CLASSIFICATION OF POTENTIAL STRUCTURES ON MINKOWSKI SPACE OVER SUBGROUPS OF THE POINCARÉ GROUP

M. A. PARINOV

ABSTRACT. We describe classes of potential structures (covector fields) on Minkowski space that admit subgroups of the Poincaré group. We describe also seven classes of Maxwell spaces that admit subgroups of the Poincaré group.

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1. Introduction

Using the classification of subgroups of the Poincaré group [1] we classified Maxwell spaces with respect these subgroups [2, 3]. We use classes of potential structures on Minkowski space, that admit the same subgroups of the Poincaré group, for obtaining representatives of Maxwell spaces classes in [3]. Some classes of potential structures were described in [4, 5], in this paper we present for the first time the classification of potential structures completely. This classification is interesting itself, moreover it helps to define more precisely some classes of Maxwell spaces. For example some classes $C_{p,q}$ in spite of [2, 3] turn out non-empty, we describe them in appendix.

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2. Formulation of the problem and method of its solution

For any smooth, real manifold $M$ we define a potential structure as a differential 1-form $A = A_i dx^i$, $A_i = A_i(x), \ x \in M$ [2].

Let $M$ be a four-dimensional manifold; let also $g = g_{ij} dx^i dx^j$ be a pseudo-Euclidean metric on $M$ of Lorentz signature (−−−+). We may understand a pair $(M, g)$ as a domain in the Minkowski space $\mathbb{R}^4_1$. Any triple $(M, g, A)$ is interpreted as a four-potential of electromagnetic field. A Maxwell space is a triple $(M, g, F)$, where

\[ F = dA = F_{ij} dx^i dx^j \quad (F_{ij} = \partial_i A_j - \partial_j A_i) \quad (2.1) \]

is a generalized symplectic structure [2]. Since 2-form $F$ is closed,

\[ dF = 0 \Leftrightarrow \partial_{[i} F_{jk]} = 0, \]

then we may understand $F_{ij}$ as a tensor of electromagnetic field\(^1\).

The problem of group classification of potential structures $(M, g, A)$ (potentials on $M \subset \mathbb{R}^4_1$) is analogous to the problem of classification Maxwell spaces over subgroups of the Poincaré group [2, 3]. For every subgroup $G_{p,q}$, corresponding to the algebra $\mathcal{L}_{p,q} = L\{\xi_1, \ldots, \xi_p\}^2$, we find the class $P_{p,q}$ of potentials $A_i$, which are invariant respectively this group; the potential $A \in P_{p,q}$ satisfies to the invariance condition

\[ L_{\xi_{\alpha}} A_i = 0 \quad (\alpha = 1, \ldots, p = \dim \mathcal{L}_{p,q}) \quad (2.2) \]

($L_{\xi}$ is the Lie derivative). Solving (2.2) for every algebra $\mathcal{L}_{p,q}$ in [1], we’ll get the complete group classification\(^3\) of potential structures.

We take the basis of the Lie algebra corresponding to the Poincaré group as follows

\[
\begin{align*}
    e_1 &= (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1), \\
    e_{12} &= (-x^2, x^1, 0, 0), \quad e_{13} = (x^3, 0, -x^1, 0), \quad e_{23} = (0, -x^3, x^2, 0), \\
    e_{14} &= (x^4, 0, 0, x^1), \quad e_{24} = (0, x^4, 0, x^2), \quad e_{34} = (0, 0, x^4, x^3).
\end{align*}
\]

Here $\{x^i\}$ are the Galilean coordinates such that

\[ g_{ij} = \text{diag}(-1, -1, -1, 1). \]

In what follows, $L\{\xi_1, \ldots, \xi_p\}$ is the linear combination of vectors $\xi_1, \ldots, \xi_p$. We suppose that components of all tensors correspond to the Galilean coordinates $\{x^i\}$ even if they are expressed as functions of other variables.

**Remark 1.** Every potential $A \in P_{p,q}$ admits the group $G_{p,q}$ or more wide subgroup of the Poincaré group.

\(^1\)If the second Maxwell equation $\nabla_k F^{ik} = -4\pi J^i$ is satisfied (if we disregard by physical restrictions, then we may understand this equation as a definition of current).

\(^2\)See the list of subgroups in [1].

\(^3\)We execute this operation only for algebras $\mathcal{L}_{p,q}$ such that $p \leq 6$. 
3. Classes of potential structures

3.1. Potentials that admit one-dimensional symmetry groups.

3.1.1. Translations. There are three types of non-conjugate in pairs, one-dimensional subgroups of translations.

3.1.1.1. Class \( P_{1,1a} \). The algebra \( \mathcal{L}_{1,1a} = L\{e_1\} \) corresponds to the one-dimensional group \( G_{1,1a} \) of translations along the space-like vector \( e_1 \). The equation (2.2) for the vector \( \xi = e_1 \) takes the form

\[
\partial_1 A_i = 0.
\]

Therefore all components of covector field \( A_i \) are independent of \( x^1 \).

**Statement 1.** The class \( P_{1,1a} \) of potentials that admit the group \( G_{1,1a} \) consists of the fields \( A_i = A_i(x^2, x^3, x^4) \).

3.1.1.2. Class \( P_{1,1b} \). The algebra \( \mathcal{L}_{1,1b} = L\{e_4\} \) corresponds to the one-dimensional group \( G_{1,1b} \) of translations along the time-like vector \( e_4 \). The equation (2.2) for the vector \( \xi = e_4 \) takes the form

\[
\partial_4 A_i = 0.
\]

Therefore all components of covector field \( A_i \) are independent of \( x^4 \).

**Statement 2.** The class \( P_{1,1b} \) of potentials that admit the group \( G_{1,1b} \) consists of the fields \( A_i = A_i(x^1, x^2, x^3) \).

3.1.1.3. Class \( P_{1,1c} \). The algebra \( \mathcal{L}_{1,1c} = L\{e_2 + e_4\} \) corresponds to the one-dimensional group \( G_{1,1c} \) of translations along the isotropic vector \( e_2 + e_4 \). The equation (2.2) for the vector \( \xi = e_2 + e_4 \) takes the form

\[
\partial_2 A_i + \partial_4 A_i = 0.
\]

Using the substitution

\[
v^1 = x^1, \quad v^2 = x^2 + x^4, \quad v^3 = x^3, \quad v^4 = x^2 - x^4,
\]

we receive the solution of equation (3.3)

\[
A_i = A_i(v^1, v^3, v^4) = A_i(x^1, x^3, x^2 - x^4),
\]

where \( A_i(v^1, v^3, v^4) \) are arbitrary functions.

**Statement 3.** The class \( P_{1,1c} \) of potentials that admit the group \( G_{1,1c} \) consists of the fields (3.5).

3.1.2. Elliptic helices. The algebra \( \mathcal{L}_{1,2} = L\{e_{13} + \lambda e_2 + \mu e_4\} \) corresponds to the one-dimensional group \( G_{1,2} \) of elliptic helices or rotations. The equation (2.2) for the vector \( \xi = e_{13} + \lambda e_2 + \mu e_4 \) takes the form

\[
x^3 \partial_1 A_i + \lambda \partial_2 A_i - x^1 \partial_3 A_i + \mu \partial_4 A_i + A_1 \delta_i^3 - A_3 \delta_i^1 = 0.
\]

Using the substitution \( \{x^i\} \mapsto \{\tilde{x}^i\} = \{r, \tilde{x}^2, \varphi, \tilde{x}^4\} \),

\[
x^1 = r \sin \varphi, \quad x^2 = \lambda \varphi + \tilde{x}^2, \quad x^3 = r \cos \varphi, \quad x^4 = \mu \varphi + \tilde{x}^4,
\]
we transform the equation \((3.6)\) to the system of equations

\[
\frac{\partial A_1}{\partial \varphi} - A_3 = 0, \quad \frac{\partial A_2}{\partial \varphi} = 0, \quad \frac{\partial A_3}{\partial \varphi} + A_1 = 0, \quad \frac{\partial A_4}{\partial \varphi} = 0. \tag{3.8}
\]

We have the following expression for the solution of the system \((3.8)\)

\[
A_1 = C_1 \cos \varphi + C_2 \sin \varphi, \quad A_2 = A_2(r, \tilde{x}^2, \tilde{x}^4),
A_3 = -C_1 \sin \varphi + C_2 \cos \varphi, \quad A_4 = A_4(r, \tilde{x}^2, \tilde{x}^4), \tag{3.9}
\]

where \(C_i = C_i(r, \tilde{x}^2, \tilde{x}^4)\) are arbitrary functions.

**Statement 4.** The class \(P_{1,2}\) of potentials that admit the group \(G_{1,2}\) consists of the fields \((3.3)\).

#### 3.1.3. Hyperbolic helices.\(^{22}\) The algebra \(\mathcal{L}_{1,3} = L\{e_{24} + \lambda e_1\}\) corresponds to the one-dimensional group \(G_{1,3}\) of hyperbolic helices or pseudo-rotations (Lorentz transformations). The equation \((2.2)\) for the vector \(\xi = e_{24} + \lambda e_1\) takes the form

\[
\lambda \partial_1 A_i + x^4 \partial_2 A_i + x^2 \partial_4 A_i + A_2 \delta_1^i + A_4 \delta_2^i = 0. \tag{3.10}
\]

Using the substitution

\[
\begin{align*}
x^1 &= \lambda \varphi + \tilde{x}^1, & x^2 &= r \cosh \varphi, & x^3 &= \tilde{x}^3, & x^4 &= r \sinh \varphi
\end{align*} \tag{3.11}
\]

we transform the equation \((3.10)\) to the system of equations

\[
\frac{\partial A_1}{\partial \varphi} = 0, \quad \frac{\partial A_2}{\partial \varphi} + A_4 = 0, \quad \frac{\partial A_3}{\partial \varphi} = 0, \quad \frac{\partial A_4}{\partial \varphi} + A_2 = 0. \tag{3.12}
\]

We have the following expression for the solution of the system \((3.12)\)

\[
\begin{align*}
A_1 &= A_1(\tilde{x}^1, r, \tilde{x}^3), & A_2 &= C_1 \cosh \varphi + C_2 \sinh \varphi,
A_3 &= A_3(\tilde{x}^1, r, \tilde{x}^3), & A_4 &= -C_1 \sinh \varphi - C_2 \cosh \varphi,
\end{align*} \tag{3.13}
\]

where \(C_i = C_i(\tilde{x}^1, r, \tilde{x}^3)\) are arbitrary functions.

**Statement 5.** The class \(P_{1,3}\) of potentials that admit the group \(G_{1,3}\) consists of the fields \((3.13)\).

#### 3.1.4. Parabolic helices.\(^{22}\) The algebra \(\mathcal{L}_{1,4} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3\}\) \((\lambda, \mu = \text{const}, \lambda \mu = 0)\) corresponds to the one-dimensional group \(G_{1,4}\) of parabolic helices or parabolic rotations. The equation \((2.2)\) for the vector \(\xi = e_{12} - e_{14} + \lambda e_2 + \mu e_3\) takes the form

\[
XA_i - A_1(\delta_1^2 + \delta_2^4) + (A_2 - A_4)\delta_1^4 = 0, \tag{3.14}
\]

where

\[
X = -(x^2 + x^4) \partial_1 + (x^1 + \lambda) \partial_2 + \mu \partial_3 - x^1 \partial_4. \tag{3.15}
\]

We consider 3 cases: a) \(\lambda = \mu = 0\); b) \(\lambda = 0, \mu \neq 0\); c) \(\lambda \neq 0, \mu = 0\).

#### 3.1.4.1. Class \(P_{1,4a}\). For \(\lambda = \mu = 0\) we use the substitution

\[
\begin{align*}
\tilde{x}^1 &= x^2 + x^4, & \tilde{x}^2 &= -x^1/(x^2 + x^4),
\tilde{x}^3 &= x^3, & \tilde{x}^4 &= \frac{1}{2}(x^1)^2 + x^2(x^2 + x^4).
\end{align*} \tag{3.16}
\]
the operator (3.15) is replaced by partial derivative with respect to $\tilde{x}^2$ and the equation (3.14) is transformed to the system of equations
\[
\frac{\partial A_1}{\partial \tilde{x}^2} + A_2 - A_4 = 0, \quad \frac{\partial A_2}{\partial \tilde{x}^2} - A_1 = 0, \quad \frac{\partial A_3}{\partial \tilde{x}^2} = 0, \quad \frac{\partial A_4}{\partial \tilde{x}^2} - A_1 = 0. \quad (3.17)
\]
We have the following expression for the solution of the system (3.17)
\[
A_1 = C_2 \tilde{x}^2 + C_3, \quad A_2 = \frac{1}{2} C_2 (\tilde{x}^2)^2 + C_3 \tilde{x}^2 + C_1, \quad A_3 = A_3(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4), \quad A_4 = A_2 + C_2,
\]
where $A_3(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4)$ and $C_k = C_k(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4)$ ($k = 1, 2, 3$) are arbitrary functions.

**Statement 6.** The class $P_{1,4a}$ of potentials that admit the group $G_{1,4a}$, corresponding to the algebra $L_{1,4}$ ($\lambda = \mu = 0$), consists of the fields (3.18).

3.1.4.2. Class $P_{1,4b}$. For $\lambda = 0$, $\mu \neq 0$ we use in place of (3.16) the substitution
\[
\tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^2 = -x^1/(x^2 + x^4),
\]
\[
\tilde{x}^3 = x^3 + \frac{\mu x^1}{x^2 + x^4}, \quad \tilde{x}^4 = \frac{1}{2}(x^1)^2 + x^2(x^2 + x^4);
\]
then the equation (3.14) is transformed to the system (3.17).

**Statement 7.** The class $P_{1,4b}$ of potentials that admit the group $G_{1,4b}$, corresponding to the algebra $L_{1,4}$ ($\lambda = 0$, $\mu \neq 0$), is defined by (3.18), where the substitution (3.16) is replaced by (3.19).

3.1.4.3. Class $P_{1,4c}$. For $\lambda \neq 0$, $\mu = 0$ we use in place of (3.16) the substitution
\[
\tilde{x}^1 = 2\lambda x^1 + (x^2 + x^4)^2, \quad \tilde{x}^2 = (x^2 + x^4)/\lambda, \quad \tilde{x}^3 = x^3,
\]
\[
\tilde{x}^4 = \lambda x^4 + x^1(x^2 + x^4) + (x^2 + x^4)^3/3\lambda,
\]
which transforms (3.14) to (3.17).

**Statement 8.** The class $P_{1,4c}$ of potentials that admit the group $G_{1,4c}$, corresponding to the algebra $L_{1,4}$ ($\lambda \neq 0$, $\mu = 0$), is defined by (3.18), where the substitution (3.16) is replaced by (3.20).

3.1.5. Proportional bi-rotations. The algebra $L_{1,5} = L\{e_{13} + \lambda e_{24}\}$ corresponds to the group $G_{1,5}$ of proportional bi-rotations. The equation (2.22) for the vector $\xi = e_{13} + \lambda e_{24}$ takes the form
\[
XA_i + A_1 \delta^i_3 + \lambda A_2 \delta^i_4 - A_3 \delta^i_1 + \lambda A_4 \delta^i_2; \quad (3.21)
\]
\[
X = x^3 \partial_1 + \lambda x^4 \partial_2 - x^1 \partial_3 + \lambda x^2 \partial_4. \quad (3.22)
\]
We use the substitution $\{x^i\} \rightarrow \{r, \rho, \theta, \varphi\}$,
\[
x^1 = r \cos(\theta - \varphi), \quad x^2 = \rho \cosh(\lambda \varphi), \quad x^3 = r \sin(\theta - \varphi), \quad x^4 = \rho \sinh(\lambda \varphi); \quad (3.23)
\]
the operator \( (3.22) \) is replaced by partial derivative with respect to \( \varphi \) and the equation \( (3.21) \) is transformed to the system of equations

\[
\begin{align*}
\frac{\partial A_1}{\partial \varphi} - A_3 &= 0, \\
\frac{\partial A_2}{\partial \varphi} + \lambda A_4 &= 0, \\
\frac{\partial A_3}{\partial \varphi} + A_1 &= 0, \\
\frac{\partial A_4}{\partial \varphi} + \lambda A_2 &= 0.
\end{align*}
\]  

(3.24)

We have the following expression for the solution of the system \( (3.24) \): 

\[
\begin{align*}
A_1 &= C_1 \cos \varphi + C_2 \sin \varphi, \\
A_2 &= C_3 \cosh \lambda \varphi + C_4 \sinh \lambda \varphi, \\
A_3 &= -C_1 \sin \varphi + C_2 \cos \varphi, \\
A_4 &= -C_3 \sinh \lambda \varphi - C_4 \cosh \lambda \varphi,
\end{align*}
\]  

(3.25)

where \( C_i = C_i(\rho, r, \theta) \) are arbitrary functions.

**Statement 9.** The class \( P_{1,5} \) of potentials that admit the group \( G_{1,5} \) consists of the fields \( (3.25) \).

### 3.2. Potentials that admit two-dimensional symmetry groups.

#### 3.2.1. Translations. There are three types of non-conjugate in pairs, two-dimensional subgroups of translations.

- **Class \( P_{2,1a} \).** The algebra \( L_{2,1a} = L\{e_1, e_2\} \) corresponds to the group \( G_{2,1a} \) of translations along the vectors of the Euclidean plane. We have \( L_{1,1a} \subset L_{2,1a} \), therefore the class \( C_{2,1a} \) is a subclass of the class \( C_{1,1a} \). The equation \( (2.2) \) for the vector \( \xi = e_2 \) takes the form

\[
\partial_2 A_i = 0,
\]  

(3.26)

Substituting \( A_i(x^2, x^3, x^4) \) for \( A_i \) in \( (3.26) \), we get the following result.

**Statement 10.** The class \( P_{2,1a} \) of potentials that admit the group \( G_{2,1a} \) consists of the fields \( A_i = A_i(x^3, x^4) \).

- **Class \( P_{2,1b} \).** The algebra \( L_{2,1b} = L\{e_2, e_4\} \) corresponds to the group \( G_{2,1b} \) of translations along the vectors of the pseudo-Euclidean plane. Since \( L_{1,1b} \subset L_{2,1b} \), we have \( P_{2,1b} \subset P_{1,1b} \). Substituting \( A_i(x^1, x^2, x^3) \) for \( A_i \) in \( (3.26) \), we get the following result.

**Statement 11.** The class \( P_{2,1b} \) of potentials that admit the group \( G_{2,1b} \) consists of the fields \( A_i = A_i(x^1, x^3) \).

- **Class \( P_{2,1c} \).** The algebra \( L_{2,1c} = L\{e_1, e_2 + e_4\} \) corresponds to the group \( G_{2,1c} \) of translations along the vectors of the isotropic plane. Since \( L_{1,1c} \subset L_{2,1c} \), we have \( P_{2,1c} \subset P_{1,1c} \). Combining \( (3.5) \) and \( (3.1) \), we get the following result.

**Statement 12.** The class \( P_{2,1c} \) of potentials that admit the group \( G_{2,1c} \) consists of the fields \( A_i = A_i(x^3, x^2 - x^4) \).
3.2.2. **Class $P_{2,2}$.** The algebra $\mathcal{L}_{2,2} = L\{e_{13} + \mu e_4, e_2\}$ corresponds to the group $G_{2,2}$ generated by elliptic helices with a time-like axis and by translations along a space-like straight line. Let $\mathcal{L}_{1,2b} = L\{e_{13} + \mu e_4\}$ be the algebra $\mathcal{L}_{1,2}$ for $\lambda = 0$. Since $\mathcal{L}_{1,2b} \subset \mathcal{L}_{2,2}$, we have $P_{2,2} \subset P_{1,2b}$. Since $\lambda = 0$, the substitution (3.7) takes the form
\[
x^1 = r \sin \varphi, \quad x^2 = \tilde{x}^2, \quad x^3 = r \cos \varphi, \quad x^4 = \mu \varphi + \tilde{x}^4.
\] (3.27)
Substituting (3.9) for $A_i$ in (3.26), we get
\[
A_1 = C_1 \cos \varphi + C_2 \sin \varphi, \quad A_2 = A_2(r, \tilde{x}^4),
\]
\[
A_3 = -C_1 \sin \varphi + C_2 \cos \varphi, \quad A_4 = A_4(r, \tilde{x}^4),
\] (3.28)
where $C_i = C_i(r, \tilde{x}^4)$ are arbitrary functions.

**Statement 13.** The class $P_{2,2}$ of potentials that admit the group $G_{2,2}$ consists of the fields (3.28).

3.2.3. **Class $P_{2,3}$.** The algebra $\mathcal{L}_{2,3} = L\{e_{13} + \lambda e_2, e_4\}$ corresponds to the group $G_{2,3}$ generated by elliptic helices with a space-like axis and by translations along a time-like straight line. By $P_{1,2a}$ denote the class $P_{1,2}$ for $\mu = 0$, $\lambda \neq 0$. The class $P_{2,3}$ is an intersection of $P_{1,1b}$ and $P_{1,2a}$. In this case, the substitution (3.7) takes the form
\[
x^1 = r \sin \varphi, \quad x^2 = \lambda \varphi + \tilde{x}^2, \quad x^3 = r \cos \varphi, \quad x^4 = \tilde{x}^4.
\] (3.29)
Since $P_{2,3} \subset P_{1,2}$, it follows that $P_{2,3}$ is defined by (3.9). If $P_{2,3} \subset P_{1,1b}$, then all components $A_i$ are independent of $x^4$, therefore we have the following result.

**Statement 14.** The class $P_{2,3}$ of potentials that admit the group $G_{2,3}$ consists of the following fields
\[
A_1 = b_1 \cos \varphi + b_2 \sin \varphi, \quad A_2 = A_2(r, \tilde{x}^2),
\]
\[
A_3 = -b_1 \sin \varphi + b_2 \cos \varphi, \quad A_4 = A_4(r, \tilde{x}^2),
\] (3.30)
where $b_k = b_k(r, \tilde{x}^2)$, $A_2(r, \tilde{x}^2)$, and $A_4(r, \tilde{x}^2)$ are arbitrary functions (the transformation of coordinates is defined by (3.24)).

3.2.4. **Class $P_{2,4}$.** The algebra $\mathcal{L}_{2,4} = L\{e_{13} + \lambda e_2, e_2 + e_4\}$ corresponds to the group $G_{2,4}$ generated by elliptic helices with a space-like axis and by translations along an isotropic straight line. The class $P_{2,4}$ is an intersection of classes $P_{1,2a}$ and $P_{1,1c}$. Substituting (3.9) for $A_i$ in (3.26), we get the following result.

**Statement 15.** The class $P_{2,4}$ of potentials that admit the group $G_{2,4}$ consists of the following fields
\[
A_1 = C_1 \cos \varphi + C_2 \sin \varphi, \quad A_2 = A_2(r, \tilde{x}^2 - \tilde{x}^4),
\]
\[
A_3 = -C_1 \sin \varphi + C_2 \cos \varphi, \quad A_4 = A_4(r, \tilde{x}^2 - \tilde{x}^4),
\] (3.31)
where $C_i = C_i(r, \tilde{x}^2 - \tilde{x}^4)$ are arbitrary functions (the transformation of coordinates is defined by (3.28)).
3.2.5. Class $P_{2.5}$. The algebra $L_{2.5} = L\{e_{24} + \lambda e_3, e_1\}$ corresponds to the group $G_{2.5}$ generated by hyperbolic helices and by translations along the space-like straight line. By $P_{1.3a}$ denote the class of potentials that admit the group $G_{1.3a}$ corresponding to the algebra

$$L_{1.3a} = L\{e_{24} + \lambda e_3\};$$

it is defined by (3.13), where the substitution (3.11) is replaced by the following one:

$$x^1 = \tilde{x}^1, \quad x^2 = r \cosh \varphi, \quad x^3 = \lambda \varphi + \tilde{x}^3, \quad x^4 = r \sinh \varphi. \quad (3.32)$$

The class $P_{2.5}$ is an intersection of classes $P_{1.3a}$ and $P_{1.1a}$. Combining (3.13), (3.32), and (3.1), we get the following result.

**Statement 16.** The class $P_{2.5}$ of potentials that admit the group $G_{2.5}$ consists of the following fields

$$A_1 = A_1(r, \tilde{x}^3), \quad A_2 = C_1 \cosh \varphi + C_2 \sinh \varphi, \quad A_3 = A_3(r, \tilde{x}^3), \quad A_4 = -C_1 \sinh \varphi - C_2 \cosh \varphi, \quad (3.33)$$

where $C_i = C_i(r, \tilde{x}^3)$ are arbitrary functions and the transformation of coordinates is defined by (3.32).

3.2.6. Class $P_{2.6}$. The algebra $L_{2.6} = L\{e_{24} + \lambda e_3, e_2 - e_4\}$ corresponds to the group $G_{2.6}$ generated by hyperbolic helices and by translations along the isotropic straight line. Since $L_{1.3a} \subset L_{2.6}$, we have $P_{2.6} \subset P_{1.3a}$. The equation (3.22) for the vector $\xi = e_2 - e_4$ takes the form

$$\partial_2 A_i - \partial_4 A_i = 0. \quad (3.34)$$

Using the substitution (3.32), we solve the equation (3.34) for the potential (3.13). We get the following result.

**Statement 17.** The class $P_{2.6}$ of potentials that admit the group $G_{2.6}$ consists of the following fields

$$A_1 = A_1(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r), \quad A_2 = C_1 \cosh \varphi + C_2 \sinh \varphi, \quad A_3 = A_3(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r), \quad A_4 = -C_1 \sinh \varphi - C_2 \cosh \varphi, \quad (3.35)$$

where

$$C_1 = a_1(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \cosh \ln r + a_2(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \sinh \ln r, \quad C_2 = a_1(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \sinh \ln r + a_2(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \cosh \ln r; \quad (3.36)$$

the transformation of coordinates is defined by (3.32).

3.2.7. Here we describe classes of potentials corresponding to the algebra $L_{2.7} = L\{e_{12} - e_{14} + \lambda e_3 + \mu e_3, e_2 - e_4\}$ ($\lambda \mu = 0$) for various $\lambda$ and $\mu$. The corresponding group $G_{2.7}$ is generated by parabolic helices and by translations along the isotropic straight line. The algebra $L_{2.7}$ is an extension of $L_{1.4}$ by means of the vector $\xi = e_2 - e_4$; therefore corresponding classes $P_{2.7a}$, $P_{2.7b}$, and $P_{2.7c}$ are restrictions of classes $P_{1.4a}$, $P_{1.4b}$, and $P_{1.4c}$ by the condition (3.34).
3.2.7.1. **Class P_{2,7a}**. For \( \lambda = \mu = 0 \) the substitution (3.16) transforms the equation (3.34) to the form \( \tilde{x}^1 \partial A_i / \partial \tilde{x}^4 = 0 \); therefore all components of potential are independent of \( \tilde{x}^4 \).

**Statement 18.** The class \( P_{2,7a} \) of potentials that admit the group \( G_{2,7a} \), corresponding to the algebra \( L_{2,7} \) (\( \lambda = \mu = 0 \)), consists of the following fields

\[
A_1 = C_2 \tilde{x}^2 + C_3, \quad A_2 = \frac{1}{2} C_2 (\tilde{x}^2)^2 + C_3 \tilde{x}^2 + C_1, \\
A_3 = A_3(\tilde{x}^1, \tilde{x}^3), \quad A_4 = A_2 + C_2,
\]

(3.37)

where \( A_3(\tilde{x}^1, \tilde{x}^3) \) and \( C_k = C_k(\tilde{x}^1, \tilde{x}^3) \) (\( k = 1, 2, 3 \)) are arbitrary functions and the transformation of coordinates is defined by (3.16).

3.2.7.2. **Class P_{2,7b}**. For \( \lambda = 0, \mu \neq 0 \) we use the substitution (3.19) instead of (3.16). We have the following result.

**Statement 19.** The class \( P_{2,7b} \) of potentials that admit the group \( G_{2,7b} \), corresponding to the algebra \( L_{2,7} \) (\( \lambda = 0, \mu \neq 0 \)), consists of the fields (3.37), where the transformation of coordinates is defined by (3.19).

3.2.7.3. **Class P_{2,7c}**. For \( \lambda \neq 0 \) and \( \mu = 0 \) the substitution (3.20) transforms the equation (3.34) to the form: \( -\lambda \partial A_i / \partial \tilde{x}^4 = 0 \); therefore all components of potential are independent of \( \tilde{x}^4 \).

**Statement 20.** The class \( P_{2,7c} \) of potentials that admit the group \( G_{2,7c} \), corresponding to the algebra \( L_{2,7} \) (\( \lambda \neq 0, \mu = 0 \)), consists of the fields (3.37), where the transformation of coordinates is defined by (3.20).

3.2.8. **Class P_{2,8}**. The algebra \( L_{2,8} = L\{e_{12} - e_{14} + \lambda e_2, \ e_3\} \) corresponds to the group \( G_{2,8} \) generated by parabolic helices and by translations along a space-like straight line. Since \( L_{1,4c} \subset L_{2,8} \), then \( P_{2,8} \subset P_{1,4c} \). The class \( P_{2,8} \) is a restriction of the class \( P_{1,4c} \) by the condition (2.22) for the vector \( e_3 \)

\[
\partial_3 A_i = 0.
\]

(3.38)

The substitution (3.20) transforms the equation (3.38) to the form: \( \partial A_i / \partial \tilde{x}^3 = 0 \); thus we have the following result.

**Statement 21.** The class \( P_{2,8} \) of potentials that admit the group \( G_{2,8} \) consists of the following fields

\[
A_1 = C_2 \tilde{x}^2 + C_3, \quad A_2 = \frac{1}{2} C_2 (\tilde{x}^2)^2 + C_3 \tilde{x}^2 + C_1, \\
A_3 = A_3(\tilde{x}^1, \tilde{x}^4), \quad A_4 = A_2 + C_2,
\]

(3.39)

where \( A_3(\tilde{x}^1, \tilde{x}^4) \) and \( C_k = C_k(\tilde{x}^1, \tilde{x}^4) \) (\( k = 1, 2, 3 \)) are arbitrary functions and the transformation of coordinates is defined by (3.20).
3.2.9. **Class $P_{2.9}$**. The algebra $L_{2.9} = L\{e_{13} + \lambda e_{24}, e_2 - e_4\}$ corresponds to the group $G_{2.9}$ generated by proportional bi-rotations and by translations along an isotropic straight line. Since $L_{1.5} \subset L_{2.9}$, then the class $P_{2.9}$ is a subclass of the class $P_{1.5}$. The class $P_{2.9}$ is a restriction of the class $P_{1.5}$ by the condition (3.33) (2.2) for the vector $e_2 - e_4$. The substitution (3.23) transforms the equation (3.34) to the form:
\[
\frac{\partial A_i}{\partial \varphi} + \frac{\partial A_i}{\partial \theta} - \lambda \rho \frac{\partial A_i}{\partial \rho} = 0; \tag{3.40}
\]
thus we have the following result.

**Statement 22.** The class $P_{2.9}$ of potentials that admit the group $G_{2.9}$ consists of the following fields
\[
\begin{align*}
A_1 &= C_1 \cos \varphi + C_2 \sin \varphi, \\
A_2 &= \rho \Phi_3 e^{\lambda \varphi}, \\
A_3 &= -C_1 \sin \varphi + C_2 \cos \varphi, \\
A_4 &= -\rho \Phi_3 e^{\lambda \varphi},
\end{align*} \tag{3.41}
\]
\[
\begin{align*}
C_1 &= \Phi_1 \cos \frac{\ln \rho}{\lambda} + \Phi_2 \sin \frac{\ln \rho}{\lambda}, \\
C_2 &= -\Phi_1 \sin \frac{\ln \rho}{\lambda} + \Phi_2 \cos \frac{\ln \rho}{\lambda},
\end{align*} \tag{3.42}
\]
where $\Phi_k = \Phi_k(r, \lambda \theta + \ln \rho)$ are arbitrary functions and the transformation of coordinates is defined by (3.23).

3.2.10. **Class $P_{2.10}$**. The algebra $L_{2.10} = L\{e_{13}, e_{24}\} = L\{e_{13} + e_{24}, e_{13}\}$ corresponds to the group $G_{2.10}$ generated by rotations and pseudo-rotations or, equivalently, by rotations and proportional bi-rotations for $\lambda = 1$. As $L_{1.5} \subset L_{2.10}$, then $C_{2.10} \subset C_{1.5}$ ($\lambda = 1$). In this case (3.23) takes the form
\[
\begin{align*}
A_1 &= C_1 \cos \varphi + C_2 \sin \varphi, \\
A_2 &= C_3 e^\varphi + C_4 e^{-\varphi}, \\
A_3 &= -C_1 \sin \varphi + C_2 \cos \varphi, \\
A_4 &= -C_3 e^\varphi + C_4 e^{-\varphi}.
\end{align*} \tag{3.43}
\]
Substituting (3.43) for $A_i$ in the equation (3.6) for $\lambda = \mu = 0$
\[
\begin{align*}
x^3 \partial_1 A_i - x^1 \partial_3 A_i + A_1 \delta^1_i - A_3 \delta^3_i = 0,
\end{align*} \tag{3.44}
\]
we obtain the following result.

**Statement 23.** The class $P_{2.10}$ of potentials that admit the group $G_{2.10}$ consists of the following fields
\[
\begin{align*}
A_1 &= -t_1 \sin(\theta - \varphi) + t_2 \cos(\theta - \varphi), \\
A_2 &= t_3 e^\varphi + t_4 e^{-\varphi}, \\
A_3 &= t_1 \cos(\theta - \varphi) + t_2 \sin(\theta - \varphi), \\
A_4 &= -t_3 e^\varphi + t_4 e^{-\varphi},
\end{align*} \tag{3.45}
\]
where $t_k = t_k(r, \rho)$ are arbitrary functions and the transformation of coordinates is defined by (3.23) for $\lambda = 1$:
\[
\begin{align*}
x^1 &= r \cos(\theta - \varphi), \\
x^2 &= \rho \cosh \varphi, \\
x^3 &= r \sin(\theta - \varphi), \\
x^4 &= \rho \sinh \varphi.
\end{align*} \tag{3.46}
\]
3.2.11. Here we describe classes of potentials corresponding to the algebra $L_{2,11} = L\{e_{12} - e_{34} + \lambda e_1 + \mu e_3, e_{23} + e_{34} - \mu e_1 + \lambda e_3\}$ ($\lambda = 0, \mu \neq 0 \sim \lambda \neq 0, \mu = 0$). The case $\lambda = \mu = 0$ is required for description some following classes.

3.2.11.1. Class $P_{2,11}$ ($\lambda = 0, \mu \neq 0$). The algebra $L_{2,11}$ corresponds to the group $G_{2,11}$ generated by two one-dimensional subgroups of parabolic helices with different axises. Since $L_{1,4b} \subset L_{2,11}$, then $P_{2,11} \subset P_{1,4b}$. For description the class $P_{2,11}$ we substitute (3.18)–(3.19) for $A_i$ in equation (2.2) for the basis vectors

$$XA_i - A_2 \delta^3_i + A_3 (\delta^2_i + \delta^4_i) + A_4 \delta^3_i = 0,$$

(3.47)

$$Xf = -\mu \partial_1 f - x^3 \partial_2 f + (x^2 + x^4) \partial_3 f + x^3 \partial_4 f =$$

$$= \frac{\mu}{\bar{x}^1} \frac{\partial f}{\partial \bar{x}^2} + \frac{(\bar{x}^1)^2 - \mu^2}{\bar{x}^1} \frac{\partial f}{\partial \bar{x}^3} - \bar{x}^1 \frac{\partial f}{\partial \bar{x}^4}$$

(3.48)

We use the substitution

$$u = \frac{\bar{x}^1 \bar{x}^3}{(\bar{x}^1)^2 - \mu^2}, \quad v = \frac{1}{2} (\bar{x}^1 \bar{x}^3)^2 + \bar{x}^4 (\bar{x}^1)^2 - \mu^2$$

(3.49)

to solve these equations; we obtain the following result.

Statement 24. The class $P_{2,11}$ of potentials that admit the group $G_{2,11}$ consists of the following fields

$$A_1 = \Phi \bar{x}^2 + \Psi, \quad A_2 = \frac{1}{2} \Phi (\bar{x}^2)^2 + \Psi \bar{x}^2 + \Xi,$$

(3.50)

$$A_3 = \Upsilon, \quad A_4 = \frac{1}{2} \Phi (\bar{x}^2)^2 + \Psi \bar{x}^2 + \Xi + \Phi,$$

$$\Psi = \frac{\mu u}{\bar{x}^1} \Phi + C_1, \quad \Upsilon = -\Phi u + C_2,$$

$$\Xi = \frac{\mu^2 + (\bar{x}^1)^2}{2 (\bar{x}^1)^2} \Phi u^2 - \frac{\mu C_1 + \bar{x}^1 C_2}{\bar{x}^1} u + C_3,$$

(3.51)

where $\Phi = \Phi(\bar{x}^1, v)$ and $C_k = C_k(\bar{x}^1, v)$ are arbitrary functions and transformations of variables are defined by (3.19) and (3.49).

3.2.11.2. Class $P_{2,11a}$ ($\lambda = \mu = 0$). The algebra

$$L_{2,11a} = L\{e_{12} - e_{14}, e_{23} + e_{34}\}$$

corresponds to the group $G_{2,11a}$ generated by two one-dimensional subgroups of parabolic rotations. We write the equation (2.2) for the basis vectors $e_{12} - e_{14}$ and $e_{23} + e_{34}$:

$$XA_1 + A_2 - A_4 = 0, \quad XA_2 - A_1 = 0, \quad XA_3 = 0, \quad XA_4 - A_1 = 0,$$

$$X = -(x^2 + x^4) \partial_1 + x^1 \partial_2 - x^1 \partial_4,$$

(3.52)
and

\[ Y A_1 = 0, \quad Y A_2 + A_3 = 0, \quad Y A_3 - A_2 + A_4 = 0, \quad Y A_4 + A_3 = 0, \]
\[ Y = -x^2 \partial_2 + (x^2 + x^4) \partial_3 + x^3 \partial_4. \]  

(3.53)

We use the substitution

\[ \tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^2 = -x^1/(x^2 + x^4), \quad \tilde{x}^3 = x^3/(x^2 + x^4), \]
\[ \tilde{x}^4 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2; \]  

(3.54)

the operator \(X\) is replaced by partial derivative with respect to \(\tilde{x}^2\), the operator \(Y\) — by partial derivative with respect to \(\tilde{x}^3\). We have the following solution of the system (3.52)–(3.53):

\[ A_1 = -\tilde{x}^2 \Phi + \Psi, \quad A_2 = -\frac{1}{2} \Phi((\tilde{x}^2)^2 + (\tilde{x}^3)^2) + \tilde{x}^2 \Psi - \tilde{x}^3 \Xi + \Theta, \]
\[ A_3 = \tilde{x}^3 \Phi + \Xi, \quad A_4 = A_2 - \Phi, \]  

(3.55)

where \(\Phi = \Phi(\tilde{x}^1, \tilde{x}^4), \quad \Psi = \Psi(\tilde{x}^1, \tilde{x}^4), \quad \Xi = \Xi(\tilde{x}^1, \tilde{x}^4)\) and \(\Theta = \Theta(\tilde{x}^1, \tilde{x}^4)\) are arbitrary functions.

**Statement 25.** The class \(P_{2,11a}\) of potentials that admit the group \(G_{2,11a}\) is defined by (3.55) and (3.54).

3.2.12. Class \(P_{2,12}\). The algebra \(L_{2,12} = L\{e_{12} - e_{14}, \ e_{24} + \lambda e_3\}\) corresponds to the group \(G_{2,12}\) generated by parabolic rotations and by hyperbolic helices. Since \(L_{1,4a} \subset L_{2,12}\) and \(L_{1,3a} \subset L_{2,12}\), then \(P_{2,12} = P_{1,4a} \cap P_{1,3a}\). For description the class \(P_{2,12}\) we substitute (3.18)–(3.16) for \(A_i\) in equation (2.2) for the vector \(\xi = e_{24} + \lambda e_3\)

\[ x^4 \partial_2 A_i + \lambda \partial_3 A_i + x^2 \partial_1 A_i + A_2 \delta_i^1 + A_4 \delta_i^2 = 0. \]  

(3.56)

The equation (3.56) is a system

\[XA_1 = 0, \quad XA_2 + A_3 = 0, \quad XA_3 = 0, \quad XA_4 + A_2 = 0,\]  

(3.57)

where the operator \(X\) by substitution (3.16) is replaced to the form:

\[X f = x^4 \partial_2 f + \lambda \partial_3 f + x^2 \partial_4 f = \tilde{x}^1 \frac{\partial f}{\partial \tilde{x}^1} - \tilde{x}^2 \frac{\partial f}{\partial \tilde{x}^2} + \lambda \frac{\partial f}{\partial \tilde{x}^3} + (\tilde{x}^1)^2 \frac{\partial f}{\partial \tilde{x}^4}.\]

Substituting (3.18) for \(A_i\) in (3.57), we obtain some differential equation; taking into account a linear independence of the functions \((\tilde{x}^2)^2\),...
Using the substitution

\[ \Phi_k(3.16) \]

where transformations of variables are defined by

we integrate the system (3.58); the result is

Euclidean space \( Ox \) and \( G \) the group \( O_3 \) of dimensional groups of translations.

3.3.1. Here we describe classes of potentials corresponding to three-dimensional symmetry groups.

**Statement 26.** The class \( P_{2,12} \) of potentials that admit the group \( G_{2,12} \) is defined by

\[
\begin{align*}
A_1 &= \bar{x}^1 \bar{x}^2 \Phi_1 + \Phi_3, \\
A_2 &= \frac{\bar{x}^1}{2} \left( \left( \frac{\bar{x}^2}{\bar{x}^1} \right)^2 - 1 \right) \Phi_1 + \bar{x}^2 \Phi_3 + \frac{\Phi_2}{\bar{x}^1}, \\
A_3 &= \Phi_4, \\
A_4 &= \frac{\bar{x}^1}{2} \left( \left( \frac{\bar{x}^2}{\bar{x}^1} \right)^2 + 1 \right) \Phi_1 + \bar{x}^2 \Phi_3 + \frac{\Phi_2}{\bar{x}^1},
\end{align*}
\]

(3.61)

where transformations of variables are defined by (3.18) and (3.59).

3.3. Potentials that admit three-dimensional symmetry groups.

3.3.1. Here we describe classes of potentials corresponding to three-dimensional groups of translations.

3.3.1.1. Class \( P_{3,1a} \) The algebra \( \mathcal{L}_{3,1a} = L\{e_1, e_2, e_3\} \) corresponds to the group \( G_{3,1a} \) of translations along the vectors of three-dimensional Euclidean space \( Ox^1 x^2 x^3 \). Since \( \mathcal{L}_{2,1a} \subset \mathcal{L}_{3,1a} \), then the class \( P_{3,1a} \) is a subclass of \( P_{2,1a} \). Substituting \( A_i(x^3, x^4) \) for \( A_i \) in the equation (3.58) ((2.2) for the vector \( \xi = e_3 \)), we have the following result.

**Statement 27.** The class \( P_{3,1a} \) of potentials that admit the group \( G_{3,1a} \) consists of the fields \( A_i = A_i(x^4) \).
3.3.1.2. **Class** $P_{3,1b}$. The algebra $\mathcal{L}_{3,1b} = L\{e_1, e_2, e_4\}$ corresponds to the group $G_{3,1b}$ of translations along the vectors of three-dimensional pseudo-Euclidean space $Ox^1x^2x^4$. Since $\mathcal{L}_{2,1b} \subset \mathcal{L}_{3,1b}$, then the class $C_{3,1b}$ is a subclass of $C_{2,1b}$. Substituting $A_i(x^1, x^3)$ for $A_i$ in the equation (3.2), we have the following result.

**Statement 28.** The class $P_{3,1b}$ of potentials that admit the group $G_{3,1b}$ consists of the fields $A_i = A_i(x^3)$.

3.3.1.3. **Class** $P_{3,1c}$. The algebra $\mathcal{L}_{3,1c} = L\{e_1, e_3, e_2 + e_4\}$ corresponds to the group $G_{3,1c}$ of translations along the vectors of a three-dimensional isotropic space. Since $\mathcal{L}_{2,1c} \subset \mathcal{L}_{3,1c}$, then the class $P_{3,1c}$ is a subclass of $P_{2,1c}$. Substituting $A_i(x^3, x^2 - x^4)$ for $A_i$ in (3.38), we have the following result.

**Statement 29.** The class $P_{3,1c}$ of potentials that admit the group $G_{3,1c}$ consists of the fields $A_i = A_i(x^2 - x^4)$.

3.3.2. **Class** $P_{3,2}$. The algebra $\mathcal{L}_{3,2} = L\{e_{13} + \lambda e_2, e_1, e_3\}$ ($\lambda \neq 0$) corresponds to the group $G_{3,2}$ generated by elliptic helices with a space-like axis and by translations along the vectors of the two-dimensional Euclidean plane. We obtain the class $P_{3,2}$ as a solution of the system of equations (3.1), (3.38), and (3.6) for $\mu = 0$:

$$x^3 \partial_1 A_i + \lambda \partial_2 A_i - x^1 \partial_3 A_i + A_i \delta_3^1 - A_3 \delta_1^3 = 0. \quad (3.62)$$

The solution of the system (3.1)–(3.38) is $A_i = A_i(x^2, x^4)$. Substituting $A_i(x^2, x^4)$ for $A_i$ in equation (3.62), we have

$$\lambda \partial_2 A_1 - A_3 = 0, \quad \lambda \partial_2 A_2 = 0, \quad \lambda \partial_2 A_3 + A_1 = 0, \quad \lambda \partial_2 A_4 = 0. \quad (3.63)$$

We obtain the solution of the system (3.63) for $\lambda \neq 0$ in the form

$$A_1 = C_1(x^4) \sin \frac{x^2}{\lambda} + C_2(x^4) \cos \frac{x^2}{\lambda}, \quad A_2 = A_2(x^4),$$

$$A_3 = C_1(x^4) \cos \frac{x^2}{\lambda} - C_2(x^4) \sin \frac{x^2}{\lambda}, \quad A_4 = A_4(x^4), \quad (3.64)$$

where $C_1(x^4), C_2(x^4), A_2(x^4), A_4(x^4)$ are arbitrary functions.

For $\lambda = 0$ the group $G_{3,2}$ is a motion group of the two-dimensional Euclidean plane; in this case we have the solution of the system (3.63) in the form

$$A_1 = A_3 = 0, \quad A_2 = A_2(x^2, x^4), \quad A_4 = A_4(x^2, x^4). \quad (3.65)$$

**Statement 30.** For $\lambda \neq 0$ the class $P_{3,2}$ of potentials that admit the group $G_{3,2}$ consists of the fields (3.64); for $\lambda = 0$ this class defined by (3.65).
3.3.3. Class $P_{3,3}$. The algebra $\mathcal{L}_{3,3} = L\{e_{13} + \mu e_4, e_1, e_3\}$ ($\mu \neq 0$) corresponds to the group $G_{3,3}$ generated by elliptic helices with a time-like axis and by translations along the vectors of the two-dimensional Euclidian plane. We obtain the class $P_{3,3}$ as a solution of the system of equations (3.1), (3.38) and (3.6) for $\lambda = 0$:

$$x^3 \partial_1 A_i - x^1 \partial_3 A_i + \mu \partial_4 A_i + A_1 \delta_i^3 - A_3 \delta_i^1 = 0.$$  \hspace{1cm} (3.66)

We have the following result.

**Statement 31.** The class $P_{3,3}$ of potentials that admit the group $G_{3,3}$ consists of the fields

$$A_1 = C_1(x^2) \sin \frac{x^4}{\mu} + C_2(x^2) \cos \frac{x^4}{\mu}, \quad A_2 = A_2(x^2),$$

$$A_3 = C_1(x^2) \cos \frac{x^4}{\mu} - C_2(x^2) \sin \frac{x^4}{\mu}, \quad A_4 = A_4(x^2),$$ \hspace{1cm} (3.67)

where $C_1(x^2)$, $C_2(x^2)$, $A_2(x^2)$, and $A_4(x^2)$ are arbitrary functions.

3.3.4. Class $P_{3,4}$. The algebra $\mathcal{L}_{3,4} = L\{e_{13} + \lambda(e_2 + e_4), e_1, e_3\}$ corresponds to the group $G_{3,4}$ generated by elliptic helices with an isotropic axis and by translations along the vectors of the two-dimensional Euclidian plane $Ox^1x^3$. We obtain the class $P_{3,4}$ as a solution of the system of equations (3.1), (3.38) and (3.6) for $\lambda = \mu \neq 0$:

$$x^3 \partial_1 A_i + \lambda \partial_2 A_i - x^1 \partial_3 A_i + \lambda \partial_4 A_i + A_1 \delta_i^3 - A_3 \delta_i^1 = 0.$$  \hspace{1cm} (3.68)

Substituting $A_i(x^2, x^4)$ for $A_i$ in (3.68), we have

$$\lambda(\partial_2 + \partial_4)A_1 - A_3 = 0, \quad \lambda(\partial_2 + \partial_4)A_2 = 0,$$

$$\lambda(\partial_2 + \partial_4)A_3 + A_1 = 0, \quad \lambda(\partial_2 + \partial_4)A_4 = 0.$$  \hspace{1cm} (3.69)

Using the substitution

$$u = x^2 + x^4, \quad v = x^2 - x^4,$$ \hspace{1cm} (3.70)

we get the solution of the system (3.69):

$$A_1 = C_1(v) \sin \frac{u}{2\lambda} + C_2(v) \cos \frac{u}{2\lambda}, \quad A_2 = A_2(v),$$

$$A_3 = C_1(v) \cos \frac{u}{2\lambda} - C_2(v) \sin \frac{u}{2\lambda}, \quad A_4 = A_4(v),$$ \hspace{1cm} (3.71)

where $C_1(v)$, $C_2(v)$, $A_2(v)$, and $A_4(v)$ are arbitrary functions.

**Statement 32.** The class $P_{3,4}$ of potentials that admit the group $G_{3,4}$ is defined by (3.71) and (3.70).
3.3.5. **Class \(P_{3,5}\).** The algebra \(L_{3,5} = L\{e_{24}, e_1, e_3\}\) corresponds to the group \(G_{3,5}\) generated by pseudo-rotations in the plane \(Ox^2x^4\) and by translations along the vectors of the Euclidean plane \(Ox^1x^3\). We obtain the class \(P_{3,5}\) as a solution of the system of equations (3.76) in the form (3.77) for \(\lambda = 0\):

\[
x^4 \partial_2 A_i + x^2 \partial_4 A_i + A_2 \delta_i^4 + A_4 \delta_i^2 = 0.
\]  

Substituting \(A_i(x^2, x^4)\) for \(A_i\) in (3.72), we get

\[
XA_1 = 0, \quadXA_2 + A_4 = 0, \quadXA_3 = 0, \quadXA_4 + A_2 = 0,
\]  

where \(X = x^4 \partial_2 + x^2 \partial_4\). Using the substitution

\[
x^2 = \rho \cosh \varphi, \quad x^4 = \rho \sinh \varphi,
\]  

we obtain the solution of the system (3.76) in the form

\[
A_1 = A_1(\rho), \quad A_2 = C_1(\rho) \cosh \varphi + C_2(\rho) \sinh \varphi,
\]

\[
A_3 = A_3(\rho), \quad A_4 = -C_1(\rho) \sinh \varphi - C_2(\rho) \cosh \varphi,
\]  

where \(C_1(\rho), C_2(\rho), A_1(\rho),\) and \(A_3(\rho)\) are arbitrary functions.

**Statement 33.** The class \(P_{3,5}\) of potentials that admit the group \(G_{3,5}\) is defined by (3.75) and (3.72).

3.3.6. **Class \(P_{3,6}\).** The algebra \(L_{3,6} = L\{e_{24} + \lambda e_3, e_2, e_4\}\) corresponds to the group \(G_{3,6}\) generated by hyperbolic helices and translations along the vectors of the pseudo-Euclidean plane. The algebra \(L_{3,6}\) is an extension of \(L_{2,16}\) by means of the vector \(\xi = e_{24} + \lambda e_3\), therefore \(P_{3,6} \subset P_{2,16}\). Substituting \(A_i(x^1, x^3)\) for \(A_i\) in the equation (3.75) (2.22) for the vector \(\xi = e_{24} + \lambda e_3\), we get

\[
\lambda \partial_3 A_1 = 0, \quad \lambda \partial_3 A_2 + A_4 = 0, \quad \lambda \partial_3 A_3 = 0, \quad \lambda \partial_3 A_4 + A_2 = 0.
\]  

For \(\lambda \neq 0\) we have the following solution of the system (3.76):

\[
A_1 = A_1(x^1), \quad A_2 = C_1(x^1) \cosh \frac{x^3}{\lambda} + C_2(x^1) \sinh \frac{x^3}{\lambda},
\]

\[
A_3 = A_3(x^1), \quad A_4 = -C_1(x^1) \sinh \frac{x^3}{\lambda} - C_2(x^1) \cosh \frac{x^3}{\lambda},
\]  

where \(C_1(x^1), C_2(x^1), A_1(x^1),\) and \(A_3(x^1)\) are arbitrary functions.

For \(\lambda = 0\) \(G_{3,6}\) is a motion group of two-dimensional pseudo–Euclidean plane; in this case we have the solution of the system (3.76) in the form

\[
A_1 = A_1(x^1, x^3), \quad A_2 = 0, \quad A_3 = A_3(x^1, x^3), \quad A_4 = 0.
\]  

**Statement 34.** The class \(P_{3,6}\) of potentials that admit the group \(G_{3,6}\) is defined by (3.77) for \(\lambda \neq 0\); for \(\lambda = 0\) it is defined by (3.78).
3.3.7. Class $P_{3,7}$. The algebra $\mathcal{L}_{3,7} = L\{e_{24} + \lambda e_3, e_1, e_2 - e_4\}$ corresponds to the group $G_{3,7}$ generated by hyperbolic helices and by translations along the vectors of an isotropic plane. The algebra $\mathcal{L}_{3,7}$ is an extension of $\mathcal{L}_{2,6}$ by means of the vector $e_1$, therefore $P_{3,7} \subset P_{2,6}$. Since $A_i$ satisfies to the equation (3.1), then it is independent of $x^1 = \tilde{x}^1$; thus we have the following result.

**Statement 35.** The class $P_{3,7}$ of potentials that admit the group $G_{3,7}$ consists of the fields

$$
A_1 = A_1(u), \quad A_2 = C_1 \cosh \varphi + C_2 \sinh \varphi,
A_3 = A_3(u), \quad A_4 = -C_1 \sinh \varphi - C_2 \cosh \varphi,
$$

(3.79)

where

$$
C_1 = a_1(u) \cosh \ln r + a_2(u) \sinh \ln r,
C_2 = a_1(u) \sinh \ln r + a_2(u) \cosh \ln r,
$$

(3.80)

$u = \tilde{x}^3 - \lambda \ln r$, and the transformation of coordinates is defined by (3.82).

3.3.8. Class $P_{3,8}$. The algebra $\mathcal{L}_{3,8} = L\{e_{12} - e_{14} + \lambda e_2, e_3, e_2 - e_4\}$ corresponds to the group $G_{3,8}$ generated by parabolic helices and by translations along the vectors of an isotropic plane. The algebra $\mathcal{L}_{3,8}$ is an extension of $\mathcal{L}_{2,7c}$ and $\mathcal{L}_{2,8}$, therefore $P_{3,8} = P_{2,7c} \cap P_{2,8}$. We have the following result.

**Statement 36.** The class $P_{3,8}$ of potentials that admit the group $G_{3,8}$ consists of the fields

$$
A_1 = C_2 \tilde{x}^2 + C_3, \quad A_2 = \frac{1}{2} C_2 (\tilde{x}^2)^2 + C_3 \tilde{x}^2 + C_1,
A_3 = A_3(\tilde{x}^1), \quad A_4 = A_2 + C_2,
$$

(3.81)

where $A_3(\tilde{x}^1)$ and $C_k = C_k(\tilde{x}^1)$ ($k = 1, 2, 3$) are arbitrary functions and $\tilde{x}^1 = 2\lambda x^1 + (x^2 + x^4)^2$.

3.3.9. Here we describe classes of potentials corresponding to the algebra $\mathcal{L}_{3,9} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3, e_1, e_2 - e_4\}$ ($\lambda \mu = 0$) for various $\lambda$ and $\mu$. The corresponding group $G_{3,9}$ is generated by parabolic helices and by translations along the vectors of an isotropic plane. The algebra $\mathcal{L}_{3,9}$ is an extension of $\mathcal{L}_{2,7}$ by means of the vector $e_1$, therefore the corresponding classes $P_{3,9a}$, $P_{3,9b}$, and $P_{3,9c}$ are restrictions of classes $P_{2,7a}$, $P_{2,7b}$, and $P_{2,7c}$ by the condition (3.1). We consider three cases:

a) $\lambda = \mu = 0$;

b) $\lambda = 0$, $\mu \neq 0$;

c) $\lambda \neq 0$, $\mu = 0$.

3.3.9.1. Class $P_{3,9b}$. For $\lambda = 0$, $\mu \neq 0$ we use the substitution (3.19), the equation (3.1) is transformed to

$$
-\frac{1}{\tilde{x}^1} \frac{\partial A_i}{\partial \tilde{x}^2} + \mu \frac{\partial A_i}{\tilde{x}^1 \tilde{x}^3} - \tilde{x}^1 \tilde{x}^2 \frac{\partial A_i}{\partial \tilde{x}^4} = 0.
$$

(3.82)
Substituting (3.37) for \( A_i \) in (3.34), we get some equation; using a linear independence of functions \((\tilde{x}^2)^2, \tilde{x}^2, \) and 1, we obtain the following equations

\[
\frac{\mu}{\partial \tilde{x}^3} C_2 = 0, \quad \frac{\mu}{\partial \tilde{x}^3} C_3 - C_2 = 0, \quad \frac{\mu}{\partial \tilde{x}^3} C_1 - C_3 = 0, \quad \frac{\mu}{\partial \tilde{x}^3} A_3 = 0
\]

(3.83)

for the functions \( C_k(\tilde{x}^1, \tilde{x}^3) \) and \( A_3(\tilde{x}^1, \tilde{x}^3) \). We have the solution of (3.83) in the form:

\[
C_1 = \frac{(\tilde{x}^3)^2}{2\mu^2} \Phi(\tilde{x}^1) + \frac{\tilde{x}^3}{\mu} \Psi(\tilde{x}^1) + \Xi(\tilde{x}^1), \quad C_2 = \Phi(\tilde{x}^1),
\]

\[
C_3 = \frac{\tilde{x}^3}{\mu} \Phi(\tilde{x}^1) + \Psi(\tilde{x}^1), \quad A_3 = A_3(\tilde{x}^1),
\]

(3.84)

where \( A_3(\tilde{x}^1) \), \( \Phi(\tilde{x}^1) \), \( \Psi(\tilde{x}^1) \) and \( \Xi(\tilde{x}^1) \) are arbitrary functions and

\[
\tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^3 = x^3 + \frac{\mu x^1}{x^2 + x^4}.
\]

**Statement 37.** The class \( P_{3.96} \) of potentials that admit the group \( G_{3.96} \), corresponding to the algebra \( L_{3.9} \) \((\lambda = 0, \mu \neq 0)\), consists of the fields

\[
A_1 = C_2 \tilde{x}^2 + C_3, \quad A_2 = \frac{1}{2} C_2 (\tilde{x}^2)^2 + C_3 \tilde{x}^2 + C_1,
\]

\[
A_3 = A_3(\tilde{x}^1), \quad A_4 = A_2 + C_2,
\]

(3.85)

where \( C_k \) are defined by (3.84).

3.3.9.2. Class \( P_{3.9a} \). Let now \( \mu = \lambda = 0 \), then \( P_{3.9a} \subset P_{2.7a} \); we obtain the following solution of the system (3.83)

\[
C_1 = C_1(\tilde{x}^1, \tilde{x}^3), \quad C_2 = C_3 = 0, \quad A_3 = A_3(\tilde{x}^1, \tilde{x}^3).
\]

(3.86)

**Statement 38.** The class \( P_{3.9a} \) of potentials that admit the group \( G_{3.9a} \), corresponding to the algebra \( L_{3.9} \) \((\lambda = \mu = 0)\), consists of the fields

\[
A_1 = 0, \quad A_2 = A_4 = C_1(\tilde{x}^1, \tilde{x}^3), \quad A_3 = A_3(\tilde{x}^1, \tilde{x}^3),
\]

(3.87)

where \( C_1(\tilde{x}^1, \tilde{x}^3) \) and \( A_3(\tilde{x}^1, \tilde{x}^3) \) are arbitrary functions and

\[
\tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^3 = x^3.
\]

3.3.9.3. Class \( P_{3.9c} \). For \( \lambda \neq 0, \mu = 0 \) the algebra

\[
L_{3.9c} = \langle e_{12} - e_{14} + \lambda e_2, e_1, e_2 - e_4 \rangle
\]

includes the algebra \( L_{2.7c} \), therefore \( P_{3.9c} \subset P_{2.7c} \). Here we use the substitution (3.20), the equation (3.1) takes the form

\[
2\lambda \frac{\partial A_i}{\partial \tilde{x}^1} + \lambda x^2 \frac{\partial A_i}{\partial \tilde{x}^4} = 0.
\]

(3.88)

Since \( A_i \) is independent of \( \tilde{x}^4 \), we have the following result.
Statement 39. The class $P_{3,9c}$ of potentials that admit the group $G_{3,9c}$, corresponding to the algebra $\mathcal{L}_{3,9c}$, consists of the fields

$$\begin{align*}
A_1 &= C_2\tilde{x}^2 + C_3, \\
A_2 &= \frac{1}{2}C_2(\tilde{x}^2)^2 + C_3\tilde{x}^2 + C_1, \\
A_3 &= A_3(\tilde{x}^3), \\
A_4 &= A_2 + C_2,
\end{align*}$$

(3.89)

where $A_3(\tilde{x}^3)$ and $C_k = C_k(\tilde{x}^3)$ ($k = 1, 2, 3$) are arbitrary functions and $\tilde{x}^2 = (x^2 + x^4)/\lambda$, $\tilde{x}^3 = x^3$.

3.3.10. Here we describe classes of potentials corresponding to the algebra $\mathcal{L}_{3,10} = L\{e_{12} - e_{14} + \lambda e_2, e_1 + \mu e_3, e_2 - e_4\}$ for various $\lambda$ and $\mu$. The corresponding group $G_{3,10}$ is generated by parabolic helices or parabolic rotations and by translations along the vectors of an isotropic plane. If $\mu = 0$, then $\mathcal{L}_{3,10} = \mathcal{L}_{3,9c}$. We consider two cases: a) $\lambda \neq 0$, $\mu \neq 0$; b) $\lambda = 0, \mu \neq 0$.

3.3.10.1. Class $P_{3,10a}$. Let $\lambda \neq 0, \mu \neq 0$. The algebra $\mathcal{L}_{3,10a} = \mathcal{L}_{3,10}$ is an extension of $\mathcal{L}_{2,7c}$ by means of the vector $e_1 + \mu e_3$, therefore $P_{3,10a} \subset P_{2,7c}$. The equation (2.2) for $\xi = e_1 + \mu e_3$ takes the form

$$\partial_1 A_i + \mu \partial_3 A_i = 0.$$  

(3.90)

Using the substitution (3.20) we transform (3.90) to the form

$$2\lambda \frac{\partial A_i}{\partial \tilde{x}^1} + \mu \frac{\partial A_i}{\partial \tilde{x}^3} + \lambda \tilde{x}^2 \frac{\partial A_i}{\partial \tilde{x}^4} = 0.$$  

(3.91)

Substituting (3.37) for $A_i$ in (3.91), we get the following result.

Statement 40. The class $P_{3,10a}$ of potentials that admit the group $G_{3,10a}$, corresponding to the algebra $\mathcal{L}_{3,10a}$, consists of the fields

$$\begin{align*}
A_1 &= C_2\tilde{x}^2 + C_3, \\
A_2 &= \frac{1}{2}C_2(\tilde{x}^2)^2 + C_3\tilde{x}^2 + C_1, \\
A_3 &= A_3(\mu \tilde{x}^1 - 2\lambda \tilde{x}^3), \\
A_4 &= A_2 + C_2,
\end{align*}$$

(3.92)

where $A_3(\mu \tilde{x}^1 - 2\lambda \tilde{x}^3)$ and $C_k = C_k(\mu \tilde{x}^1 - 2\lambda \tilde{x}^3)$ ($k = 1, 2, 3$) are arbitrary functions and $\tilde{x}^1 = 2\lambda \tilde{x}^1 + (x^2 + x^4)^2$, $\tilde{x}^3 = x^3$.

3.3.10.2. Class $P_{3,10b}$. Let $\lambda = 0, \mu \neq 0$. The algebra

$$\mathcal{L}_{3,10b} = L\{e_{12} - e_{14}, e_1 + \mu e_3, e_2 - e_4\}$$

is an extension of $\mathcal{L}_{2,7a}$ by means of the vector $e_1 + \mu e_3$, hence $P_{3,10b} \subset P_{2,7a}$. By means of substitution (3.16) equation (3.90) is transformed to the form

$$- \frac{1}{\tilde{x}^1} \frac{\partial A_i}{\partial \tilde{x}^2} + \mu \frac{\partial A_i}{\partial \tilde{x}^3} = 0.$$  

(3.93)

Substituting (3.37) for $A_i$ in (3.93), we get some equation; using a linear independence of functions $(\tilde{x}^2)^2$, $\tilde{x}^2$, and 1, we obtain the following
equations
\[ \mu \frac{\partial C_2}{\partial \tilde{x}^3} = 0, \quad \mu \frac{\partial C_3}{\partial \tilde{x}^3} - \frac{C_2}{\tilde{x}^1} = 0, \quad \mu \frac{\partial C_1}{\partial \tilde{x}^3} - \frac{C_3}{\tilde{x}^1} = 0, \quad \mu \frac{\partial A_3}{\partial \tilde{x}^3} = 0 \] (3.94)
for the functions \( C_k(\tilde{x}^1, \tilde{x}^3) \) and \( A_3(\tilde{x}^1, \tilde{x}^3) \). We have the solution of (3.94) in the form:
\[ C_1 = \frac{(\tilde{x}^3)^2}{2\mu^2(\tilde{x}^1)^2} \Phi(\tilde{x}^1) + \frac{\tilde{x}^3}{\mu \tilde{x}^1} \Psi(\tilde{x}^1) + \Xi(\tilde{x}^1), \quad C_2 = \Phi(\tilde{x}^1), \quad C_3 = \frac{\tilde{x}^3}{\mu \tilde{x}^1} \Phi(\tilde{x}^1) + \Psi(\tilde{x}^1), \quad A_3 = A_3(\tilde{x}^1), \] (3.95)
where \( A_3(\tilde{x}^1), \Phi(\tilde{x}^1), \Psi(\tilde{x}^1), \) and \( \Xi(\tilde{x}^1) \) are arbitrary functions.

**Statement 41.** The class \( P_{3,10b} \) of potentials that admit the group \( G_{3,10b} \), corresponding to the algebra \( L_{3,10b} \), consists of the fields (3.95), where \( C_k \) are defined by (3.95) and
\[ \tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^2 = -\frac{x^1}{x^2 + x^4}, \quad \tilde{x}^3 = x^3. \]

3.3.11. **Class \( P_{3,11} \).** The algebra \( L_{3,11} = L\{e_{13} + \lambda e_{24}, e_1, e_3\} \) corresponds to the group \( G_{3,11} \) generated by proportional bi-rotations and by translations along the vectors of two-dimensional Euclidean plane. The class \( P_{3,11} \), corresponding to the algebra \( L_{3,11} \), is a subclass of the class \( P_{1,5} \). For description of it we substitute (3.25) for \( A_i \) in equations
\[ \cos(\theta - \varphi) \frac{\partial A_i}{\partial r} - \frac{\sin(\theta - \varphi)}{r} \frac{\partial A_i}{\partial \theta} = 0 \] (3.96)
and
\[ \sin(\theta - \varphi) \frac{\partial A_i}{\partial r} + \frac{\cos(\theta - \varphi)}{r} \frac{\partial A_i}{\partial \theta} = 0 \] (3.97)
(equations (3.1) and (3.38), transformed by substitution (5.28)); we have the following solution
\[ A_1 = C_1(\rho) \cos \varphi + C_2(\rho) \sin \varphi, \]
\[ A_2 = C_3(\rho) \cosh \lambda \varphi + C_4(\rho) \sinh \lambda \varphi, \]
\[ A_3 = -C_1(\rho) \sin \varphi + C_2(\rho) \cos \varphi, \]
\[ A_4 = -C_3(\rho) \sinh \lambda \varphi - C_4(\rho) \cosh \lambda \varphi, \] (3.98)
where \( C_k = C_k(\rho) \) are arbitrary functions.

**Statement 42.** The class \( P_{3,11} \) of potentials that admit the group \( G_{3,11} \) consists of the fields (3.98).
3.3.12. **Class** \( P_{3,12} \). The algebra \( \mathcal{L}_{3,12} = L\{e_{13} + \lambda e_{24}, e_2, e_4\} \) corresponds to the group \( G_{3,12} \) generated by proportional bi-rotations and by translations along the vectors of two-dimensional pseudo-Euclidean plane. The algebra \( \mathcal{L}_{3,12} \) is an extension of \( \mathcal{L}_{1,5} \) by means of the vectors \( e_2 \) and \( e_4 \), therefore \( P_{3,12} \subset P_{1,5} \). For description of it we substitute (3.25) for \( A_i \) in equations

\[
\cosh(\lambda \varphi) \frac{\partial A_i}{\partial \rho} - \frac{\sinh(\lambda \varphi)}{\lambda \rho} \frac{\partial A_i}{\partial \theta} - \frac{\sinh(\lambda \varphi)}{\lambda \rho} \frac{\partial A_i}{\partial \varphi} = 0 \quad (3.99)
\]

and

\[
- \sinh(\lambda \varphi) \frac{\partial A_i}{\partial \rho} + \cosh(\lambda \varphi) \frac{\partial A_i}{\partial \theta} + \cosh(\lambda \varphi) \frac{\partial A_i}{\partial \varphi} = 0 \quad (3.100)
\]

(equations (3.26) and (3.2), transformed by substitution (3.23)). Multiplying (3.99) by \( \cosh(\lambda \varphi) \), (3.100) by \( \sinh(\lambda \varphi) \) and summing received equations, we get

\[
\frac{\partial A_i}{\partial \rho} = 0; \quad (3.101)
\]

therefore \( A_i \) and \( C_k \) are independent of \( \rho \): \( C_k = C_k(r, \theta) \). Further, we have the following consequence of equations (3.99) and (3.101):

\[
\frac{\partial A_i}{\partial \theta} + \frac{\partial A_i}{\partial \varphi} = 0. \quad (3.102)
\]

The system (3.99)–(3.100) is equivalent to the system (3.101)–(3.102). Substituting (3.25) for \( A_i \) in (3.101)–(3.102), we get a result of calculations:

\[
\begin{align*}
A_1 &= a_1(r) \sin(\theta - \varphi) + a_2(r) \cos(\theta - \varphi), \\
A_2 &= a_3(r) \sinh[\lambda(\theta - \varphi)] + a_4(r) \cosh[\lambda(\theta - \varphi)], \\
A_3 &= -a_1(r) \cos(\theta - \varphi) + a_2(r) \sin(\theta - \varphi), \\
A_4 &= a_3(r) \cosh[\lambda(\theta - \varphi)] + a_4(r) \sinh[\lambda(\theta - \varphi)],
\end{align*}
\]

where \( a_k = a_k(r) \) are arbitrary functions.

**Statement 43.** The class \( P_{3,12} \) of potentials that admit the group \( G_{3,12} \) consists of the fields (3.103) (the transformation of coordinates is defined by (3.23)).

3.3.13. **Class** \( P_{3,13} \). The algebra \( \mathcal{L}_{3,13} = L\{e_{13}, e_{24}, e_2 - e_4\} \) is an extension of the algebra \( \mathcal{L}_{2,10} \) by means of the vector \( e_2 - e_4 \), therefore \( P_{3,13} \subset P_{2,10} \). By substitution (3.46) the equation (3.34) (2.2) for the vector \( \xi = e_2 - e_4 \) takes the form

\[
\frac{\partial A_i}{\partial \varphi} + \frac{\partial A_i}{\partial \theta} - \rho \frac{\partial A_i}{\partial \rho} = 0. \quad (3.104)
\]

Substituting (3.45) for \( A_i \) in the equation (3.104), we obtain the following result.
Statement 44. The class $P_{3,13}$ of potentials $A_i$ that admit the group $G_{3,13}$ consists of the fields

\begin{align*}
A_1 &= -t_1(r) \sin(\theta - \varphi) + t_2(r) \cos(\theta - \varphi), \\
A_2 &= \rho C(r)e^{\varphi} + \frac{D(r)}{\rho}e^{-\varphi}, \\
A_3 &= t_1(r) \cos(\theta - \varphi) + t_2(r) \sin(\theta - \varphi), \\
A_4 &= -\rho C(r)e^{\varphi} + \frac{D(r)}{\rho}e^{-\varphi},
\end{align*}

(3.105)

where $t_1(r), t_2(r), C(r),$ and $D(r)$ are arbitrary functions. (the transformation of coordinates is defined by (3.46)).

3.3.14. Class $P_{3,14}$. The algebra \( L_{3,14} = L\{e_{12} - e_{14} + \lambda e_3, e_{23} + e_{34} + \nu e_1 + \lambda e_3, e_2 - e_4 \} \)
corresponds to the group $G_{3,14}$ generated by two one-dimensional subgroups of parabolic helices and by translations along an isotropic straight line. The equation (2.2) for basis vectors of the algebra $L_{3,14}$ take the following forms

\begin{align*}
XA_i - A_1(\delta_3^2 + \delta_4^2) + (A_2 - A_4)\delta_1^2 &= 0, \\
X &= (\lambda - x^2 - x^4)\partial_1 + \mu\partial_3, \\
YA_i - (A_2 - A_4)\delta_3^3 + A_3(\delta_2^2 + \delta_4^1) &= 0, \\
Y &= \nu\partial_1 + (\lambda + x^2 + x^4)\partial_3,
\end{align*}

(3.106) (3.107)

and (3.34). We have the solution of the equation (3.34) in the form

\begin{equation}
A_i = A_i(x^1, x^2 + x^4, x^3).
\end{equation}

(3.108)

Substituting (3.108) for $A_i$ in equations (3.106) and (3.107) and transforming theirs by means the substitution

\begin{equation}
u = x^2 + x^4, \quad \varphi = \frac{\mu x^1 + (u - \lambda)x^3}{u^2 - \lambda^2 + \mu \nu}, \quad \psi = \frac{\nu x^3 - (u + \lambda)x^1}{u^2 - \lambda^2 + \mu \nu},
\end{equation}

(3.109)

we obtain two systems

\begin{align*}
\frac{\partial A_1}{\partial \psi} + A_2 - A_4 &= 0, \quad \frac{\partial A_2}{\partial \psi} = A_1 = 0, \quad \frac{\partial A_3}{\partial \psi} = 0, \quad \frac{\partial A_4}{\partial \psi} - A_1 = 0, \\
\frac{\partial A_1}{\partial \varphi} &= 0, \quad \frac{\partial A_2}{\partial \varphi} + A_3 = 0, \quad \frac{\partial A_3}{\partial \varphi} - A_2 + A_4 = 0, \quad \frac{\partial A_4}{\partial \varphi} + A_3 = 0.
\end{align*}

(3.110) (3.111)

We have the total solution of the system (3.110) in the form

\begin{align*}
A_1 &= \psi C_2(u, \varphi) + C_3(u, \varphi), \\
A_2 &= \frac{1}{2} \psi^2 C_2(u, \varphi) + \psi C_3(u, \varphi) + C_1(u, \varphi), \\
A_3 &= A_3(u, \varphi), \quad A_4 = A_2 + C_2(u, \varphi).
\end{align*}

(3.112)
Finally, substituting (3.112) for $A_i$ in (3.111), we obtain

$$A_1 = C_3(u), \quad A_2 = A_4 = \psi C_3(u) + C_1(u), \quad A_3 = 0,$$

where $C_1(u)$ and $C_3(u)$ are arbitrary functions.

**Statement 45.** The class $P_{3,14}$ of potentials that admit the group $G_{3,14}$ consists of the fields defined by (3.113) and (3.109).

3.3.15. Class $P_{3,15}$. The algebra $\mathcal{L}_{3,15} = L\{e_{12} - e_{14}, \quad e_{24}, \quad e_3\}$ corresponds to the group $G_{3,15}$ generated by parabolic rotations, by pseudo-rotations, and by translations along a space-like straight line. The algebra $\mathcal{L}_{3,15}$ is an extension of the algebra $\mathcal{L}_{2,12a}$ ($\mathcal{L}_{2,12}$ for $\lambda = 0$) by means of the vector $e_3$, therefore the class $P_{3,15}$ is a subclass of $P_{2,12a}$ ($P_{2,12}$ for $\lambda = 0$). Substituting (3.61)–(3.59)–(3.16) for $A_i$ in equation (3.38), which means independence $A_i$ of $\tilde{x}^3$, we obtain the following result.

**Statement 46.** The class $P_{3,15}$ of potentials that admit the group $G_{3,15}$ consists of the fields defined by (3.61), where

$$\Phi_k = \Phi_k(v) = \Phi_k\left(\tilde{x}^4 - \frac{1}{2} (\tilde{x}^1)^2\right)$$

are arbitrary functions and the transformation of coordinates is defined by (3.16).

3.3.16. Class $P_{3,16}$. The algebra $\mathcal{L}_{3,16} = L\{e_{12} - e_{14}, \quad e_{24} + \lambda e_1 + \mu e_3, \quad e_2 - e_4\}$ corresponds to the group $G_{3,16}$ generated by parabolic rotations, by hyperbolic helices, and by translations along an isotropic straight line. The algebra $\mathcal{L}_{3,16}$ is an extension of the algebra $\mathcal{L}_{2,7a} = L\{e_{12} - e_{14}, \quad e_2 - e_4\}$ by means of the vector $e_{24} + \lambda e_1 + \mu e_3$, therefore $P_{3,16} \subset P_{2,7a}$. The equation (2.2) for the vector $L$ the class $\xi$ of potentials that admit the group $G_{3,16}$ takes the form

$$XA_i + A_2 \delta_i^2 + A_4 \delta_i^2 = 0,$$

(3.114)

$$X f = \lambda \partial_1 f + x^4 \partial_2 f + \mu \partial_3 f + x^2 \partial_4 f = -\lambda \left(\frac{1}{\tilde{x}^1} \frac{\partial f}{\partial \tilde{x}^2} + \tilde{x}^1 \tilde{x}^2 \frac{\partial f}{\partial \tilde{x}^4}\right) +$$

$$+ \tilde{x}^1 \frac{\partial f}{\partial \tilde{x}^1} - \tilde{x}^2 \frac{\partial f}{\partial \tilde{x}^2} + \mu \frac{\partial f}{\partial \tilde{x}^3} + (\tilde{x}^1)^2 \frac{\partial f}{\partial \tilde{x}^4}$$

(3.115)

(here we use the substitution (3.16) ). Substituting (3.37) for $A_i$ in (3.114), we get some equation; using a linear independence of functions $(\tilde{x}^3)^2$, $\tilde{x}^3$, and 1, we obtain the following equations

\begin{align*}
\tilde{x}^1 \frac{\partial C_2}{\partial \tilde{x}^1} + \mu \frac{\partial C_2}{\partial \tilde{x}^3} - C_2 &= 0, \\
\tilde{x}^1 \frac{\partial C_3}{\partial \tilde{x}^1} + \mu \frac{\partial C_3}{\partial \tilde{x}^3} - \frac{\lambda}{\tilde{x}^1} C_2 &= 0,
\end{align*}

\begin{align*}
\tilde{x}^1 \frac{\partial C_1}{\partial \tilde{x}^1} + \mu \frac{\partial C_1}{\partial \tilde{x}^3} + C_1 + C_2 - \frac{\lambda}{\tilde{x}^1} C_3 &= 0, \\
\tilde{x}^1 \frac{\partial A_3}{\partial \tilde{x}^1} + \mu \frac{\partial A_3}{\partial \tilde{x}^3} &= 0
\end{align*}

(3.116)
for the functions $C_k(\tilde{x}^1, \tilde{x}^3)$ and $A_3(\tilde{x}^1, \tilde{x}^3)$. Using the substitution

$$u = \tilde{x}^3 - \mu \ln \tilde{x}^1, \quad v = \ln \tilde{x}^1,$$

(3.117)

we transform (3.116); the solution of received system takes the form

$$C_1 = \Phi_3(u) e^{-v} - \frac{1}{2} \Phi_1(u) e^v + v e^{-v} \left[ \frac{1}{2} \lambda^2 v \Phi_1(u) + \lambda \Phi_2(u) \right],$$

(3.118)

$$C_2 = \Phi_1(u) e^v, \quad C_3 = \lambda v \Phi_1(u) + \Phi_2(u), \quad A_3 = \Phi_4(u),$$

where $\Phi_k(u)$ are arbitrary functions. Thus we have the following result.

**Statement 47.** The class $P_{3,16}$ of potentials that admit the group $G_{3,16}$ consists of the fields

$$A_1 = C_2 \tilde{x}^2 + C_3, \quad A_2 = \frac{1}{2} C_2(\tilde{x}^2)^2 + C_3 \tilde{x}^2 + C_1,$$

$$A_3 = \Phi_4(u), \quad A_4 = A_2 + C_2,$$

(3.119)

where $C_1 = C_1(u, v)$ are defined by (3.118)–(3.117), $\Phi_k(u)$ are arbitrary functions, and

$$\tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^2 = -\frac{x^1}{x^2 + x^4}, \quad \tilde{x}^3 = x^3.$$

3.3.17. **Class $P_{3,17}$.** The algebra $\mathcal{L}_{3,17} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{24}\}$ corresponds to the group $G_{3,17}$ generated by two one-dimensional subgroups of parabolic helices and by hyperbolic rotations. The algebra $\mathcal{L}_{3,17}$ is an extension of the algebra $\mathcal{L}_{2,11a}$ by means of the vector $e_{24}$, hence $P_{3,17} \subset P_{2,11a}$. The equation (2.22) for the vector $\xi = e_{24}$ takes the form

$$XA_1 = 0, \quadXA_2 + A_4 = 0, \quadXA_3 = 0, \quadXA_4 + A_2 = 0,$$

(3.120)

where $X = x^4 \partial_2 + x^2 \partial_4$. By substitution (3.94) we replace $X$ to the form

$$X = \tilde{x}^1 \frac{\partial}{\partial \tilde{x}^1} - \tilde{x}^2 \frac{\partial}{\partial \tilde{x}^2} - \tilde{x}^3 \frac{\partial}{\partial \tilde{x}^3}.$$

(3.121)

Substituting (3.55) for $A_4$ in (3.120)–(3.121), we get some equation; using a linear independence of functions $(\tilde{x}^2)^2, \tilde{x}^2, (\tilde{x}^3)^2, \tilde{x}^3$ and 1, we obtain some system of equations; solving this system, we get the following result.

**Statement 48.** The class $P_{3,17}$ of potentials that admit the group $G_{3,17}$ consists of the fields defined by (3.55), where

$$\Phi = \tilde{x}^1 C_1(\tilde{x}^4), \quad \Psi = C_2(\tilde{x}^4), \quad \Xi = C_3(\tilde{x}^4),$$

$$\Theta = \frac{\tilde{x}^1}{2} C_1(\tilde{x}^4) + \frac{C_4(\tilde{x}^4)}{\tilde{x}^1},$$

(3.122)

$C_k(\tilde{x}^4)$ are arbitrary functions, and the transformation of coordinates is defined by (3.51).
3.3.18. The algebra $\mathcal{L}_{3,18} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda(e_2 - e_4)\}$ corresponds to the group $G_{3,18}$ generated by two one-dimensional subgroups of parabolic rotations and by elliptic helices with an isotropic axis (or rotations for $\lambda = 0$). The algebra $\mathcal{L}_{3,18}$ is an extension of the algebra $\mathcal{L}_{2,11\alpha}$ by means of the vector $e_{13} + \lambda(e_2 - e_4)$, hence $P_{3,18} \subset P_{2,11\alpha}$.

The equation (2.2) for the vector $\xi$ to the form (3.127) for $\Phi, \Psi, \Xi, \Theta$ in (3.55), we get the following result.

$X A_1 - A_3 = 0, \quad X A_2 = 0, \quad X A_3 + A_1 = 0, \quad X A_4 = 0, \quad (3.123)$

where $X = x^3 \partial_1 + \lambda \partial_2 - x^1 \partial_3 - \lambda \partial_4$. By substitution (3.54) we replace $X$ to the form

$$X = -\tilde{x}^3 \frac{\partial}{\partial \tilde{x}^2} + \tilde{x}^2 \frac{\partial}{\partial \tilde{x}^3} + 2\lambda \tilde{x}^1 \frac{\partial}{\partial \tilde{x}^4}. \quad (3.124)$$

Substituting (3.55) for $A_i$ in (3.123)–(3.124), we get some equation; using a linear independence of variables $\tilde{x}^2, \tilde{x}^3$, and their powers, we obtain the system of equations:

$$\lambda \frac{\partial \Phi}{\partial \tilde{x}^4} = 0, \quad 2\lambda \tilde{x}^1 \frac{\partial \Psi}{\partial \tilde{x}^4} - \Xi = 0, \quad \lambda \frac{\partial \Theta}{\partial \tilde{x}^4} = 0, \quad 2\lambda \tilde{x}^1 \frac{\partial \Xi}{\partial \tilde{x}^4} + \Psi = 0. \quad (3.125)$$

3.3.18.1. **Class $P_{3,18\alpha}$**. For $\lambda \neq 0$ the solution of equations (3.125) takes the form

$$\Psi = C_1(\tilde{x}^1) \cos \frac{\tilde{x}^4}{2\lambda \tilde{x}^1}, \quad \Phi = C_3(\tilde{x}^1),$$

$$\Xi = -C_1(\tilde{x}^1) \sin \frac{\tilde{x}^4}{2\lambda \tilde{x}^1} + C_2(\tilde{x}^1) \cos \frac{\tilde{x}^4}{2\lambda \tilde{x}^1}, \quad \Theta = C_4(\tilde{x}^1), \quad (3.126)$$

where $C_k(\tilde{x}^1)$ are arbitrary functions.

**Statement 49.** The class $P_{3,18\alpha}$ of potentials that admit the group $G_{3,18\alpha}$ corresponding to the algebra $\mathcal{L}_{3,18}$ ($\lambda \neq 0$) is defined by (3.55) and (3.126) (the transformation of coordinates is defined by (3.54)).

3.3.18.2. **Class $P_{3,18\beta}$**. For $\lambda = 0$ the solution of equations (3.125) takes the form

$$\Phi = \Phi(\tilde{x}^1, \tilde{x}^4), \quad \Psi = \Xi = 0, \quad \Theta = \Theta(\tilde{x}^1, \tilde{x}^4), \quad (3.127)$$

where $\Phi(\tilde{x}^1, \tilde{x}^4)$ and $\Theta(\tilde{x}^1, \tilde{x}^4)$ are arbitrary functions. Substituting (3.127) for $\Phi, \Psi, \Xi, \Theta$ in (3.55), we get the following result.

**Statement 50.** The class $P_{3,18\beta}$ of potentials that admit the group $G_{3,18\beta}$ corresponding to the algebra $\mathcal{L}_{3,18\beta} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13}\}$ ($\mathcal{L}_{3,18}$ for $\lambda = 0$) is defined by the following formulae

$$A_1 = -\tilde{x}^2 \Phi(\tilde{x}^1, \tilde{x}^4), \quad A_2 = -\frac{1}{2}((\tilde{x}^2)^2 + (\tilde{x}^3)^2)\Phi(\tilde{x}^1, \tilde{x}^4) + \Theta(\tilde{x}^1, \tilde{x}^4),$$

$$A_3 = \tilde{x}^3 \Phi(\tilde{x}^1, \tilde{x}^4), \quad A_4 = A_2 - \Phi(\tilde{x}^1, \tilde{x}^4) \quad (3.128)$$

(the transformation of coordinates is defined by (3.54)).
3.3.19. Class $P_{3,19}$. The algebra
\[ \mathcal{L}_{3,19} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{24}\} \quad (\lambda \neq 0) \]
corresponds to the group $G_{3,19}$ generated by two one-dimensional subgroups of parabolic rotations and by one-dimensional subgroup of be-rotations. The algebra $\mathcal{L}_{3,19}$ is an extension of the algebra $\mathcal{L}_{2,11a}$ by means of the vector $e_{13} + \lambda e_{24}$, hence $P_{3,19} \subset P_{2,11a}$. The equation (2.2) for the vector $\xi = e_{13} + \lambda e_{24}$ takes the form
\[
\begin{align*}
XA_1 - A_3 &= 0,\quad XA_2 + \lambda A_4 = 0, \\
XA_3 + A_1 &= 0,\quad XA_4 + \lambda A_2 = 0,
\end{align*}
\] (3.129)
where $X = x^3\partial_1 + \lambda x^4\partial_2 - x^4\partial_3 + \lambda x^2\partial_4$. By substitution (3.54) we replace $X$ to the form
\[
X = \lambda \tilde{x}^1 \frac{\partial}{\partial \tilde{x}^1} - (\tilde{x}^3 + \lambda \tilde{x}^2) \frac{\partial}{\partial \tilde{x}^2} + (\tilde{x}^2 - \lambda \tilde{x}^3) \frac{\partial}{\partial \tilde{x}^3},
\]
(3.130)
Substituting (3.55) for $A_i$ in (3.129)—(3.130), we get some equation; using a linear independence of variables $\tilde{x}^2$, $\tilde{x}^3$, and their powers, we obtain the system of equations:
\[
\begin{align*}
\lambda \left( \tilde{x}^1 \frac{\partial \Phi}{\partial \tilde{x}^1} - \Phi \right) &= 0, \quad \lambda \tilde{x}^1 \frac{\partial \Psi}{\partial \tilde{x}^1} - \Xi = 0, \\
\lambda \left( \tilde{x}^1 \frac{\partial \Theta}{\partial \tilde{x}^1} + \Theta - \Phi \right) &= 0, \quad \lambda \tilde{x}^1 \frac{\partial \Xi}{\partial \tilde{x}^1} + \Psi = 0.
\end{align*}
\]
(3.131)
The solution of equations (3.131) takes the form
\[
\begin{align*}
\Phi &= \tilde{x}^1 C_1(\tilde{x}^4), \\
\Theta &= \frac{\tilde{x}^1 C_1(\tilde{x}^4)}{2} + \frac{C_2(\tilde{x}^4)}{\tilde{x}^1}, \\
\Psi &= C_3(\tilde{x}^4) \cos \frac{\ln \tilde{x}^1}{\lambda} + C_4(\tilde{x}^4) \sin \frac{\ln \tilde{x}^1}{\lambda}, \\
\Xi &= -C_3(\tilde{x}^4) \sin \frac{\ln \tilde{x}^1}{\lambda} + C_4(\tilde{x}^4) \cos \frac{\ln \tilde{x}^1}{\lambda},
\end{align*}
\] (3.132)
where $C_k(\tilde{x}^4)$ are arbitrary functions.

**Statement 51.** The class $P_{3,19}$ of potentials that admit the group $G_{3,19}$ is defined by formulae (3.55) and (3.132) (the transformation of coordinates is defined by (3.54)).

3.3.20. Class $P_{3,20}$. The algebra $\mathcal{L}_{3,20} = L\{e_{12}, e_{13}, e_{23}\}$ corresponds to the group $G_{3,20} = SO(3)$ of rotations over origin $O$ in the three-di-imensional subspace $\mathbb{R}_0^3 = \{x \in \mathbb{R}_4^1 : x^4 = 0\}$ of Minkowski space. Since for $\lambda = \mu = 0$ $\mathcal{L}_{1,2} \subset \mathcal{L}_{3,20}$ and the algebra $\mathcal{L}_{3,20}$ is an extension of the algebra $L\{e_{13}\}$ by means of the vectors $e_{12}$ and $e_{23}$, hence $P_{3,20} \subset P_{1,2}$ (for $\lambda = \mu = 0$). The equation (2.2) for the vectors $e_{12}$ and $e_{23}$ takes the forms
\[
\begin{align*}
XA_1 + A_2 &= 0, \quad XA_2 - A_1 = 0, \quad XA_3 = 0, \quad XA_4 = 0
\end{align*}
\]
(3.133)
(X = −x^2∂_1 + x^1∂_2) and
\[ Y A_1 = 0, \ Y A_2 + A_3 = 0, \ Y A_3 - A_2 = 0, \ Y A_4 = 0 \quad (3.134) \]

(Y = −x^3∂_2 + x^2∂_3). By substitution (3.7) for \( \lambda = \mu = 0 \)
\[ x^1 = r \sin \varphi, \ x^2 = \tilde{x}^2, \ x^3 = r \cos \varphi, \ x^4 = \tilde{x}^4 \quad (3.135) \]
we replace \( X \) and \( Y \) to the forms:
\[ X = -\tilde{x}^2 \sin \varphi \frac{\partial}{\partial r} + r \sin \varphi \frac{\partial}{\partial \tilde{x}^2} - \frac{\tilde{x}^2 \cos \varphi}{r} \frac{\partial}{\partial \varphi}, \quad (3.136) \]
\[ Y = \tilde{x}^2 \cos \varphi \frac{\partial}{\partial r} - r \cos \varphi \frac{\partial}{\partial \tilde{x}^2} - \frac{\tilde{x}^2 \sin \varphi}{r} \frac{\partial}{\partial \varphi}. \quad (3.137) \]

Substituting (3.9) for \( A \) in equations (3.133)–(3.136) and (3.134)–(3.137), we get some equations; dividing variables in them, we obtain the system of equations:
\[ -\tilde{x}^2 \frac{\partial C_2}{\partial r} + r \frac{\partial C_2}{\partial \tilde{x}^2} + A_2 = 0, \ -\tilde{x}^2 \frac{\partial C_1}{\partial r} + r \frac{\partial C_1}{\partial \tilde{x}^2} + \frac{C_1 \tilde{x}^2}{r} = 0, \]
\[ -\frac{C_2 \tilde{x}^2}{r} + A_2 = 0, \ -\frac{\tilde{x}^2 \frac{\partial A_2}{\partial r} + r \frac{\partial A_2}{\partial \tilde{x}^2} - C_2}{r} = 0, \ C_1 = 0, \quad (3.138) \]
\[ -\frac{\tilde{x}^2 \frac{\partial C_1}{\partial r} + r \frac{\partial C_1}{\partial \tilde{x}^2} + \frac{C_2 \tilde{x}^2}{r}}{r} = 0, \ -\tilde{x}^2 \frac{\partial A_4}{\partial r} + r \frac{\partial A_4}{\partial \tilde{x}^2} = 0. \]

The solution of the system (3.138) takes the form
\[ C_1 = C_2 = A_2 = 0, \ A_4 = A_4(\rho, \ x^4), \quad (3.139) \]
where \( \rho = \sqrt{r^2 + (\tilde{x}^2)^2} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \). Substituting (3.139) for \( C_k \) and \( A_i \) in (3.9), we get the following result.

**Statement 52.** The class \( P_{3,20} \) of potentials that admit the group \( G_{3,20} \) takes the form
\[ A_i = (0, \ 0, \ 0, \ A_4(\rho, \ x^4)) \quad (3.140) \]
where \( A_4(\rho, \ x^4) \) is an arbitrary function.

3.3.21. **Class \( P_{3,21} \).** The algebra \( L_{3,21} = L\{e_{12}, \ e_{14}, \ e_{24}\} \) corresponds to the group \( G_{3,21} \) generated by rotations and by pseudo-rotations. The algebra \( L_{3,21} \) is an extension of the algebra \( L_{1,3} \) (\( \lambda = 0 \)) by means of the vectors \( e_{12} \) and \( e_{14} \), hence \( P_{3,21} \subset P_{1,3} \) (for \( \lambda = 0 \)). The equation (2.2) for the vector \( e_{12} \) takes the form (3.133), and for the vector \( e_{14} \) — as follows
\[ x^4 \partial_1 A_i + x^1 \partial_4 A_i + A_1 \delta_i^4 + A_4 \delta_i^1 = 0. \quad (3.141) \]
By substitution (3.11) for \( \lambda = 0 \)
\[ x^1 = \tilde{x}^1, \ x^2 = r \cos \varphi, \ x^3 = \tilde{x}^3, \ x^4 = r \sin \varphi \quad (3.142) \]
we replace the operator \( X = -x^2 \partial_1 + x^1 \partial_2 \) to the form:
\[ X = -r \cos \varphi \frac{\partial}{\partial \tilde{x}^1} + \tilde{x}^1 \cos \varphi \frac{\partial}{\partial r} - \frac{\tilde{x}^1 \sin \varphi}{r} \frac{\partial}{\partial \varphi}. \quad (3.143) \]
Substituting (3.15) for \(A_i\) in equations (3.133)–(3.143), we get some equations; dividing variables in them, we obtain the system of equations:

\[
C_2 = 0, \quad r \frac{\partial A_1}{\partial x^1} - \tilde{x}^1 \frac{\partial A_1}{\partial r} - C_1 = 0,
\]

\[
r \frac{\partial C_1}{\partial x^1} - \tilde{x}^1 \frac{\partial C_1}{\partial r} + A_1 = 0, \quad A_1 + \frac{\tilde{x}^1 C_1}{r} = 0, \quad (3.144)
\]

\[
r \frac{\partial A_3}{\partial x^1} - \tilde{x}^1 \frac{\partial A_3}{\partial r} = 0, \quad r \frac{\partial C_1}{\partial x^1} - \tilde{x}^1 \frac{\partial C_1}{\partial r} + \frac{\tilde{x}^1 C_1}{r} = 0.
\]

The solution of the system (3.144) takes the form

\[
A_1 = A_2 = A_4 = 0, \quad A_3 = A_3(u, x^3), \quad (3.145)
\]

where \(u = \sqrt{r^2 + (\tilde{x}^1)^2} = \sqrt{(x^1)^2 + (x^2)^2 - (x^4)^2}\). Substituting (3.145) for \(A_i\) in (3.141), we get an identity. Thus we obtain the following result.

**Statement 53.** The class \(P_{3,21}\) of potentials that admit the group \(G_{3,21}\) takes the form (3.145), where \(A_3(u, x^3)\) is an arbitrary function.

### 3.4. Potentials that admit four-dimensional symmetry groups.

#### 3.4.1. Class \(P_{4,1}\).

The algebra \(L_{4,1} = L\{e_1, e_2, e_3, e_4\}\) corresponds to the group of translations of Minkowski space \(R_4^4\). The algebra \(L_{4,1}\) is an extension of the algebra \(L_{3,1a}\) by means of the vector \(e_4\), hence \(P_{4,1} \subset P_{3,1a}\). Substituting \(A_i(x^4)\) for \(A_i\) in (3.141), we get

**Statement 54.** The class \(P_{4,1}\) of potentials that admit the group \(G_{4,1}\) consists of the fields \(A_i\), which are constant in Galilean coordinates:

\(A_i = \text{const}\).

#### 3.4.2. Class \(P_{4,2}\).

The algebra \(L_{4,2} = L\{e_{13} + \mu e_4, e_1, e_2, e_3\}\) \((\mu \neq 0)\) is an extension of the algebra \(L_{3,3}\) by means of the vector \(e_2\), therefore \(P_{4,2} \subset P_{3,3}\). Substituting (3.67) for \(A_i\) in (3.26), we get

\[
A_1 = C_1 \sin \frac{x^4}{\mu} + C_2 \cos \frac{x^4}{\mu}, \quad A_2 = C_3,
\]

\[
A_3 = C_1 \cos \frac{x^4}{\mu} - C_2 \sin \frac{x^4}{\mu}, \quad A_4 = C_4 \quad (C_k = \text{const}). \quad (3.146)
\]

**Statement 55.** The class \(P_{4,2}\) of potentials that admit the group \(G_{4,2}\) consists of the fields (3.146).

#### 3.4.3. Class \(P_{4,3}\).

The algebra \(L_{4,3} = L\{e_{13} + \lambda e_2, e_1, e_3, e_4\}\) \((\lambda \neq 0)\) is an extension of the algebra \(L_{3,2}\) by means of the vector \(e_4\), therefore \(P_{4,3} \subset P_{3,2}\). Substituting (3.64) for \(A_i\) in (3.2), we get

\[
A_1 = C_1 \sin \frac{x^2}{\lambda} + C_2 \cos \frac{x^2}{\lambda}, \quad A_2 = C_3,
\]

\[
A_3 = C_1 \cos \frac{x^2}{\lambda} - C_2 \sin \frac{x^2}{\lambda}, \quad A_4 = C_4. \quad (3.147)
\]
where $C_k = \text{const}$. For $\lambda = 0$ potentials of the class $P_{3,2}$ are defined by (3.63). Substituting (3.65) for $A_i$ in (3.2), we get

$$A_1 = A_3 = 0, \quad A_2 = A_2(x^2), \quad A_4 = A_4(x^2). \quad (3.148)$$

**Statement 56.** For $\lambda \neq 0$ the class $P_{4,3}$ of potentials that admit the group $G_{4,3}$ consists of the fields (3.147); for $\lambda = 0$ it consists of the fields (3.148).

3.4.4. Class $P_{4,4}$. The algebra $L_{4,4} = L\{e_{13} + \lambda e_2, \ e_1, e_3, e_2 + e_4\}$ is an extension of the algebra $L_{3,1c}$ by means of the vector $e_{13} + \lambda e_2$, therefore $P_{4,4} \subset P_{3,1c}$. Substituting $A_i(x^2 - x^4)$ for $A_i$ in (3.62) (2.2) for the vector $\xi = e_{13} + \lambda e_2$, we obtain the system (3.63); for $\lambda \neq 0$ the solution of (3.63) takes the form

$$A_1 = C_1 \sin \frac{x^2 - x^4}{\lambda} + C_2 \cos \frac{x^2 - x^4}{\lambda}, \quad A_2 = C_3,$$

$$A_3 = C_1 \cos \frac{x^2 - x^4}{\lambda} - C_2 \sin \frac{x^2 - x^4}{\lambda}, \quad A_4 = C_4, \quad (3.149)$$

where $C_k = \text{const}$. For $\lambda = 0$ and $A_i = A_i(x^2 - x^4)$ the solution of (3.63) takes the form

$$A_1 = A_3 = 0, \quad A_2 = A_2(x^2 - x^4), \quad A_4 = A_4(x^2 - x^4), \quad (3.150)$$

where $A_2(x^2 - x^4)$ and $A_4(x^2 - x^4)$ are arbitrary functions.

**Statement 57.** For $\lambda \neq 0$ the class $P_{4,4}$ of potentials that admit the group $G_{4,4}$ consists of the fields (3.149); for $\lambda = 0$ it consists of the fields (3.150).

3.4.5. Class $P_{4,5}$. The algebra $L_{4,5} = L\{e_{24}, e_1, e_3, e_2 + e_4\}$ is an extension of the algebra $L_{3,1c}$ by means of the vector $e_{24}$, hence $P_{4,5} \subset P_{3,1c}$. Substituting $A_i(x^2 - x^4)$ for $A_i$ in (3.120) (2.2) for the vector $\xi = e_{24}$, we obtain

$$(x^4 - x^2)A'_1 = 0, \quad (x^4 - x^2)A'_2 + A_4 = 0,$$

$$(x^4 - x^2)A'_3 = 0, \quad (x^4 - x^2)A'_4 + A_2 = 0. \quad (3.151)$$

The solution of (3.151) takes the form

$$A_1 = C_1, \quad A_2 = C_2 \cdot (x^2 - x^4) + \frac{C_4}{x^2 - x^4}, \quad (3.152)$$

$$A_3 = C_3, \quad A_4 = C_2 \cdot (x^2 - x^4) - \frac{C_4}{x^2 - x^4}, \quad (C_k = \text{const}).$$

**Statement 58.** The class $P_{4,5}$ of potentials that admit the group $G_{4,5}$ consists of the fields (3.152).
3.4.6. Class $P_{4,6}$. The algebra $\mathcal{L}_{4,6} = L\{e_{24} + \lambda e_3, e_1, e_2, e_4\}$ is an extension of the algebra $\mathcal{L}_{3,6}$ by means of the vector $e_1$, hence $P_{4,6} \subset P_{3,6}$. Substituting (3.77) for $A_i$ in (3.1) (3.2) for the vector $\xi = e_1$, we obtain for $\lambda \neq 0$ the following result:

$$A_1 = C_1, \quad A_2 = C_2 \cosh \frac{x^3}{\lambda} + C_4 \sinh \frac{x^3}{\lambda},$$

$$A_3 = C_3, \quad A_4 = -C_2 \sinh \frac{x^3}{\lambda} - C_4 \cosh \frac{x^3}{\lambda},$$

where $C_k = \text{const}$. Let be $\lambda = 0$. Substituting (3.78) for $A_i$ in (3.1), we get

$$A_1 = A_1(x^3), \quad A_2 = 0, \quad A_3 = A_3(x^3), \quad A_4 = 0. \quad (3.154)$$

**Statement 59.** For $\lambda \neq 0$ the class $P_{4,6}$ of potentials that admit the group $G_{4,6}$ consists of the fields (3.153); for $\lambda = 0$ it consists of the fields (3.154).

3.4.7. Class $P_{4,7}$. The algebra $\mathcal{L}_{4,7} = L\{e_{13} + \lambda e_{24}, e_1, e_3, e_2 + e_4\}$ ($\lambda \neq 0$) is an extension of the algebra $\mathcal{L}_{3,1c}$ by means of the vector $e_{13} + \lambda e_{24}$, hence $P_{4,7} \subset P_{3,1c}$. Substituting $A_i(x^2 - x^4)$ for $A_i$ in (3.21)–(3.22) for the vector $\xi = e_{13} + \lambda e_{24}$, we obtain

$$\lambda(x^4 - x^2)A'_1 - A_3 = 0, \quad \lambda(x^4 - x^2)A'_2 + \lambda A_4 = 0,$$

$$\lambda(x^4 - x^2)A'_3 + A_1 = 0, \quad \lambda(x^4 - x^2)A'_4 + \lambda A_2 = 0. \quad (3.155)$$

The solution of (3.155) takes the form

$$A_1 = C_1 \cos \frac{\ln(x^2 - x^4)}{\lambda} + C_3 \sin \frac{\ln(x^2 - x^4)}{\lambda},$$

$$A_2 = C_2(x^2 - x^4) + \frac{C_4}{x^2 - x^4},$$

$$A_3 = C_1 \sin \frac{\ln(x^2 - x^4)}{\lambda} - C_3 \cos \frac{\ln(x^2 - x^4)}{\lambda},$$

$$A_4 = C_2(x^2 - x^4) - \frac{C_4}{x^2 - x^4}, \quad (C_k = \text{const}). \quad (3.156)$$

**Statement 60.** The class $P_{4,7}$ of potentials that admit the group $G_{4,7}$ consists of the fields (3.156).

3.4.8. Class $P_{4,8}$. The algebra $\mathcal{L}_{4,8} = L\{e_{12} - e_{14} + \lambda e_3, e_1, e_2, e_4\}$ is an extension of the algebra $\mathcal{L}_{3,1b}$ by means of the vector $e_{12} - e_{14} + \lambda e_3$, hence $P_{4,8} \subset P_{3,1b}$. Substituting $A_i(x^3)$ for $A_i$ in the equation (2.2) for the vector $\xi = e_{12} - e_{14} + \lambda e_3$

$$XA_i - A_1(\delta_i^2 + \delta_i^4) + (A_2 - A_4)\delta_i^1 = 0, \quad (3.157)$$

$$X = -(x^2 + x^4)\partial_1 + x^1\partial_2 + \lambda \partial_3 - x^1\partial_4, \quad (3.158)$$

we obtain

$$\lambda A'_1 + A_2 - A_4 = 0, \quad \lambda A'_2 - A_1 = 0, \quad \lambda A'_3 = 0, \quad \lambda A'_4 - A_1 = 0. \quad (3.159)$$
For \( \lambda \neq 0 \) the solution of (3.159) takes the form
\[
A_1 = \frac{C_2}{\lambda} x^3 + C_3, \quad A_2 = \frac{C_2}{2\lambda^2} (x^3)^2 + \frac{C_3}{\lambda} x^3 + C_4,
\]
\[
A_3 = C_1, \quad A_4 = A_2 + C_2 \quad (C_k = \text{const});
\]
for \( \lambda = 0 \) the same is
\[
A_1 = 0, \quad A_2 = A_4 = \Phi(x^3), \quad A_3 = \Psi(x^3),
\]
where \( \Phi(x^3) \) and \( \Psi(x^3) \) are arbitrary functions.

**Statement 61.** For \( \lambda \neq 0 \) the class \( P_{4,8} \) of potentials that admit the group \( G_{4,8} \) consists of the fields (3.160); for \( \lambda = 0 \) it consists of the fields (3.161).

### 3.4.9. Class \( P_{4,9} \)

The algebra \( \mathcal{L}_{4,9} = \{e_{12} - e_{14} + \lambda e_2, e_1, e_3, e_2 - e_4\} \) corresponds to the group \( G_{4,9} \) generated by parabolic helices and by translations along the vectors of an isotropic hyperplane. The solution of the system \( L_{e_1} A_i = L_{e_3} A_i = L_{e_2 - e_4} A_i = 0 \) takes the form
\[
A_i = A_i(x^2 + x^4).
\]
Substituting (3.162) for \( A_i \) in (2.2) for the vector \( e_{12} - e_{14} + \lambda e_2 \)
\[
XA_i - A_1(\delta_i^2 + \delta_i^4) + (A_2 - A_4)\delta_i^1 = 0,
\]
\[
X = -(x^2 + x^4)\partial_1 + (x^1 + \lambda)\partial_2 - x^1\partial_4,
\]
we get the system (3.159). For \( \lambda \neq 0 \) and \( A_i = A_i(x^2 + x^4) \) the solution of (3.159) takes the form
\[
A_1 = \frac{C_2}{\lambda} (x^2 + x^4) + C_3, \quad A_2 = \frac{C_2}{2\lambda^2} (x^2 + x^4)^2 + \frac{C_3}{\lambda} (x^2 + x^4) + C_4,
\]
\[
A_3 = C_1, \quad A_4 = A_2 + C_2 \quad (C_k = \text{const});
\]
for \( \lambda = 0 \) the same is
\[
A_1 = 0, \quad A_2 = A_4 = \Phi(x^2 + x^4), \quad A_3 = \Psi(x^2 + x^4),
\]
where \( \Phi(x^2 + x^4) \) and \( \Psi(x^2 + x^4) \) are arbitrary functions.

**Statement 62.** For \( \lambda \neq 0 \) the class \( P_{4,9} \) of potentials that admit the group \( G_{4,9} \) consists of the fields (3.165); for \( \lambda = 0 \) it consists of the fields (3.166).

### 3.4.10. Class \( P_{4,10} \)

The algebra \( \mathcal{L}_{4,10} = L\{e_{13}, e_{24}, e_1, e_3,\} \) is an extension of the algebra \( \mathcal{L}_{3,5} \) by means of the vector \( e_{13} \), hence \( P_{4,10} \subset P_{3,5} \). Substituting (3.173) for \( A_i \) in (2.2) for the vector \( \xi = e_{13} \)
\[
x^3\partial_1 A_i - x^1\partial_3 A_i + A_1\delta_i^3 - A_3\delta_i^1 = 0,
\]
we get some system; using the substitution (3.174), we obtain the following result
\[
A_1 = 0, \quad A_2 = C_1(\rho) \cosh \varphi + C_2(\rho) \sinh \varphi,
\]
\[
A_3 = 0, \quad A_4 = -C_1(\rho) \sinh \varphi - C_2(\rho) \cosh \varphi,
\]
where \( C_1(\rho) \) and \( C_2(\rho) \) are arbitrary functions.

**Statement 63.** The class \( P_{4,10} \) of potentials that admit the group \( G_{4,10} \) is defined by (3.168) and (3.74).

3.4.11. **Class \( P_{4,11} \).** The algebra \( \mathcal{L}_{4,11} = L\{e_{13}, e_{24}, e_2, e_4, \} \) is an extension of the algebra \( \mathcal{L}_{3,6} \) (for \( \lambda = 0 \)) by means of the vector \( e_{13} \), hence \( P_{4,11} \subset P_{3,6} \) (\( \lambda = 0 \)). Substituting (3.78) for \( A_i \) in (3.167), we get some system; using the substitution

\[
x^1 = r \sin \phi, \quad x^3 = r \cos \phi,
\]

we obtain the following result

\[
A_1 = C_1(r) \sin \phi + C_2(r) \cos \phi, \quad A_2 = 0,
\]

\[
A_3 = C_1(r) \cos \phi - C_2(r) \sin \phi, \quad A_4 = 0,
\]

(3.170)

where \( C_1(r) \) and \( C_2(r) \) are arbitrary functions.

**Statement 64.** The class \( P_{4,11} \) of potentials that admit the group \( G_{4,11} \) is defined by (3.170) and (3.169).

3.4.12. **Class \( P_{4,12} \).** The algebra

\[
\mathcal{L}_{4,12} = L\{e_{12} - e_{14} + \mu e_3, \quad e_{23} + e_{34} + \nu e_2, \quad e_1, \quad e_2 - e_4 \}
\]
corresponds to the group \( G_{4,12} \) generated by two one-dimensional subgroups of parabolic helices and by translations along the vectors of an isotropic two-dimensional plane. The total solution of the system of equations (2.2) for \( e_1 \) and \( e_2 - e_4 \) takes the form

\[
A_i = A_i(x^2 + x^4, x^3).
\]

(3.171)

The equation (2.2) for the vector \( \xi = e_{12} - e_{14} + \mu e_3 \) takes the form

\[
XA_i - A_1(\delta^2_i + \delta^4_i) + (A_2 - A_4)\delta^1_i = 0,
\]

\[
X = -(x^2 + x^4)\partial_1 + x^1(\partial_2 - \partial_4) + \mu \partial_3;
\]

(3.172)

(3.173)

for the vector \( \xi = e_{23} + e_{34} + \nu e_2 \) the same is

\[
YA_i + A_3(\delta^2_i + \delta^4_i) - (A_2 - A_4)\delta^3_i = 0,
\]

\[
Y = \nu \partial_2 + (x^2 + x^4)\partial_3 - x^3(\partial_2 - \partial_4).
\]

(3.174)

(3.175)

Substituting (3.171) for \( A_i \) in (3.172)–(3.173) and (3.174)–(3.175), we get

\[
\mu \frac{\partial A_1}{\partial x^3} + A_2 - A_4 = 0, \quad \mu \frac{\partial A_2}{\partial x^3} - A_1 = 0,
\]

\[
\mu \frac{\partial A_3}{\partial x^3} = 0, \quad \mu \frac{\partial A_4}{\partial x^3} - A_1 = 0
\]

(3.176)
and
\[
\nu \frac{\partial A_1}{\partial x^2} + (x^2 + x^4) \frac{\partial A_1}{\partial x^3} = 0, \quad \nu \frac{\partial A_2}{\partial x^2} + (x^2 + x^4) \frac{\partial A_2}{\partial x^3} + A_3 = 0, \\
\nu \frac{\partial A_3}{\partial x^2} + (x^2 + x^4) \frac{\partial A_3}{\partial x^3} - A_2 + A_4 = 0, \\
\nu \frac{\partial A_4}{\partial x^2} + (x^2 + x^4) \frac{\partial A_4}{\partial x^3} + A_3 = 0. \tag{3.177}
\]

Solving the system of equations (3.176)–(3.177), we obtain the result.

**Statement 65.** The class \(P_{4,12}\) of potentials that admit the group \(G_{4,12}\) is defined by the following formulae:

a) for \(\mu = \nu = 0\)

\[
A_1 = 0, \quad A_3 = \Psi(x^2 + x^4), \quad A_2 = A_4 = \Phi(x^2 + x^4) - \frac{x^3 \Psi(x^2 + x^4)}{x^2 + x^4}; \tag{3.178}
\]

b) for \(\mu = 0, \nu \neq 0\)

\[
A_1 = 0, \quad A_2 = A_4 = \Phi(u) - \frac{x^2 + x^4}{\nu} \Psi(u), \quad A_3 = \Psi(u) \left( u = x^3 - \frac{(x^2 + x^4)^2}{2\nu} \right); \tag{3.179}
\]

c) for \(\mu \neq 0, \nu = 0\)

\[
A_1 = \Phi(x^2 + x^4), \quad A_3 = -\frac{x^2 + x^4}{\mu} \Phi(x^2 + x^4), \quad A_2 = A_4 = \frac{x^3}{\mu} \Phi(x^2 + x^4) + \Psi(x^2 + x^4); \tag{3.180}
\]

d) for \(\mu \neq 0, \nu \neq 0\)

\[
A_1 = A_3 = 0, \quad A_2 = A_4 = \text{const} \tag{3.181}
\]

(\(\Phi\) and \(\Psi\) are arbitrary functions of one variable).

3.4.13. **Class \(P_{4,13}\).** The algebra

\[\mathcal{L}_{4,13} = L\{e_{12} - e_{14}, e_{24} + \lambda e_1, e_3, e_2 - e_4\}\]
corresponds to the group \(G_{4,13}\) generated by parabolic rotations, by hyperbolic helices, and by translations along the vectors of an isotropic two-dimensional plane. The solution of the system of equations (2.2) for \(e_3\) and \(e_2 - e_4\) takes the form

\[
A_i = A_i(x^1, x^2 + x^4). \tag{3.182}
\]

Substituting (3.182) for \(A_i\) in equation (2.2) for the vector \(e_{12} - e_{14}\)

\[
XA_i - A_i(\delta_i^2 + \delta_i^4) + (A_2 - A_4)\delta_i^1 = 0, \tag{3.183}
\]

\[
X = -(x^2 + x^4)\partial_1 + x^1(\partial_2 - \partial_4). \tag{3.184}
\]
we have the result:
\[
A_1 = -\frac{x^4C_1}{x^4 + x^2}, \quad A_2 = \frac{(x^4)^2C_1}{2(x^2 + x^4)} - \frac{x^1C_2}{x^2 + x^4} + C_3, \quad (3.185)
\]
\[
A_3 = C_4, \quad A_4 = A_2 + C_1 \ (C_k = C_k(x^2 + x^4)).
\]

Substituting (3.185) for \( A \) in (3.10) (the equation (2.2) for the vector \( e_{24} + \lambda e_1 \)), we obtain
\[
C_1 = K_1(x^2 + x^4), \quad C_2 = K_1\lambda \ln(x^2 + x^4) + K_2,
\]
\[
C_3 = \frac{K_1\lambda^2 \ln^2(x^2 + x^4) + 2K_2\lambda \ln(x^2 + x^4)}{2(x^2 + x^4)} - \frac{K_1}{2}(x^2 + x^4) + \frac{K_3}{x^2 + x^4}, \quad C_4 = K_4 \ (K_i = \text{const}).
\]

\textbf{Statement 66.} The class \( P_{4,13} \) of potentials that admit the group \( G_{4,13} \) is defined by (3.185) and (3.186).

3.4.14. The algebra
\[
\mathcal{L}_{4,14} = L\{e_{12} - e_{14}, e_{24} + \lambda e_3, e_1 + \nu e_3, e_2 - e_4\}
\]
corresponds to the group \( G_{4,14} \) generated by parabolic rotations, by hyperbolic helices, and by translations along the vectors of an isotropic two-dimensional plane (the group \( G_{4,14} \) is not conjugated to \( G_{4,13} \)). The solution of the system of equations (2.2) for \( e_1 + \nu e_3 \) and \( e_2 - e_4 \) takes the form
\[
A_i = A_i(x^2 + x^4, x^3 - \nu x^1). \quad (3.187)
\]

3.4.14.1. Class \( P_{4,14a} \). Let be \( \nu \neq 0 \). Substituting (3.187) for \( A_i \) in (3.183)–(3.184) (the equation (2.2) for the vector \( e_{12} - e_{14} \)), we obtain
\[
A_1 = \frac{(x^3 - \nu x^1)C_1}{\nu(x^2 + x^4)} + C_2,
\]
\[
A_2 = \frac{(x^3 - \nu x^1)^2C_1}{2\nu^2(x^2 + x^4)^2} + \frac{(x^3 - \nu x^1)C_2}{\nu(x^2 + x^4)} + C_3, \quad (3.188)
\]
\[
A_3 = C_4, \quad A_4 = A_2 + C_1 \ (C_k = C_k(x^2 + x^4)).
\]

Substituting (3.188) for \( A_i \) in equation
\[
x^4\partial_2 A_i + \nu\partial_3 A_i + x^2\partial_4 A_i + A_2\delta^4_i + A_4\delta^2_i = 0 \quad (3.189)
\]
(2.2) for the vector \( \xi = e_{24} + \nu e_3 \), we get
\[
C_1 = K_1(x^2 + x^4), \quad C_2 = -\frac{K_1\lambda}{\nu} \ln(x^2 + x^4) + K_2,
\]
\[
C_3 = \frac{K_1\lambda^2 \ln^2(x^2 + x^4) - 2K_2\lambda\nu\ln(x^2 + x^4)}{2\nu^2(x^2 + x^4)} - \frac{K_1}{2}(x^2 + x^4) + \frac{K_3}{x^2 + x^4}, \quad C_4 = K_4 \ (K_i = \text{const}).
\]
Statement 67. The class \( P_{4,14a} \) of potentials that admit the group \( G_{4,14a} \) is defined by (3.188) and (3.190).

3.4.14.2. Class \( P_{4,14b} \). For \( \nu = 0 \) we have the following result.

Statement 68. The class \( P_{4,14b} \) of potentials that admit the group \( G_{4,14b} \) \((G_{4,14} \text{ for } \nu = 0)\), is defined by the formulae

\[
\begin{align*}
A_1 &= 0, \quad A_3 = \Psi(x^3 - \lambda \ln(x^2 + x^4)), \\
A_2 &= A_4 = \frac{\Phi(x^3 - \lambda \ln(x^2 + x^4))}{x^2 + x^4},
\end{align*}
\]

(3.191)

where \( \Phi \) and \( \Psi \) are arbitrary functions of one variable.

3.4.15. Class \( P_{4,15} \). The algebra \( L_{4,15} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{24} + \lambda e_1, e_2 - e_4\} \) is an extension of the algebra \( L_{2,11a} \) by means of the vectors \( e_{24} + \lambda e_1 \) and \( e_2 - e_4 \), hence \( P_{4,15} \subset P_{2,11a} \). Substituting (3.55)–(3.54) for \( A_i \) in equation

\[ L_{e_2 - e_4}A_i = \partial_2 A_i - \partial_4 A_i = 2x^1 \frac{\partial A_i}{\partial x^4} = 0, \]

we get

\[
\begin{align*}
A_1 &= \frac{x^1 C_1}{x^2 + x^4} + C_2, \\
A_2 &= -\frac{(x^1)^2 + (x^3)^2}{2(x^2 + x^4)^2} - \frac{x^1 C_2 + x^3 C_4}{x^2 + x^4} + C_4, \\
A_3 &= \frac{x^3 C_1}{x^2 + x^4} + C_3, \\
A_4 &= A_2 - C_1 \quad (C_k = C_k(x^2 + x^4)).
\end{align*}
\]

(3.193)

Substituting (3.193) for \( A_i \) in (3.10) (equation (2.2) for \( e_{24} + \lambda e_1 \)), we obtain

\[
\begin{align*}
C_1 &= K_1(x^2 + x^4), \\
C_2 &= -K_1 \lambda \ln(x^2 + x^4) + K_2, \\
C_3 &= K_3, \\
C_4 &= \frac{-K_1 \lambda^2 \ln^2(x^2 + x^4) + 2K_2 \lambda \ln(x^2 + x^4)}{2(x^2 + x^4)} + \frac{K_1}{2}(x^2 + x^4) + \frac{K_4}{x^2 + x^4}, \quad (K_i = \text{const}).
\end{align*}
\]

(3.194)

Statement 69. The class \( P_{4,15} \) of potentials that admit the group \( G_{4,15} \) is defined by (3.193) and (3.194).

3.4.16. Class \( P_{4,16} \). The algebra

\[ L_{4,16} = L\{e_{12} - e_{14} + \lambda e_3, e_{23} + e_{34} + \lambda e_1, e_{13}, e_2 - e_4\} \]

is an extension of the algebra \( L_{3,14} \) by means of the vector \( e_{13} \); therefore \( P_{4,16} \subset P_{3,14} \) for corresponding values of parameters. The substitution (3.109) is replaced by

\[
\begin{align*}
\varphi &= \frac{\lambda x^1 + u x^3}{u^2 + \lambda^2}, \\
\psi &= \frac{\lambda x^3 - x^1 u}{u^2 + \lambda^2}.
\end{align*}
\]

(3.195)

Substituting (3.113)–(3.195) for \( A_i \) in (3.167) (equation (2.2) for the vector \( e_{13} \)), we obtain the following result.

\footnote{If we replace \( \mu \mapsto \lambda, \nu \mapsto \lambda, \lambda \mapsto 0 \).}
Statement 70. The class $P_{4,16}$ of potentials that admit the group $G_{4,16}$ is defined by the formulae

$$A_1 = A_3 = 0, \quad A_2 = A_4 = \Phi(x^2 + x^4),$$  \hspace{1cm} (3.196)

where $\Phi(u)$ is an arbitrary function.

3.4.17. Class $P_{4,17}$. The algebra

$$\mathcal{L}_{4,17} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{24}, e_2 - e_4\}$$

is an extension of the algebra $\mathcal{L}_{3,19}$ by means of the vector $e_2 - e_4$; therefore $P_{4,17} \subset P_{3,19}$. Substituting (3.135)–(3.132) for $A_i$ in (3.192), we get result

$$A_1 = K_1 x^4 + \Psi, \quad A_2 = -\frac{K_1 ((x^1)^2 + (x^3)^2)}{2(x^2 + x^4)} + \frac{x^1 \Psi + x^3 \Xi}{x^2 + x^4} + \frac{K_1}{2} (x^2 + x^4) + \frac{K_2}{2} \frac{x^2}{x^2 + x^4},$$  \hspace{1cm} (3.197)

$$A_3 = K_1 x^3 + \Xi, \quad A_4 = A_2 - K_1 (x^2 + x^4),$$

where

$$\Psi = K_3 \cos \frac{\ln(x^2 + x^4)}{\lambda} + K_4 \sin \frac{\ln(x^2 + x^4)}{\lambda},$$

$$\Xi = -K_3 \sin \frac{\ln(x^2 + x^4)}{\lambda} + K_4 \cos \frac{\ln(x^2 + x^4)}{\lambda} \quad (K_i = \text{const}).$$  \hspace{1cm} (3.198)

Statement 71. The class $P_{4,17}$ of potentials that admit the group $G_{4,17}$ is defined by the formulae (3.197) and (3.198).

3.4.18. Class $P_{4,18}$. The algebra $\mathcal{L}_{4,18} = L\{e_{12}, e_{13}, e_{23}, e_4\}$ is an extension of the algebra $\mathcal{L}_{3,20}$ by means of the vector $e_4$, hence $P_{4,18} \subset P_{3,20}$. Substituting (3.140) for $A_i$ in (3.12), we get the following result.

Statement 72. The class $P_{4,18}$ of potentials that admit the group $G_{4,18}$ takes the form

$$A_i = (0, 0, 0, A_4(\rho)), \quad (3.199)$$

where $\rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, and $A_4(\rho)$ is an arbitrary function.

3.4.19. Class $P_{4,19}$. The algebra $\mathcal{L}_{4,19} = L\{e_{12}, e_{14}, e_{24}, e_3\}$ is an extension of the algebra $\mathcal{L}_{3,21}$ by means of the vector $e_3$, hence $P_{4,19} \subset P_{3,21}$. Substituting (3.145) for $A_i$ in (3.38), we get the following result.

Statement 73. The class $P_{4,19}$ of potentials that admit the group $G_{4,19}$ takes the form

$$A_i = (0, 0, C(u), 0), \quad (3.200)$$

where $u = \sqrt{(x^1)^2 + (x^2)^2 - (x^4)^2}$, and $C(u)$ is an arbitrary function.
3.4.20. Class $P_{4,20}$. The algebra $\mathcal{L}_{4,20} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13}, e_{24}\}$ is an extension of the algebra $\mathcal{L}_{3,17}$ by means of the vector $e_{13}$, hence $P_{4,20} \subset P_{3,17}$. Substituting (3.55)–(3.122) for $A_i$ in (3.123)–(3.124) (for $\lambda = 0$), we get a solution; returning to coordinates $\{x^i\}$, we obtain the result.

**Statement 74.** The class $P_{4,20}$ of potentials that admit the group $G_{4,20}$ takes the form

\begin{align*}
A_1 &= x^1 C(\tilde{x}^4), \quad A_3 = x^3 C(\tilde{x}^4), \\
A_2 &= \frac{C(\tilde{x}^4)}{2} \left( x^2 + x^4 - \frac{(x^1)^2 + (x^3)^2}{x^2 + x^4} \right) + \frac{D(\tilde{x}^4)}{x^2 + x^4}, \quad (3.201) \\
A_4 &= A_2 - (x^2 + x^4)C(\tilde{x}^4),
\end{align*}

where $\tilde{x}^4 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2$, and $C(\tilde{x}^4), D(\tilde{x}^4)$ are arbitrary functions.
3.5. Potentials that admit five-dimensional symmetry groups.

3.5.1. Class $P_{5,1}$. The algebra $L_{5,1} = L\{e_{24}, e_1, e_2, e_3, e_4\}$ is an extension of the algebra $L_{4,1}$ by means of the vector $e_{24}$, hence $P_{5,1} \subset P_{4,1}$. Substituting $A_i = \text{const}$ for $A_i$ in (3.10) (for $\lambda = 0$), we obtain the result.

**Statement 75.** The class $P_{5,1}$ of potentials that admit the group $G_{5,1}$ takes the form

$$A_i = (0, A, 0, B) \quad (A, B = \text{const}). \quad (3.202)$$

3.5.2. Class $P_{5,2}$. The algebra $L_{5,2} = L\{e_{13} + \lambda e_{24}, e_1, e_2, e_3, e_4\}$ is an extension of the algebra $L_{4,1}$ by means of the vector $e_{13} + \lambda e_{24}$, hence $P_{5,2} \subset P_{4,1}$. Substituting $A_i = \text{const}$ for $A_i$ in (3.14)–(3.15) (for $\lambda \neq 0$), we obtain $A_i = 0$, i.e. the class $P_{5,2}$ is empty.

3.5.3. Class $P_{5,3}$. The algebra $L_{5,3} = L\{e_{12} - e_{14}, e_1, e_2, e_3, e_4\}$ is an extension of the algebra $L_{4,1}$ by means of the vector $e_{12} - e_{14}$, hence $P_{5,3} \subset P_{4,1}$. Substituting $A_i = \text{const}$ for $A_i$ in (3.14)–(3.15) (for $\lambda = \mu = 0$), we obtain the result.

**Statement 76.** The class $P_{5,3}$ of potentials that admit the group $G_{5,3}$ takes the form

$$A_i = (0, A, B, A) \quad (A, B = \text{const}). \quad (3.203)$$

3.5.4. Class $P_{5,4}$. The algebra $L_{5,4} = L\{e_{13}, e_{24}, e_1, e_2, e_3 + e_4\}$ is an extension of the algebra $L_{4,5}$ by means of the vector $e_{13}$, hence $P_{5,4} \subset P_{4,5}$. Substituting (3.152) for $A_i$ in (3.6) (for $\lambda = \mu = 0$), we obtain

$$A_1 = 0, \quad A_2 = K_1(x^2 - x^4) + \frac{K_2}{x^2 - x^4}, \quad A_3 = 0, \quad A_4 = K_1(x^2 - x^4) - \frac{K_2}{x^2 - x^4}, \quad (3.204)$$

where $K_1$ and $K_2$ are arbitrary constants.

**Statement 77.** The class $P_{5,4}$ of potentials that admit the group $G_{5,4}$ is defined by (3.204).

3.5.5. Class $P_{5,5}$. The algebra

$$L_{5,5} = L\{e_{12} - e_{14}, e_{23} + e_{34} + \lambda e_2, e_1, e_3, e_2 - e_4\}$$

is an extension of the algebra $L_{4,9}$ (for $\lambda = 0$) by means of the vector $e_{23} + e_{34} + \lambda e_2$, hence $P_{5,5} \subset P_{4,9}$. Substituting (3.166) for $A_i$ in (3.174)–(3.175) ($\nu \mapsto \lambda$), we obtain the following result.

**Statement 78.** For $\lambda = 0$ the class $P_{5,5}$ of potentials that admit the group $G_{5,5}$ takes the form

$$A_i = (0, \Phi(x^2 + x^4), 0, \Phi(x^2 + x^4)) \quad (3.205)$$
where $\Phi$ is an arbitrary function; for $\lambda \neq 0$ the same is

$$A_1 = 0, \quad A_2 = A_4 = -\frac{K_1}{\lambda} (x^2 + x^4) + K_2, \quad A_3 = K_1,$$

(3.206)

where $K_1$ and $K_2$ are arbitrary constants.

3.5.6. Class $P_{5,6}$. The algebra $L_{5,6} = L\{e_{12} - e_{14}, e_{24} + \lambda e_3, e_1, e_2, e_4\}$ is an extension of the algebra $L_{4,6}$ by means of the vector $e_{12} - e_{14}$, hence $P_{5,6} \subset P_{4,6}$. Substituting (3.153) or (3.154) for $A_i$ in (3.14)–(3.15) ($\lambda = \mu = 0$), we get

$$A_1 = 0, \quad A_2 = A_4 = K_1 e^{-x^3/\lambda}, \quad A_3 = K_2 (K_1, K_2 = \text{const})$$

(3.207)

or

$$A_i = (0, 0, \Phi(x^3), 0).$$

(3.208)

**Statement 79.** For $\lambda \neq 0$ the class $P_{5,6}$ of potentials that admit the group $G_{5,6}$ is defined by (3.207); for $\lambda = 0$ the same is (3.208).

3.5.7. Class $P_{5,7}$. The algebra $L_{5,7} = L\{e_{12} - e_{14}, e_{24}, e_1, e_3, e_2 - e_4\}$ is an extension of the algebra $L_{4,13}$ by means of the vector $e_1$, hence $P_{5,7} \subset P_{4,13}$. Substituting (3.185)–(3.186) for $A_i$ in (3.1), we obtain the following result.

**Statement 80.** The class $P_{5,7}$ of potentials that admit the group $G_{5,7}$ takes the form

$$A_i = \left(0, \frac{B}{x^2 + x^4}, C, \frac{B}{x^2 + x^4}\right),$$

(3.209)

where $B$ and $C$ are arbitrary constants.

3.5.8. Class $P_{5,8}$. The algebra

$L_{5,8} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{24} + \lambda e_3, e_1, e_2 - e_4\}$

is an extension of the algebra $L_{4,12a}$ ($L_{4,12}$ for $\mu = \nu = 0$) by means of the vector $e_{24} + \lambda e_3$, hence $P_{5,8} \subset P_{4,12a}$. Substituting (3.178) for $A_i$ in (3.50), we obtain the following result.

**Statement 81.** The class $P_{5,8}$ of potentials that admit the group $G_{5,8}$ takes the form

$$A_i = (0, \Phi(x^2 + x^4), 0, \Phi(x^2 + x^4)),$$

(3.210)

where $\Phi$ is an arbitrary function of one variable.

3.5.9. Class $P_{5,9}$. The algebra

$L_{5,9} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13}, e_{24}, e_2 - e_4\}$

is an extension of the algebra $L_{4,20}$ by means of the vector $e_2 - e_4$, hence $P_{5,9} \subset P_{4,20}$. Substituting (3.201) for $A_i$ in (3.34), we get

$$A_1 = Cx^1, \quad A_2 = \frac{C}{2} \left(x^2 + x^4 - \frac{(x^1)^2 + (x^3)^2}{x^2 + x^4}\right) + \frac{D}{x^2 + x^4},$$

$$A_3 = Cx^3, \quad A_4 = A_2 - C(x^2 + x^4) (C, D = \text{const}).$$

(3.211)
Statement 82. The class \( P_{5,9} \) of potentials that admit the group \( G_{5,9} \) is defined by (3.211).

3.6. Potentials that admit six-dimensional symmetry groups.

3.6.1. Class \( P_{6,1} \). The algebra \( \mathcal{L}_{6,1} = L\{e_{12}, e_{13}, e_{23}, e_{14}, e_{24}, e_{34}\} \) corresponds to the Lorentz group. It is an extension of the algebra \( \mathcal{L}_{3,20} \) by means of the vectors \( e_{14}, e_{24}, \) and \( e_{34} \), hence \( P_{6,1} \subset P_{3,20} \). Substituting (3.140) for \( A_i \) in (3.10) \((\lambda = 0)\), we obtain \( A_i = 0 \), i. e. the class of potentials that admit the Lorentz group \( G_{6,1} \) is empty.

3.6.2. Class \( P_{6,2} \). The algebra \( \mathcal{L}_{6,2} = L\{e_{13}, e_{24}, e_{12}, e_{23}, e_{34}, e_{4}\} \) is an extension of the algebra \( \mathcal{L}_{5,1} \) by means of the vector \( e_{13} \), hence \( P_{6,2} \subset P_{5,1} \). Substituting (3.202) for \( A_i \) in (3.3) \((\lambda = \mu = 0)\), we obtain \( A_i = 0 \), i. e. the class of potentials that admit the group \( G_{6,2} \) is empty.

3.6.3. Class \( P_{6,3} \). The algebra \( \mathcal{L}_{6,3} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{12}, e_{23}, e_{34}, e_{4}\} \) is an extension of the algebra \( \mathcal{L}_{5,3} \) by means of the vector \( e_{23} + e_{34} \), hence \( P_{6,3} \subset P_{5,3} \). Substituting (3.203) for \( A_i \) in (3.53), we obtain the following result.

Statement 83. The class \( P_{6,3} \) of potentials that admit the group \( G_{6,3} \) takes the form

\[
A_i = (0, A, 0, A), \quad A = \text{const}. \tag{3.212}
\]

3.6.4. Class \( P_{6,4} \). The algebra \( \mathcal{L}_{6,4} = L\{e_{12} - e_{14}, e_{24}, e_{12}, e_{23}, e_{34}, e_{4}\} \) is an extension of the algebra \( \mathcal{L}_{5,1} \) by means of the vector \( e_{12} - e_{14} \), hence \( C_{6,4} \subset C_{5,1} \). Substituting (3.202) for \( A_i \) in (3.14), (3.15) \((\lambda = \mu = 0)\), we get the following result.

Statement 84. The class \( P_{6,4} \) of potentials that admit the group \( G_{6,4} \) is defined by (3.212).

Remark 2. Statements 83 and 84 involve the potential (3.212) admits more wide symmetry group than \( G_{6,3} \) and \( G_{6,4} \).

3.6.5. Class \( P_{6,5} \). The algebra

\[
\mathcal{L}_{6,5} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{2}, e_{12}, e_{3}, e_{2} - e_{4}\}
\]

is an extension of the algebra \( \mathcal{L}_{5,5} \) \((\lambda = 0)\) by means of the vector \( e_{13} + \lambda e_{2} \), hence \( P_{6,5} \subset P_{5,5}(\lambda = 0) \). Substituting (3.205) for \( A_i \) in (3.6) \((\mu = 0)\), we get \( \lambda \Phi' = 0 \). We have a result.

Statement 85. For \( \lambda = 0 \) the class \( P_{6,5} \) of potentials that admit the group \( G_{6,5} \) is defined by (3.205); for \( \lambda \neq 0 \) the same is (3.212).
3.6.6. Class $P_{6,6}$. The algebra
\[
\mathcal{L}_{6,6} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{24}, e_1, e_3, e_2 - e_4\}
\]
is an extension of the algebra $\mathcal{L}_{5.5}$ ($\lambda = 0$) by means of the vector $e_{24}$, hence $P_{6,6} \subset P_{5.5(\lambda=0)}$. Substituting (3.205) for $A_i$ in (3.10) ($\lambda = 0$), we get
\[
A_i = \left(0, \frac{B}{x^2 + x^4}, 0, \frac{B}{x^2 + x^4}\right), \quad B = \text{const.} \quad (3.213)
\]

**Statement 86.** The class $P_{6,6}$ of potentials that admit the group $G_{6,6}$ is defined by (3.213).

3.6.7. Class $P_{6,7}$. The algebra
\[
\mathcal{L}_{6,7} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{24}, e_1, e_3, e_2 - e_4\}
\]
is an extension of the algebra $\mathcal{L}_{5.5}$ ($\lambda = 0$) by means of the vector $e_{13} + \lambda e_{24}$, hence $P_{6,7} \subset P_{5.5(\lambda=0)}$. Substituting (3.205) for $A_i$ in (3.21)–(3.22), we get the result.

**Statement 87.** The class $P_{6,7}$ of potentials that admit the group $G_{6,7}$ is defined by (3.213).

Remark 3. Statements 86 and 87 involve the potential (3.213) admits more wide symmetry group than $G_{6,6}$ and $G_{6,7}$.

3.6.8. Class $P_{6,8}$. The algebra $\mathcal{L}_{6,8} = L\{e_{12}, e_{13}, e_{23}, e_1, e_2, e_3\}$ is an extension of $\mathcal{L}_{3.20}$ by means of $e_1, e_2$, and $e_3$, hence $P_{6,8} \subset P_{3.20}$. Substituting (3.140) for $A_i$ in equations $\partial_1 A_i = \partial_2 A_i = \partial_3 A_i = 0$, we get the result.

**Statement 88.** The class $P_{6,8}$ of potentials that admit the group $G_{6,8}$ takes the form $A_i = (0, 0, 0, \Phi(x^4))$, where $\Phi(x^4)$ is an arbitrary function.

3.6.9. Class $P_{6,9}$. The algebra $\mathcal{L}_{6,9} = L\{e_{12}, e_{14}, e_{24}, e_1, e_2, e_4\}$ is an extension of $\mathcal{L}_{3.21}$ by means of $e_1, e_2$, and $e_4$, hence $P_{6,9} \subset P_{3.21}$. Substituting (3.145) for $A_i$ in equations $\partial_1 A_i = \partial_2 A_i = \partial_4 A_i = 0$, we get the result.

**Statement 89.** The class $P_{6,9}$ of potentials that admit the group $G_{6,9}$ takes the form $A_i = (0, 0, \Phi(x^3), 0)$, where $\Phi(x^3)$ is an arbitrary function.

4. Appendix. Seven classes of Maxwell spaces that admit subgroups of the Poincaré group.

Using the group classification of potential structures, we define more precisely classes of Maxwell spaces that admit subgroups of the Poincaré group [2,3]. Here we describe classes of Maxwell spaces that correspond to algebras $\mathcal{L}_{3.19}$, $\mathcal{L}_{4.16}$, $\mathcal{L}_{4.17}$, $\mathcal{L}_{4.20}$, $\mathcal{L}_{5.9}$, $\mathcal{L}_{6.5}$, and $\mathcal{L}_{6.7}$ (according to [2] these classes are empty).
1°. Class $C_{3,19}$. For the algebra

$$L_{3,19} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{24}\} \ (\lambda \neq 0)$$

we have the result.

**Statement 90.** A Maxwell space of the class $C_{3,19}$ is defined by the tensor $F_{ij}$ such that

\[
F_{12} = -\frac{\tilde{x}^1}{2} \Phi_2 \left(1 + (\tilde{x}^2)^2 - (\tilde{x}^3)^2\right) - \tilde{x}^1 \tilde{x}^2 \tilde{x}^3 \Phi_1 + \frac{\Phi_3}{\tilde{x}^1} - \tilde{x}^2 \Phi_5,
\]

\[
F_{13} = \tilde{x}^1 (\tilde{x}^2 \Phi_1 - \tilde{x}^3 \Phi_2), \quad F_{14} = F_{12} + \tilde{x}^1 \Phi_2,
\]

\[
F_{23} = -\frac{\tilde{x}^1}{2} \Phi_1 \left(1 - (\tilde{x}^2)^2 + (\tilde{x}^3)^2\right) - \tilde{x}^1 \tilde{x}^2 \tilde{x}^3 \Phi_2 - \frac{\Phi_4}{\tilde{x}^1} - \tilde{x}^3 \Phi_5,
\]

\[
F_{24} = \tilde{x}^1 (\tilde{x}^2 \Phi_2 + \tilde{x}^3 \Phi_1) + \Phi_5, \quad F_{34} = -F_{23} - \tilde{x}^1 \Phi_1,
\]

where $\Phi_k = \Phi_k(\tilde{x}^4)$ ($k = 1, 2, 5$) are arbitrary functions,

\[
\Phi_3 = \Phi_3(\tilde{x}^4) = \int \left(\frac{1}{2\lambda} \Phi_1(\tilde{x}^4) - \frac{\tilde{x}^4}{2} \Phi'_2(\tilde{x}^4) - \Phi_2(\tilde{x}^4)\right) d\tilde{x}^4,
\]

\[
\Phi_4 = \Phi_4(\tilde{x}^4) = \int \left(-\frac{1}{2\lambda} \Phi_2(\tilde{x}^4) + \frac{\tilde{x}^4}{2} \Phi'_1(\tilde{x}^4) + \Phi_1(\tilde{x}^4)\right) d\tilde{x}^4,
\]

and the transformation of coordinates is defined by (3.54).

**Example 1.** If we replace $C_1$, $C_2$, $C_3$ by zero and $C_4$ by $\varphi = \varphi(\tilde{x}^4)$ in (3.55) – (3.132), we get the potential

\[
A_1 = \varphi(\tilde{x}^4) \sin \frac{\ln \tilde{x}^1}{\lambda}, \quad A_3 = \varphi(\tilde{x}^4) \cos \frac{\ln \tilde{x}^1}{\lambda},
\]

\[
A_2 = A_4 = \tilde{x}^2 \varphi(\tilde{x}^4) \sin \frac{\ln \tilde{x}^1}{\lambda} - \tilde{x}^3 \varphi(\tilde{x}^4) \cos \frac{\ln \tilde{x}^1}{\lambda}.
\]
Substituting (4.3)–(3.54) for \( A \) in (2.1), we get

\[
F_{12} = - \left( \frac{2(x^1)^2 \varphi' + \varphi}{x^2 + x^4} + 2x^2 \varphi' \right) \frac{\ln(x^2 + x^4)}{\lambda} - \frac{2\lambda x^1 x^3 \varphi' + \varphi}{\lambda(x^2 + x^4)} \cos \frac{\ln(x^2 + x^4)}{\lambda},
\]

\[
F_{13} = 2\varphi' \left( x^1 \cos \frac{\ln(x^2 + x^4)}{\lambda} - x^3 \sin \frac{\ln(x^2 + x^4)}{\lambda} \right),
\]

\[
F_{14} = F_{12} + 2(x^2 + x^4) \varphi' \sin \frac{\ln(x^2 + x^4)}{\lambda},
\]

\[
F_{23} = \left( \frac{2(x^3)^2 \varphi' + \varphi}{x^2 + x^4} + 2x^2 \varphi' \right) \frac{\ln(x^2 + x^4)}{\lambda} + \frac{2\lambda x^1 x^3 \varphi' - \varphi}{\lambda(x^2 + x^4)} \sin \frac{\ln(x^2 + x^4)}{\lambda},
\]

\[
F_{24} = -2\varphi' \left( x^1 \sin \frac{\ln(x^2 + x^4)}{\lambda} + x^3 \cos \frac{\ln(x^2 + x^4)}{\lambda} \right),
\]

\[
F_{34} = -F_{23} + 2(x^2 + x^4) \varphi' \cos \frac{\ln(x^2 + x^4)}{\lambda}. \tag{4.4}
\]

**Statement 91.** If \( \varphi' = \varphi' (\tilde{x}^4) \neq 0 \), then the Maxwell space defined by the tensor (4.1) admits the three-dimensional group \( G_S = G_{3.19} \).

2°. Class \( C_{4.16} \). For the algebra

\[
L_{4.16} = L\{e_{12} - e_{14} + \lambda e_3, e_{23} + e_{34} + \lambda e_1, e_{13}, e_2 - e_4 \}
\]

we have the result.

**Statement 92.** A Maxwell space of the class \( C_{4.16} \) is defined by the tensor \( F_{ij} \) such that

\[
F_{12} = F_{14} = -\varphi \Phi_1(u) + \psi \Phi_2(u), \quad F_{13} = \Phi_1(u),
\]

\[
F_{23} = -F_{34} = \varphi \Phi_2(u) + \psi \Phi_1(u), \quad F_{24} = \Phi_2(u), \tag{4.5}
\]

where \( \Phi_1(u) = K/(u^2 + \lambda^2) \), \( K = \text{const} \), \( \Phi_2(u) \) is an arbitrary function, and

\[
u = x^2 + x^4, \quad \varphi = \frac{\lambda x^1 + u x^3}{u^2 + \lambda^2}, \quad \psi = \frac{\lambda x^3 - x^1 u}{u^2 + \lambda^2}. \tag{4.6}
\]

**Example 2.** Substituting 0 for \( \Phi_2(u) \) in (4.5)–(4.6), we get

\[
F_{12} = F_{14} = -K(\lambda x^1 + (x^2 + x^4) x^3) / ((x^2 + x^4)^2 + \lambda^2)^2, \quad F_{13} = \frac{K}{(x^2 + x^4)^2 + \lambda^2},
\]

\[
F_{23} = -F_{34} = \frac{K(\lambda x^3 - x^1 (x^2 + x^4))}{((x^2 + x^4)^2 + \lambda^2)^2}, \quad F_{24} = 0. \tag{4.7}
\]

**Statement 93.** If \( K \neq 0 \), then the Maxwell space defined by the tensor (4.7) admits the four-dimensional group \( G_S = G_{4.16} \).
3°. Class $C_{4,17}$. For the algebra
\[ \mathcal{L}_{4,17} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{24}, e_2 - e_4\} \]
we have the result.

**Statement 94.** A Maxwell space of the class $C_{4,17}$ is defined by the tensor $F_{ij}$ such that
\[
F_{12} = F_{14} = \frac{1}{x^2 + x^4} \left( A \cos \frac{\ln(x^2 + x^4)}{\lambda} + B \sin \frac{\ln(x^2 + x^4)}{\lambda} + C x^1 \right),
\]
\[
F_{23} = -F_{34} = \frac{1}{x^2 + x^4} \left( B \cos \frac{\ln(x^2 + x^4)}{\lambda} - A \sin \frac{\ln(x^2 + x^4)}{\lambda} - C x^3 \right),
\]
\[ F_{13} = 0, \quad F_{24} = C \quad (A, B, C = \text{const}). \] (4.8)

**Statement 95.** If 1) $C \neq 0$ and 2) $A \neq 0$ (or $B \neq 0$), then the Maxwell space defined by the tensor $F_{ij}$ admits the four-dimensional group $G_S = G_{4,17}$.

4°. Class $C_{4,20}$. For the algebra $\mathcal{L}_{4,20} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13}, e_{24}\}$ we have the result.

**Statement 96.** A Maxwell space of the class $C_{4,20}$ is defined by the tensor $F_{ij}$ such that
\[
F_{12} = F_{14} = \frac{x^1 \Phi}{x^2 + x^4}, \quad F_{13} = 0, \quad F_{23} = -F_{34} = \frac{-x^3 \Phi}{x^2 + x^4}, \quad F_{24} = \Phi \quad (\Phi = \Phi(\tilde{x}^4) = \Phi((x^1)^2 + (x^3)^2 + (x^4)^2)).
\] (4.9)

**Statement 97.** If $\Phi'(\tilde{x}^4) \neq 0$, then the Maxwell space defined by the tensor $F_{ij}$ admits the four-dimensional group $G_S = G_{4,20}$.

**Remark 4.** In $\mathcal{L}_{4,20}$ the class $C_{4,20}$ is not empty, but this is more narrow than above.

5°. Class $C_{5,9}$. For the algebra
\[ \mathcal{L}_{5,9} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13}, e_{24}, e_2 - e_4\} \]
we have the result.

**Statement 98.** A Maxwell space of the class $C_{5,9}$ is defined by the tensor $F_{ij}$ such that
\[
F_{12} = F_{14} = \frac{C x^1}{x^2 + x^4}, \quad F_{13} = 0, \quad F_{24} = C, \quad F_{23} = -F_{34} = \frac{-C x^3}{x^2 + x^4}, \quad (C = \text{const}).
\] (4.10)

**Statement 99.** If $C \neq 0$, then the Maxwell space defined by the tensor $F_{ij}$ admits the five-dimensional group $G_S = G_{5,9}$. 
6°. Class $C_{6,5}$. For the algebra
\[ L_{6,5} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{2}, e_{1}, e_{3}, e_{2} - e_{4}\} \]
we have the result.

**Statement 100.** For $\lambda \neq 0$ a Maxwell space of the class $C_{6,5}$ is defined by the tensor $F_{ij}$ such that
\[
\begin{align*}
F_{12} &= F_{14} = C_1 \sin \frac{x^2 + x^4}{\lambda} + C_2 \cos \frac{x^2 + x^4}{\lambda}, \\
F_{23} &= -F_{34} = C_1 \cos \frac{x^2 + x^4}{\lambda} - C_2 \sin \frac{x^2 + x^4}{\lambda}, \\
F_{13} &= F_{24} = 0 \ (C_1, C_2 = \text{const});
\end{align*}
\]
if $\lambda = 0$, then $F_{ij} = 0$.

**Statement 101.** If $C_1 \neq 0$ or $C_2 \neq 0$, then the Maxwell space defined by the tensor (4.11) admits the six-dimensional group $G_S = G_{6,5}$.

7°. Class $C_{6,7}$. For the algebra
\[ L_{6,7} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_{13} + \lambda e_{24}, e_{1}, e_{3}, e_{2} - e_{4}\} \]
we have the result.

**Statement 102.** A Maxwell space of the class $C_{6,7}$ is defined by the tensor $F_{ij}$ such that
\[
\begin{align*}
F_{12} &= F_{14} = \Phi, \quad F_{13} = F_{24} = 0, \quad F_{23} = -F_{34} = \Psi, \\
\Phi &= \frac{1}{x^2 + x^4} \left( a_1 \cos \ln \frac{x^2 + x^4}{\lambda} - a_2 \sin \ln \frac{x^2 + x^4}{\lambda} \right), \\
\Psi &= \frac{1}{x^2 + x^4} \left( a_1 \sin \ln \frac{x^2 + x^4}{\lambda} + a_2 \cos \ln \frac{x^2 + x^4}{\lambda} \right),
\end{align*}
\]
(4.12)

where
\[ a_1, a_2 = \text{const}. \]

**Statement 103.** If $a_1 \neq 0$ or $a_2 \neq 0$, then the Maxwell space defined by the tensor (4.12)–(4.13) admits the six-dimensional group $G_S = G_{6,5}$.

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Ivanovo State University. 153025, Russia, Ivanovo, ul. Ermaka, 39
E-mail address: parinov@ivanovo.ac.ru