Quasiparticle Hall Transport of d-wave Superconductors in Vortex State

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We present a theory of quasiparticle Hall transport in strongly type-II superconductors within their vortex state. We establish the existence of integer quantum spin Hall effect in clean unconventional $d_{x^2-y^2}$ superconductors in the vortex state from a general analysis of the Bogoliubov-de Gennes equation. The spin Hall conductivity $\sigma_{xy}^s$ is shown to be quantized in units of $\frac{e^2}{4\pi h}$. This result does not rest on linearization of the BdG equations around Dirac nodes and therefore includes inter-nodal physics in its entirety. In addition, this result holds for a generic inversion-symmetric lattice of vortices as long as the magnetic field $B$ satisfies $H_{c1} \ll B \ll H_{c2}$. We then derive the Wiedemann-Franz law for the spin and thermal Hall conductivity in the vortex state. In the limit of $T \to 0$, the thermal Hall conductivity satisfies $\kappa_{xy} = \frac{8\pi^2}{3\hbar^2} T \sigma_{xy}^s$. The transitions between different quantized values of $\sigma_{xy}^s$ as well as relation to conventional superconductors are discussed.

I. INTRODUCTION

One of the fundamental characteristics of high temperature superconductors (HTS) is the apparent applicability of the d-wave BCS based phenomenology to the broad range of quasiparticle properties in the superconducting state $\mathbb{C}$. This is far from trivial property for materials known to exhibit strong electron correlations. Although the horizon is still not entirely clear and there remain few unresolved issues, examples being the temperature dependence of quasiparticle lifetimes or the penetration depth in the under-doped regime, $\mathbb{C}$ the cumulative weight of evidence indicates that the low energy properties of cuprate superconductors are indeed governed by nodal quasiparticles with Dirac like dispersion, as seen in assorted spectroscopic $\mathbb{C}$ and transport measurements $\mathbb{C}$. These and other experiments serve as the foundation for the “BCS-like d-wave paradigm” as it is currently used in both theory and interpretation of experiments.

A natural question is how does this picture hold together in a mixed phase, in presence of an external magnetic field and an array of superconducting vortices and, if it does, are there some special features of the d-wave quasiparticle phenomenology which could be used to deepen our understanding of high temperature superconductivity. Recent activity on the experimental front appears most encouraging in this regard. In particular, measurements of the thermal Hall conductivity $\kappa_{xy}$ in cuprate superconductors are currently viewed as especially informative probes of quasiparticle dynamics. These measurements provide a clean way of extracting quasiparticle contribution to $\kappa_{xy}$ as phonons, the other source of significant thermal conduction, do not couple to the magnetic field by virtue of being neutral. In addition, contrary to what takes place in an ordinary electrical Hall conductivity experiment, vortices do not experience strong Lorenz force since there is only heat current and no net electrical supercurrent to which vortices are strongly coupled. Thus, vortices tend to remain stationary and their transport does not serve as a significant channel for heat conduction.

Recently, measurements of $\kappa_{xy}$ were conducted by Ong’s group $\mathbb{C}$ on YBCO samples with a very long mean free path. These experiments were carried out over a wide range of magnetic fields (up to $\sim 14T$) and at temperatures from $T \sim 12.5 K$ to above the superconducting transition $T_c \sim 90 K$. Unfortunately, the experiments are resolution limited below 12.5 K as signal becomes too weak. At temperatures up to 25K and for a (T dependent) range of magnetic fields $\sqrt{H/Tesla} < T/25 K$ the experiments seem to suggest a rather simple scaling form $\mathbb{C}$ for $\kappa_{xy}$:

$$\kappa_{xy}(H, T) = \text{const.} \times \sqrt{H T}. \quad (1.1)$$

On the theoretical front, the initial interest, largely inspired by Gorkov and Schrieffer $\mathbb{C}$ and, in a somewhat different context, by Anderson $\mathbb{C}$, was directed at the formation of “Dirac Landau levels” and their signatures in the quasiparticle thermodynamics and transport $\mathbb{C}$. This picture of Dirac Landau level quantization, while theoretically elegant and appealing, relies on the minimal coupling of the nodal BCS-like quasiparticles to the electromagnetic vector potential, $A$. The assumption of such minimal coupling seems innocent but it is not, on fundamental grounds. The physics behind the interaction of nodal quasiparticles with the external magnetic field and vortices was elucidated by Franz and Tešanović $\mathbb{C}$. These authors devised a singular gauge transformation which allows one to recast the original problem of BCS-like quasiparticles, moving under the combined influence of an external magnetic field and a superflow arising from vortex array, into that of Bloch particles moving in an effective non-uniform and periodic magnetic field, the spatial average of which equals zero. This approach clearly demonstrates that the low energy por-
tion of the quasiparticle spectrum can be described as that of a relativistic Dirac particle minimally coupled to a fictitious $U(1)$ gauge potential, i.e. the “Berry” gauge field $a$, which supplies the needed $\pm \pi$ winding in the quantum mechanical phase of a quasiparticle as it encircles a vortex. Such half-flux Aharonov-Bohm scattering arises entirely through interaction of quasiparticles with vortices and it does not involve the external magnetic field explicitly. Thus, the cyclotron motion in a Dirac cone is caused exclusively by a time-reversal invariant “Berry” gauge field and cannot lead to any Dirac Landau level quantization. Further progress came through the work of Marinelli, Halperin and Simon [13] who analyzed the quasiparticle excitation spectra of different vortex lattices and provided analytic symmetry arguments regarding the presence of nodal (zero energy) points in such spectra. These authors also devised a perturbation theory in the vicinity of nodal points which can be used to derive various results by analytic means. Various analytic results were also derived in the large anisotropy limit \cite{14,15}. Finally, the intricacies of Dirac equation in presence of half-flux Bohm-Aharonov scattering were addressed in Ref. \cite{16}, where the tight-binding regularization was introduced to restore the exact singular gauge symmetry in numerical calculations using the low energy (linearized) theory. Such regularization describes a lattice $d$-wave superconductor and is a natural choice for cuprates: after all, most of the microscopic theories of HTS start from tight-binding effective Hamiltonians. We should note that similar results for the quasiparticle spectra were also found in fully self-consistent calculations on the original Bogoliubov-deGennes (BdG) equations, using the basis formed by eigenstates of the magnetic translation group \cite{17,18}.

In this paper, we present a detailed analytical study of the quasiparticle Hall transport in a vortex state complemented by explicit numerical calculations \cite{19,20}. We consider a lattice $d$-wave superconductor of Ref. \cite{16} – This is important and necessary since the straightforward linearization of BdG equations drops curvature terms and results in $\kappa_{xy} = 0$ \cite{21,22}. We employ the Franz and Tešanović (FT) transformation so that we can use the familiar Bloch representation of the translation group in which the overall chirality of the problem vanishes. This should be contrasted with the original problem where the overall chirality is finite and the magnetic translation group states must be used instead. Naively, it might appear that after an FT singular gauge transformation the effects of the magnetic field have somehow been transformed away since the new problem is found to have zero average effective magnetic field. Of course, this is not true. The presence of magnetic field in the original problem reveals itself fully in the FT transformed quasiparticle wavefunctions. Alternatively, there is an “intrinsic” chirality imposed on the system which cannot be transformed away by a choice of the basis. One manifestation of this chirality is the Hall effect. The utility of the singular gauge transformation in the calculation of the electrical Hall conductivity in the normal $2D$ electron gas in a (non-uniform) magnetic field was realized by Nielsen and Hedegård \cite{23}. They demonstrated that using singular gauge transformed wavefunctions one still obtains the correct result, giving the electrical Hall conductance quantized in units of $e^2/h$ if the chemical potential lies in the energy gap. In a superconductor, the question of Hall response becomes rather interesting as there is a strong mixing between particles and holes. Evidently, the electrical Hall response is very different from the normal state, since charge is not conserved in the state with broken $U(1)$ symmetry. Therefore, as pointed out in Ref. \cite{24}, charge cannot be transported by diffusion. On the other hand, the spin is still a good quantum number \cite{24} and it is natural to ask what is the spin Hall conductivity in the vortex state of an extreme type-II superconductor \cite{25}. Moreover, every channel of spin conduction simultaneously transports entropy \cite{26,27} and we would expect some variation on Wiedemann-Franz law to hold between spin and thermal conductivity.

As one of our main results, we derive the Wiedemann-Franz law connecting the spin and thermal Hall transport in the vortex state of a $d$-wave superconductor. In the process, we show that the spin Hall conductivity, $\sigma_{xy}^s$, just like the electrical Hall conductivity of a normal state in a magnetic field, is topological in nature and can be explicitly evaluated as a first Chern number characterizing the eigenstates of our singularly gauge transformed problem \cite{28,29}. Consequently, as $T \to 0$, the spin Hall conductivity is quantized in the units of $\hbar/8\pi$ when the energy spectrum is gapped, which, combined with the Wiedemann-Franz law, implies the quantization of $\kappa_{xy}/T$. We then explicitly compute the quantized values of $\sigma_{xy}^s$ for a sequence of gapped states using our lattice $d$-wave superconductor model in the case of an inversion-symmetric vortex lattice. Within this model one is naturally led to consider the $BCS$-Hofstadter problem: the BCS pairing problem defined on a uniformly frustrated tight-binding lattice. We find a sequence of plateau transitions, separating gapped states characterized by different quantized values of $\sigma_{xy}^s$. At a plateau transition, level crossings take place and $\sigma_{xy}^s$ changes by an even integer \cite{27}. Both the origin of the gaps in quasiparticle spectra and the sequence of values for $\sigma_{xy}^s$ are rather different than in the normal state, i.e. in the standard Hofstadter problem \cite{28}. In a superconductor, the gaps are strongly affected by the pairing and the interactions of quasiparticles with a vortex array. The sequence of $\sigma_{xy}^s$ changes as a function of the pairing strength (and therefore interactions), measured by the maximum value of the gap function $\Delta$ \cite{28}. Finally, we discuss the relation of our results to those obtained within the continuum low energy (linearized) theory.
II. BOGOLIUBOV-DEGENNES HAMILTONIAN

The experimental evidence points towards well defined d-wave quasiparticles in cuprate superconductors in the absence of the external magnetic field. This suggests that to zeroth order fluctuations can be ignored and that one can think in terms of an effective BCS Hamiltonian, the simplest of which is written on the 2-D tight-binding lattice with the nearest neighbor interaction thus naturally implementing $d_{x^2-y^2}$ pairing. In question is then the response of such a superconductor to an externally applied magnetic field $B$. All high temperature superconductors are extreme type-II forming a vortex state in a wide range of magnetic fields. This immediately sets up the contrast between $B = 0$ and $B \neq 0$ situations: first, the problem is not spatially uniform and therefore momentum is not a good quantum number and second, the array of $\frac{hc}{2e}$ vortex fluxes poses topological constraint on the quasi-particles encircling the vortices. Therefore, despite ignoring any fluctuations, the problem is far from trivial and demands careful examination.

The natural starting point is therefore the mean-field BCS Hamiltonian written in second quantized form [3]:

$$H = \int dx \psi^\dagger(x) \left( \frac{1}{2m^*}(p - \frac{e}{c}A)^2 - \mu \right) \psi(x) + \int dx \int dy \Delta(x,y) \psi^\dagger(x) \psi^\dagger(y) + \Delta^*(x,y) \psi_1(y) \psi_1(x)$$  \hspace{1cm} (2.1)

where $A(x)$ is the vector potential associated with the uniform external magnetic field $B$, single electron energy is measured relative to the chemical potential $\mu$, $\psi_\alpha(x)$ is the fermion field operator with spin index $\alpha$, and $\Delta(x,y)$ is the pairing field. For convenience we will define an integral operator $\hat{\Delta}$ such that:

$$\hat{\Delta}\psi(x) = \int dy \Delta(x,y) \psi(y).$$  \hspace{1cm} (2.2)

In the strictest sense, on the mean field level this problem must be solved self-consistently which renders any analytical solution virtually intractable. On the other hand, in the case at hand the vortex lattice is dilute for a wide range of magnetic fields, and by the very nature of cuprate superconductors having short coherence length, the size of the vortex core can be ignored relative to the distance between the vortices. Thus, to the first approximation, all essential physics is captured by fixing the amplitude of the order parameter $\Delta$ while allowing vortex defects in its phase. Moreover, on a tight-binding lattice the vortex flux is concentrated inside the plaquette and thus the length-scale associated with the core is implicitly the lattice spacing $\delta$ of the underlying tight-binding lattice. As shown in Ref. [10], under these approximations the d-wave pairing operator in the vortex state can be written as a differential operator:

$$\Delta = \Delta_0 \sum_\delta \eta_\delta e^{i\phi(x)/2} e^{i\delta p} e^{i\phi(x)/2}. \hspace{1cm} (2.3)$$

The sums are over nearest neighbors and on the square tight-binding lattice $\delta = \pm x, \pm y$; the vortex phase fields satisfy $\nabla \times \nabla \phi(x) = 2\pi \sum_\delta \delta(x-x_\delta)$ with $x_\delta$ denoting the vortex positions and $\delta(x-x_\delta)$ being a 2D Dirac delta function; $p$ is a momentum operator, and

$$\eta_\delta = \begin{cases} 1 & \text{if } \delta = \pm x \\ -1 & \text{if } \delta = \pm y \end{cases}. \hspace{1cm} (2.4)$$

The operator $\eta_\delta$ follows from the d-wave pairing: $\Delta = 2\Delta_0[\cos(k_\delta x) - \cos(k_\delta y)]$. For notational convenience we will use units where $\hbar = 1$ and return to the conventional units when necessary.

It is straightforward to derive the continuum version of the tight binding lattice operator $\Delta$ (see Ref. [10]):

$$\hat{\Delta} = \frac{1}{2p_F^2}\{\partial_x, \{\partial_x, \Delta(x)\}\} - \frac{1}{2p_F^2}\{\partial_y, \{\partial_y, \Delta(x)\}\} + \frac{i}{8p_F^2}\Delta(x)[(\partial^2_x \phi) - (\partial^2_y \phi)], \hspace{1cm} (2.5)$$

but for convenience we will keep the lattice definition throughout. One can always define continuum as an appropriate limit of the tight-binding lattice theory. With the above definitions, the Hamiltonian (2.1) can now be written in the Nambu formalism as

$$H = \int dx \Psi^\dagger(x) \hat{H}_0 \Psi(x) \hspace{1cm} (2.6)$$

where the Nambu spinor $\Psi^\dagger = (\psi_1^\dagger, \psi_1)$ and the matrix differential operator

$$\hat{H}_0 = \begin{pmatrix} \hbar & \Delta \\ \Delta^* & -\hbar \end{pmatrix}. \hspace{1cm} (2.7)$$

In the continuum formulation $\hat{\hbar} = \frac{1}{2m^*}(p - \frac{e}{c}A)^2 - \mu$, while on the tight-binding lattice:

$$\hat{\hbar} = -t \sum_\delta e^{i\delta x} e^{i\delta y} (p - \frac{e}{c}A)^2 - \mu. \hspace{1cm} (2.8)$$

$t$ is the hopping constant and $\mu$ is the Fermi energy. The equations of motion of the Nambu fields $\Psi$ are then:

$$i\hbar \dot{\Psi} = [\Psi, H] = \hat{H}_0 \Psi. \hspace{1cm} (2.9)$$

Note, that the Hamiltonian in Eq. (2.1) is our starting unperturbed Hamiltonian. In order to compute the linear response to externally applied perturbations we will have to add terms to Eq. (2.3). In particular, we will consider two types of perturbations in the later sections: First, partly for theoretical convenience, we will consider a weak gradient of magnetic field ($\nabla B$) on top of the uniform $B$ already taken into account fully by Eq. (2.1).
The $\nabla B$ term induces spin current in the superconductor \[11\]. The response is then characterized by spin conductivity tensor $\sigma^s$ which in general has non-zero off-diagonal components. Second, we consider perturbing the system by pseudo-gravitational field, which formally induces flow of energy (see \[12\, 15\]) and allows us to compute thermal conductivity $\kappa_{xy}$ via linear response. The advantage of these formal considerations are made clear in Section IV.

A. Particle-Hole Symmetry

The equations of motion (2.9) for stationary states lead to Bogoliubov-de Gennes equations \[10\]

$$\hat{H}_0 \Phi_n = \epsilon_n \Phi_n. \quad (2.10)$$

The solution of these coupled differential equations are quasi-particle wavefunctions that are rank two spinors in the Nambu space, $\Phi^T(x) = (u(x), v(x))$. The single particle excitations of the system are completely specified once the quasi-particle wavefunctions are given, and as discussed later, transverse transport coefficients can be computed solely on the basis of $\Phi$'s. It is a general symmetry of the BdG equations that if $(u_n(x), v_n(x))$ is a solution with energy $\epsilon_n$, then there is always another solution $(-v_n^*(x), u_n^*(x))$ with energy $-\epsilon_n$ (see for example Ref. \[30\]).

In addition, on the tight-binding lattice, if the chemical potential $\mu = 0$ in the above BdG Hamiltonian \[27\], then there is a particle-hole symmetry in the following sense: if $(u_n(x), v_n(x))$ is a solution with energy $\epsilon_n$, then there is always another solution $e^{i \pi (r_x + r_y)}(u_n(x), v_n(x))$ with energy $-\epsilon_n$. Thus we can choose:

$$\begin{pmatrix} u_n^-(x) \\ v_n^-(x) \end{pmatrix} = e^{i \pi (r_x + r_y)} \begin{pmatrix} u_n^+(x) \\ v_n^+(x) \end{pmatrix}, \quad (2.11)$$

where $+$ $(-)$ corresponds to a solution with positive (negative) energy eigenvalue. We will refer to this as particle hole transformation $P_H$.

B. Franz-Tešanović Transformation and Translation Symmetry

In order to elucidate another important symmetry of the Hamiltonian \[2.7\], we follow FT \[12\, 16\] and perform a “bipartite” singular gauge transformation on the Bogoliubov-de Gennes Hamiltonian \[2.10\],

$$\hat{H}_0 \rightarrow U^{-1} \hat{H}_0 U, \quad U = \begin{pmatrix} e^{i \phi_c(x)} & 0 \\ 0 & e^{-i \phi_h(x)} \end{pmatrix}, \quad (2.12)$$

where $\phi_c(x)$ and $\phi_h(x)$ are two auxiliary vortex phase functions satisfying

$$\phi_c(x) + \phi_h(x) = \phi(x). \quad (2.13)$$

This transformation eliminates the phase of the order parameter from the pairing term of the Hamiltonian. The phase fields $\phi_c(x)$ and $\phi_h(x)$ can be chosen in a way that avoids multiple valuedness of the wavefunctions. The way to accomplish this is to assign the singular part of the phase field generated by any given vortex to either $\phi_c(x)$ or $\phi_h(x)$, but not both. Physically, a vortex assigned to $\phi_c(x)$ will be seen by electrons and be invisible to holes, while vortex assigned to $\phi_h(x)$ will be seen by holes and be invisible to electrons. For periodic Abrikosov vortex array, we implement the above transformation by dividing vortices into two groups $A$ and $B$, positioned at $\{ r^A_i \}$ and $\{ r^B_i \}$ respectively (see Fig. 1). We then define two phase fields $\phi^A(x)$ and $\phi^B(x)$ such that

$$\nabla \times \nabla \phi^\alpha(x) = 2\pi \sum_i \delta(x - r^\alpha_i), \quad \alpha = A, B, \quad (2.14)$$

and identify $\phi_c = \phi^A$ and $\phi_h = \phi^B$. On the tight-binding lattice the transformed Hamiltonian becomes

$$\hat{H}_N = \sum_\delta \left\{ \sigma_3 (-t e^{i r^\alpha \cdot \delta} \eta \sigma_1 - \sigma_1 \eta \sigma_3) d^\dagger e^{i \delta \cdot p} - \mu \right\} + \sigma_1 \Delta_0 \eta \delta e^{i r^\alpha \cdot \delta} d^\dagger e^{i \delta \cdot p} \quad (2.15)$$

where

$$\eta = \frac{1}{2} \nabla \phi - \frac{e}{c} A; \quad \sigma_1 A = \frac{1}{2} (\nabla \phi^A - \nabla \phi^B), \quad (2.16)$$

$\sigma_1$ and $\sigma_3$ are Pauli matrices, and the sum is again over the nearest neighbors. Note that the integrand of Eq. (2.17) is proportional to the superfluid velocities

$$v^\alpha_s = \frac{1}{m^*} (\nabla \phi^\alpha - \frac{e}{c} A), \quad \alpha = A, B. \quad (2.17)$$
and is therefore explicitly gauge invariant as are the off-diagonal pairing terms.

From the perspective of quasiparticles $\mathbf{v}_A$ and $\mathbf{v}_B$ can be thought of as effective vector potentials acting on electrons and holes respectively. Corresponding effective magnetic field seen by the quasiparticles is $\mathbf{B}_\text{eff} = -\frac{\alpha}{\hbar c} \delta(\mathbf{r} - \mathbf{r}_n)$, and can be expressed using Eqs. (2.14) and (2.13) as

$$\mathbf{B}_\text{eff} = \mathbf{B} - \phi_0 \hat{z} \sum_i \delta(\mathbf{r} - \mathbf{r}_n), \quad \alpha = A, B,$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ is the physical magnetic field and $\phi_0 = \hbar c/e$ is the flux quantum. We observe that quasi-electrons and quasi-holes propagate in the effective field which consists of (almost) uniform physical magnetic field $\mathbf{B}$ and an array of opposing delta function “spikes” of unit fluxes associated with vortex singularities. The latter are different for electrons and holes. As discussed in [13, 14] this choice guarantees that the effective magnetic field vanishes on average, i.e. $\langle \mathbf{B}_\text{eff} \rangle = 0$ since we have precisely one flux spike (of $A$ and $B$ type) per flux quantum of the physical magnetic field. Flux quantization guarantees that the right hand side of Eq. (2.18) vanishes when averaged over a vortex lattice unit cell containing two physical vortices. It also implies that there must be equal numbers of $A$ and $B$ vortices in the system.

The essential advantage of the choice with vanishing $\langle \mathbf{B}_\text{eff} \rangle$ is that $\mathbf{v}_A$ and $\mathbf{v}_B$ can be chosen periodic in space with periodicity of the magnetic unit cell containing an integer number of electronic flux quanta $\hbar c/e$. Notice that vector potential of a field that does not vanish on average can never be periodic in space. Condition $\langle \mathbf{B}_\text{eff} \rangle = 0$ is therefore crucial in this respect. The singular gauge transformation (2.12) thus maps the original Hamiltonian of fermionic quasiparticles in finite magnetic field onto a new Hamiltonian which is formally in zero average field and has only “neutralized” singular phase windings in the off-diagonal components.

The resulting new Hamiltonian now commutes with translations spanned by the magnetic unit cell i.e.

$$[\hat{T}_R, \hat{H}_N] = 0,$$

where the translation operator $\hat{T}_R = \exp(i\mathbf{R} \cdot \mathbf{p})$. We can therefore label eigenstates with a “vortex” crystal momentum quantum number $\mathbf{k}$ and use the familiar Bloch states as the natural basis for the eigenproblem. In particular we seek the eigensolution of the BdG equation $\hat{H}_N \psi = \epsilon \psi$ in the Bloch form

$$\psi_{nk}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \Phi_{nk}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \left( \frac{U_{nk}(\mathbf{r})}{V_{nk}(\mathbf{r})} \right),$$

where $(U_{nk}, V_{nk})$ are periodic on the corresponding unit cell, $n$ is a band index and $\mathbf{k}$ is a crystal wave vector. Bloch wavefunction $\Phi_{nk}(\mathbf{r})$ satisfies the “off-diagonal” Bloch equation $\hat{H}(\mathbf{k}) \Phi_{nk} = \epsilon_{nk} \Phi_{nk}$ with the Hamiltonian of the form

$$\hat{H}(\mathbf{k}) = e^{-i\mathbf{k} \cdot \mathbf{r}} \hat{H}_N e^{i\mathbf{k} \cdot \mathbf{r}}.$$  

Note, that the dependence on $\mathbf{k}$, which is bounded to lie in the first Brillouin zone, is continuous. This will become important when topological properties of spin transport are discussed in the Section III C.

### III. SPIN CONDUCTIVITY

Within the framework of linear-response theory [36], spin dc conductivity can be related to the spin current-current retarded correlation function $D^{R}_{\mu\nu}$ through :

$$\sigma_{\mu\nu}^s = \lim_{\Omega \to 0} \lim_{q_1, q_2 \to 0} -\frac{1}{i\Omega} \left( D^{R}_{\mu\nu}(q_1, q_2, \Omega) - D^{R}_{\mu\nu}(q_1, q_2, 0) \right).$$

The retarded correlation function $D^{R}_{\mu\nu}(\Omega)$ can in turn be related to the the Matsubara finite temperature correlation function

$$D_{\mu\nu}(i\Omega) = -\int_0^\beta e^{i\tau_{\Omega} (T_\tau j^s_{\mu\nu}(\tau) j^s_{\mu\nu}(0))} d\tau$$

as

$$\lim_{q_1, q_2 \to 0} D^{R}_{\mu\nu}(q_1, q_2, \Omega) = D_{\mu\nu}(i\Omega \to \Omega + i0).$$

In the Eq. 3.2 the spatial average of the spin current $j^s(\tau)$ is implicit, since we are looking for dc response of spatially inhomogeneous system. In the next section we derive the spin current and evaluate the above formulae.

#### A. Spin Current

In order to find the dc spin conductivity, we must first find the spin current. More precisely, since we are looking only for the spatial average of the spin current $j^s(\tau)$ we just need its $\mathbf{k} \to 0$ component. In direct analogy with $\mathbf{B} = 0$ situation [24], we can define the spin current by the continuity equation:

$$\dot{\rho}^s + \nabla \cdot \mathbf{j}^s = 0$$

where $\rho^s = \frac{\hbar}{2} (\psi_\uparrow \psi_\uparrow - \psi_\downarrow \psi_\downarrow)$ is the spin density projected onto z-axis. We can then use equations of motion for the spin fields (2.9) and compute the current density $\mathbf{j}^s$ from (3.4).

In the limit of $q \to 0$ the spin current can be written as (see Appendix B)

$$j_{\mu}^s = \frac{\hbar}{2} \Psi^\dagger V^s_{\mu} \Psi,$$

where the Nambu field $\Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger)$ and the generalized velocity matrix operator $V^s_{\mu}$ satisfies the following commutator identity

$$V^s_{\mu} = \frac{1}{i\hbar} [x_{\mu}, \hat{H}_0].$$
The equation (3.6) is a direct restatement of the fact that spin can be transported by diffusion, i.e. it is a good quantum number in a superconductor, and that the average velocity of its propagation is just the group velocity of the quantum mechanical wave.

In the clean limit, the transverse spin conductivity \( \sigma_{xy}^s \) defined in Eq. (3.3) is

\[
\sigma_{xy}^s(T) = \frac{\hbar^2}{4i} \sum_{m,n} (f_m - f_n) \frac{V_{ymn}V_{xnm}}{(\epsilon_n - \epsilon_m + i0)^2}, \tag{3.7}
\]

where \( f_m = (1 + \exp(\beta\epsilon_m))^{-1} \) is the Fermi-Dirac distribution function evaluated at energy \( \epsilon_m \). For details of the derivation see Appendix B. The indices \( m \) and \( n \) label quantum numbers of particular states. The matrix elements \( V_{\mu}^{mn} \) are

\[
V_{\mu}^{mn} = \langle m|V_\mu|n \rangle = \int dx \langle u_m^s,v_n^s \rangle V_\mu \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \tag{3.8}
\]

where the particle-hole wavefunctions \( u_m^s, v_n \) satisfy the Bogoliubov-deGennes equation (2.10). Note that unlike the longitudinal dc conductivity, transverse conductivity becomes automatically set to zero.

where the particle-hole wavefunctions \( u_m^s, v_n \) satisfy the Bogoliubov-deGennes equation (2.10). Note that unlike the longitudinal dc conductivity, transverse conductivity is well defined even in the absence of impurity scattering. This demonstrates the fact that the transverse conductivity is not dissipative in origin. Rather, as will be discussed in Section III C, its nature is topological.

In the limit of \( T \to 0 \) the expression (3.7) for \( \sigma_{xy}^s \) becomes

\[
\sigma_{xy}^s = \frac{\hbar^2}{4i} \sum_{\epsilon_n < 0} \frac{V_{ymn}V_{ynm} - V_{ynm}V_{ymn}}{(\epsilon_n - \epsilon_m)^2}. \tag{3.9}
\]

The summation extends over all states below and above the Fermi energy which, by the nature of the superconductor, is automatically set to zero.

\[ \text{B. Vanishing of the Spin Conductivity at Half Filling (} \mu = 0) \]

It is useful to contrast the semiclassical approach with the full quantum mechanical treatment of transverse spin conductivity. In semiclassical analysis the starting unperturbed Hamiltonian is usually defined in the absence of magnetic field \( \mathbf{B} \). One then assumes semiclassical dynamics and no inter-band transitions. In this picture, if there is particle-hole symmetry in the original \((\mathbf{B} = 0)\) Hamiltonian, then there will be no transverse spin (thermal) transport, since the number of carriers with a given spin (energy) will be the same in opposite directions. In this context, similar argument was put forth in Ref. [3]. However, the problem of a d-wave superconductor is not so straightforward. As pointed out in Ref. [4], in the nodal (Dirac fermion) approximation, the vector potential is solely due to the superflow while the uniform magnetic field enters as a Doppler shift i.e. Dirac scalar potential. Semiclassical analysis must then be started from this vantage point and the above conclusions are not straightforward, since the quasiparticle motion is irreducibly quantum mechanical.

Here we present an argument for the full quantum mechanical problem, without relying on the semiclassical analysis. We show that spin conductivity tensor (3.7) vanishes at \( \mu = 0 \) due to particle-hole symmetry (2.11). First note that the Fermi-Dirac distribution function satisfies \( f(\epsilon) = 1 - f(-\epsilon) \). Therefore, the factor \( f_m - f_n \) changes sign under the particle-hole transformation \( \tilde{P}_H \) (2.11) while the denominator \( (\epsilon_n - \epsilon_m)^2 \) clearly remains unchanged. In addition, each of the matrix elements \( V_{\mu}^{mn} \) changes sign under \( \tilde{P}_H \). Thus the double summation over all states in Eq. (3.7) yields zero.

Consequently the spin transport vanishes for a clean strongly type-II BCS d-wave superconductor on a tight binding lattice at half filling. Due to Wiedemann-Franz law, which we derive in the next section, thermal Hall conductivity also vanishes at half filling at sufficiently low temperatures. Note that this result is independent of the vortex arrangement i.e. it holds even for disordered vortex array and does not rely on any approximation regarding inter- or intra- nodal scattering.

\[ \text{C. Topological Nature of Spin Hall Conductivity at } T=0 \]

In order to elucidate the topological nature of \( \sigma_{xy}^s \), we make use of the translational symmetry discussed in Section II B and formally assume that the vortex arrangement is periodic. However, the detailed nature of the vortex lattice will not be specified and thus any vortex arrangement is allowed within the magnetic unit cell. The conclusions we reach are therefore quite general.

We will first rewrite the velocity matrix elements \( V_{\mu}^{mn} \) using the singularly gauge transformed basis as discussed in Section II B. Inserting unity in the form of the FT gauge transformation (2.12)

\[
V_{\mu}^{mn} = \langle m|U^{-1}V_\mu U U^{-1}|n \rangle. \tag{3.10}
\]

The transformed basis states \( U^{-1}|n \rangle \) can now be written in the Bloch form as \( e^{ik \cdot r} m_k \) and therefore the matrix element becomes

\[
V_{\mu}^{mn} = \langle m_k|e^{-ik \cdot r} U^{-1}V_\mu U e^{ik \cdot r} |n_k \rangle = \langle m_k|V_\mu(k)|n_k \rangle. \tag{3.11}
\]

We used the same symbol \( k \) for both bra and ket because the crystal momentum in the first Brillouin zone is conserved. The resulting velocity operator can now be simply expressed as

\[
V_\mu(k) = \frac{1}{\hbar} \frac{\partial \hat{H}(k)}{\partial k_\mu}. \tag{3.12}
\]
where \( \hat{H}_0(\mathbf{k}) \) was defined in [21]. Furthermore the matrix elements of the partial derivatives of \( \hat{H}(\mathbf{k}) \) can be simplified according to
\[
\langle m_k | \frac{\partial \hat{H}(\mathbf{k})}{\partial k_\mu} | n_k \rangle = (\epsilon_k^m - \epsilon_k^n) \langle m_k | \frac{\partial n_k}{\partial k_\mu} | n_k \rangle = -(\epsilon_k^m - \epsilon_k^n) \langle \frac{\partial m_k}{\partial k_\mu} | n_k \rangle.
\]
\[(3.13)\]
for \( m \neq n \). Utilizing Eqs. (3.12) and (3.13), Eq. (3.1) for \( \sigma_{xy}^s \) can now be written as
\[
\sigma_{xy}^s = \frac{\hbar}{4i} \int \frac{d\mathbf{k}}{(2\pi)^2} \sum_{\epsilon_m < 0 < \epsilon_n} \langle \frac{\partial m_k}{\partial k_x} | n_k \rangle \langle n_k | \frac{\partial m_k}{\partial k_y} | n_k \rangle - \langle \frac{\partial m_k}{\partial k_y} | n_k \rangle \langle n_k | \frac{\partial m_k}{\partial k_x} | n_k \rangle.
\]
\[(3.14)\]
The identity \( \sum_{\epsilon_m < 0 < \epsilon_n} \langle \frac{\partial m_k}{\partial k_x} | n_k \rangle \langle n_k | \frac{\partial m_k}{\partial k_y} | n_k \rangle = 1 \), can be further used to simplify the above expression to read
\[
\sigma_{xy}^{s,m} = \frac{\hbar}{8\pi i} \int d\mathbf{k} \left( \langle \frac{\partial m_k}{\partial k_x} | \frac{\partial m_k}{\partial k_y} \rangle - \langle \frac{\partial m_k}{\partial k_y} | \frac{\partial m_k}{\partial k_x} \rangle \right)
\]
\[(3.15)\]
where \( \sigma_{xy}^{s,m} \) is a contribution to the spin Hall conductance from a completely filled band \( m \), well separated from the rest of the spectrum. Therefore the integral extends over the entire magnetic Brillouin zone that is topologically a two-torus \( T^2 \). Let us define a vector field \( \tilde{A} \) in the magnetic Brillouin zone as
\[
\tilde{A}(\mathbf{k}) = \langle m_k | \nabla_k | m_k \rangle,
\]
\[(3.16)\]
where \( \nabla_k \) is a gradient operator in the \( \mathbf{k} \) space. From (3.15) this contribution becomes
\[
\sigma_{xy}^{s,m} = \frac{\hbar}{8\pi i} \int d\mathbf{k} | \nabla_k \times \tilde{A}(\mathbf{k}) |_z,
\]
\[(3.17)\]
where \([ \ ] \) represents the third component of the vector. The topological aspects of the quantity in (3.17) were extensively studied in the context of integer quantum Hall effect (see e.g. [37]) and it is a well known fact that
\[
\frac{1}{2\pi i} \int d\mathbf{k} | \nabla_k \times \tilde{A}(\mathbf{k}) |_z = C_1
\]
\[(3.18)\]
where \( C_1 \) is a first Chern number that is an integer. Therefore, a contribution of each filled band to \( \sigma_{xy}^s \) is
\[
\sigma_{xy}^{s,m} = \frac{\hbar}{8\pi} N
\]
\[(3.19)\]
where \( N \) is an integer. The assumption that the band must be separated from the rest of the spectrum can be relaxed. If two or more fully filled bands cross each other the sum total of their contributions to spin Hall conductance is quantized even though nothing guarantees the quantization of the individual contributions. The quantization of the total spin Hall conductance requires a gap in the single particle spectrum at the Fermi energy. As discussed in the Section V, the general single particle spectrum of the d-wave superconductor in the vortex state with inversion-symmetric vortex lattice is gapped and therefore the quantization of \( \sigma_{xy}^s \) is guaranteed.

IV. THERMAL CONDUCTIVITY

Before discussing the nature of the quasi-particle spectrum, we will establish a Wiedemann-Franz law between spin conductivity and thermal conductivity for a d-wave superconductor. This relation is naturally expected to hold for a very general system in which the quasi-particles form a degenerate assembly i.e. it holds even in the presence of elastically scattering impurities.

Following Luttinger [32], and Smrˇ cka and Stˇ reda [34] we introduce a pseudo-gravitational potential \( \chi = \mathbf{x} \cdot \mathbf{g} / e^2 \) into the Hamiltonian (2.6) where \( \mathbf{g} \) is a constant vector. The purpose is to include a coupling to the energy density on the Hamiltonian level. This formal trick allows us to equate statistical \( (T \nabla (1/T)) \) and mechanical \( (\mathbf{g}) \) forces so that the thermal current \( j^Q \), in the long wavelength limit given by
\[
j^Q = L^Q(T) \left( T \nabla \frac{1}{T} - \nabla \chi \right),
\]
\[(4.1)\]
will vanish in equilibrium. Therefore it is enough to consider only the dynamical force \( \mathbf{g} \) to calculate the phenomenological coefficient \( L^Q_{\mu\nu} \). Note that thermal conductivity \( \kappa_{xy} \) is
\[
\kappa_{\mu\nu}(T) = \frac{1}{T} L^Q_{\mu\nu}(T).
\]
\[(4.2)\]
When the BCS Hamiltonian \( H \) introduced in Eq. (2.1) becomes perturbed by the pseudo-gravitational field, the resulting Hamiltonian \( H_T \) has the form
\[
H_T = H + F
\]
\[(4.3)\]
where \( F \) incorporates the interaction with the perturbing field:
\[
F = \frac{1}{2} \int d\mathbf{x} \ \Psi^\dagger(\mathbf{x}) \left( \hat{H}_0 \chi + \chi \hat{H}_0 \right) \Psi(\mathbf{x}).
\]
\[(4.4)\]
Since \( \chi \) is a small perturbation, to the first order in \( \chi \) the Hamiltonian \( H_T \) can be written as
\[
H_T = \int d\mathbf{x} \ (1 + \frac{\chi}{2}) \Psi^\dagger(\mathbf{x}) \hat{H}_0 (1 + \frac{\chi}{2}) \Psi(\mathbf{x})
\]
\[(4.5)\]
i.e. the application of the pseudo-gravitational field results in rescaling of the fermion operators:
\[
\Psi \rightarrow \Psi = (1 + \frac{\chi}{2}) \Psi.
\]
\[(4.6)\]
If we measure the energy relative to the Fermi level, the transport of heat is equivalent to the transport of energy. In analogy with the Section III A, we define the heat current $j^Q$ through diffusion of the energy-density $h_T$. From conservation of the energy-density the continuity equation follows

$$h_T + \nabla \cdot j^Q = 0. \quad (4.7)$$

In the limit of $q \to 0$ the thermal current is

$$j^Q_{\mu} = \frac{i}{2} \left( \mathbf{\hat{w}}^\dagger V_{\mu} \mathbf{\hat{w}} - \mathbf{\hat{w}} V_{\mu} \mathbf{\hat{w}} \right). \quad (4.8)$$

For details see Appendix C. Note that the quantum statistical average of the current has two contributions, both linear in $\chi$,

$$\langle j^Q_{\mu} \rangle = \langle j^Q_{0\mu} \rangle + \langle j^Q_{1\mu} \rangle = -(K^Q_{\mu\nu} + M_{\mu\nu}) \partial \nu \chi. \quad (4.9)$$

The first term is the usual Kubo contribution to $L_{\mu\nu}$ while the second term is related to magnetization of the sample for transverse components of $\kappa_{\mu\nu}$ and vanishes for the longitudinal components. In Appendix C we show that at $T = 0$ the term related to magnetization cancels the Kubo term and therefore the transverse component of $\kappa_{\mu\nu}$ is zero at $T = 0$. To obtain finite temperature response, we perform Sommerfeld expansion and derive Wiedemann-Franz law for spin and thermal Hall conductivities.

As shown in the Appendix C

$$L^Q_{\mu\nu}(T) = -\left( \frac{2}{\hbar} \right)^2 \int d\xi \xi^2 \frac{df(\xi)}{d\xi} \tilde{\sigma}_{\mu\nu}(\xi) \quad (4.10)$$

where

$$\tilde{\sigma}_{\mu\nu}(\xi) = \frac{\hbar^2}{4i} \sum_{m} \frac{V_{nm} V_{nm} - V_{nm} V_{nm}}{(\epsilon_m - \epsilon_n)^2}. \quad (4.11)$$

Note that $\tilde{\sigma}_{\mu\nu}(\xi = 0) = \sigma^s_{\mu\nu}(T = 0)$. For a superconductor at low temperature the derivative of the Fermi-Dirac distribution function is

$$-\frac{df(\xi)}{d\xi} = \delta(\xi) + \frac{\pi^2}{6}(k_B T)^2 \frac{d^2}{d\xi^2} \delta(\xi) + \cdots \quad (4.12)$$

Substituting (4.12) into (4.10) we obtain

$$L^Q_{\mu\nu}(T) = \frac{4\pi^2}{3h}(k_B T)^2 \sigma^s_{\mu\nu}, \quad (4.13)$$

where $\sigma^s_{\mu\nu}$ is evaluated at $T = 0$. Finally, using (4.2), in the limit of $T \to 0$

$$\kappa_{\mu\nu}(T) = \frac{4\pi^2}{3} \left( \frac{k_B}{\hbar} \right)^2 T \sigma^s_{\mu\nu}. \quad (4.14)$$

We recognize the Wiedemann-Franz law for the spin and thermal conductivity in the above equation. As mentioned, this relation is quite general in that it is independent of the spatial arrangement of the vortex array or elastic impurities. Thus, quantization of the transverse spin conductivity $\sigma^s_{xy}$ implies quantization of $\kappa_{xy}/T$ in the limit of $T \to 0$.

V. QUASIPARTICLE SPECTRUM AND QUANTIZED CONDUCTIVITY

General features of the quasi-particle spectrum can be understood on the basis of symmetry alone. Since the time-reversal symmetry is broken, the Bogoliubov-de Gennes Hamiltonian $H_0$ [2.11] must be, in general, complex. According to the “non-crossing” theorem of von Neumann and Wigner [38], a complex Hamiltonian can have degenerate eigenvalues unrelated to symmetry only if there are three parameters which can be varied simultaneously.

Since the system is two dimensional, with the vortices arranged on the lattice, there are two parameters in the Hamiltonian $\hat{H}(k)$ (2.21): vortex crystal momenta $k_x$ and $k_y$ which vary in the first Brillouin zone. Therefore, we should not expect any degeneracy to occur, in general, unless there is some symmetry which protects it. Away from half-filling ($\mu \neq 0$) and with unspecified arrangement of vortices in the magnetic unit cell there is not enough symmetry to cause degeneracy. There is only global Bogoliubov-de Gennes symmetry relating quasiparticle energy $\epsilon_k$ at some point $k$ in the first Brillouin zone to $-\epsilon_{-k}$.

In order for every quasiparticle band to be either completely below or completely above the Fermi energy, it is sufficient for the vortex lattice to have inversion symmetry. This can be readily seen by the following argument: Consider a vortex lattice with inversion symmetry. Then, by the very nature of the superconducting vortex carrying flux, there must be even number of vortices per magnetic unit cell and we are then free to choose Franz-Tesanovic labels $A$ and $B$ in such a way that $\psi^A(-r) = -\psi^B(r)$. To see this note that the explicit form of the superfluid velocities can be written as [10]:

$$v^A(r) = \frac{2\pi \hbar}{m^*} \int \frac{d^2k}{(2\pi)^2} \frac{\epsilon}{k^2} \sum_i e^{i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_i^\alpha)}, \quad (5.1)$$

where $\alpha = A$ or $B$ and $\mathbf{r}_i^\alpha$ denotes the position of the vortex with label $\alpha$. If the vortex lattice has inversion symmetry then for every $\mathbf{r}_i^A$ there is a corresponding $-\mathbf{r}_i^B$ such that $\mathbf{r}_i^A = -\mathbf{r}_i^B$. Therefore, under space inversion $I$

$$Iv^A(r) = v^A(-\mathbf{r}) = -v^B(r). \quad (5.2)$$

Recall that the tight-binding lattice Bogoliubov-de Gennes Hamiltonian written in the Bloch basis (2.21) reads:

$$\hat{H}(k) = \sum_{\delta} \left\{ \sigma_3 \left( -\frac{te}{\hbar} \int \rho^{+\delta}(\mathbf{a} - \mathbf{r}_1\mathbf{v}) \cdot d\mathbf{l} \mathbf{e}^{i \delta(\mathbf{k} + \mathbf{p})} - \mu \right) 
+ \sigma_1 \Delta_0 \delta \epsilon^{+\delta} \int \rho^{+\delta}(\mathbf{a} + \mathbf{l} \mathbf{e}^{i \delta(\mathbf{k} + \mathbf{p})}) \right\} \quad (5.3)$$

where

$$v(r) = \frac{1}{2}(v^A(r) + v^B(r)); \quad a(r) \equiv \frac{1}{2}(v^A(r) - v^B(r)). \quad (5.4)$$
As before $\sigma_1$ and $\sigma_3$ are Pauli matrices operating in Nambu space and the sum is again over the nearest neighbors. It can be easily seen that upon applying the space inversion $I$ to $\hat{H}(\mathbf{k})$ followed by complex conjugation $C$ and $i\sigma_2$ we have a symmetry that for every $\epsilon_\mathbf{k}$ there is $-\epsilon_\mathbf{k}$, that is:

$$-i\sigma_2CIT\hat{H}(\mathbf{k})ICi\sigma_2 = -\hat{H}(\mathbf{k})$$  \hspace{1cm} (5.5)

which holds for every point in the Brillouin zone. Therefore, in order for the spectrum not to be gapped, we would need band crossing at the Fermi level. But by the non-crossing theorem this cannot happen in general. Thus, the generic quasi-particle spectrum of inversion symmetric vortex lattice is gapped. As was established in the previous section, gapped quasi-particle spectrum implies quantization of the transverse spin conductivity $\sigma_{xy}$, as well as $\kappa_{xy}/T$ for $T$ sufficiently low.

Precisely at half-filling ($\mu = 0$) $\sigma_{xy}^0$ must vanish on the basis of particle-hole symmetry (see Section III B). We can then vary the chemical potential so that $\mu \neq 0$ and break particle-hole symmetry. Hence, the chemical potential $\mu$ can serve as the third parameter necessary for creating the accidental degeneracy, i.e. at some special values of $\mu^*$ the gap at the Fermi level will close (see Fig. 3). This results in a possibility of changing the quantized value of $\sigma_{xy}^0$ by an integer in units of $\hbar/8\pi$ (Fig. 3). By the very nature of the superconducting state, we achieve plateau dependence on the chemical potential. This is to be contrasted to the plateaus in the “ordinary” integer quantum Hall effect which are due to the presence of disorder. In our case, the system is clean and the plateaus are due to the magnetic field induced gap and superconducting pairing.

Similarly, we can change the strength of the electron-electron attraction, which is proportional to the maximum value of the superconducting order parameter $\Delta_0$ while keeping the chemical potential $\mu$ fixed. Again, as can be seen in Fig. 3, at some special values $\Delta_0^*$ the spectrum is gapless and the quantized Hall conductance undergoes a transition.

### VI. SCALING FUNCTIONS

As shown in the previous section, the quasi-particle spectrum of a d-wave superconductor in vortex state is gapped, in general. Therefore, at temperatures $T$ which are much less than the $\Delta_m$ (magnetic field $B$ induced gap), $T$-dependence of thermal conductivity $\kappa_{xy}$ can be determined uniquely. Moreover, $B$-dependence of $\kappa_{xy}$ comes entirely from the spin conductivity $\sigma_{xy}^s(B)$. That is:

$$\kappa_{xy} = \frac{4\pi^2}{3} \left( \frac{k_B}{\hbar} \right)^2 T \sigma_{xy}^s(B).$$  \hspace{1cm} (6.1)

**FIG. 2.** The mechanism for changing the quantized spin Hall conductivity is through exchanging the topological quanta via ("accidental") gap closing. The upper panel displays spin Hall conductivity $\sigma_{xy}$ as a function of the chemical potential $\mu$. The lower panel shows the magnetic field induced gap $\Delta_m$ in the quasiparticle spectrum. Note that the change in the spin Hall conductivity occurs precisely at those values of chemical potential at which the gap closes. Hence the mechanism behind the changes of $\sigma_{xy}$ is the exchange of the topological quanta at the band crossings. The parameters for the above calculation were: square vortex lattice, magnetic length $l = 4\delta$, $\Delta = 0.1t$ or equivalently the Dirac anisotropy $\alpha_D = 10$.

**FIG. 3.** The upper panel displays spin Hall conductivity $\sigma_{xy}$ as a function of the maximum superconducting order parameter $\Delta_0$. The lower panel shows the magnetic field induced gap in the quasiparticle spectrum. The change in the spin Hall conductivity occurs at those values of $\Delta_0$ at which the gap closes. The parameters for the above calculation were: square vortex lattice, magnetic length $l = 4\delta$, $\mu = 2.2t$. 


Curiously, if the above equation is naively combined with Simon and Lee scaling \(^{(6.2)}\)
\[ \kappa_{xy}(T, B) = T^2 F_{xy} \left( \frac{\sqrt{B}}{T} \right). \]
the scaling function would be determined up to a proportionality constant \(C\):
\[ \kappa_{xy}(T, B) = C T \sqrt{B}. \]

In the recent experiments of Ong et. al.\(^{(5)}\) this is precisely the scaling seen in the temperature range up to \(\sim 25K\). Although tempting, the above arguments are unlikely explanation for the experiments since the measurements are done at rather high temperatures.

**VII. CONTINUUM VS. LATTICE THEORY**

The previous discussion concentrated on the tight-binding formulation of the problem which is important if the magnetic field is relatively large and if there is a strong interaction between the underlying ionic lattice and the quasiparticles. In usual experimental situations, however, the magnetic length is much longer than the inter-ionic spacing and we would expect that the length-scale associated with the ionic lattice becomes unimportant at low energies. This leads to a continuum formulation of the theory (see Section I).

**A. Linearized Continuum Theory**

On the basis of the \(B = 0\) problem, we expect that the low energy physics is confined in \(k\)-space around four nodal points on the Fermi surface. In order to treat the effect of the magnetic field and of the vortex lattice on the low energy properties of the spectrum, we would ideally like to construct a Hamiltonian which treats each of the four nodes independently. A natural way to do this was suggested by Simon and Lee \(^{(42)}\) and later extensively used by others \(^{(40,41)}\).

As was pointed out in Ref. \(^{(16)}\) there is a subtle problem associated with numerical implementation of the continuum linearized version of the theory. On the physical grounds any given problem involving charge \(e\) (quasi)particle interacting with a +\(\frac{hc}{2e}\) vortex must lead to an identical eigenvalue spectrum as for a -\(\frac{hc}{2e}\) vortex as long as the wavefunction of the particle vanishes precisely at the vortex. This is an exact singular gauge symmetry of our physical problem and therefore it should be present at all stages of any approximation. Extensive numerical study of the linearized approximation \(^{(16)}\) pointed out that this symmetry is weakly violated in the quasiparticle spectra. The problem persists irrespective of the techniques chosen to diagonalize the linearized Hamiltonian i.e. it is present in real space as well as momentum space representation. The issue was resolved in Ref. \(^{(10)}\) by regularizing the problem on the tight-binding lattice which offered many advantages, some of which are utilized in this paper. On the other hand, the concept of single node physics is lost in a tight binding formulation as the latter naturally describes the physics of all four nodes. It would be desirable to formulate a properly regularized Hamiltonian describing single node physics only. From the work of Franz and Tesanovic \(^{(43)}\), the most natural candidate is a free, anisotropic, massless Dirac Hamiltonian
\[ \mathcal{H}_0 = \begin{pmatrix} v_F \hat{p}_x + v_\Delta \hat{p}_y & v_\Delta \hat{p}_y - v_F \hat{p}_x \end{pmatrix} \] perturbed by scalar and vector potentials defined in Eq. \(^{(2.17)}\):
\[ \mathcal{H}' = m \begin{pmatrix} v_F v_A^x & \frac{1}{2} v_\Delta (v_A^x - v_B^y) \\ \frac{1}{2} v_\Delta (v_A^y - v_B^x) & v_F v_A^y \end{pmatrix}. \]

\(v_F\) and \(v_\Delta\) are Fermi and gap velocities respectively \(^{(12)}\). We wish to argue that the above Hamiltonian is well defined and does not suffer from singular gauge asymmetry, provided that suitable boundary conditions and their transformations are also specified. The discrepancies encountered in the numerics are believed to result from neglecting the boundary conditions and the effect of the singular gauge transformation on them.

In the context of high energy physics, a similar problem was studied extensively. It was found that the Dirac-type equations in the presence of a single Aharonov-Bohm string require self-adjoint extensions which enlarge the Hilbert space by the wavefunctions with \(r^{-\frac{1}{2}}\) radial behavior. In addition, the wavefunctions depend on a dimensionless parameter \(\Theta\) which specifies the boundary conditions at the core. A consistent procedure enabling one to construct the needed self-adjoint extension involves the theory of von Neumann deficiency indices \(^{(10,11)}\).

The above considerations can be illustrated on a well studied problem of a Dirac particle in the field of a single string \(^{(42,43)}\). We will restrict ourselves to the case of a massless Dirac particle and \(\frac{hc}{2e}\) flux carried by the string. The Hamiltonian
\[ (i \not{\partial} - e \not{A}) \psi = 0 \] in this case is axially symmetric and after separation of variables
\[ \psi(r, \phi) = \begin{pmatrix} \chi^1 e^{i\phi} \\ \chi^2 e^{in\phi} \end{pmatrix} \] one obtains the equation for the radial wavefunctions:
\[ H_r \chi = \begin{pmatrix} 0 & -i \left( \frac{d}{dr} + \frac{n+1}{r} \right) \\ -i \left( \frac{d}{dr} - \frac{n}{r} \right) & 0 \end{pmatrix} \chi(r) = E \chi(r) \]
where $\nu = \frac{1}{2} + n$ is a half-integer. The solutions of this equation can be expressed through Bessel functions as

$$\chi_r \propto \left( \frac{J_{\nu}(|E|r)}{iJ_{(\nu+1)}(|E|r)} \right)$$  \(7.6\)

where $\epsilon = \pm 1$. The square integrability condition determines sign of $\epsilon$ for all $\nu$ except $\nu = -1/2$. Both choices of $\epsilon$ result in a wavefunction diverging as $1/\sqrt{r}$ while still remaining square integrable. The requirement of regularity for all the solutions at $r = 0$ turns out to be too restrictive and results in numerous pathologies such as incompleteness of the basis. The problem is solved by using von-Neumann theory of self-adjoint extensions $^{42,43}$. Since the radial $H_r$ operator in $(7.3)$ defined in the domain of regular functions has deficiency indices $(1,1)$, one is forced to extend the Hilbert space by relaxing the condition of regularity and allowing for the wavefunctions $^{42}$

$$\chi(r) \propto \frac{1}{r^{\mu/2}} \left( \frac{i \sin \Theta}{\cos \Theta} \right).$$  \(7.7\)

The angle $\Theta \in (\pi/4, \pi/4 + \pi)$, determined by the details of the vortex core, parameterizes the self-adjoint extension.

The energy eigenstates for $\nu = -1/2$ are given by

$$\left( \sin \mu J_{-1/2}(|E|r) + (-1)^n \cos \mu J_{1/2}(|E|r) \right)$$

$$\left( \sin \mu J_{1/2}(|E|r) - (-1)^n \cos \mu J_{-1/2}(|E|r) \right)$$  \(7.8\)

where the parameter $\mu = \Theta$ if $n$ is even, while for $n$ odd $\mu = \pi - \Theta$. In addition, for some values of $\Theta$ there is a bound state $^{42}$.

We encounter a peculiar situation in which the long distance physics governed by Eq. $(7.3)$ still depends on boundary conditions imposed at the core. Linearization of the original BdG problem neglects terms “small” in conventional sense which, nevertheless, remain effectively present in $\Theta$ dependence of Eq. $(7.8)$. Although the linearized equation $(7.3)$ has no length scale, the core physics, now effectively shrunk to a circle with vanishing radius around the singularity, manifests itself in the wavefunctions $(7.7)$ with the long power-law tails. In general, there are only two values of $\Theta$ corresponding to the pure $\delta$-function magnetic field $^{45}$. However, in the case of other contact interactions present in the core, the parameter $\Theta$ can be arbitrary $^{45}$.

Various regularizations of the problem were studied leading to different choices of the self-adjoint extension $^{46}$. One possibility is to put an impenetrable cylinder of small but finite radius $\rho$ around a vortex thus forcing the wavefunctions to vanish on the surface of the cylinder. Such a boundary condition specification will immediately restore the $\pm \frac{\pi}{2}$ vortex invariance because the value of the wavefunction at the surface of the cylinder will not transform under singular gauge transformation. Of course, the true boundary conditions must descend from the original physical problem, in this case from the self-consistent solution of the full Bogoliubov-de Gennes equation including the core region. The quasi-particle wavefunctions do not, in general, vanish around the vortex $^{17}$. Therefore, not only the quasi-particle wavefunctions but also the constraints imposed on them must be properly transformed under singular gauge transformation.

B. Beyond the Linearized Theory

As pointed out by Simon and Lee $^{7}$ and then later by Ye $^{20}$, linearized theory predicts vanishing transverse conductivities. In order to include the description of Hall effects, the curvature terms must be included. Without regularization, the perturbative analysis of the curvature terms are far from straightforward. This has to do with the fast rise of both wavefunctions and perturbations around the core. If the regularization is taken into account and the wavefunctions are forced to vanish at some radius $\rho$ around the core, the new length scale $\rho$ will break the scale invariance of the Dirac equation. Moreover, the perturbative analysis of the continuum equation would predict that the contribution of each node to $\sigma_{xy}^s$ can be either 0 or $\pm \frac{1}{2}$ in units of $\hbar/8\pi$ $^{19}$. Thus, if all four nodes are included, the only possible values of $\sigma_{xy}^s$ are 0 or $\pm 2$. On the other hand, explicit evaluation of $\sigma_{xy}^s$ via the tight-binding regularization indicates that a wide range of values for $\sigma_{xy}^s$ is possible (see Figs. 2 and 3). An interesting question remaining for future study is how to implement the picture presented in Ref. $^{19}$, of individual Dirac nodes changing $\sigma_{xy}$ by $\pm 1$ through action of the perturbatively determined mass term as an “elementary building block” of more complicated band crossings.

One way to reach the continuum limit is to make the magnetic length $l$ much longer than the tight binding lattice constant $\delta$. The drawback associated with the tight-binding regularization of the problem stems from the fact that the continuum limit is not smooth and unless an exact analytical solution is found it is complicated to analyze numerically. The similar situation occurs in a tight-binding Hofstadter problem of a 2D normal electron gas on the lattice.

VIII. CONVENTIONAL SUPERCONDUCTORS

Although the previous analysis was focused on unconventional d-wave superconductors, the main results can be directly generalized to conventional 2D s-wave superconductors in the vortex state: the transverse spin Hall conductivity $\sigma_{xy}^s$ is quantized in units of $\hbar/8\pi$ and by Wiedemann-Franz law $\kappa_{xy} = \frac{4\pi^2}{3} \left( \frac{k_B}{\hbar} \right)^2 T \sigma_{xy}^s$ as $T \to 0$. At low magnetic fields, the quasiparticle
spectrum is gapped by the virtue of the lowest Caroli-Matricon-de Gennes vortex core bound state having energy \( \sim \Delta^2/\epsilon_F \). In the vortex lattice the CMdG bound states are extended and form bands. If the vortices are disordered, the states in the band tails will be localized because the localized states do not contribute to transport. As the magnetic field is increased, the spectrum becomes progressively altered and at some critical field \( B^* \) it develops nodal points (see Ref. [48]). The transition between states with two different quantized values of \( \sigma_{xy}^s \) happens at the induced gap closings. Therefore, a series of transitions between different spin Hall states is predicted to occur in the conventional superconductors as well.

**IX. CONCLUSIONS**

In conclusion, we examined a general problem of 2D type-II superconductors in the vortex state with inversion symmetric vortex lattice. The single particle excitation spectrum is typically gapped and results in quantization of transverse spin conductivity \( \sigma_{xy}^s \) in units of \( \hbar/8\pi \). The topological nature of this phenomenon is discussed. The size of the magnetic field induced gap \( \Delta_m \) in unconventional d-wave superconductors is not universal and in principle can be as large as several percent of the maximum superconducting gap \( \Delta_0 \). By virtue of the Wiedemann-Franz law, which we derive for the d-wave Bogoliubov-de Gennes equation in the vortex state, the thermal conductivity \( \kappa_{xy} = \frac{\pi^2}{3} \left( \frac{k_B}{\hbar} \right)^2 T \sigma_{xy}^s \) as \( T \to 0 \).

Thus at \( T \ll \Delta_m \) the quantization of \( \kappa_{xy} / T \) will be observable in clean samples with negligible Lande \( g \) factor and with well ordered Abrikosov vortex lattice. In conventional superconductors, the size of \( \Delta_m \) is given by Matricon-Caroli-de Gennes vortex bound states \( \sim \Delta^2/\epsilon_F \).

In real experimental situations, Lande \( g \) factor is not necessarily small. In fact it is close to 2 in cuprates [49]. The Zeeman effect must therefore be included in the analysis. Nevertheless, detailed numerical examination of the quasiparticle spectra reveals that the spectrum remains gapped for a wide range of physically realizable parameters even if \( g = 2 \). Thus, although the Zeeman splitting is a competing effect, in general it is not strong enough to prevent the quantization of \( \sigma_{xy}^s \) and consequently of \( \kappa_{xy} / T \).

We have explicitly evaluated the quantized values of \( \sigma_{xy}^s \) on the tight-binding lattice model of \( d_{x^2-y^2} \)-wave BCS superconductor in the vortex state and showed that in principle a wide range of integer values can be obtained. This should be contrasted with the notion that the effect of a magnetic field on a d-wave superconductor is solely to generate a \( d + id \) state for the order parameter, as in that case \( \sigma_{xy}^s = \pm 2 \) in units of \( \hbar/8\pi \). In the presence of a vortex lattice, the situation appears to be more complex.

By varying some external parameter, for instance the strength of the electron-electron attraction which is proportional to \( \Delta_0 \) or the chemical potential \( \mu \), the gap closing is achieved i.e. \( \Delta_m = 0 \). The transition between two different values of \( \sigma_{xy}^s \) occurs precisely when the gap closes and topological quanta are exchanged. The remarkable new feature is the plateau dependence of \( \sigma_{xy}^s \) on the \( \Delta_0 \) or \( \mu \). It is qualitatively different from the plateaus in the ordinary integer quantum Hall effect which are essentially due to disorder. In superconductors, the plateaus happen in a clean system because the gap in the quasiparticle spectrum is generated by the superconducting pairing interactions.

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APPENDIX A: GREEN’S FUNCTIONS: DEFINITIONS AND IDENTITIES

The one-particle Green’s function matrix \( \hat{G} \) is defined as

\[
\hat{G}_{\alpha\beta}(r_1; r_2; \tau_2) = -(T_\tau \Psi_\alpha(1)^\dagger \Psi_\beta(2)) \tag{A1}
\]
where \( T_\tau \) denotes imaginary time ordering operator and \( \alpha = 1, 2 \) denotes components of a Nambu spinor \( \Psi_\dagger(1) \) which is a shorthand for \( \Psi_\dagger(r_1, \tau_1) = (\psi_{\uparrow}(r_1, \tau_1), \psi_{\downarrow}(r_1, \tau_1)) \).

Due to the time independence of the Hamiltonian \( H_0 \), the Green’s function \( \hat{G} \) depends only on the imaginary time difference \( \tau = \tau_1 - \tau_2 \). Therefore, its Fourier transform is given by

\[
\hat{G}(r_1, r_2; i\omega) = \int_0^\beta e^{i\omega\tau} \hat{G}(r_1, r_2, \tau) d\tau
\]

\[
\hat{G}(r_1, r_2; \tau) = \frac{1}{\beta} \sum_{i\omega} e^{-i\omega\tau} \hat{G}(r_1, r_2; i\omega). \tag{A2}
\]

Here \( \beta = 1/(k_B T) \), \( T \) is temperature, and the fermionic frequency \( \omega = (2l + 1)\pi/\beta, \ l \in \mathbb{Z} \).

Using the above relations, it is straightforward to derive the spectral representation of the following correlation functions between \( \Psi \) and its imaginary time derivatives \( \partial_T \Psi \equiv \hat{\Pi} \):

\[
\begin{align*}
\langle T_\tau \Psi_\alpha(1)^\dagger \Psi_\beta(2) \rangle &= -\frac{1}{\beta} \sum_{i\omega} e^{-i\omega\tau} \int d\epsilon \frac{\hat{A}(r_1, r_2; \epsilon)}{i\omega - \epsilon} \left\{ \begin{array}{c}
+1 \\
-\epsilon \\
+\epsilon \\
-\epsilon^2
\end{array} \right. \\
\langle T_\tau \Psi_\alpha(1)^\dagger \Psi_\beta(2) \rangle &= -\frac{1}{\beta} \sum_{i\omega, n} e^{-i\omega\tau} \Phi_{\alpha n}(r_1) \Phi_{n\beta}(r_2) \left\{ \begin{array}{c}
+1 \\
-\epsilon_n \\
+\epsilon_n \\
-\epsilon_n^2
\end{array} \right. \tag{A3}
\end{align*}
\]

The spectral function \( \hat{A}(r_1, r_2; \epsilon) \) can be written in terms of eigenfunctions \( \Phi_n(r) = (u_n(r), v_n(r))^T \) of the Bogoliubov-deGennes Hamiltonian \( H_0 \) in the form:

\[
\hat{A}(r_1, r_2; \epsilon) = \sum_n \delta(\epsilon - \epsilon_n) \Phi_{\alpha n}(r_1) \Phi_{n\beta}(r_2) \tag{A4}
\]

where \( \epsilon_n \) is the eigen-energy associated with an eigenstate labeled by the quantum number \( n \). Substituting \( A4 \) into \( A3 \), we can write the above correlation functions solely in terms of the eigenfunctions \( \Phi_n(r) \):

\[
\begin{align*}
\langle T_\tau \Psi_\alpha(1)^\dagger \Psi_\beta(2) \rangle &= -\frac{1}{\beta} \sum_{i\omega, n} e^{-i\omega\tau} \Phi_{\alpha n}(r_1) \Phi_{n\beta}(r_2) \left\{ \begin{array}{c}
+1 \\
-\epsilon_n \\
+\epsilon_n \\
-\epsilon_n^2
\end{array} \right. \\
\langle T_\tau \Psi_\alpha(1)^\dagger \Psi_\beta(2) \rangle &= -\frac{1}{\beta} \sum_{i\omega, n} e^{-i\omega\tau} \Phi_{n\alpha n}(r_1) \Phi_{n\beta}(r_2) \left\{ \begin{array}{c}
+1 \\
-\epsilon_n \\
+\epsilon_n \\
-\epsilon_n^2
\end{array} \right. \tag{A5}
\end{align*}
\]

The above equations are straightforward to derive in continuum, while on the tight-binding lattice Eqs. \( A11 \) and \( A12 \) imply a symmetric definition of the lattice divergence operator.

The identities \( A11, A12 \) explicitly ensure that the generalized velocity operator \( \hat{V}_\mu \) is Hermitian i.e it satisfies the following identity:

\[
\int d\mathbf{r} \mathbf{V}^\dagger(r) \Psi(r) = \int d\mathbf{r} \mathbf{V}^\dagger(r) \Psi(r) \tag{A13}
\]

APPENDIX B: SPIN CURRENT AND SPIN CONDUCTIVITY TENSOR

The time derivative of spin density \( \rho_s = \frac{i}{\hbar}(\psi_\dagger \psi_\uparrow - \psi_\dagger \psi_\downarrow) \) can be written in Nambu formalism as

\[
\rho_s = \frac{i}{\hbar}[H, \rho_s] = \frac{\hbar}{2}(\Psi^\dagger \Psi + \Psi^\dagger \Psi) \tag{B1}
\]
Using equations of motion (2.9) together with the explicit form of $\dot{H}_0$ operator (2.7) we obtain

$$\dot{\rho}^s = \frac{i}{2} \left( \frac{\pi^2}{2m} \Psi_1' - \Psi_1' \frac{\pi^2}{2m} \Psi_1 + \Delta \Psi_1' \right) \Psi_2 - \Psi_1' \Delta \Psi_2$$

$$+ \Delta^* \Psi_1' \Psi_1 - \Psi_2' \Delta^* \Psi_1' - \Psi_2' \frac{\pi^2}{2m} \Psi_2 + \Psi_2' \frac{\pi^2}{2m} \Psi_2) .$$

(B2)

Using the identities (A11), (A12) it is easy to show that

$$\dot{\rho}^s = -\frac{h}{4} \nabla \cdot (\Psi V^\dagger V + V^\dagger \Psi) = -\nabla \cdot j^s$$

(B3)

where the generalized velocity operator $V^\mu$ is defined in Eq. (A3), and the last equality follows from the continuity equation (3.4) relating the spin density $\rho^s$ and the spin current $j^s$. Upon spatial averaging and utilizing Eq. (A13) we find the $q \to 0$ limit of the spin current

$$j^s_\mu = \frac{h}{2} \Psi V^\dagger V \Psi .$$

(B4)

The evaluation of the spin current-current correlation function (3.2) is straightforward and yields:

$$D_{\mu\nu}(i\Omega) = -\frac{h^2}{4} \int_0^\beta e^{i\Omega [V^\mu(2)V^\nu(4) \times (T, \Psi(1) \Psi(2)\Psi(3)\Psi(4))]} \cdot \nabla \cdot j^Q = 0 .$$

(C3)

The Hamiltonian density $h_T$ follows from Eq. (C1) and reads

$$h_T = \frac{1}{2m^s}(\pi^s_{\mu} \dot{\Psi}^s_{\mu} - \pi^s_{\mu} \dot{\Psi}^s_{\mu} - \pi^s_{\mu} \dot{\Psi}^s_{\mu}) + \frac{1}{2} \left( \Psi_{\alpha}^s \Delta_{\alpha\beta} \Psi_{\beta} + \Delta_{\alpha\beta} \Psi_{\alpha} \dot{\Psi}_{\beta} \right)$$

(C4)

where $\Psi = (1 + \tilde{\Psi})$. Taking the time derivative of the Hamiltonian density $h_T$ and using Eqs. (A11) and equations of motion (C2) we obtain

$$\dot{h}_T = i[H_T, h_T] = \frac{i}{2m} \nabla \cdot \left( \Psi^{\dagger} \Pi_{\mu} \Psi - \Pi_{\mu} \Psi^{\dagger} \Psi \right)$$

(C5)

where we introduced the matrix operators

$$\Delta_{\alpha\beta} = \begin{pmatrix} 0 & \Delta^* \\ \Delta & 0 \end{pmatrix} , \quad \Pi^s_{\mu} = \begin{pmatrix} \pi_{\mu}^s & 0 \\ 0 & \pi_{\mu}^s \end{pmatrix} .$$

(C6)

Here $\tilde{\Psi} = (1 + \tilde{\Psi})$ and $\pi_{\mu}^s$ is defined in Eq. (A11). Finally we use (A12) to extract $j^Q$ from (C5). Upon
spatial averaging and using the Hermiticity of $V_\mu$ \((A13)\) the thermal current $j^Q$ reads
\[
j^Q = i \frac{1}{2} \left( \langle \Psi^\dagger V_\mu \dot{\Psi} - \dot{\Psi} V_\mu \Psi \rangle \right).
\]
(C7)

Note that the expression for $j^Q$ contains two terms
\[
j^Q = j_0^Q + j_1^Q
\]
(C8)

where $j_0^Q$ is independent of $\chi$ and $j_1^Q$ is linear in $\chi$. Explicitly:
\[
j_0^Q(r) = \frac{1}{2} \Psi^\dagger \{ \mathbf{V}, \hat{H}_0 \} \Psi
\]
and
\[
j_1^Q(r) = -i \frac{1}{4} \partial_\nu \chi \Psi^\dagger (V_\mu V_\nu - V_\nu V_\mu) \Psi
\]
\[+ \frac{\partial_\nu \chi}{4} \Psi^\dagger (x_\nu V_{\mu x} + 3V_{x x_\nu} \hat{H}_0 + \hat{H}_0 (3x_\nu V_\nu + V_\nu x_\nu)) \Psi
\]
(C10)

where \(\{ a, b \} = ab + ba\). Analogously to the situation in the normal metal, the thermal average of $j_1^Q$ does not in general vanish in the presence of the magnetic field \((B8)\).

The linear response of the system to the external perturbation can be described by
\[
\langle j^Q_\beta \rangle = \langle j^Q_0 \rangle + \langle j^Q_1 \rangle = -(K_{\mu \nu} + M_{\mu \nu}) \partial_\nu \chi \equiv -L^Q_{\mu \nu} \partial_\nu \chi
\]
(C11)

where
\[
K_{\mu \nu} = -\frac{\delta \langle j_{\mu \beta}^Q \rangle}{\delta \partial_\nu \chi} = -\lim_{\Omega \to 0} \frac{P^{R\mu \nu}(\Omega) - P^{R\mu \nu}(0)}{i \Omega}
\]
(C12)

is the standard Kubo formula for a dc response \((B3)\), $P^{R\mu \nu}(\Omega)$ being the retarded current-current correlation function, and
\[
M_{\mu \nu} = \frac{\delta \langle j_{\mu \beta}^Q \rangle}{\delta \partial_\nu \chi} = -\sum_n \epsilon_n f_n \langle \{ V_\mu, x_\nu \} | n \rangle + \sum_n \frac{i}{4} f_n \langle n | [ V_\mu, V_\nu ] | n \rangle
\]
(C13)

is a contribution from “diathermal” currents \((B8)\). Note that the latter vanishes for the longitudinal response while it remains finite for the transverse response. As we will show later in this section, at $T = 0$ there is an important cancellation between \((C12)\) and \((C13)\) which renders the thermal conductivity $\kappa_{\mu \nu}$ well-behaved and prevents the singularity from the temperature denominator in \((4.2)\).

The retarded thermal current-current correlation function $P^{R\mu \nu}(\Omega)$ can be expressed in terms of the Matsubara finite temperature correlation function
\[
P^{\mu \nu}(i\Omega) = -\int_0^\beta d\tau e^{i\Omega \tau} \left\langle T_r j_{\mu}(r, \tau) j_{\nu}(r', 0) \right\rangle
\]
(C14)
as
\[
P^{R\mu \nu}(\Omega) = P^{\mu \nu}(i\Omega \to \Omega + i0).
\]
(C15)

The current $j_\mu(\mathbf{r}, \tau)$ in Eq. \((C14)\) is given by
\[
j_\mu(\mathbf{r}, \tau) = \frac{1}{2} \langle \Psi^\dagger(\mathbf{r}) V_\mu \partial_\tau \Psi(\tau) - \partial_\tau \Psi^\dagger(\tau) V_\mu \Psi(\tau) \rangle
\]
(C16)

As pointed out by Ambegaokar and Griffin \((A1)\), the time ordering operator $T_r$ and time derivative operators $\partial_\tau$ do not commute and neglecting this subtlety can lead to formally divergent frequency summations \((B4)\). Taking heed of this subtlety, substitution of \((C16)\) into \((C14)\) amounts to
\[
P^{\mu \nu}(i\Omega) = -\frac{1}{4} \sum_{nm} \langle \{ V_\mu, m \} | \langle m V_n^\dagger \rangle | \langle n + \epsilon \rangle \rangle \langle \langle \epsilon + \epsilon \rangle \rangle S_{nm}(i\Omega)
\]
(C18)

where $S_{nm}$ is given by Eq. \((A7)\). Analytically continuing $i\Omega \to \Omega + i0$ we obtain
\[
P^{R\mu \nu}(\Omega) = \frac{1}{4} \sum_{nm} \frac{(\epsilon_n + \epsilon_m)2}{\epsilon_n - \epsilon_m + \Omega + i0} \langle f_n - f_m \rangle.
\]
(C19)

Note that the only difference between the Kubo contribution to the thermal response \((C19)\) and the spin response \((B7)\) is the value of the coupling constant. In the case of thermal response \((C19)\), the coupling constant is $(\epsilon_n + \epsilon_m)/2$ which is eigenstate dependent, while in the case of spin response \((B7)\) it is $\hbar/2$ and eigenstate independent.

Using Eq. \((C12)\) we find that the Kubo contribution to the thermal transport coefficient is given by
\[
K_{\mu \nu} = -\frac{i}{4} \sum_{nm} \frac{(\epsilon_n + \epsilon_m)^2}{(\epsilon_n - \epsilon_m + i0)^2} \langle V_{nm}^\dagger V_{mn} \rangle \langle f_n - f_m \rangle
\]
(C20)

This can be written as
\[
K_{\mu \nu} = K_{\mu \nu}^{(1)} + K_{\mu \nu}^{(2)}
\]
(C21)
where
\[
K_{\mu \nu}^{(1)} = -\frac{i}{4} \sum_{nm} \frac{4\epsilon_n \epsilon_m}{(\epsilon_n - \epsilon_m + i0)^2} \langle V_{nm}^\dagger V_{mn} \rangle \langle f_n - f_m \rangle
\]
(C22)
and
\[ K^{(2)}_{\mu\nu} = -\frac{i}{4} \sum_{nm} V_{\mu}^{nm} V_{\nu}^{mn} (f_n - f_m) \] (C23)

Similarly, we can separate the “diathermal” contribution (C13) as
\[ M_{\mu\nu} = M_{\mu\nu}^{(1)} + M_{\mu\nu}^{(2)} \] (C24)

where \( M_{\mu\nu}^{(1)} \) and \( M_{\mu\nu}^{(2)} \) refer to the first and second term in (C13) respectively. Using the completeness relation \( \sum_m |m\rangle \langle m| = 1 \) it is easy to show that
\[ M_{\mu\nu}^{(2)} = \frac{i}{4} \sum_{nm} (f_n - f_m) V_{\mu}^{nm} V_{\nu}^{mn}. \] (C25)

Comparison of (C22) and (C25) yields \( M_{\mu\nu}^{(2)} + K_{\mu\nu}^{(2)} = 0 \). Therefore the thermal response coefficient is given by
\[ L_{\mu\nu}^{(T)} = K_{\mu\nu}^{(1)} + M_{\mu\nu}^{(1)}. \] (C26)

Utilizing commutation relationships (3.6), \( M_{\mu\nu}^{(1)} \) can be expressed in the form:
\[ M_{\mu\nu}^{(1)} = \int \eta f(\eta) \text{Tr} \left( \delta(\eta - \tilde{H}_0)(x^{\mu}V^{\nu} - x^{\nu}V^{\mu}) \right) d\eta \] (C27)
where the integral extends over the entire real line. The rest of the section follows closely Smrčka and Středa [34].

We define the resolvents \( G^\pm \):
\[ G^\pm \equiv (\eta \pm i0 - \tilde{H}_0)^{-1} \] (C28)

and operators
\[
\begin{align*}
A(\eta) &= i \text{Tr} \left( V_{\mu}^{\alpha \beta} \frac{dG^+}{d\eta} \right) \\
B(\eta) &= i \text{Tr} \left( V_{\mu}^{\alpha \beta} G^+ \delta(\eta - \tilde{H}_0) \right)
\end{align*}
\] (C29)

To facilitate Sommerfeld expansion we note that response coefficients \( K_{\mu\nu}^{(1)}(T) \), \( M_{\mu\nu}^{(1)}(T) \), \( \sigma_{\mu\nu}^s(T) \) have generic form
\[ L(T) = \int f(\eta) l(\eta) d\eta, \] (C30)
and after integration by parts
\[ L(T) = -\int \frac{df}{d\eta} \tilde{L}(\eta) d\eta. \] (C31)

Here \( \tilde{L}(\xi) \) is defined as
\[ \tilde{L}(\xi) \equiv \int_{-\infty}^{\xi} l(\eta) d\eta. \] (C32)

Note that \( L(T = 0) = \tilde{L}(\xi = 0) \) and in particular the spin conductivity at \( T = 0 \) satisfies \( \sigma_{\mu\nu}^s(T = 0) = \tilde{\sigma}_{\mu\nu}^s(\xi = 0) \).

Identities (C31) and (C32) will enable us to express the coefficients at finite temperature through the coefficients at zero temperature. For example, it follows from Eq. (C27) that
\[ M^{(1)}(\xi) = \int_{-\infty}^{\xi} \eta \text{Tr} \left( \delta(\eta - \tilde{H}_0)(x^{\mu}V^{\nu} - x^{\nu}V^{\mu}) \right) \] (C33)
and from Eqs. (B8, C29)
\[ \tilde{\sigma}_{\mu\nu}^s(\xi) = \frac{1}{4} \int_{-\infty}^{\xi} A(\eta) d\eta \] (C34)

Similarly, coefficient \( K_{\mu\nu}^{(1)} \) from (C22) can be written as
\[ K_{\mu\nu}^{(1)}(\xi) = -i \int d\eta f(\eta) \eta \sum \delta(\eta - \epsilon_n) \epsilon_m \times \left( \frac{V_{\mu}^{nm} V_{\nu}^{mn}}{(\eta - \epsilon_m + i0)^2} - \frac{V_{\mu}^{mn} V_{\nu}^{nm}}{(\eta - \epsilon_m - i0)^2} \right) \] (C35)
so that
\[ \tilde{K}_{\mu\nu}^{(1)}(\xi) = -i \int_{-\infty}^{\xi} d\eta \eta \sum \delta(\eta - \epsilon_n) \epsilon_m \times \left( \frac{V_{\mu}^{nm} V_{\nu}^{mn}}{(\eta - \epsilon_m + i0)^2} - \frac{V_{\mu}^{mn} V_{\nu}^{nm}}{(\eta - \epsilon_m - i0)^2} \right) \] (C36)

Using definitions (C29), \( \tilde{K}^{(1)}(\xi) \) can be expressed as
\[ \tilde{K}_{\mu\nu}^{(1)}(\xi) = \int_{-\infty}^{\xi} \eta^2 A(\eta) d\eta + \int_{-\infty}^{\xi} \eta B(\eta) d\eta. \] (C37)

After integration by parts obtains
\[ \tilde{K}_{\mu\nu}^{(1)}(\xi) = \xi^2 \int_{-\infty}^{\xi} A(\eta) d\eta + \frac{\xi^2}{2} \left( A(\eta) - \frac{1}{2} \frac{dB}{d\eta} \right) d\eta. \] (C38)

As shown in Ref. (34) the last term in this expression is exactly compensated by \( \tilde{M}^{(1)}(\xi) \): This becomes evident after noting that definitions in Eq. (C26) imply
\[ A - \frac{1}{2} \frac{dB}{d\eta} = \frac{1}{2} \text{Tr} \left[ \frac{d\delta(\eta - \tilde{H}_0)}{d\eta} \left( x^{\mu}V^{\nu} - x^{\nu}V^{\mu} \right) \right]. \] (C39)

Substituting the last identity into the Eq. (C38) and integrating the second term by parts we obtain
\[ \tilde{K}_{\mu\nu}^{(1)}(\xi) = \xi^2 \int_{-\infty}^{\xi} A(\eta) d\eta \]
\[ - \frac{\xi}{2} \int_{-\infty}^{\xi} \eta \text{Tr} \left( \delta(\eta - \tilde{H}_0)(x^{\mu}V^{\nu} - x^{\nu}V^{\mu}) \right) d\eta. \] (C40)
The second term here is equal to $-M_{\mu\nu}^{(1)}$ from (C33). After the cancellation the result simply reads:

$$L_{\mu\nu}^Q(\xi) = \tilde{L}_{\mu\nu}^Q(\xi) + \tilde{M}_{\mu\nu}^{(1)}(\xi) = \xi^2 \int_{-\infty}^{\xi} A(\eta)d\eta.$$  \hspace{1cm} (C41)

Or, using (C34)

$$L_{\mu\nu}^Q(\xi) = \left(\frac{2\xi}{\hbar}\right)^2 \tilde{\sigma}_{\mu\nu}(\xi).$$  \hspace{1cm} (C42)

Finally, from Eq.(C31) we find

$$L_{\mu\nu}^Q(T) = -\frac{4}{\hbar^2} \int \frac{df(\eta)}{d\eta} \eta^2 \tilde{\sigma}_{\mu\nu}(\eta)d\eta.$$  \hspace{1cm} (C43)

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