Zero point energy on extra dimensions: Noncommutative Torus

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In this paper we calculate the zero point energy density experienced by observers on $\mathcal{M}^4$ due to a massless scalar field defined throughout $\mathcal{M}^4 \times T_2^F$, where $T_2^F$ are fuzzy extra dimensions. Using the Green’s function approach we calculate the energy density for the commutative torus and the fuzzy torus. We calculate then the energy density for the fuzzy torus using the Hamiltonian approach. Agreement is shown between Green’s function and Hamiltonian approaches.

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I. INTRODUCTION

The zero-point energy of the Universe remains one of the fundamental mysteries of physics. The Universe is known to be accelerating, and so far the cause of this phenomenon (called dark energy) has defied explanation in terms of conventional classical or quantum physics. Since the Universe’s zero-point energy is one the characteristics determining its fate, it is important to determine the factors which may contribute to the value of the zero-point energy. A possible source of vacuum energy is the Casimir effect. The contribution of this effect to the vacuum energy depends upon the topology of spacetime, the number of spacetime dimensions, the type of field existing in the spacetime and possibly other phenomena, such as the commutativity or noncommutativity of spacetime. The present paper is part of a series of attempts to determine under which conditions the effects described above can contribute to the vacuum energy density of the Universe.

In a previous paper [1] we explored the possibility that dark energy is a manifestation of the Casimir energy from extra dimensions with the topology of a noncommutative $S^2$. We found that the value of the energy density present on $\mathcal{M}^4 \times S_2^F$ is positive, i.e. it provides dark energy, and we calculated the radius of $S_2^F$ for a chosen value of the size of the representation of the noncommutative algebra. An exciting approach to noncommutative extra dimensions, in which the said dimensions are even dynamically generated (and are spheres) is described in [2].

The purpose of this paper is to repeat the calculation with the extra dimensions being the fuzzy torus $T_2^F$ instead of $S_2^F$. In [3],[4] the study of a self-coupled massive scalar field on $\mathcal{M}^{1,D} \times T_2^F$ has been considered. We consider a massless scalar field. We take two different approaches to the problem: the first is based on the Green’s function method used by R. Kantowski and K. Milton in [5], and the second is based on the Hamiltonian approach, as done in [6]. With the use of the Green’s function we perform the calculations for both the commutative and fuzzy cases: $T_2^F$ which has finite dimensional representations. In [6], using the Hamiltonian approach, the energy density was calculated for $\mathbb{R} \times T_2^F$; we generalize the manifold to the $\mathcal{M}^4 \times T_2^F$ case using the same approach. In the case of the fuzzy torus, the energy density calculated using the Green’s function agrees with that calculated using the Hamiltonian formalism.

Regardless of the method, the energy density present on $\mathcal{M}^4$ due to the extra dimensions with topology of $T_2^F$ turns out to be negative for each possible value of the size of the representation $N$ of the algebra for the torus and therefore it cannot be considered as a source for dark energy. The

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energy density we obtain with the two different approaches has the same form as obtained in the case $\mathcal{M}^4 \times S^2_F$ \[1\]

$$u_\rho = \frac{\alpha}{\rho^4} \ln \frac{\rho}{b}$$  \hspace{1cm} (1)

where $\rho$ is the radius of the torus (the two radii are chosen to be the same), $b$ a momentum cutoff and $\alpha$ is a negative constant. In the fuzzy case the value of $\alpha$ turns out to be the same using either the Green’s function or the Hamiltonian approach. We also calculate the numerical value of the energy density in the case of maximum noncommutativity $N = 2$.

II. THE NONCOMMUTATIVE TORUS

The noncommutative 2-torus is defined in terms of operators $\hat{U}_i$ $(i = 1, 2)$ satisfying

$$\hat{U}_i \hat{X}_j \hat{U}_i^{-1} = X_j + \delta_{ij} 2\pi R_j 1$$  \hspace{1cm} (2)

$$U_i U_j = e^{2\pi \theta \epsilon_{ij}} U_j U_i$$  \hspace{1cm} (3)

where $X_i$ are the coordinate operators, $R_i$ the compactification radii (set both equal to $\rho$) and the dimensionless noncommutativity parameter $\theta_{ij} = \theta \epsilon_{ij}$. From the definition above it follows that

$$\hat{U}_i = e^{i \hat{\sigma}_i}, \quad \hat{\sigma}_i = \frac{\hat{X}_i}{\rho}$$  \hspace{1cm} (4)

where the $\hat{\sigma}_i$ are dimensionless and satisfy

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2\pi \theta \epsilon_{ij}.$$  \hspace{1cm} (5)

The $\lim \theta \to 0$ (the commutative limit) gives $[\hat{X}_i, \hat{X}_j] = 0$. The definition of $\hat{U}_i$ allows a function defined on the $T^2_\theta$ to be expanded in an operator Fourier series \[6\] \[7\] \[8\]

$$\phi(\hat{\sigma}_i, \hat{\sigma}_j) = \sum_{k,p=0}^{\infty} c_{kp} \hat{U}_k \hat{U}_p$$  \hspace{1cm} (6)

For the commutative case, using the symbols corresponding to the $\hat{U}_i$ operators, we can write an equivalent expression to the above

$$\phi(\sigma_i, \sigma_j) = \sum_{k,p=-\infty}^{\infty} c_{kp} e^{i(\frac{k}{p}X_1 + \frac{p}{k}X_2)} \quad X_i \in \mathbb{R}$$  \hspace{1cm} (7)

In the rational (fuzzy) case, we can take $\theta = \frac{1}{N}$ without losing generality, where $N$ is the (finite) size of the representation and the theory on the fuzzy torus is equivalent to a lattice theory with $X \to X_n = \frac{2\pi}{N} n$. Eq. (7) becomes \[9\]

$$\phi(n, m) = \sum_{k,p=0}^{N-1} d_{kp} e^{i\frac{2\pi}{N}(kn + pm)}.$$  \hspace{1cm} (8)

We will use the equation above to expand the reduced Green’s function (see below) $g(y, y', k^\lambda k_\lambda)$ on the fuzzy torus.
III. GREEN’S FUNCTION TECHNIQUE

To obtain the energy density on $M^4 \times T^2$, we first calculate the energy-momentum tensor of a massless scalar field defined by

$$t_{AB} = \partial_A \varphi \partial_B \varphi - \frac{1}{2} g_{AB} \partial_C \varphi \partial^C \varphi \quad (A, B = 0 \ldots 5).$$  \hfill (9)

In terms of the Green’s function the energy density on $M^4 \times T^2$ can be written [5] as

$$u(\rho) = -\frac{i V_{T^2}}{2(2\pi)^4} \int \frac{d\vec{k}}{c} \int dw \, w^2 g(y, y, k^\lambda k_\lambda)$$  \hfill (10)

where $g(y, y, k^\lambda k_\lambda)$ is the reduced Green’s function defined on the extra dimensions ($x_\mu \in M^4$ and $y = y_1, y_2$ are the two coordinates for the torus). To find the expression for $g(y, y, k^\lambda k_\lambda)$ we solve the equation of motion satisfied by $g(y, y', k^\lambda k_\lambda)$. The equations of motion for a commutative torus differ from that of $T^2_f$. We first perform the calculations for the commutative case.

A. Green’s function for the commutative torus

Our goal is to calculate the contribution to the vacuum energy density from the part of the manifold which is a noncommutative strip with opposite sides identified, which is topologically a torus. To do this we first calculate the periodic Green’s function for the commutative torus. We were unable to find the explicit form for such a Green’s function in the literature, and so we include this calculation for completeness. We will then adapt this calculation to the fuzzy torus in the next section.

When $y_1$ and $y_2$ are continuous variables, the equation of motion is the usual Klein-Gordon equation with $k^\lambda k_\lambda$ representing the Kaluza-Klein mass term and $\nabla^2_{T^2}$ the Laplacian operator defined on a torus, i.e the ordinary Laplacian defined on $\mathbb{R}^2$ but acting on a periodic function

$$(\nabla^2_{T^2} + k^\mu k_\mu) g(y, y', k^\lambda k_\lambda) = -\delta_P^2 (y - y'),$$  \hfill (11)

with $\delta_P$ being a “periodic” delta function (see Appendix). To find $g(y, y', k^\lambda k_\lambda)$ we expand it on the basis $U_i$ and specify $y \rightarrow (y_1, y_2) \in T^2$:

$$g(y, y', k^\lambda k_\lambda) = \sum_{k, p = -\infty}^{\infty} c_{kp} e^{i \left(\frac{k}{\rho} y_1 + \frac{p}{\rho} y_2\right)} e^{-i \left(\frac{k}{\rho} y'_1 + \frac{p}{\rho} y'_2\right)}.$$  \hfill (12)

In order to find $c_{kp}$, we substitute Eq. (12) into Eq. (11) and with $\nabla^2_{T^2} U_i = -\frac{k^2 + p^2}{\rho^2} U_i$ we obtain

$$\sum_{k, p = -\infty}^{\infty} c_{kp} \left( -\frac{k^2 + p^2}{\rho^2} + k^\lambda k_\lambda \right) e^{i \left(\frac{k}{\rho} y_1 + \frac{p}{\rho} y_2\right)} e^{-i \left(\frac{k}{\rho} y'_1 + \frac{p}{\rho} y'_2\right)} = -\delta_P (y_1 - y'_1) \delta_P (y_2 - y'_2)$$  \hfill (13)

A property of $\delta_P (y - y')$ is

$$\frac{1}{(2\pi \rho)^2} \sum_{k, p = -\infty}^{\infty} e^{i \left(\frac{k}{\rho} y_1 + \frac{p}{\rho} y_2\right)} e^{-i \left(\frac{k}{\rho} y'_1 + \frac{p}{\rho} y'_2\right)} = \delta_P (y_1 - y'_1) \delta_P (y_2 - y'_2).$$  \hfill (14)

The expression for $c_{kp}$ is therefore

$$c_{kp} = \frac{1}{(2\pi \rho)^2 \left(\frac{k^2 + p^2}{\rho^2} - k^\lambda k_\lambda\right)}$$  \hfill (15)
We finally obtain for the energy density (GC: Green’s function approach, Commutative case)

\[
    u_{GC}(\rho) = -\frac{i}{(2\pi)^4} \int d\vec{k} \int dw \, w^2 \sum_{k,p=-\infty}^{\infty} \frac{1}{\frac{k^2+p^2}{\rho^2} + \vec{k}^2 - w^2}.
\]  

(16)

It should be obvious from the context that we integrate only over the wave vector $\vec{k}$ while summing over the integer $k$. To calculate the expression above, we first perform the integration using the contour argument in [5]. The $d\vec{k}$ term gives $4\pi k^2 dk$, then we perform the change of coordinates $k = R \cos(\theta)$ and $\omega = i R \sin(\theta)$, and the integral becomes

\[
    u_{GC}(\rho) = -\frac{i}{(2\pi)^4} \left( \frac{4\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \right) \frac{R^2 \cos^2(\theta) \sin^2(\theta)}{\frac{k^2+p^2}{\rho^2} + R^2}.
\]  

(17)

In order to obtain a finite result we impose a cutoff on the variable $R$: $R_{\text{max}} = \frac{1}{b}$ with $b \simeq L\text{Planck}$. We also discard the two infinite terms, which are not proportional to $\ln(\frac{\rho}{b})$ as done previously in [1], and we obtain

\[
    u_{GC}(\rho) = -\frac{1}{32\pi^2 \rho^4} \sum_{k,p \neq 0,0} (k^2 + p^2)^2 \ln \left[ 1 + \frac{\rho^2}{b^2 (k^2 + p^2)} \right].
\]  

(18)

We first notice that this quantity is always negative and therefore considering the case of the extra dimensions to be a two dimensional commutative torus does not provide a valid model for dark energy. The summation above does not contain the $k = p = 0$ term because we have previously discarded it; it corresponded to the $b^{-4}$ divergence contained in Eq (17). Also note that the sum is even in $k$ and $p$. In order to evaluate the sum present in Eq. (18) we introduce a cutoff on the number of modes allowed. We call this cut off $N$ and introduce the new variables $x_k = \frac{2\pi \rho}{N} k$ and $y_p = \frac{2\pi \rho}{N} p$. In the large $N$ limit these variables become continuous, and the sum can be approximated by an integral \[ \sum_{k,p} \rightarrow \frac{N^2}{(2\pi \rho)^4} \int \]. We obtain

\[
    u_{GC}(\rho) = -\frac{1}{32\pi^2 \rho^4} \left( \frac{2\pi \rho}{N b} \right)^4 \int_{0}^{2\pi \rho} dx \, dy \, \left( x^2 + y^2 \right)^2 \ln \left[ 1 + \left( \frac{2\pi \rho}{N b} \right)^2 \frac{\rho^2}{x^2 + y^2} \right].
\]  

(19)

The integral converges due to the ultraviolet behavior of the torus factor dominating that of the flat $M^4$. The regime in which commutative effective field theory is valid is $x > b$ thus, respecting this condition for the toroidal theory amounts to satisfying $\frac{\rho}{b} \sim N$. The further change of variable $x' = \frac{x}{2\rho}$ allows the integral to be evaluated numerically with the result

\[
    u_{GC}(\rho) = -\alpha_P \frac{N^6}{32\pi^2 \rho^4},
\]  

(20)

where $\alpha_P \simeq 418$ [9].

B. Green’s function for the fuzzy torus

The assumption that $\theta$ is rational (equal to $\frac{1}{N}$ for simplicity), implies that the theory becomes equivalent to a lattice theory with $y \rightarrow y_n = \frac{2\pi \rho}{N} n$. The discrete version of the Laplacian operator used in Eq. (11) can be evaluated by performing the variation of the action below (similar to Eq. [5])
(4.7) in \([6]\) and considering only the two terms relevant to the torus.\[ S[\Phi] = \frac{V T^2}{2} \sum_{n,m=0}^{N-1} \int_{M^4} d^4z \left\{ (\partial_\mu \Phi)^2 - \frac{1}{(2\pi \rho)^2} (\delta_n \Phi)^2 - \frac{1}{(2\pi \rho)^2} (\delta_m \Phi)^2 \right\} \tag{21} \]

where \(\delta_n \Phi(z, n, m) \equiv \Phi(z, n + 1, m) - \Phi(z, n, m)\) and similarly for \(\delta_m\). With the notation: \(y_1 \to n, y_2 \to m\), the equation of motion satisfied by the reduced Green’s function is\[ \left( - \frac{N^2}{(2\pi \rho)^2} [(\delta_{n-1} - \delta_n) + (\delta_{m-1} - \delta_m)] + k^2 k_\lambda \right) g(n, n', m, m') = - \frac{N^2}{(2\pi \rho)^2} \delta_{nm} \delta_{mn'} \tag{22} \]

and the expansion of the reduced Green’s function is\[ g(n, n', m, m') = \sum_{k,p=0}^{N-1} d_{kp} e^{i \frac{2\pi}{N} (kn + pm)} e^{-i \frac{2\pi}{N} (kn' + pm')} \tag{23} \]

The action of the discrete Laplacian on the basis is \(((\delta_{n-1} - \delta_n)U^n = (2 - 2 \cos(\frac{2\pi k}{N}))U^n\) and \(d_{kp}\) can be found from the relation\[ \sum_{k,p=0}^{N-1} d_{kp} \left( - \frac{4N^2}{(2\pi \rho)^2} \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right] + k^2 k_\lambda \right) e^{i \frac{2\pi}{N} (kn + pm)} e^{-i \frac{2\pi}{N} (kn' + pm')} = - \frac{N^2}{(2\pi \rho)^2} \delta_{mm'} \delta_{nn'} \tag{24} \]

Using\[ \sum_{k} e^{i \frac{2\pi}{N} (kn - n')} = N \delta_{nn'} \equiv \delta_{Fmn'}, \tag{25} \]

we find the expression for the coefficient to be\[ d_{kp} = \frac{1}{(2\pi \rho)^2 \left( \frac{4N^2}{(2\pi \rho)^2} \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right] - k^2 k_\lambda \right)}, \tag{26} \]

and we obtain the energy density due to the fuzzy torus to be (GF: Green’s function approach, Fuzzy torus)\[ u_{GF}(\rho) = - \frac{i}{(2\pi)^4} \int d\vec{k} \int dw w^2 \sum_{k,p=0}^{N-1} \frac{1}{4N^2 \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right] + \vec{k}^2 - w^2}. \tag{27} \]

We calculate the integral in the same way as in the commutative case. From Eq. (10) \(u_{GF}(\rho)\) is obtained\[ u_{GF}(\rho) = - \frac{N^4}{64\pi^6 \rho^4} \sum_{k,p=0}^{N-1} \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right]^2 \ln \left( 1 + \frac{4\pi^2 \rho^2}{b^2N^2[\sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N}]^2} \right). \tag{28} \]

We now simplify the expression above considering the approximation \(1 < N \ll \frac{\rho}{b}\) and obtain\[ u_{GF}(\rho) \approx - \frac{N^4}{64\pi^6 \rho^4} \sum_{k,p=0}^{N-1} \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right]^2 \ln \left( \frac{\rho^2}{b^2N^2[\sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N}]^2} \right). \tag{29} \]

Eq. (28) in the case of maximum noncommutativity \((N = 2)\) can be expressed as\[ u_{GF}(\rho) = - \frac{\alpha_{GF}}{\rho^4} \ln \left( \frac{\rho}{b} \right) = - \frac{3}{\pi^6} \ln \left( \frac{\rho}{b} \right) \approx - \frac{0.00312}{\rho^4} \ln \left( \frac{\rho}{b} \right). \tag{30} \]
IV. HAMILTONIAN TECHNIQUE

In [6] the Casimir energy on $\mathbb{R} \times T^2_F$ has been evaluated by expanding the scalar field $\phi$ into the creation and annihilation operators with the following result

$$u(\rho) = \frac{N}{16\pi^3 \rho^3} \sum_{k,p=0}^{N-1} \sqrt{\frac{\sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N}}{N}}.$$ (31)

In order to compare with the result given by Eq.(29) we generalize Eq.(31) to the $\mathcal{M}^4 \times T^2_F$ case. We notice first that the definition of $\omega$, Eq.(4.8) in [6], needs to be modified in order to include the dependence on the wave vector $\vec{k}$

$$\omega_{k,p}^2 = \frac{N^2}{\pi^2 \rho^2} \left( \frac{\sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N}}{N} \right) + \vec{k}^2.$$ (32)

The vacuum expectation value multiplied by the volume of the two-torus $V_{T^2} = (2\pi \rho)^2$ only, gives the energy density on $\mathcal{M}^4$ (HF: Hamiltonian approach, Fuzzy torus)

$$u_{HF}(\rho) = \frac{N \cdot V_{T^2}}{16\pi^3 \rho^3} \sum_{k,p=0}^{N-1} \int \frac{2d\vec{k}}{(2\pi)^2} \sqrt{\left( \frac{\sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N}}{N} \right) + \frac{\vec{k}^2 \pi^2 \rho^2}{N^2}}.$$ (33)

To evaluate (33) we first perform the integration over $d\vec{k} = 4\pi k^2 dk$. The integral diverges in this case also and needs to be regularized by 1) introducing a cutoff on $k$ ($k_{max} = \frac{\rho}{b}$ with $b \sim L_{Planck}$),

2) discarding again two infinite terms which are not proportional to $\ln(\frac{\rho}{b})$. The energy density given by the expansion of the field in terms of creation and annihilation operators is

$$u_{HF}(\rho) = \frac{-N^4}{64\pi^6 \rho^4} \sum_{k,p=0}^{N-1} \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right]^2 \ln \left( \frac{\rho^2}{b^2 N^2 \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right]} \right).$$ (34)

Again considering $1 < N \ll \frac{\rho}{b}$ we obtain:

$$u_{HF}(\rho) \simeq \frac{-N^4}{64\pi^6 \rho^4} \sum_{k,p=0}^{N-1} \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right]^2 \ln \left( \frac{\rho^2}{b^2 N^2 \left[ \sin^2 \frac{\pi k}{N} + \sin^2 \frac{\pi p}{N} \right]} \right),$$ (35)

which agrees with Eq.(29) obtained using the Green’s function approach.

In the particular case of $N = 2$, the energy density given by Eq.(34) is

$$u_{HF}(\rho) = -\frac{\alpha_{HF}}{\rho^4} \ln \left( \frac{\rho}{b} \right) = -\frac{3}{\pi^6} \ln \left( \frac{\rho}{b} \right) \simeq -0.00312 \frac{\rho^4}{\rho^4} \ln \left( \frac{\rho}{b} \right).$$ (36)

Which agrees with the result found in Eq.(30): $\alpha_{GF} = \alpha_{HF}$.

V. CONCLUSION

We have explored the Casimir energy density experienced by observers on $\mathcal{M}^4$ due to extra fuzzy dimensions, $T^2_F$. Different approaches yield consistent results where they are both valid. We have adhered to general field theory lore concerning the approach to regularization of the Casimir energy,
extracting the part which depends only logarithmically on the Planck scale. However, there are still some interesting subtleties which we will pursue in a forthcoming paper. Chief among these is the proof that \( u(\theta) \) is well behaved as a function of \( \theta \). Because of the simplifying assumption that rational \( \theta \) is in the form \( \frac{1}{N} \), the large degree of freedom limit coincides with the commutative one. More generally \( \theta = M/N \). So we can have a large number of degrees of freedom in the noncommutative theory without going directly to the commutative limit. This is important since it would be unpleasant to have the Casimir energy jump by a finite amount whenever \( \theta \) passes from a rational to an irrational value. In a theory in which \( \theta \) is dynamical, this defect would zero-out the matrix model contribution (it would be measure zero). Thus it remains to show that a sequence of matrix model energy densities \( u(\theta = M/N) \) converge to the \( \theta \) irrational result, which is in turn Morita equivalent \(^8\) to the commutative torus (which was analyzed here).

VI. APPENDIX

The delta function on the torus must respect periodicity. Our candidate for \( \delta_P(t) \) is given by the limit as \( N \to \infty \) of \( F_N(t) \), where \( F_N(t) \) is a periodic function with period \( 2\pi \) in the \( t = \frac{\theta}{\rho} \) variable

\[
F_N(t) = \sum_{k=-N}^{N} e^{ikt} \frac{\sin((2N + 1)\frac{t}{2})}{\sin(\frac{t}{2})}.
\]

We prove now the defining properties of a delta function for \( t \in [0, 2\pi] \). Considering the following integral of \( F_N(t) \)

\[
\int_{0}^{2\pi} F_N(t) dt = \int_{0}^{2\pi} \frac{\sin((N + \frac{1}{2})t)}{\sin(\frac{t}{2})} dt = \int_{0}^{2\pi(N + \frac{1}{2})} \frac{\sin(u)}{(N + \frac{1}{2}) \sin\left(\frac{u}{2N+1}\right)} du
\]

with \( u = (N + \frac{1}{2})t \). In the \( \lim_{N \to \infty} \) the integral above of \( F_N(t) \) becomes

\[
\lim_{N \to \infty} \int_{0}^{2\pi} F_N(t) dt = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin(u)}{u} du = 1. \quad (39)
\]

Now we act on a periodic test function \( f(t) \)

\[
\lim_{N \to \infty} \int_{0}^{2\pi} F_N(t) f(t) dt = \lim_{N \to \infty} \int_{0}^{2\pi(N + \frac{1}{2})} \frac{\sin(u)}{(N + \frac{1}{2}) \sin\left(\frac{u}{2N+1}\right)} f\left(\frac{u}{N + \frac{1}{2}}\right) du
\]

\[
= \frac{2}{\pi} \int_{0}^{\infty} f(0) \frac{\sin(u)}{u} du = f(0) \quad (40)
\]

where \( u \) was defined earlier.

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