GLOBAL WEAK SOLUTIONS TO LANDAU-LIFSHTIZ SYSTEMS WITH SPIN-POLARIZED TRANSPORT

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Abstract. In this paper, we consider the Landau-Lifshitz-Gilbert systems with spin-polarized transport from a bounded domain in $\mathbb{R}^3$ into $S^2$ and show the existence of global weak solutions to the Cauchy problems of such Landau-Lifshtiz systems. In particular, we show that the Cauchy problem to Landau-Lifshtiz equation without damping but with diffusion process of the spin accumulation admits a global weak solution. The Landau-Lifshtiz system with spin-polarized transport into a compact Lie algebra is also discussed and some similar results are proved. The key ingredients of this article consist of the choices of test functions and approximate equations.

1. Introduction. In physics, the Landau-Lifshitz equation, which is deduced in 1935 by Landau and Lifshitz [23], is a fundamental equation describing the evolution of ferromagnetic spin chain and was proposed on the phenomenological ground in studying the dispersive theory of magnetization of ferromagnets. The equation describes the Hamiltonian dynamics corresponding to the Landau-Lifshitz energy, which is defined as follows.

Let $\Omega$ be a bounded domain in the Euclidean space $\mathbb{R}^3$ and $u$, denoting magnetization vector, be a mapping from $\Omega$ into a unit sphere $S^2 \subset \mathbb{R}^3$. The energy of map $u$ is defined by

$$E(u) = \frac{1}{2} \int_\Omega |\nabla_R^3 u|^2 dx,$$

where the $\nabla_{\mathbb{R}^3}$ and $\Delta_{\mathbb{R}^3}$ denotes the gradient operator and Laplace operator on $\mathbb{R}^3$ respectively, and $dx$ is the volume element. The well-known Landau-Lifshitz equation is

$$u_t = -u \times \Delta_{\mathbb{R}^3} u.$$

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Here $\times$ is vector cross product in $\mathbb{R}^3$. The Landau-Lifshitz equation with dissipation, which can be written as

$$u_t = -u \times \Delta_{\mathbb{R}^3} u + \alpha u \times u_t,$$

was proposed by Gilbert in 1955 [16]. Here $\alpha$ is the damping parameter, which is characteristic of the material, and $\alpha$ is usually called the Gilbert damping coefficient. Hence the Landau-Lifshitz equation with damping term is also called the Landau-Lifshitz-Gilbert (LLG) equation in the literature.

Generally, in physics the Landau-Lifshitz functional is defined by

$$E(u) := \int_{\Omega} \Phi(u) \, dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\Omega} h_d \cdot u \, dx.$$

In the above functional, the first and second terms are the anisotropy and exchange energies, respectively. $\Phi(u)$ is a real function on $S^2$. If one only considers uniaxial materials with easy axis parallel to the OX-axis, for which $\Phi(u) = u_1^2 + u_3^2$. The last term is the self-induced energy, and $h_d = -\nabla w$ is the demagnetizing field. The magnetostatic potential, $w$, solves the differential equation $\Delta w = \text{div}(u\chi_{\Omega})$ in $\mathbb{R}^3$ in the sense of distributions. The solution to this equation is

$$w(x) = \int_{\Omega} \nabla N(x - y) u(y) \, dy,$$

where $N(x) = -\frac{1}{4\pi|x|}$ is the Newtonian potential in $\mathbb{R}^3$.

In the absence of spin currents, the relaxation process of the magnetization distribution is described by the following

$$u_t = -u \times h + \alpha u \times u_t,$$

with $|u| = 1$ and Neumann boundary condition:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega,$$

where $\nu$ represents the outward unit normal on $\partial \Omega$. The local field $h$ of $E(u)$ is just

$$h := -\frac{\delta E(u)}{\delta u} = \Delta u + h_d - \nabla u \Phi.$$

People also care much about another new physical model for the spin magnetization system which takes into account the diffusion process of the spin accumulation through the multilayer and has been presented by Zhang et al. [37, 28]. Later, in [13] García-Cervera and Wang considered an extension to three dimensions of the model derived in [37], and studied this model for spin-polarized transport. For convenience, we call this model as Landau-Lifshitz equation with spin accumulation which is given by

$$\begin{cases}
\partial_t s = -\text{div} J_s - D_0(x) \cdot s - D_0(x) \cdot s \times u, & x \in \Omega_0, \\
\partial_t u = -u \times (h + s) + \alpha \cdot u \times \partial_t u, & x \in \Omega,
\end{cases}$$

(1)

with initial-boundary conditions:

$$\begin{cases}
s(\cdot, 0) = s_0 : \Omega_0 \rightarrow \mathbb{R}^3, & \frac{\partial s}{\partial \nu} \mid_{\partial \Omega_0} = 0, \\
u(\cdot, 0) = u_0 : \Omega \rightarrow S^2, & \frac{\partial u}{\partial \nu} \mid_{\partial \Omega_0} = 0.
\end{cases}$$

Here, $\Omega_0$ is a bounded domain and $\Omega \subset \Omega_0 \subset \mathbb{R}^3$, $s = (s_1, s_2, s_3) \in \mathbb{R}^3$ is the spin accumulation, $u = (u_1, u_2, u_3) \in S^2$ is the precession of the magnetization, $D_0(x)$ is
the diffusion coefficient of the spin accumulation which is a nonnegative measurable function bounded from above, \( \theta \in (0, 1) \) is the spin polarization parameter, \( \alpha \in (0, \infty) \) is the Gilbert damping parameter, and \( J_s \) is the spin current given by
\[
J_s := u \otimes J_e - D_0 \cdot \{ \nabla s - \theta \cdot u \otimes (\nabla \cdot u) \}
\]
where \( J_e \) is the applied electric current.

The spin accumulation \( s \) is defined on \( \Omega_0 \) and the magnetization \( u \) is defined on the magnetic domain \( \Omega \) and extended as zero outside. We will assume that the boundaries of \( \Omega_0 \) and \( \Omega \) are smooth and may intersect each other.

If the spin accumulation \( s \) is not considered, obviously, the system (1) reduces to LLG equation. The LLG equation is interesting in both mathematics and physics, not only because it is closely related to the heat flow of harmonic maps (formally when the Gilbert damping parameter \( \alpha \to \infty \)) and to the Schrödinger flow on the sphere (when the Gilbert damping parameter \( \alpha \to 0 \)) [11], but also because it is of concrete physics background in the study of the magnetization in ferromagnets.

On the other hand, if we set \( D_0 \equiv 1 \) and \( h = \Delta u \) in the above equation, then the equation for the spin accumulation \( s \) in (1) can be written as
\[
\partial_t s = \text{div}(A(u)\nabla s - u \otimes J_e) - s - s \times u,
\]
where the coefficient matrix is expressed as
\[
A(u) := \begin{pmatrix}
1 - \theta \cdot u_1^2 & -\theta \cdot u_1 u_2 & -\theta \cdot u_1 u_3 \\
-\theta \cdot u_1 u_2 & 1 - \theta \cdot u_2^2 & -\theta \cdot u_2 u_3 \\
-\theta \cdot u_1 u_3 & -\theta \cdot u_2 u_3 & 1 - \theta \cdot u_3^2
\end{pmatrix}.
\] (2)

Moreover, the second equation of (1) can be easily written as
\[
\partial_t u - \alpha \cdot u \times \partial_t u = -u \times (\Delta u + s).
\]

Recall that Arnold and Khesin had ever proposed in [3] considering the so-called Landau-Lifshitz model associated with a Lie algebra. In fact, Ding, Wang and Wang in [12] have ever discussed the existence of global weak solution to the Landau-Lifshitz systems from a closed Riemannian manifold into the unit sphere of a compact Lie algebra. In this paper, we are intend to extend these models to the case the unknown functions \( u \) are of Lie algebra value. For this, we need to summarize some fundamental facts on compact Lie algebra.

Let \( G \) be a compact Lie group and \( \mathfrak{g} \) be its associate Lie algebra. It is well-known that there is an \( Ad(G) \)-invariant inner product induced by the Killing form on \( \mathfrak{g} \), denoted by \( \langle \cdot, \cdot \rangle \) (sometimes we omit it and sometimes we denote it by “\(,\)”), such that, for any \( X,Y,Z \in \mathfrak{g} \), there holds
\[
\langle Y, [X,Z] \rangle + \langle [X,Y], Z \rangle = 0,
\]
where \( [\cdot, \cdot] \) is the Lie bracket. It follows that we have the following identity
\[
\langle X, [X,Z] \rangle = 0.
\]
In fact, \( \mathbb{R}^3 \) with cross product is just a compact Lie algebra. For the details we refer to the chapter 4 of [34] or [12].

Now we are going to generalize the above model with spin accumulation. Let \( (\mathfrak{g},[\cdot,\cdot]) \) denote an \( m \)-dimensional compact Lie algebra associated with a compact Lie group \( G \) and \( S_\mathfrak{g}(1) \) denote the unit sphere in \( \mathfrak{g} \) centered at the origin. First, we
would like to consider the following system which takes value in a Lie algebra \( \mathfrak{g} \)
\[
\begin{aligned}
\begin{cases}
\partial_s s = \text{div}(A(x, u) \nabla s - u \otimes J_e) - D_0(x) \cdot s - D_0(x) \cdot [s, u], & x \in \Omega_0, \\
\partial_t u - \alpha[u, \partial_t u] = -[u, h + s], & x \in \Omega,
\end{cases}
\end{aligned}
\]
with initial-boundary conditions:
\[
\begin{aligned}
\begin{cases}
s(\cdot, 0) = s_0 : \Omega_0 \rightarrow \mathfrak{g}, & \frac{\partial s}{\partial \nu_1} \big|_{\partial \Omega_0} = 0, \\
u(\cdot, 0) = u_0 : \Omega \rightarrow S^g_\mathfrak{g}(1), & \frac{\partial u}{\partial \nu_2} \big|_{\partial \Omega} = 0.
\end{cases}
\end{aligned}
\]
Here, \( \Omega \) and \( \Omega_0 (\Omega \subset \Omega_0) \) are two bounded domains in Euclidean space \( \mathbb{R}^n \) with boundaries \( \partial \Omega_0 \) and \( \partial \Omega \) respectively, \( \nu_1 \) and \( \nu_2 \) are the corresponding outward unit normal vectors \( \partial \Omega_0 \) and \( \partial \Omega \). It is worthy to point out that \( \text{dim}(\Omega) = \text{dim}(\mathfrak{g}) \) is needed. Otherwise, the definition of \( h_\alpha \) does not make sense.

On the other hand, we can also make the following extension of the Landau-Lifshitz equation with spin accumulation. Let \((T, \mathfrak{g})\) be an \( n \)-dimensional closed Riemannian manifold with metric \( h = (h_{ij}) \). We propose to consider the following Cauchy problem:
\[
\begin{aligned}
\begin{cases}
\partial_t s = \text{div}(A(x, u) \nabla s - u \otimes J_e) - D_0(x) \cdot s - D_0(x) \cdot [s, u], & (x, t) \in T \times \mathbb{R}^+, \\
\partial_t u - \alpha[u, \partial_t u] = -[u, \Delta u + s], & (x, t) \in T \times \mathbb{R}^+,
\end{cases}
\end{aligned}
\]
with initial conditions:
\[
\begin{aligned}
\begin{cases}
s(\cdot, 0) = s_0 : T \rightarrow \mathfrak{g}, \\
u(\cdot, 0) = u_0 : T \rightarrow S^g_\mathfrak{g}(1),
\end{cases}
\end{aligned}
\]
where \((u, s) : T \times \mathbb{R}^+ \rightarrow S^g_\mathfrak{g}(1) \times \mathfrak{g}\) is an unknown mapping, the coefficient matrix \( A(x, u) \) is symmetric and measurable \( \dim(\mathfrak{g}) \times \dim(\mathfrak{g}) \) matrix defined on \( T \times S^g_\mathfrak{g}(1) \), and \( J_e \) maps \( T \times \mathbb{R}^+ \) onto the tangent bundle \( T \mathbb{T} \) of \( T \) which belongs to \( L^2(T \times \mathbb{R}^+, TT) \).

In recent years, there has been lots of interesting studies for the Landau-Lifshitz equation, concerning its existence, uniqueness and regularities of various kinds of solutions. Before moving on to the next step, we list only a few of the literature that are closely related to our work in the present paper.

First, let us recall some results of Landau-Lifshitz equation with spin accumulation. Garcia-Cervera and Wang [13] applied Galerkin approximation to construct a global weak solution of (1), provided \( \alpha > 0, D_0 \) and \( J_e \) meet some suitable conditions. Following the seminal work of Struwe for the heat flow of harmonic maps, Pu and Wang [27] gave a unique global weak solution to the simplified system (1) from \( \mathbb{R}^2 \) into \( S^2 \). They proved the uniqueness of solution to this problem under the help of Littlewood-Paley theory and the techniques of Besov spaces.

Besides, one also discussed the so-called spin-vector drift-diffusion equations which can be derived from the spinor Boltzmann equation by assuming a moderate spin-orbit coupling [18] and the scattering rates are supposed to be scalar quantities. The existence of global weak solutions to a coupled spin drift-diffusion and Maxwell-Landau-Lifshitz system is proved in [36]. Assuming that the scattering rates are positive definite Hermitian matrices, a more general matrix drift-diffusion model was derived in [35]. The global existence of weak solutions to this model was shown in [21].
The equations are considered in a two-dimensional magnetic layer structure and are supplemented with Dirichlet-Neumann boundary conditions. The spin drift-diffusion model for the charge density and spin density vector is the diffusion limit of a spinorial Boltzmann equation for a vanishing spin polarization constant. The Maxwell-Landau-Lifshitz system consists of the time-dependent Maxwell equations for the electric and magnetic fields and of the Landau-Lifshitz-Gilbert equation for the local magnetization, involving the interaction between magnetization and spin density vector. The existence proof is based on a regularization procedure, $L^2$-type estimates, and Moser-type iterations which yield the boundedness of the charge and spin densities. Furthermore, the free energy is shown to be nonincreasing in time if the magnetization-spin interaction constant in the Landau-Lifshitz equation is sufficiently small.

Next, we also retrospect the work relative to the following LL equation

$$\frac{\partial}{\partial t}u - \alpha [u, \partial_t u] = -[u, \Delta u], \quad x \in M. \tag{5}$$

In case $\alpha = 0$ and $[,]$ is cross product in $\mathbb{R}^3$, for the above system Wang has established the existence of global solution to LL equation without Gilbert damping term defined on a closed Riemannian manifold in \cite{33}. Later, in \cite{12} the authors proved the existence of global weak solutions to (5) on a closed Riemannian manifold $\mathcal{T}$ or $\mathcal{T} = \mathbb{R}^n$ for the case $\alpha = 0$ and $[,]$ is a compact Lie algebra.

For the case $\Omega$ is a bounded domain in $\mathbb{R}^3$, Carbou and Fabrie studied a model of ferromagnetic material governed by a nonlinear Landau-Lifshitz equation coupled with Maxwell equations and proved the existence of weak solutions in \cite{6} (also see \cite{7}). Later, Tilioua \cite{29} employed the penalized method to show the existence of the weak solution to the following Cauchy problem:

$$\begin{cases}
  u_t - \alpha u \times u_t = -\gamma u \times [\Delta u - \mathbb{P}(u) + \beta (J \cdot \nabla u)] & \text{in } \Omega \times (0, T), \\
  u(0, \cdot) = u_0 : \Omega \to \mathbb{S}^2,
\end{cases}$$

with parameters $\alpha > 0$, $\gamma > 0$, $\beta > 0$, where $\mathbb{P}$ denotes the orthogonal projector onto the closure of the space of gradients of smooth functions in $L^2$-topology.

In this paper, we follow the idea in \cite{33} to approach the existence problems of systems (1) and (4). One of the crucial ingredients for the presented analysis here is the choice of auxiliary approximation equations and test functions. It is the aim of the paper at hand to present a proof of the existence of global weak solution. Now we are in the position to present our main conclusions.

**Theorem 1.1.** Let $\Omega_0 \subset \mathbb{R}^3$ is a bounded domain and $\Omega \subset \Omega_0$. Suppose that the following conditions are met:

1. the initial value maps $s_0 \in L^2(\Omega_0, \mathbb{R}^3)$ and $u_0 \in H^1(\Omega, \mathbb{S}^2)$ satisfying $\frac{\partial u_0}{\partial \nu} \bigg|_{\partial \Omega} = 0$, where $\nu$ denotes the corresponding outward unit normal on $\partial \Omega_0$ and $\partial \Omega$.
2. $J_\nu \in L^2(\Omega_0 \times \mathbb{R}^+, \mathbb{R}^3)$, $J_\nu|_{\partial \Omega_0} \in L^2(\partial \Omega_0 \times \mathbb{R}^+, \mathbb{R}^3)$, and $D_0 : \Omega_0 \to \mathbb{R}^+$ is a measurable function and there exist positive constants such that $0 < c \leq D_0(x) \leq C$, a.e. $x \in \Omega_0$.

Then, for any $\alpha \geq 0$, the system (1) admits a global weak solution $(s, u)$ with initial value map $(s_0, u_0)$.

**Theorem 1.2.** Let $\mathcal{T}$ be an $n$-dimensional closed manifolds and $\mathfrak{g}$ be a $m$-dimensional compact Lie algebra. Suppose that the following conditions are met:

1. $A(x, u)$ is a symmetric and measurable $\dim(\mathfrak{g}) \times \dim(\mathfrak{g})$ function matrix defined on $\mathcal{T} \times S^0(1)$ and is Lipshitz continuous with respect to $u$, moreover, there
exist two positive constants \( \theta_1 \) and \( \theta_2 \) such that for any \( (x, u) \in \mathbb{T} \times S_{g}(1) \) and any \( Y \in g \) there holds

\[
\theta_1 \cdot |Y|^2 \leq \langle A(x, u)Y, Y \rangle \leq \theta_2 \cdot |Y|^2;
\]

(2) the initial value maps \( s_0 \in L^2(\mathbb{T}, g) \) and \( u_0 \in H^1(\mathbb{T}, g) \) with \( |u_0| = 1 \) a.e. on \( \mathbb{T} \);

(3) \( J_\varepsilon \in L^2(\mathbb{T} \times \mathbb{R}^+, \mathbb{T} \mathbb{T}) \), \( D_0 : \mathbb{T} \to R^+ \) is a measurable function and there exist positive constants such that \( 0 < c \leq D_0(x) \leq C \), a.e. \( x \in \mathbb{T} \).

Then, for any \( \alpha \geq 0 \), the system (4) admits a global weak solution \((s, u)\) with initial value map \((s_0, u_0)\).

As a direct corollary, we have

**Corollary 1.3.** Let \( \mathbb{T} \) be a \( n \)-dimensional closed manifolds. Suppose that the following conditions hold true

(1) \( A(x, u) \) is a symmetric and measurable \( 3 \times 3 \) function matrix defined on \( \mathbb{T} \times S^2 \) and is Lipschitz continuous with respect to \( u \), moreover, there exist two positive constants \( \theta_1 \) and \( \theta_2 \) such that for any \( (x, u) \in \mathbb{T} \times S^2 \) and any \( Y \in \mathbb{R}^3 \) there holds

\[
\theta_1 \cdot |Y|^2 \leq \langle A(x, u)Y, Y \rangle \leq \theta_2 \cdot |Y|^2;
\]

(2) \( s_0 \in L^2(\mathbb{T}, \mathbb{R}^3) \) and \( u_0 \in H^1(\mathbb{T}, \mathbb{R}^3) \) with \( |u_0| = 1 \) a.e. on \( \mathbb{T} \);

(3) \( J_\varepsilon \in L^2(\mathbb{T} \times \mathbb{R}^+, \mathbb{T} \mathbb{T}) \), \( D_0 : \mathbb{T} \to R^+ \) is a measurable function and there exist positive constants such that \( 0 < c \leq D_0(x) \leq C \), a.e. \( x \in \mathbb{T} \).

Then, for any \( \alpha \geq 0 \), the LL equation with accumulation (4) admits a global weak solution \((s, u)\) with initial value map \((s_0, u_0)\).

2. **Some notations and definitions.** Let \((\mathbb{T}, h)\) be a Riemannian manifold equipped with a metric \( h \). In local coordinates \((x^1, ..., x^n)\) on coordinates \((\mathbb{T}, h)\) and \((u_1, ..., u_m)\) on \( g \),

\[
\Delta u = (\Delta u_1, ..., \Delta u_m)
\]

where

\[
\Delta u_\alpha = \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^i} (h^{ij} \sqrt{h} \frac{\partial u_\alpha}{\partial x^j}), \quad \text{for} \quad \alpha = 1, 2, ..., m,
\]

and \((h^{ij})\) is the inverse of \((h_{ij})\). On the other hand, for a smooth function \( f \) defined on \( \mathbb{T} \), we denote

\[
\nabla f \cdot \nabla u = (\nabla f \cdot \nabla u_1, ..., \nabla f \cdot \nabla u_m)
\]

where

\[
\nabla f \cdot \nabla u_\alpha = \frac{\partial f}{\partial x^i} \frac{\partial u_\alpha}{\partial x^p} h^{ip}, \quad \text{for} \quad \alpha = 1, 2, ..., m.
\]

We define the Sobolev space of the functions with compact Lie algebra value

\[
L^2(M, g) = \{ u : \int_M |u(x)|^2 \, dM < \infty \},
\]

\[
H^1(M, g) = \{ u : |u|, |\nabla u| \in L^2(M) \}
\]

and for maps from \( M \) into \( N \) by

\[
H^1(M, N) = \{ u : u \in H^1(M, \mathbb{R}^K), u(x) \in N \quad \text{a.e.} \ M \}.
\]

Moreover, we define

\[
W^{r,s}_p(M, N) = \{ u : u \in W^{r,s}_p(M, \mathbb{R}^K), u(x) \in N \quad \text{a.e.} \ M \}.
\]
Similarly,

\[ H^1(T, S_g(1)) = \{ u : u \in H^1(T, g), u(x) \in S_g(1) \; \text{a.e.} \; T \}, \]

and \( u \in L^\infty_{loc}(\mathbb{R}^+, H^1(T, S_g(1))) \) means that \( u \in L^\infty([0, T], H^1(T, S_g(1))) \) for any \( T > 0 \).

Let \( \Omega \subset \mathbb{R}^n \) be a smooth domain. The operator is defined by

\[ h_d(u^N) = -\nabla(\nabla N * u^N) : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n, \mathbb{R}^n), \]

where \( N(|x - y|) \) is the classical Newton potential. We will use the following properties of the above operator. For \( u^N, u^M \in L^2(\Omega, \mathbb{R}^n) \) there hold

\[ \int_{\mathbb{R}^n} |h_d(u^N)|^2 \, dx \leq \int_{\Omega} |u^N|^2 \, dx, \]

and

\[ \int_{\mathbb{R}^n} |h_d(u^N) - h_d(u^M)|^2 \, dx \leq \int_{\Omega} |u^N - u^M|^2 \, dx. \]

The fields \( h_d(u^N) \) can be defined equivalently by

\[ h_d(u^N) = -\nabla w^N, \]

where

\[ \Delta w^N = \text{div}(u^N \chi_{\Omega}) \; \text{in} \; \mathbb{R}^n \]

in the sense of distributions. Multiplying the equation by any \( v \in H^1(\mathbb{R}^n) \) integrating by parts, we obtain

\[ \int_{\mathbb{R}^n} \nabla w^N \cdot \nabla v = \int_{\Omega} u^N \cdot \nabla v. \]

Taking \( v = w^N \) in the above identity we get

\[ \int_{\mathbb{R}^n} |\nabla w^N|^2 = \int_{\Omega} u^N \cdot \nabla w^N \leq \left( \int_{\mathbb{R}^n} |\nabla w^N|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |u^N|^2 \right)^{\frac{1}{2}}. \]

In fact, the following lemma was shown in [6] and [13] although they only need to consider the case \( \dim(M) = n = 3 \) therein. For more details we refer to page 196 in [22].

**Lemma 2.1.** For any \( u^N, u^M \in L^2(\Omega, \mathbb{R}^n) \), the operator \( h_d \) defined as above satisfies

\[ \int_{\mathbb{R}^n} |h_d(u^N)|^2 \, dx \leq \int_{\Omega} |u^N|^2 \, dx, \]

and

\[ \int_{\mathbb{R}^n} |h_d(u^N) - h_d(u^M)|^2 \, dx \leq \int_{\Omega} |u^N - u^M|^2 \, dx. \]

Moreover, if \( u^N \) belongs to \( W^{1,p}(\Omega) \) and \( p \in (0, +\infty) \), the restriction of \( h_d(u^N) \) to \( \Omega \) belongs to \( W^{1,p}(\Omega) \) and there exists a constant \( C \) such that

\[ \|h_d(u^N)\|_{W^{1,p}(\Omega)} \leq C\|u^N\|_{W^{1,p}(\Omega)}. \]

Next, we give two definitions on weak solutions.
Definition 2.2. In the case $\alpha > 0$, the function pair $(s, u)$ with
\[ s \in L^2([0, T], H^1(\Omega_0, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\Omega_0, \mathbb{R}^3)) \]
and
\[ u \in L^\infty([0, T], H^1(\Omega, S^2)) \cap W^{1,1}_2(\Omega \times [0, T], S^2) \]
is called a weak solution to system (1) with initial values $(s_0, u_0)$ if $(s, u)$ satisfies
\[
\int_{\Omega_0} \langle s(T), \phi(T) \rangle \, dx - \int_{\Omega_0} \langle s_0, \phi(0) \rangle \, dx - \int_0^T dt \int_{\Omega_0} \langle (s, \partial_t \phi) + (u, \nabla_j \phi) \rangle \, dx \\
= - \int_0^T dt \int_{\partial \Omega_0} \langle J_e, \nu \rangle \cdot \langle u, \phi \rangle \, d\sigma - \int_0^T dt \int_{\Omega_0} D_0(x) \langle A(u) \nabla s, \nabla \phi \rangle \, dx \\
+ \int_0^T dt \int_{\Omega_0} \langle (u \times s, \phi) - (s, \phi) \rangle D_0(x) \, dx,
\]
and
\[
\int_{\Omega} \langle u(T), \varphi(T) \rangle \, dx - \int_{\Omega} \langle u_0, \varphi(0) \rangle \, dx - \alpha \int_0^T \int_{\Omega} \langle u \times u_t, \varphi \rangle \, dxdt \\
= \int_0^T \int_{\Omega} \langle u, \varphi_t \rangle \, dxdt + \int_0^T \int_{\Omega} \langle u \times \nabla u, \nabla \varphi \rangle \, dxdt \\
- \int_0^T \int_{\Omega} \langle u \times \{h_d(u) - \nabla_u \Phi(u) + s\}, \varphi \rangle \, dxdt,
\]
for all $\varphi \in C^\infty(\Omega \times [0, T], \mathbb{R}^3)$ and $\phi \in C^\infty(\Omega_0 \times [0, T], \mathbb{R}^3)$. Here $d\sigma$ is the volume element of $\partial \Omega_0$ and $A(u)$ is given by (2).

In the case $\alpha = 0$, we say that the function pair $(s, u)$, which satisfies $s \in L^2([0, T], H^1(\Omega_0, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\Omega_0, \mathbb{R}^3))$ and $u \in L^\infty([0, T], H^1(\Omega, S^2))$, is a weak solution to system (1) with initial values $s_0$ and $u_0$ if there hold true
\[
\int_{\Omega_0} \langle s(T), \phi(T) \rangle \, dx - \int_{\Omega_0} \langle s_0, \phi(0) \rangle \, dx - \int_0^T dt \int_{\Omega_0} \langle (s, \partial_t \phi) + (u, \nabla_j \phi) \rangle \, dx \\
= - \int_0^T dt \int_{\partial \Omega_0} \langle J_e, \nu \rangle \cdot \langle u, \phi \rangle \, d\sigma - \int_0^T dt \int_{\Omega_0} D_0(x) \langle A(u) \nabla s, \nabla \phi \rangle \, dx \\
+ \int_0^T dt \int_{\Omega_0} \langle (u \times s, \phi) - (s, \phi) \rangle D_0(x) \, dx,
\]
and
\[
\int_{\Omega} \langle u(T), \varphi(T) \rangle \, dx - \int_{\Omega} \langle u_0, \varphi(0) \rangle \, dx - \int_0^T \int_{\Omega} \langle u \times u_t, \varphi \rangle \, dxdt \\
= \int_0^T \int_{\Omega} \langle u \times \nabla u, \nabla \varphi \rangle \, dxdt - \int_0^T \int_{\Omega} \langle u \times \{h_d(u) - \nabla_u \Phi(u) + s\}, \varphi \rangle \, dxdt.
\]

Definition 2.3. In the case $\alpha > 0$, the function pair $(s, u)$ of Lie algebra value, which satisfies
\[ s \in L^2([0, T], H^1(\mathbb{T}, g)) \cap L^\infty([0, T], L^2(\mathbb{T}, g)) \]
and
\[ u \in L^\infty([0, T], H^1(\mathbb{T}, S^1(1))) \cap W^{1,1}_2(\mathbb{T} \times [0, T], g), \]

is called a weak solutions to system (4) with initial values \((s_0, u_0)\) if it satisfies
\[
\int_T \langle s(T), \phi(T) \rangle dT - \int_T \langle s_0, \phi(0) \rangle dT - \int_0^T dt \int_T \langle s, \partial_t \phi \rangle dT
- \int_0^T dt \int_T \langle u, \nabla J_s \phi \rangle = - \int_0^T dt \int_T \sum_{p=1}^n \langle A(u) \nabla e_p s, \nabla e_p \phi \rangle
+ \int_0^T dt \int_T (\langle [u, s], \phi \rangle - \langle s, \phi \rangle) D_0(x) dT,
\]
and
\[
\int_T \langle u(T), \varphi(T) \rangle dT - \int_T \langle u_0, \varphi(0) \rangle dT - \alpha \int_0^T \int_T \langle [u, \partial_t u], \varphi \rangle dT dt
= \int_0^T \int_T \langle u, \varphi_1 \rangle dT dt + \int_0^T \int_T \langle [u, \nabla e_p u], \nabla e_p \varphi \rangle dT dt - \int_0^T \int_T \langle [u, s], \varphi \rangle dT dt,
\]
for all \(\varphi, \phi \in C^\infty(T \times [0, T], \mathfrak{g})\). Here \(\{e_p : 1 \leq p \leq n\}\) is a local orthonormal frame on \(T\).

**Definition 2.4.** In the case \(\alpha = 0\), we say that the function pair \((s, u)\) of Lie algebra value on \(T \times [0, T]\), which satisfies \(s \in L^2([0, T], H^1(T, \mathfrak{g})) \cap L^\infty([0, T], L^2(T, \mathfrak{g}))\) and \(u \in L^\infty([0, T], H^1(T, S_0(1)))\) is a weak solution to system (4) with initial value map \((s_0, u_0)\) if there hold true
\[
\int_T \langle s(T), \phi(T) \rangle dT - \int_T \langle s_0, \phi(0) \rangle dT - \int_0^T dt \int_T \langle s, \partial_t \phi \rangle dT
- \int_0^T dt \int_T \langle u, \nabla J_s \phi \rangle = - \int_0^T dt \int_T \sum_{p=1}^n \langle A(u) \nabla e_p s, \nabla e_p \phi \rangle
+ \int_0^T dt \int_T (\langle [u, s], \phi \rangle - \langle s, \phi \rangle) D_0(x) dT,
\]
and
\[
\int_T \langle u(T), \varphi(T) \rangle dT - \int_T \langle u_0, \varphi(0) \rangle dT + \int_0^T \int_T \langle [u, \partial_t u], \varphi \rangle dT dt
= \int_0^T \int_T \langle u, \varphi_1 \rangle dT dt + \int_0^T \int_T \langle [u, \nabla e_p u], \nabla e_p \varphi \rangle dT dt,
\]
for all \(\varphi, \phi \in C^\infty(T \times [0, T], \mathfrak{g})\). Here \(\{e_p : 1 \leq p \leq n\}\) is a local orthonormal frame on \(T\).

3. Initial-boundary problem on a bounded domains in \(\mathbb{R}^3\). In this section, we discuss the following
\[
\begin{align*}
\partial_t s &= -\text{div} J_s - D_0(x) \cdot s - D_0(x) \cdot s \times u, \quad x \in \Omega_0, \\
\partial_t u &= -u \times (h + s) + \alpha \cdot u \times \partial_t u, \quad x \in \Omega,
\end{align*}
\]
with initial-boundary conditions:
\[
\begin{align*}
s(\cdot, 0) &= s_0 : \Omega_0 \longrightarrow \mathbb{R}^3, \quad \frac{\partial s}{\partial u} \bigg|_{\partial \Omega_0} = 0, \\
u(\cdot, 0) &= u_0 : \Omega \longrightarrow S^2, \quad \frac{\partial u}{\partial u} \bigg|_{\partial \Omega} = 0.
\end{align*}
\]
Here,
\[ J_s = J_s(x, u) := u \otimes J_e - D_0 \cdot \{ \nabla s - \theta \cdot u \otimes (\nabla s \cdot u) \}, \]
and
\[ h := \Delta u + h_d(u) - \nabla u \Phi(u), \]
with
\[ h_d(u) := -\nabla(N^* u), \]
\[ N(x) := -\frac{1}{4\pi|x|}. \]
\( \Omega \subset \Omega_0 \) are two bounded domains in Euclidean space \( \mathbb{R}^3 \) with boundaries \( \partial \Omega \) and \( \partial \Omega_0 \) respectively, \( \nu \) denotes the corresponding outward unit normal on \( \partial \Omega_0 \) and \( \partial \Omega \).

We employ the following auxiliary approximation system
\[
\begin{aligned}
\partial_t s &= -\text{div}J_s(x, \mathcal{J}(u)) - D_0(x) \cdot s - D_0(x) \cdot s \times \mathcal{J}(u), \quad x \in \Omega, \\
\partial_t u &= \varepsilon \Delta u - \mathcal{J}(u) \times \{ \Delta u + h_d(u) + \nabla u \Phi(\mathcal{J}(u)) + s \} + \alpha \mathcal{J}(u) \times \partial_t u, \quad x \in \Omega,
\end{aligned}
\]
with initial-boundary conditions:
\[
\begin{aligned}
s(\cdot, 0) &= s_0 : \Omega_0 \rightarrow \mathbb{R}^3, \quad \frac{\partial s}{\partial \nu} \big|_{\partial \Omega_0} = 0, \\
u(\cdot, 0) &= u_0 : \Omega \rightarrow S^2, \quad \frac{\partial u}{\partial \nu} \big|_{\partial \Omega} = 0.
\end{aligned}
\]
Here, we have set
\[ \mathcal{J}(u) = \frac{u}{\max\{|u|, 1\}}. \]

It should be pointed out that the above \( \Phi(u) \) has been extended to the closed ball \( \overline{B}(1) \subset \mathbb{R}^3 \). In fact, we can extend \( \Phi(z) \) by
\[
\tilde{\Phi}(z) = \begin{cases} 
\zeta(|z|^2) \Phi \left( \frac{z}{\max\{|\delta_0, |z|^2\}} \right), & |z|^2 > \delta_0, \\
0, & |z|^2 \leq \delta_0,
\end{cases}
\]
where \( \zeta(t) : [0, 1] \rightarrow [0, 1] \) is a \( C^2 \)-smooth function with \( \zeta(t) \equiv 0 \) on \([0, 2\delta_0] \) \((2\delta_0 < 1)\) and \( \zeta(1) = 1 \). It is easy to see that \( \tilde{\Phi} \) is \( C^2 \)-smooth on \( \overline{B}(1) \). For simplicity, we still denote \( \tilde{\Phi} \) by \( \Phi \).

3.1. **Galerkin approximation: A priori estimates.** For convenience, we set \( \Omega_1 := \Omega \). Let \( \lambda_i^j (j = 0, 1 \text{ and } i = 1, 2, \cdots) \) \((0 = \lambda_1^1 \leq \lambda_2^1 \leq \cdots \leq \lambda_i^1 \leq \cdots)\) be the eigenvalues of the operator \(-\Delta\) on the domain
\[
X_j := \left\{ \omega \in H^2(\Omega_j) : \frac{\partial \omega}{\partial \nu} \big|_{\partial \Omega_j} = 0 \right\},
\]
and \( \omega_j^i \) is the normalized eigenfunction corresponding to \( \lambda_i^j \). That is to say,
\[
\begin{aligned}
-\Delta \omega_j^i &= \lambda_i^j \cdot \omega_j^i, \\
\frac{\partial \omega_j^i}{\partial \nu} \big|_{\partial \Omega_j} &= 0.
\end{aligned}
\]

According to Galerkin approximation, we define
\[
u^N(x, t) := \sum_{i=1}^{N} \beta_i^N(t) \omega_i^1(x) \quad \text{and} \quad s^N(x, t) := \sum_{i=1}^{N} \beta_i^N(t) \omega_i^0(x). \]

\[ J_s = J_s(x, u) := u \otimes J_e - D_0 \cdot \{ \nabla s - \theta \cdot u \otimes (\nabla s \cdot u) \}, \]
and
\[ h := \Delta u + h_d(u) - \nabla u \Phi(u), \]
with
\[ h_d(u) := -\nabla(N^* u), \]
\[ N(x) := -\frac{1}{4\pi|x|}. \]
Here \( \{ \beta_i^N(t), \rho_i^N(t) \} \) are unknown functions and assumed to satisfy the following ODE:

\[
\begin{aligned}
\frac{d \rho_i^N}{dt} &+ \sum_{k=1}^N \rho_k^N \int_{\Omega_0} D_0 \omega_0^k \omega_i^1 + \sum_{k=1}^N \rho_k^N \times \int_{\Omega_0} D_0 \omega_0^k \omega_0^l \mathcal{J} \left( \sum_{l=1}^N \beta_i^N \omega_1^l \right) \\
&= -\sum_{k=1}^N \int_{\Omega_0} D_0 \cdot A \left( \mathcal{J} \left( \sum_{l=1}^N \beta_i^N \omega_1^l \right) \right) \rho_k^N \cdot \nabla \omega_0^k \cdot \nabla \omega_0^i \\
&\quad + \int_{\Omega_0} \mathcal{J} \left( \sum_{l=1}^N \beta_i^N \omega_1^l \right) \cdot \nabla \omega_0^i - \int_{\partial \Omega_0} \mathcal{J} \left( \sum_{k=1}^N \beta_k^N \omega_1^k \right) \cdot \omega_0^i \cdot \langle J_e, \nu \rangle, \\
\frac{d \beta_i^N}{dt} &- \alpha \sum_{k=1}^N \int_{\Omega_1} \mathcal{J} \left( \sum_{l=1}^N \beta_l^N \omega_1^l \right) \times \frac{d \beta_k^N}{dt} \omega_1^k \omega_1^i - \varepsilon \int_{\Omega_1} \beta_k^N (-\lambda \omega_1^k) \omega_1^i \\
&= -\int_{\Omega_1} \mathcal{J} \left( \sum_{l=1}^N \beta_l^N \omega_1^l \right) \times \left( \sum_{k=1}^N \beta_k^N (-\lambda \omega_1^k) + \sum_{k=1}^N \rho_k^N \cdot \omega_0^k \right) \omega_1^i \\
&\quad - \int_{\Omega_1} \mathcal{J} \left( \sum_{l=1}^N \beta_l^N \omega_1^l \right) \times \nabla \Phi \left( \mathcal{J} \left( \sum_{l=1}^N \beta_l^N \omega_1^l \right) \right) \omega_1^i \\
&\quad + \int_{\Omega_1} \mathcal{J} \left( \sum_{l=1}^N \beta_l^N \omega_1^l \right) \times \nabla \left( \nabla N \cdot \sum_{k=1}^N \beta_k^N \omega_1^k \right) \omega_1^i,
\end{aligned}
\] (8)

where \( A(u) \) is given by (2). It is easy to check that (8) admits a local solution existing in the interval \([0, \tau]\) for some \( \tau > 0 \). So we get

\[
\begin{aligned}
\int_{\Omega_0} \partial_t s^N \cdot \omega_0^i + \int_{\Omega_0} D_0 \cdot s^N \cdot \omega_0^i &- \int_{\Omega_0} D_0 \cdot \mathcal{J}(u^N) \times s^N \cdot \omega_0^i \\
&= -\int_{\Omega_0} D_0 \cdot A(\mathcal{J}(u^N)) \nabla s^N \cdot \nabla \omega_0^i + \int_{\Omega_0} \mathcal{J}(u^N) \cdot \nabla \omega_0^i \\
&\quad - \int_{\partial \Omega_0} \mathcal{J}(u^N) \cdot \omega_0^i \cdot \langle J_e, \nu \rangle, \\
\int_{\Omega_1} \partial_t u^N \cdot \omega_1^i - \alpha \int_{\Omega_1} \mathcal{J}(u^N) \times \partial_t u^N \cdot \omega_1^i + \int_{\Omega_1} \mathcal{J}(u^N) \times h_d(u^N) \cdot \omega_1^i \\
&= -\int_{\Omega_1} \mathcal{J}(u^N) \times (\Delta u^N + s^N) \cdot \omega_1^i + \varepsilon \int_{\Omega_1} \Delta u^N \cdot \omega_1^i \\
&\quad - \int_{\Omega_1} \mathcal{J}(u^N) \times \nabla \Phi(\mathcal{J}(u^N)) \cdot \omega_1^i,
\end{aligned}
\] (9)

with initial conditions

\[
u^N(0) := \sum_{j=1}^N \left( \int_{\Omega_1} u_0 \cdot \omega_1^j \right) \omega_1^j, \quad s^N(0) := \sum_{j=1}^N \left( \int_{\Omega_0} s_0 \cdot \omega_0^j \right) \omega_0^j.
\]
Multiplying both sides of the first equation of (9) by $\rho_1^N$ and summing $i$ from 1 to $N$ yield
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} |s^N|^2 + \int_{\Omega_0} D_0 \cdot \langle A(u^N) \nabla s^N, \nabla s^N \rangle + \int_{\Omega_0} D_0 \cdot |s^N|^2 = \int_{\Omega_0} \langle \nabla s^N, J(u^N) \rangle - \int_{\partial \Omega_0} \langle J(u^N), s^N \rangle \cdot (J_e, \nu). \tag{10}
\]
From (2) it follows that
\[
(1 - \theta) \cdot |\nabla s^N|^2 \leq \langle A(J(u^N)) \nabla s^N, \nabla s^N \rangle.
\]
Combining the assumption $0 < c \leq D_0(x) \leq C$, a.e. $x \in \Omega_0$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} |s^N|^2 + (1 - \theta)c \int_{\Omega_0} |\nabla s^N|^2 + c \int_{\Omega_0} |s^N|^2 \leq \int_{\Omega_0} \langle \nabla s^N, J(u^N) \rangle - \int_{\partial \Omega_0} \langle J(u^N), s^N \rangle \cdot (J_e, \nu)
\leq \int_{\Omega_0} |J_e| \cdot |\nabla s^N| + \int_{\partial \Omega_0} |s^N| \cdot |J_e|
\leq \eta \int_{\Omega_0} |\nabla s^N|^2 + \frac{1}{4\eta} \int_{\Omega_0} |J_e|^2 + \eta \int_{\partial \Omega_0} |s^N|^2 + \frac{1}{4\eta} \int_{\partial \Omega_0} |J_e|^2
\leq \eta \int_{\Omega_0} |\nabla s^N|^2 + \frac{1}{4\eta} \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right)
+ \eta \cdot \tilde{c} \left( \int_{\Omega_0} |s^N|^2 + \int_{\Omega_0} |\nabla s^N|^2 \right)
= \frac{1}{4\eta} \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right) + \eta \cdot (1 + \tilde{c}) \int_{\Omega_0} |\nabla s^N|^2 + \eta \cdot \tilde{c} \int_{\Omega_0} |s^N|^2,
\]
where $\tilde{c}$ is the Sobolev constant of the imbedding $H^1(\Omega_0) \hookrightarrow L^2(\partial \Omega_0)$. For the details of the boundary trace imbedding theorem we refer to Theorem 5.36 in page 164 of [1]. We integrate the above inequality on $[0, t]$ and then pick $\eta = \frac{(1 - \theta)c}{2(1 + \tilde{c})}$ to derive the following
\[
\int_{\Omega_0} |s^N(t)|^2 + (1 - \theta)c \int_0^t \int_{\Omega_0} |\nabla s^N|^2 + c \int_0^t \int_{\Omega_0} |s^N|^2 \tag{11}
\leq \int_{\Omega_0} |s_0|^2 + \frac{1}{2\eta} \int_0^t d\tau \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right)
= \sum_{i=1}^N \int_{\Omega_0} s_0 \cdot \omega_0 \cdot \omega_0^i \int_0^t d\tau \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right)
\leq \sum_{i=1}^N \int_{\Omega_0} s_0 \cdot \omega_0 \cdot \omega_0^i \int_0^t d\tau \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right)
= \int_{\Omega_0} |s_0|^2 + \frac{1}{2\eta} \int_0^t d\tau \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right).
\]
Similarly, multiplying both sides of the second equation of (9) by $\beta_1^N$ and summing $i$ from 1 to $N$ and integrating by parts, we can derive
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |u^N|^2 + \varepsilon \int_{\Omega_1} |\nabla u^N|^2 = 0.
\]
So, we have
\[ \int_{\Omega_1} |u^N|^2 + 2\varepsilon \int_0^T dt \int_{\Omega_1} |\nabla u^N|^2 = \int_{\Omega_1} |u_0^N|^2 \leq \int_{\Omega_1} |u_0|^2 = \text{vol}(\Omega_1), \] (12)
where \( \text{vol}(\Omega_1) \) is the volume of \( \Omega_1 \).

Since
\[ \int_{\Omega_1} |u^N(t)|^2 = \int_{\Omega_1} \left( \sum_{i=1}^N \beta_i^N(t) \omega_i^1, \sum_{j=1}^N \beta_j^N(t) \omega_j^1 \right) = \sum_{i=1}^N |\beta_i^N(t)|^2 \]
and
\[ \int_{\Omega_0} |s^N(t)|^2 = \sum_{i=1}^N |\rho_i^N(t)|^2, \]
then, from the above estimates on \( \|u^N\|_{L^2} \) and \( \|s^N\|_{L^2} \) we know that \( \rho_i^N(t) \) and \( \beta_i^N(t) \) can be extended to \([0, T]\) for any \( T > 0 \) and \( i \). That is to say, \( \{s^N\} \) and \( \{u^N\} \) can be extended to \([0, T]\).

From the first equation of (9) it follows that
\[
\left. \begin{array}{l}
\int_0^T dt \int_{\Omega_0} \partial_t s^N \cdot \omega_0^i \\
= -\int_0^T dt \int_{\Omega_0} D_0 \cdot s^N \cdot \omega_0^i + \int_0^T dt \int_{\Omega_0} \mathcal{J}(u^N) \times s^N \cdot \omega_0^i \cdot D_0 \\
- \int_0^T dt \int_{\Omega_0} D_0 \cdot A(\mathcal{J}(u^N)) \nabla s^N \cdot \nabla \omega_0^i + \int_0^T dt \int_{\Omega_0} \mathcal{J}(u^N) \cdot \nabla J_s \omega_0^i, \\
- \int_0^T dt \int_{\partial \Omega_0} \mathcal{J}(u^N) \cdot \omega_0^i \cdot (J_\varepsilon, \nu) \\
\end{array} \right\} \leq C \left( \begin{array}{c}
\frac{C}{2} \int_{\Omega_0} |\nabla s^N|^2 + \frac{1 + C}{2} \cdot T \int_{\Omega_0} |\nabla \omega_0^i|^2 + \frac{T}{2} \int_{\partial \Omega_0} |\omega_0^i|^2 \\
+ C \int_{\Omega_0} |s^N|^2 + C \cdot T + \frac{1}{2} \int_{\Omega_0} dt \left( \int_{\Omega_0} |J_s|^2 + \int_{\partial \Omega_0} |J_e|^2 \right) \\
\end{array} \right) \\
+ \left( \begin{array}{c}
\frac{C}{2c} + \frac{C}{4(1 - \theta)c} \int_{\Omega_0} |s_0|^2 + \frac{1 + C + \tilde{c}}{2} \cdot T \cdot \int_{\Omega_0} |\nabla \omega_0^i|^2 + (C + \tilde{c}/2) \cdot T \\
\end{array} \right) \int_0^T dt \left( \int_{\Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_e|^2 \right)
\]
where we have used (11) and the fact that \( ||\omega_0^i||_{L^2(\Omega_0)} = 1 \). This leads to
\[ \int_0^T dt \int_{\Omega_0} \partial_t s^N \cdot \omega_0^i \leq C(\tilde{c}, \theta, s_0, T, J_\varepsilon, \Omega_0, c, C) \cdot ||\omega_0^i||_{H^1(\Omega_0)}. \] (13)

Therefore, \( \{\partial_t s^N\} \) is uniformly bounded in \( L^2([0, T], H^{-1}(\Omega_0, \mathbb{R}^3)) \).

Multiplying both sides of the second equation of (9) by \(-\lambda_i \beta_i^N\) and summing \( i \) from 1 to \( N \), we get
\[
\int_{\Omega_1} \langle u_i^N, \Delta u^N \rangle - \alpha \int_{\Omega_1} \langle \mathcal{J}(u^N) \times u_i^N, \Delta u^N \rangle + \int_{\Omega_1} \langle \mathcal{J}(u^N) \times h_d(u^N), \Delta u^N \rangle \\
= \varepsilon \int_{\Omega_1} |\Delta u^N|^2 - \int_{\Omega_1} \langle \mathcal{J}(u^N) \times s^N, \Delta u^N \rangle - \int_{\Omega_1} \langle \mathcal{J}(u^N) \times \nabla \Phi(\mathcal{J}(u^N)), \Delta u^N \rangle.
\]
This implies the following

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |\nabla u_N|^2 + \varepsilon \int_{\Omega_t} |\Delta u_N|^2 = -\alpha \int_{\Omega_t} (\mathcal{J}(u_N) \times u_N, \Delta u_N) + \int_{\Omega_t} (\mathcal{J}(u_N) \times s_N, \Delta u_N). \tag{14}
\]

Multiplying both sides of the second equation of (9) by \(\frac{\partial u_N}{\partial t}\) and integrating by parts give

\[
\int_{\Omega_t} \alpha u_N \frac{\partial u_N}{\partial t} + \varepsilon \int_{\Omega_t} |\nabla u_N|^2 = \int_{\Omega_t} (\mathcal{J}(u_N) \times u_N, \Delta u_N + s_N + h_d(u_N) + \nabla_u \mathcal{J}(u_N)) \tag{15}
\]

Multiplying the two sides of (15) by \(\alpha\) and then adding both sides of (14) to the two sides of (15), we obtain:

\[
\alpha \int_{\Omega_t} |u_N|^2 + \frac{\alpha \varepsilon + 1}{2} \frac{d}{dt} \int_{\Omega_t} |\nabla u_N|^2 + \varepsilon \int_{\Omega_t} |\Delta u_N|^2 \leq \int_{\Omega_t} (\mathcal{J}(u_N) \times (\alpha u_N - \Delta u_N), s_N + h_d(u_N) + \nabla_u \mathcal{J}(u_N)) \]

\[
\leq \left( \frac{1}{2\varepsilon} + \frac{\alpha}{2} \right) \int_{\Omega_t} \left( |s_N|^2 + |h_d(u_N)|^2 + |\nabla_u \mathcal{J}(u_N)|^2 \right) \]

\[
+ \frac{\alpha}{2} \int_{\Omega_t} |u_N|^2 + \varepsilon \int_{\Omega_t} |\Delta u_N|^2 \leq \frac{\alpha}{2} \int_{\Omega_t} \left( |s_N|^2 + |u_N|^2 + C(\Phi) \right),
\]

where we have used Lemma 2.1 and the constant \(C(\Phi)\) depends upon the values of \(\Phi, \nabla_u \Phi \) and \(\nabla^2_u \Phi\), which are restricted in the unit closed ball \(B(1)\). Therefore, we infer from the above

\[
\frac{\alpha}{2} \int_{\Omega_t} |u_N|^2 + \frac{\alpha \varepsilon + 1}{2} \frac{d}{dt} \int_{\Omega_t} |\nabla u_N|^2 + \varepsilon \int_{\Omega_t} |\Delta u_N|^2 \leq \left( \frac{\alpha}{2} + \frac{1}{2\varepsilon} \right) \int_{\Omega_t} \left( |s_N|^2 + |u_N|^2 + C(\Phi) \right).
\]

Integrating the above inequality on \([0, t]\) gives

\[
\frac{\alpha}{2} \int_0^t d\tau \int_{\Omega_{\tau}} |u_N|^2 + \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_{\tau}} |\nabla u_N|^2 + \varepsilon \int_0^t d\tau \int_{\Omega_{\tau}} |\Delta u_N|^2 \leq \left( \frac{\alpha}{2} + \frac{1}{2\varepsilon} \right) \int_0^t d\tau \int_{\Omega_{\tau}} \left( |s_N|^2 + |u_N|^2 + C(\Phi) \right) + \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_{\tau}} |\nabla u_0|^2 \]

\[
\leq \left( \frac{\alpha}{2} + \frac{1}{2\varepsilon} \right) \cdot \left\{ \int_{\Omega_0} |s_0|^2 + \frac{1}{2\eta} \cdot \int_0^t d\tau \left( \int_{\Omega_0} |J_\epsilon|^2 + \int_{\partial\Omega_0} |J_\epsilon|^2 \right) \right\} + \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_{\tau}} |\nabla u_0|^2 + \text{vol}(\Omega_{\tau}) \cdot t \cdot (1 + C(\Phi)) \cdot \left( \frac{\alpha}{2} + \frac{1}{2\varepsilon} \right). \]

From the inequality, (11), (12) and (13), we conclude that

- \(\{s_N\}\) is a bounded sequence in \(L^2([0, T], H^1(\Omega_0, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\Omega_0, \mathbb{R}^3))\);
• \{\partial_t s^N\} is a bounded sequence in \(L^2([0,T],H^{-1}(\Omega_0, \mathbb{R}^3))\);
• \{u^N\} is a bounded sequence in \(L^\infty([0,T],H^1(\Omega_1, \mathbb{R}^3))\);
• \{u_t^N\} is a bounded sequence in \(L^2([0,T], L^2(\Omega_1, \mathbb{R}^3))\);
• \{\Delta u^N\} is a bounded sequence in \(L^2([0,T], L^2(\Omega_1, \mathbb{R}^3))\);
• \{\nabla u^N\} is a bounded sequence in \(L^\infty([0,T], L^2(\Omega_1, \mathbb{R}^3 \otimes \mathbb{R}^3))\).

Furthermore, by the property of weak limits and Aubin-Lions Lemma, there exist two functions \(v^\varepsilon \in W^{2,1}_2(\Omega_1 \times [0,T], \mathbb{R}^3)\) and

\[ s \in L^2([0,T], H^1(\Omega_0, \mathbb{R}^3)) \bigcap L^\infty([0,T], L^2(\Omega_0, \mathbb{R}^3)) \]

and a subsequence of \(\{u^N, s^N\}\) which is also denoted by \(\{u^N, s^N\}\) such that

• \(u^N \rightharpoonup v^\varepsilon\) weakly * in \(L^\infty([0,T], H^1(\Omega_1, \mathbb{R}^3))\);
• \(u^N \rightharpoonup v^\varepsilon\) strongly in \(L^2([0,T], L^2(\Omega_1, \mathbb{R}^3))\);
• \(u^N \rightarrow v^\varepsilon\) a.e. \(\Omega_1 \times [0,T]\);
• \(u^N \rightarrow v^\varepsilon\) a.e. \(\partial \Omega_1 \times [0,T]\);
• \(u_t^N \rightharpoonup v_t^\varepsilon\) weakly in \(L^2([0,T], L^2(\Omega_1, \mathbb{R}^3))\);
• \(\Delta u^N \rightharpoonup \Delta v^\varepsilon\) weakly in \(L^2([0,T], L^2(\Omega_1, \mathbb{R}^3))\);
• \(\nabla u^N \rightharpoonup \nabla v^\varepsilon\) weakly * in \(L^\infty([0,T], L^2(\Omega_1, \mathbb{R}^3 \otimes \mathbb{R}^3))\);
• \(h_d(u^N) \rightarrow h_d(v^\varepsilon)\) strongly in \(L^\infty([0,T], L^2(\mathbb{R}^3, \mathbb{R}^3))\),

here we have used Lemma 2.1:

• \(h_d(u^N) \rightarrow h_d(v^\varepsilon)\) a.e. \(\mathbb{R}^3 \times [0,T]\).
• \(s^N \rightharpoonup s\) weakly * in \(L^\infty([0,T], L^2(\Omega_0, \mathbb{R}^3))\);
• \(s^N \rightarrow s\) weakly in \(L^2([0,T], H^1(\Omega_0, \mathbb{R}^3))\);
• \(s_t^N \rightharpoonup s_t\) weakly in \(L^2([0,T], H^{-1}(\Omega_0, \mathbb{R}^3))\);
• \(s^N \rightarrow s\) strongly in \(L^2([0,T], L^2(\Omega_0, \mathbb{R}^3))\);
• \(s^N \rightarrow s\) a.e. \(\Omega_0 \times [0,T]\);
• \(\nabla s^N \rightharpoonup \nabla s\) weakly in \(L^2([0,T], L^2(\Omega_0, \mathbb{R}^3 \otimes \mathbb{R}^3))\).

It is easy to check that, for any \(\phi \in C^\infty(\Omega_0 \times [0,T], \mathbb{R}^3)\), \(s\) satisfies the following equations

\[
\int_{\Omega_0} \langle s(T), \phi(T) \rangle \, dx - \int_{\Omega_0} \langle s_0, \phi(0) \rangle \, dx \\
- \int_0^T \int_{\Omega_0} \langle (s, \partial_t \phi) + \langle J(v^\varepsilon), \nabla J_s \phi \rangle \rangle \, dx \, dt
= - \int_0^T \int_{\partial \Omega_0} \langle J_s, \nu \rangle \cdot \langle J(v^\varepsilon), \phi \rangle \, d\sigma - \int_0^T \int_{\Omega_0} D_0(x) \langle A(J(v^\varepsilon)) \nabla s, \nabla \phi \rangle \, dx \, dt
+ \int_0^T \int_{\Omega_0} \langle (J(v^\varepsilon) \times s, \phi) - \langle s, \phi \rangle \rangle D_0(x) \, dx, \tag{16}
\]

with \(s(0, \cdot) = s_0\).

From the previous argument we can see easily that there hold

\[ ||v^\varepsilon||_{L^\infty([0,T], H^1(\Omega_1, \mathbb{R}^3))} \leq \tilde{C}_{12}, \]
for some constant $\tilde{C}_{12}$ and $u^N \rightharpoonup v^\varepsilon$ weakly* in $L^\infty([0, T], H^1(\Omega_1, \mathbb{R}^3))$. Similarly, we also have
\[
\alpha \int_0^T dt \int_{\Omega_1} |v_\varepsilon|^2 \leq \tilde{C}_{13}
\]
for some constant $\tilde{C}_{13}$. So, there hold true
\[
\alpha \mathcal{J}(u^N) \times u^N_i \rightharpoonup \alpha \mathcal{J}(v^\varepsilon) \times v_\varepsilon^i
\]
weakly in $L^2([0, T], L^2(\Omega_1, \mathbb{R}^3))$,
\[
\mathcal{J}(u^N) \times \Delta u^N \rightharpoonup \mathcal{J}(v^\varepsilon) \times \Delta v^\varepsilon
\]
weakly in $L^2([0, T], L^2(\Omega_1, \mathbb{R}^3))$, and
\[
\mathcal{J}(u^N) \times s^N \rightharpoonup \mathcal{J}(v^\varepsilon) \times s
\]
a.e. $\Omega_1 \times [0, T]$.

Fixing $r \in \mathbb{Z}^+$ and taking any $N \geq r$, we multiply two sides of the second equation of (9) by $\eta_i(t)$, which belongs to $C^\infty([0, T], \mathbb{R}^3)$, and integrate it on $[0, T]$. Then, we sum the obtained identities corresponding to $i$ from 1 to $r$ to derive
\[
\int_0^T dt \int_{\Omega_1} \langle u^N_i, \Phi^r \rangle - \alpha \int_0^T dt \int_{\Omega_1} \langle \mathcal{J}(u^N) \times u^N_i, \Phi^r \rangle
\]
\[
= - \int_0^T dt \int_{\Omega_1} \langle \mathcal{J}(u^N) \times (\Delta u^N + h_d(u^N) - \nabla u \Phi(\mathcal{J}(u^N)) + s^N), \Phi^r \rangle
\]
\[
+ \varepsilon \int_0^T dt \int_{\Omega_1} \langle \Delta u^N, \Phi^r \rangle
\]
where
\[
\Phi^r(x, t) = \sum_{i=1}^r \omega^i(x) \eta_i(t).
\]
Letting $N$ tends to $\infty$, for any $r$ we get
\[
\int_0^T dt \int_{\Omega_1} \langle v_\varepsilon^r, \Phi^r \rangle - \alpha \int_0^T dt \int_{\Omega_1} \langle \mathcal{J}(v^\varepsilon) \times v_\varepsilon^r, \Phi^r \rangle
\]
\[
= - \int_0^T dt \int_{\Omega_1} \langle \mathcal{J}(v^\varepsilon) \times \{\Delta v^\varepsilon + h_d(v^\varepsilon) - \nabla u \Phi(\mathcal{J}(v^\varepsilon)) + s\}, \Phi^r \rangle
\]
\[
+ \varepsilon \int_0^T dt \int_{\Omega_1} \langle \Delta v^\varepsilon, \Phi^r \rangle.
\]
It is easy to see the functions $\Phi^r(x, t)$ defined as above are dense in $L^2([0, T], L^2(\Omega_1, \mathbb{R}^3))$. Hence, we conclude that, in the sense of distribution, there holds
\[
v_\varepsilon^r - \alpha \mathcal{J}(v^\varepsilon) \times v_\varepsilon^r = \varepsilon \Delta v^\varepsilon - \mathcal{J}(v^\varepsilon) \times \{\Delta v^\varepsilon + h_d(v^\varepsilon) - \nabla u \Phi(\mathcal{J}(v^\varepsilon)) + s\}
\]
(17)
with $v^\varepsilon(0, \cdot) = u_0$.

Next, we would like to show that $\frac{\partial v^\varepsilon}{\partial t} \big|_{\partial \Omega_1} = 0$. Indeed, for any $\phi \in C^\infty(\Omega_1)$, we have
\[
\int_{\Omega_1} \Delta u^N \cdot \phi + \int_{\Omega_1} \nabla u^N \cdot \nabla \phi = 0,
\]
since $\frac{\partial u^N}{\partial r}|_{\partial \Omega_1} = 0$. Letting $N$ tends to $\infty$ yields
\[ \int_{\Omega_1} \Delta v^\varepsilon \cdot \phi + \int_{\Omega_1} \nabla v^\varepsilon \cdot \nabla \phi = 0. \]
The arbitrariness of $\phi$ implies $\frac{\partial u^N}{\partial r}|_{\partial \Omega_1} = 0$.

**Lemma 3.1.** If $v^\varepsilon \in W^{2,1}_0(\Omega_1 \times [0, T], \mathbb{R}^3)$ is a solution to (17) in the sense of distribution, then we will have $|v^\varepsilon| \leq 1$ a.e. $\Omega_1 \times [0, T]$.

**Proof.** We choose the following
\[ v^\varepsilon - v^\varepsilon \min\{1, |v^\varepsilon|]\]
as a test function of the above equation (17) to obtain
\[ \int_{\Omega_1} \langle v^\varepsilon_t, v^\varepsilon - v^\varepsilon \min\{1, |v^\varepsilon|\} \rangle = \varepsilon \int_{\Omega_1} \langle \Delta v^\varepsilon, v^\varepsilon - v^\varepsilon \min\{1, |v^\varepsilon|\} \rangle. \]
So, it follows
\[ \frac{1}{2} \frac{d}{dt} \int_{|v^\varepsilon| > 1} |v^\varepsilon|^2 \left( 1 - \frac{1}{|v^\varepsilon|} \right) = \frac{1}{2} \int_{|v^\varepsilon| > 1} \langle v^\varepsilon, v^\varepsilon \rangle \left( 1 - \frac{1}{|v^\varepsilon|} \right) - \varepsilon \int_{|v^\varepsilon| > 1} \frac{|\nabla v^\varepsilon|^2}{|v^\varepsilon|^3}. \]
Taking
\[ \frac{v^\varepsilon(max\{|v^\varepsilon|, 1\} - 1)}{|v^\varepsilon|(|v^\varepsilon| - 1 + \delta)} \]
as another test function of (17), we get:
\[ \int_{|v^\varepsilon| > 1} \frac{\langle v^\varepsilon, v^\varepsilon \rangle}{|v^\varepsilon|} \cdot \frac{|v^\varepsilon| - 1}{|v^\varepsilon| - 1 + \delta} = -\varepsilon \int_{\Omega_1} \langle \nabla v^\varepsilon, \nabla \left( v^\varepsilon(max\{|v^\varepsilon|, 1\} - 1) \right) \rangle \]
\[ = -\varepsilon \int_{|v^\varepsilon| > 1} \frac{|\nabla v^\varepsilon|^2}{|v^\varepsilon|(|v^\varepsilon| - 1 + \delta)} \]
\[ - \varepsilon \int_{|v^\varepsilon| > 1} \frac{|(\nabla v^\varepsilon, v^\varepsilon)|^2}{|v^\varepsilon|^2 \min\{1, |v^\varepsilon|\} - 1 + \delta} \]
By the controlled convergence theorem, letting $\delta \to 0$ we get:
\[ \int_{|v^\varepsilon| > 1} \frac{\langle v^\varepsilon, v^\varepsilon \rangle}{|v^\varepsilon|} = -\varepsilon \int_{|v^\varepsilon| > 1} \frac{|\nabla v^\varepsilon|^2}{|v^\varepsilon|^3} + \varepsilon \int_{|v^\varepsilon| > 1} \frac{|(\nabla v^\varepsilon, v^\varepsilon)|^2}{|v^\varepsilon|^3}, \]
Substituting the above identity into (18) yields
\[ \frac{d}{dt} \int_{|v^\varepsilon| > 1} |v^\varepsilon|^2 \left( 1 - \frac{1}{|v^\varepsilon|} \right) \leq 0. \]
This means that the following function
\[ q(t) := \int_{|v^\varepsilon| > 1} |v^\varepsilon|^2 \left( 1 - \frac{1}{|v^\varepsilon|} \right) \]
is decreasing non-negative function. Noting $|v^\varepsilon(\cdot, 0)| = |u_0| = 1$, i.e. $q(0) = 0$, we can see that $q(t) \equiv 0$ for any $t > 0$. Therefore, we have $|v^\varepsilon| \leq 1$ a.e. on $\Omega_1 \times [0, T]$. This completes the proof. □
From Lemma 3.1, it follows that (17) and (16) can be written respectively as
\begin{equation}
\begin{align*}
v_i^\varepsilon - \alpha v^\varepsilon \times v_i^\varepsilon &= \varepsilon \Delta v^\varepsilon - v^\varepsilon \times \{ \Delta v^\varepsilon + h_d(v^\varepsilon) - \nabla u \Phi(v^\varepsilon) + s \} \quad (19)
\end{align*}
\end{equation}
with \( v^\varepsilon(\cdot, 0) = u_0 \), and
\begin{equation}
\begin{align*}
\int_{\Omega_0} (s(T), \phi(T)) \, dx &- \int_{\Omega_0} (s_0, \phi(0)) \, dx \\
&- \int_0^T \int_{\Omega_0} \langle (s, \partial_t \phi) + \langle v^\varepsilon, \nabla J, \phi \rangle \rangle \, dx \\
&= - \int_0^T \int_{\partial \Omega_0} \langle J_e, \nu \rangle \cdot \langle v^\varepsilon, \phi \rangle \, d\sigma - \int_0^T \int_{\Omega_0} D_0(x) \langle A(v^\varepsilon) \nabla s, \nabla \phi \rangle \, dx \\
&+ \int_0^T \int_{\Omega_0} \langle (v^\varepsilon \times s, \phi) - (s, \phi) \rangle D_0(x) \, dx,
\end{align*}
\end{equation}
with \( s(\cdot, 0) = s_0 \), for any \( \phi \in C^\infty(\Omega_0 \times [0, T], \mathbb{R}^3) \).

**Lemma 3.2.** If \( v^\varepsilon \) and \( s \) satisfy (19) and (20) respectively, then there exist two positive constants \( M_1(T) \) and \( M_2(T) \), which are independent of \( \varepsilon \) and \( \alpha \) as \( \varepsilon \) and \( \alpha \) are small enough, such that
\begin{equation}
\| v^\varepsilon \|_{L^\infty([0, T], H^1(\Omega_1, \mathbb{R}^3))} \leq M_1(T)
\end{equation}
and
\begin{equation}
\sqrt{\alpha} \cdot \| v_i^\varepsilon \|_{L^2([0, T], L^2(\Omega_1, \mathbb{R}^3))} \leq M_2(T).
\end{equation}

**Proof.** Multiplying the both sides of (19) by \( v^\varepsilon \) and then integrating (19) on \( \Omega_1 \times [0, T] \) yield
\begin{equation}
\int_{\Omega_1} (|v^\varepsilon(t)|^2 - 1) + 2\varepsilon \int_0^t \int_{\Omega_1} |\nabla v^\varepsilon|^2 = 0. 
\end{equation}
Note that we have gotten
\( s^N \rightharpoonup s \) weakly * in \( L^\infty([0, T], L^2(\Omega_0, \mathbb{R}^3)) \)
and
\( s^N \rightharpoonup s \) weakly in \( L^2([0, T], H^1(\Omega_0, \mathbb{R}^3)) \).
By the property of weak limits and (11), we obtain
\begin{equation}
\int_{\Omega_0} |s(t)|^2 + (1 - \theta)c \cdot \int_0^t \int_{\Omega_0} |\nabla s|^2 + c \int_0^t \int_{\Omega_0} |s|^2 \\
\leq 2 \int_0^t \int_{\Omega_0} |s_0|^2 + \frac{1}{2\eta} \cdot \int_0^t \int_{\Omega_0} \left( \int_{\partial \Omega_0} |J_e|^2 + \int_{\partial \Omega_0} |J_i|^2 \right).
\end{equation}
Multiplying the both sides of (19) by \( v_i^\varepsilon \) and then integrating (19) on \( \Omega_1 \times [0, t] \) we can derive
\begin{equation}
\int_0^t \int_{\Omega_1} |v_i^\varepsilon|^2 + \varepsilon t \int_{\Omega_1} |\nabla v^\varepsilon(t)|^2 \\
= \frac{\varepsilon}{2} \int_{\Omega_1} |\nabla u_0|^2 - \int_0^t \int_{\Omega_1} v^\varepsilon \times \{ \Delta v^\varepsilon + h_d(v^\varepsilon) - \nabla u \Phi(v^\varepsilon) + s \} : v_i^\varepsilon.
\end{equation}
Multiplying the two sides of (19) by $\Delta v^\varepsilon$ and then integrating (19) on $\Omega_1 \times [0,t]$, we are led to
\[
\frac{1}{2} \int_{\Omega_1} |\nabla v^\varepsilon(t)|^2 - \frac{1}{2} \int_{\Omega_1} |\nabla u_0|^2 + \varepsilon \int_0^t d\tau \int_{\Omega_1} |\Delta v^\varepsilon|^2 
= -\alpha \int_0^t d\tau \int_{\Omega_1} v^\varepsilon \times u_t^\varepsilon \cdot \Delta v^\varepsilon + \int_0^t d\tau \int_{\Omega_1} v^\varepsilon \times \{h_d(v^\varepsilon) - \nabla u_\Phi(v^\varepsilon) + s\} \cdot \Delta v^\varepsilon.
\] (24)

Hence, from (23) we have
\[
\alpha \int_0^t d\tau \int_{\Omega_1} v^\varepsilon \times u_t^\varepsilon = -\alpha \int_0^t d\tau \int_{\Omega_1} |v_t^\varepsilon|^2 - \frac{\alpha \varepsilon}{2} \int_{\Omega_1} |\nabla v^\varepsilon(t)|^2 
+ \frac{\alpha \varepsilon}{2} \int_{\Omega_1} |\nabla u_0|^2 - \alpha \int_0^t d\tau \int_{\Omega_1} v^\varepsilon \times \{h_d(v^\varepsilon) - \nabla u_\Phi(v^\varepsilon) + s\} \cdot v_t^\varepsilon.
\] (25)

Substituting this identity into (24) we have
\[
\frac{\alpha \varepsilon + 1}{2} \int_{\Omega_1} |\nabla v^\varepsilon(t)|^2 + \varepsilon \int_0^t d\tau \int_{\Omega_1} |\Delta v^\varepsilon|^2 + \alpha \int_0^t d\tau \int_{\Omega_1} |v_t^\varepsilon|^2 
= \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_1} |\nabla u_0|^2 + \alpha \int_0^t d\tau \int_{\Omega_1} v^\varepsilon \times \{h_d(v^\varepsilon) - \nabla u_\Phi(v^\varepsilon) + s\} \cdot (\Delta v^\varepsilon - \alpha v_t^\varepsilon)
+ \int_0^t d\tau \int_{\Omega_1} v^\varepsilon \times \{\nabla^2 w^\varepsilon + \nabla^2 u_\Phi(v^\varepsilon) \cdot \nabla v^\varepsilon - \nabla s \cdot \nabla v^\varepsilon\}
\leq \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_1} |\nabla u_0|^2 + \frac{3\alpha}{2} \int_0^t d\tau \int_{\mathbb{R}^3} |h_d(v^\varepsilon)|^2 + \frac{3\alpha}{2} C(\Phi) \cdot \text{vol}(\Omega_1) \cdot t
+ \frac{3\alpha}{2} \int_0^t d\tau \int_{\Omega_1} |\nabla v^\varepsilon|^2 + \frac{1}{2} \int_0^t d\tau \int_{\Omega_1} |\nabla^2 w^\varepsilon|^2
+ \{1 + C(\Phi)\} \int_0^t d\tau \int_{\Omega_1} |\nabla v^\varepsilon|^2 + \frac{1}{2} \int_0^t d\tau \int_{\Omega_0} |\nabla s|^2,
\]
where we have used the fact that $h_d(v^\varepsilon) = -\nabla w^\varepsilon$.

By Lemma 2.1 we know that
\[
\int_{\mathbb{R}^3} |h_d(v^\varepsilon)|^2 \leq \int_{\Omega_1} |v^\varepsilon|^2 \leq \int_{\Omega_1} = \text{vol}(\Omega_1),
\]
and
\[
\int_{\Omega_1} |\nabla^2 w^\varepsilon|^2 \leq C \int_{\Omega_1} (|\nabla v^\varepsilon|^2 + |v^\varepsilon|^2).
\]

Hence, it follows
\[
\frac{\alpha \varepsilon + 1}{2} \int_{\Omega_1} |\nabla v^\varepsilon(t)|^2 + \varepsilon \int_0^t d\tau \int_{\Omega_1} |\Delta v^\varepsilon|^2 + \frac{\alpha}{2} \int_0^t d\tau \int_{\Omega_1} |v_t^\varepsilon|^2 
\leq \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_1} |\nabla u_0|^2 + \frac{3\alpha}{2} \{1 + C(\Phi)\} \cdot \text{vol}(\Omega_1) \cdot t
+ \frac{3\alpha}{2} \int_0^t d\tau \int_{\Omega_1} |\nabla v^\varepsilon|^2 + \frac{1}{2} \int_0^t d\tau \int_{\Omega_0} |\nabla s|^2
+ \{C(\Phi) + (2 + C)/2\} \int_0^t d\tau \int_{\Omega_1} |\nabla v^\varepsilon|^2 + \frac{1}{2} \int_0^t d\tau \int_{\Omega_0} |\nabla s|^2 + C\text{vol}(\Omega_1) \cdot t.
\]
\[
\leq \{C(\Phi) + (2 + C)/2\} \int_0^t dt \int_{\Omega_1} |\nabla v^\varepsilon|^2 + C(\alpha, t),
\]

where
\[
C(\alpha, t) \equiv \frac{\alpha \varepsilon + 1}{2} \int_{\Omega_1} |\nabla u_0|^2 + \left(\frac{3\alpha}{2} \{1 + C(\Phi)\} + C\right) \cdot \text{vol}(\Omega_1) \cdot t
\]
\[
+ \left(\frac{3\alpha}{2c} + \frac{1}{2(1-\theta)c}\right) \cdot \left\{ \int_{\Omega_0} |s_0|^2 + \frac{1}{2\eta} \int_0^t d\tau \left( \int_{\Omega_0} |J_\varepsilon|^2 + \int_{\partial\Omega_0} |J_\varepsilon|^2 \right) \right\}
\]
is non-decreasing function with respect to \(t\). The Gronwall inequality tells us that there holds true
\[
\int_{\Omega_1} |\nabla v^\varepsilon(t)|^2 \leq M_1(\alpha, t).
\]
Furthermore, we have
\[
\alpha \int_0^t dt \int_{\Omega_1} |v_t^\varepsilon|^2 \leq M_2(\alpha, t).
\]
So, the desired estimates follows immediately. This completes the proof of the lemma.

Proof of Theorem 1.1. We need to deal with two cases:

Case 1. \(\alpha > 0\). From Lemma 3.2 and Aubin-Lions Lemma, it follows that there is a function \(u\) and a subsequence of \(\{v^\varepsilon\}\), which is also denoted by \(\{v^\varepsilon\}\), such that:

- \(v^\varepsilon \rightharpoonup u\) weakly * in \(L^\infty([0,T], H^1(\Omega_1, \mathbb{R}^3))\);
- \(v^\varepsilon \to u\) a.e. \(\Omega_1 \times [0,T]\);
- \(v^\varepsilon \to u\) a.e. \(\partial\Omega_1 \times [0,T]\);
- \(v^\varepsilon \to u\) strongly in \(L^\infty([0,T], L^2(\Omega_1, \mathbb{R}^3))\);
- \(v_t^\varepsilon \to u_t\) weakly in \(L^2([0,T], L^2(\Omega_1, \mathbb{R}^3))\).

Letting \(\varepsilon\) in (21) tends to 0, we have:
\[
\int_{\Omega_1} (|u(x,t)|^2 - 1) = 0.
\]

On the other hand, for any \(t\) we have \(|u| \leq 1\) for a.e. \(x \in \Omega_1\), which is implied by the fact: for any \(t \in [0,T]\) there holds true \(|v^\varepsilon| \leq 1\) for a.e. \(x \in \Omega_1\). Hence, we deduce from the above that
\[
|u(x,t)| = 1, \quad \text{a.e. } x \in \Omega_1
\]
for all \(t \in [0,T]\).

Since there holds true that for any \(\varphi \in C^\infty(\Omega_1 \times [0,T], \mathbb{R}^3)\)
\[
\int_{\Omega_1} v^\varepsilon(T) \cdot \varphi(T) - \int_{\Omega_1} u_0 \cdot \varphi(0) - \alpha \int_0^T dt \int_{\Omega_1} v^\varepsilon \times v^\varepsilon_t \cdot \varphi
\]
\[
= \int_0^T dt \int_{\Omega_1} v^\varepsilon \varphi_t + \int_0^T dt \int_{\Omega_1} v^\varepsilon \times \nabla v^\varepsilon \cdot \nabla \varphi
\]
\[
- \int_0^T dt \int_{\Omega_1} v^\varepsilon \times \{s + h_\varepsilon(v^\varepsilon) - \nabla u \Phi(v^\varepsilon)\} \cdot \varphi - \varepsilon \int_0^T dt \int_{\Omega_1} \nabla v^\varepsilon \cdot \nabla \varphi,
\]
we let $\varepsilon \to 0$ to derive
\[
\int_{\Omega_t} u(T) \cdot \varphi(T) - \int_{\Omega_t} u_0 \cdot \varphi(0) - \alpha \int_0^T dt \int_{\Omega_t} u \times u_t \cdot \varphi
\]
\[
= \int_0^T dt \int_{\Omega_t} u \varphi_t + \int_0^T dt \int_{\Omega_t} u \times \nabla u \cdot \nabla \varphi
\]
\[
- \int_0^T dt \int_{\Omega_t} u \times \{s + h_d(u) - \nabla_u \Phi(u)\} \cdot \varphi.
\]
Moreover, as $\varepsilon \to 0$ we deduce from (20)
\[
\int_{\Omega_0} \langle s(T), \phi(T) \rangle \, dx - \int_{\Omega_0} \langle s_0, \phi(0) \rangle \, dx - \int_0^T dt \int_{\Omega_0} \langle (s, \partial_t \phi) \rangle
\]
\[
\quad + \langle u, \nabla J_\alpha \rangle \, dx
\]
\[
= - \int_0^T dt \int_{\Omega_0} (1, \nu) \cdot (u, \phi) \, d\sigma - \int_0^T dt \int_{\Omega_0} D_0(x) (A(u) \nabla s, \nabla \phi) \, dx
\]
\[
\quad + \int_0^T dt \int_{\Omega_0} ((u \times s, \phi) - \langle s, \phi \rangle) D_0(x) \, dx.
\]
Therefore, $(s, u)$ is the weak solution to system (1) by the definition.

**Case 2.** $\alpha = 0$. When $\alpha > 0$, we know that $u^\alpha$ is uniformly bounded in $L^\infty_{loc} (\mathbb{R}^+, H^1(\Omega, S^2))$, where $(u^\alpha, s)$ is the weak solution in Case 1. Therefore, there exists a subsequence of $u^\alpha$, which is still denoted by $u^\alpha$, converging to $u$ weakly $*$ in $L^\infty_{loc} (\mathbb{R}^+, H^1(\Omega, S^2))$ and converging to $u$ a.e. $\Omega_t \times \mathbb{R}^+$. Besides, we also have $u^\alpha \to u$ a.e. $\partial \Omega_t \times \mathbb{R}^+$ and $h_d(u^\alpha) \to h_d(u)$ a.e. $\mathbb{R}^3 \times \mathbb{R}^+$. On the other hand, for any $\varphi \in C^\infty(\Omega_t \times [0, T], \mathbb{R}^3)$ there holds true
\[
\int_{\Omega_t} u^\alpha(T) \cdot \varphi(T) - \int_{\Omega_t} u_0 \cdot \varphi(0) - \alpha \int_0^T dt \int_{\Omega_t} u^\alpha \times u^\alpha_t \cdot \varphi - \int_0^T dt \int_{\Omega_t} u^\alpha \varphi_t
\]
\[
= \int_0^T dt \int_{\Omega_t} u^\alpha \times \nabla u^\alpha \cdot \nabla \varphi - \int_0^T dt \int_{\Omega_t} u^\alpha \times \{s + h_d(u^\alpha) - \nabla_u \Phi(u^\alpha)\} \cdot \varphi.
\]
Noting that
\[
\alpha \int_0^t dt \int_{\Omega_t} |u^\alpha_t|^2 \leq M_2(\alpha, t)
\]
implies
\[
\alpha \int_0^T dt \int_{\Omega_t} u^\alpha \times u^\alpha_t \cdot \varphi \to 0,
\]
we let $\alpha \to 0$ in (27) to derive
\[
\int_{\Omega_t} u(T) \cdot \varphi(T) - \int_{\Omega_t} u_0 \cdot \varphi(0) - \int_0^T dt \int_{\Omega_t} u \varphi_t
\]
\[
= \int_0^T dt \int_{\Omega_t} u \times \nabla u \cdot \nabla \varphi - \int_0^T dt \int_{\Omega_t} u \times \{s + h_d(u) - \nabla_u \Phi(u)\} \cdot \varphi.
\]
Moreover, since
\[
\int_{\Omega_0} \langle s(T), \phi(T) \rangle \, dx - \int_{\Omega_0} \langle s_0, \phi(0) \rangle \, dx - \int_0^T dt \int_{\Omega_0} \langle (s, \partial_t \phi) + \langle u^\alpha, \nabla J_\alpha \phi \rangle \rangle \, dx
\]
where We employ the following Cauchy problem of auxiliary equation:

\[ \int_0^T dt \int_{\partial \Omega_0} \langle J_e, \nu \rangle \cdot \langle u^\alpha, \phi \rangle \, d\sigma - \int_0^T dt \int_{\Omega_0} D_0(x) (A(u^\alpha) \nabla s, \nabla \phi) \, dx \]

\[ + \int_0^T dt \int_{\Omega_0} (\langle u^\alpha \times s, \phi \rangle - \langle s, \phi \rangle) D_0(x) \, dx, \]

As \( \alpha \to 0 \) in the above identity we deduce that (26) still holds true. Therefore, \((s, u)\) is the weak solution to system (1) by the definition. This completes the proof. \( \square \)

4. Initial problems on a closed manifold. For the sake of simplification we set that \( D_0(x) \equiv 1 \) and one will see that there is no loss of generality. First of all, we need to extend the domain of \( \tilde{A} \) to \( T \times g \). For arbitrary \( u \in g \) and \( u \neq 0 \), we define

\[ \tilde{A}(x, u) := (1 - |u|) \cdot \text{Id} + |u| \cdot A(x, u/|u|); \]

if \( u = 0 \), we define \( \tilde{A}(x, u) = \text{Id} \), where \( \text{Id} \) is the identity matrix on \( g \).

It is easy to check that \( \tilde{A} \) is also symmetric and measurable on \( T \times g \) and there exist two positive constants \( \theta \) and \( \tilde{\theta} \) such that for any \((x, u) \in T \times B_g(1) \) and \( Y \in g \),

\[ \theta \cdot |Y|^2 \leq \langle \tilde{A}(x, u) Y, Y \rangle \leq \tilde{\theta} \cdot |Y|^2, \]

where \( B_g(1) \) is the closure of the unit ball \( B_g(1) \) in the Lie algebra \( g \). Suppose that the eigenvalues of \( \tilde{A}(x, u) \) are

\[ \lambda_1(x, u) \leq \lambda_2(x, u) \leq \cdots \leq \lambda_m(x, u). \]

From (28) it follows that

\[ 0 < \theta \leq \lambda_1(x, u) \leq \lambda_2(x, u) \leq \cdots \leq \lambda_m(x, u) \leq \tilde{\theta} < \infty. \]

Since \( \tilde{A}(x, u) \) is a symmetric matrix, we have

\[ |\langle \tilde{A}(x, u) Y_1, Y_2 \rangle| \leq \lambda_m(x, u) \cdot |Y_1| \cdot |Y_2| \leq \tilde{\theta} \cdot |Y_1| \cdot |Y_2|, \]

for any \( u \in B_g(1) \) and \( Y_1, Y_2 \in g \).

For the sake of convenience, we still set

\[ J(u) = \frac{u}{\max\{|u|, 1\}}. \]

We employ the following Cauchy problem of auxiliary equation:

\[
\begin{cases}
\partial_t s + s - [J(u), s] - \text{div}(\tilde{A}(x, J(u)) \nabla s - J(u) \otimes J_e) = 0, \\
\partial_t u - \alpha [J(u), \partial_t u] = - [J(u), \Delta u + s] + \varepsilon \Delta u, \\
s(\cdot, 0) = s_0, \quad u(\cdot, 0) = u_0.
\end{cases}
\]

to approach the the Cauchy problem (4), where \( \varepsilon \) is a positive constant. From now on, without confusions we always denote \( \tilde{A} \) by \( A \).

Let \( \lambda_i (i = 1, 2, \ldots) \) be the eigenvalues of the operator \(-\Delta \) on the domain \( H^2(T) \). \( \omega^i \) is the normalized eigenfunction corresponding to \( \lambda_i \). That is to say,

\[ -\Delta \omega^i = \lambda_i \omega^i. \]

According to Galerkin approximation, we define

\[ u^N(x, t) := \sum_{i=1}^{N} \beta^N_i(t) \omega^i(x) \quad \text{and} \quad s^N(x, t) := \sum_{i=1}^{N} \beta^N_i(t) \omega^i(x). \]
Here \( \{\beta^N_1(t), \rho^N_1(t)\} \) are unknown functions taking values in \( g \) and assumed to satisfy the following ODE:

\[
\begin{align*}
\frac{d\rho^N_1}{dt} + \rho^N_1 + \sum_{k=1}^N \sum_{l=1}^N [\rho^N_k, \beta^N_l] \int_T \omega^k \omega^l / \max\{|\sum_{j=1}^N \beta^N_j \omega^j|, 1\} \, d\tau \\
= -\sum_{k=1}^N \int_T A(x, J(u^N)) \rho^N_k \cdot \sum_{p=1}^n \nabla_{e_p} \omega^k \cdot \nabla_{e_p} \omega^i \, d\tau \\
+ \sum_{j=1}^N \beta^N_j \cdot \int_T (\omega^j / \max\{|\sum_{j=1}^N \beta^N_j \omega^j|, 1\}) \cdot \nabla_{J^i} \omega^i \, d\tau,
\end{align*}
\]

(30)

It is easy to check that (30) admits a local solution existing in the interval \([0, \tau]\) for some \( \tau > 0 \). So we get

\[
\begin{align*}
\int_T \partial_t s^N \cdot \omega^i \, d\tau + \int_T s^N \cdot \omega^i \, d\tau - \int_T [J(u^N), s^N] \cdot \omega^i \, d\tau \\
= -\int_T A(x, J(u^N)) \sum_{p=1}^n \nabla_{e_p} s^N \cdot \nabla_{e_p} \omega^i \, d\tau + \int_T J(u^N) \cdot \nabla_{J^i} \omega^i \, d\tau,
\end{align*}
\]

(31)

Multiplying both sides of the first equation of (31) by \( \rho^N_1 \) and summing \( i \) from 1 to \( N \) yield

\[
\frac{1}{2} \frac{d}{dt} \int_T |s^N|^2 \, d\tau + \sum_{p=1}^n \int_T \langle A(x, J(u^N)) \nabla_{e_p} s^N, \nabla_{e_p} s^N \rangle \, d\tau
\]

(32)

From (28) it follows that

\[
\theta \cdot |\nabla s^N|^2 \leq \sum_{p=1}^n \langle A(x, J(u^N)) \nabla_{e_p} s^N, \nabla_{e_p} s^N \rangle.
\]

Substituting the above inequality into (32) leads to

\[
\frac{1}{2} \frac{d}{dt} \int_T |s^N|^2 \, d\tau + \theta \cdot \int_T |\nabla s^N|^2 \, d\tau + \int_T |s^N|^2 \, d\tau
\]
where \( \text{vol}(T) \) is the volume of \( T \).

Since

\[
\int_T |u^N(t)|^2 \, dT = \int_T \left( \sum_{i=1}^{N} \beta_i^N(t) \omega^i \right) \left( \sum_{j=1}^{N} \theta_j^N(t) \omega^j \right) \, dT = \sum_{i=1}^{N} |\beta_i^N(t)|^2
\]

and

\[
\int_T |s^N(t)|^2 \, dT = \sum_{i=1}^{N} |\rho_i^N(t)|^2;
\]

then, from the above estimates on \( \|u^N\|_{L^2} \) and \( \|s^N\|_{L^2} \) we know that \( \rho_i^N(t) \) and \( \beta_i^N(t) \) can be extended to \( [0, T] \) for any \( T > 0 \) and \( i \). That is to say, \( \{s^N\} \) and \( \{u^N\} \) can be extended to \( [0, T] \).

From the first equation of (31) it follows that

\[
\int_0^T dt \int_T \partial_t s^N \cdot \omega^i \, dT
\]

\[
= - \int_0^T dt \int_T s^N \cdot \omega^i \, dT + \int_0^T dt \int_T [\mathcal{J}(u^N), s^N] \cdot \omega^i \, dT
\]

\[
- \int_0^T dt \int_T A(x, \mathcal{J}(u^N)) \sum_{p=1}^{n} \nabla_{e_p} s^N \cdot \nabla_{e_p} \omega^i \, dT
\]

It implies the following

\[
\int_T |s^N(t)|^2 \, dT + \theta \cdot \int_0^t dt \int_T |\nabla s^N|^2 \, dT + 2 \int_0^t dt \int_T |s^N|^2 \, dT = \sum_{i=1}^{N} \left( \int_T s_0 \cdot \omega^i \, dT \right)^2 + 1 \cdot \int_0^t dt \int_T |J_\varepsilon|^2 \, dT
\]

\[
= \sum_{i=1}^{N} \left( \int_T s_0 \cdot \omega^i \, dT \right)^2 + 1 \cdot \int_0^t dt \int_T |J_\varepsilon|^2 \, dT
\]

\[
\leq \sum_{i=1}^{\infty} \left( \int_T s_0 \cdot \omega^i \, dT \right)^2 + 1 \cdot \int_0^t dt \int_T |J_\varepsilon|^2 \, dT
\]

\[
= \int_T |s_0|^2 \, dT + 1 \cdot \int_0^t dt \int_T |J_\varepsilon|^2 \, dT.
\]

Similarly, multiplying both sides of the second equation of (31) by \( \beta_i^N \) and summing \( i \) from 1 to \( N \) and integrating by parts, we can derive

\[
\frac{1}{2} \int_T |u^N|^2 \, dT + \varepsilon \int_T |\nabla u^N|^2 \, dT = 0.
\]
Here we have used the property of inner product induced by its Killing form.  

\[ \int_0^T dt \int_T \mathcal{J}(u^N) \cdot \nabla_J \omega^i \, d\tau, \]

\[
\leq \frac{\bar{\theta}}{2} \int_0^T dt \int_T |\nabla s^N|^2 \, d\tau + \frac{1 + \bar{\theta}}{2} \cdot \int_T |\nabla \omega^i|^2 \, d\tau \\
+ \int_0^T dt \int_T |s^N|^2 \, d\tau + T + \frac{1}{2} \cdot \int_0^T dt \int_T |\mathcal{J}_e|^2 \, d\tau \\
\leq \left( \frac{1}{2} + \frac{\bar{\theta}}{2\bar{g}} \right) \cdot \left\{ \int_T |s_0|^2 \, d\tau + \frac{1}{\bar{g}} \cdot \int_0^T dt \int_T |\mathcal{J}_e|^2 \, d\tau \right\} \\
+ \frac{1 + \bar{\theta}}{2} \cdot T \cdot \int_T |\nabla \omega|^2 \, d\tau + T + \frac{1}{2} \cdot \int_0^T dt \int_T |\mathcal{J}_e|^2 \, d\tau 
\]

where we have used (29), (33) and the fact that $||\omega^i||_{L^2} = 1$. This leads to

\[
\int_0^T dt \int_T \partial_t s^N \cdot \omega^i \, d\tau \leq C(\bar{\theta}, \theta, s_0, T, \mathcal{J}_e, T) \cdot ||\omega^i||_{H^1}. \tag{35}
\]

Therefore, $\{ \partial_t s^N \}$ is uniformly bounded in $L^2([0, T], H^{-1}(T, g))$.

Multiplying both sides of the second equation of (31) by $-\lambda_i \beta_i^N$ and summing $i$ from 1 to $N$, we get

\[
\int_T \langle u^N_i, v^N_i \rangle \, d\tau = \int_T \langle \mathcal{J}(u^N_i), v^N_i \rangle \, d\tau \\
\leq \varepsilon \int_T |\Delta u^N|^2 \, d\tau - \int_T \langle \mathcal{J}(u^N_i), s^N_i \rangle, \Delta u^N \rangle \, d\tau.
\]

This implies the following

\[
\frac{d}{dt} \int_T |
abla u^N|^2 \, d\tau + \varepsilon \int_T |\Delta u^N|^2 \, d\tau \leq -\alpha \int_T \langle \mathcal{J}(u^N_i), u^N_i \rangle, \Delta u^N \rangle \, d\tau + \int_T \langle \mathcal{J}(u^N_i), s^N \rangle, \Delta u^N \rangle \, d\tau.
\]

Multiplying the both sides of the second equation of (31) by $\frac{d\beta_i^N}{dt}$ and summing $i$ from 1 to $N$ and integrating by parts give

\[
\int_T |u^N_i|^2 \, d\tau + \varepsilon \frac{d}{dt} \int_T |\nabla u^N|^2 \, d\tau = \int_T \langle \mathcal{J}(u^N_i), u^N_i \rangle, \Delta u^N + s^N \rangle \, d\tau.
\]

Here we have used the property of inner product induced by its Killing form.

Multiplying the two sides of (37) by $\alpha$ and then adding both sides of (36) to the two sides of (37), we obtain:

\[
\frac{\alpha}{2} \int_T |u^N_i|^2 \, d\tau + \frac{\alpha \varepsilon + \frac{1}{2}}{2} \int_T |\nabla u^N|^2 \, d\tau + \varepsilon \int_T |\Delta u^N|^2 \, d\tau \\
= \frac{\alpha}{2} \int_T \langle \mathcal{J}(u^N_i), u^N_i \rangle, s^N \rangle \, d\tau - \int_T \langle \mathcal{J}(u^N_i), \Delta u^N \rangle, s^N \rangle \, d\tau \\
\leq \frac{\alpha}{2} \int_T |u^N_i|^2 \, d\tau + \frac{\alpha}{2} \cdot \int_T |s^N|^2 \, d\tau + \frac{\varepsilon}{2} \int_T |\Delta u^N|^2 \, d\tau + \frac{1}{2} \varepsilon \int_T |s^N|^2 \, d\tau.
\]

Therefore, we infer from the above

\[
\frac{\alpha}{2} \int_T |u^N_i|^2 \, d\tau + \frac{\alpha \varepsilon + \frac{1}{2}}{2} \int_T |\nabla u^N|^2 \, d\tau + \frac{\varepsilon}{2} \int_T |\Delta u^N|^2 \, d\tau \leq \left( \frac{\alpha}{2} + \frac{1}{2\varepsilon} \right) \int_T |s^N|^2 \, d\tau.
\]
Integrating the above inequality on \([0, t]\) gives
\[
\frac{\alpha}{2} \int_0^t d\tau \int_T |u_N|^2 d\tau + \frac{\alpha \varepsilon + 1}{2} \int_0^t d\tau \int_T |\nabla u_N|^2 d\tau + \frac{\varepsilon}{2} \int_0^t d\tau \int_T |\Delta u_N|^2 d\tau
\]
\[
\leq \left( \frac{\alpha}{2} + \frac{1}{2\varepsilon} \right) \int_0^t d\tau \int_T |s_N|^2 d\tau + \frac{\alpha \varepsilon + 1}{2} \int_0^t d\tau \int_T |\nabla u_0|^2 d\tau
\]
\[
\leq \left( \frac{\alpha}{4} + \frac{1}{4\varepsilon} \right) \cdot \left\{ \int_T |s_0|^2 d\tau + \frac{1}{\theta} \cdot \int_0^t d\tau \int_T |J_\varepsilon|^2 d\tau \right\} + \frac{\alpha \varepsilon + 1}{2} \int_T |\nabla u_0|^2 d\tau.
\]
From the inequality, (33), (34) and (35), we conclude that
\begin{itemize}
  \item \(\{s^N\}\) is a bounded sequence in \(L^2([0, T], H^1(\mathbb{T}, g)) \cap L^\infty([0, T], L^2(\mathbb{T}, g))\);
  \item \(\{\partial_t s^N\}\) is a bounded sequence in \(L^2([0, T], H^{-1}(\mathbb{T}, g))\);
  \item \(\{u^N\}\) is a bounded sequence in \(L^\infty([0, T], H^1(\mathbb{T}, g))\);
  \item \(\{u^N_t\}\) is a bounded sequence in \(L^2([0, T], L^2(\mathbb{T}, g))\);
  \item \(\{\Delta u^N\}\) is a bounded sequence in \(L^2([0, T], L^2(\mathbb{T}, g))\);
  \item \(\{\nabla u^N\}\) is a bounded sequence in \(L^\infty([0, T], L^2(\mathbb{T}, TT \otimes g))\), where \(TT\) is the tangent bundle of \(\mathbb{T}\).
\end{itemize}
Furthermore, by the property of weak limits and Aubin-Lions Lemma, there exist two functions
\[
v^\varepsilon \in W^{2,1}_2(\mathbb{T} \times [0, T], g)
\]
and
\[
s \in L^2([0, T], H^1(\mathbb{T}, g)) \cap L^\infty([0, T], L^2(\mathbb{T}, g))
\]
and a subsequence of \(\{u^N, s^N\}\) which is also denoted by \(\{u^N, s^N\}\) such that
\begin{itemize}
  \item \(u^N \rightharpoonup v^\varepsilon\) weakly* in \(L^\infty([0, T], H^1(\mathbb{T}, g))\);
  \item \(u^N \rightharpoonup v^\varepsilon\) strongly in \(L^\infty([0, T], L^2(\mathbb{T}, g))\);
  \item \(u^N \rightharpoonup v^\varepsilon\) s.a.e. \(\mathbb{T} \times [0, T]\);
  \item \(u^N_t \rightharpoonup v^\varepsilon_t\) weakly in \(L^2([0, T], L^2(\mathbb{T}, g))\);
  \item \(\Delta u^N \rightharpoonup \Delta v^\varepsilon\) weakly in \(L^2([0, T], L^2(\mathbb{T}, g))\);
  \item \(\nabla u^N \rightharpoonup \nabla v^\varepsilon\) weakly* in \(L^\infty([0, T], L^2(\mathbb{T}, TT \otimes g))\);
  \item \(s^N \rightharpoonup s\) weakly * in \(L^\infty([0, T], L^2(\mathbb{T}, g))\);
  \item \(s^N \rightharpoonup s\) weakly in \(L^2([0, T], H^1(\mathbb{T}, g))\);
  \item \(s^N_t \rightharpoonup s_t\) weakly in \(L^2([0, T], H^{-1}(\mathbb{T}, g))\);
  \item \(s^N \rightharpoonup s\) strongly in \(L^2([0, T], L^2(\mathbb{T}, g))\);
  \item \(s^N \rightharpoonup s\) a.e. \(\mathbb{T} \times [0, T]\);
  \item \(\nabla s^N \rightharpoonup \nabla s\) weakly in \(L^2([0, T], L^2(\mathbb{T}, TT \otimes g))\).
\end{itemize}
It is easy to check that, for any $\phi \in C^\infty(\mathbb{T} \times [0,T], g)$, $s$ satisfies the following equations

$$\int_0^T dt \int_\mathbb{T} \langle \partial_t s, \phi \rangle d\mathbb{T} + \int_0^T dt \int_\mathbb{T} \langle s, \phi \rangle d\mathbb{T} - \int_0^T dt \int_\mathbb{T} \langle \mathcal{J}(v^\varepsilon), s \rangle d\mathbb{T} \quad (38)$$

$$= - \int_0^T dt \int_\mathbb{T} \langle A(x, \mathcal{J}(v^\varepsilon)) \sum_{p=1}^2 \nabla e_p s, \nabla e_p \phi \rangle d\mathbb{T} + \int_0^T dt \int_\mathbb{T} \langle \mathcal{J}(v^\varepsilon), \nabla J \phi \rangle d\mathbb{T},$$

with $s(0, \cdot) = s_0$.

From the previous argument we can see easily that there hold

$$\|v^\varepsilon\|_{L^\infty([0,T], H^1(\mathbb{T}, g))} \leq \tilde{C}_{12},$$

since

$$\|u^N\|_{L^\infty([0,T], H^1(\mathbb{T}, g))} \leq \tilde{C}_{12}$$

for some constant $\tilde{C}_{12}$ and $u^N \rightharpoonup v^\varepsilon$ weakly in $L^\infty([0,T], H^1(\mathbb{T}, g))$. Similarly, we also have

$$\int_0^T \int_\mathbb{T} \|v_i^\varepsilon\|^2 d\mathbb{T} dt \leq \tilde{C}_{13}$$

for some constant $\tilde{C}_{13}$. So, there hold true

$$[\mathcal{J}(u^N), u_i^N] \rightharpoonup [\mathcal{J}(v^\varepsilon), v_i^\varepsilon]$$

weakly in $L^2([0,T], L^2(\mathbb{T}, g))$,

$$[\mathcal{J}(u^N), \Delta u^N] \rightharpoonup [\mathcal{J}(v^\varepsilon), \Delta v^\varepsilon]$$

weakly in $L^2([0,T], L^2(\mathbb{T}, g))$, and

$$[\mathcal{J}(u^N), s^N] \rightharpoonup [\mathcal{J}(v^\varepsilon), s]$$

a.e. $\mathbb{T} \times [0,T]$.

Fixing $r \in \mathbb{Z}^+$ and taking any $N \geq r$, we multiply two sides of the second equation of (31) by $\eta^i(t)$, which belongs to $C^\infty([0,T], g)$, and integrate it on $[0,T]$. Then, we sum the obtained identities corresponding to $i$ from 1 to $r$ to derive

$$\int_0^T \int_\mathbb{T} \langle u_i^N, \Phi^\varepsilon \rangle d\mathbb{T} dt - \alpha \int_0^T \int_\mathbb{T} \langle \mathcal{J}(u^N), u_i^N \rangle, \Phi^\varepsilon \rangle d\mathbb{T} dt$$

$$\varepsilon \int_0^T \int_\mathbb{T} \langle \Delta u^N, \Phi^\varepsilon \rangle d\mathbb{T} dt - \int_0^T \int_\mathbb{T} \langle \mathcal{J}(u^N), \Delta u^N + s^N \rangle, \Phi^\varepsilon \rangle d\mathbb{T} dt,$$

where

$$\Phi^\varepsilon(x,t) = \sum_{i=1}^r \omega^i(x)\eta^i(t).$$

Letting $N$ tends to $\infty$, for any $r$ we get

$$\int_0^T \int_\mathbb{T} \langle v_i^\varepsilon, \Phi^\varepsilon \rangle d\mathbb{T} dt - \alpha \int_0^T \int_\mathbb{T} \langle \mathcal{J}(v^\varepsilon), v_i^\varepsilon \rangle, \Phi^\varepsilon \rangle d\mathbb{T} dt$$

$$= - \int_0^T \int_\mathbb{T} \langle [\mathcal{J}(v^\varepsilon), \Delta v^\varepsilon + s], \Phi^\varepsilon \rangle d\mathbb{T} dt + \varepsilon \int_0^T \int_\mathbb{T} \langle \Delta v^\varepsilon, \Phi^\varepsilon \rangle d\mathbb{T} dt.$$

It is easy to see the functions $\Phi^\varepsilon(x,t)$ defined as above are dense in $L^2([0,T], L^2(\mathbb{T}, g))$. Hence, we conclude that, in the sense of distribution, there holds

$$v_i^\varepsilon - \alpha [\mathcal{J}(v^\varepsilon), v_i^\varepsilon] = \varepsilon \Delta v^\varepsilon - [\mathcal{J}(v^\varepsilon), \Delta v^\varepsilon + s]$$

(39)

with $v^\varepsilon(0, \cdot) = u_0$. 
Lemma 4.1. If $v^\varepsilon \in W^{2,1}_2(T \times [0, T], g)$ is a solution to (39) in the sense of distribution, then we will have $|v^\varepsilon| \leq 1$ a.e. $T \times [0, T]$.

Proof. By the same way as in Lemma (3.1), we choose the following

$$v^\varepsilon - v^\varepsilon \min\{1, |v^\varepsilon|\}$$

and

$$v^\varepsilon \max\{|v^\varepsilon|, 1\} - 1\over |v^\varepsilon| - 1 + \delta$$

as the test functions of the above equation (39) to obtain

$$\frac{d}{dt} \int_{|v^\varepsilon| > 1} |v^\varepsilon|^2 \left(1 - \frac{1}{|v^\varepsilon|}\right) dT \leq 0.$$ 

This means that the following function

$$q(t) := \int_{|v^\varepsilon| > 1} |v^\varepsilon|^2 \left(1 - \frac{1}{|v^\varepsilon|}\right) dT$$

is decreasing non-negative function. Noting $|v^\varepsilon(\cdot, 0)| = |u_0| = 1$, i.e. $q(0) = 0$, we can see that $q(t) \equiv 0$ for any $t > 0$. Therefore, we have $|v^\varepsilon| \leq 1$ a.e. on $T \times [0, T]$. This completes the proof. 

From Lemma 4.1, it follows that (39) and (38) can be written respectively as

$$v^\varepsilon_\alpha - \alpha [v^\varepsilon, v^\varepsilon_\alpha] = \varepsilon \Delta v^\varepsilon - [v^\varepsilon, \Delta v^\varepsilon + s], \quad \text{with} \quad v^\varepsilon(\cdot, 0) = u_0, \quad (40)$$

and

$$\int_0^T dt \int_T \langle \partial_t s, \phi \rangle dT + \int_0^T dt \int_T \langle s, \phi \rangle dT - \int_0^T dt \int_T \langle [v^\varepsilon, s], \phi \rangle dT$$

$$= - \int_0^T dt \int_T \langle A(x, v^\varepsilon) \sum_{p=1}^n \nabla e_p s, \nabla e_p \phi \rangle dT + \int_0^T dt \int_T \langle v^\varepsilon, \nabla L \phi \rangle dT, \quad (41)$$

with $s(\cdot, 0) = s_0$, for any $\phi \in C^\infty(T \times [0, T], g)$.

Lemma 4.2. If $v^\varepsilon$ and $s$ satisfy (40) and (41) respectively, then there exist two positive constants $M_1(T)$ and $M_2(T)$, which are independent of $\varepsilon$ and $\alpha$ as $\varepsilon$ and $\alpha$ are small enough, such that

$$\|v^\varepsilon\|_{L^\infty([0, T], H^1(T, g))} \leq M_1(T)$$

and

$$\sqrt{\alpha} \cdot \|v^\varepsilon\|_{L^2([0, T], L^2(T, g))} \leq M_2(T).$$

Proof. Multiplying both sides of (40) by $v^\varepsilon$ and then integrating (40) on $T \times [0, T]$ yield

$$\int_T (|v^\varepsilon(t)|^2 - 1) dT + 2\varepsilon \int_T |\nabla v^\varepsilon|^2 dT dt = 0. \quad (42)$$

Note that we have gotten

$$s^N \rightharpoonup s \quad \text{weakly} \quad \text{in} \quad L^\infty([0, T], L^2(T, g))$$

and

$$s^N \rightharpoonup s \quad \text{weakly} \quad \text{in} \quad L^2([0, T], H^1(T, g)).$$
By the property of weak limits and (33), we obtain
\[
\frac{1}{t} \int_T |s(t)|^2 \, d\mathbb{T} + \theta \cdot \int_0^t \int_T |\nabla s|^2 \, d\mathbb{T} + 2 \int_0^t \int_T |s|^2 \, d\mathbb{T} \leq \int_T |s_0|^2 \, d\mathbb{T} + \frac{1}{\theta} \cdot \int_0^t \int_T |J_e|^2 \, d\mathbb{T}.
\] (43)

Multiplying both sides of (40) by \( v_i^\varepsilon \) and then integrating (40) on \( T \times [0, t] \) we can derive
\[
\frac{1}{2} \int_T |\nabla v_i^\varepsilon(t)|^2 \, d\mathbb{T} - \frac{1}{2} \int_T |\nabla u_0|^2 \, d\mathbb{T} = -\alpha \int_0^t \int_T [v_i^\varepsilon, \Delta v^\varepsilon] \cdot \Delta v^\varepsilon \, d\mathbb{T} - \varepsilon \int_0^t \int_T \Delta v^\varepsilon |^2 \, d\mathbb{T} + \frac{\alpha \varepsilon}{2} \int_T [\nabla v_i^\varepsilon(t)]^2 \, d\mathbb{T} \] (44)

Hence, from (44) we have
\[
\frac{\alpha \varepsilon + 1}{2} \int_T [\nabla v_i^\varepsilon(t)]^2 \, d\mathbb{T} + \frac{\varepsilon}{2} \int_0^t \int_T |\Delta v^\varepsilon|^2 \, d\mathbb{T} = -\alpha \int_0^t \int_T [v_i^\varepsilon]^2 \, d\mathbb{T} - \frac{\alpha \varepsilon}{2} \int_T [\nabla v_i^\varepsilon(t)]^2 \, d\mathbb{T} \] (46)

Substituting this identity into (45) we have
\[
\frac{\alpha \varepsilon + 1}{2} \int_T [\nabla v_i^\varepsilon(t)]^2 \, d\mathbb{T} + \frac{\varepsilon}{2} \int_0^t \int_T |\Delta v^\varepsilon|^2 \, d\mathbb{T} = -\alpha \int_0^t \int_T [v_i^\varepsilon]^2 \, d\mathbb{T} + \frac{\alpha \varepsilon + 1}{2} \int_T [\nabla u_0]^2 \, d\mathbb{T} - \alpha \int_0^t \int_T [v_i^\varepsilon, s] \cdot v_i^\varepsilon \, d\mathbb{T} + \int_0^t \int_T [v_i^\varepsilon, s] \cdot \Delta v^\varepsilon \, d\mathbb{T} \
\leq \frac{1}{2} \int_0^t \int_T [\nabla v_i^\varepsilon]^2 \, d\mathbb{T} + \frac{\alpha \varepsilon + 1}{2} \int_T [\nabla u_0]^2 \, d\mathbb{T} + \frac{\alpha \varepsilon}{2} \int_T [v_i^\varepsilon]^2 \, d\mathbb{T} + \frac{\alpha \varepsilon + 1}{2} \int_T [\nabla u_0]^2 \, d\mathbb{T} + \frac{\alpha \varepsilon}{2} \int_0^t \int_T |s|^2 \, d\mathbb{T} + \frac{1}{2} \int_0^t \int_T |\nabla s|^2 \, d\mathbb{T}.
\]

It follows
\[
\frac{\alpha \varepsilon + 1}{2} \int_T [\nabla v_i^\varepsilon(t)]^2 \, d\mathbb{T} + \varepsilon \int_0^t \int_T |\Delta v^\varepsilon|^2 \, d\mathbb{T} + \frac{\alpha \varepsilon}{2} \int_0^t \int_T [v_i^\varepsilon]^2 \, d\mathbb{T} \leq \frac{1}{2} \int_0^t \int_T [\nabla v_i^\varepsilon]^2 \, d\mathbb{T} + C(\alpha, t),
\]

where
\[
C(\alpha, t) \equiv \frac{\alpha \varepsilon + 1}{2} \int_T |\nabla u_0|^2 \, d\mathbb{T} + \left( \frac{\alpha}{4} + \frac{1}{2\theta} \right) \cdot \left( \int_T |s_0|^2 \, d\mathbb{T} + \frac{1}{\theta} \int_0^t \int_T |J_e|^2 \, d\mathbb{T} \right)
\]
is non-decreasing function with respect to $t$. The Gronwall inequality tells us that there holds true
\[ \int_T |\nabla v^\varepsilon(t)|^2 \, d\mathbb{T} \leq M_1(\alpha, t). \]
Furthermore, we have
\[ \alpha \int_0^t \, d\tau \int_T |v^\varepsilon|_2^2 \, d\mathbb{T} \leq M_2(\alpha, t). \]
So, the desired estimates follows immediately. This completes the proof of the lemma. □

Proof of Theorem 1.2. We need to deal with two cases:

Case 1. $\alpha > 0$. From Lemma 4.2 and Aubin-Lions Lemma, it follows that there is a function $u$ and a subsequence of $\{v^\varepsilon\}$, which is also denoted by $\{v^\varepsilon\}$, such that:

- $v^\varepsilon \rightharpoonup u$ weakly* in $L^\infty([0, T], H^1(\mathbb{T}, g))$;
- $v^\varepsilon \rightarrow u$ strongly in $L^\infty([0, T], L^2(\mathbb{T}, g))$;
- $v^\varepsilon_t \rightharpoonup u_t$ weakly in $L^2([0, T], L^2(\mathbb{T}, g))$.

Letting $\varepsilon$ in (42) tends to 0, we have:
\[ \int_T (|u(x,t)|^2 - 1) \, d\mathbb{T} = 0. \]
On the other hand, for any $t$ we have $|u| \leq 1$ for a.e. $x \in \mathbb{T}$, which is implied by the fact: for any $t \in [0, T]$ there holds true $|v^\varepsilon| \leq 1$ for a.e. $x \in \mathbb{T}$. Hence, we deduce from the above that
\[ |u(x,t)| = 1, \quad a.e. \ x \in \mathbb{T} \]
for all $t \in [0, T]$.

Since there holds true that for any $\varphi \in C^\infty([0, T] \times \mathbb{T}, g)$
\[ \int_T v^\varepsilon(T) \cdot \varphi(T) \, d\mathbb{T} - \int_T u_0 \cdot \varphi(0) \, d\mathbb{T} - \alpha \int_0^T \int_T [v^\varepsilon, v^\varepsilon_t] \cdot \varphi \, d\mathbb{T} \, dt \]
\[ = \int_0^T \int_T v^\varepsilon \varphi_t \, d\mathbb{T} \, dt + \int_0^T \int_T \sum_{p=1}^n [v^\varepsilon, \nabla e_p v^\varepsilon] \cdot \nabla e_p \varphi \, d\mathbb{T} \, dt - \int_0^T \int_T [v^\varepsilon, s] \cdot \varphi \, d\mathbb{T} \, dt \]
\[ - \varepsilon \int_0^T \int_T \sum_{p=1}^n \nabla e_p v^\varepsilon \cdot \nabla e_p \varphi \, d\mathbb{T} \, dt, \]
we let $\varepsilon \rightarrow 0$ to derive
\[ \int_T u(T) \varphi(T) \, d\mathbb{T} - \int_T u_0 \varphi(0) \, d\mathbb{T} - \alpha \int_0^T \int_T [u, u_t] \varphi \, d\mathbb{T} \, dt \]
\[ = \int_0^T \int_T u \varphi_t \, d\mathbb{T} \, dt + \int_0^T \int_T \sum_{p=1}^n [u, \nabla e_p u] \nabla e_p \varphi \, d\mathbb{T} \, dt - \int_0^T \int_T [u, s] \varphi \, d\mathbb{T} \, dt. \]
Moreover, as $\varepsilon \to 0$ we deduce from (41)
\[
\int_T s(T)\phi(T)\,dT - \int_T s_0\phi(0)\,dT - \int_0^T dt \int_T s\partial_t\phi\,dT - \int_0^T dt \int_T \langle u, \nabla J_e\phi \rangle\,dT = -\int_0^T dt \int_T \sum_{p=1}^n \langle A(x,u)\nabla e_p s, \nabla e_p \phi \rangle - \int_0^T dt \int_T s\phi\,dT + \int_0^T dt \int_T [u, s]\phi\,dT. 
\]
(47)

Therefore, $(s,u)$ is the weak solution to system (4) by the definition.

**Case 2.** $\alpha = 0$. When $\alpha > 0$, we know that $u^\alpha$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^+, H^1(\mathbb{T}, \mathfrak{g}(1)))$, where $(u^\alpha, s)$ is the weak solution in Case 1. Therefore, there exists a subsequence of $u^\alpha$, which is still denoted by $u^\alpha$, converging to $u$ weakly * in $L^\infty_{loc}(\mathbb{R}^+, H^1(\mathbb{T}, \mathfrak{g}(1)))$ and converging to $u$ a.e. $\mathbb{T} \times \mathbb{R}^+$. On the other hand, for any $\varphi \in C^\infty(\mathbb{T} \times [0,T], \mathfrak{g})$ there holds true
\[
\int_0^T \int_T u^\alpha \varphi\,dT\,dt - \alpha \int_0^T \int_T [u^\alpha, u^\alpha_t]\varphi\,dT\,dt = \int_0^T \int_T \sum_{p=1}^n [u^\alpha, \nabla e_p u^\alpha]\nabla e_p \varphi\,dT\,dt - \int_0^T \int_T [u^\alpha, s]\varphi\,dT\,dt. 
\]
(48)

Noting that, as $\alpha \to 0$,
\[
\alpha \int_0^T \int_T |u^\alpha_t|^2\,dT \leq M_2(\alpha, t)
\]
implies
\[
\alpha \int_0^T \int_T [u^\alpha, u^\alpha_t]\varphi\,dT\,dt \to 0,
\]
we let $\alpha \to 0$ in (48) to derive
\[
\int_T u(T)\varphi(T)\,dT - \int_T u_0\varphi(0)\,dT = \int_0^T \int_T u\varphi_t\,dT\,dt + \int_0^T \int_T \sum_{p=1}^n [u, \nabla e_p u]\nabla e_p \varphi\,dT\,dt - \int_0^T \int_T [u, s]\varphi\,dT\,dt.
\]

Moreover, since
\[
\int_T s(T)\phi(T)\,dT - \int_T s_0\phi(0)\,dT - \int_0^T dt \int_T s\partial_t\phi\,dT - \int_0^T dt \int_T \langle u^\alpha, \nabla J_e\phi \rangle\,dT = -\int_0^T dt \int_T \sum_{p=1}^n \langle A(x,u^\alpha)\nabla e_p s, \nabla e_p \phi \rangle - \int_0^T dt \int_T s\phi\,dT + \int_0^T dt \int_T [u^\alpha, s]\phi\,dT,
\]
as $\alpha \to 0$ in the above identity we deduce that (47) still holds true. Therefore, $(s,u)$ is the weak solution to system (4) by the definition. This completes the proof. \(\square\)

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