Multiplicity of positive solutions for \((p, q)\)-Laplace equations with two parameters

Vladimir Bobkov
Department of Mathematics and NTIS, Faculty of Applied Sciences, University of West Bohemia
Univerzitní 8, 301 00 Plzeň, Czech Republic
Institute of Mathematics, Ufa Federal Research Centre, RAS
Chernyshevsky str. 112, 450008 Ufa, Russia
e-mail: bobkov@kma.zcu.cz

Mieko Tanaka
Department of Mathematics, Tokyo University of Science
Kagurazaka 1-3, Shinjyuku-ku, Tokyo 162-8601, Japan
e-mail: miekotanaka@rs.tus.ac.jp

Abstract
We study the zero Dirichlet problem for the equation
\[-\Delta_p u - \Delta_q u = \alpha |u|^{p-2} u + \beta |u|^{q-2} u\]
in a bounded domain \(\Omega \subset \mathbb{R}^N\), with \(1 < q < p\). We investigate the relation between two critical curves on the \((\alpha, \beta)\)-plane corresponding to the threshold of existence of special classes of positive solutions. In particular, in certain neighbourhoods of the point \((\alpha, \beta) = \left(\frac{\|\nabla \varphi_p\|_p}{\|\varphi_p\|_p}, \frac{\|\nabla \varphi_p\|_q}{\|\varphi_p\|_q}\right)\), where \(\varphi_p\) is the first eigenfunction of the \(p\)-Laplacian, we show the existence of two and, which is rather unexpected, three distinct positive solutions, depending on a relation between the exponents \(p\) and \(q\).

Keywords: \((p, q)\)-Laplacian, positive solutions, fibered functional, mountain pass theorem, local minimum, S-shaped bifurcation, three solutions.

MSC2010: 35P30, 35B09, 35B32, 35B34, 35J62, 35J20

1. Introduction and main results

We consider the boundary value problem
\[
\begin{cases}
-\Delta_p u - \Delta_q u = \alpha |u|^{p-2} u + \beta |u|^{q-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
\((D_{\alpha, \beta})\)

where the operator \(\Delta_r\), formally defined as \(\Delta_r u = \text{div} (|\nabla u|^{r-2} \nabla u)\) for \(r = p, q > 1\), is the \(r\)-Laplacian, \(\alpha, \beta \in \mathbb{R}\) are parameters, and \(\Omega \subset \mathbb{R}^N\) is a bounded domain, \(N \geq 1\). In the case \(N \geq 2\), we require the boundary \(\partial \Omega\) of \(\Omega\) to be \(C^2\)-smooth. Throughout the text, we always assume \(q < p\), which involves no loss of generality.

The differential operator in the problem \((D_{\alpha, \beta})\) is usually called the \((p, q)\)-Laplacian, and thereby \((D_{\alpha, \beta})\) can be formally understood as the corresponding eigenvalue problem. Although the presence of two spectral parameters \((\alpha\) and \(\beta)\) is not typical in nonlinear spectral theories (cf. [3, 18]), such choice appears to be more convenient for our particular problem since it provides a separate control of the influence of the \((p - 1)\)- and \((q - 1)\)-homogeneous parts. Considered independently, these parts correspond to the eigenvalue problems for the \(p\)- and
$q$-Laplacians, and it is thus natural to anticipate a strong dependence of the structure of the solution set of $(D_{\alpha,\beta})$ on the spectrum of both $p$- and $q$-Laplacians. Indeed, the problem $(D_{\alpha,\beta})$ has been investigated in a few works, where certain nontrivial dependences of this kind were obtained, see, e.g., [14, 15, 22, 27, 30, 40], the works [5, 6, 7, 8] of the present authors, and a survey [26]. In the present article, we continue our investigation of the problem $(D_{\alpha,\beta})$ by establishing several nontrivial multiplicity results, mainly in special neighbourhoods of the point

$$\left(\alpha, \beta\right) = \left(\frac{\|\nabla \varphi_p\|_p^p}{\|\varphi_p\|_p^p}, \frac{\|\nabla \varphi_p\|_q^q}{\|\varphi_p\|_q^q}\right),$$

where $\varphi_p$ is the first eigenfunction of the $p$-Laplacian. In particular, we discover the formation of an $S$-shaped bifurcation diagram when $p > 2q$, see Figure 1.

Prior to the rigorous description of main results, let us mention that various problems with the $(p, q)$-Laplacian, whose motivation arises from both mathematical and physical premises, are actively studied in the contemporary literature. Among physical origins of the $(p, q)$-Laplacian, one can think of it as a formal two-term Taylor approximation of more complex differential operators, see, e.g., [41] for the Zakharov equation describing in a simplified way long-wave oscillations of a plasma, [4] for a higher-dimensional generalization of the sine-Gordon equation which possesses soliton-type solutions, and [9] for an approximation of the electrostatic Born-Infeld equation with a superposition of point charges. Let us also point out a model in the theory of crystal growth containing the one-dimensional $(1,2)$-Laplacian which was studied in [31]. Among mathematical origins, the $(p, q)$-Laplacian occurs, e.g., in the procedure of elliptic regularization which consists in the inclusion of the regularizing term $\varepsilon^2 \Delta$, $\varepsilon \in \mathbb{R}$, in a nonlinear equation, with a view to obtain better properties of the augmented equation, see, for instance, [1, 28]. An investigation of variational functionals with nonstandard $(p, q)$-growth conditions, mainly in connection with the Lavrentiev gap phenomenon, has been performed, e.g., in [16, 42]. Finally, we refer the interested reader to a nonexhaustive list of works [12, 13, 14, 27, 34, 35] for a development of the existence theory for various problems with the $(p, q)$-Laplacian.

### 1.1. Several notations

Hereinafter, we denote the Sobolev space $W_0^{1,r}(\Omega)$ shortly by $W_0^{1,r}$, where $r > 1$. The standard norm of the Lebesgue space $L^r(\Omega)$ will be denoted by $\| \cdot \|_r$. A function $u \in W_0^{1,p}$ is called a (weak) solution of $(D_{\alpha,\beta})$ if the following equality is satisfied for any test function $\varphi \in W_0^{1,p}$:

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \varphi \, dx = \alpha \int_{\Omega} |u|^{p-2} u \varphi \, dx + \beta \int_{\Omega} |u|^{q-2} u \varphi \, dx. \quad (1.1)$$

The energy functional $E_{\alpha,\beta} : W_0^{1,p} \to \mathbb{R}$ associated with $(D_{\alpha,\beta})$ is given by

$$E_{\alpha,\beta}(u) = \frac{1}{p} H_{\alpha}(u) + \frac{1}{q} G_{\beta}(u),$$

where

$$H_{\alpha}(u) := \|\nabla u\|_p^p - \alpha \|u\|_p^p \quad \text{and} \quad G_{\beta}(u) := \|\nabla u\|_q^q - \beta \|u\|_q^q.$$ 

Since $p > q > 1$, we have $E_{\alpha,\beta} \in C^1(W_0^{1,p}, \mathbb{R})$, and hence weak solutions of $(D_{\alpha,\beta})$ are in one-to-one correspondence with critical points of $E_{\alpha,\beta}$. 
Remark 1.1. Using the Moser iteration process (see, e.g., [29, Appendix A]), one can show that any solution \( u \) of \( (D_{\alpha, \beta}) \) belongs to \( L^\infty(\Omega) \). Then, the regularity up to the boundary given by [24, Theorem 1] and [25, p. 320] ensures that \( u \in C^{1, \gamma}_0(\Omega) \) for some \( \gamma \in (0,1) \). Moreover, if \( u \) is a nonzero nonnegative solution, then the strong maximum principle and the boundary point lemma (see, e.g., [38, Theorems 5.4.1 and 5.5.1]) guarantee that \( u \) is positive and belongs to
\[
\text{int } C^1_0(\Omega)_+ := \left\{ u \in C^1_0(\Omega) : \ u(x) > 0 \text{ for all } x \in \Omega, \frac{\partial u}{\partial \nu}(x) < 0 \text{ for all } x \in \partial \Omega \right\},
\]
the interior of the positive cone of \( C^1_0(\Omega) \). Here \( \nu \) is the exterior unit normal vector to \( \partial \Omega \).

Finally, we denote by \( \lambda_1(r) \) the first eigenvalue of the \( r \)-Laplacian, i.e.,
\[
\lambda_1(r) = \inf \left\{ \frac{\| \nabla u \|^r_p}{\| u \|^r_p} : u \in W^{1,r}_0(\Omega) \setminus \{0\} \right\},
\]
and by \( \varphi_r \) the corresponding first eigenfunction. Notice that \( \varphi_r \) has a constant sign in \( \Omega \), and we will assume, without loss of generality, that \( \varphi_r > 0 \) in \( \Omega \) and \( \| \nabla \varphi_r \|_r = 1 \). Moreover, for such \( \varphi_r \) we have \( \varphi_r \in \text{int } C^1_0(\Omega)_+ \). Furthermore, since \( p > q \), \( \varphi_p \) cannot simultaneously be an eigenfunction of the \( q \)-Laplacian, see [7, Proposition 13].

1.2. Overview of known results

Let us divide the \((\alpha, \beta)\)-plane into four open quadrants by the lines \( \{ \lambda_1(p) \} \times \mathbb{R} \) and \( \mathbb{R} \times \{ \lambda_1(q) \} \) (see Figures 1, 2). We recall several known facts about the existence, nonexistence, and multiplicity of positive solutions of \((D_{\alpha, \beta})\) in these quadrants, as well as on their boundaries \( \{ \lambda_1(p) \} \times \mathbb{R} \) and \( \mathbb{R} \times \{ \lambda_1(q) \} \).

**Proposition 1.2** ([5, Proposition 1] and [7, Proposition 13]). Let \( \alpha \leq \lambda_1(p) \) and \( \beta \leq \lambda_1(q) \). Then \((D_{\alpha, \beta})\) has no nonzero solution.

**Proposition 1.3** ([5, Propositions 2 and 6] and [7, Remark 1]; see also [27, Lemma 2.2] for a related result). Let \( \alpha < \lambda_1(p) \) and \( \beta > \lambda_1(q) \). Then \((D_{\alpha, \beta})\) has at least one positive solution. Moreover, any nonzero solution \( u \) of \((D_{\alpha, \beta})\) satisfies \( E_{\alpha, \beta}(u) < 0 \). Furthermore, if \( \alpha \leq 0 \), then the positive solution is unique.

**Proposition 1.4** ([5, Propositions 2 and 6]). Let \( \alpha > \lambda_1(p) \) and \( \beta < \lambda_1(q) \). Then \((D_{\alpha, \beta})\) has at least one positive solution. Moreover, any nonzero solution \( u \) of \((D_{\alpha, \beta})\) satisfies \( E_{\alpha, \beta}(u) > 0 \).

In order to discuss the existence in the remaining quadrant \((\lambda_1(p), +\infty) \times (\lambda_1(q), +\infty)\), we introduce the threshold curve
\[
\beta_{ps}(\alpha) := \sup \{ \beta \in \mathbb{R} : (D_{\alpha, \beta}) \text{ has at least one positive solution} \}
\]
for \( \alpha \geq \lambda_1(p) \). Define also the values
\[
\alpha_* = \frac{\| \nabla \varphi_q \|^p_p}{\| \varphi_q \|^p_p} \quad \text{and} \quad \beta_* = \frac{\| \nabla \varphi_p \|^q_q}{\| \varphi_p \|^q_q}.
\]
It was proved in [5, Proposition 3] (see also [7, Remark 4]) that \( \beta_{ps}(\alpha) < +\infty \) for any \( \alpha > \lambda_1(p) \), \( \beta_{ps}(\lambda_1(p)) \geq \beta_* \), \( \beta_{ps}(\cdot) \) is continuous and nonincreasing on \((\lambda_1(p), +\infty)\), and \( \beta_{ps}(\alpha) = \lambda_1(q) \) for all \( \alpha \geq \alpha_* \). Moreover, \( \alpha_* > \lambda_1(p) \) and \( \beta_* > \lambda_1(q) \), see [7, Lemma 2.1].
Theorem 1.5 ([5, Theorem 2.2 and Proposition 4]). Let $\alpha \in (\lambda_1(p), \alpha_*)$ and $\beta \in (-\infty, \beta_{ps}(\alpha)]$. Then $( D_{\alpha, \beta} )$ has at least one positive solution.

However, the properties of $\beta_{ps}(\alpha)$ for $\alpha \in [\lambda_1(p), \alpha_*)$ are far from being completely understood. In particular, the asymptotic behaviour of $\beta_{ps}(\alpha)$ as $\alpha$ approaches $\lambda_1(p)$ was substantially unclear until the recent work [8], where the following results have been established by obtaining a nontrivial generalization of the classical Picone inequality [2] and using the generalized Picone inequalities [10, Proposition 2.9], [20, Lemma 1], and a radial symmetry result of [11].

Theorem 1.6 ([8, Theorem 3.3]). We have $\beta_* \leq \beta_{ps}(\lambda_1(p)) < +\infty$. Moreover, $( D_{\lambda_1(p), \beta} )$ has at least one positive solution if $\lambda_1(q) < \beta < \beta_{ps}(\lambda_1(p))$. Furthermore, if $\beta_{ps}(\lambda_1(p)) > \beta_*$, then $( D_{\lambda_1(p), \beta} )$ has at least one positive solution if and only if $\lambda_1(q) < \beta \leq \beta_{ps}(\lambda_1(p))$.

Theorem 1.7 ([8, Theorem 3.2]). Assume that one of the following assumptions is satisfied:

(i) $p \in I(q)$, where

$$I(q) := \{ p > 1 : (q - 1)s^p + qs^{p-1} - (p - q)s + (q - p + 1) \geq 0 \text{ for all } s \geq 0 \};$$

(ii) $p \leq q + 1$ and $\Omega$ is an $N$-ball.

Then $( D_{\lambda_1(p), \beta} )$ has no positive solution for $\beta > \beta_*$, that is, $\beta_{ps}(\lambda_1(p)) = \beta_*$. Moreover, if $p < q + 1$ and $\Omega$ is an $N$-ball, then $( D_{\lambda_1(p), \beta} )$ has no positive solution also for $\beta = \beta_*$.

Remark 1.8. We recall that $q < p$ in Theorem 1.7 by default. The set $I(q)$ is characterized in [8, Lemma 1.6]. In particular, it is known that for each $q > 1$ there exists $\tilde{p} \in (\max\{2, q\}, q+1)$ such that $[2, \tilde{p}] \subset I(q)$ and $(\tilde{p}, +\infty) \cap I(q) = \emptyset$.

Theorem 1.7 generates a natural question on whether $\beta_{ps}(\lambda_1(p)) > \beta_*$ if either the assumption (i) or (ii) of this theorem is violated. Nontriviality of this question is supported by the fact that the behaviour of the energy functional $E_{\alpha, \beta}$ at the point $(\lambda_1(p), \beta_*)$ crucially depends on the relation between $p$ and $q$.

Theorem 1.9 ([7, Theorem 2.6 (ii) and Remark 5]). We have the following assertions:

(i) If $p < 2q$, then $\inf_{W^1_0} E_{\lambda_1(p), \beta_*} = -\infty$.

(ii) If $p = 2q$, and $\partial \Omega$ is connected when $N \geq 2$, then $\inf_{W^1_0} E_{\lambda_1(p), \beta_*} \in (-\infty, 0)$.

(iii) If $p > 2q$, and $\partial \Omega$ is connected when $N \geq 2$, then $\inf_{W^1_0} E_{\lambda_1(p), \beta_*} \in (-\infty, 0)$ and the infimum is attained by a positive solution of $( D_{\lambda_1(p), \beta_*} )$.

Let us remark that the connectedness of $\partial \Omega$ is required in the proof of Theorem 1.9 (ii), (iii) due to the usage of the improved Poincaré inequality obtained in [17]. It is conjectured in [17, Section 3.1], however, that this assumption on $\partial \Omega$ can be omitted for sufficiently regular domains. To the best of our knowledge, this conjecture is still open.

Theorem 1.9 suggests that not only the behaviour of $E_{\lambda_1(p), \beta_*}$ but also the structure of the solution set of $( D_{\alpha, \beta} )$ in a neighbourhood of the point $(\lambda_1(p), \beta_*)$ is different in the cases $p < 2q$, $p = 2q$, and $p > 2q$, which possibly affects the relation between $\beta_{ps}(\lambda_1(p))$ and $\beta_*$. 


Indeed, this turns out to be true, and the precise results will be formulated in Section 1.3 below.

A finer existence result in the quadrant \((\lambda_1(p), +\infty) \times (\lambda_1(q), +\infty)\) can be obtained if we introduce the following family of critical values for \(\alpha \geq \lambda_1(p)\):

\[
\beta_\star(\alpha) := \inf \left\{ \frac{\|\nabla u\|_q^q}{\|u\|_q^q} : u \in W_0^{1,p} \setminus \{0\} \text{ and } H_\alpha(u) \leq 0 \right\},
\]

(1.2)
or, equivalently,

\[
\beta_\star(\alpha) = \inf \left\{ \frac{\|\nabla u\|_q^q}{\|u\|_q^q} : u \in W_0^{1,p} \setminus \{0\} \text{ and } \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \leq \alpha \right\}.
\]

It is known that \(\beta_\star(\cdot)\) is continuous and nonincreasing on \([\lambda_1(p), +\infty)\), \(\beta(\lambda_1(p)) = \beta_\star > \beta_\star(\alpha) \geq \lambda_1(q)\) for \(\alpha > \lambda_1(p)\), and \(\beta_\star(\alpha) > \lambda_1(q)\) if and only if \(\alpha < \alpha_\star\), see [7, Proposition 7].

**Theorem 1.10** ([7, Theorem 2.7]). Let \(\lambda_1(p) < \alpha < \alpha_\star\) and \(\lambda_1(q) < \beta \leq \beta_\star(\alpha)\). Then \((D_{\alpha,\beta})\) has at least two positive solutions \(u_1\) and \(u_2\) such that \(E_{\alpha,\beta}(u_1) < 0\), \(E_{\alpha,\beta}(u_2) > 0\) if \(\beta < \beta_\star(\alpha)\), and \(E_{\alpha,\beta}(u_2) = 0\) if \(\beta = \beta_\star(\alpha)\). Moreover, \(u_1\) is the global least energy solution, and if \(\beta < \beta_\star(\alpha)\), then \(u_2\) has the least energy among all solutions \(w\) of \((D_{\alpha,\beta})\) with \(E_{\alpha,\beta}(w) > 0\).

In particular, Theorem 1.10 yields

\[
\beta_\star(\alpha) \leq \beta_{ps}(\alpha) \quad \text{for all } \alpha \geq \lambda_1(p).
\]

(1.3)
Moreover, \(\beta_\star(\alpha) = \beta_{ps}(\alpha) = \lambda_1(q)\) for all \(\alpha \geq \alpha_\star\). The most essential open question here was whether the strict inequality in (1.3) holds. This issue is addressed in the present article, see the following subsection.

Finally, in accordance with the results described above, the only place on the \((\alpha, \beta)\)-plane where it remains to discuss the existence of positive solutions of \((D_{\alpha,\beta})\) is the interval \([\alpha_\star, +\infty) \times \{\lambda_1(q)\}\). It was proved in [5, Proposition 4 (ii)] that \((D_{\alpha,\lambda_1(q)})\) has no positive solution whenever \(\alpha > \alpha_\star\). In fact, the proof of [5, Proposition 4 (ii)] can be slightly updated in order to show that the nonexistence persists also in the case \(\alpha = \alpha_\star\). Indeed, it follows from the proof of [5, Proposition 4 (ii)] that if \((D_{\alpha,\lambda_1(q)})\) possesses a positive solution \(u\), then \(u = k \varphi_q\) for some \(k > 0\). However, this is impossible in view of [7, Proposition 13]. Thus, thanks to Proposition 1.2 and Theorem 1.5, \((D_{\alpha,\lambda_1(q)})\) possesses a positive solution if and only if \(\alpha \in (\lambda_1(p), \alpha_\star)\).

### 1.3. Statements of main results

For convenience, we introduce the following hypothesis:

(H) \(p > 2q\), and if \(N \geq 2\), then \(\partial \Omega\) is connected.

Our first result is devoted to the relation between \(\beta_{ps}(\alpha)\) and \(\beta_\star(\alpha)\).

**Theorem 1.11.** Let \(\alpha \in [\lambda_1(p), \alpha_\star)\), and assume (H) if \(\alpha = \lambda_1(p)\). Then there exists \(\bar{\beta}(\alpha) > \beta_\star(\alpha)\) such that \((D_{\alpha,\beta})\) possesses a positive solution \(u\) for any \(\beta \in (\beta_\star(\alpha), \bar{\beta}(\alpha)]\). Moreover, \(u\) is a local minimum point of \(E_{\alpha,\beta}\) and \(E_{\alpha,\beta}(u) < 0\).
The case \( p > 2q \). The behaviour of \( \beta_*(\alpha) \), \( \beta_{ps}(\alpha) \), \( \alpha_*(\beta) \), and alleged “minimal” bifurcation diagrams for the \( L^\infty \)-norms of positive solutions of \( (D_{\alpha,\beta}) \) with respect to \( \beta \) for several fixed \( \alpha \)'s. Light grey - two positive solutions, one of which is with positive energy and another one is with negative energy; grey - two positive solutions with negative energy; dark grey - three positive solutions with negative energy.

The idea of the proof of Theorem 1.11 is based on the recent work [21], where the authors obtained a local continuation of the branch of least energy solutions of an elliptic problem with indefinite nonlinearity using an original variational argument of a constrained minimization type.

Theorem 1.11 implies, in particular, that

\[
\beta_*(\alpha) < \beta_{ps}(\alpha) \quad \text{for all } \alpha \in (\lambda_1(p), \alpha_*),
\]

and that \( \beta_* < \beta_{ps}(\lambda_1(p)) \) provided the additional assumption (H) is satisfied. On the other hand, we know from Theorem 1.7 that \( \beta_* = \beta_{ps}(\lambda_1(p)) \) if either \( p \in I(q) \) or \( p \leq q + 1 \) and \( \Omega \) is an \( N \)-ball. Nevertheless, it remains unknown whether \( \beta_* = \beta_{ps}(\lambda_1(p)) \) for all \( p \leq 2q \) regardless of assumptions on \( \Omega \). Moreover, we do not know whether \( (D_{\lambda_1(p),\beta_{ps}(\lambda_1(p))}) \) possesses a positive solution provided \( \beta_{ps}(\lambda_1(p)) = \beta_* \), except in the case discussed in Theorem 1.7, cf. Theorem 1.5.

Theorem 1.11 in combination with the behaviour of \( E_{\alpha,\beta} \) investigated in [7] allows us to state the following two multiplicity results, among which Theorem 1.13 is perhaps the most surprising since it indicates the occurrence of an \( S \)-shaped bifurcation diagram in the case \( p > 2q \), see Figure 1.

**Theorem 1.12.** Let \( \alpha \in [\lambda_1(p), \alpha_*) \) and \( \beta \in (\beta_*(\alpha), \beta_{ps}(\alpha)) \), and assume (H) if \( \alpha = \lambda_1(p) \). Then \( (D_{\alpha,\beta}) \) has at least two positive solutions \( u_1 \) and \( u_2 \) satisfying \( E_{\alpha,\beta}(u_1) < 0 \) and \( E_{\alpha,\beta}(u_2) < 0 \).
Figure 2: $p, q$ as in Theorem 1.7. The behaviour of $\beta_*(\alpha)$, $\beta_{ps}(\alpha)$, and alleged “minimal” bifurcation diagrams for the $L^\infty$-norms of positive solutions of $(D_{\alpha,\beta})$ with respect to $\beta$ for several fixed $\alpha$’s. Light grey - two positive solutions, one of which is with positive energy and another one is with negative energy; grey - two positive solutions with negative energy.

**Theorem 1.13.** Assume (H). Then for every $\beta \in (\beta_*, \beta_{ps}(\lambda_1(p)))$ there exists $\alpha_*(\beta) \in (0, \lambda_1(p))$ such that $(D_{\alpha,\beta})$ has at least three positive solutions for any $\alpha \in (\alpha_*(\beta), \lambda_1(p))$.

**Remark 1.14.** All three solutions obtained in Theorem 1.13 have negative energy, see Proposition 1.3.

Let us recall that if $\alpha \leq 0$, then the positive solution of $(D_{\alpha,\beta})$ is unique (see Proposition 1.3), and it was unclear (see [7, Remark 1]) whether a difficulty to extend the uniqueness to $\alpha \in (0, \lambda_1(p))$ lies only in the limitation of the method of the original proof, or a multiplicity of positive solutions can actually occur. Theorem 1.13 answers this question in a nontrivial way. We emphasize that this multiplicity result does not depend on the domain, as it happens, e.g., in the case of superlinear problems of the type $-\Delta u = |u|^{p-2}u$, cf. [32]. Moreover, there is no simple *a priori* intuition about such multiplicity based on the behaviour of fiber functions of $E_{\alpha,\beta}$, since these functions have at most one critical point which is the point of global minimum, see Section 2. Finally, let us mention that the $S$-shaped bifurcation diagram indicated by Theorem 1.13 clarifies the shape of the bifurcation diagrams (A) or (B) in [22] obtained for the one-dimensional version of $(D_{\lambda,\lambda})$. See Figure 3 for some numerical results in the one-dimensional case.

**Remark 1.15.** Properties of the family of critical points $\alpha_*(\beta)$, such as the continuity, monotonicity, etc., are mostly unknown. We anticipate that the set of parameters $\alpha, \beta$ corresponding to the existence of three positive solutions of $(D_{\alpha,\beta})$ can be extended to a larger region as depicted by the dashed line on Figure 1.

The rest of the article is structured as follows. In Section 2, we introduce a few additional notations and provide an auxiliary lemma needed for the proof of Theorem 1.11 which we
establish in Section 3. Section 4 provides auxiliary results needed to prove Theorems 1.12 and 1.13. These theorems are established in Section 5. Appendix A contains a “$W_1^{1,p}$ versus $C_0^1$ local minimizers”-type result for general problems with the $(p,q)$-Laplacian, which we also apply in Section 5.

2. Auxiliary results I. The fibered functional $J_{\alpha,\beta}$

Take any $u \in W_0^{1,p}$ satisfying $H_\alpha(u) \cdot G_\beta(u) < 0$ and consider the fiber function $t \mapsto E_{\alpha,\beta}(tu)$ for $t > 0$. It is not hard to observe that this function has a unique critical point $t_{\alpha,\beta}(u)$ given by

$$t_{\alpha,\beta}(u) = \left( \frac{-G_\beta(u)}{H_\alpha(u)} \right)^\frac{1}{p-q} = \frac{|G_\beta(u)|^{\frac{1}{p-q}}}{|H_\alpha(u)|^{\frac{1}{p-q}}},$$

see [5, Proposition 6]. Moreover,

$$J_{\alpha,\beta}(u) := E_{\alpha,\beta}(t_{\alpha,\beta}(u)u) = -\text{sign}(H_\alpha(u)) \frac{p-q}{pq} \frac{|G_\beta(u)|^{\frac{1}{p-q}}}{|H_\alpha(u)|^{\frac{1}{p-q}}}.$$  

In particular, if $G_\beta(u) < 0 < H_\alpha(u)$, then $t_{\alpha,\beta}(u)$ is the point of global minimum of the function $t \mapsto E_{\alpha,\beta}(tu)$, i.e.,

$$\min_{t>0} E_{\alpha,\beta}(tu) = E_{\alpha,\beta}(t_{\alpha,\beta}(u)u) \equiv J_{\alpha,\beta}(u) = -\frac{p-q}{pq} \frac{|G_\beta(u)|^{\frac{1}{p-q}}}{|H_\alpha(u)|^{\frac{1}{p-q}}} < 0. \quad (2.3)$$

The functional $J_{\alpha,\beta}$ is called fibered functional [36], it is 0-homogeneous, and if $u$ is a critical point of $J_{\alpha,\beta}$ satisfying $H_\alpha(u) \cdot G_\beta(u) < 0$, then $t_{\alpha,\beta}(u)u$ is a critical point of $E_{\alpha,\beta}$.

By $\mathcal{N}_{\alpha,\beta}$ we denote the Nehari manifold associated to $E_{\alpha,\beta}$, that is,

$$\mathcal{N}_{\alpha,\beta} = \left\{ v \in W_0^{1,p} \setminus \{0\} : \langle E_{\alpha,\beta}'(v), v \rangle = H_\alpha(v) + G_\beta(v) = 0 \right\}.$$ 

Clearly, this set contains all nonzero critical points of $E_{\alpha,\beta}$. Notice that if we take any $u \in W_0^{1,p}$ satisfying $H_\alpha(u) \cdot G_\beta(u) < 0$, then $t_{\alpha,\beta}(u)u \in \mathcal{N}_{\alpha,\beta}$, see [7, Proposition 10].
The following auxiliary result will be used in the proof of Theorem 1.11 in Section 3 below.

**Lemma 2.1.** Let \( \alpha \in [\lambda_1(p), \alpha_*] \), \( \{\beta_n\} \subset [\beta_*(\alpha), +\infty) \) be a sequence converging to \( \beta \geq \beta_*(\alpha) \), and \( \mu \in (\lambda_1(q), \beta_*(\alpha)) \). Assume that a sequence \( \{w_n\} \subset W^{1,p}_0 \) satisfies \( \|\nabla w_n\|_p = 1 \) for all \( n \in \mathbb{N} \), and let \( w_0 \in W^{1,p}_0 \) be such that \( \{w_n\} \) converges weakly in \( W^{1,p}_0 \) and strongly in \( L^p(\Omega) \) to \( w_0 \) as \( n \to +\infty \). Assume, moreover, that

\[
G_\mu(w_n) \leq 0 < H_\alpha(w_n) \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad -\infty \leq \liminf_{n \to +\infty} J_{\alpha,\beta_n}(w_n) < 0. \quad (2.4)
\]

Then \( w_0 \neq 0 \) in \( \Omega \), and we have

\[
G_\beta(w_0) < G_\mu(w_0) \leq 0 < H_\alpha(w_0) \quad \text{and} \quad -\infty < J_{\alpha,\beta}(w_0) \leq \liminf_{n \to +\infty} J_{\alpha,\beta_n}(w_n).
\]

**Proof.** First we show that \( w_0 \neq 0 \) in \( \Omega \). Suppose, by contradiction, that \( w_0 \equiv 0 \) in \( \Omega \). That is, \( \|w_n\|_p \to 0 \) and \( \|w_n\|_q \to 0 \) as \( n \to +\infty \). In particular, we have \( H_\alpha(w_n) = 1 - o(1) \). Moreover, since \( G_\mu(w_n) \leq 0 \), we see that \( \|\nabla w_n\|_q \to 0 \), which yields \( G_{\beta_n}(w_n) \to 0 \), and, consequently, \( J_{\alpha,\beta_n}(w_n) \to 0 \). However, this is a contradiction to (2.4), and hence \( w_0 \neq 0 \) in \( \Omega \).

By the weak lower semicontinuity, we readily get

\[
G_\mu(w_0) \leq \liminf_{n \to +\infty} G_\mu(w_n) \leq 0,
\]

which implies that

\[
\frac{\|\nabla w_0\|_p^p}{\|w_0\|_q^q} \leq \mu < \beta_*(\alpha).
\]

Due to the definition (1.2) of \( \beta_*(\alpha) \), we conclude that \( H_\alpha(w_0) > 0 \). On the other hand, by our assumptions, we have \( \beta > \mu \) and \( \beta_n > \mu \) for all \( n \), which implies that \( G_{\beta}(w_0) < 0 \) and \( G_{\beta_n}(w_n) < 0 \) for all \( n \). Therefore, the weak lower semicontinuity of \( G_{\beta_n} \) and \( H_\alpha \) yields

\[
-\infty < J_{\alpha,\beta}(w_0) \leq \liminf_{n \to +\infty} J_{\alpha,\beta_n}(w_n),
\]

which completes the proof.

\[\square\]

3. **Beyond \( \beta_*(\alpha) \). The proof of Theorem 1.11**

In this section, we prove Theorem 1.11. Throughout the section, we assume \( \alpha \in [\lambda_1(p), \alpha_*] \) to be fixed, and we require hypothesis (H) if \( \alpha = \lambda_1(p) \).

The proof of Theorem 1.11 will rely on the consideration of the following minimization problem:

\[
\mathcal{J}(\beta, \mu) := \inf \left\{ J_{\alpha,\beta}(u) : u \in W^{1,p}_0, \ G_\mu(u) < 0 < H_\alpha(u) \right\}, \quad (3.1)
\]

where we assume \( \beta \geq \beta_*(\alpha) \) and \( \mu \in (\lambda_1(q), \beta_*(\alpha)) \), and \( J_{\alpha,\beta} \) is the fibered functional defined by (2.2). Notice that the index of \( G_\beta \) presented in \( J_{\alpha,\beta} \) is, in general, different from the index of \( G_\mu \) presented in the constraint. To the best of our knowledge, the idea of introduction of such constraints was originated in the work [21].

Let us discuss several general properties of (3.1). The admissible set for \( \mathcal{J}(\beta, \mu) \) is nonempty because \( \mu > \lambda_1(q) \) and \( \alpha < \alpha_* \) yield \( G_\mu(\varphi_q) < 0 < H_\alpha(\varphi_q) \). Consequently, we always have

\[
\mathcal{J}(\beta, \mu) \leq J_{\alpha,\beta}(\varphi_q) < 0, \quad (3.2)
\]
since $\beta \geq \beta_*(\alpha) > \lambda_1(q)$. If we let $\beta = \mu$, then $J(\beta, \mu)$ translates to the usual minimization problem of finding the least energy solution to $(D_{\alpha, \beta})$, see, e.g., [5, 7]. In particular, $J(\beta_*(\alpha), \beta_*(\alpha))$ is attained, and if $u_*$ is a corresponding minimizer, then $t_{\alpha, \beta}(u_*)u_*$ is a solution of $(D_{\alpha, \beta}(\alpha))$, see Theorem 1.10 in the case $\lambda_1(p) < \alpha < \alpha_*$ and Theorem 1.9 in the case $\alpha = \lambda_1(p)$. Let us define

$$
\mu_0 = \mu_0(\alpha) := \sup \left\{ \frac{\|\nabla u_*\|^q}{\|u_*\|^q} : u_* \text{ is a minimizer of } J(\beta_*(\alpha), \beta_*(\alpha)) \right\}. \quad (3.3)
$$

**Proposition 3.1.** $\mu_0 < \beta_*(\alpha)$.

**Proof.** It is clear that $\mu_0 \leq \beta_*(\alpha)$, since otherwise $G_{\beta_*(\alpha)}(u_*) > 0$ for some minimizer $u_*$ of $J(\beta_*(\alpha), \beta_*(\alpha))$, which is impossible, see (3.1) with $\beta = \mu = \beta_*(\alpha)$. Suppose, contrary to our claim, that $\mu_0 = \beta_*(\alpha)$. That is, there exists a sequence of minimizers $\{u_k\}$ of $J(\beta_*(\alpha), \beta_*(\alpha))$ such that $\|\nabla u_k\|^q \to \beta_*(\alpha)$. Since $J_{\alpha, \beta}$ is 0-homogeneous, we may assume, without loss of generality, that $\|\nabla u_k\|_p = 1$ for each $k$. Thus, the latter convergence yields $G_{\beta_*(\alpha)}(u_k) \to 0$.

On the other hand, since $J_{\alpha, \beta}(u_k) = J(\beta_*(\alpha), \beta_*(\alpha)) < 0$ by (3.2), we get from (2.3) that

$$
H_\alpha(u_k) = \left(\frac{p - q}{pq}\right)^{\frac{q}{p}} \frac{|G_{\beta_*(\alpha)}(u_k)|^\frac{q}{p}}{(-J(\beta_*(\alpha), \beta_*(\alpha)))^\frac{q}{p}} \text{ for all } k \in \mathbb{N}. \quad (3.4)
$$

Substituting (3.4) into (2.1), we deduce, in view of the default assumption $p > q$, that $t_{\alpha, \beta_*(\alpha)}(u_k) \to +\infty$. Moreover, by considering $|u_k|$ if necessary, we may assume that $u_k \geq 0$ in $\Omega$ for all $k$. Recall now that $t_{\alpha, \beta_*(\alpha)}(u_k)u_k$ is a solution of $(D_{\alpha, \beta_*(\alpha)})$, and hence

$$
\langle H_\alpha'(u_k), \varphi \rangle + t_{\alpha, \beta_*(\alpha)}(u_k)^{\gamma - p} \left\langle G_{\beta_*(\alpha)}'(u_k), \varphi \right\rangle = 0 \quad \text{for all } \varphi \in W_0^{1,p},
$$

which implies that $u_k \to \varphi_p$ (strongly) in $W_0^{1,p}$, up to a subsequence, and $\alpha = \lambda_1(p)$, see [6, Lemma 3.3]. Therefore, if we fixed $\alpha > \lambda_1(p)$, then we get a contradiction, and, consequently, the proposition follows. Assume that we fixed $\alpha = \lambda_1(p)$. Notice that in this case we require $(H)$. Considering the $L^2$-orthogonal decomposition $u_k = \gamma_k \varphi_p + v_k$, where $\gamma_k = \|\varphi_p\|_2^{-2} \int_{\Omega} u_k \varphi_p \, dx$ and $\int_{\Omega} v_k \varphi_p \, dx = 0$, we see that $\gamma_k \to 1$ and $\|\nabla v_k\|_p \to 0$. Employing now the improved Poincaré inequality from [17] along the same lines as in the proof of [7, Proposition 11] (see, more precisely, [7, pp. 1233-1234]), we deduce that $J_{\lambda_1(p), \beta_*(\alpha)}(u_k) \to 0$, which contradicts (3.2). Hence the proof is complete. \hfill \Box

In general, if $J(\beta, \mu)$ is attained, then the corresponding minimizer generates a solution of $(D_{\alpha, \beta})$. We detail this fact as follows.

**Proposition 3.2.** Let $\beta \geq \beta_*(\alpha)$ and assume that $u_0 \in W_0^{1,p}$ is a minimizer of $J(\beta, \mu)$ for some $\mu \in (\lambda_1(q), \beta_*(\alpha)]$. Then $t_{\alpha, \beta}(u_0)u_0$ is a local minimum point of $E_{\alpha, \beta}$ and

$$
E_{\alpha, \beta}(t_{\alpha, \beta}(u_0)u_0) \equiv J_{\alpha, \beta}(u_0) = J(\beta, \mu) < 0.
$$

**Proof.** Suppose, by contradiction, that there exists a sequence $\{u_n\}$ convergent to $\tilde{u}_0 := t_{\alpha, \beta}(u_0)u_0$ in $W_0^{1,p}$ such that

$$
E_{\alpha, \beta}(u_n) < E_{\alpha, \beta}(\tilde{u}_0) \quad \text{for all } n \in \mathbb{N}.
$$
Using the fact that \( u_0 \) is a minimizer of \( J(\beta, \mu) \), we have \( G_\mu(\tilde{u}_0) < 0 < H_\alpha(\tilde{u}_0) \), and hence
\[
G_\beta(u_n) \leq G_\mu(u_n) < 0 < H_\alpha(u_n)
\]
for all sufficiently large \( n \), which means that any such \( u_n \) is an admissible function for \( J(\beta, \mu) \). But then, using (2.3), we get the following contradiction:
\[
E_{\alpha, \beta}(\tilde{u}_0) = J_{\alpha, \beta}(u_0) = J(\beta, \mu) \leq J_{\alpha, \beta}(u_n) = E_{\alpha, \beta}(t_{\alpha, \beta}(u_n) u_n) \leq E_{\alpha, \beta}(u_n) < E_{\alpha, \beta}(\tilde{u}_0)
\]
for all sufficiently large \( n \).

Let us now discuss the existence of a minimizer of \( J(\beta, \mu) \) required in Proposition 3.2.

**Lemma 3.3.** Let \( \beta \geq \beta_*(\alpha) \) and \( \mu \in (\lambda_1(q), \beta_*(\alpha)) \). Then there exists a nonnegative function \( u_0 \in W^{1, p}_0 \) satisfying \( \|\nabla u_0\|_p = 1 \) such that
\[
G_\mu(u_0) \leq 0 < H_\alpha(u_0) \quad \text{and} \quad J_{\alpha, \beta}(u_0) \leq J(\beta, \mu) < 0. \tag{3.5}
\]

**Proof.** First, we recall that \( J(\beta, \mu) < 0 \) by (3.2). Let \( \{u_n\} \) be a minimizing sequence for \( J(\beta, \mu) \). Since \( J_{\alpha, \beta} \) is \( 0 \)-homogeneous and even, we can assume, without loss of generality, that \( \|\nabla u_n\|_p = 1 \) and \( u_n \geq 0 \) in \( \Omega \) for all \( n \in \mathbb{N} \) by considering \( |u_n| \) if necessary. Therefore, there exists a nonnegative function \( u_0 \in W^{1, p}_0 \) such that \( u_n \rightharpoonup u_0 \) in \( W^{1, p}_0 \) and \( u_n \to u_0 \) in \( L^p(\Omega) \), up to an appropriate subsequence. Applying Lemma 2.1 (with \( \beta_n = \beta \)), we get (3.5). Using again the \( 0 \)-homogeneity of \( J_{\alpha, \beta} \), we can assume that \( \|\nabla u_0\|_p = 1 \), which completes the proof.

If the function \( u_0 \) obtained in Lemma 3.3 satisfies \( G_\mu(u_0) < 0 \), then \( u_0 \) is a minimizer of \( J(\beta, \mu) \). Consequently, Proposition 3.2 in combination with Remark 1.1 implies that \( t_{\alpha, \beta}(u_0) u_0 \) is a positive solution of (\( D_{\alpha, \beta} \)). Thus, the proof of Theorem 1.11 reduces to the search of such \( \beta > \beta_*(\alpha) \) and \( \mu \in (\lambda_1(q), \beta_*(\alpha)) \) that \( G_\mu(u_0) < 0 \). The details are as follows.

**Proof of Theorem 1.11.** Let us fix any \( \mu \in (\mu_0, \beta_*(\alpha)) \), where \( \mu_0 \) is defined in (3.3) and \( \mu_0 < \beta_*(\alpha) \) by Proposition 3.1. Denote by \( u_0 = u_0(\beta) \) a normalized nonnegative function given by Lemma 3.3. We are going to obtain the existence of \( \beta(\alpha) > \beta_*(\alpha) \) such that \( G_\mu(u_0(\beta)) < 0 \) for any \( \beta \in (\beta_*(\alpha), \beta(\alpha)) \). Suppose, contrary to our claim, that there exists a sequence \( \{\beta_n\} \) such that \( \beta_n \searrow \beta_*(\alpha) \) and \( G_\mu(u_0(\beta_n)) = 0 \) for all \( n \in \mathbb{N} \). We will reach a contradiction by showing that the corresponding sequence \( \{u_0(\beta_n)\} \) converges in \( W^{1, p}_0 \), up to a subsequence, to a minimizer of \( J(\beta_*(\alpha), \beta_*(\alpha)) \), which is impossible in view of the definition of \( \mu_0 \).

Since \( \|\nabla u_0(\beta_n)\|_p = 1 \) for all \( n \), there exists \( \bar{u} \geq 0 \) such that \( u_0(\beta_n) \) converges to \( \bar{u} \) weakly in \( W^{1, p}_0 \) and strongly in \( L^p(\Omega) \) and \( L^q(\Omega) \), up to an appropriate subsequence. At the same time, in view of (3.5), we have
\[
G_\mu(u_0(\beta_n)) = 0 < H_\alpha(u_0(\beta_n)) \quad \text{for all} \quad n \in \mathbb{N},
\]
and
\[
-\infty \leq \liminf_{n \to +\infty} J_{\alpha, \beta_n}(u_0(\beta_n)) \leq \liminf_{n \to +\infty} J(\beta_n, \mu) < 0, \tag{3.6}
\]
where the last inequality in (3.6) follows from the uniform bound (3.2). Consequently, applying Lemma 2.1 (with \( \beta = \beta_*(\alpha) \)) to the sequence \( \{u_0(\beta_n)\} \), we deduce that
\[
G_{\beta_*(\alpha)}(\bar{u}) < G_\mu(\bar{u}) \leq 0 < H_\alpha(\bar{u}) \tag{3.7}
\]
and
\[ J(\beta_*(\alpha), \beta_*(\alpha)) \leq J_{\alpha, \beta_*(\alpha)}(\bar{u}) \leq \liminf_{n \to +\infty} J_{\alpha, \beta_*(\alpha)}(u_0) \leq \liminf_{n \to +\infty} J(\beta_n, \mu), \] (3.8)
where the first inequality in (3.8) follows from the fact that \( \bar{u} \) is an admissible function for \( J(\beta_*(\alpha), \beta_*(\alpha)) \), see (3.7). On the other hand, noting that any minimizer \( u_* \) of \( J(\beta_*(\alpha), \beta_*(\alpha)) \) satisfies
\[ G_{\beta_*}(u_*) < G_\mu(u_*) < 0 < H_\alpha(u_*) \]
by the definition of \( \mu_0 \) and the fact that \( \beta_n > \beta_*(\alpha) > \mu > \mu_0 \), we get
\[ J(\beta_*, \mu) \leq J_{\alpha, \beta_*(\alpha)}(u_*) + o(1) = J(\beta_*(\alpha), \beta_*(\alpha)) + o(1). \] (3.9)
Therefore, combining (3.8) with (3.9), we conclude that \( J(\beta_*(\alpha), \beta_*(\alpha)) = J_{\alpha, \beta_*(\alpha)}(\bar{u}) \), which means that \( \bar{u} \) is a minimizer of \( J(\beta_*(\alpha), \beta_*(\alpha)) \). Moreover, since (3.8) and (3.9) also imply
\[ J_{\alpha, \beta_*(\alpha)}(\bar{u}) = \liminf_{n \to +\infty} J_{\alpha, \beta_*}(u_0(\beta_*)), \] (3.10)
we get \( u_0(\beta_*) \to \bar{u} \) in \( W^{1,p}_0 \), up to a subsequence. Indeed, if we suppose that there is no strong convergence, then \( \|\nabla \bar{u}\|_p < \liminf_{n \to +\infty} \|\nabla u_0(\beta_*)\|_p \), and hence \( 0 < H_\alpha(\bar{u}) < \liminf_{n \to +\infty} H_\alpha(u_0(\beta_*)) \), which implies a contradiction to the equality in (3.10). Finally, let us notice that the strong convergence of \( \{u_0(\beta_*)\} \) gives \( G_\mu(\bar{u}) = 0 \), see (3.7). However, this contradicts the definition of \( \mu_0 \) and the fact that \( \mu > \mu_0 \).

\[ \square \]

4. Auxiliary results II. Mountain pass type arguments

In this section, we prepare several results related to the mountain pass theorem, which will be used to prove Theorems 1.12 and 1.13 in Section 5 below. Since our aim is to find positive solutions of \( (D_{\alpha, \beta}) \), in the arguments of this section it will be convenient to consider the \( C^1 \)-functional
\[ \widetilde{E}_{\alpha, \beta}(u) = \frac{1}{p} \widetilde{H}_\alpha(u) + \frac{1}{q} \widetilde{G}_\beta(u), \quad u \in W^{1,p}_0, \]
where
\[ \widetilde{H}_\alpha(u) := \|\nabla u\|_p^p - \alpha \|u_+\|_p^p \quad \text{and} \quad \widetilde{G}_\beta(u) := \|\nabla u\|_q^q - \beta \|u_+\|_q^q, \]
and \( u_+ := \max\{u, 0\} \). The functional \( \widetilde{E}_{\alpha, \beta} \) differs from \( E_{\alpha, \beta} \) in that if \( u \in W^{1,p}_0 \) is an arbitrary critical point of \( \widetilde{E}_{\alpha, \beta} \), then \( u \) is a nonnegative solution of \( (D_{\alpha, \beta}) \), which can be easily seen by taking \( u_- := \max\{-u, 0\} \) as a test function. Moreover, \( u \) is a positive solution belonging to \( \text{int} C^1_0(\Omega)_+ \) provided \( u \neq 0 \), see Remark 1.1.

Now we discuss the assumptions under which \( \widetilde{E}_{\alpha, \beta} \) satisfies the Palais–Smale condition.

Lemma 4.1. Let \( (\alpha, \beta) \neq (\lambda_1(p), \beta_*) \). Then \( \widetilde{E}_{\alpha, \beta} \) satisfies the Palais–Smale condition.

Proof. Let us take any Palais–Smale sequence \( \{u_n\} \) for \( \widetilde{E}_{\alpha, \beta} \). According to the \((S_+)-\) property of the operator \(-\Delta_p - \Delta_q \) (see, e.g., [6, Remark 3.5]), the desired Palais–Smale condition for \( \widetilde{E}_{\alpha, \beta} \) will follow if \( \{u_n\} \) is bounded. Suppose, by contradiction, that \( \|\nabla u_n\|_p \to +\infty \) as \( n \to +\infty \), up to a subsequence. Considering the sequence of normalized functions \( v_n := \frac{u_n}{\|\nabla u_n\|_p} \) and arguing in much the same way as in [6, Lemma 3.3], we derive that \( \{v_n\} \) converges in
Therefore, we deduce that $v_0 \geq 0$ in $\Omega$. This yields $\alpha = \lambda_1(p)$ and $v_0 = \varphi_p$, since $\varphi_p$ is the only constant-sign eigenfunction of the $p$-Laplacian and we assumed that $\|\nabla \varphi_p\|_p = 1$. Thus, if $\alpha \neq \lambda_1(p)$, then we get a contradiction, and hence the Palais–Smale condition for $\tilde{E}_{\alpha,\beta}$ holds for any $\beta \in \mathbb{R}$. On the other hand, in the case $\alpha = \lambda_1(p)$ and $\beta \neq \beta_*$, we get

$$o(1) = \frac{1}{\|\nabla u_n\|_p} \left( p\tilde{E}_{\alpha,\beta}(u_n) - \left( \tilde{E}_{\alpha,\beta}(u_n), u_n \right) \right) = \left( \frac{p}{q} - 1 \right) \tilde{G}_{\beta}(u_n).$$

This yields $0 = \tilde{G}_{\beta}(\varphi_p) = G_{\beta}(\varphi_p)$, which contradicts the assumption $\beta \neq \beta_*$. \hfill \Box

Before providing a mountain pass-type result, we give the following auxiliary lemma.

**Lemma 4.2.** Let $u_1$ be a local minimum point of $\tilde{E}_{\alpha,\beta}$ such that

$$\inf_{u \in \mathcal{N}_{\alpha,\beta}} E_{\alpha,\beta}(u) < \tilde{E}_{\alpha,\beta}(u_1) < 0. \tag{4.1}$$

Then there exists a continuous path $\eta \in C([0,1], W_0^{1,p})$ such that

$$\eta(0) = u_1, \quad \tilde{E}_{\alpha,\beta}(\eta(1)) < \tilde{E}_{\alpha,\beta}(u_1), \quad \text{and} \quad \max_{s \in [0,1]} \tilde{E}_{\alpha,\beta}(\eta(s)) < 0. \tag{4.2}$$

**Proof.** Noting that $u_1 \in \text{int} \, C_0^1(\overline{\Omega})_+$ (see Remark 1.1) and $u_1 \in \mathcal{N}_{\alpha,\beta}$, we get

$$0 > \tilde{E}_{\alpha,\beta}(u_1) = E_{\alpha,\beta}(u_1) = \frac{p-q}{pq} G_{\beta}(u_1) = -\frac{p-q}{pq} H_\alpha(u_1),$$

and hence

$$\tilde{G}_{\beta}(u_1) = G_{\beta}(u_1) < 0 < H_\alpha(u_1) = \tilde{H}_\alpha(u_1).$$

According to (4.1), we can find $v_1 \in \mathcal{N}_{\alpha,\beta}$ such that $E_{\alpha,\beta}(v_1) < \tilde{E}_{\alpha,\beta}(u_1) < 0$. Moreover, since $H_\alpha$ and $G_{\beta}$ are even, we may assume, by considering $|v_1|$ if necessary, that $v_1 \geq 0$ in $\Omega$. Therefore,

$$\tilde{G}_{\beta}(v_1) = G_{\beta}(v_1) < 0 < H_\alpha(v_1) = \tilde{H}_\alpha(v_1) \quad \text{and} \quad \tilde{E}_{\alpha,\beta}(v_1) < \tilde{E}_{\alpha,\beta}(u_1) < 0.$$

Let us consider the path

$$\xi(s) = ((1-s)v_1^q + sv_1^q)^{1/q} \quad \text{for} \ s \in [0,1].$$

The hidden convexity of $\xi$ (see, e.g., [39, Lemma 2.4] or [10, Proposition 2.6]) implies that

$$G_{\beta}(\xi(s)) \leq (1-s)G_{\beta}(v_1) + sG_{\beta}(v_1) \leq \max\{G_{\beta}(v_1), G_{\beta}(v_1)\} < 0 \quad \text{for all} \ s \in [0,1]. \tag{4.3}$$

Assume first that $H_\alpha(\xi(s)) > 0$ for all $s \in [0,1]$, and define the new path $\eta(s) = t_{\alpha,\beta}(\xi(s))\xi(s)$, $s \in [0,1]$, where $t_{\alpha,\beta}(\xi(s))$ is given by (2.1). Noting that $t_{\alpha,\beta}(\xi(0)) = t_{\alpha,\beta}(\xi(1)) = 1$ in view of $u_1, v_1 \in \mathcal{N}_{\alpha,\beta}$, and that

$$\tilde{E}_{\alpha,\beta}(\eta(s)) = E_{\alpha,\beta}(\eta(s)) = J_{\alpha,\beta}(\xi(s)) = -\frac{p-q}{pq} \frac{|G_{\beta}(\xi(s))|^{\frac{q}{p-q}}}{H_\alpha(\xi(s))^{\frac{p-q}{p-q}}} < 0 \quad \text{for all} \ s \in [0,1]$$


by (2.3), we readily see that \( \eta \) satisfies (4.2).

Recalling that \( H_\alpha(\xi(0)) > 0 \) and \( H_\alpha(\xi(1)) > 0 \), assume now that there exists \( s_0 \in (0,1) \) such that \( H_\alpha(\xi(s_0)) = 0 \). Without loss of generality, we may set

\[
s_0 = \inf\{ s \in (0,1) : H_\alpha(\xi(s)) \leq 0 \},
\]

and so \( H_\alpha(\xi(s)) > 0 \) for all \( s \in (0, s_0) \). This implies that \( J_{\alpha,\beta}(\xi(s)) \to -\infty \) as \( s \nearrow s_0 \), thanks to (4.3). Thus, there exists some \( s_1 \in (0, s_0) \) such that

\[
J_{\alpha,\beta}(\xi(s_1)) < \tilde{E}_{\alpha,\beta}(u_1).
\]

Considering the path \( \eta = t_{\alpha,\beta}(\xi(s_1)) \xi(s_1) \) for \( s \in [0,1] \), we complete the proof. \( \square \)

**Theorem 4.3.** Let \( (\alpha, \beta) \neq (\lambda_1(p), \beta_*) \). Assume that \( u_1 \) is a local minimum point of \( \tilde{E}_{\alpha,\beta} \) such that

\[
\inf_{u \in \mathcal{N}_{\alpha,\beta}} E_{\alpha,\beta}(u) < \tilde{E}_{\alpha,\beta}(u_1) < 0.
\]

Then there exists another critical point \( u_2 \) of \( \tilde{E}_{\alpha,\beta} \) satisfying

\[
\tilde{E}_{\alpha,\beta}(u_1) \leq \tilde{E}_{\alpha,\beta}(u_2) < 0.
\]

**Proof.** Let \( \eta \) be a path given by Lemma 4.2. Since \( u_1 \) is a local minimum point of \( \tilde{E}_{\alpha,\beta} \), there exists \( r \in (0, \| \nabla (u_1 - \eta(1)) \|_p) \) such that

\[
\tilde{E}_{\alpha,\beta}(u_1) \leq \tilde{E}_{\alpha,\beta}(u) < 0 \quad \text{for every } u \in B_r(u_1),
\]

where \( B_r(u_1) = \{ u \in W_0^{1,p} : \| \nabla (u_1 - u) \|_p \leq r \} \). Therefore, the generalized mountain pass theorem [37, Theorem 1] in combination with Lemma 4.1 implies that

\[
c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \tilde{E}_{\alpha,\beta}(\gamma(s)) \geq \tilde{E}_{\alpha,\beta}(u_1)
\]

is a critical level of \( \tilde{E}_{\alpha,\beta} \), and there exists a critical point \( u_2 \) on the level \( c \) which is different from \( u_1 \). Here

\[
\Gamma := \left\{ \gamma \in C([0,1], W_0^{1,p}) : \gamma(0) = u_1, \gamma(1) = \eta(1) \right\}.
\]

The properties (4.2) of the admissible path \( \eta \) yield \( c < 0 \), which gives (4.5). \( \square \)

### 5. Multiplicity. The proofs of Theorems 1.12 and 1.13

In this section, we prove Theorems 1.12 and 1.13 using the results of Section 4. A local minimum point of \( \tilde{E}_{\alpha,\beta} \) will be obtained by the super- and subsolution method. Let us denote, for brevity,

\[
f_{\alpha,\beta}(u) = \alpha |u|^{p-2}u + \beta |u|^{q-2}u,
\]

and recall that a function \( u \in W^{1,p} \) is called supersolution (resp. subsolution) of \( (D_{\alpha,\beta}) \) if \( u \geq 0 \) (resp. \( \leq 0 \)) on \( \partial \Omega \) in the sense of traces and

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \varphi \, dx \geq \int_{\Omega} f_{\alpha,\beta}(u) \varphi \, dx \quad (\text{resp.} \leq 0)
\]
for any nonnegative \( \varphi \in W^{1,p}_0 \). If, in addition, the strict inequality in (5.1) is satisfied for any nonnegative and nonzero \( \varphi \), then \( u \) is called strict supersolution (resp. strict subsolution) of \((D_{\alpha,\beta})\).

Taking any \( v, w \in L^\infty(\Omega) \) such that \( v \leq w \) a.e. in \( \Omega \), we introduce the truncation

\[
f_{\alpha,\beta}^{[v,w]}(x,t) = \begin{cases} 
\varphi(x) & \text{if } t \leq v(x), \\
\varphi(t) & \text{if } v(x) < t < w(x), \\
\varphi(w(x)) & \text{if } t \geq w(x),
\end{cases}
\]

and define the corresponding \( C^1 \)-functional

\[
E_{\alpha,\beta}^{[v,w]}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^q \, dx - \int_\Omega \int_0^{u(x)} f_{\alpha,\beta}^{[v,w]}(x,t) \, dt \, dx, \quad u \in W^{1,p}_0.
\]

If \( v \) and \( w \) are sub- and supersolutions of \((D_{\alpha,\beta})\), respectively, then critical points of \( E_{\alpha,\beta}^{[v,w]} \) are solutions of \((D_{\alpha,\beta})\) and they belong to the ordered interval \([v, w]\), see, e.g., [5, Remark 2].

We will make use of the following two lemmas.

**Lemma 5.1** ([5, Lemma 6 and Remark 2]). Let \( \alpha \in \mathbb{R} \) and \( \beta > \lambda_1(q) \), and let \( w \in \text{int} C^1(\overline{\Omega})_+ \) be a positive supersolution of \((D_{\alpha,\beta})\). Then \( \inf_{W^{1,p}_0} E_{\alpha,\beta}^{[0,w]} < 0 \), the infimum is attained, and the corresponding global minimum point \( u \in [0,w] \) satisfies \((D_{\alpha,\beta})\) and belongs to \( \text{int} C^1(\overline{\Omega})_+ \).

**Lemma 5.2.** Let \( \alpha \geq 0 \) and \( \beta > \lambda_1(q) \). Let \( w \in \text{int} C^1(\overline{\Omega})_+ \) be a positive strict supersolution of \((D_{\alpha,\beta})\), that is,

\[
\langle E'_{\alpha,\beta}(w), \varphi \rangle > 0 \quad \text{for any nonnegative and nonzero } \varphi \in W^{1,p}_0.
\]

Let \( u \in \text{int} C^1(\overline{\Omega})_+ \) be a global minimum point of \( E_{\alpha,\beta}^{[0,w]} \) given by Lemma 5.1. Then \( u \in (0,w) \) and \( u \) is a local minimum point of both \( E_{\alpha,\beta} \) and \( \tilde{E}_{\alpha,\beta} \) in \( C^1(\overline{\Omega}) \)-topology.

**Proof.** Noting that \( u \in (0,w) \) and \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} < 0 \) on \( \partial\Omega \), we will prove that

\[
u < w \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial x} > \frac{\partial w}{\partial y} \quad \text{on } \partial\Omega.
\]

This fact directly implies the desired results. Indeed, if (5.3) holds, then \( w - u \in \text{int} C^1(\overline{\Omega})_+ \) and hence, taking a sufficiently small \( \kappa > 0 \), we get \( u + v \in [0,w] \) for any \( v \in C^1(\overline{\Omega}) \) satisfying \( \|v\|_{C^1(\overline{\Omega})} < \kappa \), whence

\[
E_{\alpha,\beta}(u) = \tilde{E}_{\alpha,\beta}(u) = E_{\alpha,\beta}^{[0,w]}(u) = \inf_{W^{1,p}_0} E_{\alpha,\beta}^{[0,w]}(u + v) \leq E_{\alpha,\beta}^{[0,w]}(u) = \tilde{E}_{\alpha,\beta}(u) = E_{\alpha,\beta}(u + v).
\]

To establish (5.3), we first show that \( u < w \) in a neighbourhood of \( \partial\Omega \), and then we derive that \( u < w \) in the remaining part of \( \Omega \). The details are as follows.

For a sufficiently small \( \delta > 0 \), we define

\[
\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial\Omega) < \delta \}.
\]
Since \( u, w \in \text{int} C^1_0(\Omega)_+ \), one can find \( \varepsilon, \delta > 0 \) such that \(|\nabla ((1 - s)u + sw)| > \varepsilon\) in \( \overline{\Omega}_\delta \) for all \( s \in [0, 1] \). Indeed, suppose, by contradiction, that for any \( n \in \mathbb{N} \) there exist \( x_n \in \Omega_{1/n} \) and \( s_n \in [0, 1] \) such that \(|\nabla ((1 - s_n)u(x_n) + s_n w(x_n))| \leq \frac{1}{n} \). Passing to appropriate subsequences, we get \( x_n \to x_0 \in \partial \Omega, s_n \to s_0 \in [0, 1] \), and \(|\nabla ((1 - s_0)u(x_0) + s_0 w(x_0))| = 0 \). However, this contradicts the fact that \( \frac{\partial u}{\partial \nu}(x_0), \frac{\partial w}{\partial \nu}(x_0) < 0 \).

Let us denote, for short,

\[
A(a) := |a|^{p - 2}a + |a|^{q - 2}a \quad \text{for } a \in \mathbb{R}^N,
\]

and define the linealization \( N \times N\)-matrix \( A(a) \) as

\[
A(a) = |a|^{p - 2}
\left[
I + (p - 2)\frac{a \otimes a}{|a|^2}
\right] + |a|^{q - 2}
\left[
I + (q - 2)\frac{a \otimes a}{|a|^2}
\right)
\]

for \( a \in \mathbb{R}^N \setminus \{0\} \),

where \( I \) is the identity matrix and \( \otimes \) denotes the Kronecker product, see, e.g., [33, Appendix A.2]. Consider now \( v := w - u \). Clearly, \( v \geq 0 \) in \( \Omega \). Recalling that \( w \) is a strict supersolution of \( (D_{\alpha,\beta}) \), we subtract (1.1) from (5.2) and deduce, according to the mean value theorem, that \( v \) satisfies

\[
-\text{div} \left( \int_0^1 A(\nabla ((1 - s)u + sw)) \, ds \right) \nabla v = -\text{div}(A(\nabla w) - A(\nabla u))
\]

\[
> f_{\alpha,\beta}(w) - f_{\alpha,\beta}(u) \geq 0 \quad \text{in } \Omega_\delta,
\]

in the weak sense, where the last inequality follows from the fact that \( \alpha \) and \( \beta \) are nonnegative and \( w \geq u > 0 \) in \( \Omega \). Applying the estimates [33, (A.10)] to the matrix \( A \), we get

\[
\left( \min\{1, p - 1\}|a|^{p - 2} + \min\{1, q - 1\}|a|^{q - 2} \right) \leq \langle A(a)\xi, \xi \rangle_{\mathbb{R}^N}
\]

\[
\leq \left( \max\{1, p - 1\}|a|^{p - 2} + \max\{1, q - 1\}|a|^{q - 2} \right) |\xi|^2
\]

for any \( \xi \in \mathbb{R}^N \) and \( a \in \mathbb{R}^N \setminus \{0\} \). Here, for clarity, we denote by \( \langle \cdot, \cdot \rangle_{\mathbb{R}^N} \) the usual scalar product in \( \mathbb{R}^N \). Recalling that \(|\nabla ((1 - s)u + sw)| > \varepsilon\) in \( \overline{\Omega}_\delta \) for all \( s \in [0, 1] \), we employ the inequalities [33, (A.4) and (A.6)] to see that for any \( r > 1 \) there exist \( C_1, C_2 > 0 \) such that

\[
C_1 \left( \max_{s \in [0, 1]} |\nabla((1 - s)u + sw)| \right)^{-r} \leq \int_0^1 |\nabla((1 - s)u + sw)|^{-2} \, ds \leq C_2 \left( \max_{s \in [0, 1]} |\nabla((1 - s)u + sw)| \right)^{-r} \in \overline{\Omega}_\delta.
\]

Thus, taking \( a = \nabla((1 - s)u + sw) \) in (5.5) and using (5.6) with \( r = p \) and \( r = q \), we conclude that there exist \( C_3, C_4 > 0 \) satisfying

\[
C_3 |\xi|^2 \leq \left( \int_0^1 A(\nabla((1 - s)u + sw)) \, ds \right)_{\mathbb{R}^N} \xi, \xi \right)_{\mathbb{R}^N} \leq C_4 |\xi|^2 \quad \text{in } \overline{\Omega}_\delta, \text{ for any } \xi \in \mathbb{R}^N.
\]

That is, the differential operator in (5.4) is uniformly elliptic in \( \overline{\Omega}_\delta \). Therefore, in view of the strict inequality in (5.4), the strong maximum principle yields \( v > 0 \) in \( \Omega_\delta \) and \( \frac{\partial v}{\partial \nu} < 0 \) on \( \partial \Omega \).

Consequently, \( u < w \) in \( \Omega_\delta \) and \( \frac{\partial u}{\partial \nu} > \frac{\partial w}{\partial \nu} \) on \( \partial \Omega \).
Let us now fix some $\delta' \in (0, \delta)$ and a sufficiently small $C > 0$ such that $u + C \leq w$ on $\partial \Omega' \cap \Omega$. Denoting $z = u + C$, we see that
\[
\int_{\Omega} |\nabla z|^{p-2} \nabla z \nabla \varphi \, dx + \int_{\Omega} |\nabla z|^{q-2} \nabla z \nabla \varphi \, dx = \int_{\Omega} f_{\alpha,\beta}(u) \varphi \, dx \quad \text{for any } \varphi \in W^{1,p}_0. \tag{5.7}
\]
Therefore, subtracting (5.2) from (5.7) and taking $\varphi = \max\{z - w, 0\}$ in $\Omega \setminus \Omega'$ and $\varphi = 0$ in $\Omega'$, we derive that
\[
0 \leq \int_{\{z > w\} \cap (\Omega \setminus \Omega')} (|\nabla z|^{p-2} \nabla z - |\nabla w|^{p-2} \nabla w) (\nabla z - \nabla w) \, dx
+ \int_{\{z > w\} \cap (\Omega \setminus \Omega')} (|\nabla z|^{q-2} \nabla z - |\nabla w|^{q-2} \nabla w) (\nabla z - \nabla w) \, dx
\leq \int_{\{z > w\} \cap (\Omega \setminus \Omega')} (f_{\alpha,\beta}(u) - f_{\alpha,\beta}(w)) (z - w) \, dx \leq 0,
\]
which implies that $\{z > w\} = \emptyset$ in $\Omega \setminus \Omega'$. Thus, $u + C \leq w$ and, consequently, $u < w$ in $\Omega \setminus \Omega'$. Recalling that $u < w$ in $\Omega_{\delta}$, we conclude that $u \in (0, w)$ in $\Omega$. Thus, (5.3) is satisfied, which completes the proof. \qed

5.1. Proof of Theorem 1.12

Fix any $\alpha \in [\lambda_1(p), \alpha_*]$ and $\beta \in (\beta_*(\alpha), \beta_{ps}(\alpha))$. Choosing an arbitrary $\beta' \in (\beta, \beta_{ps}(\alpha)]$, we denote by $w \in \text{int} C^1_0(\Omega)$ a positive solution of $(D_{\alpha,\beta})$, see Theorem 1.5 in the case $\alpha > \lambda_1(p)$ and Theorem 1.6 in the case $\alpha = \lambda_1(p)$ for the existence result. Clearly, $w$ is a strict supersolution of $(D_{\alpha,\beta})$. Hence, thanks to Lemma 5.1, we can find a global minimum point $u_1 \in \text{int} C^1_0(\Omega)$ of $E_{\alpha,\beta}^{[0,w]}$ such that $E_{\alpha,\beta}(u_1) = E_{\alpha,\beta}^{[0,w]}(u_1) < 0$, and $u_1$ is a positive solution of $(D_{\alpha,\beta})$. Moreover, according to Lemma 5.2, $u_1$ is a local minimum point of $\tilde{E}_{\alpha,\beta}$ in $C^1_0(\Omega)$-topology. Therefore, applying Theorem A.1 with $f(x, t) = \alpha t^{p-1} + \beta t^{q-1}$, we see that $u_1$ is a local minimum point of $\tilde{E}_{\alpha,\beta}$ in $W^{1,p}_0$.

On the other hand, it was shown in [7, Theorem 2.5] that $\inf_{u \in X_{\alpha,\beta}} E_{\alpha,\beta}(u) = -\infty$ provided $\beta > \beta_*(\alpha)$. Consequently, (4.4) holds, whence Theorem 4.3 yields the existence of the second positive solution $u_2$ of $(D_{\alpha,\beta})$ which satisfies (4.5). \qed

5.2. Proof of Theorem 1.13

Fix any $\beta \in (\beta_*, \beta_{ps}(\lambda_1(p))]$ and denote by $w \in \text{int} C^1_0(\Omega)$ a positive solution of $(D_{\lambda_1(p),\beta})$, see Theorem 1.6 for the existence of $w$. Evidently, $w$ is a strict supersolution of $(D_{\alpha,\beta})$ for any $\alpha \in [0, \lambda_1(p))$. Therefore, arguing as in the proof of Theorem 1.12 above, we can find a local minimum point $u_1 = u_1(\alpha) \in \text{int} C^1_0(\Omega)$ of $\tilde{E}_{\alpha,\beta}$ in $W^{1,p}_0$ such that
\[
u_1(\alpha) \in (0, w) \quad \text{and} \quad \tilde{E}_{\alpha,\beta}(u_1(\alpha)) = E_{\alpha,\beta}(u_1(\alpha)) < 0 \quad \text{for any } \alpha \in [0, \lambda_1(p)). \tag{5.8}
\]

Thus, $u_1(\alpha)$ is the first positive solution of $(D_{\alpha,\beta})$. Moreover, in view of the uniform $L^\infty$-bound of $u_1(\alpha)$ in (5.8), we get
\[
\inf \{E_{\alpha,\beta}(u_1(\alpha)) : \alpha \in [0, \lambda_1(p)]\} > -\infty. \tag{5.9}
\]
Let \( u_2 = u_2(\alpha) \in \text{int} C_0^1(\Omega)_+ \) be a global minimum point of \( E_{\alpha,\beta} \) for \( \alpha < \lambda_1(p) \) obtained in [7, Proposition 1]. It is proved in [7, Proposition 2 (i)] that

\[
E_{\alpha,\beta}(u_2(\alpha)) \to -\infty, \quad \|u_2(\alpha)\|_p \to +\infty, \quad \text{and} \quad \frac{u_2(\alpha)}{\|u_2(\alpha)\|_p} \to \frac{\varphi_p}{\|\varphi_p\|_p} \text{ in } W_0^{1,p} \quad (5.10)
\]
as \( \alpha \searrow \lambda_1(p) \). Comparing (5.9) and (5.10), we derive the existence of \( \alpha,\beta \in [0, \lambda_1(p)) \) such that

\[
E_{\alpha,\beta}(u_2(\alpha)) < E_{\alpha,\beta}(u_1(\alpha)) \quad \text{for any } \alpha \in (\alpha_*(\beta), \lambda_1(p)). \quad (5.11)
\]

Hence, \( u_2(\alpha) \neq u_1(\alpha) \) whenever \( \alpha \in (\alpha_*(\beta), \lambda_1(p)) \). Moreover, we note that, in fact, \( \alpha_*(\beta) \in (0, \lambda_1(p)) \) due to the uniqueness result in Proposition 1.3. On the other hand, in view of (5.11), Theorem 4.3 provides us with the existence of the third positive solution \( u_3(\alpha) \) of \( (D_{\alpha,\beta}) \) for any \( \alpha \in (\alpha_*(\beta), \lambda_1(p)) \), and \( u_3(\alpha) \) is different from \( u_1(\alpha) \) and \( u_2(\alpha) \).

A. \( W_0^{1,p} \) versus \( C_0^1 \) local minimizers

Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be any Carathéodory function and let \( F(x,u) = \int_0^u f(x,v) \, dv \) be the primitive of \( f \). Along this section, we assume that \( f \) satisfies the following subcritical growth condition:

(G) There exist \( C > 0 \) and \( r \in [1,p^*) \) such that

\[
|f(x,t)| \leq C(1 + |t|^{r-1}) \quad \text{for every } t \in \mathbb{R} \text{ and a.e. } x \in \Omega,
\]

where \( p^* = \frac{pN}{N-p} \) if \( N > p \), and \( p^* = +\infty \) if \( N \leq p \).

It is well known that the functional

\[
I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^q \, dx - \int_\Omega F(x,u) \, dx, \quad u \in W_0^{1,p},
\]
is weakly lower semicontinuous and of class \( C^1 \) under the assumption (G).

The following result can be obtained in much the same way as [19, Theorem 1.2] or [29, Theorem 23], see also [23] for a generalization. For the convenience of the reader we sketch its proof based on [29, Theorem 23].

**Theorem A.1.** Let \( u_0 \in W_0^{1,p} \) be a local minimum point of \( I \) in \( C_0^1(\Omega) \)-topology, namely, there exists \( \varepsilon > 0 \) such that

\[
I(u_0) \leq I(u_0 + h) \quad \text{for any } h \in C_0^1(\Omega) \text{ satisfying } \|h\|_{C_0^1(\Omega)} < \varepsilon. \quad (A.1)
\]

Then \( u_0 \) is also a local minimum point of \( I \) in \( W_0^{1,p} \)-topology.

**Proof.** Since \( \langle I'(u_0), h \rangle = 0 \) for every \( h \in C_0^1(\Omega) \) and since \( C_0^1(\Omega) \) is dense in \( W_0^{1,p} \), we deduce that \( u_0 \) is a critical point of \( I \). That is,

\[
-\Delta_p u_0 - \Delta_q u_0 = f(x,u_0) \quad \text{in } \Omega, \quad (A.2)
\]

18
in the weak sense. Moreover, one can show that $u_0 \in C^{1,\nu}_0(\Omega)$ for some $\nu \in (0, 1)$, cf. Remark 1.1 or [26, Section 2.4].

Suppose, by contradiction, that $u_0$ is not a local minimum point of $I$ in $W^{1,p}_0$-topology. Then for any sufficiently small $\varepsilon > 0$ we have

$$m_\varepsilon := \inf \left\{ I(u_0 + h) : h \in 2\tilde{B}_\varepsilon(0) \right\} < I(u_0),$$

where $\tilde{B}_\varepsilon(0) = \{ v \in W^{1,p}_0 : \| v \|_p \leq \varepsilon \}$ and $r \in [1, p^*)$ is given in the assumption (G). Let $\{ h_n \}$ be a minimizing sequence for $m_\varepsilon$ with a fixed $\varepsilon > 0$. Thanks to (G), $\{ h_n \}$ is bounded in $W^{1,p}_0$. Therefore, $m_\varepsilon$ is attained by some $h_\varepsilon \in \tilde{B}_\varepsilon(0)$, since $I$ is weakly lower semicontinuous on $W^{1,p}_0$ and $\tilde{B}_\varepsilon(0)$ is weakly closed in $W^{1,p}_0$. Then, due to the Lagrange multipliers rule, there exists $\lambda_\varepsilon \leq 0$ such that

$$-\Delta_p(u_0 + h_\varepsilon) - \Delta_q(u_0 + h_\varepsilon) = f(x, u_0 + h_\varepsilon) + \lambda_\varepsilon |h_\varepsilon|^{r-2}h_\varepsilon \quad \text{in } \Omega. \quad (A.3)$$

Denoting now $\tilde{A}(x, y) = |\nabla u_0(x) + y|^p - 2(\nabla u_0(x) + y) + |\nabla u_0(x) + y|^q - 2(\nabla u_0(x) + y) - |\nabla u_0(x)|^{p-2}\nabla u_0(x) - |\nabla u_0(x)|^{q-2}\nabla u_0(x)$, we subtract (A.2) from (A.3) and get

$$-\text{div} \tilde{A}(x, \nabla h_\varepsilon) = f(x, u_0 + h_\varepsilon) - f(x, u_0) + \lambda_\varepsilon |h_\varepsilon|^{r-2}h_\varepsilon \quad \text{in } \Omega.$$ 

Recalling that $\lambda_\varepsilon \leq 0$ and using the Moser iteration method (see, e.g., [29, Theorem C]), we can find $M_1 > 0$ independent of $\varepsilon$ such that $\| h_\varepsilon \|_\infty \leq M_1$ for every $\varepsilon > 0$. Then, applying the regularity result of [25] to the solution $u_0 + h_\varepsilon$ of (A.3), we obtain that $u_0 + h_\varepsilon \in C^{1,\theta}_0(\Omega)$, and so $h_\varepsilon \in C^{1,\theta}_0(\Omega)$ for every $\varepsilon > 0$.

Finally, it can be shown as in [29, Theorem 23] that there exists $d_0 > 0$ such that

$$|\lambda_\varepsilon| |h_\varepsilon(x)|^{r-2}h_\varepsilon(x)| \leq d_0 \quad \text{for every } x \in \Omega \text{ and } \varepsilon > 0.$$ 

This implies that $f(x, u) + \lambda_\varepsilon |h_\varepsilon(x)|^{r-2}h_\varepsilon(x)$ is bounded on $\Omega \times [-M_1 - \|u_0\|_\infty, M_1 + \|u_0\|_\infty]$ uniformly in $\varepsilon > 0$. Thus, applying again the regularity result of [25] to the solution $u_0 + h_\varepsilon$ of (A.3), we deduce the existence of $\theta \in (0, 1)$ and $M_2 > 0$, both independent of $\varepsilon$, such that $u_0 + h_\varepsilon \in C^{1,\theta}_0(\Omega)$ and $\| u_0 + h_\varepsilon \|_{C^{1,\theta}_0(\Omega)} \leq M_2$ for every $\varepsilon > 0$. Since $C^{1,\theta}_0(\Omega)$ is embedded compactly into $C^1_0(\Omega)$, we infer that $u_0 + h_\varepsilon \to u_0$ as $\varepsilon \searrow 0$ in $C^1_0(\Omega)$ by noting that $h_\varepsilon \to 0$ in $L^r(\Omega)$ as $\varepsilon \searrow 0$. Consequently, we get the following contradiction between (A.1) and (A.2):

$$I(u_0 + h_\varepsilon) = m_\varepsilon < I(u_0) \leq I(u_0 + h_\varepsilon) \quad \text{for all sufficiently small } \varepsilon > 0. \quad \square$$

Acknowledgements

V. Bobkov was supported in the framework of executing the development program of Scientific Educational Mathematical Center of Privolzhsky Federal Area, additional agreement no. 075-02-2020-1421/1 to agreement no. 075-02-2020-1421. M. Tanaka was supported by JSPS KAKENHI Grant Number JP 19K03591.
References

[1] Agudelo, O., Restrepo, D., & Vélez, C. (2020). On the Morse index of least energy nodal solutions for quasilinear elliptic problems. Calculus of Variations and Partial Differential Equations, 59(2), 1-35. DOI:10.1007/s00526-020-1730-x

[2] Allegretto, W., & Huang, Y. (1998). A Picone’s identity for the $p$-Laplacian and applications. Nonlinear Analysis: Theory, Methods & Applications, 32(7), 819-830. DOI:10.1016/S0362-546X(97)00530-0

[3] Appell, J., De Pascale, E., & Vignoli, A. (2008). Nonlinear spectral theory (Vol. 10). Walter de Gruyter. DOI:10.1515/9783110199260

[4] Benci, V., Fortunato, D., & Pisani, L. (1998). Soliton like solutions of a Lorentz invariant equation in dimension 3. Reviews in Mathematical Physics, 10(3), 315-344. DOI:10.1142/S0129055X98000102

[5] Bobkov, V., & Tanaka, M. (2015). On positive solutions for $(p, q)$-Laplace equations with two parameters. Calculus of Variations and Partial Differential Equations, 54(3), 3277-3301. DOI:10.1007/s00526-015-0903-5

[6] Bobkov, V., & Tanaka, M. (2016). On sign-changing solutions for $(p, q)$-Laplace equations with two parameters. Advances in Nonlinear Analysis, 8(1), 101-129. DOI:10.1515/anona-2016-0172

[7] Bobkov, V., & Tanaka, M. (2018). Remarks on minimizers for $(p, q)$-Laplace equations with two parameters. Communications on Pure and Applied Analysis, 17(3), 1219-1253. DOI:10.3934/cpaa.2018059

[8] Bobkov, V., & Tanaka, M. (2020). Generalized Picone inequalities and their applications to $(p, q)$-Laplace equations. Open Mathematics, 18(1), 1030-1044. DOI:10.1515/math-2020-0065

[9] Bonheure, D., Colasuonno, F., & Földes, J. (2019). On the Born-Infeld equation for electrostatic fields with a superposition of point charges. Annali di Matematica Pura ed Applicata (1923-), 198(3), 749-772. DOI:10.1007/s10231-018-0796-y

[10] Brasco, L., & Franzina, G. (2014). Convexity properties of Dirichlet integrals and Picone-type inequalities. Kodai Mathematical Journal, 37(3), 769-799. DOI:10.2996/kmj/1414674621

[11] Brock, F., & Takáč, P. (2019). Symmetry and stability of non-negative solutions to degenerate elliptic equations in a ball. arXiv:1912.09324

[12] Candito, P., Marano, S. A., & Perera, K. (2015). On a class of critical $(p, q)$-Laplacian problems. Nonlinear Differential Equations and Applications NoDEA, 22(6), 1959-1972. DOI:10.1007/s00030-015-0353-y

[13] Cherfils, L., & Il’yasov, Y. (2005). On the stationary solutions of generalized reaction diffusion equations with $p$-$q$-Laplacian. Communications on Pure and Applied Analysis, 4(1), 9-22. DOI:10.3934/cpaa.2005.4.9

[14] Cingolani, S., & Degiovanni, M. (2005). Nontrivial solutions for $p$-Laplace equations with right-hand side having $p$-linear growth at infinity. Communications in Partial Differential Equations, 30(8), 1191-1203. DOI:10.1080/03605300500257594

[15] Colasuonno, F., & Squassina, M. (2016). Eigenvalues for double phase variational integrals. Annali di Matematica Pura ed Applicata (1923-), 195(6), 1917-1959. DOI:10.1007/s10231-015-0542-7

[16] Colombo, M., & Mingione, G. (2015). Regularity for double phase variational problems. Archive for Rational Mechanics and Analysis, 215(2), 443-496. DOI:10.1007/s00205-014-0785-2
[17] Fleckinger-Pellé J., & Takáč, P. An improved Poincaré inequality and the p-Laplacian at resonance for $p > 2$. Advances in Differential Equations, 7(8), 951-971. http://projecteuclid.org/euclid.ade/1356651685
[18] Fučík, S., Nečas, J., Souček, J., & Souček, V. (2006). Spectral analysis of nonlinear operators (Vol. 346). Springer. DOI:10.1007/BFb059360
[19] García Azorero, J. P., Peral Alonso, I., & Manfredi, J. J. (2000). Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations. Communications in Contemporary Mathematics, 2(03), 385-404. DOI:10.1142/S0219199700000190
[20] Il’yasov, Y. (2001). On positive solutions of indefinite elliptic equations. Comptes Rendus de l’Académie des Sciences-Series I-Mathematics, 333(6), 533-538. DOI:10.1016/S0764-4442(01)01924-3
[21] Ilyasov, Y., & Silva, K. (2018). On branches of positive solutions for p-Laplacian problems at the extreme value of the Nehari manifold method. Proceedings of the American Mathematical Society, 146(7), 2925-2935. DOI:10.1090/proc/13972
[22] Kajikiya, R., Tanaka, M., & Tanaka, S. (2017). Bifurcation of positive solutions for the one-dimensional $(p,q)$-Laplace equation. Electronic Journal of Differential Equations, 2017(107), 1-37. https://ejde.math.txstate.edu/Volumes/2017/107/kajikija.pdf
[23] Khan, A. A., & Motreanu, D. (2013). Local minimizers versus $X$-local minimizers. Optimization Letters, 7(5), 1027-1033. DOI:10.1007/s11590-012-0474-8
[24] Lieberman, G. M. (1988). Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Analysis: Theory, Methods & Applications, 12(11), 1203-1219. DOI:10.1016/0362-546X(88)90053-3
[25] Lieberman, G. M. (1991). The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations. Communications in Partial Differential Equations, 16(2-3), 311-361. DOI:10.1080/03605309108820761
[26] Marano, S., & Mosconi, S. (2017). Some recent results on the Dirichlet problem for $(p,q)$-Laplace equations. Discrete and Continuous Dynamical Systems - Series S, 11(2), 279-291. DOI:10.3934/dcdss.2018015
[27] Marano, S. A., & Papageorgiou, N. S. (2013). Constant-sign and nodal solutions of coercive $(p,q)$-Laplacian problems. Nonlinear Analysis: Theory, Methods & Applications, 77, 118-129. DOI:10.1016/j.na.2012.09.007
[28] Marcellini, P., & Miller, K. (1997). Elliptic versus parabolic regularization for the equation of prescribed mean curvature. Journal of Differential Equations, 137(1), 1-53. DOI:10.1006/jdeq.1997.3247
[29] Miyajima, S., Motreanu, D., & Tanaka, M. (2012). Multiple existence results of solutions for the Neumann problems via super-and sub-solutions. Journal of Functional Analysis, 262(4), 1921-1953. DOI:10.1016/j.jfa.2011.11.028
[30] Motreanu, D., & Tanaka, M. (2016). On a positive solution for $(p,q)$-Laplace equation with indefinite weight. Minimax Theory and its Applications, 1(1), 1-20. http://www.heldermann.de/MTA/MTA01/MTA011/mta01001.htm
[31] Mucha, P. B., & Rybka, P. (2012). A note on a model system with sudden directional diffusion. Journal of Statistical Physics, 146(5), 975-988. DOI:10.1007/s10955-012-0446-5
[32] Nazarov, A. I. (2005). On solutions to the Dirichlet problem for an equation with $p$-Laplacian in a spherical layer. Translations of the American Mathematical Society-Series 2, 214, 29-58. DOI:10.1090/trans2/214/03
[33] Padial, J. F., Takáč, P., & Tello, L. (2010). An antimaximum principle for a degenerate parabolic problem. Advances in Differential Equations, 15(7/8), 601-648. https://projecteuclid.org/euclid.ade/1355854621

[34] Papageorgiou, N. S., Vetro, C., & Vetro, F. (2020). Multiple solutions with sign information for a $(p,2)$-equation with combined nonlinearities. Nonlinear Analysis, 192, 111716. DOI:10.1016/j.na.2019.111716

[35] Perera, K., & Squassina, M. (2018). Existence results for double-phase problems via Morse theory. Communications in Contemporary Mathematics, 20(02), 1750023. DOI:10.1142/S0219199717500237

[36] Pohozaev, S. I. (2008). Nonlinear variational problems via the fibering method. Handbook of differential equations: stationary partial differential equations, 5, 49-209. DOI:10.1016/S1874-5733(08)80009-5

[37] Pucci, P., & Serrin, J. (1985). A mountain pass theorem. Journal of Differential Equations, 60(1), 142-149. DOI:10.1016/0022-0396(85)90125-1

[38] Pucci, P., & Serrin, J. B. (2007). The maximum principle (Vol. 73). Springer. DOI:10.1007/978-3-7643-8145-5

[39] Takáč, P., Tello, L., & Ulm, M. (2002). Variational problems with a $p$-homogeneous energy. Positivity, 6(1), 75-94. DOI:10.1023/A:1012088127719

[40] Tanaka, M. (2014). Generalized eigenvalue problems for $(p,q)$-Laplacian with indefinite weight. Journal of Mathematical Analysis and Applications, 419(2), 1181-1192. DOI:10.1016/j.jmaa.2014.05.044

[41] Zakharov, V. E. (1972). Collapse of Langmuir waves. Journal of Experimental and Theoretical Physics, 35(5), 908-914. http://jetp.ac.ru/cgi-bin/dn/e_035_05_0908.pdf

[42] Zhikov, V. V. (2011). On variational problems and nonlinear elliptic equations with nonstandard growth conditions. Journal of Mathematical Sciences, 173(5), 463-570. DOI:10.1007/s10958-011-0260-7