On low degree $k$-ordered graphs

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Abstract

A simple graph $G$ is $k$-ordered (respectively, $k$-ordered hamiltonian) if, for any sequence of $k$ distinct vertices $v_1,\ldots,v_k$ of $G$, there exists a cycle (respectively, a hamiltonian cycle) in $G$ containing these $k$ vertices in the specified order. In 1997 Ng and Schultz introduced these concepts of cycle orderability, and motivated by the fact that $k$-orderedness of a graph implies $(k-1)$-connectivity, they posed the question of the existence of low degree $k$-ordered hamiltonian graphs. We construct an infinite family of graphs, which we call bracelet graphs, that are $(k-1)$-regular and are $k$-ordered hamiltonian for odd $k$. This result provides the best possible answer to the question of the existence of low degree $k$-ordered hamiltonian graphs for odd $k$. We further show that for even $k$, there exist no $k$-ordered bracelet graphs with minimum degree $k-1$ and maximum degree less than $k+2$, and we exhibit an infinite family of bracelet graphs with minimum degree $k-1$ and maximum degree $k+2$ that are $k$-ordered for even $k$. A concept related to $k$-orderedness, namely that of $k$-edge-orderedness, is likewise strongly related to connectivity properties. We study this relation in both undirected and directed graphs, and give bounds on the connectivity necessary to imply $k$-(edge-)orderedness properties.

1 Introduction

The concept of $k$-ordered graphs was introduced in 1997 by Ng and Schultz [11]. A simple graph $G$ is a graph without loops or multiple edges, and it is called hamiltonian if there exists a cycle (called a hamiltonian cycle) that contains all vertices of $G$. In this paper we consider only finite simple graphs. A simple graph $G$ is called $k$-ordered (respectively, $k$-ordered hamiltonian) if, for any sequence of $k$ distinct vertices $v_1,v_2,\ldots,v_k$ of $G$, there exists a cycle (respectively, a hamiltonian cycle) in $G$ containing these $k$ vertices in the specified order. Previous results concerning cycle orderability mainly regard minimum degree and forbidden subgraph conditions that imply $k$-orderedness or $k$-ordered hamiltonicity [3, 5, 6, 7, 9]. A comprehensive survey of results can be found in [4].

A notion related to $k$-orderedness, that of $k$-edge-orderedness, has been studied in [2]. A simple graph $G$ is $k$-edge-ordered (respectively, $k$-edge-ordered eulerian) if, for any sequence of $k$ distinct edges $e_1,e_2,\ldots,e_k$ of $G$, there exists a tour (respectively, an eulerian tour, that is, a tour containing each edge of $G$) in $G$ containing these $k$ edges in the specified order. It is natural to explore analogous notions in directed graphs. A directed graph $D$ is $k$-ordered (hamiltonian) if, for any sequence of $k$ distinct vertices $v_1,v_2,\ldots,v_k$ of $D$, there exists a directed (hamiltonian) cycle in $D$ containing these $k$ vertices in the specified order. Furthermore, $D$ is $k$-edge-ordered (eulerian) if, for any sequence of $k$ distinct edges $e_1,e_2,\ldots,e_k$ of $D$, there exists a directed (eulerian) tour in $D$ containing these $k$ edges in the specified order.
As $k$-orderedness implies $(k - 1)$-connectivity, a natural question to pose is the existence of low degree $k$-ordered graphs. The question of the existence of 3-regular 4-ordered graphs was posed in [11] and answered in the affirmative in [10]. In Section 2, we answer the more general question of the existence of $(k - 1)$-regular $k$-ordered graphs for odd $k$; in particular, we exhibit an infinite family of graphs, called bracelet graphs, that are $(k - 1)$-regular and $k$-ordered hamiltonian. We also exhibit sufficient conditions for a bracelet graph to be $k$-ordered.

In Section 3 we exhibit a bound on the diameter of a $k$-ordered graph, and we show that the bound is almost tight for the bracelet graph that we constructed in Section 2. In Section 4 we continue investigating low degree $k$-ordered graphs for even $k$, and we show that for even $k$ there are no $k$-ordered bracelet graphs with minimum degree $k - 1$ and maximum degree less than $k + 2$; however, we also exhibit an infinite family of bracelet graphs with minimum degree $k - 1$ and maximum degree $k + 2$ that are $k$-ordered for even $k$. This construction partially answers the question of the existence of low degree $k$-ordered graphs for even $k$.

In Section 5 we consider $k$-orderedness properties of directed graphs, exhibiting an infinite family of $(k - 1)$-diregular graphs that are $k$-ordered hamiltonian. In Sections 6 and 7 we establish a relation between connectivity and $k$-(edge-)orderedness in undirected as well as directed graphs. We conclude our paper by posing open questions.

## 2 2k-regular $(2k + 1)$-ordered hamiltonian graphs

As observed in [11], a $k$-ordered graph $G$ is also $(k - 1)$-connected, and hence has minimum degree at least $k - 1$. The question of the existence of an infinite family of 3-regular 4-ordered graphs was raised in [11] and answered in [10] by constructing such a family. More generally, we are interested in whether there exists an infinite family of $(k - 1)$-regular $k$-ordered graphs. In this section we answer this question in the case where $k$ is odd, exhibiting an infinite family of $(k - 1)$-regular $k$-ordered hamiltonian graphs for all odd $k \geq 3$.

We call a graph $G$ a bracelet graph if its vertex set $V$ can be partitioned into $V_1 \cup V_2 \cup \cdots \cup V_m$, $m \geq 3$, with $V_i$ nonempty for all $i \in [m]$ (we denote the set $\{1, 2, \ldots, m\}$ by $[m]$), such that $v$ is adjacent to $u$ in $G$ if and only if $v \in V_i$ and $u \in V_j$ and $i - j \equiv 1$ or $-1 \pmod{m}$. We call $V_i$, for $i \in [m]$, a part of $G$, and denote its cardinality by $|V_i|$. We say that two parts $V_i$ and $V_j$ are adjacent if $i - j \equiv 1$ or $-1 \pmod{m}$. We also say that parts $V_i$ and $V_j$ are at distance $d$ if there is a path from a vertex in $V_i$ to a vertex in $V_j$ such that it contains $d$ edges and there are no 2 vertices on the path from the same part. Note that as bracelet graphs are “cyclic” there are two options for the distance between two parts; in general, it will be clear from the context which of the two distances we mean.

Throughout this paper we will frequently want to construct a cycle or path through vertices in a specified order. We will refer to these specified vertices as marked vertices. We also use the idea of free vertices in the course of the paper; we shall define free vertices in the statement of Lemma 1, which we will use for proving Theorem 2.1.

Let $G_{k,2m}$ be a bracelet graph with parts $V_1, V_2, \ldots, V_{2m}$, $m \geq 2$, such that $|V_i| = k$ for $i \in [2m]$. It is clear that $G$ is simple and 2$k$-regular by construction.

**Lemma 1.** Given $2k + 1$ marked vertices $v_1, v_2, \ldots, v_{2k+1}$ in $G_{k,2m}$, there exists a set of $2m$ vertices, which we call free vertices, satisfying the following two properties: (i) there is exactly one free vertex in each part of $G_{k,2m}$, and (ii) there exists some $i$ such that the marked vertices $v_i$ and
$v_{i+1}$ are in the set of free vertices (indices taken modulo $2k+1$) and no other marked vertices are in the set of free vertices.

Proof. If each part of $G_{k,2m}$ contains at most $k - 1$ marked vertices let $B_1$ any part that contains a marked vertex, and if there is exactly one part that has $k$ marked vertices, let $B_1$ be that part. Take a vertex $v_i$ from $B_1$ such that $v_{i+1} \notin B_1$. Then we can take vertices $v_i, v_{i+1}$, and one unmarked vertex from each of the parts not containing $v_i$ or $v_{i+1}$ as the free vertices.

If there are exactly two parts, $B_1$ and $B_2$, that have $k$ marked vertices, then it is not hard to see that there are two consecutive vertices $v_i$ and $v_{i+1}$ (indices taken modulo $2k + 1$) such that $v_i \in B_1$ and $v_{i+1} \in B_2$. Then we can take vertices $v_i, v_{i+1}$, and one unmarked vertex from each of the parts not containing $v_i$ or $v_{i+1}$ as the free vertices. This completes the proof.

Theorem 2.1. For every $k \geq 1$, there exists an infinite family of $2k$-regular graphs that are $(2k+1)$-ordered hamiltonian.

Proof. We prove that the bracelet graphs $G_{k,2m}$ introduced above are $(2k+1)$-ordered hamiltonian for $k \geq 1$ and $m \geq 2$. In fact, we will prove more: given any $2k+1$ vertices $v_1, v_2, \ldots, v_{2k+1}$ in $G_{k,2m}$, there exists a hamiltonian cycle $H_{k,2m}$ of $G_{k,2m}$ that traverses the vertices in order, and satisfies the following condition ($\star$): for any two adjacent parts $B_1$ and $B_2$ of $G_{k,2m}$, there exists an edge of $H_{k,2m}$ with one vertex in each of $B_1$ and $B_2$. We will proceed by induction on $k$.

Base case: $k = 1$.

It is clear that $G_{1,2m}$ for $m \geq 2$ is just a cycle, and it follows that $G_{1,2m}$ is 3-ordered hamiltonian; furthermore we can take $H_{1,2m} = G_{1,2m}$.

Inductive step.

Suppose that $G_{k-1,2m}$ is $(2k-1)$-ordered hamiltonian for $m \geq 2$ and given $2k-1$ vertices in $G_{k-1,2m}$ there is a hamiltonian cycle $H_{k-1,2m}$ satisfying condition ($\star$). Consider the $2k+1$ marked vertices $v_1, v_2, \ldots, v_{2k+1}$ through which we wish to construct a hamiltonian cycle. By Lemma 1 it is possible to find $2m$ free vertices in $G_{k,2m}$, one in each part. Without loss of generality we can suppose that the two marked vertices among the $2m$ free vertices are $v_{2k}$ and $v_{2k+1}$. Note that the graph induced by the $2m$ free vertices is a cycle, $C$, and the graph induced on the vertices of $G_{k,2m}$ without the free vertices is (isomorphic to) $G_{k-1,2m}$. Therefore, by the induction hypothesis, there exists a hamiltonian cycle $H_{k-1,2m}$ through the $2k-1$ marked vertices $v_1, v_2, \ldots, v_{2k-1}$ in $G_{k-1,2m}$ satisfying condition ($\star$).

We show that given $v_1, v_2, \ldots, v_{2k+1}$ in $G_{k,2m}$ there is a hamiltonian cycle $H_{k,2m}$ containing the $2k+1$ vertices in the specified order, and such that for any two adjacent parts $B_1$ and $B_2$ of $G_{k,2m}$ there exists an edge of $H_{k,2m}$ with one vertex in each of $B_1$ and $B_2$. This will also show, in particular, that $G_{k,2m}$ is $(2k+1)$-ordered hamiltonian for $m \geq 2$. We will examine cases depending on the positions of the $2k+1$ specified vertices and show how to construct the desired hamiltonian cycle in each case.

Case 1. Suppose $v_{2k-1}$ and $v_{2k}$ are in different parts. In this case the hamiltonian cycle $H_{k,2m}$ in $G_{k,2m}$ is as follows. Follow $H_{k-1,2m}$ in $G_{k-1,2m}$ from $v_1$ until reaching $v_{2k-1}$. If $v_{2k}$ is in a part adjacent to the part of $v_{2k-1}$, go to $v_{2k}$ from $v_{2k-1}$ and continue going to the free vertices in the not yet visited adjacent parts along the cycle $C$ so that we reach $v_{2k+1}$. If $v_{2k}$ is not in a part adjacent to the part containing $v_{2k-1}$, then continue going to the free vertices in the not yet visited adjacent parts along the cycle $C$ so that we reach $v_{2k}$ first and then $v_{2k+1}$. In both cases (whether or not $v_{2k}$ is in a part adjacent to the part containing $v_{2k-1}$) continue along $C$ after meeting $v_{2k+1}$ until reaching the free vertex of the part containing $v_{2k-1}$. After this, go to the vertex that is adjacent
to \(v_{2k-1}\) in \(H_{k-1,2m}\) when going from \(v_{2k-1}\) to \(v_1\), and continue on \(H_{k-1,2m}\) until \(v_1\). It is clear that the hamiltonian cycle \(H_{k,2m}\) that we have constructed has the property that for any two adjacent parts of \(G_{k,2m}\) there exists an edge of \(H_{k,2m}\) with one vertex in each of the two adjacent parts.

**Case 2.** Suppose \(v_{2k-1}\) and \(v_{2k}\) are in the same part.

Let the part containing \(v_{2k-1}\) and \(v_{2k}\) be \(B_1\), and the part containing \(v_{2k+1}\) be \(B_2\). Note that \(B_1 \neq B_2\) since \(v_{2k}\) and \(v_{2k+1}\) are in different parts. Let \(u\) be the vertex adjacent to \(v_{2k-1}\) in the traversal of \(H_{k-1,2m}\) from \(v_{2k-1}\) to \(v_1\).

**Case 2.1.** Suppose \(u \notin B_2\). Follow \(H_{k-1,2m}\) in \(G_{k-1,2m}\) from \(v_1\) to \(v_{2k-1}\). From \(v_{2k-1}\) go to an unmarked free vertex in the part containing \(u\). From this free vertex go to \(v_{2k}\), and after this continue going to the free vertices in the not yet visited adjacent parts along \(C\). During this we meet \(v_{2k+1}\), and continue along \(C\) until we are at the free vertex of the part adjacent to the part containing \(u\). Go to \(u\) and continue on \(H_{k-1,2m}\) until \(v_1\). The obtained hamiltonian cycle is \(H_{k,2m}\).

**Case 2.2.** Suppose \(u \in B_2\). In this case, \(B_1\) and \(B_2\) are adjacent. Follow \(H_{k-1,2m}\) in \(G_{k-1,2m}\) from \(v_1\) to \(v_{2k-1}\). From \(v_{2k-1}\) go to an unmarked free vertex in the adjacent part that is not \(B_2\). From here go to \(v_{2k}\), then to \(v_{2k+1}\), then to the free vertex in the next adjacent part and back to \(u\). Continue on \(H_{k-1,2m}\) until reaching \(v_1\). When \(m = 2\) this is a hamiltonian cycle in \(G_{k,2m}\), but if \(m > 2\), this is not a hamiltonian cycle. We now show that we can reroute the path from \(v_1\) to \(v_{2k-1}\) so that we pick up all the missing vertices.

Indeed, note the following. If there is an edge \(ab\) in the hamiltonian cycle \(H_{k-1,2m}\) in \(G_{k-1,2m}\) and \(c\) and \(d\) are free unmarked vertices, such that \(a\) and \(c\) are in the same part and \(b\) and \(d\) are in the same part, then replacing edge \(ab\) by edges \(ad, dc\), and \(cb\), preserves the ordering, and includes \(c\) and \(d\) in the cycle. Call this operation of rerouting \(\alpha\).

By the inductive hypothesis, \(H_{k-1,2m}\) in \(G_{k-1,2m}\) is such that for any two adjacent parts \(B_1\) and \(B_2\) in \(G_{k-1,2m}\) there exists an edge of \(H_{k-1,2m}\) with one vertex in each of \(B_1\) and \(B_2\). Because the number of parts is even, we can pair up adjacent parts (without the part containing \(v_{2k}\), the part containing \(v_{2k+1}\) and the two parts adjacent to these) and perform the rerouting operation \(\alpha\) as explained in the preceding paragraph for the \(m - 2\) part pairs. The hamiltonian path \(H_{k,2m}\) satisfies condition \((*)\), concluding the proof.

\[\square\]

**Corollary 2.2.** The graphs \(G_{k,m}\) are \((2k + 1)\)-ordered for all \(m \geq 4\).

**Proof.** Note that in the proof of Theorem 2.1 the assumption about the even number of parts was used only in Case 2.2 where we rerouted the cycle using \(\alpha\). To prove \((2k + 1)\)-orderedness, we do not require the rerouting, thus \(G_{k,m}\) is \((2k + 1)\)-ordered for all \(m \geq 4\). \[\square\]

**Theorem 2.3.** Any bracelet graph with at least 4 parts, at least \(k\) vertices in each part, and at least \(2k + 1\) total vertices in every pair of parts at distance 2 is \((2k + 1)\)-ordered.

**Proof.** Let \(G\) be a bracelet graph with parts \(V_1, V_2, \ldots, V_m\), such that each part has at least \(k\) vertices and there are at least \(2k + 1\) vertices in any two parts at distance 2. Let \(v_1, v_2, \ldots, v_{2k+1}\) be any \(2k + 1\) specified vertices. We show that there exists a cycle in \(G\) containing \(v_1, v_2, \ldots, v_{2k+1}\) in this order, and therefore \(G\) is \((2k + 1)\)-ordered.

We will prove the statement by induction on \(k\).

**Base case:** \(k = 1\). It is not hard to see that any bracelet graph with at least 1 vertex per part and at least 3 vertices in any two parts at distance 2 is 3-ordered.

**Inductive step.**
**Case 1.** Suppose each part contains at most $k$ marked vertices. Let $V'_i \subset V_i$ for all $i \in [m]$ such that each $V'_i$ contains $k$ vertices, and all of the marked vertices are in $V' = V'_1 \cup V'_2 \cup \cdots \cup V'_m$. As the graph induced by $V'$ is isomorphic to $G_{k,m}$, it follows by Corollary 2.3 that we can find the desired cycle.

**Case 2.** Suppose there is a part $B$ that contains $k + l$ marked vertices, $l \geq 1$. As there are a total of $2k + 1$ marked vertices, there can be only one such part.

**Case 2.1.** Suppose $B$ contains $2k + 1$ marked vertices. As the two adjacent parts contain at least $2k + 1$ vertices, it is not hard to see that there is a cycle containing the $2k + 1$ vertices in the specified order.

**Case 2.2.** Suppose $B$ contains fewer than $2k + 1$ marked vertices. Then there exists a different part $B'$ such that $v_i \in B$ and $v_{i+1} \in B'$ (indices taken modulo $2k + 1$) for some $i$. Without loss of generality $v_{2k} \in B$ and $v_{2k+1} \in B'$. In all of the other parts choose one vertex that is not marked (note that this is possible as all of the other parts contain at most $k - 1$ marked vertices), and call it a free vertex. Consider the graph $H$ induced on the vertices of $G$ without $v_{2k}, v_{2k+1}$, and the free vertices. This graph is $(2k-1)$-ordered by the inductive hypothesis. Let $D$ be the cycle containing vertices $v_1, v_2, \ldots, v_{2k-1}$ in this order in $H$. Let $C$ be the cycle induced on $v_{2k}, v_{2k+1}$, and the free vertices.

Now we show how to construct a cycle in $G$ containing the vertices $v_1, v_2, \ldots, v_{2k+1}$ in this order.

**Case 2.2.1.** Suppose $v_{2k-1}$ and $v_{2k}$ are in different parts. In this case the cycle in $G$ is as follows. Follow $D$ in $H$ from $v_1$ to $v_{2k-1}$. From $v_{2k-1}$ go to a free vertex $f$ in one of the adjacent parts, choosing the adjacent part so that either $f = v_{2k}$ or $f$ is unmarked. If $f \neq v_{2k}$, then continue going to the free vertices in the not yet visited adjacent parts along the cycle $C$ so that we reach $v_{2k}$ first and then $v_{2k+1}$. Continue along $C$ after meeting $v_{2k+1}$ until we reach the free vertex of the part containing $v_{2k-1}$. After this, go to the vertex that is adjacent to $v_{2k-1}$ in $D$ when going from $v_{2k-1}$ to $v_1$, and continue along $D$ until $v_1$.

**Case 2.2.2.** Suppose $v_{2k-1}$ and $v_{2k}$ are in the same part. Denote the part containing $v_{2k-1}$ and $v_{2k}$ by $B_1$, and denote the part containing $v_{2k+1}$ by $B_2$. Let $u$ be the end vertex of the edge incident to $v_{2k-1}$ that is an edge in the segment of $D$ from $v_{2k-1}$ to $v_1$.

**Case 2.2.2.1** Suppose $u \not\in B_2$. Follow $D$ in $H$ from $v_1$ to $v_{2k-1}$. From $v_{2k-1}$ go to an unmarked free vertex in the part containing $u$. From the unmarked free vertex in the part containing $u$ go to $v_{2k}$, and after this continue going to the free vertices in the not yet visited adjacent parts along $C$. During this we meet $v_{2k+1}$, and we continue along $C$ until we reach the free vertex of the part adjacent to the part of $u$. Go to $u$ and continue on $D$ until $v_1$.

**Case 2.2.2.2.** Suppose $u \in B_2$. Then $B_1$ and $B_2$ are adjacent. Follow $D$ in $H$ from $v_1$ to $v_{2k-1}$. From $v_{2k-1}$ go to an unmarked free vertex in the adjacent part that is not $B_2$. From here go to $v_{2k}$, then to $v_{2k+1}$, then to the free vertex in the adjacent part and back to $u$. Continue on $D$ until $v_1$.

**Corollary 2.4.** Any bracelet graph with at least 4 parts and at least $k + 1$ vertices in each part is $(2k + 1)$-ordered.

In Corollary 2.5 we cannot replace the requirement of each part having at least $k + 1$ vertices with each part having at least $k$ vertices. For a a subset $S$ of vertices let $N(S)$ denote the set of vertices adjacent to some vertex in $S$, not including vertices in $S$. If $v_1, v_2, \ldots, v_{2k+1}$ are independent vertices (there are no edges between any of them), and if there is a cycle containing $v_1, v_2, \ldots, v_{2k+1}$ in this
order then \(|N(\{v_1, v_2, \ldots, v_{2k+1}\})| \geq 2k + 1\). Consider the example of a bracelet graph \(G\) with three adjacent parts \(B_1, B_2,\) and \(B_3\) such that \(|B_1| = |B_3| = k\) and \(|B_2| = 2k + 1\). Specify \(v_1, v_2, \ldots, v_{2k+1}\) to be in \(B_2\). Then \(v_1, v_2, \ldots, v_{2k+1}\) are independent, and then \(|N(\{v_1, v_2, \ldots, v_{2k+1}\})| = 2k\), which shows that such a bracelet graph cannot be \((2k + 1)\)-ordered. We also cannot weaken the condition in Theorem 2.3 that every two parts at distance 2 have at least \(2k + 1\) vertices total.

**Lemma 2.** Given two vertices \(u\) and \(v\) in a bracelet graph \(G\) with an even number of parts, the parity of the number of vertices on any path from \(u\) to \(v\) is the same. Moreover, any cycle in a bracelet graph \(G\) with an even number of parts has even length.

Although Lemma 2 is an easy observation, it will be a convenient tool for proving non-existence of bracelet graphs with certain properties.

**Corollary 2.5.** There is no \(k\)-ordered Hamiltonian bracelet graph \(G\) with an even number of parts and an odd number of vertices.

### 3 The diameter of a \(k\)-ordered graph

In this section we give an upper bound on the diameter of \(k\)-ordered graphs, and we study the tightness of this bound. It has been observed by Denis Chebikin (personal communication) that if \(n\) is the number of vertices in a 4-ordered graph, then the diameter of the graph is at most \(\frac{n}{4} + 2\). Indeed, as shown in the following proposition and corollary, a \(k\)-ordered graph has diameter not more than roughly \(\frac{n}{2k}\).

**Proposition 3.1.** Given a \(2k\)-ordered graph \(G\) on \(n\) vertices, the diameter \(d\) of \(G\) is at most \(\left\lfloor \frac{n-3}{2k} \right\rfloor + 2\).

**Proof.** Take two vertices \(a\) and \(b\) of \(G\) at distance \(d\). As \(G\) is \(2k\)-ordered, it is \((2k - 1)\)-connected, so in particular both \(a\) and \(b\) have at least \(2k - 1\) neighbors. Let \(a_1, a_2, \ldots, a_{k-1}\) be \(k - 1\) distinct neighbors of \(a\), and let \(b_1, b_2, \ldots, b_{k-1}\) be \(k - 1\) distinct neighbors of \(b\). Note that the distance between \(a_i\) and \(b\) is at least \(d - 1\) for all \(i \in [k-1]\), and likewise the distance between \(b_i\) and \(a\) is at least \(d - 1\) for all \(i \in [k-1]\). Furthermore, the distance between \(a_i\) and \(b_j\) is at least \(d - 2\) for all \(i, j \in [k-1]\). Consider any cycle containing the \(2k\) vertices \(a, b_1, a_1, b_2, a_2, \ldots, b_{k-1}, a_{k-1}, b\) in the specified order. It contains at least \(2k + (d - 2) + (2k - 3)(d - 3) + (d - 2) + (d - 1)\) vertices, so \(n \geq 2kd - 4k + 4\), and \(d \leq \frac{n}{2k} + 2 - \frac{4}{2k}\). The result follows.

**Corollary 3.2.** Given a \((2k + 1)\)-ordered graph \(G\) on \(n\) vertices, the diameter \(d\) of \(G\) does not exceed \(\left\lfloor \frac{n-3}{2k} \right\rfloor + 2\).

**Proof.** The statement follows from Proposition 3.1 as a \((2k + 1)\)-ordered graph is \(2k\)-ordered as well.

**Theorem 3.3.** There exist infinitely many \((2k + 1)\)-ordered graphs on \(n\) vertices with diameter \(\left\lfloor \frac{n-3}{2k} \right\rfloor + 1\) for all \(k \geq 2\).

**Proof.** Consider graphs \(G_{k,2m}\) as constructed in Section 2. In this case \(n = 2mk\), and it is not hard to see that the diameter of \(G_{k,2m}\) is \(m = \left\lfloor \frac{2mk - 3}{2k} \right\rfloor + 1\).
4 Low degree $2k$-ordered graphs

In this section we focus on low degree $2k$-ordered graphs. Since $2k$-orderedness of a graph $G$ implies $(2k-1)$-connectivity, and thus a $2k$-ordered graph $G$ has minimum degree at least $2k-1$, a question of interest is the existence of $(2k-1)$-regular $2k$-ordered graphs, or low degree $2k$-ordered graphs in general. Since $(2k+1)$-ordered graphs are $2k$-ordered as well, it follows from Theorem 2.1 that there is an infinite family of $2k$-regular graphs that are $2k$-ordered. In this section we present a stronger result (Theorem 4.3), exhibiting an infinite family of $2k$-ordered bracelet graphs such that the minimum degree of these graphs is $2k-1$, and the maximum degree is $2k+2$. We also show that this result is the best possible for bracelet graphs.

Note that the construction in Theorem 2.1 is specific to $2k$-regular graphs. For example, consider an analogue of the graphs $G_{k,m}$ that are $(2k-1)$-regular. Note that in this case the number of vertices in all the parts cannot be the same. Thus, consider the graph $H_{k,4m}$ to be a graph with $4m$ parts having a repeating pattern of two parts with $k-1$ vertices followed by two parts with $k$ vertices. Note that $H_{k,4}$ is the bipartite graph $K_{2k-1,2k-1}$, which was shown in [11] to be $2k$-ordered hamiltonian. However, if $m > 1$, then $H_{2,4m}$ is not $4$-ordered, as it contains a square (4-cycle), and by [10] a 3-regular 4-ordered graph on more than 6 vertices cannot contain a square.

In fact, it is easy to see that $H_{k,4m}$ is not $2k$-ordered for any $m > 1$, because $2k$-orderedness implies $(2k-1)$-connectivity and the deletion of two non-adjacent parts of $k-1$ vertices would disconnect $H_{k,4m}$.

Lemma 3. Let $G$ be a $2k$-ordered graph. If $s \leq k$, then there exists no subset $V_1$ of the vertices of $G$ such that $|V_1| = s$, $|V \setminus (N(V_1) \cup V_1)| \geq s$, and $|N(V_1)| < 2s$.

Proof. Suppose that $G$ is a graph such that there is a subset $V_1$ of the vertices of $G$ with $|V_1| = s$, $|V \setminus (N(V_1) \cup V_1)| \geq s$, and $|N(V_1)| < 2s$. Specify $2s$ vertices $v_1, \ldots, v_{2s}$ such that for odd $i$, the $v_i$ are in $V_1$, and for even $i$, the $v_i$ are in $V \setminus (N(v_1) \cup V_1)$. Then it would be impossible to have a cycle containing them in the specified order, because each vertex in $V_1$ would have to be adjacent to 2 distinct vertices in $N(V_1)$. Thus, $G$ could not have been $2s$-ordered, nor $2k$-ordered for $k \geq s$. □

Corollary 4.1. Given a $2k$-ordered bracelet graph $G$ with more than 5 parts, there exists no part $B$ in $G$ such that $|B| \leq k$ and $N(B) < 2|B|$, and there also exists no part $B'$ in $G$ such that $|B'| > k$ and $N(B') < 2k$.

Proof. Consider 6 consecutive parts $B_1, B_2, \ldots, B_6$ of a $2k$-ordered bracelet graph $G$. Since $2k$-orderedness implies $(2k-1)$-connectivity, the total number of vertices in parts $B_4$ and $B_6$ is at least $2k-1$.

Suppose that there exists a part $B$ in $G$ such that $|B| \leq k$ and $N(B) < 2|B|$, and let $B = B_2$ without loss of generality. Taking $V_1$ as described in Lemma 3 to be the vertices in $B_2$ it follows that $G$ cannot be $2k$-ordered, contradicting our assumption.

Suppose that there exists a part $B'$ in $G$ such that $|B'| > k$ and $N(B') < 2k$, and let $B' = B_2$, without loss of generality. Taking $V_1$ as described in Lemma 3 to be some $k$ vertices in $B_2$ it follows that $G$ cannot be $2k$-ordered, contradicting our assumption. □

Proposition 4.2. There is no $2k$-ordered bracelet graph with more than 6 parts that has minimum degree $2k-1$ and maximum degree less than $2k+2$.
Proof. Suppose that there exists a $2k$-ordered bracelet graph $G$ with more than 6 parts that has minimum degree $2k - 1$ and maximum degree less than $2k + 2$.

Claim. There is no part containing fewer than $k - 1$ vertices.

Let $B$ be a part with the minimum number of vertices, $b$, in bracelet graph $G$. Let $B, B_1, B_2, B_3, B_4$ be a sequence of 5 adjacent parts. Then $B_2$ contains $2k + 1 - b$, $2k - b$ or $2k - 1 - b$ vertices, while $B_1$ contains $b, b + 1$ or $b + 2$ vertices (since $B$ was a part with the minimum number of vertices). Thus, because $G$ is $(2k - 1)$-connected and there are at least 6 parts, $b + (b + 2) \geq 2k - 1$, implying that $b \geq k - 1$.

Therefore, since there exists a vertex in $G$ with degree $2k - 1$, there exist parts $B'$ and $B'_2$ distance 2 apart, with $k - 1$ and $k$ vertices respectively. Applying Corollary 4.1 to $B'$, the part between $B'$ and $B'_2$, we see that $B'_1$ also has $k - 1$ vertices. Also, since the graph is $(2k - 1)$-connected, every part other than $B'$ and $B'_1$ must have at least $k$ vertices.

Suppose part $C$ contains $k$ vertices, and $C$ is not $B'_2$ or the part adjacent to $B'$ (that is not $B'_1$). Let $C_1$ be the part following $C$, in the same direction that $B'_1$ follows $B'$. Choose $v_2, v_3, \ldots, v_{2k-2}$ in $B'$, choose $v_1, v_3, \ldots, v_{2k-1}$ in $B'_2$, and choose $v_{2k}$ in $C_1$. Then it is not hard to see that there can be no cycle visiting the $2k$ vertices in order, as there could be no path leading from $v_{2k}$ to $v_1$ once we have traversed $v_1, v_2, \ldots, v_{2k}$. Thus, the only parts that might have fewer than $k + 1$ vertices are $B', B'_1, B'_2$ and the parts adjacent to $B'$. Since there are at least 7 parts, it follows that there is a vertex with degree at least $2k + 2$, contradicting our assumptions. \hfill \Box

By arguments analogous to those in Proposition 4.2, it is not hard to see that there is no $(2k - 1)$-regular $2k$-ordered bracelet graph on more than 4 parts, and that there is no $2k$-ordered bracelet graph with minimum degree $2k - 1$ and maximum degree $2k$ on more than 5 parts. Also, the $k$-orderedness of bracelet graphs on at most 6 parts can be easily studied. In the following theorem we show that Proposition 4.2 is the best possible by exhibiting an infinite family of $2k$-ordered graphs with minimum degree $2k - 1$ and maximum degree $2k + 2$.

**Theorem 4.3.** There exists an infinite family of $2k$-ordered graphs $P_{k,m}$ with minimum degree $2k - 1$ and maximum degree $2k + 2$.

Proof. For $m \geq 5$, let $P_{k,m}$ be the bracelet graph with $m$ parts, $V_1, V_2, \ldots, V_m$, satisfying $|V_1| = |V_2| = k - 1$, $|V_3| = k$, and $|V_i| = k + 1$ for all $i \geq 3$. We will show by induction on $k$ that the graphs $P_{k,m}$ are $2k$-ordered for $k \geq 2$ and $m \geq 5$.

Base case: $k = 2$.

Case 1. Suppose one of the parts contains 3 of the 4 marked vertices. Call this part $B$. Without loss of generality we may assume that it contains vertices $v_1, v_2,$ and $v_3$. As there is a part adjacent to $B$ with 3 vertices, it is possible to go from $v_1$ to $v_3$ through $v_2$, meeting two vertices from that adjacent part, and then it is possible to go from $v_3$ by going through all the parts to $v_4$ and on to $v_1$.

Case 2. Suppose one of the parts contains 2 marked vertices. Call such a part $B$. Without loss of generality we may assume that it contains vertices $v_1$ and $v_2$.

Case 2.1. Suppose $B$ contains vertices $v_1$ and $v_2$.

Using one vertex (that is not $v_3$ or $v_4$) from an adjacent part with 3 vertices we can go from $v_1$ to $v_2$, and then, regardless of where $v_3$ and $v_4$ are, it is not hard to see that the desired cycle exists.

Case 2.2. Suppose $B$ contains vertices $v_1$ and $v_3$. We can divide this case up into cases depending on whether $v_2$ or $v_4$ are in parts with 1 or 3 vertices, or whether they are both in the part with 2 vertices. We can easily find the desired cycles in each case.
Case 3. Suppose all four vertices are in different parts. Let \( v_1 \in V_{i_1}, v_2 \in V_{i_2}, v_3 \in V_{i_3}, \) and \( v_4 \in V_{i_4}. \) Without loss of generality we can suppose that \( 0 < i_2 - i_1 < i_3 - i_1 < i_4 - i_1 \) or \( 0 < i_2 - i_1 < i_4 - i_1 < i_3 - i_1 \) (as by symmetry we can rotate and reflect an ordered cycle), where we consider subtraction modulo \( m \) taking results between 0 and \( m - 1. \) In the case \( 0 < i_2 - i_1 < i_3 - i_1 < i_4 - i_1 \) it is clear that there is a cycle containing the four vertices in the specified order, and if \( 0 < i_2 - i_1 < i_4 - i_1 < i_3 - i_1 \) analysis shows the existence of the desired cycle.

Inductive step. Suppose the claim is true for all numbers less than \( k. \) We shall show that \( P_{k,m} \) is still \( 2k \)-ordered for \( m \geq 5. \)

Suppose there is no part with all vertices marked or there is exactly one part with all vertices marked. Then we can find two different parts containing two consecutive vertices, \( v_{2k-1} \) and \( v_{2k} \) without loss of generality, such that all other parts have a free vertex. By arguments analogous to those in Theorems 2.1 and 2.3 we can show how to find a cycle in \( P_{k,m} \) using the cycle in \( P_{k-1,m}. \)

If there are exactly two parts with all vertices marked, then one of the following cases occurs:

(i) both parts have exactly \( k - 1 \) vertices,

(ii) one part has exactly \( k - 1 \) vertices and the other has exactly \( k \) vertices, or

(iii) one of the parts has exactly \( k - 1 \) vertices and the other has exactly \( k + 1 \) vertices.

If either case (ii) or case (iii) occurs, then there exist marked vertices \( v_i \) and \( v_{i+1} \), one in either part with all vertices marked. The result then follows from arguments similar to those in Theorems 2.1 and 2.3.

If, however, the two parts with all vertices marked both have \( k - 1 \) vertices and there are no two vertices in them adjacent (if there are, then we are done by arguments as before), then, without loss of generality, the two parts with all vertices marked contain the vertices \( v_2, v_3, \ldots, v_k \) and \( v_{k+2}, v_{k+3}, \ldots, v_{2k} \) respectively. It is not hard to see that in this case there exists a cycle containing \( v_1, v_2, \ldots, v_{2k} \) in this order, regardless of where \( v_1 \) and \( v_{k+1} \) are.

\[ \square \]

5 \( k \)-Ordered directed graphs

In this section we address the existence of low degree \( k \)-ordered directed graphs. This inquiry is motivated by the analogous questions for undirected graphs.

Consider a directed graph \( D, \) and denote its set of vertices by \( V(D). \) A directed graph \( D \) is said to be strongly connected if given any two vertices \( u \) and \( v \) in \( D, \) there exists a directed path from \( u \) to \( v. \) A vertex cut of a digraph \( D \) is a set \( S \subset V(D) \) such that \( D-S \) is not strongly connected.

**Proposition 5.1.** If a directed graph \( D \) is \( k \)-ordered, then every vertex cut has at least \( k - 1 \) vertices.

The proof is analogous to the proof of the undirected case in [11].

**Corollary 5.2.** If a directed graph \( D \) is \( k \)-ordered, then for every vertex \( v \) we have \( \text{indeg}(v), \text{outdeg}(v) \geq k - 1. \)

**Theorem 5.3.** For every \( k \geq 2 \) there exists an infinite family \( \mathcal{F}_k \) of \((k - 1)\)-diagonal graphs that are \( k \)-ordered hamiltonian.

**Proof.** Given \( k, \) consider the undirected bracelet graphs \( G_{k-1,l}, \) \( l \geq 3, \) as defined in Section 2. We define the directed graph \( \overrightarrow{G_{k-1,l}} \) on the same vertex set with parts \( V_1, V_2, \ldots, V_l, \) each with \( k - 1 \)
vertices. Edge \( \overrightarrow{uv} \) is in \( \overrightarrow{G_{k-1,l}} \) if and only if \( u \in V_i \) and \( v \in V_{i+1} \) where indices are taken modulo \( l \).

The graph \( \overrightarrow{G_{k-1,l}} \) is \((k - 1)\)-diregular, and we now prove that it is \(k\)-ordered hamiltonian.

Consider marked vertices \( v_1, v_2, \ldots, v_k \). We will construct a hamiltonian cycle containing these vertices in this order. It is easy to see that there have to be two consecutive vertices among \( v_1, v_2, \ldots, v_k \), without loss of generality \( v_1 \) and \( v_2 \), that are in different parts. We can also suppose without loss of generality that \( v_1 \in V_1 \). Write the vertices of \( \overrightarrow{G_{k-1,l}} \) as a grid where the \( i^{th} \) column contains the vertices in the \( i^{th} \) part for \( i \in [l] \), and where the first row contains the two marked vertices \( v_1 \) and \( v_2 \), and where the \( j^{th} \) row contains \( v_j \) for \( j = 2, 3, \ldots, k - 1 \). Then a hamiltonian cycle containing \( v_1, v_2, \ldots, v_k \) in this order is as follows. Start at \( v_1 \) going from left to right across the first row, then from the rightmost element in the first row go to the leftmost element in the second row, and so on until reaching the lower rightmost vertex, from which we close off the hamiltonian cycle by going back to \( v_1 \).

\[ \square \]

6 Connectivity, linkage, and \(k\)-edge-orderedness

Connectivity, linkage and \(k\)-orderedness appear to be related concepts. As noted in [4], a \(k\)-linked graph \( G \) is also \(k\)-ordered. Let \( f(k) \) be the minimum connectivity of a graph \( G \) that implies \(k\)-orderedness; the existence of the function \( f(k) \) has been shown in [4]. By a result in [11], \(2k\)-connected graphs are \(k\)-linked, and thus are \(k\)-ordered as well, leading to the upper bound \( f(k) \leq 22k \) observed in [4].

It is natural to pose analogous questions for edge-orderedness and directed graphs. In this section we consider edge-orderedness, while in the following sections we consider directed graphs.

A graph \( G \) is said to be weakly \(k\)-linked if, given \(2k\) vertices (not necessarily distinct) \( s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k \), there exist edge-disjoint paths from \( s_i \) to \( t_i \), for \( i \in [k] \).

**Lemma 4.** If a graph \( G \) is weakly \(2k\)-linked, then \( G \) is \(k\)-edge-ordered.

**Proof.** Consider distinct edges \( e_1, \ldots, e_k \). Let \( v_i \) and \( u_i \) be the end vertices of \( e_i \). As \( G \) is weakly \(2k\)-linked, there are edge-disjoint paths from \( v_1 \) to \( u_1 \), \( v_1 \) to \( u_2 \), \ldots, \( v_k \) to \( u_k \), and \( u_k \) to \( v_1 \). If for all \( i \) the path we chose between \( v_i \) and \( u_i \) is the edge \( v_i u_i \), then we are done. If the path from \( v_i \) to \( u_i \) is not the edge \( e_i \), but the edge \( e_i \) has not been used in any path, then we can just replace the path from \( v_i \) to \( u_i \) with the edge \( e_i \). On the other hand, if the edge \( e_i \) has been used in some other path, then it has been used by exactly one of them, say \( p \). In this case we can replace the edge \( e_i \) in path \( p \) by the path that was between \( v_i \) and \( u_i \), and we can replace the path from \( v_i \) to \( u_i \) with the edge \( e_i \). Repeating this process as necessary, we obtain a tour containing \( e_1, \ldots, e_k \) in this order, and thus \( D \) is \(k\)-edge-ordered.

Let \( g(k) \) be the minimum edge-connectivity of a graph \( G \) that implies \(k\)-edge-orderedness of \( G \).

**Proposition 6.1.** The upper bound \( g(k) \leq 2k + 2 \) holds.

**Proof.** It is known that \((k + 2)\)-edge-connectivity implies weakly \(k\)-linked [8]. Thus, the statement of the proposition is a corollary of Lemma 6.1 and [8].

It is easy to see that \( g(k) \geq k - 1 \).
Lemma 5. If a graph $G$ is $2k$-edge-ordered, then it is weakly $k$-linked.

Proof. Consider vertices $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$. Since $2k$-edge-ordered implies $2k$-edge-connected [2], the degree of every vertex is at least $2k$, and therefore there exist edges $s_1s_1', s_2s_2', \ldots, s_k's_k', t_1t_1', t_2t_2', \ldots$ such that

$$\{s_1', s_2', \ldots, s_k', t_1', t_2', \ldots, t_k'\} \cap \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\} = \emptyset.$$  

Then, as $G$ is $2k$-edge-ordered, it readily follows that it is also weakly $k$-linked. \hfill \Box

Lemmas 6.1 and 6.3 exhibit a relation between weak linkage and edge-orderedness, but tightness of these lemmas remains uncertain.

7 Connectivity, diameter and orderedness

In the previous section we considered the relation between connectivity and edge-orderedness in undirected graphs. In this section we pose the question: What connectivity implies $k$-(edge-ordered)ness in directed graphs?

A digraph $D$ is $k$-connected if every vertex cut has at least $k$ vertices. The minimum size of a vertex cut is the connectivity of $D$. For $S, T \subset V(D)$, let $[S, T]$ be the set of edges from $S$ to $T$. An edge cut is the set $[S, \overline{S}]$ for some nonempty $S \subset V(D)$. A digraph $D$ is $k$-edge-connected if every edge cut has at least $k$ edges. The minimum size of an edge cut is the edge-connectivity of $D$ ([3], Section 4).

It is not immediately clear that any connectivity in directed graphs implies any orderedness property. Indeed, it has been shown by Thomassen [1] that for every natural number $k$ there exists a strongly $k$-connected digraph $D_k$ containing two vertices not lying on a cycle. This implies that there is no connectivity that would guarantee even the $2$-(edge-)orderedness of a directed graph. When the diameter is small, however, we can prove some positive results.

Theorem 7.1. If a digraph $D$ is $g(k)$-edge-connected with diameter $d$ and $g(k) \geq (2k - 1)\lceil \frac{d}{2} \rceil + 1$, then $D$ is $k$-edge-ordered.

Proof. Suppose that a digraph $D$ is $g(k)$-edge-connected and it has diameter $d$. Let $e_1 = v_1u_1$, \ldots, $e_k = v_ku_k$. As $g(k) \geq 1$, there exists a directed path between $v_1$ and $u_1$ not longer than $d$. Delete the edges of this path from $D$, which decreases the edge-connectivity by at most $\lceil \frac{d}{2} \rceil$. Note that $D$ still has edge-connectivity at least $1$, and indeed the edge-connectivity will allow us to repeat the same argument $2k$ times for paths from $v_1$ to $u_1$, $u_1$ to $v_2$, \ldots, $v_k$ to $u_k$, and $u_k$ to $v_1$. At the last step the connectivity will be greater than or equal to $g(k) - (2k - 1)\lceil \frac{d}{2} \rceil \geq 1$. Therefore, if $g(k) \geq (2k - 1)\lceil \frac{d}{2} \rceil + 1$, then we can obtain an oriented tour through the vertices $v_1, u_1, \ldots, v_k, u_k$ in this order.

If for all $i$ the directed path we chose between $v_i$ and $u_i$ is the edge $v_iu_i$, then we are done. If the path from $v_i$ to $u_i$ is not the edge $v_iu_i$, but the edge $v_iu_i$ has not been used in any path, then we can just replace the path from $v_i$ and $u_i$ with the edge $v_iu_i$. On the other hand, if the edge $v_iu_i$ has been used in some other path, then it has been used by exactly one of them, say $p$. In this case we can replace the edge $v_iu_i$ in path $p$ by the path that was between $v_i$ and $u_i$ and we can replace the path from $v_i$ to $u_i$ with the edge $v_iu_i$. Repeating this process as necessary, we obtain a tour containing $e_1, \ldots, e_k$ in this order, and thus $D$ is $k$-edge-ordered. \hfill \Box
Theorem 7.2. If a digraph $D$ is $g(k)$-connected with diameter $d \geq 1$, where $g(k) \geq (k - 1)d$, then $D$ is $k$-ordered.

Proof. Suppose that a digraph $D$ is $g(k)$-connected and that it has diameter $d$. Choose marked vertices $v_1, v_2, \ldots, v_k$.

Since $g(k) \geq k - 1$, removing vertices $v_3, v_4, \ldots, v_k$ would not disconnect $D$, thus there exists a directed path from $v_1$ to $v_2$ not containing $v_3, v_4, \ldots, v_k$ and furthermore this path has length at most $d$. Delete the vertices of the path from $v_1$ to $v_2$, except $v_1$ and $v_2$. The connectivity decreases by at most $d - 1$, and we can repeat the same process $k$ times for paths from $v_1$ to $v_2$, $v_2$ to $v_3, \ldots, v_k$ to $v_1$. At the last step the connectivity will be greater than or equal to $g(k) - (k - 1)(d - 1) \geq k - 1$, as required. Therefore, if $g(k) \geq (k - 1)d$, then we can obtain a cycle through the vertices $v_1, v_2, \ldots, v_k$ in this order. This shows $k$-orderedness.

Using analogous methods, similar results can be obtained for undirected graphs. Indeed, we get a bound $g(k) \geq (2k - 1)d + 1$ for a statement analogous to Theorem 7.1 for undirected graphs, and the same bound in the analogue of Theorem 7.2.

8 Conclusion

We conclude by giving an overview of questions motivated by this paper. In Section 2 we constructed an infinite family of $2k$-regular $(2k + 1)$-ordered hamiltonian bracelet graphs, and in Section 3 we showed that there are no $2k$-ordered bracelet graphs with minimum degree $2k - 1$ and maximum degree less than $2k + 2$. We constructed an infinite family of $2k$-ordered bracket graphs with minimum degree $2k - 1$ and maximum degree $2k + 2$. The following question, however, remains open.

Question 8.1. Is there an infinite family of $2k$-ordered hamiltonian bracelet graphs with minimum degree $2k - 1$ and maximum degree $2k + 2$ for all $k \geq 2$?

In Theorem 2.3 we gave a sufficient condition for a bracelet graph to be $(2k + 1)$-ordered. Note that Theorem 2.3 only applies when each part has at least $k$ vertices. It is not hard to see from connectivity properties that any $(2k + 1)$-ordered bracelet graph with at least 5 parts has at most two parts with fewer than $k$ vertices, and if it has two such parts then they must be adjacent.

Question 8.2. What are the necessary and sufficient conditions for a bracelet graph to be $(2k + 1)$-ordered?

Naturally, one can also pose this question for $k$-ordered graphs in general.

In Section 3 we showed that a $2k$-ordered graph has diameter at most $\left\lceil \frac{n-3}{2k} \right\rceil + 2$, where $n$ is the number of the vertices of the graph. We have also shown that there exists an infinite family of $(2k + 1)$-ordered hamiltonian graphs that have diameter $\left\lceil \frac{n-3}{2k} \right\rceil + 1$. It is natural to pose the following question.

Question 8.3. Is there a $(2k + 1)$-ordered or $2k$-ordered graph that has diameter $\left\lceil \frac{n-3}{2k} \right\rceil + 2$?

Note that one can ask the analogues of these questions for directed graphs.
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