Exponential decay of correlation for the Stochastic Process associated to the Entropy Penalized Method

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Abstract

In this paper we present an upper bound for the decay of correlation for the stationary stochastic process associated with the Entropy Penalized Method. Let $L(x,v) : T^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a $C^1$ Lagrangian of the form

$$L(x,v) = \frac{1}{2} |v|^2 - U(x) + \langle P, v \rangle.$$ 

We point out that we do not assume more differentiability of $L$ according the the dimension of the torus $T^n$.

For each value of $\epsilon$ and $h$, consider the operator

$$G[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^n} e^{-\frac{hL(x,v) + \phi(x+h,v)}{\epsilon h}} dv \right],$$

as well as the reversed operator

$$\bar{G}[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^n} e^{-\frac{hL(x+h,v) - \phi(x,v)}{\epsilon h}} dv \right],$$

both acting on continuous functions $\phi : T^n \rightarrow \mathbb{R}$. Denote by $\phi_{\epsilon,h}$ the solution of $G[\phi_{\epsilon,h}] = \phi_{\epsilon,h} + \lambda_{\epsilon,h}$, and by $\bar{\phi}_{\epsilon,h}$ the solution of $\bar{G}[\phi_{\epsilon,h}] = \bar{\phi}_{\epsilon,h} + \lambda_{\epsilon,h}$. Let $\theta_{\epsilon,h}(x) = e^{-\frac{L(x,v) + \phi_{\epsilon,h}(x)}{\epsilon h}}$. From [Gy], it is known that

$$\mu_{\epsilon,h}(x,v) = \theta_{\epsilon,h}(x) \gamma_{\epsilon,h}(x,v) = \theta_{\epsilon,h}(x) e^{-\frac{hL(x,v) + \phi_{\epsilon,h}(x+h,v) - \phi_{\epsilon,h}(x) - \lambda_{\epsilon,h}}{\epsilon h}},$$

is a solution to the entropy penalized problem: $\min \{ \int_{T^n \times \mathbb{R}^n} L(x,v) d\mu(x,v) + \epsilon S[\mu] \}$, where the entropy $S$ is given by

$$S[\mu] = \int_{T^n \times \mathbb{R}^n} \mu(x,v) \ln \frac{\mu(x,v)}{\int_{\mathbb{R}^n} \mu(x,w) dw} dxdv,$$

and the minimization is made over all holonomic probability densities on $T^n \times \mathbb{R}^n$, that is probabilities that satisfy $\int \varphi(x+v) - \varphi(x) \mu(x,v) dxdv = 0$, for all $\varphi \in C^1(T^n)$. The density $\gamma_{\epsilon,h}(x,v)$ defines a Markovian transition kernel on $(T^n)^N$. The invariant initial density in $T^n$ is $\theta_{\epsilon,h}(x)$. In order to analyze the decay of correlation for this process we show that the operator $L[\varphi](x) = \int e^{-\frac{hL(x,v)}{\epsilon h}} \varphi(x+h,v) dv$, has a maximal eigenvalue isolated from the rest of the spectrum.
1 Definitions and the set up of the problem

Let $T^n$ be the $n$-dimensional torus. In this paper we assume that the Lagrangian, $L(x, v): T^n \times \mathbb{R}^N \to \mathbb{R}$ has the form

$$L(x, v) = \frac{1}{2} |v|^2 - U(x) + \langle P, v \rangle,$$

where $U \in C^1(T^n)$, and $P \in \mathbb{R}^n$ is constant.

We consider here the discrete time Aubry-Mather problem [Gom] and the Entropy Penalized Mather method which provides a way to obtain approximations by continuous densities of the Aubry-Mather measure. We refer the reader to [Gom] and the last section of [GLM] for some of the main properties of Aubry-Mather measures, subactions, Peierl’s barrier, etc...

The Entropy Penalized Mather problem (see [GV] for general properties of this problem) can be used to approximate Mather measures [CI] by means of absolutely continuous densities $\mu_{\epsilon, h}(x, v)$, when $\epsilon, h \to 0$, both in the continuous case or in the discrete case. In [GLM] it is presented a Large Deviation principle associated to this procedure. We briefly mention some definitions and results.

Consider, for each value of $\epsilon$ and $h$, the operators acting on continuous functions $\phi$:

$$G[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^N} e^{-hL(x, v) + \phi(x + hv) + \phi(x)} \epsilon h \, dv \right],$$

and

$$\bar{G}[\phi](x) := -\epsilon h \ln \left[ \int_{\mathbb{R}^N} e^{-hL(x-hv, v) + \phi(x-hv) + \phi(x)} \epsilon h \, dv \right].$$

Denote by $\phi_{\epsilon, h}$ the solution of $G[\phi_{\epsilon, h}] = \phi_{\epsilon, h} + \lambda_{\epsilon, h}$, and by $\bar{\phi}_{\epsilon, h}$ the solution of $\bar{G}[\phi_{\epsilon, h}] = \bar{\phi}_{\epsilon, h} + \lambda_{\epsilon, h}$. Let

$$\theta_{\epsilon, h}(x) = e^{-\frac{\phi_{\epsilon, h}(x) + \phi_{\epsilon, h}(x)}{\epsilon h}}.$$

By adding a suitable constant to $\phi_{\epsilon, h}$ or $\bar{\phi}_{\epsilon, h}$, we can assume that $\theta_{\epsilon, h}(x)$ is a probability density on $T^N$. From D. Gomes and E. Valdinoci, it is known that the probability measure on $T^N \times \mathbb{R}^N$

$$\mu_{\epsilon, h}(x, v) = \theta_{\epsilon, h}(x) e^{-\frac{hL(x, v) + \phi_{\epsilon, h}(x + hv) - \phi_{\epsilon, h}(x) - \lambda_{\epsilon, h}}{\epsilon h}},$$

is a solution to the entropy penalized Mather problem:

$$\min_{M_h} \left\{ \int_{T^N \times \mathbb{R}^N} L(x, v) d\mu(x, v) + \epsilon S[\mu] \right\},$$

where the entropy $S$ is given by

$$S[\mu] = \int_{T^N \times \mathbb{R}^N} \mu(x, v) \ln \frac{\mu(x, v)}{\int_{\mathbb{R}^N} \mu(x, w) \, dw} \, dx \, dv,$$

and

$$M_h := \left\{ \mu \in \mathcal{M} : \int_{T^N \times \mathbb{R}^N} \varphi(x + hv) - \varphi(x) \, d\mu = 0, \forall \varphi \in C(T^N) \right\}. \quad (1)$$
Here $\mathcal{M}$ denotes the set of probability densities on $\mathbb{T}^N \times \mathbb{R}^N$ and we will call $\mu \in \mathcal{M}_h$ a holonomic probability measure.

We will be interested in measures that minimize the functional below (under the holonomic constrain)

$$
\int_{\mathbb{T}^N \times \mathbb{R}^N} L(x, v) d\mu(x, v) + \epsilon S[\mu].
$$

(2)

Note that, for a probability $\mu(x, v)$ the value

$$
-S[\mu] = - \int_{\mathbb{T}^N \times \mathbb{R}^N} \mu(x, v) \ln \frac{\mu(x, v)}{\int_{\mathbb{R}^N} \mu(x, w) \, dw} \, dx \, dv
$$

is not necessarily positive.

This is the entropy penalized version of the discrete time Aubry-Mather problem, see [Gom], where we look for probability measures $\mu \in \mathcal{M}_h$ that minimize the action

$$
\int_{\mathbb{T}^N \times \mathbb{R}^N} L(x, v) d\mu(x, v)
$$

(3)

**Definition 1:** The forward (non-normalized) Perron operator $L$ is defined

$$
x \to \varphi(x) \Rightarrow x \to L(\varphi)(x) = \int e^{-\frac{L(x,v)}{\epsilon}} \varphi(x + h v) \, dv,
$$

In [GV] it is shown that $L$ has a unique eigenfunction $e^{-\frac{\phi_{\epsilon,h}}{\epsilon \pi}}$ with eigenvalue $e^{-\frac{\lambda_{\epsilon,h}}{\epsilon \pi}}$.

**Definition 2:** The backward operator $N$ is given by

$$
x \to \varphi(x) \Rightarrow x \to N(\varphi)(x) = \int e^{-\frac{L(x-hv,v)}{\epsilon}} \varphi(x - h v) \, dv,
$$

In [GV] it is shown that $N$ has a unique eigenfunction $e^{-\frac{\bar{\phi}_{\epsilon,h}}{\epsilon \pi}}$ with eigenvalue $e^{-\frac{\lambda_{\epsilon,h}}{\epsilon \pi}}$.

**Definition 3:** The operator

$$
g(x) \to F(g)(x) = \int e^{\frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h}} g(x + hv) \, dv,
$$

is the normalized forward Perron operator.

From [GV] we have that given a continuous function $g : \mathbb{T}^n \to \mathbb{R}$, then $F^m(g)$ converges to the unique eigenfunction $k$ as $m \to \infty$. We show in this paper that for $\epsilon$ and $h$ fixed, the convergence is exponentially fast.

Our notation:

$$
\theta = \theta_{\epsilon,h}(x) = e^{\frac{\phi_{\epsilon,h}(x)+\phi_{\epsilon,h}(x)}{\epsilon h}},
$$

$$
\gamma(x, v) = \gamma_{\epsilon,h}(x, v) = e^{\frac{hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h}},
$$

in such way that $\mu_{\epsilon,h} = \theta_{\epsilon,h}(x) \gamma_{\epsilon,h}(x, v)$. 

3
2 Reversed Markov Process and Adjoint Operator

In this section we define the reversed Markov process and compute the adjoint of $F$ in $L^2(\theta)$. We assume $h = 1$ from now on.

We can consider the stationary forward Markovian process $X_n$ according to the initial probability $\theta(x)$ and transition $\gamma(x,v)$. For example

$$P(X_0 \in A_0) = \int_{x \in T^n \cap A_0} \theta(x) dx,$$

$$P(X_0 \in A_0, X_1 \in A_1) = \int_{x \in T^n \cap A_0, (x+v) \in A_1} \theta(x) \gamma(x,v) dx dv,$$

and so on. Define the backward transfer operator $F^*$ acting on continuous functions $f(x)$ by

$$F^*(f)(x) = \int \frac{\theta(x-v) \gamma(x-v,v)}{\theta(x)} f(x-v) dv.$$

The backward transition kernel is given by

$$Q(x,v) = \frac{\theta(x-v) \gamma(x-v,v)}{\theta(x)}.$$

The fact that for any $x$ we have $\int Q(x,v) dv = 1$ follows from Theorem 32 in [GV]. We will show in Corollary 1 that $\theta$ is an invariant measure for the process with transition kernel $Q$, more precisely, that

$$\int g d\theta = \int F^* (g) d\theta,$$

for any $g \in L^2(d\theta)$.

**Theorem 1.** $F^*$ is the adjoint of $F$ in $L^2(\theta)$, that is for all $f, g \in L^2(\theta)$ then

$$\int f(x) F g(x) \theta(x) dx = \int g(x) F^* f(x) \theta(x) dx.$$

**Proof.** Consider $f, g \in L^2(\theta)$, then

$$\int g(x) \left[ F^*(f)(x) \right] \theta(x) dx = \int g(x) \left[ \int \frac{\theta(x-v) \gamma(x-v,v)}{\theta(x)} f(x-v) dv \right] \theta(x) dx$$

$$= \int g(x) \left[ \int \theta(x-v) \gamma(x-v,v) f(x-v) dv \right] dx$$

$$= \int \left[ \int g(x) \theta(x-v) \gamma(x-v,v) f(x-v) dx \right] dv$$

$$= \int \left[ \int g(x+v) \theta(x) \gamma(x,v) f(x) dx \right] dv$$

$$= \int f(x) \left[ \int \gamma(x,v) g(x+v) dv \right] \theta(x) dx$$

$$= \int f(x) \left[ \int e^{-\frac{L(x,v)+\phi_{1}(x+v)-\phi_{1}(x)-\lambda_{1}}{\epsilon}} g(x+v) dv \right] \theta(x) dx$$

$$= \int f(x) \left[ F(g)(x) \right] \theta(x) dx.$$
where we use above the change of coordinates $x \rightarrow x - v$ and the fact that $\mu$ is holonomic.

**Corollary 1.** Consider the inner product $\langle \cdot, \cdot \rangle$ in $L^2(\theta)$. Then $\mathcal{F}$ leaves invariant the orthogonal space to the constant functions: $\{ g \mid \langle g, 1 \rangle = \int g \, d\theta = 0 \}$. Furthermore

$$\int g \, d\theta = \int \mathcal{F}^*(g) \, d\theta.$$ 

**Proof.** Note that $\mathcal{F}(1) = 1$, therefore

$$\int g \, d\theta = \int g \mathcal{F}(1) \, d\theta = \int \mathcal{F}^*(g) \, d\theta.$$ 

Thus if $\int g \, d\theta = 0$ it follows $\int \mathcal{F}^*(g) \, d\theta = 0$. 



### 3 Spectral gap, exponential convergence and decay of correlations

From [GV] it is known that $\mathcal{L}$ has a unique (normalized) eigenfunction $e^{-\frac{\phi}{h}}$ corresponding to the largest eigenvalue $e^{-\frac{\lambda}{h}}$, in the next theorem we prove the this eigenvalue is separated from the rest of the spectrum.

**Theorem 2.** The largest eigenvalue of $\mathcal{L}$ is at a positive distance from the rest of the spectrum.

**Proof.** We will prove the result for the normalized operator

$$g(x) \rightarrow \mathcal{F}(g)(x) = \int e^{\frac{-hL(x,v)+\phi_{\epsilon,h}(x+hv)-\phi_{\epsilon,h}(x)-\lambda_{\epsilon,h}}{\epsilon h}} g(x+hv) \, dv.$$ 

Recall from [GV] that the functions $\phi_{\epsilon,h}(x)$ and $\tilde{\phi}_{\epsilon,h}(x)$ are differentiable. In this way we consider a new Lagrangian (adding $\phi_{\epsilon,h}(x+hv) - \phi_{\epsilon,h}(x) - \lambda_{\epsilon,h}$) in such way $\mathcal{L} = \mathcal{F}$. We also assume $\epsilon = 1$ and $h = 1$ from now on.

Therefore,

$$g(x) \rightarrow \mathcal{F}(g)(x) = \int e^{-L(x,v)} g(x+v) \, dv,$$

the eigenvalue is 1, and, by the results in [GV], the corresponding eigenspace is one-dimensional and is generated by the constant functions.

Suppose there exist a sequence of $f_p \in L^2(\theta)$, $p \in \mathbb{N}$, such that

$$\mathcal{F}(f_p) = \lambda_p(f_p),$$

$$\langle f_p, 1 \rangle = 0, \lambda_p \rightarrow 1$$ and $||f_p|| = 1$. If the operator is compact, then the theorem follows from the classical argument: through a subsequence $f_p \rightarrow f$, and since $\lambda_p \rightarrow 1$ we have $\mathcal{F}(f) = f$. Furthermore, since $\langle f_p, 1 \rangle = 0$, it follows $\langle f, 1 \rangle = 0$, which is a contradiction. Therefore we proceed to establish the compactness of the operator $\mathcal{F}$.
To establish compactness, consider \( g \in L^2(\theta) \). We claim that \( f = F(g) \) is in the Sobolev space \( H^1 \) (see [E] for definition and properties). Indeed, for a fixed \( x \), we will compute the derivative of \( f \). Integrating by parts we have

\[
\frac{d}{dx} f(x) = \frac{d}{dx} (F(g)(x)) = \int \left( \frac{d}{dx} g(x + v) \right) e^{-L(x,v)} - L(x,v) \left( \frac{d}{dx} e^{-L(x,v)} \right) g(x + v) \, dv
\]

From the hypothesis about \( L \), if \( g \in L^2(\theta) \), then indeed \( \frac{d}{dx} f \) is also in \( L^2(\theta) \) (with the above derivative).

Note that, for \( v \) uniformly in a bounded set

\[
\left\| \frac{d}{dx} f \right\|_2 \leq \left\| \frac{d}{dx} f \right\|_\infty \leq \left\| \frac{d}{dv} e^{-L(x,v)} - L(x,v) \left( \frac{d}{dx} e^{-L(x,v)} \right) \right\|_2 \|g\|_2.
\]

Therefore, \( f \) is in the Sobolev space \( H^1 \).

By iterating the procedure described above, we have that \( g_j = F^j(g) \in H^1 \).

It is known that if \( j > \frac{n}{2} \), where \( n \) is the dimension of the torus \( \mathbb{T}^n \), then \( g_j \) is continuous Hölder continuous [E]. Thus the operator \( F \) is compact and \( g_j \) is differentiable for a much more larger \( j \). From the reasoning described before, \( f_p \to f \), and \( F(f) = f \), \( \langle f, 1 \rangle = 0 \) and \( f \) is differentiable. It is easy to see that the modulus of concavity of \( f \) is bounded (the iteration by \( F \) does not decrease it). We can add a constant to \( f \) and by linearity of \( F \) we also get a new fixed point for \( F \) (note that \( F(1) = 1 \)). Therefore, we can assume \( f = e^{-g} \) for some \( g \).

In this way, we obtain a contradiction with the uniqueness in Theorem 26 in [GV]. \( \square \)

Suppose \( \int g(x) \theta(x) dx = 0 \). For \( \epsilon, h \) fixed, then it follows from above that \( F^\mu(g) \to 0 \) with exponential velocity (according to the spectral gap).

Consider the backward stationary Markov process \( Y_n \) according to the transition \( Q(x,v) \) and initial probability \( \theta \) as above.

**Theorem 3.** Given \( f(x), g(x) \) with \( \int f(x) \theta(x) dx = \int g(x) \theta(x) dx = 0 \), it follows

\[
\int g(Y_0) f(Y_n) \, dP \to 0,
\]

with exponential velocity.
Proof. Note that
\[\int g(Y_0) f(Y_1) dP = \int g(x) \left( \int Q(x, v) f(x - v) dv \right) \theta(x) dx = \int g(x) \left[ \mathcal{F}^n(f)(x) \right] \theta(x) dx.\]

In the same way, for any \(n\)
\[\int g(Y_0) f(Y_n) dP = \int f(x) \left[ \mathcal{F}^n(g)(x) \right] \theta(x) dx.\]

The exponential decay of correlation follows from this. \(\square\)

**Theorem 4.** Let \(f, g \in L^2(\theta)\) be such that \(\int f(x) \theta(x) dx = \int g(x) \theta(x) dx = 0\). Then
\[\int g(X_0) f(X_n) dP \to 0,\]
with exponential velocity.

Proof. Now, for analyzing the decay of the forward system, \(X_n\), with transition \(\gamma(x, v)\), we have to consider the backwark operator \(\mathcal{F}^*\), use the fact that its exponential convergent, that is \((\mathcal{F}^*)^n(g) \to 0,\) if \(\int g(x) \theta(x) dx = 0\), and the result follows in the same way. \(\square\)

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