Quantum critical scaling and holographic bound for transport coefficients near Lifshitz points

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The transport behavior of strongly anisotropic systems is significantly richer compared to isotropic ones. The most dramatic spatial anisotropy at a critical point occurs at a Lifshitz transition. The present study uses scaling arguments and the gauge-gravity duality to investigate universal bounds appearing in strongly-coupled quantum anisotropic systems near a Lifshitz point. Explicit examples are merging Dirac or Weyl points or Lifshitz points near the superconductor-insulator quantum phase transition. Using scaling arguments we propose a generalization of both the shear-viscosity to entropy-density ratio and the charge-diffusivity bounds to the anisotropic case. We find that the electric conductivity and viscosity of the same material vanish along certain directions yet diverge along others. Thus, at such a quantum Lifshitz point the non-quasi-particle transport in the strong coupling limit is both metallic and insulating, depending on the electric field direction. We investigate the strongly-coupled phase of such systems in a gravitational Einstein-Maxwell-dilaton model with a linear massless scalar. The holographic computation demonstrates that some elements of the viscosity tensor can be related to the ratio of the electric conductivities. From the IR critical geometry, we express the charge diffusion constants in terms of the square butterfly velocities. The proportionality factor turns out to be direction-independent, linear in the inverse temperature, and related to the critical exponents which parametrize the anisotropic scaling of the dual field theory.

I. INTRODUCTION

Bounds on transport coefficients are an important tool to quantify the strength of correlations in quantum many-body systems. If one can identify a theoretical value for a minimal electrical conductivity or viscosity, then one can judge how strongly-interacting a system is. A highly influential bound for momentum conserving scattering of quantum fluids was proposed by Kovtun, Son, and Starinets1 (KSS) for the ratio of the shear viscosity and entropy density

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}.$$  

(1)

It is obeyed in systems like the quark gluon plasma2 or cold atoms in the unitary scattering limit3. Graphene at charge neutrality is another example that is expected to be close to this bound1. Within the Boltzmann transport theory one finds that a bound for $\eta/s$ can be related to the ratio $l_{\text{mfp}}/\lambda$ of the mean-free path $l_{\text{mfp}}$ and the mean distance $\lambda$ between carriers. However, Eq.(1) is valid even for systems that cannot be described in terms of the quasi-classical Boltzmann theory. Indeed, the bound is saturated for quantum field theories in the strong coupling limit as was shown in Ref.1 using the holographic duality of conformal field theory and gravity in anti-de-Sitter spacetime5–7.

Limiting bounds for the charge transport like the electrical conductivity are somewhat more subtle. A much discussed example is the Mott-Ioffe-Regel limit8–10 that corresponds to a threshold value of the electrical resistivity when $l_{\text{mfp}}/\lambda \sim O(1)$. While some systems clearly show a saturation of the resistivity once $\lambda/l_{\text{mfp}}$ reaches unity, materials like the cuprate or iron-based superconductors violate this limit11. For a detailed discussion of correlated materials that obey or systematically violate the Mott-Ioffe-Regel bound, see Ref.12. Transport properties in quantum critical systems were argued under certain circumstances to be governed by a Planckian relaxation rate $\hbar\tau^{-1} \approx k_B T^{13,14}$, which would also limit the electrical conductivity at quantum critical points. A bound on charge transport that is less restrictive and theoretically better justified than the Mott-Ioffe-Regel limit was proposed in Ref.15. It constrains the value of the charge diffusivity as determined by the Einstein relation:

$$D_c = \frac{\sigma}{\chi_e} \geq C_D \frac{\hbar v^2}{k_B T},$$  

(2)

with $C_D$ is a numerical coefficient of order unity. Here $v$ is a characteristic velocity of the problem, $\sigma$ is the electrical conductivity, and $\chi_e = \partial \rho / \partial \mu$ the charge susceptibility with particle density $\rho$ and chemical potential $\mu$. The latter is related to the charge compressibility since $\chi_e = -\frac{\partial v}{\partial \rho}$. If $v^2 \chi_e$ stays constant as $T \to 0$, the electrical resistivity cannot vanish slower than linearly in $T^{15}$. Ref.16,17 proposed the butterfly velocity $v = v_B$ as the characteristic velocity. $v_B$ follows from the analysis of out-of-time-order (OTOC) correlations $C(x,t) = -\langle [A(x,t), B(0,0)]^2 \rangle$ that are discussed in the context of chaos and information scrambling18–22. It
can be obtained from the long-distance behavior, e.g. via

\[ C(x, t) \sim e^{2\lambda_L(t-\frac{x}{v})}. \] (3)

The scrambling rate \( \lambda_L \) that enters the OTOC is also subject to the bound \( \lambda_L \geq 2\pi k_B T/\hbar \). While the interpretation of \( \lambda_L \) and its relation to transport and thermalization rates is not always correct\(^{23-28} \), the butterfly velocity seems to yield a natural scale for the characteristic velocity of a system, even if no clear quasiparticle description is available. A caveat applies when a symmetry of the system is weakly broken and triggers a sound-to-diffusion crossover: in this case, the resulting diffusivity is more naturally expressed in terms of the sound velocity and the gap\(^{25,29,30} \).

The focus of this paper is the investigation of anisotropic systems, where the conductivity tensor \( \sigma_{\alpha\beta} \) and the viscosity tensor \( \eta_{\alpha\beta\gamma\delta} \) exhibit a more complex structure with potentially different temperature dependencies for distinct tensor elements\(^{31,32} \). The anisotropy that we consider is most naturally expressed in terms of the relation between characteristic energies and momenta along different directions. For a system with two space dimensions, it holds then that:

\[ \omega \sim |k_x|^z/\phi \]
\[ \omega \sim |k_y|^z \] (4)

with dynamical exponent \( z \). We characterize the anisotropy in terms of the exponent \( \phi \) that relates typical momenta along the two directions according to

\[ |k_x| \sim |k_y|^{\phi/\gamma} \] (5)

A single particle dispersion that is consistent with such scaling would be \( \varepsilon(k) \sim |k_x|^{z/\phi} + a |k_y|^z \) that corresponds to a system at a Lifshitz point\(^{33-40} \). However, our conclusions do not require the existence of well defined quasiparticles with this dispersion relation.

Anisotropic systems, that obey scaling behavior of a Lifshitz transition were recently shown to violate the viscosity bound\(^{32,41-48} \). In Ref.\(^{32} \) a model of anisotropic Dirac fermions that emerged from two ordinary Dirac cones was analyzed as an explicit condensed matter realization\(^{49} \). Within a quasiparticle description of the transport processes and a Boltzmann equation approach, the conductivity anisotropy was found to diverge: one direction is metallic and another one insulating. Based on the quasiparticle transport theory, a modified bound was conjectured, that involves not just the viscosity tensor elements \( \eta_{\alpha\beta\gamma\delta} \) and the entropy density \( s(T) \), but also the conductivities\(^{32} \):

\[ \frac{\eta_{\alpha\beta\gamma\delta} \sigma_{\beta\gamma}}{s} \geq \frac{\hbar}{4\pi k_B}. \] (6)

Here, no summation over repeated indices is implied. Other tensor elements like \( \eta_{\alpha\beta\gamma\alpha} \) continue to obey Eq.(1).

The origin for this combined viscosity-conductivity bound is the different scaling behavior of the typical velocities \( v_\alpha \) for different directions. Candidate materials with Lifshitz transitions are the organic conductor \( \alpha-(\text{BEDT-TTF})_2\text{I}_3 \) under pressure\(^{50} \), and the heterostructure of the \( 5/3\text{TiO}_2/\text{VO}_2 \) supercell\(^{51,52} \). Moreover, the surface modes of topological crystalline insulators with unpinned surface Dirac cones\(^{53} \) and quadratic double Weyl fermions\(^{54} \) are expected to exhibit such a behavior. The analysis of Ref.\(^{32} \) was based on the Boltzmann equation and did not allow to explicitly analyze a model that satisfies this bound or determine the precise numerical coefficient in Eq.(6), i.e. the factor \( 1/4\pi \). This can only be done within a formalism that addresses transport in strongly-coupled non-quasi-particle many-body systems. In the same context it is of interest to address the related question of whether the diffusivity bound, Eq.(2), is also modified for anisotropic systems.

In this paper we perform a holographic analysis of anisotropic transport, exploiting the duality between strongly coupled quantum field theories in \( d+1 \) dimensions and gravity theories in one additional dimension\(^{6} \). The calculation is based on an Einstein-Maxwell-dilaton (EMD) action, where the anisotropy is generated by massless scalars, linear in the boundary spatial coordinates. See Refs.\(^{41-44,46-48,55-60} \) for previous studies of these holographic systems. As a consequence, the scalars also break translations, momentum is not conserved and the viscosity cannot be interpreted as a hydrodynamic coefficient.

It is well known that in such holographic frameworks the KSS bound is violated\(^{41-48,61-66} \). We will choose a geometry where the momentum is conserved along one of the spatial directions, say the \( \beta \)-direction. Thus, the stress tensor elements \( T_{\alpha\beta} \) serves as currents of the conserved momentum density along the direction \( \beta \). Consequently, the viscosity elements \( \eta_{\alpha\beta\gamma\delta} \) maintain their meaning as hydrodynamic coefficients, for all \( \alpha \) and \( \gamma \). In particular the model obeys

\[ \frac{\eta_{\alpha\beta\gamma\delta}}{s} = \frac{\hbar}{4\pi k_B} \frac{\sigma_{\alpha\beta}}{\sigma_{\gamma\delta}}. \] (7)

The generalized bound Eq.(7) has to be understood as a relation between hydrodynamic coefficients. Moreover, the combination \( \frac{\eta_{\alpha\beta\gamma\delta}}{s} \frac{\sigma_{\alpha\beta}}{\sigma_{\gamma\delta}} \) serves as an indicator of strong coupling behavior in anisotropic systems. In Fig.1 we show typical temperature dependencies for these transport coefficients for a specific value of the crossover exponent \( \phi \) that characterizes the anisotropy.

In addition, we determine the anisotropic butterfly velocity \( v_{B,\alpha} \) (see Refs.\(^{23,45,48,55,67-70} \) for previous studies) and the compressibility, and obtain for the anisotropic diffusivity the generalization of Eq.(2)

\[ D_{\alpha,\beta} = \frac{d_{\text{eff}} - \theta}{\Delta x} \frac{\hbar v_B^2}{2\pi k_B T}. \] (8)
between the different spatial directions \( k \) the crossover exponent that characterizes the anisotropy be-

\[ \phi, z, \]

ficient that now depends on the exponents bound, the anisotropy only changes the universal coef-

\[ \Delta \]

bility. Thus, the bound of Eq.(2) can be generalized and \( \Delta \) Eq.(11) below,

\[ d \]

where \( d_{\text{eff}} \) is the effective spatial dimensionality – see Eq.(11) below, \( \theta \) the hyperscaling violating exponent, and \( \Delta_A \) the scaling dimension of the charge suscepti-

ability. Thus, the bound of Eq.(2) can be generalized to anisotropic systems. In distinction to the viscosity bound, the anisotropy only changes the universal coef-

\[ \phi, z, \]

ficient that now depends on the exponents \( \phi, z, \) and \( \theta \). Furthermore, (8) recovers the limit of isotropic charge neutral theories.\(^{16}\) In Ref.\(^{55}\), the thermal diffusivity was computed in anisotropic setups and also found to obey a relation similar to (8). See Ref.\(^{45}\) for an alternative proposal to (8) at an anisotropic QCP.

Before we present the theories that yield these results, we give some general scaling arguments, assuming charge and momentum conservation. This analysis motivates us to consider the appropriate combinations of transport quantities that enter Eq.(6) and Eq.(8). The scaling analysis is then followed by a holographic analysis of the combined viscosity-conductivity bound, the charge suscepti-

bility, and the butterfly velocity within an anisotropic gravity theory.

II. SCALING ARGUMENTS

We consider the scaling behavior of transport coefficients in anisotropic systems near a quantum critical Lifshitz point. As we will see, scaling arguments can be efficiently used to make statements about transport bounds. Once a combination of physical observables has scaling dimen-

sion zero, it naturally approaches a universal value in the limit \( T, \mu, \omega \cdots \to 0 \), that corresponds to an underlying quantum critical state. If one can argue, usually based on an analysis of conservation laws, that this value is neither zero nor infinity, it should be some dimensionless num-

ber times the natural unit of the observable. In other words, this combination should be insensitive to irrele-

vant deformations of the quantum critical point. As an example we consider the electrical conductivity at zero density. For isotropic systems its scaling dimension is \( d = 2 \), a result that follows from single-parameter scaling and charge conservation. Thus the conductivity of a zero density two-dimensional system is expected to reach a universal value in units of the natural scale \( e^2/h \). Under the same conditions, both the viscosity and the entropy density have scale dimension \( d \) such that their ratio has scaling dimension zero. Then \( \eta/s \) should approach a uni-

versal value times \( h/k_B \) which yields the correct physical unit. This observation helps to rationalize a result like Eq.(1). As an aside, these scaling considerations also offer a natural explanation why the bound Eq.(1), while applicable, is not very relevant for Fermi liquids. Here, the existence of a large Fermi surface gives rise to hyper-

scaling violating exponents.\(^{71}\) If one performs the appropriate scaling near the Fermi surface,\(^{72}\) then it seems more natural to use \( \eta s^2 \) as the natural bound, a quantity that approaches a constant value as \( T \to 0 \).

The conclusions of this section require that scaling relations are valid, i.e that the system under consideration behaves critical and is below its upper critical dimension. In the remainder of this section we assume that this is the case. To be specific, we analyze a \( d \)-dimensional system and allow for one direction to be governed by a character-

istic length scale with a different scaling dimension \( \phi \neq 1 \) than the other spatial directions, see Eqs.(4,5) above. In addition, the temporal direction is characterized by a dyna-

mical scaling exponent \( z \). Let us then consider a physical observable \( O(\mathbf{k}, \omega) \). By assumption the observable obeys the scaling relation

\[ O(k_\perp, k_\parallel, \omega) = b^{-\Delta_O} O(b^\phi k_\perp, b^z k_\parallel, b\omega). \quad (9) \]

Here \( \Delta_O \) is the scaling dimension of the observable. The \( d \)-dimensional momentum vector \( \mathbf{k} = (k_\perp, k_\parallel) \) consists of one component \( k_\perp \) that is governed by the exponent \( \phi \) and a \( d - 1 \) dimensional component \( k_\parallel \). In the sub-

sequent holographic analysis we focus on a system with two spatial coordinates and use the notation \( k_\perp = k_x \) and \( k_\parallel = k_y \). While the scaling analysis presented here cannot determine the values of the exponents, it allows for rather general conclusions once those exponents are known. For an explicit model with nontrivial exponents \( z \) and \( \phi \), see Ref.\(^{32}\).
A. Scaling of thermodynamic quantities

We begin our discussion of scaling laws with thermodynamic quantities. For the free-energy density of the system holds the following scaling law:

\[ F(T, \mu) = b^{-d_s} \omega^{-z} F\left(b^z T, b^\gamma \mu\right), \]

with effective dimension

\[ d_{\text{eff}} = d - 1 + \phi. \]

As an energy density, \( F \) should scale like unit energy per unit volume. To obtain its scaling dimension it is then easiest to start from the usual result \( d + z \) for isotropic systems\(^{14}\) and replace \( d \) by \( d_{\text{eff}} \). This takes into account the different weight of the directions \( k \parallel \) and \( k \perp \). With \( s = -\partial F/\partial T \) and \( \rho = \partial F/\partial \mu \) we obtain immediately the scaling dimensions

\[ \Delta_s = \Delta_\rho = d_{\text{eff}} \]

for the entropy density \( s \) and particle density \( \rho \), respectively. Away from zero density, the relation \( \Delta_\rho = d_{\text{eff}} \) generally does not hold\(^{12,15,74} \). The second derivative of the free energy with respect to the chemical potential yields charge susceptibility

\[ \chi^{(\rho)}(T, \mu) = b^{-\Delta_s} \chi^{(\rho)}\left(b^z T, b^\gamma \mu\right) \]

with \( \Delta_s = d_{\text{eff}} - z \). We can now use these thermodynamic relations to determine the scaling behavior of the conductivity and viscosity. To do so is possible because of the restrictions that follow from charge and momentum conservation.

B. Scaling of transport coefficients

The conductivity is determined via a Kubo formula from the current-current correlation function, e.g.

\[ \text{Re } \sigma_{\alpha \beta}(\omega) = \frac{\text{Im } \Pi_{\alpha \beta}(\omega)}{\omega}. \]

At zero density, the system has a finite d.c. conductivity. \( \Pi_{\alpha \beta}(\omega) \) is the Fourier transform of the retarded current-current correlation function

\[ \Pi_{\alpha \beta}(t) = -i\theta(t) \langle \hat{J}_{\alpha}(t) \hat{J}_{\beta} \rangle. \]

In order to exploit the implications of charge conservation we use the continuity equation

\[ \partial_t \rho + \partial_{\alpha} J_{\alpha} = 0 \]

and obtain the well known relation between the longitudinal conductivity \( \sigma_{\alpha \alpha}(\omega) \) and the density-density correlation \( \chi^{(\rho)}(k, \omega) \)

\[ \sigma_{\alpha \alpha}(\omega) = \lim_{k \to 0} \frac{\omega}{k^2} \chi^{(\rho)}(k, \omega). \]

Here \( \chi^{(\rho)}(k, \omega) \) is the temporal Fourier transform of \( \chi^{(\rho)}(k, t) = -i\theta(t) \langle \rho(k, t), \rho(-k, 0) \rangle \), where \( \rho(k, t) \) is the spatial Fourier transform of the density \( \rho(x, t) \). Since \( \chi^{(\rho)} = \lim_{k \to 0} \chi^{(\rho)}(k, \omega = 0) \), the scaling dimension of \( \chi^{(\rho)} \) is also \( \Delta_s \), given below Eq.\(13\). Thus we find

\[ \Delta_{s,\parallel} = \Delta_s + 2 = d_{\text{eff}} - 2, \]

\[ \Delta_{s,\perp} = \Delta_s + 2 - 2\phi = d_{\text{eff}} - 2\phi, \]

for the conductivities along the two directions. This yields for the conductivities:

\[ \sigma_{\parallel}(T, \omega) = b^{3-\phi-d_s} \sigma_{\parallel}\left(b^z T, b^\gamma \omega\right), \]

\[ \sigma_{\perp}(T, \omega) = b^{\phi+1-d_s} \sigma_{\perp}\left(b^z T, b^\gamma \omega\right). \]

If we return to the isotropic limit, where \( \phi = 1 \), both components of the conductivity behave the same with usual conductivity scaling dimension \( d - 2 \). Interestingly, in the anisotropic case, this continues to be the dimension of the geometric mean \( \sqrt{\sigma_{\parallel} \sigma_{\perp}} \). Distinct scaling exponents for the tensor elements imply a different temperature dependency of the conductivity for different directions. Thus, a more insulating behavior along one direction will force the other direction to be more metallic. For a two-dimensional system, one direction will have to be insulating and the other then has to be metallic as long as \( \phi \neq 1 \). Finally, the ratio \( \sigma_{\parallel}/\sigma_{\perp} \) of the conductivity is governed by \( \Delta_{s,\parallel} - \Delta_{s,\perp} = 2(\phi - 1) \), i.e.

\[ \frac{\sigma_{\parallel}(T)}{\sigma_{\perp}(T)} = b^{-2(\phi-1)} \frac{\sigma_{\parallel}(b^z T)}{\sigma_{\perp}(b^z T)}. \]

We can perform a similar analysis for the viscosity tensor. It is given by a different Kubo formula

\[ \text{Re } \eta_{\alpha \beta \gamma \delta}(\omega) = \frac{\text{Im } \Pi_{\alpha \beta \gamma \delta}(\omega)}{\omega}, \]

with \( \Pi_{\alpha \beta \gamma \delta}(\omega) \) the Fourier transform of the retarded stress-tensor correlation function

\[ \Pi_{\alpha \beta \gamma \delta}(t) = -i\theta(t) \langle [T_{\alpha \beta}(t), T_{\gamma \delta}] \rangle. \]

Momentum conservation gives rise to the continuity equation for the momentum density \( g_{\alpha} \):

\[ \partial_t g_{\alpha} + \partial_{\alpha} T_{\alpha \beta} = 0. \]

We are considering a system without rotation invariance. In this case it is important to keep track of the order of the tensor indices as \( T_{\alpha \beta} \) cannot be brought into a symmetric form\(^{75} \). From the continuity equation for the momentum follows for the viscosity

\[ \eta_{\alpha \beta \gamma \delta}(\omega) = \lim_{k \to 0} \frac{\omega}{k_\alpha k_\beta k_\gamma k_\delta} \chi^{(g)}_{\beta \delta}(k, \omega), \]

with momentum-density correlation function \( \chi^{(g)}_{\beta \delta}(k, \omega) \), i.e. the Fourier transform of \( \chi^{(g)}_{\beta \delta}(k, t) = -i\theta(t) \langle g_{\beta}(k, t), g_{\delta}(-k, 0) \rangle \). Thus, we only need to
know the scaling dimension of $\chi^{(g)}_{x_\alpha}$ to determine the behavior of the viscosity. The easiest way to obtain this scaling dimension is to realize that under a boost operation, a velocity field is thermodynamically conjugate to the momentum density. A velocity has scaling dimension $z - 1$ for the directions along $k_\parallel$ and $z - \phi$ for $k_\perp$. To capture all the options we write this as $z - \varphi_\alpha$ where $\varphi_\alpha = 1$ for all directions but along $k_\perp$ where we have $\varphi_\alpha = \phi$. Thus, it holds

$$\chi^{(g)}_{x_\alpha}(k_\parallel, k_\parallel, \omega) = b^{-\Delta_{\alpha, \beta \gamma \delta}} \chi^{(g)}_{x_\alpha}(b^b k_\parallel, b^e k_\parallel, b^c \omega).$$ \hspace{1cm} (23)

with $\Delta_{\alpha, \beta \gamma \delta} = d_{\text{eff}} - z + \varphi_\beta + \varphi_\delta$. In the Appendix A we obtain the same behavior from an analysis of strain generators, following Refs.\textsuperscript{75,76}. Using $\Delta_{\alpha, \beta \gamma \delta}$ allows us to determine the scaling behavior of the viscosity tensor

$$\eta_{\alpha \beta \gamma \delta}(T) = b^{-\Delta_{\alpha, \beta \gamma \delta}} \eta_{\alpha \beta \gamma \delta}(b^b T).$$ \hspace{1cm} (24)

with

$$\Delta_{\alpha, \beta \gamma \delta} = \Delta_{\beta, \gamma \delta} + z - \varphi_\alpha - \varphi_\gamma = d_{\text{eff}} - \varphi_\alpha + \varphi_\beta - \varphi_\gamma + \varphi_\delta. \hspace{1cm} (25)$$

For isotropic systems, this gives the well known result that the scaling dimension of the viscosity is $d$, i.e. the same as for the entropy or particle density. For an anisotropic system the scaling dimensions of the viscosity and the entropy density can still be the same. This is the case whenever $\varphi_\alpha + \varphi_\gamma = \varphi_\beta + \varphi_\delta$. Examples are $\eta_{\parallel \parallel \parallel \parallel}$, $\eta_{\parallel \parallel \parallel \perp}$, $\eta_{\parallel \perp \perp \perp}$, or $\eta_{\perp \perp \perp \perp}$, where $\alpha, \beta$ etc. stand for components of $k_\parallel$.

However, the scaling dimension of the viscosity can also differ from the one of the entropy density. This is the case for

$$\eta_{\alpha \perp \perp \perp}(T) = b^{-((d - 3)\varphi_\alpha + 3\varphi_\gamma)} \eta_{\alpha \perp \perp \perp}(b^b T),$$

$$\eta_{\parallel \perp \perp \perp}(T) = b^{-((d + 1)\varphi_\alpha - \varphi_\gamma)} \eta_{\parallel \perp \perp \perp}(b^b T). \hspace{1cm} (26)$$

If we now take the ratio of the viscosity to entropy density, we find

$$\frac{\eta_{\alpha \perp \perp \perp}(T)}{s(T)} = b^{-2(\varphi_\alpha - 1)} \frac{\eta_{\alpha \perp \perp \perp}(b^b T)}{s(b^b T)},$$

$$\frac{\eta_{\parallel \perp \perp \perp}(T)}{s(T)} = b^{2(\varphi_\alpha - 1)} \frac{\eta_{\parallel \perp \perp \perp}(b^b T)}{s(b^b T)}. \hspace{1cm} (27)$$

Thus, for $\phi \neq 1$ there is always one tensor element of the viscosity, where $\eta_{\alpha \perp \perp \perp}/s$ diverges as $T \rightarrow 0$ and another one that vanishes. The latter will then obviously violate any bound for the ratio of a viscosity to entropy density. In Ref.\textsuperscript{32} it was shown that precisely these tensor elements turn out to be important for the hydrodynamic Poiseuille flow of anisotropic fluids.

The origin of unconventional scaling of both the conductivities and the viscosities is geometric, i.e. rooted in the anisotropic scaling of spatial coordinates at the Lifshitz point. If one combines Eqs.\textsuperscript{(19)} and (27), it is straightforward to see that the combinations that enter Eq.\textsuperscript{(6)} always have scaling dimension zero. While it certainly does not offer a proof of Eq.\textsuperscript{(6)} this is necessary for such quantity to approach a universal, constant low-temperature value.

Finally we comment on the scaling behavior of the diffusivity bound, Eq.\textsuperscript{(8)}. To check whether this bound even makes sense for an anisotropic system, we consider the quantity

$$X_{\alpha} = k_B T D_{c, \alpha}/\hbar v_\alpha^2 \hspace{1cm} (28)$$

where $v_\alpha$ is the characteristic velocity along the $\alpha$-th direction and $D_{c, \alpha} = \sigma_{\alpha \alpha}/\chi_c$ the diffusivity along this direction. It obviously holds

$$\Delta X_\alpha = z + \Delta_{\sigma, \alpha} - \Delta \chi - 2(z - \varphi_\alpha) \hspace{1cm} (29)$$

where we used again that a velocity scales as $z - \varphi_\alpha$. If we now insert our above results, it follows

$$\Delta X_\parallel = \Delta X_\perp = 0. \hspace{1cm} (30)$$

This implies that $X_\alpha$ should approach a universal constant times $\hbar/k_B$. Thus, we expect Eq.\textsuperscript{(2)} to be valid even for anisotropic systems, which yields Eq.\textsuperscript{(8)}. In this sense is this bound even more general than the original viscosity bound of Eq.\textsuperscript{(1)}.

## III. HOLOGRAPHIC DERIVATION OF THE VISCOSETY-CONDUCTIVITY BOUND

The correspondence between gravity theories and quantum field theories, as it occurs in the anti-de Sitter space/conformal field theory duality\textsuperscript{5-7}, is a powerful tool to analyze the universal properties of strongly-coupled field theories. In what follows we analyze an anisotropic bulk geometry in order to determine the relationships between distinct transport coefficients of anisotropic quantum many-body problems in the strong-coupling limit. To this end we use the membrane paradigm\textsuperscript{77} to express boundary theory transport coefficients in terms of geometric quantities at the horizon\textsuperscript{78}. To be specific, we consider a system of two space dimensions, i.e. with $D = 2 + 1$ space-time coordinates at the boundary. The relation between the generating functional of the quantum field theory and the gravity action for imaginary time is given by\textsuperscript{6,79}

$$\langle e^{-\int d^3 x \Phi_0 O} \rangle = e^{-S[\Phi]} \bigg|_{\Phi(r \rightarrow \infty) = \Phi_0}, \hspace{1cm} (31)$$

where $O$ is an operator of the field theory, $\Phi_0$ a conjugate source, $\Phi$ the dual field, and $S$ a gravitational action in the $D + 1$ dimensional bulk, with additional coordinate $r$. 
We start from the Einstein-Maxwell-Dilaton action
\begin{equation}
S = \int d^{d+1}x \sqrt{-g} (R + \mathcal{L}_M),
\end{equation}
with Lagrangian
\begin{equation}
\mathcal{L}_M = \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) - \frac{1}{2} Y(\varphi) (\nabla \psi)^2 - \frac{Z(\varphi)}{4} F^2. \tag{35}
\end{equation}
\varphi \text{ is referred to as the dilaton. It is a scalar field which enters the action modifying all the couplings involved.}
\begin{equation}
V(\varphi) \text{ is its own potential. We include the dilaton as it will allow us to consider anisotropic geometries that arise near the horizon of near-extremal black holes. In the absence of the dilaton field, the model reduces to the usual AdS system with electromagnetic field, i.e V(0) = 2\Lambda} \text{ with cosmological constant } \Lambda = -3/\ell^2, Z(0) = 1 \text{ and } Y(0) = 0. \ell \text{ is the radius of curvature of the AdS space.}
\end{equation}
As we will shortly see, by considering a bulk profile that depends linearly on one of the boundary spatial coordinates, the massless scalar \psi \text{ will break the rotation and translation symmetries of the dual field theory.}
For related work on this family of holographic models, see Refs.29, 41–44, 48, 57–59, 61, 65, 66, 84–87
\section{F2 is the Maxwell Lagrangian with } F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} \text{ the usual field tensor with vector potential } A_{\mu}. \text{ This term is needed to implement a U(1) global symmetry in the boundary theory and to determine the electrical conductivity.}

We summarize the field equations of motion that follow from Eq.(34) varying the action with respect to the fields \( g_{\mu\nu}, A_{\mu}, \psi, \varphi \).

Varying the metric, we obtain the Einstein equations
\begin{equation}
R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = - \frac{1}{\sqrt{-g}} \delta(\sqrt{-g}\mathcal{L}_M) \delta g^{\mu\nu}. \tag{36}
\end{equation}
The variation of the gauge field yields the Maxwell equations
\begin{equation}
\partial_\mu(\sqrt{-g} Z(\varphi) F^{\mu\nu}) = 0. \tag{37}
\end{equation}
Notice that both scalar fields are neutral such that the Maxwell equations are bulk conservation equations for the two-form \( F \). Ultimately, this will let us evaluate the boundary charge current at the black hole horizon. Finally, the wave equations for the two scalars are:
\begin{equation}
\partial_\mu(\sqrt{-g} Y(\varphi) \partial^\mu \psi) = 0, \tag{38}
\end{equation}
\begin{equation}
\partial_\mu(\sqrt{-g} \partial^\mu \varphi) = \partial_\varphi V_{\text{eff}}, \tag{39}
\end{equation}
where
\begin{equation}
V_{\text{eff}} = \sqrt{-g} V(\varphi) + \frac{Y(\varphi)}{2} (\partial \psi)^2 + \frac{Z(\varphi)}{4} F^2. \tag{40}
\end{equation}
In the absence of external perturbations, we use the following ansatz
\begin{equation}
ds^2 = -g_{tt}(t) dt^2 + g_{rr}(r) dr^2 + \sum_{\alpha} g_{\alpha\alpha}(r) dx_{\alpha}^2 \quad \varphi = \varphi(r), \quad A = A_t(t) dt, \quad \psi = \psi(t), \tag{41}
\end{equation}
\begin{equation}
\text{for } \alpha = x, y. \tag{42}
\end{equation}
\begin{equation}
\text{for } \alpha = x, y. \tag{42}
\end{equation}
where $a$ is real and $\alpha = \{x, y\}$. The Ansatz for $\psi$ is consistent with the field equations and preserves the homogeneity of the other fields. Indeed, $\psi$ back-reacts on the equations of motion only through gradients so that all dependence on $y$ drops out of the field equations. However, translations along the $y$-direction are broken and momentum is dissipated at a strength set by $a$. On the other hand, momentum along $x$ direction is conserved which allows us to perform a hydrodynamic analysis of the viscosity tensor elements $\nu_{ax}$. The metric in (41) describes anisotropic bulk geometries since, in general, $g_{ax}(r) \neq g_{by}(r)$. The coefficient $a$ determines the temperature scale $T_0$ below which the anisotropy effects are large. Setting $a = 0$ restores both rotations and translations in the dual field theory.

In its more general formulation, the holographic correspondence maps the RG-flow of the dual (strongly coupled) field theory to the evolution along the radial direction \cite{2} – see Fig.2. The near boundary region captures the UV of the dual field theory, while the near horizon region describes the IR. In the UV ($r \to \infty$) the geometry is assumed to be asymptotically AdS$_4$:

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2(dx^2 + dy^2) + \ldots \quad (42)$$

where the dots denote subleading terms as $r \to +\infty$. This requires

$$V_{UV} \equiv V(0) = -6, \quad Y_{UV} \equiv Y(0) = 0, \quad Z_{UV} \equiv Z(0) = 1 \quad (43)$$

with the dilaton vanishing like $\varphi = \varphi_r^{\Delta_2 - 3} + \varphi_r^{-\Delta_2} + \ldots$ coming from the near boundary expansion of Eq.(39). $\Delta_2 < 3$ is the largest solution of $M^2 = \Delta_2 (\Delta_2 - 3)$, $M$ being the mass of the field. The dilaton field is thus dual to a relevant deformation of the UV CFT, with source $\varphi_r$ and vacuum expectation value $\varphi_v$ \cite{82}.

From this discussion, we also see that the bulk field $\psi$ sources a marginal deformation of the UV CFT. Similarly, $A_t = \mu - p r^{-1} + \ldots$ with chemical potential $\mu$ and charge density $\rho$. In the following we analyze the charge neutral case $A_t = 0$.

Since both scalars are dual to relevant/marginal deformations of the UV CFT, we expect the system to be able to flow to a non-trivial quantum critical phase. This IR endpoint of the RG flow is represented in the bulk by a power law geometry, which arises in the near horizon region at very low temperatures compared to the sources of the UV CFT. To find such geometries, we assume that the dilaton runs logarithmically in the IR ($r \to 0$) $\varphi = 2\kappa \log(r)$ and that the scalar potentials take the following form \cite{73,88}

$$V_{IR} = -V_0 e^{\delta \varphi}, \quad Y_{IR} = e^{\kappa \varphi}, \quad Z_{IR} = e^{\delta \varphi}. \quad (44)$$

The critical scaling of the previous section is holographically realized by a Lifshitz geometry of the form

$$ds^2 = r^\theta \left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + \frac{dx^2}{r^\alpha} + \frac{dy^2}{r^\beta} \right), \quad (45)$$

which is covariant under the scale transformation $(t, r, x, y) \rightarrow (b^{-\theta}t, b^{-1}r, b^{-\alpha}x, b^{-\beta}y)$, up to a conformal factor $ds^2 \rightarrow b^{-\theta} ds^2$. Therefore, $\phi$ and $z$ coincide with the anisotropic and dynamical exponents, and $\theta$ quantifies the violation of scale invariance in the metric \cite{45,71,89}. All the parameters involved are real and $V_0, \delta, L > 0$. The explicit derivation of such a solution can be found in Appendix B, for the (marginally) relevant single axion case, which has $z = \phi \neq 1$, the marginally double axion case, which has $z > 1$, $\phi \neq 1$, and the irrelevant single axion case, which has $z = 1$, $\phi = 1$ (and where rotations/translations along $x$ are only broken away from the IR endpoint through the irrelevant deformation).

A finite temperature can be introduced via the embalckening factor

$$f(r) = 1 - \left(\frac{r_+}{r_0}\right)^{\delta_0} \quad (46)$$

where $r_+$ denotes the location of the event horizon and $\delta_0 = 1 + \phi + z - \theta$. The Hawking temperature is

$$4\pi T = \frac{\left|\delta_0\right|}{L} r_+^{-\frac{\phi + 1}{z}} \quad (47)$$

and satisfies $T \to b^2 T$, consistently with the scaling analysis. The fact that scaling stops at a finite value of the flow is reflected in the event horizon at finite $r_+$. The entropy density follows from the area of the horizon $s = 4\pi r_+^{\phi + 1}$.

Thus, with an appropriate choice of $V(\varphi)$, and $Y(\varphi)$ we can “engineer” a holographic dual that generates a desired crossover exponent $\phi$. Without more constructive statements about the field theory-gravity dual, it is not possible to determine the values of $\phi$ for a given quantum field theory. However, we can make statements about a number of physical observables for a given value of $\phi$.

\section{Analysis of the conductivity}

In this section we review the results of Ref.\cite{56,59,78} to express the electric conductivities in terms of IR quantities. In particular, we calculate the d.c. conductivity along the $\alpha$-direction

$$\sigma_{\alpha\alpha} = \lim_{r \to 0} \frac{\text{Im} \ j^\alpha(r, \omega)}{\omega A_\alpha(r, \omega)}, \quad (48)$$

working directly at zero frequency and switching on a constant and small electric field. $A_\alpha$ is the fluctuation respect to which we linearize the gauge equations, and $j^\alpha$ is the associated canonical momentum.

Within the homogeneous ansatz (41), Maxwell equations assume the form

$$\partial_t (\sqrt{-g} Z(\varphi) F^{\mu \nu}) = 0. \quad (49)$$
The quantity in brackets coincides with the conjugate momentum of the gauge field \( j^\mu = \delta S/\delta (\partial_t A_\mu) \). From the holographic dictionary (32) follows that the boundary value \( j^\mu (r = \infty) \) of this quantity is the electric current density of the dual field theory. From (49) \( j^\mu \) is radially conserved, i.e. \( \partial_r j^\mu = 0 \). Thus, we can determine the current at the boundary from the behavior of \( j^\mu \) at the horizon

\[
j^\mu (r = \infty, x, t) = \lim_{r \to r^+} j^\mu (r, x, t). \tag{50}
\]

In the absence of external fields, the only non zero component of \( j^\mu \) is the temporal one \( j^t = \sqrt{-g} Z (\varphi) F^{tr} \) which corresponds to the charge density \( \rho \) of the field theory. In the following we focus on the charge neutral case \( \rho = 0 \).

In order to determine the conductivity, we add a small electric field \( E_\alpha = F_{\alpha t} \) in the \( \alpha \)-direction, e.g. via

\[
A_\alpha^{\text{ext}} = -E_\alpha t. \tag{51}
\]

This electric field will polarize the system and therefore induce small corrections to the metric and matter fields. We parametrize those corrections via

\[
A_\alpha = -E_\alpha t + \delta A_\alpha (r),
\]

\[
g_{\alpha\alpha} = \delta g_{\alpha\alpha} (r),
\]

\[
g_{\alpha\gamma} = g_{\alpha\alpha} (r) \delta h_{\alpha\gamma} (r), \tag{52}
\]

and \( \psi = ay + \psi (r) \) if \( \alpha = y \).

All terms \( \delta A_\alpha \) etc. are assumed to be of first order in the electric field. They can be related to each other through a perturbative solution of the field equations. The above ansatz yields at first order and for zero density \( \rho = 0 \):

\[
j^\alpha = -\sqrt{-g} Z (\varphi) \frac{\partial_r \delta A_\alpha}{g_{\alpha\alpha}}. \tag{53}
\]

This result further simplifies our analysis as we only need to determine \( \delta A_\alpha \). To this end we perform a transformation to a set of coordinates that is free of singularities at the horizon. This is accomplished by the Eddington-Finkelstein (EF) coordinates \(^{90,91} t^\prime = t + r_\ast (r), \) where \( dr_\ast = dr/\gamma (r) \) is the tortoise coordinate, \( \gamma (r) = \sqrt{g_{tt} (r)/g_{rr} (r)} \). In these variables holds that near the horizon

\[
A_\alpha = -E_\alpha t^\prime + E_\alpha r_\ast (r) + \delta A_\alpha. \tag{54}
\]

If we now demand regularity of \( A_\alpha \) in the EF coordinates it follows for the leading, singular contribution:

\[
\delta A_\alpha (r \to r^+) = -E_\alpha r_\ast (r). \tag{55}
\]

It is now straightforward to determine the conductivities

\[
\sigma_{\alpha\alpha} = \lim_{r \to r^+} j^\alpha / E_\alpha = \sqrt{\frac{\delta g_{\alpha\alpha}}{g_{\alpha\alpha}}} Z (\varphi) \bigg|_{r^+}, \tag{56}
\]

where \( \overline{x} = y \) and \( \overline{y} = x \).

### B. Analysis of the viscosity

In order to compute the correlation function (20), we act on the bulk-metric field which is dual to the boundary stress tensor\(^1\). To get the shear viscosity components, we switch on small off-diagonal fluctuations of the spatial sector

\[
ds^2 \to ds^2 + e^{-i\omega t} \delta h_{xy} (r) dx dy. \tag{57}
\]

In the following we linearize Einstein equations with respect to the one-index-up parametrization \( h_\alpha^\beta = g_\beta^\alpha \delta h_{\alpha\beta} \) and compute the viscosity through

\[
\eta_{\alpha\beta\gamma\delta} = \lim_{r \to \infty} \frac{1}{\omega} \Im \frac{\Pi_3^{\gamma\delta} (r, \omega)}{h_\alpha^\beta (r, \omega)}, \tag{58}
\]

where \( \Pi_3^{\gamma\delta} \) is the associated radial momentum\(^92\). Since the model is anisotropic, there will be two fluctuations satisfying different equations of motion.

We start with the simpler case to review the standard derivation of the viscosity, and consider \( \delta h_{xy} = g_{xx} h^y_x \). The Einstein equations (36) yield

\[
\partial_\mu \left( \sqrt{-g} N \partial^\mu h^y_x \right) = 0, \tag{59}
\]

which describes the dynamics of a massless scalar with radial dependent coupling \( N (r) = g_{yy} (r) g^{xx} (r) \). The canonical momentum is

\[
\Pi^{y}_x = \sqrt{-g} N \partial^r h^x_y, \tag{60}
\]

satisfying \( N \partial_r \Pi^{y}_x = -\omega^2 \sqrt{-g} h^y_x \). In the low frequency limit, i.e. \( \omega \to 0 \) keeping \( \omega h^y_x \Pi^{y}_x \) and \( N \) fixed, both the fluctuation and the momentum are radially conserved allowing to perform a near horizon limit in Eq.(58). Here the fluctuation satisfies the in-falling conditions

\[
h^y_x (r, \omega) \to h_0 (r) e^{-i\omega \tau_\ast (r)}. \tag{61}
\]

\( h_0 \) is the real solution to the frequency independent wave equation, which asymptotes to a constant at the boundary and is regular at the horizon. Due to the radial conservation \( h_0 (r) \equiv 1 \). We then obtain

\[
\frac{\eta_{yxx}}{s} = \frac{1}{4\pi g_{yy}} \frac{g_{xx}}{r_\ast}, \tag{62}
\]

which reproduces the bound of Eq.(1) in the isotropic limit \( g_{xx} = g_{yy} \). These results, together with our findings of Eq.(56) for the conductivities immediately yield the expression Eq.(7) given in the introduction. This is one of the key results of this paper.

For the \( y \)-index-up parametrization we find

\[
\partial_\mu \left( \sqrt{-g} N \partial^\mu h^x_y \right) = \sqrt{-g} N m^2 h^x_y \tag{63}
\]
with radial mass \( m^2(r) = a^2 Y(\varphi) g^{yy}(r) \) arising due to the breaking of translations along \( y \). As before, we define the conjugate momentum via

\[
\Pi^x_y = \sqrt{-g} N \partial^y h^x_y, \tag{64}
\]

with \( \partial_r \Pi^x_y = \sqrt{-g} N (m^2 - \omega^2 g^{yy}) h^x_y \). The non vanishing mass makes the evolution along \( r \) non trivial even at zero frequency. However, from the equations follows that \( \text{Im} \left[ \Pi^x_y h^x_y \right] \) is radially conserved\(^{12} \), \( h^x_y \) denoting the complex conjugate fluctuation. In particular we can switch to the near horizon limit in the numerator

\[
\eta_{xyxy} = \lim_{\omega \to 0} \lim_{r \to r_+} \text{Im} \left[ \Pi^x_y h^x_y \right]. \tag{65}
\]

Using the in-falling conditions in the numerator we obtain

\[
\eta_{xyxy}^s = \frac{1}{s} \frac{g_{yy}}{4\pi g_{xx} \left| h^2_x(r_+) \right|}. \tag{66}
\]

\( h_0(r_+) \) denotes the horizon value assumed by \( h^2_x(r) \). A similar result obtains in isotropic backgrounds with momentum relaxation\(^{61,62,64} \). \( h_0(r_+) \) originates from the simultaneous breaking of rotations and translations along \( y \) caused by the massless scalar. Since it has a non-trivial radial evolution, we expect that it will differ from unity as temperature decreases, i.e. as the system flows away from the UV AdS$_4$.

We can also discuss the temperature dependence of \( h_0(r_+) \) at low temperatures. First, we discuss the case where the massless scalar \( \psi \) vanishes faster than other bulk fields towards the extremal horizon. Then, the IR endpoint enjoys rotation and translation symmetries, which are broken only through an irrelevant deformation sourced by \( \psi \). We expect that \( h_0(r_+) \) goes to a constant typically less than unity, as in Ref.\(^{61} \).

Alternatively, the translation/rotation breaking field \( \psi \) can source a marginal deformation at \( T = 0 \). In this case, there is no notion of momentum, although of course we can still compute the response to shear strain using the Kubo formula. But then the object we are computing does not have the interpretation of a shear viscosity. Its temperature dependence follows from an asymptotic analysis near the boundary of the IR region and yields:

\[
\sigma_{yy} \eta_{xyxy} \sim T^{-\frac{3a^2 - 2(\epsilon - 1)}{2}} \left( 1 + \left( \frac{2a L}{\pi - 2(\epsilon - 1)} \right)^2 \right). \tag{67}
\]

The sign of the exponent is not fixed, hence the tensor element can vanish or diverge - for details on the parameter range see Appendix B. This result is still valid when two axions are taken into account (B6). The isotropic limit of this last case is consistent with Ref.\(^{65,66} \) at charge neutrality.

In any case, the viscosity-conductivity bound stated through the scaling analysis is holographically realized at least for one of the \( \eta/s \)-tensor elements.

### IV. HOLOGRAPHIC DERIVATION OF THE CHARGE-DIFFUSIVITY BOUND

The charge diffusivity in the \( \alpha \)-direction is determined by the electrical conductivity and the charge susceptibility via the Einstein relation \( D_{c,\alpha} = \sigma_{\alpha \alpha}/\chi_c \). In section II we demonstrated that the combination

\[
X_\alpha = \frac{k_B T D_{c,\alpha}}{\hbar^2 a^3} \tag{68}
\]

has scaling dimension zero, which suggests that it approaches at low temperatures a universal value. In the subsequent sections we will use our result Eq.\((56) \) for the conductivity, obtained through the holographic approach and determine, within the same theory, the charge susceptibility and the butterfly velocity of the system. Without loss of generality we set \( \epsilon = 0 \), as in the charge neutral case the exponent of \( Z_{IR} = e^{\kappa \varphi} \) is not constrained – see Appendix B. We then obtain the result that

\[
X_\alpha = \frac{1}{2} \left( 1 + \phi - \theta \right) \tag{69}
\]

independent on the space direction \( \alpha \) leads to Eq.\((8) \).

#### A. Analysis of the diffusivity

An important ingredient for the bound on the diffusivity in Eq.\((2) \) is the isothermal charge susceptibility \( \chi_c \equiv \langle \partial \rho/\partial \mu \rangle_T \). In order to derive the correspondent holographic relation, we formally solve Maxwell equations \((49) \):

\[
A_t(r) = A_t(r_+) + \rho \int_{r_+}^r \frac{dr}{\sqrt{-g Z(\varphi) g^{rr} g^{tt}}}. \tag{70}
\]

As mentioned, \( A_t \) yields the chemical potential near the boundary and vanishes at the horizon, therefore

\[
\chi_c^{-1} = \int_{r_+}^\infty \frac{dr}{\sqrt{-g Z(\varphi) g^{rr} g^{tt}}}, \tag{71}
\]

see also Ref.\(^{78} \). Due to the non locality of the above formula, the integral can only be worked out by explicitly solving the RG flow from the boundary to the horizon. Keeping in mind that \( r_+ \propto T^{-1/2} \), we observe that the near horizon geometry contribution scales as \( T^{-\Delta_x/z} \).

Within a low temperature analysis, this is the dominant term if \( \Delta_x/z > 0 \) and the charge diffusion is uniquely controlled by the IR physics, in accord with the isotropic analysis of Ref.\(^{16,17} \). In this case we obtain

\[
\chi_c^{-1} = -\frac{L}{\Delta_x Z(\varphi)} \bigg|_{r_+} . \tag{72}
\]
We can alternatively Taylor-expand the integrand \(i(r)\) of (71) near the horizon. From the IR scaling behavior follows the recursion rule

\[
i^{(n)}(r) = \frac{(-1)^n}{r^n} \left[ \prod_{k=1}^{n} (k - \Delta \chi) \right] i(r), \tag{73}\]

\(i^{(n)}(r)\) denotes the \(n\)-th radial derivative of \(i(r)\). Plugging this expression into the Taylor expansion we find

\[
i(r) = i(r_+) \sum_{n=0}^{\infty} \left( \frac{n - \Delta \chi}{n} \right) \left( 1 - \frac{r}{r_+} \right)^n \tag{74}\]

Performing the binomial series we obtain \(i(r) = i(r_+)(r/r_+)^{\Delta \chi - 1}\), which yields the same result of the previous analysis.

The susceptibility together with the holographic conductivities (56) yields the diffusion constants

\[
D_{c,\alpha} = -\frac{L}{\Delta \chi \, g_{\alpha \alpha}(r)} \left. \right|_{r_+} \tag{75}.
\]

The above results are still valid in the \(\zeta \neq 0\) case.

### B. Analysis of the butterfly velocity in anisotropic systems

Following Ref.\(^{18}\), we determine the butterfly velocity for an anisotropic holographic system using a shock-wave analysis. As mentioned in the introduction, the butterfly velocity can be thought of as the velocity of growth of out-of-time-order correlation functions of local operators. Holographically, it can be calculated from the back-reaction of the metric due to a massless particle falling towards the horizon of the black hole. The velocity of growth of this back-reaction can then be identified as the butterfly velocity.

For the subsequent analysis it is convenient to use Kruskal-coordinates

\[
uv = -e^{\gamma'(r_+) r_+}, \quad u/v = -e^{-\gamma'(r_+) t}, \tag{76}\]

where \(\gamma'\) denotes the radial derivative. \(uv = 0\) and \(uv = -1\) correspond to the horizon and to the boundary respectively – see Fig.3. The anisotropic metric (41) takes the form

\[
ds^2 = -g_{uv}(uv) \, du \, dv + \sum_{\alpha} g_{\alpha \alpha}(uv) \, dx_{\alpha}^2. \tag{77}\]

Next we perturb the system by adding \(\delta T_{uv} \propto Ee^{2\pi T_{uv}}(u)\delta(x)\delta(y)\) to the holographic stress-energy tensor, which represents a particle of energy \(E\) released at the left boundary at time \(t_w\) in the past and propagating towards the \(u = 0\) horizon\(^{18,33}\). The perturbed metric can then be expressed in the following shock-wave form

\[
ds^2 = -g_{uv}(uv) \, du \, dv + g_{uv}(uv) h(x, y) du^2 + \sum_{\alpha} g_{\alpha \alpha}(uv) \, dx_{\alpha}^2. \tag{78}\]

The equation of motion for \(h(x, y)\) follows from Einstein equations at near the \(u = 0\) horizon:

\[
\left( \sum_{\alpha} \frac{\partial^2}{c_{\alpha}^2} - m_h^2 \right) h(x, y) = b \delta(x) \delta(y), \tag{79}\]

with \(c_{\alpha} = \sqrt{g_{\alpha \alpha}(0)}\), \(b \propto Ee^{2\pi T_{uv}}/g_{uv}(0)\) and mass

\[
m_h^2 = \left. \frac{1}{g_{uv}} \, \left( \frac{\partial}{\partial(u)} \log(g_{xx}g_{yy}) \right) \right|_{u=0}. \tag{80}\]

Eq.(78) is consistent with the isotropic case of Ref.\(^{17}\). The solution can be expressed in terms of the 0th modified Bessel function of the second kind \(K_0\) as

\[
h(x, y) \propto -\frac{b c_x c_y}{2\pi} K_0(m_h \theta), \tag{81}\]

where \(\theta^2 = c_x^2 x^2 + c_y^2 y^2\). At large values of \(\theta\), i.e. at large spatial distances, this gives

\[
h(x, y) \propto \frac{1}{\sqrt{\theta}} \exp \left[ 2\pi T \left( t_w - \frac{m_h}{2\pi T} \theta \right) \right]. \tag{82}\]

From the exponent we can extract the direction-averaged scale for the velocity

\[
\bar{v}_B = \frac{2\pi T}{m_h}. \tag{83}\]
In order to switch to the original system of coordinates, we use the identity \( uv g_{uv}(uv) = g_u(r) / \partial_r g_u(r)^2 \) and obtain
\[
\hat{v}_B^2 = -\frac{2\pi TL}{d_{\text{eff}} - \theta^2} \frac{\theta}{\bar{v}_B^z}. \tag{83}
\]
To determine the butterfly velocity along \( x \), we consider the case where \( y = 0 \) and we move in the \( x \)-direction. This gives
\[
\frac{\theta}{v_B} = \frac{e_x |x|}{|x|} = \frac{|x|}{v_{B,\bar{x}}}. \tag{84}
\]
It then follows for the velocity along the \( \alpha \)-direction
\[
\nu_{B,\alpha} = \frac{v_B}{\sqrt{g_{\alpha\alpha}(r_+)}}, \tag{85}
\]
in accord with Ref\(^ {23,45,48,55,67,68} \). This result violates the upper bound of the isotropic case pointed out in Ref\(^ {94} \), consistently with Ref\(^ {69,70} \).

Considering the ratio between the diffusion constant (75) and the square butterfly velocity we finally obtain
\[
\frac{D_{\alpha,\alpha}}{v_{B,\alpha}^2} = \frac{d_{\text{eff}} - \theta}{\Delta x} \frac{1}{2\pi T}, \tag{86}
\]
which yields the result Eq.(8) for the diffusivity bound, another key result of this paper.

V. CONCLUSIONS

In this paper we analyzed transport coefficients at a quantum Lifshitz point in the strong coupling limit, using scaling arguments and exploiting the duality between quantum field theories and gravity theories. We have focused on particle-hole symmetric theories at charge neutrality which admit a gravitational dual description. We have shown that bounds on transport coefficients of the isotropic case can be generalized to the anisotropic one. We analyzed the behavior of several observables after a spacetime dilatation, emphasizing that the scale dimensionless ones must approach a constant value for low temperatures. It turned out that some elements of the \( \eta/s \)-tensor have a nonzero dimension while the diffusivity still exhibits the scaling of the rotational invariant case. In order to address the former, we included the electric transport, multiplying the ratio by a specific combination of conductivities such that the dimension of the resulting quantity is zero.

Within the Einstein-Maxwell-dilaton model considered, translational symmetry is broken along the \( y \) direction by a massless scalar in the bulk with a bulk profile linear in \( y \). Thus, the \( x \)-component of the momentum is still conserved and \( T_{\alpha x} \) continues to be the current of a conserved quantity. Therefore, the viscosity tensor elements \( \eta_{\alpha x \beta x} \) maintain their meaning as hydrodynamic transport coefficients. Since we can find solutions of the field equations that yield either \( \phi < 1 \) or \( \phi > 1 \), we can always construct an anisotropic geometry that violates the isotropic viscosity bound for at least one tensor element, while fulfilling the generalized bound given in Eq.(7).

In the direction where translations are broken, momentum relaxes at a rate \( 1/\tau_{\text{mr}} \). Provided \( 1/\tau_{\text{mr}} \ll \Lambda \), where \( \Lambda \) is a UV cutoff, there is a range of intermediate times \( 1/\tau_{\text{mr}} \ll t \ll \Lambda \) where momentum is approximately conserved. In this regime, the viscosity can be defined from the shear Kubo formula, yet is still found to violate the viscosity-to-entropy-density-ratio bound. Alternatively, the diffusivity of transverse momentum can be considered, and has been reported\(^ {93} \) to obey a bound of the kind (2).

Differently from the other quantities, the diffusivity is not solely given by data on the horizon and is expressed through an integral over the radial direction. Although we do not have the full expression of bulk fields, we have derived a near horizon formula for the compressibility, and could relate the diffusion constant to the horizon data in a simple fashion. Indeed, the near IR geometry dominates at low temperature\(^ {16,25,26} \). On the other hand we have calculated the butterfly velocities by moving to the Kruskal system of coordinates and using a generalization of the shock-wave technique. We have computed the proportionality factor between the diffusivity to the square butterfly velocity ratio and the inverse temperature, finding that it can be expressed in terms of the critical exponents \( z, \phi \), and \( \theta \).

Eventually, both the viscosity and the diffusivity bounds could be analyzed within higher-derivative gravitational backgrounds\(^ {95,96} \) to take into account finite coupling effects. Moreover, particle-hole symmetry breaking could be taken into account as well. Thus, we conclude that the transport properties of a strongly-interacting many-body system near a quantum Lifshitz point can be efficiently described using holographic methods and requires a generalization of the viscosity bound obtained in isotropic theories.

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Appendix A: Scaling of the viscosity tensor

In this appendix we offer an alternative derivation of the scaling dimension, Eq. (25) of the viscosity tensor. The analysis leads to results identical to those presented in Section II of the paper.

Since the viscosity tensor describes the linear response to the temporal change of an externally-applied strain field, we can also define it using the strain generators $J_{\alpha\beta}$. The strain generators describe the deformation of the coordinate systems due to an applied external strain and are given by $J_{\alpha\beta} = x_\alpha k_\beta + \frac{1}{2} \delta_{\alpha\beta}$. Hence, the viscosity tensor is defined as

$$\eta_{\alpha\beta\gamma\delta}(\omega) = \omega \text{Im} \chi^{(J)}_{\alpha\beta\gamma\delta}(\omega), \tag{A1}$$

with $\chi^{(J)}_{\alpha\beta\gamma\delta}(\omega)$ being the Fourier transform of $\chi^{J}_{\alpha\beta\gamma\delta}(t) = -i\delta(t)[[J_{\alpha\beta}(t), J_{\gamma\delta}(0)]]$, where $J_{\alpha\beta}$ is the density of the strain generator $J_{\alpha\beta}$. In order to obtain the scaling dimension of the correlation function, we assume for the strain generator density the same dimensionality as the particle density $\Delta = d_{eff}$ times the scaling dimension of the momentum coordinates $k_\parallel$, $k_\perp$ and the spatial coordinates $x_\alpha$, $x_\gamma$, which have the dimensionality of the inverse momentum. We find for the correlation function of the two strain generators

$$\chi^{(J)}_{\alpha\beta\gamma\delta}(k_\parallel, k_\perp, \omega) = b^{-\Delta_{J,\alpha\beta\gamma\delta}} \chi^{(J)}_{\alpha\beta\gamma\delta}(b^\parallel k_\parallel, b^\perp k_\perp, b^\gamma \omega), \tag{A2}$$

with $\Delta_{J,\alpha\beta\gamma\delta} = d_{eff} - z - \varphi_\alpha + \varphi_\beta - \varphi_\gamma + \varphi_\delta$. Here we used the same notation as in the main paper, where $\varphi_\alpha = 1$ if the $\alpha$-component is aligned along the direction of $k_\parallel$ and $\varphi_\alpha = \phi$ for the direction of $k_\perp$. Using $\Delta_{J,\alpha\beta\gamma\delta}$ allows us to determine the scaling behavior of the viscosity tensor

$$\eta_{\alpha\beta\gamma\delta}(T, \omega) = b^{-\Delta_{\eta,\alpha\beta\gamma\delta}} \eta_{\alpha\beta\gamma\delta}(b^\gamma T, b^\gamma \omega). \tag{A3}$$

with

$$\Delta_{\eta,\alpha\beta\gamma\delta} = \Delta_{J,\alpha\beta\gamma\delta} + z = d_{eff} - \varphi_\alpha + \varphi_\beta - \varphi_\gamma + \varphi_\delta, \tag{A4}$$

which is in agreement with Eq. (25) of the main part of the paper.
Appendix B: IR models

In order to analyze the IR metric (45), we derive the hyperscaling-violating solutions in the presence of both one and two axion fields. It is worth to emphasize that the radial coordinate parameterizing the IR geometry (45) does not coincide with the one in the UV region (42). To be specific, we consider the matter Lagrangian

\[ \mathcal{L}_M = -\frac{1}{2} (\nabla \varphi)^2 + V_0 r^{2\kappa \delta} - \sum_{\alpha=1}^{p} \frac{r^{2\kappa \lambda_\alpha}}{2} (\nabla \psi_\alpha)^2 - \frac{r^{2\kappa \zeta}}{4} F^2, \]

where \( p \) is the number of axions and \( \psi_\alpha = a_\alpha x_\alpha \), with no index summation. In the \( p = 1 \) case it reduces to (35).

The effective dilaton potential (40) looks like

\[ V_{\text{eff}}(r) = \frac{1}{\sqrt{-g}} \frac{1}{2} \sum_{\alpha=1}^{p} a_\alpha^2 r^{2\Lambda_\alpha - \theta} - V_0 r^{2\delta \kappa}, \]

where \( \Lambda_1 = \kappa \lambda_x + \phi \) and \( \Lambda_2 = \kappa \lambda_y + 1 \).

1. Marginally relevant case

In order to avoid radial dependences coming from the \( a_\alpha \)-terms, we set \( 2\Lambda_\alpha = 2\kappa \delta + \theta \). This corresponds to take the axions as marginal deformations of the IR fixed point. Furthermore, setting \( \theta + 2\delta \kappa = 0 \) yields a set of algebraic equations in both the cases \( p = 1, 2 \).

Let us start with the one single axion case \( p = 1 \) – we omit the subscript \( \alpha = 1 \) everywhere. The solution to the field equations is given by:

\[
\begin{align*}
    z &= \phi, \\
    2\kappa \delta &= -\theta, \\
    4\kappa^2 &= \theta^2 - 2\theta \phi + 2\phi - 2, \\
    L^2 &= (\theta - 2\phi - 1) (\theta - 2\phi) / V_0, \\
    a^2 &= \frac{2V_0(1 - \phi)}{\theta - 2\phi}. \tag{B3}
\end{align*}
\]

Note how a low momentum dissipation limit \( (a \to 0) \) always restores the isotropy of the system \( (\phi = 1) \). In order to get a realistic solution, we demand the positivity of the squared quantities and the specific heat \( c = T \partial_T s \). In addition, we require the vanishing of the line element in the IR at \( T = 0 \), obtaining the following set of conditions:

\[
\begin{align*}
    \theta &< 2, \quad \theta^2 + 2\phi > 2\theta \phi + 2, \quad \phi > 1, \tag{B4} \\
    \theta &> 2, \quad \theta^2 + 2\phi > 2\theta \phi + 2, \quad \theta \phi < \phi^2 + \phi. \tag{B5}
\end{align*}
\]

In the former the IR is at \( r = \infty \), in the latter at \( r = 0 \). The null energy condition (NEC) turns out to be fulfilled.

In the \( p = 2 \) case we find

\[
\begin{align*}
    2\kappa \delta &= -\theta, \\
    \kappa \lambda_x &= -\phi, \\
    \kappa \lambda_y &= -1, \\
    4\kappa^2 &= \theta(\theta - 2z) - 2\phi(\phi - 2z) - 2(1 - z), \\
    L^2 &= (\theta - 2z)(\theta - \phi - z - 1) / V_0, \\
    a_x^2 &= \frac{2V_0(\phi - z)}{\theta - 2z}, \\
    a_y^2 &= \frac{2V_0(1 - z)}{\theta - 2z}, \tag{B6}
\end{align*}
\]

which reproduces Eq. (B3) in the \( a_x = 0 \) case. The consistency conditions follows form analogue considerations and are depicted in Fig. 4 – the NEC is automatically satisfied. Even in this case, sending the momentum dissipation to zero restores the isotropy of the system.

One can easily check that the above solution reproduces the single axion one when \( a_x = 0 \).
2. Irrelevant case

Now we wish to investigate the $p = 1$ case, where the axion acts as an irrelevant deformation of the IR endpoint. Details on the $p = 2$ mixed case can be found in Ref.$^{26,55}$. We firstly determine the solution when $a = 0$ and then consider perturbations of the form:

$$\Phi = \Phi_{a=0} (1 + c_{\phi} a^2 e^{2\Delta_a}) .$$  \hspace{1cm} (B7)

$\Phi$ stands for the metric elements or the dilaton field, and $c_{\phi}$ are numerical coefficients that follow from the $O(a^2)$ fields equations. Such corrections are expressed in terms of $a^2$ as the axion enters quadratically the field equations. Moreover, such irrelevant perturbations must grow towards the boundary of the IR region, hence we set $\Delta_a > 0$. The leading solution is given by

$$z = \phi = 1, \quad 4a^2 = \theta (\theta - 2), \quad L^2 = (\theta - 2)(\theta - 3)/V_0,$$  \hspace{1cm} (B8)

provided that $\theta + 2b\kappa = 0$. Moreover we obtain

$$2\Delta_a = 2 + \frac{\kappa \lambda}{2}$$  \hspace{1cm} (B9)

in accord with Ref.$^{26}$. The consistency conditions read

$$\theta < 0, \quad \Delta_a \leq 0,$$  \hspace{1cm} (B10)

where the last inequality depends on the location of the IR.

Appendix C: The holographic dual of out-of-time-order correlation functions

In the context of the butterfly velocity, we consider out-of-time-order correlation functions (OTOCs) of the form

$$C(\vec{x}, t_w) = -\langle [A(\vec{x}, t_w), B(0, 0)]^2 \rangle,$$  \hspace{1cm} (C1)

where $A$ and $B$ are hermitian local operators. In order to translate such functions to the holographic language, it is convenient to regularize them by rotating one of the commutators halfway around the thermal circle.$^{22}$ This results in

$$C(\vec{x}, t_w) = -\text{tr} [\hat{g} [A(\vec{x}, t_w), B(0, 0), [\hat{g} A(\vec{x}, t_w), B(0, 0)]],$$  \hspace{1cm} (C1)

where $\hat{g}$ is the squareroot of the density matrix. Next, we introduce the thermofield-double (TFD) state

$$|\beta\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R ,$$  \hspace{1cm} (C2)

with the partition function $Z$ and the inverse temperature $\beta$. This state lies in the product space of two copies of the Hilbert space and $|n\rangle_L$ and $|n\rangle_R$ denote energy Eigenstates with Eigenvalues $E_n$ in the respective copies. Operators acting on the two copies are defined as $O_L = O^T \otimes 1$ and $O_R = 1 \otimes O$. With these definitions, the regularized OTOC can be written as an expectation value in the TFD state, i.e.

$$C(\vec{x}, t_w) = -\langle \beta | [B_L(0, 0), A_L(\vec{x}, t_w)] \cdot [A_R(\vec{x}, t_w), B_R(0, 0)] |\beta\rangle.$$  \hspace{1cm} (C3)

Furthermore, we note that the TFD state is invariant under time translations generated by $H_{\text{tot}} = H_L - H_R$. To proceed, we need to investigate the transition amplitudes prepared by $|\beta\rangle$ in order to identify the spacetime connecting the L and the R system. For two given states $|\xi\rangle$ and $|\zeta\rangle$, these transition amplitudes are given by

$$\langle \xi |_R \langle \zeta |_L |\beta\rangle \propto \langle \zeta | e^{-\beta H/2} |\xi\rangle ,$$  \hspace{1cm} (C4)

where the conjugate state $|\bar{\xi}\rangle$ is defined such that $\langle n |\xi\rangle = \langle \xi | n \rangle$ for all states $|n\rangle$. This definition is only well-defined if the states $|n\rangle$ are redefined by $|n\rangle \to e^{-i\text{arcc}(\zeta | n) \ |n\rangle}$ in order to make the scalar products real. Using the fact that the Hamilton operator $H$ is obtained from the Hamilton density by integrating over the position space $P$, the transition amplitude shows that the L and R systems are connected by the spacetime

$$B = [0, \beta/2] \times P.$$  \hspace{1cm} (C5)

According to the holographic dictionary, this spacetime is the boundary of its holographic dual. It was shown in$^{97}$, that the holographic dual of the TFD state is given by a two-sided black hole spacetime. For simplicity, we will demonstrate this for the case of a one-dimensional position space $P$, but the results hold in any dimension. We first consider a Euclidean black hole in three dimensions, whose metric can be written in the two equivalent forms

$$ds^2 = (r^2 - r_+^2)dr^2 + \frac{1}{r^2 - r_+^2} dr^2 + g_{xx}(r)dx^2,$$  \hspace{1cm} (C6)

$$ds^2 = \frac{4}{(1- zz^*)^2} dz dz^* + g_{xx}(z z^*) dx^2 .$$  \hspace{1cm} (C7)

The coordinate $z$ is restricted to the position space $P$ and the two expressions are related by $z = e^{i \frac{\pi}{2} (r, r - r_+)}$ with the tortoise coordinate. Here, $g_{tt}(r) = g_{rr}(r) = r^2 - r_+^2$ and $g_{tt}(r) = 1/(r^2 - r_+^2)$.  

If $\tau$ is restricted to the Euclidean time interval $[0, \beta/2]$, the boundary of the Euclidean black hole is equal to $B$. This can be achieved by cutting the spacetime along the Im$(z) = 0$ surface. Furthermore, the metric is invariant under time translations of the form $z \to z \ e^{-i \frac{\pi}{2} \Delta \tau}$. Such time translations change the position of the Im$(z) = 0$ surface, but leave the distance between its two boundary points invariant.
This metric is invariant under Lorentzian time translations giving an analytic extension of the Lorentzian black hole to the Euclidean spacetime along the \( t \) direction. This is glued to the \( \text{Im}(z) = 0 \) surface of the Euclidean spacetime along the \( t = 0 \) surface. The boundary regions \( L \) and \( R \) are separated along the thermal circle \( \beta \) of inverse mass and can be identified as the holographic dual of the TFD state. In the massive case, this field is entirely fixed by the field equations and in the massless case, it can be interpreted as a gauge degree of freedom. The stress-energy tensor for a massless particle is given by

\[
T_{\mu\nu}(x^\rho) = \frac{2}{\sqrt{-g(x^\rho)}} \frac{\delta S}{\delta g^{\mu\nu}} = \int d\lambda \frac{1}{e(\lambda)} \frac{dz_\mu}{d\lambda} \frac{dz_\nu}{d\lambda} \sqrt{-g(x^\rho)} \delta(4)(x^\rho - z^\rho(\lambda)) \frac{(x^\rho - z^\rho(\lambda))}{\sqrt{-g(x^\rho)}}. \tag{C11}
\]

A light-like infalling geodesic requires \( dt/dr = -1/\gamma(r) \). If the particle is inserted at the boundary at time \( -t_w \), the resulting geodesic is given by

\[
(z^\rho(\lambda))^T = (-r_*(\tilde{r}(\lambda)) - t_w, \tilde{r}(\lambda), 0, 0), \tag{C12}
\]

where \( \tilde{r}(\lambda) \) denotes the radial coordinate at position \( \lambda \). If \( \lambda \) is identified with the radial coordinate, \( e(\lambda) \) has units of inverse mass and can be identified as the inverse of the particle energy \( E \). Such a parameterization may not in general be a solution of the geodesic equation. However, for a different parameterization, the resulting stress-energy tensor would only change by a global factor. It is thus sufficient to assume the \( \tilde{r}(\lambda) = \lambda \) case.

The stress-energy tensor is now given by

\[
T_{\mu\nu}(x^\rho) = E \frac{\delta(x)\delta(y)\delta(t + t_w + r_*(r))}{\sqrt{-g(x^\rho)}} \delta(x) \delta(y) \delta(t + t_w + r_*(r)) \frac{\delta\mu\nu}{\gamma^2(r)} - \frac{\delta\mu\nu}{\gamma(r)} + \delta\mu\nu \frac{\delta\mu\nu}{\gamma(r)}. \tag{C13}
\]

At large \( t_w \), after switching to Kruskal coordinates, the only non-vanishing component of the stress-energy tensor for a particle inserted at the right boundary is given by

\[
T_{\nu\nu} \propto E e^{\frac{\pi}{2} t_w} \delta(x) \delta(y) \delta(v). \tag{C14}
\]

For a particle inserted at the left boundary, time is reversed \( (t \leftrightarrow -t) \), which is equivalent to \( u \leftrightarrow v \). In this case, the stress-energy tensor is given by

\[
T_{uv} \propto E e^{\frac{\pi}{2} t_w} \delta(x) \delta(y) \delta(u). \tag{C15}
\]

As discussed above, perturbing a two-sided black hole with this stress-energy tensor results in a shock wave, which can be identified as the OTOC.