THE UNIVERSALITY OF HUGHES-FREE DIVISION RINGS

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Abstract. Let $E * G$ be a crossed product of a division ring $E$ and a locally indicable group $G$. Hughes showed that up to $E * G$-isomorphism, there exists at most one Hughes-free division $E * G$-ring. The existence of a Hughes-free division $E * G$-ring $D_{E * G}$ is an open question, but it exists, for example, if $G$ is amenable or $G$ is bi-orderable. We study, whether $D_{E * G}$ is universal division ring of fractions in these cases. In Appendix we give a description of $D_{E[G]}$ when $G$ is a RFRS group, generalizing a result of Kielak.

1. Introduction

A division $R$-ring $\phi : R \rightarrow D$ is called epic if $\phi(R)$ generates $D$ as a division ring. Each division $R$-ring $D$ induces a Sylvester matrix rank function $\text{rk}_D$ on $R$. Given a ring $R$, Cohn introduced the notion of universal division $R$-ring (see, for example, [4, Section 7.2]). In the language of Sylvester rank functions, an epic division $R$-ring $D$ is universal if for every division $R$-ring $E$, $\text{rk}_D \geq \text{rk}_E$. By a result of Cohn [3, Theorem 4.4.1], it follows that the universal division $R$-ring is unique up to $R$-isomorphism. The universal division $R$-ring $D$ is called universal division ring of fractions of $R$ if $D$ is epic and $\text{rk}_D$ is faithful (that is $R$ is embedded in $D$).

If $R$ is a commutative domain, then the field of fractions $Q(R)$ is the universal division $R$-ring. The situation is much more complicated in the non-commutative setting. For example, Passman [25] gave an example of a Noetherian domain which does not have a universal division ring of fractions. Moreover, we show in Proposition 4.1 that the group algebra $Q[H]$ does not have a universal division ring of fractions if $H$ is not locally indicable. In this paper we want to study whether a group algebra or, more generally, a crossed product $E * G$, where $E$ is a division ring, has a universal division ring of fractions. Thus, from the previous observation it is natural to consider the case of group algebras and crossed products $E * G$ where $G$ is locally indicable.

Let $E$ be a division ring and $G$ a locally indicable group. Hughes [12] introduced a condition on an epic division $E * G$-rings and showed that up to $E * G$-isomorphism, there exists at most one epic division $E * G$-ring satisfying this condition. We call this division ring, the Hughes-free division $E * G$-ring and denote it by $D_{E * G}$. For simplicity, in this paper the Sylvester matrix rank function $\text{rk}_{D_{E * G}}$ is denoted by $\text{rk}_{E * G}$. We say that a locally indicable group $G$ is Hughes-free embeddable if $E * G$ has a Hughes-free division ring for every division ring $E$ and every crossed product $E * G$.

The existence of a Hughes-free division $E * G$-ring is known for several families of locally indicable groups. In the case of amenable locally-indicable groups $G$,
\[ D_{E \ast G} = Q(E \ast G) \] is the classical ring of fractions of \( E \ast G \), and in the case of bi-orderable groups \( G \), \( D_{E \ast G} \) is constructed using the Malcev-Neumann construction \cite{21,24} (see \cite{9}). The existence of \( D_{K[G]} \) is also known for group algebras \( K[G] \), where \( K \) is of characteristic 0 and \( G \) is an arbitrary locally indicable group \cite{16}.

In \cite[Theorem 8.1]{16} it is shown that if there exists a universal epic division \( E \ast G \)-ring and a Hughes-free division \( E \ast G \)-ring, they should be isomorphic as \( E \ast G \)-rings. Following Sánchez (see \cite[Definition 6.18]{20}), we say that a locally indicable group \( G \) is a Lewin group if it is Hughes-free embeddable and for all possible crossed products \( E \ast G \), where \( E \) is a division ring, \( D_{E \ast G} \) is universal (in Subsection 3.3 we will see that this definition is equivalent to the Sánchez one). We conjecture that all locally indicable groups are Lewin.

**Conjecture 1.** Let \( G \) be a locally indicable group and \( E \) a division ring. Then

(A) the Hughes-free division \( E \ast G \)-ring \( D_{E \ast G} \) exists and

(B) it is universal division ring of fractions of \( E \ast G \).

We want to notice that at this moment it is also an open problem of whether the universal division \( E \ast G \)-ring of fractions (if exists) should be Hughes-free.

In this paper we study part (B) of the conjecture in some cases where part (A) is known. Using Theorem 3.7 we can show that Conjecture 1 is valid for the following locally indicable groups.

**Theorem 1.1.** Locally indicable amenable groups, residually-(torsion-free nilpotent) groups and free-by-cyclic groups are Lewin groups.

In the case of group algebras we can prove a stronger result. The metric space \( G_n \) of marked \( n \)-generated groups consists of pairs \((G; S)\), where \( G \) is a group and \( S \) is an ordered generating set of \( G \) of cardinality \( n \). Such pairs are in 1-to-1 correspondence with epimorphisms \( F_n \to G \), where \( F_n \) is the free group of rank \( n \), and thus the set \( G_n \) can be identified with the set of all normal subgroups of \( F = F_n \). The distance between \( M_1 \) and \( M_2 \) is defined by

\[
d(M_1, M_2) = \inf \{ e^{-k} : M_1 \cap B_k(1_F) = M_2 \cap B_k(1_F) \},
\]

where \( B_k(1_F) \) denotes the closed ball of radius \( k \) and center \( 1_F \).

We say that a sequence of \( n \)-generated groups \( \{G_i\}_{i \in \mathbb{N}} \) converges to an \( n \)-generated group \( G \) if \((G_i; S_i) \in G_n\) converge to \((G; S) \in G_n\) for some generating sets \( S_i \) of \( G_i \) \((i \in \mathbb{N})\) and \( S \) of \( G \), respectively.

**Theorem 1.2.** Let \( F \) be a free group freely generated by a finite set \( S \) and \( M \) and \( \{M_i\}_{i \in \mathbb{N}} \) normal subgroups of \( F \). We put \( G = F/M \) and \( G_i = F/M_i \) and assume that \((G_i, SM_i/M_i)\) converges to \((G, SM/M)\). Assume that for all \( i \), \( G_i \) is locally indicable and \( D_{E[G_i]} \) exists. Then \( G \) is locally indicable, \( D_{E[G]} \) exists and

\[
\text{rk}_{E[G]} = \lim_{i \to \infty} \text{rk}_{E[G_i]}
\]

as Sylvester matrix rank functions on \( E[F] \).

Theorems 1.1 and 1.2 have the following immediate consequence.

**Corollary 1.3.** Let \( G \) be a residually-(locally indicable and amenable) group and let \( E \) be a division ring. Then \( D_{E[G]} \) exists and it is the universal division ring of fractions of \( E[G] \).
The corollary can be applied to RFRS groups, because they are residually poly-$\mathbb{Z}$. In Appendix we give a description of $D_{E[G]}$ when $G$ is a RFRS group that generalizes a result of Kielak \[19\].

Let us consider now the case of group algebras $K[G]$ where $K$ is a subfield of $\mathbb{C}$ and $G$ is locally indicable. In this case it was shown in \[16\] that the division closure $D(K[G], U(G))$ of $K[G]$ in the algebra of affiliated operators $U(G)$ is a Hughes-free division $K[G]$-ring. We denote by $r_k$ the von Neumann rank function (its definition is recalled in Subsection 2.6), and by $r_{\{1\}}$ the Sylvester matrix rank function on $\mathbb{Q}[G]$ induced by the homomorphism $\mathbb{Q}[G] \to \mathbb{Q}$ that sends all the elements of $G$ to 1 (in the previous notation $r_{\{1\}}$ is $r_k$). In view of Conjecture 1 it is natural to ask for which groups $G$, $r_k[G] \geq r_{\{1\}}[G]$. It follows from \[27\, Proposition 1.9\] that if a group $G$ satisfies the condition $r_k[G] \geq r_{\{1\}}[G]$, then $G$ is locally indicable.

Thus, we propose also a weak version of Conjecture 1.

**Conjecture 2.** Let $G$ be a locally indicable group. Then $r_k[G] \geq r_{\{1\}}[G]$ as Sylvester matrix rank functions on $\mathbb{Q}[G]$.

From the discussion in the paragraph before the conjecture, we conclude that Corollary 1.3 has the following consequence.

**Corollary 1.4.** Let $G$ be a residually-(locally indicable and amenable) group. Then $r_k[G] \geq r_{\{1\}}[G]$ as Sylvester matrix rank functions on $\mathbb{Q}[G]$.

Using a description of the division ring $D_{\mathbb{Q}[G]}$ for RFRS groups $G$, Kielak \[19\] obtained a criterion of virtually fiberness of RFRS groups that generalizes a celebrated result of I. Agol \[1\] showing that an irreducible 3-manifold, whose fundamental group is RFRS, is virtually fibered. Since RFRS groups are residually poly-$\mathbb{Z}$, we can apply Corollary 1.4 and together with Kielak’s criterion we obtain the following corollary.

**Corollary 1.5.** Let $G$ be a finitely generated group which is virtually RFRS. Then the following are equivalent.

1. $G$ is virtually fibered, in the sense that it admits a virtual map onto $\mathbb{Z}$ with finitely generated kernel.
2. $G$ admits a virtual map onto $\mathbb{Z}$ whose kernel has finite first Betti number.

Our next result is another consequence of Corollary 1.4 that generalizes a result of Wise from \[29\, Theorem 1.3\].

**Corollary 1.6.** Let $X$ be a compact CW-complex with $\pi_1 X$ non-trivial residually-(locally indicable and amenable) group. Then

$$b_1^{(2)}(\tilde{X}) \leq b_1(X) - 1 \quad \text{and} \quad b_p^{(2)}(\tilde{X}) \leq b_p(X) \quad \text{if} \ p \geq 2.$$
2. Preliminaries

2.1. Notation and definitions. All rings in this paper are unitary and ring homomorphisms send the identity element to the identity element. By a module we will mean a left module. Let $G$ be a group with trivial element $e$. We say that a ring $R$ is $G$-graded if $R$ is equal to the direct sum $\bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g$ and $h$ in $G$. If for each $g \in G$, $R_g$ contains an invertible element $u_g$, then we say that $R$ is a crossed product of $R_e$ and $G$ and we will write $R = S \ast G$ if $R_e = S$. In the following if $H$ is a subgroup of $G$, $S \ast H$ will denote the subring of $R$ generated by $S$ and $\{u_h : h \in H\}$.

A ring $R$ may have several different $G$-gradings. It will be always clear from the context what $G$-grading we use. However, in some situations the grading is unique. Assume that $R \cong E \ast G$, where $E$ is a division ring and $G$ is locally indicable, then by \cite{10}, the invertible elements $U(R)$ of $R$ are $\bigcup_{g \in G} R_g \setminus \{0\}$. Hence $R_e$ is the maximal subring in $U(R) \cup \{0\}$ and $G \cong U(R)/(R_e \setminus \{0\})$. Thus, $R$ has a unique grading with $R_e$ is a division ring and $G$ is locally indicable.

An $R$-ring is a pair $(S, \phi)$ where $\phi : R \to S$ is a homomorphism. We will often omit $\phi$ if it is clear from the context.

2.2. Ordered groups. A total order $\leq$ on a group $G$ is left-invariant if for any $a, b, g \in G$, if $a \leq b$ then $ga \leq gb$. It is bi-invariant if, moreover we have $ag \leq bg$.

Let $\preceq$ be a left-invariant order on a group $G$. A subgroup $H$ is called convex if $H$ contains every element $g$ lying between any two elements of $H$ ($h_1 \preceq g \preceq h_2$ with $h_1, h_2 \in H$). We say that $\preceq$ is Conradian if for all elements $f, g \succ 1$, there exists a natural number $n$ such that $fg^n \succ g$. In fact, one may actually take $n = 2$ (\cite{5} Proposition 3.2.1]). Recall that a group $G$ is locally indicable if every finitely generated non-trivial subgroup of $G$ has an infinite cyclic quotient. A useful characterization of locally indicable groups says that they are the groups admitting a Conradian order (\cite{5}). We will need the following important property of a Conradian order.

Proposition 2.1. \cite{6} Corollary 3.2.28] Let $(G, \preceq)$ be a group with a Conradian order and let $N$ be a maximal convex subgroup of $G$. Then there exists an order preserving homomorphism $\phi : G \to \mathbb{R}$ such that $N = \ker \phi$.

2.3. Hughes-free division rings. Let $E$ be a division ring and $G$ a locally indicable group. Let $\varphi : E \ast G \to D$ be a homomorphism from $E \ast G$ to a division ring $D$. We say that a division $E \ast G$-ring $(D, \varphi)$ is Hughes-free if

1. $D$ is the division closure of $\varphi(E \ast G)$ ($D$ is epic).
2. For every non-trivial finitely generated subgroup $H$ of $G$, a normal subgroup $N$ of $H$ with $H/N \cong \mathbb{Z}$, and $h_1, \ldots, h_n \in H$ in distinct cosets of $N$, the sum $D_{N,D}\varphi(u_{h_1}) + \ldots + D_{N,D}\varphi(u_{h_n})$ is direct. (Here $D_{N,D} = D(\varphi(E \ast N), D)$ is the division closure of $\varphi(E \ast N)$ in $D$.)

Hughes \cite{12} (see also \cite{7}) showed that up to $E \ast G$-isomorphism there exists at most one Hughes-free division ring. We denote it by $D_{E \ast G}$. The uniqueness of Hughes-free division rings implies that for every subgroup $H$ of $G$, $D_{H,D_{E \ast G}}$ is Hughes-free as a division $E \ast H$-ring.
Gräter showed in [9, Corollary 8.3] that $D_{E*G}$ (if it exists) is strongly Hughes-free, that it satisfies the following additional condition:

(2') For every non-trivial subgroup $H$ of $G$, a normal subgroup $N$ of $H$ and $h_1, \ldots, h_n \in H$ in distinct cosets of $N$, the sum $D_{N,D_{E,G}} \varphi(u_{h_1}) + \ldots + D_{N,D_{E,G}} \varphi(u_{h_n})$ is direct.

In particular, this implies the following result that we will use often without mentioning it explicitly.

**Proposition 2.2.** Let $G$ be a locally indicable group, $N$ a normal subgroup of $G$ and $E$ a division ring. Assume that for a crossed product $E * G$, $D_{E*G}$ exists. Then the ring $R$ generated by $D_{N,D_{E,G}}$ and $G$ has structure of a crossed product $D_{E*N} * (G/N)$. In particular,

(1) if $N$ is of finite index in $G$, then $D_{E*G} = D_{E*N} * (G/N)$ and

(2) if $G/N$ is abelian, $D_{E*G}$ is isomorphic to the classical Ore ring of fractions of $D_{E*N} * (G/N)$.

2.4. **Free division $E * G$-ring of fractions.** Let $G$ be group with a Conradian left-invariant order $\preceq$ (so, $G$ is locally indicable). Let $E$ be a division ring. Let $\varphi : E * G \rightarrow D$ be a homomorphism from a crossed product $E * G$ to a division ring $D$. We say that a division $E * G$-ring $(D, \varphi)$ is free with respect to $\preceq$ if

(1) $D$ is the division closure of $\varphi(E * G)$.

(2) For every subgroup $H$ of $G$, and the maximal convex subgroup $N$ of $H$ (which is normal by Proposition 2.1), and $h_1, \ldots, h_n \in H$ in distinct cosets of $N$, the sum $D_{N,D} \varphi(u_{h_1}) + \ldots + D_{N,D} \varphi(u_{h_n})$ is direct.

This notion was introduced by Gräter in [9].

**Remark 2.3.** Notice that in part (2) of the definition, we also can assume that $H$ is finitely generated. Indeed, assume (2) holds for finitely generated subgroups, but for some $H$ and $h_1, \ldots, h_n$, there are $d_1, \ldots, d_n \in D_{N,D}$, not all equal to zero, such that $d_1 \varphi(u_{h_1}) + \ldots + d_n(u_{h_n}) = 0$. Then we can find a finitely generated subgroup of $N'$ of $N$ such that $d_1, \ldots, d_n \in D_{N',D}$. Let $H'$ be the subgroup of $G$ generated by $h_1, \ldots, h_n$ and $N'$. Since $n \geq 2$, $N \cap H'$ is the maximal convex subgroup of $H'$. This contradicts our assumption that (2) holds for $H'$.

Gräter proved the following result.

**Proposition 2.4.** [9, Corollary 8.3] Let $G$ be group with a Conradian left-invariant order $\preceq$ and let $E$ be a division ring. A division $E * G$-ring is free with respect to $\preceq$ if and only if it is Hughes-free (and so, it is $E * G$-isomorphic to $D_{E*G}$).

2.5. **Sylvester matrix rank functions.** Let $R$ be a ring. A Sylvester matrix rank function $\text{rk}$ on $R$ is a function that assigns a non-negative real number to each matrix over $R$ and satisfies the following conditions.

(SMat1) $\text{rk}(M) = 0$ if $M$ is any zero matrix and $\text{rk}(1) = 1$;

(SMat2) $\text{rk}(M_1 M_2) \leq \min\{\text{rk}(M_1), \text{rk}(M_2)\}$ for any matrices $M_1$ and $M_2$ which can be multiplied;

(SMat3) $\text{rk} \left( \begin{array}{cc} M_1 & 0 \\ 0 & M_2 \end{array} \right) = \text{rk}(M_1) + \text{rk}(M_2)$ for any matrices $M_1$ and $M_2$;

(SMat4) $\text{rk} \left( \begin{array}{cc} M_1 & M_3 \\ 0 & M_2 \end{array} \right) \geq \text{rk}(M_1) + \text{rk}(M_2)$ for any matrices $M_1$, $M_2$ and $M_3$ of appropriate sizes.
We denote by $\mathbb{P}(R)$ the set of Sylvester matrix rank functions on $R$, which is a compact convex subset of the space of functions on matrices over $R$. If $\phi : F_1 \to F_2$ is an $R$-homomorphism between two free finitely generated $R$-modules $F_1$ and $F_2$, then $\text{rk}(\phi)$ is $\text{rk}(A)$ where $A$ is the matrix associated with $\phi$ with respect to some $R$-bases of $F_1$ and $F_2$. It is clear that $\text{rk}(\phi)$ does not depend on the choice of the bases.

A useful observation is that a ring homomorphism $\varphi : R \to S$ induces a continuous map $\varphi^\dagger : \mathbb{P}(S) \to \mathbb{P}(R)$, i.e., we can pull back any rank function $\text{rk}$ on $S$ to a rank function $\varphi^\dagger(\text{rk})$ on $R$ by just defining

$$\varphi^\dagger(\text{rk})(A) = \text{rk}(\varphi(A))$$

for every matrix $A$ over $R$. We will often abuse the notation and write $\text{rk}$ instead of $\varphi^\dagger(\text{rk})$ when it is clear that we speak about the rank function on $R$.

A division ring $D$ has a unique Sylvester matrix rank function which we denote by $\text{rk}_D$. If a Sylvester matrix rank function $\text{rk}$ on $R$ takes only integer values, then by a result of P. Malcolmson [22] there are a division ring $D$ and a homomorphism $\varphi : R \to D$ such that $\text{rk} = \varphi^\dagger(\text{rk}_D)$. Moreover, if $D$ is equal to the division closure of $\varphi(R)$ ($D$ is an epic division $R$-ring), then $\varphi : R \to D$ is defined uniquely up to isomorphisms of $R$-rings. We denote the set of integer-valued rank functions on a ring $R$ by $\mathbb{P}_{\text{div}}(R)$. In the following, if a rank function on $R$ is induced by a homomorphism to $D$ we will also use $\text{rk}_D$ to denote this rank function (in this case the homomorphism will be clear from the context).

Given two Sylvester matrix rank functions on $R$, $\text{rk}_1$ and $\text{rk}_2$, we will write $\text{rk}_1 \leq \text{rk}_2$ if for any matrix $A$ over $R$, $\text{rk}_1(A) \leq \text{rk}_2(A)$. In the case where both functions are integer-valued and come from homomorphisms $\varphi_i : R \to D_i$ ($i = 1, 2$) from $R$ to epic division rings $D_1$ and $D_2$, the condition $\text{rk}_D_1 \leq \text{rk}_D_2$ is equivalent to the existence of a specialization from $D_2$ to $D_1$ in the sense of P. M. Cohn (3 Subsection 4.1). We say that an epic division $R$-ring $D$ is universal if for every epic division $R$-ring $E$, $\text{rk}_D \geq \text{rk}_E$.

An alternative way to introduce Sylvester rank functions is via Sylvester module rank functions. A **Sylvester module rank function** $\dim$ on $R$ is a function that assigns a non-negative real number to each finitely presented $R$-module and satisfies the following conditions.

(SMod1) $\dim\{0\} = 0$, $\dim R = 1$;
(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;
(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$  

There exists a natural bijection between Sylvester matrix and module rank functions over a ring. Given a Sylvester matrix rank function $\text{rk}$ on $R$ and a finitely presented $R$-module $M \cong R^n/R^m.A$ ($A$ is a matrix over $R$), we define the corresponding Sylvester module rank function $\dim$ by means of $\dim(M) = n - \text{rk}(A)$. If a Sylvester matrix rank function $\text{rk}_D$ comes from a division $R$-ring $D$, then the corresponding Sylvester module rank function will be denoted by $\dim_D$. Then $D$ is the universal epic division $R$-ring if and only if for every epic division $R$-ring $E$ and every finitely presented $R$-module, $\dim_D(M) \leq \dim_E(M)$.

By a recent result of Li [20], any Sylvester module rank function on $R$ can be extended to arbitrary modules over $R$. In the case of an integer-valued Sylvester
module rank function \( \dim_D \) and an \( R \)-module \( M \) we simply have \( \dim_D(M) = \dim_D(D \otimes_R M) \).

2.6. **Von Neumann rank function.** Consider first the case where \( G \) is countable. Then \( G \) acts by left and right multiplication on the separable Hilbert space \( l^2(G) \). A finitely generated **Hilbert** \( G \)-module is a closed subspace \( V \leq l^2(G)^n \), invariant under the left action of \( G \). We denote by \( \text{proj}_V : l^2(G)^n \to l^2(G)^n \) the orthogonal projection onto \( V \) and we define

\[
\dim_G V := \text{Tr}_G(\text{proj}_V) := \sum_{i=1}^n ((1_i) \text{proj}_V, 1_i)_{l^2(G)^n},
\]

where \( 1_i \) is the element of \( l^2(G)^n \) having 1 in the \( i \)th entry and 0 in the rest of the entries. The number \( \dim_G V \) is the **von Neumann dimension** of \( V \).

Let \( A \in \text{Mat}_{n \times m}(C[G]) \) be a matrix over \( C[G] \). The action of \( A \) by right multiplication on \( l^2(G)^n \) induces a bounded linear operator \( \phi^A_G : l^2(G)^n \to l^2(G)^m \). We put

\[
\text{rk}_G(A) = \dim_G \text{Im} \phi^A_G.
\]

If \( G \) is not countable then \( \text{rk}_G \) can be also defined. Take a matrix \( A \) over \( C[G] \). Then the group elements that appear in \( A \) are contained in a finitely generated group \( H \). We will put \( \text{rk}_G(A) = \text{rk}_H(A) \). One easily checks that the value \( \text{rk}_H(A) \) does not depend on the subgroup \( H \).

Another obvious **Sylvester matrix rank function** on \( G \) arises from the trivial homomorphism \( G \to \{1\} \) and it is defined as

\[
\text{rk}_{\{1\}}(A) = \text{rk}_C(\mathcal{A}),
\]

where \( \mathcal{A} \) is the matrix over \( C \) obtained from \( A \) by sending all the elements of \( G \) to 1. More generally, if \( \mathcal{G} \) is a quotient of \( G \), \( \text{rk}_{\mathcal{G}}(A) \) is denoted to be \( \text{rk}_{\mathcal{G}}(\mathcal{A}) \), where \( \mathcal{A} \) is the matrix over \( C[\mathcal{G}] \) obtained from \( A \) by applying the obvious map \( C[G] \to C[\mathcal{G}] \).

2.7. **The natural extension.** Let \( R = E * G \) be a crossed product of a division ring \( E \) and a group \( G \). Let \( N \) be a normal subgroup of \( G \) such that \( G/N \) is amenable. Consider a transversal \( \mathcal{X} \) of \( N \) in \( G \). Since \( G/N \) is amenable there are finite subsets \( \mathcal{X}_k \) of \( \mathcal{X} \) such that \( \{N\mathcal{X}_k/N\} \) is a Følner sequence in \( G/N \) with respect to the right action. Put \( X_k = N\mathcal{X}_k \).

Let \( \text{rk} \) be a Sylvester rank function on \( E * N \) and assume that \( \text{rk} \) is invariant under conjugation by the elements \( \{u_g\}_{g \in G} \). Observe that if \( \text{rk} = \text{rk}_E \) for some epic division \( E * N \)-ring \( \mathcal{E} \), then the conjugation of \( E * N \) by any \( u_g (g \in G) \) can be extended to a unique automorphism of \( \mathcal{E} \). Thus one can consider the crossed product \( E * G/N \) containing \( E * G \).

Let \( A \in \text{Mat}_{n \times m}(R) \) and let \( S \) be the union of supports of the entries of \( A \). For any subset \( T \) of \( G \) we denote \( R_T = \oplus_{t \in T} R_t \). Let \( \phi_k : (R_{X_k})^n \to (R_{X_k}S)^m \) be the homomorphism of finitely generated free \( E * N \)-modules induced by the right multiplication by \( A \). Let \( \omega \) be a non-principal ultrafilter on \( N \). Then we put

\[
\tilde{\text{rk}}_\omega(A) = \lim_{\omega} \frac{\text{rk}(\phi_k)}{|X_k|}.
\]

Then \( \tilde{\text{rk}}_\omega \) is a Sylvester rank function on \( R \). The rank function \( \tilde{\text{rk}}_\omega \) has been already studied previously in different situations (see [28, 15, 16, 18]). In [18] it is shown that \( \tilde{\text{rk}}_\omega \) does not depend on \( \omega \). Therefore in the following we denote \( \tilde{\text{rk}}_\omega \) by \( \tilde{\text{rk}} \).
The Sylvester rank function $\tilde{r}_k$ is called the natural extension of $r_k$. We describe now the cases that appear in this paper.

**Proposition 2.5.** Let $G$ be a group with a normal subgroup $N$ such that $G/N$ is amenable. Let $E$ be a division ring, and assume the previous notation. Then the following holds.

1. Assume that $N$ and $G/N$ are locally indicable and $rk = rk_E$ for some epic division $E * N$-ring $E$. Then $\tilde{r}_k$ coincides with $rk_{Q(E^*(G/N))}$, where $Q(E^*(G/N))$ denotes the classical Ore ring of fractions of $E * (G/N)$.
2. Assume $E^* G = K[G]$, where $K$ is a subfield of $\mathbb{C}$ and $rk = rk_N$. Then $\tilde{r}_k$ is equal to $rk_G$.
3. Assume $E^* G = K[G]$, where $K$ is a subfield of $\mathbb{C}$ and $rk = rk_{\{1\}}$. Then $\tilde{r}_k$ is equal to $rk_{G/N}$.

**Proof.** (1) We can extend $\tilde{r}_k$ to a Sylvester matrix rank function on $E^*(G/N)$ (which we denote also by $\tilde{r}_k$) using the formula (1). Since $G/N$ is locally indicable, the ring $E^*(G/N)$ is a domain. Thus, by the definition of $\tilde{r}_k$, $\tilde{r}_k(a) = 1$ for every $0 \neq a \in E^*(G/N)$. Hence, applying [15, Proposition 5.2], we obtain that $\tilde{r}_k = rk_{Q(E^*(G/N))}$.

The statements (2) and (3) follow from [15, Theorem 12.1].

3. On the universality of $D_{E*G}$

3.1. **A general criterion of universality.** In this subsection we present a general criterion of universality of a division $R$-ring. The proof of the following lemma is immediate.

**Lemma 3.1.** Let $R$ be a ring and $E$ a division $R$-ring. Let $M$ be a finitely generated left $R$-module. Then the following are equivalent.

1. $\dim_E(M) \neq 0$.
2. $E \otimes_R M \neq 0$.
3. $\text{Hom}_R(M, E) \neq 0$.

The following proposition tells us that in order to check universality of a division $R$-ring $D$ it is enough to understand the structure of its finitely generated $R$-submodules.

**Proposition 3.2.** Let $R$ be a ring and $D$ an epic division $R$-ring. Then $rk_D$ is universal in $\mathcal{P}_{\text{div}}(R)$ if and only if for every finitely generated left $R$-submodule $L$ of $D$ and every division $R$-ring $E$, $\dim_E(L) > 0$.

**Proof.** Assume that $rk_D$ is universal. Since $\text{Hom}_R(L, D) \neq 0$, by Lemma 3.1 $\dim_D(L) > 0$ and so $\dim_E(L) \geq \dim_D(L) > 0$.

This proves the “only if” part of the proposition.

Now, consider the “if” part. We want to show that for every finitely generated left $R$-module $M$ and every division $R$-ring $E$, $\dim_E(M) \geq \dim_D(M)$. We will do it by induction on $\dim_D(M)$.

Let $\overline{M}$ be the image of the natural $R$-homomorphism $\alpha : M \to D \otimes_R M$ that sends $m \in M$ to $1 \otimes m$. Observe that, since $D \otimes_R M \cong D \otimes_R \overline{M}$, $\dim_D(M) = \dim_D(\overline{M})$. 
We have also that \( \dim_{E}(M) \leq \dim_{E}(M) \). Thus, without loss of generality, we can assume that \( \alpha \) is injective.

Now assume that \( \dim_{D}(M) = 1 \). Since \( M \) is a submodule of \( D \), then \( \dim_{E}(M) > 0 \), and so, \( \dim_{E}(M) \geq 1 = \dim_{D}(M) \). This gives us the base of induction.

Assume that the claim holds if \( \dim_{D}(M) \leq n-1 \). Consider the case \( \dim_{D}(M) = n \geq 2 \). Observe that \( \dim_{E}(M) \neq 0 \), since \( M \) has a non-trivial quotient that lies in \( D \). Hence \( E \otimes_{R} M \neq \{0\} \). Let \( m \in M \) be such that \( 1 \otimes m \) is not trivial in \( E \otimes_{R} M \). Then \( \dim_{E}(M/Rm) = \dim_{E}(M) - 1 \). Since we assume that \( \alpha \) is injective, \( 1 \otimes m \) is non-trivial in \( D \otimes_{R} M \), and so, we also have \( \dim_{D}(M/Rm) = \dim_{D}(M) - 1 \).

Applying the inductive assumption we obtain that

\[
\dim_{D}(M) = \dim_{D}(M/Rm) + 1 \leq \dim_{E}(M/Rm) + 1 = \dim_{E}(M).
\]

\( \square \)

3.2. The universality of \( D_{E \ast G} \) in the amenable case. Let \( E \) be a division ring and \( G \) a locally indicable group. Proposition 3.2 indicates that in order to prove the universality we have to understand the structure of finitely generated \( E \ast G \)-submodules of \( D_{E \ast G} \). If \( G \) is amenable, they are isomorphic to finitely generated left ideals of \( E \ast G \). The following result shows that in the latter case the condition of Proposition 3.2 holds.

**Proposition 3.3.** Let \( R = E \ast G \) be a crossed product of a division ring \( E \) and a locally indicable group \( G \). Then for every non-trivial finitely generated left ideal \( L \) of \( R \) and every division \( R \)-ring \( E \), \( \dim_{E}(L) > 0 \).

**Proof.** We denote by \( R_{g} \) the \( g \)-th component of \( R \) and let \( u_{g} \) be an invertible element of \( R_{g} \). Then \( R_{g} \cong E \). For any element \( r = \sum_{g \in G} r_{g} \in R (r_{g} \in R_{g}) \) denote by \( \text{supp}(r) \) the elements \( g \in G \) for which \( r_{g} \neq 0 \) and put \( l(r) \) to be equal to the number of non-trivial elements in \( \text{supp}(r) \). Thus, \( l(r) = 0 \) means that \( r \in R_{e} \). For a non-trivial finitely generated left ideal \( L \) of \( R \) we put

\[
l(L) = \min \{ l(r_{1}) + \ldots + l(r_{s}) : L = Rr_{1} + \ldots + Rr_{s} \}.
\]

Observe that if a set of generators \( \{r_{1}, \ldots, r_{s}\} \) of \( L \) satisfies the equality \( l(L) = l(r_{1}) + \ldots l(r_{s}) \), then for each \( i \), \( l(r_{i}) = |\text{supp}(r_{i})| - 1 \). (If not, we can change \( r_{i} \) by \( u_{g}^{-1}r_{i} \) with \( g \in \text{supp}(r_{i}) \) and obtain a contradiction.) Moreover, if all \( r_{i} \) are non-trivial and \( L \neq R \), then \( s < l(L) \). Now, we define

\[
s(L) = \max \{ s : L = Rr_{1} + \ldots + Rr_{s}, l(L) = l(r_{1}) + \ldots + l(r_{s}) \text{ and } r_{i} \text{ are non-trivial} \}.
\]

We will prove the proposition by induction on \( l(L) \). If \( l(L) = 0 \), then \( L = R \) and we are done. Now assume that the proposition holds if \( l(L) \leq n - 1 \), and consider the case \( l(L) = n \geq 1 \).

We will proceed by inverse induction on \( s(L) \). Observe that there is no \( L \) such that \( s(L) \geq l(L) + 1 \), so there is nothing to prove in this case. Assume that we can prove the proposition if \( l(L) = n \) and \( s(L) \geq k + 1 \), and consider the case \( l(L) = n \) and \( s(L) = k \).

Let \( r_{1}, \ldots r_{k} \) be a set of non-zero generators of \( L \) such that \( n = l(r_{1}) + \ldots l(r_{k}) \). Let \( H \) be the group generated by \( \cup_{i=1}^{k} \text{supp}(r_{i}) \). Since \( G \) is locally indicable there exists a surjective \( \alpha : H \to \mathbb{Z} \). Let \( N = \ker \alpha \) and \( t \in H \) such that \( \langle t \rangle N = H \). We write

\[
r_{i} = \sum_{j} u_{ij}^{t} r_{ij} \text{ with } 0 \neq r_{ij} \in E \ast N.
\]
Let \( L' \) be a left \( R \)-module generated by \( \{ r_{ij} \} \). Observe that
\[
\sum_{i,j} l(r_{ij}) \leq \sum_i l(r_i) \text{ and } |\{ r_{ij} \}| > s(L) = k.
\]
Thus, we obtain that either \( l(L') < l(L) \) or \( l(L') = l(L) \) and \( s(L') > s(L) \).
Hence we can apply the inductive hypothesis and obtain that \( \text{rk}_E(L') > 0 \). Thus \( \text{Hom}_R(L', \mathcal{E}) \neq 0 \).

Put \( S = E \ast H \). Observe that \( S \cong E \ast N[x^{\pm 1}; \tau] \), where \( \tau \) is conjugation by \( u_t \).

Let \( \tilde{E} \) be the Ore division ring of fractions of \( \mathcal{E}[x^{\pm}; \tau] \), where \( \tau \) is conjugation by \( u_t \). Then \( \tilde{E} \) has a natural \( S \)-ring structure. We denote by \( \dim_{\tilde{E}} \) the corresponding Sylvester module rank function on \( S \). By Proposition 2.5(1), \( \text{rk}_{\tilde{E}} \) is equal to the natural extension of the restriction of \( \text{rk}_E \) on \( E \ast N \).

Let \( L_0 \) and \( L'_0 \) be the left \( S \)-submodules of \( \mathcal{D} \) generated by \( \{ r_i \} \) and \( \{ r_{ij} \} \) respectively. We have that \( L_0 \leq L'_0 \). Every element \( m \) of \( L'_0 \) can be written in a unique way as \( m = \sum_j u_jm_j \), where \( m_j \in E \ast N \). We define \( \phi(m) = \sum_j x^j \phi(m_j) \).

This defines a homomorphism of left \( S \)-modules \( \phi : L'_0 \to \tilde{E} \). Since \( \phi \) is not trivial, there exists \( r_{ij} \) such that \( \phi(r_{ij}) \neq 0 \). Therefore, \( \phi(r_i) \neq 0 \). Thus, the restriction of \( \phi \) on \( L_0 \) is not trivial. Hence, by Lemma 3.1, \( \dim_{\tilde{E}}(L_0) > 0 \).

Let \( \dim_{\tilde{E}} \) be the Sylvester module rank function associated to the division \( S \)-ring \( \mathcal{E} \). Since the restrictions of \( \text{rk}_E \) and \( \text{rk}_{\tilde{E}} \) on \( E \ast N \) coincide \cite{10}, Lemma 8.3 implies that \( \text{rk}_E \leq \text{rk}_{\tilde{E}} \) as Sylvester matrix rank functions on \( E \ast H \), and so
\[
\dim_{\tilde{E}}(L_0) \geq \dim_{\tilde{E}}(L_0) > 0.
\]

Now observe that \( L \cong R \otimes_S L_0 \). Hence
\[
\dim_{\tilde{E}}(L) = \dim_{\tilde{E}}(L_0) > 0
\]
and we are done.

\[
\square
\]

**Corollary 3.4.** Let \( G \) be an amenable locally indicable group and let \( E \) be a division ring. Then \( \mathcal{D}_{E \ast G} \) is the universal division ring of fractions of \( E \ast G \).

**Proof.** Observe that \( E \ast G \) satisfies the left Ore condition and so \( \mathcal{D}_{E \ast G} \) is isomorphic as \( E \ast G \)-ring to the classical ring of fractions \( \mathcal{Q}(E \ast G) \). Since any finitely generated left submodule of \( \mathcal{Q}(E \ast G) \) is isomorphic to a left ideal of \( E \ast G \), Proposition 3.2 and Proposition 3.3 imply the desired result.

\[
\square
\]

We remark that Corollary 3.4 can be also proved using arguments similar to the ones used in the proof of \cite{11} Lemma 2.1. Also it is worth to be mentioned here that, by a result of D. Morris \cite{23}, a left orderable amenable group is always locally indicable.

### 3.3. A criterion for a group to be Lewin.

In this subsection we will show that in order to prove that a Hughes-free embeddable group \( G \) is Lewin, it is enough to consider only group algebras \( E[G] \). As before, by \( \text{rk}_E \) we denote the Sylvester matrix rank function on \( E[G] \) induced by the homomorphism \( E[G] \to E \) that sends all the group elements from \( G \) to 1.

**Proposition 3.5.** Let \( G \) be a locally indicable group and \( E \) a division ring. Assume that for every division ring \( \mathcal{E} \),
If for a crossed product $E \oplus \text{First let us show that } D\text{ the Hughes-free division ring }
\text{Clearly the ring } \tilde{g} \phi \text{ is a homomorphism. Then } w_g w_h = \phi(u_g^{-1})\phi(u_h^{-1})v_h = \phi(u_h^{-1})\phi(u_g^{-1})v_g v_h = \phi(u_g^{-1})\phi(u_h^{-1})\phi(\alpha(g,h))v_h = \phi(u_g^{-1})v_g \phi(u_h^{-1})v_h = \phi(\alpha(g,h))v_h.
Thus, we obtain that $\tilde{R} \cong E[G]$. In particular $\mathcal{D}_{E[G]}$, and so, $\mathcal{D}_{E*G}$ exist and $\tilde{\phi}(\text{rk}_{\mathcal{D}_{E*G}})$ is equal to $\text{rk}_{\mathcal{D}_{E*G}}$.
Now, we want to show that $\mathcal{D}_{E*G}$ is universal: $\text{rk}_{\mathcal{D}_{E*G}} \geq \phi(\text{rk}_{\mathcal{D}_E})$. Let $\psi : E \cong E[\mathcal{G}]$ be the map that sends all $w_g$ to 1. Denote by $\text{rk}_E^\phi$ the Sylvester matrix rank function on $E[\mathcal{G}]$ induced by $\psi$. By our assumptions, $\text{rk}_E^\phi \leq \text{rk}_{\mathcal{D}_{E*G}}$. Now observe that $\phi = \psi \circ \tilde{\phi}$. Hence
$$
\phi(\text{rk}_E) = (\psi \circ \tilde{\phi})(\text{rk}_E) = \tilde{\phi}(\phi(\text{rk}_E)) = \tilde{\phi}(\text{rk}_E^\phi) \leq \tilde{\phi}(\text{rk}_{\mathcal{D}_{E*G}}) = \text{rk}_{\mathcal{D}_{E*G}}
$$
as Sylvester matrix rank functions on $E[\mathcal{G}]$.

**Corollary 3.6.** Any subgroup of a Lewin group is Lewin.

The corollary implies that our definition of Lewin group is equivalent to the one of Sánchez ([26, Definition 6.18]).

**3.4. Proofs of Theorem 1.2 and Corollary 1.3.** Let $F$ be a free group freely generated by a finite set $S$, and let $M$ and $\{M_i\}_{i \in \mathbb{N}}$ be normal subgroups of $F$. We put $G = F/M$ and $G_i = F/M_i$ and assume that $(G_i, SM_i/M_i)$ converges to $(G, SM/M)$. Assume that for all $i$, $G_i$ is locally indicable and $\mathcal{D}_{E[G_i]}$ exists. Since $G_i$ are quotients of $F$, abusing notation, we will also refer to $\text{rk}_{E}[G_i]$ as a Sylvester matrix rank function on $E[\mathcal{F}]$.

Let $\omega$ be an arbitrary non-principal ultrafilter on $\mathbb{N}$. We put $\text{rk} = \lim_{\omega} \text{rk}_{\mathcal{D}_{E[G_i]}} \in \mathbb{P}_{\text{div}}(E[\mathcal{F}])$. 


Observe that for every $g \in M$, $\text{rk}(g - 1) = 0$. Thus, $\text{rk}$ is also a Sylvester matrix rank function on $E[G]$. We want to show that $\text{rk}$ corresponds to the Sylvester matrix rank function of a Hughes-free division $E \ast G$-ring. This will prove Theorem 1.2.

For each $i$ we fix a left-invariant Conradian order $\preceq_i$ on $G_i$. Define an order $\preceq$ on $G$ by

$$fM \preceq hM \text{ if } \{i \in \mathbb{N} : fM_i \preceq_i hM_i\} \in \omega.$$ 

The definition does not depend on the choice of representatives, because for every $m \in M$, the set $\{i \in \mathbb{N} : m \in M_i\}$ is in $\omega$. It is also clear that $\preceq$ is left-invariant and Conradian. In particular, this proves that $G$ is locally indicable.

Denote by $\alpha_j$ the canonical homomorphism $F \to G_j$ and extend it to the homomorphism $\alpha_j : E[F] \to D_{E[G_j]}$. The rank function $\text{rk}$ corresponds to the homomorphism

$$\alpha = (\alpha_i) : E[F] \to \prod_{\omega} D_{E[G_i]} := (\prod_{i \in \mathbb{N}} D_{E[G_i]})/I_\omega,$$

with $I_\omega = \{(d_i) : \lim_\omega \text{rk}_{E[G_i]}(d_i) = 0\}$. Observe that $\prod_{\omega} D_{E[G_i]}$ is a division ring. We denote by $D$ the division closure of $\alpha(E[F])$ in $\prod_{\omega} D_{E[G_i]}$. As we have observed before, for each $m \in M$, $\alpha(m - 1) = 0$. Thus, $D$ is a epic division $E[G]$-ring. We are going to show that $D$ is free with respect to $\preceq$. For simplicity, in what follows, for each $j \in \mathbb{N}$, $D_{E[G_j]}$ is denoted by $D_j$.

Let $H$ be a finitely generated subgroup of $G$ and let $N$ be the maximal convex subgroup of $H$. Let $h_1, \ldots, h_n \in H$ be in distinct cosets of $N$. We want to show that $\alpha(h_1), \ldots, \alpha(h_n)$ are $D_{N, D_{\omega}}$-linearly independent. Without loss of generality we will assume that $H = G$.

Let $L_j/M_j$ be the maximal convex subgroup of $G_j$ with respect to $\preceq_j$. By Proposition 2.1 since $\preceq_j$ is Conradian, there exists order-preserving homomorphism $\phi_j : G_j \to \mathbb{R}$ such that $\ker \phi_j = L_j/M_j$. Without loss of generality we see $\phi_j$ as an element of $H^1(F, \mathbb{R})$. We can multiply $\phi_j$ by a scalar in such way that $\max_{s \in S} |\phi_j(s)| = 1$. Let $\phi = \lim_\omega \phi_j \in H^1(F, \mathbb{R})$ and $L = \ker \phi$. Observe that $\phi$ is non-trivial, $M \leq \ker \phi$ and $\phi$ is order-preserving with respect to $\preceq$ if we consider it as a map $G \to \mathbb{R}$. In particular, $N = L/M$.

For each $i$ choose $f_i \in F$ such that $h_i = f_iM$. By way of contradiction, assume that there are $d_1, \ldots, d_n \in D_{N, D}$ such that

$$d_1 \alpha(f_1) + \ldots + d_n \alpha(f_n) = 0 \text{ in } D$$

with $d_i \neq 0$ for some $1 \leq i \leq n$.

Consider the subring $R$ of $D$ generated by $D_{[G, G], D}$ and $N$. It is a quotient of a crossed product $D_{[G, G], D} \ast (N/[G, G])$. Since $N/[G, G]$ is finitely generated abelian, $D_{[G, G], D} \ast (N/[G, G])$ is left and right Noetherian. Thus, $R$ is also left and right Noetherian. Since $R$ is a domain, $D_{N, D}$ is the classical division ring of fractions of $R$. Hence, without loss of generality we can assume that $d_i \in R$ in (2). Therefore, there are $f_{il} \in L$ and $d_{il} \in D_{[G, G], D}$ such that

$$d_i = \sum_l d_{il} \cdot \alpha(f_{il}).$$
Since \( h_1, \ldots, h_n \in H \) belong to distinct cosets of \( N \), all values \( \phi(f_1), \ldots, \phi(f_n) \) are distinct. Let \( \epsilon = \min_{i \neq i} |\phi(f_i) - \phi(f_i)| \). Since for all \( i, j, \phi(f_{ij}) = 0 \), we obtain that

\[
\{k \in \mathbb{N} : |\phi_k(f_i)| \leq \frac{\epsilon}{4} \text{ for all } i, l \text{ and } |\phi_k(f_j) - \phi_k(f_j)| \geq \frac{3\epsilon}{4} \text{ for all } i \neq j \} \in \omega.
\]

Thus, without loss of generality we assume that for every \( k \in \mathbb{N}, |\phi_k(f_{ii})| \leq \frac{\epsilon}{4} \) for all \( i, l \) and \( |\phi_k(f_{ij})| \geq \frac{3\epsilon}{4} \) for all \( i \neq j \).

Since \( d_{il} \in D_{[G,G],D} \), \( d_{il} \) are in the division closure of \( \alpha([F,F]) \). Therefore, we can write

\[
d_{il} = (d_{ilk})_k \text{ and } d_i = \left( \sum_l d_{ilk} \alpha_k(f_{il}) \right)_k \in \prod_{\omega} D_k, \text{ with } d_{ilk} \in D_{[G_i,G_j],D_j}.
\]

Since \( d_1 \alpha(f_1) + \ldots + d_n \alpha(f_n) = 0 \), we obtain that

\[
\{k \in \mathbb{N} : \sum_{i,j} d_{ilk} \alpha_k(f_{il} \cdot f_i) = 0 \} \in \omega.
\]

Thus, we can assume that \( \sum_{i,j} d_{ilk} \alpha_k(f_{il} \cdot f_i) = 0 \) for all \( k \in \mathbb{N} \). Observe that since \( |\phi_k(f_{il})| \leq \frac{\epsilon}{4} \) and \( |\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4} \), \( \phi_k(f_{il} \cdot f_i) \neq \phi_k(f_{il} \cdot f_j) \) if \( i \neq j \).

Since \( D_k \) is free with respect to \( \preceq_k \), this implies that for all \( i \),

\[
\left( \sum_l d_{ilk} \alpha_k(f_{il}) \right) \alpha_k(f_i) = \sum_l d_{ilk} \alpha_k(f_{il} \cdot f_i) = 0.
\]

Since this holds for all \( k, d_i = 0 \) for all \( i \). This shows that \( D \) is free with respect to \( \preceq \), and so it is Hughes-free by Proposition 2.4. This finishes the proof of Theorem 1.2.

**Proof Corollary 1.3.** Without loss of generality we may assume that \( G \) is finitely generated. Hence \( G \) is a limit of a collection of locally indicable amenable groups \( \{G_i\} \). Thus, by Theorem 1.3, for every division ring \( E \), there exists \( D_{E[G]} \). Moreover, since by Corollary 3.4, \( \text{rk}_{E[G_i]} \geq \text{rk}_{E} \) as Sylvester matrix rank functions on \( E[G_i] \), Theorem 1.3 also implies that \( \text{rk}_{E[G]} \geq \text{rk}_{E} \) as Sylvester matrix rank functions on \( E[G] \). Now, by Proposition 3.5, we obtain that \( D_{E[G]} \) is universal. \( \square \)

### 3.5. Examples of Lewin groups.

The following theorem shows that the groups that appear in Theorem 1.1 are Lewin.

**Theorem 3.7.** Let \( G \) be a locally indicable group.

1. If all finitely generated subgroups of \( G \) are Lewin, then \( G \) is also Lewin.
2. Any subgroup of a Lewin group is also Lewin.
3. \( G \) is Lewin if \( G \) has a normal Lewin subgroup \( N \) such that \( G/N \) is amenable and locally indicable.
4. Any limit in \( G_n \) of Lewin groups which is Hughes-free embeddable is Lewin.
5. A finite direct product of Lewin groups is Lewin.

**Proof.** The first statement follows directly from the definition of Lewin groups and the second one from Corollary 3.6. Let us prove now part (3).

First observe that \( G \) is Hughes-free embeddable by 13 (see also 26, Theorem 6.19). Let \( E \) be a division ring. Observe that the restriction of \( \text{rk}_{D_{E[G]}} \) on \( E[N] \) is
equal to \( \text{rk}_{\mathcal{D}_{E[N]}} \) and \( \mathcal{D}_{E[G]} \cong \mathcal{Q}(\mathcal{D}_{E[N]} \ast G/N) \) as \( \mathcal{E}[G] \)-rings. Thus, by Proposition 2.5(1), \( \text{rk}_{\mathcal{D}_{E[G]}} = \text{rk}_{\mathcal{D}_{E[N]}} \).

Denote by \( \text{rk}'_E \) the Sylvester matrix rank function on \( E[N] \) coming from the obvious map \( \mathcal{E}[N] \to \mathcal{E} \). Then, again by Proposition 2.5(1), we obtain that \( \text{rk}_{\mathcal{D}_{E[N]}} = \text{rk}_Q(\mathcal{E}[G/N]) = \text{rk}'_E \).

Since \( N \) is Lewin, \( \text{rk}_{\mathcal{D}_{E[N]}} \geq \text{rk}_E \), and so, \( \text{rk}_{\mathcal{D}_{E[N]}} \geq \text{rk}'_E \). Thus, \( \text{rk}_{\mathcal{D}_{E[G]}} \geq \text{rk}_{\mathcal{D}_{E[G/N]}} \) as Sylvester matrix rank functions on \( E[G] \). Since \( G/N \) is amenable and locally indicable, Corollary 3.4 implies that \( \text{rk}_{\mathcal{D}_{E[G/N]}} \geq \text{rk}_E \). Hence \( \text{rk}_{\mathcal{D}_{E[G]}} \geq \text{rk}_E \). Using Proposition 3.5 we obtain (3).

The fourth statement follows from Proposition 3.5 and Theorem 1.2.

Consider now the fifth claim. First let us prove that the direct product \( G = G_1 \times G_2 \) of two Lewin groups \( G_1 \) and \( G_2 \) is again Lewin. By \([13]\), \( G \) is Hughes-free embeddable. Let \( \mathcal{E} \) be a division ring. Consider the natural homomorphisms

\[
\phi_1 : \mathcal{E}[G] \to \mathcal{E}[G_1], \quad \phi_2 : \mathcal{E}[G_1] \to \mathcal{E} \quad \text{and} \quad \phi_3 = \phi_2 \circ \phi_1 : \mathcal{E}[G] \to \mathcal{E}.
\]

Since \( G_2 \) is Lewin,

\[
\text{rk}_{\mathcal{D}_{E[G_1][G_2]}} \geq \text{rk}_{\mathcal{D}_{E[G_1]}} \quad \text{in} \quad \mathcal{P}(\mathcal{D}_{E[G_1][G_2]}).
\]

Therefore, since \( \mathcal{D}_{E[G]} = \mathcal{D}_{\mathcal{D}_{E[G_1]}}[G_2] \),

\[
\text{rk}_{\mathcal{D}_{E[G]}} \geq \phi_1^\#(\text{rk}_{\mathcal{D}_{E[G_1]}}) \quad \text{in} \quad \mathcal{P}(\mathcal{E}[G]).
\]

Since \( G_1 \) is Lewin,

\[
\text{rk}_{\mathcal{D}_{E[G_1]}} \geq \phi_2^\#(\text{rk}_E) \quad \text{in} \quad \mathcal{P}(\mathcal{E}[G_1]).
\]

Hence, we conclude that

\[
\text{rk}_{\mathcal{D}_{E[G]}} \geq \phi_1^\#(\text{rk}_{\mathcal{D}_{E[G_1]}}) \geq \phi_1^\#(\phi_2^\#(\text{rk}_E)) = \phi_2^\#(\text{rk}_E) \quad \text{in} \quad \mathcal{P}(\mathcal{E}[G]).
\]

Since \( \mathcal{E} \) is arbitrary, applying Proposition 3.5 we obtain that \( G \) is Lewin. The case of two groups implies that (5) holds for an arbitrary finite product of Lewin groups.

\[\square\]

4. Universality of \( \text{rk}_G \)

As we have already mentioned in Introduction, when \( G \) is locally indicable \( \text{rk}_G = \text{rk}_{\mathcal{D}_{G[G]}} \). In this section we compare \( \text{rk}_G \) with other natural Sylvester matrix rank functions on \( \mathcal{C}[G] \).

4.1. The condition \( \text{rk}_G \geq \text{rk}_{\{1\}} \). In this subsection we will see several consequences of the condition \( \text{rk}_G \geq \text{rk}_{\{1\}} \). Recall that \( \text{rk}_{\{1\}} \) is an alternative expression for \( \text{rk}_C \) that has appeared in the previous sections. We start with the following useful proposition.

Proposition 4.1. Let \( H \) be a finitely generated group and assume that \( H \) is not indicable. Then \( \text{rk}_{\{1\}} \) is maximal in \( \mathcal{P}(\mathcal{Q}[H]) \). In particular, any group \( G \) for which \( \mathcal{Q}[G] \) has a universal division ring of fractions, is locally indicable.

Proof. Suppose that \( H \) has the following presentation.

\[ H = \langle x_1, \ldots, x_d | r_1, r_2, \ldots \rangle. \]
Reordering the relations \( \{ r_i \} \) of \( H \), without loss of generality we can assume that the abelianization of the group
\[
\widetilde{H} = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots, r_d \rangle
\]
is already finite.

Let \( F \) be a free group generated by \( x_1, \ldots, x_d \). For each \( 1 \leq i \leq d \), we write
\[
r_i - 1 = \sum_{j=1}^{d} a_{ij} (x_j - 1),
\]
where \( a_{ij} \in \mathbb{Z}[F] \). Let
\[
A = (a_{ij}) \in \text{Mat}_d(\mathbb{Z}[F]) \quad \text{and} \quad B = \begin{pmatrix} x_1 - 1 & \vdots & x_d - 1 \end{pmatrix} \in \text{Mat}_{d \times 1}(\mathbb{Z}[F]).
\]

Denote by \( \overline{A} \) and \( \overline{B} \) the matrices over \( \mathbb{Z}[H] \) obtained from \( A \) and \( B \), respectively, by applying the obvious homomorphism \( \mathbb{Z}[F] \to \mathbb{Z}[H] \). Since \( H \) has finite abelianization, we obtain that
\[
\text{rk}_{\{1\}}(A) = d - \dim \mathbb{Q} H_1(\widetilde{H}, \mathbb{Q}) = d.
\]
Let \( \text{rk} \in \mathbb{P}(\mathbb{Q}[H]) \) satisfy \( \text{rk} \geq \text{rk}_{\{1\}} \). In particular,
\[
\text{rk}(\overline{A}) \geq \text{rk}_{\{1\}}(\overline{A}) = \text{rk}_{\{1\}}(A) = d.
\]

Since \( AB = \begin{pmatrix} r_1 - 1 & \vdots & r_d - 1 \end{pmatrix} \), we obtain that \( \overline{A}\overline{B} = 0 \). Thus, by [14] Proposition 5.1(3), \( \text{rk}(\overline{B}) = 0 \). Therefore, \( \text{rk}(a) = 0 \) for every \( a \in I \), where \( I \) is the augmentation ideal of \( \mathbb{Q}[H] \). Since \( \mathbb{Q}[H]/I \) is a division ring and so it has only one Sylvester matrix rank function, \( \text{rk} = \text{rk}_{\{1\}} \). This shows the first part of the proposition.

Assume now that \( \mathbb{Q}[G] \) has a universal division ring of fractions \( \mathcal{D} \). Let \( H \) be a finitely generated subgroup of \( G \). If \( H \) is not indicable, then, as we have just proved, the restriction of \( \text{rk}_\mathcal{D} \) on \( \mathbb{Q}[H] \) is equal to \( \text{rk}_{\{1\}} \). Since \( \text{rk}_\mathcal{D} \) is faithful, \( H = \{1\} \). □

In the next proposition we will show that the condition \( \text{rk}_G \geq \text{rk}_{\{1\}} \) implies that \( \text{rk}_G \geq \text{rk}_{\mathcal{D}} \) for any amenable quotient \( \overline{G} \) of \( G \).

**Proposition 4.2.** Let \( G \) be a group and \( N \) a normal subgroup with \( G/N \) amenable. Let \( K \) be a subfield of \( \mathbb{C} \). Assume that \( \text{rk}_K \geq \text{rk}_{\{1\}} \) in \( \mathbb{P}(K[N]) \). Then \( \text{rk}_G \geq \text{rk}_{G/N} \) as Sylvester matrix rank functions on \( K[G] \).

**Proof.** By Proposition 2.3 \( \text{rk}_G \) is the natural extension of \( \text{rk}_N \) and \( \text{rk}_{G/N} \) is the natural extension of \( \text{rk}_{\{1\}} \). Since \( \text{rk}_N \geq \text{rk}_{\{1\}} \) in \( \mathbb{P}(K[N]) \), we obtain that \( \text{rk}_G \geq \text{rk}_{G/N} \) in \( \mathbb{P}(K[G]) \).

□

**Corollary 4.3.** Let \( G \) be a group and \( N \) a normal subgroup with \( G/N \) residually amenable. Let \( K \) be a subfield of \( \mathbb{C} \). If \( \text{rk}_G \geq \text{rk}_{\{1\}} \) in \( \mathbb{P}(K[G]) \), then \( \text{rk}_G \geq \text{rk}_{G/N} \) holds as well.

**Proof.** Without loss of generality we may assume that \( G \) is finitely generated. Then there exists a chain \( G = N_0 > N_1 > N_2 > \cdots \) of normal subgroups of \( G \) such that \( G/N_k \) is amenable and \( \cap N_k = N \). By [14] Theorem 1.3,
\[
\text{rk}_{G/N} = \lim_{k \to \infty} \text{rk}_{G/N_k} \quad \text{in} \quad \mathbb{P}(K[G]).
\]
By Proposition 4.2, \( \text{rk}_G \geq \text{rk}_{G/N_k} \) in \( \mathbb{P}(K[G]) \) for every \( k \). Hence \( \text{rk}_G \geq \text{rk}_{G/N} \) holds as well.

We conjecture that the corollary holds without the condition that \( G/N \) is residually amenable.

Conjecture 3. Let \( G \) be a group and let \( K \) be a subfield of \( \mathbb{C} \). Assume that \( \text{rk}_G \geq \text{rk}_{\{1\}} \) in \( \mathbb{P}(K[G]) \). Then \( \text{rk}_G \geq \text{rk}_{\mathbb{C}} \) in \( \mathbb{P}(K[G]) \) for any quotient \( \overline{G} \) of \( G \).

4.2. Proof of Corollary 1.5. It is clear that part (1) of Corollary 1.5 implies part (2). Kielak proved in [19] that in order to show (1), it is enough to prove that the first \( L^2 \)-Betti number of \( G \) is zero. Using Theorem 1.1, we will show that the condition (2) of Corollary 1.5 implies that the first \( L^2 \)-Betti number of \( G \) is zero.

First, let us recall the definition of RFRS groups. A group \( G \) is called residually finite rationally solvable or RFRS if there exists a chain \( G = H_0 > H_1 > \ldots \) of finite index normal subgroups of \( G \) with trivial intersection such that \( H_{i+1} \) contains a normal subgroup \( K_{i+1} \mid H_i \) satisfying that \( H_i/K_{i+1} \) is torsion-free abelian. The following proposition implies that RFRS groups are residually poly-\( \mathbb{Z} \).

Proposition 4.4. Let \( G \) be a finitely generated group, and let

\[
G = H_0 > H_1 > H_2 > \cdots > H_n > \cdots
\]

be a chain of finite index normal subgroups of \( G \) with \( \bigcap_{n=0}^{\infty} H_n = 1 \). Suppose that for every \( n \geq 0 \) there exists a subgroup \( K_{n+1} \mid H_n \) such that \( K_{n+1} \leq H_{n+1} \) and \( H_n/K_{n+1} \) is poly-\( \mathbb{Z} \). Then \( G \) is residually poly-\( \mathbb{Z} \).

Proof. A pro-\( p \) version of this result is proved in [17, Proposition 5.1]. The same proof works in our case. We include it for the convenience of the reader.

For \( n \geq 1 \) let

\[
\widetilde{K}_n = \bigcap_{g \in H/\langle H_{n-1} \rangle} gK_ng^{-1} \vartriangleleft G
\]

be the normal core of \( K_n \) in \( G \). Since the direct product of poly-\( \mathbb{Z} \)-groups is poly-\( \mathbb{Z} \) and a subgroup of a poly-\( \mathbb{Z} \)-group is poly-\( \mathbb{Z} \), the group \( H_{n-1}/\widetilde{K}_n \) is poly-\( \mathbb{Z} \) as well.

For every \( n \geq 1 \) set

\[
L_n = \bigcap_{i \leq n} \widetilde{K}_i \ vartriangleleft G
\]

and note that since \( \bigcap_{n=0}^{\infty} H_n = 1 \), this is a chain of subgroups that satisfies

\[
\bigcap_{n=1}^{\infty} L_n \subseteq \bigcap_{n=1}^{\infty} \widetilde{K}_n \subseteq \bigcap_{n=1}^{\infty} K_n \subseteq \bigcap_{n=1}^{\infty} H_{n-1} = 1.
\]

We shall argue, by induction on \( n \), that \( G/L_n \) is poly-\( \mathbb{Z} \). For \( n = 1 \) we have

\[
G/L_1 = G/\widetilde{K}_1 = H_0/\widetilde{K}_1 \text{ is poly-} \mathbb{Z}.
\]

Once \( n \geq 2 \) we have \( L_n = L_{n-1} \cap \widetilde{K}_n \), and by induction \( G/L_{n-1} \) is poly-\( \mathbb{Z} \). Thus, since an extension of two poly-\( \mathbb{Z} \) groups is poly-\( \mathbb{Z} \), it suffices to show that \( L_{n-1}/L_n \) is poly-\( \mathbb{Z} \). Indeed, since \( \widetilde{K}_{n-1} \leq H_{n-1} \), we have that

\[
L_{n-1}/L_n = L_{n-1}/L_{n-1} \cap \widetilde{K}_n \cong L_{n-1} \widetilde{K}_n/L_{n-1} \widetilde{K}_n \leq H_{n-1}/\widetilde{K}_n \text{ is poly-} \mathbb{Z}.
\]

Therefore, we conclude by recalling that a subgroup of a poly-\( \mathbb{Z} \)-group is poly-\( \mathbb{Z} \).
Now let us prove that the condition (2) of Corollary 1.5 implies that the first $L^2$-Betti number of $G$ is zero. Let $H$ be a subgroup of finite index such that there exists a normal subgroup $N$ of $H$ with $H/N \cong \mathbb{Z}$ and $H_1(N, \mathbb{Q})$ is finite-dimensional.

Assume that $H$ has the following presentation.

$$H = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots \rangle.$$  

Observe that $H_1(N, \mathbb{Q}) \cong H_1(H, \mathbb{Q}[H/N]).$

Let $F$ be a free group generated by $x_1, \ldots, x_d$ and consider $\mathbb{Q}[H/N]$ as an $F$-module. Then $H_1(F, \mathbb{Q}[H/N]) \cong \mathbb{Q}[H/N]^{d-1}$ as a $\mathbb{Q}[H/N]$-module. Since $\mathbb{Q}[H/N]$ is a PID, we can reorganize the relations $\{r_i\}$ and without loss of generality we can assume that $H_1(\hat{H}, \mathbb{Q}[\hat{H}/\hat{N}])$ is finite, where

$$\hat{H} = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots, r_{d-1} \rangle,$$

$\phi : \hat{H} \to H$ is the canonical surjection and $\hat{N} = \phi^{-1}(N).$

For each $1 \leq i \leq d - 1$, we write $r_i = 1 = \sum_{j=1}^{d} a_{ij}(x_j - 1)$, where $a_{ij} \in \mathbb{Z}[F]$. Let

$$A = (a_{ij}) \in \text{Mat}_{(d-1) \times d}(\mathbb{Z}[F])$$

and $B = \left( \begin{array}{cccc} x_1 - 1 \\ \vdots \\ x_d - 1 \end{array} \right) \in \text{Mat}_{d \times 1}(\mathbb{Z}[F]).$

Denote by $\overline{A}$ and $\overline{B}$ the matrices over $\mathbb{Z}[H]$ obtained from $A$ and $B$, respectively, by applying the obvious homomorphism $\mathbb{Z}[F] \to \mathbb{Z}[H]$. Since $H_1(\hat{H}, \mathbb{Q}[\hat{H}/\hat{N}])$ is finite, we obtain that

$$\text{rk}_{\mathbb{R}N}(\overline{A}) = \text{rk}_{\mathbb{R}N}(A) = \text{rk}_{\mathbb{R}\hat{N}}(A) = d - 1.$$

By Proposition 4.4 $H$ is residually poly-$\mathbb{Z}$. By Corollary 4.3 $\text{rk}_H \geq \text{rk}_{(1)}$ in $\mathbb{P}(\mathbb{Q}[H])$. Thus, by Corollary 4.3 $\text{rk}_H(\overline{A}) \geq \text{rk}_{\mathbb{R}N}(A) = d - 1$. Hence, since $H$ is infinite, the sequence

$$l^2(H)^d \xrightarrow{\partial_p} l^2(H)^d \xrightarrow{\phi} l^2(H) \to 0$$

is weakly exact. Therefore, the first $L^2$-Betti number of $H$ vanishes, and so the first $L^2$-Betti number of $G$ vanishes as well.

### 4.3. Proof of Corollary 1.6

Consider the cellular chain complex of $\tilde{X}$

$$C(\tilde{X}) : \ldots \rightarrow \mathbb{Z}[\mathbb{C}_{p+1}(\tilde{X})] \xrightarrow{\partial_{p+1}} \mathbb{Z}[\mathbb{C}_p(\tilde{X})] \xrightarrow{\partial_p} \mathbb{Z}[\mathbb{C}_{p-1}(\tilde{X})] \rightarrow \mathbb{Z}.$$  

Since $G$ acts freely on $\tilde{X}$ and $X = \tilde{X}/G$ is of finite type, we obtain that $\mathbb{Z}[\mathbb{C}_p(\tilde{X})] \cong \mathbb{Z}[G]^{n_p}$ is a free $\mathbb{Z}[G]$-module of finite rank and the connected morphisms $\partial_p$ are represented by multiplication by matrices $A_p$ over $\mathbb{Z}[G]$. Hence we obtain the following equivalent representation of $C(\tilde{X})$:

$$C(\tilde{X}) : \ldots \rightarrow \mathbb{Z}[G]^{n_{p+1}} \times A_{p+1} \mathbb{Z}[G]^{n_p} \times A_p \mathbb{Z}[G]^{n_{p-1}} \rightarrow \mathbb{Z}.$$  

Therefore, if $p \geq 1$ the $p$th Betti number of $X$ and the $p$th $L^2$-Betti number of $\tilde{X}$ can be expressed in the following way.

$$b_p(X) = n_p - (\text{rk}_{(1)}(A_p) + \text{rk}_{(1)}(A_{p+1}))$$

and $b_p^{(2)}(\tilde{X}) = n_p - (\text{rk}_G(A_p) + \text{rk}_G(A_{p+1})).$

Thus, Corollary 1.4 implies that $\text{rk}_{(1)}(A_1) = 1$ and $\text{rk}_{(1)}(A_1) = 0$. Therefore $b_1^{(2)}(\tilde{X}) \leq b_1(X) - 1.$
5. Appendix: The universal division ring of fractions of group rings of division rings and RFRS groups

In this section \( G \) is assumed to be a finitely generated RFRS group and \( E \) is a division ring. By Proposition 4.4 \( [1,3] \) \( G \) is residually poly-Z. Therefore, Corollary 1.3 implies that \( \mathcal{D}_{E[G]} \) exists and it is universal. In this section we will give an alternative description of \( \mathcal{D}_{E[G]} \) (see Theorem 5.10). Our proof follows essentially the argument of Kielak [19], where this description is done when \( E = \mathbb{Q} \).

5.1. Characters. A character of \( G \) is a homomorphism from \( G \) to the additive group of real numbers \( \mathbb{R} \). The set of characters \( \text{Hom}(G, \mathbb{R}) \) is denoted also by \( H^1(G; \mathbb{R}) \). A character \( \phi \) is called irrational if \( \ker \phi / [G,G] \) is a torsion group.

If \( H \) is a subgroup of finite index of \( G \) then the restriction map embeds \( H^1(G; \mathbb{R}) \) into \( H^1(H; \mathbb{R}) \). In what follows, we will often consider \( H^1(G; \mathbb{R}) \) as a subset of \( H^1(H; \mathbb{R}) \).

If \( H \) is a normal subgroup of \( G \) then \( G \) acts on \( H^1(H; R) \): for \( \phi \in H^1(H; R) \) and \( g \in G \), we denote by \( \phi^g \) the character that sends \( h \in H \) to \( \phi(ghg^{-1}) \).

Let \( G = H_0 > H_1 > H_2 > \ldots \) be a chain of subgroups of \( G \) of finite index and \( n \geq 0 \). For any \( U \subset H^1(H_n; \mathbb{R}) \) we denote
\[
U_n = U^o \quad \text{and} \quad U_{k-1} = (U_k)^o \cap H^1(H_{k-1}; \mathbb{R}) \quad \text{when} \quad 1 \leq k \leq n.
\]

We say that \( U \) is \( (G, \{H_i\}_{i \geq 0}) \)-rich if \( U_0 \) contains all the irrational characters of \( G \). When \( G \) and \( \{H_i\}_{i \geq 0} \) are clear from the context, we will simply say that \( U \) is rich.

Lemma 5.1. Let \( G = H_0 > H_1 > H_2 > \ldots \) be a chain of subgroups of \( G \) of finite index.

(1) If \( U \) is rich in \( H^1(H_n; \mathbb{R}) \) and \( g \in G \), then \( U^g \) is also rich.

(2) The intersection of two rich subsets of \( H^1(H_n; \mathbb{R}) \) is again rich.

Proof. Claim (1) is clear. Let us show the second claim.

First observe that if \( U \) and \( V \) are two open subsets of \( \mathbb{R}^k \), then
\[
(U \cap V)^o = (U)^o \cap (V)^o.
\]

Indeed, let \( x \in (U)^o \cap (V)^o \) and let \( O(x) \) be a neighborhood of \( x \) such that
\[
O(x) \subset U \cap V.
\]

Consider \( y \in O(x) \), and let \( O(y) \) be an arbitrary neighborhood of \( y \) such that
\[
O(y) \subset U \cap V.
\]

In particular, there exists \( z \in U \cap O(y) \). Recall that \( U \) is open. Consider an arbitrary neighborhood \( O(z) \) of \( z \) such that \( O(z) \subset U \cap V \). Since \( V \) is open, \( O(z) \cap U \cap V \) is not empty. Hence \( z \in U \cap V \), and so, \( y \in U \cap V \) as well. Thus, \( O(x) \subset U \cap V \) and \( x \in (U \cap V)^o \).

Now let \( U \) and \( V \) be two rich subset of \( H^1(H_n; \mathbb{R}) \) and let \( W = U \cap V \). We put
\[
U_n = U^o \quad \text{and} \quad U_{k-1} = (U_k)^o \cap H^1(H_{k-1}; \mathbb{R}) \quad \text{when} \quad 1 \leq k \leq n,
\]
and similarly we define \( V_k \) and \( W_k \).

Then we have that \( W_n = U_n \cap V_n \). Now, assume that we have proved that \( W_k = U_k \cap V_k \) for some \( k \leq n \). Then we obtain that
\[
W_{k-1} = (W_k)^o \cap H^1(H_{k-1}; \mathbb{R}) = (U_k \cap V_k)^o \cap H^1(H_{k-1}; \mathbb{R}) \subseteq U_{k-1} \cap V_{k-1}.
\]
In particular, $W_0$ contains all the irrational characters of $G$, and so, $W$ is rich. \hfill $\Box$

We will need the following criterion of richness.

**Lemma 5.2.** Let $G = H_0 > H_1 > H_2 > \ldots$ be a chain of subgroups of $G$ of finite index. Take non-negative integers $n \geq k \geq 0$. Let $U$ be an open subset of $H_1(H_k; \mathbb{R})$ and let $V$ be an open subset of $H_1(H_{n}; \mathbb{R})$. Assume that $U$ is rich and all the irrational characters of $U$ belong to $V$. Then $V$ is also rich.

**Proof.** We put $V_n = V^n$ and $V_{i-1} = (V_n)^o \cap H^1(H_{i-1}; \mathbb{R})$ when $1 \leq i \leq n$. Then by the inverse induction on $i$, we obtain that all the irrational characters of $U$ belong also to $V_i$ for $n \leq i \leq k$. Hence $\overline{U} \subseteq \overline{V_k}$. This clearly implies that $V$ is rich. \hfill $\Box$

5.2. **Novikov rings.** Let $S * G$ be a crossed product and let $\phi \in H^1(G, \mathbb{R})$. Denote by $\| \|$ a norm on $S * G$ defined by

\[ \| \sum_i s_i g_i \|_\phi = \max \{ 2^{-\phi(g_i)} : s_i \neq 0 \}. \]

Our convention is that $\| 0 \|_\phi = 0$. Let $\overline{S * G}^\phi$ be the completion of $S * G$ with respect to the metric induced by the norm $\| \|_\phi$. The ring $\overline{S * G}^\phi$ is called the **Novikov ring of** $R * G$ **with respect to** $\phi$.

Let $N = \ker \phi$. Then $\phi$ is also a character of $G/N$ and $\overline{S * G}^\phi$ is canonically isomorphic to $(\overline{S * N}) * G/N^\phi$. We will not distinguish between these two rings.

Any element of $\overline{S * G}^\phi$ can be represented in the following form $\sum_{i=1}^{\infty} a_i g_i$, where $a_i \in R * N$ and $\{ \phi(g_i) \}_{i \in \mathbb{N}}$ is an increasing sequence tending to the infinity.

We would like to construct an environment, where we can calculate the intersection $\overline{D_{E[G]}} \cap \overline{E[G]}^\phi$. In order to do this, consider the following commutative diagram of of injective homomorphisms of rings.

\[
\begin{array}{ccc}
E[G] & \hookrightarrow & \overline{D_{E[G]}} \\
\downarrow & & \downarrow \alpha_{G, \phi} \\
\overline{E[G]}^\phi & \hookleftarrow & \overline{D_{E[N]}} * G/N^\phi
\end{array}
\]

where the maps are defined as follows.

Notice that $\overline{D_{E[N]}} * G/N^\phi$ is a division ring and $\overline{D_{E[G]}}$ is the classical Ore ring of fractions of $\overline{D_{E[N]}} * G/N$. Therefore, the map $\alpha_{G, \phi}$ is the unique extension of the embedding

$\overline{D_{E[N]}} * G/N \hookrightarrow \overline{D_{E[N]}} * G/N^\phi$.

Since Hughes-free division ring is unique, for every subgroup $H$ of $G$, the division ring $\overline{D_{E[H]}}$ can be identified with the division closure of $E[H]$ in $\overline{D_{E[G]}}$. Thus, the ring $\overline{D_{E[N \cap H]}} * (H/(N \cap H))^\phi$ can be identified with the closure of

$\overline{D_{E[N \cap H]}} * (H/(N \cap H)) \cong \overline{D_{E[N \cap H]}} * (H/N) \subset \overline{D_{E[N]}} * G/N$

in $\overline{D_{E[N]}} * G/N^\phi$. Using this identifications, we obtain that $\alpha_{H, \phi}$ is the restriction of $\alpha_{G, \phi}$. Therefore, in the following we will simply write $\alpha_{\phi}$ instead of $\alpha_{G, \phi}$.
The map $\beta_{G,\phi}$ can be defined as the the continuous (with respect to $\| \cdot \|_\phi$) extension of the map

$E[G] = E[N] \ast G/N \hookrightarrow D_{E[N]} \ast G/N$.

Let $H$ be a normal subgroup of $G$ of finite index. Then the restriction of $\phi$ on $H$ is a character of $H$ and $E[H]_\phi$ can be identified with the closure of $E[H]$ in $E[G]_\phi$. It follows from the definitions that $\beta_{H,\phi}$ is the restriction of $\beta_{G,\phi}$ on $E[H]_\phi$. Thus, in the following we will simply write $\beta_\phi$ instead of $\beta_{G,\phi}$.

For any subset $S$ of $H^1(G, \mathbb{R})$ we put

\begin{equation}
D_{E[G],S} = \{ x \in D_{E[G]} : \alpha_\phi(x) \in \text{Im } \beta_\phi \text{ for every } \phi \in S \}.
\end{equation}

If $\phi \in H^1(G, \mathbb{R})$, we will simply write $D_{E[G],\phi}$ instead of $D_{E[G],\{\phi\}}$. Therefore, by our definition,

$D_{E[G],S} = \bigcap_{\phi \in S} D_{E[G],\phi}.$

**Proposition 5.3.** Let $H$ be a normal subgroup of $G$ of finite index and let $S$ be a subset of $H^1(G, \mathbb{R})$. Then $D_{E[H],S}$ is $G$-invariant and $D_{E[G],S}$ is equal to the ring generated by $D_{E[H],S}$ and $G$. In particular $D_{E[G],S}$ is a crossed product $D_{E[H],S} \ast G/H$.

**Proof.** It is clear that $D_{E[H],S}$ and $G$ are contained in $D_{E[G],S}$.

Now let $x \in D_{E[G],S}$. Let $Q$ be a transversal of $H$ in $G$. Since $D_{E[G]} = D_{E[H]} \ast G/H$, we can write

$x = \sum_{q \in Q} x_q q$

with $x_q \in D_{E[H]}$. We want to show that

\begin{equation}
x_q \in D_{E[H],S} \text{ for all } q \in Q.
\end{equation}

This will prove the proposition. Observe that this claim does not depend on the choice of $Q$, because $H \subset D_{E[H],S}$.

In order to prove (6), it is enough to show that for every $\phi \in S$, $x_q \in D_{E[H],\phi}$.

We will do it in two steps. Put $N = \text{ker } \phi$. In the first step we will assume that $G = HN$ and in the second that $N \leq H$. The combination of these two steps implies the general case of (6).

Thus, assume first that $G = HN$. Then, in this case, without loss of generality we can also assume that $Q \subset N$. Thus $Q$ is also a transversal of $N \cap H$ in $N$.

For each $r \in \text{Im } \phi$, choose, $h_r \in H$ such that $\phi(h_r) = r$. Then there are $r_1 > r_2 > r_3 > \ldots$ such that we can write

$\alpha_\phi(x_q) = \sum_{i=1}^{\infty} h_r, a_i, q$ with $a_i, q \in D_{E[N \cap H]}$.

Therefore, we obtain that

$\alpha_\phi(x) = \sum_{i=1}^{\infty} h_r, (\sum_{q \in Q} a_i, q)$.

Since $\alpha_\phi(x) \in \text{Im } \beta_\phi$, we obtain that for each $i \geq 1$,

$\sum_{q \in Q} a_i, q \in E[N]$. 

Therefore, for each \(i \geq 1\) and \(q \in Q\), \(a_{i,q} \in E[N \cap H]\). This implies, that \(\alpha_\phi(x_q) \in \text{Im} \beta_\phi\), and so, \(x_q \in \mathcal{D}_{E[H],\phi}\) for every \(q\).

The proof of the case \(N \subseteq H\) is left as an exercise.

Let \(H\) be a normal subgroup of finite index of \(G\) and let \(S\) be a subset of \(H^1(H, R)\). Then we put

\[
\mathcal{D}_{E[G],S} = \sum_{g \in G} \mathcal{D}_{E[H],S} g.
\]

In view of Proposition 5.3, this definition is coherent with the previous definition of \(\mathcal{D}_{E[G],S}\) in (8).

Observe that if \(S\) is \(G\)-invariant, then \(g^{-1} \mathcal{D}_{E[H],S} g \subseteq \mathcal{D}_{E[H],S}\) for all \(g\), and so, \(\mathcal{D}_{E[G],S}\) is equal to the subring of \(\mathcal{D}_{E[G]}\) generated by \(G\) and \(\mathcal{D}_{E[H],S}\). In this case \(\mathcal{D}_{E[G],S}\) has a structure of a crossed product \(\mathcal{D}_{E[H],S} \ast G/H\). For arbitrary \(S\), \(\mathcal{D}_{E[G],S}\) is not always a subring of \(\mathcal{D}_{E[G]}\).

Let \(\phi \in H^1(H, R)\). We denote by \(\phi^H\) the \(G\)-orbit in \(H^1(H, R)\). Then \(\mathcal{D}_{E[G],\phi}\) is a right \(\mathcal{D}_{E[G],\phi^G}\)-module. Let \(N = \ker \phi\). As in (4) we have

\[
\begin{align*}
E[H] & \leftrightarrow \mathcal{D}_{E[H],S} \left[ \phi^G \right] \\
\downarrow & \downarrow \\
\mathcal{D}_{E[H],S}^\phi & \left[ \phi^G \right] \hookrightarrow \mathcal{D}_{E[N],S}^\phi \ast H/N^\phi,
\end{align*}
\]

which induces another exact sequence

\[
\begin{align*}
E[G] & \hookrightarrow \mathcal{D}_{E[G]} \\
\downarrow & \downarrow \\
\mathcal{D}_{E[H],S}^\phi \otimes_{E[H]} E[G] & \hookrightarrow \mathcal{D}_{E[N],S}^\phi \ast H/N^\phi \otimes \mathcal{D}_{E[H],S}^\phi \mathcal{D}_{E[G],\phi^G},
\end{align*}
\]

where \(\tilde{\alpha}_\phi\) and \(\tilde{\beta}_\phi\) are homomorphisms of right \(E[G]\)-modules defined in the following way. Fix a right transversal \(Q\) of \(H\) in \(G\). Then \(\tilde{\beta}_\phi\) is defined on a basic tensor by

\[
\tilde{\beta}_\phi(b \otimes q) = \beta_\phi(b) \otimes q.
\]

In order to define \(\tilde{\alpha}_\phi\), we write an element \(a \in \mathcal{D}_{E[G]}\) as \(a = \sum_{q \in Q} a_q q\), with \(a_q \in \mathcal{D}_{E[H]}\), and define

\[
\tilde{\alpha}_\phi(a) = \sum_{q \in Q} \alpha_\phi(a_q) \otimes q.
\]

Observe that with this new notation we also have

\[
\mathcal{D}_{E[G],\phi} = \{ x \in \mathcal{D}_{E[G]} : \tilde{\alpha}_\phi(x) \in \text{Im} \tilde{\beta}_\phi \}.
\]

5.3. Continuity of \(|\cdot|_\phi\). Let \(\phi \in H^1(G, R)\) and \(x \in \mathcal{D}_{E[G]}\). Then we put

\[
|\cdot|_\phi = |\alpha_\phi(\cdot)|_\phi.
\]

**Proposition 5.4.** Let \(x \in \mathcal{D}_{E[G]}\). Then the map \(H^1(G; R) \to R\) defined by

\[
\phi \mapsto |x|_\phi
\]

is continuous.

**Proof.** Let \(G/K\) be the maximal torsion-free abelian quotient of \(G\). Let \(R\) be a subring of \(\mathcal{D}_{E[G]}\) generated by \(\mathcal{D}_{E[K]}\) and \(G\). Then the ring \(\mathcal{D}_{E[G]}\) is isomorphic to the classical Ore ring of fractions of \(R\). Thus, there are \(y \in R\) and \(0 \neq z \in R\) such that \(x = y z^{-1}\). Since \(|x|_\phi = |y|_\phi |z|_\phi^{-1}\), it is enough to prove the proposition in the case \(x \in R\). Thus, let us assume that \(x \in R\).
Let $A$ be a transversal of $K$ in $G$. Then we can write $x = \sum_{a \in A_0} x_a a$, where $A_0$ is a finite subset of $A$, and, for each $a \in A_0$, $x_a \in \mathcal{D}_{E[K]}$. Observe that
\[
\|x\|_\phi = \max\{\|a\|_\phi : a \in A_0\} = \max\{2^{-\phi(a)} : a \in A_0\}.
\]
This clearly implies that $\|x\|_\phi$ is a continuous function in $\phi$. 

5.4. Invertibility over Novikov rings. Let $H$ be a normal subgroup of $G$ of finite index and $\phi \in H_1(H; \mathbb{R})$. In this subsection we want to give a sufficient condition for $x \in \mathcal{D}_{E[G], \phi^G}$ to have its inverse in $\mathcal{D}_{E[G], \phi^G}$.

Let $G_0$ be a subgroup of $G$ containing $H$ and let $Q$ be a transversal of $H$ in $G_0$. Observe that
\[
\phi^{G_0} = \{\phi^g : g \in G_0\} = \{\phi^g : g \in Q\} = \phi^Q.
\]
We can decompose any $x \in \mathcal{D}_{E[G_0]}$ as $x = \sum_{q \in Q} x_q q$ with $x_q \in \mathcal{D}_{E[H]}$. The $(Q, \phi)$-norm of $x$ is defined by
\[
\|x\|_{\phi, Q} = \max\{\|x_q\|_{\psi, Q}\|q\|_{\phi, Q} : \psi \in \phi^Q, q \in Q\}.
\]
By the definition, $\| \|_{\phi, Q}$ has the following properties.

**Lemma 5.5.** Let $z_1, z_2 \in \mathcal{D}_{E[H]}$ and $q \in Q$. Then
\begin{enumerate}
\item $\|z_1 z_2 q\|_{\phi, Q} \leq \|z_1\|_{\phi, Q}\|z_2 q\|_{\phi, Q}$.
\item $\|z_1 q\|_{\phi, Q} = \|z_1\|_{\phi, Q}\|q\|_{\phi, Q}$.
\end{enumerate}

Observe that if $\phi \in H^1(G_0; \mathbb{R}) \subseteq H^1(H; \mathbb{R})$ is a restriction of some character of $G_0$, then $\|x\|_{\phi, Q} = \|x\|_{\phi}$, and so, in this case $\| \|_{\phi, Q}$ is multiplicative. However, if $\phi$ is an arbitrary character of $H_1(H; \mathbb{R})$, then $\| \|_{\phi, Q}$ is not multiplicative in general. This motivates the notion of the **defect of** $\| \|_{\phi, Q}$
\[
\text{def}_Q(\phi) = \max\{\|q_1 q_2\|_{\phi, Q} : q_1, q_2 \in Q\}.
\]
Observe that if $q_1 \in H$, then by Lemma 5.5, $\|q_1 q_2\|_{\phi, Q} = \|q_1\|_{\phi, Q}\|q_2\|_{\phi, Q}$. Thus, $\text{def}_Q(\phi)$ is always at least 1. We have the following consequence of Proposition 5.4.

**Corollary 5.6.** Let $H$ be a normal subgroup of finite index of $G$, $H \leq G_0 \leq G$ and $Q$ a transversal of $H$ in $G_0$. Let $x \in \mathcal{D}_{E[G_0]}$. Then the following functions on $H^1(H; \mathbb{R})$,
\[
\phi \mapsto \|x\|_{\phi, Q} \text{ and } \phi \mapsto \text{def}_Q(\phi),
\]
are continuous.

We will use the following properties of $\| \|_{\phi, Q}$.

**Proposition 5.7.** Let $H$ be a normal subgroup of finite index of $G$, $H \leq G_0 \leq G$ and $Q$ a transversal of $H$ in $G_0$. Let $\phi \in H^1(H; \mathbb{R})$. Then for every $w, z \in \mathcal{D}_{E[G_0]}$,
\[
\|z + w\|_{\phi, Q} \leq \max\{\|z\|_{\phi, Q}, \|w\|_{\phi, Q}\} \text{ and } \|z \cdot w\|_{\phi, Q} \leq \|z\|_{\phi, Q} \cdot \|w\|_{\phi, Q} \cdot \text{def}_Q(\phi).
\]

**Proof.** If $g \in G_0$, $\bar{g} \in Q$ be such that $Hg = H\bar{g}$. We write $z = \sum_{q \in Q} z_q q$ and $w = \sum_{q \in Q} w_q q$, with $z_q, w_q \in \mathcal{D}_{E[H]}$. Then
\[
z + w = \sum_{q \in Q} (z_q + w_q) q \text{ and } z \cdot w = \sum_{q \in Q} \sum_{q' \in Q} z_q (w_{q'})^{-1} q_1 q_2.
\]
Let $\psi \in \phi^Q$. Since $\|z_q + w_q\|_{\psi} \leq \max\{\|z_q\|_{\psi}, \|w_q\|_{\psi}\}$, we obtain that $\|z + w\|_{\phi, Q} \leq \max\{\|z\|_{\phi, Q}, \|w\|_{\phi, Q}\}$.

Observe that

$$\|z_q, (w_q)_{q^i}^{-1} q_1 q_2\|_{\phi, Q} \leq \|z_q\|_{\phi, Q} \|w_{q_2}\|_{\phi, Q} \|q_1 q_2\|_{\phi, Q} \leq \|z_q\|_{\phi, Q} \|q_1\|_{\phi, Q} \|w_{q_2}\|_{\phi, Q} \|q_2\|_{\phi, Q} \text{ def}_Q(\phi) \text{ Lemma 5.5}.$$ 

Therefore $\|z \cdot w\|_{\phi, Q} \leq \|z\|_{\phi, Q} \cdot \|w\|_{\phi, Q} \cdot \text{ def}_Q(\phi)$. \hfill $\Box$

**Corollary 5.8.** Let $H$ be a normal subgroup of finite index of $G$, $H \leq G_0 \leq G$ and $Q$ a transversal of $H$ in $G_0$. Let $\phi \in H^1(H, \mathbb{R})$ and let $w, y \in D_0[G_0, \phi].$ Assume that $w$ is invertible in $D_0[G_0, \phi]$ and

$$\|y\|_{\phi, Q} \cdot \|w^{-1}\|_{\phi, Q} < \text{ def}_Q(\phi)^{-2}.$$ 

Then $w + y \neq 0$ and $(w + y)^{-1} \in D_0[G_0, \phi].$

**Proof.** By Proposition 5.7,

$$(w + y)w^{-1} = 1 - z$$

with $\|z\|_{\phi, Q} < \text{ def}_Q(\phi)^{-1}$.

In particular $w + y \neq 0$.

Let us put $\epsilon = \|z\|_{\phi, Q} \text{ def}_Q(\phi)$. Then $\epsilon < 1$ and, by Proposition 5.7

$$\|z^n\|_{\phi, Q} \leq \frac{\epsilon^n}{\text{ def}_Q(\phi)}.$$ 

Thus, if we write

$$z^n = \sum_{q \in Q} z_{q, n} q,$$

then we obtain that for every $\psi \in \phi^Q$,

$$\|z_{q, n}\|_{\psi} \leq \|z^n\|_{\phi, Q} \frac{\epsilon^n}{\text{ def}_Q(\phi)}.$$ 

Consider

$$v = \sum_{q \in Q} \left(\sum_{n=0}^{\infty} z_{q, n}\right) \otimes q,$$

and observe that, by (10), $v \in \text{ Im } \beta_{\psi}$. On the one hand we have that

$$v(1 - z) = \left(\sum_{q \in Q} \left(\lim_{k \to \infty} \sum_{n=0}^{k} z_{q, n}\right) \otimes q\right) (1 - z) =$$

$$\left(\lim_{k \to \infty} \beta_{\psi} \left(\sum_{n=0}^{k} z^n\right)\right) (1 - z) = \lim_{k \to \infty} \beta_{\psi}(1 - z^{k+1}) = 1 \otimes 1.$$ 

On the other hand,

$$\alpha_{\psi}(1 - z)^{-1}(1 - z) = \alpha_{\psi}(1) = 1 \otimes 1.$$ 

Thus, $\alpha_{\psi}(1 - z)^{-1} = v$. By [9], we conclude that $(1 - z)^{-1} \in D_0[G_0, \phi]$, and so, $(w + y)^{-1} \in D_0[G_0, \phi]$. \hfill $\Box$
5.5. A description of $D_{E[G]}$. For any $x \in D_{E[G]}$ and any normal subgroup $H$ of finite index in $G$ we put

$$U_H(x) = \{ \phi \in H^1(H, \mathbb{R}) : x \in D_{E[G],\phi} \}.$$ 

Informally, $U_H(x)$ consists of the set of characters of $H$ such that $x$ can be represented as a matrix over $\overline{E[H]}^\phi$.

**Lemma 5.9.** Let $H_2 \leq H_1$ be two normal subgroups of $G$ of finite index. Let $A$ be a transversal of $H_1$ in $G$. Consider $x \in D_{E[G]}$ and write $x = \sum_{a \in A} x_a a$ with $x_a \in D_{E[H_1]}$. Then

$$U_{H_2}(x) = \bigcap_{a \in A} U_{H_2}(x_a).$$

**Proof.** Let $\phi \in H^1(H_2; \mathbb{R})$. By the definition,

$$D_{E[G],\phi} = \sum_{g \in G} D_{E[H_2],\phi g} \text{ and } D_{E[H_1],\phi} = \sum_{g \in H_1} D_{E[H_2],\phi g}.$$ 

Therefore, $D_{E[G],\phi} = \sum_{a \in A} D_{E[H_1],\phi a}$. Thus, $x \in D_{E[G],\phi}$ if and only if $x_a \in D_{E[H_1],\phi}$ for all $a \in A$. Hence, $U_{H_2}(x) = \bigcap_{a \in A} U_{H_2}(x_a)$. \qed

Since $G$ is RFRS, there exists a chain $G = H_0 > H_1 > \ldots$ of finite index normal subgroups of $G$ with trivial intersection such that $H_{i+1}$ contains a normal subgroup $K_i$ of $H_i$ satisfying $H_i/K_i$ is torsion free abelian. The chain $\{H_i\}$ satisfying this property is called **witnessing**. We fix a witnessing chain $\{H_i\}$ in $G$. Let $K_{E[G]}$ denotes the set of all $x \in D_{E[G]}$ such that for some $k \geq 0$, $U_{H_n}(x)$ is $(G, \{H_i\})$-rich for every $n \geq k$.

In this section we prove the following theorem. This is the main result of Appendix.

**Theorem 5.10.** We have that $K_{E[G]} = D_{E[G]}$.

First let us see that $K_{E[G]}$ is a subring of $D_{E[G]}$. Indeed, if $a, b \in K_{E[G]}$, using Lemma 5.1 we obtain that there exists $k \geq 0$ such that for every $n \geq k$ there is a $G$-invariant rich subset $U_n$ of $H^1(H_n; \mathbb{R})$ with $a, b \in D_{E[G],U_n}$. Since $D_{E[G],U_n}$ is a subring of $D_{E[G]}$, $a + b, ab \in D_{E[G]}$. Hence $K_{E[G]}$ a subring of $D_{E[G]}$.

Thus, in order to show that $K_{E[G]} = D_{E[G]}$, we have to prove that for any $0 \neq x \in K_{E[G]}$, $x^{-1} \in K_{E[G]}$. First let us consider the case where $x \in E[G]$.

**Proposition 5.11.** Let $x \in E[G]$. Then $x$ is invertible in $K_{E[G]}$.

**Proof.** Write $x = \sum_{g \in G} \alpha_g g$ and denote by $\text{supp } x = \{ g \in G : \alpha_g \neq 0 \}$. We will show that $x^{-1} \in K_{E[G]}$ by induction on $|\text{supp } x|$. The base of induction is clear. Let us assume that $|\text{supp } x| > 1$. There exists $k \geq 0$ such that

$$|\{ gH_k : g \in \text{supp } x \}| = 1 \text{ and } |\{ gH_{k+1} : g \in \text{supp } x \}| \geq 2.$$ 

Let $A$ be a transversal of $H_{k+1}$ in $H_k$. Hence, there exists $g \in G$ such that we can write

$$x = \sum_{a \in A} x_a ag, \text{ with } x_a \in E[H_{k+1}].$$
Since \( g, g^{-1} \in K_{E[G]} \), without loss of generality we may assume that \( g = 1 \). In particular, \( x \in E[H_k] \).

For each \( i \geq k \) we fix a transversal \( Q_i \) of \( H_i \) in \( H_k \). For any \( a \in A \) we put

\[
V_{i,a} = \{ \phi \in H^1(H_i, \mathbb{R}) : \| x - x_a \phi \|_{\phi, Q_i} \cdot \| (x_a)^{-1} \|_{\phi, Q_i} < \text{def}_{Q_i}(\phi)^{-2} \}.
\]

Let \( V_i = \bigcup_{a \in A} V_{i,a} \).

**Claim 5.12.** For each \( i \geq k \), the set \( V_i \) is rich in \( H^1(H_i, \mathbb{R}) \).

**Proof.** First observe that Corollary \( \text{C.5.6} \) implies that \( V_{i,a} \), and so \( V_i \) are open in \( H^1(H_i, \mathbb{R}) \). Let \( \phi \) be an irrational character of \( H^1(H_k, \mathbb{R}) \). Since \( \{Q_i\} \) is a witnessing chain and \( \phi \) is irrational, \( \ker \phi \leq H_{k+1} \). Therefore, there exists \( a \in A \) such that

\[
\| x - x_a \phi \|_{\phi, Q_i} = \| x - x_a \phi \|_{\phi} < \| (x_a)^{-1} \|_{\phi} = 1.
\]

Since \( \text{def}_{Q_i}(\phi) = 1 \), we obtain that \( \phi \in V_{i,a} \) for all \( i \geq k \), and so \( V_i \) contains all irrational characters of \( H_k \). Now the claim follows from Lemma \( \text{5.2} \). \( \square \)

By the inductive assumption, \( x_a \) is invertible in \( K_{E[G]} \). Thus, there exists \( n \geq k \) such that for every \( i \geq n \) and \( a \in A \), \( U_{H_i}(x_a^{-1}) \) is rich in \( H^1(H_i, \mathbb{R}) \). We put

\[
W_i = \bigcap_{q \in Q_i} \left( V_i \cap \bigcap_{a \in A} U_{H_i}((x_a)^{-1}) \right)^q.
\]

By Lemma \( \text{5.1} \) \( W_i \) is rich. Let \( \phi \in W_i \). Observe that \( W_i \) is \( H_k \)-invariant. Hence \( \phi_{Q_i} \subseteq V_i \cap \bigcap_{a \in A} U_{H_i}((x_a)^{-1}) \). There exists \( a \in A \) such that \( \phi \in V_{i,a} \). Observe that\( x - x_a, x_a, (x_a)^{-1} \in D_{E[H_k], \phi, Q_i} \). By Corollary \( \text{5.8} \) \( x^{-1} \in D_{E[H_k], \phi} \subset D_{E[G], \phi} \). Thus, \( W_i \subseteq U_{H_i}(x^{-1}) \) and we are done. \( \square \)

Now, we consider the general case.

**Proof of Theorem \( \text{5.10} \)** We will show that \( x^{-1} \in K_{E[G]} \) for every \( 0 \neq x \in K_{E[G]} \) by induction on the level \( l(x) \) of \( x \), that is defined as follows.

\[
l(x) = \min\{n - k : x \in D_{E[H_k]} \text{ and } U_{H_i}(x) \text{ is rich for every } i \geq n\}.
\]

Consider first the case \( l(x) \leq 0 \). Then \( x \in D_{E[H_k]} \) and \( U_{H_i}(x) \) is rich for every \( i \geq k \). Let \( H_k/K \) be the maximal torsion-free abelian quotient of \( H_k \). Let \( R \) be the subring of \( D_{E[H_k]} \) generated by \( D_{E[K]} \) and \( H_k \). Since \( D_{E[H_k]} \) is the classical ring of quotients of \( R \), we can write \( x = yz^{-1} \) with non-zero \( y, z \in R \). Let \( A \) be a transversal of \( K \) in \( H_k \). Then there are finite subsets \( A_0 \) and \( B_0 \) of \( A \) such that

\[
y = \sum_{a \in A_0} y_a a, \ z = \sum_{a \in B_0} z_a a \text{ with non-zero } y_a, z_a \in D_{E[K]}.
\]

Let \( \phi \) be an irrational character of \( H_k \). Observe that takes different values on the elements of \( A_0 \) and on the elements of \( B_0 \). Therefore, there are unique \( a_\phi \in A_0 \) and \( b_\phi \in B_0 \) such that

\[
\phi(a_\phi) = \min\{\phi(a) : a \in A_0\} \text{ and } \phi(b_\phi) = \min\{\phi(b) : b \in B_0\}.
\]

**Claim 5.13.** Let \( \phi \) be an irrational character of \( H_k \) and \( w = (y_{a_\phi} a_\phi)(z_{b_\phi} b_\phi)^{-1} \).

Then \( \| x \|_\phi = \| w \|_\phi > \| x - w \|_\phi \). Moreover, if \( x \in D_{E[H_k], \phi} \), then \( w \in E[H_k] \).
Proof. The claim follows directly from the definitions. □

Let

\[ T = \{ w_{a,b} = (y_{a,b})(z_{b})^{-1} : a \in A_0, b \in B_0 \} \cap E[H_k]. \]

Since \( T^{-1} \subseteq K_{E[G]} \) (Proposition 5.11), there exists \( n \) such that \( U_{H_i}(w^{-1}) \) is rich for every \( w \in T \) and \( i \geq n \).

For each \( i \geq n \) let \( Q_i \) be a transversal of \( H_i \) in \( H_k \). For each \( w \in T \) and \( i \geq n \) we put

\[ V_{i,w} = \{ \phi \in H^1(H_i; \mathbb{R}) : \| x - w \|_{\phi,Q} \cdot \| w^{-1} \|_{\phi,Q} < \text{def}_{Q_i}(\phi)^{-2} \} \]

and \( V_i = \bigcup_{w \in T} V_{i,w} \). Observe that \( V_i \) are open and if \( \phi \in H^1(H_k, \mathbb{R}) \), \( \text{def}_{Q_i}(\phi) = 1 \).

Thus, by Claim 5.13 for all \( i \geq n \), \( V_i \) contains all the irrational characters of \((U_{H_k}(x))^\phi \). Since \((U_{H_k}(x))^\phi \) is rich, Lemma 5.2 implies that \( V_i \) is rich for \( i \geq n \).

For each \( i \geq n \) we define

\[ W_i = \bigcap_{\phi \in Q_i} \left( V_i \cap U_{H_i}(x) \right) \bigcap_{w \in T} U_{H_i}(w^{-1}) \bigg)^q. \]

By Lemma 5.1 \( W_i \) is rich. Let \( \phi \in W_i \). Observe that \( W_i \) is \( H_k \)-invariant. Hence \( \phi Q_i \subseteq V_i \cap \bigcap_{w \in T} U_{H_i}(w^{-1}) \). There exists \( w \in T \) such that \( \phi \in V_{i,w} \). Observe that \( x - w, w, (w^{-1}) \in D_{E[H_k],\phi} \). By Corollary 5.8 \( x^{-1} \in D_{E[H_k],\phi} \subseteq D_{E[G],\phi} \). Thus, \( W_i \subseteq U_{H_i}(x^{-1}) \). Thus, \( x^{-1} \in K_{E[G]} \).

Now, we assume that \( l(x) > 0 \) and that the non-zero elements of \( K_{E[G]} \) of level less than that of \( l(x) \) are invertible in \( K_{E[G]} \). There are \( n \) and \( k \) such that \( l(x) = n - k \), \( x \in D_{E[H_k]} \) and \( U_{H_k}(x) \) is rich for every \( i \geq n \).

Let \( A \) be a transversal of \( H_{k+1} \) in \( H_k \). Hence, we can write

\[ x = \sum_{a \in A} x_a a, \quad \text{with} \quad x_a \in D_{E[H_{k+1}]} \]

By Lemma 5.9 for every \( a \in A \), \( x_a \in K_{E[G]} \) and \( l(x_a) < l(x) \).

For each \( i \geq k \) we fix a transversal \( Q_i \) of \( H_i \) in \( H_k \). For any \( a \in A \) we put

\[ V_{i,a} = \{ \phi \in H^1(H_i, \mathbb{R}) : \| x - x_a a \|_{\phi,Q_i} \cdot \| (x_a a)^{-1} \|_{\phi,Q_i} < \text{def}_{Q_i}(\phi)^{-2} \}. \]

Let \( V_i = \bigcup_{a \in A} V_{i,a} \). Arguing as in the proof of Claim 5.12, we obtain that all \( V_i \) are rich. By the inductive assumption, \( x_a a \) is invertible in \( K_{E[G]} \). Thus, there exists \( n \geq k \) such that for every \( i \geq n \) and \( a \in A \), \( U_{H_i}((x_a a)^{-1}) \) is rich in \( H^1(H_i, \mathbb{R}) \). We put

\[ W_i = \bigcap_{a \in A} \left( V_i \cap U_{H_i}(x) \cap \bigcup_{a \in A} U_{H_i}((x_a a)^{-1}) \bigg)^q. \]

By Lemma 5.1 \( W_i \) is rich. Let \( \phi \in W_i \). Observe that \( W_i \) is \( H_k \)-invariant. Hence \( \phi Q_i \subseteq V_i \cap \bigcup_{a \in A} U_{H_i}((x_a a)^{-1}) \). There exists \( a \in A \) such that \( \phi \in V_{i,a} \). Observe that \( x - x_a a, x_a a, (x_a a)^{-1} \in D_{E[H_k],\phi} \). By Corollary 5.8 \( x^{-1} \in D_{E[H_k],\phi} \subseteq D_{E[G],\phi} \). Thus, \( W_i \subseteq U_{H_i}(x^{-1}) \) and we are done. □
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