ABSTRACT This paper considers event-triggered disturbance rejection control for discrete time linear systems. A novel event-based disturbance rejection controller is designed, and sufficient conditions are given to guarantee that the closed-loop system is asymptotically stable. Based on these criteria, an event-triggered state feedback controller is designed and developed. Then numerical examples are given to illustrate the results.

INDEX TERMS Asymptotically stable, disturbance rejection, discrete time systems, event trigger.

I. INTRODUCTION

It is well known that event-triggered control is commonly used in modern industry systems to reduce workload of network and improve the efficiency. Instead of updating the input of controller in constant-time frequency, the input of actuator is event driven in an event-triggered type of control. When the magnitude of changes in system states reaches prescribed threshold for the performance, the controller will be updated. See details in [1]–[8]. A usually used construction is that the measured state will be sent to the event generator first, and then the results decide whether or not it is necessary to send it to the controller, some attempt work has been done in [10], [11]. In this paper, we propose event triggered disturbance rejection control for discrete systems, where a reduced order observer is addressed.

In recent years, the approach of DOBC (Disturbance observer based control) is investigated and designed for estimation of the disturbance comes from input channel, and controller maybe designed to reject the disturbance by considering its effects. Much work has been done and various types of results have been obtained, for instance, in [12], a disturbance rejection scheme for a class of nonlinear systems has been addressed, and then, a linear DOBC problem was investigated for a class of uncertain linear exosystems with time delay [13], and extensively studies have been done in many areas, see [14]–[16]. However, disturbance can not be avoid in many real applications [9], the most concern is how to balance the workload in network and disturbance rejection, especially when no exactly information is available to describe such disturbance.

In this paper, the anti-disturbance problem is investigated for a class of discrete systems in the event-triggered control framework. Firstly, a reduced-order disturbance rejection observer is designed to guarantee the closed-loop system asymptotically stable under a given event-triggered control policy. Under these conditions, design schemes are presented for the problem of event-triggered linear state feedback in the form of linear matrix inequalities (LMIs). To demonstrate its effectiveness, numerical examples are given and solved.

The remainder of this paper is organized as follows. Section 2 is the problem statement and preliminaries, in which necessary definitions and lemmas are given. Disturbance rejection observer is designed in Section 3, while stability analysis of a closed-loop system is put into effect with event triggered linear state feedback control laws and stability conditions are established. Section 4 presents the algorithms for the design of the observer-based linear state feedback laws. Two numerical simulations are provided to show the applicability of the control strategy in Section 5. Finally Section 6 concludes the paper.

Notation: Throughout the paper, the n-dimensional Euclidean space is denoted by \( \mathbb{R}^n \), \( A^T \) denotes the transpose of the matrix \( A \). A positive (negative) definite matrix \( P \) is written as \( P > 0 \) (\( P < 0 \)) while * indicates a symmetric element in a symmetric matrix.
II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a class of discrete-time linear systems,
\[ x(k + 1) = Ax(k) + Bu(k) + d_1(k), \]
where \( x(k) \in \mathbb{R}^n \) and \( u(k) \in \mathbb{R}^m \) are the state and the control input, \( A \) and \( B \) are constant matrices of appropriate dimensions and \( d_1(k) \) is an external noise which comes from system (2).

The input disturbance \( d_1(k) \) in system (1) is the output of system (2) given below:
\[
\begin{align*}
\dot{w}(k+1) &= Dw(k), \\
\dot{d}_1(k) &= Fw(k),
\end{align*}
\]
where \( D \) and \( F \) are constant matrices of appropriate dimensions.

Assumption 1: For systems (1) and (2), it holds that \((A, B)\) is controllable and \((D, BF)\) is observable.

Based on the assumption that all the system states are available, we only need to design a reduced-order observer as follows to estimate \( d_1(k) \).
\[
\begin{align*}
\dot{\hat{d}}_1(k) &= F\hat{w}(k), \\
\dot{\hat{w}}(k) &= v(k) - Lx(k), \\
\dot{\hat{x}}(k+1) &= (D + LF)(v(k) - Lx(k)) + L(Ax(k) + Bu(k)),
\end{align*}
\]
where \( \hat{d}_1(k) \) and \( \hat{w}(k) \) are estimations of \( d_1(k) \) and \( w(k) \), respectively.

The controller in this system can be designed as:
\[ u(k) = -\dot{\hat{d}}_1(k) + Kx(k). \]

Denote
\[ f(k) = w(k) - \hat{w}(k), \]
then, we have
\[ f(k+1) = (D + LF)f(k). \]

System (V-B) will be asymptotically stable under an event-triggered feedback control law designed. Let \( \hat{x}_k \) be a new signal applied to the controller in the time interval \((k, k + 1] \) with
\[
\hat{x}(k) = \begin{cases} 
  x(k) & \text{if event condition is satisfied,} \\
  \hat{x}(k-1) & \text{if event condition is not satisfied,}
\end{cases}
\]
and \( \hat{x}(k) = 0 \) for \( k \leq 0 \) with the initial time \( k_0 = 0 \). Let \( u(k) = -\dot{\hat{d}}_1(k) + K\hat{x}(k) \). The following decision condition for signal transmission is given by the event generator, which is
\[ \|\hat{x}(k-1) - x(k)\| > \sigma \|x(k)\|, \]
where \( \sigma > 0 \). That means if condition (6) is satisfied, the state signal is sent and the controller is updated. The controller is defined by
\[
\begin{align*}
u(k) = \begin{cases} 
  -\dot{\hat{d}}_1(k) + Kx(k), & \text{if } \|\hat{x}(k-1) - x(k)\| > \sigma \|x(k)\| \\
  -\dot{\hat{d}}_1(k) + K\hat{x}(k-1), & \text{if } \|\hat{x}(k-1) - x(k)\| \leq \sigma \|x(k)\|.
\end{cases}
\]

Let \( \eta^T(k) = [x^T(k) f^T(k)] \) and \( e(k) = \hat{x}(k) - x(k) \), combining systems (V-B), exogenous system(2) with (3), we obtain an error estimation system given below:
\[ \eta(k + 1) = \tilde{A}\eta(k) + \tilde{B}e(k), \]
where
\[ \tilde{A} = \begin{bmatrix} A + BK & BF \\
0 & D + LF \end{bmatrix}, \]
\[ \tilde{B} = \begin{bmatrix} BK \\
0 \end{bmatrix}. \]

Definition 1: For a given initial state \( \eta(0) \), suppose it holds that
\[ \lim_{m \to \infty} \sum_{k=0}^{m} \eta^T(k) \eta(k) | \eta(0) \rangle < \infty. \]
Then system (8) is said to be asymptotically stable and \( K \) is called the solution of the controller gain.

Our objective is design an observer based anti-disturbance controller for system (8), guarantee that the error estimation system (8) is asymptotically stable. To proceed further, some definitions are provided to develop our main results in the paper.

III. STABILITY ANALYSIS

Note that event triggered design is discussed, and the reduced order observer has been addressed. This section gives the sufficient conditions under which system (8) is asymptotically stable.

Theorem 1: Let \( \sigma \) be given. If there exist a positive definite symmetric matrix \( P \), and a constant value of \( \kappa \) satisfying
\[ \tilde{\Gamma} = \begin{bmatrix} -P + \tilde{A}^T \tilde{P} A + \kappa S & \tilde{A}^T \tilde{P} \tilde{B} \\
* & \tilde{B}^T \tilde{P} \tilde{B} - \kappa I \end{bmatrix} < 0. \]
Then system (8) is asymptotically stable.

Proof: We consider the following Lyapunov function
\[ V(\eta(k)) = \eta^T(k) P \eta(k). \]
Taking the difference of the Lyapunov function along the trajectory of system (8) yields
\[ \Delta V(\eta(k)) = V(\eta(k + 1)) - V(\eta(k)) \]
\[ = \eta^T(k + 1) P \eta(k + 1) - \eta^T(k) P \eta(k) \]
\[ = \eta^T(k) \tilde{A}^T \tilde{P} \eta(k) + \eta^T(k) \tilde{A}^T \tilde{P} \tilde{B} e(k) + e^T(k) \tilde{B}^T \tilde{P} \eta(k) + e^T(k) \tilde{B}^T \tilde{P} \tilde{B} e(k) - \eta^T(k) P \eta(k). \]
To ensure that $\Delta V(\eta(k)) \leq 0$, we need to have
\[
\hat{\Gamma}_1 = \begin{bmatrix} -P + \hat{A}^T \hat{P}A & \hat{A}^T \hat{P}B \\ \ast & \hat{B}^T \hat{P}B \end{bmatrix} \leq 0.
\tag{11}
\]

Let
\[
\xi = \min_k \lambda_{\min}(-\hat{\Gamma}_1),
\]
where $\lambda_{\min}(-\hat{\Gamma}_1)$ is the minimal eigenvalue of $-\hat{\Gamma}_1$. Then,
\[
\Delta V(\eta(k)) = -\xi \eta^T(k) \eta(k),
\]
where
\[
\eta^T(k) = \begin{bmatrix} x(k) \\ f(k) \end{bmatrix}^T.
\]

Thus,
\[
\sum_{k=0}^{T} \Delta V(\eta(k)) = V(\eta(T + 1)) - V(\eta(0)) \leq -\xi \sum_{k=0}^{T} \|\eta(k)\|^2.
\]

This, in turn, means that
\[
\sum_{k=0}^{T} \|\eta(k)\|^2 \leq \frac{1}{\xi} \{V(\eta(0)) - V(\eta(T + 1))\} \leq \frac{1}{\xi} V(\eta(0)),
\]
and hence
\[
\lim_{T \to \infty} E\{\sum_{k=0}^{T} \|\eta(k)\|^2\} \leq \frac{1}{\xi} V(\eta(0)).
\]

By Definition 1, system (8) is asymptotically stable.

On the other hand, recall the error variable $e(k) = \hat{x}(k) - x(k)$. In the time interval $(k, k + 1)$, it follows from condition (6) that the inequality $\|e(k)\| \leq \sigma \|x(k)\|$ is always satisfied, which can be constructed as:
\[
\begin{bmatrix} \eta(k)^T & e(k) \end{bmatrix} \begin{bmatrix} S & 0 \\ \ast & I \end{bmatrix} \begin{bmatrix} \eta(k) \\ e(k) \end{bmatrix} \leq 0,
\tag{12}
\]
where
\[
S = \begin{bmatrix} -\sigma^2 I & 0 \\ \ast & 0 \end{bmatrix}.
\]

Applying the lossless S-procedure [17], for a given $\kappa > 0$, combining condition (11) with (12), it follows from condition (10) that
\[
\begin{bmatrix} \eta(k)^T & e(k) \end{bmatrix} \hat{\Gamma}_1 \begin{bmatrix} \eta(k) \\ e(k) \end{bmatrix} \leq 0.
\tag{13}
\]

This completes the proof and system (8) is asymptotically stable.

**Remark 1:** Note that Theorem 1 presents sufficient conditions to ensure that the resulting system is stable with an event triggered controller subject to reduced order observer, and the disturbance will be observed and rejected. However, it is difficult to solve these inequalities by LMIs, as they are nonlinear matrices. Next, we will further discuss these inequalities so that the conditions obtained become solvable.

**IV. CONTROLLER DESIGN**

In this section, our objective is to design a feedback gain $K$ such that the closed-loop system is asymptotically stable. Based on the conditions given in Theorem 1, we can formulate this problem into the following Theorem 2.

**Theorem 2:** Let $\sigma$ be given. If there exist a positive definite symmetric matrix $Q$, and a constant value of $\kappa$ such that it holds
\[
\begin{bmatrix} -G^T - G + Q & 0 & 0 \\ * & -2I + Q & 0 \\ * & * & -G^T - G + 1 \frac{1}{\kappa} I \\ \ast & * & * \\ \ast & * & * \\ \ast & * & -1 \frac{1}{\kappa \sigma^2} \end{bmatrix} < 0.
\tag{14}
\]

Then system (8) is said to be asymptotically stable.

**Proof:** Based on the Schur complement, it follows from Constraint (10) that
\[
\begin{bmatrix} -P + \kappa S & 0 & (\hat{A})^T \\ \ast & -\kappa I & (\hat{B})^T \\ \ast & \ast & -P^{-1} \end{bmatrix} < 0.
\tag{15}
\]

Multiplying to both sides of inequalities (15) by $\text{diag} \{G^T, I, G^T, I, I\}$ and $\text{diag} \{G, I, G, I, I\}$, respectively, where $G \in \mathbb{R}^{n \times n}$ is a positive definite diagonal matrix.

Noting that
\[
(P^{-1} - G)^T P (P^{-1} - G) \geq 0,
\]
or
\[
G^T P G \geq G^T + G - P^{-1}.
\]

Let $Q = P^{-1}$, $P = \text{diag} \{P_1, P_1\}$ and $\hat{K} = KG$. Then, Condition (14) is obtained which is linear in the variables $G$, $Q$ and $\hat{K}$. Once a solution of (14) is obtained, the feedback gain can be computed as $K = \hat{K}^{-1} G^{-1}$.

**Remark 2:** It is observed that the conditions in Theorem 2 are solvable by LMIs, it shows that under the controller designed combined with rejection observer, the disturbance will be rejected and the system is asymptotically stable. Numerical examples will be given in the next section.

**V. NUMERICAL EXAMPLES**

**A. EXAMPLE 1**

We consider a discrete-time system and the parameters are given below.
\[
A = \begin{bmatrix} 1.7 & -0.23 \\ 0.25 & -0.92 \end{bmatrix}, \quad B = \begin{bmatrix} 0.31 \\ 0.25 \end{bmatrix},
\]

P. Cheng et al.: Event-Triggered Disturbance Rejection Control of Discrete Systems
P. Cheng et al.: Event-Triggered Disturbance Rejection Control of Discrete Systems

FIGURE 1. State trajectories for the case of \( \sigma = 0.08 \).

FIGURE 2. Estimation of disturbance.

\[
D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}.
\]

For the case of \( \sigma = 0.08 \), the corresponding trajectories are shown in Fig. 1. The disturbance \( d_1(k) \), the estimation disturbance \( \hat{d}_1(k) \), and the error disturbance \( d_1(k) - \hat{d}_1(k) \) are shown in Figure 2. Obviously, the system (8) with disturbance is asymptotically stable under such a controller along with a reduced order observer.

B. EXAMPLE 2

DC motor is a common actuator in control systems, and it has widely applications in providing translational motion. Choose the rotational speed and electric current as the state variables, the system will be given in state-space form as below:

\[
\begin{bmatrix} \ddot{\theta} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} -b/J & K/J \\ -K/L & R/L \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u(t),
\]

where \( J = 0.01, b = 0.1, K = 0.01, R = 1, L = 0.5 \).

Discretization this model by 0.005s, \( u(t) \) is the input of voltage source, then continuous system matrices are given as follows

\[
A = \begin{bmatrix} 0.95 & 0.00485 \\ -9.705e-05 & 0.99 \end{bmatrix}, \quad B = \begin{bmatrix} 2.45e-05 \\ 0.00995 \end{bmatrix}, \quad D = \begin{bmatrix} -0.53 & 0 \\ 0 & 0.56 \end{bmatrix}, \quad F = \begin{bmatrix} 0.16 \\ 10.17 \end{bmatrix}
\]

For the case of \( \sigma = 0.08 \), the corresponding trajectories are shown in Fig. 3. control input is shown in Fig. 4, where event triggered phenomenon is observed. The estimation of disturbance is given in Fig. 5. Note that the DC system with disturbance is asymptotically stable under such a event triggered controller with a reduced order observer.

Remark 3: Comparing with other literatures, the big challenge in our result is there are external disturbances in the system with neither distribution nor membership available, thus, a reduced order observer is very helpful to make sure that the
state of disturbance is observable and detectable, it guarantees that the controller designed by Lyapunov stability theory in terms of state feedback control can stabilize the closed-loop system without any interruption from noises.

VI. CONCLUSION

In this paper, the problem of event-based disturbance rejection control for discrete time systems is investigated. The approach of linear inequality matrix is used to established conditions under which the closed-loop system is asymptotically stable. Based on these conditions, designing an event-triggered state feedback anti-rejection controller is formulated and solved. Finally, the numerical examples show the efficiency of the proposed design procedure and the performance of the resulting closed-loop system.

REFERENCES

[1] K. E. Ariz, “A simple event-based PID controller,” presented at the 14th World Congr. IFAC, 1984, pp. 423–428.
[2] J. Lunze and D. Lehmann, “A state-feedback approach to event-based control,” Automatica, vol. 46, no. 1, pp. 211–215, Jan. 2010.
[3] X. Wang and M. D. Lemmon, “Event-triggering in distributed networked control systems,” IEEE Trans. Autom. Control, vol. 56, no. 3, pp. 586–601, Mar. 2011.
[4] W. P. M. H. Heemels, M. C. F. Donkers, and A. R. Teel, “Periodic event-triggered control for linear systems,” IEEE Trans. Autom. Control, vol. 58, no. 4, pp. 847–861, Apr. 2013.
[5] W. Hu, L. Liu, and G. Feng, “Output consensus of heterogeneous linear multi-agent systems by distributed event-triggered/self-triggered strategy,” IEEE Trans. Cybern., vol. 47, no. 8, pp. 1914–1924, Aug. 2017.
[6] J. Zhang and G. Feng, “Event-driven observer-based output feedback control for linear systems,” Automatica, vol. 50, no. 7, pp. 1852–1859, Jul. 2014.
[7] E. Tian, Z. Wang, L. Zou, and D. Yue, “Probabilistic-constrained filtering for a class of nonlinear systems with improved static event-triggered communication,” Int. J. Robust Nonlinear Control, vol. 29, no. 5, pp. 1484–1498, Mar. 2019.
[8] S. Hu, D. Yue, X. Yin, X. Xie, and Y. Ma, “Adaptive event-triggered control for nonlinear discrete-time systems,” Int. J. Robust Nonlinear Control, vol. 26, no. 18, pp. 4104–4125, Dec. 2016.
[9] E. Tian, Z. Wang, L. Zou, and D. Yue, “Chance-constrained $H_{\infty}$ control for a class of time-varying systems with stochastic nonlinearities: The finite-horizon case,” Automatica, vol. 107, pp. 296–305, Sep. 2019.
[10] Y. Yin, Z. Lin, Y. Liu, and K. L. Teo, “Event-triggered constrained control of positive systems with input saturation,” Int. J. Robust Nonlinear Control, vol. 28, no. 11, pp. 3532–3542, Jul. 2018.
[11] Y. Yin, Y. Liu, K. L. Teo, and S. Wang, “Event-triggered probabilistic robust control of linear systems with input constraints: By scenario optimization approach,” Int. J. Robust Nonlinear Control, vol. 28, no. 1, pp. 144–153, Jan. 2018.
[12] L. Guo and W.-H. Chen, “Disturbance attenuation and rejection for systems with nonlinearity via DOBC approach,” Int. J. Robust Nonlinear Control, vol. 15, no. 3, pp. 109–125, Feb. 2005.
[13] M. Chen and W.-H. Chen, “Disturbance-observer-based robust control for time delay uncertain systems,” Int. J. Control, Autom. Syst., vol. 8, no. 2, pp. 445–453, Apr. 2010.
[14] L. Guo and S. Cao, Anti-Disturbance Control for Systems With Multiple Disturbances. Boca Raton, FL, USA: CRC Press, 2014.
[15] L. Guo and S. Cao, “Anti-disturbance control theory for systems with multiple disturbances: A survey,” ISA Trans., vol. 53, no. 4, pp. 846–849, Jul. 2014.
[16] X. Yao and L. Guo, “Composite anti-disturbance control for Markovian jump nonlinear systems via disturbance observer,” Automatica, vol. 49, no. 8, pp. 2538–2545, Aug. 2013.
[17] T. Iwasaki, G. Meinsma, and M. Fu, “Generalized S-procedure and finite frequency KYP lemma,” Math. Problems Eng., vol. 6, nos. 2–3, pp. 305–320, 2000.
SONG WANG received the B.Sc. degree from the Wuhan University of Hydraulic and Electric Engineering (now Wuhan University), Wuhan, China, in 1982, and the Ph.D. degree in numerical analysis from the Trinity College Dublin, Ireland, in 1989. He was with Dublin-based hi-tech company-Tritech Ltd., University of New South Wales, Curtin University of Technology, and The University of Western Australia. He is currently a Professor and the Head of the Department of Mathematics and Statistics, Curtin University. He has published over 100 journal research articles and supervised over ten Ph.D. students in these areas. His research interests include scientific computation, numerical optimization and optimal control, optimum design, and computational finance. He is also on the editorial boards of several international journals.

FENG PAN received the Ph.D. degree in control science and control engineering from Jiangnan University, Wuxi, China. He is currently a Professor with the Institute of Automation, Jiangnan University. His research interests include production process modeling, optimization and control, and biochemical process intelligent control.