On quasi-integrable deformation scheme of the KdV system

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We propose a general approach to quasi-deform the Korteweg–De Vries (KdV) equation by deforming its Hamiltonian. The standard abelianization process based on the inherent sl(2) loop algebra leads to an infinite number of anomalous conservation laws, that yield conserved charges for definite space-time parity of the solution. Judicious choice of the deformed Hamiltonian yields an integrable system with scaled parameters as well as a hierarchy of deformed systems, some of which possibly are quasi-integrable. One such system maps to the known quasi-deformed nonlinear Schrödinger (NLS) soliton in the already known weak-coupling limit, whereas a generic scaling of the KdV amplitude \( u \to u^{1+\epsilon} \) also suggests quasi-integrability under an order-by-order expansion. In general, these deformed KdV solutions need to be parity-even for quasi-conservation that agrees with our analytical results.

Following the recent demonstration of quasi-integrability in regularized long wave (RLW) and modified regularized long wave (mRLW) systems by ter Braak et al. (Nucl Phys B 939:49–94, 2019), that are also suggests quasi-integrability under an order-by-order expansion. In general, these deformed KdV solutions need to be parity-even for quasi-conservation that agrees with our analytical results.

The dynamics of NLS solitons has been well studied including various deformations and so are the undeformed KdV counterparts. However, detailed studies of localized structures in deformed KdV systems are sparse, which could be due to its higher-order nature. Integrability may be difficult for deformations of such systems as for them the balance between dispersion and nonlinearity is more delicate. Subsequently, it becomes very difficult to obtain localized solutions for these deformed systems.

A particular motivation behind studying deformed integrable models with local solution is that real physical systems are characterized by a finite number of parameters and local deformities, prohibiting integrability in terms of the corresponding undeformed field-theoretical models. Yet these physical systems are known to admit localized excitations similar to those seen in the theoretical models. For example, some include ones that resemble particular deformations of the sine-Gordon (SG) equation. This motivates the study of continuous systems as slightly deformed integrable models with asymptotic conservation. The SG model was deformed in such a way that it supports the conservation of only a subset of the charges while the others obey an anomalous zero-curvature condition. Such deformations were obtained for the supersymmetric extension of the SG system and also for the NLS and the AB systems. Single and multisoliton solutions were numerically obtained for these deformations. As a result, these localized solutions appear to be individual solitons only when they are far away from mimicking an integrable system, given that particular symmetry conditions are satisfied by the system. Such behavior is more similar to real systems, as the effect of local physical deformities can be expected to become prominent only at a close range.

Keywords: KdV equation, NLS equation, Integrability, Quasi-integrable deformation

The 1 + 1-dimensional Korteweg–de Vries (KdV) equation is widely applicable to real-life phenomena. Despite having a dispersion odd in momentum powers, this nonlinear partial differential equation is completely integrable and admits localized soliton solutions. Solitons are well known in other nonlinear systems in 1 + 1-dimensions such as the nonlinear Schrödinger (NLS) equation. NLS solitons have been observed in Bose–Einstein condensates, cold atoms, and optics. Unlike the KdV case, the quadratic dispersion of the NLS system is more suitable for a physical visualization. The KdV equation is geometrically connected to the group of orientation-preserving diffeomorphism through its connection to the Euler equation, whereas the NLS system is related to the loop algebra. Moreover the usual Lax representation of the KdV system involves second and third-order monic differential operators \((L, A)\), whereas those for the NLS system are \(2 \times 2\) matrices. However, in a suitable weak-coupling limit, their respective solutions map to each other, including solitons.

The SG model was deformed in such a way that it supports the conservation of only a subset of the charges while the others obey an anomalous zero-curvature condition. Such deformations were obtained for the supersymmetric extension of the SG system and also for the NLS and the AB systems. Single and multisoliton solutions were numerically obtained for these deformations. As a result, these localized solutions appear to be individual solitons only when they are far away from mimicking an integrable system, given that particular symmetry conditions are satisfied by the system. Such behavior is more similar to real systems, as the effect of local physical deformities can be expected to become prominent only at a close range.

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Particularly for the NLS system, Ferreira et al. modified the corresponding potential \( V(\varphi) = |\varphi|^2 \rightarrow (|\varphi|^2)^{2+\varepsilon} \), where \( \varepsilon \) is the deformation parameter\(^{19}\). As a result, an infinite number of quasi-conserved (anomalous) charges were obtained that are conserved asymptotically corresponding to numerically calculated solitonic solutions \( \varphi \). The corresponding anomaly function responsible for the deformed curvature should have definite parity properties for the asymptotic conservation of the anomalous charges. Given its close relation to the NLS system\(^{9,10}\), the KdV equation is also expected to display such quasi-integrability. However, this third-order equation in space eludes a direct dynamical interpretation in the usual classical sense, and there are no ‘potential’ analogs here. Also, the usual KdV Lax pair is constituted by monic differential operators. All these factors complicate the quasi-deformation of the KdV system as this formalism utilizes the inherent SU(2) structure of the Lax pair to implement the effect of the modified potential term in the abelianization process through a \( sl(2) \) loop algebra\(^{14-16,18}\). However, there is a \( SU(2) \) representation of the KdV Lax pair\(^{20}\) with the appropriate grade structure\(^{21}\) for the abelianization procedure\(^{14,15}\) required for quasi-integrability. Yet, a general structure for the quasi-integrable deformation of the KdV system has not been obtained till now.

Recently, some particular non-integrable deformations of the KdV equation, namely the regularized long-wave (RLW) and the modified regularized long-wave (mRLW) equations, were shown to be quasi-integrable in an excellent work\(^{21}\). Scattering up to three solitons was numerically obtained accompanied by analytical evaluation of single solitons for particular cases, admitted by anomalous charges that recover their initial configurations after scattering. The present work, on the other hand, seeks to quasi-deform the integrable KdV system in a direct and general way in the likeness of sine-Gordon\(^{15,16}\) or NLS\(^{9,25}\) ones. We achieve this through a \( sl(2) \) loop algebraic abelianization of the KdV system starting from the deformation of the corresponding Hamiltonian that eventually leads to anomalous charges. This loop algebra further yields a crucial parity structure of the quasi-deformation anomalies required for asymptotic conservation of the charges. Finally, a few deformed solutions that suggest quasi-integrability and corresponding anomalies are analyzed. The particular cases of RLW and mRLW systems are known to support similar deformed solutions\(^{22}\) and multi-solitonic structures were numerically obtained in conformity. However, pure analytic determination (e.g. by Hirota method) of definite-parity localized solutions of these systems is still limited to a particular parameter range\(^{22}\). The perturbative approach in the present work supplies some insight into the possible general quasi-deformed KdV solutions that are found to be localized. The present formalism is also distinct from the recent Riccati-type pseudo-potential technique used to obtain infinite towers of anomalous charges for the RLW, mRLW\(^{23}\) and potential KdV\(^{24}\) models, some of which persist even for the undeformed KdV system.

Being a third-order differential system, the KdV system needs an off-shell quasi-deformation scheme, further allowing a hierarchy of higher-derivative extensions of KdV. At the simplest level, such deformations amount to the scaling of KdV parameters and thus retain integrability. Given the weak coupling relation between KdV and NLS solutions\(^{9,10}\), we could link this quasi-KdV system to the known results for quasi-NLS system\(^{16}\). Further, the quasi-deformed KdV system is compared to the non-holonomically deformed version\(^{20}\) of the KdV equation, with the latter retaining integrability. Given the known correspondence between these two deformations of the NLS system\(^{22}\), we expect a similar correspondence in the case of the KdV system.

In the following, section “Quasi-integrable deformation of KdV equation” provides a detailed loop algebraic structure of the general quasi-deformed KdV system. In section “General quasi-integrability of the KdV system and possible deformations” we obtain a general analysis of the algebraic structure of this deformation procedure, ensuring quasi-conservation of the charges for localized solutions. Some detailed results in the perturbative limit are obtained in section “Perturbative QI deformation: the NLS analogy” with particular examples. Section “Conclusions and discussion” concludes this work following discussions that highlight the remaining issues and further possibilities.

### Quasi-integrable deformation of KdV equation

**Zakharov–Shabat representation**

A systematic procedure for obtaining completely integrable finite-dimensional systems is the Adler, Kostant, and Symes (AKS) theorem\(^ {26,27} \) that applies to some Lie algebra \( \mathfrak{g} \) equipped with an ad-invariant non-degenerate bilinear form. When it is applied to loop algebra and the Fordy–Kulish decomposition scheme is invoked, the NLS\(^ {28} \) and KdV\(^ {29} \) equations can be derived. This mechanism can further construct nonlinear hierarchies\(^ {30} \). In the case of the KdV system, the most general construction coming out of the AKS procedure is a pair of coupled complex KdV equations\(^ {30} \). They originate from the Lax pair:

\[
A := Q \quad \text{and} \quad B := T + [S, Q],
\]

where,

\[
T := -Q_{xx} + [Q^+, [Q^-, Q^+]] - [Q^-, [Q^-, Q^+]] \quad \text{and} \quad S := Q_x^+ + Q_x^- + 4\epsilon (Q_x^+ + Q_x^-), \quad \epsilon \in \mathbb{R}.
\]

The involved parameters are defined as follows,

\[
Q := \begin{pmatrix} 0 & \bar{q} \\ -\bar{q} & 0 \end{pmatrix}, \quad Q^+ := \begin{pmatrix} 0 & \bar{q} \\ 0 & 0 \end{pmatrix}, \quad Q^- := \begin{pmatrix} 0 & 0 \\ -\bar{q} & 0 \end{pmatrix},
\]

\[
Q = Q^+ + Q^- \equiv \bar{q} s_+ - q s_-.
\]
where, the Pauli matrices generate the Lie algebra isomorphic to $su(2)$:

$$[\sigma_+,\sigma_-] = \sigma_3 \quad \text{and} \quad [\sigma_3,\sigma_{\pm}] = \pm 2\sigma_{\pm},$$

and $q(x, t)$, $\bar{q}(x, t)$ are mutually conjugate complex amplitudes that lead to the coupled KdV equations. An $s(2)$-loop algebra can be constructed on this canonical basis of $su(2)$, which in turn enables a complete gauge-group interpretation of this system.

Incorporating the generic representation of Eq. (3), the Lax pair takes the form

$$A = \bar{q}\sigma_+ - q\sigma_- \quad \text{and} \quad B = (\bar{q}q_x - q\bar{q}_x)\sigma_3 - (\bar{q}q_x + 2qq_x^2)\sigma_+ + (q_{xx} + 2q^2)\sigma_-,$$  

with the corresponding curvature

$$F_{tx} = (\bar{q}_t + \bar{q}q_{xx} + 6qq\bar{q}_x)\sigma_+ - (q_t + q_{xx} + 6q\bar{q}q_x)\sigma_-.$$  

The zero curvature condition then implies two coupled KdV-like equations

$$\bar{q}_t + \bar{q}q_{xx} + 6qq\bar{q}_x = 0 \quad \text{and} \quad q_t + q_{xx} + 6q\bar{q}q_x = 0,$$  

for each linearly independent constituent of the curvature matrix. Although the above equations possess a nonlinearity of higher order than the usual KdV system, a straight-forward choice of variables

$$\bar{q}(q) = 1 \quad \text{and} \quad q(\bar{q}) = u, \quad u \in \mathbb{R},$$  

immediately leads to the uncoupled (usual) KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$  

The outcome of the other possibility, $\bar{q}(q) = -u$, $q(\bar{q}) = 1$, can be transformed into the usual KdV equation through the transformation $u \rightarrow -u$. With the choice in Eq. (8) implemented, the Lax pair corresponding to Eq. (7) is

$$A = \sigma_+ - u\sigma_- \quad \text{and} \quad B = u_x\sigma_3 - 2u\sigma_+ + \left(u_{xx} + 2u^3\right)\sigma_-,$$  

which leads to the KdV equation through the on-shell vanishing of the corresponding curvature

$$F_{tx} := A_t - B_x + [A, B] \equiv - (u_t + 6uu_x + u_{xxx})\sigma_-.$$  

This algebraic structure stemming from the $su(2)$ representation allows the construction of $s(2)$ loop algebra necessary for the abelianization procedure of quasi-integrability. This would not have been possible with the more common monic Lax pair $A = \partial_x + \partial_x^3 + u$ and $B = -4\partial_x^2 - 6u\partial_x - 3u_x$ for the KdV equation. In the following, we explain the abelianization procedure in detail.

### Quasi-integrable deformation

As mentioned previously, the application of the quasi-integrability mechanism of Eq. (15–18) to the KdV system essentially requires the notion of potential, which is not straight-forward given the third-order dispersion of the KdV system. However, a well-known Hamiltonian formulation exists wherein the KdV system of Eq. (9) emerges from two different equivalent Hamiltonians. Given the order of nonlinearity appearing in the Lax pair from Eq. (10), we choose the following Hamiltonian:

$$H_1[u] = \int_{-\infty}^{\infty} dx \left( \frac{1}{2}u_x^2 - u^3 \right) \quad \text{with} \quad \frac{\delta H_1[u]}{\delta u(x)} = -3u^2 - u_{xx}.$$  

The temporal Lax component $B$ then can be re-expressed as

$$B \equiv u_x\sigma_3 - 2u\sigma_+ + \left[u_{xx} - \frac{2}{3}\left(\frac{\delta H_1[u]}{\delta u(x)} + u_{xx}\right)\right]\sigma_-,$$  

which is a general expression to implement any possible deformation at the Hamiltonian level. We impose the deformation in the nonlinear part of Hamiltonian to achieve quasi-integrability, the explicit form of which will be discussed below. Subsequently, the curvature takes the form

$$F_{tx} \equiv \left[u_t + u_{xxx} - \frac{2}{3}\partial_x \left(\frac{\delta H_1[u]}{\delta u(x)} + u_{xx}\right) + 2uu_x\right]\sigma_+ + \mathcal{F}\sigma_3.$$
with the supposed anomaly term

\[ \mathcal{A} = 2u^2 + \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \]  

(15)

that vanishes for undeformed system. Alternatively, one can very well work with the other KdV Hamiltonian:\[ H_2[u] = -\int_{-\infty}^{\infty} dx \left( \frac{1}{2} u^2(x) \right), \] with the alternative fundamental bracket defined as \[ \{u(x), u(y)\} = \left[ \partial_x^2 + 2(u_x + u\partial_x) \right] \delta(x - y). \] Then, the time component of the Lax pair will take the form: \[ B \equiv -u_x \sigma_3 - \left[ u_{xx} - 4 \frac{\delta H_1[u]}{\delta u} \right] \sigma_+ + 2u \sigma_. \] The rest will follow with the replacement:

\[ \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \rightarrow 4 \frac{\delta H_1[u]}{\delta x}. \]

In the presence of the anomaly \( \mathcal{A} \) the deformed KdV equation

\[ u_t + u_{xxx} - \frac{2}{3} \partial_x \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) + 2uu_x = 0 \]  

or

\[ u_t + 6uu_x + u_{xxx} = \mathcal{A} \]  

(16)

corresponds to a non-zero curvature. One can construct an infinite number of quasi-conserved charges through the abelianization procedure\(^{15,16,18}\) by gauge-transforming the deformed Lax components as

\[ (A, B) \rightarrow U(A, B) U^{-1} + U(x, t) U^{-1} \implies F_{tx} \rightarrow U F_{tx} U^{-1}, \]  

(17)

where \( U \) is the element of the corresponding gauge group. In doing so, the anomaly \( \mathcal{A} \) prevents the rotation of both of them into the same infinite-dimensional abelian subalgebra of the loop \( sl(2) \), resulting in an infinite set of quasi-conservation laws characterized by \( \mathcal{A} \).

**The \( sl(2) \) loop algebra**

Based on the \( su(2) \) algebraic structure an \( sl(2) \) loop algebra can be constructed for the KdV system. This algebra is spanned by the generators \( F^m, F^n \), \( m, n \in \mathbb{Z} \), that satisfy the following commutation relations:

\[ [F^m, F^n] = 2F^{m+n}, \quad [F^m, F^n] = F^{m+n+1}. \]  

(18)

This loop algebra is consistent with the definitions,

\[ F^m = \lambda^m \sigma_3, \quad F^n = \frac{\lambda^n}{\sqrt{2}} (\sigma_+ - \lambda \sigma_-) \quad \text{and} \quad F^n = \frac{\lambda^n}{\sqrt{2}} (\sigma_+ + \lambda \sigma_-), \]  

(19)

with \( \lambda \) being the spectral parameter. Such a structure is essentially the same as that for the quasi-NLS system\(^{18}\), suggesting a strong connection between the quasi-deformations of the two systems, which we will address in section "Perturbative QI deformation: the NLS analogy".

**The gauge transformation**

Following the quasi-deformation in Eq. (13) the Lax pair, however, is not suitable for the abelianization (a version of the Drinfeld–Sokolov reduction) as the spatial component \( A \) does not contain a constant semi-simple element of the \( sl(2) \) algebra that split the algebra into the corresponding Kernel and Image subspaces. In other words, this mandates the presence of the spectral parameter \( \lambda \) in the Lax pair in a particular way. A Lax pair that satisfies this algebraic requirement and also leads to the quasi-KdV equation is\(^{31}\)

\[ \tilde{A} = \sigma_+ - (u - \lambda) \sigma_- \quad \text{and} \quad \tilde{B} = u_x \sigma_3 - (2u + 4\lambda) \sigma_+ + \left\{ u_{xx} - \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u} + u_{xx} \right) + 2\lambda u - 4\lambda^2 \right\} \sigma_-, \]  

(20)

which we will adopt for the remaining analysis. The spatial Lax operator is free from the quasi-deformation, a crucial property exploited by the abelianization procedure to obtain the general form of the quasi-conserved charges. The undeformed version of the above Lax pair can be obtained from the one previously suggested in Eq. (10) through a gauge transformation corresponding to a unitary operator (It might not be the most general gauge transformation that leads to the desired Lax pair. Further, the time dependence of \( a_\pm \) may include some non-trivial extensions. As the particular form in \( \tilde{A} \) is needed, the corresponding undeformed \( \tilde{B} \) may suitably be constructed through term-by-term compensation starting with \( B \)
\[ G = \exp (a_+ \sigma_+ + a_- \sigma_-); \]
\[ a_+ = \frac{\lambda}{\sqrt{\lambda - 2u}} \int \frac{dx}{\sqrt{\lambda - 2u}}, \quad a_- = \frac{\lambda}{2} \sqrt{\lambda - 2u} \int \frac{dx}{\sqrt{\lambda - 2u}}. \]  

In terms of the \(sl(2)\) generators, the new Lax operators take the forms

\[ \bar{A} = \sqrt{2} F_0^1 - \frac{u}{\sqrt{2}} (F_+^{-1} - F_-^{-1}) \quad \text{and} \]
\[ \bar{B} = -4\sqrt{2} F_1^1 + u_4 F_0^0 - 2\sqrt{2} u F_0^0 + \frac{f(u)}{\sqrt{2}} (F_+^{-1} - F_-^{-1}), \]  

where \( f(u) = \left( \frac{u_{xx}}{3} - \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u} \right) + u_{xx} \right), \)

whose particular versions for RLW and mRLW systems were obtained by Braak et al.\textsuperscript{22} subjected to the corresponding form of \( H_1[u] \). The above structure incorporates all possibilities of quasi-deformation of the KdV equations and therefore should correspond to multiple quasi-KdV systems.

Following the general approach for abelianization\textsuperscript{15,16,18} by gauge rotation of the spatial Lax operator exclusively to the image of \(sl(2)\), we apply the gauge transformation defined by

\[ g = e^G, \quad G = \sum_{n=-\infty}^{\infty} (\alpha_n F_n^0 + \beta_n F_n^0). \]  

Here the coefficient functions \( \alpha_{-n}(x, t), \beta_{-n}(x, t) \) are to be chosen such that the transformed spatial component \( \tilde{A} = g A g^{-1} + g_x g^{-1} \) depends only on \( F_n^0 \)'s:

\[ \tilde{A} = \sum_{n=0}^{\infty} \gamma_n F_n^0. \]  

On employing the Baker–Campbell–Hausdorff (BCH) formula \( e^X e^{-X} = Y + [X, Y] + (1/2!) [X, [X, Y]] + \cdots \) the new spatial component takes the general form

\[ \tilde{A} = \tilde{A} + [G, \tilde{A}] + \frac{1}{2!} [G, [G, \tilde{A}]] + \frac{1}{3!} [G, [G, [G, \tilde{A}]]] + \cdots + G_x + \frac{1}{2!} [G, G_x] + \frac{1}{3!} [G, [G, G_x]] + \cdots. \]  

The first few of the individual commutators are
\[ [G, \bar{A}] = \sum_{n=1}^{\infty} \left[ \sqrt{2} \alpha_n \left( F_{n+1}^m - \frac{u}{2} F_n^m \right) + \sqrt{2} \beta_n \left( 2 F_n^m - u F_{n+1}^m + u F_{n+1}^{m+n} \right) \right], \]
\[
\frac{1}{2!} [G, [G, \bar{A}]] = \sqrt{2} \sum_{m,n=1}^{\infty} \left[ \left( \frac{u}{2} \alpha_m \alpha_n + 2 \beta_m \beta_n \right) F_{n+1}^m + \frac{u}{2} \alpha_m \beta_n F_{n+1}^m - \alpha_m \alpha_n F_{n+1}^m + u \beta_m \beta_n F_{n+1}^m \right] + u \beta_m \beta_n F_{n+1}^m,
\]
\[
\frac{1}{3!} [G, [G, [G, \bar{A}]]] = \frac{\sqrt{2}}{3} \sum_{l,m,n=1}^{\infty} \left[ - \alpha_l \alpha_m \alpha_n F_{l+m+n+2}^m + \alpha_l \left( \frac{u}{2} \alpha_m \alpha_n + 2 \beta_m \beta_n \right) F_{l+m+n+1}^m \right. \\
\left. - u \alpha_l \beta_m \beta_n F_{l+m+n}^m - u \alpha_l \alpha_m \beta_n F_{l+m+n+1}^m + 2 u \beta_m \beta_n F_{l+m+n+1}^m - 2 \beta_l \alpha_m \alpha_n F_{l+m+n+1}^m \right] + \beta_l \left( \alpha_m \alpha_n + 4 \beta_m \beta_n \right) F_{l+m+n+1}^m - 2 u \beta_l \beta_m \beta_n F_{l+m+n+1}^m,
\]
\[
\frac{1}{4!} [G, [G, [G, [G, \bar{A}]]]] = \frac{1}{6 \sqrt{2}} \sum_{k,l,m,n=1}^{\infty} \left[ - \alpha_k \alpha_l \alpha_m \alpha_n F_{k+l+m+n+1}^m + 2 u \alpha_k \beta_l \beta_m \beta_n F_{k+l+m+n}^m + 2 u \alpha_k \alpha_l \alpha_m \beta_n F_{k+l+m+n+1}^m \right. \\
\left. - 2 u \alpha_k \alpha_l \alpha_m \beta_n F_{k+l+m+n+1}^m + 4 u \beta_l \beta_m \beta_n F_{k+l+m+n+1}^m \right],
\]
\[ \ldots \]

and
\[
\frac{1}{2!} [G, G_x] = \sum_{m,n=1}^{\infty} \left( \beta_m \alpha_n, x - \alpha_m \beta_n, x \right) F_{n+1}^m,
\]
\[
\frac{1}{3!} [G, [G, G_x]] = \frac{1}{3} \sum_{l,m,n=1}^{\infty} \left( \beta_m \alpha_n, x - \alpha_m \beta_n, x \right) \alpha_l F_{l+m+n+1}^m + 2 \beta_l F_{l+m+n}^m,
\]
\[
\frac{1}{4!} [G, [G, [G, G_x]]] = \frac{1}{6} \sum_{k,l,m,n=1}^{\infty} \left( \beta_m \alpha_n, x - \alpha_m \beta_n, x \right) \beta_k F_{k+l+m+n+1}^m + 2 \beta_l \beta_k F_{k+l+m+n+1}^m - \alpha_k \alpha_l F_{k+l+m+n+1}^m ,
\]
\[ \ldots \]

Since the component \( \bar{A} \) contains only the powers of \( F_n^m \), the order-by-order vanishing of coefficients of \( F^n_m \), \( F^n_m \) s leads to the expressions for the expansion coefficients of the gauge operator \( g \). Starting with the vanishing of coefficient of \( F^n_m \) (the highest order possible) gives \( \alpha_{-1} = 0 \). Using that, the vanishing of coefficients of \( F^{-1}_n \) (the highest order) yields the value of \( \beta_{-1} \) to be linear in the solution \( u \). From these results, the vanishing of the coefficient of \( F^{-1}_n \) (the next order) gives \( \alpha_{-1} \) to be linear in \( \beta_{-1} \) and thus linear in \( u \). In this way, alternating between coefficients of \( F^n_m \), \( F^n_m \) for successive orders, all the coefficient functions \( \alpha_{-n}, \beta_{-n} \) can be obtained as functions of \( u \) and its derivatives. A few of them are listed below as
\[ \mathcal{O}(F^0) : \quad \alpha_{-1} = 0, \]
\[ \mathcal{O}(F^{-1}) : \quad \alpha_{-2} = \frac{u_x}{4\sqrt{2}}, \]
\[ \mathcal{O}(F^{-2}) : \quad \alpha_{-3} = \frac{u_{xxx}}{16\sqrt{2}} + \frac{3}{8\sqrt{2}} u u_x, \]
\[ \mathcal{O}(F^{-3}) : \quad \alpha_{-4} = \frac{u_{xxxx}}{64\sqrt{2}} + \frac{5}{16\sqrt{2}} u_x u_{xx} + \frac{5}{32\sqrt{2}} u u_{xxx} + \frac{11}{24\sqrt{2}} u^2 u_x, \]
\[ \vdots \]
\[ \mathcal{O}(F_{-1}^{(-1)}) : \quad \beta_{-1} = -\frac{u}{4}, \]
\[ \mathcal{O}(F_{-2}^{(-1)}) : \quad \beta_{-2} = -\frac{u_x}{16} - \frac{u^2}{8}, \]
\[ \mathcal{O}(F_{-3}^{(-1)}) : \quad \beta_{-3} = -\frac{u_{xxx}}{64} - \frac{u u_{xx}}{8} - \frac{3}{32} (u_x)^2 - \frac{u^3}{12}, \]
\[ \mathcal{O}(F_{-4}^{(-1)}) : \quad \beta_{-4} = -\frac{u_{xxxx}}{256} - \frac{5}{64} (u_{xx})^2 - \frac{15}{128} u_x u_{xxx} - \frac{3}{64} u u_{xxx} - \frac{53}{192} u (u_x)^2 \]
\[ \quad \quad - \frac{3}{16} u^2 u_x - \frac{u^4}{16}, \]
\[ \vdots \]

These immediately lead to the rotated spatial Lax component \( \tilde{A} \) as the expansion coefficients in Eq. (24) are completely determined as,
\[ \gamma_0 = \sqrt{2}, \quad \gamma_{-1} = -\frac{u}{\sqrt{2}}, \quad \gamma_{-2} = -\frac{u^2}{4\sqrt{2}}, \quad \gamma_{-3} = \frac{1}{16\sqrt{2}} (u_x)^2 - u u_{xx} - 2u^3, \]
\[ \gamma_{-4} = -\frac{u u_{xxx}}{64\sqrt{2}} - \frac{3}{32\sqrt{2}} (u_x)^2 - \frac{u^2 u_{xx}}{8\sqrt{2}} - \frac{15}{192\sqrt{2}} u^4, \]
\[ \vdots \]
\[ (28) \]

in terms of the deformed solution \( u \).

Subsequently, the temporal Lax component \( \tilde{B} \) transforms to
\[ \tilde{B} = g \tilde{B} g^{-1} + g_t g^{-1} = \tilde{B} + [G, \tilde{B}] + \frac{1}{2!} [G, [G, \tilde{B}]] + \cdots + G_t + \frac{1}{2!} [G, G_t] + \cdots \]
\[ (29) \]

with a few of the lowest-order commutators being
\[ [G, \dot{B}] = \sum_{n=-1}^{\infty} \left[ -4\sqrt{2}a_n F^{n+2} - 2\sqrt{2}u a_n F^{n+1} + \frac{f(u)}{\sqrt{2}} a_n F^n - 2u_x a_n F^n - \sqrt{2} f(u) \beta_n F^{n+1} - 8\sqrt{2} \beta_n F^{n+1} - 4\sqrt{2}u \beta_n F^n + \sqrt{2} f(u) \beta_n F^{n+1} \right] \]

\[ \frac{1}{2!} [G, [G, \dot{B}]] = \sum_{m,n=-1}^{\infty} \left[ -u_x a_m a_n F^{m+n+1} - \frac{f(u)}{\sqrt{2}} a_m \left( \alpha_n + \frac{\beta_n}{2} \right) F^{m+n+1} + 4\sqrt{2} u a_m \beta_n F^{m+n+1} - 4\sqrt{2} u a_m \beta_n F^{m+n+1} + \sqrt{2} f(u) \beta_m \beta_n F^{m+n+1} - 2u_x a_m \beta_n F^{m+n+1} - \sqrt{2} f(u) \beta_m \beta_n F^{m+n+1} \right] \]

\[ \frac{1}{3!} [G, [G, G]] = \sum_{l,m,n=-1}^{\infty} \left[ \frac{4\sqrt{2}}{3} \alpha_l a_m a_n F^{l+m+n+3} + \frac{2\sqrt{2}}{3} \alpha_l \left( u a_m a_n - 4\beta_m \beta_n \right) F^{l+m+n+2} - \frac{4\sqrt{2}}{3} u a_l a_m a_n F^{l+m+n+1} - \frac{\sqrt{2}}{3} f(u) a_l \beta_m \beta_n F^{l+m+n+1} + \frac{2}{3} f(u) a_l \beta_m \beta_n F^{l+m+n+1} - \frac{8\sqrt{2}}{3} \beta_m \beta_n a_n F^{l+m+n+2} + \frac{4\sqrt{2}}{3} \beta_l \left( u a_m a_n - 4\beta_m \beta_n \right) F^{l+m+n+1} - \frac{8\sqrt{2}}{3} u \beta_l \beta_m \beta_n F^{l+m+n+1} + \frac{2\sqrt{2}}{3} f(u) \beta_l \beta_m \beta_n F^{l+m+n+1} \right] \]

\[ \frac{1}{2!} [G, G] = \sum_{m,n=-1}^{\infty} (\beta_m a_{n,t} - a_m a_{n,t}) F^{m+n}, \]

\[ \frac{1}{3!} [G, [G, G]] = \frac{1}{3} \sum_{l,m,n=-1}^{\infty} (\beta_m a_{n,t} - a_m a_{n,t}) \left( \alpha_l F^{l+m+n+1} + 2\beta_l F^{l+m+n} \right) \]

\[ \frac{1}{4!} [G, [G, [G, G]]] = \frac{1}{6} \sum_{k,l,m,n=-1}^{\infty} (\beta_m a_{n,t} - a_m a_{n,t}) \left( 2\beta_l \beta_k F^{l+m+n+1} - \alpha_k a_l F^{l+m+n+1} \right) \]

The rotated temporal Lax component will span both Kernel and Image of sl(2) with a general form:

\[ \dot{B} = \sum_{n=1}^{\infty} \left[ a_n F^n + b_n F^n + c_n F^n \right] \]

wherein, a few of the nontrivial expansion coefficients are

\[ a_1 = 0, \quad a_0 = 2\sqrt{2}u, \quad a_{-1} = -\frac{f(u)}{\sqrt{2}} + \frac{u_{xx}}{\sqrt{2}} + 2\sqrt{2}u^2, \]

\[ a_{-2} = -\frac{u}{2\sqrt{2}} f(u) + \frac{u_x u_t}{4\sqrt{2}} + \frac{u_{xxx}}{4\sqrt{2}} + \frac{u_{xx} u}{2\sqrt{2}} + \frac{3}{2\sqrt{2}} (u_x)^2 + \frac{13}{6\sqrt{2}} u^3, \]

\[ \ldots \]

\[ b_1 = 0, \quad b_0 = \frac{u_t}{4} \quad \frac{u_{xx} u}{4} - 2uu_x, \]

\[ b_{-1} = -\frac{u_{xx} u}{16} - \frac{u u_t}{4} + \frac{f(u)}{8} - \frac{u_{xxx}}{16} - \frac{5}{4} u u_{xx} - \frac{3}{4} u_{xx} u - \frac{21}{8} u^2 u_x, \]

\[ \ldots \]

\[ c_1 = -4\sqrt{2}, \quad c_0 = -2\sqrt{2}u, \quad c_{-1} = \frac{1}{\sqrt{2}} \left( f(u) - u^2 \right), \]

\[ c_{-2} = \frac{f(u)}{2\sqrt{2}u} - \frac{(u_x)^2}{4\sqrt{2}} - \frac{u u_{xx}}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} u^3, \]

\[ \ldots \]

These are the general versions of the coefficients of the rotated temporal components for RLW and mRLW systems\textsuperscript{22}.

On the other hand, from Eq. (22), the deformed curvature has the form:
\[ F_{tx} = - \left[ u_t + u_{tx} + 2u_{ux} - \frac{2}{3} \left( \frac{\partial H_1[u]}{\partial u} + u_{xx} \right) \right] \sigma_- - \mathcal{J} \sigma_+ \equiv - \mathcal{J} F^0, \]  

wherein the coefficient of \( \sigma_- \) vanishes on-shell. Subsequently, the gauge transformation yields the rotated curvature,

\[ F_{tx} \rightarrow F_{tx} = g F_{tx} g^{-1} = F_{tx} + [G, F_{tx}] + \frac{1}{2!} [G, [G, F_{tx}]] + \frac{1}{3!} [G, [G, [G, F_{tx}]]] + \cdots, \]

where a few of the commutators have the forms:

\[ [G, F_{tx}] = 2 \mathcal{J} \sum_{n=-\infty}^{-1} \alpha_n F^n_+, \]

\[ \frac{1}{2!} [G, [G, F_{tx}]] = \mathcal{J} \sum_{m,n=-\infty}^{-1} \left( \alpha_m \alpha_n F^{m+n+1}_+ + 2 \beta_m \alpha_n F^{m+n}_+ \right), \]

\[ \frac{1}{3!} [G, [G, [G, F_{tx}]]] = \frac{2}{3} \mathcal{J} \sum_{l,m,n=-\infty}^{-1} \left( 2 \beta_l \beta_m \alpha_n F^{l+m+n}_+ - \alpha_l \alpha_m \alpha_n F^{l+m+n+1}_+ \right), \]

\[ \frac{1}{4!} [G, [G, [G, [G, F_{tx}]]]] = \frac{2}{6} \mathcal{J} \sum_{k,l,m,n=-\infty}^{-1} \left[ 2 \alpha_k \beta_l \beta_m \alpha_n F^{k+l+m+n+1}_+ - \alpha_l \alpha_m \alpha_n F^{k+l+m+n+2}_- + 4 \beta_k \beta_l \beta_m \alpha_n F^{k+l+m+n}_- - 2 \beta_k \alpha_l \alpha_m \alpha_n F^{k+l+m+n+1}_- \right]. \]

On the other hand, as \( F_{tx} \propto F^0 \), the general form of the rotated curvature looks like

\[ F_{tx} = \mathcal{J} \sum_n (f_n F^n_+ + f^+_n F^n_+ + f^-_n F^n_-). \]

Consistency between these two expressions of the rotated curvature determines its expansion coefficients in terms of the system variables such as

\[ f_0 = -1, \quad f_{-1} = 0 = f_{-2}, \quad f_{-3} = \frac{(u_x)^2}{16}, \quad f_{-4} = \frac{u_{xxx} + 3}{32} u u_x, \quad \cdots \]

\[ f^+_1 = 0, \quad f^+_2 = \frac{u_{xxx} + 5}{32} u u_x u_x, \quad f^+_3 = \frac{u_{xxx} + 5}{32} u u_x u_x, \quad f^+_4 = \frac{u_{xxxx} + 5}{32} u u_x u_x u_x + \frac{15}{16} u^2 u_x, \quad \cdots \]

\[ f^-_2 = 0, \quad f^-_3 = \frac{u_{xxx}}{8 \sqrt{2}}, \quad f^-_4 = - \frac{u_{xxx}}{32 \sqrt{2}} - \frac{u_{xxxx}}{32 \sqrt{2}} - \frac{u_{xxx}}{4 \sqrt{2}}, \quad \cdots \]

The rotated curvature can also be obtained from the corresponding rotated Lax pair \((\tilde{A}, \tilde{B})\) as,

\[ \tilde{F}_{tx} = \tilde{A}_t - \tilde{B}_x + [\tilde{A}, \tilde{B}] = \sum_n \left( (\gamma_{n,t} - c_{n,x}) F^n_+ + (b_{n,x} - 2 \gamma_n \sum_m \alpha_m F^{m+1}_+) \right) F^n_+ - \left( a_{n,x} + 2 \gamma_n \sum_m b_m F^m_+ \right). \]

The generators \( F^n_+ \) define projection onto the image of sl(2), to which \( \tilde{A} \) was rotated exclusively. In the expression for the curvature above these generators are linear in the coefficients which will be crucial for defining quasi-conserved charges.

**Quasi-conservation**

To demonstrate the deviation from integrability, it is pertinent to evaluate quantities that would have represented conservation or have themselves been conserved for the undeformed system. The gauge transformation rotates the spatial Lax component to the abelian subalgebra spanned by \( \{ F^n_+ \}, n \in \mathbb{Z} \), which is a particular Drinfeld–Sokolov reduction. This essentially linearizes the corresponding term in the curvature \( F_{tx} \) despite the general non-commutation of the Lax components. As the undeformed curvature vanishes at each spectral order, the contribution from the linearized subalgebra yields a very simple ‘continuity’ relation: \( \gamma_{n,t} - c_{n,x} = 0 \) following Eq. (38). Subsequently, the following charges:

\[ \text{Quasi-conservation} \]
$$Q^n = \int_x \gamma_n \to \frac{dQ^n}{dt} = \int_x (\gamma_{n,t} - c_{n,x}) \equiv 0$$

(39)

are conserved given $c_{n,x}$ are asymptotically well-behaved as they depend on the local solutions $u$. For the deformed system, on comparing the expressions for the curvature in Eqs. (36) and (38), these charges are not conserved in general:

$$\frac{dQ^n}{dt} = \int_x (\gamma_{n,t} - c_{n,x}) = \int_x \mathcal{F} f_n^+ \equiv \Gamma^n.$$  

(40)

However, a subset of them can still be conserved depending on particular values of $f_n^+$, like $Q^{-1}$ for example. This particular conservation essentially implies the deformed KdV equation, as can be checked by substituting for $\gamma_{-1}$ and $c_{-1}$. It also ensures locality for $u$, as a constant $Q^{-1}$ can be related to the conserved energy. In general, for a given deformed solution $u$, several $\gamma_n$s can be well localized. The anomaly $\mathcal{F}$ itself can have a certain general symmetry that annihilates a particular subset $\{\Gamma^n\}$, as shown in the next section. For a particular kind of quasi-deformations, the anomalous charges can regain conservation for $u$ being localized yet well-separated. However, such solutions for the KdV case require heavy numerics which is beyond the scope of the present work. Instead, we focus on particular deformed Hamiltonian $(H_1[u])$ forms leading to classes of possible quasi-KdV systems.

**General quasi-integrability of the KdV system and possible deformations**

To formally explore the quasi-integrability of the KdV system the $\mathbb{Z}_2$ symmetry of the $\text{sl}(2)$ loop algebra needs to be utilized, which bears a strong resemblance with the particular quasi-KdV systems of Ref. and the quasi-NLS system. It is found that the anomaly function and the relevant expansion coefficients must possess definite space-time parity for quasi-conservation of the charges, invoking close relation to $\mathcal{P}\mathcal{F}$-symmetric systems and possible attribution to the abelianization approach itself. The $\mathbb{Z}_2$ transformation is a combination of the order 2 automorphism of $\text{sl}(2)$ loop algebra:

$$\Sigma(F^n) = F^n, \quad \Sigma(F_+^n) = -F_+^n, \quad \Sigma(F_-^n) = -F_-^n$$

(41)

and parity $\mathcal{P}$: $(x - x_0, t - t_0) \to (-x + x_0, -t + t_0)$ about a point $(x_0, t_0)$ that can very well be the origin. Since $\Sigma$ affects the generators of the algebra, while the effect of $\mathcal{P}$ is exclusive to space-time coordinates, their respective actions are mutually independent. In particular $\Omega(\tilde{A}) = -\tilde{A}$, where $\Omega = \Sigma \mathcal{P}$, for $u$ being parity-even which makes sense for solitonic structures $((x_0, t_0)$ can very well be the center of the soliton). Like the KdV equation, its quasi-modification in Eq. (16) is also parity-invariant subjected to the explicit deformation(s) to be introduced in the next section, which will amount to $\mathcal{P}\mathcal{F}$ being odd in derivatives. More intuitively, as the quasi-deformed systems are known to support solitons similar to those of the undeformed systems that are parity-even, it is sensible to expect that $u$ could be such.

The $\mathbb{Z}_2$ transformation enables asymptotic vanishing of $\Gamma^n$ in a general way, ensuring the conservation of the corresponding charges. For this purpose, the generator $F_+$ serves as the semi-simple element that splits the $\text{sl}(2)$ loop algebra $\mathfrak{g} = \text{Ker} + \text{Im}$ into kernel and image subspaces such that $[F_+, \text{Ker}] = 0$ and $[F_+, \mathfrak{g}] = 1.\text{Im}$. This has been achieved through the gauge transformation with a general form

$$\mathfrak{g} = \exp \sum_{n=-1}^{\infty} \mathfrak{G}^n.$$  

(42)

Here $\mathfrak{G}^n$ is any linear combination of the generators $F^n$, $F_-^n$ that rotates to the abelian subalgebra generated by $F_+$. The gauge-rotated spatial connection

$$\tilde{A} \to \mathcal{A} = g \tilde{A} g^{-1} + g \mathbf{g}^{-1}$$

(43)

continues to be an eigenstate of $\Omega$. To see this, considering Eq. (25), we can identify contributions to $\mathcal{A}^n = \sum_n \mathcal{A}^n$ for different powers of spectral parameter $\lambda$ as
\[ \dot{A}_0 = \bar{A}_0, \]
\[ \dot{\bar{A}}_{-1} = \left[ \bar{\mathfrak{g}}^{-1}, A_0 \right] + \bar{A}_{-1} + \mathfrak{g}^{-1}, \]
\[ \dot{\bar{A}}_{-2} = \left[ \bar{\mathfrak{g}}^{-2}, A_0 \right] + \left[ \bar{\mathfrak{g}}^{-1}, \bar{A}_{-1} \right] + \frac{1}{2!} \left[ \bar{\mathfrak{g}}^{-1}, \left[ \bar{\mathfrak{g}}^{-1}, A_0 \right] \right] + \mathfrak{g}^{-2} + \frac{1}{2!} \left[ \bar{\mathfrak{g}}^{-1}, \mathfrak{g}^{-1} \right], \]
\[ \dot{\bar{A}}_{-3} = \left[ \bar{\mathfrak{g}}^{-3}, A_0 \right] + \left[ \bar{\mathfrak{g}}^{-2}, \bar{A}_{-1} \right] + \frac{1}{2!} \left[ \bar{\mathfrak{g}}^{-2}, \left[ \bar{\mathfrak{g}}^{-1}, A_0 \right] \right] + \frac{1}{2!} \left[ \bar{\mathfrak{g}}^{-1}, [\bar{\mathfrak{g}}^{-1}, A_0] \right] + \mathfrak{g}^{-3} + \frac{1}{2!} \left[ \bar{\mathfrak{g}}^{-2}, \mathfrak{g}^{-1} \right] \] (44)
\[ \vdots \]

where \( \bar{A} = \bar{A}_0 + \bar{A}_{-1} \), \( \bar{A}_0 = \sqrt{2} F^0 \), \( \bar{A}_{-1} = -\frac{u}{\sqrt{2}} \left( F_{-1} - F_{-1} \right) . \)

Herein terms with numerical prefixes are separated according to their grades (powers of \( \lambda \)). We can immediately conclude that \( \Omega (\dot{\bar{A}}_0) = -\dot{A}_0 \). From the second equation
\[ \Omega (\dot{\bar{A}}_{-1}) = -\left( \Omega \left( \bar{\mathfrak{g}}^{-1} \right) , \bar{A}_0 \right) - \bar{A}_{-1} + \Omega \left( \mathfrak{g}^{-1} \right) , \] (45)

which when added back to the second equation yields
\[ (1 + \Omega) (\dot{\bar{A}}_{-1}) = \left[ (1 - \Omega) (\bar{\mathfrak{g}}^{-1}) , \bar{A}_0 - \partial_\mu \right] . \] (46)

In the above, the LHS is exclusively in the abelian subalgebra generated by \( F^m_n \) whereas the RHS is excluded of it. Thus both vanish identically with the RHS non-trivially leading to \( \Omega (\bar{\mathfrak{g}}^{-1}) = \mathfrak{g}^{-1} \). One can similarly proceed from the third of Eq. (44) onward to obtain \( \Omega (\mathfrak{g}^{-1}) = \mathfrak{g}^{-1} \forall n \in \mathbb{Z}_+ \), eventually leading to \( \Omega (\mathfrak{g}) = \mathfrak{g} \). This finally implies \( \Omega (\dot{\bar{A}}) = -\dot{A} \) from Eq. (43).

Therefore the \( \mathbb{Z}_2 \) automorphism is preserved for the spatial connection under the abelianising gauge transformation. One can check this explicitly from the expansion coefficients obtained for the particular case of \( \bar{A} \) in the last section, which is manifested through their parity properties as
\[ \mathcal{P} (\alpha_n) = -\alpha_n \quad \text{and} \quad \mathcal{P} (\beta_n) = \beta_n . \] (47)

Subsequently, definite parity properties are imposed on the components \( f^+_n \) in the abelian subalgebra generated by \( F^m_n \). To see this we utilize the Killing form of the \( sl(2) \) loop algebra \(^{18} \),
\[ \mathcal{R} \left( F^m_n F^m_n \right) = 2 \delta_{m+n,0} , \quad \mathcal{R} \left( F^m_n F^m_\pm \right) = 2 \delta_{m+n+1,0} , \quad \mathcal{R} \left( F^m_n F^m_\mp \right) = 0 = \mathcal{R} \left( F^m_n F^m_\pm \right) ; \]
\[ \text{wherein} \quad \mathcal{R} (\bullet) = -\frac{i}{2\pi} \oint d\lambda \lambda^{-1} \mathfrak{r} (\bullet) . \] (48)

From Eqs. (33) and (36) the expansion coefficients of \( \hat{F}_{\varepsilon s} \) in the linearized subalgebra are expressed as
\[ f^+_n \equiv \frac{1}{2} \mathcal{R} \left( -\mathfrak{g} F^m_n \mathfrak{g}^{-1} F^m_{-n-1} \right) \quad \forall n \in \mathbb{Z}_+ . \] (49)

Since the Killing form is invariant under \( \Omega \) we get
\[ \mathcal{P} \left( f^+_n \right) \equiv \Omega \left( f^+_n \right) \equiv \frac{1}{2} \mathcal{R} \left( -\Omega (\mathfrak{g}) \Omega (F^m) \Omega (\mathfrak{g}^{-1}) \Omega (F^m_{-n-1}) \right) \equiv -f^+_n . \] (50)

One can explicitly verify this from the particular expressions obtained for \( f^+_n \)’s in the previous section. Therefore
\[ Q^n (t = \tilde{t}) - Q^n (t = -\tilde{t}) = \int_{-\tilde{t}}^{\tilde{t}} \int_{-\tilde{t}}^{\tilde{t}} \mathcal{P} f^+_n \equiv 0 \] (51)

given \( \mathcal{P} \) is parity-even. Here \( \pm (\tilde{t}, \tilde{t}) \) refers to the spatiotemporal infinity where all the quasi-conserved charges should vanish. This general parity property agrees in detail with those of Ref.\(^{22} \). Therein analytical derivation and numerical evolution of one-, two- and three-soliton solutions displayed distinct parity-evenness except when they were interacting. Following the close relation between the parity property and quasi-integrability of various systems,\(^{15-19} \) the preceding treatment strengthens the validity of our general approach for a quasi-KdV system.
The quasi-NLS system has a parity-odd anomaly\(^{18}\) whereas the KdV counterpart needs a parity-even \(\mathcal{X}\) coming from judicious modification to \(H_1[u]\). Since the undeformed Hamiltonian itself is parity-even, from Eq. (15), it needs a parity-even extension. For example

\[
H_1[u] \rightarrow H_1[u] = \int_{-\infty}^{\infty} \left[ \frac{1}{2} u_x^2 - u^3 + \varepsilon F(u) \right], \quad \varepsilon \in \mathbb{N},
\]

where \(F(u) = (3/4)uu_{xx}\), leads to \(\mathcal{X} = \varepsilon u_{xx}\), which is parity-even. In particular, this ensures local conservation of \(Q^{-2}\) and asymptotic conservation of all \(Q^N\)'s given \(u\) is parity-even. To the latter end, the corresponding deformed KdV equation

\[
u_t + 6uu_x + (1 - \varepsilon)u_{xxx} = 0
\]

is essentially a scaling of the undeformed one and therefore is integrable. Hence, the proposed quasi-deformation scheme can lead to some integrable deformations along with the quasi-deformed ones. Moreover, some genuine quasi-deformations may asymptotically map to a scaled version of the integrable model instead of the exact one, or even to a non-holonomic version as observed for the NLS system\(^{25}\). This equation supports single-soliton solutions of the form

\[
u = \frac{c}{2} \text{sech}^2 \left[ \sqrt{\frac{c}{4(1 - \varepsilon)}} (x - ct + x_0 + ct_0) \right], \quad c > 0
\]

moving with speed \(c\). This is expected as the choice for \(F(u)\) is just a total derivative away from the first term in \(H_1[u]\). More importantly, however, it is an opportunity to construct a hierarchy of higher-order extensions of KdV for different choices of \(F(u)\). For demonstration, we consider the following two:

\[
F(u) = \frac{3}{2m(m-1)} u^m \quad \text{and} \quad F(u) = \frac{3}{4} uu^{(2n)},
\]

where \(m \in \mathbb{Z}\) is ordinary power and \(u^{(2n)}\) stands for the \(2n\)th derivative of \(u\) with respect to \(x\) with \(n = 1, 2, \ldots\), leading to the higher-derivative equations

\[
u_t + 6uu_x + u_{xxx} = \varepsilon u^{m-2}u_x \quad \text{and} \quad \nu_t + 6uu_x + u_{xxx} = \varepsilon u^{(2n+1)}
\]

respectively. In particular, for \(n = 1\) we obtain the system in Eq. (53). It may be worthwhile to study such KdV extensions that can admit more complicated solitonic structures than that in Eq. (54). Not all of them can be quasi-integrable; beyond the obvious parity count, such systems must possess localized solutions with proper asymptotic properties. The deformed part may not be linearly isolated in the Hamiltonian unlike the potential deformations\(^{15,16,18}\). Then to identify \(\mathcal{X}\), an order-by-order approach is necessary, although \(\varepsilon\) does not need to be small\(^{18,19,25}\). In the next section, we consider such an approach utilizing the weak coupling map to the NLS system.

**Perturbative QI deformation: the NLS analogy**

Solutions for arbitrary quasi-deformed systems like those in Eq. (56) may not always be possible. However, since \(\varepsilon\) is independent, an order-by-order expansion can yield a solution like other systems\(^{15,19}\). The correspondence between the KdV and the NLS solutions in a weak coupling limit is both encouraging and fruitful here, which appears through the following parameterization of the NLS amplitude\(^{6}\):

\[
u = \xi \left( \varphi e^{i\theta} + \varphi e^{-i\theta} \right) + \frac{k_0^2}{\varepsilon} \left( \varphi^2 e^{2i\theta} + \varphi^2 e^{-2i\theta} \right) - 2 \frac{k_0^2}{\varepsilon} |\varphi|^2,
\]

where \(\theta = k_0 x + \omega_0 t, \quad 0 < \xi \ll 1, \quad \omega_0 = k_0^3 \neq 0\).

On substituting \(\nu\) in Eq. (9) the \(\mathcal{O}(\xi^3)\) contribution with phase \(e^{\pm i\theta}\) yields the NLS equation

\[
\varphi_T + i3k_0 \varphi_{XX} + i \frac{6}{k_0^2} |\varphi|^2 \varphi = 0
\]

in terms of the new coordinates \(T = \xi^2 t\) and \(X = \xi \left( x + 3k_0^2 t \right)\). Herein, the ‘time’-derivative term follows from its third derivative, and the nonlinear term comes from the nonlinearity therein. Such approximate yet direct correspondence encourages the perturbative approach to obtain the quasi-KdV system. Noticeably, this approach introduces another expansion parameter \(\xi\) that tracks the effect of quasi-deformation in the NLS sector. The bright soliton for the quasi-NLS system has the form\(^{15}\)
that produces the undeformed counterpart for $\varepsilon \to 0$. From their plots in Fig. 1a,c, the localized nature prevails over QID. The weak-coupling map of Eq. (57) now yields a soliton train for the KdV system (Fig. 1b) that gets distorted (Fig. 1d) over QID in the NLS sector. A similar exercise can be carried out with the recently obtained quasi-NLS dark-solitons through the Riccati pseudo-potential approach\(^{31}\). Though it is not guaranteed that the weak-coupling map will persist over quasi-deformation it should be noted that the map itself is an approximation. Nevertheless, since the KdV soliton train undergoes only minor local distortions, it can strongly be expected that the quasi-KdV system can support localized solutions. This assertion is supported by various one- and multi-soliton solutions obtained for particular quasi-KdV systems\(^{22-24}\).

For the more assuring scenario of a direct map from the quasi-KdV case to a known quasi-NLS solution, we consider Eq. (56) with $m = 3$ which essentially amounts to the scaling

$$u_t + u_{xxx} + (6 - \varepsilon) uu_x = 0$$  \hspace{1cm}  \text{(60)}$$

that maintains integrability. Thus the modified NLS companion is directly obtained as \(\text{(Since the nonlinear terms of KdV and NLS systems map to each other\(^{6}\), any scaling of the same will also be mapped identically)}\)

$$\varphi_T + i3k_0\varphi_{XX} + i\left(1 - \frac{\varepsilon}{6}\right)\frac{6}{k_0}|\varphi|^2\varphi = 0.$$  \hspace{1cm}  \text{(61)}$$

It is easy to see that

$$\left(1 - \frac{\varepsilon}{6}\right)|\varphi|^2 \approx \frac{\delta}{|\varphi|^2} V(\varphi) \quad \text{with} \quad V(\varphi) \equiv \frac{1}{2}|\varphi|^4(1 - \varepsilon), \quad \tilde{\varepsilon} = \frac{\varepsilon}{6}$$  \hspace{1cm}  \text{(62)}$$

since the density $|\varphi|^2$ should be small in the weak-coupling limit on physical grounds. The deformed NLS potential $V(\varphi)$ qualifies as quasi-deformed\(^{31}\). This mapping between the two deformed systems justifies the Hamiltonian deformation approach to the quasi-KdV system. The appearance of essentially the same $\mathfrak{s}(2)$ algebra for the quasi-NLS system\(^{18}\), though the linearizing subalgebra being different, further strengthens this line of argument.

Further confirmation of this assertion comes from the solutions. The single-soliton for the scaled KdV equation has the form

$$u = \frac{c}{2}\text{sech}^2\left[\frac{1}{2}\sqrt{c} \beta (x - x_0 - \beta c(t - t_0))\right], \quad \text{where} \quad \beta = 1 - \frac{\varepsilon}{3}, \quad c > 0$$  \hspace{1cm}  \text{(63)}$$

amounting to a variable scaling. The corresponding NLS soliton for Eq. (61) is

$$\varphi(X,T) = K\text{sech} \left[\Lambda_1 K (\Lambda_2 \tilde{X} - V \tilde{T})\right] \exp \left[\frac{i}{2} \Lambda_2 V \tilde{X} + \frac{i}{4} (\Lambda_2^2 K^2 - \Lambda_1^2) \tilde{T}\right],$$  \hspace{1cm}  \text{(64)}$$

where $\Lambda_1 = i\sqrt{3k_0}$, $\Lambda_2 = -i\sqrt{3k_0}$, $K \in \mathbb{R}^+$. 

\[\textbf{Fig. 1.}\] The effect of quasi-deformation on the weak-coupling map from NLS to KdV solutions. The NLS single soliton (a) maps to a KdV soliton train (b), a property retained over the quasi-deformation (c,d). Herein, $\varepsilon = 1.5$, $V = 1 = \rho$, $X_0 = 0 = T_0 \xi = 0.1$ and $\omega_0 = 1 = k_0$. 

\[\text{(a) Undeformed NLS soliton. (b) Mapped KdV soliton train. (c) Quasi-deformed NLS soliton. (d) Mapped quasi-KdV soliton train.}\]
with similar variable scaling. It strongly suggests that a quasi-KdV system can be obtained in the usual way that works for other systems, particularly through the algebraic framework of the previous sections. The anomaly $\mathcal{X}$ and the deformed solution both being parity-even should be sufficient for this purpose.

The perturbative expansion: an example
As we have seen, choosing different forms of $F(u)$ to obtain a parity-even $\mathcal{X}$ also includes some integrable systems. For example a choice of $F(u) = \kappa_3 u^3 + \kappa_4 u^4$ results in a parity-even $\mathcal{X}$. However, it leads to Gardner or modified KdV (mKdV) equations depending on the values of $\kappa_{3,4}$ which are completely integrable. As a non-trivial example, we consider a power modification of the nonlinear term as

$$H_1^{\text{def}}[u] \equiv \int_{-\infty}^{\infty} dx \left( \frac{1}{2} u_x^2 - u^{3+3\varepsilon} \right)$$ (65)

in the spirit of the quasi-NLS system\cite{18} which affects a power-scaling of the amplitude: $u^3 \to u^{3+3\varepsilon}$. The corresponding equation

$$u_t + u_{xxx} + 6u u_x = 4uu_x - 2(1 + \varepsilon)(2 + 3\varepsilon)u^{1+3\varepsilon} u_x$$ (66)

is relatively difficult to solve. We have numerically obtained a few solutions using Mathematica for different values of $\varepsilon$ in Fig. 2a–d which represent deviations from the undeformed structure. As expected for finite $\varepsilon$ the solutions do not possess definite parity which eventually distorts the anomaly function $\mathcal{X}$ resulting in locally non-conserved charges. However, these deformed solutions are still localized, which could mean strong interactions or radiation-effected solitonic structures\cite{22}, strongly suggesting asymptotic conservation.

For a better idea we undertake an order-by-order expansion in terms of $\varepsilon$. Though $\varepsilon$ need not be small always, such an order expansion is valid given the solution analytically depends on it\cite{18}. For this particular case the anomaly

$$\mathcal{X} = 2u^2 - 2(1 + \varepsilon)u^{2+3\varepsilon} = -2\varepsilon u^2 (1 + 3 \ln(u)) + \mathcal{O}(\varepsilon^2),$$ (67)

mirrors the parity of the solution. A logarithmic nonlinearity appears at $\mathcal{O}(\varepsilon)$, making the evaluation of quasi-corrections quite difficult, especially when $u \ll 1$. For finite $u$ and $\varepsilon \ll 1$, a localized solution has the form $u_{\text{app}} = (c_0/2) \text{sech}^2 \phi$ where $\phi$ satisfies the following approximation:

$$x - c_0 t \approx -\frac{2}{\sqrt{c_0}} (1 - 2\varepsilon) \phi + \frac{2}{\sqrt{c_0}} \varepsilon \left( \ln \frac{c_0}{2} + 2 \ln \text{sech} \phi \right) \coth \phi.$$ (68)

It goes to the usual KdV bright soliton for $\varepsilon = 0$. A plot for $u_{\text{app}}$ in Fig. 3a depicts a soliton train-like structure, suggesting asymptotic conservation at the least.

We assume that the solution $u$, anomaly $\mathcal{X}$, charge $Q^n$ etc to be fairly well-behaved functions of $\varepsilon$ and thus can be expanded in a power-series. In particular, the expansion for the deformed solution is

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$ (69)

with $u_0$ being the undeformed solution. Similarly, the anomaly and the rate of change of charges can be expanded as

Fig. 2. Numerical solutions $u_d$ for the deformation $u^3 \to u^{3+3\varepsilon}$. The localized and parity-even soliton for $\varepsilon = 0$ (a) gets significantly distorted (b–d) even for small values of $\varepsilon$. They remain fairly localized suggesting possible asymptotic conservation. These plots are evaluated at time $t = 1$. 

| $\varepsilon$ | Figure |
|------------|--------|
| 0         | (a)    |
| 0.1       | (b)    |
| 0.2       | (c)    |
| 0.5       | (d)    |
contain only odd derivatives of $c$. Since it contains $\ln(u)$ in the next order contribution for $u$, the first-order deformed solution $u_1$, apart from $u_0$, is also no longer parity-even. The anomaly in Eq. (67) the $\mathcal{O}(\varepsilon)$ contribution to the deformed equation has the form:

$$u_{1,t} + u_{1,xxx} + 6 (u_0 u_1)_x = -(10 + 12 \ln u_0) u_0 u_{0,x} \hspace{1cm} (71)$$

The solution $u_1$ of this equation, with $u_0$ being the KdV soliton, is depicted in Fig. 3b. It and subsequently the first-order deformed solution $u_1 = u_0 + \varepsilon u_1$ do not have definite parity. The corresponding anomaly function

$$\mathcal{X} = \varepsilon \mathcal{X}_1 + \varepsilon^2 \mathcal{X}_2 + \cdots \quad \text{and} \quad \Gamma^n = \varepsilon \Gamma^n_1 + \varepsilon^2 \Gamma^n_2 + \cdots \hspace{1cm} (70)$$

The zero-order contributions to both quantities vanish as they correspond to the undeformed system. For the anomaly in Eq. (67) the $\mathcal{O}(\varepsilon)$ contribution to the deformed equation has the form:

$$\mathcal{X} = -2\varepsilon \{1 + 3 \ln(u_0)\} u_0^3 + 3 \varepsilon^2 \{u_0 u_1 + (2u_1 + u_0) u_0 \ln(u_0)\} + \mathcal{O}(\varepsilon^3) \hspace{1cm} (72)$$

is also no longer parity-even at $\mathcal{O}(\varepsilon^2)$. As for $\Gamma^n$, the $\mathcal{O}(\varepsilon)$ contribution depends only on $u_0$ after expanding the coefficients $f^n_n$'s in $\varepsilon$ and thus vanishes. One can trivially check this for the KdV single soliton $u_0 = (c_0/2) \sech^2 \left( \sqrt{c_0} (x - c_0 t) / 2 \right)$ as $f^n_n$ contain only odd derivatives of $u$ (Eq. (37)). As a demonstration, the next order contribution for $n = 2$ is

$$\Gamma^2_2 = \frac{1}{2\sqrt{2}} \int_x [\mathcal{X}_1 u_{1,x} + \mathcal{X}_2 u_{0,x}] \quad \text{since} \quad f^2_2 = \frac{1}{2\sqrt{2}} (u_0 + \varepsilon u_1)_x \hspace{1cm} (73)$$

where $\mathcal{X}_1$ and $\mathcal{X}_2$ can be read off of Eq. (72). Clearly, $\Gamma^2_2 \neq 0$ since it contains $u_1$ and thus $Q^{-2}$ eventually will not be conserved.

In principle these deviations from integrability can exactly be calculated for all $n$. The $\mathcal{O}(\varepsilon^n)$ contributions to $\mathcal{X}$ and the integrand in $\Gamma^n_0$ are entirely constituted of the undeformed solution $u_0$ whereas the $\mathcal{O}(\varepsilon^{n+2})$ contributions contain only $u_1$ in addition. The corrections to the undeformed solution $u_0$'s can successively be calculated from the $\mathcal{O}(\varepsilon^n)$ contribution to the parent equation 66 following the evaluation of $u_{n-1}$. Thus all the deformed charges and their rates can be obtained up to all orders in principle. A good convergence of the net sum of these contributions should ensure quasi-conservation but it requires a good deal of numeric simulation which is beyond our scope presently. However, such confirmation has already been obtained for particular quasi-KdV systems.

**Connection with non-holonomic deformation**

It is fruitful to compare the quasi-deformation obtained thus far with the nonholonomic deformation of the KdV and NLS systems. The nonholonomic deformation is practically obtained by extending the temporal Lax component with local functions of various spectral grades so that the time evolution is not affected. This results in an inhomogeneous extension to the original equation while maintaining integrability through the zero-curvature condition. As a result higher order differential constraints get imposed on the deformation functions. Nonholonomic deformation had been well-analyzed for KdV and coupled complex KdV systems, through both loop-algebraic and AKNS approaches. Though the nonholonomic NLS system is locally different from its quasi-deformation, asymptotically they may converge.

The quasi-deformation affects the terms containing the dynamical variable itself. It deforms the Lax component without affecting other spectral grades, disrupting the zero curvature condition. Therefore, the two deformations are fundamentally different. However, since their charges are conserved asymptotically, a quasi-deformed system may converge to a nonholonomic one asymptotically. In the present case, the deformed
solutions maintain localization. Further, the logarithmic contribution in Eq. (67) becomes subdominant at large distances for a localized u, leaving out a pure KdV-type system with scaled constants that should be integrable. A similar property was observed for the NLS system\(^{23}\). Given the weak coupling map connecting these two systems and their quasi-deformations, one could expect the quasi-KdV system to converge to a nonholonomic variant.

In that regime the present power-series expansion in \(\varepsilon\) may be interpreted as local constraints characterizing a nonholonomic system. The quasi-KdV system further supports single- and multi-soliton-type solutions that asymptotically converge to their ideal counterparts. It will be interesting to identify the constraints with order-by-order relations in \(\varepsilon\) by evaluating the asymptotic form of the exact quasi-deformed solution.

**Conclusions and discussion**

It is seen that a comprehensive quasi-integrable deformation of the KdV system is indeed possible through the loop-algebraic generalization\(^{20}\). Since the KdV equation is not dynamical in the Galilean (like NLS) or Lorentz (like sine-Gordon) sense, the deformation has to be off-shell at the level of the Hamiltonian. In this ab-initio treatment the SU(2) Lax representation has been utilized to obtain the standard abelianization through the inherent sl(2) loop algebra. In the abelianized subspace anomalous conservation laws were obtained. The corresponding anomaly function and the coefficients of the rotated Lax pair have definite parity given the deformed solution is a parity eigenstate, subsequently ensuring asymptotic conservation of the charges. Specifically, the present treatment marks the general construction of a quasi-KdV system having a characteristic infinity of anomalous charges. It conforms to the recent treatments for particular KdV deformations\(^{22,23}\). Although there can be an additional infinity of anomalous charges even for the integrable counterpart\(^{23,24,31,33}\), the physical interest of modeling realistic KdV-like systems is sufficiently covered by the present approach.

The particular cases of local extensions and power deformation in the Hamiltonian density are considered. The prior results in a scaled KdV equation at the simplest level with a single-soliton profile. It further yields families of higher-derivative extensions with some potentially quasi-integrable members. An intuitive validation for it comes from the weak coupling correspondence between KdV and NLS systems. The present deformation of the KdV system is found to be compatible with the quasi-NLS system, supported by corresponding one-soliton solutions with variable scaling. The KdV solution corresponding to the quasi-NLS soliton has a structure similar to the KdV soliton train that corresponds to the undeformed NLS soliton. For power deformation \(u \rightarrow u^{1+\varepsilon}\) of the Hamiltonian the situation becomes more complicated with possible singularities, though soliton-like localized structures are still supported. Further, an order-by-order expansion in the deformation parameter led to localized solutions. Although devoid of definite parity, they are expected to yield conservation of the charges asymptotically. It will be worthwhile to numerically obtain particular stable solutions to the deformed KdV and higher derivative systems and to study their connection with the quasi-NLS solutions.

The present quasi-deformation scheme can be extended to related systems like KdV-type hierarchies, mKdV, and their nonlocal counterparts. Quasi-deformation of the coupled complex KdV system of Eq. (7) could be more challenging as a constant semi-simple sl(2) element in the spatial Lax component is required. A suitable deformation of this component could be

\[
A \rightarrow \left( \frac{q}{\lambda} + 1 \right) \sigma_+ - (u - \lambda) \sigma_- \equiv \frac{q - q_m}{\sqrt{2}} F_+^{-1} + \frac{q + q_m}{\sqrt{2}} F_-^{-1} + \sqrt{2} F_0^0
\]

under the sl(2) representation of Eq. (19). However, the scaling of the conjugate field \(q\) by the spectral parameter violates its relative grading for \(q\) in \(A\) in order to yield a KdV system. This relative grading is apparent from the equivalent Lax pair

\[
A \rightarrow \frac{q}{\lambda} \sigma_+ - \lambda q \sigma_- \quad \text{and}
\]

\[
B \rightarrow (q q_x - \bar{q} \bar{q}_x) \sigma_3 - \frac{1}{\lambda} \left( \bar{q} q_x + 2 q \bar{q}^2 \right) \sigma_+ + \lambda \left( u q_x + 2 q \bar{q}^2 \right) \sigma_-
\]

with explicit spectral dependence that yields the same undeformed KdV system. This hampers the abelianization scheme, more so as the Lax pair is a direct consequence of the AKS hierarchy\(^{20}\). However, some nontrivial Lax representation can yield quasi-deformation of this complex coupled KdV system requiring brute-force numerical calculations. We further aspire to analyze similar systems like mKdV and other hierarchies of the KdV system for possible quasi-deformation and to extend this study to nonlocal systems. On a related yet formal note, the role of \(P\)-symmetry in the algebraic abelianization procedure is worth venturing into, given the space-time properties of general quasi-conserved systems\(^{23}\).

**Data availability**

All data that support the findings in this study are available in the article. Additional information is available from the corresponding author upon request.

Received: 27 July 2024; Accepted: 10 January 2025
Published online: 18 January 2025
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Acknowledgements

Kumar Abhinav’s research has been funded by Mahidol University (Fundamental Fund: fiscal year 2024 by National Science Research and Innovation Fund (NSRF)). Partha Guha is grateful to Jun Nian and Vasily Pestun for interesting discussions. The authors thank Professors Luiz and A. Ferreira, Wojtek J. Zakrzewski, and Betti Hartmann for their encouragement, various useful discussions, and critical reading of the draft during the initial phases of this work. The authors greatly appreciate highly fruitful discussions with Indranil Mukherjee throughout this work.

Author contributions

K. A. co-conceived the idea, did the initial calculations, performed the analysis, made the plots, wrote and communicated the manuscript. P. G. co-conceived the idea and supervised the progress of the manuscript. Both authors reviewed the manuscript.

Declarations

Competing interests

The authors declare no competing interests.

Additional information

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