More about Kaluza-Klein Regularization

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Abstract

We study the so-called “Kaluza-Klein regularization”. We calculate one-loop corrections to the Higgs mass due to Kaluza-Klein modes explicitly in a model with SUSY breaking mass splitting between bosonic and fermionic modes. We perform the proper time cutoff at $1/\Lambda^2$ and the KK level truncation at $\ell$. It is shown that the finite result is obtained as long as $\ell \gg \Lambda R$ for the compactification radius $R$.

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Recently, many efforts have been done to study supersymmetry (SUSY) breaking in extra dimensions as well as other phenomenologies in extra dimensions, and novel aspects have been found. Among many interesting aspects, in Ref. [1]-[5] it has been found that one-loop radiative corrections to the Higgs mass are finite in the models with the Scherk-Schwarz SUSY breaking mechanism [6, 7] and also in the models with localized SUSY breaking 3-brane. In such perturbative calculations, the so-called “Kaluza-Klein (KK) regularization” is used, that is, infinite summation of all KK modes is taken first.

However, a doubt on the validity of the KK regularization has arisen [8, 9]. It is claimed that the exchange between infinite sum and infinite integral would lead to an incorrect result. In Ref. [8], the momentum cut-off Λ is introduced and the KK-tower is truncated at the level ℓ. In addition, it is also claimed that physically the level to contribute should be truncated around the momentum cut-off Λ. In Ref. [8], such physical truncation is realized by a sharp cut, ℓ ≈ ΛR, where R is the compactification radius. Then, the quadratic divergence is found to appear in the Higgs mass corrections due to finite numbers of KK modes.

Moreover, there is a debate that the result of Ref. [8] may be because of the sharp cut of the KK-tower, and such procedure spoils symmetries of the underlying theory. Instead of such truncation, in Ref. [10] the suppression by a Gaussian brane distribution is considered. There the infinite numbers of KK modes are summed, but the couplings of higher modes are suppressed by a finite width of the brane. Then the result is the same as the KK regularization, that is, the Higgs mass correction is finite. However, it is pointed out also that there are other distributions leading to a linear sensitivity with the momentum cut-off Λ. Also the Pauli-Villars regularization is discussed in Ref. [11].

In this letter, we consider the grounds for finiteness of the KK regularization. We calculate explicitly the correction to the Higgs mass due to the KK modes by performing both of the momentum cut-off and the KK level truncation. Our purpose is to show in which case the correction becomes insensitive to details of physics in the ultraviolet (UV) scale. We discuss by using the proper time regularization for the UV divergence. Because the proper time regularization does not spoil four-dimensional symmetries. Also the proper time regularization is in a sense smooth compared with the sharp momentum cut-off. Actually, the proper time regularization was used to derive the power-law behavior of gauge couplings due to KK modes [14, 15, 16]. It was shown in Ref. [16] that transition to the power-law behavior at the compactification scale appears smoothly in the proper time regularization compared with other regularization schemes, although qualitatively same results for the power-law behavior are obtained in different regularization schemes. Hence, we use here the proper time regularization as a first trial. Then it will be shown that the finite result is obtained as long as ℓ ≫ ΛR for the compactification radius R. This is seen also even when the sharp momentum cut-off is used [8].

Suppose that we have the following bosonic and fermionic KK modes,

\[ m_{b(n)} = \sqrt{\pi} \frac{n + q}{R}, \quad m_{f(n)} = \sqrt{\pi} \frac{n}{R}, \]

with \( n = 0, \pm 1, \pm 2, \cdots \), where we have normalized the compactification radius \( R \) with the factor \( \sqrt{\pi} \) for convenience of the later calculations. For simplicity, we have assumed that

\[ 1 \text{ See also Ref. [12, 13].} \]
only the bosonic modes have SUSY breaking masses. However, the following discussions are applicable for more generic case. On top of that, their coupling to the (zero-mode) Higgs field is denoted by \( g \), which is assumed to be universal between bosonic and fermionic modes, and lower and higher KK modes.

In this setup, the fermionic contribution to the Higgs mass is proportional to the following integral,

\[
g^2 \sum_{n=-\infty}^{\infty} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + \pi n^2/R^2}.
\]

(2)

Thus, how to calculate this and the corresponding bosonic integral is the point to be investigated. Here we use the Schwinger representation, that is, we use the following identities:

\[
\int_0^\infty dt e^{-A t} = \frac{1}{A}, \quad \int \frac{d^4 p}{(2\pi)^4} e^{-t p^2} = \frac{1}{16\pi^2 t^2}.
\]

(3)

Then the above integral (2) is written as

\[
g^2 \frac{16\pi^2}{\ell} \sum_{n=-\infty}^{\infty} \int_0^\infty dt \frac{1}{t^2} e^{-\pi n^2/R^2}.
\]

(4)

Here we truncate the KK-modes at the level \( \ell \) and put the cut-off \( 1/\Lambda^2 \) for the proper time integral. Then what should be evaluated is the following integral

\[
I_f = \frac{g^2}{16\pi^2} \sum_{n=-\ell}^{\ell} \int_0^\infty dt \frac{1}{t^2} e^{-\pi n^2/R^2}.
\]

(5)

Now, both of the summation and integral are finite, and we can exchange safely the summation and integration,

\[
I_f = \frac{g^2}{16\pi^2} \int_1/\Lambda^2 dt \sum_{n=-\ell}^{\ell} \frac{1}{t^2} e^{-\pi n^2/R^2}.
\]

(6)

Eq. (5) includes the suppression factor \( e^{-\pi n^2/R^2} \) for higher KK modes \( n \gg \Lambda R/\sqrt{\pi} \). Thus, naively thinking, we can replace the finite summation by the infinite summation,

\[
\int_1/\Lambda^2 dt \sum_{n=-\ell}^{\ell} \frac{1}{t^2} e^{-\pi n^2/R^2} \rightarrow \int_1/\Lambda^2 dt \sum_{n=-\infty}^{\infty} \frac{1}{t^2} e^{-\pi n^2/R^2}.
\]

(7)

We shall discuss implication of this replacement later. If the replacement (7) is allowed, the calculation is rather simple. We use the Poisson summation formula, i.e. the modular transformation property of the \( \theta_3(iA) \) function,

\[
\theta_3(iA) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi n^2 A} = \frac{1}{\sqrt{A}} \sum_{m=-\infty}^{\infty} e^{-\pi A^{-1} m^2} = \frac{1}{\sqrt{A}} \theta_3(i/A),
\]

(8)

\[2\] The replacement similar to Eq. (7) and the Poisson summation formulæ are used in order to derive the power-law behavior of gauge couplings due to KK modes [14, 15, 16].
such that we obtain

$$I_f = \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{m=-\infty}^{\infty} \frac{R}{t^2} e^{-\pi m^2 R^2 / t},$$  \quad (9)

$$= \frac{g^2}{24\pi^2} R \Lambda^3 + \frac{g^2}{8\pi \sqrt{2}} C \zeta(3) \quad (10)$$

where $C \equiv \int_{1/\Lambda^2}^{\infty} dy \sqrt{y} e^{-y}$, and that is finite. In particular, we have $C = \Gamma(3/2)$ in the limit $\Lambda^2 \to \infty$. Thus we have the $\Lambda^3$ divergence for the fermionic modes only.

Similarly, we can calculate the contribution due to bosonic modes with the masses $(m_{b(n)})^2 = \pi (n + a)^2 / R^2$. The divergent term is exactly the same as Eq. (10), although the finite part is different. Thus, if the replacement (7) is allowed, we can deduce that the Higgs mass correction due to bosonic and fermionic KK modes with the mass spectrum (4) is finite at the one-loop level of perturbative calculations.

In the above calculations, we have used the replacement (7) of the finite number of summation by the infinite number of summation. Such replacement by the infinite number of summation may be doubtable from the viewpoint of the sharp truncation. Here we discuss more about the replacement (7). First of all, each fermionic (bosonic) KK mode would have a quadratic divergent correction $\Lambda^2$ to the Higgs mass. If we have $(2k+1)$ KK modes contributing to the Higgs mass corrections, we have totally the corrections of the order of $(2k+1)\Lambda^2$. We compare this naive estimation with the result (10), whose divergence part is proportional to $\Lambda^3$. That implies that in the proper time regularization and the replacement (7) the KK modes higher than $k \approx \Lambda R$ are decoupled effectively, but not by the explicitly sharp truncation, that is, the infinite summation seems not essential to obtain the finite result, but the summation over $\ell \gg k \approx \Lambda R$ is enough. This result is also in agreement with the sharp cutoff case examined in Ref. [8].

Next we examine the replacement (7). For concreteness, we use the case with $a = 1/2$. For the moment, we require two points that 1) the positive and negative modes be treated on an equal footing and 2) the involved number of bosonic KK modes be the same as one of the corresponding fermionic modes in Eq. (5). Hence, the integral corresponding to $I_f$ (7) is obtained

$$I_b = \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{n=-\ell}^{\ell} \frac{1}{2t^2} \left[ e^{-\pi t(n+1/2)^2 / R^2} + e^{-\pi t(n-1/2)^2 / R^2} \right].$$  \quad (11)

Now we estimate what we have added in the replacement (7). We have added the following contribution for the fermionic modes:

$$I'_f = \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{n=\ell+1}^{\infty} \frac{2}{t^2} e^{-\pi t n^2 / R^2}.$$  \quad (12)

In a similar replacement of bosonic contribution, we have added the following contribution:

$$I'_b = \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{n=\ell+1}^{\infty} \frac{1}{t^2} \left[ e^{-\pi t(n+1/2)^2 / R^2} + e^{-\pi t(n-1/2)^2 / R^2} \right].$$  \quad (13)
Then the difference $I_b' - I_f'$ is written

$$I_b' - I_f' = \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{n=\ell+1}^{\infty} \frac{1}{t^2} \left[ e^{-\pi t(n+1/2)^2/R^2} + e^{-\pi t(n-1/2)^2/R^2} - 2e^{-\pi tn^2/R^2} \right]$$

$$= \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{n=\ell+1}^{\infty} \frac{1}{t^2} e^{-\pi tn^2/R^2} \left[ e^{-\pi t(n+1/4)/R^2} + e^{\pi t(n-1/4)/R^2} - 2 \right].$$  \hspace{1cm} (14)

This difference $I_b' - I_f'$ contributes to the Higgs mass if $\ell$ is finite. The terms in the bracket of Eq. (14) can be expanded

$$\left[ 1 + 1 - 2 - \frac{\pi}{R^2}(n + \frac{1}{4})t + \frac{\pi}{R^2}(n - \frac{1}{4})t + O(t^2) \right].$$  \hspace{1cm} (15)

The terms corresponding to the quadratic divergence $\Lambda^2$ seem to be cancelled, but the terms corresponding to the logarithmic divergence $\log \Lambda^2$ are not cancelled. However, note that there is the suppression factor $e^{-\pi tn^2/R^2}$ in Eq. (14) with $n \geq \ell+1$ and $t \geq 1/\Lambda^2$. Thus, if $\ell$ is large enough, such logarithmic divergence $\log \Lambda^2$ is decoupled. That is consistent with the previous result.

On the other hand, around $\ell \approx RA$, such suppression factor does not work. Then, we have the logarithmic divergence $\log \Lambda^2$ for each mode. The divergence is enhanced by the number of relevant KK modes. Thus, again, whether we have the divergence or not, depends on if we allow to take $\ell$ enough large. We need not the infinite number for $\ell$.

The splitting of the modes with the masses $n + 1/2$ and $n - 1/2$ in Eq. (11) might look artificial. The reason for such representation is that we cared the edge modes with $n = \pm(\ell + 1/2)$. However, now it is obvious that such higher modes have no contribution for sufficiently large $\ell$ because of the suppression factor $e^{-\pi \ell^2/(\Lambda R)^2}$. Actually, the integral is written

$$I_b = \frac{g^2}{16\pi^2} \int_{1/\Lambda^2}^{\infty} dt \sum_{n=\ell}^{\ell} \frac{1}{t^2} e^{-\pi t(n+1/2)^2/R^2},$$  \hspace{1cm} (16)

and the result is same, that is, we have the finite result for large $\ell \gg \Lambda R$. Even in the case that we differ the levels for truncation between the fermionic and bosonic modes, say $\ell_f$ and $\ell_b$, we have the finite result for sufficiently large $\ell_f, \ell_b \gg \Lambda R$.

So far we have not been concerned about running of the gauge coupling (and Yukawa couplings, if any). In extra dimensions the scale dependence is given by power law. We can incorporate this running coupling into the Higgs mass corrections by examining the renormalization group (RG) equations for $I_b$ (11) and $I_f$ (6). Indeed the finiteness is clearly seen in the RG point of view. Here we consider the RG equation in the Wilson sense, which are given by

$$\Lambda \frac{\partial I_f}{\partial \Lambda} = \frac{g^2(\Lambda)}{16\pi^2} \Lambda^2 \sum_{n=-\ell}^{\ell} e^{-\pi n^2/(\Lambda R)^2} \equiv \frac{g^2(\Lambda)}{16\pi^2} \Lambda^2 \frac{1}{2} \varepsilon_f(\Lambda R),$$

$$\Lambda \frac{\partial I_b}{\partial \Lambda} = \frac{g^2(\Lambda)}{16\pi^2} \Lambda^2 \sum_{n=-\ell}^{\ell} \frac{1}{2} \left[ e^{-\pi (n+a)^2/(\Lambda R)^2} + e^{-\pi (n-a)^2/(\Lambda R)^2} \right] \equiv \frac{g^2(\Lambda)}{16\pi^2} \Lambda^2 \frac{1}{2} \varepsilon_b(\Lambda R),$$  \hspace{1cm} (17)

with $a = 1/2$. Actually it is not necessary to give the beta function for the gauge coupling in order to see the finiteness of the correction. What concerns us is $\varepsilon_b(\Lambda R) - \varepsilon_f(\Lambda R)$. If
For example we take \( \ell = 10. \) Fig. 1 shows \( \varepsilon_{b}(x) \) and \( \varepsilon_{f}(x) \). Behaviors of the two curves are almost same for \( x > 0.5 \). In the region with \( x < 10 \), i.e. \( R\Lambda < \ell \), the curve of \( \varepsilon_{b,f}(x) \) behaves linearly. This behavior is consistent with the previous result (10) that the divergence behaves like \( \Lambda^{3} \) for \( R\Lambda \ll \ell \). For large \( x \), \( \varepsilon_{b,f}(x) \) goes close to the constant \( 2(2\ell+1) \).

Fig. 2 shows the difference \( \varepsilon_{f}(R\Lambda) - \varepsilon_{b}(R\Lambda) \). The difference damps rapidly above \( x \sim 1 \). That implies that for \( 1 \leq R\Lambda < \ell \), no correction arises to the Higgs mass. Therefore, as long as we keep the KK level truncation as \( R\Lambda \ll \ell \), the correction is finite and of \( O(1/R^{2}) \). It was mentioned above that we need large \( \ell \gg \Lambda R \) in order to obtain the finite result. This numerical calculation shows how large \( \ell \) is necessary, and \( \Lambda R/\ell < 0.9 \) seems sufficient, although the explicit value 0.9 has no serious meaning. That suggests a huge gap between \( \ell \) and \( \Lambda R \) is not required. There is a small lump starting from \( x \approx 10 \), i.e. \( \Lambda R \approx \ell \). That corresponds to the sum of logarithmic divergences, which is mentioned in the level-by-level calculation around Eq. (15), and that is also consistent with the calculation in Ref. [8] showing the presence of divergence, where the level truncation is taken around \( \ell \approx \Lambda R \). The difference \( \varepsilon_{f}(R\Lambda) - \varepsilon_{b}(R\Lambda) \) behaves as \( x^{-2} \) at \( x \gg \ell \). This implies that the corrections depend on the UV cut-off \( \Lambda \) logarithmically (with constant \( g \)), when we truncate the KK modes at much lower scale than \( \Lambda \).

We have calculated numerically the case with \( a = 1/2 \). We can easily extend to other cases, e.g. for other values of \( a \),

\[
I_{b} = \frac{g^{2}}{16\pi^{2}} \int_{1/\Lambda^{2}}^{\infty} dt \sum_{n=-\ell}^{\ell} \frac{1}{t^{2}} e^{-\pi t(n+a)^{2}/R^{2}},
\]

(19)

\[
\Lambda \frac{\partial I_{b}}{\partial \Lambda} = \frac{g^{2}}{16\pi^{2}} \Lambda^{2} \sum_{n=-\ell}^{\ell} e^{-\pi (n+a)^{2}/(R\Lambda)^{2}} \equiv \frac{g^{2}}{16\pi^{2}} \Lambda^{2} \frac{1}{2} \varepsilon_{b}(R\Lambda).
\]

(20)

For any value of \( a \), the difference \( \varepsilon_{f}(R\Lambda) - \varepsilon_{b}(R\Lambda) \) behaves like Fig. 2 in the region with \( x < \ell \). That implies for \( \ell > \Lambda R \) the correction to the Higgs mass is finite and of
$O(1/R^2)$. Furthermore, in $I_b$ of Eq. (13) we can replace the truncated level $\ell$ by $\ell_b$, which is independent of the truncated level of fermionic modes $\ell$. Even in this case, we have the same behavior of $\epsilon_f(R\Lambda) - \epsilon_b(R\Lambda)$ in the region with $x < \ell, \ell_b$.

We have used the proper time regularization. However we can apply also other regularization schemes, e.g. the sharp momentum cut-off. Explicitly we examine the beta function

$$\Lambda \frac{\partial I_f}{\partial \Lambda} = \frac{g^2}{16\pi^2} \sum_{n=-\ell}^{\ell} \frac{\Lambda^4 R^2}{(\Lambda R)^2 + \pi n^2} \equiv \frac{g^2}{16\pi^2} \Lambda^2 \frac{1}{2} \epsilon_f(\Lambda R)$$

and the similar function for the bosonic contributions. Then the power law behavior does not depend on the regularization scheme, except for the KK threshold corrections [16]. Therefore we obtain qualitatively the same results for the finiteness of the corrections.

Also similar discussions seem useful not only for the model with the mass spectrum (1), but more generic case. For any other mass spectra, we can define the difference $\epsilon_f(R\Lambda) - \epsilon_b(R\Lambda)$ like Eqs. (17) and (18). Then it can be used as an index for the presence of quadratic divergences.

To summarize, we have calculated explicitly the Higgs mass correction due to KK modes. The momentum cut-off $\Lambda$ and the level of truncation $\ell$ have been included. We have used the proper time regularization, which decouples effectively higher KK modes $n > \Lambda R$, and that would fit the philosophy of Ref. [8] that such higher modes should be decoupled. However, the decoupling by the proper time regularization is smooth compared with the sharp cut of Ref. [8]. Our results are in order. Finiteness or the appearance of divergences does not depend on whether we put a finite truncation of the KK modes or we sum the infinite number of KK modes, but it depends on where we take the finite truncation $\ell$. If we are allowed sufficiently large truncation compared with the momentum cut-off $\ell \gg \Lambda R$, we obtain the finite result. Numerical study shows we need a large truncation $\ell$, but not a huge gap between $\ell$ and $\Lambda R$. On the other hand, if we put $\ell \approx \Lambda R$, the divergence appears. That might be rather obvious. Because truncating around the momentum cut-off means we would see how to truncate the modes. Therefore, finiteness depends if theory allows $\ell \gg \Lambda R$. For the spectrum (1), it might be artificial to assign the bosonic state of the mass $m_{b(n)} = \sqrt{n(n+q)/R}$ with the fermionic state.
of the mass \( m_{f(n)} = \sqrt{\pi n/R} \). The corresponding bosonic state might be the state with \( m_{b(n)} = \sqrt{\pi (n - 1 + q)/R} \). For example, suppose a theory, which should be irrelevant to such artificial truncation of the edge. In this case one has to take \( \ell \gg \Lambda R \), such that low energy physics becomes insensitive to how to treat such edge. The concept of locality in extra dimensions, which are discussed in Refs. \[1\]-\[5\], might forbid the truncation of KK modes around the cut-off. On the other hand, suppose a theory, where the cut-off \( \Lambda \) has a real meaning, e.g. the string scale. In this case, we have to truncate at \( \ell \approx \Lambda R \), above which new modes might appear. For such case, we have to know initial conditions at \( \Lambda \) to discuss low energy physics.

The radiative corrections in the string theory with a large radius of compactification may be well approximated by using the effective field theory with infinite tower of the KK modes \[17\]. There the corrections are represented by a kind of proper time cutoff, with which the KK modes heavier than the string scale are decoupled. Therefore the correction to the higgs mass is supposed to be insensitive to the string scale.

Finite results for the Higgs mass correction due to KK modes have been found first for models with SUSY breaking by the Scherk-Schwarz mechanism Refs. \[1\] - \[5\]. However, what we have studied is the calculations of the correction under the mass spectrum \[1\]. Whether the spectrum is obtained by the Scherk-Schwarz mechanism or other SUSY breaking mechanisms, is irrelevant to the calculations. Thus, finite results are generic for models with the mass spectrum. To repeat, the key-point is whether models allow to take large truncation \( \ell \gg \Lambda R \).

Our discussions for finiteness are valid at the one-loop level of perturbation theory. The two-loop level \[18\] and more are beyond the scope.

Acknowledgments

The authors would like to thank D.M. Ghilencea, Y. Murakami, H. Nakano and K. Yoshiioka for useful discussions.

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