ABSTRACT

Quantum gravity is studied nonperturbatively in the case in which space has a boundary with finite area. A natural set of boundary conditions is studied in the Euclidean signature theory, in which the pullback of the curvature to the boundary is self-dual (with a cosmological constant). A Hilbert space which describes all the information accessible by measuring the metric and connection induced in the boundary is constructed and is found to be the direct sum of the state spaces of all $SU(2)$ Chern-Simons theories defined by all choices of punctures and representations on the spatial boundary $S$. The integer level $k$ of Chern-Simons theory is found to be given by $k = 6\pi G^2 \Lambda + \alpha$, where $\Lambda$ is the cosmological constant and $\alpha$ is a $CP$ breaking phase. Using these results, expectation values of observables which are functions of fields on the boundary may be evaluated in closed form. The Beckenstein bound and 't Hooft-Susskind holographic hypothesis are confirmed, (in the limit of large area and small cosmological constant) in the sense that once the two metric of the boundary has been measured, the subspace of the physical state space that describes the further information that the observer on the boundary may obtain about the interior has finite dimension equal to the exponent of the area of the boundary, in Planck units, times a fixed constant. Finally, the construction of the state space for quantum gravity in a region from that of all Chern-Simons theories defined on its boundary confirms the categorical-theoretic “ladder of dimensions picture” of Crane.
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I. Introduction

In the last years significant progress has been made towards the construction of a quantum theory of gravity in several different directions. Three of these directions, in particular, have involved the use of new ideas and mathematical structures that seem, in different ways, well suited to the problem of describing the geometry of spacetime quantum mechanically. These are string theory, topological quantum field theory, and non-perturbative quantum gravity, based on the loop representation. Furthermore, despite genuine differences, there are a number of concepts shared by these approaches, which suggests the possibility of a deeper relation between them. These include the common use of one dimensional rather than pointlike excitations, as well as the appearance of structures associated with knot theory, spin networks and duality. There are also senses in which each development seems to lead to a picture in which there is a discrete structure at short distances, corresponding to there being only a finite number of degrees of freedom per Planck volume of a system, or even per Planck area of the boundary of the system.

At the same time, each development faces certain internal difficulties, that have so far resisted solution. Further, none of these approaches has been able to overcome the great conceptual difficulties concerned with extending the quantum description to the cosmological case. Because of this situation, it seems that it may be useful to investigate the idea that a kind of unification of these different approaches, taking what is successfully achieved by string theory, topological quantum field theory and non-perturbative quantum gravity, may be the right way to achieve a quantum theory of gravity.

In this paper I would like to propose one approach to bringing together these separate developments. While still incomplete in certain aspects, this approach does show that by incorporating the methods of topological quantum field theory into non-perturbative quantum gravity certain things may be achieved and the theory may be moved forward significantly. Moreover, the key mathematical structure that makes this possible turns out to be closely related to conformal field theory, which is the basic mathematical framework for perturbative string theory.

The basic idea will be to study the quantization of the gravitational field in a context in which we impose a certain kind of boundary condition on spacetime. This boundary will have a finite spatial area, unlike the bound-
aries at infinity that are usually studied in the asymptotically flat context in general relativity. The boundary condition will represent an idealized situation, analogous to the case in which the electromagnetic field is confined to a box. We will assume that the observers, who live outside of the box, can only observe the quantum gravitational field in its interior by making measurements of fields it induced on the walls.

The main result will be that the quantum fields that the observers who measure things at the walls have access to may be described by topological quantum field theory. Or, more properly, by an infinite set of topological field theories. For what we will find is that the state space of the gravitational field of the interior is decomposed into the direct sum of an infinite number of subspaces, each of which may be labeled by eigenvalues of the operators that measure the surface areas of regions of the boundary. In each of these subspaces, further observations are described in terms of a certain topological field theory. One way to say this is that in this formulation of quantum gravity, measurements of the metric geometry of the boundary do not pick out a state in a Hilbert space; they pick out a field theory, whose states describe the possible further knowledge that the observer may gain about the interior.

While we will derive these results completely from the methods of canonical quantum gravity, they confirm the expectations of two lines of thought, coming from topological quantum field theory and string theory.

The first is the program Crane, who proposed a category-theoretic formulation of quantum gravity motivated by the problems of interpretation in quantum cosmology. Basic to this proposal is a framework that Crane calls the “ladder of dimensions” that hypothesizes the existence of certain relationships tying together diffeomorphism invariant quantum field theories in two, three and four dimensions. From the mathematical side, this framework has been so far realized in the constructions of Crane and Yetter and Crane and Frenkel. We will see that non-perturbative quantum gravity with the particular boundary conditions I mention here provides another realization of this mathematical framework. But as this arises directly from the quantization of general relativity, this may be said to confirm Crane’s conjecture that a theory of the gravitational field involving an infinite number of degrees of freedom is in fact the right object to appear on the fourth rung of the ladder of dimensions.

One aspect of Crane’s ladder of dimensions is that quantum gravity in $3 + 1$ dimensions should be described, not by a single vector space, but by a linear structure, which is spanned by basis elements that each correspond
themselves to a vector space. The idea is then that the vector spaces that may appear as the “components of vectors” in the description of the $3 + 1$ dimensional theory are the state spaces of appropriate $2 + 1$ dimensional theories. This is the basic reason that category theory is necessary for this description, because it allows us to talk about a structure in which one may take superpositions of vector spaces, or more simply to describe vectors whose elements are themselves vector spaces [34].

This picture will be realized here in that the metric of a spatial surface will turn out to label the different topological quantum field theories that may be defined on it. The physical state space that describes the 4 dimensional quantum gravitational field in a region bounded by that surface will then be constructed from the state spaces of all the topological quantum field theories that live on it.

This structure also provides a physically well defined framework for the tangle algebra introduced by Baez in [23], and ties its application in quantum gravity to its realization in topological quantum field theory.

The second development that these results support is the holographic hypothesis that has arisen in the work of ’t Hooft [24] and Susskind [25]. The basic idea of this hypothesis is that a diffeomorphism invariant quantum field theory describing the quantum geometry on the interior of a surface is best described by a quantum field theory on the boundary, rather than as a theory of local degrees of freedom in the interior.

Consider a region $\Sigma$ of space which is surrounded by a spacial surface, $S = \partial \Sigma$ with a finite area $A[S]$. ’tHooft and Susskind conjecture that in a quantum theory of gravity the state space describing the physics in $\Sigma$ should be finite dimensional, with a dimension given by

$$d(S, h_{\alpha\beta}) = e^{A[S]/c^2_{Planck}},$$

(1)

where $h_{\alpha\beta}$ is the metric on the two surface and $c$ is some constant of proportionality of order one. This is motivated from two directions, first from the conjectured Beckenstein bound [27] on the information that can be contained within any surface of finite area and second, from the behavior of string theory near horizons [25, 26].

I may note that this motivation is greatly strengthened by the recent derivation of Jacobson [28], who shows, in essence, that the Beckenstein bound must hold on the event horizons of all uniformly accelerating observers if the Einstein equations are to hold in the classical limit.

Here I will show that the Beckenstein bound and the holographic hypothesis are, at least under certain conditions and assumptions, consequences of
non-perturbative quantum gravity. They may be derived under the assumption that the space of states of the quantum gravitational field in the region $\Sigma$ must be spanned by eigenstates of observables that are functions of fields on the boundary $\mathcal{S}$. This would be the case, for example, if the surface is just above a black hole horizon. But, given that the boundary has a finite area, this is plausibly the case generally. Because of what we already have said, once the metric on the boundary is fixed, the possible states of the system that describe further measurements of the gravitational field in the interior that may be made by observers on the surface are described in terms of a particular topological field theory. We will then find that the dimension of the state space of that theory is, in the limit of small cosmological constant, given by (1).

Before beginning to describe these results in more detail, I must mention their main limitations. First, they have so far been derived under a certain set of boundary conditions. Whether there are other boundary conditions for which similar results hold is as yet unknown. Second, the results hold so far only in the case that the relevant gauge group is compact. This restricts us to the case of Euclidean signature if we use the Ashtekar formalism. This is not necessarily unphysical, as it may very well be that Euclidean quantum gravity is the right context to discuss the theory at finite temperature [29]. There is also another alternative which is to use instead the new formalism of Barbero [30], in which case general relativity with real Minkowskian signature is described in terms of a real $SU(2)$ connection. Only in the case of Euclidean signature. However, it is also not impossible that it will be possible to continue at least some of these results to Minkowskian signature in the Ashtekar formalism. In the concluding section I will discuss the prospects for overcoming both of these limitations.

As in many cases in nonperturbative quantum gravity, it is easier to explain the main results than it is to understand all the technical details involved in their derivation. Thus, I will start in the next section with a sketch of the main features of topological quantum field theory I will need, and then, in section $III$, give a summary of the main idea and results. The derivation of these results are given in the next three sections. Section $IV$ describes the classical theory with the self-dual boundary conditions I will impose. The classical hamiltonian analysis of the theory with these boundary conditions is the subject of section $V$, and the quantization, leading to the main results, is given in section $VI$. Some implications and directions for further extension of these results are discussed in the concluding section $VII$. 

6
II. Summary of results from topological quantum field theory

The basic picture of what a 3 dimensional topological field theory is was developed by Witten[2], Atiyah[3] and Segal[4]. I give here a summary of the main ideas and results I will need. I will not give the most general or complete form of the theory, but specialize to the case that will be of interest here, which is the topological quantum field theory associated with \( SU(2) \) Chern-Simons theory.

The basic object that we will be concerned with is a compact two dimensional surface \( S \), with a finite set of marked points, \( y_\alpha, \alpha = 1, ..., n \). These points are also called punctures. Each point \( y_\alpha \) is labeled by half-integers \( j_\alpha \) taken from the set \( 1/2 \leq j_\alpha \leq k/2 \). Here \( k \), which is required to be an integer, is the coupling constant, or level, of the Chern-Simons theory.

These \( j_\alpha \)'s label representations of the quantum group \( SL(2)_q \), with \( q = e^{2\pi i/k+2} \). We may note that the representations of quantum groups play an essential role in topological quantum field theory; one result of the present work will be the discovery of a role for them also in canonical quantum gravity. The fact that there is a highest spin representation is a crucial property of quantum groups (with \( q \) at a root of unity) and this will play a key role in the physical results such as the confirmation of the Beckenstein bound.

The basic idea of a topological quantum field theory is that a finite dimensional vector space \( \mathcal{H}_{S,y_\alpha,j_\alpha} \) may be associated to each set \( (S, y_\alpha, j_\alpha) \). As these vector spaces will be the central objects on which our theory is built, it will be useful to review some of their properties.

Each state space may be considered to be the quantization of \( SU(2) \) Chern-Simons theory appropriate to the spacial manifold \( S-y_\alpha \) with sources at the marked points \( y_\alpha \). Chern-Simons theory is described by the action,

\[
S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} Y_{CS}(a) \tag{2}
\]

where \( \mathcal{M} \) is a three dimensional manifold with boundary \( S = \partial \mathcal{M} \) and \( Y_{CS}(A) = \frac{1}{2}(A^i \wedge dA^i + \frac{1}{2}\epsilon_{ijk}A^i \wedge A^j \wedge A^k) \) is the Chern-Simons form\footnote{We use the notation in which \( a, b, c, ... \) are spatial indices and \( i, j, k \) are internal \( SO(3) \) indices that label the frame fields of space. We will use units here in which \( \hbar = 1 \), but \( G \) is written explicitly, so that \( G \) has dimensions of \( (\text{length})^2 \), while the cosmological constant \( \Lambda \) has dimensions of \( (\text{length})^{-4} \). The combination \( \lambda = G^2 \Lambda \), where \( G \) is Newton’s constant, is then dimensionless.}.
the marked points there are sources, so that the constraint equation arising from (2) is

$$F^i(\sigma) = \sum_{\alpha} \lambda^i_{\alpha} \delta^2(\sigma, y_\alpha)$$

(3)

where $\lambda^i_{\alpha}$ is a source, that lives in the $j_\alpha$’th representation. The treatment of the sources is crucial to the correct quantization of Chern-Simons theory, and will be discussed in Section VI.

To each such choice of surface, marked points and representation is associated the vector space $H_{S, y_\alpha, j_\alpha}^{CS}$, which may be obtained from the quantization of a phase space, which is given by

$$P^{CS} = \frac{\text{Flat, SU}(2) \text{ connections on } S - \{y_\alpha\}}{\text{SU}(2) \text{ gauge transformations}}$$

(4)

with the restrictions (3).

I will denote by $a^i_\alpha$ an $SU(2)$ connection on the two dimensional manifold $S - \{y_\alpha\}$. The basic Poisson brackets are given by,

$$\{a^i_\alpha(\sigma), a^j_\beta(\sigma')\} = \frac{2\pi}{k} \epsilon_{\alpha\beta} \delta^{ij} \delta^2(\sigma, \sigma')$$

(5)

We may note that the phase space $P^{CS}$ is compact, so that there are a finite number of independent observables. A complete set of coordinates on the phase space is given by the loop observables,

$$t[h(\alpha)] = Tr P e^{\oint_{\alpha} a}$$

(6)

which, by the constraints only depend on the homotopy class of $\alpha$, which I denote by $h(\alpha)$. These satisfy an algebra, which is given by

$$\{t[h(\alpha)], t[h(\beta)]\} = \frac{2\pi}{k} \text{Int}[h(\alpha), h(\beta)] \left(t[h(\alpha) \cdot h(\beta)] - t[h(\alpha) \cdot h(\beta^{-1})]\right)$$

(7)

We will call the algebra of the observables of each of these Chern-Simons theory by $A_{S, y_\alpha, j_\alpha}^{CS}$. A complete set of these observables is given by the $t[\alpha]$ for $\alpha \in S - \{y_\alpha\}$. These also satisfy the reality conditions

$$t[\alpha]^* = t[\alpha]$$

(8)

These observables then have representations on the finite dimensional vector spaces $H_{S, y_\alpha, j_\alpha}$ which preserves the reality conditions.
In addition, we will need to take into account the fact that the loop operator \( t[\alpha] \) of a loop \( \alpha \) that surround only one point \( y_\alpha \) are restricted by the condition (3). We will see later in section VI how this is accomplished. This will in fact be the key point in the reduction of the dimensionality of the state space to a finite value. Roughly speaking, the space of states \( \mathcal{H}_{S,y_\alpha,j_\alpha}^{CS} \) is spanned by a basis that corresponds to the independent ways that the spins \( j_\alpha \) may be combined consistently according to the rules of additional of angular momentum of the quantum group \( SL(2)_q \).

Acting also on these states and observables are also the generators of the large diffeomorphism group

\[
Diff_{L}(S - y_\alpha) = \frac{Diff(S - \{y_\alpha}\)}{Diff_0(S - \{y_\alpha\})}
\]

where \( Diff_0 \) denotes the component connected to the identity.

These representations are well understood using the technology of conformal field theory\(^2\)\(^3\)\(^1\)\(^2\)\(^3\)\(^2\)\(^3\). However, as we will not need the details of their construction here, I will not describe them further. I will need only one fact, which is that in the limit of large \( k \) the \( \mathcal{H}_{S,y_\alpha,j_\alpha} \) are given by

\[
\mathcal{H}_{S,y_\alpha,j_\alpha} \rightarrow Inv[\sigma_\alpha R_{j_\alpha}]
\]

where \( R_j \) is the spin \( j \) representation of \( SL(2)_q \) and \( Inv \) means the group invariant part. As the taking of the group invariant part involves a finite number of relations, we have, in the limit of large \( k \)

\[
dim\mathcal{H}_{S,y_\alpha,j_\alpha} \rightarrow \prod_\alpha (2j_\alpha + 1)
\]

as, in the same limit, the dimensions of the quantum group representations go over into their classical counterparts.

The second part of the definition of a topological quantum field theory is a description of the states in \( \mathcal{H}_{S,y_\alpha,j_\alpha} \). To describe this we need to define a quantum spin network\(^3\)\(^5\), which is a generalization of the spin networks introduced by Penrose\(^3\)\(^6\).

A quantum spin network is an oriented graph, which we denote also by \( \Gamma \), which is composed of smooth curve segments that meet at vertices. Each line is labeled by an integer \( l \) between 0 and \( k/2 \) denoting a representation of \( SL(2)_q \), and the rules for addition of \( q \)-angular momentum must be satisfied at the vertices. If there is more than one way for a singlet to be constructed by the direct product of the representations given by the labels of the lines
entering a vertex, then that vertex must be labeled by an index labeling these
different possibilities. There is no additional labeling for trivalent vertices.

According to the Atiyah axioms, it is the case that for every choice of
compact manifolds $\Sigma$ whose boundary is $\mathcal{S}$, together with a quantum spin
network $\Gamma$ in $\Sigma$, that meets the boundary at the marked points $y_\alpha$, so that
the labels on the lines that go into the point match their labels $j_\alpha$, there is
a quantum state in $\mathcal{H}_{\mathcal{S},y_\alpha,j_\alpha}^{CS}$. I will denote this state $|\Gamma,\Sigma >^{CS} \in \mathcal{H}_{\mathcal{S},y_\alpha,j_\alpha}^{CS}$. (Note that in cases where $\Sigma$ is fixed it will not be explicitly indicated, so
these states are simply $|\Gamma >^{CS}$.) We may note that this association of mani-
folds with imbedded $q$-spin networks and states is highly non-unique, as the
dimensionality of each of the state spaces is finite dimensional, while there
are an infinite numbers of manifolds and graphs that can match the bound-
ary. It is essentially because of this non-uniqueness that this construction
yields a useful recursive topological invariant.

There is also a duality relation on the $\mathcal{H}_{\mathcal{S},y_\alpha,j_\alpha}^{CS}$ that reverses the orienta-
tion of $\mathcal{S}$ and takes each representation $j_\alpha$ into its dual. Given this there is
an inner product on states in which

$$<\Gamma,\Sigma|\Gamma',\Sigma'>^{CS} = \int d\mu(a)e^{i\frac{4\pi}{k} \int_M Y_{\mathcal{C}S(a)}^\Sigma T[\Gamma \circ \Gamma']}$$

(12)

where $T[\Gamma \circ \Gamma']$ is the appropriate product of traces of holonomy of the $SU(2)$
connection $a$ and the integral is over connections in the manifold $\Sigma \# \Sigma'$. To define the integral the holonomies of the loops must be framed. Once this is done, this may be evaluated using either conformal field theory
as described by Witten, or by explicitly evaluating the integral, using
perturbation theory. The result is,

$$<\Gamma,\Sigma|\Gamma',\Sigma'>^{CS} = \mathcal{K}^k[\Gamma \circ \Gamma',\Sigma \# \Sigma']$$

(13)

which is the Kauffman bracket of the spin network $\Gamma \circ \Gamma'$ in the compact
manifold $\Sigma \# \Sigma'$.

Finally, this construction can also be used to construct linear maps be-
tween two different state spaces, $\mathcal{H}_{\mathcal{S},y_\alpha,j_\alpha}$, associated with two marked punctu-
tured surfaces $(\mathcal{S}, y_\alpha, j_\alpha)$ and $(\mathcal{S}', y'_\alpha, j'_\alpha)$. For every way of choosing a com-
 pact manifold $\Sigma$, whose boundary is given by $\mathcal{S} \cup \mathcal{S}'$, and for every $q$-spin
network $\Gamma$ inbedded in it that meets each boundary at the marked points, matching the representations, there is a corresponding linear map,

$$\mathcal{M}_{\Gamma, \Sigma} : \mathcal{H}_{S, y, j, \alpha}^k \rightarrow \mathcal{H}_{S', y', j', \alpha}^k.$$ (14)

### III. The basic idea

Now I will describe the basic idea of using the structure of Chern-Simons theory to describe a set of observables in non-perturbative quantum gravity. In this section I sketch and motivate the construction, in later sections the key mathematical relations that are needed to realize this idea are derived from quantum general relativity by imposing an appropriate boundary condition.

Let us assume that we are a family of observers, for whom there is a region of spacetime, $\mathcal{M}$ that we cannot directly observe. This region has a boundary $\partial \mathcal{M}$ that marks the limits of where we may probe with our measuring instruments. We will assume that $\mathcal{M} = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ will be the time direction, or that $\mathcal{M} = \Sigma \times \mathbb{S}^1$, in the case we want to do finite temperature quantum physics. It then follows that the boundary is $\partial \mathcal{M} = \partial \Sigma \times \mathbb{R}$ or $\partial \mathcal{M} = \partial \Sigma \times \mathbb{S}^1$, respectively. As in the last section, $\partial \Sigma$ will be denoted by $\mathcal{S}$, and will be assumed to be compact without boundary.

There are two kinds of situations in which this may be imagined to occur. It could be that we have constructed walls out of a very special material that is perfectly reflecting to gravitational waves, in which case $\mathcal{S}$ denotes the position of the walls. We are then able to position detectors just at the walls, but no further inside. (We may note that because of the results of [37] the walls must be constructed with material that violates the positive energy conditions, but perhaps we can do this by cleverly manipulating the vacuum of a gauge theory.)

The second case that could be relevant is that in which $\partial \mathcal{M}$ is a black hole or cosmological event horizon or a horizon that is present because we are a family of uniformly accelerating observers. In this case the spacial boundary $\mathcal{S}$ corresponds to the intersection of the null horizon with some three dimensional surface of simultaneity.

In either case we have a system, the gravitational field in the interior of $\Sigma$, whose quantization we will now consider, using the methods of nonperturbative quantum gravity.

To proceed we must say something about the gauge invariances and symmetries on the boundary $\mathcal{S}$. The simplest assumption, which we will see
below can be realized explicitly, is that in which we can realize the $SU(2)$ symmetry on the full $\Sigma$, including the boundary, but all diffeomorphisms are constrained to vanish on the spacetime boundary $\partial M$. In this case, neither the diffeomorphism or hamiltonian constraints can generate gauge transformations at the boundary.

Once we have fixed the gauge transformations on the boundary, we can ask what observables are accessible to our family of observers who cannot penetrate inside the boundary $\partial M$. Given that there are no diffeomorphisms on the boundary, we may note that the following two sets of observables will be well defined on the boundary:

- $A[R]$, the area of any finite region $R$ in the spatial boundary $\mathcal{S}$.
- $T[\alpha]$, the Wilson loop of the Ashtekar connection of any loop $\alpha \in \mathcal{S}$.

We may note that by the previous results[22] each area commutes with all the other observables in this set. However, we will see that we cannot assume that the different Wilson loops on the boundary commute with each other. We may note also that as the diffeomorphisms are broken at the surface, different regions and loops on $\mathcal{S}$ are distinct. The diffeomorphisms of the spatial boundary, denoted, $Diff(\mathcal{S})$, are not gauge transformations, instead they comprise a symmetry group. Its generators, $D(v)$, where $v^\alpha$ will denote vector fields on $\mathcal{S}$, must then also be observables.

We will denote the algebra of observables generated by these three sets, $A[R]$, $T[\alpha]$ and $D(v)$ by $A_{\text{boundary}}$. These are, by assumption, what we as observers unable to penetrate into the interior of $\Sigma$ can measure. They comprise a subalgebra of the full algebra of physical observables of the gravitational field in the manifold $M$, which I will denote by $A_{\Sigma}^{\text{phys}}$.

Let us then see how far we can get by applying the results of non-perturbative quantum gravity[8, 10, 11, 21, 22] to this situation. We will try to use these observables to learn as much as possible about the physical state space $\mathcal{H}_{\Sigma}^{\text{phys}}$ that describes the quantum gravitational field in the interior of the region $\Sigma$. We will begin, however, by studying the simpler problem of characterizing the larger space of spatially diffeomorphism invariant states, which we will denote, $\mathcal{H}_{\Sigma}^{\text{diffeo}}$.

As they commute among themselves and with the Wilson loops, and transform naturally among themselves under the $D(v)$, it is natural to begin to characterize the space in terms of the eigenstates of the area operators $A[R]$. We may then seek to use immediately the results[22] that the eigenstates of the operators that measure the area of a region in quantum gravity are the spin network states[38].

As the diffeomorphisms are broken at the boundary, but gauge invariance
is preserved, we may expect that the diffeomorphism and gauge invariant states are described by spin networks $\Gamma$ in $\Sigma$, which are distinct up to diffeomorphisms that leave the boundary fixed. I will denote such states by $|\Gamma >^g_{\text{gr}}$. We may note that the spin networks $\Gamma$ here may meet the boundary and then run along inside of it. Each spin network $\Gamma$ then has associated to it a set of marked points, $y^\Gamma_\alpha$, where it meets the boundary $S$ and labels $j^\Gamma_\alpha$ of the spins of the lines that meet the boundary.

Note that as no line can end, in the boundary or elsewhere, the total spin entering the boundary must vanish.

At this stage, we are dealing with the ordinary spin networks labeled by representations of $SU(2)$, in contrast to the quantum spin networks that play a role in Chern-Simons theory.

As we noted, the states $|\Gamma >^g_{\text{gr}}$ are all eigenstates of the area operators $\hat{A}$. The corresponding eigenvalues defined by

$$\hat{A}[S]|\Gamma >^g_{\text{gr}} = a(S, \gamma)|\Gamma >,$$

are given by

$$a(S, \gamma) = \sum_{y^\Gamma_\alpha \in S} \frac{l^2_{pl}}{2} \sqrt{j_\alpha (j_\alpha + 1)}.$$

The factor of $1/2$ in this expression comes from the fact that the edges of $\Gamma$ do not cross the boundary $S$, but meet it and then run along in it. The intersection number of each with the boundary is then $1/2$.

We may then decompose the state space of quantum gravity according to degenerate eigenspaces of these area observables. Thus, for each set of marked points $y_\alpha$ and representation labels $j_\alpha$ on $S$ there is a subspace $H_{\text{diffeo}}^{y_\alpha, j_\alpha}$ of $H_{\Sigma}^{\text{diffeo}}$, spanned by the states $|\Gamma >^g_{\text{gr}}$ where the $\Gamma$ are the spin networks that meet the boundary at those points, with lines colored with those representations. Each state $|\Gamma >^g_{\text{gr}}$ must be in one of those subspaces, so that

$$H_{\Sigma}^{\text{diffeo}} = \bigoplus_{n=0}^{\infty} \sum_{j_1, \ldots, j_n} \int d^2 y_1 \cdots d^2 y_n H_{y_\alpha, j_\alpha}^{\text{diffeo}}$$

Furthermore, we know that in any physical inner product, states in different of these subspaces must be orthogonal to each other, as they have different eigenvalues of a physical observable.

We may note that some of the observables in $A_S^{\text{boundary}}$ mix up these subspaces, while others leave them invariant. To describe this situation more precisely, let $A_{y_\alpha, j_\alpha}^{g_{\text{gr}}}$ denote the subalgebra of $A_S^{\text{boundary}}$ that leaves invariant
the subspace $\mathcal{H}_{y\alpha,j\alpha}^{\text{diff}}$. All of the area operators, $\mathcal{A}[\mathcal{R}]$ act proportionally to the identity on these subspaces. The Wilson loop operators, $T[\alpha]$, of loops in the boundary are also all in these subspaces, as they each commute with the area operators. But to avoid possible divergences from multiplying Wilson loops, we may consider the subalgebra of the $T[\alpha]$’s generated by the Wilson loops of all $\alpha \in \mathcal{S} - \{y\alpha\}$. It will then be a question of the representation and boundary conditions how to extend this to all loops including those that go through the marked points. Further, for each set of marked points, there exists a subgroup of $\text{Diff}(\mathcal{S})$ that leaves invariant each $\mathcal{H}_{y\alpha,j\alpha}^{\text{diff}}$, which is of course given by $\text{Diff}(\mathcal{S} - \{y\alpha\})$. We will then take each $\mathcal{A}_{S,y\alpha,j\alpha}^{\text{gr}}$ to be generated by the $T[\alpha]$ and $D(v)$ for all loops and vector fields in $\mathcal{S} - \{y\alpha\}$. These are equal to the full algebras $\mathcal{A}_{S,y\alpha,j\alpha}^{\text{gr}}$ of surface observables that leave invariant each $\mathcal{H}_{y\alpha,j\alpha}^{\text{diff}}$ up to technical issues involving the treatment of loops that intersect the punctures.

We have now come to the point where we can ask about a possible relationship between quantum gravity and the Chern-Simons theories. For, given any set of marked points and representations of $\text{SL}(2)_q$ on $\mathcal{S}$ we have two state spaces of interest, $\mathcal{H}_{y\alpha,j\alpha}^{\text{CS}}$ of the corresponding Chern-Simons theory and $\mathcal{H}_{y\alpha,j\alpha}^{\text{diff}}$, a subspace of the states of quantum gravity. We also have two observable algebras, $\mathcal{A}_{S,y\alpha,j\alpha}^{\text{CS}}$ of Chern-Simons theory and $\mathcal{A}_{S,y\alpha,j\alpha}^{\text{gr}}$, a subalgebra of observables of general relativity.

We may note that while the states in general relativity are described so far in terms of ordinary spin networks, for every $j \leq k/2$ there are representations of both $\text{SU}(2)$ and $\text{SL}(2)_q$. Thus, we have this correspondence for each set of $j\alpha \leq k/2$.

The question of whether there may be any relationship between these subalgebras and representation spaces depends on the choices of dynamics and boundary conditions imposed on quantum gravity. Any correspondence must also give a meaning to the level $k$ of the Chern-Simons theory, which is the deformation parameter of the quantum group, in terms of the parameters of quantum gravity.

What is the best possible correspondence that we may hope for between the states and observables of Chern-Simons theory and quantum gravity? The best that may be hoped for is that there are isomorphisms,

$$\mathcal{A}_{S,y\alpha,j\alpha}^{\text{CS}} = \mathcal{A}_{S,y\alpha,j\alpha}^{\text{gr}}. \quad (18)$$

I will show in the next three sections that there is a choice of boundary conditions that exactly achieves this. This is accomplished by requiring
that, in the spacetime language,

\[
\frac{1}{G} e^{AA'} \wedge e_{A'} - \frac{k}{2\pi} F^{AB} \partial_M = 0,
\]

(19)

where \( e^{AA'} \) is the frame field one form and \( F^{AB} \) is the left handed curvature two form, and \((...)_{\partial M} \) means a form pulled back into the boundary. When expressed in a canonical language, this will become three conditions on the initial data at each point on the boundary, \( S \),

\[
\left( \tilde{\Pi}^a_{\epsilon} \epsilon_{abc} - \frac{k}{2\pi} F^i_{bc} \right)_{S} = 0
\]

(20)

where \( \tilde{\Pi}^a_{\epsilon} \) is the momenta conjugate to the connection \( A^i_a \). In words, we are imposing the condition that the fields, pulled back to the boundary, are self-dual.

I will discuss in the next section how this condition may be imposed, and what can be said about how restrictive it is.

The parameter \( k \) will appear as the coefficient of a boundary term, and, by invariance under large gauge transformations, will be required to be an integer, just as in the case of Chern-Simons theory. But it will be constrained, by consistency with the Einstein equations, to be given by,

\[
k = \frac{6\pi}{G^2\Lambda} + \alpha
\]

(21)

where \( G \) is Newton’s constant, \( \Lambda \) is the cosmological constant, and \( \alpha \) is a topological \( CP \) breaking parameter.

The relations (20) and (21) will later be derived from the quantization of general relativity with the boundary conditions (19). For now we proceed to describe the theory that follows, once we are given these conditions.

The main result of the imposition of these boundary conditions is a reduction and deformation of the action of each of the subalgebras, \( A^{gr}_{S,j_{y_\alpha j_\alpha}} \) on the subspaces \( \mathcal{H}^{diff}_{y_{\alpha j_\alpha}} \) so that they are isomorphic to the observable algebras \( A^{CS}_{S,j_{y_\alpha j_\alpha}} \) of the corresponding Chern-Simon theories. This is because, as is straightforward to show, the canonical form of the boundary condition, (20) when imposed in the loop representation, implies directly the Gauss constraint of Chern-Simons theory with sources (3), because the \( \tilde{\Pi}^a_{\epsilon} \) integrated over the boundary, acting on a state \( |\Gamma >^{gr} \) will receive contributions only at points where the spin network \( \Gamma \) enters the boundary. We will see the details of this in section VI.
The isomorphism of the observable algebras of the boundaries has a crucial implication, which is that we must use quantum spin networks, rather than ordinary spin networks, to construct the state space of quantum gravity. This is because the boundary condition implies, as we have just said, that the subalgebras $A_{S,y_0,j_0}^{gr}$ are isomorphic to the observer algebras of the corresponding Chern-Simons theories, and they can only be realized in a space of states whose basis elements are labeled by quantum spin networks. Thus, at least the portions of the networks in $\Sigma$ that run in the boundary must be labeled in the restrictive set of $j \leq k$ that correspond to representations of $SL(2)_q$.

We may ask then if it is necessary that the spin networks remain $q$-deformed for those parts that travel in the interior of the space $\Sigma$. Perhaps an edge could be interpreted as carrying a representation of $SL(2)_q$ when it meets the boundary, while it carries a the corresponding representation of $SU(2)$ when it meets a vertex in the interior. A simple argument shows that were this to occur, it would be inconsistent with diffeomorphism invariance. Let us imagine an edge $\gamma$ that runs from the boundary to a vertex, $v$, where it meets other edges in the interior. Let us consider a family of little surfaces that intersect $\gamma$ at points between the boundary and the disk. By diffeomorphism invariance we should expect that all these surfaces have the same area, for there is no diffeomorphism invariant structure to distinguish them. Thus, a surface that approaches arbitrarily closely to the boundary, $S$ must have the same area as one that approaches arbitrarily close to the vertex, $v$. However, we know that the area of a surface depends on $q$. This means that if the edge $\gamma$ is considered to carry a representation of the quantum group as it leaves the surface, it must carry the same $j$ and $q$ when it meets the vertex; it cannot somewhere forget the value of $q$. But this means that all lines exiting the vertex must carry representations of $SL(2)_q$, otherwise the combinatorics of the addition of quantum spin would not be consistent at the vertex. One can then continue this argument to show that every edge in the interior must be labeled by a representation of $SL(2)_q$.

We may ask whether it makes sense to represent the observable algebra of quantum gravity in terms of quantum spin networks. This means that the algebra of observables of quantum gravity accessible on the surface has been deformed, with $k$ given by (21) being the deformation parameter, in each subspace $H_{\gamma_0,j_0}^{diffgeo}$, to the algebra of observables of the corresponding Chern-Simons theory. We are forced to do this by the modifications in the commutation relations that the boundary conditions impose. But we may
ask if the deformation in the observable algebra may be extended consistently to observables on the interior of \( \Sigma \), so that their action on quantum spin networks may be defined. The answer is that this can be accomplished by a suitable deformation of the loop algebra\(^3\). The result is that all the parameters of general relativity, \( G, \Lambda \) and \( \alpha \) are coded in the kinematics of the loop algebra.

But once we know that we must use quantum spin networks as the basis of the states of quantum gravity (in this context) we have at once, for every set of punctures and representations, a map from the diffeomorphism invariant state space of general relativity into the direct sum of the physical state spaces of all the Chern-Simons theories on \( \mathcal{S} \). This map is defined for every \( y_\alpha \) and \( j_\alpha \),

\[
\mathcal{N}_{y_\alpha, j_\alpha} : \mathcal{H}_{y_\alpha, j_\alpha}^{\text{diffeo}} \to \mathcal{H}_{y_\alpha, j_\alpha}^{\text{CS}}
\]

and is given by

\[
\mathcal{N}_{y_\alpha, j_\alpha} : |\Gamma >^{\text{gr}} \to |\Gamma >^{\text{CS}}.
\]

Let me emphasize that this map exists naturally by the axioms of topological quantum field theory and the fact that the isomorphism between the observable algebras at the boundary requires that quantum spin networks be used in quantum gravity as well.

We may then write

\[
\mathcal{H}_{y_\alpha, j_\alpha}^{\text{diffeo}} = \mathcal{H}_{y_\alpha, j_\alpha}^{\text{CS}} \otimes \mathcal{K}_{y_\alpha, j_\alpha}^{\text{diffeo}}
\]

where \( \mathcal{K}_{y_\alpha, j_\alpha}^{\text{diffeo}} \) is the kernel of the map \( \mathcal{N}_{y_\alpha, j_\alpha} \).

Given that the state space \( \mathcal{H}_{y_\alpha, j_\alpha}^{\text{CS}} \) of the Chern-Simons theory is finite dimensional, we may expect that, at the level of diffeomorphism invariant states, these kernels are themselves infinite dimensional. In the classical theory, we expect that there are an infinite number of diffeomorphism invariant observables that correspond to measurements made at points inside of the boundary, that must then commute with all the observables defined on the boundary. Unfortunately, we do not know many of these observers explicitly. One, which is now understood, is the volume of \( \Sigma \). The corresponding quantum operator has now been defined, and their eigenstates and eigenvalues have been characterized\(^2\). The trivalent spin networks are precisely eigenstates, while for higher valence networks the eigenstates are known to lie in the finite dimensional sectors spanned by the different intertwinings at the vertices. In both cases the eigenvalues are discrete, and on order of the Planck scale. This operator may also be defined, through the deformation of the loop algebra, to act on the quantum spin networks\(^3\).
Indeed, given the interpretation of spin network states at the kinematical level\textsuperscript{22} in which each node carries a certain unit of volume and each edge a quanta of area, we have a good understanding of what the different states $|\Gamma >_{\text{diff}co}$ mean, at least in the limit of large $k$: they are eigenstates of observables that measure the three geometry of the interior. It is clear that, even when the geometry of the surface is fixed, which determines the $y_{\alpha}$ and $j_{\alpha}$ of the intersections of $\Gamma$ with the boundary, there should still be an infinite number of diffeomorphism invariant quantum states of the gravitational field in $H_{S}^{\text{diff}co}$ for each state $|N_{y_{\alpha},j_{\alpha}} \circ \Gamma >_{\text{CS}}$ in the associated Chern-Simons theory. These correspond to different spatial diffeomorphism classes of three geometries on the interior that, however agree about the quantum fields that are induced on the boundary.

To show this we need to have more information about the diffeomorphism invariant observable algebra and its $q$ deformation. However, in the absence of this, we may still attempt to make reasonable hypotheses about the full diffeomorphism invariant and physical state spaces. I would like now to describe two.

**Hypothesis I: Existence of a complete set of diffeomorphism invariant observables**

Let us consider the diffeomorphism invariant configuration space of general relativity with $\Sigma$ fixed, given the boundary conditions (20), which I will call $C_{\text{diff}}$. It is clear that there must exist a complete set of coordinates on it

$$x_{\text{phys}} = \{A[R], x_{\Sigma}\}$$

(25)

where $x_{\Sigma}$ label all the diffeomorphism classes of three metrics on $\Sigma$ that induce the areas $A[R]$ on $S$. This is a weak assumption, the main issues to be solved to establish it are global issues.

If this is the case, then there must be in the full quantum theory a set of commuting diffeomorphism observables, which I will call $O_{I}^{l}$, where $I$ is a generic label, which have the property that an orthonormal basis of each kernel $K_{y_{\alpha},j_{\alpha}}$ exists which are the eigenstates of the $\hat{O}_{I}^{l}$. I will label these by $|\lambda^{I}, y_{\alpha}, j_{\alpha} >$, which are defined so that

$$|\lambda^{I}, y_{\alpha}, j_{\alpha} > \in K_{y_{\alpha},j_{\alpha}}$$

(26)

and

$$\hat{O}_{I}^{l} |\lambda^{I}, y_{\alpha}, j_{\alpha} >= \lambda_{I} |\lambda^{I}, y_{\alpha}, j_{\alpha} >$$

(27)
Let $|z>$ be an orthonormal basis of states in $H_{y_{\alpha},j_{\alpha}}^{CS}$ such that, under the inner product of that theory,

$$<z|z'>_{y_{\alpha},j_{\alpha}}^{CS} = \delta_{zz'}$$  \hspace{1cm} (28)

It then follows that there is a basis for diffeomorphism invariant states of the gravitational field, given by

$$|y_{\alpha},j_{\alpha},\lambda_I, z> = |z> \otimes |\lambda_I, y_{\alpha}, j_{\alpha}>$$  \hspace{1cm} (29)

By virtue of the fact that the $A[R]$ and $O^I$ must be represented by hermitian observables, these are an orthonormal basis for the Hilbert space of spatially diffeomorphism and gauge invariant states. It then follows that if any two states $|\Psi>, |\Psi'> \in H_{y_{\alpha},j_{\alpha}}^{diff}$ of the form

$$|\Psi> = |N \circ \Psi > \otimes |K \Psi >$$  \hspace{1cm} (30)

where the two factors $|N \circ \Psi >$ and $|K \Psi >$ are in $H_{y_{\alpha},j_{\alpha}}^{CS}$ and $K_{y_{\alpha},j_{\alpha}}^{diff}$, respectively, we have

$$<\Psi|\Psi'>_{diff}^{diff} = <N \circ \Psi|N \circ \Psi'>_{y_{\alpha},j_{\alpha}}^{CS} <K \Psi|K \Psi'>_{y_{\alpha},j_{\alpha}}^{CS}$$  \hspace{1cm} (31)

We will see in section VIh. below that this is realized, given only some rather weak assumptions.

Let us now consider the state $|\Gamma>^{y_{\alpha}} \in H_{S}^{diff}$ associated to the quantum spin network $\Gamma$. It then follows from what we have said, together with the isomorphism (18) that if we measure an observable $\hat{T}[\alpha]$ for a loop $\alpha \in S$ we have $|N \circ \Gamma > = |\Gamma>^{CS}$ so that,

$$\frac{<\Gamma||\hat{T}[\alpha]|\Gamma>^{diff}}{<\Gamma||\Gamma>^{diff}} = \frac{<\Gamma||\hat{T}[\alpha]|\Gamma>_{y_{\alpha},j_{\alpha}}^{CS}}{<\Gamma||\Gamma>_{y_{\alpha},j_{\alpha}}^{CS}}$$  \hspace{1cm} (32)

Thus, the expectation value in quantum gravity of an observable that measures the self-dual curvature at the surface $S$ is given by an expression in Chern-Simons theory. Furthermore, this expression may be evaluated in closed form; it is proportional to the Kauffman bracket of the knotted graph $\Gamma \circ \Gamma^* \cup \alpha$.

**Hypothesis II: The kernal is trivial in the physical state space**

The kernal $K_{y_{\alpha},j_{\alpha}}^{diff}$ contain precisely the information about the diffeomorphism invariant state of the gravitational field that is not accessible to the
observers who can only measure the system by measuring the metric and connection on the surface $\mathcal{S}$. It is clear from the construction we have just gone through that in the case we are considering diffeomorphism invariant states there is an infinite amount of such information. However, what is the situation with respect to physical states, those that are solutions to the Hamiltonian constraint, and are therefor constrained by the dynamics of the theory?

As I will show in section V, with the boundary conditions (20) the hamiltonian constraint must vanish on the boundary, so we know that the observables $\mathcal{A}^{\text{boundary}}_\mathcal{S}$ are all physical observables. We also know, from the action of the Hamiltonian constraint, in any of the forms given in [8, 10, 11, 14] that the individual spin network states $|\Gamma \rangle$ are not physical states. However, because the areas on the boundary commute with the hamiltonian constraint, the action will not mix states in different subspaces $\mathcal{H}_{y_\alpha, j_\alpha}^{gr}$.

Thus, we know that the physical state space, $\mathcal{H}_{\Sigma}^{\text{phys}}$, defined as the subspace of $\mathcal{H}_{\Sigma}^{\text{phys}}$ that is annihilated by the hamiltonian constraint, with lapse vanishing at the boundary, is given by the same decomposition

$$\mathcal{H}_{\Sigma}^{\text{phys}} = \sum_{n=0}^{\infty} \sum_{j_1 \ldots j_n} \int d^2y_1 \ldots d^2y_n \mathcal{H}_{y_\alpha, j_\alpha}^{\text{phys}}$$

(33)

where each

$$\mathcal{H}_{y_\alpha, j_\alpha}^{\text{phys}} = \mathcal{H}_{y_\alpha, j_\alpha}^{CS} \otimes \mathcal{K}_{y_\alpha, j_\alpha}^{\text{phys}}$$

(34)

It then follows that the physical kernals $\mathcal{K}_{y_\alpha, j_\alpha}^{\text{phys}}$ are each subspaces of the diffeomorphism invariant kernals $\mathcal{K}_{y_\alpha, j_\alpha}$.

This leads to the important result that the expression (32) must hold also for the physical expectation values of the physical observables $T[\alpha]$ on physical states, as long as $\alpha \in \mathcal{S}$. We are similarly able to evaluate the expectation value of any of the physical area observables. This means that we are able to evaluate the expectation values of an infinite and nontrivial set of physical observables, which characterize observations we may make on the boundary, in closed form, even in the absense of further information about the physical states.

To say more than this, we need information about the physical kernals, $\mathcal{K}_{y_\alpha, j_\alpha}^{\text{phys}}$. In the light of what we have just said I would like to propose that we consider the following simple conjecture, which is that these physical kernals are all trivial. This is equivalent to the physical hypothesis that all information about a physical state of the gravitational field in a region $\Sigma$
surrounded by a finite boundary is accessible by measuring the metric and connection induced in that boundary. Mathematically, it is equivalent to the conjecture that the physical state space is precisely,

$$\mathcal{H}_{\Sigma}^{\text{phys}} = \sum_{n=0}^{\infty} \sum_{j_1 \ldots j_n} \int d^2 y_1 \ldots d^2 y_n \mathcal{H}_{y_\alpha, j_\alpha}^{CS}(35)$$

In this case, the physical quantum states must be exactly the states $|\Gamma, \Sigma >^{CS}$ defined by the Chern-Simons path integral, but now with appropriate boundary conditions. These may be considered to be the generalizations of the Bruegmann-Gambini-Pullin states$^{[43]}$ to the case of the boundary conditions (20). If this conjecture is correct, then we come to the conclusion that the number of solutions to the Hamiltonian constraint that match the boundary conditions for each set of colored punctures, $(y_\alpha, j_\alpha)$ is equal only to the number of states of the associated Chern-Simons theory. All other states, such as the infinite numbers of states known which are associated with knot classes$^{[8, 38]}$, must fail to match the boundary conditions on the surfaces. Another way to say this is that, as proposed by Crane$^{[5]}$, all physical quantum states found by performing the loop transform of the Kodama state of the connection representation,

$$\psi_{\text{Kodama}}[A] = e^{i \frac{ik}{4\pi} \int_{\Sigma} Y_{CS}(A)}$$

with the differences in the states being due to boundary conditions imposed on the transform.

I would like to give several arguments for this proposal:

1) As already mentioned, it is a reasonable hypotheses that all physical information about the interior of a finite region should be determinable by measurements made on its boundary.

I may note that even if this is not the case, then the description in terms of (35) should apply at least to those cases in which, because of the presence of a horizon, we can only measure observables on the boundary $\mathcal{S}$. If we take an operational point of view, in which the Hilbert space is to be spanned by the eigenstates of a complete set of commuting operators corresponding to observables that we can in fact measure, then there is no sense to introducing extra factors in the Hilbert space that are distinguished only by operators that cannot be measured.

2) This does not conflict with the usual practice in conventional quantum field theory. There it is usually established that the physical Hilbert
space may be spanned by scattering states that can all be distinguished by measurements made by observables far from the region in which interactions take place. In a diffeomorphism invariant system, there is no meaning, independent of a given state, to being far from the interacting region. We may try to investigate the notion of asymptotic observables, which has a diffeomorphism invariant meaning, but there are special problems associated with this. At least in the case of spatial infinity, it is doubtful that there are more than a finite number of observables defined at spatial infinity.

The alternative is to fix a finite boundary, and define the physically meaningful states by what an observer on the boundary could measure. This seems to be the most useful diffeomorphism invariant extension of the usual practice of defining the physical Hilbert space in terms of asymptotic states, and it leads, in the case under consideration, to the conclusion that the kernels $k_{y_1, y_2}^{phys}$ must be trivial.

To get consistency with the usual practice, what is required is only that the dimensionality of the state space associated with a fixed two sphere metric on the boundary go to infinity as the area of the boundary itself is taken to infinity. This proposal satisfies that requirement, as we will see explicitly.

3) This does not mean that the state space of quantum gravity is finite dimensional. It only implies that the degenerate subspaces of states that agree about all measurements of areas of regions of the boundary are finite dimensional. The whole physical state space is infinite dimensional because it is composed of sums and integrals over finite factors according to (35). What this analysis has given us is a complete description of each of the factors as the state space of a Chern-Simons theory.

4) It may still seem that the state space given by (35) is too small to represent quantum mechanically the degrees of freedom of the gravitational field in the interior of $\Sigma$, as there are, according to the classical canonical analysis, two degrees of freedom per point. However, there is the old argument, given first by Beckenstein\textsuperscript{[27]}, and developed by Hawking\textsuperscript{[42]}, and more recently by 't Hooft\textsuperscript{[24]} and Susskind\textsuperscript{[25]}, that this drastically overcounts the actual degrees of freedom in any region surrounded by a surface of finite area. The reason is that, due to the phenomena of gravitational collapse, almost all of the configurations that one would take into account by counting two degrees per point of volume would be surrounded by an event horizon. Instead, Beckenstein conjectured that the amount of information that can be stored in a region surrounded by a boundary $\mathcal{S}$ of finite area

22
\( \mathcal{A}[S] \) is finite and is bounded by,

\[
I_S = c \frac{\mathcal{A}[S]}{l_{Pl}^2}
\]  

(37)

where \( c \) is a fixed constant. This means that the dimension of the physical state space for the corresponding quantum system must be finite and bounded by

\[
dim \mathcal{H}_S^{phy} \leq e^{I_S LN(2)}
\]  

(38)

Motivated by this, ’t Hooft and Susskind have made the “holographic hypothesis”, which is that the states in \( \mathcal{H}_S^{phy} \) can be actually represented in terms of a finite quantum field theory on \( S \)  

[24, 25].

The hypothesis (35) for the physical state space may be considered to be a realization of the holographic hypothesis. We may further show that it yields also the Beckenstein bound, at least in the case of small cosmological constant.

To see this, let us assume that we have measured the metric on the spatial boundary as accurately as we can, given the discrete nature of the quantum geometry. How much information yet remains that could code a description of the quantum state inside the boundary?

The most precise possible measurement of the metric geometry of \( S \) reduces us to a single subspace of the physical state space, \( \mathcal{H}_S^{CS, y_\alpha, j_\alpha} \). The information that is then given by specifying the exact state within this subspace is

\[
I_{S, y_\alpha, j_\alpha} = LN dim \left( \mathcal{H}_S^{CS, y_\alpha, j_\alpha} \right) = \sum_\alpha LN(2j_\alpha + 1)
\]  

(39)

At the same time the area is given in the large \( k \) limit by (16). (We may note that for finite \( k \) there are corrections to this area formula\[33\], but they are irrelevant in this case.)

To proceed further we now want to maximize both the amount of information contained in any large but finite region of \( S \) and the accuracy with which the quantum geometry approximate a classical metric. As both the area and the information (at large \( k \)) are additive quantities, each is maximized in the same way, which is when we represent the two metric by the maximum number of punctures, each of which has the smallest possible spin and so contributes the minimal possible area. To see this note that, if all punctures have the same spin, \( j \), \( I/A \approx 2LN(2j + 1)/\sqrt{j(j + 1)} \) which is
maximized for the smallest value $j = 1/2$. We then have

$$I_S = \frac{4(LN(2))^2}{\sqrt{3}} \frac{A[S]}{l_P^2}$$

(40)

Thus, given the conjecture that all the information about the quantum state in $\Sigma$ can be gotten by measuring the metric and self-dual connection induced on the surface $S$, we reproduce the Beckenstein bound that the maximum amount of information that can be contained within a surface is equal to a fixed constant times its area, once the metric of the surface is measured as accurately as possible.

We may invert this argument to conclude that, if the physical kernels $K_{\alpha_0,j_0}^{\text{phys}}$ are non-trivial, the Beckenstein bound will be violated (unless the dimensionality of the kernels also grows like $\exp(A[S])$). But this would make it impossible to give a statistical interpretation to black hole entropy.

5) In the particular cases that the spacetime boundary $\partial M$ is the horizon for a family of observers, there is an argument, due to Jacobson, which derives the Einstein equations from the following short list of assumptions: 1) the laws of thermodynamics, 2) standard special relativistic field theory holds in small regions for inertial observers 3) the information not measurable by the observers, because of the existence of the horizon, is proportional to its area.

This would suggest that there can be no more information in the quantum state of the gravitational field in the interior of $M$ than is contained in the hypothesized physical state space (35).

### IV. The classical theory with self-dual boundary conditions

All the conclusions of the previous section were based on the assumption that a quantum theory can be defined to describe the quantum physics of the gravitational field in the interior of a surface $S$ in such a way that three conditions are satisfied:

1) The self-dual boundary condition (20) is imposed.

2) The gauge transformations generated by the hamiltonian and diffeomorphism constraints leave the boundary $S$ fixed.

3) The internal gauge transformations act on the boundary.

In this section we will derive these conditions by imposing the condition (19) that the spacetime curvature be self-dual, with a cosmological constant,
at the surface $\mathcal{S}$.

The most direct way to do this is to use the Capovilla-Dell-Jacobson forms of the action \cite{14}, which is given in terms of the left handed spacetime connection one form, $A^i$, and a matrix of scalar fields, $\Phi_{ij}$, which is restricted to satisfy

$$
\Phi_{ij} = \Phi_{ji}, \quad \text{and} \quad \sum_i \Phi_{ii} = \lambda. \quad (41)
$$

where $\lambda = G^2 \Lambda$ is the cosmological constant in Planck units. We will see that it will be essential here to consider the case in which $\lambda$ is not zero. Here $i, j, k$ are the $SU(2)_{Left}$ indices. To write the action we begin with the CDJ form, which is,

$$
S_{CDJ} = \frac{1}{2} \int_{\mathcal{M}} F^i \wedge F^j (\Phi^{-1})_{ij} \quad (42)
$$

This action does not by itself give a good variational principle in the case that spacetime has boundaries. To define a consistent variational principle we must choose boundary conditions at $\partial \mathcal{M}$ and add a boundary term to the action such that the variational derivative of the total action exists. One way to do this, which leads to the self-dual boundary conditions (20) is to use the fact that at points of spacetime at which $\Phi$ is diagonal the Weyl tensor must be self-dual \cite{15}. Roughly speaking, this corresponds to solutions in which there are only left-handed gravitational waves at the boundary.

We thus proceed by imposing the boundary condition that

$$
\Phi_{ij}|_{\partial \mathcal{M}} = \frac{\lambda}{3} \delta_{ij} \quad (43)
$$

We then must add a boundary term to the action (42) whose variation will cancell the boundary term in its variation. Interestingly enough, the correct boundary term is the Chern-Simons action of the remaining, left-handed part of the curvature on the boundary \cite{15}.

Thus, the full action principle we will be interested in is,

$$
S = \frac{1}{2} \int_{\mathcal{M}} F^i \wedge F^j (\Phi^{-1})_{ij} - \frac{3}{2 \lambda} \int_{\partial \mathcal{M}} Y_{CS}(A) \quad (44)
$$

where the Chern-Simons form is given by,

$$
Y_{CS}(A) = A^i \wedge dA^i + \frac{1}{3} \epsilon_{ijk} A^i \wedge A^j \wedge A^k \quad (45)
$$
As the reader may check, with the boundary condition (43) the variation of this action is a pure volume term, giving the field equations

$$\frac{\delta S}{\delta A_{\alpha i}} = -\epsilon^{\alpha\beta\gamma\delta} D_{\beta} \left( F^{j}_{\gamma\delta}(\Phi^{-1})_{ij} \right) = 0$$ \hspace{1cm} (46)

and

$$\frac{\delta S}{\delta \phi_{ij}} = F^k \wedge F^l (\Phi^{-1})_{ki} (\Phi^{-1})_{lj} - \frac{1}{3} \delta_{kl} (\Phi^{-1})_{mn} (\Phi^{-1})^{mn} = 0.$$ \hspace{1cm} (47)

We may also add a topological term to the action (44) to get,

$$S = \frac{1}{2} \int_{\mathcal{M}} F^i \wedge F^j (\Phi^{-1})_{ij} + \frac{\alpha}{4\pi} \int_{\mathcal{M}} F^i \wedge F^i - \left( \frac{3}{2\lambda} + \frac{\alpha}{4\pi} \right) \int_{\partial\mathcal{M}} Y_{CS}(A)$$ \hspace{1cm} (48)

This makes the theory CP violating, as $\alpha$ plays the same role as the strong $CP$ breaking parameters in $QCD^{[10]}$.

To complete the definition of the theory, we must note that gauge invariance imposes an additional restriction on our theory. As is well known, the Chern-Simons action is not invariant under large gauge transformations, instead it transforms as $\int Y_{CS} \rightarrow \int Y_{CS} + 8\pi^2$. If we require that the Minkowskian path integral $e^{iS}$ is to be invariant under such transformations, this imposes the requirement that

$$k = \frac{6\pi}{\lambda} + \alpha$$ \hspace{1cm} (49)

where $k$ is an integer, and in fact will be the level of the Chern-Simons theory.

Thus, gauge invariance implies a quantization of the cosmological constant. Significantly, we see that small cosmological constant corresponds to large $k$, which is the semiclassical limit of the Chern-Simons theory.

One may be suspicious that these results are somehow an artifact of the Capovilla-Dell-Jacobson formalism. However, it is easy to see that one gets the same results using the self-dual actions given earlier in $[17]$. In the presence of the finite boundary, that action must be extended by a boundary term so that the variation is a total derivative, so we have

$$S_{SD} = \int_{\mathcal{M}} \left( \frac{1}{G} e^{A^i} \wedge e^{B^i} \wedge F_{AB}(A) + \frac{\alpha}{2\pi} F^{AB} \wedge F_{AB} + \Lambda det(E) \right) - \frac{k}{2\pi} \int_{\partial\mathcal{M}} Y_{CS}(A)$$ \hspace{1cm} (50)
When we vary this we find,

\[
\delta S_{SD} = \int_M \left( \frac{1}{G} \delta A_{AB} \wedge \mathcal{D} \wedge (e^{AA'} \wedge e^{B'}_B) + \delta e^{AA'} \wedge \left( \frac{2}{G} e_{A'}^B \wedge F_{AB} + 3 \Lambda e_{B'}^C \wedge e_{A'}^C \right) \right)
+ \int_{\partial M} \delta_{AB} \wedge \left[ \frac{1}{G} e^{AA'} \wedge e^{B'}_B - \frac{(k - \alpha)}{2\pi} F_{AB} \right]
\]

(51)

For the equations of motion to be well defined, the surface term must vanish. We see from this expression that one way to accomplish this is to fix \(A_{AB}\) on the boundary so that its variation vanishes there. We may note that if \(k - \alpha\) vanishes, this is the only alternative, as long as we want the metric on the boundary to be non-degenerate. However, there is another way to cancel the surface term which is to choice the self-dual condition (19) on the boundary, so that

\[
\frac{1}{G} e^{AA'} \wedge e^{B'}_B - \frac{(k - \alpha)}{2\pi} F_{AB} = 0
\]

(52)

This is an acceptable boundary condition, as long as it is consistent with the field equations at the boundary. This is necessary because this allows both the connection and frame field to vary at the boundary, so that the field equations must be satisfied there. If we take \(e^{B'}_A\) times (52), we see that this is equal to the frame field equation at the boundary, so long as the condition (49) on the constants is imposed. Thus, we see that an examination of the variational principle in this form leads to the same conclusions as the variational principle according to the CDJ formalism.

Before proceeding, we must say a word about the effect of imposing these boundaries on the space of solutions. It is important to note that the condition we have imposed, which is that the pull back of the curvature to the boundary satisfies the self-dual condition (52) is significantly weaker than a requirement that the whole curvature tensor, evaluated at the boundary, is self-dual. If self-duality is imposed on all components of the tensor, then it may be argued that, at least in the analytic case, the field is self-dual in the whole manifold. This may be shown directly by writing down the Bianchi equations. The conditions that are being imposed here are weaker than that, in terms of initial data they involve, as we will see in the next section, imposing the constraint (20), which involves three of the nine components of the self-dual equations. A linearized analysis of these conditions may be performed that shows that, at least locally, we may expect an infinite number of non-self-dual solutions to Einstein’s equations with these boundary conditions, this will be discussed elsewhere.
We may note also that were the whole self-dual conditions being imposed in the Minkowskian case, then there would be only one solution, as the whole solution would be self-dual, but there is only self-dual solution of Minkowskian signature, which is DeSitter spacetime. How these weaker conditions interact with the Minkowskian reality conditions is an important open question. For this reason, and because the Chern-Simon theory is simplest in the compact case, the considerations of this paper are restricted to Euclidean signature.

We now proceed to the Hamiltonian analysis, based on the CDJ form of the action.

V. Hamiltonian dynamics, including the boundary conditions

We now proceed to construct the hamiltonian dynamics that corresponds to the variational principle (44). To keep the analysis simple, we will restrict ourselves to the case that the $CP$ violating phase $\alpha$ is set to zero. When it is included, the hamiltonian analysis is a bit more complicated, following the lines described in [46].

To proceed, we will assume a 3+1 splitting, so that spatial indices, $a, b, c$ will correspond to coordinates in $\Sigma$, while $t$ will be a coordinate on $R$. We then find the canonical momenta has a boundary contribution, and is of the form

$$\tilde{\Pi}^{ai}(x) \equiv \frac{\delta S}{\delta A^{a}_{i}(x)} = \frac{1}{2} \epsilon^{abc} F_{bc}^{j} (\Phi^{-1})_{ij}(x) - \frac{3}{2 \lambda} \int_{\partial \Sigma} d^{2}S^{ab}(\sigma) A^{a}_{i}(S(\sigma)) \delta^{3}(x, S(\sigma))$$

Here $\sigma$ are the two coordinates on $\partial \Sigma$, which we denote here as $S$.

As much of what follows depends on the fact of there being a contribution to the canonical momenta from the boundary, it is important to note why this must be there. The boundary term is there to cancell an integration by parts when one takes the variation of the action by the left handed connection, $A^{i}$. However, the canonical momenta is the variation of the action by a derivative of $A^{i}$, hence the cancellation does not hold in this case and one picks up a boundary contribution.
Va. The second class constraints associated with the boundary

We must first take care of the fact that the presence of the boundary term leads to primary constraints associated with points of the boundary. We may recall that this is what happens in the actual Chern-Simons theory, when the momentum is given only by the boundary term, so that

$$\pi^\alpha_S = \epsilon^{\alpha\beta} A^i_\beta,$$

where \(\alpha\) and \(\beta\) are coordinates in the two surface. However there is an important difference, due to the fact that for Chern-Simons theory space is the two dimensional boundary, whereas in the present case it is the three dimensional space \(\Sigma\). As the momenta are three dimensional densities, if there are constraints that arise from their definition, they must be defined by integrating the definition of the momenta against smooth functions on \(\Sigma\). If we smear the definition (53) against smooth functions \(f_{ai}(x)\) we find

$$J(f) \equiv \int_\Sigma f_{ai}(\tilde{\Pi}^ai - \tilde{B}^a (\Phi^{-1})_{ij}) - \int_{\partial \Sigma} d^2 S^{ab} f_{ai}(S(\sigma)) A^a_b(S(\sigma)) = 0 \quad (54)$$

where we have used the useful notation, \(\tilde{B}^ai = \frac{1}{2}\epsilon^{abc} F^i_{bc}\).

These \(J(f)\) are primary constraints, that have a second class algebra due to the presence of the boundary. We are now about to invert those constraints, to find the Dirac brackets, which we will then use to construct the quantum theory. What follows is a bit of a technical exercise, those uninterested in the details may skip to the end of this section, where the resulting modification of the loop algebra is displayed in eq. (66). The result, as we will see, is quite intuitive: loop observables obey the algebra of observables of quantum gravity for those portions of the loops in the interior of the manifold \(\Sigma\), while they satisfy the algebra (7) of Chern-Simons theory for those portions of the loops that travel in the boundary \(S\).

To define the Poisson algebra of these constraints, we must define the brackets of the fields \(A^i\) and \(\Pi^{ai}\) with the \(\Phi\). We would like to do this in such a way that the conventional Poisson structure is returned in the absence of the boundary, or, equivalently, for all one forms \(f_{ai}\) whose pull back to the boundary vanishes. This means we must take

$$\{\tilde{\Pi}^ai(x), \tilde{E}^ai(y)\} = 0 \quad (55)$$

where

$$\tilde{E}^ai(x) \equiv \tilde{B}^a (\Phi^{-1})_{ij}, \quad (56)$$

29
since in the absence of the boundary contribution $\tilde{\Pi}^{ai}(x) = \tilde{E}^{aj}$. Using this we find the Poisson algebra,

$$\{J(f), J(g)\} = \frac{3}{\lambda} \int_{\partial\Sigma} d^2 S^{ab}(\sigma)f_{ai}(S(\sigma))g_{b\dot{c}i}(S(\sigma))$$

(57)

Thus, whenever the pull back of the one forms $f_{ai}$ and $g_{bi}$ to the boundary are nonvanishing, we have second class constraints. To define the Poisson structure of the theory we must invert these to construct Dirac brackets. To invert these brackets we must construct an appropriate regularization of this second class algebra, and then define the inversion through a suitable limit. We may do this in the following way. First let us define an appropriate set of smearing functions, associated to ribbons in $\Sigma$. These ribbons are one parameter families of curves, $\alpha_\tau(s)$, where $0 \leq \tau \leq 1$ labels the curves, each of which is parameterized by $0 \leq s \leq 1$. These curves then define a surface in $\Sigma$, which may or may not intersect the boundary. This construction mimics the strip regularization of the loop algebra [11]. We then define smearing functions.

$$g^{\dot{a}i}_\tau(x) \equiv \int ds \int d\tau \epsilon^{abc}_{\dot{a}}(\alpha_\tau^b, \alpha_\tau^c)(x, \alpha_\tau(s))e^{\dot{a}i}_a$$

(58)

where $\alpha_\tau^c = \partial \alpha_\tau^c / \partial \tau$ and $e^{\dot{a}i}_a$ is a Lie algebra element. It is then a simple calculation to show that if $J_{\alpha_\tau} = J(g_{\alpha_\tau})$,

$$\{J_{\alpha_\tau}, J_{\beta_\tau'}\} = -\frac{3}{\lambda} \text{Int}(\tilde{\alpha}, \tilde{\beta})(e^{\dot{a}i}_a e^{\dot{b}i}_b)$$

(59)

where $\tilde{\alpha}$ is the loop which is the intersection of the ribbon with the boundary $\partial \Sigma$ (if there is no intersection it is the zero loop). We may note that as the ribbon must be in $\Sigma$, its intersection of the boundary must consist of some segments of loops.

To invert these relations we are going to represent this algebra as the limit of a sequence of finite algebras, each associated with a lattice associated to the surface. To do this we impose a set of coordinates $\sigma^{\dot{a}}$ on the boundary, and using this construct a lattice of points with spacing $L$. The points of the lattice will be denoted $n^L$, and to each there are two line segments, denoted $\gamma^{L}_{\dot{a}}$, that each extend in the $\dot{a}$ direction for a distance $L/2$ to each side of the point $n$. These lines then form a family of crosses centered on the points $n^L$. To each line we construct a ribbon $\gamma^{L}_{\dot{a}\tau}$ such that $\gamma^{L}_{\dot{a}\tau=0} = \gamma^{L}_{\dot{a}}$. We also impose the requirement that as we take $L \to 0$ the whole ribbon approaches the boundary, so that

$$\lim_{L \to 0} \gamma^{L}_{\dot{a}\tau} \in \partial \Sigma$$

(60)
for all $\tau$. We then define constraints associated to each of the lines in the lattice by

$$J_{\hat{n}\hat{a}}^i = J_{\hat{n}\alpha}^i | e^i = \delta^{i}_{\hat{a}}$$

(61)

whose algebra is given by

$$\{J_{\hat{n}\hat{a}}^i, J_{\hat{m}\hat{b}}^j\} = \frac{3}{\lambda} \delta_{\hat{n}\hat{m}} \epsilon_{\hat{a}\hat{b}} \delta^{ij}.$$  

(62)

It is clear that this constitutes a regularization of the second class algebra. That is, given a one form $g_{\alpha i}$ on $\partial \Sigma$, one can easily show that,

$$\{J(g), J(g')\} = \lim_{L \to 0} \sum_{\hat{n}\hat{a}} \sum_{\hat{m}\hat{b}} g(\hat{n})_{\hat{a}} g(\hat{m})_{\hat{b}} \{J_{\hat{n}\hat{a}}^i, J_{\hat{m}\hat{b}}^j\}$$

(63)

We may then define the Dirac brackets through the corresponding limit

$$\{A, B\}_D = \lim_{L \to 0} \{A, B\}_{D,L}$$

(64)

where the regulated Dirac bracket is

$$\{A, B\}_{D,L} \equiv \{A, B\} - \frac{\lambda}{3} \sum_{\hat{n}} \epsilon^{\hat{a}\hat{b}} \{A, J_{\hat{n}\hat{a}}^i\} \{J_{\hat{n}\hat{b}}^i, B\}$$

(65)

Using these relations we can check that the loop algebra is modified in the cases that the loops have segments that run in the boundary. The basic relation, which may be derived from (64), is that

$$\{T[\alpha], T[\beta]\}_D = -\frac{\lambda}{3} \sum_{\hat{a}, \hat{b}} \text{Int}[\alpha, \beta]^S \left( T[\alpha \circ \beta] - T[\alpha \circ \beta^{-1}] \right)$$

(66)

where, here $\hat{a}$ is the intersection of the curve with the boundary, $\text{Int}[\alpha]$ is the intersection number in the two dimensional boundary and the sum is over the set of points $\hat{a} \cup \hat{b}$ where the curves intersect in the boundary.

But, (66) is exactly the loop algebra of Chern-Simons theory, with the relation (49), with the $CP$ breaking parameter $\alpha = 0$. Thus, we see that when the loops run in the boundary, the loop algebra is deformed by terms that come from Chern-Simons theory.

For loops that do not intersect the boundary, the second class constraints have no effect and the usual loop algebra is satisfied.

Having taken care of the second class constraints associated with the boundary, we may go on with the construction of the constrained Hamiltonian dynamics of our theory.
Vb. The Gauss’s law constraint, in the presence of the boundary

We next turn to consideration of the first class constraints in the theory. First of all the momenta conjugate to $A_0^i$ vanish, from which we find as secondary constraints Gauss’s law,

$$\tilde{G}^i(x) \equiv \frac{\delta S}{\delta A_0^i(x)} = D_a \left( \tilde{B}^{aj}(\Phi^{-1})_{ij} \right)$$

(67)

Because of the cancellation coming from the boundary condition this, like all equations of motion, is local in $\Sigma$. However, when we express it in terms of the canonical commenta (53) we find a boundary piece. To see this it is easiest to consider the constraint smeared with smooth functions $\Lambda^i$ on $\Sigma$,

$$\mathcal{G}(\Lambda) \equiv \int_\Sigma \tilde{G}^i \Lambda^i = 0$$

(68)

$$= \int_\Sigma d^3x \Lambda^i D_a \Pi^{ai} - \frac{3}{2\lambda} \int_{\partial\Sigma} d^2S_{ab}(\sigma) F_{ab}^i(S(\sigma)) \Lambda^i(S(\sigma))$$

$$= - \int_\Sigma d^3x (D_a \Lambda^i) \Pi^{ai} - \frac{1}{2} \int_{\partial\Sigma} d^2S_{ab}(\sigma) \Lambda^i(S(\sigma)) \left( \Pi^c \epsilon_{abc} + \frac{3}{\lambda} F_{ab}^i(S(\sigma)) \right)$$

We may note that the surface term is precisely the self-dual constraint (20) with $k = 6\pi/\lambda$.

It is interesting to look at the variation of our fields generated by this constraint. We have,

$$\delta\Lambda \int_\Sigma \Pi^{ai} f_{ai} \equiv \{\mathcal{G}(\Lambda), \int_\Sigma \Pi^{ai} f_{ai}\}$$

(69)

$$= \int_\Sigma \Lambda^i \epsilon_{ijk} f_{aj} \Pi^{ak} - \frac{3}{2\lambda} \int_{\partial\Sigma} d^2S_{ab}(\sigma) (D_b \Lambda^i(S(\sigma)) f_{ai}(S(\sigma)))$$

$$\delta\Lambda \int_\Sigma g^{ai} A_{ai} \equiv \{\mathcal{G}(\Lambda), \int_\Sigma g^{ai} A_{ai}\} = \int (D_a \Lambda_i) g^{ai}$$

(70)

This tells us that, in spite of the boundary term in the definition of $\mathcal{G}(\Lambda)$, for all smooth $\Lambda$ it generates a motion on the phase space. The specific form of the variation of the momenta is also interesting; it tells us that restricted to the boundary the canonical momenta transforms like a connection. This is exactly what must happen, because, by virtue of the second class constraints associated with the boundary, restricted to the boundary the momenta is the connection, as is the case in Chern-Simons theory.
Further, we may note that for arbitrary $\Lambda$ the algebra of the $G(\Lambda)$ is still first class. It is easily established that the additional term does not change the algebra, so that
\[
\{G(\Lambda), G(\Lambda')\} = 2G(\Lambda \times \Lambda'),
\] (71)
where the $\times$ is the $SU(2)$ product. This occurs because a possible singular term coming from the Poisson bracket of the two boundary terms vanishes by virtue of a symmetry.

Thus, we take as the set of Gauss’s law constraints all $G(\Lambda)$ including those in which $\Lambda^i$ is nonvanishing on the boundary.

\textbf{Vc. The diffeomorphism and hamiltonian constraints}

We may now go on to discuss the diffeomorphism and hamiltonian constraints of the theory. These arise as primary constraints that are imposed by the restrictions (41) on the form of $\Phi_{ij}$. Because $\Phi_{ij}$ has been fixed on the boundary, these constraints exist for every open set in the interior of $\Sigma$. Thus, we may note that for all smooth $v^a$ on $\Sigma$ the following identity holds,
\[
0 = \int_{\Sigma} \epsilon_{abc} v^a \tilde{B}^{bi} \tilde{B}^{cj} (\Phi^{-1})_{ij}
\] (72)
as a result only of the symmetry of the $\Phi_{ij}$. If we use the defining relation of the momenta, (53), we get a primary constraint, which may be written as,
\[
0 = D(v) = \int_{\Sigma} v^a F_{ab}^i \tilde{\Pi}^{bi} + \frac{3}{\lambda} \int_{\partial \Sigma} d^2S^{bc} v^a F_{ab}^i A^i_c
\] (73)
However, this does not generate a canonical transformation for all $v^a$ on $\Sigma$ because, as may be easily checked, it is not functionally differentiable unless conditions are imposed on the $v^a$. These are that
\[
v^a|_{\partial \Sigma} = 0 \quad \text{and} \quad \partial_r v^r|_{\partial \Sigma} = 0
\] (74)
where the $r$ is any coordinate direction not in the two surface.

However, it might be the case, as it is for the Gauss law constraints, that the $D(v)$ still form a closed algebra for all $v^a$, including those that do not vanish on the boundary. A simple check shows that this is not because, while the Poisson bracket of two $D(v)$’s is defined, even when smeared with
vector fields that don’t vanish on the boundary, the algebra does not close unless the vector fields do vanish there. Instead, we have,

$$\{\mathcal{D}(v), \mathcal{D}(w)\} = \int_\Sigma (L_v w^a) F_{ab} \tilde{\Pi}^{bi} + \frac{6}{\lambda} \int_{\partial \Sigma} d^2 S^{bc} F_{ab} F_{de} v^a w^c$$  \hspace{1cm} (75)

Thus, there is no meaningful sense in which the diffeomorphism group can be extended beyond those that satisfy (74). In this case the diffeomorphism constraints have the standard local form,

$$0 = \mathcal{D}(v) = \int_\Sigma v^a F_{ab} \tilde{\Pi}^{bi}. \hspace{1cm} (76)$$

Similarly, we can define the Hamiltonian constraint. Due to the constraint on the trace of $\Phi_{ij}$, it is easy to see that the following is, for all smooth $N(x)$ on $\Sigma$, an identity,

$$0 = \int_\Sigma N \left( \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{ai} \tilde{E}^{bj} \tilde{E}^{ck} + \lambda \text{det}(\tilde{E}) \right)$$ \hspace{1cm} (77)

where $\tilde{E}^{ai} \equiv \tilde{B}^{bij}(\Phi^{-1})_{ij}$. We then may attempt to write a constraint for all $N$, by using the definition of the momenta (53) and so writing, $\tilde{E}^{ai} = \tilde{\Pi}^{ai} - (3/\lambda) \tilde{r}^{ai}$, where the latter stands for the surface term,

$$\tilde{r}^{ai} = \int_{\partial \Sigma} d^2 S^{ab}(\sigma) A_{i}^{ab}(\sigma) \delta^3(x, S(\sigma))$$ \hspace{1cm} (78)

However, one quickly sees that for all $N(x)$ the resulting functional is singular, due to the quadratic and higher terms in $\tilde{r}^{ai}$. Thus, we may only take as generators of gauge transformations those constraints with $N(x)$ vanishing on the boundary. Thus, our Hamiltonian constraint is simply the standard local one,

$$0 = \mathcal{C}(N) = \int_\Sigma N \left( \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{ai} \tilde{\Pi}^{bj} \tilde{\Pi}^{ck} + \lambda \text{det}(\tilde{\Pi}) \right)$$ \hspace{1cm} (79)

with the condition that

$$N|_{\partial \Sigma} = 0 \hspace{1cm} (80)$$

**Vd. The Hamiltonian**

We may ask also if we can extend this constraint to a Hamiltonian, which will give us a nontrivial time evolution, as measured by a lapse function that reaches to the boundary. We may do this in the standard way, by noting
that we may add a boundary term to the Hamiltonian constraint, so that
the resulting functional is now functionally differentiable for all $N$ including
those that are nonvanishing at the boundary. This, not surprisingly, takes
the same form as in the asymptotically flat case,

$$
\mathcal{H}(N) \equiv \int_{\Sigma} N \left( \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{ai} \tilde{\Pi}^{bj} \tilde{\Pi}^{ck} + \lambda \det(\tilde{\Pi}) \right) - \int_{\partial \Sigma} d^2 S a N A_{a}^{k} \tilde{\Pi}^{ai} \tilde{\Pi}^{bj} \epsilon_{ijk}
$$

(81)

This boundary has the property that when taken to infinity in the presence of
asymptotically flat boundary conditions, it goes to the ADM mass. However,
in the presence of these boundary conditions there is no such simplification,
so this remains a non-linear term.

Ve. Some observables and symmetries associated with the
boundary

As I indicated in the introduction, one motivation for introducing a finite
boundary in quantum gravity is to have an infinite dimensional algebra
of observables, associated with measurements that might be made on the
boundary of the system. We have already in Chapter 3 mentioned the
algebra of observables $A_{S}^{\text{boundary}}$. Here we introduce them in the context of
the classical canonical theory we have just defined. The first are the loop
algebra $T[\alpha]$, the traces of holonomies of loops $\alpha$ in the boundary. We may
note now that, under the conditions just derived, these commute with all the
constraints of the theory, and so are a subalgebra of the algebra of physical
observables of the theory.

The algebra of these observables is defined by the Dirac brackets (66).
Further, because we are working with the Euclidean boundary conditions
this algebra is a star algebra, because

$$
T[\alpha]^* = T[\alpha]
$$

(82)

The second set of observables is the areas of regions of the boundary,
which we denote by $A[R] = \int_{R} \sqrt{h}$ where $h_{\alpha \beta}$ is the two metric induced on
the boundary $S$. These are also real.

Finally, the third set of observables is given by the generators of the
diffeomorphism group of the boundary $Diff(\partial \Sigma)$, which as we have empha-
sized is a symmetry group. The generators of $Diff(\partial \Sigma)$, may be denoted,
by $d(v)$, for $v^\alpha$ a vector field on the boundary. They are given, as in the full
theory, by

$$d(v) \equiv \int_{\partial \Sigma} v^\alpha (F^i_{\alpha \beta} \Pi^{\beta i} + F^i_{\alpha r} \Pi^{ri})$$  \hspace{1cm} (83)

Again, by the reality conditions,

$$d(v)^* = d(v)$$  \hspace{1cm} (84)

VI. Quantization in the presence of boundaries

We are finally in a position to construct the quantum theory that corre-
sponds to the classical theory we have constructed in the last two sections,
and recover the results we described in the section III.

I give here an outline of the main steps in the quantization, stressing
those aspects that are new in this case. Full details will appear elsewhere.

We proceed in the following order. First, I will review a few more of the
basics of quantum Chern-Simons theory on the two manifold with punctures,
$(S, y_\alpha, j_\alpha)$. Then we will extend this first to the kinematical and then to the
diffeomorphism invariant quantum theories on the interior of $\Sigma$.

VIa. More facts about quantum Chern-Simons theory

Before proceeding to the quantum theory in the whole manifold $\Sigma$, we need
to fill in a few of the details of quantum Chern-Simons theory on $(S, y_\alpha, j_\alpha)$
that were left out of the sketch in section 2. The main fact we will need is
that

**Spin network basis:** An orthonormal basis for the state space $H_{CS}^{y_\alpha, j_\alpha}$
is given by the set of independent $q$-spin network trees in $S$ in which one
edge emerges from each puncture $y_\alpha$ colored with spin $j_\alpha$. By independent
here we mean under the equivalence relations of quantum spin networks and
homotopy on $S - y_\alpha$. The key equivalence relation among the spin networks
are the defining relations of quantum $6-j$ symbols, which tell us that the
four valent node can be decomposed in two equivalent ways as a product
of two trivalent nodes, analogous to the usual duality diagrams in which
annihilation and creation processes are equivalent to scattering.

For finite $k$ and genus $n$, these spaces are finite dimensional.

This is a standard result of conformal field theory[31,33]. It is sometimes
stated in a language or trinions, in which the manifold $S - \{y_\alpha\}$ is decom-
posed according to trinions, colored with representations of $SL(2)_q$ on the
three boundaries, so that addition of $q$-angular momentum is satisfied\[31,33\]. The description in terms of trinions is completely equivalent to that of quantum spin networks.

To see this we may recall that a trinion is a sphere with three disks removed, which are labeled by representations. A trinion decomposition of a punctured surface without boundary may be given in which the three boundaries of each trinion are associated with either punctures or circles along which the surface has been cut. In the latter case, the two labels on the circles of each two trinions are joined must match.

Given a trinion decomposition of the surface, we may associate a spin network by drawing a graph on each trinion which consists of three lines coming from each boundary joined at one trivalent vertex. At any circle where two trinions are joined we join the two lines going out to those circle, which we may since the representations match. Correspondingly, it is straightforward to see that given any spin network on the surface we may, under the equivalence relations available for flat connections, be replaced by a sum over trivalent spin networks, so that each may be replaced by an equivalent trinion decomposition.

Given this picture of the state spaces, the action of the elements of the observable algebra $\mathcal{A}_S,y_0,j_0$ may be computed following the standard methods of conformal field theory.

This algebra is generated by the generators of $Diff(S - y_0)$ and by $t[\alpha]$, where $\alpha \in \pi_1(S - y_0)$. The algebra is given by (7), together with the action of the diffeomorphisms on homotopy classes.

VIb. Kinematical quantum theory

We now begin the construction of the kinematical quantum theory on the whole manifold $\Sigma$. We begin in the connection representation, later, when we have established the kinematical quantization we will transform to the loop representation to find the diffeomorphism invariant states, following the usual ideas for manifolds without boundary\[8,10,11,38\].

We will seek to follow the constructions that underlie the application of the loop representation to the non-perturbative quantization of diffeomorphism invariant theories in the absence of boundaries. We will, however, have to modify the construction at several points to account for the conditions on the boundaries.
VIc. The spin network basis in the connection representation

The basic idea of the connection representation is that the states are functions of $A_{ai}$ and the canonical momenta $\Pi^{ai}(x)$ are represented by $\delta/\delta A^{ai}(x)$. As usual, the crucial element in the quantization is the treatment of the Gauss’s law constraint. There are two main approaches to quantization with constraints, quantize and then apply the constraints as operator equations, or solve the constraints classically and then quantize. Most of the work that follows is performed using the first method. But, as a check, in subsection VIIf the same results are derived using the solve and then quantize approach.

We choose to first quantize, and then solve this on the quantum states. We may note that acting on states in the connection representation the Gauss’s law constraint gives, for every $\Lambda^i$ on $\Sigma$,

\[ \hat{G}(\Lambda)\Psi[A] = -\frac{1}{2} \int_{\partial\Sigma} d^2 S^{ab}(\sigma) \Lambda^i(S(\sigma)) \left( \epsilon_{abc} \frac{\delta}{\delta A^{ai}(x)} + 3 \frac{\lambda}{\lambda} F^{ab}_i(S(\sigma)) \right) \Psi[A] = 0 \]

It is important to note that because this must be true for every choice of $\Lambda^i$, the volume and surface terms must be solved separately. This is because we must solve it for every $\Lambda^i$ with support that excludes the boundary as well as for every choice of support on a region that extends an arbitrarily small distance $\epsilon$ into $\Sigma$ from the boundary.

We may then proceed to solve the volume term in the Gauss’s law as usual by expressing the states in terms of loops, corresponding to Wilson loops of the connection $A^{ai}$ on $\Sigma$. We then assume that given a set of loops $\gamma_I$ we may construct an overcomplete basis of states in the connection representation of the form

\[ \Psi_{\gamma}[A] = <A|\gamma_I> = \prod_I T[\gamma, A] \]

where the $T[\gamma_I, A]$ is the trace of the holonomy in the fundamental representation. The loops $\gamma_I$ may enter or leave the boundary $\mathcal{S}$, and run along in it.

These loop states are overcomplete because of the Mandelstam and re-tracing identities[12]. However an independent basis may be constructed[38, 48] which is in one to one correspondence with the spin networks. This is,
of course, a standard result in lattice gauge theory \cite{49} and topological field theory \cite{5,6,50}, we review the basic idea here.

Each such basis element is equal to a linear combination of loop states which is constructed in the following way (for details, see \cite{38}). A spin network diagram is a spin network, as defined in the previous section, together with an orientation which is given by an imbedding of the network in the plane. To each edge of the network, labeled by an integer \( p \), associate \( n \) copies of that segment. To each vertex of valence higher than three, there is also a label, which indicates how the routings through it are to be decomposed in terms of trivalent vertices (this is also given uniquely in terms of the imbedding in the diagram). Then combine all these segments joining them across the vertices in all possible ways, resulting in a finite set of multiloops \( \gamma_i \), in which each edge is traced the number of times given by its labeling. The spin network basis is then a sum over the basis elements labeled by these multiloops. In the connection representation it is given by

\[
\Psi_{\Gamma}[A] = \sum_{\gamma_i \in \Gamma} (-1)^{r_i} T[\gamma_i, A] \tag{87}
\]

where the \( T[\gamma_i, A] \) are the usual Wilson loops in the fundamental representation and the overall sign \( r_i \) is defined by a rule which depends on the imbedding of the graph in the plane.

As explained in \cite{38}, once a unique rule for the labeling of the higher valence vertices is given, the elements of the spin network basis are independent, in that they satisfy no further identities.

VId. Restrictions due to the Dirac brackets at the boundary

There is, however, an important restriction on the states of the form (86) or (87), when the edges of the spin network run in the boundary \( S \). This is that, because of the commutation relations (66), the connections restricted to the boundary do not all commute with each other. If we allow states to have arbitrary dependence on the connection on the boundary we will have too many states, as we should have enough to give a basis of functions on the configuration space, and not on the whole phase space.

Several different ways to accomplish this have been developed in studies of Chern-Simons theory. One which has been often used depends on the trinion decomposition we discussed in the previous section. On each trinion we may pick a complex coordinate and require that the states are holomorphic functions of \( z \). Holomorphic functions may be constructed on the whole
manifold by insuring that the holomorphic conditions can be continued across any boundary joining two trinions.

For our purposes we may make the following equivalent construction. We may make a choice of local coordinates \( r, \theta \) in a neighborhood of each puncture such that curves of constant \( r \) circle the puncture, while curves of constant \( \theta \) run radially away from it. On each trinion, these three patches may be joined consistently to yield coordinates on the whole trinion. Furthermore, the coordinates can be matched along each boundary where two trinions meet, as these are circles of constant \( r \). We may then require that the states are functions only of the \( A_i^r \) components of the connections. This means that the curves in the spin network states (87) can run on the trinions only along the radial directions, on constant \( \theta \) curves. This allows us to have a trivalent spin network on each trinion, where the three edges run from each disk radially to a point on the trinion where the three coordinate patches meet. We will call a graph \( \Gamma \) on \( S - \{ \alpha \} \), with ends on the punctures, an allowable graph, when there is a trinion decomposition of the surface, and a choice of local coordinates on each trinion, such that the edges of the graph that run in the surface only run in the radial coordinate direction in those coordinates on each trinion.

We will assume that the spin network states then depend only on allowable graphs, when they run in the surface.

Vie. Imposing the surface term in the Gauss law constraints

The next step is to impose the boundary term in the Gauss law constraint, eq. (85) on the space of states we have just described. To do this it will be useful to decompose the kinematical state space in terms of sets of points and representations, \((y_\alpha, j_\alpha)\) on \( S\), as we described in section III. We then may study the action of the surface term in (85) on states in the subspace \( H^{km}_{y_\alpha, j_\alpha} \). This consists of all spin network states of the form (87) where the network \( \Gamma \) enters the boundary only at the marked points \( y_\alpha \), and so that the edges of \( \Gamma \) that intersects \( S \) at \( y_\alpha \) carries the representation \( j_\alpha \).

We will now see that in each such sector, the action of the surface part of the Gauss’s law constraint is to to impose exactly the Gauss’s law constraint in the Chern-Simon’s theory with sources at the points \( y_\alpha \) with spins \( j_\alpha \).

We then consider, for all functions \( \Lambda^i \) with support that includes a piece of the boundary \( S \), the condition,

\[
\int_{\partial S} d^2 S^{ab}(\sigma) \Lambda^i(S(\sigma)) \left( \epsilon_{abc} \frac{\delta}{\delta A^c_L(x)} + \frac{3}{\lambda} F_{ab}^i(S(\sigma)) \right) \Psi_{\Gamma}[A] = 0 \quad (88)
\]
where the state can only have support on allowable graphs $\Gamma$. We may consider first of all the case that the support of the support of $\Lambda^i$ does not include any of the punctures $y_\alpha$. By considering all such cases, we reach the conclusion that, in the connection representation, the states must have support only on flat connections in $S - y_\alpha^\Gamma$. This is because the $\delta/\delta A_{ai}$ terms annihilate such states. The curvature $F_{ab}^i$ can be represented in terms of the holonomy of a loop that encloses no punctures, the constraint then says that the holonomy must be trivial, which means that the connection is flat.

This means that the loop function corresponding to $\Gamma$ can only be functions of the homotopy of those portions of the loops that run in $S - y_\alpha^\Gamma$. (Both of these results are found using the standard arguments developed for $2 + 1$ gravity and Chern-Simons theory[2, 3, 32, 51].) This means that the states can only depend on the spin networks up to homotopy in the boundary and the equivalence relations among spin networks.

We have then reached the following result: Fix a spin network $\Gamma'$ with open edges, in the interior of $\Sigma$ such that the open ends touch the boundary at a set of points $y_\alpha$, with edges labeled by representations $j_\alpha$. For every independent way to complete $\Gamma'$ to a spin network $\Gamma$ with the ends joined by edges running in the boundary, we will have an independent kinematical quantum state of the gravitational field in all of $\Sigma$. What we have just shown is that these independent possibilities are exactly given by the ways to draw a trivalent spin network in the surface $S$ with ends at $y_\alpha$, with edges labeled by the $j_\alpha$, up to homotopy of the loops in $S - \{y_\alpha\}$ and the $6-j$ symbol relations among trivalent spin networks. What we have done is reproduced the description of the state space of Chern-Simons theory associated to $(S, y_\alpha, j_\alpha)$, with one difference, which is that we have not so far obtained the restriction to quantum spin networks at level $k$.

To realize this we must take the last step and impose the remaining constraints, which are the surface parts of the Gauss's law in the case that $\Lambda^i$ has support on a region of the boundary that includes at least one puncture $y_\alpha$. It is, of course, sufficient to restrict $\Lambda^i$ to have support on a region that includes a single puncture $y_1$.

We should proceed with care, as $F_{\alpha\beta}$ is now a non-trivial operator acting on the states. $F_{\alpha\beta}$ involves products of operators that don't commute with each other, and a complete discussion of the relation (88) must involve a careful regularization procedure. To establish the necessity of the restriction to quantum spin networks, however, it will be sufficient to compute the simplest case, which is the case the puncture is in the fundamental representation, so $j_1 = 1/2$, in the limit of large $k$. 41
To proceed, we consider how we may define the action of a loop operator $\hat{T}[\alpha]$, for a loop $\alpha$ that goes once around the puncture at $y_1$ at constant $r$ in some choice of radial coordinates. We may note that this may be defined two ways, by realizing directly the Poisson bracket relations (66) on the states or by using the surface constraint (88) for a region $\mathcal{R}$ which is the interior of $\alpha$. The basic point, as we will see, is that the surface constraint must define the action of a loop operator that surrounds one puncture.

Note that as all the angular components, $A_i^\theta$ commute with each other there is no ordering problem in defining $\hat{T}[\alpha]$. But there are ordering issues when evaluating its effect on states based on admissible loops.

We begin with the second method. To construct an operator to represent $T[\alpha]$, we may use the non-Abelian Stokes theorem [52] to write

$$\hat{T}[\alpha] \Psi_\Gamma[A] = \text{Tr} e^{i \int_\mathcal{R} d^2S_{ab}(\sigma) U_{\eta\sigma} \hat{F}_{ab}(\sigma) U_{\eta^{-1}\sigma}} \Psi_\Gamma[A]$$

(89)

Here $\eta_\sigma$ is a loop from an arbitrary base point somewhere on $\alpha$ to the point $\sigma \in \mathcal{R}$. The actual loop generally is defined in a particular way through the construction of the Non-Abelian Stokes theorem, but using the fact that the connection is flat on $\mathcal{R} - y_1$ we may take any loop that connects them with trivial homotopy.

We may now use the surface constraint from Gauss’s law (88). Restricting to the case that there is a single $j = 1/2$ line meeting the manifold at $y_1$ in $\mathcal{R}$, we have $\Psi_\Gamma[A] = T[\gamma, A] \xi[A]$, where $\xi$ is not acted on by the loop operator $T[\alpha]$. We then want to evaluate,

$$\hat{T}[\alpha] T[\gamma, A] = \text{Tr} e^{-2\pi \frac{i}{k} \int_\mathcal{R} d^2S_c(\sigma) U_{\eta\sigma} \tau^i U_{\eta^{-1}\sigma} \delta A_i^c(\sigma)} T[\gamma, A].$$

(90)

It is straightforward to evaluate the first two terms in the expansion of the exponential. The term in $1/k$ vanishes because the trace of a symmetrization gives zero, so that the leading non-trivial term is order $1/k^2$. After a simple calculation, using the methods described in [38], we find that

$$\hat{T}[\alpha] T[\gamma, A] = \left[ 2 - \frac{3}{2} \left( \frac{2\pi}{k} \right)^2 + O(k^{-3}) \right] T[\gamma, A]$$

(91)

There is a problem with the normalization of this result, as we shall see in a moment. The basic problem is that it gives a non-zero answer for the action of the operator, even in the case in which the loop $\alpha$ surrounds no punctures. However, in order to realize the commutation relations (66) in a way that
is consistent with the restriction that the states in (87) are admissable, according to the rules we developed above, in this case the action of the operator must vanish. To see this, let us consider the case in which we want to realize the commutation relations coming from (66):

$$[\hat{T}[\alpha], \hat{T}[\gamma]] = \frac{2\pi i}{k} \text{Int}[\tilde{\alpha}, \tilde{\gamma} \left( \hat{T}[\alpha \circ \gamma] - \hat{T}[\alpha^{-1} \circ \gamma] \right)]$$

(92)
in the case that, as here, $\alpha$ is not an admissable loops, but $\gamma$ is. We may note that as $\alpha$ is not admissable, the corresponding operator $\hat{T}[\alpha]$ cannot create a loop, as in the usual definition of the Wilson loop operator. We can define this action according to the commutation relations (66), in the following way: a loop operator is treated as a $T^0$ when the loop is admissable, and therefor adds that loop, while it acts like the standard definition of a $T^1$ in the case that the loop is non-admissable (and in the surface $S$). This will guarantee that the algebra (66) is realized win all cases. We then have, in the case that only $\gamma$ is admissable,

$$\hat{T}[\alpha] \Psi_{\gamma}[A] = \frac{2\pi i}{k} \text{Int}[\tilde{\alpha}, \tilde{\gamma}] \left( \Psi_{\alpha \circ \gamma}[A] - \Psi_{\alpha^{-1} \circ \gamma}[A] \right).$$

(93)

Clearly, we can only require that this is true if it is the case that all three of the loops $\gamma$, $\alpha \circ \gamma$ and $\alpha^{-1} \circ \gamma$ are homotopic to admissable loops in the region $R - y_1$. We shall see in a moment that this is the case, if $\gamma$ is admissable.

There is, however, a subtle point, which is that the definition we have just given requires that $\hat{T}[\alpha]$ annihilate a loop state $\Psi_{\gamma}[A]$ in the case that $\alpha$ is not admissable and does not intersect with and $\gamma$ in the surface. This disagrees with the result of the non-Abelian Stokes theorem, which is (91). To make them consistent with each other, we must take into account that the classical non-Abelian Stokes theorem may have to be modified when it is applied to quantum operators that don’t commute with each other. To account for the discrepancy, it must then be that we missed a constant subtraction in the evaluation of the non-Abelian Stokes theorem coming from the fact that it involves manipulation of non-commuting operators. To make it match at leading order, we must then modify (90) so that

$$\hat{T}'[\alpha] T[\gamma, A] = \left[ \text{Tr} e^{-\frac{2\pi i}{k} \int_R d^2 \sigma \left( U_{\gamma} U_{\sigma} \frac{\delta}{\delta A_{\sigma}} \right) - 2} \right] T[\gamma, A]$$

(94)
where the $'$ indicates that a subtraction has been done in the definition of the operator so that it agrees with the requirement that when $\alpha$ is a non-admissable loop $\hat{T}[\alpha]$ must annihilates states supported on loops that have
no intersection with $\alpha$ in the boundary. We then have,

$$
\hat{T}'[\alpha]T[\gamma, A] = \left[ \frac{3}{2} \left( \frac{2\pi}{k} \right)^2 + O(k^{-3}) \right] T[\gamma, A] \tag{95}
$$

We may now compare this answer to the one we get if we directly use the definition (93) of the action of the $\hat{T}[\alpha]$ operator to evaluate the action of the loop operator on the spin network states.

Now we come to a second subtlety, which is indeed the key point of the whole calculation. This is whether or not we allow homotopic equivalence to include smooth deformations of the loops in $\mathcal{S} - \{y_\alpha\}$ that rotate one or more of the punctures by $2\pi$. Note that if we take the naive picture in which the puncture may be twisted, the loops $\gamma \circ \alpha$ and $\gamma \circ \alpha^{-1}$ are homotopic to each other and to $\gamma$ in $\mathcal{R} - y_1$. Hence, under this assumption, both the first and the second terms of (93) are equal to each other, so that the action is zero. This means that the assumption that homotopy classes include those in which the puncture is rotated is inconsistent with the surface constraints (88), because we have just performed the calculation another way that uses them and found a non-vanishing answer.

This means that the loops must be framed, so that there is a factor coming from twisting the puncture. We must then have,

$$
\Psi_{\gamma \circ \alpha}[A] = e^{\frac{2\pi i \theta}{k}} \Psi_{\gamma}[A] \\
\Psi_{\gamma \circ \alpha^{-1}}[A] = e^{-\frac{2\pi i \theta}{k}} \Psi_{\gamma}[A] \tag{96}
$$

where the factor $\theta$ must come from the operator ordering that requires the loops to be framed. We then have,

$$
\hat{T}'[\alpha]T[\gamma, A] = \frac{2\pi i}{k} (2\pi)sin\left(\frac{2\pi \theta}{k}\right) \tag{97}
$$

Comparison of this at large $k$ with (95) tells us that we must have $\theta = -3/4$. We may note that this factor may be computed directly in Chern-Simons theory\cite{2, 53} and one gets precisely this same value of $\theta$. Indeed, the action of a twist is one of the defining relations of the Kauffman bracket, so that this factor is determined directly by the Jones polynomial\cite{35}. It is non-trivial that this calculation has led to the same value of $\theta$.

We have thus shown that to leading non-trivial order in $k$, the loops must be framed in agreement with the result in Chern-Simons theory. Furthermore, we have checked directly, by taking into account the surface constraint
(88) that to leading non-trivial order in $k$, the operator associated with a loop that surrounds a puncture agrees precisely with the value in Chern-Simons theory, at least in the case that the puncture has $j = 1/2$. Thus, for this same case, we have deduced the isomorphism (18) of the observables algebra of Chern-Simons theory on $S - \{y_{\alpha}\}$ with the subspace of the surface algebra of quantum gravity $\mathcal{A}_{S, y_{\alpha}, j_{\alpha}}^{\text{gr}}$.

This derivation will be carried out in the general case elsewhere. For now, we may check that the same result follows by taking the other approach to quantizing in which we first solve the Gauss’s law constraint, and then quantize.

VII. Constraining and then quantizing

We have run into a cumbersome problem, that involved having to define an operator by doing a subtraction, because we have followed the route of imposing the Gauss law constraints on the states. While we have been able to extract the result by considering the leading terms at large $k$, there is an easier way to reach the same conclusion. In most treatments of Chern-Simons theory, the problem of defining the Gauss law constraint as an operator equation in a way that is consistent with the commutation relations (66) is avoided by first solving the constraints classically, which yields the reduced phase space, and then quantizing. When this path is followed in the case of punctures, the problem is how to describe the effect of the puncture in the classical phase space. This is usually done by adding a degree of freedom to each puncture, so that the Gauss’s law constraint becomes

$$\frac{k}{4\pi} \epsilon^{\alpha\beta} F_{\alpha\beta}(\sigma) = \sum_{\alpha} r_{\alpha} \delta^{2}(\sigma, y_{\alpha})$$

where the $r_{\alpha}$ are classical variables, valued in the Lie algebra of $SU(2)$ associated to the punctures at $y_{\alpha}$. In order that the classical variable $r_{\alpha}$ will yield a state in the $j_{\alpha}$'th representation on quantizing, it is taken to lie in the coadjoint orbit, with a symplectic structure given by a choice of representation, using the connection between symplectic structures on groups and their representations\textsuperscript{[2, 32]}. For our purposes, this means that, up to a gauge transformation, $r_{\alpha}$ must be the equal, in the Lie algebra, to the root that is the highest weight of the $j$’th representation. This means that we may take $r_{\alpha} = \omega_{\alpha} \lambda_{\alpha} \omega_{\alpha}^{-1}$, where $\omega_{\alpha} \in SU(2)$ and $\lambda_{\alpha}$ is a root associated with the spin $j_{\alpha}$ representation.
Given the classical system described by (98), quantization then yields the Chern-Simon theory as we described it above\[2, 32\].

We may follow this development by finding a suitable classical problem that may correspond to our constraint (88). The main question here, as in the case of Chern-Simon theory, is what the representation of the puncture will be in the classical theory.

There is readily available an answer to this, which is to recall that, as described in [11, 55], the classical limit of the kinematical states in the loop representation correspond to a certain kind of discrete geometry, which is analogous to the Regge calculus. However, in this case, the metric geometry is distributional, with the frame fields having support only on the edges of the graphs. We may note that this is appropriate, because the classical limit must only yield something that approximates a smooth geometry in the case that the state is based on a large graph such as the weaves described in [21]. We will see that if we use this classical picture, we reproduce the constraint (98) of Chern-Simons theory in the case of punctures.

We may then consider the classical geometry that corresponds to a spin network $\Gamma$ to be given by a distributional frame $\tilde{E}_{ai}^{\Gamma}$ defined by,

$$\tilde{E}_{ai}^{\Gamma}(x) = G \sum_l \int ds \delta^3(x, \gamma_l(s)) \dot{\gamma}_a^l(s) t^l_i(s)$$

(99)

Here, $l$ labels the edges of the graph and $t^l_i(s)$ are elements of the Lie algebra. The magnitude of these Lie algebra elements must be picked by requiring that the geometric observables computed using this classical frame field agree with the eigenvalues of the corresponding quantum operators in the spin network state $\Gamma$. It is interesting that the area and volume observables may be computed for such frame fields, in spite of their distributional character, using exactly the same regularization technique as is employed in the construction of the action of the quantum operators on the loop states\[11\].

The result is that the classical correspondence principle applied to the area observables requires that $|t_\alpha| = \sqrt{j_\alpha(j_\alpha + 1)}$. This means that the $t_\alpha$ must be taken so that

$$t_\alpha(s) = \omega(s) \lambda_\alpha \omega^{-1}(s).$$

(100)

where $\omega(s)$ is an $SU(2)$ valued field on the edge. If we now plug this into the surface term of the constraint (68) we find exactly the condition (98) with $t_\alpha = r_\alpha$.

Thus, from considerations of the classical limit of the quantum gravity theory, we arrive at the same classical version of the surface Gauss’s law
constraint which is used as the starting point of the quantization of the corresponding Chern-Simons theory. This tells us that the isomorphisms of the observable algebras (18) must hold at the classical level.

Quantization must then yield an algebra of quantum operators at the surface which is identical to the algebra of observables of the Chern-Simons theory. Thus, the two representation spaces must be identical, which means that the restriction to the quantum spin networks must apply also in the quantum gravity case, with the identification of the level \( k \) coming from quantum gravity given by (49) corresponding to the \( k \) of the classical Chern-Simons theory. Further more, as the level is renormalized in the Chern-Simons theory, but the (49) relation holds between the classical theories, this means that \( k \) will be renormalized from \( k \) to \( k + 2 \) also in the quantum gravity case.

VIg. Kinematical inner product

We may now use what we have learned to construct a kinematical inner product for the quantum theory, that respects the reality conditions of the Euclidean theory. It is sufficient to impose an inner product on each subspace \( \mathcal{H}_{\psi_{\alpha},j_{\alpha}}^{kin} \) because we know that states in different of these subspaces must be orthogonal as they are each eigenspaces of the operators that measure the areas of surfaces. The results of the last subsection mean that we may decompose each of these spaces according to the analogue of (24) where we now have a characterization of the kernal: \( K_{C,\psi_{\alpha},j_{\alpha}}^{kin} \) must be spanned by states corresponding to each quantum spin network \( \Gamma \) in the interior of \( \Sigma \) with open ends with labels \( j_{\alpha} \) that meet the boundary at the points \( y_{\alpha} \).

We may now impose a kinematical inner product \( < \mid \cdot \mid >_{C,\psi_{\alpha},j_{\alpha}}^{kin} \) in each of these subspaces. This must satisfy a requirement that every area and volume operator, associated with every surface and region in the interior of \( \Sigma \) orthogonal. But given any two distinct spin networks \( \Gamma \) and \( \Gamma' \) in the interior there will exist such operators such that the states \( \Psi_{\Gamma} \) and \( \Psi_{\Gamma'} \) are both eigenstates, but with distinct eigenvectors. This means that all states associated with distinct networks must be orthogonal to each other, so that the inner product must have the form,

\[
< \Gamma | \Gamma' >_{C,\psi_{\alpha},j_{\alpha}}^{kin} = n_{\Gamma} \delta_{\Gamma \Gamma'}
\]

The restriction to this class at the kinematical level is justified by the fact that the spin network states are eigenstates of a complete set of commuting operators.
observables, which are the volumes of arbitrary regions and the areas of arbitrary surfaces in $\Sigma$ \cite{22}. Further the distinct spin networks are completely distinguished by their respective eigenvalues. Therefore they must be orthogonal to each other in any inner product chosen to respect the hermiticity of these observables.

We may note that additional observables must be invoked to fix the coefficients $n_{\Gamma}$. In the case of Euclidean signature, with which we are now concerned, it is a simple exercise to show that imposing that the loop operators $\hat{T}^0[\gamma]$ be hermitian leads to the conclusion that all the $n_{\Gamma}$'s are equal to unity \cite{11, 13}.

To complete the description of the kinematical state space, we may write an orthonormal basis,

$$|\Gamma, z \rangle = |\Gamma \rangle \otimes |z \rangle$$ (102)

where $|z \rangle$ are a basis in $H_{CS, y_\alpha, j_\alpha}^{CS}$. But we know that a basis for the states of the Chern-Simon theory are given by the independent trivalent quantum spin networks in the surface with ends on the punctures. Thus, we have the following construction. Let $\Gamma$ be a quantum spin network in $\Sigma$ which meets the boundary $\mathcal{S}$ at points $y_\alpha$ with quantum spins $j_\alpha$ and is admissible and trivalent when it runs in the boundary. Then it induces a state $|z_\Gamma \rangle \in H_{CS, y_\alpha, j_\alpha}^{CS}$ given by the quantum spin network that is its intersection with the boundary. It also induces a quantum spin network $\tilde{\Gamma}$ that runs in the interior of $\Sigma$ and meets the boundary at the points $y_\alpha$. Then the kinematical inner product of the Euclidean theory is

$$\langle \Gamma_1 | \Gamma_2 \rangle = \langle z_{\Gamma_1}^\Gamma | z_{\Gamma_2}^\Gamma \rangle_{CS} \delta_{\Gamma_1, \Gamma_2}$$ (103)

We may note that this inner product is nonseparable. This would no doubt be a problem, were we doing anything other than quantizing a diffeomorphism invariant theory. We will see that this problem is fixed at the next step.

VIh. Diffeomorphism invariant quantum theory

Finally, we may next construct the diffeomorphism invariant states. As the diffeomorphisms are frozen on the boundary, this is straightforward, spin networks $\Gamma$ that meet the boundary are replaced by their diffeomorphism invariant classes following the logic of \cite{8}. The fact that the diffeomorphisms are fixed on the boundary means that the subspaces $H_{y_\alpha, j_\alpha}^{diffco}$ and $H_{\phi \circ y_\alpha, j_\alpha}^{diffco}$, where $\phi$ is a diffeomorphism of the boundary are physically distinct, and
are related by a symmetry transformation. We then have diffeomorphism invariant states $|\{\Gamma\}\rangle$, which correspond to an admissible, trivalent loop, running in $\mathcal{S} - \{y_\alpha^\Gamma\}$, together with a diffeomorphism equivalence class of continuations $\{\tilde{\Gamma}\}$ into the boundary, where, again, the diffeomorphisms vanish on the boundary. The diffeomorphism invariant inner product is then

$$< \{\Gamma_1\}|\{\Gamma_2\}\rangle = \delta_{\gamma_\alpha^\Gamma_1 \gamma_\beta^\Gamma_2} < z^{\Gamma_1}|z^{\Gamma_2} > CS \delta_{\{\tilde{\Gamma}_1\}|\{\tilde{\Gamma}_2\}}$$

(104)

VII. Conclusions

In closing, it may be interesting to mention several directions that may be developed beginning with the results described here.

A crucial question is whether it is possible to extend these results from the Euclidean to the Minkowskian case. As was mentioned above, there is not necessarily a problem with applying the self-dual boundary conditions in the case of Minkowskian signature, as only one third of the self-dual conditions are imposed on the boundary. What Minkowskian solutions are consistent with these conditions needs to be investigated. The possibility that these results might also apply to the Barbero formalism [30] should also be investigated.

An especially interesting suggestion is whether these or related conditions can be achieved on an apparent horizon. If so, the results of Carlip [56], who fixes boundary conditions on horizons in 2 + 1 gravity, may be brought to the physical 3 + 1 dimensional case.

The main issue to be faced in extending these results to the Minkowskian case is that the results I used from quantum Chern-Simons theory depend to some extent on the compactness of $SU(2)$. In the Minkowskian case the gauge group that will be induced on the boundary is the left handed part of the Lorentz group, which is not compact. However, it may be possible to extend some of the results by finding an appropriate notion of analytic continuation.

Some comments may be made about the appearance of quantum groups in the formalism of quantum gravity. How the loop algebra in 3 + 1 quantum gravity may in fact be deformed so that the state space has a basis given by quantum spin networks will be explained elsewhere [39]. But perhaps some possible implications of the existence of this structure may be mentioned here. As has been mentioned before [58] at finite $k$ there is an infrared cutoff in the spectra of diffeomorphism invariant states of the gravitational field. It is then not surprising that this is associated with a finite cosmological,
which is inversely proportional to $k$, as we find here. (It is suggestive also that Carlip finds a similar relation in $2 + 1$ dimensions \cite{30}.) It is possible that this infrared cutoff may play a useful role in the statistical mechanics of the spin network states aimed at understanding quantum gravity at finite temperature.

But, beyond this, the appearance of quantum spin networks opens up the possibility of qualitatively new phenomena in quantum gravity. One is the following. The states we are speaking of here are closely connected to the Jones polynomial, especially if the conjecture about the triviality of the physical kernels is correct. We may note that in such states there emerges a non-trivial duality between topology and geometry. To see this, consider a quantum state of the type we are describing here, associated to the Jones polynomial. Let us consider the evaluation of the state on some spin network of the form of $\Gamma \cup \alpha_j$, where $\alpha$ is a component that does not intersect with the rest, but may be knotted and linked to the rest. The $j$ labels the representation associated with the closed loop $\alpha$. We assume that this is being evaluated in some three manifold with topology $\Sigma$. For every representation $j$ labeling the loop $\alpha$, there is a value of the state given by $\Psi^j[\Gamma \cup \alpha_j, \Sigma]$. Now, as we have described, $j$ has a geometrical interpretation, it tells us how much area there may be in any surface that intersects only the $\alpha$ loop once. Now, it is known that there are coefficients, $C_j$, such that

$$\Psi^J[\Gamma, \Sigma'] = \sum_j c_j \Psi^j[\Gamma \cup \alpha_j, \Sigma]$$

where $\Sigma'$ is a different three manifold which is constructed by cutting out from $\Sigma$ a torus around the loop $\alpha$, twisting it and then identifying the surfaces of the torus we removed.

The meaning of this formula is that the value of the state on the spin network $\Gamma$ in one topology is related to a sum over its values, on different spin networks in a different topology, which differ from each other only by the eigenvalue of a geometric observable, the areas of surfaces pierced by $\alpha$. This suggests the existence of a kind of duality between geometry and topology.

This, of course, depends on the assumption that the physical states are those which are related to the Jones polynomial. This is the most important problem not solved here: to characterize the solutions of the quantum dynamics in the presence of the boundary conditions. We have conjectured that there will be only one unique solution to the hamiltonian constraint that matches each possible state in a Chern-Simon theory associated with a
set of punctures and representations. Related to this, as well, is the question of whether the Hamiltonian defined on the boundary in section Vd. can be defined quantum mechanically acting on the states we have described here.

We may note, finally, that if this conjecture about the triviality of the physical kernels is true, the results here may be considered to constitute a solution of quantum Euclidean general relativity in the case with these boundary conditions, in terms of the infinite dimensional state space \( \mathcal{A}^{boundary} \) and the realization on it, of the observable algebra \( \mathcal{A}^{boundary} \). The investigation of this conjecture is then a priority for further work.

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