Inert states of spin-5 and spin-6 Bose–Einstein condensates

Marcin Fizia and Krzysztof Sacha

Instytut Fizyki Imienia Mariana Smoluchowskiego and Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, Ulica Reymonta 4, PL-30-059 Kraków, Poland

E-mail: krzysztof.sacha@uj.edu.pl

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Abstract
In this paper, we consider spinor Bose–Einstein condensates with spin \( f = 5 \) and \( f = 6 \) in the presence and absence of an external magnetic field at the mean field level. We calculate all of the so-called inert states of these systems. Inert states are a very unique class of stationary states because they remain stationary while Hamiltonian parameters change. Their existence comes from Michel’s theorem. For illustration of symmetry properties of the inert states we use a method that allows for the classification of the systems as a polyhedron with \( 2f \) vertices proposed by Barnett et al (2006 Phys. Rev. Lett. 97 180412).

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(Some figures may appear in colour only in the online journal)

1. Introduction

The properties of Bose–Einstein condensates have been found mostly since 1995 when the first condensate was performed experimentally [1–5]. After optical dipole traps were developed, it was possible to create not only scalar but also spinor condensates. Computing properties of spinor condensates is more complicated than those of scalar ones [4–6]. In scalar condensates, only one \( s \)-wave channel of interaction is possible since only one kind of boson is present. On the other hand, in the spinor condensate of spin-\( f \) atoms, there are \( 2f + 1 \) internal states of atoms. Therefore, the number of interaction channels increases. Because of this complexity finding stationary states for spinor condensates is not trivial even within the mean field approximation.

In this paper, we will first briefly show how to construct the Hamiltonian for spinor condensates and how to perform calculations within mean field theory for that system [6]. After that we will present Michel’s theorem and point out how it could help us find stationary states [7]. Michel’s theorem ensures us that there have to exist so-called inert states. Their name comes from the fact that they remain stationary for every value of Hamiltonian’s parameters (such as the external magnetic field, density, etc). Finally, we will show how to use Michel’s
2. Hamiltonian and mean field theory

That part is very well described in [6] so we will briefly point a way to construct the Hamiltonian and its most important assumptions. In spinor condensates, atoms possess internal degrees of freedom characterized by a state of hyperfine structure. We should remember that usually in spinor Bose–Einstein condensate description by \( f \) we understand the total angular momentum of all electrons and nuclei of atom which we will just call spin. So the wavefunction of the spinor condensate in the mean field description should be written as

\[
\Psi(\vec{r}_1, \ldots, \vec{r}_N) = \prod_{i=1}^{N} \begin{bmatrix} \Psi_f(\vec{r}_i) \\ \vdots \\ \Psi_{-f}(\vec{r}_i) \end{bmatrix},
\]

where \( \Psi_m(\vec{r}) \) is the wavefunction component describing particles with the projection \( m \) of spin on the chosen quantization axis.

Let us denote \( \hat{\psi}^\dagger_m(\vec{r}) \) as a bosonic field operator which creates a boson at point \( \vec{r} \) with spin projection equal to \( m \). In the absence of the external trapping potential but in the presence of an external magnetic field, the energy of this system can be decomposed into kinetic energy, energy of Zeeman effects (linear and quadratic) and interaction energy. So we can write

\[
\hat{H} = \hat{H}_0 + \hat{V},
\]

where by \( \hat{H}_0 \) we understand the part of the Hamiltonian without interaction between particles. Assuming the direction of the external magnetic field \( \vec{B} \) along the \( \vec{z} \)-axis, it can be decomposed as follows:

\[
\hat{H}_0 = \int d^3 \vec{r} \sum_{m_1, m_2 = -f}^f \hat{\psi}^\dagger_m(\vec{r}) \left[ -\frac{\hbar^2 \nabla^2}{2M} - p(f_z)_{m_1 m_2} + q(f_z^2)_{m_1 m_2} \right] \hat{\psi}_{m_2}(\vec{r}),
\]

where \( \hat{\psi} \) is the mass of atoms, the \( p \sim B \) factor describes the linear Zeeman effect, the \( q \sim B^2 \) factor describes the quadratic Zeeman effect and \( (f_z)_{m_1 m_2} \) is the spin matrix (in the chosen coordinate system, it is diagonal \( (f_z)_{m_1 m_2} = m_1 \delta_{m_1, m_2} \)).

In dilute and ultra-cold atomic gases, only two-body \( s \)-wave interaction channels are important. From wavefunction symmetry considerations we can find that the only possible channels for identical bosons correspond to even values of the total spin \( F \) of two interacting particles. So the interaction part of the Hamiltonian can be decomposed in the following way:

\[
\hat{V} = \sum_{F=0,2,\ldots,2f} \hat{V}^{(F)}.
\]

We can construct (using Clebsch–Gordan coefficients) operators which annihilate pairs of bosons with total angular momentum \( F \) and its projection \( M \) on the \( \vec{z} \)-axis:

\[
\hat{A}_{FM}(\vec{r}, \vec{r}') = \sum_{m_1, m_2 = -f}^f \langle F, M | f, m_1; f, m_2 \rangle \hat{\psi}_{m_1}(\vec{r}) \hat{\psi}_{m_2}(\vec{r}').
\]
Now we can see that operators $\hat{V}^{(F)}$ can be written as

$$\hat{V}^{(F)} = \frac{1}{2} \int d^3r \int d^3r' \nu^{(F)}(\vec{r}, \vec{r}') \sum_{M=-F}^F \hat{A}_{FM}(\vec{r}, \vec{r}') \hat{A}_{FM}(\vec{r}, \vec{r}') .$$  \quad (6)$$

The function $\nu^{(F)}(\vec{r}, \vec{r}')$ describes spatial dependence of interactions between two particles in the channel $F$. This function using $s$-wave scattering length $a_F$ can be assumed to be

$$\nu^{(F)}(\vec{r}, \vec{r}') = g_F \delta^{(3)}(\vec{r} - \vec{r}') ,$$  \quad (7)

where $g_F = \frac{4\pi \hbar^2}{M a_F}$.

We can also use two relations to simplify our final form of the Hamiltonian. Namely

- the Hilbert space completeness relation:

$$\sum_{F=0,2,\ldots}^M \sum_{M=-F}^F \hat{A}^\dagger_{FM}(\vec{r}) \hat{A}_{FM}(\vec{r}) = : \hat{n}(\vec{r}) \hat{n}(\vec{r}) : ,$$  \quad (8)

where $: :$ stands for normal ordering of the creation and annihilation operators,

- the relation coming from the composition of angular momentum

$$\hat{F}(\vec{r}) \hat{F}(\vec{r}) := \sum_{F=0,2,\ldots}^M \sum_{M=-F}^F \left[ \frac{1}{2} F(F + 1) - f(f + 1) \right] \hat{A}^\dagger_{FM}(\vec{r}) \hat{A}_{FM}(\vec{r}) .$$  \quad (9)

In the particle number conserving version of the mean field theory, we have to calculate the mean value of the Hamiltonian assuming that $N$ bosons are in the same state; see equation (1). We also assume that the spinor and spatial parts of the wavefunction factorize, i.e. $|\Psi(\vec{r})\rangle = \phi_0(\vec{r})|\xi\rangle$. In the absence of an external trapping potential, the ground state mode $\phi_0 = \frac{1}{\sqrt{V}}$, where $V$ is the volume of the system, and the energy reduces to the expression

$$E(\xi) = \langle \xi | \hat{H} | \xi \rangle ,$$  \quad (10)

where $|\xi\rangle = \frac{1}{\sqrt{N}} \left( \sum_{m=-f}^{f} \xi_m \hat{a}_{m}^{\dagger} \right)^N |0\rangle$. The vector $\xi = (\xi_1, \ldots, \xi_f)^T$ will be called an order parameter. This parameter completely describes the state of spinor condensate. Neglecting terms proportional to the $1/N$ final result for the energy per particle reads [6]

$$\varepsilon(\xi) = \frac{E(\xi)}{N} = \sum_{m=-f}^{f} \left[ |\xi_m|^2 (-pm + qm^2) + \frac{1}{2} \rho_c 0 + \frac{1}{2} \rho c_1 |\vec{F}|^2 \right] + \sum_{F=0,2,\ldots}^M \frac{d_F \rho}{2} \sum_{M=-F}^F A_{FM} \hat{A}_{FM} ,$$  \quad (11)

where $\rho$ is the particle density. Parameters $c_0, c_1, d_i$ are linear combinations of coefficients $g_i$ and are presented in the appendix. Functions used in (11) have the following forms:

$$F_{\mu} = \sum_{m_1, m_2=-f}^{f} \xi_{m_1}^{*} (f_\mu)_{m_1 m_2} \xi_{m_2} ,$$  \quad (12)

$$A_{FM} = \sum_{m_1, m_2=-f}^{f} \langle F, M | f, m_1; f, m_2 \rangle \xi_{m_1} \xi_{m_2} .$$  \quad (13)

In order to calculate all stationary states of energy (11), one has to derive the corresponding Euler–Lagrange equations and solve them, which usually has to be done numerically. However, a certain class of stationary states can be found analytically by consideration of symmetries of the system. These are the so-called inert states which we present in the following sections.
3. Michel’s theorem [7]

Michel’s theorem allows us to find special stationary states. They are called inert states because they are stationary for any values of Hamiltonian’s parameters (in our case the parameters are particle density, scattering lengths and external magnetic field). The stationarity of these states is guaranteed by symmetries of a system.

When an external magnetic field is present, the symmetry group of the energy function (11) is a product group $G_{B \neq 0} = U(1)_F \times U(1)$. The part $U(1)_F$ is rotation about the field direction (i.e. the $\vec{z}$-axis) and the part $U(1)$ is just global phase changing symmetry. If an external magnetic field is not present, the symmetry group is $G_{B = 0} = SO(3) \times U(1)$. The meaning of the part $U(1)$ is the same, i.e. it is the global phase changing symmetry, whereas $SO(3)$ is the group of rotations in three-dimensional space.

For our further considerations, we should call the symmetry group by $G$ and by the $\varepsilon(\zeta)$ function defined in a finite-dimensional smooth domain $\mathcal{M}$ which has this symmetry. By orbit $G(\zeta)$ of point $\zeta$, we understand the set of points being results of the action of all group elements on point $\zeta$:

$$G(\zeta) = \{g \zeta | g \in G\} \subset \mathcal{M}.$$ 

Isotropy group of point $\zeta$ is a set of group elements which act on $\zeta$ as an identity element

$$G_\zeta = \{g \in G | g \zeta = \zeta\} \subset G.$$ 

We say that any two isotropy groups are conjugate if

$$\exists g \in G : G_\zeta = g G_\zeta' g^{-1}.$$ 

Stratum is the union of all orbits of points whose isotropy groups are conjugate

$$S(\zeta) = \bigcup_{\zeta' : G_\zeta = g G_\zeta' g^{-1}} G(\zeta') \subset \mathcal{M}.$$ 

We say that an orbit is isolated in its stratum if there exists the neighbourhood of this orbit $U \supset G(\zeta)$ so that its intersection with the stratum gives that the orbit

$$G(\zeta) = U \cap S(\zeta).$$ 

Michel proved the following theorem.

**Theorem 1** (Michel). Every function defined in a finite-dimensional smooth domain $\mathcal{M}$ and having real numbers as the co-domain which is invariant under elements of the compact Lie group $G$, i.e.

$$\forall \varepsilon : \mathcal{M} \rightarrow \mathbb{R} : \forall g \in G \forall \zeta \in \mathcal{M} : \varepsilon(\zeta) = \varepsilon(g \zeta),$$

has common orbits of extrema which are orbits being isolated in their stratum.

From this theorem, we know that for our spinor condensate system we are able to find some stationary points $\zeta$ just by finding isolated orbits. The crucial point of the proof of Michel’s theorem is to show that the gradient of function $\varepsilon(\zeta)$ is a vector which is tangential to the stratum. We know from the definition of the orbit that the gradient of function is orthogonal to the orbit. And if any orbit is isolated in its stratum we know that the gradient has to be null on it so this orbit has to be extremal (minimal, maximal or saddle) point.

Barnett et al [10] showed how to translate symmetries of spin-$f$ states $|\zeta\rangle$ into symmetries of a polyhedron with $2f$ vertices. We should not redo it here but point out the most important steps. We have to find a set of maximally polarized states, i.e. $\bar{f}|\chi\rangle = f|\chi\rangle$, which are orthogonal to $|\zeta\rangle$:

$$\langle \zeta | \chi \rangle = 0.$$ (14)
The states $|\chi\rangle$ can be parameterized as $\chi = e^{i\phi} \tan \frac{\theta}{2}$, where $\phi$ and $\theta$ denote the positions of vertices of a polyhedron on the unit sphere. For our considerations, the most important corollary is that the maximal number of found vertices is equal to $2f$ because equation (14) is actually a polynomial equation of $2f$ degree. Therefore, when we are looking for inert states of the system, we may consider only this point group whose corresponding regular polyhedron has no more than $2f$ vortices.

Now we should present a way of calculating and finding inert states [6, 8, 9]. In order to do that we should go through all subgroups of a global symmetry group of a system. For each subgroup, we should find states which are invariant under this subgroup. In that case, this subgroup is their isotropy group.

First we will show how to calculate inert states for subgroups with only one generator. In our case, such generators have the form $\hat{K} = e^{i\Omega_1 \phi}$ (i.e. they are rotation operators about the $\hat{\Omega}_1$-axis by an angle $\phi$). It is worth pointing out that if $\hat{K}$ belongs to the symmetry group of the system so does $e^{i\gamma} \hat{K}$, because in our case the symmetry group of the system is a direct product of a rotation group and the global phase-changing symmetry group. All eigenvalues of $\hat{K}$ can be written as $e^{i\gamma}$. If we find all eigenvectors of a generator $\hat{K}$ we can identify some of them to be inert states. Through eigensubspaces of $\hat{K}$ we should choose those which are non-degenerate. That means that for each eigenvalue $e^{i\gamma}$ there exists only one eigenvector $\zeta_\gamma$. Therefore, we know that the group generated by $e^{-i\gamma} \hat{K}$ is the isotropy group of this vector. Moreover, we see that there is only one orbit whose isotropy group is this group. Thus, this orbit must be isolated in its stratum because stratum consists of this orbit only. Then, Michel’s theorem tells us that $\zeta_\gamma$ is an inert state.

If some subgroup has two or more generators, as in the case of point groups, we ought to find vectors which are common eigenvectors for all generators. If there is only one common eigenvector, it is an inert state of the system. When on the other hand the common eigenspace is not one-dimensional, we still can calculate stationary states which are not inert states. To this end we have to find the extrema of the energy of the system within the common eigensubspace. The existence of such stationary states comes from another theorem of Michel [6]. However, these states depend on the Hamiltonian parameters. We will not consider them in this paper.

4. Inert states for spin-5 condensates with an external magnetic field

Global symmetry group of spinor condensates with an external magnetic field is the direct product group $G_{B} = U(1)_{F_z} \times U(1)$. The general group element has the form

$$g(\phi, \gamma) = e^{i\phi} e^{-i\gamma}$$

where $\phi$ is the global phase change and $\gamma$ is the angle of rotation about the $\hat{z}$-axis which coincides with the direction of the magnetic field. We have found 11 inert states for this case. All of them are inert states whose isotropy groups are continuous. Analysis of discrete subgroups does not reveal any additional inert state.

We present the inert states with their isotropy group’s generator and the corresponding energy. Isotropy groups are denoted by $U(1)_{F_z+\alpha(\phi)}$. We have decided to illustrate the symmetry only for one state. One way is that proposed by Barnett et al [10]. The second one is by using spherical harmonic expansion. That is, every bosonic spin state can be expanded in terms of spherical harmonics as follows:

$$\Psi(\theta, \phi) = \sum_{m=-f}^{f} \zeta_m Y_m^f(\theta, \phi).$$

Figure 1. Visualization of the state \( \zeta_{F5+} \), equation (17), with Barnett et al.'s method [10]. The circle indicates the positions of 2\( f = 10 \) vertices of a polyhedron. In the presented case, all vertices are situated on the same point (that is, indicated by the number 10 in the figure) and the polyhedron is reduced to a single point.

We can visualize states by painting surfaces \( |\Psi(\theta, \phi)|^2 = 1 \). The colour of surface should express the phase of state at a given point [10, 12].

- **\( U(1)_{F+5\phi} \) generator:** \( \{e^{i5\phi}e^{-i\phi}\hat{f}_z\} \)

  \[
  \zeta_{F5+} = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{17}
  \]

  \[
  \varepsilon(\zeta_{F5+}) = \frac{1}{2}\rho c_0 + \frac{25}{2}\rho c_1 + 25q - 5p; \tag{18}
  \]

  the state is visualized in figures 1 and 2.

- **\( U(1)_{F+4\phi} \) generator:** \( \{e^{i4\phi}e^{-i\phi}\hat{f}_z\} \)

  \[
  \zeta_{F4+} = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{19}
  \]

  \[
  \varepsilon(\zeta_{F4+}) = \frac{1}{2}\rho c_0 + 8\rho c_1 + 16q - 4p; \tag{20}
  \]

- **\( U(1)_{F+3\phi} \) generator:** \( \{e^{i3\phi}e^{-i\phi}\hat{f}_z\} \)

  \[
  \zeta_{F3+} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{21}
  \]

  \[
  \varepsilon(\zeta_{F3+}) = \frac{1}{2}\rho c_0 + \frac{8}{7}\rho c_1 + \frac{14}{72}\rho d_4 + 9q - 3p; \tag{22}
  \]

- **\( U(1)_{F+2\phi} \) generator:** \( \{e^{i2\phi}e^{-i\phi}\hat{f}_z\} \)

  \[
  \zeta_{F2+} = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T \tag{23}
  \]

  \[
  \varepsilon(\zeta_{F2+}) = \frac{1}{2}\rho c_0 + 2\rho c_1 + \frac{35}{280}\rho d_4 + \frac{84}{915}\rho d_6 + 4q - 2p. \tag{24}
  \]
Figure 2. Visualization of the state $\zeta_{F_5}$, equation (17), by means of spherical harmonics. Surface $|\Psi_1(\theta,\phi)|^2 = 1$, see equation (16), is plotted in the figure. The phase of the state at a given point is expressed in shades of grey.

- $U(1)_{F_+}^{+\phi}$ generator: $\{e^{i\phi} e^{i\phi \hat{f}_z}\}$
  \[ \zeta_{F_1^+} = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0)^T \]  
  \[ \varepsilon(\zeta_{F_1^+}) = \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{25}{5280} \rho d_2 + \frac{10}{117} \rho d_4 + \frac{14}{167} \rho d_6 + q - p. \]  

- $U(1)_{F_+}^{-\phi}$ generator: $\{e^{-i\phi} e^{-i\phi \hat{f}_z}\}$
  \[ \zeta_{F_1^+} = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0)^T \]  
  \[ \varepsilon(\zeta_{F_1^+}) = \frac{1}{2} \rho c_0 + \frac{2}{25} \rho c_0 + \frac{25}{2860} \rho d_2 + \frac{10}{117} \rho d_4 + \frac{14}{167} \rho d_6 + q - p. \]  

- $U(1)_{F^-}^{+\phi}$ generator: $\{e^{i\phi} e^{-i\phi \hat{f}_z}\}$
  \[ \zeta_{F_2^-} = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0)^T \]  
  \[ \varepsilon(\zeta_{F_2^-}) = \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{25}{5280} \rho d_2 + \frac{10}{117} \rho d_4 + \frac{14}{167} \rho d_6 + 4q + 2p. \]
by applying some elements of the group $\mathbb{G}_8$.

In this case, the global symmetry group is $G_{B=0} = SO(3) \times U(1)$. The general group element can be written as

$$g(\alpha, \beta, \gamma, \phi) = e^{i\phi} e^{-i\beta \hat{J}_z} e^{-i\gamma \hat{J}_y},$$

where $\phi$ is the global phase change and $\alpha, \beta, \gamma$ are Euler angles. This group has the same continuous subgroups as the global symmetry group considered in the previous section. So we expect that the corresponding inert states are the same. It easy to show that, in the case without magnetic field, the energies of some of these inert states are equal. That is, for $p = q = 0$, we get $\epsilon(\zeta_{F3+}) = \epsilon(\zeta_{F5-}), \epsilon(\zeta_{F3+}) = \epsilon(\zeta_{F4-}), \epsilon(\zeta_{F4+}) = \epsilon(\zeta_{F5-}), \epsilon(\zeta_{F3+}) = \epsilon(\zeta_{F2+})$ and $\epsilon(\zeta_{F1+}) = \epsilon(\zeta_{F1-})$. We can show that members of each pair can be transformed to each other by applying some elements of the group $G_{B=0}$. So they belong to the same orbit and therefore they are actually the same states. We have decided not to show these states one more time but present how to change one vector into another by action with group elements:

$$g(0, \pi, 0)\zeta_{F5+} = \zeta_{F5-}$$

$$g(0, \pi, 0)\zeta_{F4+} = \zeta_{F4-}$$

$$g(0, \pi, 0)\zeta_{F3+} = \zeta_{F3-}$$

$$g(0, \pi, 0)\zeta_{F2+} = \zeta_{F2-}$$

$$g(0, \pi, 0)\zeta_{F1+} = \zeta_{F1-}.$$

Now we should consider all discrete point groups. Let us begin with the dihedral group $D_n$. Generators of this group are $\hat{C}_{n,z}$ (rotation about the $z$-axis by angle $2\pi/n$) and $\hat{C}_{2,x}$.
(rotation about the $\vec{z}$-axis by angle $\pi$ with simultaneous phase change by $\pi$). We would like to stress that in our consideration, these generators will be supplemented with additional phase factors. This is necessary in order to obtain proper isotropy groups and we are allowed to do that because the symmetry group of our system contains global phase changing freedom. It is worth mentioning that the dihedral group with infinite $n$ is also an isotropy group of the previously found state $\zeta_{\rho}$, equation (27). In the limit $n \to \infty$, the dihedral group becomes continuous. Other inert states for the dihedral group are presented here:

- $D_{10}$ generators: \( \{ e^{i\pi \hat{C}_{2,2}}, e^{i\pi \hat{C}_{10,2}} \} \)
  \[
  \zeta_{D_{10}} = \frac{1}{\sqrt{2}} (1, 0, 0, 0, 0, 0, 0, 0, -1)^T. \tag{45}
  \]

  We can show that $\hat{C}_{2,2} \zeta_{D_{10}} = e^{-i\pi} \zeta_{D_{10}}$ and $\hat{C}_{10,2} \zeta_{D_{10}} = e^{-i\pi} \zeta_{D_{10}}$. Thus, in order to obtain the isotropy group, we have to supplement the original dihedral group generators with the phase factor $e^{i\pi}$:
  \[
  \epsilon(\zeta_{D_{10}}) = \frac{1}{2} \rho c_0 + \frac{1}{27} \rho d_0 + \frac{75}{377} \rho d_2 + \frac{9}{127} \rho d_4 + \frac{15}{1749} \rho d_6. \tag{46}
  \]

- $D_8$ generators: \( \{ e^{i\pi \hat{C}_{2,2}}, e^{i\pi \hat{C}_{8,2}} \} \)
  \[
  \zeta_{D_{10}} = \frac{1}{\sqrt{2}} (0, 1, 0, 0, 0, 0, 0, 0, -1, 0)^T \tag{47}
  \]

  \[
  \epsilon(\zeta_{D_{10}}) = \frac{1}{2} \rho c_0 + \frac{1}{27} \rho d_0 + \frac{1}{143} \rho d_2 + \frac{9}{143} \rho d_4 + \frac{96}{572} \rho d_6. \tag{48}
  \]

- $D_6$ generators: \( \{ e^{i\pi \hat{C}_{2,2}}, e^{i\pi \hat{C}_{6,2}} \} \)
  \[
  \zeta_{D_{10}} = \frac{1}{\sqrt{2}} (0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0)^T \tag{49}
  \]

  \[
  \epsilon(\zeta_{D_{10}}) = \frac{1}{2} \rho c_0 + \frac{1}{27} \rho d_0 + \frac{1}{143} \rho d_2 + \frac{9}{143} \rho d_4 + \frac{2689}{22440} \rho d_6. \tag{50}
  \]

- $D_4$ generators: \( \{ e^{i\pi \hat{C}_{2,2}}, e^{i\pi \hat{C}_{4,2}} \} \)
  \[
  \zeta_{D_{10}} = \frac{1}{\sqrt{2}} (0, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0)^T \tag{51}
  \]

  \[
  \epsilon(\zeta_{D_{4}}) = \frac{1}{2} \rho c_0 + \frac{1}{27} \rho d_0 + \frac{3}{143} \rho d_2 + \frac{9}{143} \rho d_4 + \frac{96}{572} \rho d_6. \tag{52}
  \]

- $D_2$ generators: \( \{ e^{i\pi \hat{C}_{2,2}}, e^{i\pi \hat{C}_{2,2}} \} \)
  \[
  \zeta_{D_{2}} = \frac{1}{\sqrt{2}} (0, 0, 0, 1, 0, -1, 0, 0, 0, 0)^T \tag{53}
  \]

  \[
  \epsilon(\zeta_{D_{2}}) = \frac{1}{2} \rho c_0 + \frac{1}{27} \rho d_0 + \frac{1}{143} \rho d_2 + \frac{9}{143} \rho d_4 + \frac{44}{572} \rho d_6. \tag{54}
  \]
Another point group that should be considered is the tetrahedron group $T$ with generators
\[ \hat{C}_3, x+y+z \text{ (rotation about the } \vec{x}+\vec{y}+\vec{z} \text{ axis by angle } 2\pi/3) \text{ and } \hat{C}_2, z \text{ (rotation about the } \vec{z} \text{ axis by angle } \pi). \]

There is only one inert state whose isotropy group is this group, namely
\[ \zeta_T = \frac{1}{2} (0, 1, 0, i, 0, 0, -i, 0, -1, 0)^T. \]

One can check that
\[ \hat{C}_3, x+y+z \zeta_T = e^{i \frac{2\pi}{3}} \zeta_T, \]
\[ \hat{C}_2, z \zeta_T = \zeta_T. \]

Thus, the generators of the isotropy group are \{\hat{C}_2, e^{-i2\pi/3} \hat{C}_3, x+y+z\}:
\[ \varepsilon(\zeta_T) = \frac{1}{2} \rho c_0 + \frac{6}{143} \rho d_2 + \frac{15}{286} \rho d_4 + \frac{168}{935} \rho d_6. \]

The state with tetrahedral symmetry is visualized in figures 3 and 4.

There are no states with the octahedral group as isotropy group. We also know that the maximal number of vertices of a polyhedron which represents states for spin-5 condensates is 10. Since an icosahedron has 12 vortices there cannot be any inert state having the icosahedral group as its isotropy group. Therefore, the states presented are all the inert states one can find for spin-5 condensates.

6. Inert state for the spin-6 condensate with an external magnetic field

Here we have the same global symmetry group as for the corresponding spin-5 case. The calculations are similar as before and results are as follows:

- $U(1)_{F+6\phi}$
  generator: \{e^{6\phi} e^{-i\phi} \}
  \[ \xi_{F6+} = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \]
  \[ \varepsilon(\xi_{F6+}) = \frac{1}{2} \rho c_0 + 18 \rho c_1 + 36q - 6p, \]
Figure 4. The same as in figure 2 but for the state $\xi_T$, equation (55).

- $U(1)_{F_{5}+5\phi}$
  generator: $\{e^{5i\phi}e^{-i\phi}\hat{\xi}\}$
  \[\xi_{F5+} = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T\] (61)
  \[\varepsilon(\xi_{F5+}) = \frac{1}{2}\rho c_0 + \frac{25}{2}\rho c_1 + 25q - 5p.\] (62)

- $U(1)_{F_{4}+4\phi}$
  generator: $\{e^{4i\phi}e^{-i\phi}\hat{\xi}\}$
  \[\xi_{F4+} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T\] (63)
  \[\varepsilon(\xi_{F4+}) = \frac{1}{2}\rho c_0 + 8\rho c_1 + \frac{45}{255}\rho ds + 16q - 4p.\] (64)

- $U(1)_{F_{3}+3\phi}$
  generator: $\{e^{3i\phi}e^{-i\phi}\hat{\xi}\}$
  \[\xi_{F3+} = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T\] (65)
  \[\varepsilon(\xi_{F3+}) = \frac{1}{2}\rho c_0 + \frac{9}{2}\rho c_1 + \frac{42}{335}\rho ds + \frac{12}{17}\rho ds + 9q - 3p.\] (66)

- $U(1)_{F_{2}+2\phi}$
  generator: $\{e^{2i\phi}e^{-i\phi}\hat{\xi}\}$
  \[\xi_{F2+} = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T\] (67)
  \[\varepsilon(\xi_{F2+}) = \frac{1}{2}\rho c_0 + 2\rho c_1 + \frac{245}{3357}\rho ds + \frac{252}{3357}\rho ds + \frac{18}{257}\rho ds + 4q - 2p.\] (68)
\begin{itemize}
  \item $U(1)_{F,\phi}$
    \begin{itemize}
      \item generator: $\{e^{i\theta \xi_{F,\phi}}\}$
      \begin{equation}
        \xi_{F,\phi} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,\rho}$
    \begin{itemize}
      \item generator: $\{e^{-i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,\rho} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,-\phi}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,-\phi} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,-\rho}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,-\rho} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,\phi}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,\phi} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,-\phi}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,-\phi} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,\rho}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,\rho} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,-\rho}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,-\rho} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,\phi}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,\phi} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,-\phi}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,-\phi} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,\rho}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,\rho} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
  \item $U(1)_{F,-\rho}$
    \begin{itemize}
      \item generator: $\{e^{i\phi \xi_{F,\rho}}\}$
      \begin{equation}
        \xi_{F,-\rho} = (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T
      \end{equation}
    \end{itemize}
\end{itemize}

\begin{align}
  \xi_{F,+} &= (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T \tag{69} \\
  \varepsilon(\xi_{F,+}) &= \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{21}{250} \rho d_2 + \frac{140}{243} \rho d_4 + \frac{210}{355} \rho d_6 + \frac{180}{271} \rho d_8 + q - p. \tag{70} \\
  \xi_{F,-} &= (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T \tag{71} \\
  \varepsilon(\xi_{F,-}) &= \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{21}{250} \rho d_2 + \frac{140}{243} \rho d_4 + \frac{210}{355} \rho d_6 + \frac{180}{271} \rho d_8, \tag{72} \\
  \xi_{F_2+} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{73} \\
  \varepsilon(\xi_{F_2-}) &= \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{21}{250} \rho d_2 + \frac{140}{243} \rho d_4 + \frac{210}{355} \rho d_6 + \frac{180}{271} \rho d_8 + q + p, \tag{74} \\
  \xi_{F_3+} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{75} \\
  \varepsilon(\xi_{F_3-}) &= \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{21}{250} \rho d_2 + \frac{140}{243} \rho d_4 + \frac{210}{355} \rho d_6 + \frac{180}{271} \rho d_8 + 4q + 2p. \tag{76} \\
  \xi_{F_4+} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{77} \\
  \varepsilon(\xi_{F_4-}) &= \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{21}{250} \rho d_2 + \frac{140}{243} \rho d_4 + \frac{210}{355} \rho d_6 + 9q + 3p. \tag{78} \\
  \xi_{F_5+} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{79} \\
  \varepsilon(\xi_{F_5-}) &= \frac{1}{2} \rho c_0 + \frac{1}{2} \rho c_1 + \frac{21}{250} \rho d_2 + \frac{140}{243} \rho d_4 + \frac{210}{355} \rho d_6 + 16q + 4p, \tag{80} \\
  \xi_{F_6+} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{81} \\
  \varepsilon(\xi_{F_6-}) &= \frac{1}{2} \rho c_0 + \frac{25}{2} \rho c_1 + 25q + 5p. \tag{82} \\
  \xi_{F_7+} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \tag{83} \\
  \varepsilon(\xi_{F_7-}) &= \frac{1}{2} \rho c_0 + \frac{18}{5} \rho c_1 + 36q + 6p. \tag{84}
\end{align}
7. Inert states for spin-6 condensates without an external magnetic field

Similarly as in the corresponding case of spin-5 condensates, the inert states found in the presence of an external magnetic field are also inert states without the field. Also similarly as previously, pairs of these states are actually the same states because in the absence of the field they can be transformed into each other by applying elements of the global symmetry group of the system. These transformations are shown below:

\[ g(0, \pi, 0, 0) \xi_{F6+} = \xi_{F6-} \]  
\[ g(0, \pi, 0, \pi) \xi_{F5+} = \xi_{F5-} \]  
\[ g(0, \pi, 0, 0) \xi_{F4+} = \xi_{43-} \]  
\[ g(0, \pi, 0, \pi) \xi_{F3+} = \xi_{F3-} \]  
\[ g(0, \pi, 0, 0) \xi_{F2+} = \xi_{F2-} \]  
\[ g(0, \pi, 0, \pi) \xi_{F1+} = \xi_{F1-}. \]

The state \( \xi_\rho \), equation (71), is also an inert state whose isotropy group is the continuous dihedral group \( D_\infty \). The other inert states with the dihedral groups as isotropy groups are as follows:

- **D\(_{12}\)**
  
  generators: \( \{ e^{i\pi \hat{C}_{2,1}}, e^{i\pi \hat{C}_{12,7}} \} \)

\[ \xi_{D12} = \frac{1}{\sqrt{2}} (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1)^T \]  
\[ \varepsilon(\xi_{D12}) = \frac{1}{2} \rho c_0 + \frac{1}{2} \rho d_0 + \frac{11}{36} \rho d_2 + \frac{891}{12376} \rho d_4 + \frac{11}{64} \rho d_6 + \frac{11}{6912} \rho d_8, \]  
\[ (91) \]

- **D\(_{10}\)**
  
  generators: \( \{ e^{i\pi \hat{C}_{2,1}}, e^{i\pi \hat{C}_{10,7}} \} \)

\[ \xi_{D10} = \frac{1}{\sqrt{2}} (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0)^T \]  
\[ \varepsilon(\xi_{D10}) = \frac{1}{2} \rho c_0 + \frac{1}{2} \rho d_0 + \frac{11}{364} \rho d_2 + \frac{99}{3094} \rho d_4 + \frac{275}{2584} \rho d_6 + \frac{275}{6912} \rho d_8, \]  
\[ (92) \]

- **D\(_{8}\)**
  
  generators: \( \{ e^{i\pi \hat{C}_{2,1}}, e^{i\pi \hat{C}_{8,7}} \} \)

\[ \xi_{D8} = \frac{1}{\sqrt{2}} (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0)^T \]  
\[ \varepsilon(\xi_{D8}) = \frac{1}{2} \rho c_0 + \frac{1}{2} \rho d_0 + \frac{11}{1004} \rho d_2 + \frac{1152}{17017} \rho d_4 + \frac{8}{353} \rho d_6 + \frac{3589}{19019} \rho d_8, \]  
\[ (93) \]

- **D\(_{6}\)**
  
  generators: \( \{ e^{i\pi \hat{C}_{2,1}}, e^{i\pi \hat{C}_{6,7}} \} \)

\[ \xi_{D6} = \frac{1}{\sqrt{2}} (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0)^T \]  
\[ \varepsilon(\xi_{D6}) = \frac{1}{2} \rho c_0 + \frac{1}{2} \rho d_0 + \frac{25}{3004} \rho d_2 + \frac{729}{34034} \rho d_4 + \frac{3697}{28427} \rho d_6 + \frac{3793}{66076} \rho d_8, \]  
\[ (94) \]
generators: \(\{\exp(i\pi\hat{C}_{2z}, \exp(i\pi\hat{C}_{4z})\}\)
\[\xi_{D_4} = \frac{1}{\sqrt{2}} (0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0)^T\]  
(99)

\[\epsilon(\xi_{D_4}) = \frac{1}{2} \rho c_0 + \frac{1}{26} \rho d_0 + \frac{25}{1001} \rho d_2 + \frac{537}{100472} \rho d_4 + \frac{473}{100472} \rho d_6 + \frac{601}{100472} \rho d_8.\]  
(100)

- \(D_2\) generators: \(\{\exp(i\pi\hat{C}_{2z}, \exp(i\pi\hat{C}_{2z})\}\)
\[\xi_{D_2} = \frac{1}{\sqrt{2}} (0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, 0)^T\]  
(101)

\[\epsilon(\xi_{D_2}) = \frac{1}{2} \rho c_0 + \frac{1}{26} \rho d_0 + \frac{99}{1001} \rho d_2 + \frac{1000}{1701} \rho d_4 + \frac{155}{19019} \rho d_6 + \frac{655}{19019} \rho d_8.\]  
(102)

In the case of the tetrahedron group, we have found two candidates for inert states:

- generators: \(\{\exp(i\pi/3\hat{C}_{3x+y+z})\}\)
\[\xi_{T_1} = \left(\frac{\sqrt{11}}{8}, 0, -i \sqrt{\frac{3}{4}}, 0, 0, i \sqrt{\frac{3}{4}}, 0, 0, \frac{\sqrt{11}}{8} \right)^T,\]  
(103)

\[\epsilon(\xi_{T_1}) = \frac{1}{2} \rho c_0 + \frac{72}{1001} \rho d_2 + \frac{15}{1701} \rho d_4 + \frac{224}{19019} \rho d_6 + \frac{202}{19019} \rho d_8.\]  
(104)

- generators: \(\{\exp(i\pi/3\hat{C}_{3x+y+z})\}\)
\[\xi_{T_2} = \left(\frac{\sqrt{11}}{8}, 0, -i \sqrt{\frac{3}{4}}, 0, 0, -i \sqrt{\frac{3}{4}}, 0, 0, \frac{\sqrt{11}}{8} \right)^T,\]  
(105)

\[\epsilon(\xi_{T_2}) = \frac{1}{2} \rho c_0 + \frac{72}{1001} \rho d_2 + \frac{15}{1701} \rho d_4 + \frac{224}{19019} \rho d_6 + \frac{202}{19019} \rho d_8.\]  
(106)

The energies of these two states are the same and it turns out that they belong to the same orbit
\[g(0, 0, \pi, \pi)\xi_{T_1} = \xi_{T_2}.\]  
(107)

and thus they are actually the same inert state.

Another group which has to be considered is the octahedron group. The octahedron group is the group generated by three generators \(C_{4z}\) (rotation about the \(\varepsilon\)-axis by angle \(\pi/2\)), \(C_{3x+y+z}\) (rotation about the \(x+y+z\)-axis by angle \(2\pi/3\)) and \(C_{2x+y}\) (rotation about the \(x+y\)-axis by angle \(\pi\)). For this group, we have found one inert state:

\[\xi_O = \left(0, 0, -i \sqrt{\frac{9}{4}}, 0, 0, 0, 0, -i \sqrt{\frac{9}{4}}, 0, 0 \right)^T,\]  
(108)

\[\epsilon(\xi_O) = \frac{1}{2} \rho c_0 + \frac{1}{26} \rho d_0 + \frac{1323}{19448} \rho d_4 + \frac{4}{3553} \rho d_6 + \frac{189}{988} \rho d_8.\]  
(109)

The \(\xi_O\) state is an eigenstate of the \(\hat{C}_{4z}, \hat{C}_{3x+y+z}\) and \(\hat{C}_{2x+y}\) generators corresponding to eigenvalues equal to 1. Thus, no phase supplement is necessary because the original generators form the isotropy group of \(\xi_O\).

The last group is the icosahedron group. We ought to consider that group because an icosahedron has 12 vertices and the maximal number of vertices for the illustration of spin-6 states is exactly 12. The icosahedron group generators are \(\hat{C}_{5z}\) (rotation about the \(\varepsilon\)-axis by
angle $2\pi/5$, $\hat{C}_{2,y}$ (rotation about the $y$-axis by angle $\pi$) and $\hat{C}_{2,(-1+\sqrt{5})x+(2+\sqrt{5})z}$ (rotation about the $(-1+\sqrt{5})x+(2+\sqrt{5})z$-axis by angle $\pi$). For this group, we have found one inert state

$$\xi_Y = \frac{1}{5}(0, -\sqrt{7}, 0, 0, 0, \sqrt{11}, 0, 0, 0, \sqrt{7}, 0)^T, \quad (110)$$

$$\epsilon(\xi_Y) = \frac{1}{2}\rho c_0 + \frac{1}{26}\rho d_0 + \frac{121}{646}\rho d_6. \quad (111)$$

Similarly as in the case of the octahedron group, no phase supplement is necessary because the generators $\hat{C}_{5,z}$, $\hat{C}_{2,y}$ and $\hat{C}_{2,(-1+\sqrt{5})x+(2+\sqrt{5})z}$ form the isotropy group of $\xi_Y$.

8. Conclusion

In this paper, we have presented all inert states for both spin-5 and spin-6 systems with and without an external magnetic field. We have found all these states by considerations of symmetries of the energy density functional. The idea of finding inert states was used in superconductors and Bose–Einstein condensates for lower spins earlier [13, 14, 8, 9, 6]. Some inert states for spin-5 and spin-6 were also presented in [9].

We look for the inert states employing Michel’s theorem. Yip argues that if one has found inert states for some group, they need not consider any subgroup of this group because there are no additional inert states related to the subgroups [8]. It is obviously true. We want to pay attention that any of the dihedral groups presented in our paper is the subgroup of the other. However, Mäkelä and Suominen in [9] point out that we should not forget about very important global phase-changing factors in their generators, which cause that they cannot be subgroups of the other. They consider groups $D_4$ and $D_8$. These groups without phase factors obviously obey the relation $D_8 \subset D_4$. But in our case it is not true, because

$$(e^{i\pi} \hat{C}_{8,z})^2 = \hat{C}_{4,z} \neq e^{i\pi} \hat{C}_{4,z}. \quad (112)$$

Thus, we can easily see that we cannot construct elements of group $D_4$ by using group $D_8$ elements if they are supplemented by phase factors [9]. Consequently both $D_4$ and $D_8$ have to be considered separately in the investigation of inert states of the systems. The same argumentation can be repeated for groups $D_2$ and $D_4$.

The results presented in this paper can be used as a first step for calculating all stationary states of considered systems. Changing Hamiltonian parameters (external magnetic field or scattering lengths), one can also identify quantum phase transitions when ground states of the systems change their symmetries. Dynamical properties of the phase transitions can be analysed employing the Bogoliubov approach that allows us to investigate stability properties of the stationary states [15–17]. In order to analyse stability of the inert states, one has to solve the Bogoliubov–de Gennes equations. These equations, however, cannot be solved analytically and numerical calculations are necessary. Because we do not know values of scattering lengths for any spin-5 and spin-6 condensates, numerical studies of the stability of the inert states are postponed for future research.

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Appendix

The values of coefficients of the Hamiltonian for the spin-5 system are as follows:

\[ c_0 = \frac{1}{19}(25g_8 - 6g_{10}) \]
\[ c_1 = \frac{1}{19}(g_{10} - g_8) \]
\[ d_0 = \frac{1}{19}(19g_0 + 36g_{10} - 55g_8) \]
\[ d_2 = \frac{1}{19}(19g_2 + 33g_{10} - 52g_8) \]
\[ d_4 = \frac{1}{19}(19g_4 + 26g_{10} - 45g_8) \]
\[ d_6 = \frac{1}{19}(19g_6 + 15g_{10} - 34g_8) . \]

The values of coefficients of the Hamiltonian for the spin-6 system are as follows:

\[ c_0 = \frac{1}{23}(36g_{10} - 13g_{12}) \]
\[ c_1 = \frac{1}{23}(g_{12} - g_{10}) \]
\[ d_0 = \frac{1}{23}(23g_0 + 55g_{12} - 78g_{10}) \]
\[ d_2 = \frac{1}{23}(23g_2 + 52g_{12} - 75g_{10}) \]
\[ d_4 = \frac{1}{23}(23g_4 + 45g_{12} - 68g_{10}) \]
\[ d_6 = \frac{1}{23}(23g_6 + 34g_{12} - 57g_{10}) \]
\[ d_8 = \frac{1}{23}(23g_8 + 19g_{12} - 42g_{10}) . \]

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