Relational reasoning in the Region Connection Calculus *

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Abstract

This paper is mainly concerned with the relation-algebraic aspects of the well-known Region Connection Calculus (RCC). We show that the contact relation algebra (CRA) of certain RCC model is not atomic complete and hence infinite. So in general an extensional composition table for the RCC cannot be obtained by simply refining the RCC8 relations. After having shown that each RCC model is a consistent model of the RCC11 CT, we give an exhaustive investigation about extensional interpretation of the RCC11 CT, where we attach a superscript $\times$ to a cell entry in the table if and only if extensional interpretation is impossible for this entry. More important, we show the complemented closed disk algebra is a representation for the relation algebra determined by the RCC11 table. The domain of this algebra contains two classes of regions, the closed disks and closures of their complements in the real plane, and the contact relation is standard Whiteheadean contact (i.e. $aC b$ iff $a \cap b \neq \emptyset$).

Keywords: Region Connection Calculus; Contact relation algebras; Composition table; Complemented closed disk algebra; Dual-relation set; Extensionality.

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1 Introduction

Since the mid-1970’s the relational methods has become a fundamental conceptual and methodological tool in computer science. The wide-ranging diversity and applicability of relational methods has been demonstrated by series of RelMiCS seminars (International Seminar on Relational Methods in Computer Science). Relation algebra has been used as a basis for analyzing, modeling or resolving several computer science problems such as program specification, heuristic approaches for program derivation, automatic prover design, database and software decomposition, program fault tolerance, testing, data abstraction and information coding, and last but not least qualitative spatial reasoning. For a detailed overview we invite the reader to consult [6, 21, 30, 9].

Qualitative spatial reasoning (QSR) is an important subfield of AI which is concerned with the qualitative aspects of representing and reasoning about spatial entities. A large part of contemporary qualitative spatial reasoning is based on the behavior of “part of” and “connection” (or “contact”) relations in various domains [16, 7], and the expressive power, consistency and complexity of relational reasoning has become an important object of study in QSR.

Rather than give to attention to all the various systems existing on the market, we shall focus on one of the most widely referenced formalism for QSR, the Region Connection Calculus (RCC). RCC was initially described by Randell, Cohn and Cui in [31, 33], which is intended to provide a logical framework for incorporating spatial reasoning into AI systems.

In the RCC theory, the Jointly Exhaustive and Pairwise Disjoint (JEPD) set of topological relations known as RCC8 are identified as being of particular importance. RCC8 contains relations “$x$ is disconnected from $y$”, “$x$ is externally connected to $y$”, “$x$ partially overlaps $y$”, “$x$ is equal to $y$”, “$x$ is a tangential proper part of $y$”, “$x$ is a non-tangential proper part of $y$”, and the inverses of the latter two relations. Interestingly, this classification of topological relations has been independently given by Egenhofer [15] in the context of Geographical Information Systems (GIS). Since RCC8 is JEPD, it supports a composition table. The RCC8 composition table appears first in [8] and coincides with that of [15].

Originating in Allen’s analysis of temporal relations [1, 2], the notion of a composition table (CT) has become a key technique in providing an efficient inference mechanism for a wide class of theories [41, 17, 19, 32, 34, 35]. It is worthy of note that the precise meaning of a composition table depends to some
extent on the situation where it is employed.

Generally speaking, a CT is just a mapping $CT : \text{Rels} \times \text{Rels} \rightarrow 2^{\text{Rels}}$, where \text{Rels} is a set of relation symbols [10]. For three relation symbols $R$, $S$ and $T$, we say $\langle R, T, S \rangle$ is a composition triad in CT if $T$ is in $CT(R, S)$. A model of $CT$ is then a pair $\langle U, v \rangle$, where $U$ is a set and $v$ is a mapping from $\text{Rels}$ to the set of binary relations on $U$ such that $\{v(R) : R \in \text{Rels}\}$ is a partition of $U \times U$ and $v(R) \circ v(S) \subseteq \bigcup_{T \subseteq CT(R, S)} v(T)$ for all $R, S \in \text{Rels}$, where $\circ$ is the usual relation composition. A model $\langle U, v \rangle$ is called consistent if $T \in CT(R, S) \iff (v(R) \circ v(S)) \cap v(T) \neq \emptyset$ for all $R, S, T \in \text{Rels}$ [25]. This means that, for any three relation symbols $T$, $R$ and $S$, $T$ is an entry of the cell specified by $R$ and $S$ if and only if there exist three regions $a, b, c$ in $U$ such that $R(a, b)$, $S(b, c)$ and $T(a, c)$. We call a consistent model extensional if $v(R) \circ v(S) = \bigcup_{T \subseteq CT(R, S)} v(T)$ for all $R, S \in \text{Rels}$ [25]. In such a model, if $T$ is an entry in the cell specified by $R$ and $S$, then whenever $T(a, c)$ holds, there must exist some $b$ in $U$ s.t. $R(a, b)$ and $S(b, c)$. Note if a CT has an extensional model $(U, v)$, then by a theorem given in [22], this CT is the composition table of a relation algebra and $(U, v)$ is a representation of this relation algebra. In what follows, when the interpretation mapping $v$ is clear from the context, we also write $U$ for this model.

Suppose that $\mathcal{R}$ is a JEPD set of relations on a nonempty set $U$, and $R, S \in \mathcal{R}$. Düntsch [10] defines the weak composition of $R, S$ as

$$R \circ_w S = \bigcup \{T \in \mathcal{R} : T \cap R \circ S \neq \emptyset\}.$$  

In case $\mathcal{R}$ is finite, we summarize the weak compositions in a table and call this a weak composition table. Note by definition, a model $(U, v)$ of a CT $CT : \text{Rels} \times \text{Rels} \rightarrow 2^{\text{Rels}}$ is consistent if and only if $CT$ is precisely the weak composition table of $\text{Rels}$ on $U$.\footnote{What should be addressed is, although Düntsch call the RCC11 table [10, Table 17] weak, it is not clear or at least haven’t be proven whether or not this table is precisely the weak composition table for each RCC model.}

Since the RCC theory entails the RCC8 CT, each RCC model is already a model of the RCC8 CT. But examination of the RCC8 CT reveals that an extensional interpretation is not compatible with the 1st-order RCC theory. This fact is pointed out by Bennett in [3] and [4]. To avoid this problem and hence construct an extensional composition table, Bennett suggests [3] to remove the
universal region from the domain of possible referents of the region constants. In [25], however, after an exhaustive investigation about extensional interpretation of the RCC8 CT, Li and Ying has shown that no RCC model can be interpreted extensionally anyway.

Another way to construct an extensional composition table has also been suggested by Bennett et al. [4]:

“One might further conjecture that by refining relations in a set Rel one can always arrive at a set Rel' which is more expressive than Rel and whose CT can be interpreted extensionally.”

This approach to extensional composition table relates closely to the formalism of relation algebras initiated by Tarski [39]. Moreover, noticing that the expressiveness of reasoning with basic operations on binary relations is equal to the expressive power of the three variable fragment of first order logic with at most binary relations [40], it seems worthwhile to use methods of relation algebras to study connection (or contact) relations in their own right. Note that Bennett’s question described above can be reformulated as the question of determining the relation algebra generated by the connectedness relation.

In a series of papers [10, 13, 12, 14], Düntsch and his colleagues study the relation-algebraic aspects of the RCC theory systematically. They show that the contact relation algebra contains more relations than the RCC8 relations might suggest: the RCC8 relations has been refined to RCC10, RCC11 and RCC25, and corresponding weak CTs are also given. In particular, they show [12] that each relation algebra generated by the contact relation of an RCC model contains an integral algebra \( A \) with 25 atoms as a subalgebra. In the same paper, they ask if there is an RCC model with \( A \) as its associated binary relation algebra.

Later, Mormann [27] introduces the concept of ‘Hole’ relation \( H \) in RCC and shows \( H \) and \( C \) are interdefinable. More importantly, he shows several RCC25 base relations can be split by \( H \) or some relations derived from \( H \), thus gives a negative answer to Düntsch’s question. He also suggests that the hole relation \( H \) may be used to define various infinite families of hole relations. Interestingly, by formalizing the concept of ‘Separable Proper Part’, he shows that [28], for a large class of RCC models, the relation algebra generated by the contact relation contains infinitely many elements.

Similar results are also obtained in this paper. For the RCC model \( \mathfrak{B}_\omega \) constructed in [24], which is a least RCC model in the sense that each RCC model
contains it as a sub-model, we show that the contact relation algebra of \( \mathcal{B}_\omega \) is not atomic complete, therefore not finite. For the standard RCC model associated to each \( \mathbb{R}^n \), we construct two strictly decreasing sequences of ‘hole’ relations in the associated contact relation algebra.

All these results suggest that Bennett’s conjecture is not applicable. To obtain an extensional model of the RCC8 CT, one should restrict the domain of possible regions: an RCC model might contain too much regions. Düntsch [10] has shown that the domain of closed disks of the Euclidean plane provides an extensional model of the RCC8 CT, namely, the relation algebra determined by the RCC8 CT can be represented by the closed disk algebra. The domain of connected regions bounded by Jordan curves, called Egenhofer model, also provides an extensional interpretation [26]. Interestingly, this model is in a sense a maximal extensional domain of the RCC8 CT [26]. This suggests that these disk-like regions are more suitable for the RCC8 relations. One serious problem with these two domains of regions is neither are closed under complementation. But, as noted by Stell [37], complement is a fundamental concept in spatial relations. These two domains of regions are therefore too restrictive.

In [10], with modelling complementation in mind, Düntsch refines RCC8 to RCC11: the ‘\( x \) is externally connected to \( y \)’ relation splits into two situations according to whether or not \( x \) is equal to \( y' \), the complement of \( y \); the ‘\( x \) partially overlaps to \( y' \)’ relations splits into three situations according to whether of not \( x \) is a tangential or non-tangential proper part of \( y' \). The RCC11 CT is also given and it “turns out that there is a relation algebra \( A \) whose composition is represented by the RCC11 table. \( A \), however, cannot come from an RCC model as Proposition 8.6 shows, and no representation of \( A \) is known” [10].

In the present paper, we first show that each RCC model is consistent w.r.t. the RCC11 CT and then an exhaustive investigation about extensional interpretation of the RCC11 CT is given. In fact, we attach a superscript \( \times \) to a cell entry in the table if and only if extensional interpretation is impossible for this entry.

One of the main contribution of this paper is to provide an extensional model for the RCC11 CT. Note models of the RCC11 CT are closed under complementation. Our model then contains simply two kinds of regions: the closed disks and the closures of their complements in the Euclidean plane, where two regions are connected if they have nonempty intersection. Note this domain is clearly
a sub-domain of the standard RCC model associated to $\mathbb{R}^2$. We then have two methods to introduce the RCC11 relations on this domain: the first system of relations is obtained by restriction of the RCC11 relations in the standard RCC model associated to $\mathbb{R}^2$, the second can be defined by the connectedness relation on this domain. Interestingly these two systems of relations are identical. The binary relation algebra generated by the connectedness relation, the *complemented closed disk algebra*, has 11 atoms that correspond to the RCC11 relation and the composition of this algebra is just the one specified by the RCC11 CT. In a word, the complemented closed disk algebra provides a representation of the relation algebra determined by the RCC11 CT.

Note that hand building of composition tables even for a small number of relations is an arduous and tedious work. Although there are more general methods to compute composition tables (see e.g. [23]), these methods seem not appropriate for the present purposes. Our requirements are manifold: the method should be applicable not only for determining the composition table, but also for checking the consistency and extensionality of the table. To this aim, we propose a specialized approach to reduce the calculations: by using this approach, the work needed can be reduced to nearly 1/8 of that needed by the cell-by-cell verification. For example, the work need for the RCC11 CT has been decreased to 15 calculations of compositions, contrasting with the $11 \times 11$ cell-by-cell verifications. This approach is also valid to other composition tables whose domain is closed under complementation, e.g. the RCC7 weak CT and the RCC25 weak CT.

The rest of the paper is arranged as follows. In next section, we briefly summarize some basic concepts of contact relation algebras and the RCC theory. Section 3 concerns the contact relation algebras for certain RCC models. We first show the CRA of $\mathcal{B}_\omega$ is not atomic complete and then construct two infinite chains in the CRA of $n$-dimensional Euclidean space. This fact shows that it is impossible to obtain an extensional CT for the RCC theory by simply refining the RCC8 relations. The notions of dual relation set and dual generating set for RCC relations are introduced in Section 4. Based on these notions, a very effective approach to determine the RCC weak CT is introduced. Using this approach, in Section 5, we first show each RCC model is a consistent model of the RCC11 CT and then give a complete analysis of the extensionality of the RCC11 CT. Section 6 introduces the complemented closed disk algebra $\mathcal{L}$ which is a representation of the relation algebra determined by the RCC11 composition.
2 Contact relation algebras

In this section we summarize some basic concepts of contact relation algebras and the RCC models. For contact relation algebras our references are [10, 13, 12, 11], and for RCC models [31, 33, 7, 3, 36, 14, 25].

Recall in a relation algebra (RA) \((A, +, \cdot, -, 0, 1, \sim, 1')\), \((A, +, \cdot, -)\) is a Boolean algebra, and \((A, \circ, \sim, 1')\) is a semigroup with identity \(1'\), and \(a \sim \sim = a, (a \circ b) \sim = b \circ a \sim\). In the sequel, we will usually identify algebras with their base set.

An important example of relation algebra is the full algebra of binary relations on the underlying set \(U\), written \((\text{Rel}(U), \cup, \cap, -, \emptyset, U \times U, \circ, \sim, 1')\), where \(\text{Rel}(U)\) is the set of all binary relations on \(U\), \(\circ\) is the relational composition, \(\sim\) the relation converse, and \(1'\) is the identity relation on \(U\). For \(R \in \text{Rel}(U)\), and \(x, y, z \in U\) we usually write \(x \text{R} y\) or \(R(x, y)\) if \((x, y) \in R\).

Recall a subset \(A\) of \(\text{Rel}(U)\) which is closed under the distinguished operations of \(\text{Rel}(U)\) and contains the distinguished constants is called an algebra of binary relations (BRA) on \(U\). A relation algebra \(A\) is called representable if it is isomorphic to a subalgebra of a product of full algebras of binary relations, \(A\) is called integral, if \(1'\) is an atom of \(A\).

To avoid trivialities, we always assume that the structures under consideration have at least two elements. Suppose that \(U\) is a nonempty set of regions, and that \(C\) is a binary relation on \(U\) which satisfies

(C1) \(C\) is reflexive and symmetric,
(C2) \(\forall x, y \in U\)[\(x = y \leftrightarrow \forall z \in U(C(x, z) \leftrightarrow C(y, z))\)].

Düntsch et al. [13] call a binary relation \(C\) which satisfies (C1) and (C2) a contact relation; and an RA generated by a contact relation will be called a contact RA (CRA). A contact relation \(C\) on an ordered structure \(\langle U, \leq \rangle\) is said to be compatible with \(\leq\) if \(- (C \circ - C) = \leq\). In this paper we only consider compatible contact relations on orthocomplemented lattices. Recall an orthocomplemented lattice [38] is a bounded lattice \(\langle L, 0, 1, \lor, \land \rangle\) equipped with a unary complemented operation \(': L \to L'\) such that

\[x'' = x, \quad x \land x' = 0, \quad x \leq y \iff x' \geq y'.\]
Suppose $L$ is an orthocomplemented lattice containing more than four elements and $C$ is a contact relation other than the identity. Set $U = L \setminus \{0, 1\}$. Since $1_U$ is RA definable \cite{10}, we can restrict the contact relations $C$ and other relations definable by $C$ on $U$. The following relations can then be defined from $C$ on $U$:

\[
\begin{align*}
DC &= -C \\
1' &= P \cdot P^\sim \\
O &= P^\sim \circ P \\
EC &= C \cdot \neg O \\
NTPP &= PP \cdot \neg TPP \\
T &= -(P \circ P^\sim) \\
POD &= O \cdot \neg T \\
ECN &= EC \cdot \neg ECD \\
DN &= DR \cdot \neg ECD
\end{align*}
\]

\[
\begin{align*}
ECD &= -O \cdot \neg T \\
PODZ &= ECD \circ NTPP \\
PODY &= POD - PODZ
\end{align*}
\]

We have the following systems of JEPD relations on $U$ \cite{10}:

RCC5 relations

$R_5 = \{1', PP, PP^\sim, PO, DR\}$;

RCC7 relations

$R_7 = \{1', PP, PP^\sim, PO, POD, ECD, DN\}$;

RCC8 relations

$R_8 = \{DC, EC, PO, 1', TPP, NTPP, TPP^\sim, NTPP^\sim\}$;

RCC11 relations

$R_{11} = \{1', TPP, TPP^\sim, NTPP, NTPP^\sim, PON, PODY, PODZ, ECN, ECD, DC\}$.

We summarize some characterizations of these RCC relations.

**Lemma 2.1.** Suppose $L$ is an orthocomplemented lattice $L$ with $|L| > 4$ and $C$ is a compatible contact relation on $L$ other than the identity. Then for any $x, y \in U = L \setminus \{0, 1\}$, we have the following results:

1. $xPONY$ iff $x \land y > 0$, $x \lor y < 1$, $x \land y' > 0$ and $x' \land y > 0$;
2. $xPODY$ iff $x \land y > 0$, $x \lor y = 1$;
3. $xPPY$ iff $x < y$;
4. $xECDY$ iff $x = y'$;
5. $xECDY$ iff $x < y'$ and $xCy$;
\[(6) \quad x \text{TPP} y \iff x < y \text{ and } x \text{Cy}'; \]
\[(7) \quad x \text{NTPP} y \iff x < y \text{ and } x \text{DCy}'; \]
\[(8) \quad x \text{PODY} y \iff y' < x \text{ and } x' \text{Cy}'; \]
\[(9) \quad x \text{PODZ} y \iff y' < x \text{ and } x' \text{DCy}'. \]

In what follows, we shall often write respectively \(-x, x + y, x - y\) for \(x', x \lor y\) and \(x \land y'\).

### 2.1 Models of the RCC axioms

The Region Connection Calculus (RCC) was originally formulated by Randel, Cui and Cohn [33]. There are several equivalent formulations of RCC [36, 10], we adopt in this paper the one in terms of Boolean connection algebra (BCA) [36].

**Definition 2.1.** A model of the RCC is a structure \(\langle A, C \rangle\) such that

A1. \(A = \langle A; 0, 1, ', \lor, \land \rangle\) is a Boolean algebra with more than two elements.

A2. \(C\) is a symmetric and reflexive binary relation on \(A \setminus \{0\}\).

A3. \(C(x, x')\) for any \(x \in A \setminus \{0, 1\}\).

A4. \(C(x, y \lor z) \iff C(x, y) \text{ or } C(x, z)\) for any \(x, y, z \in A \setminus \{0\}\).

A5. For any \(x \in A \setminus \{0, 1\}\), there exists some \(w \in A \setminus \{0, 1\}\) such that \(C(x, w)\) doesn’t hold.

Stell [36] calls such a construction a *Boolean connection algebra* (BCA), this conception is stronger than the *Boolean contact algebra* given by Düntsch [10]. In particular, the connection in a BCA satisfies Condition (C2) and hence is a contact relation in Düntsch’s sense.

Given a regular connected space \(X\), write \(\text{RC}(X)\) for the regular closed algebra of \(X\). Then with the standard Whiteheadean contact (i.e. \(a C b \iff a \cap b \neq \emptyset\)), \(\langle \text{RC}(X), C \rangle\) is a model of the RCC [20]. These models are called *standard RCC models* [10]. Later we shall refer the standard model associated to a regular connected space \(X\) simply \(\text{RC}(X)\).

If an RCC model \(A\) satisfies the following interpolation property [29, 3] (INT for short):

\[
x \text{NTPP} y \rightarrow \exists z (x \text{NTPP} z \land z \text{NTPP} y)
\]

we call it a *strong* RCC model. Standard RCC models associated to \(\mathbb{R}^n\) are strong models. There are also RCC models which are not strong, e.g., the least RCC model \(\mathfrak{B}_\omega\) constructed in [24] (see Section 3.1. of this paper).
Note in general some of the RCC11 relations, e.g. TPP, generated by some contact relation on an orthocomplemented lattice will be empty. But for RCC models, all these relations are nonempty. Düntsch et al. also refined the RCC11 relations and obtained 25 JEPD topological relations, namely, the RCC25 relations. These set of relations are contained in the CRA of each RCC model.

3 RCC models and their contact relation algebras

In this section we shall show that the CRA of $B_\omega$ is atomic incomplete and the CRA of $RC(\mathbb{R}^n)$ is infinite and hence not generated by a finite number of atoms. This gives a negative answer to Bennett’s conjecture depicted in the introduction of this paper. A sufficient condition for these relation algebras to be integral is also given.

3.1 The CRA of a least RCC model $B_\omega$

In [24], Li and Ying constructed a countable RCC model $B_\omega$ which is least in the sense that each RCC model contains a sub-model isomorphic to $B_\omega$. We recall some basic facts about this model.

Let $\Sigma = \{0, 1\}$ and let $\Sigma^*$ be the set of finite strings over $\Sigma$ with $\epsilon$ the empty string. Now for each string $s \in \Sigma^*$, we associate a left-closed-and-right-open sub-interval of $[0, 1)$ as follows: Take $x_\epsilon = [0, 1); x_0 = [0, 1/2), x_1 = [1/2, 1);$ $x_{00} = [0, 1/4), x_{01} = [1/4, 1/2), x_{10} = [1/2, 3/4), x_{11} = [3/4, 1).$

In general, suppose $x_s$ has been defined for a string $s \in \{0, 1\}^*$, we define $x_{s0}$ to be the first half left-closed-and-right-open sub-interval of $x_s$, and $x_{s1}$ the second half.

Write $B$ the subalgebra of the powerset algebra of $[0, 1)$ generated by all $x_s$. Clearly, $B$ is a countable atomless Boolean algebra. Define a connection $C_\omega$ on $U = B \setminus \{\emptyset, x_\epsilon\}$ as follows: for two regions $a, b \in U$, $C_\omega(a, b)$ if and only if either $a \cap b \neq \emptyset$ or there exist $s, t, s_1 \in \Sigma^*$ and some $n \geq 0$ with $\{s, t\} = \{s_10\underbrace{1 \cdots 1}_{n}, s_11\underbrace{1 \cdots 1}_{n}\}$ and $x_s \subseteq a$, $x_t \subseteq b$.

Recall the following proposition in [24].
Proposition 3.1. (i) For any string $s$ and any $n \geq 1$, $\text{NTPP}_\omega(x_{s0}, x_s)$ and $\text{NTPP}_\omega(x_{s0}, x_s \cup x_s \{1 \ldots n\})$;

(ii) For any nonempty $a \in \mathbb{B}$ and any string $s \neq \epsilon$, $\text{NTPP}_\omega(a, x_s)$ if and only if $a \subseteq x_s - x_s \{1 \ldots n\}$ for some $n \geq 1$.

By above proposition, we have shown in [25] that $\text{NTPP}_\omega \circ \text{NTPP}_\omega = \text{NTPP}_\omega$ doesn’t hold. Moreover, if we write inductively $\text{NTPP}^{n+1} = \text{NTPP}_\omega \circ \text{NTPP}^n$, then, the following theorem shows

$$\text{NTPP}_\omega, \text{NTPP}^2, \ldots, \text{NTPP}^n, \ldots$$

is a strict decreasing chain.

Theorem 3.1. In the RCC model $\mathbb{B}_\omega$, we have $\text{NTPP}^n \neq \text{NTPP}^{n+1}$ for any positive integer $n$, and $\bigcap_{n \in \mathbb{N}} \text{NTPP}^n = \emptyset$.

Proof. Note for any two regions $a, b \in \mathbb{U}$, if we set $a^* = \bigcup \{x_{0s} : x_s \subseteq a\}$ and $b^* = \bigcup \{x_{0s} : x_s \subseteq b\}$, then we have $\text{NTPP}_\omega(a, b)$ if and only if $\text{NTPP}_\omega(a^*, b^*)$. This can be easily proved by entreating the definitions of $C_\omega$ and $\text{NTPP}_\omega$.

Suppose there exist two regions $a, b \in \mathbb{U}$ with $(a, b) \in \bigcap_{n \in \mathbb{N}} \text{NTPP}^n$. By above observation, we also have $(a^*, b^*) \in \bigcap_{n \in \mathbb{N}} \text{NTPP}^n$. Since there is a string $s = 0l_1l_2 \cdots l_k$ ($l_i \in \{0, 1\}$ for $i = 1, \ldots, k$) such that $x_s \subseteq a^* \subseteq b^* \subseteq x_0$, we have $(x_s, x_0) \in \bigcap_{n \in \mathbb{N}} \text{NTPP}^n$. We now show how to obtain a contradiction by proving that $(x_s, x_0) \notin \text{NTPP}^{k+1}$.

In general, given a string $t = t_1 0 1 \cdots 1$ and a region $a \in \mathbb{U}$ with $\text{NTPP}_\omega(x_t, a)$, we claim there exists some $p \geq 0$ such that $x_{\nu'} \subseteq a$, where $t' = t_1 1 \cdots 1$. Recall $a$ is a sum of finite many base regions, $x_{s_i}$ for instance, suppose $n$ the largest one of the lengths of these $s_i$. Then for any string $s$ with length bigger than or equal to $n$, we have either $x_s \subseteq a$ or $x_s \cap a = \emptyset$. Suppose for some $p$ bigger enough we have $x_{\nu'} \cap a = \emptyset$ with $t'$ as above. Then since $x_{\nu'}$ is externally connected to $x_t$, we shall have $x_{\nu'}$ is also externally connected to $a$. This contradicts the assumption that $\text{NTPP}_\omega(x_t, a)$.

For a string $t$, set $\lambda(t)$ as the total number of occurrences of $0$ in $t$. The above result then can be reformulated as follows: for a string $t = 0t_1$ and a region $a \subseteq x_0$ with $\text{NTPP}_\omega(x_t, a)$, then there exists another string $t'$ with $\lambda(t') = \lambda(t) - 1$ and $x_{\nu'} \subseteq a$. 
Now suppose there exist \( a_1, a_2, \ldots, a_k, a_{k+1} = x_0 \) such that

\[
x_s \text{NTPP}_\omega a_1 \text{NTPP}_\omega a_2 \cdots a_k \text{NTPP}_\omega a_{k+1} = x_0,
\]

where \( s = 0l_1l_2 \cdots l_k \) as above. Suppose \( \lambda(s) = m > 0 \). By above observation, we shall have some \( s_1 \) such that \( \lambda(s_1) = \lambda(s) - 1 \) and \( x_{s_1} \subseteq a_1 \). By assumption that \( a_1 \text{NTPP}_\omega a_2 \) we shall have \( x_{s_1} \text{NTPP}_\omega a_2 \). Continuing this procedure, since \( 1 < m \leq k+1 \), we shall obtain a string \( t = 01 \cdots 1 \) \((p \geq 0)\) such that \( x_t \subseteq a_k \), and therefore \( \text{NTPP}(x_t, x_0) \). This cannot be true since \( x_1 \) is externally connected to both \( x_t \) and \( x_0 \). As a result we have \( (x_s, x_0) \notin \text{NTPP}_\omega^{k+1} \) for any \( s = 0l_1l_2 \cdots l_k \) \((l_i \in \{0, 1\})\). This suggests \( \bigcap_{n \in \mathbb{N}} \text{NTPP}_\omega^n = \emptyset \).

On the other hand, note if we set \( s_i = 00 \cdots 0 \) for \( i \geq 0 \), we have

\[
x_{s_i} \text{NTPP}_\omega x_{s_i} \subseteq x_{s_i-1} \text{NTPP}_\omega x_{s_i} \cdots x_s \text{NTPP}_\omega x_0 = x_0.
\]

Combining these two observations, we shall have \( (x_{s_k}, x_0) \) is in \( \text{NTPP}_\omega^k \) but not in \( \text{NTPP}_\omega^{k+1} \). Therefore we have shown \( \text{NTPP}_\omega^k \neq \text{NTPP}_\omega^{k+1} \) for any positive integer \( k \).

This result shows that the CRA of countable RCC model \( \mathfrak{B}_\omega \) is not atomic complete, hence infinite. As a result, the weak composition table for RCC8 relations (and all its finite refinements closed under inverse) cannot be extensional w.r.t. the RCC theory.

In next section we shall show that the CRA of standard model of RC(\( \mathbb{R}^n \)) contains two strictly decreasing sequences of relations. The proof of this result relies on a binary hole relation defined by Mormann [27].

### 3.2 Hole relations

To show the contact relation algebra of an RCC model may contain more relations than that given in [12], Mormann introduces the concept of Hole relation [27]. This definition captures the intuitive concept “hole”.

**Definition 3.1.** Let \( \langle A, C \rangle \) be a model of RCC. Then the relation \( \text{H} \) on \( U = A \setminus \{0, 1\} \) is defined as: \( \text{H} = \text{EC} \cap -(\text{EC} \circ \text{O}) \).

Note any region \( a \in U \) is always a hole of its complement \( a' \). It is natural to exclude these situations from the definition of “hole”. Mormann also introduces
the following restricted version of hole relation \cite{27}: \( H' = ECN \cap H \), recall where 
\( ECN = \{(x, y) : EC(x, y), x \neq y'\} \). In case \( aH'b \), we call \( a \) a non-trivial hole of \( b \).

We summarize some basic properties of these two hole relations.

Lemma 3.1. \cite{27} Let \( \langle A, C \rangle \) be a model of RCC and set \( U = A \setminus \{0, 1\} \). Then

1. \( H \) and \( H' \) are nonempty relations on \( U \).
2. \( H(x, y) \) iff \( EC(x, y) \) and \( NTPP(x, x \vee y) \).
3. \( H(x, y) \) iff there is some \( z \in U \) such that \( NTPP(x, z) \) and \( y = z - x \).
4. The relation \( ECNB \) splits as \( ECNB = H' \cup H'\sim \).

The last result of above lemma shows that the contact relation algebra of any RCC model contains a JEPD set of relations which refines RCC25.

Proposition 3.2. \cite{28} For standard models of RCC one has \( H^2 = H^1 \). This implies \( H^i = H^{i+2} \) for \( i \geq 2 \).

Proposition 3.3. Let \( \langle A, C \rangle \) be a model of RCC and set \( U = A \setminus \{0, 1\} \). If 
\( U = A \setminus \{0, 1\} \) contains a solid region \( a \), that is, there is no region which is a non-trivial hole of \( a \), then the contact relation algebra of \( A \) is not integral.

Proof. Note there are \( b, c \in U \) with \( bH'c \), hence \( cH'\sim b \) and \( (c, c) \in H' \sim \circ H' \).
By the assumption that \( a \) is a solid region, we know that \( (a, a) \notin H' \sim \circ H \). Set
\( G_1 = (H' \sim \circ H') \cap 1' \) and \( G_2 = 1' - G_1 \). Then \( G_1 \) and \( G_2 \) forms a partition of identity relation \( 1' \). This shows that the contact RA of \( A \) is not integral. \( \square \)

Proposition 3.4. Let \( \langle A, C \rangle \) be a model of RCC and set \( U = A \setminus \{0, 1\} \). Then
\( H^n, \bigcap_{i=1}^n H^{2i-1} \) and \( \bigcap_{i=1}^n H^{2i} \) are all nonempty for \( n \geq 1 \).

Proof. Note for any region \( a \in U \), by the definition of RCC model, there exists some region \( b \in U \) with \( aDCb \), hence \( aNTPP - b \). We have a sequence of regions \( a_1, a_2, \ldots, a_k, \ldots \) such \( aiNTPPa_{i+1} \) for any \( i \geq 1 \). Write \( b_1 = a_1 \), and
\( b_i = a_i - a_{i-1} \) for \( i \geq 2 \). By \( aiNTPPa_{i+1} \), we have \( ECN(a_i, b_{i+1}) \) for \( i \geq 1 \).
Moreover, since \( a_i = b_i + a_{i-1} \) and \( aiNTPPa_i \), we have \( ECN(b_i, b_{i+1}) \) for \( i \geq 1 \).

Define \( c_i \) inductively as follows: \( c_1 = a_1 \), \( c_i = a_i - c_{i-1} \) for \( i \geq 2 \). Note for \( k \geq 1 \) we have \( c_{2k+1} = \sum_{i=0}^k b_{2i+1} \) and \( c_{2k} = \sum_{i=1}^k b_{2k} \). Then we have \( c_iH'c_{i+1} \) for \( i \geq 1 \).
This is because that \( c_i \leq aiNTPPa_{i+1} = c_i + c_{i+1} \) and, by \( ECN(b_i, b_{i+1}) \), we have \( ECN(c_i, c_{i+1}) \). This suggests that \( H^n \) is nonempty for any \( n \geq 1 \), one
instance is \((c_1, c_{n+1})\). At the same time, note \(c_1 \text{ECN} c_2\) and \(c_2 = b_2 \leq c_2\) for \(i \geq 1\), we also have \(c_1 \text{H'} c_2\) for \(i \geq 1\). This suggests \((c_1, c_{2n}) \in \bigcap_{i=1}^n \text{H'}^{2i-1}\) and \((c_1, c_{2n+1}) \in \bigcap_{i=1}^n \text{H'}^{2i}\) for \(n \geq 1\). \(\square\)

In what follows we shall show in the CRA of standard model \(\text{RC}(\mathbb{R}^n)\),

\[
\text{H'}, \text{H'} \cap \text{H'}^3, \text{H'} \cap \text{H'}^3 \cap \text{H'}^5, \ldots
\]

and

\[
\text{H'}^2, \text{H'}^2 \cap \text{H'}^4, \text{H'}^2 \cap \text{H'}^4 \cap \text{H'}^6, \ldots
\]

are two strictly decreasing sequences of relations. To this aim, we need the following lemma.

**Lemma 3.2.** Suppose \(X\) is a regular connected space and \(a, b \neq X\) are two nonempty regular closed sets. If \(a^o \cap b^o = \emptyset\), \(a \cap b = \emptyset\) and \(a \subset (a \cup b)^o\), then \(\partial a \subset \partial b\), \(\partial(a \cup b) = \partial b - \partial a\). Moreover, \(\partial a = \partial b\) if and only if \(a = b\).

**Proof.** To begin with, note \((a \cup b)^o - b\) is an open subset contained in \(a\), it is also contained in \(a^\circ\). For any \(p \in \partial a\), by \(a \subset (a \cup b)^o\), we have \(p \in b\) for otherwise \(p \in (a \cup b)^o - b \subseteq a^o\). Clearly \(p\) cannot be an interior point of \(b\) since any neighborhood of \(p\) containing some points in \(a^o\). As a result we have \(p \in \partial b\), hence \(\partial a \subset \partial b\).

Next we show \(\partial(a \cup b) = \partial b - \partial a\). For any \(p \in \partial(a \cup b)\), we have \(p \in X - a\) since \(a \subset (a \cup b)^o\). Now for any neighborhood \(U\) of \(p\), since \(U - a\) is also a neighborhood of \(p\), we have \((U - a) \cap (a \cup b)^o \neq \emptyset\), hence \(U \cap b^o \neq \emptyset\). Therefore \(p\) is a boundary point of \(b\). On the other hand, if \(p \in \partial b - \partial a\), then we have \(p \notin a\). But by \(p \notin b^o = (a \cup b)^o - a\), we have \(p \notin (a \cup b)^o\), hence \(p \in \partial(a \cup b)\).

In case \(\partial a = \partial b\), we have \(\partial(a \cup b) = \partial b - \partial a = \emptyset\). This holds if and only if \(a \cup b = X\) since \(X\) is connected. \(\square\)

By above lemma, note in a standard model of RCC, \(a \text{EC} b\) if and only if \(a^o \cap b^o = \emptyset\), \(a \cap b \neq \emptyset\), \(a \text{NTPP}(a + b)\) if and only if \(a \subset (a \cup b)^o\), we have the following

**Corollary 3.1.** For a standard model \(\text{RC}(X)\) of RCC, if \(\text{H}(a, b)\), then \(\partial a \subset \partial b\), \(\partial(a \cup b) = \partial b - \partial a\). Moreover, \(a \text{H'} b\) only if \(\partial a \subset \partial b\).

Note if \(a \text{H'} b\), then \(\partial b\) can be separated into two nonempty closed subsets, viz. \(\partial a\) and \(\partial b - \partial a\). This suggests \(\partial b\) contains more connected components than...
Theorem 3.2. For the CRA of the standard RCC model $\text{RC}(\mathbb{R}^n)$, we have $\bigcap_{i=1}^{k} H^{2i-1} \neq \bigcap_{i=1}^{k+1} H^{2i-1}$ and $\bigcap_{i=1}^{k} H^{2i} \neq \bigcap_{i=1}^{k+1} H^{2i}$ for $k \geq 1$.

Proof. For $k \geq 1$, we show there exist two regions $a, b$ such that $(a, b)$ is in $\bigcap_{i=1}^{k} H^{2i-1}$ but not in $\bigcap_{i=1}^{k+1} H^{2i-1}$. To this end, we construct two regions $a, b$ such that $(a, b) \in \bigcap_{i=1}^{k} H^{2i-1}$ and $\partial b$ contains $2k - 1$ more connected components than $\partial a$ does. By Corollary 3.4, if $(a, b) \in \bigcap_{i=1}^{k+1} H^{2i-1}$, then $\partial b$ should contain at least $2k + 1$ more connected components than $\partial a$ does. We shall obtain a contradiction.

We now construct two such regions. For $n = 1$, we set $b_0 = (-\infty, 0]$, $b_i = [i - 1, i]$ for $i \geq 1$. For $k \geq 1$, write $a_{2k-1} = \Sigma_{i=1}^{k} b_{2i-1}$ and $a_{2k} = \Sigma_{i=0}^{k} b_{2i}$. Similar to the argument given in Proposition 3.4, we have $a_i H' a_{i+1}$ and $a_1 H' a_2$ for $i \geq 1$, hence $(a_1, a_{2k}) \in \bigcap_{i=1}^{k} H^{2i-1}$. Now, since $a_1$ contains 2 end points and $a_{2k} = b_0 + b_2 + \cdots + b_{2k}$ contains only $2k + 1$ end points, $(a_1, a_{2k}) \not\in \bigcap_{i=1}^{k+1} H^{2i-1}$.

For $n \geq 2$, set $a_i$ as the $n$-ball $B(o, i)$ which has radius $i$ and is centered at $o$ for $i \geq 1$. Define $c_1 = a_1$ and $c_i = a_i - c_{i-1}$ for $i \geq 2$. Then we have $c_i H' c_{i+1}$ and $c_1 H' c_{2i}$, hence $(c_1, c_{2k}) \in \bigcap_{i=1}^{k} H^{2i-1}$. But since each $\partial c_i$ contains only $i$ connected components, $(c_1, c_{2k}) \not\in \bigcap_{i=1}^{k+1} H^{2i-1}$. \qed

4 Dual relation sets and RCC composition tables

In this section we shall propose a specialized approach for reducing the computational work of establishing an RCC $\text{CT}$. This approach can also be applied in determining the consistency and extensionality of an RCC $\text{CT}$.

4.1 Dual relation set and dual generating set

Definition 4.1. Let $\langle L, \cdot \rangle$ be an orthocomplemented lattice with $|L| > 4$ and let $U = L \setminus \{0, 1\}$. For two relation $R$, $S$ on $U$, if $(\forall x, y \in U) x S y \leftrightarrow x R y'$, then $S$ is called the right dual of $R$ and is denoted by $R^d$. If $(\forall x, y \in U) x S y \leftrightarrow x \rightarrow R y$, then we call $S$ the left dual of $R$ and denote it by $dR$.

The right dual and the left dual are just two unitary operations on $\text{Rel}(U)$. For any $X \subseteq \text{Rel}(U)$, we call the relation set $X$ a dual relation set on $U$ if $X$
is closed under the right dual and the left dual. Clearly Rel(U) itself is a dual relation set on U and intersection of dual relation sets on U is also dual on U.

We define the dualization of a relation set X, denoted by d(X), to be the least dual relation set containing X as a subset. For a dual relation set R, we can find a minimal subset S of R such that R = S ∪ S^d = dS ∪ S. We call S a dual generating set of R.

The following lemma summarize some basic properties of these two dual operations and can be easily checked.

**Lemma 4.1.** Let ⟨L, ’⟩ be an orthocomplemented lattice with |L| > 4 and let U = L\{0, 1\}. Suppose R, S are two relations on U. Then the following conditions hold:

1. \( R^d = R \circ \text{ECD}, \quad dR = \text{ECD} \circ R; \)
2. \( R^{dd} = R, \quad ddR = R, \quad (dR^d)^d = (dR)^d; \)
3. \( R^{-d'} = dR, \quad (d(R^-))^\sim = R^d; \)
4. \( R^d \cap S \neq \emptyset \iff R \cap S^d \neq \emptyset; \)
5. \( dR \cap S \neq \emptyset \iff R \cap dS \neq \emptyset; \)
6. For all \( x, y \in U, \ (x, y) \in (dR^d) \iff (x', y') \in R. \)

**Theorem 4.1.** Let ⟨L, ’⟩ be an orthocomplemented lattice with |L| > 4 and let U = L\{0, 1\}. Suppose C is a compatible contact relation of U other than the identity and R is a JEPD set of relations in the CRA of U. Then for any M, N ∈ R, we always have the following equations, where \( \circ_w \) denotes the weak composition, namely \( M \circ_w N = \bigcup\{R \in R : R \cap M \circ N \neq \emptyset\} \):

1. \( (M \circ N)^\sim = N^\sim \circ M^\sim; \)
2. \( (M \circ N)^d = M \circ N^d, \quad (d(M \circ N)) = dM \circ N, \quad (dM \circ N)^d = (dM \circ N)^d; \)
3. \( (M \circ_w N)^\sim = N^\sim \circ_w M^\sim; \)
4. Suppose \( R \) is a dual relation set on U, then \( (M \circ_w N)^d = M \circ_w N^d, \quad d(M \circ_w N)^d = d(M \circ_w N)^d. \)

**Proof.** The proofs of (1), (2) and (3) are direct. For (4), since \( R \) is a dual relation set on A, we have \( M \circ \text{ECD} = M \circ_w \text{ECD} \) and \( \text{ECD} \circ M = \text{ECD} \circ_w M \) for each \( M \in R \). Now applying (2) and Lemma 4.1 (4), we have \( (M \circ_w N)^d = \bigcup\{R^d : R \cap M \circ N \neq \emptyset\} = \bigcup\{R \cap (M \circ N)^d \neq \emptyset\} = \bigcup\{R \cap (M \circ N)^d \neq \emptyset\} = M \circ_w N^d. \) Similarly we have \( d(M \circ_w N) = dM \circ_w N. \) The last equation now follows from these two equations.
Theorem 4.1 shows that the work can be simplified. Therefore we only need to check Case (1). This can be further simplified. Suppose we reduce (2) to (1). Similarly, Case (3) and Case (4) can be reduced to (1).

4.2 An approach for reducing the calculations of weak composition table

The above theorem suggests that, for a dual relation set $\mathcal{R}$, the work of constructing the weak composition table can be simplified drastically.

Suppose $\mathcal{R}$ is a dual relation set which is closed under inverse and contains 1’. Let $\mathcal{S}$ be a dual generating set of $\mathcal{R}$ which is also closed under inverse. Denote $\mathcal{M} = \{ R \in \mathcal{S} : R = R^\sim \text{ and } R \neq 1’ \}$ and $\mathcal{N} = \{ R \in \mathcal{S} : R \neq R^\sim \}$. Write $r, s, m, n$ to be the number of relations in $\mathcal{R}$, $\mathcal{S}$, $\mathcal{M}$, $\mathcal{N}$ respectively. Then $s = m + n + 1$ and $n = 2k$ for some $k \in \mathbb{N}$.

To construct the weak CT, one should compute $M \circ_\omega N$ for each $M, N \in \mathcal{R}$. Theorem 4.1 shows that the work can be simplified.

There are four cases, namely, (1) $M, N \in \mathcal{S}$; (2) $M \in \mathcal{S}$ and $N \not\in \mathcal{S}$; (3) $M \not\in \mathcal{S}$ and $N \in \mathcal{S}$; (4) $M, N \not\in \mathcal{S}$.

For Case (2), since $\mathcal{S}$ is a dual generating set of $\mathcal{R}$, we can choose $R \in \mathcal{S}$ such that $R^d = N$. Then $M \circ_\omega N = M \circ_\omega R^d = (M \circ_\omega R)^d$ by (4) of Theorem 4.1. We reduce (2) to (1). Similarly, Case (3) and Case (4) can be reduced to (1). Therefore we only need to check Case (1). This can be further simplified. Suppose $\mathcal{M} = \{ M_1, M_2, \cdots, M_m \}$ and $\mathcal{N} = \{ N_1, N_1^\sim, N_2, N_2^\sim, \cdots, N_k, N_k^\sim \}$.

- For $M, N \in \mathcal{M}$, note $M_i \circ_\omega M_j = (M_j \circ_\omega M_i)^\sim$. The work needed in this case is $(m \times (m + 1))/2$;

- For $M \in \mathcal{M}$, $N \in \mathcal{N}$ or $M \in \mathcal{N}$, $N \in \mathcal{M}$, note $M_i \circ_\omega N_j^\sim = (N_j \circ_\omega M_i)^\sim$ and $N_j^\sim \circ_\omega M_i = (M_i \circ N_j)^\sim$. The work needed in this case is $2m \times k$;

- For $M, N \in \mathcal{N}$, note the following equations hold:
  
  $N_i \circ_\omega N_j = (N_j^\sim \circ_\omega N_i^\sim)^\sim$, $N_i \circ_\omega N_j^\sim = (N_j \circ_\omega N_i^\sim)^\sim$, $N_i^\sim \circ_\omega N_j = (N_j^\sim \circ_\omega N_i)^\sim$. 

17
### Table 1: Dual operations on RCC7.

| R  | PP | PP<sup>~</sup> | PON | POD | DN | ECD | i\(^{t}\) |
|----|----|---------------|-----|-----|----|-----|-------|
| R\(^{d}\) | DN | POD | PON | PP<sup>~</sup> | PP | 1<sup>t</sup> | ECD |
| dR | POD | DN | PON | PP | PP<sup>~</sup> | 1<sup>t</sup> | ECD |
| 4R<sup>d</sup> | PP<sup>~</sup> | PP | PON | DN | POD | ECD | 1<sup>t</sup> |

The work needed in this case is 2\(k^2 + k\).

Therefore the total work needed to construct the weak CT is \(T = (m + n)(m + n + 1)/2 = s(s - 1)/2\).

### 4.3 Dual relations of RCC systems

In this subsection we assume \(\langle L, \langle \rangle \rangle\) is an orthocomplemented lattice with \(|L| > 4\) and let \(U = L \setminus \{0, 1\}\). We also suppose \(C\) is a compatible contact relation on \(U\) other than the identity.

**Example 4.1.** RCC5, RCC8 and RCC10 are not dual on \(L\). Note that PP\(^{d}\) is not in RCC5, TPP\(^{d}\) is not in RCC8, and POD\(^{d}\) is not in RCC10. But by Tables 1 and 2, RCC7 and RCC11 are clearly dual relation sets.

Moreover, for RCC7 and RCC11, we have

\[S_7 = \{1', PP, PP^\sim, PON\}\]

is a dual generating set of \(R_7\); and

\[S_{11} = \{1', TPP, TPP^\sim, NTPP, NTPP^\sim, PON\}\]

is a dual generating set of \(R_{11}\).

By Tables 1 and 2, \(S_7\) and \(S_{11}\) are closed under inverse and \(dR^d = R^\sim\) for \(R \in S_7\) or \(R \in S_{11}\). Moreover, for \(M, N \in S_7\) or \(S_{11}\), by \(dM \circ N^d = (dM^d) \circ (dN^d) = M^\sim \circ N^\sim\), we have the following

**Proposition 4.2.** For \(M, N \in S_7\) or \(S_{11}\), we have \(dM \circ N^d = M^\sim \circ N^\sim\).
By this proposition and Theorem 4.1, we have the following equations:

1. $PODY \circ PODY = TPP \circ TPP$;
2. $PODY \circ PODZ = TPP \circ NTPP$;
3. $PODY \circ ECN = TPP \circ TPP$;
4. $PODY \circ DC = TPP \circ NTPP$;
5. $PODZ \circ PODY = NTPP \circ TPP$;
6. $PODZ \circ PODZ = NTPP \circ NTPP$;
7. $PODZ \circ ECN = NTPP \circ TPP$;
8. $PODZ \circ DC = NTPP \circ NTPP$;
9. $ECN \circ PODY = TPP \circ TPP$;
10. $ECN \circ PODZ = TPP \circ NTPP$;
11. $ECN \circ ECN = TPP \circ TPP$;
12. $ECN \circ DC = TPP \circ NTPP$;
13. $DC \circ PODY = NTPP \circ TPP$;
14. $DC \circ PODZ = NTPP \circ NTPP$;
15. $DC \circ ECN = NTPP \circ TPP$;
16. $DC \circ DC = NTPP \circ NTPP$.

Note by Proposition 4.1, the relational composition $\circ$ in above equations can be replaced by weak composition $\circ_w$.

We now apply the approach described in Section 4.2 to RCC7 and RCC11. Set $t = T/n^2$ to be the ratio of the work needed in our approach to that using the cell-by-cell checking.

RCC7 $r = 7$, $s = 4$, $m = 1$, $n = 2$, $T = 6$ and $t = 6/49 < 1/8$;

RCC11 $r = 11$, $s = 6$, $m = 1$, $n = 4$, $T = 15$ and $t = 15/121 < 1/8$;

Remark 4.1. For RCC25, we assume $\langle A, C \rangle$ is a model of the RCC axioms and set $U = A \setminus \{0, 1\}$. We can show that RCC25 is closed under left and right dual operations. Moreover, set

$S_{25} = \{1', TPP, TPP^\sim, TPPB, TPPB^\sim, NTPP, NTPP^\sim, PONXA1, PONYA1, PONYA1^\sim, PONYA2, PONYA2^\sim, PONZ, PONYB, PONYB^\sim\}.$

Then we can show $S_{25}$ is a dual generating set of the RCC25 relations which is closed under inverse. Applying the approach described in Section 4.2 to RCC25, we need to check 105 cells from the total $25 \times 25$ cells.
5 Consistency and extensionality of RCC11 CT

In this section, we consider the consistency and extensionality of the RCC11 CT. In what follows, we write by \( \tau_{11} : \mathcal{R}_{11} \times \mathcal{R}_{11} \to 2^{\mathcal{R}_{11}} \) the (abstract) RCC11 CT given in [10].

5.1 Each RCC model is a consistent model of RCC11 CT

Although the RCC11 CT has been established in [10] and Düntsch calls this a weak composition table, it is not clear or at least haven’t been proven whether or not this table is precisely the weak composition table for each RCC model. Namely, our question is: Is each RCC model a consistent model of the RCC11 CT?

In this subsection we shall show this by constructing the weak RCC11 composition table for each RCC model. To this aim, suppose \( A \) is an RCC model and let \( U = A \setminus \{0, 1\} \). Applying the approach described in Section 4.2, we can simplify the computation. Recall \( S_{11} = \{1', \text{TPP}, \text{TPP}^\sim, \text{NTPP}, \text{NTPP}^\sim, \text{PON}\} \). Let \( \mathcal{M}_{11} = \{\text{PON}\}, \mathcal{N}_{11} = \{\text{TPP}, \text{TPP}^\sim, \text{NTPP}, \text{NTPP}^\sim\} \). We need only to calculate the 15 weak compositions appeared in Table 3.

Proposition 5.1. Suppose \( A \) is an RCC model and \( R, S, T \) are three RCC11 relations on \( U = A \setminus \{0, 1\} \). Then \( T \) is in \( \tau_{11}(R, S) \) if and only if \( T \cap R \circ S \neq \emptyset \).

Proof. Note \( \tau_{11} \) satisfies the following conditions: \( \tau_{11}(R, S^d) = (\tau_{11}(R, S))^d \), \( \tau_{11}(dR, S) = d(\tau_{11}(R, S)) \), \( \tau_{11}(dR, S^d) = d(\tau_{11}(R, S))^d \), \( \tau_{11}(S^\sim, R^\sim) = (\tau_{11}(R, S))^{\sim} \).

Applying the approach specified in Section 4.2, we need only to consider cells with a symbol ‘?’ in Table 3. Thus there are only 15 cases to be checked. We take \( \langle \text{TPP}^\sim, \text{TPP} \rangle \) as an example. The rest are similar.

Note if \( a \text{TPP} > b \text{TPP} \), then \( a \land c \geq b > 0 \), hence \( a \) and \( c \) cannot be related by either ECD or ECN or DC. Moreover, if \( a \) is a non-tangential proper part...
of $c$, then so is $b$ since $b < a$. This contradicts $b_{\text{TPP}}c$, hence $a_{\text{NTPP}}c$ cannot hold. For the same reason, $a_{\text{NTPP}}^{-}c$ is also impossible. This shows if $T \cap \text{TPP}^{-} \circ \text{TPP} \neq \emptyset$, then $T$ is in $\tau_{11}(\text{TPP}^{-}, \text{TPP})$. On the other hand, suppose $T$ is in $\tau_{11}(\text{TPP}^{-}, \text{TPP})$. Take $x_i (i = 1, 2, 3)$ with $\text{DC}(x_i, x_j)$ for $i \neq j$, take $y_{\text{NTPP}}x_1$. For $T = 1'$, set $a = c = x_1 + x_2, b = x_1$; For $T = \text{TPP}$, set $a = x_1 + x_2 + x_3, b = x_1, c = x_1 + x_3$; For $T = \text{TPP}^{-}$, set $a = x_1 + x_3, b = x_1, c = x_1 + x_3 + x_2$; For $T = \text{PON}$, set $a = x_1 + x_2, b = x_1, c = x_1 + x_3$; For $T = \text{PODY}$, set $a = 1 - x_2 - x_3 - y = (x_1 - y) + (1 - x_1 - x_2 - x_3), b = x_1 - y, c = x_1 + x_2 + x_3$, then $a_{\text{TPP}}^{-}b_{\text{TPP}}c$ and $a_{\text{TPP}}^{-}1 - x_1 - x_2 - x_3$. Note $c' = 1 - x_1 - x_2 - x_3$, we have $a_{\text{PODY}}c$; For $T = \text{PODZ}$, take $a = x_1, b = x_1 - y, c = 1 - y$, then $a_{\text{TPP}}^{-}b_{\text{TPP}}c$ and $a_{\text{NTPP}}^{-}c'$ hold, hence $a_{\text{PODZ}}c$. \hfill $\square$

Recall $R \circ w S$ is defined to be the union of all $T$ with $T \cap R \circ S \neq \emptyset$. By this proposition we have the following

**Theorem 5.1.** Each RCC model is a consistent model of the RCC11 CT $\tau_{11}: R_{11} \times R_{11} \rightarrow 2^{R_{11}}$.

### 5.2 When is a composition triad extensional?

For an RCC model $A$, or more general, a contact structure $\langle L, C \rangle$ on an orthocomplemented lattice, we say a composition triad $\langle R, T, S \rangle$ in $\tau_{11}$ is **extensional** if $T \subseteq R \circ S$. In [10], D"untsch has shown that in general the RCC11 CT is not extensional. As a matter of fact, he has determined for each cell $\langle R, S \rangle$ whether or not $R \circ w S = R \circ S$ is true for all RCC models. Our intention now is to give an exhaustive investigation of the extensionality of the RCC11 table. We want to indicate, for each triad $\langle R, S, T \rangle$ with $T$ an entry in the cell specified by the pair $\langle R, S \rangle$, whether or not the following condition

$$T(x, y) \rightarrow \exists z(R(x, z) \land S(z, y))$$

hold for all RCC models.\footnote{A similar and more detailed interpretation for RCC8 CT has been given in [23].}

To make the calculations simple, we consider only strong RCC models, namely those models which satisfy the INT property. This cannot be too restrictive since stand RCC models of the Euclidean spaces are strong.

We summarize the results in Table 6 where a cell entry $T$ (in the cell specified by $\langle R, S \rangle$) is attached a superscript $\times$ if and only if $T \not\subseteq R \circ S$. In this way, we
Table 4: Reduced ‘extensional’ RCC11 CT, where T= TPP, N=NTPP, Ti=TPP~, Ni=NTPP~, PN=PON, PDY=PODY, PDZ=PODZ.

indicate for which triad the composition is extensional, and when it need not to be.

The following proposition suggests the approach specified in Section 4.2 can be used to reduce the calculations.

Proposition 5.2. Suppose A is an RCC model and R,S,T are three RCC11 relations on U = A \ {0, 1}. Then the following conditions are equivalent:
(1) \( T \subseteq R \circ_w S \);
(2) \( T^d \subseteq R \circ_w S^d \);
(3) \( dT \subseteq dR \circ_w S \);
(4) \( dT^d \subseteq dR \circ_w S^d \);
(5) \( T^\sim \subseteq S^\sim \circ_w T^\sim \).

Proof. The proofs are straightforward and leave to the reader.

So we need only to calculate the 15 weak compositions appeared in Table 3. The results are given in Table 4.

The verifications are similar to that given in [25] for RCC8 weak CT. Moreover, constructions given in [25] Table 4, 5] can also be applied for the RCC11 weak compositions. As a matter of fact, for any cell entry R in Table 4 which is other than PODY, PODZ, ECD, we have: (1) if a × is attached to R, the construction given in Table 4 of [25] for corresponding RCC8 cell entry is still valid; (2) if this is not the case, entertaining the counter-example constructed in Table 5 of [25] will be enough. In particular, for strong RCC models, we have by Table 3 of [25]:

| \( \circ_w \) | T | Ti | N | Ni | PN |
|---|---|---|---|---|---|
| T | T,N | T′, T, Ti, DC \( \times \), ECN \( \times \) | N | Ti\( \times \), Ni, PN\( \times \) \( \times \), DC | T, N, PN, ECN, DC |
| Ti | T′, T, Ti, PN\( \times \) \( \times \), PDY \( \times \), PDZ | T\( \times \), N, PN\( \times \) \( \times \), PDY \( \times \), PDZ | Ti, Ni, PN, PDY, PDZ |
| N | N | N | N | T′, T, Ti, Ni \( \times \), PN, ECN, DC | T, N, PN, ECN, DC |
| Ni | | T′, T, Ti, N, Ni \( \times \), PN, PDY, PDZ | Ti, Ni, PN, PDY, PDZ |
| PN | | | | T′, T, Ti, N, Ni, PN, DC \( \times \), PDY, PDZ, ECN, ECD |
two classes of regions: the closed disks and their complements in $\mathcal{R}_C(\mathbb{R}^2)$. Denote by $D_1$ the class of closed disks, by $D_2$ the class of their complements and call for convenience regions in $D_2$ complement disks. Define a binary relation $C$ on $D$ as follows: for two regions $a, b \in D$, $aCb$ if $a \cap b \neq \emptyset$. Clearly this relation is a contact relation on $U$. In contrast with the closed disk algebra for RCC8 table given in [13, 10], we call the contact relation algebra on this domain the complemented closed disk algebra, written $\mathcal{L}$. In what follows we shall show this

| $(\mathcal{T}PP^\sim, \mathcal{PODZ}, \mathcal{TPP})$ | Set $b = a \land c$ |
|----------------|------------------|
| $(\mathcal{T}PP^\sim, \mathcal{PODZ}, \mathcal{NTPP})$ | Take $m$ with $c' \mathcal{NTPP} m \mathcal{NTPP} a$, set $b = a - m$ |
| $(\mathcal{T}PP^\sim, \mathcal{PODY}, \mathcal{PON})$ | Take $m = c'$, $n \mathcal{NTPP} (a \land c)$, set $b = m + n$ |
| $(\mathcal{T}PP^\sim, \mathcal{PODY}, \mathcal{PON})$ | Take $m \mathcal{NTPP} c'$, $n = a \land c$, set $b = m + n$ |
| $(\mathcal{NTPP}^\sim, \mathcal{PODY}, \mathcal{PON})$ | Take $m \mathcal{NTPP} c'$, $n \mathcal{NTPP} (a \land c)$, set $b = m + n$ |
| $(\mathcal{NTPP}^\sim, \mathcal{PODZ}, \mathcal{PON})$ | Take $m \mathcal{NTPP} c'$, $n \mathcal{NTPP} (a \land c)$, set $b = m + n$ |
| $(\mathcal{PON}, \mathcal{PODY}, \mathcal{PON})$ | Take $m \mathcal{NTPP} c'$, $n \mathcal{NTPP} a'$, set $b = m + n$ |
| $(\mathcal{PON}, \mathcal{PODZ}, \mathcal{PON})$ | Take $m \mathcal{NTPP} c'$, $n \mathcal{NTPP} a'$, set $b = m + n$ |
| $(\mathcal{PON}, \mathcal{ECD}, \mathcal{PON})$ | Take $m \mathcal{NTPP} c'$, $n \mathcal{NTPP} a'$, set $b = m + n$ |

Table 5: Positive RCC11 weak compositions and instances of the region $b$

\[
\mathcal{TPP} \circ \mathcal{NTPP} = \mathcal{NTPP} \circ \mathcal{TPP} = \mathcal{NTPP} \circ \mathcal{NTPP} = \mathcal{NTPP};
\]

\[
\mathcal{TPP} \circ \mathcal{TPP} = \mathcal{TPP} \cup \mathcal{NTPP};
\]

\[
\mathcal{NTPP} \circ \mathcal{NTPP}^\sim = \mathcal{I}' \cup \mathcal{TPP} \cup \mathcal{TPP}^\sim \cup \mathcal{NTPP} \cup \mathcal{PON} \cup \mathcal{ECN} \cup \mathcal{DC};
\]

\[
\mathcal{NTPP}^\sim \circ \mathcal{NTPP} = \mathcal{I}' \cup \mathcal{TPP} \cup \mathcal{TPP}^\sim \cup \mathcal{PON} \cup \mathcal{PODY} \cup \mathcal{PODZ}.
\]

There are still 11 cell entries to be settled. For the two negative triads, $(\mathcal{TPP}^\sim, \mathcal{PODY}^\times, \mathcal{TPP})$ and $(\mathcal{TPP}^\sim, \mathcal{PODY}^\times, \mathcal{NTPP})$, take $p, q \in U$ with $p \mathcal{NTPP} q$, set $a = q$, $c' = q - p$, then $a \land c = p$. Note by $a \mathcal{TPP}^\sim c'$ we have $a \mathcal{PODY} c$, but there cannot exist a region $b$ with $a \mathcal{TPP}^\sim b$ and $b \leq c$ since $a \land c = p$ is already a non-tangential proper part of $a$. For the rest positive composition triads, we can choose a region $b$ with the desired property. These constructions are summarized in Table 5.

## 6 Complemented closed disk algebra

This section shall provide a representation for the relation algebra determined by the RCC11 CT. Recall $\mathcal{R}_C(\mathbb{R}^2)$, the standard RCC model associated to the Euclidean plane, contains all regular closed subsets of $\mathbb{R}^2$, and two (nonempty) regions are said to be connected provided that they have nonempty intersection.

Our domain of regions, denoted by $D$, is a sub-domain of $\mathcal{R}_C(\mathbb{R}^2)$ and contains two classes of regions: the closed disks and their complements in $\mathcal{R}_C(\mathbb{R}^2)$. We denote by $D_1$ the class of closed disks, by $D_2$ the class of their complements and call for convenience regions in $D_2$ complement disks. Define a binary relation $C$ on $D$ as follows: for two regions $a, b \in D$, $aCb$ if $a \cap b \neq \emptyset$. Clearly this relation is a contact relation on $U$. In contrast with the closed disk algebra for RCC8 table given in [13, 10], we call the contact relation algebra on this domain the complemented closed disk algebra, written $\mathcal{L}$. In what follows we shall show this
Table 6: Extensionality of RCC11 CT, where T=TPP, N=NTPP, Ti=TPP⁻, Ni=NTPP⁻, PN=PON, PY=PODY, PZ=PODZ.
CRA is finite and contains RCC11 as its atoms, and it is indeed a representation of the relation algebra determined by the RCC11 CT. Interestingly, the RCC11 relations on this domain can be equivalently determined by the 9-intersection principle of Egenhofer and Herring [16].

6.1 Topological characterization of RCC11 relations in \( \mathcal{L} \)

Write \( L = D \cup \{\emptyset, \mathbb{R}^2\} \). Then \( L \) with the usual inclusion ordering is an orthocomplemented lattice. Based on the contact relation \( C \) on \( D \), we can define RCC11 relations on \( D \) (see Section 2 of this paper).

The following theorem gives a topological characterization of these relations:

**Theorem 6.1.** The RCC11 relations on \( D \) has the following characterization:

1. \( x1' \ y \iff x = y \);
2. \( xTPPy \iff x \subseteq y, x \neq y \) and \( \partial x \cap \partial y \neq \emptyset \);
3. \( xTPP^y \iff x \supseteq y, x \neq y \) and \( \partial x \cap \partial y \neq \emptyset \);
4. \( xNTPy \iff x \subseteq y, x \neq y \) and \( \partial x \cap \partial y = \emptyset \);
5. \( xNTPP^y \iff x \supseteq y, x \neq y \) and \( \partial x \cap \partial y = \emptyset \);
6. \( xPONy \iff x^o \cap y^o \neq \emptyset, x \not\subseteq y, y \not\subseteq x, \) and \( x \cup y \neq \mathbb{R}^2 \);
7. \( xPODYy \iff x^o \cap y^o \neq \emptyset, \partial x \cap \partial y \neq \emptyset \) and \( x \cup y = \mathbb{R}^2 \);
8. \( xPODZy \iff x^o \cap y^o \neq \emptyset, \partial x \cap \partial y = \emptyset \) and \( x \cup y = \mathbb{R}^2 \);
9. \( xECNy \iff x^o \cap y^o = \emptyset, x \cap y \neq \emptyset \) and \( x \cup y \neq \mathbb{R}^2 \);
10. \( xECDy \iff x^o \cap y^o = \emptyset, x \cap y \neq \emptyset \) and \( x \cup y = \mathbb{R}^2 \);
11. \( xDCy \iff x \cap y = \emptyset \).

**Proof.** The proofs are routine and leave to the reader. \( \square \)

From this theorem we know that these relations on \( D \) are precisely the restrictions of corresponding RCC11 relations in \( RC(\mathbb{R}^2) \) to \( D \). The corresponding configurations are illustrated in Figure 1 where we figure closed disks as shaded circles and their complements as hollowed circles. Because \( TPP^\sim \) and \( NTPP^\sim \) are inverse relations of \( TPP \) and \( NTPP \) respectively, we give 9 figures of the 11 relations.

6.2 9-Intersection relations on \( D \)

Interestingly, these 11 relations on \( D \) can be classified by the 9-intersection principle posed by Egenhofer and Herring [16]. According to the 9-intersection principle, the binary topological relation \( R \) between two regions, \( x \) and \( y \), is based
Figure 1: Illustration of RCC11 relations in the complemented closed disk algebra.
upon the intersections of $x$’s interior ($x^\circ$), boundary ($\partial x$), and exterior ($x^-$) with $y$’s interior ($y^\circ$), boundary ($\partial y$), and exterior ($y^-$), which is concisely represented as a $3 \times 3$-matrix
\[
\begin{pmatrix}
  x^\circ \cap y^\circ & x^\circ \cap \partial y & x^\circ \cap y^- \\
  \partial x \cap y^\circ & \partial x \cap \partial y & \partial x \cap y^- \\
  x^- \cap y^\circ & x^- \cap \partial y & x^- \cap y^-
\end{pmatrix}.
\]

For the 9-intersection mode, the content of the nine intersections was identified as a simple and most general topological invariant, it characterizes each of the nine intersections by a value empty (0) or nonempty (1). The sequence of the nine intersections, from left to right and from top to bottom, will always be in ordering \textit{(interior, boundary, exterior)}.

The nine empty/nonempty intersections describe a set of relations that provides a complete coverage–any set is either empty or not empty and tertium non datur. Furthermore, these relations are Jointly Exhaustive and Pairwise Disjoint (JEPD).

By applying the 9-intersection principle to our domain of regions $D$, we find there are 11 JEPD relations on $D$. Moreover, these 11 relations are just the same as the RCC11 relations on $D$. This fact follows from the topological characterization of the relations. We describe these RCC11 relations by $3 \times 3$ matrixes as follows:

\[
1' = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}, \quad
\text{TPP} = \begin{pmatrix}
  1 & 0 & 0 \\
  1 & 1 & 0 \\
  1 & 1 & 1
\end{pmatrix}, \quad
\text{TPP}^\sim = \begin{pmatrix}
  1 & 1 & 1 \\
  0 & 1 & 1 \\
  0 & 0 & 1
\end{pmatrix},
\]
\[
\text{NTPP} = \begin{pmatrix}
  1 & 0 & 0 \\
  1 & 0 & 0 \\
  1 & 1 & 1
\end{pmatrix}, \quad
\text{NTPP}^\sim = \begin{pmatrix}
  1 & 1 & 1 \\
  0 & 0 & 1 \\
  0 & 0 & 1
\end{pmatrix}, \quad
\text{PON} = \begin{pmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  0 & 0 & 1
\end{pmatrix},
\]
\[
\text{PODY} = \begin{pmatrix}
  1 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}, \quad
\text{PODZ} = \begin{pmatrix}
  1 & 0 & 0 \\
  1 & 0 & 0 \\
  1 & 1 & 1
\end{pmatrix}, \quad
\text{ECN} = \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 1 \\
  1 & 1 & 1
\end{pmatrix},
\]
\[
\text{ECD} = \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0
\end{pmatrix}, \quad
\text{DC} = \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 1 & 1
\end{pmatrix}.
\]

### 6.3 The composition of the complemented closed disk algebra

Now we shall show that the composition operation of $\mathcal{L}$ is precisely that one specified by the RCC11 CT. What we should do is to indicate, for each triad $\langle R, T, S \rangle$ with $T$ an entry in the cell specified by the pair $\langle R, S \rangle$, whether or not
the following condition hold

\[ T(x, y) \rightarrow (\exists z \in D)(R(x, z) \land S(z, y)). \]

Note the approach described in Section 4.2 is also valid for the present purpose. This is since (i) RCC11 relations on \( D \) is a dual relation set which contains \( 1' \) and is closed under inverse; (ii) \( S_{11} = \{1', \text{TPP, TPP}^\sim, \text{NTPP, NTPP}^\sim, \text{PON}\} \) is a dual generating set which is also closed under inverse; (iii) Proposition 5.2 is still valid for \( L \). As a result, we need only to calculate the 15 compositions appeared in Table 3.

To begin with, we first show the \text{NTPP} relation on \( D \) satisfies the interpolation property.

\textbf{Lemma 6.1.} Given any two regions \( a, c \) in \( D \) with \( a\text{NTPP}c \), there exists another region \( b \in D \) with \( a\text{NTPP}b\text{NTPP}c \).

\textit{Proof.} By the topological characterization of the \text{NTPP} relation given in Theorem 6.1, we know that \( a\text{NTPP}c \) if and only if \( a \subseteq c^\partial \). There are three cases:

Case I: \( a, c \) are closed disks. In this case, \( \partial a \) and \( \partial c \) are two non-tangential circles and \( \partial a \) is inside \( \partial c \). Then we can find another circle \( B \) between these two circles. Taking \( b \) as the closed disk bounded by \( B \), then \( b \) satisfies the desired property.

Case II: \( a, c \) are complement disks. In this case, \( \partial a \) and \( \partial c \) are two non-tangential circles and \( \partial c \) is inside \( \partial a \). Then we can find another circle \( B \) between these two circles. Taking \( b \) as the complement disk bounded by \( B \), then \( b \) satisfies the desired property.

Case III: \( a \) is a closed disk and \( c \) is a complement disk, \( \partial a \) and \( \partial c \) are two separated circles and the distance between them is non-zero. Then we can find another circle \( B \) such that \( \partial a \) is inside \( B \) and \( B \) is separated from \( \partial c \). Taking \( b \) as the closed disk bounded by \( B \), then \( b \) satisfies the desired property. \( \square \)

\textbf{Proposition 6.1.} In the complemented closed disk algebra \( L \), the following equations \( \text{NTPP} \circ \text{NTPP} = \text{NTPP}, \text{TPP} \circ \text{NTPP} = \text{NTPP} \) and \( \text{NTPP} \circ \text{TPP} = \text{NTPP} \) hold.

\textit{Proof.} Note the “\( \subseteq \)” part of these equations follow directly from the definitions and the first equation is then clear by above lemma.

For the second equation, suppose \( a\text{NTPP}c \) in \( D \), we want to find \( b \) such that \( a\text{TPP}b\text{NTPP}c \). There are three cases:
Case I: \(a, c\) are closed disks. In this case, \(\partial a\) and \(\partial c\) are two non-tangential circles and \(\partial a\) is inside \(\partial c\). Then we can find another circle \(B\) such that \(\partial a\) is internally tangent to \(B\) and \(B\) is inside the circle \(\partial c\). Taking \(b\) as the closed disk bounded by \(B\), then \(b\) satisfies the desired property.

Case II: \(a, c\) are complement disks. In this case, \(\partial a\) and \(\partial c\) are two non-tangential circles and \(\partial c\) is inside \(\partial a\). Then we can find another circle \(B\) such that \(B\) is internally tangent to \(\partial a\) and \(\partial c\) is inside \(B\). Taking \(b\) as the complement disk bounded by \(B\), then \(b\) satisfies the desired property.

Case III: \(a\) is a closed disk and \(c\) is a complement disk, \(\partial a\) and \(\partial c\) are two separated circles and the distance between them is non-zero. Then we can find another circle \(B\) such that \(\partial a\) is internally tangent to \(B\) and \(B\) is separated from \(\partial c\). Taking \(b\) as the closed disk bounded by \(B\), then \(b\) satisfies the desired property.

The proof of the last equation is similar.

The following proposition proves the remainder 12 equations in CCA.

**Proposition 6.2.** In the complemented closed disk algebra \(L\), the following composition equations hold.

\begin{align*}
(C-1) \quad &\text{TPP} \circ \text{TPP} = \text{TPP} \cup \text{NTPP}; \\
(C-2) \quad &\text{TPP} \circ \text{TPP}^\sim = 1' \cup \text{TPP} \cup \text{TPP}^\sim \cup \text{PON} \cup \text{ECN} \cup \text{DC}; \\
(C-3) \quad &\text{TPP} \circ \text{NTPP}^\sim = \text{TPP}^\sim \cup \text{NTPP}^\sim \cup \text{PON} \cup \text{ECN} \cup \text{DC}; \\
(C-4) \quad &\text{TPP} \circ \text{PON} = \text{TPP} \cup \text{NTPP} \cup \text{PON} \cup \text{ECN} \cup \text{DC}; \\
(C-5) \quad &\text{TPP}^\sim \circ \text{TPP} = 1' \cup \text{TPP} \cup \text{TPP}^\sim \cup \text{PON} \cup \text{PODY} \cup \text{PODZ}; \\
(C-6) \quad &\text{TPP}^\sim \circ \text{NTPP} = \text{TPP} \cup \text{NTPP} \cup \text{PON} \cup \text{PODY} \cup \text{PODZ}; \\
(C-7) \quad &\text{TPP}^\sim \circ \text{PON} = \text{TPP}^\sim \cup \text{NTPP}^\sim \cup \text{PON} \cup \text{PODY} \cup \text{PODZ}; \\
(C-8) \quad &\text{NTPP} \circ \text{NTPP}^\sim = 1' \cup \text{TPP} \cup \text{TPP}^\sim \cup \text{NTPP} \cup \text{NTPP}^\sim \cup \text{PON} \cup \text{ECN} \cup \text{DC}; \\
(C-9) \quad &\text{NTPP} \circ \text{PON} = \text{TPP} \cup \text{NTPP} \cup \text{PON} \cup \text{ECN} \cup \text{DC}; \\
(C-10) \quad &\text{NTPP}^\sim \circ \text{NTPP} = 1' \cup \text{TPP} \cup \text{TPP}^\sim \cup \text{NTPP} \cup \text{NTPP}^\sim \cup \text{PON} \cup \text{PODY} \cup \text{PODZ}; \\
(C-11) \quad &\text{NTPP}^\sim \circ \text{PON} = \text{TPP}^\sim \cup \text{NTPP}^\sim \cup \text{PON} \cup \text{PODY} \cup \text{PODZ}; \\
(C-12) \quad &\text{PON} \circ \text{PON} = 1' \cup \text{TPP} \cup \text{TPP}^\sim \cup \text{NTPP} \cup \text{NTPP}^\sim \cup \text{PON} \cup \text{PODY} \cup \text{PODZ} \cup \text{ECN} \cup \text{ECD} \cup \text{DC}.
\end{align*}

**Proof.** Since regions in \(D\) are either closed disks or the complement of closed disks, the above equations can be verified using elementary theory for circles (such as, internally tangent, externally tangent, containment, disjoint, etc.). For
each cell entry in the reduced ‘extensional’ RCC11 CT (Table 4) which is attached
a superscript $\times$, we give an illustration for visual reference in Figure 2. Similar
proofs can also be given to the other 61 triads.

As a result, we know that the complemented closed disk algebra has 11 atoms
and it’s composition is just as the one given in the RCC11 CT.

**Theorem 6.2.** The relation algebra determined by the RCC11 CT can be represen-
ted by the complemented closed disk algebra.

7 Summary and outlook

This paper explored several important relation-algebraic questions arising in the
RCC theory. We have shown that the contact relation algebra of $B_\omega$, a least
RCC model, is not atomic complete; and the contact relation algebra of the $n$-
dimensional Euclidean space is infinite and not integral. These results suggest
that in general we cannot obtain an extensional composition table for the RCC
theory by simply refining the RCC8 relations.

In order to obtain an extensional CT, one should restrict the domain of re-
gions: RCC models, in particular the regular closed algebra of a regular connected
space, might contain too much regions. This has been partially demonstrated by
the exhaustive investigation of extensionality of RCC8 and RCC11 CT given
respectively in [24] and Section 5 of the present paper. There are also positive
demonstrations. For the RCC8 table, the closed disk algebra given in [10] and
the Egenhofer model provides two extensional models which are arising from re-
striction of regions in the real plane [26]. For the RCC11 table, we have shown
in Section 6 of this paper the complemented closed disk algebra, whose domain
contains only the closed disks and closures of their complements in the real plane,
is an extensional model. Restricting the regions to connected regions bounded
by Jordan curves and closures of their complements seems also provide such a
model, this shall be investigated later.

Future work will investigate the contact relation algebra of various small do-
 mains of regions which admits more operations than complementation, e.g., finite
unions or finite intersections. In particular, the (complemented) Worboys-Bofakos
model [42] deserves a detailed study with the tools of relation algebra. Note that
the 9-intersection principle can be applied to these domains, we can compare the
expressivity of RA logic with that of the 9-intersection model.
Figure 2: Negative composition triads in Table 4 are extensional on $D$. 

[Image of diagrams explaining the composition triads]
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