Prescribing oscillation behavior of solutions to the heat equation on $\mathbb{R}^n$ via the initial data and its average integral

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March 12, 2021

Abstract

Motivated by a classical stabilization result for solution to the Cauchy problem of the heat equation $\partial_t u = \Delta u$ on $\mathbb{R}^n$, we consider its oscillation behavior with radial initial data $\varphi(x) = \varphi(|x|) \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Given four arbitrary finite numbers $r < \alpha < \beta < s$, one can construct a radial $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ so that $\varphi$ together with its corresponding solution $u(x,t)$ satisfy the oscillation behavior:

$$\liminf_{\tau \to \infty} \varphi(\tau) = r < \liminf_{t \to \infty} u(0, t) = \alpha < \limsup_{t \to \infty} u(0, t) = \beta < \limsup_{\tau \to \infty} \varphi(\tau) = s.$$ 

Similarly, given $p < \alpha < \beta < q$ with $p+q = \alpha + \beta$, one can find a radial $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ so that its average integral $H(\tau)$ and $u(x,t)$ satisfy

$$\liminf_{\tau \to \infty} H(\tau) = p < \liminf_{t \to \infty} u(0, t) = \alpha < \limsup_{t \to \infty} u(0, t) = \beta < \limsup_{\tau \to \infty} H(\tau) = q.$$ 

Here $H(\tau), \tau \in (0, \infty)$, is given by

$$H(\tau) = \frac{1}{|B(0, \tau)|} \int_{B(0, \tau)} \varphi(y) \, dy, \quad \tau \in (0, \infty)$$

and $B(0, \tau)$ is the open ball with radius $\tau > 0$ centered at the origin of $\mathbb{R}^n$.

1 Introduction.

This article is a continuation of our previous ones [TN, CT] and proves several new interesting results. Consider the initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \\
&u(x, 0) = \varphi(x), \quad x \in \mathbb{R}^n,
\end{aligned} \tag{1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a given continuous function. It is known that if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous bounded function, then the function given by convolution integral

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \varphi(y) \, dy, \quad x \in \mathbb{R}^n, \quad t > 0 \tag{2}$$

Mathematics Subject Classification: 35K05, 35K15.
is a smooth solution of the heat equation on $\mathbb{R}^n \times (0, \infty)$ with $\lim_{(x,t) \to (x_0,0)} u(x,t) = \varphi(x_0)$ for any $x_0 \in \mathbb{R}^n$. Due to the example by Tychonoff in 1935 (see the book [1], Chapter 7), it is known that the Cauchy problem for the heat equation (1) has no unique solution on $\mathbb{R}^n \times (0, \infty)$ even if the initial data $\varphi(x)$ is bounded (unless we impose certain growth condition of $u(x,t)$ as $|x| \to \infty$ for $t > 0$). From now on, when we say $u(x,t)$ is "the solution" of the heat equation with $u(x,0) = \varphi(x)$, $x \in \mathbb{R}^n$, we always mean that it is the solution given by the convolution integral (2). As a consequence, if $|\varphi| \leq M$ on $\mathbb{R}^n$ for some constant $M > 0$, we also have $|u| \leq M$ on $\mathbb{R}^n \times [0, \infty)$.

In this paper, we shall always assume that $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and use $H(\tau), \tau \in [0, \infty)$, to denote the average integral of $\varphi$ over the open ball $B(0, \tau) \subset \mathbb{R}^n$ centered at $x = 0$ with radius $\tau > 0$, i.e.

$$H(\tau) = \frac{1}{|B(0, \tau)|} \int_{B(0,\tau)} \varphi(y) \, dy = \frac{1}{\omega(n) \tau^n} \int_{B(0,\tau)} \varphi(y) \, dy, \quad \tau \in (0, \infty),$$

(3)

where $\omega(n) = |B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^n$. By continuity, if we define $H(0) = \varphi(0)$, then $H \in C^0[0, \infty) \cap L^\infty[0, \infty)$.

The role played by $H(\tau)$ in the Cauchy problem (1) is the following beautiful stabilization result:

**Theorem 1** (See [2, 3, 4].) Assume $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $u(x,t)$ is the solution of the heat equation (1) given by (2). Then

$$\lim_{t \to \infty} u(0,t) = 0 \quad \text{if and only if} \quad \lim_{\tau \to \infty} \left( \frac{1}{|B(0, \tau)|} \int_{B(0,\tau)} \varphi(y) \, dy \right) = 0.$$  

(4)

Moreover,

$$\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}^n} |u(x,t)| \right) = 0$$

(5)

if and only if

$$\lim_{\tau \to \infty} \left( \sup_{x \in \mathbb{R}^n} \left| \frac{1}{|B(x, \tau)|} \int_{B(x,\tau)} \varphi(y) \, dy \right| \right) = 0.$$  

(6)

Here $|B(x, \tau)|$ is the volume of the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $\tau > 0$.

**Remark 2** Similar result holds if we replace the limit value 0 in (4) by any real number $c$.

**Remark 3** As demonstrated in [CT], Theorem 1 fails if the initial data $\varphi$ is unbounded. In this paper, we will always assume $\varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

**Remark 4** Since $|\varphi| \leq M$ on $\mathbb{R}^n$ for some constant $M > 0$, the convolution solution $u(x,t)$ satisfies the following gradient estimate

$$|\nabla_x u(x,t)| \leq \frac{c(n) M}{\sqrt{t}}, \quad \forall (x,t) \in \mathbb{R}^n \times (0, \infty),$$

(7)

where $c(n)$ is a constant depending only on $n$. Hence if we have $\lim_{t \to \infty} u(0,t) = 0$, we also have $\lim_{t \to \infty} u(x,t) = 0$ for all $x \in \mathbb{R}^n$ and the convergence is uniform in $x \in K$ where $K \subset \mathbb{R}^n$ is any compact set.

By Theorem 1 and Remark 2 the $\lim_{\tau \to \infty} H(\tau)$ does not exist if and only if $\lim_{t \to \infty} u(0,t)$ does not exist. In this case, we have the following comparison result concerning the oscillation of $u(0,t)$ and $H(\tau)$ as $t \to \infty$ and $\tau \to \infty$ (see [CT]), namely

$$\liminf_{\tau \to \infty} H(\tau) \leq \liminf_{t \to \infty} u(0,t) < \limsup_{t \to \infty} u(0,t) \leq \limsup_{\tau \to \infty} H(\tau).$$

(8)
The key point in the proof of (8) is to use the representation formula (see Lemma 4 in [CT]):
\[ u(0, t) = \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-z^2} z^{n+1} H\left(\sqrt{4t}z\right) dz, \quad \forall \ t \in (0, \infty) \]  
(9)
together with the classical Fatou’s Lemma. Same as in [CT], we shall use \( p \leq \alpha < \beta \leq q \) to denote the four finite limit values in (8) in this paper.

By analogy, we can also compare the oscillation of \( u(0, t), \ t \in [0, \infty), \) with the oscillation of the initial data \( \varphi(x) \). Since \( \varphi(x) \) is a function defined on \( \mathbb{R}^n \), it would be quite tricky unless we assume \( \varphi(x) = \varphi(|x|), \ \tau = |x| \in [0, \infty), \ x \in \mathbb{R}^n, \) is a radial function. With this, both \( u(0, t) \) and \( \varphi(\tau) \) are one-variable functions defined on \( [0, \infty) \) and we can compare their oscillations as \( t \to \infty \) and \( \tau \to \infty \) respectively. From now on, we shall assume that \( \varphi(x) = \varphi(|x|), \ x \in \mathbb{R}^n, \) is a radial function.

Clearly, the limit \( \lim_{\tau \to \infty} \varphi(\tau) = 0 \) will imply \( \lim_{t \to \infty} u(0, t) = 0 \), but not conversely. For example, the solution to the one-dimensional heat equation \( u_t = u_{xx} \) with radial initial data \( \varphi(x) = \cos x, \ x \in [0, \infty), \) is given by \( u(x, t) = e^{-t}\cos x \), which satisfies \( \lim_{t \to \infty} u(0, t) = 0. \) But we have \( \liminf_{x \to \infty} \varphi(x) = -1, \ \limsup_{x \to \infty} \varphi(x) = 1. \)

For radial \( \varphi \in C^0([0, \infty) \cap L^\infty([0, \infty) \), the identities (2) (at \( x = 0 \)) and (3) will become
\[ u(0, t) = \frac{n\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-z^2} z^{n-1} \varphi(\sqrt{4t}z) dz, \quad \forall \ t \in (0, \infty) \]  
(10)
and
\[ H(\tau) = \frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr, \quad \tau \in (0, \infty), \quad H(0) = \varphi(0). \]  
(11)
In particular, we note the similarity between (10) and (9).

Let \( r = \liminf_{\tau \to \infty} \varphi(\tau) \) and \( s = \limsup_{\tau \to \infty} \varphi(\tau) \). We have the following general inequalities among the six numbers \( \alpha, \beta, p, q, r, s \):
\[ r \leq p \leq \alpha \leq \beta \leq q \leq s, \]  
(12)
where the first and last inequalities in (12) can be derived from the identity \( H(\tau) = \frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr \) and the definition of \( \liminf \) and \( \limsup \). Note that Theorem 1 says that \( \alpha = \beta \) if and only if \( p = q \). However, \( \alpha = \beta \) does not necessarily imply \( r = s \) (as the above simple example shows), i.e. the stabilization of \( u(0, t) \) as \( t \to \infty \) does not imply the stabilization of \( \varphi(\tau) \) as \( \tau \to \infty \).

In this paper we are interested in prescribing the oscillation values \( \alpha < \beta \) of \( u(0, t) \) as \( t \to \infty \) by choosing suitable radial \( \varphi(\tau) \) in (10) or by choosing suitable average integral \( H(\tau) \) in (9) (by (21) below, one can determine \( \varphi(\tau) \) as long as \( H(\tau) \) is chosen). There are many choices for such \( \varphi(\tau) \) and \( H(\tau) \). The interesting results here are that one can also prescribe the oscillation values of \( \varphi(\tau) \) and \( H(\tau) \) as \( \tau \to \infty \).

More precisely, the results in this paper are:

1. To prescribe, as \( \tau \to \infty \) and \( t \to \infty \), the oscillation behavior of \( H(\tau) \) and \( u(0, t) \) for four arbitrary finite numbers \( p \leq \alpha < \beta \leq q \) satisfying the symmetry condition
\[ p + q = \alpha + \beta. \]  
(13)

2. To prescribe, as \( \tau \to \infty \) and \( t \to \infty \), the oscillation behavior of \( \varphi(\tau) \) and \( u(0, t) \) for four arbitrary finite numbers \( r \leq \alpha \leq \beta \leq s \).

See Theorem 5 and Theorem 8 in the next section.
2 Main results and their proofs.

2.1 Prescribing $H(τ)$ and $u(0, t)$.

The discussion of prescribing $H(τ)$ and $u(0, t)$ is motivated by Theorem 1. Denote the four limit values in $(8)$ as $p$, $α$, $β$, $q$ respectively. In [CT], with the help of the formula $(9)$, we have constructed two bounded radial functions $φ(x) = φ(|x|)$, $x ∈ \mathbb{R}^n$, so that the following are satisfied:

(a). $p = -1 < α = -√A^2 + B^2 < β = √A^2 + B^2 < q = 1$. Here $A$, $B$ are the values of certain convergent improper integrals satisfying $0 < A^2 + B^2 < 1$.

(b). $p = α < β = q$ for arbitrary two finite numbers $α < β$.

In the above two examples, we have the identity

\[ p + q = α + β, \quad \text{where} \quad α < β. \] (14)

Our new result in the following is that, as long as the symmetry condition $(14)$ is satisfied, we can prescribe them. For four numbers $p ≤ α < β ≤ q$ to satisfy $(14)$, we must have either $p < α < β < q$ or $p = α < β = q$. Since the second case has been done, it suffices to look at the first case. We have:

**Theorem 5** Let $p < α < β < q$ be four arbitrary finite numbers satisfying the identity $p + q = α + β$. One can find a continuous bounded radial function $φ(x) = φ(|x|)$, $x ∈ \mathbb{R}^n$, so that its average integral $H(τ)$ and the convolution solution $(2)$ of the problem $(1)$ satisfy

\[ \liminf_{τ→∞} H(τ) = p < \liminf_{t→∞} u(0, t) = α < \limsup_{t→∞} u(0, t) = β < \limsup_{τ→∞} H(τ) = q. \] (15)

More precisely, we can choose the initial data $φ(x) = φ(|x|)$, $x ∈ \mathbb{R}^n$, as

\[ φ(τ) = \frac{q - p}{2} \sin(m \log(τ + 1)) + \frac{q - p}{2} \frac{mτ}{n(τ + 1)} \cos(m \log(τ + 1)) + \frac{q + p}{2}, \] (16)

where $τ ∈ [0, ∞)$ and $m ∈ (0, ∞)$ is a number satisfying the identity

\[ \left(\frac{2ω(n)}{π^{n/2}} \int_0^∞ e^{-z^2} z^{n+1} \cos(m \log z) dz\right)^2 + \left(\frac{2ω(n)}{π^{n/2}} \int_0^∞ e^{-z^2} z^{n+1} \sin(m \log z) dz\right)^2 = \left(\frac{β - α}{q - p}\right)^2 ∈ (0, 1). \] (17)

**Remark 6** Without the condition $p + q = α + β$, we do not know how to achieve the result $(15)$ in general. However, we can construct some specific example with $p + q ≠ α + β$. See Section 3.1.

**Remark 7** Note that if the initial data $φ(x) = φ(|x|)$ is radial, the convolution solution $u(x, t)$ of the heat equation will also be radial in $x ∈ \mathbb{R}^n$ for each fixed time $t > 0$.

2.2 Prescribing $φ(τ)$ and $u(0, t)$.

It is due to Theorem 1 that we study the oscillation behavior of $H(τ)$ and its effect on $u(0, t)$. Instead, since $φ(τ)$ is the initial data, it is perhaps more natural to study the oscillation relation between $φ(τ)$ and $u(0, t)$. However, there is a major difference here. Since we always assume that $φ(τ) ∈ C^0[0, ∞) \cap L^∞[0, ∞)$ is continuous and bounded, its average integral function $H(τ)$ will also be continuous and bounded and satisfy $H'(τ) = O(1/τ)$ as $τ → ∞$, which we call it a slow
oscillation if we have \( p < q \) (see (22) and Remark 14 below). Any such oscillation will cause \( u(0,t) \) to have a slow oscillation too (by Theorem 1 we have \( p < q \) if and only if \( \alpha < \beta \)).

On the other hand, \( \varphi(\tau) \in C^0[0,\infty) \bigcap L^\infty[0,\infty) \) may not have a slow oscillation (i.e. it may not satisfy \( \varphi'(\tau) = O(1/\tau) \) as \( \tau \to \infty \) even if it is differentiable) and so its oscillation (with \( r < s \)) may not pass to \( u(0,t) \). In fact, for any \( 2\pi \)-periodic regular oscillation function \( \varphi(\tau), \tau \in [0,\infty), \) the Cauchy problem (11) with \( u(x,0) = \varphi(|x|), x \in \mathbb{R}^n, \) will have the convergence \( \lim_{t \to \infty} u(0,t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau) \, d\tau \) (see Lemma 17 below). However, due to the similarity between the two representation formulas (9) and (10), if \( \varphi(\tau) \) has a slow oscillation as \( \tau \to \infty \) with \( r < s \), then it has the chance of passing to the function \( u(0,t) \) as \( t \to \infty \) (see Lemma 18 below).

The major result for prescribing \( \varphi(\tau) \) and \( u(0,t) \) is the following:

**Theorem 8** Let \( r < \alpha < \beta < s \) be four arbitrary finite numbers. One can find a continuous bounded radial function \( \varphi(x) = \varphi(|x|), x \in \mathbb{R}^n, \) so that it and the convolution solution (2) of the problem (1) satisfy

\[
\liminf_{\tau \to \infty} \varphi(\tau) = r < \liminf_{t \to \infty} u(0,t) = \alpha < \limsup_{t \to \infty} u(0,t) = \beta < \limsup_{\tau \to \infty} \varphi(\tau) = s.
\]

More precisely, we can choose the initial data \( \varphi(x) = \varphi(|x|), x \in \mathbb{R}^n, \) as

\[
\varphi(\tau) = [C_1 \sin(m \log(\tau + 1)) + C_2] + \psi(\tau), \quad \tau \in [0,\infty),
\]

for some suitable constants \( m > 0, C_1 > 0, C_2 \) depending on \( r, \alpha, \beta, s \) and some \( 2\pi \)-periodic continuous nonzero function \( \psi(\tau) \) satisfying \( \frac{1}{2\pi} \int_0^{2\pi} \psi(\tau) \, d\tau = 0 \).

**Remark 9** Theorem 8 is actually valid for four arbitrary finite numbers \( r \leq \alpha \leq \beta \leq s \). We will deal with the remaining cases in Theorem 22 below.

**Remark 10** This is a comparison between Theorem 5 and Theorem 8. For any initial data \( \varphi \in C^0(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n), \) no matter it is radial or not, the function \( H(\tau) \) is always a slow oscillation function as \( \tau \to \infty \). Therefore, the function space for \( H(\tau) \) is smaller than the function space for \( \varphi(\tau) \), which may explain why we have more general result in Theorem 8. Note that the function \( \varphi(\tau) \) in (12) does not have a slow oscillation as \( \tau \to \infty \) due to the term \( \psi(\tau) \).

### 2.2.1 Proof of Theorem 5

The proof is an interesting modification of the argument in Section 3.1.1 of [CT], which can achieve much more general result. Some motivation should be explained at the beginning. First, we choose \( \varphi(x) = \varphi(|x|) \) to be a radial function in order to simplify the computation. We think a non-radial function can also be found as long as we can overcome the computational complexity. Second, we choose the average integral function \( H(\tau) \) first and then go back to find its corresponding \( \varphi(|x|) \). The function \( H(\tau) \) has to be oscillatory so that we have \( \liminf_{\tau \to \infty} H(\tau) < \limsup_{\tau \to \infty} H(\tau) \). Moreover, since we require \( \varphi \) to be bounded and continuous (in view of Theorem 1), its average integral \( H(\tau) \) will satisfy the derivative estimate \( H'(\tau) = O(1/\tau) \) as \( \tau \to \infty \) (see (22) below). Therefore, a natural choice is roughly like the function \( \sin(\log \tau), \tau \in (0,\infty) \). After suitable modification of the function \( \sin(\log \tau) \), the proof can be achieved.

Given \( \varphi(x) = \varphi(|x|) \in C^0[0,\infty) \bigcap L^\infty[0,\infty), \) its average integral function

\[
H(\tau) = \frac{1}{\omega(n) \tau^n} \int_{B(0,\tau)} \varphi(y) \, dy = \frac{n}{\tau^n} \int_0^\tau \varphi(r) \, r^{n-1} \, dr, \quad \tau \in (0,\infty), \quad H(0) = \varphi(0),
\]

is a bounded continuous function on \([0,\infty), \) differentiable in \((0,\infty), \) and satisfies the identity

\[
H'(\tau) = -\frac{n}{\tau} H(\tau) + \frac{n}{\tau} \varphi(\tau), \quad \forall \tau \in (0,\infty).
\]
Hence it has the asymptotic behavior
\[ H'(\tau) = O\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to \infty. \]  

Moreover, the radial function \( \varphi(\tau) \) can be determined from \( H(\tau) \) by the identity (21).

Let \( m \in (0, \infty) \) be a fixed number. Motivated by our principle stated above, we temporarily choose \( H(\tau) \) (will modify it later on) to be equal to
\[ H(\tau) = \frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr = \sin (m \log (\tau + 1)), \quad \tau \in (0, \infty), \quad H(0) = 0, \]  
which, by (21), gives
\[ \varphi(\tau) = \sin (m \log (\tau + 1)) + \frac{m \tau}{n (\tau + 1)} \cos (m \log (\tau + 1)), \quad \tau \in (0, \infty). \]  

That is, if we choose \( \varphi(\tau) \) to be the function given by (24) and define \( \varphi(0) = 0 \), then the function \( \varphi(x) = \varphi(|x|) \), \( x \in \mathbb{R}^n \), will be a radial function defined on \( \mathbb{R}^n \), lying in the space \( C^0(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n) \), and its average integral is given by \( H(\tau) = \sin (m \log (\tau + 1)) \) for all \( \tau \in [0, \infty) \). In particular, we have \( p = -1, \ q = 1 \).

To find the values of \( \alpha, \beta \), we use the representation formula (9). We have
\[ u(0,t) = \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \sin \left( m \log \left( \sqrt{4t} z + 1 \right) \right) dz \]
\[ = \left\{ \begin{array}{l}
\left[ \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \cos \left( m \log \left( z + \frac{1}{\sqrt{4t}} \right) \right) dz \right] \sin \left( m \log \sqrt{4t} \right) \\
+ \left[ \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \sin \left( m \log \left( z + \frac{1}{\sqrt{4t}} \right) \right) dz \right] \cos \left( m \log \sqrt{4t} \right)
\end{array} \right\}, \quad t \in (0, \infty) \]  
and the Lebesgue Dominated Convergence Theorem implies
\[ \lim_{t \to \infty} \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \cos \left( m \log \left( z + \frac{1}{\sqrt{4t}} \right) \right) dz = \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \cos (m \log z) dz := A(m) \in (-1, 1) \]  
and
\[ \lim_{t \to \infty} \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \sin \left( m \log \left( z + \frac{1}{\sqrt{4t}} \right) \right) dz = \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \sin (m \log z) dz := B(m) \in (-1, 1), \]  
where we can use the identity \( \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} dz = 1 \) to know that \( A(m), B(m) \in (-1, 1) \). By Hölder inequality, the constant \( A(m) \) satisfies the estimate
\[ A^2(m) = \left( \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty \sqrt{e^{-s^2 z^{n+1}} e^{-s^2 z^{n+1}} \cos (m \log z) dz} \right)^2 \]
\[ < \left( \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} dz \right) \left( \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \cos^2 (m \log z) dz \right) \]
\[ = \frac{2\omega(n)}{\pi^{n/2}} \int_0^\infty e^{-s^2 z^{n+1}} \cos^2 (m \log z) dz \]
The idea next is to view $m \in (0, \infty)$ as a variable. We observe the following:

**Lemma 11** We have

$$
\lim_{m \to \infty} A(m) = \lim_{m \to \infty} B(m) = 0, \quad m \in (0, \infty).
$$

**Proof.** For any $\varepsilon > 0$, one can find small number $\delta > 0$ and large number $M > 0$, both are independent of $m \in (0, \infty)$, such that

$$
\left| \int_0^\delta e^{-z^2} z^{n+1} \cos (m \log z) \, dz \right| + \left| \int_M^\infty e^{-z^2} z^{n+1} \cos (m \log z) \, dz \right| < \varepsilon. \tag{32}
$$

On the other hand, by the change of variables, we have

$$
\int_0^M e^{-z^2} z^{n+1} \cos (m \log z) \, dz = \int_{\log \delta}^{\log M} F(x) \cos (mx) \, dx, \quad x = \log z,
$$

where $F(x) = e^{-e^{2x+(n+2)x}}$. The familiar **Riemann-Lebesgue lemma** in analysis (or just use integration by parts) implies

$$
\lim_{m \to \infty} \int_{\log \delta}^{\log M} F(x) \cos (mx) \, dx = 0. \tag{33}
$$

which, together with (32), implies $\lim_{m \to \infty} A(m) = 0$. The proof of $\lim_{m \to \infty} B(m) = 0$ is similar.\(\square\)

Since estimate (32) is valid for all $m \in (0, \infty)$, the two improper integrals

$$
\int_0^\infty e^{-z^2} z^{n+1} \cos (m \log z) \, dz, \quad \int_0^\infty e^{-z^2} z^{n+1} \sin (m \log z) \, dz
$$

**converge uniformly** with respect to $m \in (0, \infty)$. As a consequence, both $A(m)$ and $B(m)$ are **continuous functions** of $m \in (0, \infty)$, with

$$
\lim_{m \to \infty} A(m) = \lim_{m \to \infty} B(m) = 0, \quad \lim_{m \to 0} A(m) = 1, \quad \lim_{m \to 0} B(m) = 0. \tag{35}
$$
where the last two limits in (35) are due to the Lebesgue Dominated Convergence Theorem. Moreover, the intermediate value theorem implies the existence of a number \( m \in (0, \infty) \) satisfying
\[
\sqrt{A^2(m) + B^2(m)} = \frac{\beta - \alpha}{q - p} \in (0, 1).
\] (36)

Now we choose \( H(\tau) \) as
\[
H(\tau) = \frac{q - p}{2} \sin (m \log (\tau + 1)) + \frac{q + p}{2}, \quad H(0) = \frac{q + p}{2}, \quad \tau \in [0, \infty),
\] (37)
where \( m \in (0, \infty) \) is the number satisfying (36). By (23) and (24), its corresponding initial bounded radial function \( \varphi(x) = \varphi(|x|), x \in \mathbb{R}^n \), is given by
\[
\varphi(\tau) = \begin{cases} 
\frac{q - p}{2} \sin (m \log (\tau + 1)) \\
+ \frac{q + p}{2} \frac{m \tau}{n(\tau + 1)} \cos (m \log (\tau + 1)) + \frac{q + p}{2}, \quad \varphi(0) = \frac{q + p}{2}, \quad \tau \in [0, \infty).
\end{cases}
\] (38)

Similar to (29), we have
\[
\lim_{t \to \infty} \left| u(0, t) - \left\{ \frac{q - p}{2} \left[ A(m) \sin (m \log \sqrt{4t}) + B(m) \cos (m \log \sqrt{4t}) \right] + \frac{q + p}{2} \right\} \right| = 0,
\] (39)
which gives
\[
\lim_{t \to \infty} u(0, t) = -\frac{q - p}{2} \sqrt{A^2(m) + B^2(m)} + \frac{q + p}{2} = -\frac{\beta - \alpha}{2} + \frac{\beta + \alpha}{2} = \alpha
\]
and
\[
\lim_{t \to \infty} \sup u(0, t) = \frac{q - p}{2} \sqrt{A^2(m) + B^2(m)} + \frac{q + p}{2} = \frac{\beta - \alpha}{2} + \frac{\beta + \alpha}{2} = \beta.
\]
Since we clearly have \( \lim_{\tau \to \infty} H(\tau) = p, \lim_{\tau \to \infty} H(\tau) = q \), the proof of Theorem 3 is now complete. \( \square \)

**Remark 12** The proof of Theorem 3 also reveals the following interesting observation. For fixed \( q - p > 0 \), if we have small \( \beta - \alpha \), then by (36) and (37), we will have large \( m \in (0, \infty) \) and the function \( \varphi(\tau) \) in (38) will tend to be unbounded due to the term \( m \tau / (n(\tau + 1)) \). This may suggest that we cannot find a bounded radial function \( \varphi(\tau) \) satisfying \( p < \alpha = \beta < q \). This matches with the result in Theorem 4, which implies that if we have \( p < \alpha = \beta < q \), then the initial data \( \varphi \in C^0(\mathbb{R}^n) \) must be unbounded.

**Remark 13** By (7), for a given initial data \( \varphi \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), if we have
\[
\lim_{t \to \infty} \inf u(0, t) = \alpha < \lim_{t \to \infty} \sup u(0, t) = \beta,
\]
then for any fixed \( x \in \mathbb{R}^n \) we also have
\[
\lim_{t \to \infty} \inf u(x, t) = \alpha < \lim_{t \to \infty} \sup u(x, t) = \beta.
\] (40)

**Remark 14** It has been shown in [CT] (see its equations (16) and (37)) that both functions \( H(\tau) \) and \( u(0, t) \) satisfy
\[
H'(\tau) = O \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to \infty, \quad u_t(0, t) = O \left( \frac{1}{t} \right) \quad \text{as} \quad t \to \infty,
\] (41)
as long as the initial data \( \varphi \) lies in the space \( C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) (no matter it is radial or not). By (41), we may say that any oscillation in \( H(\tau) \) (with \( p < q \)) or in \( u(0, t) \) (with \( \alpha < \beta \)) is a slow oscillation.
As a consequence of Theorem 5, we have the following two corollaries, which say that we can prescribe the oscillation of \( H(\tau) \) and \( u(0,t) \) for three arbitrary different numbers.

**Corollary 15** For any three different numbers, denoted as \( p < \alpha < \beta \), one can find a continuous bounded radial function \( \varphi(x) = \varphi(|x|) \), \( x \in \mathbb{R}^n \), so that its average integral and the solution (2) of the problem (1) satisfies

\[
\lim_{\tau \to \infty} H(\tau) = p < \lim_{t \to \infty} u(0,t) = \alpha < \lim_{t \to \infty} u(0,t) = \beta < \lim_{\tau \to \infty} H(\tau) = q, \tag{42}
\]

where \( q = \alpha + \beta - p \). Similarly, for any three different numbers, denoted as \( \alpha < \beta < q \), one can find a bounded radial function satisfying (42), where now \( p = \alpha + \beta - q \).

**Corollary 16** For any three different numbers, denoted as \( p < \alpha < q \), with \( \alpha < (p+q)/2 \), the same result as in Corollary 15 holds, where now \( \beta = p+q-\alpha \). Similarly, for any three different numbers, denoted as \( p < \beta < q \), with \( \beta > (p+q)/2 \), the same result as in Corollary 15 holds, where now \( \alpha = p+q-\beta \).

### 2.2.2 Proof of Theorem 8

To prove Theorem 8, we first need the following:

**Lemma 17** Assume \( \varphi(\tau) \) is a \( 2\pi \)-periodic radial function defined on \( \tau \in [0, \infty) \). We have the convergence

\[
\lim_{t \to \infty} u(0,t) = \lim_{\tau \to \infty} H(\tau) = \lim_{\tau \to \infty} \frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\tau) d\tau. \tag{43}
\]

**Proof.** Since \( \varphi \in C^0[0, \infty) \cap L^\infty[0, \infty) \), by Remark 2 in Theorem 1, it suffices to prove the identity for \( H(\tau) \). The proof is quite straightforward for the case \( n = 1 \) (see Remark 6 in [TN]), but may need a trick for \( n > 1 \). For large \( \tau > 0 \), we can express it as \( \tau = 2m\pi + R \) for some \( m \in \mathbb{N} \) and \( R \in [0, 2\pi) \), with the understanding that both \( m \) and \( R \) depend on \( \tau \) and as \( \tau \to \infty \) we have \( m \to \infty \), hence we obtain

\[
\frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr = \frac{n}{(2m\pi + R)^n} \left( \int_0^{2\pi} \varphi(r) r^{n-1} dr + \int_{2\pi}^{4\pi} \varphi(r) r^{n-1} dr + \cdots + \int_{2(2m-1)\pi}^{2m\pi} \varphi(r) r^{n-1} dr \right). \tag{44}
\]

If we do the change of variables

\[
\int_{2\pi}^{4\pi} \varphi(r) r^{n-1} dr = \int_0^{2\pi} \varphi(s) (s + 2\pi)^{n-1} ds, \quad r = s + 2\pi,
\]

and

\[
\int_{4\pi}^{6\pi} \varphi(r) r^{n-1} dr = \int_0^{2\pi} \varphi(s) (s + 4\pi)^{n-1} ds, \quad r = s + 4\pi,
\]

..., etc., (44) becomes

\[
\frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr
\]

\[
= \frac{n}{(2m\pi + R)^n} \left( \int_0^{2\pi} \varphi(s) \left[ (s^{n-1} + (s + 2\pi)^{n-1} + (s + 4\pi)^{n-1} + \cdots + (s + 2(m-1)\pi)^{n-1} \right] ds \right)
\]

\[
:= I + II. \tag{45}
\]
For the second term \( II \) in (45), we have
\[
\lim_{m \to \infty} |II| \leq \lim_{m \to \infty} \left( \frac{n}{(2m\pi + R)^n} \cdot 2\pi \max_{s \in [0,2\pi]} |\varphi(s)| \cdot (2\pi + 2m\pi)^{n-1} \right) = 0. \tag{46}
\]
To estimate the first term \( I \) in (45), we first look at the integral
\[
\frac{n}{(2m\pi)^n} \int_0^{2\pi} \varphi(s) \left[ s^{n-1} + (s + 2\pi)^{n-1} + (s + 4\pi)^{n-1} + \cdots + (s + 2(m-1)\pi)^{n-1} \right] ds
= \frac{1}{2\pi} \int_0^{2\pi} \varphi(s) \frac{n}{m^n(2\pi)^{n-1}} \left[ s^{n-1} + (s + 2\pi)^{n-1} + (s + 4\pi)^{n-1} + \cdots + (s + 2(m-1)\pi)^{n-1} \right] ds,
\]
where the underlined term \( \cdots \) is equal to
\[
\cdots = \frac{n}{m^n} \left[ \left( \frac{s}{2\pi} \right)^{n-1} + \left( \frac{s}{2\pi} + 1 \right)^{n-1} + \left( \frac{s}{2\pi} + 2 \right)^{n-1} + \cdots + \left( \frac{s}{2\pi} + (m-1) \right)^{n-1} \right]
\]
and for \( s \in [0,2\pi] \) we have
\[
\frac{n}{m^n} \left( 0^{n-1} + 1^{n-1} + 2^{n-1} + \cdots + (m-1)^{n-1} \right)
\leq \cdots \leq \frac{n}{m^n} \left( 1^{n-1} + 2^{n-1} + 3^{n-1} + \cdots + m^{n-1} \right). \tag{47}
\]
By the inequality
\[
\frac{n}{m^n} \left( 0^{n-1} + 1^{n-1} + 2^{n-1} + \cdots + (m-1)^{n-1} \right)
\leq \frac{n}{m^n} \int_0^m x^{n-1} dx = 1 < \frac{n}{m^n} \left( 1^{n-1} + 2^{n-1} + 3^{n-1} + \cdots + m^{n-1} \right) \tag{48}
\]
with
\[
\lim_{m \to \infty} \frac{n}{m^n} \left( 1^{n-1} + 2^{n-1} + 3^{n-1} + \cdots + m^{n-1} \right) = 1
\]
we must have
\[
\lim_{m \to \infty} \frac{n}{m^n} \cdot \frac{n}{m^n} \left( 0^{n-1} + 1^{n-1} + 2^{n-1} + \cdots + (m-1)^{n-1} \right) = 1
\]
and hence \( \lim_{m \to \infty} \cdots = 1 \), which implies
\[
\lim_{m \to \infty} \frac{n}{(2m\pi)^n} \int_0^{2m\pi} \varphi(r) r^{n-1} dr = \frac{1}{2\pi} \int_0^{2\pi} \varphi(s) \left( \lim_{m \to \infty} \cdots \right) ds = \frac{1}{2\pi} \int_0^{2\pi} \varphi(s) ds. \tag{49}
\]
Finally we have
\[
\lim_{m \to \infty} I = \lim_{m \to \infty} \left( \frac{2m\pi}{2m\pi + R} \right)^n \frac{n}{(2m\pi)^n} \int_0^{2m\pi} \varphi(r) r^{n-1} dr = \frac{1}{2\pi} \int_0^{2\pi} \varphi(s) ds.
\]
The proof is done.

As we have said in the paragraph before Theorem 8, if \( \varphi(\tau) \) has a slow oscillation as \( \tau \to \infty \) with \( r < s \), then it has the chance of passing to the function \( u(0,t) \). The following gives a simple way to find a slow oscillation function \( \varphi(\tau) \) on \([0,\infty)\) which will cause \( \alpha < \beta \).
Lemma 18 Let $g (\tau)$ be a $2\pi$-periodic non-constant $C^2$ function defined on $[0, \infty)$ and let $G (\tau) = g (\log (\tau + 1))$, $\tau \in [0, \infty)$. Then the radial function $\varphi (\tau) = \frac{\tau}{n} G' (\tau) + G (\tau)$, $\tau \in [0, \infty)$, has a slow oscillation as $\tau \to \infty$, lies in the space $C^0[0, \infty) \cap L^\infty[0, \infty)$, and its average integral $H (\tau)$, $\tau \in [0, \infty)$, satisfies $H (\tau) = g (\log (\tau + 1))$, $\tau \in [0, \infty)$, with

$$p = \liminf_{\tau \to \infty} g (\tau) < \limsup_{\tau \to \infty} g (\tau) = q,$$

which, by Theorem 1, will also imply $\alpha < \beta$.

Proof. By

$$\left\{ \begin{array}{l} \varphi (\tau) = \frac{\tau}{n} g'(\log(\tau+1)) + g (\log (\tau + 1)), \quad \varphi (0) = g (0), \\ \varphi' (\tau) = \left(1 + \frac{1}{n} \right) g'(\log(\tau+1)) + \frac{\tau n}{\tau+1} g''(\log(\tau+1)) - g'(\log(\tau+1)), \quad \tau \in [0, \infty), \end{array} \right.$$}

we see that $\varphi$ is a slow oscillation function on $[0, \infty)$, lying in the space $C^0[0, \infty) \cap L^\infty[0, \infty)$. Now

$$H (\tau) = \frac{n}{\tau n} \int_0^\tau \varphi (r) r^{n-1} dr = \frac{n}{\tau n} \int_0^\tau n \left( \frac{r}{n} G' (r) + G (r) \right) r^{n-1} dr = \frac{1}{\tau n} \int_0^\tau \frac{d}{dr} (r^n G (r)) dr = G (\tau) = g (\log (\tau + 1)),$$

and $H (0) = G (0) = g (0)$. Therefore, the inequality (50) follows. \hfill \Box

With the help of Lemma 17, we are ready to prove Theorem 8. For any $r < \alpha < \beta < s$, we divide the proof into two cases.

Case 1: $r + s = \alpha + \beta$, $r < \alpha < \beta < s$.

In this case we can use the representation formula (10) and, similar to (37), choose

$$\varphi (\tau) = \frac{s - r}{2} \sin (m \log (\tau + 1)) + \frac{s + r}{2}, \quad \varphi (0) = \frac{s + r}{2}, \quad \tau \in [0, \infty),$$

and perform the same argument as in the proof of Theorem 5 to prescribe them. More precisely, here we need to choose $m \in (0, \infty)$ to satisfy $\sqrt{A^2 (m) + \tilde{B}^2 (m)} = \frac{\beta - \alpha}{s - r} \in (0, 1)$, where now

$$\begin{align*}
\tilde{A} (m) &= \frac{n \omega (n)}{\pi^{n/2}} \int_0^\infty e^{-z^2} z^{n-1} \cos (m \log z) \, dz \in (-1, 1), \\
\tilde{B} (m) &= \frac{n \omega (n)}{\pi^{n/2}} \int_0^\infty e^{-z^2} z^{n-1} \sin (m \log z) \, dz \in (-1, 1),
\end{align*}$$

with

$$0 < \tilde{A}^2 (m) + \tilde{B}^2 (m) < \frac{n \omega (n)}{\pi^{n/2}} \int_0^\infty e^{-z^2} z^{n-1} \left[ \cos^2 (m \log z) + \sin^2 (m \log z) \right] \, dz = 1.$$ (53)

Note that in this case the chosen function $\varphi (\tau)$, like $H (\tau)$ in (37), has a slow oscillation on $[0, \infty)$.

Case 2: $r + s \neq \alpha + \beta$, $r < \alpha < \beta < s$.

In this case, we may assume $r + s > \alpha + \beta$ (the treatment for the case $r + s < \alpha + \beta$ is similar). We will choose $\varphi (\tau) \in C^0[0, \infty) \cap L^\infty[0, \infty)$ to be the sum of two functions $\varphi_1 (\tau) + \varphi_2 (\tau)$, where $\varphi_1 (\tau)$ has a slow oscillation, but $\varphi_2 (\tau)$ has a regular oscillation (which is $2\pi$-periodic).
Let $\lambda$ be the number satisfying $r + \lambda = \alpha + \beta$. Since $r < \alpha < \beta < s$ and $r + s > \alpha + \beta$, we have $\beta < \lambda < s$. We can write $\lambda$ as $\varepsilon + \delta$ for some small $\varepsilon > 0$ and $\delta \in \mathbb{R}$ so that we have
\[
r + \varepsilon < \alpha < \beta < \delta < s.
\] (54)

Now we have $(r + \varepsilon) + \delta = \alpha + \beta$ and by Case 1, we can find a radial $\varphi_1 \in C^0[0, \infty) \cap L^\infty[0, \infty)$ such that it and its convolution solution $u_1(x, t)$ satisfy
\[
\liminf_{\tau \to \infty} \varphi_1(\tau) = r + \varepsilon < \liminf_{t \to \infty} u_1(0, t) = \alpha < \limsup_{t \to \infty} u_1(0, t) = \beta < \limsup_{\tau \to \infty} \varphi_1(\tau) = \delta.
\] (55)

Next let $\varphi_2(\tau)$ be a $2\pi$-periodic continuous function on $\tau \in [0, \infty)$ which satisfies
\[
\frac{1}{2\pi} \int_0^{2\pi} \varphi_2(\tau) \, d\tau = 0, \quad \min_{\tau \in [0, 2\pi]} \varphi_2(\tau) = -\varepsilon < 0, \quad \max_{\tau \in [0, 2\pi]} \varphi_2(\tau) = s - \delta > 0.
\] (56)

Such a function $\varphi_2$ clearly exists and, together with its convolution solution $u_2(x, t)$, they satisfy
\[
\liminf_{\tau \to \infty} \varphi_2(\tau) = -\varepsilon < \liminf_{t \to \infty} u_2(0, t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_2(\tau) \, d\tau = 0 < \limsup_{\tau \to \infty} \varphi_2(\tau) = s - \delta,
\] (57)
due to the result in Lemma 17. Finally, we set $\varphi(\tau) = \varphi_1(\tau) + \varphi_2(\tau), \tau \in [0, \infty)$. We have $\varphi \in C^0[0, \infty) \cap L^\infty[0, \infty)$ and its convolution solution $u(x, t)$ satisfy $u(x, t) = u_1(x, t) + u_2(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Since $u_2(0, t) \to 0$ as $t \to \infty$, we have
\[
\begin{cases}
\liminf_{t \to \infty} u(0, t) = \liminf_{t \to \infty} u_1(0, t) = \alpha \\
\limsup_{t \to \infty} u(0, t) = \limsup_{t \to \infty} u_1(0, t) = \beta.
\end{cases}
\] (58)

It remains to look at $\varphi(\tau)$ for $\tau \to \infty$. By (51) we know $\varphi_1(\tau)$ is a slow oscillation function on $[0, \infty)$ with the form
\[
\varphi_1(\tau) = \frac{\delta - (r + \varepsilon)}{2} \sin (m \log (\tau + 1)) + \frac{\delta + (r + \varepsilon)}{2}, \quad \varphi_1(0) = \frac{\delta + (r + \varepsilon)}{2}, \quad \tau \in [0, \infty)
\] (59)

for some fixed $m \in (0, \infty)$ satisfying $\sqrt{\hat{A}^2(m) + \hat{B}^2(m)} = \frac{\beta - \alpha}{\delta - (r + \varepsilon)} \in (0, 1)$. On the other hand, $\varphi_2(\tau)$ is a regular oscillation function (2$\pi$-periodic function) on $[0, \infty)$, which oscillates between $-\varepsilon$ and $s - \delta$. It is not difficult to see that there exists a sequence $\tau_j \to \infty$ so that
\[
\sin (m \log (\tau_j + 1)) \to -1 \quad \text{and} \quad \varphi_2(\tau_j) \to -\varepsilon = \min_{\tau \in [0, 2\pi]} \varphi_2(\tau) < 0
\] (60)
as $j \to \infty$. Similarly, there exists a sequence $\tau_k \to \infty$ so that
\[
\sin (m \log (\tau_k + 1)) \to 1 \quad \text{and} \quad \varphi_2(\tau_k) \to s - \delta = \max_{\tau \in [0, 2\pi]} \varphi_2(\tau) > 0
\] (61)
as $k \to \infty$. By (59), (61), and (60), we have
\[
\liminf_{\tau \to \infty} \varphi(\tau) = \liminf_{\tau \to \infty} (\varphi_1(\tau) + \varphi_2(\tau)) = \liminf_{\tau \to \infty} \varphi_1(\tau) + \liminf_{\tau \to \infty} \varphi_2(\tau) = (r + \varepsilon) - \varepsilon = r
\] (62)
and
\[
\limsup_{\tau \to \infty} \varphi(\tau) = \limsup_{\tau \to \infty} (\varphi_1(\tau) + \varphi_2(\tau)) = \limsup_{\tau \to \infty} \varphi_1(\tau) + \limsup_{\tau \to \infty} \varphi_2(\tau) = \delta + (s - \delta) = s.
\] (63)

The proof is done due to (58), (62), and (63).

Combining Case 1 and Case 2, the proof of Theorem 8 is now complete.
Remark 19 This is to explain the existence of a sequence $\tau_k \to \infty$ so that (61) holds. We first note that, for given $\varepsilon > 0$, there exists $\delta > 0$ (depending only on $\varepsilon$), such that if $m \log (\tau + 1)$ (here $m > 0$ is fixed) lies in the interval $(2k\pi + \frac{\pi}{2} - \delta, 2k\pi + \frac{\pi}{2} + \delta)$ for some $k \in \mathbb{N}$, then we have $|\sin (m \log (\tau + 1)) - 1| < \varepsilon$, which is equivalent for $\tau + 1$ to lie in the interval
\[
\tau + 1 \in \left( \exp \left( \frac{2k\pi + \frac{\pi}{2} - \delta}{m} \right), \exp \left( \frac{2k\pi + \frac{\pi}{2} + \delta}{m} \right) \right). \tag{64}
\]

We can choose $k \in \mathbb{N}$ sufficiently large so that the length of the interval in (64) is sufficiently large. By this observation, we clearly have (61). The reason for (60) to hold is similar.

2.3 The remaining cases not covered by Theorem 8.

For prescribing general $r \leq \alpha \leq \beta \leq s$, the remaining cases not covered by Theorem 8 are discussed in the following:

Theorem 20 For any four arbitrary finite numbers $r$, $\alpha$, $\beta$, $s$ in one of the following cases:
\[
\begin{align*}
(1) &. \ r = \alpha < \beta < s, & (2) &. \ r < \alpha = \beta < s, & (3) &. \ r < \alpha < \beta = s, \\
(4) &. \ r = \alpha = \beta < s, & (5) &. \ r = \alpha < \beta = s, & (6) &. \ r < \alpha = \beta = s,
\end{align*}
\tag{65}
\]

one can choose a suitable radial initial data $\varphi(x) = \varphi(|x|) \in C^0[0, \infty) \cap L^\infty[0, \infty)$ satisfying
\[
\liminf_{\tau \to \infty} \varphi(\tau) = r, \quad \limsup_{\tau \to \infty} \varphi(\tau) = s \tag{66}
\]
and its corresponding convolution solution $u(x, t)$ in (4) satisfies
\[
\liminf_{t \to \infty} u(0, t) = \alpha, \quad \limsup_{t \to \infty} u(0, t) = \beta. \tag{67}
\]

Remark 21 The case $r = \alpha = \beta = s$ is trivial. Just choose $\varphi(x)$ to be a constant function.

Proof. (2). For $r < \alpha = \beta < s$, we can choose $\varphi(\tau)$, $\tau \in [0, \infty)$, to be any $2\pi$-periodic function with average value $\alpha = \beta$, maximum value $s$, minimum value $r$. By Lemma 17, it can be achieved.

(5). For $r < \alpha = \beta = s$, we choose $\varphi(\tau)$ to be the extremely slow oscillation function
\[
\varphi(\tau) = \frac{\beta - \alpha}{2} \sin \left[ \log \left( \log (\tau + 2) \right) \right] + \frac{\beta + \alpha}{2}, \quad \tau \in [0, \infty), \tag{68}
\]
which has
\[
\liminf_{\tau \to \infty} \varphi(\tau) = \alpha, \quad \limsup_{\tau \to \infty} \varphi(\tau) = \beta. \tag{69}
\]

By (68), its corresponding $u(0, t)$ is given by
\[
u(0, t) = \frac{\beta - \alpha}{2} \frac{n \omega(n)}{\pi^{n/2}} \int_0^\infty e^{-z^2} z^{n-1} \sin \left[ \log \left( \log \left( \frac{\sqrt{4t} z + 2}{\sqrt{4t}} \right) \right) \right] dz + \frac{\beta + \alpha}{2}, \tag{70}
\]
where one can write
\[
\sin \left[ \log \left( \log \left( \frac{\sqrt{4t} z + 2}{\sqrt{4t}} \right) \right) \right] = \sin \left[ \log \left( \log \frac{\sqrt{4t}}{4t} \right) + g(t, z) \right],
\]
with
\[
g(t, z) = \log \left( 1 + \frac{1}{\log \sqrt{4t}} \log \left( z + \frac{2}{\sqrt{4t}} \right) \right), \quad t, z \in (0, \infty).
\]
We see that \( \lim_{t \to \infty} g(t, z) = 0 \) for fixed \( z \in (0, \infty) \). Hence the Lebesgue Dominated Convergence Theorem implies

\[
\lim_{t \to \infty} \left| u(0, t) - \left[ \frac{\beta - \alpha}{2} \sin \left( \log \left( \log \sqrt{4t} \right) \right) + \frac{\beta + \alpha}{2} \right] \right| = 0
\]  

(71)

and so

\[
\liminf_{t \to \infty} u(0, t) = \alpha, \quad \limsup_{t \to \infty} u(0, t) = \beta.
\]

(72)

(4). For \( r = \alpha = \beta < s \), we choose \( \varphi \in C^0[0, \infty) \cap L^\infty[0, \infty) \) with \( \liminf_{\tau \to \infty} \varphi(\tau) = r = \alpha \), \( \limsup_{\tau \to \infty} \varphi(\tau) = s \), and for most \( \tau \in [0, \infty) \) the value of \( \varphi(\tau) \) is equal to \( \alpha \) and for \( \varphi(\tau) \) not equal to \( \alpha \), it looks like a thin bump with height \( s - \alpha \) and the supports of these bumps are spreading further and further apart. By this choice of \( \varphi \) we have

\[
\lim_{\tau \to \infty} H(\tau) = \lim_{\tau \to \infty} \frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr = \alpha,
\]

which implies \( \lim_{t \to \infty} u(0, t) = \alpha \) due to Theorem [1]. Hence, the case \( r = \alpha = \beta < s \) is achieved.

(6). For \( r < \alpha = \beta = s \), the construction of \( \varphi \) is similar to that in (4) except that we reverse the role of \( r \) and \( s \).

(1). For \( r = \alpha < \beta < s \), we first choose \( \varphi_1(\tau) \) to be the function given by (68), which, together with its corresponding \( u_1(0, t) \), will satisfy (69) and (72). Next, we choose \( \varphi_2(\tau) \). Let \( \varepsilon > 0 \) be a fixed small number and for each \( m \in \mathbb{N} \), let

\[
\tau_m = \exp \left( \exp \left( 2m\pi + \frac{\pi}{2} \right) \right) - 2, \quad \tilde{\tau}_m = \exp \left( \exp \left( 2m\pi + \frac{3\pi}{2} \right) \right) - 2,
\]

where we note that \( \varphi_1(\tilde{\tau}_m) = \alpha \), \( \varphi_1(\tau_m) = \beta \). We require \( \varphi_2(\tau) \) to be a nonnegative function satisfying

\[
\varphi_2(\tau) = \begin{cases} 
\frac{\beta - \alpha}{\varepsilon} (\tau - (\tau_m - \varepsilon)), & \tau \in [\tau_m - \varepsilon, \tau_m] \\
-\frac{\beta - \alpha}{\varepsilon} (\tau - (\tau_m + \varepsilon)), & \tau \in [\tau_m, \tau_m + \varepsilon] \\
0, & \text{otherwise}, \quad \tau \in [0, \infty).
\end{cases}
\]

It satisfies \( \liminf_{\tau \to \infty} \varphi_2(\tau) = 0 \) and \( \limsup_{\tau \to \infty} \varphi_2(\tau) = s - \beta \) and

\[
\lim_{\tau \to \infty} H_2(\tau) = \lim_{\tau \to \infty} \frac{n}{\tau^n} \int_0^\tau \varphi_2(r) r^{n-1} dr = 0.
\]

Hence its corresponding \( u_2(0, t) \) satisfies \( \lim_{t \to \infty} u_2(0, t) = 0 \) due to Theorem [1]. Now set \( \varphi = \varphi_1 + \varphi_2 \in C^0[0, \infty) \cap L^\infty[0, \infty) \). We will have \( u(0, t) = u_1(0, t) + u_2(0, t) \) with

\[
\begin{cases} 
\liminf_{t \to \infty} u(0, t) = \liminf_{t \to \infty} u_1(0, t) = \alpha \\
\limsup_{t \to \infty} u(0, t) = \limsup_{t \to \infty} u_1(0, t) = \beta.
\end{cases}
\]

Also, by \( \varphi(\tilde{\tau}_m) = \varphi_1(\tilde{\tau}_m) + \varphi_2(\tilde{\tau}_m) = \alpha + 0 = \alpha \) and \( \varphi(\tau_m) = \varphi_1(\tau_m) + \varphi_2(\tau_m) = \beta + (s - \beta) = s \), we have

\[
\begin{cases} 
\liminf_{\tau \to \infty} \varphi(\tau) = \liminf_{\tau \to \infty} (\varphi_1(\tau) + \varphi_2(\tau)) = \alpha \\
\limsup_{\tau \to \infty} \varphi(\tau) = \limsup_{\tau \to \infty} (\varphi_1(\tau) + \varphi_2(\tau)) = \beta + (s - \beta) = s.
\end{cases}
\]

(3) For \( r < \alpha < \beta = s \), the construction of \( \varphi \) is similar to that in (1).

The proof of Theorem [20] is now complete. \( \square \)
3 Some side issues.

3.1 An example of prescribing four different numbers not satisfying (13) in Theorem 5.

This is related to Theorem 4. Until now, we still do not know how to prescribe the oscillation of $H(\tau)$ and $u(0, t)$ for four arbitrary numbers $p < \alpha < \beta < q$ not satisfying the condition (13). This will be an interesting problem to explore. However, it is not difficult to construct a particular example not satisfying (13). We have:

Lemma 22 There exists a radial function $\varphi \in C^0[0, \infty) \cap L^\infty[0, \infty)$ so that its average integral $H(\tau)$ and the convolution solution (3) of the problem (1) satisfy $p < \alpha < \beta < q$ with $p + q \neq \alpha + \beta$.

Proof. The idea of breaking the symmetry, unlike (23), is to choose $H(\tau)$ to be equal to the sum of several (at least two) different slow oscillation functions. For simplicity, we only look at the case $n = 1$ (the construction for $n > 1$ is similar, but we cannot find numerical values involving a general variable $n$) and require $H(\tau)$ to be equal to the sum of two slow oscillation functions, given by

$$H(\tau) = \frac{n}{\tau^n} \int_0^\tau \varphi(r) r^{n-1} dr = \frac{1}{\tau} \int_0^\tau \varphi(r) dr \quad (n = 1)$$

$$= \sin(\log(\tau + 1)) + \sin(2 \log(\tau + 1)), \quad \tau \in (0, \infty), \quad H(0) = 0. \quad (73)$$

Similar to (24), one can find the corresponding initial radial function $\varphi \in C^0[0, \infty) \cap L^\infty[0, \infty)$. With the help of Maple software, we can evaluate

$$\left\{ \begin{array}{l}
p = \lim \inf_{\tau \to \infty} H(\tau) = \min_{x \in [0, 2\pi]} (\sin x + \sin 2x) \approx -1.760172593 \\
q = \lim \sup_{\tau \to \infty} H(\tau) = \max_{x \in [0, 2\pi]} (\sin x + \sin 2x) \approx 1.760172593.
\end{array} \right.$$

By the representation formula $u(0, t) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-z^2} z^2 H(\sqrt{4t} z) dz, \quad t \in (0, \infty)$, we have

$$u(0, t) = \left\{ \begin{array}{l}
A(t) \sin(\log \sqrt{4t}) + B(t) \cos(\log \sqrt{4t}) \\
+ C(t) \sin(2 \log \sqrt{4t}) + D(t) \cos(2 \log \sqrt{4t}), \quad t \in (0, \infty),
\end{array} \right.$$

where

$$\left\{ \begin{array}{l}
A(t) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-z^2} z^2 \cos\left(\log\left(z + \frac{1}{\sqrt{4t}}\right)\right) dz \\
B(t) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-z^2} z^2 \sin\left(\log\left(z + \frac{1}{\sqrt{4t}}\right)\right) dz \\
C(t) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-z^2} z^2 \sin\left(2 \log\left(z + \frac{1}{\sqrt{4t}}\right)\right) dz \\
D(t) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-z^2} z^2 \cos\left(2 \log\left(z + \frac{1}{\sqrt{4t}}\right)\right) dz, \quad t \in (0, \infty).
\end{array} \right.$$

As $t \to \infty$, we have

$$\left\{ \begin{array}{l}
\lim_{t \to \infty} A(t) = A \approx 0.892253317, \quad \lim_{t \to \infty} B(t) = B \approx 0.030945895 \\
\lim_{t \to \infty} C(t) = C \approx 0.649173672, \quad \lim_{t \to \infty} D(t) = D \approx 0.099535090,
\end{array} \right.$$

and so

$$\lim_{t \to \infty} \left| u(0, t) - \left[ A \sin(\log \sqrt{4t}) + B \cos(\log \sqrt{4t}) + C \sin(2 \log \sqrt{4t}) + D \cos(2 \log \sqrt{4t}) \right] \right| = 0,$$
which gives

\[
\begin{align*}
\alpha &= \lim_{t \to \infty} \inf_{x \in [0,2\pi]} u(x,0) = \min_{x \in [0,2\pi]} (A \sin x + B \cos x + C \sin 2x + D \cos 2x) \approx -1.369211837 \\
\beta &= \lim_{t \to \infty} \sup_{x \in [0,2\pi]} u(x,0) = \max_{x \in [0,2\pi]} (A \sin x + B \cos x + C \sin 2x + D \cos 2x) \approx 1.328017886
\end{align*}
\]

and we find that \( p + q \neq \alpha + \beta \). \( \square \)

**Remark 23** We have used different software to evaluate the above numerical values of \( p, \alpha, \beta, q \) and obtain the same values.

### 3.2 The behavior of \( u(x,t) \) as \( |x| \to \infty \).

In Theorem 5 and Theorem 8 we only look at the behavior of \( u \) for fixed \( x \) with \( t \to \infty \). We may as well look at the behavior of \( u(x,t) \) for fixed \( t \) with \( |x| \to \infty \). In the following lemma, we discuss such property for a particular radial initial data. We choose the function in (24) with \( m \) to be the function in (24) with \( \beta \).

**Lemma 24** Let \( u(x,t) \) be the solution of the problem (1) given by (2) with \( u(x,0) = \varphi(|x|) \), \( x \in \mathbb{R}^n \), where \( \varphi(\tau) \) is given by

\[
\varphi(\tau) = \sin(\log(\tau + 1)) + \frac{\tau}{n(\tau + 1)} \cos(\log(\tau + 1)), \quad \tau \in [0,\infty).
\]

Then for each fixed \( t > 0 \) we have

\[
\lim_{|x| \to \infty} \left| u(x,t) - \left( \sin(\log |x|) + \frac{1}{n} \cos(\log |x|) \right) \right| = 0.
\]

**Remark 25** Note that for fixed \( t > 0 \), we have

\[
\bar{\alpha} := \lim_{|x| \to \infty} \inf_{\tau \to \infty} u(x,t) = \lim_{\tau \to \infty} \inf \varphi(\tau) = r = -\sqrt{1 + \frac{1}{n^2}} < \lim_{\tau \to \infty} H(\tau) = p = -1
\]

and

\[
\bar{\beta} := \lim_{|x| \to \infty} \sup_{\tau \to \infty} u(x,t) = \lim_{\tau \to \infty} \sup \varphi(\tau) = s = \sqrt{1 + \frac{1}{n^2}} > \lim_{\tau \to \infty} H(\tau) = q = 1.
\]

Thus the space oscillation of \( u(x,t) \) is, in general, not necessarily bounded by the oscillation of \( H(\tau) \) (however, it is always bounded by the oscillation of \( \varphi(\tau) \) due to the maximum principle). This is different from the behavior for time oscillation, where we always have \( r \leq p \leq \alpha \leq \beta \leq q \leq s \).

**Proof.** By Remark 7, we know that \( u(x,t) \) is radial in \( x \in \mathbb{R}^n \) for each \( t > 0 \). Therefore, it suffices to look at the oscillation behavior of \( u(x,t) \) for \( |x| \to \infty \). The representation formulas (12) and (13) cannot be used here since they are valid only for \( u(0,t) \). Instead, we use the convolution formula (2) to get

\[
u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} \varphi(|y|) dy = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} \varphi\left(|x + \sqrt{4t}z|\right) dz,
\]

where by (74) we have

\[
\varphi\left(|x + \sqrt{4t}z|\right) = \left\{ \begin{array}{l}
\sin(\log(\sqrt{|x + \sqrt{4t}z|} + 1)) \\
+ \frac{|x + \sqrt{4t}z|}{n(|x + \sqrt{4t}z| + 1)} \cos(\log(\sqrt{|x + \sqrt{4t}z|} + 1)) \end{array} \right\}.
\]
By the Lebesgue Dominated Convergence Theorem and the identity
\[
\sin\left(\log\left(|x + \sqrt{4tz}| + 1\right)\right) = \begin{cases} 
\sin (\log |x|) \cos \left(\log \left(\frac{|x + \sqrt{4tz}| + 1}{|x|}\right)\right), \\
+ \cos (\log |x|) \sin \left(\log \left(\frac{|x + \sqrt{4tz}| + 1}{|x|}\right)\right),
\end{cases}
\]

where
\[
\lim_{|x| \to \infty} \frac{|x + \sqrt{4tz}| + 1}{|x|} = 1, \quad \text{for fixed } t \text{ and } z,
\]
we have
\[
\lim_{|x| \to \infty} \left| \sin \left(\log \left(|x + \sqrt{4tz}| + 1\right)\right) - \sin (\log |x|) \right| = 0
\]
and so
\[
\lim_{|x| \to \infty} \left| \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} \sin \left(\log \left(|x + \sqrt{4tz}| + 1\right)\right) dz - \sin (\log |x|) \right| = 0. \quad (77)
\]

Similarly, we have
\[
\lim_{|x| \to \infty} \left| \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} \left(\frac{|x + \sqrt{4tz}|}{n(|x + \sqrt{4tz}| + 1)} \cos \left(\log \left(\frac{|x + \sqrt{4tz}|}{n(|x + \sqrt{4tz}| + 1)}\right)\right)\right) dz - \frac{1}{n} \cos (\log |x|) \right| = 0 \quad (78)
\]
and (75) follows. The proof is done. \qed

Acknowledgement. Research supported by NCTS (National Center for Theoretical Sciences) and MoST (Ministry of Science and Technology) of Taiwan with grant number 108-2115-M-007-013-MY2.

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