STABLE PAIRS AND GOPAKUMAR-VAFA TYPE INVARIANTS
FOR CALABI-YAU 4-FOLDS

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Abstract. As an analogy to Gopakumar-Vafa conjecture on CY 3-folds, Klemm-Pandharipande
defined GV type invariants on CY 4-folds using GW theory and conjectured their integrality.
In this paper, we define stable pair type invariants on CY 4-folds and use them to interpret
these GV type invariants. Examples are computed for both compact and non-compact CY
4-folds to support our conjectures.

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0. Introduction

0.1. Background. Gromov-Witten invariants are rational numbers counting stable maps from
complex curves to algebraic varieties (or symplectic manifolds). They are not necessarily integers
because of multiple cover contributions. In [20], Klemm-Pandharipande gave a definition of
Gopakumar-Vafa type invariants on Calabi-Yau 4-folds using GW theory and conjectured that
they are integers. For dimensional reasons, GW invariants for genus \( g \geq 2 \) always vanish on Calabi-Yau 4-folds, so the integrality conjecture only applies in genus 0 and 1. In our previous paper \[12\], we gave a sheaf-theoretic interpretation of \( g = 0 \) GV type invariants using DT\(_4\) invariants \[10\] \[4\] of one-dimensional stable sheaves, analogous to the work of Katz for 3-folds \[18\].

In this paper, we propose a sheaf-theoretic approach to both genus 0 and 1 GV type invariants using stable pairs on CY 4-folds. For CY 3-folds, a Pairs/GV conjecture was first developed in work of Pandharipande and Thomas \[30, 32\]. Our paper may be viewed as an analogue of their work in the setting of CY 4-folds.

0.2. GV type invariants on CY 4-folds. Let \( X \) be a smooth projective CY 4-fold. As mentioned above, Gromov-Witten invariants vanish for genus \( g \geq 2 \) for dimensional reasons, so we only consider the genus 0 and 1 cases.

The genus 0 GW invariants on \( X \) are defined using insertions: for integral classes \( \gamma_i \in H^{m_i}(X, \mathbb{Z}) \), \( 1 \leq i \leq n \), one defines

\[
GW_{0,\beta}(\gamma_1, \ldots, \gamma_n) = \int_{[M_{0,n}(X,\beta)]^\text{vir}} \prod_{i=1}^{n} \text{ev}_i^*(\gamma_i),
\]

where \( \text{ev}_i : \overline{M}_{0,n}(X,\beta) \to X \) is the i-th evaluation map.

The invariants
\[
(0.1) \quad n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}
\]
are defined in \[20\] by the identity

\[
\sum_{\beta > 0} GW_{0,\beta}(\gamma_1, \ldots, \gamma_n)q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma_1, \ldots, \gamma_n) \sum_{d=1}^{\infty} d^{n-3} q^{d\beta}.
\]

For genus 1, virtual dimensions of GW moduli spaces without marked points are zero, so the GW invariants
\[
GW_{1,\beta} = \int_{[M_{1,0}(X,\beta)]^\text{vir}} 1 \in \mathbb{Q}
\]
can be defined without insertions. The invariants
\[
(0.2) \quad n_{1,\beta} \in \mathbb{Q}
\]
are defined in \[20\] by the identity

\[
\sum_{\beta > 0} GW_{1,\beta}q^\beta = \sum_{\beta > 0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta > 0} n_{0,\beta}(c_2(X)) \log(1 - q^\beta) - \frac{1}{24} \sum_{\beta_1, \beta_2} m_{\beta_1, \beta_2} \log(1 - q^{\beta_1 + \beta_2}),
\]

where \( \sigma(d) = \sum_{d|n} \varphi(n) \) and \( m_{\beta_1, \beta_2} \in \mathbb{Z} \) are called meeting invariants which can be inductively determined by genus 0 GW invariants. In \[20\], both of the invariants \[11\], \[12\] are conjectured to be integers, and GW invariants on \( X \) are computed to support the conjectures in many examples by either localization techniques or mirror symmetry.

0.3. Our proposal. The aim of this paper is to give a sheaf-theoretic interpretation for the above GV-type invariants \[11\] \[12\] via stable pairs, using Donaldson-Thomas theory for CY 4-folds introduced by Cao-Leung \[10\] and Borisov-Joyce \[4\].

We consider the moduli space \( P_n(X,\beta) \) of stable pairs \( (s : \mathcal{O}_X \to F) \) with \( \text{ch}(F) = (0, 0, 0, \beta, n) \). By Theorem \[1.4\] one can construct a virtual class
\[
(0.3) \quad [P_n(X,\beta)]^\text{vir} \in H_{2n}(P_n(X,\beta), \mathbb{Z}),
\]
which depends on the choice of orientation of a certain (real) line bundle over \( P_n(X,\beta) \). On each connected component of \( P_n(X,\beta) \), there are two choices of orientation, which affect the corresponding contribution to the virtual class \( [0.3] \) by a sign (for each connected component).

When \( n = 0 \), the virtual dimension of the virtual class \( [0.3] \) is zero. By integrating, we define the stable pair invariant
\[
P_{0,\beta} := \int_{[P_0(X,\beta)]^\text{vir}} 1 \in \mathbb{Z},
\]
When \( n = 1 \), the (real) virtual dimension of the virtual class \((\mathcal{O}_X, \beta)\) is two. We use insertions to define invariants as follows. For integral classes \( \gamma_i \in H^{m_i}(X, \mathbb{Z}) \), \( 1 \leq i \leq n \), let
\[
(0.4) \quad \tau : H^m(X) \to H^{m-2}(P_1(X, \beta)), \quad \tau(\gamma) = \pi_{P_1}^{*}(\pi_X^{*}\gamma \cup \text{ch}_3(F)),
\]
where \( \pi_X, \pi_P \) are projections from \( X \times P_1(X, \beta) \) to corresponding factors, \( l = (\pi_X^{*}\mathcal{O}_X \to F) \) is the universal pair, and \( \text{ch}_3(F) \) is the Poincaré dual to the fundamental cycle of \( F \).

Then we define stable pair invariants
\[
P_{1, \beta}(\gamma_1, \ldots, \gamma_n) := \int_{\mathcal{P}_1(X, \beta)^{\text{vir}}} \prod_{i=1}^{n} \tau(\gamma_i).
\]

We propose the following interpretation of \((0.1), (0.2)\) using stable pair invariants.

**Conjecture 0.1.** (Conjecture \([1.5]\) For a suitable choice of orientation, we have
\[
P_{1, \beta}(\gamma_1, \ldots, \gamma_n) = \sum_{\beta_1 + \beta_2 = \beta, \beta_1, \beta_2 \geq 0} n_{0, \beta}(\gamma_1, \ldots, \gamma_n) \cdot P_{0, \beta},
\]
where the sum is over all possible effective classes, and we set \( n_{0, 0}(\gamma_1, \ldots, \gamma_n) := 0 \) and \( P_{0, 0} := 1 \).

In particular, when \( \beta \) is irreducible,
\[
P_{1, \beta}(\gamma_1, \ldots, \gamma_n) = n_{0, \beta}(\gamma_1, \ldots, \gamma_n).
\]

**Conjecture 0.2.** (Conjecture \([1.6]\) For a suitable choice of orientation, we have
\[
\sum_{\beta \geq 0} P_{0, \beta} q^{\beta} = \prod_{\beta \geq 0} M(q)^{n_{1, \beta}},
\]
where \( M(q) = \prod_{k \geq 1} (1 - q^k)^{-k} \) is the MacMahon function and \( P_{0, 0} := 1 \).

For instance, when Picard number of \( X \) is one, for an irreducible curve class \( \beta \), the above identity implies
\[
P_{0, \beta} = n_{1, \beta},
\]
\[
P_{0, 2\beta} = n_{1, 2\beta} + 3n_{1, \beta} + \binom{n_{1, \beta}}{2},
\]
\[
P_{0, 3\beta} = n_{1, 3\beta} + n_{1, \beta} \cdot n_{1, 2\beta} + 6n_{1, \beta} + 6 \binom{n_{1, \beta}}{2} + \binom{n_{1, \beta}}{3},
\]
by comparing coefficients of \( q^\beta \), \( q^{2\beta} \) and \( q^{3\beta} \).

One issue with our current proposal (as in our earlier conjecture \([12]\)) is that we do not have a general mechanism for choosing the orientation in the above conjectures. Currently, in the cases we examine in this paper, we choose orientations on a case-by-case basis to show the correct matching. It would be very interesting to construct canonical choices of orientation for these moduli spaces and study our conjectures using them.

Our proposal is based on a heuristic argument given in Section \([1.5]\), where we show Conjecture \([0.1], 0.2\) assuming the CY 4-fold \( X \) to be ‘ideal’, i.e. curves in \( X \) deform in some family of expected dimensions. Apart from that, we verify our conjecture in examples as follows.

### 0.4. Verifications of the conjecture I: compact examples.

We first prove our conjectures for some special compact Calabi-Yau 4-folds.

**Sextic 4-folds.** Let \( X \subseteq \mathbb{P}^5 \) be a degree six smooth hypersurface and \([l] \in H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^5, \mathbb{Z}) \) be the line class. We check our conjectures for \( \beta = [l] \) and \( 2[l] \).

**Proposition 0.3.** (Proposition \([2.1], 2.2\) Let \( X \) be a smooth sextic 4-fold and \([l] \in H_2(X, \mathbb{Z}) \) be the line class. Then Conjecture \([0.1], 0.2\) are true for \( \beta = [l] \) and \( 2[l] \).

**Elliptic fibrations.** We consider a projective CY 4-fold \( X \) which admits an elliptic fibration
\[
\pi : X \to \mathbb{P}^3,
\]
given by a Weierstrass model \([2.1]\). Let \( f \) be a general fiber of \( \pi \) and \( h \) be a hyperplane in \( \mathbb{P}^3 \), set
\[
B = \pi^{*}h, \quad E = t(\mathbb{P}^3) \in H_6(X, \mathbb{Z}),
\]
where \( t \) is a section of \( \pi \). Then we have
Proposition 0.4. (Proposition 2.4) 2.0

(1) Conjecture 0.1 is true for fiber class \( \beta = [f] \) and \( \gamma = B^2 \) or \( B \cdot E \).

(2) Conjecture 0.2 is true for multiple fiber classes \( \beta = r[f] \) (\( r \geq 1 \)).

In the above cases, we can directly compute the pair invariants and check the compatibility with the computation of GW invariants in [20].

Product of elliptic curve and Calabi-Yau 3-fold. Let \( X = Y \times E \) be a product of a Calabi-Yau 3-fold and an elliptic curve \( E \). We check our conjectures when the curve class comes from either \( Y \) or \( E \).

Theorem 0.5. (Theorem 2.13, 2.15 Proposition 2.17) Let \( X = Y \times E \) be given as above. Then

1. Conjecture 0.1 is true for any irreducible curve class \( \beta \in H_2(Y) \subseteq H_2(X) \), provided that \( Y \) is a complete intersection in a product of projective spaces.
2. Conjecture 0.2 is true for any irreducible curve class \( \beta \in H_2(Y) \subseteq H_2(X) \).
3. Conjecture 0.3 is true for classes \( \beta = r[E] \) (\( r \geq 1 \)).

The proof of these results is briefly reviewed here.

For (1), when \( \beta \in H_2(Y) \subseteq H_2(X) \) is an irreducible curve class, we have an isomorphism

\[ P_n(X, \beta) \cong P_n(Y, \beta) \times E. \]

The corresponding virtual class satisfies (see Proposition 2.11):

\[ [P_n(X, \beta)]^\text{vir} = [P_n(Y, \beta)]^\text{pair}_\text{vir} \otimes [E], \]

for certain choice of orientation in defining the LHS, where the virtual class of \( P_n(Y, \beta) \) is defined using the deformation-obstruction theory of pairs (Lemma 2.9) instead of the deformation-obstruction theory of complexes in the derived category used by [30].

In this case, we have a forgetful morphism

\[ f : P_1(Y, \beta) \to M_{1, \beta}(Y), \quad (O_Y \to F) \mapsto F, \]

to the moduli space \( M_{1, \beta}(X) \) of 1-dimensional stable sheaves \( F \) with \( [F] = \beta \) and \( \chi(F) = 1 \). We show that the map satisfies Manolache’s virtual push-forward formula (Proposition 2.10),

\[ \int [P_1(Y, \beta)]^\text{pair}_\text{vir} 1 = \int [M_{1, \beta}(Y)]^\text{vir} 1. \]

Then Conjecture 0.1 can be reduced to Katz’s conjecture on CY 3-fold \( Y \) (Corollary 2.12). Combining with our previous proof of Katz’s conjecture for primitive classes [12, Cor. A.6], we can conclude (1) of Theorem 0.5.

For (2), this is one of few cases where we can compute non-primitive curve classes and form generating series. The point is to identify pair moduli spaces on \( X \) with Hilbert schemes of points on \( Y \) and use computation of zero dimensional DT invariants of \( Y \).

Hyperkähler 4-folds. When the CY 4-fold \( X \) is a hyperkähler 4-fold, GW invariants vanish, and so do the GV type invariants. To verify our conjectures, we are left to prove the vanishing of pair invariants. A cosection map from the (trace-free) obstruction space is constructed and shown to be surjective and compatible with Serre duality (Proposition 2.13). We expect the following result then follows.

Claim 0.6. (Claim 2.13) Let \( X \) be a projective hyperkähler 4-fold and \( P_n(X, \beta) \) be the moduli space of stable pairs with \( n \neq 0 \) or \( \beta \neq 0 \). Then the virtual class satisfies

\[ [P_n(X, \beta)]^\text{vir} = 0. \]

At the moment, a Kiem-Li type theory of cosection localization for D-manifolds is not available in the literature. We believe that when such a theory is established, our claim should follow automatically. Nevertheless, we have the following evidence for the claim.

1. At least when \( P_n(X, \beta) \) is smooth, Proposition 2.13 gives the vanishing of virtual class.
2. If there is a complex analytic version of (-2)-shifted symplectic geometry [33] and the corresponding construction of virtual classes [1], one could prove the vanishing result as in GW theory, i.e. taking a generic complex structure in the \( S^2 \)-twistor family of the hyperkähler 4-fold which does not support coherent sheaves and then vanishing of virtual classes follows from their deformation invariance.
0.5. Verifications of the conjecture II: local 3-folds and surfaces. For a Fano 3-fold \( Y \), we consider the non-compact CY 4-fold

\[ X = K_Y. \]

In this case, the stable pair moduli space \( P_n(X, \beta) \) is compact (Proposition 3.3), so we can formulate Conjecture 0.1 here (even though the target is not projective).

When the curve class \( \beta \in H_2(X) \) is irreducible, we study this as follows. Similar to the case of the product of a CY 3-fold and an elliptic curve, for a certain choice of orientation, the virtual class of \( P_n(X, \beta) \) satisfies (Proposition 3.3)

\[ [P_n(X, \beta)]_{\text{vir}} = [P_n(Y, \beta)]_{\text{vir}}, \]

under the isomorphism

\[ P_n(X, \beta) \cong P_n(Y, \beta). \]

And we have a virtual push-forward formula (Proposition 3.2)

\[ f_*[P_1(Y, \beta)]_{\text{pair}} = [M_1, \beta(Y)]_{\text{vir}}, \]

where \( f : P_1(Y, \beta) \to M_{1, \beta}(Y), (\mathcal{O}_X \to F) \mapsto F \) is the morphism forgetting the section, \( M_{1, \beta}(Y) \) is the moduli scheme of 1-dimensional stable sheaves \( E \) on \( Y \) with \( [E] = \beta \) and \( \chi(E) = 1. \) Then Conjecture 0.1 is easily reduced to our previous conjecture [12, Conjecture 0.2]. Combined with computations in [5], we have

**Theorem 0.7.** (Proposition 3.4, 3.5) Let \( X = K_Y \) be given as above. Then

1. Conjecture 0.1 is true for any irreducible curve class \( \beta \in H_2(X) \cong H_2(Y) \), provided that
   - (i) \( Y \subseteq \mathbb{P}^4 \) is a smooth hypersurface of degree \( d \leq 4 \), or
   - (ii) \( Y = S \times \mathbb{P}^1 \) for a toric del Pezzo surface \( S \).

2. Conjecture 0.2 is true for an irreducible curve class \( \beta \in H_2(X) \cong H_2(Y) \) when \( Y = \mathbb{P}^3 \).

Similarly for a smooth projective surface \( S \), we consider the non-compact CY 4-fold

\[ X = \text{Tot}_S(L_1 \oplus L_2), \]

where \( L_1, L_2 \) are line bundles on \( S \) satisfying \( L_1 \oplus L_2 \cong K_S \). In particular, when \( \beta \) is irreducible and \( L_1, \beta < 0 (i = 1, 2) \), the moduli space \( P_n(X, \beta) \) of stable pairs on \( X \) is compact and smooth (Lemma 3.6, Proposition 3.7). So pair invariants are well-defined and we can also study our conjectures in this case. In particular, we have

**Proposition 0.8.** (Proposition 3.8) Let \( S \) be a del-Pezzo surface and \( L_1^{-1}, L_2^{-1} \) be two ample line bundles on \( S \) such that \( L_1 \oplus L_2 \cong K_S \). Denote \( \beta \in H_2(X, Z) \cong H_2(S, \mathbb{Z}) \) to be an irreducible curve class on \( X = \text{Tot}_S(L_1 \oplus L_2) \). Then Conjecture 0.1, 0.2 are true for \( \beta \).

In fact such a del Pezzo surface must be \( \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \) (see the proof of Proposition 3.8), and the corresponding \( X \) is given by

\[ \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \quad \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1)). \]

By using computations due to Kool and Monavari [21], one can check Conjecture 0.1, 0.2 for small degree curve classes on such \( X \) (see Section 3.4 for details).

0.6. Verifications of the conjecture III: local curves. Let \( C \) be a smooth projective curve. We consider a CY 4-fold \( X \) given by

\[ X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3), \]

where \( L_1, L_2, L_3 \) are line bundles on \( C \) satisfying \( L_1 \oplus L_2 \oplus L_3 \cong \omega_C \). The three dimensional complex torus \( T = (\mathbb{C}^*)^3 \) acts on \( X \) fiberwise over \( C \). The \( T \)-equivariant GW invariants

\[ GW_{g,d}(X) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3) \]

can be defined via equivariant residue. Here \( \lambda_i \) are the equivariant parameters with respect to the \( T \)-action.

On the other hand, there is a two dimensional subtorus \( T_0 \subseteq (\mathbb{C}^*)^3 \) which preserves the CY 4-form on \( X \). We may define equivariant pair invariants

\[ P_{n,d}(X) \in \mathbb{Q}(\lambda_1, \lambda_2) \]

as rational functions in terms of equivariant parameters of \( T_0 \) following a localization principle for DT invariants (see Section 1.2, 10, 12 Sect. 4.2). When \( C = \mathbb{P}^1 \) and \( X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3) \), we explicitly determine \( P_{1,d}(X) \) for \( d \leq 2 \) (Proposition 3.5). Note in this case \( P_{0,|\mathbb{P}^1|}(X) = 0 \) and there are no insertions, so an equivariant analogue of Conjecture 0.1 is given by the following conjecture:
Conjecture 0.9. (Conjecture 0.6) Let $X = \mathcal{O}_{\mathbb{P}^1}(l_1, l_2, l_3)$ for $l_1 + l_2 + l_3 = -2$. Then
\[ \text{GW}_{0,2}(X) = P_{1,2}[\varpi](X) + \frac{1}{8}\text{PT}_{0,2}(\mathbb{P}^1)(X). \]

We can verify the above equivariant conjecture in a large number of examples.

Theorem 0.10. (Theorem 4.7) Conjecture 0.2 is true if $|l_1| \leq 10$ and $|l_2| \leq 10$.

When $C$ is an elliptic curve and $L_i$'s are general degree zero line bundles on $C$, one can define pair invariants and explicitly compute them.

Theorem 0.11. (Theorem 4.10) Let $C$ be an elliptic curve, $L_i \in \text{Pic}^0(C)$ $(i = 1, 2, 3)$ general line bundles satisfying $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$ and $X = \text{Tot}_{C}(L_1 \oplus L_2 \oplus L_3)$.

Then stable pair invariants $P_{0,d}[C](X)$ are well-defined and fit into the generating series
\[ \sum_{d \geq 0} P_{0,d}[C](X) q^d = M(q), \]
where $M(q) := \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function.

Similarly, if we have $n_{1, \beta}$ $(\beta \in H_2(X, \mathbb{Z}))$ many such elliptic curves, then they contribute to pair invariants according to the formula:
\[ \sum_{\beta \geq 0} P_{0, \beta} q^\beta = \prod_{\beta > 0} M(q^{\beta})^{n_{1, \beta}}. \]

This calculation arises in the heuristic argument for our genus one conjecture (Conjecture 0.2) in the ‘ideal’ situation as families of rational curves do not contribute to pair invariants $P_{0, \beta}$’s (see Section 1.5 for more details).

0.7. Speculation on the generating series of stable pair invariants. As before, if we allow insertions, we can use the virtual class [3] and insertions to define stable pair invariants of $P_n(X, \beta)$ for any $n$.

For $\gamma \in H^1(X, \mathbb{Z})$, $\tau(\gamma) \in H^2(P_n(X, \beta), \mathbb{Z})$, so we may define
\[ P_{n, \beta}(\gamma) := \int_{(P_n(X, \beta))^{vir}} \tau(\gamma)^n. \]

Our computations and geometric arguments indicate that we may have the following formula, which generalizes the formula in Conjecture 0.1
\[ P_{n, \beta}(\gamma) = \sum_{\beta_0 + \beta_1 + \cdots + \beta_n = \beta \atop \beta_0, \beta_1, \cdots, \beta_n \geq 0} P_{0, \beta_0} \prod_{i=1}^n n_{0, \beta_i}(\gamma). \]

To group these invariants into a generating series, we introduce notation
\[ \text{PT}(X)(\exp(\gamma)) := \sum_{n, \beta} \frac{P_{n, \beta}(\gamma)}{n!} q^n y^\beta. \]

Assuming Conjecture 0.2 then (0.6) is equivalent to the following Gopakumar-Vafa type formula:
\[ \text{PT}(X)(\exp(\gamma)) = \prod_{\beta} \left( \exp(y^{\beta})^{n_{0, \beta}(\gamma)} \cdot M(q^{\beta})^{n_{1, \beta}} \right), \]
where $n_{0, \beta}(\gamma)$ and $n_{1, \beta}$ are genus 0 and 1 GV type invariants of $X$ (0.1), (0.2) respectively and $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function. As mentioned before, GW invariants on CY 4-folds vanish for $g > 1$, so they do not form a nice generating series as in the 3-folds case. Here the advantage of considering stable pair invariants is we can use them to form a generating series which is conjecturally of GV form.

A heuristic explanation of the formula will be given in Section 1.5. Some more analysis will be pursued in a future work.

0.8. Notation and convention. In this paper, all varieties and schemes are defined over $\mathbb{C}$. For a morphism $\pi : X \to Y$ of schemes, and for $F, G \in D^b(\text{Coh}(X))$, we denote by $R\text{Hom}_x(F, G)$ the functor $R\pi_* R\text{Hom}_Y(F, G)$. We also denote by $\text{ext}^1(F, G)$ the dimension of $\text{Ext}^1_x(F, G)$.

A class $\beta \in H_2(X, \mathbb{Z})$ is called irreducible (resp. primitive) if it is not the sum of two non-zero effective classes (resp. if it is not a positive integer multiple of an effective class).
Acknowledgement. We are very grateful to Martijn Kool and Sergej Monavari for helpful discussions on stable pairs on local surfaces and generously sharing their computational results. We would like to thank Dominic Joyce for helpful comments on our preprint and a responsible referee for very careful reading of our paper and helpful suggestions which improves the exposition of the paper. Y. C. is partly supported by the Royal Society Newton International Fellowship, the World Premier International Research Center Initiative (WPI), MEXT, Japan and JSPS KAKENHI Grant Number JP19K23397. D. M. is partly supported by NSF FRG grant DMS-1159265. Y. T. is supported by World Premier International Research Center Initiative (WPI), MEXT, Japan, and Grant-in Aid for Scientific Research grant (No. 26287002) from MEXT, Japan.

1. Definitions and conjectures

Throughout this paper, unless stated otherwise, $X$ is always denoted to be a smooth projective Calabi-Yau 4-fold, i.e. $K_X \cong \mathcal{O}_X$.

1.1. GW/GV conjecture on CY 4-folds. Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of genus $g$, $n$-pointed stable maps to $X$ with curve class $\beta$. Its virtual dimension is given by

$$-K_X \cdot \beta + (\dim X - 3)(1 - g) + n = 1 - g + n.$$  \hspace{1cm} (1.1)

For integral classes $\gamma_i \in H^{m_i}(X, \mathbb{Z})$, $1 \leq i \leq n$,

$$GW_{g, \beta}(\gamma_1, \ldots, \gamma_n) = \int_{[\overline{M}_{g,n}(X, \beta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\gamma_i).$$ \hspace{1cm} (1.2)

where $ev_i : \overline{M}_{g,n}(X, \beta) \to X$ is the $i$-th evaluation map.

For $g = 0$, the virtual dimension of $\overline{M}_{0,n}(X, \beta)$ is $n + 1$, so no insertions are needed. The genus one GW invariant $GW_{1, \beta} = \int_{[\overline{M}_{1,0}(X, \beta)]^{vir}} \in \mathbb{Q}$ is also expected to be described in terms of certain integer valued invariants.

In analogy with the Gopakumar-Vafa conjecture for CY 3-folds [14], Klemm-Pandharipande [20] defined invariants $n_{0, \beta}(\gamma_1, \ldots, \gamma_n)$ on CY 4-folds by the identity

$$\sum_{\beta > 0} GW_{0, \beta}(\gamma_1, \ldots, \gamma_n)q^\beta = \sum_{\beta > 0} n_{0, \beta}(\gamma_1, \ldots, \gamma_n) \sum_{d=1}^{\infty} d^{n-3} q^{d \beta},$$

and conjecture the following

Conjecture 1.1. ([20 Conjecture 0]) The invariants $n_{0, \beta}(\gamma_1, \ldots, \gamma_n)$ are integers.

For $g = 1$, the virtual dimension of $\overline{M}_{1,0}(X, \beta)$ is zero, so no insertions are needed. The genus one GW invariant

$$GW_{1, \beta} = \int_{[\overline{M}_{1,0}(X, \beta)]^{vir}} \in \mathbb{Q}$$

is also expected to be described in terms of certain integer valued invariants.

Let $S_1, \ldots, S_k$ be a basis of the free part of $H^4(X, \mathbb{Z})$ and

$$\sum_{i,j} g^{ij}[S_i \otimes S_j] \in H^8(X \times X, \mathbb{Z})$$

be the $(4,4)$-component of K"unneth decomposition of the diagonal. For $\beta_1, \beta_2 \in H_2(X, \mathbb{Z})$, the meeting number

$$m_{\beta_1, \beta_2} \in \mathbb{Z}$$

is introduced in [20] as a virtual number of rational curves of class $\beta_1$ meeting rational curves of class $\beta_2$. They are uniquely determined by the following rules:

(i) The meeting invariants are symmetric, $m_{\beta_1, \beta_2} = m_{\beta_2, \beta_1}$.

(ii) If either deg($\beta_1$) $\leq$ 0 or deg($\beta_2$) $\leq$ 0, we have $m_{\beta_1, \beta_2} = 0$. 
(iii) If $\beta_1 \neq \beta_2$, then

$$m_{\beta_1, \beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i)g^{ij}n_{0,\beta_2}(S_j) + m_{\beta_1, \beta_2 - \beta_1} + m_{\beta_1 - \beta_2, \beta_2}.$$ 

(iv) If $\beta_1 = \beta_2 = \beta$, we have

$$m_{\beta, \beta} = n_{0,\beta}(c_2(X)) + \sum_{i,j} n_{0,\beta}(S_i)g^{ij}n_{0,\beta}(S_j) - \sum_{\beta_1 + \beta_2 = \beta} m_{\beta_1, \beta_2}.$$ 

The invariants $n_{1,\beta}$ are uniquely defined by the identity

$$\sum_{\beta > 0} \frac{GW}{\chi} q^\beta = \sum_{\beta > 0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta}$$

$$+ \frac{1}{24} \sum_{\beta > 0} n_{0,\beta}(c_2(X)) \log(1 - q^\beta)$$

$$- \frac{1}{24} \sum_{\beta_1, \beta_2} m_{\beta_1, \beta_2} \log(1 - q^{\beta_1 + \beta_2}),$$

where $\sigma(d) = \sum_{i|d} i$.

**Conjecture 1.2.** ([20 Conjecture 1]) The invariants $n_{1,\beta}$ are integers.

For $g \geq 2$, GW invariants vanish for dimension reasons, so the GW/GV type integrality conjecture on CY 4-folds only applies for genus 0 and 1. In [20], GW invariants are computed directly in many examples using localization or mirror symmetry to support the conjectures.

1.2. Review of DT$_4$ invariants. Let us first introduce the set-up of DT$_4$ invariants. We fix an ample divisor $\omega$ on $X$ and take a cohomology class $v \in H^*(X, \mathbb{Q})$.

The coarse moduli space $M_\omega(v)$ of $\omega$-Gieseker semistable sheaves $E$ on $X$ with $\text{ch}(E) = v$ exists as a projective scheme. We always assume that $M_\omega(v)$ is a fine moduli space, i.e. any point $[E] \in M_\omega(v)$ is stable and there is a universal family

$$E \in \text{Coh}(X \times M_\omega(v)).$$

In [4, 10], under certain hypotheses, the authors construct a DT$_4$ virtual class

$$[M_\omega(v)]^\text{vir} \in H_{2-\chi(v,v)}(M_\omega(v), \mathbb{Z}),$$

where $\chi(-, -)$ is the Euler pairing. Notice that this class may not necessarily be algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point $[E] \in M_\omega(v)$, a half-dimensional real subspace

$$\text{Ext}^2_+(E, E) \subset \text{Ext}^2(E, E),$$

of the usual obstruction space $\text{Ext}^2(E, E)$, on which the quadratic form $Q$ defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of form

$$\kappa_* = \pi_+ \circ \kappa : \text{Ext}^1(E, E) \to \text{Ext}^2_+(E, E),$$

where $\kappa$ is a Kuranishi map of $M_\omega(v)$ at $E$ and $\pi_+$ is the projection according to the decomposition $\text{Ext}^2(E, E) = \text{Ext}^2_+(E, E) \oplus \sqrt{-1} \cdot \text{Ext}^2_-(E, E)$.

In [10], local models are glued in three special cases:

1. when $M_\omega(v)$ consists of locally free sheaves only;
2. when $M_\omega(v)$ is smooth;
3. when $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme.

And the corresponding virtual classes are constructed using either gauge theory or algebro-geometric perfect obstruction theory.

The general gluing construction is due to Borisov-Joyce [3] based on Panetev-Töen-Vaqué-Vezzosi’s theory of shifted symplectic geometry [33] and Joyce’s theory of derived $C^\infty$-geometry. The corresponding virtual class is constructed using Joyce’s D-manifold theory (a machinery similar to Fukaya-Oh-Ohta-Ono’s theory of Kuranishi space structures used in defining Lagrangian Floer theory).

In this paper, all computations and examples will only involve the virtual class constructions in situations (2), (3), mentioned above. We briefly review them as follows:

---

1One needs to assume that $M_\omega(v)$ can be given a (-2)-shifted symplectic structure as in Claim 3.29 [14] to apply their constructions. In the stable pairs case, we show this can be done in Lemma [15].
• When $M_\omega(v)$ is smooth, the obstruction sheaf $Ob \to M_\omega(v)$ is a vector bundle endowed with a quadratic form $Q$ via Serre duality. Then the DT$_4$ virtual class is given by
$$[M_\omega(v)]^{vir} = \text{PD}(e(Ob, Q)).$$

Here $e(Ob, Q)$ is the half-Euler class of $(Ob, Q)$ (i.e., the Euler class of its real form $Ob_+$), and PD$(-)$ is its Poincaré dual. Note that the half-Euler class satisfies
$$e(Ob, Q)^2 = (-1)^{\frac{\text{rk}(Ob)}{2}} e(Ob),$$
if $\text{rk}(Ob)$ is even,
$$e(Ob, Q) = 0,$$
if $\text{rk}(Ob)$ is odd.

• When $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme, roughly speaking, this means that at any closed point $[F] \in M_\omega(v)$, we have Kuranishi map of type
$$\kappa: \text{Ext}^1(F, F) \to \text{Ext}^2(F, F) = V_F \oplus V_F^*,$$
where $\kappa$ factors through a maximal isotropic subspace $V_F$ of $(\text{Ext}^2(F, F), Q)$. Then the DT$_4$ virtual class of $M_\omega(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_F\}_{F \in M_\omega(v)}$. When $M_\omega(v)$ is furthermore smooth as a scheme, then it is simply the Euler class of the vector bundle $\{V_F\}_{F \in M_\omega(v)}$ over $M_\omega(v)$.

On orientations. To construct the above virtual class (1.5) with coefficients in $\mathbb{Z}$ (instead of $\mathbb{Z}_2$), we need an orientability result for $M_\omega(v)$, which is stated as follows. Let
$$L := \det(\text{RHom})_M(E, E) \in \text{Pic}(M_\omega(v)), \quad \pi_M: X \times M_\omega(v) \to M_\omega(v),$$
be the determinant line bundle of $M_\omega(v)$, equipped with a symmetric pairing $Q$ induced by Serre duality. An orientation of $(L, Q)$ is a reduction of its structure group (from $O(1, \mathbb{C})$) to $SO(1, \mathbb{C}) = \{1\}$; in other words, we require a choice of square root of the isomorphism
$$Q: L \otimes L \to \mathcal{O}_{M_\omega(v)}$$
(1.7)
to construct the virtual class (1.5). An orientability result was first obtained for $M_\omega(v)$ when the CY 4-fold $X$ satisfies $\text{Hol}(X) = \text{SU}(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$ [11, Theorem 2.2] and it has recently been generalized to arbitrary CY 4-folds by [7]. Notice that, if an orientation exists, the set of orientations forms a torsor over $H^0(M_\omega(v), \mathbb{Z}_2)$.

1.3. Stable pair invariants on CY 4-folds. The notion of stable pairs on a CY 4-fold $X$ can be defined similarly as in the case of threefolds [30]. It consists of data
$$(F, s), \quad F \in \text{Coh}(X), \quad s: \mathcal{O}_X \to F$$
where $F$ is a pure one dimensional sheaf and $s$ is surjective in dimension one.

For $\beta \in H^2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, let
$$P_n(X, \beta)$$
(1.8)
be the moduli space of stable pairs $(F, s)$ on $X$ such that $[F] = \beta$, $\chi(F) = n$. It is a projective scheme parametrizing two-term complexes
$$I = (\mathcal{O}_X \to F) \in D^b(\text{Coh}(X))$$
in the derived category of coherent sheaves on $X$.

Similar to moduli spaces of stable sheaves, the stable pair moduli space (1.8) admits a deformation-obstruction theory, whose tangent, obstruction and ‘higher’ obstruction spaces are given by
$$\text{Ext}^1(I, I)_0, \quad \text{Ext}^2(I, I)_0, \quad \text{Ext}^3(I, I)_0,$$
where $(-)_0$ denotes the trace-free part. Note that Serre duality gives an isomorphism $\text{Ext}^1_0 \cong (\text{Ext}^3_0)^\vee$ and a non-degenerate quadratic form on $\text{Ext}^2_0$. Moreover, we have

Lemma 1.3. The stable pair moduli space $P_n(X, \beta)$ can be given the structure of a $(-2)$-shifted symplectic derived scheme in the sense of Pantev-Täen-Vaquic-Vezzosi [33].

Proof. By [33, Theorem 2.7], $P_n(X, \beta)$ is a disjoint union of connected components of the moduli stack of perfect complexes of coherent sheaves of trivial determinant on $X$, whose $(-2)$-shifted symplectic structure is constructed by [33, Theorem 0.1] (see [33, Sect. 3.2, pp. 48] for pull-back to determinant fixed substack).

\[\square\]
Let \( I = (\mathcal{O}_X \to P_n(X, \beta) \to \mathbb{F}) \) be the universal pair, the determinant line bundle
\[
\mathcal{L} := \det(\mathcal{R}\text{Hom}_{\pi_P}(I, I)) \in \text{Pic}(P_n(X, \beta))
\]
is endowed with a non-degenerate quadratic form \( Q \) defined by Serre duality, where \( \pi_P : X \times P_n(X, \beta) \to P_n(X, \beta) \) is the projection. Similarly as before, the orientability issue for the pair moduli space \( P_n(X, \beta) \) is whether the structure group of the quadratic line bundle \( (\mathcal{L}, Q) \) can be reduced from \( O(1, \mathbb{C}) \) to \( \text{SO}(1, \mathbb{C}) = \{1\} \). By \cite{1}, these moduli spaces are always orientable.

**Theorem 1.4.** Let \( X \) be a CY 4-fold and \( \beta \in H_2(X, \mathbb{Z}) \) and \( n \in \mathbb{Z} \) be an integer. Then \( P_n(X, \beta) \) has a virtual class
\[
(1.9) \quad [P_n(X, \beta)]^{\text{vir}} \in H_{2n}(P_n(X, \beta), \mathbb{Z}),
\]
in the sense of Borisov-Joyce \cite{2}, depending on the choice of orientation.

**Proof.** By Lemma 1.3, \( P_n(X, \beta) \) has a \((-2)\)-shifted symplectic structure. By \cite{3}, \( P_n(X, \beta) \) is orientable in the sense stated above. Then we may apply \cite{4} Thm. 1.1 to \( P_n(X, \beta) \).

When \( n = 0 \), the virtual dimension of the virtual class \( 1.9 \) is zero. We define the stable pair invariant
\[
P_{0, \beta} := \int_{[P_0(X, \beta)]^{\text{vir}}} 1 \in \mathbb{Z},
\]
as the degree of the virtual class.

When \( n = 1 \), the (real) virtual dimension of the virtual class \( 1.9 \) is two, so we consider insertions as follows. For integral classes \( \gamma_i \in H^{m_i}(X, \mathbb{Z}) \), \( 1 \leq i \leq n \), let
\[
\tau : H^m(X) \to H^{m-2}(P_1(X, \beta)), \quad \tau(\gamma) := (\pi_P)_*(\pi_X^*\gamma \cup \text{ch}_2(\mathcal{F})),
\]
where \( \pi_X, \pi_P \) are projections from \( X \times P_1(X, \beta) \) to corresponding factors, \( \mathcal{L} = (\pi_X^*\mathcal{O}_X \to \mathbb{F}) \) is the universal pair and \( \text{ch}_2(\mathcal{F}) \) is the Poincaré dual to the fundamental cycle of \( \mathcal{F} \).

We define the stable pair invariant
\[
P_{1, \beta}(\gamma_1, \ldots, \gamma_n) := \int_{[P_1(X, \beta)]^{\text{vir}}} \prod_{i=1}^n \tau(\gamma_i).
\]

### 1.4. Relations with GW/GV conjecture on CY 4

We use the stable pair invariants defined in Section 1.3 to give a sheaf-theoretic approach to the GW/GV conjecture in Section 1.1.

**Conjecture 1.5.** (Genus 0) For a suitable choice of orientation, we have
\[
P_{1, \beta}(\gamma_1, \ldots, \gamma_n) = \sum_{\beta_1 + \beta_2 = \beta, \beta_1, \beta_2 \geq 0} n_{0, \beta_1}(\gamma_1, \ldots, \gamma_n) \cdot P_{0, \beta_2},
\]
where the sum is over all possible effective classes, and we set \( n_{0, 0}(\gamma_1, \ldots, \gamma_n) := 0 \), \( P_{0, 0} := 1 \).

**Conjecture 1.6.** (Genus 1) For a suitable choice of orientation, we have
\[
\sum_{\beta \geq 0} P_{0, \beta} q^\beta = \prod_{\beta > 0} M(q)^{\beta^{n_1, \beta}},
\]
where \( M(q) = \prod_{k \geq 1} (1 - q^k)^{-k} \) is the MacMahon function and \( P_{0, 0} := 1 \).

### 1.5. Heuristic approach to conjectures

In this subsection, we give a heuristic argument to explain why we expect Conjecture 1.5 (and equality \cite{1} to be true. Even in this heuristic discussion, we ignore questions of orientation.

Let \( X \) be an ‘ideal’ CY 4 in the sense that all curves of \( X \) deform in families of expected dimensions, and have expected generic properties, i.e.

1. any rational curve in \( X \) is a chain of smooth \( \mathbb{P}^1 \)'s with normal bundle \( \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0) \), and moves in a compact 1-dimensional smooth family of embedded rational curves, whose general member is smooth with normal bundle \( \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0) \).
2. any elliptic curve \( E \) in \( X \) is smooth, super-rigid, i.e. the normal bundle is \( L_1 \oplus L_2 \oplus L_3 \) for general degree zero line bundle \( L_i \) on \( E \) satisfying \( L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_E \). Furthermore any two elliptic curves are disjoint and disjoint from all families of rational curves on \( X \).
3. there is no curve in \( X \) with genus \( g \geq 2 \).
$P_0(X, \beta)$ and genus 1 conjecture. Under our ideal assumptions, a one-dimensional Cohen-Macaulay scheme $C$ supported in one of our families of rational curves has $\chi(O_C) \geq 1$, so for any stable pair $I = (O_X \to F) \in P_0(X, \beta)$, the sheaf $F$ can only be supported on some rigid elliptic curves in $X$. For a rigid elliptic curve $E$ with $[E] = \beta$ and ‘general’ normal bundle (i.e. direct sum of three degree zero general line bundles on $E$), its contribution to the pair invariant is

$$\sum_{m \geq 0} P_{0,m}[E]q^m = M(q), \text{ where } M(q) = \prod_{k \geq 1}(1 - q^k)^{-k},$$

by a localization calculation (see Theorem 1.10). Similarly, if we have $n_1, \beta (\beta \in H_2(X, Z))$ many such elliptic curves, then they contribute to pair invariants according to the formula:

$$\sum_{\beta \geq 0} P_{0,\beta}q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_1, \beta}.$$ 

$P_1(X, \beta)$ and genus 0 conjecture. Given a stable pair $I = (O_X \to F) \in P_1(X, \beta)$, $F$ may be supported on a union of rational curves and elliptic curves. Let $C := \text{supp}(F)$, then $C = C_1 \sqcup C_2$ is a disjoint union of ‘rational curve components’ and ‘elliptic curve components’. Note a Cohen-Macaulay scheme $D$ in $\text{Tot}^F_{\mathbb{Z}}(-1, -1, 0)$ (resp. in $\text{Tot}^E_{\mathbb{Z}}(L_1 \oplus L_2 \oplus L_3)$, where $E$ a smooth elliptic curve and $L_i$’s are degree zero general line bundles on $E$) satisfies $\chi(O_D) \geq 1$ (resp. $\chi(O_D) \geq 0$).

Thus from the exact sequence

$$0 \to O_C \to F \to Q \to 0,$$

we know if $C_1 \neq \emptyset$, then $Q = 0$ and $F \cong O_{C_1 \sqcup C_2}$ (with $\chi(O_{C_1}) = 1$, $\chi(O_{C_2}) = 0$). Note that when $C_1 = \emptyset$, i.e. when $F$ is supported on elliptic curves, once we include insertions, these stable pairs do not contribute to the invariant

$$\int_{[P_1(X, \beta)]^{vir}} \tau(\gamma).$$

So we only consider the case when $F \cong O_{C_1 \sqcup C_2}$ with $C_1$ supported on rational curves in a one-dimensional family $\{C_t\}_{t \in T}$. We may further assume the support of $C_1$ is smooth with normal bundle $O_{P^1}(-1, -1, 0)$ due to the presence of insertions, at which point it must have multiplicity 1 as well.

Since the families of rational curves are disjoint from the elliptic curves, the moduli space $P_1(X, \beta)$ of stable pairs is a disjoint union of product of rational curve families (with curve class $\beta_1$) and $P_0(X, \beta_2)$ (where $\beta_1 + \beta_2 = \beta$). And a direct calculation shows the corresponding virtual class factors as the product of the fundamental class of those rational curve families and $[P_0(X, \beta_2)]^{vir}$. For $\gamma \in H^4(X)$, we then have

$$\int_{[P_1(X, \beta)]^{vir}} \tau(\gamma) = \sum_{\beta_1 + \beta_2 = \beta, \beta_1, \beta_2 \geq 0} n_{0, \beta_1}(\gamma) \cdot P_{0, \beta_2}.$$ 

$P_n(X, \beta)$ and generating series. For the moduli space $P_n(X, \beta)$ of stable pairs with $n \geq 1$, we want to compute

$$\int_{[P_n(X, \beta)]^{vir}} \tau(\gamma)^n, \quad \gamma \in H^4(X, Z),$$

when $X$ is an ideal CY 4-fold. Let $\{Z_i\}_{i=1}^n$ be 4-cycles which represent the class $\gamma$. For dimension reasons, we may assume for any $i \neq j$ the rational curves which meet with $Z_i$ are disjoint from those with $Z_j$. The insertions cut out the moduli space and pick up stable pairs whose support intersects with all $\{Z_i\}_{i=1}^n$. We denote the moduli space of such ‘incident’ stable pairs by

$$Q_n(X, \beta; \{Z_i\}_{i=1}^n) \subseteq P_n(X, \beta).$$

Then we claim that

$$(1.10) \quad Q_n(X, \beta; \{Z_i\}_{i=1}^n) = \prod_{\beta_0 + \beta_1 + \cdots + \beta_n = \beta} P_0(X, \beta_0) \times Q_1(X, \beta_1; Z_1) \times \cdots \times Q_1(X, \beta_n; Z_n),$$

where $Q_1(X, \beta_i; Z_i)$ is the moduli space of stable pairs supported on rational curves (in class $\beta_i$) which meet with $Z_i$.

Indeed let us take a stable pair $(O_X \to F)$ in $Q_n(X, \beta; \{Z_i\}_{i=1}^n)$. Then $F$ decomposes into a direct sum $\oplus_{i=0}^n F_i$, where $F_i$ is supported on elliptic curves and each $F_i$ for $1 \leq i \leq n$ is supported on rational curves which meet with $Z_i$. As explained before, a Cohen-Macaulay scheme $C$ supported in the family of rational curves (resp. elliptic curves) satisfies $\chi(O_C) \geq 1$
The above arguments give a heuristic explanation for the formula (resp. \( \chi(\mathcal{O}_C) \geq 0 \)), so \( \chi(F_0) \geq 0 \) and \( \chi(F_i) \geq 1 \) for \( 1 \leq i \leq n \). Hence \( \chi(F_0) = 0 \) and \( \chi(F_i) = 1 \) for \( 1 \leq i \leq n \). Therefore \( \text{1.10} \) holds.

Moreover each \( Q_i(X, \beta; Z_i) \) consists of finitely many rational curves which meet with \( Z_i \), whose number is exactly \( n_{0, \beta}(\gamma) \). By counting the number of points in \( P_0(X, \beta_0) \) and \( Q_i(X, \beta_i; Z_i)'s \), we obtain

\[
P_{n, \beta}(\gamma) := \int_{[P_n(X, \beta)]^{vir}} \tau(\gamma)^n = \int_{[Q_n(X, \beta; \gamma)]^{vir}} 1 = \sum_{\beta_0 + \beta_1 + \cdots + \beta_n = \beta} P_{0, \beta_0} \prod_{i=1}^{n} n_{0, \beta_i}(\gamma).
\]

The above arguments give a heuristic explanation for the formula

\[
\sum_{n, \beta} \frac{P_{n, \beta}(\gamma)}{n!} y^n q^\beta = \prod_{\beta} \left( \exp(yq^\beta n_{0, \beta}(\gamma), M(q^\beta)^{n_{1, \beta}}) \right)
\]

mentioned in Section 0.4.

### 2. Compact examples

In this section, we verify Conjectures \( \text{[1.5]} \) and \( \text{[1.6]} \) for certain compact Calabi-Yau 4-folds.

#### 2.1. Sextic 4-folds

Let \( X \) be a smooth sextic 4-fold, i.e., a smooth degree six hypersurface of \( \mathbb{P}^5 \). By the Lefschetz hyperplane theorem, \( H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^5, \mathbb{Z}) \cong \mathbb{Z}. \) In order to verify our conjectures, we may use deformation invariance and assume \( X \) is general in the (projective) space \( \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(6))) \) of degree six hypersurfaces.

**Genus 0.** For the genus 0 conjecture, we have:

**Proposition 2.1.** Let \( X \) be a smooth sextic 4-fold and \( [l] \in H_2(X, \mathbb{Z}) \) be the line class.

Then Conjecture \( \text{[1.5]} \) is true for \( \beta = [l] \) and \( 2[l] \).

**Proof.** In such cases, \( P_{0, \beta}(X) = 0 \) by Proposition \( \text{[2.2]} \). So we only need to show

\[
P_{1, \beta}(X)(\gamma_1, \ldots, \gamma_n) = n_{0, \beta}(\gamma_1, \ldots, \gamma_n).
\]

We consider \( \beta = 2[l] \) as the degree one case follows from the same argument. A Cohen-Macaulay curve \( C \) in \( X \) with \( [C] = \beta \) has \( \chi(\mathcal{O}_C) = 1 \). For a stable pair \( (\mathcal{O}_X \to F) \in P_1(X, \beta) \), there is an exact sequence

\[
0 \to \mathcal{O}_C \to F \to Q \to 0,
\]

where \( C \) is the support of \( F \). Since \( 1 = \chi(F) = \chi(\mathcal{O}_C) + \chi(Q) \), we must have \( Q = 0 \) and \( F \cong \mathcal{O}_C \).

When \( X \) is a general sextic, \( C \) is either a smooth conic or a pair of distinct intersecting lines (see e.g., \( \text{[6]} \) Proposition 1.4). The morphism \( P_1(X, \beta) \to M_{1, \beta}(X) \), \( I = (\mathcal{O}_X \to F) \mapsto F \), to the moduli space \( M_{1, \beta}(X) \) of one dimensional stable sheaves, with \( [F] = \beta \) and \( \chi(F) = 1 \), is an isomorphism. Furthermore, under the isomorphism, we have identifications

\[
\text{Ext}^1(I, I)_0 \cong \text{Ext}^1(F, F)_0 \cong \mathcal{C},
\]

\[
\text{Ext}^2(I, I)_0 \cong \text{Ext}^2(F, F)_0 = 0,
\]

of deformation and obstruction spaces (ref. \( \text{[6]} \) Proposition 2.2). So one can identify virtual classes

\[
[P_1(X, \beta)]^{vir} = [M_{1, \beta}(X)]^{vir},
\]

for a certain choice of orientation. Then Conjecture \( \text{[1.5]} \) reduces to our previous conjecture \( \text{[12]} \) Conjecture 0.2, which has been verified in this setting in \( \text{[6]} \) Theorem 2.4. \( \square \)

**Genus 1.** From \( \text{[20]} \) Table 2, pp. 33], we know genus one GV type invariants of \( X \) are zero for degree one and two classes. In these cases, pair invariants are obviously zero.

**Proposition 2.2.** Let \( X \) be a smooth sextic 4-fold and \( [l] \in H_2(X, \mathbb{Z}) \) be the line class.

Then Conjecture \( \text{[1.6]} \) is true for \( \beta = [l] \) and \( 2[l] \).

\( ^2 \)The map is well-defined as \( \mathcal{O}_C \) is stable (\( \text{[5]} \) Proposition 2.2).
Proof. Let \( \beta = [t] \) or \( 2[t] \). For a stable pair \((\mathcal{O}_X \to F) \in P_0(X, \beta)\), there is an exact sequence
\[
0 \to \mathcal{O}_C \to F \to Q \to 0,
\]
where \( C \) is the support of \( F \) and \( Q \) is zero dimensional. A Cohen-Macaulay curve \( C \) in \( X \) with \([C] = \beta \) has \( \chi(\mathcal{O}_C) \geq 1 \), contradicting with \( \chi(F) = 0 \). So \( R_0(X, \beta) = \emptyset \). \( \Box \)

2.2. Elliptic fibration. For \( Y = \mathbb{P}^3 \), we take general elements
\[
\begin{aligned}
 u &\in H^0(Y, \mathcal{O}_Y(-4K_Y)), \\
v &\in H^0(Y, \mathcal{O}_Y(-6K_Y)).
\end{aligned}
\]
Let \( X \) be a CY 4-fold with an elliptic fibration
\[
(2.1)
\]
given by the equation
\[
zy^2 = x^3 + ux^2z + vz^3
\]
in the \( \mathbb{P}^2 \)-bundle
\[
\mathbb{P}(\mathcal{O}_Y(-2K_Y) \oplus \mathcal{O}_Y(-3K_Y) \oplus \mathcal{O}_Y) \to Y,
\]
where \([x : y : z]\) is the homogeneous coordinate of the above projective bundle. A general fiber of \( \pi \) is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve. Moreover, \( \pi \) admits a section \( \iota \) whose image corresponds to fiber point \([0 : 1 : 0]\).

Let \( h \) be a hyperplane in \( \mathbb{P}^3 \), \( f \) be a general fiber of \( \pi : X \to Y \) and set
\[
(2.2) B = \pi^*h, \quad E = \iota(\mathbb{P}^3) \in H_0(X, \mathbb{Z}).
\]

Genus 0. We consider the stable pair moduli space \( P_1(X, [f]) \) for the fiber class of \( \pi \) and verify Conjecture 13 in this case.

Lemma 2.3. Let \([f]\) be the fiber class of the elliptic fibration \([23]\). Then we have an isomorphism
\[
P_1(X, [f]) \to X,
\]
under which the virtual class satisfies
\[
[P_1(X, [f])]^\text{vir} = \pm \text{PD}(c_3(X)),
\]
where the sign corresponds to the choice of orientation in defining the LHS.

Proof. Since \([f]\) is irreducible, we have a morphism
\[
\phi : P_1(X, [f]) \to M_1([f])(X) \cong X,
\]
to the moduli space \( M_1([f])(X) \) of 1-dimensional stable sheaves on \( X \) with Chern character \((0, 0, 0, [f], 1)\) (which is isomorphic to \( X \) by [12] Lem. 2.1]). The fiber of \( \phi \) over \( F \) is \( \mathbb{P}(H^0(X, F)) \) (ref. [22] pp. 270]).

By [12] Lem. 2.2, any \( F \in M_1([f])(X) \) is scheme-theoretically supported on a fiber, and \( F = (i_t)_*m_x^\vee \) for some \( x \in X_t := \pi^{-1}(t) \), where \( i_t : X_t \to X \) is the inclusion and \( m_x \) is the maximal ideal sheaf of \( x \) in \( X_t \). By Serre duality, we have
\[
H^1(X, F) \cong H^1(X_t, m_x^\vee) \cong H^0(X_t, m_x) = 0.
\]
Hence \( H^0(X, F) \cong \mathbb{C} \), and \( \phi \) is an isomorphism.

Next, we compare the obstruction theories. Let \( I = (\mathcal{O}_X \to F) \in P_1(X, [f]) \) be a stable pair. By applying \( \mathbf{R}\text{Hom}_X(-, F) \) to \( I \to \mathcal{O}_X \to F \), we obtain a distinguished triangle
\[
\mathbf{R}\text{Hom}_X(F, F) \to \mathbf{R}\text{Hom}_X(\mathcal{O}_X, F) \to \mathbf{R}\text{Hom}_X(I, F),
\]
whose cohomology gives an exact sequence
\[
(2.3) 0 \to H^1(X, F) \to \text{Ext}_X^1(I, F) \to \text{Ext}_X^2(F, F) \to H^2(X, F) = 0.
\]
From the distinguished triangle
\[
F \to I[1] \to \mathcal{O}_X[1],
\]
we have the diagram

\[
\begin{array}{ccc}
R\Gamma(O_X)[1] & \xrightarrow{\varphi} & R\Gamma(O_X)[1] \\
R\text{Hom}_X(I, F) & \xrightarrow{\varphi} & R\text{Hom}_X(I, I)[1] \\
& & \xrightarrow{\varphi} R\text{Hom}_X(I, O_X)[1] \\
R\text{Hom}_X(I, I)[0] & \xrightarrow{\varphi} & R\text{Hom}_X(F, O_X)[2],
\end{array}
\]

where the horizontal and vertical arrows are distinguished triangles. By taking cones, we obtain a distinguished triangle

\[R\text{Hom}_X(I, F) \to R\text{Hom}_X(I, I)[0] \to R\text{Hom}_X(F, O_X)[2],\]

whose cohomology gives an exact sequence

\[0 \to \text{Ext}^1_X(I, F) \to \text{Ext}^2_X(I, I)[0] \to H^1(X, F) = 0.\]

Combining (2.3) and (2.4), we can identify the obstruction spaces

\[
\text{Ext}^2_X(I, I)[0] = \text{Ext}^2_X(F, F).
\]

Then under the isomorphism \(\phi\), their virtual classes can be identified. The identification of the virtual class of \(M_{1,|f|}(X)\) with the Poincaré dual of the third Chern class of \(X\) can be found in \([12, \text{Lem. 2.1}].\)

Then by \([12, \text{Prop. 2.3}],\) we have the following

**Proposition 2.4.** Let \(\pi : X \to Y\) be the elliptic fibration \([2.4].\) Then Conjecture \(1.5\) is true for fiber class \(\beta = [f]\) and \(\gamma = B^2\) or \(B : E\) \([2.3].\)

**Genus 1.** We consider the stable pair moduli space \(P_0(X, r[f])\) for multiple fiber classes \(r[f]\) \((r \geq 1)\) of \(\pi\) and confirm Conjecture \(1.6\) in this case.

**Lemma 2.5.** For any \(r \in \mathbb{Z}_{\geq 2},\) there exists an isomorphism

\[P_0(X, r[f]) \cong \text{Hilb}^r(\mathbb{P}^3),\]

under which the virtual class is given by

\[[P_0(X, r[f])]^{\text{vir}} = (-1)^r \cdot [\text{Hilb}^r(\mathbb{P}^3)]^{\text{vir}},\]

for certain choice of orientation in defining the LHS, where \([\text{Hilb}^r(\mathbb{P}^3)]^{\text{vir}}\) is the DT \(3\) virtual class \(3.3\).

**Proof.** The proof is similar to the one in \([38, \text{Proposition 6.8}].\) We show that the natural morphism

\[\pi^* : \text{Hilb}^r(\mathbb{P}^3) \to P_0(X, r[f])\]

is an isomorphism. Let \((s : O_X \to F) \in P_0(X, r[f])\) be a stable pair. By the Harder-Narasimhan and Jordan-Hölder filtrations, we have

\[0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F,\]

where the quotient \(E_i = F_i/F_{i-1}\)'s are non-zero stable sheaves with decreasing slopes

\[\chi(E_1)_{r_1} > \chi(E_2)_{r_2} > \cdots > \chi(E_n)_{r_n}.\]

Here the slope of a zero dimensional sheaf is defined to be infinity.

Since \(F\) is a pure one dimensional sheaf, so \(E_1 = F_1\) can not be zero dimensional \((r_1 \geq 1)\). Therefore \(\chi(E_i) = (0, 0, 0, r_i[f], \chi(E_i))\) for some \(r_i \geq 1\). The stability of \(E_i\) implies that it is scheme theoretically supported on some fiber \(X_{p_i} = \pi^{-1}(p_i)\) of \(\pi\), i.e. \(E_i = (\iota_{p_i})_*(E_i')\) for some \(\iota_{p_i} : X_{p_i} \hookrightarrow X\) and stable sheaf \(E_i' \in \text{Coh}(X_{p_i}).\)

Since \(s : O_X \to F\) is surjective in dimension one, so is the composition \(O_X \to F \to E_n.\) By adjunction, there is an isomorphism

\[\text{Hom}_X(O_X, E_n) \cong \text{Hom}_{X_{p_i}}(O_{X_{p_i}}, E_n') \neq 0,\]

which implies that \(\chi(E_n') \geq 0,\) hence \(\chi(E_n) \geq 0.\) Then

\[0 = \chi(F) = \sum_{i=1}^{n} \chi(E_i) \geq 0\]
implies that $\chi(E_i) = 0$ for any $i$, and hence $E'_n \cong \mathcal{O}_{X_{p_n}}$ \cite{15} Proposition 1.2.7.

By diagram chasing, we obtain a morphism $I_{X_{p_n}} \to F_{n-1}$ for the ideal sheaf $I_{X_{p_n}} \subseteq \mathcal{O}_X$ of $X_{p_n}$, which is surjective in dimension one. Then so is the composition

$$I_{X_{p_n}} \to F_{n-1} \to E_{n-1}.$$  

We have the isomorphism

$$\text{Hom}_X(I_{X_{p_n}}, E_{n-1}) \cong \text{Hom}_{X_{p_n-1}}(I^*_{p_n-1}I_{X_{p_n}}, E_{n-1}) \neq 0.$$  

Notice that $I_{X_{p_n}} \cong \pi^* I_{p_n}$ for ideal sheaf $I_{p_n} \subseteq \mathcal{O}_{\mathbb{P}^3}$ of $p_n \in \mathbb{P}^3$ by the flatness of $\pi$, so

$$I^*_{p_n-1}I_{X_{p_n}} \cong \pi^* N^\vee_{(p_{n-1})/\mathbb{P}^3} \cong (O_{X_{p_n-1}})^{\oplus 3}, \ \text{if} \ \ p_{n-1} = p_n,$$

$$I^*_{p_n-1}I_{X_{p_n}} \cong O_{X_{p_n-1}}, \ \text{if} \ \ p_{n-1} \neq p_n.$$  

In either case, similarly as before, we have $E'_{n-1} \cong O_{X_{p_n-1}}$. Moreover the morphism (2.7) is a pull-back of a surjection $I_{p_n} \to O_{p_n-1}$ by $\pi^*$.

By repeating the above argument, we see that each $E_i$ is isomorphic to $O_{X_{p_i}}$, so $O_{X} \to F$ is surjective and given by a pull back of a surjection $O_{\mathbb{P}^3} \to O_Z$ by $\pi^*$ for some zero dimensional subscheme $Z \subseteq \mathbb{P}^3$ with length $n$. Using the section $\iota$ of $\pi: X \to \mathbb{P}^3$, we have the morphism $\iota^*: P_0(X, r[f]) \to \text{Hilb}^r(\mathbb{P}^3)$, which gives an inverse of (2.6). Therefore the morphism (2.6) is an isomorphism.

It remains to compare the virtual classes. We take $I_Z \in \text{Hilb}^r(\mathbb{P}^3)$ and use the spectral sequence

$$\text{Ext}^r_{\mathbb{P}^3}(I_Z, I_Z \otimes R^*\pi_4\mathcal{O}_X) \Rightarrow \text{Ext}^r_X(\pi^*I_Z, \pi^*I_Z),$$

where $R^*\pi_4\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^3} \oplus K_{\mathbb{P}^3}[-1]$. This gives canonical isomorphisms

$$\text{Ext}^r_X(\pi^*I_Z, \pi^*I_Z) \cong \text{Ext}^r_{\mathbb{P}^3}(I_Z, I_Z),$$

$$\text{Ext}^r_X(\pi^*I_Z, \pi^*I_Z) \cong \text{Ext}^r_{\mathbb{P}^3}(I_Z, I_Z) \oplus \text{Ext}^r_{\mathbb{P}^3}(I_Z, I_Z)^\vee.$$  

Furthermore, Kuranishi maps for deformations of $\pi^*I_Z$ on $X$ can be identified with Kuranishi maps for deformations of $I_Z$ on $\mathbb{P}^3$. Similar to \cite{11} Theorem 6.5, we are done. \hfill $\square$

**Proposition 2.6.** Let $\pi: X \to Y$ be the elliptic fibration \cite{22} and $[f]$ be the fiber class. Then Conjecture \cite{17} is true for $\beta = r[f]$ ($r \geq 1$), i.e.

$$\sum_{r=0}^{\infty} P_{0, r[f]} q^r = M(q)^{-20},$$

for certain choice of orientation in defining the LHS, where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function and we define $P_{0, 0[f]} = 1$.

**Proof.** Combining Lemma 2.5 (where we choose sign to be $(-1)^r$ according to the parity of $r$) and the generating series for zero-dimensional DT invariants \cite{25} \cite{24} \cite{23}, we obtain the formula. Notice from \cite{20} Table 7, we have $n_{1,[f]} = -20$ and $n_{1,k[f]} = 0$ for $k \neq 1$ (which can also be checked from GW theory). \hfill $\square$

### 2.3. Quintic fibration

We consider a compact Calabi-Yau 4-fold $X$ which admits a quintic 3-fold fibration structure

$$\pi: X \to \mathbb{P}^1,$$

i.e. $\pi$ is a proper morphism whose general fiber is a smooth quintic 3-fold $Y \subseteq \mathbb{P}^4$. Examples of such CY 4-fold contain resolution of degree 10 orbifold hypersurface in $\mathbb{P}^5(1, 1, 2, 2, 2, 2)$ and hypersurface of bidegree $(2, 5)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ (see \cite{20} pp. 33-37).

In this section, we discuss the irreducible curve class in a quintic fiber for these two examples. Here we only consider genus 1 invariants.

**Genus 1.** Conjecture \cite{13} predicts that for an irreducible class $\beta$ and a suitable choice of orientation, we have

$$P_{0, \beta} = n_{1, \beta} := GW_{1, \beta} + \frac{1}{24} GW_{0, \beta}(c_2(X)).$$

Note that the genus 1 invariants $n_{1, \beta}$ for irreducible $\beta$ are zero for both quintic fibration examples in \cite{20}, where their computations of $GW_{1, \beta}$ are based on BCOV theory \cite{3}. The pair invariant $P_{0, \beta}$ is obviously zero in this case since we have:

**Lemma 2.7.** Let $\beta \in H_2(X, \mathbb{Z})$ be an irreducible class. The pair moduli space $P_0(X, \beta)$ is empty if and only if any curve $C \in \text{Chow}_\beta(X)$ in the Chow variety is a smooth rational curve.
Then the universal stable pair is given by $O$ where\footnote{map identifies $P$ $F$ whose fiber over $0$}
\[ \text{h}^1(C, O_C) = 0, \]
i.e. $h^1(C, O_C) = 0$, which implies that $C$ is a smooth rational curve.

With this lemma, we can verify Conjecture \ref{conj:stable-pairs} for irreducible classes in more examples.

**Proposition 2.8.** Conjecture \ref{conj:stable-pairs} is true for irreducible class $\beta \in H_2(X, \mathbb{Z})$ when $X$ is either
(1) one of the quintic fibrations in \cite{24};
(2) a smooth complete intersection in a projective space;
(3) one of the complete intersections in Grassmannian varieties in \cite{13}.

**Proof.** In all above cases, any curve $C$ in an irreducible class $[C] = \beta$ is a smooth $\mathbb{P}^1$, by Lemma \ref{lem:curve-class} $P_{0, \beta}(X) = 0$ and hence $P_{0, \beta} = 0$. Meanwhile for those examples in (1) and (3), Klemm-Pandharipande \cite{20} and Gerhardus-Jockers \cite{13} used BCOV theory \cite{3} to compute genus 1 GW invariants and found that $n_{1, \beta} = 0$. As for (2), we have Popa’s computation of genus 1 GW invariants using hyperplane principle developed by Li-Zinger \cite{31, 59}. \hfill $\square$

\subsection*{2.4. Product of elliptic curve and CY 3-fold}

In this subsection, we consider a CY 4-fold of type $X = Y \times X$, where $Y$ is a projective CY 3-fold and $E$ is an elliptic curve.

**Genus 0.** We study Conjecture \ref{conj:stable-pairs} for an irreducible curve class of $X = Y \times E$. If $\beta = [E]$, $P_{r, \beta} = 0$, the conjecture is obviously true (in fact for any $r \geq 1$, one can show Conjecture \ref{conj:stable-pairs} is true for $\beta = r[E]$). Below we consider curve classes coming from the CY 3-fold.

**Lemma 2.9.** Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a CY 3-fold $Y$.

Then the pair deformation-obstruction theory of $P_n(Y, \beta)$ is perfect in the sense of $\Xi \Xi$.

Hence we have an algebraic virtual class\footnote{\[ P_n(Y, \beta) \] is a smooth rational curve.}
\[ [P_n(Y, \beta)]_{\text{vir}} \in A_{n-1}(P_n(Y, \beta), \mathbb{Z}). \]

**Proof.** For any stable pair $I_Y = (s : O_Y \to F) \in P_n(Y, \beta)$ with $\beta$ irreducible, we know $F$ is stable (ref. \cite{92}, pp. 270)), hence
\[ \text{Ext}^1_Y(F, F) \cong \text{Hom}_Y(F, F)^{\vee} \cong C. \]

Applying $R\text{Hom}_Y(-, F)$ to $I_Y \to O_Y \to F$, we obtain a distinguished triangle
\[ R\text{Hom}_Y(F, F) \to R\text{Hom}_Y(O_Y, F) \to R\text{Hom}_Y(I_Y, F), \]
whose cohomology gives an exact sequence\footnote{\[ \text{Ext}^i_Y(I_Y, F) = 0 \]
for $i \geq 3$.}
\[ 0 \to H^2(Y, F) \to \text{Ext}^2_Y(I_Y, F) \to \text{Ext}^2_Y(F, F) \to 0 \to \text{Ext}^3_Y(I_Y, F) \to 0. \]

Hence $\text{Ext}^2_Y(I_Y, F) = 0$ for $i \geq 3$ and $\text{Ext}^2_Y(F, F) \cong \text{Ext}^2_Y(F, F) \cong C$. By truncating $\text{Ext}^2_Y(I_Y, F) = C$, the pair deformation theory is perfect. \hfill $\square$

In particular, when $n = 1$, the virtual class $[P_1(Y, \beta)]_{\text{vir}}$ has zero degree. We show the following virtual push-forward formula.

**Proposition 2.10.** Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a CY 3-fold $Y$. Then
\[ \int_{[P_1(Y, \beta)]_{\text{vir}}} 1 = \int_{[M_{1, \beta}(Y)]_{\text{vir}}} 1, \]
where $M_{1, \beta}(Y)$ is the moduli scheme of 1-dimensional stable sheaves on $Y$ with Chern character $(0, 0, 0, \beta, 1)$.

**Proof.** Since $\beta$ is irreducible, there is a morphism
\[ f : P_1(Y, \beta) \to M_{1, \beta}(Y), \quad (O_Y \to F) \mapsto F, \]
whose fiber over $[F]$ is $\mathbb{P}(H^0(Y, F))$. Let $F \to M_{1, \beta}(Y) \times Y$ be the universal sheaf. Then the above map identifies $P_1(Y, \beta)$ with $\mathbb{P}(\pi_M, F)$ where $\pi_M : M_{1, \beta}(Y) \times Y \to M_{1, \beta}(Y)$ is the projection. Then the universal stable pair is given by
\[ I = (O_Y \times P_1(Y, \beta)) \to F^\dagger, \quad F^\dagger := (\text{id}_Y \times f)^* F \otimes O(1), \]
where $O(1)$ is the tautological line bundle on $\mathbb{P}(\pi_M, F)$ and $s$ is the tautological map.
Let \( \pi_P : P_I(Y, \beta) \times Y \to P_I(Y, \beta) \) be the projection, there exists a distinguished triangle

\[
(R\hom_{\pi_P}(F^+, F^+)[1])^\vee \to (R\hom_{\pi_P}(I, F^+))^\vee \to (R\hom_{\pi_P}(O_Y \times P_I(Y, \beta), F^+))^\vee.
\]

By considering a derived extension of the morphism \( f \), the first two terms in (2.10) are the restriction of cotangent complexes of the corresponding derived schemes to the classical underlying schemes. They are obstruction theories (see [35, Sect. 1.2]), which fit into a commutative diagram

\[
\begin{array}{c}
\tau^{\geq 1} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee & \to & \tau^{\geq 1} (R\hom_{\pi_P}(I, F^+))^\vee \\
\tau^{\geq 1} (R\hom_{\pi_P}(O_Y \times P_I(Y, \beta), F^+))^\vee \\
\end{array}
\]

where the bottom vertical arrows are truncation functors.

Note the above obstruction theories are not perfect. To kill \( h^{-2} \), as in [16, Sect. 4.4], we consider the top part of trace map

\[
t : R\hom_{\pi_P}(F^+, F^+)[1] \to R^3\pi_P*(O_Y \times P_I(Y, \beta)[-2],
\]

whose cone is \( \tau^{\leq 1} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee \). Then we have a commutative diagram

\[
\begin{array}{c}
R^3\pi_P*(O_Y \times P_I(Y, \beta)[-2])^\vee & \to & R^3\pi_P*(O_Y \times P_I(Y, \beta)[-2])^\vee \\
\tau^{\leq 1} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee \\
\end{array}
\]

By taking cones, we obtain a distinguished triangle

\[
\tau^{\leq 1} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee \to \text{Cone} (\alpha) \to (R\hom_{\pi_P}(O_Y \times P_I(Y, \beta), F^+))^\vee.
\]

Since \( R^3\pi_P*(O_Y \times P_I(Y, \beta)[-2])^\vee \) is a vector bundle concentrated in degree \(-2\) and \( \tau^{\geq 1} (\cdot) \) has cohomology in degree greater than \(-2\), so we have a commutative diagram

\[
\begin{array}{c}
\tau^{\leq 1} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee & \to & \text{Cone} (\alpha) \\
\tau^{\geq 1} (f^*L_{M_1, \beta}(Y)) \\
\end{array}
\]

To kill \( h^1 \) of the left upper term, we consider the inclusion

\[
O_{P_I(Y, \beta)}[1] \to \tau^{\leq 1} (R\hom_{\pi_P}(F^+, F^+)[1]),
\]

whose restriction to a closed point \( I = (O_Y \to F) \) induces an isomorphism \( C \to \hom(F, F) \).

The cone of the inclusion is \( \tau^{[0,1]} (R\hom_{\pi_P}(F^+, F^+)[1]) \).

Then we have a commutative diagram

\[
\begin{array}{c}
\text{Cone} (\beta) & \to & \tau^{[0,1]} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee \\
\tau^{[0,1]} (R\hom_{\pi_P}(O_Y \times P_I(Y, \beta), F^+))^\vee [-1] & \to & \tau^{\leq 1} (R\hom_{\pi_P}(F^+, F^+)[1])^\vee \\
\beta \\
\end{array}
\]

\[
(O_{P_I(Y, \beta)}[1])^\vee & \to & (O_{P_I(Y, \beta)}[1])^\vee.
\]
As \((O_{P_1(Y,\beta)}[1])^\vee\) is a vector bundle concentrated on degree 1, so we get commutative diagram
\[
\begin{array}{ccc}
\tau^\beta-1(f^\ast L_{\mathcal{M}_1,\beta}(Y)) & \xrightarrow{\phi_1} & \tau^\beta-1\mathcal{L}_{P_1(Y,\beta)}
\\ 
\phi_2 & & \phi_3
\\ 
\end{array}
\]

It is easy to see that \(\phi_1\) and \(\phi_2\) define perfect obstruction theories. By diagram chasing on cohomology, \(\phi_3\) defines a perfect relative obstruction theory. Then we apply Manolache’s virtual push-forward formula \([26]\):
\[
f_*[P_1(Y,\beta)]^{vir}_{\text{pair}} = c \cdot [M_1,\beta(Y)]^{vir},
\]
where the coefficient \(c\) is the degree of the virtual class of the relative obstruction theory \(\phi_3\) and can be shown to be 1 by base-change to a closed point.

Now we come back to CY 4-fold \(X = Y \times E\) and show the virtual class \([P_1(Y,\beta)]^{vir}_{\text{pair}}\) defined using pair deformation-obstruction theory naturally arises in this setting.

**Proposition 2.11.** Let \(X = Y \times E\) be a product of a CY 3-fold \(Y\) with an elliptic curve \(E\). For an irreducible curve class \(\beta \in H_2(Y,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})\), we have an isomorphism
\[
P_n(X, \beta) \cong P_n(Y, \beta) \times E.
\]

The virtual class of \(P_n(X, \beta)\) satisfies
\[
[P_n(X, \beta)]^{vir} = [P_n(Y, \beta)]^{vir}_{\text{pair}} \otimes [E],
\]
for certain choice of orientation in defining the LHS. Here \([P_n(Y, \beta)]^{vir}_{\text{pair}} \in A_{n-1}(P_n(Y, \beta),\mathbb{Z})\) is the virtual class defined in Lemma \([22, 27]\).

**Proof.** As \(\beta\) is irreducible, for \(I_X = (s : O_X \to E) \in P_n(X, \beta)\), \(E\) is stable (ref. \([22\text{ pp. 270}]\)), hence \(E\) is scheme theoretically supported on some \(Y \times \{t\}, t \in E\) (e.g. \([12\text{ Lem. 2.2}]\)).

Let \(i_t : Y \times \{t\} \to X\) be the inclusion, then \(E = (i_t)_*F\) for some \(F \in \text{Coh}(Y)\). By adjunction, we have
\[
\text{Hom}_X(O_X, E) \cong \text{Hom}_Y(O_Y, F).
\]

Hence, the morphism
\[
(\mathcal{I}_Y := (s : i_t^*O_X \to F), t) \mapsto (s : O_X \to (i_t)_*F) := I_X
\]
is bijective on closed points. Next, we compare their deformation-obstruction theories.

Denote \(i = i_t\). From the distinguished triangle
\[
i_*F \to I_X[1] \to O_X[1],
\]
we have the diagram
\[
\begin{array}{ccc}
\text{RHom}_X(I_X, i_*F) & \xrightarrow{\text{RHom}_X(i_*F)} & \text{RHom}_X(I_X, O_X)[1] \\
\downarrow & & \downarrow \\
\text{RHom}_X(I_X, i_*F)[1] & \xrightarrow{\text{RHom}_X(i_*F)} & \text{RHom}_X(i_*F, O_X)[1] \\
\downarrow & & \downarrow \\
\text{RHom}_X(I_X, I_X)_0[1] & \xrightarrow{\text{RHom}_X(i_*F)} & \text{RHom}_X(i_*F, O_X)[2],
\end{array}
\]
where the horizontal and vertical arrows are distinguished triangles. By taking cones, we obtain a distinguished triangle
\[
\text{RHom}_X(I_X, i_*F) \to \text{RHom}_X(I_X, I_X)_0[1] \to \text{RHom}_X(i_*F, O_X)[2].
\]

On the other hand, from the distinguished triangle
\[
I_X \to O_X \to i_*F,
\]
and the isomorphism (see e.g. \([12\text{ Proposition-Definition 3.3}]\)):
\[
Li^*I_X \cong F \oplus (F \otimes N_{Y \times \{t\}/X}[1]), \quad \text{where } N_{Y \times \{t\}/X} = O_{Y \times \{t\}},
\]
we can obtain the isomorphism
\[
Li^*I_X \cong I_Y \oplus F,
\]
which implies that
\[ \mathbf{R} \text{Hom}_X(I_X, i_* F) \cong \mathbf{R} \text{Hom}_Y(I_Y, F) \oplus \mathbf{R} \text{Hom}_Y(F, F). \]
Therefore by (2.13), we have the distinguished triangle
\[ \mathbf{R} \text{Hom}_Y(I_Y, F) \oplus \mathbf{R} \text{Hom}_Y(F, F) \to \mathbf{R} \text{Hom}_X(I_X, I_X)_0[1] \to \mathbf{R} \text{Hom}_X(i_* F, \mathcal{O}_X)[2]. \]
It follows that we have the distinguished triangle
\[ (2.15) \quad \mathbf{R} \text{Hom}_Y(I_Y, F) \to \mathbf{R} \text{Hom}_X(I_X, I_X)_0[1] \to T, \]
where \( T \) fits into the distinguished triangle
\[ (2.16) \quad \mathbf{R} \text{Hom}_Y(F, F) \to T \to \mathbf{R} \text{Hom}_X(i_* F, \mathcal{O}_X)[2]. \]
By Serre duality, adjunction and degree shift, (2.16) becomes
\[ T \to \mathbf{R} \text{Hom}_Y(\mathcal{O}_Y, F) \vert^2[-2] \to \mathbf{R} \text{Hom}_Y(F, F) \vert^2[-2], \]
whose dual gives a distinguished triangle
\[ (2.17) \quad \mathbf{R} \text{Hom}_Y(F, F)[2] \to \mathbf{R} \text{Hom}_Y(\mathcal{O}_Y, F)[2] \to T'. \]
Combining (2.18), (2.17), we obtain
\[ T \cong \mathbf{R} \text{Hom}_Y(I_Y, F)'[-2]. \]
Combining with (2.15) and taking the cohomological long exact sequence, we have
\[ 0 \to \mathbf{Ext}^1_Y(I_Y, F) \to \mathbf{Ext}^2_X(I_X, I_X)_0 \to \mathbf{Ext}^1_Y(I_Y, F)' \to . \]
We claim the above exact sequence breaks into short exact sequences
\[ 0 \to \mathbf{Ext}^0_Y(I_Y, F) \to \mathbf{Ext}^1_X(I_X, I_X)_0 \to \mathbb{C} \to 0, \]
\[ 0 \to \mathbf{Ext}^1_Y(I_Y, F) \to \mathbf{Ext}^2_X(I_X, I_X)_0 \to \mathbf{Ext}^1_Y(I_Y, F)' \to 0, \]
since \( \mathbf{Ext}^2_Y(I_Y, F) \cong \mathbb{C} \) (see the proof of Lemma 2.9) and a dimension counting by Riemann-Roch. The first exact sequence above implies that the map (2.11) induces an isomorphism on tangent spaces. The second exact sequence implies that the obstructions of deforming stable pairs on LHS of (2.11) vanish if and only if those on RHS of (2.11) vanish. Therefore, the map (2.11) induces an isomorphism on formal completions of structure sheaves of both sides at any closed point. So (2.11) must be a scheme theoretical isomorphism.

Next, we show \( \mathbf{Ext}^1_Y(I_Y, F) \subseteq \mathbf{Ext}^2_X(I_X, I_X)_0 \) is a maximal isotropic subspace with respect to the Serre duality pairing on \( \mathbf{Ext}^2_X(I_X, I_X)_0 \). For \( u \in \mathbf{Ext}^1_Y(I_Y, F) \), the corresponding element in \( \mathbf{Ext}^2_X(I_X, I_X)_0 \) is given by the composition
\[ I_X \xrightarrow{\alpha} i_* I_Y \xrightarrow{i_* F[1]} i_* F[2] \xrightarrow{i_* F[3]} I_X[4], \]
where the morphism \( \alpha \) is the canonical morphism and \( \beta \) is given by (2.12). For another \( u' \in \mathbf{Ext}^1_Y(I_Y, F) \), it is enough to show the vanishing of the composition
\[ (2.18) \quad I_X \xrightarrow{\alpha \gamma} i_* I_Y \xrightarrow{i_* F[1]} i_* F[2] \xrightarrow{i_* F[3]} i_* F[4]. \]
Since \( \mathbf{Ext}^0_Y(F, I_Y \otimes K_Y) \cong \mathbf{Ext}^3_Y(I_Y, F) = 0 \) (see the proof of Lemma 2.10), the composition \( i_* F[1] \xrightarrow{i_* F[2]} i_* I_Y[2] \) can be written as \( i_* \gamma \). Therefore the composition
\[ i_* I_Y \xrightarrow{i_* F[1]} i_* F[2] \xrightarrow{i_* F[3]} i_* F[4] \]
vanishes, again by \( \mathbf{Ext}^3_Y(I_Y, F) = 0 \).

Moreover, a local Kuranishi map of \( P_n(X, \beta) \) at \( I_X \) can be identified as
\[ (\kappa_{I_X}, 0) : \mathbf{Ext}^0_Y(I_Y, F) \times T_0 E \to \mathbf{Ext}^1_Y(I_Y, F), \]
where \( \kappa_{I_X} \) is a local Kuranishi map of \( P_n(Y, \beta) \) at \( I_Y \). Similarly as [10, Thm. 6.5], we have the desired equality on virtual classes.

Combining the above result with Proposition 2.10, our genus zero conjecture can be reduced to Katz’s conjecture [13].

**Corollary 2.12.** Let \( X = Y \times E \) be a product of a CY 3-fold \( Y \) with an elliptic curve \( E \).

Then Conjecture 1.3 holds for an irreducible curve class \( \beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}) \) if and only if Katz’s conjecture holds for \( \beta \).
Proof. To have non-trivial invariants, we only need to consider insertions of form
\[ \gamma = (\gamma_1, [pt]) \in H^2(Y, \mathbb{Z}) \otimes H^2(E, \mathbb{Z}). \]
By Proposition 2.10 and 2.11 we have
\[ P_{1, \beta}(\gamma) = (\gamma_1 \cdot \beta) \int_{[P_1(Y, \beta)]^{vir}} 1 = (\gamma_1 \cdot \beta) \int_{[M_{1, \beta}(Y)]^{vir}} 1. \]
Then Conjecture 1.5 reduces to Katz’s conjecture. □

Katz’s conjecture has been verified for primitive classes in complete intersection CY 3-folds \[ [12, \text{Cor. A.6}] \]. So we obtain

**Theorem 2.13.** Let \( Y \) be a complete intersection CY 3-fold in a product of projective spaces, \( X = Y \times E \) be the product of \( Y \) with an elliptic curve \( E \). Then Conjecture 1.5 is true for an irreducible curve class \( \beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}) \).

**Genus 1.** Similar to Lemma 2.3 for \( X = Y \times E \) and \( \beta = r[E] \), we have

**Lemma 2.14.** For any \( r \in \mathbb{Z}_{\geq 1} \), there exists an isomorphism
\[ P_0(X, r[E]) \cong \text{Hilb}^r(Y), \]
under which the virtual class is given by
\[ [P_0(X, r[E])]^{vir} = (-1)^r \cdot [\text{Hilb}^r(Y)]^{vir}, \]
for certain choice of orientation in defining the LHS.
Furthermore, their degrees fit into the generating series
\[ \sum_{r=0}^{\infty} P_{0, r[E]} q^r = M(q)^{\chi(Y)}, \]
where \( M(q) = \prod_{k \geq 1} (1 - q^k)^{-k} \) is the MacMahon function and we define \( P_{0, 0[E]} = 1 \).

We check Conjecture 1.4 for this case.

**Theorem 2.15.** Let \( X = Y \times E \) be the product of a CY 3-fold \( Y \) with an elliptic curve \( E \). Then Conjecture 1.4 is true for \( \beta = r[E] \in H_2(X, \mathbb{Z}) \) for any \( r \geq 1 \).

Proof. By Lemma 2.14 we are left to show \( n_{1,[E]} = \chi(Y) \) and \( n_{1,r[E]} = 0 \) if \( r \geq 2 \). Since genus zero Gromov-Witten invariants \( GW_{0,r[E]}(X) = 0 \) for any \( r \geq 1 \), this is equivalent to
\[ \sum_{r=1}^{\infty} GW_{1,r[E]}(X) q^r = \chi(Y) \cdot \sum_{d=1}^{\infty} \sigma(d) d^{-1} q^d, \]
where \( \sigma(d) = \sum_{i \mid d} i \). We have an isomorphism
\[ \overline{M}_{1,0}(X, r[E]) \cong \overline{M}_{1,0}(E, r[E]) \times Y \]
for moduli space \( \overline{M}_{1,0}(X, r[E]) \) of genus 1 stable maps to \( X \). Note that \( \overline{M}_{1,0}(E, r[E]) \) is smooth of expected dimension and consists of \( \frac{r}{2} \) points (modulo automorphisms) (see e.g. [29]). And the genus one invariant for constant map to \( Y \) is \( \chi(Y) \). So \( GW_{1,r[E]}(X) = \chi(Y) \cdot \frac{\sigma(r)}{r} \).

When the curve class \( \beta \in H_2(Y) \subseteq H_2(X) \) comes from \( Y \), we have

**Lemma 2.16.** Let \( X = Y \times E \) be the product of a CY 3-fold \( Y \) with an elliptic curve \( E \). Then for \( \beta \in H_2(Y) \subseteq H_2(X) \), we have
\[ GW_{0,\beta}(\gamma) = \deg([\overline{M}_{0,0}(Y, \beta)]^{vir}) \cdot \int_{\beta} \gamma_1 \cdot \int_E \gamma_2, \quad \text{if } \gamma = \gamma_1 \otimes \gamma_2 \in H^2(Y) \otimes H^2(E); \]
\[ GW_{0,\beta}(\gamma) = 0, \quad \text{if } \gamma \in H^4(Y) \subseteq H^4(X); \quad GW_{1,\beta} = 0. \]

Proof. We have an isomorphism
\[ (2.19) \quad \overline{M}_{0,1}(X, \beta) \cong \overline{M}_{0,1}(Y, \beta) \times E, \]
under which the virtual class satisfies
\[ [\overline{M}_{0,1}(X, \beta)]^{vir} \cong [\overline{M}_{0,1}(Y, \beta)]^{vir} \otimes [E]. \]
By divisor equation, one can compute genus zero GW invariants. \( \overline{M}_{1,0}(X, \beta) \) has a similar product structure as \( 2.19 \). The obstruction sheaf has a trivial factor \( TE = O_E \) in \( E \) direction. So genus one GW invariants vanish. □
Then it is easy to show the following:

**Proposition 2.17.** Let $X = Y \times E$ be the product of a CY 3-fold $Y$ with an elliptic curve $E$. Then Conjecture 1.6 is true for any irreducible class $\beta \in H_2(Y) \subseteq H_2(X)$.

**Proof.** By Lemma 2.16 we know $n_{1,\beta} = 0$. By Proposition 2.11 the virtual dimension of $[P_0(Y, \beta)]^{vir}_{\text{pair}}$ is negative, so $P_{0,\beta} = 0$.

\[ \square \]

2.5. Hyperkähler 4-folds. When the CY 4-fold $X$ is hyperkähler, GW invariants on $X$ vanish as they are deformation-invariant and there are no holomorphic curves for generic complex structures in the $S^2$-twistor family. Another way to see the vanishing is via the cosection localization technique developed by Kiem-Li [19].

Roughly speaking, given a perfect obstruction theory [1 25] on a Deligne-Mumford moduli stack $M$, the existence of a cosection

$$\varphi : \text{Ob}_M \to \mathcal{O}_M$$

of the obstruction sheaf $\text{Ob}_M$ makes virtual class of $M$ localize to closed subspace $Z(\varphi) \subseteq M$ where $\varphi$ is not surjective. In particular, if $\varphi$ is surjective everywhere (in GW theory this is guaranteed by the existence of holomorphic symplectic forms), then the virtual class of $M$ vanishes. Moreover, by truncating the obstruction theory to remove the trivial factor $\mathcal{O}_M$, one can define a reduced obstruction theory and reduced virtual class.

To verify Conjectures 1.4 and 1.5 for hyperkähler 4-folds, we only need to show the vanishing of stable pair invariants of $P_0(X, \beta)$ and $P_1(X, \beta)$.

Cosection and vanishing of DT$_4$ virtual classes. Fix a stable pair $I \in P_n(X, \beta)$, by taking wedge product with square $\text{At}(I)^2$ of the Atiyah class and contraction with the holomorphic symplectic form $\sigma$, we get a surjective map

$$\phi : \text{Ext}^2(I, I)_0 \xrightarrow{\Delta(I)^2} \text{Ext}^4(I, I \otimes \Omega_X^2) \xrightarrow{\sigma} \text{Ext}^4(I, I) \xrightarrow{\tau} H^4(X, \mathcal{O}_X).$$

In fact, we have

**Proposition 2.18.** Let $X$ be a projective hyperkähler 4-fold, $I$ be a perfect complex on $X$ and $Q$ be the Serre duality quadratic form on $\text{Ext}^2(I, I)_0$. Then the composition map

$$\phi : \text{Ext}^2(I, I)_0 \xrightarrow{\Delta(I)^2} \text{Ext}^4(I, I \otimes \Omega_X^2) \xrightarrow{\sigma} \text{Ext}^4(I, I) \xrightarrow{\tau} H^4(X, \mathcal{O}_X).$$

is surjective if either $\text{ch}_2(I) \neq 0$ or $\text{ch}_4(I) \neq 0$. Moreover,

1. if $\text{ch}_4(I) \neq 0$, then we have a $Q$-orthogonal decomposition

$$\text{Ext}^2(I, I)_0 = \text{Ker}(\phi) \oplus C(\text{At}(I)^2 \cdot \sigma),$$

where $Q$ is non-degenerate on each subspace;

2. if $\text{ch}_4(I) = 0$ and $\text{ch}_3(I) \neq 0$, then we have a $Q$-orthogonal decomposition

$$\text{Ext}^2(I, I)_0 = C(\text{At}(I)^2 \cdot \sigma, \kappa_X \circ \text{At}(I)) \oplus (C(\text{At}(I)^2 \cdot \sigma, \kappa_X \circ \text{At}(I)))^\perp,$$

where $Q$ is non-degenerate on each subspace. Here $\kappa_X$ is the Kodaira-Spencer class which is Serre dual to $\text{ch}_3(I)$.

**Proof.** See the proof of [12] Prop. 2.9.

We claim that the surjectivity of cosection maps leads to the vanishing of virtual classes for stable pair moduli spaces (it also applies to other moduli spaces, e.g. Hilbert schemes of curves/points used in DT/PT correspondence [3 9]).

**Claim 2.19.** Let $X$ be a projective hyperkähler 4-fold and $P_n(X, \beta)$ be the moduli space of stable pairs with $n \neq 0$ or $\beta \neq 0$. Then the virtual class satisfies

$$[P_n(X, \beta)]^{vir} = 0.$$
3. Non-compact examples

3.1. Irreducible curve classes on local Fano 3-folds. Let $Y$ be a Fano 3-fold. When $Y$ embeds into a CY 4-fold $X$, the normal bundle of $Y \subseteq X$ is the canonical bundle $K_Y$ of $Y$. By the negativity of $K_Y$, there exists an analytic neighbourhood of $Y$ in $X$ which is isomorphic to an analytic neighbourhood of $Y$ in $K_Y$. Here we simply consider non-compact CY 4-folds of form $X = K_Y$.

Similar to Lemma 2.11 we have

**Lemma 3.1.** Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a Fano 3-fold $Y$.

Then the pair deformation-obstruction theory of $P_n(Y, \beta)$ is perfect in the sense of [12, 25]. Hence we have an algebraic virtual class

$$[P_n(Y, \beta)]^{\text{vir}}_{\text{pair}} \in A_n(P_n(Y, \beta), \mathbb{Z}).$$

**Proof.** For any stable pair $I_Y = (s : O_Y \to F) \in P_n(Y, \beta)$ with $\beta$ irreducible, we know $F$ is stable (ref. [22 pp. 270]), hence

$$\text{Ext}^i_Y(F, F) \cong \text{Hom}_Y(F, F \otimes K_Y)^\vee = 0.$$

Applying $R\text{Hom}_Y(–, F)$ to $I_Y \to O_Y \to F$, we obtain a distinguished triangle

$$(3.1) \quad R\text{Hom}_Y(F, F) \to R\text{Hom}_Y(O_Y, F) \to R\text{Hom}_Y(I_Y, F),$$

whose cohomology gives an exact sequence

$$0 = H^2(Y, F) \to \text{Ext}^2_Y(I_Y, F) \to \text{Ext}^2_Y(F, F) \to 0 \to \text{Ext}^3_Y(I_Y, F) \to 0.$$

Hence $\text{Ext}^i_Y(I_Y, F) = 0$ for $i \geq 2$. Then we can apply the construction of [12, 25]. □

When $n = 1$, similar to Proposition 2.11 we have

**Proposition 3.2.** Let $\beta \in H_2(Y, \mathbb{Z})$ be an irreducible curve class on a Fano 3-fold $Y$. Then

$$f_*[P_1(Y, \beta)]^{\text{vir}}_{\text{pair}} = [M_{1, \beta}(Y)]^{\text{vir}},$$

where $f : P_1(Y, \beta) \to M_{1, \beta}(Y)$, $(O_X \to F) \mapsto F$ is the morphism forgetting the section, $M_{1, \beta}(Y)$ is the moduli scheme of 1-dimensional stable sheaves $E$ on $Y$ with $[E] = \beta$ and $\chi(E) = 1$.

Now we come back to CY 4-fold $X = K_Y$. Similar to Proposition 2.11 we have

**Proposition 3.3.** Let $Y$ be a Fano 3-fold and $X = K_Y$. For an irreducible curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$, we have an isomorphism

$$P_n(X, \beta) \cong P_n(Y, \beta).$$

The virtual class of $P_n(X, \beta)$ satisfies

$$[P_n(X, \beta)]^{\text{vir}} = [P_n(Y, \beta)]^{\text{vir}}_{\text{pair}},$$

for certain choice of orientation in defining the LHS, where $[P_n(Y, \beta)]^{\text{vir}}$ is the virtual class defined in Lemma 3.1.

**Proof.** The proof is the same as the proof of Proposition 2.11. Just note that as in (2.13), there is a distinguished triangle

$$R\text{Hom}_X(I_X, i_* F) \to R\text{Hom}_X(I_X, I_X)_0[1] \to R\text{Hom}_X(i_* F, O_X)[2],$$

where the cohomology of $R\text{Hom}_X(I_X, I_X)_0[1]$ is finite dimensional as $F$ has compact support (although $X$ is non-compact) and we may work with a compactification of $X$. □

**Genus 0.** Combining Proposition 3.2 and Proposition 3.3 Conjecture 1.5 for irreducible curve classes on $K_Y$ is equivalent to the genus zero GV/DT$_3$ conjecture [12 Conjecture 0.2] on $K_Y$ (see also [3, Conjecture 1.2]), which has been verified in the following cases (ref. [3, Prop. 2.1, 2.3, Thm. 2.7]).

**Proposition 3.4.** Conjecture 1.1 is true for any irreducible curve class $\beta \in H_2(K_Y, \mathbb{Z}) \cong H_2(\mathbb{P}^4, \mathbb{Z})$ provided that (i) $Y \subseteq \mathbb{P}^4$ is a smooth hypersurface of degree $d \leq 4$, or (ii) $Y = S \times \mathbb{P}^1$ for a toric del Pezzo surface $S$.

**Genus 1.** When any curve $C$ in an irreducible class $\beta \in H_2(Y)$ is a smooth rational curve, $P_0(Y, \beta) = \emptyset$ by Lemma 2.7, so $P_0,\beta(X) = 0$ (by Proposition 3.3). In this case, to verify Conjecture 1.6 we are reduced to compute GW invariants and show $n_{1, \beta} = 0$. 
Proposition 3.5. Let $Y = \mathbb{P}^3$ and $X = K_Y$. Then Conjecture 3.6 is true for any irreducible curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$.

Proof. When $Y = \mathbb{P}^3$, $n_{1, \beta} = 0$ by [20] Table 1, pp. 31.

3.2. Irreducible curve classes on local surfaces. Let $(S, \mathcal{O}_S(1))$ be a smooth projective surface and

\begin{equation}
\pi: X = \text{Tot}_S(L_1 \oplus L_2) \rightarrow S
\end{equation}

be the total space of direct sum of two line bundles $L_1$, $L_2$ on $S$. Assuming that

\begin{equation}
L_1 \otimes L_2 \cong K_S,
\end{equation}

then $X$ is a non-compact CY 4-fold. For a curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$, we can consider the moduli space $P_n(X, \beta)$ of stable pairs on $X$, which is in general non-compact. In this section, we restrict to the case when the curve class $\beta$ is irreducible such that $L_1 \cdot \beta < 0$, in which case $P_n(X, \beta)$ is compact and smooth.

Lemma 3.6. Let $S$ be a smooth projective surface and $\beta \in H_2(S, \mathbb{Z})$ be an irreducible curve class such that $K_S \cdot \beta < 0$. Then the moduli space $P_n(S, \beta)$ of stable pairs on $S$ is smooth.

Proof. Similar to the proof of Lemma 3.1 for any stable pair $I_S = (s: \mathcal{O}_S \rightarrow F) \in P_n(S, \beta)$ with $\beta$ irreducible, $F$ is stable, hence $\text{Ext}^2_S(F, F) = \text{Hom}_S(F, F \otimes K_S) = 0$.

Applying $\text{RHom}_S(-, F)$ to $I_S \rightarrow \mathcal{O}_S \rightarrow F$, we obtain a distinguished triangle

\begin{equation}
\text{RHom}_S(F, F) \rightarrow \text{RHom}_S(\mathcal{O}_S, F) \rightarrow \text{RHom}_S(I_S, F),
\end{equation}

whose cohomology gives an exact sequence

\begin{align*}
0 & \rightarrow \text{Hom}_S(F, F) \rightarrow H^0(F) \rightarrow \text{Hom}_S(I_S, F) \rightarrow \text{Ext}^1_S(F, F) \rightarrow \\
& \rightarrow H^1(F) \rightarrow \text{Ext}^1_S(I_S, F) \rightarrow \text{Ext}^2_S(F, F) = 0,
\end{align*}

and $\text{Ext}^i_S(I_S, F) = 0$ for $i \geq 2$. We claim the map $\text{Ext}^1_S(F, F) \rightarrow H^1(F)$ above is surjective, then $\text{Ext}^1_S(I_S, F) = 0$ follows from the exact sequence (so the smoothness of moduli follows).

In fact, we only need to show the surjectivity of

\begin{equation}
H^1(\mathcal{O}_C) \xrightarrow{id} H^1(\text{Hom}_S(F, F)) \subseteq \text{Ext}^1_S(F, F) \rightarrow H^1(F),
\end{equation}

where $C$ is the scheme theoretical support of $F$. However, the above map is simply the multiplication by the section $s$, which fits into an exact sequence

\begin{equation}
H^1(\mathcal{O}_C) \xrightarrow{s} H^1(F) \rightarrow H^1(Q) = 0,
\end{equation}

where $Q \cong F/s(\mathcal{O}_S)$ is zero dimensional.

Proposition 3.7. Let $S$ be a smooth projective surface and $L_1$, $L_2$ be two line bundles on $S$ such that $L_1 \otimes L_2 \cong K_S$. Then for any irreducible curve class $\beta \in H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$ such that $L_i \cdot \beta < 0$ ($i = 1, 2$), we have an isomorphism

\begin{equation}
P_n(X, \beta) \cong P_n(S, \beta).
\end{equation}

And the virtual class satisfies

\begin{equation}
[P_n(X, \beta)]^{\text{vir}} = [P_n(S, \beta)] \cdot \left( - \text{RHom}_{\pi_{P_S}}(F, F \otimes L_1) \right),
\end{equation}

for certain choice of orientation in defining the LHS. Here $I_S = (\mathcal{O}_{S \times P_n(S, \beta)} \rightarrow \mathbb{F}) \in D^b(S \times P_n(S, \beta))$ is the universal stable pair and $\pi_{P_S}: S \times P_n(S, \beta) \rightarrow P_n(S, \beta)$ is the projection.

Proof. Under assumption $L_i \cdot \beta < 0$ and $\beta$ is irreducible, as in the proof of [12] Prop. 3.1, one can show, for zero section $i: S \rightarrow X$, the morphism

\begin{equation}
P_n(S, \beta) \rightarrow P_n(X, \beta),
\end{equation}

$I_S := (s: i^* \mathcal{O}_X \rightarrow F) \rightarrow (s: \mathcal{O}_X \rightarrow i_* F) := I_X$

is bijective on closed points. And we have distinguished triangles

\begin{equation}
i_* F \rightarrow I_X[1] \rightarrow \mathcal{O}_X[1],
\end{equation}

\begin{equation}
\text{RHom}_X(I_X, i_* F) \rightarrow \text{RHom}_X(I_X, I_X)[1] \rightarrow \text{RHom}_X(i_* F, \mathcal{O}_X)[2],
\end{equation}

\text{RHom}_X(I_X, i_* F) \rightarrow \text{RHom}_X(I_X, I_X)[1] \rightarrow \text{RHom}_X(i_* F, \mathcal{O}_X)[2].
where the last isomorphism is deduced similarly as (2.14).

It follows that we have a distinguished triangle

\[
\text{RHom}_S(I_S, F) \oplus \text{RHom}_S(F, F \otimes L_1) \rightarrow \text{RHom}_X(I_X, I_X)_0[1] \rightarrow T, 
\]

where \( T \) fits into the distinguished triangle

\[
\text{RHom}_S(F, F \otimes L_2) \oplus \text{RHom}_S(F, F \otimes K_S)[1] \rightarrow T \rightarrow \text{RHom}_X(i_*i^*O_X)[2]. 
\]

By Serre duality, degree shift and taking dual, (3.8) becomes

\[
\text{RHom}_S(F, F \otimes L_1)[1] \oplus \text{RHom}_S(F, F)[2] \rightarrow \text{RHom}_S(O_S, F)[2] \rightarrow T^\vee. 
\]

Combining with (3.4), we obtain a distinguished triangle

\[
\text{RHom}_S(F, F \otimes L_1)[1] \rightarrow \text{RHom}_S(I_S, F)[2] \rightarrow T^\vee, 
\]

whose dual is

\[
T \rightarrow \text{RHom}_S(I_S, F)^\vee[-2] \rightarrow \text{RHom}_S(F, F \otimes L_1)^\vee[-1]. 
\]

By taking cohomology of (3.9), we obtain exact sequences

\[
0 \rightarrow H^0(T) \rightarrow \text{Ext}^2_S(I_S, F)^\vee \rightarrow \text{Ext}^1_S(F, F \otimes L_1)^\vee \rightarrow H^1(T) \rightarrow \text{Ext}^1_S(I_S, F)^\vee \rightarrow \text{Hom}_S(F, F \otimes L_1)^\vee = 0, 
\]

where \( \text{Ext}^1_S(I_S, F) = 0 \) by the proof of Lemma 5.6. Hence

\[
H^0(T) = 0, \quad H^1(T) \cong \text{Ext}^1_S(F, F \otimes L_1)^\vee. 
\]

By taking cohomology of (3.7), we obtain

\[
\text{Ext}^0_S(I_S, F) \cong \text{Ext}^1_X(I_X, I_X)_0, 
\]

\[
0 \rightarrow \text{Ext}^1_S(F, F \otimes L_1) \rightarrow \text{Ext}^2_X(I_X, I_X)_0 \rightarrow H^1(T) \rightarrow \text{Ext}^2_S(I_S, F) \oplus \text{Ext}^2_S(F, F \otimes L_1) = 0, 
\]

hence also the exact sequence

\[
0 \rightarrow \text{Ext}^1_S(F, F \otimes L_1) \rightarrow \text{Ext}^2_X(I_X, I_X)_0 \rightarrow \text{Ext}^1_S(I_S, F \otimes L_1)^\vee \rightarrow 0. 
\]

By the first isomorphism above, we know the map (3.5) induces an isomorphism on tangent spaces. Moreover since \( P_n(S, \beta) \) is smooth (Lemma 4.6) and (3.5) is bijective on closed points, so the map (3.5) is an isomorphism.

As in Proposition 5.3, we can show \( \text{Ext}^1_S(F, F \otimes L_1) \) is a maximal isotropic subspace of \( \text{Ext}^2_X(I_X, I_X)_0 \) with respect to the Serre duality pairing on \( \text{Ext}^2_X(I_X, I_X)_0 \).

Since \( \text{Ext}^1_S(F, F \otimes L_1) = \text{Ext}^1_S(F, F \otimes L_1) = 0, \text{Ext}^2_S(F, F \otimes L_1) \) is constant over \( P_n(S, \beta) \), so it forms a maximal isotropic subbundle of the obstruction bundle of \( P_n(X, \beta) \) whose fiber over \( I_X \in P_n(X, \beta) \) is \( \text{Ext}^2_X(I_X, I_X)_0 \). Then the virtual class has the desired property \([10]\). \( \square \)

It is easy to check Conjecture [1,2,7] for irreducible curve classes on \( \text{Tot}_S(L_1 \oplus L_2) \) in the following setting.

**Proposition 3.8.** Let \( S \) be a del Pezzo surface and \( L_1, L_2 \) be two ample line bundles on \( S \) such that \( L_1 \otimes L_2 \cong K_S \). Denote \( \beta \in H_2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \) to be an irreducible curve class on \( X = \text{Tot}_S(L_1 \oplus L_2) \). Then Conjecture [1,2,7] is true for \( \beta \).

**Proof.** We claim that \( S \) does not contain any \((-1)\) curve. In fact, if \( C \) is a \((-1)\) curve, then

\[
-2 \geq \text{deg}(L_1|C) + \text{deg}(L_2|C) = \text{deg}(K_S|C) = -1. 
\]

So \( S \) is either \( S_2 \) or \( S_1 \times S_1 \), and any curve in an irreducible class is a smooth rational curve. By Lemma 2.6, \( P_0(S, \beta) = \emptyset \), so \( [P_0(X, \beta)]^{\text{vir}} = 0 \) by Proposition 5.4. This matches with Klemm-Pandharipande’s computation [20] pp. 22, 24, i.e. Conjecture [1,6] is true for \( \beta \).

As for the genus 0 conjecture, for any stable pair \((s : O_S \rightarrow F) \in P_1(S, \beta), F \) is stable and supported on some \( C \cong \mathbb{P}^1 \) in \( S \). Then \( F = O_C \) and the morphism \( \phi : P_1(S, \beta) \rightarrow M_{1,1}(S) \rightarrow (O_S \rightarrow F) \rightarrow F \) to the moduli space \( M_{1,1}(S) \) of 1-dimensional stable sheaves \( F \)’s on \( S \) with \([F] = \beta \) and \( \chi(F) = 1 \) is an isomorphism.

As for the moduli space \( \overline{M}_{0,0}(X, \beta) \) of stable maps, we have isomorphisms

\[
\overline{M}_{0,0}(X, \beta) \cong \overline{M}_{0,0}(S, \beta) \cong M_{1,1}(S), 
\]

where the first isomorphism is by the negativity of \( L_i \) (\( i = 1, 2 \)) and the second one is defined by mapping \( f : \mathbb{P}^1 \rightarrow S \) to \( \mathcal{O}_{f(\mathbb{P}^1)} \).
Next, we compare obstruction theories. By Proposition 5.7, the "half" obstruction space of $P_1(X, \beta)$ at $(s: \mathcal{O}_X \rightarrow \mathcal{O}_C)$ is $\text{Ext}^1_{\mathcal{O}}(\mathcal{O}_C, \mathcal{O}_C \otimes L_1)$ which fits into the exact sequence

$$0 \rightarrow H^1(C, L_1|C) \rightarrow \text{Ext}^1_{\mathcal{O}}(\mathcal{O}_C, \mathcal{O}_C \otimes L_1) \rightarrow H^0(C, L_1|C \otimes N_{C/S}) \rightarrow 0.$$ 

Since $S$ is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and $\beta$ is irreducible, all stable maps are embedding. The obstruction space of $M_{0,0}(X, \beta)$ at $f: \mathbb{P}^1 \rightarrow S$ is $H^1(C, N_{C/X}) \cong H^1(\mathbb{P}^1, f^*TX)$ with $C = f(\mathbb{P}^1)$, which fits into the exact sequence

$$0 = H^1(\mathbb{P}^1, f^*TS) \rightarrow H^1(\mathbb{P}^1, f^*TX) \rightarrow H^1(\mathbb{P}^1, f^*(L_1 \oplus L_2)) \rightarrow 0.$$ 

Note that

$$H^1(\mathbb{P}^1, f^*(L_1 \oplus L_2)) \cong H^1(C, L_1|C) \oplus H^0(C, L_1|C \otimes N_{C/S}).$$

A family version of these computations shows the virtual classes satisfy

$$[P_1(X, \beta)]^{\text{vir}} = [M_{0,0}(X, \beta)]^{\text{vir}},$$

up to a sign (for each connected component of the moduli space). It is easy to match the insertions and then verify Conjecture 1.5. More specifically, when $S = \mathbb{P}^2$, $P_{1,1}([pt]) = n_{0,1}([pt]) = -1$ and when $S = \mathbb{P}^1 \times \mathbb{P}^1$, $P_{1,1}([pt]) = n_{0,1}([pt]) = P_{1,1}([pt]) = n_{0,1}([pt]) = 1$ for certain choices of orientations. □

3.3. Small degree curve classes on local surfaces. We learned from discussions with Kool and Monavari [21] that by using relative Hilbert schemes and techniques developed in Kool-Thomas [22], one can do explicit computations of pair invariants in small degrees for non-compact CY 4-folds

$$\text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \quad \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1)).$$

We list the results as follows (where pair invariants are defined with respect to certain of choices of orientation).

Let $X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$, then

1. $P_{0,1} = P_{0,2} = 0$, $P_{0,3} = -1$, $P_{0,4} = 3$,
2. $P_{1,1}([pt]) = -1$, $P_{1,2}([pt]) = 1$, $P_{1,3}([pt]) = -1$, $P_{1,4}([pt]) = 3$.

Let $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$, then

1. $P_{0,2} = 1$, $P_{0,3} = 2$, $P_{0,4} = 5$, $P_{0,5} = 10$,
2. $P_{1,2}([pt]) = 2$, $P_{1,3}([pt]) = 5$.

Comparing with [20] pp. 22, 24, we know our Conjecture 1.5 hold in all above cases.

4. Local curves

Let $C$ be a smooth projective curve of genus $g(C) = g$, and

$$p: X = \text{Tot}_{C}(L_1 + L_2 + L_3) \rightarrow C$$

be the total space of a split rank three vector bundle on it. Assuming that

$$L_1 \otimes L_2 \otimes L_3 \cong \omega_C,$$

then the variety [10] is a non-compact CY 4-fold. Below we set $l_i := \text{deg} L_i$ and may assume that $l_1 \geq l_2 \geq l_3$ without loss of generality.

Let $T = (\mathbb{C}^*)^3$ be the three dimensional complex torus which acts on the fibers of $X$. Its restriction to the subtorus

$$T_0 = \{ t_1 t_2 t_3 = 1 \} \subset T$$

preserves the CY 4-form on $X$ and also the Serre duality pairing on $P_{\ast}(X, \beta)$. In this section, we aim to define equivariant virtual classes of $P_{\ast}(X, \beta)$ using a localization formula with respect to the $T_0$-action [10] [11], and investigate their relations with equivariant GW invariants.

Let $\bullet$ be the point $\text{Spec} \mathbb{C}$ with trivial $T$-action, $\mathbb{C} \otimes t_i$ be the one dimensional $T$-representation with weight 1, and $\lambda_i \in H^2_T(\bullet)$ be its first Chern class. They are generators of equivariant cohomology rings:

$$H^2_T(\bullet) = \mathbb{C}[\lambda_1, \lambda_2, \lambda_3], \quad H^0_{T_0}(\bullet) = \frac{\mathbb{C}[\lambda_1, \lambda_2, \lambda_3]}{(\lambda_1 + \lambda_2 + \lambda_3)} \cong \mathbb{C}[\lambda_1, \lambda_2].$$
4.1. Localization for GW invariants. Let \( j : C \hookrightarrow X \) be the zero section of the projection \( \pi \). We have

\[
H_2(X, Z) = \mathbb{Z}[C],
\]

where \([C]\) is the fundamental class of \( j(C) \). For \( m \in \mathbb{Z}_{>0} \), we consider the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & C \\
\pi \downarrow & & \downarrow \\
\overline{\mathcal{M}}_h(C, m[C]),
\end{array}
\]

where \( \mathcal{C} \) is the universal curve and \( f \) is the universal stable map.

The \( T \)-equivariant GW invariant of \( X \) is defined by

\[
GW_{h,d|C}(X) = GW_{h,d}(X) := \int_{\overline{\mathcal{M}}_h(C, d[C])} e(-R h_* f^* N) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),
\]

where \( N \) is the \( T \)-equivariant normal bundle of \( j(C) \subset X \):

\[
(4.4) \quad N = (L_1 \otimes t_1) \oplus (L_2 \otimes t_2) \oplus (L_3 \otimes t_3).
\]

If \( g(C) > 0 \), the vanishing of genus zero GW invariants

\[
GW_{0,d}(X) = 0, \quad g(C) > 0, \quad d \in \mathbb{Z}_{>0}
\]

follows from \( \overline{\mathcal{M}}_0(C, d[C]) = \emptyset \).

If \( g(C) = 0 \), we have

\[
GW_{0,d}(X) = \int_{[\mathbb{P}^1, d]} e(-R h_* f^* (O_{\mathbb{P}^1}(l_1)t_1 \oplus O_{\mathbb{P}^1}(l_2)t_2 \oplus O_{\mathbb{P}^1}(l_3)t_3)).
\]

For example in the \( d = 1 \) case, \( \overline{\mathcal{M}}_0(\mathbb{P}^1, 1) \) is one point and

\[
(4.5) \quad GW_{0,1}(X) = \lambda_1^{-1} - \lambda_2^{-1} \lambda_3^{-1} - l_1^{-1}.
\]

In the \( d = 2 \) case, a straightforward localization calculation with respect to the \((\mathbb{C}^*)^2\)-action on \( \mathbb{P}^1 \) gives

\[
GW_{0,2}(X) = \frac{1}{8} \lambda_1^{-2} \lambda_2^{-2} \lambda_3^{-2} \left\{ (t_2 - (t_1 - 1)^2 + \cdots) \lambda_1^{-2} + (t_2 - (t_1 - 1)^2 + \cdots) \lambda_2^{-2} + (t_3 - (t_1 - 1)^2 + \cdots) \lambda_3^{-2} + l_1 l_2 l_3 - l_2 l_3 - l_1 l_3 + l_1 l_2 l_3 \right\}.
\]

Here we write \( \mathbb{I} = l \) for \( l \geq 0 \) and \( \mathbb{I} = -l - 1 \) for \( l < 0 \).

4.2. Localization for stable pairs. Similarly, for \( m \in \mathbb{Z}_{\geq 0} \), we want to define (equivariant) stable pair invariant

\[
(4.7) \quad P_{n,m[C]}(X) = [P_n(X, m[C])^{T_n}]^{vir} \cdot e(R \text{Hom}_{\mathbb{C}}(\mathbb{I}, \mathbb{I}^0)^{\text{mov}})^{1/2},
\]

where \( \mathcal{I} = (\mathcal{O}_X \times P_n(X, m[C])) \to \mathbb{P}^r \) is a universal stable pair and \( \pi_p : X \times P_n(X, m[C]) \to P_n(X, m[C]) \) is the projection. Of course, the above equality is not a definition as the virtual class of the fixed locus as well as the square root needs justification.

We will make this precise in specific cases where we compare with GW invariants of \( X \).

Let us first describe stable pairs \((s : \mathcal{O}_X \to F) \in P_n(X, m[C])^T\) which are fixed by the fulltorus \( T \): decompose \( F \) into \( T \)-weight space

\[
p_* F = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^3} F^{i_1, i_2, i_3},
\]

where the \( T \)-weight of \( F^{i_1, i_2, i_3} \) is \((i_1, i_2, i_3)\). We denote an index set

\[
(4.8) \quad \Delta := \{(i_1, i_2, i_3) \in \mathbb{Z}_{\geq 0}^3 : F^{-i_1, -i_2, -i_3} \neq 0\}.
\]

We also have the decomposition

\[
p_* \mathcal{O}_X = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}_{\geq 0}^3} L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3}
\]

into direct sum of weight \((-i_1, -i_2, -i_3)\) factor \( L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \).
The $T$-equivariance of $s$ induces morphisms
\[ s^{i_1, i_2, i_3} : L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \to F^{i_1, i_2, i_3} \]
in $\text{Coh}(C)$ which are surjective in dimension one. It follows that each $F^{i_1, i_2, i_3}$ is either zero or can be written as
\[ F^{i_1, i_2, i_3} = L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \otimes \mathcal{O}_C(Z_{i_1, i_2, i_3}) \]
for some effective divisor $Z_{i_1, i_2, i_3} \subset C$. Moreover, the $p_i \mathcal{O}_X$-module structure on $F$ gives a morphism
\[ F^{i_1, i_2, i_3} \otimes L_1^{-1} \to F^{i_1-1, i_2, i_3} \]
which commutes with $s^{i_1, i_2, i_3}$ and $s^{i_1+1, i_2, i_3}$. Similar morphisms replacing $L_1$ by $L_2, L_3$ exist and have similar commuting property. Hence, for $(i_1, i_2, i_3) \in \Delta$, we have
\[ Z_{i_1-1, i_2, i_3}, Z_{i_1, i_2-1, i_3}, Z_{i_1, i_2, i_3-1} \subseteq Z_{i_1, i_2, i_3}, \]
as divisors in $C$. So the set $\Delta$ is a three dimensional Young diagram, which is finite by the coherence of $F$.

In general, it is difficult to explicitly determine $T_0$-fixed stable pairs. In fact, a $T_0$-fixed stable pair is not necessarily $T$-fixed. Nevertheless, for a $T_0$-fixed stable pair $(s: \mathcal{O}_X \to F)$, $\mathcal{O}_{C^F} := \text{Im} s$ and the corresponding ideal sheaf $I_{C^F}$ are actually $T$-fixed.

**Lemma 4.1.** Let $I = (s: \mathcal{O}_X \to F) \in P_n(X, m[C])^{T_0}$ be a $T_0$-fixed stable pair and $\mathcal{O}_{C^F} := \text{Im} s \subseteq F$. Then the ideal sheaf $I_{C^F} \subseteq \mathcal{O}_X$ is $T$-fixed.

**Proof.** Since $I_{C^F} = \mathcal{H}^0(I)$, it is $T_0$-fixed. For $t \in T$, we have the diagram
\[ \begin{array}{ccc}
0 & \to & I_{C^F} \to \mathcal{O}_X \to \mathcal{O}_{C^F} \to 0 \\
& \downarrow \cong & \\
0 & \to & t^* I_{C^F} \to t^* \mathcal{O}_X \to t^* \mathcal{O}_{C^F} \to 0.
\end{array} \]

The above diagram induces the morphism $u \in \text{Hom}(I_{C^F}, t^* \mathcal{O}_{C^F})$. It is enough to show $u = 0$. For a general point $c \in C$, let $X_c = p^{-1}(c) = C^3$ be the fiber of $p$ at $c$. Then $I_{C^F} |_{X_c}$ is an ideal sheaf of $T_0$-fixed zero dimensional subscheme of $C^3$. Then it is also $T$-fixed by [2, Lemma 4.1]. This implies that the morphism restricted to $X_c$ is a zero map. Then $\text{Im} u \subseteq t^* \mathcal{O}_{C^F}$ is zero on the general fiber of $p$, hence $\text{Im} u = 0$ by the purity of $C^F$.

Another convenient way to determine $T_0$-fixed stable pairs is in the case when $P_n(X, m[C])^T$ is smooth and $\text{Hom}_X(I, F)^T = \text{Hom}_X(I, F)^T$ for any $I = (\mathcal{O}_X \to F) \in P_n(X, m[C])^T$ (see e.g. Sect. 3.3 on toric 3-folds). Then one has $P_n(X, m[C])^T = P_n(X, m[C])^{T_0}$. In the examples below, we will explicitly determine the $T_0$-fixed locus mainly using Lemma 4.1.

### 4.3. $P_{1, m[C]}(X)$ and genus zero conjecture

Let $C = \mathbb{P}^1$ be a smooth rational curve and $X = \mathbb{P}_{\mathbb{P}^1}(l_1, l_2, l_3)$ with $l_1 + l_2 + l_3 = -2$. This serves as the local model for a neighbourhood of a rational curve in a CY 4-fold.

For some special choice of $(l_1, l_2, l_3)$, we can determine $P_{1, m[C]}(X)$ for all $m$.

**Proposition 4.2.** If $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, then $P_{1, m[\mathbb{P}^1]}(X)$ is well-defined and satisfies
\[ P_{1, m[\mathbb{P}^1]}(X) = \pm \lambda_3^{-1}, \quad P_{1, m[\mathbb{P}^1]}(X) = 0, \quad \text{when } m > 1. \]
If $X = \mathcal{O}_{\mathbb{P}^1}(-2, 0, 0)$, then $P_{1, m[\mathbb{P}^1]}(X)$ is well-defined and satisfies
\[ P_{1, m[\mathbb{P}^1]}(X) = \pm \frac{\lambda_1}{\lambda_2 \lambda_3}, \quad P_{1, m[\mathbb{P}^1]}(X) = 0, \quad \text{when } m > 1. \]

**Proof.** Let $(s: \mathcal{O}_X \to F)$ be a $T_0$-fixed stable pair and $\mathcal{O}_{C^F} = \text{Im}(s)$. Then
\[ 1 = \chi(F) = \chi(\mathcal{O}_{C^F}) + \chi(F/\mathcal{O}_{C^F}). \]

So $\chi(\mathcal{O}_{C^F}) = 1$ or 0. By Lemma 4.1 $(s: \mathcal{O}_X \to \mathcal{O}_{C^F})$ is $T$-fixed. From the characterization of $T$-fixed stable pairs, it is of the form
\[ \mathcal{O}_X \to \bigoplus_{(l_1, l_2, l_3) \in \Delta} \mathcal{O}_{\mathbb{P}^1}(l_1)^{-l_1} \otimes \mathcal{O}_{\mathbb{P}^1}(l_2)^{-l_2} \otimes \mathcal{O}_{\mathbb{P}^1}(l_3)^{-l_3} \otimes \mathcal{O}_{\mathbb{P}^1}(Z_{i_1, i_2, i_3}). \]
If $(l_1, l_2, l_3) = (-1, -1, 0)$ or $(-2, 0, 0)$, it is obvious that the only possibility is $\chi(\mathcal{O}_{C^F}) = 1$ (so $F \cong \mathcal{O}_{C^F}$) and $C^F$ is the zero section of $X$. So $P_{1}(X, m[\mathbb{P}^1]) = 0$ unless $m = 1$.
By \([1,7]\), we have

\[
P_{1,\{P^1\}}(X) = \pm \sqrt{(-1)^{i_0} \xi_{\text{ext}}^2(I_{P^1}, I_{P^1})}, \quad e_{T_0}(\text{Ext}^2_X(I_{P^1}, I_{P^1}))
\]

\[
e_{T_0}(\text{Ext}^1_X(I_{P^1}, I_{P^1})) = \pm e_{T_0}(H^1(P^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))
\]

Then the calculation is straightforward. □

By comparing the above computations with the corresponding GW invariants, we obtain the following equivariant analogue of Conjecture \([1,5]\) (note from the above proof, we know \(P_{0, m|C|}(X) = 0\) (m > 1) since \(P_0(X, m|C|) = 0\).

**Corollary 4.3.** Let \(X = O_{P^1}(-1, -1, 0)\) or \(O_{P^1}(-2, 0, 0)\). Then

\[
GW_{0, m}(X) = \sum_{k | m, k \geq 1} \frac{1}{k} P_{1, \{m/k\}|P^1}(X),
\]

for suitable choices of orientation in defining the RHS.

**Proof.** If \(X = O_{P^1}(-1, -1, 0)\), by Aspinwall-Morrison formula, we have

\[
GW_{0, m}(X) = \frac{1}{m^3} \lambda_3^{-1}.
\]

If \(X = O_{P^1}(-2, 0, 0)\), from GW invariants of \(K_{P^1}\) (e.g. \([27]\) Thm 1.1)), we can conclude

\[
GW_{0, m}(X) = \frac{1}{m^3} \frac{m^2}{\lambda_1 \lambda_2 \lambda_3}
\]

Comparing with Proposition 4.2 we are done. □

For general local curve \(X = O_{P^1}(l_1, l_2, l_3)\), we study \(P_{1}(X, m|P^1|)\) for \(m = 1, 2\) as follows.

**Degree one class.** When \(m = 1\), it is easy to show the canonical section

\((s : O_X \to O_{P^1})\)

gives the only \(T_0\)-fixed stable pair in \(P_1(X, |P^1|)\). Similar to Proposition 4.2 we have

\[
P_{1,\{P^1\}}(X) = \frac{e_{T_0}(H^1(P^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))}{e_{T_0}(H^0(P^1, L_1 \otimes t_1 \oplus L_2 \otimes t_2 \oplus L_3 \otimes t_3))}
\]

which coincides with corresponding GW invariant \([1,5]\). Here we have chosen the plus sign in defining \(P_{1,\{P^1\}}(X)\).

**Degree two class.** When \(m = 2\), let \((s : O_X \to F) \in P_1(X, 2|P^1|)\) be a \(T_0\)-fixed stable pair. Then \(F\) is thickened into one of the \(L_i\)-direction, i.e.

\[p_i, F = F_0 \oplus (F_i \otimes t_i^{-1})\]

where \(F_0, F_i\) are line bundles on \(P^1\), hence \(F\) is also \(T\)-fixed. As the \(T\)-weight of \(F_i\) is not of form \((l, l, l)\), so \(T_0\)-invariant sections of \(F\) are also \(T\)-invariant. So we have a commutative diagram

\[
\begin{array}{ccc}
O_{P^1} \otimes L_i^{-1} & \xrightarrow{s^0} & F_0 \otimes L_i^{-1} \\
\downarrow & & \downarrow \phi \\
L_i^{-1} & \xrightarrow{s^i} & F_i \otimes t_i^{-1} \\
\end{array}
\]

where \(s^0\) and \(s^i\) are injective, and surjective in dimension one, \(\phi\) defines the \(p_i, O_X\)-module structure (which is also injective, and surjective in dimension one by the diagram). Denote \(F_0 = O_{P^1}(d_0)\) and \(F_i = F_i \otimes L_i = O_{P^1}(d_i)\), the above diagram is equivalent to a commutative diagram

\[
\begin{array}{ccc}
O_{P^1} & \xrightarrow{s^0} & O_{P^1}(d_0) \\
\downarrow & & \downarrow \phi \\
O_{P^1} & \xrightarrow{s^i} & O_{P^1}(d_i) \\
\end{array}
\]
where \( s^0, s^i \) and \( \phi \) are injective. These impose conditions
\[
0 \leq d_0 \leq d_i, \quad d_0 + d_i = l_i - 1,
\]
where the last equality is because \( \chi(F) = 1 \). It is not hard to show the following

**Lemma 4.4.** We have the following isomorphism

\[
(4.9) \quad P_1(X, 2[\mathbb{P}^1])_{T_0} \cong \prod_{i=1}^{3} \prod_{(d_i, d_i) \in \mathbb{Z}^2} \Pic^{(d_0, d_i)}(\mathbb{P}^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d_0)));
\]

where \( \Pic^{(a,b)}(\mathbb{P}^1) \) denotes the moduli space of triples
\[
(L, L', t), \quad (L, L') \in \Pic^a(\mathbb{P}^1) \times \Pic^b(\mathbb{P}^1), \quad t : L \hookrightarrow L',
\]
and \( t \) is an inclusion of sheaves.

To determine the virtual class \( [P_1(X, 2[\mathbb{P}^1])_{T_0}]^{\vir} \) and the square root in (4.7), we take a \( T_0 \)-fixed stable pair \( I = (s : \mathcal{O}_X \to F) \) and view it as an element in the \( T_0 \)-equivariant \( K \)-theory of \( X \). Then
\[
\chi(I, I)_0 = \chi(F, F) - \chi(\mathcal{O}_X, F) - \chi(F, \mathcal{O}_X) \in K_{T_0}(pt),
\]
where both sides of the equality can be written using grading into \( T_0 \)-weight space.

Similar to [12], Sect. 4.4, we set

\[
(4.10) \quad \chi(F, F)^{1/2} := \chi(j_*F_0, j_*F_0) + \chi(j_*F_0, j_*F_i)t_i^{-1}, \quad \chi(I, I)^{1/2}_0 := \chi(F, F)^{1/2} - \chi(\mathcal{O}_X, F)
\]
where \( F = F_0 + F_i \otimes t_i^{-1} \in K_{T_0}(X) \) and \( j \) is the inclusion of zero section of \( X \).

The \( T_0 \)-fixed and movable part satisfies
\[
\begin{align*}
\chi(I, I)_0^{1/2, \text{fix}} &= \chi(F, F)^{1/2, \text{fix}} - \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d_0)), \\
\chi(I, I)_0^{1/2, \text{mov}} &= \chi(F, F)^{1/2, \text{mov}} - \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d_i - l_i)) \cdot t_i^{-1},
\end{align*}
\]
where \( \chi(F, F)^{1/2, \text{fix}} \) and \( \chi(F, F)^{1/2, \text{mov}} \) was computed in [12], Sect. 4.4. In particular,
\[
\dim \mathbb{C} \chi(F, F)^{1/2, \text{fix}} = 1 - d_i + d_0, \quad \dim \mathbb{C} \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d_0)) = d_0 + 1.
\]
So \( \dim \mathbb{C}(-\chi(I, I)_0^{1/2, \text{fix}}) = d_i \) is the dimension of \( P_1(X, 2[\mathbb{P}^1])_{T_0} \). Thus the virtual class of the associated \( T_0 \)-fixed locus \( P_1(X, 2[\mathbb{P}^1])_{T_0} \) may be defined to be its usual fundamental class.

We can now give a definition of \( P_1,2[\mathbb{P}^1](X) \in \mathbb{Q}(\lambda_1, \lambda_2) \) based on the localization formula [17] and the above discussion. Denote
\[
(F_0, F_i, t), \quad t : F_0 \hookrightarrow F_i
\]
to be the universal object on \( \Pic^{(d_0, d_i)}(\mathbb{P}^1) \times \mathbb{P}^1 \), where \( F_0, F_i \) are line bundles on \( \Pic^{(d_0, d_i)}(\mathbb{P}^1) \times \mathbb{P}^1 \) and \( t \) is the universal injection. Let \( F_i := F_i \boxtimes L_i^{-1} \), and consider its push-forward
\[
\pi : F_i \in \text{Coh}(\Pic^{(d_0, d_i)}(\mathbb{P}^1) \times X), \quad i = 1, 2, 3.
\]
From [17] and [11], we define \( P_1,2[\mathbb{P}^1](X) \) as an element in \( \mathbb{Q}(\lambda_1, \lambda_2) \) by

\[
(4.11) \quad P_{1,2}[\mathbb{P}^1](X) := \sum_{i=1}^{3} \sum_{(d_0, d_i) \in \mathbb{Z}^2} \left( \int_{\Pic^{(d_0, d_i)}(\mathbb{P}^1)} e_{T_0}(N_1) \cdot \int_{\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d_0)))} e_{T_0}(N_2) \right),
\]
where
\[
N_1 := \mathbb{R}\text{Hom}_{\mathbb{P}^1}(j_*F_0, j_*F_0)^{\text{mov}} + \mathbb{R}\text{Hom}_{\mathbb{P}^1}(j_*F_0, j_*F_i \cdot t_i^{-1})^{\text{mov}},
\]
\[
N_2 := -\mathbb{R}(\pi)_*(\mathcal{O}_{\mathbb{P}^1}(1, d_i - l_i) \cdot t_i^{-1})^{\text{mov}}.
\]
Here \( \pi_1 : \Pic^{(d_0, d_i)}(\mathbb{P}^1) \times X \to \Pic^{(d_0, d_i)}(\mathbb{P}^1) \) and \( \pi_2 : \mathbb{P}^d \times \mathbb{P}^1 \to \mathbb{P}^d \) are natural projections. and we have used the isomorphism (4.10).

In the above definition, the second integration can be easily shown to be 1 and the first one has been explicitly determined before [12] Corollary 4.9. So we obtain
Proposition 4.5. Let $X = \mathcal{O}_P(l_1, l_2, l_3)$ with $l_1 + l_2 + l_3 = -2$ and $l_1 \geq l_2 \geq l_3$. Then
\[ P_{1,2}[P^1](X) = -\lambda_1^{-2l_1-2} \lambda_2^{-2l_2-2} (\lambda_1 + \lambda_2)^{-2l_3-2} \]
\[ \cdot \left( \sum_{1 \leq k \leq l_1, k \equiv l_1 \pmod{2}} A(l_1, l_2, l_3, k) + \sum_{1 \leq k \leq l_2, k \equiv l_2 \pmod{2}} B(l_1, l_2, l_3, k) \right), \]
where
\[ A(l_1, l_2, l_3, k) := \text{Res}_{h=0} \left\{ h^{-k}(-\lambda_1 + h)^2(\lambda_2 + h)^{k+l_2} \right\}, \]
\[ (-\lambda_1 - \lambda_2 + h)^{k+l_3}(-\lambda_1 + h)^{l_1-l_2-k}(-2\lambda_1 - \lambda_2 + h)^{l_1-l_3-k}(2\lambda_1 + h)^{k-2-l_1} \right\}, \]
\[ B(l_1, l_2, l_3, k) := \text{Res}_{h=0} \left\{ h^{-k}(-\lambda_2 + h)^2(\lambda_1 + h)^{k+l_3} \right\}, \]
\[ (-\lambda_2 - \lambda_1 + h)^{k+l_1}(-\lambda_2 + h)^{l_2-l_1-k}(-2\lambda_2 - \lambda_1 + h)^{l_2-l_3-k}(2\lambda_2 + h)^{k-2-l_2} \right\}. \]

We pose the following equivariant version of Conjecture 4.6 (note in this case $P_{0, [P^1]}(X) = 0$ as $\chi(\mathcal{O}_P) > 0$). It is consistent with our previous conjecture on one dimensional stable sheaves [12] Conj. 4.10.

Conjecture 4.6. Let $X = \mathcal{O}_P(l_1, l_2, l_3)$ for $l_1 + l_2 + l_3 = -2$. Then
\[ GW_{0,2}(X) = P_{1,2}[P^1](X) + \frac{1}{8} P_{1,2}[P^1](X). \]

Combining Proposition 4.5 and [12] Thm. 4.12, we can verify the conjecture in a large number of examples.

Theorem 4.7. Conjecture 4.6 is true if $|l_1| \leq 10$ and $|l_2| \leq 10$.

4.4. $P_{0, n[C]}(X)$ and genus one conjecture. To complete the heuristic argument for our genus one conjecture in Section 4.3, we consider $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$ where $C$ is an elliptic curve and $L_1 \otimes L_2 \otimes L_3 \cong \omega_C \cong \mathcal{O}_C$.

Lemma 4.8. Let $I \subset \mathcal{O}_X$ be the ideal sheaf of a closed subscheme $Z \subset X$ with $\dim Z \leq 1$. Then we have canonical isomorphisms
\[ \text{Ext}^1_X(I, I)_0 \cong H^0(X, \text{Ext}^1_X(I, I)), \]
\[ \text{Ext}^2_X(I, I)_0 \cong H^0(X, \text{Ext}^1_X(I, I)) \oplus H^1(X, \text{Ext}^1_X(I, I)). \]

Furthermore, if $p_* \mathcal{E}xt^1_X(I, I)$ and $p_* \mathcal{E}xt^2_X(I, I)$ are locally free, then
\[ H^1(X, \text{Ext}^1_X(I, I)) \cong H^0(X, \text{Ext}^2_X(I, I))^\vee. \]

And $H^0(X, \text{Ext}^2_X(I, I))$, $H^1(X, \text{Ext}^1_X(I, I))$ are maximal isotropic subspaces of $\text{Ext}^2_X(I, I)_0$ with respect to Serre duality pairing.

Proof. We have the local to global spectral sequence
\[ E_2^{p,q} = H^p(X, \text{Ext}^q_X(I, I)_0) \Rightarrow \text{Ext}^{p+q}_X(I, I)_0. \]
And
\[ \text{Ext}^0_Y(I, I)_0 = 0, \quad \text{Ext}^{1+1}_Y(I, I)_0 \cong \text{Ext}^{2+1}_Y(I, I) \]
are supported on $Z$. Therefore we have $E_2^{p,0} = 0$ and $E_2^{p,q} = 0$ for $p \geq 2$, $q \geq 1$. Then the above spectral sequence degenerates and [14,12] holds. The latter statement follows from the adjunction
\[ \text{Ext}^1_X(p^* \mathcal{O}_C, \text{Ext}^2_Y(I, I)) = \text{Ext}^1_C(\mathcal{O}_C, p_* \mathcal{E}xt^2_X(I, I)), \]
and the Grothendieck duality
\[ R p_* R \text{Hom}_X(I, I)_0[4] \cong R \text{Hom}_C(R p_* R \text{Hom}_X(I, I)_0, \omega_C[1]) \]
for the projection $p : X \to C$ [11]. □

We describe the torus fixed locus $P_0(X, m[C])^{T_0}$ as follows.
Lemma 4.9. Let $C$ be an elliptic curve and $L_i \in \text{Pic}^0(C)$. Then $(\mathcal{O}_X \to F) \in P_0(X, m[\mathcal{C}])$ is $T_0$-fixed if and only if it is of the form

$$
\mathcal{O}_X \twoheadrightarrow \bigoplus_{(i_1, i_2, i_3) \in \Delta} L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3}
$$

for some three dimensional Young diagram $\Delta \subset \mathbb{Z}^3_{\geq 0}$. In particular, we have

$$
P_0(X, m[\mathcal{C}])^T = P_0(X, m[\mathcal{C}])^{T_0}
$$

in this case.

Proof. The stable pair (4.13) is obviously $T$-fixed, hence $T_0$-fixed. Conversely, for $T_0$-fixed stable pair $(s: \mathcal{O}_X \to F)$ with $\chi(F) = 0$, we denote $\mathcal{O}_Z \rightarrow \text{Im} s$ and then $I_Z$ is $T$-fixed by Lemma 4.1. It follows that $\mathcal{O}_X \to \mathcal{O}_Z$ is of the form

$$
\mathcal{O}_X \twoheadrightarrow \bigoplus_{(i_1, i_2, i_3) \in \Delta} L_1^{-i_1} \otimes L_2^{-i_2} \otimes L_3^{-i_3} \otimes \mathcal{O}_C(Z_{i_1, i_2, i_3}).
$$

Since $c_1(L_i) = 0$ and $F/\mathcal{O}_Z$ is zero dimensional, then

$$
0 = \chi(F) = \chi(\mathcal{O}_Z) + \chi(F/\mathcal{O}_Z) \geq \chi(F/\mathcal{O}_Z) \geq 0.
$$

So $F \cong \mathcal{O}_Z$ and $Z_{i_1, i_2, i_3} = 0$. \qed

We determine stable pair invariants for $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$ when line bundles $L_i \in \text{Pic}^0(C)$ over the elliptic curve $C$ are general.

Theorem 4.10. Let $C$ be an elliptic curve, $L_i \in \text{Pic}^0(C)$ $(i = 1, 2, 3)$ general line bundles satisfying $L_1 \otimes L_2 \otimes L_3 \cong \omega_C$ and $X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3)$.

Then stable pair invariants $P_{0, m[C]}(X)$ (4.17) are well-defined and fit into generating series

$$
\sum_{m \geq 0} P_{0, m[C]}(X) q^m = M(q),
$$

where $M(q) := \prod_{k \geq 1}(1 - q^k)^{-k}$ is the MacMahon function.

Proof. By Lemma 4.9 $P_0(X, m[C])^{T_0}$ is the finite set of three dimensional partitions of $m$, and any $(s: \mathcal{O}_X \to F) \in P_0(X, m[C])^{T_0}$ satisfies $F \cong \mathcal{O}_W$ for some Cohen-Macaulay curve $W$ in $X$. We denote $I$ to be the ideal sheaf of $W$.

Let $U \subset C$ be an open subset on which $L_i$ are trivial. Then $p^{-1}(U) \cong U \times \mathbb{C}^4$ and $I_{p^{-1}(U)}$ is isomorphic to $\pi^* I_Z$ for the $T$-fixed zero-dimensional subscheme $Z \subset \mathbb{C}^3$ corresponding to $\Delta$.

Therefore we have an isomorphism of $T$-equivariant sheaves on $U$

$$
p_* \text{Ext}^k_X(I, I)|_U \cong \text{Ext}^k_{\mathcal{O}_X}(I_Z, I_Z) \otimes \mathcal{O}_U.
$$

Let

$$
\text{Ext}^k_{\mathcal{O}_X}(I_Z, I_Z) = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^3} V_{i_1, i_2, i_3}^{k} \otimes t_1^{i_1} t_2^{i_2} t_3^{i_3}
$$

be the decomposition into $T$-weight spaces. By (4.14), we have

$$
p_* \text{Ext}^k_X(I, I) \cong \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^3} V_{i_1, i_2, i_3}^{k} \otimes t_1^{i_1} t_2^{i_2} t_3^{i_3}.
$$

The relation (4.12) and Lemma 4.8 imply that

$$
\text{Ext}^1_X(I, I)|_{T_0} = \bigoplus_{i \in \mathbb{Z}} V_{i,i,i}^{1}, \quad \text{Ext}^2_X(I, I)|_{T_0} = \bigoplus_{i \in \mathbb{Z}} V_{i,i,i}^{1} \oplus V_{i,i,i}^{2}.
$$

By \cite{[1]} Lemma 4.1, we have $V_{i,i,i}^{1} = 0$ (note $(V_{i,i,i}^{2})^\vee \cong V_{-i, -i, -i}^{1}$). Therefore

$$
[P_0(X, m[C])^{T_0}]^{\text{vir}} = [P_0(X, m[C])].
$$

For the movable part, there are decompositions

$$
\text{Ext}^1_X(I, I)|_{\text{mov}} = \bigoplus_{(i_1, i_2, i_3) \neq (0,0)} V_{i_1, i_2, i_3}^{1} \otimes H^{0}(L_1^{i_1-i_3} \otimes L_2^{i_2-i_3}) \otimes t_1^{i_1-i_3} t_2^{i_2-i_3},
$$

$$
\text{Ext}^2_X(I, I)|_{\text{mov}} = \bigoplus_{(i_1, i_2, i_3) \neq (0,0)} (V_{i_1, i_2, i_3}^{1} \otimes V_{i_1, i_2, i_3}^{2}) \otimes H^{0}(L_1^{i_1-i_3} \otimes L_2^{i_2-i_3}) \otimes t_1^{i_1-i_3} t_2^{i_2-i_3}.
$$
For a general choice of \((L_1, L_2)\), we have \(H^0(L_1^a \otimes L_2^b) = H^1(L_1^a \otimes L_2^b) = 0\) for any \((a, b) \neq (0, 0)\), so the movable part also vanishes. Thus
\[
P_{0,m}[C](X) = \sharp(P_0(X, m[C])^{T_e}),
\]
which is the number of three dimensional partitions of \(m\).

\[
\square
\]

5. Appendices

5.1. Stable pairs and one dimensional sheaves for irreducible curve classes. When \(\beta \in H_2(X, \mathbb{Z})\) be an irreducible curve class on a smooth projective CY 4-fold \(X\), we have a morphism
\[
\phi_n : P_n(X, \beta) \rightarrow M_{n, \beta}(X)
\]
to the moduli scheme of 1-dimensional stable sheaves with Chern character \((0, \beta, n)\) (e.g. [32 pp. 270]), whose fiber over \([F]\) is \(\mathbb{P}(H^0(X, F))\). Note that \(M_{n, \beta}(X)\) is in general a stack instead of scheme when \(\beta\) is arbitrary. The virtual dimension of \(M_{n, \beta}(X)\) satisfies
\[
\text{vir.dim}_\mathbb{R}(M_{n, \beta}(X)) = 2,
\]
by [1][12]. One could use the virtual class to define invariants.

For integral classes \(\gamma_i \in H^{m_i}(X, \mathbb{Z})\), \(1 \leq i \leq \ell\), let
\[
\tau : H^m(X) \rightarrow H^{m-2}(M_{n, \beta}(X)), \quad \tau(\gamma) = \pi_\ast \chi^\gamma \circ \chi_3(\mathbb{F}),
\]
where \(\pi_X, \pi_M\) are projections from \(X \times M_{n, \beta}(X)\) to corresponding factors, \(\mathbb{F} \rightarrow X \times M_{n, \beta}(X)\) is the universal sheaf, and \(\chi_3(\mathbb{F})\) is the Poincaré dual to the fundamental cycle of \(\mathbb{F}\).

Then we define DT\(_4\) invariant
\[
\text{DT}_4(n, \beta \mid \gamma_1, \ldots, \gamma_\ell) := \int_{M_{n, \beta}(X)} \prod_{i=1}^\ell \tau(\gamma_i).
\]
We propose the following conjecture.

**Conjecture 5.1.** For an irreducible class \(\beta \in H_2(X, \mathbb{Z})\), the invariants
\[
\text{DT}_4(n, \beta \mid \gamma_1, \ldots, \gamma_\ell)
\]
are independent of the choice of \(n\) for certain choices of orientation in defining them.

In all compact examples studied in this paper, one can check Conjecture [5][5] holds in these cases. In particular, when \(X = Y \times E\) is the product of a CY 3-fold \(Y\) with an elliptic curve \(E\) and the irreducible class \(\beta \in H_2(Y, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})\) sits inside \(Y\), then Conjecture [5][5] reduces to a special case of the multiple cover formula ([17] Conjecture 6.20), ([38] Conjecture 6.3):
\[
N_{n, \beta} = \sum_{k \geq 1, k|n(\beta)} \frac{1}{k^2} N_{1, \beta/k},
\]
for any \(\beta\) in a CY 3-fold \(Y\), where \(N_{n, \beta} \in \mathbb{Q}\) is the generalized DT invariant [17] which counts one dimensional semistable sheaves \(E\) on \(Y\) with \([E] = \beta\), \(\chi(E) = n\). The above formula is proved when \(\beta\) is primitive in [37] Lemma 2.12 (see also [12] Appendix A.1).

It is an interesting question to define ‘generalized DT\(_4\) type invariant’ counting semistable sheaves on CY 4-folds and search for similar multiple cover formula on CY 4-folds.

5.2. An orientability result for moduli spaces of stable pairs on CY 4-folds.

Let \(X\) be a smooth projective CY 4-fold and \(c \in H^{even}(X)\). For a moduli stack \(M_c\) of coherent sheaves on \(X\) with Chern character \(c\), we define
\[
\mathcal{L} := \text{det}(R\text{hom}(\mathbb{F}, \mathbb{F})),
\]
to be the determinant line bundle, where \(\mathbb{F} \rightarrow M_c \times X\) is the universal sheaf of \(M_c\) and \(p_M : M_c \times X \rightarrow M_c\) is the projection. By Serre duality, we have a non-degenerate pairing
\[
Q : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{O}_{M_c},
\]
which defines a \(O(1, \mathbb{C})\)-structure on \(\mathcal{L}\). The quadratic line bundle \((\mathcal{L}, Q)\) is called orientable if its structure group can be reduced to \(SO(1, \mathbb{C}) = \{1\}\). An orientability result is recently proved on arbitrary CY 4-folds [4], where the proof uses involved tools like semi-topological \(K\)-theory.

To be self-contained, we include here a simpler proof of an orientability result for CY 4-folds with technical assumptions \(\text{Hol}(X) = SU(4)\) and \(H^{odd}(X, \mathbb{Z}) = 0\).

**Lemma 5.2.** Let \(X\) be a CY 4-fold with \(\text{Hol}(X) = SU(4)\) and \(H^{odd}(X, \mathbb{Z}) = 0\). Let \(M_c\) be a finite type open substack of the moduli stack of coherent sheaves with Chern character \(c \in H^{even}(X)\). Then the quadratic line bundle \((\mathcal{L}, Q)\) is orientable.
Proof. By the work of Joyce-Song \cite[Thm. 5.3]{JS}, the moduli stack $M_c$ is 1-isomorphic to a finite type moduli stack of holomorphic vector bundles on $X$ via Seidel-Thomas twists, under which the universal family can be identified (so is the determinant line bundle and Serre duality pairing). Thus we may assume $M_c$ to be a moduli stack of (rank $n$) holomorphic bundles without loss of generality.

Fix a base point $x_0 \in X$, a framing $\phi$ of a vector bundle $E$ is an isomorphism
\[ \phi : E|_{x_0} \cong \mathbb{C}^n. \]
There is a natural $GL(n, \mathbb{C})$-action on $\phi$ changing the framing.

Let $M_c^{framed}$ denote the moduli stacks of framed holomorphic bundles with Chern character $c$, on which $GL(n, \mathbb{C})$ acts by changing framings. Note that $M_c^{framed}$ is a scheme as the stabilizer is trivial and we have a 1-isomorphism
\[ [M_c^{framed}/GL(n, \mathbb{C})] \cong M_c, \quad (E, \phi) \mapsto E, \]
of Artin stacks. The universal family
\[ \mathcal{E} \to M_c^{framed} \times X \]
descends to the universal sheaf $F$ of $M_c$. Let
\[ \mathcal{L} := det(Rp_*\mathcal{H}om(\mathcal{E}, \mathcal{E})) \]
be the determinant line bundle of $M_c^{framed}$, where $p : M_c^{framed} \times X \to M_c^{framed}$ is the projection. One may reduce the orientability problem of $M_c$ to the orientability of $(\mathcal{L}, Q)$, where $Q$ is the quadratic form on $\mathcal{L}$ defined by Serre duality.

We view a holomorphic bundle as an integrable $\overline{\nabla}$ connection on its underlying topological bundle. Then there is a natural embedding of $M_c^{framed}$ (with induced complex analytic topology)
\[ M_c^{framed} \hookrightarrow \mathcal{B}_E \]
into the space $\mathcal{B} := A \times g E_{x_0}$ of framed (not necessarily integrable) connections on the underlying topological bundle $E$. The determinant line bundle $\mathcal{L}$ on $M_c^{framed}$ is the pull-back of a line bundle $\mathcal{L}_B$ on $\mathcal{B}_E$ defined as the determinant of the index bundle of certain twisted Dirac operators and the quadratic form $Q$ on $\mathcal{L}$ extends to $\mathcal{L}_B$ defined using the spin structure of $X$ (see \cite[pp. 50-51]{Joyce}). By \cite[Thm. 1.3]{Joyce}, the quadratic line bundle $(\mathcal{L}_B, Q)$ is orientable. Hence we are done. \hfill $\square$

Then orientability for moduli spaces of stable pairs follows from the orientability of moduli stacks of one dimensional sheaves.

**Theorem 5.3.** Let $X$ be a CY 4-fold with $\text{Hol}(X) = SU(4)$ and $H^{odd}(X, \mathbb{Z}) = 0$. Then for any $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the quadratic line bundle $(\mathcal{L}, Q)$ over $P_n(X, \beta)$ is orientable.

Proof. There is a morphism
\[ \phi : P_n(X, \beta) \to M(0, 0, 0, \beta, n), \]
\[ \phi(F, s) = F \]
to the moduli stack of $1$-dimensional sheaves on $X$ with Chern character $(0, 0, 0, \beta, n)$.

Parallel to the quadratic line bundle $(\mathcal{L} = det(R(\pi_P)_*\mathcal{R}Hom(\mathbb{I}, \mathbb{I}))_n, Q)$ over $P_n(X, \beta)$, there exists a determinant line bundle
\[ \mathcal{L}_M = det(R(\pi_M)_*\mathcal{R}Hom(F, F)) \]
with a quadratic form $Q_M$ over $M(0, 0, 0, \beta, n)$, where $\pi_M : M(0, 0, 0, \beta, n) \times X \to M(0, 0, 0, \beta, n)$ is the projection, and we use $F$ to denote universal sheaf for both $P_n(X, \beta)$ and $M(0, 0, 0, \beta, n)$.

Via the morphism $\phi$, we have an isomorphism
\[ \phi^* \mathcal{L}_M \cong det(R(\pi_P)_*\mathcal{R}Hom(F, F)), \]
where $\pi_P : X \times P_n(X, \beta) \to P_n(X, \beta)$ is the projection.

Since $\mathbb{I} = (O_{X \times P_n(X, \beta)} \to F)$, we have a distinguished triangle
\[ \mathcal{R}Hom(F, F) \to \mathcal{R}Hom(O_{X \times P_n(X, \beta)}, F) \to \mathcal{R}Hom(\mathbb{I}, F), \]
which gives an isomorphism
\[ det(R(\pi_P)_*\mathcal{R}Hom(F, F)) \otimes det(R(\pi_P)_*\mathcal{R}Hom(\mathbb{I}, F)) \cong det(R(\pi_P)_*\mathcal{R}Hom(O_{X \times P_n(X, \beta)}, F)) \]
between determinant line bundles.

Similarly, from the distinguished triangle
\[ \mathcal{R}Hom(\mathbb{I}, F) \to \mathcal{R}Hom(\mathbb{I}, \mathbb{I})_n[1] \to \mathcal{R}Hom(F, O_{X \times P_n(X, \beta)})[2], \]
we have an isomorphism
\[(5.3)\]
\[
\det(R(\pi_P), \mathcal{R}Hom(I, F)) \otimes \det(R(\pi_P), \mathcal{R}Hom(F, O_X \times P_n(X, \beta))) \cong \left( \det(R(\pi_P), \mathcal{R}Hom(I, I)) \right)^{-1}.
\]
Combining (5.2), (5.3) and Serre duality, we obtain
\[
\det(R(\pi_P), \mathcal{R}Hom(I, I)) \cong \det(R(\pi_P), \mathcal{R}Hom(F, F)),
\]
under which the natural quadratic forms on them are identified.

By Lemma 5.2, the structure group of quadratic line bundle \((L_M, Q_M)\) can be reduced to \(SO(1, \mathbb{C})\), so is \((L, Q)\) via the pull-back (5.1). \(\square\)

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