Symmetry and Supersymmetry in Nuclear Physics

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Summary. — A survey of algebraic approaches to various problems in nuclear physics is given. Examples are chosen from pairing of many-nucleon systems, nuclear structure, fusion reactions below the Coulomb barrier, and supernova neutrino physics to illustrate the utility of group-theoretical and related algebraic methods in nuclear physics.

PACS 03.65.Fd – Algebraic methods.
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PACS 26.30.-k – Nucleosynthesis in novae, supernovae, and other explosive environments.

1. – Introduction

Symmetry concepts play a very important role in all of physics. Originally symmetries utilized in physics did not change the particle statistics: such symmetries either transform bosons into bosons or fermions into fermions. Natural mathematical tools to explore symmetries are Lie algebras, i.e. the set of operators closing under commutation relations, schematically shown as

\[ [G_B, G_B] = G_B. \]
Lie algebras are associated with Lie groups. One could think of Lie groups as the exponentiation of Lie algebras.

On the other hand, supersymmetries transform bosons into bosons, fermions into fermions, and bosons into fermions and vice versa. Natural mathematical tools to explore them are superalgebras and supergroups: Superalgebras, sets containing bosonic ($G_B$) as well as fermionic ($G_F$) operators, close under commutation and anticommutation relations as shown below:

\[
\begin{align*}
[G_B, G_B] &= G_B, \\
[G_B, G_F] &= G_F, \\
\{G_F, G_F\} &= G_B
\end{align*}
\]

The simplest superalgebra can be easily worked out. Consider three dimensional harmonic oscillator creation and annihilation operators and define

\[
K_0 = \frac{1}{2} \left( b_i^\dagger b_i + \frac{3}{2} \right) \quad K_+ = \frac{1}{2} b_i^\dagger b_i^\dagger = (K_-)^\dagger.
\]

It is easy to show that the operators defined in Eq. (3) satisfy

\[
[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0.
\]

The algebra depicted in Eq. (4) is the SU(1,1) algebra [1]. Casimir operators are operators that satisfy the condition

\[
[C(G), K_0] = 0 = [C(G), K_\pm]
\]

Casimir operators obtained by multiplying one, two, three elements of the algebra are called linear, quadratic, cubic Casimir operators. For SU(1,1) the quadratic Casimir op. is

\[
C_2 = K_0^2 - \frac{1}{2} (K_+ K_- + K_- K_+)
\]

The concept of dynamical symmetries found many applications in nuclear physics (see e.g. [2]). Consider a chain of algebras (or associated groups):

\[
G_1 \supset G_2 \supset \cdots \supset G_n
\]

If a given Hamiltonian can be written in terms of the Casimir operators of the algebras in this chain, then such a Hamiltonian is said to possess a dynamical symmetry:

\[
H = \sum_{i=1}^{n} [\alpha_i C_1(G_i) + \beta_i C_2(G_i)].
\]
Obviously all these Casimir operators commute with each other. Consequently, finding the energy eigenvalues of this Hamiltonian reduces to the problem of reading the eigenvalues of the Casimir operators off the existing tabulations.

Introducing spin (i.e., fermionic) degrees of freedom represented by the Pauli matrices as well as the bosonic (harmonic oscillator) ones and defining

\[ F_+ = \frac{1}{2} \sum_i \sigma_i b_i^\dagger, \quad F_- = \frac{1}{2} \sum_i \sigma_i b_i, \]

one can write the following additional commutation and anticommutation relations:

\[
\begin{align*}
[K_0, F_\pm] &= \pm \frac{1}{2} F_\pm, \\
[K_+, F_+] &= 0 = [K_-, F_-], \\
[K_\pm, F_\mp] &= \mp F_\pm, \\
\{F_\pm, F_\pm\} &= K_\pm, \\
\{F_+, F_-\} &= K_0
\end{align*}
\]

The operators \( K_+, K_-, K_0, F_+, \) and \( F_- \) generate the Osp(1/2) superalgebra, which is non-compact [1, 3]. Since the operators \( K_+, K_-, K_0 \) alone generate the SP(2) \( \sim \) SU(1,1) subalgebra we can write the group chain

\[
\text{Osp}(1/2) \supset \text{SU}(1,1) \supset \text{SO}(2).
\]

(Note that the operator \( K_0 \) can be viewed as the Casimir operator of the SO(2) subalgebra of SU(1,1)). The Casimir operators of Osp(1/2) and Sp(2) \( \sim \) SU(1,1) are given by

\[
\begin{align*}
C_2 (\text{Osp}(1/2)) &= \frac{1}{4} \left( L + \frac{\sigma}{2} \right)^2 = \frac{1}{4} J^2, \\
C_2 (\text{Sp}(2)) &= \frac{1}{2} L^2 - \frac{3}{16},
\end{align*}
\]

where \( L \) is the angular momentum carried by the oscillator, i.e. \( L_i = \epsilon_{ijk} r_j p_k = i \epsilon_{ijk} b^\dagger_k b_j \). It can easily be shown that a harmonic oscillator Hamiltonian with a constant spin-orbit coupling

\[
H = \frac{1}{2} \left( p^2 + r^2 \right) + \lambda \left( \sigma \cdot L + \frac{3}{2} \right)
\]

can be written in terms of the Casimir operators of the group chain given in Eq. (10):

\[
H = 4\lambda C_2 (\text{Osp}(1/2)) - 4\lambda C_2 (\text{Sp}(2)) + 2K_0.
\]

This provides perhaps the simplest example of a dynamical supersymmetry.
2. – Fermion pairing

Pairing is a salient property of multi-fermion systems and as such it has a long history in nuclear physics (for a recent review see [4]). Already in 1950 Meyer suggested that short-range attractive nucleon-nucleon interaction yields nuclear ground states with angular momentum zero [5]. Mean field calculations with effective interactions describe many nuclear properties, however they cannot provide a complete solution of the underlying complex many-body problem. For example, after many years of investigations, we now that the structure of low-lying collective states in medium-heavy to heavy nuclei are determined by pairing correlations with L=0 and L=2. This was exploited by many successful models of nuclear structure such as the Interacting Boson Model of Arima and Iachello [6, 7, 8].

Pairing plays a significant role not only in finite nuclei, but also in nuclear matter and can directly effect related observables. For example, we know that neutron superfluidity is present in the crust and the inner part of a neutron star. Pairing could significantly effect the thermal evolution of the neutron star by suppressing neutrino (and possibly exotics such as axions) emission [9].

Charge symmetry implies that interactions between two protons and two neutrons are very similar; hence proton-proton and neutron-neutron pairs play a similar role in nuclei. In addition isospin symmetry implies that proton-neutron interaction is also very similar to the proton-proton and neutron-neutron interactions. Currently there is very little experimental information about neutron-proton pairing in heavier nuclei as one needs to study proton rich nuclei to achieve this goal. But one expects that data from the current and future radiative beam facilities will change this picture.

First microscopic theory of pairing was the Bardeen, Cooper, Schriffer (BCS) theory [10]. Soon after its introduction, the BCS theory was applied to nuclear structure [11, 12, 13]. Application of the BCS theory to nuclear structure has a main drawback: BCS wave function is not an eigenstate of the number operator. Several solutions were offered to remedy this shortcoming such as adding Random-Phase Approximation (RPA) to the BCS theory [14], projection of the particle number after variation [15], or projection of the particle number before the variation. The last technique was recently much utilized in nuclei (see e.g. [16]). Pairing correlations may also play an interesting role in halo nuclei [17]. It should also be noted that the theory of pairing in nuclear physics has many parallels with the theory of ultrasmall metallic grains in condensed matter physics (see e.g. [18]).

2.1. Quasi-Spin Algebra. – The concept of seniority was introduced by Racah to aid the classification of atomic spectra [19]. Seniority quantum number is basically the number of unpaired particles in the $j^n$ configuration. Seniority-conserving pairing interactions are a very limited class, however such interactions make an interesting case study. Kerman introduced the quasi-spin scheme to treat such cases [20]. In this scheme nucleons are placed at time-reversed states $|jm\rangle$ and $(−1)^{(j−m)}|j+m\rangle$. Introducing the creation and annihilation operators for nucleons at level $j$, $a_{j m}^\dagger$ and $a_{j m}$, the quasi-spin operators are
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\[ \hat{S}_j^+ = \sum_{m>0} (-1)^{(j-m)} a_j^\dagger m a_j^{-m}, \]

(15)

\[ \hat{S}_j^- = \sum_{m>0} (-1)^{(j-m)} a_j^{-m} a_j m, \]

(16)

and

\[ \hat{S}_j^0 = \frac{1}{2} \sum_{m>0} \left( a_j^\dagger m a_j m + a_j^{-m} a_j^{-m} - 1 \right) \]

(17)

These operators form a set of mutually commuting SU(2) algebras:

\[ \left[ \hat{S}_i^+, \hat{S}_j^- \right] = 2 \delta_{ij} \hat{S}_j^0, \quad \left[ \hat{S}_i^0, \hat{S}_j^\pm \right] = \pm \delta_{ij} \hat{S}_j^\pm. \]

(18)

The operator \( \hat{S}_j^0 \) can be related to the number operator

\[ \hat{N}_j = \frac{1}{2} \hat{N}_j - \frac{1}{2} \Omega_j, \]

(19)

where \( \Omega_j = j + \frac{1}{2} \) is the maximum number of pairs that can occupy the level \( j \) and the number operator is

\[ \hat{N}_j = \frac{1}{2} \sum_{m>0} \left( a_j^\dagger m a_j m + a_j^{-m} a_j^{-m} \right). \]

(20)

Since \( 0 < \hat{N}_j < \Omega_j \) these SU(2) algebras are realized in the representation with the total angular momentum quantum number \( \frac{1}{2} \Omega_j \).

The most general Hamiltonian for nucleons interacting with a pairing force can be written as

\[ \hat{H} = \sum_{jm} \epsilon_j a_j^\dagger m a_j m - \left| G \right| \sum_{jj'} c_{jj'} \hat{S}_j^+ \hat{S}_{j'}^-, \]

(21)

where \( \epsilon_j \) are the single particle energy levels, \( |G| \) is the pairing interaction strength with dimensions of energy, and \( c_{jj'} \) are dimensionless parameters describing the distribution of this strength between different orbitals. When the latter are separable \( (c_{jj'} = c_j^* c_{jj'}) \) we get

\[ \hat{H} = \sum_{jm} \epsilon_j a_j^\dagger m a_j m - \left| G \right| \sum_{jj'} c_j^* c_{jj'} \hat{S}_j^+ \hat{S}_{j'}^-. \]

(22)
There are a number of approximations one can make to simplify the Hamiltonians in Eqs. (21) or (22). If we assume that the NN interaction is determined by a single parameter (usually chosen to be the scattering length), all \( c_j \)'s are the same and we get

\[
\hat{H} = \sum_{jm} \epsilon_j a_j^\dagger a_j - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^-.
\]

This case was solved by Richardson [21].

If we assume that the energy levels are degenerate then the first term is a constant for a given fixed number of pairs. This case can be solved by using the quasispin algebra since \( H \propto S^+ S^- \). A list of the exactly solvable cases can then be given as follows:

- **Quasi-spin limit** [20]:
  \[
  \hat{H} = -|G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^-.
  \]

- **Richardson’s limit** [21]:
  \[
  \hat{H} = \sum_{jm} \epsilon_j a_j^\dagger a_j - |G| \sum_{jj'} \hat{S}_j^+ \hat{S}_{j'}^-.
  \]

- **An algebraic solution developed by Gaudin** [22] which is sketched out in the next section (section 2.2). Gaudin’s model is closely related to Richardson’s limit.

- **The limit in which the energy levels are degenerate (when the first term becomes a constant for a given number of pairs):**
  \[
  \hat{H} = -|G| \sum_{jj'} c_j^* c_{j'} \hat{S}_j^+ \hat{S}_{j'}^-.
  \]

Different aspects of this solution was worked out in Refs. [23, 24, 25].

- **Most general separable case with two shells** [26].

Clearly we can solve the pairing problem numerically in the quasispin basis [27]. Note that it is also possible to utilize the quasi-spin concept for mixed systems of bosons and fermions in a supersymmetric framework [28].

### 2.2. Gaudin Algebra

To study the problem of interacting spins on a lattice Gaudin introduced a method [22] closely related to the Richardson’s solution. Our presentation of Gaudin’s method here follows that given in Ref. [29]. This method starts with three operators \( J^\pm (\lambda), J^0 (\lambda), \) parametrized by one parameter \( \lambda \), satisfying the commutation relations

\[
[J^+ (\lambda), J^- (\mu)] = 2\frac{J^0 (\lambda) - J^0 (\mu)}{\lambda - \mu}.
\]
\[ [J^0(\lambda), J^\pm(\mu)] = \pm \frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu}, \]

and

\[ [J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0. \]

The Lie algebra depicted above is referred to as the rational Gaudin algebra. A possible realization of this algebra is given by

\[ J^0(\lambda) = \sum_{i=1}^{N} \frac{S^0_i}{\epsilon_i - \lambda} \quad \text{and} \quad J^\pm(\lambda) = \sum_{i=1}^{N} \frac{S^\pm_i}{\epsilon_i - \lambda}, \]

where \( \epsilon_i \) are, in general, arbitrary parameters. In Eq. (30), instead of the quasi-spin algebra one can obviously use any mutually-commuting \( N \) SU(2) algebras (in fact, copies of any other algebra). The operator

\[ H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2} J^+(\lambda)J^-(\lambda) + \frac{1}{2} J^-(-\lambda)J^+(\lambda) \]

is not the Casimir operator of the Gaudin algebra, but such operators commute for different values of the parameter:

\[ [H(\lambda), H(\mu)] = 0. \]

Lowest weight vector, \(|0\rangle\), is chosen to satisfy the conditions

\[ J^-(\lambda)|0\rangle = 0, \quad \text{and} \quad J^0(\lambda)|0\rangle = W(\lambda)|0\rangle. \]

Hence it is an eigenstate of the Hamiltonian given in Eq. (31):

\[ H(\lambda)|0\rangle = [W(\lambda)^2 - W'(\lambda)]|0\rangle. \]

To find other eigenstates consider the state \(|\xi\rangle \equiv J^+(\xi)|0\rangle\) for an arbitrary complex number \( \xi \). Since

\[ [H(\lambda), J^+(\xi)] = \frac{2}{\lambda - \xi} \left( J^+(\lambda)J^0(\xi) - J^+(\xi)J^0(\lambda) \right), \]

we conclude that if \( W(\xi) = 0 \), then \( J^+(\xi)|0\rangle \) is an eigenstate of \( H(\lambda) \) with the eigenvalue

\[ E_1(\lambda) = [W(\lambda)^2 - W'(\lambda)] - 2 \frac{W(\lambda)}{\lambda - \xi}. \]

Gaudin showed that this approach can be generalized and a state of the form

\[ |\xi_1, \xi_2, \ldots, \xi_n \rangle \equiv J^+(\xi_1)J^+(\xi_2) \ldots J^+(\xi_n)|0\rangle \]
is an eigenvector of $H(\lambda)$ if the numbers $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ satisfy the so-called Bethe Ansatz equations:

\begin{equation}
W(\xi_\alpha) = \sum_{\beta=1, \beta \neq \alpha}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n.
\end{equation}

Corresponding eigenvalue is

\begin{equation}
E_n(\lambda) = \left[ W(\lambda)^2 - W'(\lambda) \right] - 2 \sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}.
\end{equation}

To make a connection to Richardson’s solution we define so-called $R$-operators as

\begin{equation}
\lim_{\lambda \to \epsilon_k} (\lambda - \epsilon_k)H(\lambda) = R_k.
\end{equation}

In the realization of Eq. (30), these operators take the form

\begin{equation}
R_k = -2 \sum_{i \neq j} S_k \cdot S_j.
\end{equation}

Taking the limits $\mu \to \epsilon_k$ first and $\lambda \to \epsilon_j$ second in Eq. (32), one easily obtains

\begin{equation}
[H(\lambda), R_k] = 0, \quad [R_j, R_k] = 0.
\end{equation}

One can also prove the equalities

\begin{equation}
\sum_i R_i = 0,
\end{equation}

and

\begin{equation}
\sum_i \epsilon_i R_i = -2 \sum_{i \neq j} S_i \cdot S_j.
\end{equation}

A careful examination of the Gaudin algebra in Eqs. (27), (28), and (29) indicates that not only the operators $J(\lambda)$, but also the operators $J(\lambda) + c$ satisfy this algebra for a constant $c$. In this case the conserved quantity is replaced by

\begin{equation}
H(\lambda) = J(\lambda) \cdot J(\lambda) \Rightarrow H(\lambda) + 2c \cdot J(\lambda) + c^2
\end{equation}

which has the same eigenstates. One can define Richardson operators, $R_k$, in an analogous way to the $R_k$ defined in Eq. (40):

\begin{equation}
\lim_{\lambda \to \epsilon_k} (\lambda - \epsilon_k) (H(\lambda) + 2c \cdot S) = R_k.
\end{equation}
which implies

\[ R_k = -2c \cdot S_k - 2 \sum_{j \neq k} \frac{S_k \cdot S_j}{\epsilon_k - \epsilon_j}. \]  \hspace{1cm} (47)

Eq. (42) is then replaced by

\[ [H(\lambda) + 2c \cdot S, R_k] = 0 \quad [R_j, R_k] = 0, \]  \hspace{1cm} (48)

with the conditions

\[ \sum_i R_i = -2c \cdot \sum_k S_k, \]  \hspace{1cm} (49)

and

\[ \sum_i \epsilon_i R_i = -2 \sum_i \epsilon_i c \cdot S_i - 2 \sum_{i \neq j} S_i \cdot S_j. \]  \hspace{1cm} (50)

Rewriting the Hamiltonian of Eq. (25) in the form

\[ H = \sum_j \epsilon_j S_j^0 - |G| \left( \left( \sum_i S_i \right) \cdot \left( \sum_i S_i \right) - \left( \sum_i S_i^0 \right)^2 + \left( \sum_i S_i^0 \right) \right) \]  \hspace{1cm} (51)

+ constant terms,

and choosing the constant vector of Eq. (45) to be

\[ c = (0, 0, -1/2|G|) \]  \hspace{1cm} (52)

one immediately obtains

\[ \frac{H}{|G|} = \sum_i \epsilon_i R_i + |G|^2 \left( \sum_i R_i \right)^2 - |G| \sum_i R_i + \cdots \]  \hspace{1cm} (53)

Since all \( R_k \) and \( H(\lambda) + 2c \cdot S \) mutually commute (cf. Eq. (48)), they have the same eigenvalues. Eq. (53) then tells us that they are also the eigenvalues of the Richardson Hamiltonian, Eq. (25).

2.3. Exact Solution for Degenerate Spectra. – If the single particle spectrum is degenerate, then it is possible to find the eigenvalues and eigenstates of the Hamiltonian in Eq. (22). If all the single particle energies are the same, the first term in Eq. (22) is a constant and can be ignored. Defining the operators

\[ \hat{S}^+(0) = \sum_j \epsilon_j^* \hat{S}_j^+ \quad \text{and} \quad \hat{S}^- (0) = \sum_j \epsilon_j \hat{S}_j^-, \]  \hspace{1cm} (54)
the Hamiltonian of Eq. (22) can be rewritten as

\begin{equation}
\hat{H} = -|G|\hat{S}^+(0)\hat{S}^-(0) + \text{constant.}
\end{equation}

(The operators in Eq. (54) are defined with argument 0 for reasons explained in the following discussion). In the 1970’s Talmi showed that, under certain assumptions, a state of the form

\begin{equation}
\hat{S}^+(0)|0\rangle = \sum_j c^*_j \hat{S}^+_j |0\rangle,
\end{equation}

with \(|0\rangle\) being the particle vacuum, is an eigenstate of a class of Hamiltonians including the one above [30]. A direct calculation yields the eigenvalue equation

\begin{equation}
\hat{H}\hat{S}^+(0)|0\rangle = \left(-|G| \sum_j \Omega_j |c_j|^2\right) \hat{S}^+(0)|0\rangle.
\end{equation}

However, there are other one-pair states besides the one in Eq. (57). For example for two levels \(j_1\) and \(j_2\), the orthogonal state

\begin{equation}
\left(\frac{c_{j_2}}{\Omega_{j_1}} \hat{S}^+_1 - \frac{c_{j_1}}{\Omega_{j_2}} \hat{S}^+_2\right)|0\rangle,
\end{equation}

is also an eigenstate with \(E=0\). In Ref. [23] it was shown that there is a systematic way to derive these states. To see their solution we define the operators

\begin{equation}
\hat{S}^+(x) = \sum_j \frac{c^*_j}{1 - |c_j|^2 x} \hat{S}^+_j \quad \text{and} \quad \hat{S}^-(x) = \sum_j \frac{c_j}{1 - |c_j|^2 x} \hat{S}^-_j.
\end{equation}

Note that if one substitutes \(x = 0\) in the operators of Eq. (59), one obtains the operators in Eq. (54). Further defining the operator

\begin{equation}
\hat{K}^0(x) = \sum_j \frac{1}{1/|c_j|^2 - x} \hat{S}^0_j,
\end{equation}

one can prove the following commutation relations:

\begin{equation}
[\hat{S}^+(x), \hat{S}^-(0)] = [\hat{S}^+(0), \hat{S}^-(x)] = 2\hat{K}^0(x)
\end{equation}

\begin{equation}
[\hat{K}^0(x), \hat{S}^\pm(y)] = \pm \frac{\hat{S}^\pm(x) - \hat{S}^\pm(y)}{x - y}
\end{equation}
These commutators are very similar to, but not the same as, those of the Gaudin algebra described in Eqs. (27), (28), and (29). Using the commutators in Eqs. (61) and (62) one can easily show that

\[ \hat{S}^\pm (0) \hat{S}^\pm (z_1^{(N)}) \ldots \hat{S}^\pm (z_{N-1}^{(N)}) |0\rangle \]

is an eigenstate of the Hamiltonian in Eq. (55) if the following Bethe ansatz equations are satisfied:

\[ -\frac{\Omega_j}{2} \frac{1}{|c_j|^2 - z_m^{(N)}} = \frac{1}{z_m^{(N)}} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z_k^{(N)} - z_k^{(N)}} \quad m = 1, 2, \ldots N - 1. \]

The energy of the state in Eq. (63) is

\[ E_N = -|G| \left( \sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z_k^{(N)}} \right). \]

The authors of Ref. [23] used a Laurent expansion of the operators given in Eq. (59) around \( x = 0 \) followed by an analytic continuation argument to show the validity of their results in the entire complex plane except some singular points. The derivation sketched above instead utilizes the algebra depicted in Eqs. (61) and (62); it is significantly simpler. The state in Eq. (63) is an eigenstate if the shell is at most half full. Similarly the state

\[ \hat{S}^\pm (x_1^{(N)}) \hat{S}^\pm (x_2^{(N)}) \ldots \hat{S}^\pm (x_N^{(N)}) |\bar{0}\rangle \]

is an eigenstate with zero energy if the following Bethe ansatz equations are satisfied:

\[ -\frac{\Omega_j}{2} \frac{1}{|c_j|^2 - x_m^{(N)}} = \sum_{k=1(k\neq m)}^{N} \frac{1}{x_m^{(N)} - x_k^{(N)}} \quad \text{for every} \quad m = 1, 2, \ldots , N \]

Again this is an eigenstate if the shell is at most half full.

To figure out what happens if the available states are more than half full we note that there are degeneracies in the spectra. Let us denote the state where all levels are completely filled by \( |\bar{0}\rangle \). It is easy to show that the state \( |\bar{0}\rangle \) and the particle vacuum, \( |0\rangle \) are both eigenstates of the Hamiltonian of Eq. (55) with the same energy, \( E = -|G| \sum_j \Omega_j |c_j|^2 \). This suggests that if the shells are more than half full then one should start with a state of the form

\[ \hat{S}^- (z_1^{(N)}) \hat{S}^- (z_2^{(N)}) \ldots \hat{S}^- (z_{N-1}^{(N)}) |\bar{0}\rangle \]

in order to find the correct eigenstate.
Indeed the state in Eq. (68) is an eigenstate of the Hamiltonian in Eq. (55) with the energy

\[ E = -G \left( \sum_j \Omega_j |c_j|^2 - \sum_{k=1}^{N-1} \frac{2}{z^{(N)}_k} \right) \]

if the following Bethe ansatz equations are satisfied [24, 25]:

\[ \sum_j \frac{-\Omega_j/2}{1/|c_j|^2 - z^{(N)}_m} = \frac{1}{z^{(N)}_m} + \sum_{k=1(k\neq m)}^{N-1} \frac{1}{z^{(N)}_m - z^{(N)}_k}. \]

In Eq. (70) \( N_{max} + 1 - N \) is the number of particle pairs. If the available particle states are more than half full, then there are no zero energy states. Note that the Bethe ansatz conditions in Eqs. (70) and (64) as well the energies in Eqs. (69) and (65) are identical. This particle-hole degeneracy is due to a hidden supersymmetry [25] and is illustrated in Table I. This supersymmetry is described in Section 3.2.

2.4. Exact Solutions with two shells. – Consider the most general pairing Hamiltonian with only two shells:

\[ \hat{H} = \sum_j 2\xi_j \hat{S}_j^0 - \sum_{j,j'} c_{j'}^* c_j \hat{S}_j^+ \hat{S}_{j'}^- + \sum_j \xi_j \Omega_j, \]

with \( \xi_j = \epsilon_j/|G| \). It turns out that this problem can be solved using techniques illustrated in the previous sections. Eigenstates of the Hamiltonian in Eq. (71) can be written using the new step operators [26]

\[ \mathcal{J}^+(x) = \sum_j \frac{c_j^*}{2\xi_j - |c_j|^2 x} \hat{S}_j^+ \]

as

\[ \mathcal{J}^+(x_1)\mathcal{J}^+(x_2)\cdots \mathcal{J}^+(x_N)|0\rangle. \]
Introducing the definitions

\[ \beta = 2 \frac{\varepsilon_{j_1} - \varepsilon_{j_2}}{|c_{j_1}|^2 - |c_{j_2}|^2} \]

\[ \delta = 2 \frac{\varepsilon_{j_2}|c_{j_1}|^2 - \varepsilon_{j_1}|c_{j_2}|^2}{|c_{j_1}|^2 - |c_{j_2}|^2}. \]

we obtain the energy eigenvalue to be

\[ E_N = -\sum_{n=1}^{N} \frac{\delta x_n}{\beta - x_n} \]

if the parameters \(x_k\) satisfy the Bethe ansatz equations

\[ \sum_j \Omega_j = \frac{2}{|c_{j_1}|^2 - |c_{j_2}|^2} \beta - x_k + \sum_{n=1}^{N} \frac{2}{x_n - x_k}. \]

2.5. Solutions of Bethe Ansatz equations. – Various solutions of the pairing problem discussed in the previous sections are typically considered as semi-analytical solutions since one still needs to find the solutions of the Bethe ansatz equations. This a task which, quite often, needs to be tackled numerically. However, in certain limits it is possible to find solutions of the Bethe ansatz equations analytically. The method outlined here was first presented in Ref. [31].

Consider the Bethe ansatz equations for the degenerate single particle levels and zero energy eigenstate given in Eq. (67). Introducing new variables, \(\eta_i\),

\[ x_i^{(N)} = \frac{1}{|c_{j_2}|^2} + \eta_{i}^{(N)} \left( \frac{1}{|c_{j_1}|^2} - \frac{1}{|c_{j_2}|^2} \right) \]

Eq. (67) can be rewritten as

\[ \sum_{k=1}^{N} \frac{\Omega_j}{\eta_{i}^{(N)} - \eta_{k}^{(N)}} \left( \frac{1}{|c_{j_1}|^2} - \frac{1}{|c_{j_2}|^2} \right) = 0. \]

It can be easily shown that the polynomial admitting the solutions of Eq. (78) as zeros

\[ p_N(z) = \prod_{i=1}^{N} (z - \eta_{i}^{(N)}) \]

satisfies the hypergeometric equation [32]

\[ z(1-z)p''_N + [-\Omega_{j_2} + (\Omega_{j_1}\Omega_{j_2}) z]p'_N + N(N - \Omega_{j_1} - \Omega_{j_2} - 1)p_N = 0. \]

Consequently the problem of finding the solutions of the Bethe ansatz equation (67) reduces to calculating the roots of hypergeometric functions. For analytical expressions of the energy eigenvalues obtained in this manner the reader is referred to Ref [24].
3. – Supersymmetric Quantum Mechanics in Nuclear Physics

Consider two Hamiltonians

\[ H_1 = G^\dagger G, \quad H_2 = GG^\dagger, \]  

(81)

where \( G \) is an arbitrary operator. The eigenvalues of these two Hamiltonians

\[
\begin{align*}
G^\dagger G|1, n\rangle &= E_n^{(1)}|1, n\rangle, \\
GG^\dagger|2, n\rangle &= E_n^{(2)}|2, n\rangle
\end{align*}
\]

(82)

are the same:

\[ E_n^{(1)} = E_n^{(2)} = E_n \]  

(83)

and the eigenvectors are related:

\[
|2, n\rangle = G \left[ G^\dagger G \right]^{-1/2} |1, n\rangle.
\]

(84)

This works for all cases except when \( G|1, n\rangle = 0 \), which should be the ground state energy of the positive-definite Hamiltonian \( H_1 \).

To see why this is called supersymmetry we define

\[
Q^\dagger = \begin{pmatrix} 0 & 0 \\ G^\dagger & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix},
\]

(85)

Then

\[ H = \{Q, Q^\dagger\} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}. \]

(86)

with

\[ [H, Q] = 0 = [H, Q^\dagger]. \]

(87)

Clearly the operators \( H, Q, \) and \( Q^\dagger \) close under the commutation and anticommutation relations of Eq. (86) and (87), forming a very simple superalgebra. The two Hamiltonians depicted in Eq. (81) are said to form a system of supersymmetric quantum mechanics [33]. Many aspects of the supersymmetric quantum mechanics has been investigated in detail [34, 35, 36]. In the following sections two applications of supersymmetric quantum mechanics to nuclear physics are summarized.
3.1. Application of Supersymmetric Quantum Mechanics to Pseudo-Orbital Angular Momentum and Pseudospin. – The nuclear shell model is a mean-field theory where the single particle levels can be taken as those of a three-dimensional harmonic oscillator (labeled with SU(3) quantum numbers) for the lowest \((A \leq 20)\) levels. For heavier nuclei with more than 20 protons or neutrons, different parity orbitals mix. The Nilsson Hamiltonian of the spherical shell model is \[ H = \omega b_i^\dagger b_i - 2kL \cdot S - k\mu L^2, \] where the second term mixes opposite parity orbitals and the last term mocks up the deeper potential felt by the nucleons as \(L\) increases.

Fits to data indicate that \(\mu \approx 0.5\), hence there exists degeneracies in the single particle spectra. In the 50–82 shell (the SU(3) label or the principal harmonic oscillator quantum number of which is \(N = 4\)) the \(s_{1/2}\) and \(d_{3/2}\) orbitals and further \(d_{5/2}\) and \(g_{7/2}\) orbitals are almost degenerate. It is possible to give a phenomenological account of this degeneracy by introducing a second SU(3) algebra called the pseudo-SU(3) \([38, 39]\). Assuming that those orbitals belong to the \(N = 3\) (with \(\ell = 1, 3\)) representation of the latter SU(3) algebra, the quantum numbers of the SO(3) algebra included in this new SU(3) are called pseudo-orbital-angular momentum (\(\ell = 1, 3\) in this case). We can also introduce a “pseudo-spin” \((s = 1/2)\). One can easily show that \(j = 1/2\) and 3/2 orbitals (and also \(j = 5/2\) and 7/2 orbitals) are degenerate if pseudo-orbital angular momentum and pseudo-spin coupling vanishes. This degeneracy follows from the supersymmetric quantum mechanical nature of the problem.

It can be shown that two Hamiltonians written in the SU(3) and the pseudo-SU(3) bases are supersymmetric partners of each other \([40]\). The operator that transforms these two bases into one another is \([41]\)

\[ U = G \left[ G^\dagger G \right]^{-1/2} = \sqrt{2} F_- \left( K_0 + [F_+, F_-] \right)^{-1/2} \]

yielding the supersymmetry transformation between the pseudo-SU(3) Hamiltonian \(H'\) and the SU(3) Hamiltonian \(H\):

\[ H' = U H U^\dagger = b_i^\dagger b_i - 2k(2\mu - 1) L \cdot S - k\mu L^2 + [1 - 2k(\mu - 1)]. \]

The transformation depicted in Eq. (90) can also be expressed in terms of the generators of the orthosymplectic superalgebra Osp(1/2) \([40]\).

3.2. Supersymmetric Quantum Mechanics and Pairing in Nuclei. – Let us consider the separable pairing Hamiltonian with degenerate single-particle spectra given in Eq. (55):

\[ \hat{H}_{SC} \sim -|G| \hat{S}^+(0) \hat{S}^-(0), \]
and introduce the operator

\[
\hat{T} = \exp \left( -i \frac{\pi}{2} \sum_i (\hat{S}^+_i + \hat{S}^-_i) \right)
\]

This operator transforms the empty shell, \( |0\rangle \), to the fully occupied shell, \( |\bar{0}\rangle \):

\[
\hat{T} |0\rangle = |\bar{0}\rangle.
\]

To establish the connection to the supersymmetric quantum mechanics we define the operators

\[
\hat{B}^- = \hat{T}^\dagger \hat{S}^-(0), \quad \hat{B}^+ = \hat{S}^+(0)\hat{T}.
\]

Supersymmetric quantum mechanics tells us that the partner Hamiltonians \( \hat{H}_1 = \hat{B}^+\hat{B}^- \) and \( \hat{H}_2 = \hat{B}^-\hat{B}^+ \) have identical spectra except for the ground state of \( \hat{H}_1 \). It can easily be shown that in this case two Hamiltonians \( \hat{H}_1 \) and \( \hat{H}_2 \) are actually identical and equal to the pairing Hamiltonian (Eq. (91)). Hence the role of the supersymmetry is to connect the states \( |1, n\rangle \) and \( |2, n\rangle \). These are the “particle” and “hole” states. Hence if the shell is less than half-full (“particles”) there is a zero-energy state (note that the Hamiltonian in Eq. (55) negative definite, hence this is the highest energy state). If the shell is more than half-full (“holes”) this state disappears. Otherwise the spectra for particle and hole states are the same as the rules of the supersymmetric quantum mechanics implies [25].

4. – Dynamical Supersymmetries in Nuclear Physics

Dynamical supersymmetries in nuclear physics start with the algebraic model of nuclear collectivity called Interacting Boson Model [6, 7, 8]. In this model, low-lying quadrupole collective states of even-even nuclei are generated as states of a system of bosons occupying two levels, one with angular momentum zero (s-boson) and one with angular momentum two (d-boson). It is discussed elsewhere in great detail in these proceedings [42, 43]. There are three exactly solvable limits of the simplest form of the Interacting Boson Model where neutron and proton bosons are not distinguished:

- **Vibrational Limit:** \( SU(6) \supset SU(5) \supset SO(5) \supset SO(3) \).
- **Rotational Limit:** \( SU(6) \supset SU(3) \supset SO(3) \).
- **Gamma-Unstable Limit:** \( SU(6) \supset SO(6) \supset SO(5) \supset SO(3) \).

It is possible to extend the Interacting Boson Model to describe odd-even and odd-odd nuclei [44]. Dynamical supersymmetries arise out of certain exactly solvable limits of this Interacting Boson-Fermion Model [45, 46].

In an odd-even nucleus, in addition to the correlated nucleon pairs (s and d bosons) we need the degrees of freedom of the unpaired fermions. If those unpaired fermions are...
in the $j_1, j_2, j_3, \ldots$ orbitals then the fermionic sector of the theory is represented by the fermionic algebra $SU_F\left(\sum_i (2j_i+1)\right)$ and the resulting $SU(6)_B \times SU_F\left(\sum_i (2j_i+1)\right)$ algebra is embedded in the superalgebra $SU(6/\sum_i (2j_i+1))$. In the first example of dynamical supersymmetry worked out the unpaired fermion was in a $j = 3/2$ ($d_{3/2}$) orbital coupled to the nuclei described by the gamma-unstable ($SO(6)$) limit of the Interacting Boson Model. The resulting $SU(6/4)$ supersymmetry was used to describe many properties of nuclei in the Os-Pt region [47]. Immediately afterwards this supersymmetry was extended to the $SU(6/12)$ superalgebra including fermions in $s_{1/2}, d_{3/2},$ and $d_{5/2}$ orbitals [48]. Theoretical implications of nuclear supersymmetries were extensively investigated by many authors [49, 50, 51, 52, 53, 54]. The existence of dynamical supersymmetries in nuclei is experimentally established [55, 56, 57, 58, 59, 60].

There exists other symmetries of nuclei describing phase transitions between different dynamical symmetry limits. For these so-called critical point symmetries zeros of wavefunctions in confining potentials of the geometric model of nuclei give rise to point groups symmetries [2]. This scheme is generally applicable to the spectra of systems undergoing a second-order phase transition between the dynamical symmetry limits $SU(n-1)$ and $SO(n)$. The resulting symmetries are either named after the discrete subgroups of the Euclidean group $E(5)$, or $X(5)$. A review of the experimental searches for critical point symmetries at nuclei in the $A=130$ and $A=150$ regions is given in Ref. [61]. It is also possible to couple Bohr Hamiltonian of the geometric model with a five-dimensional square well potential to a fermion using the five-dimensional generalization of the spin-orbit interaction. In doing so the $E(5)$ symmetry of the even-even nuclei goes into the $E(5/4)$ supersymmetry for odd-even nuclei near the critical point [62]. Initial experimental tests of this latter supersymmetry are encouraging [63].

5. – Application of Symmetry Techniques to Subbarrier Fusion

In the study of nuclear reactions one needs to describe translational motion coupled with internal degrees of freedom representing the structure of colliding nuclei. The Hamiltonian for such a multidimensional quantum problem is taken to be

\begin{equation}
H = -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) + H_0(q) + H_{\text{int}}(r, q),
\end{equation}

where $r$ is the relative coordinate of the target-projectile pair and $q$ represents any internal degrees of freedom of the system. To study the effect of the structure of the target nuclei on fusion cross sections near and below the Coulomb barrier one can take $V(r)$ to be the potential barrier and the third term in Eq. (95) represents the internal structure of the target nuclei. All the dynamical information about the system can be obtained by solving the evolution equation

\begin{equation}
\frac{i\hbar}{\partial t} \hat{U} = \hat{H} \hat{U}
\end{equation}
with the initial condition \( \dot{U}(t_i) = 1 \). If part of the Hamiltonian in Eq. (95), say \( H_0(q) + H_{\text{int}}(r, q) \), is an element of a particular Lie algebra, then symmetry methods can significantly simplify the solution of the problem. In such a case, if one considers a single trajectory \( r(t) \), the \( q \)-dependent part of the evolution operator becomes an element of the Lie group associated with the Lie Algebra mentioned above. (It is possible to make this statement rigorous in the context of path-integral formalism [64]).

It is by now well-established that heavy-ion fusion cross-sections below the Coulomb barrier are several orders of magnitude larger than one would expect from a one-dimensional barrier penetration picture, an enhancement which is attributed to the coupling of the translational motion to additional degrees of freedom such as nuclear and Coulomb excitation, nucleon transfer, or neck formation [65, 66]. The multidimensional barrier penetration problem inherent in subbarrier fusion can be addressed in the coupled-channels formalism and state of the art coupled-channel codes are currently available (see e.g. [67]). Although several puzzles remain (such as the large values of the surface diffuseness parameter in the nuclear potential required to fit the data [68, 69]; very steep fall-off of the fusion data for some systems at extreme subbarrier energies [70]; or inadequacy of the standard nuclear potentials to simultaneously reproduce fusion and elastic scattering measurements [71]) there is overall good agreement between coupled-channels calculations and the experimental data.

An alternative approach is to formulate the problem algebraically by using a model amenable to such an approach for describing the nuclear structure, such as the Interacting Boson Model [72]. Not only this approach fits the data well, it can also be used for transitional nuclei, the treatment of which could be more complicated for coupled-channels calculations [72, 73, 74, 75].

6. – Application of Algebraic Techniques in Nuclear Astrophysics

Nuclear astrophysics has been very successful exploring the origin of elements. As our understanding of the heavens evolved, it was realized that nuclear data was needed for astrophysics calculations, such as nucleosynthesis, stellar evolution, the Big Bang cosmology, x-ray bursts, and supernova dynamics. Nuclear astrophysics initially started with reaction rate measurements, but with the recent rapid growth of the observational data, expanding computational capabilities, and availability of exotic nuclear beams, interest in all aspects of nuclear physics relevant to astrophysical phenomena has significantly increased [76]. In the rest of this section an application of algebraic techniques to a nuclear astrophysics problem is presented.

Light nuclei are formed during the big-bang nucleosynthesis era and nuclei up to the iron, nickel and cobalt group are formed during the stellar evolution. A good fraction of the nuclei heavier than iron were formed in the rapid neutron capture (r-process) nucleosynthesis. The astrophysical location of the r-process is expected to be where explosive phenomena are present since a large number of interactions are required to take place at this location during a rather short time interval. Core-collapse supernovae which occur following the stages of nuclear burning during stellar evolution after the
formation of an iron core is such a site. Neutrino interactions play a very important role in the evolution of core-collapse supernovae [77]. Almost all (99%) of the gravitational binding energy ($10^{53}$ ergs) of the progenitor star is released in the neutrino cooling of the neutron star formed after the collapse. It was suggested that neutrino-neutrino interactions could play a potentially very significant role in core-collapse supernovae [78, 79, 80, 81, 82].

A key quantity for determining the r-process yields is the neutron to seed-nucleus ratio (or equivalently neutron-to-proton ratio). Interactions of the neutrinos and antineutrinos streaming out of the core both with nucleons and seed nuclei determine the neutron-to-proton ratio. Before these neutrinos reach the r-process region they undergo matter-enhanced neutrino oscillations as well as coherently scatter over other neutrinos. Many-body behavior of this neutrino gas is not completely understood, but may have a significant impact on r-process nucleosynthesis.

It was shown that neutrinos moving in matter gain an effective potential due to forward scattering from the background particles such as the electrons [83, 84, 85]. One can write the Hamiltonian describing neutrino transport in dense matter for two neutrino flavors as

$$H_\nu = \int dp \left( \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2}G_F N_e \right) J_0(p) + \frac{1}{2} \int dp \frac{\delta m^2}{2p} \sin 2\theta (J_+(p) + J_-(p)),$$

where $N_e$ is the electron density of the medium (assumed to be charge neutral and unpolarized), $\theta$ is the mixing angle between neutrino flavors, $\delta m^2 = m_2^2 - m_1^2$ is the difference between squares of the neutrino masses, and $G_F$ is the Fermi coupling strength of the weak interactions. In the above equation the operators $J^\pm$, $J^0$ has been written in terms of the neutrino creation and annihilation operators:

$$J_+(p) = a_e^\dagger(p)a_e(p), \quad J_-(p) = a_\mu^\dagger(p)a_\mu(p), \quad J_0(p) = \frac{1}{2} (a_e^\dagger(p)a_e(p) - a_\mu^\dagger(p)a_\mu(p)).$$

Here $a_e(p)$ and $a_\mu(p)$ are the creation and annihilation operators for the electron neutrino with momentum $p$ and either muon or tau neutrino with momentum $p$, respectively. The operators in Eq. (98) form as many mutually commuting SU(2) algebras as the number of allowed values of neutrino momenta:

$$[J_+(p), J_-(q)] = 2\delta^3(p-q)J_0(p), \quad [J_0(p), J_\pm(p)] = \pm\delta^3(p-q)J_\pm(p).$$

If, instead of two neutrino flavors, one considers all three active neutrino flavors then one should use copies of SU(3) algebras. The contribution of neutrino-neutrino forward scattering terms to the neutrino Hamiltonian is given by

$$H_{\nu\nu} = \sqrt{2}G_F \int dp \int dq \left( 1 - \cos \theta_{pq} \right) \mathbf{J}(p) \cdot \mathbf{J}(q).$$
where $\vartheta_{pq}$ is the angle between neutrino momenta $\mathbf{p}$ and $\mathbf{q}$. It is easy to include neutrino-antineutrino and antineutrino-antineutrino scattering terms in this Hamiltonian [81], but we ignore them here to keep the presentation simple.

Since a large number of neutrinos ($10^{58}$) are emitted during a typical core-collapse it is very difficult to evaluate neutrino evolution exactly using the two-body Hamiltonian of Eq. (100). Instead one uses a mean field approximation where the product of two commuting arbitrary operators $\hat{O}_1$ and $\hat{O}_2$ can be approximated as

$$
\hat{O}_1 \hat{O}_2 \sim \hat{O}_1 \langle \xi | \hat{O}_2 | \xi \rangle + \langle \xi | \hat{O}_1 | \xi \rangle \hat{O}_2 - \langle \xi | \hat{O}_1 | \xi \rangle \langle \xi | \hat{O}_2 | \xi \rangle,
$$

(101)

provided that the condition

$$
\langle \xi | \hat{O}_1 \hat{O}_2 | \xi \rangle = \langle \xi | \hat{O}_1 | \xi \rangle \langle \xi | \hat{O}_2 | \xi \rangle
$$

(102)

is satisfied. In Eqs. (101) and (102) One can then replace the exact Hamiltonian of Eq. (100) with the approximate expression

$$
H_{\nu \nu} \sim 2\sqrt{2}G_F V \int dp dq \ R_{pq} \left( J_0(p)\langle J_0(q) \rangle + \frac{1}{2} J_+(p) \langle J_-(q) \rangle + \frac{1}{2} J_-(p) \langle J_+(q) \rangle \right)
$$

where $R_{pq} = (1 - \cos \vartheta_{pq})$ and the averages are calculated over the entire ensemble of neutrinos. Systematic corrections to this expression are explored in [81]. Calculations using this approximation do not yield conditions (i.e. large enough neutron to seed nucleus ratio) favorable to r-process nucleosynthesis [78]. There is encouraging progress in numerical calculations using the exact Hamiltonian of Eq. (100) [79, 80]. However an algebraic solution to this problem is currently lacking.

7. – Conclusions

In many-fermion physics, mean field approaches can describe many properties; but are inadequate to describe the whole picture; pairing correlations play a crucial role. In fact, pairing correlations are not only necessary to understand the structure of rare-earths and actinides, but, since they are essential for the description of the neutrino gas in a core-collapse supernova where many nuclei are produced, also necessary to understand the existence of such nuclei in the first place. Models exploiting symmetry properties and pairing correlations have been very successful. These also gave rise to dynamical supersymmetries. We showed that using algebraic techniques it is possible to solve the s-wave pairing problem almost exactly (i.e., reducing it to Bethe ansatz equations) at least for a number of simplified cases.

* * *

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REFERENCES

[1] Balantekin A. B., *Annals Phys.*, **164** (1985) 277.
[2] Iachello F., *Phys. Rev. Lett.*, **85** (2000) 3580; **44** (1980) 772.
[3] Balantekin A. B., Schmitt H. A. and Barrett B. R., *J. Math. Phys.*, **29** (1988) 1634.
[4] Dean D. J. and Hjorth-Jensen M., *Rev. Mod. Phys.*, **75** (2003) 607 [arXiv:nucl-th/0210033].
[5] Mayer M. G., *Phys. Rev.*, **78** (1950) 22.
[6] Balantekin A. B., Schmitt H. A. and Barrett B. R., *J. Math. Phys.*, **29** (1988) 1634.
[7] Page D., Prakash M., Lattimer J. M. and Steiner A., *Phys. Rev. Lett.*, **85** (2000) 1225 [arXiv:hep-ph/0005094].
[8] Bardeen J., Cooper L. N. and Schrieffer J. R., *Phys. Rev.*, **108** (1957) 1175.
[9] Bohr A., Mottelson B. and Pines D., *Phys. Rev.*, **110** (1958) 936.
[10] Belyaev S., *Mat. Fys. Medd. K. Dan. Vidensk. Selsk.*, **31** (1959) 641.
[11] Migdal A., *Nucl. Phys.*, **13** (1959) 655.
[12] Unna I. and Weneser J., *Phys. Rev.*, **137** (1965) B1455.
[13] Kerman A. K., Lawton R. D. and Macfarlane M. H., *Phys. Rev.*, **124** (162) 1961.
[14] Hagino K. and Bertsch G. F., *Nucl. Phys. A*, **679** (2000) 163 [arXiv:nucl-th/0003016];
[15] Hagino K., Sagawa H., Carbonell J. and Schuck P., *Phys. Rev. Lett.*, **99** (2007) 022506 [arXiv:nucl-th/0611064].
[16] von Delft J. and Braun F., arXiv:cond-mat/9911058; von Delft J., Zaikin A. D., Golubev D. S. and Tichy W., *Phys. Rev. Lett.*, **77** (1996) 3189.
[17] Racah G., *Phys. Rev.*, **63** (1943) 367.
[18] Kerman A. K., *Ann. Phys. (NY)*, **12** (1961) 300.
[19] Richardson R. W., *Phys. Lett.*, **3** (1963) 277; **14** (1965) 325; *J. Math. Phys.*, **6** (1965) 1034; **18** (1967) 1802; *Phys. Rev.*, **141** (1966) 949; **144** (1966) 874.
[20] Gaudin M., *J. Physique*, **37** (1976) 1087; *La Fonction d'onde de Bethe, Collection du Commissariat a l'énergie atomique, Masson, Paris, 1983.*
[21] Pan F., Draayer J. P. and Ormand W. E., *Phys. Lett. B*, **422** (1998) 1 [arXiv:nucl-th/9709036].
[22] Balantekin A. B., de Jesus J. H., and Pehlivan Y., *Phys. Rev. C*, **75** (2007) 064304 [arXiv:nucl-th/0702059].
[23] Balantekin A. B. and Pehlivan Y., *J. Phys. G*, **34** (2007) 1783 [arXiv:0705.1318 [nucl-th]].
[24] Balantekin A. B. and Pehlivan Y., *Phys. Rev. C*, **76** (2007) in press [arXiv:0710.3941 [nucl-th]].
[25] Volya A., Brown B. A. and Zelevinsky V., *Phys. Lett. B*, **509** (2001) 37 [arXiv:nucl-th/0011079].
[26] Schmitt H. A., Halse P., Barrett B. R. and Balantekin A. B., *Phys. Lett. B*, **210** (1988) 1.
[27] Balantekin A. B., Dereli T. and Pehlivan Y., *J. Phys. A*, **38** (2005) 5697 [arXiv:math-ph/0505071].
[28] Talini T., *Nucl. Phys.*, **A172** (1971) 1.
[29] Balantekin A. B., Dereli T. and Pehlivan Y., *J. Phys. G*, **30** (2004) 1225 [arXiv:nucl-th/0407006]; *Int. J. Mod. Phys. E*, **14** (2005) 47 [arXiv:nucl-th/0505023].
[30] Stieltjes T. J., *Sur Quelques Theoremes d'Algebre, Oeuvres Completes, V. 11* (Groningen:Noordhoff) 1914.
[31] Witten E., *Nucl. Phys. B*, **188** (1981) 513.
[34] Cooper F., Khare A. and Sukhatme U., Phys. Rept., 251 (1995) 267 [arXiv:hep-th/9405029].
[35] Cooper F., Ginocchio J. N. and Khare A., Phys. Rev. D, 36 (1987) 2458.
[36] Fricke S.H., Balantekin A.B., Hatchell P. J. and Uzer T., Phys. Rev. A, 37 (1988) 2797.
[37] Nilsson S. G., Mat. Fys. Medd. K. Dan. Vidensk. Selsk., 29 (1955) 16.
[38] Ratna Raju R. D., Draayer J. P. and Hecht K. T., Nucl. Phys. A, 202 (1973) 433.
[39] Arima A., Harvey M. and Shimizu K., Phys. Lett. B, 30 (1969) 517.
[40] Balantekin A. B., Castanos O. and Moshinsky M., Phys. Lett. B, 284 (1992) 1.
[41] Castanos O., Moshinsky M. and Quesne C., Phys. Lett. B, 277 (1992) 238.
[42] Casten R., These proceedings.
[43] van Isacker P., These proceedings.
[44] Arima A. and Iachello F., Phys. Rev. C, 14 (1976) 761.
[45] Iachello F., Phys. Rev. Lett., 44 (1980) 772.
[46] Iachello F. and Scholten O., Phys. Rev. Lett., 43 (1979) 679.
[47] Balantekin A. B., Bars I. and Iachello F., Phys. Rev. Lett., 47 (1981) 19; Nucl. Phys. A, 370 (1981) 284.
[48] Balantekin A. B., Bars I., Bijker R. and Iachello F., Phys. Rev. C, 27 (1983) 1761.
[49] Balantekin A.B. and Paar V., Phys. Rev. C, 34 (1986) 1917.
[50] Schmitt H. A., Halse P., Balantekin A. B. and Barrett B. R., Phys. Rev. C, 39 (1989) 2419.
[51] Navratiil P., Geyer H. B. and Dobaczewski J., Nucl. Phys. A, 607 (1996) 23 [arXiv:nucl-th/9606043]; Cejnar P. and Geyer H. B., Phys. Rev. C, 68 (2003) 054324 [arXiv:nucl-th/0306058].
[52] Barea J., Bijker R., Frank A. and Loyola G., Phys. Rev. C, 64 (2001) 064313 [arXiv:nucl-th/0107048].
[53] Leviatan A., Phys. Rev. Lett., 92 (2004) 202501 [Erratum-92 (2004) 219902] [arXiv:nucl-th/0312018]; Int. J. Mod. Phys. E, 14 (2005) 111 [arXiv:nucl-th/0407107].
[54] Barea J., Alonso C. E., Arias J. M. and Jolie J., Phys. Rev. C, 71 (2005) 014314.
[55] Mauthofer A. et al., Phys. Rev. C, 34 (1986) 1958; 39 (1989) 1111.
[56] Rotbard G. et al., Phys. Rev. C, 55 (1997) 1200.
[57] Metz A., Eisermann Y., Golwitzzer A., Hertenberger R., Valnion B. D., Graw G. and Jolie J., Phys. Rev. C, 61 (2000) 064313 [Erratum-67 (2003) 049901].
[58] Algora A., Jolie J., Dombrazi Z., Sohler D., Podolyak Z. and Fenyes T., Phys. Rev. C, 67 (2003) 044303.
[59] Barea J., Bijker R. and Frank A., Phys. Rev. Lett., 94 (2005) 152501 [arXiv:nucl-th/0412090].
[60] Wirth H. F. et al., Phys. Rev. C, 70 (2004) 014610.
[61] McCutchen E. A., Zamfir N. V. and Casten R. F., J. Phys. G, 31 (2005) S1485.
[62] Caprio M. A. and Iachello F., Nucl. Phys. A, 781 (2007) 26 [arXiv:nucl-th/0610026].
[63] Fetea M. S. et al., Phys. Rev. C, 73 (2006) 051301.
[64] Balantekin A. B. and Takigawa N., Annals Phys., 160 (1985) 441.
[65] Balantekin A. B. and Takigawa N., Rev. Mod. Phys., 70 (1998) 77 [arXiv:nucl-th/9708036].
[66] Dasgupta M., Hinde D. J., Rowley N. and Stefanini A. M., Ann. Rev. Nucl. Part. Sci., 48 (1998) 401.
[67] Hagino K., Rowley N. and Kruppa A. T., Comput. Phys. Commun., 123 (1999) 143 [arXiv:nucl-th/9903074].
[68] Hagino K., Rowley N. and Dasgupta M., Phys. Rev. C, 67 (2003) 054603 [arXiv:nucl-th/0302025].
[69] Hagino K., Takehi T., Balantekin A. B. and Takigawa N., Phys. Rev. C, 71 (2005) 044612 [arXiv:nucl-th/0412044].
[70] C. L. Jiang, et al., Phys. Rev. C, 69 (2004) 014604; Phys. Rev. Lett., 89 (1992) 052701.
[71] Mukherjee A., Hinde D. J., Dasgupta M., Hagino K, Newton J. O. and Butt R. D., Phys. Rev. C, 75 (2007) 044608.
[72] Balantekin A. B., Bennett J. R. and Takigawa N., Phys. Rev. C, 44 (1991) 145.
[73] Balantekin A. B, Bennett J. R., DeWeerd A. J. and Kuyucak S., Phys. Rev. C, 46 (1992) 2019.
[74] Balantekin A. B., Bennett J. R. and Kuyucak S., Phys. Rev. C, 48 (1993) 1269; 49 (1994) 1079; 49 (1994) 1294; Phys. Lett. B, 335 (1994) 295 [arXiv:nucl-th/9407037].
[75] Balantekin A. B. and Kuyucak S., J. Phys. G, 23 (1997) 1159 [arXiv:nucl-th/9706068].
[76] Langanke K., These proceedings.
[77] Balantekin A. B. and Fuller G. M., J. Phys. G, 29 (2003) 2513 [arXiv:astro-ph/0309519].
[78] Balantekin A. B. and Yuksel H., New J. Phys., 7 (2005) 51 [arXiv:astro-ph/0411159].
[79] Duan H., Fuller G. M. and Qian Y. Z., Phys. Rev. D, 74 (2006) 123004 [arXiv:astro-ph/0511275]; 76 (2007) 085013 [arXiv:0706.4293 [astro-ph]].
[80] Duan H., Fuller G. M., Carlson J. and Qian Y. Z., Rev. Lett., 97 (2006) 241101 [arXiv:astro-ph/0608050]; Phys. Rev. D, 74 (2006) 105014 [arXiv:astro-ph/0606616]; 75 (2007) 125005 [arXiv:astro-ph/0703776].
[81] Balantekin A. B. and Pehlivan Y., J. Phys. G, 34 (2007) 47 [arXiv:astro-ph/0607527].
[82] Hannestad S., Raffelt G. G., Sigl G. and Wong Y. Y., Phys. Rev. D, 74 (2006) 105010 [Erratum-76 (2007) 029901] [arXiv:astro-ph/0608695]; Raffelt G. G. and Smirnov A. Y., Phys. Rev. D, 76 (2007) 081301 [arXiv:0705.1830 [hep-ph]].
[83] Wolfenstein L., Phys. Rev. D, 17 (1978) 2369.
[84] Mikheev S. P. and Smirnov A. Y., Sov. J. Nucl. Phys., 42 (1985) 913 [Yad. Fiz., 42 (1985) 1441].
[85] Mikheev S. P. and Smirnov A. Y., Nuovo Cim. C, 9 (1986) 17.