Spherically symmetric Yang-Mills solutions in a 5-dimensional (Anti-) de Sitter space-time

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Abstract

We consider an Einstein-Yang-Mills Lagrangian in a five dimensional space-time including a cosmological constant. Assuming all fields to be independent of the extra coordinate, a dimensional reduction leads to an effective (3+1)-dimensional Einstein-Yang-Mills-Higgs-dilaton model where the cosmological constant induces a Liouville potential in the dilaton field. We construct spherically symmetric solutions analytically in specific limits and study the generic solutions for vanishing dilaton coupling numerically. We find that in this latter case the solutions bifurcate with the branch of (Anti-) de Sitter-Reissner-Nordström ((A)dSRN) solutions.
1 Introduction

The scalar dilaton field arised as companion of the metric tensor in (super)string theories and is associated with the scale invariance of these theories [1]. Thus it is interesting to study classical field theory solutions coupled to a dilaton. In most studies, the dilaton was assumed to be massless while, however, from the viewpoint of a realistic theory the dilaton should be massive in order to avoid long-range scalar forces. In [2] a dilaton potential of Liouville type was introduced to take into account the effects of a specific symmetry-breaking mechanism which gives mass to the dilaton. This type of potential has a constant prefactor which in the limit of vanishing dilaton coupling reduces to a cosmological constant. It was found that there exist no asymptotically flat/ de Sitter/ Anti-de Sitter solutions for non-vanishing potential [3]. Rotating generalisations of the black hole solutions found in [2] have been constructed in [4].

Volkov argued recently [5] that if $\frac{\partial}{\partial x^4}$ is a symmetry of the Einstein-Yang-Mills (EYM) system in (4+1) dimensions, where $x_4$ is the coordinate associated with the 5th dimensions, than the (4+1)-dimensional EYM system reduces effectively to a (3+1)-dimensional EYMHD system with a specific coupling between the dilaton field and the Higgs field. The generalisation of this (3+1)-dimensional EYMHD model was consequently studied in [6].

In this paper, we study spherically symmetric solutions of the (3+1)-dimensional EYMHD model deduced from the (4+1)-dimensional EYM system including a cosmological constant. The dimensional reduction then leads to a Liouville-type potential in the (3+1)-dimensional model. In Section 2, we present both the five-dimensional model and the from this deduced and then generalised (3+1)-dimensional EYMHD model. In Section 3, we discuss the solutions for the case of vanishing dilaton coupling, especially, we present our numerical results for the generic solutions in this case. In section 4, we discuss possible solutions for the generic case of non-vanishing dilaton coupling. Our conclusions are presented in Section 5.

2 The model

We start with the Einstein-Yang-Mills Lagrangian in five dimensions including a cosmological constant given by:

$$ S = \int \left( \frac{1}{16\pi G_5} (R - 2\Lambda_5) - \frac{1}{4\tilde{e}^2} F_{MN}^a F^{aMN} \right) \sqrt{g^{(5)}} d^5x $$

with the SU(2) Yang-Mills field strength $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + \epsilon_{abc} A_M^b A_N^c$, the gauge index $a = 1, 2, 3$ and the space-time index $M = 0, 1, 2, 3, 4$. $G_5$ and $\tilde{e}$ denote respectively the 5-dimensional Newton’s constant and the coupling constant of the gauge field theory. $G_5$ is related to the 5-dimensional Planck scale $M_{Pl(5)}$ by $G_5 = M_{Pl(5)}^3$. $\Lambda_5$ is the 5-dimensional cosmological constant.
If both the matter functions and the metric functions are independent on $x_4$, the 5-dimensional fields can be parametrized as follows [5]:

$$g^{(5)}_{MN} dx^M dx^N = e^{-\zeta} g^{(4)}_{\mu\nu} dx^\mu dx^\nu + e^{2\zeta} (dx^4)^2 , \quad \mu, \nu = 0, 1, 2, 3$$

(2)

and

$$A^a_M dx^M = A^a_\mu dx^\mu + \Phi^a dx^4 ,$$

(3)

where $g^{(4)}$ is the 4-dimensional metric tensor and $\zeta$ plays the role of the dilaton.

In [5] it was shown that for $\Lambda_{(5)} = 0$ the classical equations are equivalent to those of a four-dimensional Einstein-Yang-Mills-Higgs dilaton theory. In this paper, we consider the case with a cosmological constant. We then choose the generalised $(3 + 1)$-dimensional action to be:

$$S = S_G + S_M = \int L_G \sqrt{-g^{(4)}} d^4 x + \int L_M \sqrt{-g^{(4)}} d^4 x .$$

(4)

with the gravity Lagrangian:

$$L_G = \frac{1}{16\pi G} R ,$$

(5)

and $G$ denoting the 4-dimensional Newton’s constant. The matter Lagrangian $L_M$ reads:

$$L_M = -\frac{1}{4} e^{2\kappa \Psi} F^a_{\mu\nu} F^{\mu\nu,a} - \frac{1}{2} \partial_\mu \Psi \partial^\mu \Psi - \frac{1}{2} e^{-4\kappa \Psi} D_\mu \Psi^a D^\mu \Phi^a - e^{-2\kappa \Psi} V(\Phi^a) - \frac{\bar{\Lambda}}{2} e^{-2\kappa \Psi} ,$$

(6)

with the Higgs potential

$$V(\Phi^a) = \frac{\lambda}{4} (\Phi^a \Phi^a - v^2)^2 ,$$

(7)

the non-abelian field strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu ,$$

(8)

and the covariant derivative of the Higgs field in the adjoint representation

$$D_\mu \Phi^a = \partial_\mu \Phi^a + \epsilon_{abc} A^b_\mu \Phi^c .$$

(9)

The gauge field coupling constant is denoted $e$, $\lambda$ is the Higgs field coupling constant and $v$ the vacuum expectation value of the Higgs field.

Note that we have introduced a coupling $\kappa$ specific to the dilaton field by setting $\zeta = 2\kappa \Psi$. This will allow to study the influence of the dilaton systematically. We remark that the 5-dimensional cosmological constant has through dimensional reduction led to a Liouville potential in the dilaton field with coupling constant $\bar{\Lambda}$. For $\kappa = 0$, $\bar{\Lambda}$ is proportional to the four-dimensional cosmological constant.
2.1 The Ansatz

For the metric the spherically symmetric Ansatz in Schwarzschild-like coordinates reads \([7, 8]\):

\[
ds^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu = -A^2(r)N(r)dt^2 + N^{-1}(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2
\]

with

\[
N(r) = 1 - \frac{2m(r)}{r}
\]

In these coordinates, \(m(\infty)\) denotes the (dimensionful) mass of the field configuration.

For the gauge and Higgs fields, we use the purely magnetic hedgehog ansatz \([11]\)

\[
A_{r}^a = A_{t}^a = 0,
A_\theta^a = \frac{1 - K(r)}{e} e^\phi a, \quad A_\phi^a = -\frac{1 - K(r)}{e} \sin \theta e_\theta a,
\]

\[
\Phi^a = vH(r)e_r a.
\]

The dilaton is a scalar field depending only on \(r\)

\[
\Psi = \Psi(r).
\]

Inserting the Ansatz into the Lagrangian and varying with respect to the matter fields yields the Euler-Lagrange equations, while variation with respect to the metric yields the Einstein equations.

2.2 Classical field equations

With the introduction of dimensionless coordinates and fields

\[
x = evr, \quad \mu = evm, \quad \phi = \frac{\Phi}{v}, \quad \psi = \frac{\Psi}{v}
\]

the Lagrangian and the resulting set of differential equations depend on the following coupling constants:

\[
\alpha = \sqrt{G}v = \frac{M_W}{eM_{Pl}}, \quad \beta = \sqrt{\Lambda} = \frac{M_H}{\sqrt{2}M_W}, \quad \gamma = \kappa v = \frac{\kappa M_W}{e}, \quad \Lambda = 2\alpha^2\tilde{\Lambda},
\]

where \(M_W = ev, M_H = \sqrt{2}\lambda v\) and \(M_{Pl} = 1/\sqrt{G}\). With the rescalings (16) and (17), the dimensionless mass of the solution is given by \(\frac{\mu(\infty)}{\alpha^2}\). Note that we have rescaled the cosmological constant in order to obtain the equations of a conventional (3+1)-dimensional Einstein-Yang-Mills-Higgs model including a cosmological constant in the limit of vanishing dilaton coupling.

With (16) and (17) the Euler-Lagrange equations read:

\[
(e^{2\gamma\psi}ANK')' = A(e^{2\gamma\psi} \frac{K(K^2 - 1)}{x^2} + e^{-4\gamma\psi}H^2K),
\]
\[(e^{-4\gamma\psi}x^2 AN'H')' = AH(2e^{-4\gamma\psi}K^2 + \beta^2 x^2 e^{-2\gamma\psi}(H^2 - 1)) , \quad (19)\]

\[(x^2 AN\psi')' = 2\gamma A[e^{2\gamma\psi}(N(K')^2 + (K^2 - 1)^2) - \frac{\Lambda}{4\alpha^2} x^2 e^{-2\gamma\psi} - e^{-2\gamma\psi} \frac{\beta^2 x^2}{4}(H^2 - 1)^2 - 2e^{-4\gamma\psi}(\frac{1}{2}N(H')^2 x^2 + H^2 K^2)] , \quad (20)\]

where the prime denotes the derivative with respect to \(x\), while we use the following combination of the Einstein equations

\[G_{tt} = 2\alpha^2 T_{tt} = -2\alpha^2 A^2 N L_M , \quad (21)\]

\[g^{xx} G_{xx} - g^{tt} G_{tt} = -4\alpha^2 N \frac{\partial L_M}{\partial N} \quad (22)\]

to obtain two differential equations for the two metric functions:

\[\mu' = \alpha^2 \left(e^{2\gamma\psi} N(K')^2 + \frac{1}{2} N x^2 (H')^2 e^{-4\gamma\psi} + \frac{1}{2} x^2 (K^2 - 1)^2 e^{2\gamma\psi} + K^2 H^2 e^{-4\gamma\psi} \right. \]
\[+ \left. \frac{\beta^2}{4} x^2 (H^2 - 1)^2 e^{-2\gamma\psi} + \frac{1}{2} N x^2 (\psi')^2 \right) + \frac{\Lambda}{4} x^2 e^{-2\gamma\psi} , \quad (23)\]

\[A' = \alpha^2 x A \left(\frac{2(K')^2}{x^2} e^{2\gamma\psi} + e^{-4\gamma\psi} (H')^2 + (\psi')^2 \right) . \quad (24)\]

Note that the equations of the original five dimensional theory are recovered by using the following specific choice of the coupling constants:

\[\alpha^2 = 3\gamma^2 , \quad \Lambda = \Lambda_{(5)} , \quad \beta = 0 . \quad (25)\]

The case \(\Lambda = 0\) was previously studied in [5, 6]. If in addition \(\gamma = 0\), the equations of the Einstein-Yang-Mills-Higgs equations are recovered [7, 8]. Choosing \(\Lambda = \alpha = 0\) (assuming \(\Lambda/\alpha^2 = 0\) as well), the model reduces to the Yang-Mills-Higgs-dilaton system studied in [9].

### 3 Spherically symmetric solutions for \(\gamma = 0\)

We will first discuss the solutions in the case \(\gamma = 0\). The equation of the dilaton field can then be decoupled and \(\psi(x) \equiv 0\). We will study solutions of this system which are regular at the origin this implies the following conditions

\[K(0) = 1 , \quad H(0) = 0 , \quad \mu(0) = 0 . \quad (26)\]

Finiteness of the ADM mass requires that the fields approach particular values asymptotically, namely:

\[K(\infty) = 0 , \quad H(\infty) = 1 , \quad A(\infty) = 1 . \quad (27)\]

For \(\Lambda > 0\) the metric function \(N(x)\) has a zero at a finite value of \(x\), say \(x = x_c\). This is the so-called “cosmological horizon”. The value \(x_c\) depends on the actual values of the coupling constants.
3.1 (Anti-) De Sitter-Reissner-Nordström ((A)dSRN) solutions

Setting $\gamma = 0$ the system admits embedded abelian solutions, the so-called (Anti-) de Sitter-Reissner-Nordström solutions:

$$K(x) = 0 \ , \ H(x) = 1 \ , \ \psi(x) = 0 \ , \ A(x) = 1 \ , \ N(x) = 1 - \frac{1}{6} \Lambda x^2 - \frac{2 \mu_{\infty}}{x} + \frac{\alpha^2}{x^2} .$$

(28)

The metric function $N(x)$ has a physical singularity at the origin $x = 0$ which is evident from the Kretschmann scalar $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$:

$$K = \frac{2}{3 x^8} \left( \Lambda^2 x^8 + 72 \mu_{\infty}^2 x^2 - 144 \mu_{\infty} \alpha^2 x + 84 \alpha^2 \right) .$$

(29)

Depending on the choice of the sign of the cosmological constant, up to 4 zeros of $N(x)$ can exist. 3 of the 4 zeros correspond to horizons since the first zero has always negative value and thus has no physical meaning. The two inner horizons $x_-, x_+$ with $x_- \leq x_+$ correspond to the well known Cauchy, respectively event horizon of the Reissner-Nordström solution, while the third outer horizon $x_c > x_+$ exists only for positive cosmological constant.

Extremal black hole solutions - like in the asymptotically flat space - are possible. Then, we have $x_-=x_+=x_h$ with $N(x_h) = N'|_{x=x_h} = 0$. This leads to the equation:

$$\Lambda x_h^4 - 2 x_h^2 + 2 \alpha^2 = 0 .$$

(30)

This is solved by:

$$x_{h/c} = \frac{1}{\sqrt{\Lambda}} \sqrt{1 \pm \sqrt{1 - 2 \alpha^2 \Lambda}} \quad \text{for} \quad \frac{1}{2 \alpha^2} \geq \Lambda > 0$$

(31)

and

$$x_h = \frac{1}{\sqrt{|\Lambda|}} \sqrt{-1 + \sqrt{1 - 2 \alpha^2 \Lambda}} \quad \text{for} \quad \Lambda < 0 .$$

(32)

For $\Lambda > 0$, the solution with the plus sign is the outer, cosmological horizon $x_c$, while the inner, event horizon $x_h$ is the solution with the minus sign. Obviously, the appearance of horizons in dS space is restricted by $\alpha^2 \leq \frac{1}{2 \Lambda}$. The corresponding mass of the extremal solution is given by:

$$\mu_{\infty} = \frac{2 \alpha^2}{3 x_h} + \frac{x_h}{3} .$$

(33)

Apparently, the $\Lambda = 0$ limit is ill-defined. However, for $0 < \Lambda \ll 1$ we find

$$\mu_{\infty} = \alpha - \frac{\alpha^3}{9} \Lambda + O(\Lambda^2) , \quad x_h = \alpha + \frac{\alpha^3}{3} \Lambda + O(\Lambda^2)$$

(34)

which for $\Lambda \to 0$ obviously leads to the corresponding values of the well-known asymptotically flat Reissner-Nordström solution.
3.2 de Sitter (dS) gravitating monopoles

Since gravitating monopoles in Anti-de Sitter space have been studied previously \[10\], we concentrate here on monopoles in de Sitter space. To our knowledge, these type of solutions have not been studied previously.

In the absence of a cosmological constant, the flat space magnetic monopole \[11\] is deformed by gravity and exist up to a critical value of \(\alpha = \alpha_{cr}\) where the solution bifurcates with the branch of extremal Reissner-Nordström solutions \[7\]. For instance in the BPS limit \((\beta = 0)\) the gravitating monopole bifurcates with this branch at \(\alpha_{cr} \approx 1.386\).

Now analysing the equations in the presence of a cosmological constant, we were able to construct dS-gravitating monopoles. They are characterised by a cosmological horizon at \(x = x_c\) with \(N(x = x_c) = 0\). The behaviour of the function \(N(x)\) is illustrated in Fig. 1 for \(\alpha = 0.8\) and different values of \(\Lambda\). We find that \(x_c\) is decreasing with the increase of \(\Lambda\): \(x_c \sim 108\) for \(\Lambda \sim 0.0005\) and \(x_c \sim 77\) for \(\Lambda = 0.001\). As is obvious from the figure, the solutions have a local minimum at some value of the radial coordinate \(x = x_{min}(\Lambda)\).

The main aim of this study was to determine the domain of coupling constants in which dS-gravitating monopoles exist. Fixing \(\beta\) and \(\Lambda\) our analysis demonstrates that dS-gravitating monopoles bifurcate with the branch of extremal dSRN solutions described in the previous section at a critical value of \(\alpha\). Since we limited our analysis to small values of \(\Lambda\) the critical value of \(\alpha\) where the bifurcation occurs hardly differs from the corresponding one in the asymptotically flat case.

The way how the extremal dSRN solution is approached is illustrated in Fig. 2 for \(\Lambda = 0.001\) and \(\beta = 0.1\). This clearly shows that the value of the local minimum of the function \(N(x)\) decreases while \(\alpha\) increases. We find that solutions exist up to a maximal value of the gravitational coupling \(\alpha = \alpha_{max} \approx 1.382\). There another branch of non abelian solutions exist which bifurcates with the branch of dSRN solutions at a critical value of \(\alpha = \alpha_{cr} \approx 1.378\). At this point, a degenerate horizon forms at \(x = x_h\). The critical solution can be described by the dS-RN solution with horizons \(31\) for \(x \geq x_h\), while for \(x_h > x \geq 0\), it is non-singular and non-trivial. Compared to the case \(\Lambda = 0\) \[7\], the values of \(\alpha_{max}\) and \(\alpha_{cr}\) are smaller when \(\Lambda > 0\). Moreover, the interval of \(\alpha\) on which two solutions exist decreases. This can be related to the increased cosmological expansion for \(\Lambda > 0\).

4 Spherically symmetric solutions for \(\gamma \neq 0\)

In the case of Einstein-Maxwell-dilaton theory, the Liouville potential leads to the fact that the solutions are neither asymptotically flat nor de Sitter nor Anti-de Sitter \[2\]. As far as our numerical simulations suggest, this holds also true for the case of non-abelian gauge fields, since we were not able to construct asymptotically flat/ de Sitter/ Anti- de Sitter solutions. However, in a specific limit, namely the embedded abelian case, analytic solutions are available.
4.1 The case $H(x) \equiv 1$, $K(x) \equiv 0$

Setting $H \equiv 1$ and $K \equiv 0$ for all $x$, we find the following solutions of the system of equations:

$$A(x) = a_0 x^{\frac{\gamma^2}{\gamma^2 + \alpha^2}}, \quad \psi(x) = \psi_0 + \frac{1}{\gamma} \ln(x) \quad (35)$$

and

$$N(x) = n_0 - n_1 x^{\frac{(\gamma^2 + \alpha^2)}{\gamma^2}} \text{ with } n_0 = \frac{\gamma^4}{\alpha^2 + \gamma^2} \left( e^{2\gamma\psi_0} - \frac{\Lambda}{2\alpha^2} e^{-2\gamma\psi_0} \right) . \quad (36)$$

The cosmological constant is given by:

$$\Lambda = 2\alpha^2 \left( \frac{1}{\alpha^2 - \gamma^2} e^{2\gamma\psi_0} - \frac{\gamma^2 + \alpha^2}{\alpha^2 - \gamma^2} e^{4\gamma\psi_0} \right) . \quad (37)$$

This solution has a single event horizon for $n_1 > 0$. Moreover, it can be seen, that this solution is ill-defined for $\alpha = \gamma$. Note that these are generalisations of the solutions constructed in [2]. For $\alpha = 1$, the above solution corresponds to one of the solutions found in [2]. In Fig. 3, we show qualitative profiles of the functions for the choice of parameters which corresponds to the 5-dimensional limit (25). In addition, we choose $\gamma = \psi_0 = a_0 = n_1 = 1$. It is obvious from this figure that the solution has a horizon (here at $x = x_h \approx 0.655$) and thus represents a black hole.

If we choose instead the limit $\alpha = 0$, the function $A(x)$ becomes constant $= a_0$. The metric function $N(x) = 1 - n_1 x^{-1}$ in this limit.

5 Conclusions

In a previous paper [5], it was shown that a Einstein-Yang-Mills model in 5 dimensions can be reduced to an effective $(3+1)$-dimensional Einstein-Yang-Mills-Higgs-dilaton model under certain symmetry conditions - spherical symmetry and independence on the coordinate associated with the 5th dimension. One of the main results of the present paper shows that the reduction of a 5 dimensional de Sitter (dS)/Anti-de Sitter (AdS) Einstein-Yang-Mills system to an effective $(3+1)$-dimensional action (with the same symmetry assumptions as in [5]) leads to a self-interaction of the dilaton field via a Liouville potential.

Previous considerations of an Einstein-Maxwell-dilaton model including a Liouville potential [2] have revealed that no asymptotically flat/ de Sitter/ Anti-de Sitter solutions can be constructed [4]. All our attempts to construct numerically solutions of the non-abelian counterpart have failed. Thus, we believe that the absence of asymptotically flat/ de Sitter/ Anti-de Sitter solutions holds also true in the case of non-abelian gauge fields. However, considering the limit of vanishing dilaton coupling, we were able to recover the AdS gravitating monopoles studied previously [10] and to produce previously not studied solutions, namely the dS gravitating monopoles.
We show for the latter solutions that they bifurcate with the branch of dS-Reissner-Nordström (dSRN) solutions at a critical value of the gravitational coupling. Finally, considering the limit $K(x) \equiv 0$ and $H(x) \equiv 1$ for non-vanishing Liouville potential, we were able to construct generalisations of the solutions found in [2].

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**References**

[1] G. Gibbons and K. Maeda, Nucl. Phys. **B298** (1988), 741; D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev. **D43** (1991), 371.

[2] K. C. K. Chan, J. H. Horne and R. B. Mann, Nucl. Phys. **B447** (1995), 441.

[3] S. Poletti and D. Wiltshire, Phys. Rev. **D50** (1994) 7260.

[4] T. Ghosh and P. Mitra, Class. Quantum Grav. **20** (2003), 1403.

[5] M. S. Volkov, Phys. Lett. **B524** (2002), 369.

[6] Y. Brihaye and B. Hartmann, Phys. Lett. B 534 (2002) 137.

[7] P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. **B383** (1992), 357; P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. **B442** (1995), 126.

[8] K. Lee, V. P. Nair and E. J. Weinberg, Phys. Rev. **D45** (1992) 2751.

[9] P. Forgacs and J. Gyueruesi, Phys. Lett. **B366** (1996), 205.

[10] A. R. Lugo, E. F. Moreno and F. A. Shaposnik, Phys. Lett. **B 473** (2000) 35.

[11] G. ‘t Hooft, Nucl. Phys. **B79** (1974), 276; A. M. Polyakov, JETP Lett. **20** (1974), 194.
Figure 1: The metric function $N(x)$ is shown for the gravitating monopoles with $\alpha = 0.8$ and three different choices of the cosmological constant $\Lambda$. 
Figure 2: The metric function $N(x)$ of the de Sitter gravitating monopoles is shown for $\Lambda = 0.001$, $\beta = 0.1$ and different choices of $\alpha$ including $\alpha \approx \alpha_{cr}$. For comparison also the corresponding de Sitter-Reissner Nordström (dSRN) solution is shown.
Figure 3: The qualitative profiles of the metric functions $A(x)$ and $N(x)$ and the dilaton function $\psi(x)$ are shown for the 5-dimensional parameter limit with $H(x) \equiv 1$ and $K(x) \equiv 0.$