From polarized gravitational waves to analytically solvable electromagnetic beams

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Abstract

Using the correspondence between solutions of gravitational and gauge theories (the so-called classical double copy conjecture) some analytically solvable electromagnetic fields with vortices are constructed. The starting point is a certain class of plane gravitational waves exhibiting the conformal symmetry. The notion of the Niederer transformation, crucial for the solvability, is analysed in the case of the Lorentz force equation on the curved spacetimes. Furthermore, the models discussed recently in the context of the intense laser beams are constructed from their gravitational counterparts, with the special emphasis put on the focusing property, and new solvable examples are presented.

1 Introduction

Gravitational waves have been intensively studied since the invention of general relativity. Recently they have gained a new interest, both due to their direct observations \cite{1,2} as well as because of new theoretical ideas like the memory effect and soft graviton theorems \cite{3-10} and references therein). The circularly polarized ones seems to be particularly interesting as they may arise as the effect of coalescing black holes, neutron star merger or can be observed from the astrometric data \cite{2,11}. The linearly polarized waves are, in turn, distinguished by their relative simplicity. Far from the source in the neighbourhood of the detector one can approximate gravitational waves by the exact plane ones (assuming that the back reaction of the detector is negligible). Finally, the so-called impulsive gravitational waves seem to be of some importance \cite{12-17}.

On the other hand in the case of the mentioned above exact gravitational waves there are some special classes; they are defined by the maximal (in the non-flat case), seven-dimensional,
conformal symmetry \[18, 19, 20\]. Such classes describe, among others, linearly or circularly polarized gravitational pulses; moreover they can be used to model impulsive gravitational waves with Dirac delta profile. Furthermore, it turns out that for such exact gravitational waves some phenomena can be analytically discussed (such as singularities, focusing, classical cross section and the velocity memory effect) also in the Dirac delta limit \[17, 21\]. Moreover, it was shown \[21\] that the explicit solvability of the geodesic equations is strictly related to the notion of the so-called Niederer transformation \[22, 23\] (see also Sec. 2 for some details).

In this work, based on the mentioned above polarized gravitational pulses, we study the interaction of the electromagnetic field with a charged particle. The bridge between the gravitational waves and electromagnetic fields is provided by the idea of the classical double copy (a part of the colour-kinematic duality, see \[24]-\[28\] and references therein). At the quantum level (analysed as the tree and few loops levels) this conjecture concerns the problem of how scattering amplitudes in gravity can be obtained from those in the gauge theory by replacing colour structure with kinematical one. At the classical level, it consists in the mapping of the solutions of the Einstein equations into the solutions of the Yang-Mills equations. This approach is applied in Sec. 3 to polarized gravitational pulses defined by the existence of the proper conformal transformations; in this way we obtain analytically solvable electromagnetic backgrounds. The latter ones exhibit vortices and generalize the one described in Ref. \[29\], which seems to be of some importance for singular optics and trapping problems (the confinement mechanism of particles by electromagnetic or gravitational fields, see \[30, 31\] for the electromagnetic case and \[32, 33, 34\] for their gravitational counterparts). Since this solvability is strictly related to the notion of the Niederer transformation in Sec. 4 we discuss a geometric extension of the Niederer map to the case of the plane gravitational spacetimes endowed with some, crossed, electromagnetic fields.

In Sec. 5 we slightly modify the Kerr-Schild type relation of the double copy \[24\] (the linear structure of the Einstein equations for the Kerr-Schild metrics corresponds to Abelian gauge fields) to produce non-null electromagnetic fields. Such electromagnetic fields have been discussed in Refs. \[35, 36\]; they capture some essential features of the transverse magnetic beam of laser light near the beam axis. Next, we apply this procedure to the linearly polarized gravitational case and obtain an explicitly solvable model described in Ref. \[35\]. Moreover, relying on a member of the family of circularly polarized gravitational pulses we construct a transversally solvable electromagnetic background. However, in contrast to the previous one it has zeros; this fact essentially modifies the focusing conditions important for intense or ultra-short laser pulses (when the paraxial approximation may be no longer valid, see e.g. \[37, 38\]). In the general case, we give some criteria which, applied to the cases under consideration, again explicitly confirm the correlation between zeros of the electromagnetic field and focusing conditions. Finally, in Sec. 6 we give a summary with an outlook for further studies.
2 Plane gravitational waves and conformal symmetry

Let us consider a subclass of the pp-waves, the so-called generalized plane gravitational waves, of the following form

$$g = x \cdot H(u)x du^2 + 2dudv + dx \cdot dx,$$

where $H$ is assumed, without loss of generality, to be a symmetric matrix. In general, they are solutions to Einstein’s field equations in which the only source of gravity is some kind of radiation i.e. the source is a null fluid. The weak energy condition implies $\text{tr}(H) \leq 0$ and the scalar curvature vanishes. If $\text{tr}(H) = 0$ then $g$ satisfies the vacuum Einstein equations and, consequently, describe a plane gravitational wave (exact gravitational wave). The geodesic equations for the metric (2.1) reduce to the following ones

$$\ddot{x} = Hx,$$

$$\ddot{v} = -\frac{1}{2} x \cdot \dot{H}x - 2x \cdot H\dot{x},$$

where dot refers to derivative with respect to $u$. Furthermore, eq. (2.3) can be directly integrated yielding

$$v(u) = -\frac{1}{2} x \cdot \dot{x} + C_1 u + C_2,$$

where $x$ is a solution to (2.2). Thus for the (generalized) plane gravitational waves the solution to geodesic equations is obtained by solving the set of eqs. (2.2). Moreover, the latter are particularly interesting in the case of the exact gravitational waves since they coincide with the deviation equations and enter the transformation rules to the so-called Baldwin-Jeffery-Rosen coordinates [5, 6]. Although this set of equations cannot, in general, be explicitly solved, there are some special cases when the solution is accessible (see e.g., the classical paper [39]). Another, more geometric approach to this problem is related to the symmetry of the metric; the most interesting cases are the ones exhibiting the maximal symmetry. Let us start with the isometry groups. It is well known that the generic dimension of the isometry group of plane gravitational waves is five. There exist two exceptional families admitting the six-dimensional isometry groups. The first one is defined by the matrix $H$ of the form

$$H^{(0)}(u) = \left( \begin{array}{cc} \cos(\kappa u) & \sin(\kappa u) \\ \sin(\kappa u) & -\cos(\kappa u) \end{array} \right);$$

it admits the explicit solutions to the geodesic equations (cf. [16, 40] and references therein). The second family describes geodesically incomplete manifolds defined by $H \sim u^{-2}$; it is intensively used in the context of the Penrose limit [41, 42].

The situation becomes more interesting if we take into account the conformal symmetry. Let us recall that the maximal dimension of the conformal group of the, non-conformally flat,
metric is seven [13]. Note that the metric (2.1) admits a homothetic vector field. Therefore, the two, mentioned above, families exhibit the seven-dimensional conformal symmetry (six isometries and homothety). It appears that there exists only one subclass of the plane gravitational waves with the seven-dimensional conformal group consisting of five-dimensional isometry, homothety and non-homothetic conformal transformations (cf. [18, 19, 20] and [44, 45, 46]). This special subclass consists of two families describing linearly and circularly polarized plane gravitational waves. In the following, we concentrate on them (actually, on the geodesically complete cases as the most interesting ones). The first family, linearly polarized, is defined by the metric \( g^{(1)} \) with the profile

\[
H^{(1)}(u) = \frac{a}{(u^2 + \epsilon^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(2.6)

where \( \epsilon > 0 \) and \( a \) is an arbitrary number (excluding the trivial Minkowski case and changing \( x^1 \) and \( x^2 \) one can assume \( a > 0 \)). Moreover, let us note that taking \( a \approx \epsilon^3 \) one obtains the impulsive gravitational wave with the Dirac delta profile (as \( \epsilon \) tends to zero).

The second family \( g^{(2)} \) is an example of the circularly polarized plane gravitational waves. It is defined by the following profile

\[
H^{(2)}(u) = \frac{a}{(u^2 + \epsilon^2)^2} \begin{pmatrix} \cos(\phi(u)) & \sin(\phi(u)) \\ \sin(\phi(u)) & -\cos(\phi(u)) \end{pmatrix},
\]

(2.7)

where

\[
\phi(u) = \frac{2\gamma}{\epsilon} \tan^{-1}(u/\epsilon),
\]

(2.8)

\( \epsilon, \gamma > 0 \) and \( a \) can be chosen as above (for \( \gamma = 0 \), eq. (2.7) reduces to the previous case; however, for physical and mathematical reasons we shall consider linear and circular polarizations separately).

Some properties of the gravitational waves \( g^{(1,2)} \) were discussed in [17, 21]. Among others it was noticed that the geodesic equations can be explicitly solved; furthermore, it was shown that this fact can be simply explained in terms of the so-called Niederer transformation (see also Sec. [4]). In the case of \( g^{(1)} \) the transversal part of the geodesics reads

\[
x^i(u) = C_1^i \sqrt{u^2 + \epsilon^2} \sin(\lambda_i \tan^{-1}(u/\epsilon)) + C_2^i,
\]

(2.9)

where

\[
\lambda_i = \sqrt{1 + (-1)^i \frac{a}{\epsilon^2}} \quad i = 1, 2.
\]

(2.10)

Moreover, the initial conditions

\[
\dot{x}(-\infty) = 0, \quad x(-\infty) = x_{in},
\]

(2.11)

lead to the observation that only the second component \( x^2(u) \) exhibits focusing. In the case of \( g^{(2)} \), the solutions are of the form

\[
x(u) = \frac{\epsilon R(\tilde{u})y(\tilde{u})}{\cos(\tilde{u})}, \quad \tilde{u} = \tan^{-1}(u/\epsilon),
\]

(2.12)
where

\[ R(\bar{u}) = \begin{pmatrix} \cos(\omega \bar{u}) & -\sin(\omega \bar{u}) \\ \sin(\omega \bar{u}) & \cos(\omega \bar{u}) \end{pmatrix} \quad \omega = \frac{\gamma}{\epsilon}, \]  

and \( y \)'s are solutions to the following set of differential equations with constant coefficients

\begin{align*}
(y^2)'' + 2\omega (y^1)' + \Omega_- y^2 &= 0, \\
(y^1)'' - 2\omega (y^2)' + \Omega_+ y^1 &= 0,
\end{align*}

with

\[ \Omega_{\pm} = 1 - \omega^2 \mp \Omega, \quad \Omega = \frac{a}{\epsilon^2}; \]

here primes refer to the derivatives with respect to \( \bar{u} \). These results form a setup for our further considerations.

### 3 Exactly solvable electromagnetic vortices

Let us consider, in the Minkowski spacetime and light-cone coordinates, the following electromagnetic one-form

\[ A = -x \cdot A(u) x du. \]  

Such a potential yields the (crossed) electromagnetic fields

\[ \vec{E} = (f_1, f_2, 0), \quad \vec{B} = (-f_2, f_1, 0), \]  

where

\[ f = (f_1, f_2) = (\sqrt{2}A_{1i}x^i, \sqrt{2}A_{2i}x^i). \]  

The fields \( \vec{E} \) and \( \vec{B} \) satisfy Maxwell’s equations with the following null current

\[ j^\mu = \sqrt{2} \text{tr}(A)(1, 0, 0, 1), \quad j^\mu j_\mu = 0, \]  

(cf. [47, 48]).

The case \( \text{tr}(A) = 0 \) corresponds to the electromagnetic field satisfying the vacuum Maxwell equations; however, such fields are not, in general, electromagnetic plane waves. Furthermore, the following conditions

\[ \vec{E}^2 - \vec{B}^2 = 0, \quad \vec{E} \cdot \vec{B} = 0, \]  

hold and imply that the electromagnetic field is the pure radiation (the energy density and the Poynting vector form a null four-vector). In general, conditions (3.5) (equivalently the vanishing of the square of the Riemann-Silberstein vector) can be used to describe the vortex of the electromagnetic field [49]. However, in the null fluid case, i.e. when (3.5) vanishes identically, the notion of the vortex can be simplified; it is then related to the condition that all components of the electromagnetic field vanish (see [50]). In this approach the electromagnetic field (3.2) carries a straight vortex line along the \( z \)-axis.
Now let us note that for the potential (3.1) the Lorentz force equations
\[ m \frac{d^2x}{dt^2} = e F_{\mu \nu} \frac{dx^\nu}{dt}, \]  
(3.6)
give
\[ m \frac{d^2x}{dt^2} = e f \frac{d}{dt} (x^0 - x^3), \]  
(3.7)
\[ \frac{1}{\sqrt{2}} \left( \frac{dx^0}{dt} - \frac{dx^3}{dt} \right) = - \frac{du}{d\tau} = - \frac{p_v}{m} > 0; \]  
(3.8)
\((p_v < 0 \text{ is the light-cone momentum of the particle})\) and consequently eqs. (3.6) can be expressed in terms of the \( u \) coordinate as follows
\[ \ddot{x} = \frac{2e}{-p_v} A \dot{x}, \]  
(3.9)
\[ \ddot{v} = \frac{2e}{p_v} x \cdot A \dot{x}, \]  
(3.10)
(see also [47, 48]). Thus, up to the constant \( \frac{2e}{-p_v} \), the transverse part of the equation of motion (3.9) has the same form (in contrast to the longitudinal direction, see the discussion below) as for the plane gravitational waves (cf. eqs. (2.2)).

The above properties of the vacuum electromagnetic field given by the potential (3.1) can be considered as a manifestation of the idea of double copy approach. At the classical level [24]-[28], it consists in the mapping the solutions of the Einstein equations into solutions of Yang-Mills equations. One of the main examples of such a situation is provided by the plane gravitational waves and non-plane electromagnetic fields if we identify the matrices \( A = H \). In particular this concerns a vortex proposed in Ref. [29] which can act as a beam guide for charged particles; moreover, it is analytically solvable (see also [47]) as well as an approximation to more realistic beams [51]. Such a vortex corresponds to the plane gravitational waves defined by the profile (2.5) [32]. In what follows we show that it has solvable extensions based on the gravitational waves related to the proper conformal symmetry discussed in the previous section.

Following the above mentioned idea of classical double copy we put \( A^{(1,2)} = H^{(1,2)} \); then, by virtue of (2.6), (2.7) and (3.1)-(3.3), one obtains the following electromagnetic fields in the Minkowski spacetime

\[ \vec{E}^{(1)}(x) = \frac{\sqrt{2}a}{(u^2 + \epsilon^2)^2} \left( x^1, -x^2, 0 \right), \quad \vec{B}^{(1)}(x) = \frac{\sqrt{2}a}{(u^2 + \epsilon^2)^2} \left( x^2, x^1, 0 \right); \]  
(3.11)
\[ \vec{E}^{(2)}(x) = \frac{\sqrt{2}a}{(u^2 + \epsilon^2)^2} \left( x^1 \cos(\phi(u)) + x^2 \sin(\phi(u)), x^1 \sin(\phi(u)) - x^2 \cos(\phi(u)), 0 \right), \]  
(3.12)
\[ \vec{B}^{(2)}(x) = \frac{\sqrt{2}a}{(u^2 + \epsilon^2)^2} \left( -x^1 \sin(\phi(u)) + x^2 \cos(\phi(u)), x^1 \cos(\phi(u)) + x^2 \sin(\phi(u)), 0 \right), \]
where \( \phi \) is given by (2.8) and \( u = (x^3 - x^0)/\sqrt{2} \). As it has been indicated in Sec. 2 the geodesics equations for \( g^{(1,2)} \) are analytically solvable; in consequence one immediately (after replacing \( a \mapsto \frac{2 e}{p_0} a \), see (2.2) and (3.9)) obtains the solution to the transverse Lorentz force equations with \( \vec{E}^{(1,2)} \) and \( \vec{B}^{(1,2)} \) (cf. eqs. (2.9) and (2.12)-(2.15)). Finally, taking an appropriate limit (cf. [17, 21]) one gets the case of electromagnetic field with the profile proportional to the Dirac delta function \( \delta(u) \).

Now, let us consider the longitudinal direction. In contrast to the gravitational case eq. (3.10) can be directly integrated only once

\[
\dot{v} = -\frac{1}{2} \dot{x} \cdot \dot{x} + D_1. \tag{3.13}
\]

However, in what follows we show that for the field given by (3.11) and (3.12) it is also possible to find explicitly the \( v \) coordinate. Indeed, integrating (3.13) by substitution \( \tilde{u} = \tan^{-1}(u/\epsilon) \) one obtains

\[
v(u) = D_1 u + D_2 - \frac{1}{2\epsilon} \int \dot{x}'(\tilde{u}) \cdot \dot{x}'(\tilde{u}) \cos^2(\tilde{u}) d\tilde{u}. \tag{3.14}
\]

Now, for the fields \( \vec{E}^{(1)}, \vec{B}^{(1)} \) using (2.9) one gets, after some computations, the following final form of the longitudinal coordinate

\[
v(u) = -\frac{\epsilon}{2} \sum_i (C_i^2) \left( \frac{1}{2} (b_i^2 - 1) \tilde{u} + \sin^2(b_i \tilde{u} + C_2^i) \tan(\tilde{u}) + \frac{1 + b_i^2}{4b_i} \sin(2(b_i \tilde{u} + C_2^i)) \right)
+ D_2 + D_1 u, \tag{3.15}
\]

where \( \tilde{u} = \tan^{-1}(u/\epsilon) \) and \( b_i = \sqrt{1 - (-1)^i \frac{2 e a}{\epsilon^2 p_0}} \).

The second case is slightly more involved. First, we express the integral in (3.14) in terms of \( y \)'s satisfying eqs. (2.14) (with \( \Omega = \frac{2 e a}{-p_0 \epsilon^2} \)) and, subsequently, we extract a total time derivative term; in consequence we arrive at the following equalities

\[
-\frac{1}{\epsilon^2} \int \dot{x}'(\tilde{u}) \cdot \dot{x}'(\tilde{u}) \cos^2(\tilde{u}) d\tilde{u} =
\int \left[ ((y^1)' - \omega y^2)^2 + ((y^2)' + \omega y^1)^2 + 2 \tan(\tilde{u})((y^1)' y^1 + (y^2)' y^2) + \tan^2(\tilde{u})((y^1)^2 + (y^2)^2) \right] d\tilde{u}
= (y^2)' y^2 + (y^1)' y^1 + ((y^1)^2 + (y^2)^2) \tan(\tilde{u}) + \Omega \int [(y^2)^2 - (y^1)^2] d\tilde{u}. \tag{3.16}
\]

Let us note that \( y \)'s are combinations of the trigonometric or hyperbolic functions (cf. (2.14)); thus the last integral in (3.16) is an elementary one and can be explicitly computed. Finally, substituting \( \tilde{u} = \tan^{-1}(u/\epsilon) \) in (3.14) one obtains the form of \( v(u) \) which gives the analytical solvability of the Lorentz force equations in the electromagnetic backgrounds (3.11) and (3.12) (extending in this way the results obtained in Ref. [23]).
4 Niederer’s map and Lorentz’s force equation

It turns out that \[21\] the existence of analytical solutions to the geodesic equations for the gravitational waves \( g^{(1,2)} \) can be simply explained in terms of the so-called Niederer transformation \[22, 23\]. We shall show that the similar situation holds also when the electromagnetic fields \[3.11\] or \[3.12\] are switched on. To this end let us recall some facts. The Niederer transformation

\[
\begin{align*}
\tilde{u} &= \epsilon \tan(\tilde{\nu}), \\
\tilde{x} &= \frac{\epsilon \tilde{\nu}}{\cos(\tilde{\nu})},
\end{align*}
\]

relates the free motion \( \ddot{x} = 0 \) (for our purpose we consider the 2-dimensional case) on the whole real axis \((-\infty < u < \infty)\) to the half of period motion \((-\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2})\) of the attractive harmonic motion \( \ddot{\tilde{x}} = -\tilde{x} \); as above dot and prime refer to the derivatives with respect to \( u \) and \( \tilde{u} \), respectively (this equivalence continues also to hold at the quantum level \[22\]). Of course, the above observation have a local character; however, it reflects a similarity between both systems and brings some useful information. Various local quantities can be directly related; this concerns even the global ones (for instance, Feynman propagators) if sufficient care is exercised (see, e.g., \[22, 52, 53\]). In particular, the maximal symmetry groups of both systems are isomorphic and one obtains the explicit relation between symmetry generators as well as solutions of the both systems \[54\].

On the other hand eqs. \[2.2\] describe, in fact, a linear oscillator (in general) with time-dependent frequencies; however, it turns out that in some cases the Niederer mapping can be also applied to relate them to a harmonic (or a more simpler linear) oscillator \[21\]. Namely, under the Niederer transformation eqs. \[2.2\] are transformed into the following ones

\[
\ddot{\tilde{x}} = \tilde{H}(\tilde{u})\tilde{x},
\]

where

\[
\tilde{H}(\tilde{u}) = \frac{\epsilon^2 H(\epsilon \tan(\tilde{u}))}{\cos^4(\tilde{u})} - I.
\]

In particular, if the matrix \( H \) is of the form

\[
H(u) = \frac{a}{(\epsilon^2 + u^2)^2} G(u),
\]

where \( G \) is a symmetric matrix, then, by virtue of eq. \[4.3\], the equations of motion \[4.2\] read

\[
\ddot{\tilde{x}} = \tilde{H}(\tilde{u})\tilde{x} = \left( \frac{a}{\epsilon^2} G(\epsilon \tan(\tilde{u})) - I \right) \tilde{x}.
\]

For example, the (non-singular) time-dependent linear oscillator defined by a constant matrix \( G \) is mapped under \[4.1\] to a part of motion of the harmonic oscillator. In consequence the

\[\text{---}^2\text{For further considerations we adopt } u\text{-notation. There exists a hyperbolic counterpart of Niederer’s transformation leading to the repulsive case.}\]
Niederer transformation can be useful to solve some geodesic equations for plane gravitational waves (or other problems where time dependent linear oscillators occur). Moreover, such properties of the Niederer transformation have a reflection in a geometric picture obtained by means of the Eisenhart-Duval lift \[23, 55\] (see also \[56, 57\] and references therein). Namely, extending the Niederer map by adding the following transformation rule
\[
v = \epsilon \tilde{v} - \frac{\epsilon \tan(\tilde{u})}{2}\tilde{x}^2,
\]
(4.6)
one gets the identity
\[
g \equiv x \cdot H(u) x du^2 + 2dudv + dx^2 = \frac{\epsilon^2}{\cos^2(u)}(\tilde{x} \cdot \tilde{H}(\tilde{u}) \tilde{x} d\tilde{u}^2 + 2d\tilde{u}d\tilde{v} + d\tilde{x}^2) \equiv \frac{\epsilon^2}{\cos^2(u)}\tilde{g},
\]
(4.7)
where \(H\) and \(\tilde{H}\) are connected by eq. \(4.3\). For the particular case, \(H = 0\), \(4.7\) reduces to the well-known relation between the Bargmann spacetimes corresponding to the free \((u, x, v)\) and the half-oscillatory period \((\tilde{u}, \tilde{x}, \tilde{v})\) motions in the Eisenhart-Duval lift language \[23\] (see also \[58, 59\]).

In view of eq. \(4.7\), by means of the Niederer transformation one can associate with the metric \(g\) the new one \(\tilde{g}\), conformally related to \(g\), belonging to the same class (generalized plane gravitational waves). Of course, at most only one of the metrics \(g\) and \(\tilde{g}\) describes the vacuum solution; moreover, the geodesic equations for \(g\) and \(\tilde{g}\) are not equivalent (except those for null geodesics). However, from the reasoning underlying the Niederer transformation it follows that the transversal geodesic equations \(2.2\) for \(g\) are mapped into the transversal geodesic equations \(4.2\) for the metric \(\tilde{g}\), which may be more tractable than those for \(g\). The non-equivalence of the geodesic equations for \(g\) and \(\tilde{g}\) is reduced to the non-equivalence of equations determining \(v\) and \(\tilde{v}\) (however, the latter can be easily solved, cf. eq. \(2.4\)). Such a situation holds, for instance, for the conformally distinguished metric families: \(g^{(1,2)}\) and the one defined by \(G \sim I\) (see \[17, 21\]).

Now let us analyse the action of Niederer’s transformation on the spacetime described by the metric \(2.1\) (in particular for, \(H = 0\) the Minkowski one) endowed with the electromagnetic field given by the potential \(3.1\). First let us note that the condition \(\text{tr}(A) = 0\) implies that, as in the Minkowski case, the electromagnetic field under consideration satisfies the vacuum Maxwell equations on the spacetime defined by a plane gravitational wave (this fact is valid even in the case of an arbitrary pp-wave spacetime). Furthermore, in this case the equations of motion of a charged test particle
\[
\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\mu} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} = \frac{q}{m} g^{\alpha\nu} F_{\nu\mu} \frac{dx^\mu}{d\tau},
\]
(4.8)
reduce to the following ones
\[
\dot{x} = (H - \frac{2e}{p_v} A)x
\]
(4.9)
\[
\dot{v} = -\frac{1}{2} x \cdot \dot{H} x - 2x \cdot (H - \frac{e}{p_v} A) \dot{x}
\]
(4.10)
(for the Minkowski spacetime they coincide with the eqs. (3.9) and (3.10)).

From the above we see that the transversal part of the Lorentz force equation is still decoupled and can be considered separately. Thus we can try to extend the geometric interpretation and applications of Niederer’s transformation to electromagnetic fields in curved spacetimes. To this end let us note that under the Niederer transformation the transverse set of equations (4.9) is mapped to following ones

\[ \tilde{x}'' = \left( \tilde{H}(\tilde{u}) + \frac{2e}{p_v} \tilde{A}(\tilde{u}) \right) \tilde{x}, \]  

(4.11)

where \( \tilde{H} \) is given by (4.3) while

\[ \tilde{A}(\tilde{u}) = \frac{\epsilon^2}{\cos^4(\tilde{u})} A(\epsilon \tan(\tilde{u})). \]  

(4.12)

The solutions to eqs. (4.11) corresponding the purely gravitational case, i.e. \( A = 0 \) and \( g = g^{(1,2)} \), have been discussed in [17, 21] (see Sec. 2), while for the Minkowski spacetime endowed with the electromagnetic field given by (3.11) and (3.12) in Sec. 3. However, by considering these gravitational and electromagnetic backgrounds altogether (i.e. \( \vec{E}^{(i)}, \vec{B}^{(i)} \) on the spacetime \( g^{(i)} \), for \( i = 1, 2 \) respectively) one finds that in these cases the transverse Lorentz force equations, given by (4.11), are also analytically solvable.

Next, by virtue of (4.7) the Niederer map transforms the metric \( g \) conformally into the metric \( \tilde{g} \) of the same type, i.e. we do not leave out the class of the generalized plane gravitational waves; for the electromagnetic potential (3.1) one obtains

\[ \tilde{A}(\tilde{u}) = -x \cdot A(u) x du = \epsilon x \cdot \tilde{A}(\tilde{u}) x d\tilde{u}, \]  

(4.13)

where \( \tilde{A} \) is defined by (4.12). Thus the potential \( \tilde{A} \), in the new coordinates, is also of the same type as \( A \) and, consequently, yields also a crossed electromagnetic field. Moreover, the pair \( (\tilde{g}, \tilde{A}) \) is the one for which the transverse part of the Lorentz force equations is given by eq. (4.11) (let us note that since \( \tilde{g} \) is conformally related to \( g \) the electromagnetic field arising from the potential \( \tilde{A} \) is a vacuum solution to the Maxwell equation with respect to \( \tilde{g} \); in contrast to the metric case where the vacuum solution \( g \) is mapped to the non-vacuum null-fluid solution \( \tilde{g} \)). In this way we extend the notion of the Niederer transformation, originally established for the non-relativistic dynamical systems, to the one including electromagnetic fields on curved spacetimes; namely with the pair \( (g, A) \) we associate the new one \( (\tilde{g}, \tilde{A}) \) such that the transverse part of the Lorentz force equations for \( (g, A) \) transform, by means of the Niederer map, into the one for \( (\tilde{g}, \tilde{A}) \) (moreover \( g, \tilde{g} \) and \( A, \tilde{A} \) belong to the same classes: the generalized plane gravitational waves and vacuum crossed electromagnetic fields, respectively).

5 Light-matter interaction

In the context of the interaction between laser and matter the following electromagnetic fields

\[ \vec{E}(x) = \mathcal{E}(u)(x^1, x^2, \sqrt{2}u), \quad \vec{B}(x) = \mathcal{E}(u)(-x^2, x^1, 0), \]  

(5.1)
were considered in Ref. [35], where the function $E(u)$ is picked, at least, at $u = 0$. Such fields capture some essential features of the transverse magnetic beams near the beam axis, such as the polarization structure, the local rise of the transverse fields, and the suppression of the longitudinal field as well as satisfies $\vec{E} \cdot \vec{B} = 0$ while, in contrast to (3.5), $\vec{E}^2 - \vec{B}^2 > 0$ (see also [35]).

First, we shall show that such fields can emerge from the gravitational metrics if we slightly modify the correspondence between gravitational, given by (2.1), and electromagnetic fields. As before we put $A = H$ in the one-form (3.1); however, this time we add a new term with an arbitrary function $F$

$$A = -x \cdot H(u) du + F(u) dv. \tag{5.2}$$

Then one gets

$$\vec{E} = (f_1, f_2, \dot{F}), \quad \vec{B} = (-f_2, f_1, 0), \tag{5.3}$$

$$j^\mu = \sqrt{2}(\text{tr}(H) + \frac{1}{2} \dot{F})(1, 0, 0, 1), \quad j^\mu j_\mu = 0, \tag{5.4}$$

where $f_1$ and $f_2$ are given by (3.3). Now, repeating the previous considerations, one finds the following relation

$$m \frac{du}{d\tau} = -eF(u) - D, \tag{5.5}$$

and, consequently, the Lorentz equations in terms of the $u$ coordinate read (cf. [47])

$$\ddot{x}(eF + D) + e\dot{F}\dot{x} = 2eHx, \tag{5.6}$$

$$\dot{v}(eF + D) + 2e\dot{F}\dot{v} = -2e\dot{x} \cdot H\dot{x}. \tag{5.7}$$

Let us consider the diagonal profile $H_{11} = H_{22}, H_{12} = H_{21} = 0$ and the function

$$F(u) = 2 \int_u^{-\infty} \bar{u}H_{11}(\bar{u}) d\bar{u} \tag{5.8}$$

(we assume here the vanishing of the gravitational profile at plus/minus null infinity in such a way that the function $F$ is well defined). Consequently, one gets the electromagnetic field given by (5.1) with

$$E(u) = \sqrt{2}H_{11}(u). \tag{5.9}$$

Of course, the function $F$ can be chosen up to a (irrelevant) constant, and our choice $F(-\infty) = 0$, due to the asymptotic vanishing of the gravitational profile and (5.5), gives

$$D = -p_v, \tag{5.10}$$

where $p_v < 0$ (cf. (3.8)). Finally, let us note that the choice of the matrix $H$, $\text{tr}(H) \neq 0$, corresponds to the, non-vacuum, generalized plane gravitational waves (here, in general, we do not assume that the weak energy condition holds) and, consequently, gives non-vacuum electromagnetic fields.
Eq. (5.8) together with diagonal form of the matrix $H$ imply that $x_i^i(u) = u$ is a solution to the transverse Lorentz force equation (5.6). Then the general solution is a linear combination of $x_i^1$ and the second solution of the form

$$x_2^i(u) = u \int \frac{du}{u^2(eF(u) - pv)}; \tag{5.11}$$

(in the case of the longitudinal direction eq. (5.7) can be once integrated [35]).

Now, taking $H_{11} = H_{11}^{(1)}$ given by (2.6) one gets

$$F(u) = -\frac{a}{u^2 + \epsilon^2}, \tag{5.12}$$

and the following electromagnetic vector field

$$\vec{E} = \frac{\sqrt{2}a}{(u^2 + \epsilon^2)^2} (x^1, x^2, \sqrt{2}u), \quad \vec{B} = \frac{\sqrt{2}a}{(u^2 + \epsilon^2)^2} (-x^2, x^1, 0), \tag{5.13}$$

which was studied, in another gauge, in [35]. More precisely, it was shown there that the transverse Lorentz equations are explicitly solvable; moreover, there exist solutions exhibiting the focusing property. Namely, for the electromagnetic field (5.13) the conditions (2.11) lead to the following solutions to eqs. (5.6)

$$x(u) = x_m \left[ 1 + \frac{ug}{\epsilon\sqrt{1 - g}} \left( \frac{\pi}{2} + \tan^{-1} \left( \frac{u}{\epsilon\sqrt{1 - g}} \right) \right) \right], \tag{5.14}$$

where

$$g \equiv \frac{ea - pv\epsilon^2}{p^2}, \tag{5.15}$$

satisfies $g < 1$ (for $g > 1$ the solutions are singular and thus they are skipped here). Now, one can show that for $g < 0$ there is a point $u_0$ such that $x^i(u_0) = 0$, i.e. focusing; for $g > 0$ there is no focusing. For example, taking $a = -\text{sgn}(\epsilon)$ one gets the focusing case $g = \frac{|\epsilon|}{p^2\epsilon^2} < 0$.

However, as it was suggested in [35], it would be interesting to analyise explicitly an example of the electric field (5.1) with multiple field oscillations (in contrast to (5.13)). To this end we consider a counterpart of the circularly polarized plane gravitational waves (2.7). Namely, let us put $\gamma = r\epsilon$, $r > 0$ in $H^{(2)}$ (see (2.7) and (2.8)) and consider the profile

$$H_{11} = H_{22} = \frac{a}{(u^2 + \epsilon^2)^2} \cos(2r\tan^{-1}(u/\epsilon)), \quad H_{12} = H_{21} = 0. \tag{5.16}$$

Then the function $F$ is as follows: for $r = 1$

$$F(u) = \frac{au^2}{(u^2 + \epsilon^2)^2}, \tag{5.17}$$

and for $r \neq 1$

$$F(u) = a\frac{2e^2u^2 \sin(2r\tan^{-1}(u/\epsilon)) + (\epsilon^2 - u^2) \cos(2r\tan^{-1}(u/\epsilon))}{2e^2(r^2 - 1)(u^2 + \epsilon^2)} + \frac{a}{2e^2(r^2 - 1)} \cos(\pi r). \tag{5.18}$$
Let us begin with the case of \( r = 1 \). Then
\[
\mathcal{E}(u) = \sqrt{2a} \frac{e^2 - u^2}{(u^2 + e^2)^{3/2}},
\] (5.19)
and the transverse part of the electromagnetic field is picked at \( u = 0 \) and has two zeros. We shall show that the trajectories can also be explicitly written down.

Let us consider three cases distinguished by the value of the constant \( g \) given by eq. (5.15).

First, let \( g > 0 \). Imposing (2.11) one gets the solution
\[
x(u) = x_{in} \left[ 1 + \frac{gu}{4g + g^2} \left( \frac{1}{\sqrt{A_-}} \left( \frac{\pi}{2} + \tan^{-1} \left( \frac{u}{\sqrt{A_-}} \right) \right) - \frac{1}{\sqrt{A_+}} \left( \frac{\pi}{2} + \tan^{-1} \left( \frac{u}{\sqrt{A_+}} \right) \right) \right) \right],
\] (5.20)
where
\[
A_\pm \equiv \frac{e^2}{2} (2 + g \pm \sqrt{4g + g^2}) > 0.
\] (5.21)

Using \( g > 0 \) one can show, after some calculations, that there is no focusing.

Next, when \( 0 > g > -4 \), we have
\[
x(u) = x_{in} \left[ 1 + \frac{gu}{4B} \left( \pi + \tan^{-1} \left( \frac{u + \epsilon \sqrt{-g/4}}{B} \right) + \tan^{-1} \left( \frac{u - \epsilon \sqrt{-g/4}}{B} \right) \right) \right.
\]
\[
- \frac{\sqrt{-gu}}{2\epsilon} \tanh^{-1} \left( \frac{\epsilon \sqrt{-gu}}{u^2 + \epsilon^2} \right) \right],
\] (5.22)
where
\[
B = \epsilon \sqrt{1 + g/4}.
\] (5.23)

Then there is a point \( u_0 \) where \( x(u_0) = 0 \) for all initial points, i.e. focusing. Let us note that taking \( a = -\text{sgn}(\epsilon) \) one gets \( g = \frac{|\epsilon|}{p_v} < 0 \); however, only for some values of the particle parameters the condition \( g > -4 \) (and consequently the focusing property) holds (in contrast to the electromagnetic field (5.13), non-vanishing in the transverse direction). Finally, for \( g \leq -4 \) the solutions can also be expressed in terms of elementary functions; however, they exhibit some singularities thus we skip them here.

For arbitrary function \( \mathcal{F} \) the integral in (5.11) cannot be explicitly computed; even when the functions \( \mathcal{E} \) and \( \mathcal{F} \) are both rational ones (e.g., \( \mathcal{F} \) given by (5.16) with the non-negative integer \( r \)). Thus, in what follows we give some sufficient conditions to ensure the focusing property of the electromagnetic field (5.1) and next we apply them to the case (5.16).

To begin with, we should assume that the denominator in the integral (5.11) does not vanish anywhere
\[
e\mathcal{F}(u) - p_v \neq 0;
\] (5.24)
this can be achieved by a suitable choice of the range of parameters \( e \) and/or \( p_v \) (since \( \mathcal{F} \) tends to zero at the null infinities). Next, we rewrite the general solution in the following form
\[
x^i(u) = C^i_1 x_1^i(u) + C^i_2 x_2^i(u) = C^i_1 u + \frac{C^i_1 p_v}{e\mathcal{F}(0) - p_v} + C^i_2 \int_{-\infty}^{u} \frac{G(\bar{u})d\bar{u}}{\bar{u}^2(e\mathcal{F}(\bar{u}) - p_v)},
\] (5.25)
where
\[ G(u) = \frac{-ep_v(F(0) - F(u))}{eF(0) - p_v}. \] (5.26)

Then the function under the integral (5.25) is well defined on the whole real line since \( G(0) = 0 \) and, by virtue of (5.8), \( \dot{G}(0) = 0 \). Now imposing the conditions (2.11) one obtains \( C_{1i} = 0 \) and, by virtue of L'Hospital's rule, the following form of the solution
\[ x(u) = x_{in} \left( -u \int_{-\infty}^{u} \frac{G(\bar{u})d\bar{u}}{\bar{u}^2(eF(\bar{u}) - p_v)} - \frac{p_v}{eF(0) - p_v} \right). \] (5.27)

Hence, the inequality
\[ \frac{1}{(eF(0) - p_v)} \int_{-\infty}^{\infty} \frac{e(F(0) - F(u))du}{u^2(eF(u) - p_v)} > 0, \] (5.28)

together with (5.24) imply a focusing point. For instance, the condition (5.28) is satisfied when the function \( eF(u) \) has the global maximum at \( u = 0 \),
\[ eF(0) \geq eF(u). \] (5.29)

One can check that in the case of the electromagnetic field defined by (5.13) and (5.19) (equivalently, (5.16) with \( r = 0,1 \)) the criteria (5.24) and (5.29) yield the conditions obtained above. Furthermore, applying these criteria to the electromagnetic field defined by (5.16) with \( 0 < r < 1 \) we arrive at the following focusing conditions
\begin{align*}
\text{for } & 0 < r \leq \frac{1}{2}, \quad g < 0; \quad (5.30) \\
\text{for } & \frac{1}{2} < r < 1, \quad \frac{2(1 - r^2)}{\cos(\pi r) + r \sin(\frac{\pi}{2})} < g < 0. \quad (5.31)
\end{align*}

Taking into account the above discussed case \( r = 1 \) we see explicitly that when the profile of the electromagnetic field vanishes at some points, see (5.9) and (5.16) with \( \frac{1}{2} < r \leq 1 \), then there are some additional restrictions for the particle parameters (see (5.31)). Finally, it is worth to notice that (5.31) gives a new Kober's-type inequality [60] which we leave for further study.

### 6 Conclusions and outlook

Let us summarize. In the present work, using the idea of the classical double copy, we construct null electromagnetic fields which are explicitly solvable and directly generalize the electromagnetic vortices considered, in the context of singular optics, in Ref. [29]. Since these results are strictly related to the notion of Niederer's transformation we analyse the geometric extension of the latter in the case of the plane gravitational spacetimes endowed with the crossed electromagnetic fields (3.2) and (3.3); in this approach the transverse part
of the Lorentz force equations transformed by the Niederer map can be obtained by means of a new metric (conformally related to the initial one) and a new, crossed, electromagnetic field. In consequence, for some special cases, related to conformal symmetry of spacetimes, the transverse Lorentz equations on a curved spacetime can be also analytically solvable.

In the second part we showed that the electromagnetic backgrounds which capture some essential features of the transverse magnetic laser beam near the beam axis, proposed in Ref. [35], can also emerge from some gravitational counterparts. Moreover, one of the latter ones leads to the electromagnetic profile with zeros and is analytically solvable in the transverse directions. The focusing properties of such electromagnetic fields seem to be especially interesting; thus we gave some criteria and apply them to the electromagnetic counterparts of some gravitational metrics.

The results obtained can be extended in various directions. As usual the exact solvability at the classical level should have its reflection at the quantum level. Thus it would be interesting to consider the quantum picture of the results obtained here, including both gravitational and electromagnetic backgrounds (see e.g. [35] [61]). This is especially interesting in view of the double copy and recent results presented in Ref. [62] for gravitational sandwiches. Furthermore, let us recall that the Penrose limit of spacetimes yields the plane gravitational waves; thus, the question is which of ones correspond to the distinguished plane wave spacetimes $g^{(1,2)}$ (cf. [9] [11] [42]). Moreover, the analytical solutions obtained can be also useful in the recent studies concerning some aspects of optical effects in the nonlinear plane gravitational waves [63] [64] as well as trapping problems in gravity [32] [33] [34]. Finally, following Refs. [37] [38] and [65] - [66], we hope that they can be also useful in the study of the light-matter interaction, especially for strong focusing of short or intense laser pulses.

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