Black holes and Hawking radiation in spacetime and its analogues

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Abstract These notes introduce the fundamentals of black hole geometry, the thermality of the vacuum, and the Hawking effect, in spacetime and its analogues. Stimulated emission of Hawking radiation, the trans-Planckian question, short wavelength dispersion, and white hole radiation in the setting of analogue models are also discussed. No prior knowledge of differential geometry, general relativity, or quantum field theory in curved spacetime is assumed. The discussion attempts to capture the essence of these topics without oversimplification.
1 Spacetime geometry and black holes

In this section I explain how black holes are described in general relativity, starting with the example of a spherical black hole, and followed by the 1+1 dimensional generalization that figures in many analogue models. Next I discuss how symmetries and conservation laws are formulated in this setting, and how negative energy states arise. Finally, I introduce the concepts of Killing horizon and surface gravity, and illustrate them with the Rindler or acceleration horizon, which forms the template for all horizons.

1.1 Spacetime geometry

The line element or metric $ds^2$ assigns a number to any infinitesimal displacement in spacetime. In a flat spacetime in a Minkowski coordinate system it takes the form

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2),$$

where $t$ is the time coordinate, $x, y, z$ are the spatial Cartesian coordinates, and $c$ is the speed of light. Hereafter I will mostly employ units with $c = 1$ except when discussing analogue models (for which $c$ may depend on position and time when using the Newtonian $t$ coordinate). When $ds^2 = 0$ the displacement is called lightlike, or null. The set of such displacements at each event $p$ forms a double cone with vertex at $p$ and spherical cross sections, called the light cone or null cone (see Fig. 1). Events outside the light cone are spacelike related to $p$, while events inside the cone are either future timelike or past timelike related to $p$. For timelike displacements, $ds^2$ determines the square of the corresponding proper time interval.

The metric also defines the spacetime inner product $g(v,w)$ between two 4-vectors $v$ and $w$, that is,
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\[ g(v, w) = ds^2(v, w) = c^2 dt(v) dt(w) - [dx(v) dx(w) + dy(v) dy(w) + dz(v) dz(w)] = c^2 dt(v) dt(w) - \left[ dx(v) dx(w) + dy(v) dy(w) + dz(v) dz(w) \right]. \] (2)

Here \( dt(v) = v^a \partial_a t = v^t \) is the rate of change of the \( t \) coordinate along \( v \), etc.

In a general curved spacetime the metric takes the form

\[ ds^2 = g_{\alpha \beta}(x) dx^\alpha dx^\beta, \] (3)

where \( \{x^\alpha\} \) are coordinates that label the points in a patch of a spacetime (perhaps the whole spacetime), and there is an implicit summation over the values of the indices \( \alpha \) and \( \beta \). The metric components \( g_{\alpha \beta} \) are functions of the coordinates, denoted \( x \) in (3). In order to define a metric with Minkowski signature, the matrix \( g_{\alpha \beta} \) must have one positive and three negative eigenvalues at each point. Then local inertial coordinates can be chosen in the neighborhood any point \( p \) such that (i) the metric has the Minkowski form (1) at \( p \) and (ii) the first partial derivatives of the metric vanish at \( p \). In two spacetime dimensions there are \( 9 \) independent second partials of the metric at a point. These can be modified by a change of coordinates \( x^\mu \rightarrow x'^\mu \), but the relevant freedom resides in the third order Taylor expansion coefficients \( \left( \partial^3 x'^\mu / \partial x^\alpha \partial x^\beta \partial x^\gamma \right)_p \), of which only \( 8 \) are independent because of the symmetry of mixed partials. The discrepancy \( 9 - 8 = 1 \) measures the number of independent second partials of the metric that cannot be set to zero at \( p \), which is the same as the number of independent components of the Riemann curvature tensor at \( p \). So a single curvature scalar characterizes the curvature in a two dimensional spacetime. In four dimensions the count is \( 100 - 80 = 20 \).

\subsection*{1.2 Spherical black hole}

The Einstein equation has a unique (up to coordinate changes) spherical solution in vacuum for each mass, called the Schwarzschild spacetime.

\subsection{1.2.1 Schwarzschild coordinates}

The line element in so-called Schwarzschild coordinates is given by

\[ ds^2 = \left( 1 - \frac{r_s}{r} \right) dt^2 - \left( 1 - \frac{r_s}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (4)

Here \( r_s = 2GM/c^2 \) is the Schwarzschild radius, with \( M \) is the mass, and \( c \) is set to 1. Far from the black hole, \( M \) determines the force of attraction in the Newtonian limit, and \( Mc^2 \) is the total energy of the spacetime.

The spherical symmetry is manifest in the form of the line element. The coordinates \( \theta \) and \( \phi \) are standard spherical coordinates, while \( r \) measures \( 1/2\pi \) times the circumference of a great circle, or the square root of \( 1/4\pi \) times the area of a sphere. The value \( r = r_s \) corresponds to the event horizon, as will be explained, and
the value $r = 0$ is the “center”, where the gravitational tidal force (curvature of the spacetime) is infinite. Note that $r$ should not to be thought of as the radial distance to $r = 0$. That distance isn’t well defined until a spacetime path is chosen. (A path at constant $\bar{t}$ does not reach any $r < r_s$.)

The coordinate $\bar{t}$ is the Schwarzschild time. It measures proper time at $r = \infty$, whereas at any other fixed $r, \theta, \phi$ the proper time interval is $\Delta \tau = \sqrt{1 - r_s/r} \, dt$. The coefficients in the line element are independent of $\bar{t}$, hence the spacetime has a symmetry under $\bar{t}$ translation. This is ordinary time translation symmetry at $r = \infty$, but it becomes a lightlike translation at $r = r_s$, and a space translation symmetry for $r < r_s$, since the coefficient of $d\bar{t}^2$ is negative there. The defining property of the Schwarzschild time coordinate, other than that it measures proper time in the rest frame of the black hole at infinity, is that surfaces of constant $\bar{t}$ are orthogonal, in the spacetime sense, to the direction of the time-translation symmetry, i.e. to the lines of constant $(r, \theta, \phi)$: there are no off-diagonal terms in the line element. But this nice property is also why $\bar{t}$ is ill behaved at the horizon.

**Redshift and horizon**

Suppose a light wave is generated with coordinate period $\Delta \bar{t}$ at some radius $r_a$, and propagates to another radius $r_b$ (see Fig. 2). Because of the time translation symmetry of the spacetime, the coordinate period of the wave at $r_b$ will also be $\Delta \bar{t}$. The ratio of the proper time periods will thus be $\Delta \tau_a / \Delta \tau_b = \sqrt{1 - r_s/r_a} / \sqrt{1 - r_s/r_b}$, and the ratio of the frequencies will the the reciprocal. This is the gravitational redshift. Note that as $r_a \rightarrow r_s$, the redshift is infinite. The infinite redshift surface $r = r_s$.
of the spherical black hole is the (stationary) event horizon. The same is true of the 1+1 dimensional black holes we focus on later in these notes.

It is worth emphasizing that for a non-spherical stationary black hole, for instance a rotating black hole, the infinite redshift surface, where the time-translation symmetry becomes lightlike, is generally not the event horizon, because it is a timelike surface. A timelike surface can be crossed in either direction. In order to be a horizon, a surface must be tangent to the local light cone at each point, so that it cannot be crossed from inside to outside without going faster than light. At each point of such a null surface there is one null tangent direction, and all other tangent directions are spacelike and orthogonal to the null direction (see Fig. 3). Therefore the null tangent direction is orthogonal to all directions in the surface, i.e. the null tangent is also the normal. If the horizon is a constant $r$ surface, then the gradient $\nabla_\alpha r$ is also orthogonal to all directions in the surface, so it must be parallel to the null normal. This means that it is a null (co)vector, hence $g^{\alpha\beta}\nabla_\alpha r\nabla_\beta r = g^{rr} = 0$ at the horizon.

1.2.2 Painlevé-Gullstrand coordinates

A new time coordinate $t$ that is well behaved at the horizon can be defined by $t = \bar{t} + h(r)$ for a suitable function $h(r)$ whose bad behavior at $r_+$ cancels that of $\bar{t}$. This property of course leaves a huge freedom in $h(r)$, but a particularly nice choice is defined by

$$dt = d\bar{t} + \frac{\sqrt{r}}{r-1}dr, \quad \text{i.e.} \quad t = \bar{t} - 2\sqrt{r} + \ln \left( \frac{\sqrt{r} + 1}{\sqrt{r} - 1} \right)$$

(5)
where now I have adopted units with $r_s = 1$. It is easy to see that the $t\cdot r$ part of the Schwarzschild line element takes the form

$$ds^2 = dt^2 - \left( dr + \sqrt{\frac{1}{r}} \ dt \right)^2 - r^2(d\theta^2 + \sin^2 \theta \ d\phi^2)$$ \hspace{1cm} (6)

$$= \left( 1 - \frac{1}{r} \right) dt^2 - \frac{2}{\sqrt{r}} dt \ dr - r^2 - r^2(d\theta^2 + \sin^2 \theta \ d\phi^2)$$ \hspace{1cm} (7)

The new coordinate $t$ is called the Painlevé-Gullstrand (PG) time. At $r = 1$ the metric coefficients are all regular, and indeed the coordinates are all well behaved there. According to (7), we have $ds^2 = 0$ along a line of constant $(r = 1, \theta, \phi)$, so such a line is lightlike. Such lines generate the event horizon of the black hole. The PG time coordinate has some remarkable properties:

- the constant $t$ surfaces are flat, Euclidean spaces;
- the radial worldlines orthogonal to the constant $t$ surfaces are timelike geodesics (free-fall trajectories) along which $dt$ is the proper time.

For some practice in spacetime geometry, let me take you through verifying these properties. Setting $dt = 0$ in the line element we see immediately that $\{r, \theta, \phi\}$ are standard spherical coordinates in Euclidean space. To find the direction orthogonal to a constant $t$ surface we could note that the gradient $\nabla_a t$ has vanishing contraction with any vector tangent to this surface, which implies that the contravariant vector $g^{a\beta} \nabla_{\beta} t$, formed by contraction with the inverse metric $g^{a\beta}$, is orthogonal to the surface. Alternatively, we need not compute the inverse metric, since the form of the line element (6) allows us to read off the orthogonal direction “by inspection” as follows. Consider the inner product of two 4-vectors $v$ and $w$ in this metric,

$$g(v, w) = dt(v)dt(w) - \left( dr + \sqrt{\frac{1}{r}} \ dt \right)(v) \left( dr + \sqrt{\frac{1}{r}} \ dt \right)(w)$$

$$- r^2 d\theta(v)d\theta(w) - r^2 \sin^2 \theta d\phi(v)d\phi(w),$$ \hspace{1cm} (8)

using the notation of Eq. (2). If the vector $v$ is tangent to the constant $t$ surface, then $dt(v) = 0$, so the first term vanishes. The remaining terms will vanish if \(dr + \sqrt{1/r} \ dt)(w) = d\theta(w) = d\phi(w) = 0\). Thus radial curves with $dr + \sqrt{1/r} \ dt = d\theta = d\phi = 0$ are orthogonal to the surface, and along them $ds^2 = dt^2$, i.e. $dt$ measures proper time along those curves. Moreover, any other timelike curve connecting the same two spacetime points will have shorter proper time, because the negative terms in $ds^2$ will contribute. The proper time is thus stationary with respect to first order variations of the curve, which is the defining property of a geodesic. \footnote{Even if the other terms in the line element (6) had not been negative, they would not contribute to the first order variation in the proper time away from a path with $(dr + \sqrt{1/r} \ dt) = d\theta^2 = d\phi^2 = 0$, since the line element is quadratic in these terms. Thus the curve would still have been a geodesic (although the metric signature would not be Lorentzian).}
1.2.3 Spacetime diagram of the black hole

The nature of the unusual geometry of the black hole spacetime can be grasped rather easily with the aid of a spacetime diagram (see Fig. 4). For the Schwarzschild black hole, we may exploit the spherical symmetry and plot just a fixed value of the spherical angles \((\theta, \phi)\), and we may plot the lines of constant \(r\) vertically and the lines of constant PG time \(t\) horizontally. Then the time translation symmetry corresponds to a vertical translation symmetry of the diagram.

The diagram comes alive when the light cones are plotted. At a given event, the light cone is determined by \(ds^2 = 0\), which for radial displacements corresponds to the two slopes

\[
dt/dr = \frac{1}{\pm 1 - \sqrt{1/r}} \quad \text{(radial lightrays)} \quad (10)
\]

Far from the horizon these are the outgoing and incoming lightrays \(dt/dr \rightarrow \pm 1\). The ingoing slope is negative and gets smaller in absolute value as \(r\) decreases, approaching 0 as \(r \rightarrow 0\). The outgoing slope grows as \(r\) decreases, until reaching infinity at the horizon at \(r = 1\). Inside the horizon it is negative, so an “outgoing” lightray actually propagates to smaller values of \(r\). The outgoing slope also approaches 0 as \(r \rightarrow 0\).
1.2.4 Redshift of outgoing waves near the horizon

An outgoing wave is stretched as it climbs away from the horizon. The lines of constant phase for an outgoing wave satisfying the relativistic wave equation are just the outgoing lightrays (10). The rate of change of a wavelength $\lambda$ is given by

$$\frac{d\lambda}{dt} = (d/dr)(dr/dt)\lambda,$$

The relative stretching rate is thus given by

$$\kappa \equiv \frac{d\lambda}{dt} = \frac{d}{dr} \frac{dr}{dt} = \frac{c}{2r_s},$$

where in the second step the expression is evaluated at the horizon, and the dimensionful constants are restored to better illustrate the meaning. This rate is called the “surface gravity” $\kappa$ of the horizon. Later I will explain different ways in which the surface gravity can be defined and calculated.

We can go further and use the lightray equation (10) to obtain an approximate expression for the wave phase near the horizon. Consider an outgoing wave of the form $e^{i\phi}$, with $\phi = -\omega t + \int k(r')dr'$. (This simple harmonic $t$ dependence is exact because the metric is independent of $t$.) Along an outgoing lightray the phase is constant: $0 = d\phi = -\omega dt + k(r)dr$, so

$$k(r) = \frac{\omega}{1 - r^{-1/2}} \sim \frac{2\omega}{r - 1} = \frac{\omega/\kappa}{r - r_s}$$

where in the second step a near horizon approximation is used, and in the last step the dimensionful constants are again restored. The wave thus has the near-horizon form

$$e^{-i\omega t} e^{i(\omega/\kappa)\ln(r-r_s)}.$$  
Note that the surface gravity appears in a ratio with the wave frequency, and there is a logarithmic divergence in the outgoing wave phase at the horizon.

1.3 Effective black hole and white hole spacetimes

Many black hole analogues can be described with one spatial dimension, and I will focus on those here. They are simple generalizations of the radial direction for a spherical black hole.

Waves or quasiparticles in a stationary 1+1 dimensional setting can often be described by a relativistic field in an effective spacetime defined by a metric of the form

$$ds^2 = c(x)^2 dt^2 - [dx - v(x)dt]^2 = [c(x)^2 - v(x)^2]dt^2 + 2v(x)dt \ dx - dx^2.$$  

In fact, any stationary two dimensional metric can be put in this form, with $c(x) = 1$, by a suitable choice of coordinates (see e.g. Appendix A in Ref. [1] for a proof
of this statement). If $c(x) = 1$ this corresponds to the PG metric, with $x \leftrightarrow r$ and $v(x) \leftrightarrow -1/\sqrt{r}$. A horizon exists in the spacetime \([14]\) if $|v(x)| > |c(x)|$ somewhere.

The metric \([14]\) would arise for example in a Newtonian setting of a fluid, with velocity $v(x)$ in a “laboratory frame”, with $c(x) = c$ a constant speed of sound. In that example, the coordinate $x$ would measure distance in the lab at fixed Newtonian time $t$, and the metric would describe the effective spacetime for waves in the fluid that propagate at speed $c$ relative to the local rest frame of the fluid. If the wave speed in the frame of the medium depends on some ambient local conditions then $c(x)$ will depend on position.

### Moving texture

In some models the medium may be at rest in the lab, but the local conditions that determine the wave speed may depend on both time and space in a “texture” that moves. (If the motion is uniform then in the frame of the texture this is equivalent to the previous case.) An example of a line element of this sort is $[c(y - wt)]^2 dt^2 - dy^2$. Here again $y$ measures proper distance in the lab at Newtonian time $t$, and the texture moves in the $y$ direction with constant speed $w$. The line element may not look stationary, but it has a symmetry under $t \rightarrow t + \Delta t$ combined with $y \rightarrow y + w\Delta t$.

### Black hole – white hole pair

An example that often arises has $v(x) < -c(x) < 0$ in a finite interval $(x_-, x_+)$. Then $x_+$ is a black hole horizon, analogous to the one previously discussed for the PG spacetime, and $x_-$ is a white hole horizon: no waves can escape from the region $x < x_+$ into the region $x > x_+$, and no waves can enter the region $x > x_-$ from the region $x < x_-$. The region between the horizons is of finite size and nonsingular.

Fig. [5] is a spacetime diagram of this scenario. Black hole horizon on the right and white hole horizon on the left. The vertical arrows depict the Killing vector, which is spacelike in the ergoregion between the horizons and timelike outside.

### 1.4 Symmetries, Killing vectors, and conserved quantities

Each symmetry of the background spacetime and fields leads to a corresponding conservation law. The most transparent situation is when the metric and any other background fields are simply independent of some coordinate. This holds for example with the Schwarzschild metric \([4]\), which is independent of both $t$ and $\phi$. Of course the spherical symmetry goes beyond just $\phi$ translations, but the other rotational symmetries are not manifest in this particular form of the line element. They could be made manifest by a change of coordinates however, but not all at once. To be able to talk about symmetries in a way that is independent of whether or not they are manifest it is useful to introduce the notion of a Killing vector field. The flow
of the spacetime along the integral curves of a Killing vector is a symmetry of the spacetime.

Suppose translation by some particular coordinate $x^{\hat{\alpha}}$ (\(\hat{\alpha}\) indicates one particular value of the index \(\alpha\)) is a manifest symmetry. The metric components satisfy $g_{\mu\nu,\hat{\alpha}} = 0$, where the comma notation denotes partial derivative with respect to $x^{\hat{\alpha}}$. The corresponding Killing vector, written in these coordinates, is $\chi^\mu = \delta^\mu_{\hat{\alpha}}$, i.e. the vector with all components zero except the $\hat{\alpha}$ component which is 1. Then the symmetry is expressed by the equation $g_{\mu\nu,\hat{\alpha}}\chi^{\hat{\alpha}} = 0$. This holds only in special coordinate systems adapted to the Killing vector. It is not a tensor equation, since the partial derivative of the metric is not a tensor.

It may be helpful to understand that this condition is equivalent to the covariant, tensor equation for a Killing vector,

$$\chi^{\alpha;\beta} + \chi^{\beta;\alpha} = 0,$$

where the semicolon denotes the covariant derivative. This is called Killing’s equation. One way to see the equivalence is to use the fact that in a local inertial coordinate system at a point $p$, the covariant derivative reduces to the partial derivative, and the partials of the metric are zero. Thus Killing’s equation at the point $p$ becomes $\chi^{\alpha;\beta}\eta_{\alpha\alpha} + \chi^{\beta;\alpha}\eta_{\alpha\beta} = 0$, where $\eta_{\alpha\tau}$ is the Minkowski metric. This implies that the infinitesimal flow $x^\sigma \to x^\sigma + \epsilon \chi^\sigma(x)$ generated by $\chi^\alpha$ is, to lowest order, a translation plus a Lorentz transformation, i.e. a symmetry of the metric.$^2$

For a more computational proof, note that since Killing’s equation is a tensor equation it holds in all coordinate systems if it holds in one. In a coordinate system for which $\chi^\mu = \delta^\mu_{\hat{\alpha}}$ we have $\chi_{\alpha\beta} = g_{\alpha\mu}\chi^{\mu\beta}$, $g_{\alpha\mu}\Gamma^\mu_{\beta\sigma}\chi^\sigma = \frac{1}{2}(g_{\alpha\beta,\sigma} + g_{\alpha\sigma,\beta} - g_{\beta\sigma,\alpha})\chi^\sigma$. If $\chi^\mu$ is a Killing vector the first term vanishes in this adapted coordinate system, and the remaining expression is antisymmetric in

\[\text{Fig. 5} \quad \text{Black hole horizon on the right and white hole horizon on the left. The vertical arrows depict the Killing vector, which is spacelike in the ergoregion between the horizons and timelike outside.}\]
A simple example is the Euclidean plane with line element \( ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2 \) in Cartesian and polar coordinates respectively. The rotation Killing vector about the origin in polar coordinates is just \( \partial_\phi \), with components \( \delta^\alpha_\phi \), as the metric components are independent of \( \phi \). The same Killing vector in Cartesian coordinates is \( x \partial_y - y \partial_x \). This satisfies Killing’s equation since \( \chi_{x,x} = 0 = \chi_{y,y} \), and \( \chi_{x,y} + \chi_{y,x} = -1 + 1 = 0 \).

1.4.1 Ergoregions

It is of paramount importance in black hole physics that a Killing field may be timelike in some regions and spacelike in other regions of a spacetime. For example in the Schwarzschild spacetime, say in PG coordinates \( 6 \), or the 1+1 dimensional generalization \( 14 \) the Killing vector \( \partial_t \) is timelike outside the horizon, but it is lightlike on the horizon and spacelike inside. For the black hole-white hole pair discussed above, it is the region between the black and white hole horizons (see Fig. 5). This is evident because the coefficient of \( dt^2 \) in the line element becomes negative.

A region where an otherwise timelike Killing vector becomes spacelike is called an ergoregion. (The reason for the name will become clear below.) The boundary of this region is called the ergosurface, and it is a surface of infinite redshift, since the norm of the time translation Killing vector vanishes there. An ergoregion need not lie behind a horizon. For instance it occurs outside the horizon (as well as inside) of a spinning black hole. In analogue models, ergoregions can arise for example around a vortex \( 2 \) or in a moving soliton in superfluid \( ^3 \)He-A \( 3 \). For the Schwarzschild black hole, and the 1+1 dimensional generalization \( 14 \), however, the ergoregion always corresponds to the region inside the horizon.

1.4.2 Conserved quantities

Particle trajectories (both timelike and lightlike) can be determined by the variational principle \( \delta \int L d\lambda = 0 \) with Lagrangian \( L = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\alpha \dot{x}^\nu \). Here \( \lambda \) is a path parameter and the dot denotes \( d/d\lambda \). The Euler-Lagrange equation is the geodesic equation for motion in the metric \( g_{\mu\nu} \) with affine parameter \( \lambda \). If the metric is independent of \( x^\alpha \) then the corresponding conjugate momentum \( p_\alpha = \partial L / \partial \dot{x}^\alpha = g_{\mu\beta} \dot{x}^\mu \) is a constant of motion. Note that this momentum can also be expressed as the inner product of the 4-velocity \( u^\nu = \dot{x}^\nu \) with the Killing field, \( u \cdot \chi = g_{\mu\nu} \dot{x}^\mu \chi^\nu = g_{\mu\alpha} \dot{x}^\mu \).

\( \alpha \) and \( \beta \), so adding \( \chi_{\beta,\alpha} \) yields zero. Conversely, if Killing’s equation holds, the entire expression is antisymmetric in \( \alpha \) and \( \beta \), so the first term must vanish.
Killing energy and ergoregions

The conserved momentum conjugate to a particular timelike Killing field is called Killing energy. For a particle with rest mass $m$, the physical 4-momentum would be $p = mu$, so the Killing energy as defined above is actually the Killing energy per unit rest mass. For a massless particle, the physical 4-momentum is proportional to the lightlike 4-velocity, scaled so that the time component in a given frame is the energy in that frame. In both cases, the true Killing energy is the inner product of the 4-momentum and the Killing vector,

$$E_{\text{Killing}} = p \cdot \chi. \quad (16)$$

The 4-momentum of a massive particle is timelike, while that of a massless particle is lightlike. In both cases, for a physical state (i.e. an allowable excitation of the vacuum), stability of the local vacuum implies that the energy of the particle is positive as measured locally in any rest frame. This is equivalent to the statement that $p$ is a future pointing 4-vector.

The importance of ergoregions stems from the fact that negative Killing energy physical states exist there. This happens because a future pointing 4-momentum can of course have a negative inner product with a spacelike vector (see Fig. 6). In an ergoregion, the Killing energy is what would normally be called a linear momentum component, and there is of course no lower limit on the linear momentum of a physical state.

Penrose [4, 5] realized that the existence of an ergoregion outside a spinning black hole implies that energy can be extracted from the black hole by a classical process, at the cost of lowering the angular momentum. This is the Penrose process, whose existence led to the discovery of black hole thermodynamics. For a non-spinning black hole the ergoregion lies inside the horizon, so no classical process can exploit it to extract energy, but the Hawking effect is a quantum process by which energy is extracted.

What do the negative Killing energy states “look like”? A particle with negative Killing energy cannot escape from the ergoregion, nor can it have fallen freely into the ergoregion, because Killing energy is conserved along a geodesic and it must have positive Killing energy if outside the ergoregion. For example, in the 1+1 black
hole, or in the radial direction of the Schwarzschild solution, a massless particle with negative Killing energy inside the horizon must be “outgoing” as seen by a local observer (see Fig. 7).

1.5 Killing horizons and surface gravity

An event horizon can be defined purely in terms of the causal structure of a spacetime, and is meaningful even when the spacetime is not stationary, i.e. has no time translation symmetry. A Killing horizon on the other hand is a lightlike hypersurface (surface of one less dimension than the whole spacetime) generated by the flow of a Killing vector. This is sometimes called the horizon generating Killing vector.

The Schwarzschild event horizon is a Killing horizon with respect to the Killing vector $\partial_t$, as is the horizon of the 1+1 black hole. A distinction arises in the case of a stationary black hole with spin. Then the Killing vector $\partial_t$ that is a time translation at spatial infinity becomes lightlike at the boundary of the ergoregion, which lies outside the event horizon. However that boundary is timelike, so the ergosurface is not a Killing horizon. The event horizon of a spinning black hole is nevertheless a Killing horizon, but for a Killing vector $\partial_t + \Omega_H \partial_\phi$ that is a linear combination of the time translation and rotation Killing vectors, $\Omega_H$ being the angular velocity of the horizon. In the effective spacetime of a moving texture in superfluid $^3$He-A, the horizon generating Killing vector has the similar form $\partial_t + w \partial_x$, where $\partial_t$ and $\partial_x$ are time and space translation Killing vectors, and the constant $w$ can be thought of as the transverse velocity of the horizon [3].
Lines of constant $\eta$ are radial from the origin, and $\eta$ measures the hyperbolic opening angle of the shaded wedge.

**Rindler (acceleration) horizon**

A simple yet canonical example of a Killing horizon is the Rindler horizon in Minkowski spacetime. The relevant Killing symmetry here is Lorentz boosts in a certain direction. Geometrically, these are just hyperbolic rotations. For example, using the Minkowski coordinates of (1) a boost Killing vector is

$$\chi_B = x \partial_t + t \partial_x.$$  \hspace{1cm} (17)

This has covariant components $(\chi_B)_\alpha = \eta_{\alpha\beta} \chi_B^\beta = (x, -t)$ and so obviously satisfies Killing’s equation (15). It can also be made manifest by changing from Minkowski to polar coordinates:

$$dt^2 - dx^2 = \ell^2 d\eta^2 - d\ell^2.$$  \hspace{1cm} (18)

Then the boost symmetry is just rotation of the hyperbolic angle $\eta$, i.e.

$$\chi_B = \partial_\eta.$$  \hspace{1cm} (19)

The flow lines of the Killing field are hyperbolas (see Fig. 8). Note that the polar coordinate system covers only one “Rindler wedge”, e.g. $x > |t|$ of the Minkowski spacetime. The full Killing horizon is the set $|x| = |t|$.

### 1.5.1 Surface gravity

Associated to a Killing horizon is a quantity $\kappa$ called the *surface gravity*. There are many ways to define, calculate, and think of the surface gravity. It was already introduced in Sec. 1.2.4 as the relative rate of stretching of outgoing wavelengths near
the horizon. I will mention here several other definitions, which are given directly in terms of the geometry of the horizon.

Geometrically, the simplest definition of surface gravity may be via

\[ [\chi_{\alpha\beta}]^{\alpha\beta} |_{H} = -2\kappa^2, \tag{20} \]

horizon the square bracket on indices denotes antisymmetrization, and the subscript \( H \) indicates that the quantity is evaluated on the horizon. That is, \( \kappa \) is the magnitude of the infinitesimal Lorentz transformation generator. However the meaning of this is probably not very intuitive.

The conceptually simplest definition might be the rate at which the norm of the Killing vector vanishes as the horizon is approached from outside. That is,

\[ \kappa = \|\chi\| \big|_{\alpha \vert H}. \tag{21} \]

the horizon limit of the norm of the gradient of the norm of \( \chi \). Notice that if the Killing vector is rescaled by a constant multiple \( \chi \rightarrow \alpha \chi \), then it remains a Killing vector, and the surface gravity for this new Killing vector is \( \alpha \kappa \). This illustrates the important point that the intrinsic structure of a Killing horizon alone does not suffice to define the surface gravity. Rather, a particular normalization of the Killing vector is required. The symmetry implies that \( \kappa \) is constant along a particular null generator of the horizon, but in general it need not be the same on all generators. For a discussion of conditions under which the surface gravity can be proved to be constant see [6].

The surface gravity (21) has the interesting property that it is conformally invariant. That is, it is unchanged by a conformal rescaling of the metric \( g_{ab} \rightarrow \Omega^2 g_{ab} \), provided the conformal factor \( \Omega \) is regular at the horizon [7]. This follows simply because \( |\chi| \) is rescaled by \( \Omega \), while the norm of its gradient is rescaled by \( \Omega^{-1} \), and the contribution from \( d\Omega \) vanishes since it is multiplied by \( |\chi|_{H} \) which vanishes.

For the metric (14) and the Killing vector \( \chi = \partial_t \) we have \( |\chi| = \sqrt{c^2 - v^2} \), which depends on \( x \) and not \( t \). Thus \( \kappa = (-g^{xx} \partial_x |\chi| \partial_x |\chi|)^{1/2} \), and the minus sign arises because the gradient is spacelike outside the horizon. At a horizon where \( v = c \) this evaluates to \( |\partial_x (v - c)|_{H} \), while at a horizon where \( v = -c \) it would instead be \( |\partial_x (v + c)|_{H} \).

In case \( c = constant \), the surface gravity is thus just the gradient of the flow speed at the horizon. A covariant and more general version of this can be formulated. Any observer falling freely across a horizon can define the velocity of the static frame relative to himself, and can evaluate the spatial gradient of this velocity in his frame. If he has unit Killing energy \( (u \cdot \chi = 1) \) then it can be shown that this gradient, evaluated at the horizon, agrees with the surface gravity [8]. Another interesting observation is that this velocity gradient has a sort of “cosmological” interpretation as the local fractional rate of expansion (“Hubble constant”) of the distances separating a family of freely falling observers stretched along the direction of the Killing frame velocity [8]. At the horizon, for unit energy observers, this expansion rate is the same as the surface gravity.
Computationally, a somewhat simpler definition of surface gravity is via

\[ \partial_\alpha (\chi^2) = -2\kappa \chi_\alpha |_H. \] (22)

This is at least well-defined: since \( \chi^2 \) vanishes everywhere on the Killing horizon, its gradient has zero contraction with all vectors tangent to the horizon. The same is true for \( \chi_a = g_{\alpha\beta} \chi^\beta \), so these two co-vectors must be parallel. If using a coordinate component of this equation to evaluate \( \kappa \), it is important that the coordinate system be regular at the horizon. For the metric (14), we may just evaluate the \( x \) component of this equation:

\[ \partial_x (c^2 - v^2) = -2\kappa \chi_x = -2\kappa g_{xt} = -2\kappa v, \]

which on the horizon \( v = c \) yields \( \kappa = \left| \partial_x (v - c) \right|_H \) as before. (Note that this definition does not come with an absolute value. At a horizon \( v = -c \) it yields \( \kappa = \left| \partial_x (v + c) \right|_H \).)

**Surface gravity of the Rindler horizon**

The surface gravity of the Rindler horizon can be computed for example using the polar coordinates to evaluate (21). Then the norm of the Killing vector is just \( \ell \), so \( \partial_\alpha |_{\chi_B} = \delta_\alpha^a \), which has norm 1. Thus the boost Killing vector has unit surface gravity. Alternatively, we may use the \( x \) component of (22):

\[ \partial_x \chi_B^2 = x^2 - t^2 = 2x, \]

and

\[ -2\kappa (\chi_B)_x = 2\kappa, \]

so \( \kappa = (x/t)_H = \pm 1 \). On the future horizon \( x = t \) and this is positive, while on the past horizon it is negative. Usually one is only interested in the absolute value.

Finally, it is sometimes of interest to use the proper time along a particular hyperbola rather than the hyperbolic angle as the coordinate. On the hyperbola located at \( \ell = \ell_0 \) the proper time is \( d\tau = \ell_0 d\eta \). The Minkowski line element can be written in terms of the time coordinate \( \tau = \ell_0 \eta \) as \( ds^2 = \ell_0^2 d\tau^2 - dt^2 \). The scaling of the Killing field \( \partial_\tau = (1/\ell_0) \partial_\eta \) that generates proper time flow on this particular hyperbola has surface gravity \( \kappa = 1/\ell_0 \). This is also equal to the acceleration of the hyperbolic worldline. The relation between the surface gravity and acceleration can be shown quite generally using coordinate free methods, but here let’s just show it by direct computation using Cartesian coordinates. The 4-velocity of the hyperbola is the unit vector \( u = \ell_0^{-1} (x, t, 0, 0) \), and the acceleration of this worldline is \( (u \cdot \nabla) u = \ell_0^{-2} (x \partial_t + t \partial_x) (x, t, 0, 0) = \ell_0^{-2} (t, x, 0, 0) \). The norm of the spacelike vector \( (t, x, 0, 0) \) is \( \ell_0 \), so the norm of the acceleration is \( 1/\ell_0 \).

**2 Thermality of the vacuum**

The subject of the rest of these notes is the Hawking effect, i.e. the emission of thermal radiation from a black hole. The root of the Hawking effect is the thermality of the vacuum in flat spacetime. This thermality is known as the Unruh, or Fulling-Davies-Unruh, effect [9]. In its narrowest form, this is the fact that a probe with uniform proper acceleration \( a \), moving through the vacuum of a quantum field in flat spacetime, is thermally excited at the Unruh temperature.
\[ T_U = \hbar a / 2\pi c. \] (23)

(I’ve restored \( c \) here to show where it enters, but will immediately revert to units with \( c = 1 \).) When described this way, however, too much attention is focused on the probe and its acceleration.

Underlying the response of the probe is a rather amazing general fact: when restricted to a Rindler wedge, the vacuum of a relativistic quantum field is a canonical thermal state with density matrix
\[ \rho_R \propto \exp(-2\pi H_\eta / \hbar), \] (24)

where \( H_\eta \) is the “boost Hamiltonian” or “Rindler hamiltonian” generating shifts of the hyperbolic angle coordinate \( \eta \) defined in (18). In terms of Minkowski coordinates \((t, x, y, z)\), \( H_\eta \) is given on a \( t = 0 \) surface of the Rindler wedge by
\[ H_\eta = \int_{\Sigma_R} T_{ab} \chi_a^b d\Sigma^b = \int x T_{tt} dxdydz, \] (25)

where \( T_{ab} \) is the energy-momentum tensor. The “temperature” of the thermal state (24) is
\[ T_R = \hbar / 2\pi. \] (26)

Like a rotation angle, the hyperbolic angle is dimensionless, so the boost generator and temperature have dimensions of angular momentum.

Note that the thermal nature of the vacuum in the wedge does not refer to any particular acceleration, and it characterizes the state even on a single time slice. Nevertheless it does directly predict the Unruh effect. A localized probe that moves along a particular hyperbolic trajectory at proper distance \( \ell_0 \) from the vertex of the wedge has proper time interval \( d\tau = \ell_0 d\eta \) (cf. 18). When scaled to generate translations of this proper time the field Hamiltonian is thus \( H_\tau = \ell_0^{-1} H_\eta \), and the corresponding temperature is \( T_0 = \ell_0^{-1} h / 2\pi \). The proper acceleration of that hyperbola is \( \ell_0^{-1} \), so the probe will be excited at the Unruh temperature (23).

The thermality of the vacuum in one wedge is related to entanglement between the quantum states in the right and left wedges. It can be understood using a simple, but abstract and formal, argument that employs the path integral expression for the ground state. Because the result is so central to the subject, I think this argument deserves to be explained.

The vacuum \( \left| 0 \right> \) is the ground state of the field Hamiltonian \( H \), and can therefore be projected out of any state \( \left| \chi \right> \) as \( \left| 0 \right> \propto \lim_{t \to \pm \infty} e^{-itH} \left| \chi \right> \), as long as \( \left< 0 \left| \chi \right> \right> \neq 0 \). The operator \( e^{-itH} \) can be thought of as the time evolution operator for an imaginary time \(-it\), and its matrix elements can be represented by a path integral over fields \( \phi \) on Euclidean space. This yields a path integral representation for the vacuum wave functional,
\[ \Psi_0[\phi] \propto \lim_{t \to \pm \infty} \left< \phi \left| e^{-itH} \right| \chi \right> \propto \int_{\phi(t=0)=\phi} \int_{\phi(t=-\infty)=\chi} D\phi \, e^{-S/\hbar}, \] (27)
Euclidean Minkowski space with boundary at $t=0$. When the path integral is sliced by constant $t$ surfaces it presents the vacuum wave-functional. When sliced by constant angle surfaces, it presents matrix elements of the operator $\exp(-\pi H_\eta)$, where $H_\eta$ is the Lorentz boost generator.

where $S$ is the Euclidean action corresponding to the Hamiltonian $H$. The standard demonstration of this path integral expression for matrix elements of $e^{-tH}$ proceeds by slicing the Euclidean space into steps of constant Euclidean time, and exploits the time translation invariance of the Hamiltonian. If the original Hamiltonian is also Lorentz boost invariant, then the Euclidean action is also rotationally invariant. This extra symmetry leads to an alternate interpretation of the path integral.

Fixing a particular rotational symmetry, e.g. around the origin in the Euclidean $tx$ plane, we may choose to slice the Euclidean space into steps of constant angle around the corresponding vertex (see Fig. 9). This vertex divides the time slice $t=0$ into two halves, and the final field configuration $\phi$ restricts to some $\phi_L$ and $\phi_R$ on the left and right sides respectively. These configurations define Dirac “bras” $\langle \phi_L|$ and $\langle \phi_R|$ in the duals of the left and right side Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$. The full Hilbert space is the tensor product $\mathcal{H}_L \otimes \mathcal{H}_R$.

With this angular slicing, (and not worrying about boundary conditions at the vertex), we can think of the path integral as producing the matrix element of the operator $\exp(-\pi H_\eta)$ between $\phi_L$, regarded now as an initial state, and the final state $\phi_R$,

$$\Psi[\phi_L, \phi_R] \propto \langle \phi_R | e^{-\pi H_\eta} J | \phi_L \rangle.$$  \hspace{1cm} (28)

Here $H_\eta$ is the boost Hamiltonian, which is the generator of angle shifts, and $\pi$ is the rotation angle in the Euclidean plane. (The rotation angle is to the boost angle as the Euclidean time is to the Minkowski time.) The final state bra $\langle \phi_L|$ is replaced by a “corresponding” initial state ket $J | \phi_L \rangle$ that can be identified with a state in $\mathcal{H}_R$. Here $J = CTP$ is the operator of charge conjugation, time reversal, and reflection across the Rindler plane, which is a symmetry of all Lorentz invariant quantum field theories.$^3$

$^3$ For a configuration eigenstate of a real field, the ket $J | \phi_L \rangle$ can just be identified with the same function $\phi_L$, reflected by an operator $P^1$ across the Rindler plane. More generally, $J$ includes $CT$ to undo the conjugation of the $\langle \text{bra} \rangle \rightarrow | \text{ket} \rangle$ duality.
The vacuum wave-functional \((28)\) can also be represented as a vector in the Hilbert space \(\mathcal{H}_L \otimes \mathcal{H}_R\), by multiplying the amplitudes \((28)\) by the corresponding kets and integrating over the fields:

\[
|0\rangle \propto \int D\phi_L D\phi_R |\phi_L\rangle |\phi_R\rangle e^{-\pi H_R J} |\phi_L\rangle
\]

\[= \int D\phi_L |\phi_L\rangle e^{-\pi H_R J} |\phi_L\rangle\]

\[= \sum_n e^{-\pi E_n} |n\rangle_L |\bar{n}\rangle_R. \tag{31}\]

In the last line the state is expressed in terms of eigenstates \(|n\rangle\) of the boost Hamiltonian with boost energy \(E_n\) (with additional implicit quantum numbers). It is obtained via \(J|\phi_L\rangle = \sum_n |n\rangle \langle n| \phi_L\rangle = \sum_n \langle \phi_L|n\rangle |n\rangle\), using the anti-linearity of \(J\). Then the integral over \(\phi_L\) yields the identity operator, and the result follows since \(H_R\) commutes with \(J\). The state \(|\bar{n}\rangle\) stands for the “antiparticle state” \(J|n\rangle\).

This exhibits the precise sense in which the quantum field degrees of freedom in the left and right Rindler wedges are entangled in the vacuum state. This entanglement is the origin of the correlations between the Hawking quanta and their partners, and it produces the entanglement entropy for quantum fields outside a horizon. Tracing over the state in the left wedge we obtain the reduced density matrix for the state restricted to right wedge,

\[
\rho_R = \text{Tr}_L |0\rangle \langle 0| \propto \sum_n e^{-2\pi E_n} |n\rangle \langle n|. \tag{32}\]

This is the canonical thermal state \((24)\) mentioned above. The horizon entanglement entropy is the entropy of this thermal state. It diverges as the horizon area times the square of the momentum cutoff.

### 3 Hawking effect

The essence of the Hawking effect \([10]\) is that the correlated vacuum fluctuations described in the previous section exist near the horizon of a black hole, which is locally equivalent to a Rindler horizon. The crucial difference from flat space is that tidal effects of curved spacetime peel apart the correlated partners. The outside quanta sometimes escape to infinity and sometimes fall backwards into the black hole, while the inside ones fall deeper into the black hole. The escaping quanta have a thermal spectrum with respect to the analogue of the boost Hamiltonian, that is, with respect to the Hamiltonian for the horizon-generating symmetry. If the horizon generating Killing vector is normalized to have unit surface gravity, like the boost Killing vector, the temperature is again the Rindler temperature \(T_R = \hbar / 2\pi\).

---

4 Its matrix elements could also have been obtained directly using the wave functional \((28)\), via

\[
\int D\phi_L D\phi_R |\phi_L\rangle |\phi_R\rangle e^{-2\pi H_R J} |\phi_L\rangle \langle \phi_R| e^{-2\pi H_R J} |\phi_L\rangle = \langle \phi_L| e^{-2\pi H_R J} |\phi_L\rangle.
\]
(26). However, for a quantum that escapes from the black hole region, the natural
definition of energy is the generator of asymptotic time translations. For defining
this energy we normalize the time translation Killing vector at infinity. Then the
black hole horizon has a surface gravity $\kappa$, and the temperature is the Hawking
temperature,

$$T_H = \frac{\hbar \kappa}{2\pi}. \quad (33)$$

Note that the Unruh temperature (23) can be expressed in exactly the same way as
the Hawking temperature since, as explained in Sec. 1.5.1 when the boost Killing
field is normalized to unity on a given hyperbola the surface gravity of the Rindler
horizon is precisely the acceleration of that hyperbola.

For a rotating black hole, as explained in Sec. 1.5, the horizon generating Killing
vector is $\partial_t + \Omega_H \partial_\phi$. The eigenvalues of the Hamiltonian corresponding to this
Killing vector are $E - \Omega_H L$, where $E$ and $L$ are the energy “at infinity” and angular
momentum respectively. Thus the Boltzmann factor for the Hawking radiation is
$$e^{-\frac{(E - \Omega_H L)}{T_H}}.$$ The angular velocity $\Omega_H$ plays the role of a chemical potential for
the angular momentum.

Missing from this explanation of the Hawking effect is the specification of the
incoming state. In principle, there are two places where the state can “come in”
from: spatial infinity, and the horizon. The state coming from the horizon is deter-
mined to be the local vacuum by a regularity condition, since anything other than
the vacuum would be singular as a result of infinite blueshift when followed back-
wards in time toward the horizon. This is what accounts for the universality of the
thermal emission. However the state coming in from infinity has freedom. If it is
the vacuum, the state is called the “Unruh state”, while if it is a thermal state, as
appropriate for thermal equilibrium of a black hole with its surroundings, it is the
“Hartle-Hawking” state. In the neighborhood of the intersection of past and future
horizons, the Hartle-Hawking state is close to the local Minkowski vacuum.

For black holes in general relativity, the above description of the Hawking ef-
fect is, in a sense, the complete story. For analogue models, however, one wants a
derivation that does not assume Lorentz invariance, and that shows the way to the
modifications brought about by the lack thereof. Also, it is important to be able to
allow for experimental conditions that determine different incoming states. More-
over, in the analogue case the horizon state need not be the vacuum, since in the
presence of Lorentz violating dispersion a different state can exist without entail-
ing anything singular on the horizon. Thus we now take a very different viewpoint,
analyzing the vacuum “mode by mode”. It is this approach that Hawking originally
followed when he discovered black hole radiation. It should be emphasized at the
outset however that, unlike the previous treatment, this approach will apply only to
free field theory, with uncoupled modes satisfying a linear field equation.

---

5 The sign of the $L$ term is opposite to that of the $E$ term because $\partial_\phi$ is spacelike while $\partial_t$ is timelike.


3.1 Mode solutions

My aim here is to convey the essence of the Hawking effect, using a language that is easily adapted to analogue models in which dispersive effects play a role. Hence I will discuss only a system with one spatial dimension, and will highlight the role of the dispersion relation, using WKB methods.

Consider a scalar field $\phi$ that satisfies the wave equation $\partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\partial_{\beta}\phi) = 0$. For the metric (14) we have $\sqrt{-g} = c$ and $g^{tt} = 1/c^2$, $g^{tx} = v/c^2$, $g^{xx} = (v^2 - c^2)/c^2$. Since the metric is independent of $t$ we can find solutions with definite Killing frequency, $\phi = e^{-i\omega t}u(x)$. Because of the redshift effect an outgoing solution has very rapid spatial oscillations of $u(x)$ near the horizon. We can thus find an approximate solution near the horizon by neglecting all terms in which there is not at least one derivative of $u(x)$. This yields the equation

$$\partial_x[(v^2/c - c)\partial_x u] = (2i\omega/v)c \partial_x u.$$  

(34)

Near a horizon $x = x_H$ where $v = -c$ we have the expansions $v/c = -1 + O(x - x_H)$ and $v^2/c - c = -2\kappa(x - x_H) + O((x - x_H)^2)$. Thus at the lowest order in $x - x_H$ the near horizon approximation of (34) becomes

$$\partial_x[(x - x_H)\partial_x u] = (i\omega/\kappa)\partial_x u,$$  

(35)

whose solutions have the form

$$u \sim (x - x_H)^{i\omega/\kappa} = e^{i(\omega/\kappa)\ln(x - x_H)}.$$  

(36)

The logarithmic divergence in the phase justifies the dominance of spatial derivatives of $\phi$ near the horizon. Note that this mode has the same form as (13), which we inferred in Sec. 1.2.4 using the equation of outgoing lightrays to propagate the phase of the wave in the near horizon region.

Now let’s see how to arrive at the same approximate solution using the dispersion relation with the fluid picture. First, a mode solution in a homogeneous fluid has the form $\phi \sim e^{-i\omega t}e^{ikx}$, where $x$ is the position in the fluid frame and the dispersion relation is $\omega^2 = F(k)^2$ for some function $F(k)$. For instance, for a nondispersive wave with speed $c$ we have simply $F(k) = ck$. If the fluid is flowing with speed $v$ relative to the “lab” then $x = x_f + vt$, where $x_f$ is at rest with respect to the fluid. In terms of $x_f$ the mode is $e^{-i(\omega - vk)t}e^{ikx_f}$, which allows us to read off the frequency as measured in the fluid frame, $\omega_f = \omega - vk$. The dispersion relation holds in the fluid frame, so we have $\omega - vk = \pm F(k)$.

If the flow velocity $v(x)$ is not uniform, $\omega_f = \omega - v(x)k$ is locally accurate provided the change of $v(x)$ over a wavelength is small compared to $v(x)$ itself. The local dispersion relation then becomes

$$\omega - v(x)k = \pm F(k),$$  

(37)
which for a fixed Killing frequency yields a position-dependent wavevector, $k_\omega(x)$. It should be emphasized that the Killing frequency $\omega$ is a well-defined global constant for a solution, even if the Killing vector is not everywhere timelike.

An approximate, WKB mode solution, taking into account only the phase factor, is thus

$$u(x) \sim \exp \left( i \int \omega(x') dx' \right).$$

(38)

Finally, if the local wave velocity $c(x)$ also depends on position in the fluid (but is time independent in the lab frame), then the function $F(k,x)$ also depends on position. If $c(x)$ changes slowly over a wavelength, then the mode of the same form is again a good approximation. For the case of relativistic dispersion $F(k,x) = c(x)k$ we obtain $k_\omega = \omega/(c + v)$ for the outgoing mode. Expanding around the horizon this yields $k_\omega(x) = (\omega/\kappa)(x-x_H)^{-1}$, and so the mode takes the same form as (36) derived above.

### 3.2 Positive norm modes and the local vacuum

When the field is quantized, the Hilbert space is constructed as a Fock space built from single particle states corresponding to (complex) solutions to the field equation with positive conserved “norm”. The norm can be identified using a conserved inner product, the existence of which follows from global phase invariance of the action. Here I will not attempt to explain the details of this construction, which can be found in many expositions,[6] but instead will try to provide a simple argument that captures the essence of the story. In this section the relativistic case will be explained, and in the last section I will make some brief comments about what happens when there is Lorentz violating dispersion for short wavelengths. The quantum field is taken to be a hermitian scalar, which arises from quantization of a real scalar field.

Positive norm modes that are localized can be recognized as those that have positive frequency in the fluid frame. In the relativistic case, this amounts to positive frequency in any freely falling frame. The time derivative in the fluid frame is $(\partial_t)_f = \partial_t + v \partial_x$. For a mode of the form (36) near the horizon, this is dominated by the second term, and $v \approx -c$, hence for such modes positive frequency with respect to $t$ in the fluid frame is the same as positive frequency with respect to $x$. (There are two minus signs that cancel: $v = -c < 0$ at the horizon, but the conventional definition of “positive frequency” is $e^{-i\omega t}$ with $\omega > 0$ for temporal frequency, and $e^{ikx}$ with $k > 0$ for spatial frequency.)

The mode (36) with logarithmic phase divergence at the horizon can be analytically continued across the horizon to make either a positive or a negative frequency solution. To see how this works, let’s first simplify the notation a bit and set $x_H = 0$, so the horizon lies at $x = 0$. Now a positive $x$-frequency function has the form $\int_0^\infty df(k)e^{ikx}$, which is analytic in the upper-half complex $x$-plane since

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[6] For a pedagogical introduction see, e.g. [10], or references therein.
addition of a positive imaginary part to \( x \) leaves the integral convergent. Similarly, a negative \( x \)-frequency function is analytic in the lower half \( x \)-plane. The argument of the logarithm is \( x = e^{i\theta} |x| \), so \( \ln x = i\theta + \ln |x| \). Continuing to \(-x\) in the upper or lower half plane thus gives \( (\ln x)_\pm = \pm i\pi + \ln |x| \) respectively, hence
\[
e^{i(\omega/\kappa)\ln x} \rightarrow e^{\mp \pi\omega/\kappa} e^{i(\omega/\kappa)\ln |x|}.
\] (39)

We can thus write down positive and negative frequency continuations,
\[
q_+ = u + e^{-\pi\omega/\kappa} \bar{u}
\]
\[
q_- = e^{-\pi\omega/\kappa} u + \bar{u},
\]
(40)
(41)

where \( u = \theta(x)e^{i(\omega/\kappa)\ln x} \) and \( \bar{u} = \theta(-x)e^{i(\omega/\kappa)\ln |x|} \), and \( N \) is a normalization factor. (The negative frequency continuation \( q_- \) has been multiplied by \( e^{-\pi\omega/\kappa} \) to better reflect the symmetry and thus simplify the following discussion.)

We can now express \( u \) as a superposition of positive and negative norm parts,
\[
u = u_+ + u_- \propto q_+ - e^{-\pi\omega/\kappa} q_-.
\] (42)

From the symmetry of the construction, the norms of \( q_+ \) and \( q_- \) are equal up to a sign, hence the ratio of the squared norms (denoted \( \langle \cdot, \cdot \rangle \)) of the negative and positive norm parts of \( u \) is
\[
\frac{\langle u_-, u_- \rangle}{\langle u_+, u_+ \rangle} = e^{-2\pi\omega/\kappa} = e^{-E/T_H}.
\] (43)

In the last equality I’ve defined the energy \( E = \hbar \omega \), and \( T_H = \hbar \kappa / 2\pi \) is the Hawking temperature. This “thermal ratio” is the signature of the Hawking effect, as indicated via the mode \( u \) outside the horizon. Note that this ratio is a property of the classical solution to the wave equation, and is determined by the ratio of the frequency to the surface gravity. Planck’s constant enters only when we express the result in terms of the energy quantum \( \hbar \omega \). Note also that if the Killing vector is rescaled, then the Killing frequency \( \omega \) and surface gravity \( \kappa \) are rescaled in the same way, so that the ratio \( \omega / \kappa \) is unchanged.

The presence of the negative frequency part \( u_- \) in \( u \) (42) is unexpected from the WKB viewpoint. It corresponds to a negative wavevector, whereas when we solved the local dispersion relation we found \( k_\omega(x) = (\omega/\kappa)(x - x_H)^{-1} \). Since the support of \( u \) lies outside the horizon at \( x > x_H \), it might seem that this dispersion relation implies that \( k_\omega(x) \) is positive, and thus that the frequency is purely positive. However this is a misconception, because a function with support on a half line cannot have purely positive frequency. The concept of a definite local wavevector must therefore have broken down. Indeed, if we examine the change of \( k \) over a wavelength we find \( (dk/dx)/k \sim (\kappa/\omega)k \), which is not much smaller than \( k \) unless \( \omega \gg \kappa \). This resolves the puzzle.\footnote{However, it raises another one: why did the WKB type mode \( \sim \exp(i \int^x k_\omega(x')dx') \) agree so well with the mode function \( \text{(36)} \)? The answer is that \( \text{(35)} \) is a first order equation, not a second order one, once an overall \( \partial_x \) derivative is peeled off.}

\( \partial_x \)
The local outgoing vacuum

The local outgoing vacuum contains no outgoing excitations. More precisely, it is the ground state in the Fock space of outgoing positive norm modes. The outgoing modes we have been discussing are not themselves localized, but one can form localized wavepackets from superpositions of them with different frequencies. Hence we may characterize the local outgoing vacuum by the requirement that it be annihilated by the annihilation operators \( a(q^+) \) and \( a(q^-) \) for all positive norm modes.

These operators can be expressed in terms of the annihilation and creation operators corresponding to \( u \) and \( \tilde{u} \) using (i) linearity, (ii) equations (40) and (41), and (iii) the relation \( a(f) = -a^\dagger(f^*) \) which should be used if \( f \) has negative norm. For example, \( a(q_+) = a(u) + e^{-\pi\omega/\kappa}a(\tilde{u}) = a(u) - e^{-\pi\omega/\kappa}a(\tilde{u}^*) \). The vacuum conditions

\[
\begin{align*}
a(q_+)|0\rangle &= 0 \quad (44) \\
a(q^-)|0\rangle &= 0 \quad (45)
\end{align*}
\]

thus amount to

\[
\begin{align*}
a(u)|0\rangle &= e^{-\pi\omega/\kappa}a^\dagger(\tilde{u}^*)|0\rangle \quad (46) \\
a(\tilde{u}^*)|0\rangle &= e^{-\pi\omega/\kappa}a^\dagger(u)|0\rangle. \quad (47)
\end{align*}
\]

If we normalize the mode \( u \), then the commutation relation \([a(u), a^\dagger(u)] = 1\) holds and implies that, in effect, \( a(u) = \partial / \partial a^\dagger(u) \), and similarly for \( \tilde{u} \). Thus (46) can be solved to find the vacuum state for these particular modes of frequency \( \omega \),

\[
|0\rangle \propto \exp\left(e^{-\pi\omega/\kappa}a^\dagger(u)a^\dagger(\tilde{u}^*)\right)|0_L0_R\rangle, \quad (49)
\]

where \(|0_L0_R\rangle\) is the state with no \( u \) or \( \tilde{u}^* \) excitations on either side of the horizon, \( a(u)|0_L0_R\rangle = 0 = a(\tilde{u}^*)|0_L0_R\rangle \). In flat space \(|0_L0_R\rangle\) is called the (outgoing factor of the) “Rindler vacuum”, while in a black hole spacetime it is the “Boulware vacuum”.

Expanding the exponential in (49) we obtain another expression for the vacuum

\[
|0\rangle \propto \sum e^{-n\pi\omega/\kappa}|n_Ln_R\rangle, \quad (50)
\]

where \( n_L \) and \( n_R \) are the number of particles in the given mode. Taking the product over all frequencies, we then arrive at an expression for the local vacuum of a free field theory near the horizon that has the same form as the general thermal result

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8 What I am calling the annihilation operator here is related to the field operator \( \phi \) by \( a(f) = \langle f, \phi \rangle \), where \( f \) is a positive norm mode. If \( f \) is not normalized this is actually \( \langle f, f \rangle \) times a true annihilation operator.

9 The minus sign comes from the conjugation of a factor of \( i \) in the definition of the norm, which I will not explain in detail here.

10 Here I’ve use the relation \( (a^\dagger)^n|0\rangle = \sqrt{n!}|n\rangle \).
obtained earlier using the path integral. The results look different only because here the energies of free field states with $n$ quanta are given by $n\hbar\omega$, and because here the Killing vector is not normalized to unit surface gravity.

3.3 Stimulated emission of Hawking radiation

So far I spoke only of the Hawking effect arising from the local vacuum at the horizon. For a real black hole this is probably the only relevant condition, but for analoguemodels it is possible, and even unavoidable because of thermal fluctuations, noise, or coherent excitations, that the in-state is not the vacuum. Then what arises is stimulated emission of Hawking radiation [11], just as the decay of an excited atomic state can be stimulated by the presence of a photon.

To quantify this process, instead of imposing the vacuum condition (44) we can assume the quantum field is in an excited state,

$$a^\dagger(\hat{q}_+)a(\hat{q}_+)|\Psi\rangle = n_+|\Psi\rangle$$  
$$a^\dagger(\hat{q}_-)a(\hat{q}_-)|\Psi\rangle = n_-|\Psi\rangle,$$

where the $\hat{q}_\pm$ are normalized versions of (40,41). A simple way to diagnose the emission is via the expectation value of the occupation number of the normalized mode $u$. Using (42) and (43) we find

$$\langle \Psi | a^\dagger(u)a(u) |\Psi\rangle = \langle \Psi | a^\dagger(u_+)a(u_+) + a(u_+)a^\dagger(u_-) |\Psi\rangle$$  
$$= \langle u_+, u_+ \rangle [n_+ + e^{-2\pi\omega/\kappa}(n_- + 1)]$$  
$$= n_+ + n_- + 1$$

where $\langle u_+, u_+ \rangle = 1/(1 - e^{-2\pi\omega/\kappa})$. Thus both $n_+$ and $n_-$ stimulate Hawking emission, while only $n_+$ shows up in the non-thermal spectrum. Had the state been a coherent state, the occupation numbers would be replaced by squared amplitudes. Something analogueous to this occurs in the surface wave white hole radiation experiments [12], although those waves do not have a relativistic dispersion relation. In the case of a Bose condensate, the appropriate in-state would presumably be more like a thermal state [13].

4 The trans-Planckian question

The sonic black hole was originally conceived by Unruh [14] in part to address what has come to be called the trans-Planckian question: Can the derivation of Hawking radiation be considered reliable given that it refers to arbitrarily high frequency field modes? If one assumes local Lorentz invariance at arbitrarily large boosts, then any
high frequency mode can be Doppler shifted to low frequency, so one might argue that there is nothing to be concerned about. Sometimes the point is raised that there is an arbitrarily large invariant center of mass energy in the collision between ingoing and outgoing modes in the vacuum outside a horizon. However, this is true even in flat spacetime. We never see the effects of such collisions because they concern the “internal structure” of the ground state. We could presumably see this quantum gravity structure of the vacuum only with probes that have Planckian invariant energy. Hence it is not clear to me that there is anything to worry about in the derivation, provided one is willing to assume local Lorentz symmetry at boost factors arbitrarily far beyond anything that will ever be tested.

Even without assuming exact Lorentz symmetry, one can infer the Hawking effect by assuming that the outgoing modes are in their local ground state near the horizon for free-fall frequencies high compared to, say, the light-crossing time of the black hole, but small compared to the Planck frequency \[^15\]. Validity of this assumption is highly plausible since the black hole formation, and field propagation in the black hole background, is very slow compared to frequencies much higher than the light crossing time. One would thus expect that whatever is happening in the vacuum, it remains unexcited, and the outgoing modes would emerge in their ground state in the near horizon region. The sonic model and other analogues allow this hypothesis to be tested in well-understood material systems that break Lorentz symmetry.

Thus one is led to consider Hawking radiation in the presence of high frequency/short wavelength dispersion, both because of the possibility that spacetime is Lorentz violating (LV), and because of the fact that analogue models are LV. However, given the very strong observational constraints on Lorentz violation \[^16\], as well as the difficulty of accounting for low energy Lorentz symmetry in a theory that is LV in the UV \[^17\], the possibility of fundamental LV seems rather unlikely. Hence the main motivations for considering LV dispersion are to understand condensed matter analogues, and to have an example—probably unphysical from a fundamental viewpoint—in which the vacuum has strong UV modifications and the existence of Hawking radiation can be checked.

The central issue in my view is the origin of the outgoing modes \[^18\]. In a condensed matter model with a UV cutoff these must arise from somewhere other than the near horizon region, either from “superluminal” modes behind the horizon, from “subluminal” modes that are dragged towards the horizon and then released, or from no modes at all. The last scenario refers to the possibility that modes “assemble” from microscopic degrees of freedom in the near horizon region. This seems most likely the closest to what happens near a spacetime black hole, and for that reason deserves to be better understood. Other than a linear model that has been studied in the cosmological context \[^19\], and a linear model of quantum field theory on a 1+1 dimensional growing lattice \[^20\], I don’t know of any work focusing on how to characterize or study such a process.
5 Short wavelength dispersion

In this concluding section, I discuss what becomes of the Hawking effect when the dispersion relation is Lorentz invariant (“relativistic”) for long wavelengths but not for short wavelengths, as would be relevant for many analogue models. First I summarize results on the robustness of the “standard” black hole radiation spectrum, and then I describe the phenomena of stimulated emission and white hole radiation.

Dispersion relations of the form \( \omega^2 = c^2(k^2 \pm k^4/\Lambda^2) \) have been exhaustively studied. The plus sign gives “superluminal” propagation at high wavevectors, while the minus sign gives “subluminal” propagation. Roughly speaking, a horizon (for long wavelengths) will emit thermal Hawking radiation in a given mode provided that there is a regime near the horizon in which the mode is relativistic and in the locally defined vacuum state. This much was argued carefully in Ref. [15], and much subsequent work has gone into determining the precise conditions under which this will happen, and the size of the deviations from the thermal spectrum, for specific types of dispersion relations. The dispersion determines how the outgoing modes arise, that is whether they come from inside or outside the horizon, and what quantum state they would be found in if the initial state were near the ground state of the field, as in Hawking’s original calculation.

The most recent and most complete analysis of the effects of dispersion on the spectrum can be found in Ref. [21], in which many references to earlier work can also be found. The basic technique used there is that of matched asymptotic expansions, pioneered in Refs. [22, 23] as applied to Hawking radiation for dispersive fields. The dispersive modes have associated eikonal trajectories with a turning point outside or inside the horizon for the sub- and super-luminal cases respectively. Away from the turning point approximate solutions can be found using WKB methods. If the background fluid velocity (or its analogue) has a linear form \( v(x) = -1 + \kappa x \) to a good approximation out beyond the turning point, then one can match a near horizon solution to WKB solutions, and use this to find the Hawking radiation state and correlation functions. The near horizon solution is most easily found in \( k \) space, because while the mode equation is of higher order in \( x \) derivatives, \( v(x) = -1 + i\kappa\partial_x \) is linear in \( k \) derivatives, so the mode equation is second order in \( \partial_k \). Further simplifications come about because a linear \( v(x) \) in fact corresponds to de Sitter spacetime, which has an extra symmetry that produces factorized modes. One factor is independent of the dispersion and has a universal \( \omega \) dependence, while the other factor is independent of \( \omega \) and captures the dispersion dependence.

The result, for dispersion relations of the form \( \omega^2 = c^2(k \pm k^{2n+1}/\Lambda^{2n})^2 \) (chosen for convenience to be a perfect square), is that the relative deviations from the thermal spectrum are no greater than of order \( (\kappa/\Lambda)(\kappa x_{\text{lin}})^{-(1+1/2n)} \) times a polynomial in \( \omega/\kappa \). Here the horizon is at \( x = 0 \), and \( x_{\text{lin}} \) is the largest \( x \) for which \( v(x) \) has the linear form to a good approximation. Thus while it is important that the Lorentz violation wavevector scale \( \Lambda \) be much greater than the surface gravity \( \kappa \), this may not

\[\text{[11] For frequencies of order the surface gravity, this quantity can also be expressed as } (x_{\text{tp}}/x_{\text{lin}})^{1+1/2n}, \text{ where } x_{\text{tp}} \text{ is the (}\omega\text{-dependent) WKB turning point.}\]
be good enough to ensure agreement with the relativistic Hawking spectrum if the linear regime of the velocity extends over a distance much shorter than the inverse surface gravity.

At the other extreme, when the surface gravity is much larger than the largest frequency for which the turning point falls in the linear region, the spectrum of created excitations has been found to be proportional to $1/\omega$, at least for dispersion relations of the form $\omega^2 = c^2(k^2 \pm k^4/\Lambda^2)$. This is the low frequency limit of a thermal spectrum, but the temperature is set not by the surface gravity but by $\sim \Lambda(\kappa_{\text{lin}})^{3/2}$. This result applies even in the limit of an abrupt “step” at which the velocity changes discontinuously from sub- to supersonic [24, 25].

### 5.1 Stimulated Hawking radiation and dispersion

For a relativistic free field, the ancestors of Hawking quanta can be traced backwards in time along the horizon to the formation of the horizon, and then out to infinity. They are thus exponentially trans-Planckian. In the presence of dispersion, blueshifting is limited by the scale of dispersion, so that ancestors can be traced back to incoming modes with wave vectors of order $\Lambda$. If the dispersion is subluminal, those modes come from outside the black hole horizon, while if it is superluminal, they come from behind the horizon. Either way, they are potentially accessible to the control of an experiment. Instead of being in their ground state, they might be intentionally populated in an experiment, or they might be inadvertently thermally populated. Either way, they can lead to stimulated emission of Hawking radiation, as discussed in Section 3.3.

This opportunity to probe the dependence of the emitted radiation on the incoming state is useful to experiments, and it can amplify the Hawking effect, making it easier to detect. Note however that when the Hawking radiation is stimulated rather than spontaneous, it is less quantum mechanical, and if the incoming mode is significantly populated it is essentially purely classical.

### 5.2 White hole radiation

A white hole is the time reverse of a black hole. Just as nothing can escape from a black hole horizon without going faster than light, nothing can enter a white hole horizon without going faster than light. Einstein’s field equation is time reversal invariant, so it admits white hole solutions. In fact the Schwarzschild solution is time reversal symmetric: when taken in its entirety it includes a white hole. A black hole that forms from collapse is of course not time reversal invariant, but the time reverse of this spacetime is also a solution to Einstein’s equation. It is not a solution we expect to see in Nature, however, both because we don’t expect the corresponding initial condition to occur, and because, even if it did, the white hole would be
gravitationally unstable to forming a black hole due to accretion of matter [26, 27].
Moreover, even if there were no matter to accrete, the horizon would be classically and quantum mechanically unstable due to an infinite blueshift effect, as will be explained below.

White hole analogues, on the other hand, can be engineered in a laboratory, and are amenable to experimental investigation. For example, one could be realized by a fluid flow with velocity decreasing from supersonic to subsonic in the direction of the flow. Sound waves propagating against the flow would slow down and blueshift as they approach the sonic point, but the blueshifting would be limited by short wavelength dispersion, so the white hole horizon might be stable. If the horizon is stable, then the time reverse of the Hawking effect will take place on a white hole background, and the emitted radiation will be thermal, at the Hawking temperature of the white hole horizon [24] (see also Appendix D of Ref. [13]). Underlying this relation is the fact that the modes on the white hole background are the time reverse of the modes on the time-reversed black hole background. Note that this means that the roles of the in and out modes are swapped. In particular, the incoming vacuum relevant to the Hawking radiation consists of low wavenumber modes propagating against the flow.

When such a mode with positive norm approaches the white hole horizon, it is blocked and begins blueshifting. At this stage, it has become a superposition of positive and negative co-moving frequency (and therefore positive and negative norm) parts. If it were relativistic at all scales, it would continue blueshifting without limit. It would also be unentangled with the other side of the horizon, so would evidently be in an excited state, not the co-moving ground state. Hence there would be a quantum instability of the vacuum in which the state becomes increasingly singular on the horizon. A classical perturbation would behave in a similarly unstable fashion.

In the presence of dispersion, however, the blueshifting is arrested when the it reaches the dispersion scale. At that stage, if the mode becomes superluminal, it accelerates and both parts propagate across the horizon. If instead it becomes subluminal, then it slows down and both parts get dragged back out with the flow. In either case, the positive and negative norm parts are in an entangled, excited state that is thermal when tracing over one of the pair. Thus, a dispersive wave field exhibits Hawking radiation from a white hole horizon, but with two marked differences when compared to black hole radiation: the Hawking quanta have high wavevectors even when the Hawking temperature is low, and the entangled partners propagate on the same side of the horizon (inside for superluminal, outside for subluminal dispersion). While on the same side, the partners can separate, since in general they have different group velocities.

There is an important potential complication with this story of white hole radiation. Although the singularity that would arise in the relativistic case is cured by dispersion, an avatar of it emerges in the form of a zero Killing frequency standing wave. This has been shown to arise from the zero frequency limit of the Hawking radiation. In that limit, the emission rate diverges as \(1/\omega\), leading to a state with macroscopic occupation number that grows in time [28, 21]. This process can also be seeded by classical perturbations, and it grows until nonlinear effects saturate the
The resulting standing wave, which is a well-known phenomenon in other contexts, is referred to in the white hole setting as an “undulation”. It is composed of short wavelengths that are well into the dispersive regime. Depending on the nature of the flow and the saturation mechanism, it could disrupt the flow and prevent a smooth horizon from forming.

To conclude, I will now describe what was seen in the Vancouver experiment [12]. That experiment involved a flow of water in a flume tank with a velocity profile that produced a white hole horizon for long wavelength, shallow water, surface waves (which are dispersionless over a uniform bottom). When blueshifted those waves convert to deep water waves, with a lower group velocity, which behave like the “subluminal” case described above. In the experiment coherent, long waves with nine different frequencies were launched from downstream, propagating back upstream towards the white hole horizon, and the resulting conversion to short waves was observed. The squared norm ratio of the negative and positive norm components of the corresponding frequency eigenmode was consistent with the thermal ratio [12]. This can be understood as coherently stimulated emission of Hawking radiation (see Appendix C of Ref. [13] for a general discussion of this process). It is strictly classical, but it is governed by the same mode conversion amplitudes that would produce spontaneous emission if the system could be prepared in the ground state.

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12 The relevant norm is determined by the action for the modes. This has been worked out assuming irrotational flow [29], which is a good approximation although the flow does develop some vorticity. The predicted Hawking temperature was estimated, but it is difficult to evaluate accurately because of the presence of the undulation, the fact that it depends on the flow velocity field that was not precisely measured, and the presence of vorticity which has not yet been included in an effective metric description.
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