Vertical and complete lifts of sections of a (dual) vector bundle and Legendre duality

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Abstract

Supplementary comments about generalized Lie algebroids are presented and a new point of view over the construction of the Lie algebroid generalized tangent bundle of a (dual) vector bundle is introduced. Using the general theory of exterior differential calculus for generalized Lie algebroids, a covariant derivative for exterior forms of a (dual) vector bundle is introduced. Using this covariant derivative, the complete lift of an arbitrary section of a (dual) vector bundle is discovered. A theory of Legendre type and Legendre duality between vertical and complete lifts is presented. Finally, a duality between Lie algebroids structures is developed.

1 Introduction

It is well-known that the lift of geometrical objects such as functions, vector fields and 1-forms defined on the base of the usual Lie algebroid

$$(TM, τ_M, M), [·]_{TM}, (Id_{TM}, Id_M),$$

has an important role in the geometry of the Lie algebroid

$$(TTM, τ_{TM}, TM), [·]_{TTM}, (Id_{TTM}, Id_{TM}).$$

Using these lifts, it is possible to introduce the lift of a (pseudo) Riemannian metric structure. The Sasaki lift of a Riemannian metric structure on $M$ is an important example of metric structures on $TM$ used in differential geometry with many applications in physics [10]. Lift of geometrical structures of the Lie algebroid

$$(TM, τ_M, M), [·]_{TM}, (Id_{TM}, Id_M),$$

to the Lie algebroid

$$(TTM, τ_{TM}, TM), [·]_{TTM}, (Id_{TTM}, Id_{TM}),$$

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were introduced and studied by several authors [8, 9, 17]. In many papers such as [5, 11, 14, 15], the authors studied the lifts to the second order tangent bundle, tensor bundle and jet bundle.

The Lie algebroids are important issues in physics and mechanics since the extension of Lagrangian and Hamiltonian systems to their entity [6, 7, 12] and catching the Poisson structure [13]. Several authors presented and studied the lift of geometrical objects of a Lie algebroid \((F, \nu, N, ([.,]_F, (\rho, Id_N))\) to the Lie algebroid prolongation. Using the vertical and complete lifts of sections of a Lie algebroid, the first author presented important results about Lie symmetry and horizontal lifts in the general framework of prolongation Lie algebroid [10].

Extending the notion of Lie algebroid from one base manifold to a pair of diffeomorphic base manifolds, the second author introduced the generalized Lie algebroid [1, 2]. Using the lift of a differentiable curve defined on the base of a generalized Lie algebroid, he developed a new theory of mechanical systems with many applications in physics [4]. The space used for developing this theory of mechanical systems is the Lie algebroid generalized tangent bundle \(((\rho, \eta)_T E, ([.,]_E, (\rho, \eta))\).

This paper is arranged as follows. Some notions and results about exterior differential algebra of a vector bundle and information about generalized Lie algebroids are presented in Section 1. Using a vector bundle \((E, \pi, M)\) anchored by a generalized Lie algebroid \(((F, \nu, N), ([.,]_F, (\rho, \eta))\)) and a vector bundle morphism \((g, h)\) we obtain a new point of view over construction of the Lie algebroid generalized tangent bundle

\[\(((\rho, \eta)_T E, (\rho, \eta)_E, E, ([.,]_E, (\rho, \eta))\),\]

of a generalized Lie algebroid \(((F, \nu, N), ([.,]_F, (\rho, \eta))\)).

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\[\(((\rho, \eta)_T E, (\rho, \eta)_E, E, ([.,]_E, (\rho, \eta))\),\]

in Section 2. Using the exterior differential calculus of the exterior algebra of the generalized Lie algebroid \(((F, \nu, N), ([.,]_F, (\rho, \eta))\)) presented in [3], in Section 3, we introduce a Lie covariant derivative for the exterior algebra of the vector bundle \((E, \pi, M)\). Using this Lie covariant derivative, we introduce in Theorem 11 the complete \((g, h)\)-lift \(u^c \in \Gamma((\rho, \eta)_T E, (\rho, \eta)_E)\) of an arbitrary section \(u \in \Gamma(E, \pi, M)\). Using the complete \((g, h)\)-lift of a function \(f \in F(N)\) we obtain new results for vertical and complete \((g, h)\)-lifts. In the final of Section 3, we introduced the complete and vertical \((g, h)\)-lifts

\[u^C, u^V \in \Gamma((\rho, \eta)_T E, (\rho, \eta)_E),\]

of a section \(u \in \Gamma(E, \pi, M)\) and important results are presented in Theorem 18. Also, using the dual vector bundle \((\hat{E}, \hat{\pi}, M)\) anchored by a generalized Lie algebroid \(((\hat{F}, \hat{\nu}, N), ([.,]_{\hat{F}}, (\hat{\rho}, \hat{Id}_N))\)) and a vector bundle morphism \((\hat{g}, h)\), in Section 4, we obtain a new point of view over construction of the Lie algebroid generalized tangent bundle

\[\(((\rho, \eta)_T \hat{E}, (\rho, \eta)_{\hat{E}}, \hat{E}, ([.,]_{\hat{E}}, (\rho, \eta))\),\]

\[\hat{E}, ([.,]_{\hat{E}}, (\rho, \eta))\).
A dual theory for the vertical and complete lifts is presented in Section 5 and similar results are obtained. A general presentation of Lagrange (Finsler) and Hamilton (Cartan) fundamental functions and a theory of Legendre type are presented in Section 6. Using the tangent \((\rho, \eta)\)-application of the Legendre bundle morphism associated to a Lagrange respectively Hamilton fundamental function, we obtain new results about duality between vertical and complete \((g, h)\)-lifts in Section 7. New results about duality between Lie algebroids structures and the Legendre \((\rho, \eta)\)-equivalence between the vector bundle \((E, \pi, M)\) and its dual \((\hat{E}, \hat{\pi}, M)\) are presented in Section 8.

2 Preliminaries

Let \((E, \pi, M)\) be an arbitrary vector bundle. If \(\Gamma(E, \pi, M)\) is the set of sections of the vector bundle \((E, \pi, M)\) and \(\mathcal{F}(M)\) is the set of differentiable real-valued functions on \(M\), then \(\Gamma(E, \pi, M)\) is a \(\mathcal{F}(M)\)-module.

For any \(q \in \mathbb{N}\) we denote by \((\Sigma_q, \circ)\) the permutations group of the set \(\{1, 2, ..., q\}\). We denoted by \(\Lambda^q(E, \pi, M)\) the set of \(q\)-linear applications

\[
\Gamma(E, \pi, M)^q \rightarrow \mathcal{F}(M),
\]

\[
(z_1, ..., z_q) \mapsto \omega(z_1, ..., z_q),
\]

such that

\[
\omega(z_{\sigma(1)}, ..., z_{\sigma(q)}) = sgn(\sigma) \cdot \omega(z_1, ..., z_q),
\]

for any \(z_1, ..., z_q \in \Gamma(E, \pi, M)\) and for any \(\sigma \in \Sigma_q\). The elements of \(\Lambda^q(E, \pi, M)\) will be called differential forms of degree \(q\) or differential \(q\)-forms. It is known that \((\Lambda^q(E, \pi, M), +, \cdot)\) is a \(\mathcal{F}(M)\)-module [3].

**Definition 1** If \(\omega \in \Lambda^q(E, \pi, M)\) and \(\theta \in \Lambda^r(E, \pi, M)\), then the \((q + r)\)-form \(\omega \wedge \theta\) defined by

\[
\omega \wedge \theta(u_1, ..., u_{q+r}) = \sum_{\sigma(1) < ... < \sigma(q), \sigma(q+1) < ... < \sigma(q+r)} sgn(\sigma)\omega(u_{\sigma(1)}, ..., u_{\sigma(q)})\theta(u_{\sigma(q+1)}, ..., u_{\sigma(q+r)})
\]

\[
= \frac{1}{q! r!} \sum_{\sigma \in \Sigma_{q+r}} sgn(\sigma)\omega(u_{\sigma(1)}, ..., u_{\sigma(q)})\theta(u_{\sigma(q+1)}, ..., u_{\sigma(q+r)})
\]

for any \(u_1, ..., u_{q+r} \in \Gamma(E, \pi, M)\), will be called the exterior product of the forms \(\omega\) and \(\theta\).

Using the previous definition, we obtain

**Theorem 2** Let \(\omega, \sigma \in \Lambda^q(E, \pi, M)\), \(\theta \in \Lambda^r(E, \pi, M)\), \(\eta, \xi \in \Lambda^s(E, \pi, M)\) and \(f \in \mathcal{F}(M)\). Then

\[
\omega \wedge \theta = (-1)^q sgn(\theta) \wedge \omega, \quad (\omega \wedge \theta) \wedge \eta = \omega \wedge (\theta \wedge \eta),
\]

\[
(\omega + \sigma) \wedge \eta = \omega \wedge \eta + \sigma \wedge \eta, \quad \omega \wedge (\eta + \xi) = \omega \wedge \eta + \omega \wedge \xi,
\]

\[
(\omega \wedge \theta = f(\omega \wedge \theta) = \omega \wedge (f \theta).
\]
We set
\[ \Lambda(E, \pi, M) := \bigoplus_{q \geq 0} \Lambda^q(E, \pi, M). \]
Then it is easy to see that \( \Lambda(E, \pi, M) \) is a \( \mathcal{F}(M) \)-algebra. This algebra will be called the exterior differential algebra of the vector bundle \( (E, \pi, M) \).

Now let \( (F, \nu, N) \) be another vector bundle and \( (\varphi, \varphi_0) \) is a vector bundles morphism from \( (E, \pi, M) \) to \( (F, \nu, N) \) such that \( \varphi_0 \) is an isomorphism from \( M \) to \( N \). Then using the operation
\[ \mathcal{F}(M) \times \Gamma(F, \nu, N) \longrightarrow \Gamma(F, \nu, N), \]
\[ (f, z) \longmapsto f \circ \varphi_0^{-1} \cdot z, \]

it results that \( (\Gamma(F, \nu, N), +, \cdot) \) is a \( \mathcal{F}(M) \)-module and we obtain the modules morphism
\[ \Gamma(E, \pi, M) \xrightarrow{\varphi, \varphi_0} \Gamma(F, \nu, N), \]
\[ u = u^a s_a \longmapsto (u^a \circ \varphi_0^{-1}) (\varphi^a \circ \varphi_0^{-1}) t_a. \]

**Definition 3** Let \( (\varphi, \varphi_0) \) be a vector bundles morphism from \( (E, \pi, M) \) to \( (F, \nu, N) \) such that \( \varphi_0 \) is an isomorphism from \( M \) to \( N \). Then we define the pull-back application
\[ \Lambda^q(F, \nu, N) \xrightarrow{(\varphi, \varphi_0)^*} \Lambda^q(E, \pi, M), \]
\[ \omega \longmapsto (\varphi, \varphi_0)^* \omega, \]
by
\[ ((\varphi, \varphi_0)^* \omega)(u_1, ..., u_q) = \omega(\Gamma(\varphi, \varphi_0) (u_1), ..., \Gamma(\varphi, \varphi_0) (u_q)), \]
for any \( u_1, ..., u_q \in \Gamma(E, \pi, M) \). If \( f \in \mathcal{F}(N) \), then \((\varphi, \varphi_0)^*(f) = f \circ \varphi_0\).

**Remark 4** If \( (\rho, \eta) \) and \( (Th, h) \) are two vector bundles morphisms given by the diagrams
\[
\begin{array}{cccc}
F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\
\downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N
\end{array}
\]

then we obtain the modules morphism \( \Gamma(Th \circ \rho, h \circ \eta) \) different by \( \Gamma(Th, h) \circ \Gamma(\rho, \eta) \) defined by
\[ \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha)(f) = z^\alpha \rho^j_i \left( \frac{\partial (f \circ h)}{\partial z^j} \right) \circ h^{-1}, \]
for any \( z^\alpha t_\alpha \in \Gamma(F, \nu, N) \) and \( f \in \mathcal{F}(N) \).

**Definition 5** A generalized Lie algebroid is a vector bundle \( (F, \nu, N) \) given by the diagrams:
\[
\begin{array}{cccc}
(F, [,]_F, h) & \xrightarrow{\rho} & (TM, [,]_TM) & \xrightarrow{Th} & (TN, [,]_{TN}) \\
\downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N
\end{array}
\]
where $h$ and $\eta$ are arbitrary isomorphisms, $(\rho, \eta)$ is a vector bundles morphism from $(F, \nu, N)$ to $(TM, \tau_M, M)$ and the operation

$$\Gamma(F, \nu, N) \times \Gamma(F, \nu, N) \xrightarrow{[,]_{F,h}} \Gamma(F, \nu, N),$$

satisfies in

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma;'(Th \circ \rho, h \circ \eta)(u)f \cdot v, \quad \forall f \in \mathcal{F}(N),$$

such that the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [,]_{F,h})$ is a Lie $\mathcal{F}(N)$-algebra.

We denote by $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$ the generalized Lie algebroid defined in the above. Moreover, the couple $((F, \nu, N), (\rho, \eta))$ is called the generalized Lie algebroid structure. In particular, if $\eta = Id_M = h$, then we obtain the definition of Lie algebroid. So, any Lie algebroid can be regarded as a generalized Lie algebroid.

**Remark 6** The modules morphism $\Gamma(Th \circ \rho, h \circ \eta)$ is Lie algebras morphism from $\Gamma(F, \nu, N)$ to $\Gamma(TM, \tau_N, N)$.

If we take local coordinates $(x^i)$ and $(\chi^i)$ on open sets $V \subset M$ and $W \subset N$, respectively, then we have the corresponding local coordinates $(x^i, y^i)$ and $(\chi^i, z^i)$ on $TM$ and $TN$, respectively, where $i, j \in \{1, \ldots, m\}$. Moreover, we consider $(\chi^i, z^i)$ as the local coordinates on $F$, where $\alpha \in \{1, \ldots, p\}$. If $\{t_\alpha\}$ is a local basis for module of sections of $(F, \nu, N)$, then we put $[t_\alpha, t_\beta]_{F,h} = L^\gamma_{\alpha\beta} t_\gamma$, where $L^\gamma_{\alpha\beta}$ are local functions on $N$ and $\alpha, \beta, \gamma \in \{1, \ldots, p\}$. It is easy to see that $L^\gamma_{\alpha\beta} = -L^\gamma_{\beta\alpha}$. Using Remark 6 we obtain

$$
(L^\gamma_{\alpha\beta} \circ h)(\rho^k \circ h) = (\rho^i \circ h) \frac{\partial(\rho^k \circ h)}{\partial x^i} - (\rho^j \circ h) \frac{\partial(\rho^k \circ h)}{\partial x^j}.
$$

(1)

The local functions $L^\gamma_{\alpha\beta}$ introduced in the above are called the structure functions of the generalized Lie algebroid $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$.

A morphism from $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$ to $((F', \nu', N'), [,]_{F',\nu'}, (\rho', \eta'))$ is a morphism $(\phi, \phi_0)$ from $(F, \nu, N)$ to $(F', \nu', N')$ such that $\phi_0$ is an isomorphism from $N$ to $N'$, and the modules morphism $\Gamma(\phi, \phi_0)$ is a Lie algebras morphism from $\Gamma(F, \nu, N)$ to $\Gamma(F', \nu', N')$. Thus, we can discuss about the category of generalized Lie algebroids.

### 2.1 The generalized tangent bundle of a vector bundle

We consider the following diagrams:

$$
\begin{array}{cccccc}
E & \xrightarrow{g} & (F, [\cdot]_{F,h}) & \xrightarrow{\rho} & TM & \xrightarrow{Th} & (TN, [\cdot]_{TN}) \\
\downarrow \pi & & \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
M & \xrightarrow{h} & N & \xrightarrow{\eta} & M & \xrightarrow{h} & N
\end{array},
$$

(2)
where \(((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))\) is a generalized Lie algebroid, \((E, \pi, M)\) is a vector bundle and \((g, h)\) is a vector bundle morphism from \((E, \pi, M)\) to \((F, \nu, N)\) with components \(g^a_\alpha, \alpha \in \{1, 2, \cdots, n\}\) and \(b \in \{1, 2, \cdots, r\}\).

If \(v = v^\alpha t^\alpha\) is a section of \((F, \nu, N)\), then we define its corresponding section \(X = X^\alpha T^\alpha\) in the pull-back vector bundle \(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E)\) given by

\[
X(u_x) = ((g^a_\alpha \circ h^{-1})(u^b \circ h^{-1})t^a_\alpha)(h \circ \pi(u_x)) = g^a_\alpha(x)u^b(x)t^a_\alpha(h(x))
\]

for any \(u_x \in E\). If we define

\[
\begin{align*}
(h \circ \pi)^* F & \xrightarrow{(\rho, \eta)^*} TE, \\
X^\alpha T^\alpha(u_x) & \xrightarrow{g^a_\alpha \circ \pi} (u^b \circ \pi)(\rho^a_\alpha \circ h \circ \pi)\partial_i(u_x),
\end{align*}
\]

then \(((\rho, \eta)^* E, Id_E)\) is a vector bundles morphism from \(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E)\) to \((TE, \tau_E, E)\). Moreover, the operation

\[
\Gamma(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E))^2 \xrightarrow{[\cdot, \cdot]_{(h \circ \pi)^* F}} \Gamma(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E)),
\]

defined by

\[
\begin{align*}
[T^\alpha, T^\beta]_{(h \circ \pi)^* F} &= (L^\alpha_{\gamma \beta} \circ h \circ \pi)T^\gamma, \\
[T^\alpha, fT^\beta]_{(h \circ \pi)^* F} &= f(L^\gamma_{\alpha \beta} \circ h \circ \pi)T^\gamma + (\rho^a_\alpha \circ h \circ \pi)\partial_i(f)T^\beta, \\
[fT^\alpha, T^\beta]_{(h \circ \pi)^* F} &= -[T^\beta, fT^\alpha]_{(h \circ \pi)^* F},
\end{align*}
\]

for any \(f \in \mathcal{F}(E)\), is a Lie bracket on \(\Gamma(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E)\). It is easy to check that

\[
(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E), [\cdot, \cdot]_{(h \circ \pi)^* F} \circ (\rho, \eta)^* E, Id_E))
\]

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid \(((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))\).

Let \((\partial_i, \hat{\partial}_a)\) be the base sections for the Lie \(\mathcal{F}(E)\)-algebra

\[
\Gamma(TE, \tau_E, E), +, [\cdot, \cdot]_{TE}.
\]

Then

\[
X^\alpha \hat{\partial}_a := X^\alpha(T^\alpha \circ (\rho^a_\alpha \circ h \circ \pi)\partial_i) + \hat{X}^a(0_{(h \circ \pi)^* F} \oplus \hat{\partial}_a)
\]

\[
= X^\alpha T^\alpha \oplus (X^\alpha(\rho^a_\alpha \circ h \circ \pi)\partial_i + \hat{X}^a \hat{\partial}_a),
\]

making from \(X^\alpha S^a \in \Gamma(((h \circ \pi)^* F, (h \circ \pi)^* \nu, E)\) and \(\hat{X}^a \hat{\partial}_a \in \Gamma(VTE, \tau_E, E)\) is a section of \(((h \circ \pi)^* F \oplus TE, \hat{\pi}, E)\). Moreover, it is easy to see that the sections \(\hat{\partial}_1, \cdots, \hat{\partial}_r\) are linearly independent. Now, we consider the vector subbundle \(((\rho, \eta)TE, \rho, \eta)\tau_E, E)\) of the vector bundle \(((h \circ \pi)^* F \oplus TE, \hat{\pi}, E)\), for which the \(\mathcal{F}(E)\)-module of sections is the \(\mathcal{F}(E)\)-submodule of

\[
\Gamma(((h \circ \pi)^* F \oplus TE, \hat{\pi}, E), +, \cdot),
\]

6
So, for any vector local \( (f, \hat{a}) \). The base sections \((\partial_\alpha, \hat{\partial}_\alpha)\) are called the natural \((\rho, \eta)\)-base. Now, we consider the vector bundles morphism \((\tilde{\rho}, \Id_E)\) from \(((\rho, \eta)\tau_E, E)\) to \((TE, \tau_E, E)\), where

\[
(\rho, \eta) TE \xrightarrow{\tilde{\rho}} TE,
\]

\[
(X^\alpha \partial_\alpha + \tilde{X}^a \tilde{\partial}_a)(u_x) \mapsto (X^\alpha (\rho_1^a \circ h \circ \pi) \partial_1 + \tilde{X}^a \tilde{\partial}_a(u_x)).
\]

Moreover, define the Lie bracket \([,]_{(\rho, \eta)TE}\) as follows

\[
[X^\alpha \partial_\alpha + \tilde{X}^a \tilde{\partial}_a, X^\beta \partial_\beta + \tilde{X}^b \tilde{\partial}_b]_{(\rho, \eta)TE} = [X^\alpha T_\alpha, X^\beta T_\beta]_{(h \circ \pi)^* F} \oplus [(X^\alpha (\rho_1^a \circ h \circ \pi) \partial_1 + \tilde{X}^a \tilde{\partial}_a, X^\beta (\rho_2^b \circ h \circ \pi) \partial_2 + \tilde{X}^b \tilde{\partial}_b)]_{TE}.
\]

Easily we obtain that \((\rho, \eta)\tau_E, E\) is a Lie algebroid structure for the vector bundle \(((\rho, \eta)\tau_E, (\rho, \eta)\tau_E, E)\) which is called the generalized tangent bundle.

### 3 Vertical and complete \((g, h)\)-lifts of sections of a vector bundle

In this section, we consider the diagram \((2)\) for the generalized Lie algebroid \(((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta))\). Also, we admit that \((g, h)\) is a vector bundles morphism locally invertible from \((E, \pi, M)\) to \((F, \nu, N)\) with components

\[
g^\alpha_b, \quad \alpha \in \{1, \ldots, n\}, \quad b \in \{1, \ldots, r\}.
\]

So, for any vector local \((m + r)\)-chart \((V, t_V)\) of \((E, \pi, M)\), there exist the real functions

\[
V \xrightarrow{\tilde{g}^\beta_a} \mathbb{R}, \quad \alpha \in \{1, \ldots, n\}, \quad b \in \{1, \ldots, r\},
\]

such that \(\tilde{g}^\alpha_b(x) \cdot g^\alpha_b(x) = \delta^\alpha_b\) and \(\tilde{g}^\beta_a(x) \cdot g^\beta_a(x) = \delta^\beta_a\), for any \(x \in V\).

Thus, we can discuss about vector bundles morphism \((g^{-1}, h^{-1})\) from \((F, \nu, N)\) to \((E, \pi, M)\) with components

\[
g^\beta_a \circ h^{-1}, \quad \alpha \in \{1, \ldots, n\}, \quad b \in \{1, \ldots, r\}.
\]

**Definition 7** If \(f \in \mathcal{F}(N)\) (respectively \(f \in \mathcal{F}(M)\)), then the real function \(f^\vee = f \circ h \circ \pi\) (respectively \(f^\vee = f \circ \pi\)) is called the vertical lift of the function \(f\).

It is remarkable that since

\[
(\Gamma(Th \circ \rho, h \circ \eta)\Gamma(g, h)(u^a s_a))(f) = (g^\alpha_b u^b) \circ h^{-1} \rho_1^a \frac{\partial (f \circ h)}{\partial x^i} \circ h^{-1} ,
\]

then using the above definition

\[
(\Gamma(Th \circ \rho, h \circ \eta)\Gamma(g, h)(u^a s_a))(f)^\vee = ((g^\alpha_b u^b \rho_1^a \circ h) \circ \pi) \partial_i (f \circ h \circ \pi). \quad (3)
\]
**Definition 8** If \( u = u^a s_a \) is a section of \((E, \pi, M)\), then we introduce the vertical lift of \( u \) as section of \( \Gamma(TE, \tau E, E) \) given by

\[
u^\nu = (u^b \circ \pi) \partial_a.
\]

If \( \{ s_a \} \) be a basis of sections of \( \Gamma(E, \pi, M) \), then using the above equation we have

\[s_a^\nu = \partial_a.
\]

Using the locally expression of \( u^\nu \) we can deduce

**Lemma 9** If \( u \) and \( v \) are sections of \( E \) and \( f \in \mathcal{F}(M) \), then

\[
\begin{align*}
(u + v)^\nu &= u^\nu + v^\nu, \\
(fu)^\nu &= f^\nu u^\nu, \\
u^\nu(f^\nu) &= 0.
\end{align*}
\]

For any \( z \in \Gamma(F, \nu, N) \), the \( \mathcal{F}(N) \)-multilinear application

\[
\Lambda(F, \nu, N) \xrightarrow{L_z} \Lambda(F, \nu, N),
\]

defined by

\[
L_z(f) = [\Gamma(Th \circ \rho, h \circ \eta)z](f), \quad \forall f \in \mathcal{F}(N),
\]

and

\[
L_z \theta(z_1, ..., z_q) = \Gamma(Th \circ \rho, h \circ \eta)z(\omega(z_1, ..., z_q)) \quad - \sum_{i=1}^{q} \theta((z_1, ..., [z, z_i]_{F,h}, ..., z_q)),
\]

for any \( \theta \in \Lambda^q(F, \nu, N) \) and \( z_1, ..., z_q \in \Gamma(F, \nu, N) \), will be called the covariant Lie derivative with respect to the section \( z \). Also for any \( u \in \Gamma(E, \pi, M) \), the \( \mathcal{F}(M) \)-multilinear application

\[
\Lambda(E, \pi, M) \xrightarrow{(g,h)L_u} \Lambda(E, \pi, M),
\]

defined by

\[
(g,h)L_u(f) = (g,h)^*\{\Gamma(Th \circ \rho, h \circ \eta)(\Gamma(g,h)u)(g^{-1}, h^{-1})^*f\}, \quad \forall f \in \mathcal{F}(M),
\]

and

\[
(g,h)L_u \omega(u_1, ..., u_q)
= (g,h)^*\{L_{\Gamma(g,h)u}(g^{-1}, h^{-1})^*\omega(\Gamma(g,h)u_1, ..., \Gamma(g,h)u_q)\}
= (g,h)^*\{\Gamma(Th \circ \rho, h \circ \eta)[\Gamma(g,h)u](g^{-1}, h^{-1})^*\omega(\Gamma(g,h)u_1, ..., \Gamma(g,h)u_q)\}
- (g,h)^*\{(g^{-1}, h^{-1})^*\omega(\Gamma(g,h)u_1, ..., \Gamma(g,h)u_q)\},
\]

for any \( \omega \in \Lambda^q(E, \pi, M) \) and \( u_1, ..., u_q \in \Gamma(E, \pi, M) \), will be called the covariant Lie \((g,h)\)-derivative with respect to the section \( u \).
Definition 10: Let $u \in \Gamma(TE, \tau_E, E)$ be a section of $(E, \pi, M)$, then there exists a unique vector field $u^c \in \Gamma(TE, \tau_E, E)$, the complete $(g, h)$-lift of $u$, satisfying the following conditions:

(i) $u^c$ is $(h \circ \pi)$-related with $\Gamma(T(Th \circ \rho, h \circ \eta)(\Gamma(g, h)u))$, i.e.,

$$T(h \circ \pi)(u^c) = \{\Gamma(T(Th \circ \rho, h \circ \eta)(\Gamma(g, h)u))\}(h \circ \pi(v_x)),$$

(ii) $u^c(\hat{\omega}) = (g, h)\Lo\omega, \quad \forall \omega \in \Lambda^1(E, \pi, M)$.

Proof. At first we let that there exists $u^c$ such that satisfies in (i) and (ii). Since $u^c$ is a vector field on $E$, then we can write it as follows:

$$u^c = A^i \partial_i + B^a \hat{\partial}_a,$$

where $A^i, B^a \in \mathcal{F}(E)$. We have

$$T(h \circ \pi)(\partial_i u^c)(f) = T\pi(\partial_i v_x)(f \circ h) = \partial_i v_x(f \circ h \circ \pi),$$

and

$$T(h \circ \pi)(\hat{\partial}_a u^c)(f) = T\pi(\hat{\partial}_a v_x)(f \circ h) = \hat{\partial}_a v_x(f \circ h \circ \pi) = 0.$$

From two above equations we obtain

$$T(h \circ \pi)(u^c) = A^i(v_x) \partial_i v_x(f \circ h \circ \pi).$$

On the other hand we have

$$\Gamma(T(Th \circ \rho, h \circ \eta)(\Gamma(g, h)u) \circ \pi(v_x)) = ((g^a_c \circ \pi)(u^c \circ \pi)(\rho^c_\alpha \circ h \circ \pi))(v_x) \partial_i v_x(f \circ h \circ \pi).$$

Condition (i) give us

$$A^i = (g^a_c u^c \rho^c_\alpha \circ h) \circ \pi.$$

Therefore we have

$$u^c = ((g^a_c u^c \rho^c_\alpha \circ h) \circ \pi) \partial_i + B^a \hat{\partial}_a.$$
Now, let \( \omega = \omega_b s^b \in \Lambda^1(E, \pi, M) \). Then we get

\[
u^c(\omega) = U^b((g_a^b u^c \rho_a^i \circ h) \circ \pi) \partial_i (\omega_b \circ \pi) + B^b(\omega_b \circ \pi).
\]

Now, let \( K^\gamma_a(u) = \Gamma g, h u, \Gamma g, h s_a |_{F, h} \). Then using (2) we get

\[
K^\gamma_a(u) = (g_a^b u^c \circ h \cdot \rho^\gamma_b \partial x^\gamma \circ h^{-1} - (g_a^b \circ h^{-1}) \rho^\gamma_b \partial x^\gamma \circ h^{-1}) + (g_a^b u^c) \circ h^{-1} L^\gamma_{\alpha \beta}(g_a^\beta \circ h^{-1}).
\]

On the other hand we have

\[
(g, h) \mathcal{L}_\omega (s_a) = (g, h)^* \{ \Gamma (Th \circ \rho, h \circ \eta)(\Gamma g, h u)(g^{-1}, h^{-1})^* \omega (\Gamma g, h s_a) \}
\]

\[
-(g, h)^* \{ (g^{-1}, h^{-1})^* \omega (K^\gamma_a(u) t^\gamma) \}
\]

\[
= g_a^b u^b (\rho^a_i \circ h) \frac{\partial x^a}{\partial x^b} - \gamma^b_a \omega_b K^\gamma_a(u) \circ h
\]

\[
= g_a^b u^b (\rho^a_i \circ h) \frac{\partial x^a}{\partial x^b} - \gamma^b_a \omega_b g_a^b u^c \circ h \frac{\partial x^a}{\partial x^b} \gamma^a_c \omega_c
\]

\[
- g_a^b (\rho^a_i \circ h) \frac{\partial x^a}{\partial x^b} \gamma^b_i \omega_b + g_a^b u^c (L^\gamma_{\alpha \beta} \circ h) g_a^\beta \gamma^b \omega_b.
\]

Thus we have

\[
(g, h) \mathcal{L}_\omega - U^a((g, h) \mathcal{L}_\omega (s_a)) \circ \pi
\]

\[
= U^a(g_a^b u^b (\rho^a_i \circ h) \frac{\partial x^a}{\partial x^b} - \gamma^b_a \omega_b K^\gamma_a(u) \circ h) \circ \pi
\]

\[
= U^a((g_a^b u^b (\rho^a_i \circ h) \frac{\partial x^a}{\partial x^b} - \gamma^b_a \omega_b g_a^b u^c \circ h \frac{\partial x^a}{\partial x^b} \gamma^a_c \omega_c) \circ \pi
\]

\[
+ (-g_a^b (\rho^a_i \circ h) \frac{\partial x^a}{\partial x^b} \gamma^b_i \omega_b + g_a^b u^c (L^\gamma_{\alpha \beta} \circ h) g_a^\beta \gamma^b \omega_b) \circ \pi.
\]

But condition (ii) gives us

\[
B^b(\omega_b \circ \pi) = -U^a(K^\gamma_a(u) \circ h \circ \pi)(\gamma^b_i \omega_b) \circ \pi.
\]

Since \( \omega \) is arbitrary, then we suppose that \( \omega = s^b \). Thus we have \( \omega_b = 1 \) and \( \omega_a = 0 \), for any \( a \neq b \). Therefore we obtain

\[
B^b = -U^a(K^\gamma_a(u) \circ h \circ \pi)(\gamma^b_i \omega_b) \circ \pi.
\]

So, for \( u^c \) we can obtain the following locally expression:

\[
u^c = (g_a^b u^c \rho_a^i \circ h) \circ \pi \partial_i - U^a(K^\gamma_a(u) \circ h \circ \pi)(\gamma^b_i \circ \pi) \partial_i.
\]

The above relation prove the existence and uniqueness of the complete lift. ■

**Definition 12** The complete \((g, h)\)-lift of a function \( f \in \mathcal{F}(N) \) into \( \mathcal{F}(E) \) is the function

\[
f^c : E \to \mathbb{R},
\]

defined by

\[
f^c|_{\pi^{-1}(V)} = U^a(g_a^b \circ \pi)(\rho_b^i \circ h \circ \pi) \partial_i (f \circ h \circ \pi)|_{\pi^{-1}(V)},
\]

where \((V, s_V)\) is an arbitrary vector local \((m + r)\)-chart.
Lemma 13 If $u$ is a section of $(E, \pi, M)$ and $f, f_1, f_2 \in \mathcal{F}(N)$, then

(i) $(f_1 + f_2)^c = f_1^c + f_2^c$,
(ii) $(f_1 f_2)^c = f_1^c f_2^c + f_1^c f_2^c$,
(iii) $u^\vee(f^c) = \{ \Gamma(Th \circ \rho, h \circ \eta)(\Gamma(g, h)u)(f) \}^\vee$.

Proof. We only prove (iii). Using (9), (11) and (12) we obtain

$$u^\vee(f^c) = (u^b \circ \pi)\mathcal{H}_h(U^b(g^a \circ \pi)(\rho^a \circ h \circ \pi)\partial_i(f \circ h \circ \pi)) = (g^a \circ \pi)(u^b \circ \pi)(\rho^a \circ h \circ \pi)\partial_i(f \circ h \circ \pi) = ((g^a \circ h^{-1})(u^b \circ h^{-1})(\rho^a)((\partial_i(f \circ h)) \circ h^{-1})) \circ h \circ \pi = \{ \Gamma(Th \circ \rho, h \circ \eta)(\Gamma(g, h)u)(f) \}^\vee.$$
Definition 15 The complete \((g,h)\)-lift \(u^C\) of a section \(u \in \Gamma(E,\pi,M)\) is the section of \(\langle (\rho,\eta)T\rangle E, (\rho,\eta)\pi_E, E \rangle\) given by

\[
 u^C = (g^c_α \circ \pi)(u^c \circ \pi)T_α + (g^c_α \circ \pi)(ρ^c_α \circ h \circ π)\partial_ι \tag{7}
\]

\[-U^α(K^α_1(u) \circ h \circ π)(0_{(h_0π^*)E} \oplus \hat{∂}_b)\]

\[= (g^c_α \circ π)(u^c \circ π)\partial_ι - U^α(K^α_1(u) \circ h \circ π)(\hat{g}^b_κ \circ π)\hat{∂}_b.\]

Using the above definition, we can obtain

\[\Gamma(\hat{ρ}, T_E)(u^c) = u^c.\]

In the particular case of Lie algebroids, \((g,\eta, h) = (Id_E, Id_M, Id_M)\), the complete lifts are given by the equality:

\[
u^c = \{(u^αρ^b_α \circ π)\partial_ι + y^b\{(ρ^b_α\partial_ι u^α + u^d L^b_α) \circ π\}\hat{∂}_b, \]

\[u^C = (u^α \circ π)\hat{∂}_b + y^j\{(ρ^j_β\partial_ι u^α + u^d L^j_α) \circ π\}\hat{∂}_b,\]

and in the classical case, \(ρ = Id_{T_M}\), the complete lifts are given by the equality:

\[u^C = (X^i \circ π)\partial_ι + y^j(\partial_ι X^i \circ π)\partial_ι = u^c.\]

Definition 16 If \(u = u^αs_α\) is a section of \((E,\pi,M)\), then we introduce the vertical \((g,h)\)-lift of \(u\) as section of \(\langle (\rho,\eta)T\rangle E, (\rho,\eta)\pi_E, E \rangle\) given by

\[u^V = 0_{(h_0π^*)E} \oplus u^V.\]

If \(u = u^ασ_α \in \Gamma(E,\pi,M)\), then in the locally expressions we get

\[u^V = 0_{(h_0π^*)E} \oplus (u^α \circ π)\hat{∂}_b = (u^α \circ π)(0_{(h_0π^*)E} \oplus \hat{∂}_b) = (\hat{∂}_b \circ π)\hat{∂}_b.\]

In particular, we have \(s^V_α = \hat{∂}_b.\)

Remark 17 Using the almost tangent \((g,h)\)-structure \(J_{(g,h)}\) given by

\[\Gamma((\rho,\eta)T\rangle E, (\rho,\eta)\pi_E, E) \xrightarrow{J_{(g,h)}} \Gamma((\rho,\eta)T\rangle E, (\rho,\eta)\pi_E, E),\]

\[Z^α\hat{∂}_α + Y^b\hat{∂}_b \longrightarrow (\hat{g}^b_κ \circ π)Z^α\hat{∂}_b,\]

it results that \(J_{(g,h)}(u^c) = u^V.\)

Theorem 18 The Lie brackets of vertical and complete \((g,h)\)-lifts satisfy the following equalities:

\[i) \ [u^V, v^V]_{(\rho,η)T\rangle E} = 0,\]

\[ii) \ [u^V, v^C]_{(\rho,η)T\rangle E} = \{Γ(g^{-1}, h^{-1})[Γ(g,h)u, Γ(g,h)v]_{F,h}\}^V,\]

\[iii) \ [u^C, v^C]_{(\rho,η)T\rangle E} = \{Γ(g^{-1}, h^{-1})[Γ(g,h)u, Γ(g,h)v]_{F,h}\}^C.\]
Proof. Direct calculation gives us

\[ [u^V, v^C]_{(\rho, \eta)TE} = -\{((u^a\hat{g}^b) \circ \pi)((g_\alpha^c v^d (\rho_\beta^a \circ h)) \circ \pi)\partial_i (g_\alpha^c),
\]

\[- (g_\alpha^a(\rho_\beta^c \circ h)) \circ \pi)\partial_i ((g_\gamma^c v^d \circ \pi) + ((g_\alpha^c v^d g_\alpha^d (L_{\alpha,\beta}^\gamma \circ h)) \circ \pi))
\]

\[ + ((g_\alpha^a v^d (\rho_\beta^a \circ h)) \circ \pi)\partial_i (u^b \circ \pi))\hat{h}. \tag{9} \]

On the other hand, we have

\[ [\Gamma(g, h)(u), \Gamma(g, h)(v)]_{F,h} = \{((g^\alpha_a u^a) \circ h^{-1})t_\alpha, ((g^\beta_b v^b) \circ h^{-1})t_\beta\}_{F,h}
\]

\[ = ((g^\alpha_a u^a) \circ h^{-1})\Gamma(Th \circ \rho, h \circ \eta)(t_\alpha)((g^\beta_b v^b) \circ h^{-1})t_\beta
\]

\[ - ((g^\beta_b v^b) \circ h^{-1})\Gamma(Th \circ \rho, h \circ \eta)(t_\beta)((g^\alpha_a u^a) \circ h^{-1})t_\alpha
\]

\[ + ((g^\alpha_a u^a g^\beta_b v^b) \circ h^{-1})L_{\alpha,\beta}^\gamma t_\gamma
\]

\[ = \{((g^\alpha_a u^a) \circ h^{-1})\rho_\alpha^i \partial (g_\alpha^a v^b) \circ h^{-1} - ((g^\beta_b v^b) \circ h^{-1})\rho_\beta^i \partial (g_\gamma^c u^a) \circ h^{-1}
\]

\[ + ((g^\alpha_a u^a g^\beta_b v^b) \circ h^{-1})L_{\alpha,\beta}^\gamma t_\gamma\}. \tag{9} \]

The above equation gives us

\[ \Gamma(g^{-1}, h^{-1})([\Gamma(g, h)(u), \Gamma(g, h)(v)]_{F,h}) = \hat{g}_\gamma^a g^\alpha_a u^a (\rho_\alpha^i \circ h) \partial (g_\gamma^c v^b) \circ h^{-1}
\]

\[ - g^\beta_b v^b (\rho_\beta^c \circ h) \partial (g_\gamma^c u^a) \circ h^{-1} + g^\alpha_a u^a g^\beta_b v^b (L_{\alpha,\beta}^\gamma \circ h) s_d. \]

Thus

\[ (\Gamma(g^{-1}, h^{-1})[\Gamma(g, h)(u), \Gamma(g, h)(v)]_{F,h})^V = (\hat{g}_\gamma^a \circ \pi)\{(v^b g_\alpha^c u^a (\rho_\alpha^i \circ h)) \circ \pi)\partial_i (g_\alpha^c \circ \pi)
\]

\[ + ((g^\alpha_a g^\beta_b (\rho_\beta^a \circ h)) \circ \pi)\partial_i (v^b \circ \pi) - ((u^a g^\beta_b v^b (\rho_\beta^a \circ h)) \circ \pi)\partial_i (g_\alpha^c \circ \pi)
\]

\[ - ((g^\alpha_a u^a (\rho_\beta^c \circ h)) \circ \pi)\partial_d (u^a \circ \pi) + ((g^\alpha_a u^a g^\beta_b v^b (L_{\alpha,\beta}^\gamma \circ h)) \circ \pi)\hat{d}. \tag{10} \]

From (9) and (10) we get (ii). Now we prove (iii). we have

\[ \{\Gamma(g^{-1}, h^{-1})[\Gamma(g, h)u, \Gamma(g, h)v]_{F,h}\}^C = \mathcal{A}^V \hat{\partial}_x + \mathcal{B}^V \hat{\partial}_r, \tag{11} \]

where

\[ \mathcal{A}^V = ((g^\alpha_a u^a (\rho_\beta^a \circ h)) \circ \pi)\partial_i ((g^\beta_b v^b \circ \pi) - ((g^\alpha_a v^d (\rho_\beta^a \circ h)) \circ \pi)\partial_i ((g_\gamma^c u^a) \circ \pi)
\]

\[ + ((g^\alpha_a u^a g^\beta_b v^b (L_{\alpha,\beta}^\gamma \circ h)) \circ \pi), \tag{12} \]

\[ \mathcal{B}^V = (g^\alpha_a u^a (\rho_\beta^a \circ h)) \circ \pi)\partial_i ((g^\beta_b v^b \circ \pi) - ((g^\alpha_a v^d (\rho_\beta^a \circ h)) \circ \pi)\partial_i ((g_\gamma^c u^a) \circ \pi)
\]

\[ + ((g^\alpha_a u^a g^\beta_b v^b (L_{\alpha,\beta}^\gamma \circ h)) \circ \pi), \tag{12} \]
Using (7) and direct calculation we get
\[
B^r = -U^d(g_γ^r \circ π) \left\{ (g_α^a u^a((ρ_α^i ρ_α^j) \circ h)) \circ π) \partial_i ((g_λ^b v^b) \circ π) \partial_j (g_γ^r \circ π) \right. \\
- (g_λ^b v^b((ρ_β^i ρ_β^j) \circ h)) \circ π) \partial_j ((g_α^a u^a) \circ π) \partial_i (g_γ^r \circ π) \\
+ (g_α^a u^a g_β^b ((L_α^β ρ_α^i) \circ h)) \circ π) \partial_j (g_γ^r \circ π) \\
- (g_λ^b ((ρ_α^i ρ_α^j) \circ h) \circ π) \partial_j ((g_α^a u^a) \circ π) \partial_i (g_γ^r \circ π) \\
- (g_λ^b (ρ_α^i ρ_γ^r) \circ h)) \circ π) \partial_j ((g_α^a g_β^b v^b) \circ π) \partial_i (g_γ^r u^a) \circ π) \\
+ (g_α^a (ρ_β^i ρ_γ^r) \circ h) \circ π) \partial_j ((g_α^a u^a) \circ π) \partial_i (g_γ^r \circ π) \\
- (g_λ^b (ρ_α^i g_β^b v^b) (L_α^β ρ_α^i) \circ h)) \circ π) \partial_j (g_γ^r \circ π) \\
- (g_α^a (ρ_γ^r ρ_γ^r) \circ h) \circ π) \partial_j (g_γ^r \circ π) \\
+ (g_α^a g_β^b ((L_α^β ρ_α^i) \circ h)) \circ π) \partial_j ((g_α^a u^a) \circ π) \\
+ (g_α^a g_β^b ((L_α^β ρ_α^i) \circ h)) \circ π) \partial_j ((g_α^a u^a) \circ π) \partial_i (g_γ^r \circ π) \\
\left. + (g_α^a g_β^b (L_α^β ρ_γ^r) \circ h)) \circ π) \partial_j ((g_α^a u^a) \circ π) \partial_i (g_γ^r \circ π) \right\}. \tag{13}
\]

Using (7) and direct calculation we get
\[
[u^C, v^C]_{(ρ, π) T E} = A^γ \partial_γ + (B^r + C^r) \partial_r, \tag{14}
\]
where

\[ C^r = -U^d(\bar{\partial}_r \circ \pi) \left\{ -((\bar{g}_x^a g_c^a u^c g_c^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) 
+ ((\bar{g}_x^b g_c^b v^c g_c^b u^c ((\rho_\mu^b \rho_\lambda^b) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) 
+ ((\bar{g}_x^c g_c^c v^c g_c^c u^c ((\rho_\mu^c \rho_\lambda^c) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) 
- ((\bar{g}_x^b g_d^b g_c^b u^c ((\rho_\mu^b \rho_\lambda^b) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) 
- ((\bar{g}_x^c g_d^c g_c^c u^c ((\rho_\mu^c \rho_\lambda^c) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) 
+ ((\bar{g}_x^d g_c^d v^c g_c^d u^c ((\rho_\mu^d \rho_\lambda^d) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) \right\} 
- U^d((g_c^a u^c g_c^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) 
+ U^d((g_c^a u^c g_c^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a v^c \circ \pi) 
- U^d((g_c^a u^c g_c^a v^c g_d^a \partial_i(\bar{\partial}_r \circ \pi)((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) 
+ U^d((g_c^a v^c g_d^a \partial_i(\bar{\partial}_r \circ \pi)((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) \partial_i(g_a^a \circ \pi). \quad (15) \]

On the other hand, we have

\[ g_d^a = \tilde{g}_x^a \tilde{g}_a^a \tilde{g}_d^a. \]

Derivative of the above expression with respect to \( j \), we get

\[ \partial_j(g_d^a) = -\partial_j(g_a^a) g_d^a \tilde{g}_d^a. \]

Using the above equation, we obtain

\[ -U^d((g_c^a u^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(\bar{\partial}_r \circ \pi) \partial_j(g_d^a \circ \pi) \]
\[ = U^d((g_c^a u^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(\bar{\partial}_r \circ \pi) \partial_j(g_a^a \circ \pi)(g_d^a \circ \pi)(g_a^a \circ \pi) \]
\[ = -U^d((g_c^a u^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(\bar{\partial}_r \circ \pi) \tilde{g}_a^a \partial_j(\tilde{g}_a \circ \pi)(\tilde{g}_d \circ \pi) \]
\[ = U^d((g_c^a u^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \tilde{g}_a^a \partial_i(\bar{\partial}_r \circ \pi) \partial_j(\tilde{g}_a \circ \pi)(\tilde{g}_a \circ \pi)(\tilde{g}_d \circ \pi)(\tilde{g}_d \circ \pi). \quad (16) \]

Similarly, we have

\[ U^d((g_a^a u^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(\bar{\partial}_r \circ \pi) \partial_j((g_c^a v^c \circ \pi) \partial_i(g_a^a \circ \pi). \]
\[ = -U^d((g_c^a v^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(\bar{\partial}_r \circ \pi) \partial_j((g_c^a v^c \circ \pi) \partial_i(g_a^a \circ \pi) \partial_i(g_a^a \circ \pi). \quad (17) \]

\[ U^d((g_a^a u^c \tilde{g}_x^a v^c g_a^a \tilde{g}_x^a \tilde{g}_d^a \tilde{g}_a^a \partial_i(g_a^a \circ \pi)((L_{\lambda \mu} \rho_\lambda^a) \circ h)) \circ \pi) \]
\[ = -U^d((g_a^a \tilde{g}_x^a \partial_i(g_a^a \circ \pi)((L_{\lambda \mu} \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(g_a^a \circ \pi), \quad (18) \]

\[ U^d(\tilde{g}_x^a \circ \pi)((g_c^a g_a^a u^c \tilde{g}_x^a v^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_j(g_d^a \circ \pi) \partial_i(g_a^a \circ \pi) \]
\[ = -U^d((g_a^a v^c \tilde{g}_x^a u^c ((\rho_\mu^a \rho_\lambda^a) \circ h)) \circ \pi) \partial_i(g_a^a \circ \pi) \partial_j(g_a^a \circ \pi). \quad (19) \]
If we define $U^d((g^\alpha v^\gamma g^\lambda ((\rho^\beta \rho^\delta_\lambda_\beta) \circ h)) \circ \pi) \partial_i((g^\alpha u^\nu) \circ \pi)$

$$= -U^d((\check{g}^\alpha \circ \pi)((\check{g}^\alpha g^\nu g^\mu ((\rho^\beta \rho^\delta_\lambda_\beta) \circ h)) \circ \pi) \partial_j((g^\alpha u^\nu) \circ \pi) \partial_i((g^\alpha \circ \pi), \quad (20)$$

and

$$U^d((g^\alpha v^\gamma g^\lambda h^\alpha g^\mu \rho^\lambda_\mu \cdot \cdot \cdot ) \circ \pi) = -U^d((\check{g}^\alpha \circ \pi)((\check{g}^\alpha g^\nu g^\mu v^\nu ((\rho^\beta \rho^\delta_\lambda_\beta) \circ h)) \circ \pi) \partial_j((g^\alpha \circ \pi). \quad (21)$$

Setting (17)-(21) in (15) we deduce that $C^r = 0$. This equation together with (11) and (14) give us (iii). ■

4 The generalized tangent bundle of a dual vector bundle

We consider the following diagrams:

$$
\begin{array}{cccc}
\hat{E} & \rightarrow & (F, [\cdot], F, h) & \rightarrow \rho \rightarrow TM & \rightarrow T h \rightarrow (T N, [\cdot], T N) \\
\downarrow \pi & & \downarrow \nu & \downarrow \tau M & \downarrow \tau N \\
M & \rightarrow h \rightarrow N & \rightarrow \eta \rightarrow M & \rightarrow h \rightarrow N
\end{array}
$$

where $((F, \nu, N), [\cdot], F, h, (\rho, \eta))$ is a generalized lie algebroid, $(E, \pi, M)$ is a vector bundle and $(\hat{g}, h)$ is a vector bundle morphism from $(\hat{E}, \hat{\pi}, M)$ to $(F, \nu, N)$ with components $g^{ab}, \alpha \in \{1, 2, \ldots, n\}$ and $b \in \{1, 2, \ldots, r\}$.

Setting $(x^i, p_a)$ as the canonical local coordinates on $(\hat{E}, \hat{\pi}, M)$, where $i \in 1, \cdots, m$, $a \in 1, \cdots, r$, and $(x^i, p_a) \rightarrow (x^i, p_a)$, then the coordinates $p_a$ change to $p_{a'}$ according to the rule

$p_{a'} = M^a_{a'} p_a$.

If $v = v^a t_a$ is a section of $(F, \nu, N)$, then we define its corresponding section $X = X^a T_a$ in the pull-back vector bundle $((h \circ \hat{\pi})^* F, (h \circ \hat{\pi})^* \nu, \hat{E})$ given by

$X(\hat{v}_x) = ((g^{ab} \circ h^{-1})(u_b \circ h^{-1})t_a)((h \circ \hat{\pi}(\hat{v}_x))g^{ab}(x)u_b(x)t_a(h(x)), \forall \hat{v}_x \in \hat{E}$.

Let $(\hat{\partial_1}, \hat{\partial_2})$ be the base sections for the Lie $\mathcal{F}(\hat{E})$-algebra

$$(\Gamma(T E, \tau, \hat{E}), +, [\cdot], [\cdot])_{\mathcal{F}(\hat{E})}.$$ 

If we define

$$
\begin{array}{c}
(h \circ \hat{\pi})^* F \xrightarrow{(h \circ \hat{\pi})^* F} T \hat{E}, \\
X^a T_a(\hat{v}_x) \quad \rightarrow \quad (g^{ab} \circ \hat{\pi})(u_b \circ \hat{\pi})(\rho^a \circ h \circ \hat{\pi}) \partial_i(\hat{v}_x),
\end{array}
$$

16
Moreover, we define the Lie bracket $[[\vec{\rho}, \vec{\eta}]]_{(h \circ \vec{\pi})^*F, (h \circ \vec{\pi})^*\nu, \vec{E}}$, which is called the pull-back Lie algebra of the generalized Lie algebroid $((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta))$.

Now, we consider the vector subbundle $((\rho, \eta)T\vec{E}, (\rho, \eta)\tau_{\vec{E}}, \vec{E})$ of the vector bundle $((h \circ \vec{\pi})^*F \oplus T\vec{E}, \vec{\pi}, \vec{E})$, for which the $\mathcal{F}(\vec{E})$-module of sections is the $\mathcal{F}(\vec{E})$-submodule of $(\Gamma((h \circ \pi)^*F \oplus T\vec{E}, \vec{\pi}, \vec{E}), +, \cdot)$, generated by the set of sections $(\tilde{\partial}_a, \tilde{\partial}^a)$, where

\[
\tilde{\partial}_a = T_a \dot{\rho} \circ h \circ \vec{\pi}, \quad \tilde{\partial}^a = 0_{(h \circ \vec{\pi})^*F} \oplus \dot{\rho}^a.
\]

The base sections $(\dot{\partial}_a, \dot{\partial}^a)$ are called the natural $(\rho, \eta)$-base. Now consider the vector bundles morphism $(\dot{\rho}, Id_{\vec{E}})$ from $((\rho, \eta)T\vec{E}, (\rho, \eta)\tau_{\vec{E}}, \vec{E})$ to $(T\vec{E}, \tau_{\vec{E}}, \vec{E})$, where

\[
(\rho, \eta)T\vec{E} \xrightarrow{\tilde{\rho}} T\vec{E},
\]

\[
(X^a \dot{\partial}_a + X^\alpha \dot{\partial}^\alpha)_{\vec{u}_x} \mapsto (X^a (\dot{\rho}^b \circ h \circ \vec{\pi}) \partial_i + \dot{X}_a \dot{\partial}^a)_{\vec{u}_x}.
\]

Moreover, we define the Lie bracket $[\cdot, \cdot]_{(\rho, \eta)T\vec{E}}$ as follows

\[
[X_1^a \dot{\partial}^a + X_2^\alpha \dot{\partial}^\alpha, X_3^b \dot{\partial}_b + \dot{X}_3^a \dot{\partial}^a]_{(\rho, \eta)T\vec{E}} = [X_1^a T_a + X_2^\alpha T^\alpha_{1 \beta}, X_3^b (\dot{\rho}^b \circ h \circ \vec{\pi}) \partial_j + \dot{X}_3^a \dot{\partial}^a]_{T\vec{E}}.
\]
Easily we obtain that \(([,\cdot]_{(\rho,\eta)^*E}(\hat{\rho},\text{Id}_E))\) is a Lie algebroid structure for the vector bundle \((\rho, \eta)^*T_E, (\rho, \eta)^*E)\) which is called the generalized tangent bundle.

5 Vertical and complete \((\hat{g},\hat{h})\)-lifts of sections of a dual vector bundle

In this section, we consider the following diagrams

\[
\begin{array}{cccc}
\hat{E} & \xrightarrow{\hat{g}} & (F, [,], F.h) & \xrightarrow{\rho} & TM & \xrightarrow{\tau}\tau_E & (TN, [,], TN) \\
\downarrow \pi & & \downarrow \nu & & \downarrow \tau & & \downarrow \tau_N \\
M & \xrightarrow{h} & N & \xrightarrow{\eta} & M & \xrightarrow{h} & N
\end{array}
\]

where \(((F,\nu,N),[,],F.h,\rho,\eta))\) is a generalized Lie algebroid. We admit that \((\hat{g},\hat{h})\) is a vector bundle morphism locally invertible from \((\hat{E},\pi,M)\) to \((F,\nu,N)\) with components \(g^{\alpha b}, \alpha \in \{1,\cdots,n\}, b \in \{1,\cdots,r\}\).

So, for any vector local \((m+r)\)-chart \((V,t_V)\) of \((\hat{E},\pi,M)\), there exists the real functions \(\tilde{g}_{\alpha b} : V \rightarrow \mathbb{R}, \alpha \in \{1,\cdots,n\}, b \in \{1,\cdots,r\}\), such that

\[
\tilde{g}_{\alpha b}(x) \cdot g^{\alpha a}(x) = \delta^a_b, \quad g^{\alpha a}(x)\tilde{g}_{a\beta}(x) = \delta^\alpha_\beta,
\]

for any \(x \in V\). So, we can discuss about vector bundles morphism \((\tilde{g}^{-1},\tilde{h}^{-1})\) from \((F,\nu,N)\) to \((\hat{E},\pi,M)\) with components \(\tilde{g}_{\alpha b},\alpha \in \{1,\cdots,n\}, b \in \{1,\cdots,r\}\).

**Definition 19** If \(f \in \mathcal{F}(N)\) (respectively \(f \in \mathcal{F}(M)\)), then the real function \(f^\lor = f \circ h \circ \pi\) (respectively \(f^\lor = f \circ \pi\)) is called the vertical lift of the function \(f\).

**Remark 20** Since

\[
(\Gamma(Th \circ \rho, h \circ \eta))\Gamma(\hat{g},\hat{h})(u_{a}s^a))(f) = (g^{\alpha b}u_b) \circ h^{-1}\rho^i_\alpha \frac{\partial (f \circ \pi)}{\partial x^i} \circ h^{-1},
\]

then using the above definition we obtain

\[
(\Gamma(Th \circ \rho, h \circ \eta))\Gamma(\hat{g},\hat{h})(u_{a}s^a)(f))^\lor = ((g^{\alpha b}u_b)^i_\alpha \circ h \circ \pi)^\lor \partial_i(f \circ h \circ \pi).
\]
Definition 21 If \( \hat{u} = u_{\alpha} s^\alpha \) is a section of \((E, \pi, M)\), then we introduce the vertical lift of \( \hat{u} \) as a section of \((TE, \tau, \hat{E})\) given by,

\[
\hat{u}^\vee = (u_{\alpha} \circ \pi) \hat{\partial}^\alpha.
\]

If \( \{s^\alpha\} \) be a basis of sections of \( \Gamma(E, \pi, M) \), then using the above equation we have

\[
(s^\alpha)^\vee = \hat{\partial}^\alpha.
\]

Using the locally expression of \( \hat{u}^\vee \) we can deduce

Lemma 22 If \( \hat{u} \) and \( \hat{v} \) are sections of \((E, \pi, M)\) and \( f \in \mathcal{F}(M) \), then

\[
(\hat{u} + \hat{v})^\vee = \hat{u}^\vee + \hat{v}^\vee, \quad (f \hat{u})^\vee = f^\vee \hat{u}^\vee, \quad \hat{u}^\vee (f^\vee) = 0.
\]

Definition 23 For any \( \hat{u} \in \Gamma(E, \pi, M) \), the \( \mathcal{F}(M) \)-multilinear application

\[
\Lambda(E, \pi, M) \xrightarrow{(g, h) L_u} \Lambda(E, \pi, M),
\]

defined by

\[
(g, h) L_u^*(f) = (g, h)^* \{ \Gamma(T h \circ \rho, h \circ \eta)[\Gamma(\hat{g}, \hat{h}) \hat{u}] (g^{-1}, h^{-1})^* f \}, \quad \forall f \in \mathcal{F}(M),
\]

and

\[
(g, h) L_u^*(u_1, \ldots, u_q) = (g, h)^* \{ \mathcal{L}_{(g, h) u} (g^{-1}, h^{-1})^* \omega(\Gamma(\hat{g}, \hat{h}) \hat{u}_1, \ldots, \Gamma(\hat{g}, \hat{h}) \hat{u}_q) \}
\]

\[
= (g, h)^* \{ \Gamma(T h \circ \rho, h \circ \eta)[\Gamma(\hat{g}, \hat{h}) \hat{u}] (g^{-1}, h^{-1})^* \omega(\Gamma(\hat{g}, \hat{h}) \hat{u}_1, \ldots, \Gamma(\hat{g}, \hat{h}) \hat{u}_q) \}
\]

\[
- (g, h)^* \{ (g^{-1}, h^{-1})^* \omega(\Gamma(\hat{g}, \hat{h}) \hat{u}_1, \ldots, \Gamma(\hat{g}, \hat{h}) \hat{u}_q), \Gamma(\hat{g}, \hat{h}) \hat{u}, \Gamma(\hat{g}, \hat{h}) \hat{u}_i \}_{F,h}, \ldots, \Gamma(\hat{g}, \hat{h}) \hat{u}_q) \},
\]

for any \( \omega \in \Lambda^q (E, \pi, M) \) and \( \hat{u}_1, \ldots, \hat{u}_q \in \Gamma(E, \pi, M) \), will be called the covariant Lie \( (g, h) \)-derivative with respect to the section \( \hat{u} \).

Definition 24 For any \( a = 1, \ldots, r \), we consider the real function \( U_a \) on \( E \) such that

\[
U_a \big|_{\pi^{-1}(V)} (\hat{u}_x) = p_a,
\]

where the real numbers \( p_1, \ldots, p_r \) are the fibre components of the point \( \hat{u}_x \) in the arbitrary vector local \((m + r)\)-chart \((V, s_V)\).

Remark 25 Using the above definition we have \( \hat{\partial}^b(U_a) = \delta_a^b \) and \( \partial_i(U_a) = 0 \), where \( a \in \{1, \ldots, n\}, b \in \{1, \ldots, r\} \).
Definition 26 If $\hat{\omega} = \omega^a s_a \in \Lambda^1 (\hat{E}, \hat{\tau}, \hat{M})$, then we consider the real function $\hat{\omega}$ defined by
\[
\hat{\omega}|_{\pi^{-1}(V)} = U_a(\omega^a \circ \hat{\pi})|_{\pi^{-1}(V)},
\]
where $(V, s_V)$ is an arbitrary vector local $(m + r)$-chart.

Theorem 27 Let $\hat{\bar{u}}$ be a section of $(\hat{E}, \hat{\tau}, \hat{M})$. Then there exists a unique vector field $\hat{\bar{u}}^c \in \Gamma(\hat{T}\hat{E}, \tau_{\hat{E}}, \hat{E})$, the complete $(\hat{g}, \hat{h})$-lift of $\hat{\bar{u}}$, satisfying the following conditions:
i) $\hat{\bar{u}}^c$ is $(h \circ \hat{\pi})$-related, i.e.,
\[
T(h \circ \hat{\pi})(\hat{\bar{u}}^c) = \{\Gamma(\hat{T}h \circ \rho, h \circ \eta)(\Gamma(\hat{g}, \hat{h})((\hat{\bar{u}})^c))(h \circ \hat{\pi}(\hat{\bar{u}}))\},
\]
ii) $\hat{\bar{u}}^c(\hat{\omega}) = (\hat{\bar{g}}, \hat{\bar{h}})\hat{\Lambda}_u \hat{\omega}$, for any $\hat{\omega} \in \Lambda^1 (\hat{E}, \hat{\tau}, \hat{M})$.

Proof. Similar to the proof of Theorem [11] we obtain the following locally expression for $\hat{\bar{u}}$ that show the existence and uniqueness of it.
\[
\hat{\bar{u}}^c = ((g^a e_\rho \circ h) \circ \hat{\pi})^e - U_a(K^{\gamma a}(\hat{u}) \circ h \circ \hat{\pi})(\hat{g}_{\beta} \circ \hat{\pi})^e, \]
where
\[
K^{\gamma a}(\hat{u}) = (g^{a e} e_\rho \partial (g^{\gamma a}) - g^{a e} \rho_i \partial (g^{\gamma e} u_c) + g^{a e} u_c L^{\gamma}_{\alpha \beta} g^{\beta a} \circ h^{-1}.
\]

Definition 28 The complete $(\hat{g}, \hat{h})$-lift of a function $f \in \mathcal{F}(N)$ into $\mathcal{F}(\hat{E})$ is the function
\[
f^c : \hat{E} \longrightarrow \mathbb{R},
\]
defined by
\[
f^c|_{\pi^{-1}(V)} = U_a(g^{a e} \circ \hat{\pi})(\rho_i \circ h \circ \hat{\pi})\partial_i(f \circ h \circ \hat{\pi})|_{\pi^{-1}(V)},
\]
where $(V, s_V)$ is an arbitrary vector local $(m + r)$-chart.

Similar to the Lemmas [13] and [14], we have

Lemma 29 If $\hat{\bar{u}}$ is a section of $(\hat{E}, \hat{\tau}, \hat{M})$ and $f, f_1, f_2 \in \mathcal{F}(N)$, then
\[
(i) (f_1 + f_2)^c = f_1^c + f_2^c,
(ii) (f_1 f_2)^c = f_1^c f_2^c + f_1^c f_2^c,
(iii) \hat{\bar{u}}^c(f^c) = \{\Gamma(\hat{T}h \circ \rho, h \circ \eta)(\Gamma(\hat{g}, \hat{h})f)\},
(iv) \hat{\bar{u}}^c(f^c) = \{\Gamma(\hat{T}h \circ \rho, h \circ \eta)(\Gamma(\hat{g}, \hat{h})f)\}^c.
\]
Definition 30 The complete \((\hat{g}, h)\)-lift \(\hat{u}^C\) of a section \(\hat{u} \in \Gamma(E, \pi, M)\) is the section of \(((\rho, \eta)T_E, (\rho, \eta)\tau_E, E)\) given by
\[
\hat{u}^C = ((g^\alpha \circ \pi)(u_e \circ \pi))\tilde{\partial}_\alpha - U_a(K^\gamma_a(\hat{u}) \circ h \circ \pi)(\hat{g}_\gamma \circ \pi)\hat{\partial}^b.
\] (22)

It is easy to check that \(\Gamma(\hat{\rho}, Id_E)(\hat{u}^C) = \hat{u}^c\).

Definition 31 If \(\hat{u} = u^a e_a\) be a section of \((\hat{\rho}, \eta), M)\), then we introduce the vertical lift of \(\hat{u}\) as section of \(((\rho, \eta)T\hat{E}, (\rho, \eta)\tau\hat{E}, \hat{E})\) given by
\[
\hat{u}^V = 0_{(ho\hat{\rho})T\hat{E}} \oplus \hat{u}^V.
\]
If \(u = u^a e_a \in \Gamma(E, \pi, M)\), then in the locally expressions we get
\[
\hat{u}^V = (u^a \circ \pi)\hat{\partial}^a,
\] (23)
which gives us \((s^a)^V = \hat{\partial}^a\).

Remark 32 Using the almost tangent \((\hat{g}, h)\)-structure \(\hat{J}(\hat{g}, h)\) given by
\[
\Gamma((\rho, \eta)T\hat{E}, (\rho, \eta)\tau\hat{E}, \hat{E}) \xrightarrow{\hat{J}(\hat{g}, h)} \Gamma((\rho, \eta)T\hat{E}, (\rho, \eta)\tau\hat{E}, \hat{E})
\]
\[
Z^\alpha \hat{\partial}_\alpha + Y^b \hat{\partial}_b \mapsto (\hat{g}_\alpha \circ \pi)Z^\alpha \hat{\partial}_b
\]
it results that \(\hat{J}(\hat{g}, h)(\hat{u}^C) = \hat{u}^V\).

Similar to Theorem 18 we can deduce the following

Theorem 33 The Lie brackets of generalized vertical lifts and generalized complete \((\hat{g}, h)\)-lifts satisfy the following equalities:

i) \(\left[u^V, \hat{v}^V\right]_{(\rho, \eta)T\hat{E}} = 0\),

ii) \(\left[u^V, \hat{v}^C\right]_{(\rho, \eta)T\hat{E}} = \{\Gamma(\hat{g}^{-1}, h^{-1})[\Gamma(\hat{g}, h)u, \Gamma(\hat{g}, h)v]_{F,h}\}^V\),

iii) \(\left[\hat{u}^C, \hat{v}^C\right]_{(\rho, \eta)T\hat{E}} = \{\Gamma(\hat{g}^{-1}, h^{-1})[\Gamma(\hat{g}, h)u, \Gamma(\hat{g}, h)v]_{F,h}\}^C\).

6 Legendre transformation

Definition 34 A Lagrange fundamental function on the vector bundle \((E, \pi, M)\) is a function \(E \xrightarrow{L} \mathbb{R}\) which satisfies the following conditions:

L1. \(L \circ u \in C^\infty(M)\), for any \(u \in \Gamma(E, \pi, M) \setminus \{0\}\);

L2. \(L \circ 0 \in C^0(M)\), where 0 means the null section of \((E, \pi, M)\).
Remark 35 If \((U, s_U)\) is a local vector \((m + r)\)-chart, then we obtain the following real functions defined on \(\pi^{-1}(U)\):

\[
L_i = \frac{\partial L}{\partial x^i}, \quad L_{ib} = \frac{\partial^2 L}{\partial x^i \partial y^b}, \quad L_a = \frac{\partial L}{\partial y^a}, \quad L_{ab} = \frac{\partial^2 L}{\partial y^a \partial y^b}.
\]

Definition 36 If \(L\) is a Lagrange fundamental function such that

\[
\text{rank} \|L_{ab}(u_x)\| = r,
\]

for any \(u_x \in \pi^{-1}(U) \setminus \{0_x\}\), then we will say that the Lagrange fundamental function \(L\) is regular and we obtain the real functions \(\tilde{L}_{ab}\) locally defined by

\[
\pi^{-1}(U) \xrightarrow{\tilde{L}_{ab}} \mathbb{R}, \quad u_x \mapsto \tilde{L}_{ab}(u_x),
\]

where \(\|\tilde{L}_{ab}(u_x)\| = \|L_{ba}(u_x)\|^{-1}\), for any \(u_x \in \pi^{-1}(U) \setminus \{0_x\}\).

Definition 37 If \(L\) is a Lagrange fundamental function, then we build the Legendre bundles morphism

\[
E \xrightarrow{\varphi_L} \dot{E}, \quad \pi \downarrow \downarrow \dot{\pi}, \quad M \xrightarrow{Id_M} M
\]

locally defined

\[
\pi^{-1}(U) \xrightarrow{\varphi_L} \pi^{-1}(U), \quad u_x = u^a(x)s_a(x) \mapsto u^a(x)L_{ab}(u_x)s^b(x),
\]

for any vector local \((m + r)\)-charts \((U, s_U)\) and \((U, \dot{s}_U)\) of \((E, \pi, M)\) and \((\dot{E}, \dot{\pi}, M)\) respectively.

Using the above definition, we deduce that if \(u = u^a s_a\) belongs to \(\Gamma(E, \pi, M)\), then we obtain its Legendre transformation

\[
\Gamma(\varphi_L, Id_M)(u) = (u^a(L_{ab} \circ u))s^b,
\]

belongs to \(\Gamma(\dot{E}, \dot{\pi}, M)\).

Definition 38 If \(L\) is a Lagrange fundamental function positively homogenous of degree two, namely

\[
F_1. \quad \text{\(L\) is positively 2-homogenous on the fibres of vector bundle \((E, \pi, M)\);} \quad F_2. \quad \text{\(\|L_{ab}(u_x)\|\) is positively define for any} \ u_x \in \pi^{-1}(U) \setminus \{0_x\}, \text{\(then \(L\) will be called Finsler fundamental function.}
\]
Proposition 39 If \( L \) is a Finsler fundamental function on the vector bundle \((E, \pi, M)\), then

\[
\varphi_L(u_x) = L_b(u_x) s^b(x), \quad \forall u_x \in E.
\]

Proof. From \((24)\) we have \(\varphi_L(u_x) = u^a(x)L_{ab}(u_x)s^b(x)\). But, the Finsler fundamental function \(L\) satisfies

\[
u^a(x)L_{ab}(u_x) = L_b,
\]

because \(L\) is positively 2-homogenous on the fibres of \((E, \pi, M)\). This completes the proof. \(\square\)

Definition 40 A Hamilton fundamental function on the dual vector bundle \((\mathring{E}, \mathring{\pi}, M)\) is a function \(\mathring{E} \rightarrow \mathbb{R}\) which satisfies the following conditions:

- \(H_1\): \(H \circ \mathring{u} \in C^\infty(M)\), for any \(\mathring{u} \in \Gamma(\mathring{E}, \mathring{\pi}, M) \setminus \{0\}\);
- \(H_2\): \(H \circ 0 \in C^0(M)\), where \(0\) means the null section of \((\mathring{E}, \mathring{\pi}, M)\).

Remark 41 If \((U, \mathring{s}_U)\) is a local vector \((m + r)\)-chart, then we obtain the following real functions defined on \(\pi^{-1}(U)\):

\[
H_i = \frac{\partial H}{\partial x^i}, \quad H^b_i = \frac{\partial^2 H}{\partial x^i \partial p^b},
\]

\[
H^a = \frac{\partial H}{\partial p^a}, \quad H^{ab} = \frac{\partial^2 H}{\partial p^a \partial p^b}.
\]

Definition 42 If \(H\) is a Hamilton fundamental function such that \(\text{rank} \|H^{ab}(\mathring{u}_x)\| = r\), for any \(\mathring{u}_x \in \pi^{-1}(U) \setminus \{0_x\}\), then we will say that the Hamilton fundamental function \(H\) is regular and we obtain the real functions \(\mathring{H}_{ab}\) locally defined by

\[
\begin{align*}
\pi^{-1}(U) & \xrightarrow{\mathring{s}_U} \mathring{E}, \\
\mathring{u}_x & \mapsto \mathring{H}_{ab}(\mathring{u}_x),
\end{align*}
\]

where \(\|\mathring{H}_{ab}(\mathring{u}_x)\| = \|H^{ab}(\mathring{u}_x)\|^{-1}\), for any \(\mathring{u}_x \in \pi^{-1}(U) \setminus \{0_x\}\).

Definition 43 If \(H\) is a Hamilton fundamental function on the vector bundle \((\mathring{E}, \mathring{\pi}, M)\), then we build the Legendre bundles morphism

\[
\begin{align*}
\mathring{E} & \xrightarrow{\varphi_H} E, \\
\pi & \downarrow \downarrow \pi, \\
\mathring{M} & \xrightarrow{Id_M} M.
\end{align*}
\]
where \( \varphi_H \) is locally defined

\[
\pi^{-1}(U) \xrightarrow{\varphi_H} \pi^{-1}(U)
\]

for any vector local \((m+r)\)-chart \((U, s_U)\) of \((E, \pi, M)\) and for any vector local \((m+r)\)-chart \((U, s_U)\) of \((E, \pi, M)\).

Using the above definition, we deduce that if \( \ast u = u_a s^a \) belongs to \( \Gamma(\ast E, \ast \pi, M) \), then we obtain its Legendre transformation

\[
\Gamma(\varphi_H, Id_M)(\ast u) = (u_a (H^{ab} \circ u)) s_b,
\]

belongs to \( \Gamma(E, \pi, M) \).

**Definition 44** If \( H \) is Hamilton fundamental function positively homogeneous of degree two, namely

\begin{itemize}
  \item \( C_1 \). \( H \) is positively 2-homogeneous on the fibres of vector bundle \((\ast E, \ast \pi, M)\);
  \item \( C_2 \). For any vector local \((m+r)\)-chart \((U, s_U)\) of \((\ast E, \ast \pi, M)\), the hessian:
    \[
    \| H^{ab}(\ast u_x) \|
    \]
    is positively define for any \( \ast u_x \in \ast \pi^{-1}(U) \setminus \{0_x\} \), then \( H \) will be called Cartan fundamental function.
\end{itemize}

Similar to proposition (39), we have the following

**Proposition 45** If \( H \) is a Cartan fundamental function, then

\[
\varphi_H(\ast u_x) = H^b(\ast u_x) s_b(x), \forall \ast u_x \in \ast E.
\]

**Theorem 46** If \( L \) is a Lagrange fundamental function on the vector bundle \((E, \pi, M)\) and \( H \) is a Hamiltonian on the dual vector bundle \((\ast E, \ast \pi, M)\), then:

\begin{itemize}
  \item i) \( \varphi_H \circ \varphi_L = Id_{\pi^{-1}(U)} \) if and only if \( L \) is regular and \( \tilde{H}^{ab} = H^{ab} \circ \varphi_L \);
  \item ii) \( \varphi_L \circ \varphi_H = Id_{\ast \pi^{-1}(U)} \) if and only if \( H \) is regular and \( \tilde{H}_{ab} = L_{ab} \circ \varphi_H \).
\end{itemize}

**Proof.** Using definition (37) and (43), we deduce that

\[
\varphi_H \circ \varphi_L (u_x) = \varphi_H \left( u^a(x) L_{ab}(u_x) s^b(x) \right) = u^a(x) L_{ab}(u_x) H^{bc}(\varphi_L(u_x)) s_c(x) = Id_{\pi^{-1}(U)}(u_x),
\]

if and only if

\[
L_{ab}(u_x) H^{bc}(\varphi_L(u_x)) = \delta_c^a(u_x),
\]

for any \( u_x \in \pi^{-1}(U) \). Thus we have (i). Similar, we can prove (ii).
Definition 47 If \( L \) is a Lagrange fundamental function on the vector bundle \((E, \pi, M)\), then the Hamilton fundamental function \( H \), locally defined by

\[
\ast \pi^{-1}(U) \xrightarrow{H} \mathbb{R}, \quad u_x = u_a(x)s^a(x) \mapsto u_a(x)u^a(x) - L(u_x),
\]

for any vector local \((m+r)\)-chart \((U, s_U)\) of \((E, \pi, M)\), where \(u^a(x), a \in \{1, \cdots, r\}\), are the components of the solution of the system of differentiable equations

\[
\begin{align*}
  u_1(x) &= u^a(x)L_{a1}(u_x), \\
  \vdots &= \vdots, \\
  u_r(x) &= u^a(x)L_{ar}(u_x),
\end{align*}
\]

will be called the Legendre transformation of the Lagrangian \( L \).

It is remarkable that in the general case, if \( L \) is a Lagrange fundamental function on the vector bundle \((E, \pi, M)\) and \( H \) is its Legendre transformation, then \( H \circ \varphi_L \neq L \), but in particular, if \( L \) is a Finsler fundamental function on the vector bundle \((E, \pi, M)\) and \( H \) is its Legendre transformation, then \( H \circ \varphi_L = L \).

Definition 48 If \( H \) is a Hamilton fundamental function on the dual vector bundle \((\ast E, \ast \pi, M)\), then the Lagrange fundamental function \( L \), locally defined by

\[
\pi^{-1}(U) \xrightarrow{L} \mathbb{R}, \quad u_x = u^a(x)s^a(x) \mapsto u_a(x)u^a(x) - H(u_x),
\]

for any vector local \((m+r)\)-chart \((U, s_U)\) of \((E, \pi, M)\), where \((u_a(x), a \in \{1, \cdots, r\})\) are the components of the solution of the system of differentiable equations

\[
\begin{align*}
  u^1(x) &= u_a(x)H_{a1}(u_x) \\
  \vdots &= \vdots, \\
  u^r(x) &= u_a(x)H_{ar}(u_x)
\end{align*}
\]

will be called the Legendre transformation of the Hamiltonian \( H \).

In general, if \( H \) is a Hamilton fundamental function on the vector bundle \((\ast E, \ast \pi, M)\) and \( L \) is its Legendre transformation, then \( L \circ \varphi_H \neq H \), but in particular, if \( H \) is a Cartan fundamental function on the vector bundle \((\ast E, \ast \pi, M)\) and \( L \) is its Legendre transformation, then \( L \circ \varphi_H = H \).

Remark 49 The Hamilton fundamental function \( H \) is the Legendre transformation of the Lagrange fundamental function \( L \) if and only if the Lagrange fundamental function \( L \) is the Legendre transformation of the Hamilton fundamental function \( H \).
7 Duality between vertical and complete lifts

Let $L$ be a Lagrangian on the vector bundle $(E, \pi, M)$ and let $H$ be its Legendre transformation.

Using the Legendre bundles morphism $(\varphi_L, Id_M)$, we build the vector bundles morphism $((\rho, \eta) T\varphi_L, \varphi_L)$ given by the diagram

$$
\begin{array}{ccc}
(p, \eta) \tau_E & \xrightarrow{(p, \eta) T\varphi_L} & (p, \eta) T E^* \\
\downarrow (p, \eta) T\tau_E & & \downarrow (p, \eta) T\tau_E^* \\
E & \xrightarrow{\varphi_L} & E^*
\end{array}
$$

such that

$$
\Gamma((p, \eta) T\varphi_L, \varphi_L)(Z^\alpha \hat{\partial}_\alpha) = (Z^\alpha \circ \varphi_H \hat{\partial}_\alpha) + [(\rho^i_\alpha \circ h \circ \pi) Z^\alpha L_{ib}] \circ \varphi_H \hat{\partial}_b,
$$

for any $Z^\alpha \hat{\partial}_\alpha + Y^a \hat{\partial}_a \in \Gamma((p, \eta) TE, (p, \eta) T\tau_E, E)$. The vector bundles morphism $((p, \eta) T\varphi_L, \varphi_L)$ will be called the tangent $(p, \eta)$-application of the Legendre bundles morphism associated to the Lagrangian $L$. Using this application together with (7), (8) we deduce the following theorems.

**Theorem 50** If $u = u^a s_a \in \Gamma(E, \pi, M)$ such that

$$
\Gamma((p, \eta) T\varphi_L, \varphi_L)(u^V) = \Gamma(\varphi_L, Id_M)(u),
$$

then

$$
u^a \circ \pi \circ \varphi_H = u^a \circ \pi, \quad L_{ab} \circ \varphi_H = L_{ab} \circ u \circ \pi.
$$

**Theorem 51** If $u = u^a s_a \in \Gamma(E, \pi, M)$ such that

$$
\Gamma((p, \eta) T\varphi_L, \varphi_L)(u^C) = \Gamma(\varphi_L, Id_M)(u),
$$

then

$$
(g^{\alpha_c} u_c) \circ \pi = (g^{\alpha_c} u_c) \circ \pi \circ \varphi_H,
$$

$$
U_a [K^{\gamma a} (u) \circ h \circ \pi] (\tilde{g}_{\beta \gamma} \circ \pi) = \left\{ U^{\alpha} (K^{\alpha a}_c (u) \circ h \circ \pi) (\tilde{g}_{\beta \gamma} \circ \pi) L_{ib} \right\} \circ \varphi_H - \left\{ (\rho^i_\alpha \circ h \circ \pi) ((g^{\alpha c} u^c) \circ \pi) L_{ib} \right\} \circ \varphi_H.
$$

Using the bundles morphism $(\varphi_H, Id_M)$, we build the vector bundles morphism $((p, \eta) T\varphi_H, \varphi_H)$ given by the diagram

$$
\begin{array}{ccc}
(p, \eta) T E^* & \xrightarrow{(p, \eta) T\varphi_H} & (p, \eta) T E \\
\downarrow (p, \eta) T\tau_E & & \downarrow (p, \eta) T\tau_E^* \\
E^* & \xrightarrow{\varphi_H} & E
\end{array}
$$
such that
\[
\Gamma((\rho, \eta)T\varphi_H, \varphi_H)(Z^a) = (Z^a \circ \varphi_L)\tilde{\partial}_a + [(\rho_\alpha^a \circ \varphi_H^\alpha)Z^a H_b^\alpha] \circ \varphi_L \tilde{\partial}_b,
\]
\[
\Gamma((\rho, \eta)T\varphi_H, \varphi_H)(Y_a \tilde{\partial}) = (Y_a H^{ab}) \circ \varphi_L \tilde{\partial}_b,
\]
for any \(Z^a \tilde{\partial}_a + Y_a \tilde{\partial} \in \Gamma((\rho, \eta)T\tilde{E}, (\rho, \eta)\tilde{\tau}_E^*, \tilde{E})\). The vector bundles morphism \((\rho, \eta) T\varphi_H, \varphi_H\) will be called the tangent \((\rho, \eta)\)-application of the Legendre bundles morphism associated to the Hamiltonian \(H\). Using this application together with (22) and (23) we deduce the following theorems.

**Theorem 52** If \(u = u_a s^a \in \Gamma(\tilde{E}, \tilde{\pi}, M)\) such that

\[
\Gamma((\rho, \eta)T\varphi_H, \varphi_H)(u^V) = \Gamma(\varphi_H, Id_M)(u),
\]

then

\[
u_a \circ \tilde{\pi} \circ \varphi_L = u_a \circ \pi, \quad H^{ab} \circ \varphi_L = H^{ab} \circ u \circ \pi.
\]

**Theorem 53** If \(u = u_a s^a \in \Gamma(\tilde{E}, \tilde{\pi}, M)\) such that

\[
\Gamma((\rho, \eta)T\varphi_H, \varphi_H)(u^C) = \Gamma(\varphi_H, Id_M)(u),
\]

then

\[
g^{\alpha e}_u \circ \pi = (g^{\alpha e} u_e) \circ \tilde{\pi} \circ \varphi_L,
\]

\[
U^a(K^\alpha(u) \circ h \circ \pi)(\tilde{g}^\alpha_t \circ \pi) = \{U_a(K^\alpha(u) \circ h \circ \pi)(\tilde{g}^\alpha_t \circ \tilde{\pi})H^{cb} \circ \varphi_L
\]

\[-(p_\alpha^i h \circ \tilde{\pi})(g^{\alpha e}_u \circ \tilde{\pi})H^b_i \circ \varphi_L.
\]

### 8 Duality between Lie algebroids structures

**Theorem 54** If the vector bundles morphism \((\rho, \eta) T\varphi_L, \varphi_L\) is a morphism of Lie algebroids, then we obtain:

\[
(L^\gamma_{\alpha \beta} \circ h \circ \pi) \circ \varphi_H = L^\gamma_{\alpha \beta} \circ h \circ \tilde{\pi},
\]

\[
((L^\gamma_{\alpha \beta} \rho^k_\gamma) \circ h \circ \pi \circ L_{kb}) \circ \varphi_H = \rho^i_\alpha \circ \varphi_H^\alpha \circ \rho^j_\beta \circ \varphi_H^\beta - \rho^i_\alpha \circ \varphi_H^\alpha \circ \rho^j_\beta \circ \varphi_H^\beta
\]

\[-\rho^i_\alpha \circ \varphi_H^\alpha \circ \rho^j_\beta \circ \varphi_H^\beta + (\rho^i_\alpha \circ \varphi_H^\alpha \circ L_{ia}) \circ \varphi_H \circ \rho^j_\beta \circ \varphi_H^\beta + (\rho^i_\alpha \circ \varphi_H^\alpha \circ L_{ia}) \circ \varphi_H \circ \rho^j_\beta \circ \varphi_H^\beta
\]

\[-(\rho^i_\alpha \circ \varphi_H^\alpha \circ L_{ia}) \circ \varphi_H \circ \rho^j_\beta \circ \varphi_H^\beta + (\rho^i_\alpha \circ \varphi_H^\alpha \circ L_{ia}) \circ \varphi_H \circ \rho^j_\beta \circ \varphi_H^\beta.
\]

0 = \(L^\gamma_{\alpha \beta} \circ h \circ \pi \circ \frac{\partial}{\partial \varphi_H} \circ (L_{ba} \circ \varphi_H) + (\rho^i_\alpha \circ \varphi_H^\alpha \circ L_{bc}) \circ \varphi_H \circ \frac{\partial}{\partial \varphi_H} \circ (L_{ia} \circ \varphi_H)
\]

\[-L_{ba} \circ \varphi_H \circ \frac{\partial}{\partial \varphi_H} \circ (L_{ia} \circ \varphi_H) \circ \varphi_H),
\]

and

\[
0 = L_{ac} \circ \varphi_H \circ \frac{\partial}{\partial \varphi_H} \circ (L_{bd} \circ \varphi_H) - L_{bd} \circ \varphi_H \circ \frac{\partial}{\partial \varphi_H} \circ (L_{ac} \circ \varphi_H).
\]
Proof. Developing the following equalities

\[ \Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_\alpha, \tilde{\partial}_\beta((\rho, \eta)TE) \]

\[ = [\Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_\alpha, \Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_\beta]((\rho, \eta)TE) \]

\[ \Gamma((\rho, \eta)T\varphi_L, \varphi_L)[\tilde{\partial}_a, \tilde{\partial}_b]((\rho, \eta)TE) \]

\[ = [\Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_a, \Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_b]((\rho, \eta)TE) \]

and

\[ \Gamma((\rho, \eta)T\varphi_L, \varphi_L)[\tilde{\partial}_a, \tilde{\partial}_b]((\rho, \eta)TE) \]

\[ = [\Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_a, \Gamma((\rho, \eta)T\varphi_L, \varphi_L)\tilde{\partial}_b]((\rho, \eta)TE) \]

it results the conclusion of the theorem. ■

Corollary 55 In the particular case of Lie algebroids, \((\eta, h) = (Id_M, Id_M)\), we obtain:

\[ (L^\alpha_{\beta} \circ \pi) \circ \varphi_H = L^\alpha_{\beta} \circ \pi, \]

\[ ((L^\alpha_{\beta} \rho^k) \circ \pi \cdot L_{kb}) \circ \varphi_H = \rho^d_{\alpha} \circ \pi \cdot \frac{\partial}{\partial \pi^d}((\rho^j_{\beta} \circ \pi \cdot L_{jb}) \circ \varphi_H) \]

\[ -\rho^j_{\beta} \circ \pi \cdot \frac{\partial}{\partial \pi^j}((\rho^l_{\alpha} \circ \pi \cdot L_{lb}) \circ \varphi_H) + (\rho^l_{\alpha} \circ \pi \cdot L_{la}) \circ \varphi_H \cdot \frac{\partial}{\partial \pi^l}((\rho^j_{\beta} \circ \pi \cdot L_{jb}) \circ \varphi_H) \]

\[ - (\rho^l_{\alpha} \circ \pi \cdot L_{ja}) \circ \varphi_H \cdot \frac{\partial}{\partial \pi^l}((\rho^j_{\beta} \circ \pi \cdot L_{jb}) \circ \varphi_H), \]

\[ 0 = \rho^d_{\alpha} \circ \pi \cdot \frac{\partial}{\partial \pi^d}((L_{ba} \circ \varphi_H) + (\rho^l_{\alpha} \circ \pi \cdot L_{bc}) \circ \varphi_H \cdot \frac{\partial}{\partial \pi^l}((L_{ia} \circ \varphi_H) \]

\[ - L_{bc} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^c}((\rho^l_{\alpha} \circ \pi \cdot L_{ia}) \circ \varphi_H), \]

and

\[ 0 = L_{ac} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^c}((L_{bd} \circ \varphi_H) - L_{bc} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^c}((L_{ad} \circ \varphi_H). \]

In the classical case, \((\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)\), we obtain:

\[ 0 = \frac{\partial}{\partial x^j}(\frac{\partial^2 L}{\partial x^j \partial y^k} \circ \varphi_H) - \frac{\partial}{\partial x^j}(\frac{\partial^2 L}{\partial x^j \partial y^k} \circ \varphi_H) \]

\[ + \frac{\partial^2 L}{\partial x^j \partial y^k} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^j}((\rho^l_{\alpha} \circ \pi \cdot L_{ia}) \circ \varphi_H) \]

\[ - \frac{\partial^2 L}{\partial x^j \partial y^k} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^j}((\rho^l_{\alpha} \circ \pi \cdot L_{ia}) \circ \varphi_H), \]

and

\[ 0 = \frac{\partial^2 L}{\partial y^j \partial y^k} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^j}((\rho^l_{\alpha} \circ \pi \cdot L_{ia}) \circ \varphi_H) \]

\[ - \frac{\partial^2 L}{\partial y^j \partial y^k} \circ \varphi_H \cdot \frac{\partial}{\partial \pi^j}((\rho^l_{\alpha} \circ \pi \cdot L_{ia}) \circ \varphi_H). \]
Theorem 56 If the vector bundles morphism \(((\rho, \eta)T\varphi_H, \varphi_H)\) is a morphism of Lie algebroids, then we obtain:

\[(L^\gamma_{\alpha \beta} \circ h \circ \pi) \circ \varphi_L = L^\gamma_{\alpha \beta} \circ h \circ \pi,\]

\[((L^\gamma_{\alpha \beta} \circ h \circ \pi \cdot H_k^b) \circ \varphi_L = \rho^a_\alpha \circ \rho \circ \pi \cdot \frac{\partial}{\partial x^a}((\rho^b_\beta \circ \rho \circ \pi \cdot H_k^b) \circ \varphi_L) - \rho^b_\beta \circ \rho \circ \pi \cdot \frac{\partial}{\partial y^b}((\rho^a_\alpha \circ \rho \circ \pi \cdot H_k^b) \circ \varphi_L) + \rho^a_\alpha \circ \rho \circ \pi \cdot \frac{\partial}{\partial y^a}((\rho^b_\beta \circ \rho \circ \pi \cdot H_k^b) \circ \varphi_L)
\]

\[-(\rho^a_\alpha \circ \rho \circ \pi \cdot H_k^a) \circ \varphi_L \cdot \frac{\partial}{\partial y^a}((\rho^b_\beta \circ \rho \circ \pi \cdot H_k^b) \circ \varphi_L),\]

\[0 = \rho^a_\alpha \circ \rho \circ \pi \cdot \frac{\partial}{\partial y^a}((H^{ba} \circ \varphi_L) + (\rho^b_\beta \circ \rho \circ \pi \cdot H^{bc}) \circ \varphi_L \cdot \frac{\partial}{\partial y^b}((H^{ba} \circ \varphi_L) - L^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^b}((H^{ad} \circ \varphi_L).
\]

Proof. Developing the equalities

\[\Gamma((\rho, \eta)T\varphi_H, \varphi_H)[\hat{\partial}^a, \hat{\partial}^b]_{(\rho, \eta)TE} = \Gamma((\rho, \eta)T\varphi_H, \varphi_H)[\hat{\partial}^a, \hat{\partial}^b]_{\rho, \eta)TE,\]

and

\[\Gamma((\rho, \eta)T\varphi_H, \varphi_H)[\hat{\partial}^a, \hat{\partial}^b]_{(\rho, \eta)TE} = \Gamma((\rho, \eta)T\varphi_H, \varphi_H)[\hat{\partial}^a, \hat{\partial}^b]_{\rho, \eta)TE,\]

it results the conclusion of the theorem. ■

Corollary 57 In the particular case of Lie algebroids, \((\eta, h) = (Id_M, Id_M),\) we obtain

\[(L^\gamma_{\alpha \beta} \circ \pi) \circ \varphi_L = L^\gamma_{\alpha \beta} \circ \pi,\]

\[((L^\gamma_{\alpha \beta} \circ \pi \cdot H_k^b) \circ \varphi_L = \rho^a_\alpha \circ \pi \cdot \frac{\partial}{\partial x^a}((\rho^b_\beta \circ \pi \cdot H_k^b) \circ \varphi_L) - \rho^b_\beta \circ \pi \cdot \frac{\partial}{\partial y^b}((\rho^a_\alpha \circ \pi \cdot H_k^b) \circ \varphi_L) + \rho^a_\alpha \circ \pi \cdot \frac{\partial}{\partial y^a}((\rho^b_\beta \circ \pi \cdot H_k^b) \circ \varphi_L)
\]

\[-(\rho^a_\alpha \circ \pi \cdot H_k^a) \circ \varphi_L \cdot \frac{\partial}{\partial y^a}((\rho^b_\beta \circ \pi \cdot H_k^b) \circ \varphi_L),\]

\[0 = \rho^a_\alpha \circ \pi \cdot \frac{\partial}{\partial y^a}((H^{ba} \circ \varphi_L) + (\rho^b_\beta \circ \pi \cdot H^{bc}) \circ \varphi_L \cdot \frac{\partial}{\partial y^b}((H^{ba} \circ \varphi_L) - L^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^b}((H^{ad} \circ \varphi_L),
\]

and

\[0 = H^{ac} \circ \varphi_L \cdot \frac{\partial}{\partial y^a}(H^{bd} \circ \varphi_L) - H^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^b}(H^{ad} \circ \varphi_L).\]
In the classical case, \((\rho, \eta, h) = (Id_T M, Id_M, Id_M)\), we obtain

\[
0 = \frac{\partial}{\partial x^i} \left( \frac{\partial^2 H}{\partial p^j \partial p^k} \circ \varphi_L \right) - \frac{\partial^2 H}{\partial x^i \partial p^k} \circ \varphi_L \cdot \frac{\partial}{\partial y^i} \left( \left( \frac{\partial^2 H}{\partial x^j \partial p^j} \circ \varphi_L \right) \circ \varphi_L \right),
\]

\[
0 = \frac{\partial}{\partial x^k} \left( \frac{\partial^2 H}{\partial p^i \partial p^j} \circ \varphi_L \right) + \frac{\partial^2 H}{\partial p^i \partial p^j} \circ \varphi_L \cdot \frac{\partial}{\partial y^i} \left( \left( \frac{\partial^2 H}{\partial x^k \partial p^j} \circ \varphi_L \right) \circ \varphi_L \right),
\]

and

\[
0 = \frac{\partial^2 H}{\partial p^i \partial p^j} \circ \varphi_L \cdot \frac{\partial}{\partial y^i} \left( \frac{\partial^2 H}{\partial p^j \partial p^k} \circ \varphi_L \right) - \frac{\partial^2 H}{\partial p^j \partial p^k} \circ \varphi_L \cdot \frac{\partial}{\partial y^j} \left( \frac{\partial^2 H}{\partial p^i \partial p^k} \circ \varphi_L \right).
\]

**Definition 58** If \(((\rho, \eta) T \varphi_L, \varphi_L)\) and \(((\rho, \eta) T \varphi_H, \varphi_H)\) are Lie algebroids morphisms, then we will say that \((E, \pi, M)\) and \((\ast E, \ast \pi, M)\) are Legendre \((\rho, \eta)\)-equivalent and we will write

\[
(E, \pi, M) \overset{L}{\sim}_{(\rho, \eta)} (\ast E, \ast \pi, M).
\]

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**References**

[1] C. M. Arcuş, *The generalized Lie algebroids and their applications*, arXiv:1007.1541v2, (2010), 206 pages.

[2] C. M. Arcuş, *Generalized Lie algebroids and connections over pair of diffeomorphic manifolds*, J. Gen. Lie Theory Appl., 7 (2013), 32 pages.

[3] C. M. Arcuş, *Intersection between the geometry of generalized Lie algebroids and some aspects of interior and exterior differential systems*, arXiv: 1311.1147v1, (2013), 30 pages.

[4] C. M. Arcuş, *Mechanical systems in the generalized Lie algebroids framework*, Int. J. Geom. Methods Mod. Phys., 11 (2014), 40 pages.

[5] E. Esin and S. Civelek, *The lifts on the second order tangent bundles*, J. Math. Stat. Fac. Art. Sc. Gazi Univ., 2 (1989), 117-135.

[6] Manuel de León, Juan C. Marrero, Eduardo Martínez, *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A: Math. Gen. 38 (2005), 241-308.

[7] E. Martínez, *Lagrangian mechanics on Lie algebroids*, Acta Appl. Math. 67(2001), 295-320.
[8] T. Omran, A. Sharfuddin and S. I. Husain, Lifts of structures on manifolds, Publications De L’institut Math. 36, 50 (1984), 93-97.

[9] M. Özkan, Prolongations of golden structures to tangent bundles, Differential Geometry-Dynamical Systems, 16 (2014), 227-238.

[10] E. Peyghan, Models of Finsler Geometry on Lie algebroids, arXiv: 1310.7393v1, (2013), 90 pages. E. Peyghan, Models of Finsler Geometry on Lie algebroids, arXiv: 1310.7393v1, (2013), 90 pages.

[11] E. Peyghan, H. Nasrabadi and A. Tayebi, The homogenous lift of the \((1, 1)\)-tensor bundle of a Riemannian metric, Int. J. Geom. Methods Mod. Phys., 10 (2013), 18 pages.

[12] L. Popescu, The geometry of Lie algebroids and its applications to optimal control, arXiv:1302.5212v2 [Math.DG] 25 Feb 2013.

[13] L. Popescu, A note on Poisson-Lie algebroids, J. Geom. Symmetry Phys., 12 (2008), 63-73.

[14] A. A. Salimov and A. Magden, Complete lifts of tensor fields on a pure cross-section in the tensor bundle \(T^1_q(M^n)\), Note di Matematica, 18 (1998), 27-37.

[15] W. Sarlet and G. Waeyaert, Lifting geometric objects to the dual of the first jet bundle of a bundle fibred over \(\mathbb{R}\), J. Geom and Phys., 74, (2013), 109-118.

[16] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. J. (I, 10 (1958) 338-354; II, 14 (1962) 146-155).

[17] K. Yano and S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker 1973.