Large deviations for Markov processes with resetting
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Introduction

Outline

- Simple examples of resetting
- Mathematical formulation
- Main result
- Derivation of main result
- Application to the Ornstein-Uhlenbeck process (← LDT)
- Outlook
Examples of resetting

Discrete-space

- birth-death process with catastrophes (population dynamics)
- clearing of queues on failure of server (queueing theory)
- search strategies (computer science)

Continuous-space

- protein searchers on DNA (microbiology)
- foraging of animals (ecology)
- motion of molecular motor along biofilament
Brownian motion with resetting

**Figure** – Brownian motion $X_t$ reset at rate $r$ to zero.
Mathematical formulation

Stochastic differential equation (SDE)

\[ dX_t = F(X_t)dt + \sigma dW_t \]  

- \( X_t \) - position at time \( t \)
- \( F(\cdot) \) - drift function
- \( \sigma \) - noise strength
- \( dW_t \) - standard Brownian motion

In infinitesimal time interval \( dt \):
- reset to \( x_r \) with probability \( rdt \)
- evolve according to Eq. (1) with probability \( 1 - rdt \)
Definitions

Observables of $X_t$ having time-additive form:

$$A_T = \frac{1}{T} \int_0^T f(X_t) dt$$  \hspace{1cm} (2)

Generating function:

$$G_r(x, k, t) = E_x[e^{tkA_t}]$$  \hspace{1cm} (3)

$E_x[.]$ denotes expectation w.r.t. $X_t$ with resetting started at $x$.

Laplace transform of Generating function in time

$$\tilde{G}_r(x, k, s) = \int_0^\infty e^{-st} G_r(x, k, t) dt$$  \hspace{1cm} (4)

subscript $r$ refers to process with resetting. Objects are defined in an analogous way for no reset ($r=0$).
Main Result

Simple equation relating reset and no reset processes

\[ \tilde{G}_r(x, k, s) = \frac{\tilde{G}_0(x, k, s + r)}{1 - r\tilde{G}_0(x_r, k, s + r)} \]  

(5)
Reset periods

Assume $n$ resets in $[0, T]$ with duration $\tau_1, \tau_2, \ldots, \tau_n$ s.t.

$$T = \sum_{i=1}^{n+1} \tau_i$$

(6)

where $\tau_{n+1}$ is period after last reset. Can rewrite $A_T$:

$$A_T = \frac{1}{T} \sum_{i=1}^{n+1} \int \sum_{j=1}^{i} \tau_j f(X_s) ds$$

(7)

where we define $\tau_0 = 0$. 
Renewal representation of $G_r(x, k, s)$

**Derivation**

Due to independent diffusion segments:

$$G_r(x, k, T) = \sum_{n=0}^{\infty} \int_0^T d\tau_1 r e^{-r\tau_1} G_0(x_r, k, \tau_1) \int_0^T d\tau_2 r e^{-r\tau_2} G_0(x, k, \tau_2)$$

$$\times \ldots \int_0^T d\tau_{n+1} e^{-r\tau_{n+1}} G_0(x_r, k, \tau_{n+1}) \delta(T - \sum_{i=1}^{n+1} \tau_i) \quad (8)$$

Taking the Laplace transform:

$$\tilde{G}_r(x, k, s) = \tilde{G}_0(x, k, s + r) \sum_{n=0}^{\infty} r^n \tilde{G}_0(x_r, k, s + r)^n \quad (9)$$

Assuming $r \tilde{G}_0(x_r, k, s + r) < 1$ gives the result.
Mathematical formulation

Stochastic differential equation (SDE)

\[ dX_t = -\gamma X_t \, dt + \sigma dW_t \]  

- \( X_t \) - position at time \( t \)
- \( -\gamma X_t \) - drift function
- \( \sigma \) - noise strength
- \( dW_t \) - standard Brownian motion

Consider additive observable

\[ A_T = \frac{1}{T} \int_0^T X_t \, dt \]
Spectral decomposition

Generating function for reset free process:

\[
G_0(x, k, T) = \sum_{i=0}^{\infty} \psi_{k,i}(x) e^{\lambda_{0,i}(k)T}
\]  

where

\[
\lambda_{0,i} = \frac{k^2 \sigma^2}{2\gamma^2} - i\gamma, \quad i = 0, 1, \ldots
\]

and

\[
\psi_{k,i}(x) = \frac{(-1)^i \gamma^{3i/2} (k\sigma)^i e^{kx/\gamma - 3k^2\sigma^2/(4\gamma^3)} H_i\left(\frac{\sqrt{\gamma}x}{\sigma} - \frac{k\sigma}{\gamma^{3/2}}\right)}{\sqrt{2^i i! \sqrt{(2i)!!}}}
\]

with \(H_i\) \(i^{th}\) Hermite polynomial.
Large deviation theory

Largest eigenvalue (SCGF):

\[ \lambda_0(k) = \frac{k^2 \sigma^2}{2\gamma^2} \]  \hspace{1cm} (15)

corresponds to rate function:

\[ l_0(a) = \frac{\gamma^2 a^2}{2\sigma^2} \]  \hspace{1cm} (16)

for reset process:

\[ G_r(x, k, T) \sim e^{\lambda_r(k)T} \]  \hspace{1cm} (17)

In Laplace space:

\[ \tilde{G}_r(x, k, s) \sim \frac{1}{s - \lambda_r(k)} \]  \hspace{1cm} (18)

Conclusion

SCGF for OU process with reset is largest pole in \( s \) of main result Eq. (5).
Finite $n$ truncation

Taking only the first term in Eq. (12) gives invalid SCGF but by including more terms converge to valid SCGF

**Figure** — Largest pole of Eq. (5) for various truncation orders (left). Rate function for process without reset (dotted) and with reset (solid) with $r = 1$ (right). Both with reset position at $x_r = 0$. 
Non-zero reset position

\[ \lambda_r(k) \]

\[ I_{0,r}(a) \]

**Figure** — Largest pole of Eq. (5) for various truncation orders (left). Rate function for process without reset (dotted) and with reset (solid) with \( r = 1 \) (right). Both with reset position at \( x_r = 0.5 \).
Comments

- small fluctuations - many resets - Gaussian fluctuations in rate function with modified variance $\lambda'_r(k)$
- large fluctuations come from no/few resets suggesting

$$\lambda_r(k) \approx \lambda_0(k) - r \quad \text{as} \quad |k| \to \infty \quad (19)$$

and

$$l_r(a) \approx l_0(a) + r \quad \text{as} \quad |a| \to \infty \quad (20)$$

- competing effects: drift $-\gamma X_t$ vs. resetting rate $r$ gives non-parabolic rate function
- zero of rate function $a^*$ changes linearly in reset position $X_r$
- $a^*$ approaches $x_r$ as $r \to \infty$ given that $\gamma$ is fixed
Outlook

Possibly interesting questions

- Can resetting introduce large deviation principle?
- Can this be generalized to current type observables?
- Which other process can this be applied to? (e.g. Reflected Brownian motion?)
- Could we pick the reset position from a distribution?
- Can we introduce a random time penalty at each reset event?
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[2] M. R. Evans and S. N. Majumdar.
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