Flux Tubes in Effective Field Theory

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Abstract

Quark-antiquark bound states are examined in the long-range strong-coupling limit with the minimal area law of lattice gauge theory assumed as input. Matrix element relations are established which in the effective theory obtain dynamical equations equivalent to a formulation of the flux-tube model.
1 Introduction

The success of QCD as the correct theory of strong interactions has rested primarily upon perturbative calculations. The prediction of hadronic properties is inherently non-perturbative and has been much slower to develop. Most off lattice progress has turned out to be largely irrelevant to hadron dynamics. An exception has been the low quark velocity Wilson loop expansion [1]. The aim of this paper is to extend this method to arbitrary quark velocity. Our principal result is that under natural assumptions QCD provides a framework for meson dynamics that is identical with the relativistic flux tube (RFT) model [2, 3].

The RFT model assumes that a chromoelectric flux tube stretches in a straight line between the quark and the antiquark in a meson. The tube rest frame energy per unit length $a$ contributes to rotational energy, momentum, and angular momentum in the meson rest frame. Also, as pointed out by Buchmüller [4], a pure chromoelectric field in the tube rest frame implies no long range spin-spin correlation. In the last few years the RFT model has been demonstrated to be numerically calculable for any quark mass case [3]. The RFT model provides intuitive physical pictures for relativistic corrections in agreement with QCD and becomes a Nambu-Goto string for high rotational states [2].

The QCD Lagrangian density is a fundamental object in the theory of strongly interacting particles. For a meson it is given by the minimally transformed free Lagrangian

$$p_\mu \rightarrow p_\mu - g A_\mu ,$$

$$\mathcal{L}_{free} \rightarrow \mathcal{L}_{QCD} = \sum_j \bar{q}_j (\not{D} - m_j) q_j - \frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} ,$$

from which follow the Euler-Lagrange equations

$$(\not{D} - m_j) q_j = 0 ,$$

$$D^\mu F_{\mu \nu} = g \sum_j \bar{q}_j T_{\gamma \nu} q_j .$$
These are non-linear and amenable only as a perturbation expansion in small coupling $g$. Wilson’s observation \cite{6} over two decades ago has led to some of the most fruitful work in the non-perturbative regime. The simple picture offered in his area law for static quarks is compelling: Contributions to the $q\bar{q}$ propagator fall off exponentially in the area swept out by lines of gauge invariant chromoelectric flux joining the constituents’s world lines; in this way, widely separated paths are suppressed. Subsequent lattice simulations verify the same qualitative behavior for dynamical quarks.

Progress in obtaining a consistent description of confinement for realistic off-lattice calculations has been largely uncertain. The variety of semi-relativised confinement models based on the above simple idea is striking and is partly due to the limited theoretical input available beyond the static limit. Notable exceptions are in works of Eichten and Feinberg, Gromes and later those of Brambilla and Prosperi \cite{1} where analytic expressions are derived from the minimal area law to first order in the inverse heavy quark mass. More recently, a relativistic QCD-string Lagrangian for spinless quarks \cite{7} has been deduced.

Here we derive an effective Hamiltonian for a quark-antiquark system in the confinement region with the minimal area (MA) law of lattice gauge theory as input. In this arrangement, the gauge field’s non-Abelian character is manifest through the minimal area law; complete non-Abelian expressions are therefore given only where useful, e.g., to clarify a point. Our method consists in expressing the usual canonical matrix elements from \cite{2} in terms of Wilson loop expectation values derived from the MA law. In the effective theory these become first-quantized operators acting on state vectors. The results are fully relativistic.
2 Wilson loop expectation values

We begin with the Lagrangian density assuming that the MA analysis has been carried out; hence fermion and gauge fields are interrelated at the outset, effectively reducing the degrees of freedom in (2). The idea is simple: gauge fields specified by MA relations are taken to be "external" to the Lagrangian; Euler-Lagrange non-linearities (3-4) are thereby hidden within undetermined matrix elements.

From the energy-momentum tensor of (2) the conserved quantities are the usual

\[
H = \sum_j \int d^3x \bar{q}_j [\gamma \cdot (-i \nabla - gA) + \gamma^0 gA_0 + m_j]q_j ,
\]

(5)

\[
P = \sum_j \int d^3x q_j^\dagger (-i \nabla) q_j ,
\]

(6)

\[
J = \sum_j \int d^3x q_j^\dagger [\vec{x} \times (-i \nabla) + \frac{1}{2} \sigma] q_j ,
\]

(7)

which with the MA law,

\[
i \ln \langle W(C) \rangle = aS_{\text{min}} ,
\]

(8)

define our bound state problem. Eventually, we take independent variations of (8) along the paths \( C \).

We consider the Wilson loop of a straight slice out of the \( q\bar{q} \) minimal world sheet from time \( \tau' \),

\[
\langle W(C) \rangle = \frac{1}{3} \langle \text{Tr} \ P \exp(i g \int_C dx^\mu A_\mu(x)) \rangle
\]

\[
= \frac{1}{3} \langle \text{Tr} \ P \exp(-i g \int_{\tau_1'}^{\tau_1''} d\tau [A_0(z_1) - \dot{z}_1 \cdot A(z_1)]) U(z_1', z_1'') \rangle
\]

\[
\times \langle P \exp(-i g \int_{\tau_2'}^{\tau_2''} d\tau [A_0(z_2) - \dot{z}_2 \cdot A(z_2)]) U(z_2', z_2'') \rangle .
\]

The \( U \)’s here are straight-line path ordered exponentials, and the average is taken over gauge fields,

\[
\langle \vartheta \rangle = \frac{\int \mathcal{D}A \exp(i S[A]) \vartheta[A]}{\int \mathcal{D}A \exp(i S[A])} ,
\]

(9)
where the pure gauge action \( S[A] \) includes a gauge fixing term.

The c-number path variables for a given gauge field (\( z_i \) and \( \dot{z}_i \)) are related to coordinate and mechanical-momentum matrix elements,

\[
\begin{align*}
z_i &\equiv z_i(\tau; A) = \langle q_i(\tau; A)| \hat{x} |q_i(\tau; A)\rangle, \\
m_i\gamma_i \dot{z}_i &\equiv \langle q_i(\tau; A)| (\hat{p} - gA(\hat{x})) |q_i(\tau; A)\rangle, \\
\gamma_i &\equiv \frac{1}{\sqrt{1 - \dot{z}_i^2}}.
\end{align*}
\]

where \( q_i \) is either a particle or antiparticle Euler-Lagrange solution \((3-4)\) with a classical gauge field. These valence or quenched solutions prevent the possibility of time-backtracking \([7]\) not present in the MA picture. If \( \chi \) is the meson wavefunction in a product space of \( |q_1\rangle \) and \( |q_2\rangle \), a reasonable ansatz is the auxiliary condition

\[
\chi \to \Lambda_+^{(1)} \chi \Lambda_-^{(2)} - \Lambda_-^{(1)} \chi \Lambda_+^{(2)},
\]

where \( \Lambda_{\pm}^{(i)} \) are positive or negative energy projection operators. In this approximation, the Casimir operators reappear in the Hamiltonian \((5)\).

On the contrary, we point out that the quark degrees of freedom have not yet entered the dynamics except as a means to specify the physical gauge field configurations. The above identifications are made for later use.

The enclosed area we parameterize in the Nambu-Goto form,

\[
S = a \int_{\tau_f}^{\tau_i} d\tau \int_0^1 d\sigma \left[ -\dot{x}^2 x'^2 + (\dot{x} \cdot x')^2 \right]^{1/2}
= a \int_{\tau_f}^{\tau_i} d\tau \int_0^1 d\sigma S
\]

with \( x_{\mu} = x_{\mu}(\tau, \sigma) \), \( x_0 = \tau \), \( \dot{x}_{\mu} = \frac{\partial x_{\mu}}{\partial \tau} \), and \( x'_{\mu} = \frac{\partial x_{\mu}}{\partial \sigma} \). At the boundary, \( x(\tau, 1) = \langle z_1(\tau) \rangle \) and \( x(\tau, 0) = \langle z_2(\tau) \rangle \). In the usual straight-line and equal-times approximations, which we assume here, the minimum is given by

\[
x_{\text{min}}(\tau, \sigma) = \sigma \langle z_1(\tau) \rangle + (1 - \sigma) \langle z_2(\tau) \rangle.
\]
A general path variation of (8) for fixed endpoints and fixed time,
\[ \langle i \delta W(C) \rangle = \frac{-i}{\langle W(C) \rangle} \cdot i \int_{\tau'}^{\tau_f} d\tau \frac{1}{3} \left\langle \text{Tr} P \exp \left( ig \oint_C dx^\mu A_\mu(x) \right) \times g \left\{ (\delta z_1 \cdot \frac{\partial}{\partial z_1} + \delta \dot{z}_1 \cdot \frac{\partial}{\partial \dot{z}_1})[A_0(z_1) - \dot{z}_1 \cdot A(z_1)] \right\} \right. \\
\left. \quad - (\delta z_2 \cdot \frac{\partial}{\partial z_2} + \delta \dot{z}_2 \cdot \frac{\partial}{\partial \dot{z}_2})[A_0(z_2) - \dot{z}_2 \cdot A(z_2)] \right\} \right\rangle \]
\[ = a \int_{\tau'}^{\tau_f} d\tau \int_0^1 d\sigma \left[ \frac{\partial S}{\partial x^\mu} \delta x^\mu + \frac{\partial S}{\partial \dot{x}^\mu} \delta \dot{x}^\mu \right]_{x_{\text{min}}} , \quad (16) \]
specified to \( \delta z_1 = \delta z_2 \), obtains in the center of momentum (see Appendix for details)
\[ \left\langle \langle g A(z) \rangle \right\rangle \equiv \frac{1}{\langle W(C) \rangle} \frac{1}{3} \left\langle \text{Tr} P \exp(ig \oint_C dx^\mu A_\mu(x)) g A(z) \right\rangle \]
\[ = a \int_0^{(z)} dx \gamma_{\perp} \hat{x}_1 \equiv p_t , \quad (17) \]
which are the desired Wilson loop expectation values with reference points chosen at the origin, \( A(0) = 0 \), and "\( \perp \)" defined relative to the straight line, \( \mathbf{v}_\perp \equiv \mathbf{v} - (\mathbf{v} \cdot \hat{r})\hat{r} \). It should be clear that the gauge fields quantized in this average carry spatial dependence in the radial coordinate only. Evidently, this procedure selects the physically realizable transverse polarizations. The time component is obtained by simple differentiation of (8) with respect to \( \tau \),
\[ \frac{d}{d\tau} i \ln \langle W(C) \rangle = \frac{-i}{\langle W(C) \rangle} \cdot \frac{1}{3} \left\langle \text{Tr} P \exp \left( ig \oint_C dx^\mu A_\mu(x) \right) \times g \left\{ A_0(z_1) - A_0(z_2) - [\dot{z}_1 \cdot A(z_1) - \dot{z}_2 \cdot A(z_2)] \right\} \right\rangle \]
\[ = -a \int_0^1 d\sigma S_{x_{\text{min}}} , \quad (18) \]
yielding (using the procedure described in the Appendix)
\[ \langle \langle g A_0(z) \rangle \rangle = a \int_0^{(z)} dx \gamma_{\perp} \equiv H_t , \quad (19) \]
and also
\[ \langle \langle \mathbf{z} \times g A(z) \rangle \rangle = a \int_0^{(z)} dx (\mathbf{x} \times \gamma_{\perp} \hat{x}_1) \equiv L_t . \quad (20) \]
3 Relation to Flux Tube Model

With the Wilson loop expectation values we re-express conserved quantities (5-7) in terms of coordinate and velocity matrix elements. In the effective theory these are promoted to noncommuting quantum operators, observables requiring symmetrization. We give the final expressions

\[ H = \sum_{i=1}^{2} \alpha_i \cdot [m_i \gamma_i \dot{x}_i]_{\text{sym}} + \left[ \langle\langle gA_0(x_i) \rangle\rangle \right]_{\text{sym}} + \beta m_i , \quad (21) \]

\[ P = \sum_{i=1}^{2} [m_i \gamma_i \dot{x}_i]_{\text{sym}} + \left[ \langle\langle gA(x_i) \rangle\rangle \right]_{\text{sym}} = 0 , \quad (22) \]

\[ J = \sum_{i=1}^{2} [x_i \times m_i \gamma_i \dot{x}_i]_{\text{sym}} + \left[ \langle\langle x_i \times gA(x_i) \rangle\rangle \right]_{\text{sym}} + \frac{\sigma_i}{2} . \quad (23) \]

This compares well with other results from the Wilson loop. Both spin and spin-independent Hamiltonians of [1], for example, are reproduced on semi-relativistic reduction of (22). Equations (21-23) in fact define the relativistic flux tube model [8] as formulated by the present authors in [2, 3]. There, dynamical relations are derived from a heuristic Lagrangian in which the flux tube is fashioned as a simple constant energy density. The "tube operators" are equivalent to the Wilson loop expectation values above. This straightforward derivation serves to clarify the model’s close relation to the underlying QCD.

4 Conclusion

We have demonstrated that under the same assumptions as in previous work [1, 7] that the Wilson loop area law and the QCD Lagrangian yields a relativistically valid picture of dynamical confinement. We emphasize that we have not “solved” QCD, but that our ignorance can be distilled into three loop expectation values each of which is equal to that of the mechanical RFT model. We find that

\[ \langle\langle gA_0 \rangle\rangle = H_t , \quad (24) \]
for the tube energy, momentum, and angular momentum respectively. In addition we have shown that the natural equation of motion is the Salpeter equation in which the “covariant tube substitution” \[3\] has been made.

The verification of the RFT model structure promises more than to legitimize a physically reasonable model. The close relation of the RFT model to lattice QCD should shed light on both subjects. In addition, a systematic program of improving the RFT model to include field fluctuations \[8\] can now be envisioned.

A Appendix

Equation (17) follows from the physical gauge field’s spatial dependence in the straight line and equal times approximations

\[
A(r) \sim \sum_k \exp(-i k \cdot r) \rightarrow \sum_k \exp(-i k r) .
\] (27)

Then small angular variations of a given path at fixed time leave \(A\) unchanged. It will suffice to consider the transverse part of (16) with

\[
\delta z_i(\tau; A) = \delta z_i(\tau; A') \equiv \delta z_i ,
\] (28)

for all gauge fields at a given \(\tau\). Naturally,

\[
\delta \langle z_i \rangle = \delta z_i .
\] (29)

We write the Wilson loop in discrete form,

\[
W(C) = \frac{1}{3} \text{Tr} \prod_{n=0}^{N} \exp(i g \Delta t_n \left[ A_0(z_{1n}) - \dot{z}_{1n} \cdot A(z_{1n}) \right]) U(z_2', z_1')
\times \exp(-i g \Delta t_n \left[ A_0(z_{2n}) - \dot{z}_{2n} \cdot A(z_{2n}) \right]) U(z_1 f, z_2 f) ,
\] (30)
and take the path variations of (8) according to

\[ \delta(i \ln \langle W(C) \rangle) = \frac{i}{\langle W(C) \rangle} \delta \langle W(C) \rangle \]

\[ = \frac{i}{\langle W(C) \rangle} \left\langle \sum_{n=0}^{N} (\delta z_{1n} \cdot \frac{\partial}{\partial z_{1n}} + \delta \dot{z}_{1n} \cdot \frac{\partial}{\partial \dot{z}_{1n}} \rangle \right. \]

\[ + \left. \delta z_{2n} \cdot \frac{\partial}{\partial z_{2n}} + \delta \dot{z}_{2n} \cdot \frac{\partial}{\partial \dot{z}_{2n}} \right) \langle W(C) \rangle. \]

(31)

giving equation (16),

\[ g \langle W(C) \rangle \int_{\tau}^{\tau'} d\tau \frac{1}{3} \langle \text{Tr P exp}(i g \oint_C dx^\mu A_\mu(x)) \sum_{i=1}^{2} (\delta z_i \cdot \frac{\partial}{\partial z_i} + \delta \dot{z}_i \cdot \frac{\partial}{\partial \dot{z}_i}) f(z_i, \dot{z}_i) \rangle \]

\[ = a \int_{\tau}^{\tau'} d\tau \int_{0}^{1} d\sigma \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial}{\partial \sigma} \delta x^\mu + \frac{\partial S}{\partial \dot{x}^\mu} \frac{\partial}{\partial \tau} \delta x^\mu \right]_{x_{min}}, \] (32)

where

\[ f(z_i, \dot{z}_i) = (-1)^i [z_i \cdot gA(z_i) - gA_0(z_i)] . \] (33)

The temporal derivative is transferred from the variations by partial integration, so that

\[ \frac{1}{\langle W(C) \rangle} \int_{\tau}^{\tau'} d\tau \sum_i \frac{1}{3} \langle \text{Tr P exp}(i g \oint_C dx^\mu A_\mu(x)) \rangle (-\frac{d}{d\tau} \frac{\partial}{\partial z_i} + \frac{\partial}{\partial \dot{z}_i}) f(z_i, \dot{z}_i) \rangle \cdot \delta z_i \]

\[ = a \int_{\tau}^{\tau'} d\tau \int_{0}^{1} d\sigma \left[ - \left( \frac{\partial S}{\partial x^\mu} \right)_{x_{min}} \cdot (\delta z_1 - \delta z_2) + \frac{d}{d\tau} \left( \frac{\partial S}{\partial \dot{x}^\mu} \right)_{x_{min}} \cdot \left[ \sigma \delta z_1 + (1 - \sigma) \delta z_2 \right] \right], \]

(34)

yielding in the \( \delta z_1 = \delta z_2 (\equiv \delta z_\perp) \) case of interest

\[ - \sum_i \langle \langle \frac{\partial}{\partial z_i} f(z_i, \dot{z}_i) \rangle \rangle = a \int_{0}^{1} d\sigma \left( \frac{\partial S}{\partial x} \right)_{x_{min}}, \] (35)

or

\[ \langle \langle gA(z_1) - gA(z_2) \rangle \rangle = a \langle \langle z_1 \rangle \rangle - \langle \langle z_2 \rangle \rangle \int_{0}^{1} d\sigma \frac{\sigma \langle \dot{z}_{1\perp} \rangle + (1 - \sigma) \langle \dot{z}_{2\perp} \rangle}{1 - \sigma \langle \dot{z}_{1\perp} \rangle + (1 - \sigma) \langle \dot{z}_{2\perp} \rangle}^{1/2}. \] (36)

This obtains (17) when reference points are chosen at the origin of coordinates.
Also, with the $A_0$ reference point chosen at the origin, $A_0(0) = 0$, and (18) can be written
\[
1\langle W(C) \rangle = \frac{1}{(W(C))^{1/3}} \left( \frac{\text{Tr} \int dA_0 g[A_0(z_{A_0}) - \hat{z}_{A'_t} A_t(z_{A_0})] \int \mathcal{D}A \exp(i\oint C dx \mu A_{\mu}) \exp(is[A])}{\int \mathcal{D}A \exp(is[A])} \right) \\
= a \langle z_t \rangle \int_0^1 ds \left[ 1 - \sigma^2 \langle \dot{z}_t \rangle_\perp \right]^{1/2},
\]
with
\[
\langle \dot{z}_t \rangle = \frac{\int dA_0 \dot{z}_{A_t} \int \mathcal{D}A \exp(is[A])}{\int \mathcal{D}A \exp(is[A])}. \tag{38}
\]
Then, the functional derivative of (37) with respect to $\dot{z}_{A'_t}$ is
\[
\frac{1}{(W(C))^{1/3}} \left( \frac{\text{Tr} \int dA_0 \dot{z}_{A'_t} A_t(z_{A_0}) \int \mathcal{D}A \exp(i\oint C dx \mu A_{\mu}) \exp(is[A])}{\int \mathcal{D}A \exp(is[A])} \right) \\
= a \langle z_t \rangle \int_0^1 ds \left[ 1 - \sigma^2 \langle \dot{z}_t \rangle_\perp \right]^{-1/2} \int \mathcal{D}A \exp(is[A]).
\]
Taking the scalar product of the above with $\dot{z}_{A'_t}$, and integrating over $A'_t$, gives
\[
\langle \langle z \cdot gA \rangle \rangle = a \int^{(z)} dx \hat{x}_\perp \gamma_\perp, \tag{40}
\]
which is used in obtaining (19). Taking the vector product of (39) with $\dot{z}_{A'_t}$ yields (20) after integration over $A'_t$,
\[
\langle \langle z \times gA(z) \rangle \rangle = a \int^{(z)} dx (x \times \gamma_\perp \hat{x}_\perp). \tag{41}
\]

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