Universality of semisuper-Efimov effect

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We study the semisuper-Efimov effect, which is found for four identical bosons with a resonant three-body interaction in 2D, in various systems. Based on solutions of bound-state and renormalization-group equations, we first demonstrate an emergence of the semisuper-Efimov effect in mass-imbalanced bosons in 2D. Compared with the Efimov and the super-Efimov effects, the mass ratio-dependent scaling parameter is unexpectedly found to take on a finite value even for extremely mass-imbalanced situations, where the mass ratio is 0 or ∞. By a renormalization-group analysis, we also show that a weak two-body interaction sustains the semisuper-Efimov effect. Finally, we liberate the universality of the semisuper-Efimov effect from 2D by showing that bosons with linear-dispersion relation support the semisuper-Efimov effect in 1D.

I. INTRODUCTION

Efimov effect is one of prominent examples of universal phenomena that appear in resonantly interacting few-body systems. In the seminal paper by Efimov [1], an emergence of an infinite series of self-similar trimer states is predicted for three-body systems with resonant s-wave interaction, irrespective of the specific form of the twobody interaction. Reflecting its self-similar binding energies \( E_{n+1}/E_n \approx (22.7)^{-2} \), the universality of the Efimov effect is represented by the renormalization-group limit cycle [2–5], which refers to a periodic renormalization-group flow [6, 7]. Due to its universality and self similarity, the Efimov effect is extensively studied in variety of physical systems including nucleons [8], mass-imbalanced fermions [9, 10], particles in mixed dimensions [11], magnons [12], and macromolecules [13]. In particular, experimental observations of the Efimov effect in ultracold atoms [14–24] provide a renewed interest to this subject 40 years after the Efimov’s prediction. Among the theoretical studies in ultracold atoms is a prediction of super-Efimov effect [25] in a two-dimensional system of three-spinless fermions with resonant p-wave interaction, where the three-body binding energy \( E_n \) exhibits a double exponential growth \( E_n \propto \exp(\pi n) \). The super-Efimov effect is shown to emerge also for mass-imbalanced systems in 2D provided that the two-body interaction is dominated by the p-wave contribution [26].

In this paper, we focus on yet another few-body clusters proposed by Nishida [27]. Motivated by theoretical proposals [28–31] for realizing a three-body interaction in absence of a two-body interaction, Nishida investigates four identical bosons with a resonant three-body interaction in 2D. He finds that the system supports the semisuper-Efimov tetrarers where the four-body binding energy \( E_n \) grows as \( E_n \propto e^{(\pi n)^2} \). Despite its qualitatively distinct feature from the Efimov and the super-Efimov effects, the semisuper-Efimov effect in other systems than identical bosons in 2D is yet to be explored. Therefore, we here extend the universality to mass-imbalanced systems and, in particular, demonstrate a qualitative difference of the mass-ratio dependent scaling parameter from those of the Efimov and the super-Efimov effects: The scaling parameter of the semisuper-Efimov effect is unexpectedly found to be stable against a variation of the mass ratio and takes on a finite value even in extremely mass-imbalanced situations where the mass-ratio takes on 0 or ∞. To further demonstrate universality of the semisuper-Efimov effect, we also liberate the semisuper-Efimov effect from two-dimensional systems by showing an emergence of the semisuper-Efimov effect for bosonic particles with a linear-dispersion relation in 1D.

We organize the paper in the following manner: In Sec. II, we study mass-imbalanced bosons in 2D. Firstly, we investigate two-component mass-imbalanced bosons in 2D, as the simplest extension. Analytical solutions of bound-state and renormalization-group equations show an emergence of the semisuper-Efimov effect with a mass-ratio dependent scaling parameter. Effects of two-body interactions are then evaluated quantitatively by a renormalization-group analysis. As another extension, we also investigate three-component mass-imbalanced bosons. In Sec. III, we further extend the universality to 1D by investigating identical bosons with a linear-dispersion relation. Similarly to Sec. II, we first show an emergence of the semisuper-Efimov effect in 1D by solving a bound-state equation and then discuss its stability against a weak two-body interaction by a renormalization-group analysis. Sec. IV is devoted to the summary and discussion of this paper.

II. MASS-IMBALANCED BOSONS IN 2D

A. Two-component bosons

We first consider a system of two-component bosons in 2D with a resonant three-body interaction. With a model Hamiltonian, we first solve a four-body bound-state equation analytically and show that the system supports the semisuper-Efimov effect with a mass ratio-dependent scaling parameter. We then support the analytical result and discuss its stability against two-body interactions by a renormalization-group analysis.
where $\psi_A^\dagger$ and $\psi_B^\dagger$ ($\psi_A$ and $\psi_B$) represent the creation (annihilation) operators of the two species $A$ and $B$ of bosons, respectively. The parameter $\alpha = \frac{m_A}{m_B} \in (0, \infty)$ is the mass-ratio between the two species of bosons. Here we employ the units $\hbar = m_B = 1$. We note that the Jacobi coordinate is employed in the three-body-interaction term since it facilitates loop-momentum integrals that appear in calculating $T$ matrices. The function $\chi$ represents a separable potential which is tractable due to its separability, and we here choose the following Lorentzian function:

$$\chi(q_x, q_y) = \frac{\Lambda^2}{\alpha^2 q_x^2 + \frac{2\alpha + 1}{\alpha(\alpha + 1)} q_y^2 + \Lambda^2}. \tag{2}$$

To ensure the system to be Galilean invariant, we choose $\chi$ depending only on the relative momenta. In the limit of taking $\Lambda \rightarrow \infty$ ($\chi \rightarrow 1$), the three-body interaction in Eq. (1) reduces to the contact interaction. To put it another way, the separable potential $\chi$ plays the role of the short-range regulator of the system with the ultraviolet cutoff $\Lambda$.

Based on the model, we calculate the three- and the four-body $T$ matrices analytically by summing up the ladder-type Feynman diagrams which are the only non-vanishing diagrams in the particle vacuum. Concerning the three-body sector, an exact $T$-matrix $T^{(3)}$ is given by

$$T^{(3)}(p^0, p, q_x, q_y, q_z, q_y') = \frac{\chi(q_x, q_y') \chi(q_z, q_y)}{\frac{1}{g} - \frac{1}{32\pi^2} \alpha^2 q^2 \frac{\alpha^2}{2\alpha + 1} f \left( \frac{\epsilon}{\Lambda^2} \right)}, \tag{3}$$

$$f(x) = \frac{1}{1 - x} \left( \frac{1}{1 - x} \ln x + 1 \right) \sim \ln x, \tag{4}$$

where $\epsilon = ip^0 + \frac{p^2}{2\alpha + 1}$ is the total energy of the three particles, and $g$ is the renormalized three-body coupling constant which is related to the bare coupling $\lambda$ via

$$\frac{1}{\lambda} = \frac{1}{32\pi^2} \frac{\alpha^2}{2\alpha + 1} \Lambda^2 = \frac{1}{g}. \tag{5}$$

A three-body bound state can be obtained as a pole of $T^{(3)}$ in Eq. (3), and for a large-positive $g$, we find a shallow bound state whose energy $\epsilon$ satisfies $\epsilon \ln \frac{\epsilon}{\Lambda^2} = \frac{1}{g} - \frac{1}{32\pi^2} \frac{\alpha^2}{2\alpha + 1} \Lambda^2 = -\frac{1}{g}$. The resonance condition is, therefore, achieved by tuning $1/g = 0$ so that the three-body binding energy $\epsilon$ vanishes. As a complement, we calculate the $T$ matrix $T^{(3)}$ with different separable potentials $\chi$ from Eq. (2) and check that the asymptotic functional form of $f \left( \frac{\epsilon}{\Lambda^2} \right)$ does not depend on the specific choice of $\chi$ at sufficiently low-energy $\epsilon/\Lambda^2 \ll 1$.

We then turn to the four-body sector where we consider the scattering of three identical bosons $\psi_A$ with a

\[ H = \int \frac{d^2 q}{(2\pi)^2} q^2 \psi_A^\dagger q \psi_A q + \int \frac{d^2 q}{(2\pi)^2} q^2 \psi_B^\dagger q \psi_B q - \frac{\lambda}{4} \int \frac{d^2 Q d^2 q_x d^2 q_y d^2 q_z d^2 q_y'}{(2\pi)^4} \chi(q_x, q_y') \chi(q_z, q_y) \]
In the integrand is replaced by
\[ Z \left( \frac{q}{Z} \right) = \frac{2(2\alpha + 1)^2}{\alpha + 1} \frac{\chi \left( l, \frac{\alpha}{2\alpha + 1} q' + l' \right) \chi \left( l, \frac{\alpha}{2\alpha + 1} q' + l' + q \right)}{\alpha + 1} \left[ E + \frac{1}{\alpha} (q'^2 + l'^2) + \frac{1}{\alpha + 1} (q'^2 + l'^2) \right] \]
\[ \times \left[ -\frac{1}{32\pi^2} \frac{\alpha^2}{2\alpha + 1} \left( E + \frac{3\alpha + 1}{\alpha(2\alpha + 1)} l'^2 \right) f \left( E + \frac{3\alpha + 1}{\alpha(2\alpha + 1)} l'^2 \right) \right]^{-1} Z(q), \tag{6} \]
which is solved as
\[ Z_0(\xi) = s \left[ A \cdot J_1(s) + B \cdot N_1(s) \right], \tag{10} \]
\[ s := \sqrt{\frac{8(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} \xi, \tag{11} \]
where \( J_1 \) is the Bessel function of the first kind, \( N_1 \) is the Neumann function and \( A \) and \( B \) are the constants of integration. The solution Eq. (10) is further restricted by the following two boundary conditions obtained by substituting \( \xi' = \delta \) or \( \xi' = \ln \frac{A}{E} \) into Eq. (7):
\[ Z_0(\delta) = \frac{2(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)} \int_{\delta}^{\ln \frac{A}{E}} d\xi Z_0(\xi), \tag{12} \]
\[ \ln \frac{A^2}{E} = \frac{2(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)} \int_{\delta}^{\ln \frac{A}{E}} d\xi Z_0(\xi). \tag{13} \]
In the low-energy limit \( \frac{E}{A^2} \to 0 \), the first boundary condition Eq. (12) relates \( A/B \) to the short-range non-universal parameter \( \delta \) via
\[ A/B = -\frac{\sigma \cdot N_1(\sigma) - \frac{\sigma}{\alpha} \int_{\sigma}^{\infty} d\sigma N_1(s)}{\sigma \cdot J_1(\sigma) - \frac{\sigma}{\alpha} \int_{\sigma}^{\infty} d\sigma J_1(s)}, \tag{14} \]
\[ \sigma := \sqrt{\frac{8(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} \delta. \tag{15} \]
The other boundary condition Eq. (13) determines the values of the binding energy \( E \) via
\[ A/B = -\frac{\theta \cdot N_1(\theta) - \frac{\theta}{\alpha} \int_{\theta}^{\infty} d\theta N_2(\theta)}{\theta \cdot J_1(\theta) - \frac{\theta}{\alpha} \int_{\theta}^{\infty} d\theta J_2(\theta)}, \tag{16} \]
\[ \theta := \sqrt{\frac{8(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} \ln \frac{A^2}{E}. \tag{17} \]
It is straightforward to see that Eq. (16) in the low-energy limit \( \theta \to \infty \) is satisfied only for \( \theta = \theta^* + n\pi \) \((n \in \mathbb{Z})\) since the left-hand side is a \( \theta \)-independent constant. We thus obtain the quantized binding energy \( E_n \) of the tetramers as

\[
E_n = \Lambda^2 \exp \left[ -\frac{(\alpha + 1)(3\alpha + 1)}{8(2\alpha + 1)^2} (n\pi + \theta^*)^2 \right],
\]

or equivalently,

\[
\sqrt{\ln \frac{\Lambda^2}{E_{n+1}}} - \sqrt{\ln \frac{\Lambda^2}{E_n}} = \pi \sqrt{\frac{(\alpha + 1)(3\alpha + 1)}{8(2\alpha + 1)^2}},
\]

which is nothing but the energy spectrum of the semisuper-Efimov effect.

In Eq. (18), the scaling parameter \( (\alpha + 1)(3\alpha + 1)/8(2\alpha + 1)^2 \) is unexpectedly found to be stable under the variation of the mass-ratio \( \alpha \). In particular, major qualitative differences of the present result from Efimov and super-Efimov effects can be found in the extremely mass-imbalanced regimes \( \alpha \to 0 \) and \( \alpha \to \infty \): Compared to the Efimov and the super-Efimov effects, we observe the nonvanishing scaling parameter in the limit of \( \alpha \to 0 \) and the non-diverging scaling parameter in the limit of \( \alpha \to \infty \).

2. Universality

To discuss the universality of the analytically calculated binding energy Eq. (18) in mass-imbalanced bosons, here we perform a renormalization-group analysis. We first deal with the situation with vanishing two-body interactions to reproduce Eq. (18), and then we discuss the stability of the solution in the presence of the two-body interactions. For this purpose, we consider the following effective-field theory by introducing an auxiliary field of a composite of three bosons:

\[
L = \psi_A^\dagger \left( i\partial_t + \frac{\nabla^2}{\alpha} \right) \psi_A + \psi_B^\dagger \left( i\partial_t + \frac{\nabla^2}{\alpha} \right) \psi_B
\]

\[+ \phi^\dagger \left( i\partial_t + \frac{\nabla^2}{2\alpha + 1} - \epsilon_0 \right) \phi + \frac{h}{2} (\phi^\dagger \psi_A \psi_B + \text{h.c.})
\]

\[+ v_A \phi^\dagger \psi_A^\dagger \psi_A \phi + v_B \phi^\dagger \psi_B^\dagger \psi_B \phi + g_A \phi^\dagger \psi_A^\dagger \psi_A \psi_A
\]

\[+ \frac{g_B}{4} \psi_B^\dagger \psi_B \psi_B + g_{AB} \psi_A^\dagger \psi_B^\dagger \psi_B \psi_A + \frac{g_\phi}{4} \phi^\dagger \phi \phi,
\]

(20)

where we introduce all the symmetry-preserving relevant couplings consisting of the two species \( \psi_A, \psi_B \) of bosons and the trimer \( \phi \). For the four-body problem of three \( A \) bosons and a \( B \) boson, the couplings \( v_A, g_A \) are decoupled and do not enter the renormalization-group equations. To ensure that the system is at the three-body resonance, we tune \( \epsilon_0 \) at each renormalization-group energy scale of \( \mu \) as

\[
\epsilon_0 = -\frac{\hbar^2 \ln 4}{32\pi^2} \frac{\alpha^2}{2\alpha + 1} (\Lambda^2 - \mu^2),
\]

(21)

where \( \Lambda \) is an intrinsic ultraviolet cutoff. Beta functions of renormalization-group equations are then given by coefficients of logarithmically divergent Feynman diagrams [25, 27] which are collected in Fig. 3. We thus obtain the following renormalization-group equations:

\[
\frac{dg_A}{ds} = -\frac{\alpha g_A^2}{8\pi}, \quad \frac{dg_B}{ds} = -\frac{g_B^2}{8\pi}, \quad \frac{dg_{AB}}{ds} = -\frac{g_{AB}^2}{2\pi} \frac{\alpha}{\alpha + 1}.
\]

(22)

\[
\frac{dh}{ds} = \frac{\hbar^4}{32\pi^2} \frac{h^2}{2\alpha + 1} - \frac{g_A h}{2\pi} - \frac{g_B h}{2\pi} \frac{\alpha}{\alpha + 1},
\]

(23)

\[
\frac{dv_A}{ds} = -\frac{v_A h}{16\pi^2} \frac{\alpha^2}{2\alpha + 1} - \frac{v_A^2}{2\pi} \frac{\alpha}{3\alpha + 1} - \frac{h^2}{2\pi} \frac{\alpha}{\alpha + 1},
\]

(24)

where \( s := \ln \Lambda/\mu \) is the renormalization time.

To verify the result in Eq. (18), we first consider the situation where two-body interactions are absent: \( g_A = g_B = g_{AB} = 0 \). In the situation, \( h \) becomes

\[
h(s)^2 = \left( \frac{1}{h(0)^2} + \frac{s}{16\pi^2} \frac{\alpha^2}{2\alpha + 1} \right)^{-1},
\]

(25)

which leads to \( h(s)^2 \to \left( \frac{s}{16\pi^2} \frac{\alpha^2}{2\alpha + 1} \right)^{-1} \) in the low-energy limit \( s \to \infty \). By substituting this expression into Eq. (24), we obtain

\[
v_A = \frac{4\pi}{\alpha} \sqrt{\frac{3\alpha + 1}{\alpha + 1}} \frac{1}{\sqrt{s}} \tan \left( -\sqrt{\frac{8(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} \sqrt{2s + C} \right),
\]
where $C$ is a non-universal constant that depends on the initial value of $v_4(0)$ (see Appendix A for a detailed calculation).

Since a divergence of a coupling constant is a fingerprint of an emergence of a bound state, we assign $\mu = \mu_n (n \in \mathbb{Z})$ at which $v_4 \left( \ln \frac{\Lambda}{\mu} \right)$ diverges. Due to the periodic nature of $\sqrt{s} v_4(s)$ with respect to $\sqrt{2s} = \sqrt{\ln \frac{\Lambda^2}{\mu^2}}$, we obtain

$$\sqrt{\ln \frac{\Lambda^2}{\mu_{n+1}}} - \sqrt{\ln \frac{\Lambda^2}{\mu_n}} = \pi \sqrt{\frac{(\alpha + 1)(3\alpha + 1)}{8(2\alpha + 1)^2}},$$

(27)

which is in perfect agreement with the period of the obtained energy spectrum Eq. (19) of the semisuper-Efimov effect.

To discuss the stability of the semisuper-Efimov states against the two-body interaction, we then consider the situation where the two-body couplings $g_A, g_B$ and $g_{AB}$ take on nonzero values. Although Nishida discuss in Ref. [27] that the semisuper-Efimov effect vanishes if we introduce a two-body interaction, we find it is not the case. Even in presence of two-body interactions, we obtain the same solution as Eq. (26) under the following conditions:

$$\left| \frac{1}{g_A(0)} \right| \gg \frac{\alpha s}{8\pi}, \left| \frac{1}{g_{AB}(0)} \right| \gg \frac{s}{2\pi \alpha + 1},$$

(28)

$$h(0)^2 \gg 32\pi^2 \frac{2\alpha + 1}{\alpha^2} \left( \frac{g_A(0)}{\alpha + 1} + \frac{\alpha g_A(0)}{8\pi} \right),$$

(29)

$$s \gg \frac{16\pi^2 (2\alpha + 1)}{\alpha^2 h(0)^2}.$$

(30)

In Appendix B, we explicitly derive Eq. (26) under the conditions Eqs. (28), (29) and (30); here we discuss physical significance of these conditions. Firstly, in Eq. (28), the inverse two-body coupling constants $1/|g_A(0)|$ and $1/|g_{AB}(0)|$ determine the upper bound of $s = \ln \Lambda/\mu$ (the lower bound of $\mu$) below which the semisuper-Efimov states are present, i.e., $1/|g_A(0)|$ and $1/|g_{AB}(0)|$ serve as infrared cutoffs of the spectrum of the semisuper-Efimov tetramers. Physically, the result suggests that an arbitrarily large quantum halo is prohibited by the two-body interactions and the possible size of the largest semisuper-Efimov tetramer is given by $\min\{\frac{1}{\alpha} e^{-\frac{\Lambda^2}{2\pi\alpha \mu(0)}}, \frac{1}{\alpha} e^{-\frac{\Lambda^2}{\pi\alpha \mu(0)}}\}$. In other words, the minimum value of the binding energy of the semisuper-Efimov tetramer is given by $\max\{\Lambda^2 e^{-\frac{\Lambda^2}{2\pi\alpha \mu(0)}}, \Lambda^2 e^{-\frac{\Lambda^2}{\pi\alpha \mu(0)}}\}$. It is reasonable to consider that semisuper-Efimov tetramers are absent below the energy scales $\Lambda^2 e^{-\frac{\Lambda^2}{2\pi\alpha \mu(0)}}$ and $\Lambda^2 e^{-\frac{\Lambda^2}{\pi\alpha \mu(0)}}$, which are the binding energies of two-body states. The second condition Eq. (29) means that the three-body coupling constant $h(0)$ must be sufficiently larger than the two-body couplings $g_A(0)$ and $g_{AB}(0)$, so that the resonant three-body interaction overwhelms the weak two-body interactions (quantum fluctuation is dominated by the loop corrections originating in the three-body interaction). The final condition Eq. (30) ensures that the semisuper-Efimov effect occur at a sufficiently low-energy regime where the only non-vanishing energy scale is the intrinsic ultraviolet cutoff $\Lambda$. We note that the three conditions Eqs. (28), (29) and (30) are compatible with each other.

In conclusion, we verify the analytically obtained energy spectrum Eq. (18) by comparing the spectrum with a solution Eq. (26) of a renormalization-group equation. Moreover, by clarifying the conditions under which the solution Eq. (26) is stable, we show the stability of the semisuper-Efimov effect in presence of weak two-body interactions.

### B. Three-component bosons

As another system to investigate, we here consider a three-component bosons in which two bosons of the three species have an identical mass. Since the calculation procedure is almost same as that performed for two-component bosons, we here present the results. The Hamiltonian we consider is the following:

$$H = \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{\alpha} \psi^\dagger_{\uparrow, q} \psi_{\uparrow, q} + \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{\alpha} \psi^\dagger_{\downarrow, q} \psi_{\downarrow, q} + \int \frac{d^2q}{(2\pi)^2} q^2 \psi^\dagger_{B, q} \psi_{B, q} - \lambda \int \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \psi^\dagger_{B, q} \psi_{B, q} - \lambda \int \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \psi^\dagger_{B, q} \psi_{B, q}.$$

(31)

Here we employ the units $\hbar = 2m = 1$, where $m$ is the mass of a boson $\psi_\mu$. The annihilation (creation) operators $\psi_\uparrow$ and $\psi_\downarrow$ (\psi_{\uparrow}^\dagger$ and $\psi_{\downarrow}^\dagger$) represent two species of bosons who have an identical mass of $\alpha/2$. To see the effects of the quantum statistics of the particles, here we assume that the three-body interaction among the three species of bosons is tuned to its resonance, and the other three-body interactions are not. In the three-body inter-
action, we employ the separable interaction potential $\chi$ in Eq. (2).

For the system, we perform the same analysis as two-component mass-imbalanced bosons, i.e. we solve the Bethe-Salpeter equation analytically. Since the procedure employed here is same as that of Sec. II A 1, we here just list up the results. Firstly, the three-body $T$-matrix $T^{(3)}$ of the three-distinguishable bosons is obtained by summing up the ladder-type Feynman diagrams depicted in Fig. 4. Consequently, we obtain

$$T^{(3)}(p^0, p; q_x', q_y', q_x, q_y) = \frac{\lambda(q_x', q_y')\chi(q_x, q_y)}{-\frac{1}{2} - \frac{1}{16\pi^2}\frac{\alpha^2x}{2\alpha + 1}f\left(\frac{\epsilon}{\Lambda}\right)},$$

where the function $f$ is given in Eq. (4). So that the system is on its three-body resonance, we renormalize the three-body coupling constant $\lambda$ as $\frac{1}{2} - \frac{1}{16\pi^2}\frac{\alpha^2x}{2\alpha + 1}\Lambda^2 = 0$.

Using the three-body $T$-matrix, we then turn to the four-body sector where we consider the scattering of the three-distinguishable bosons and an additional boson of $\psi_T$. The four-body Bethe-Salpeter equation is then obtained by seeing a pole structure of the Skornyakov-Ter-Martirosian-type integral equation depicted in Fig. 5. Corresponding to Eq. (7), we finally obtain the nonvanishing Bethe-Salpeter equation for the $s$-wave sector $Z_0$ as

$$Z_0(\xi') = \frac{(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)} \times \left[ \int_{\xi}^{\xi'} d\xi Z_0(\xi) + \int_{\xi'}^{\xi} d\xi \frac{2}{\xi}Z_0(\xi) \right].$$

By following the same procedure as Sec. II A 1, we finally arrive at the following energy spectrum of tetramers:

$$E_n = \Lambda^2 \exp \left[ - \frac{(\alpha + 1)(3\alpha + 1)}{4(2\alpha + 1)^2}(n\pi + \theta^*)^2 \right],$$

$$\sqrt{\ln \frac{\Lambda^2}{E_{n+1}}} - \sqrt{\ln \frac{\Lambda^2}{E_n}} = \pi \sqrt{\frac{(\alpha + 1)(3\alpha + 1)}{4(2\alpha + 1)^2}},$$

where $\theta^*$ is a nonuniversal constant.

Again we find an emergence of the semisuper-Efimov effect with a mass-ratio-dependent scaling parameter. In particular, presence of the tetramer states in extremely mass-imbalanced situations ($\alpha \gg 1$ and $\alpha \ll 1$) is observed.

III. PARTICLES WITH LINEAR-DISPERSION RELATION IN 1D

To further extend the universality of the semisuper-Efimov effect, we here consider a one-dimensional system of identical bosons that have a linear-dispersion relation.

A. model analysis

We consider the following model Hamiltonian:

$$H = \int \frac{dq}{2\pi} \left[ q\psi_\uparrow^\dagger \psi_\uparrow - \frac{\lambda}{(3!)^2} \right] \frac{dQdq'dq''dq'dq''dq'dq''}{(2\pi)^5} \chi(q_x', q_y')\chi(q_x, q_y) \times \psi_\uparrow^\dagger \psi_\uparrow \psi_\downarrow^\dagger \psi_\downarrow + q_x \psi_\uparrow^\dagger \psi_\uparrow q_x' - q_x' \psi_\downarrow^\dagger \psi_\downarrow q_x + q_x',$$

(36)
where $\psi^\dagger (\psi)$ is the creation (annihilation) operator of a boson. Here we employ the units $\hbar = v = 1$, where $v$ is the velocity of the linearly dispersing boson. As a separable potential $\chi$, we choose the following sharp-cutoff function:

$$\chi(q_x, q_y) = \Theta(\Lambda - q_x) \Theta(2\Lambda - q_y),$$

where $\Theta$ is the Heaviside unit-step function.

In the system, we first consider the three-body sector in which the three-body $T$-matrix $T^{(3)}$ is the summation of the ladder-type Feynman diagrams depicted in Fig. 6. Consequently, we obtain

$$T^{(3)}(p^0, p; q_x^0, q_y^0, q_z^0) = -\chi(q_x^0, q_y^0)\Gamma(p^0, p)\chi(q_x, q_y),$$

$$-\Gamma(p^0, p) = \frac{6}{\lambda} + \frac{4 \ln 2 + 1}{2\pi^2} \Lambda + \frac{3ip\nu}{2\pi^2} \ln \frac{p^0 + |p|}{\Lambda} + \frac{2p\nu}{\pi^2} \ln \left(\frac{4}{\pi} |p| + \frac{3}{2} + \frac{1}{4p^0 + |p|}\right),$$

and $\Lambda = 0$. We note that the dominant term in Eq. (39) is the logarithmic function $\frac{3ip\nu}{2\pi^2} \ln \frac{p^0 + |p|}{\Lambda}$ in a sufficiently low-energy regime $p^0, |p| \ll \Lambda$.

As discussed in Sec. II A 1, the bound-state equation of the four-body sector is obtained by comparing the residue of the Skornyakov-Ter-Martirosian-type integral equation depicted in Fig. 7. We note that there are two partial wave sectors in 1D labeled by the the parity quantum number. The bound-state equation with the binding energy $E$ can be obtained as

$$Z(q) = \int_0^\infty dl \ln \left[ \frac{2\Lambda}{E + 2q + 2l} \frac{2\Lambda}{E + q + l + |q - l|} \right] \frac{2Z(l)}{(E + l) \ln \frac{E + 2l}{\Lambda}},$$

for the even-parity ($Z(-q) = Z(q)$) sector and

$$Z(q) = \int_0^\infty dl \ln \left[ \frac{E + q + l + |q - l|}{E + q + l + |q + l|} \right] \frac{-2Z(l)}{(E + l) \ln \frac{E + 2l}{\Lambda}},$$

for the odd-parity ($Z(-q) = -Z(q)$) sector. For the odd parity sector, we find no bound state. Concerning the even-parity sector, we employ the leading-logarithmic approximation together with the change of variables $\xi := \ln \frac{\Lambda}{E + 2l}$ and $\xi' := \ln \frac{\Lambda}{E + 2q}$. Then we have

$$\psi(\xi') = 4 \int_\delta^{\xi'} d\xi Z(\xi) + 4 \int_\xi^{\xi'} \frac{d\xi}{\xi} \frac{Z(\xi)}{\xi},$$

Similarly to the solution of Eq. (7), we obtain the following energy spectrum of tetramers:

$$E_n = \Lambda \exp \left( -\frac{1}{16} (n\pi + \theta^*)^2 \right),$$

$$\sqrt{\ln \frac{\Lambda}{E_{n+1}}} - \sqrt{\ln \frac{\Lambda}{E_n}} = \frac{\pi}{4},$$

where $\theta^*$ is a nonuniversal constant determined by short-range details of a three-body interaction. We note that in Eqs. (43) and (44), a momenta and an energy have the same dimension due to the linearity of the dispersion relation. We thus show that the semisuper-Efimov effect occurs even in one dimension if a particle exhibits the linear dispersion relation.
B. universality

We here discuss the universality of the results by following the same procedure as Sec. II A 2. To perform a renormalization-group analysis, we consider an effective field theory with the following Action:

\[
S = \int \frac{d^2q}{(2\pi)^2} \bar{\psi}(q)(i\gamma^0 + |q|)\psi(q) + \int \frac{d^2q}{(2\pi)^2} \phi^\dagger(q)(i\gamma^0 - \epsilon)\phi(q) + \int d^2x \frac{g_2}{4} \bar{\psi}(x)\psi^\dagger(x)\phi(x)\psi(x) + \frac{h}{6} [\phi^\dagger(x)\psi(x)\psi(x)\psi(x) + \text{h.c.}] + g_4\phi^\dagger(x)\psi^\dagger(x)\psi(x)\phi(x),
\]

which is in perfect agreement with the energy spectrum Eq. (44).

Similarly to Sec. II A 2 (see also Appendix B), the solution Eq. (50) is stable even if we introduce a finite two-body interaction $g_2(0)$. Specifically, the same solution Eq. (50) for the four-body coupling constant $g_4$ under the following conditions:

\[
\begin{align*}
\left| \frac{1}{g_2(0)} \right| & \gg \frac{s}{4\pi}, \\
h(0)^2 & \gg 24\pi g_2(0), \\
s & \gg \frac{32\pi^2}{h(0)^2}.
\end{align*}
\]

In conclusion, we obtain a consistent result Eq. (51) with the energy spectrum Eq. (44) of the semisuper-Efimov effect. Furthermore, we argue that a weak two-body interaction sustains the semisuper-Efimov tetraters in the parameter region given by Eqs. (52), (53) and (54).

IV. SUMMARY AND DISCUSSION

We here summarize the discussion in the main text. Firstly, for mass-imbalanced bosons in 2D, we demonstrate an emergence of the semisuper-Efimov effect and derive mass-ratio dependent scaling factor. The scaling factor is unexpectedly found to be stable under the variation of the mass-ratio and, in particular, we find that the scaling factor remains finite in extremely mass-imbalanced situation where the mass ratio takes 0 or \(\infty\). This is in clear contrast with the Efimov and the super-Efimov effects where the scaling factors vanish or diverge for an extreme-mass imbalance. A renormalization-group analysis is then performed and a consistent renormalization-group flow with the semisuper-Efimov effect is obtained. Furthermore, by introducing a finite two-body interaction, we show that the semisuper-Efimov effect is stable even in the presence of a weak two-body interaction. The semisuper-Efimov effect is then shown to emerge also in the system of three-component bosons. Finally, we liberate the semisuper-
Efimov effect from 2D, by considering bosonic particles with linear-dispersion relation in 1D.

Because of the presence of the semisuper-Efimov effect in an extremely mass-imbalanced situation, the effect might be of relevance in impurity problems where an impurity has a large inertial mass. For example, consider a system of identical bosons with a spatially localized external potential such as a narrow square-well potential. If an interaction between two identical bosons is tuned to be on its resonance only inside the potential well, the two-body interaction between identical bosons can effectively be regarded as a three-body interaction between two identical bosons and the external potential:

\[ V(r_A, r_B) \propto \delta(r_P - r_A)\delta(r_P - r_B), \]

where \( r_A \) and \( r_B \) refers to the positions of identical bosons and \( r_P \) is the position of the external potential. Experimentally, a spatial control of interaction is already realized in Ref. [33]. If the optical control of interaction introduced in Ref. [33] is implemented in a system with single-atom resolution, the above effective short-range three-body interaction might be realized. Analogous to the Efimov effect, a resonance of the atomic loss will be a fingerprint of the semisuper-Efimov effect.

\[ \frac{dv_4}{ds} = -\frac{v_4}{s} - \frac{v_2^2}{2\pi} \frac{\alpha(2\alpha + 1)}{3\alpha + 1} - \frac{8\pi}{s} \frac{2\alpha + 1}{\alpha(\alpha + 1)} \]  

(A1)

Since we expect that the solution \( v_4(s) \) is periodic with respect to the variable \( \sqrt{s} \), we introduce convenient variables \( s := t^2 \) and \( g_4 := tv_4 \). Consequently, we obtain

\[ \frac{dg_4}{dt} = -\frac{g_4}{t} - \frac{g_2^2}{3\alpha + 1} - 16\pi \frac{2\alpha + 1}{\alpha(\alpha + 1)} \]  

(A2)

In solving the equation, we assume that the first term on the right-hand side does not provide a dominant contribution in the low-energy limit \( t \gg 1 \) and the term will be neglected hereafter. The assumption is justified by substituting the obtained solution of Eq. (26) into Eq. (A2). Namely, at sufficiently low-energy \( t \gg 1 \), the first term on the right-hand side of Eq. (A2) is \( 1/t \)-times smaller than the rest two terms. We thus arrive at the following simple equation:

\[ -\frac{dg_4}{dt} = \frac{2\alpha(2\alpha + 1)}{\pi(3\alpha + 1)} + 16\pi \frac{2\alpha + 1}{\alpha(\alpha + 1)} = dt, \]

(A3)

which can be easily integrated to

\[ \arctan \frac{g_4(t)}{2\pi \sqrt{\frac{3\alpha + 1}{\alpha + 1}}} = -4\sqrt{\frac{(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} t + \text{const.} \]

(A4)

We thus obtain Eq. (26).

As a complement, we note that an exact solution of Eq. (A2) is given by

\[ g_4(t) = -\frac{4\pi}{\alpha} \sqrt{\frac{3\alpha + 1}{\alpha + 1}} J_1 \left( 4\sqrt{\frac{(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} t \right) + N_1 \left( 4\sqrt{\frac{(2\alpha + 1)^2}{(\alpha + 1)(3\alpha + 1)}} t \right) A \]

(A5)
where $A$ is a constant. From the solution, we can also obtain Eq. (26) by using asymptotic forms of the Bessel’s functions $J_{\nu}$ and $N_{\nu}$.

**Appendix B: Effects of two-body interaction**

In Sec. II A 2, we discuss an emergence of the semisuper Efimov states in presence of two-body interactions. Here we show the derivation of Eq. (26) under the conditions Eqs. (28), (29) and (30). Since the same discussion applies to systems of three-component bosons and bosons in 1D, we here consider the system of two-component mass-imbalanced bosons. Firstly, under the condition of Eq. (28), the solutions of the renormalization-group equations Eq. (22) of the two-body sector become

$$g_{A}(s) = \left( \frac{1}{g_{A}(0)} + \frac{\alpha s}{8\pi} \right)^{-1} \approx g_{A}(0), \quad (B1)$$

$$g_{AB}(s) = \left( \frac{1}{g_{AB}(0)} + \frac{\alpha s}{8\pi} \right)^{-1} \approx g_{AB}(0). \quad (B2)$$

By substituting these solutions into the renormalization-group equation Eq. (23) of the three-body coupling constant $h(s)$, we obtain

$$h(s)^{2} = \frac{16\pi^{2}C(2\alpha+1)h(0)^{2}}{c_{s}^{2} + \frac{16\pi^{2}C(2\alpha+1)}{\alpha^{2}h(0)^{2}}}, \quad (B3)$$

$$C := \frac{\alpha g_{A}(0)}{4\pi} + \frac{2\alpha}{\alpha + 1}, \quad (B4)$$

with $C \ll 1$ and $\alpha^{2}h(0)^{2} \gg 16\pi^{2}C(2\alpha+1)$ due to the conditions of Eqs. (28) and (29). Consequently, $h(s)$ behaves as

$$h(s)^{2} \approx \frac{16\pi^{2}2\alpha+1}{s + \frac{16\pi^{2}(2\alpha+1)}{\alpha^{2}h(0)^{2}}} \quad (B5)$$

We immediately notice that the solution is equal to Eq. (25), which is the solution of $h(s)$ with vanishing two-body interaction. Therefore, $h(s)$ asymptotically behaves as $h(s)^{2} \approx \frac{16\pi^{2}2\alpha+1}{s}$ in the parameter region of Eq. (30). Following the discussion in Appendix A we finally arrive at the solution Eq. (26) even in presence of the two-body interactions.

As we noted in the main article, here the physical consequence of the two-body coupling constants are the infrared cutoffs of the semisuper-Efimov effect due to the condition Eq. (28).

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