TRANSLATIONS IN AFFINE WEYL GROUPS AND THEIR APPLICATIONS IN DISCRETE INTEGRABLE SYSTEMS

Yang Shi College of Science and Engineering, Flinders at Tonsley, Flinders University, SA 5042, Australia yang.shi@flinders.edu.au

Abstract. In this paper, we review the properties and representations of the Weyl groups relevant in the study of discrete integrable systems. Previously in [18, 30], properties of Weyl groups of type $ADE$ (known as simply-laced) were shown to be useful in characterizing and establishing relations between different integrable systems. Here we extend the formulations and discussions to include non-simply-laced types, giving special attention to developing formulas related to the translational elements of the affine Weyl groups. As applications, we show how these are used to clarify the natures of some integrable systems of type $E_8$ [16] and $F_4$ [4] appeared recently in the literature.

Contents

1. Introduction
2. Weyl group
3. Affine Weyl group
   The longest element.
   3.1. Normalizer
   3.2. A dual representation
      A linear map from $V^{(1)}$ to $V^{(1)}^*$
   3.3. Coroots
   3.4. Translations
   3.5. A normal subgroup of translations, $W^{(1)} = W \times Q$
   3.6. Translations in $W(B_3^{(1)})$.
   3.7. Translations in $W(C_3^{(1)})$
   3.8. Translations in $W(F_4^{(1)})$.
   3.9. Translations in $W(G_2^{(1)})$.
4. Extended affine Weyl groups
   4.1. A normal subgroup of translations, $\overline{W}^{(1)} = W \rtimes P = A \rtimes W^{(1)}$
   4.2. Translations on the weight lattice of $\overline{W}(B_3^{(1)}) = W(B_3) \rtimes P = A \times W(B_3^{(1)})$.
   4.3. Translations on the weight lattice $\overline{W}(C_3^{(1)}) = W(C_3) \times P = A \times W(C_3^{(1)})$.
5. Applications
   5.1. Two discrete Painlevé systems of type $E_8^{(1)}$.
Many integrable equations and their discrete analogs admit Coxeter group or Weyl group symmetries. Examples include Hirota’s discrete analog of the KP equation on an octahedron \([13]\),
\[
\alpha \tau_1 \tau_{23} + \beta \tau_2 \tau_{13} + \gamma \tau_3 \tau_{12} = 0,
\]
where \(\tau = \tau(n, m, l)\) for \(n, m, l \in \mathbb{Z}\), and \(\alpha, \beta, \gamma\) are constant parameters. We have used the convention \(\tau(n+1, m, l) = \tau_1\), and \(\tau(n, m+1, l+1) = \tau_{23}\) etc. Equation (1) has Weyl symmetry of type \(A_3\).

There is the classification of Adler, Bobenko, and Suris \([2]\) for discrete analogs of KdV type equations on quadrilaterals, or ABS’s quad-equations. It contains Adler’s discretization of the Krichever-Novikov equation \([1]\) as the Q4 equation,
\[
\text{sn}(\alpha)(uu_1 + u_2 u_{12}) - \text{sn}(\beta)(uu_2 + u_1 u_{12})
- \text{sn}(\alpha - \beta)(u_1 u_2 + uu_{12} - \text{sn}(\alpha)\text{sn}(\beta)(1 + k^2 uu_1 u_2 u_{12})) = 0,
\]
written in its elliptic form \([12]\). We have \(u = u(m, n), u_1 = u(m + 1, n), u_2 = u(m, n + 1), u_{12} = u(m + 1, n + 1)\), for \(n, m \in \mathbb{Z}\), \(\alpha\) and \(\beta\) are constant parameters, and \(k\) is the modulus of the Jacobi sn function. Quad-equations all have Weyl symmetry of type \(B_2\). Equations (1) and (2) are examples of integrable partial difference equations (PΔEs).

Following Okamoto’s geometric description of the Painlevé equations \([27]\), Sakai gave a classification of 22 types of discrete Painlevé equations, these are 2nd-order integrable ordinary difference equations (OΔEs) \([28]\). They are listed by their affine Weyl symmetry types in Figure \([17]\). For a comprehensive overview of the current geometric theory of the discrete Painlevé equations see, a recent survey \([21]\).

The master equation of Sakai’s list, in the sense that the other 21 types can be obtained from it in some degeneration limits, is the elliptic Painlevé equation of \(E_8^{(1)}\) type, \(e-P(E_8^{(1)})\), shown in Equation (3).
It is given as a system of two 1st-order OΔEs for \( f(t), g(t) \) in the variable \( t \). (\( f(t), g(t) \)) is being considered as inhomogeneous coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \). The \( 2 \times 2 \) matrices, given in terms of Weierstrass’s \( \wp \) function, represent PGL(2)-action on \( \mathbb{P}^1 \), i.e., \( w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \) means \( w = (az + b)/(cz + d) \) and we have,

\[
\overline{g} = g(t + \lambda) = M \left( f, c_7, c_8, t - \frac{1}{4} \sum_{i=1}^{6} c_i \right) M \left( f, c_5, c_6, t - \frac{1}{4} \sum_{i=1}^{4} c_i \right) M \left( f, c_3, c_4, t - \frac{1}{4} (c_1 + c_2) \right) \times M(f, c_1, c_2, t) g,
\]

\[
f = f(t - \lambda) = M \left( g, d_7, d_8, t - \frac{1}{4} \sum_{i=1}^{6} d_i \right) M \left( g, d_5, d_6, t - \frac{1}{4} \sum_{i=1}^{4} d_i \right) M \left( g, d_3, d_4, t - \frac{1}{4} (d_1 + d_2) \right) \times M(g, d_1, d_2, t) f,
\]

where

\[
M(h, \kappa_1, \kappa_2, s)
= \begin{pmatrix}
\wp \left( 2s - \frac{\kappa_1 + \kappa_2}{2} \right) \wp \left( 2s - \frac{\kappa_1 \kappa_2}{2} \right) \\
\wp \left( 2s - \frac{\kappa_1 + \kappa_2}{2} \right) \wp \left( 2s - \frac{\kappa_1 \kappa_2}{2} \right) \\
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
(h - \wp (\kappa_2)) (\wp(2s) - \wp(2s - \kappa_2)) (\wp(2s - \frac{\kappa_1 + \kappa_2}{2}) - \wp(2s - \frac{\kappa_1 - \kappa_2}{2})) \\
0 (h - \wp (\kappa_1)) (\wp(2s) - \wp(2s - \kappa_1)) (\wp(2s - \frac{\kappa_1 + \kappa_2}{2}) - \wp(2s - \frac{\kappa_1 - \kappa_2}{2})) \\
1 -\wp(2s - \kappa_1) \\
1 -\wp(2s - \kappa_2)
\end{pmatrix}
\]

Here, \( \lambda = \frac{1}{2} \sum_{i=1}^{8} b_i \), \( c_i = b_i + t \), \( d_i = t - b_i \), and \( b_i \ (1 \leq i \leq 8) \) are constant parameters.

By saying that Equation (3) has Weyl group symmetry of type \( E_8^{(1)} \), \( W(E_8^{(1)}) \), we mean the equation is left invariant by the transformations that form a birational representation of \( W(E_8^{(1)}) \), given in Equation (3c).

The element \( T_1 \) of \( W(E_8^{(1)}) \) that gives rise to the map in Equation (3a),

\[
T_1 : (b_i, t, f, g) \mapsto (b_i, t + \lambda, \overline{f}, \overline{g}), \quad 1 \leq i \leq 8,
\]

acts on the simple roots \( \{ \alpha_j \ | \ 0 \leq j \leq 8 \} \) of \( W(E_8^{(1)}) \) by

\[
T_1 : \{ \alpha_1, \alpha_3 \} \mapsto \{ \alpha_1 - 2\delta, \alpha_3 + \delta \},
\]

where only non-trivial actions are shown. \( T_1 \) is a translation by \( \alpha_1 \) (with \( |\alpha_1|^2 = 2 \), the shortest weight length) on the \( E_8 \) root/weight lattice, it has an expression as a product of
58 simple reflections of $\text{W}(E_8^{(1)})$,

$$T_1 = s_3 s_4 s_2 s_5 s_6 s_7 s_8 s_0 s_2 s_4 s_5 s_6 s_7 s_8 s_3 s_4 s_5 s_6 s_7 s_2 s_4 s_5 s_6 s_3 s_4 s_5 s_2 s_4 s_3 s_1$$

In fact, equations in Sakai’s list [28], referred to as the canonical forms of the discrete Painlevé equations, are all given by a translation of the shortest weight vector of its Weyl type, which we refer to as a basic translation.

The symmetry description has led to some natural generalisations of discrete integrable systems, such as the Painlevé equations of type $A_{n-1}^{(1)}$ [25]; and type $D_n^{(1)}$ generalisations [29, 22] of the $d$-$\text{P}(D_4^{(1)})$ and $q$-$\text{P}(D_5^{(1)})$ equations. It also allows one to put different kinds of discrete systems on the same footing [10, 3, 18, 24]. For example, the relation between a system of ABS’s quad-equations on a $n$-cube (type $B_n$) and a Painlevé equation of type $A_{n-1}^{(1)}$ was found in [17] using the relation between the two weight lattices of these Weyl types. Another example using the fact that the tau functions of the $e$-$\text{P}(E_8^{(1)})$ equation are indexed by $E_8$ weight vectors [26], Noumi [24] showed that Equation (3) is equivalent to a system of 7560 octahedron equations (1).

It is clear that symmetry plays an important role in the study of discrete integrable systems. Having integrable equations that come as some birational representations of affine Weyl groups such as those given in [27, 28, 25] means some behaviours of the nonlinear systems can be studied as certain properties of the Weyl group by using a linear representation, such as describing the discrete Painlevé equations as translations on the weight lattice.

A question that has been of considerable interest concerns the nature of certain 2nd-order OΔEs which were shown to admit symmetries that do not appear explicitly in Sakai’s list [31, 20, 4, 9]. In our previous works [18, 30], we discussed properties and representations of the affine Weyl group useful for establishing connections between such systems and Sakai’s equations for the simply-laced Weyl groups. The formulas allowed us to reduce the problem of finding potentially complicated birational transformations that relates the two OΔEs to some simple manipulations in linear algebra. Moreover, we found that different kinds of equations amount to translations of different lengths on the weight lattice.

Here, we extend our previous discussions [18, 30] to non-simply-laced types, giving a detailed exposition on the affine Weyl group, paying particular attention to describing its elements of translation. By employing a linear map $\pi$ from the vector space $V^{(1)}$ on which the Weyl groups act as groups of reflections to a dual vector space $V^{(1)*}$, we realise translational actions of the Weyl groups on an affine plane of this dual space, where translations by vectors of different lengths can be explicitly discussed. Although all formulas discussed here may be found one way or another in classical texts such as Bourbaki [6] or Humphreys [15], we
believe having explicit formulas for the actions of the affine Weyl group in some well-chosen representation can be useful in studying the integrable systems with Weyl symmetries in general.

As applications, two examples from the integrable system literature are chosen to illustrate how the formulas and properties of the Weyl groups discussed here can be used in this context.

First, we derive the relation between a “new” elliptic difference equation with $W(E_8^{(1)})$ symmetry found recently in [16] and Sakai’s $e\cdot P(E_8^{(1)})$ equation (3). As this particular equation takes almost three pages to write down we refer the interested reader to Equation (3.27) in [16]. This equation is said to be new as the element $T_{J,1} \in W(E_8^{(1)})$ that gives rise to its discrete evolution is a translation of squared length 4 rather than 2 (as for $T_1$ of Sakai’s $e\cdot P(E_8^{(1)})$ equation), that is not a basic translation in $W(E_8^{(1)})$, and can not be conjugated to a basic translation under the actions of $W(E_8)$. In particular, $T_{J,1}$ acts on the simple roots $\{\alpha_j | 0 \leq j \leq 8\}$ of $W(E_8^{(1)})$ by

$$T_{J,1} : \{\alpha_1, \alpha_6\} \mapsto \{\alpha_1 - 2\delta, \alpha_6 + \delta\}. \quad (5)$$

Using the dual representation of $W(E_8^{(1)})$, we show that $T_{J,1}$ can be obtained by a composition of $T_1$ (Equation (4c)) and another basic translation in $W(E_8^{(1)})$, thus clarify the relation between these two elliptic difference equations of type $E_8^{(1)}$.

Our second example concerns a subsystem of type $E_4^{(1)}$ of the $e\cdot P(E_8^{(1)})$ equation (3) discussed in [4]. It is obtained from the Q4 equation (2) by imposing the condition $u_1 = u_2$,

$$\text{cn} (\gamma_n) \text{dn} (\gamma_n) \left(1 - k^2 \text{sn}^4 (z_n)\right) u_n (u_{n+1} + u_{n-1})$$

$$- \text{cn} (z_n) \text{dn} (z_n) \left(1 - k^2 \text{sn}^2 (z_n) \text{sn}^2 (\gamma_n)\right) \left(u_{n+1} u_{n-1} + u_n^2\right)$$

$$+ \left(\text{cn}^2 (z_n) - \text{cn}^2 (\gamma_n)\right) \text{cn} (z_n) \text{dn} (z_n) \left(1 + k^2 u_n^2 u_{n+1} u_{n-1}\right) = 0, \quad (6)$$

where $n \in \mathbb{Z}$ now is the only independent variable, $u_n = u(n)$ is the dependent variable, $k$ is the modulus of the Jacobi sn function, and $\gamma_e, \gamma_o$ are constant complex parameters with

$$z_n = (\gamma_e + \gamma_o) n + z_0, \quad \gamma_n = \begin{cases} \gamma_e & \text{for } n = 2j \\
\gamma_o & \text{for } n = 2j + 1. \end{cases}$$

The element $\varphi_a$ of $W(E_8^{(1)})$ that gives rise to the discrete evolution in Equation (6),

$$\varphi_a : n \mapsto n + 1, \quad (7a)$$

acts on the simple roots $\{\alpha_j | 0 \leq j \leq 8\}$ of $W(E_8^{(1)})$ by

$$\varphi_a : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\} \mapsto \{\alpha_1 + \delta, -\alpha_2, \alpha_{23445} - \delta, -\alpha_4, -\alpha_5, -\alpha_{12334456} + \delta, -\alpha_7, -\alpha_8, -\alpha_0\}, \quad (7b)$$
whereas we have,
\[
\varphi^2_a : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\} \\
\mapsto \{\alpha_1 + 2\delta, \alpha_2, \alpha_3 - 2\delta, \alpha_4, \alpha_5, \alpha_6 + \delta, \alpha_7, \alpha_8, \alpha_0\}.
\]
That is, \(\varphi_a\) is not a translation while \(\varphi^2_a\) is. We call elements such as \(\varphi_a\) quasi-translations.
\(\varphi_a\) has the following expression in terms of simple reflections of \(W(E_8^{(1)})\),
\[
\varphi_a = \varphi^2_a, \quad \text{where} \quad \varphi_s = s_6s_5s_4s_2s_7s_6s_5s_4s_2s_8s_7s_6s_5s_4s_2s_8s_7s_6s_5s_4s_8s_7s_6s_5s_4s_8s_7s_6s_5s_4s_3.
\]
It was verified (using MAGMA [5]) that \(\varphi_a\) is an element of a \(F_4^{(1)}\) type subgroup of \(W(E_8^{(1)})\). Here, we show that this \(F_4^{(1)}\) subgroup in fact arises as part of a normalizer in \(W(E_8^{(1)})\). Moreover, we give the sub-root system of \(E_8^{(1)}\) on which the generators of this \(F_4^{(1)}\) subgroup (involutions in general) can be realised as reflections, thereby allowing translational (or quasi-translational) type elements to be constructed. Finally, we show that \(\varphi_a\) is an element of quasi-translation in the \(F_4^{(1)}\) subgroup.

The paper is organised as follows. We lay down some general facts about the finite Weyl group \(W\) in Section 2. In particular, we discuss in detail the root system of \(W\)—one of the most useful tools for studying the Weyl groups. Examples of root systems of type \(B_3, C_3, F_4\) and \(G_2\) are used to highlight the various properties missing from our previous discussions on simply-laced groups [18, 30]. In Section 3, after listing some general properties of the affine Weyl group \(W^{(1)}\), a dual space \(V^{(1)*}\) is introduced along with the coroots and fundamental weights. Here, we show that \(W^{(1)}\) contains a normal subgroup of translations on the root lattice \(Q\), \(W^{(1)} = W \ltimes Q\), by studying the dual representation. Moreover, we write down explicitly the elements of translation by both long and short coroots for non-simply-laced types. In Section 4, we construct a certain extension of \(W^{(1)}\), such that it contains a normal subgroup of translations on the weight lattice \(P\), that is \(\hat{W}^{(1)} = W \ltimes P\). We write down explicitly the expressions for translations on the weight lattice of types \(B_3\) and \(C_3\). In Section 5, we discuss two examples from integrable systems using the expressions derived earlier, and finally give some concluding remarks and future prospects in Section 6.

## 2. Weyl group

Let \(\Gamma\) be a Dynkin diagram with \(n\) nodes. We associate to it a reflection group or Coxeter group,
\[
W = W(\Gamma) = \langle s_i \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, 1 \leq i, j \leq n \rangle,
\]
given by the Coxeter presentation, that is a generating set satisfying some defining relations. When parameter \(m_{ij}\) takes value in \(\{2, 3, 4, 6\}\), known as the crystallographic condition, \(W\) is called a Weyl group. Weyl groups are classified by the associated types of Dynkin diagrams.
(see Figure 1), their defining relations encoded in the corresponding Dynkin diagram $\Gamma$ with the following rules. Each node of the diagram represents a generator $s_i$ (of order 2) for $1 \leq i \leq n$. The order of the product of generators $s_i$ and $s_j$, that is $m_{ij}$ takes the value of: 2, 3, 4 or 6 when two nodes labeled $i$ and $j$ are: disconnected, joined by a single, a double, or a triple edge, respectively. As an example, see the fundamental relations (21) of $W(B_3)$ corresponding to $\Gamma(B_3)$ in Figure 2. For each $\Gamma$, there is a corresponding Cartan matrix,

$$C(\Gamma) = (a_{ij}), \quad \text{where} \quad a_{ii} = 2 \quad \text{for all} \quad 1 \leq i, j \leq n.$$  

(9)

The values of $a_{ij}$, for $i \neq j$, can be read off from $\Gamma$ depending on the connection between nodes $i$ and $j$ following

| $a_{ij}$ | $a_{ji}$ | $m_{ij}$ | $i$ | $j$ |
|----------|----------|----------|-----|-----|
| 0        | 0        | 2        | $\circ$ | $\circ$ |
| -1       | -1       | 3        | $\circ$ | $\circ$ |
| -2       | -1       | 4        | $\circ$ | $\circ$ |
| -3       | -1       | 6        | $\circ$ | $\circ$ |

**Table 1.** For connections in the Dynkin diagram between vertices $i$ and $j$ in the last column, we give $a_{ij}$, entries of the Cartan matrix for $i \neq j$; $m_{ij}$, and the order of the product $s_i s_j$.

Diagrams of type $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$ contain only single edges are called *simply-laced*. Otherwise, they are *non-simply-laced* types, that is $B_n$, $C_n$, $F_4$ and $G_2$.

Let $V$ be an $n$-dimensional real vector space. Each node $i$ of $\Gamma$ can be associated with a vector $\alpha_i \in V$, called *a simple root*. The set of simple roots $\Delta = \Delta(\Gamma) = \{\alpha_i \mid 1 \leq i \leq n\}$, is the *simple system*, and it forms a basis for $V$. The vector space $V$ is equipped with a symmetric positive definite bilinear form given by

$$\alpha_i \cdot \alpha_j = |\alpha_i||\alpha_j| \cos \left( \frac{\pi - |\alpha_i||\alpha_j|}{2m_{ij}} \right) = \frac{|\alpha_i|^2 a_{ij}}{2}, \quad \text{for all} \quad 1 \leq i, j \leq n,$$

(10)

where the quantity $\alpha_i \cdot \alpha_i = |\alpha_i|^2$ gives the usual interpretation of squared length of $\alpha_i$, its value depends on the type of Dynkin diagram.

The crystallographic condition of $W$ means that the entries of $C(\Gamma)$,

$$a_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}$$

(11)

take integer values. Moreover, from Equation (11), we have

$$\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{a_{ij}}{a_{ji}}, \quad \text{for all} \quad 1 \leq i, j \leq n.$$
Figure 1. Dynkin diagrams of affine Weyl groups.
For simply-laced types $\Gamma$, $C(\Gamma)$ is symmetric $a_{ij} = a_{ji}$ (see Table 1). Then by Equation (12), there is only one root length, usually normalised to $|\alpha_i|^2 = 2$, for all $1 \leq i \leq n$.

For non-simply-laced types, $C(\Gamma)$ is not symmetric and Equation (12) implies that we have roots of two different lengths, referred to as being long and short, respectively. The difference in lengths of the simple roots is indicated in a non-simply-laced Dynkin diagram by the arrow on a multiple edge: arrow points from a long root to a short root (see Figure 1).

We give a chosen normalisation of root lengths in the following remark.

**Remark 1.**

- $B_n$ The long roots are normalised to have squared length 2. That is
  \[ |\alpha_j|^2 = 2, \quad 1 \leq j \leq n - 1 \quad \text{which implies that} \quad |\alpha_n|^2 = 1, \quad (13a) \]
  since \[ \frac{|\alpha_{n-1}|^2}{|\alpha_n|^2} = \frac{a_{n-1,n}}{a_{n-1,n-1}} = 2. \]

- $C_n$ The short roots are normalised to have squared length 2. That is
  \[ |\alpha_j|^2 = 2, \quad 1 \leq j \leq n - 1 \quad \text{which implies that} \quad |\alpha_n|^2 = 4, \quad (13b) \]
  since \[ \frac{|\alpha_{n-1}|^2}{|\alpha_n|^2} = \frac{a_{n-1,n}}{a_{n-1,n-1}} = \frac{1}{2}. \]

- $F_4$ The long roots are normalised to have squared length 2. That is
  \[ |\alpha_1|^2 = |\alpha_2|^2 = 2 \quad \text{which implies that} \quad |\alpha_3|^2 = |\alpha_4|^2 = 1, \quad (13c) \]
  since \[ \frac{|\alpha_2|^2}{|\alpha_3|^2} = \frac{a_{12}}{a_{32}} = 2. \]

- $G_2$ The short roots are normalised to have squared length 2. That is
  \[ |\alpha_1|^2 = 2, \quad \text{which implies that} \quad |\alpha_2|^2 = 6, \quad (13d) \]
  since \[ \frac{|\alpha_2|^2}{|\alpha_1|^2} = \frac{a_{12}}{a_{11}} = 3. \]

$W(\Gamma)$ can be realised as a group of reflections in $V$ as follows. The generator $s_j = s_{\alpha_j}$ of $W$ is realised as the reflection along the hyperplane orthogonal to the simple root $\alpha_j \in \Delta$. In particular, it acts on the $\Delta$ basis of $V$ by the usual formula of reflection,

\[ s_j(\alpha_i) = \alpha_i - \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} \alpha_j = \alpha_i - a_{ij} \alpha_j \quad \text{for all} \quad i, j \in \{1, ..., n\}, \quad (14) \]

where $a_{ij}$ is the $(i, j)$-entry of $C(\Gamma)$. The element $s_j$ is called a simple reflection.

Define the root system of $W(\Gamma)$ to be the set of vectors we get by acting $W(\Gamma)$ on $\Delta$,

\[ \Phi = \Phi(\Gamma) = W(\Gamma) \Delta. \quad (15) \]
Remark 2. From Equation (14), and the fact that $a_{ij}$ are integers, we have that all elements of $\Phi$, known as roots, are integer combinations of the simple system $\Delta$. We have
\[
\frac{2\alpha \cdot \beta}{\beta \cdot \beta} \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in \Phi.
\] (16)

$\Phi$ is finite whenever $W$ is finite. Moreover, we have
\[
\Phi = \Phi_+ \cup \Phi_-,
\] (17)
where
\[
\Phi_+ = \{\alpha = \sum_{i=1}^{n} \lambda_i \alpha_i \mid \mathbb{Z} \ni \lambda_i \geq 0, \alpha_i \in \Delta\} \quad \text{and} \quad \Phi_- = \{-\alpha \mid \alpha \in \Phi_+\}.
\] (18)
$\sum \lambda_i$ is called the height of the root $\alpha$, and $\Phi_+$ is the positive root system. We define $-\alpha$ to have the same height as $\alpha$, for all $\alpha \in \Phi_+$. Equations (17) and (18) say that a root is some integer combinations of $\Delta$, with coefficients being either all positive or all negative, moreover $|\Phi| = 2|\Phi_+|$. For each finite root system $\Phi$, there exists a highest root
\[
\tilde{\alpha} = \sum_{i=1}^{n} c_i \alpha_i.
\] (19)
We list the coefficients $c_i$ ($1 \leq i \leq n$) for all Weyl groups in Table 2 of Appendix A (see Figure 1 for the corresponding numbering of nodes in the Dynkin diagrams). For $\Phi$ of non-simply-laced type, where there are long and short roots, $\tilde{\alpha}$ is used to denote the highest long root, and we use $\tilde{\alpha}_s$ for the short root in $\Phi_+$ with maximum height.

Remark 3. The finite Weyl group $W$ acts transitively on roots of the same length. That is, all long (respectively short) roots of $\Phi$ form a single $W$-orbit. In general, vectors in the same $W$-orbit will have the same length (note that the opposite is not true). We list $|\Phi_+|$ and $|W|$ for Weyl groups of all types in Table 2 of Appendix A.

We now illustrate the properties of $W$ discussed so far for the $B_3$, $C_3$, $F_4$ and $G_2$ types.

Example 1. Finite Weyl group of type $B_3$, $W(B_3)$. Given the Dynkin diagram of type $B_3$, $\Gamma(B_3)$ in Figure 2

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
1 & 2 & 3 \\
\end{array}
\]

Figure 2. Dynkin diagram of finite type $B_3$, $\Gamma(B_3)$.

the corresponding Cartan matrix,
\[
C(B_3) = (a_{ij})_{1 \leq i,j \leq 3} = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2 \\
\end{pmatrix},
\] (20)
and the defining relations for the Weyl group,

\[ W(B_3) = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^2 = (s_2 s_3)^4 = 1 \rangle, \tag{21} \]

can be written down using \( \Gamma(B_3) \) with the rules given in Table 1. The simple system \( \Delta = \{ \alpha_1, \alpha_2, \alpha_3 \} \) is a basis of an 3-dimensional real vector space \( V \), equipped with a symmetric bilinear form,

\[(\alpha_i \cdot \alpha_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} |\alpha_i|^2 & a_{ij} \\ a_{ij} & |\alpha_j|^2 \end{pmatrix}_{1 \leq i,j \leq 3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \tag{22} \]

on which \( W(B_3) \) acts as a group of reflections.

From Equation (22) we have

\[ |\alpha_i|^2 = 2 \quad \text{for} \quad (i = 1,2) \quad \text{and} \quad |\alpha_3|^2 = 1. \tag{23} \]

That is, roots \( \alpha_1, \alpha_2 \) are long and \( \alpha_3 \) is short, as indicated by how the arrow on the multiple edge points in \( \Gamma(B_3) \), see Figure 2.

The simple reflection \( s_j \in W(B_3) \) acts on the \( \Delta \) basis of \( V \) by Equation (14) with \( a_{ij} \) being the \((i, j)\)-entry of \( C(B_3) \) from Equation (20). The \( B_3 \) root system is given by \( \Phi(B_3) = W(B_3)\Delta = \Phi^+ \cup \Phi^- \), where \( \Phi^+ \) has six long roots and three short roots. We list them below, starting with the highest roots (long \( \tilde{\alpha} \) and short \( \tilde{\alpha}_s \), respectively). We give also the corresponding element in \( W \) takes the highest root to that root.

| Long                  | Short                  |
|-----------------------|------------------------|
| 1 : \( \alpha_1 + 2\alpha_2 + 2\alpha_3 = \tilde{\alpha} \), | 1 : \( \alpha_1 + \alpha_2 + \alpha_3 = \tilde{\alpha}_s \), |
| \( s_2 \) : \( \alpha_1 + \alpha_2 + 2\alpha_3 \),        | \( s_1 \) : \( \alpha_2 + \alpha_3 \),             |
| \( s_1 s_2 \) : \( \alpha_2 + 2\alpha_3 \),                    | \( s_2 s_1 \) : \( \alpha_3 \),                        |
| \( s_3 s_2 \) : \( \alpha_1 + \alpha_2 \),                      | \( s_1 s_3 s_2 \) : \( \alpha_2 \),                  |

We see that, of the simple system \( \Delta \), the long roots \( \alpha_1, \alpha_2 \) belong to the same \( W \)-orbit of \( \tilde{\alpha} \), whereas the short simple root \( \alpha_3 \) is in the \( W \)-orbit of \( \tilde{\alpha}_s \).

**Remark 4.** In general, for \( B_n \) type root system, we have \( n^2 = n(n-1) + n \) positive roots in \( \Phi^+ \), with \( n(n-1) \) long and \( n \) short.
Remark 5. For simplicity, we sometimes adopt the notation: \( \alpha_i + \ldots + \alpha_j = \alpha_{i..j} \) to express the sum of simple roots, and \( s_{i..j} = s_{i..j} \) for product of simple reflections. For example, we now write \( \tilde{\alpha} \) and \( \tilde{\alpha}_s \) as \( \alpha_{12333} \) and \( \alpha_{1233} \), respectively. The element \( s_2 s_3 s_2 s_3 \in W(B_3) \), can now be written as \( s_{2323} \).

Example 2. Finite Weyl group of type \( C_3 \), \( W(C_3) \). Given the Dynkin diagram \( \Gamma(C_3) \) in Figure 3.

![Dynkin diagram](image)

Figure 3. Dynkin diagram of finite type \( C_3 \), \( \Gamma(C_3) \).

the corresponding Cartan matrix,

\[
C(C_3) = (a_{ij})_{1 \leq i,j \leq 3} = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -2 & 2
\end{pmatrix},
\]

and the defining relations for the Weyl group,

\[
W(C_3) = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^2 = (s_2 s_3)^4 = 1 \rangle,
\]

(26)
can be written down using \( \Gamma(C_3) \) with the rules given in Table 1.

The simple system \( \Delta = \{ \alpha_1, \alpha_2, \alpha_3 \} \) is a basis of an 3-dimensional real vector space \( V \), equipped with a symmetric bilinear form given by

\[
(\alpha_i \cdot \alpha_j)_{1 \leq i,j \leq 3} = \begin{pmatrix}
\frac{|\alpha_j|^2}{2} & a_{ij} \\
a_{ij} & -1 & 2 & -2 \\
0 & -2 & 4
\end{pmatrix},
\]

(27)
on which \( W(C_3) \) acts as a group of reflections.

From Equation (27) we have

\[
|\alpha_i|^2 = 2 \quad \text{for} \quad (i = 1, 2) \quad \text{and} \quad |\alpha_3|^2 = 4.
\]

(28)
That is, \( \alpha_1, \alpha_2 \) are short and \( \alpha_3 \) is long, as indicated by the way the arrow is pointing in \( \Gamma(C_3) \), see Figure 3.

Actions of \( s_i \in W(C_3) \) on \( \Delta \) are given by Equation (14) with \( a_{ij} \) being the \( (i,j) \)-entry of \( C(C_3) \) from Equation (25). The \( C_3 \) root system is \( \Phi = W(C_3) \Delta = \Phi^+ \cup \Phi^- \). In general, for \( C_n \) type root system there are \( n^2 = n(n-1) + n \) positive roots in \( \Phi^+ \), with \( n(n-1) \) short and
n long. Hence for Φ+ of C_3 type we have the following six short roots and three long roots,

| Short | Long |
|-------|------|
| 1 : α_{1223} = \tilde{\alpha}_s, | 1 : α_{11223} = \tilde{\alpha}, |
| s_2 : α_{123}, | s_1 : α_{223}, |
| s_1s_2 : α_{23}, | s_2s_1 : α_3, |
| s_3s_2 : α_{12}, | |

\begin{equation}
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix},
\end{equation}

(29)

Example 3. Finite Weyl group of type F_4, W(F_4). Given the Dynkin diagram Γ(F_4) in Figure 4.

Figure 4. Dynkin diagram of finite type F_4, Γ(F_4).

we have the corresponding Cartan matrix,

\begin{equation}
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix},
\end{equation}

(30)

and the defining relations for the Weyl group,

\begin{align*}
W(F_4) &= (s_1, s_2, s_3, s_4 \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1 s_3)^2 = (s_1 s_4)^2 = (s_2 s_4)^2 = 1, \\
(s_1 s_2)^3 &= (s_2 s_3)^4 = (s_3 s_4)^3 = 1 \tag{31}
\end{align*}

using the rules given in Table 4.

The simple system $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of an 4-dimensional real vector space $V$, on which $W(F_4)$ acts as a group of reflections. $V$ is equipped with a symmetric bilinear form,

\begin{equation}
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 1 & -1/2 \\
0 & 0 & -1/2 & 1
\end{pmatrix}.
\end{equation}

(32)
From Equation (32) we have

\[ |\alpha_i|^2 = 2 \quad \text{for} \quad (i = 1, 2) \quad \text{and} \quad |\alpha_j|^2 = 1, \quad j = 3, 4. \tag{33} \]

That is, \( \alpha_1, \alpha_2 \) are long, while \( \alpha_3 \) and \( \alpha_4 \) are short, as indicated by how the arrow points in \( \Gamma(F_4) \), see Figure 4. Actions of \( s_i \in W(F_4) \) on \( \Delta \) are given by Equation (14) with \( a_{ij} \) being the \((i, j)\)-entry of \( C(F_4) \) from Equation (30). The \( F_4 \) root system is \( \Phi = W(F_4)\Delta = \Phi^+ \cup \Phi^- \). There are 24 positive roots in \( \Phi^+ \), with 12 long and 12 short. In particular, we have,

\[ \tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \tag{34} \]

and

\[ \tilde{\alpha}_s = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4. \tag{35} \]

**Example 4.** Finite Weyl group of type \( G_2 \), \( W(G_2) \). The Dynkin diagram of type \( G_2 \), \( \Gamma(G_2) \) is given in Figure 5.

![Figure 5. Dynkin diagram of finite type \( G_2 \), \( \Gamma(G_2) \).](image)

The corresponding Cartan matrix,

\[ C(G_2) = (a_{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \tag{36} \]

and the defining relations for the Weyl group,

\[ W(G_2) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^6 = 1 \rangle, \tag{37} \]

can be written down using \( \Gamma(G_2) \) with the rules given in Table 1. The simple system \( \Delta(G_2) = \Delta = \{\alpha_1, \alpha_2\} \) forms a basis of an 2-dimensional real vector space \( V \) on which \( W(G_2) \) acts as a group of reflections. The vector space \( V \) is equipped with a symmetric bilinear form,

\[ (\alpha_i \cdot \alpha_j)_{1 \leq i,j \leq 2} = \begin{pmatrix} |\alpha_j|^2 \cr 2 a_{ij} \end{pmatrix}_{1 \leq i,j \leq 2} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}. \tag{38} \]

We have

\[ |\alpha_1|^2 = 2, \quad \text{and} \quad |\alpha_2|^2 = 6. \tag{39} \]

That is, roots \( \alpha_1 \) is short and \( \alpha_2 \) is long as indicated by how the arrow on the multiple edge points in \( \Gamma(G_2) \), see Figure 5.

The simple reflection \( s_j \in W(G_2) \) acts on the \( \Delta \) basis of \( V \) by Equation (14) with \( a_{ij} \) being the \((i, j)\)-entry of \( C(G_2) \) given by Equation (36). The \( G_2 \) root system is \( \Phi(G_2) = W(G_2)\Delta = \Phi^+ \cup \Phi^- \). \( \Phi^+ \) has three long roots and three short roots. We list them below, starting with the
highest roots (long \(\tilde{\alpha}\) and short \(\tilde{\alpha}_s\), respectively). We give also the corresponding element in \(W(G_2)\) that takes the highest long/short root to that root.

\[
\begin{array}{cccc}
\text{Long} & & \text{Short} \\
1: & 3\alpha_1 + 2\alpha_2 = \tilde{\alpha}, & 1: & 2\alpha_1 + \alpha_2 = \tilde{\alpha}_s, \\
s_2: & 3\alpha_1 + \alpha_2, & s_1: & \alpha_1 + \alpha_2, \\
s_1s_2: & \alpha_2, & s_2s_1: & \alpha_1.
\end{array}
\]

(40)

3. Affine Weyl group

Let \(\Gamma^{(1)}\) be the affine extension of \(\Gamma\), having an extra node labelled 0, associated with the affine reflection \(s_0\) (or the affine simple root \(\alpha_0\)), see diagrams in Figure 1. The corresponding affine Weyl group,

\[
W^{(1)} = \langle s_i \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, 1 \leq i, j \leq n \rangle,
\]

(41)

has its defining relations encoded by \(\Gamma^{(1)}\) with rules given in Table IV. \(W^{(1)}\) is a group of infinite order, to which \(W = \langle s_i \mid 0 \leq i \leq n \rangle\) is a finite standard parabolic subgroup. The corresponding Cartan matrix, \(C(\Gamma^{(1)}) = (a_{ij})_{1 \leq i,j \leq n, 0}\), the generalised Cartan matrix, is degenerate. Here, we arrange the entries of \(C(\Gamma^{(1)})\) such that \((a_{i0})\) and \((a_{0j})\) are its last column and row, respectively. That is, we have \(C(\Gamma) = (C(\Gamma^{(1)}))_{1 \leq i,j \leq n}\).

The simple affine system

\[
\Delta^{(1)} = \Delta \cup \{\alpha_0\} = \{\alpha_i \mid 0 \leq i \leq n\}
\]

(42)

forms a basis for an \(n + 1\)-dimensional real vector space \(V^{(1)}\), on which one can define a semidefinite symmetric bilinear form using \(C(\Delta^{(1)})\),

\[
\alpha_i \cdot \alpha_j = \frac{|\alpha_j|^2a_{ij}}{2}, \quad \text{for} \quad i, j \in \{0, 1, ..., n\}.
\]

(43)

The bilinear form on \(V^{(1)}\) being semidefinite implies that there is a 1-dimensional subspace in \(V^{(1)}\), the radical of \(V^{(1)}\): \(\text{Rad}(V^{(1)})\), is spanned by a vector \(\delta \in V^{(1)}\) called the null root, such that

\[
\alpha_i \cdot \delta = 0 \quad \text{for all} \quad 0 \leq i \leq n.
\]

(44)

In particular, we have

\[
\delta = \alpha_0 + \tilde{\alpha} = \alpha_0 + \sum_{i=1}^{n} c_i \alpha_i = \sum_{i=0}^{n} c_i \alpha_i,
\]

(45)

recall that \(\tilde{\alpha}\) is the highest long root of \(W\) defined in Equation (19). By Equation (45), we see that the set of vectors \(\{\alpha_1, \alpha_2, \ldots, \alpha_n, \delta\}\) forms another basis of \(V^{(1)}\). Moreover, we see
from the $\Gamma^{(1)}$ diagrams given in Figure[1] that the root $\alpha_0 \in \Delta^{(1)}$ is always a long root. That is,

$$|\alpha_0|^2 = \alpha_0 \cdot \alpha_0 = (\delta - \tilde{\alpha}) \cdot (\delta - \tilde{\alpha}) = \tilde{\alpha} \cdot \tilde{\alpha} = |\tilde{\alpha}|^2.$$  \hfill (46)

Recall that $W^{(1)}$ is a group of infinite order. The corresponding affine root system

$$\Phi^{(1)} = \Phi(\Gamma^{(1)}) = W^{(1)}\Delta^{(1)},$$  \hfill (47)

is also infinite. $W^{(1)}$ acts transitively on roots of $\Phi^{(1)}$ (of the same length). Moreover, we have

$$\Phi^{(1)} = \Phi^+_1 \cup \Phi^-_1,$$  \hfill (48)

where

$$\Phi^+_1 = \{ \alpha = \sum_{i=0}^n \lambda_i \alpha_i \mid \mathbb{Z} \ni \lambda_i \geq 0, \alpha_i \in \Delta^{(1)} \} \quad \text{and} \quad \Phi^-_1 = \{-\alpha \mid \alpha \in \Phi^+_1\}. \hfill (49)$$

The concept of the height of a root was discussed for a finite Weyl group can be extended to the affine case.

**Definition 1. Height of a root.** For a root in the positive system, $\alpha \in \Phi^+_1$, where $\alpha = \sum_{i=0}^n \lambda_i \alpha_i$, for $\alpha_i \in \Delta^{(1)}$ and $\mathbb{Z} \ni \lambda_i \geq 0$, the height of $\alpha$ is given by $\sum_{i=0}^n \lambda_i$. We define $-\alpha$ to have the same height as $\alpha$, for all $\alpha \in \Phi^+_1$.

Finally, it happens that all roots of the affine root system have the form

$$\Phi^{(1)} = \{ \alpha + m\delta \mid \alpha \in \Phi, m \in \mathbb{Z} \},$$  \hfill (50)

where $\Phi = W\Delta$ is the finite root system in a subspace of $V^{(1)}$, $V = \text{Span}(\Delta)$.

For $0 \leq j \leq n$, the generator $s_j \in W^{(1)}$ can be realised as the reflection along the simple root $\alpha_j \in \Delta^{(1)}$, its action on $\Delta^{(1)}$ is given by

$$s_j(\alpha_i) = s_{\alpha_j}(\alpha_i) = \alpha_i - a_{ij}\alpha_j, \quad \text{for all} \quad i, j \in \{0, 1, ..., n\}, \hfill (51)$$

where $a_{ij}$ is now the $(i, j)$-entry of $C(\Gamma^{(1)})$.

In general, the element of reflection along any root $\beta \in \Phi^{(1)}$, $s_\beta \in W^{(1)}$ acts on $V^{(1)}$ by,

$$s_\beta(v) = v - \frac{2v \cdot \beta}{\beta \cdot \beta} \beta, \quad \text{and we have} \quad s_{-\beta} = s_\beta \hfill (52)$$

for all $v \in V^{(1)}$. Furthermore, $s_\beta$ is related to a simple reflection $s_i$ by a conjugation,

$$s_\beta = s_{w(\alpha_i)} = ws_iw^{-1}, \quad \text{where} \quad w(\alpha_i) = \beta, \quad w \in W^{(1)}, \quad \alpha_i \in \Delta^{(1)}, \quad \beta \in \Phi^{(1)}. \hfill (53)$$

By Equations (52) and (14) we have

$$s_i(\delta) = \delta - \frac{2\delta \cdot \alpha_i}{\alpha_i \cdot \alpha_i} \alpha_i = \delta \quad \text{for all} \quad 0 \leq i \leq n. \hfill (54)$$

That is

$$w(\delta) = \delta \quad \text{for all} \quad w \in W^{(1)}. \hfill (55)$$
We now recall some useful properties and functions of the Weyl groups, which are also true for Coxeter groups in general.

**Definition 2.** For each \( w \in W^{(1)} \), define

\[
N(w) = \{ \alpha \in \Phi_+^{(1)} \mid w(\alpha) \in \Phi_-^{(1)} \}.
\]

That is, \( N(w) \) is the set of positive roots that \( w \) takes to some negative roots.

**Definition 3.** Length of an element. Each \( w \in W^{(1)} \) can be expressed as a product of simple reflections. The least number of simple reflections expression is called reduced, and this number is defined to be the length of \( w, l(w) \). Moreover, if \( w_1, w_2 \in W^{(1)} \) and \( N(w_2) \subset N(w_1) \) then

\[
l(w_1w_2^{-1}) = l(w_1) - l(w_2).
\]

The general theory of Coxeter groups says that, for \( w \in W^{(1)} \) and \( \alpha \in \Delta^{(1)} \) we have,

\[
l(ws_{\alpha}) = \begin{cases} l(w) + 1, & \text{if } w(\alpha) \in \Phi_+^{(1)} \\ l(w) - 1, & \text{if } w(\alpha) \in \Phi_-^{(1)}. \end{cases}
\]

Then we have \( |N(w)| = l(w) = k \), where \( k \) is a non-negative integer. That is, Equation (58) can be applied repeatedly until we can write \( w \) as a product of \( k \) simple reflections,

\[
w_{s_{l_1}}...s_{l_k} = 1, \quad l_1, ..., l_k \in \{0,1,...,n\},
\]

or

\[
w = s_{l_k}...s_{l_1},
\]

where we have used \( s_j^2 = 1 \ (j \in \{0,1,...,n\}) \).

Now as an example, we show how by knowing the actions of an element \( T_1 \in W(E_8^{(1)}) \) on the \( E_8^{(1)} \) simple system given in Equation (4c), one can use Equation (58) to write \( T_1 \) as a product of simple reflections as given in Equation (4c).

**Example 5.** Let \( \Delta^{(1)} = \{ \alpha_j \mid 0 \leq j \leq 8 \} \) be the \( E_8^{(1)} \) simple system, we have \( W(E_8^{(1)}) = \{ s_i \mid 0 \leq i \leq 8 \} \) and \( \Phi^{(1)} = W(E_8^{(1)})\Delta^{(1)} = \Phi_+^{(1)} \cup \Phi_-^{(1)} \) is the \( E_8^{(1)} \) root system. The actions of \( s_i \in W(E_8^{(1)}) \ (1 \leq i \leq 8) \) on \( \Delta^{(1)} \) are given by Equation (51) with \( a_{ij} \) being the \((i,j)\)-entry of \( C(E_8^{(1)}) \) given in Equation (340), and the \( \delta \) of \( W(E_8^{(1)}) \) is given in Equation (341).

Given the action of an element \( T_1 \) of \( W(E_8^{(1)}) \) on its simple system \( \Delta^{(1)} = \{ \alpha_j \mid 0 \leq j \leq 8 \} \),

\[
T_1 : \{ \alpha_1, \alpha_3 \} \mapsto \{ \alpha_1 - 2\delta, \alpha_3 + \delta \},
\]

where only non-trivial actions are shown after that.

We see that \( T_1(\alpha_i) \in \Phi^{(1)} \) only when \( i = 1 \). That is \( l(T_1s_1) = l(T_1) - 1 \) by Equation (58).

Compute the action of \( T_1s_1 \) on \( \Delta^{(1)} \) we have,

\[
T_1s_1 : \{ \alpha_1, \alpha_3 \} \mapsto \{ -\alpha_1 + 2\delta, \alpha_3 - \delta \},
\]
that is $T_1s_1(\alpha_i) \in \Phi_\Delta^{(1)}$ only when $i = 3$, so we have $l(T_1s_1s_3) = l(T_1s_1) - 1$. Calculate the action of $T_1s_1s_3$ on $\Delta^{(1)}$ we have,

$$T_1s_1s_3 : \{\alpha_1, \alpha_3, \alpha_4\} \mapsto \{\alpha_3 + \delta, -\alpha_1 - \alpha_3 + \delta, \alpha_1 + \alpha_3 + \alpha_4 - \delta\},$$

that is $T_1s_1s_3(\alpha_i) \in \Phi_\Delta^{(1)}$ only when $i = 4$, that is $l(T_1s_1s_3s_4) = l(T_1s_1s_3) - 1$. Proceed in this fashion, we have $T_1s_1s_3s_4 \cdots s_{l_k} = 1$, where $l_1, \ldots, l_k \in \{0, 1, \ldots, 8\}$, where we found $l_1 = 1$, $l_2 = 3$, $l_3 = 4$, ..., $l_{58} = 3$ and $k = 58$. Rewriting this using $s^2 = 1$ (for $j \in \{0, 1, \ldots, 8\}$) we have,

$$T_1 = s_{l_1} \cdots s_{l_k}s_{l_2}s_{l_1} = s_3 \cdots s_{48}s_3s_1,$$

which gives us the expression for $T_1$ as a product of 58 simple reflections in Equation (4c).

The longest element. For a finite Weyl group of rank $n$, $W = W(\Gamma)$ with the simple system $\Delta = \Delta(\Gamma)$ and the corresponding finite positive root system $\Phi_\Gamma^+ = WT \Gamma$ whereby slight abuse of notation we denote both the Dynkin diagram and its type by $\Gamma$, there exist a unique element of maximal length $w_\Gamma \in W$, its longest element, where $w_\Gamma^2 = 1$ and $N(w_\Gamma) = \Phi_\Gamma^+$. Then we have $l(w_\Gamma) = |\Phi_\Gamma^+|$. Moreover, $w_\Gamma \Delta = -\Delta = \{-\alpha \mid \alpha \in \Delta\}$. In fact for a permutation $\sigma$ on the index set of $\Gamma$ we have, $w_\Gamma(\alpha_i) = -\alpha_{\sigma(i)}$. For finite Weyl groups $\sigma$ is the identity for $A_1$, $B_n$, $C_n$, $D_n$ ($n$ even), $F_4$, $E_7$, $E_8$. For the other types, $\sigma$ corresponds to the unique symmetry of order 2 of the Dynkin diagram.

Example 6. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ be the simple system of type $A_3$, and $W(A_3) = \langle s_1, s_2, s_3 \rangle$. From Table 2 we see that $|\Phi_{A_3}^+| = 6$, hence $l(w_{A_3}) = 6$. The longest element of $W(A_3)$ is given by

$$w_{A_3} = s_2s_1s_3s_2s_1s_3,$$

it acts on the $A_3$ simple system by:

$$w_{A_3} : \{\alpha_1, \alpha_2, \alpha_3\} \mapsto \{-\alpha_3, -\alpha_2, -\alpha_1\}.$$

3.1. Normalizer. Let $W^{(1)}$ be an affine Weyl group with a simple system $\Delta^{(1)}$. For a subset $J \subset \Delta^{(1)}$, the group $W_J = \{s_i \mid \alpha_i \in J\}$ is call the standard parabolic subgroup of $W^{(1)}$. The Normalizer of $W_J$ in $W^{(1)}$ is defined by $N_W(W_J) = \{g \in W^{(1)} \mid g^{-1}W_Jg = W_J\} = N_J \ltimes W_J$. In [14] it was shown that, $N_J = \{w \in W^{(1)} \mid wJ = J\}$. That is, $N_J$ the set wise stabilizer of $J$-its element either fixes or permutes the elements of $J$. The group $N_J$ is generated by $R$-elements and $M$-elements. The R-elements, also known as quasi-reflections, are involutions that act permutatively on the subset $J$, whereas the M-elements permute the R-elements. Together, they generate a group of extended affine Weyl type for which elements of translation (or
quasi-translations in \( W^{(1)} \) can be constructed. Here we briefly explain how to find the R-elements when computing a normalizer of \( W_J \) in \( W^{(1)} \). For the full theory of normalizers in Coxeter groups see \[13, 15\].

Let \( I \) and \( J \) be disjoint subsets of \( \Delta^{(1)} \), and let \( L = I \cup J \subset \Delta^{(1)} \). Then there is a unique element \( v[I, J] \) of \( W_L \) given by,

\[
v[I, J] = w_L w_J,
\]

where \( w_L \) and \( w_J \) are the longest elements of the parabolic subgroups \( W_L \) and \( W_J \), respectively. By Equation (57) we have

\[
l(v[I, J]) = l(w_L) - l(w_J) = |\Phi_L^+| - |\Phi_J^+|.
\]

In particular, when \( |I| = 1 \), elements \( v[a, J] \) (for \( a \in \Delta^{(1)} \setminus J \)) which are involutions are called the \( R \)-elements. It is also useful for us to consider the case when \( |I| = 2 \), that is elements of the type \( v\{a, b \}, J \) (for \( a, b \in \Delta^{(1)} \setminus J \)). \( v\{a, b \}, J \) has exactly two standard expressions:

\[
v[a_1, J_1]...v[a_n J_n] = v[b_1, K_1]...v[b_n K_n],
\]

where \( a_n = a, b_n = b, J_n = K_n = J \). Furthermore, \( a_i, b_j \in J \cup \{a, b\} \) and \( J_i, K_j \subseteq J \cup \{a, b\} \) for all \( i, j \). Equation (68) allows one to compute the orders for the pairwise products of the generators of \( N_J \). Let \( V_J = \text{Span}(J) \), and \( V_J^j \) be the orthogonal complement of \( V_J \) in \( V^{(1)} \), that is, \( V^{(1)} = V_J \oplus V_J^j \). Coxeter groups \( W^{(1)}/W_J/N_J \) act on \( V^{(1)}/V_J/V_J^j \) as reflection groups.

**Example 7.** Given the finite Weyl group of type \( A_3 \), \( W(A_3) = \langle s_1, s_2, s_3 \rangle \) and its simple system \( \Delta = \{\alpha_1, \alpha_2, \alpha_3\} \). Let \( J = \{\alpha_1, \alpha_3\} \cong 2A_1 \), then we have \( W_J = \langle s_1, s_3 \rangle \cong W(2A_1) \). The normalizer of \( W_J \) in \( W(A_3) \) is given by

\[
N(W_J) = N_J \ltimes W_J = (s_2 s_2 s_1) = \langle s_1, s_2 \rangle \cong W(A_1) \ltimes W(2A_1).
\]

**Proof.** To see this, recall that \( N_J \) is generated by the R- and M-elements. Since \( J = \{\alpha_1, \alpha_3\} \) is the only type \( 2A_1 \) subset of \( \Delta \) there are no M-elements and we need only to find the R-elements, that is elements of the form \( v[a, J] \) (for \( a \in \Delta^{(1)} \setminus J \)) which are involutions, and we see that there is only one possibility, \( v[\alpha_2, J] \). First, let us find the length of this element.

By Equation (67) we have,

\[
l(v[\alpha_2, J]) = l(w_{\alpha_2}) - l(w_{\alpha_3}) = l(w_{\alpha_3}) - l(w_{2A_1}) = |\Phi_{\alpha_3}^+| - 2 |\Phi_{\alpha_1}^+| = 6 - 2 = 4.
\]

Moreover, since \( W_J \cong W(2A_1) \), we have \( w_J = w_{2A_1} = s_1 s_3 \), and

\[
w_{2A_1} : \{\alpha_1, \alpha_2, \alpha_3\} \mapsto \{-\alpha_1, \alpha_2, -\alpha_3\}.
\]
Then by Equations (66) and (64) we have,

$$v[\alpha_2, J] = w_{J, (\alpha_2)} w_J,$$

$$= w_{A_3} w_{2A_1},$$

$$= s_2 s_3 s_1 s_2 s_1 s_3 s_1 s_3,$$

$$= s_2 s_3 s_1 s_2,$$

$$= s_2 s_3 s_1 s_2.$$

(71)

Its action on $\Delta$ is obtained by composing the actions of $w_{2A_1}$ and $w_{A_3}$ given in Equations (70) and (65),

$$v[\alpha_2, J]: \{\alpha_1, \alpha_2, \alpha_3\} \mapsto \{\alpha_3, \alpha_2, \alpha_1\}.$$

That is, $v[\alpha_2, J]$ permutes $\alpha_1$ and $\alpha_3$. That is, it is an involution hence an R-element.

Remark 6. For an arbitrary group $W$, the Normaliser of its subgroup usually is no more than the subgroup itself. However, when $W$ is a Coxeter group and the subgroup a parabolic one, one can have some very non-trivial $N_J$.

3.2. A dual representation. It is well-known that $W^{(1)}$ contains an abelian subgroup of translations on the root lattice. To this end, it is useful to introduce a dual space $V^{(1)*}$ on which the translations are realised.

Definition 4. Let $V^{(1)*}$ be an $(n+1)$-dimensional real vector space, and $\langle , \rangle : V^{(1)} \times V^{(1)*} \rightarrow \mathbb{R}$ be a bilinear pairing between $V^{(1)}$ and $V^{(1)*}$. Let $\{h_1, \ldots, h_n, h_\delta\}$ be a basis of $V^{(1)*}$ dual to $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \delta\} \subset V^{(1)}$, that is we have,

$$\langle \alpha_i, h_j \rangle = \delta_{ij},$$

(72a)

$$\langle \alpha_i, h_\delta \rangle = \langle \delta, h_j \rangle = 0, \quad \text{for } 1 \leq i, j \leq n,$$

(72b)

$$\langle \delta, h_\delta \rangle = 1.$$

(72c)

The group $W^{(1)}$ acts on $V^{(1)*}$ via the contragredient action:

$$\langle w^{-1}v, h \rangle = \langle v, wh \rangle, \quad \text{for } v \in V^{(1)}, h \in V^{(1)*}, w \in W^{(1)}.$$

(73)

A useful consequence of Definition 4 is

$$\langle \alpha_0, h_j \rangle = \langle \delta - \sum_{i=1}^n c_i \alpha_i, h_j \rangle = -c_j, \quad 1 \leq j \leq n.$$

(74)

Definition 5. Define an $n$-dimensional hyperplane in $V^{(1)*}$,

$$X_k = \{ h \in V^{(1)*} | \langle \delta, h \rangle = k \}.$$

(75)
Since $w(\delta) = \delta$ for all $w \in W^{(1)}$, the set $X_k$ is preserved by $W^{(1)}$ for any $k \in \mathbb{R}$. Whenever $k \neq 0$, the set $X_k$ can be regarded as an $n$-dimensional affine space on which $W^{(1)}$ acts as affine transformations, such as translations. Moreover, we have $X_k = kh_\delta + X_0$.

For $k = 0$, $X_0$ is an $n$-dimensional subspace of $V^{(1)*}$. We see that the set \{h_j | 1 \leq j \leq n\} forms a basis for $X_0$ from the second equality in Equation (72d) of Definition 4.

Definition 6. Vectors $h_j \ (1 \leq j \leq n)$ are called the fundamental weights of $W^{(1)}$. The weight lattice is the integer span of fundamental weights:

$$P = \bigoplus_{i=1}^{n} \mathbb{Z}h_i. \quad (76)$$

A linear map from $V^{(1)}$ to $V^{(1)*}$. We introduce a linear mapping $\pi$ from $V^{(1)}$ to its dual $V^{(1)*}$ which enables us to quantitatively analyse different types of translational elements of $W^{(1)}$ and its extensions. This is particularly relevant in the context of discrete integrable systems, since the dynamics of such systems arise from different types of translations in the Weyl group.

Definition 7. For each $u \in V^{(1)}$ there is a linear map $\pi(u) : V^{(1)} \to \mathbb{R}$ defined by $v \mapsto v \cdot u$ for all $v \in V^{(1)}$. That is, $\pi(u)$ is an element of $V^{(1)*}$ satisfying

$$v \cdot u = \langle v, \pi(u) \rangle \quad \text{for all} \quad v \in V^{(1)}. \quad (77)$$

For all $u, v \in V^{(1)}$, $w \in W^{(1)}$ we have

$$\langle v, w\pi(u) \rangle = \langle w^{-1}(v), \pi(u) \rangle = w^{-1}(v) \cdot u = v \cdot w(u) = \langle v, \pi(w(u)) \rangle,$$

that is

$$w(\pi(u)) = \pi(w(u)). \quad (78)$$

Thus $\pi$ is a $W^{(1)}$-homomorphism from $V^{(1)}$ to $V^{(1)*}$. We now discuss some useful properties of the map $\pi$.

Let $\alpha + m\delta = \beta \in \Phi^{(1)}$, for $\alpha \in \Phi$ and $0 \neq m \in \mathbb{Z}$, we have

$$\langle v, \pi(\beta) \rangle = v \cdot \beta = v \cdot (\alpha + m\delta) = v \cdot \alpha = \langle v, \pi(\alpha) \rangle,$$

that is

$$\pi(\beta) = \pi(\alpha). \quad (79)$$

So the kernel of $\pi$ is $\text{Rad}(V^{(1)})$ and its image is $X_0$.

For each $\beta \in \Phi^{(1)}$, define

$$\beta^\vee = \frac{2\beta}{\beta \cdot \beta} = \frac{2\beta}{|\beta|^2}. \quad (80)$$
then by Equation (43), we have
\[ a_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \alpha_i \cdot \alpha_j = \langle \alpha_i, \pi(\alpha_j) \rangle, \quad \text{for} \quad 0 \leq i, j \leq n, \] (81)
where \( a_{ij} = (C(\Gamma^{(1)}))_{ij} \).

**Proposition 1.** \( \{ \pi(\alpha_j) \mid 1 \leq j \leq n \} \) is a basis of \( X_0 \). Moreover, we have,
\[ \pi(\alpha_j) = \sum_{k=1}^{n} a_{kj} h_k = \sum_{k=1}^{n} (C(\Gamma)^T)_{jk} h_k, \quad \text{for} \quad 1 \leq j \leq n. \] (82)

Or we have,
\[ h_i = \sum_{k=1}^{n} (C(\Gamma)^T)^{-1}_{ik} \pi(\alpha_k), \quad \text{for} \quad 1 \leq i \leq n. \] (83)

**Proof.** Recall that \( \{ h_j \mid 1 \leq j \leq n \} \) is a basis of \( X_0 \). Since for all \( 1 \leq i \leq n \) we have,
\[ \langle \alpha_i, \pi(\alpha_j) \rangle = \langle \alpha_i, \sum_{k=1}^{n} (\pi(\alpha_j)) h_k \rangle, \]
\[ = \langle \alpha_i, \sum_{k=1}^{n} a_{kj} h_k, \rangle \]
\[ = \langle \alpha_i, \sum_{k=1}^{n} (C(\Gamma)^T)_{jk} h_k, \rangle, \]
where \( 1 \leq j \leq n \), and we have used Equation (72b) and Definition 4. That is,
\[ \pi(\alpha_j) = \sum_{k=1}^{n} a_{kj} h_k = \sum_{k=1}^{n} (C(\Gamma)^T)_{jk} h_k, \quad \text{for} \quad 1 \leq j \leq n. \]

As \( C(\Gamma) \) is nondegenerate, the set \( \{ \pi(\alpha_j) \mid 1 \leq j \leq n \} \) is also a basis of \( X_0 \). Equation (83) follows from Equation (82). \( \square \)

**Definition 8.** The root lattice of \( W^{(1)} \) is the integer span of simple coroots of the dual root system in \( V^{(1)*} \):
\[ Q = \bigoplus_{i=1}^{n} \mathbb{Z} \pi(\alpha_i^\vee). \] (84)

We now look at the actions of \( W^{(1)} \) on the \( \{ \pi(\alpha_j^\vee) \mid 1 \leq j \leq n \} \) basis of \( X_0 \).

**Proposition 2.** For \( 0 \leq k, j \leq n \), we have \( s_k \in W^{(1)} \) acts on \( \{ \pi(\alpha_j^\vee) \mid 1 \leq j \leq n \} \) by,
\[ s_k(\pi(\alpha_j^\vee)) = \pi(\alpha_j^\vee) - a_{kj} \pi(\alpha_k^\vee) = \pi(\alpha_j^\vee) - (C(\Gamma^{(1)})^T)_{jk} \pi(\alpha_k^\vee). \] (85)
Proof. For any \(0 \leq i, k, j \leq n\) we have,

\[
\langle \alpha_i, s_k(\pi(\alpha_j^\vee)) \rangle = (s_k(\alpha_i), \pi(\alpha_j^\vee)),
\]

\[
= (\alpha_i - a_{ik} \alpha_k, \pi(\alpha_j^\vee)),
\]

\[
= (\alpha_i, \pi(\alpha_j^\vee)) - a_{ik}(\alpha_k, \pi(\alpha_j^\vee)),
\]

\[
= (\alpha_i, \pi(\alpha_j^\vee)) - (\alpha_i, \pi(\alpha_k^\vee))\pi(\alpha_j^\vee),
\]

\[
= (\alpha_i, \pi(\alpha_j^\vee) - a_{kj} \alpha_k).
\]

where we have used Equations (73), (51), and (9).

\[\square\]

Remark 7. In fact, under the contragredient actions of \(W\), the set \(\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq n\}\) generates a dual root system \(\Phi^\vee\) in \(X_0 \subset V(1)^*\), its elements are called coroots. When \(\Gamma\) is of a non-simply-laced type, the Dynkin diagram for the dual root system is obtained from that of \(\Gamma\) by reversing the direction of the arrow on the multiple edge of \(\Gamma\), which corresponds to the appearance of \(C(\Gamma^{(1)})^T\) in Proposition 2. For simply-laced ADE type systems, we have \(C(\Gamma^{(1)})^T = C(\Gamma^{(1)})\).

Proposition 3. In general, the element \(s_\beta \in W^{(1)}\) (for any \(\beta \in \Phi^{(1)}\)) acts on \(V^{(1)*}\) by

\[
s_\beta(f) = f - \langle \beta, f \rangle \pi(\beta^\vee), \quad \text{for any} \quad f \in V^{(1)*}.
\]  

(86)

Proof. By Equations (52), (77) and the contragredient action of \(W^{(1)}\) we have,

\[
\langle v, s_\beta(f) \rangle = \langle s_\beta(v), f \rangle,
\]

\[
= \langle v - v \cdot \beta^\vee \beta, f \rangle,
\]

\[
= \langle v, f \rangle - v \cdot \beta^\vee \langle \beta, f \rangle,
\]

\[
= \langle v, f \rangle - \langle v, \pi(\beta^\vee) \rangle \langle \beta, f \rangle,
\]

\[
= \langle v, f - \langle \beta, f \rangle \pi(\beta^\vee) \rangle.
\]

\[\square\]

3.3. Coroots. Proposition 3 says that in order to understand how \(W^{(1)}\) acts on the dual space \(V^{(1)*}\) it is worth working out what the coroot of \(\beta\), \(\pi(\beta^\vee)\) is in \(X_0\), for any \(\beta \in \Phi^{(1)}\).

We establish some properties of \(\pi(\beta^\vee)\) via the following six propositions in Section 3.3.

Proposition 4. For any \(\beta = \alpha + m\delta \in \Phi^{(1)}\), \(\alpha \in \Phi\) and \(m \in \mathbb{Z}\),

\[
\pi(\beta^\vee) = \pi(\alpha^\vee).
\]

(88)
Proof. We know that the image of the function $\pi$ is in $X_0$. Moreover, for any $\beta = \alpha + m\delta \in \Phi^{(1)}$, $\alpha \in \Phi$ and $m \in \mathbb{Z}$, we have

$$\langle \beta, h_\delta \rangle = m, \quad \text{and} \quad \beta \cdot \beta = (\alpha + m\delta) \cdot (\alpha + m\delta) = \alpha \cdot \alpha,$$

that is

$$\pi(\beta^\vee) = \pi\left(\frac{2\beta}{\beta \cdot \beta}\right) = \frac{2\pi(\beta)}{\beta \cdot \beta} = \frac{2\pi(\alpha)}{\alpha \cdot \alpha} = \pi\left(\frac{2\alpha}{\alpha \cdot \alpha}\right) = \pi(\alpha^\vee),$$

where we have used Equation (79).

Proposition 5. Let $\alpha \in \Phi$, then $\pi(\alpha^\vee)$ belongs to the dual root system $\Phi^\vee$ generated by $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq n\}$ under the contragredient action of $W$. That is, for $\alpha \in \Phi$ where we have

$$w(\alpha_j) = \alpha \quad \text{for some} \quad w \in W, \alpha_j \in \Delta,$$

then,

$$w\left(\pi(\alpha_j^\vee)\right) = \pi(\alpha^\vee).$$

Proof. Let $\alpha \in \Phi$, then we have $w(\alpha_j) = \alpha$ for some $w \in W$, $\alpha_j \in \Delta$, moreover we have $|\alpha_j| = |\alpha|$. Then

$$w\left(\pi(\alpha_j^\vee)\right) = w\left(\pi\left(\frac{2\alpha_j}{|\alpha_j|^2}\right)\right) = \frac{2w(\pi(\alpha_j))}{|\alpha_j|^2} = \frac{2\pi(\alpha_j)}{|\alpha|^2} = \frac{2\pi(\alpha)}{|\alpha|^2} = \pi(\alpha^\vee).$$

This means that $\pi(\alpha^\vee)$ can be written as some integer combinations of $\pi(\alpha_j^\vee) (1 \leq j \leq n)$, just as $\alpha \in \Phi$ can be written as some integer combinations of $\alpha_j (1 \leq j \leq n)$. The integer coefficients of $\alpha_j$ in $\alpha \in \Phi$ and those of $\pi(\alpha_j^\vee)$ in $\pi(\alpha^\vee) \in \Phi^\vee$ are related by the following proposition.

Proposition 6. If $\alpha \in \Phi$, that is $\alpha = \sum_{i=1}^n m_i\alpha_i$, for some $m_i \in \mathbb{Z}$ and $\alpha_i \in \Delta$, then the coroot of $\alpha$, $\pi(\alpha^\vee)$ is an element in $\Phi^\vee$.

$$\pi(\alpha^\vee) = \sum_{i=1}^n k_i\pi(\alpha_i^\vee), \quad \text{where} \quad k_i = m_i \frac{|\alpha_i|^2}{|\alpha|^2} \quad \text{or} \quad m_i = \frac{|\alpha|^2}{|\alpha_i|^2} k_i, \quad \text{for} \quad 1 \leq i \leq n.$$
\[
\pi(\alpha^\vee) = \pi \left( \frac{2}{|\alpha|^2} \sum_{i=1}^{n} m_i \alpha_i \right) \\
= \sum_{i=1}^{n} m_i \pi \left( \frac{2 \alpha_i \alpha_i^\vee}{|\alpha|^2 |\alpha|^2} \right) \\
= \sum_{i=1}^{n} \frac{\alpha_i^2}{|\alpha|^2} \pi(\alpha_i^\vee) \\
= \sum_{i=1}^{n} k_i \pi(\alpha_i^\vee).
\]

\[\square\]

**Remark 8.** Proposition \( \mathbf{6} \) says that for simply-laced type root systems, where \( |\alpha_i|^2 = |\alpha|^2 = 2 \) (for all \( \alpha_i, \alpha \in \Phi \)), we have \( m_i = k_i \ (1 \leq i \leq n) \). For these types we can identify the simple coroots with the simple roots \( \pi(\alpha_i^\vee) = \alpha_j \ (1 \leq j \leq n) \). For non-simply-laced-types, \( m_i \) and \( k_i \) are not the same for some values of \( i \), \( \pi(\alpha_i^\vee) \) is identified with \( \frac{2 \alpha_i}{|\alpha_i|^2} \). The values of the \( k_i \)'s in \( \pi(\tilde{\alpha}^\vee) \) for non-simply-laced-types are listed in Table \( \mathbf{2} \) of Appendix \( \mathbf{A} \).

Now we show that \( \pi(\beta^\vee) \in X_0 \) for any \( \beta \in \Phi^{(1)} \) is an element of the weight lattice \( P \).

**Proposition 7.** For any \( \beta \in \Phi^{(1)} \), let \( b_k = \langle \alpha_k, \pi(\beta^\vee) \rangle \) for \( 0 \leq k \leq n \). We have
\[
\pi(\beta^\vee) = \sum_{j=1}^{n} b_j h_j = \sum_{j=1}^{n} \langle \alpha_j, \pi(\beta^\vee) \rangle h_j. \tag{95}
\]
Moreover,
\[
\sum_{i=0}^{n} c_i b_i = 0, \quad \text{or} \quad b_0 = -\sum_{i=1}^{n} c_i b_i, \tag{96}
\]
where \( c_i \) is the coefficient of \( \alpha_i \) in \( \delta \). In particular, the coefficients \( b_i \) (for \( 0 \leq i \leq n \)) are all integral. That is, \( \pi(\beta^\vee) \in X_0 \) is an element of the weight lattice \( P = \bigoplus_{i=1}^{n} \mathbb{Z} h_i \).

**Proof.** Recall that \( \{h_j \mid 1 \leq j \leq n\} \) is a basis of \( X_0 \), and \( \pi(\beta^\vee) \in X_0 \) for any \( \beta \in \Phi^{(1)} \). For any \( 0 \leq i \leq n \) we have,
\[
\langle \alpha_i, \pi(\beta^\vee) \rangle,
\]
\[
= \langle \alpha_i, \sum_{j=1}^{n} \langle \alpha_j, \pi(\beta^\vee) \rangle h_j \rangle,
\]
\[
= \langle \alpha_i, \sum_{i=1}^{n} b_j h_j \rangle,
\]
that is,
\[
\pi(\beta^\vee) = \sum_{j=1}^{n} b_j h_j = \sum_{j=1}^{n} \langle \alpha_j, \pi(\beta^\vee) \rangle h_j.
\]

To show that Equation (96) is true, observe that

\[ 0 = \langle \delta, \pi(\beta^\vee) \rangle = \sum_{i=0}^{n} c_i \alpha_i, \pi(\beta^\vee) \rangle = \sum_{i=0}^{n} c_i b_i, \quad \text{or} \quad b_0 = - \sum_{i=1}^{n} c_i b_i, \]  

(97)
since \( c_0 = 1 \) by Equation (45).

To show that \( b_i \) for \( 0 \leq i \leq n \) are all integers, first for \( 1 \leq i \leq n \) we have,

\[ b_i = \langle \alpha_i, \pi(\beta^\vee) \rangle = \langle \alpha_i, \pi(\alpha^\vee) \rangle = \alpha_i \cdot \alpha^\vee = \frac{2 \alpha_i \cdot \alpha}{\alpha \cdot \alpha}, \]

which are integers for all \( \alpha_i, \alpha \in \Delta \) by Remark 2. By Equation (97) and the fact that \( b_i \) and \( c_i \) (\( 1 \leq i \leq n \)) are all integers we have \( b_0 \) is also an integer. \( \square \)

**Proposition 8.** For each \( 0 \leq j \leq n \), \( \pi(\alpha_j^\vee) \) can be expressed as a linear combination of \( \{h_k \mid 1 \leq k \leq n\} \) with coefficients from the \( j \)-th column of \( C(\Gamma^{(1)}) \),

\[ \pi(\alpha_j^\vee) = \sum_{k=1}^{n} (C(\Gamma^{(1)}))_{kj} h_k = \sum_{k=1}^{n} a_k h_k. \]  

(98)

Moreover, for any \( 0 \leq j \leq n \) we have

\[ \sum_{i=0}^{n} c_i a_{ij} = 0, \]  

(99)

where \( c_i \) is the coefficient of \( \alpha_i \) in \( \delta \), for \( 0 \leq i \leq n \).

**Proof.** In Proposition 7 let \( \beta = \alpha_j \), for \( 0 \leq j \leq n \) we have,

\[ b_k = \langle \alpha_k, \pi(\alpha_j^\vee) \rangle = a_{kj}, \quad 0 \leq k \leq n, \]  

(100)

and

\[ \pi(\alpha_j^\vee) = \sum_{k=1}^{n} b_k h_k = \sum_{k=1}^{n} \langle \alpha_k, \pi(\alpha_j^\vee) \rangle h_k = \sum_{k=1}^{n} a_k h_k = \sum_{k=1}^{n} (C(\Gamma^{(1)}))_{kj} h_k. \]  

(101)

For \( 1 \leq j \leq n \) we recover Equation (82). When \( j = 0 \), we have,

\[ \pi(\alpha_0^\vee) = \sum_{i=1}^{n} \langle \alpha_i, \pi(\alpha_0^\vee) \rangle h_i = \sum_{i=1}^{n} a_{i0} h_i \]  

(102)

where \( a_{i0} \) (\( 1 \leq i \leq n \)) are the first \( n \) entries of the last column of \( C(\Gamma^{(1)}) \).

Equation (99) is just a special case of Equation (96) on letting \( \beta = \alpha_j \) (\( 0 \leq j \leq n \)). \( \square \)

We give \( \pi(\tilde{\alpha}^\vee) \) and \( \pi(\tilde{\alpha}_j^\vee) \) for all Weyl group types in terms of the fundamental weights in Table 2 of Appendix A.

Now, introducing a bilinear form on \( X_0 \subset V^{(1)*} \), we discuss the lengths of coroots and fundamental weights in \( X_0 \).
Definition 9. Let \((\cdot, \cdot) : X_0 \times X_0 \to \mathbb{R}\) be a symmetric positive definite bilinear form on \(X_0 \subset V^{(1)*}\) such that,
\[
(\pi(\alpha_i^\vee), \pi(\alpha_j^\vee)) = \frac{2}{|\alpha_i|} a_{ij}, \quad 1 \leq i, j \leq n. \tag{103}
\]
Using \(|\pi(\beta)|\) to denote the length of the vector \(\pi(\beta) \in X_0\) for any \(\beta \in V^{(1)}\), from Equation (103) we have,
\[
|\pi(\alpha_i^\vee)|^2 = (\pi(\alpha_i^\vee), \pi(\alpha_i^\vee)) = \frac{2}{|\alpha_i|^2} a_{ii} = \frac{4}{|\alpha_i|^2}, \quad 1 \leq i \leq n. \tag{104}
\]
Hence for any \(\alpha \in \Phi\) we have,
\[
|\pi(\alpha^\vee)|^2 = \begin{cases} 
4, & \text{for } |\alpha|^2 = 1, \\
2, & \text{for } |\alpha|^2 = 2, \\
1, & \text{for } |\alpha|^2 = 4, \\
\frac{2}{3}, & \text{for } |\alpha|^2 = 6.
\end{cases} \tag{105}
\]

Proposition 9. The bilinear form \((\cdot, \cdot)\) given in Definition 9 on \(X_0\) is related to a restriction of the bilinear pairing between \(V^{(1)}\) and \(V^{(1)*}\) \((\cdot, \cdot)\) given in Definition 4 by
\[
\pi(\alpha_i), \pi(\alpha_j^\vee) = a_{ij} = \langle \alpha_i, \pi(\alpha_j^\vee) \rangle, \quad 1 \leq i, j \leq n. \tag{106}
\]

Proof. For \(1 \leq i, j \leq n\) we have
\[
(\pi(\alpha_i), \pi(\alpha_j^\vee)) = \left(\frac{|\alpha_i|^2 \pi(\alpha_i^\vee)}{2}, \pi(\alpha_j^\vee)\right),
\]
\[
= \frac{|\alpha_i|^2}{2} \left(\pi(\alpha_i^\vee), \pi(\alpha_j^\vee)\right),
\]
\[
= a_{ij},
\]
\[
= \alpha_i \cdot \alpha_j^\vee,
\]
\[
= \langle \alpha_i, \pi(\alpha_j^\vee) \rangle,
\]
where we have used Definition 9 and Equation (103). \(\square\)

Proposition 10. The bilinear form \((\cdot, \cdot)\) in the \(\{h_j \mid 1 \leq j \leq n\}\) basis of \(X_0\) is
\[
(h_i, h_j) = \sum_{k=1}^{n} \left(C(\Gamma)^T\right)^{-1}_{ik} \frac{2}{|\alpha_k|^2} \delta_{kj}, \quad 1 \leq i, j \leq n. \tag{107}
\]
Proof. From Equation (83) we have

\[
(h_i, h_j) = \left( \sum_{k=1}^{n} (C(\Gamma)^T)^{-1}_{ik} \pi(\alpha_k^\vee), h_j \right),
\]

\[
= \sum_{k=1}^{n} \left( (C(\Gamma)^T)^{-1}_{ik} \frac{2}{|\alpha_k|^2} \pi(\alpha_k), h_j \right),
\]

\[
= \sum_{k=1}^{n} \left( (C(\Gamma)^T)^{-1}_{ik} \frac{2}{|\alpha_k|^2} \langle \alpha, h_j \rangle \right),
\]

\[
= \sum_{k=1}^{n} (C(\Gamma)^T)^{-1}_{ik} \frac{2}{|\alpha_k|^2} \delta_{kj},
\]

where we have used Equation (80), Proposition 9 and Equation (72b). □

Remark 9. In Proposition 9, observe that for simply-laced type (ADE) root systems, where $|\alpha_k|^2 = 2$ (1 ≤ $k$ ≤ $n$) and $C(\Gamma)$ is symmetric, $|h_j|^2$ (1 ≤ $j$ ≤ $n$) are just the diagonal entries of $C(\Gamma)^{-1}$. We list $|h_j|^2$ (1 ≤ $j$ ≤ $n$) for all Weyl groups in Table 2 of Appendix A.

We now illustrate the properties of $W^{(1)}$ discussed so far for $B_3$, $C_3$, $F_4$ and $G_2$ type systems.

Example 8. Affine Weyl group of type $B_3$, $W(B_3^{(1)})$. The Dynkin diagram of type $B_3^{(1)}$, $\Gamma(B_3^{(1)})$ is given in Figure 6. The corresponding generalized Cartan matrix of type $B_3^{(1)}$,

\[
C(B_3^{(1)}) = (a_{ij})_{1 \leq i, j \leq 3} = (\alpha_i \cdot \alpha_j^\vee)_{1 \leq i, j \leq 3} = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix},
\]

and the defining relations for the Weyl group $W(B_3^{(1)}) = \langle s_i \mid 0 \leq i \leq 3 \rangle$,

\[
s_1^2 = s_2^2 = s_3^2 = 1, \quad (s_1s_2)^3 = 1, \quad (s_1s_3)^2 = 1, \quad (s_2s_3)^4 = 1, \quad s_0^2 = 1, \quad (s_0s_2)^3 = 1, \quad (s_0s_3)^2 = (s_0s_1)^2 = 1,
\]

Figure 6. Dynkin diagram of affine $B_3$ type, $\Gamma(B_3^{(1)})$.
can be written down using $\Gamma(B_3^{(1)})$ with the rules given in Table 7. The $B_3^{(1)}$ simple system
\[ \Delta^{(1)} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \]
forms a basis for an 4-dimensional real vector space $V^{(1)}$. Generators
$s_j$ act on $V^{(1)}$ by Equation (51), with $a_{ij}$ being the $(i,j)$-entry of $C(B_3^{(1)})$ given in Equation (108). The dual space $V^{(1)*}$ and its hyperplanes $X_k$ are given by Definitions 4 and 5, respectively. The set of simple coroots of $B_3$, $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee), \pi(\alpha_3^\vee)\}$ form a dual system of $C_3$ type (see Figure 7) and is a basis of $X_0 \subset V^{(1)*}$.

\[ \begin{array}{ccc}
\pi(\alpha_1^\vee) & \pi(\alpha_2^\vee) & \pi(\alpha_3^\vee) \\
\end{array} \]

**Figure 7.** Dynkin diagram for the dual system.

The group $W(B_3^{(1)})$ acts on $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 3\}$ by Proposition 2. By Equation (82), $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 3\}$ can be expressed in terms of the fundamental weights $\{h_j \mid 1 \leq j \leq 3\}$ by
\[
\begin{pmatrix}
\pi(\alpha_1^\vee) \\
\pi(\alpha_2^\vee) \\
\pi(\alpha_3^\vee)
\end{pmatrix} = C(B_3)^T
\begin{pmatrix}
h_1 \\
h_2 \\
h_3
\end{pmatrix}
= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}
\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},
\]

where $C(B_3)$ is given by Equation (20), or we have
\[
\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = (C(B_3)^T)^{-1}
\begin{pmatrix}
\pi(\alpha_1^\vee) \\
\pi(\alpha_2^\vee) \\
\pi(\alpha_3^\vee)
\end{pmatrix}
= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ 1 & 2 & \frac{3}{2} \end{pmatrix}
\begin{pmatrix}
\pi(\alpha_1^\vee) \\
\pi(\alpha_2^\vee) \\
\pi(\alpha_3^\vee)
\end{pmatrix}.
\]

The matrix of symmetric bilinear form $(\ , \ )$ on subspace $X_0$ in $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee), \pi(\alpha_3^\vee)\}$ basis
is given by Equation (103):
\[
\left((\pi(\alpha_i^\vee), \pi(\alpha_j^\vee))\right)_{1 \leq i,j \leq 3} = \left(\frac{2}{|\alpha_i|^2}a_{ij}\right)_{1 \leq i,j \leq 3}
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}
= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix}.
\]

The diagonal entries of the last matrix in Equation (112) tell us that $|\pi(\alpha_i^\vee)|^2$ for $1 \leq j \leq 3$ are 2, 2, and 4, respectively. The bilinear form in $\{h_j \mid 1 \leq j \leq 3\}$ basis of $X_0$ is given by Equation (107):
\[
\left((h_i, h_j)\right)_{1 \leq i,j \leq 3} = (C(B_3)^T)^{-1}
\left(\frac{2}{|\alpha_k|^2} \delta_{kj}\right)_{1 \leq k,j \leq 3}
= \left(\frac{2}{|\alpha_k|^2} \delta_{kj}\right)_{1 \leq k,j \leq 3},
\]

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ 1 & 2 & \frac{3}{2} \end{pmatrix}
= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}.
\]
The diagonal entries of the last matrix in Equation (113) tell us that $|h_j|^2$ for $1 \leq j \leq 3$ are 1, 2 and 3, respectively. That is, $h_1$ with $|h_1|^2 = 1$ is the shortest of the fundamental weights of the $B_3$ system.

Now let us consider the coroots of some non-simple roots of the finite $B_3$ root system (given in Equation (24)) using Proposition 7. In particular, consider the highest short and long roots of the $B_3$ system: $\tilde{\alpha}_s = \alpha_{123}$ and $\tilde{\alpha} = \alpha_{12233}$.

For $\alpha_s = \alpha_{123}$, we have $m_i = 1$ for $1 \leq i \leq 3$. Moreover, we have $|\alpha_1|^2 = |\alpha_2|^2 = 2$, and $|\alpha_3|^2 = |\tilde{\alpha}_s|^2 = 1$ from Equations (22) and (24). Then by Proposition 6 we have,

$$\pi(\tilde{\alpha}_s) = \pi(\alpha_{123}^\vee) = 2\pi(\alpha_1^\vee) + 2\pi(\alpha_2^\vee) + \pi(\alpha_3^\vee) = 2h_1,$$

(114)

where $|\pi(\tilde{\alpha}_s)|^2 = 4|h_1|^2 = 4$. That is, $\pi(\tilde{\alpha}_s)$ is a long root in the dual $C_3$ root system in $X_0$.

For $\tilde{\alpha} = \alpha_{12233}$, we have $m_1 = 1$, $m_2 = m_3 = 2$, and $|\alpha_{12233}|^2 = 2$. By Proposition 6 we have

$$\pi(\tilde{\alpha}) = \pi(\alpha_{12233}^\vee) = \frac{|\alpha_1|^2}{2} \pi(\alpha_1^\vee) + 2\frac{|\alpha_2|^2}{2} \pi(\alpha_2^\vee) + 2\frac{|\alpha_3|^2}{2} \pi(\alpha_3^\vee) = \pi(\alpha_1^\vee) + 2\pi(\alpha_2^\vee) + \pi(\alpha_3^\vee) = h_2,$$

(115)

where $|\pi(\tilde{\alpha})|^2 = |h_2|^2 = 2$. That is, $\pi(\tilde{\alpha})$ is a short root in the dual $C_3$ system in $X_0$. To express $\pi(\tilde{\alpha}_s)$ and $\pi(\tilde{\alpha})$ in terms of the fundamental weights we have used Equation (114). The last expression of Equation (115) can also be obtained using Proposition 8 and Equation (91),

$$\pi(\alpha_0^\vee) = \sum_{k=1}^3 (C(B_3^{(1)})_{k0} h_k = 0.h_1 + (-1).h_2 + 0.h_3 = -h_2,$$

(116)

with $C(B_3^{(1)})$ given by Equation (108).

Remark 10. Comparing the expressions for $\pi(\tilde{\alpha}_s)$ and $\pi(\tilde{\alpha})$ given in Equations (114) and (115) with roots of the finite $C_3$ system given in Equation (29), we see that the map $\pi$ takes $\tilde{\alpha}_s$ and $\tilde{\alpha}$ of $B_3$ type to the highest long and short root of the $C_3$ type dual system generated by $\{\pi(\alpha_i^\vee) | 1 \leq j \leq 3\}$, respectively.

Example 9. Affine Weyl group of type $C_3$, $W(C_3^{(1)})$. The Dynkin diagram of type $C_3^{(1)}$, $\Gamma(C_3^{(1)})$ is given in Figure 8.

\[ \begin{array}{ccc}
0 & \rightarrow & 1 \\
& \rightarrow & \leftarrow \\
& 2 & \leftarrow \end{array} \]

Figure 8. Dynkin diagram of affine $C_3$ type, $\Gamma(C_3^{(1)})$. 

30
The corresponding generalized Cartan matrix of type $C_3^{(1)}$, 

$$C(C_3^{(1)}) = (a_{ij})_{1 \leq i,j \leq 3,0} = (\alpha_i \cdot \alpha_j^\vee)_{1 \leq i,j \leq 3,0} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}, \quad (117)$$

and the defining relations for the Weyl group $W(C_3^{(1)}) = \langle s_i \mid 0 \leq i \leq 3 \rangle$,

$$s_1^2 = s_2^2 = s_3^2 = 1, \quad (s_1s_2)^2 = 1, \quad (s_1s_3)^2 = 1, \quad (s_2s_3)^4 = 1, \quad (s_0s_1)^4 = 1, \quad (s_0s_2)^2 = (s_0s_3)^2 = 1, \quad (118)$$

can be written down using $\Gamma(C_3^{(1)})$ with the rules given in Table 7. The $C_3^{(1)}$ simple system $\Delta^{(1)} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\}$ forms a basis for an 4-dimensional real vector space $V^{(1)}$. Generators $s_j$ act on $V^{(1)}$ by Equation (51), where $a_{ij}$ is the $(i,j)$-entry of $C(C_3^{(1)})$ given in Equation (117). The dual space $V^{(1)*}$ and its hyperplanes $X_k$ are given by Definitions 4 and 7 respectively. The set of simple coroots of $C_3$, $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee), \pi(\alpha_3^\vee)\}$ form a dual system of $B_3$ type in $X_0 \subset V^{(1)*}$ (see Figure 9). The group $W(C_3^{(1)})$ acts on $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 3\}$ by Proposition 2.

**Figure 9.** Dynkin diagram for the dual system.

By Equation (82), $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 3\}$ can be expressed in terms of the fundamental weights by

$$\begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \\ \pi(\alpha_3^\vee) \end{pmatrix} = C(C_3)^T \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad (120)$$

where $C(C_3)$ is given by Equation (25). Moreover, we have

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = (C(C_3)^T)^{-1} \begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \\ \pi(\alpha_3^\vee) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \frac{1}{2} & 1 & 3/2 \end{pmatrix} \begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \\ \pi(\alpha_3^\vee) \end{pmatrix}. \quad (121)$$

The matrix of symmetric bilinear form on the subspace $X_0 \subset V^{(1)*}$ in $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee), \pi(\alpha_3^\vee)\}$ basis is given by Equation (103):

$$\begin{pmatrix} \pi(\alpha_i^\vee), \pi(\alpha_j^\vee) \end{pmatrix}_{1 \leq i,j \leq 3} = \left(\begin{array}{ccc} \frac{2}{|\alpha_i|^2}a_{ij} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{array}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (122)$$
The diagonal entries of the last matrix in Equation (122) tell us that $|\pi(\alpha_j^\vee)|^2$ for $1 \leq j \leq 3$ are 2, 2, and 1, respectively. The bilinear form in the $\{h_j \mid 1 \leq j \leq 3\}$ basis of $X_0$ is given by Equation (107):

$$((h_i, h_j))_{1 \leq i, j \leq 3} = (C(C_3^T)^T)^{-1} \left( \frac{2}{|\alpha_k|^2} \delta_{kj} \right)_{1 \leq k, j \leq 3},$$

$$= \left( C(C_3^T)^T \right)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{4} \end{pmatrix}.\tag{123}$$

The diagonal entries of the matrix on the right in Equation (123) tell us that $|h_j|^2$ for $1 \leq j \leq 3$ are 1, 2 and $\frac{3}{4}$, respectively. Hence $h_3$ with $|h_3|^2 = \frac{3}{4}$ is the shortest of the fundamental weights of a $C_3$ system.

Using Proposition 6, we now compute the coroots of the highest short and long roots of the finite $C_3$ system, $\tilde{\alpha}_s = \alpha_{1223}$ and $\tilde{\alpha} = \alpha_{11223}$, respectively. For $\tilde{\alpha}_s = \alpha_{1223}$, we have $m_1 = 1$, $m_2 = 2$ and $m_3 = 1$. From Equations (28), and (29) we know that $|\alpha_1|^2 = |\alpha_2|^2 = 2$, $|\alpha_3|^2 = 4$ and $|\alpha_{1223}|^2 = 2$. Then by Proposition 7 we have,

$$\pi(\tilde{\alpha}_s^\vee) = \pi(\alpha_{1223}^\vee) = \pi(\alpha_1^\vee) + 2\pi(\alpha_2^\vee) + 2\pi(\alpha_3^\vee) = h_2,\tag{124}$$

where $|\pi(\tilde{\alpha}_s^\vee)|^2 = |h_2|^2 = 2$. That is $\pi(\tilde{\alpha}_s^\vee)$ is a long root in the dual $C_3$ system in $X_0$. For $\tilde{\alpha} = \alpha_{11223}$, we have $m_1 = m_2 = 2$, $m_3 = 1$ and $|\alpha_{11223}|^2 = 4$. Then by Proposition 7 we have

$$\pi(\tilde{\alpha}^\vee) = \pi(\alpha_{11223}^\vee) = 2\frac{|\alpha_1|^2}{4}\pi(\alpha_1^\vee) + 2\frac{|\alpha_2|^2}{4}\pi(\alpha_2^\vee) + \frac{|\alpha_3|^2}{4}\pi(\alpha_3^\vee) = \pi(\alpha_1^\vee) + \pi(\alpha_2^\vee) + \pi(\alpha_3^\vee) = h_1,\tag{125}$$

where $|\pi(\tilde{\alpha}^\vee)|^2 = |h_1|^2 = 1$. That is $\pi(\tilde{\alpha}^\vee)$ is a short root in the dual $B_3$ system in $X_0$. To express $\pi(\tilde{\alpha}_s^\vee)$ and $\pi(\tilde{\alpha}^\vee)$ in terms of the fundamental weight we have used Equation (121). The last expression on the right of Equation (125) can also be obtained using Proposition 8 and Equation (91),

$$\pi(\alpha_0^\vee) = \sum_{k=1}^{3} \left( C(C_3^{(1)}) \right)_{k0} h_k = (-1) h_1 + 0 h_2 + 0 h_3 = -h_1,\tag{126}$$

with $C(C_3^{(1)})$ given by Equation (117).
Example 10. Affine Weyl group of type $F_4$, $W(F_4^{(1)})$. The Dynkin diagram of type $F_4^{(1)}$, $\Gamma(F_4^{(1)})$ is given in Figure 1. The corresponding generalized Cartan matrix of type $F_4^{(1)}$,

$$C(F_4^{(1)}) = (a_{ij})_{1\leq i,j\leq 4,0} = (\alpha_i \cdot \alpha_j)_{1\leq i,j\leq 4,0} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix},$$

and the defining relations for the Weyl group $W(F_4^{(1)}) = \{s_i \mid 0 \leq i \leq 4, \}$

$$s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1 s_3)^2 = (s_1 s_4)^2 = 1, \quad (s_1 s_2)^3 = (s_2 s_3)^4 = (s_3 s_4)^3 = 1$$

$$s_0^2 = 1, \quad (s_0 s_1)^3 = 1, \quad (s_0 s_2)^2 = (s_0 s_3)^2 = (s_0 s_4)^2 = 1,$$

can be written down using $\Gamma(F_4^{(1)})$ with the rules given in Table 7. The $F_4^{(1)}$ simple system $\Delta^{(1)} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0\}$ forms a basis for an 5-dimensional real vector space $V^{(1)}$. Generators $s_j$ act on $V^{(1)}$ by Equation (51), where $a_{ij}$ is the $(i,j)$-entry of $C(F_4^{(1)})$ given in Equation (127). The dual space $V^{(1)*}$ and its hyperplanes $X_k$ are given by Definitions 4 and 5 respectively. The set of simple coroots of $F_4$, $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee), \pi(\alpha_3^\vee), \pi(\alpha_4^\vee)\}$ form a dual system again of $F_4$ type (see Figure 10) and is a basis of $X_0 \subset V^{(1)*}$. The group $W(F_4^{(1)}$)

![Dynkin diagram](image)

**Figure 10.** Dynkin diagram for the dual system.

acting on $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 4\}$ given by Proposition 7. By Equation (62), $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 4\}$ can be expressed in terms of the fundamental weights $\{h_j \mid 1 \leq j \leq 4\}$ by

$$\begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \\ \pi(\alpha_3^\vee) \\ \pi(\alpha_4^\vee) \end{pmatrix} = C(F_4)^T \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix},$$

where $C(F_4)$ is given by Equation (30), or we have

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = (C(F_4)^T)^{-1} \begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \\ \pi(\alpha_3^\vee) \\ \pi(\alpha_4^\vee) \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \\ \pi(\alpha_3^\vee) \\ \pi(\alpha_4^\vee) \end{pmatrix}. $$
The matrix of symmetric bilinear form $\langle \cdot , \cdot \rangle$ on subspace $X_0$ in $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee), \pi(\alpha_3^\vee), \pi(\alpha_4^\vee)\}$ basis is given by Equation (103):

$$
\left(\pi(\alpha_i^\vee), \pi(\alpha_j^\vee)\right)_{1 \leq i,j \leq 4} = \left(\frac{2}{|\alpha_i|^2} a_{ij}\right)_{1 \leq i,j \leq 4},
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
= \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}.
$$

The diagonal entries of the last matrix in Equation (131) tell us that $|\pi(\alpha_j^\vee)|^2$ (for $1 \leq j \leq 4$) are 2, 2, 4 and 4, respectively. That is, $\pi(\alpha_3^\vee)$ and $\pi(\alpha_4^\vee)$ are long while $\pi(\alpha_1^\vee)$ and $\pi(\alpha_2^\vee)$ are short as indicated by the Dynkin diagram in Figure 17.

The bilinear form in $\{h_j \mid 1 \leq j \leq 4\}$ basis of $X_0$ is given by Equation (107):

$$
\left(h_i, h_j\right)_{1 \leq i,j \leq 4} = (C(F_4)^T)^{-1} \left(\frac{2}{|\alpha_k|^2} \delta_{kj}\right)_{1 \leq k,j \leq 4},
$$

$$
= (C(F_4)^T)^{-1} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
= \begin{pmatrix}
2 & 3 & 2 & 1 \\
3 & 6 & 4 & 2 \\
4 & 8 & 6 & 3 \\
2 & 4 & 3 & 2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
= \begin{pmatrix}
2 & 3 & 4 & 2 \\
3 & 6 & 8 & 4 \\
4 & 8 & 12 & 6 \\
2 & 4 & 6 & 4
\end{pmatrix}.
$$

The diagonal entries of the last matrix in Equation (132) tell us that $|h_j|^2$ for $1 \leq j \leq 4$ are 2, 6, 12 and 4, respectively. That is, $h_1$ with $|h_1|^2 = 2$ is the shortest of the fundamental weights of the $F_4$ system.

Now let us consider the coroots of some non-simple roots of the finite $F_4$ root system using Proposition 4. In particular, consider the highest short and long roots of the $F_4$ system: $\check{\alpha}_s = \alpha_{12233344}$ and $\check{\alpha} = \alpha_{11222333344}$. For $\check{\alpha}_s = \alpha_{12233344}$, we have $m_i$ for $1 \leq i \leq 4$ are 1, 2, 3, 2, respectively. Moreover, we have $|\alpha_1|^2 = |\alpha_2|^2 = 2$, $|\alpha_3|^2 = |\alpha_4|^2 = |\check{\alpha}_s|^2 = 1$ from Equation (32). Then by Proposition 4, we have

$$
\pi(\check{\alpha}_s^\vee) = \pi(\alpha_{12233344}^\vee) = (1 \cdot 2)\pi(\alpha_1^\vee) + (2 \cdot 2)\pi(\alpha_2^\vee) + (3 \cdot 1)\pi(\alpha_3^\vee) + (2 \cdot 1)\pi(\alpha_4^\vee)
= 2\pi(\alpha_1^\vee) + 4\pi(\alpha_2^\vee) + 3\pi(\alpha_3^\vee) + 2\pi(\alpha_4^\vee) = h_4.
$$

Hence we have $|\pi(\check{\alpha}_s^\vee)|^2 = |h_4|^2 = 4$, that is $\pi(\check{\alpha}_s^\vee)$ is a long root in the dual $F_4$ system in $X_0$.

For $\check{\alpha} = \alpha_{11222333344}$, we have $|\check{\alpha}|^2 = 2$, and $m_i$ for $1 \leq i \leq 4$ are 2, 3, 4, 2, respectively. By Proposition 4, we have

$$
\pi(\check{\alpha}^\vee) = \pi(\alpha_{11222333344}^\vee) = (2 \cdot 1)\pi(\alpha_1^\vee) + (3 \cdot 1)\pi(\alpha_2^\vee) + (4 \cdot 1/2)\pi(\alpha_3^\vee) + (2 \cdot 1/2)\pi(\alpha_4^\vee)
= 2\pi(\alpha_1^\vee) + 3\pi(\alpha_2^\vee) + 2\pi(\alpha_3^\vee) + \pi(\alpha_4^\vee) = h_1.
$$
and $|\pi(\tilde{\alpha}^\vee)|^2 = |h_1|^2 = 2$. That is, $\pi(\tilde{\alpha}^\vee)$ is a short root in the dual $F_4$ system in $X_0$. To express $\pi(\tilde{\alpha}^\vee)$ and $\pi(\tilde{\alpha}^\vee)$ in terms of the fundamental weights we have used Equation (130). The last expression of Equation (134) can also be obtained using Proposition 8 and Equation (91),

$$\pi(\alpha_0^\vee) = \frac{4}{\sum_{k=0}^4 (C(F_4^{(1)})) h_k = (-1)h_1 + 0h_2 + 0h_3 + 0h_4 = -h_1, \quad (135)$$

with $C(F_4^{(1)})$ given in Equation (127).

**Remark 11.** Comparing the expressions for $\pi(\tilde{\alpha}_4^\vee)$ and $\pi(\tilde{\alpha}^\vee)$ given in Equations (133) and (134) with roots of the finite $F_4$ system in Example 3, we see that the map $\pi$ takes $\tilde{\alpha}_4^\vee$ and $\tilde{\alpha}^\vee$ of $F_4$ type to the highest long and short root of the $F_4$ type dual system generated by $\{\pi(\alpha_j^\vee) \mid 1 \leq j \leq 4\}$, respectively.

**Example 11.** Affine Weyl group of type $G_2$, $W(G_2^{(1)})$. The Dynkin diagram of type $G_2^{(1)}$, $\Gamma(G_2^{(1)})$ is given in Figure 11. The corresponding generalized Cartan matrix of type $G_2^{(1)}$,

$$C(G_2^{(1)}) = (a_{ij})_{1 \leq i,j \leq 2,0} = (\alpha_i \cdot \alpha_j^\vee)_{1 \leq i,j \leq 2,0} = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad (136)$$

and the defining relations for the Weyl group $W(G_2^{(1)}) = \langle s_i \mid 0 \leq i \leq 2 \rangle$,

$$s_1^2 = s_2^2 = 1, \quad (s_1s_2)^6 = 1, \quad (137)$$

$$s_2^2 = 1, \quad (s_0s_1)^2 = 1, \quad (s_0s_2)^3 = 1, \quad (138)$$

can be read off from $\Gamma(G_2^{(1)})$ with the rules given in Table 4. The $G_2^{(1)}$ simple system $\Delta^{(1)} = \{\alpha_1, \alpha_2, \alpha_0\}$ forms a basis for an 3-dimensional real vector space $V^{(1)}$. Generators $s_j$ act on $V^{(1)}$ by Equation (51), where $a_{ij}$ is the $(i,j)$-entry of $C(G_2^{(1)})$ given in Equation (136). The dual space $V^{(1)*}$ and its hyperplanes $X_k$ are given by Definitions 4 and 8 respectively. The set of simple coroots of $G_2$, $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee)\}$ form a dual system also of $G_2$ type (see Figure 11) and is a basis of $X_0 \subset V^{(1)*}$. The group $W(G_2^{(1)})$ acts on $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee)\}$ by Proposition

[4] By Equation (52), $\{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee)\}$ can be expressed in terms of the fundamental weights by

$$\begin{pmatrix} \pi(\alpha_1^\vee) \\ \pi(\alpha_2^\vee) \end{pmatrix} = C(G_2) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad (139)$$

**Figure 11.** Dynkin diagram for the dual system.
where \( C(G_2) \) is given by Equation (36). Moreover, we have

\[
\begin{pmatrix}
    h_1 \\
    h_2
\end{pmatrix} = (C(G_2)^T)^{-1} \begin{pmatrix}
    \pi(\alpha_1^\vee) \\
    \pi(\alpha_2^\vee)
\end{pmatrix} = \begin{pmatrix}
    2 & 3 \\
    1 & 2
\end{pmatrix} \begin{pmatrix}
    \pi(\alpha_1^\vee) \\
    \pi(\alpha_2^\vee)
\end{pmatrix}. \tag{140}
\]

The matrix of symmetric bilinear form on the subspace \( X_0 \subset V^{(1)*} \) in \( \{\pi(\alpha_1^\vee), \pi(\alpha_2^\vee)\} \) basis is given by Equation (103):

\[
\left( (\pi(\alpha_i^\vee), \pi(\alpha_j^\vee)) \right)_{1 \leq i,j \leq 2} = \left( \frac{2}{|\alpha_i|^2} a_{ij} \right)_{1 \leq i,j \leq 2} = \begin{pmatrix}
    1 & 0 \\
    0 & \frac{4}{3}
\end{pmatrix} \begin{pmatrix}
    2 & -1 \\
    -3 & 2
\end{pmatrix} = \begin{pmatrix}
    2 & -1 \\
    -1 & \frac{2}{3}
\end{pmatrix}. \tag{141}
\]

The diagonal entries of the last matrix in Equation (141) tell us that \( |\pi(\alpha_1^\vee)|^2 = 2 \) and \( |\pi(\alpha_2^\vee)|^2 = 2/3 \). The bilinear form in the \( \{h_1, h_2\} \) basis of \( X_0 \) is given by Equation (107):

\[
\left( (h_i, h_j) \right)_{1 \leq i,j \leq 2} = (C(G_2)^T)^{-1} \begin{pmatrix}
    1 & 0 \\
    0 & \frac{4}{3}
\end{pmatrix} = \begin{pmatrix}
    2 & 3 \\
    1 & 2
\end{pmatrix} \begin{pmatrix}
    1 & 0 \\
    0 & \frac{4}{3}
\end{pmatrix} = \begin{pmatrix}
    2 & 1 \\
    1 & \frac{2}{3}
\end{pmatrix}. \tag{142}
\]

The diagonal entries of the matrix on the right in Equation (142) tell us that \( |h_1|^2 = 2 \) and \( |h_2|^2 = 2/3 \). Hence \( h_2 \) is the shortest fundamental weight of the \( G_2 \) system.

Using Proposition 6 we compute the coroots of the highest short root \( \tilde{\alpha}_s = \alpha_{112} \), and long root \( \tilde{\alpha} = \alpha_{11122} \), of the finite \( G_2 \) system given in Equation (40).

For \( \tilde{\alpha}_s = \alpha_{112} \), we have \( m_1 = 2 \) and \( m_2 = 1 \). From Example 4 we know that \( |\alpha_1|^2 = 2 \), \( |\alpha_2|^2 = 6 \) and \( |\alpha_{1112}|^2 = 2 \). Then by Proposition 7 we have

\[
\pi(\tilde{\alpha}_s^\vee) = \pi(\alpha_{112}^\vee) = (2 \cdot 1)\pi(\alpha_1^\vee) + (1 \cdot 3)\pi(\alpha_2^\vee) = 2\pi(\alpha_1^\vee) + 3\pi(\alpha_2^\vee) = h_1, \tag{143}
\]

so \( |\pi(\tilde{\alpha}_s^\vee)|^2 = |h_1|^2 = 2 \). That is \( \pi(\tilde{\alpha}_s^\vee) \) is a long root in the dual \( G_2 \) system in \( X_0 \). For \( \tilde{\alpha} = \alpha_{11122} \), we have \( m_1 = 3 \), \( m_2 = 2 \) and \( |\alpha_{11122}|^2 = 6 \). Then by Proposition 7 we have

\[
\pi(\tilde{\alpha}^\vee) = \pi(\alpha_{11122}^\vee) = (3 \cdot \frac{1}{3})\pi(\alpha_1^\vee) + (2 \cdot 1)\pi(\alpha_2^\vee) = \pi(\alpha_1^\vee) + 2\pi(\alpha_2^\vee) = h_2, \tag{144}
\]

where \( |\pi(\tilde{\alpha}^\vee)|^2 = |h_2|^2 = 2/3 \). That is \( \pi(\tilde{\alpha}^\vee) \) is a short root in the dual \( G_2 \) system in \( X_0 \). To express \( \pi(\tilde{\alpha}_s^\vee) \) and \( \pi(\tilde{\alpha}^\vee) \) in terms of the fundamental weight we have used Equation (140). The last expression on the right of Equation (143) can also be obtained using Proposition 8 and Equation (91),

\[
\pi(\alpha_0^\vee) = \sum_{k=1}^{2} \left( C(G_2^{(1)}) \right)_{k0} h_k = 0. h_1 + (-1). h_2 = -h_2, \tag{145}
\]

with \( C(G_2^{(1)}) \) given by Equation (36).

3.4. Translations. We are now ready to investigate actions of \( W^{(1)} \) on the dual space \( V^{(1)*} \). In particular, we construct certain elements of \( W^{(1)} \) which act as translations on the affine subspace \( X_1 \subset V^{(1)*} \).
Proposition 11. For any root of the affine root system: \( \beta = \alpha + m\delta \in \Phi^{(1)} \), where \( 0 \neq m \in \mathbb{Z} \), and \( \alpha \in \Phi \), we form the element \( t_{\beta} = s_{\alpha}s_{\beta} \in W^{(1)} \). We have
\[
t_{\beta}(h) = h, \quad \text{(146)}
\]
for all \( h \in X_0 \). That is, \( t_{\beta} \in W^{(1)} \) fixes all elements of \( X_0 \).

Proof. Let \( t_{\beta} = s_{\alpha}s_{\beta} \) for any \( \beta = \alpha + m\delta \in \Phi^{(1)} \) \((0 \neq m \in \mathbb{Z}, \text{ and } \alpha \in \Phi)\). First, observe that for \( \alpha \in \Phi \) we have
\[
s_{\alpha}(h) \in X_0, \quad h \in X_0, \quad \text{(147)}
\]
since
\[
\langle \delta, s_{\alpha}(h) \rangle = \langle s_{\alpha}(\delta), h \rangle = \langle \delta, h \rangle = 0, \quad \text{(148)}
\]
that is \( s_{\alpha} \in W \) preserves \( X_0 \).

Now by Equation (86), for \( h \in X_0 \) we have
\[
s_{\beta}(h) = h - \langle \alpha + m\delta, h \rangle \pi(\beta^\vee) = h - \langle \alpha + m\delta, h \rangle \pi(\alpha^\vee) = h - \langle \alpha, h \rangle \pi(\alpha^\vee) = s_{\alpha}(h). \quad \text{(149)}
\]

Then we have,
\[
t_{\beta}(h) = s_{\alpha}s_{\beta}(h) = s_{\alpha}s_{\alpha}(h) = h, \quad \text{(150)}
\]
for all \( h \in X_0 \). \( \square \)

Proposition 12. For any \( \beta = \alpha + m\delta \in \Phi^{(1)} \) \((0 \neq m \in \mathbb{Z}, \text{ and } \alpha \in \Phi)\), the element \( t_{\beta} \in W^{(1)} \) acts on \( h_\delta \in V^{(1)*} \) by
\[
t_{\beta}(h_\delta) = h_\delta + \langle \beta, h_\delta \rangle \pi(\beta^\vee). \quad \text{(151)}
\]

Proof. By Equation (3) we have,
\[
s_{\alpha}(h_\delta) = h_\delta - \langle \alpha, h_\delta \rangle \pi(\beta^\vee) = h_\delta, \quad \text{(152)}
\]
where \( \langle \alpha, h_\delta \rangle = 0 \) by Definition (4) since \( \alpha \in \Phi \) is a linear combination of \( \{\alpha_i|1 \leq i \leq n\} \). Moreover,
\[
s_{\beta}(h_\delta) = h_\delta - \langle \beta, h_\delta \rangle \pi(\beta^\vee). \quad \text{(153)}
\]
By Equation (86) we have
\[
s_{\alpha}(\pi(\beta^\vee)) = \pi(\beta^\vee) - \langle \alpha, \pi(\beta^\vee) \rangle \pi(\beta^\vee)
= \pi(\beta^\vee) - \langle \alpha, \pi(\alpha^\vee) \rangle \pi(\beta^\vee)
= \pi(\beta^\vee) - 2\pi(\beta^\vee)
= -\pi(\beta^\vee),
\]

37
where we have used \( \langle \alpha, \pi(\alpha^\vee) \rangle = \alpha \cdot \alpha^\vee = \frac{2\alpha \cdot \alpha}{\alpha \cdot \alpha} = 2 \). Hence we found that,
\[
t_\beta(h_\delta) = (s_\alpha s_\beta)(h_\delta) = s_\alpha(h_\delta - \langle \beta, h_\delta \rangle \pi(\beta^\vee)) = h_\delta - \langle \beta, h_\delta \rangle s_\alpha(\pi(\beta^\vee)) = h_\delta + \langle \beta, h_\delta \rangle \pi(\beta^\vee). \tag{154}
\]

The dual space \( V^{(1)*} \) is the direct sum \( \mathbb{R} h_\delta \oplus X_0 \), actions of \( t_\beta \) on \( V^{(1)*} \) and \( V^{(1)} \) (by the contragredient action) in general are given by the following Proposition.

**Proposition 13.** The element \( t_\beta = s_\alpha s_\beta \in W^{(1)} \) for any \( \beta = \alpha + m \delta \in \Phi^{(1)} \) \((0 \neq m \in \mathbb{Z}, \text{ and } \alpha \in \Phi)\) acts on \( V^{(1)*} \) and \( V^{(1)} \) by
\[
t_\beta(f) = f + \langle \delta, f \rangle \langle \beta, h_\delta \rangle \pi(\beta^\vee) = f + \langle \delta, f \rangle m \pi(\alpha^\vee), \quad \text{for all } f \in V^{(1)*},
\]
and
\[
t_\beta(v) = v - \langle \beta, h_\delta \rangle \langle v, \pi(\beta^\vee) \rangle \delta = v - m \langle v, \pi(\alpha^\vee) \rangle \delta, \quad \text{for all } v \in V^{(1)},
\]
respectively.

*Proof.* Let \( f \) be an arbitrary element of \( V^{(1)*} \) then \( f_0 = f - \langle \delta, f \rangle h_\delta \) is an element of the subspace \( X_0 \) since \( \langle \delta, f_0 \rangle = \langle \delta, f \rangle - \langle \delta, f \rangle \langle \delta, h_\delta \rangle = 0 \). It follows from Propositions 11 and 12 that for \( \beta = \alpha + m \delta \in \Phi^{(1)} \) \((0 \neq m \in \mathbb{Z}, \text{ and } \alpha \in \Phi)\),
\[
t_\beta(f) = t_\beta(f_0) + \langle \delta, f \rangle t_\beta(h_\delta),
\]
\[
= f_0 + \langle \delta, f \rangle \langle h_\delta + \langle \beta, h_\delta \rangle \pi(\beta^\vee) \rangle,
\]
\[
= f + \langle \delta, f \rangle \langle \beta, h_\delta \rangle \pi(\beta^\vee),
\]
\[
= f + \langle \delta, f \rangle m \pi(\alpha^\vee), \tag{157}
\]
for all \( f \in V^{(1)*} \).

Action of \( t_\beta \) on \( v \in V^{(1)} \) can be obtained using Equation (157) and the contragredient action of \( W^{(1)} \). That is,
\[
\langle t_\beta^{-1}(v), f \rangle = \langle v, t_\beta(f) \rangle,
\]
\[
= \langle v, f + \langle \delta, f \rangle m \pi(\alpha^\vee) \rangle,
\]
\[
= \langle v, f \rangle + \langle \delta, f \rangle m \langle v, \pi(\alpha^\vee) \rangle,
\]
\[
= \langle v + m \langle v, \pi(\alpha^\vee) \rangle, \delta, f \rangle,
\]
so \( t_\beta^{-1}(v) = v + m \langle v, \pi(\alpha^\vee) \rangle \delta \) or
\[
t_\beta(v) = v - m \langle v, \pi(\alpha^\vee) \rangle \delta, \tag{158}
\]
for all \( v \in V^{(1)} \).  \( \square \)
Recall that we have $n$-dimensional affine spaces $X_k = k h + X_0$ in $V^{(1)*}$ on which $t_\beta$ acts as a translation by Equation (155). Since for every non-zero value of $k$, $X_k$ behaves like every other, we henceforth consider only $X_1$. Moreover, it is useful to have the actions of $t_\beta$ on the \{$\alpha_1, \alpha_2, ..., \alpha_0$\} basis of $V^{(1)}$. Hence we have the following.

**Proposition 14.** The element $t_\beta = s_\alpha s_\beta \in W^{(1)}$ for any $\beta = \alpha + m\delta \in \Phi^{(1)}$ ($0 \neq m \in \mathbb{Z}$, and $\alpha \in \Phi$) acts on $f \in X_1$ by

$$t_\beta(f) = f + m \sum_{j=1}^{n} b_j h_j,$$

and on the \{$\alpha_1, \alpha_2, ..., \alpha_0$\} basis of $V^{(1)}$ by

$$t_\beta(\alpha_i) = \alpha_i - mb_i \delta,$$

where $b_i = \langle \alpha_i, \pi(\beta^\vee) \rangle$ for $0 \leq i \leq n$.

**Proof.** By Proposition 13 having $\langle f, \delta \rangle = 1$ for all $f \in X_1$, gives us

$$t_\beta(f) = f + \langle \beta, h_\delta \rangle \pi(\beta^\vee) = f + m \sum_{j=1}^{n} b_j h_j, \quad b_j = \langle \alpha_j, \pi(\beta^\vee) \rangle,$$

where we have used Proposition 7. The actions of $t_\beta$ on the \{$\alpha_1, \alpha_2, ..., \alpha_0$\} basis are given by Equation (156),

$$t_\beta(\alpha_i) = \alpha_i - m\langle \alpha_i, \sum_{j=1}^{n} b_j h_j \rangle \delta = \alpha_i - mb_i \delta, \quad \text{for all} \quad 0 \leq i \leq n.$$

$$\square$$

3.4.1. **Translations by simple coroots.** Here we apply Proposition 13 to write down some explicit examples of translations in $W^{(1)}$.

**Proposition 15.** Let

$$t_j = t_{\beta_j} = s_{\alpha_j} s_{\beta_j}, \quad \text{where} \quad \beta_j = \alpha_j + \delta \quad \text{for} \quad 1 \leq j \leq n, \quad \text{and} \quad \beta_0 = \alpha_0.$$

Element $t_j$ (0 $\leq j \leq n$) acts on $f \in X_1$ by

$$t_j(f) = f + \pi(\alpha_j^\vee) = f + \sum_{k=1}^{n} a_{kj} h_k,$$

and on \{$\alpha_1, \alpha_2, ..., \alpha_0$\} basis of $V^{(1)}$ by

$$t_j(\alpha_i) = \alpha_i - a_{ij} \delta,$$

where $0 \leq i, j \leq n$, and $a_{ij}$ is the $(i, j)$-entry of $C(\Gamma^{(1)})$.

**Proof.** In Proposition 14 for $\beta_j$ (1 $\leq j \leq n$) we have $\beta = \alpha_j + \delta$, that is, $\alpha = \alpha_j$ and $m = 1$, and $t_j = t_{\beta_j} = t_{\alpha_j + \delta} = t_\beta = s_\alpha s_\beta = s_{\alpha_j} s_{\beta_j}$. 

39
For $\beta_0 = \alpha_0$, we have $\beta = \alpha_0 = -\tilde{\alpha} + \delta$, that is, $\alpha = -\tilde{\alpha}$, $m = 1$ and

$$t_0 = t_{\alpha_0} = t_\beta = s_\beta s_\alpha = s_{-\tilde{\alpha}} s_{\alpha_0} = s_{\tilde{\alpha}} s_{\alpha_0} = s_{\tilde{\alpha}} s_0.$$  

(166)

Element $t_j$ $(0 \leq j \leq n)$ acts on $f \in X_1$ by

$$t_j(f) = f + \pi(\alpha_j') = f + \sum_{k=1}^{n} a_{kj} h_k,$$  

(167)

where we have used Proposition 8. Its action on $\{\alpha_1, \alpha_2, ..., \alpha_0\}$ basis of $V(1)$ is given by Equation (160), we have

$$t_j(\alpha_i) = \alpha_i - (\alpha_i, \pi(\alpha_j')) \delta = \alpha_i - a_{ij} \delta, \quad \text{for all } 0 \leq i, j \leq n. \quad (168)$$

\[\square\]

Remark 12. Recall that the group $W(1)$ acts transitively on roots in $\Phi(1)$ (of the same length) and hence also on the coroots (of the same length) via the contragredient action. That is, it is enough to write down an element of translation by a simple coroot of the simply-laced system, and two simple coroots of different lengths for the non-simply-laced system. In particular, elements of translations are related by the formula given in Equation (166).

We end this section with a discussion on the expression of $T_1 \in W(E_8(1))$ given earlier in Equation (160) for Sakai’s $e \cdot P(E_8(1))$ equation employing the formulas discussed in Equation (3.4).

Example 12. Let $\Delta(1) = \{\alpha_j \mid 0 \leq j \leq 8\}$ be the $E_8$ simple system with the numbering on $\Gamma(E_8(1))$ in Figure 1. We have $W(E_8(1)) = \langle s_i \mid 0 \leq i \leq 8 \rangle$ and $\Phi(1)$ is the $E_8(1)$ root system. An element $T_1$ of $W(E_8(1))$ given as a product of 58 simple reflections,

$$T_1 = s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20} s_{21} s_{22} s_{23} s_{24} s_{25} s_{26} s_{27} s_{28} s_{29} s_{30} s_{31} s_{32} s_{33} s_{34} s_{35} s_{36} s_{37} s_{38} s_{39} s_{40} s_{41} s_{42} s_{43} s_{44} s_{45} s_{46} s_{47} s_{48} s_{49} s_{50} s_{51} s_{52} s_{53} s_{54} s_{55} s_{56} s_{57} s_{58} s_{59} s_{60} s_{61} s_{62} s_{63} s_{64} s_{65} s_{66} s_{67} s_{68} s_{69} s_{70} s_{71} s_{72} s_{73} s_{74} s_{75} s_{76} s_{77} s_{78} s_{79} s_{80} s_{81},$$

(169)

is said to be a translation by $\alpha_1$ in $W(E_8(1))$. The actions of $T_1$ on the simple roots $\{\alpha_j \mid 0 \leq j \leq 8\}$ can be computed using Equation (169) by composing the actions of simple reflections given in Equation (160) with $C(E_8(1))$ from Equation (3.4), where the compositions are taken from right to left,

$$T_1 : \{\alpha_1, \alpha_3\} \mapsto \{\alpha_1 - 2\delta, \alpha_3 + \delta\},$$

(170)

and they coincide with those given in Equation (160). On the other hand, by Proposition 12, for $j = 1$ we have translation of $\alpha_1$ in $W(E_8(1))$ given by,

$$t_1 = t_{\alpha_1 + \delta} = s_{\alpha_1} s_{\alpha_1 + \delta},$$

(171)

where $\delta$ is given by Equation (3.4). The action of $t_1$ on $\Delta(1)$ is given by Equation (165) with the $\alpha_{ij}$’s from $C(E_8(1))$. Since the only non-zero entries in the first column of $C(E_8(1))$
are $a_{11} = 2$ and $a_{31} = -1$, we have

$$t_1(\alpha_1) = \alpha_1 - 2\delta, \quad t_1(\alpha_3) = \alpha_3 + \delta, \quad \text{and} \quad t_1(\alpha_i) = \alpha_i, \quad \text{for} \quad i \neq 1, 3,$$

(172) which agrees with the action computed in Equation (170).

The expression for $T_1$ in Equation (169) is given as a product of simple reflections of the form: $58 = 28 + 1 + 28 + 1$. On the other hand, our formula for $t_1$ in Equation (171) is given as a product of two reflections, along the two roots $\alpha_1$, and $\alpha_1 + \delta \in \Phi^{(1)}$. The element $t_1$ given in Equation (171) can be rewritten into the form given in Equation (169) following a series of observations. First, we have

$$t_1 = t_{\alpha_1 + \delta} = t_{\alpha_1 - \delta}^{-1} = s_{\alpha_1} s_{\alpha_1 - \delta}^{-1} = s_{\alpha_1 - \delta} s_{\alpha_1} = s_{\delta - \alpha_1} s_{\alpha_1},$$

(173) where we have used Equation (13). Next, we find an expression for $s_{\delta - \alpha_1}$. Recall that $W^{(1)}$ acts transitively on the roots of $\Phi^{(1)}$, that is there is an $w \in W(E_8^{(1)})$ such that

$$w(\alpha_1) = \delta - \alpha_1.$$  

(174) By [7, Lemma 4.4], we have the length of $w$ is equal to the height of $\delta - \alpha_1$ minus 1. Recall that the height of a root is given earlier in Definition 1. That is, the height of $\delta - \alpha_1$ is given by,

$$1 + \sum_{i=1}^{8} c_i - 1 = 1 + 29 - 1 = 29.$$  

(175) So we have,

$$l(w) = 29 - 1 = 28.$$  

(176) An algorithm for finding an expression of $w$ is also given in [7] and we have,

$$w = s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_2 s_6 s_7 s_6 s_5 s_4 s_3,$$

(177) where $l(w) = 28$ and $w^2 = 1$. Then by Equation (53) we have $s_{\delta - \alpha_1} = w s_1 w^{-1}$, and finally,

$$t_1 = s_{\delta - \alpha_1} s_{\alpha_1} = w s_1 w^{-1} s_1 = w s_1 w s_1,$$

(178) the second expression from the right coincides with the expression given in Equation (169).

**Remark 13.** Note that a different, geometrically motivated expression for $t_1 \in W(E_8^{(1)})$ is given in [19].

**Remark 14.** The relation between the height of a root $\beta \in \Phi^{(1)}_+$ and the length of the element $w$ such that $w(\alpha_i) = \beta$ (where $\alpha_i$ is a simple root) is given for general Coxeter groups and their root systems in [7], where the concept of the height of a root is generalised to the depth of a root.
3.5. **A normal subgroup of translations**, \(W^{(1)} = W \ltimes Q\). In the previous section, we constructed elements of translation \(t_\beta\) (for any \(\beta \in \Phi^{(1)}\)) and looked at in particular translations by simple coroots of \(W^{(1)}\).

Now, using a representation \(R\) of \(W^{(1)}\) on the dual vector space \(V^{(1)*}\) we investigate the properties of set of all translations in \(W^{(1)}\) and show that it forms an abelian normal subgroup of \(W^{(1)}\).

**Definition 10.** Let \(R\) be a representation of \(W^{(1)}\) on \(V^{(1)*}\). That is, if \(w \in W^{(1)}\) then \(R(w)\) is the linear transformation of \(V^{(1)*}\) given by

\[(R(w))(f) = w(f) \quad \text{for all } f \in V^{(1)*}. \quad (179)\]

The translational transformations of \(R\), \(F_h : V^{(1)*} \to V^{(1)*}\) are given by

\[F_h(f) = f + \langle \delta, f \rangle h \quad \text{for each } h \in X_0 \text{ and for all } f \in V^{(1)*}. \quad (180)\]

Comparing Equation (180) with the action of \(t_\beta\) on \(f \in V^{(1)*}\) given in Equation (155) we see that

\[R(t_\beta) = F_h, \quad \text{with} \quad h = \langle \beta, h_\delta \rangle \pi(\beta^\vee) = m\pi(\alpha^\vee), \quad (181)\]

for \(\beta = \alpha + m\delta \in \Phi^{(1)}, \alpha \in \Phi, m \in \mathbb{Z}\).

**Proposition 16.** Let

\[T = \{t_\beta \mid \beta \in \Phi^{(1)}\}. \quad (182)\]

\(T\) is a finitely generated abelian normal subgroup of translations on the root lattice of \(W^{(1)}\),

\[W^{(1)} = W \ltimes T = W \ltimes \langle t_j \mid 1 \leq j \leq n \rangle = W \ltimes Q, \quad (183)\]

where \(t_j\) are given by Proposition 15.

**Proof.** Let \(h, k \in X_0\) and \(f \in V^{(1)*}\) then

\[(F_h F_k)(f) = F_h(f + \langle \delta, f \rangle k) \]
\[= (f + \langle \delta, f \rangle k) + \langle \delta, f + \langle \delta, f \rangle k \rangle h \]
\[= f + \langle \delta, f \rangle k + \langle \delta, f \rangle h \]
\[= F_{h+k}(f), \]

where we used \(\langle \delta, k \rangle = 0\), since \(k \in X_0\). That is,

\[F_h F_k = F_{h+k} = F_k F_h. \quad (184)\]

Moreover, \(F_h\) is the identity if and only if \(h = 0\) and

\[F_{nh} = (F_h)^n \quad \text{for all } n \in \mathbb{Z}. \quad (185)\]
If \( w \in W^{(1)} \) and \( h \in X_0 \) then for all \( v \in V^{(1)*} \) and \( f \in V^{(1)*} \) we have

\[
\langle v, w(F_h(w^{-1}(f))) \rangle = \langle w^{-1}v, F_h(w^{-1}(f)) \rangle \\
= \langle w^{-1}(v), w^{-1}(f) \rangle + \langle \delta, w^{-1}(f) \rangle h \\
= \langle w^{-1}(v), w^{-1}(f) \rangle + \langle \delta, w^{-1}(f) \rangle \langle w^{-1}(v), h \rangle \\
= \langle v, f \rangle + \langle w(\delta), f \rangle \langle v, w(h) \rangle \\
= \langle v, f + \langle \delta, f \rangle w(h) \rangle \\
= \langle v, F_{w(h)}(f) \rangle,
\]

where we have used \( w(\delta) = \delta \). So we have,

\[
w(F_h(w^{-1}(f))) = F_{w(h)}(f),
\]

that is, a translation by \( w(h) \) (for \( h \in X_0, w \in W^{(1)} \)) on \( X_1 \) is related to a translation by \( h \) on \( X_1 \) by a conjugation of \( w \). In particular, if we have some \( w \in W^{(1)} \) such that \( w(\alpha_j + \delta) = \beta \), then we have

\[
t_\beta = wt_jw^{-1},
\]

where \( t_j \) is the translation on \( X_1 \) by \( \pi(\alpha_j^\gamma) \). Since

\[
R(t_\beta) = R(wt_jw^{-1}) = F_{\pi(w(\alpha_j)^\nu)} = F_{w(\pi(\alpha_j^\gamma))}, \quad 1 \leq j \leq n, \quad w \in W^{(1)},
\]

where we have used Equation (78).

So far, we have shown that \( T \) is an abelian normal subgroup of \( W^{(1)} \). To prove that it is finitely generated let us define

\[
T_j = F_{\pi(\alpha_j^\nu)} = R(t_\beta) = R(t_j), \quad \text{where} \quad \beta_j = \alpha_j + \delta \quad \text{for each} \quad 1 \leq j \leq n.
\]

Then for any \( \beta \in \Phi^{(1)} : \beta = \alpha + m\delta = \sum_{i=1}^n m_i\alpha_i + m\delta, \ (m, m_i \in \mathbb{Z}, \alpha \in \Phi, \alpha_i \in \Delta) \) we have

\[
R(t_\beta) = F_{m\pi(\alpha^\nu)} = F_{m\sum_{i=1}^n k_i\pi(\alpha_i^\nu)} = T_1^{m_{k_1}}T_2^{m_{k_2}}...T_n^{m_{k_n}} = R(\prod_{i=1}^n t_i^{m_{k_i}}),
\]

where \( Z \ni k_i = m_i|\alpha_i|^2 \), and we have used Proposition 6 Equations (181), (184) and (185).

That is, \( T \) is finitely generated by \( \{t_j \mid 1 \leq j \leq n\} \). Finally, the root lattice \( Q \) (given by Definition (8)) is isomorphic to \( T \) by

\[
\sum_{i=1}^n k_i\pi(\alpha_i^\nu) \mapsto \prod_{i=1}^n t_i^{k_i}, \quad k_i \in \mathbb{Z} \quad \text{for} \quad 1 \leq i \leq n.
\]

We illustrate properties of \( W^{(1)} \) discussed in Sections 3.4 and 3.5 for \( B_3, C_3, F_4 \) and \( G_2 \) type systems.
3.6. **Translations in** $W(B_3^{(1)})$. Recall that in $\Delta(B_3^{(1)})$, $\alpha_i$ for $i = 0, 1, 2$ are long while $\alpha_3$ is short. Moreover, $\delta = \alpha_0 + \tilde{\alpha} = \alpha_0 + 2\alpha_3$ and $\tilde{\alpha} = \alpha_1 + \alpha_2$.

(1) Translations associated with long roots of $\Phi(B_3^{(1)})$. We first write down an explicit expression for $t_0 = t_{\alpha_0}$ in terms of simple reflections of $W(B_3^{(1)})$. Expressions for $t_1 = t_{\alpha_1 + \delta}$ and $t_2 = t_{\alpha_2 + \delta}$ or any other long roots of $\Phi(B_3^{(1)})$ can be obtained by using either Proposition 15 or Equation (186) by the fact that root $\alpha_0$ is a long root of $\Phi(B_3^{(1)})$ and is in the same orbit of any long root.

From Equation (166) we have $t_{\alpha_0} = s_{\tilde{\alpha}_0} s_0$. By Equation (24), we know that $s_{\tilde{\alpha}_0} = s_{\tilde{\alpha}_0} s_0 s_{\tilde{\alpha}_0} s_0$, where we have used Equation (53). We have then,

$$t_{\alpha_0} = s_{\tilde{\alpha}_0} s_0 = s_{\tilde{\alpha}_0} s_23212320.$$  \(193\)

Action of $t_{\alpha_0}$ on $X_1$ is given by Equation (164):

$$t_{\alpha_0}(f) = f - \pi(\tilde{\alpha}_0^\vee) = f - h_2, \quad f \in X_1,$$

where we have used Equation (115) for the last equality.

On $\{\alpha_1, \alpha_2, \alpha_3, \alpha_0\}$ basis of $V^{(1)}$ actions of $t_{\alpha_0}$ are given by Equation (165) and $C(B_3^{(1)})$ in Equation (108),

$$t_{\alpha_0} : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1, \alpha_2 + \delta, \alpha_3, \alpha_0 - 2\delta\}. \quad (195)$$

If we are using Proposition 13 to get an expression for $t_1 = t_{\alpha_1 + \delta}$ it is easier to consider first its inverse $t_1^{-1} = t_{-\alpha_1 - \delta}$ and observe that

$$s_{232}(\alpha_0) = s_{232}(\delta - \tilde{\alpha}) = -\alpha_1 + \delta,$$

that is, we have,

$$s_{-\alpha_1 - \delta} = s_{232}(\alpha_0) = s_{232} s_0 s_{232}. \quad (197)$$

Then by Proposition 13 let $\beta = -\alpha_1 + \delta$, that is $\alpha = -\alpha_1$ and $m = 1$, we have

$$t_1^{-1} = t_{-\alpha_1 - \delta} = t_0 = s_{\alpha} s_{\beta}, \quad (198)$$

$$= s_{\alpha} s_{\alpha_1 + \delta}, \quad (198)$$

$$= s_1 s_{2320232} = s^{12320232}. \quad (199)$$

Finally, we have

$$t_1 = (s_{-\alpha_1} s_{-\alpha_1 + \delta})^{-1} = s_{2320232}. \quad (199)$$
Actions of $t_1$ on $X_1$ and on $V^{(1)}$ are given by Equations (164) and (165), respectively, for $j = 1$,
\[ t_1(f) = f + \pi(\alpha_1^\vee) = f + 2h_1 - h_2, \quad f \in X_1 \] (200)
and
\[ t_1 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1 - 2\delta, \alpha_2 + \delta, \alpha_3, \alpha_0\}, \] (201)
where we have used $C(B_3^{(1)})$ given in Equation (108).

(2) Translations associated with short roots of $\Phi(B_3^{(1)})$. We look at two such elements, $t_{\tilde{\alpha}_s + \delta}$ and $t_3$.

First observe that $t_{\tilde{\alpha}_s + \delta} = t_{-\tilde{\alpha}_s + \delta}$ and
\[ -\tilde{\alpha}_s + \delta = -\alpha_{123} + \alpha_{012233} = \alpha_{023}. \] (202)
Moreover, we have
\[ s_{12}(\alpha_3) = \alpha_{123}, \quad \text{and} \quad s_{02}(\alpha_3) = \alpha_{023}, \] (203)
that is,
\[ s_{\alpha_{123}} = s_{12}s_3s_{21}, \quad \text{and} \quad s_{\alpha_{023}} = s_{02}s_3s_{20}. \] (204)
Then by Proposition 13 on letting $\beta = -\tilde{\alpha}_s + \delta = \alpha_{023}$, that is $\alpha = -\tilde{\alpha}_s = -\alpha_{123}$ and $m = 1$, we have
\[ t_{-\tilde{\alpha}_s + \delta} = t_{\beta} = s_{\alpha}s_{\beta} = s_{-\tilde{\alpha}_s}s_{-\tilde{\alpha}_s + \delta} = s_{\alpha_{123}}s_{\alpha_{023}} = s_{12321}s_{02320}. \] (205)

Finally, we have
\[ t_{\tilde{\alpha}_s + \delta} = t_{-\tilde{\alpha}_s + \delta}^{-1} = (s_{-\tilde{\alpha}_s}s_{-\tilde{\alpha}_s + \delta})^{-1} = (s_{\alpha_{123}}s_{\alpha_{023}})^{-1} = s_{\alpha_{023}}s_{\alpha_{123}} = s_{02320}s_{12321}. \] (206)

Actions of $t_{\tilde{\alpha}_s + \delta}$ on $X_1$ are given by Equation (155):
\[ t_{\tilde{\alpha}_s + \delta}(f) = f + \pi(\tilde{\alpha}_s^\vee) = f + 2h_1, \quad f \in X_1, \] (207)
where we have used Equation (114) for the last expression. On $V^{(1)}$ $t_{\tilde{\alpha}_s + \delta}$ acts by Equation (160):
\[ t_{\tilde{\alpha}_s + \delta} : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1 - 2\delta, \alpha_2, \alpha_3, \alpha_0 + 2\delta\}. \] (208)

To obtain an expression for $t_3$, we make use of Equation (186). Recall from Example 1 that $\tilde{\alpha}_s = \alpha_{123}$ and $\alpha_3$ are both short roots of $\Phi(B_3^{(1)})$ hence belong to the same $W(B_3^{(1)})$-orbit:
\[ s_{21}(\alpha_s) = s_{21}(\alpha_{123}) = \alpha_3 \] (209)
We have,
\[ t_3 = t_{\alpha_3 + \delta} = t_{s_{12}(\alpha_3 + \delta)} = s_{21}t_{\alpha_3 + \delta}s_{12} = s_{21}s_{02320}s_{12321}s_{12} = s_{20123}s_{20123}, \]
(210)
where we have used the defining relations of \( W(B_3^{(1)}) \) given in Equation (109) to simplify the last expression. Actions of \( t_3 \) on \( X_1 \) and \( V^{(1)} \) are given by Equations (164) and (165) for \( j = 3 \):
\[ t_3(f) = f + \pi(\alpha_3'') = f - 2h_2 + 2h_3, \quad f \in X_1, \]
(211)
and
\[ t_3 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1 + 2\delta, \alpha_2 - 2\delta, \alpha_3 - 2\delta, \alpha_0\}. \]
(212)

### 3.7. Translations in \( W(C_3^{(1)}) \)

Recall that in \( \Delta(C_3^{(1)}) \), \( \alpha_0 \) and \( \alpha_3 \) are long, \( \alpha_1 \) and \( \alpha_2 \) are short. Moreover, \( \tilde{\delta} = \alpha_0 + \tilde{\alpha} = \alpha_{011223} \) and \( \tilde{\alpha}_s = \alpha_{1223} \).

(1) Translations associated to long roots in \( \Phi(C_3^{(1)}) \). We write done explicit expressions for \( t_0 = t_{\alpha_0} \) and \( t_1 = t_{\alpha_3 + \delta} \). By Equation (166), we have \( t_{\alpha_0} = s_{\tilde{\alpha}}s_0 \). From Example 2 we know that
\[ s_{12}(\alpha_3) = \tilde{\alpha} = \alpha_{11223}. \]
(213)
That is,
\[ s_{\tilde{\alpha}} = s_{s_{12}(\alpha_3)} = s_{12}s_3s_{21} = s_{12321}, \]
(214)
so we have
\[ t_{\alpha_0} = s_{\tilde{\alpha}}s_0 = s_{123210}. \]
(215)
Action of \( t_{\alpha_0} \) on \( X_1 \) is given by Equation (164):
\[ t_{\alpha_0}(f) = f + \pi(\alpha_0'') = f - h_1, \quad f \in X_1, \]
(216)
where we have used Equation (125) for the last expression. On \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \) basis of \( V^{(1)} \) its action is given by Equation (165):
\[ t_{\alpha_0} : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1 + \delta, \alpha_2, \alpha_3, \alpha_0 - 2\delta\}, \]
(217)
with \( C(C_3^{(1)}) \) given in Equation (117).

To write done an expression for \( t_3 = t_{\alpha_3 + \delta} \) it is easier to consider first its inverse \( t_3^{-1} = t_{-\alpha_3 + \delta} \) and observe that
\[ s_{21}(\alpha_0) = \alpha_{01122} = -\alpha_3 + \tilde{\delta}. \]
(218)
That is,
\[ s_{-\alpha_3 + \delta} = s_{s_{21}(\alpha_0)} = s_{21}s_0s_{12}. \]
(219)
Then by Proposition 13, letting \( \beta = -\tilde{\alpha}_s + \delta = \alpha_{0122} \), that is \( \alpha = -\alpha_3 \) and \( m = 1 \), we have

\[
t_{\tilde{\alpha}_s + \delta}^{-1} = t_{-\alpha_3 + \delta} = t_{\beta} = s_\alpha s_\beta,
\]

\[
= s_{-\alpha_3} s_{-\alpha_3 + \delta},
\]

\[
= s_3 s_{21012} = s_{321012}.
\]

That is

\[
t_3 = (s_{-\alpha_3} s_{-\alpha_3 + \delta})^{-1} = s_{210123}.
\]

Actions of \( t_3 \) on \( X_1 \) are given by Equation (164) for \( j = 3 \):

\[
t_3(f) = f + \pi(\alpha_3^\vee) = f - h_2 + 2h_3, \quad f \in X_1
\]

and on \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \) basis of \( V^{(1)} \) by Equation (165),

\[
t_3 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1, \alpha_2 + \delta, \alpha_3 - 2\delta, \alpha_0\}.
\]

Any other translations associated with a long root of \( \Phi(C_3) \) can be obtained from these expressions of \( t_3 \) or \( t_0 \) by using Equation (186).

(2) Translations associated to short roots of \( \Phi(C_3^{(1)}) \). First, observe that

\[
-\tilde{\alpha}_s + \delta = -\alpha_{1223} + \alpha_{01122} = \alpha_{01},
\]

and from Equation (29) from Example 2 we had \( s_{232}(\tilde{\alpha}_s) = \alpha_1 \), then

\[
s_{232}(\alpha_1) = \alpha_{1223} = \tilde{\alpha}_s, \quad \text{and} \quad s_0(\alpha_1) = \alpha_{01},
\]

so

\[
s_{\tilde{\alpha}_s} = s_{2321232}, \quad \text{and} \quad s_{\alpha_0} = s_{010}.
\]

In Proposition 13 on letting \( \beta = -\tilde{\alpha}_s + \delta = \alpha_{01} \), that is \( \alpha = -\tilde{\alpha}_s = -\alpha_{1223} \) and \( m = 1 \), we have

\[
t_{-\tilde{\alpha}_s + \delta} = t_{\beta} = s_\alpha s_\beta = s_{-\tilde{\alpha}_s} s_{-\tilde{\alpha}_s + \delta},
\]

\[
= s_{01223} s_{001} = s_{2321232} s_{010}.
\]

Then

\[
t_{\tilde{\alpha}_s + \delta} = t_{-\tilde{\alpha}_s + \delta}^{-1} = (s_{-\tilde{\alpha}_s} s_{-\tilde{\alpha}_s + \delta})^{-1} = s_{010} s_{2321232}.
\]

Action of \( t_{\tilde{\alpha}_s + \delta} \) on \( X_1 \) is given by Equation (155):

\[
t_{\tilde{\alpha}_s + \delta}(f) = f + \pi(\tilde{\alpha}_s^\vee) = f + h_2, \quad f \in X_1,
\]
where we have used Equation (124). On \( \{ \alpha_1, \alpha_2, \alpha_3, \alpha_0 \} \) basis of \( V^{(1)} \) its action is given by Equation (160):

\[
t_{\tilde{\alpha}_s + \delta} : \{ \alpha_1, \alpha_2, \alpha_3, \alpha_0 \} \mapsto \{ \alpha_1, \alpha_2 - \delta, \alpha_3, \alpha_0 + 2\delta \}.
\]

(230)

For \( t_1 \), using \( s_{232}(\tilde{\alpha}_s) = \alpha_1 \) and Equation (186) we have

\[
t_1 = t_{\alpha_1 + \delta} = s_{232}t_{\tilde{\alpha}_s + \delta}s_{232} = s_{232}s_{010}s_{232}s_{232} = s_{232}s_{010}s_{232}.
\]

(231)

The actions of \( t_1 \) on \( X_1 \) and \( V^{(1)} \) are given by Equations (164) and (165), respectively, for \( j = 1 \):

\[
t_1(f) = f + \pi(\alpha_1^\vee) = f + 2h_1 - h_2, \quad f \in X_1,
\]

(232)

and

\[
t_1 : \{ \alpha_1, \alpha_2, \alpha_3, \alpha_0 \} \mapsto \{ \alpha_1 - 2\delta, \alpha_2 + \delta, \alpha_3, \alpha_0 \}.
\]

(233)

Translation \( t_2 = t_{\alpha_2 + \delta} \) can be similarly obtained using \( s_{132}(\tilde{\alpha}_s) = \alpha_2 \) from Equation (29) in Example 2 and we have,

\[
t_2 = t_{\alpha_2 + \delta} = t_{s_{132}(\tilde{\alpha}_s + \delta)} = s_{132}t_{\tilde{\alpha}_s + \delta}s_{231},
\]

(234)

with \( t_{\tilde{\alpha}} \) given in Equation (228).

3.8. Translations in \( W(F_4^{(1)}) \). Recall that in \( \Delta(F_4^{(1)}) \), \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are long while \( \alpha_3 \) and \( \alpha_4 \) are short. Recall that we have \( \delta = \alpha_0 + \tilde{\alpha} = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \) and \( \tilde{\alpha}_s = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \).

(1) Translations associated with long roots of \( \Phi(F_4^{(1)}) \). We look at two such elements, \( t_0 = t_{\alpha_0} \) and \( t_2 \).

From Equation (166) we have \( t_{\alpha_0} = s_{\tilde{\alpha}}s_0 \). It can be checked that

\[
s_{1232143}(\alpha_2) = \tilde{\alpha}.
\]

(235)

That is,

\[
s_{\tilde{\alpha}} = s_{s_{1232143}(\alpha_2)} = s_{1232143}s_2s_312321,
\]

(236)

where we have used the relation given in Equation (33). Finally,

\[
t_{\alpha_0} = s_{\tilde{\alpha}}s_0 = s_{1232143}s_2s_312321s_0.
\]

(237)

Action of \( t_{\alpha_0} \) on \( X_1 \) is given by Equations (164):

\[
t_{\alpha_0}(f) = f - \pi(\tilde{\alpha}^\vee) = f - h_1, \quad f \in X_1,
\]

(238)

where we have used Equation (134) for the last equality. That is, \( t_{\alpha_0} \) translate on \( X_1 \) by \(-h_1\).

On \( \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0 \} \) basis of \( V^{(1)} \) actions of \( t_{\alpha_0} \) are given by Equation (165) with
\( C(F_4^{(1)}) \) from Equation (127),

\[
t_{\alpha_0} : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0\} \mapsto \{\alpha_1 + \delta, \alpha_2, \alpha_3, \alpha_4, \alpha_0 - 2\delta\}.
\]

(239)

Before using Proposition 13 to get an expression for \( t_2 = t_{\alpha_2 + \delta} \) it is easier to consider first its inverse \( t_2^{-1} = t_{-\alpha_2 + \delta} \) and observe that

\[
s_{3412321}(\alpha_0) = s_{3412321}(\delta - \tilde{\alpha}) = -\alpha_2 + \delta,
\]

(240)

that is, we have,

\[
s_{-\alpha_2 + \delta} = s_{3412321}(\alpha_0) = s_{3412321}s_0s_{232143}.
\]

Then by Proposition 13 let \( \beta = -\alpha_2 + \delta \), that is \( \alpha = -\alpha_2 \) and \( m = 1 \), we have

\[
t_2^{-1} = t_{-\alpha_2 + \delta} = t_\beta = s_\alpha s_\beta
\]

\[
= s_{-\alpha_2}s_{-\alpha_2 + \delta}
\]

\[
= s_2s_{3412321}s_0s_{232143}.
\]

(242)

Hence we have

\[
t_2 = (s_{-\alpha_2}s_{-\alpha_2 + \delta})^{-1} = s_{2341232}s_0s_{232143}.
\]

(243)

Actions of \( t_2 \) on \( X_1 \) and on \( V^{(1)} \) are given by Equations (164) and (165), respectively, for \( j = 2 \),

\[
t_2(f) = f + \pi(\alpha_2) = f - h_1 + 2h_2 - h_3, \quad f \in X_1
\]

(244)

and

\[
t_2 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0\} \mapsto \{\alpha_1 + \delta, \alpha_2 - 2\delta, \alpha_3 - \delta, \alpha_4, \alpha_0\},
\]

(245)

where we have used \( C(F_4^{(1)}) \) given in Equation (127).

(2) Translations associated with short roots of \( \Phi(F_4^{(1)}) \). We look at two such elements, \( t_{\tilde{\alpha}_s + \delta} \) and \( t_4 \).

First observe that \( t_{\tilde{\alpha}_s + \delta} = t_{-\alpha_2 + \delta}^{-1} \) and

\[
-\tilde{\alpha}_s + \delta = -\alpha_1223334 + \alpha_011223334 = \alpha_{0123}.
\]

(246)

It can be checked that,

\[
s_{4321323}(\alpha_4) = \tilde{\alpha}_s,
\]

(247)

and

\[
s_{012}(\alpha_3) = \alpha_{0123},
\]

(248)

hence we have

\[
s_{\tilde{\alpha}_s} = s_{4321323} s_4 s_{3231234},
\]

(249)
and

\[ s_{0123} = s_{012838210}. \]  

(250)

Then by Proposition 13 on letting \( \beta = -\tilde{\alpha}_s + \delta = \alpha_{0123} \), that is \( \alpha = -\tilde{\alpha}_s \) and \( m = 1 \), we have

\[ t_{-\tilde{\alpha}_s + \delta} = t_{-\tilde{\alpha}_s + \delta} = s_{-\tilde{\alpha}_s} s_{-\tilde{\alpha}_s + \delta}, \]
\[ = s_{\tilde{\alpha}_s} s_{0123} = s_{432132348 s_{321234} s_{012838210}}. \]  

(251)

Finally, we have

\[ t_{\tilde{\alpha}_s + \delta} = t_{-\tilde{\alpha}_s + \delta}^{-1} = (s_{\tilde{\alpha}_s} s_{-\tilde{\alpha}_s + \delta})^{-1} = (s_{\tilde{\alpha}_s} s_{0123})^{-1} = s_{0123210} s_{432132348 s_{321234}}. \]  

(252)

Actions of \( t_{\tilde{\alpha}_s + \delta} \) on \( X_1 \) are given by Equation (155):

\[ t_{\tilde{\alpha}_s + \delta}(f) = f + \pi(\tilde{\alpha}_s^\gamma) = f + h_4, \quad f \in X_1, \]  

(253)

where we have used Equation (133) for the last equality. That is, \( t_{\tilde{\alpha}_s + \delta} \) translates on \( X_1 \) by \( h_4 \). On \( V^{(1)} \) \( t_{\tilde{\alpha}_s + \delta} \) acts by Equation (160):

\[ t_{\tilde{\alpha}_s + \delta} : \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0 \} \mapsto \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 - \delta, \alpha_0 + 2\delta \}. \]  

(254)

To obtain an expression for \( t_4 \), we make use of Equation (186). By Equation (247), we have

\[ s_{4321323}(\alpha_4) = \tilde{\alpha}_s \quad \text{or} \quad \alpha_4 = s_{321234}(\tilde{\alpha}_s). \]  

(255)

Then,

\[ t_4 = t_{\alpha_4 + \delta} = t_{s_{321234}(\tilde{\alpha}_s + \delta)} = s_{321234} t_{\tilde{\alpha}_s + \delta} s_{4321323}, \]  

(256)

with \( t_{\tilde{\alpha}_s + \delta} \) given by Equation (252). Action of \( t_4 \) on \( X_1 \) and \( V^{(1)} \) are given by Equations (164) and (165) for \( j = 4 \):

\[ t_4(f) = f + \pi(\alpha_4^\gamma) = f - h_3 + 2h_4, \quad f \in X_1, \]  

(257)

where we have used Equation (130) and

\[ t_4 : \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_0 \} \mapsto \{ \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4 - 2\delta, \alpha_0 \}, \]  

(258)

where we have used \( C(F^{(1)}_4) \) given in Equation (127).

3.9. **Translations in** \( W(G_2^{(1)}) \). Recall that in \( \Delta(G_2^{(1)}) \), \( \alpha_0 \) and \( \alpha_2 \) are long while \( \alpha_1 \) is short, moreover we have \( \delta = \alpha_0 + \tilde{\alpha} = \alpha_0 + 3\alpha_1 + 2\alpha_2 \) and \( \tilde{\alpha}_s = 2\alpha_1 + \alpha_2 \).

(1) Translations associated with long roots of \( \Phi(G_2^{(1)}) \). We look at two such elements, \( t_0 = t_{\alpha_0} \) and \( t_2 \).
From Equation (166) we have \( t_{\alpha_0} = s_\tilde{\alpha}s_0 \). From Equation (40) in Example 4 we know that

\[
  s_{12}(\tilde{\alpha}) = \alpha_2, \quad \text{or} \quad \tilde{\alpha} = s_{21}(\alpha_2).
\]  

(259)

That is,

\[
  s_\tilde{\alpha} = s_{s_{21}(\alpha_2)} = s_{21}s_2s_{12},
\]  

(260)

where we have used Equation (53). Finally,

\[
  t_{\alpha_0} = s_\tilde{\alpha}s_0 = s_{21212}.
\]  

(261)

Action of \( t_{\alpha_0} \) on \( X_1 \) is given by Equation (164):

\[
  t_{\alpha_0}(f) = f - \pi(\tilde{\alpha}^\vee) = f - h_2, \quad f \in X_1,
\]  

(262)

where we have used Equation (144) for the last equality. That is, \( t_{\alpha_0} \) translate on \( X_1 \) by \(-h_2\).

On \( \{\alpha_1, \alpha_2, \alpha_0\} \) basis of \( V^{(1)} \) actions of \( t_{\alpha_0} \) are given by Equation (165) with \( C(G^{(1)}_2) \) from Equation (136),

\[
  t_{\alpha_0} : \{\alpha_1, \alpha_2, \alpha_0\} \mapsto \{\alpha_1, \alpha_2 + \delta, \alpha_0 - 2\delta\}.
\]  

(263)

Instead of using Proposition 13 to get an expression for \( t_2 = t_{\alpha_2 + \delta} \) we make use of Equations (186) and (259). Observe that

\[
  t_{\tilde{\alpha} + \delta} = (t_{-\tilde{\alpha} + \delta})^{-1} = t_0^{-1},
\]  

(264)

and

\[
  t_2 = t_{\alpha_2 + \delta} = t_{s_{12}(\tilde{\alpha} + \delta)} = s_{12}t_0^{-1}s_{21}.
\]  

(265)

Then by Equation (261) we have,

\[
  t_2 = s_{12}t_0^{-1}s_{21} = s_{12}s_0s_{21}s_1 = s_{12012},
\]  

(266)

where we have used \( s_1^2 = s_2^2 = 1 \) to simplify the last expression.

Actions of \( t_2 \) on \( X_1 \) and on \( V^{(1)} \) are given by Equations (164) and (165), respectively, for \( j = 2 \),

\[
  t_2(f) = f + \pi(\alpha_2^\vee) = f - h_1 + 2h_2, \quad f \in X_1
\]  

(267)

and

\[
  t_2 : \{\alpha_1, \alpha_2, \alpha_0\} \mapsto \{\alpha_1 + \delta, \alpha_2 - 2\delta, \alpha_0 - \delta\},
\]  

(268)

where we have used \( C(G^{(1)}_2) \) from Equation (136).

(2) Translations associated with short roots of \( \Phi(G^{(1)}_2) \). We look at two such elements, \( t_{\tilde{\alpha} + \delta} \) and \( t_1 \).
First observe that $t_{\tilde{\alpha}_s+\delta} = t^{-1}_{-\tilde{\alpha}_s+\delta}$ and

$$-\tilde{\alpha}_s + \delta = -\alpha_{112} + \alpha_{01122} = \alpha_{012}. \quad (269)$$

Moreover, we have

$$s_{12}(\alpha_1) = \tilde{\alpha}_s, \quad (270)$$

and

$$s_{02}(\alpha_1) = \alpha_{012}, \quad (271)$$

hence

$$s_{\tilde{\alpha}_s} = s_{12}s_1s_{21}, \quad (272)$$

and

$$s_{\alpha_{012}} = s_{02}s_1s_{20}. \quad (273)$$

Then by Proposition 13 on letting $\beta = -\tilde{\alpha}_s + \delta = \alpha_{012}$, that is $\alpha = -\tilde{\alpha}_s$ and $m = 1$, we have

$$t_{-\tilde{\alpha}_s+\delta} = t_{\beta} = s_{\alpha}\beta = s_{-\tilde{\alpha}_s}s_{-\tilde{\alpha}_s+\delta} = s_{\tilde{\alpha}_s}s_{\alpha_{012}} = s_{12121}s_{02120}. \quad (274)$$

Finally, we have

$$t_{\tilde{\alpha}_s+\delta} = t^{-1}_{-\tilde{\alpha}_s+\delta} = (s_{-\tilde{\alpha}_s}s_{-\tilde{\alpha}_s+\delta})^{-1} = (s_{\tilde{\alpha}_s}s_{\alpha_{012}})^{-1} = s_{\alpha_{012}}s_{\tilde{\alpha}_s} = s_{02120}s_{12121}. \quad (275)$$

Actions of $t_{\tilde{\alpha}_s+\delta}$ on $X_1$ are given by Equation (155):

$$t_{\tilde{\alpha}_s+\delta}(f) = f + \pi(\tilde{\alpha}_s') = f + h_1, \quad f \in X_1, \quad (276)$$

where we have used Equation (143) for the last equality. That is, $t_{\tilde{\alpha}_s+\delta}$ translates on $X_1$ by $h_1$. On $V^{(1)}$ $t_{\tilde{\alpha}_s+\delta}$ acts by Equation (160):

$$t_{\tilde{\alpha}_s+\delta} : \{\alpha_1, \alpha_2, \alpha_0\} \mapsto \{\alpha_1 - \delta, \alpha_2, \alpha_0 + 3\delta\}. \quad (277)$$

For an expression of $t_1 = t_{\alpha_1+\delta}$ we make use of Equation (186). Recall that,

$$\alpha_1 = s_{21}(\tilde{\alpha}_s). \quad (278)$$

Then we have

$$t_1 = t_{\alpha_1+\delta} = t_{s_{21}(\tilde{\alpha}_s+\delta)} = s_{21}t_{\tilde{\alpha}_s+\delta}s_{21} = s_{21}s_{02120}s_{12121}s_{12} = s_{2102120}s_{12121}. \quad (279)$$

where we had the expression of $t_{\tilde{\alpha}_s+\delta}$ given by Equation (275), and $s_1^2 = s_2^2 = 1$ was used to simplify the last equality.
Actions of \( t_1 \) on \( X_1 \) and \( V^{(1)} \) are given by Equations (164) and (165) for \( j = 1 \):
\[
\begin{align*}
t_1(f) &= f + \pi(\alpha_j) = f + 2h_1 - 3h_2, & f \in X_1,
\end{align*}
\] (280)
where we have used Equation (140) and
\[
\begin{align*}
t_1 : \{\alpha_1, \alpha_2, \alpha_0\} &\rightarrow \{\alpha_1 - 2\delta, \alpha_2 + 3\delta, \alpha_0\},
\end{align*}
\] (281)
with \( C(C_2^{(1)}) \) given by Equation (136).

4. Extended affine Weyl groups

Previously in Section 3, we have seen how \( W^{(1)} \) decomposes into a semi-direct product of the finite Weyl group \( W \) and an abelian group of translations on the root lattice \( Q \), \( W^{(1)} = W \ltimes Q \). Generators of \( Q \), \( t_j \) are associated with translation on \( X_1 \) by \( \pi(\alpha_j) \) (1 \( \leq \) \( j \) \( \leq \) \( n \)). In order to describe translations on the weight lattice \( P \) we construct certain extensions of \( W^{(1)} \), \( \overline{W}^{(1)} = W \ltimes \overline{P} \).

For affine Weyl groups of type \( A_n, B_n, C_n, D_n, E_6 \) and \( E_7 \), \( P/Q \) amounts to some group \( A \) of diagram automorphisms of the affine Dynkin diagram \( \Gamma^{(1)} \). The group \( A \) preserves the simple system \( \Delta^{(1)} \) and hence normalises \( W^{(1)} \). That is, we have \( \overline{W}^{(1)} = W \ltimes P = A \ltimes W^{(1)} \).

For types \( E_8, F_4 \) and \( G_2 \), \( P \) is isomorphic to \( Q \) since the affine Dynkin diagrams for these types have no non-trivial diagram automorphisms.

**Definition 11.** For each 1 \( \leq \) \( j \) \( \leq \) \( n \), define a linear transformation
\[
\begin{align*}
U_j &= R(u_j) = F_{h_j} : V^{(1)*} \rightarrow V^{(1)*} \\
\end{align*}
\]
by,
\[
\begin{align*}
(R(u_j))(f) &= F_{h_j}(f) = f + \{\delta, f\}h_j \quad \text{for each} \quad h_j \in X_0 \quad 1 \leq j \leq n, \tag{282}
\end{align*}
\]
for all \( f \in V^{(1)*} \). Furthermore, let \( U \) be a free abelian group generated by \( \{u_j \mid 1 \leq j \leq n\} \),
\[
\begin{align*}
U = \langle u_j \mid 1 \leq j \leq n \rangle. \tag{283}
\end{align*}
\]
The weight lattice \( P \), which is the integer span of \( \{h_j \mid 1 \leq j \leq n\} \), is isomorphic to \( U \) by the map:
\[
\begin{align*}
\sum_{i=1}^n n_i h_i \mapsto \prod_{i=1}^n u_i^{n_i}, \quad n_i \in \mathbb{Z}. \tag{284}
\end{align*}
\]
We define an action of \( W^{(1)} \) on \( U \) by requiring
\[
\begin{align*}
R(wu_jw^{-1}) &= F_{w(h_j)} = F_{\Sigma_{i=1}^n n_{ij} h_i} = \prod_{i=1}^n U_i^{n_{ij}} = R\left(\prod_{i=1}^n u_i^{n_{ij}}\right) \tag{285}
\end{align*}
\]
for 1 \( \leq \) \( j \) \( \leq \) \( n \), and \( w \in W^{(1)} \). That is, if we have \( w(h_j) = \Sigma_{i=1}^n n_{ij} h_i \) for some \( n_{ij} \in \mathbb{Z} \) then,
\[
\begin{align*}
w u_j w^{-1} = \prod_{i=1}^n u_i^{n_{ij}}. \tag{286}
\end{align*}
\]
Definition 12. Let the extended affine Weyl group $\overline{W}(1)$ be the semi-direct product of $U$ by $W$,

$$\overline{W}(1) = W \rtimes U = W \rtimes \langle u_j \mid 1 \leq j \leq n \rangle = W \rtimes P,$$

where $R$ is extended to a representation of $\overline{W}(1)$ on $V^{(1)*}$. Equation (282) tells us how $u_j$ acts on $V^{(1)*}$.

Proposition 17. Action of $u_j \in \overline{W}(1)$ on $V^{(1)*}$ is given by

$$u_j(f) = f + \langle \delta, f \rangle h_j, \quad 1 \leq j \leq n,$$

for all $f \in V^{(1)*}$. In particular, actions of $u_j$ on $\{h_1, h_2, \ldots, h_n, h_\delta\}$ basis of $V^{(1)*}$ are particularly simple. We have

$$u_j(h_i) = h_i, \quad 1 \leq i \leq n \quad \text{and} \quad u_j(h_\delta) = h_\delta + h_j, \quad 1 \leq j \leq n.$$

Proof. For Equation (289) we have used the facts $\langle \delta, h_i \rangle = 0 \ (1 \leq i \leq n)$ and $\langle \delta, h_\delta \rangle = 1$. $\square$

It remains to determine, via the contragredient actions of $R$, how the generators $u_j$ of $U$ act on $V^{(1)}$.

Proposition 18. Action of $u_j \in \overline{W}(1)$ on $v \in V^{(1)}$ is given by

$$u_j(v) = v - \langle v, h_j \rangle \delta, \quad 1 \leq j \leq n.$$

In particular, we have for $1 \leq i, j \leq n$,

$$u_j(\alpha_i) = \begin{cases} \alpha_i, & \text{for} \quad i \neq j, \\ \alpha_j - \delta, & \text{for} \quad i = j, \end{cases}$$

and

$$u_j(\alpha_0) = \alpha_0 + c_j \delta,$$

where $c_j$ is the coefficient of $\alpha_j$ in $\delta$.

Proof. For each $j$ and $v \in V^{(1)}$ we have,

$$\langle u_j^{-1}(v), f \rangle = \langle v, R(u_j)(f) \rangle,$$

$$= \langle v, F_{h_j}(f) \rangle,$$

$$= \langle v, f + \langle \delta, f \rangle h_j \rangle,$$

$$= \langle v, f + \langle \delta, f \rangle \langle v, h_j \rangle \rangle,$$

$$= \langle v + \langle v, h_j \rangle \delta, f \rangle,$$

for all $f \in V^{(1)*}$. Equation (282) tells us how $u_j$ acts on $V^{(1)*}$. $\square$
that is, \( u_j^{-1}(v) = v + \langle v, h_j \rangle \delta \), or
\[
  u_j(v) = v - \langle v, h_j \rangle \delta. \tag{293}
\]

Actions of \( u_j \) on \( \{\alpha_1, \alpha_2, ..., \alpha_0\} \) basis of \( V^{(1)} \) are given by,
\[
  u_j(\alpha_i) = \alpha_i - \langle \alpha_i, h_j \rangle \delta, \quad \text{for all } 0 \leq i \leq n, \ 1 \leq j \leq n. \tag{294}
\]

For Equations (291) and (292) we have used the facts \( \langle \alpha_i, h_j \rangle = 0 \) for \( i \neq j \), \( \langle \alpha_j, h_j \rangle = 1 \), and \( \langle \alpha_0, h_j \rangle = -c_j \) for \( 1 \leq i, j \leq n \).

\[\square\]

**Proposition 19.** \( P \) is preserved by \( W^{(1)} \). That is
\[
  w(h_j) = \sum_{i=1}^{n} n_{ij} h_i, \quad w \in W^{(1)}, \quad \text{for some } n_{ij} \in \mathbb{Z}. \tag{295}
\]

In particular we have for \( 1 \leq i, j \leq n \),
\[
  s_i(h_j) = \begin{cases} h_j, & \text{for } i \neq j, \\ h_j - \sum_{k=1}^{n} a_{ki} h_k, & \text{for } i = j, \end{cases} \tag{296}
\]
and
\[
  s_0(h_j) = h_j + c_j \sum_{k=1}^{n} a_{k0} h_k, \tag{297}
\]
where \( c_i \)'s are the coefficients of \( \alpha_i \) in \( \delta \), and \( a_{ij} \) are the entries of \( C(\Gamma^{(1)}) \).

**Proof.** From Equation (86) for \( 1 \leq j \leq n, \ 0 \leq i \leq n \) we have,
\[
  s_i(h_j) = h_j - \langle \alpha_i, h_j \rangle \pi(\alpha_i^\gamma), \tag{298}
\]
with \( \pi(\alpha_i^\gamma) \) expressed in terms of \( h_k \) \((1 \leq k \leq n)\) given by Proposition 8. For Equations (296) and (297) we have used \( \langle \alpha_i, h_j \rangle = 0 \) for \( i \neq j \); \( \langle \alpha_j, h_j \rangle = 1 \), and \( \langle \alpha_0, h_j \rangle = -c_j \) for \( 1 \leq j \leq n \). The coefficients \( c_i \) and \( a_{ij} \) are all integers. That is, the generators \( s_i \) \((0 \leq i \leq n)\) and hence \( W^{(1)} \) preserve \( P \). \[\square\]

Earlier in Section 3.4 for discussions on \( t_j \in W^{(1)} \), that is elements of translations on \( X_1 \) by \( \pi(\alpha_j^\gamma), \) we used the fact that all long (short) roots belong to the same \( W \)-orbit of \( \tilde{\alpha} \) (\( \tilde{\alpha}_s \)). For discussions on \( u_j \in \tilde{W}^{(1)} \), that is elements of translations on \( X_1 \) by \( h_j \) we compute the orbit of \( h_j \) under the finite Weyl group \( W \) in the following proposition.

**Proposition 20.** Each \( h_j \) forms a single \( W \)-orbit, \( Wh_j, \) for \( 1 \leq j \leq n \). In particular,
\[
  |Wh_j| = \frac{|W|}{|W_{h_j}|}, \tag{299}
\]
where \( W_{h_j} = \langle s_{\alpha_i} \mid \alpha_i \in \Delta, \text{for } i \neq j \rangle \) is the stabilizer of \( h_j \) in \( W \).

**Proof.** First, let us recall the coset theorem of a group.
A group $G$ acts transitively on a set $X$. For any $x \in X$, let $Gx$ be the orbit of $x$ under $G$, and $G_x$ denotes the stabilizer of $x$ in $G$, that is $G_x = \{ w \in G \mid wx = x, x \in G \}$. We have

$$|G| = |Gx|/|G_x|,$$

where $|G|$ denotes the order of the group $G$.

Now let $G = W$, $x = h_j$ and we have $|W| = |Wh_j||W_{h_j}|$, where $W_{h_j}$ is the stabilizer of $h_j$ in $W$ and $Wh_j$ is the $W$-orbit of $h_j$. Computations of $W_{h_j}$ are particularly easy since by Equation (296), $h_j$ is stabilised by all the simple reflections associated with nodes that are not connected to the node $j$ in the Dynkin diagram. Then different $h_j$ can not have the same stabilizer hence each $h_j$ belongs to a different $W$-orbit, $Wh_j$, for $1 \leq j \leq n$. □

Since $W$ preserves length, all vectors in the same weight orbit have the same length. The other direction of this statement is not true, that is, two vectors of the same length may or may not be in the same weight orbit. Recall that by Proposition 10, squared lengths of $h_j$'s are given by the diagonal entries of $C(\Gamma^{(1)})$, which are listed in Table 2 of Appendix A. For example, we see that the two fundamental weights of $W(E_6)$, $h_2$ and $h_6$ both have squared lengths of $4/3$ but they belong to different orbits under the actions of $W(E_6)$.

**Remark 15.** Shortest weights and basic translations. Consider the shortest weight/s given in Table 2 of Appendix A. We see that for type $A_n$, they are $h_1$ and $h_n$; for type $B_n$, it is $h_1$; for type $C_n$, it is $h_n$; for type $D_n$, they are $h_{n-1}$ and $h_n$; for type $E_6$, they are $h_2$ and $h_6$; for type $E_7$, it is $h_7$; for type $E_8$, it is $h_8$; for type $F_4$, it is $h_1$; and for type $G_2$, it is $h_2$.

**Definition 13.** Let $h_j$ (for some $j$ or $j$'s) be the shortest of the fundamental weights of a type $\Gamma^{(1)}$ system, the corresponding element of translation is $u_j \in \overline{W}(\Gamma^{(1)})$. We call a translation $T \in \overline{W}(\Gamma^{(1)})$ basic if

$$T(f) = f + h, \quad f \in X_1, \quad h \in Wh_j.$$  

A translation is basic if it translates by an element of the $W$-orbit of $h_j$. Moreover, since $h \in Wh_j$ there is some $w \in W$, such that $w(h_j) = h$ and we have $T = wu_jw^{-1}$.

Translation, $T$, is “basic” in the sense that any other translations on the weight and root lattices of type $\Gamma^{(1)}$ system can be obtained from $T$ by conjugation and compositions of elements of $\overline{W}(\Gamma^{(1)})$. All mappings that give rise to discrete Painlevé equations in Sakai’s classification [28] correspond to some basic translations. We discuss such a system of $E_8^{(1)}$ type in Section 5.1.

Now we give an example of how weight orbits in a Weyl group can be computed and used.

**Example 13.** Previously in Remark 4, we stated that there are $2n(n - 1)$ long roots and $2n$ short roots in the root system $\Phi$ of $B_n$ type. Now we make use of Proposition 27 to show this.
For long roots, first observe that the orbit of \( \pi(\tilde{\alpha}^\vee) = h_2 \) (see Table 2 of Appendix A) under \( W(B_n) \) has the same carnality as the orbit of \( \tilde{\alpha} \) under \( W(B_n) \) (that is, the number of long roots in \( \Phi \)). We compute the orbit of \( h_2 \) under \( W(B_n) \) using Proposition 20. First, the stabilizer of \( h_2 \) in \( W(B_n) \) is given by

\[
W(B_n)_{h_2} = (s_{\alpha_i} \mid \alpha_i \in \Delta, \text{ for } i \neq 2),
\]

\[
\cong W(A_1) \times W(B_{n-2}),
\]

(302)

where \( \Delta \) is the simple system of \( B_n \). Then,

\[
|W(B_n)_{h_2}| = \frac{|W(B_n)|}{|W(A_1) \times W(B_{n-2})|},
\]

\[
= \frac{2^n n!}{2 \times 2^{n-2}(n-2)!},
\]

(303)

\[
= 2n(n-1),
\]

where we have used the orders of Weyl groups of type \( A_n \) and \( B_n \) given in Table 2 of Appendix A. That is, there are \( 2n(n-1) \) long roots in \( \Phi \) of \( B_n \) type.

For short roots in \( \Phi \), recall that \( \pi(\tilde{\alpha}^\vee_s) = 2h_1 \). So one just needs to find the size of the orbit of \( h_1 \) under \( W(B_n) \). The stabilizer of \( h_1 \) in \( W(B_n) \) is given by

\[
W(B_n)_{h_1} = (s_{\alpha_i} \mid \alpha_i \in \Delta, \text{ for } i \neq 2)
\]

\[
\cong W(B_{n-1}),
\]

(304)

so we have

\[
|W(B_n)_{h_1}| = \frac{|W(B_n)|}{|W(B_{n-1})|},
\]

\[
= \frac{2^n n!}{2^{n-1}(n-1)!},
\]

(305)

\[
= 2n.
\]

That is, there are \( 2n \) long roots in \( \Phi \) of \( B_n \) type.

4.1. A normal subgroup of translations, \( \overline{W}^{(1)} = W \rtimes P = A \rtimes \overline{W}^{(1)} \). It can be shown that \( \overline{W}^{(1)} \) is decomposed as a semidirect product of \( W^{(1)} \) by the group of Dynkin diagram automorphisms \( A \), that is \( \overline{W}^{(1)} = A \rtimes W^{(1)} \). That is, elements of \( \overline{W}^{(1)} \) can be written in the form \( aw \) with \( a \in A \) and \( w \in W^{(1)} \). To explicitly write down translations \( u_j \in \overline{W}^{(1)} \) on the weight lattice \( P \), one needs to work out the elements of \( A \). Definition 2 can be extended to the case of \( \overline{W}^{(1)} \).
Definition 14. For each \( u \in \overline{W}(1) \), define
\[
N(u) = \{ \alpha \in \Phi_+^{(1)} \mid u(\alpha) \in \Phi_-^{(1)} \}.
\] (306)

That is, \( N(u) \) is the set of positive roots that \( u \) takes to some negative roots. The group \( A \) preserves the simple system \( \Delta^{(1)} \), acting on the simple roots as permutations. Hence elements of \( A \subset \overline{W}(1) \) are characterised by having the property, that \( N(a) \) is the empty set for any \( a \in A \).

Definition 14 and Equation (308) provide us with a strategy for writing any element \( u \in \overline{W}(1) \) in the form
\[
u = aw = as_{l_k}...s_{l_1}, \quad l_1, ..., l_k \in \{0, 1, ..., n\}, \quad a \in A, \quad w \in W^{(1)},
\] (307)
where we have \( l(w) = k \) and \( s_j \) are the simple reflections of \( W^{(1)} \). We now illustrate properties of \( \overline{W}(1) \) discussed in Section 4 for \( B_3 \) and \( C_3 \) type systems. Recall that for types \( F_4 \) and \( G_2 \), the two lattices \( P \) and \( Q \) are isomorphic so discussions on translations have been done in Sections 3.8 and 3.9 respectively. \( \overline{W}(1) = W \ltimes P = A \ltimes W^{(1)} \)

![Figure 12. Dynkin diagram of affine B_3 type with the diagram automorphism, \( \hat{\Gamma}(B_3^{(1)}) \).](image)

4.2. Translations on the weight lattice of \( \overline{W}(B_3^{(1)}) = W(B_3) \ltimes P = A \ltimes W(B_3^{(1)}) \). Here we find the group \( A \) and write down explicit expressions for \( u_j \) \((1 \leq j \leq 3)\) in terms of the generators of \( \overline{W}(B_3^{(1)}) \). Recall that from Equation (113) in Example 8 we have \( |h_j|^2 \) for \( 1 \leq j \leq 3 \) are 1, 2 and 3, respectively. Moreover, by Equation (112), the two squared lengths for the coroots are 2 and 4.

It is useful to recall that \( \delta = \alpha_{012233} \), that is \( c_1 = 1, c_2 = 2, c_3 = 2 \) for the applications of Proposition 18 in this section.

(1) Action of \( u_1 \) on \( X_1 \) is given by Proposition 17,
\[
u_1(f) = f + h_1, \quad f \in X_1.
\] (308)

By Proposition 18 \( u_1 \) acts on \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \) basis of \( V^{(1)} \) by,
\[
u_1 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \to \{\alpha_1 - \delta, \alpha_2, \alpha_3, \alpha_0 + \delta\}.
\] (309)
We see that $u_1(\alpha_1) = \alpha_1 - \delta = -\alpha_{02233} \in \Phi_+^{(1)}$, hence $l(u_1 s_1) = l(u_1) - 1$ by Equation (58). For $u_1 s_1$, we have

$$u_1 s_1 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_2, \alpha_{0233}, -\alpha_{0233}, \alpha_3, \alpha_0 + \delta\},$$

where the actions of $s_1$ are given by Equation (51) with $a_{ij}$ being the $(i,j)$-entry of $C(B_3^{(1)})$ given by Equation (108). Here we see that $u_1 s_1(\alpha_2) = -\alpha_{0233} \in \Phi_+^{(1)}$, hence $l(u_1 s_1 s_2) = l(u_1 s_1) - 1$ by Equation (58). Continuing in this way we have,

$$u_1 s_1 s_2 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_2, \alpha_{0233}, -\alpha_{0233}, \alpha_0, \alpha_0 + \delta\},$$

(310)

where we have used the fundamental relations of $W(B_3^{(1)})$ given in Equation (109). However, before launching into the length-reducing procedure we recall the following fact about $h_2$. By Equations (113) and (91) in Example 8 we have,

$$h_2 = \pi(\alpha_1^\gamma) + 2\pi(\alpha_2^\gamma) + \pi(\alpha_3^\gamma) = \pi(\tilde{\alpha}^\gamma) = -\pi(\alpha_0^\gamma),$$

(318)

which means that $u_2$ is a translation on the root lattice $Q$, in particular by Equation (139) we have,

$$u_2 = t_{\alpha_0}^{-1} t_0^{-1} = s_{02321232}.$$

(319)

(3) Action of $u_3$ on $X_1$ is given by,

$$u_3(f) = f + h_3, \quad f \in X_1,$$

(320)

and by Proposition 18 $u_3$ acts on $\{\alpha_1, \alpha_2, \alpha_3, \alpha_0\}$ basis of $V^{(1)}$ by,

$$u_3 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1, \alpha_2, \alpha_3 - \delta, \alpha_0 + 2\delta\}.$$
Applying the same length-reducing procedure we find that,

\[ u_3 s_3 s_2 s_1 s_3 s_2 s_0 s_3 s_2 s_1 = \sigma \cdot \{ \alpha_1, \alpha_2, \alpha_3, \alpha_0 \} \mapsto \{ \alpha_0, \alpha_2, \alpha_3, \alpha_1 \}. \]  (322)

That is,

\[ u_3 = \sigma s_{123023123}. \]  (323)

Finally, we have

\[ \bar{W}(B_3^{(1)}) = W(B_3) \rtimes P = W(B_3) \rtimes (u_j \mid 1 \leq j \leq 3) = A \rtimes W(B_3^{(1)}) = (\sigma \mid \sigma^2 = 1) \rtimes W(B_3^{(1)}). \]  (324)

4.3. **Translations on the weight lattice** \( \bar{W}(C_3^{(1)}) = W(C_3) \rtimes P = A \rtimes W(C_3^{(1)}) \). Here we find the group \( A \) and write down explicit expressions for \( u_j (1 \leq j \leq 3) \) in terms of the generators of \( \bar{W}(C_3^{(1)}) \). Recall that from Equation (123) in Example 9, we have \( h_j \) for \( 1 \leq j \leq 3 \) are 1, 2 and \( \frac{2}{3} \), respectively. Moreover, by Equation (122), the two squared lengths for the coroots are 2 and 1.

It is useful to recall that \( \delta = \alpha_{01223} \), that is we have \( c_1 = 2, c_2 = 2, c_3 = 1 \) for the applications of Proposition 18.

\[ 0 \begin{array}{c} \sigma \end{array} 1 \begin{array}{c} \sigma' \end{array} 2 \begin{array}{c} \sigma'' \end{array} 3 \]

**Figure 13.** Dynkin diagram of affine \( C_3 \) type with the diagram automorphism, \( \Gamma(C_3^{(1)}) \).

1) Action of \( u_1 \) on \( X_1 \) is given by Proposition 17

\[ u_1(f) = f + h_1, \quad f \in X_1. \]  (325)

By Proposition 18 \( u_1 \) acts on \( \{ \alpha_1, \alpha_2, \alpha_3, \alpha_0 \} \) basis of \( V^{(1)} \) by,

\[ u_1 : \{ \alpha_1, \alpha_2, \alpha_3, \alpha_0 \} \mapsto \{ \alpha_1 - \delta, \alpha_2, \alpha_3, \alpha_0 + 2\delta \}. \]  (326)

By Equations (125) and (91) in Example 9 we have

\[ h_1 = \pi(\alpha_1^\vee) + \pi(\alpha_2^\vee) + \pi(\alpha_3^\vee) = \pi(\tilde{\alpha}^\vee) = -\pi(\alpha_0^\vee), \]  (327)

Which means that \( u_1 \) is a translation on the root lattice \( Q \), in particular by Equation (215) in Example 3.7 we have,

\[ u_1 = t_{\alpha_0}^{-1} t_0^{-1} = (s_{123210})^{-1} = s_{012321}. \]  (328)

2) Action of \( u_2 \) on \( X_1 \) is given by Proposition 17

\[ u_2(f) = f + h_2, \quad f \in X_1. \]  (329)
By Proposition 18, \( u_2 \) acts on \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \) basis of \( V^{(1)} \) by,
\[
u_2 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1, \alpha_2 - \delta, \alpha_3, \alpha_0 + 2\delta\}.
\]

By Equation (124) we have
\[
h_2 = \pi(\alpha_1^\vee) + 2\pi(\alpha_2^\vee) = \pi(\alpha_3^\vee) = \pi(\tilde{\alpha}_s^\vee),
\]

which means that \( u_2 \) is a translation on the root lattice \( Q \), in particular by Equation (68) in Section 3.7 we have,
\[
u_2 = t_{\tilde{\alpha}_s+\delta} = s_{010}s_{2321232}.
\]

(3) Action of \( u_3 \) on \( X_1 \) is given by Proposition 17,
\[
u_3(f) = f + h_3, \quad f \in X_1.
\]

By Proposition 18, \( u_3 \) acts on \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \) basis of \( V^{(1)} \) by,
\[
u_3 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_1, \alpha_2, \alpha_3 - \delta, \alpha_0 + \delta\}.
\]

Applying the length-reducing procedure using Equation (58) we find that
\[
u_3s_3s_2s_3s_1s_2s_3 = \sigma : \{\alpha_1, \alpha_2, \alpha_3, \alpha_0\} \mapsto \{\alpha_2, \alpha_1, \alpha_0, \alpha_1\}.
\]

That is, the element \( \sigma = u_3s_3s_2s_3s_1s_2s_3 \) switches \( \alpha_0 \) and \( \alpha_3 \); \( \alpha_1 \) and \( \alpha_2 \) (see Figure 13), hence \( \sigma^2 = 1 \), and it is the generator of \( A \), and
\[
u_3 = \sigma s_{321323}.
\]

Finally, we have
\[
\overline{W}(C_3^{(1)}) = W(C_3) \rtimes P = W(C_3) \rtimes \langle u_j \mid 1 \leq j \leq 3 \rangle,
\]
\[
= A \rtimes W(C_3^{(1)}) = \langle \sigma \mid \sigma^2 = 1 \rangle \rtimes W(C_3^{(1)}).
\]

5. Applications

Here, we illustrate the power of the formulas developed in the earlier sections of the paper in studying different integrable systems recently appeared in the literature.

5.1. Two discrete Painlevé systems of type \( E_8^{(1)} \). In [16], an elliptic Painlevé equation associated with a translation \( T_{J,1} \in W(E_8^{(1)}) \) was constructed. \( T_{J,1} \) acts on the \( \{\alpha_j \mid 0 \leq j \leq 8\} \) basis of \( V^{(1)} \) by
\[
T_{J,1} : \{\alpha_1, \alpha_6\} \mapsto \{\alpha_1 - 2\delta, \alpha_6 + \delta\}.
\]
On the other hand, the element \( T_1 \in W(E_8^{(1)}) \) which gives Sakai’s \( e-P(E_8^{(1)}) \) equation (3) acts on \( \{\alpha_j \mid 0 \leq j \leq 8\} \) by

\[
T_1 : \{\alpha_1, \alpha_3\} \mapsto \{\alpha_1 - 2\delta, \alpha_3 + \delta\}.
\]

In Equation (12), we have shown that \( T_1 \) as defined in Equation (169) is indeed a translation by \( \alpha_1 \) in \( W(E_8^{(1)}) \).

Here we show that \( \alpha_1 \), a vector of squared length 2 is in the \( W(E_8) \)-orbit of \( h_8 \), the shortest of fundamental weights of \( E_8 \) type, hence \( T_1 \) and any of its conjugations by \( W(E_8) \) correspond to a translation by one of the 240 roots of the \( W(E_8) \) root system and hence are basic (by Definition 13). Then we identify the vector associated with the translation \( T_{J,1} \) and express it as a vector sum of \( \alpha \), and another root from the \( W(E_8) \) root system. This allows us to express \( T_{J,1} \) as a composition of two basic translations in \( W(E_8^{(1)}) \). However, before doing so we need to lay down some basic properties of the group \( W(E_8^{(1)}) \).

The Dynkin diagram of type \( E_8^{(1)} \), \( \Gamma(E_8^{(1)}) \) is given in Figure 1. The corresponding \( E_8^{(1)} \) simple system, \( \Delta^{(1)} = \{\alpha_j \mid 0 \leq j \leq 8\} \) forms a basis for a 9-dimensional real vector space \( V^{(1)} \) equipped with a semidefinite symmetric positive bilinear form given by Equation (13) using the generalised Cartan matrix of type \( E_8^{(1)} \),

\[
C(E_8^{(1)}) = (a_{ij})_{1 \leq i, j \leq 8, 0} = (\alpha_i \cdot \alpha_j^\vee)_{1 \leq i, j \leq 8, 0} = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0
\end{pmatrix}.
\]

The defining relations of \( W(E_8^{(1)}) = \{s_i \mid 0 \leq i \leq 8\} \) can be read off from \( \Gamma(E_8^{(1)}) \) with the rules given in Table 1. Generators \( s_j \in W(E_8^{(1)}) \) act on \( V^{(1)} \) by Equation (51), where \( a_{ij} \) is the \( (i,j) \)-entry of \( C(E_8^{(1)}) \) from Equation (340). We have the null root and the highest root given by,

\[
\delta = \alpha_0 + \tilde{\alpha} = \alpha_0 + \sum_{i=1}^{8} c_i \alpha_i = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8.
\]

where \( c_i \ (1 \leq i \leq 8) \) of \( E_8^{(1)} \) type can be found in Table 2 of Appendix 1. A dual space \( V^{(1)*} \) with basis \( \{h_1, \ldots, h_8, h_8\} \) is given by Definition 4 in which we consider the subspace \( X_0 \) with basis \( \{\pi(\alpha_j^\vee) \mid 1 \leq j \leq n\} \) and the hyperplane \( X_1 = h_8 + X_0 \) (see Definition 75). Since \( \Gamma(E_8^{(1)}) \) is simply-laced, and \( C(E_8) \) is symmetric, we can identify the simple coroots \( \pi(\alpha_j^\vee) \)
with the simple roots $\alpha_j \,(1 \leq j \leq 8)$. Then, by Equation (82) we have

$$\alpha_j = \sum_{k=1}^{8} (C(E_8))_{kj} h_k = \sum_{k=1}^{8} a_{kj} h_k, \quad 1 \leq j \leq 8. \quad (342)$$

By Proposition 10, the bilinear form $(\cdot, \cdot)$ in $\{h_j \mid 1 \leq j \leq 8\}$ basis of $X_0$ is

$$((h_i, h_j))_{1 \leq i, j \leq 8} = C(E_8)^{-1} = \begin{pmatrix}
4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\
5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\
7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\
10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\
8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\
6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\
4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\
2 & 3 & 4 & 6 & 5 & 4 & 3 & 2
\end{pmatrix}. \quad (343)$$

Moreover, by Equation (83) we have

$$h_i = \sum_{k=1}^{8} C(E_8)^{-1}_{ik} \alpha_k \quad \text{for} \quad 1 \leq i \leq n, \quad (344)$$

with $C(E_8)^{-1}$ given in Equation (343).

As it was discussed earlier, the weight lattice and the root lattice of $W(E_8^{(1)})$ are isomorphic. The weight lattice is generated by translations of the fundamental weights $\{u_j \mid 1 \leq j \leq 8\}$ (given by Propositions 17 and 18) or equivalently by translations of the simple coroots $\{t_j \mid 1 \leq j \leq 8\}$ (given by Proposition 15). Moreover, it is enough to know an element of translation by a shortest weight (a basic translation) - any other translations on the weight lattice can be obtained by conjugations or compositions of the basic translations. We see from Equation (343) that the shortest weight is $h_8$ with $|h_8|^2 = 2$. By Equation (344), entries of the last row of $C(E_8)^{-1}$ in Equation (343) gives the coefficients of $\alpha_k \,(1 \leq k \leq 8)$ in $h_8$. Compare these coefficients with those of $\alpha_k \,(1 \leq k \leq 8)$ in $\tilde{\alpha}$ given in Equation (341) we see that $h_8 = \tilde{\alpha}$. That is the $W(E_8)$-orbit of $h_8$ coincides with $\Phi$, the finite $E_8$ root system.

Let us compute the size of orbit of $h_8$ under $W(E_8)$ using Proposition 20. First, the stabilizer of $h_8$ in $W(E_8)$ is given by

$$W(E_8)_{h_8} = \{s_{\alpha_i} \mid \alpha_i \in \Delta, \quad \text{for} \quad i \neq 8\},$$

$$\cong W(E_7). \quad (345)$$
Then the \( W(E_8) \)-orbit of \( h_8 \) is given by,

\[
|W(E_8)h_8| = \frac{|W(E_8)|}{|W(E_7)|} = \frac{2^{14}3^55^27}{2^63^45^27} = 2^4 \cdot 3 \cdot 5 \cdot 7, 
\]

\[= 240, \quad (346)\]

where we have used the orders of Weyl groups of type \( E_7 \) and \( E_8 \) given in Table 2. That is, there are 240 roots in \( \Phi \) of \( E_8 \) type, as we know from again Table 2. Looking at the diagonal entries of the matrix in Equation (343), we see that the weight of the next weight length is \( h_1 \) with \( |h_1|^2 = 4 \). As it will be relevant to the problem we are discussing, we compute also the size of \( W \)-orbit of \( h_1 \). First, the stabilizer of \( h_1 \) in \( W(E_8) \) is given by

\[
W(E_8)h_1 = \langle s_{\alpha_i} \mid \alpha_i \in \Delta, \text{ for } i \neq 1 \rangle, 
\]

\[\cong W(D_7). \quad (347)\]

Then the \( W(E_8) \)-orbit of \( h_1 \) is given by,

\[
|W(E_8)h_1| = \frac{|W(E_8)|}{|W(D_7)|} = \frac{2^{14}3^55^27}{2^63^45^27} = 2160, 
\]

\[= 2160, \quad (348)\]

where we have used the orders of Weyl groups of type \( D_7 \) and \( E_8 \) given in Table 2. That is, there are 2160 weight vectors in the \( W(E_8) \)-orbit of \( h_1 \).

Now we are ready to consider the relationship between the two mappings given by \( T_{1,1} \) and \( T_1 \). By Proposition 18 and the actions of \( T_{1,1} \) and \( T_1 \) given in Equations (338) and (339), we recognise that \( T_{1,1} \) and \( T_1 \) are translations by \( 2h_1 - h_6 \) and \( 2h_1 - h_3 \) on \( X_1 \), respectively.

Next, we compute the squared lengths of \( 2h_1 - h_6 \) and \( 2h_1 - h_3 \) using the symmetric bilinear form given in Equation (343). We have \( |2h_1 - h_6|^2 = 4 \) and \( |2h_1 - h_3|^2 = 2 \).

By Proposition 20 we see that \( 2h_1 - h_3 \) must be in the \( W(E_8) \)-orbit of the shortest weight \( h_8 \), or equivalently \( \Phi \). Using Equation (342) and an inspection of the Cartan matrix in Equation (340) tells us that \( \alpha_1 = 2h_1 - h_3 \), that is \( T_1 \) is the translation by the simple root \( \alpha_1 \),

\[
T_1 = t_1 = t_{\alpha_1 + \delta}, \quad (349)
\]

with \( t_1 \) given by \( j = 1 \) in Proposition 15. Moreover, \( T_1 \) and any of its 240 conjugations under the actions of \( W(E_8) \) are all basic translations.
Figure 14. Addition of the roots in $\Phi$ that give rise to the vector $2h_1 - h_6 = \alpha_{11233445}$ associated with $T_{J,1}$.

Again by Proposition 20, the vector $2h_1 - h_6$ (of squared length 4) is in the $W(E_8)$-orbit of $h_1$ (see Equation (348)). Moreover, it must be a vector sum of two orthogonal roots in $\Phi$, since by Pythagoras theorem we have $(\sqrt{2})^2 + (\sqrt{2})^2 = 2^2 = 4$ (See Figure 14).

Using Equation (344) and $C(E_8)^{-1}$ in Equation (343) we first write $2h_1 - h_6$ as a sum of simple roots, and then as a sum of two orthogonal roots as follows,

$$2h_1 - h_6 = \alpha_{11233445} = \alpha_1 + \alpha_{1233445} = 2h_1 - h_3 + \alpha_{1233445}.$$ (350)

The orthogonality between $\alpha_1$ and $\alpha_{1233445}$ can be checked using the bilinear form given in Equation (340).

The vector $\alpha_{1233445}$ is a root, and hence is in $\Phi$, the $W(E_8)$-orbit of $\alpha_1$. We found the element in $W(E_8)$ that takes $\alpha_1$ to $\alpha_{1233445}$,

$$w(\alpha_1) = s_{345243}(\alpha_1) = \alpha_{1233445},$$ (351)

where we have used the convention of writing roots and products of simple reflections introduced in Remark 5. Then by Equation (187), the element of translation associated to the root $\alpha_{1233445}$ is given by $wT_1w^{-1}$. Finally, the element of translation associated to $\alpha_{11233445} = 2h_1 - h_6$ is given by a composition of $T_1$ and $wT_1w^{-1}$,

$$T_{J,1} = T_1wT_1w^{-1},$$ (352)

with $w$ given in Equation (351). Moreover, the Painlevé equation given by $T_{J,1}$ in [16] is equivalent to those given by any of its 2160 conjugates (size of the orbit of $h_1$, computed in Equation (348)) under the actions of $W(E_8)$.

5.2. A subsystem of type $F_4^{(1)}$ in $E_8^{(1)}$. Let $\Delta^{(1)} = \{\alpha_j \mid 0 \leq j \leq 8\}$ be the $E_8^{(1)}$ simple system with the numbering on $\Gamma(E_8^{(1)})$ given in Figure 11 and $h_j$ ($1 \leq j \leq 8$) the fundamental weights. We have $V^{(1)} = \text{Span}(\Delta^{(1)})$, the real vector space on which $W(E_8^{(1)}) = \{s_i \mid 0 \leq i \leq 8\}$ acts as a group of reflections.
A subsystem of type $F_{4}^{(1)}$, Equation (3), was found for Sakai’s $\text{e-P}(E_{8}^{(1)})$ equation (3) in [4]. It was verified (using MAGMA [5]) that the element $\varphi_{a}$ (given in Equation (7d)), that gives rise to the discrete evolution in Equation (3), is an element of a $F_{4}^{(1)}$ type subgroup of $W(E_{8}^{(1)})$.

Now we explain how such a subgroup arises from $W(E_{8}^{(1)})$ using the normalizer theory of Coxeter groups [8]. We shall see that although the generators of this subgroup, which are the involutions in general, satisfy the defining relations of a Weyl group of type $F_{4}^{(1)}$ they can not be realised as reflections on $V^{(1)}$. We find a subspace in $V^{(1)}$ on which they can be realised as simple reflections of $F_{4}^{(1)}$ type, hence allowing the construction of translational type elements. Finally we show that $\varphi_{a}$ is a translation in this $F_{4}^{(1)}$ subgroup, and an element of quasi-translation in $W(E_{8}^{(1)})$.

**Proposition 21.** Take $J = \{\alpha_{2}, \alpha_{5}, \alpha_{7}, \alpha_{0}\} \subset \Delta^{(1)}$. The normalizer of $W_{J} = \langle s_{2}, s_{5}, s_{7}, s_{0} \rangle$ in $W(E_{8}^{(1)})$ is given by

$$N(W_{J}) = N_{J} \ltimes W_{J} = \langle b_{i} \mid 0 \leq i \leq 4 \rangle \ltimes \langle s_{2}, s_{5}, s_{7}, s_{0} \rangle \cong W(F_{4}^{(1)}) \ltimes W(4A_{1}), \quad (353)$$

with

$$b_{0} = s_{8708}, \quad b_{1} = s_{6576}, \quad b_{2} = s_{4254}, \quad b_{3} = s_{3}, \quad b_{4} = s_{1}. \quad (354)$$

**Proof.** Recall from Section 3.1 that $N(W_{J}) = N_{J} \ltimes W_{J}$, where $N_{J}$ generated by the R- and M-elements, is the group of all elements of $W(E_{8}^{(1)})$ that act permutatively on the set $J$. Here we have $J = \{\alpha_{2}, \alpha_{5}, \alpha_{7}, \alpha_{0}\} \cong 4A_{1}$, and $W_{J} = \langle s_{2}, s_{5}, s_{7}, s_{0} \rangle \cong W(4A_{1})$. As $J$ is not conjugated to any other type $4A_{1}$ subsets of $\Delta^{(1)}$ under the actions of $W(E_{8}^{(1)})$, there is no M-elements, and we have $N_{J} = \langle b_{i} \mid 0 \leq i \leq 4 \rangle$ with

$$b_{0} = v[\alpha_{8}, J] = w_{J,0}[\alpha_{8}]w_{J} = w_{A_{3}+2A_{1}}w_{A_{3}}w_{A_{1}} = w_{A_{3}+2A_{1}}w_{7,8,0}w_{7,0}, \quad (355)$$

$$b_{1} = v[\alpha_{6}, J] = w_{J,0}[\alpha_{6}]w_{J} = w_{5,6,7}w_{5,7}, \quad (355)$$

$$b_{2} = v[\alpha_{4}, J] = w_{J,0}[\alpha_{4}]w_{J} = w_{2,4,5}w_{2,5}, \quad (355)$$

$$b_{3} = v[\alpha_{3}, J] = s_{3}, \quad b_{4} = v[\alpha_{1}, J] = s_{1}. \quad (355)$$

For the evaluations of $b_{i}$ ($0 \leq i \leq 2$) we have used the fact that when two of the simple roots in $J$ are orthogonal to the $a$ in $v[a, J]$ we have $v[a, J] = w_{J,0}w_{J} = w_{A_{3}+2A_{1}}w_{A_{3}}w_{A_{1}}$. Then we can use the expression for $w_{A_{3}}w_{2A_{1}}$ we have found from Example 7 from which we also know that $b_{0}, b_{1}, b_{2}$ are involutions and their actions on $J$. That is, written as permutations on the index set of $J$ we have,

$$b_{0} = (70), \quad b_{1} = (57), \quad b_{2} = (25). \quad (356)$$
That is, all the generators of $N_j$ are involutions, $b_i^2 = 1$ ($0 \leq i \leq 4$), with $b_3 = s_3$ and $b_4 = s_1$
being also reflections. Elements $b_i$ ($0 \leq i \leq 2$) cannot be realised as reflections (as defined in Equation (52)) in $V^{(1)}$, as they do not fix point wisely a co-dimension one hyperplane in $V^{(1)}$. We now proceed to find the orders of the pairwise products of the generators of $N_j$ to identify its Coxeter type.

First of all, from $\Gamma(E_8^{(1)})$ it is obvious that $(b_i b_j)^3 = (s_1 s_3)^3 = 1$. Moreover, $(b_i b_2)^2 = (b_0 b_2)^2 = 1$ and $(b_i b_j)^2 = 1$ for $i = 3, 4$ and $j = 1, 0$, since elements that correspond to disjoint
nodes (or products of nodes which are disjoint) in the Dynkin diagram commute. For the
expressions of $v$ in (357), we have from Equation (67),

$$l(v[\alpha_3, \{\alpha_2, \alpha_5\}]) = l(w_{D_4}) - l(w_{2A_1}) = |\Phi_{D_4}^+| - 2 |\Phi_{A_1}^+| = 12 - 2 = 10,$$

where the value of $|\Phi_{D_4}^+|$ can be found in Table 2. By Equation (68), the two standard
expressions of $v[\{\alpha_3, \alpha_4\}, \{\alpha_2, \alpha_5\}]$ are given by

$$v[\alpha_3, \{\alpha_2, \alpha_5\}]v[\alpha_4, \{\alpha_2, \alpha_5\}] = v[\alpha_4, \{\alpha_2, \alpha_5\}]v[\alpha_3, \{\alpha_2, \alpha_5\}],$$

since

$$l(v[\alpha_3, \{\alpha_2, \alpha_5\}]) + l(v[\alpha_4, \{\alpha_2, \alpha_5\}]) + l(v[\alpha_3, \{\alpha_2, \alpha_5\}])$$

$$+ l(v[\alpha_4, \{\alpha_2, \alpha_5\}]) = l(v[\alpha_4, \{\alpha_2, \alpha_5\}]) + l(v[\alpha_3, \{\alpha_2, \alpha_5\}]) + l(v[\alpha_4, \{\alpha_2, \alpha_5\}])$$

$$+ l(v[\alpha_3, \{\alpha_2, \alpha_5\}]),$$

or $1 + 4 + 1 + 4 = 4 + 1 + 4 + 1$, that is we have $(b_3 b_2)^4 = 1$.

To find the order of $b_2 b_1$ we consider the subset $\{\alpha_4, \alpha_6, \alpha_2, \alpha_5, \alpha_7\} \cong A_5$, noticing that

$$v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}] = s_6576 = b_1, \quad \text{and} \quad v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}] = s_{4254} = b_2.$$  

We have

$$l(v[\{\alpha_4, \alpha_6\}, \{\alpha_2, \alpha_5, \alpha_7\}]) = l(w_{A_5}) - l(w_{3A_1}) = |\Phi_{A_5}^+| - 3 |\Phi_{A_1}^+| = 15 - 3 = 12.$$

The two standard expressions of $v[\{\alpha_4, \alpha_6\}, \{\alpha_2, \alpha_5, \alpha_7\}]$ are given by,

$$v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}]v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}]v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}]$$

$$= v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}]v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}]v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}],$$
since
\[
\begin{aligned}
l(v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}])l(v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}])l(v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}])
\end{aligned}
\]
\[
= l(v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}])l(v[\alpha_6, \{\alpha_2, \alpha_5, \alpha_7\}])l(v[\alpha_4, \{\alpha_2, \alpha_5, \alpha_7\}]),
\]
or \[4 + 4 + 4 = 4 + 4 + 4,\] that is we have \((b_2b_1)^3 = 1.\]

Similarly, by considering the subset \(\{\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\} \cong A_5\) one can show that \((b_1b_0)^3 = 1.\) Orders for the \(\binom{6}{2}\) pairwise products of the generators of \(N_J\) thus obtained are summarized by the Dynkin diagram in Figure [15] using the rules given in Table [I]. That is, we found that the generators of \(N_J = \{b_i \mid 0 \leq i \leq 4\} = \{s_{8708}, s_{6576}, s_{4254}, s_3, s_1\}\) satisfy the defining relations (given in Equation (128)) for a Weyl group of type \(F_4(1).\]

5.2.1. **Sub-root system and translations for** \(N_J \cong \tilde{W}(F_4(1))\). Here we find the under-laying root system of type \(F_4(1)\) for \(N_J\). It spans a subspace in \(V(1)\) on which \(b_i \in N_J \quad (0 \leq i \leq 4)\) can be realised as simple reflections of \(F_4(1)\) type.

**Proposition 22.** Recall that \(V_J = \text{Span}(J)\), and \(V_J^\perp\) is the orthogonal complement of \(V_J\) in \(V(1)\), that is, \(V(1) = V_J \oplus V_J^\perp\). The root system for \(N_J\) given in Equation (354) is generated by \(\beta = \{\beta_i \mid 0 \leq i \leq 4\}\) with
\[
\beta_0 = \alpha_{7880}, \quad \beta_1 = \alpha_{5667}, \quad \beta_2 = \alpha_{2445}, \quad \beta_3 = \alpha_3, \quad \beta_4 = \alpha_1,
\]
is of \(F_4(1)\) type, and \(V_J^\perp = \text{Span}(\beta)\). The group \(N_J = \{b_i \mid 0 \leq i \leq 4\}\) with the \(b_i\)'s given by Equation (355) can be realised as a reflection group of type \(F_4(1)\) on \(V_J^\perp\), with \(b_i\) actings as the reflection along the root \(\beta_i.\)

**Proof.** Let \(\beta = \{\beta_i \mid 0 \leq i \leq 4\}\) with the \(\beta_i\)'s as given in Equation (361). The bilinear form on \(V(1)\) in the \(\Delta^{(1)}\) basis, given by Equation (13) with \(a_{ij}\) being the \((i,j)\)-entry of \(C(E_8^{(1)})\) from Equation (340), is used to check that the \(\beta\)-system is orthogonal to \(V_J\). That is, \(V_J^\perp = \text{Span}(\beta)\). Moreover, we find \(|\beta_i|^2\) for \(0 \leq i \leq 4\) to be 4, 4, 4, 2 and 2, respectively.

Compute \((\beta_i \cdot \beta_j)\gamma\) again using this bilinear form and we have,
\[
C(F_4^{(1)}) = (\beta_i \cdot \beta_j)_{1 \leq i, j \leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix},
\]
which is the generalized Cartan matrix of type \(F_4^{(1)}\) (see Equation (127) in Example [10]). That is, \(\beta\) forms a simple system of \(F_4^{(1)}\) type. This is summarised by the Dynkin diagram in Figure [15] The null root of this \(F_4^{(1)}\) subsystem is given by,
\[ \delta_{F_4} = \beta_0 + \bar{\beta}_{F_4} = \beta_0 + 2\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4. \]  
(363)

\[ = a_{7880} + 2a_{5667} + 3a_{2445} + 4a_3 + 2a_1. \]

\[ = a_0 + 2a_1 + 3a_2 + 4a_3 + 6a_4 + 5a_5 + 4a_6 + 3a_7 + 2a_8, \]

\[ = a_0 + \alpha_{E_8} = \delta_{E_8} = \delta, \]

where \( \bar{\beta}_{F_4} \) and \( \alpha_{E_8} \) denote the highest roots of root systems of type \( F_4 \) and \( E_8 \), respectively. The coefficients of simple roots in \( \bar{\beta}_{F_4} \) and \( \alpha_{E_8} \) can be found in Table 2. We see that \( \delta_{F_4} = \delta_{E_8} = \delta \).

Now, let us look at the actions of \( b_i \) (\( 0 \leq i \leq 4 \)) on the \( \beta \cup J = \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \} \) basis of \( V^{(1)} \). These are computed using Figure 354 by composing the actions of \( s_j \in W(E_8^{(1)}) \) on \( V^{(1)} \) given by Equation (51) and \( C(E_8^{(1)}) \). We have:

\[ b_1 : \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \} \mapsto \]

\[ \{ -\beta_1, \beta_2 + \beta_1, \beta_3, \beta_4, \beta_0 + \beta_1, \alpha_2, \alpha_7, \alpha_5, \alpha_0 \}, \]

\[ b_2 : \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \} \mapsto \]

\[ \{ \beta_1 + \beta_2, -\beta_2, \beta_3 + \beta_2, \beta_4, \beta_0, \alpha_5, \alpha_2, \alpha_7, \alpha_0 \}, \]

\[ b_3 : \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \} \mapsto \]

\[ \{ \beta_1, \beta_2 + 2\beta_3, -\beta_3, \beta_4 + \beta_3, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \}, \]

\[ b_4 : \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \} \mapsto \]

\[ \{ \beta_1, \beta_2, \beta_3 + \beta_4, -\beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \}, \]

\[ b_0 : \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0 \} \mapsto \]

\[ \{ \beta_1 + \beta_0, \beta_2, \beta_3, \beta_4, -\beta_0, \alpha_2, \alpha_5, \alpha_0, \alpha_7 \}. \]

(364)

In the \( \beta \cup J \) basis it is easy to see that \( b_i \) act on the \( \beta \)-system exactly as the reflection along the root \( \beta_i \) of \( F_4^{(1)} \) type (for \( 0 \leq i \leq 4 \)), while acting permutatively on \( J \) as given in Equation 356.

Recall that the element \( \varphi_a \) which gives rise to the discrete evolution in Equation (6), is given as \( \varphi_a = \varphi_s^2 \) with \( \varphi_s \) expressed as a product of simple reflections of \( W(E_8^{(1)}) \) in Equation (7d). The action of \( \varphi_s \) on the simple system \( \Delta^{(1)} \) of \( W(E_8^{(1)}) \) can be computed by composing
the actions of \( s_j \in W(E_8^{(1)}) \) on \( V^{(1)} \) given by Equation (51) and \( C(E_8^{(1)}) \),

\[
\varphi_s : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\} \\
\mapsto \{-\alpha_{123456} + \delta, -\alpha_7, -\alpha_{12345678}, -\alpha_8, -\alpha_{013456780}, \alpha_2, \alpha_4, \alpha_5\},
\]

while its action on the \( \beta \cup J = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0\} \) basis is given by,

\[
\varphi_s : \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0\} \\
\mapsto \{\beta_{123456} - \beta_0, -\beta_{1234}, \beta_{1123456}, \beta_2, -\alpha_7, -\alpha_0, \alpha_2, \alpha_5\}.
\]

We can now use Equation (58) on the \( \beta \)-\( J \)-system to write \( \varphi_s \) as a product of generators of the normalizer \( N(W_J) = N_J \ltimes W_J \) given in Equation (353), and we have,

\[
\varphi_s = b_1 b_0 b_2 b_1 b_4 b_2 b_5 s_5 s_2.
\]  

(366)

That is, \( \varphi_s \) is an element of \( N(W_J) \), and so is \( \varphi_a \).

Recall that by Proposition 16, an affine Weyl group has the following decomposition

\[
W(F_4^{(1)}) = W(F_4) \ltimes Q = \langle b_j \mid 1 \leq j \leq 4 \rangle \ltimes \langle t_j \mid 1 \leq j \leq 4 \rangle,
\]

(367)

where \( t_j \) given by Proposition 15 is the translation by simple coroot \( \pi(\beta_i^\vee) \) \( (1 \leq j \leq 4) \) and \( Q = \langle t_j \mid 1 \leq j \leq 4 \rangle \) is the root lattice of the \( \beta \)-system. Since for \( F_4^{(1)} \) type Weyl group the weight lattice is isomorphic to the root lattice we have also,

\[
W(F_4^{(1)}) = W(F_4) \ltimes P = \langle b_j \mid 1 \leq j \leq 4 \rangle \ltimes \langle U_j \mid 1 \leq j \leq 4 \rangle,
\]

(368)

where \( U_j \) given in Definition 11 is the translation by the fundamental weight \( H_j \) \( (1 \leq j \leq 4) \) of the \( \beta \)-system, and \( P = \langle U_j \mid 1 \leq j \leq 4 \rangle \) is the weight lattice.

From Equation (364), we see that \( b_i \) \( (0 \leq i \leq 2) \) cannot be reflections on the whole of \( V^{(1)} \) since they act permutatively on \( \{\alpha_2, \alpha_5, \alpha_7, \alpha_0\} \). For this reason, we do not expect an element of translation in this \( F_4^{(1)} \) type subsystem to be also a translation in the original \( W(E_8^{(1)}) \) group in general, but rather a quasi-translation. However, first, we need to take a closer look at the coroots and weights of the \( \beta \)-system.

The Dynkin diagram for the dual root system \( \{\pi(\beta_i^\vee) \mid 1 \leq i \leq 4\} \), given in Figure 16, is obtained by reversing the direction of the arrow in the \( F_4^{(1)} \) Dynkin diagram of Figure 15. The lengths of the simple coroots are,

\[
|\pi(\beta_i^\vee)|^2 = |\pi(\beta_i^\vee)|^2 = 1, \quad \text{and} \quad |\pi(\beta_3^\vee)|^2 = |\pi(\beta_4^\vee)|^2 = 2
\]

(369)

computed by Equation (105) using \( |\beta_i|^2 = |\beta_2|^2 = 4 \) and \( |\beta_4|^2 = |\beta_5|^2 = 2 \) found earlier.
\[
\pi(\beta_1^\vee) \quad \pi(\beta_2^\vee) \quad \pi(\beta_3^\vee) \quad \pi(\beta_4^\vee)
\]

**Figure 16.** Dynkin diagram for the dual \(\{\pi(\beta_i^\vee) \mid 0 \leq i \leq 4\}\) system, where we have \(|\pi(\beta_i^\vee)|^2 = |\pi(\beta_i^\vee)|^2 = 1\) and \(|\pi(\beta_3^\vee)|^2 = |\pi(\beta_4^\vee)|^2 = 2\).

From Equations (129) and (130) we have

\[
\begin{pmatrix}
\pi(\beta_1^\vee) \\
\pi(\beta_2^\vee) \\
\pi(\beta_3^\vee) \\
\pi(\beta_4^\vee)
\end{pmatrix} = C(F_4)^T
\begin{pmatrix}
H_1 \\
H_2 \\
H_3 \\
H_4
\end{pmatrix} =
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\begin{pmatrix}
H_1 \\
H_2 \\
H_3 \\
H_4
\end{pmatrix},
\]

(370)

and

\[
\begin{pmatrix}
H_1 \\
H_2 \\
H_3 \\
H_4
\end{pmatrix} = (C(F_4)^T)^{-1}
\begin{pmatrix}
\pi(\beta_1^\vee) \\
\pi(\beta_2^\vee) \\
\pi(\beta_3^\vee) \\
\pi(\beta_4^\vee)
\end{pmatrix} =
\begin{pmatrix}
2 & 3 & 2 & 1 \\
3 & 6 & 4 & 2 \\
4 & 8 & 6 & 3 \\
2 & 4 & 3 & 2
\end{pmatrix}
\begin{pmatrix}
\pi(\beta_1^\vee) \\
\pi(\beta_2^\vee) \\
\pi(\beta_3^\vee) \\
\pi(\beta_4^\vee)
\end{pmatrix},
\]

(371)

respectively. The matrix of symmetric bilinear form \((\ ,\ )\) in \(\{\pi(\beta_i^\vee) \mid 0 \leq i \leq 4\}\) basis is given by Equation (103):

\[
((\pi(\beta_i^\vee), \pi(\beta_j^\vee)))_{1 \leq i,j \leq 4} = \left(\frac{2}{|\beta_i|^2} a_{ij}\right)_{1 \leq i,j \leq 4},
\]

(372)

where we have used the fact that \(|\beta_i|^2\) for \(1 \leq i \leq 4\) are \(4, 4, 2\) and \(2\).

The bilinear form in \(\{H_j \mid 1 \leq j \leq 4\}\) basis is given by Equation (107):

\[
((H_i, H_j))_{1 \leq i,j \leq 4} = (C(F_4)^T)^{-1}
\left(\frac{2}{|\beta_k|^2} \delta_{kj}\right)_{1 \leq k,j \leq 4},
\]

\[
= (C(F_4)^T)^{-1}
\begin{pmatrix}
1/2 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 3 & 2 & 1 \\
3 & 6 & 4 & 2 \\
4 & 8 & 6 & 3 \\
2 & 4 & 3 & 2
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 3/2 & 2 & 1 \\
3/2 & 3 & 4 & 2 \\
2 & 4 & 6 & 3 \\
1 & 2 & 3 & 2
\end{pmatrix}.
\]

(373)

The diagonal entries of the last matrix in Equation (373) tell us that \(|H_j|^2\) for \(1 \leq j \leq 4\) are \(1, 3, 6,\) and \(2\), respectively. That is, \(H_1\) (with \(|H_1|^2 = 1\)) is the shortest of the fundamental weights of this \(F_4^{(1)}\) type \(\beta\)-system, which means that \(U_1\) is a basic translation of the \(\beta\)-system. Recall that for non-simply-laced systems we can identify \(\pi(\beta_i^\vee)\) with \(\frac{\beta_i}{|\beta_i|^2}\), so that
we can express the simple coroots of the $F_4^{(1)}$ system in terms of the simple roots of the $E_8^{(1)}$ system,
\[
\pi(\beta_0^\vee) = \frac{\alpha_{70}}{2} + \alpha_8, \quad \pi(\beta_1^\vee) = \frac{\alpha_{57}}{2} + \alpha_6, \quad \pi(\beta_2^\vee) = \frac{\alpha_{25}}{2} + \alpha_4, \quad \pi(\beta_3^\vee) = \alpha_3, \quad \pi(\beta_4^\vee) = \alpha_1. \tag{374}
\]
Moreover, from Equation 135 in Example 10 we found
\[
\text{In particular from Equation (237) we have the translation by}
\]
\[
\pi(h) \text{ where we have used Equation (342) for the expression in terms of the fundamental weights system. On the other hand, we have,}
\[
\text{moreover,}
\]
\[
\text{where we see that } H_1 \text{ of the } F_4^{(1)} \text{ type subsystem is neither a root nor weight of the } E_8^{(1)} \text{ system. On the other hand, we have,}
\]
\[
2H_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7,
\]
\[
= h_6 - 2h_8, \tag{376}
\]
where we have used Equation (372) for the expression in terms of the fundamental weights $h_j$ $(1 \leq j \leq 8)$, of the original $E_8^{(1)}$ system.

Earlier in Section 3.8 we discussed some elements of translations by coroots in $W(F_4^{(1)})$. In particular from Equation (237) we have the translation by $\pi(\beta_0^\vee)$ is given by,
\[
t_{\beta_0} = b_{1232143}b_2b_{3412321}b_0. \tag{377}
\]
Moreover, from Equation 135 in Example 10 we found $\pi(\beta_0^\vee) = -H_1$. That is, $t_{\beta_0}$ is the translation by $-H_1$, then we have
\[
U_1 = t_{-\beta_0}^{-1} = b_0b_{1232143}b_2b_{3412321}. \tag{378}
\]
$U_1$ acts on the $\Delta^{(1)}$ basis of $V^{(1)}$ by
\[
U_1 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\} \mapsto \{\alpha_1, \alpha_5, \alpha_3, \alpha_4, \alpha_2, \alpha_{123444555666778} - \delta, \alpha_0, \alpha_8 + \delta, \alpha_7\}, \tag{379}
\]
moreover,
\[
U_1^2 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\} \mapsto \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 - \delta, \alpha_7, \alpha_8 + 2\delta, \alpha_0\}. \tag{380}
\]
By Proposition 18 we recognise that $U_1^2$ is a translation in $W(E_8^{(1)})$ by $h_6 - 2h_8$, and that $|h_6 - 2h_8|^2 = |2H_1|^2 = 4|H_1|^2 = 4$. The nature of $U_1$ becomes more evident in the $\beta \cup J$ basis of $V^{(1)}$, we have
\[
U_1 : \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0\} \mapsto \{\beta_1 - \delta, \beta_2, \beta_3, \beta_4, \beta_0 + 2\delta, \alpha_5, \alpha_2, \alpha_0, \alpha_7\}. \tag{381}
\]
where we see that although $U_1$ acts on the $\beta$-system like a translation, it also acts on the subset $J$ as a permutation of order two, (25)(07). We call the element $U_1$, given by Equation (378), a quasi-translation of order two.

In [4], the discrete dynamics of equation (6) is given by $\varphi_a \in W(E_8^{(1)})$ (see Equation (7d)), whose actions on $\Delta^{(1)}$ are,

$$\varphi_a : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\}$$

$$\mapsto \{\alpha_1 + \delta, -\alpha_2, \alpha_23445 - \delta, -\alpha_4, -\alpha_5, -\alpha_12334456 + \delta, -\alpha_7, -\alpha_8, -\alpha_0\},$$

moreover, we have,

$$\varphi_a^2 : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_0\}$$

$$\mapsto \{\alpha_1 + 2\delta, \alpha_2, \alpha_3 - 2\delta, \alpha_4, \alpha_5, \alpha_6 + \delta, \alpha_7, \alpha_8, \alpha_0\}.$$

That is, $\varphi_a^2$ is a translation by $-2h_1 + 2h_3 - h_6$ and $\varphi_a$ a quasi-translation of order two. It can be computed using the symmetric bilinear form given in Equation (343) that $| -2h_1 + 2h_3 - h_6 |^2 = 4$. That is, $U_1$ and $\varphi_a$ are both quasi-translations of order two for some translations by vectors of squared length 4.

**Proposition 23.** $U_1^2$ is related to $\varphi^2$ by a conjugation of an element of $W(F_4) = \langle b_j \mid 1 \leq j \leq 4 \rangle$,

$$XU_1^2X^{-1} = \varphi_a^2, \quad \text{where} \quad X = b_{124321}.$$  

**Proof.** First we know that

$$|h_6 - 2h_8|^2 = |2H_1|^2 = 4|H_1|^2 = 4 = | -2h_1 + 2h_3 - h_6 |^2.$$  

That is $-2h_1 + 2h_3 - h_6$ must be in the same $W(F_4)$-orbit of $2H_1$. Let us compute the size of orbit of $H_1$ under $W(F_4)$, $|W(F_4)H_1|$, using Proposition 20. The stabilizer of $H_1$ in $W(F_4)$ is given by

$$W(F_4)_{H_1} = \langle b_2, b_3, b_4 \rangle,$$

$$\cong W(C_3).$$

Then by Proposition 20 we have,

$$|W(F_4)H_1| = \frac{|W(F_4)|}{|W(C_3)|},$$

$$= \frac{2^73}{2^33!},$$

$$= 2^43 = 24,$$

(387)
where we have used the orders of Weyl groups of type $C_3$ and $F_4$ given in Table 2. We find that,

$$X(2H_1) = X(h_6 - 2h_8) = b_{412321}(h_6 - 2h_8) = -2h_1 + 2h_3 - h_6.$$  \hspace{1cm} (388)

Then by Equation (286) we have Equation (384).

\begin{flushright}
\Box
\end{flushright}

**Proposition 24.** Any element of the form $yXU_1X^{-1}$ ($y \in \langle b_2, b_0 \rangle \times \langle s_2s_5, s_7s_0 \rangle$) has the property that it iterated twice is a translation by $-2h_1 + 2h_3 - h_6$. In particular, we have

$$s_{0752}b_0b_2XU_1X^{-1} = \varphi_a.$$  \hspace{1cm} (389)

**Proof.** Let $y$ be an involution in $N(W_J)$ that commutes with $XU_1X^{-1}$. That is,

$$yXU_1X^{-1} = XU_1X^{-1}y, \quad \text{and} \quad y^2 = 1,$$

then we have,

$$XU_1^2X^{-1} = XU_1X^{-1}XU_1X^{-1},$$

$$= XU_1X^{-1}y^2XU_1X^{-1},$$

$$= XU_1X^{-1}yXU_1X^{-1},$$

$$= (yXU_1X^{-1})^2,$$

$$= \varphi_a^2.$$  \hspace{1cm} (391)

Now it is left to find all involutions in $N(W_J)$ that commute with $XU_1X^{-1}$. From the actions of $XU_1X^{-1}$ on $\beta \cup J$,

$$XU_1X^{-1} : \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_0, \alpha_2, \alpha_5, \alpha_7, \alpha_0\} \mapsto \{\beta_1 + \delta, \beta_2, \beta_3 - \delta, \beta_4 + \delta, \beta_0, \alpha_5, \alpha_2, \alpha_0, \alpha_7\},$$

we see that $XU_1X^{-1}$ commutes with $b_2, b_0$. Moreover, we have

$$XU_1X^{-1}s_2 = s_5XU_1X^{-1}, \quad \text{and} \quad XU_1X^{-1}s_7 = s_0XU_1X^{-1},$$

hence elements $s_2s_5$ and $s_7s_0$ and their products are all involutions that commute with $XU_1X^{-1}$. In particular, we find that

$$s_{0752}b_0b_2XU_1X^{-1} = \varphi_a.$$  \hspace{1cm} (393)

That is, $\varphi_a$ is an element of quasi-translation of order two in a normalizer of $W(4A_1)$ in $W(E_8^{(1)})$ given in Proposition 21. \hspace{1cm} \Box
6. Conclusion

In this work, we reviewed some formulations and properties of the affine Weyl group and demonstrated how they can be used in studying discrete integrable systems with affine Weyl symmetries. We exploited the fact that a Weyl group defined abstractly as a Coxeter system, (that is, a generating set with defining relations) has, on the one hand, a birational representation (symmetries for a nonlinear equation), while on the other, classical linear representations known for studying the many remarkable properties of the group. In particular, we showed that some behaviours of integrable equations are manifestations of certain properties of the Weyl group, such as translation and quasi-translation, hence can be dealt with effectively using a well-chosen linear representation. Computations that may be difficult in the context of the integrable system, such as finding a birational transformation between two nonlinear equations amounted to relating two vectors in a linear representation of the affine Weyl group, as shown for our two examples of systems of type $E_8^{(1)}$ and $F_4^{(1)}$. This approach is useful for understanding and clarifying the nature of the plethora of discrete integrable equations that appear in the literature, since unless by construction, equations rarely come with the full symmetries of their types (as listed in Figure 17) and are even less likely to be in the canonical form.

Finally, properties discussed here for the affine Weyl group have their generalisations in the theory of Coxeter groups, which is a rich and well-developed area of mathematics, with major breakthroughs still being made [11]. It occupies a special place in the theory of groups in that certain questions which are undecidable for general groups such as the normalizer problem [8] can be answered for Coxeter groups. Integrable system occupies a similar position in the theory of equations. It would be interesting and profitable to explore further implications of the many remarkable properties of the Coxeter groups in the context of the integrable system.

Acknowledgement

The author would like to express her gratitude to R. B. Howlett for the enlightening discussions related to the Coxeter groups.
APPENDIX A.

Below, we collect some data from the Weyl groups relevant for our discussions. Except when specifically stated, the index $i$ runs from 1 to $n$. We have: $|\Phi_+|$ is the number of positive roots in a finite root system; $c_j$ are coefficients of $\alpha_j$ in $\tilde{\alpha}$; $|W|$ is the order of the group; $k_j$ are coefficients of $\pi(\alpha_j')$ in $\pi(\tilde{\alpha}')$; the $-$ indicates that the $k_j$’s are the same as the $c_j$’s of that row; $|h_i|^2$ are the squared lengths of the fundamental weights. Finally, we write the coroots for the highest long and short root in terms of the fundamental weights.(Lengths of $\pi(\tilde{\alpha}')$ and $\pi(\tilde{\alpha}'_s)$ are given in Equation (105)).

| Type | $|\Phi_+|$ | $c_j$ | $|W|$ | $k_j$ | $|h_i|^2$ | $\pi(\tilde{\alpha}')$ | $\pi(\tilde{\alpha}'_s)$ |
|------|-----------|------|-------|-------|---------|---------------|----------------|
| $A_n$ | $\frac{n(n+1)}{2}$ | 1, 1, ..., 1 | $(n + 1)!$ | - | $\frac{i(n+1-i)}{(n+1)}$ | $h_1 + h_n$ | - |
| $B_n$ | $n^2$ | 1, 2, 2, ..., 2 | $2^n n!$ | 1, 2, ..., 2 | $i$ | $h_1$ | $2h_2$ |
| $C_n$ | $n^2$ | 2, 2, ..., 2, 1 | $2^n n!$ | 1, 1, 1 | $i$ | $2h_1 h_2$ | $h_2$ |
| $D_n$ | $n(n - 1)$ | 1, 2, ..., 2, 1, 1 | $2^{n-1} n!$ | - | $i$ | $2h_1 h_2$ | $h_2$ |
| $E_6$ | 36 | 1, 2, 3, 2, 1, 2 | $2^7 3^3 5$ | - | $2, \frac{4}{3}, \frac{11}{2}, \frac{11}{2}, \frac{7}{3}$ | $h_1$ | - |
| $E_7$ | 63 | 2, 2, 3, 4, 3, 2, 1 | $2^{10} 3^4 57$ | - | $2, \frac{7}{2}, 6, 12, \frac{15}{2}, 4, \frac{3}{2}$ | $h_1$ | - |
| $E_8$ | 120 | 2, 3, 4, 6, 5, 4, 3, 2 | $2^{14} 3^5 5^2 7$ | - | $4, 8, 14, 30, 20, 12, 6, 2$ | $h_8$ | - |
| $F_4$ | 24 | 2, 3, 4, 2 | $2^7 3^2$ | 2, 4, 3, 2 | 2, 6, 12, 4 | $h_1$ | $h_4$ |
| $G_2$ | 6 | 3, 2 | 12 | 2, 3 | 2, $\frac{2}{3}$ | $h_2$ | $h_1$ |

Table 2. Some useful data of the Weyl groups.

APPENDIX B.

B.1. A birational representation of $W(E_8^{(1)})$. Transformations on the variables $t$, $f$, $g$, and parameters $b_i (1 \leq i \leq 8)$ of Sakai’s $\mathbf{e-P}(E_8^{(1)})$ equation (3) given in Equation (395) satisfy the defining relations of $W(E_8^{(1)}) = \langle s_i \mid 0 \leq i \leq 8 \rangle$ [23]. That is, the transformations generate a birational representation of $W(E_8^{(1)})$.

$$s_2 : \begin{pmatrix} b_1, b_2, b_3, b_4 \\ b_5, b_6, b_7, b_8 \end{pmatrix} \begin{pmatrix} t, f, g \end{pmatrix} \quad (395a)$$

$$\mapsto \begin{pmatrix} b_1 - 3t^2 + \frac{b_1 + b_2}{4}, b_2 - 3t^2 + \frac{b_1 + b_2}{4}, b_3 + \frac{2t + b_1 + b_2}{4}, b_4 + \frac{2t + b_1 + b_2}{4} \\ b_5 + \frac{2t + b_1 + b_2}{4}, b_6 + \frac{2t + b_1 + b_2}{4}, b_7 + \frac{2t + b_1 + b_2}{4}, b_8 + \frac{2t + b_1 + b_2}{4} \end{pmatrix} \begin{pmatrix} t - 2t + b_1 + b_2, t, f, g \end{pmatrix} ;$$

$$s_1 : \begin{pmatrix} t, f, g \end{pmatrix} \mapsto \begin{pmatrix} -t, g, f \end{pmatrix}, \quad s_i : \begin{pmatrix} b_{i-1}, b_i \end{pmatrix} \mapsto \begin{pmatrix} b_i, b_{i-1} \end{pmatrix} \quad (i = 3, \ldots, 7),$$

$$s_8 : \begin{pmatrix} b_1, b_2 \end{pmatrix} \mapsto \begin{pmatrix} b_2, b_1 \end{pmatrix}, \quad s_0 : \begin{pmatrix} b_7, b_8 \end{pmatrix} \mapsto \begin{pmatrix} b_8, b_7 \end{pmatrix}, \quad (395b)$$

where $g$ is given by
\[
\frac{\bar{g} - \varphi(2t - (b_1 - b_2)/2)}{\bar{g} - \varphi(2t - (-b_1 + b_2)/2)} = \frac{f - \varphi(b_2 + t)}{f - \varphi(b_1 + t)} \varphi(t - (b_1 + b_2)/2) - \varphi(2t - (b_1 - b_2)/2) \varphi(2t) - \varphi(t - b_2) g - \varphi(t - b_1).
\] (395c)

The actions of \( T_1 \in W(E_8^{(1)}) \) on \( t, f, g \) and \( b_i (1 \leq i \leq 8) \), given in Equations (3) and (4a), can be computed using the expression of \( T_1 \) given in Equation (4c) by composing the actions of the \( s_i \)'s given in Equation (395) from left to right.

### B.2. Discrete Painlevé equations

Sakai’s classification of 22 types of discrete Painlevé equations \([28]\) listed by their symmetry types are shown in Figure 17. They arise naturally as three classes of difference equations: elliptic-difference (\( e^- \)), q-difference (\( q^- \)) and additive difference (\( d^- \)).

**Figure 17.** Sakai’s classification of 22 types of discrete Painlevé equations listed by symmetry types.

### References

[1] V. Adler. Backlund transformation for the Krichever-Novikov equation. *International Mathematics Research Notices*, 1:1–4, 1998.

[2] V. E. Adler, A. Bobenko, and Y. B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm.Math.Phys.*, 233(3):513–543, 2003.

[3] J. Atkinson. On the lattice-geometry and birational group of the six-point multi-ratio equation. *Proc. A.*, 471(2173):20140612, 16, 2015.

[4] J. Atkinson, P. Howes, N. Joshi, and N. Nakazono. Geometry of an elliptic difference equation related to Q4. *J. Lond. Math. Soc. (2)*, 93(3):763–784, 2016.

[5] W. BOSMA, J. CANNON, and C. PLAYOUST. The magma algebra system i: The user language. *Journal of Symbolic Computation*, 24(3):235–265, 1997.
[6] N. Bourbaki. *Groupes et algèbres de Lie, Chapitre IV–VI*. Hermann, Paris, 1968.

[7] B. Brink. The set of dominance-minimal roots. *J. Algebra*, 206(2):371–412, 1998.

[8] B. Brink and R. B. Howlett. Normalizers of parabolic subgroups in Coxeter groups. *Invent. Math.*, 136(2):323–351, 1999.

[9] A. S. Carstea, A. Dzhumay, and T. Takenawa. Fiber-dependent deinautonomization of integrable 2D mappings and discrete Painlevé equations. *J. Phys. A: Math. Theor.*, 50:405202, (41pp), 2017.

[10] A. Doliwa. Desargues maps and their reductions. *Nonlinear and Modern Mathematical Physics: AIP Conf. Proc.*, 1562(9):30–42, 2013.

[11] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)*, 180(3):1089–1136, 2014.

[12] J. Hietarinta. Searching for CAC-maps. *J. Nonlinear Math. Phys*, 12(sup2):223–230, 2005.

[13] R. Hirota. Discrete analogue of a generalized Toda equation. *J. Phys. Soc. Japan*, 50(11):3785–3791, 1981.

[14] R. B. Howlett. Normalizers of parabolic subgroups of reflection groups. *J. London Math. Soc. (2)*, 21(1):62–80, 1980.

[15] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[16] N. Joshi and N. Nakazono. Elliptic Painlevé equations from next-nearest-neighbor translations on the $E_8^{(1)}$ lattice. *Journal of Physics A: Mathematical and Theoretical*, 50(30):305205, 2017.

[17] N. Joshi, N. Nakazono, and Y. Shi. Geometric reductions of ABS equations on an $n$-cube to discrete Painlevé systems. *J. Phys. A*, 47(50):505201, 16, 2014.

[18] N. Joshi, N. Nakazono, and Y. Shi. Reflection groups and discrete integrable systems. *Journal of Integrable Systems*, 1(1):xyw006 (37 pages), 2016.

[19] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada. $10E_9$ solution to the elliptic Painlevé equation. *J.Phys.A*, (36):L263–L272, 2003.

[20] K. Kajiwara, N. Nakazono, and T. Tsuda. Projective reduction of the discrete painlevé system of type $(a_2 + a_1)(1)$. *International Mathematics Research Notices*, 2011(4):930–966, 2011.

[21] K. Kajiwara, M. Noumi, and Y. Yamada. Geometric aspects of painlevé equations. *Journal of Physics A: Mathematical and Theoretical*, 50(7):073001, jan 2017.

[22] T. Masuda. A $q$-analogue of the higher order Painlevé type equations with the affine Weyl group symmetry of type $D$. *Funkcial. Ekvac.*, 58(3):405–430, 2015.

[23] M. Murata, H. Sakai, and J. Yoneda. Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type $c_{E_8}^{(1)}$. *Journal of Mathematical Physics*, 44(3):1396–1414, 2003.

[24] M. Noumi. Remarks on $\tau$-functions for the difference Painlevé equations of type $E_8$. In *Representation theory, special functions and Painlevé equations—RIMS 2015*, volume 76 of *Adv. Stud. Pure Math.*, pages 1–65. Math. Soc. Japan, Tokyo, 2018.

[25] M. Noumi and Y. Yamada. Affine Weyl groups, discrete dynamical systems and Painlevé equations. *Comm. Math. Phys.*, 199(2):281–295, 1998.

[26] Y. Ohta, A. Ramani, and B. Grammaticos. An affine Weyl group approach to the eight-parameter discrete Painlevé equation. *J. Phys. A*, 34(48):10523–10532, 2001. Symmetries and integrability of difference equations (Tokyo, 2000).
[27] K. Okamoto. Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, espaces des conditions initiales. Science Council of Japan. Jap. J. Math. New Series, 5:1–79, 1979.

[28] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. Comm.Math.Phys., 220(1):165–229, 2001.

[29] Y. Sasano. Higher order Painlevé equations of type $D_l^{(1)}$. RIMS Kōkyūroku, 1473:143–163, 2006.

[30] Y. Shi. Two variations on $(A_3 \times A_1 \times A_1)^{(1)}$ type discrete Painlevé equations. Proc. A., 475(2229):20190299, 18, 2019.

[31] T. Takenawa. Weyl group symmetry of type $D_5^{(1)}$ in the $q$-Painlevé V equation. Funkcial. Ekvac., 46(1):173–186, 2003.