k} \ldots \frac{1}{\partial - S_1}.
\]

(A.1)

Our next task is to calculate the commutator \([L^2_+, L] \). Obviously, we have

\[
[L^2_+, L] = \partial^2 + \frac{2}{N} a_1, \partial^N + \sum_{l=1}^{N-1} a_l \partial^{N-l-1}
\]

After a straightforward calculation we obtain

\[
[L^2_+, L] = [\partial^2 + \frac{2}{N} a_1, \partial^N + \sum_{l=1}^{N-1} a_l \partial^{N-l-1}]
\]

\[
= 2 \sum_{l=1}^{N-1} a'_l \partial^{N-l} + \sum_{l=1}^{N-1} a''_l \partial^{N-l-1} - \frac{2}{N} \sum_{r=1}^{N} \left( N \begin{array}{c} \cdot \end{array} \right) a^{(r)}_1 \partial^{N-r}
\]

\[
= \frac{2}{N} \sum_{l=1}^{N-2} a_l \sum_{r=1}^{N-1} \left( \begin{array}{c} N - l - 1 \end{array} \right) a^{(r)}_1 \partial^{N-r-l-1},
\]

\[
[\partial^2 + \frac{2}{N} a_1, a_{N+l-1}] = 2 a'_{N+l-1} \partial + a''_{N+l-1}
\]

\[
= 2 a'_{N+l-1} (\partial - S_l) + 2 a'_{N+l-1} S_l + a''_{N+l-1},
\]

\[
[\partial^2 + \frac{2}{N} a_1, \partial - S_k] = -2 S'_k \partial - S''_k - \frac{2}{N} a'_1
\]

\[
= - (\partial - S_k) 2 S'_k + S''_k - 2 S'_k - \frac{2}{N} a'_1,
\]

\[
[\partial^2 + \frac{2}{N} a_1, \frac{1}{\partial - S_k}] = - \frac{1}{\partial - S_k} [\partial^2 + \frac{2}{N} a_1, \partial - S_k] \frac{1}{\partial - S_k}
\]

\[
= 2 S'_k \frac{1}{\partial - S_k} + \frac{1}{\partial - S_k} \left( \frac{2}{N} a'_1 + 2 S'_k S''_k \right) \frac{1}{\partial - S_k}.
\]

The last two formulas enable us to calculate

\[
[L^2_+, L] + \sum_{l=1}^{M} a_{N+l-1} \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \ldots \frac{1}{\partial - S_1}
\]

\[
= 2 a'_N + 2 \sum_{l=1}^{M-1} a'_{N+l-1} \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \ldots \frac{1}{\partial - S_1}
\]

32
\[ + \sum_{l=1}^{M} \left( a_{N+l-1} + 2a_{N+l-1}S_l \right)^{\prime} \frac{1}{\partial - S_l} \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1} \]  
(A.3)

\[ + \sum_{l=1}^{M} \sum_{k=1}^{l} a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_{k+1}} \left( \frac{2}{N} a_k' + 2S_k' - S_k'' \right) \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \]

The last term here is not in the standard form: we have to move \( S_k' \) to the left

\[ + \sum_{l=2}^{M} \sum_{k=1}^{l-1} a_{N+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_{k+1}} \left( 2S_k' \right) \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \]

(B.1)

\[ \sum_{l=2}^{M} \sum_{k=1}^{l-1} 2a_{N+l-1} \left[ \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_{k+1}} \right] \left[ 2S_k' \right] \frac{1}{\partial - S_k} \cdots \frac{1}{\partial - S_1} \]

Combining eqs. (A.2), (A.3), and (A.4), and comparing with eq (A.1), we obtain eqs. (B.1a–B.1d), which are the first non–trivial flow equations in the hierarchy (1.3).

**B Some simple \( W(N, M) \) algebras**

In this Appendix we give explicit expressions for a few simple \( W(N, M) \) algebras.

**\( W(1, 2) \) algebra**

\[ \{a_1, a_1\} = (a_1 \partial + \partial a_1)\delta(x - y), \quad \{a_1, a_2\} = (a_2 \partial + 2\partial a_2)\delta(x - y), \]

\[ \{a_1, S_1\} = (\partial^2 + S_1 \partial)\delta(x - y), \quad \{a_1, S_2\} = (2\partial^2 + S_2 \partial)\delta(x - y), \]

\[ \{a_2, a_2\} = [2a_2' + 4a_2S_2 - 2a_2S_1] \partial + a_2'' + (2a_2S_2 - a_2S_1) \delta(x - y), \quad (B.1) \]

\[ \{a_2, S_2\} = (a_1 + (\partial + S_2)(\partial + S_2 - S_1))\delta(x - y), \]

\[ \{a_2, S_1\} = 0, \quad \{S_1, S_j\} = (\delta_{ij} + 1)\delta(x - y), \quad i, j = 1, 2. \]

The associated Lax operator is

\[ L = \partial + a_1 \frac{1}{\partial - S_1} + a_2 \frac{1}{\partial - S_2} \frac{1}{\partial - S_1}. \quad (B.2) \]
This algebra generates the $W_{\infty}$ algebra through the transformation (4.4). For this reason it is called the 4–boson representation of $W_{\infty}$ algebra.

**$W(2,1)$ algebra**

\[
\{a_1, a_1\} = \left( \frac{1}{2} \partial^3 + a_1 \partial + \partial a_1 \right) \delta(x - y), \\
\{a_2, a_1\} = \left( \partial^2 a_2 - a_2 \partial^2 + 2a_2 S_2 \partial + 2\partial a_2 S_2 \right) \delta(x - y), \\
\{a_1, a_2\} = (a_2 \partial + 2\partial a_2) \delta(x - y), \\
\{a_1, S_2\} = (\frac{3}{2} \partial^2 + S_2 \partial) \delta(x - y), \\
\{a_2, S_2\} = \left( a_1 + (\partial + S_2)^2 \right) \delta'(x - y), \\
\{S_2, S_2\} = \frac{3}{2} \delta'(x - y).
\]

The associated scalar Lax operator is

\[
L = \partial^2 + a_1 \partial + a_2 \frac{1}{\partial - S_2}.
\]

**$W(3,0) = W_3$ algebra**

The $W(3,0)$ algebra is nothing but the $W_3$ algebra

\[
\{a_1, a_1\} = (2\partial^3 + a_1 \partial + \partial a_1) \delta(x - y), \\
\{a_1, a_2\} = (a_2 \partial + 2\partial a_2 - \partial^2 a_1 - \partial^4) \delta(x - y), \\
\{a_1, S_1\} = (\partial^2 + S_1 \partial) \delta(x - y), \\
\{a_1, S_3\} = (3\partial^2 + S_3 \partial) \delta(x - y), \\
\{a_2, a_2\} = \left( \partial^3 a_2 - a_2 \partial^2 + 2a_2 \partial + 2\partial a_2 + a_2(2S_2 - S_1) \partial + a_2(2S_2 - S_1) \right) \delta(x - y), \\
\{a_2, S_3\} = \left( 3\partial^3 a_3 - a_3 \partial^2 - 2S_1 \partial a_3 - a_3 \partial S_1 + 2S_2 \partial a_3 - \partial a_3 S_2 \\
+ 2a_3 S_3 \partial + 3\partial a_3 S_3 \right) \delta(x - y), \\
\{a_2, S_1\} = 0, \\
\{a_3, a_3\} = \left( a_3 + 3\partial^2 + (3S_3 - S_3 - S_1) \partial + (S_3^2 - S_3 S_1 + S_3' - S_3' - S_3' + 2S_3' \right) \delta'(x - y), \\
\{a_3, a_3\} = \left( a_3 \partial^3 + \partial^3 a_3 + a_3 \partial a_3 + a_3(1 + S_2 - 3S_3) \partial^2
\right)
\]

34
The Lax operator is
\[ L = \partial + a_1 \frac{1}{\partial - S_1} + a_2 \frac{1}{\partial - S_2} \frac{1}{\partial - S_3} + a_3 \frac{1}{\partial - S_3} \frac{1}{\partial - S_2} \frac{1}{\partial - S_1}. \]

This algebra also generates the \( W_{\infty} \) algebra, and is called the 6-field representation of the \( W_{\infty} \) algebra.

**\( W(2,2) \) algebra**

\[
\begin{align*}
{a_1, a_1} &= \left( \frac{1}{2} \partial^3 + a_1 \partial + \partial a_1 \right) \delta(x-y), \\
{a_1, a_2} &= (a_2 \partial + 2 \partial a_2) \delta(x-y), \quad {a_1, a_3} = (a_3 \partial + 3 \partial a_3) \delta(x-y), \\
{a_1, S_2} &= \left( 3 \partial^2 + S_2 \partial \right) \delta(x-y), \quad {a_1, S_3} = \left( \frac{5}{2} \partial^2 + S_3 \partial \right) \delta(x-y), \\
{a_2, a_2} &= \left( \partial^2 a_2 - a_2 \partial^2 + 2 a_3 \partial + 2 \partial a_3 + 2 a_2 S_2 \partial + 2 \partial a_2 S_2 \right) \delta(x-y), \\
{a_2, S_2} &= \left( a_1 + (\partial + S_2)^2 \right) \delta'(x-y), \quad (B.9) \\
{a_2, S_3} &= \left( a_1 + 3 \partial^2 + (S_2 + 3 S_3) \partial + (S_3^2 + 2 S_3^t - S_2^t) \right) \delta'(x-y), \\
{a_2, a_3} &= \left( 3 \partial^2 a_3 - a_3 \partial^2 + 2 S_2 \partial a_3 - \partial a_3 S_2 + 2 a_3 S_3 \partial + 3 \partial a_3 S_3 \right) \delta(x-y), \\
{a_3, a_3} &= \left( a_3 \partial^3 + \partial^3 a_3 + a_3 \partial a_1 + a_1 \partial a_3 + a_3 (S_2 - 3 S_3) \partial^2 - \partial^2 a_3 (S_2 - 3 S_3) \right. \\
&\quad \left. + a_3 (3 S_3^2 - 2 S_2 S_3 - 3 S_3^t) \partial + \partial a_3 (3 S_3^2 - 2 S_2 S_3 - 3 S_3^t) \right) \delta'(x-y) \\
{a_3, S_3} &= \left( a_1 \partial + a_2 + a_1 (S_3 - S_2) + (\partial + S_3)^2 (\partial + S_3 - S_2) \right) \delta'(x-y), \\
{a_3, S_2} &= 0, \quad \{S_i, S_j\} = (\delta_{ij} + \frac{1}{2}) \delta'(x-y), \quad i, j = 2, 3.
\end{align*}
\]

In this case the scalar Lax operator is
\[ L = \partial^2 + a_1 + a_2 \frac{1}{\partial - S_2} + a_3 \frac{1}{\partial - S_3} \frac{1}{\partial - S_2}. \]

**\( W(3,1) \) algebra**

\[
\begin{align*}
{a_1, a_1} &= \left( 2 \partial^3 a_1 \partial + \partial a_1 \right) \delta(x-y), \\
{a_1, a_2} &= (a_2 \partial + 2 \partial a_2 - \partial^2 a_1 - \partial^4) \delta(x-y), \\
{a_1, a_3} &= (a_3 \partial + 3 \partial a_3) \delta(x-y), \quad {a_1, S_3} = \left( 2 \partial^2 + S_3 \partial \right) \delta(x-y), \\
{a_2, a_2} &= \left( \partial^2 a_2 - a_2 \partial^2 + 2 a_3 \partial + 2 \partial a_3 - \frac{2}{3} (a_1 + \partial^2) \partial (\partial^2 + a_1) \right) \delta(x-y),
\end{align*}
\]
(4.8)

\{a_2, a_3\} = (3\partial^2 a_3 - a_3 \partial^2 + 2a_3 S_3 \partial + 3\partial a_3 S_3) \delta(x - y), \quad (B.11)

\{a_2, S_3\} = \left(\frac{2}{3} a_2 + \frac{8}{3} \partial^2 + 3S_3 \partial + (S_3^2 + 2S_3')\right) \delta'(x - y),

\{a_3, a_3\} = \left(a_3 \partial^3 + \partial^3 a_3 + a_3 a_1 + a_1 \partial a_3 - 3a_3 S_3 \partial^2 + 3\partial^2 a_3 S_3 + 3a_3(S_3^2 - S_3') \partial + 3\partial a_3(S_3^2 - S_3')\right) \delta'(x - y),

\{a_3, S_3\} = \left(a_2 + a_1 (\partial + S_3) + (\partial + S_3)^3\right) \delta'(x - y),

\{S_3, S_3\} = \frac{4}{3} \delta'(x - y).

with Lax operator

\[ L = \partial^3 + a_1 \partial + a_2 + a_3 \frac{1}{\partial - S_3}. \] \quad (B.12)

**W(4,0) = W_4 algebra**

At last let us give here the \(W_4\) algebra

\{a_1, a_1\} = (5\partial^3 a_1 \partial + \partial a_1) \delta(x - y),

\{a_1, a_2\} = (a_2 \partial + 2\partial a_2 - 2\partial a_1 - 5\partial^4) \delta(x - y),

\{a_1, a_3\} = \left(a_3 \partial + 3\partial a_3 + \frac{3}{2} (\partial^5 + \partial^3 a_1 - \partial^2 a_2)\right) \delta(x - y),

\{a_2, a_2\} = \left(\partial^2 a_2 - a_2 \partial^2 + 2a_3 \partial + 2\partial a_3 - a_1 \partial a_1 - 2a_1 \partial^3 - 2\partial^3 a_1 - 6\partial^5\right) \delta(x - y), \quad (B.13)

\{a_2, a_3\} = \left(3\partial^2 a_3 - a_3 \partial^2 + \frac{1}{2} (a_1 + 4\partial^2) \partial (\partial a_1 - a_2 + \partial^3)\right) \delta(x - y),

\{a_3, a_3\} = \left(a_3 \partial^3 + \partial^3 a_3 + a_3 \partial a_1 + a_1 \partial a_3 + \frac{3}{4} (a_2 + a_1 \partial + \partial^3) \partial (\partial^3 + \partial a_1 - a_2)\right) \delta'(x - y).

It is well-known that this algebra is associated with the scalar Lax operator

\[ L = \partial^4 + a_1 \partial^2 + a_2 \partial + a_3. \] \quad (B.14)

**C Some W(N,M) algebras in the \((q,r)\)-gauge**

**W(2,1) algebra**

\{a_1, a_1\} = \left(\frac{1}{2} \partial^3 + a_1 \partial + \partial a_1\right) \delta(x - y),

\{a_1, q\} = (q \partial + \frac{1}{2} \partial q) \delta(x - y), \quad \{a_1, r\} = (r \partial + \frac{1}{2} \partial r) \delta(x - y),

\{q, q\} = -\frac{3}{2} q \partial^{-1} q \delta(x - y), \quad \{r, r\} = -\frac{3}{2} r \partial^{-1} r \delta(x - y),

\{q, r\} = \left(\partial^2 + a_1 + \frac{3}{2} q \partial^{-1} r\right) \delta(x - y).
The associated scalar Lax operator

\[ L = \partial^2 + a_1 + q\partial^{-1}r. \]

**W(2, 2) algebra**

\[
\begin{aligned}
\{a_1, a_1\} &= \left(\frac{1}{2}\partial^3 + a_1\partial + \partial a_1\right)\delta(x - y), \\
\{a_1, r_1\} &= (r_1\partial + \frac{1}{2}\partial r_1)\delta(x - y), \quad \{a_1, r_2\} = (r_2\partial + \frac{3}{2}\partial r_2)\delta(x - y), \\
\{a_1, q_1\} &= (q_1\partial + \frac{1}{2}\partial q_1)\delta(x - y), \quad \{a_1, q_2\} = (q_2\partial + \frac{1}{2}\partial q_2)\delta(x - y), \\
\{r_1, r_2\} &= -\left(\delta_{ij} + \frac{1}{2}\right)r_ar_j\delta(x - y), \quad i, j = 1, 2, \quad \{q_2, r_1\} = \frac{1}{2}q_2\partial^{-1}r_1\delta(x - y), \\
\{q_1, r_1\} &= \left(\partial^2 + a_1 + \frac{3}{2}q_1\partial^{-1}r_1\right)\delta(x - y), \\
\{q_2, r_2\} &= \left(q_1r_1 + \frac{a_1}{r_1}\partial r_1 + \partial^2\frac{1}{r_1}\partial r_1 + \frac{3}{2}q_2\partial^{-1}r_2\right)\delta(x - y), \\
\{q_1, q_2\} &= \left(-\frac{3}{2}q_1\partial^{-1}q_1 + \frac{2q_2}{r_1}\partial q_2 + \frac{2r_2}{r_1}\partial q_2\right)\delta(x - y) \\
\{q_1, q_2\} &= \left(\frac{1}{r_1}\partial^2 q_2 - \frac{2}{r_1}\partial q_2\partial - \frac{1}{2}\partial r_1^{-1}q_2 - \frac{q_2}{r_1}a_1\right)\delta(x - y), \\
\{q_2, q_2\} &= \frac{3}{2}q_2\partial^{-1}q_2\delta(x - y).
\end{aligned}
\]

In this case the scalar Lax operator is

\[ L = \partial^2 + a_1 + q_1\partial^{-1}r_1 + q_2\partial^{-1}r_2\partial^{-1}r_1. \]

**D  Identities satisfied by the polynomials \( e_k \)**

We devote this Appendix to proving the identities (7.6) and (7.7a–7.7f).

**Proof of eq.(7.6)**

Let us define

\[
G(p; S_1, S_2, \ldots, S_M) = \frac{1}{(p - S_1)(p - S_2) \cdots (p - S_M)},
\]

which is the generating function of the symmetric polynomials \( e_k \),

\[
G = \sum_{l=0}^{\infty} e_l p^{-l-M}.
\]

(D.1)

Taking one time derivative with respect to, say, \( S_1 \) for \( G \), then expanding it in the powers of \( p \), we obtain (7.6).
Likewise we can prove eqs. (7.7d, 7.7f).

Proof of eq. (7.7d)

\[
LHS \rightarrow \sum_{l=1}^{M} \sum_{\beta=0}^{i-1} \sum_{\tilde{\alpha}=0}^{i-\tilde{\alpha}} S_1^{\beta} e_{i-\tilde{\alpha}} e_{j+\tilde{\alpha}}
\]
\[
= \sum_{\tilde{\alpha}=0}^{i} \sum_{\beta=0}^{i-\tilde{\alpha}} S_1^{\beta} e_{i-\tilde{\alpha}} e_{j+\tilde{\alpha}}
\]
\[
= \sum_{\alpha=1}^{i} (i - \alpha + M) e_{i-\alpha} e_{j+\alpha} = RHS.
\]

In the last step we have used eq. (7.7f). The proof of eq. (7.7d) is the same as this.

Proof of eq. (7.7f)

Since eq. (7.7f) is an operatorial equation, we can cut it into two pieces.

\[
\sum_{l=1}^{M} \left( \frac{\partial}{\partial S_l} e_{i+l} \right) \left( \frac{\partial}{\partial S_l} e_{j+l} \right) = (j + M) e_i e_j + \sum_{l=1}^{i} (j - i + 2l) e_{i-l} e_{j+l}.
\] (D.2a)

\[
\sum_{l=1}^{M} \left( \frac{\partial}{\partial S_l} e_{i+l} \right)' = (j + M) e_i e_j' + \sum_{l=1}^{i} (l e_{i+l} e_{i-l}' + (j - i + l) e_{i-l} e_{j+l}').
\] (D.2b)

Now let us consider first eq. (D.2a)

\[
LHS = \sum_{l=1}^{M} \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j+\alpha} e_{i-\alpha} e_{j-\beta} S_l^{\alpha+\beta} = \sum_{l=1}^{M} \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j+\alpha} e_{i-\alpha} e_{j+\beta} S_l^{\beta}
\]
\[
= \sum_{l=1}^{M} \sum_{\alpha=0}^{i} \left( \sum_{\beta=0}^{j+\alpha} - \sum_{\beta=0}^{j+\alpha-1} \right) e_{i-\alpha} e_{j+\alpha} S_l^{\beta}
\]
\[
= \sum_{\alpha=0}^{i} (j + \alpha + M) e_{i-\alpha} e_{j+\alpha} - \sum_{l=1}^{i-1} \sum_{\alpha=0}^{j+\alpha} e_{i-\alpha} e_{j+\beta} S_l^{\beta}
\]
\[
= \sum_{\alpha=0}^{i} (j + \alpha + M) e_{i-\alpha} e_{j+\alpha} - \sum_{\alpha=1}^{i} (i - \alpha + M) e_{i-\alpha} e_{j+\alpha} = RHS.
\]

The LHS of (D.2b) can be divided into two pieces the first being

\[
A = \sum_{k,l=1}^{M} \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j-1} \sum_{\mu=0}^{j-\beta-1} S_k S_l^{\mu} S_1^{\alpha+\beta} e_{i-\alpha} e_{j-\beta-\mu-1}
\]
\[
= \sum_{k=1}^{M} \sum_{\mu=0}^{i} \sum_{\alpha=0}^{j-1} \sum_{\beta=0}^{j-\alpha-1} S_k S_l^{\mu} S_1^{\alpha+\beta} e_{i-\alpha} e_{j-\beta-\mu-1},
\]

Splitting one of the above summations as follows
\[
\sum_{\beta=0}^{\alpha-1} = \sum_{\beta=0}^{\alpha-1} - \sum_{\beta=0}^{\alpha-1},
\]
and using eqs. (D.7c, D.7d), we get

\[ A = \sum_{\mu=0}^{M} \sum_{\alpha=1}^{j-1} \left( \sum_{\beta=0}^{i} (j + \alpha - \mu - 1 + M) - \sum_{\alpha=1}^{M} (i - \alpha + M) \right) e_{i-\alpha} e_{j+\alpha-\mu-1} S_k^\mu S_k^\nu \]

\[ = (j + M) e_i P_j + \sum_{\alpha=1}^{M} (j - i + \alpha) e_{i-\alpha} e_{j+\alpha} - \sum_{\alpha=1}^{M} (\mu + 1) e_{i-\alpha} e_{j+\alpha-\mu-1} S_k^\mu S_k^\nu \]

\[ + \sum_{\beta=0}^{M} \sum_{\alpha=1}^{i} \sum_{\beta=0}^{j-1} (\alpha - \beta - 1) e_{i-\beta} e_{j+\beta-\alpha-1} S_k^\beta S_k^\nu. \quad (D.3) \]

The other term in (D.2b) is

\[ B = \sum_{\beta=0}^{M} \sum_{\alpha=1}^{i} \sum_{\beta=0}^{j-1} \beta e_{i-\alpha} e_{j+\beta} S_k^\alpha S_k^\beta S_k^\nu \]

\[ = \sum_{\beta=0}^{M} \sum_{\alpha=1}^{i} (\beta + 1) e_{i-\alpha} e_{j+\beta-1} S_k^\beta S_k^\nu + \sum_{\beta=0}^{M} \sum_{\alpha=1}^{i} (\beta - \alpha + 1) e_{i-\alpha} e_{j+\alpha-1} S_k^\beta S_k^\nu. \quad (D.4) \]

Combining (D.3) and (D.4) and making use of eq. (7.7d), we obtain

\[ A + B = (j + M) e_i e_j + \sum_{\alpha=1}^{i} (j - i + \alpha) e_{i-\alpha} e_{j+\alpha} \]

\[ + \sum_{\alpha=1}^{M} \sum_{\beta=0}^{i} \sum_{\beta=0}^{j-1} (\alpha - \beta - 1) e_{i-\alpha} e_{j+\alpha-\beta-1} S_k^\beta S_k^\nu \]

\[ - \sum_{\beta=0}^{M} \sum_{\alpha=1}^{i} (j - i + 2\alpha - \beta - 1) e_{i-\alpha} e_{j+\alpha-1} S_k^\beta S_k^\nu \]

\[ = (j + M) e_i e_j + \sum_{\alpha=1}^{i} (\alpha e_{i-\alpha} e_{j+\alpha} + (j - i + \alpha) e_{i-\alpha} e_{j+\alpha}). \]

The last term in the intermediate expression vanishes as one can see in the following way. Let us split the term into two parts

\[ \sum_{\alpha=1}^{M} \sum_{\beta=0}^{i} (j + \alpha - \beta - 1) e_{i-\alpha} e_{j+\alpha-\beta-1} S_k^\beta S_k^\nu, \quad \sum_{\alpha=1}^{M} \sum_{\beta=0}^{j-1} (\alpha - i) e_{i-\alpha} e_{j+\alpha-\beta-1} S_k^\beta S_k^\nu, \]

and redefine the summation parameter for the first part as follows

\[ \tilde{\alpha} = i - j + \alpha + \beta + 1, \quad \tilde{\beta} = \beta, \]

After changing the order of the summations the first part it is exactly the same as the second except for the different sign. This completes our proof of eq. (D.2b).

References

[1] D. Gross and A. Migdal, Phys.Rev.Lett64(1990)127;
E. Brezin and V. Kazakov, Phys.Lett.B236(1990)144;
M. Douglas and S. Shenker, Nucl.Phys.B335(1990)635.
[2] E. Witten, Nucl.Phys.\textbf{B340}(1990)281;  
E. Witten, Nucl.Phys.\textbf{B342}(1990)486;  
E. Witten, Surveys in Diff. Geom.\textbf{1}(1991)243.  
[3] M. Kontsevich, Commun. Math. Phys.\textbf{147}(1992)1.  
[4] M. Douglas, Phys.Lett.\textbf{B238}(1990)176;  
J. Goeree, Nucl. Phys.\textbf{B358}(1991)737.  
[5] L. Bonora and C. S. Xiong, Phys.Lett.\textbf{B285}(1992)191.  
L. Bonora and C. S. Xiong, Int.J.Math.Phys.\textbf{A8}(1993) 2973.  
[6] F. Yu and Y.-S. Wu, Phys. Rev. Lett.\textbf{68}(1992)2996;  
H. Aratyn, L. Ferreira, J. Gomez and A. Zimerman, preprint, IFT–P/020/92; IFT–P/038/93.  
Y. Cheng, J. Maths. Phys.\textbf{33}(1992)3774.  
[7] F. Guil and M. Mañas, \textit{AKNS Hierarchy, Self–Similarity, String Equations and the Grassmannian} \texttt{hep-th 93mmxx}  
[8] L. Bonora and C. S. Xiong, Nucl.Phys.\textbf{B405}(1993)191.  
[9] L. Bonora and C. S. Xiong, Phys.Lett.\textbf{B317}(1993)329.  
[10] L. Bonora and C. S. Xiong, \textit{Correlation functions of the two–matrix models}, SISSA 172/93/EP, BONN–HE/45/93  
[11] L. Dickey, \textit{Soliton equations and Hamiltonian systems}, World Scientific, 1991.  
O. Babelon and C. Viallet, Lecture notes in SISSA(1989).  
P. van Moerbeke \textit{Integrable Foundations of String Theory} Lecture notes, University of Louvain (1993).  
[12] B.A.Kuperschmidt, Comm.Math.Phys.\textbf{99}(1985)51.  
[13] F. Magri, J.Math.Phys. 19(1978) 1156.  
[14] W. Oevel and W. Strampp \textit{Constrained KP hierarchy and bi–Hamiltonian structures}, Loughborough Univ. (Math. Rep. A168).  
[15] M.A. Semenov–Tian–Shansky, Funct.An.Appl.\textbf{17}(1983)259.  
[16] V.G. Drinfeld and V.V. Sokolov, J.Sov.Math.\textbf{30}(1984)1975.  
[17] Q. P. Liu and C. S. Xiong, $W_{3}^{(2)}$ \textit{algebra and the related hierarchies}, preprint, ASITP–93–48.  
[18] I. Bakas and D. Depireux, Mod.Phys.Lett.\textbf{A6}(1991)1561;  
F. Bais, T. Tjin and P. van Driel, Nucl. Phys.\textbf{B357}(1991)632.