SYMMETRY AND OBSERVABLES IN INDUCED QCD

Gordon W. Semenoff and Nathan Weiss

Department of Physics, University of British Columbia,
Vancouver, British Columbia, Canada V6T 1Z1

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Abstract

We review some of the basic features of the Kazakov-Migdal model of induced QCD. We emphasize the role of $Z_N$ symmetry in determining the observable properties of the model and also argue that it can be broken explicitly without ruining the solvability of induced QCD in the infinite $N$ limit. We outline the sort of critical behavior which the master field must have in order that the model is still solvable. We also review some aspects of the $D = 1$ version of the model where the partition function can be obtained analytically.

I. INDUCING QCD

Quantum chromodynamics is presently accepted as the only viable theory of the strong interactions. It describes many of the quantitative features of the interactions of hadrons at high energies. However, at energies lower than the hadronic scale it gives only qualitative information. Part of the reason is that perturbative QCD has only one dimensionless constant, the gauge coupling which is changed by renormalization from a dimensionless constant into a mass scale. The perturbative regime is where all external momenta of Feynman diagrams are greater than this mass scale. It is thus impossible to address the infrared structure of QCD, such as details of its spectrum and low energy interactions using conventional perturbation theory.
Another parameter of QCD which could be varied is the number of colors of quarks. QCD is known to simplify somewhat in the limit where the number of colors, $N$, is large. In this limit, only planar graphs contribute to scattering amplitudes which consequently exhibit some of the qualitative features of the strong interactions. However, so far no explicit solution of QCD in dimensions greater than 2 is available in the large $N$ limit and it has thus led to very few quantitative results.

There are some features of the large $N$ limit which are particularly appealing. First, there is every indication that the large $N$ limit is quite smooth and that it exhibits confinement and dynamical chiral symmetry breaking which is found in the actual QCD for $N=3$. Also, $1/N^2 = 1/9$ is an expansion parameter which is less than one, so some large $N$ results should have reasonable accuracy. Furthermore, the parameter $1/N$ is not dependent on the mass scale, so once a solution is found it is good at both high and low energies. This might also shed some light on the relationship between lattice and continuum QCD.

Recently, Kazakov and Migdal have proposed a novel approach to the large $N$ limit which has the hope of providing an exact solution of infinite $N$ QCD. They consider induced QCD which is obtained by integrating over the scalar fields in the lattice gauge theory with the partition function

$$Z_{KM} = \int d\phi [dU] \exp \left( -N \sum_x \text{Tr} V[\phi(x)] + N \sum_{<x,y>} \text{Tr} \phi(x)U(xy)\phi(y)U^\dagger(xy) \right) \tag{1}$$

where $\phi(x)$ are $N \times N$ Hermitean matrices which reside on lattice sites $x$, $U(xy)$ are unitary $N \times N$ matrices which reside on links $<xy>$ between neighboring sites $x$ and $y$, $d\phi$ is the Euclidean integration measure for Hermitean matrices, $[dU]$ is the invariant Haar measure for integration over the unitary group $U(N)$ and $V[\phi]$ is a potential for the scalars. This model is invariant under the gauge transformations

$$\phi(x) \to \omega(x)\phi(x)\omega^\dagger(x) \tag{2}$$

$$U(xy) \to \omega(x)U(xy)\omega^\dagger(y) \tag{3}$$

where $\omega(x)$ is an element of $U(N)$. By restricting the trace of $\phi$ to zero and the determinant of $U$ to one in (2), we could also consider a model with $SU(N)$ gauge symmetry.

The partition function of the Kazakov-Migdal model can be regarded as the $1/g^2 \to \infty$ limit of lattice scalar QCD with action

$$Z = \int d\phi [dU] \exp \left( -N \sum_x \text{Tr} V[\phi(x)] + N \sum_{<x,y>} \text{Tr} \phi(x)U(xy)\phi(y)U^\dagger(xy) + \frac{N}{g^2} \sum_{\Box} \left( \text{Tr} U(\Box) + \text{Tr} U^\dagger(\Box) \right) \right) \tag{4}$$

This action differs from that of the Kazakov-Migdal model by the addition of the Wilson term,

$$\sum_{\Box} \left( \text{Tr} U(\Box) + \text{Tr} U^\dagger(\Box) \right) \tag{5}$$
The Wilson term is the trace of a product of link operators around an elementary plaquette of the lattice. This term is the naive latticization of the continuum Yang-Mills action, \( \text{Tr} F_{\mu\nu} F_{\mu\nu} \). In (3) \( \square \) denotes plaquettes of the lattice and \( U(\square) \) a product of the link operators on the links on the boundary of \( \square \).

Asymptotic freedom implies that the continuum limit of the lattice theory (3) is obtained by taking the bare coupling constant to zero,

\[
\frac{1}{g^2} \to \infty \quad (6)
\]

In fact, if instead of a lattice cutoff we had a large momentum cutoff \( \Lambda \) the bare coupling which would be necessary to insure one loop renormalizability of QCD is

\[
\frac{1}{g^2} = \frac{11}{48\pi^2} \ln(\Lambda^2/\mu^2) \quad (7)
\]

The hypothesis of the Kazakov-Migdal model is that the scalar QCD might still find a way to arrange things so that a continuum limit exists in the opposite limit, where the bare coupling constant is infinite

\[
\frac{1}{g^2} \to 0 \quad (8)
\]

in (3) they present a naive argument to show how this might be possible. They begin with QCD coupled to scalars and without a kinetic term for the gluon field. The Yang-Mills action is induced by the vacuum polarization of the scalar fields in the cutoff theory. The one-loop result is

\[
\frac{1}{g_{\text{ind}}^2} = \frac{1}{96\pi^2} \ln(\Lambda^2/m^2) \quad (9)
\]

where \( m \) is the scalar mass. This can produce the \( g^2 \) in (2) necessary to obtain a continuum limit for the gauge field sector of the theory with the ultraviolet cutoff replaced by the scalar mass if we take the mass of the scalar to be

\[
m^2 = \mu^2 \left( \frac{\Lambda^2}{\mu^2} \right)^{1/21} \quad (10)
\]

In this way, by giving up on finiteness of the scalar mass we can, at least at one-loop order, induce a renormalizable action for QCD. Of course this is only a rough argument. Higher order corrections from hard gluons will change this result significantly. They can only be compensated by some strong self-interactions of the scalar field. The resulting picture is one of a complicated, strongly interacting theory. It also requires that we have the ability to arrange that the scalar mass goes to infinity with a slower exponent than the cutoff in the continuum limit. This is possible if the scalar field theory has a second order phase transition and the accompanying critical behavior. The appealing feature of this model is that one may be able to solve it in the large \( N \) limit.

Recently we have proposed a slight modification of this idea [1]. The solvability of the large \( N \) limit comes about through the absence of a kinetic term for the gauge fields in the
bare Lagrangian. To leading order in $N$ this property is also there if the kinetic term is not zero but is sub-leading in large $N$, or

$$\frac{1}{g^2} = \frac{\lambda}{N}$$

(11)

where $\lambda \sim 1$ and $1/g^2 \sim 1/N$ as $N \to \infty$ In [1] we argued that, by tuning $\lambda$ appropriately we could still produce QCD with a string tension which is finite in the continuum limit.

The key to the solvability of the Kazakov-Migdal model is the fact that the single–link Itzykson–Zuber integral can be done analytically [25,26]

$$I_{IZ} = \int [dU] e^{N \sum \phi_i \chi_j \left| U_{ij} \right|^2} = \frac{\det(\phi) \Delta[\chi]}{\Delta[\phi]}$$

(12)

where

$$\Delta[\phi] = \det(\phi)_{ij} = \prod_{i<j}(\phi_i - \phi_j)$$

(13)

is the Vandermonde determinant for $\phi$.

This allows us to express the partition function (1) as an integral over the eigenvalues of the scalar field $\phi$ at each site

$$Z_{KM} \propto \int \prod_{x,i} d\phi_i(x) \Delta^2[\phi(x)] e^{-N \sum_{x,i} V[\phi_i(x)]} \prod_{<xy>} \frac{\det_{ij} e^{N \phi_i(x) \phi_j(y)}}{\Delta[\phi(x)] \Delta[\phi(y)]}$$

(14)

Here, the integral over $U$ matrices can be obtained explicitly only because the Wilson kinetic term is absent from the action. If the action has a Wilson term with coefficient $\lambda$ which is of order one in the infinite $N$ limit, the Wilson term is of order $N$ whereas all other terms in the action are of order $N^2$. Then, the Wilson term can be ignored and the effective action for the scalar field, to leading order in $N$ is still given by (14).

The eigenvalues $\phi_i$ behave like a master field since the large $N$ limit in (14) is the classical limit and the integral can be performed by saddle point approximation. Migdal [4] has derived integral equations which are obeyed by the eigenvalue density and has given an expression for the asymptotics of the solution. Corrections to the classical behavior and the spectrum of elementary excitations can also be computed [3]. (This is of course neglecting the corrections which would arise from the presence of the Wilson term, which should begin to contribute at this order.) This model has been considered further in [6] - [22].

If the model (1) has a second order phase transition and if the fluctuations in the vicinity of the critical point are non-Gaussian, one might expect that the critical behavior should be represented by QCD, the only known nontrivial four dimensional field theory with non-Abelian gauge symmetry. Since the rough argument leading to (14) indicates that the scalar mass should scale to infinity slower than the lattice scale in the continuum limit, it is necessary that the scalar field exhibits critical behavior, i.e. that the effective scalar field theory in (14) has a second order phase transition itself.

In order to familiarize the reader with the Kazakov-Migdal model, we shall begin by reviewing how it can be solved in the simple case of a lattice with a single site and subsequently the case where the lattice is one-dimensional and periodic. These simple models are interesting in that one can obtain critical behavior for the scalar fields when the scalar field action is quadratic.
II. KAZAKOV-MIGDAL MODEL ON A SINGLE SITE

It is instructive to consider the Kazakov–Migdal model on a single site. To this end consider the following integral

\[ Z = \int \mathcal{D}\phi \mathcal{D}U \ e^{-m^2 \text{Tr}(\phi^2) + \text{Tr}(\phi U \phi U^{-1})} \]  

(15)

The evaluation of this integral follows very closely the method of D’Adda et al. [10]. The integral over \( U \) can be done using the Itzykson–Zuber formula [26], [25] and the result depends only on the eigenvalues of \( \phi \). We thus have that

\[ Z \propto \int \prod_i d\phi_i \Delta^2(\phi) e^{-m^2 \sum_i \phi_i^2 \det(e^{\phi_i \phi_{\sigma(i)}})} \]  

(16)

The determinant can be written explicitly as a sum over permutations

\[ Z \propto \sum_{\sigma \in S_N} \epsilon(\sigma) \int \prod_i d\phi_i \ e^{-m^2 \sum_i \phi_i^2} e^{\sum_i \phi_i \phi_{\sigma(i)}} \]  

(17)

where \( \epsilon(\sigma) \) is the sign of the permutation \( \sigma \). This integral is a Gaussian integral which can be done explicitly. The integrand can be simplified by introducing two real quantities \( a \) and \( b \) such that

\[ a^2 + b^2 = m^2 \quad \text{and} \quad ab = 1 \]  

(18)

so that

\[ Z \propto \sum_{\sigma \in S_N} \epsilon(\sigma) \int \prod_i d\phi_i \ e^{-\frac{1}{2}(a\phi_i + b\phi_{\sigma(i)})^2} \]  

(19)

and

\[ a, b = m^2 \pm \sqrt{m^4 - 1} \]  

(20)

Note that Eq. (19) is symmetric under the interchange of \( a \) and \( b \).

The calculation proceeds by a change of variables from the eigenvalues \( \phi_i \) to

\[ \xi_i = a\phi_i + b\phi_{\sigma(i)} \]  

(21)

The transformation is linear and the Jacobian in independent of \( \phi \). To evaluate the Jacobian first note that any permutation \( \sigma \) can be written as a product of \( r \)-cycles of the form

\[ c_1, c_2, \ldots, c_r \rightarrow c_2, c_3, \ldots, c_r, c_1 \]  

(22)

Consider a fixed permutation \( \sigma \) and write \( \sigma \) as a product of \( n_1 \) cycles of size 1, \( n_2 \) cycles of size 2, \( \ldots \), \( n_k \) cycles of size \( k \), \( \ldots \). Clearly

\[ \sum_{j=1}^{N} jn_j = N \]  

(23)
It is also straightforward to prove by recursion that the Jacobian for an r-cycles is

\[
\det \begin{pmatrix}
  a & -b & 0 & \ldots \\
  0 & a & -b & 0 & \ldots \\
  \vdots & 0 & a & -b & \ldots \\
  -b & 0 & \ldots & 0 & a
\end{pmatrix} = a^r - b^r
\]  

(24)

where the above is an \( r \times r \) matrix.

Using the change of variables (21) in Eq. (19) we find that

\[
Z \propto \sum_{\sigma \in S_N} \epsilon(\sigma) Z_{\sigma}
\]

(25)

with

\[
Z_{\sigma} = \int \prod_i d\phi_i e^{-\xi_i^2/2} = \int \prod_{\text{cycles}} d\xi_i \frac{1}{a^r - b^r} e^{-\xi_i^2/2}
\]

(26)

Performing the Gaussian integral we see that

\[
Z_{\sigma} = \pi^{N/2} \prod_{i=1}^{N} \left( \frac{1}{a^i - b^i} \right)^n \propto \prod_{i=1}^{\infty} \left( \frac{1}{a^i - b^i} \right)^n
\]

(27)

since \( n_k = 0 \) for \( k > N \).

The next step is to do the sum over permutations (25). To this end note that there are

\[
N! \prod_{j=1}^{N} \frac{1}{n_j! j^{n_j}}
\]

(28)

distinct permutations which can be described the a given set \( (n_1, \ldots n_N, \ldots) \) (with \( n_k = 0 \) for \( k > N \)). Thus

\[
Z \propto \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \cdots \delta(N - \sum_{k=1}^{\infty} k n_k) \prod_{j=1}^{\infty} \left( \frac{1}{a^j - b^j} \right)^{n_j} (-1)^{(j+1)n_j} \frac{1}{k^{n_k} n_k!}
\]

(29)

where we have used the fact that

\[
\epsilon(\sigma) = \prod_{k=1}^{\infty} (-1)^{(k+1)n_k}
\]

(30)

Using the integral representation of the \( \delta \) function \( \delta(x) = \int_0^{2\pi} d\theta \exp(-ix\theta) \) we find that

\[
Z \propto \int d\theta \ e^{-iN\theta} \prod_{k=1}^{\infty} \left( \sum_{n_k=0}^{\infty} \frac{\epsilon^{|k|n_k} (-1)^{(k+1)n_k}}{[k(a^k - b^k)]^{n_k} n_k!} \right)
\]

(31)

The sum in Eq. (31) can be summed to an exponential. The remaining product then becomes a sum over the exponents so that
\[ Z \propto \int d\theta \ e^{-iN\theta} \exp \left[ \sum_{k=1}^{\infty} \frac{(-ae^{i\theta})^k}{k} \frac{1}{1 - a^{2k}} \right] \]  

(32)

where we use have used the fact that \( b = 1/a \). Without loss of generality we assume that \( a < 1 \) (i.e. \( a = m^2 - \sqrt{m^4 - 1} \)). The term \( 1/(1 - a^{2k}) \) can now be expanded in a geometric series and the resulting sum over \( k \) sums to a logarithm. Thus

\[
Z \propto \int d\theta \ e^{-iN\theta} \prod_{n=0}^{\infty} \exp \left[ \log \left( 1 + e^{i\theta} a^{2n+1} \right) \right] = \int d\theta \ e^{-iN\theta} \prod_{n=0}^{\infty} \left( 1 + e^{i\theta} a^{2n+1} \right) 
\]

(33)

The final step is to perform the integration over \( \theta \). To do this note that only terms in the product which are proportional to \( \exp(iN\theta) \) will give a nonzero integral. Thus

\[
Z \propto a^N \sum_{n_1=0}^{\infty} \sum_{n_2=n_1+1}^{\infty} \cdots \sum_{n_N=n_{N-1}+1}^{\infty} (a^2)^{n_1+\cdots+n_N} 
\]

(34)

The sums can be done one at a time. They are all geometric series. The result is

\[
Z \propto \frac{1}{a^N} \prod_{k=1}^{N} \frac{(a^2)^k}{1 - (a^2)^k} = (a^2)^{N^2/2} \prod_{k=1}^{N} \frac{1}{1 - (a^2)^k} 
\]

(35)

In fact if we define \( q = a^2 = (m^2 - \sqrt{m^4 - 1})^2 \) then

\[
Z \propto q^{N^2/2} \left( \prod_{k=1}^{N} \frac{1}{1 - q^k} \right) 
\]

(36)

Notice that apart from the factor \( q^{N^2/2} \) this is just the expression for a q-factorial. It is interesting to note that a similar expression is obtained for the partition function of a system bosons on a circle at nonzero temperature. In fact the partition function for a bosonic string on a circle (suitably restricted to the singlet sector) is precisely given by the above formula.

III. KAZAKOV-MIGDAL MODEL ON A CIRCLE

It is straightforward to generalize the above calculation to the evaluation of the partition function of the Kazakov–Migdal Model on a circle. This calculation is discussed in Ref. \[10\] and \[7\]. We consider the partition function

\[
Z = \int \prod_{x=1}^{L} D\phi_{x} \prod_{x=1}^{L} DU_{x,x+1} \ e^{-m^2 \sum_{i} \text{Tr} (\phi_{x}^2) + \sum_{x} \text{Tr} (\phi_{x} U_{x,x+1} \phi_{x+1} U_{x,x+1}^{-1})} 
\]

(37)

where the fields \( \phi_{x} \) live on the sites \( x \) of a circle with periodic boundary conditions \( \phi_{L+1} = \phi_{1} \) and the \( U_{x,x+1} \) live on the links of the circle. There are two distinct ways of calculating this partition function. The first method is to eliminate almost all the \( U_{x,x+1} \)'s by a gauge transformation leaving only one \( U \) which cannot be eliminated. This can be chosen to be the \( U \) on the first link i.e. \( U = U_{1,2} \). The result is a partition function which involves an integral over only a single link variable, and a Gaussian integral over all \( N \) Hermetian
matrices. The second method is to integrate over all the link variables $U_{x,x+1}$ explicitly using the same formula as for the single link case above. This second method generalizes the method presented in the previous section for the single link integral. The details of this calculation are discussed by D’Adda and Panzeri [10]. The basic idea is to perform all the $U$ integrations in Eq. (37). The result is

$$Z \propto \int \prod_{x=1}^{L} \left( \prod_{i=1}^{N} d\phi_{x}^{i} \right) \Delta^{2}(\phi_{x}) \left( \prod_{x=1}^{L} \frac{\det e^{\phi_{x}^{i}\phi_{x+1}^{i}}}{\Delta(\phi_{x})\Delta(\phi_{x+1})} \right)$$

(38)

where $\phi_{x}^{i}$ is the $i$'th eigenvalue of the matrix $\phi_{x}$. Note that the Vandermonde determinants $\Delta(\phi)$ precisely cancel leaving us with

$$Z \propto \int \prod_{x=1}^{L} \left( \det e^{\phi_{x}^{i}\phi_{x+1}^{i}} e^{-m^{2}\sum_{i=1}^{N} (\phi_{x}^{i})^{2}} \right)$$

(39)

This is now a Gaussian integral of precisely the same form as the integral for the K-M model on a point. It is evaluated in Ref. [10]. The result is

$$Z \propto q^{LN^{2}/2} \left( \prod_{k=1}^{N} \frac{1}{1 - q^{L}} \right)$$

(40)

with

$$q = m^{2} - \sqrt{m^{4} - 1}$$

(41)

Note that this is obtained from the single link integral by a simple replacement of $q$ by $q^{L}$.

IV. SYMMETRY AND OBSERVABLES

It was pointed out in [3] that, like all adjoint lattice models, the Kazakov-Migdal model (1) has an extra gauge symmetry which is not a symmetry of continuum QCD. The action in (1) is invariant under redefining any of the gauge matrices by an element of the center of the gauge group,

$$U(xy) \rightarrow z(xy)U(xy)$$

$$\phi(x) \rightarrow \phi(x)$$

(42)

(43)

where $z(xy) \in U(1)$ if the gauge group is $U(N)$ and $z(xy) \in Z_{N}$ if the gauge group is $SU(N)$. (We shall call the symmetry a $Z_{N}$ gauge symmetry in either case.) It was subsequently pointed out by Gross [11] and by Boulatov [19] that there is a larger symmetry of this kind: one could redefine $U$ and $U^{\dagger}$ by any element which commutes with the matrix $\phi$. The $Z_{N}$ symmetry in (13) is the maximal subgroup of the transformations discussed by Gross and Boulatov which can be implemented with field independent elements $z(xy)$. Because of this symmetry the conventional Wilson loop observables of lattice gauge theory have vanishing average unless they have either equal numbers of $U$ and $U^{\dagger}$ operators on each link or else, in the case of $SU(N)$, unless they have an integer multiple of $N U$’s or $N U^{\dagger}$’s.
In conventional QCD, the expectation value of the Wilson loop operator gives the free energy for a process which creates a heavy quark-antiquark pair, separates them for some time and lets them annihilate. From the asymptotics for large loops, one extracts the interaction potential for the quarks. If the expectation value of the Wilson loop behaves asymptotically like $e^{-\alpha A}$ where $A$ is the area of a minimal surface whose boundary is the loop, the quark-antiquark potential grows linearly with separation at large distances and quarks are confined. The parameter $\alpha$ is the string tension. On the other hand if the expectation value of the Wilson loop goes like the exponential its perimeter then the potential is not confining.

In the Kazakov-Migdal model (1), due to the $Z_N$ symmetry, the expectation value of the Wilson Loop is identically zero for all loops with non-zero area. We can interpret this as giving an area law with infinite string tension, $\alpha = \infty$, and no propagation of colored objects is allowed at all. (An exception is the baryon ($U_N$) loops in the case of $SU(N)$ where the correct statement is that $N$-ality cannot propagate.) It is for this reason that the original Kazakov-Migdal model has difficulty describing pure gluodynamics.

The $Z_N$ symmetry of the pure Kazakov-Migdal model is broken explicitly by the introduction of a Wilson term in (4). However, if the Wilson term has vanishingly small coefficient and is negligible in the large $N$ limit, one might expect the problem of $Z_N$ symmetry to remain - the string tension would still be infinite. It has been argued in (1) that this need not be so. It was shown that, if the scalar fields exhibit a particular kind of critical behavior, it is possible that the presence of an infinitesimally small Wilson term is sufficient to give a finite string tension.

There are currently several other points of view on how to avoid the constraints of $Z_N$ symmetry. In (1) it was suggested that if there is a phase transition so that the $Z_N$ symmetry is represented in a Higgs phase, the resulting large distance theory would resemble conventional QCD. This approach has been pursued in (9), (12), (13). An alternative, which was advocated in (6), is to use unconventional observables such as filled Wilson loops which reduce to the usual Wilson loop in the naive continuum limit but which are invariant under $Z_N$. The third possibility is to break the $Z_N$ symmetry explicitly. This was suggested by Migdal (13) in his mixed model in which he breaks the $Z_N$ symmetry by introducing into the model heavy quarks in the fundamental representation of the gauge group.

In (1) we considered explicit $Z_N$ symmetry breaking using a Wilson term. We showed that it is equivalent to using the filled Wilson loop observables. We used that fact that the filled Wilson loops arise naturally from ordinary Wilson loops in a modified version of the Kazakov-Migdal model which has additional explicit symmetry breaking terms. We argued that one version of this modified model should be solvable in the large $N$ limit.

V. FILLED WILSON LOOPS

We begin with a brief review of the properties of the filled Wilson loop operators which were introduced in (6) and discussed in detail in (7). These are a special class of correlation functions which survive the $Z_N$ symmetry of the original Kazakov-Migdal model (1). They are defined by considering an oriented closed curve $\Gamma$ made of links of the lattice. The ordinary Wilson loop operator on $\Gamma$ is given by
\[ W[\Gamma] = \text{Tr} \left\{ \prod_{<xy> \in \Gamma} U(xy) \right\}. \tag{44} \]

For any surface \( S \) which is made of plaquettes such that the boundary of \( S \) is the curve \( \Gamma \) we define

\[ W_F[\Gamma, S] = W[\Gamma] \prod_{\square \in S} W^\dagger[\square] \tag{45} \]

where \( \square \) denotes an elementary plaquette in the surface \( S \). The filled Wilson loop for \( \Gamma \) is now defined as

\[ W_F[\Gamma] = \sum_S \mu(S) W_F[\Gamma, S] \tag{46} \]

where the sum is over all surfaces \( S \) whose boundary is the loop \( \Gamma \) with some (yet to be specified) weight function \( \mu(S) \). Notice that for each plaquette \( \square \in S \) we have inserted the negatively oriented Wilson loop \( W^\dagger[\square] \). Thus for arbitrary weight functional \( \mu(S) \) the filled Wilson loop operator is invariant under the local \( Z_N \) gauge symmetry since it has equal numbers of \( U \) and \( U^\dagger \) operators on each link. Although we have assumed that the loop is filled with elementary plaquettes this can be easily generalized to other fillings (the other extreme case being the adjoint loop \( W[\Gamma] W^\dagger[\Gamma] \)). We can also define the ‘filled correlator” of more than one loop by summing over all surfaces whose boundary is given by those loops.

In ref. [7] it was shown that computing the expectation value of \( W_F[\Gamma] \) is equivalent to computing the partition function of a certain statistical model on a random two–dimensional lattice. When computing \( Z_N \) gauge invariant correlation functions of \( U \)–matrices in the master field approximation the \( \phi \)–integral is evaluated by substituting the master field \( \bar{\phi} = \text{diag}(\bar{\phi}_1, \ldots, \bar{\phi}_N) \) for the eigenvalues of \( \phi \).

\begin{align*}
\langle U_{i_1 j_1} \ldots U^\dagger_{k_1 l_1} \ldots \rangle &= \frac{\int d\hat{\phi}[dU] e^{-N \sum V[\phi] - \sum \phi U \phi U^\dagger} U_{i_1 j_1} \ldots U^\dagger_{k_1 l_1} \ldots}{\int d\phi[dU] e^{-N \sum V[\phi] - \sum \phi U \phi U^\dagger}} \tag{47} \\
&\approx \frac{\int d\hat{\phi}[dU] e^{N \sum \phi U \phi U^\dagger} U_{i_1 j_1} \ldots U^\dagger_{k_1 l_1} \ldots}{\int d\phi[dU] e^{N \sum \phi U \phi U^\dagger}} \\
\text{In this integral, the scalar field is written at } \phi &= V \phi_D V^\dagger \text{ with } \phi_D \text{ a diagonal matrix. The eigenvalues of } \phi \text{ are fixed at the value of the master field } \bar{\phi} \text{ and, in order to obtain gauge invariance of the correlator, the angular matrices } V \text{ are still integrated}, \\
\int d\phi &\equiv \Delta(\phi) DV \tag{48} \\
\text{In any gauge invariant correlator, the matrices } V \text{ can be absorbed by redefining } U.
\end{align*}

If we consider for the moment surfaces which are not self-intersecting so that the filled Wilson loop correlator has at most one \( UU^\dagger \) pair on any link we need to consider only the two field correlator \( \langle U_{ij} U^\dagger_{kl} \rangle \). Gauge invariance implies that [7]

\begin{align*}
\langle U_{ij} U^\dagger_{kl} \rangle &= C_{ij} \delta_{ik} \delta_{jk} \quad \text{with} \quad C_{ij} = \frac{\int d\hat{\phi}[dU] e^{N \sum \phi U \phi U^\dagger} |U_{ij}|^2}{\int d\phi[dU] e^{N \sum \phi U \phi U^\dagger}} \tag{49} \\
\end{align*}
Thus, in the master field approximation, the expectation value of the filled Wilson loop is given by

\[
< W_F[\Gamma] > = \sum_S \mu(S) \left[ \prod_{\text{sites } x \in S} \sum_{i(x)=1}^{N} \prod_{\text{links } <xy> \in S} C_{i(x) \ j(x)} \right] \tag{50}
\]

This is a generalized Potts model on a random surface in which \(N\)-component spins reside at each site and the Boltzmann weights \(C_{ij}\) for the bonds are correctly normalized to be conditional probabilities; \(\sum_i C_{ij} = 1, \sum_j C_{ij} = 1\).

### VI. COMPUTING CORRELATION FUNCTIONS

Techniques for evaluating \(C_{ij}\) for general \(N\) and for arbitrary \(\bar{\phi}\) are presented in \([17]\) and \([18]\). An explicit formula for \(SU(2)\) is given in \([7]\). Although the general formula for \(C_{ij}\) in \(SU(N)\) is quite difficult to deal with, it is still possible to estimate the surface dependence of the statistical model partition function in \((50)\) when the master field \(\bar{\phi}\) is homogeneous by considering two different limits.

#### A. Low Temperature

First, consider the case where \(\bar{\phi}_i\) are large. We also assume that the eigenvalues \(\bar{\phi}_i\) are not too close to each other in the sense that

\[
\sum_{i \neq j} \frac{1}{(\bar{\phi}_i - \bar{\phi}_j)^2} \ll N \tag{51}
\]

(Note that this is a single sum over \(j\) for fixed \(i\).) The integral in \((12)\) is known to be exact in the semi-classical approximation (see \([7]\) for a discussion). The classical equation of motion is

\[
[U \bar{\phi} U^\dagger, \bar{\phi}] = 0 \tag{52}
\]

which, since \(\bar{\phi}\) is diagonal, is solved by any \(U\) of the form \(U_0 = DP\) where \(D\) is a diagonal unitary matrix and \(P\) is a matrix which permutes the eigenvalues,

\[
(P \bar{\phi} P)_{ij} = \delta_{ij} \bar{\phi}_{P(i)} \tag{53}
\]

Also, when \(N\) is large and \(\bar{\phi}\) is not too small the identity permutation gives the smallest contribution to the action in \((12)\) and therefore is the dominant classical solution. In this case we use this minimum to evaluate the correlators,

\[
I_{IZ}^{-1} \int [dU] e^{N \sum \bar{\phi}_i \phi_j [U_{ij}]^2} U_{i_1 j_1} \ldots U_{i_n j_n} U_{k_1 l_1}^\dagger \ldots U_{k_n l_n}^\dagger = \delta_{i_1 j_1} \ldots \delta_{k_n l_n} S_{k_1 \ldots k_n} \tag{54}
\]

where we have written the normalized integral over diagonal matrices.
\[ S_{k_1\ldots k_n}^{i_1\ldots i_n} = \int \prod_{\ell} d\theta_\ell \prod_{p<q} \sin^2(\theta_p - \theta_q) e^{i(\theta_{i_1} + \ldots + \theta_{i_n} - \theta_{k_1} - \ldots - \theta_{k_n})} / \int \prod_{\ell} d\theta_\ell \prod_{p<q} \sin^2(\theta_p - \theta_q) \]

\[
= \begin{cases} 
1 & \text{if } i_1 \ldots i_n \text{ is a permutation of } k_1 \ldots k_n \\
0 & \text{otherwise}
\end{cases}
\] (55)

We have decomposed the integration over unitary matrices into an integration over the diagonals and an integration over the unitary group modulo diagonals [27]. The diagonals are the ‘zero modes’ for the semiclassical integral and must be integrated exactly. The unitary modulo diagonal integral is damped by the integrand and is performed by substituting the classical configuration. Of course, to get the next to leading order the latter integration must be done in a Gaussian approximation. It can be done for the first few correlators, The result is

\[ C_{ii} = 1 - \frac{1}{N} \sum_{k \neq i} \frac{1}{(\bar{\phi}_i - \bar{\phi}_k)^2} \quad \text{when } i \neq j \quad C_{ij} = \frac{1}{N} \frac{1}{(\bar{\phi}_i - \bar{\phi}_j)^2} \]

We remark that similar calculations can be easily done for correlators of more than two \( U \)'s

\[ C_{ij,kl} = i_{IZ}^{-1} \int [dU] e^N \sum \bar{\phi}_i \bar{\phi}_j |U_{ij}|^2 |U_{kl}|^2 \]

\[ i = j , \ k = \ell \quad C_{ii,kk} = 1 - \frac{1}{N} \sum_{n \neq i} \left[ \frac{1}{(\bar{\phi}_i - \bar{\phi}_n)^2} + \frac{1}{(\bar{\phi}_k - \bar{\phi}_n)^2} \right] \]

\[ i \neq j , \ k = \ell \quad C_{ij,kl} = \frac{1}{N^2 (\bar{\phi}_i - \bar{\phi}_j)^2 (\bar{\phi}_k - \bar{\phi}_\ell)^2} \]

where the next corrections will be of the form

\[ \frac{1}{N^2} \frac{1}{(\bar{\phi}_i - \bar{\phi}_j)^2} \sum_{m \neq k} \frac{1}{(\bar{\phi}_k - \bar{\phi}_m)^2} \] (58)

We call this limit of large \( \bar{\phi} \) the “Low Temperature” limit since in this limit

\[ C_{ij}^{LT} = \delta_{ij} + \ldots \] (59)

and the value of the spin at each site is equal. In this case the \( C_{ij} \) represent the Bolzman weight for a perfectly ordered system. These two cases lead to profoundly different behavior for the filled Wilson loop. We shall assume that, by choosing the potential for the scalar field in (14) appropriately, either of these limits could be obtained (the eigenvalue repulsion due to the Vandermonde determinants in (14) and the possibility of adding repulsive central potentials makes the low temperature limit more natural).
B. High Temperature

The other limit is where $\bar{\phi}$ is small. There, we can obtain the correlators by Taylor expansion,

$$C_{ij} = \int [dU] \left( 1 + N \text{Tr} \bar{\phi} U \bar{\phi} U^\dagger + \ldots \right) |U_{ij}|^2 = \frac{1}{N} + \bar{\phi}_i \bar{\phi}_j + \ldots$$  \hspace{1cm} (60)

We call this the “High Temperature” limit since in this limit is independent of $i$ and $j$. It thus represents the Bolzman weights for a highly disordered system.

C. Renormalization of the String Tension

We begin by estimating the value of the filled Wilson loop for a fixed surface $S$. In the “high temperature” case the statistical model is disordered. The sums over configurations at the various sites are independent and they contribute an overall factor $N^V$ (where $V$ is the number of vertices on the surface) to the expectation value of the filled Wilson loop. Furthermore each link contributes a factor $C_{ij}=1/N$ so that the links contribute a total factor of $N^{-L}$ where $L$ is the total number of links. It follows that the expectation value of the filled Wilson loop goes like

$$< W_F[\Gamma, S] > \sim N^{V-L} = N^{2-2g(S)} N^{-A(S)}$$  \hspace{1cm} (61)

where $A(S)$ is the area and $g(S)$ is the genus of the surface $S$ (i.e. the number of plaquettes comprising $S$) and we have used Euler’s theorem, $\chi \equiv 2 - 2g = V - L + A$. We thus get the renormalization of the string tension $\delta \alpha_{\text{HT}} = \log N$. Notice also that higher genus surfaces are suppressed and that the loop (genus) expansion parameter is $1/N^2$. This is precisely what is obtained in the conventional strong coupling expansion of Wilson’s lattice gauge theory which is known to describe a string theory with extra degrees of freedom associated with self-intersections of the string [23].

In the “low temperature” case, the statistical system is ordered. The spins on all the sites are frozen at a uniform value. In this case the partition function is proportional to the degeneracy of the ground state,

$$< W_F[\Gamma, S] > \sim N$$  \hspace{1cm} (62)

Note that in this case the statistical model gives no contribution to the string tension ($\delta \alpha_{\text{LT}} \approx 0$) and there is no suppression of higher genus surfaces.

In order to proceed to the evaluation of the filled Wilson loop we need to choose a weight function $\mu(S)$ in order to perform the sum over surfaces. The most reasonable criterion for choosing such a weight function is our desire to get a finite physical string tension in the continuum limit. In order to accomplish this goal we must choose a weight function $\mu(S)$ which depend on the area of the surface differently in the low and in the high temperature cases. It is known that the number of closed surfaces with a given area grows exponentially as

$$n(A) \sim A^{\kappa(g)} e^{\mu_0 A}$$  \hspace{1cm} (63)
where $\kappa(g)$ is a universal constant which depends only on the genus of the surface and $\mu_0$ is a non-universal, regulator dependent constant which will lead to a renormalization of the string tension. In our case, although the surfaces are open, the above formula should still be valid for surfaces whose area is much larger than the area of the minimal surface bounded by $\Gamma$. If the continuum limit of our theory is realized in the “high temperature” phase we should use the weight function $\mu_{HT}(S) \sim N^{A(S)} e^{-\mu_0 A(S)}$. This leads to a vanishing string tension in the lattice theory which is a necessary condition for having a finite string tension in the continuum limit. To accomplish the same goal in the “low temperature” phase we should use $\mu_{LT}(S) \sim e^{-\mu_0 A(S)}$. Although these choices of $\mu(S)$ give the desired result, it is rather unnatural to have to choose $\mu(S)$ in such an ad hoc fashion.

D. Filled Wilson Loops from the Wilson Action

Fortunately there is a very natural way to obtain the sum over surfaces in (50). Consider the following expectation value

$$< W_F[\Gamma] > = \frac{< W[\Gamma] e^{\lambda \sum_\Box (W[\Box] + W^\dagger[\Box])} >}{< e^{\lambda \sum_\Box (W[\Box] + W^\dagger[\Box])} >}$$  \hspace{1cm} (64)

where $W[\Gamma]$ is the conventional Wilson loop. Remember that the average is weighted by the Kazakov–Migdal action as in (47): In the master field limit it is computed by integrating only over $U$–matrices with $\phi = \bar{\phi}$ and with the Kazakov–Migdal action. Note that the exponent in (64) is simply the conventional Wilson kinetic term for the gauge fields in lattice gauge theory. If we expand the right hand side of (64) in $\lambda$ the non–vanishing terms are all of those surfaces which fill the Wilson loop. The result is thus a filled Wilson loop with a surface weight $\mu(S) = \lambda^{A(S)}$.

It is clear that we would obtain exactly the same expression (in the master field approximation) by evaluating the expectation value of the ordinary Wilson loop operator in the modified version of the Kazakov–Migdal model in which a conventional Wilson term $(\lambda \sum_\Box (W[\Box] + W^\dagger[\Box]))$ is added to the action. This term breaks the $Z_N$ gauge symmetry explicitly and allows Wilson loop operators with non–zero area to have non–zero expectation values. We would expect that it is necessary to keep $\lambda$ small if one is to maintain the successes of the Kazakov–Migdal model. We shall now argue that in the “low temperature” limit this picture is self–consistent in the sense that the physical string tension is finite when $\lambda$ is small and consequently the saddle point solution of the original model is unchanged. We shall also see that this is not the case in the “high temperature” phase.

Let us begin by determining how $\lambda$ should behave in the continuum limit if we are to have a finite physical string tension. As discussed above a necessary condition for having a finite physical string tension is that the string tension in lattice units should vanish. It is thus necessary for the bare string tension $-\ln \lambda$ to be chosen so as to precisely cancel the renormalization of the string tension due to both the statistical model to the sum over surfaces. It is straightforward to check that in the “high temperature” phase we must choose $\lambda_{HT} = Ne^{-\mu_0}$, whereas in the “low temperature” phase we must choose $\lambda_{LT} = e^{-\mu_0}$. Notice that this $\lambda$ is proportional to $N$ in the “high temperature” phase and thus cannot be assumed small in large $N$. 

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In the large $N$ limit of conventional lattice gauge theory the coefficient of the Wilson term must be proportional to $N$ if one is to obtain a consistent large $N$ expansion. In our case we see that this is true for the “high temperature” phase in which case the Wilson term is of the same order as the Kazakov–Migdal term and it thus plays an important role in the infinite $N$ limit. One can say that in this phase we have ordinary QCD. Unfortunately it is impossible to preserve the master field solution of the Kazakov-Migdal model in this limit since the Wilson term, being of order $N$, would modify the large $N$ solution, ruining the self-consistency of the mean-field approximation as described here.

The situation is much more appealing in the “low temperature” phase. In this case the required coefficient of the Wilson term is of order one. It is subdominant and therefore negligible in the large $N$ limit. Thus, Migdal’s solution \([4]\) of the Kazakov-Migdal model in the large $N$ limit should still apply to our proposed modification of the action. In fact the only reason that the Wilson term is important at all in the large $N$ limit of the “low temperature” phase is related to the collective phenomenon which orders the statistical system on the surfaces. It effectively makes the statistical model’s contribution to the string tension much smaller than would be expected from naive counting of powers of $1/N$ and a truly infinitesimal breaking of the $Z_N$ gauge symmetry ($\lambda_{LT}/N \to 0$ as $N \to \infty$) is sufficient to make the averages of Wilson loop operators non-vanishing. The self-consistency of this picture can also be demonstrated by computing the contribution of the Wilson term to the free energy. This can be computed in a small $\lambda$ expansion. For a cubic lattice the result is:

$$Z = e^{\lambda \text{Tr} \left( \sum_\square \text{Tr} \left( W[\square] + W^\dagger[\square] \right) \right)} = Z_{KM} \exp \left( NV \frac{D(D-1)}{2} \left( \lambda_{LT}^2 + 2\lambda_{LT}^6 + \ldots \right) \right)$$

(65)

is of order $N$ (where $V$ is the volume, $D$ is the dimension). This should be compared with the free energy in the pure Kazakov-Migdal model which is proportional to $N^2$. Here, the first term in the free energy is the contribution of the doubled elementary plaquette and the second term is due to the two orientations of the elementary cube. It is interesting that, to order 6, there is no energy of interaction of doubled elementary plaquettes with each other. We conjecture that the interaction energy of surfaces is absent to all orders and the free energy obtains contributions from all possible topologically distinct surfaces which can be built from elementary plaquettes. This suggests a free string picture of the “low temperature” limit of the Kazakov-Migdal model at lattice scales.

E. Self-Intersecting Surfaces

We have thus far neglected the self-intersecting surfaces in the sum (50) which are generated by the expansion of (54) in $\lambda$. In order to evaluate the contribution of these surfaces we need to compute the correlator of $n$ $UU^\dagger$ pairs on the same link. The computation of these correlators in full generality is quite complicated. In the Appendix we compute them in the “low temperature” (ordered) phase. We find that

$$< U_{i_1j_1} \ldots U_{i_nj_n} U^\dagger_{k_1l_1} \ldots U^\dagger_{k_nl_n} > = \delta_{i_1j_1} \ldots \delta_{k_nl_n} S_{k_1 \ldots k_n}^{i_1 \ldots i_n}$$

(66)

where $S_{k_1 \ldots k_n}^{i_1 \ldots i_n}$ is the tensor which is one if $i_1 \ldots i_n$ is a permutation of $k_1 \ldots k_n$ and is zero otherwise. It is now evident that in this limit the $U$–matrices are replaced by unit matrices
which freeze together the spin degrees of freedom on the various intersecting surfaces. As a special case we can consider a single, connected, self–intersecting surface. In this case all the spin indices on the surface are equal and since $S_{i_1\ldots i_k}^{1\ldots k} = 1$ when all arguments are equal the partition function of the statistical model corresponding to that surface is simply $N$ just as it was for a non-intersecting surface. Thus just as the statistical model does not contribute to the string tension it also does not contribute to the interaction energy of self–intersecting surfaces. This implies that in the “low temperature” limit, the sum over connected surfaces which have a common boundary behaves like a Nambu-Goto string theory with no internal degrees of freedom.

VII. DISCUSSION

In summary, the self-consistency of the “low temperature” limit leads us to a new large $N$ limit of the conventional lattice gauge theory coupled to scalars:

$$Z = \int d\phi [dU] \exp \left( -N \sum_x \text{Tr} V[\phi(x)] + N \sum_{<x,y>} \text{Tr} \phi(x) U(xy) \phi(y) U^\dagger(xy) + \lambda \sum_\square (W(\Box) + W^\dagger(\Box)) \right)$$

The conventional large $N$ limit occurs when $\lambda$ is of order $N$ and describes scalar QCD. The other limit occurs when $N \to \infty$ with $\lambda$ of order one. This model is soluble using the Kazakov–Migdal approach.

It is the latter case in which $\lambda$ remains constant that is of special interest to us. In this case we saw that the large $N$ expansion corresponds to a string theory with some unusual features. The partition function and the Wilson loop expectation value can be described as a sum over surfaces. What is unusual is that the genus of the surfaces is not suppressed in the large $N$ limit, as it is in continuum QCD. (We do of course expect the higher corrections in $1/N$ to suppress higher genus terms.) For a continuum string theory this sum over the genus is badly divergent. This, together with the presence of tachyons, suggests that the true ground state of the string theory is some sort of condensate. This could pose a complication for the present version of the Kazakov-Migdal model in the continuum limit and deserves further attention. It is still a mystery to us how the sum over all surfaces at the lattice scale should turn into the sum over planar diagrams in the continuum theory of QCD.

An alternative to the model presented here is the mixed model which was invented by Migdal [15] to solve the problem of $Z_N$ symmetry. It contains heavy quarks in the fundamental representation of the gauge group. Despite the obvious differences between our model and Migdal’s mixed model they have many features in common. As in all cases when there are fields in the fundamental representation, the asymptotics of the Wilson loops in the mixed model exhibit a perimeter law. In conventional QCD one would expect that if the quarks are heavy enough, there is an area law for small enough loops, i.e. there would exist a size scale which is far enough into the infrared region that the quark potential is linear but the interaction energy is not yet large enough that it is screened by producing quark-antiquark pairs. Thus, in QCD we expect that adding heavy quarks would not ruin the area law for Wilson loops smaller then some scale.
The mixed model has just the opposite scenario, it is possible to get an area law only when the heavy quarks are light enough. This is a result of the fact that, in the Kazakov-Migdal model, no Wilson loops are allowed at all unless the $Z_N$ symmetry is explicitly broken. In the mixed model, the $Z_N$ charge of links in a Wilson loop must be screened by the heavy quarks. This can happen in two ways. First, the Wilson loop can just bind a heavy quark to form an adjoint loop - giving a perimeter law for the free energy of the loop. This is the leading behavior if the fermion mass, $M$, is large. The free energy would go like $1/M^P$ where $P$ is the perimeter. The only way an area law might arise is when the fermions are light enough that their propagators could from a filled Wilson loop with free energy $1/M^{2L}$ where $L \approx 2A$ is the number of links. Then, since the entropy for filled loops is much larger than that for adjoint loops, these configurations would be important if $M^4 < e^{\mu_0}$. Then, the asymptotics behavior of the Wilson loop would still have a perimeter law but there would be loops with $4A - P < \mu_0/\ln M$ where there would be an approximate area law.

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[27] Despite the simple form of $S$ its tensor structure is quite complicated. For example

$$S^{i_1 i_2}_{k_1 k_2} = \delta^{i_1}_{k_1} \delta^{i_2}_{k_2} + \delta^{i_1}_{k_2} \delta^{i_2}_{k_1} - \delta^{i_1}_{i_2} \delta^{k_1}_{k_2}$$