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We construct charged asymptotically flat black hole solutions in Einstein-Maxwell-Weyl (EMW) gravity. These solutions can be interpreted as generalizations of two different groups: Schwarzschild black hole (SBH) and non-Schwarzschild black hole (NSBH) solutions. In addition, we discuss the thermodynamic properties of two groups of numerical solutions in detail, and show that they obey the first law of thermodynamics.

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\section{I. INTRODUCTION}

As well known, the general relativity can be viewed as an effective low-energy field theory in string theory. It’s also the first term in an infinite series of gravitational corrections built from powers of the curvature tensor and its derivatives \cite{1}. In addition, the general relativity leaves some open fundamental questions, including the problems of singularity, non-renormalizability, the dark matter and dark energy. In order to answer these questions, Stelle et.al asserted one can add higher derivatives terms to the Einstein-Hilbert action \cite{2}. In four-dimensional spacetime, the most general theory up to the second order in the curvature is given by \cite{3–5}

\begin{equation}
I = \int d^4x \sqrt{-g} \left[ \gamma R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2 \right]
\end{equation}

where $\alpha$, $\beta$ and $\gamma$ are constants, and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Notice that for any static, spherically symmetric black-hole solution, the term quadratic in Ricci scalar $R$ makes no contribution to the corresponding field equations \cite{6}. As a result, this setting of $\beta = 0$ and $\gamma = 1$ has been often applied for simplicity in the Einstein-Weyl (EW) gravity, and this theory reduces to the pure Einstein-Weyl gravity. By fixed $\alpha = 1/2$, Lü et.al \cite{4, 5} derived a new numerical non-Schwarzschild
black hole (NSBH) in the region of $0.876 < r_0 < 1.143$ for horizon radius $r_0$ in the EW gravity, besides the Schwarzschild black hole (SBH). Different from SBH, this NSBH has some remarkable properties: i) it admits positive and negative values of black hole mass; ii) when the horizon radius $r_0$ approaches some extremal value, this new black hole approaches the massless state \cite{7}. In terms of the continued fractions, Kokkotas et al. \cite{8} further constructed this NSBH solution in the analytical form. Very recently, the new black hole solutions have been also derived by adopting a new form of the metric in the pure Einstein-Weyl gravity \cite{9}. Under test scalar field perturbation, the stabilities of NSBH and SBH have been also separately investigated \cite{10, 11}, which recovered that the quasinormal modes for NSBH share larger real oscillation frequency and larger damping rate than the SBH branch. In particular, undamped oscillations (called quasi-resonances) also emerged for the NSBH, if perturbed scalar field possesses sufficiently large masses.

Inspired by above issues, Lin et al. further explored the Einstein-Maxwell-Weyl (EMW) gravity consisting of pure EW term and electromagnetic field \cite{12}, where it presented two groups (Groups I and II) of new charged black holes with fixed $\beta = 0$, $\gamma = 1$ and $\alpha = 1/2$. Actually, the two groups solutions could be separately viewed as a charged generalization of the higher derivative curvature for SBH and a charged generalization of NSBH. However, Reissner-Nordstr"om (RN) black hole solution is not a solution in the EMW gravity.

Comparing with the NSBH in the bound $0.876 < r_0 < 1.143$ \cite{4, 5}, new NSBH solutions have been obtained within an extended bound $0.363 < r_0 < 1.143$ in pure EW gravity \cite{7}. With regard to the EMW gravity, we can also construct new charged black hole solutions according to the new non-charged ‘seed’ solutions. Then, we will further discuss the thermodynamic properties of these new charged black holes in the EMW gravity.

This paper is organized as follows. In Sec. II we numerically derive new charged asymptotically flat black hole solutions. Then, some related thermodynamic properties of these new charged black holes are explored in Sec. III. Finally, we end the paper with conclusions and discussions in Sec. IV.

\section{II. EINSTEIN-MAXWELL-WEYL GRAVITY AND NUMERICAL SOLUTIONS}

The Einstein-Maxwell-Weyl action including pure Einstein-Weyl term and electromagnetic field is given by \cite{12}

\[ I = \int d^4x \sqrt{-g} \left[ R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \kappa F_{\mu\nu}F^{\mu\nu} \right], \]  

(2)
where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the electromagnetic tensor. Then, the equations of motion are obtained as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 4\alpha B_{\mu\nu} - 2\kappa T_{\mu\nu} = 0,$$

$$\nabla_\mu F^{\mu\nu} = 0,$$

where the trace-free Bach tensor $B_{\mu\nu}$ and energy-momentum tensor of electromagnetic field $T_{\mu\nu}$ are defined as

$$B_{\mu\nu} = \left(\nabla^\rho \nabla_\rho + \frac{1}{2} R_{\rho\sigma}\right) C_{\mu\nu\rho\sigma}, \quad T_{\mu\nu} = F_{\alpha\mu} F^\alpha_\nu - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}.$$

In order to construct new charged black hole solution, we assume a new metric ansatz

$$ds^2 = -f(r)e^{-2\delta(r)}dt^2 + \frac{1}{f(r)}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

with $h(r) = f(r)e^{-2\delta(r)}$ and $f(r) = 1 - \frac{2m(r)}{r}$. Substituting the ansatz [eq.(5)] into eq.(3), one get

$$2r^3 m' - Q^2 r + 2r^3 (r - 2m)\delta' + \alpha \left[ 4r (2m - r) (r - 3m + rm' + r(r - 2m)\delta') \delta'' + 16 (2m - r)m' + 4 (4r^2 m'^2 - 2r^2 + 2m^2 + 5mr + (r^2 - 10mr)m') \delta' 
+ (2r (28m^2 - 16rm + r^2) + 24r^2 (r - 2m)) \delta'^2 + 8r^2 (r - 2m)^2 \delta'^3 \right] = 0,$$

$$2r^3 m' - Q^2 r + 2r^3 (r - 2m)\delta' + \alpha \left[ 4r (r - 3m + rm' + r(r - 2m)\delta') m'' + 8 (m - r)m' + 8m^2 r + 4 (r^2 m'^2 - 2mr + 5m^2) + 2 (r^2 - 3mr)m') \delta' 
+ (2r (20m^2 - 16rm + 3r^2) + 8r^2 (r - 2m)) \delta'^2 + 4r^2 (r - 2m)^2 \delta'^3 \right] = 0,$$

$$A'_t - \frac{Qe^{-\delta}}{r^2} = 0,$$

where the prime (') denotes differentiation with respect to $r$, and parameter $Q$ denotes the electric charge.

Influenced by the functional form of the electric charge in the metric, the charged black holes have more than one horizon in general. For example, RN black hole has one event horizon and one Cauchy horizon. However, we suppose that the spacetime has only one horizon to make it easier for the expansion of $m(r)$ and $\delta(r)$ around the event horizon $r_0$.

$$m(r) = \frac{r_0}{2} + m_1 (r - r_0) + m_2 (r - r_0)^2 + \ldots,$$

$$\delta(r) = \delta_0 + \delta_1 (r - r_0) + \delta_2 (r - r_0) + \ldots,$$

$$A'_t(r) = A_{t1} (r - r_0) + A_{t2} (r - r_0)^2 + \ldots.$$
Substituting these expansions into eq.(3), the coefficients $\delta_i$, $A_{ti}$ (for $i = 1$) and $m_i$ (for $i = 2$) can be solved in terms of the three non-trivial free parameters $r_0$, $m_1$ and $\delta_0$. For example, $m_2$, $A_{t1}$ and $\delta_1$ can be obtained as

$$m_2 = -\frac{m_1}{r_0} - \frac{6m_1r_0^2 - 3Q^2}{16\alpha(2m_1 - 1)r_0}, \quad \delta_1 = \frac{2m_1r_0^2 - Q^2}{4\alpha(2m_1 - 1)^2r_0}, \quad A_{t1} = \frac{e^{-\delta_0}Q}{r_0^2}. \quad (12)$$

Here the coefficients of expansion can be also presented by $m_1 = -\frac{\delta}{2}$, and $\delta_0 = \frac{1}{2}\ln\left(\frac{1 + \delta}{\delta_{c0}}\right)$ in refs. [4, 5].

At the other asymptotic regime, that of radial infinity ($r \to 1$), the metric functions and the Maxwell field may be expanded in power series, this time in terms of $1/r$. The metric components reduce to

$$m(r) = M - \frac{Q^2}{2r} + \ldots, \quad \delta(r) = \frac{2\alpha Q^2}{r^4} + \ldots, \quad A_i(r) = \Phi + \frac{Q}{r} + \ldots, \quad (13)$$

where the parameter $M$ and $Q$ are associated with the mass and charge of black hole, and $\Phi$ is the electric potential.

Firstly, we reconsider non-Schwarzschild black hole solution ($Q = 0$) in the EW gravity. Throughout this paper, we also take $\alpha = \frac{1}{2}$ and $\kappa = 1$ for simplicity [4, 5]. The signal for a good black hole solution is that the functions $f(r)$ and $h(r)$ should approach very close to 1 as $r$ increases. Comparing with the previous bound ($0.876 < r_0 < 1.143$) for the horizon radius $r_0$ [4, 5], we numerically derive an extended bound $0.363 < r_0 < 1.143$. The values 0.363 and 1.143 of horizon radius $r_0$ denote the disappearance of temperature $T = 0$ and massless state $M = 0$ for the non-Schwarzschild black hole, respectively. If $r_0$ is smaller than 0.363, the temperature of non-Schwarzschild black hole is negative. When $r_0$ is larger than 1.143, the black hole mass would become negative [4]. The corresponding bound for the parameter $m_1$ is $0.46 > m_1 > -0.422$. At $r_0 \approx 0.876$, the Schwarzschild and the non-Schwarzschild black holes 'coalesce' with $m_1 = 0$. In other words, for any selected value of $r_0$ in the above extended bounded interval, there exists only one value of $m_1$ that allows for a healthy non-Schwarzschild black hole. In Fig.1, we plot the metric functions $f$ and $h$ for the NSBH with horizon radius $r_0 = 0.6$ and 1.

Now, we turn to discuss the charged black hole in the Einstein-Maxwell-Weyl gravity. It’s worth noticing that the ref. [12] asserted the metric functions $f(r)$ and $h(r)$ for the charged black hole solutions in the Group II presented a peak outside the event horizon. It implied the presence of a unphysical negative effective mass [Fig.3 in ref. [12]]. Here we reconstruct four different charged black hole solutions of groups I and II on both sides of the coalesce point $r_0 \approx 0.876$ of Schwarzschild and non-Schwarzschild black holes, see Figs. 2 and 3. Obviously, the peak of the
functions $f(r)$ and $h(r)$ vanishes, which means that new charged black holes of the Group II could possess positive masses. We will verify it in the next section.

FIG. 2: Numerical charged solutions for $f(r)$, $h(r)$ and $A_t(r)$ and some values of $Q = 0.16$ in Group I. In each plot the function $h(r)$ with $\delta_0 = 0.423$ is chosen to approach to $4/5$ instead of $1$ for clarity. The purple dashed line represents the unity.

### III. THERMODYNAMIC PROPERTIES OF CHARGED BLACK HOLES

With the numerical charged black holes of Groups I and II, it’s interesting to study the thermodynamic properties of these solutions. In order to do this, we need to collect the numerical results for a sequence of charged black-hole solutions with different values of $Q$.

In Fig. 4 we show the mass $M$ as a function of Hawking temperature $T$ for the SBH and NSBH in the pure Einstein-Weyl gravity. At high temperature region, SBH and NSBH take small masses $M$, coalesce at $T \approx 0.091 (r_0 \approx 0.876)$, and then take reverse behaviors at lower temperatures.

Take the charged black holes with $r_0 = 1.3$ of the Group I as an example: starting from SBH
and holding the horizon radius $r_0 = 2$, the mass of charged black hole becomes larger, while the temperature decreases as the increase of charge $Q$, and then vanishes at $Q \approx 1.86$, see Fig. 4(a). Here the arrow indicates the increase of charge $Q$. On the other hand, the temperature increases and mass decreases when the charge $Q$ becomes larger for the new charged black holes bifurcation from the right-hand part of coalesce point. There also exist a bound $0 < Q \leq 1.035$ for the charged black hole with $r_0 = 0.5$. Interestingly, if $Q$ is increased beyond $Q_c = 1.035$, one can still obtain a charged black hole solution for an appropriate choice of $m_1$, but now the mass is actually negative, see Fig. 5(a).

The similar phenomenon occurs for charged black holes in the Group II. We construct charged black holes starting from a sequence of non-Schwarzschild black holes with horizon radius in the region of $0.363 < r_0 < 1.143$. see Fig. 4(b). Considering $M \geq 0$ and $T \geq 0$, the charge $Q$ of charged black holes need to satisfy $0 < Q < 0.305$ for $r_0 = 1.1$, $0 < Q < 0.4$ for $r_0 = 0.5$ and $0 < Q < 0.65$ for $r_0 = 0.7$, see Fig. 5(b).

FIG. 3: Numerical charged solutions for $f(r)$, $h(r)$ and $A_t(r)$ and $Q = 0.16$ in Group II. In each plot the function $h(r)$ is chosen to approach $2/5$ for clarity. The purple dashed line represents the unity.

FIG. 4: Mass $M$ versus temperature $T$ relations for charged, SBH and NSBH. The arrow indicates the increase of charge $Q$. 
Due to the higher-derivative theory, the entropy cannot be simply given by one quarter of the area of the event horizon, which equals to \( S = \pi r_0^2 + 8\pi \alpha m_1 \). The entropy \( S \) as a function of mass \( M \) of these charged black holes in the Groups I and II are shown in Figs. 6.

Now we check the first law of thermodynamics of charged black holes by utilizing the numerical values for thermodynamic quantities \( M, S, Q, T \) and \( \Phi \). We only consider the charged black hole with \( r_0 = 0.5 \) in the Group I and II. These discrete values of these thermodynamical quantities are shown in the TABLE I. In the table on the left (charged black hole in Group I), forward differences of mass \( M \) entropy \( S \) and charge \( Q \) can be written as

\[
\Delta M \equiv \frac{M[i+2] - M[i]}{2},
\]

\[
= \{0.0002635, 0.0013918, 0.0026031, 0.003350, 0.0042297, 0.0060, 0.0068047\},
\]

\[
\Delta S \equiv \frac{S[i+2] - S[i]}{2},
\]

\[
= \{0.0044782, 0.023697, 0.044434, 0.0574451, 0.07291, 0.104281, 0.119241\},
\]

\[
\Delta Q \equiv \frac{Q[i+2] - Q[i]}{2} = \{-0.015, -0.03, -0.035, -0.03, -0.03, -0.035, -0.035\}, \quad i = 1, \ldots, 7.
\]
Then the expression \( dM - (TdS + \Phi dQ) \) in the form of discrete points is given by
\[
\Delta M[i] - (T[i + 1] \cdot \Delta S[i] + \Phi[i + 1] \cdot \Delta Q[i]) \\
= \{-1.5 \times 10^{-4}, -5.8 \times 10^{-4}, 3.6 \times 10^{-4}, 1.6 \times 10^{-5}, 2.0 \times 10^{-5}, -2.6 \times 10^{-4}, 3.2 \times 10^{-4}\}, \quad i = 1..7.
\]
(15)

From the Table on the right (charged black hole in Group II), we can also calculate \( dM - (TdS + \Phi dQ) \) by using the finite difference method
\[
\Delta M[i] - (T[i + 1] \cdot \Delta S[i] + \Phi[i + 1] \cdot \Delta Q[i]) \\
= \{6.8 \times 10^{-5}, 2.2 \times 10^{-5}, 6.6 \times 10^{-5}, 2.1 \times 10^{-6}, 6.6 \times 10^{-5}, 1.2 \times 10^{-6}, 6.0 \times 10^{-5}\}, \quad i = 1..7.
\]
(16)

Thus the charged black holes obey the first law \( dM = TdS + \Phi dQ \) to quite a high precision.

| No. | Q   | M    | S    | T    | \Phi |
|-----|-----|------|------|------|------|
| 1   | 0   | 0.25 | 0.78539 | 0.159155 | 0    |
| 2   | 1/100 | 0.24994 | 0.78440 | 0.15919 | 0.0200 |
| 3   | 3/100 | 0.24947 | 0.77644 | 0.15949 | 0.06001 |
| 4   | 7/100 | 0.24715 | 0.73700 | 0.16097 | 0.14023 |
| 5   | 10/100 | 0.24426 | 0.68757 | 0.16283 | 0.20066 |
| 6   | 13/100 | 0.24045 | 0.62211 | 0.16530 | 0.26143 |
| 7   | 16/100 | 0.23580 | 0.54175 | 0.16835 | 0.32261 |
| 8   | 20/100 | 0.22845 | 0.41355 | 0.17325 | 0.40497 |
| 9   | 23/100 | 0.22220 | 0.30327 | 0.17749 | 0.46737 |

| No. | Q   | M    | S    | T    | \Phi |
|-----|-----|------|------|------|------|
| 1   | 0   | 0.61944 | 5.71036 | 0.01348 | 0    |
| 2   | 2/100 | 0.619849 | 5.71391 | 0.013413 | 0.031039 |
| 3   | 35/1000 | 0.62054 | 5.72123 | 0.01328 | 0.054268 |
| 4   | 5/100 | 0.62168 | 5.73247 | 0.013079 | 0.07744 |
| 5   | 7/100 | 0.62380 | 5.75344 | 0.012705 | 0.10821 |
| 6   | 9/100 | 0.62662 | 5.78105 | 0.012218 | 0.13877 |
| 7   | 105/1000 | 0.62917 | 5.80592 | 0.01178 | 0.16152 |
| 8   | 12/100 | 0.63208 | 5.8342 | 0.01129 | 0.18409 |
| 9   | 14/100 | 0.63653 | 5.87694 | 0.010576 | 0.21388 |

TABLE I: The discrete values of thermodynamical quantities \( M, T, S \) and \( \Phi \) for charged black holes with \( r_+ = 0.5 \) in the Group I (Left) and Group II (Right).

It is important to explore the free energy \( F = M - TS \) as a function of temperature. According to the left-hand side of joint point \( (T \approx 0.091) \) in Fig(1)(b), it can be seen that free energies of the charged black holes of Group II are always larger than that of non-Schwarzschild black hole, but smaller than that of Schwarzschild black hole at a given temperature \( T \). Nevertheless, the charged black holes located on the right-hand side of joint point \( (T \approx 0.091) \) always possess larger values for fixed \( T \). The charged black holes in Group I display more complicated properties in Fig(1)(a). The free energy of charged black hole for left-hand side of joint point \( (T \approx 0.091) \) is larger than that of Schwarzschild black hole within some region of \( T(> T_c) \), while becomes smaller when the temperature satisfies \( T > T_c \) \( [T_c \approx 0.023 \text{ for } r_0 = 1.3 \text{ and } T_c \approx 0.017 \text{ for } r_0 = 2] \).
IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we obtained numerically charged asymptotically flat black hole solutions in Einstein-Maxwell-Weyl gravity. These solutions were separated in two groups according to their seed solutions: Schwarzschild and non-Schwarzschild black hole solutions. Since the Schwarzschild and the non-Schwarzschild black holes ‘coalesce’ as $r_0 = r_0^{\text{min}} \approx 0.876$, we constructed the charged numerical black hole solutions from both sides of this joint point. For each value of $r_0$, there is a corresponding value $m_1 = m_1^*$ that yields the charged black hole solution as the increase of charge $Q$ without a singularity at spatial infinity. Later, the thermodynamic proprieties of these charged black holes were discussed in detail, and show that they obey the first law of thermodynamics.

Notice that the quasinormal modes of non-Schwarzschild solutions have been investigated in refs. [10, 11], which shows that the non-Schwarzschild black hole is stable. Therefore it is necessary to investigate the quasinormal modes and stability of the charged black hole solutions in the Einstein-Maxwell-Weyl theory. Another interesting possibility is (Anti-) de Sitter charged black hole solutions in Einstein-Maxwell-Weyl gravity. They have shown in the Einstein-Hilbert theory of gravity with additional quadratic curvature terms [17]. Beside the analytic Schwarzschild (Anti-) de Sitter solutions, two groups of non-Schwarzschild (Anti-) de Sitter solutions were also obtained numerically. Their thermodynamic properties of the two groups of numerical solutions deserve a new work in future.

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