Periodic Steiner Networks Minimizing Length

Jerome Alex · Karsten Grosse-Brauckmann

Received: 25 April 2018 / Revised: 7 August 2023 / Accepted: 12 August 2023 / Published online: 9 October 2023
© The Author(s) 2023

Abstract
We study a problem of geometric graph theory: We determine the triply periodic graph in Euclidean 3-space which minimizes length among all graphs spanning a fundamental domain with the same volume. The minimizer is the so-called srs network with quotient the complete graph on four vertices $K_4$. For comparison we consider a competing topological class, also with a quotient on four vertices, and determine the minimizing tths networks.

Keywords Triply periodic network · Length minimizer · srs network · Laves network · Gyroid

Mathematics Subject Classification 05C10 · 53A10 · 49Q10

1 Introduction

Given a finite set of points, the Steiner problem is to find a tree of minimal length connecting them [8]. While this is a classical problem for the plane, the case of dimension 3 and higher has received less attention. For any dimension, trees minimizing length usually have further vertices which necessarily are of degree 3, where the incident edges are coplanar and meet at $120^\circ$-angles. We call this property the Steiner condition.

We study infinite graphs $N$ without terminal vertices in $n$-dimensional Euclidean space. We assume they are periodic, that is, invariant under the translations of some lattice $\Lambda \subset \mathbb{R}^n$ with rank $n$. If, moreover, their quotient $N/\Lambda$ is finite we call these...
graphs networks. The fundamental domain $\mathbb{R}^n / \Lambda$ is a flat $n$-torus with volume $V$, and the network quotient $N / \Lambda$ has a length $L$. We usually refer to $V$ and $L$ as the volume and length of the network $N$ itself. See Sect. 2 for details.

We are interested in shortest networks. Since scaling can reduce the length of $N$, a well-posed variational problem is: Minimize the network length $L$ subject to the constraint $V = 1$ among all networks. Equivalently, one can minimize the scaling-invariant ratio $L^3 / V$. Each minimizer necessarily is what we call a Steiner network: at all vertices, the degree is 3 and the Steiner condition holds. For the planar, doubly periodic case $n = 2$, the hexagonal network is seen to be the minimizer, see Sect. 3 and Fig. 4.

In Euclidean 3-space, a prominent example of a Steiner network is the srs network, see Fig. 1. It is maximally symmetric in the sense that its symmetry group $I4_132$ acts transitively on vertices and (directed) edges. Its quotient under the body-centred cubic lattice is the complete graph on four vertices $K_4$, see Fig. 2. By calling the network srs, a name which refers to the strontium silicide SrSi$_2$ crystal, we follow a crystallographic convention (see [7] and also rcsr.net). Other names for the network are Laves [2] or (10, 3)-a [17] (see also [12, 16]).

Triply periodic Steiner networks exist for other lattices and with quotients other than $K_4$. Our main result is that the srs network is the unique minimizer:

**Theorem A** Length and volume of a triply periodic network in $\mathbb{R}^3$ satisfy

$$\frac{L^3}{V} \geq \frac{27}{\sqrt{2}} \approx 19.09.$$  \hfill (1)

Equality holds exactly for the srs network.

Here, the terminology is as in Sect. 2. In particular, uniqueness is always up to removal of redundant vertices of degree 2.

Let us summarize how we prove Theorem A. Section 2 contains our results for arbitrary dimension. In Theorem 2.3 we use a compactness argument to establish the existence of an embedded Steiner network minimizing $L^3 / V$ among all networks with a prescribed lattice. The quotient is a graph on $2n - 2$ vertices. It can be shown to be simple for $n \geq 3$, see Theorem 2.6.

Therefore, for $n = 3$ the quotient must be $K_4$, the complete graph on 4 vertices (Proposition 4.1). An obvious approach to prove estimate (1) would be to parameterize $K_4$ networks by their lattices, and to minimize the ratio $L^3 / V$ over all lattices. However, the space of lattices is inconvenient to parameterize and Steiner networks are not unique for their lattice.

Instead we show with Lemma 4.2 that the six edge lengths of $N / \Lambda$ suffice to parameterize the space of all Steiner networks with quotient $K_4$. In terms of these parameters, lattice generators become linear functions, and length $L$ and volume $V$ are explicit, see Lemma 4.3. Thus it remains to solve, by Theorem 4.4, a finite dimensional optimization problem under constraints. In fact, we prove (1) by maximizing $V$ under the constraint $L = 1$. Using Lagrange parameters we find the maximum is attained with all lengths equal.
Fig. 1 We identify the srs network (top) as the minimizer of $L^3/V$ among all triply periodic networks. As indicated by the colouring, the quotient has four vertices; it is the graph $K_4$. Triply periodic Steiner networks on four vertices can also have the graph $D_1 \square D_2$ as a quotient. For this topological class, a family of networks minimizes. A minimizer with maximal symmetry is shown on the bottom. The longer edges define zigzag curves which are contained in perpendicular planes while the short edges are contained in lines of intersection of these planes.
Steiner networks whose quotient has the minimal number of four vertices can attain only one more topological type: The quotient graph can be $D_1 \square D_2$, the Cartesian graph product of the dipole graphs $D_1$ and $D_2$, see Fig. 2. Arguably, the two most important networks of degree 3 are those based on $K_4$ and $D_1 \square D_2$ see O’Keeffe et al [10]. Nevertheless, $D_1 \square D_2$ is less symmetric than $K_4$, and its single and double edges imply network symmetries cannot be transitive on edges. We can ask for the optimal networks with this topology and how the length ratio $L^3/V$ compares with srs.

Applying the strategy we outlined above for $K_4$ we find an entire 1-parameter family of Steiner networks attaining the minimal length ratio, see Theorem 5.2:

**Theorem B** A triply periodic network in $\mathbb{R}^3$ with quotient $D_1 \square D_2$ satisfies

$$\frac{L^3}{V} \geq \frac{81}{4} = 20.25.$$  

Equality holds exactly for a one-parameter family of (non-similar) Steiner networks, all with the same body-centered tetragonal lattice $\Lambda$, generated by $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, \sqrt{3})/2$ up to similarity. Moreover, each of these minimizing networks can be homotoped into a Steiner network of smaller length covering $K_4$, such that the length is non-increasing and the lattice remains fixed.

We call the minimizers ths networks, again making use of a crystallographic name, this time referring to the thorium silicide ThSi$_2$ crystal. The network is also known as (10, 3)-b [17]. This labelling should be taken with care, however, as usually it is applied to the case of all edges having the same length (see [10] or rcsr.net). The symmetry group for that case, $I4_1/amd$, is attained by precisely one of our ths networks, distinguished in the family by the two short edges attaining the same length (smaller than the remaining four equal edge lengths). Only for this particular network, depicted in Fig. 1, the symmetries are transitive on vertices. All other networks in our family have lower symmetry, with two kinds of vertices and three kinds of edges. The claimed homotopy is established in Proposition 5.3 and visualized in Fig. 3.

Our results should be compared with the work of Sunada and Kotani [9] (see also [14, 15]). Instead of minimizing the length $L = \sum_{i=1}^{m} x_i$ of a network with edge lengths $x_1, \ldots, x_m$, they minimize the quadratic energy $E = \sum_{i=1}^{m} (x_i)^2$; they also impose a volume constraint. The energy $E$ models a physical crystal with harmonic oscillators along the edges of the network. For various topological types of networks, Sunada and Kotani determine energy minimizers, essentially by solving a linear algebra problem. Unlike for our setting, minimizers are unique for given topology. The
A ths network (a) can continuously be deformed into a triply periodic network with quotient $K_4$ (c). The homotopy preserves the lattice but decreases length. The change in topology occurs when two vertices coincide (b).

The srs network is also the energy minimizer, while for the case of $D_1 \square D_2$, Sunada and Kotani exhibit a unique energy minimizer, not contained in our length minimizing ths family. At this place we would like to recommend Sunada’s book [15] as a comprehensive introduction to the geometric theory of networks and graphs, in particular to a systematic treatment of their topology.

Our own motivation for the present study comes from surface theory [3, 5]. There are various self-assembling biological and chemical systems which give rise to triply periodic interfaces. As pointed out for instance in [3], the most prevalent geometry is the gyroid, a triply periodic embedded surface with a quotient of genus 3. The surface is a handlebody and the srs network is its spine. In fact, it was the srs network which let Alan Schoen discover the gyroid minimal surface in the 1970s [11]. On the other hand, we are not aware that the existence of a minimal surface for ths has been shown (Evolver experiments are mentioned in [3]).

Numerical studies have shown the gyroid to be superior to certain well-known surfaces, for instance with respect to surface area per fundamental cell, Gauss curvature variance, or channel diameter variance [4, 13]. Nevertheless, it seems completely out of reach to show optimality in comparison to arbitrary periodic surfaces, that is, without specifying a lattice. We regard networks as a simplified model for surfaces with a given spine, with network length $L$ (but not the quadratic energy $E$) relating to surface area of constant mean curvature or minimal surfaces. Our simple variational characterization of the srs network provides further support for the optimality of the gyroid and its symmetries.

Our results raise the question whether the well-known most symmetric networks of higher degree similarly minimize length, provided their degree $d$ is prescribed. Affirmative results for $d = 4, 5, 6$ are contained in a companion paper [1]. Also it remains to determine optimal Steiner networks in higher dimensions $n \geq 4$; while our reasoning generalizes in principle, the number of admissible combinatorial graphs strongly increases in $n$.

The results presented in this paper are part of the first author’s PhD thesis. We thank one of the referees in particular for remarks that led to substantial improvements.

2 Steiner Networks

For our purposes it is convenient to use the term network in the following sense:
Fig. 4 Doubly periodic Steiner networks and their fundamental domains. The underlying abstract graph of the first two networks is the dipole graph of order 3 (cf. Fig. 7). The hexagonal network on the left minimizes length for given area of its fundamental domain. The network on the right has the quotient $D_1 \Box D_2$

**Definition 2.1** An \((n\text{-periodic})\) network \(N\) is a connected simple graph, immersed with straight edges of positive length into \(\mathbb{R}^n, n \geq 2\), subject to the following:

- \(N\) is invariant under a lattice \(\Lambda\) of rank \(n\).
- The quotient \(N/\Lambda\) is a finite graph \(\Gamma\), possibly with loops and multiple edges.

We call \(V = V(\mathbb{R}^n/\Lambda)\) the \((spanned)\) volume of \(N\) and \(L = L(N/\Lambda)\) the length of \(N\).

Let us explain our terminology. If a graph is mapped injectively to \(\mathbb{R}^n\) then we call the map an **embedding**. We call it an **immersion** if for each vertex the restriction to the union of the incident edges is injective, i.e., the star of each vertex is embedded.

A **lattice** (of rank \(n\)) is a set \(\Lambda = \{\sum_{i=1}^{n} a_i g_i : a_i \in \mathbb{Z}\} \subset \mathbb{R}^n\), where the vectors \(g_1, \ldots, g_n \in \mathbb{R}^n\) are linearly independent. The ambient space quotient \(\mathbb{R}^n/\Lambda\) is an \(n\)-dimensional flat torus. It can be represented by a parallelepiped (up to identifications of the boundary) spanned by the vectors \(g_i\). We refer to the quotient or its representing epiped as a **fundamental domain**.

We need the rank of the first homology group \(H_1(\Gamma, \mathbb{Z})\) of the quotient graph, or more generally of a connected finite graph \(\Gamma\),

\[
\text{rank } \Gamma := 1 - \# \text{ vertices of } \Gamma + \# \text{ edges of } \Gamma. \quad (2)
\]

This non-negative integer counts the number of homologically independent cycles and is called **circuit rank** or **cyclomatic number** of \(\Gamma\). Clearly a tree has circuit rank 0, and for a 3-regular graph we have

\[
\text{rank } \Gamma = 1 + \frac{1}{2} \# \text{ vertices of } \Gamma. \quad (3)
\]

It is convenient to define the circuit rank of a network \(N\) in terms of its quotient, \(\text{rank } N := \text{rank } (N/\Lambda)\).

We remark that given a connected finite graph \(\Gamma\) we can describe a network \(N\) as an immersion of an abelian covering of \(\Gamma\) (see Sunada [15]). For \(N\) to be \(n\)-periodic we must require \(\text{rank } \Gamma \geq n\) since \(\Gamma\) needs at least \(n\) independent cycles to map to the generators of a lattice \(\Lambda\).

Our variational problem is to minimize the length \(L = L(N/\Lambda)\) of a network \(N\) subject to the constraint that the \(n\)-dimensional volume \(V = V(\mathbb{R}^n/\Lambda)\) of a fundamental domain is fixed to 1. Equivalently, we may minimize the scaling-invariant
length ratio $L_n/V$. Suppose a network $N$ is critical for length. Then the first variation formula gives for each vertex $p \in N$ which has three adjacent vertices $q_1, q_2, q_3$:

$$\frac{p - q_1}{|p - q_1|} + \frac{p - q_2}{|p - q_2|} + \frac{p - q_3}{|p - q_3|} = 0.$$  (4)

Equivalently, the three edges incident to $p$ meet at $120^\circ$-angles. We refer to (4) as the Steiner condition or as balancing.

**Definition 2.2** A Steiner network is an $n$-periodic network where all vertices have degree 3 (the network is 3-regular) and satisfy the Steiner condition (4) at each vertex.

For fixed lattice, Steiner networks result from minimizing the length ratio:

**Theorem 2.3** Among $n$-periodic networks in $\mathbb{R}^n$ with fixed lattice $\Lambda$ there exists a network minimizing the length $L$. Any such minimizing network $N$ is an embedded Steiner network such that $N/\Lambda$ has $2n - 2$ vertices, up to vertices of degree 2 with collinear incident edges.

**Remark** 1. From now on we assume that redundant vertices of degree 2 are removed from minimizers. For $n \geq 2$ this must leave a nonzero number of vertices.
2. For given lattice $\Lambda$ and quotient graph, Steiner networks need not be unique. Possibly, the immersions attain different homotopy classes.

We will obtain the minimizing network of the theorem as the limit of a minimizing sequence $(N_k)$ of networks. The following lemma asserts that we can assume each $N_k$ to have rank $n$ and be 3-regular.

**Lemma 2.4** Let $N$ be an $n$-periodic network with lattice $\Lambda \subset \mathbb{R}^n$.

(i) If $\text{rank } N > n$ then there exists an $n$-periodic network $N'$, also with lattice $\Lambda$, which has smaller length, $L(N') < L(N)$, and rank $\text{rank } N' = n$.

(ii) Suppose $N$ contains a vertex with degree $d \geq 4$ or $d = 1$, or with degree $d = 2$ and non-collinear edges. Then there exists an $n$-periodic network $N'$ with lattice $\Lambda$ and rank $\text{rank } N' = \text{rank } N$, such that $N'$ is 3-regular with smaller length, $L(N') < L(N)$.

**Proof** (i) Since $N/\Lambda$ is not a tree there exists an edge whose removal does not disconnect the graph. Then length decreases, the rank (2) drops by 1, and removal can be iterated.

(ii) We describe the modification leading from $N$ to $N'$ in terms of a finite number of reduction steps on the quotient level $N/\Lambda$.

Clearly we decrease length by successively removing all vertices of degree $d = 1$ together with their incident edges from $N/\Lambda$. If the resulting graph contains vertices of degree $d = 2$ incident to a pair of non-collinear edges we replace the pair by a single edge to reduce length.

If the result contains a vertex $p$ of degree $d \geq 4$ we use a well-known argument to reduce length (see for instance [8, p. 120f.]). Then the star at $p$ contains two edges with endpoints $q_1, q_2$ which make a non-zero angle of less than 120 degrees. To define $N'$, we replace these two edges in $N/\Lambda$ with a tripod which connects the triple $p, q_1, q_2$.
with a further point in the same plane, chosen such that the length decreases. Thereby the degree at \( p \) changes from \( d \) to \( d - 1 \). Upon iteration we can reduce the degree to \( d \leq 3 \) at all vertices of \( N/\Lambda \).

Our operations preserve \( n \)-periodicity. However, the replacements for \( d = 2 \) and \( d \geq 4 \) possibly create a star of a vertex which contains a pair of edges in the same direction, hence violating the immersion property. Clearly, we can merge a pair of coinciding edges of \( N/\Lambda \) to a single edge; if two edges overlap partly, for instance, \( e = \overline{pq} \) strictly contains \( f = \overline{pr} \), we keep \( f \), connect it with a new edge \( \overline{rq} \), and discard \( e \). In both cases we reduce length strictly. If necessary we can iterate this construction to obtain a network of shorter length, all of its stars are embedded.

Finally, we merge collinear edges incident to vertices of degree 2 to form a single edge, leaving \( L \) unchanged, and let \( N' \) be the resulting network. Note that all our operations preserve the circuit rank. Hence \( N' \) has the properties claimed. \( \square \)

Proof of the Theorem Consider a length minimizing sequence \( (N_k) \), that is, \( \lim_{k \to \infty} L(N_k) = \inf L(N) \), where the infimum is taken over all \( n \)-periodic networks with fixed lattice \( \Lambda \).

Applying Lemma 2.4 we may assume first that rank \( N_k = n \), and then that \( N_k \) is 3-regular, keeping the rank. By (3) the number of vertices then is \( 2n - 2 \). Combinatorially, there are only finitely many such graphs. Thus by passing to a subsequence we can assume all \( N_k/\Lambda \) have the same combinatorial type \( \Gamma \).

The set of \( 2n - 2 \) vertices of \( N_k/\Lambda \) is compact in \( \mathbb{R}^n/\Lambda \), and the connecting edges are geodesics with uniformly bounded length. Hence vertices and edges of a further subsequence of \( (N_k) \) converge to a limit \( N \). We claim that all edges of \( N \) attain positive length. Otherwise we can regard \( N \) as a network with at least one vertex of degree \( d \geq 4 \). But Lemma 2.4(ii) implies \( N \) does not minimize length over all combinatorial types, contradicting the fact that \( (N_k) \) is a minimizing sequence.

Let us show \( N \) is embedded. Suppose two edges intersect in an interval of positive length. Then comparison with a network where the intersection set is replaced by a single edge shows that \( N \) cannot be minimizing. Similarly, supposing \( N \) has an isolated point of intersection, we compare \( N \) with a network where this point is a vertex of degree \( d \geq 4 \). Then \( N \) cannot minimize by Lemma 2.4(ii). Finally, since \( N \) is a minimizer with positive edge lengths, the first variation formula (4) shows \( N \) is Steiner. \( \square \)

Remark The proof indicates that our results do not change if we drop the connectivity assumption in the definition of \( n \)-periodic networks (but still require that the cycles of the underlying possibly disconnected quotient graph span a lattice of rank \( n \)). Indeed, if a minimizer was disconnected, we could use translations to move one component as to intersect the other. Again this contradicts Lemma 2.4.

Next we derive consequences of the Steiner condition. First we rule out loops in the quotient, making the graphs shown in Fig. 5 impossible. Second, we show a pair of double edges in the quotient lifts to a zigzag curve, namely a planar polygonal curve such that every other edge has the same length and is parallel. The Steiner condition implies that edges incident to the zigzag curve are still coplanar with it. See Fig. 6 and compare with the right network of Fig. 1.
Proposition 2.5  Let $N \subset \mathbb{R}^n$ be an $n$-periodic Steiner network.
(i) Then $N/\Lambda$ cannot contain any loops.
(ii) Let $\Gamma_0 \subset N/\Lambda$ be a double edge together with the two incident edges (which possibly coincide). Then each connected lift of $\Gamma_0$ in $N$ is contained in a 2-plane $P \subset \mathbb{R}^n$ and consists of a zigzag curve together with edges pointing pairwise in opposite directions.

Proof (i) A loop at a vertex $p \in N/\Lambda$ lifts to an edge of $N$ which is bounded by two lifts of $p$. Thus $N$ contains a vertex of degree 3 with a collinear pair of incident edges, thereby contradicting balancing.

(ii) As in Fig. 6, let $p, q \in \Gamma_0$ be vertices connected by a pair of double edges, and $r, s$ be the vertices of further edges incident to $p, q$; possibly $r = s$. To define the plane $P$, consider a lift $p_0$ of $p$ together with its three coplanar edges in $N$. The two vertices $q_0, q_1$, adjacent to $p_0$ in $N$, are lifts of the same point $q$, meaning that the network is invariant under translation by the vector $q_1 - q_0$. So the double edge forms a cycle which lifts to a zigzag curve contained in $P$.

Coplanarity also implies that the plane $P$ contains all edges incident to the vertices of the zigzag curve, such as $p_0r_0$ and $q_0s_0$. Furthermore, the 120° angles of the Steiner condition give that the edges incident to the zigzag curve are parallel and alternate in opposite directions; for instance $r_0 - p_0$ is a positive multiple of $s_0 - q_0$.  

We now show that if the quotient of a network has double edges, then this network cannot be the minimizer of Theorem 2.3 among all networks for $n \geq 3$. Thus triply periodic networks with quotient $D_1 \Box D_2$ cannot be minimizing in this sense.

Theorem 2.6  Suppose $N \subset \mathbb{R}^n$ is an $n$-periodic network which minimizes $L^n/V$ among all networks with prescribed lattice $\Lambda$. Then $N/\Lambda$ is a 3-regular graph on $2n - 2$ vertices without loops, which for $n \geq 3$ is simple.

Proof By Theorem 2.3 the network $N$ is Steiner. So its quotient $N/\Lambda$ is 3-regular, with no loops by Proposition 2.5. Suppose that such an $N/\Lambda$ is not simple.
If two vertices in $N/\Lambda$ are joined by a triple edge, then the connected graph $N/\Lambda$ is the dipole graph $D_3$ (see Fig. 7. It has rank 2 and so $n = 2$.

It remains to prove that for $n \geq 3$ a pair of vertices $p \neq q$ which is doubly connected contradicts the minimizing property. We do this by constructing a homotopy $t \mapsto N'$ of $N = N_0$ to a network $N_1$, with fixed lattice $\Lambda$. The homotopy preserves length, so that $N_1$ is also a minimizer among all networks. However, since $N_1/\Lambda$ has a vertex of degree 4 this contradicts Lemma 2.4(ii).

To define $N'$ use the planar subnetworks of Proposition 2.5(ii) as follows. Move the vertices $p_0$ and $q_0$ simultaneously within the plane $P$, namely replace them with $p_0' := p_0 + t(r_0 - p_0)$ and $q_0' := q_0 + t(r_0 - p_0)$ for $0 \leq t < 1$.

Then let $N'$ be the $n$-periodic Steiner network obtained by applying the same modification to all lifts of $p$, $q$. Then indeed the homotopy leaves the network length invariant, $L(N') = L(N)$, as $|p_0' - r_0| + |q_0' - s_0|$ is independent of $t$ by Proposition 2.5(ii), and the length of the zigzag curve is preserved by translation.

**Remark** 1. The number of 3-regular simple graphs on $2n - 2$ vertices, i.e., cubic graphs, is rapidly growing in $n$, see oeis.org.

2. While in dimension $n = 3$ the minimizing network must have a $K_4$ quotient, in higher dimension the topology of a minimizer possibly depends on the lattice. Indeed, this happens for $n = 3$ and prescribed degree 5, where two different graphs are attained as the quotients of minimizers for different lattices, see [1].

### 3 Doubly Periodic Steiner Networks

We first determine the minimizing graph for the two-periodic case $n = 2$, before studying $n = 3$. For a prescribed lattice, the topology of the quotient graph can be determined as follows. By Theorem 2.3 the graph is a Steiner graph on 2 vertices, therefore 3-regular, and by Proposition 2.5(i) it has no loops. The only connected 3-regular graph on 2 vertices without loops is $D_3$, see Fig. 7. Writing $A$ for the area of $\Lambda$ we obtain:

**Lemma 3.1** A doubly periodic network $N \subset \mathbb{R}^2$ minimizing the length ratio $L_2^2 / A$ for prescribed lattice $\Lambda$ is Steiner on 2 vertices with the dipole graph $D_3$ as a quotient.

Since the edges of a Steiner network enclose $120^\circ$-angles, a minimizing network can be described in terms of three edge lengths alone:

**Lemma 3.2** Up to isometry, a doubly periodic Steiner network $N \subset \mathbb{R}^2$ with quotient $D_3$ is uniquely determined by its three edge lengths $x_1, x_2, x_3 > 0$. Its length and spanned area are

$$L = x_1 + x_2 + x_3 \quad \text{and} \quad A = \frac{\sqrt{3}}{2}(x_1x_2 + x_1x_3 + x_2x_3).$$
The two vertices of $D_3$ correspond in $N$ to a vertex $p_0$ and its incident vertices $p_1, p_2, p_3$, where the labelling relates to the lengths as in Fig. 7. Then the lattice $\Lambda$ of $N$ is spanned, for instance, by $g_1 := p_1 - p_3$ and $g_2 := p_2 - p_3$. Specifically, we may assume that up to isometry we have

$$p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p_1 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p_2 = x_2 \begin{pmatrix} -\frac{1}{2} \\ \sqrt{\frac{3}{2}} \end{pmatrix}, \quad p_3 = x_3 \begin{pmatrix} -\frac{1}{2} \\ -\sqrt{\frac{3}{2}} \end{pmatrix},$$

and so the lattice $\Lambda$ has area

$$A = |\det(g_1, g_2)| = \frac{\sqrt{3}}{2} (x_1 x_2 + x_1 x_3 + x_2 x_3).$$

As might be expected, the optimal doubly periodic network is given by the tesselation of $\mathbb{R}^2$ with regular hexagons:

**Proposition 3.3** For each doubly periodic network $N \subset \mathbb{R}^2$ we have

$$\frac{L^2}{A} \geq 2\sqrt{3}. \tag{5}$$

Equality holds if $N$ has the quotient $D_3$ and the three edge lengths of $N$ are equal; then the lattice is hexagonal.

**Proof** For a prescribed lattice $\Lambda$, Lemma 3.1 asserts the existence of a Steiner network $N_0$ with quotient $D_3$ which is a minimizer, $(L^2/A)(N) \geq (L^2/A)(N_0)$, where the inequality is strict if the quotient is not $D_3$. According to Lemma 3.2, the edge lengths $x_1, x_2, x_3 > 0$ determine $N_0$, and the area $A(N_0)$ is a multiple of the elementary symmetric polynomial of degree 2 in three variables. Thus Maclaurin’s inequality (20) implies (5) for $N_0$:

$$A(N_0) = \frac{\sqrt{3}}{2} (x_1 x_2 + x_1 x_3 + x_2 x_3) \leq \frac{3\sqrt{3}}{2} \left( \frac{x_1 + x_2 + x_3}{3} \right)^2 = \frac{1}{2\sqrt{3}} (L(N_0))^2.$$

In particular, (5) follows for $N$. To discuss the equality case, note that for $N$ with quotient $D_3$ and $x_1 = x_2 = x_3$, equality in (5) is obvious. But by the above and Lemma 5.4 the equality can only hold for this case. \(\square\)
4 The srs Network Covering the $K_4$ Graph

In three dimensions, Theorems 2.3 and 2.6 uniquely determine the topology of a Steiner network which minimizes length for its lattice:

**Proposition 4.1** For each lattice $\Lambda \in \mathbb{R}^3$ there exists a triply periodic Steiner network $N$ of topology $K_4$, minimizing $L^3/V$ among all triply periodic networks. Each such minimizer has this topology.

To find the minimizing network over all lattices, we consider all networks with quotient $K_4$, and show we can use their edge lengths to parameterize them. The Steiner condition implies coplanarity of a vertex with its three incident vertices. We call the containing unoriented affine plane the tangent plane of a vertex in a Steiner network. For each Steiner network with quotient $K_4$ the set of four tangent planes is completely determined up to motion:

**Lemma 4.2** Let $N \subset \mathbb{R}^3$ be a triply periodic Steiner network with quotient graph $K_4$. Then the four tangent planes of $N/\Lambda$ are tangent to the four faces of a regular tetrahedron. Consequently, up to isometry of $\mathbb{R}^3$, the network $N$ is uniquely defined by its six edge lengths $x_1, \ldots, x_6 > 0$.

**Proof** From $N$ we pick a connected subgraph which contains a vertex $p_0$ and its three neighbours $p_1, p_2, p_3$, representing the vertices of $N/\Lambda$. Without loss of generality we may assume $p_0$ to be the origin, $p_1$ to lie on the $x$-axis and $p_2, p_3$ to lie in the $xy$-plane. In view of the Steiner condition this means we assume

$$p_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = x_2 \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}, \quad p_3 = x_3 \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix}, \quad (6)$$

where $x_i > 0$ is the edge length of the edge incident to $p_i$.

Let $p_6 \neq p_0$ be a vertex incident to $p_1$, compare Fig. 8. The tangent plane at $p_1$ is a rotation $A_\varphi$ about the $x$-axis of the tangent plane at $p_0$, by some angle $-\pi < \varphi < \pi$, that is,
\[ p_6 - p_1 = \frac{x_6}{x_2} A_\phi (p_0 - p_2) = x_6 A_\phi \left( \begin{array}{c} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{array} \right) = x_6 \left( \begin{array}{c} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \cos \phi \\ -\frac{\sqrt{3}}{2} \sin \phi \end{array} \right). \] (7)

Then \( \min \{|\phi|, \pi - |\phi|\} \) represents the dihedral angle between the two tangent planes at \( p_0 \) and \( p_1 \).

In the quotient \( N/\Lambda \), the vertex \( p_6 \) must be identified with one of the four vertices \( p_0, \ldots, p_3 \). Since the shortest cycle in \( K_4 \) consists of three edges this vertex must be either \( p_2 \) or \( p_3 \). Suppose \( p_6 \) is identified with \( p_2 \). The tangent planes at these two points agree as vector spaces. Hence the balancing equation (4) implies that the vectors \( p_2 - p_0 \) and \( p_6 - p_1 \) enclose 120 degrees, and the sum of the two unit vectors pointing into these directions must be a unit vector:

\[ 1 = \left| \frac{p_0 - p_2}{|p_0 - p_2|} + \frac{p_1 - p_6}{|p_1 - p_6|} \right| = \left| \frac{0}{\sqrt{3}} \left( -1 + \cos \phi \right) \right| = \sqrt{\frac{3}{2}} \sqrt{1 - \cos \phi}. \]

The other case is that \( p_6 \) is identified with \( p_3 \). Then, similarly,

\[ 1 = \left| \frac{p_1 - p_6}{|p_1 - p_6|} + \frac{p_0 - p_3}{|p_0 - p_3|} \right| = \left| \frac{0}{\sqrt{3}} \left( 1 + \cos \phi \right) \right| = \sqrt{\frac{3}{2}} \sqrt{1 + \cos \phi}. \]

From both cases we conclude \( |\cos \phi| = 1/3 \), and so the dihedral angle of the tangent planes at \( p_0 \) and \( p_1 \) is the tetrahedral angle \( \arccos(1/3) \approx 70.53^\circ \).

In \( K_4 \), each pair of vertices is connected by an edge, and so the same argument applies to all pairs of vertices \( p_i, p_j \) of \( N/\Lambda \). But four planes in \( \mathbb{R}^3 \) can only have pairwise dihedral angles \( \arccos(1/3) \) if they are parallel to the faces of a regular tetrahedron. Finally, lengths and tangent planes determine a Steiner network completely up to isometry.

\[ \square \]

**Remark** 1. The srs network is chiral, meaning that up to proper rigid motion there are two srs networks with the same set of lengths. They are related by a mirror reflection, corresponding to a sign change of \( \phi \) in (7).

2. Those permutations of the six edge lengths which are induced by a permutation of the four vertices lead to isometric networks.

For the next statement we label the six edges \( e_1, \ldots, e_6 \), such that the edges \( e_i \) and \( e_{i+3} \) do not have endpoints in common, see Fig. 8. We let \( x_i \) be the length of \( e_i \).
Lemma 4.3 Let $N$ be a triply periodic Steiner network with quotient $K_4$. Then $N$ has length $L = \sum_i x_i$ and the spanned volume is

$$V = \frac{1}{\sqrt{2}} \left( x_1 x_2 x_4 + x_1 x_2 x_5 + x_1 x_2 x_6 + x_1 x_3 x_4 + x_1 x_3 x_5 + x_1 x_3 x_6 + x_1 x_4 x_5 + x_1 x_4 x_6 + x_2 x_3 x_4 + x_2 x_3 x_5 + x_2 x_3 x_6 + x_2 x_4 x_5 + x_2 x_4 x_6 + x_2 x_5 x_6 + x_3 x_4 x_6 + x_3 x_5 x_6 + x_4 x_5 x_6 \right).$$  

(8)

Note (8) extends over all possible products of three edge lengths except for those relating to triples of concurrent edges. Thus $L$ and $V$ are invariant under relabelling induced by a symmetry of the $K_4$ graph.

Proof After isometry of $\mathbb{R}^3$ we assume the coordinates are as in (6). For $i = 1, 2, 3$ let $A_i^g \in SO(3)$ be the rotation fixing $p_i$ with an angle $\varphi = \arccos(1/3)$. In view of the first remark above we replace $\varphi$ by $-\varphi$ if necessary. We obtain that the three vectors

$$g_1 := (p_0 - p_2) + (p_1 - p_0) + \frac{x_6}{x_2} A_1^g (p_0 - p_2) = \begin{pmatrix} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_6 \\ -\frac{\sqrt{3}}{2} x_2 - \frac{1}{2\sqrt{3}} x_6 \\ -\frac{\sqrt{3}}{3} x_6 \end{pmatrix},$$

$$g_2 := (p_0 - p_3) + (p_2 - p_0) + \frac{x_4}{x_3} A_2^g (p_0 - p_3) = \begin{pmatrix} \frac{\sqrt{3}}{2} x_2 + \frac{\sqrt{3}}{2} x_3 + \frac{1}{\sqrt{3}} x_4 \\ -\frac{\sqrt{3}}{5} x_3 \\ -\frac{\sqrt{3}}{5} x_4 \end{pmatrix},$$

$$g_3 := (p_0 - p_1) + (p_3 - p_0) + \frac{x_5}{x_1} A_3^g (p_0 - p_1) = \begin{pmatrix} -x_1 - \frac{1}{2} x_3 - \frac{1}{2} x_5 \\ -\frac{\sqrt{3}}{2} x_3 - \frac{1}{2\sqrt{3}} x_5 \\ -\frac{\sqrt{3}}{2} x_5 \end{pmatrix}.$$ 

are linearly independent and span the lattice $\Lambda$. To verify (8), calculate

$$\det(g_1, g_2, g_3) = -\frac{1}{2\sqrt{2}} \det \begin{pmatrix} 2x_1 + x_2 + x_6 & -x_2 + x_3 & -2x_1 - x_3 - x_5 \\ -x_2 - \frac{1}{3} x_6 & x_2 + x_3 + \frac{2}{3} x_4 & -x_3 - \frac{1}{3} x_5 \\ x_6 & x_4 & x_5 \end{pmatrix}. \Box$$

Theorem 4.4 Let $N$ be a triply periodic Steiner network in $\mathbb{R}^3$ with quotient $K_4$. Then

$$\frac{L^3}{V} \geq \frac{27}{\sqrt{2}} \approx 19.09,$$  

(9)

where equality holds if and only if all edge lengths of $N$ coincide and the lattice is body-centered cubic.

Springer
By Lemma 4.3 we can consider $L$ and $V$ to be functions of the six positive edge lengths $(x_1, \ldots, x_6)$. These functions are still defined and continuous on $[0, \infty)^6$. The minimization problem for $L^3/V$ is equivalent to maximization of the function $V$ under the constraint $L = 1$. On the set $[0, \infty)^6 \setminus \{0\}$, this problem has a solution $z = (x_1, \ldots, x_6)$, as $L^{-1}(1) \subset [0, \infty)^6 \setminus \{0\}$ is compact and the function $V$ is continuous. We claim that $z$ cannot be a boundary point of $[0, \infty)^6$, meaning that all lengths $x_i$ are strictly positive.

Assume on the contrary that at least one component of $z$ vanishes. Then $z$ corresponds to a network $N_0$ whose lengths are the nonzero components of $z$. Note that $N_0$ must contain a vertex of degree at least 4. Also, $N_0$ is immersed and triply periodic (the limiting cycles still generate). Let $\Lambda_0$ be its lattice.

By construction, $N_0$ maximizes the scaling invariant quotient $V/L^3$. Equivalently, $N_0$ minimizes $L^3/V$, in particular it does so among networks with fixed lattice $\Lambda_0$. But this contradicts Proposition 4.1: A minimizing triply periodic network for $\Lambda_0$ must have topology $K_4$.

This proves our claim.

We conclude that under the constraint $L = 1$ the function $V$ takes a maximum at an interior point $z$ of $[0, \infty)^6$.

Thus there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\nabla V(z) = \lambda \nabla L(z).$$

Equivalently, by (8),

$$\frac{1}{\sqrt{2}} \begin{pmatrix} x_2 x_4 + x_3 x_4 + x_2 x_5 + x_3 x_5 + x_4 x_5 + x_2 x_6 + x_3 x_6 + x_4 x_6 \\ x_1 x_4 + x_3 x_4 + x_1 x_5 + x_3 x_5 + x_4 x_5 + x_1 x_6 + x_3 x_6 + x_5 x_6 \\ x_1 x_4 + x_2 x_4 + x_1 x_5 + x_2 x_5 + x_1 x_6 + x_2 x_6 + x_4 x_6 + x_5 x_6 \\ x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_5 + x_2 x_5 + x_1 x_6 + x_3 x_6 + x_5 x_6 \\ x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_1 x_6 + x_3 x_6 + x_4 x_6 \\ x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_3 x_4 + x_2 x_5 + x_3 x_5 + x_4 x_5 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{10}$$

We claim this implies $z$ is unique and satisfies $x_1 = \cdots = x_6 = \frac{1}{6}$. To see that, consider dot products of (10) with four different vectors and use that $z$ has positive coordinates. For $(1, 0, -1, 1, 0, -1)$ we obtain $-2x_1 x_4 + 2x_3 x_6 = 0$, and $(0, 1, -1, 0, 1, -1)$ gives $-2x_2 x_5 + 2x_3 x_6 = 0$. Equivalently, $x_1 x_4 = x_2 x_5 = x_3 x_6$ or, since none of the coordinates can vanish, $x_4 = x_3 x_6/x_1$ and $x_5 = x_3 x_6/x_2$. Using this, we conclude

$$0 = \langle \nabla V(z), (1, -1, 0, 0, 0, 0) \rangle = (x_2 - x_1) \frac{x_6 (x_1 x_2 + x_1 x_3 + x_2 x_3 + x_3 x_6)}{x_1 x_2},$$

$$0 = \langle \nabla V(z), (0, 0, 1, -1, 0, 0) \rangle = (x_6 - x_1) \frac{x_1 x_2 + x_1 x_3 + x_2 x_3 + x_3 x_6}{x_1},$$

$$0 = \langle \nabla V(z), (0, 0, 0, 1, -1) \rangle = (x_2 - x_3) \frac{x_6 (x_1 x_2 + x_1 x_3 + x_2 x_3 + x_3 x_6)}{x_1 x_2}. $$
Therefore $x_1 = x_2 = x_3 = x_6$ and, using $x_1x_4 = x_2x_5 = x_3x_6$, these must agree with $x_4 = x_5$. This proves the claim.

We have shown there is a unique critical point in $z \in (0, \infty)^6$ for $V$ under the constraint $L = 1$, where $V$ attains the maximal value $V = 16/(6^3 \sqrt{2}) = \sqrt{2}/3^3$. This implies the inequality (9) for $L = 1$ and, by scaling invariance, in general; the uniqueness of $z$ implies that equality in (9) holds if and only if all lengths are equal. That the lattice is body-centred cubic is well-known for srs, and can be verified using the lattice vectors of the proof of Lemma 4.3. □

Remark The remark following Theorem 5.2 indicates an alternative proof that $z$ is an interior point. It establishes the explicit lower bound $L^3/V \geq 81/4$ along the boundary of $[0, \infty)^6$, larger than (9).

We would like to draw a further consequence of Lemma 4.2.

Proposition 4.5 For each choice of edge lengths $x_1, \ldots, x_6 > 0$ there exists a Steiner network in $\mathbb{R}^3$ with quotient $K_4$. Up to isometry, its vertices $p_1, \ldots, p_6$ are uniquely given by (6) as well as

\[
p_4 = p_2 + x_4 \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}, \quad p_5 = p_3 + x_5 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}, \quad p_6 = p_1 + x_6 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix},
\]

and the lattice is $\Lambda = (p_6 - p_2)\mathbb{Z} + (p_4 - p_3)\mathbb{Z} + (p_5 - p_1)\mathbb{Z}$.

Setting, for instance, $x_1 = x_2 = x_3 = 1$ and $x_4 = x_5 = x_6 = 3$ gives

\[
g_1 = \begin{pmatrix} 3 \\ -\sqrt{3} \\ -\sqrt{6} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 2\sqrt{3} \\ -\sqrt{6} \end{pmatrix}, \quad g_3 = \begin{pmatrix} -3 \\ -\sqrt{3} \\ -\sqrt{6} \end{pmatrix}.
\]
These vectors have the same length and are orthogonal so that the lattice is primitive cubic. Moreover, $L^3/V = 16\sqrt{2} \approx 22.63$. See Fig. 9.

5 Triply Periodic Steiner Networks Covering $D_1 \Box D_2$

Besides $K_4$ only the graph $D_1 \Box D_2$ is 3-regular on four vertices with no loops. We determine length minimizers for this topology. For each lattice, their length ratio must be larger than for $K_4$, as predicted by Theorem 2.6.

From Proposition 2.5 we know that each pair of double edges of $D_1 \Box D_2$ lifts to a set of zigzag curves. So each pair defines a set of containing parallel planes, with pairs of planes from different sets meeting at a dihedral angle $\alpha$; a vanishing angle relates to a doubly periodic network. To parameterize the set of all networks, we can use the six edge lengths as well as the angle $\alpha$:

**Lemma 5.1** Let $N \subset \mathbb{R}^3$ be a triply periodic Steiner network with quotient $D_1 \Box D_2$. Then, up to isometry, the network $N$ is uniquely determined by its six edge lengths $x_1, \ldots, x_6 > 0$ and an angle $\alpha \in (0, \pi/2]$. Moreover, $N$ has length $L = \sum_i x_i$ and, for a labelling of the edge lengths as in Fig. 10, the spanned volume is

$$V = \frac{3}{4} \sin(\alpha) \left( x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + (x_5 + x_6)(x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4) \right).$$

**Proof** Consider a connected subgraph $\tilde{N} \subset N$ with seven vertices $p_0, \ldots, p_6$ as in Fig. 10 such that $p_0$, $p_4$, $p_6$, as well as $p_3$, $p_5$ are identified in the quotient. We may assume $p_1$ is the origin, $p_2$ is on the $x$-axis, and $p_0$, $p_4$ are in the $xy$-plane. Balancing then implies

$$p_0 = x_1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = x_5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_4 = x_2 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}. \quad (12)$$
The tangent plane at $p_2$ must be a rotation about the $x$-axis of the tangent plane at $p_1$ by an angle $\alpha \in [0, 2\pi)$. Let $A_\alpha \in SO(3)$ denote such a rotation. The Steiner condition then implies that $p_3 - p_2$ points in the same direction as $p_1 - p_0$ rotated by $A_\alpha$. The same applies to $p_5 - p_2$ and $p_1 - p_4$. That is,

$$p_3 = p_2 + x_3 A_\alpha \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_3 + 2x_5 \\ x_3 \sqrt{3} \cos \alpha \\ x_3 \sqrt{3} \sin \alpha \end{pmatrix},$$

(13)

$$p_5 = p_2 + x_4 A_\alpha \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_4 + 2x_5 \\ -x_4 \sqrt{3} \cos \alpha \\ -x_4 \sqrt{3} \sin \alpha \end{pmatrix}.$$  

Triple periodicity implies $\alpha \neq 0 \mod \pi$. We may assume $\alpha \in (0, \pi/2]$ by a change of labelling induced by a graph symmetry: exchanging $p_3$ with $p_5$ replaces $\alpha$ with $\alpha \pm \pi$, and exchanging $p_1$ with $p_2$ replaces $\alpha$ by $\pi - \alpha$.

In the proof of Proposition 2.5(ii) we showed that the double edges $e_3$ and $e_4$ of $D_1 \square D_2$ lift to a zigzag curve of $N$, contained in a plane that also contains lifts of the edges $e_5$ and $e_6$. The statement of the Proposition implies the vector $p_6 - p_3$ points in the same direction as $p_2 - p_1$. So we have

$$p_6 = p_3 + x_6 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_3 + 2x_5 + 2x_6 \\ x_3 \sqrt{3} \cos \alpha \\ x_3 \sqrt{3} \sin \alpha \end{pmatrix}.$$  

(14)

The three vectors

$$g_1 := p_4 - p_0 = \frac{1}{2} \begin{pmatrix} x_1 - x_2 \\ \sqrt{3}(x_1 + x_2) \\ 0 \end{pmatrix},$$

$$g_2 := p_5 - p_3 = \frac{1}{2} \begin{pmatrix} x_4 - x_3 \\ -\sqrt{3}(x_3 + x_4) \cos \alpha \\ -\sqrt{3}(x_3 + x_4) \sin \alpha \end{pmatrix},$$

(15)

$$g_3 := p_6 - p_0 = \frac{1}{2} \begin{pmatrix} x_1 + x_3 + 2x_5 + 2x_6 \\ \sqrt{3}(x_1 + x_3 \cos \alpha) \\ x_3 \sqrt{3} \sin \alpha \end{pmatrix}$$

span the lattice $\Lambda$; indeed, an inspection of Fig. 10 shows they correspond to minimal cycles in the abstract graph. Then $|\det(g_1, g_2, g_3)|$ can be computed to give (11). \qed

To determine optimal networks with quotient $D_1 \square D_2$ we follow the same strategy as for $K_4$. This time, however, our arguments for the boundary case are explicit. Interestingly enough, up to similarity of $\mathbb{R}^3$ the result is a one-parameter family of optimal networks, meaning that these networks are not strictly stable.
Theorem 5.2  Let $N \subset \mathbb{R}^3$ be a triply periodic Steiner network with quotient $D_1 \Join D_2$. Then

$$\frac{L^3}{V} \geq \frac{81}{4},$$

(16)

where equality holds if and only if

$$x_1 = x_2 = x_3 = x_4 = 2x_5 + 2x_6 \quad \text{and} \quad \alpha = \frac{\pi}{2}.$$ \hspace{1cm} (17)

In the equality case the lattice is generated, up to similarity, by $g_1 = (0, \sqrt{3}, 0)$, $g_2 = (0, 0, -\sqrt{3})$, $g_3 = (3, \sqrt{3}, \sqrt{3})/2$.

Here the lattice vectors are resulting from (15) for the choice of (17) with $x_1 = 1$. After relabelling coordinates and scaling, the lattice attains the form stated in Theorem B.

Proof  Admitting vanishing edge lengths, we will show the inequality in a form implying (16), namely

$$V \leq \frac{4}{81} L^3 \quad \text{for all } x \in [0, \infty)^6 \text{ and } \alpha \in (0, \pi),$$

(18)

with equality precisely for (17).

For fixed $x = (x_1, \ldots, x_6)$ clearly $L$ is independent of $\alpha$, while (11) gives that $V$ is maximal exactly at $\alpha = \pi/2$.

Moreover, both $V$ and $L$ depend on $x_5$, $x_6$ only through $y := x_5 + x_6$. Thus in order to establish (18) we may fix $\alpha$ to $\pi/2$ and consider the functions induced by $L$ and $V$ on the domain $[0, \infty)^5 \ni (x_1, x_2, x_3, x_4, y)$. For the remainder of the proof we denote these continuous functions again by $L$ and $V$.

We claim that (18) holds along the boundary of $[0, \infty)^5$. Trivially, this is true at 0. Otherwise let $(x_1, x_2, x_3, x_4, y)$ be a point where at least one coordinate vanishes. In case $y = 0$ the volume is

$$V = \frac{3}{4} (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4).$$

The right-hand side contains the elementary symmetric polynomial of degree $k = 3$ in $m = 4$ variables and so indeed, by Maclaurin’s inequality (20),

$$V \leq 3 \left( \frac{x_1 + x_2 + x_3 + x_4}{4} \right)^3 = \frac{3}{64} L^3 < \frac{4}{81} L^3.$$

The other case is that some $x_i$ vanishes for $i = 1, 2, 3, or 4$. In view of symmetries it suffices to consider the case $x_1 = 0$. Then Maclaurin’s inequality gives

$$V = \frac{3}{4} x_2 (x_3 x_4 + x_3 y + x_4 y) \leq \frac{1}{4} x_2 (x_3 + x_4 + y)^2.$$
and our claim follows from the estimate on geometric and arithmetic mean,

\[ V \leq x_2 \frac{x_3 + x_4 + y}{2} \frac{x_3 + x_4 + y}{2} \leq \frac{1}{27} (x_2 + x_3 + x_4 + y)^3 = \frac{1}{27} L^3 < \frac{4}{81} L^3. \]

The continuous function \( V \) attains its maximum on the compact set \( L^{-1}(1) \subset [0, \infty)^5 \setminus \{0\} \) at some point \( z := (x_1, \ldots, x_4, y) \). One easily verifies \( V = 4 L^3 / 81 \) if \( (17) \) holds, so that our claim implies that in fact \( z \in (0, \infty)^5 \). For the set \((0, \infty)^5\), the point \( z \) is critical for \( V \) under the constraint \( L = 1 \), and so \( \nabla V(z) = \lambda \nabla L(z) \) with a Lagrange multiplier \( \lambda \in \mathbb{R} \). Equivalently,

\[
\begin{pmatrix}
\frac{3}{4} (x_2 x_3 + x_2 x_4 + x_3 x_4 + x_3 y + x_4 y) \\
\frac{3}{4} (x_1 x_3 + x_1 x_4 + x_3 x_4 + x_3 y + x_4 y) \\
\frac{3}{4} (x_1 x_2 + x_1 x_4 + x_2 x_4 + x_1 y + x_2 y) \\
\frac{3}{4} (x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 y + x_2 y) \\
\frac{3}{4} (x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4)
\end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{19}
\]

We claim this implies \( x_1 = x_2 = x_3 = x_4 = 2y \). For the proof, again we consider dot products. The product with \((1, -1, 0, 0, 0)\) gives \((x_2 - x_1)(x_3 + x_4) = 0\), and the product with \((0, 0, 1, -1, 0)\) gives \((x_1 + x_2)(x_4 - x_3) = 0\), implying \( x_1 = x_2 \) and \( x_3 = x_4 \). Thus the product with \((0, 1, 0, -1, 0)\)

\[
gives \quad (x_3 - x_1)(x_1 + x_3 + 2y) = 0 \quad \text{and so} \quad x_1 = x_3, \quad \text{and} \quad (0, 1, 0, 0, -1)
\]

gives \( x_1(2y - x_1) = 0 \) implying, finally, \( 2y = x_1 \). Reasoning literally as in the proof of Theorem 4.4 concludes the proof. \( \square \)

**Remark** The preceding proof leads to a quantitative argument that for \( K_4 \) the maximum for the constrained extremum of \( V \) cannot be attained at a boundary point \( z \) of \([0, \infty)^6\). So assume that at least one component \( x_i \) of \( z \) vanishes. By \( K_4 \) symmetries, we may assume that \( x_6 = 0 \). According to (8) then

\[
V = \frac{1}{\sqrt{2}} (x_1 x_2 x_4 + x_1 x_2 x_5 + x_1 x_3 x_4 + x_1 x_3 x_5 \\
\quad + x_1 x_4 x_5 + x_2 x_3 x_4 + x_2 x_3 x_5 + x_2 x_4 x_5).
\]

This expression matches the volume (11) of a \( D_1 \Box D_2 \) network obtained with \( \alpha = \arccos(1/3) = \arcsin(2\sqrt{2}/3) \) by contracting the edge \( e_6 \) so that \( x_6 \) becomes 0, and after exchanging the indices 3 and 5. Note that in fact the two network quotients limit to the same graph, see Fig. 11. Using (18) in the limit \( x_6 = 0 \) this proves, as desired,

\[
L^3 \geq \frac{81}{4} V > \frac{27}{\sqrt{2}} V.
\]

We conclude this section by establishing the continuous deformation claimed in Theorem B. It initiates from our locally minimizing one-parameter family of \( \alpha \) networks satisfying (17) and ends in a \( K_4 \) network with the same lattice. The length ratio monotonously decreases along the deformation.
To state it, note we can assume that our family of ths networks has the same tetragonal lattice $\Lambda$. Let $N^1$ be the $K_4$ network which minimizes length for this lattice $\Lambda$, obtained from Theorem 2.3. It suffices to let the deformation start at the limiting network $N^0$ of the ths family with $x_6 = 0$. Note that $N^0$ is not Steiner, but contains a vertex of degree 4.

**Proposition 5.3** There exists a continuous family of triply periodic networks $N^t \subset \mathbb{R}^3$ for $t \in [0, 1]$ from the limit $N^0$ of ths networks to the Steiner network $N^1$ with the following properties:

- Each $N^t$ has the same lattice $\Lambda$ and so the same volume $V$.
- $N^t$ is a network with quotient graph $K_4$ for $0 < t \leq 1$.
- The length $t \mapsto L(N^t)$ is decreasing and $L(N^0) > L(N^1)$.

See Figs. 11 and 12 for the combinatorial picture, and Fig. 3 for the geometry.

**Proof** We specify the vertices of $N^t$ as the convex combination of $N^0$ and $N^1$. That is, if $p_i^0$ are the vertices of $N^0$, and $p_i^1$ those of $N^1$ then $N^t$ has vertices $p_i^t := (1-t)p_i^0 + tp_i^1$ for $i = 1, \ldots, 6$. Moreover, we connect the vertices of $N^t$ with edges as prescribed by the third image of Fig. 12.

We claim that $L(N^t)$ is a decreasing function of $t \in [0, 1]$. This follows from the following facts. First, summing over all edges, we see the length

$$t \mapsto L(N^t) = \sum |p_i^t - p_j^t|$$
is convex on \([0, 1]\) as each distance function \(t \mapsto |p_i^t - p_j^t|\) is convex. Second, the minimizing \(K_4\) network \(N^1\) is Steiner, hence critical for \(t \mapsto L(N^t)\) under our variation.

Consequently, \(L(N^t)\) is strictly decreasing from \(t = 0\) to the minimum critical point \(t = 1\) (once again this shows that \(N^0\) cannot minimize). \(\square\)

**Remark** Our \(K_4\) networks and their limit \(N^0\) are mirror-symmetric with respect to reflection in the planes containing zigzag curves, whereas networks with quotient \(K_4\) are chiral (they do not admit a mirror reflection). For instance, the piece shown in Fig. 3a admits reflection in the plane containing the yellow and green vertices, but this symmetry is broken in (c). As two referees suggested our homotopy can be extended to define a path between the \(srs\) network and its mirror image, such that the largest length ratio is attained at \(N^0\). Thus it can be conjectured that in fact \(N^0\) is a minimax (mountain-pass) point for the length ratio of paths between the pair of mirror-related \(srs\)-networks.

**Funding** Open Access funding enabled and organized by Projekt DEAL.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**Appendix: On Maclaurin’s Inequality**

We state and prove a well-known inequality which was used in the proofs of Proposition 3.3 and Theorem 5.2. The Lagrange multiplier technique used in Theorems 5.2 and 4.4 leads to a proof which is shorter than the standard proof in terms of Newton’s inequality, see [6]. Let

\[
P_k(x_1, \ldots, x_m) := \sum_{1 \leq j_1 < \ldots < j_k \leq m} x_{j_1} \cdots x_{j_k}, \quad 1 \leq k \leq m,
\]

be the elementary symmetric polynomial of degree \(k\), and set \(P_0(x_1, \ldots, x_m) := 1\).

**Lemma 5.4 (Maclaurin’s inequality)** Let \(2 \leq k \leq m\) and \(x_1, \ldots, x_m \geq 0\). Then

\[
P_k(x_1, \ldots, x_m) \leq \binom{m}{k} \left(\frac{x_1 + \ldots + x_m}{m}\right)^k,
\]

where equality holds if and only if \(x_1 = \ldots = x_m\).

**Proof** For each \(k \geq 2\) we prove (20) by induction over \(m\). The base case is \(m = k\), where \(P_k(x) = x_1 \cdots x_m\). Then (20) is the estimate on geometric and arithmetic mean.
For the step suppose \( m > k \). First we claim that (20) holds strictly if some but not all \( x_i \) vanish. In view of the symmetry of (20) we may assume \( x_m = 0 \). Note that \( P_k(x_1, \ldots, x_{m-1}, 0) \) is an elementary symmetric polynomial of degree \( k \) in \( m - 1 \) variables, and so the induction hypothesis gives

\[
P_k(x_1, \ldots, x_{m-1}, 0) \leq \left( \frac{m-1}{k} \right) \left( \frac{x_1 + \ldots + x_{m-1}}{m-1} \right)^k.
\]

We estimate the right hand side, using the strict inequality

\[
\left( \frac{m-1}{k} \right) \frac{1}{(m-1)^k} = \frac{1}{k!} \frac{m-1}{m} \ldots \frac{m-k+1}{m} = \left( \frac{m}{k} \right) \frac{1}{m^k}.
\]

Not all \( x_i \) vanish and so this yields strict inequality in (20), as claimed.

It remains to prove the step if all \( x_i \) are positive. Since (20) is scaling invariant it is sufficient to prove this inequality under the length constraint \( L(x) := x_1 + \ldots + x_m = 1 \). The continuous function \( P_k \) attains a maximum over the compact set \( L^{-1}(1) \subset [0, \infty)^m \) at some point \( z = (z_1, \ldots, z_m) \). Note that \( z \neq 0 \), that we have equality in (20) for \( x = (\frac{1}{m}, \ldots, \frac{1}{m}) \), and that the claim, making use of the induction hypothesis, yields strict inequality in (20) on \( \partial([0, \infty)^m) \setminus \{0\} \).

Thus \( z \) must be an interior point of \([0, \infty)^m\). Since \( z \) is critical for \( P_k \) under the smooth constraint \( L = 1 \) we obtain the necessary condition

\[
\nabla P_k(z) = \lambda \nabla L(z) = \lambda(1, \ldots, 1)
\]

with \( \lambda \in \mathbb{R} \) a Lagrange multiplier. It remains to show this implies \( z_1 = \ldots = z_m \).

Since \( z \) assigns equality to (20) and as \( z \) was chosen maximally, then the proof of the induction step is complete.

For any \( i \neq j \) we can write the elementary symmetric polynomial \( P_k \) in the form

\[
P_k(x) = Q_0(x) + (x_i + x_j)Q_1(x) + x_i x_j Q_2(x),
\]

where \( Q_\ell \) is a polynomial in \( k - \ell \geq 0 \) variables, independent of \( x_i \) and \( x_j \). At the interior maximum \( z \) we conclude from (21) that \( \partial_i P_k(z) = \partial_j P_k(z) \) for all \( i, j \), or

\[
Q_1(z) + z_j Q_2(z) = Q_1(z) + z_i Q_2(z).
\]

But \( z \) has all components positive and so \( Q_2(z) \) does not vanish. Thus indeed \( z_i = z_j \) for all \( i, j \). \( \square \)

References

1. Alex, J., Grosse-Brauckmann, K.: Periodic networks of fixed degree minimizing length, arXiv:1911.01792 (2019)
2. Coxeter, H.S.M.: On Laves’ graph of girth ten. Can. J. Math. 8, 18–23 (1955)
3. de Campo, L., Delgado-Friedrichs, O., Hyde, S.T., O’Keeffe, M.: Minimal nets and minimal minimal surfaces. Acta Cryst. A 69, 483–489 (2013)
4. Grosse-Brauckmann, K.: On gyroid interfaces. J. Colloid Interface Sci. 187, 418–428 (1997)
5. Grosse-Brauckmann, K., Wohlgemuth, M.: The gyroid is embedded and has constant mean curvature companions. Calc. Var. Partial Diff. Eq. 4, 499–523 (1996)
6. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1952)
7. Hyde, S.T., O’Keeffe, M., Proserpio, D.M.: A short history of an elusive yet ubiquitous structure in chemistry, materials, and mathematics. Angew. Chem. Int. Ed. 47, 7996–8000 (2008)
8. Ivanov, A., Tuzhilin, A.: Minimal Networks. The Steiner problem and Its Generalizations, CRC Press, Boca Raton (1994)
9. Kotani, M., Sunada, T.: Standard realizations of crystal lattices via harmonic maps. Trans. Am. Math. Soc. 353, 1–20 (2001)
10. O’Keeffe, M., Eddaoudi, M., Li, H., Reineke, T., Yaghi, O.M.: Frameworks for extended solids: geometrical design principles. J. Solid State Chem. 152, 3–20 (2000)
11. Schoen, A. H.: Infinite periodic minimal surfaces without self-intersections. NASA Technical Note TN D-5541 (1970)
12. Schoen, A.H.: On the graph (10–3)-a. Notices Am. Math. Soc. 55, 663 (2008)
13. Schröder-Turk, G., Fogden, A., Hyde, S.T.: Bicontinuous geometries and molecular self-assembly: comparison of local curvature and global packing variations in genus-three cubic, tetragonal and rhombohedral surfaces. Eur. Phys. J. B 54, 509–524 (2006)
14. Sunada, T.: Crystals that nature might miss creating. Notices Am. Math. Soc. 8, 208–2015 (2008)
15. Sunada, T.: Topological crystallography: with a view towards discrete geometric analysis. Surveys and Tutorials in the Applied Mathematical Sciences, Springer (2012)
16. Sunada, T.: Diamond twin, arXiv:1904.07230 (2019)
17. Wells, A.F.: The geometrical basis of crystal chemistry. XII. Review of structures based on three-dimensional 3-connected nets. Acta Cryst. 32, 2619–2626 (1976)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.