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Analysis and equivalent circuit for accurate wideband calculations of the impedance for a piezoelectric transducer having loss

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ABSTRACT
Others have analyzed the operation of piezoelectric transducers by defining two complex parameters which have the units of frequency to allow for the effects of loss. The present paper presents an analysis in which this procedure is extended to include harmonics as well as the fundamental frequency. In this extension it is seen that both positive and negative extrema in both the resistance and the conductance occur at a series of harmonics. Equivalent circuits are also presented with examples for both high-Q and low-Q materials showing agreement with these simulations.

I. INTRODUCTION

Berlincourt et al. and Meeker derived expressions for the electrical impedance of a piezoelectric disk transducer in the thickness modes when the loss is negligible. Sherrit et al. wrote this equation in the following form, and Sherrit and Mukerjee presented other equations having a similar form for modes with other types of piezoelectric transducers.

\[ Z(\omega) = \frac{t}{j\omega e_{33}} \left[ 1 - \frac{4k^2 F_p}{\omega} \tan \left( \frac{\omega}{4F_p} \right) \right] \]  \hspace{1cm} (1)

Here \( \omega \) is the angular frequency, and the relevant material constants and calculated parameters for Eq. (1) that were defined and evaluated for the first two examples by Sherrit et al. are given in Tables I and II. For clarity we use the symbol \( F_p \) which is defined in Table II instead of \( f_p \), where the symbols \( f_p \) and \( f_s \) are defined and used later in the present derivation.

Berlincourt et al. and Meeker assumed zero loss so they used real values for all of the material parameters which requires that the impedance is purely reactive with singularities at frequencies that are odd integer multiples of \( F_p \). However, Sherrit et al. used measured values for the material parameters and dimensions that are complex to include the effects of loss and are given in Table I which we have used in examples. Table II shows the corresponding calculated parameters.

II. ANALYSIS SHOWING THE RESONANCES

Equation (1) may be written as the difference of two separate terms. The first, which is referred to as the “baseline impedance”, varies inversely with the frequency while the second term has a series of resonances that are superimposed on the baseline impedance. The ratio of the second term to the baseline impedance is given by the following expression:

\[ \text{Ratio}(\omega) = -\frac{4k^2 F_p}{\omega} \tan \left( \frac{\omega}{4F_p} \right) \]  \hspace{1cm} (2)

Figure 1 shows the magnitude of this ratio for the first 13 resonances (n = 1, 3, 5, . . ., 25) calculated using the material constants and calculated parameters for the high-Q and low-Q examples in Tables I and II. Note that the ratio decreases monotonically with the multiplier and the low-Q example has no peak beyond the 10th harmonic. In narrowband applications, such as with a piezoelectric resonator, it may be sufficient to consider only the first resonance. However, now we generalize and extend the equivalent circuit model...
to a much greater bandwidth than the single resonance studied by Sherrit et al. The IEEE/ANSI standard on piezoelectricity defines two parameters, \( f_p \) and \( f_s \), as the real values for the frequencies at maximum resistance and maximum conductance, respectively. These two parameters enable defining an equivalent circuit to provide the resonant frequencies for the cases of parallel and series excitation. We follow the extension of Eq. (1) which was made by Sherrit et al. to a much greater bandwidth than the single resonance studied by Sherrit et al.

In searches of the complex frequency plane that we made using Eq. (1) and the corresponding equation for the admittance we have found a sequence of sharply-defined maxima for both the real part of the impedance \( \text{Re}[Z] \), and the real part of the admittance \( \text{Re}[Y] \). We define the complex parameters \( \gamma \) and \( \alpha \) such that \( f_p = \gamma f_p \) and \( f_s = \alpha f_p \), respectively at the maxima for \( \text{Re}[Z] \) and \( \text{Re}[Y] \) where \( f_p \) was defined in Table II.

### A. Effect of parameter \( \gamma \) on the real part of the complex impedance

For the case of zero loss, at which \( Z \) is purely imaginary, there is a singularity when \( \gamma \) is an odd real integer. Thus, to provide continuity to the case of low loss, we examine the behavior of Eq. (1) for the impedance at \( \gamma = n + \delta \) when \( n \) is an odd integer and \( \delta \) is small and may be complex.

\[
Z(y) = \frac{t}{j2\pi A y F_p e_3^2} \left[ 1 - k_i^2 \left( \frac{2}{n \pi} \right) \tan \left( \frac{n \pi}{2} \right) \right] \tag{3}
\]

\[
= \frac{t}{j2\pi A (n + \delta) F_p e_3^2} \left[ 1 - \frac{2k_i^2}{\pi (n + \delta)} \tan \left( \frac{n \pi}{2} + \frac{n \delta}{2} \right) \right] \tag{4}
\]

\[
= \frac{t}{j2\pi A (n + \delta) F_p e_3^2} \left[ 1 + \frac{2k_i^2}{\pi (n + \delta)} \cot \left( \frac{n \delta}{2} \right) \right] \tag{5}
\]

\[
\approx \frac{4k_i^2 t}{jA2\pi^2 n^2 F_p e_3^2 \delta} \tag{6}
\]

Equation (6) shows that there is a singularity in the impedance which occurs as the modulus \(|\delta| \to 0\) regardless of the argument of \( \delta \). This is consistent with our numerical simulations and confirms Sherrit’s use of \( F_p \) as \( f_p \) when he only considered the first resonance.

The singularity in the resistance \( R \) at the frequency \( f = f_p \) requires that the susceptance \( B \) must be zero at this frequency. This may be seen by examining the relationship of the impedance to the admittance. For \( Y = G + jB, \text{Re}[Z] = G/(G^2 + B^2) \) is bounded at all values of \( G \) when \( B \) is non-zero, but \( \text{Re}[Z] \) may be singular when \( B = 0 \).
Conversely, for $Z = R + jX$, then $\text{Re}[Y] = R/(R^2 + X^2)$ is bounded for all values of $R$ when $X$ is non-zero, but $\text{Re}[Y]$ may be singular when $X = 0$.

Figure 2 shows the real part of the complex impedance $Z$ in the region of the complex frequency plane near the first resonance for the high-Q example in Tables I and II. The imaginary part of $\gamma$ has the range of $-10^{-9}$ to $10^{-9}$ with zero at the midpoint. The real part of $\gamma$ has the range of $1-10^{-9}$ to $1+10^{-9}$ with 1 is at the midpoint. Figure 2 shows the neighborhood of the singularity near $\gamma = 1$ for the first resonance.

B. Effect of parameter $\alpha$ on the real part of the complex admittance

Our search of the complex frequency plane shows that near $f_s$, where $\text{Re}[Y]$ has a maximum, the reactance $\text{Im}[Z]$ is non-zero so that $\text{Re}[Y]$ is not singular. Thus, we see large positive and negative values for $\text{Re}[Y]$ when the complex frequency is near $f_s$ but these values for $\text{Re}[Y]$ are bounded.

Figure 3 shows the real part of the complex admittance in the region of the complex frequency plane near the first resonance for the high-Q example in Tables I and II. The center of the complex-Y plane in this figure is at $f_s = \alpha F_p$, where $\text{Re}[\alpha] = 0.8964968342$ and $\text{Im}[\alpha] = 0.0022944067$, where each of these parts is specified to 10 decimal places because the values of $Y$ were determined at increments of $10^{-10}$ in both $\text{Re}[\alpha]$ and $\text{Im}[\alpha]$. The imaginary part of $\alpha$ has a range of $0.0022944067-10^{-9}$ to $0.0022944067+10^{-9}$, and the real part of $\alpha$ has a range of $0.8964968342 -10^{-9}$ to $0.8964968342 +10^{-9}$. Figure 3 verifies that the susceptance $\text{Re}[Y]$ has a sharp maximum but is non-singular.

In generating the data for Fig. 3, $\alpha_t$ was evaluated by maximizing $\text{Re}[Y(\alpha_t)]$ based on Eq. (1), but now a simpler method for evaluating $\alpha$ will be presented. Equation (1) may be inverted to obtain the following expression for the admittance where trial values of $\alpha$, labeled as $\alpha_t$, may be used to maximize $\text{Re}[Y(\alpha_t)]$ to determine $\alpha$:

$$Y(\alpha_t) = -\frac{j4AF_p\varepsilon_0}{t} \left[ \frac{\pi}{2} - k^2 \tan \left(\frac{\pi}{2} \cdot \frac{\alpha_t}{t} \right) \right]$$

It is not possible for the denominator in Eq. (7) to be exactly zero—for which $Y$ would be singular. However, $\alpha$ may be approximated by minimizing $\left\lvert \frac{\pi}{2} - k^2 \tan \left(\frac{\pi}{2} \cdot \frac{\alpha_t}{t} \right) \right\rvert$.

Figure 4 shows the imaginary part of $\alpha$ as a function of the harmonic (multiplier) minus the real part of $\alpha$ for the first five values of $\alpha$. The harmonics are in decreasing order from left to right so the abscissa is $9 - \alpha_R$ for the high-Q and low-Q examples at the LHS and $1 - \alpha_R$ at the RHS. This figure shows that $\alpha_I \approx C(n - \alpha_R)$ for all of the harmonics for a particular piezoelectric resonator where $C$ is a constant that depends on the properties of that device.

FIG. 2. Real part of the complex impedance as a function of the complex parameter $\gamma$ at the first resonance for the high-Q example in Tables I and II.

FIG. 3. Real part of the complex admittance as a function of the complex parameter $\alpha$ at the first resonance for the high-Q example in Tables I and II.

FIG. 4. Relationship of the imaginary and real parts of $\alpha$ determined for the high-Q and low-Q examples in Tables I and II.
We have shown that accurate and efficient determination of $\alpha$ for multiple resonances with a single device is possible by using the technique described in the previous paragraph with the approximate relationship for the real and imaginary parts of $\alpha$.

Sherrit et al. have made the approximation of determining $f_p$ and $f_s$ by maximizing $\text{Re}[Z]$ and $\text{Re}[Y/f]$ respectively. This simplifies the calculations but this procedure may be questioned because multiplying or dividing by the complex frequency causes a rotation in the complex plane. We have found that this approximation causes a small error which may generally be neglected, but all of the solutions for $f_p$ and $f_s$ in this paper were determined using Eq. (1) and its equivalent for admittance without further approximations.

### III. EXTENSION OF THE SHERRIT EQUIVALENT CIRCUIT

The Butterworth-Van Dyke model for a piezoelectric resonator has a resistor, inductor, and capacitor in series (R-L-C), all shunted by a second capacitor $C_0$ as shown in Fig. 5. Van Dyke was the first to propose this equivalent circuit, and he suggested extending the model by having multiple R-L-C circuits in parallel to include the effects of multiple resonances as shown in Fig. 6. The Sherrit model is shown in Fig. 7 and our proposed extension of the Sherrit model for multiple resonances is shown in Fig. 8.

Table III gives the parameters for Butterworth-Van Dyke (BVD) and Sherrit equivalent circuits for the high-Q and low-Q examples.

Figures 9 and 10 show the resistance and reactance simulated with these two equivalent circuits and with the analytical solution from Eq. (1). These calculations were only made for the high-Q example in Tables I and II. Figure 9 shows that the resistance for the Sherrit equivalent circuit is consistent with the analytical solution with the exception of missing the higher-order resonances. However, while the BVD model is accurate near the first resonance the resistance is too small above and below the resonance. This may be understood because well below the single resonance the resistance remains approximately constant since the current is divided by $C_0$ and $C_1$, and the only loss is in the resistor $R_1$ which is in series with $C_1$. The BVD model is also inaccurate well above resonance where the inductor $L_1$ causes a greater fraction of the current to flow through capacitor $C_0$ instead of through resistor $R_1$.

Figure 10 shows that the reactance calculated for the BVD and Sherrit equivalent circuits is in good agreement with that from Eq. (1), with the exception of the higher-order resonances that are seen in the analytical solution. This may be understood because well below the single resonance the reactance is approximately that for $C_0$ and $C_1$ or $C_0'$ and $C_1'$ in parallel, and well above the resonance the inductor causes the reactance to approximate that of $C_0$, or $C_0'$, and the resonance is quite sharp for the high-Q example. Figures 9 and 10, are log-log plots to clarify that the resistive component of the impedance has two components as was previously stated; a baseline impedance varying inversely with the frequency as well as superimposed resonances.

We acknowledge that others have also used single and multi-branch equivalent circuits, but Sherrit et al. derived complex functions for the circuit elements from Eq. (1), and simulations using their equivalent circuit are consistent with Eq. (1) except for neglecting the higher-order resonances. The introduction of complex expressions for the circuit elements correctly associates the different types of loss with each circuit element. The following discussion considers our proposed extension of the Sherrit model which is shown in Fig. 8.

We begin by requiring that at low-frequencies the impedance of the equivalent circuit must agree with Eq. (1), so that now we must require the following:

\[
Z(\omega) = \frac{t(1 - k_i^2)}{j\omega\varepsilon^2} = \frac{1}{j\omega(C_0' + C_1' + \cdots + C_{N'})} \tag{8}
\]

where $N$ is the number of branches in the equivalent circuit. Thus, once the capacitance in each branch has been determined we evaluate $C_0$ with the following expression:
TABLE III. Circuit parameters for equivalent circuit models allowing only for the first resonance.

| Model       | Component | High-Q (Q = 100) | Low-Q (Q = 10) |
|-------------|-----------|------------------|-----------------|
| BVD         | C₀, F     | 1.95x10⁻⁹        | 1.94x10⁻⁹      |
|             | C₁, F     | 4.77x10⁻¹⁰       | 4.99x10⁻¹⁰      |
|             | L₁, H     | 5.54x10⁻⁵        | 5.42x10⁻⁵       |
|             | R₁, Ω     | 3.83             | 37.7            |
| Sherrit     | C₀', F    | 1.95x10⁻⁹/1.20°  | 1.95x10⁻⁹/1.42° |
|             | C₁', F    | 4.76x10⁻¹⁰/2.69° | 4.71x10⁻¹⁰/9.59°|
|             | L₁', H    | 5.55x10⁻⁵/1.82°  | 5.98x10⁻⁵/21.9° |

\[ C₀' = \frac{AεS_{33}}{t(1 - k^2)} - \left( C₁' + \cdots + C_N' \right) \]  

(9)

The IEEE/ANSI standard on piezoelectricity defines the parameters \( f_p \) and \( f_s \) as real values of the frequencies for maximum resistance and maximum conductance, respectively. Sherrit et al. appear to be the first to treat \( f_p \) and \( f_s \) as complex variables to be compatible with lossy piezoelectric resonators and use this procedure to predict multiple resonances; in this case the analysis was for the radial mode instead of the thickness mode which we have studied.

Table IV gives the complex values of \( f_p \) and \( f_s \) that we calculated for the first 5 resonances using iteration with Eq. (1):

To avoid confusion, we have labeled the parameter \( f_p \) used by Sherrit et al. as \( F_p \) in this paper. The parameter \( F_p \) in Eq. (1) is a function of the material parameters which is used in Eq. (1) to determine the impedance at any frequency. Notice that \( f_{p1} \) is not exactly equal to \( F_p \), but rather \( f_{p1} \) must also be determined by the process that has just been defined and used to prepare Table V. This difference is more pronounced with the low-Q example.

Now that \( f_{pn} \) and \( f_{sn} \) have been determined in Table V, the components \( C_n' \) and \( L_n' \) in the proposed extension of the Sherrit model may be evaluated with Eqs. (10) and (11), and then \( C₀' \) may be determined with Eq. (9).

\[ C_n' = \frac{AεS_{33}}{t(1 - k^2)} \left[ 1 - \left( \frac{f_{pn}'}{f_{pn}} \right)^2 \right] \quad \text{for } n = 1 \text{ to } N \]  

(10)

\[ L_n' = \frac{1}{(2πf_{sn}')^2 C_n'} \quad \text{for } n = 1 \text{ to } N \]  

(11)

The admittance of the equivalent circuit may be calculated as follows:

\[ Y(ω) = jωC₀' + \frac{jωC₁'}{1 - ω^2L₁'C₁'} + \frac{jωC₂'}{1 - ω^2L₂'C₂'} + \cdots + \frac{jωC_N'}{1 - ω^2L_NC_N'} \]  

(12)

or

\[ Y(ω) = -\frac{jωAεS_{33}}{t(1 - k^2)} + \sum_{n=1}^{N} \frac{jωL_n'C_n'^2}{1 - ω^2L_n'C_n'^2} \]  

(13)

We have used Eq. (13) to determine the impedance of our extension of the Sherrit model for the equivalent circuit because the modular nature of this equation makes it possible to determine the contribution for each branch separately and then these values may be combined to obtain the solution for various values of N. Figure 11 shows the impedance calculated for the equivalent circuit with \( N = 5 \) (circles) and using Eq. (1) for the analytical solution (solid lines).
IV. CONCLUSIONS

1. The complex frequency for parallel resonance \( (f_p) \) is equal to an integer multiple of the parameter which we call \( F_p \) and need not be found by the general method of iteration.

2. The complex frequency for series resonance \( (f_s) \) must be determined by iteration since the real part of \( Y \) is bounded at this point in the complex frequency plane, which occurs because the imaginary part of \( Z \) is non-zero at that same point. However, simpler means have been determined for these iterations.

3. An equivalent circuit has been developed which accurately predicts the impedance from DC through the first five resonances, in agreement with the equation for the impedance, and may be extended for use over a greater frequency range.

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