Multiple Inflationary Stages with Varying Equation of State

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We consider a model of inflation consisting a single fluid with a time-dependent equation of state. In this phenomenological picture, two periods of inflation are separated by an intermediate non-inflationary stage which can be either a radiation dominated, matter dominated or kinetic energy dominated universe, respectively, with the equation of state $w = 1/3$, 0 or 1. We consider the toy model in which the change in $w$ happens instantaneously. Depending on whether the mode of interest leaves the horizon before or after or between the phase transitions, the curvature power spectrum can have non-trivial sinusoidal modulations. This can have interesting observational implications for CMB anisotropies and for primordial black-hole formation.

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\section{I. INTRODUCTION}

Inflation has emerged as the leading paradigm for early universe cosmology and structure formation. Basics predictions of inflation are in good agreement with cosmological observations. Namely, simplest models of inflation predict almost scale invariant, almost Gaussian and almost adiabatic fluctuations on cosmic microwave background (CMB) which are accurately measured in recent cosmological observations \cite{1}. Nonetheless, it is interesting to consider more elaborate models of inflation which can predict observable deviations from these simple predictions. In particular, models of inflation with local features may be interesting. Observationally, these models are employed to address the glitches in the CMB angular power spectrum on scales $\ell \sim 20 - 40$. Theoretically, one can construct different scenarios which can contain local features \cite{2,21}. Models with local features may originate from high energy physics, particle creations, field annihilations, change in sound speed or time variations of the Newton constant during inflation \cite{22,36}. Many of these models are based on multi-field or multi-fluid scenarios. As a consequence, there are always iso-curvature perturbations which may be constrained from CMB observations.

In this work we consider a phenomenological model with multiple inflationary stages. Different stages of inflation are separated by an intermediate non-inflationary period. In our model, these multiple inflationary stages are realized by changes in the equation of state $w$ for a single fluid. During inflation $w \simeq -1$ while in the intermediate non-inflationary stage we have $1 + 3w > 0$. Particular interests are the cases in which the intermediate non-inflationary stage has the equation of state $w = w_2 = 1/3$, 0 or 1, corresponding respectively to a radiation, matter or kinetic energy dominated universe. Having this said, we should emphasis that this is a phenomenological study and a dynamical mechanism causing the jump in $w$ has yet to be constructed. We shall briefly present a simple scalar field model which can provide a simple dynamical mechanism for changing $w$. Idea similar to this line of thought was studied in \cite{37} in the context of MSSM inflation.

The rest of the paper is organized as follows. In section \textbf{II} we present our setup and background equations. In section \textbf{III} we present the general perturbation equations with appropriate matching conditions. The resulting transfer function of the outgoing perturbations for arbitrary $w_2 \neq 0$ is given in Section \textbf{IV}. The special case of $w_2 = 0$ is considered in section \textbf{V}. The conclusion and discussions are given in section \textbf{VI} and some technical issues are relegated to appendices.

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II. THE BACKGROUND

Here we present the background evolution of a universe filled with a single perfect fluid with an arbitrary but constant $w = P/\rho$ where $\rho$ and $P$ are the energy density and pressure, respectively. The background space-time is assumed to be a flat FLRW universe,

$$ds^2 = -dt^2 + a(t)^2dx^2 = a^2(\eta)(-d\eta^2 + dx^2),$$

(1)

where $\eta$ defined by $d\eta = dt/a(t)$ is the conformal time.

To be specific we have the following picture in mind. We have three distinct stages of an expanding background in which two inflationary periods are separated by an intermediate non-inflationary stage. The first inflationary stage continues till $\eta = \eta_{12}$ and during this period the fluid driving inflation has a constant equation of state $w = w_1$. In order to support inflation we require $1 + 3w_1 < 0$. We assume that at $\eta = \eta_{12}$ the equation of state changes sharply from $w_1$ to $w_2$ such that $1 + 3w_2 > 0$ and the first stage of inflation is terminated. As specific examples we shall consider the important cases of $w_2 = 1/3, 0$ and $1$, corresponding respectively to radiation, matter and kinetic energy dominated universes. The third expanding stage starts at $\eta = \eta_{23}$ when $w$ goes a second abrupt change from $w_2$ to $w_3$. In order to support the final stage of inflation we assume that $1 + 3w_3 < 0$. The time when the inflation ends is set to be $\eta = \eta_e = 0$ followed by a (p)reheating era. In summary, we have two inflationary stages with equations of state $w = w_1$ and $w_3$ separated by an intermediate non-inflationary stage with $w = w_2$.

One may wonder how dynamically these jumps in $w$ can be realized in a consistent way. In Discussions Section we present a simple scalar field model which can mimic this behavior. However, in this and the following sections, we shall proceed phenomenologically assuming that there exists a dynamical mechanism which can cause these changes in $w$. For our analytical analysis we proceed with the arbitrary sharp changes in $w$. We note that physically it is expected that the process in which $w$ undergoes large changes will take some finite lapse of time. Therefore we also consider numerically the case when there is a short but finite duration of the phase transitions. We shall also compare our analytical results with sudden changes in $w$ to those obtained numerically in which the change in $w$ takes a finite lapse of time.

With this picture in mind, now we present the background equations. Using the energy conservation equation and denoting the initial conditions with subscript 0, the evolution of energy density is given by

$$\rho = \rho_0 \left( \frac{a}{a_0} \right)^{-3(1+w)}.$$  

(2)

Here and below the subscript 0 collectively denotes the time of phase transitions, so it corresponds to either $\eta_{12}$ or $\eta_{23}$ depending on which period is studied (see Eq. (7) below for further details). The Friedmann equation for a flat universe is

$$3M_{\text{Pl}}^2H^2 = a^2\rho,$$

(3)

where $H = a'/a$ is the conformal Hubble parameter and the prime denotes the derivative with respect to conformal time $\eta$. The Friedmann equation for $w \neq -1/3$ can be integrated to

$$a(\eta) = a_0 \left( \frac{H_0}{\beta}(\eta - \eta_0) + 1 \right)^{\beta},$$

(4)

where $a_0 \equiv a(\eta_0)$ and

$$\beta = \frac{2}{3w + 1}.$$  

(5)

Taking the conformal time derivative of Eq. (4), one obtains

$$H = \frac{H_0}{1 + \frac{H_0}{\beta}(\eta - \eta_0)}.$$  

(6)

It is easy to see that both the scale factor $a(\eta)$ and the conformal Hubble parameter $H$ must be continuous at the time of phase transition $\eta = \eta_0$ when $w$ undergoes a sudden change.
We label quantities at the three stages with 1, 2 and 3. For example $\mathcal{H}_1(\eta)$ is the conformal Hubble parameter during the first stage whereas $\mathcal{H}_{12}$ is the value of $\mathcal{H}(\eta)$ at the time of first phase transition $\eta = \eta_{12}$, $\mathcal{H}_{12} = \mathcal{H}(\eta_{12})$, and so on. As a result one has

$$
\mathcal{H}(\eta) = \begin{cases} 
\frac{\mathcal{H}_{12}}{1 + \mathcal{H}_{12}(\eta - \eta_{12})/\beta_1} & \text{for } \eta < \eta_{12}, \\
\frac{\mathcal{H}_{12}}{1 + \mathcal{H}_{12}(\eta - \eta_{12})/\beta_2} & \text{for } \eta_{12} < \eta < \eta_{23}, \\
\frac{\mathcal{H}_{23}}{1 + \mathcal{H}_{23}(\eta - \eta_{23})/\beta_3} & \text{for } \eta > \eta_{23},
\end{cases}
$$

(7)

in which $\beta_i$ are defined as in Eq. [5] with $w$ replaced by $w_i$ of each stage. Here we assume that the first stage of inflation starts at $\eta \to -\infty$ and the second stage of inflation ends at $\eta_e = 0$. Note that the Hubble parameters at two phase transitions are related by

$$
\mathcal{H}_{23} = \frac{\mathcal{H}_{12}}{1 + \mathcal{H}_{12}(\eta_{23} - \eta_{12})/\beta_2},
$$

(8)

while the Hubble parameter at the end of inflation is given by

$$
\mathcal{H}_e = \frac{\mathcal{H}_{23}}{1 - \mathcal{H}_{23}\eta_{23}/\beta_3}.
$$

(9)

Note that, as long as the intermediate non-inflationary stage corresponds to a universe dominated by an ordinary matter ($w > 0$) one has

$$
\mathcal{H}_{12} > \mathcal{H}_{23}, \quad \mathcal{H}_e > \mathcal{H}_{23}.
$$

(10)

Alternatively, it may be useful to work with the number of $e$-folds as the clock $dn = H dt = \mathcal{H} d\eta$. With the scale factor given by Eq. [4] one obtains

$$
n(\eta) = n_0 + \beta \ln \left(1 + \beta^{-1}\mathcal{H}(\eta - \eta_0)\right).
$$

(11)

In particular, the number of $e$-folds of the second non-inflationary stage $\Delta N_2 = n_{23} - n_{12}$, where $n_{23} = n(\eta_{23})$ and $n_{12} = n(\eta_{12})$, and that of the third inflationary stage $\Delta N_3 = n_e - n_{23}$, where $n_e = n(\eta_e) = n(0)$, we obtain

$$
\frac{\mathcal{H}_{12}}{\mathcal{H}_{23}} = 1 + \beta_2^{-1}\mathcal{H}_{12}(\eta_{23} - \eta_{12}) = e^{\Delta N_2/\beta_2}, \quad \frac{\mathcal{H}_{23}}{\mathcal{H}_e} = 1 - \beta_3^{-1}\mathcal{H}_{23}\eta_{23} = e^{\Delta N_3/\beta_3}.
$$

(12)

### III. THE PERTURBATIONS

In this section we study the perturbation equations in details. We study the behaviors of the comoving curvature perturbations $\mathcal{R}$ or the Bardeen potential $\Phi$ which are gauge invariant. Some technical details are described in Appendix[A]. For a review see, e.g. [38].

For a universe filled with a single fluid with the known equation of state parameter $w$ and sound speed $c_s$ one obtains the following equation for the Fourier space mode function of the comoving curvature perturbation:

$$
\mathcal{R}''_k + \frac{(z^2)'}{z^2} \mathcal{R}'_k + c_s^2 k^2 \mathcal{R}_k = 0,
$$

(13)

where

$$
z = a(\eta) M_{Pl} \sqrt{3(1 + w/c_s^2)}.
$$

(14)

Note that $c_s$ is defined as $\delta P_c = c_s^2 \delta \rho_c$ in which the subscript $c$ indicates that the corresponding quantities are measured on the comoving hypersurface (on which the fluid 4-velocity coincides with the unit normal to the hypersurface).

One can easily solve Eq. [13] in each phase with constant values of $w$ and $c_s$. At the time of transition, we need two matching conditions in order to match the outgoing solutions to the incoming solutions [39].
The first matching condition is the continuity of the curvature perturbation itself,

$$[\mathcal{R}_k]^+_\eta = 0, \quad (15)$$

where $[X]^+_\eta$ denotes the difference in the value of quantity $X$ after and before the transition: $[X]^+_\eta = X(\eta_+) - X(\eta_-)$. Geometrically, the continuity of $\mathcal{R}$ can be interpreted as the continuity of the extrinsic and intrinsic curvature at the three-dimensional spatial hyper-surfaces located at $\eta = \eta_{12}$ and $\eta = \eta_{23}$, to be consistent with the Bianchi identity.

We also need another matching condition for the time derivative of $\mathcal{R}$. Note that the Eq. (13) can be rewritten by

$$\frac{d}{d\eta} \left( \frac{a^2(1 + w)\mathcal{R}'_k}{c_s^2} \right) + a^2(1 + w)k^2\mathcal{R}_k = 0. \quad (16)$$

By integrating the above equation in a small range around the phase transition, the last term vanishes and one obtains the second matching condition by

$$\left[ \frac{1 + w}{c_s^2} \mathcal{R}'_k \right]_{\eta_1}^{\eta_2} = 0. \quad (17)$$

Alternatively, one can obtain the matching condition (17) in a different way. From the continuity of the extrinsic and intrinsic three-dimensional hyper-surface at the time of phase transition, we also conclude that the curvature perturbation on the shear-free hypersurface (Newton gauge) $\Phi$ is continuous across the transition surface,

$$[\Phi]^+_\eta = 0. \quad (18)$$

Then by noting the relations between $\Phi$ and $\mathcal{R}$ (see Appendix A for details)

$$\mathcal{R} = \frac{2\Phi' + (5 + 3w)\mathcal{H}\Phi}{3\mathcal{H}(1 + w)}, \quad (19)$$

and

$$\Phi = -\frac{3(1 + w)\mathcal{H}}{2c_s^2k^2} \mathcal{R}', \quad (20)$$

we see from Eq. (20) that the matching condition (17) implies the continuity of $\Phi$.

Now we solve the equation of motion for $\mathcal{R}_k$. For a constant $w$ and $c_s$, we have $z'/z = a'/a = \mathcal{H}$. Hence Eq. (18) simplifies to

$$\left[ \frac{d^2}{dx^2} + \frac{2\beta}{x} \frac{d}{dx} + 1 \right] \mathcal{R}_k = 0, \quad (21)$$

where

$$x = \frac{\beta}{[\beta]} c_s k(\eta - \eta_0 + \beta \mathcal{H}_0^{-1}). \quad (22)$$

The solution is given by

$$\mathcal{R}_k = x^\nu \left[ C_1 H^{(1)}_{\nu}(x) + D_1 H^{(2)}_{\nu}(x) \right], \quad \nu \equiv \frac{1}{2} - \beta = \frac{3(w - 1)}{2(3w + 1)}, \quad (23)$$

where $C_1$ and $D_1$ are constant of integrations, and $H^{(1)}_{\nu}(x)$ and $H^{(2)}_{\nu}(x)$ are the Hankel functions of the first and second kinds, respectively. Note that during inflation $\beta < 0$ (for slow-roll inflation $\beta \approx -1$) and the above general definition of $x$ yields

$$x = x_1(\eta) \equiv -c_s k(\eta - \eta_1 + \beta \mathcal{H}_1^{-1}), \quad (24)$$

during the first period of inflation.

Our goal is to find the curvature perturbation $\mathcal{R}_k$ at the end of inflation $\eta = 0$. The power spectrum $P_\mathcal{R}$ is defined by

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle \equiv (2\pi)^3 P_\mathcal{R}(k) \delta^3(k + k'), \quad P_\mathcal{R} \equiv \frac{k^3}{2\pi^2} P_\mathcal{R}(k); \quad P_\mathcal{R}(k) = |\mathcal{R}_k|^2, \quad (25)$$
where $\mathcal{R}_k$ is the normalized positive frequency mode function. At sufficiently early times, the solution should approach the Minkowski positive frequency mode function. That is, for $\eta \to -\infty$,

$$
\mathcal{R}_k \to e^{-i c_s k \eta} \frac{z(\eta)}{\sqrt{2 c_s k}} \quad \text{for } \eta \to -\infty.
$$

(26)

Imposing this initial condition on the solution (23) and using the asymptotic form of the Hankel function given by Eq. (31), we find at the first stage of inflation, $D_1 = 0$ and

$$
\mathcal{R}_k(\eta) = C_1 x_1(\eta) H^{(1)}_{\nu_1}(x_1(\eta)); \quad \eta < \eta_{12},
$$

(27)

where

$$
C_1 \equiv \frac{-1}{2 M_{\text{Pl}} a(\eta_{12})} \left( \frac{i \pi c_s \nu_1}{3 k (1 + w_1)} \right)^{1/2} x_1(\eta_{12})^{1/2 - \nu_1}.
$$

(28)

Considering the slow-roll limit in which $w_1 = -1 + 2 \epsilon_1/3$ and $\nu_1 \approx 3/2$ the above equation results in the following power spectrum at $\eta = \eta_{12}$ for the modes which leave the horizon during the first stage of inflation:

$$
\mathcal{P}_R(\eta_{12}) \simeq \frac{H^2_{12}}{8 \pi^2 M_{\text{Pl}}^2 c_s \epsilon_1},
$$

(29)

where the slow-roll parameter $\epsilon$ is defined by

$$
\epsilon \equiv \frac{-\dot{H}}{H^2} = \frac{3}{2} (1 + w).
$$

(30)

For those modes that remain superhorizon until the end of inflation, the curvature perturbation is conserved, and we have

$$
\mathcal{P}_R(\eta_e) = \mathcal{P}_R(\eta_{12}) \simeq \frac{H^2_{12}}{8 \pi^2 M_{\text{Pl}}^2 c_s \epsilon_1}.
$$

(31)

This is the standard results for single field inflation [40].

Applying the general solution (23) to the second and third stages, we have

$$
\mathcal{R}_k(\eta) = C_2 x_2^{\nu_2} H^{(1)}_{\nu_2}(x_2(\eta)) + D_2 x_2^{\nu_2} H^{(2)}_{\nu_2}(x_2(\eta)); \quad \eta_{12} < \eta < \eta_{23},
$$

(32)

$$
\mathcal{R}_k(\eta) = C_3 x_3^{\nu_3} H^{(1)}_{\nu_3}(x_3(\eta)) + D_3 x_3^{\nu_3} H^{(2)}_{\nu_3}(x_3(\eta)); \quad \eta_{23} < \eta,
$$

(33)

where $x_i(\eta)$ ($i = 2, 3$) are defined in accordance with the general definition (22).

$$
x_2 \equiv c_2 k (\eta - \eta_{12} + \beta_2 H^{-1}_{12}); \quad \nu_2 = \frac{1}{2} - \beta_2 = \frac{3(w_2 - 1)}{2(3w_2 + 1)},
$$

$$
x_3 \equiv c_3 k (\eta - \eta_{23} + \beta_3 H^{-1}_{23}); \quad \nu_3 = \frac{1}{2} - \beta_3 = \frac{3(w_3 - 1)}{2(3w_3 + 1)}.
$$

(34)

Here we assume that $w_2 = c_{22}^2 \neq 0$, so the intermediate non-inflationary stage is not a matter-dominated universe. The case when the intermediate stage is matter dominated with $w = c_{22}^2 = 0$ is considered separately in Section [V].

As usual, we are interested in modes which are super-horizon at the end of inflation $x_3(\eta_e) \ll 1$ where $\eta_e \to 0$. Using the asymptotic form of the Hankel function, we obtain

$$
\mathcal{R}_k(\eta \to 0) \simeq \frac{i 2 \nu_3}{\pi} \Gamma(\nu_3) (C_3 - D_3).
$$

(35)

It is useful to define the transfer function $T$ for the power spectrum as

$$
\mathcal{P}_R(\eta = 0) = T \mathcal{P}_R(\eta = 0),
$$

(36)

where $\mathcal{P}_R(\eta = 0)$ is the power spectrum at the end of inflation if there were no transition and $w = w_1$ throughout the inflationary stage, as calculated in Eq. (31). Assuming $\nu_1 \approx \nu_3 \approx 3/2$ we obtain

$$
T \simeq \left| \frac{C_3 - D_3}{C_1^2} \right|.
$$

(37)

Thus any non-trivial effect due to change in $w$ is captured by a non-trivial transfer function $T \neq 1$. The details of the calculation of the coefficients $C_2$, $D_2$, $C_3$ and $D_3$ are given in Appendix [C].
In this section we assume that the intermediate non-inflationary stage is not matter dominated so \( w_2 = c_{s2}^2 \neq 0 \) and our formulas \((C1)-(C4)\) are valid. The case in which \( w_2 = c_{s2}^2 = 0 \) is studied separately in Section IV. Also we provide the general formula for arbitrary \( w_2 \neq 0 \) and then consider the particular example \( w_2 = 1/3 \) and \( w_2 = 1 \) corresponding, respectively, to a radiation dominated and kinetic energy dominated universe.

The behaviors of the power spectrum depends on the ratio \( k\mathcal{H}/c_s \). Depending on this value different situations arise. To be specific let us define

\[
k_{12} \equiv \frac{-\mathcal{H}_{12}}{\beta_1 c_s}, \quad k_{23} \equiv \frac{-\mathcal{H}_{23}}{\beta_3 c_s}.
\]

Here \( k_{12} \) represents the mode which leaves the horizon at the time of the first phase transition whereas \( k_{23} \) is the mode which leaves the horizon at time of the second phase transition. Note that \( k_{12} > k_{23} \) as long as the second stage is non-inflationary. With the above definitions of the characteristic wave numbers, there are three categories of the modes. The behavior of \( \mathcal{H}/c_s \) as well as that of the ratio of \( \mathcal{H}/c_s \) to \( k \) for three typical values of \( k \) is depicted as a function of the number of \( e \)-folds in Fig. 1.

The first category contains the modes \( k > k_{12} \), which remain sub-horizon until the second stage of inflation. For this category we have \( x_1(\eta_{12}), x_2(\eta_{12}), x_2(\eta_{23}), x_3(\eta_{23}) \gg 1 \).

The second category contains the modes \( k_{23} < k < k_{12} \), which leave the horizon during the first inflationary stage, re-enter the horizon during the second non-inflationary stage, and finally exit the horizon during the second inflationary stage. For this category we have \( x_1(\eta_{12}), x_2(\eta_{12}) \ll 1 \) and \( x_2(\eta_{23}), x_3(\eta_{23}) \gg 1 \).

The third category contain the modes \( k < k_{23} \), which leave the horizon during the first stage and always remain super-horizon until the end of inflation. For this category we have \( x_1(\eta_{12}), x_2(\eta_{12}), x_2(\eta_{23}), x_3(\eta_{23}) \ll 1 \).

Now we study each category of the modes in turn. Let us start with the first category, modes which remain subhorizon until the second phase of inflation, \( k > k_{12} \) (for which \( x_1(\eta_{12}), x_2(\eta_{12}), x_2(\eta_{23}), x_3(\eta_{23}) \gg 1 \)). The coefficients \( C_2, D_2, C_3 \) and \( D_3 \) are calculated in Appendix C. The amplitude of the curvature perturbation at the end of inflation is obtained as

\[
|\mathcal{R}_k(\eta = 0)| \simeq f_2 \frac{C_1}{\pi} 2^{\nu_3 - \frac{1}{2}} \Gamma(\nu_3) \frac{x_1(\eta_{12})^{\nu_3 - \frac{1}{2}}}{x_3(\eta_{23})^{\nu_3 - \frac{1}{2}}} \left( \frac{x_2(\eta_{23})}{x_2(\eta_{12})} \right)^{\nu_2 - \frac{1}{2}} \times \sqrt{1 + \sin \left( 2x_3(\eta_{23}) - \pi \nu_3 \right) \sin \left( x_2(\eta_{12}) - x_2(\eta_{23}) \right)}.
\]
Then the transfer function $T$ is calculated to be

$$T^{1/2} \simeq \sqrt{2} f_{23} \frac{x_1(\eta_{12})^{\nu_1-1/2}}{x_3(\eta_{29})^{\nu_3-1/2}} \left( \frac{x_2(\eta_{23})}{x_2(\eta_{12})} \right)^{\nu_2-\frac{1}{2}} \sqrt{1 + \sin(2x_3(\eta_{23}) - \pi \nu_3) \sin(x_2(\eta_{12}) - x_2(\eta_{23}))}$$

$$\simeq \sqrt{2} f_{23} \left( \frac{k}{k_{12}} \right)^{\nu_1-\frac{1}{2}} \left( \frac{k}{k_{23}} \right)^{\nu_3-\frac{1}{2}} \beta \left( \frac{\beta_2 c_{23} k_{12}}{\beta_3 c_{33} k_{23}} \right)^{\nu_2-\frac{1}{2}}$$

$$\times \sqrt{1 + \sin \left( \frac{2k}{k_{23}} - \pi \nu_3 \right)} \sin \left( \frac{3k}{k_{12}} \right) \left[ 1 - \frac{\beta_1 c_{13} k_{12}}{\beta_3 c_{33} k_{23}} \right],$$

where we have defined

$$f_{ij} = \frac{\text{sgn}(1 + 3w_i) (1 + w_i) c_{ij}}{\text{sgn}(1 + 3w_j) (1 + w_j) c_{jj}},$$

in which $\text{sgn}(x)$ is the sign function; $\text{sgn}(x) = +1 (-1)$ for $x > 0 (x < 0)$. Note that in the continuous limit where $w_i \rightarrow w_j$ we obtain the expected result that $f_{ij} \rightarrow 1$ corresponding to no sharp transition. Since a change in $w$ naturally causes a change in $c_s$ too, it is the combination $f_{ij}$ which controls whether or not we have a non-trivial phase transition.

Now consider the second category of the modes $k_{23} < k < k_{12}$, which leave the horizon at the first stage of inflation, re-enter during the intermediate stage and cross the horizon again during the second stage of inflation (for which $x_1(\eta_{12}) = x_2(\eta_{23}) < 1$ while $x_2(\eta_{23}), x_3(\eta_{29}) \gg 1$). From the result given in Appendix C we obtain

$$|R_k(0)| \simeq \left| C_1 \right| \frac{f_{23} 2^{\nu_3 - \nu_2 + \nu_3} \Gamma(\nu_1 - \nu_2 + 1) x_3(\eta_{23})^{-\nu_3 + \frac{1}{2}} x_2(\eta_{23})^{\nu_2 - \frac{1}{2}}}{\pi} \sqrt{(1 - \sin(2x_2(\eta_{23}) - 3\nu_2 \pi)) (1 + \sin(2x_3(\eta_{23}) - \pi \nu_3))}.$$  \hspace{1cm} (42)

The transfer function is given by

$$T^{1/2} \simeq \frac{2^{\nu_3 - \nu_2}}{\pi} f_{23} \Gamma(1 - \nu_2) x_3(\eta_{23})^{-\nu_3 + \frac{1}{2}} x_2(\eta_{23})^{\nu_2 - \frac{1}{2}}$$

$$\times \sqrt{(1 - \sin(2x_2(\eta_{23}) - 3\nu_2 \pi)) (1 + \sin(2x_3(\eta_{23}) - \pi \nu_3))}$$

$$\simeq \frac{2^{\nu_3 - \nu_2}}{\pi} f_{23} \Gamma(1 - \nu_2) \left( \frac{k}{k_{23}} \right)^{-\nu_3 + \frac{1}{2}} \left( \frac{\beta c_{23} k}{\beta c_{33} k_{23}} \right)^{\nu_2 - \frac{1}{2}}$$

$$\times \sqrt{1 - \sin \left( \frac{2\beta c_{23} k}{k_{23}} - 3\nu_2 \pi \right)} \left[ 1 + \sin \left( \frac{2k}{k_{23}} - \nu_3 \pi \right) \right].$$  \hspace{1cm} (43)

Finally, consider the third category, $k < k_{23}$, corresponding to the modes which leave the horizon during the first stage of inflation and never re-enter the horizon until the end of inflation. Since the curvature perturbation is conserved on superhorizon scales, the amplitude of these modes are simply given by the standard result given in Eq. (31), and the transfer function is trivial; $T \simeq 1$.

The results obtained above show that neither the amplitude nor the scale-dependence of the power spectrum is the same as the standard case for the modes $k > k_{23}$. A typical example of the transfer function is shown in Fig. 2. The power spectrum is highly oscillatory as a function of momentum. This non-trivial behavior is a result of the scattering of the initial wave function by the two phase transitions which lead to a mixing of the negative frequency modes, which are absent initially. As a result the subhorizon modes at the second phase of inflation are no longer purely positive frequency.

We note that in the limit $k \gg k_{12}$ we obtain the scaling property $T \propto k^{2\nu_1 - 2\nu_3}$ from Eq. (40). The non-decaying sinusoidal modulation on top of this mild scale-dependence is because of the assumption that the changes in $w$ take place abruptly. Under this assumption the small scale modes, no matter how deep inside the horizon they are, are all affected. However, if we allow a finite time-scale for the change in $w$, say $\Delta \tau = \delta$, then the sinusoidal modulations on the power spectrum die out for frequencies bigger than $\delta^{-1}$ and the power spectrum reaches its almost scale-invariant value at the end of inflation. This behavior is seen in Fig. 3 which is obtained numerically for an example of smooth changes in $w$.

Before closing this section, let us explore the dependence of the power spectrum enhancement as a function of the duration of the intermediate non-inflationary stage, $\Delta N_2$. Here we propose two methods to see this behavior in
FIG. 2: The transfer function $T$ is plotted for the case where $w_2 = 1/3$, i.e. the non-inflationary stage is radiation dominated. The numerical parameters are $w_1 = -0.99$, $w_3 = -0.93$, $c_{s1} = c_{s3} = 1$, $c_{s2} = \sqrt{w_2}$. Three distinct categories are recognized as discussed below Eq. (38), corresponding to $k > k_{12}$, $k_{23} < k < k_{12}$ and $k < k_{23}$. The blue curve is from our analytical solution (37) while the red dots are from the full numerical result.

FIG. 3: The transfer function $T$ (the red dashed line) for the same situation as in Fig. 2 above but with a relatively mild phase transition in which the change in $w$ takes place in $1/3$ of an e-fold. As expected, for sufficiently small scales the sinusoidal modulations disappear. The blue solid curve is the analytic expression for the sharp phase transition.
which each has its own advantages. We especially concentrate on Eq. (43) which to good approximation shows the behavior we are looking for. Firstly note that for a fixed value of $H_e$ and assuming $w_3 \simeq -1$, the value of $k_{23}$ is nearly determined by $\Delta N_2$ and as a result we would like to compare the values of $T$ for different $\Delta N_2$ at the mode $k = k_{23}$. This is the extreme limit of the validity of Eq. (43) which holds for the modes in the range $k_{23} \ll k \ll k_{12}$. Obviously, one should have $k_{23} \ll k_{12}$ if one uses the above approximation, which is the case when $\Delta N_2$ is sufficiently large.

The first approach is to set the parameters $w_1$ and $w_3$ to constant values and look for the $\Delta N_2$ dependence in the transfer function. An inspection of Eq. (43) shows that for the scale $k = k_{23}$, there is no $\Delta N_2$ dependence on the amplitude of the transfer function. As a result, the transfer function will not vary much as a function of $\Delta N_2$. This argument is supported by Fig. 5, in which the value of the transfer function at $k = k_{23}$ is shown as a function of $\Delta N_2$.

The second approach is to set the power spectrum of the two inflationary stages equal to each other and vary $\Delta N_2$. In this case we impose the condition,

$$\frac{H^2_{12}}{\epsilon_1 c_{s1}} \simeq \frac{H^2_{23}}{\epsilon_3 c_{s3}}.$$  \hspace{1cm} (44)

As a result, we should change e.g. $\epsilon_3$ for fixed values of $\epsilon_1$ and $c_{s1}$, when we vary $\Delta N_2$, since the Hubble parameter changes considerably during the intermediate stage. In fact, using the above equality, one has

$$\epsilon_3 \simeq \epsilon_1 e^{-2\Delta N_2 (1 + 1/\beta_2)}$$  \hspace{1cm} (45)

Noting that $1 + w_3 = 2\epsilon_3/3$, and $f_{23} \propto 1/(1 + w_3)$, one can conclude from Eq. (43) that the transfer function behaves as $T \propto e^{4\Delta N_2 (1 + 1/\beta_2)}$, i.e. the enhancement is exponentially larger for larger values of $\Delta N_2$, which is a very interesting phenomenon. This feature is also supported from the exact numerical result shown in Fig. 6.

V. MATTER-DOMINATED INTERMEDIATE STAGE

The analysis in the previous sections are valid as long as the sound speed and equation of state in each stage do not vanish. However, if the intermediate stage is a matter-dominated universe then the previous results are not applicable since the matching conditions as well as the equation of motion of curvature perturbation are singular. In this section we obtain the power spectrum of curvature perturbation for this case.
FIG. 5: $T$ at the scale $k = k_{23}$ as a function of $\Delta N_2$, for the fixed values of $w_1$, $w_2$ and $c_{s1}$, $c_{s3}$. Except $\Delta N_2$, the other parameters are the same as the previous plots for $w_2 = 1/3$.

FIG. 6: $\ln T$ at the scale $k = k_{23}$ as a function of $\Delta N_2$, for the case with equal values of power spectrum in two stages of inflation. The dashed red line is a line proportional to $e^{4N_2(1+1/\beta_2)}$. 
In this special case, it is better to work with the curvature perturbation on the Newtonian slice \( \Phi \) (i.e., the so-called Bardeen potential), since the equations behave properly when written in terms of \( \Phi \). The relations between \( R \) and \( \Phi \) are given in Eqs. (19) and (20). Eliminating \( R \) from these formulas results in the following second order differential equation for \( \Phi \):

\[
\Phi'' + 3H(1 + c_w^2)\Phi' + [c_w^2 k^2 - 3(w - c_w^2)H^2]\Phi = 0 ,
\]

(46)

where

\[
c_w^2 \equiv \frac{P'}{\rho'} = w - \frac{w' + w}{3(1 + w)H} .
\]

(47)

It is crucial to realize that \( c_w \) is not the same as \( c_v \) in general. Note that \( c_v \) is defined as \( \delta P_c/c^2 \delta \rho_c \) where \( \delta P_c \) and \( \delta \rho_c \) are the adiabatic pressure and energy density perturbations on comoving slices. Only for a universe dominated by a perfect fluid, we have \( c_w = c_v \).

Note that the equation of motion for the Newtonian potential is well-defined even for the case in which both \( w \) and \( c_v \) vanish. In order to solve the above equation we need two matching conditions for \( \Phi \) and \( \Phi' \). As before, the continuity of the intrinsic and extrinsic curvatures at the surface of phase transition implies that \( \Phi \) is continuous as given in Eq. (18). On the other hand, using Eq. (19) and noting that both \( R \) and \( \Phi \) are continuous across the transition one obtains the second matching condition,

\[
\left[ \frac{\Phi'}{(1 + w)H} + \frac{\Phi}{(1 + w)} \right]_{x = 0, 1} \pm = 0 .
\]

(48)

Now we have the necessary information to obtain the solution for \( \Phi \). At the first stage, the solution is

\[
\Phi_1 = A_1 x_1^{\nu_1 - 1} H^{(1)}_{\nu_1 - 1}(x_1) .
\]

(49)

with the same definition of \( x_1(\eta) \) and \( \nu_1 \) as it is in Eq. (22). Using the relations (20) and (27) one has

\[
A_1 = -\frac{3}{2} \beta_1 (1 + w_1) C_1 .
\]

(50)

For the second stage with \( w_2 = c_{v_2} = 0 \), one has

\[
\Phi_2 = A_2 \left[ \frac{H_{12}}{2} (\eta - \eta_{12}) + 1 \right]^{-5} + B_2 .
\]

(51)

Finally, for the third stage one has

\[
\Phi_3 = x_3^{\nu_3 - 1} \left[ A_3 H^{(1)}_{\nu_3 - 1}(x_3) + B_3 H^{(2)}_{\nu_3 - 1}(x_3) \right] .
\]

(52)

Since we are interested in the power spectrum of the comoving curvature perturbation at the final stage, we use Eq. (19) to obtain

\[
R_k = -\frac{2}{3(1 + w_3)\beta_3} x_3^{\nu_3} \left[ A_3 H^{(1)}_{\nu_3}(x_3) + B_3 H^{(2)}_{\nu_3}(x_3) \right] .
\]

(53)

From this the transfer function is obtained as

\[
T^{1/2} = \frac{2|A_3 - B_3|}{3|\beta_3|(1 + w_3)|C_1|} .
\]

(54)

The explicit expressions of \( A_3 \) and \( B_3 \) in terms of \( C_1 \) are computed in Appendix C.

In order to obtain an approximate expression for the transfer function, firstly note that \( (H_{23}/H_{12})^5 \ll 1 \) for \( w_2 \geq 0 \). Besides that all of \( \beta_i \) and \( w_i \) are of the order of unity. Using this information as well as the asymptotic behavior of the Hankel functions, we find for the first category, \( k > k_{12} \),

\[
T^{1/2} \sim \left( \frac{k}{k_{23}} \right)^{-\nu_3 + 3/2} \left( \frac{k}{k_{12}} \right)^{\nu_1 - 1/2} \frac{\sqrt{2}}{5\beta_3(1 + w_3)} \sqrt{1 + \sin \left( \frac{2k}{k_{23}} - \nu_3 \right)} ; \quad k > k_{12} .
\]

(55)
and for the second category, $k_{23} < k < k_{12}$,

$$T^{1/2} \approx \frac{2^\nu_3 - 2}{3\sqrt{\pi}} \beta_3 (5 + 3w_3) \Gamma(\nu_3 - 1) \left( \frac{k}{k_{23}} \right)^{-\nu_3 + 3/2} \sqrt{1 + \sin \left( \frac{2k}{k_{23}} - \pi\nu_3 \right)}; \quad k_{23} < k < k_{12}. \quad (56)$$

In Fig. 7 both numerical and analytic results are shown, which are in very good agreement with each other. Interestingly, the power spectrum on small scales is highly enhanced due to the sharp phase transition to the matter-dominated era. This agrees with the result obtained in [41], and will have interesting implications for the primordial black hole formation [42–45].

**VI. DISCUSSIONS AND CONCLUSIONS**

In this work we considered a universe in which inflation has multiple stages, separated by intermediate non-inflationary stages. To be specific, we considered two inflationary stages separated by an intermediate non-inflationary universe. To simplify the calculation, we also assumed sharp transitions between the stages, and studied the case when the intermediate stage is either radiation dominated, matter dominated or kinetic energy dominated.

To read off the final outgoing curvature perturbations, one has to perform the appropriate matching conditions at the surface of phase transitions. We have provided a consistent mechanism as how to do these matching conditions, given in Eqs. (15) and (17) when $w \neq 0$ and in Eq. (45) when $w = 0$. Furthermore, one has to take into account a non-trivial interplay between $c_s$ and $w$ when imposing the matching conditions which is captured by the parameter $f_{ij}$ as given in Eq. (48) and the fact that $c_s \neq c_w$ as given in Eq. (46). Some of these technical aspects of performing the matching conditions properly and the effects of $c_s$ have not been addressed in the previous works conclusively.

As well known, in the absence of isocurvature or entropy perturbations, the comoving curvature perturbations are frozen on superhorizon scales, $\dot{R} = 0$. So the transitions do not affect superhorizon modes. However, sub-horizon modes are significantly affected. Hence the final amplitude of the comoving curvature perturbation depends crucially on the equation of state parameter of the intermediate stage, $w = w_2$, as well as on the time when the mode of interest leaves the horizon.

For modes which leave the horizon at the second stage of inflation, we see a sinusoidal modulations. If the phase transition is arbitrarily sharp, then these sinusoidal modulations persist down to infinitely small scales. This is an artifact of our assumption that the change in $w$ is instantaneous. In a realistic situation in which the change in $w$
takes some finite time scale, then on sufficiently small scales the power spectrum retains its pure vacuum form with no sinusoidal modulations.

We found that the maximum enhancement in the power spectrum occurs for modes which leave the horizon either at the time of the first phase transition $k = k_{23}$ or at the time of the second phase transition $k = k_{12}$. For the case of a matter-dominated intermediate stage, $w_2 = 0$, we found that the power spectrum increases towards small scales. This can have interesting implications for the primordial black-hole formation. It would be very interesting to study the possibility of the over-production of primordial black-holes in our model.

In our analysis so far we have not provided a dynamical mechanism which causes a rapid change in the equations of state. Here it may be worth mentioning a simple toy model which can mimic this situation. Consider a single scalar field with the canonical kinetic term which is slowly rolling down the potential. This corresponds to the first inflationary stage. When the inflaton field reaches near the minimum of the potential and starts to oscillate, the intermediate non-inflationary stage commences. The equation of state of this stage depends on the shape of the potential near its minimum. For a simple quadratic potential, the universe behaves as matter dominated.

Now we add a small cosmological constant to the potential which would not affect the dynamics of the first inflationary stage. After the energy density of the scalar field is diluted sufficiently due to damped oscillations, the cosmological constant starts to dominate and the second stage of inflation begins.

Note that this is a single field scenario, so our previous analysis are applicable here. However in order to terminate inflation, one may need an auxiliary field. Nevertheless, one can assume that this field is sufficiently heavy so that it does not affect the large scale (CMB scale) perturbations, as in the hybrid inflation scenario [46, 47].

The above model can be realized by the potential as simple as

$$V = \frac{1}{2} m^2 \phi^2 + V_0.$$  \hspace{1cm} (57)

We have used this for the numerical plots in Figs. 8 and 9. As it is clear from the figure, the first phase transition can be quite sharp, while the second transition turns out to be relatively mild. Note that this case is different from what we have plotted in Fig. 7 not only because of the mild second phase transition but also because the sound speed is equal to unity throughout all the stages, even for the intermediate stage. This is because for the scalar field with the canonical kinetic term the sound speed is equal to unity.

If we relax the assumptions we adopted in the present paper, and consider a multi-field or multi-fluid system, then we may be able to construct more realistic models. However the price to pay is the complication of the equations of motion and the possible appearance of significant entropy/isocurvature perturbations which are severely constrained by the current observational data. Nevertheless it would be interesting to look for such models.

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FIG. 9: Two numeric results for two different phase transitions. The solid blue line corresponds to a sharp phase transition while for the dashed red line the second transition is mild. In both cases we set \( c_{s1} = c_{s2} = c_{s3} = 1 \). The other parameters are the same as in [2].

Appen
dix A: Relations between \( \Phi \) and \( R \)

In this appendix we obtain the relations between \( \Phi \) and \( R \) as given in Eqs. (19) and (20). The scalar perturbation of the metric is

\[
 ds^2 = - (1 + 2A) dt^2 + 2a \partial_i \dot{B} dx^i \, dt + a^2 [(1 + 2F) \delta_{ij} + 2 \partial_i \partial_j E] dx^i dx^j, \tag{A1}
\]

The 00, ii, 0i and \( i \neq j \) components of Einstein equations are

\[
 3H(\ddot{F} - HA) + k^2 \frac{a^2}{\alpha^2} [F - H\theta] = \frac{\delta \rho}{2M_P^2}, \tag{A2}
\]

\[
 (2\dot{H} + 3H^2)A + HA - \dddot{F} - 3H \dot{F} = \frac{\delta P}{2M_P^2} \tag{A3}
\]

\[
 - \ddot{F} + HA = - \frac{\rho + p}{2M_P^2} v \tag{A4}
\]

\[
 \dot{\theta} + H\theta - A - F = 0 \tag{A5}
\]

in which \( \delta \rho \) and \( \delta P \) are the energy density and the pressure perturbations. We have defined

\[
 \theta \equiv a^2 (\dot{E} - B/a), \tag{A6}
\]

and \( v \) is the fluid velocity scalar potential defined as \( u_\mu = \partial_\mu v \) in which \( u^\mu \) is the fluid four-velocity vector. A hypersurface on which \( v = 0 \) is called the comoving slice.
The gauge invariant variables Newtonian potential $\Psi$ and the Bardeen Potential $\Phi$ are given by

$$\Psi = A - \dot{\theta}, \quad \Phi = F - H\dot{\theta}. \quad (A7)$$

Furthermore, the curvature perturbation on constant energy hypersurface $\zeta$ and the curvature perturbation on co-moving hypersurface $\mathcal{R}$ are given by

$$\zeta = F - \frac{H}{\dot{\rho}} \delta \rho, \quad \mathcal{R} = F + H\nu. \quad (A8)$$

We work in an isotropic background with no anisotropic pressure perturbation, so from Eq. (A5) one concludes that $\Phi = -\Psi$.

To find Eqs. (19) and (20) it is very helpful to go to the comoving slice on which $v = 0$ and $F = \mathcal{R}$. In this gauge from Eq. (A4) we obtain $A = \dot{\mathcal{R}}/H$. Plugging this into Eq. (A7) to remove $\theta$ and using $\Phi = -\Psi$ yields

$$\mathcal{R} = \Phi - \frac{H}{\dot{H}} \left( \dot{\Phi} + H\Phi \right). \quad (A9)$$

Using $\ddot{H} = -3(1+w)H^2/2$ and going to the conformal time one can easily check that Eq. (A9) is equivalent to Eq. (19). To obtain Eq. (20) we have to use the remaining two Einstein equations, Eqs. (A2) and (A3). The key point is that the sound speed of perturbations $c_s$ is defined on the comoving slice as

$$\delta P_{c}^{\delta \rho} = c_s^2 \delta \rho_{c}$$

where $\delta P_{c}$ and $\delta \rho_{c}$ are the pressure and energy density perturbations on the comoving slice. Using this relation in Eqs. (A2) and (A3) and noting that $A = \dot{\mathcal{R}}/H$ on this slice, one obtains

$$\Phi = \frac{a^2 H^2}{H k^2 c_s^2} \mathcal{R}. \quad (A10)$$

As above, using $\ddot{H} = -3(1+w)H^2/2$ and going to the conformal time one can easily check the equivalence between Eq. (A10) and Eq. (20).

**Appendix B: Asymptotic behavior of the Hankel function**

Here we recapitulate the asymptotic behavior of the Hankel functions. For large values of the argument one has

$$H_{\nu}^{(1)}(Z \gg 1) \simeq \sqrt{\frac{2}{i\pi Z}} e^{i(Z-\pi\nu/2)}, \quad (B1)$$

whereas for small values of the argument,

$$H_{\nu}^{(1)}(Z \ll 1) \simeq -\Gamma(|\nu|) \left( i \pi \right) \left( \frac{2}{Z} \right)^{|\nu|} e^{-i\pi/2 (\nu - |\nu|)}. \quad (B2)$$

Also the following identities are helpful:

$$H_{\nu}'(Z) = H_{\nu-1}(Z) - \frac{\nu}{Z} H_{\nu}(Z), \quad (B3)$$

and

$$H_{\nu}^{(1)}(Z)H_{\nu}^{(2)'}(Z) - H_{\nu}^{(1)'}(Z)H_{\nu}^{(2)}(Z) = -\frac{4i}{\pi Z}. \quad (B4)$$

**Appendix C: Transfer Function for Outgoing Solutions**

By solving the matching conditions here we calculate the coefficients of outgoing solutions $C_2, D_2, C_3$ and $D_3$ for $w_2 \neq 0$ and $A_2, B_2, A_3$ and $B_3$ for $w_2=0$ in terms of $C_1$. We consider each case separately.
1. \( w_2 \neq 0 \)

Consider the case where the intermediate non-inflationary stage is not matter dominated, \( w_2 = c_{s2}^2 \neq 0 \). The general solution for the comoving curvature perturbation at the intermediate non-inflationary and second inflationary stages are given in Eqs. (32) and (33), respectively. With the help of the identities (33) and (34) and imposing the matching conditions (15) and (17) at \( \eta_{32} \) and \( \eta_{33} \), we obtain

\[
C_2 = -\frac{\pi x_1(\eta_{32})}{4i x_2(\eta_{32})} C_1 \left[ H^{(1)}_{\nu_1}(x_1(\eta_{32})) H^{(2)}_{\nu_2-1}(x_2(\eta_{32})) - f_{12} H^{(1)}_{\nu_1-1}(x_1(\eta_{32})) H^{(2)}_{\nu_2}(x_2(\eta_{32})) \right],
\]

(31)

\[
D_2 = \frac{\pi x_1(\eta_{32})}{4i x_2(\eta_{32})} C_1 \left[ H^{(1)}_{\nu_1}(x_1(\eta_{32})) H^{(2)}_{\nu_2-1}(x_2(\eta_{32})) - f_{12} H^{(1)}_{\nu_1-1}(x_1(\eta_{32})) H^{(2)}_{\nu_2}(x_2(\eta_{32})) \right],
\]

(32)

\[
C_3 = -\frac{\pi x_2(\eta_{33})}{4i x_3(\eta_{33})} \left[ C_2 \left( H^{(1)}_{\nu_2}(x_2(\eta_{33})) H^{(2)}_{\nu_3-1}(x_3(\eta_{33})) - f_{23} H^{(1)}_{\nu_2-1}(x_2(\eta_{33})) H^{(2)}_{\nu_3}(x_3(\eta_{33})) \right) + D_2 \left( H^{(2)}_{\nu_2}(x_2(\eta_{33})) H^{(1)}_{\nu_3-1}(x_3(\eta_{33})) - f_{23} H^{(2)}_{\nu_2-1}(x_2(\eta_{33})) H^{(1)}_{\nu_3}(x_3(\eta_{33})) \right) \right],
\]

(33)

\[
D_3 = \frac{\pi x_2(\eta_{33})}{4i x_3(\eta_{33})} \left[ C_2 \left( H^{(1)}_{\nu_2}(x_2(\eta_{33})) H^{(2)}_{\nu_3-1}(x_3(\eta_{33})) - f_{23} H^{(1)}_{\nu_2-1}(x_2(\eta_{33})) H^{(2)}_{\nu_3}(x_3(\eta_{33})) \right) + D_2 \left( H^{(2)}_{\nu_2}(x_2(\eta_{33})) H^{(1)}_{\nu_3-1}(x_3(\eta_{33})) - f_{23} H^{(2)}_{\nu_2-1}(x_2(\eta_{33})) H^{(1)}_{\nu_3}(x_3(\eta_{33})) \right) \right],
\]

(34)

where we have defined

\[
f_{ij} = \frac{\text{sgn}(1 + 3w_i)(1 + w_i)c_{ij}}{\text{sgn}(1 + 3w_j)(1 + w_j)c_{ai}}.
\]

(C5)

Note that in the continuous limit where \( w_1 \to w_2 \) so \( f_{12} \to 1 \), we obtain the expected results that \( C_2 = C_1 \) and \( D_2 = 0 \). It is interesting to note that it is the combination \( f_{ij} \) of the changes in \( w \) and \( c_0 \) that controls whether or not we have a non-trivial phase transition.

We would like to calculate the power spectrum of the modes which are super-horizon at the end of inflation, \( \eta_e = 0 \), corresponding to \( x_3(\eta_e) \ll 1 \). The transfer function is defined in Eq. (37). As described in the text we divide the modes of interest into three categories. The first category is defined by \( k > k_{12} \), the second category by \( k_{23} < k < k_{12} \) and the third category by \( k < k_{23} \).

First consider the first category, \( k > k_{12} \). For these modes we have \( x_1(\eta_{12}), x_2(\eta_{12}), x_2(\eta_{23}), x_3(\eta_{23}) \gg 1 \). Using the large value approximation of the Hankel functions we obtain

\[
C_2 \simeq i D_2 \left( \frac{f_{12} + 1}{-f_{12} + 1} \right) e^{-2i(x_2(\eta_{12}) - \pi \nu_2)/2} \simeq \frac{C_1 x_1(\eta_{23})}{2 x_2(\eta_{12})} e^{(i x_1(\eta_{12}) - x_2(\eta_{12}) - \pi (\nu_1 - \nu_2)/2)},
\]

(35)

\[
C_3 \simeq i D_3 \frac{I}{J} e^{-2i(x_3(\eta_{23}) - \pi \nu_3)/2}
\]

(36)

\[
\simeq \frac{I}{J} \frac{C_2 x_2(\eta_{23})}{2 x_3(\eta_{23})} e^{i(x_2(\eta_{23}) - x_3(\eta_{23}) - \pi (\nu_2 - \nu_3)/2)},
\]

where

\[
I \equiv (1 + f_{23}) \left[ 1 + \frac{1 - f_{23}}{1 + f_{23}} e^{2i(x_2(\eta_{12}) - x_2(\eta_{23}))} \right] \simeq f_{23} \left[ 1 - e^{2i(x_2(\eta_{12}) - x_2(\eta_{23}))} \right],
\]

(37)

\[
J \equiv (1 - f_{23}) \left[ 1 + \frac{f_{23}}{1 - f_{23}} e^{2i(x_2(\eta_{12}) - x_2(\eta_{23}))} \right] \simeq -I.
\]

(38)

The last approximate equality is valid for the case in which \( w_1 \sim w_3 \simeq -1 \) and as a result \( f_{12} \ll 1 \) and \( f_{23} \gg 1 \). Plugging these values of \( C_3 \) and \( D_3 \) in the formulas for the final curvature perturbation amplitude (35) and the transfer function (37), we obtain Eqs. (39) and (40).

Now consider the second category, \( k_{23} < k < k_{12} \). For these modes we have \( x_1(\eta_{12}), x_2(\eta_{12}) \ll 1 \) while \( x_2(\eta_{23}), x_3(\eta_{23}) \gg 1 \). For this category, we obtain

\[
C_2 \simeq D_2 e^{2i\nu_2} \simeq \frac{-i}{\pi} 2^{\nu_1 - \nu_2 - 1} C_1 \Gamma(\nu_1) \Gamma(-\nu_2 + 1) e^{-i \nu_2},
\]

(39)

\[
C_3 \simeq D_3 \frac{I}{J} e^{-2i(x_3 - \pi \nu_3/2)} \simeq \frac{I}{J} \frac{C_2 x_3(\eta_{23})}{2 x_2(\eta_{23})} e^{i(x_2(\eta_{23}) - x_3(\eta_{23}) - \pi (\nu_2 - \nu_3)/2)},
\]

(40)
where
\[
\mathcal{I}' = (f_{23} + 1) + i(1 - f_{23})e^{-2i(x_3(\eta_2) - 3\pi\nu_2/2)} \simeq f_{23} \left(1 - ie^{-2i(x_2(\eta_2) - 3\pi\nu_2/2)}\right),
\]
\[
\mathcal{J}' = (-f_{23} + 1) + i(1 + f_{23})e^{-2i(x_2(\eta_2) - 3\pi\nu_2/2)} \simeq -\mathcal{I}'.
\]
Again the second approximate equality holds for the case \(f_{23} \gg 1\). Now plugging these expressions of \(C_3\) and \(D_3\) into the formulas for the final curvature perturbation amplitude \(23\) and the transfer function \(17\), we obtain Eqs. \(42\) and \(43\).

2. Matter Dominated Intermediate Era

Now consider the case where the intermediate non-inflationary stage is a matter-dominated universe, so \(w_2 = c_{s,2} = 0\). As described in the text for this case we work with the Bardeen potential \(\Phi\) while for the final curvature perturbation power spectrum we can switch to \(R\) as given by Eq. \(23\) by using Eq. \(19\). Our task is to find the expressions of \(A_3\) and \(B_3\) in Eq. \(23\) in terms of \(A_1\), which is expressed in terms of \(C_1\) in Eq. \(50\).

Imposing the matching conditions \(18\) and \(48\) at \(\eta = \eta_{12}\) and \(\eta = \eta_{23}\) yields
\[
A_2 = -\frac{2w_1 A_1 x_1(\eta_{12})^{\nu_1-1}}{5(1 + w_1)} \left[\frac{x_1(\eta_{12})}{\beta_1 w_1} H_{\nu_1-2}^{(1)}(x_1(\eta_{12})) - H_{\nu_1-1}^{(1)}(x_1(\eta_{12}))\right],
\]
\[
B_2 = A_1 x_1(\eta_{12})^{\nu_1-1} H_{\nu_1-1}^{(1)}(x_1(\eta_{12})) - A_2,
\]
\[
A_3 = -\frac{\pi}{8} x_3(\eta_{23})^{1-\nu_3} \left[2x_3(\eta_{23}) \left(B_2 + A_2 \left(\frac{\mathcal{H}_{23}}{\mathcal{H}_{12}}\right)^{5}\right) H_{\nu_3-2}^{(2)}(x_3(\eta_{23}))\right] + \beta_3 \left(-2w_3 B_2 + (5 + 3w_3) A_2 \left(\frac{\mathcal{H}_{23}}{\mathcal{H}_{12}}\right)^{5}\right) H_{\nu_3-1}^{(1)}(x_3(\eta_{23}))\right],
\]
\[
B_3 = \frac{\pi}{8} x_3(\eta_{23})^{1-\nu_3} \left[2x_3(\eta_{23}) \left(B_2 + A_2 \left(\frac{\mathcal{H}_{23}}{\mathcal{H}_{12}}\right)^{5}\right) H_{\nu_3-2}^{(1)}(x_3(\eta_{23}))\right] + \beta_3 \left(-2w_3 B_2 + (5 + 3w_3) A_2 \left(\frac{\mathcal{H}_{23}}{\mathcal{H}_{12}}\right)^{5}\right) H_{\nu_3-1}^{(1)}(x_3(\eta_{23}))\right].
\]

As before, we evaluate \(A_3\) and \(B_3\) for the three different categories of the modes separately. For the first category, \(k > k_{12}\), we have
\[
A_2 \simeq B_2 \simeq -\frac{2i A_1 x_1(\eta_{12})^{\nu_1-1/2}}{5\beta_1 (1 + w_1)} \sqrt{\frac{2i}{\pi}} e^{i(x_1 - \pi\nu_1/2)},
\]
\[
A_3 \simeq -ie^{-2i(x_3(\eta_{23}) - \pi\nu_3/2)} B_3 \simeq \sqrt{\frac{2}{\pi}} \frac{B_2 x_3(\eta_{23})^{-\nu_3 + 3/2}}{8} e^{-i(x_3(\eta_{23}) - \pi\nu_3/2)}.
\]

For the second category, \(k_{23} < k < k_{12}\), we have
\[
A_2 \simeq \frac{2}{3 + 5w_1} B_2 \simeq -\frac{i2w_1}{5\pi (1 + w_1)} A_1 \Gamma(\nu_1 - 1),
\]
\[
A_3 \simeq -ie^{-2i(x_3(\eta_{23}) - \pi\nu_3/2)} B_3 \simeq \frac{\pi}{4} \sqrt{\frac{2i}{\pi}} B_2 x_3(\eta_{23})^{-\nu_3 + 3/2} e^{-i(x_3(\eta_{23}) - \pi\nu_3/2)}.
\]

Plugging these into the transfer function formula \(54\) yields Eqs. \(55\) and \(56\).

Appendix D: Necessary relations for numeric calculations

In order to have an efficient and accurate code, it is much better to change the variable from the conformal time to the number of e-folds,
\[
dn = \mathcal{H}d\eta.
\]
Using the Friedmann equation one has

$$\mathcal{H}(n) = \mathcal{H}_e \exp \left( \frac{1}{2} \int_n^{n_e} (1 + 3w) \, dn' \right),$$  \hspace{1cm} (D2)

where $n_e - n$ is the total number of $e$-folds from the time $n$ until the end of inflation.

The curvature perturbation obeys the equation of motion in terms of $n$,

$$\frac{d^2 R}{dn^2} + \left( \frac{3}{2} (1 - w) + \frac{d}{dn} \ln \left( (1 + w) / \epsilon_s^2 \right) \right) \frac{dR}{dn} + \frac{c_s^2 k^2}{H^2} R = 0.$$  \hspace{1cm} (D3)

Similarly one has

$$\frac{d^2 \Phi}{dn^2} + \left[ \frac{5}{2} + 3(c_w^2 - w/2) \right] \frac{d\Phi}{dn} + \left[ \frac{c_s^2 k^2}{H^2} - 3(w - c_w^2) \right] \Phi = 0.$$  \hspace{1cm} (D4)

To model smooth transitions, we use the error function. For example the equation of state is given by

$$w = w_1 + (-w_1 + w_2) \left( 1 + \text{erf}[d_1(n - n_1)] \right) / 2 + (-w_2 + w_3) \left( 1 + \text{erf}[d_2(n - (n_1 + n_2))] \right) / 2,$$  \hspace{1cm} (D5)

where $d_1$ and $d_2$ determine the sharpness of the first and second phase transitions, respectively.

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