1 Introduction.

In this paper we analyze the following problem: to find a non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$, on $\mathbb{R}_+ = [0, +\infty)$, $M(0) = 0$, which is left-continuous on $(0, +\infty)$, and such that

$$\int_{\mathbb{R}_+} x^n dM(x) = S_n, \quad n \in \mathbb{Z},$$

(1)

where $\{S_n\}_{n \in \mathbb{Z}}$ is a prescribed sequence of Hermitian $(N \times N)$ complex matrices (moments), $N \in \mathbb{N}$. This problem is said to be a strong matrix Stieltjes moment problem. The problem is said to be determinate if it has a unique solution and indeterminate in the opposite case.

The scalar ($N = 1$) strong Stieltjes moment problem (in a slightly different setting) was introduced in 1980 by Jones, Thron and Waadeland [1]. Necessary and sufficient conditions for the existence of a solution with an infinite number of points of increase and for the uniqueness of such a solution were established [1, Theorem 6.3]. Also necessary and sufficient conditions for the existence of a unique solution with a finite number of points of increase were obtained [1, Theorem 5.2]. The approach of Jones, Thron and Waadeland’s investigation was made through the study of special continued fractions related to the moments.

In 1995, Njåstad described some classes of solutions of the scalar strong Stieltjes moment problem [2], [3]. He used properties of some associated Laurent polynomials.

In 1996, Kats and Nudelman obtained necessary and sufficient conditions for the existence of a solution of the scalar strong Stieltjes moment problem [4, Theorem 1.1]. The degenerate case was studied in full: in this case the solution is unique, given explicitly and it has a finite number of points of increase. In the non-degenerate case, conditions for the determinacy were given and the unique solution was presented. In the (non-degenerate) indeterminate case a Nevanlinna-type parameterization for all solutions of the scalar strong Stieltjes moment problem was obtained [4, Theorem 4.1]. Canonical solutions and Weyl-type lunes were studied, as well. Kats and Nudelman used the results of Krein on the semi-infinite string theory.

Various other results on the scalar strong Stieltjes moment problem can be found in papers [5], [6], [7], [8], [9] (see also References therein).
The moment problem \( (1) \) where the half-axis \( \mathbb{R}_+ \) is replaced by the whole axis \( \mathbb{R} \) is said to be the **strong matrix Hamburger moment problem**. The scalar \((N = 1)\) strong matrix Hamburger moment problem has been intensively studied since 1980-th, see a survey [7], a recent paper [10] and References therein. For the matrix case, see papers [11], [12] and papers cited there.

The aim of our present investigation is threefold. Firstly, we obtain necessary and sufficient conditions for the solvability of the strong matrix Stieltjes moment problem \( (1) \). Consider the following block matrices constructed by moments:

\[
\Gamma_n = (S_{i+j})_{i,j=-n}^n = \begin{pmatrix}
S_{-2n} & \ldots & S_{-n} & \ldots & S_0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
S_{-n} & \ldots & S_0 & \ldots & S_n \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
S_0 & \ldots & S_n & \ldots & S_{2n}
\end{pmatrix}, \quad n \in \mathbb{Z},
\]

\( (2) \)

\[
\tilde{\Gamma}_n = (S_{i+j+1})_{i,j=-n}^n = \begin{pmatrix}
S_{-2n+1} & \ldots & S_{-n+1} & \ldots & S_1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
S_{-n+1} & \ldots & S_1 & \ldots & S_{n+1} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
S_1 & \ldots & S_{n+1} & \ldots & S_{2n+1}
\end{pmatrix}, \quad n \in \mathbb{Z},
\]

\( (3) \)

We shall prove that conditions

\[
\Gamma_n \geq 0, \quad \tilde{\Gamma}_n \geq 0 \quad n = 0, 1, 2, \ldots,
\]

are necessary and sufficient for the solvability of the moment problem \( (1) \).

Secondly, we obtain an analytic description of all solutions of the moment problem \( (1) \). We shall use an abstract operator approach similar to the "pure operator" approach of Szőkefalvi-Nagy and Koranyi to the Nevanlinna-Pick interpolation problem, see [13], [14]. We shall need some properties of generalized \( \Pi \)-resolvents of non-negative operators and generalized \( sc \)-resolvents of Hermitian contractions, established by Krein and Ovcharenko in [15], [16]. As a by-product, we present a description of generalized \( \Pi \)-resolvents of a non-negative operator which does not use improper elements or relations as it was done in the original work of Krein [17] and in the paper of Derkach and Malamud [18]. Here we adapt some ideas from [19] of Chumakin who studied generalized resolvents of isometric operators.
Thirdly, we obtain necessary and sufficient conditions for the strong matrix Stieltjes moment problem to be determinate.

Notations. As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \), the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. The space of \( n \)-dimensional complex vectors \( a = (a_0, a_1, \ldots, a_{n-1}) \), we denote by \( \mathbb{C}^n \), \( n \in \mathbb{N}. \) If \( a \in \mathbb{C}^n \), then \( a^* \) means the complex conjugate vector. By \( P_L \) we denote the space of all complex Laurent polynomials, i.e. functions \( \sum_{k=a}^b \alpha_k x^k \), \( a, b \in \mathbb{Z} : a \leq b, \alpha_k \in \mathbb{C}. \)

Let \( M(x) \) be a left-continuous non-decreasing matrix function \( M(x) = (m_{k,l}(x))_{k,l=0}^{N-1} \) on \( \mathbb{R} \), \( M(-\infty) = 0 \), and \( \tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x) \); \( \Psi(x) = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1} \). By \( L^2(M) \) we denote a set (of classes of equivalence) of vector-valued functions \( f : \mathbb{R} \to \mathbb{C}^N, f = (f_0, f_1, \ldots, f_{N-1}), \) such that (see, e.g., [20])

\[
\|f\|_{L^2(M)}^2 := \int_{\mathbb{R}} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.
\]

The space \( L^2(M) \) is a Hilbert space with a scalar product

\[
(f,g)_{L^2(M)} := \int_{\mathbb{R}} f(x)\Psi(x)g^*(x)d\tau_M(x), \quad f,g \in L^2(M).
\]

We denote \( \tilde{e}_k = (\delta_{0,k}, \delta_{1,k}, \ldots, \delta_{N-1,k}), \) \( 0 \leq k \leq N-1, \) where \( \delta_{j,k} \) is the Kronecker delta.

If \( H \) is a Hilbert space then \( (\cdot, \cdot)_H \) and \( \| \cdot \|_H \) mean the scalar product and the norm in \( H \), respectively. Indices may be omitted in obvious cases. For a linear operator \( A \) in \( H \), we denote by \( D(A) \) its domain, by \( R(A) \) its range, by \( \text{Ker} \ A \) its kernel, and \( A^* \) means the adjoint operator if it exists. If \( A \) is invertible then \( A^{-1} \) means its inverse. \( \overline{A} \) means the closure of the operator, if the operator is closable. If \( A \) is self-adjoint, by \( R_s(A) \) we denote the resolvent of \( A, z \in \mathbb{C} \setminus \mathbb{R}. \) If \( A \) is bounded then \( \|A\| \) denotes its norm. For an arbitrary set of elements \( \{x_n\}_{n \in \mathbb{Z}} \) in \( H \), we denote by Lin\( \{x_n\}_{n \in \mathbb{Z}} \) and span\( \{x_n\}_{n \in \mathbb{Z}} \) the linear span and the closed linear span (in the norm of \( H \)), respectively. For a set \( M \subseteq H \) we denote by \( \overline{M} \) the closure of \( M \) in the norm of \( H \). By \( E_H \) we denote the identity operator in \( H \), i.e. \( E_H x = x, x \in H. \) If \( H_1 \) is a subspace of \( H \), then \( P_{H_1} = P_{H_1}^H \) is an operator of the orthogonal projection on \( H_1 \) in \( H. \) By \( [H] \) we denote a set of all bounded linear operators \( A \) in \( H, D(A) = H. \)
The solvability of the strong matrix Stieltjes moment problem.

In this section we are going to establish the following theorem.

**Theorem 1.** Let the strong matrix Stieltjes moment problem (1) with a set of moments \( \{S_n\}_{n \in \mathbb{Z}} \) be given. The moment problem has a solution if and only if conditions (4) are satisfied.

**Proof. Necessity.** Let the strong matrix Stieltjes moment problem (1) has a solution \( M(x) \). Choose an arbitrary vector function \( f(x) = \sum_{k=-n}^{N-1} \sum_{j=0}^{N-1} f_{j,k} x^k \vec{e}_j \), \( f_{j,k} \in \mathbb{C} \). This function belongs to \( L^2(M) \) and

\[
0 \leq \int_{\mathbb{R}} f(x) x^s dM(x) f^*(x) = \sum_{k,r=-n}^{n} \sum_{j,l=0}^{N-1} f_{j,k} \overline{f_{l,r}} \int_{\mathbb{R}} x^{k+r+s} \vec{e}_j dM(x) \vec{e}_l^*
\]

\[
= \sum_{k,r=-n}^{n} \sum_{j,l=0}^{N-1} f_{j,k} \overline{f_{l,r}} S_{k+r+s} = \sum_{k,r=-n}^{n} (f_{0,k}, f_{1,k}, \ldots, f_{N-1,k}) S_{k+r+s}
\]

\[
* (f_{0,r}, f_{1,r}, \ldots, f_{N-1,r})^* = \begin{cases} v \Gamma_n v^*, & s = 0 \\ v \Gamma_n v^*, & s = 1 \end{cases}
\]

where \( v = (f_{0,-n}, f_{1,-n}, \ldots, f_{N-1,-n}, f_{0,-n+1}, f_{1,-n+1}, \ldots, f_{N-1,-n+1}, \ldots, f_{0,n}, f_{1,n}, \ldots, f_{N-1,n}) \). Here we make use of the rules for the multiplication of block matrices.

**Sufficiency.** Let the strong matrix Stieltjes moment problem (1) be given and conditions (4) be satisfied. Let \( S_j = (S_{j;k,l})_{k,l=0}^{N-1} \), \( S_{j;k,l} \in \mathbb{C} \), \( j \in \mathbb{Z} \). Consider the following infinite block matrix:

\[
\Gamma = (S_{i+j})_{i,j=-\infty}^{\infty} = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & S_{-2n} & S_{-n} & S_0 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & S_{-n} & S_0 & S_n & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & S_0 & S_n & S_{2n} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}, \quad (5)
\]

where the element in the box corresponds to the indices \( i = j = 0 \).

We assume that the left upper entry of the element in the box stands in row 0, column 0. Let us numerate rows (columns) in the increasing order to
the bottom (respectively to the right). Then we numerate rows (columns) in the decreasing order to the top (respectively to the left). Thus, the matrix \( \Gamma \) may be viewed as a numerical matrix: \( \Gamma = (\gamma_{k,l})_{k=-\infty}^{\infty}, \gamma_{k,l} \in \mathbb{C} \). Observe that the following equalities hold

\[
\gamma_{rN+j,tN+n} = S_{r+t,j,n}, \quad r, t \in \mathbb{Z}, \quad 0 \leq j, n \leq N - 1. \tag{6}
\]

From conditions (4) it easily follows that

\[
(\gamma_{k,l})^r_{k,l} \geq 0, \quad (\gamma_{k+N,l})^r_{k,l} \geq 0, \quad \forall r \in \mathbb{Z}_+. \tag{7}
\]

The first inequality in the latter relation implies that there exist a Hilbert space \( H \) and a set of elements \( \{x_n\}_{n \in \mathbb{Z}} \) in \( H \) such that

\[
(x_n, x_m)_H = \gamma_{n,m}, \quad n, m \in \mathbb{Z}, \tag{8}
\]

and \( \text{span}\{x_n\}_{n \in \mathbb{Z}} = H \), see Lemma in [14, p. 177]. The latter fact is well known and goes back to a paper of Gelfand, Naimark [21]. By (6) we get

\[
\gamma_{a+N,b} = \gamma_{a,b}, \quad a, b \in \mathbb{Z}. \tag{9}
\]

Set \( L = \text{Lin}\{x_n\}_{n \in \mathbb{Z}} \). Choose an arbitrary element \( x \in L \). Let

\[
x = \sum_{k=-\infty}^{\infty} \alpha_k x_k, \quad x = \sum_{k=-\infty}^{\infty} \beta_k x_k, \quad \alpha_k, \beta_k \in \mathbb{C}.
\]

Here only a finite number of coefficients \( \alpha_k, \beta_k \) are non-zero. In what follows, this will be assumed in analogous situations with elements of the linear span. By (8), (9) we may write

\[
\left( \sum_{k=-\infty}^{\infty} \alpha_k x_{k+N}, x_l \right) = \sum_{k=-\infty}^{\infty} \alpha_k \gamma_{k+N,l} = \sum_{k=-\infty}^{\infty} \alpha_k \gamma_{k,l+N} =
\]

\[
= \sum_{k=-\infty}^{\infty} \alpha_k \left( x_k, x_{l+N} \right) = \left( \sum_{k=-\infty}^{\infty} \alpha_k x_k, x_{l+N} \right) = (x, x_{l+N}), \quad l \in \mathbb{Z}.
\]

Similarly we conclude that \( \left( \sum_{k=-\infty}^{\infty} \beta_k x_{k+N}, x_l \right) = (x, x_{l+N}), \quad l \in \mathbb{Z} \). Since \( \overline{L} = H \), we get \( \sum_{k=-\infty}^{\infty} \alpha_k x_{k+N} = \sum_{k=-\infty}^{\infty} \beta_k x_{k+N} \). Set

\[
A_0 x = \sum_{k=0}^{\infty} \alpha_k x_{k+N}, \quad x \in L, \quad x = \sum_{k=-\infty}^{\infty} \alpha_k x_k, \quad \alpha_k \in \mathbb{C}. \tag{10}
\]

The above considerations ensure us that the operator \( A_0 \) is defined correctly. Choose arbitrary elements \( x, y \in L, \quad x = \sum_{k=-\infty}^{\infty} \alpha_k x_k, \quad y = \sum_{n=-\infty}^{\infty} \beta_n x_n, \quad \alpha_k, \beta_n \in \mathbb{C} \), and write

\[
(A_0 x, y)_H = \left( \sum_{k=-\infty}^{\infty} \alpha_k x_{k+N}, \sum_{n=-\infty}^{\infty} \beta_n x_n \right)_H = \sum_{k,n=-\infty}^{\infty} \alpha_k \beta_n (x_{k+N}, x_n)_H =
\]
Moreover, we have

\[ (A_0 x, x)_H = \sum_{k, n = -\infty}^{\infty} \alpha_k \beta_n (x_k + N, x_n)_H = \sum_{k, n = -\infty}^{\infty} \alpha_k \beta_n \gamma_{k+N, n} \geq 0. \tag{11} \]

Thus, the operator \( A_0 \) is a non-negative symmetric operator in \( H \). Set \( A = \widetilde{A}_0 \). The operator \( A \) always has a non-negative self-adjoint extension \( \widetilde{A} \) in a Hilbert space \( \widetilde{H} \supseteq H \) \cite[Theorem 7, p.450]{15} . We may assume that \( \text{Ker} \widetilde{A} = \{0\} \). In the opposite case, since \( \text{Ker} \widetilde{A} \perp R(\widetilde{A}) \), \( R(\widetilde{A}) \supseteq L \), we conclude that \( \text{Ker} A \perp H \). Therefore the operator \( A \), restricted to \( \widetilde{H} \subseteq \text{Ker} A \), also will be a self-adjoint extension of the operator \( A \), with a null kernel. Let \( \{\widetilde{E}_\lambda\}_{\lambda \in \mathbb{R}} \) be the left-continuous orthogonal resolution of unity of the operator \( A \). By the induction argument it is easy to check that

\[ x_{rN+j} = A^r x_j, \quad r \in \mathbb{Z}, \quad 0 \leq j \leq N - 1. \]

By (6), (8) we may write

\[ S_{r,j,n} = \gamma_{rN+j,n} = (x_{rN+j}, x_n)_H = (A^r x_j, x_n)_H = (\widetilde{A}^r x_j, x_n)_\widetilde{H} = \int_{\mathbb{R}^+} \lambda^r d\widetilde{E}_\lambda x_j x_n, 0 \leq n \leq N - 1. \]

Therefore we get

\[ S_r = \int_{\mathbb{R}^+} \lambda^r d\widetilde{M}(\lambda), \quad r \in \mathbb{Z}, \tag{12} \]

where \( \widetilde{M}(\lambda) := \left( (P_H^\lambda \widetilde{E}_\lambda x_j, x_n)_H \right)^{N-1}_{j,n=0} \). Therefore the matrix function \( \widetilde{M}(\lambda) \) is a solution of the moment problem (13). (From the properties of the orthogonal resolution of unity it easily follows that \( \widetilde{M}(\lambda) \) is left-continuous on \((0, +\infty)\), non-decreasing and \( \widetilde{M}(0) = 0 \).)

\[ \square \]

3 An analytic description of solutions of the strong matrix Stieltjes moment problem.

Let \( A \) be an arbitrary closed Hermitian operator in a Hilbert space \( H \), \( D(A) \subseteq H \). Let \( \widetilde{A} \) be an arbitrary self-adjoint extension of \( A \) in a Hilbert
Denote by \( \{ \hat{E}_\lambda \}_{\lambda \in \mathbb{R}} \) its orthogonal resolution of unity. Recall that an operator-valued function \( R_z = P_\hat{H} R_z(\hat{A}) \) is said to be a **generalized resolvent** of \( A \), \( z \in \mathbb{C} \setminus \mathbb{R} \). A function \( E_\lambda = P_\hat{H} \hat{E}_\lambda \), \( \lambda \in \mathbb{R} \), is said to be a **spectral function** of \( A \). There exists a bijective correspondence between generalized resolvents and left-continuous (or normalized in some other way) spectral functions established by the following relation (\([22]\)):

\[
(R_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(E_\lambda f, g)_H, \quad f, g \in H, \ z \in \mathbb{C} \setminus \mathbb{R}. \tag{13}
\]

If the operator \( A \) is densely defined symmetric and non-negative (\( A \geq 0 \)), and the extension \( \hat{A} \) is self-adjoint and non-negative, then the corresponding generalized resolvent \( R_z \) and the spectral function \( E_\lambda \) are said to be a **generalized \( \Pi \)-resolvent** and a **\( \Pi \)-spectral function** of \( A \). Relation (13) establishes a bijective correspondence between generalized \( \Pi \)-resolvents and left-continuous \( \Pi \)-spectral functions.

If the operator \( A \) is a Hermitian contraction (\( \|A\| \leq 1 \)), and the extension \( \hat{A} \) is self-adjoint contraction, then the corresponding generalized resolvent \( R_z \) and the spectral function \( E_\lambda \) are said to be a **generalized \( sc \)-resolvent** and a **\( sc \)-spectral function** of \( A \). Relation (13) establishes a bijective correspondence between generalized \( sc \)-resolvents and left-continuous \( sc \)-spectral functions, as well.

If a generalized \( \Pi \)-resolvent (a generalized \( sc \)-resolvent) is generated by an extension \( \text{inside } H \), i.e. \( \hat{H} = H \), then it is said to be a **canonical \( \Pi \)-resolvent** (respectively a **canonical \( sc \)-resolvent**).

Firstly, we shall obtain a description of solutions of the strong matrix Stieltjes moment problem by virtue of \( \Pi \)-spectral functions.

**Theorem 2.** Let the strong matrix Stieltjes moment problem (1) be given and condition (4) be satisfied. Suppose that the operator \( A = A_0 \) in a Hilbert space \( H \) is constructed for the moment problem by (10) and the preceding procedure. All solutions of the moment problem have the following form

\[
M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (E_\lambda x_k, x_j)_H, \tag{14}
\]

where \( E_\lambda \) is a left-continuous \( \Pi \)-spectral function of the operator \( A \). On the other hand, each left-continuous \( \Pi \)-spectral function of the operator \( A \) generates by (14) a solution of the moment problem. Moreover, the correspondence between all left-continuous \( \Pi \)-spectral functions of the operator \( A \) and all solutions of the moment problem is bijective.
Proof. Let $E_{\lambda}$ be an arbitrary $\Pi$-spectral function of the operator $A$. It corresponds to a self-adjoint operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$. Then we repeat considerations after (11) to obtain that $M(\lambda)$, given by (14), is a solution of the moment problem (1).

Let $\hat{M}(x) = (\hat{m}_{k,l}(x))_{k,l=0}^{N-1}$ be an arbitrary solution of the moment problem (1). Consider the space $L^2(\hat{M})$ and let $Q$ be the operator of multiplication by an independent variable in $L^2(\hat{M})$. A set (of classes of equivalence) of functions $f \in L^2(\hat{M})$ such that (the corresponding class includes) $f = (f_0, f_1, \ldots, f_{N-1})$, $f_j \in \mathbb{P}_L$, we denote by $P_L(\hat{M})$. Set $L^2_L(\hat{M}) := P_L(\hat{M})$.

For an arbitrary vector Laurent polynomial $f = (f_0, f_1, \ldots, f_{N-1})$, $f_j \in \mathbb{P}_L$, there exists a unique representation of the following form:

$$f(x) = \sum_{k=0}^{N-1} \sum_{j=-\infty}^{\infty} \alpha_{k,j} x^j \tilde{e}_k, \quad \alpha_{k,j} \in \mathbb{C},$$

(15)

where all but finite number of coefficients $\alpha_{k,j}$ are zero. Choose another vector Laurent polynomial $g$ with a representation

$$g(x) = \sum_{l=0}^{N-1} \sum_{r=-\infty}^{\infty} \beta_{l,r} x^r \tilde{e}_l, \quad \beta_{l,r} \in \mathbb{C}.$$  

(16)

We may write

$$(f,g)_{L^2(\hat{M})} = \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \int_{\mathbb{R}^+} x^{j+r} \tilde{e}_k \overline{\tilde{e}_l} \, d\hat{M}(x)$$

$$= \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \int_{\mathbb{R}^+} x^{j+r} \overline{\hat{m}_{k,l}(x)} = \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} S_{j+r,k,l}.\quad (17)$$

On the other hand, we have

$$\left( \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_j x_{N+k}, \sum_{r=-\infty}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_r x_{N+l} \right)_H = \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}}$$

$$\ast (x_{jN+k}, x_{rN+l})_H = \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \gamma_j x_{N+k} x_{N+l} = \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} S_{j+r,k,l}.\quad (18)$$

8
By (17), (18) we get

\[
(f, g)_{L^2(\hat{M})} = \left( \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=-\infty}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H.
\] (19)

Set

\[
V f = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k},
\]

(20)

for a vector Laurent polynomial \( f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{j} \). If \( f, g \) are vector Laurent polynomials with representations (15), (16), such that

\[
\|f - g\|_{L^2(\hat{M})} = 0,
\]

then by (19) we may write

\[
\|V f - V g\|_H^2 = (V(f - g), V(f - g))_H = (f - g, f - g)_{L^2(\hat{M})} = \|f - g\|_{L^2(\hat{M})}^2 = 0.
\]

Thus, \( V \) is correctly defined as an operator from \( \mathbb{P}^2(\hat{M}) \) to \( H \). Relation (19) shows that \( V \) is an isometric transformation from \( \mathbb{P}^2(\hat{M}) \) on \( L \). We extend \( V \) by continuity to an isometric transformation from \( L^2(\hat{M}) \) on \( H \). Observe that

\[
V x^j \vec{e}_k = x_{jN+k}, \quad j \in \mathbb{Z}; \quad 0 \leq k \leq N - 1.
\]

(21)

Let \( L^2_1(\hat{M}) := L^2(\hat{M}) \oplus L^2_2(\hat{M}) \), and \( U := V \oplus E_{L^2_1(\hat{M})} \). The operator \( U \) is an isometric transformation from \( L^2(\hat{M}) \) on \( H \oplus L^2_1(\hat{M}) \) =: \( \hat{H} \). Set

\[
\hat{A} := UQU^{-1}.
\]

The operator \( \hat{A} \) is a self-adjoint operator in \( \hat{H} \). Let \( \{\hat{E}_\lambda\}_{\lambda \in \mathbb{R}} \) be its left-continuous orthogonal resolution of unity. Notice that

\[
UQU^{-1} x_{jN+k} = VQV^{-1} x_{jN+k} = VQ x^j \vec{e}_k = V x^{j+1} \vec{e}_k = x_{(j+1)N+k} = x_{jN+k+N} = Ax_{jN+k}, \quad j \in \mathbb{Z}; \quad 0 \leq k \leq N - 1.
\]

By linearity we get: \( UQU^{-1} x = Ax, \ x \in L \), and therefore \( \hat{A} \supseteq A \). Choose an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \) and write

\[
\int_{\mathbb{R}^+} \frac{1}{\lambda - z} d(\hat{E}_\lambda x_k, x_j)_{\hat{H}} = \left( \int_{\mathbb{R}^+} \frac{1}{\lambda - z} d\hat{E}_\lambda x_k, x_j \right)_{\hat{H}}
\]

\[
= \left( U^{-1} \int_{\mathbb{R}^+} \frac{1}{\lambda - z} d\hat{E}_\lambda x_k, U^{-1} x_j \right)_{L^2(\hat{M})}
\]
\[ \left( \int_{\mathbb{R}^+} \frac{1}{\lambda-z} dU^{-1} \hat{E}_\lambda U e_k, e_j \right)_{L^2(\hat{M})} = \left( \int_{\mathbb{R}^+} \frac{1}{\lambda-z} dE_\lambda e_k, e_j \right)_{L^2(\hat{M})} = \left( \int_{\mathbb{R}^+} \frac{1}{\lambda-z} dE_\lambda \hat{e}_k, \hat{e}_j \right)_{L^2(\hat{M})} \]

\[ = \left( (Q-z)^{-1} \hat{e}_k, \hat{e}_j \right)_{L^2(\hat{M})} = \int_{\mathbb{R}^+} \frac{1}{\lambda-z} \hat{e}_k d\hat{M}(\lambda) \hat{e}_j = \int_{\mathbb{R}^+} \frac{1}{\lambda-z} d\hat{m}_{k,j}(\lambda), \]

where \( E_\lambda \) is a left-continuous orthogonal resolution of unity of the operators \( Q \). By the Stieltjes-Perron inversion formula (e.g. [23]) we conclude that

\[ \hat{m}_{k,j}(\lambda) = \left( P_H^\dagger \tilde{E}_\lambda x_k, x_j \right)_H, \quad \lambda \in \mathbb{R}. \]

Thus, \( \hat{M} \) is generated by a \( \Pi \)-spectral function of \( A \).

Let us check that an arbitrary element \( u \in L \) can be represented in the following form

\[ u = u_z + u_0, \quad u_z \in H_z, \quad u_0 \in L_N, \quad (22) \]

where \( L_N := \text{Lin}\{x_n\}_{n=0}^{N-1}, H_z := (A - zE_H)D(A) \). Let \( u = \sum_{k=-\infty}^{\infty} c_k x_k, \) \( c_k \in \mathbb{C} \), and choose a number \( z \in \mathbb{C} \setminus \mathbb{R} \). Suppose that \( c_k = 0 \), if \( k \leq r \) or \( k \geq R \), where \( r \leq -2; R \geq N+1 \). Set \( d_k := 0 \), if \( k \leq r \) or \( k \geq R - N \). Then we set

\[ \begin{align*}
d_k & := \frac{1}{z}(d_{k-N} - c_k), \quad k = r + 1, \ldots, -1; \\
d_{k-N} & := zd_k + c_k, \quad k = R - 1, R - 2, \ldots, N.
\end{align*} \]

Set \( v := \sum_{k=-\infty}^{\infty} d_k x_k \in L \). Then we directly calculate that

\[ (A - zE_H)v - u = \sum_{k=0}^{N-1} (d_{k-N} - zd_k - c_k) x_k, \]

and relation (22) holds. From the latter equality it easily follows that the deficiency index of \( A \) is equal to \( (n, n) \), \( 0 \leq n \leq N \).

Let us check that different left-continuous \( \Pi \)-spectral functions of the operator \( A \) generate different solutions of the moment problem (1). Suppose that two different left-continuous \( \Pi \)-spectral functions generate the same solution of the moment problem. This means that there exist two self-adjoint operators \( A_j \supseteq A \), in Hilbert spaces \( H_j \supseteq H \), such that \( P_H^{H_j} E_{1,\lambda} \neq P_H^{H_j} E_{2,\lambda} \), and

\[ (P_H^{H_j} E_{1,\lambda} x_k, x_j)_H = (P_H^{H_j} E_{2,\lambda} x_k, x_j)_H, \quad 0 \leq k, j \leq N - 1, \quad \lambda \in \mathbb{R}, \]

where \( \{ E_{n,\lambda} \}_{\lambda \in \mathbb{R}} \) are orthogonal left-continuous resolutions of unity of operators \( A_n, n = 1, 2 \). By the linearity we get

\[ (P_H^{H_j} E_{1,\lambda} x, y)_H = (P_H^{H_j} E_{2,\lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{R}. \quad (23) \]
Set $R_{n,\lambda} := P_H^n R_\lambda (A_n)$, $n = 1, 2$. By (23), (13) we get
\[(R_{1,\lambda} x, y)_H = (R_{2,\lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (24)\]
Since
\[R_z (A_j) (A - zE_H)x = (A_j - zE_H)^{-1} (A_j - zE_H)x = x, \quad x \in L = D(A_0),\]
then $R_z (A_1) u = R_z (A_2) u \in H, u \in H_z$;
\[R_{1,z} u = R_{2,z} u, \quad u \in H_z, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (25)\]
We may write
\[(R_{n,z} x, u)_H = (R_z (A_n) x, u)_{H_n} = (x, R_{n,z} (A_n) u)_{H_n} = (x, R_{n,z} u)_H,\]
where $x \in L_N, u \in H_z, n = 1, 2$, and therefore
\[(R_{1,z} x, u)_H = (R_{2,z} x, u)_H, \quad x \in L_N, u \in H_z. \quad (26)\]
By (22) an arbitrary element $y \in L$ can be represented in the following form $y = y_z + y', y_z \in H_z, y' \in L_N$. Using (24) and (26) we obtain
\[(R_{1,z} x, y)_H = (R_{1,z} x, y_z + y')_H = (R_{2,z} x, y_z + y')_H = (R_{2,z} x, y)_H,\]
where $x \in L_N, y \in L$. Since $L = H$, we obtain
\[R_{1,z} x = R_{2,z} x, \quad x \in L_N, z \in \mathbb{C} \setminus \mathbb{R}. \quad (27)\]
For arbitrary $x \in L, x = x_z + x', x_z \in H_z, x' \in L_N$, using relations (25), (27) we get
\[R_{1,z} x = R_{1,z} (x_z + x') = R_{2,z} (x_z + x') = R_{2,z} x, \quad x \in L, z \in \mathbb{C} \setminus \mathbb{R},\]
and therefore $R_{1,z} = R_{2,z}, z \in \mathbb{C} \setminus \mathbb{R}$. By (13) this means that the corresponding II-spectral functions coincide. The obtained contradiction completes the proof. \(\square\)

We shall use some known important facts about sc-resolvents, see [16]. Let $B$ be an arbitrary Hermitian contraction in a Hilbert space $H$. Set $D = D(B)$, $\mathcal{R} = H \ominus D$. A set of all self-adjoint contractive extensions of $B$ inside $H$, we denote by $B_H(B)$. A set of all self-adjoint contractive extensions of $B$ in a Hilbert space $\tilde{H} \supseteq H$, we denote by $B_{\tilde{H}}(B)$. By Krein’s theorem [15, Theorem 2, p. 440], there always exists a self-adjoint
extension $\tilde{B}$ of the operator $B$ in $H$ with the norm $\|B\|$. Therefore the set $\mathcal{B}_H(B)$ is non-empty. There are the "minimal" element $B^\mu$ and the "maximal" element $B^M$ in this set, such that $\mathcal{B}_H(B)$ coincides with the operator segment

$$B^\mu \leq \tilde{B} \leq B^M.$$  

(28)

In the case $B^\mu = B^M$ the set $\mathcal{B}_H(B)$ consists of a unique element. This case is said to be determinate. The case $B^\mu \neq B^M$ is called indeterminate. The indeterminate case can be always reduced to the completely indeterminate. If $\mathcal{R}_0 = \{x \in \mathcal{R} : B^\mu x = B^M x\}$, we may set

$$B_{e} x = B x, \ x \in D; \ B_{e} x = B^\mu x, \ x \in \mathcal{R}_0.$$  

(29)

The sets of generalized sc-resolvents for $B$ and for $B_e$ coincide ([16, p. 1039]). Elements of $\mathcal{B}_H(B)$ are canonical (i.e. inside $H$) extensions of $B$ and their resolvents are said to be canonical sc-resolvents of $B$. On the other hand, elements of $\mathcal{B}_{\tilde{H}}(B)$ for all possible $\tilde{H} \supseteq H$ generate generalized sc-resolvents of $B$ (here the space $\tilde{H}$ is not fixed). The set of all generalized sc-resolvents we denote by $\mathcal{R}^c(B)$. Set

$$C = B^M - B^\mu,$$  

(30)

$$Q_{\mu}(z) = \left( \frac{C^\frac{\nu}{2} R_{\mu}^\nu C^\frac{\nu}{2}}{R_{\nu}} + E_H \right) \bigg|_{\mathcal{R}}, \quad z \in \mathbb{C}\setminus[-1, 1],$$  

(31)

where $R_{\nu}^\mu = (B^\mu - zE_H)^{-1}$.

An operator-valued function $k(z)$ with values in $[\mathcal{R}]$ belongs to the class $R_{\mathcal{R}}[-1, 1]$ if

1) $k(z)$ is analytic in $z \in \mathbb{C}\setminus[-1, 1]$ and

$$\frac{\text{Im} k(z)}{\text{Im} z} \leq 0, \quad z \in \mathbb{C} : \text{Im} z \neq 0;$$

2) For $z \in \mathbb{R}\setminus[-1, 1]$, $k(z)$ is a self-adjoint non-negative contraction.

Notice that functions from the class $R_{\mathcal{R}}[-1, 1]$ admit a special integral representation, see [16].

**Theorem 3.** ([16, p. 1053]). Let $B$ be a Hermitian contraction in a Hilbert space $H$ with $D(B) = D$; $\mathcal{R} = H \ominus D$. Suppose that for $B$ it takes place the completely indeterminate case and the corresponding operator $C$, as an operator in $\mathcal{R}$, has an inverse in $[\mathcal{R}]$. Then the following equality:

$$\widetilde{R}^c_{\nu} = R_{\nu}^\mu - R_{\nu}^\mu C^\nu \left(E_{\mathcal{R}} + (Q_{\mu}(z) - E_{\mathcal{R}})k(z)\right)^{-1} C^\nu R_{\nu}^\mu,$$  

(32)
where \( k(z) \in R_R[-1,1], \ R_z^c \in R^c(B) \), establishes a bijective correspondence between the set \( R_R[-1,1] \) and the set \( R^c(B) \).

Moreover, the canonical resolvents correspond in (32) to the constant functions \( k(z) \equiv K, K \in [0, E_R] \).

Let \( A \) be an arbitrary non-negative symmetric operator in a Hilbert space \( H, D(A) = H \). We are going to obtain a formula for the generalized \( H \)-resolvents of \( A \), by virtue of Theorem 3. Set

\[
T = (E_H - A)(E_H + A)^{-1} = -E_H + 2(E_H + A)^{-1}, \quad D(T) = (A + E_H)D(A).
\]

(33)

Then

\[
A = (E_H - T)(E_H + T)^{-1} = -E_H + 2(E_H + T)^{-1}, \quad D(A) = (T + E_H)D(T).
\]

(34)

The latter transformations were introduced and intensively studied by Krein [15]. The operator \( T \) is a Hermitian contraction in \( H \). In fact, for an arbitrary \( h = (A + E_H)f, f \in D(A) \) we may write

\[
\|Th\|_H^2 = \|(-E_H + 2(E_H + A)^{-1})(A + E_H)f\|_H^2 = \| -Af + f\|_H^2
\]

\[
= \|Af\|_H^2 + \|f\|_H^2 - 2(Af, f)_H \leq \|Af\|_H^2 + \|f\|_H^2 + 2(Af, f)_H = \|h\|_H^2.
\]

Let \( \tilde{A} \supseteq A \) be a non-negative self-adjoint extension of \( A \) in a Hilbert space \( \tilde{H} \supseteq H \). Then the operator

\[
\tilde{T} = (E_{\tilde{H}} - \tilde{A})(E_{\tilde{H}} + \tilde{A})^{-1} = -E_{\tilde{H}} + 2(E_{\tilde{H}} + \tilde{A})^{-1}, \quad D(\tilde{T}) = (\tilde{A} + E_{\tilde{H}})D(\tilde{A}),
\]

(35)

is a self-adjoint contraction \( \tilde{T} \supseteq T \) in \( \tilde{H} \), and

\[
\tilde{A} = (E_{\tilde{H}} - \tilde{T})(E_{\tilde{H}} + \tilde{T})^{-1} = -E_{\tilde{H}} + 2(E_{\tilde{H}} + \tilde{T})^{-1}, \quad D(\tilde{A}) = (\tilde{T} + E_{\tilde{H}})D(\tilde{T}).
\]

(36)

Consider the following fractional linear transformation:

\[
z = \frac{1 - \lambda}{1 + \lambda} = -1 + 2 \frac{1}{1 + \lambda}; \quad \lambda = \frac{1 - z}{1 + z} = -1 + 2 \frac{1}{1 + z}.
\]

(37)

Choose an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \) and set \( \lambda := \frac{1 - z}{1 + z} \). Observe that \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Then

\[
R_{z}(\tilde{T}) = (\tilde{T} - zE_{\tilde{H}})^{-1} = \left(-E_{\tilde{H}} + 2(E_{\tilde{H}} + \tilde{A})^{-1} - \frac{1 - \lambda}{1 + \lambda}E_{\tilde{H}}\right)^{-1}
\]

\[
= \left(\frac{-2}{1 + \lambda}(E_{\tilde{H}} + \tilde{A})(E_{\tilde{H}} + \tilde{A})^{-1} + 2(E_{\tilde{H}} + \tilde{A})^{-1}\right)^{-1}
\]

13
\[
\begin{align*}
&\left(\frac{2\lambda}{1+\lambda}E_\tilde{H} - \frac{2}{1+\lambda}\tilde{A}\right)(E_\tilde{H} + \tilde{A})^{-1} \\
&= -\frac{\lambda+1}{2} \left( (\tilde{A} - \lambda E_\tilde{H}) (E_\tilde{H} + \tilde{A})^{-1} \right)^{-1} = -\frac{\lambda+1}{2}(E_\tilde{H} + \tilde{A})(\tilde{A} - \lambda E_\tilde{H})^{-1} \\
&= -\frac{(\lambda+1)^2}{2}(\tilde{A} - \lambda E_\tilde{H})^{-1} - \frac{\lambda+1}{2}E_\tilde{H} = -\frac{(\lambda+1)^2}{2}R_\lambda(\tilde{A}) - \frac{\lambda+1}{2}E_\tilde{H}.
\end{align*}
\]

Therefore
\[
R_\lambda(\tilde{A}) = -\frac{2}{(\lambda+1)^2}R_{\frac{1}{1+i\lambda}}(\tilde{T}) - \frac{1}{\lambda+1}E_\tilde{H}, \quad \forall \lambda \in \C \setminus \R. \tag{38}
\]

Applying the orthogonal projection on \(H\), we get
\[
R_\lambda(A) = -\frac{2}{(\lambda+1)^2}R_{\frac{1}{1+i\lambda}}(T) - \frac{1}{\lambda+1}E_H, \quad \forall \lambda \in \C \setminus \R. \tag{39}
\]

Here \(R_\lambda(A)\) is the generalized \(\Pi\)-resolvent corresponding to \(\tilde{A}\), and \(R_\lambda(T)\) is the generalized \(sc\)-resolvent corresponding to \(\tilde{T}\). Thus, an arbitrary generalized \(\Pi\)-resolvent of \(A\) can be constructed by a generalized \(sc\)-resolvent of \(T\) by relation (39).

On the other hand, choose an arbitrary \(sc\)-resolvent \(R_\lambda'(T)\) of \(T\). It corresponds to a self-adjoint contractive extension \(\tilde{T} \supseteq T\) in a Hilbert space \(\tilde{H} \supseteq H\). Observe that
\[
\text{Ker}(E_\tilde{H} + \tilde{T}) \perp R(E_\tilde{H} + \tilde{T}) \supseteq R(E_H + T) = D(A),
\]
and therefore \(\text{Ker}(E_\tilde{H} + \tilde{T}) \perp H\). We may assume that \(H_1 := \text{Ker}(E_\tilde{H} + \tilde{T}) = \{0\}\), since in the opposite case one may consider the operator \(\tilde{T}\) restricted to \(\tilde{H} \cap H_1 \supseteq H\). Then we set
\[
\tilde{A} = (E_\tilde{H} - \tilde{T})(E_\tilde{H} + \tilde{T})^{-1} = -E_\tilde{H} + 2(E_\tilde{H} + \tilde{T})^{-1}, \quad D(\tilde{A}) = (\tilde{T} + E_\tilde{H})D(\tilde{T}). \tag{40}
\]

The operator \(\tilde{A}\) is densely defined since \(\tilde{A} \supseteq A\), and it is self-adjoint. For an arbitrary \(u \in D(\tilde{T})\) we may write
\[
(\tilde{A}(\tilde{T} + E_\tilde{H})u, (\tilde{T} + E_\tilde{H})u)_{\tilde{H}} = (-\tilde{T}u + u, \tilde{T}u + u)_{\tilde{H}} = \|u\|_{\tilde{H}}^2 - \|\tilde{T}u\|_{\tilde{H}}^2 \geq 0.
\]

Thus, the operator \(\tilde{A}\) is non-negative. Observe that
\[
\tilde{T} = (E_\tilde{H} - \tilde{A})(E_\tilde{H} + \tilde{A})^{-1} = -E_\tilde{H} + 2(E_\tilde{H} + \tilde{A})^{-1}. \tag{41}
\]
Repeating the considerations after relation (37), we obtain that

$$R_\lambda'(A) = -\frac{2}{(\lambda + 1)^2} R_{\lambda/1+\lambda}^\mu (T) - \frac{1}{\lambda + 1} E_H, \quad \forall \lambda \in \mathbb{C}\setminus \mathbb{R},$$

(42)

gives a generalized Π-resolvent of $A$ (corresponding to $\hat{A}$).

Consequently, the relation (39) establishes a bijective correspondence between the set of all $sc$-resolvents of $T$ and the set of all Π-resolvents of $A$. It is not hard to see that the canonical $sc$-resolvents are related to the canonical Π-resolvents.

They say that for the operator $A$ it takes place a **completely indeterminate case**, if for the corresponding operator $T$ it takes place the completely indeterminate case [24].

It is known that all self-adjoint contractive extensions of $T$ are extensions of the extended operator $T_e$ defined by (29) [16, Theorem 1.4]. Set

$$A_e = (E_H - T_e)(E_H + T_e)^{-1} = -E_H + 2(E_H + T_e)^{-1}, \quad D(A_e) = (T_e + E_H)D(T_e).$$

(43)

It is easily seen that the above operator $\tilde{A}$ is an extension of $A_e$. Therefore the sets of generalized Π-resolvents for $A$ and for $A_e$ coincide.

**Theorem 4.** Let $A$ be a non-negative symmetric operator in a Hilbert space $H$, $\mathcal{D}(A) = H$. Suppose that for $A$ it takes place the completely indeterminate case. Let $T$ be given by (33); $\mathcal{D} = \mathcal{D}(T)$, $\mathcal{R} = H \oplus \mathcal{D}$. Suppose that the corresponding operator $C = T^M - T^\mu$, as an operator in $\mathcal{R}$, has an inverse in $\mathcal{R}$. Then the following equality:

$$R_\lambda(A) = -\frac{2}{(\lambda + 1)^2} R_{\lambda/1+\lambda}^\mu \left( \frac{1}{\lambda + 1} C_{\lambda} \right) (E_H + (Q_\mu(\lambda) - E_H)k(\lambda))^{-1} C_{\lambda} R_{\lambda/1+\lambda}^\mu,$$

(44)

where $Q_\mu(\lambda) = Q_\mu \left( \frac{1}{1+\lambda} \right)$, $k(\lambda) = k \left( \frac{1}{1+\lambda} \right)$; $k(\cdot) \in R_{\mathcal{R}}[-1, 1]$, establishes a bijective correspondence between the set $R_{\mathcal{R}}[-1, 1]$ and the set of all generalized Π-resolvents of $A$. Here $Q_\mu$ is defined by (31) for $T$, $R^\mu_z = (T^\mu - zE_H)^{-1}$, and $R_\lambda(A)$ is a generalized Π-resolvent of $A$.

Moreover, the canonical resolvents correspond in (44) to the constant functions $k(z) \equiv K$, $K \in [0, E_R]$.

**Proof.** It follows directly from the preceding considerations, formula (39) and by applying Theorem 3.

□
Let the strong matrix Stieltjes moment problem be given and conditions (4) hold. Consider an arbitrary Hilbert space $H$ and a sequence of elements $\{x_n\}_{n \in \mathbb{Z}}$ in $H$, such that relation (8) holds. Let $A = A_0$, where the operator $A_0$ is defined by (10). Denote $L_N = \text{Lin}\{x_k\}_{k=0}^{N-1}$. Define a linear transformation $G$ from $\mathbb{C}^N$ onto $L_N$ by the following relation:

$$G\vec{u}_k = x_k, \quad k = 0, 1, ..., N - 1,$$

where $\vec{u}_k = (\delta_{0,k}, \delta_{1,k}, ..., \delta_{N-1,k})$.

**Theorem 5.** Let the strong matrix Stieltjes moment problem (1) be given and conditions (4) be satisfied. Let $\{x_n\}_{n \in \mathbb{Z}}$ be a sequence of elements of a Hilbert space $H$ such that relation (8) holds. Let $A = A_0$, where the operator $A_0$ is defined by relation (10). Let $T = -E_H + 2(E_H + A)^{-1}$. The following statements are true:

1) If $T^\mu = T^M$, then the moment problem (1) has a unique solution. This solution is given by

$$M(t) = (m_{j,n}(t))_{j,n=0}^{N-1}, \quad m_{j,n}(t) = (E^\mu_j x_j, x_n)_H, \quad 0 \leq j, n \leq N - 1,$$

where $\{E^\mu_j\}$ is the left-continuous orthogonal resolution of unity of the operator $A^\mu = -E_H + 2(E_H + T^\mu)^{-1}$.

2) If $T^\mu \neq T^M$, define the extended operator $T_e$ by (29); $R_e = H \ominus D(T_e)$, $C = T^M - T^\mu$, and $R^\mu_e = (T^\mu - zE_H)^{-1}$, $Q_{\mu,e}(z) = \left(C^\frac{1}{2}R^\mu_e C^\frac{1}{2} + E_H\right)_{R_e}$, $z \in \mathbb{C} \setminus [-1, 1]$. An arbitrary solution $M(\cdot)$ of the moment problem can be found by the Stieltjes-Perron inversion formula from the following relation

$$\int_{R_+} \frac{1}{t-z}dM^T(t) = \mathcal{A}(z) - C(z)k(z)(E_{R_e} + D(z)k(z))^{-1}\mathcal{B}(z),$$

where $k(\lambda) = k\left(\frac{1-\lambda}{1+\lambda}\right)$, $k(z) \in R_{R_e}[-1,1]$, and on the right-hand side one means the matrix of the corresponding operator in $\mathbb{C}^N$. Here $\mathcal{A}(z)$, $\mathcal{B}(z)$, $\mathcal{C}(z)$, $\mathcal{D}(z)$ are analytic operator-valued functions given by

$$\mathcal{A}(z) = -\frac{2}{(\lambda + 1)^2} G^* R^\mu_{\frac{1}{1+\lambda}} G - \frac{1}{\lambda + 1} G^* G : \mathbb{C}^N \to \mathbb{C}^N,$$

$$\mathcal{B}(z) = C^\frac{1}{2} R^\mu_{\frac{1}{1+\lambda}} G : \mathbb{C}^N \to R_e,$$
\[ C(z) = \frac{2}{(\lambda + 1)^2} G^* R^\mu \frac{1}{1+\lambda} C^\frac{3}{2} : \mathcal{R}_e \to \mathbb{C}^N, \quad (50) \]

\[ D(z) = Q_{\mu,e} \left( \frac{1-\lambda}{1+\lambda} \right) - E_{\mathcal{R}_e} : \mathcal{R}_e \to \mathcal{R}_e. \quad (51) \]

Moreover, the correspondence between all solutions of the moment problem and \( k(z) \in R_{\mathcal{R}_e}[-1,1] \) is bijective.

**Proof.** Consider the case 1). In this case all self-adjoint contractions \( \tilde{T} \supset T \) in a Hilbert space \( \tilde{H} \supseteq H \) coincide on \( H \) with \( T^\mu \), see [16, p. 1039]. Thus, the corresponding sc-spectral functions are spectral functions of the self-adjoint operator \( T^\mu \), as well. However, a self-adjoint operator has a unique (normalized) spectral function. Thus, a set of sc-spectral functions of \( T \) consists of a unique element. Therefore the set of \( \Pi \)-resolvents of \( A \) consists of a unique element, as well. This element is the spectral function of \( A^\mu \).

Consider the case 2). By Theorem 3 and relation (13) it follows that an arbitrary solution \( M(t) = (m_{j,n}(t))_{j,n=0}^{N-1} \) of the moment problem (1) can be found from the following relation:

\[ \int_{\mathbb{R}_+} \frac{1}{t-z} dm_{j,n}(t) = (R_x j, x_n)_H, \quad 0 \leq j, n \leq N-1; \ z \in \mathbb{C} \setminus \mathbb{R}, \]

where \( R_x \) is a generalized \( \Pi \)-resolvent of the operator \( A \). Moreover, the correspondence between the set of all generalized \( \Pi \)-resolvents of \( A \) (which is equal to the set of all generalized \( \Pi \)-resolvents of \( A_e \)) and the set of all solutions of the moment problem is bijective. Notice that \( T^\mu = T^e_\mu \) and \( T^M = T_e^M \). By Theorem 4 (applied to the operator \( A_e \)) we may rewrite the latter relation in the following form:

\[ \int_{\mathbb{R}_+} \frac{1}{t-z} dm_{j,n}(t) = \left\{ -\frac{2}{(\lambda + 1)^2} R^\mu \frac{1}{1+\lambda} G^* G - \frac{1}{\lambda + 1} E_H \right\} \quad (50) \]

\[ + \frac{2}{(\lambda + 1)^2} R^\mu \frac{1}{1+\lambda} C^\frac{3}{2} k(\lambda) (E_{\mathcal{R}_e} + (Q_{\mu,e}(\lambda) - E_{\mathcal{R}_e}) k(\lambda))^{-1} C^\frac{1}{2} R^\mu \frac{1}{1+\lambda} \left\{ x_j, x_n \right\} \quad (52) \]

where \( k(\lambda) = k \left( \frac{1}{1+\lambda} \right) \), \( k(z) \in R_{\mathcal{R}_e}[-1,1] \), \( Q_{\mu,e}(\lambda) = Q_{\mu,e} \left( \frac{1}{1+\lambda} \right) \). Then

\[ \int_{\mathbb{R}_+} \frac{1}{t-z} dm_{j,n}(t) = \left\{ -\frac{2}{(\lambda + 1)^2} G^* R^\mu \frac{1}{1+\lambda} G - \frac{1}{\lambda + 1} G^* G + \frac{2}{(\lambda + 1)^2} G^* \right\} x_j, x_n \quad (53) \]

\[ \ast R^\mu \frac{1}{1+\lambda} C^\frac{3}{2} k(\lambda) (E_{\mathcal{R}_e} + (Q_{\mu,e}(\lambda) - E_{\mathcal{R}_e}) k(\lambda))^{-1} C^\frac{1}{2} R^\mu \frac{1}{1+\lambda} G \left\{ u_j, u_n \right\} \quad (54) \]

Introducing functions \( A(z), B(z), C(z), D(z) \) by formulas (45)-(51) one easily obtains relation (17).
Theorem 6. Let the strong matrix Stieltjes moment problem \((1)\) be given and conditions \((4)\) be satisfied. Let \(\{x_n\}_{n \in \mathbb{Z}}\) be a sequence of elements of a Hilbert space \(H\) such that relation \((8)\) holds. Let \(A = A_0\), where the operator \(A_0\) is defined by relation \((10)\). The moment problem is determinate if and only if \(T^\mu = T^M\), where \(T^\mu, T^M\) are the extremal extensions of the operator \(T = -E_H + 2(E_H + A)^{-1}\).

**Proof.** The sufficiency follows from Statement 1 of Theorem 5. The necessity follows from Statement 2 of Theorem 5, if we take into account that the class \(R_e([-1, 1])\), where \(\dim R_e > 0\), has at least two different elements. In fact, from the definition of the class \(R_e([-1, 1])\) it follows that \(k_{1}(z) \equiv 0\), and \(k_{1}(z) \equiv E_{R_e}\), belong to \(R_e([-1, 1])\). \(\Box\)

**Example 3.1.** Consider the moment problem \((1)\) with \(N = 2\) and

\[
S_n = \begin{pmatrix} 1 & \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & 1 \end{pmatrix}, \quad n \in \mathbb{Z}.
\]

In this case we have

\[
\Gamma = (S_{i+j})_{i,j=-\infty}^{\infty} = (\gamma_{n,m})_{n,m=-\infty}^{\infty},
\]

where

\[
\gamma_{2k,2l} = \gamma_{2k+1,2l+1} = 1, \quad \gamma_{2k,2l+1} = \gamma_{2k+1,2l} = \frac{3}{\sqrt{10}}, \quad k, l \in \mathbb{Z}.
\]

Consider the space \(C^2\) and elements \(u_0, u_1 \in C^2\):

\[
u_0 = \frac{1}{\sqrt{2}}(1, 1), \quad u_1 = \frac{1}{\sqrt{5}}(1, 2).
\]

Set

\[
x_{2k} = u_0, \quad x_{2k+1} = u_1, \quad k \in \mathbb{Z}.
\]

Then relation \((8)\) holds. Define by \((10)\) the operator \(A_0\). In this case \(A = A_0 = E_{C^2}\). Therefore the operators \(A\) and \(T = -E_H + 2(E_H + A)^{-1}\) are self-adjoint and have unique spectral functions. Hence, \(T^M = T^\mu\), and by Theorem 6 we conclude that the moment problem has a unique solution. By Theorem 2 it has the following form

\[
M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (E_\lambda x_k, x_j)_H,
\]

where \(E_\lambda\) is the left-continuous spectral function of the operator \(E_{C^2}\). Consequently, the matrix function \(M(t)\) is equal to 0, for \(t \leq 1\), and \(M(t) = \begin{pmatrix} 1 & 3 \\ 3 \sqrt{10} & 1 \end{pmatrix}\), for \(t > 1\).
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The strong matrix Stieltjes moment problem.

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In this paper we study the strong matrix Stieltjes moment problem. We obtain necessary and sufficient conditions for its solvability. An analytic description of all solutions of the moment problem is derived. Necessary and sufficient conditions for the determinateness of the moment problem are given.