OVERBARRIER RESONANCES AS SOLUTIONS OF SET INHOMOGENEOUS SHRÖDINGER EQUATIONS

Il-Tong Cheon
Department of Physics, Yonsei University, Seoul 120 – 749, Korea

G.Kim
Institute of Nuclear Physics, Tashkent, Uzbekistan
Department of Physics, Yonsei University, Seoul 120 – 749, Korea

A.V. Khugaev
Institute of Nuclear Physics, Tashkent, Uzbekistan

Abstract

In the paper the Schrödinger equation for quasibound resonance state with complex energy is considered. The system of inhomogeneous differential equations is obtained for the real and imaginary parts of wave function. On the base of known solution of corresponding homogeneous equation, the inhomogeneous system is solved with help of iteration procedure. The single-particle neutron 2p-state in the Woods - Saxon potential is analyzed for $^{13}$C nucleus.

I. INTRODUCTION

Many papers, published since 1980 and devoted to investigation of the structure of nuclei with $A = 13$, show that for description of the ground states of these nuclei, it is necessary to take the $2p$ - shell into account in a ground state [1 - 17]. Recently nuclear structure calculation appeared in [11], which described nuclei with $A = 4 – 16$ within a full $(0+2)\hbar\omega$ shell model space, i.e. configuration mixing, is not only restricted to the $1p$ - shell, but includes contributions from the $2s$-, $1d$-, $2p$- and $1f$-shells as well. These wave functions provide a good overall description of the relevant energy spectra. Furthermore,

1Fax: 82-2-392-1592, E-mail: itcheon@phya.yonsei.ac.kr  Tel: (02)361 - 2610
from the analysis of the $M1$ form factor of electron elastic scattering on the nucleus $^{13}C$ [3 - 11,14], it is followed that the $2p$ - shell is the most important one among the $2s$-, $1d$-, $2p$- and $1f$- shells. The similar conclusion regarding $2p$ - shell admixtures was obtained for the processes with pion [12,13,15]: pion photoproduction, radiative pion capture on $^{13}C$ and pion single charge exchage on $^{13}C$.

If we choose as an average field the oscillator potential, parameters of which are obtained in accordance with the binding energy of nucleon in $1p$ - shell, the binding energy of nucleon in $2p$ - shell becomes positive. Thereby, at the value $b = 1.663$ $fm$ for oscillator parameter and $E_{\text{bind}}(1p) = -4.947$ $MeV$ for the neutron binding energy on $1p$ - shell of $^{13}C$, we find the binding energy in $2p$ - shell $E_{\text{bind}}(2p) = 25.016$ $MeV$ in the same oscillator well. (Somewhat less value $E_{\text{bind}}(2p_{1/2}) = 14.65$ $MeV$ was obtained in work [18] from the analysis of the spectra $^{12}C$, $^{13}C$ and $^{11}C$ nuclei.) And if the $2p$ - state with such positive energy is bound in the oscillator potential because of the depth of oscillator well being infinity, then for the realistic potential of Woods - Saxon, the $2p$ - state represents itself as an overbarrier resonance [19]. One the other hand, the $2s$-, $1d$ - states are subbarrier Gamov resonances, which can be considered, for example, by $R$ - matrix theory [20], or in the framework of the potential model with taking into account the continuum [21] into account. Regarding to the overbarrier resonances, such resonances do not exist in the complete theory as far as we know. At present the quasi-classical approach [19] and the complex scaling method [23] are developed for the calculation of characteristics of such resonances. Therefore, the determination of the wave function, the energy and the width of the overbarrier resonance in the real nuclei with the realistic average field is a nontrivial problem. This paper is devoted to find a solution of this problem.

II. FORMALISM

For determination of the resonance energy $E_{\text{res}}$ and the width $\Gamma$ of the resonant $2p$ - state, it is necessary to solve the time-dependent Schrödinger equation [24]:

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left[ \frac{\hbar^2}{2\mu} \Delta + U(r) \right] \Psi(\vec{r}, t).$$  (1)
And if we use the substitution $\Psi(\vec{r}, t) = \tilde{\varphi}(\vec{r}) \exp(-i\frac{E}{\hbar}t)$, then this partial differential equation can be converted into an ordinary one:

$$\left[ \hat{E} + \frac{\hbar^2}{2\mu} \Delta - U(r) \right] \tilde{\varphi}(\vec{r}) = 0. \quad (2)$$

Taking into account spherical symmetry of the potential $U(r)$ and substituting the expression $\tilde{\varphi}_{nlj}(r) = \chi_{nlj}(r)/r$ for radial part of wave function, we find:

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{r^2} + \frac{\hbar^2}{2\mu} l(l+1) + U(r) - \hat{E}_{nlj} \right] \chi_{nlj}(r) = 0, \quad (3)$$

where $\mu$ is the nucleon reduced mass, $l$ and $j$ are orbital and total angular momenta of the nucleon; $n$ is the principal quantum number, which denotes a number of knots in the radial wave function and is connected with a spectroscopic principal quantum number $n_c$ given by $n_c = n + 1$. For the potential $U(r) = V(r) + V_{S.O.} + V_{Coul.}$, we have the following expressions \[16\]:

the Woods - Saxon potential

$$V(r) = -V_0 \left\{ 1 + \exp \left( \frac{r - R_0}{a} \right) \right\}^{-1}, \quad (4)$$

and the spin - orbital interaction

$$V_{S.O.} = -\kappa \cdot (\vec{\sigma} \cdot \vec{l}) \cdot \frac{1}{r} \frac{d}{dr} V(r), \quad (5)$$

where

$$(\vec{\sigma} \cdot \vec{l}) = \begin{cases} 
  l, & j = l + \frac{1}{2}, \\
  -(l+1), & j = l - \frac{1}{2}.
\end{cases} \quad (6)$$

The nucleus radius is determined as $R_0 = r_0 A^{\frac{1}{3}}$. Here $r_0$ is the nucleon radius, $a$ is the diffuseness of potential and $\kappa = 0.263 \text{fm}^2$ is the parameter for the spin - orbital interaction. The depth of potential $V_0$ is calculated in accordance with the experimental value of the single-particle binding energy $E_{bind} = 4.947 \text{MeV}$ for the neutron on $1p$ - shell.
Although the Coulomb interaction does not need to be involved within the present framework of model description of $^{13}\text{C}$, it should participate to the case of the proton playing a role. It is given in the form

$$V_{\text{Coul.}}(r) = \frac{(Z - 1)e^2}{r} \cdot \begin{cases} \frac{3r}{2R_0} - \frac{1}{2} \left( \frac{r}{R_0} \right)^3, & r < R_0 \\ 1, & r \geq R_0. \end{cases}$$  

(7)

The solution of equation (3) will be searched with

$$\chi_{nlj}(r) = u_{nlj}(r) + iv_{nlj}(r).$$  

(8)

As $r \to \infty$, this solution must satisfy the following boundary condition:

$$\chi_{nlj}(r) \to \left(-\frac{i}{2}\right) \exp \{i(\bar{k}r - \eta \ln 2\bar{k}r - \frac{l\pi}{2} + \delta_l)\},$$  

(9)

with $\delta_l = \text{arg}\Gamma(l + 1 + i\eta)$, where $\eta$ is Coulomb parameter.

From eq (9), we have

$$\lim_{r \to \infty} \frac{d\ln \chi_{nlj}(r)}{dr} = i\bar{k} = \frac{i\sqrt{2\mu\tilde{E}_{nlj}}}{\hbar},$$  

(10)

and this condition selects the discrete complex value

$$\tilde{E}_{nlj} = (E_{\text{res}})_{nlj} - \frac{i}{2} \Gamma_{nlj}.$$  

(11)

Substituting these expressions for the energy (11) and wave function (8) into the radial part of the Schrödinger equation (3), and separating the real and imaginary parts, we obtain a set of two inhomogeneous differential equations:

\begin{equation}
\begin{cases}
\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + V_{\text{S.O.}} + V_{\text{Coul.}} + V(r) - (E_{\text{res}})_{nlj} \right] u_{nlj}(r) = \frac{1}{2} \Gamma_{nlj} \cdot v_{nlj}(r), \\
\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + V_{\text{S.O.}} + V_{\text{Coul.}} + V(r) - (E_{\text{res}})_{nlj} \right] v_{nlj}(r) = -\frac{1}{2} \Gamma_{nlj} \cdot u_{nlj}(r).
\end{cases}
\end{equation}

(12)

For $\Gamma_{nlj} = 0$, we have $v_{nlj}(r) = 0$ and $\chi_{nlj}(r) = u_{nlj}(r) \equiv u_{nlj}^{(o)}(r)$. Then, the second equation vanishes and the first one converts into homogeneous type as

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + V_{\text{S.O.}} + V_{\text{Coul.}} + V(r) \right] u_{nlj}^{(o)}(r) = E_{nlj} \cdot u_{nlj}^{(o)}(r).$$  

(13)
This homogeneous equation with the Wood-Saxon potential cannot be solved by the ordinary method. Therefore, it should be solved via the expansion on the Sturm-Liouville functions [25] (the detailed description of this method is given in Appendix).

The solution of inhomogeneous differential equation of the second order can be expressed through that of the appropriate homogeneous equation $u^{(o)}_{nlj}(r)$ [25]:

$$u_{nlj}(r) = C_{u1} \cdot u^{(o)}_{nlj}(r) + C_{u2} \cdot u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} d\eta + u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} \left( \int^{\eta} \frac{2\mu}{\hbar^2} \Gamma_{nlj} \cdot v_{nlj}(\xi) u^{(o)}_{nlj}(\xi) d\xi \right) d\eta; (14.1)$$

$$v_{nlj}(r) = C_{v1} \cdot u^{(o)}_{nlj}(r) + C_{v2} \cdot u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} d\eta - u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} \left( \int^{\eta} \frac{2\mu}{\hbar^2} \Gamma_{nlj} \cdot u^{(o)}_{nlj}(\xi) u^{(o)}_{nlj}(\xi) d\xi \right) d\eta. (14.2)$$

Here all integrals are indefinite ones. If $\Gamma_{nlj} = 0$, it follows that $u_{nlj}(r) = u^{(o)}_{nlj}(r)$ and $v_{nlj}(r) = 0$. Then, from the first equation (14.1) one obtains, $C_{u1} = 1$ and $C_{u2} = 0$. Similarly, $C_{v1} = C_{v2} = 0$ follows from the second (14.2). Therefore, we have

$$u_{nlj}(r) = u^{(o)}_{nlj}(r) + G u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} \left( \int^{\eta} v_{nlj}(\xi) u^{(o)}_{nlj}(\xi) d\xi \right) d\eta; (15.1)$$

$$v_{nlj}(r) = -G u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} \left( \int^{\eta} u^{(o)}_{nlj}(\xi) u^{(o)}_{nlj}(\xi) d\xi \right) d\eta, (15.2)$$

where $G = \frac{2\mu}{\hbar^2} \Gamma_{nlj}$.

Substitution of the second equation (15.2) into the first one (15.1) leads to the integral equation for $u_{nlj}(r)$. The solution of this integral equation is used to determine $v_{nlj}(r)$ by the second equation (15.2). After all, one find

$$u_{nlj}(r) = u^{(o)}_{nlj}(r) - G^2 u^{(o)}_{nlj}(r) \int^{r} \frac{1}{[u^{(o)}_{nlj}(\eta)]^2} \int^{\eta} [u^{(o)}_{nlj}(\xi)]^2 d\eta.$$


\[
\times \int_\xi^\xi \frac{1}{[u_{nlj}^{(0)}(\nu)]^2} \int_\nu^\tau u_{nlj}^{(0)}(\nu) u_{nlj}(\tau) d\nu d\xi d\eta; \quad (16.1)
\]

\[
v_{nlj}(r) = -G u_{nlj}^{(0)}(r) \int^r \frac{1}{[u_{nlj}^{(0)}(\eta)]^2} \left( \int_\eta^\tau u_{nlj}^{(0)}(\xi) u_{nlj}(\xi) d\xi \right) d\eta. \quad (16.2)
\]

These integral equations can be solved by means of an iteration procedure (of consistent approximations): the initial approximation is chosen as

\[
\begin{cases}
u_{nlj}^{(0)}(r) - \text{the solution of the Sturm – Liouville problem}; \\
v_{nlj}^{(0)}(r) = 0;
\end{cases} \quad (17)
\]

and, then, the first iteration gives

\[
\begin{cases}
u_{nlj}^{(1)}(r) = \nu_{nlj}^{(0)}(r) + \Delta u_{nlj}^{(0)}(r); \\
u_{nlj}^{(1)}(r) = \Delta v_{nlj}^{(0)}(r);
\end{cases} \quad (18)
\]

where

\[
\Delta u_{nlj}^{(0)}(r) = -G^2 u_{nlj}^{(0)}(r) \int^r \frac{1}{[u_{nlj}^{(0)}(\eta)]^2} \int_\eta^\tau \frac{1}{[u_{nlj}^{(0)}(\xi)]^2} \int_\xi^\tau \frac{1}{[u_{nlj}^{(0)}(\nu)]^2} 
\times \int^\tau [u_{nlj}^{(0)}(\tau)]^2 d\tau d\nu d\xi d\eta; \quad (19a)
\]

\[
\Delta v_{nlj}^{(0)}(r) = -G u_{nlj}^{(0)}(r) \int^r \frac{1}{[u_{nlj}^{(0)}(\eta)]^2} \int_\eta^\tau u_{nlj}^{(0)}(\xi) u_{nlj}^{(1)}(\xi) d\xi d\eta. \quad (19b)
\]

The second iteration yields

\[
\begin{cases}
u_{nlj}^{(2)}(r) = \nu_{nlj}^{(1)}(r) + \Delta u_{nlj}^{(1)}(r); \\
u_{nlj}^{(2)}(r) = \Delta v_{nlj}^{(1)}(r);
\end{cases} \quad (20)
\]

where

\[
\Delta u_{nlj}^{(1)}(r) = G^2 u_{nlj}^{(1)}(r) \int^r \frac{1}{[u_{nlj}^{(0)}(\eta)]^2} \int_\eta^\tau \frac{1}{[u_{nlj}^{(0)}(\xi)]^2} \int_\xi^\tau \frac{1}{[u_{nlj}^{(0)}(\nu)]^2} 
\times \int^\tau [u_{nlj}^{(0)}(\tau)]^2 \cdot \Delta u_{nlj}^{(0)}(\tau) d\tau d\nu d\xi d\eta; \quad (21a)
\]

\[
\Delta v_{nlj}^{(1)}(r) = -G u_{nlj}^{(1)}(r) \int^r \frac{1}{[u_{nlj}^{(0)}(\eta)]^2} \int_\eta^\tau u_{nlj}^{(0)}(\xi) \cdot \Delta u_{nlj}^{(1)}(\xi) d\xi d\eta; \quad (21b)
\]

and etc.
The boundary conditions (9) and (10) can be rewritten as

\[
\begin{align*}
\lim_{r \to \infty} u_{nlj}(r) &= \frac{1}{2} \exp(\tilde{k}_I r + \eta \frac{\psi}{2}) \sin(\tilde{k}_R r - \eta \ln \frac{2\sqrt{2\mu\rho}}{\hbar} r - \frac{l\pi}{2} + \delta_l), \\
\lim_{r \to \infty} v_{nlj}(r) &= -\frac{1}{2} \exp(\tilde{k}_I r + \eta \frac{\psi}{2}) \cos(\tilde{k}_R r - \eta \ln \frac{2\sqrt{2\mu\rho}}{\hbar} r - \frac{l\pi}{2} + \delta_l),
\end{align*}
\]

(9a)  

(9b)

\[
\begin{align*}
\lim_{r \to \infty} \left\{ [u'_{nlj}(r)u_{nlj}(r) + v'_{nlj}(r)v_{nlj}(r)]/[u^2_{nlj}(r) + v^2_{nlj}(r)] \right\} &= \tilde{k}_I, \\
\lim_{r \to \infty} \left\{ [u_{nlj}(r)v'_{nlj}(r) - u'_{nlj}(r)v_{nlj}(r)]/[u^2_{nlj}(r) + v^2_{nlj}(r)] \right\} &= \tilde{k}_R.
\end{align*}
\]

(10a)  

(10b)

In these expressions, \( \tilde{k} = \tilde{k}_R - i\tilde{k}_I \) and

\[
\begin{align*}
\tilde{k}_R &= \frac{\sqrt{2\mu\rho}}{\hbar} \cos \frac{\psi}{2}, \quad \tilde{k}_I &= \frac{\sqrt{2\mu\rho}}{\hbar} \sin \frac{\psi}{2}, \\
\rho &= \sqrt{(E_{res})^2_{nlj} + \Gamma^2_{nlj}/4}, \quad \tan \psi = \frac{\Gamma_{nlj}}{2(E_{res})_{nlj}}.
\end{align*}
\]

(22)  

(23)

The inverse transformation gives

\[
(E_{res})_{nlj} = \frac{\hbar^2}{2\mu}(\tilde{k}_R^2 - \tilde{k}_I^2), \quad \Gamma_{nlj} = \frac{\hbar^2}{\mu}\tilde{k}_R \cdot \tilde{k}_I.
\]

(24)

Thus, by having \( u^{(c)}_{nlj}(r) \) and \( (E_{res})_{nlj} \) from the solution of the Sturm - Liouville problem, the wave functions \( u_{nlj}(r), v_{nlj}(r) \) and the width \( \Gamma_{nlj} \) can be calculated by means of the iteration procedure. And, this iteration process must converge under the execution of the boundary conditions (9a,b) and (10a,b).
III. RESULTS AND DISCUSSION

The above formalism has been used for calculation of the wave functions $u_{nlj}(r)$, $v_{nlj}(r)$, the resonance energy $(E_{\text{res}})_{nlj}$ and the width $\Gamma_{nlj}$ for $2p_{1/2}$ - state of neutron in $^{13}C$ nucleus. For this case, the Coulomb interaction does not play any role, and, thereby, it should disappear from the formalism given in the previous section.

For the Woods - Saxon potential the following parameters were used: $r_0 = 1.46$ fm, $a = 0.67$ fm, $V_0 = 32.3$ MeV ($E_{\text{bind}}(1p_{1/2}) = -4.947$ MeV). The wave function $u_{nlj}(r)$ and the energy $(E_{\text{res}})_{nlj}$ were obtained by means of solving the Sturm - Liouville problem \[16,17\] (see Appendix). The number of the expansion $M$ in sum (A.12) was taken equal to 30 so as to reach the enough good convergence on energy $|E_{2p_{1/2}}^{(M)} - E_{2p_{1/2}}^{(M-1)}| \leq 0.1$ MeV. The values $E_{2p_{1/2}}^{(M)}$ for the different values $M$ are shown in Fig. 1. The value $E_{2p_{1/2}} = 8.1$ MeV was obtained at $M = 30$. The value of the expanding coefficients $d_{n1_{1/2}}$ for the neutron wave function $u_{2p_{1/2}}^{(c)}(r)$ are presented in Table 1.

For $\Gamma_{n,lj} = \Gamma_{2p_{1/2}} = 22.244$ MeV, the wave functions $u_{2p_{1/2}}(r)$ and $v_{2p_{1/2}}(r)$ after 5th iteration were tailed smoothly together with their asymptotical expressions, i.e. the boundary conditions (9a,b) and (10a,b) were fulfilled.

For the time dependent part of wave function, we have

$$\Psi(t) = \exp\left(-\frac{i}{\hbar} \tilde{E}_{nlj} \cdot t\right) = \exp\left(-\frac{i}{\hbar} (E)_{nlj} \cdot t\right) \cdot \exp\left(-\frac{\Gamma_{nlj}}{2\hbar} \cdot t\right) =$$

$$= \exp\left(-\frac{\hbar}{2\mu} (\tilde{k}_R^2 - \tilde{k}_I^2) \cdot t\right) \cdot \exp\left(-\frac{\hbar}{2\mu} \tilde{k}_R \cdot \tilde{k}_I \cdot t\right), \quad (25)$$

where the first factor is the wave function oscillating over time, and the second factor is the descending function over time. There were used relations (11) and (24).

Notice that the full radial wave function is

$$\tilde{\Psi}_{nlj}(r,t) = \chi_{nlj}(r) \cdot \Psi(t). \quad (26)$$

If this wave function is considered in the 3 - dimensional space $(z, r, t)$, where $z = \text{Re} \tilde{\Psi}_{nlj}(r,t)$ or $\text{Im} \tilde{\Psi}_{nlj}(r,t)$, then one obtains a complicated surface
of the second order.
In the plane \((z, r)\), at \(t = 0\), the \(u_{nlj}(r)\) and \(v_{nlj}(r)\) are oscillating and increasing in amplitude (i.e. diverging) when \(r \to \infty\). In the plane \((z, t)\), at \(r = 0\), the real and imaginary parts of \(\chi_{nlj}(r = 0) \cdot \Psi(t)\) are oscillating and decreasing in amplitude (i.e. convergent) when \(t \to \infty\). And if the full wave function is considered in the plane \((z, t)\) for \(t = r/v\), where \(v = \frac{\hbar k_N}{\mu}\) is nucleon speed at the resonance energy \((E_{res})_{nlj}\) of 2\(p_{1/2}\) - state \((k_N = \sqrt{2\mu(E_{res})_{nlj}}/\hbar\) – the wave vector for nucleon in 2\(p_{1/2}\) - state), then

\[
\tilde{\Psi}(r, t(r)) = \chi_{nlj}(r) \cdot \exp \left( -\frac{i(\hat{k}_R^2 - \hat{k}_I^2)}{2k_N} \cdot r \right) \cdot \exp \left( -\frac{\hat{k}_R \cdot \hat{k}_I}{k_N} \cdot r \right). \tag{27}
\]

Such a wave function at \(r \to \infty\) is convergent and can be normalized to 1. The Fig. 2 shows the real and imaginary parts of the normalized radial wave function for the neutron in \((2p_{1/2})\) - state of \(^{13}\text{C}\). With such wave functions, the root-mean-square radius (r.m.s.), \(< r^2_{2p_{1/2}} >^{1/2} = 11.92 \text{ fm}\), for 2\(p_{1/2}\) - state of neutron was calculated. For the neutron in \(1p_{1/2}\) - state, we have \(< r^2_{1p_{1/2}} >^{1/2} = 3.40 \text{ fm}\). The value of the r.m.s. radius of neutron in 2\(p_{1/2}\) - state is much larger than one in 1\(p_{1/2}\) -state. Therefore, the neutron 2\(p_{1/2}\) - state may be regarded as the ”halo - like” state.

Generally, the resonance wave function obtained from the time-independent Schrödinger equation is exponentially divergent at an asymptotic distance. Therefore, it is not easy to calculate the r.m.s. radius with such an unnormalizable resonance state wave function. One way to calculate the r.m.s. radius is to use the complex scaling method[22] or to introduce a convergence factor such as \(\exp^{-\alpha r^2}\) provided \(\alpha \to \infty\) at the end[23]. It should be noticed that the r.m.s. radius calculated by the last method[23], i.e. convergence factor method, is a complex number for the resonance state, e.g. \(< r^2_{3S_{1/2}} > = 9.41 + i4.52 \text{ fm}\) for the overbarrier resonance state \(3S_{1/2}\) with \(E_r = 1.63\text{MeV}\) and \(\Gamma = 0.246\text{MeV}\) (the barrier top is 1.50 MeV). Consequently, the quadrupole moments of nuclei will be complex values, too, in their approach. This sequence seems to require further consideration and deeper investigation. On the other hand, our method of calculation for the r.m.s. radius takes into account the time-dependent part of wave function by means of \(t = \mu r/\hbar k_N\). As is explained above, this time-dependent part makes the full wave function convergent. Thus, the r.m.s. radius
can be calculated without any difficulty.
IV. CONCLUSION

In this paper, we suggested the method of solving the system of inhomogeneous differential equations of the second order with help of preliminary determined solution of corresponding homogeneous equation. Such an approach allows to obtain the wave functions, energy and width for the overbarrier resonance state.

This method was used for analyzing $2p_{1/2}$ - resonance state of the neutron in $^{13}C$ nucleus. The value of root-mean-square radius of neutron in this state is larger than that in the $1p_{1/2}$ - state.

In the framework of this method, we propose to calculate neutron and proton resonance $2p$ - states in $^{13}C$ and $^{13}N$ nuclei and then to explore them for analyzing the electron scattering data and $\beta$ - transition. It can also be a good application of our method to analyze the elastic proton scattering from $^{13}C$ at 1 GeV [27].

Besides, we suggest in a new fashion to look at the structure of the halo - nuclei, in which an existence of the sub - and over - barrier resonant single-particle states, as well as the correlated multi-particle states can be displayed.

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APPENDIX A

The common theory of the expansion on the Sturm - Liouville functions was developed long ago [28,29,30]. It is actually the generalization of the expansion in the Fourier series. Examples of application of this theory to the atomic and nuclear problems are given in [25]. The Sturm - Liouville equation for eigenvalues $\alpha_{nlj}$ and eigenfunction $\varphi_{nlj}(r)$ is
\[
\left[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + V_{S.O.} + V_{Coul.} + \alpha_{nlj} \cdot V(r) \right] \varphi_{nlj}(r) = E_{0lj}\varphi_{nlj}(r), \quad (A1)
\]

where \( E_{0lj} \) is the fixed negative number (in our case this is the binding energy of nucleon in 1\(p\) - shell).

The functions \( \varphi_{nlj}(r) \) must satisfy the following boundary condition,

\[
\varphi_{nlj}(r) \to 0 \quad \text{at} \quad r \to 0 \quad \text{and} \quad r \to \infty.
\]

If \( \alpha_{0lj} = 1 \) and \( E_{0lj} \) is the eigenvalue of the Schrödinger equation, then the solution of this equation will be the first eigenfunction of the Sturm - Liouville problem, i.e. \( \varphi_{0lj}(r) \) is the ordinary physical wave function. For fixed values \( l \) and \( j \), the depth of potential will grow with increasing \( \alpha_{nlj} \). If we find such \( \alpha_{nlj} \), that energy of \( n \) - state coincides with \( E_{0lj} \), then the obtained wave function will be the eigenfunction of the Sturm - Liouville problem. The eigenvalues \( \alpha_{nlj} \) are infinite succession of positive discrete numbers, which satisfy the condition:

\[
\alpha_{0lj} < \alpha_{1lj} < \alpha_{2lj} < ... \alpha_{n-1, lj} < \alpha_{n,l} < ...
\]

and \( \alpha_{nlj} \to \infty \) at \( n \to \infty \). It follows from the Hermite conjugate of the Hamiltonian operator for the Sturm - Liouville problem that the functions \( \varphi_{nlj}(r) \) are orthogonal with the weight \( V(r) \) at the fixed \( l, j \) and \( E_{0lj} \), i.e.

\[
\int_0^\infty \varphi_{n'lj}V(r)\varphi_{nlj}(r)dr = -\delta_{n',n}.
\]

\[ (A2) \]

Other mathematical aspects of this method (e.g. the problems of sign of the eigenvalues, of the completeness and the convergence of the expansion at \( r \to \infty \)) are considered in works [25] in detail.

It is very convenient to present the eigenfunctions \( \varphi_{nlj}(r) \) in an analytical form with parameters calculated by means of modifying quasi-classical method [31]:
\[ \varphi_{nlj}(r) = N_{nlj}(S')^{-\frac{1}{2}} \exp\left(-\frac{S^2}{2}\right) H_n(S), \quad (A3) \]

where \(N_{nlj}\) is coefficient of norm, \(H_n(S)\) is Hermite polynomial, \(S' = dS/dr\), and \(S(r)\) is the correct function to be found by means of numerical calculations.

As the comparison equation, the equation of the harmonic oscillator type is chosen,

\[ \frac{d^2\Phi}{dS^2} + (2n + 1 - S^2)\Phi = 0, \quad (A4) \]

and the integral relation is obtained for the definition \(S(r)\) [25]:

\[ \int_{-\sqrt{2n+1}}^{S}(2n + 1 - \sigma^2)^{\frac{1}{2}}d\sigma = \int_{r_1}^{r} p(\xi)d\xi, \quad (A5) \]

where

\[ p(r) = \sqrt{\frac{2\mu}{\hbar^2}(E_{nlj} - \alpha_{nlj}V(r)) - V_{S.O.} - V_{Coul.} - \frac{(l + \frac{1}{2})^2}{r^2}} \quad (A6) \]

is the momentum analogous to the quasi-classical one, \(r_1\) is the least root of equation \(p(r) = 0\).

In our work, we express \(S(r)\) in term of

\[ S(r) = \frac{A_0}{2} + \sum_{k=1}^{k_{max}} A_k T_k\left(\frac{2r - a - b}{b - a}\right), \quad (A7) \]

\[ A_k = \frac{2}{k_{max} + 1} \sum_{j=0}^{k_{max}} S(r_j) \cos\left(\frac{(2j + 1)k\pi}{2k_{max} + 2}\right), \quad (A8) \]

\[ r_j = \frac{a + b}{2} + \frac{b - a}{2} \cos\left(\frac{(2j + 1)\pi}{2k_{max} + 2}\right), \quad (A9) \]

where \(T_k(\xi)\) is Chebyshev polynomial of \(k\) power; \(a = r_1(E_{nlj})\) and \(b = r_2(E_{nlj})\) are the turning points, which are defined by the equation:

\[ E_{nlj} - \alpha_{nlj} \cdot V(r) - V_{S.O.} - V_{Coul.} - \frac{\hbar^2 (l + \frac{1}{2})^2}{2\mu r^2} = 0, \quad (A10) \]
$k_{\text{max}}$ is maximum number of expansion (in our case, $k_{\text{max}} = 30$ was took for reaching the necessary uniformity).

Having such a Sturm - Liouville functions, we can solve the homogeneous Schrödinger equation

$$\left[ \frac{-\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + V_{\text{S.O.}} + V_{\text{Coul.}} + V(r) \right] u^{(\circ)}_{nlj}(r) = E_{nlj} \cdot u^{(\circ)}_{nlj}(r) \quad (A11)$$

with the boundary condition $\chi_{nlj} \to 0$ for $r \to 0$ and $r \to \infty$, by means of expansion $u^{(\circ)}_{nlj}(r)$ with respect to $\varphi_{nlj}(r)$:

$$u^{(\circ)}_{nlj}(r) = \sum_{n'=0}^{\infty} d_{n'lj} \cdot \varphi_{n'lj}(r). \quad (A12)$$

If we substitute this expansion into equation (A11), add and subtract $\alpha_{nlj}V(r)$, multiply on left-side to $\varphi_{nlj}(r)V(r)$, and carry out integration over $r$, then the following equation can be obtained,

$$(E - E_{0lj})d_{nlj} + \sum_{n'=0}^{\infty} d_{n'lj}(1 - \alpha_{n'lj}) \int \varphi_{nlj}V^2(r)\varphi_{n'lj}dr = 0, \quad (A13)$$

which is an infinite set of equations. In order to solve this set of equations, it is necessary to cut off the summation at some fixed number $M$. Then, from the condition for solving this set of equations, one finds the approximate eigenvalue $E^{(M)}_{nlj} \equiv \left( E_{\text{res}} \right)_{nlj}$ and the coefficients $d^{M}_{nlj}$, accordingly the eigenfunction $u^{(\circ)(M)}_{nlj}(r)$.

At $M \to \infty$, $E^{(M)}_{nlj} \to E^{\text{exact}}_{nlj} \equiv \left( E_{\text{res}} \right)_{nlj}$.  

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TABLES

TABLE 1. The values of the expanding coefficients $d_{n1/2}$ for the neutron wave function $u_{2p_{1/2}}^{(c)}(r)$ in $^{13}C$.

| n  | $d_{n1/2}$ | n  | $d_{n1/2}$ | n  | $d_{n1/2}$ |
|----|------------|----|------------|----|------------|
| 1  | .67484     | 11 | -.02874    | 21 | -.00491    |
| 2  | .62089     | 12 | .02375     | 22 | .00408     |
| 3  | -.29481    | 13 | -.01974    | 23 | -.00337    |
| 4  | .18248     | 14 | .01651     | 24 | .00275     |
| 5  | -.12831    | 15 | -.01386    | 25 | -.00221    |
| 6  | .09306     | 16 | .01166     | 26 | .00174     |
| 7  | -.06997    | 17 | -.00984    | 27 | -.00132    |
| 8  | .05489     | 18 | .00829     | 28 | .00096     |
| 9  | -.04343    | 19 | -.00699    | 29 | -.00064    |
| 10 | .03521     | 20 | .00587     | 30 | .00036     |
FIGURES

FIG. 1. The energy values $E_{2p_{1/2}}^{(M)}$ versus the number of expansion, $M$, for the neutron wave function, i.e. $M$ is the number of terms taken in the summation (A12).

FIG. 2. Radial wave function for neutron in $2p_{1/2}$ - state:
a) real and b) imaginary parts.
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