SOLUBLE GROUPS WITH FEW ORBITS UNDER AUTOMORPHISMS

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Abstract. Let $G$ be a group. The orbits of the natural action of $\text{Aut}(G)$ on $G$ are called “automorphism orbits” of $G$, and the number of automorphism orbits of $G$ is denoted by $\omega(G)$. We prove that if $G$ is a soluble group with finite rank such that $\omega(G) < \infty$, then $G$ contains a torsion-free characteristic nilpotent subgroup $K$ such that $G = K \rtimes H$, where $H$ is a finite group. Moreover, we classify the mixed order soluble groups of finite rank such that $\omega(G) = 3$.

1. Introduction

Let $G$ be a group. The orbits of the natural action of $\text{Aut}(G)$ on $G$ are called “automorphism orbits” of $G$, and the number of automorphism orbits of $G$ is denoted by $\omega(G)$. It is interesting to ask what can we say about “$G$” only knowing $\omega(G)$. It is obvious that $\omega(G) = 1$ if and only if $G = \{1\}$, and it is well known that if $G$ is a finite group then $\omega(G) = 2$ if and only if $G$ is elementary abelian. In [3], T. J. Laffey and D. MacHale proved that if $G$ is a finite non-soluble group with $\omega(G) \leq 4$, then $G$ is isomorphic to $\text{PSL}(2, \mathbb{F}_q)$ with $q \in \{4,7,8,9\}$. Later, M. Stroppel, in [8], has shown that the only finite non-abelian simple groups $G$ with $\omega(G) \leq 5$ are the groups $\text{PSL}(2, \mathbb{F}_q)$ with $q \in \{4,7,8,9\}$, $\text{PSL}(3, \mathbb{F}_4)$ or $\text{ASL}(2, \mathbb{F}_4)$ (answering a question of M. Stroppel, cf. [3 Problem 2.5]).

Some aspects of automorphism orbits are also investigated for infinite groups. M. Schwachhöfer and M. Stroppel in [7, Lemma 1.1], have shown that if $G$ is an abelian group with finitely many automorphism orbits, then $G = \text{Tor}(G) \oplus D$, where $D$ is a characteristic torsion-free divisible subgroup of $G$ and $\text{Tor}(G)$ is the set of all torsion elements in $G$. In [1, Theorem A], the authors proved that if $G$ is a FC-group with finitely many automorphism orbits, then the derived subgroup $G'$

\begin{flushright}
2010 Mathematics Subject Classification. 20E22; 20E36.

Key words and phrases. Extensions; Automorphisms; Soluble groups.

This work was partially supported by FAPDF - Brazil.
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is finite and \( G \) admits a decomposition \( G = \text{Tor}(G) \times A \), where \( A \) is a divisible characteristic subgroup of \( Z(G) \). For more details concerning automorphism orbits of groups see [8].

If \( G \) is a group and \( r \) is a positive integer, then \( G \) is said to have finite rank \( r \) if each finitely generated subgroup of \( G \) can be generated by \( r \) or fewer elements and if \( r \) is the least such integer. The next result can be viewed as a generalization of the above mentioned results from [1] and [7].

**Theorem A.** Let \( G \) be a soluble group of finite rank. If \( \omega(G) < \infty \), then \( G \) has a torsion-free radicable nilpotent subgroup \( K \) such that \( G = K \rtimes H \), where \( H \) is a finite subgroup.

We do not know whether the hypothesis that \( G \) has finite rank is really needed in Theorem A. The proof that we present here uses this assumption in a very essential way.

In [3], T. J. Laffey and D. MacHale showed that \( G \) is a finite group in which the order \( |G| \) is not prime power and \( \omega(G) = 3 \) if and only if \( |G| = pq^n \), the Sylow \( q \)-subgroup \( Q \) is a normal elementary abelian subgroup of \( G \) and \( P \) is a Sylow \( p \)-subgroup which acts fixed-point-freely on \( Q \). See also [5] for groups with \( \omega(G) \leq 3 \) (almost homogeneous groups).

Recall that a group \( G \) has mixed order if it contains non-trivial elements of finite order and also elements of infinite order. We obtain the following classification.

**Theorem B.** Let \( G \) be a mixed order soluble group with finite rank. We have \( \omega(G) = 3 \) if and only if \( G = A \rtimes H \) where \( |H| = p \) for some prime \( p \), \( H \) acts fixed-point-freely on \( A \) and \( A = \mathbb{Q}^n \) for some positive integer \( n \).

### 2. Proofs

A well-known result in the context of extensions of finite groups, due to I. Schur, states that if \( G \) is a finite group and \( N \) is a normal abelian subgroup with \((|N|, |G : N|) = 1 \), then there exists a complement \( K \) of \( N \). Recall that \( K \) is a complement of (a normal subgroup) \( N \) in \( G \) if \( N \cap K = 1 \) and \( G = NK \). In particular, \( G = N \rtimes K \). Now, we prove that Schur’s theorem holds under a more general assumption that \( G \) contains a divisible abelian subgroup of finite index. The proof presented here is adapted from the ideas of the finite case (cf. [6] 9.1.2).

**Lemma 2.1.** Let \( A \) be a divisible normal abelian subgroup of finite index of a group \( G \). Then there exists a subgroup \( H \) of \( G \) such that \( G = A \rtimes H \).
Proof. Let $B = G/A$. From each coset $x$ in $B$ we choose a representative $t_x$, so that the set $T = \{t_x \mid x \in B\}$ is a transversal to $A$ in $G$. Since $t_xt_yA = t_{xy}A$, there is an element $c(x, y)$ of $A$ such that $t_xt_y = t_{xy}c(x, y)$. Then
\[
(t_xt_y)t_z = t_{xy}c(x, y)t_z = t_{xy}t_zc(x, y)^z = t_{xy}c(xy, z)c(x, y)^z,
\]
and $c(xy, z)c(x, y)^z = c(x, yz)c(y, z)$, for each $x, y \in B$. Consider the element $d(y) = \prod_{x \in B}c(x, y) \in A$. As $A$ is an abelian group $d(z)d(y)^z = d(yz)c(y, z)^n$, where $n = |B|$. We obtain that $d(yz)^n = d(y)\cdot d(z)c(y, z)^{-n}$.

Now, since $A$ is a divisible group, there exists $e(y) \in A$ such that $e(y)^n = d(y)^{-1}$ for each $y \in B$. Hence
\[
e(z)^{-n} = (e(y)^z e(z)c(y, z))^{-n}.
\]

Since $A$ is torsion-free, it follows that $e(z) = (e(y)^z e(z)c(y, z))$. Define $s_x = t_x e(x)$, then
\[
s_y s_z = t_y t_z e(y)^z e(z) = t_y c(y, z) (e(y)^z e(z)) = t_y c(y, z) = s_y s_z.
\]
Thus $x \mapsto s_x$ defines a homomorphism $\phi : B \rightarrow G$. Now $s_x = 1$ implies that $t_x \in A$ and $x = A = 1_B$. From this we conclude that $H = B^\phi$ is the desired complement.

Now, we consider groups with finitely many automorphism orbits with a characteristic torsion-free soluble subgroup of finite index (see also Schur-Zassenhaus Theorem [6, 9.1.2]).

Lemma 2.2. Let $n$ be a positive integer. Let $G$ be a group such that $\omega(G) < \infty$ and $A$ a torsion-free characteristic subgroup of $G$ with finite index $n$. If $A$ is soluble, then there exists a subgroup $H$ of $G$ such that $G = A \rtimes H$.

Proof. If $A$ is abelian, then the result is immediate by Lemma 2.1. Assume that $A$ is non-abelian. Set $d$ the derived length of $A$. First we prove that $A/A^{(d-1)}$ is torsion-free. The subgroup $A^{(d-1)}$ is a torsion-free abelian divisible subgroup since $A$ is torsion-free and has finitely many automorphism orbits. If $A/A^{(d-1)}$ is not torsion-free, then we can find an element $a \in A$ such that $\langle a, A^{(d-1)} \rangle$ has a torsion-free divisible group of finite index. Then by Lemma 2.1 the subgroup $\langle a, A^{(d-1)} \rangle$ has elements of finite order. That is a contradiction. Thus $A/A^{(d-1)}$ is torsion-free.

Now, we complete the proof arguing by induction on the derived length of $A$. Consider the quotient group $\tilde{G} = G/A^{(d-1)}$. By induction we deduce that there exists a finite subgroup $\tilde{B}$ of order $n$ in $\tilde{G}$ such that $\tilde{G} = \tilde{A} \rtimes \tilde{B}$. Set $B$ the inverse image of $\tilde{B}$. Clearly $A^{(d-1)} \leq B$. 

and $A^{(d-1)}$ has finite index $n$ in $B$. Therefore, by Lemma 2.1 $B$ has a subgroup $H$ of order $n$ and so such a subgroup is a complement of $A$ in $G$. The result follows. □

The following lemma is well-known. We supply the proof for the reader’s convenience.

**LEMMA 2.3.** Let $G$ be an abelian group of finite rank. If $\omega(G) < \infty$, then the torsion subgroup $\text{Tor}(G)$ is finite.

*Proof.* Since $G$ has finitely many automorphism orbits, it follows that the exponent $\exp(\text{Tor}(G))$ is bounded. As $G$ has finite rank we have that $\text{Tor}(G)$ is finitely generated. We deduce that $\text{Tor}(G)$ is finite, which completes the proof. □

The following result provides a description of radicable nilpotent groups of finite rank (see [4, Theorem 5.3.6] for more details). Recall that a group $G$ is said to be radicable if each element is an $n$th power for every positive integer $n$.

**LEMMA 2.4.** Let $G$ be a soluble group with finite rank. Then the following are equivalent:

(i) $G$ has no proper subgroups of finite index;
(ii) $G = G^m$ for all $m > 0$;
(iii) $G$ is radicable and nilpotent.

We are now in a position to prove Theorem A.

*Proof of Theorem A.* We argue by induction on derived length of $G$.

Assume that $G$ is abelian. By Schwachhöfer-Stroppel’s result [4], $G = D \oplus T$, where $D$ is a characteristic torsion free divisible subgroup and $T$ is the torsion subgroup of $G$. By Lemma 2.3 the torsion subgroup $T = \text{Tor}(G)$ is a finite subgroup of $G$, the result follows.

Now, we assume that $G$ is non-abelian. Set $d$ the derived length of $G$. Arguing as in the previous paragraph, we deduce that $G^{(d-1)} = D_1 \oplus T_1$, where $D_1$ is a characteristic torsion-free divisible subgroup and $T_1$ is the torsion subgroup of $G^{(d-1)}$ and so, $T_1$ is finite. By induction $G/G^{(d-1)}$ has the desired decomposition. More precisely, $G^{(d-1)} = D_1 \oplus T_1$ and $G/G^{(d-1)} = \bar{A} \rtimes \bar{B}$ where $\bar{A}$ is torsion-free and $\bar{B}$ is finite. Note that $\bar{A}^n = \bar{A}$ for any positive integer $n$, since the quotient groups $\bar{A}^{(i)}/\bar{A}^{(i+1)}$ are torsion-free divisible groups (we can use Lemma 2.2 to conclude that each quotient is torsion-free).

Note that the centralizer $C_G(T_1)$ is a subgroup of finite index in $G$, because $G/C_G(T_1)$ embeds in the automorphism group of $T_1$ which has finite order. Let $\bar{A}$ be the inverse image of $\bar{A}$. As $\bar{A}$ is torsion-free
and \( A^n = A \) for any positive integer \( n \), we have \( A \leq C_G(T_1) \). Thus \( T_1 \leq Z(A) \) and Tor(\( A \)) = \( T_1 \). Set \( K = A^e \), where \( e = \exp(T_1) \). Then \( K \) is torsion-free and has finite index in \( G \). Therefore, by Lemma 2.2, there exists a finite subgroup \( H \) such that \( G = K \rtimes H \). According to Lemma 2.4, we deduce that \( K \) is a radicable nilpotent group (the subgroup \( K \) has no proper subgroups of finite index). The proof is complete. \( \square \)

Now we will deal with Theorem B: Let \( G \) be a mixed order soluble group with finite rank. We have \( \omega(G) = 3 \) if and only if \( G = A \rtimes H \) where \( A = Q^n \) for some positive integer \( n \), \( |H| = p \) for some prime \( p \) and \( H \) acts fixed-point-freely on \( A \).

**Proof of Theorem B.** First assume that \( G \) is a mixed order soluble group of finite rank and have \( \omega(G) = 3 \). By Theorem A, \( G = A \rtimes H \) where \( A \) is a torsion-free radicable nilpotent subgroup and \( H \) is a finite group. Since \( \omega(G) = 3 \) and \( G \) has mixed order, it follows that \( A \) must be abelian (so that \( A = Q^n \)) and \( H \) is an elementary abelian \( p \)-subgroup. On the other hand, since \( A \) is characteristic, we deduce that all elements in \( G \setminus A \) have order \( p \) and then \( H \) acts fixed-point-freely on \( A \). Thus using the identity \((xy)^p = x^n(y^{p-1}) \ldots y^xy\), we obtain

\[
(h^ja)^p = h^ja^h \ldots a^{h^{p-1}j}) = 1
\]

for all \( a \in Q^n \). So all elements of \( G \setminus A \) have order exactly \( p \) and act fixed-point-freely on \( A \).

Now, let \( b, c \in A \setminus \{1\} \) and \( \alpha, \beta \in G \setminus A \). By Cyclic Decomposition Theorem, there exist \( b_1, b_2, \ldots, b_t, c_1, c_2, \ldots, c_t \in A \) such that

\[
\{b_1, b_1^\alpha, \ldots, b_1^{\alpha^{p-2}}, \ldots, b_t, b_t^\alpha, \ldots, b_t^{\alpha^{p-2}}\}
\]

and

\[
\{c_1, c_1^\beta, \ldots, c_1^{\beta^{p-2}}, \ldots, c_t, c_t^\beta, \ldots, c_t^{\beta^{p-2}}\}
\]
are bases of $A$. Without loss of generality we can assume that $b = b_1$
and $c = c_1$. Thus the map given by

$$b_i \mapsto c_i \text{ and } \alpha \mapsto \beta,$$

where $i = 1, \ldots, t$ extends to an automorphism of $G$. Hence all non-
trivial elements of $A$ belong to the same orbit under the action of
$\text{Aut}(G)$, and all elements in $G \setminus A$ are in the same orbit under the
action of $\text{Aut}(G)$. The proof is complete. \hfill $\square$

References

[1] R. Bastos and A. C. Dantas, FC-groups with finitely many automorphism or-
bits, J. Algebra, 516 (2018) pp. 401–413.
[2] R. Bastos, A. C. Dantas and M. Garonzi, Finite groups with six or seven au-
tomorphism orbits, J. Group Theory, 21 (2017) pp. 945–954.
[3] T. J. Laffey and D. MacHale, Automorphism orbits of finite groups, J. Austral.
Math. Soc. Ser. A, 40(2) (1986) pp. 253–260.
[4] J. C. Lennox and D. J. S. Robinson, The Theory of Infinite Soluble Groups,
Clarendon Press, Oxford, 2004.
[5] H. Mäurer and M. Stroppel, Groups that are almost homogeneous, Geom. Dedicata,
68 (1997) pp. 229–243.
[6] D. J. S. Robinson, A course in the theory of groups, 2nd edition, Springer-
Verlag, New York, 1996.
[7] M. Schwachhöfer and M. Stroppel, Finding representatives for the orbits under
the automorphism group of a bounded abelian group, J. Algebra, 211 (1999)
pp. 225–239.
[8] M. Stroppel, Locally compact groups with few orbits under automorphisms,
Top. Proc., 26(2) (2002) pp. 819–842.