Supersymmetric Gauge Theories
with Classical Groups via M Theory Fivebrane

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Abstract

We study the moduli space of vacua of four dimensional $N = 1$ and $N = 2$ supersymmetric gauge theories with the gauge groups $Sp(2N_c)$, $SO(2N_c)$ and $SO(2N_c + 1)$ using the M theory fivebrane. Higgs branches of the $N = 2$ supersymmetric gauge theories are interpreted in terms of the M theory fivebrane and the type IIA $s$-rule is realized in it. In particular we construct the fivebrane configuration which corresponds to a special Higgs branch root. This root is analogous to the baryonic branch root in the $SU(N_c)$ theory which remains as a vacuum after the adjoint mass perturbation to break $N = 2$ to $N = 1$. Furthermore we obtain the monopole condensations and the meson vacuum expectation values in the confining phase of $N = 1$ supersymmetric gauge theories using the fivebrane technique. These are in complete agreement with the field theory results for the vacua in the phase with a single confined photon.
1 Introduction

Recently the D(ichlet)-brane in flat spacetime has provided very interesting tools to understand strong coupling dynamics of gauge theory in various dimensions. This approach was initiated in the work of Hanany and Witten in type IIB setup [1].

The four dimensional gauge theory is realized on the worldvolume of D branes which are suspended between the two NS 5-branes in type IIA string theory [2] [3]. It is shown that one can give an exact low energy description of the $N = 2$ $SU(N_c)$ gauge theory by reinterpreting the brane configuration from M theory point of view where the D4-branes and NS 5-branes of the type IIA theory are unified into the fivebrane of M theory [4]. In this framework, the Higgs phase of $N = 2$ $SU(N_c)$ gauge theory is investigated intensively [5] [6].

The $N = 1$ perturbation by the mass term of $\Phi$, which is the scalar chiral multiplet in the adjoint representation of the gauge group, induces the rotation of $N = 2$ fivebrane configuration [7]. In this way one can study the strong coupling dynamics of $N = 1$ $SU(N_c)$ gauge theory [5] [8]-[15]. Furthermore one can construct the fivebrane configuration that describes the $N = 1$ $SU(N_c)$ gauge theory with the superpotential $\Delta W = \sum \mu_{2k} \text{Tr} \Phi^{2k}/2k$ by imposing the appropriate boundary conditions on the fivebrane configuration [19]. The fivebrane contains the complete informations of the the monopole condensations and the meson vacuum expectation values (vev) in the confining phase of corresponding $N = 1$ supersymmetric gauge theory.

In this article we first study the Higgs phase of $N = 2$ gauge theory with the classical gauge groups in the fivebrane framework. To carry out this, we introduce the orientifold in M theory fivebrane configuration [3] [16]-[18]. We find the agreement between the field theory results of the Higgs phase of $N = 2$ gauge theory [20] [21] and the results from the fivebrane picture. Next we extend the analysis of [19] to the theory with the classical gauge groups. The monopole and meson vevs are computed using the fivebrane configuration in the vacua where photons are confined. We again find the agreement with the field theory results.

Section 2 is devoted to field theory analysis of the theory with the gauge group $Sp(2N_c)$. In particular we study the monopole and meson vevs of the $N = 1$ theory obtained
by adding to the $N = 2$ superpotential an $N = 1$ perturbation of the form $\Delta W = \sum \mu_{2k} \text{Tr} \Phi^{2k}/2k$. In section 3, we construct the fivebrane configuration describing the Higgs branches of the $N = 2$ supersymmetric QCD with the gauge group $Sp(2N_c)$. In section 4, we rotate the $N = 2$ fivebrane configuration corresponding to the adjoint mass perturbation in such a way that the resulting fivebrane configuration describes the $N = 1$ theory with the superpotential $\Delta W$. Then the fivebrane results are compared to those obtained in section 2. In sections 5 and 6, we consider the $SO(2N_c)$ and $SO(2N_c+1)$ gauge theories respectively. Finally we draw our conclusions.

2 Field theory analysis of $Sp(2N_c)$ gauge theory

Following the paper of [5] and [19], we consider the case of the gauge group $Sp(2N_c)$ in section 2,3 and 4. In this section the field theory analysis of the theory with the gauge group $Sp(2N_c)$ is performed. The results obtained in this section will be compared to the results from the point of view of M theory.

2.1 $N = 2$ Moduli space of vacua

Let us consider $N = 2$ supersymmetric gauge theory with the gauge group $Sp(2N_c)$ and $N_f$ quark hypermultiplets in the fundamental representation $Q^i_a$, $i = 1, \ldots, 2N_f$. In term of the $N = 1$ superfields, the vector multiplet consists of a field strength chiral multiplet $(W^a)_b$ and a scalar chiral multiplet $\Phi^b$ both in the adjoint representation of the gauge group. Here $a, b = 1, \ldots, 2N_c$ are color indices. The $2N_c \times 2N_c$ tensor $\Phi$ is subject to $t\Phi = J\Phi J$ with the symplectic metric $J = \text{diag}(i\sigma_2, \cdots, i\sigma_2)$ where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The $N = 2$ superpotential takes the form

$$W = \sqrt{2} Q^i_a J_{ab} \Phi_b^c Q^i_c + \sqrt{2} m^{ij} Q^i_a J_{ab} Q^j_b,$$

where $m$ is quark mass matrix. From the $N = 2$ supersymmetry we can take $m = \text{diag}(im_1\sigma_2, \cdots, im_N\sigma_2)$.

If the quark mass $m$ vanishes, the classical global symmetry groups are the flavor symmetry $O(2N_f) = SO(2N_f) \times \mathbb{Z}_2$ in addition to $SU(2)_R \times U(1)_R$ R-symmetry group. The theory is asymptotically free for $N_f < 2N_c + 2$. The $U(1)_R$ symmetry is anomalous.
and is broken down to $\mathbb{Z}_{2N_c+2-N_f}$ since the instanton factor is proportional to $\Lambda_{N=2}^{2N_c+2-N_f}$ where $\Lambda_{N=2}$ is the dynamically generated scale. Note that $\mathbb{Z}_2$ of the flavor symmetry is also broken down because the instanton factor changes its sign by the action of the $\mathbb{Z}_2$.

The moduli space of the vacua consists of the Coulomb and Higgs branches. The Coulomb branch is $N_c$ complex dimensional and is parametrized by the gauge invariant order parameters

$$u_{2k} = \left\langle \frac{1}{2k} \text{Tr} \left( \Phi^{2k} \right) \right\rangle, \quad k = 1, \ldots, N_c.$$  

(2)

The Coulomb branch parametrize a family of genus $N_c$ hyperelliptic curves [22]

$$xy^2 = \left( x B_{2N_c}(x, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \right)^2 - \Lambda_{N=2}^{2(2N_c+2-N_f)} \prod_{i=1}^{N_f} (x - m_i^2),$$  

(3)

where $B_{2N_c}$ is a degree $N_c$ polynomial in $x$ with coefficients that depend on the gauge invariant order parameters $u_{2k}$ and $m_i$. For $N_f < N_c + 2$ the polynomial $B_{2N_c}$ is given by

$$B_{2N_c}(x) = \sum_{i=0}^{N_c} s_{2i} x^{N_c-i},$$  

(4)

where $s_{2k}$ and $u_{2k}$ are related by the Newton formula

$$ks_{2k} + \sum_{j=1}^{k} js_{2(k-j)} u_{2j} = 0, \quad k = 1, 2, \ldots, N_c,$$  

(5)

where $s_0 = 1$. We also define $s_{2k} = 0$ for $k > N_c$, $k < 0$. From these we can see that the relation

$$\frac{\partial s_{2j}}{\partial u_{2k}} = -s_{2(j-k)},$$  

(6)

holds.

In the rest of this section we will consider the case with the vanishing bare quark mass. There is only one type of the gauge invariants which are constructed from $Q$, the meson fields $M^{ij} = Q^i J Q^j$. The Higgs branches are classified by an integer $r$ such that $0 \leq r \leq \min\{N_c, \frac{N_f}{2}\}$ [20]. The $r$-th Higgs branch has complex dimension $2r(2N_f - 2r - 1)$ and the root of this branch is its $N_c - r$ dimensional submanifold of intersection with the Coulomb branch. The effective theory along the root of the $r$-th Higgs branch is $Sp(2r) \times U(1)^{N_c-r}$ with $N_f$ massless quarks which are neutral with respect to any $U(1)$ factor. The curve at the $r$-th Higgs branch root takes the form

$$y^2 = x^{2r+1} \left( B_{2(N_c-r)}(x)^2 - \Lambda_{N=2}^{2(2N_c+2-N_f)} x^{N_f-2r-2} \right),$$

(7)
where $B_{2(N_c-r)}$ is a degree $(N_c - r)$ polynomial in $x$. At the special points along the root, additional massless hypermultiplets charged under the $U(1)$’s exist. In particular in the $r^*$-th Higgs branch where $r^* = N_f - N_c - 2$, the maximal singularity occurs at which the curve takes the form

$$y^2 = x^{2r^*+1} \left( x^{N_c-r^*} + \frac{1}{4} \Lambda^{2(N_c-r^*)} \right)^2,$$

and $N_c - r^*$ hypermultiplets become simultaneously massless.

### 2.2 Breaking $N = 2$ to $N = 1$

Next we consider the perturbation $\Delta W$ to the $N = 2$ superpotential (1)

$$W = \sqrt{2} Q J \Phi Q + \sqrt{2} m Q J Q + \Delta W,$$

where

$$\Delta W = \sum_{k=1}^{N_c} \mu_{2k} \frac{1}{2k} \text{Tr}(\Phi^{2k}),$$

which break the $N = 2$ supersymmetry to the $N = 1$ supersymmetry. If only the mass perturbation, say $\Delta W = \mu_2 \frac{1}{2} \text{Tr}\Phi^2$, is present, only the points in the $r^*$-th Higgs branch root at which the curve takes the form (8) are not lifted [20]. Moreover even in $N_f < N_c + 2$ case the point in the Coulomb branch where the curve degenerates to the genus zero curve such as $y^2 = x^{2n} \prod_{i=1}^{N_c-n} (x - a_i)^2(x - b)$ where $n$ is some integer is also expected to remain as vacua. Because if we take the limit $m_i \to \infty$ for all $i$ the theory becomes the pure Yang-Mills theory and the $N = 2$ curve degenerates to the genus zero curve at some discrete points in the moduli space of the pure Yang-Mills theory.

Now we consider $N = 2$ pure $Sp(2N_c)$ Yang-Mills theory perturbed by the superpotential $\Delta W$ ([1]). Near points where $N_c$ or less mutually local dyons are massless, the low energy effective superpotential is

$$W = \sqrt{2} \sum_{i=1}^{N_c} \tilde{M}_i A_i M_i + \sum_{k=1}^{N_c} \mu_{2k} U_{2k},$$

where $A_i$ are the $N = 1$ chiral superfields in the $N_c N = 2 U(1)$ vector multiplets, $M_i, \tilde{M}_i$ are the dyon hypermultiplets and $U_{2k}$ represent the superfields corresponding to Casimirs $\frac{1}{2k} \text{Tr}(\Phi^{2k})$ [23] [24]. We will use lower-case letters to denote the lowest components of the corresponding upper-case superfields.
As in the $SU(N_c)$ case [19], the holomorphy and global symmetries guarantee that the superpotential (11) is exact. The list of charges of the parameters and fields under $U(1)_R$ and $U(1)_J \subset SU(2)_R$ is given by

\[ \begin{array}{ccc}
U(1)_R & U(1)_J \\
A_i & 2 & 0 \\
\tilde{M}_i M_i & 0 & 2 \\
\mu_{2k} & 2 - 4k & 2 \\
U_{2k} & 4k & 0 \\
\Lambda_{N=2} & 2 & 0 \\
\end{array} \]  

(12)

We find the vacua of the low energy theory by imposing the vanishing D-term constraints $|m_i| = |\tilde{m}_i|$ and the equation of motion

\[ - \frac{\mu_{2k}}{\sqrt{2}} = \sum_{i=1}^{N_c} \frac{\partial a_i}{\partial u_{2k}} m_i \tilde{m}_i, \quad k = 1, \ldots, N_c, \]  

(13)

and

\[ a_i m_i = a_i \tilde{m}_i = 0, \quad i = 1, \ldots, N_c. \]  

(14)

This implies that the points of the moduli space of the vacua are lifted by the perturbation if there are no massless dyons.

Next we consider a point in the moduli space where $l$ mutually local dyons are massless. This means that some of the one-cycles shrink to zero and that the genus $N_c$ curve (3) degenerates to a genus $N_c - l$ curve. Thus the curve (3) takes the form

\[ y^2 = x B_{2N_c}(x)^2 + 2 B_{2N_c}(x) \Lambda_{N=2}^{2N_c+2} = \prod_{i=1}^{l} (x - p_i^2)^2 \prod_{j=1}^{2N_c+1-2l} (x - q_j^2) \]  

(15)

with $p_i$ and $q_j$ distinct. The equation of motion implies that $m_i = \tilde{m}_i = 0$ for $i = l + 1, \ldots, N_c$ while $m_i, \tilde{m}_i$ for $i = 1, \ldots, l$ are unconstrained. Since the matrix $\partial a_i/\partial u_{2k}$ is non-degenerate (see (17)) and there are $N_c - l$ free parameters in the form (15), there will be a complex $N_c - l$ dimensional moduli space of $N = 1$ vacua which remains after the perturbation.

The matrix $\partial a_i/\partial u_{2k}$ can be explicitly evaluated using the relation

\[ \frac{\partial a_i}{\partial s_{2k}} = \oint_{\beta_i} \frac{x^{N_c-k} dx}{y}, \]  

(16)
where the rhs is the integral of a holomorphic one form on the curve (3). At a point where the l dyons become massless the cycles $\beta_i \to 0$ ($i = 1 \ldots l$) and (16) becomes to a contour integrals around $x = p_i^2$ where $i = 1, \ldots, l$. Then we obtain

$$\frac{\partial a_i}{\partial s_{2k}} = \frac{p_i^{2(N_c - k)}}{\prod_{l \neq i} (p_i^2 - p_l^2) \prod_m (p_i^2 - q_m^2)^{1/2}}. \quad (17)$$

From (6), (13) and (17) we find the relation between the parameter $s_{2k}$ and the dyon vevs $m_i \tilde{m}_i$ as

$$\frac{-\mu_{2k}}{\sqrt{2}} = \sum_{i=1}^{l} \sum_{j=1}^{N_c} (-s_{2(j-k)}) p_i^{2(N_c-j)} \frac{m_i \tilde{m}_i}{\prod_{l \neq i} (p_i^2 - p_l^2) \prod_m (p_i^2 - q_m^2)^{1/2}}. \quad (18)$$

For later convenience we define

$$\omega_i = \sqrt{2} m_i \tilde{m}_i \prod_{l \neq i} (p_i^2 - p_l^2) \prod_m (p_i^2 - q_m^2)^{1/2}, \quad (19)$$

and rewrite the generating function $\sum_{k=1}^{N_c} \mu_{2k} v^{2k-1}$ for the $\mu_{2k}$ as the form

$$\sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} = \sum_{k=1}^{N_c} \sum_{l=1}^{N_c} \sum_{j=1}^{N_c} v^{2k-1} s_{2(j-k)} p_i^{2(N_c-j)} \omega_i$$

$$= \sum_{i=1}^{l} \sum_{j=1}^{N_c} B_{2N_c} (v^2) v^{2(j-N_c)-1} p_i^{2(N_c-j)} \omega_i + \mathcal{O}(v^{-1})$$

$$= v B_{2N_c} (v^2) \sum_{i=1}^{l} \frac{\omega_i}{(v^2 - p_i^2)} + \mathcal{O}(v^{-1}), \quad (20)$$

where we have set $x$ as $v^2$.

When perturbation parameters $\mu_{2k}$ are given and a point in the $N = 2$ moduli space of vacua is specified by the set $p_i, q_j$ of (15), the equation (20) determines whether this point remains as an $N = 1$ vacuum after the perturbation and fixes the vevs of the dyon fields $m_i \tilde{m}_i$. This is so since on the lhs of (20) we have $N_c$ couplings $\mu_{2k}$ ($k = 1, \ldots, N_c$), while on the rhs there are $l$ dyon condensates $\omega_i$ and $(N_c - l)$ independent parameters among $p_i$’s and $q_j$’s as is seen from (13).

We remark here that in the perturbation $\Delta W$ (10) we choose a particular basis of Casimirs $u_{2k}$. It may be allowed to take different basis $\tilde{u}_{2k} = \tilde{u}_{2k}(u_{2i})$. In fact this is an useful ambiguity as we will observe when discussing the meson vev in view of the confining
phase and the fivebrane in M theory. If we adopt the perturbation \( \Delta W = \sum_{k=1}^{N_c} \mu_{2k} \tilde{u}_{2k} \), We define \( \tilde{\mu}_{2k} \) from the equation
\[
\left[ \frac{\partial \Delta W}{\partial \Phi} \right] = \sum_{k=1}^{N_c} \mu_{2k} \sum_{n=1}^{N_c} \phi^{2n-1} \frac{\partial \tilde{u}_{2k}}{\partial u_{2n}} = \sum_{n=1}^{N_c} \tilde{\mu}_{2n} \phi^{2n-1}.
\]
(21)
Thus as in the previous considerations, the dyon condensation is expressed as
\[
- \frac{\mu_{2k}}{\sqrt{2}} = \sum_{i=1}^{N_c} \frac{\partial a_i}{\partial \tilde{u}_{2k}} m_i \tilde{m}_i, \quad k = 1, \ldots, N_c.
\]
(22)
And if we define \( w_i \) as Eq. (19), we obtain that
\[
\mu_{2k} = - \sum_{i=1}^{N_c} \sum_{j=1}^{N_c} \sum_{m=1}^{N_c} \frac{\partial s_{2j}}{\partial \tilde{u}_{2k}} p_i^{2(N_c-j)} w_i = \sum_{i=1}^{N_c} \sum_{j=1}^{N_c} \sum_{m=1}^{N_c} s_{2(j-m)} \frac{\partial u_{2m}}{\partial \tilde{u}_{2k}} p_i^{2(N_c-j)} w_i.
\]
(23)
Therefore we find that
\[
\sum_{n=1}^{N_c} \tilde{\mu}_{2n} v^{2n-1} = \sum_{k=1}^{N_c} \sum_{n=1}^{N_c} \mu_{2k} \frac{\partial \tilde{u}_{2k}}{\partial u_{2n}} v^{2n-1} = \sum_{i=1}^{N_c} \sum_{j=1}^{N_c} \sum_{m=1}^{N_c} s_{2(j-m)} v^{2n-1} p_i^{2(N_c-j)} w_i = v B_{2N_c} \left( \frac{\omega_i}{v^2 - p_i^2} \right) + O(v^{-1}),
\]
(24)
which is the same form as (20).

There are \( N_c + 1 \) points in the moduli space related to each other by the action of the discrete \( \mathbb{Z}_{2N_c+2} \) \( R \)-symmetry group (12), where \( N_c \) mutually local dyons are massless. At these points \( a_i = 0, i = 1, \ldots, N_c \) and the curve (3) degenerates to a genus zero curve. These points correspond to the \( N_c + 1 \) massive vacua of \( N = 1 \) pure Yang-Mills theory where the discrete \( \mathbb{Z}_{2N_c+2} \) \( R \)-symmetry is spontaneously broken to \( \mathbb{Z}_2 \). Equations (13) can be solved for generic \( \mu_{2k} \) and these \( N = 1 \) vacua are generically not lifted.

Now we introduce \( N_f \) flavors in the \( Sp(2N_c) \) gauge theory. As in the pure Yang-Mills case, the perturbation (14) lifts the non singular locus of the \( N = 2 \) Coulomb branch. The computation of the dyon vevs along the singular locus which does not become Higgs branch root is similar to the pure Yang-Mills case.

In the following we will compute the dyon vevs at the roots of the Higgs branches. The effective theory along the root of the \( r \)-th Higgs branch is \( Sp(2r) \times U(1)^{N_c-r} \) with \( N_f \) massless quark multiplets \( Q \). We assume here that the superpotential describing \( \tilde{M}_i, M_i \)
and the vector multiplet $A_i$ for each $U(1)$ factor at these points is

$$W = \sqrt{2} \sum_{i=1}^{N_c-r} \tilde{M}_i A_i M_i + \sum_{k=1}^{N_c-r} \mu_k U_k,$$

where we set $\mu_{2k} = 0$ ($k \geq N_c - r + 1$) as in [19]. Then from the equation of motion, most of the Higgs branch roots are lifted unless some of the hypermultiplets $\tilde{M}_i, M_i$ become massless. Thus we will consider the point where the hypermultiplets $\tilde{M}_i, M_i$ ($1 \leq i \leq l$) are massless. Taking the coordinate $\tilde{y} = \frac{y}{x^r}$, the curve (7) becomes the form

$$\tilde{y}^2 = x \left( B_2(N_c-r)(x)^2 - \Lambda_{N=2}^{2N_c+2-N_f} x^{N_f-2r-2} \right) = \prod_{i=1}^{l} (x - p_i^2)^{2(N_c-r-l)+1} \prod_{j=1}^{N_f} (x - q_j^2).$$

Repeat the analysis of the pure $Sp(2N_c-2r)$ Yang-Mills case with the relation

$$\frac{\partial a_i}{\partial s_{2k}} = \oint_{\beta_i} \frac{x^{N_c-k}dx}{y} = \oint_{\beta_i} \frac{x^{N_c-k-r}dx}{\tilde{y}},$$

then we obtain

$$\sum_{k=1}^{N_c-r} \mu_{2k} v^{2k-1} = 2H(v^2) \frac{v B_2(N_c-r)(v^2)}{\prod_{i=1}^{l}(v^2 - p_i^2)} + \mathcal{O}(v^{-1}).$$

### 2.3 Meson vev

We now compute the vev of the meson field $QJQ$ along the singular locus of the Coulomb branch. This vev is generated by the non perturbative dynamics of the $N = 1$ theory, and was clearly zero in the $N = 2$ theory before the perturbation (10).

Using the technique of the confining phase superpotentials [25]-[31], we can determine a low-energy effective superpotential for the phase with a confined photon. From that, we determine the meson vevs. We take a tree-level superpotential [29]

$$W = \sum_{n=1}^{N_c-1} \mu_{2n} u_{2n} + \mu_{2N_c}s_{2N_c} + \sqrt{2}QJ\Phi Q + \sqrt{2}mQJQ.$$  

Note here that we choose $s_{2N_c}$ instead of $u_{2N_c}$ as the top Casimir. Due to this modification, the theory with the superpotential (29) does not allow vacua with an unbroken $Sp(2) \times U(1)^{N_c-1}$ gauge symmetry classically and recovers the $N = 2$ curve correctly. Then we take the classical vacua with $Q = 0$ and an unbroken $SU(2) \times U(1)^{N_c-1}$ gauge symmetry which corresponds to the phase with a confined photon. Including the contributions from
the instanton effect in the broken part of the gauge group and the gaugino condensation in the low energy pure $SU(2)$ Yang-Mills theory, we obtain the effective superpotential

$$W(\mu_{2k}, m_i) = \sum_{n=1}^{N_c-1} \mu_{2n} u_{2n}^e + \mu_{2N_c} s_{2N_c}^e + \frac{\mu_{2N_c}}{p_1^2} \Lambda_{N=2}^{2N_c+2-2N_f} \left(i^{N_f} \prod_{i=1}^{N_f} m_i \pm \prod_{i=1}^{N_f} (p_1^2 - m_i^2)^{1/2} \right),$$

(30)

where $p_1^2 = \frac{\mu_{2(Nc-1)}}{\mu_{2N_c}}$ and the normalization of $\Lambda_{N=2}$ is fixed by the $N = 2$ curve (3). Note that $p_1$ is identical to the one in the degenerate curve (15) with $l = 1$. Here we assume that the effective superpotential (30) is exact [32] [25]. Defining $M^i$ as the $i$-th eigenvalue of the meson field $M$ and using $\sqrt{2} \langle M^i \rangle = \partial W(\mu_{2k}, m_i)/\partial m_i$, we find the meson vev

$$\langle M^i \rangle = \frac{\mu_{2N_c}}{\sqrt{2}p_1^2} \Lambda_{N=2}^{2N_c+2-2N_f} \left(i^{N_f} \prod_{k=1}^{N_f} m_k \pm \prod_{i=1}^{N_f} (p_1^2 - m_i^2)^{1/2} \right).$$

(31)

This method seems to apply for the phase with more confined photons. But for example in the case of the gauge group $SU(4)$ broken to $SU(3) \times U(1)$ or $SU(2) \times SU(2) \times U(1)$ classically, the low-energy effective superpotential obtained in this method can not derive the singular locus of the $N = 2$ theory. This can be seen from the explicit calculation. Furthermore we see that the meson vev computed from this low-energy effective superpotential is not correct for the case of the gauge group $SU(N_c)$ [19]. Thus for the phase with more confined photons, we expect this method is not reliable and we will not discuss the meson vev from field theory analysis. Later this will be computed from the fivebrane configuration in M theory.

3 $N = 2$ Higgs branch of $Sp(2N_c)$ theory and M theory

In this section we study the moduli space of vacua of $N = 2$ supersymmetric QCD with the gauge group $Sp(2N_c)$ and its deformations by the superpotential (10) by using the fivebranes in M theory.

Let us consider first the type IIA string theory on a flat space-time. Following the paper of [17], we consider the type IIA picture of the $N = 2$ gauge theory with the gauge group $Sp(2N_c)$. Consider the brane configuration which consists of two NS 5-branes with world-volume coordinates $x^0, x^1, x^2, x^3, x^4, x^5$ at $x^7 = x^8 = x^9 = 0$, $N_c$ D4-branes suspended
between them with worldvolume coordinates \(x^0, x^1, x^2, x^3, x^6\) at \(x^7 = x^8 = x^9 = 0\) and \(N_f\) D6-branes with worldvolume coordinates \(x^0, x^1, x^2, x^3, x^7, x^8, x^9\). In addition to this we introduce an orientifold four plane parallel to the D4-branes. Corresponds to this, we mod out the spacetime by the reflection \((x^4, x^5, x^7, x^8, x^9) \rightarrow (-x^4, -x^5, -x^7, -x^8, -x^9)\), together with the gauging of world sheet parity \(\Omega\). In order to obtain the \(Sp(2N_c)\) gauge group, the Chan-Paton wavefunction of the vector is taken to be symmetric. Each object which is not invariant under the reflection induced by the orientifold plane has a mirror partner. We take the configuration in which the all D6-branes have their mirror partners but the NS 5-branes do not have mirror partners since we will not move the NS 5-branes.

Another important aspect of the orientifold is its RR charge. In the normalization, where a D4-brane and its mirror partner carries one unit of this charge, the charge of the orientifold plane is 1 in the D4-brane sector. Note that if we take the \(SO\) gauge group, the charge of the orientifold plane is \(-1\).

Since the D4-branes are finite in the \(x^6\) direction, the effective world volume theory of the D4-branes becomes the four dimensional \(\mathcal{N} = 2\) supersymmetric QCD with the gauge group \(Sp(2N_c)\) and \(N_f\) flavors. The classical \(U(1)_R\) and \(SU(2)_R\) R-symmetry groups of the four-dimensional theory on the D4-branes worldvolume are interpreted as the rotations in the \(x^4, x^5\) and \(x^7, x^8, x^9\) directions.

The Coulomb branch of the theory is parametrized by the motions of D4-branes along the NS 5-branes. If some D4-branes lie on two or more D6-branes, we find that the phase of the theory can shift to the Higgs branch to break the D4-branes on the D6-branes and have them suspended between the D6-branes. Motions of the D4-branes along the D6-branes describe the Higgs branch. The location of a D4-brane between two D6-branes is parametrized by two complex parameters, the \(x^7, x^8, x^9\) coordinates together with the gauge field component \(A_6\) in the \(x^6\) coordinate.

The \(r\)-th Higgs branch corresponds to \(2(N_c - r)\) D4-branes suspended between the two NS 5-branes and \(2r\) D4-branes broken on the D6-branes. Since the \(s\)-rule \([\ref{footnote:brane-boundary}]\) does not allow more than one D4-brane to be suspended between a NS 5-brane and a D6-brane, \(r\) can not be greater than \([N_f/2]\). Since the Coulomb branch in the brane picture corresponds to D4-branes moving along the two NS 5-branes, the \(r\)-th Higgs branch shares \((N_c - r)\) complex dimensions with the Coulomb branch, corresponding to the the gauge
group $Sp(2(N_c - r))$.

The complex dimension of the $r$-th Higgs branch in the Higgs direction is determined by counting the number of the D4-branes suspended between the D6-branes [3]. If we ignore the effect of the orientifold, the configuration with the $2r$ D4-branes broken on the D6-branes is identical with the one appearing in the $2r$-th non-baryonic branch of the $SU(2N_c)$ gauge theory with $2N_f$ flavors. In this configuration the $j$-th D4-brane ($1 \leq j \leq 2r$) is broken on the $i$-th D6-branes ($j \leq i \leq 2N_f + 1 - j$) and the $k$-th D4-brane ($2r + 1 \leq k \leq 2N_f$) is not broken. Here the $(2n - 1)$-th and the $2n$-th D6-branes are interchanged under the reflection due to the orientifold. Now taking into account the effect of the orientifold, the components of the broken D4-branes must be paired. Then in this configuration the components of the D4-branes which are not paired are fixed on the orientifold and the components of the $j$-th pair of D4-branes ($1 \leq j \leq r$) are suspended between the $i$-th D6-branes where $2j \leq i \leq 2N_f + 1 - 2j$. Thus we obtain the dimension to be

$$2 \sum_{l=1}^{r} [2N_f - (4l - 3) - 2] = 2r(2N_f - 2r - 1)$$

in agreement with the field theory results.

Now we consider the effect of the intersection of branes and orientifold as in [39]. The charge of the orientifold is +1 in the part between the two NS 5-branes and it is −1 in the part outside the NS 5-branes. Following [39], we expect that between the two NS 5-branes there are two D4-branes whose degrees of freedom corresponding to the motion away from the orientifold are frozen. Furthermore these fixed D4-branes are expected to be absent between the mirror pair of the D6-branes. On the other hand in the case of the $SO(2N_c)$ gauge theory, the charge of the orientifold is −1 in the part between the two NS 5-branes and it is +1 in the part outside the NS 5-branes. Thus we expect that outside the NS 5-branes and between the mirror pair of the D6-branes, there are two fixed D4-branes.

In the above consideration of counting the dimension of the Higgs branch, we have thought that the components of the D4-branes between the $k$-th D6-brane and the $k+1$-th D6-brane are paired under the reflection. If $k$ is odd there remains a component of the D4-brane which is not paired. This remaining component has no degrees of freedom.
if fixed D4-branes exist we suppose that this remaining component of the D4-brane can be paired with the fixed D4-brane and may have the degrees of freedom. Although we have no convincing arguments about how this phenomenon occurs, we will assume this hereafter and see the validity of this assumption. In the case of the $Sp(2N_c)$ gauge theory, this assumption does not alter the above consideration of counting the dimension of the Higgs branch. However in the case of the $SO(1)$ gauge theory, we easily see by virtue of our assumption that the dimension of the Higgs branch is $2N_f$. This coincides with the dimension of the moduli space of vacua of the theory of $2N_f$ free chiral superfields which is equivalent to the $SO(1)$ gauge theory. Later we will see that the correct dimensions of the Higgs branches are obtained in the case of the $SO(n)$ gauge theory ($n > 1$) under our assumption. This consideration can also apply to counting the dimension of the Higgs phase of the $N = 1$ supersymmetric QCD with classical groups. The results are in agreement with the field theory results.

In [13] and [14] the points in the moduli space where all the eigenvalues of meson vev are equal after the $N = 1$ perturbation are interpreted as belonging to the $r$-th Higgs branch with $r = N_c$. In the light of the $SU(N_c)$ result based on the global symmetry [3] and the field theory result of the $Sp(2N_c)$ theory [20], however, we think it reasonable to identify these points with those in the Coulomb branch where the curve degenerates to genus zero curve as stated in subsection 2.2. As we shall see in the following our interpretation naturally fits into the M theory consideration.

### 3.1 Fivebrane configuration in M theory

Let us describe how the above type IIA brane configuration is embedded in M theory in terms of a fivebrane whose worldvolume is $\mathbb{R}^{1,3} \times \Sigma$. The curve $\Sigma$ is identified with the Seiberg-Witten curve that determines the solutions to Coulomb branch of the field theory. As usual, we write $s = (x^6 + ix^{10})/R$, $t = e^{-s}$ where $x^{10}$ is the eleventh coordinate of M theory which is compactified on a circle of radius $R$. Then the curve $\Sigma$ which describes the $N = 2$ $Sp(2N_c)$ gauge theory with $N_f$ flavors is given by an equation in $(v, t)$ space
\[ t^2 - 2 \left( v^2 B_{2N_c}(v^2, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} N_f \prod_{i=1}^{N_f} (v^2 - m_i^2) \right) t + \Lambda_{N=2}^{2(2N_c+2-N_f)} N_f \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0, \quad (33) \]

where \( B_{2N_c}(v^2, u_{2k}) \) is a degree 2\( N_c \) polynomial in \( v \) which is even under \( v \to -v \), and the coefficients depend on the moduli \( u_{2k} \) as well as the quark mass \( m_i \).

In M theory, the D6-branes are Kaluza-Klein monopoles described by a multi-Taub-NUT space \([33]\). To construct the Seiberg-Witten curves we do not need the full details of the metric of this multi-Taub-NUT space but only a description of the space in terms of its one complex structure. Such a description has been constructed in \([34], [17]\) and the result is

\[ tz = \Lambda_{N=2}^{2(2N_c+2-N_f)} N_f \prod_{i=1}^{N_f} (v^2 - m_i^2) \quad (34) \]

in \( \mathbb{C}^3 \) for \( Sp(2N_c) \) where one of the coordinates of \( \mathbb{C}^3 \) is taken to be \( t \) of the curve (3) for simplicity. The Riemann surface \( \Sigma \) is embedded as a curve in this surface and given by

\[ t + z = 2 \left( v^2 B_{2N_c}(v^2, u_{2k}) + \Lambda_{N=2}^{2N_c+2-N_f} N_f \prod_{i=1}^{N_f} (v^2 - m_i^2) \right). \quad (35) \]

The surface (34) becomes singular when some \( m_i \) take the same value. Physically this corresponds to the coincident D6-branes in the \( x^4, x^5 \) directions but they can be separated in the \( x^6 \) direction \([1]\). The separations of the D6-branes in the \( x^6 \) direction correspond to the resolution of the singularities \([4]\).

When all bare masses are turned off, the surface (34) develops singularities of type \( A_{2N_f-1} \) at the point \( t = z = v = 0 \). By succession of blowing ups, we obtain a smooth complex surface which isomorphically maps onto the singular surface (34) except at the inverse image of the singular points. Over each singular point, there exist \( 2N_f - 1 \) rational curves \( \mathbb{CP}^1 \)'s called the exceptional curves on this smooth surface. We denote the rational curves by \( C_1, C_2, \cdots, C_{2N_f-1} \). This resolved surface is described as follows. It is covered by \( 2N_f \) complex planes \( U_1, U_2, \cdots, U_{2N_f} \) with coordinates \( (t_1 = t, z_1), (t_2, z_2), \cdots, (t_{2N_f}, z_{2N_f} = z) \) which are mapped to the singular surface.

\(^1\)I would like to thank K. Hori for his pointing out an error in our original argument based on \([13]\).
by

\[ U_i \ni (t_i, z_i) \mapsto \begin{cases} 
  t = t_i^i z_i^{i-1} \\
  z = t_i^{2N_f-i} z_i^{2N_f+1-i} \\
  v = t_i z_i 
\end{cases} \]  

(36)

The planes \( U_i \) are glued together by \( z_i t_{i+1} = 1 \) and \( t_i z_i = t_{i+1} z_{i+1} \). We define the exceptional curve \( C_i \) by the locus of \( t_i = 0 \) in \( U_i \) and \( z_{i+1} = 0 \) in \( U_{i+1} \). The spacetime reflection due to the orientifold can be extended to the resolved surface by considering the action \( t_i \rightarrow (-1)^{i+1} t_i, z_i \rightarrow (-1)^i z_i \). Under the reflection, \( C_{2n-1} \) parametrized by \( z_{2n-1} \) is rotated while \( C_{2n} \) is invariant. The positions of the D6-branes may be interpreted as the \( 2N_f \) intersection points of the exceptional curves.

### 3.2 Higgs Branch

Now we would like to study the Higgs branch when all the bare masses are turned off. In M theory, the Higgs branch appears when the fivebrane intersects with the D6-branes. This is possible only when the image \( \text{(33)} \) in the \( t-z-v \) space of the curve passes through the singular point \( t = z = v = 0 \). Thus, we must have \( v^2 B_{2N_c}(v^2) = 0 \) at \( v = 0 \) but this condition is always satisfied due to the vanishing bare mass. However the extra \( v^2 \) factor has its origin in the bending and connecting fivebrane at \( v = 0 \) by the effect of the charge of the orientifold \( \text{(37)} \). Therefore we must have \( B_{2N_c}(v^2) \) in the factorized form

\[ B_{2N_c}(v^2) = v^{2r}(v^{2(N_c-r)} + s_2 v^{2(N_c-r-2)} + \cdots + s_2(N_c-r)) , \]  

(37)

where \( r > 0 \).

We will examine the curve by separating it into the one near the singularity \( t = z = v = 0 \), and the one away from it. Away from the singular point \( t = z = v = 0 \), we can consider the curve as embedded in the original \( t-z-v \) space because there is no distinction from the resolved surface in this region. Because \( v \) never vanishes in this region of the curve, we can safely divide the coordinates \( t \) and \( z \) by some power of \( v \). If \( 2r \leq N_f \) they can be divided by \( v^{2r} \), and we see that this piece of the curve is equivalent with the generic curve for the \( Sp(2(N_c-r)) \) gauge theory with \( (N_f-2r) \) flavors and thus has genus \( (N_c-r) \).

Near \( t = z = v = 0 \), we should consider the resolved \( A_{2N_f-1} \) surface. Thus, we must describe the curve in the \( 2N_f \) patches as described above. It is useful to remark that the
higher order terms $v^{2(r+2)}, v^{2(r+3)}, \ldots$ are negligible near $v = 0$ compared to $v^{2(r+1)}$. Thus we can replace the defining equation $t + z = v^2 v^{2r} (s_{2(N_r-r)} + \cdots) = 0$ by $t + z = v^2 v^{2r}$ where we set $s_{2(N_r-r)} = 1$ for notational simplicity. On the $i$-th patch $U_i$, the equation of the curve $\Sigma$ becomes
\begin{equation}
t_i^i z_i^{i-1} + t_i^{2N_f-i} z_i^{2N_f+1-i} = t_i^r z_i^r (t_i z_i)^2.
\end{equation}

Thus we have this equation factorize as
\begin{align*}
t_i v_i^{i-1} (1 + v_i^{2N_f-2i} z_i^2 - v_i^{2r-i} z_i (v_i)^2) = 0, & \quad i = 1, \ldots, 2r \\
v_i^{2r} (t_i v_i^{i-2r-1} + v_i^{2N_f-2r-i} z_i - (v_i)^2) = 0, & \quad i = 2r + 1, \ldots, 2N_f - 2r \\
v_i^{2N_f-i} z_i (t_i^r v_i^{2i-2N_f-2} + 1 - t_i v_i^{i-2N_f+2r-1} (v_i)^2) = 0, & \quad i = 2N_f - 2r + 1, \ldots, 2N_f.
\end{align*}

where we define $v_i = t_i z_i$. However we can further factor out $t_i, t_i^2 z_i$, $(t_i z_i)^2$, $t_i z_i^2$ and $z_i$ for $i = 2r + 1, i = 2r + 2, 2r + 3 \leq i \leq 2N_f - 2r - 2, i = 2N_f - 2r - 1$ and $i = 2N_f - 2r$ respectively from (39). These factors are interpreted as the fivebrane fixed at $v = 0$, which may have no contribution to the dimension of the Higgs branch.

In addition to the above components, the curve consists of several components which correspond to the D4-branes as a consequence of the factorized form (39). One component, which we call $C$, is the zero of the last factor of (39). This extends to the one in the region away from $t = z = v = 0$ which we have already considered. The other components are the rational curves $C_1, \ldots, C_{2N_f-1}$ with multiplicities. As is evident from the factorized form (39), the component $C_i$ has multiplicity $\ell_i$ where $\ell_i = i$ for $i = 1, \ldots, 2r$, $\ell_i = 2r$ for $i = 2r + 1, \ldots, 2N_f - 2r - 1$, and $\ell_i = 2N_f - i$ for $i = 2N_f - 2r, \ldots, 2N_f - 1$. Note that the component $C$ intersects with $C_{2r+2}$ and $C_{2N_f-2r-2}$ and the unmoving $\mathbb{CP}^1$ components are aligned from $C_{2r+1}$ to $C_{2N_f-2r-1}$. We also note that the branes configuration in the type IIA theory with the $s$-rule is verified by the above M theory fivebrane configuration.

To count the dimension of the Higgs branch, remember that once the curve degenerates and $\mathbb{CP}^1$ components are generated, they can move in the $x^7, x^8, x^9$ directions [4]. This motion together with the integration of the chiral two-forms on such $\mathbb{CP}^1$'s parameterizes the Higgs branch of the four-dimensional theory. As in the type IIA picture, taking into account the orientifolding, we expect that $\mathbb{CP}^1$'s move in pairs. The $\mathbb{CP}^1$'s which can not become pairs are fixed. Thus we find that the complex dimension of the $r$-th Higgs
branch is
\[ 2 \sum_{i=1}^{2N_f-1} \left[ \frac{l_i}{2} \right] = 4 \left( 2 \sum_{i=1}^{r-1} i + r \right) + 2r(2N_f - 1 - 4r) = 2r(2N_f - 2r - 1) \] (40)
in agreement with (32).

For the case \( N_f \geq N_c + 2 \), there is a special point in the Higgs branch root in which the maximal massless dyons appear \[20\]. This special point in the Higgs branch root is in the \( r^* \)-th Higgs branch root where \( r^* = N_f - N_c - 2 \) and takes the color Casimirs such that
\[ B_{2N_c}(u_{2N_c}, v^2) = v^{2N_c} + \frac{1}{4} v^{2r^*} \Lambda_{N=2}^{2(N_c-r^*)} = v^{2r^*} \left( v^{2(N_c-r^*)} + \frac{1}{4} \Lambda_{N=2}^{2(N_c-r^*)} \right). \] (41)
Then at this point the equation (35) factorizes as
\[ \frac{1}{t} \left( t - 2v^{2(N_c+1)} \right) \left( t - \frac{1}{2} \Lambda_{N=2}^{2(N_c-r^*)} v^{2(r^*+1)} \right) = 0. \] (42)
Namely, the infinite curve \( C \) factorizes into two rational curves \( C_L \) and \( C_R \) corresponding to \( t = 2v^{2(N_c+1)} \) and \( t = \frac{1}{2} \Lambda_{N=2}^{2(N_c-r^*)} v^{2(N_f-N_c-1)} \) respectively. We note that these curves are invariant under the orientifold projection \( v \to -v, t \to t \). It is also noted that the two curves \( C_L \) and \( C_R \) intersect at \( (2N_c + 2 - N_f) \) points.

4 \( N = 1 \) moduli space of \( Sp(2N_c) \) theory from M theory

In this section we deform the fivebrane configuration in M theory for \( N = 2 \) field theory to the one for \( N = 1 \) field theory.

First we will study the configuration of fivebrane in M theory which corresponds to adding only a mass term to the adjoint chiral multiplet in the \( N = 2 \) vector multiplet. In the Type IIA picture, this corresponds to rotating the one of the NS 5-branes.

Next we will consider the configuration of fivebrane in M theory which corresponds to more general perturbation \( (10) \). In the Type IIA picture, this corresponds to adding plural NS 5-branes and changing the relative orientation of the NS 5-branes.
4.1 Rotated configurations

Here we study the adjoint mass perturbation. Let us introduce a complex coordinate

$$w = x^8 + ix^9.$$  \hfill (43)

Before breaking the $N = 2$ supersymmetry, the fivebrane is located at $w = 0$. Now we rotate only the left NS 5-brane toward the $w$ direction. From the behavior of two asymptotic regions which correspond to the left and right NS 5-brane with $v \to \infty$, this rotation leads to the boundary conditions

$$w \to \mu v \text{ as } v \to \infty, \quad t \sim v^{2N_c+2},$$

$$w \to 0 \text{ as } v \to \infty, \quad t \sim \Lambda_{N=2}^{2(N_c+N_f)-2} v^{2(N_f-N_c-1)}. \hfill (44)$$

We can identify $\mu$ as the mass of the adjoint chiral multiplet by using the $R$-symmetries.

Such rotation is only possible at points in the moduli space at which all $\beta$-cycles on the curve $\Sigma$ are degenerate. This is possible at the special points in the Coulomb branch or in the $r^*$-th Higgs branch root. The possible fivebrane configuration which corresponds to the points in the Coulomb branch is studied in \[13\].

Here we consider the fivebrane configuration corresponding to the special point in the $r^*$-th Higgs branch root which plays the important role in the arguments of demonstrating the $N = 1$ Non Abelian duality \[20\].

The rotation of the curve at this point is straightforward. The component $C$ at this point factorizes into two pieces — $C_L$ described by $t = 2v^{(N_c+1)}$, $w = 0$ and $C_R$ described by $t = \frac{1}{2} \Lambda_{N=2}^{2(N_c-N_f+2)} v^{2(N_f-N_c-1)}$, $w = 0$. The rotation can be done just by replacing $w = 0$ for $C_L$ by $w = \mu v$. The curve is explicitly given by

$$\tilde{C}_L \left\{ \begin{array}{l} t = 2v^{(N_c+1)} \\ w = \mu v \end{array} \right., \quad C_R \left\{ \begin{array}{l} t = \frac{1}{2} \Lambda_{N=2}^{2(N_c-N_f+2)} v^{2(N_f-N_c-1)} \\ w = 0. \end{array} \right. \hfill (45)$$

Note that the curves (45) are invariant under the orientifold reflection $v \to -v, t \to t, w \to -w$.

We can take the limit $\mu \to \infty$ which corresponds to the $N = 1$ supersymmetric QCD in view of field theory. Following \[13\], we should rescale $t$ by a factor $\mu^{2N_c+2}$ and introduce a new variable

$$\bar{t} = \mu^{2(N_c+1)} t. \hfill (46)$$
Using the rescaled variable, the spacetime is described by

\[
\tilde{t}z = \mu^{2(N_c+1)} \Lambda_{N=2}^{4N_c+4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2).
\] (47)

This equation defines a smooth surface in the limit \( \mu \to \infty \) provided the constant \( \Lambda_{N=1} \) given by

\[
\Lambda_{N=1}^{2(3(N_c+1)-N_f)} = \mu^{2(N_c+1)} \Lambda_{N=2}^{2(2N_c+2-N_f)}
\] (48)

which is kept finite. This relation is the same as the renormalization matching condition of the corresponding field theory.

We can take the limit \( \mu \to \infty \) of the curves of the remaining Higgs branch root (45) and the curves become

\[
C_L \begin{cases} \tilde{t} = 2w^{2(N_c+1)} \\ v = 0 \end{cases} \quad C_R \begin{cases} \tilde{t} = \frac{1}{2} \Lambda_{N=1}^{2(3(N_c+1)-N_f)} v^{2(N_f-N_c-1)} \\ w = 0. \end{cases}
\] (49)

Note that in this limit the extra rational curves may appear and the dimension of the Higgs branch is increased as in field theory.

### 4.2 Brane configuration of more general \( N = 1 \) theory

Let us now consider the configuration of fivebrane in M theory which corresponds to more general perturbation of the form \( \Delta W \).

In the type IIA picture, this corresponds to the configuration consists of \( N_c \) NS' 5-branes and their mirror pairs located right to a NS 5-brane which does not have mirror pair. We will take the \( N_c \) NS' 5-branes to stretch in the \((v, w)\) coordinates. In field theory the eigenvalues of the vev of \( \Phi \) are labeled by the minima of the superpotential classically and they label the separation of the NS' branes in the \( v \) direction.

Next we will construct the configuration of M theory fivebrane which corresponds to the perturbation \( \Delta W \). The left NS 5-brane corresponds to the asymptotic region \( v \to \infty, t \sim v^{2(N_c+1)} \), the right \( N_c \) NS' 5-branes correspond to the asymptotic region \( v \to \infty, t \sim \Lambda_{N=2}^{2(2N_c+2-N_f)} v^{2(N_f-N_c-1)} \). The boundary conditions that we will impose are

\[
w \to \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} \quad \text{as} \quad v \to \infty, \quad t \sim \Lambda_{N=2}^{2(2N_c+2-N_f)} v^{2(N_f-N_c-1)}
\]

\[
w \to 0 \quad \text{as} \quad v \to \infty, \quad t \sim v^{2(N_c+1)}.
\] (50)
Alternatively, if the $N_c$ NS$'$ 5-branes were located at the left and the NS 5-brane at the right the boundary conditions would read

$$
w \to \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} \quad \text{as } v \to \infty, \ t \sim v^{2(N_c+1)}
$$

$$
w \to 0 \quad \text{as } v \to \infty, \ t \sim \Lambda_{N=2}^{2(2N_c+2-N_f)} v^{2(N_f-N_c-1)}.
$$

The motion of the D4-brane between the NS-NS$'$ 5-branes is the degree of freedom corresponding to the adjoint scalar field $\Phi$. However this motion is not possible because NS 5-branes and NS$'$ 5-branes are not parallel. This will give rise to a potential which gives to the mass to $\Phi$, say $M_\Phi$. It is expected that

$$
\frac{\partial w(v)}{\partial v} \sim M_\Phi \quad \text{at } v = \langle \Phi \rangle,
$$

where $w(v)$ is the position of the one of NS$'$ 5-branes. For $\Delta W = \frac{1}{2} \mu \text{Tr} \Phi^2$, it is clear that (52) holds since $w = \mu v$. Eq. (52) is also valid for more general $\Delta W$ and consistent with boundary condition (50) since $M_\Phi$ is given by $\Delta W''(\Phi)$.

In the fivebrane configuration $SU(2)_{7,8,9}$ is broken to $U(1)_{1,8,9}$ if the parameter $\mu_{2k}$ is assigned the $U(1)_{4,5} \times U(1)_{8,9}$ charge $(2 - 4k; 2)$. We list below the charges of the coordinates and parameters.

$$
\begin{array}{ccc}
  & U(1)_{4,5} & U(1)_{8,9} \\
v & 2 & 0 \\
w & 0 & 2 \\
t & 2(N_c+1) & 0 \\
z & 2(N_c+1) & 0 \\
\mu_k & 2(1-2k) & 2 \\
\Lambda_{N=2} & 2 & 0 \\
\end{array}
$$

(53)

We can identify $U(1)_{45} = U(1)_R$ and $U(1)_{89} = U(1)_L$.

From the field theory results, we expect to be able to construct the fivebrane configuration for only special values of $u_k$'s. Let us consider a perturbation of the form

$$
\sum_{k=1}^{N_c-l'+1} \mu_{2k} \frac{1}{2k} \text{Tr}(\Phi^{2k}).
$$

The point in the moduli space of vacua that remains as a vacuum after this perturbation is the singular locus of the $N = 2$ Coulomb branch where $l$ or more mutually local dyons become massless in field theory point of view. In the M-theory picture, it is possible to construct the corresponding fivebrane only when the $(v, t)$ curve degenerates to a genus $g \leq 2N_c - 2l'$ curve. It can be seen by that the boundary conditions

19
(50) or (54) mean that \( w \) is a meromorphic function of the \( (v, t) \) which has a pole of order \( 2N_c - 2l' + 1 \) at one point. Such a function exists only when the \( (v, t) \) curve is equivalent to a genus \( g \leq 2N_c - 2l' \) curve.

Following the paper of [19], we will find the possible fivebrane configurations with the boundary conditions (50). Here it is assumed that the equation defining the \( N = 2 \) curve,

\[
t^2 - 2C(v^2, u_{2k})t + G(v^2, u_{2k}) = 0,
\]

where \( C = v^2B_{2N_c} + \Lambda_{N=2}^{2N_c+2-N_l} \prod_{j=1}^{N_l} m_j \) and \( G = \Lambda_{N=2}^{2N_c+2-N_l} \prod_{i=1}^{N_f}(v^2 - m_i^2) \), remains unchanged. We also assume that \( w \) will be a rational function of \( t \) and \( v \).

Using (54), we can rewrite \( w(v, t) \) in the form

\[
w(t, v) = \frac{a(v)t + b(v)}{c(v)t + d(v)},
\]

(55)

Let us denote the two solutions of (54) by \( t_{\pm}(v) \) and consider a point in the \( N = 2 \) moduli space of vacua where the \( (x, t) \) curve degenerates to a genus \( N_c - l \) curve \( (l \geq 1) \). Hence we have that

\[
v^2y^2 = C^2(v^2) - G(v^2) \equiv S^2(v)T(v^2),
\]

(56)

where

\[
S = v \prod_{i=1}^{l} (v^2 - p_i^2), \quad T = \prod_{j=1}^{2N_c+1-2l} (v^2 - q_j^2).
\]

(57)

Note that there is no poles for a finite value of \( v \) since there are no other infinite NS 5-branes in the type IIA picture as in the \( SU(N_c) \) case. Thus, according to [19], we get

\[
w = N + \frac{H}{S}(t - C),
\]

(58)

where \( N \) and \( H \) are some polynomials in \( v \) and should not depend on \( t \), or

\[
w(t_{\pm}(v), v) = N(v) \pm H(v)\sqrt{T(v)}.
\]

(59)

In the case of \( Sp(2N_c) \) gauge group theory, there is the orientifold and (58) has to be invariant under the reflection \( w \rightarrow -w, v \rightarrow -v, t \rightarrow t \). We see that \( C, S \) and \( T \) transform to \( C, -S \) and \( T \) under this reflection respectively. Thus we require that under this reflection \( H \) and \( N/v \) are invariant, in other words \( H = H(v^2) \).
Next we must impose the boundary conditions on (58). As $v \to \infty$ and $t = t_-(v) \sim v^{2(N_c+1)}$, we want that $w \to 0$. This completely fixes $N$ as

$$N(v) = [H(v^2)\sqrt{T(v^2)}]_+, \quad (60)$$

where $[f(v)]_+$ denotes the part of $f(v)$ with non-negative powers of $v$, in a power series expansion around $v = \infty$. This form of $N$ is indeed odd degree in $v$ due to the fact that $\sqrt{T} = v^{2N_c+1-2l} \prod_i (1 - b_i^2/v^2)$. 

In the other asymptotic region, $v \to \infty$ and $t = t_+(v) \sim v^{2(N_f-N_c-1)}$, we see $w$ behaves as

$$w = [2H(v^2)\sqrt{T(v^2)}]_+ + O(v^{-1}). \quad (61)$$

If we take the $N = 1$ perturbation of the form $\sum_{k=1}^{N_c-l'} \mu_{2k} \text{Tr} \Phi^{2k}/2k$ and $H$ as the degree 2s polynomial in $v$, the equation (61) implies $s = l - l'$. As a consequence of this and $s \geq 0$, we find $l \geq l'$. This clearly shows the relation between the genus of the degenerate Riemann surface, and the minimal power needed in the superpotential. In particular $l = 1$ is the lowest values of $l$.

We note that imposing the other boundary condition (51) simply corresponds to the choice $N(v) = -[H(v^2)\sqrt{T(v^2)}]_+$. Finally, we note that $w$ satisfies the following important equation

$$w^2 - 2Nw + N^2 - TH^2 = 0. \quad (62)$$

### 4.3 Comparison to field theory

In the following we explicitly construct the brane configuration of the superpotential perturbation $\Delta W$ (10) of the $N = 2$ theory and compare the results to the field theory analysis in section 2.

We will start with the $Sp(2N_c)$ pure Yang-Mills theory. We already know that the most general deformation of the fivebrane is

$$w = N(v) + H(v^2) \frac{t - C(v^2)}{v \prod_{i=1}^l (v^2 - p_i^2)}, \quad (63)$$

where $H(v^2)$ and $N(v)$ are arbitrary polynomials of $v$. Consider the deformation of the right NS' 5-branes. We must impose the boundary conditions (50). As shown in the
previous subsection, the second boundary condition implies that $N$ has to be given by (60).

\[
N = \left[ H \prod_{j=1}^{2N_c+1-2l} (v^2 - q_j^2)^{1/2} \right]_+,
\]

where $[f(v)]_+$ denotes the part of $f(v)$ with non-negative powers of $v$. From the first boundary condition in (50) the relation between $H(v^2)$ and the values of $\mu_{2k}$ can be determined by expanding $w$ as given in (61) in powers of $v$. Using that $t = 2C(v^2) + \mathcal{O}(v^{-2(N_c+1)})$ we find

\[
w = 2H(v^2) \frac{v^2 B_{2N_c}(v^2) + i^{N_f} \prod_{i=1}^{N_f} m_i}{v \prod_{i=1}^{l} (v^2 - p_i^2)} + \mathcal{O}(v^{-1}) = \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} + \mathcal{O}(v^{-1}),
\]

which determines $H(v^2)$ in terms of $\mu_{2k}$. Note that the term proportional to $\prod_{i=1}^{N_f} m_i$ is of order $\mathcal{O}(v^{-3})$ and will be ignored in (65).

Expand $H(v^2)$ as

\[
\frac{2H(v^2)}{\prod_{i=1}^{l} (v^2 - p_i^2)} = \sum_{i=1}^{l} \frac{\omega_i}{(v^2 - p_i^2)},
\]

then (65) agrees with the field theory result (20). Eq. (65) together with (57) determines the $N = 1$ moduli space of vacua after the perturbation and the dyon vevs. We see that the M theory fivebrane describes correctly the fact that only the singular locus of the $N = 2$ Coulomb branch is not lifted and rederive the equations that determine the vevs of the massless dyons along the singular locus.

In the case of $Sp(2N_c)$ with $N_f$ flavors, we can compute the dyon vevs along the singular locus which is not at the Higgs branch root similarly to the above consideration and the results agree with the field theory results if $N_f \leq N_c + 2$ since $t = 2C(v^2) + \mathcal{O}(v^{2(N_f-N_c-1)})$. However, this does not contradict (20), because that result assumes the form of the curve (3) which is not valid for $N_f > N_c + 2$. We have not considered the case $N_f > N_c + 2$ in detail.

We can compute the dyon vevs at the roots of the Higgs branches, for the cases with $N_f - 2r - 2 \leq N_c - r$. Let us consider a point at the $r$-th Higgs branch root where the curve take the form (27) following (19). Imposing the boundary conditions (50), we easily get

\[
w = \sum_{k=1}^{N_c-r} \mu_{2k} v^{2k-1} + \mathcal{O}(v^{-1}) = 2H(v^2) \frac{v B_{2(N_c-r)}(v^2)}{\prod_{i=1}^{l} (v^2 - p_i^2)} + \mathcal{O}(v^{-1}),
\]

22
which determines $H(v^2)$. Eq. (67) agrees with the field theory result (28).

At the $r^\ast$-th Higgs branch, the curve is represented by the fivebrane (15) and remains two rational curves after the perturbation (10) as

$$\tilde{C}_L \begin{cases} t = 2v^2(N_c+1) \\ w = 0 \end{cases} \quad \tilde{C}_R \begin{cases} t = \frac{1}{2} \Lambda_{N=2}^{2(2N_c-N_f+2)}v^2(N_f-N_c-1) \\ w = \sum_{k=1}^{N_c} \mu_{2k}v^{2k-1}. \end{cases}$$

(68)

Thus we can see from the brane picture that the $r^\ast$-th Higgs branch root is not lifted for arbitrary values of the parameters $\mu_{2k}$.

Here we will give the geometrical interpretation of the dyon vevs according to [19]. Note that the dyon vev $m_i\bar{m}_i$ is equal up to a factor of $\sqrt{2}$ to the difference between the two finite values of $w$ (59) as we take $v^2 = p_i^2$. And the singular $N = 2$ curve (15), (33) has a double point at $v^2 = p_i^2, t = C(p_i^2)$. After the perturbation $\Delta W$ (10) this double point splits into two separate points in $(v, t, w)$ space, and the distance between the points in the $w$ direction is exactly the vev of the dyon that became massless at this point in the $N = 2$ theory. This provides a simple geometrical interpretation of the dyon vevs in the brane picture.

The meson vevs are also obtained as the values of $w$ at $t = 0, v^2 = m_i^2$ following [3] [19]. Let us now compute the finite values of $w$ at $t = 0, v^2 = m_i^2$ and compare to the meson vevs for the case with one massless dyon, in other words $l = 1$ and $H$ is a constant. Therefore we have that

$$C(v^2)^2 - \Lambda_{N=2}^{2(2N_c+2-N_f)} \prod_{i=1}^{N_f} (v^2 - m_i^2) = v^2(v^2 - p^2)^2T(v^2).$$

(69)

and the function $w$ is given by

$$w = [H\sqrt{T(v^2)}]_+ \pm H\sqrt{T(v^2)}.$$

(70)

Assuming $N_f \leq N_c + 2$, we simplify the $\sqrt{T(v^2)}$ as

$$\sqrt{T(v^2)} = \frac{C(v^2)}{v(v^2 - p^2)} + \mathcal{O}(v^{-1}).$$

(71)

We can always decompose

$$C(v^2) = C(0) + \frac{v^2}{p^2}(C(p^2) - C(0)) + v^2(v^2 - p^2)\tilde{C}(v^2),$$

(72)
for some $2N_c - 2$ degree polynomial $\tilde{C}$, and we see
\[
\sqrt{T(v^2)} = v \tilde{C}(v^2) + \mathcal{O}(v^{-1}) \quad \rightarrow \quad \sqrt{T(v^2)} = v \tilde{C}(v^2).
\] (73)

Using (69) and (72) we get
\[
\sqrt{T(m_i^2)} = \frac{p^2 C(0) + m_i^2 (C(p^2) - C(0))}{p^2 m_i (m_i^2 - p^2)} + m_i \tilde{C}(m_i^2).
\] (74)

Thus we obtain the finite value of $w(i)$ as
\[
w(i) \equiv w(v \rightarrow m_i) = \frac{H}{p^2} \left( \frac{C(0)}{m_i} - \frac{m_i}{m_i^2 - p^2} C(p^2) \right).
\] (75)

It is noted that the reflection due to the orientifold which change the sign of $w$ and $m_i$ changes the sign of $w(i)$. We can evaluate $C(0)$ and $C(p^2)$ from (69). We can also compute $p$ and $H$ from the asymptotic behavior of $w$ for large $v$ and $t \sim v^{2(N_c + 1)}$ and the results are $2H = \mu_{2N_c}$, $p^2 = \frac{\mu_{2(N_c - 1)}}{\mu_{2N_c}} - s_2$. Then we find
\[
w(i) = \frac{1}{2} \mu_{2N_c} \frac{1}{p^2} \Lambda_{N=2}^{2N_c + 2 - N_f} \left( i^{N_f} \prod_{j=1}^{N_f} m_j \pm \frac{m_i}{p^2 - m_i^2} \prod_{j=1}^{N_f} (p^2 - m_j^2)^{\frac{1}{2}} \right).
\] (76)

Here we will assume that if we take the perturbation $\Delta W = \sum_{i=1}^{N_c} \mu_{2k} \tilde{u}_{2k}$ we should replace $\mu_{2k}$ for boundary conditions (34) by $\tilde{\mu}_{2k}$. This assumption is natural in the sense of the arguments in subsection 4.2. Therefore taking the superpotential (29), we can easily see that $p^2 = \frac{\mu_{2(N_c - 1)}}{\mu_{2N_c}}$ and the dyon condensation derived from the fivebrane in M theory still agrees with field theory result. Then comparing (76) and (31) we see that up to a factor of $\sqrt{T}$, the values of $w$ at $t = 0$, $v^2 = m_i^2$ are exactly the eigenvalues of meson vevs derived from the field theory taking the appropriate basis of Casimirs and assuming that the low energy effective superpotential (31) is exact.

We can compute the meson vev for the case with more than one massless dyon. Following [19], we find that
\[
w(i) = \Lambda_{N=2}^{2N_c + 2 - N_f} \left( \frac{H(0)}{p^2 k} \prod_{j=1}^{N_f} (-m_j^2)^{\frac{1}{2}} \right)
\]
\[
\pm \sum_{j=1}^{N_f} \frac{H(p_j^2)}{p_j^2 \prod_{k \neq j} (p_j^2 - p_k^2)} \left( \prod_{j=1}^{N_f} (p_j^2 - m_j^2)^{\frac{1}{2}} \right)
\]
\[
= - \frac{\partial}{\partial m_i} \left( \Lambda_{N=2}^{2N_c + 2 - N_f} \sum_{j=1}^{N_f} \omega_j \frac{1}{p_j^2} \left( \det(-m)^{\frac{1}{2}} \pm \det(p_j - m)^{\frac{1}{2}} \right) \right).
\] (77)
We can also obtain the fivebrane configuration at the point in the moduli space of vacua with maximal number of mutually local massless dyons which implies \( l = N_c \) as in [19], but here we omit this discussion for the brevity.

5 Theories with \( SO(2N_c) \) gauge group

The procedure discussed above can be also applied to the other classical gauge groups. In this section we study the \( SO(2N_c) \) gauge group case.

5.1 Field theory analysis of \( SO(2N_c) \) gauge theory

Let us consider \( N = 2 \) supersymmetric gauge theory with the gauge group \( SO(2N_c) \) and \( N_f \) quark hypermultiplets in the fundamental representation \( Q^i_a, i = 1, \ldots , 2N_f \). Here \( a = 1, \ldots , 2N_c \) is color indice. The scalar chiral multiplet \( \Phi_{ab} \) is the \( 2N_c \times 2N_c \) antisymmetric tensor and the \( N = 2 \) superpotential takes the form

\[
W = \sqrt{2}J_{ij}Q^i_a\Phi_{ab}Q^j_b + \sqrt{2}m_{ij}Q^i_aQ^j_a, \tag{78}
\]

where \( J = \text{diag}(i\sigma_2, \ldots , i\sigma_2) \). From the \( N = 2 \) supersymmetry we can take the quark mass matrix as

\[
m = \text{diag}(m_1\sigma_1, \ldots , m_{N_f}\sigma_1) \text{ where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The flavor symmetry is \( Sp(2N_f) \) in addition to \( SU(2)_R \times U(1)_R \) R-symmetry group for massless quarks. The instanton factor is proportional to \( \Lambda_{N=2}^{4N_c-4-2N_f} \) and the \( U(1)_R \) symmetry is anomalous and is broken down to \( \mathbb{Z}_{4N_c-4-2N_f} \).

The Coulomb branch is \( N_c \) complex dimensional and is parametrized by the gauge invariant order parameters

\[
u_{2k} = \left\langle \frac{1}{2k}\text{Tr} \left( \Phi^{2k} \right) \right\rangle, \quad k = 1, \ldots , N_c - 1, \tag{79}
\]

\[
P = \left\langle \text{Pf} \Phi \right\rangle.
\]

Hereafter we use \( u_{2k} = \text{Tr}(\Phi^{2N_c}) = \text{Pf}\Phi^2 + \ldots \) instead of \( \text{Pf}\Phi \) because it is difficult to interpret \( \text{Pf}\Phi \) as the boundary conditions of the fivebrane in M theory. The Coulomb branch parametrizes a family of genus \( N_c \) hyperelliptic curves [23] [33]

\[
y^2 = x \left( B_{2N_c}(x, u_{2k})^2 - x^2\Lambda_{N=2}^{2(N_c-2)} - 2N_f \prod_{i=1}^{N_f} (x - m_i^2) \right), \tag{80}
\]

25
where $B_{2N_c}$ is a degree $N_c$ polynomial in $x$ with coefficients that depend on the gauge invariant order parameters $u_{2k}$ and $m_i$. For $N_f < N_c - 2$ the polynomial $B_{2N_c}$ is given by

$$B_{2N_c}(x) = \sum_{i=0}^{N_c} s_{2i} x^{N_c-i}. \quad (81)$$

Hereafter we will consider the case with the vanishing bare quark mass for simplicity.

There are two types of the gauge invariants which are constructed from $Q$, the meson fields $M^{ij} = Q^i J Q^j$ and the baryon fields $B^{[i_1...i_{N_f}]} = Q_{a_1}^{i_1} \cdots Q_{a_{N_f}}^{i_{N_f}} \epsilon_{a_1...a_{N_f}}$. The baryon field is defined for $N_f \geq N_c$. The Higgs branches are classified by an integer $r$ such that $0 \leq r \leq \min\{N_c, \frac{N_f}{2}\}$ \[20\]. The $r$-th Higgs branch has complex dimension $2r(2N_f - 2r + 1)$ and emanating from the $N_c - r$ dimensional submanifold of the Coulomb branch. The baryon field has non vanishing vev only when $r = N_c \leq N_f/2$. The effective theory along the root of the $r$-th Higgs branch is $SO(2r) \times U(1)^{N_c-r}$ with $N_f$ massless quarks which are neutral with respect to any $U(1)$ factor. There are special points along the root where such massless matters exist. In particular in $r^*$-th Higgs branch where $r^* = N_f - N_c + 2$, the maximal singularity occurs at which the curve takes the form

$$y^2 = x^{2r+1} \left( x^{N_c-r} + \frac{1}{4} A^{2(N_c-r)} \right)^2, \quad (82)$$

and $N_c - r^*$ hypermultiplets become simultaneously massless. Note that in the $r^*$-th Higgs branch the baryon field always has vanishing vev.

We consider the perturbation $\Delta W$ \[10\] to the $N = 2$ superpotential \[78\]

$$W = \sqrt{2} JQ \Phi Q + \sqrt{2} mQQ + \Delta W. \quad (83)$$

If only the mass perturbation is present, only the special points in the $r^*$-th Higgs branch root and the point in the Coulomb branch where the curve degenerate to the genus zero curve also remains as vacua as in the case of $Sp(2N_c)$.

Now we consider $N = 2$ pure $SO(2N_c)$ Yang-Mills theory perturbed by the superpotential $\Delta W$ \[10\]. Near the singularities of the moduli space of vacua, the low energy superpotential is also given by \[10\] which is exact from the holomorphy and global symmetry arguments. The list of charges under $U(1)_R$ and $U(1)_J \subset SU(2)_R$ is same as \[12\] in $Sp(2N_c)$ case. And the equation of motion \[13\] and \[14\] also hold in this case. Let us
consider a point in the moduli space where \( l \) mutually local dyons are massless. At this point the curve takes the form

\[
y^2 = x \left( B_{2N_c}(x)^2 - x^2 \Lambda_{N=2}^{4(N_c-1)} \right) = \prod_{i=1}^l (x - p_i^2)^2 \prod_{j=1}^{2N_c-2l+1} (x - q_j^2)
\]

with \( p_i \) and \( q_j \) distinct. The equation of motion implies that there will be a complex \( N_c - l \) dimensional moduli space of \( N = 1 \) vacua which remains after the perturbation assuming the matrix \( \partial a_i/\partial u_{2k} \) is non-degenerate.

Computation of the matrix \( \partial a_i/\partial u_{2k} \) can be done as in \( Sp(2N_c) \) case and using this we find the relation between the parameters \( \mu_{2k} \) and the dyon vevs \( m_i \bar{m}_i \) as

\[
\sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} = v B_{2N_c}(v^2) \frac{\sum_{i=1}^l \omega_i}{\prod_{i=1}^l (v^2 - p_i^2)} + O(v^{-1})
= v B_{2N_c}(v^2) \frac{2H(v^2)}{\prod_{i=1}^l (v^2 - p_i^2)} + O(v^{-1}),
\]

where we identify \( x \) as \( v^2 \). Here we define \( \omega_i \) and \( H \) from (19) and (66).

In \( N = 1 \) pure Yang-Mills theory with the gauge group \( SO(2N_c) \), there are \( 2N_c - 2 \) massive vacua where the discrete \( \mathbb{Z}_{4N_c-4} R \)-symmetry is spontaneously broken to \( \mathbb{Z}_2 \). This vacuum corresponds to the genus zero curve and generically is not lifted.

The computations of the dyon vevs in the case of the \( SO(2N_c) \) supersymmetric QCD are straightforward and the results are similar to the \( Sp(2N_c) \) case. The vev of the meson field \( QQ \) along the singular locus of the Coulomb branch is generated by the non-perturbative dynamics of the \( N = 1 \) theory and become nonzero. We will compute this vev below.

We take a tree-level superpotential as

\[
W = \sum_{n=1}^{N_c-1} \mu_{2n} u_{2n} + \mu_{2N_c} s_{2N_c} + \sqrt{2} JQ \Phi Q + \sqrt{2} m QQ.
\]

Here we choose \( s_{2N_c} \) instead of \( u_{2N_c} \) as in \( Sp(2N_c) \) case. Due to this choice, we can easily obtain the low energy effective superpotential as

\[
W(\mu_{2k}, m_i) = \sum_{n=1}^{N_c-1} \mu_{2n} u_{2n}^d + \mu_{2N_c} s_{2N_c}^d \pm \mu_{2(N_c-1)} \Lambda_{N=2}^{2N_c-2-N_f} \prod_{i=1}^{N_f} (p_i^2 - m_i^2) \frac{1}{2},
\]
where \( p_1^2 = \frac{\mu_{2(N_c - 1)}}{\mu_{2N_c}} \). Indeed the vev of \( s_{2k} \) calculated from (87) is
\[
\langle s_{2k} \rangle = s_{2k}^d(g) \pm \delta_{k,N_c-1} \frac{2A(p_1) + p_1^2 A'(p_1)}{2\sqrt{A(p_1)}} \mp \delta_{k,N_c-1} \frac{p_1^2 A'(a_1)}{2\sqrt{A(p_1)}}.
\] (88)

where \( A(v) \equiv \Lambda^{4(N_c - 1) - 2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2) = A(-v) \) and \( A'(v) = \frac{\partial}{\partial v} A(v) \) and these vev reproduces the singularities in the moduli space of the \( N = 2 \) curves correctly. Therefore we get the vev of the meson to derivate (87) by \( m_i \) as
\[
\langle M^{ii} \rangle = \mp \frac{1}{\sqrt{2}} \mu_{2N_c} p_1^2 \Lambda_{N_c=2}^{2N_c-2-2N_f} m_i \prod_{i=1}^{N_f} \left( \frac{p_1^2 - m_i^2}{p_1^2 - m_i^2} \right)^{\frac{1}{2}}.
\] (89)

The calculation of the dyon condensation under the perturbation (86) is similar to the \( Sp(2N_c) \) case and the only modification is replacing \( \mu_{2k} \) by \( \tilde{\mu}_{2k} \).

### 5.2 \( N = 2 \) Higgs branch of \( SO(2N_c) \) theory from M theory

We will consider the moduli space of vacua of \( N = 2 \) supersymmetric QCD with the gauge group \( SO(2N_c) \) and its deformation by the superpotential (10) using the M theory fivebrane.

The brane configuration in type IIA in which the four dimensional \( SO(2N_c) \) gauge theory is realized is almost same as \( Sp(2N_c) \) case but the Chan-Paton wavefunction of the vector is taken to be antisymmetric.

The \( r \)-th Higgs branch corresponds to \( 2(N_c - r) \) D4-branes suspended between the two NS 5-branes and \( 2r \) D4-branes broken on the D6-branes. The \( r \)-th Higgs branch shares \( (N_c - r) \) complex dimensions with the Coulomb branch, corresponding to the gauge group \( SO(2(N_c - r)) \). Following what we have discussed in section 3, we obtain the complex dimension of the \( r \)-th Higgs branch to be
\[
2 \sum_{l=1}^{r} [2N_f - (4l - 3)] = 2r(2N_f - 2r + 1)
\] (90)
in agreement with the field theory results.

We embed the type IIA brane configuration in M theory fivebrane configuration. The curve \( \Sigma \), describing the \( N = 2 \) \( SO(2N_c) \) gauge theory with \( N_f \) flavors, is given by an equation in \((v, t)\) space
\[
v^2 t^2 - 2B_{2N_c}(v^2, u_{2k}) t + \Lambda_{N_c=2}^{2(N_c - 2 - N_f)} u_f N_f \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0,
\] (91)
where \( B_{2N_c}(v^2, u_{2k}) \) is a degree \( 2N_c \) polynomial in \( v \), and has the form \( v^{2N_c} + \cdots \) with only even degree in \( v \).

We include the D6-branes to the above M theory fivebrane configuration. This situation is described by the surface

\[
tz = \Lambda_{N=2}^{2(2N_c-2-N_f)} \prod_{i=1}^{N_f} (v^2 - m_i^2)
\]  

(92)

in \( \mathbb{C}^3 \) for \( SO(2N_c) \). The Riemann surface \( \Sigma \) is embedded as a curve in this curved surface and is given by

\[
v^2(t + z) = 2B_{2N_c}(v^2, u_{2k}).
\]  

(93)

Note that the \( v^2 \) factor in (93) has its origin in the fivebrane which is bent and infinitely extended along the orientifold by the effect of the charge of the orientifold [17]. The resolution of the singularities of (92) is needed when some \( m_i \) are coincident. We can carry out this resolution as in \( Sp(2N_c) \) case.

Next we will study the Higgs branch when all the bare masses are turned off. The Higgs branch appears when the fivebrane intersects with the D6-branes. Thus we have \( B_{2N_c}(v^2) \) in the factorized form

\[
B_{2N_c}(v^2) = v^{2r}(v^{2(N_c-r)} + s_2v^{2(N_c-r-2)} + \cdots + s_{2(N_c-r)})
\]  

(94)

where \( r > 0 \).

Away from the singular point \( t = z = v = 0 \), the curve is equivalent with the generic curve for the \( SO(2(N_c - r)) \) gauge theory with \( (N_f - 2r) \) flavors and thus has genus \( (N_c - r) \) if \( 2r \leq N_f \). From (93) and (94), one of the components of the curve, \( C_0 \), lies on \( v = 0 \).

Near \( t = z = v = 0 \), we can replace the defining equation \( v^2(t + z) = v^{2r}(s_{2(N_c-r)} + \cdots) = 0 \) by \( v^2(t + z) = v^{2r} \). On the \( i \)-th patch \( U_i \), the equation of the curve \( \Sigma \) becomes

\[
(t_i z_i)^2(t_i^{j-1}z_i^{j-1} + t_i^{2N_f-j-i}z_i^{2N_f-j+1-i}) = t_i^{r}z_i^{r}.
\]  

(95)

Thus we have this equation factorize as

\[
t_i^{j-1}(v_i^2(1 + z_i^{2N_f-2j}) - v_i^{2r-i}z_i) = 0, \quad i = 1, \ldots, 2r
\]
\[ v_i^{2r}(v_i^2(t_i v_i^{i-2r-1} + v_i^{2N_f-2r-i} z_i)) - 1 = 0, \quad i = 2r + 1, \ldots, 2N_f - 2r \]
\[ v_i^{2N_f-i} z_i(v_i^2(t_i^2 z_i^{2i-2N_f-2} + t_i v_i^{i-2N_f+2r-1})) - 1 = 0, \quad i = 2N_f-2r + 1, \ldots, 2N_f, \]

where we define \( v_i = t_i z_i \). However we can further factor out \( t_i^2 z_i^2, t_i z_i^2, z_i, t_i^2 z_i \) and \( t_i^2 z_i^2 \) for \( i \leq 2r - 2, i = 2r - 1, i = 2r, i = 2N_f - 2r + 1, i = 2N_f - 2r + 2 \) and \( i \geq 2N_f - 2r + 3 \) respectively from (96). The components represented by these factors together with \( C_0 \) are interpreted as the infinitely extending fivebrane along the orientifold. These components of the curve may have no contribution to the dimension of the Higgs branch.

Thus we find the curve consists of \( C \), which is the zero of the last factor of (96) and extends to infinity, and the rational curves \( C_1, \ldots, C_{2N_f-1} \) with multiplicities. From the factorized form (96), the component \( C_i \) has multiplicity \( \ell_i \) where \( \ell_i = i \) for \( i = 1, \ldots, 2r \), \( \ell_i = 2r \) for \( i = 2r + 1, \ldots, 2N_f - 2r - 1 \) and \( \ell_i = 2N_f - i \) for \( i = 2N_f - 2r, \ldots, 2N_f - 1 \). Note that the component \( C \) intersects with \( C_{2r-2} \) and \( C_{2N_f-2r+2} \).

As in the IIA picture, it is expected that an additional \( \text{CP}^1 \) are appeared in \( C_k \) where \( k \) is an odd integer. Taking into account the orientifolding, \( \text{CP}^1 \)'s must move in pairs. Therefore if we define \( o_k \) as \( o_k = 0 \) for \( k \) even and \( o_k = 1 \) for \( k \) odd, the complex dimension of the \( r \)-th Higgs branch is obtained as

\[ 4 \sum_{i=1}^{2N_f-1} \left[ \frac{l_i + o_i}{2} \right] = 8 \sum_{i=1}^{r} i + 2r(2N_f - 1 - 4r) = 2r(2N_f - 2r + 1) \]

in agreement with (97). It is noted that the fivebrane configuration subjected to the \( s \)-rule in the type IIA theory is verified by the above M theory fivebrane configuration.

For the case of \( N_f \geq N_c - 2 \), there is a special point in the Higgs branch root. This special point in the Higgs branch root is in the \( r^* \)-th Higgs branch root where \( r^* = N_f - N_c + 2 \) and takes the color Casimirs such that

\[ B_{2N_c}(u_{2N_c}, v^2) = u^{2N_c} + \frac{1}{4} v^{2r^*} \Lambda_{N=2}^{2(N_c-r^*)} = v^{2r^*} \left( u^{2(N_c-r^*)} + \frac{1}{4} \Lambda_{N=2}^{2(N_c-r^*)} \right) \]

Then at this point the equation (93) factorizes as

\[ \frac{v^2}{t} (t - 2v^{2N_c-2}) (t - \frac{1}{2} \Lambda_{N=2}^{2(N_c-r^*)} v^{2(r^*-1)}) = 0. \]
5.3 $N = 1$ moduli space of $SO(2N_c)$ theory from M theory

In the same way as the case with $Sp(2N_c)$ group, we will consider the deformation of the fivebrane configuration in M theory for the $N = 2$ $SO(2N_c)$ gauge field theory to the one for the $N = 1$ field theory.

First we study the adjoint mass perturbation. The boundary conditions we impose are

$$ w \to \mu v \quad \text{as} \quad v \to \infty, \quad t \sim v^{2N_c-2} $$
$$ w \to 0 \quad \text{as} \quad v \to \infty, \quad t \sim \Lambda_{N=2}^{2(2N_c-2-N_f)} v^{2(N_f-N_c+1)}. \quad (100) $$

We can identify $\mu$ as the mass of the adjoint chiral multiplet by using the $R$-symmetries.

These conditions are only satisfied at points in the moduli space at which all $\beta$-cycles on the curve are degenerate. This is possible at the special points in the Coulomb branch or in the $r^*$-th Higgs branch root. The possible fivebrane configuration which corresponds to the points in the Coulomb branch is studied in [14]. The rotated curve for the $r^*$-th Higgs branch root is explicitly given by

$$ \tilde{C}_L \left\{ \begin{array}{l}
{\tilde{t}} = 2w^{2(N_c-1)} \\
{\tilde{w}} = \mu v
\end{array} \right. \quad \tilde{C}_R \left\{ \begin{array}{l}
{\tilde{t}} = \frac{1}{2} \Lambda_{N=2}^{2(2N_c-2-N_f)} v^{2(N_f-N_c+1)} \\
{\tilde{w}} = 0
\end{array} \right. \quad (101) $$

The $N = 1$ supersymmetric QCD limit, $\mu \to \infty$, can be taken by introducing a new variable $\tilde{t} = \mu^{2(N_c-1)} t$. The equation (103) defines a smooth surface in the limit $\mu \to \infty$ provided the constant $\Lambda_{N=1}$ given by

$$ \Lambda_{N=1}^{3(2N_c-2)-2N_f} = \mu^{2(N_c-1)} \Lambda_{N=2}^{2(2N_c-1)-N_f} \quad (102) $$

is kept finite and this relation agrees with the field theory one. We also take the limit $\mu \to \infty$ of the curves of the remaining Higgs branch root (101) and the curves become

$$ C_L \left\{ \begin{array}{l}
\tilde{t} = 2w^{2(N_c-1)} \\
v = 0
\end{array} \right. \quad C_R \left\{ \begin{array}{l}
\tilde{t} = \frac{1}{2} \Lambda_{N=1}^{3(2N_c-2)-2N_f} v^{2(N_f-N_c+1)} \\
w = 0
\end{array} \right. \quad (103) $$

Next let us consider the configuration of fivebrane in M theory which corresponds to more general perturbation of the form $\Delta W$ (10) in a similar way. The type IIA brane configuration is same as the $Sp(2N_c)$ case.
We will construct the configuration of M theory fivebrane corresponding to it. The boundary conditions which we will impose are

\[
\begin{align*}
    w &\to \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} \quad \text{as } v \to \infty, \quad t \sim \frac{A_{N_c=2}^{2(Nc-2-N_f)}}{v^{2(N_f-N_c+1)}} \\
    w &\to 0 \quad \text{as } v \to \infty, \quad t \sim v^{2(N_c-1)}.
\end{align*}
\]

(104)

Under the assumption that the equation defining the \( N = 2 \) curve remains unchanged and that \( w \) will be a rational function of \( t \) and \( v \), we can proceed in the \( SO(2N_c) \) case along the consideration of [19]. At the point in the moduli space where \( l \) mutually local dyons are massless, we can factorize the equation

\[
y^2/v^6 = C^2 - G = S^2T \quad \text{where } C = B_{2N_c}/v^2, \quad G = \frac{\Lambda_{N=2}^{2(Nc-1)-N_f}}{N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2) \]

and

\[
S = \frac{1}{v^2} \prod_{i=1}^{l} (v^2 - p_i^2), \quad T = \prod_{j=1}^{2N_c-2l} (v^2 - q_j^2).
\]

(105)

Then we find

\[
w = N + \frac{H'}{S}(t - C),
\]

(106)

where \( N \) and \( H' \) are some polynomials in \( v \). Note that there are no poles for a finite value of \( v \). Since \( C, S \) and \( T \) are invariant under the reflection \( w \to -w, v \to -v, t \to t \), we require that \( H \equiv H' / v = H(v^2) \) and \( N \) transforms to \(-N\) under this reflection.

Now we must impose the boundary conditions (104) on the fivebrane configuration. As \( v \to \infty \) and \( t = t_-(v) \sim v^{2(N_c-1)} \), we require that \( w \to 0 \). This determines \( N \) as

\[
N(v) = [vH(v^2)\sqrt{T(v^2)}]_+.
\]

(107)

In the other asymptotic region, \( v \to \infty \) and \( t = t_+(v) \sim v^{2(N_f-N_c+1)} \), \( w \) behaves as

\[
w = [2vH(v^2)\sqrt{T(v^2)}]_+ + O(v^{-1}).
\]

(108)

We note that \( w \) satisfies the following important equation

\[
w^2 - 2Nw + N^2 - T v^2 H^2 = 0.
\]

(109)

Let us focus on the \( SO(2N_c) \) pure Yang-Mills theory. Using that \( T = (t - C)/S \) and \( t = 2C(v^2) + O(v^{-2(Nc-1)}) \), the first boundary condition in (104) reads

\[
w = 2vH(v^2) \frac{B_{2N_c}(v^2)}{\prod_{i=1}^{l} (v^2 - p_i^2)} + O(v^{-1}) = \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} + O(v^{-1}),
\]

(110)
which determines $H(v^2)$ in terms of $\mu_{2k}$. The equation (110) agree with the field theory equation (83).

In the case of $SO(2N_c)$ with $N_f$ flavors, we can also compute the dyon vevs along the singular locus as in the previous calculations. It is seen that at the $r^*$-th Higgs branch root the curves (101) remain two rational curves after the rotation as

$$
\tilde{C}_L \begin{cases} 
t = 2v^2(N_c-1) \\
w = 0 
\end{cases} \quad \tilde{C}_R \begin{cases} 
t = \frac{1}{2}\Lambda_{N=2}^{2(N_c-N_f-2)}v^2(N_f-N_c+1) \\
w = \sum_{k=1}^{N_c} \mu_{2k}v^{2k-1}.
\end{cases}
$$

Thus we see from the brane picture that the $r^*$-th Higgs branch root is not lifted for arbitrary values of the parameters $\mu_{2k}$.

The meson vevs can be read from the values of $w$ at $t = 0, v^2 = m_i^2$. We can compute the finite values of $w$ at $t = 0, v^2 = m_i^2$ and compare these to the meson vevs for the case with one massless dyon, in other words $l = 1$ and $H$ is a constant. Assuming $N_f < N_c - 1$, we find

$$
w(i) = \pm \frac{1}{2}\mu_{2N_c}^{2N_c-2-N_f} \prod_{j=1}^{N_f} (p_j^2 - m_i^2)^{\frac{1}{2}}, \quad p^2 = \frac{\mu_{2(N_c-1)}}{\mu_{2N_c}},
$$

which is agree with (89) up to a factor $\sqrt{2}$. Here we change the $\Delta W$ as (86) and the boundary conditions as in the case of $Sp(2N_c)$. We also compute the meson vev for the case with more than one massless dyon. Following the previous calculations, we can easily obtain that

$$
w(i) = \pm \Lambda_{N=2}^{2N_c-2-N_f} m_i \sum_{j=1}^{l} \frac{p_j^2 H(p_j^2)}{\prod_{k \neq j} (p_j^2 - p_k^2)} \prod_{j=1}^{N_f} \frac{(p_j^2 - m_f^2)^{\frac{1}{2}}}{p_j^2 - m_f^2} = \pm \frac{\partial}{\partial m_i} \left( \Lambda_{N=2}^{2N_c-2-N_f} \sum_{j=1}^{l} \det(p_j - m)^{\frac{1}{2}} w_j p_j^2 \right).
$$

6 \hspace{1em} Theories with $SO(2N_c + 1)$ gauge group

Let us now turn to the case of $SO(2N_c + 1)$ gauge group. We will proceed in parallel with the case of $SO(2N_c)$.  

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6.1 Field theory analysis of $SO(2N_c + 1)$ gauge theory

Consider $N = 2$ supersymmetric gauge theory with the gauge group $SO(2N_c + 1)$ and $N_f$ quark hypermultiplets in the fundamental representation $Q_a^i$, $i = 1, \ldots, 2N_f$. The $N = 2$ superpotential takes the form (78). The instanton factor is proportional to $\Lambda_{N=2}^{2(2N_c-1-N_f)}$ and the $U(1)_R$ symmetry is anomalous and is broken down to $Z_{2(2N_c-1-N_f)}$.

The Coulomb branch is $N_c$ complex dimensional and is parametrized by the gauge invariant order parameters

$$u_{2k} = \left\langle \frac{1}{2k} \text{Tr} \left( \Phi^{2k} \right) \right\rangle, \quad k = 1, \ldots, N_c. \quad (114)$$

The Coulomb branch parametrize a family of genus $N_c$ hyperelliptic curves

$$y^2 = x B_{2N_c}(x, u_{2k})^2 - x^2 \Lambda_{N=2}^{2(2N_c-1-N_f)} \prod_{i=1}^{N_f} (x - m_i^2), \quad (115)$$

where $B_{2N_c}$ is a degree $N_c$ polynomial in $x$ with coefficients that depend on the gauge invariant order parameters $u_{2k}$ and $m_i$. For $N_f < N_c - 1$ the polynomial $B_{2N_c}$ is given by

$$B_{2N_c}(x) = \sum_{i=0}^{N_c} s_{2i} x^{N_c-i}. \quad (116)$$

There are two types of the gauge invariants which are constructed from $Q$, the meson fields and the baryon fields, however, the baryon field is defined for $N_f \geq N_c + 1$. The Higgs branches are classified by an integer $r$ such that $0 \leq r \leq \min\{N_c, \frac{N_f-1}{2}\}$ [20]. The $r$-th Higgs branch has complex dimension $2(2r + 1)(N_f - r)$ and emanating from the $N_c - r$ dimensional submanifold of the Coulomb branch. Classically the baryon field has non vanishing vev only when $r = N_c \leq \frac{N_f-1}{2}$. The effective theory along the root of the $r$-th Higgs branch is $SO(2r + 1) \times U(1)^{N_c-r}$ with $N_f$ massless quarks which are neutral with respect to any $U(1)$ factor. There are special points along the root where additional massless matters exist. In particular in $r^*$-th Higgs branch where $r^* = N_f - N_c + 1$, the maximal singularity occurs at which the curve takes the form

$$y^2 = x^{2r^*+1} \left( x^{N_c-r^*} + \frac{1}{4} \Lambda_{N=2}^{2(N_c-r^*)} \right)^2, \quad (117)$$

and $N_c - r^*$ hypermultiplets become simultaneously massless.
We now consider the perturbation $\Delta W$ (10) to the $N = 2$ superpotential (78) with the gauge group $SO(2N_c + 1)$. The effect of the mass perturbation is same as $SO(2N_c)$ case.

First we study $N = 2$ pure $SO(2N_c + 1)$ Yang-Mills theory perturbed by the superpotential $\Delta W$ (10). Let us consider a point in the moduli space where $l$ mutually local dyons are massless where the curve takes the form

$$y^2 = x \left(B_{2N_c}(x)^2 - x \Lambda_{N=2}^{2(2N_c-1)} \right) = \prod_{i=1}^l(x - p_i^2)^{2N_c-2l+1} \prod_{j=1}^l(x - q_j^2)$$  \hspace{1cm} (118)

with $p_i$ and $q_j$ distinct. The equation of motion implies that there will be a complex $N_c - l$ dimensional moduli space of the $N = 1$ vacua which remains after the perturbation assuming the matrix $\partial a_i / \partial u_{2k}$ is non-degenerate.

We can find the relation between the parameters $\mu_{2k}$ and the dyon vevs $m_i \tilde{m}_i$ as

$$\sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} = vB_{2N_c}(v^2) \sum_{i=1}^l \frac{\omega_i}{(v^2 - p_i^2)} + O(v^{-1})$$

$$= vB_{2N_c}(v^2) \frac{2H(v^2)}{\prod_{i=1}^l(v^2 - p_i^2)} + O(v^{-1}),$$  \hspace{1cm} (119)

where we identify $x$ as $v^2$. Here we define $\omega_i$ and $H$ from (66) and (19). It is seen also in this case that the vacua correspond to the curve (115) degenerates to a genus zero curve and generically is not lifted. The computations of the dyon vevs in the the case of the $SO(2N_c + 1)$ supersymmetric QCD are straightforward and the results are similar to the $Sp(2N_c)$ case.

The vev of the meson field $QQ$ along the singular locus of the Coulomb branch is generated by the non perturbative dynamics of the $N = 1$ theory and becomes nonzero. We take a tree-level superpotential as

$$W = \sum_{n=1}^{N_f-1} \mu_{2n} u_{2n} + \mu_{2N_c} s_{2N_c} + \sqrt{2} JQ\Phi Q + \sqrt{2} mQQ.$$

Here we choose $s_{2N_c}$ instead of $u_{2N_c}$ as in the $Sp(2N_c)$ case. Due to this, we can easily obtain the low energy effective superpotential as

$$W(\mu_{2k}, \tilde{m}_i) = \sum_{n=1}^{N_c-1} \mu_{2n} u_{2n}^d + \mu_{2N_c} s_{2N_c} + \mu_{2N_c}p_1 \Lambda_{N=2}^{2N_c-1-N_f} \prod_{i=1}^{N_f} (p_i^2 - m_i^2)^\frac{1}{2},$$  \hspace{1cm} (121)
where \( p_1^2 = \frac{\mu_2(N_c-1)}{\mu_2N_c} \). Therefore we find the vev of the meson to derivate (121) by \( m_i \) as

\[
\langle M^{ii} \rangle = \pm \frac{1}{\sqrt{2}} \mu_2 \Lambda^{2N_c-1-2N_f}_{N=2} m_i \prod_{i=1}^{N_f} (p_i^2 - m_i^2)^\frac{1}{2} \frac{p_1^2 - m_i^2}{p_1^2 - m_i^2}.
\]  

(122)

### 6.2 \( N = 2 \) Higgs branch of \( SO(2N_c+1) \) theory from M theory

We will consider the moduli space of vacua of \( N = 2 \) supersymmetric QCD with the gauge group \( SO(2N_c+1) \) and its deformations by the superpotential (10) by analyzing M theory fivebrane.

The brane configuration in type IIA in which four dimensional \( SO(2N_c+1) \) gauge theory is realized is almost same as \( SO(2N_c) \) case. Only difference between two cases is that there is an extra D4-brane on the orientifold four plane in the \( SO(2N_c+1) \) case. This D4-brane can not move since it does not have a mirror partner.

The \( r \)-th Higgs branch corresponds to \( 2(N_c-r) \) D4-branes suspended between the two NS 5-branes and \( 2r+1 \) D4-branes broken on the D6-branes. The \( r \)-th Higgs branch shares \( (N_c-r) \) complex dimensions with the Coulomb branch, corresponding to the gauge group \( SO(2(N_c-r)) \). On the basis of our arguments in section 3, we get the complex dimension of the \( r \)-th Higgs branch

\[
2 \sum_{l=1}^{r} [2N_f - (4l - 3)] + 2(N_f - 2r) = 2(2r + 1)(N_f - r) \tag{123}
\]

in agreement with the field theory results. The last term on the lhs of (123) comes from the extra D4-brane.

The curve \( \Sigma \) in M theory, describing \( N = 2 \) \( SO(2N_c+1) \) gauge theory with \( N_f \) flavors, is given by an equation in \( (v,t) \) space [17]

\[
v^2 t^2 - 2vB_{2N_c}(v^2, u_{2k}) t + \Lambda^{2(2N_c-1-N_f)}_{N=2} v^2 \prod_{i=1}^{N_f} (v^2 - m_i^2) = 0, \tag{124}
\]

where \( B_{2N_c}(v^2, u_{2k}) \) is a degree \( 2N_c \) polynomial in \( v \), and has the form \( v^{2N_c} + \cdots \) with only even degree in \( v \).

Including the D6-branes to the above M theory fivebrane configuration is described by the surface

\[
tz = \Lambda^{2(2N_c-1-N_f)}_{N=2} \prod_{i=1}^{N_f} (v^2 - m_i^2) \tag{125}
\]
in $\mathbb{C}^3$ for $SO(2N_c + 1)$. The Riemann surface $\Sigma$ is embedded in the surface and given by

$$v^2(t + z) = 2vB_{2N_c}(v^2, u_{2k}).$$

(126)

Note that the factor $v^2$ on the lhs of (126) has its origin in the fivebrane which is bent and infinitely extended along the orientifold by the effect of the charge of the orientifold and the factor of $v$ on the rhs of (126) has its origin in the D4-brane without a mirror partner [17]. The resolution of the singularities of (126) is needed when some $m_i$ are coincident. We can done this resolution as in $Sp(2N_c)$ case. However the spacetime reflection due to the orientifold is different from $Sp(2N_c)$ case and it may be extended to the resolved surface by considering the action $t_i \rightarrow (-1)^it_i$, $z_i \rightarrow (-1)^{i+1}z_i$. In this case, $C_n$ is rotated by reflection due to the orientifold.

We now consider the Higgs branch where $B_{2N_c}(v^2)$ factorizes as

$$B_{2N_c}(v^2) = v^{2r}(s_{2(N_c-r)} + \cdots + s_{2(N_c-r)}).$$

(127)

Away from the singular point $t = z = v = 0$, we see that the curve is equivalent with the generic curve for the $SO(2(N_c - r))$ gauge theory with $(N_f - 2r - 1)$ flavors and thus has genus $(N_c - r)$ if $2r \leq N_f$. Note that the one of the components of the curve, $C_0$, lies on $v = 0$.

Near $t = z = v = 0$, we can replace the defining equation $v^2(t + z) = v^{2r+1}(s_{2(N_c-r)} + \cdots) = 0$ by $v^2(t + z) = v^{2r+1}$. On the $i$-th patch $U_i$, the equation of the curve $\Sigma$ becomes

$$(t_i z_i)^2(t_i^{1/z_i^1} + t_i^{2N_f-i}z_i^{2N_f+1-i}) = t_i^r z_i^r(t_i z_i).$$

(128)

Thus we have this equation factorize as

$$t_i v_i^{j-1} \left( v_i^2(1 + v_i^{2N_f-2j}z_i^2) - v_i^{2r-j}z_i(v_i) \right) = 0, \quad i = 1, \ldots, 2r$$

$$v_i^{2r} \left( v_i^2(t_i v_i^{j-2r-1} + v_i^{2N_f-2r-j}z_i) - (v_i) \right) = 0, \quad i = 2r + 1, \ldots, 2N_f - 2r$$

$$v_i^{2N_f-j}z_i \left( v_i^2(t_i^2 v_i^{2i-2N_f-2} + 1) - t_i v_i^{i-2N_f+2r-1} \right) = 0, \quad i = 2N_f - 2r + 1, \ldots, 2N_f,$$

(129)

where we define $v_i = t_i z_i$. However we can further factor out $t_i^2 z_i^2$, $t_i z_i^2$, $t_i z_i$, $t_i^{2} z_i$ and $t_i^{2} z_i^2$ for $i \leq 2r - 1$, $i = 2r$, $2r + 1 \leq i \leq 2N_f - 2r$, $i = 2N_f - 2r + 1$ and $i \geq 2N_f - 2r + 2$ respectively from (129). Among these components, the middle components $C_i$ ($2r + 1 \leq$
\[ i \leq 2N_f - 2r - 1 \] correspond to the extra D4-brane. The other components represented by these factors together with \( C_0 \) are interpreted as the infinitely extending fivebrane along the orientifold which may have no contribution to the dimension of the Higgs branch.

Therefore we obtain the curve consists of \( C \), which extends to infinity, and the rational curves \( C_1, \ldots, C_{2N_f-1} \) with multiplicities. From the factorized form, the component \( C_i \) has multiplicity \( \ell_i \) where \( \ell_i = i \) for \( i = 1, \ldots, 2r \), \( \ell_i = 2r + 1 \) for \( i = 2r + 1, \ldots, 2N_f - 2r - 1 \) and \( \ell_i = 2N_f - i \) for \( i = 2N_f - 2r, \ldots, 2N_f - 1 \). Here the \( \mathbb{C}P^1 \)'s coming from the extra D4-brane are included in the multiplicity \( \ell_i \). Note that the component \( C \) intersects with \( C_{2r-1} \) and \( C_{2N_f-2r+1} \). From this we calculate the complex dimension of the \( r \)-th Higgs branch

\[ 4 \sum_{i=1}^{2N_f-1} \left[ \frac{\ell_i + \alpha_i}{2} \right] = 2(2r + 1)(N_f - r), \tag{130} \]

which agrees with \((123)\).

For the case of \( N_f \geq N_c - 1 \), there is the special point in the \( r^* \)-th Higgs branch root where \( r^* = N_f - N_c + 1 \) and the color Casimirs are given by

\[ B_{2N_c}(u_{2N_c}, v^2) = v^{2N_c} + \frac{1}{4} v^{2r^*} \Lambda_{N=2}^{2(N_c-r^*)} = v^{2r^*} \left( v^{2(N_c-r^*)} + \frac{1}{4} \Lambda_{N=2}^{2(N_c-r^*)} \right). \tag{131} \]

Then at this point the equation \((126)\) factorizes as

\[ \frac{v^2}{t} \left( t - 2v^{2N_c-1} \right) \left( t - \frac{1}{2} \Lambda_{N=2}^{2(N_c-r^*)} v^{2r^*-1} \right) = 0. \tag{132} \]

6.3 \( N = 1 \) moduli space of \( SO(2N_c+1) \) theory from M theory

As in the case of \( SO(2N_c) \), we consider the deformation of the fivebrane configuration in M theory for the \( N = 2 \) \( SO(2N_c+1) \) gauge field theory to the one for the \( N = 1 \) theory.

Let us start with the mass perturbation \( \mu \text{Tr} \Phi^2/2 \). The boundary conditions we impose are

\[ w \to \mu v \quad \text{as} \quad v \to \infty, \quad t \sim v^{2N_c-1} \]

\[ w \to 0 \quad \text{as} \quad v \to \infty, \quad t \sim \Lambda_{N=2}^{2(N_c-1-N_f)} v^{2(N_f-N_c)+1}. \tag{133} \]

The conditions \((133)\) are only satisfied at points in the moduli space where the curve becomes completely degenerate. This is possible at the special points in the Coulomb
branch or in the $r^*$-th Higgs branch. The possible fivebrane configuration which corresponds to the points in the Coulomb branch is studied in [14]. The rotated curves for the $r^*$-th Higgs branch are explicitly given by

\[
\tilde{C}_L \begin{cases} 
    t = 2v^{2N_c-1} \\
    w = \mu v 
\end{cases} \quad \text{and} \quad \tilde{C}_R \begin{cases} 
    t = \frac{1}{2} \Lambda_{N=2}^{2(2N_c-N_f)} v^{2(N_f-N_c)+1} \\
    w = 0. 
\end{cases} \quad (134)
\]

As in the $SO(2N_c)$ case the $N = 1$ supersymmetric QCD limit can be taken if we define $\Lambda_{N=1}$ as

\[
\Lambda_{N=1}^{2(2N_c-1)-2N_f} = \mu^{2N_c-1} \Lambda_{N=2}^{2(2N_c-1-N_f)}. \quad (135)
\]

This relation agrees with the field theory one. We also take the limit $\mu \to \infty$ of the curves of the remaining Higgs branch root (134) and the curves become

\[
C_L \begin{cases} 
    \tilde{t} = 2w^{2N_c-1} \\
    v = 0 
\end{cases} \quad \text{and} \quad C_R \begin{cases} 
    \tilde{t} = \frac{1}{2} \Lambda_{N=1}^{3(2N_c-1)-2N_f} v^{2(N_f-N_c)+1} \\
    w = 0. 
\end{cases} \quad (136)
\]

We will construct the configuration of M theory fivebrane corresponding to more general perturbation of the form $\Delta W$ (10). The boundary conditions which we will impose are

\[
\begin{align*}
    w &\to \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1} \text{ as } v \to \infty, \quad t \sim \Lambda_{N=2}^{2(2N_c-1-N_f)} v^{2(N_f-N_c)+1} \\
    w &\to 0 \text{ as } v \to \infty, \quad t \sim v^{2N_c-1}. \quad (137)
\end{align*}
\]

At the point in the moduli space where $l$ mutually local dyons are massless, we can factorize the equation $y^2/v^4 = C^2 - G = S^2T$ where $C = \frac{B_{2N_c}}{v}$, $G = \Lambda_{N=2}^{2(2N_c-1-N_f)} \prod_{i=1}^{N_f} (v^2 - m_i^2)$ and

\[
S = \frac{1}{v} \prod_{i=1}^{l} (v^2 - p_i^2), \quad T = \prod_{j=1}^{2N_c-2l} (v^2 - q_j^2). \quad (138)
\]

Under the assumption concerning the projected curve on $v, t$ space and the form of $w$, we find

\[
w = N + \frac{H'}{S} (t - C), \quad (139)
\]

where $N$ and $H'$ are some polynomials in $v$. Remember that the fivebrane configuration has to be invariant under the reflection $w \to -w, v \to -v, t \to -t$ [17], we see that $C, S$ and $T$ transform to $-C, -S$ and $T$ under this reflection respectively. Thus we require that $H \equiv H'/v = H(v^2)$, and $N$ transforms to $-N$. The boundary conditions (137) completely fix $N$ as (107) and the behavior of $w$ in the asymptotic region $t \sim v^{2N_c-1}$ as (108). Note that $w$ satisfies (109) in the $SO(2N_c+1)$ case also.
We now consider the $SO(2N_c + 1)$ pure Yang-Mills theory. From the first boundary condition in (137) we find

$$w = 2vH(v^2)\frac{B_{2N_c}(v^2)}{\prod_{i=1}^{2N_c}(v^2 - p_i^2)} + \mathcal{O}(v^{-1}) = \sum_{k=1}^{2N_c} \mu_{2k} v^{2k-1} + \mathcal{O}(v^{-1}),$$

(140)

which determines $H(v^2)$ in terms of $\mu_{2k}$. The equation (140) agrees with the field theory equation (119). In the case of $SO(2N_c + 1)$ with $N_f$ flavors, we can also compute the dyon vevs along the singular locus as in the previous calculations. At the $r^*$-th Higgs branch the curves (134) remain two rational curves after the rotation as

$$\tilde{C}_L \left\{ \begin{array}{c} t = 2v^{2N_c-1} \\ w = 0 \end{array} \right. \quad \tilde{C}_R \left\{ \begin{array}{c} t = \frac{1}{2} A_{N=2}^{2(2N_c-N_f-1)} v^{2(N_f-N_c)+1} \\ w = \sum_{k=1}^{N_c} \mu_{2k} v^{2k-1}. \end{array} \right.$$

(141)

Thus from the brane picture we can observe that the $r^*$-th Higgs branch root is not lifted for arbitrary values of the parameters $\mu_{2k}$.

We will compute the finite values of $w$ at $t = 0$, $v^2 = m_i^2$ which correspond to the eigen values of the meson vev and compare these to the field theory results for the case with one massless dyon. Assuming $N_f < N_c - 1$, we find

$$w_{(i)} = \pm \frac{1}{2} \mu_{2N_c} A_{N=2}^{2N_c-N_f} \sum_{j=1}^{N_f} \frac{\prod_{j=1}^{N_f}(p_j^2 - m_j^2)^{\frac{1}{2}}}{m_j^2 - p_j^2}, \quad p_j^2 = \frac{\mu^2 (N_c-1)}{2N_c},$$

(142)

which agrees with (122) up to a factor $\sqrt{2}$. Here we change the perturbation $\Delta W$ and the boundary conditions (137) as in the case of $Sp(2N_c)$.

The meson vev for the case with more than one massless dyon is calculated as

$$w_{(i)} = \pm \Lambda_{N=2}^{2N_c-1-N_f} \sum_{j=1}^{N_f} \frac{p_j H(p_j^2)}{\prod_{k\neq j} \left( p_j^2 - p_k^2 \right)^{\frac{1}{2}}} \prod_{j=1}^{N_f} \left( p_j^2 - m_j^2 \right)^{\frac{1}{2}}$$

$$= \pm \frac{\partial}{\partial m_i} \left( \Lambda_{N=2}^{2N_c-2-N_f} \sum_{j=1}^{N_f} \det (p_j - m)^{\frac{1}{2}} w_j p_j \right).$$

(143)

7 Conclusions

We have obtained the descriptions of the moduli space of vacua of the four dimensional $N = 1$ and $N = 2$ supersymmetric gauge theories with the gauge groups $Sp(2N_c)$, $SO(2N_c)$ and $SO(2N_c + 1)$ using the M theory fivebrane and the orientifold plane.
First of all, we have constructed the fivebrane configuration corresponding to the Higgs branches of the $N = 2$ supersymmetric gauge theories, especially, the $r^*$-th Higgs branch root which is analogous to the baryonic branch root of $SU(N_c)$ theory. From the field theory analysis, it is known that this root is not lifted after the adjoint mass perturbation to break $N = 2$ to $N = 1$ supersymmetry. This phenomenon has been verified from the explicit construction of the corresponding M theory fivebrane configuration. It is important that in M theory we should view the orientifold plane as the deformation of the fivebrane induced by the presence of the RR charges.

We have also studied the monopole condensations and the meson vev in the $N = 1$ supersymmetric gauge theories by considering the rotated NS 5-branes configurations. The results agree with the field theory results for the vacua in the phase with a single confined photon.

In the fivebrane framework, $N = 1$ gauge theories with the Landau-Ginzburg type superpotential [36] [29] [30] and the non trivial fixed points of the $N = 1$ supersymmetric gauge theories [24] [26] [31] are also studied along the lines of [19] . It will be interesting to study these issues for the gauge groups $Sp(2N_c)$ and $SO(N)$.

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Note added:

As this article was being completed, we received the preprints [37] [38] which overlap parts of the present work. However the boundary conditions (50), (104) and (137) for the M theory fivebrane we impose are different from those in [37] [38].
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