New rational solutions
of Yang-Baxter equation and deformed Yangians

Alexander Stolin
Department of Mathematics, Royal Institute of Technology
S-100 44 Stockholm, Sweden

Petr P. Kulish
St.Petersburg Branch of Steklov Mathematical Institute
Fontanka 27, St.Petersburg 191011, Russia

Abstract. In this paper a class of new quantum groups is presented: deformed Yangians. They arise from rational solutions of the classical Yang-Baxter equation of the form $\frac{\theta}{q^2} + const$. The universal quantum R-matrix for a deformed Yangian is described. Its image in finite-dimensional representations of the Yangian gives new matrix rational solutions of the Yang-Baxter equation (YBE).

1. Introduction

The term “quantum groups” and the algebraic constructions associated with them appeared approximately 10 years ago in [D], [D2], [J1]. One of the starting points for such constructions was the classification of trigonometric solutions of the classical Yang-Baxter equation (CYBE) obtained in [BD]. In particular $U_q(sl(n))$ can be viewed as a quantization
of the Lie bialgebra arising from the Drinfeld-Jimbo solution of CYBE. Another “quantum
group” was called Yangian ([D2]) and it arose from a rational solution of CYBE, a so-called
Yang solution. Here we present an attempt to define new quantum groups, which arise from
other rational solutions of CYBE.

Now let $g$ be a simple Lie algebra over $\mathbb{C}$.

**Definition.** Let $X(u, v) = \frac{c_2}{u-v} + r(u, v)$ be a function from $\mathbb{C}^2$ to $g \otimes g$. We say that $X(u, v)$ is a rational solution of the classical Yang-Baxter equation (CYBE) if:

1. $c_2 = \sum_{\mu} I_\mu \otimes I_\mu$, where $\{I_\mu\}$ is an orthogonal basis of $g$ with respect to the Killing form;
2. $r(u, v)$ is a polynomial in $u, v$;
3. $X(u, v) = -X(v, u)^\sigma$, where $\sigma$ interchanges factors in $g \otimes g$;
4. $[X^{12}(u_1, u_2), X^{13}(u_1, u_3)] + [X^{12}(u_1, u_2), X^{23}(u_2, u_3)] + [X^{13}(u_1, u_3), X^{23}(u_2, u_3)] = 0$.

Here $[X^{12}, X^{13}]$ is the usual commutator in the associative algebra $U(g)^{\otimes 3}$. The other
two summands are defined in the same way.

**Definition.** We say that two rational solutions $X_1(u, v)$ and $X_2(u, v)$ are gauge
equivalent if there exists an automorphism $\lambda$ of algebra $g[u]$ such that $(\lambda \otimes \lambda)X_1(u, v) = X_2(u, v)$.

It turns out that the degree of the polynomial part of a rational solution of CYBE can
be estimated. More exactly, the following result was proved in [S]:

**Theorem 1.** Let $X(u, v) = \frac{c_2}{u-v} + r(u, v)$ be a rational solution. Then there exists
a rational solution $X_1(u, v)$, which is gauge equivalent to $X(u, v)$ and such that

$$X_1(u, v) = \frac{c_2}{u-v} + a_0 + b_1 u + b_2 v + cuv .$$

Here $a_0, b_1, b_2, c \in g^{\otimes 2}$.

In the present paper we will be dealing with the case $X(u, v) = \frac{c_2}{u-v} + r_0$, where
$r_0 \in g^{\otimes 2}$. Clearly $X(u, v)$ is a solution of CYBE if and only if $r_0$ itself is a solution of
CYBE.

Let $K = \mathbb{C}((u^{-1}))$. One can define the following non-degenerate ad-invariant inner
product on $g \otimes K$: $(x, y) = \text{Res}_{u=0} \text{tr}(adx \cdot ady)$. Denote $g \otimes C[[u^{-1}]]$ by $g[[u^{-1}]]$.

**Theorem 2** (see [S]).

1) There is a 1-1 correspondence between the set of rational solutions of CYBE of the
form $\frac{c_2}{u-v} + r_0$ and subalgebras $W \subset g \otimes K$ such that:
(i) \( u^{-2}g[[u^{-1}]] \subset W \subset g[[u^{-1}]] \).
(ii) \( W^\perp = W \) with respect to the form \( (\ , \ ) \) introduced above.
(iii) \( W \oplus g[u] = g \otimes K \).

2) Any \( W \) satisfying conditions (i-iii) above defines a subalgebra \( L \subset g \) and a non-degenerate 2-cocycle \( B \) on \( L \). In other words \( B \) is skew-symmetric and satisfies

\[ B([x, y], z) + B([z, x], y) + B([y, z], x) = 0 \]

for any \( x, y, z \in L \). Moreover, \( r_0 \) is contained in \( \Lambda^2 L \), is a non-degenerate 2-tensor and \( r_0^{-1} = B \in \Lambda^2 L^* \).

A Lie algebra with a non-degenerate 2-cocycle is called quasi-Frobenius.

3) Conversely, any pair \( (L, B) \) such that \( L \) is a subalgebra of \( g \) and \( B \) is a non-degenerate 2-cocycle on \( L \), defines a rational solution of the form \( \frac{c_2}{u-v} + r_0 \).

Our approach to quantization of a rational solution of CYBE of the form \( \frac{c_2}{u-v} + r_0 \) is based on the following result borrowed from [D1].

**Theorem 3.** Let \( r_0 \in L \otimes L \subset g \otimes g \) satisfy CYBE. Then there exists an element \( F \in (U(L)[[h]]) \otimes 2 \subset (U(g)[[h]]) \otimes 2 \) such that:

1) \( (\Delta_0 \otimes 1) F \circ F^{12} = (1 \otimes \Delta_0) F \circ F^{23} \), where \( \Delta_0 : U(g) \rightarrow U(g) \otimes 2 \) is the usual cocommutative comultiplication.
2) \( F = 1 \otimes 1 + \frac{1}{2} hr_0 + \sum_2^\infty F_i h^i \).
3) \( R = (F^{21})^{-1} F \in (U(L)[[h]]) \otimes 2 \subset (U(g)[[h]]) \otimes 2 \) satisfies YBE and is of the form \( R = 1 \otimes 1 + hr_0 + \sum_2^\infty R_i h^i \). Here \( F^{21} = F^\sigma \), where \( \sigma \) interchanges factors in \( (U(g)[[h]]) \otimes 2 \).

2. Deformation of Yangians

Now we return to rational solutions of CYBE. The simplest rational solution is \( X_0(u, v) = \frac{c_2}{u-v} \), i.e., \( r(u, v) \equiv 0 \). Yangians were introduced by Drinfeld in [D2] in order to obtain a "sophisticated quantization" of \( X_0(u, v) \).

**Definition.** Let \( g \) be a simple Lie algebra over \( \mathbb{C} \), given by generators \( \{I_\alpha\} \) and relations \( [I_\alpha, I_\beta] = \epsilon_{\alpha\beta\gamma} I_\gamma \), where \( \{I_\gamma\} \) is an orthonormal basis with respect to the Killing form. Then Yangian \( Y(g) \) is an associative algebra with 1, generated by elements \( \{I_\alpha\} \) and \( \{T_\alpha\} \) and the following relations

\[ [I_\alpha, I_\beta] = C^\gamma_{\alpha\beta} I_\gamma ; \quad [I_\alpha, T_\beta] = C^\gamma_{\alpha\beta} T_\gamma ; \]

\[ [[T_\lambda, T_\mu], I_\nu] - [I_\lambda, [T_\mu, T_\nu]] = \epsilon_{\lambda\mu\nu} \{I_\alpha, I_\beta, I_\gamma\} , \]

3
here $a_{\lambda\mu\nu} = \frac{1}{24} C_{\lambda\alpha}^i C_{\mu\beta}^j C_{\nu\gamma}^k C_{ij}^k$ and \{ $x_1, x_2, x_3$ \} = $\sum_{i\neq j\neq k} x_i x_j x_k$.

(3) $\Delta I_\lambda = I_\lambda \otimes 1 + 1 \otimes I_\lambda$

(4) $\Delta T_\lambda = T_\lambda \otimes 1 + 1 \otimes T_\lambda + \frac{1}{2} C_{\lambda\mu}^\nu I_\nu \otimes I_\mu$.

For any $a \in \mathbb{C}$ define an automorphism $T_a$ of $Y(g)$ by formulas:

$T_a(I_\lambda) = I_\lambda; \quad T_a(T_\lambda) = T_\lambda + aI_\lambda$.

As usual, we denote by $\Delta'$ the opposite comultiplication.

**Theorem 4** ([D2]). There exists a unique $R(u) = 1 \otimes 1 + \sum_{k=1}^{\infty} R_k u^{-k}$, $R_k \in Y(g)^{\otimes 2}$ such that

1) $(\Delta \otimes 1)R(u) = R^{13}(u)R^{23}(u)$;

2) $(T_a \otimes 1)\Delta'(a) = R(u)((T_a \otimes 1)\Delta(a))R(u)^{-1}$ for all $a \in Y(g)$;

3) $(T_a \otimes T_b)R(u) = R(u + a - b)$;

4) $R^{12}(u)R^{21}(-u) = 1 \otimes 1$;

5) $R^{12}(u_1 - u_2)R^{13}(u_1 - u_3)R^{23}(u_2 - u_3) = R^{23}(u_2 - u_3)R^{13}(u_1 - u_3)R^{12}(u_1 - u_2)$;

6) $R_1 = c_2$.

The identity 2) means that $Y(g)$ is a pseudotriangular Hopf algebra. Consider $Y(g)[[h]] = Y_h(g)$. Clearly $Y_h(g)$ contains $U(g)[[h]]$ as a Hopf subalgebra. Let $F$ satisfy condition 1 of Theorem 3 and we can view $F$ as an element of $(Y_h(g))^{\otimes 2}$. Obviously, one can extend the Hopf algebra structure to $Y_h(g)$. Let us define a new algebra $\tilde{Y}_h(g)$, which has the same multiplication as $Y_h(g)$ but comultiplication is defined as $\tilde{\Delta}(a) = F^{-1}\Delta(a)F$.

The main result of this paper is the following:

**Theorem 5.**

1) The algebra $\tilde{Y}_h(g)$ is a Hopf algebra.

2) Define $\tilde{R}(u)$ to be $\tilde{R}(u) = (F^{21})^{-1}R(u)F$.

Then $(\tilde{\Delta} \otimes 1)\tilde{R}(u) = \tilde{R}^{13}(u)\tilde{R}^{23}(u)$;

3) $(T_a \otimes T_b)\tilde{R}(u) = \tilde{R}(u + a - b)$;

4) $\tilde{R}^{12}(u)\tilde{R}^{21}(-u) = 1 \otimes 1$;

5) $(T_a \otimes 1)\tilde{\Delta}'(a) = \tilde{R}(u)((T_a \otimes 1)\tilde{\Delta}(a))\tilde{R}(u)^{-1}$ for all $a \in \tilde{Y}_h(g)$;

6) $\tilde{R}^{12}(u_1 - u_2)\tilde{R}^{13}(u_1 - u_3)\tilde{R}^{23}(u_2 - u_3) = \tilde{R}^{23}(u_2 - u_3)\tilde{R}^{13}(u_1 - u_3)\tilde{R}^{12}(u_1 - u_2)$;

7) $\tilde{R}(\frac{a}{h}) = 1 \otimes 1 + h \left( \frac{c_2}{u} + r \right) + 0(h)$;
Proof. 1) We must prove that $\tilde{\Delta}$ is a coassociative operation. This is straightforward from coassociativity $\Delta$ and the defining identity for $F$.

5) By the definition of $\tilde{\Delta}$ we have: $(T_u \otimes 1)\tilde{\Delta}'(a) = (T_u \otimes 1)((F^{21})^{-1}\Delta'(a)F^{21})$.

We note that $(T_u \otimes T_b)F = F$ since $F \in (U(g)[h])^{S2}$. Hence,

$$(T_u \otimes 1)\tilde{\Delta}'(a) = (F^{21})^{-1}R(u)((T_u \otimes 1)\Delta(a))R(u)^{-1}F^{21} = ((F^{21})^{-1}R(u)F)((T_u \otimes 1)\tilde{\Delta}(a))((F^{21})^{-1}R(u)F)^{-1} = \tilde{R}(u)((T_u \otimes 1)\tilde{\Delta}(a)\tilde{R}(u)^{-1}$ by Theorem 4.

2) If $Y(g)$ were a triangular Hopf algebra, all would follow from results [D3]. It turns out that the pseudotriangular structure does not affect considerations similar to ones of [D3].

We have:

$$(\tilde{\Delta} \otimes 1)\tilde{R}(u) = (F^{12})^{-1}((\Delta \otimes 1)((F^{21})^{-1}R(u)F))F^{12} = (F^{12})^{-1}(\Delta \otimes 1)(F^{21})^{-1}((\Delta \otimes 1)R(u))(\Delta \otimes 1)F \circ F^{12} = (F^{12})^{-1}(\Delta \otimes 1)(F^{21})^{-1}(R^{13}(u)R^{23}(u))(1 \otimes \Delta)F \circ F^{23}.

Again since $(T_u \otimes T_b)F = F$, it follows from Theorem 4 that $R^{23}(u)((1 \otimes \Delta)F) = ((1 \otimes \Delta')F)R^{23}(u)$. On the other hand $(1 \otimes \Delta')F = ((\Delta \otimes 1)F)^{32}F^{13}(F^{32})^{-1}$, where $(a \otimes b \otimes c)^{32} = a \otimes c \otimes b$. Further, $R^{13}(u)((\Delta \otimes 1)F)^{32} = ((\Delta' \otimes 1)F)^{32}(F^{31})^{-1}$. It remains to show, that

$$(F^{12})^{-1}((\Delta \otimes 1)(F^{21})^{-1}((\Delta' \otimes 1)F)^{32} = (F^{31})^{-1}

which is true by the defining relation for $F$.

3), 4) and 7) are straightforward from the corresponding statements of Theorem 4. Let us deduce 6). It follows from 2) that $(T_a \otimes 1 \otimes 1)((\tilde{\Delta} \otimes 1)\tilde{R}(x)) = \tilde{R}^{13}(x + a)\tilde{R}^{23}(x)$. Hence,

$$\tilde{R}(a)(T_a \otimes 1 \otimes 1)((\tilde{\Delta} \otimes 1)\tilde{R}(x)) = \tilde{R}^{12}(a)\tilde{R}^{13}(x + a)\tilde{R}^{23}(x).$$

On the other hand 5) implies that

$$\tilde{R}(a)(T_a \otimes 1 \otimes 1)((\tilde{\Delta} \otimes 1)\tilde{R}(x)) = (T_a \otimes 1 \otimes 1)((\tilde{\Delta}' \otimes 1)\tilde{R}(x))\tilde{R}(a).$$

Since $(\tilde{\Delta}' \otimes 1)\tilde{R}(x) = \tilde{R}^{23}(x)\tilde{R}^{13}(x)$, we find that $(T_a \otimes 1 \otimes 1)((\tilde{\Delta} \otimes 1)\tilde{R}(x)) = \tilde{R}^{23}(x)\tilde{R}^{13}(x + a)$, which completes the proof.

The results from [KST] show that the problem of finding explicit formulas leads to rather difficult computations even in the simplest case of $s\ell(2)$ with $F$ found in [CGG]. Our
aim is to present a number of cases when a rational R-matrix for \( s\ell(n) \) and \( o(n) \) can be computed explicitly in the corresponding fundamental n-dimensional representations. We need the following corrolary to Theorem 5:

**Corrolary 1.** Let \( \frac{c_2}{u-v} + r_0 \) be a rational solution of CYBE for \( s\ell(n) \), \( F \in U(s\ell(n))^{\otimes 2} \) be the corresponding “quantizing element” and \( R \in \text{Mat}(n, \mathbb{C})^{\otimes 2} \) be the image of the quantum R-matrix \( (F^{21})^{-1}F \) in the fundamental n-dimensional representation of \( s\ell(n) \). If \( P \in \text{Mat}(n, \mathbb{C})^{\otimes 2} \) is the permutation matrix, which acts in \( \mathbb{C}^n \otimes \mathbb{C}^n \) as \( P(a \otimes b) = b \otimes a \), then \( uR + P \) satisfies YBE.

**Proof.** Let us consider the R-matrix \( \tilde{R}(u) = (F^{21})^{-1}R(u)F \), where \( R(u) \) is Drinfeld’s R-matrix for \( Y(s\ell(n)) \). It was proved in [D2] that the image of \( R(u) \) in the n-dimensional representation is \( 1 \otimes 1 + \frac{P}{u} \) up to a scalar factor. It is easy to see that \( (T^{21})^{-1}PT = P \) for any invertible \( T \in \text{Mat}(n, \mathbb{C})^{\otimes 2} \). This observation completes the proof.

**Remark.** It is worth noticing that we have proved that if \( R \in \text{Mat}(n, \mathbb{C})^{\otimes 2} \) satisfies YBE and is unitary, i.e. \( R^{21}R = 1 \otimes 1 \), then \( R + \frac{P}{u} \) is a rational solution of YBE because according to [D1] any such \( R \) comes from some \( F \in U(g\ell(n))^{\otimes 2} \). Of course, knowing the answer it is not difficult to check that \( (a \otimes b)P = P(b \otimes a) \) for any \( a, b \in \text{Mat}(n, \mathbb{C}) \). However the general approach provides rational solutions in any finite-dimensional representation of \( Y(s\ell(n)) \).

**Example 1.** We would like to expose a number of unitary R-matrices not involving complicated computations. According to Theorem 2 we have to indicate a pair \((L, B)\), where \( L \subset s\ell(n) \) and \( B \) is the corresponding non-degenerate 2-cocycle. Put

\[
L = \{(a_{ij}) : a_{ij} = 0 \text{ for } i > j; \quad a_{ii} = -a_{n+1-i, n+1-i}\}, \quad B(x, y) = f([x, y])
\]

where \( f([a_{ij}]) = \sum_{i+j=n+1} a_{ij} \).

Let \( E_{ij} \in \text{Mat}(n) \) be the set of matrix units. Let us denote \( E_{ii} - E_{n+1-i, n+1-i} \) by \( H_i \).

Then the corresponding classical r-matrix \( r \in \text{Mat}(n)^{\otimes 2} \) has of the following form:

\[
r = \frac{1}{2} \left( \sum_i (H_i \otimes E_{i,n+1-i} - E_{i,n+1-i} \otimes H_i) + \sum_{i<j<n+1} (E_{ij} \otimes E_{j,n+1-i} - E_{j,n+1-i} \otimes E_{ij}) \right)
\]

Direct computations show that \( r^3 = 0 \). It is known (see [CGG]) that in this case \( R = 1 \otimes 1 + r + \frac{1}{2}r^2 \) is a unitary solution of YBE. Corrolary 1 implies that

\[
u(1 \otimes 1 + r + \frac{1}{2}r^2) + P
\]
is a rational solution of YBE.

Let \( o(N, \mathbb{C}) \) be an orthogonal Lie algebra consisting of all matrices \( A \in \text{Mat}(N, \mathbb{C}) \) such that \( A^t = -A \). Let \( K \in \text{Mat}(N, \mathbb{C}) \otimes \mathbb{C}^2 \) be the matrix obtained from \( P \in \text{Mat}(N, \mathbb{C}) \otimes \mathbb{C}^2 \) by the transposition in the first factor. It was proved in \([D2, KS]\) that the image of \( R(u) \in Y(o(N)) \otimes \mathbb{C}^2 \) in the \( N \)-dimensional representation of \( Y(o(N)) \) is, up to a scalar factor

\[
u_1 \otimes 1 + P - \frac{u}{k+u} K, \quad k = \frac{1}{2}(N-2)
\]

**Corollary 2.** Let \( r \in o(N) \otimes \mathbb{C}^2 \) be a classical r-matrix and \( F \in \text{U}(o(N)) \otimes \mathbb{C}^2 \) be the corresponding quantizing element. Let us denote by \( F_0 \) (respectively \( R_0 \)) the image of \( F \) (respectively \( (F^{21})^{-1}F \)) in \( \text{Mat}(N) \otimes \mathbb{C}^2 \).

Then \( uR_0 + P - \frac{u}{k+u}(F_0^{21})^{-1}KF_0 \) satisfies YBE.

**Proof.** The statement can be proved exactly as Corollary 1.

**Example 2.** Now we need another realization of \( o(N, \mathbb{C}) \), namely

\[o(N) = \{(a_{ij}) \in \text{Mat}(N) : a_{ij} = -a_{N+1-j,N+1-i}\}\]

Let \( T \) be any element of \( \text{GL}(N, \mathbb{C}) \) which conjugates the first form of \( o(N) \) to the second one. Denote \( E_{11} - E_{NN} \) by \( H \) and \( E_{12} - E_{N-1,N} \) by \( E \). Clearly \( e = TET^{-1} \) and \( h = THT^{-1} \) are skew-symmetric matrices. Further we have \([h, e] = e\) since \([H, E] = E\) and \( r_0 = h \otimes e - e \otimes h \) satisfies CYBE (for \( N > 3 \)). The corresponding quantizing element \( F \) was found in \([CGG]\) and is of the form:

\[F = 1 \otimes 1 + \sum_n \frac{1}{n!} h(h+1)...(h+n-1) \otimes e^n \subset \text{U}(o(N)) \otimes \mathbb{C}^2\]

Clearly \( r_1 = H \otimes E - E \otimes H, \quad N > 3 \) also satisfies CYBE and therefore, we can compute the corresponding matrix solution of YBE, which is

\[1 \otimes 1 + r_1 - E_{N-1,N} \otimes E_{12} \subset \text{Mat}(N) \otimes \mathbb{C}^2\]

since the image of \( E^2 \) is 0 in \( \text{Mat}(N) \). Finally we obtain that the following element of \( \text{Mat}(N) \otimes \mathbb{C}^2 \) is a rational solution of YBE:

\[(1 \otimes 1 + r_0 - e_- \otimes e_+)u + P - \frac{u}{k+u} (1 \otimes 1 - e \otimes h) K (1 \otimes 1 + h \otimes e), \quad k = \frac{1}{2}(N-2)\]

where \( e_- = T E_{N-1,N} T^{-1} \) and \( e_+ = T E_{12} T^{-1} \).
**Remark.** It is interesting to point out that the construction of the new solutions to the YBE (Theorem 5) preserves the regularity property \([KS]\) of the initial \(R\)-matrix: \(R(0) = P\). Therefore one can obtain series of integrable models with local Hamiltonians corresponding to these new \(R\)-matrices. In the simplest case of \(Y(s\ell(2))\) with non-standard quantization of \(s\ell(2)\) (see \([KST]\)) the spin-1/2 analog of the \(XXX\)-model on one-dimensional chain is given by the Hamiltonian

\[
H = \sum_n ((\sigma_n, \sigma_{n+1}) + \xi^2 \sigma_n^x \sigma_{n+1}^x + \xi (\sigma_n^- - \sigma_{n+1}^-)),
\]

where \(\xi\) is a deformation parameter, \(\sigma_n^x, \sigma_n^y, \sigma_n^z\) are Pauli sigma-matrices acting in \(\mathbb{C}^2\) related to the \(n\)-th site of the chain and \(\sigma_n^- = \frac{1}{2} (\sigma_n^x - i\sigma_n^y)\).

**Acknowledgements.** The authors are thankful to Professors J. Lukiersky and A. Molev for valuable discussions. The results of this paper were delivered by the first author at the Colloquium “Quantum Groups and Integrable Systems” in Prague, June 1996. The visit was supported by Swedish Natural Science Research Council.

**References**

[B] Belavin, A.A., Drinfeld, V.G., Funct. Anal. Appl., 16 (1982) 159.

[CGG] Coll, V., Gerstenhaber, M., Giaquinto, A., Israel Math. Conf. Proc., vol. 1, Weizmann Science Press, (1989).

[D] Drinfeld, V.G., Proc. ICM-86 (Berkeley), vol. 1 (1986) 798.

[D1] Drinfeld, V.G., Soviet Math. Dokl., 27 (1983) 68.

[D2] Drinfeld, V.G., Soviet Math. Dokl., 32 (1985) 254.

[D3] Drinfeld, V.G., Leningrad Math. J., 1 (1990) 1459.

[J1] Jimbo, M., Lett. Math. Phys., 10 (1985) 63.

[KS] Kulish, P.P., Sklyanin, E.K., Zap. Nauch. Semin. LOMI, 95 (1980) 129.

[KST] Khoroshkin, S., Stolin, A., Tolstoy, V., preprint TRITA-MAT-1995-MA-17, KTH, Stockholm (1995).

[S] Stolin, A., Math. Scand., 69 (1991) 56.