Singular SQCD Vacua and Confinement

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Abstract

We revisit the study of confining vacua in the softly broken $\mathcal{N}=2$ supersymmetric QCD, in the light of some recent developments in our understanding of the dynamics of $\mathcal{N}=2$ gauge theories. These vacua are characterized by an effective magnetic $SU(r)$ gauge group ($0 \leq r \leq N_f/2$) and are referred to sometimes as the $r$ vacua. We further clarify the meaning of $r \leftrightarrow N_f - r$ duality arising from the matching of semi-classical and quantum vacua. A particular attention is paid to certain singular SCFT’s of $\mathcal{N}=2$ SQCD, driven into confinement phase by the adjoint mass deformation $\mu \Phi^2$. In some cases they occur as a result of coalescence of different $r$ vacua as the bare mass is tuned to a critical value.
## 1 Introduction

A considerable progress is being made in our understanding of the dynamics of non-Abelian gauge theories in four dimensions. A recent remarkable development concerns the better understanding of $\mathcal{N} = 2$ superconformal theories (SCFT) [1]-[4]. Also many new results on the exact BPS spectra...
in the strongly coupled gauge systems are now available (see e.g. [5, 6]). Another venue in which considerable development has occurred is the investigations of soliton vortex and monopoles of non-Abelian type [7]-[15]. Together, it is quite plausible that these developments help clarifying many issues left still to be elucidated, even after the discovery of Seiberg-Witten solutions of $\mathcal{N} = 2$ gauge theories and the developments which followed.

One such important problem concerns the possible types of strongly-coupled gauge systems in confinement phase. It is the purpose of this paper to make a few remarks on the confining vacua occurring in the softly broken $\mathcal{N} = 2$ SQCD. To fix an idea and for definiteness, we stick to the $\mathcal{N} = 2$ supersymmetric QCD like theories with $SU(N)$ or $USp(2N)$ gauge theories with $N_f$ flavors of quark hypermultiplets, deformed by the adjoint mass perturbation, $\mu \text{Tr} \Phi^2$.

The paper will be organized as follows. In Section 2 we discuss the simplest and well-understood case of $SU(2)$ SQCD, describing the ideas that will play a crucial role in our analysis. In Section 3 we review the physics of the $r$ vacua and the mechanism of confinement which takes place when the $\mathcal{N} = 1$ perturbation is turned on. In Section 4 we clarify the mechanism at the basis of the two-to-one map relating semiclassical and quantum vacua observed in [16]. This correspondence links $r$ and $N_f - r$ vacua and is reminiscent of Seiberg’s duality. In Section 5 we discuss the low energy physics at a large class of fixed points in $\mathcal{N} = 2$ SQCD. These SCFTs become confining when we turn on the $\mathcal{N} = 1$ perturbation and we discuss the mechanism of confinement occurring in these special cases. We conclude with a discussion in Section 6.

## 2 Softly broken $\mathcal{N} = 2$ $SU(2)$ SQCD

The softly broken $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 1, 2, 3$ quark hypermultiplets has extensively been studied starting from the pioneering work by Seiberg and Witten [17]-[19]. In order to introduce some of the issues discussed later let us briefly review the physics of the $N_f = 2$ theory. There are four singularities in the $u = \langle \text{Tr} \Phi^2 \rangle$ plane where some hypermultiplets become massless. The effective theory at these points is Abelian, dual $U(1)$ gauge theory. For small, nearly equal bare quark masses, $m_1 \simeq m_2 \ll \Lambda$, the singularities group into two pairs of nearby singularities. The massless hypermultiplets in these two singularities are the Abelian monopoles in one or the other of the spinor representations

$$(2, 1) \quad \text{or} \quad (1, 2)$$

of the flavor symmetry group $SO(4) \sim SU(2) \times SU(2)$.

When the perturbation $\mu \text{Tr} \Phi^2 (\mu \ll \Lambda)$ is added in the system, the monopole, say in $(2, 1)$,
condenses,
\[ \langle M_1 \rangle \sim \sqrt{\mu \Lambda}, \quad (2.2) \]
the dual \( U_D(1) \) gauge group is Higgsed, and the system is in confinement phase. An interesting feature of this case is that the confinement order parameter at the same time breaks the global symmetry as
\[ SU(2) \times SU(2) \rightarrow SU(2). \quad (2.3) \]
The Seiberg-Witten effective action correctly describes the low-energy excitations: the exactly massless Nambu-Golstone bosons of the symmetry breaking \((2.3)\) and their superpartners. Unlike the light flavored standard QCD, the massless Nambu-Goldstone bosons do not carry the quantum numbers of the remaining unbroken \( SU(2) \). There are also light but massive dual photon and dual photino of the order of \( \sqrt{\mu \Lambda} \), which arise as a result of the dual Higgs mechanism.

All these light particles are gauge invariant states (they are asymptotic states); the presence of the original quarks degrees of freedom can be detected in the flavor quantum numbers \([20]\).

The low energy system is a dual Abelian \( U(1) \) gauge theory broken by the magnetic monopole condensation \((2.2)\). The ANO vortex of this system, with tension \( \sim \mu \Lambda \), carries the (Abelianized) chromoelectric flux. The fact that the underlying \( SU(2) \) theory is simply connected, means that such a vortex must end: the endpoints are the quarks (and squarks) of the underlying theory. Quarks are confined.

An important point we want to stress is the fact that the particles becoming massless at each Abelian singularity of the Seiberg-Witten curve are pure magnetic monopoles even though they carry distinct labels \( \{n_{mi}, n_{ei}\} \) \((i = 1, 2, \ldots, N - 1)\) and coupled to different “magnetic duals”, \( n_m A_D \mu + n_e A_\mu \). In the case of \( SU(2) \) theory where there is only \( U(1) \) gauge interactions at low energies this fact is easily seen \([21]\). At a singularity of the quantum moduli space where the \((n_m, n_e)\) dyon becomes massless
\[ n_m a_D + n_e a = 0, \quad (2.4) \]
the exact SW solution tells us that
\[ \frac{n_m (da_D/du) + n_e (da/du)}{(da/du)} = 0, \quad (2.5) \]
due to a logarithmic singularity in the denominator. Thus
\[ \theta_{\text{eff}} = \Re \frac{da_D}{da} \pi = -\frac{n_e}{n_m} \pi \quad (2.6) \]
and the electric charge of a \((n_m, n_e)\) “dyon” is \[22\]

\[
\frac{2}{g}Q_e = n_e + \frac{\theta_{\text{eff}}}{\pi}n_m = 0. \tag{2.7}
\]

In the case of the \(SU(2)\) theory with \(N_f = 2\) with small masses, the massless dyons (which condense upon \(\mu \Phi^2\) perturbation) carry \((n_m, n_e)\) charges

\[
(n_m, n_e) = (1, 0), \quad \theta_{\text{eff}} = 0 , \tag{2.8}
\]
in one doublet of singularities, and

\[
(n_m, n_e) = (1, 1), \quad \theta_{\text{eff}} = -\pi , \tag{2.9}
\]
in the other. Thus in all vacua the quarks (with charges \((n_m^1, n_e^1) = (0, 1)\)) are confined, carrying a relative nonzero Dirac unit

\[
D = n_m^1n_e^2 - n_m^2n_e^1 \mod 2, \tag{2.10}
\]
with respect to the condensed fields, \((n_m^2, n_e^2)\). In the \(m \to 0\) limit, a \(\mathbb{Z}_2\) symmetry ensures that the physics at the two vacua look identical, even though the light monopoles (dyons) are coupled locally to two different magnetic duals.

As was shown in [18], all massless “dyons” in \(SU(2)\) theory with various \(N_f\) carry \(n_m = 1\). Their condensation upon the \(\mu \Phi^2\) perturbation leads to quark confinement. The only exception occurs [18] in one of the vacua of \(N_f = 3\) theory, where massless (2, 1) dyons appear as the infrared degrees of freedom. This vacuum (where \(\theta_{\text{eff}} = -1/2\)) survives the \(\mu \Phi^2\) deformation, the (2, 1) dyons condense, but quarks are unconfined: it is in ’t Hooft’s oblique confinement phase [23]. The phase of the pure (non supersymmetric) \(SU(2)\) Yang-Mills theory with \(\theta = -\pi\) is believed to be in such a phase, where the composite of the (1, 0) monopole and the (1, 1) dyon with charges \(\mp \frac{1}{2}\) condenses.

### 3 The quantum \(r\) vacua

The classical and quantum moduli space of the vacua of the \(N = 2\) supersymmetric \(SU(N)\) QCD has been first studied systematically by Argyres, Plessier and Seiberg and others [24, 25, 16]. Of particular interest are the \(r\)-vacua characterized by an effective low-energy \(SU(r) \times U(1)^{N-r}\) gauge symmetry, with massless monopoles carrying the charges shown in the Table 1 (taken from [23]). When the adjoint scalar mass \(\mu \text{Tr} \Phi^2\) term, which breaks supersymmetry to \(\mathcal{N} = 1\),
is added the massless Abelian \((M_k)\) and non-Abelian monopoles \((\mathcal{M})\) all condense, bringing the system to a confinement phase. The form of the effective action describing these light degrees of freedom is dictated by the \(\mathcal{N} = 2\) supersymmetry and the gauge and flavor symmetries. The effective superpotential has the form \([24, 25, 16]\)

\[
W_{r-vacua} = \sqrt{2} \text{Tr}(\mathcal{M}\phi\tilde{\mathcal{M}}) + \sqrt{2} a_{D0} \text{Tr}(\mathcal{M}\tilde{\mathcal{M}}) + \sqrt{2} \sum_{k=1}^{N-r-1} a_{Dk} M_k \tilde{M}_k +
\]

\[+ \mu \left( \Lambda \sum_{k=0}^{N-r-1} c_k a_{Dk} + \frac{1}{2} \text{Tr}\phi^2 \right), \tag{3.1}\]

where \(\phi\) and \(a_{D0}\) are the adjoint scalar fields in the \(\mathcal{N} = 2\) SU\((r) \times U(1)\) vector multiplet, \(a_{Dk}, k = 1, 2, \ldots, N-r-1\) are the adjoint scalars of the Abelian \(U(1)^{N-r-1}\) gauge multiplets. \(M_k\)’s are the Abelian monopoles, each carrying one of the magnetic \(U(1)\) charges, whereas \(\mathcal{M}\) (with \(r\) color components and in the fundamental representation of the flavor \(SU(N_f)\) group) are the non-Abelian monopoles. The terms linear in \(\mu\) is generated by the microscopic \(\mathcal{N} = 1\) perturbation \(\mu \text{Tr}\Phi^2\), written in terms of the infrared degrees of freedom \(a_{Dk}\) and \(\phi\), and \(c_k\) are some dimensionless constants of order of unity. These quantum \(r\)-vacua are known to exist only for \(r \leq \left\lfloor \frac{N_f}{2} \right\rfloor\).

When small, generic bare quark mass terms

\[
W_{\text{masses}} = m_i Q_i \tilde{Q}_i, \tag{3.2}
\]

are added in the microscopic theory, the infrared theory gets modified further by the addition

\[
\Delta W_{\text{masses}} = m_i \mathcal{M}^i \tilde{\mathcal{M}}_i + \sum_{k=1}^{N-r-1} S_j^k m_j M_k \tilde{M}_k, \quad (i, j = 1, 2, \ldots, N_f), \tag{3.3}
\]

where \(S_j^k\) are the \(j\)-th quark number carried by the \(k\)-th monopole. Supersymmetric vacua are found by minimizing the potential following from Eq. (3.1) with Eq. (3.3), and by vanishing of the \(D\)-term potential.

| \(SU(r)\) | \(U(1)_0\) | \(U(1)_1\) | \(\ldots\) | \(U(1)_{N-r-1}\) | \(U(1)_B\) |
|---|---|---|---|---|---|
| \(n_f \times \mathcal{M}\) | \(1\) | \(0\) | \(\ldots\) | \(0\) | \(0\) |
| \(M_1\) | \(1\) | \(0\) | \(1\) | \(\ldots\) | \(0\) | \(0\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\ddots\) | \(\vdots\) | \(\vdots\) |
| \(M_{N-r-1}\) | \(1\) | \(0\) | \(0\) | \(\ldots\) | \(1\) | \(0\) |

Table 1: The massless non-Abelian and Abelian monopoles and their charges at the \(r\) vacua at the root of a “non-baryonic” \(r\)-th Higgs branch.
The part of the decoupled $U(1)^{N-r-1}$ theory involving Abelian monopoles is trivial and gives the VEV's

$$a_{Dk} \sim O(m_i); \quad M_k = \bar{M}_k \sim \sqrt{\mu \Lambda}, \quad (3.4)$$

as in the $SU(2)$ theories.

The equations for the $SU(r) \times U(1)$ sector (see Eq. (B.1)-Eq. (B.7)) are less trivial. The equations look rather similar to the semiclassical equations of the microscopic $SU(N)$ theory, which are valid for $\mu \gg \Lambda, |m_i| \gg \Lambda$ (Appendix A), but there are a few crucial differences.

One is that the effective gauge group $SU(r) \times U(1)$ is not simply connected and the low-energy system generates vortex solutions, while the microscopic theory cannot possess such solitons. Secondly, the massless hypermultiplets in the system describe magnetically charged particles, in contrast to those in the original ultraviolet Lagrangian. Finally, the range of validity of the effective theory is limited to the excitations of energies much less than the dynamical scale $\Lambda$, as the particles of masses of the order of $\Lambda$ or larger have been integrated out in obtaining it.

This last fact makes the identification of the correct solutions of Eq. (B.1)-Eq. (B.7) somewhat a subtle task (i.e., fake solutions involving VEVs of the order of $\Lambda$ must be disregarded): the solutions are given by [16]:

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -m_1 - \sqrt{2} \psi_0 \\ \vdots \\ -m_r - \sqrt{2} \psi_0 \end{pmatrix}, \quad (3.5)$$

$$M^i_a = \begin{pmatrix} d_1 \\ \vdots \\ d_r \\ 0 \end{pmatrix}, \quad \tilde{M}^a_i = \begin{pmatrix} \tilde{d}_1 \\ \vdots \\ \tilde{d}_r \\ 0 \end{pmatrix}, \quad (3.6)$$

where $d_i, \tilde{d}_i$'s and $\psi_0$ are given by

$$\psi_0 = -\frac{1}{\sqrt{2} r} \sum_i m_i, \quad (3.7)$$

$$d_i \tilde{d}_i = \mu \left( m_i - \frac{1}{r} \sum_j m_j \right) - \frac{\mu \Lambda}{\sqrt{2} r}. \quad (3.8)$$

In the limit $m_i \to 0$ the monopole VEV's tend to

$$\langle M^i_a \rangle = \delta^i_a \sqrt{\frac{\mu \Lambda}{\sqrt{2} r}}, \quad i, a = 1, 2, \ldots, r, \quad \langle M^i_{r+a} \rangle = 0, \quad i = r + 1, \ldots, N_f \quad (3.9)$$

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The system is in a color-flavor locked phase of the dual $SU(r)$ gauge theory. The flavor $SU(N_f) \times U(1)$ symmetry of the underlying SQCD is dynamically broken as

$$SU(N_f) \times U(1) \rightarrow U(r) \times U(N_f - r).$$

The fact that $\langle M_i a \rangle$ is nonvanishing in the limit $m_i \rightarrow 0$ means that the symmetry breaking is dynamical, and this property distinguishes the $r$ vacua appearing at these nonbaryonic roots from the vacua at the baryonic root.

Until now we have discussed only the case of degenerate (or slightly unequal) bare masses $m_i$ for the flavors. An important observation is the fact that, when the $m_i$’s are generic, each $r$ vacuum splits in $(N_f r)$ Abelian vacua. Taking into account the Witten effect as in Section 2, it is easy to generalize the argument we gave for $SU(2)$ to $SU(N)$ and conclude, as we will now see, that the particles becoming massless in each one of these vacua are magnetic monopoles and not dyons. Consider a Cartan basis for $SU(N)$:

$$[H_i, H_k] = 0, \quad (i, k = 1, 2, \ldots, r); \quad [H_i, E_\alpha] = \alpha_i E_\alpha; \quad [E_\alpha, E_-\alpha] = \alpha^t H_i;$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad (\alpha + \beta \neq 0).$$

where $\alpha$’s are the root vectors. 3($N-1$) generators can be grouped into $SU(2)$ subsets of generators,

$$[H_i, E_\alpha] = \alpha_i E_\alpha; \quad [E_\alpha, E_-\alpha] = \alpha^t H_i,$n

containing $N-1$ diagonal $U(1)$ generators.

Assuming Abelianization the magnetic monopoles are the ’t Hooft-Polyakov monopoles living in these broken $SU(2)$ groups. Each of the $SU(2)$ group acquires a $\theta$ term,

$$\frac{\theta}{32\pi^2} \sum_{j=1}^3 F_{\mu
u}^j \tilde{F}^{j \mu\nu} = \frac{\theta}{8\pi^2} \sum_{j=1}^3 E^j \cdot B^j,$$}

The $i$-th magnetic monopole contributes to the electromagnetic static energy

$$\frac{\theta}{8\pi^2} \sum_{j=1}^3 E^j \cdot B^j = \frac{\theta}{8\pi^2} (-\nabla\phi) \cdot \nabla^2 g_m \frac{g_m}{r} = \frac{\theta}{8\pi^2} g_m \frac{\nabla^2 g_m}{r} = -\frac{\theta}{2\pi} g_m \phi \delta^3(r).$$

$^1$ The vacua at the baryonic root, present only for $N_f > N$, are interesting as they are characterized by the low-energy effective $SU(\tilde{N})$ gauge group, $\tilde{N} \equiv N_f - N$. It was indeed argued that these might be relevant for the understanding of the Seiberg duality in the $N = 1$ SQCD, and some further observations on this point were made recently. These vacua at the baryonic root are however nonconfining in the limit $m \rightarrow 0, \mu \neq 0$. 

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thus carries the $i$-th “electric” $U(1)$ charge, $-\frac{\theta_i}{2\pi}g_m$. Of course, the monopole of the $i$-th $SU(2)/U(1)$ sector ($i = 1, 2, \ldots N - 1$) is neutral with respect to all other $U(1)$'s.

Under the dynamical hypothesis of Abelianization, thus each $U(1)$ factor has its own Witten effect. The argument made in the $SU(2) \to U(1)$ theories works here too.

In the equal mass limit, the $r$ vacua with their nonAbelian sector are recovered. The massless and light particles in various Abelian vacua now (nontrivially) recombine into multiplets of the $SU(r)$ gauge group. We conclude that these are nonAbelian monopoles and dual gauge fields.

### 3.1 Low-energy excitations and nonAbelian chromoelectric vortices

The obvious low-energy excitations of this system are the massless and light particles described by the effective Lagrangian described above. These can be found by expanding around the vacua (3.5)-(3.9). They contain massless Nambu-Goldstone bosons of the breaking (3.10) and their superpartners, as well as light pseudo Nambu-Goldstone particles of the $SU_R(2)$ breaking. Also, the dual $SU(r) \times U(1)^{N-r}$ gauge bosons and gauginos form light massive multiplets.

What is perhaps not so well known (however, see [26] for related remarks) is the fact that, apart from these elementary excitations, the system described by Eqs.(3.5) and (3.3) has low-energy nonAbelian excitations of a different sort. As $\Pi_1(U(r) \times U(1))^{N-r-1} = \mathbb{Z}^{N-r}$, the low-energy system possesses soliton vortices. In the vacuum (3.9) the minimum vortex configuration (see Eq. (C.1)) breaks the color-flavor diagonal symmetry to $SU(r-1) \times U(1)$: it is a nonAbelian vortex [7]-[15]. The fluctuation of the orientational modes of

$$CP^{r-1} = SU(r)/SU(r-1) \times U(1)$$

is described by a vortex worldsheet sigma model,

$$S_{1+1} = 2\beta \int dt dz \ tr \ \{X^{-1}\partial_\alpha B^\dagger Y^{-1}\partial_\alpha B \}$$

$$= 2\beta \int dt dz \ tr \ \{(1 + B^\dagger B)^{-1} \partial_\alpha B^\dagger (1_{r-1} + BB^\dagger)^{-1} \partial_\alpha B \} ,$$

where $B$, a $r - 1$ component vector, represents the inhomogeneous coordinates of $CP^{r-1}$ (see Eq. (C.6)) and $\beta$ is a constant. The low energy system has also $N - r - 1$ distinct Abelian (Abrikosov-Nielsen-Olesen) vortices, as the dual $U(1)^{N-r-1}$ theory is in the Higgs phase (see Eq. (3.4)).

The point of crucial importance is the fact that the underlying $SU(N)$ theory, being simply connected, does not support a vortex solution. It means that both the nonAbelian vortex (3.17)
and the Abelian vortices of the $U(1)^{N-r-1}$ sectors must end. These vortices in the dual, magnetic theory carry chromoelectric fluxes. The endpoints are quarks of the fundamental theory, which, being relatively nonlocal to the low-energy effective degrees of freedom, and also having dynamical masses of the order of $\Lambda$, are not explicitly visible in the low-energy effective action. The quarks are confined.

The system produces more than one kinds of confining strings as the $SU(N)$ gauge symmetry of the ultraviolet theory is dynamically broken to $U(r) \times U(1)^{N-r-1}$ at low energies; the mesons appear in various Regge trajectories of different slopes. The only exception is the case of $SU(3)$ theory, where the only nontrivial $r$ vacua ($r = 2$) corresponds to a low energy $U(2)$ theory. Since $\Pi_1(U(2)) = \mathbb{Z}$ there is a unique universal Regge trajectory.

4 $r \leftrightarrow N_f - r$ duality

One of the remarkable facts noted in [16], [27] (see also more recent observations [28, 26] on this point) is the fact that semiclassical $r$ vacua (defined at large and equal quark masses $m_i$) are related nontrivially to quantum $r$ vacua. First of all, while there are semi-classical $r$ vacua with $r = 0, 1, 2, \ldots, \min\{N_c - 1, N_f\}$ there are quantum $r$ vacua only up to $r = N_f/2$. This latter fact can be understood as due to the renormalization group behavior of the magnetic monopoles: the dual $SU(r)$ group is only infrared-free as long as $r < N_f/2$. In Ref.[16, 27] the two-to-one correspondence (both $r$ and $N_f - r$ classical vacua flow into quantum $r$ vacua) was suggested by the counting of the number of vacua having the same global symmetries.

Such a correspondence implies that the low energy physics of the classical $r$ and $N_f - r$ vacua is the same. To investigate this point, we need somehow a nonperturbative definition of classical $r$ vacua and this was achieved in [29]. Each vacuum can be obtained placing $N_f$ poles on the $\mathcal{N} = 1$ curve at $z = -m/\sqrt{2}$; some on the first sheet and the others on the second one. If we take the large $m$ limit, it is easy to see that vacua characterized by $r$ poles on the first sheet match precisely with classical $r$ vacua (this is what we mean by non perturbative definition).

In the semiclassical regime (for $m \gg \Lambda$), classical $r$ vacua with $r < \frac{N_f}{2}$ are characterized by a nonAbelian $SU(r)$ gauge group which is infrared free: the Higgs mechanism that breaks $SU(N)$ to $SU(r) \times U(1)^{N-r-1}$ occurs at high energies ($\sim m$), where the coupling costant is very

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2Not all effects related to the underlying quarks are invisible at low energies, however. The zero-energy quark fermion modes are indeed responsible for giving the flavor quantum numbers to the monopoles [20] as in Table 1.

3Of course, as the underlying theory contains scalars in the fundamental representation there are no distinct phases between the confinement and Higgs phase in these theories (complementarity).
small. The theory remains at weak coupling at all energy scales and we do not expect quantum corrections to change this picture dramatically. As we let $m$ decrease, quantum corrections will become relevant at energy scales of order $\sim \Lambda$, but the physics in the infrared will not be modified. In the massless limit we then obtain precisely one of the quantum $r$ vacua described in [24, 16].

If we instead consider a classical $r$ vacuum with $r > \frac{N_f}{2}$, from the equations of motion alone we cannot say much about the behavior at low energies: the theory at energy scales below $m$ is now asymptotically free and the coupling constant grows in the infrared. Our claim is that this theory admits in the infrared a weakly coupled dual description equivalent to the low energy effective action for $r' = N_f - r < \frac{N_f}{2}$ vacua. In order to prove this, we must show that the SW curve factorizes in the same way for $r$ and $N_f - r$ classical vacua, as we will now explain. In [24] it was shown that the vacua which are not lifted by the $\mathcal{N} = 1$ perturbation $\mu \langle \text{Tr} \Phi \rangle^2$ are all the points in the moduli space such that the SW curve factorizes as

$$y^2 = (x + m)^{2r} Q_N^2 \left( \frac{x - a}{z} \right) (x - \alpha) (x - \beta), \quad r \leq \frac{N_f}{2},$$

and in such a vacuum the effective low energy theory includes an Abelian $U(1)^{N - r - 1}$ sector and a nonAbelian one which is an infrared free $U(r)$ gauge theory with $N_f$ massless flavors. From the above discussion it is clear that classical $r$ vacua with $r < \frac{N_f}{2}$ fall in this class. In order to prove our claim, we thus need to show that the SW curve factorizes as above both for classical $r$ and $N_f - r$ vacua.

In order to determine the form of the SW curve at classical $r$ vacua, our starting point will be the equation relating the SW curve and the chiral condensates [30]:

$$P_N(z) = z^N e^{-\sum_i U_i^2 |z|^2} + \Lambda^{2N - N_f} \left( \frac{z + m}{z^N} \right)^{N_f} e^{\sum_i U_i^2 |z|^2},$$

where $U_i$ are the vacuum expectation values $U_i = \frac{1}{N} \langle \text{Tr} \Phi^i \rangle$ and the symbol $\ldots |^\gamma$ indicates that only terms with nonnegative powers of $z$ are kept ($P_N(z)$ is thus a polynomial). These in turn can be computed from the generalized Konishi anomaly relations [30] adapted to the case of $SU(N)$ SQCD with the $\mathcal{N} = 1$ adjoint scalar mass term $\mu \langle \text{Tr} \Phi^2 \rangle$ [29]:

$$\langle \text{Tr} \frac{1}{z - \Phi} \rangle = \sum_{i \geq 0} \langle \text{Tr} \Phi^i \rangle \frac{N_f - 2r}{2} \frac{\sqrt{\mu^2 (a + m)^2 - 4S \mu}}{(z + m) \sqrt{\mu^2 (z - a)^2 - 4S \mu}} + \frac{N_f / 2}{z + m} + \frac{\mu (N - N_f / 2)}{\sqrt{\mu^2 (z - a)^2 - 4S \mu}},$$

where $S$ and $a$ are the gaugino and meson condensates

$$S = \frac{g^2}{32 \pi^2} \langle W^\alpha W_\alpha \rangle; \quad a = \frac{\sqrt{2}}{N \mu} \langle \hat{Q}_i Q_i \rangle.$$
These can be determined from the Dijkgraaf-Vafa superpotential. The corresponding equations, determining at once $S$ and $a$ both for $r$ and $N_f - r$ classical vacua, are

\[
\left( \frac{N - r}{N_f - 2r} a - \frac{r}{N_f - 2r} \right)^{N-r} \left( \frac{N_f - N - r}{N_f - 2r} a + \frac{N_f - r}{N_f - 2r} \right)^{N+r-N_f} = \Lambda^{2N-N_f},
\]

\[
S = \mu \left( \frac{N - r}{N_f - 2r} a - \frac{r}{N_f - 2r} \right) \left( \frac{N_f - N - r}{N_f - 2r} a + \frac{N_f - r}{N_f - 2r} \right).
\]

Of the $2N - N_f$ solutions $N - r$ describe classical $r$ vacua and the remaining $N + r - N_f$ correspond to $N_f - r$ vacua. We can distinguish the two groups of solutions exploiting the fact that $S$ tends to infinity in the large $m$ limit only for classical $r$ vacua with $r < N_f/2$.

Solving these equations in general is very hard and we will not attempt to do so. However, in the massless limit they greatly simplify making it possible to check our claim. In the massless case the equation for $a$ can be easily solved, leading to the $2N - N_f$ solutions

\[ a = \text{const.} \omega_{2N-N_f}^k \Lambda; \quad k = 1, \ldots, 2N - N_f, \]

where $\omega_{2N-N_f}$ is the $2N - N_f$-th root of unity. If we consider two roots $a$ and $a'$ such that $a' = \omega_{2N-N_f}^j a$ for some integer $j$, we have from Eq. (4.3)

\[
\sum_{i \geq 0} \frac{\langle \text{Tr} \Phi^i \rangle(a')}{(z')^{i+1}} = \frac{1}{\omega_{2N-N_f}^j} \sum_{i \geq 0} \frac{\langle \text{Tr} \Phi^i \rangle(a)}{z^{i+1}}, \quad \text{or} \quad \sum_i U_i(a') = \sum_i U_i(a),
\]

where we have defined $z' = \omega_{2N-N_f}^j z$. Eq. (4.2) tells then that the SW curve factorizes in the same way in all $N - r$ vacua of a given $r$, and in all vacua with $r' = N_f - r$. Since we know that in all $r$ vacua with $r < N_f/2$ the low energy physics can be described as an infrared free SQCD with $r$ colors for any value of $m$ (see the above discussion), we know from [24] that the SW curve factorizes precisely as in (4.1). This proves our claim.

We have also checked the above relations, for generic $m$, in the case of $SU(5)$ theory with $N_f = 6$. We have verified by solving Eqs. (4.2)-(4.5) with Mathematica that both $r = 2$ and $r = 4$ classical vacua are described by the identical singularity of the SW curve

\[ y^2 = P_N(z)^2 - 4\Lambda^{2N-N_f}(z + m)^{N_f} \sim (z + m)^4, \]

corresponding to the quantum $r = 2$ theory of Section 3. The expression for $P_5(x)$ from equation (4.2) is rather lengthy and we shall not write it. However, there are two limiting cases worth

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As explained in [29], the equations obtained by extremizing the DV superpotential can be brought in this form only for $r \geq N_f - N$. This will be enough for our purpose of discussing the $r \leftrightarrow N_f - r$ correspondence. The equations in [29] look slightly different from the ones given here. This is due to a different normalization used: if $m'$ and $\Lambda'$ denote the parameters used in [29] the following relations hold: $m = m'/\sqrt{2}$ and $\Lambda^{2N-N_f} = \sqrt{2}^{N_f}(\Lambda')^{2N-N_f}$.
mentioning: the massless and the semiclassical ($\Lambda \to 0$) one. In the massless limit we get the following four solutions for $P_5(x)$:

\[ x^5 - \sqrt{\frac{4}{3}} i\Lambda^2 x^3 \pm \frac{16}{9} \left( -\frac{1}{3} \right)^{1/4} \Lambda^3 x^2, \]
\[ x^5 + \sqrt{\frac{4}{3}} i\Lambda^2 x^3 \pm \frac{16}{9} \left( -\frac{1}{3} \right)^{1/4} i\Lambda^3 x^2. \]

One can easily check that the SW curve satisfies the factorization condition \((4.1)\) with $r = 2$. In the $\Lambda \to 0$ limit we expect instead to recover the semiclassical result: three $r = 2$ classical vacua ($-m$ is a root of $P_5(x)$ with multiplicity two) and one $r = 4$ vacuum ($-m$ is a root of $P_5(x)$ with multiplicity four). Defining $z = x + m$ we find in fact the four solutions

\[ \left( z - \frac{5}{3}m \right)^3 z^2, \quad \left( z - \frac{5}{3}m \right)^3 z^2, \quad \left( z - \frac{5}{3}m \right)^3 z^2, \quad (z - 5m)z^4. \]

Notice that this result is obtained discarding all subleading terms (higher orders in $\Lambda/m$); in the exact solution all the polynomials are divisible just by $z^2$. The point is that the coefficients for the cubic and quadratic terms in $z$ for the fourth polynomial are negligible in this limit.

5 Singular points and colliding $r$ Vacua

The physics of the local $r$ vacua represents a beautiful example of confining vacuum which is dual Higgs system of nonAbelian variety. But perhaps even more interesting is the situation in which singular SCFT’s recently studied are deformed by an $\mathcal{N} = 1$ adjoint scalar mass term $\mu \text{Tr}\Phi^2$. In this section we discuss a few examples of the vacua of this type.

5.1 SCFT point in the massless $USp(2N)$ theory and the Gaiotto, Seiberg, Tachikawa (GST) dual

It was pointed out in \cite{31} that in the massless limit of $\mathcal{N} = 2$, $USp(2N)$ theory with $N_f = 2n$ matter hypermultiplets new SCFTs emerge, different from those seen in $SU(N)$ theory. Note that in the massive, equal mass ($m_i = m \neq 0$) theory, the SCFT vacua occurring in the $USp(2N)$ theory are the same $r$ vacua of $SU(N)$ theory, $r = 0, 1, 2, \ldots, N_f/2$, exemplifying the universality of SCFTs. In the $m_i \to 0$ limit, however, the $r$ vacua collapse into a singular SCFT (“Tcheby- shev” point) \cite{10} with a larger global symmetry $SO(2N_f)$. A recent study \cite{28} by one of us, done following closely the analysis by Gaiotto, Seiberg, Tachikawa \cite{2}, has shown that the relevant
SCFT can be analyzed by introducing two different scalings for the scalar VEVs $u_i \equiv \langle \Phi^i \rangle$ (the Coulomb branch coordinates) around the singular point. The infrared physics of this system is

(i) $U(1)^{N-n}$ Abelian sector, with massless particles charged under each $U(1)$ subgroup.

(ii) The (in general, non-Lagrangian) A sector with global symmetry $SU(2) \times SO(4n)$.

(iii) The B sector is free and describes a doublet of hypermultiplets. The flavor symmetry of this system is $SU(2)$.

(iv) $SU(2)$ gauge fields coupled weakly to the last two sectors.

For general $N_f$ these involve non-Lagrangian SCFT theories, and it is not easy to analyze the effects of $\mu \text{Tr} \Phi^2$ deformation. In a particular case $n = 2$ ($USp(2N)$ theory with $N_f = 4$), however, the A sector becomes free and describes four doublets of $SU(2)$. Let us consider the effect of $\mu \Phi^2$ deformation of this particular theory focusing on the nonAbelian sector. The superpotential for a hypermultiplet $Q_0$ and four hypermultiplets $Q_i$’s, coupled to $SU(2) \times U(1)$ gauge fields (only $Q_0$ carrying the $U(1)$ charge) is

$$Q_0 A_D \tilde{Q}^0 + Q_0 \phi \tilde{Q}^0 + \sum_{i=1}^{4} Q_i \phi \tilde{Q}^i + \mu A_D \Lambda + \mu \text{Tr} \phi^2 . \quad (5.1)$$

The vacuum equations are

$$Q_0 \tilde{Q}^0 + \mu \Lambda = 0 ; \quad (5.2)$$

$$(\phi + A_D) \tilde{Q}^0 = Q_0 (\phi + A_D) = 0 ; \quad (5.3)$$

$$\frac{1}{2} \sum_{i=1}^{4} Q_i \tilde{Q}_i - \frac{1}{4} (Q_i \tilde{Q}^i) \delta^a_i + \frac{1}{2} Q_0^a \tilde{Q}_0 - \frac{1}{4} (Q_0 \tilde{Q}^0) \delta^a_i + \mu \phi^a_i = 0 ; \quad (5.4)$$

$$\phi \tilde{Q}^i = Q_i \phi = 0 , \quad \forall i . \quad (5.5)$$

The first tells that $Q_0 \neq 0$. By gauge choice

$$Q_0^1 = \tilde{Q}^{01} = \sqrt{-\mu \Lambda} \neq 0 ; \quad Q_0^2 = \tilde{Q}^{02} = 0 . \quad (5.6)$$

A solution with

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \neq 0$$
would necessarily imply
\[ Q_i = \tilde{Q}_i = 0 ; \quad \forall i , \quad A_D = -a , \quad a = \frac{\Lambda}{4} , \]
(5.7)
Such a solution involves a fluctuation (\( \sim \Lambda \)) beyond the validity of the effective action: it is an artefact of the low-energy effective action and must be disregarded. We must therefore choose
\[ \phi = 0 , \quad A_D = 0 . \]
(5.8)
The contribution from \( Q_i \)'s must then cancel that of \( Q_0 \) in Eq. (5.4). By flavor rotation the nonzero VEV can be attributed to \( Q_1, \tilde{Q}_1 \), i.e., either of the form
\[ (Q_1)_1^1 = (\tilde{Q}_1)_1^1 = \sqrt{\mu \Lambda} , \quad Q_i = \tilde{Q}_i = 0 , \quad i = 2, 3, 4. \]
(5.9)
or
\[ (Q_1)_2^2 = (\tilde{Q}_1)_2^2 = -\sqrt{\mu \Lambda} , \quad Q_i = \tilde{Q}_i = 0 , \quad i = 2, 3, 4. \]
(5.10)
The vacuum is thus given by Eq. (5.6), Eq. (5.8) and Eq. (5.9) or Eq. (5.10). This means that the flavor symmetry is broken as
\[ SO(8) \rightarrow U(1) \times SO(6) = U(1) \times SU(4), \]
(5.11)
where the \( U(1) \) factor is the color-flavor diagonal \( SO(2) \) left unbroken by Eq. (5.9) or Eq. (5.10). This pattern of symmetry breaking is precisely what is expected from the result known at large \( \mu \gg \Lambda \) [16], showing the consistency of the whole picture.

5.2 Singular points in \( SU(N) \) SQCD

In this section we study two types of singular points in \( SU(N) \) SQCD (with an even number \( N_f = 2n \) of flavors) which are relevant for the breaking to \( N = 1 \). In the first case the singular points arise from the collision of different \( r \) vacua. Dynamical flavor symmetry breaking does not occur in this case. The second class of singular points arise in a “degeneration limit” of \( r = n \) vacua, in which the SW curve becomes more singular. Vacua with different \( r \) are not involved in this case and the pattern of flavor symmetry breaking remains to be \( U(N_f) \rightarrow U(n) \times U(n) \).

5.2.1 Colliding \( r \)-vacua of the \( SU(N) \) SQCD

It has been noted recently [29, 28] that when the (equal) quark mass parameter is fine-tuned to a particular value of the order of \( \Lambda \),
\[ m = m^* \equiv \omega^k \frac{2N - N_f}{N} \Lambda , \quad (k = 1, \ldots, 2N - N_f , \quad \omega^{2N-N_f} = 1) , \]
(5.12)
all the $r$-vacua with $r = 0, 1, \ldots, \frac{N_f}{2}$ (more precisely, one representative from each $r$ vacua) coalesce to form a single vacuum

$$y^2 \sim (x + m^*)^{N_f + 1}.$$ (5.13)

This corresponds to the SCFT of the highest criticality \[31\] (called EHIY point in \[2\]) for

$$N = n + 1 \quad (N_F = 2n).$$ (5.14)

Now the EHIY points in general $SU(N)$ theories with $N_F = 2n$ flavors have recently been reanalyzed by Gaiotto, Seiberg and Tachikawa \[2\]. According to these authors, the low-energy system is an infrared free $SU(2)$ gauge theory, weakly coupled to two separate SCFT’s. One (the A sector) is a strongly coupled (in general, without local Lagrangian description) theory with $SU(2) \times SU(N_f)$ flavor symmetry, the other (the sector B) is the most singular superconformal point of $SU(N - n + 1)$ theory with two flavors, with $SU(2) \times U(1)$ flavor symmetry. The diagonal combination of the $SU(2)$ flavors is weakly gauged. It is easy to see that the low energy physics at the singular point of interest for us can be described as \[28\]

(i) A $U(1)^{N-n-1}$ Abelian sector, with massless particles charged under each $U(1)$ subgroup.

(ii) The (in general, non-Lagrangian) A sector with global symmetry $SU(2) \times SU(N_f)$.

(iii) The B sector is the most singular superconformal point of $SU(2)$ theory with two flavors (or the $D_3$ Argyres-Douglas theory), with $SU(2) \times U(1)$ flavor symmetry.

(iv) $SU(2)$ gauge fields coupled to the last two sectors.

The presence of the $\mu \Phi^2$ term, breaking $SU_R(2)$ explicitly, is expected to generate nonvanishing gaugino condensate through anomaly, and induce the symmetry breaking

$$\mathbb{Z}_{2N-N_f} \rightarrow \mathbb{Z}_2.$$ (5.15)

We are not able to deduce such a result directly with the GST dual description. However, what happens in the colliding $r$ vacua in general $SU(N)$ gauge theories with even $N_f$ flavors, perturbed by the adjoint mass term, can be exactly determined by the generalized Konishi anomaly relations \[30\] and by use of the Dijkgraaf-Vafa superpotential. The analysis of \[29\] shows that the meson and gaugino condensates are of the form (see Eq. (4.4) and Eq. (4.5))

$$\langle \tilde{Q}^i Q_i \rangle \sim \mu \Lambda, \quad \text{(indep. of } i \text{)}; \quad \langle \lambda \lambda \rangle \sim \mu \Lambda \neq 0.$$ (5.16)

The $SU(N_f) \times U(1)$ symmetry remains unbroken, in contrast to what happens (for generic $m$) in single $r$ vacua, \[31\].

In \[32\] an analogous phenomenon was studied, but by using an appropriate $\mathcal{N} = 1$ superpotential $W(\Phi)$ and selecting particular vacua.
5.2.2 Higher order singularity

The vacuum arising from the collision of $r$ vacua is not the only higher singular point in softly broken $\mathcal{N} = 2 \, SU(N)$ SQCD. For illustration let us consider the simplest example, namely $SU(4)$ theory with $N_f = 4$: the SW curve is

$$y^2 = (x^4 - u_2 x^2 - u_3 x - u_4)^2 - 4\Lambda^4 (x + m)^4.$$ 

In this case the $r = 2$ vacuum can be found easily and the curve assumes the form

$$y^2 = (x + m)^4(x - m)^2(x - m - 2\Lambda)(x - m + 2\Lambda).$$

From here we easily see that when $m = \pm\Lambda$ the curve can be approximated as $y^2 \approx (x + m)^5$ and we recover the case studied before. On the other hand, in the limit $m = 0$ the curve becomes more singular and reduces to $y^2 \approx x^6$. Of course, this singular point exists in the general case, as long as $n < N - 1$ (this was already noticed in [16]): in an $r = \frac{N_f}{2}$ vacuum the SW curve assumes the form

$$y^2 = (x + m)^{N_f} Q^2_{N-n-1} (x^2 - \alpha)(x - \beta).$$

The roots of $Q_{N-n-1}$ have multiplicity one and are located at (see [16])

$$x = \frac{N_f}{2N - N_f} m + 2\Lambda \cos \left( \frac{2k\pi}{2N - N_f} \right); \quad k = 1, \ldots, N - \frac{N_f}{2} - 1. \quad (5.17)$$

When the bare mass is chosen in such a way that $-m$ coincides with one of these roots the SW curve can be approximated as $y^2 \approx (x + m)^{N_f+2}$.

From the analysis performed in [2] we can conclude that the low energy physics at this singular point can be described as follows:

(i) A $U(1)^{N-n-2}$ Abelian sector, with massless particles charged under each $U(1)$ subgroup.

(ii) The A sector with global symmetry $SU(2) \times SU(N_f)$ described in the previous section.

(iii) The B sector is the most singular point of $SU(3)$ theory with two flavors (or the $D_4$ Argyres-Douglas theory), with $SU(2) \times U(1)$ flavor symmetry.

(iv) $SU(2)$ gauge fields coupled to the last two sectors.

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6Actually, it was recently shown in [33] that in this case the flavor symmetry enhances to $SU(3)$. However, an $SU(2)$ subgroup is gauged and the manifest flavor symmetry is the commutant of $SU(2)$ inside $SU(3)$, which is $U(1)$.
From the analysis performed in [29] it is easy to see that the pattern of symmetry breaking (once the superpotential for the adjoint field is turned on) is $U(N_f) \rightarrow U(N_f/2) \times U(N_f/2)$, the same as in the $r = N_f/2$ vacuum. In contrast to the case discussed in the previous paragraph 5.2.1 this vacuum does not arise from the coalescence of different $r$ vacua.

5.2.3 Breaking to $\mathcal{N} = 1$ in the singular vacua

In this section we wish to test our proposal for the low-energy effective description at the singular points by reproducing the correct pattern of flavor symmetry breaking occurring once the $\mu \text{Tr}\Phi^2$ perturbation is turned on.

As we have seen, for generic $m$ the most singular point in the moduli space is the $r = N_f/2$ vacuum $\mathcal{I}$ (in which the SW curve can be approximated as $y^2 \approx (x + m)^{N_f}$). Its low-energy physics involves a scale invariant sector with $SU(N_f/2)$ gauge group, whose coupling constant depends on $m$. For special values $m^*$ of order $\Lambda$ (or zero) the SW curve degenerates further ($y^2 \approx (x + m)^{N_f+1}$ or $y^2 \approx (x + m)^{N_f+2}$, as we have seen above). It is easy to see that, as we approach these critical values the coupling constant of the $SU(N_f/2)$ theory diverges [34]. In this limit it is convenient to adopt the Argyres-Seiberg dual description [1] in which an $SU(2)$ gauge group emerges, coupled to a hypermultiplet in the doublet (B sector) and to a strongly coupled interacting sector which coincides precisely with the A sector introduced above.

In order to describe the low energy physics at the most singular point in a neighbourhood of the critical values $m^*$, it is thus convenient to introduce these two sectors. As we approach the critical value $m^*$ the curve becomes more singular and the B sector, which is free in the $r$ vacuum, becomes interacting ($D_3$ or $D_4$ Argyres-Douglas theory for the two classes of singular points we are interested in). In the process the A sector is just a spectator.

Finding the effective low energy description at these singular points once the $\mu \text{Tr}\Phi^2$ term is turned on is in general rather difficult. However, in the $N_f = 4$ case the problem is greatly simplified, since the A sector is free. For $m$ close to $m^*$ the low energy physics in the $r = 2$ vacuum admits a Lagrangian description analogous to (5.1). The only difference is that the A sector describes three doublets of $SU(2)$ instead of four. Imposing the F-term equations we find as before a non vanishing condensate for the $Q_0$ and $Q_1$ fields (we use the same notation as in (5.1)), reproducing the correct pattern of symmetry breaking:

$$U(1) \times SO(6) \rightarrow U(1) \times U(1) \times SO(4) \simeq U(2) \times U(2).$$

\[7\] As is well known, the curve can become even more singular. However, such points are not relevant from the point of view of the breaking to $\mathcal{N} = 1$. 

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Clearly, if the condensate for $Q_0$ vanishes, the one for $Q_1$ vanishes as well, restoring the full $U(N_f)$ flavor symmetry of the theory. The $Q_0$ condensate can be determined focusing on the B sector only, which is the most singular point in $SU(2)$ or $SU(3)$ theory with two flavors in the cases of interest for us. The problem is thus reduced to computing the Abelian condensates in $SU(2)$ or $SU(3)$ theories. The result of the direct analysis (see Appendix D) is that the $Q_0$ condensate vanishes in the $SU(2)$ case but not in the $SU(3)$ one, reproducing the expected pattern of flavor symmetry breaking discussed in the two paragraphs 5.2.1 and 5.2.2 respectively.

6 Discussion

Altogether, what we learned from the softly broken $\mathcal{N} = 2$ SQCD, with quark supermultiplets in the fundamental representation, supports a fairly standard picture of confinement: magnetically charged particles condense, leading to quark confinement [35]. The details of how this occurs, however, depends on the parameters and particular vacuum considered in various ways, and as soon as we stray away from the well understood cases of Abelian-dual-superconductor systems, we appear suddenly to find ourselves in a foreign land full of unfamiliar phenomena.

The physics of $r$ vacua for small masses $m$ in the $SU(N)$ SQCD is characterized by the (dual-) color-flavor locking condensates of nonAbelian magnetic monopoles. The dual gauge symmetry is completely broken (confinement), at the same time leaving the global $SU(r)$ symmetry intact. This echoes the properties of the color-flavor locked vacuum found at the $r$ (or $N_f - r$) semiclassical singularity, for large $m, \mu$, with the same global symmetry. Such a matching is absolutely indispensable to realize a complete Higgsing of the dual gauge symmetry - confinement - yet keeping the same global symmetry as the corresponding semiclassical theory in Higgs phase. The flow from the semiclassical (large $m \gg \mu \gg \Lambda$) to quantum ($m \sim \mu \ll \Lambda$) regions is smooth as our theory involves scalars in the fundamental representation (another way of seeing it is the fact that the $\mathcal{N} = 1$ supersymmetry maintained throughout guarantees a holomorphic dependence on $m$ and $\mu$ so that no discontinuous change of physics is possible between the two regions).

The “nonAbelian vortex” carrying the $CP^{r-1}$ moduli fluctuations in the quantum $r$ vacuum, Eq. (3.17), is nothing but the fluctuations of the chromoelectric, confining string flux: the gauge symmetry is dynamically broken to $SU(r) \times U(1)^{N_f - r}$. The system thus presents a rare instance in which the properties of the confining string can be analytically studied, thanks to the existence of a weakly coupled dual (but still local) description.

A more intriguing situation occurs when the bare mass is tuned to a special value of the
order of $\Lambda$, $m \to m^*$. For the particular choice of the critical mass, Eq. (5.12), all the $r$ vacua ($r = 0, 1, \ldots, N_f/2$) collide to form a single vacuum, which corresponds to one of the singular SCFT which have been given much attention recently, starting from the work by Argyres and Seiberg. The change of the global symmetry in the limit of coalescing vacua, from $U(r) \times U(N_f-r)$ to $U(N_f)$, indicates a different set of condensates inducing confinement, i.e., a new mechanism of confinement. The system is now described as a deformation of a singular, non-Langrangian SCFT; the direct analysis of this phenomenon in terms of the low-energy degrees of freedom, is still beyond our ability, even though the exact answer on the symmetry breaking pattern can be inferred from the $\mathcal{N} = 1$ side, via the generalized Konishi anomaly and the Dijkgraaf-Vafa superpotential [29].

For a still another choice of the critical mass, Eq. (5.17), the highest $r$ vacua, with $r = \frac{N_f}{2}$ becomes more singular, without however colliding with other lower $r$ vacua. In this case the pattern of symmetry breaking remains to be $U\left(\frac{N_f}{2}\right) \times U\left(\frac{N_f}{2}\right)$.

We have given, for particular cases of $N_f = 4$ theories, justifications for these results [16 28] by using the direct analysis of the singular vacua [1 2].

Finally, in $USp(2N)$ theory the collision of the $r$ vacua occurs when the bare quark mass approaches zero, when the global symmetry of the underlying theory gets enhanced from $SU(N_f) \times U(1)$ to $SO(2N_f)$. A study of the corresponding semi-classical system (at large $\mu \gg \Lambda$) shows that the global symmetry is broken to $U(N_F)$, indicating a new set of condensates forming in the limit. In the case of $USp(2N)$ theory with $N_f = 4$, where the GST dual becomes sets of free hypermultiplets weakly coupled to $SU(2)$ gauge fields, we have been able to reproduce such a result directly from the GST dual description, showing the consistency of our reasonings.

To conclude, a proper description of the conformal theories appearing as the infrared fixed-point theories as found recently by Seiberg et. al. [1 2], is indispensable for the understanding of confinement, because in systems discussed here the latter occurs as the result of small deformation of conformally invariant systems. This means that the nature of the degrees of freedom and their interactions in the conformal limit determine how confinement is induced by the deformation.

A series of papers by Shifman and Yung have discussed many related issues in $U(N)$ theories [26].

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The superpotential has the form
\[ W = \mu \text{Tr} \Phi^2 + \sqrt{2} \tilde{Q}_a^i \Phi^b Q^i_b + m_i \tilde{Q}_a^i Q^i_a. \] (A.1)

The vacuum equations read
\[ [\Phi, \Phi^\dagger] = 0 ; \] (A.2)
\[ \nu \delta^b_a = Q^i_a (Q^\dagger)^b_i - (\tilde{Q}^i)^i_a \tilde{Q}^b_i ; \] (A.3)
\[ Q^i_a \tilde{Q}^b_i - \frac{1}{N} \delta^b_a (Q^c)^i_c + \sqrt{2} \mu \Phi^b_a = 0 ; \] (A.4)
\[ Q^i_a m_i + \sqrt{2} \Phi^b_a Q^i_b = 0 \quad \text{(no sum over } i) ; \] (A.5)
\[ m_i \tilde{Q}^a_i + \sqrt{2} \tilde{Q}^b_i \Phi^b_a = 0 \quad \text{(no sum over } i) . \] (A.6)

By gauge rotation \( \Phi \) can be taken as
\[ \Phi = \text{diag} (\phi_1, \phi_2, \ldots, \phi_N) , \quad \sum \phi_a = 0 . \] (A.7)

\( Q^i_a \) and \( \tilde{Q}^b_i \) are either nontrivial eigenvectors of the matrix \( \Phi \) with possible eigenvalues \( m_i \), or null vectors. The solution with eigenvalues \( m_1, m_2, \ldots, m_r \) is
\[ \Phi = \frac{1}{\sqrt{2}} \text{diag} (-m_1, -m_2, \ldots, -m_r, c, \ldots, c) ; \quad c = \frac{1}{N - r} \sum_{k=1}^{r} m_k . \] (A.8)

\[ Q^i_a = \begin{pmatrix} f_1 & & & \\ & \ddots & & \\ & & f_r & \\ 0 & & & 0 \end{pmatrix} , \quad \tilde{Q}^a_i = \begin{pmatrix} \tilde{f}_1 & & & \\ & \ddots & & \\ & & \tilde{f}_r & \\ 0 & & & 0 \end{pmatrix} . \] (A.9)
where
\[ r = 0, 1, \ldots, \min \{N_f, N - 1\}, \]  
(A.10)
The solution for \( f_i, \tilde{f}_i \) is (see [16] for more details)
\[ f_i \tilde{f}_i = \mu m_i + \frac{1}{N - r} \mu \sum_{k=1}^{r} m_k, \quad f_i^2 = |\tilde{f}_i|^2, \quad (f_i > 0). \]  
(A.11)
The number of the quark flavors “used” to make solutions define various classical \( r \)-vacua. As the solution with a given \( r \) leaves a local \( SU(N - r) \) invariance it counts as a set of \( N - r \) solutions (Witten’s index). In all there are precisely
\[ \mathcal{N} = \sum_{r=0}^{\min \{N_f, N-1\}} (N - r) \binom{N_f}{r} \]  
(A.12)
classical solutions for generic \( m_i \)’s and \( \mu \neq 0 \).

**B Equations determining VEVs in the quantum \( r \) vacua**

The D-tem potential gives
\[ 0 = [\phi, \phi^\dagger]; \]  
(B.1)

\[ \nu \delta_a^b = q_i^a (q^\dagger_i)^b - (\tilde{q}^\dagger_i)^a q_i^b; \]  
(B.2)

\[ 0 = q_i^a (q^\dagger_i)^a - (\tilde{q}^\dagger_i)^a q_i^a; \]  
(B.3)

while the F-term equations are
\[ q_i^a \tilde{q}_i^b - \frac{1}{r} \delta_a^b (q^\dagger_i \tilde{q}^c_i) + \sqrt{2} \mu \phi_a^b = 0; \]  
(B.4)

\[ 0 = \sqrt{2} \phi_a^b \tilde{q}_i^b + q_i^a (m_i + \sqrt{2} a_{D0}); \quad \text{(no sum over } i, \ a) \]  
(B.5)

\[ 0 = \sqrt{2} \tilde{q}_i^b \phi_a^b + (m_i + \sqrt{2} a_{D0}) \tilde{q}_i^b \quad \text{(no sum over } i, \ a). \]  
(B.6)

\[ \sqrt{2} \text{Tr}(q\tilde{q}) + \mu \Lambda = 0. \]  
(B.7)
The \( SU(r) \) adjoint scalars can be diagonalized by color rotations,
\[ \text{diag } \phi = (\phi_1, \phi_2, \ldots, \phi_r), \quad \sum \phi_a = 0. \]  
(B.8)
C Non-Abelian vortex in the $r$ vacua

\[ \mathcal{M} = \begin{pmatrix} e^{i\theta} \phi_1(\rho) & 0 \\ 0 & e^{i\theta} \phi_2(\rho) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{e^{i\theta} \phi_1(\rho) + \phi_2(\rho)}{2} \begin{pmatrix} 1 & \frac{e^{i\theta} \phi_1(\rho) - \phi_2(\rho)}{2} \end{pmatrix}, \]

which is oriented to a specific direction. In (C.1)

\[ T \equiv \text{diag}(1, -1) \]

and $z, \rho, \theta$ are cylindrical coordinates. The profile functions $\phi_{1,2}(\rho), f(\rho), f_{NA}(\rho)$ satisfy the boundary conditions

\[ \phi_{1,2}(\infty) = \frac{v}{\sqrt{2N}}, \quad f(\infty) = f_{NA}(\infty) = 0, \quad \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{NA}(0) = 1. \]

The vortex oriented in a generic direction in color-flavor space can be written as

\[ \mathcal{M} = U \begin{pmatrix} \phi_1(\rho) & 0 \\ 0 & \phi_2(\rho) \end{pmatrix} U^{-1} = \frac{\phi_1(\rho) + \phi_2(\rho)}{2} \begin{pmatrix} 1 & \frac{\phi_1(\rho \rho) - \phi_2(\rho)}{2} \end{pmatrix} UTU^{-1}, \]

\[ A_i = -\frac{1}{2} \xi_{ij} \frac{x^j}{r^2} \left[(1 - f(\rho)) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (1 - f_{NA}(\rho)) T \right], \]

where

\[ T \equiv \text{diag}(1, -1) \]

The matrix $U$ represents the coset

\[ SU(r)/SU(r-1) \times U(1) \sim CP^{r-1}, \]

and is expressed in terms of an $r - 1$ dimensional complex vector $B$ as

\[ U = \begin{pmatrix} 1 & -B^\dagger \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & Y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} = \begin{pmatrix} X^{-\frac{1}{2}} & -B^\dagger Y^{-\frac{1}{2}} \\ BX^{-\frac{1}{2}} & Y^{-\frac{1}{2}} \end{pmatrix}, \]

where the matrices $X$ and $Y$ are defined by

\[ X \equiv 1 + B^\dagger B, \quad Y \equiv 1_{r-1} + BB^\dagger, \]

This form of the unitary $SU(r)$ matrices containing only the coset coordinates $B$ is known as the reducing matrix.
D  Monopole condensates in $SU(2)$ and $SU(3)$ $N_f = 2$ theories

The SW curve for the $SU(2)$ theory with two flavors can be written as

$$y^2 = (x^2 - u)^2 - 4\Lambda^2(x + m)^2,$$

and if we set $u = m^2$ it degenerates to

$$y^2 = (x + m)^2(x - m - 2\Lambda)(x - m + 2\Lambda).$$  \(\text{(D.1)}\)

The low energy physics at this point is described by an Abelian $U(1)$ theory with two massless electrons and when we turn on the $\mathcal{N} = 1$ deformation $\mu \text{Tr}\Phi^2$, the corresponding effective action includes the superpotential

$$\sqrt{2} \tilde{Q}_1 A Q_1 + \sqrt{2} \tilde{Q}_2 A Q_2 + \mu U; \quad U \equiv \langle \text{Tr}\Phi^2 \rangle.$$

The equations of motion thus impose the constraint

$$\langle \tilde{Q}_1 Q_1 \rangle + \langle \tilde{Q}_2 Q_2 \rangle = -\frac{\mu}{\sqrt{2}} \frac{\partial U}{\partial A}.$$

In order to compute the condensate we now have to evaluate $\partial U / \partial A$. This can be done noticing that

$$\frac{\partial U}{\partial A}^{-1} = \frac{\partial A}{\partial U} = \int_{\gamma} \frac{\partial \lambda}{\partial U},$$

where the contour $\gamma$ is a small circle surrounding the point $x = -m$. We can now exploit the fact that the SW differential for $SU(N)$ SQCD satisfies the relation [19]

$$\frac{\partial \lambda}{\partial U} = \frac{dx}{y} x^{N-2}.$$

From (D.1) we then obtain

$$\frac{\partial U}{\partial A} \propto \sqrt{(\Lambda + m)(\Lambda - m)}.$$

This quantity vanishes for $m = \pm \Lambda$, which are precisely the values such that the SW curve degenerates further and we encounter the $D_3$ Argyres-Douglas point. This shows that the $Q_0$ condensate (in the notation of 5.1) vanishes at this point.

The computation for $SU(3)$ is similar: the SW curve in this case is

$$y^2 = (x^3 - U x - V)^2 - 4\Lambda^4(x + m)^2.$$
and setting $U = 2\Lambda^2 + \frac{3}{4}m^2$, $V = 2m\Lambda^2 - \frac{m^3}{4}$ we reach the $r = 1$ vacuum, the point we are looking for. The SW curve at this point factorizes as

$$y^2 = (x + m)^2(x - \frac{m}{2})^2(x^2 - mx + \frac{m^2}{4} - 4\Lambda^2),$$

(D.2)

and the low energy effective action describes an Abelian $U(1)^2$ theory with two massless hypermultiplets charged under one $U(1)$ factor and another hypermultiplet charged under the second one. In the $m \to 0$ limit the curve degenerates further and we find the maximally singular point. In order to find the $Q_0$ condensate we have to evaluate as before

$$\frac{\partial A}{\partial U} = \int_\gamma \frac{\partial \lambda}{\partial U},$$

where the contour $\gamma$ is again a circle around the point $x = -m$. The crucial difference with respect to the $SU(2)$ case is the fact that now

$$\frac{\partial \lambda}{\partial U} = \frac{x dx}{y},$$

leading to the relation

$$\frac{\partial A}{\partial U} \propto \left( \sqrt{4\Lambda^2 - \frac{9}{4}m^2} \right)^{-1}.$$ 

This quantity remains finite in the $m \to 0$ limit (notice that $\partial A/\partial V$ diverges instead). The computation of $\partial U/\partial A$, the quantity we are interested in, is slightly more delicate with respect to the $SU(2)$ case, since the Coulomb branch has now complex dimension two and the $A$ cycle is a function of both $U$ and $V$. It is convenient to introduce the homology cycle $B$, which satisfies the equation

$$\frac{\partial B}{\partial U} = \int_{\gamma'} \frac{\partial \lambda}{\partial U}, \quad \frac{\partial B}{\partial V} = \int_{\gamma'} \frac{\partial \lambda}{\partial V},$$

where $\gamma'$ is a loop around the point $x = \frac{m}{2}$. We can take $A$ and $B$ as a basis of “electric” cycles. From the above formulas it is clear that $\partial A/\partial U$ and $\partial B/\partial U$ are both finite in the massless limit, whereas $\partial A/\partial V$ and $\partial B/\partial V$ are both proportional to $\sim \frac{1}{m}$ for small $m$. Considering now the equations

$$\frac{\partial A}{\partial U} \frac{\partial U}{\partial A} + \frac{\partial A}{\partial V} \frac{\partial V}{\partial A} = 1; \quad \frac{\partial B}{\partial U} \frac{\partial U}{\partial A} + \frac{\partial B}{\partial V} \frac{\partial V}{\partial A} = 0,$$

we can easily see that they cannot be satisfied if $\partial U/\partial A$ vanishes. This guarantees that the $Q_0$ condensate does not vanish.