Contact skew CR-warped product submanifolds of Sasakian manifolds

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Abstract In this paper, we study warped products of contact skew-CR submanifolds, called contact skew CR-warped products. We establish an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle. The equality case in the statement of the inequality is investigated and some applications of derived inequality are given. Furthermore, we provide a non-trivial example of such submanifolds.

Keywords warped products · slant · semi-slant submanifolds · pseudo-slant submanifolds · contact skew CR-submanifolds · Sasakian manifolds

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1 Introduction

The concept of skew CR-submanifolds of almost Hermitian manifolds was given by G. S. Ronsse [23] to unify and generalize the concepts of holomorphic, totally real, CR, slant, semi-slant and pseudo-slant (hemi-slant in the sense of B. Sahin [24]) submanifolds by exploiting the behavior of the bounded symmetric linear operator. Later, this idea is extended to the contact geometry by Tripathi in [26] with the name of almost semi-invariant submanifolds.

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as a generalized class of invariant, anti-invariant, slant, contact CR, bi-slant submanifolds of contact metric manifolds.

On the other hand, the warped products of skew CR-submanifolds of Kähler manifolds were studied by B. Sahin in [25] as a generalization of CR-warped products introduced by B.-Y. Chen in his seminal work [9,10,11,12] and of warped product hemi-slant submanifolds, studied by B. Sahin in [24]. Later on, the contact version of skew CR-warped products of cosymplectic manifolds appeared in [17]. Recently, we studied warped product skew CR-submanifolds of Kenmotsu manifolds in [21]. For up-to-date survey on warped product manifolds and warped product submanifolds we refer to B.-Y. Chen’s books [13,15] and his survey article [14].

In this paper, we study the contact skew CR-warped product submanifolds by considering the base manifold of the warped product as a semi-slant product submanifold i.e., the Riemannian product of invariant and proper slant submanifolds of a Sasakian manifold and the fiber of warped product is an anti-invariant submanifold.

The paper is organized as follows: In Section 2, we give some basic formulas and definitions for almost contact metric manifolds and their submanifolds. In Section 3, we recall the definitions of warped products and a basic lemma which is useful to this study. In this, section, we find some useful relations for contact skew CR-warped products those are essential to derive our main result. Moreover, we construct a non-trivial example of contact skew CR-warped products. In Section 4, we derive a lower bound relation for the squared norm of the second fundamental form in terms of components of the gradient of warping function along both factors of base a manifold. The equality case is also considered. In Section 5, we give applications of our derived inequality.

2 Preliminaries

In this section we recall the notion of Riemannian manifolds, submanifolds and give some definitions as a brief review of the literature on submanifolds of contact metric manifolds.

A \((2m+1)\)-dimensional differentiable manifold \(\tilde{M}\) is called an almost contact manifold if there is an almost contact structure \((\varphi, \xi, \eta)\) consisting of a \((1,1)\) tensor field \(\varphi\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying [3]

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0.
\]  

(1)

where \(I: T\tilde{M} \to T\tilde{M}\) is the identity mapping. From the definition it follows that the \((1,1)\)-tensor field \(\varphi\) has constant rank \(2m\) (cf. [3]). An almost contact manifold \((\tilde{M}, \varphi, \eta, \xi)\) is said to be normal when the tensor field \(N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi\) vanishes identically, where \([\varphi, \varphi]\) is the Nijenhuis torsion of \(\varphi\). It is known that any almost contact manifold \((\tilde{M}, \varphi, \eta, \xi)\) admits a Riemannian metric \(\tilde{g}\) such that

\[
\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y)
\]  

(2)
for any \( X, Y \in \Gamma(T\tilde{M}) \), where the \( \Gamma(T\tilde{M}) \) is the Lie algebra of vector fields on \( \tilde{M} \). This metric \( \tilde{g} \) is called a compatible metric and the manifold \( \tilde{M} \) together with the structure \((\varphi, \xi, \eta, \tilde{g})\) is called an almost contact metric manifold.

As an immediate consequence of (2), one has \( \eta(X) = \tilde{g}(X, \xi), \; \eta(\xi) = 1 \) and \( \tilde{g}(\varphi X, Y) = -\tilde{g}(X, \varphi Y) \). Hence the fundamental 2-form \( \Phi \) of \( \tilde{M} \) is defined \( \Phi(X, Y) = \tilde{g}(X, \varphi Y) \) and the manifold is said to be contact metric manifold if \( \Phi = d\eta \). If \( \xi \) is a Killing vector field with respect to \( \tilde{g} \), the contact metric structure is called a contact structure. A normal contact metric manifold is said to be a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

\[
(\tilde{\nabla}_X \varphi) Y = \tilde{g}(X, Y) \xi - \eta(Y) X
\]  

(3)

for all \( X, Y \in \Gamma(T\tilde{M}) \), where \( \tilde{\nabla} \) is the Levi-Civita connection of \( \tilde{g} \). From the formula (3), it follows that \( \tilde{\nabla}_X \xi = -\varphi X \). A Sasakian manifold is always a \( K \)-contact manifold and the converse is true in the dimension three.

Let \( M \) be a submanifold of a Riemannian manifold \( \tilde{M} \) equipped with a Riemannian metric \( \tilde{g} \). We use the same symbol \( g \) for both the metrics \( \tilde{g} \)of \( \tilde{M} \) and the induced metric \( g \) on the submanifold \( M \). Let \( \Gamma(TM) \) the Lie algebra of vector fields on \( M \) and \( \Gamma(T^\perp M) \), the set of all vector fields normal to \( M \). If we denote by \( \nabla \), the Levi-Civita connection of \( \tilde{M} \), then the Gauss and Weingarten formulas are respectively given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),
\]

(4)

\[
\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,
\]

(5)

for any vector field \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), where \( \nabla^\perp \) is the normal connection in the normal bundle, \( \sigma \) is the second fundamental form and \( A_N \) is the shape operator (corresponding to the normal vector field \( N \)) for the immersion of \( M \) into \( \tilde{M} \). They are related by \( g(\sigma(X, Y), N) = g(A_N X, Y) \).

A submanifold \( M \) is said to be totally geodesic if \( \sigma = 0 \) and totally umbilical if \( \sigma(X, Y) = g(X, Y) H, \; \forall X, Y \in \Gamma(TM) \), where \( H = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i) \) is the mean curvature vector of \( M \). For any \( x \in M \) and \( \{e_1, \cdots, e_n, \cdots, e_{2m+1}\} \) is an orthonormal frame of the tangent space \( T_x \tilde{M} \) such that \( e_1, \cdots, e_n \) are tangent to \( M \) at \( x \). Then, we set

\[
\sigma_{ij} = g(\sigma(e_i, e_j), e_r), \quad i, j \in \{1, \cdots, n\}, \quad r \in \{n+1, \cdots, 2m+1\},
\]

(6)

\[
\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)).
\]

(7)

According to the behavior of the tangent bundle of a submanifold under the action of the almost contact structure tensor \( \varphi \) of the ambient manifold, there are two well-known classes of submanifolds, namely, \( \varphi \)-invariant submanifolds and \( \varphi \)-anti-invariant submanifolds. In the first case the tangent space of the
submanifold remains invariant under the action of the almost contact structure tensor $\varphi$ whereas in the second case it is mapped into the normal space.

Later, A. Bejancu [1] generalized the concept of invariant and anti-invariant submanifolds into a semi-invariant submanifold (also known as contact CR-submanifold [18], [31]). A submanifold $M$ tangent to the structure vector field $\xi$ of an almost contact metric manifold $\tilde{M}$ is called a contact CR-submanifold if there exists a pair of orthogonal distributions $D: x \rightarrow D_x$ and $D^\perp: x \rightarrow D^\perp_x$, $\forall x \in M$ such that $TM = D \oplus D^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the 1-dimensional distribution spanned by the structure vector field $\xi$ with $D$ is invariant, i.e., $\varphi D = D$ and $D^\perp$ is anti-invariant, i.e., $\varphi D^\perp \subseteq T^\perp M$. Obviously, invariant and anti-invariant submanifolds are contact CR-submanifolds with $D^\perp = \{0\}$ and $D = \{0\}$, respectively.

Slant submanifolds in complex geometry were defined and studied by B.-Y. Chen [7,8]. In [20], A. Lotta introduced the contact version of slant submanifolds. Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$. Let $\mathfrak{D}$ be a differentiable distribution on $M$. For any non-zero vector $X \in \mathfrak{D}_x$, the angle $\theta_{\mathfrak{D}}(X)$ between $\varphi X$ and $\mathfrak{D}_x$ is a slant angle of $X$ with respect to the distribution $\mathfrak{D}$. If the slant angle $\theta_{\mathfrak{D}}(X)$ is constant, i.e., it is independent of the choice $x \in M$ and $X \in \mathfrak{D}_x$, then $\mathfrak{D}$ is called a $\theta$-slant distribution and $\theta_{\mathfrak{D}}(X) = \theta_0$ is called the slant angle of the distribution $\mathfrak{D}$. A submanifold $M$ tangent to $\xi$ is said to be slant if for any $x \in M$ and any $X \in T_x M$, linearly independent to $\xi$, the angle between $\varphi X$ and $T_x M$ is a constant $\theta \in [0, \pi/2]$, called the slant angle of $M$ in $\tilde{M}$. Invariant and anti-invariant submanifolds are $\theta$-slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called proper slant. For more details, we refer to [8,5].

For any vector field $X \in \Gamma(TM)$, we have

$$\varphi X = TX + FX,$$

where $TX$ and $FX$ are the tangential and normal components of $\varphi X$, respectively. For a slant submanifold of almost contact metric manifolds we have the following useful result.

**Theorem 1** [5] Let $M$ be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in \Gamma(TM)$. Then $M$ is slant if and only if there exists a constant $\lambda \in [0,1]$ such that

$$T^2 = \lambda(-I + \eta \otimes \xi).$$

Furthermore, if $\theta$ is slant angle, then $\lambda = \cos^2 \theta$.

Following relations are straightforward consequence of (9).

$$g(TX,TY) = \cos^2 \theta[g(X,Y) - \eta(X)\eta(Y)]$$

$$g(FX,FY) = \sin^2 \theta[g(X,Y) - \eta(X)\eta(Y)]$$

where $TX$ and $FX$ are the tangential and normal components of $\varphi X$, respectively.
for any $X,Y \in \Gamma(TM)$.

Beside these classes of submanifolds of almost contact metric manifolds there are some other submanifolds. J.L. Cabrerizo et al. defined and studied semi-slant submanifolds of Sasakian manifolds in [11]. A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be a semi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ on $M$ such that $\mathcal{D}$ is $\varphi$-invariant and $\mathcal{D}^\theta$ is proper slant with slant angle $\theta$ with $TM = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle$.

Pseudo-slant submanifolds were defined by Carriazo in [10] under the name of anti-slant submanifolds as a particular class of bi-slant submanifolds. Later, he called these classes of submanifolds as pseudo-slant submanifolds (see also [19]). A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ on $M$ such that $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle$ with $\mathcal{D}^\perp$ is anti-invariant, that is, $\varphi(\mathcal{D}^\perp) \subset T^\perp M$ and $\mathcal{D}^\theta$ is a proper slant distribution with angle $\theta$.

To generalize the above classes of submanifolds Tripathi [26] introduced the concept of contact skew CR-submanifolds under the name almost semi-invariant submanifolds by exploiting the behavior of a natural bounded symmetric linear operator $T^2 = Q$ on the submanifold. From (2) and (8), it is easy to see that $g(TX,Y) = -g(X,TY)$, for any $X,Y \in \Gamma(TM)$, which implies that $g(QX,Y) = g(X,QY)$, i.e., $Q$ is a symmetric operator, therefore its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by $-1$ and $0$.

Since $\xi \in \Gamma(TM)$, then we have $TM = \langle \xi \rangle \oplus (\langle \xi \rangle)\perp$ where $\langle \xi \rangle$ is the distribution spanned by $\xi$ and $(\langle \xi \rangle)\perp$ is the orthogonal complementary distribution of $\langle \xi \rangle$ in $M$. For any $x \in M$, we may write

$$\mathcal{D}^\perp_x = ker (Q + \lambda^2(x)I) x,$$

where $I$ is the identity transformation and $\lambda(x) \in [0, 1]$ such that $-\lambda^2(x)$ is an eigenvalue of $Q(x)$. We note that $\mathcal{D}^\perp = ker F$ and $\mathcal{D}^\theta = ker T$. $\mathcal{D}^\perp$ is the maximal $\varphi$-invariant subspace of $T_xM$ and $\mathcal{D}^\theta$ is the maximal $\varphi$-anti-invariant subspace of $T_xM$. From now on, we denote the distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by $\mathcal{D} \oplus \langle \xi \rangle$ and $\mathcal{D}^\perp$, respectively. Since $Q_x$ is symmetric and diagonalizable, for some integer $k$ if $-\lambda_1^2(x), \ldots, -\lambda_k^2(x)$ are the eigenvalues of $Q$ at $x \in M$, then $(\langle \xi \rangle^\perp)_x$ can be decomposed as direct sum of mutually orthogonal eigenspaces, i.e.

$$(\langle \xi \rangle^\perp)_x = \mathcal{D}^{\lambda_1} \oplus \mathcal{D}^{\lambda_2} \cdots \oplus \mathcal{D}^{\lambda_k}.$$  

Each $\mathcal{D}^{\lambda_i}$, $1 \leq i \leq k$, is a $T$-invariant subspace of $T_xM$. Moreover if $\lambda_i \neq 0$, then $\mathcal{D}^{\lambda_i}$ is even dimensional. We say that a submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is a generic submanifold if there exists an integer $k$ and functions $\lambda_i$, $1 \leq i \leq k$ defined on $M$ with values in $(0, 1)$ such that

(1) Each $-\lambda_i^2(x)$, $1 \leq i \leq k$ is a distinct eigenvalue of $Q$ with

$$T_xM = \mathcal{D}_x \oplus \mathcal{D}_x^\perp \oplus \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_k} \oplus \langle \xi \rangle_x$$

for any $x \in M$.  

Moreover, if each $\lambda_i$ is constant on $M$, then $M$ is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with $k = 0$, $\mathcal{D} \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$. And slant submanifolds are also a particular class of skew CR-submanifolds with $k = 1$, $\mathcal{D} = \{0\}$, $\mathcal{D}^\perp = \{0\}$ and $\lambda_1$ is constant. Moreover, if $\mathcal{D}^\perp = \{0\}$, $\mathcal{D} \neq 0$ and $k = 1$, then $M$ is a semi-slant submanifold. Furthermore, if $\mathcal{D} = \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $k = 1$, then $M$ is a pseudo-slant (or hemi-slant) submanifold.

A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be a Contact skew CR-submanifold of order 1 if $M$ is a skew CR-submanifold such that $k = 1$ and $\lambda_1$ is constant. In this case, the tangent bundle of $M$ is decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle$$

The normal bundle $T^\perp M$ of a contact skew CR-submanifold $M$ is decomposed as

$$T^\perp M = \varphi \mathcal{D}^\perp \oplus F \mathcal{D}^\theta \oplus \nu,$$

where $\nu$ is a $\varphi$-invariant normal subbundle of $T^\perp M$.

3 Contact skew CR-warped product submanifolds

Let $(B,g_B)$ and $(F,g_F)$ be two Riemannian manifolds and $f$ be a positive smooth function on $B$. Consider the product manifold $B \times F$ with canonical projections $\pi_1 : B \times F \to B$ and $\pi_2 : B \times F \to F$. Then the manifold $M = B \times_f F$ is said to be warped product if it is equipped with the following warped metric

$$g(X,Y) = g_B(\pi_1*(X),\pi_1*(Y)) + (f \circ \pi_1)^2 g_F(\pi_2*(X),\pi_2*(Y))$$

for all $X,Y \in \Gamma(TM)$ and $\pi_1^*$ stands for derivation maps. The function $f$ is called the warping function and a warped product manifold $M$ is said to be trivial or simply a Riemannian product of $B$ and $F$ if $f$ is constant.

**Proposition 1** [2] For $X,Y \in \Gamma(TB)$ and $Z,W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$ that

(i) $\nabla_X Y \in \Gamma(TB)$,

(ii) $\nabla_X Z = \nabla_Z X = X(ln f)Z$,

(iii) $\nabla_Z W = \nabla^\prime_Z W - \frac{g(Z,W)}{f} \nabla f$,

where $\nabla$ and $\nabla'$ denote the Levi-Civita connections on $M$ and $F$, respectively and $\nabla f$ is the gradient of $f$ defined by $g(\nabla f,X) = X(f)$. 

Now for a smooth function $f$ on an $n$-dimensional manifold $M$, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{m} (e_i(f))^2$$

(13)

for the given orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ on $M$.

Remark 1 It is also important to note that for a warped product $M = B \times_f F$; $B$ is totally geodesic and $F$ is totally umbilical in $M$ [24].

In this section, we study warped products of contact skew CR-submanifolds of order 1 of a Sasakian manifold $\tilde{M}$ which we define as: A warped product submanifolds of the form $M = B \times_f M_{\perp}$ is called a contact skew CR-warped product submanifold if $B = M_T \times M_\theta$ is the product of $M_T$ and $M_\theta$, called semislant product, where $M_T$, $M_{\perp}$ and $M_\theta$ are invariant, anti-invariant and proper slant submanifolds of $M$, respectively. Throughout this paper, we assume the structure vector field $\xi$ tangent to the submanifold. For this reason, on a contact skew CR-warped product $M = B \times_f M_{\perp}$, two case arise either $\xi$ is tangent to $M_{\perp}$ or $\xi$ is tangent to $B$. When, $\xi \in \Gamma(TM_{\perp})$, then we have the following non-existence result.

**Theorem 2** Let $M = B \times_f M_{\perp}$ be a contact skew CR-warped product submanifold with $B = M_T \times M_\theta$ of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $M_{\perp}$. Then $M$ is simply a Riemannian product submanifold of $\tilde{M}$.

**Proof** For any $U_1 + U_2 = U \in \Gamma(TB)$, where $U_1 \in \Gamma(TM_T)$ and $U_2 \in \Gamma(TM_\theta)$, we have

$$\tilde{\nabla}_U \xi = -\phi U = -\phi U_1 - TU_2 - FU_2.$$  

Using (4) and equating the tangential components, we derive

$$\nabla_U \xi = -\phi U_1 - TU_2.$$  

Then using Proposition 1(ii), we get

$$U((\ln f)) \xi = -\phi U_1 - TU_2.$$  

Taking the inner product with $\xi$ in the above relation, we find that $U((\ln f)) = 0$, i.e., $f$ is constant, which proves the theorem completely.

From now, for the simplicity we denote the tangent spaces of $M_T$, $M_{\perp}$ and $M_\theta$ by the same symbols $\mathcal{D}$, $\mathcal{D}_{\perp}$ and $\mathcal{D}_\theta$, respectively.

First, we construct the following non-trivial example of contact skew CR-warped submanifolds in Euclidean spaces.
Example 1 Consider the Euclidean space $\mathbb{R}^{13}$ with the cartesian coordinates $(x_1, \ldots, x_6, y_1, \ldots, y_6, z)$ and the almost contact structure

$$
\varphi \left( \frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial y_i}, \quad \varphi \left( \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad \varphi \left( \frac{\partial}{\partial z} \right) = 0, \quad 1 \leq i, j \leq 6.
$$

It is clear that $\mathbb{R}^{13}$ is an almost contact metric manifold with respect to the given structure and standard Euclidean metric tensor of $\mathbb{R}^{13}$. Let $M$ be a submanifold of $\mathbb{R}^{13}$ defined by the immersion $\psi$ as follows

$$
\psi(u, v, w, r, s, t, z) = (u \cos(w + r), u \sin(w + r), v \cos(w - r), v \sin(w - r), k(u + v), s + t, v \cos(w + r), v \sin(w + r), u \cos(w - r), u \sin(w - r), -k(u - v), -s + t, z)
$$

for non-zero vectors and a scalar $k \neq 0$. If the tangent space of $M$ is spanned by the following vectors

$$
X_1 = \cos(w + r) \frac{\partial}{\partial x_1} + \sin(w + r) \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_5} + \cos(w - r) \frac{\partial}{\partial y_3} + \sin(w - r) \frac{\partial}{\partial y_4} - k \frac{\partial}{\partial y_5},
$$

$$
X_2 = \cos(w - r) \frac{\partial}{\partial x_3} + \sin(w - r) \frac{\partial}{\partial x_4} + k \frac{\partial}{\partial x_5} + \cos(w + r) \frac{\partial}{\partial y_1} + \sin(w + r) \frac{\partial}{\partial y_2} + k \frac{\partial}{\partial y_5},
$$

$$
X_3 = -u \sin(w + r) \frac{\partial}{\partial x_1} + u \cos(w + r) \frac{\partial}{\partial x_2} - v \sin(w - r) \frac{\partial}{\partial x_3} + v \cos(w - r) \frac{\partial}{\partial x_4} - v \sin(w + r) \frac{\partial}{\partial y_1} + v \cos(w + r) \frac{\partial}{\partial y_2} - u \sin(w - r) \frac{\partial}{\partial y_3} + u \cos(w - r) \frac{\partial}{\partial y_4},
$$

$$
X_4 = -u \sin(w + r) \frac{\partial}{\partial x_1} + u \cos(w + r) \frac{\partial}{\partial x_2} + v \sin(w - r) \frac{\partial}{\partial x_3} - v \cos(w - r) \frac{\partial}{\partial x_4} - v \sin(w + r) \frac{\partial}{\partial y_1} + v \cos(w + r) \frac{\partial}{\partial y_2} + u \sin(w - r) \frac{\partial}{\partial y_3} - u \cos(w - r) \frac{\partial}{\partial y_4},
$$

$$
X_5 = \frac{\partial}{\partial x_6} - \frac{\partial}{\partial y_6}, \quad X_6 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_6}, \quad X_7 = \frac{\partial}{\partial z}
$$

then, we find

$$
\varphi X_1 = -\cos(w + r) \frac{\partial}{\partial y_1} - \sin(w + r) \frac{\partial}{\partial y_2} - k \frac{\partial}{\partial y_5} + \cos(w - r) \frac{\partial}{\partial x_3} + \sin(w - r) \frac{\partial}{\partial x_4} - k \frac{\partial}{\partial x_5},
$$

$$
\varphi X_2 = -\cos(w - r) \frac{\partial}{\partial y_3} - \sin(w - r) \frac{\partial}{\partial y_4} - k \frac{\partial}{\partial y_5} + \cos(w + r) \frac{\partial}{\partial x_1} + \sin(w + r) \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_5}.
$$
For any $X,Y$ for any lemma. For the second part, we have of $\tilde{B}$ = manifold of order 2 of $\xi$. Hence, $\varphi_X$ is tangent to $\xi$. Hence, $\varphi_X$ = $\varphi_Y$ = $\varphi_Z$. Then, we see that $\varphi_X$ and $\varphi_X$ are orthogonal to $TM$ and hence the distribution $\mathcal{D}^\perp = \text{Span}\{X_3,X_4\}$ is an invariant distribution. It is easy to see that $\mathcal{D} = \text{Span}\{X_5,X_6\}$ is an invariant distribution and $\mathcal{D}^\theta = \text{Span}\{X_1,X_2\}$ is a slant distribution with slant angle $\theta = \cos^{-1}\left(\frac{\sqrt{1+2k^2}}{1+k^2}\right)$. Hence, $M$ is a proper skew CR-submanifold of order 1 of $\mathbb{R}^{13}$ such that $\xi = \frac{\partial}{\partial y}$ is tangent to $M$. It is easy to observe that each distribution is integrable. If we denote the integral manifolds of $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by $M_T$, $M_\theta$ and $M_\perp$, respectively, then the induced metric tensor $g$ of $M$ is given by

$$dS^2 = 2(1+k^2)(du^2 + dv^2) + 2(ds^2 + dt^2) + dz^2 + 2(u^2 + v^2)(du^2 + dv^2)$$

$$g = g_M + 2(u^2 + v^2)g_{M_\perp}.$$ 

Hence, $M$ is a skew CR-warped product submanifold of $\mathbb{R}^{13}$ with the warping function $f = \sqrt{2(u^2 + v^2)}$ and the warped product metric $g$ such that $(M_1,g_1) = (M_T \times M_\theta,g_1)$ with product metric $g_1 = 2(1+k^2)(du^2 + dv^2) + 2(ds^2 + dt^2) + dz^2$.

Now, if we consider $\xi \in \Gamma(TM)$, then there are two possibilities that either $\xi$ is tangent to $M_T$ or tangent to $M_\theta$. For this, we have the following useful results.

**Lemma 1** Let $M = B \times fM_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $B$ and $B = M_T \times M_\theta$, where $M_T$ and $M_\theta$ are invariant and proper slant submanifolds of $M$, respectively. Then, we have

(i) $\xi(\ln f) = 0$,
(ii) $g(\sigma(X,Y),\varphi Z) = 0$,
(iii) $g(\sigma(X,V),\varphi Z) = -g(\sigma(X,Z),F V) = 0$,

for any $X,Y \in \Gamma(\mathcal{D})$, $V \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

**Proof** For any $Z \in \Gamma(\mathcal{D}^\perp)$, we have $\nabla_Z \xi = -\varphi Z$, by using (1), we find that $\nabla_Z \xi = 0, \sigma(Z,\xi) = -\varphi Z$. Using Proposition (1) we get the first part of the lemma. For the second part, we have

$$g(\sigma(X,Y),\varphi Z) = g(\nabla_X Y,\varphi Z) = -g(\nabla_X \varphi Y, Z) + g((\nabla_X \varphi) Y, Z).$$
for any $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Using (3) and the orthogonality of vector fields, we derive
\[
g(\sigma(X, Y), \varphi Z) = g(\tilde{\nabla}_X Z, \varphi Y) = g(\nabla_X Z, \varphi Y).
\]

Again, using Proposition 1, we find that $g(\sigma(X, Y), \varphi Z) = X(\ln f)g(Z, \varphi Y) = 0$, which is (ii). Similarly, for any $X \in \Gamma(\mathcal{D}), V \in \Gamma(\mathcal{D}^0)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have
\[
g(\sigma(X, V), \varphi Z) = g(\tilde{\nabla}_X V, \varphi Z) = -g(\tilde{\nabla}_X \varphi V, Z) + g((\tilde{\nabla}_X \varphi)V, Z).
\]

Again, from (3), (8) and the orthogonality of vector fields, we obtain
\[
g(X, V) = g(\tilde{\nabla}_X V, \varphi Z) = g(\nabla_X Z, \varphi V)\, \nabla_X Z, \varphi V)= 0, which is (ii). Similarly, for any $X \in \Gamma(\mathcal{D}), V \in \Gamma(\mathcal{D}^0)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have
\[
g(\sigma(X, V), \varphi Z) = g(\tilde{\nabla}_X V, \varphi Z) = -g(\tilde{\nabla}_X \varphi V, Z) + g((\tilde{\nabla}_X \varphi)V, Z).
\]

Again, using the structure equation of Sasakian manifold, the orthogonality of vector fields and Proposition 1, we get $\sigma(X, V) = g(\tilde{\nabla}_X V, \varphi Z) = 0$, which is the second equality. Hence, the proof is complete.

**Lemma 2** Let $M = B \times \mathcal{I}M_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $B$. Then
\[
g(\sigma(U, V), \varphi Z) = g(\sigma(U, Z), FV) \quad (14)
\]
for any $U, V \in \Gamma(\mathcal{D}^0)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

**Proof** For any $U, V \in \Gamma(\mathcal{D}^0)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have
\[
g(\sigma(U, V), \varphi Z) = g(\tilde{\nabla}_U V, \varphi Z) = -g(\tilde{\nabla}_U \varphi V, Z) + g((\tilde{\nabla}_U \varphi)V, Z).
\]

Using (3), (5) and the orthogonality of vector fields, we find
\[
g(\sigma(U, V), \varphi Z) = g(\tilde{\nabla}_U V, \varphi Z) - g(\tilde{\nabla}_U \varphi V, Z) - g(\varphi \nabla_U V, TV) - g(A_{FV} U, Z).
\]

By Proposition 1 and the orthogonality of vector field, we obtain
\[
g(\sigma(U, V), \varphi Z) = g(\sigma(U, Z), FV),
\]
which proves the lemma completely.

**Lemma 3** Let $M = B \times \mathcal{I}M_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $B$. Then, we have
\[
g(\sigma(\varphi X, Z), \varphi W) = X(\ln f)g(Z, W) \quad (15)
\]
for any $X \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$. 

Proof For any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have
\[ g(\sigma(X, Z), \varphi W) = g(\tilde{\nabla}_Z X, \varphi W) = -g(\tilde{\nabla}_Z \varphi X, W) + g((\tilde{\nabla}_Z \varphi)X, W). \]
Using Proposition 1 structure equation (3) and the orthogonality of vector fields, we find
\[ g(\sigma(X, Z), \varphi W) = -\varphi X(\ln f)g(Z, W) - \eta(X)g(Z, W). \quad (16) \]
Interchanging $X$ by $\varphi X$ and using (1), we find (15), which completes the proof.

A warped product $M = B \times f F$ is said to be mixed totally geodesic if $\sigma(X, Z) = 0$, for any $X \in \Gamma(TB)$ and $Z \in \Gamma(TF)$. From Lemma 3 we have the following consequence for a mixed totally geodesic warped product.

**Theorem 3** Let $M = B \times f M_{\perp}$ be a warped product submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $B$. Then $M$ is simply a Riemannian product manifold.

**Proof** The proof of this theorem follows from (15) and the mixed totally geodesic condition.

**Lemma 4** Let $M = B \times f M_{\perp}$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $B$. Then

(i) $g(\sigma(Z, W), FV) - g(\sigma(Z, V), \varphi W) = (TV(ln f) + \eta(V))g(Z, W),$
(ii) $g(\sigma(Z, W), FTV) - g(\sigma(Z, TV), \varphi W) = -\cos^2 \theta V(ln f)g(Z, W)$

for any $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D}^0)$.

**Proof** For any $V \in \Gamma(\mathfrak{D}^0)$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have
\[ g(\sigma(Z, V), \varphi W) = g(\tilde{\nabla}_Z V, \varphi W) = -g(\tilde{\nabla}_Z \varphi V, W) + g((\tilde{\nabla}_Z \varphi)V, W). \]
Using (3) and (8), we derive
\[ g(\sigma(Z, V), \varphi W) = -g(\tilde{\nabla}_Z TV, W) - g(\tilde{\nabla}_Z FV, W) - \eta(V)g(Z, W), \]
which on using Proposition 1 (ii) implies that
\[ g(\sigma(Z, W), FV) - g(\sigma(Z, V), \varphi W) = (TV(ln f) + \eta(V))g(Z, W), \]
which is (i). Interchanging $V$ by $TV$ in (i) and using Theorem 1 we find (ii), which ends the proof.
4 Inequality for contact skew CR-warped products

In this section, we derive an inequality for the squared norm of the second fundamental form of contact skew CR-warped product immersion in terms of the components of the gradient vector field of the warping function.

First, we construct the following frame fields for the contact skew CR-warped product submanifold $M$ of Sasakian manifold $\tilde{M}$. Let $M = B \times_f M_\perp$ be a $n$-dimensional contact skew CR-warped product submanifold of a $(2m + 1)$-dimensional Sasakian manifold $M$ with $B = M_T \times M_\theta$ and $\xi$ is tangent to $B$ where $M_T$, $M_\perp$ and $M_\theta$ are invariant, anti-invariant and proper slant submanifolds of $M$ with their real dimensions as $\dim M_T = m_1$, $\dim M_\perp = m_2$ and $\dim M_\theta = m_3$, respectively. Then, clearly we have $n = m_1 + m_2 + m_3$.

We denote the tangent bundle of $\tilde{M}$ warped product submanifold the components of the gradient vector field of the warping function.

Now, using the above orthonormal frame field and some results of previous sections, we derive the following main result of this paper.

**Theorem 4** Let $M = B \times_f M_\perp$ be a $\mathcal{D}^\perp - \mathcal{D}^\theta$ mixed totally geodesic contact skew CR-warped product submanifold of order 1 of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $B$ and $B = M_T \times M_\theta$, where $M_T$, $M_\perp$ and $M_\theta$ are invariant, anti-invariant and proper slant submanifolds of $\tilde{M}$ with their real dimensions $m_1$, $m_2$ and $m_3$, respectively. Then we have:

(i) If $\xi$ is tangent to $M_T$, then

$$\|\sigma\|^2 \geq 2m_2(\|\nabla^T (\ln f)\|^2 + 1) + m_2 \cot^2 \theta \|\nabla^\theta (\ln f)\|^2.$$  

(ii) If $\xi$ is tangent to $M_\theta$, then

$$\|\sigma\|^2 \geq 2m_2\|\nabla^T (\ln f)\|^2 + m_2 \cot^2 \theta \|\nabla^\theta (\ln f)\|^2,$$

where $\nabla^T (\ln f)$ and $\nabla^\theta (\ln f)$ are the gradient components along $M_T$ and $M_\theta$, respectively.

(iii) If the equality sign holds in above inequalities, then $B$ is totally geodesic and $M_\perp$ is a totally umbilical in $\tilde{M}$.
Proof From the definition of the second fundamental from $\sigma$, we have

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=1}^{2m+1} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), e_r).$$

According to the constructed frame filed, the above relation takes the from

$$\|\sigma\|^2 = \sum_{r=n+1}^{n+m_2} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), e_r)^2 + \sum_{r=n+2}^{n+m_2+m_3} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), e_r)^2$$

$$+ \sum_{r=n+m_2+1}^{2m+1} \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), e_r)^2. \quad (17)$$

The last term in the right hand side of (17) has the $\mu-$ components only and we couldn’t find any relation for contact skew CR-warped product in terms of $\mu-$ components. Therefore, leaving this positive term, we can split the above relation for the orthogonal spaces as follows

$$\|\sigma\|^2 \geq \sum_{r=1}^{m_1} \sum_{i,j=1}^{m_2} g(\sigma(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g(\sigma(e_i, \tilde{e}_i), \tilde{e}_r)^2$$

$$+ \sum_{r=1}^{m_2} \sum_{i,j=1}^{m_3} g(\sigma(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{i=1}^{m_2} \sum_{j=1}^{m_3} g(\sigma(e_i, \tilde{e}_i), \tilde{e}_r)^2$$

$$+ \sum_{r=m_2+1}^{m_2+m_3} \sum_{i,j=1}^{m_1} g(\sigma(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{i=1}^{m_2+m_3} \sum_{j=1}^{m_1} g(\sigma(e_i, \tilde{e}_i), \tilde{e}_r)^2$$

$$+ \sum_{r=m_2+1}^{m_2+m_3} \sum_{i,j=1}^{m_2} g(\sigma(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{i=1}^{m_2+m_3} \sum_{j=1}^{m_2} g(\sigma(e_i, \tilde{e}_i), \tilde{e}_r)^2$$

$$+ \sum_{r=m_2+1}^{m_2+m_3} \sum_{i,j=1}^{m_3} g(\sigma(e_i, e_j), \tilde{e}_r)^2 + 2 \sum_{i=1}^{m_2+m_3} \sum_{j=1}^{m_3} g(\sigma(e_i, \tilde{e}_i), \tilde{e}_r)^2. \quad (18)$$

Using the hypothesis of theorem as $M$ is $\mathcal{D}^+ - \mathcal{D}^0$ mixed totally geodesic, the forth and tenth terms in the right hand side of (18) vanish identically. First and fifth terms are identically zero by using Lemma [1](ii) and Lemma [2] with mixed totally geodesic condition, respectively. Also, sixth and eighth terms vanish identically by using Lemma [1](iii). On the other hand, we couldn’t find any relation for the skew CR-warped product for the third, seventh, eleventh and twelfth terms, so leaving these terms. Then the remaining second and
ninth terms can be written as

\[
\|\sigma\|^2 \geq 2 \sum_{r=1}^{m_2} \sum_{p=1}^{m_2} g(\sigma(e_i, e_j), \varphi e_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{i=1}^{m_2} p g(\sigma(\varphi e_i, e_j), \varphi e_r)^2 \\
+ 2 \sum_{r=1}^{m_2} \sum_{j=1}^{m_2} g(\sigma(e_{2p+1}, e_j), \varphi e_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{i=1}^{m_2} p \sum_{j=1}^{m_2} g(\sigma(e_i, e_j), \csc \theta F e_r)^2 \\
+ \sum_{r=1}^{m_2} \sum_{i=1}^{m_2} g(\sigma(e_i, e_j), \csc \theta \sec \theta F T e_r)^2.
\] (19)

Since, for a submanifold \( M \) of a Sasakian manifold \( \sigma(U, \xi) = -\varphi U \), for any \( U \in \Gamma(TM) \), using this fact in the third term of \( (19) \). Also, using Lemma 3 and Lemma 4 with the \( \mathcal{D}^\perp - \mathcal{D}^\theta \) mixed totally geodesic condition, we derive

\[
\|\sigma\|^2 \geq 2 \sum_{r=1}^{m_2} \sum_{j=1}^{m_2} (\varphi e_i (\ln f) + \eta(e_i))^2 g(e_j, e_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{j=1}^{m_2} (\eta(e_i))^2 g(e_j, e_r)^2 \\
+ 2 \sum_{j=1}^{m_2} g(\sigma(e_j, \varphi e_r)^2 + \csc^2 \theta \sum_{r=1}^{m_2} s \sum_{r=1}^{m_2} (Te_r^* (\ln f) + \eta(e_r))^2 g(e_i, e_j)^2 \\
+ \cot^2 \theta \sum_{r=1}^{m_2} s \sum_{r=1}^{m_2} (e_r^* (\ln f))^2 g(e_i, e_j)^2.
\] (20)

Now, we consider both cases: (i) When \( \xi \in \Gamma(\mathcal{D}) \), then we have

\[
\|\sigma\|^2 \geq 2 m_2 \sum_{i=1}^{2p+1} (e_i(\ln f))^2 - 2 m_2 (e_{2p+1}(\ln f))^2 + 2 m_2 \\
+ m_2 \csc^2 \theta \sum_{r=1}^{m_3} (Te_r^* (\ln f))^2 + m_2 \cot^2 \theta \sum_{r=1}^{m_3} (e_r^* (\ln f))^2 \\
- m_2 \csc^2 \theta \sum_{r=s+1}^{m_3} (Te_r^* (\ln f))^2.
\]

Now, using (13) and Lemma 1 (i), we find

\[
\|\sigma\|^2 \geq 2 m_2 (\|\nabla^T (\ln f)\|^2 + 1) + m_2 \csc^2 \theta \|\nabla^\theta (\ln f)\|^2 \\
+ m_2 \cot^2 \theta \sum_{r=1}^{s} (e_r^* (\ln f))^2 - m_2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{s} g(T e_r^*, T \nabla^\theta (\ln f))^2 \\
= 2 m_2 (\|\nabla^T (\ln f)\|^2 + 1) + m_2 \csc^2 \theta \|\nabla^\theta (\ln f)\|^2,
\]
which is inequality (i). If \( \xi \in \Gamma(\mathcal{D}^\theta) \), then from (19), we obtain
\[
\|\sigma\|^2 \geq 2m_2\|\nabla^T (\ln f)\|^2 + m_2 \csc^2 \theta \sum_{r=1}^{m_2} g(e^*_r, T\nabla^\theta (\ln f))^2 + m_2 \csc^2 \theta \\
+ m_2 \cot^2 \theta \sum_{r=1}^{s} (e^*_r(\ln f))^2 - m_2 \csc^2 \theta \sum_{r=1}^{s} g(e^*_{r+s}, T\nabla^\theta (\ln f))^2 - m_2 \csc^2 \theta \\
= 2m_2\|\nabla^T (\ln f)\|^2 + m_2 \csc^2 \theta \|T\nabla^\theta (\ln f)\|^2 \\
+ m_2 \cot^2 \theta \sum_{r=1}^{s} (e^*_r(\ln f))^2 - m_2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^{s} g(T e^*_r, T\nabla^\theta (\ln f))^2 \\
= 2m_2\|\nabla^T (\ln f)\|^2 + m_2 \cot^2 \theta \|\nabla^\theta (\ln f)\|^2,
\]
which is inequality (ii). For the equality case, we have by \( \mathcal{D}^\perp - \mathcal{D}^\theta \) mixed totally geodesic condition, we have
\[
\sigma(\mathcal{D}^\perp, \mathcal{D}^\theta) = 0. \tag{21}
\]
Also, from the leaving third term of (17), we find that
\[
\sigma(\mathcal{D}, \mathcal{D}), \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp), \sigma(\mathcal{D}^\theta, \mathcal{D}^\theta), \sigma(\mathcal{D}^\perp, \mathcal{D}^\theta), \sigma(\mathcal{D}, \mathcal{D}^\theta) \perp \mu. \tag{22}
\]
From Lemma 1 (i) and first part of (22), we obtain
\[
\sigma(\mathcal{D}, \mathcal{D}) = 0. \tag{23}
\]
Also, from Lemma 2 with the \( \mathcal{D}^\perp - \mathcal{D}^\theta \) mixed totally geodesic condition and the leaving eleventh term in (18) with third relation of (22), we deduce that
\[
\sigma(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0. \tag{24}
\]
On the other hand, from the leaving twelfth term in (18) with Lemma 1 (iii) and fifth relation of (22), we obtain
\[
\sigma(\mathcal{D}, \mathcal{D}^\theta) = 0. \tag{25}
\]
Then, from (21), (23), (24), (25) with the Remark 1, we conclude that \( B \) is totally geodesic in \( \tilde{M} \). Also, from Lemma 1 (ii) with the sixth relation of (22), we find
\[
\sigma(\mathcal{D}, \mathcal{D}^\perp) \subseteq \varphi \mathcal{D}^\perp. \tag{26}
\]
While from the leaving third term in (18) with second relation of (22), we deduce that
\[
\sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) \subseteq \varphi \mathcal{D}^\theta. \tag{27}
\]
Thus, from Lemma 3 with (26) and Lemma 4 with (27) and with $D^\perp - D^\theta$ mixed totally geodesic condition, we find
\[
g(\sigma(\varphi X, Z), \varphi W) = X(\ln f)g(Z, W), \\
g(\sigma(Z, W), FTV) = -\cos^2 \theta V(\ln f)g(Z, W)
\] (28)
for any $X \in \Gamma(D)$, $V \in \Gamma(D^\theta)$ and $Z, W \in \Gamma(D^\perp)$. Then, by Remark 1 with (21), we deduce that $M^\perp$ is totally umbilical in $\tilde{M}$. Hence, the theorem is proved completely.

5 Applications of Theorem 4

There are two well known applications of Theorem 4 as given below:

1. If $D^\theta = \{0\}$ i.e., $\dim M_\theta = 0$ in a contact skew CR-warped product, then it reduces to contact CR-warped products of the form $M = M_T \times_f M_\perp$ studied in [18]. In this case, the statement of Theorem 4 will be: Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi$ is tangent to $M_T$, where $M_T$ and $M_\perp$ are invariant and anti-invariant submanifolds of $\tilde{M}$ with their real dimensions $m_1, m_2$, respectively. Then we have:

(i) The squared norm of the second fundamental from $\sigma$ satisfies
\[
||\sigma||^2 \geq 2m_2 \left(||\nabla^T(\ln f)||^2 + 1\right).
\]
where $\nabla^T(\ln f)$ is the gradient of $\ln f$ along $M_T$.

(ii) If the equality sign holds in above inequality, then $M_T$ is totally geodesic and $M_\perp$ is a totally umbilical in $\tilde{M}$.

Which is the main result of [18].

2. On the other hand, if $D = \{0\}$ in a contact skew CR-warped product, then it will change into a pseudo-slant warped product of the form $M = M_\theta \times_f M_\perp$ studied in [30]. In this case, Theorem 4.2 of [30] is a particular case of Theorem 4 as follows:

Corollary 1 (Theorem 4.2 of [30]) Let $M = M_\theta \times_f M_\perp$ be a mixed totally geodesic warped product submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(D^\theta)$, where $M_\theta$ is a proper slant submanifold and $M_\perp$ is an $m_2$-dimensional anti-invariant submanifold of $\tilde{M}$. Then we have:

(i) The squared norm of the second fundamental form of $M$ satisfies
\[
||\sigma||^2 \geq m_2 \cot^2 \theta ||\nabla^\theta(\ln f)||^2
\]
where $\nabla^\theta \ln f$ is the gradient of $\ln f$ along $M_\theta$. 
(ii) If the equality sign in (i) holds identically, then $M_\theta$ is totally geodesic in $\tilde{M}$ and $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

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