A characterization theorem and its applications for $d$-orthogonality of Sheffer polynomial sets

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Abstract. The purpose of this paper is to find the characterization of the Sheffer polynomial sets satisfying the $d$-orthogonality conditions. The generating function form of these polynomial sets is given in Theorem 2.2. As applications of the Theorem 2.2, we revisit the $d$-orthogonal polynomial sets exist in the literature and discover new $d$-orthogonal polynomial sets. Moreover, we obtain the $d$-dimensional functional vector ensuring the $d$-orthogonality of these new polynomial sets.

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1. Introduction

Recently, the generalization of orthogonal polynomials called "$d$-orthogonal polynomials" have attracted so much attention from many authors. The well-known properties of orthogonal polynomials such as recurrence relations, Favard theorem, generating function relations and differential equations have found correspondence in this new notion. New polynomial sets which contain classical orthogonal polynomials have been created so far. Let us give a brief summary of $d$-orthogonal polynomials.

Let $P$ be the vector space of polynomials with complex coefficients and $P'$ be the vector space of all linear functionals on $P$ called the algebraic dual. $\langle u, f \rangle$ is the representation of the effect of any linear functional $u \in P'$ to the polynomial $f \in P$. Let $\{P_n\}_{n \geq 0}$ be a polynomial set (deg $(P_n) = n$ for all non-negative integer $n$), and corresponding dual sequence $(u_n)_{n \geq 0}$ for polynomials taken from this set can be given by

$$\langle u_n, P_k \rangle = \delta_{nk}, \quad n, k = 0, 1, ..., \$$

where $\delta_{nk}$ is the Kronecker delta.
A polynomial set \( \{ P_n \}_{n \geq 0} \) in \( P \) is said to be \( d \)-orthogonal polynomial set with respect to the \( d \)-dimensional functional vector \( \Gamma = (u_0, u_1, \ldots, u_{d-1}) \) if the following orthogonality conditions are hold

\[
\begin{align*}
\langle u_k, P_n P_m \rangle &= 0, & m > nd + k, \\
\langle u_k, P_n P_{nd+k} \rangle &\neq 0, & n \geq 0,
\end{align*}
\] (1.1)

where \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), \( d \) is a positive integer and \( k \in \{0, 1, \ldots, d-1\} \) (see [1–2]). Characterization of these polynomials by recurrence relations and Favard type theorem was also given in [1–2]. A polynomial set \( \{ P_n \}_{n \geq 0} \) is a \( d \)-orthogonal polynomial set if and only if it fulfills a \((d + 1)\)-order recurrence relation of the type

\[
x P_n (x) = \sum_{k=0}^{d+1} \alpha_{k,d} (n) P_{n-d+k} (x), \quad n \in \mathbb{N}_0 ,
\] (1.2)

with the regularity conditions \( \alpha_{d+1,d} (n) \alpha_{0,d} (n) \neq 0 \), \( n \geq d \) and by convention \( P_{-n} (x) = 0 \), \( n \geq 1 \). Taking \( d = 1 \) in (1.1) and (1.2) leads us to the celebrated notion of orthogonal polynomials (see [3]).

The recurrence relation of order \( d + 1 \) has been the main reason of deriving many \( d \)-orthogonal polynomial sets as an extension of known ones in orthogonal polynomials. Classical orthogonal polynomials such as Laguerre, Hermite and Jacobi polynomials, discrete orthogonal polynomials like Charlier, Meixner polynomials and so on were extended to the \( d \)-orthogonality notion and many basic properties linking with these polynomials were stated by various authors ([4–23]). Especially, in [14], the authors described a useful method for checking whether if a given polynomial set \( \{ P_n \}_{n \geq 0} \) is \( d \)-orthogonal or not. This method will be described and used in the sequel.

A polynomial set \( \{ P_n \}_{n \geq 0} \) is called Sheffer polynomial set if and only if it has the generating function of the form

\[
A (t) e^{xH(t)} = \sum_{n=0}^{\infty} P_n (x) \frac{t^n}{n!}
\] (1.3)

where \( A (t) \) and \( H (t) \) have the power series expansions as following

\[
A (t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0 \neq 0 , \quad H (t) = \sum_{k=0}^{\infty} h_k t^{k+1} , \quad h_0 \neq 0 .
\]

This means that \( A (t) \) is invertible and \( H (t) \) has a compositional inverse. There are numerous polynomial sets belong to the class of Sheffer polynomials (see [24–25]). Note that, for \( H (t) = t \), we meet the definition of Appell polynomial sets [25] from the aspect of generating functions. That is to say, Appell polynomials can be defined by generating function of the type

\[
A (t) e^{xt} = \sum_{n=0}^{\infty} P_n (x) \frac{t^n}{n!}
\]
with \( A(t) = \sum_{k=0}^{\infty} a_k t^k \) \((a_0 \neq 0)\). In this contribution, our aim is to find the exact form of \(d\)-orthogonal polynomial sets which are at the same time Sheffer polynomial set. Then, we try to derive new \(d\)-orthogonal polynomial sets and find some of them’s \(d\)-dimensional functional vector for which promises the \(d\)-orthogonality. Also, we revisit some known \(d\)-orthogonal polynomial sets exist in the literature.

2. Main Results

Characterization problems related to Sheffer polynomial set have a deep history. Both Meixner [26] and Sheffer [27] interested in the same problem: what is the all possible forms of polynomial sets which are at the same time orthogonal and Sheffer polynomials. They stated that \( A(t) \) and \( H(t) \) of (1.3) should satisfy the following conditions

\[
\frac{1}{H'(t)} = (1 - \alpha t)(1 - \beta t),
\]
\[
\frac{A'(t)}{A(t)} = \frac{\lambda_2 t}{(1 - \alpha t)(1 - \beta t)}.
\]

If we discuss all possible cases in view of these two conditions, then we face with the known orthogonal polynomial sets listed below:

**Case 1:** \(\alpha = \beta = 0\) \(\Rightarrow\) Hermite polynomials.

**Case 2:** \(\alpha = \beta \neq 0\) \(\Rightarrow\) Laguerre polynomials.

**Case 3:** \(\alpha \neq 0, \beta = 0\) \(\Rightarrow\) Charlier polynomials.

**Case 4:** \(\alpha \neq \beta, (\alpha, \beta \in \mathbb{R})\) \(\Rightarrow\) Meixner polynomials.

**Case 5:** \(\alpha \neq \beta, (\text{complex conjugate of each other})\) \(\Rightarrow\) Meixner-Pollaczek polynomials.

For more information see [28]. Similar investigation was made in [29] for 2-orthogonal polynomials. In order to solve a characterization problem for \(d\)-orthogonality, we need the following lemma.

**Lemma 2.1.** ([14]) Let \(\{P_n\}_{n \geq 0}\) be a polynomial set defined by the following relation

\[
G(x, t) = A(t)G_0(x, t) = \sum_{k=0}^{\infty} P_n(x) \frac{t^n}{n!}
\]

with \(G_0(0, t) = 1\) and let \(\hat{X}_t\) and \(\hat{\sigma} := \hat{T}_x\) be the transform operator of the multiplication operator by \(x\) and \(t\), respectively. Thus,

\[
\begin{align*}
\hat{X}_t G(x, t) &= xG(x, t), \\
\hat{T}_x G(x, t) &= t G(x, t).
\end{align*}
\]

(2.1)
Then, \( \{ P_n \} \) is a \( d \)-orthogonal polynomial set if and only if \( \hat{X}_t \in \mathcal{V}_{d+2}^{(-1)} \) where the action of the set of operators \( \tau \in \mathcal{V}_r^{(-1)} \), \( r \geq 2 \), to \( t^n \) is

\[
\begin{align*}
\tau (1) &= \sum_{k=1}^{r-1} \alpha_{k-1}^{(k)} t^{k-1}, \\
\tau (t^n) &= \sum_{k=0}^{r-1} \alpha_n^{(k)} t^{n+k-1}, \quad n \geq 1.
\end{align*}
\]

Here, \( r \) complex sequences \( \{ \alpha_n^{(k)} \} \) appear for \( k = 0, 1, ..., r - 1 \) with the condition \( \alpha_0^{(0)} \alpha_n^{(r-1)} \neq 0 \). Moreover, the \( d \)-dimensional functional vector which ensures the \( d \)-orthogonality is given by

\[
\langle u_i, f \rangle = \frac{1}{i!} \left[ \sigma_i A(\sigma) f(x) \right]_{x=0}, \quad i = 0, 1, ..., d - 1, \quad f \in \mathcal{P}.
\]

From (2.1), it is obvious that \( \sigma := \hat{T}_x \) is the lowering operator of \( \{ P_n \} \). Now, we can state our main theorem.

**Theorem 2.2.** Let \( \gamma_d(t) = \sum_{k=0}^{d} \beta_k t^k \) be a polynomial of degree \( d \) \( (\beta_d \neq 0) \) and \( \sigma_{d+1}(t) = \sum_{k=0}^{d+1} \alpha_k t^k \) be a polynomial of degree less than or equal to \( d + 1 \). The only polynomial sets \( \{ P_n \} \), which are \( d \)-orthogonal and also Sheffer polynomial set, are generated by

\[
\exp \left[ \int_0^t \frac{\gamma_d(s)}{\sigma_{d+1}(s)} ds \right] \exp \left[ x \int_0^t \frac{1}{\sigma_{d+1}(s)} ds \right] = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \quad (2.3)
\]

with the conditions

\[
\alpha_0 (n \alpha_{d+1} - \beta_d) \neq 0, \quad n \geq 1. \quad (2.4)
\]

**Proof.** Let \( \{ P_n \} \) be a Sheffer polynomial set defined by the generating function (1.3). Taking the derivative of the both sides of the following equality

\[
G(x, t) = A(t) e^{xH(t)}
\]

with respect to \( t \) leads to

\[
\left[ \frac{1}{H'(t)} D_t - \frac{A'(t)}{A(t) H'(t)} \right] G(x, t) = xG(x, t)
\]

where \( D_t = \frac{d}{dt} \). Thus, from (2.1)

\[
\hat{X}_t = \frac{1}{H'(t)} D_t - \frac{A'(t)}{A(t) H'(t)}.
\]
If \( \{ P_n \}_{n \geq 0} \) is a \( d \)-orthogonal polynomial set, according to Lemma 2.1, \( \hat{X}_t \) should belong to the set of operators \( \vee_{d+2}^{(-1)} \). This means that following equal- 
ities must be satisfied

\[
\begin{align*}
\frac{1}{H(t)} &= \sum_{k=0}^{d+1} \alpha_k t^k = \sigma_{d+1}(t) \\
\frac{A'(t)}{A(t)H(t)} &= \sum_{k=0}^{d} \beta_k t^k = \gamma_d(t), \quad \beta_d \neq 0,
\end{align*}
\tag{2.5}
\]

with the conditions \( \text{(2.4)} \). Solving equations \( \text{(2.5)} \) allows us to get the desired 
result given by \( \text{(2.3)} \).

Conversely, Let \( \{ P_n \}_{n \geq 0} \) be a Sheffer polynomial set generated by \( \text{(2.3)} \) 
with the conditions \( \text{(2.4)} \). Thus,

\[
G(x, t) = \exp \left[ \int_0^t \frac{\gamma_d(s)}{\sigma_{d+1}(s)} ds \right] \exp \left[ x \int_0^t \frac{1}{\sigma_{d+1}(s)} ds \right].
\]

If we apply derivative operator \( D_t \) to the both sides of the above equality, then we obtain

\[
[\sigma_{d+1}(t) D_t - \gamma_d(t)] G(x, t) = xG(x, t).
\]

In view of Lemma 2.1 and the conditions \( \text{(2.4)} \)

\[
\hat{X}_t = \sigma_{d+1}(t) D_t - \gamma_d(t) \in \vee_{d+2}^{(-1)},
\]

so \( \{ P_n \}_{n \geq 0} \) is a \( d \)-orthogonal polynomial set. \( \square \)

Remark 2.3. This characterization of \( d \)-orthogonal Sheffer polynomial sets 
seems to be new in this notion. For \( d = 1 \), these results reduce to the ones 
obtained for orthogonal polynomials which are summarized in the beginning 
of this section.

Next, as applications of Theorem 2.2, we revisit some known \( d \)- 
orthogonal polynomial sets which are at the same time Sheffer polynomial 
sets. Also, we derive new \( d \)-orthogonal polynomial sets and find their \( d \)- 
dimensional functional vector.

(i) Laguerre type \( d \)-orthogonal polynomial sets

During the last decade, authors have paid so much attention to extend 
Laguerre polynomials to \( d \)-orthogonality. Thus, there are several extensions of 
Laguerre polynomials in the context of \( d \)-orthogonality(see \[7\], \[14\] and \[22\]). 

Now, taking Theorem 2.2 into account, we revisit some of them which are explicitly obtained from the couple of polynomials \( [\gamma_d(t), \sigma_{d+1}(t)] \).

Application 1: Let \( \{ P_n \}_{n \geq 0} \) be a Sheffer polynomial set generated by 
\( \text{(2.3)} \) with the following couple of polynomials

\[
[\gamma_d(t), \sigma_{d+1}(t)] = \left[ - (\alpha + 1)(1-t)^d, \frac{-1}{d} (1-t)^{d+1} \right].
\]
where $\alpha \neq -1$. After some calculations, thanks to Theorem 2.2, we obtain a $d$-orthogonal polynomial set of the form

$$(1 - t)^{-(\alpha + 1)d} \exp \left\{ -x \left[ (1 - t)^{-d} - 1 \right] \right\} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

with the conditions $\frac{1}{d} + \alpha + 1 \neq 0$. The $d$-orthogonality of this polynomial set deeply investigated in [22]. Also, the authors stated basic properties of these polynomials.

**Application 2:** Assume that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set which has the generating function of the form (2.3) due to the couple of polynomials given below

$$[\gamma_d(t) , \sigma_2(t)] = \left[ - (1 - t)^2 \pi_{d-1}'(t) - (\alpha + 1) (1 - t) , - (1 - t)^2 \right].$$

Here $\pi_{d-1}(t) = \sum_{k=0}^{d-1} a_k t^k$ with $a_{d-1} \neq 0$. According to Theorem 2.2, necessary computations lead us to get $d$-orthogonal polynomial sets of the type

$$e^{\pi_{d-1}(t)} (1 - t)^{-\alpha - 1} \exp \left( \frac{-xt}{1 - t} \right) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.$$

$a_{d-1} \neq 0$ is enough to satisfy the conditions (2.4). These $d$-orthogonal polynomial sets found and studied in [14].

**Remark 2.4.** These two polynomial sets (2.6) and (2.7) are the generalizations of Laguerre polynomials to the $d$-orthogonal polynomials since we meet Laguerre polynomials for $d = 1$.

Now, we present a new $d$-orthogonal polynomial set for $d \geq 2$. It seems that this polynomial set is a Laguerre type $d$-orthogonal polynomial set but the difference is Laguerre polynomials are not generated hence $d \geq 2$.

**Application 3:** Suppose that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set possessing the generating function (2.3) relate to the couple of polynomials

$$[\gamma_d(t) , \sigma_3(t)] = \left[ - (1 - t)^3 \pi_{d-2}'(t) - (\alpha + 1) (1 - t)^2 , - (1 - t)^3 \right].$$

where $\pi_{d-2}(t) = \sum_{k=0}^{d-2} a_k t^k$ with $a_{d-2} \neq 0$. Then, taking Theorem 2.2 into account, we have a new $d$-orthogonal polynomial set generated by

$$e^{\pi_{d-2}(t)} (1 - t)^{-\alpha - 1} \exp \left( \frac{x t^2 - 2t}{2(1 - t)^2} \right) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.$$

for $d \geq 2$. Similarly, $a_{d-2} \neq 0$ guarantees the conditions (2.4). Now, we deal with the case $d = 2$ i.e.: a new 2-orthogonal polynomial set. Thus

$$(1 - t)^{-\alpha - 1} \exp \left( \frac{x t^2 - 2t}{2(1 - t)^2} \right) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

(2.8)
with conditions $\alpha + n + 1 \neq 0$, $n \geq 0$. Before finding the corresponding linear functionals $u_0$ and $u_1$ of this new 2-orthogonal polynomial set, we need to state following useful lemma.

**Lemma 2.5.** (30) Let $A(t)$ and $H(t)$ be two power series given as in (1.3) and

$$H^*(t) = \sum_{k=0}^{\infty} h_k^* t^{k+1}$$

is the compositional inverse of $H(t)$ such that

$$H(H^*(t)) = H^*(H(t)) = t$$ .

(i) The lowering operator $\sigma := \hat{T}_x$ of the polynomial set $\{P_n\}_{n \geq 0}$ generated by (1.3) is given with

$$\sigma = H^*(D) , \quad D = \frac{d}{dx} .$$

(ii) The lowering operator $\sigma := \hat{T}_x$ of the polynomial set $\{P_n\}_{n \geq 0}$ generated by

$$A(t) \left(1 + \omega H(t)\right) \frac{\omega}{x} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

is given by

$$\sigma = H^*(\Delta_\omega) , \quad \Delta_\omega [f(x)] = \frac{f(x+\omega) - f(x)}{\omega} .$$

**Theorem 2.6.** The polynomial set $\{P_n\}_{n \geq 0}$ generated by (2.8) are 2-orthogonal for $\alpha > -1$ with respect to the following linear functionals

$$\langle u_0, f \rangle = \int_{0}^{\infty} \Psi_{\alpha,f}(x) e^{-x} dx , \quad f \in \mathcal{P} ,$$

$$\langle u_1, f \rangle = \int_{0}^{\infty} [\Psi_{\alpha,f}(x) - \Psi_{\alpha+1,f}(x)] e^{-x} dx , \quad f \in \mathcal{P} ,$$

where

$$\Psi_{\alpha,f}(x) = \frac{x^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k! \left(\frac{\alpha+2}{2}\right)_k} \left(\frac{x^2}{2}\right)^k ,$$

$\Gamma$ is the widely known Gamma function and $(a)_n$ is the Pochhammer’s symbol defined by the rising factorial

$$\begin{align*}
(a)_n &= a(a+1)\ldots(a+n-1) , \quad n \geq 1 , \\
(a)_0 &= 1 .
\end{align*}$$

**Proof.** (2.2) yields that

$$\langle u_i, f \rangle = \frac{1}{i!} \left[ \frac{\sigma^i}{A(\sigma)} f(x) \right]_{x=0} , \quad i = 0, 1 , \quad f \in \mathcal{P} ,$$
where $\sigma$ is the lowering operator of 2-orthogonal polynomial set generated by $\sigma$. The lowering operator $\sigma$ of this polynomial set is

$$H(t) = \frac{t^2 - 2t}{2(1-t)^2} \Rightarrow \sigma = H^* (D) = 1 - (1 - 2Dx)^{-1/2}$$

where we use Lemma 2.5. Then, for $i = 0$ and $A(t) = (1-t)^{-\alpha-1}$, we obtain

$$\langle u_0, f \rangle = \left[ (1 - 2Dx)^{-(\frac{\alpha+1}{2})} f(x) \right]_{x=0}$$

$$= \sum_{k=0}^{\infty} \left( \frac{\alpha+1}{2} \right)_k \frac{2^k}{k!} f^{(k)}(0)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 2k + 1)}{\Gamma(\alpha + 1)} \frac{f^{(k)}(0)}{k!}$$

$$= \int_0^{\infty} \left[ x^\alpha \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \left( \frac{x^2}{2} \right)^k \right] e^{-x} dx .$$

Furthermore, we calculate in a similar manner for $i = 1$

$$\langle u_1, f \rangle = \left[ \sum_{r=0}^{1} \left( \frac{1}{r} \right) (-1)^r (1 - 2Dx)^{-(\frac{\alpha+r+1}{2})} f(x) \right]_{x=0}$$

$$= \sum_{r=0}^{1} \left( \frac{1}{r} \right) (-1)^r \sum_{k=0}^{\infty} \left( \frac{\alpha+r+1}{2} \right)_k \frac{2^k}{k!} f^{(k)}(0)$$

$$= \sum_{r=0}^{1} \left( \frac{1}{r} \right) (-1)^r \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + r + 2k + 1)}{\Gamma(\alpha + r + 1)} \frac{f^{(k)}(0)}{k!} \left( \frac{x^2}{2} \right)^k$$

$$= \int_0^{\infty} \left[ \sum_{r=0}^{1} \left( \frac{1}{r} \right) (-1)^r \frac{x^\alpha r}{\Gamma(\alpha + r + 1)} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \left( \frac{x^2}{2} \right)^k \right] e^{-x} dx .$$

This finishes the proof. \qed

(ii) Hermite type $d$-orthogonal polynomial sets

Hermite type $d$-orthogonal polynomials were the first example of $d$-orthogonal polynomials which obtained constructively by Douak [16]. He discovered these polynomials as solution of the problem: Find all $d$-orthogonal polynomial sets which are at the same time Appell polynomials.

Application 4: Let $\{ P_n \}_{n \geq 0}$ be a Sheffer polynomial set due to the generating function (2.3) and the following couple of polynomials

$$[\gamma_d(t), \sigma_0(t)] = \left[ \frac{\pi'_{d+1}}{2} (t), 1 \right]$$
where $\pi_{d+1} (t) = \sum_{k=0}^{d+1} a_k t^k$ with $a_{d+1} \neq 0$. Theorem 2.2 allows us to present a $d$-orthogonal set generated by

$$e^{\pi_{d+1} (t)} \exp (x t) = \sum_{n=0}^{\infty} P_n (x) \frac{t^n}{n!}$$

under the conditions $a_{d+1} \neq 0$ from (2.4). The only polynomials which are $d$-orthogonal and Appell polynomials at the same time are generated by (2.9). Also, these polynomial sets are the generalization of Gould-Hopper polynomials [31]. For $d = 1$, we meet again the Hermite polynomials. The properties of these polynomial sets were intensively studied by Douak in [16].

(iii) **Charlier type $d$-orthogonal polynomial sets**

Also, Charlier polynomials extended to the notion of $d$-orthogonality. Now, we revisit these polynomial sets owing to Theorem 2.2.

**Application 5:** Suppose that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set generated by (2.3) associated with the couple of polynomials

$$[\gamma_d (t), \sigma_1 (t)] = \left[ (1 + \omega t) \pi_d' (t), 1 + \omega t \right]$$

where $\pi_d (t) = \sum_{k=0}^{d} a_k t^k$ with $a_d \neq 0$. From Theorem 2.2, we have a $d$-orthogonal polynomial set of the form

$$e^{\pi_d (t)} (1 + \omega t) \pi_d (t) = \sum_{n=0}^{\infty} P_n (x) \frac{t^n}{n!}$$

with the conditions $a_d \neq 0$ from (2.4). The $d$-orthogonal polynomial set generated by (2.10) was found in [13]. A similar characterization problem stated in (ii) for discrete case was solved by the authors. It is obvious that the generating function relation (2.10) yields Charlier polynomials for $(d, \omega) = (1, 1)$.

(iii) **Meixner type $d$-orthogonal polynomial sets**

Another important member of discrete orthogonal polynomial sets called Meixner polynomials were also generalized in the context of $d$-orthogonality [14]. The authors discovered these polynomials by means of a form of generating function and they found the linear functions $u_0$ and $u_1$ for the case $d = 2$. Following application of Theorem 2.2 shows that these polynomial sets can be obtained by the special case of the couple of polynomials $[\gamma_d (t), \sigma_{d+1} (t)]$.

**Application 6:** Assume that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set having the generating function of the type (2.3) for the couple of polynomials

$$[\gamma_d (t), \sigma_2 (t)] = \left[ \frac{1}{c-1} (c-t) (1-t) \pi_{d-1}' (t) + \frac{\beta}{c-1} (c-t), \frac{1}{c-1} (c-t) (1-t) \right].$$

Here $\pi_{d-1} (t) = \sum_{k=0}^{d-1} a_k t^k$ with $a_{d-1} \neq 0$ and $c \neq \{0, 1\}$. Thanks to Theorem 2.2 and this couple of polynomials, (2.3) generates the $d$-orthogonal
polynomial sets given below

\[ e^\pi d^{-1} (t) (1 - t)^{-\beta} \left( 1 + \frac{c - 1}{c} \frac{t}{1 - t} \right)^x = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} . \quad (2.11) \]

\( a_{d-1} \neq 0 \) and \( c \neq \{0, 1\} \) are sufficient enough the conditions (2.4) hold true. One can find detailed information of these polynomial sets in [14]. It is easily seen that we face with the Meixner polynomial set by taking \( d = 1 \) in (2.11). Recently, a generalization of \( d \)-orthogonal Meixner polynomial sets via quantum calculus has been given in [32]. Next, we express a new Meixner type \( d \)-orthogonal polynomial set and we find its \( d \)-dimensional functional vector.

**Application 7:** Let \( \{P_n\}_{n \geq 0} \) be a Sheffer polynomial set generated by (2.3) according to the couple of polynomials

\[ [\gamma_d (t), \sigma_{d+1} (t)] = \left[ \frac{dc\beta}{c-1} \left( 1 - t \right)^d + \frac{c - 1}{dc} \left[ 1 - (1 - t)^d \right] \right], \]

\[ \frac{c}{c-1} (1 - t) \left( 1 - t \right)^d + \frac{c - 1}{dc} \left[ 1 - (1 - t)^d \right] \]

with the restrictions

\[ \left\{ \begin{array}{l}
    c \neq \{0, \frac{1}{d}, 1\} \\
    \beta \neq -\frac{n}{d}, \ n \geq 0.
\end{array} \right. \quad (2.12) \]

After some computations, Theorem 2.2 allows us to introduce the following new \( d \)-orthogonal polynomial set with

\[ (1 - t)^{-\beta d} \left( 1 + \frac{c - 1}{dc} \left[ \frac{1}{(1 - t)^d} - 1 \right] \right)^x = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} . \quad (2.13) \]

The conditions (2.4) are satisfied from restrictions (2.12). It seems that this Meixner type \( d \)-orthogonal polynomial set is the first explicit one among others in the literature.

**Theorem 2.7.** The \( d \)-dimensional functional vectors, which the \( d \)-orthogonality of the polynomial set generated by (2.13) holds, are

\[ \langle u_r, f \rangle = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{j=0}^{\infty} \frac{(\beta + \frac{i}{d})_j}{\left( 1 - \frac{dc}{c-1} \right)^{\beta + \frac{i}{d} + j}} \frac{1}{j!} f(j) \quad (2.14) \]

where \( r = 0, 1, ..., d - 1 \) and \( f \in \mathcal{P} \).

**Proof.** Lemma 2.5 helps us to find the lowering operators of the \( d \)-orthogonal polynomial set generated by (2.13) with

\[ H(t) = \frac{c - 1}{dc} \left[ \frac{1}{(1 - t)^d} - 1 \right] \Rightarrow \sigma = H^*(\Delta) = 1 - \left( 1 - \frac{dc\Delta}{1 - c} \right)^{-\frac{1}{d}} \]
where $\Delta f(x) = f(x+1) - f(x)$. Thus, by using Lemma 2.1 and (2.2), we conclude that for $r = 0, 1, ..., d - 1$ and $f \in \mathcal{P}$

$$
\langle u_r, f \rangle = \frac{1}{r!} \left[ \frac{\sigma^r}{A(\sigma)} f(x) \right]_{x=0} = \frac{1}{r!} \left[ \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left(1 - \frac{dc\Delta}{1 - c}\right)^{-\left(\beta + \frac{i}{d}\right)} f(x) \right]_{x=0} = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{k=0}^{\infty} \frac{(\beta + \frac{i}{d})_k}{k!} \left(\frac{dc}{1 - c}\right)^k \Delta^k f(0) \quad (2.15)
$$

Substituting the fact

$$
\Delta^k f(0) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(j)
$$

into (2.15) and after shifting indices, we obtain

$$
\langle u_r, f \rangle = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{j=0}^{\infty} \binom{\beta + \frac{i}{d}}{k+j} \left(\frac{dc}{1 - c}\right)^k \Delta^k f(0)
$$

(2.16)

The equality (2.16) leads us to get the desired result by applying the following property of the Pochhammer’s symbol

$$
\left(\beta + \frac{i}{d}\right)_{k+j} = \left(\beta + \frac{i}{d}\right)_j \left(\beta + \frac{i}{d} + j\right)_k
$$

□

**Remark 2.8.** For $d = 1$, (2.13) reduces to the well known generating function of Meixner polynomial set and (2.14) becomes the following linear functional for Meixner polynomials

$$
\langle u_0, f \rangle = (1 - c)^\beta \sum_{j=0}^{\infty} \frac{(\beta)_j c^j}{j!} f(j) \quad (2.17)
$$

with $0 < c < 1$ and $\beta > 0$. Meixner polynomial set is orthogonal with respect to the linear functional given by (2.17).

**Application 8:** Suppose that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set represented by (2.23) associated to the couple of polynomials

$$
[\gamma_d(t), \sigma_3(t)] = \left[ -\frac{c}{c-1} (1-t) \left( (1-t)^2 + \frac{c-1}{2c} \left[ (1-t)^2 - 1 \right] \right) \right] \pi'_{d-2}(t) - \frac{\beta c}{c-1} \left[ (1-t)^2 + \frac{c-1}{2c} \left[ (1-t)^2 - 1 \right] \right],
$$

$$
- \frac{c}{c-1} (1-t) \left[ (1-t)^2 + \frac{c-1}{2c} \left[ (1-t)^2 - 1 \right] \right]
$$
where $\pi_{d-2}(t) = \sum_{k=0}^{d-2} a_k t^k$ with $a_{d-2} \neq 0$ and $c \neq \{0, \frac{1}{3}, 1\}$. Taking Theorem 2.2 into account, we derive a new $d$-orthogonal polynomial set for $d \geq 2$ with

$$e^{\pi_{d-2}(t)} (1 - t) - \beta \left(1 + \frac{c - 1}{2c} \frac{t^2 - 2t}{(1 - t)^2}\right)^x = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$  \hspace{1cm} (2.18)

The conditions (2.4) are satisfied since $a_{d-2} \neq 0$ and $c \neq \{0, \frac{1}{3}, 1\}$. These $d$-orthogonal polynomial sets (2.18) can not generate an orthogonal polynomial set since $d \geq 2$. But for $d = 2$, one can study the properties of this Meixner type 2-orthogonal polynomial set.

### 2.1. Concluding Remarks

Theorem 2.2 is the generalization of the characterization problem related to the orthogonality of Sheffer polynomial set. The version of this problem corresponding to $d = 1$ and $d = 2$ already exist in the literature. Then, it is expected to find similar results for $d$-orthogonality. Although the results obtained in Theorem 2.2 are expected and natural, this theorem motivates us to derive new $d$-orthogonal polynomial sets as mentioned in this paper. One can generate more $d$-orthogonal polynomial sets which are Sheffer polynomial set at the same time with the help of Theorem 2.2.

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