Polarization Tensors and Their Applications

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Abstract. We provide a survey of important properties of the generalized polarization tensors which are the basic building blocks for the full asymptotic expansions of the boundary voltage perturbations due to the presence of a small conductivity inclusion inside a conductor. These properties may be used to design efficient algorithms for reconstructing conductivity inclusions of small volume.

1. Introduction

In this paper we review some properties of the generalized polarization tensors (GPT’s) associated with a bounded Lipschitz domain $B$ and a conductivity $k$. The GPT’s are the basic building blocks for the full asymptotic expansions of the boundary voltage perturbations due to the presence of a small conductivity inclusion $D$ of the form $D = \rho B + z$ with conductivity $k$ inside a conductor $\Omega$ with conductivity 1.

It is then important from an imaging point of view to precisely characterize these GPT’s and derive some of their properties, such as symmetry, positivity, and optimal bounds on their elements, for developing efficient algorithms for reconstructing conductivity inclusions of small volume, see \cite{3}. The GPT’s seem to contain significant information on the domain $B$ and its conductivity $k$ which is yet to be investigated.

On the other hand, the use of these GPT’s leads to stable and accurate algorithms for the numerical computations of the steady-state voltage in the presence of small conductivity inclusions. It is known that small size features cause difficulties in the numerical solution of the conductivity problem by the finite element or finite difference methods. This is because such features require refined meshes in their neighborhoods, with their attendant problems \cite{28}.

The concepts of higher-order polarization tensors generalize those of classical Pólya–Szegö polarization tensors which have been extensively studied in the literature by many authors for various purposes \cite{10, 8, 9, 6, 11, 15, 19, 29, 30, 31, 39, 38, 40, 41, 43, 14}. The notion of Pólya–Szegö polarization tensor appeared in problems of potential theory related to certain problems arising in hydrodynamics and in electrostatics. If the conductivity $k$ is zero, namely, if $B$ is insulated, the polarization tensor of Pólya–Szegö is called the virtual mass. The concept of polarization tensors also occurs in several other interesting contexts, in particular in asymptotic models of dilute composites \cite{37, 6, 8, 16, 20, 34} and in low-frequency scattering of acoustic and electromagnetic waves \cite{29, 13}.

Our plan of this paper is as follows. We first give the definition of the GPT’s and give explicit formulae for the GPT’s in the disk and ball cases. We then list three important applications
of the GPT’s and provide a survey of important symmetric properties and positivity of the GPT’s and derive bounds satisfied by the tensor elements of the GPT’s. We also discuss how the knowledge of all the GPT’s uniquely determines the domain and the constitutive parameter. The paper ends with a discussion section.

Note that similar results have been established for the (generalized) anisotropic polarization tensors in [26]. These tensors are defined in the same way as the GPT’s. However, they occur due to not only the presence of discontinuity, but also the difference of the anisotropy. In a way analogous to GPT’s, elastic moment tensors (EMT’s) have been introduced and their properties investigated in [5]. The first-order EMT was given by Maz’ya and Nazarov [35].

2. Definition

Let \( B \) be a Lipschitz bounded domain in \( \mathbb{R}^d \), \( d = 2, 3 \), and let the conductivity of \( B \) be \( k \), \( 0 < k \neq 1 < +\infty \). Denote \( \lambda := (k + 1)/(2(k - 1)) \). Introduce the singular integral operator \( K_B^* \) by

\[
K_B^* \phi(x) = \frac{1}{\omega_d} \text{p.v.} \int_{\partial B} \frac{\langle x - y, \nu_x \rangle}{|x - y|^d} \phi(y) \, d\sigma(y).
\]

Here p.v. denotes the Cauchy principal value. The operator \( K_B^* \) is bounded on \( L^2(\partial B) \). See [12]. Moreover, from [18] we know that \( \lambda I - K_B^* : L^2(\partial B) \to L^2(\partial B) \) is invertible for any \( |\lambda| > 1/2 \).

**Definition 2.1** For a multi-index \( i = (i_1, \ldots, i_d) \in \mathbb{N}^d \), let \( \partial^i f = \partial_1^{i_1} \cdots \partial_d^{i_d} f \) and \( x^i := x_1^{i_1} \cdots x_d^{i_d} \). For \( i, j \in \mathbb{N}^d \), we define the generalized polarization tensor \( M_{ij} \) by

\[
M_{ij}(B, k) := \int_{\partial B} y^j \phi_i(y) \, d\sigma(y),
\]

where \( \phi_i \) is given by

\[
\phi_i(y) := (\lambda I - K_B^*)^{-1} \left( \nu_x \cdot \nabla x^i \right)(y), \quad y \in \partial B.
\]

The GPT’s can be explicitly computed for disks in the plane and balls in three-dimensional space. From [33] the following results hold.

**Theorem 2.2**

- Let \( B \) be the disk of radius \( r \) and center 0. Suppose that \( a_i \) and \( b_j \) are constants such that \( f = \sum a_i y^i, g = \sum b_j y^j \) are harmonic polynomials of homogeneous degrees \( n \) and \( m \), respectively. Then

\[
\sum a_i b_j M_{ij}(B, k) = \begin{cases} \frac{n\pi r^{2n}}{\lambda} \delta_{nm} & \text{if } f = g, \\ 0 & \text{otherwise.} \end{cases}
\]

- Let \( B \) be the ball of radius \( r \) and center 0. Then

\[
\sum a_i b_j M_{ij}(B, k) = \frac{(k - 1)n(2n + 1)}{nk + n + 1} r^{2n+1} \delta_{nm} \delta_{ll'},
\]

where \( \sum a_i y^i = |y|^n Y_{n,1}(\frac{x}{|y|}) \) and \( \sum b_j y^j = |y|^m Y_{m,1}(\frac{x}{|y|}) \). Here \( \{Y_{n,1}, \ldots, Y_{n,2n+1}\} \) is a set of orthonormal harmonics of degree \( n \).

The reader is referred once again to [33] for explicit calculations of the GPT’s in the case of ellipses.
If \(|i| = |j| = 1\), we denote \(M_{ij}\) by \((m_{pq})_{1 \leq p, q \leq 1}\) and call \(M = (m_{pq})_{1 \leq p, q \leq 1}\) the polarization tensor of Pólya–Szegő. Note that the definition (1) of GPT’s is valid even when \(k = 0\) or \(\infty\). If \(k = 0\), namely, if \(B\) is insulated, then

\[
M_{ij} := \int_{\partial B} y^j \left( -\frac{1}{2} I - K_B^* \right)^{-1} (\nu_y \cdot \nabla y^i)(y) \, d\sigma(y),
\]

while if \(k = \infty\), namely, if \(B\) is perfectly conducting, then

\[
M_{ij} := \int_{\partial B} y^j \left( \frac{1}{2} I - K_B^* \right)^{-1} (\nu_y \cdot \nabla y^i)(y) \, d\sigma(y).
\]

When \(|i| = |j| = 1\), these definitions exactly match those introduced by Pólya–Szegő [41] and Schiffer and Szegő [43].

3. Applications

We now discuss three important applications of the GPT’s.

3.1. Effective conductivity of dilute composites

Let \(Y = ]-1/2, 1/2[^d\) denote the unit cell and \(D = \rho B, \rho < 1\), where \(B\) is a reference domain containing 0 whose volume, \(|B|\), is 1. Let \(\gamma\) be periodic with the periodic cell \(Y\), and on \(Y\) we set \(\gamma = 1 + (k - 1)\chi(B)\) where \(0 < k \neq 1 < +\infty\) and \(\chi(B)\) is the characteristic function of \(B\). For a small parameter \(\epsilon\), \(\gamma(x/\epsilon)\) makes a highly oscillating conductivity and represent the material property of the composite. The problem is to determine the effective property of the composite with the conductivity \(\gamma(x/\epsilon)\), or the limit of \(\gamma(x/\epsilon)\) as \(\epsilon \to 0\).

The determination of the effective or macroscopic property of a two-phase medium has been one of the classical problems in physics. When \(B\) is a disk in the two-dimensional case, the effective electrical conductivity, \(\gamma^*\), of the composite medium is given by the well-known Maxwell-Garnett formula\(^2\) [42]

\[
\gamma^* = 1 + f \frac{2(k - 1)}{k + 1} + 2 f^2 \frac{(k - 1)^2}{(k + 1)^2} + o(f^2),
\]

where \(f\) is the volume fraction of the inclusions, \(i.e., f = |D|\).

This formula has been generalized in many directions: To include higher power terms of \(f\) for spherical inclusions, [24, 42]; To include other shape of the inclusion such as ellipses, [44, 15, 20, 34, 16, 25, 21, 37, 8]; To include the case when \(f = O(1)\), see [36] and the references therein. Quite recently, the Maxwell-Garnett formula has been extended to include inclusions of general shape with Lipschitz boundaries [6]. The formula is given by

\[
\gamma^* = I + f M \left( I - \frac{f}{d} M \right)^{-1} + O(f^3),
\]

where \(M\) is the Pólya-Szegő polarization tensor associated with \(B\) and the conductivity \(k\). When \(B\) is a disk, \(M = 2(k - 1)/(k + 1) I\), and hence it is the same as the Maxwell-Garnett formula. In [6], the authors also derive higher-order terms of the formula in terms of GPT’s defined in (1).

\(^1\) When \(k = 0\), it is called the virtual mass.

\(^2\) To this formula several different pairs of names are attached. For this see [36].
3.2. Far field expansion

Define the fundamental solution to the Laplacian by

\[
\Gamma(x) = \begin{cases} 
\frac{1}{2\pi} \ln |x|, & d = 2, \\
\frac{1}{(2 - d)\omega_d} |x|^{2-d}, & d \geq 3,
\end{cases}
\]

where \(\omega_d\) denotes the area of the \((d - 1)\)-dimensional unit sphere.

We now show that the perturbation of electrical potential due to the presence of the inclusion is completely described by the GPT’s.

**Theorem 3.1** Let \(H\) be a harmonic function in \(\mathbb{R}^d\), and let \(u\) be the solution to the following problem:

\[
\nabla \cdot ((1 + (k - 1)\chi(B))\nabla u) = 0 \quad \text{in} \ \mathbb{R}^d, \\
u(x) - H(x) = O(|x|^{1-d}) \quad \text{as} \ |x| \to \infty.
\]

Then we have

\[
u(x) = H(x) + \sum_{|i|,|j|=1}^{\infty} \frac{1}{i!j!} \partial^i \Gamma(x) \partial^j H(0) M_{ji}, \quad |x| \to \infty.
\]

3.3. Reconstruction of conductivity inclusions

Consider a conductor \(\Omega\) in \(\mathbb{R}^d\) with conductivity 1. Suppose that \(\Omega\) contains a small inclusions \(D\) of the form \(D = \rho B + z, \ z \in \Omega\) and which has conductivity \(k, 0 < k \neq 1 < +\infty\). The following theorem from [1] holds.

**Theorem 3.2** Let \(g \in L^2(\Omega), \int_{\partial\Omega} g = 0\), be an applied current and let the potential \(u\) be the solution to the following problem:

\[
\nabla \cdot ((1 + (k - 1)\chi(D))\nabla u) = 0 \quad \text{in} \ \Omega, \\
\frac{\partial u}{\partial \nu} = g \quad \text{on} \ \partial\Omega, \ \int_{\partial\Omega} u = 0.
\]

Suppose that \(\text{dist}(z, \partial\Omega) \geq c > 0\). Then we have the following (uniform) high-order asymptotic formula for the boundary potential perturbations

\[
(u - U)(x) = \sum_{|i|,|j|=1}^{d} \frac{\rho^{i+j+d-2}}{i!j!} \partial^i N(x, z) \partial^j U(z) M_{ji} + O(\rho^{2d}), \quad x \in \partial\Omega,
\]

where \(U\) is the background solution in the absence of the inclusion and \(N\) is the Neumann function, that is,

\[
\begin{cases} 
\Delta_x N(x, z) = -\delta_z, & \text{in} \ \Omega, \\
\frac{\partial N}{\partial \nu_z} \big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|} \int_{\partial\Omega} N(x, z) \, d\sigma(x) = 0 \quad \text{for} \ z \in \Omega.
\end{cases}
\]

The GPT’s can be applied to obtain accurate reconstructions of small conductivity inclusions from a small number of boundary measurements. Based on Theorem 3.2 a number of non-iterative algorithms for recovering the location of the inclusion and some of its geometric properties have been designed, see for example [7, 27, 3] and the references therein.

The GPT’s seem to carry important geometric and potential theoretic properties of the domain \(B\). In the following sections we investigate these properties.
4. Symmetry and positivity

We now consider the symmetry and positivity of GPT’s. When $|i| = |j| = 1$, these properties were proved first in [11]. For symmetry we have the following theorem from [2].

**Theorem 4.1** Let $I$ and $J$ be finite sets of multi-indices and let $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ be such that $\sum_{i \in I} a_i y^i$ and $\sum_{j \in J} b_j y^j$ are harmonic polynomials. Then

$$
\sum_{i \in I, j \in J} a_i b_j M_{ij}(B,k) = \sum_{i \in I, j \in J} a_i b_j M_{ji}(B,k) .
$$

**Proof.** Note that

$$
\sum_{i,j} a_i b_j M_{ij} = \int_{\partial B} \sum_j b_j y^j \sum_i a_i \phi_i(y) d\sigma(y) .
$$

Put $f(y) = \sum_i a_i y^i$, $g(y) = \sum_j b_j y^j$, $\phi = \sum_i a_i \phi_i = (\lambda I - K_B^*)^{-1}(\frac{\partial f}{\partial \nu})$, and $\psi = (\lambda I - K_B^*)^{-1}(\frac{\partial g}{\partial \nu})$. Then $S_B\phi$ and $S_B\psi$ satisfy the transmission conditions

$$
\frac{\partial}{\partial \nu} S_B\phi|_+ - k \frac{\partial}{\partial \nu} S_B\phi|_- = (k - 1) \frac{\partial f}{\partial \nu}
$$

and

$$
\frac{\partial}{\partial \nu} S_B\psi|_+ - k \frac{\partial}{\partial \nu} S_B\psi|_- = (k - 1) \frac{\partial g}{\partial \nu}
$$
on $\partial B$. Recall that

$$
\sum_{i,j} a_i b_j M_{ij} = \int_{\partial B} g \phi d\sigma \quad \text{and} \quad \sum_{i,j} a_i b_j M_{ji} = \int_{\partial B} f \psi d\sigma .
$$

By the transmission condition, we have

$$
\int_{\partial B} g \phi d\sigma = \int_{\partial B} g \left[ \frac{\partial S_B\phi}{\partial \nu} |_+ - \frac{\partial S_B\phi}{\partial \nu} |_- \right] d\sigma = (k - 1) \int_{\partial B} g \frac{\partial}{\partial \nu} (S_B\phi + f) |_- d\sigma .
$$

We then immediately obtain

$$
\int_{\partial B} g \phi d\sigma = (k - 1) \int_{\partial B} (S_B\psi + g) \frac{\partial}{\partial \nu} (S_B\phi + f) |_- d\sigma
$$

$$
- \int_{\partial B} S_B\psi \frac{\partial}{\partial \nu} S_B\phi |_- d\sigma + \int_{\partial B} S_B\psi \frac{\partial}{\partial \nu} S_B\phi |_+ d\sigma
$$

$$
= (k - 1) \int_B \nabla (S_B\psi + g) \cdot \nabla (S_B\phi + f) dx
$$

$$
+ \int_{\mathbb{R}^n \setminus B} \nabla S_B\psi \cdot \nabla S_B\phi dx + \int_B \nabla S_B\psi \cdot \nabla S_B\phi dx .
$$

Symmetry property (3) follows from the above identity.

Suppose that $f = g$ in the proof of Theorem 4.1. It then follows from (4) that

$$
\int_{\partial B} f \phi d\sigma = (k - 1) \int_{\partial B} \frac{\partial f}{\partial \nu} (S_B\phi + f) d\sigma .
$$
On the other hand, it follows from the transmission condition that
\[\int_{\partial B} f \phi d\sigma = (k - 1) \int_{\partial B} (S_B \phi + f) \frac{\partial}{\partial \nu} (S_B \phi + f) d\sigma \]
\[- (k - 1) \int_{\partial B} S_B \phi \frac{\partial}{\partial \nu} S_B \phi d\sigma - (k - 1) \int_{\partial B} S_B \phi \frac{\partial f}{\partial \nu} d\sigma \]
\[= (k - 1) \int_{\partial B} (S_B \phi + f) \frac{\partial}{\partial \nu} (S_B \phi + f) d\sigma \]
\[- \left(1 - \frac{1}{k}\right) \int_{\partial B} S_B \phi \frac{\partial}{\partial \nu} S_B \phi d\sigma - \left(1 - \frac{1}{k}\right) \int_{\partial B} S_B \phi \frac{\partial f}{\partial \nu} d\sigma .\]

Define quadratic forms \(Q_D(u)\) by
\[Q_D(u) := \int_D |\nabla u|^2 dx ,\] (6)
where \(D\) is a Lipschitz domain in \(\mathbb{R}^d\). Then, by equating (5) and (4), we obtain
\[\int_{\partial B} S_B \phi \frac{\partial f}{\partial \nu} d\sigma = \frac{k}{k + 1} Q_B(S_B \phi + f) + \frac{1}{k + 1} Q_{\mathbb{R}^d \setminus B}(S_B \phi) - \frac{k}{k + 1} Q_B(f) .\]

Substituting this identity into (5), we get
\[\int_{\partial B} f \phi d\sigma = \frac{k(k - 1)}{k + 1} Q_B(S_B \phi + f) + \frac{k - 1}{k + 1} Q_{\mathbb{R}^d \setminus B}(S_B \phi) \]
\[+ \frac{k - 1}{k + 1} Q_B(f) .\]

So we obtain the following theorem of positivity, which was first proved in [2].

**Theorem 4.2** Suppose that \(a_i, i \in I\), where \(I\) is a finite index set, are constants such that \(f(y) = \sum_{i \in I} a_i y_i^e\) is a harmonic polynomial. Let \(\phi = (\lambda I - K_B)^{-1}(\frac{\partial f}{\partial \nu})\). Then
\[\sum_{i,j \in I} a_i a_j M_{ij}(B,k) = \frac{k - 1}{k + 1} \left[k Q_B(S_B \phi + f) + Q_{\mathbb{R}^d \setminus B}(S_B \phi) + Q_B(f) \right] .\] (7)

Theorem 4.2 says that if \(k > 1\), then GPT’s are positive definite, and they are negative definite if \(0 < k < 1\).

5. Bounds

The variational characterization (7) will be exploited in this section to derive isoperimetric inequalities for \(\sum_{i,j \in I} a_i a_j M_{ij}(B,k)\). Set \(w_B = S_B \phi\). Then
\[\begin{cases}
\nabla \cdot (1 + (k - 1)\chi(B))\nabla (w_B + f) = 0 & \text{in } \mathbb{R}^d , \\
w_B = O(|x|^{d-1}) & \text{as } |x| \to +\infty ,
\end{cases}\] (8)

since \(f_{\partial B} \phi = 0\), which yields
\[\int_{\mathbb{R}^d} \left(1 + (k - 1)\chi(B)\right) \left(\nabla w_B + (1 - \frac{1}{k})\chi(B)\nabla f\right) \cdot \nabla v = 0 , \quad \forall v \in H^1(\mathbb{R}^d) .\] (9)
Therefore, \( w_B \) is the minimizer of the functional
\[
I_B(v) := \int_{\mathbb{R}^d} (1 + (k - 1)\chi(B)) \left| \nabla v + (1 - \frac{1}{k})\chi(B)\nabla f \right|^2,
\]
\[
= \int_{\mathbb{R}^d} (1 + (k - 1)\chi(B))|\nabla v|^2 + \left( \frac{(k - 1)^2}{k} \right) \int_B |\nabla f|^2 + 2(k - 1) \int_B \nabla v \cdot \nabla f,
\]
(10)
namely,
\[
I_B(w_B) = \inf_{v \in H^1(\mathbb{R}^d)} I_B(v).
\]
It then follows from (7) that
\[
\sum_{i,j \in I} a_i a_j M_{ij} = I_B(w_B) + (1 - \frac{1}{k}) \int_B |\nabla f|^2 = \inf_{v \in H^1(\mathbb{R}^d)} I_B(v) + (1 - \frac{1}{k}) \int_B |\nabla f|^2.
\]
(11)
Substituting \( v = w_B \) in (9), we get
\[
-\int_{\mathbb{R}^d} (1 + (k - 1)\chi(B))|\nabla w_B|^2 = (k - 1) \int_B \nabla f \cdot \nabla w_B.
\]
Thus,
\[
I_B(w) = -\int_{\mathbb{R}^d} (1 + (k - 1)\chi(B))|\nabla w_B|^2 + \left( \frac{(k - 1)^2}{k} \right) \int_B |\nabla f|^2,
\]
and we therefore arrive at
\[
\sum_{i,j \in I} a_i a_j M_{ij}(B,k) = -\int_{\mathbb{R}^d} (1 + (k - 1)\chi(B))|\nabla w_B|^2 + (k - 1) \int_B |\nabla f|^2.
\]
(12)
Combining (11) and (12) we obtain the following theorem, which is a particular case of the results shown in [4].

**Theorem 5.1** Suppose that \( a_i, i \in I \), where \( I \) is a finite index set, are constants such that \( f(y) = \sum_{i \in I} a_i y^i \) is a harmonic polynomial. Then
\[
(1 - \frac{1}{k}) \int_B |\nabla f|^2 \leq \sum_{i,j \in I} a_i a_j M_{ij}(B,k) \leq (k - 1) \int_B |\nabla f|^2.
\]
Based on the above inequalities the harmonic moments of the inclusion \( B \) can be estimated from the GPT’s.

6. Monotonocity
We prove in this section that the GPT’s are monotone as a function of the domain. The following holds.

**Theorem 6.1** Let \( B' \subset B, B \neq B' \). Suppose that \( a_i, i \in I \), where \( I \) is a finite index set, are constants such that \( f(y) = \sum_{i \in I} a_i y^i \) is a harmonic polynomial. Then
\[
\sum_{i,j \in I} a_i a_j M_{ij}(B,k) > \sum_{i,j \in I} a_i a_j M_{ij}(B',k) \quad \text{if } k > 1,
\]
and
\[
\sum_{i,j \in I} a_i a_j M_{ij}(B,k) < \sum_{i,j \in I} a_i a_j M_{ij}(B',k) \quad \text{if } 0 < k < 1.
\]
Suppose that $k > 1$. We may argue in a similar way when $0 < k < 1$. From

$$
\sum_{i,j \in I} a_i a_j \left( M_{ij}(B, k) - M_{ij}(B', k) \right) = I_B(w_B) - I_B(w_{B'}) + \left( 1 - \frac{1}{k} \right) \int_{B \setminus B'} |\nabla f|^2
\geq I_B(w_B) - I_B(w_{B'}) + \left( 1 - \frac{1}{k} \right) \int_{B \setminus B'} |\nabla f|^2.
$$

We may prove by making use of (10) that

$$
\sum_{i,j \in I} a_i a_j \left( M_{ij}(B, k) - M_{ij}(B', k) \right) \geq (k - 1) \left( \int_{B \setminus B'} |\nabla w_B|^2 + 2 \int_{B \setminus B'} \nabla w_B \cdot \nabla f + \int_{B \setminus B'} |\nabla f|^2 \right),
$$

which yields the desired monotonicity result since using the fact that $w_B + f$ satisfies (8) we can easily prove that $\int_{B \setminus B'} |\nabla (w_B + f)|^2 > 0$.

### 7. Uniqueness result

In this section we discuss how the knowledge of all the GPT’s uniquely determines the geometry and the constitutive parameter of the inclusion. In fact, we relate the GPT’s to the Dirichlet-to-Neumann (DtN) map. We prove that we can recover the DtN map from all the GPT’s and hence, by a uniqueness result due to Isakov [23] (see also Druskin [17]), $B$ and $k$ are uniquely determined from all the GPT’s.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ compactly containing $B$. Recall that the DtN map $\Lambda : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$ corresponding to $k$ and $B$ is defined by, for $f \in H^{1/2}(\partial \Omega)$,

$$
\Lambda(f) := \left. \frac{\partial u}{\partial \nu} \right|_{\partial \Omega},
$$

where $u$ is the unique weak solution of

$$
\begin{cases}
\nabla \cdot \left( 1 + (k - 1) \chi(B) \right) \nabla u = 0 & \text{in } \Omega, \\
u|_{\partial \Omega} = f.
\end{cases}
$$

Let $M_{ij}(k, B)$ denote the GPT’s associated with the domain $B$ and conductivity $k$. The following theorem from [2] asserts that we can recover the DtN map and hence $B$ and $k$ from all the GPT’s.

**Theorem 7.1** Let $k_1, k_2$ be positive numbers different from 1, and let $B_1, B_2$ be bounded Lipschitz domains in $\mathbb{R}^d$. Let $\Omega$ be a domain compactly containing $B_1 \cup B_2$, and let $\Lambda_p$ be the DtN map corresponding to $k_p$ and $B_p$, $p = 1, 2$, on $\partial \Omega$. If $M_{ij}(B_1, k_1) = M_{ij}(B_2, k_2)$ for all multi-indices $i$ and $j$, then $\Lambda_1 = \Lambda_2$, and hence $k_1 = k_2$ and $B_1 = B_2$.

### 8. Open questions

In this paper we have examined symmetry, positivity, and monotonicity properties of the GPT’s along with their relationship to the harmonic moments of the inclusion. It has been further seen that the knowledge of all the GPT’s uniquely determines the DtN map, and hence the shape and the conductivity of the inhomogeneity. Many important questions still remain. It would be interesting to know how much information one can get from the knowledge of a finite number of these GPT’s. Of similar importance would be the question of the quantification of the stability of the inverse of the map $(B, k) \to M_{ij}(B, k)$. 
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