Automorphism groups and derivation algebras of finitely generated vertex operator algebras

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Abstract

We investigate the general structure of the automorphism group and the Lie algebra of derivations of a finitely generated vertex operator algebra. The automorphism group is isomorphic to an algebraic group. Under natural assumptions, the derivation algebra has an invariant bilinear form and the ideal of inner derivations is nonsingular.

1 Introduction

A study of the automorphism group of a vertex operator algebra (VOA) is a natural part of the study of vertex operator algebras.

Definition 1.1. Let $V$ be a vertex operator algebra. We say that $a \in GL(V)$ is an automorphism of $V$ if and only if it leaves the vacuum element and the principal Virasoro element fixed ($a1 = 1$ and $a\omega = \omega$) and preserves all $V$-compositions, i.e., for all $m \in \mathbb{Z}$ and $u, v \in V$, we have $a(u_m v) = a(u)_m a(v)$. It follows that an automorphism fixes all the $V_i$ since they are eigenspaces for an operator in the series for the principal Virasoro element.

The set of all automorphisms is a group, denoted $Aut(V)$.

In the definition, it suffices to restrict $u$ and $v$ to homogeneous elements.

In some definitions of VOA automorphism, there is no requirement that the principal Virasoro element be fixed.

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So far, we know the automorphism groups explicitly for relatively few vertex operator algebras, such as \( V^\# \) [FLM], vertex operator algebra \( V_L \) for a positive definite even lattice \( L \) [DN], certain vertex operator algebras with central charge 1 ([DG], [DGR]), vertex operator algebras associated to highest weight representations for affine algebras (cf. [DLY]), vertex operator algebras associated to codes [M] and a few special cases, e.g., [G].

The determination of each of these automorphism groups has its own story and depends heavily on the specifics of the auxiliary object used to construct the VOA, such as a lattice, Lie algebra, or code. Nevertheless, one can observe that all these automorphism groups have similarities.

We denote by \((V, k^{th})\) the algebra with underlying vector space \(V\) and product \(a_k b\) for \(a, b \in V\), where \(a_k\) is the coefficient at \(z^{-k-1}\) in the vertex operator for \(a\). The linear subspace \(V_m\) is closed under this product if \(m - k = 1\). This algebra is denoted \((V_m, (m-1)^{th})\). The cases \(m = 1\) and 2 are especially interesting.

For simplicity of exposition, let us assume that \(V\) is a simple vertex operator algebra of CFT type (see [DLMM], or definition 2.8 of this article). Then \((V_1, 0^{th})\) is a Lie algebra with bracket \([u, v] = u_0 v\) for \(u, v \in V_1\) where \(Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}\). The endomorphism \(u_0\) is a derivation of \(V\) in the sense that \(u_0 1 = 0\), \(u_0 \omega = 0\) and

\[ u_0(Y(v, z)w) = Y(u_0v, z)w + Y(u, z)u_0w \]

for any \(v, w \in V\) (cf. [DN]). Moreover, the exponential \(e^{u_0}\) is an automorphism of \(V\) (\(e^{u_0}\) is a well defined operator on \(V\) as each \(V_n\) is finite dimensional). Denote by \(Aut_1(V)\) the subgroup of the automorphism group \(Aut(V)\) of \(V\) generated by \(e^{u_0}\) for \(u \in V_1\). Then \(Aut_1(V)\) is a finite dimensional connected algebraic normal subgroup of \(Aut(V)\). In all the examples discussed above, \(Aut(V)/Aut_1(V)\) is a finite group. We think that this is probably a general phenomenon for rational vertex operator algebras. There are counterexamples when \(V\) is not rational (cf. [DM1] and Examples 2.6 and 4.1 in this article).

It has been a feeling for a while that any rational vertex operator algebra of CFT type is finitely generated. If \(V\) is regular in the sense that any weak module is a direct sum of ordinary modules (see [DLM2]), it is proved in [KL] and [L3] (also see [GN]) that \(V\) is finitely generated. It is also felt that rational vertex operator algebras must be regular. Interest in the category of modules is motivation to study automorphism groups of finitely generated vertex operator algebras.
The main result of this paper is that the automorphism group $\text{Aut}(V)$ of a finitely generated vertex operator algebra is isomorphic to a finite dimensional algebraic group. It is well known that a finite dimensional algebraic group $G$ has only finitely many connected components and so $G/G^0$ is a finite group, where $G^0$ is the connected component of $G$ containing the identity. We expect that the normal subgroup $\text{Aut}_1(V)$ of $\text{Aut}(V)$ is exactly $\text{Aut}(V)^0$ for all rational vertex operator algebras $V$ of CFT type. This property holds for all examples discussed before.

There is a close relation between the automorphism group and the Lie algebra of derivations of a vertex operator algebra. If $d$ is a derivation of a vertex operator algebra $V$ then $e^d$ is an automorphism of $V$ (see Section 3). If $V$ is a finitely generated rational vertex operator algebra of CFT type, equality of $\text{Aut}_1(V)$ and $\text{Aut}(V)^0$ is equivalent to all the derivations of $V$ being given by $u_0$ for $u \in V_1$.

The paper is organized as follows. In Section 2 we prove that the automorphism group of a finitely generated vertex operator algebra is a finite dimensional algebraic group. We also give an example of a non finitely generated vertex operator algebra whose automorphism group is not isomorphic to an algebraic group. In Section 3 we study derivations of vertex operator algebras. For $v \in V$ we define a linear operator $o(v)$ by the conditions that $o(v) = v_{wv^{-1}}$ if $v$ is homogeneous. We show in Section 3 that $o(v)$ is a derivation of $V$ if and only if $v \in V_1$. We also show that the Lie algebra $V_1$ is an ideal of the Lie algebra of the derivations and has an orthogonal complement with respect to a suitable invariant symmetric bilinear form. In Section 4 we discuss an example of a nonsimple finitely generated vertex operator algebra and its automorphism group.

2 Automorphism groups

We suppose that the VOA $V$ is finitely generated (cf. [FHL]). This is equivalent to existence of $n \in \mathbb{Z}$ so that $U = \oplus_{m \leq n}V_m$ generates $V$ in the sense that

$$V = \text{span}\{u_{i_1}^1 \cdots u_{i_s}^s u \mid w^j, u \in U, s \in \{0, 1, 2, \ldots\}, i_j \in \mathbb{Z}\}.$$ 

For a subset $A$ of $V$, set $A^{r+1} := A \times \cdots \times A$ ($r+1$ times) and $A^\infty := \bigcup_{r \geq 0}A^{r+1}$. An element of $A^\infty$ is a finite length vector $\vec{x} = (x_0, \ldots, x_r)$ and we call $r+1$ the length of $\vec{x}$. For every nonempty finite sequence $\vec{m} := (m_1, \ldots, m_r)$ of integers, we define the function, called $\vec{m}$-composition $\mu :=$
\( \mu \vec{m} : V^{r+1} \to V \) by \( \mu(x_0, \ldots, x_r) := (x_0)_{m_1}(x_2)_{m_2} \cdots (x_{r-1})_{m_r}x_r \). Call \( r + 1 \) the length of \( \mu \). Thus, \( \mu(\vec{x}) \) is defined if and only if \( \mu \) and \( \vec{x} \) have the same lengths, in which case we say that \( (\mu, \vec{x}) \) is an admissible pair. In case the entries of such \( \vec{x} \) are restricted to a subset \( A \) of \( V \), we call the pair an \( A \)-admissible pair. If the coordinates of \( \vec{x} = (x_0, \ldots, x_r) \) are homogeneous, then \( \mu(\vec{x}) \) is homogeneous and we define the weight of \( (\mu, \vec{x}) \) to be \( \sum_{i=0}^{r} wt(x_i) - \sum_{i=1}^{r-1} (m_i + 1) \). This is just the weight of \( \mu(\vec{x}) \) if \( \mu(\vec{x}) \) is nonzero.

Such a function, for some \( \vec{m} \), is called a \( V \)-composition. For a subset \( A \) of \( V \), the restriction of \( \mu \) to tuples of elements in \( A \) is denoted \( \mu_A \).

**Remark 2.1.** The property that \( U \) generates \( V \) means that for each integer \( m \) there exists a finite set \( S \) of \( V \)-compositions so that \( V_m = \sum_{\mu \in S} Im(\mu_U) \cap V_m \).

We choose a basis \( B \) of \( U \) consisting of homogeneous elements, including \( 1 \). Let \( Q \) be the set of \( B \)-admissible pairs. Define \( Q_m \) to be the set of pairs in \( Q \) of weight \( m \). Then \( V_m \) is spanned by a finite set of “monomials” in \( B \) of weight \( m \), i.e., elements of certain \( Im(\mu_B) \).

There is a finite set \( R_m \) of pairs \((\mu, \vec{x})\) so that the set \( \mathcal{B}_m := \{\mu(\vec{x}) \mid (\mu, \vec{x}) \in R_m \} \) forms a basis for \( V_m \). Choose \( R_0 = \{(\mu_0, 1)\} \) (\( \mu_0 \) is the trivial length \( 1 \) composition) and set \( R := \bigcup_{m \in \mathbb{Z}} R_m \).

We write \( \text{res} \) for the restriction homomorphism \( \text{Aut}(V) \to GL(U) \). Since \( U \) generates \( V \), \( \text{res} \) is injective. We consider the question of when \( g \in GL(U) \) is in the image of \( \text{res} \).

We shall define a set function \( e : GL(U) \to End(V) \) as follows. For \( g \in GL(U) \), define \( e(g) \in End(V) \) by its action on the basis elements \( \mu(\vec{x}) \), \((\mu, \vec{x}) \in R :\)

\[
e(g)(\mu(\vec{x})) := \mu(g(\vec{x})). \tag{2.1}
\]

This endomorphism will turn out to be invertible in cases of interest to us.

Now consider the following set of conditions on \( e(g) \in End(V) \).

\[
e(g)(\mu(\vec{u})) = \mu(g(\vec{u})) \tag{2.2}
\]

\[
e(g)e(g^{-1})(\mu(\vec{u})) = \mu(\vec{u}) = e(g^{-1})e(g)(\mu(\vec{u})), \tag{2.3}
\]

for all \( U \)-admissible pairs \((\mu, \vec{u})\).
We may assume that the components of $\vec{x}$ are homogeneous elements, and even $(\mu, \vec{u}) \in Q_m$. Both sides of (2.2) are expanded in the basis $B_m$. Equating the coefficients of both sides gives polynomial conditions on the entries of $(g_{ij})$, the matrix representing $g$ with respect to $B$. A similar discussion applies to (2.3).

There is an ideal $I_{(\mu, \vec{x})}$ in the ring $\mathbb{C}[x_{ij}, det^{-1}|i, j = 1, \ldots, dim(U)]$ of polynomial functions on $GL(U)$ associated to the above set of conditions.

Finally, for $u \in U$, define the ideal $I_u$ by the condition $gu = u$. Set $I := \sum_{(\mu, \vec{x}) \in \mathcal{G}} I_{(\mu, \vec{x})} + I_1 + I_\omega$ and

$$G_U = \{ g \in GL(U) \mid p(g) = 0, \text{ for all } p \in I \}.$$  

Then $G_U$ is a variety contained in $GL(U)$. Clearly, $res(Aut(V)) \leq G_U$.

**Lemma 2.2.** $G_U$ is a subgroup of $GL(U)$ defined as the zero set for the ideal $I$. That is, $G_U$ is an algebraic group. Also, $e$ is a homomorphism.

**Proof.** First, $1 \in G_U$. Observe that for $g \in G_U$, $e(g)$ and $e(g^{-1})$ are invertible since their restrictions to each $V_i$ are invertible. Moreover, they form an inverse pair, whence $e(g^{-1}) = e(g)^{-1}$. We now show that $g^{-1}$ satisfies (2.2). Let $(\mu, \vec{y})$ be a $U$-admissible pair of length $r + 1$. Since $g \in G$, we have, for all $\vec{y}$, $e(g)\mu(g^{-1}(\vec{y})) = \mu(\vec{y})$ and so $\mu(g^{-1}(\vec{y})) = e(g)^{-1}\mu(\vec{y}) = e(g^{-1})\mu(\vec{y})$.

Since $g^{-1}$ satisfies (2.2), we have $G_U = G_U^{-1}$.

To prove closure under products, we let $(\mu, \vec{u})$ be a $U$-admissible pair and $g, h \in G_U$. We must show that $e(gh)\mu(\vec{u}) = \mu(gh(\vec{u}))$. Write $\mu(\vec{u}) = \sum_{(\nu, \vec{y}) \in \mathcal{R}} a_{(\nu, \vec{y})}\nu(\vec{y})$, for unique scalars $a_{(\nu, \vec{y})}$, almost all zero. Then,

$$e(gh)\mu(\vec{u}) = \sum_{(\nu, \vec{y}) \in \mathcal{R}} a_{(\nu, \vec{y})}e(gh)\nu(\vec{y}) = \sum_{(\nu, \vec{y}) \in \mathcal{R}} a_{(\nu, \vec{y})}\nu(gh(\vec{y})), $$

by definition of $e(gh)$. Since $g \in G_U$, this equals

$$\sum_{(\nu, \vec{y}) \in \mathcal{R}} a_{(\nu, \vec{y})}e(g)\nu(\vec{y}) = \sum_{(\nu, \vec{y}) \in \mathcal{R}} a_{(\nu, \vec{y})}e(h)\nu(\vec{y}) = e(g)e(h)\mu(\vec{u}).$$

Also $\mu(gh(\vec{u})) = e(g)\mu(h(\vec{u})) = e(g)e(h)\mu(\vec{u})$ because $g, h \in G_U$. We conclude that $gh \in G_U$ and so $G_U$ is a group.

Since the $\mu(\vec{u})$ span $V$, we also deduce that $e(gh) = e(g)e(h)$, whence $e$ is a homomorphism. $\square$
Lemma 2.3. For all $u, v \in V$, $n \in \mathbb{Z}$, we have

$$e(g)(u_n v) = (e(g)u)_n(e(g)v).$$

That is, $\text{Im}(e) \subseteq \text{Aut}(V)$.

Proof. We may assume that $u$ is “monomial”, i.e., has the form $\mu(\vec{x})$, for a $U$-admissible pair $(\mu, \vec{x})$. We argue by induction on the length of $(\mu, \vec{x})$. First, we assume that the length is 1. We may also assume that $v$ is monomial, so $v = \nu(\vec{y})$, for a $U$-admissible pair $(\nu, \vec{y})$. Say $\nu$ is an $\vec{m}$-composition, $\vec{m} = (p, \ldots, q)$ and $\vec{y} = (y_1, \ldots, y_t)$. Then, $u_nv = u_n(y_1)p\cdots q(y_t)$ and $e(g)(u_nv) = (gu)_n(gy_1)p\cdots q(gy_t)$, by 2.2. By 2.2 applied to $(\nu, \vec{y})$, we deduce $e(g)(u_nv) = (gu)_n(e(g)\nu(\vec{y})) = (gu)_n(e(g)v)$. Finally, since $e(g)x = gx$ for $x \in U$, this is $(e(g)u)_n(e(g)v)$.

Suppose next that the length is $r \geq 2$ and that $\mu$ is an $\vec{m}$-composition, $\vec{m} = (m_1, \ldots, m_r)$. Set $k = m_1, b = x_1, a = \nu(\vec{y})$, where $y = (x_2, \ldots, x_r)$ and $\nu$ is the $V$-composition associate to the $(r-1)$-tuple $(m_2, \ldots, m_r)$. Then $u = bka$.

We now do a residue calculation to verify that

$$e(g)(Y(u, z)v) = Y(e(g)u, z)(e(g)v).$$

Extracting the coefficient at $z^{-1}$ will give the Lemma.

Since $u = bka$, we have from the Jacobi identity for vertex operators (see the formula before (3.3) of [D]) that

$$Y(u, z)v = \text{Res}_w\{(w-z)^KY(b, w)Y(a, z)v - (-z + w)^KY(a, w)Y(b, z)v\}.$$

Write $h$ for $e(g)$. Then

$$h[Y(u, z)v] = \text{Res}_w\{(w-z)^k h[Y(b, w)Y(a, z)v] - (-z + w)^k h[Y(a, w)Y(b, z)v]\}.$$

By several uses of induction on length, applied to $b$ and $a$, plus the above consequence of the Jacobi identity, we deduce that this equals

$$\text{Res}_w\{(w-z)^k [Y(hb, w)h[Y(a, z)v] - (-z + w)^k Y(ha, w)h[Y(b, z)v]]\}$$

$$= \text{Res}_w\{(w-z)^k Y(hb, w)Y(ha, z)](hv)$$

$$- (-z + w)^k [Y(ha, w)Y(hb, z)](hv)\}$$

$$= Y((hb)_k(ha), z)(hv)$$

$$= Y(h(ba), z)(hv)$$

$$= Y(hu, z)(hv),$$

as desired. □
Theorem 2.4. The two maps

\[ \text{res} : \text{Aut}(V) \to G_U \quad \text{and} \quad e : G_U \to \text{Aut}(V) \]

form a pair of inverse isomorphisms. Therefore, \( \text{Aut}(V) \) is isomorphic to the algebraic group \( G_U \).

Proof. Since \( U \) generates \( V \), \( \text{res} \) is a monomorphism. Since \( \text{Im}(e) \) is contained in \( \text{Aut}(V) \) and \( \text{res} \circ e = \text{Id}_{G_U} \), \( \text{res} \) is an epimorphism hence an isomorphism. Since the set map \( e \) is a one sided inverse of an isomorphism (hence a two-sided inverse), it is an isomorphism of groups. (We proved before that \( e \) is a homomorphism, but do not need to quote that result here.) \( \square \)

Remark 2.5. The most well known vertex operator algebras are finitely generated. For examples, Heisenberg vertex operator algebras \([FLM]\) and affine vertex operator algebras (cf. \([DL]\), \([FZ]\), \([L2]\)) are generated by their weight one subspaces, Virasoro vertex operator algebras (cf. \([FZ]\), \([L2]\)) and the moonshine vertex operator algebra (see \([B]\), \([FLM]\)) are generated by weight 2 subspaces. The lattice vertex operator algebra \( V_L \) (see \([B]\), \([FLM]\)) is generated by \( \bigoplus_{m \leq n} (V_L)_m \) where \( n \) is any positive integer such that \( L \) has a direct sum decomposition \( L = \bigoplus_{i=1}^{n} \mathbb{Z} \alpha_i \) satisfying \( \langle \alpha_i, \alpha_i \rangle / 2 \leq n \). In fact, \( V_L \) is generated by \( e^{\pm \alpha_i} \) for \( i = 1, \ldots, n \).

Example 2.6. If \( G \) is not finitely generated, \( \text{Aut}(V) \) is not an algebraic group in general. Here is an example. Let \((U,Y,1,\omega)\) be a vertex operator algebra with infinitely many irreducible modules \( U^i = (U^i,Y^i) \) \( (i = 1,2,\ldots) \) not isomorphic to \( U \) such that \( U^i = \bigoplus_{n \geq 0} U_{\lambda_i+n}^i \) with \( U_{\lambda_i}^i \neq 0 \) and \( \lambda_1 < \lambda_2 < \cdots \). Set \( V = U \oplus \bigoplus_{i \neq 0} U_i \). Then \( V \) has a vertex operator algebra structure with vertex operator \( Y' \) defined in the following way (see \([L1]\)). Since \( V \) is a \( U \)-module, \( Y'(u,z)v \) for \( u \in U \) and \( v \in V \) is defined in an obvious way. Using the idea of skew symmetry, for \( v \in U^i \) and \( v \in U \), we define \( Y'(v,z)u := e^{zL(-1)}Y'(u,-z)v \). Finally we define \( Y'(v,z)w := 0 \) for all \( v,w \in \bigoplus_i U_i \). We refer the reader to \([L1]\) for the proof that \((V,Y',1,\omega)\) is indeed a vertex operator algebra.

For \( k, \lambda \in \mathbb{C} \) let \( L(k,\lambda) \) be the irreducible highest weight module for the Virasoro algebra with central charge \( k \) and highest weight \( \lambda \). Then \( L(1,0) \) is a vertex operator algebra and \( L(1,\lambda) \) is an irreducible \( L(1,0) \)-module for any \( \lambda \) (cf. \([FZ]\)). Now we take \( U = L(1,0) \) and \( U^i = L(1,i) \), for \( i = 1,2,\ldots \). Let \( u^i \) be a nonzero highest weight vector of \( U^i \) (which is unique up to a scalar).
Then $V$ is generated by $\omega$ and $u^i$ for $i > 0$. Clearly, $V$ is not finitely generated since a finite set of generators would lie in the sum of $U$ and finitely many $U^i$. Note also that the sum of any set of the $U^i$ is an ideal.

**Proposition 2.7.** The automorphism group of the VOA $\bigoplus_{n=0}^{\infty} L(1, n)$ is isomorphic to the infinite direct product $\prod_{i=1}^{\infty} \mathbb{C}_i^\times$ where $\mathbb{C}_i^\times$ is a copy of multiplicative group $\mathbb{C}^\times$ acting faithfully on $U^i$, trivially on $U^j$ for $j \neq i$ and trivially on $U$. In particular, $\text{Aut}(V)$ is not an algebraic group.

**Proof.** Let $\lambda = (\lambda_1, \lambda_2, \ldots) \in \prod_{i=1}^{\infty} \mathbb{C}_i^\times$. We define a $U$-module homomorphism $g_\lambda \in \prod_{i=1}^{\infty} \mathbb{C}_i^\times$ on $V := \bigoplus_{n=0}^{\infty} L(1, n)$ such that $g_\lambda 1 = 1$ and $g_\lambda u^i = \lambda_i u^i$. It is easy to see from the definition of $Y$ that $g_\lambda$ is an automorphism of $V$. On the other hand any automorphism $g$ is identity on $U$ as $U$ is generated by the Virasoro element. So $g$ preserves the space of highest weight vectors which is spanned by $1$ and $u^i$ for $i > 0$. Since the weights of any two highest weight vectors are different, we immediately have that $g u^i = \lambda_i u^i$ for some nonzero constant $\lambda_i$ for all $i$. Set $\lambda = (\lambda_1, \lambda_2, \ldots) \in \prod_{i=1}^{\infty} \mathbb{C}_i^\times$. Then $g = g_\lambda$. Clearly, $\lambda \mapsto g_\lambda$ is an isomorphism. \qed

Next we discuss the automorphism group of $V$ for a “nice” vertex operator algebra. We need more definitions.

**Definition 2.8.** A vertex operator algebra $V$ has CFT type if $V_n = 0$ for $n < 0$ and $\dim(V_0) = 1$ (so $V_0 = \mathbb{C} 1$).

In the following definition we use the notion of admissible modules as introduced in [Z] and [DLM2]. We refer the reader to [DLM2] for details.

**Definition 2.9.** A vertex operator algebra $V$ is rational if any admissible module is a direct sum of irreducible admissible modules.

**Definition 2.10.** A vertex operator algebra $V$ is $C_k$-cofinite if $\dim(V/C_k(V))$ is finite where $C_k(V)$ is the subspace of $V$ spanned by $u_{-k}v$ for $u, v \in V$.

The $C_2$-cofinite condition in the literature was called the $C_2$-finite condition or $C_2$-condition (as in [Z]). In the case of vertex operator algebras associated to highest weight modules for affine Lie algebras and the Virasoro algebras, $V/C_k(V)$ are the spaces of coinvariants (cf. [FF], [FKLMM]). This should explain why we are changing the terminology. In this paper only the $C_2$-cofinite condition is used.
As we have already mentioned that if $V$ is of CFT type then $(V_1, 0^h)$ is a Lie algebra under $[u, v] = u_0 v$. The part (i) of the following theorem can be found in [DM2]. The rest follows from the general structure of algebraic groups.

**Theorem 2.11.** Let $V$ be a simple, $C_2$-cofinite rational vertex operator algebra of CFT type with $L(1)V_1 = 0$. Then,

1. $V_1$ is a reductive Lie algebra; write $V_1 = \mathfrak{s} \oplus \mathfrak{t}$, where the first summand is semisimple and the second is toral.

2. $G := \text{Aut}(V)$ contains the connected component $G^0$ of the identity with finite index and satisfies $G^0 = G_1 C_1$ (central product) where $G_1 := \langle \exp(x_0) \mid x \in V_1 \rangle$ and $C_1 := C_G(V_1)^0$. We have $G_1 = S_1 T_1$, where $S_1 := \langle \exp(x_0) \mid x \in \mathfrak{s} \rangle$ and $T_1 := \langle \exp(x_0) \mid x \in \mathfrak{t} \rangle$. Also, $T_1 = (C_1 \cap G_1)^0$ and there is a connected group $K_1$ which is normal in $G$ and has the properties $C_1 = T_1 K_1$, $[T_1, K_1] = 1$ and $T_1 \cap K_1$ is finite.

We remark that the condition $L(1)V_1 = 0$ is not a strong assumption. It seems that all rational vertex operator algebras of CFT type satisfy this condition. For example, well known rational vertex operator algebras associated to highest weight integral modules for affine algebras (cf. [L2]), to minimal series for the Virasoro algebras (the weight one space is zero in this case), or to positive definite even lattices satisfy the condition (cf. [FLM]). It is proved in [L1] that for a simple vertex operator algebra of CFT type, the condition $L(1)V_1 = 0$ is equivalent to that there is a nondegenerate symmetric invariant bilinear form on $V$ in the sense of [FHL].

## 3 Derivations

There is a close relation between automorphisms and derivations for a vertex operator algebra. In this section we discuss the Lie algebra of the derivations of a vertex operator algebra.

Define a linear map $\phi$ on $V$ by setting $\phi(v) = v_{\text{wt} - 1}$ for homogeneous elements $v$. Then $\phi(V_n) \subset V_n$ for all $n$.

A \textit{derivation} of the vertex operator algebra $V$ is an endomorphism $d$ of $V$ which satisfies $d1 = 0$, $d\omega = 0$ and $[d, Y(u, z)] = Y(du, z)$. Since $d\omega = 0$, $d$ preserves all the $V_n$, which are the eigenspaces of an operator in $Y(\omega, z)$, whence $d$ is locally finite. The derivation $d$ is an \textit{inner derivation} if there is
$v \in V$ so that $o(v) = d$ (see [3.1]). Since the $V_k$ are finite dimensional, any endomorphism preserving the graded pieces is locally finite.

Since $d$ is a locally finite derivation of $V$, the exponential $e^d$ is an automorphism of $V$. On the other hand, $Aut(V)^0$ (when $V$ is finitely generated) is a connected Lie group and its Lie algebra acts on $V$ as derivations.

Set $IDer(V) := o(V) \cap Der(V)$, the space of inner derivations.

Let $V$ be of CFT type such that $L(1)V_1 = 0$. Then $V$ is a direct sum of irreducible modules for $\text{span}\{L(-1), L(0), L(1)\} \cong sl(2, \mathbb{C})$, the principal $sl_2$ [DLinM]. For homogeneous $v$, since $o(L(-1)v) = -(wt(v) - 1)v_{wt(v) - 1}$, we have equality of $\{o(v) \mid v \in V\}$ and $\{o(v) \mid v \in Ker(L(1))\}$.

Let $QV := Ker(L(1))$, the space of quasi primary vectors.

**Lemma 3.1.** We have $o(v) \in Der(V)$ for $v \in V_1$.

**Proof.** Since $[o(v), Y(u, z)] = [v_0, Y(u, z)] = Y(v_0 u, z) = Y(o(v)u, z)$ for $v \in V_1$ and $u \in V$, the result is clear. □

**Lemma 3.2.** Assume that $V$ has CFT type. If $v = \sum_{i \geq 2} v^i$, with $v^i \in V_i \cap QV$ and $o(v) \in Der(V)$, then $v = 0$.

**Proof.** Since $o(v) = \sum_{i \geq 2} v^i_{i-1}$, we have

$$[o(v), Y(u, z)] = \sum_{i \geq 2} [v^i_{i-1}, Y(u, z)]$$

$$= \sum_{i \geq 2} \sum_{j \geq 0} \binom{i - 1}{j} Y(v^i_j u, z)z^{i-1-j}$$

$$= \sum_{i \geq 2} Y(v^i_{i-1} u, z).$$

It follows that

$$\sum_{i \geq 2} \binom{i - 1}{j} \sum_{j = 0}^{i-2} Y(v^i_j u, z)z^{i-1-j} = 0$$

and

$$\lim_{z \to 0} \left\{ \sum_{i \geq 2} \binom{i - 1}{j} \sum_{j = 0}^{i-2} Y(v^i_j u, z)z^{i-1-j-1} \right\} = 0.$$
for all $u$. Thus
\[
\sum_{i \geq 2} \binom{i-1}{i-2} v_{i-2}^i = 0
\]
on $V$. Since the $v^i$ are quasi-primary vectors, $[L(1), v_{i-2}^i] = iv_{i-1}^i$. As a result we have
\[
\sum_{i \geq 2} (i - 1)iv_{i-1}^i = 0.
\]

By Theorem 2.2 of [DLMM], we get
\[
\sum_{i \geq 2} i(i - 1)v_{i-1}^i = 0.
\]

The following corollary is immediate:

**Corollary 3.3.** For $V$ of CFT type, $\text{IDer}(V) = o(V_1) = \{o(v) | v \in V_1\}$.

Recall from [DLMM] that the radical $J(V)$ of $V$ consists of those vectors $v \in V$ such that $o(v) = 0$. We need a result from [DM2].

**Lemma 3.4.** Let $V$ be a $C_2$-cofinite rational vertex operator algebra of CFT type. Then $J(V) = (L(-1) + L(0))V$.

From now on we assume that $V$ is a $C_2$-cofinite rational vertex operator algebra of CFT type. Then $g = (V_1, 0^{th})$ is a reductive Lie algebra and each $V_n$ is a finite dimensional $g$-module via $v \mapsto o(v)$. Define invariant symmetric bilinear form $(,)_M$ on $g$ for any $g$-module $M$ by $(u, v)_M = \text{tr}_M(uv)$ for $u, v \in g$.

Recall that $g = s \oplus t$ where $s$ is semisimple and $t$ is abelian. Then each finite dimensional module for $g$ is a direct sum of indecomposable modules which are tensor products of irreducible modules for $s$ and indecomposable modules for $t$.

**Lemma 3.5.** Let $M$ be a finite dimensional $g$-module such that $M$ contains $s$ as an $s$-module. Let $s = s^1 \oplus \cdots \oplus s^p$ be the decomposition into simple ideals. Write $s^0 := t$. Then $(s^i, s^j)_M = 0$ if $i \neq j$ and if $i > 0$, the restriction of the form to each $s^i$ is nondegenerate.

**Proof.** First we prove that the restriction of the form to each $s^i$ is nondegenerate. Note that as an $s^i$-module, $M$ is completely reducible and $s^i$ is an irreducible submodule.

Let $i > 0$. It is well known that for each irreducible $s^i$-module $W$ the corresponding invariant symmetric bilinear form $(,)_W$ is a nonnegative multiple.
of $(\cdot, \cdot)_\mathfrak{s}$, the Killing form on $\mathfrak{s}^i$, which is nondegenerate (cf. [H]). As a result, $(\cdot, \cdot)_M$ is nondegenerate when restricted to $\mathfrak{s}^i$.

In order to prove that $(\mathfrak{s}^i, \mathfrak{s}^j)_M = 0$ if $i \neq j$, we may assume that $M$ is irreducible. Then $t$ acts as scalars on $M$ and $M|_{\mathfrak{s}} = M^0 \otimes \cdots \otimes M^p$ is a tensor product of irreducible modules $M^i$ for $\mathfrak{s}^i$.

Let $i \geq 0$ be any index and let $x \in \mathfrak{s}^i$. Then $M^i$ is a direct sum of generalized eigenspaces under $x$. So, we can choose a basis $B_k$ for $M^k$ consisting of generalized eigenvectors for the action of $x$ on $M^i$. Let $\lambda := \sum_{k \neq j} \lambda_k$. Now fix $j > 0$ and assume $j \neq i$. Let $y \in \mathfrak{s}^j$. Think of a matrix for the action of $y$ written in a basis taken from the subspaces of the form $m^0 \otimes m^1 \otimes \cdots \otimes m^{j-1} \otimes M^j \otimes m^{j+1} \cdots \otimes m^p$.

Note that $M$ can be written as a direct sum of such subspaces. These spaces are not invariant by $x$ but the action of $xy$ has block-triangular form with respect to this direct sum.

The contribution to the trace $\text{tr}_M xy$ from the subspace $m^0 \otimes m^1 \otimes \cdots \otimes m^{j-1} \otimes M^j \otimes m^{j+1} \cdots \otimes m^p$ is equal to $\lambda \text{tr}_{M^j} y$ for $y \in \mathfrak{s}^j$. Since $\mathfrak{s}^j$ is simple and $[\mathfrak{s}^j, \mathfrak{s}^j] = \mathfrak{s}^j$ we see that $\text{tr}_{M^j} y = 0$. Thus $(\mathfrak{s}^i, \mathfrak{s}^j)_M = 0$ since $M$ is a direct sum as described above.

For convenience we denote the bilinear form $(\cdot, \cdot)_V$ on $\mathfrak{g}$ by $(\cdot, \cdot)_n$ for $n \geq 0$. We need the following result from [DM2].

**Proposition 3.6.** Let $V$ be a rational, $C_2$-cofinite vertex operator algebra of CFT type and $L(1)V_1 = 0$. Then for any $u \in V_1$ there exist $n \geq 0$ and $v \in V_1$ such that $(u, v)_n \neq 0$.

The next result sharpens Proposition 3.6.

**Theorem 3.7.** Let $V$ be as in Proposition 3.6. Then there exists $n$ such that $(\cdot, \cdot)_n$ is nondegenerate.

**Proof.** Take $n$ large enough so that $\sum_{m=0}^n V_m$ generates $V$. We claim that $(\cdot, \cdot)_n$ is nondegenerate.

Recall that $\mathbb{C}L(-1) + \mathbb{C}L(0) + \mathbb{C}L(1)$ is isomorphic to Lie algebra $sl(2, \mathbb{C})$. Let $M(\lambda)$ is the irreducible highest weight module for $sl(2, \mathbb{C})$ with highest weight $\lambda$. Then

$$M(\lambda) = \oplus_{m \geq 0} M(\lambda)_{\lambda + m}$$

such that $M(\lambda)_{\lambda}$ is spanned by a highest weight vector $v_\lambda$ and $M(\lambda)_{\lambda + m}$ is spanned by $L(-1)^m v_\lambda$. If $\lambda = 0$ then $M(\lambda)$ is trivial. If $\lambda > 0$ each $M(\lambda)$ is a Verma module and $M(\lambda)_{\lambda + m} \neq 0$ for all $m$.
First we prove that the representation of \( g \) on \( V_n \) is faithful. Assume that \( u_0 = 0 \) on \( V_n \). Since \( V \) is CFT type and \( L(1)V_1 = 0 \) it follows from Corollary 3.2 of [DLinM] that \( V \) is a direct sum of copies of \( M(\lambda) \) for \( m \geq 0 \) and the multiplicity of \( M(0) \) in the decomposition is 1. Note that \([L(i), u_0] = 0\) for \( i = -1, 0, 1 \) and \( u \in g \). Let \( M(\lambda) \) occurs in the decomposition of \( V \) such that \( 0 \neq m \leq n \). Then \( u_0 \) on \( M(\lambda) \). Also note that \( u_0 V_0 = 0 \). Thus \( u_0 = 0 \) on \( \bigoplus_{m=0}^n V_m \). Since \( u_0 \) is a derivation on \( V \) and \( V \) is generated by \( \bigoplus_{m=0}^n V_m \), we immediately see that \( u_0 = 0 \) on \( V \). This contradicts Proposition 3.6.

So we can identify \( g \) with its image \( o(g)_n = \{ u_0|_{V_n} \} \). If the form \(( , )_n\) is degenerate then by Lemma 3.3, there exists \( x \in t \) such that \((x, y)_n = 0\) for all \( y \in g \). By Lemma 4.3 of [H], \( x_0 \) is nilpotent on \( V_n \). In particular, all the eigenvalues of \( x_0 \) on \( V_n \) are zero. A similar argument as above then shows that \( x_0 \) has only zero eigenvalues on \( \bigoplus_{m=0}^n V_m \). Since \( \bigoplus_{m=0}^n V_m \) generates \( V \) we see immediately that \( x_0 \) has only zero eigenvalues on \( V_m \) for all \( m \) as

\[
 u_0 v_1^{m_1} \cdots v_k^{m_k} v = \sum_{j=1}^k v_1^{j_1} \cdots (u_0 v_j)_{m_j} \cdots v_k^{j_k} V + v_1^{j_1} \cdots v_k^{j_k} u_0 v
\]

for \( v^j, v \in V \) and \( m_j \in \mathbb{Z} \).

Note that \( t \) is an abelian Lie algebra and all irreducible modules are one-dimensional. So, \( \text{tr}_{V_n} x_0 y_0 = 0 \) for all \( y \in g \) and \( m \in \mathbb{Z} \). Again, by Proposition 3.6, this is impossible. \( \square \)

**Theorem 3.8.** Let \( V \) be a \( C_2 \)-cofinite rational vertex operator algebra of CFT type such that \( L(1)V_1 = 0 \). Let \( n \geq 0 \) such that \( \sum_{m=0}^n V_m \) generates \( V \). Then \( \text{Der}(V) \) is a direct sum of ideals \( o(g) \) and \( g^\perp \) where \( g^\perp \) consists of \( d \in D \) such that \( \text{tr}_{V_n} o(u) d = 0 \) for all \( u \in V_1 \).

**Proof.** Let \( V \) be as in Lemma 3.5 and Theorem 3.7. Let \( n > 0 \) be as in the proof of Theorem 3.7. Then the action of \( D := \text{Der}(V) \) is also faithful on \( V_n \) and \((d, d')_n = \text{tr}_{V_n} dd'\) defines a symmetric invariant bilinear form on \( D \). Then restriction of the form to \( o(g) \) is nondegenerate by Theorem 3.7. Let \( g^\perp \) be the orthogonal complement of \( o(g) \). Then the intersection of \( g^\perp \) and \( o(g) \) must be zero. On the other hand, \([d, u_0] = (du)_0\) tells us that \( o(g) \) is an ideal of \( D \) and so is \( g^\perp \). Thus \([d, u_0] = (du)_0 = 0\) for \( d \in g^\perp \) and \( u \in V_1 \). Since the action of \( D \) is faithful on \( V_n \) we have \( du = 0 \). \( \square \)
4 Example

In this section we give an example which shows that a finitely generated VOA with an infinite descending chain of ideals can still have a reductive group of automorphisms. Our example is $V^G_L$, below. In this example, we find all ideals and find that they form a countable descending chain.

**Example 4.1.** We consider the vertex operator algebra $V = V_L = L(\Lambda_0)$ where $L = \mathbb{Z}\alpha$ such that $(\alpha, \alpha) = 2$ and $L(\Lambda_0)$ is the fundamental representation for the affine algebra $A_1^{(1)}$. Then the automorphism group of $V_L$ is isomorphic to $PSL(2, \mathbb{C})$ (see [DLY] and [DN]).

Let $L(c, h)$ be the highest weight module for the Virasoro algebra with central charge $c$ and highest weight $h$. Let $W_m$ for $m \geq 1$ be the irreducible module for $sl(2, \mathbb{C})$ of dimension $m$. Then

$$V_L = \bigoplus_{m \geq 0} L(1, m^2) \otimes W_{2m+1}$$

(cf. [DG]) and $SL(2, \mathbb{C})$ acts on $V_L$ by acting on $W_{2m+1}$. Moreover, $W_{2m+1}$ regarded as a $SL(2, \mathbb{C})$-submodule of $V_L$ is generated by the highest weight vector $e^{ma}$.

Consider the subgroup $G = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{C} \right\}$ of $SL(2, \mathbb{C})$. Clearly $G$ is not compact. Then the space of $G$-invariants $W^G_{2m+1}$ is spanned by $e^{ma}$. As a result, we have a direct sum decomposition of the fixed point set for $G$ into irreducible modules for the Virasoro algebra:

$$V^G_L = \bigoplus_{m \geq 0} L(1, m^2),$$

where the highest weight module $L(1, m^2)$ for the Virasoro algebra is generated by $e^{ma}$. For distinct $m$, these are pairwise nonisomorphic. It is not hard to see that for any $n \geq 1$, $\sum_{m \geq n} L(1, m^2)$ is an ideal of $V^G_L$ as $u_k v \in L(1, (s+t)^2)$ for any $u \in L(1, s^2)$ and $v \in L(1, t^2)$ and $k \in \mathbb{Z}$. We now prove that all the ideals of $V^G_L$ are given in this way. Let $I$ be a nonzero ideal of $V^G_L$. Then $I$ is a module for the Virasoro algebra, so is a sum of a family of the above $L(1, n^2)$. Let $n \geq 1$ be the smallest positive integer such that $L(1, n^2)$ is a subspace of $I$. Then $e^{-n(\alpha, \alpha)} e^{n \alpha} = e^{(n+1)\alpha} \in I$ (see [FLM]) and $L(1, (n+1)^2)$ is contained in $I$. Continuing in this way we see that $I = \sum_{m \geq n} L(1, m^2)$. It was proved in [DLM1] that if $H$ is a compact Lie group which acts continuously on a simple vertex operator algebra $V$ then
$V^H$ is also a simple vertex operator algebra. Since our $G$ is not compact, $V^G_L$ is permitted to be nonsimple, and it is.

We now show that the automorphism group of $V^G_L$ is isomorphic to $\mathbb{C}^\times$. Note that $V^G_L$ is generated by $\omega$ and $e^\alpha$. In fact, we have already proved that $e^{(n+1)\alpha}$ can be generated from $e^\alpha$ and $e^{n\alpha}$. So all the highest weight vectors can be generated from $e^\alpha$. Using $\omega$ then generates the whole space. For any $\lambda \in \mathbb{C}^\times$ we define a linear isomorphism $\sigma_\lambda$ of $V^G_L$ such that $\sigma_\lambda$ acts on $L(1, m^2)$ as $\lambda^m$. From the discussion before it is clear that $\sigma_\lambda$ is an automorphism of $V^G_L$. On the other hand, any automorphism $\sigma$ of $V^G_L$ maps $e^\alpha$ to $\lambda e^\alpha$ for some nonzero constant $\lambda$ as $e^\alpha$ is the only highest weight vector with highest weight 1 (up to a constant) for the Virasoro algebra. Since $V^G_L$ is generated by $\omega$ and $e^\alpha$ we immediately see that $\sigma = \sigma_\lambda$.

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