Multigrid as an exact solver

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Abstract

We provide an alternative Fourier analysis for multigrid applied to the Poisson problem in 1D, based on explicit derivation of spectra of the iteration matrix. The new Fourier analysis has advantages over the existing one. It is easy to understand and enables us to write the error equation in terms of the eigenvector of the stiffness matrix. When weighted-Jacobi is used as a smoother with two different weights, multigrid is an exact solver.

Keywords:
Multigrid, Fourier analysis, smoother, weighted-Jacobi

1. Introduction

We consider the Poisson problem with Dirichlet boundary conditions given by

\begin{equation}
\begin{cases}
-u''(x) = f(x), & 0 < x < 1, \\
u(0) = u(1) = 0.
\end{cases}
\end{equation}

The domain of the problem is partitioned into \(n - 1\) uniform subintervals using the grid points \(x_j = jh\) where \(h = 1/(n-1)\) is the grid size, and \(n\) is an odd integer. Discretization of the Poisson problem (1) with central finite difference scheme gives the linear systems of equations,

\[ Au = f \]

where \(A = 1/h^2\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{(n-2)\times(n-2)}\). The matrix \(A\) assumes the eigenvectors \(v_j^k = \sin\left(\frac{jk\pi}{n-1}\right)\) with the corresponding eigenvalues \(\lambda_k(A) = \frac{4}{n^2} \sin^2\left(\frac{k\pi}{2(n-1)}\right)\) for \(k = 1, \ldots, n - 2\). Note that \(v_j^k\) represent the \(j\)-th entry of the eigenvector \(v_k\).
2. Smoother

We use weighted-Jacobi relaxation as a smoother. Let \( a_{i,j}, i, j = 1, \ldots, n - 2 \), represent the entries of \( A \). We split the matrix \( A \) as follows.

\[
A = D + K
\]

where \( D \) is the diagonal matrix with entries \( d_{i,i} = a_{i,i}, (i = 1, \ldots, n - 2) \). Weighted-Jacobi is defined as

\[
x^{k+1} = (I - \omega D^{-1}A)x^k + \omega D^{-1}f, \quad (2)
\]

where \( \omega \) is the weight to be determined. There is no restriction on \( \omega \) because it is not necessary for a smoother to be convergent for a multigrid method to be convergent. Let \( R^J_\omega \) represent the iteration matrix of the weighted-Jacobi given in (2).

\[
R^J_\omega \equiv I - \omega D^{-1}A.
\]

We can apply the weighted-Jacobi more than one with different \( \omega \)'s to further accelerate the convergence. To this end, we use the following notation.

\[
S^m_J \equiv S_J(\omega_1, \ldots, \omega_m) \equiv R^\omega_1 \ldots R^\omega_m,
\]

which means that the weighted-Jacobi is applied \( m \) times with the parameters \( \omega_i, i = 1, \ldots, m \). Note that the matrix \( S^m_J \) assumes the same eigenvectors with the matrix \( A \). The best way to find the optimal weights of the weighted-Jacobi method as a smoother, is to do spectral analysis. To this end, we carry out a spectral analysis based on explicit derivation of the spectra of the iteration matrix of the two-grid.

3. Other elements of the multigrid method and derivation of the spectrum of the iteration matrix of the two-grid

In this section, we introduce interpolation and restriction operators and show some equalities related to them which are necessary to obtain the spectrum of the iteration matrix of the two-grid. We start by setting \( A = A^h, \ u = u^h \) and \( f = f^h \) where the superscript \( h \) which is equivalent to the grid size \( h \), stands for the fine grid. The two-grid iteration matrix with only pre-smoothing with damped Jacobi relaxation is given by [2]

\[
R^{TG} = (I - I_2^h(2A^h)^{-1}I_1^hA^h)S^m_J.
\]
Our aim is to find the spectrum of $R^{TG}$. Just for easiness of our analysis, we assumed that $n$ is an odd integer. In Equation (3), the prolongation (interpolation) operator $I_{2h}^h$ is the linear interpolation which has the matrix form

$$I_{2h}^h = \frac{1}{2} \begin{bmatrix} 1 & 2 & \cdots & 1 \\ 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 \end{bmatrix} \in \mathbb{R}^{(n-2)\times(n-3)/2}$$

and restriction operator $I_{h}^{2h}$ is the transpose of the prolongation operator

$$I_{h}^{2h} = (I_{2h}^h)^T.$$  

Coarse grid matrix $A^{2h}$ is defined by Galerkin projection

$$A^{2h} = I_{h}^{2h} A_{2h} I_{2h}^h.$$  

(4)

From the above definition, it is easy to show that $A^{2h}$ is also symmetric. We apply only pre-smoothing and do not apply post-smoothing.

The prolongation operator $I_{2h}^h$ satisfies

$$I_{2h}^h v_{k}^{2h} = \cos^2 \left( \frac{k \pi}{2(n-1)} \right) v_k^h - \sin^2 \left( \frac{k \pi}{2(n-1)} \right) v_{n-1-k}^h, \quad 1 \leq k \leq \frac{n-3}{2}$$

(5)

where

$$v_{k,j}^{2h} = \sin \left( \frac{2j \pi}{n-1} \right), \quad 1 \leq j \leq \frac{n-3}{2}.$$  

(6)

The restriction operator $I_{h}^{2h}$ which is the transpose of the prolongation operator has the following properties.

$$I_{h}^{2h} v_{k}^{2h} = 2 \cos^2 \left( \frac{k \pi}{2(n-1)} \right) v_{k}^{2h} \quad \text{for} \quad 1 \leq k \leq \frac{n-3}{2}.$$  

(7)
and
\[ I_h^{2h}v_{n-1-k} = -2\sin^2\left(\frac{k\pi}{2(n-1)}\right)v_k^{2h} \quad \text{for} \quad 1 \leq k \leq \frac{n-3}{2}. \] (8)

As we stated before, the coarse grid matrix is obtained by Galerkin projection. That is,
\[ A^{2h} = I_h^{2h}A^hI_h^{2h}. \]

Using this definition and properties of the restriction and prolongation operators in (5), (7) and (8), we obtain the spectrum of the coarse matrix \( A^{2h} \).
\[ A^{2h}v_k^{2h} = \left(2\lambda_k(A^h)\cos^2(k\pi h/2) + 2\lambda_{n-1-k}(A^h)\sin^2(k\pi h/2)\right)v_k^{2h}, \quad 1 \leq k \leq \frac{n-3}{2}. \] (9)

From the above observations, it is very reasonable to expect that the eigenvectors of the matrix \( R^{TG} \) are linear combinations of \( v_k \) and \( v_{n-1-k} \). We assume that \( b_k = v_k + cv_{n-1-k} \) is an eigenvector of the matrix \( R^{TG} \) where \( c \) is to be determined. Imposing \( b_k \) into the definition of \( R^{TG} \) in (3) and using the properties of the smoother, prolongation, restriction operators and coarse grid matrix we end up with
\[
R^{TG}b_k = v_k \left( \lambda_k(S_j^m) - \frac{2\cos(k\pi h/2)\lambda_k(A^h)\lambda_k(S_j^m)}{\lambda_k(A^{2h})} + c \frac{2\sin^2(k\pi h/2)\cos^2(k\pi h/2)\lambda_{n-1-k}(A^h)\lambda_{n-1-k}(S_j^m)}{\lambda_k(A^{2h})} \right) \\
+ cv_{n-1-k} \left( \lambda_{n-1-k}(S_j^m) - \frac{2\sin^2(k\pi h/2)\lambda_{n-1-k}(A^h)\lambda_{n-1-k}(S_j^m)}{\lambda_k(A^{2h})} + \frac{1}{c} \frac{2\sin^2(k\pi h/2)\cos^2(k\pi h/2)\lambda_k(A^h)\lambda_k(S_j^m)}{\lambda_k(A^{2h})} \right).
\]

Using \( \lambda_k(A^{2h}) \) given in (9) and equating the coefficients of \( v_k \) and \( cv_{n-1-k} \) in above equation, we get the following quadratic equation.
\[
c^2 \left(2\sin^2(k\pi h/2)\cos^2(k\pi h/2)\lambda_{n-1-k}(A^h)\lambda_{n-1-k}(S_j^m)\right) \\
+ c \left(2\sin^2(k\pi h/2)\lambda_{n-1-k}(A^h)\lambda_k(S_j^m) - 2\cos^2(k\pi h/2)\lambda_k(A^h)\lambda_{n-1-k}(S_j^m)\right) \\
- 2\sin^2(k\pi h/2)\cos^2(k\pi h/2)\lambda_k(A^h)\lambda_k(S_j^m) = 0.
\]

Solving the above equation for \( c \), we obtain
\[
c_1 = \frac{\cos^2(k\pi h/2)\lambda_k(A^h)}{\sin^2(k\pi h/2)\lambda_{n-1-k}(A^h)} \quad \text{(10)}
\]
\[ c_2 = -\frac{\sin^2(k\pi h/2)\lambda_k(S^m_j)}{\cos^2(k\pi h/2)\lambda_{n-1-k}(S^m_j)}. \] (11)

Note that eigenvalues associated to \( c_2 \) are all zero. More precisely, the two-grid iteration matrix \( R_{TG} \) assumes the eigenvectors
\[
b_k = \begin{cases} 
  v_k + c_1 v_{n-1-k}, & 1 \leq k \leq \frac{n-1}{2} \\
  v_k + c_2 v_{n-1-k}, & \frac{n-1}{2} < k \leq n-2 
\end{cases} \tag{12}
\]
with the corresponding eigenvalues
\[
\lambda_k(R_{TG}) = \begin{cases} 
  \frac{\sin^4(k\pi h/2)\lambda_k(A^h)\lambda_{n-1-k}(A^h)}{\lambda(A^h)}, & 1 \leq k \leq \frac{n-1}{2} \\
  0, & \frac{n-1}{2} < k \leq n-2 
\end{cases} \tag{13}
\]

Note that the coarse grid matrix \( A^{2h} \) obtained by Galerkin projection, is just a constant multiple of the original matrix which is obtained by rediscretization of the problem (1) on coarse grid. Since \( \lambda_k(A^h) = \frac{4}{h^2} \sin^2(k\pi h/2) \) and \( \lambda_{n-1-k}(A^h) = \frac{4}{h^2} \cos^2(k\pi h/2) \), Equation (9) reduces to
\[
A^{2h}v^h_k = \left( \frac{8}{h^2} \sin^2(k\pi h/2) \cos^2(k\pi h/2) \right) v^h_k = \frac{2}{h^2} \sin^2(k\pi h) v^h_k, \quad 1 \leq k \leq \frac{n-3}{2} \tag{14}
\]
where \( v^h_k \) is given in (6). Using explicit expressions of \( \lambda_k(A^h) \) and \( \lambda_{n-1-k}(A^h) \), it is easy to show that \( c_1 \) given in (10), is equal to one. Hence, in a more compact form, \( R_{TG} \) assumes the eigenvectors
\[
b_k = \begin{cases} 
  v_k + v_{n-1-k}, & 1 \leq k \leq \frac{n-1}{2} \\
  v_k + c_2 v_{n-1-k}, & \frac{n-1}{2} < k \leq n-2 
\end{cases} \tag{15}
\]
with the corresponding eigenvalues
\[
\lambda_k(R_{TG}) = \begin{cases} 
  \lambda_k(S^m_j) \sin^2(k\pi h/2) + \lambda_{n-1-k}(S^m_j) \cos^2(k\pi h/2), & 1 \leq k \leq \frac{n-1}{2} \\
  0, & \frac{n-1}{2} < k \leq n-2 
\end{cases} \tag{15}
\]
where \( c_2 \) is given in (11).
Substituting $\omega$ First, let us observe what happens if we apply weighted-Jacobi with the optimal weight found for $\omega$ value when $k$ range $n$. The difference is that our derivation is totally algebraic. Furthermore, for $\sin$ Note that Lemma 1. The following equality holds

$$3(\sin^4(x) + \cos^4(x)) - 2(\sin^6(x) + \cos^6(x)) = 1,$$

for all $x \in \mathbb{R}$. 

**Proof.**

$$3(\sin^4(x) + \cos^4(x)) - 2(\sin^6(x) + \cos^6(x)) = 3(\sin^4(x) + \cos^4(x)) - 2(\sin^2(x) + \cos^2(x))(\sin^4(x) - \sin^2(x)\cos^2(x) + \cos^4(x)) = \sin^4(x) + \cos^4(x) + 2\sin^2(x)\cos^2(x) = (\sin^2(x) + \cos^2(x))^2 = 1.$$ 

First, let us observe what happens if we apply weighted-Jacobi with the optimal weight found for $m = 1$, two times. Substituting $\omega_1 = \omega_2 = \frac{1}{2}$ into (16), we get $\lambda_k(R^{TG}) = 1 - \frac{7}{2} = 0.7$ for all $k$. That is, $\rho(R^{TG}) = 0.7$. Now, we look for different $\omega$’s for which the spectral radius of $R^{TG}$ is reduced further. By the Lemma 1 for the choices $\omega_1 = 1$ and $\omega_2 = \frac{1}{2}$, the eigenvalues in (16) become all zero. This means that all eigenvalues of $R^{TG}$ are zero. In other
words, two-grid is an exact solver with only one iteration. Moreover, since the coarse matrix obtained by Galarkin
projection is just a constant multiple of the original matrix on coarse grid, multigrid is also an exact solver with only
one iteration. Eigenvalues of the smoothers $S_j(\frac{1}{2}, \frac{1}{2})$ and $S_j(1, \frac{1}{2})$ for $n = 33$ are presented in Figure 1. Although
for $S_j(1, \frac{1}{2})$, the two-grid method is an exact solver, we see from Figure 1 that corresponding eigenvalues of the
oscillatory modes are not zero. Furthermore, the maximum eigenvalue of $S_j(1, \frac{1}{2})$ in magnitude in oscillatory
region, is $|\lambda_{21}(S_j(1, \frac{1}{2}))| = 0.124581$ which is grater than the maximum eigenvalue of $S_j(\frac{1}{2}, \frac{1}{2})$ in magnitude in oscillatory
region, which is $\lambda_{31}(S_j(\frac{1}{2}, \frac{1}{2})) = 0.1$.

4. The error equation

Since we have explicit expressions for the eigenvalues of the two-grid iteration matrix $R^{TG}$ in (13) and of the
corresponding eigenvectors in (12), which are linear combination of the eigenvectors of $A$, we can see which modes
are damped more rapidly. To this end, we write the error $\mathbf{e}$ in terms of the eigenvectors of $R^{TG}$.

$$\mathbf{e} = \sum_{k=1}^{n-2} d_k \mathbf{b}_k = \sum_{k=1}^{(n-1)/2} d_k (\mathbf{v}_k + c_1(k)\mathbf{v}_{n-1-k}) + \sum_{k=(n+1)/2}^{n-2} d_k (\mathbf{v}_k + c_2(k)\mathbf{v}_{n-1-k})$$

where $d_k$ is any constant, $c_1 = c_1(k)$ and $c_2 = c_2(k)$ which are given in (10) and (11), respectively. Since $\lambda_1(R^{TG}) = 0$
for $k = (n + 1)/2...n - 2$, after $m$ iterations, the error becomes

$$\mathbf{e}^m = \sum_{k=1}^{(n-1)/2} d_k l_k^m(R^{TG})(\mathbf{v}_k + c_1(k)\mathbf{v}_{n-1-k}) = \sum_{k=1}^{(n-1)/2} d_k l_k^m(R^{TG})\mathbf{v}_k + \sum_{k=(n+1)/2}^{n-2} l_k^m(R^{TG})c_1(n - 1 - k)\mathbf{v}_k$$

where $\mathbf{e}^m$ stands for the error after $m$ iterations and $l_k$ is any constant. In above equation on the right, the first sum
contains the smooth modes and the second sum contains oscillatory modes. The first eigenvalue $\lambda_1(R^{TG})$ is associated
with the smoothest and the most oscillatory mode. The eigenvalue $\lambda_{(n-1)/2}(R^{TG})$ is associated only with the eigenvector $v_{(n-1)/2}$.

5. Conclusion

In this work, we provided an alternative Fourier analysis for multigrid applied to the Poisson problem in 1D. We related multigrid with the exact solver.

Note: This work is not going to be submitted to any journal. It is free to download and disseminate it.

References

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