Abstract
A first order theory \( T \) is tight iff for any deductively closed extensions \( U \) and \( V \) of \( T \) (both of which are formulated in the language of \( T \)), \( U \) and \( V \) are bi-interpretable iff \( U = V \). By a theorem of Visser, \( \text{PA} \) (Peano Arithmetic) is tight. Here we show that \( \mathbb{Z}_2 \) (second order arithmetic), \( \text{ZF} \) (Zermelo-Fraenkel set theory), and \( \text{KM} \) (Kelley-Morse theory of classes) are also tight theories.

1 Introduction
The source of inspiration for this paper is located in a key result of Albert Visser [V, Corollaries 9.4 & 9.6] concerning a curious interpretability-theoretic feature of \( \text{PA} \) (Peano arithmetic), namely:

1.1. Theorem. (Visser) Suppose \( U \) and \( V \) are deductively closed extensions of \( \text{PA} \) (both of which are formulated in the language of \( \text{PA} \)). Then \( U \) is a retract of \( V \) if \( V \subseteq U \). In particular, \( U \) and \( V \) are bi-interpretable iff \( U = V \).

A natural reaction to Theorem 1.1 is to ask whether the exhibited interpretability-theoretic feature of \( \text{PA} \) is shared by other theories. As shown here, the answer to this question is positive, in particular:

1.2. Theorem\(^2\). Theorem 1.1 remains valid if \( \text{PA} \) is replaced throughout by \( \mathbb{Z}_2 \) (second order arithmetic); or by \( \text{ZF} \) (Zermelo-Fraenkel set theory); or by \( \text{KM} \) (Kelley-Morse theory of classes).

\(^1\)The \( \text{ZF} \)-case of Theorem 1.2 was established independently in unpublished work of Albert Visser and Harvey Friedman. I am thankful to Albert for bringing this to my attention.

\(^2\)See Remark 2.8 for a more complete version of this theorem.
In the remainder of this section we review some basic notions and results of interpretability theory in order to clarify and contextualize Theorems 1.1 & 1.2.

1.3. Definitions. Suppose $U$ and $V$ are first order theories, and for the sake of notational simplicity, let us assume that $U$ and $V$ are theories that support a definable pairing function and are formulated in relational languages. We use $\mathcal{L}_U$ and $\mathcal{L}_V$ to respectively designate the languages of $U$ and $V$.

(a) An interpretation $\mathcal{I}$ of $U$ in $V$, written:

$$\mathcal{I} : U \rightarrow V$$

is given by a translation $\tau$ of each $\mathcal{L}_U$-formula $\varphi$ into an $\mathcal{L}_V$-formula $\varphi^\tau$ with the requirement that $V \vdash \varphi^\tau$ for each $\varphi \in U$, where $\tau$ is determined by an $\mathcal{L}_V$-formula $\delta(x)$ (referred to as a domain formula), and a mapping $P \mapsto A_P$ that translates each $n$-ary $\mathcal{L}_U$-predicate $P$ into some $n$-ary $\mathcal{L}_V$-formula $A_P$. The translation is then lifted to the full first order language in the obvious way by making it commute with propositional connectives, and subject to:

$$(\forall x \varphi)^\tau = \forall x (\delta(x) \rightarrow \varphi^\tau) \text{ and } (\exists x \varphi)^\tau = \exists x (\delta(x) \land \varphi^\tau).$$

Note that each interpretation $\mathcal{I} : U \rightarrow V$ gives rise to an inner model construction that uniformly builds a model $\mathcal{M}^\mathcal{I} \models U$ for any $\mathcal{M} \models V$.

(b) $U$ is interpretable in $V$, written $U \preceq V$, iff there is an interpretation $\mathcal{I} : U \rightarrow V$. $U$ and $V$ are mutually interpretable when $U \preceq V$ and $V \preceq U$.

(c) We indicate the universe of each structure with the corresponding Roman letter, e.g., the universes of structures $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{M}^*$ are respectively $M$, $N$, and $M^*$. Given an $\mathcal{L}$-structure $\mathcal{M}$ and $X \subseteq M^n$ (where $n$ is a positive integer), we say that $X$ is $\mathcal{M}$-definable iff $X$ is parametrically definable in $\mathcal{M}$, i.e., iff there is an $n$-ary formula $\varphi(x_1, \ldots, x_n)$ in the language $\mathcal{L}_M$ obtained by augmenting $\mathcal{L}$ with constant symbols $m$ for each $m \in M$ such that $X = \varphi^\mathcal{M}$, where $\varphi^\mathcal{M} = \{(a_1, \ldots, a_n) \in M^n : (\mathcal{M}, m) \models \varphi(\overline{a_1}, \ldots, \overline{a_n})\}$.

(d) Suppose $\mathcal{N}$ is an $\mathcal{L}_U$-structure and $\mathcal{M}$ is an $\mathcal{L}_V$-structure. We say that $\mathcal{N}$ is parametrically interpretable in $\mathcal{M}$, written $\mathcal{N} \preceq_{\text{par}} \mathcal{M}$ (equivalently: $\mathcal{M} \succeq_{\text{par}} \mathcal{N}$) iff the universe of discourse of $\mathcal{N}$, as well as all the $\mathcal{N}$-interpretations of $\mathcal{L}_U$-predicates are $\mathcal{M}$-definable. Note that $\preceq_{\text{par}}$ is a transitive relation.

(e) $U$ is a retract of $V$ iff there are interpretations $\mathcal{I}$ and $\mathcal{J}$ with $\mathcal{I} : U \rightarrow V$ and $\mathcal{J} : V \rightarrow U$, and a binary $U$-formula $F$ such that $F$ is, $U$-verifiably, an
isomorphism between id_U (the identity interpretation on U) and J ◦ I. In model-theoretic terms, this translates to the requirement that the following holds for every \( \mathcal{M} \models U \):

\[
F^\mathcal{M} : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^* := (\mathcal{M}^J)^I.
\]

(f) \( U \) and \( V \) are bi-interpretable\(^3\) iff there are interpretations \( \mathcal{I} \) and \( \mathcal{J} \) as above that witness that \( U \) is a retract of \( V \), and additionally, there is a \( V \)-formula \( G \), such that \( G \) is, \( V \)-verifiably, an isomorphism between id_V and \( \mathcal{I} \circ \mathcal{J} \). In particular, if \( U \) and \( V \) are bi-interpretable, then given \( \mathcal{M} \models U \) and \( \mathcal{N} \models V \), we have

\[
F^\mathcal{M} : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^* := (\mathcal{M}^J)^I \text{ and } G^\mathcal{N} : \mathcal{N} \xrightarrow{\cong} \mathcal{N}^* := (\mathcal{N}^I)^J.
\]

We conclude this section with salient examples. In what follows ACA_0 (arithmetical comprehension with limited induction) and GB (Gödel-Bernays theory of classes) are the well-known subsystems of Z_2 and KM (respectively) satisfying: ACA_0 is a conservative extension of PA, and GB is a conservative extension of ZF.

1.4. Theorem. (Folklore) PA \( \preceq \) ACA_0 and ZF \( \preceq \) GB; but ACA_0 \( \not\preceq \) PA and GB \( \not\preceq \) ZF.

Proof Outline. The first two statements have routine proofs; the last two follow by combining (a) the finite axiomatizability of ACA_0 and GB, (b) the reflexivity of PA and ZF (i.e., they prove the consistency of each finite fragment of themselves), and (c) Gödel’s second incompleteness theorem. □

By classical results of Ackermann and Mycielski, the structures \((V_\omega,\in)\) and \((\mathbb{N},+,:)\) are bi-interpretable, where \( V_\omega \) is the set of hereditarily finite sets. The two interpretations at work can be used to show Theorem 1.5 below. In what follows, ZF_{fin} is the theory obtained by replacing the axiom of infinity

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\(^3\)The notion of bi-interpretability has been informally around for a long time, but according to Hodges \([H]\) it was first studied in a general setting by Ahlbrandt and Ziegler \([AZ]\). A closely related concept (dubbed sometimes as synonymy, and other times as definitional equivalence) was introduced by de Bouvère \([D]\). Synonymy is a stronger form of bi-interpretation; however, by a result of Friedman and Visser \([FV]\), in many cases synonymy is implied by bi-interpretability, namely, when the two theories involved are sequential, and the bi-interpretability between them is witnessed by a pair of one-dimensional, identity preserving interpretations.
by its negation in the usual axiomatization of ZF and TC is the sentence asserting “every set has a transitive closure”

1.5. Theorem. (Ackermann [A], Mycielski [M], Kaye-Wong [KW]) PA and ZF fin + TC are bi-interpretable.

1.6. Theorem. (E-Schmerl-Visser [ESV, Theorem 5.1]) ZF fin and PA are not bi-interpretable; indeed ZF fin is not even a sentential retract\(^{5}\) of ZF fin + TC.

2 Solid Theories

The notions of solidity, neatness, and tightness encapsulated in Definition 2.1 below are only implicitly introduced in Visser’s paper [V]. It is not hard to see that a solid theory is neat, and a neat theory is tight. Hence to establish Theorems 1.1 and 1.2 it suffices to verify the solidity of PA, Z\(_2\), ZF, and KM. This is precisely what we will accomplish in this section. The proof of Theorem 1.1 is presented partly as an exposition of Visser’s original proof which is rather indirect since it is couched in terms of series of technical general lemmata, and partly because it provides a warm-up for the proof of Theorem 2.5 which establishes the solidity of Z\(_2\). The proof of Theorem 2.6 establishing the solidity of ZF, on the other hand, requires a brand new line of argument. The proof of Theorem 2.7, which establishes the solidity of KM is the most complex among the proofs presented here; it can be roughly described as using a blend of ideas from the proofs of Theorems 2.5 and 2.6.

2.1. Definition. Suppose T is a first order theory.
(a) T is solid iff the following property (*) holds for all models \(M, M^*,\) and \(N\) of T:

\[ (*) \quad \text{If } M \geq_{\text{par}} N \geq_{\text{par}} M^* \text{ and there is an } M\text{-definable isomorphism } i_0 : M \to M^*, \text{ then there is an } M\text{-definable isomorphism } i : M \to N. \]

\(^4\)More explicitly; the axioms of ZF\(_{\text{fin}}\) consists of the axioms of Extensionality, Empty Set, Pairs, Union, Power set, Foundation, and ¬Infinity, plus the scheme of Replacement. Note that ZF\(_{\text{fin}}\) has also been used in the literature (e.g., by the Prague school) to denote the stronger theory in which the Foundation axiom is strengthened to the Foundation scheme; the latter theory is deductively identical to ZF\(_{\text{fin}} + \text{TC}\) in our notation.

\(^5\)The notion of a sentential retract is the natural weakening of the notion of a retract in which the requirement of the existence of a definable isomorphism between \(M\) and \(M^*\) is weakened to the requirement that \(M\) and \(M^*\) be elementarily equivalent.
(b) $T$ is neat iff for any two deductively closed extensions $U$ and $V$ of $T$ (both of which are formulated in the language of $T$), $U$ is a retract of $V$ if $V \subseteq U$.

(c) $T$ is tight iff for any two deductively closed extensions $U$ and $V$ of $T$ (both of which are formulated in the language of $T$), $U$ and $V$ are bi-interpretable iff $U = V$.

2.1.1. Remark. A routine argument shows that if $T$ and $T'$ are bi-interpretable, and $T$ is solid, then $T'$ is also solid.

2.2. Theorem. (Visser [V]) $\text{PA}$ is solid.

Proof. Suppose $M$, $M^*$, and $N$ are models of $\text{PA}$ such that:

$$M \supseteq_{\text{par}} N \supseteq_{\text{par}} M^*,$$

there is an $M$-definable isomorphism $i_0 : M \to M^*$. A key property$^6$ of $\text{PA}$ is that if $M$ is a model of $\text{PA}$ and $N$ is a model of the fragment of $\text{PA}$ known as (Robinson’s) $Q$, then as soon as $N \subseteq_{\text{par}} M$ there is an $M$-definable initial embedding $j : M \to N$, i.e., an embedding $j$ such that the image $j(M)$ of $M$ is (1) a submodel of $N$, and (2) an initial segment of $N$. Hence there is an $M$-definable initial embedding $j_0 : M \to M^*$ and an $N$-definable initial embedding $j_1 : N \to M^*$.

We claim that both $j_0$ and $j_1$ are surjective. To see this, suppose not. Then $j(M)$ is a proper initial segment of $M^*$, where $j$ is the $M$-definable embedding $j : M \to M^*$ given by $j := j_1 \circ j_0$. But then $i_0^{-1}(j(M))$ is a proper $M$-definable initial segment of $M$ with no last element. This is a contradiction since $M$ is a model of $\text{PA}$, and therefore no proper initial segment of $M$ is $M$-definable. Hence $j_0$ and $j_1$ are both surjective; in particular $j_0$ serves as the desired $M$-definable isomorphism between $M$ and $N$.

□

2.2.1. Corollary. $\text{ZF}_{\text{fin}} + \text{TC}$ is solid.

Proof. In light of Remark 2.1.1, this is an immediate consequence of coupling Theorem 2.2 and Theorem 1.5. □

Before presenting the proof of solidity of $\text{Z}_2$ we need to state two propositions concerning theories that prove the full scheme of induction over some specified choice of ‘numbers’. The proofs of Propositions is a straightforward

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$^6$This important property seems to have been first noted by Feferman [F], who used it in his proof of $\Pi_1$-conservativity of $\neg\text{Con(PA)}$ over $\text{PA}$. 

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adaptation of the well-known proof for the special case of PA, so it is only presented in outline form.

2.3. Proposition. Let $T$ be a theory formulated in a language $\mathcal{L}$ such that $T$ interprets $Q$ via an interpretation whose domain formula for ‘numbers’ is $\mathbb{N}(x)$. Furthermore assume the following two hypotheses:

(a) $T \vdash \text{Ind}_{\mathbb{N}}(\mathcal{L})$, where $\text{Ind}_{\mathbb{N}}(\mathcal{L})$ is the scheme of induction over $\mathbb{N}$ whose instances are universal closures of $\mathcal{L}$-formulae of the form below:

$$(\theta(0) \land \forall x (\mathbb{N}(x) \land \theta(x) \rightarrow \theta(x^+))) \rightarrow \forall x (\mathbb{N}(x) \rightarrow \theta(x)),$$

where $x^+$ is shorthand for the successor of $x$, and $\theta$ is allowed to have suppressed parameters; these parameters are not required to lie in $\mathbb{N}$.

(b) $K \models T$ and $K \models \text{par} N \models Q$.

Then there is a $K$-definable initial embedding $j : \mathbb{N}^K \rightarrow N$.

**Proof outline.** Since $\mathbb{N}^K \models Q + \text{Ind}_{\mathbb{N}}(\mathcal{L})$, the following definition by recursion produces the desired $j$.

$$j(0^{\mathbb{N}^K}) = 0^N \text{ and } j((x^+)^{\mathbb{N}^K}) = (j(x))^N.$$ 

□

2.4. Proposition. Suppose $K$ is an $\mathcal{L}$-structure that interprets a model of $Q$ via an interpretation whose domain formula for ‘numbers’ is $\mathbb{N}(x)$ and $\text{Ind}_{\mathbb{N}}(\mathcal{L})$ holds in $K$. Then every $K$-definable proper initial segment of $\mathbb{N}^K$ has a last element.

**Proof.** Easy: the veracity of $\text{Ind}_{\mathbb{N}}(\mathcal{L})$ in $K$ immediately implies that any $K$-definable initial segment of $\mathbb{N}^K$ with no last element coincides with $\mathbb{N}^K$. □

2.5. Theorem. $\mathbb{Z}_2$ is solid.

**Proof.** Following standard practice (as in [S]) models of $\mathbb{Z}_2$ are represented as two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M} \models \text{PA}$, $\mathcal{A}$ is a collection of subsets of $\mathcal{M}$, and $(\mathcal{M}, \mathcal{A})$ satisfies the full comprehension scheme. Suppose $(\mathcal{M}, \mathcal{A})$, $(\mathcal{M}^*, \mathcal{A}^*)$, and $(\mathcal{N}, \mathcal{B})$ are models of $\mathbb{Z}_2$ such that:

$$(\mathcal{M}, \mathcal{A}) \geq_{\text{par}} (\mathcal{N}, \mathcal{B}) \geq_{\text{par}} (\mathcal{M}^*, \mathcal{A}^*),$$

and there is an $(\mathcal{M}, \mathcal{A})$-definable isomorphism

$$\hat{i}_0 : (\mathcal{M}, \mathcal{A}) \rightarrow (\mathcal{M}^*, \mathcal{A}^*).$$

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Note that \( \hat{i}_0 \) is naturally induced by \( i_0 \), where:

\[
i_0 := \hat{i}_0 \restriction M : \mathcal{M} \to \mathcal{M}^*,
\]

since \( \hat{i}_0(A) = \{i_0(m) : m \in A\} \) for \( A \in \mathcal{A}^* \).

It is clear that \( \mathcal{Z}_2 \vdash Q^N + \text{Ind}_N(\mathcal{L}) \) for \( \mathcal{L} := \mathcal{L}_{\mathcal{Z}_2} \), so by Proposition 2.3, we may conclude:

1) There is an \((\mathcal{M}, \mathcal{A})\)-definable initial embedding \( j_0 : \mathcal{M} \to \mathcal{N} \), and

2) There is an \((\mathcal{N}, \mathcal{B})\)-definable initial embedding \( j_1 : \mathcal{N} \to \mathcal{M}^* \).

Similar to the proof of Theorem 2.2 we now argue that both \( j_0 \) and \( j_1 \) are surjective since otherwise the \((\mathcal{M}, \mathcal{A})\)-definable embedding \( j : \mathcal{M} \to \mathcal{M}^* \) given by \( j := j_1 \circ j_0 \) will have the property that \( j(M) \) is a proper initial segment of \( \mathcal{M}^* \), which in turn implies that \( i_0^{-1}(j(M)) \) is an \((\mathcal{M}, \mathcal{A})\)-definable proper initial segment of \( \mathcal{M} \) with no last element, which contradicts Proposition 2.4. Hence (1) and (2) can be strengthened to:

1) There is an \((\mathcal{M}, \mathcal{A})\)-definable isomorphism \( k_0 : \mathcal{N} \to \mathcal{M} \), and

2) There is an \((\mathcal{N}, \mathcal{B})\)-definable isomorphism \( k_1 : \mathcal{M}^* \to \mathcal{N} \).

Let \( \hat{k}_0 : (\mathcal{N}, \mathcal{B}) \to (\mathcal{M}, \mathcal{A}) \) be the natural extension of \( k_0 \), i.e., \( \hat{k}_0(n) := k_0(n) \) for \( n \in N \), and \( \hat{k}_0(B) = \{k_0(n) : n \in B\} \) for \( B \in \mathcal{B} \). Note that the \((\mathcal{M}, \mathcal{A})\)-definability of \( k_0 \), along with the veracity of the comprehension scheme in \((\mathcal{M}, \mathcal{A})\) assures us that \( \hat{k}_0(B) \in \mathcal{A} \) for each \( B \in \mathcal{B} \). Therefore \( \hat{k}_0 \) is an embedding. Using an identical reasoning, since \((\mathcal{M}^*, \mathcal{A}^*)\) is \((\mathcal{N}, \mathcal{B})\)-definable by assumption, we can extend \( k_1 \) to an embedding \( \hat{k}_1 : (\mathcal{M}^*, \mathcal{A}^*) \to (\mathcal{N}, \mathcal{B}) \). Let \( \hat{k} := \hat{k}_0 \circ \hat{k}_1 \circ \hat{i}_0 \). Then:

3) \( \hat{k} : (\mathcal{M}, \mathcal{A}) \to (\mathcal{M}, \mathcal{A}) \) and \( \hat{k} \) is an \((\mathcal{M}, \mathcal{A})\)-definable embedding.

The proof of Theorem 2.5 will be complete once we verify that \( \hat{k}_0 \) is surjective. Since we already know that \( k_0 \) is surjective, it suffices to check that \( \mathcal{A} = \hat{k}_0(\mathcal{B}) := \{\hat{k}_0(B) : B \in \mathcal{B}\} \). Observe that the restriction \( k : \mathcal{M} \to \mathcal{M} \) of \( \hat{i} \) to ‘numbers’ is an automorphism of \( \mathcal{M} \), thanks to (1+), (2+), and the assumption that \( \hat{i}_0 \) is an isomorphism. But since \( \hat{k} \) is \((\mathcal{M}, \mathcal{A})\)-definable \( i(m) = m \) for all \( m \in \mathcal{M} \), thanks to the veracity of \( \text{Ind}_N(\mathcal{L}) \) in \((\mathcal{M}, \mathcal{A})\), for \( \mathcal{L} = \mathcal{L}_{\mathcal{Z}_2} \), which in turn implies that \( \hat{k} \) is just the identity automorphism on \((\mathcal{M}, \mathcal{A})\). Hence \( \hat{k}_0 \) and \( \hat{k}_1 \) are both surjective. \( \square \)

In the following corollary, \( \exists \mathcal{F} \) is the result of substituting the Replacement scheme in the usual axiomatization of \( \mathcal{Z} \mathcal{F} \) (e.g., as in [K]) with the scheme of Collection, whose instances consist of universal generalizations
of formulae of the form \((\forall x \in a \exists y \varphi(x,y)) \rightarrow (\exists b \forall x \in a \exists y \in b \varphi(x,y))\), where the parameters of \(\varphi\) are suppressed.

2.5.1. Corollary. The following theory \(T\) is solid:

\[
T := \overline{\text{ZF}}\setminus\{\text{Power Set}\} + \forall x \ |x| \leq \aleph_0.
\]

Proof. In light of Remark 2.1.1, this is an immediate consequence of Theorem 2.5 and the well-known bi-interpretability of \(T\) with \(Z_2 + \Pi^1_{\infty}\)-AC, where \(\Pi^1_{\infty}\)-AC is the scheme of choice.\footnote{This bi-interpretability was first explicitly noted by Mostowski in the context of the so-called \(\beta\)-models of \(Z_2 + \Pi^1_{\infty}\)-AC (which correspond to well-founded models of \(T\)). See \cite[Theorem VII.3.34]{S} for a refined version of this bi-interpretability result.}

2.6. Theorem. \(ZF\) is solid.

Proof. Suppose \(M, M^*,\) and \(N\) are models of \(ZF\) such that:

\[
M \geq_{\text{par}} N \geq_{\text{par}} M^*,
\]

and there is an \(M\)-definable isomorphism \(i_0 : M \to M^*.\) Since \(M\) injects \(M\) into \(M^*\) via \(i_0,\) and \(M^* \subseteq N,\) we have:

1. \(N\) is a proper class as viewed from \(M.\)

Let \(E := \in^{M^*}\). \(E\) is both extensional and well-founded as viewed from \(N;\) extensionality trivially follows from the assumption that \(M^* \models ZF,\) and well-foundedness can be easily verified using the assumptions that \(M \geq_{\text{par}} N\) and \(i_0\) is an \(M\)-definable isomorphism between \(M\) and \(M^*.\) We wish to show that \(E := \in^{M^*}\) is set-like\footnote{In the context of \(ZF,\) the extension of a binary formula \(R(x,y)\) is set-like iff for every set \(s\) there is a set \(t\) such that \(t = \{x : R(x,s)\}.\)} as viewed from \(N,\) i.e., for every \(c \in M^*,\) \(c_E\) is a set (as opposed to a proper class) of \(N,\) where \(c_E := \{x \in M^* : xEc\}.\)

This will take some effort to establish. We will present the argument in full detail, especially because a natural adaptation of the same argument will also work in one of the stages of the proof of Theorem 2.7 (establishing the solidity of \(KM\)), and will therefore be left to the reader. We will first show that \(E\) is set-like when restricted to \(\text{Ord}^{M^*}\). To this end, let \(\delta \in \text{Ord}^{M^*},\) \(\delta_E := \{m \in M^*: mE\delta\},\) and consider the \(N\)-definable ordered structure \(\Delta:\)

\[
\Delta := (\delta_E, E \cap \delta_E^2).
\]
It is clear, thanks to $i_0$, that $N$ views $\Delta$ as a well-founded linear order in the strong sense that every nonempty $N$-definable subclass of $\delta$ has an $E$-least member. In particular, $\Delta$ is a linear order in which every element other than the last element (if it exists) has an immediate successor. Given $\gamma \in \text{Ord}^N$ let $o(\Delta) \geq \gamma$ be an abbreviation for the statement:

“there is some set $f$ such that $f$ is the (graph of) an order preserving function between $(\gamma, \in)$ and an initial segment of $\Delta$,”

and let $o(\Delta) \geq \text{Ord}$ abbreviate “$\forall \gamma \in \text{Ord} \ o(\Delta) \geq \gamma$”. We wish to show that the statement $o(\Delta) \geq \text{Ord}$ does not hold in $N$. Suppose it does. Then arguing in $N$, for each $\gamma \in \text{Ord}$ there is an order-preserving map $f_\gamma$ which embeds $(\gamma, \in)$ onto an initial segment of $\Delta$. Moreover, such an $f_\gamma$ is unique since it is a theorem of ZF that no ordinal has a nontrivial automorphism. Hence if $\gamma \in \gamma'$, then $f_\gamma \subseteq f_{\gamma'}$ and therefore $f := \bigcup \{f_\gamma : \gamma \in \text{Ord}\}$ serves as an order-preserving $N$-definable injection of $\text{Ord}^N$ onto an initial segment of $\Delta$. Invoking the assumption $M \models \text{par}_N$ this shows that $M$ must view $N$ as well-founded because the map $\rho^N : (N, \in^N) \rightarrow (\text{Ord}, \in)^N$ (where $\rho$ is the usual rank function) is $\in^N$-preserving and $N$-definable, and therefore $M$-definable since $M \models \text{par}_N$. This allows us to conclude that:

(2) $M$ views $(N, \in^N)$ as a well-founded extensional structure of ordinal height at most $i_0^{-1}(\delta) \in \text{Ord}^M$.

At this point we wish to invoke an appropriate form of Mostowski’s collapse theorem in order to show that (2) implies that $N$ is a set from the point of view of $M$. To this end, consider $\text{KP}$ (Kripke-Platek set theory) whose axioms consist of Extensionality, Empty Set, Pairs, Union, $\Pi_1$-Foundation, and $\Sigma_0$-Collection\footnote{It is well-known that $\Sigma_1$-Collection is provable in $\text{KP}$, which enables $\text{KP}$ to carry out $\Sigma_1$-recursions. Also note that the formulation of $\text{KP}$ in many references (including Barwise’s monograph [B]) that focus on admissible set theory includes the full scheme of Foundation since admissible sets are transitive and automatically satisfy $\Pi_\infty$-Foundation. Our formulation of $\text{KP}$ is taken from Mathias’ paper [Mathias].}. It is well-known that $\text{KP}$ is finitely axiomatizable, and that, provably in $\text{KP}$, $\rho$ (the rank function) is an $\in$-homomorphism of the universe onto the class $\text{Ord}$ of ordinals. Let $\text{KPR}$ (Kripke-Platek set theory with ranks) be the strengthening of $\text{KP}$ with the axiom that states that $\{x : \rho(x) < \alpha\}$ is a set for each $\alpha \in \text{Ord}$. Theorem 2.6.1 below can be either seen as a scheme of theorems of $\text{ZF}$, or a single theorem of Gödel-Bernays theory of classes.

2.6.1. Theorem. If $\text{KPR}$ holds in $N$, and $\text{Ord}^N \cong \alpha \in \text{Ord}$, then $N$ is isomorphic to a transitive substructure of $(V_\alpha, \in)$.
Proof outline. Let \( h : \alpha \to \text{Ord}^N \) witness the isomorphism of \( \alpha \) and \( \text{Ord}^N \), and for \( \gamma < \alpha \) let \( N_\gamma := (V_h(\gamma), \in)^N \). A routine induction on \( \gamma < \alpha \) shows that there is a unique embedding \( j_\gamma : N_\gamma \to (V, \in) \) whose range is transitive. This implies that if \( \delta < \gamma < \alpha \), then \( j_\delta \subseteq j_\gamma \). It is then easy to verify that \( j_\alpha : N \to (V, \in) \) is an embedding with a transitive range, where \( j_\alpha := \cup \{ j_\gamma : \gamma < \alpha \} \). □

By coupling (2) with Theorem 2.6.1 we can conclude that \( N \) forms a set in \( M \), thus contradicting (1). This concludes our verification of the failure of \( o(\Delta) \geq \text{Ord} \) within \( N \).

The failure of \( o(\Delta) \geq \text{Ord} \) in \( N \) allows us to choose \( \gamma_0 \in \text{Ord}^N \) such that \( N \) views \( \gamma_0 \) to be the first ordinal \( \gamma \) such that \( o(\Delta) \geq \gamma \) is false. We claim that \( \gamma_0 \) is a successor ordinal of \( \text{Ord}^N \). If not, then, arguing in \( N \), for each \( \beta \in \gamma_0 \) there is a unique order-preserving map \( f_\beta \) which maps \( (\beta, \in) \) onto an initial segment of \( \Delta \), and \( f_\beta \subseteq f_{\beta'} \) whenever \( \beta \in \beta' \in \gamma_0 \), then \( f_\beta \subseteq f_{\beta'} \). Therefore \( \cup \{ f_\gamma : \gamma \in \gamma_0 \} \) serves as an order-preserving map between \( (\gamma_0, \in) \) and an initial segment of \( \Delta \), contradicting the choice of \( \gamma_0 \). Hence \( \gamma_0 = \beta_0 + 1 \) for some \( \beta_0 \in \text{Ord}^N \). This makes it clear that:

(3) \( f_{\beta_0} \) is a bijection between \( \beta_0 \) and \( \delta_E \),

since if the range of \( f_{\beta_0} \) is not all of \( \delta_E \), then the range \( \text{ran}(f_{\beta_0}) \) of \( f_{\beta_0} \) is a proper initial segment of \( \Delta \), and \( f_{\beta_0} \) could be extended to an order-preserving map \( f_{\gamma_0} \) with domain \( \gamma_0 \) by setting:

\[
f_{\gamma_0}(\beta_0) = \min(\delta_E \setminus \text{ran}(f_{\beta_0})).
\]

Thanks to (3), we now know that, as viewed by \( N \), \( E \) is set-like when restricted to \( \text{Ord}^{M^*} \). To verify the set-likeness of \( E \) in \( N \) it is sufficient to show that \( s_E \) forms a set in \( N \), where \( s_E := \{ m \in M^* : mEs \} \) and \( s := V_\delta^{M^*} \) for some \( \delta \in \text{Ord}^{M^*} \) such that KPR holds in \( V_\delta^{M^*} \), since such ordinals \( \delta \) are cofinal in \( \text{Ord}^{M^*} \) by the Reflection Theorem of ZF. Consider the \( N \)-definable structure \( \Sigma : \)

\[
\Sigma := (s_E, E \cap s_E^2).
\]

Since \( \Sigma \) is a model of KPR whose set of ordinals is isomorphic to \( \beta_0 \), by Theorem 2.6.1 (applied within \( N \)) there is an \( N \)-definable embedding of \( \Sigma \) onto a (transitive) subset of \( V_{\beta_0}^N \). This makes it evident that \( s_E \) forms a set in \( N \). Combined with (2) this allows us to conclude:

(4) \( E \) is extensional, set-like, and well-founded within \( N \).
At this point we invoke the Class-form of Mostowski’s Collapse Theorem:

**2.6.2. Theorem.** \([K, \text{Theorem 5.14}]\) Suppose \(E\) is a well-founded, set-like class, and extensional on a class \(M^*\); then there is a transitive class \(S\) and a 1-1 map \(G\) from \(M^*\) onto \(S\) such that \(G\) is an isomorphism between \((M^*, E)\) and \((S, \in)\).

Theorem 2.6.2 together with (4) assure us of the existence of an \(N\)-definable \(S \subseteq N\) such that \(S\) is transitive from the point of view of \(N\), and which has the property that there is an \(N\)-definable isomorphism \(i_1\), where

\[
i_1 : M^* \rightarrow (S, \in)^N.
\]

Finally, we verify that \(S = N\). We first note that \(S\) must be a proper class in the sense of \(N\), since otherwise \(N\) would be able to define the satisfaction predicate for \((S, \in)^N\), which coupled with the assumption that \(N\) is interpretable in \(M\), and \(i := i_1 \circ i_0\) is an \(M\)-definable isomorphism between \(M\) and \((S, \in)^N\), would result in \(M\) being able to define a satisfaction predicate for itself, which contradicts (an appropriate version of) Tarski’s *Undecidability of Truth Theorem*\(^{11}\). The transitivity of \(S\) coupled with the fact that \(S\) is a proper class in \(N\) together imply that \(S\) contains all of the ordinals of \(N\). Therefore, if \(S \neq N\), then arguing in \(N\), let \(V^S_\alpha\) be \(V_\alpha\) in the sense of \((S, \in)^N\) and let

\[
o_0 = \text{the first ordinal } \alpha \text{ such that } V_\alpha = V^S_\alpha, \text{ but } V_{\alpha+1}\backslash V^S_{\alpha+1} \neq \emptyset.
\]

This makes it clear, in light of the assumption that \(M \models_{\text{par}} N\), and the fact that \(i\) is an isomorphism between \(M\) and \((S, \in)^N\), that we have a contradiction at hand since \(M\) believes that \(N\) sees a ‘new subset’ of \(V^S_{i-1(\alpha_0)}\) of \(M\) that is missing from \(M\). Hence \(S = N\) and we may conclude that \(i\) is an \(M\)-definable isomorphism between \(M\) and \(N\). \(\square\)

**2.7. Theorem.** \(KM\) is solid.

**Proof.** Models of \(KM\) can be represented as two-sorted structures of the form \((M, A)\), where \(M \models ZF\); \(A\) is a collection of subsets of \(M\); and \((M, A)\) satisfies the full comprehension scheme. Suppose \((M, A)\), \((M^*, A^*)\), and \((N, B)\) are models of \(KM\) such that:

\(^{11}\)For a structure \(M\) let:

\[
\text{Th}^+(M) = \{ \langle \sigma^M, a \rangle : M \models \sigma(a) \}, \quad \text{and} \quad \text{Th}^-(M) = \{ \langle \sigma^M, a \rangle : M \models \neg \sigma(a) \}.
\]

With the above notation in mind, the version of Tarski’s theorem that is invoked here says that if \(M\) is a structure that interprets \(Q\) and is endowed with a pairing function, then \(\text{Th}^+(M)\) and \(\text{Th}^-(M)\) are \(M\)-inseparable, i.e., there is no \(M\)-definable \(D\) such that \(\text{Th}^+(M) \subseteq D\) and \(\text{Th}^-(M) \cap D = \emptyset\).
\((\mathcal{M}, \mathcal{A}) \succeq \text{par} (\mathcal{N}, \mathcal{B}) \succeq \text{par} (\mathcal{M}^*, \mathcal{A}^*)\),

and there is an \((\mathcal{M}, \mathcal{A})\)-definable isomorphism

\(\widehat{i}_0 : (\mathcal{M}, \mathcal{A}) \to (\mathcal{M}^*, \mathcal{A}^*)\).

As in the proof of Theorem 2.5 we note that \(\widehat{i}_0\) is naturally induced by \(i_0\), where:

\[i_0 := \widehat{i}_0 \upharpoonright M : M \to \mathcal{M}^*,\]

since \(\widehat{i}_0(A) = \{i_0(m) : m \in A\}\) for \(A \in \mathcal{A}^*\).

\(N\) forms a proper class in \((\mathcal{M}, \mathcal{A})\) since if \(N\) forms a set, then so does \(\mathcal{B}\), and \(\widehat{i}_0\) is an \((\mathcal{M}, \mathcal{A})\)-definable bijection between \(M \cup \mathcal{A}\) and a subset of \(N \cup \mathcal{B}\). Let \(E := \in^{\mathcal{M}^*}\). Clearly \(E\) is extensional. Furthermore, with the help of \(i_0\) and the assumption \((\mathcal{M}, \mathcal{A}) \succeq \text{par} (\mathcal{N}, \mathcal{B})\) it is easy to see that \(E\) is well-founded from the point of view of \((\mathcal{N}, \mathcal{B})\). The reader is asked to verify that an argument very similar to the one used in the proof of Theorem 2.6 shows that \(E\) is also set-like in the sense of \((\mathcal{N}, \mathcal{B})\). Theorem 2.6.2 can then be invoked to obtain an \((\mathcal{N}, \mathcal{B})\)-definable isomorphism

\[k_1 : \mathcal{M}^* \to (S, \in^N)\]

for some \((\mathcal{N}, \mathcal{B})\)-definable transitive \(S \subseteq N\). The verification that \(S = N\) is identical to the corresponding part in the proof of Theorem 2.6 (and in particular uses Tarski’s undefinability of truth theorem). Let \(k_0 := i_0^{-1} \circ k_1^{-1}\).

Clearly:

(5) \(k_0 : \mathcal{N} \to \mathcal{M}\) is an \((\mathcal{M}, \mathcal{A})\)-definable isomorphism, and

(6) \(k_1 : \mathcal{M}^* \to \mathcal{N}\) is an \((\mathcal{N}, \mathcal{B})\)-definable isomorphism.

Borrowing a notation from the proof of Theorem 2.5, let \(\hat{k}_0 : (\mathcal{N}, \mathcal{B}) \to (\mathcal{M}, \mathcal{A})\) be the natural extension of \(k_0\), and \(\hat{k}_1 : (\mathcal{M}^*, \mathcal{A}^*) \to (\mathcal{N}, \mathcal{B})\) be the natural extension of \(k_1\). Note that both \(\hat{k}_0\) and \(\hat{k}_1\) are embeddings. Let \(\hat{k} := \hat{k}_0 \circ \hat{k}_1 \circ i_0\); it is clear that:

(7) \(\hat{k} : (\mathcal{M}, \mathcal{A}) \to (\mathcal{M}, \mathcal{A})\) and \(\hat{k}\) is an \((\mathcal{M}, \mathcal{A})\)-definable embedding.

Observe that (5) and (6), together with the assumption that \(i_0\) is an isomorphism imply that the restriction \(k : \mathcal{M} \to \mathcal{M}\) of \(\hat{k}\) to ‘sets’ is an automorphism of \(\mathcal{M}\). But since \(\hat{k}\) is \((\mathcal{M}, \mathcal{A})\)-definable, \(k(m) = m\) for all \(m \in M\),
thanks to the veracity of the scheme of \( \epsilon \)-induction\(^{12}\) in KM. This shows that \( \tilde{k} \) is the identity map and in particular it is surjective, which in turn implies that \( \tilde{k}_0 \) and \( \tilde{k}_1 \) are both surjective. This makes it clear that there is an \( (\mathcal{M}, \mathcal{A}) \)-definable isomorphism between \( (\mathcal{M}, \mathcal{A}) \) and \( (\mathcal{N}, \mathcal{B}) \).

Recall that \( \tilde{ZF} \) was defined earlier, just before Corollary 2.5.1.

2.7.1. Corollary. The following theory \( T \) is solid:

\[
T := \tilde{ZF} \setminus \{\text{Power Set}\} + \exists \kappa (\kappa \text{ is strongly inaccessible, and } \forall x \, |x| \leq \kappa).
\]

**Proof.** In light of Remark 2.1.1, this follows from Theorem 2.8 and the well-known bi-interpretability of \( T \) with \( KM + \Pi_{1,\infty} -\text{AC} \), where \( \Pi_{1,\infty} -\text{AC} \) is the scheme of Choice. \( \square \)

2.8. Remark. An examination of the proofs in this section make it clear that for each positive integer \( n \), the theories \( Z_n \) (\( n \)-th order arithmetic) and \( KM_n \) (\( n \)-th order Kelley-Morse theory of classes) are solid theories (where \( Z_1 := PA \), and \( KM_1 := ZF \)). This observation, in turn, implies that the theory of types \( Z_\omega \) (with full comprehension) whose level-zero objects form a model of \( PA \) (equivalently \( ZF_{\text{fin}} + \text{TC} \)), and the theory of types \( KM_\omega \) whose level-zero objects form a model of \( ZF \) are also solid theories. Thus, the list of theories whose solidity is established in this section can be described (up to bi-interpretability) as \( \{Z_n : 1 \leq n \leq \omega\} \cup \{KM_n : 1 \leq n \leq \omega\} \).

3 Examples and Questions

All of the theories \( T \) whose solidity was established in Section 2 are sequential\(^{14}\) theories which have an interpretation \( \mathbb{N} \) for ‘numbers’ for which the full scheme \( \text{Ind}_{\mathbb{N}}(\mathcal{L}_T) \) of induction is \( T \)-provable, so one may ask whether the

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\(^{12}\)The scheme of \( \epsilon \)-induction consists of the universal closures of formulas of the form \( \forall y (\forall x : y(x) \rightarrow \theta(x)) \rightarrow \forall z \theta(z) \), where the parameters in \( \theta \) are suppressed. It is easy to see that the scheme of \( \epsilon \)-induction is equivalent to the class-form of Foundation, which asserts that every nonempty definable collection of sets has an \( \epsilon \)-minimal element. The class-form of Foundation follows from the set-form of Foundation and the comprehension scheme of KM: suppose a class \( C \) is nonempty, and let \( \alpha_0 \) be the first ordinal \( \alpha \) such that \( V_\alpha \cap C \neq \emptyset \). Then an \( \epsilon \)-minimal member of \( V_\alpha \cap C \) is also an \( \epsilon \)-minimal member of \( C \).

\(^{13}\)This bi-interpretability was first noted by Mostowski; a modern account is given in a recent paper of Antos & Friedman [AF] section 2), where \( KM + \Pi_{1,\infty} -\text{AC} \) is referred to as \( MK^* \), and \( T \) is referred to as \( \text{SetMK}^* \).

\(^{14}\)A sequential theory is a theory that has access to a definable ‘\( \beta \)-function’ for coding finite sequences of objects in the domain of discourse.
$T$-provability of $\text{Ind}_{\mathbb{N}}(\mathcal{L}_T)$ within a sequential theory is a sufficient condition for solidity. A simple counterexample gives a negative answer: let $\text{PA}(G)$ be the natural extension of $\text{PA}$ in which the induction scheme is extended to formulae in the language obtained by adding a unary predicate $G$ to the language of arithmetic. To see that $\text{PA}(G)$ is not solid, consider the extensions $T_1$ and $T_2$ of $\text{PA}(G)$, where:

$$T_1 := \text{PA}(G) + \forall x(G(x) \leftrightarrow x = 1) \text{ and } T_2 := \text{PA}(G) + \forall x(G(x) \leftrightarrow x = 2).$$

Clearly the deductive closures of $T_1$ and $T_2$ are distinct, and yet it is easy to see that $T_1$ and $T_2$ are bi-interpretable. This shows that $\text{PA}(G)$ is not tight, and therefore not solid.

With the help of [ESV, Theorem 4.9 & Remark 4.10] one can also show that the theory $\text{ZF}_{\text{fin}}$ is not tight, even though as shown in Corollary 2.2.1 its strengthening by $\text{TC}$ is a solid theory. Another example of a theory that fails to be tight is $\text{ZF}\setminus\{\text{Foundation}\}$. To see this, consider $T_1 := \text{ZF}$, and

$$T_2 := \text{ZF}\setminus\{\text{Foundation}\} + \exists! x(x = \{x\} + \forall t \exists \alpha(t \in V_\alpha(x))),$$

where $\alpha$ ranges over ordinals, and

$$V_0(x) := x, V_{\alpha+1}(x) := \mathcal{P}(V_\alpha(x)), \text{ and } V_\alpha(x) := \bigcup_{\beta<\alpha} V_\beta(x) \text{ for limit } \alpha.$$

Then $T_1$ and $T_2$ are extensions of $\text{ZF}\setminus\{\text{Foundation}\}$ with distinct deductive closures, and yet, the bi-interpretability of $T_1$ and $T_2$ can be established by well-known methods: the relevant interpretations are $\mathcal{I}$ and $\mathcal{J}$, where $\mathcal{I}$ is the classic von Neumann interpretation of $\text{ZF}$ in $\text{ZF}\setminus\{\text{Foundation}\}$, and $\mathcal{J}$ is the classic Rieger-Bernays interpretation that adds a single ‘Quine atom’ (i.e., a set $s$ such that $s = \{s\}$) to a model of $\text{ZF}$.

However, we do not know whether $T \vdash Q^\mathbb{N} + \text{Ind}_{\mathbb{N}}(\mathcal{L}_T)$ for every solid sequential theory (for an appropriate choice of numbers $\mathbb{N}$). This motivates the following question, since by a general result of Montague [Mo] the $T$-provability of $Q^\mathbb{N} + \text{Ind}_{\mathbb{N}}(\mathcal{L}_T)$ implies that $T$ is not finitely axiomatizable.

**3.1. Question.** Is there a consistent sequential finitely axiomatized theory that is solid?

The question below arises from reflecting on the results of Section 2 and noting that the proofs of solidity of each of the theories $T$ established in Section 2 uses the ‘full power’ of $T$.

**3.2. Question.** Is there an example $T$ of one of the theories whose solidity is established in Theorem 1.2, and some solid $T_0 \subseteq T$ such that the deductive closure of $T_0$ is a proper subset of the deductive closure of $T$?
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