Edge Scaling Limit of the Spectral Radius for Random Normal Matrix Ensembles at Hard Edge

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Abstract

We investigate local statistics of eigenvalues for random normal matrices, represented as 2D determinantal Coulomb gases, in the case when the eigenvalues are forced to be in the support of the equilibrium measure associated with an external field. For radially symmetric external fields with sufficient growth at infinity, we show that the fluctuations of the spectral radius around a hard edge tend to follow an exponential distribution as the number of eigenvalues tends to infinity. As a corollary, we obtain the order statistics of the moduli of eigenvalues.

Keywords Random normal matrices · Hard edge · Spectral radius · Universality · 2D Coulomb gases

Mathematics Subject Classification 60B20 · 60G55 · 82B21

1 Introduction and Results

In random matrix theory, there have been numerous studies of the spectral radius of large size matrices. The limiting distributions of the largest eigenvalue of classical random matrix ensembles, Gaussian orthogonal, unitary, and symplectic ensembles, were studied by Forrester [20] and Tracy-Widom [39,40]. More generally, a type of universality for Wigner random matrices was proved by Soshnikov [38]. The study of the scaling limit of correlation functions at the edge of the spectrum allowed to prove the universality of the largest eigenvalue distribution for some invariant ensembles [17].

The edge behavior of the spectrum of a random normal matrix is different from that of a random hermitian matrix, which is expressed in terms of Painlevé II. In the random normal matrix model, one considers random normal matrices of size $n$ with a probability measure of the form

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where $Q : \mathbb{C} \to \mathbb{R} \cup \{ +\infty \}$ is an external potential, $dM$ is the surface measure on complex-valued $n \times n$ matrices $M$ with $M^*M = MM^*$ induced from the standard metric on $\mathbb{C}^{n \times n}$, and $Z_n$ is a normalizing constant. The system of eigenvalues is represented as a 2D Coulomb gas model (one-component plasma) at a specific temperature, subjected to the external potential $Q$. If the external potential grows sufficiently fast at infinity, the eigenvalues accumulate on a compact set called the droplet as the size of matrix $n$ goes to infinity. In the case of $Q(z) = |z|^2$, the system of eigenvalues is represented by the complex Ginibre ensemble [23] and the droplet is the closed unit disk. The maximum of the moduli of the eigenvalues for the complex Ginibre ensemble has the same distribution as that of a set of independent random variables [28], and with a proper scaling, the limit law of its spectral edge follows the Gumbel distribution [34]. This result has been generalized to a class of radially symmetric potentials in [13].

In this paper, we study random normal matrix ensembles with a hard edge. By localizing a potential to the droplet, we obtain a system of eigenvalues contained completely in the droplet. The growth of local droplets associated with a potential localized to subsets was studied in [24] in connection with the 2D Coulomb gas and Laplacian growth. The hard edge Ginibre ensemble, defined by localizing the potential $Q(z) = |z|^2$ to the unit disk so that all eigenvalues are constrained to range in the disk, was studied in [6]. For more general potentials, edge behaviors of hard edge ensembles were investigated in [7] for both regular and singular boundary points, e.g., cusps or double points. We also mention [37,42] for earlier works on this type of hard edge ensembles.

We study limiting behaviors of the spectral radius for power potentials beyond the Ginibre case by finding a suitable rescaling factor and giving a proof based on the central limit theorem. We also prove the universality of this edge behavior for a class of radially symmetric potentials by applying Laplace’s method from [13].

### 1.1 The Random Normal Matrix Ensemble

We consider an external potential $Q : \mathbb{C} \to \mathbb{R} \cup \{ +\infty \}$ which is lower semi-continuous. We assume that $Q < +\infty$ on a set of positive area and $Q$ satisfies the growth condition

\[
\liminf_{z \to \infty} \frac{Q(z)}{\log |z|} > 2. \tag{1.2}
\]

In the random normal matrix model with potential $Q$, defined by the probability measure (1.1), the system of eigenvalues admits the joint probability density

\[
P_n(z_1, \cdots, z_n) = \frac{1}{Z_n} \prod_{j \neq k} |z_j - z_k| e^{-n \sum_{j=1}^n Q(z_j)} \tag{1.3}
\]

with the normalization constant

\[
Z_n = \int_{\mathbb{C}^n} \prod_{j \neq k} |z_j - z_k| e^{-n \sum_{j=1}^n Q(z_j)} \prod_{j=1}^n dA(z_j),
\]

where $dA(z) = dx dy/\pi$ is the normalized area measure on $\mathbb{C}$. We refer to [14,19,41] for earlier works on the random normal matrix model and its correlation structure. The joint probability distribution (1.3) agrees with that of 2D Coulomb particles in the external potential
$Q$, and the eigenvalues (particles) $\{z_j\}_{j=1}^n$ form a determinantal point process on $\mathbb{C}$. For integers $k = 1, \ldots, n$, its $k$-point correlation function $R_{n,k}$ is expressed as a determinant

$$R_{n,k}(z_1, \ldots, z_k) = \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} P_n(z_1, \ldots, z_n) \, dA(z_{k+1}) \cdots dA(z_n)$$

$$= \det \left[ K_n(z_i, z_j) \right]_{i,j=1}^k$$

with the correlation kernel

$$K_n(z, w) := \sum_{k=0}^{n-1} p_k(z) \bar{p}_k(w) e^{-n(Q(z)+Q(w))/2},$$

where $p_k$ is an orthonormal polynomial of degree $k$ with respect to the inner product

$$\langle p, q \rangle_{nQ} := \int_{\mathbb{C}} p(z) \bar{q}(z) e^{-nQ(z)} \, dA(z). \tag{1.4}$$

This correlation structure described in terms of the reproducing kernel for the space of analytic polynomials with the inner product in (1.4) plays an important role in the study of global properties including convergence of fluctuations of the spectral measure to gaussian fields [5] and universality for local statistics in the bulk [5,9,10] and at the edge [6,25].

### 1.2 Droplets and Equilibrium Measures

Following [36], we define the weighted logarithmic energy $I_Q(\mu)$ of a probability measure $\mu$ on $\mathbb{C}$ by

$$I_Q(\mu) = \iint_{\mathbb{C}^2} \log \frac{1}{|z - \zeta|} \, d\mu(\zeta) d\mu(z) + \int_{\mathbb{C}} Q \, d\mu.$$ 

For a fixed potential $Q$ with the growth assumption (1.2), there is a unique probability measure $\sigma_Q$ which minimizes the weighted logarithmic energy among all compactly supported Borel probability measures on $\mathbb{C}$. The minimizer $\sigma_Q$ is called Frostman’s equilibrium measure associated with $Q$ and the droplet $S = \text{supp} \sigma_Q$ is compact. It is also known that if $Q$ is smooth in a neighborhood of $S$, then $d\sigma_Q = 1_S \bar{\partial} \partial Q \, dA$. Here, we write $\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ for $z = x + iy$.

It is known that the eigenvalues of the random normal matrix ensemble tend to condensate to the droplet as $n$ goes to infinity. More precisely, the empirical distribution $\frac{1}{n} \sum_{j=1}^n \delta_{z_j}$ of the eigenvalues $z_j$ converges weakly to the equilibrium measure $\sigma_Q$. See [24,33].

### 1.3 Main Results

We localize the external potential $Q$ to the droplet $S$ and write

$$Q^S(z) = \begin{cases} Q(z), & z \in S \\ +\infty, & z \in S^c. \end{cases}$$

There still exists the equilibrium measure $\sigma_{Q^S}$ associated with $Q^S$ and it holds that $\sigma_{Q^S} = \sigma_Q$. For more details, see [24, Section 5].

We call the random matrix model with the localized potential $Q^S$ the hard edge ensemble and the one without localization the free boundary ensemble. In contrast to the free boundary case, eigenvalues of the hard edge ensemble tend to distribute completely inside the droplet.
For the special case when $Q(z) = |z|^2$, the equilibrium measure is the uniform measure on the unit disk. Fig. 1 shows the graphs of the 1-point density $\frac{1}{n} R_{n,1}$ for potentials $Q$ and $Q^S$, respectively. In both cases, the density $\frac{1}{n} R_{n,1}$ eventually converges to the equilibrium density $1_S$ as $n$ goes to infinity. However, on the local scale, the eigenvalues of the hard edge model are more densely distributed near the boundary of the droplet than those of the free boundary case.

We rescale the spectral radius at the boundary of the droplet in the inward direction. Let $\{z_j\}_1^n$ be the eigenvalues for the hard edge ensemble associated with $Q^S$ and $|z|^n = \max_{1 \leq j \leq n} |z_j|$.

We first consider the case of the power potential $Q(z) = |z|^{2d}$ ($d > 0$). In this case, the droplet is the disk $S = \{ z \in \mathbb{C} : |z| \leq d^{\frac{1}{2d}} \}$ and the equilibrium measure is

$$d\sigma(z) = d^2 |z|^{2d-2} 1_S(z) \, dA(z).$$

**Theorem 1.1** Let $Q(z) = |z|^{2d}$ with $d > 0$. Let $\omega_n$ be the rescaled spectral radius of the hard edge ensemble associated with $Q^S$, which is defined by

$$\omega_n = n d^{\frac{1}{2d}} (|z|^n - 1) \log 4.$$

Then $\omega_n$ converges in distribution to the exponential distribution:

$$\lim_{n \to \infty} P[\omega_n \leq \xi] = e^\xi, \quad \xi \leq 0,$$

uniformly for $\xi$ in every compact subset of $\mathbb{R}^-$. 

We extend this result to general radially symmetric potentials. From now on, we assume that the potential is radially symmetric: $Q(z) = q(|z|)$ where $q$ is a smooth function on $\mathbb{R}^+$. Also assume that $Q$ is subharmonic in $\mathbb{C}$ and strictly subharmonic in a neighborhood of the outer boundary of $S$. Due to the rotational symmetry, the boundary of the droplet $S$ consists of circles centered at the origin. We rescale the spectral radius $|z|^n$ about the radius of the outer boundary of $S$.

**Theorem 1.2** Let $R_0$ be the radius of the outer boundary of $S$ and $\delta = \partial \partial^* Q(R_0)$. Let $\omega_n$ be the rescaled spectral radius defined by

$$\omega_n = R_0 n \delta (|z|^n - R_0) \log 4.$$
Then, the following convergence holds:
\[
\lim_{n \to \infty} P[\omega_n \leq \xi] = e^\xi, \quad \xi \leq 0,
\]
uniformly for \(\xi\) in every compact subset of \(\mathbb{R}^-\).

For finite \(l\), we consider the distribution of the \(l\)-th largest modulus of eigenvalues. We write the limit law of the rescaled spectral radius in Theorem 1.2 as follows:
\[
F_{\text{hard}}(\xi) := e^\xi \quad \text{for} \quad \xi \leq 0.
\]
(1.5)

For the limit law \(F_{\text{hard}}\), define
\[
F_{\text{hard}}^{(l)}(\xi) := F_{\text{hard}}(\xi) \sum_{k=0}^{l-1} \frac{1}{k!} \left[ -\log F_{\text{hard}}(\xi) \right]^k = e^\xi \sum_{k=0}^{l-1} \frac{(-\xi)^k}{k!} \quad \text{for} \quad \xi \leq 0,
\]
which can be written as a Gamma distribution with parameter \(l,\)
\[
F_{\text{hard}}^{(l)}(\xi) = \frac{\Gamma(l, -\xi)}{\Gamma(l)}, \quad \xi \leq 0.
\]
Here, \(\Gamma(l, x) = \int_x^\infty t^{l-1} e^{-t} dt\) is the upper incomplete gamma function.

**Theorem 1.3** Let \(\{z_k\}_1^n\) be the eigenvalues of the hard edge ensemble associated with \(Q^S\) and \(|z_n^{(l)}|\) be the \(l\)-th largest \(|z_k|\). Let \(R_0\) be the radius of the outer boundary of the droplet and \(\delta = \partial \bar{\partial} Q(R_0)\). Then, for \(\xi \leq 0,\)
\[
\lim_{n \to \infty} P[R_0 n \delta \left( |z_n^{(l)}| - R_0 \right) \log 4 \leq \xi] = F_{\text{hard}}^{(l)}(\xi).
\]

**Remark** The hard edge ensemble can be interpreted as a Wigner jellium model, a Coulomb gas system for which the external potential is the electrostatic potential created by a background charge of opposite sign. In particular, our hard edge model agrees with a Wigner jellium for which the background distribution is given by the electrostatic equilibrium \(\sigma_Q\) with a constraint that the particles live in the droplet \(S\). See [12,42].

**Remark** In the theory of random Hermitian matrices, the terminology “hard edge” is usually used for the case associated with the Bessel kernel; for example, in the Laguerre ensemble the hard edge appears at the origin. However, the “hard edge” in this note represents a slightly different situation, and a one-dimensional analogue can be found in [15], which is called “soft/hard edge” explaining the situation that a soft edge meets a hard edge.

**Remark** For the hard edge Ginibre ensemble, it has been shown in [6] that the system of rescaled eigenvalues \(\tilde{z}_j = \sqrt{n}(z_j - 1)\) converges to the determinantal point field on the left half plane \(\mathbb{C} \cap \{\text{Re } z \leq 0\}\) with the correlation kernel
\[
H(z + \bar{w}) 1\{z\} 1\{w\} e^{z \bar{w} - |z|^2/2 - |w|^2/2},
\]
where \(H\) is the hard edge plasma function (see Fig. 2)
\[
H(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \frac{e^{-(z-t)^2/2}}{F(t)} dt; \quad F(t) := \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\xi^2/2} d\xi = \frac{1}{2} \text{erfc} \left( \frac{t}{\sqrt{2}} \right)
\]
while in the free boundary case the erfc-kernel appears:
\[
F(z + w) e^{z \bar{w} - |z|^2/2 - |w|^2/2}.
\]
Fig. 2 The graphs of the free boundary plasma function $F$ (left) and the hard edge plasma function $H$ (right), restricted to $\mathbb{R}$.

We also refer to [21, Section 15.3] and [37] for the calculation of the hard edge kernel. This result generalizes to the case of radially symmetric potentials in [2]. The normalizing constant $\log 2$ in Theorems 1.1 - 1.3 has a relation with the hard edge plasma function $H$. By simple calculation, one finds that

$$H(0) = -\int_{-\infty}^{0} \frac{d}{dt} \log \left( \int_{t}^{\infty} e^{-\xi^2/2} d\xi \right) dt = \log 2.$$  

1.4 Gap Probabilities

In this subsection, we briefly present some preliminaries on gap probabilities which will be needed in this paper. Let \( \{z_j\}_1^n \) be the random normal matrix ensemble associated with the potential \( Q \). Write \( \Omega = \Omega_x = \{z \in \mathbb{C} : |z| > x\} \) for \( x \geq 0 \). Adapting the calculation of gap probabilities in [18,31] to our case, we calculate the gap probability that none of the eigenvalues are contained in \( \Omega \) as

$$P[|z|_n \leq x] = \frac{1}{Z_n} \int \left( \prod_{j=1}^{n} \left( 1 - \mathbf{1}_{\Omega}(z_j) \right) \right) e^{-n \sum Q(z_j)} \prod_{j \neq k} |z_j - z_k| \prod_{j=1}^{n} dA(z_j)$$

$$= \prod_{j=0}^{n-1} \left( 1 - \int_{x}^{\infty} |p_j(r)|^2 e^{-nQ(r)} 2r dr \right).$$  

(1.6)

Here, \( p_j \) is an orthonormal polynomial of degree \( j \) with respect to \( e^{-nQ} dA \), which is chosen to be a monomial. Note that the gap probability (1.6) also follows from the observation of Kostlan [28] for the Gaussian case, which can be generalized to the case of radially symmetric potentials. More precisely, the maximum of moduli \(|z_1|, \ldots, |z_n|\) has the same distribution as that of a set of independent random variables \( X_0, \ldots, X_{n-1} \) with \( X_j \) of density proportional to \( r \mapsto r^{2j+1} e^{-nQ(r)} \). See also [13]. In the hard edge case, we take \( Q^S \) instead of \( Q \).

To find a limit of \( \log P[|z|_n \leq x] \) as \( n \to \infty \), we shall use the following lemma, which is obtained directly from the Taylor series expansion.

**Lemma 1.4** Let \( X_n \) be a subset of \( \mathbb{C} \) such that \( X_n = \{x_{n,1}, \ldots, x_{n,n}\} \). If \( X_n \) satisfies the conditions

(a) \( \sum_{j=1}^{n} x_{n,j} = O(1) \)
(b) \( x_{n,j} = o(1) \) uniformly for all \( 1 \leq j \leq n \)

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as $n \to \infty$, then we have
\[
\sum_{j=1}^{n} \log \left( 1 - x_{n,j} \right) = - \sum_{j=1}^{n} x_{n,j} + o(1) \quad \text{as} \quad n \to \infty.
\]

1.5 Comments and Related Works

In the free boundary case, the eigenvalues are admitted to be outside the droplet in contrast to the hard edge case. It is known that the spectral radius fluctuates slightly away from the droplet [13,34]. For example, in the free boundary Ginibre case, the spectral radius $|z|_n$ is approximated by
\[
|z|_n \simeq 1 + \sqrt{\frac{\gamma_n}{4n}} + \frac{1}{\sqrt{4n\gamma_n}} G
\]
where $\gamma_n = \log(n/2\pi) - 2 \log \log n$ and $G$ has a standard Gumbel distribution with distribution function $P[G \leq x] = e^{-e^{-x}}$ for $x \in \mathbb{R}$. On the other hand, the spectral radius of the hard edge ensemble fluctuates in a narrow range inside the droplet within a distance of order $n^{-1}$ from the boundary.

When the external potential does not satisfy the growth condition (1.2), the eigenvalues may not accumulate on a compact set. In the case of potential $Q(z) = \frac{n+1}{n} \log(1 + |z|^2)$, also known as the spherical ensemble [26], the support of the equilibrium is the whole complex plane $\mathbb{C}$ and its density has a heavy tail. A class of potentials for which fluctuations of the largest modulus of Coulomb particles are heavy-tailed has been studied in [11,12]. The fluctuations of the spectral radius and the regime of intermediate fluctuations are investigated in [29] for potentials belonging to three different universality classes: the finite $n$ density vanishes either faster than a power law, as a power law as $|z| \to \infty$ or has a finite edge beyond which it is exactly zero.

A hard edge appears in a context of truncated unitary matrices. The eigenvalues of truncated unitary matrices are represented as 2D Coulomb gases confined to the unit disk. See [43]. In the case when the original matrix is of size $n \times n$ and the truncation is of size $(n-1) \times (n-1)$, the largest modulus of eigenvalues fluctuates according to the exponential distribution $F_{\text{hard}}$ in (1.5) at the boundary. A family of 2D Coulomb gases confined to an ellipse has been studied in [32], which generalizes the results on the truncated unitary matrix model in [43]. We also refer to [27] for a study of the spectral radius of non-hermitian matrices including truncated unitary matrices, spherical ensembles, and product ensembles. In [22], edge fluctuations of 2D Coulomb gases have been studied in the case when the equilibrium measure is the uniform measure on the circle. In particular, it has been shown that with a hard edge constraint, the fluctuation follows an exponential distribution at speed $n^2$.

Global properties of 2D Coulomb gases at any temperature have been the focus of several works, e.g., the global fluctuation of particles about the equilibrium has been studied in [8] and [30]. The separation of Coulomb particles has been analyzed in [3], and a crossover behavior of them in a high temperature regime has been studied in [1]. The distance between the extremal particle and the droplet has been investigated for general potentials in [4].

The hard edge setting described in present paper can be applied to a Coulomb gas in any dimension. We refer to [16] which considers the large deviations and the transition between the “pushed” and “pulled” phases for Coulomb gases in any dimension. The “hard edge” in this paper is exactly the critical situation where the transition occurs. The distribution for the fluctuations of the spectral radius obtained in this paper does not match the large
deviation function computed in [16]. Hence, one naturally expects a regime of intermediate fluctuations, similarly to what was obtained in [29] in the case of truncated unitary matrices. Finally, a natural question is to ask about the fluctuation of the extremal particle around the hard edge in any dimension at any temperature.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof is mainly based on the normal approximation to a gamma distribution. This approach is also found in the study of the free boundary ensembles [13].

Consider the case when \( Q(z) = |z|^{2d} \). The corresponding droplet is the disk \( S = \{ z \in \mathbb{C} : |z| \leq d^{-1/2d} \} \), and the localized potential \( Q^S \) is defined by

\[
Q^S(z) = \begin{cases} 
|z|^{2d}, & |z| \leq d^{-1/2d} \\
+\infty, & |z| > d^{-1/2d}.
\end{cases}
\]

The orthonormal polynomial of degree \( j \) with respect to \( e^{-nQ^S} dA \) is

\[
p_j(z) = \left( \frac{d n^{j+1}}{\gamma(j+1, \frac{n}{d})} \right)^{\frac{1}{2}} z^j,
\]

where \( \gamma(s, \alpha) = \int_0^\alpha t^{s-1} e^{-t} dt \) is the lower incomplete gamma function. Now we define the rescaled spectral radius by

\[
\omega_n = d n \left( \frac{1}{2} \log |z| n - 1 \right) \log 4
\]

and the cumulative distribution function by \( F_n(\xi) = P[\omega_n \leq \xi] \) for \( \xi \leq 0 \). Write \( x = d^{-\frac{1}{2d}} \left( 1 + \frac{\xi}{d n \log 4} \right) \). By (1.6), we calculate \( \log F_n(\xi) \) as follows:

\[
\log F_n(\xi) = \log \prod_{j=0}^{n-1} \left( 1 - \int_{x < |z| \leq d^{-1/2d}} |p_j(z)|^2 e^{-n|z|^{2d}} dA(z) \right) \\
= \sum_{j=0}^{n-1} \log \left( 1 - \int_{\frac{x}{\pi n g^2} + o(1)} (t + \frac{n}{d})^{j+1} e^{-(t+\frac{n}{d})} \frac{\gamma(j+1, \frac{n}{d})}{\gamma(j+1, \frac{n}{d})} dt \right), \tag{2.1}
\]

where \( o(1) \to 0 \) uniformly for \( \xi \) in every compact subset of \( \mathbb{R}^+ \) as \( n \to \infty \).

Let \( \Phi \) and \( \phi \) denote the cumulative distribution function and the probability density function of the standard gaussian distribution respectively, i.e.,

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt ; \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{2.2}
\]

Throughout the section, let \( n_0 \) denote \( \sqrt{n} \log n \).

Lemma 2.1 For \( 0 \leq k \leq n_0 \), we have the following approximations:

\[
\frac{\gamma \left( \frac{n-k}{d}, \frac{n}{d} \right)}{\Gamma \left( \frac{n-k}{d} \right)} = \Phi \left( \frac{k}{\sqrt{(n-k)d}} \right) + O \left( n^{-1/2} \right),
\]
\[
(t + \frac{n}{d})^{\frac{n-k-1}{d}} \frac{e^{-(t+\frac{c}{\sqrt{n}})^{2}}}{\Gamma(\frac{n-k}{d})} = \sqrt{\frac{d}{n-k}} \frac{k}{\sqrt{(n-k)d}} (1 + O(n^{-1/2} \log^3 n))
\]
as \(n \to \infty\), where the error terms are uniform in \(k\) with \(0 \leq k \leq n_0\) and \(t\) on every compact subset of \(\mathbb{R}^+\). Furthermore, \(\Gamma(\frac{n-k}{d}) / \gamma(\frac{n-k}{d}, \frac{n}{d})\) is uniformly bounded for all \(k\) with \(0 \leq k \leq n - 1\) and sufficiently large \(n\).

**Proof** Let \(U_m\) be a random variable which follows a Gamma distribution \(\Gamma(\frac{n}{d}, 1)\). If we write \(G_m\) and \(g_m\) for the cumulative distribution function and the probability density function of \(U_m\) respectively, then we have

\[
G_{n-k} \left(\frac{n}{d}\right) = \frac{\gamma(\frac{n-k}{d}, \frac{n}{d})}{\Gamma(\frac{n-k}{d})}; \quad g_{n-k} \left(\frac{n}{d} + t\right) = \left(\frac{n}{d} + t\right)^{\frac{n-k-1}{d}} e^{-\frac{(t+\frac{c}{\sqrt{n}})^2}{\sqrt{n}}},
\]

Fix an integer \(k\) with \(0 \leq k \leq \sqrt{n} \log n\). Since \(Y = \frac{1}{\sqrt{n-k}} \left(U_{n-k} - \frac{n-k}{d}\right)\) is asymptotically normal, we have the following approximation by the Berry-Esseen theorem:

\[
G_{n-k} \left(\frac{n}{d}\right) = P \left[\sqrt{\frac{n-k}{d}} Y < \frac{k}{d}\right] = \Phi \left(\frac{k}{\sqrt{(n-k)d}}\right) + O(n^{-1/2})
\]
as \(n \to \infty\). Here, note that the \(O\)-constant can be taken to be uniform in \(k\) with \(0 \leq k \leq \sqrt{n} \log n\). Now write \(f_Y\) for the probability density function of the random variable \(Y\). By the Edgeworth expansion for \(f_Y\), we obtain

\[
f_Y(x) = \phi(x) \left(1 + \frac{c}{\sqrt{n}} H_3(x) + O(n^{-1})\right),
\]

where \(H_3\) is the Hermite polynomial of order 3 and \(c\) is a constant. This gives

\[
g_{n-k} \left(\frac{n}{d} + t\right) = \sqrt{\frac{d}{n-k}} \phi \left(\frac{k + td}{\sqrt{(n-k)d}}\right) \left(1 + \frac{c}{\sqrt{n}} H_3 \left(\frac{k + td}{\sqrt{(n-k)d}}\right) + O(n^{-1})\right).
\]

which proves the first statement of the lemma.

The uniform boundedness of \(\Gamma(\frac{n-k}{d}) / \gamma(\frac{n-k}{d}, \frac{n}{d})\) follows from

\[
P \left[U_j < \frac{n}{d}\right] \geq P \left[U_{j+1} < \frac{n}{d}\right], \quad 1 \leq j \leq n - 1
\]

and

\[
\lim_{n \to \infty} P \left[U_n < \frac{n}{d}\right] = P[Z < 0] = \frac{1}{2},
\]

where \(Z\) is the standard normal distribution. \(\square\)

Returning to (2.1), we consider the following sums:

\[
S_n := \sum_{0 \leq k \leq n_0} \left(\frac{n}{d}\right)^{\frac{n-k-1}{d}} e^{-\frac{(t+\frac{c}{\sqrt{n}})^2}{\sqrt{n}}} \gamma(\frac{n-k}{d}, \frac{n}{d});
\]

\[
\epsilon_n := \sum_{n_0 < k < n} \left(\frac{n}{d}\right)^{\frac{n-k-1}{d}} e^{-\frac{(t+\frac{c}{\sqrt{n}})^2}{\sqrt{n}}} \gamma(\frac{n-k}{d}, \frac{n}{d}).
\]
Lemma 2.2 As $n \to \infty$, we have $S_n \to d \log 2$ and $\epsilon_n \to 0$ locally uniformly for $t \in \mathbb{R}^+$. 

Proof By Lemma 2.1, we have 

$$ S_n = \sum_{0 \leq k \leq n_0} \left( \frac{\Gamma(n-k)}{\Gamma(n-k+\frac{n}{d})} \frac{(t + \frac{n}{d})^{\frac{n-k}{d}} - 1 - e^{-(t + \frac{n}{d})}}{\Gamma(n-k)} \right) $$

$$ = \sum_{0 \leq k \leq n_0} \sqrt{\frac{d}{n}} \left( \Phi \left( \frac{k}{\sqrt{nd}} \right) \right)^{-1} \phi \left( \frac{k}{\sqrt{nd}} \right) (1 + o(1)) $$

as $n \to \infty$. By the Riemann sum approximation, we obtain

$$ \lim_{n \to \infty} S_n = \frac{d}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \Phi(x) \, dx. $$

Integration by parts gives

$$ \frac{d}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \Phi(x) \, dx = d \int_0^\infty d \log \Phi(s) \, ds = d \log 2, $$

which proves the convergence of $S_n$ to $d \log 2$.

Now fix an integer $k$ with $n_0 < k \leq n - 1$. We observe that there exists a positive constant $C$ such that

$$ g_{n-k} \left( t + \frac{n}{d} \right) \leq C g_{n-n_0} \left( t + \frac{n}{d} \right). \quad (2.5) $$

Indeed, the inequality

$$ \frac{g_{n-n_0}(t + \frac{n}{d})}{g_{n-k}(t + \frac{n}{d})} = \left( t + \frac{n}{d} \right)^{\frac{k-n_0}{d}} \frac{\Gamma(n-k)}{\Gamma(n-n_0)} $$

$$ \geq \frac{1}{\int_0^{t+\frac{n}{d}} x^{\frac{n-n_0}{d}} - 1 e^{-x} \, dx} \frac{1}{\Gamma(n-n_0)} $$

implies that for sufficiently large $n$,

$$ g_{n-n_0} \left( t + \frac{n}{d} \right) \geq \left( \frac{1}{2} + o(1) \right) g_{n-k} \left( t + \frac{n}{d} \right), $$

where $o(1)$ is uniform for $k$ with $n_0 < k \leq n - 1$ and for $t$ in every compact subset of $\mathbb{R}^+$. By (2.4), we have

$$ g_{n-n_0} \left( t + \frac{n}{d} \right) \leq c_1 n^{-1/2} e^{-c_2 (\log n)^2} $$

for some positive constants $c_1$ and $c_2$. By (2.5) and the uniform boundedness of $(G_{n-k}(n/d))^{-1}$ shown in Lemma 2.1, there exists a constant $C'$ such that

$$ \epsilon_n = \sum_{n_0 < k < n} \frac{g_{n-k} \left( t + \frac{n}{d} \right)}{G_{n-k} \left( \frac{n}{d} \right)} \leq C' n^{1/2} e^{-c_2 (\log n)^2}, $$

which proves the convergence of $\epsilon_n$ to 0. \hfill \square
Proof of Theorem 1.1} Returning to the sum in (2.1), we observe from Lemma 2.2 that
\[
\sum_{j=0}^{n-1} \int_0^1 \left( t + \frac{n}{d} \right)^{j+1} \frac{e^{-t-n/\delta}}{\gamma(j+1, n/\delta)} \, dt = \int_0^1 S_n + \epsilon_n \, dt \to -\xi \quad (2.6)
\]
as \(n \to \infty\). We also obtain that
\[
\lim_{n \to \infty} \int_0^1 \left( t + \frac{n}{d} \right)^{j+1} \frac{e^{-t-n/\delta}}{\gamma(j+1, n/\delta)} \, dt = 0 \quad (2.7)
\]
uniformly in \(j\) with \(0 \leq j \leq n - 1\) by Lemmas 2.1 and 2.2. By Lemma 1.4, it follows from (2.6) and (2.7) that \(\lim_{n \to \infty} \log F_n(\xi) = \xi\) uniformly on compact subsets of \(\mathbb{R}^-\). \(\square\)

3 Universality for the Hard Edge Ensembles

In this section, we prove Theorem 1.2. We consider a radially symmetric potential \(Q(z) = q(|z|)\) with the assumptions: (i) \(q : \mathbb{R}^+ \to \mathbb{R}\) is smooth, (ii) \(Q\) is subharmonic on \(\mathbb{C}\), and (iii) \(Q\) is strictly subharmonic on a neighborhood of the outer boundary of \(S\).

Since \(\Delta Q(r) \geq 0\) on \((0, \infty), r q'(r)\) is increasing on \((0, \infty)\). Following Saff and Totik [36, Section IV.6], let \(r_0\) be the smallest number that \(q'(r) > 0\) for all \(r > r_0\) and \(R_0\) be the smallest solution to \(R_0 q'(R_0) = 2\). Then the droplet is the ring \(S = \{z \in \mathbb{C} : r_0 \leq |z| \leq R_0\}\).

Also note that \(\{r q'(r) \mid r_0 \leq r \leq R_0\} = [0, 2]\), which implies that for each \(\alpha \in [0, 2]\), there exists \(r_\alpha \in [r_0, R_0]\) such that \(r_\alpha q'(r_\alpha) = \alpha\).

Let \(\omega_n\) be the rescaled spectral radius defined by \(\omega_n = n C_0 (|z|_n - R_0)\) where \(C_0 = R_0 \delta \log 4\) and \(\delta = \partial \partial Q(R_0) > 0\). Write \(F_n\) for the cumulative distribution function of \(\omega_n\) i.e., \(F_n(\xi) = \mathbb{P}[\omega_n \leq \xi]\) for \(\xi \leq 0\).

Now we consider the functions \(V\) and \(V_k\) where
\[
V(r) = q(r) - 2 \log r, \quad V_k(r) = q(r) - \left(2 - \frac{2k + 1}{n}\right) \log r.
\]
Setting \(k = (n - 1) - j\), we have
\[
F_n(\xi) = \prod_{j=0}^{n-1} \left(1 - \int_x^{R_0} |p_j(r)|^2 e^{-nq(r)} 2r \, dr\right) = \prod_{k=0}^{n-1} \left(1 - \frac{\int_x^{R_0} e^{-nV_k(r)} \, dr}{\int_{r_0}^{R_0} e^{-nV_k(r)} \, dr}\right),
\]
where \(x = R_0 + (n C_0)^{-1} \xi\). We write
\[
x_{n,k} := \frac{\int_x^{R_0} e^{-nV_k(r)} \, dr}{\int_{r_0}^{R_0} e^{-nV_k(r)} \, dr} \quad (3.1)
\]
and show that \(\{x_{n,k}\}\) satisfies the conditions in Lemma 1.4. As in the previous section, we use the notation \(n_0 = \sqrt{n} \log n\) throughout this section.

Lemma 3.1 We have
\[
\sum_{0 \leq k < n_0} x_{n,k} = -\xi + o(1) \quad (3.2)
\]
where \(o(1) \to 0\) uniformly for \(\xi\) in every compact subset of \(\mathbb{R}^-\) as \(n \to \infty\).
We set \( f(r) = rq'(r) \). Then, \( f \) is increasing in \((0, \infty)\) and \( f' > 0 \) in a neighborhood of \( R_0 \). For each \( k \) with \( 0 \leq k \leq n_0 \), there exists a unique \( t_k \in [r_0, R_0] \) such that

\[
f(t_k) = 2 - \frac{2k + 1}{n},
\]

and \( t_k \) is decreasing in \( k \). Note that \( f'(R_0) = R_0 \Delta Q(R) = 4\delta R_0 \). From the Taylor series expansion

\[
f(r) - f(R_0) = 4\delta R_0 (r - R_0) + O(|r - R_0|^2), \quad r \to R_0,
\]

we obtain

\[
R_0 - t_k = \frac{1}{4\delta R_0} \frac{2k + 1}{n} + O\left(\frac{k}{n}\right)^2.
\]

Again, Taylor’s theorem gives that there exists \( t^*_{k,r} \in (r, t_k) \) such that

\[
V_k(r) = V_k(t_k) + \frac{1}{2} \Delta Q(t_k)(r - t_k)^2 + \frac{1}{6} V''_{k}(t^*_{k,r})|r - t_k|^3.
\]

Here we note that

\[
\Delta Q(t_k) = \Delta Q(R_0) + O(n^{-1/2} \log n),
\]

where the error term is uniform for \( k \) with \( 0 \leq k \leq n_0 \). Since

\[
V'''_{k}(r) = V'''(r) + \left(\frac{2k + 1}{n}\right) \frac{2}{r^3},
\]

we can choose sufficiently small \( \epsilon > 0 \) such that there exists a constant \( M \) satisfying \( \sup_{r \in [R_0 - \epsilon, R_0]} |V'''_{k}(r)| < M \) for all \( k \) with \( 0 \leq k \leq n_0 \). Write \( \epsilon_n = n^{-1/2} \log n \) and

\[
\xi_k = \frac{2k + 1}{2R_0 \sqrt{n \delta}}.
\]

By (3.4), (3.5) and (3.6), we obtain that for \( k \) with \( 0 \leq k \leq n_0 \)

\[
\int_{t_k - \epsilon_n}^{R_0} e^{-nV_k(r)} \, dr = e^{-nV_k(t_k)} \int_{\xi_k}^{\epsilon_k} e^{-s^2/2} \, ds \left(1 + o(1)\right),
\]

where \( o(1) \to 0 \) uniformly in \( k \) as \( n \to \infty \). On the other hand, since \( V'_k(t_k) = 0 \) and \( V'_k(r) \leq 0 \) for \( r \leq t_k \), there exist positive constants \( c \) and \( C \) such that

\[
\int_{t_k - \epsilon_n}^{R_0} e^{-n(V_k(r) - V_k(t_k))} \, dr \leq Ce^{-n(V_k(t_k - \epsilon_n) - V_k(t_k))} \leq Ce^{-c(\log n)^2}.
\]

Combining (3.7) and (3.8), we obtain the following asymptotics:

\[
\int_{r_0}^{R_0} e^{-nV_k(r)} \, dr = \sqrt{\frac{\pi}{2n\delta}} e^{-nV_k(t_k)} \Phi(\xi_k) \left(1 + o(1)\right),
\]

where \( \Phi(\cdot) \) is the standard normal probability density

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},
\]

and \( \Delta Q \) is the error term.
where $\Phi$ is the cumulative distribution function of the standard gaussian distribution defined in (2.2) and $o(1) \to 0$ uniformly in $k$ as $n \to \infty$.

Now we observe that by (3.4), (3.5), and (3.6),
\[
\int_{R_0}^{R_0 + \frac{\xi}{\sqrt{4nC_0}}} e^{-nV_k(r)} \, dr = e^{-nV_k(t_k)} \int_{\frac{\xi}{\sqrt{nC_0}}}^{\xi_k} e^{-s^2/2} \int_{\sqrt{4nC_0} \xi} \frac{ds}{\sqrt{4n\delta}} \left(1 + o(1)\right)
\]
\[
= -(nc_0)^{-1} \xi \cdot e^{-nV_k(t_k)-\xi_k^2/2} \left(1 + o(1)\right), \tag{3.10}
\]
where $o(1) \to 0$ uniformly for $k$ with $0 \leq k \leq n_0$ and for $\xi$ in every compact subset of $\mathbb{R}^-$ as $n \to \infty$. Combining the asymptotics (3.9) and (3.10) gives
\[
\sum_{0 \leq k \leq n_0} x_{n,k} = -\frac{\sqrt{2\delta}}{\sqrt{n\pi} C_0} (1 + o(1)) \sum_{k=0}^{n_0} e^{-\xi_k^2/2} \Phi(\xi_k).
\]
By the Riemann sum approximation with step length $(R_0\sqrt{n\delta})^{-1}$, we obtain
\[
\frac{1}{R_0\sqrt{n\delta}} \sum_{0 \leq k \leq n_0} \frac{e^{-\xi_k^2/2}}{\Phi(\xi_k)} = \int_0^\infty \frac{e^{-t^2/2}}{\Phi(t)} \, dt + o(1) = \sqrt{2\pi} \log 2 + o(1)
\]
as $n \to \infty$. Recall that the constant $C_0$ is defined by
\[
C_0 = R_0\delta \log 4,
\]
and we conclude that (3.2) holds. \qed

The sum of $x_{n,k}$ over $k$ from $n_0$ to $n - 1$ vanishes as $n \to \infty$. The following lemma gives the desired estimate.

**Lemma 3.2** We have
\[
\sum_{n_0 < k \leq n-1} x_{n,k} = o(1),
\]
where $o(1) \to 0$ locally uniformly for $\xi \in \mathbb{R}^-$ as $n \to \infty$.

**Proof** Write $n_1$ for the number $\frac{1}{2} \sqrt{n \log n}$ and $t_{n_1}$ for the unique solution to
\[
f(t) = 2 - \frac{2n_1 + 1}{n}
\]
as defined in (3.3). Write $\epsilon_n$ for the number $n^{-1/2} \log n$. It follows from (3.4) that
\[
t_{n_1} - t_{n_0} \geq c_0 \epsilon_n
\]
for some positive constant $c_0$. Using this, we obtain that for $n_0 < k \leq n - 1$
\[
\int_{R_0}^{R_0} e^{-nV_k(r)} \, dr \geq \int_{t_{n_0}}^{t_{n_1}} e^{-nV_k(r)} \, dr \geq c_0 \epsilon_n e^{-nV_k(t_{n_1})}, \tag{3.11}
\]
where the constant $c_0$ is uniform in $k$.

On the other hand, we observe that for $k$ with $n_0 < k \leq n - 1$
\[
V_k(r) - V_k(t_{n_1}) = V_{n_1}(r) - V_{n_1}(t_{n_1}) - \frac{2(k - n_1)}{n} \log \frac{t_{n_1}}{r}, \tag{3.12}
\]
Since \( R_0 - t_{n_1} = c_1 \epsilon_n + O(\epsilon_n)^2 \) for some positive constant \( c_1 \), there exists a constant \( c_2 \) such that for \( s \) in a compact subset in \( \mathbb{R}^- \)

\[
V_k \left( R_0 + \frac{s}{n} \right) - V_k(t_{n_1}) \geq V_{n_1} \left( R_0 + \frac{s}{n} \right) - V_{n_1}(t_{n_1}) \geq c_2 \epsilon_n^2
\]

by (3.12). It follows from (3.11) that for all \( k \) with \( n_0 < k \leq n - 1 \) and \( x = R_0 + (nC_0)^{-1} \xi \)

\[
x_{n,k} \leq (c_0 \epsilon_n)^{-1} \int_{R_0} e^{-n(V_k(r) - V_k(t_{n_1}))} \, dr \leq Cn^{-1/2}e^{-c(\log n)^2}
\]

for some constants \( c, C \) which is uniform for \( \xi \) in every compact subset in \( \mathbb{R}^- \). Thus we have

\[
\sum_{n_0 \leq k \leq n-1} x_{n,k} = o(1), \quad n \to \infty,
\]

which completes the proof. \( \square \)

**Proof of Theorem 1.2** Combining Lemma 3.1 and Lemma 3.2, we obtain that

\[
\sum_{k=0}^{n-1} x_{n,k} = -\xi + o(1),
\]

where \( o(1) \to 0 \) locally uniformly for \( \xi \in \mathbb{R}^- \) as \( n \to \infty \). It also follows from Lemma 3.2 that \( x_{n,k} \to 0 \) uniformly for all \( k \) with \( n_0 \leq k \leq n - 1 \) as \( n \to \infty \). For all \( k \) with \( 0 \leq k \leq n_0 \), we have a uniform error bound \( x_{n,k} = O(n^{-1/2}) \) by (3.9) and (3.10). Hence, by Lemma 1.4, we conclude that

\[
\log F_n(\xi) = \xi + o(1),
\]

where \( o(1) \to 0 \) locally uniformly in \( \mathbb{R}^- \). \( \square \)

**4 Order Statistics**

In this section, we examine the limit law of the \( l \)-th modulus of eigenvalues of random normal ensembles with a hard edge. Our approach is inspired by the earlier work in [35], where the order statistics for the free boundary Ginibre ensemble was studied.

**Proof of Theorem 1.3** Let \( \{z_j\}_1^n \) be the random normal matrix ensemble associated with the potential \( Q^S \). We consider the probability \( p_{n,k}(x) \) that exactly \( k \) eigenvalues are in the region \( \{z \in \mathbb{C} : |z| > x \} \). It is computed as follows:

\[
p_{n,k}(x) = \frac{1}{k!} \left( \frac{d}{d\lambda} \right)^k \int_{-1}^{\lambda} \mathbf{E} \prod_{j=1}^{n} (1 + \lambda \mathbf{1}_{|z|>x}(z_j)).
\]

See [18, Section 4.3]. It follows from the computation in Section 1.4 that

\[
\mathbf{E} \prod_{j=1}^{n} (1 + \lambda \mathbf{1}_{|z|>x}(z_j)) = \prod_{j=0}^{n-1} \left( 1 + \lambda \int_{|z|>x} |p_j(z)|^2 e^{-nQ^S(z)} \, dA(z) \right).
\]
where $p_j$ is an orthonormal polynomial of degree $j$ with respect to $e^{-nQ^S} \, dA$. For $\xi \leq 0$ and $x = R_0 + \xi/(nC_0)$ with $C_0 = R_0 \delta \log 4$, we write

$$x_{n,j} = \int_{|z| > x} |p_j(z)|^2 e^{-nQ^S(z)} \, dA(z)$$

and $g_n(\lambda) = \prod_{j=0}^{n-1} (1 + \lambda x_{n,j})$ as a function of $\lambda$ in $\mathbb{C}$. We observe that the distribution function of the $l$-th largest modulus $|z|_l$ can be written as follows:

$$P \left[ |z|_l \leq x \right] = \sum_{k=0}^{l-1} p_{n,k}(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \left( \frac{d}{d\lambda} \right)^k \bigg|_{\lambda=-1} g_n(\lambda).$$

It is sufficient to show that

$$\lim_{n \to \infty} \left( \frac{d}{d\lambda} \right)^k \bigg|_{\lambda=-1} g_n(\lambda) = F_{\text{hard}}(\xi) \left[ - \log F_{\text{hard}}(\xi) \right]^k,$$

(4.1)

where $F_{\text{hard}}(\xi) = e^\xi$ as defined before Lemma 1.3. In Sect. 3, we have shown that $\{x_{n,j}\}$ satisfies the conditions (a), (b) in Lemma 1.4 and $\sum_{j=0}^{n-1} x_{n,j}$ converges to $- \log F_{\text{hard}}(\xi)$. By applying Lemma 1.4 to $\{\lambda x_{n,j}\}$, we obtain

$$g_n(\lambda) \to e^{-\lambda \log F_{\text{hard}}(\xi)}$$

uniformly on every compact subset of $\mathbb{C}$ as $n \to \infty$. Since $g_n$ is an analytic function, Cauchy integral formula implies

$$g_n^{(k)}(\lambda) \to \left[ - \log F_{\text{hard}}(\xi) \right]^k e^{-\lambda \log F_{\text{hard}}(\xi)}$$

uniformly on every compact subset of $\mathbb{C}$. Taking $\lambda = -1$, we prove (4.1). \qed

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