COMPLEX POLYNOMIAL BOHNENBLUST-HILLE INEQUALITY WITH POLYNOMIAL BOUNDS

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ABSTRACT. The Bohnenblust-Hille inequality and its variants have found applications in several areas of Mathematics and related fields. The control of the constants for the variant for complex $m$-homogeneous polynomials is of special interest for applications in Harmonic Analysis and Number Theory. Up to now, the best known estimates for its constants are dominated by $\kappa (1 + \varepsilon)^m$, where $\varepsilon > 0$ is arbitrary and $\kappa > 0$ depends on the choice of $\varepsilon$. In this note we show that for a wide family of polynomials the above estimates can be improved to a polynomial bound.

1. Introduction

In the investigation of complex $m$-homogeneous polynomials whose monomials have a uniform bound $M$ on the number of variables, Carando, Defant, and Sevilla-Peris have shown (in [5]) that the optimal constants of the Bohnenblust-Hille inequality are dominated by a polynomial on $m$ whenever $M$ is fixed. Maia, Nogueira and Pellegrino ([7]) substantially improved these results, proving that the optimal constants have a universal bound (i.e. they are bounded by a constant dependent solely on $M$) under the same hypotheses. Nevertheless, there has not been any noticeable progress for these bounds as $M$ increases with $m$. In this note, we recover the polynomial domination of [5] above as a particular case of a more general inequality that allows indexes in an arbitrary set. This is achieved by employing a deep tool recently proved by Bayart [2] which involves the concept of combinatorial dimension. In doing this, we point to a new direction in research by showing that the crucial hypothesis for having the optimal constants under polynomial growth lies in a more general setting. We now turn to the detailed presentation of what has just been outlined.

Let $\alpha = (\alpha_j)_{j=1}^\infty$ be a sequence in $\mathbb{N} \cup \{0\}$ and $x = (x_j)_{j=1}^\infty$ be a sequence in $c_0 := c_0(\mathbb{C})$, and define $|\alpha| = \sum \alpha_j$ and $x^\alpha := \prod x_j^{\alpha_j}$. An $m$-homogeneous polynomial $P : c_0 \to \mathbb{C}$ is denoted by

$$P(x) = \sum_{|\alpha|=m} c_\alpha(P) x^\alpha$$

and the norm of $P$ is given by $\|P\| := \sup_{x \in B_{c_0}} |P(x)|$.

The Bohnenblust–Hille inequality [4] for complex $m$-homogeneous polynomials reads as follows: there is a constant $C_m \geq 1$ such that

$$\left( \sum_{|\alpha|=m} |c_\alpha(P)|^{2m/(m+1)} \right)^{m+1/2m} \leq C_m \|P\|$$

for all continuous $m$-homogeneous polynomials $P : c_0 \to \mathbb{C}$ of the form $P(x) = \sum_{|\alpha|=m} c_\alpha(P) x^\alpha$. The exact control of the growth of the constant $C_m$ plays a crucial role in a vast number of applications. In 2011, it was proved in [6] that $C_m$ can be chosen with exponential growth, and in 2014 ([3]) the result was improved as follows: for any $\varepsilon > 0$ there is a constant $\kappa > 0$ such that

$$C_m \leq \kappa (1 + \varepsilon)^m.$$
In order to improve the estimate \((1)\) to a polynomial bound the authors of \([5]\) have shown that for integers \(m, M\) with \(M \leq m\), we have

\[
\left( \sum_{\alpha \in \Delta_M} |c_\alpha(P)|^{2m \over m+1} \right)^{m+1 \over 2m} \leq 2^M M^{M+1 \over m+1} \|P\|
\]

for all continuous \(m\)-homogeneous polynomials \(P : c_0 \to \mathbb{C}\), where

\[w(\alpha) = \text{card} \{j : \alpha_j \neq 0\}\]

and

\[\Delta_M = \{\alpha : |\alpha| = m, \ w(\alpha) \leq M\}.
\]

In other words, the constant \(C_m\) can be taken with a polynomial bound provided that the sum is restricted to monomials with uniformly bounded number of variables \(M\). In \([7]\) the result of \([5]\) was improved, using techniques of \([1]\), by showing that under the same hypothesis the constant \(C_m\) can be replaced by a universal constant depending just on \(M\).

In this note we show that the crucial condition to have a polynomial bound for \(C_m\) is not to restrict the sum over the coefficients of the monomials with uniformly bounded number of variables \(M\), but to have a control on the combinatorial dimension of the size of the coefficients that are summed.

To illustrate, for \(j \in \mathbb{N}\) let \(\sigma_j : \mathbb{N} \to \mathbb{N}\) be injections such that for each \(j,k\) with \(j \neq k\) one has

\[\text{card} \{i \in \mathbb{N} : \sigma_j(i) \neq \sigma_k(i)\} = \aleph_0\ (\text{e.g. define} \ \sigma_j(n) := p_j^n, \ \text{where} \ p_j \ \text{is the} \ j\text{-th prime number}).\]

Notice that the results of \([2, 7]\) are useless for a continuous \(m\)-homogeneous polynomial

\[P(x) = \sum_{i=1}^\infty c_i x^{\sigma_1(i)} x^{\sigma_2(i)} \cdots x^{\sigma_m(i)},\]

because in this case \(M = m\).

Our result will show, for instance, that this case behaves like \(M = 1\) when the combinatorial dimension is considered (although it does not make sense at this point). The main tool used by us is a deep result recently proved by F. Bayart in his seminal paper \([2]\).

If \(x_1^{a_1} \cdots x_m^{a_m}\) is a monomial, we associate the index \((1, \ldots, 2, \ldots, 2, \ldots, m, \ldots, m)\) where each integer \(i = 1, \ldots, m\) is repeated \(\alpha_i\) times. If \(\Gamma \subset \mathbb{N}^m\) is an arbitrary set, we consider \(\Lambda_\Gamma\) the set of all indexes associated to the elements of \(\Gamma\).

We use notations and notions from combinatorial dimension, as presented in \([2]\). For \(\Lambda \in \mathbb{N}^m\) and \(n \geq 0\), define

\[\psi_\Lambda(n) := \max \{\text{card}((A_1 \times \cdots \times A_m) \cap \Lambda) : A_i \subset \mathbb{N}, \ \text{card}(A_i) \leq n\}.
\]

The combinatorial dimension of \(\Lambda\), denoted by \(\dim(\Lambda)\), is defined as

\[\dim(\Lambda) := \limsup_{n \to +\infty} \frac{\log \psi_\Lambda(n)}{\log n} = \inf \{s > 0; \ \exists C > 0, \ \psi_\Lambda(n) \leq C n^s \ \text{for all} \ n \in \mathbb{N}\}.
\]

Our main result reads as follows:

**Theorem 1.1.** For all positive integers \(m\) and all sets \(\Gamma \subset \mathbb{N}^m\), there is a constant \(C_{\Lambda\Gamma}\) such that

\[
\left( \sum_{\alpha \in \Gamma} |c_\alpha(P)|^{2m \over m+1} \right)^{m+1 \over 2m} \leq e^{\dim \Lambda_\Gamma} \left(C_{\Lambda\Gamma} m^m\right)^{\dim \Lambda_\Gamma \over m} \left(2 \over \sqrt{\pi} \right)^{(m-1) \dim \Lambda_\Gamma \over m} \|P\|
\]

for all continuous \(m\)-homogeneous polynomials \(P : c_0 \to \mathbb{C}\).

**Remark 1.2.** If \(\Gamma = \Delta_M\), we have \(\dim \Lambda_\Gamma = M\) and \(C_{\Lambda\Gamma} = 1\) and we obtain a polynomial domination similar to that of \([2]\).
2. The proof of Theorem 1.1

By [21 Theorem 2.1] we know that there is a constant $C_{\Lambda \Gamma} > 0$ such that

$$
\left( \sum_{(i_1, \ldots, i_m) \in \Lambda \Gamma} |T(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1 + \dim \Lambda \Gamma}{2 \dim \Lambda \Gamma}} \leq C_{\Lambda \Gamma} \prod_{k=1}^{m} \left( \sum_{i_k} \left( \sum_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_m} |T(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \right)
$$

for all continuous $m$–linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{C}$. It is not difficult to prove that using the Khinchine's inequality, for Steinhaus variables, we have

$$
\left( \sum_{i_k} \left( \sum_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_m} |T(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \right)^{1} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} \|T\|
$$

for all $k \in \{1, \ldots, m\}$ and for all continuous $m$–linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{C}$. In this way, there is a constant $C_{\Lambda \Gamma} > 0$ such that

$$
\left( \sum_{(i_1, \ldots, i_m) \in \Lambda \Gamma} |T(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1 + \dim \Lambda \Gamma}{2 \dim \Lambda \Gamma}} \leq m \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} C_{\Lambda \Gamma} \|T\|
$$

for all continuous $m$–linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{C}$.

Let $\hat{P}$ be the symmetric $m$–linear form associated to $P$. Note that

$$
\sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^2 \leq \sum_{(i_1, \ldots, i_m) \in \Lambda \Gamma} (m!) \left( \sum_{i_1 \ldots i_{m-1} i_{m+1} \ldots i_m} |\hat{P}(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1 + \dim \Lambda \Gamma}{2 \dim \Lambda \Gamma}}.
$$

Therefore

$$
\left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^2 \right)^{\frac{1 + \dim \Lambda \Gamma}{2 \dim \Lambda \Gamma}} \leq m! \left( \sum_{(i_1, \ldots, i_m) \in \Lambda \Gamma} |\hat{P}(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{2 \dim \Lambda \Gamma}{1 + \dim \Lambda \Gamma}}.
$$

and

$$
\left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^2 \right)^{\frac{1 + \dim \Lambda \Gamma}{2 \dim \Lambda \Gamma}} \leq (m!) \left( m \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} C_{\Lambda \Gamma} \right) \|\hat{P}\|
$$

$$
\leq (m!) \left( m \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} C_{\Lambda \Gamma} \right) e^{m\|P\|}.
$$

Since we are dealing only with complex scalars, by the Maximum Modulus Principle, we have

$$
\left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{|\alpha| = m} |c_{\alpha}(P)|^2 \right)^{\frac{1}{2}} \leq \|P\|.
$$

Since

$$
\frac{1}{2m} + \frac{1 - \theta}{2} = \frac{\theta}{2m} + \frac{1}{2}
$$

with

$$\theta = \frac{\dim \Lambda \Gamma}{m},$$

we have

$$
\|c_{\alpha}(P)\| \leq \left( \sum_{|\alpha| = m} |c_{\alpha}(P)|^2 \right)^{\frac{1}{2}} \leq \|P\|.
$$
by Hölder’s inequality, we have

\[
\left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \left[ \left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^{\frac{2 \dim \Lambda}{1 + \dim \Lambda}} \right)^{\frac{1 + \dim \Lambda}{2 \dim \Lambda}} \right]^{\frac{\dim \Lambda}{m}} \left[ \left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{m}} \left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^m \right)^{\frac{\dim \Lambda}{m}} \left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^{2m} \right)^{\frac{1}{m+1}} \left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^{2m+1} \right)^{\frac{1}{m+1}}.
\]

Thus

\[
\left( \sum_{\alpha \in \Gamma} |c_{\alpha}(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq e^{\dim \Lambda \Gamma} (C_{\Lambda \Gamma} m m!)^{\frac{\dim \Lambda \Gamma}{m}} \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{(m-1) \dim \Lambda \Gamma}{m}} \|P\|^{\frac{1}{m} - \frac{\dim \Lambda \Gamma}{m}}.
\]

\[\text{Remark 2.1.}\] If there is a constant \(C\) such that \(C_{\Lambda \Gamma} \leq C\) for all \(m\), then the constant in the above inequality is dominated asymptotically by

\[
\left( \frac{2C}{\sqrt{\pi}} \right)^{\dim \Lambda \Gamma} (m)^{\dim \Lambda \Gamma}.
\]

In fact, by Stirling’s Formula we have that

\[
e^{\dim \Lambda \Gamma} (C_{\Lambda \Gamma} m m!)^{\frac{\dim \Lambda \Gamma}{m}} \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{(m-1) \dim \Lambda \Gamma}{m}} \leq (Ce)^{\dim \Lambda \Gamma} (m m!)^{\frac{\dim \Lambda \Gamma}{m}} \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{(m-1) \dim \Lambda \Gamma}{m}}
\]

\[
\sim \left( \frac{2Ce}{\sqrt{\pi}} \right)^{\dim \Lambda \Gamma} (m!)^{\frac{\dim \Lambda \Gamma}{m}} \dim \Lambda \Gamma
\]

\[
\sim \left( \frac{2Ce}{\sqrt{\pi}} \right)^{\dim \Lambda \Gamma} \left( \frac{m^{1+\frac{1}{2m}}}{e} \right)^{\dim \Lambda \Gamma}
\]

\[
\sim \left( \frac{2Ce}{\sqrt{\pi}} \right)^{\dim \Lambda \Gamma} \left( \frac{m^{1+\frac{1}{2m}} \dim \Lambda \Gamma}{e} \right)
\]

\[
\sim \left( \frac{2Ce}{\sqrt{\pi}} \right)^{\dim \Lambda \Gamma} (m)^{\dim \Lambda \Gamma}.
\]

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