On a Lagrangian formulation of the incompressible Euler equation

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Abstract

In this paper we show that the incompressible Euler equation on the Sobolev space $H^s(\mathbb{R}^n)$, $s > n/2+1$, can be expressed in Lagrangian coordinates as a geodesic equation on an infinite dimensional manifold. Moreover the Christoffel map describing the geodesic equation is real analytic.

1 Introduction

The initial value problem for the incompressible Euler equation in $\mathbb{R}^n$, $n \geq 2$, reads as:

$$\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla p \\
\text{div } u &= 0 \\
(1) \\
u(0) &= u_0
\end{align*}$$

where $u(t, x) = (u_1(t, x), \ldots, u_n(t, x))$ is the velocity of the fluid at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^n$, $u \cdot \nabla = \sum_{k=1}^n u_k \partial_k$ acts componentwise on $u$, $\nabla p$ is the gradient of the pressure $p(t, x)$, $\text{div } u = \sum_{k=1}^n \partial_k u_k$ is the divergence of $u$ and $u_0$ is the value of $u$ at time $t = 0$ (with assumption $\text{div } u_0 = 0$).

The system (1) (going back to Euler [8]) describes a fluid motion without friction. The first equation in (1) reflects the conservation of momentum. The second equation in (1) says that the fluid motion is incompressible, i.e. that the volume of any fluid portion remains constant during the flow.

The unknowns in (1) are $u$ and $p$. But as we will see later one can express
$\nabla p$ in terms of $u$. Thus the evolution of system (1) is completely described by $u$. Therefore we will speak in the sequel of the solution $u$ instead of the solution $(u, p)$.

Consider now a fluid motion determined by $u$. If one fixes a fluid particle which at time $t = 0$ is located at $x \in \mathbb{R}^n$ and whose position at time $t \geq 0$ we denote by $\varphi(t, x) \in \mathbb{R}^n$, we get the following relation between $u$ and $\varphi$

$$\partial_t \varphi(t, x) = u(t, \varphi(t, x)),$$

i.e. $\varphi$ is the flow-map of the vectorfield $u$. The second equation in (1) translates to the well-known relation $\det(d\varphi) \equiv 1$, where $d\varphi$ is the Jacobian of $\varphi$ – see Majda, Bertozzi [17]. In this way we get a description of system (1) in terms of $\varphi$. The description of (1) in the $\varphi$-variable is called the Lagrangian description of (1), whereas the description in the $u$-variable is called the Eulerian description of (1). One advantage of the Lagrangian description of (1) is that it leads to an ODE formulation of (1). This was already used in Lichtenstein [16] and Gunter [11] to get local well-posedness of (1).

To state the result of this paper we have to introduce some notation. For $s \in \mathbb{R}_{\geq 0}$ we denote by $H^s(\mathbb{R}^n)$ the Hilbert space of real valued functions on $\mathbb{R}^n$ of Sobolev class $s$ and by $H^s(\mathbb{R}^n; \mathbb{R}^n)$ the vector fields on $\mathbb{R}^n$ of Sobolev class $s$ – see Adams [1] or Inci, Topalov, Kappeler [12] for details on Sobolev spaces. We will often need the fact that for $n \geq 1$, $s > n/2$ and $0 \leq s' \leq s$ multiplication

$$H^s(\mathbb{R}^n) \times H^{s'}(\mathbb{R}^n) \to H^{s'}(\mathbb{R}^n), \quad (f, g) \mapsto f \cdot g$$

(2)

is a continuous bilinear map.

The notion of solution for (1) we are interested in are solutions which lie in $C^0([0, T]; H^s(\mathbb{R}^n; \mathbb{R}^n))$ for some $T > 0$ and $s > n/2 + 1$. This is the space of continuous curves on $[0, T]$ with values in $H^s(\mathbb{R}^n; \mathbb{R}^n)$. To be precise we say that $u, \nabla p \in C^0([0, T]; H^s(\mathbb{R}^n; \mathbb{R}^n))$ is a solution to (1) if

$$u(t) = u_0 + \int_0^t -(u(\tau) \cdot \nabla)u(\tau) - \nabla p(\tau) \, d\tau \quad \forall 0 \leq t \leq T$$

(3)

and $\text{div} \, u(t) = 0$ for all $0 \leq t \leq T$ holds. As $s - 1 > n/2$ we know by the Banach algebra property of $H^{s-1}(\mathbb{R}^n)$ that the integrand in (3) lies in $C^0([0, T]; H^{s-1}(\mathbb{R}^n; \mathbb{R}^n))$. Due to the Sobolev imbedding and the fact $s >$
The solutions considered here are \( C^1 \) (in the \( x \)-variable slightly better than \( C^1 \)) and are thus solutions for which the derivatives appearing in (1) are classical derivatives.

The discussion above shows that in this paper the state-space of (1) in the Eulerian description is \( H^s(\mathbb{R}^n; \mathbb{R}^n) \), \( s > n/2 + 1 \). The state-space of (1) in the Lagrangian description is given by

\[
\mathcal{D}^s(\mathbb{R}^n) = \{ \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n \mid \varphi - \text{id} \in H^s(\mathbb{R}^n; \mathbb{R}^n) \text{ and } \det d_x \varphi > 0, \forall x \in \mathbb{R}^n \}
\]

where \( \text{id} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the identity map. Due to the Sobolev imbedding and the condition \( s > n/2 + 1 \) the space of maps \( \mathcal{D}^s(\mathbb{R}^n) \) consists of \( C^1 \)-diffeomorphisms – see Palais [18] – and can be identified via \( \mathcal{D}^s(\mathbb{R}^n) - \text{id} \subseteq H^s(\mathbb{R}^n; \mathbb{R}^n) \) with an open subset of \( H^s(\mathbb{R}^n; \mathbb{R}^n) \). Thus \( \mathcal{D}^s(\mathbb{R}^n) \) has naturally a real analytic differential structure (for real analyticity we refer to Whittlesey [21]) with the natural identification of the tangent space

\[
TD^s(\mathbb{R}^n) \simeq \mathcal{D}^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n; \mathbb{R}^n).
\]

Moreover it is known that \( \mathcal{D}^s(\mathbb{R}^n) \) is a topological group under composition and that for \( 0 \leq s' \leq s \) the composition map

\[
H^{s'}(\mathbb{R}^n) \times \mathcal{D}^s(\mathbb{R}^n) \rightarrow H^{s'}(\mathbb{R}^n), \quad (f, \varphi) \mapsto f \circ \varphi
\]

is continuous – see Cantor [3] and Inci, Topalov, Kappeler [12]. That \( \mathcal{D}^s(\mathbb{R}^n) \) is the right choice as configuration space for (1) in Lagrangian coordinates is justified by the fact that every \( u \in C^0([0, T]; H^s(\mathbb{R}^n; \mathbb{R}^n)) \), \( s > n/2 + 1 \), integrates uniquely to a \( \varphi \in C^1([0, T]; \mathcal{D}^s(\mathbb{R}^n)) \) fulfilling

\[
\partial_t \varphi(t) = u(t) \circ \varphi(t) \quad \text{for all } 0 \leq t \leq T
\]

– see Fischer, Marsden [9] or Inci [13] for an alternative proof.

For the rest of this section we assume \( n \geq 2 \), \( s > n/2 + 1 \) and for \( X, Y \) real Banach spaces we use the notation \( L^2(X; Y) \) for the real Banach space of continuous bilinear maps from \( X \times X \) to \( Y \). With this we can state the main result of this paper:

**Theorem 1.1.** Let \( n \geq 2 \) and \( s > n/2 + 1 \). Then there is a real analytic map

\[
\Gamma : \mathcal{D}^s(\mathbb{R}^n) \rightarrow L^2(H^s(\mathbb{R}^n; \mathbb{R}^n); H^s(\mathbb{R}^n; \mathbb{R}^n)), \quad \varphi \mapsto [(v, w) \mapsto \Gamma \varphi(v, w)]
\]
called the Christoffel map for which the geodesic equation
\[ \partial_t^2 \varphi = \Gamma_\varphi \partial_t \varphi, \partial_t \varphi; \quad \varphi(0) = \text{id}, \partial_t \varphi(0) = u_0 \in H^s(\mathbb{R}^n; \mathbb{R}^n) \]  
(5)

is a description of (1) in Lagrangian coordinates. More precisely, any \( \varphi \) solving (5) on \([0, T]\), \( T > 0 \), with \( \text{div} u_0 = 0 \) generates a solution to (1) by the formula \( u := \partial_t \varphi \circ \varphi^{-1} \) and on the other hand any \( u \) solving (1) on \([0, T]\) integrates to a \( \varphi \) solving (5) on \([0, T]\).

By ODE theory – see Dieudonné [6] – and the continuity of the composition map (4) we immediately get the following corollary (this result, using a different method, goes back to Kato [14])

**Corollary 1.2.** Let \( n \geq 2 \) and \( s > n/2 + 1 \). Then (1) is locally well-posed in \( H^s(\mathbb{R}^n) \).

Connected to a geodesic equation like (5) is the notion of an exponential map – see Lang [15]. The domain of definition for the exponential map is the set \( U \subseteq H^s(\mathbb{R}^n; \mathbb{R}^n) \) consisting of initial values \( u_0 \in H^s(\mathbb{R}^n; \mathbb{R}^n) \) for which the geodesic equation (5) has a solution on the interval \([0, 1]\). It turns out that \( U \) is star-shaped with respect to 0 and is an open neighborhood of 0. With this we define

**Definition 1.1.** The exponential map is defined as
\[ \exp : U \to \mathcal{D}^s(\mathbb{R}^n), \quad u_0 \mapsto \varphi(1; u_0) \]

where \( \varphi(1; u_0) \) denotes the value of the solution \( \varphi \) of (5) at time \( t = 1 \) for the initial condition \( \partial_t \varphi(0) = u_0 \).

By ODE theory we know that \( \exp \) is a real analytic map. Moreover we can describe every solution of (5) by considering the curves \( t \mapsto \exp(tu_0) \) as is usual for geodesic equations. A further corollary of Theorem 1.1 is

**Corollary 1.3.** The trajectories of the fluid particles moving according to (1) are analytic.

**Proof of Corollary 1.3.** Fix \( x \in \mathbb{R}^n \) and define \( \varphi(t) := \exp(tu_0) \). Then the trajectory of the fluid particle which starts at time \( t = 0 \) at \( x \) is given by \( t \mapsto \varphi(t, x) \). By Theorem 1.1 we know that \( [0, T] \mapsto H^s(\mathbb{R}^n; \mathbb{R}^n), \quad t \mapsto \varphi(t) - \text{id} \)
is analytic. Here $T > 0$ is any time up to which the fluid motion exists for sure. As $s > n/2 + 1$ we know by the Sobolev imbedding that the evaluation map at $x \in \mathbb{R}^n$

$$H^s(\mathbb{R}^n) \to \mathbb{R}, \quad f \mapsto f(x)$$

is a continuous linear map. Thus $t \mapsto \varphi(t, x) - x$ is analytic. Hence the claim.

Related work: To use an ODE-type approach for (1) via a Lagrangian formulation is already present in the works of Lichtenstein [16] and Gunter [11]. One can also get analyticity in Lagrangian coordinates by using their successive approximation procedure.

The idea to express (1) as a geodesic equation on the ”Lie group” $\mathcal{D}$, the group of diffeomorphisms, goes back to Arnold [2]. In Ebin, Marsden [7], Ebin and Marsden worked out Arnold’s idea by proving the analog of Theorem 1.1 for the Sobolev spaces $H^s(M)$, where $M$ is a compact, smooth and oriented manifold of dimension $n$ and $s > n/2 + 1$, with the difference that they proved the Christoffel map $\Gamma$ to be smooth and not analytic (it is not so clear to us whether $\Gamma$ is analytic for all these $M$). Later Cantor [4] showed the analog of Theorem 1.1 for weighted Sobolev spaces on the whole space $H^s_w(\mathbb{R}^n)$, $s > n/2 + 1$ (Cantor stated it with $\Gamma$ smooth, but one can show that his $\Gamma$ is analytic). In Serfati [19] the analog of Theorem 1.1 was shown for $C^{k, \alpha}$-spaces over $\mathbb{R}^n$, $k \geq 1$ and $0 < \alpha < 1$. Most recently analytic dependence in the Lagrangian coordinates was shown to be true in the case of Sobolev spaces $H^s(\mathbb{T}^n)$, $s > n/2 + 1$, in Shnirelman [20] and in the case of Hölder spaces $C^{1, \alpha}(\mathbb{T}^n)$, $0 < \alpha < 1$, in Frisch, Zheligovsky [10] for fluid motion in the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

This paper is more or less an excerpt from the thesis Inci [13]. So omitted proofs or references where they can be found are given in Inci [13].

2 Alternative Eulerian description

The goal of this section is to give an alternative formulation of (1) by replacing $\nabla p$ with an expression in $u$. For this we will use an idea of Chemin [5]. Throughout this section we assume $n \geq 2$ and $s > n/2 + 1$.

To motivate the approach, we apply $\text{div}$ to the first equation in (1) and use
\[
\text{div } u = 0 \quad \text{to get}
\]
\[- \Delta p = \sum_{j,k=1}^{n} \partial_j u_k \partial_k u_j = \sum_{j,k=1}^{n} \partial_j \partial_k (u_j u_k). \quad (6)
\]

In order to invert the Laplacian \( \Delta \) we will use a cut-off in Fourier space. For this we denote by \( \chi \) the characteristic function of the closed unit ball in \( \mathbb{R}^n \), i.e. \( \chi(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \chi(\xi) = 0 \) otherwise. The continuous linear operator \( \chi(D) \) on \( L^2(\mathbb{R}^n) := L_2^2(\mathbb{R}^n) \) is defined by
\[
\chi(D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad f \mapsto \mathcal{F}^{-1} [\chi(\xi) \mathcal{F}[f](\xi)]
\]
where \( \mathcal{F} \) is the Fourier transform and \( \mathcal{F}^{-1} \) its inverse. We define the Fourier transform of \( g \in L^1(\mathbb{R}^n) \) as the following complex-valued function \( \mathcal{F}[g] : \mathbb{R}^n \to \mathbb{C} \) (with the usual extension to \( L^2(\mathbb{R}^n) \))
\[
\mathcal{F}[g](\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} g(x) \, dx, \quad \xi \in \mathbb{R}^n
\]
where \( x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n \) is the Euclidean inner product in \( \mathbb{R}^n \). We have for \( s_1, s_2 \geq 0 \)
\[
||| \chi(D)f |||_{s_1+s_2} \leq 2^{s_2/2}||| f |||_{s_1}, \quad \forall f \in H^{s_1}(\mathbb{R}^n) \quad (7)
\]
where \( ||| g |||_{s'} := ||| (1 + |\xi|^{s'}/2) |\mathcal{F}[g](\xi)| |||_{L^2} \) for \( g \in H^{s'}(\mathbb{R}^n), s' \geq 0 \). We use \((6)\) to rewrite \(-\nabla p\)
\[
- \nabla p = \nabla \left( \Delta^{-1}(1 - \chi(D)) \sum_{j,k=1}^{n} \partial_j u_k \partial_k u_j + \Delta^{-1} \chi(D) \sum_{j,k=1}^{n} \partial_j \partial_k (u_j u_k) \right).
\]

Using this expression we replace \((\overline{1})\) by
\[
\partial_t u + (u \cdot \nabla)u = \nabla B(u, u), \quad u(0) = u_0 \in H^{s}(\mathbb{R}^n; \mathbb{R}^n) \quad (8)
\]
where \( B(v, w) = B_1(v, w) + B_2(v, w) \) for \( v, w \in H^{s}(\mathbb{R}^n; \mathbb{R}^n) \) with
\[
B_1(v, w) = \Delta^{-1}(1 - \chi(D)) \sum_{j,k=1}^{n} \partial_j v_k \partial_k w_j
\]
and

\[ B_2(v, w) = \Delta^{-1}(D) \sum_{j,k=1}^{n} \partial_j \partial_k (v_j w_k). \]

As \( \Delta^{-1}(1 - \chi(D)) : H^{s-1}(\mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n) \) is a continuous linear map we get by the Banach algebra property of \( H^{s-1}(\mathbb{R}^n) \) that

\[ B_1 : H^s(\mathbb{R}^n; \mathbb{R}^n) \times H^s(\mathbb{R}^n; \mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n) \]

\[ (v, w) \mapsto \Delta^{-1}(1 - \chi(D)) \sum_{j,k=1}^{n} \partial_j v_k \partial_k w_j \]

is a continuous bilinear map. And as \( \Delta^{-1} \chi(D) \partial_j \partial_k : H^{s}(\mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n) \) is a continuous linear map for any \( 1 \leq j, k \leq n \) we get by the Banach algebra property of \( H^s(\mathbb{R}^n) \) that

\[ B_2 : H^s(\mathbb{R}^n; \mathbb{R}^n) \times H^s(\mathbb{R}^n; \mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n) \]

\[ (v, w) \mapsto \Delta^{-1} \chi(D) \sum_{j,k=1}^{n} \partial_j \partial_k (v_j w_k) \]

is a continuous bilinear map. Altogether we see that

\[ \nabla B : H^s(\mathbb{R}^n; \mathbb{R}^n) \times H^s(\mathbb{R}^n; \mathbb{R}^n) \to H^s(\mathbb{R}^n; \mathbb{R}^n) \]

is a continuous bilinear map. Equation (8) is to be understood in the sense that \( u \) is a solution to (8) on \([0, T]\) for some \( T > 0 \) if \( u \in C_0^0([0, T]; H^s(\mathbb{R}^n; \mathbb{R}^n)) \) with

\[ u(t) = u_0 + \int_0^t \nabla B(u(\tau), u(\tau)) - (u(\tau) \cdot \nabla)u(\tau) \, d\tau \] \quad (9)

for any \( 0 \leq t \leq T \). By the Banach algebra property of \( H^{s-1}(\mathbb{R}^n) \) the integrand in (9) lies in \( C_0^0([0, T]; H^{s-1}(\mathbb{R}^n; \mathbb{R}^n)) \).

To consider (8) instead of (1) is justified by the following proposition

**Proposition 2.1.** Let \( n \geq 2 \) and \( s > n/2 + 1 \). If \( u \in C^0([0, T]; H^s(\mathbb{R}^n; \mathbb{R}^n)) \), \( T > 0 \), is a solution to (1) then it is also a solution to (8). Conversely, let \( u_0 \in H^s(\mathbb{R}^n; \mathbb{R}^n) \) with \( \text{div} u_0 = 0 \). Then if \( u \in C^0([0, T]; H^s(\mathbb{R}^n; \mathbb{R}^n)) \) is a solution to (8) then it is also a solution to (1) with \( p = -B(u, u) \).

Proposition 2.1 shows in particular that for solutions of (8) the condition \( \text{div} u(t) = 0 \) is preserved if it is true for \( t = 0 \). The proof of Proposition 2.1 can be found in Inci [13].
3 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. To do that we will formulate the alternative equation (8) in Lagrangian coordinates. As usual we assume \( n \geq 2 \) and \( s > n/2 + 1 \). To motivate the approach consider \( u \) solving (8) and \( \varphi \) its flow, i.e. \( \varphi \) is determined by the relation \( \partial_t \varphi = u \circ \varphi \). Taking the \( t \)-derivative in this relation we get

\[
\partial^2_t \varphi = (\partial_t u + (u \cdot \nabla) u) \circ \varphi = \nabla B(u, u) \circ \varphi.
\]

Replacing \( u \) by \( u = \partial_t \varphi \circ \varphi - 1 \) we get

\[
\partial^2_t \varphi = \nabla B(\partial_t \varphi \circ \varphi - 1, \partial_t \varphi \circ \varphi - 1) \circ \varphi.
\]

So our candidate for the \( \Gamma \) in Theorem 1.1 is

\[
\Gamma \varphi(v, w) := \nabla B(v \circ \varphi - 1, w \circ \varphi - 1) \circ \varphi. \tag{10}
\]

The key ingredient for the proof of Theorem 1.1 is the following proposition

**Proposition 3.1.** The map

\[
\Gamma : D^s(\mathbb{R}^n) \to L^2(H^s(\mathbb{R}^n; \mathbb{R}^n); H^s(\mathbb{R}^n; \mathbb{R}^n)), \quad \varphi \mapsto [(v, w) \mapsto \Gamma \varphi(v, w)]
\]

with \( \Gamma \varphi(v, w) \) as in (10) is real analytic.

Before we proof Proposition 3.1 we have to make some preparation. We introduce the following subspace of \( L^2(\mathbb{R}^n) \)

\[
H^\infty_\Xi(\mathbb{R}^n) := \{ g \in L^2(\mathbb{R}^n) \mid \text{supp } F[g] \subseteq \Xi \}
\]

where \( \Xi \subseteq \mathbb{R}^n \) is the closed unit ball and \( \text{supp } f \) denotes the support of a function \( f \). The space \( H^\infty(\mathbb{R}^n) \) is a closed subspace of \( L^2(\mathbb{R}^n) \), lies in \( \cap_{s' \geq 0} H^{s'}(\mathbb{R}^n) \) and consists of entire functions (i.e. analytic functions on \( \mathbb{R}^n \) with convergence radius \( R = \infty \)). Note that \( \chi(D) \) maps \( H^{s'}(\mathbb{R}^n) \), \( s' \geq 0 \), into \( H^\infty(\mathbb{R}^n) \). In the sequel we will also use the vector-valued analog \( H^\infty_\Xi(\mathbb{R}^n; \mathbb{R}^n) = \{(f_1, \ldots, f_n) \mid f_k \in H^\infty(\mathbb{R}^n), \forall 1 \leq k \leq n \} \). The space \( H^\infty_\Xi \) has good properties with regard to the composition map (in contrast to its bad behaviour in the \( H^s \) space – see Inci [13]):

Denoting by \( L(X; Y) \), \( X, Y \) real Banach spaces, the real Banach space of continuous linear maps from \( X \) to \( Y \) we have
Lemma 3.2. Let $n \geq 2$ and $s > n/2 + 1$. Then
$$D^s(\mathbb{R}^n) \to L(\mathcal{H}_\infty(\mathbb{R}^n); H^s(\mathbb{R}^n)), \quad \varphi \mapsto [f \mapsto f \circ \varphi]$$
is real analytic.
We also have
Lemma 3.3. Let $n \geq 2$, $s > n/2 + 1$ and $0 \leq s' \leq s$. Then
$$D^s(\mathbb{R}^n) \to L(H^{s'}(\mathbb{R}^n); \mathcal{H}_\infty(\mathbb{R}^n); H^s(\mathbb{R}^n)), \quad \varphi \mapsto [f \mapsto \chi(D)(f \circ \varphi^{-1})]$$
is real analytic.

The proofs of Lemma 3.2 and Lemma 3.3 can be found in Inci [13]. We split the proof of Proposition 3.1 according to $B = B_1 + B_2$ into two lemmas. In the sequel we will use the notation $R_\varphi$ for the right-composition, i.e. $R_\varphi f := f \circ \varphi$. Note that $R_\varphi^{-1} = R_{\varphi^{-1}}$.

Lemma 3.4. Let $n \geq 2$ and $s > n/2 + 1$. Then
$$D^s(\mathbb{R}^n) \to L^2(H^s(\mathbb{R}^n); H^s(\mathbb{R}^n)) \quad \varphi \mapsto [(v, w) \mapsto \nabla B_1(v \circ \varphi^{-1}, w \circ \varphi^{-1}) \circ \varphi]$$
is real analytic.

Proof of Lemma 3.4. Recall that $\nabla B_1(v, w)$ is given by
$$\nabla B_1(v, w) = \nabla \left( \Delta^{-1}(1 - \chi(D)) \sum_{j,k=1}^{n} \partial_j v_k \partial_k w_j \right).$$
It will be convenient to write $\Delta^{-1}(1 - \chi(D))$ as
$$\Delta^{-1}(1 - \chi(D)) = (\chi(D) + \Delta(1 - \chi(D)))^{-1} - \chi(D). \quad (11)$$
In a first step we will prove that for $A := \chi(D) + \Delta(1 - \chi(D))$
$$D^s(\mathbb{R}^n) \to L(H^s(\mathbb{R}^n); H^{s-2}(\mathbb{R}^n)), \quad \varphi \mapsto [f \mapsto R_\varphi AR_\varphi^{-1} f]$$
is real analytic. From Lemma 3.2 and 3.3 we know that
$$D^s(\mathbb{R}^n) \to L(H^s(\mathbb{R}^n); H^s(\mathbb{R}^n)), \quad \varphi \mapsto [f \mapsto R_\varphi \chi(D) R_\varphi^{-1} f]$$
is real analytic. The same is of course true if we replace above $\chi(D)$ by $1 - \chi(D)$. To proceed we prove that for any $1 \leq s' \leq s$ and $1 \leq k \leq n$

$$D^s(\mathbb{R}^n) \rightarrow L(H^{s'}(\mathbb{R}^n); H^{s'-1}(\mathbb{R}^n)), \quad \varphi \mapsto [f \mapsto R_\varphi \partial_k R_\varphi^{-1} f]$$

(12)
is real analytic. We clearly have

$$R_\varphi \partial_k R_\varphi^{-1} f = \sum_{j=1}^n \partial_j f C_{jk}$$

where $(C_{jk})_{1 \leq j, k \leq n} = [d\varphi]^{-1}$, i.e the inverse matrix of the jacobian of $\varphi$. Note that the entries of $(C_{jk})_{1 \leq j, k \leq n}$ are polynomial expressions of the entries of $[d\varphi]$ divided by $\det(d\varphi)$. As $H^{s'-1}$ is a Banach algebra and division by $\det(d\varphi)$ an analytic operation – see Inci [13] – we get by (12) that $\varphi \mapsto R_\varphi \partial_k R_\varphi^{-1}$ is real analytic as claimed. Writing

$$R_\varphi \Delta R_\varphi^{-1} = \sum_{k=1}^n R_\varphi \partial_k R_\varphi^{-1} R_\varphi \partial_k R_\varphi^{-1}$$

we thus see that $\varphi \mapsto R_\varphi \Delta R_\varphi^{-1}$ is also real analytic. Finally writing

$$R_\varphi A R_\varphi^{-1} = R_\varphi \chi(D) R_\varphi^{-1} + R_\varphi \Delta R_\varphi^{-1} R_\varphi (1 - \chi(D)) R_\varphi^{-1}$$

(13)
we get that $\varphi \mapsto R_\varphi A R_\varphi^{-1}$ is real analytic. Denoting by $X, Y$ real Banach spaces and by $GL(X; Y) \subseteq L(X; Y)$ the open subset of invertible continuous linear operators from $X$ to $Y$ we know by the Neumann series – see Dieudonné [6] – that

$$\text{inv} : GL(X; Y) \rightarrow GL(Y; X), \quad T \mapsto T^{-1}$$
is real analytic. Therefore we get from the analyticity of (13) that

$$D^s(\mathbb{R}^n) \rightarrow L(H^{s-2}(\mathbb{R}^n); H^s(\mathbb{R}^n)), \quad \varphi \mapsto R_\varphi A^{-1} R_\varphi^{-1} = (R_\varphi A R_\varphi^{-1})^{-1}$$
is real analytic. This implies by (11) that

$$D^s(\mathbb{R}^n) \rightarrow L(H^{s-2}(\mathbb{R}^n); H^s(\mathbb{R}^n)), \quad \varphi \mapsto R_\varphi \Delta^{-1}(1 - \chi(D)) R_\varphi^{-1}$$
is real analytic. By letting $\Delta^{-1}(1 - \chi(D))$ act componentwise we write

$$\nabla B_1(v \circ \varphi^{-1}, w \circ \varphi^{-1}) \circ \varphi =$$

$$(R_\varphi \Delta^{-1}(1 - \chi(D)) R_\varphi^{-1})(R_\varphi \nabla R_\varphi^{-1}) \sum_{j,k=1}^n (R_\varphi \partial_j R_\varphi^{-1} v_k)(R_\varphi \partial_k R_\varphi^{-1} w_j)$$
and we get from the considerations above that
\[
\mathcal{D}^s(\mathbb{R}^n) \to L^2(H^s(\mathbb{R}^n; \mathbb{R}^n); H^s(\mathbb{R}^n; \mathbb{R}^n))
\]
\[
\varphi \mapsto [(v, w) \mapsto \nabla B_1(v \circ \varphi^{-1}, w \circ \varphi^{-1}) \circ \varphi]
\]
is real analytic.

**Lemma 3.5.** Let \( n \geq 2 \) and \( s > n/2 + 1 \). Then
\[
\mathcal{D}^s(\mathbb{R}^n) \to L^2(H^s(\mathbb{R}^n; \mathbb{R}^n); H^s(\mathbb{R}^n; \mathbb{R}^n))
\]
\[
\varphi \mapsto [(v, w) \mapsto \nabla B_2(v \circ \varphi^{-1}, w \circ \varphi^{-1}) \circ \varphi]
\]
is real analytic.

**Proof of Lemma 3.5.** We write
\[
\nabla B_2(v \circ \varphi^{-1}, w \circ \varphi^{-1}) \circ \varphi = \sum_{j,k=1}^n R_{\varphi} \nabla \Delta^{-1} \partial_j \partial_k \chi(D) R_{\varphi}^{-1}(v_j w_k).
\]
(14)
By Lemma 3.3 we know that \( \varphi \mapsto \chi(D) R_{\varphi}^{-1} \) is real analytic with values in \( L(H^s(\mathbb{R}^n); H^\infty(\mathbb{R}^n)) \). Moreover for any \( 1 \leq j, k \leq n \)
\[
\nabla \Delta^{-1} \partial_j \partial_k : H^\infty(\mathbb{R}^n) \to H^\infty(\mathbb{R}^n; \mathbb{R}^n)
\]
is a continuous linear map. By Lemma 3.2 we then see that the expression (14) is real analytic in \( \varphi \) showing the claim.

**Proof of Proposition 3.1.** As \( B = B_1 + B_2 \) the proof follows from Lemma 3.4 and Lemma 3.5.

Now we can prove the main theorem.

**Proof of Theorem 1.1.** The analyticity statement for \( \Gamma \) follows from Proposition 3.1. To prove the first part of the second statement consider \( \varphi \in C^2([0,T]; \mathcal{D}^s(\mathbb{R}^n)) \), \( T > 0 \), solving
\[
\partial_t^2 \varphi = \Gamma_{\varphi}(\partial_t \varphi, \partial_t \varphi), \quad \varphi(0) = \text{id}, \partial_t \varphi(0) = u_0 \in H^s(\mathbb{R}^n; \mathbb{R}^n).
\]
(15)
We define \( u := \partial_t \varphi \circ \varphi^{-1} \). By the continuity of the group operations in \( \mathcal{D}^s(\mathbb{R}^n) \) and by (4) we know that \( u \in C^0([0,T]; H^s(\mathbb{R}^n; \mathbb{R}^n)) \). By the Sobolev
imbedding we have $\varphi, \partial_t \varphi \in C^1([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. By the inverse function theorem we also have $\varphi^{-1} \in C^1([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. Hence $u \in C^1([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. Taking the pointwise $t$-derivative in the relation $\partial_t \varphi(t, x) = u(t, \varphi(t, x))$ leads to

$$\partial_t^2 \varphi = (\partial_t u + (u \cdot \nabla) u) \circ \varphi. \tag{16}$$

Using the expression (10) corresponding to $\Gamma_\varphi(\partial_t \varphi, \partial_t \varphi)$ and using $u = \partial_t \varphi \circ \varphi^{-1}$ we get pointwise (for any $(t, x) \in [0, T] \times \mathbb{R}^n$ without writing the argument explicitly)

$$B(u, u) \circ \varphi = (\partial_t u + (u \cdot \nabla) u) \circ \varphi.$$

Skipping the composition by $\varphi$ on both sides, we get by the fundamental lemma of calculus for any $(t, x) \in [0, T] \times \mathbb{R}^n$ (without writing the $x$-argument)

$$u(t) = u_0 + \int_0^t B(u(\tau), u(\tau)) - (u(\tau) \cdot \nabla) u(\tau) \, d\tau. \tag{17}$$

The integrand in (17) lies in $C^0([0, T]; H^{s-1}(\mathbb{R}^n; \mathbb{R}^n))$ so that (17) is actually an identity in $H^{s-1}$, which shows that $u$ is a solution to the alternative formulation (8).

Now it remains to prove the other direction. We take $u$ solving the alternative formulation (8). We know that there is a unique $\varphi \in C^1([0, T]; D^s(\mathbb{R}^n))$ solving

$$\partial_t \varphi = u \circ \varphi, \quad \varphi(0) = \text{id}.$$

The claim is that $\varphi$ solves the geodesic equation (15). First note that by the fact that $u$ is a solution to the alternative formulation (8) and by the Sobolev imbedding we have $u, \varphi \in C^1([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. Thus we also have $\partial_t \varphi \in C^1([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. Taking the $t$-derivative in $\partial_t \varphi = u \circ \varphi$ we get the same expression as in (16). Using that $u$ is a solution to the alternative formulation (8) we get by the fundamental lemma of calculus pointwise for any $(t, x) \in [0, T] \times \mathbb{R}^n$ (dropping the $x$-argument)

$$\partial_t \varphi(t) = u_0 + \int_0^t B(\partial_t \varphi(\tau) \circ \varphi(\tau)^{-1}, \partial_t \varphi(\tau) \circ \varphi(\tau)^{-1}) \circ \varphi(\tau) \, d\tau$$

$$= u_0 + \int_0^t \Gamma_{\varphi(\tau)}(\partial_t \varphi(\tau), \partial_t \varphi(\tau)) \, d\tau.$$

But as the integrand is a continuous curve in $H^s(\mathbb{R}^n; \mathbb{R}^n)$ we see that $t \mapsto \varphi(t)$ solves the geodesic equation (15). This completes the proof. \qed
In view of the condition \( \text{div } u = 0 \), the state space of \( \Pi \) in Lagrangian coordinates is actually \( D^s_\mu(\mathbb{R}^n) \subseteq D^s(\mathbb{R}^n) \), the subgroup of volume-preserving diffeomorphisms, i.e.

\[
D^s_\mu(\mathbb{R}^n) := \{ \varphi \in D^s(\mathbb{R}^n) \mid \det(d\varphi) \equiv 1 \}.
\]

One has – see Inci [13] for the proof

**Theorem 3.1.** Let \( n \geq 2 \) and \( s > n/2 + 1 \). Then \( D^s_\mu(\mathbb{R}^n) \) is a closed real analytic submanifold of \( D^s(\mathbb{R}^n) \).

So the dynamics of \( \Pi \) in Lagrangian coordinates is real analytic on \( D^s_\mu(\mathbb{R}^n) \) or expressed with the exponential map

**Corollary 3.2.** Let \( n \geq 2 \) and \( s > n/2 + 1 \). Then

\[
\exp : U \cap H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n) \rightarrow D^s_\mu(\mathbb{R}^n)
\]

is real analytic.

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