The Geometric Invariants of Null Cartan Curves Under The Similarity Transformations

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Abstract

In this paper, we study the differential geometry of null Cartan curves under the similarity transformations in the Minkowski space-time. Besides, we extend the fundamental theorem for a null Cartan curve according to a similarity motion. We find the equations of all self-similar null curves which is given its shape Cartan curvatures.

Keywords : Lorentzian Similarity Geometry, Similarity Transformation, Similar null curves, Cartan curves.

MSC 2010 : 14H50, 14H81, 53A35, 53A55, 53B30.

1 Introduction

A similarity transformation (or similitude) of Euclidean space, which consists of a rotation, a translation and an isotropic scaling, is an automorphism preserving the angles and ratios between lengths. The structure, which forms geometric properties unchanged by similarity transformations, is called the similarity geometry. The whole Euclidean geometry can be considered as a glass of similarity geometry. The similarity transformations are studying in most area of the pure and applied mathematics.

Curve matching is an important research area in the computer vision and pattern recognition, which can help us determine what category the given curve belongs to. S. Li [23, 24] presented a system for matching and pose estimation of 3D space curves under the similarity transformation. Brook et al. [2] discussed various problems of image processing and analysis by using the similarity transformation. Sahbi [9] investigated a method for shape description based on kernel principal component analysis (KPCA) in the similarity invariance of KPCA. On the other hand, Chou and Qu [12] showed that the motions of curves in two, three and n-dimensional (n > 3) similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies by using the similarity invariants of curves.

The idea of self-similarity is one of the most basic and fruitful ideas in mathematics. A self-similar object is exactly similar to a part of itself, which in turn remains similar to a smaller part of itself, and so on. In the last few decades, the self-similarity notion led to the areas such as fractal geometry, dynamical systems, computer networks and statistical physics. Mandelbrot presented the first description of self-similar sets, namely sets that may be expressed as unions of rescaled copies of themselves. He called these sets fractals, which are systems that present such self-similar behavior and the examples in nature are many. The Cantor set, the von Koch snowflake curve and the Sierpinski gasket are some of the most famous examples of such sets (see [7, 11, 13, 18]).
When the Euclidean space is endowed with the Lorentzian inner product, we obtain the Lorentzian similarity geometry. Aristeid [25] investigated the closed Lorentzian similarity manifolds. Kamishima [28] studied the properties of compact Lorentzian similarity manifolds using developing maps and holonomy representations. The geometric invariants of null curves in the Lorentzian similarity geometry have not been considered so far.

Bonnor [26] introduced the Cartan frame to study the behaviors of a null curve and proved the fundamental existence and congruence theorems in Minkowski space-time. Bejancu [1] represented a method for the general study of the geometry of null curves in Lorentz manifolds and, more generally, in semi-Riemannian manifolds (see also the book [14]). Ferrandez, Gimenez and Lucas [4] gave a reference along a null curve in an n-dimensional Lorentzian space. They showed the fundamental existence and uniqueness theorems and described the null helices in higher dimensions. Cöken and Ciftci [3] studied null curves in the Minkowski space-time and characterized pseudo-spherical null curves and Bertrand null curves.

The study of the geometry of null curves has a growing importance in the mathematical physics. The null curves use at the solution of some equations in the classical relativistic string theory (see [15, 16, 20]). Moreover, there exists a geometric particle model associated with the geometry of null curves in the Minkowski space-time (see [5, 6]).

Berger [17] represented the broad content of similarity transformations in the arbitrary dimensional Euclidean spaces. Encheva and Georgiev [21, 22] studied the differential geometric invariants of curves according to a similarity in the finite dimensional Euclidean spaces. The main idea of this paper is to introduce the differential geometry of a null curve under the pseudo-similarity mapping and determine the self-similar null curves in the Lorentzian similarity geometry.

The scope of paper is as follows. First, we give basic informations about null Cartan curves. Then, we introduce a new parameter, which is called pseudo-de Sitter parameter and is invariant under the similarity transformation, to study null curves in Lorentzian similarity geometry. We represent the differential geometric invariants of a null Cartan curve, which are called shape Cartan curvatures, according to the group of similarity transformations in the Minkowski space-time. We prove the uniqueness theorem which states that two null Cartan curves having same the shape Cartan curvatures are equivalent according to a similarity. Furthermore, we show the existence theorem that is a process for constructing a null Cartan curve by the shape Cartan curvatures under some initial conditions. Lastly, we obtain equations of all self-similar null Cartan curves, whose shape Cartan curvatures are real constant.

2 Preliminaries

Let \( u = (u_1, u_2, u_3, u_4) \) and \( v = (v_1, v_2, v_3, v_4) \) be two arbitrary vectors in Minkowski space-time \( \mathbb{M}^4 \). The Lorentzian inner product of \( u \) and \( v \) can be stated as \( u \cdot v = u^I v^T \) where \( I^* = \text{diag}(-1, 1, 1, 1) \). We say that a vector \( u \) in \( \mathbb{M}^4 \) is called spacelike, null (lightlike) or timelike if \( u \cdot u > 0 \), \( u \cdot u = 0 \) or \( u \cdot u < 0 \), respectively. The norm of the vector \( u \) is represented by \( \|u\| = \sqrt{u \cdot u} \).

We can describe the pseudo-hyperspheres in \( \mathbb{M}^4 \) as follows: The hyperbolic 3-space is defined by

\[
H^3(-1) = \{ u \in \mathbb{M}^4 : u \cdot u = -1 \}
\]
and de Sitter 3-space is defined by

\[
S^3_1 = \{ u \in \mathbb{M}^4 : u \cdot u = 1 \} \quad (8 \ 27).
\]

A basis \( B = \{ L, N, W_1, W_2 \} \) of \( \mathbb{M}^4 \) is said to pseudo-orthonormal if it satisfies the following...
where $\kappa$ and positively oriented. The functions values are given as following equations are satisfied: $t$ for all Cartan curve has the same orientation with $F$ γ said that a null curve is linearly independent for any $t$ vector of the null curve is not null, the pseudo-arc parametrization becomes as the following γ, ε, ζ where equations: $\phi$ mapping $\phi$ [26] defined the mapping $\phi$ transformation between two sets of natural coordinate functions whose values coincide at for the pseudo-arc parameter $t$ it can be seen the materials [1], [4], [14] and [26] for more information for the pseudo-arc parameter $t$. It can be seen the materials [1], [4], [14] and [26] for more information about the geometry of null curves.

\[ \mathbf{L} \cdot \mathbf{L} = \mathbf{N} \cdot \mathbf{N} = 0, \quad \mathbf{L} \cdot \mathbf{N} = 1, \]
\[ \mathbf{L} \cdot \mathbf{W}_i = \mathbf{N} \cdot \mathbf{W}_i = \mathbf{W}_1 \cdot \mathbf{W}_2 = 0, \]
\[ \mathbf{W}_i \cdot \mathbf{W}_i = 1 \]

where $i \in \{1, 2\}$ ([14]).

Now, we consider the mapping $\varphi : (\bar{\mathbf{L}}, \bar{\mathbf{N}}, \bar{\mathbf{W}}_1, \bar{\mathbf{W}}_2) \to (\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \mathbf{W}_2)$ of one pseudo-orthonormal basis onto another at any point $P$ in $\mathbb{M}^4$, defined by

\[
\begin{bmatrix}
\mathbf{L} \\
\mathbf{N} \\
\mathbf{W}_1 \\
\mathbf{W}_2
\end{bmatrix} =
\begin{bmatrix}
\lambda & 0 & 0 & 0 \\
-\frac{1}{2} \lambda (\varepsilon^2 + \zeta^2) & \lambda^{-1} & -\varepsilon & \zeta \\
\lambda \varepsilon \cos \theta + \lambda \zeta \sin \theta & 0 & \cos \theta & -\sin \theta \\
\lambda \sin \theta - \lambda \zeta \cos \theta & 0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\bar{\mathbf{L}} \\
\bar{\mathbf{N}} \\
\bar{\mathbf{W}}_1 \\
\bar{\mathbf{W}}_2
\end{bmatrix}
\]  

(1)

where $\gamma, \varepsilon, \zeta$ and $\theta$ are real constants and $\lambda \neq 0$. The image of pseudo-orthonormal basis under the mapping $\varphi$ is a pseudo-orthonormal basis. Moreover, the orientation is preserved by (1). Bonnor [26] defined the mapping $\varphi$ as a null rotation. A null rotation at $P$ is equivalent to a Lorentzian transformation between two sets of natural coordinate functions whose values coincide at $P$.

A curve locally parameterized by $\gamma : J \subset \mathbb{R} \to \mathbb{M}^4$ is called a null curve if $\gamma'(t) \neq 0$ is a null vector for all $t$. We know that a null curve $\gamma(t)$ satisfies $\gamma''(t) \cdot \gamma''(t) \geq 0$ ([14]). If $\gamma''(t) \cdot \gamma''(t) = 0$, then it is said that a null curve $\gamma(t)$ in $\mathbb{M}^4$ is parameterized by pseudo-arc. If we assume that the acceleration vector of the null curve is not null, the pseudo-arc parametrization becomes as the following

\[ s = \int_{t_0}^{t} (\gamma''(u) \cdot \gamma''(u))^{1/2} du \quad ([3, 26]). \]  

(2)

A null curve $\gamma(t)$ in $\mathbb{M}^4$ with $\gamma''(t) \cdot \gamma''(t) \neq 0$ is a Cartan curve if $F_\gamma := \{ \gamma'(t), \gamma''(t), \gamma^{(3)}(t), \gamma^{(4)}(t) \}$ is linearly independent for any $t$. There exists a unique Cartan frame $C_\gamma := \{ \mathbf{L}, \mathbf{N}, \mathbf{W}_1, \mathbf{W}_2 \}$ of the Cartan curve has the same orientation with $F_\gamma$ according to pseudo arc-parameter $t$, such that the following equations are satisfied:

\[
\begin{align*}
\gamma' &= \mathbf{L}, \\
\mathbf{L}' &= \mathbf{W}_1, \\
\mathbf{N}' &= \kappa \mathbf{W}_1 + \tau \mathbf{W}_2 \\
\mathbf{W}_1' &= -\kappa \mathbf{L} - \mathbf{N} \\
\mathbf{W}_2' &= -\tau \mathbf{L}
\end{align*}
\]

(3)

where $\mathbf{N}$ is a null vector, which is called null transversal vector field, and $C_\gamma$ is pseudo-orthonormal and positively oriented. The functions $\kappa$ and $\tau$ are called the Cartan curvatures of $\gamma(t)$ and their values are given as

\[
\begin{align*}
\kappa(t) &= \frac{1}{2} \left( \gamma^{(3)}(t) \cdot \gamma^{(3)}(t) \right) \\
\tau(t) &= -\sqrt{\gamma^{(4)}(t) \cdot \gamma^{(4)}(t) - \left( \gamma^{(3)}(t) \cdot \gamma^{(3)}(t) \right)^2}
\end{align*}
\]

(4)

for the pseudo-arc parameter $t$. It can be seen the materials [1], [4], [14] and [26] for more information about the geometry of null curves.
3 Geometric Invariants of Null Curves in Lorentzian Similarity Geometry

Now, we define a pseudo-similarity transformation for null curves in $\mathbb{M}^4$. A pseudo-similarity (p-similarity) of Minkowski space-time is a composition of a Lorentzian transformation (or null rotation), translation and a scaling. Any p-similarity map $f : \mathbb{M}^4 \to \mathbb{M}^4$ is determined by

$$f(x) = \mu \varphi(x) + b,$$

where $\mu \neq 0$ is a real constant, $\varphi$ is a null rotation and $b$ is a translation vector. The p-similarity transformations are a group under the composition of maps and denoted by $\text{Simp}(\mathbb{M}^4)$. This group is a fundamental group of the Lorentzian similarity geometry spanned by the pseudo-orthonormal basis. The p-similarity transformations in $\mathbb{M}^4$ preserve the orientation.

Let $\gamma(t) : J \subset \mathbb{R} \to \mathbb{M}^4$ be a null curve in $\mathbb{M}^4$. We denote image of $\gamma$ under $f \in \text{Simp}(\mathbb{M}^4)$ by $\beta$. Then, the null curve $\beta$ can be stated as

$$\beta(t) = \mu \varphi \gamma(t) + b, \quad t \in J. \quad (6)$$

The pseudo-arc length function $\beta$ starting at $t_0 \in J$ is

$$s^*(t) = \int_{t_0}^{t} (\beta''(u) \cdot \beta''(u))^{1/4} du = \sqrt{\mu} s(t) \quad (7)$$

where $s \in I \subset \mathbb{R}$ is pseudo-arc parameter of $\gamma : I \to \mathbb{M}^4$. From now on, we will denote by a prime "" the differentiation with respect to $s$. We can compute the Cartan curvatures $\kappa_\beta(\sqrt{\mu}s)$ and $\tau_\beta(\sqrt{\mu}s)$ of $\beta$ by using (4) as

$$\kappa_\beta = \frac{1}{\mu} \kappa_\gamma, \quad \tau_\beta = \frac{1}{\mu} \tau_\gamma. \quad (8)$$

We define $W_2$-indicatrix $\gamma_{W_2}$ of the null curve $\gamma$ parameterized by $\gamma_{W_2}(s) = W_2(s)$. The $W_2$-indicatrix is a pseudo-hyperspherical curve lies on the de Sitter 3-space $S_3^{1}$. Since the curve $\gamma_{W_2}$ is a null curve, the pseudo-arc parameter $\sigma_\gamma$ of $\gamma_{W_2}$ can be given as $d\sigma_\gamma = \sqrt{\tau_\gamma} ds$ by using the equation (2). The parameter $\sigma_\gamma$ is invariant under the p-similarity transformation since it can be easily found $d\sigma_\beta = d\sigma_\gamma$, where $\sigma_\beta$ is the pseudo-de Sitter parameter of $\beta$. Therefore, we can reparametrize a null curve with the pseudo-de Sitter parameter in order to study differential geometry of a null curve under the p-similarity transformation. The parameter $\sigma_\gamma$ is called pseudo-de Sitter parameter of $\gamma$.

The derivative formulas of $\gamma$ and $C_\gamma$ with respect to $\sigma_\gamma$ are given by

$$\frac{d\gamma}{d\sigma_\gamma} = \frac{1}{\sqrt{\tau_\gamma}} L, \quad \frac{d^2\gamma}{d\sigma_\gamma^2} = -\frac{d\tau_\gamma}{2\tau_\gamma} \frac{d\gamma}{d\sigma_\gamma} + \frac{1}{\tau_\gamma} W_1 \quad (9)$$

and

$$\frac{dL}{d\sigma_\gamma} = \frac{1}{\sqrt{\tau_\gamma}} W_1, \quad \frac{dN}{d\sigma_\gamma} = \frac{\kappa_\gamma}{\sqrt{\tau_\gamma}} W_1 + \sqrt{\tau_\gamma} W_2, \quad \frac{dW_1}{d\sigma_\gamma} = -\frac{\kappa_\gamma}{\sqrt{\tau_\gamma}} L - \frac{1}{\sqrt{\tau_\gamma}} N, \quad \frac{dW_2}{d\sigma_\gamma} = -\sqrt{\tau_\gamma} L. \quad (10)$$
Similarly, we can find the same formulas (9) and (10) for the null curve $\beta$.

Now, we construct a new frame corresponding to p-similarity transformation for a null curve. Let’s denote the functions

$$\tilde{\tau}_\gamma := \frac{-d\tau_\gamma}{2\tau_\gamma d\sigma_\gamma} \text{ and } \tilde{\kappa}_\gamma := \frac{\kappa_\gamma}{\tau_\gamma},$$

respectively. The functions $\tilde{\tau}_\gamma$ and $\tilde{\kappa}_\gamma$ are invariant under the p-similarity because of $\tilde{\tau}_\beta = \tilde{\tau}_\gamma$ and $\tilde{\kappa}_\beta = \tilde{\kappa}_\gamma$. Let be

$$L_{sim} = \sqrt{\tau_\gamma} L, \quad N_{sim} = \frac{1}{\sqrt{\tau_\gamma}} N,$$

such that the equality $L_{sim} \cdot N_{sim} = 1$ is satisfied and $C_{sim} := \{L_{sim}, N_{sim}, W_{1 sim}, W_{2 sim}\}$ is a pseudo-orthonormal frame of $\gamma$. Then, the derivative formulas for $C_{sim}$ are

$$\frac{d}{d\sigma_\gamma} \left( C_{sim} \right)^T = P \left( C_{sim} \right)^T \tag{11}$$

where

$$P = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & \tilde{\kappa}_\gamma & 1 \\
-\tilde{\kappa}_\gamma & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}.$$

We consider the pseudo-orthogonal frame $C^H_{\gamma} := \{H^1_\gamma, H^2_\gamma, H^3_\gamma, H^4_\gamma\}$ for the null curve $\gamma$ where

$$H^1_\gamma = \frac{1}{\tau_\gamma} L^\gamma_{sim}, \quad H^2_\gamma = \frac{1}{\tau_\gamma} N^\gamma_{sim}, \quad H^3_\gamma = \frac{1}{\tau_\gamma} W^1_{\gamma sim} \text{ and } H^4_\gamma = \frac{1}{\tau_\gamma} W^2_{\gamma sim}.$$

Since we can obtain $f(H^i_\gamma) = H^i_{\beta}, i = 1, \cdots, 4$, from (5), the pseudo-orthogonal frame $C^H_{\gamma}$ is invariant according to p-similarity map. Then, using (9) and (11), we get the derivative formulas of $C^H_{\gamma}$ as the following

$$\frac{d}{d\sigma_\gamma} \left( C^H_{\gamma} \right)^T = \tilde{P} \left( C^H_{\gamma} \right)^T \tag{12}$$

where

$$\tilde{P} = \begin{bmatrix}
2\tilde{\tau}_\gamma & 0 & 1 & 0 \\
0 & 2\tilde{\tau}_\gamma & \tilde{\kappa}_\gamma & 1 \\
-\tilde{\kappa}_\gamma & -1 & 2\tilde{\tau}_\gamma & 0 \\
-1 & 0 & 0 & 2\tilde{\tau}_\gamma
\end{bmatrix}.$$

We can think the equation (12) as the structure equation of a null curve $\gamma$ according to the pseudo-orthogonal moving frame $C^H_{\gamma}$ and the p-similarity group $\text{Simp}(M^4)$. As a result, the following lemma is obtained.

**Lemma 1** Let $\gamma : I \to M^4$ be a null Cartan curve with pseudo-de Sitter parameter $\sigma$ and $\{\kappa_\gamma, \tau_\gamma\}$ be Cartan curvatures of $\gamma$ with the Cartan frame $C_\gamma$. Then, the functions

$$\tilde{\tau}_\gamma := \frac{-d\tau_\gamma}{2\tau_\gamma d\sigma_\gamma}, \quad \tilde{\kappa}_\gamma := \frac{\kappa_\gamma}{\tau_\gamma} \tag{13}$$

and the pseudo-orthogonal frame $C^H_{\gamma}$ are invariant under the p-similarity transformation in the Minkowski space-time and the derivative formulas of $C^H_{\gamma}$ with respect to $\sigma$ are given by the equation (12).
Definition 2 The functions $\tilde{\tau}_\gamma = \frac{d\tau_\gamma}{d\gamma}$, $\tilde{\kappa}_\gamma = \frac{\kappa_\gamma}{\tau_\gamma}$ and the pseudo-orthogonal frame $C^H_\gamma$ are called shape Cartan curvatures and shape Cartan frame of a null Cartan curve $\gamma$, respectively.

Remark 3 We consider the $W_1$-indicatrix $\gamma_{W_1}$ of null curve $\gamma$ parameterized by $\gamma_{W_1}(s) = W_1(s)$, where $s$ is a pseudo-arc parameter of $\gamma$. The curve $\gamma_{W_1}$ is a pseudo-hyperspherical spacelike curve if $\kappa_\gamma > 0$ or pseudo-hyperspherical timelike curve if $\kappa_\gamma < 0$ on $S^3_0$. If $u$ is a arc-parameter of $\gamma_{W_1}$, then we can find $du = \sqrt{2\kappa_\gamma}ds$. The parameter $u$ is invariant according to $p$-similarity transformation; therefore, it can also be used this parametrization for a null Cartan curve in Lorentzian similarity geometry.

Remark 4 We take $W_1^{\text{sim}} = \frac{dL^{\text{sim}}}{d\sigma_\gamma}$ instead of $W_1^{\text{sim}} = W_1$ in the frame $C^{\text{sim}}_\gamma$. The derivative formulas for a new frame are

$$\frac{d}{d\sigma_\gamma} (C^{\text{sim}}_\gamma)^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \tilde{\xi}_\gamma & 1 \\ -\tilde{\xi}_\gamma & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} (C^{\text{sim}}_\gamma)^T$$

where $\tilde{\xi}_\gamma = -\tilde{\tau}_\gamma^2 + \tilde{\kappa}_\gamma^2$. It may be considered an alternative frame for a null Cartan curve in the Lorentzian similarity geometry. However, the problem is that although $W_1^{\text{sim}}$ is a unit spacelike vector, the new frame is not the pseudo-orthogonal due to $\langle W_1^{\text{sim}}, N^{\text{sim}} \rangle \neq 0$.

4 The Fundamental Theorem for a Null Curve in Lorentzian Similarity Geometry

The existence and uniqueness theorems are shown by [1, 4] and [26] for a null Cartan curve under the Lorentz transformations. This notion can be extended with respect to $\text{Simp}(M^4)$ for the null Cartan curves parameterized by the pseudo-de Sitter parameter.

Theorem 5 (Uniqueness Theorem) Let $\gamma, \beta : I \to M^4$ be two null Cartan curves parameterized by the same pseudo-de Sitter parameter $\sigma$, where $I \subset \mathbb{R}$ is an open interval. Suppose that $\gamma$ and $\beta$ have the same shape Cartan curvatures $\tilde{\tau}_\gamma = \tilde{\tau}_\beta$ and $\tilde{\kappa}_\gamma = \tilde{\kappa}_\beta$ for any $\sigma \in I$. Then, there exists a $f \in \text{Simp}(M^4)$ such that $\beta = f \circ \gamma$.

Proof. Let $\kappa_\gamma, \tau_\gamma$ and $\kappa_\beta, \tau_\beta$ be the Cartan curvatures and also $s$ and $s^*$ be the pseudo-arc length parameters of $\gamma$ and $\beta$, respectively. Using the equality $\tilde{\tau}_\gamma = \tilde{\tau}_\beta$, we get $\tau_\gamma = \mu \tau_\beta$ for some real constant $\mu > 0$. Then, the equality $\tilde{\kappa}_\gamma = \tilde{\kappa}_\beta$ implies $\kappa_\gamma = \mu \kappa_\beta$. On the other hand, we can write $ds = \frac{1}{\sqrt{\mu}}ds^*$ from the definition of pseudo-de Sitter parameter $\sigma$.

Let’s consider the map $\Psi : M^4 \to M^4$ defined by $\Psi(x) = \frac{1}{\mu} \varphi(x)$ where $\varphi$ is a null rotation. Using the equation (3), the null Cartan curves $\alpha = \Psi(\beta)$ and $\gamma$ have the same Cartan curvatures. Then, there exists a Lorentzian transformation $\phi : M^4 \to M^4$ according to the uniqueness theorem for the null Cartan curves (see [1, 4]) such that $\phi(\gamma) = \alpha$. Therefore, we have a transformation $f = \Psi^{-1} \circ \phi : M^4 \to M^4$ which is a $p$-similarity and $f(\gamma) = \beta$. \hfill \Box

The following theorem shows that every two functions determine a null Cartan curve according to a $p$-similarity under some initial conditions.

Theorem 6 (Existence Theorem) Let $z_i : I \to \mathbb{R}$, $i = 1, 2$ be two functions and $L^{\text{sim}}, N^{\text{sim}}, W_1^{\text{sim}}, W_2^{\text{sim}}$ be a pseudo-orthonormal frame at a point $x_0$ in the Minkowski space-time. According to a
p-similarity with the center \( x_0 \) there exists a unique null Cartan curve \( \gamma : I \to \mathbb{M}^4 \) parameterized by a pseudo-de Sitter parameter \( \sigma \) such that \( \gamma \) satisfies the following conditions:

(i) There exists \( \sigma_0 \in I \) such that \( \gamma (\sigma_0) = x_0 \) and the shape Cartan frame of \( \gamma \) at \( x_0 \) is \( L^{0 \text{sim}}, N^{0 \text{sim}}, W_1^{0 \text{sim}}, W_2^{0 \text{sim}} \).

(ii) \( \kappa_\gamma (\sigma) = z_1 (\sigma) \) and \( \tilde{\tau}_\gamma (\sigma) = z_2 (\sigma) \), for any \( \sigma \in I \).

**Proof.** Let us consider the following system of differential equations with respect to a matrix-valued function \( K (\sigma) = (L^{\sim}, N^{\sim}, W_1^{\sim}, W_2^{\sim})^T \)

\[
\frac{dK}{d\sigma} (\sigma) = M (\sigma) K (\sigma)
\]

with a given matrix

\[
M (\sigma) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & z_1 & 1 \\
-z_1 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}.
\]

The system \((14)\) has a unique solution \( W (\sigma) \) which satisfies the initial conditions

\[
K (\sigma_0) = (L^{0 \text{sim}}, N^{0 \text{sim}}, W_1^{0 \text{sim}}, W_2^{0 \text{sim}})^T
\]

for \( \sigma_0 \in I \). If \( K^T (\sigma) \) is the transposed matrix of \( K (\sigma) \), then

\[
\frac{d}{d\sigma} (J^T K^T J^T K) = J^* \frac{d}{d\sigma} K^T J^T K + J^* K^T J^T \frac{d}{d\sigma} K
\]

\[
= J^* K^T M^T J^T K + J^* K^T J^T M K
\]

\[
= J^* K^T (M J^T + J^T M) K = 0
\]

since we have the equation \( M J^* + J^T M = [0]_{4 \times 4} \) where

\[
J^* = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Also, we have \( J^* W^T (\sigma_0) J^* W (\sigma_0) = I \) where \( I \) is the unit matrix since \( L^{0 \text{sim}}, N^{0 \text{sim}}, W_1^{0 \text{sim}}, W_2^{0 \text{sim}} \) is the pseudo-orthonormal 4-frame. As a result, we find \( J^* X^T (\sigma) J^* X (\sigma) = I \) for any \( \sigma \in I \). It means that the vector fields \( L^{\sim}, N^{\sim}, W_1^{\sim}, \) and \( W_2^{\sim} \) form pseudo-orthonormal frame field in the Minkowski space-time.

Let \( \gamma : I \to \mathbb{M}^4 \) be the null curve given by

\[
\gamma (\sigma) = x_0 + \int_{\sigma_0}^{\sigma} e^{2 \int_{\sigma_0}^{\sigma} ds} L^{\sim} (\sigma) d\sigma, \quad \sigma \in I.
\]

Using the equality \((14)\), we get that \( \gamma (\sigma) \) is a null Cartan curve in Minkowski space-time with shape Cartan curvatures \( \kappa_\gamma (\sigma) = z_1 (\sigma) \) and \( \tilde{\tau}_\gamma (\sigma) = z_2 (\sigma) \). Also, we find \( d\sigma = e^{-\int_{\sigma_0}^{\sigma} ds} \) by using \((2)\) and \((14)\), where \( s \) is a pseudo-arc parameter; thus, \( \sigma \) is the pseudo-de Sitter parameter of the null Cartan curve \( \gamma \). Besides, the pseudo-orthonormal 4-frame \( \{L^{\sim}, N^{\sim}, W_1^{\sim}, W_2^{\sim}\} \) is a Cartan frame of the null Cartan curve \( \gamma \) under the p-similarity transformation.
Corollary 7 In case of $\tau_\gamma (\sigma) = 0$, the Cartan curvature $\tau_\gamma = c$ is a non-zero real constant. Then, the parametrization of a null curve $\gamma : I \to \mathbb{M}^4$ with $\tilde{\kappa}_\gamma (\sigma) = 0$ with respect to pseudo-de Sitter parameter $\sigma$ is given by
\[
\gamma (\sigma) = x_0 + \frac{1}{c} \int_{\sigma_0}^{\sigma} L^{\text{sim}} (\sigma) \, d\sigma, \quad \sigma \in I
\] from the equation (9) and (15).

Example 8 Let shape Cartan curvatures of a null curve $\gamma : I \to \mathbb{M}^4$ be $\tilde{\kappa}_\gamma = 0$ and $\tilde{\tau}_\gamma = \frac{1}{\sigma}$. Choose the initial conditions
\[
L_{0\text{sim}} = \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0 \right), \quad N_{0\text{sim}} = \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0 \right),
\]
\[
W_{1\text{sim}} = \left( 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \quad W_{2\text{sim}} = \left( 0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right).
\]
Then, the system (14) determine a null vector $L^{\text{sim}}$ given by
\[
L^{\text{sim}} (\sigma) = \frac{1}{\sqrt{2}} (\cosh \sigma, \sinh \sigma, \cos \sigma, \sin \sigma)
\] with $L^{\text{sim}} (0) = L_{0\text{sim}}$, in $\mathbb{M}^4$. Solving the equation (15) we obtain the null Cartan curve $\gamma$ parameterized by
\[
\gamma (\sigma) = \frac{1}{\sqrt{2}} ((\sigma^2 + 2) \sinh \sigma - 2\sigma \cosh \sigma, (\sigma^2 + 2) \cosh \sigma - 2\sigma \sinh \sigma, \nonumber
\]
\[
(\sigma^2 - 2) \sin \sigma + 2\sigma \cos \sigma, (2 - \sigma^2) \cos \sigma + 2\sigma \sin \sigma)
\]
for any $\sigma \in I$.

5 Self-similar Null Cartan Curves

A null Cartan curve $\gamma : I \to \mathbb{M}^4$ is called self-similar if any p-similarity $f \in G$ conserve globally $\gamma$ and $G$ acts transitively on $\gamma$ where $G$ is a one-parameter subgroup of $\text{Simp} (\mathbb{M}^4)$. This means that shape Cartan curvatures $\tilde{\kappa}_\gamma$ and $\tilde{\tau}_\gamma$ are constant. In fact, let $p = \gamma (s_1)$ and $q = \gamma (s_2)$ be two different points lying on $\gamma$ for any $s_1, s_2 \in I$. Since $G$ acts transitively on $\gamma$, there is a p-similarity $f \in G$ such that $f (p) = q$. Then, we find $\tilde{\kappa}_\gamma (s_1) = \tilde{\kappa}_\gamma (s_2)$ and $\tilde{\tau}_\gamma (s_1) = \tilde{\tau}_\gamma (s_2)$, which it implies the invariability of shape Cartan curvatures.

Now, we determine the parametrizations of all self-similar null curves by means of the constant shape Cartan curvatures in the Minkowski space-time. It can be separated to the four different cases as the following. We can take the initial conditions (17) in the example 8 for the all cases.

Case 1: Let’s take $\tilde{\kappa}_{\gamma_1} = 0$ and $\tilde{\tau}_{\gamma_1} = 0$. Then, using the equation (14) we find the null vector $L^{\text{sim}}$ in the equation (18). From the equation (16), we obtain the self-similar null curve parameterized by
\[
\gamma_1 (\sigma) = \frac{1}{c \sqrt{2}} (\sinh \sigma, \cosh \sigma, \sin \sigma, -\cos \sigma).
\]

Case 2: Let’s take $\tilde{\kappa}_{\gamma_2} = 0$ and $\tilde{\tau}_{\gamma_2} = b \neq 0$. Then, using the equation (14) we find the null vector $L^{\text{sim}}$
\[
L^{\text{sim}} (\sigma) = \frac{e^{2b\sigma}}{2\sqrt{2}} (\cosh \sigma, \sinh \sigma, \cos \sigma, \sin \sigma)
\]
and from the equation (15), we get the self-similar null curve given by

$$
\gamma_2 (\sigma) = \frac{1}{2\sqrt{2}} \left( \frac{\cosh (w_1 \sigma) + \sinh (w_1 \sigma)}{\cosh (w_1 \sigma) + \sinh (w_1 \sigma)} + \frac{\cosh (w_2 \sigma) + \sinh (w_2 \sigma)}{2b - 1} \right),
$$

where $w_1 = (2b + 1)$ and $w_2 = (2b - 1)$.

**Case 3:** Let’s take $\kappa_3 = a \neq 0$ and $\tau_3 = 0$. Then, using the equation (14) and (16) it can be obtained the self-similar null curve given by

$$
\gamma_3 (\sigma) = \frac{1}{\sqrt{c}} \left( \sinh (q_1 \sigma), \cosh (q_1 \sigma), \sin (q_2 \sigma), -\cos (q_2 \sigma) \right)
$$

where $q_1 = \sqrt{-a + \sqrt{a^2 + 1}}$ and $q_2 = \sqrt{a + \sqrt{a^2 + 1}}$.

**Case 4:** Let’s take $\kappa_4 = a \neq 0$ and $\tau_4 = b \neq 0$. Then, using the equation (14) and (15) we obtain the self-similar null curve given as

$$
\gamma_4 (\sigma) = \frac{1}{\sqrt{2}} \left( \frac{\cosh (m_1 \sigma) + \sinh (m_1 \sigma)}{m_1} + \frac{\cosh (m_2 \sigma) + \sinh (m_2 \sigma)}{m_2} \right),
$$

where $m_1 = 2b + q_1$, and $m_2 = 2b - q_1$.

A null curve is called a null helix if it has the constant Cartan curvatures not both zero in $\mathbb{M}^4$. The equations of null helices satisfying $\tau \neq 0$ are expressed by

$$
\alpha (s) = \sqrt{\frac{1}{v^2 + r^2}} \left( \frac{1}{v} \sinh vs, \frac{1}{v} \cosh vs, \frac{1}{r} \sin rs, -\frac{1}{r} \cos rs \right),
$$

where $v = \sqrt{\sqrt{\kappa^2 + \tau^2} - \kappa}$ and $r = \sqrt{\sqrt{\kappa^2 + \tau^2} + \kappa}$ (26)). In case of $\kappa = 0$, the equation (19) reduces to

$$
\alpha_0 (s) = \frac{1}{\tau \sqrt{2}} \left( \sinh (\sqrt{\tau}s), \cosh (\sqrt{\tau}s), \sin (\sqrt{\tau}s), -\cos (\sqrt{\tau}s) \right).
$$

The Cartan curvatures of the self-similar null curve $\gamma_1$ can be given by $\kappa = 0$ and $\tau = c \neq 0$. Moreover, the Cartan curvatures of $\gamma_3$ are $\kappa = ac$ and $\tau = c \neq 0$ in the Case 3. Then, the self-similar null curves $\gamma_1$ and $\gamma_3$ are null helices which correspond to the null curves $\alpha_0$ and $\alpha$, respectively. Hence, we can say that null helices satisfying $\tau \neq 0$ are a class of self-similar null curves in $\mathbb{M}^4$. Also, we can characterize the null helices by means of the shape Cartan curvatures in $\mathbb{M}^4$.

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