Feature selection with gradient descent on two-layer networks in low-rotation regimes

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Abstract

This work establishes low test error of gradient flow (GF) and stochastic gradient descent (SGD) on two-layer ReLU networks with standard initialization, in three regimes where key sets of weights rotate little (either naturally due to GF and SGD, or due to an artificial constraint), and making use of margins as the core analytic technique. The first regime is near initialization, specifically until the weights have moved by $O(\sqrt{m})$, where $m$ denotes the network width, which is in sharp contrast to the $O(1)$ weight motion allowed by the Neural Tangent Kernel (NTK); here it is shown that GF and SGD only need a network width and number of samples inversely proportional to the NTK margin, and moreover that GF attains at least the NTK margin itself, which suffices to establish escape from bad KKT points of the margin objective, whereas prior work could only establish nondecreasing but arbitrarily small margins. The second regime is the Neural Collapse (NC) setting, where data lies in extremely-well-separated groups, and the sample complexity scales with the number of groups; here the contribution over prior work is an analysis of the entire GF trajectory from initialization. Lastly, if the inner layer weights are constrained to change in norm only and can not rotate, then GF with large widths achieves globally maximal margins, and its sample complexity scales with their inverse; this is in contrast to prior work, which required infinite width and a tricky dual convergence assumption. As purely technical contributions, this work develops a variety of potential functions and other tools which will hopefully aid future work.

1 Introduction

This work studies standard descent methods on two-layer networks of width $m$, specifically stochastic gradient descent (SGD) and gradient flow (GF) on the logistic and exponential losses, with a goal of establishing good test test error (low sample complexity) via margin theory (see Section 1.2 for full details on the setup). The analysis considers three settings where weights rotate little, but rather their magnitudes change a great deal, whereby the networks select or emphasize good features.

The motivation and context for this analysis is as follows. While the standard promise of deep learning is to provide automatic feature learning, by contrast, standard optimization-based analyses typically utilize the Neural Tangent Kernel (NTK) (Jacot et al., 2018), which suffices to establish low training error (Du et al., 2018; Allen-Zhu et al., 2018; Zou et al., 2018), and even low testing error (Arora et al., 2019; Li and Liang, 2018; Ji and Telgarsky, 2020b), but ultimately the NTK is equivalent to a linear predictor over fixed features given by a Taylor expansion, and fails to achieve the aforementioned feature learning promise.

Due to this gap, extensive mathematical effort has gone into the study of feature learning, where the feature maps utilized in deep learning (e.g., those given by Taylor expansions) may change drastically, as occurs in practice. The promise here is huge: even simple tasks such as 2-sparse parity

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(learning the parity of two bits in $d$ dimensions) require $d^2/\epsilon$ samples for a test error of $\epsilon > 0$ within the NTK or any other kernel regime, but it is possible for ReLU networks beyond the NTK regime to achieve an improved sample complexity of $d/\epsilon$ \cite{wei2018}. This point will be discussed in detail throughout this introduction, though a key point is that prior work generally changes the algorithm to achieve good sample complexity; as a brief summary, many analyses either add noise to the training process \cite{shi2022,wei2018}, or train the first layer for only one step and thereafter train only the second \cite{daniely2020,abbe2022,barak2022}, and lastly a few others idealize the network and make strong assumptions to show that global margin maximization occurs, giving the desired $d/\epsilon$ sample complexity \cite{chizat2020}. As will be expanded upon briefly, and indeed is depicted in Figure 1 and summarized in Table 1 in the case of the aforementioned 2-sparse parity, the present work will achieve the optimal kernel sample complexity $d^2/\epsilon$ with a standard SGD (cf. Theorem 2.1), and a beyond-kernel sample complexity of $d/\epsilon$ with an inefficient and somewhat simplified GF (cf. Theorem 3.3).

The technical approach in the present work is to build upon the rich theory of margins in machine learning \cite{boser1992,schapire2012}. While the classical theory provides that descent methods on linear models can maximize margins \cite{zhang2005,telgarsky2013,soudry2017}, recent work in deep networks has revealed that GF eventually converges monotonically to local optima of the margin function \cite{lyu2019,ji2020a}, and even that, under a variety of conditions, margins may be globally maximized \cite{chizat2020}. As mentioned above, global margin maximization is sufficient to beat the NTK sample complexity for many well-studied problems, including 2-sparse parity \cite{wei2018}.

The contributions of this work fall into roughly two categories.

1. **Section 2: Margins at least as good as the NTK.** The first set of results may sound a bit unambitious, but already constitute a variety of improvements over prior work. Throughout these contributions, let $\gamma_{ntk}$ denote the NTK margin, a quantity defined and discussed in detail in Section 1.2.

   (a) **Theorem 2.1** $\tilde{O}(1/(\epsilon \gamma_{ntk}^2))$ steps of SGD (with one example per iteration) on a network of width $m = \Omega(1/\gamma_{ntk}^2)$ suffice to achieve test error $\epsilon > 0$. For 2-sparse parity, this suffices to achieve the optimal within-kernel sample complexity $d^2/\epsilon$, and in fact the computational cost and sample complexity improve upon all existing prior work on efficient methods (cf. Table 1).

   (b) **Theorem 2.2** with similar width and samples, GF also achieves test error $\epsilon > 0$. Arguably more importantly, this analysis gives the first guarantee in a general setting that constant margins (in fact $\gamma_{ntk}/4096$) are achieved; prior work only established nondecreasing but arbitrarily small margins \cite{lyu2019}. Furthermore, this result suffices to imply that GF can escape bad local optima of the margin objective (cf. Proposition 2.6).

   (c) These proofs are not within the NTK: the NTK requires weights to move at most $O(1)$, whereas weights can move by $O(\sqrt{m})$ within these proofs; in fact, the first gradient step of SGD is shown to have norm at least $\gamma \sqrt{m}$ (and the step size is $O(1)$). These proofs will therefore hopefully serve as the basis of other proofs outside the NTK in future work.

2. **Section 3: Margins surpassing the NTK.** The remaining margin results are able to beat the preceding NTK margins, however they pay a large price: the network width is exponentially large, and the method is GF, not the discrete-time algorithm SGD. Even so, the results already require new proof techniques, and again hopefully form the basis for improvements in future work.
(a) Theorem 3.2: the first result is for Neural Collapse (NC), a setting where data lies in \( k \) tightly packed clusters which are extremely far apart; in fact, each cluster is correctly labeled by some vector \( \beta_k \), and no other data may fall in the halfspace defined by \( \beta_k \). Despite this, it is an interesting setting with a variety of empirical and theoretical works, for instance showing various asymptotic stability properties (Papyan et al., 2020); the contribution here is to analyze the entire GF trajectory from initialization, and show that it achieves a margin (and sample complexity) at least as good as the sparse ReLU network with one ReLU pointing at each cluster.

(b) Theorem 3.3: under a further idealization that the inner weights are constrained to change in norm only (meaning the inner weights can not rotate), global margin maximization occurs. As mentioned before, this suffices to establish sample complexity \( d/\epsilon \) in 2-sparse parity, which beats the \( d^2/\epsilon \) of kernel methods. As a brief comparison to prior work, similar idealized networks were proposed by Woodworth et al. (2020), however a full trajectory analysis and relationship to global margin maximization under no further assumptions was left open. Additionally, without the no-rotation constraint, Chizat and Bach (2020) showed that infinite-width networks under a tricky dual convergence assumption also globally maximize margins; the proof technique here is rooted in a desire to drop this dual convergence assumption, a point which will be discussed extensively in Section 3.

(c) New potential functions: both Theorem 3.2 and Theorem 3.3 are proved via new potential arguments which hopefully aid future work.

This introduction will close with further related work (cf. Section 1.1), notation (cf. Section 1.2), and detailed definitions and estimates of the various margin notions (cf. Section 1.2). Margin maximization at least as good as the NTK is presented in Section 2, margins beyond the NTK are in Section 3, and open problems and concluding remarks appear in Section 4. Proofs appear in the appendices.

1.1 Further related work

Margin maximization. The concept and analytical use of margins in machine learning originated in the classical perceptron convergence analysis of Novikoff (1962). The SGD analysis in Theorem 2.1 as well as the training error analysis in Lemma 2.3 were both established with a variant of the perceptron proof; similar perceptron-based proofs appeared before (Ji and Telgarsky, 2020b; Chen et al., 2019), however they required width \( 1/\gamma^8 \), unlike the \( 1/\gamma^2 \) here, and moreover the proofs themselves were in the NTK regime, whereas the proof here is not.

Works focusing on the implicit margin maximization or implicit bias of descent methods are more recent. Early works on the coordinate descent side are (Schapire et al., 1997; Zhang and Yu, 2005; Telgarsky, 2013); the proof here of Lemma 2.4 uses roughly the proof scheme in (Telgarsky, 2013). More recently, margin maximization properties of gradient descent were established, first showing global margin maximization in linear models (Soudry et al., 2017; Ji and Telgarsky, 2018b), then showing nondecreasing smoothed margins of general homogeneous networks (including multi-layer ReLU networks) (Lyu and Li, 2019), and the aforementioned global margin maximization result for 2-layer networks under dual convergence and infinite width (Chizat and Bach, 2020). The potential functions used here in Theorems 3.2 and 3.3 use ideas from (Soudry et al., 2017; Lyu and Li, 2019; Chizat and Bach, 2020, but also the shallow linear and deep linear proofs of Ji and Telgarsky, 2019, 2018a).
Figure 1. Two runs of approximate GF (GD with small step size) on 2-sparse parity with $d = 20$ and $n = 64$ and $m \in \{16, 256\}$: specifically, a new data point $x$ is sampled uniformly from the hypercube corners $\{\pm 1/\sqrt{d}\}^d$, and the label $y$ for simplicity is the parity of the first two bits, meaning $y = dx_1 x_2$. The first two dimensions of the data and the trajectories of the $m$ nodes are depicted in Figures 1a and 1b for $m \in \{16, 256\}$. Figure 1c shows per-node rotation over time in sorter order, meaning $(\frac{\|v_j(0)\|}{\|v_j(t)\|}, \frac{\|v_j(t)\|}{\|v_j(0)\|})^{m}_{j=1}$. Figure 1d shows per-node relative norms in sorted order, meaning $\left(\frac{\|a_j v_j\|_{\max_k \|a_k v_k\|}}{\|a_j v_j\|_{\max_k \|a_k v_k\|}}\right)^m_{j=1}$. Due to projection onto the first two coordinates, the 64 data points land in 4 clusters, and are colored red if negatively labeled, blue if positively labeled. Individual nodes are colored red or blue based on the sign of their output weights. The shades of red and blue darken for nodes whose total norm is larger. Comparing Figure 1c and Figure 1d, large norm and large rotation go together. While Figure 1a is highly noisy, the behavior of Figures 1b to 1d was highly regular across training. The trend of larger width leading to less rotation and greater norm imbalance was consistent across runs. This somewhat justifies the lack of rotation with exponentially large width in Theorem 3.3 and the overall terminology choice of feature selection.
Feature learning. There are many works in feature learning; a few also carrying explicit guarantees on 2-sparse parity are summarized in Table 1. An early work with high relevance to the present work is (Wei et al., 2018), which in addition to establishing that the NTK requires $\Omega(d^2/\epsilon)$ samples whereas $O(d/\epsilon)$ suffice for the global maximum margin solution, also provided a noisy Wasserstein Flow (WF) analysis which achieved the maximum margin solution, albeit using noise, infinite width, and continuous time to aid in local search. The global maximum margin work of Barak et al. (2022) was mentioned before, and will be discussed in Section 3. The work of Chizat and Bach (2020) was a two phase algorithm: the first step has a large minibatch and effectively learns the support of the parity in an unsupervised manner, and thereafter only the second layer is trained, a convex problem which is able to identify the signs within the parity; as in Table 1, this work stands alone in terms of the narrow width it can handle. The work of (Abbe et al., 2022) uses a similar two-phase approach, and while it can not learn precisely the parity, it can learn an interesting class of “staircase” functions, and presents many valuable proof techniques. Another work which operates in two phases and can learn an interesting class of functions which excludes the parity (specifically due to a Jacobian condition) is the recent work of (Damian et al., 2022). Other interesting feature learning works are (Shi et al., 2022; Bai and Lee, 2019).

1.2 Notation

Architecture and initialization. With the exception of Theorem 3.3, the architecture will be a 2-layer ReLU network of the form $x \mapsto F(x; w) = \sum_j a_j \sigma(v_j^T x) = a^T \sigma(V x)$, where $\sigma(z) = \max\{0, z\}$ is the ReLU, and where $a \in \mathbb{R}^m$ and $V \in \mathbb{R}^{m \times d}$ have initialization roughly matching the variances of pytorch default initialization, meaning $a \sim \mathcal{N}_m/\sqrt{m}$ ($m$ iid Gaussians with variance $1/m$) and $V \sim \mathcal{N}_{m \times d}/\sqrt{d}$ ($m \times d$ iid Gaussians with variance $1/d$). These parameters $(a, V)$ will be collected into a tuple $W = (a, V) \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \equiv \mathbb{R}^{m \times (d+1)}$, and for convenience per-node tuples $w_j = (a_j, v_j) \in \mathbb{R} \times \mathbb{R}^d$ will often be used as well.

Given a pair $(x, y)$ with $x \in \mathbb{R}^d$ and $y \in \{\pm 1\}$, the prediction or unnormalized margin mapping is $p(x, y; W) = y F(x; W) = y a^T \sigma(V x)$; when examples $((x_i, y_i))_{i=1}^n$ are available, a simplified notation $p_i(W) := p(x_i, y_i; W)$ is often used, and moreover define a single-node variant $p_i(w_j) := y_i a_j \sigma(v_j^T x_i)$. Throughout this work, $\|x\| \leq 1$.

It will often be useful to consider normalized parameters within proofs: define $\overline{v}_j := v_j/\|v_j\|$ and $\overline{a}_j := \text{sgn}(a_j) := a_j/|a_j|$.
**SGD and GF.** The loss function $\ell$ in this work will always be either the exponential loss $\ell_{\exp}(z) := \exp(-z)$, or the logistic loss $\ell_{\log}(z) := \ln(1 + \exp(-z))$; the corresponding empirical risk $\widehat{R}$ is

$$
\widehat{R}(p(W)) := \frac{1}{n} \sum_{i=1}^{n} \ell(p_i(W)),
$$

which used $p(W) := (p_1(W), \ldots, p_n(W)) \in \mathbb{R}^n$. With $\ell$ and $\widehat{R}$ in hand, the descent methods are defined as

$$
W_{t+1} := W_t - \eta \partial_t \ell(p_t(W)), \quad \text{stochastic gradient descent (SGD),} \quad (1.1)
$$

$$
\dot{W}_t := \frac{d}{dt} W_t = -\bar{\partial}_W \widehat{R}(p(W)), \quad \text{gradient flow (GF),} \quad (1.2)
$$

where $\partial_t$ and $\bar{\partial}$ are appropriate generalizations of subgradients for the present nonsmooth nonconvex setting, detailed as follows. For SGD, $\partial_t$ will denote any valid element of the Clarke differential (i.e., a measurable selection); for example, $\partial F(x;W) = \left(\sigma(Vx), \sum_j a_j \sigma'(v_j^T x)e_jx^T \right)$, where $e_j$ denotes the $j$th standard basis vector, and $\sigma'(v_j^T x) \in [0,1]$ is chosen in some consistent and measurable way. This approach is able to model what happens in a standard software library such as *pytorch*. For GF, $\partial_t$ will denote the unique minimum norm element of the Clarke differential; typically, GF is defined as a differential inclusion, which agrees with this minimum norm Clarke flow almost everywhere, but here the minimum norm element is used to define the flow purely for fun. Since at this point there is a vast array of ReLU network literature using Clarke differentials, it is merely asserted here that chain rules and other basic properties work as required, and $\partial_t$ and $\bar{\partial}$ are used essentially as gradients, and details are deferred to the detailed discussions in [Lyu and Li, 2019; Ji and Telgarsky, 2020a; Lyu et al., 2021].

Many expressions will have a fixed time index $t$ (e.g., $W_t$), but to reduce clutter, it will either appear as a subscript, or as a function (e.g., $W(t)$), or even simply dropped entirely.

**Margins and dual variables.** Firstly, for convenience, often $\ell_t(W) = \ell(p_t(W))$ and $\ell'_t(W) := \ell'(p_t(W))$ are used; since $\ell_t'$ is negative, often $|\ell'_t|$ is written.

The start of developing margins and dual variables is the observation that $F$ and $p_t$ are $2$-homogeneous in $W$, meaning $F(x; cW) = ca^T \sigma(cVx) = c^2 F(x; w)$ for any $x \in \mathbb{R}^d$ and $c \geq 0$ (and $p_t(cW) = c^2 p_t(W)$). It follows that $F(x; W) = \|W\|^2 F(x; W/\|W\|)$, and thus $F$ and $p_t$ scale quadratically in $\|W\|$, and it makes sense to define a *normalized* prediction mapping $\tilde{p}_t$ and margin $\gamma$ as

$$
\tilde{p}_t(W) := \frac{p_t(W)}{\|W\|^2}, \quad \gamma(W) := \min_i \frac{p_t(W)}{\|W\|^2} = \min \tilde{p}_t(W).
$$

Due to nonsmoothness, $\gamma$ can be hard to work with, thus, following [Lyu and Li, 2019], define the *smoothed margin* $\tilde{\gamma}$ and the *normalized smoothed margin* $\hat{\gamma}$ as

$$
\tilde{\gamma}(W) := \ell^{-1} \left( n\widehat{R}(W) \right) = \ell^{-1} \left( \sum_i \ell(p_i(W)) \right), \quad \hat{\gamma}(W) := \frac{\tilde{\gamma}(W)}{\|W\|^2},
$$

where a key result is that $\hat{\gamma}$ is eventually nondecreasing [Lyu and Li, 2019]. These quantities may look complicated and abstract, but note for $\ell_{\exp}$ that $\tilde{\gamma}(W) := -\ln \left( \sum_i \exp(-p_i(W)) \right)$. 


When working with gradients of losses and of smoothed margins, it will be convenient to define dual variables \( (q_i)_{i=1}^n \)

\[
q := \nabla_p \ell^{-1} \left( \sum_i \ell(p_i) \right) = \frac{\nabla_p \sum_i \ell(p_i)}{\ell'(\ell^{-1}(\sum_i \ell(p_i)))} = \frac{\nabla_p \sum_i \ell(p)}{\ell'(\gamma(p))},
\]

which made use of the inverse function theorem. Correspondingly define \( Q := \ell'(\gamma(p)) \), whereby \(-\ell'_i = q_i Q; \) for the exponential loss, \( Q = \sum_i \exp(-p_i) \) and \( \sum_i q_i = 1 \), and while these quantities are more complicated for the logistic loss, they eventually satisfy \( \sum_i q_i \geq 1 \) (Ji and Telgarsky [2019] Lemma 5.4, first part, which does not depend on linear predictors).

**Margin assumptions.** It is easiest to first state the global margin definition, used in Section 3. The first version is stated on a finite sample.

**Assumption 1.1.** For given examples \( ((x_i, y_i))_{i=1}^n \), there exists a scalar \( \gamma_\delta > 0 \) and parameters \( ((\alpha_k, \beta_k))_{k=1}^r \) with \( \|\alpha\|_1 = 1 \) and \( \beta_k \in \mathbb{R}^d \) with \( \|\beta_k\|_2 = 1 \), where

\[
\min_i y_i \sum_{k=1}^r \alpha_k \sigma(\beta_k^T x_i) \geq \gamma_\delta.
\]

This definition can also be extended to hold over a distribution, whereby it holds for all finite samples almost surely.

**Assumption 1.2.** There exists \( \gamma_\delta > 0 \) and parameters \( ((\alpha_k, \beta_k))_{k=1}^r \) so that Assumption 1.1 holds almost surely for any \( ((x_i, y_i))_{i=1}^n \) drawn iid for any \( n \) from the underlying distribution (with the same \( \gamma_\delta \) and \( ((\alpha_k, \beta_k))_{k=1}^r \)).

Assumptions 1.1 and 1.2 are used in this work to approximate the best possible margin amongst all networks of any width; in particular, the formalism here is a simplification of the max-min characterization given in (Chizat and Bach [2020], Proposition 12, optimality conditions). As will be seen shortly in Proposition 1.6, this definition suffices to achieve sample complexity \( d/\epsilon \) with gradient flow on 2-sparse parity, as in Table 1. Lastly, while the appearance of an \( \ell_1 \) norm may be a surprise, it can be seen to naturally arise from 2-homogeneity:

\[
\frac{p_i(W)}{\|W\|^2} = \frac{\sum_j p_i(w_j)}{\|W\|^2} = \sum_j \alpha_j \overline{p}_i(w_j), \quad \text{where } \alpha_j := \frac{\|w_j\|^2}{\|W\|^2}, \text{ thus } \|\alpha\|_1 = 1.
\]

Next comes the definition of the NTK margin \( \gamma_{ntk} \). As a consequence of 2-homogeneity, \( \langle W, \partial_W p_i(W) \rangle = 2p_i(W) \), which can be interpreted as a linear predictor with weights \( W \) and features \( \partial_W p_i(W)/2 \). Decoupling the weights and features gives \( \langle W, \partial_W p_i(W_0) \rangle \), where \( W_0 \) is at initialization, and \( W \) is some other choice. To get the full definition from here, \( W \) is replaced with an infinite-width object via expectations; similar definitions originated with the work of Nitanda and Suzuki [2019], and were then used in (Ji and Telgarsky [2020b] [Chen et al. 2019]).

**Assumption 1.3.** For given examples \( ((x_i, y_i))_{i=1}^n \), there exists a scalar \( \gamma_{ntk} > 0 \) and a mapping \( \theta : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} \) with \( \theta(w) = 0 \) whenever \( \|w\| \geq 2 \) so that

\[
\min_{\bar{v}} \mathbb{E}_{w \sim \mathcal{N}_0} \langle \theta(w), \bar{\theta}_w p_i(w) \rangle \geq \gamma_{ntk},
\]

where \( w = (a, v) \sim \mathcal{N}_0 \) means \( a \sim \mathcal{N} \) and \( v \sim \mathcal{N}_d/\sqrt{d} \).
Similarly to the global maximum margin definition, Assumption 1.3 can be restated over the distribution.

Assumption 1.4. There exists $\gamma_{ntk} > 0$ and a transport $\theta : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ so that Assumption 1.3 holds almost surely for any $((x_i, y_i))_{i=1}^n$ drawn iid for any $n$ from the underlying distribution (with the same $\gamma_{ntk}$ and $\theta$).

Before closing this section, here are a few estimates of $\gamma_{ntk}$ and $\gamma_{gl}$. Firstly, both function classes are universal approximators, and thus the assumption can be made to work for any prediction problem with pure conditional probabilities (Ji et al., 2020). Next, as a warmup, note the following estimates of $\gamma_{ntk}$ and $\gamma_{gl}$, for linear predictors, with an added estimate of showing the value of working with both layers in the definition of $\gamma_{ntk}$.

Proposition 1.5. Let examples $((x_i, y_i))_{i=1}^n$ be given, and suppose they are linearly separable: there exists $\Vert \bar{u} \Vert = 1$ and $\hat{\gamma} > 0$ with $\min_i y_i x_i^T \bar{u} \geq \hat{\gamma}$.

1. Choosing $\theta(a, v) := (0, \text{sgn}(a) \bar{u}) \cdot 1[\Vert (a, v) \Vert \leq 2]$, then Assumption 1.3 holds with $\gamma_{ntk} \geq \frac{\hat{\gamma}}{32}$.
2. Choosing $\theta(a, v) := (\text{sgn}(\bar{u}^T v), 0) \cdot 1[\Vert (a, v) \Vert \leq 2]$, then Assumption 1.3 holds with $\gamma_{ntk} \geq \frac{\hat{\gamma}}{16d}$.
3. Choosing $\alpha = (1/2, -1/2)$ and $\beta = (\bar{u}, -\bar{u})$, then Assumption 1.1 holds with $\gamma_{gl} \geq \frac{\hat{\gamma}}{2}$.

Margin estimates for 2-sparse parity are as follows; the key is that $\gamma_{ntk}$ scales with $1/d$ whereas $\gamma_{gl}$ scales with $1/\sqrt{d}$, which suffices to yield the separations in Table 1. The bound on $\gamma_{ntk}$ is also necessarily an upper bound, since otherwise the estimates due to Ji and Telgarsky (2020b), which are within the NTK regime, would beat NTK lower bounds (Wei et al., 2018).

Proposition 1.6. Suppose 2-sparse parity data, meaning inputs are supported on $H_d := \{\pm 1/\sqrt{d}\}^d$, and for any $x \in H_d$, the label is the product of two fixed coordinates $d x_a x_b$ with $a \neq b$.

1. Assumption 1.4 holds with $\gamma_{ntk} \geq \frac{1}{\text{sgd}}$.
2. Assumption 1.2 holds with $\gamma_{gl} \geq \frac{1}{\sqrt{\text{sgd}}}$.

2 Margins at least as good as the NTK

This section collects results which depend on the NTK margin $\gamma_{ntk}$ (cf. Assumptions 1.3 and 1.4). SGD is presented first in Section 2.1 with GF following in Section 2.2. The SGD results will not establish large margins, only low test error, whereas the GF proofs establish both. As mentioned before, these results yield the good computation and sample complexity for 2-sparse parity in Table 1 and are also enough to establish escape from bad KKT points in Section 2.2.

2.1 Stochastic gradient descent

The only SGD guarantee in this work is as follows.

Theorem 2.1. Suppose the data distribution satisfies Assumption 1.4 for some $\gamma_{ntk} > 0$, let time $t$ be given, and suppose width $m$ and step size $\eta$ satisfy

$$m \geq \left( \frac{64 \ln(t/\delta)}{\gamma_{ntk}} \right)^2,$$

$$\eta \in \left[ \frac{\gamma_{ntk}}{10 \sqrt{m}}, \frac{\gamma_{ntk}^2}{6400} \right].$$
Then, with probability at least $1 - 7\delta$, the SGD iterates $(W_s)_{s \leq t}$ with logistic loss $\ell = \ell_{\text{log}}$ satisfy

$$
\min_{s \leq t} \Pr \left[ p(x, y; W_s) \leq 0 \right] \leq \frac{8 \ln(1/\delta)}{t} + \frac{2560}{\ell_{\text{ntk}}} \left( 80\eta \sqrt{m} \right),
$$

(test error bound),

$$
\max_{s \leq t} \| W_s - W_0 \| \leq \frac{80\eta \sqrt{m}}{\gamma_{\text{ntk}}},
$$

(norm bound).

Note that while $\max_{s \leq t} \| W_s - W_0 \| \leq 80\eta \sqrt{m}/\gamma_{\text{ntk}}$ is only an upper bound, in fact, by Lemma A.3, the first gradient has norm $\gamma_{\text{ntk}} \sqrt{m}$, thus one step (with maximal step size $\eta = \gamma_{\text{ntk}}^2/6400$) is enough to exit the NTK regime. As another incidental remark, note that this proof requires the logistic loss $\ell_{\text{log}}$, and in fact breaks with the exponential loss $\ell_{\text{exp}}$. Lastly, for the 2-sparse parity, $\gamma_{\text{ntk}} \geq 1/(50d)$ as in Proposition 1.6, which after plugging in to Theorem 2.1 gives the corresponding row of Table 1.

As discussed previously, the width is only $1/\gamma_{\text{ntk}}^2$, whereas prior work has $1/\gamma_{\text{ntk}}^8$ (Ji and Telgarsky 2020b, Chen et al. 2019). The proof of Theorem 2.1 is in Appendix B.1 but here is a sketch.

1. **Sampling a good finite-width comparator $\tilde{W}$.** Perhaps the heart of the proof is showing that the parameters $\tilde{\theta} \in \mathbb{R}^{m \times (d + 1)}$ given by $\tilde{\theta}_j := \theta(w_j)$ satisfy

$$
\langle \tilde{\theta}, \partial p_s(W_0) \rangle \geq \frac{\gamma_{\text{ntk}} \sqrt{m}}{2}, \quad \forall i.
$$

More elaborate versions of this are used in the proof, and sometimes require a bit of surprising algebra (cf. Lemma A.3), which for instance seem to be able to treat $\sigma$ as though it were smooth, and without the usual careful activation-accounting in ReLU NTK proofs.

2. **Standard expand-the-square.** As is common in optimization proofs, the core potential is a squared Euclidean norm to a good comparator (in this case denoted by $\tilde{W}$, and defined in terms of $W_0$ and $\tilde{\theta}$), whereby expanding the square gives

$$
\| W_{s+1} - \tilde{W} \|^2 = \| W_s - \eta \partial \ell_s(w) - \tilde{W} \|^2
$$

$$
= \| W_s - \tilde{W} \|^2 - 2\eta \langle \partial \ell_s(W_s), W_s - \tilde{W} \rangle + \eta^2 \| \partial \ell_s(W_s) \|^2
$$

$$
= \| W_s - \tilde{W} \|^2 + 2\eta \ell_s(W_s) \langle \partial p_s(W_s), \tilde{W} - W_s \rangle + \eta^2 \ell_s(W_s)^2 \| \partial p_s(W_s) \|^2.
$$

Applying $\sum_s$ to both sides and telescoping, the (summations of the) last two terms will need to be controlled. A worrisome prospect is $\| \partial p_s(W_s) \|^2$; by the form of $p_s$ and the large initial weight norm, this can be expected to scale as $\mathcal{O}(m)$, but $\eta^2 = \mathcal{O}(1)$; how can this term be swallowed? This point will be returned to shortly.

Another critical aspect of the proof is that, in order to control various terms, it will be necessary to maintain $\max_{s \leq t} \| W_s - W_0 \| = \mathcal{O}(\sqrt{m})$ throughout. The proof handles this in a way which is common in deep network optimization proofs (albeit with a vastly larger norm here): by carefully choosing the parameters of the proof, one can let $\tau$ denote the first iteration the bound is violated, and then derive that $\tau > t$, and all the derivations with the assumed bound in fact hold unconditionally.

The middle term $2\eta \ell_s(W_s) \langle \partial p_s(W_s), \tilde{W} - W_s \rangle$ will be handled by a similar trick to one used in (Ji and Telgarsky 2020b); convexity can still be applied to $\ell_s$ (just not to $\ell$ composed with $p$), and the remaining expression can be massaged via homogeneity and the choice of $\tilde{W}$.
3. **Controlling $\ell'_s(W_s)^2\|\hat{p}_s(W_s)\|^2$: the perceptron argument.** Using the first point above, it is possible to show

$$\|W_t - W_0\| \geq \frac{1}{2} \left\langle -\vartheta, W_t - W_0 \right\rangle \geq \eta \sum_{s < t} |\ell'_s(W_s)| \frac{\gamma \sqrt{m}}{4},$$

which can then be massaged to control the aforementioned squared gradient term. Interestingly, this proof step is reminiscent of the perceptron convergence proof, and the quantity $\sum_{s < t} |\ell'_s(W_s)|$ is exactly the analog of the mistake bound quantity central in perceptron proofs \cite{Novikoff1962}. Indeed, this proof never ends up caring about the loss terms, and derives the final test error bound (a zero-one loss!) via this perceptron term.

### 2.2 Gradient flow

Whereas SGD gave a test error guarantee for free, producing a comparable test error bound with GF in this section will require much more work. This section also sketches the main steps of the proof, and then closes with a discussion of escaping bad KKT points.

**Theorem 2.2.** Suppose the data distribution satisfies Assumption 1.4 for some $\gamma_{ntk} > 0$, and the GF curve $(W_s)_{s \geq 0}$ uses $\ell \in \{\ell_{\exp}, \ell_{\log}\}$ on an architecture whose width $m$ satisfies

$$m \geq \left( \frac{640 \ln(n/\delta)}{\gamma_{ntk}} \right)^2.$$

Then, with probability at least $1 - 15\delta$, there exists $t$ with $\|W_t - W_0\| = \gamma_{ntk} \sqrt{m}/32$, and

$$\gamma(W_s) \geq \frac{\gamma_{ntk}^2}{4096} \quad \text{and} \quad \Pr[p(x, y; W_s) \leq 0] \leq \mathcal{O} \left( \frac{\ln(n)^3}{n \gamma_{ntk}^4} + \frac{\ln \frac{1}{\delta}}{n} \right) \quad \forall s \geq t,$$

and moreover the specified iterate $W_t$ satisfies an improved bound

$$\Pr \left[p(x, y; W_t) \leq 0 \right] \leq \mathcal{O} \left( \frac{\ln(n)^3}{n \gamma_{ntk}^2} + \frac{\ln \frac{1}{\delta}}{n} \right).$$

A few brief remarks are as follows. Firstly, for $W_t$, the sample complexity matches the SGD sample complexity in Theorem 2.1, though via a much more complicated proof. Second, this proof can handle $\{\ell_{\log}, \ell_{\exp}\}$ and not just $\ell_{\log}$. Lastly, an odd point is that a bit of algebra grants a better generalization bound at iterate $W_t$, but it is not clear if this improved bound holds for all time $s \geq t$; in particular it is not immediately clear that the nondecreasing margin property established by Lyu and Li \cite{LyuLi2019} can be applied.

One interesting comparison is to a leaky ReLU convergence analysis on a restricted form of linearly separable data due to Lyu et al. \cite{Lyu2021}. That work, through an extremely technical and impressive analysis, establishes convergence to a solution which is equivalent to the best linear predictor. By contrast, while the work here does not recover that analysis, due to $\gamma_{ntk}$ being a constant multiple of the linear margin (cf. Proposition 1.5), the sample complexity is within a constant factor of the best linear predictor, thus giving a sample complexity comparable to that of \cite{Lyu2021} via a simpler proof in a more general setting.

To prove Theorem 2.2, the first step is essentially the same as the proof of Theorem 2.1 however it yields only a training error guarantee, not a test error guarantee.
Lemma 2.3. Suppose the data distribution satisfies Assumption I.A for some $\gamma_{ntk} > 0$, let time $t$ be given, and suppose width $m$ satisfies

$$m \geq \left( \frac{640 \ln(t/\delta)}{\gamma_{ntk}} \right)^2.$$ 

Then, with probability at least $1 - 7\delta$, the GF curve $(W_s)_{s \in [0,t]}$ on empirical risk $\hat{R}$ with loss $\ell \in \{\ell_{log}, \ell_{exp}\}$ satisfies

$$\hat{R}(W_t) \leq \frac{1}{5t},$$

(training error bound),

$$\sup_{s < t} \|W_s - W_0\| \leq \frac{\gamma_{ntk} \sqrt{m}}{80},$$

(norm bound).

Note that this bound is morally equivalent to the SGD bound in Theorem 2.1 after accounting for the $\gamma_{ntk}^2$ “units” arising from the step size.

The second step of the proof of Theorem 2.2 is an explicit margin guarantee, which is missing from the SGD analysis.

Lemma 2.4. Let data $((x_i, y_i))_{i=1}^n$ be given satisfying Assumption I.B with margin $\gamma_{ntk} > 0$, and let $(W_s)_{s \geq 0}$ denote the GF curve resulting from loss $\ell \in \{\ell_{log}, \ell_{exp}\}$. Suppose the width $m$ satisfies

$$m \geq \frac{256 \ln(n/\delta)}{\gamma_{ntk}^2},$$

fix a distance parameter $R := \gamma_{ntk} \sqrt{m}/32$, and let time $\tau$ be given so that $\|W_\tau - W_0\| \leq R/2$ and $R(W_\tau) < \ell(0)/n$. Then, with probability at least $1 - 7\delta$, there exists a time $t$ with $\|W_t - W_0\| = R$ so that for all $s \geq t$,

$$\|W_s - W_0\| \geq R \quad \text{and} \quad \hat{\gamma}(W_s) \geq \frac{\gamma_{ntk}^2}{4096},$$

and moreover the rebalanced iterate $\tilde{W}_t := (a_t/\sqrt{\gamma_{ntk}}, v_t \sqrt{\gamma_{ntk}})$ satisfies $p(x, y; W_t) = p(x, y; \tilde{W}_t)$ for all $(x, y)$, and

$$\hat{\gamma}(\tilde{W}_t) \geq \frac{\gamma_{ntk}}{4096}.$$ 

Before discussing the proof, a few remarks are in order. Firstly, the final large margin iterate $W_t$ is stated as explicitly achieving some distance from initialization; needing such a claim is unsurprising, as the margin definition requires a lot of motion in a good direction to clear the noise in $W_0$. In particular, it is unsurprising that moving $O(\sqrt{m})$ is needed to achieve a good margin, given that the initial weight norm is $O(\sqrt{m})$; analogously, it is not surprising that Lemma 2.3 can not be used to produce a meaningful lower bound on $\hat{\gamma}(W_\tau)$ directly.

Regarding the proof, surprisingly it can almost verbatim follow a proof scheme originally designed for margin rates of coordinate descent [Telgarsky, 2013]. Specifically, noting that $\partial_\omega \gamma(W_s)$ and $\tilde{W}_s$ are colinear, the fundamental theorem of calculus (adapted to Clarke differentials) gives

$$\gamma(W_t) - \gamma(W_\tau) = \int_\tau^t \frac{d}{ds} \gamma(W_s) \, ds = \int_\tau^t \left< \partial_\omega \gamma(W_s), \tilde{W}_s \right> \, ds = \int_\tau^t \|\partial_\omega \gamma(W_s)\| \cdot \|\tilde{W}_s\| \, ds,$$

and now the terms can be controlled separately. Assuming the exponential loss for simplicity and recalling the dual variable notation from Section 1.2, then $\partial_\omega \gamma(W_s) = \sum_i q_i \partial_\omega p_i(W_s)$. Consequently, using the same good property of $\partial$ discussed in Section 2.1 gives

$$\|\partial_\omega \gamma(W_s)\| = \|\sum_i q_i \partial_\omega p_i(W_s)\| \geq \sum_i q_i \left< \frac{\partial}{\|\partial\|}, \partial_\omega p_i(W_s) \right> \geq \frac{\gamma_{ntk} \sqrt{m}}{4}.$$
Figure 2. An arrangement of positively labeled points (the blue x’s) where the margin objective has multiple KKT points, and the gradient flow is able to avoid certain bad ones. Specifically, as the two cones of data $S_1$ and $S_2$ are rotated away from each other, the linear predictor $u$ may still achieve a positive margin, but it will become arbitrarily small. By contrast, pointing two ReLUs at each of $S_1$ and $S_2$ achieves a much better margin. Lemma 2.3 is strong enough to establish this occurs, at least for some arrangements of the cones, as detailed in Proposition 2.6. This construction is reminiscent of other bad KKT constructions in the literature (Lyu et al., 2021; Vardi et al., 2021).

This leaves the other term of the integral, which is even easier:

$$\int_{\tau}^{t} \|\dot{W}_s\| \, ds \geq \int_{\tau}^{t} \|\dot{W}_s\| \, ds = \|W_t - W_\tau\| \geq \frac{R}{2},$$

which completes the proof for the exponential loss. For the logistic loss, the corresponding elements $q_i$ do not form a probability vector, and necessitate the use of a 2-phase analysis which warm-starts with Lemma 2.3.

To finish the proof of Theorem 2.2 it remains to relate margins to test error, which in order to scale with $1/n$ rather than $1/\sqrt{n}$ makes use of a beautiful refined margin-based Rademacher complexity bound due to Srebro et al. (2010, Theorem 5), and thereafter uses the special structure of 2-homogeneity to treat the network as an $\ell_1$-bounded linear combination of nodes, and thereby achieve no dependence, even logarithmic, on the width $m$.

**Lemma 2.5.** With probability at least $1 - \delta$ over the draw of $((x_i, y_i))_{i=1}^n$, for every width $m$, every choice of weights $W \in \mathbb{R}^{m \times (d+1)}$ with $\tilde{\gamma}(W) > 0$ satisfies

$$\Pr[p(x, y; W) \leq 0] \leq O\left(\frac{\ln(n)^3}{n\tilde{\gamma}(W)^2} + \frac{1}{n}\right).$$

Combining the preceding pieces yields the proof of Theorem 2.2. To conclude this section, note that these margin guarantees suffice to establish that GF can escape bad KKT points of the margin objective. The construction appears in Figure 2 and is elementary, detailed as follows. Consider data, all of the same label, lying in two narrow cones $S_1$ and $S_2$. If $S_1$ and $S_2$ are close together, the global maximum margin network corresponds to a single linear predictor. As the angle between $S_1$ and $S_2$ is increased, eventually the global maximum margin network chooses two separate ReLUs, one pointing towards each cone; meanwhile, before the angle becomes too large, if $S_1$ and $S_2$ are sufficiently narrow, there exists a situation whereby a single linear predictor still has positive margin, but worse than the 2 ReLU solution, and is still a KKT point.
The resulting data $S$ then, with probability at least $1$, satisfies Assumption 3.1. Suppose the data distribution satisfies Assumption 3.1 for some $r, \gamma_{nc}, \epsilon$, and let $\ell = \ell_{\text{exp}}$ be given. If the network width $m$ satisfies

$$m \geq 2 \left( \frac{2}{\epsilon} \right)^d \ln \frac{r}{\delta},$$

then, with probability at least $1 - 3\delta$, the GF curve $(W_s)_{s \geq 0}$ for all large times $t$ satisfies

$$\hat{\gamma}(W_t) \geq \frac{\gamma_{nc} - \epsilon}{8r}$$

and

$$\Pr[p(x, y; W_t) \leq 0] = O \left( \frac{r^2 \ln(n)^3}{n(\gamma_{nc} - \epsilon)^2} + \frac{\ln \frac{1}{\delta}}{n} \right).$$

Note that Theorem 3.2 only implies that GF selects a predictor with margins at least as good as the NC solution, and does not necessarily converge to the NC solution (i.e., rotating all ReLUs to point in the directions $(\beta_k)_{k=1}^n$ specified by Theorem 3.2). In fact, this may fail to be true, and Proposition 2.6 and Figure 2 already gave one such construction; moreover, this is not necessarily bad, as GF may converge to a solution with better margins, and potentially better generalization. Overall, the relationship of NC to the bias of 2-layer network training in practical regimes remains open.

### 3 Margins beyond the NTK

This section develops two families of bounds beyond the NTK, meaning in particular that the final margin and sample complexity bounds depend on $\gamma_{sl}$, rather than $\gamma_{ntk}$ as in Section 2. On the downside, these bounds all require exponentially large width, GF, and moreover Theorem 3.3 forces the inner layer to never rotate. These results are proved with $\ell_{\text{exp}}$ for convenience, though the same techniques handling $\ell_{\log}$ with GF in Section 2.2 should also work here.

#### 3.1 Neural collapse (NC)

The NC setting has data in groups which are well-separated (Papyan et al., 2020); in particular, data is partitioned into cones, and all data points outside a cone live within the convex polar to that cone (Hiriart-Urruty and Lemaréchal, 2001). The formal definition is as follows.

**Assumption 3.1.** There exist $(\beta_k)_{k=1}^n$ with $\|\beta_k\| = 1$ and $\alpha_k \in \{\pm 1/k\}$ and $\gamma_{nc} > 0$ and $\epsilon \in (0, \gamma_{nc})$ so that almost surely over the draw of any data $(x_i, y_i)_{i=1}^n$, then for any particular $(x_i, y_i, \beta_k)$:

- either $\beta_k^T x_i y_i \geq \gamma_{nc}$ and $\|(I - \beta_k \beta_k^T) x_i\| \leq \gamma_{nc} \sqrt{\epsilon/2}$ (example $i$ lies in a narrow cone around $\beta_k$),
- or $\beta_k^T x_i y_i \leq -\epsilon$ (example $i$ lies in the polar of the cone around $\beta_k$).

It follows that Assumption 3.1 implies Assumption 1.2 with margin $\gamma_{sl} \geq \gamma_{nc}/k$, but the condition is quite a bit stronger. The corresponding GF result is as follows.

**Theorem 3.2.** Suppose the data distribution satisfies Assumption 3.1 for some $(r, \gamma_{nc}, \epsilon)$, and let $\ell = \ell_{\text{exp}}$ be given. If the network width $m$ satisfies

$$m \geq 2 \left( \frac{2}{\epsilon} \right)^d \ln \frac{r}{\delta},$$

then, with probability at least $1 - 3\delta$, the GF curve $(W_s)_{s \geq 0}$ for all large times $t$ satisfies

$$\hat{\gamma}(W_t) \geq \frac{\gamma_{nc} - \epsilon}{8r}$$

and

$$\Pr[p(x, y; W_t) \leq 0] = O \left( \frac{r^2 \ln(n)^3}{n(\gamma_{nc} - \epsilon)^2} + \frac{\ln \frac{1}{\delta}}{n} \right).$$

Overall, the relationship of NC to the bias of 2-layer network training in practical regimes remains open.
The proof of Theorem 3.2 proceeds by developing a potential functions that asserts that either mass grows in the directions \((\beta_k)_k\), or their margin is exceeded. Within the proof, large width ensures that the mass in each good direction is initially positive, and thereafter Assumption 3.1 is used to ensure that the fraction of mass in these directions is increasing. The proof of Theorem 3.2 and of Theorem 3.3 invoke the same abstract potential function lemma, and discussion is momentarily deferred until after the presentation of Theorem 3.3.

One small point is worth explaining now, however. It may have seemed unusual to use \(\|a_j v_j\|\) as a (squared!) norm in Figure 1. Of course, layers asymptotically balance, thus asymptotically not only is there the Fenchel inequality \(2\|a_j v_j\| \leq a_j^2 + \|v_j\|^2 = \|w_j^2\|\), but also a reverse inequality \(2\|a_j v_j\| \gtrsim \|w_j\|^2\). Despite this fact, the disagreement between \(2\|a_j v_j\|\) and \(\|w_j\|^2\), namely the imbalance between \(a_j^2\) and \(\|v_j\|^2\), can cause real problems, and one solution used within the proofs is to replace \(\|w_j\|^2\) with \(\|a_j v_j\|\).

### 3.2 Global margin maximization

The final theorem will be on stylized networks where the inner layer is forced to not rotate. Specifically, the networks are of the form

\[
x \mapsto \sum_j a_j \sigma(b_j v_j^i x),
\]

where \(((a_j, b_j))_j^m = 1\) are trained, but \(v_j\) are fixed at initialization; the new scalar parameter \(b_j\) is effectively the norm of \(v_j\) (though it is allowed to be negative). As a further simplification, \(a_j\) and \(b_j\) are initialized to have the same norm; this initial balancing is common in many implicit bias proofs, but is impractical and constitutes a limitation to improve in future work. While it is clearly unpleasant that \((v_j)_j^m = 1\) can not rotate, Figure 1 provides some hope that this is approximated in networks of large width.

**Theorem 3.3.** Suppose the data distribution satisfies Assumption 1.2 for some \(\gamma_{\delta} > 0\) with reference architecture \(((\alpha_k, \beta_k))_k^r\). Consider the architecture \(x \mapsto \sum_j a_j \sigma(b_j v_j^i x)\) where \(((a_j(0), b_j(0)))_j^m = 1\) are sampled uniformly from the two choices \(\pm 1/m^{1/4}\), and \(v_j(0)\) is sampled from the unit sphere (e.g., first \(v_j^i \sim \mathcal{N}_d / \sqrt{d}\), then \(v_j(0) := v_j^i / \|v_j^i\|\), and

\[
m \geq 2 \left(\frac{4}{\gamma_{\delta}}\right)^d \ln \frac{r}{\delta}.
\]

Then, with probability at least \(1 - 3\delta\), for all large \(t\), GF on \(((a_j, b_j))_j^m = 1\) with \(\ell_{\exp}\) satisfies

\[
\hat{\gamma}_t((a(t), b(t))) \geq \frac{\gamma_{\delta}}{4} \quad \text{and} \quad \Pr[p(x, y; (a(t), b(t))) \leq 0] = O\left(\frac{\ln(n)^3}{n \gamma_{\delta}^2} + \frac{\ln 1}{n}\right).
\]

The main points of comparison for Theorem 3.3 are the global margin maximization proofs of [Wei et al., 2018] and [Chizat and Bach, 2020]. The analysis by [Wei et al., 2018] is less similar, as it heavily relies upon the benefits to local search arising from weight re-initialization, whereas the analysis here in some sense is based on the technique in [Chizat and Bach, 2020], but diverges sharply due to dropping the two key assumptions therein. Specifically, [Chizat and Bach, 2020] requires infinite width and dual convergence, meaning \(q(t)\) converges, which is open even for linear predictors in general settings. The infinite width assumption is also quite strenuous: it is used to ensure not just that weights cover the sphere at initialization (a consequence of exponentially large width), but in fact that they cover the sphere for all times \(t\).
The proof strategy of Theorem 3.3 (and Theorem 3.2) is as follows. The core of the proof scheme in (Chizat and Bach, 2020) is to pick two weights \( w_j \) and \( w_k \), where \( w_k \) achieves better margins than \( w_j \) in some sense, and consider

\[
\frac{d}{dt} \frac{\|w_j\|^2}{\|w_k\|^2} = 4Q \left( \frac{\|w_j\|^2}{\|w_k\|^2} \right) \sum_i q_i [\tilde{p}_i(w_j) - \tilde{p}_i(w_k)];
\]

as a purely technical aside, it is extremely valuable that this ratio potential automatically normalizes the margins, leading to the appearance of \( \tilde{p}_i(w_j) \) not \( p_i(w_j) \), and a similar idea is used in the proofs here, albeit starting from \( \ln \|w_j\| - \ln \|w_k\| \), the idea of \( \ln(\cdot) \) also appearing in the proofs by Lyu and Li (2019). Furthermore, this expression already shows the role of dual convergence: if we can assume every \( q_i \) converges, then we need only pick nodes \( w_k \) for which the margin surrogate \( \sum_i q_i(\infty)\tilde{p}_i(w_k(\infty)) \) is very large, and the above time derivative becomes negative if \( w_j \) has bad margin, which implies mass accumulates in directions with good margin. This is the heart of the proof scheme due to (Chizat and Bach, 2020), and circumventing or establishing dual convergence seems tricky.

The approach here is to replace \( \|w_j\| \) and \( \|w_k\| \) with other quantities which can be handled without dual convergence. First, \( \|w_j\| \) is replaced with \( \|W\| \), the norm of all nodes, which can be controlled in an elementary way for arbitrary \( L \)-homogeneous networks: as summarized in Lemma A.5, as soon as \( \|W\| \) becomes large, then \( \sum_i q_i(\infty)\tilde{p}_i(W) \approx \gamma(W) \), essentially by properties of \( \ln \sum \exp \).

Replacing \( \ln \|w_k\| \) is much harder, since \( q_i(t) \) may oscillate and thus the notion of nodes with good margin seems to be time-varying. If there is little rotation, then nodes near the reference directions \( (\beta_k)_{r=1}^R \) can be swapped with \( (\beta_k)_{r=1}^{R_1} \), and the expression \( \sum_i q_i\tilde{p}_i(w_k) \) can be swapped with \( \gamma_i \). A potential function that replaces \( \ln \|w_k\| \) and allows this swapping need only satisfy a few abstract but innocuous conditions, as summarized in Lemma A.6. Unfortunately, verifying these conditions is rather painful, and handling general settings (without explicitly disallowing rotation) seems to still need quite a few more ideas.

4 Concluding remarks and open problems

This work provides settings where SGD and GF can select good features, but many basic questions and refinements remain.

Figure 1 demonstrated low rotation with 2-sparse parity; can this be proved, thereby establishing Theorem 3.3 without forcing nodes to not rotate?

Theorem 2.1 and Theorem 2.2 achieve the same sample complexity for SGD and GF, but via drastically different proofs, the GF proof being weirdly complicated; is there a way to make the two more similar?

Looking to Table 1 for 2-sparse parity, the approaches here fail to achieve the lowest width; is there some way to achieve this with SGD and GF, perhaps even via margin analyses?

The approaches here are overly concerned with reaching a constant factor of the optimal margins; is there some way to achieve slightly worse margins with the benefit of reduced width and computation? More generally, what is the Pareto frontier of width, samples, and computation in Table 1?

The margin analysis here for the logistic loss, namely Theorem 2.2, requires a long warm start phase. Does this reflect practical regimes? Specifically, does good margin maximization and feature learning occur with the logistic loss in this early phase?
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A Technical preliminaries

As follows are basic technical tools used throughout.

A.1 Estimates of $\gamma_{ntk}$ and $\gamma_{gl}$

This section provides estimates of $\gamma_{ntk}$ and $\gamma_{gl}$ in various settings. The first estimate is of linear predictors.

Proof of Proposition 1.3. The proof considers the three settings separately.
1. For any \( i \), first note that
\[
E_{w \sim N_p} \left< \theta(w), \hat{\partial}_w p_i(w) \right> = E_{(a,v) \sim N_p} |a| \bar{u}^T x_i y_i \sigma'(v^T x_i) 1[\| (a,v) \| \leq 2] \\
\geq \gamma E_{(a,v) \sim N_p} |a| \sigma'(v^T x_i) 1[\| (a,v) \| \leq 2].
\]
To control the expectation, note that with probability at least \( 1/2 \), then \( 1/4 \leq |a| \leq \sqrt{2} \), and thus by rotational invariance
\[
E_{(a,v) \sim N_p} |a| \sigma'(v^T x_i) 1[\| (a,v) \| \leq 2] \\
\geq \frac{1}{8} E_{(a,v) \sim N_p} \sigma'(v^T x_i) 1[\| v \| \leq \sqrt{2}] \\
\geq \frac{1}{32}.
\]

2. For convenience, fix any example \((x, y) \in ((x_i, y_i))_{i=1}^n\), and write \((a,v) = w\), whereby \( w \sim N_p \) means \( a \sim N_{a_p} \) and \( v \sim N_v \). With this out of the way, define orthonormal matrix \( M \in \mathbb{R}^{d \times d} \) where the first column is \( \bar{u} \), the second column is \((I - \bar{u} \bar{u}^T)x/\|(I - \bar{u} \bar{u}^T)x\|\), and the remaining columns are arbitrary so long as \( M \) is orthonormal, and note that \( Mu = e_1 \) and \( Mx = e_1 \bar{u}^T x + e_2 r_2 \) where \( r_2 := \sqrt{\| x \|^2 - (\bar{u}^T x)^2} \). Then, using rotational invariance of the Gaussian,
\[
E_w \left< \theta(w), \hat{\partial}_p (w) \right> = y E_{w=(a,v)} \text{sgn}(\bar{u}^T v) \sigma(v^T x) 1[|v| \leq 2] \\
= y E_{\| (a,v) \| \leq 2} \alpha(Mv) \sigma(v^T M^T x) \\
= E_{\| (a,v) \| \leq 2} y \text{sgn}(v_1) \sigma(v_1 \bar{u}^T xy^2 + v_2 r_2) \\
= E_{\| (a,v) \| \leq 2} y \text{sgn}(v_1) \sigma(v_1 |v_1 \bar{u}^T xy + v_2 r_2| - \sigma(-|v_1 \bar{u}^T xy - v_2 r_2|) \\
\quad \quad \quad + \sigma(|v_1 |v_1 \bar{u}^T xy - v_2 r_2| - \sigma(-|v_1 |v_1 \bar{u}^T xy - v_2 r_2|).
\]
Considering cases, the first ReLU argument is always positive, exactly one of the second and third is positive, and the fourth is negative, whereby
\[
y E_{\| (a,v) \| \leq 2} \alpha(v) \sigma(v^T x) = \mathbb{E}_{\| (a,v) \| \leq 2} \left[ |v_1 \bar{u}^T xy + v_2 r_2| + |v_1 \bar{u}^T xy - v_2 r_2| \\
\quad \quad \quad + \mathbb{E}_{|v_2| \leq 1} \text{sgn}(v_1) |v_1| |v_1 \bar{u}^T xy - v_2 r_2|.
\]
where \( \text{Pr}[\| (a,v) \| \leq 2] \geq 1/4 \) since (for example) the \( \chi^2 \) random variables corresponding to \( |a|^2 \) and \( \| v \|^2 \) have median less than one, and the expectation term is at least \((4/\sqrt{d})\) by standard Gaussian computations [Blum et al., 2017 Theorem 2.8].
3. For any pair \((x_i, y_i)\),

\[
2y_i \sum_{j=1}^{2} \alpha_j \sigma(\beta_j^T x_i) = y_i \sigma(\bar{u}^T x_i) - y_i \sigma(-\bar{u}^T x_i)
\]

\[
= \mathbb{1}[y_i = 1] \sigma(y_i \bar{u}^T x_i) + \mathbb{1}[y_i = -1] \sigma(y_i \bar{u}^T x_i)
\]

\[
= y_i \bar{u}^T x_i
\]

\[
\geq \hat{\gamma}.
\]

Next, the construction for 2-sparse parity. As is natural in maximum margin settings, but in contrast with most studies of sparse parity, only the support of the distribution matters (and the labeling), but not the marginal distribution of the inputs.

**Proof of Proposition 1.6.** This proof shares ideas with (Wei et al., 2018; Ji and Telgarsky, 2020b), though with some adjustments to exactly fit the standard 2-sparse parity setting, and to shorten the proofs.

Without loss of generality, due to the symmetry of the data distribution about the origin, suppose \(a = 1\) and \(b = 2\), meaning for any \(x \in H_d\), the correct label is \(dx_1x_2\), the product of the first two coordinates. Both proofs will use the global margin construction (the parameters for \(\gamma_{\text{gl}}\)), given as follows:

\[
p(x, y; (\alpha, \beta)) = y \sum_{j=1}^{4} \alpha_j \sigma(\beta_j^T x),
\]

where \(\alpha = (1/4, -1/4, -1/4, 1/4)\) and

\[
\beta_1 := \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0 \right) \in \mathbb{R}^d,
\]

\[
\beta_2 := \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0 \right) \in \mathbb{R}^d,
\]

\[
\beta_3 := \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0 \right) \in \mathbb{R}^d,
\]

\[
\beta_4 := \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0 \right) \in \mathbb{R}^d.
\]

Note moreover that for any \(x \in H_{d}\), then \(\beta_j^T x > 0\) for exactly one \(j\), which will be used for both \(\gamma_{\text{latk}}\) and \(\gamma_{\text{gl}}\). The proof now splits into the two different settings, and will heavily use symmetry within \(H_{d}\) and also within \((\alpha, \beta)\).

1. Consider the transport mapping

\[
\theta((a, v)) = \left( 0, \frac{\text{sgn}(a)}{2} \sum_{j=1}^{4} \beta_j \mathbb{1}[\beta_j^T v \geq 0] \right);
\]

note that this satisfies the condition \(\|\theta(w)\| \leq 1\) thanks to the factor \(1/2\), since each \(\beta_j\) gets a hemisphere, and \((\beta_1, \beta_4)\) together partition the sphere once, and \((\beta_2, \beta_3)\) similarly together partition the sphere once.

Now let any \(x\) be given, which as above has label \(y = x_1x_2\). By rotational symmetry of the data and also the transport mapping, suppose suppose \(\beta_1\) is the unique choice with \(\beta_1^T x > 0\), which
implies \( y = 1 \), and also \( \beta_1 x = 0 = \beta_2 x = 0 \), however \( \beta_4 x = -\beta_4 x \). Using these observations, and also rotational invariance of the Gaussian,

\[
E_{a,v} \langle \theta(a,v), \partial p(x,y;w) \rangle = E_{a,v} \frac{|a|}{2} \sum_{j=1}^{4} \beta_j x \mathbf{1}[\beta_j^T v \geq 0] \cdot \mathbf{1}[v^T x \geq 0]
\]

\[
= \beta_1 x \left( E_a \frac{|a|}{2} \right) \cdot (E_v \mathbf{1}[\beta_1^T v \geq 0] \cdot \mathbf{1}[v^T x \geq 0] - E_v \mathbf{1}[-\beta_1^T v \geq 0] \cdot \mathbf{1}[v^T x \geq 0]).
\]

Now consider \( E_v \mathbf{1}[\beta_1^T v \geq 0] \cdot \mathbf{1}[v^T x \geq 0] \). A standard Gaussian computation is to introduce a rotation matrix \( M \) whose first column is \( \beta_1 \), whose second column is \( (I - \beta_1 \beta_1^T) x/\| (I - \beta_1 \beta_1^T) x \| \), and the rest are orthogonal, which by rotational invariance and the calculation \( \beta_1^T x = \sqrt{2/d} \) gives

\[
E_v \mathbf{1}[\beta_1^T v \geq 0] \cdot \mathbf{1}[v^T x \geq 0] = E_v \mathbf{1}[\beta_1^T M v \geq 0] \cdot \mathbf{1}[v^T M x \geq 0]
\]

\[
= E_v \mathbf{1}[v_1 \geq 0] \cdot \mathbf{1}[v_1 \beta_1^T x + v_2 \sqrt{1 - (\beta_1^T x)^2} \geq 0]
\]

\[
= E_v \mathbf{1}[v_1 \geq 0] \cdot \mathbf{1}[v_1 + v_2 \sqrt{d/2 - 1} \geq 0].
\]

Performing a similar calculation for the other term (arising from \( \beta_2^T x \)) and plugging all of this back in,

\[
E_{a,v} \langle \theta(a,v), \partial p(x,y;w) \rangle = \sqrt{\frac{2}{d}} \left( E_a \frac{|a|}{2} \right) \cdot E_v \mathbf{1}[v_1 \geq 0] \left( \mathbf{1}[v_1 + v_2 \sqrt{d/2 - 1} \geq 0] - \mathbf{1}[-v_1 + v_2 \sqrt{d/2 - 1} \geq 0] \right).
\]

To finish, a few observations suffice. Whenever \( v_1 \geq 0 \) (which is enforced by the common first term), then \( -v_1 + v_2 \sqrt{2} \leq v_1 + v_2 \sqrt{d/2 - 1} \), so the first indicator is 1 whenever the second indicator is 1, thus their difference is nonnegative, and to lower bound the overall quantity, it suffices to assess the probability that \( v_1 + v_2 \sqrt{d/2 - 1} \geq 0 \) whereas \( -v_1 + v_2 \sqrt{d/2 - 1} \leq 0 \). To lower bound this event, it suffices to lower bound

\[
\Pr[v_1 \geq 0 \land v_2 \geq 0 \land v_1 \geq v_2 \sqrt{d/2 - 1}] \geq \Pr[v_1 \geq 1/2] \cdot \Pr[0 \leq v_2 \leq 1/d].
\]

The first term is at least 1/5, and the second can be calculated via brute force:

\[
\Pr[v_2 \geq 1/\sqrt{d}] = \frac{1}{\sqrt{2\pi}} \int_0^{1/\sqrt{d}} \exp(-x^2) \, dx \geq \frac{1}{\sqrt{2\pi}} \int_0^{1/\sqrt{d}} \exp(-1/d) \, dx \geq \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{d}} \right) \frac{1}{e},
\]

which completes the proof after similarly using \( E_a |a| \geq 1 \), and simplifying the various constants.

2. Let any \( x \in H_d \) be given, and as above note that \( \beta_j x > 0 \) for exactly one \( j \). By symmetry, suppose it is \( \beta_1 \), whereby \( y = x_1 x_2 = 1 \), and

\[
\gamma_{kl} = p(x,y; (\alpha, \beta)) = \sum_j \alpha_j \sigma(\beta_j^T x) = |\alpha_1| \cdot \beta_1^T x = \frac{1}{4} \left( \frac{2}{\sqrt{2d}} \right) = \frac{1}{\sqrt{8d}}.
\]
Lastly, an estimate of $\gamma_{\text{atk}}$ in a simplified version of Assumption 3.1, which is used in the proof of Proposition 2.6.

**Lemma A.1.** Suppose Assumption 1.1 holds for data $((x_i, y_i))_{i=1}^n$ with reference solution $((\alpha_k, \beta_k))_{k=1}^p$ and margin $\gamma_{dL} > 0$, and additionally $\alpha_k > 0$ and $y_i = +1$ and $\|x_i\| = 1$. Then Assumption 1.3 holds with margin $\gamma_{\text{atk}} \geq \gamma_{dL}/(8\sqrt{d})$.

**Proof.** Define $\theta(a, v) := \langle \sum_{k=1}^p \alpha_k \sigma'(\beta_k v), 0 \rangle \mathbf{1}[\|a, v\| \leq 2]$. Fix any $x \in (x_i)_{i=1}^n$, and for each $k$ define orthonormal matrix $M_k$ with first column $\beta_k$ and second column $(I - \beta_k \beta_k^T)x/\|1 - \beta_k \beta_k^T)x\|$, whereby $M_k \beta_k = e_1$ and $M_k x = e_1 \beta_k^T x + e_2 r_j$ where $r_j := \sqrt{1 - (\beta_k^T x)^2}$. Then, using rotational invariance of the Gaussian and Jensen’s inequality applied to the ReLU,

$$
\begin{align*}
\gamma \bar{E}_{w \sim \mathcal{N}_d} \langle \theta(w, \bar{\sigma}p_1(w)) &= \mathbb{E}_{\|w\| \leq 2} \sum_{k=1}^p \alpha_k \sigma'(\beta_k^T v) \sigma(v^T x) \\
&= \sum_k \alpha_k \mathbb{E}_{\|M_k v\| \leq 2} \sigma'(v^T M_k u_k) \sigma(v^T M_k x) \\
&= \sum_k \alpha_k \mathbb{E}_{\|v\| \leq 1} \sigma(v_1 \beta_k^T x + v_2 r_2) \\
&\geq \sum_k \alpha_k \sigma \mathbb{E}_{\|v\| \leq 1} u_1 \beta_k^T x + v_2 r_2 \\
&= \sum_k \alpha_k \beta_k^T x \sigma \mathbb{E}_{\|v\| \leq 1} u_1 \\
&\geq \gamma_{dL} \mathbb{E}_{\|v\| \leq 1} u_1,
\end{align*}
$$

which is bounded below by $\gamma_{dL}/(8\sqrt{d})$ via similar arguments to those in the proof of Proposition 1.5.

**A.2 Gaussian Concentration**

The first concentration inequalities are purely about the initialization.

**Lemma A.2.** Suppose $a \sim \mathcal{N}_m/\sqrt{m}$ and $V \sim \mathcal{N}_m \times d/\sqrt{d}$.

1. With probability at least $1 - \delta$, then $\|a\| \leq 1 + \sqrt{2\ln(1/\delta)/m}$; similarly, with probability at least $1 - \delta$, then $\|V\| \leq \sqrt{m} + \sqrt{2\ln(1/\delta)/d}$.

2. Let examples $(x_1, \ldots, x_n)$ be given with $\|x_i\| \leq 1$. With probability at least $1 - 4\delta$,

$$\max_i \left| \sum_j a_j \sigma(v_j^T x_i) \right| \leq 4 \ln(n/\delta).$$

**Proof.** 1. Rewrite $\tilde{a} := a/\sqrt{m}$, so that $\tilde{a} \sim \mathcal{N}_m$. Since $\tilde{a} \mapsto \|\tilde{a}\|/\sqrt{m} = \|a\|$ is $(1/\sqrt{m})$-Lipschitz, then by Gaussian concentration, [Wainwright 2019, Theorem 2.26],

$$\begin{align*}
\|a\| &= \|\tilde{a}\|/\sqrt{m} \\
&\leq \mathbb{E}\|\tilde{a}\|/\sqrt{m} + \sqrt{2\ln(1/\delta)/m} \\
&\leq \sqrt{\mathbb{E}\|\tilde{a}\|^2/\sqrt{m} + 2\ln(1/\delta)/m} \\
&= 1 + \sqrt{2\ln(1/\delta)/m}.
\end{align*}$$
Similarly for $V$, defining $\tilde{V} := V\sqrt{d}$ whereby $\tilde{V} \sim \mathcal{N}_{m \times d}$, Gaussian concentration grants

$$\|V\| = \|\tilde{V}\|/\sqrt{d} \leq \sqrt{m} + \sqrt{2\ln(1/\delta)/d}. $$

2. Fix any example $x_i$, and constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ to be optimized at the end of the proof, and define $d_i := d/\|x_i\|^2$ for convenience. By rotational invariance of Gaussians and since $x_i$ is fixed, then $\sigma(Vx_i)$ is equivalent in distribution to $\|x_i\|\sigma(g)/\sqrt{d} = \sigma(g)/\sqrt{d_i}$ where $g \sim \mathcal{N}_m$. Meanwhile, $g \mapsto \|\sigma(g)\|/\sqrt{d_i}$ is $(1/\sqrt{d_i})$-Lipschitz with $\mathbb{E}\|\sigma(g)\| \leq \sqrt{m}$, and so, by Gaussian concentration (Wainwright, 2019, Theorem 2.26),

$$\Pr[\|\sigma(Vx_i)\| \geq \epsilon_1 + \sqrt{m}] = \Pr[\|\sigma(g)\|/\sqrt{d_i} \geq \epsilon_1 + \sqrt{m}] \leq \exp\left(\frac{-d_i\epsilon_1^2}{2}\right).$$

Next consider the original expression $a^T\sigma(Vx_i)$. To simplify handling of the $1/m$ variance of the coordinates of $a$, define another Gaussian $h := a\sqrt{m}$, and a new constant $c_i := md_i$ for convenience, whereby $a^T\sigma(Vx_i)$ is equivalent in distribution to equivalent in distribution to $h^T\sigma(g)/\sqrt{c_i}$ since $a$ and $V$ are independent (and thus $h$ and $V$ are independent). Conditioned on $g$, since $\mathbb{E}h = 0$, then $\mathbb{E}[h^T\sigma(g)g] = 0$. As such, applying Gaussian concentration to this conditioned random variable, since $h \mapsto h^T\sigma(g)/\sqrt{c_i}$ is $(\|\sigma(g)\|/\sqrt{c_i})$-Lipschitz, then

$$\Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2 \mid g] \leq \exp\left(\frac{-c_i\epsilon_2^2}{2\|\sigma(g)\|^2}\right).$$

Returning to the original expression, it can now be controlled via the two preceding bounds, conditioning, and the tower property of conditional expectation:

$$\Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2]$$

$$\leq \Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2 \mid \|\sigma(g)\|/\sqrt{d_i} \leq \epsilon_1 + \sqrt{m}] \cdot \Pr[\|\sigma(g)\|/\sqrt{d_i} \leq \epsilon_1 + \sqrt{m}]$$

$$+ \Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2 \mid \|\sigma(g)\|/\sqrt{d_i} > \epsilon_1 + \sqrt{m}] \cdot \Pr[\|\sigma(g)\|/\sqrt{d_i} > \epsilon_1 + \sqrt{m}]$$

$$= \mathbb{E}\left[\Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2 \mid g] \mid \|\sigma(g)\|/\sqrt{d_i} \leq \epsilon_1 + \sqrt{m}\right] \Pr[\|\sigma(g)\|/\sqrt{d_i} \leq \epsilon_1 + \sqrt{m}]$$

$$+ \Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2 \mid \|\sigma(g)\|/\sqrt{d_i} > \epsilon_1 + \sqrt{m}] \Pr[\|\sigma(g)\|/\sqrt{d_i} > \epsilon_1 + \sqrt{m}]$$

$$\leq \mathbb{E}\left[\exp\left(\frac{-c_i\epsilon_2^2}{2\|\sigma(g)\|^2}\right) \mid \|\sigma(g)\|/\sqrt{d_i} \leq \epsilon_1 + \sqrt{m}\right] + \exp\left(-d_i\epsilon_1^2/2\right)$$

$$\leq \exp\left(\frac{-c_i\epsilon_2^2}{4d_i\epsilon_1^2 + 4d_i}m\right) + \exp\left(-d_i\epsilon_1^2/2\right).$$

As such, choosing $\epsilon_2 := 4\ln(n/\delta)\sqrt{md_i}/c_i = 4\ln(n/\delta)$ and $\epsilon_1 := \sqrt{2\ln(n/\delta)/d_i}$ gives

$$\Pr[a^T\sigma(Vx_i) \geq \epsilon_2] = \Pr[h^T\sigma(g)/\sqrt{c_i} \geq \epsilon_2] \leq \frac{\delta}{n} + \frac{\delta}{n},$$

which is a sub-exponential concentration bound. Union bounding over the reverse inequality and over all $n$ examples and using max $\|x_i\| \leq 1$ gives the final bound. 

\[\square\]
Next comes a key tool in all the proofs using $\gamma_{ntk}$: guarantees that the infinite-width margin assumptions imply the existence of good finite-width networks. These bounds are stated for Assumption 1.3 however they will also be applied with Assumption 1.4 since Assumption 1.4 implies that Assumption 1.3 holds almost surely over any finite sample.

**Lemma A.3.** Let examples $((x_i, y_i))_{i=1}^n$ be given, and suppose Assumption 1.3 holds, with corresponding $\theta: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ and $\gamma_{ntk} > 0$ given.

1. With probability at least $1 - \delta$ over the draw of $(w_j)_{j=1}^m$, defining $\bar{\theta}_j := \theta(w_j)/\sqrt{m}$, then

$$\min_i \sum_j \left\langle \bar{\theta}_j, \hat{p}_i(w_j) \right\rangle \geq \gamma_{ntk} \sqrt{m} - \sqrt{32 \ln(n/\delta)}.$$ 

2. With probability at least $1 - 7\delta$ over the draw of $W$ with rows $(w_j)_{j=1}^m$ with $m \geq 2 \ln(1/\delta)$, defining rows $\bar{\theta}_j := \theta(w_j)/\sqrt{m}$ of $\bar{\theta} \in \mathbb{R}^{m \times (d+1)}$, then for any $W'$ and any $R \geq \|W - W'\|$ and any $r_\theta \geq 0$ and $r_w \geq 0$,

$$\left\langle r_\theta \bar{\theta} + r_w W, \hat{p}_i(W') \right\rangle - r_w p_i(W') \geq \gamma_{ntk} r_\theta \sqrt{m} - r_\theta \left[ \sqrt{32 \ln(n/\delta)} + 8R + 4 \right] - r_w \left[ 4 \ln(n/\delta) + 2R + 2R \sqrt{m} + 4/\sqrt{m} \right],$$

and moreover, writing $W = (a, V)$, then $\|a\| \leq 2$ and $\|V\| \leq 2 \sqrt{m}$. For the particular choice $r_\theta := R/8$ and $r_w = 1$, if $R \geq 8$ and $m \geq (64 \ln(n/\delta)/\gamma_{ntk})^2$, then

$$\left\langle r_\theta \bar{\theta} + W, \hat{p}_i(W') \right\rangle - p_i(W') \geq \frac{\gamma_{ntk} r_\theta \sqrt{m}}{2} - 160 r_\theta^2.$$ 

**Proof.** 1. Fix any example $(x_i, y_i)$, and define

$$\mu := \mathbb{E}_w \left\langle \theta(w), \hat{p}_i(w) \right\rangle,$$

where $\mu \geq \gamma_{ntk}$ by assumption. By the various conditions on $\theta$, it holds for any $(a, v) := w \in \mathbb{R}^{d+1}$ and corresponding $(\bar{a}, \bar{v}) := \theta(w) \in \mathbb{R}^{d+1}$ that

$$\left| \left\langle \theta(w), \hat{p}_i(w) \right\rangle \right| \leq |\bar{a} \sigma(v^T x_i)| + \left| \left\langle \bar{v}, ax_i \sigma'(v^T x_i) \right\rangle \right| \leq |\bar{a}| \cdot 1[\|v\| \leq 2] \cdot \|v\| \cdot \|x_i\| + \|\bar{v}\| \cdot |a| \cdot 1[|a| \leq 2] \cdot \|x_i\| \leq 4.$$ 

and therefore, by Hoeffding’s inequality, with probability at least $1 - \delta/n$ over the draw of $m$ iid copies of this random variable,

$$\sum_j \left\langle \theta(w_j), \hat{p}_i(w_j) \right\rangle \geq m \mu - \sqrt{32m \ln(n/\delta)} \geq m \gamma_{ntk} - \sqrt{32m \ln(n/\delta)},$$

which gives the desired bound after dividing by $\sqrt{m}$, recalling $\bar{\theta}_j := \theta(w_j)/\sqrt{m}$, and union bounding over all $n$ examples.
2. First, suppose with probability at least $1 - 7\delta$ that the consequences of Lemma A.2 and the preceding part of the current lemma hold, whereby simultaneously $\|a\| \leq 2$, and $\|V\| \leq 2\sqrt{m}$, and

$$\min_i p_i(W) \geq -4\ln(n/\delta), \quad \text{and} \quad \min_i \sum_j \left\langle \tilde{\theta}_j, \hat{\partial} p_i(w_j) \right\rangle \geq \gamma_{\text{atk}} \sqrt{m} - 32\ln(n/\delta).$$

The remainder of the proof proceeds by separately lower bounding the two right hand terms in

$$\left\langle r_\theta \tilde{\theta} + r_w W, \hat{\partial} p_i(W') \right\rangle - r_w p_i(W') = r_\theta \left[ \left\langle \tilde{\theta}, \hat{\partial} p_i(W) \right\rangle + \left\langle \tilde{\theta}, \hat{\partial} p_i(W') - \hat{\partial} p_i(W) \right\rangle \right]$$

$$+ r_w \left[ \left\langle W, \hat{\partial} p_i(W') \right\rangle - r_w p_i(W') \right].$$

For the first term, writing $(\tilde{a}, \tilde{V}) = \tilde{\theta}$ and noting $\|\tilde{a}\| \leq 2$ and $\|\tilde{V}\| \leq 2$, then for any $W' = (a', V')$,

$$\left| \left\langle \tilde{\theta}, \hat{\partial} p_i(W') - \hat{\partial} p_i(W) \right\rangle \right| \leq \left| \sum_j \tilde{a}_j \left( \sigma(x_i^T v_j) - \sigma(v'_j x_i) \right) \right|$$

$$+ \left| \sum_j x_i^T \tilde{\nu}_j \left( a'_j \sigma(x_i^T v'_j) - a_j \sigma' (x_i^T v_j) \right) \right|$$

$$\leq \sqrt{\sum_j \tilde{a}_j^2} \sqrt{\sum_j \left( \sigma(x_i^T v'_j) - \sigma(v'_j x_i) \right)^2}$$

$$+ \sum_j |x_i^T \tilde{\nu}_j| \cdot \left| a'_j \sigma(x_i^T v'_j) - a_j \sigma' (x_i^T v_j) \right|$$

$$+ \sum_j |x_i^T \tilde{\nu}_j| \cdot \left| a_j \sigma' (x_i^T v'_j) - a_j \sigma' (x_i^T v'_j) \right|$$

$$\leq \|\tilde{a}\| \cdot \|V' - V\| + \|a' - a\| \cdot \|\tilde{V}\| + \|a\| \cdot \|\tilde{V}\|$$

$$\leq 4R + 4.$$

For the second term,

$$\left| \left\langle W, \hat{\partial} p_i(W') \right\rangle - p_i(W') \right| = \left| \left\langle a, \hat{\partial}_a p_i(W') \right\rangle + \left\langle V, \hat{\partial}_V p_i(W') \right\rangle - \left\langle V', \hat{\partial}_V p_i(W') \right\rangle \right|$$

$$\leq \left| \sum_j a_j \sigma(x_i^T v'_j) \right| + \sum_j a'_j \left| \left\langle v_j - v'_j, x_i \right\rangle \sigma'(x_i^T v'_j) \right|$$

$$\leq p_i(w) + y_i \sum_j a_j \left( \sigma(x_i^T v'_j) - \sigma(x_i^T v_j) \right) + \sum_j |a'_j| \cdot \|v_j - v'_j\|$$

$$\leq 4\ln(n\delta) + \|a\| \cdot \|V - V'\| + \|a' - a + \|V - V'\|$$

$$\leq 4\ln(n\delta) + 4R + R^2.$$

Multiplying through by $r_\theta$ and $r$ and combining these inequalities gives, for every $i$,

$$\left| r_\theta \tilde{\theta} + r_w W, \hat{\partial} p_i(W') \right| - r_w p_i(W') \geq \gamma_{\text{atk}} r_\theta \sqrt{m} - r_\theta \left[ \sqrt{32\ln(n/\delta) + 4R + 4} \right]$$

$$- r_w \left[ 4\ln(n/\delta) + 4R + R^2 \right].$$
which establishes the first inequality. For the particular choice $r_\theta := R/8$ with $R \geq 8$ and $r_w = 1$, and using $m \geq (64 \ln(n/\delta)/\gamma_{ntk})^2$, the preceding bound simplifies to
\[
\left\langle r_{\theta} \bar{v} + r_w W, \hat{\partial} p_i(W') \right\rangle - r_w p_i(W') \geq \gamma_{ntk} r_\theta \sqrt{m} - r_\theta \left[ \frac{\gamma_{ntk} \sqrt{m}}{8} + 32 r_\theta + 32 r_\theta \right] \\
- \left[ \frac{\gamma_{ntk} \sqrt{m}}{16} + 32 r_\theta + 64 r_\theta^2 \right] \\
\geq \frac{\gamma_{ntk} r_\theta \sqrt{m}}{2} - 160 r_\theta^2.
\]

A.3 Basic properties of $L$-homogeneous predictors

This subsection collects a few properties of general $L$-homogeneous predictors in a setup more general than the rest of the work, and used in all large margin calculations. Specifically, suppose general parameters $u_t$ with some unspecified initial condition $u_0$, and thereafter given by the differential equation
\[
\dot{u}_t = -\hat{\partial}_u \tilde{R}(p(u_t)), \quad p(u) := (p_1(u), \ldots, p_n(u)) \in \mathbb{R}^n,
\]
\[
p_i(u) := y_i F(x_i; u),
\]
\[
F(x_i; cu) = c^\ell F(x_i; u) \quad \forall c \geq 0.
\]

The first property is that norms increase once there is a positive margin.

Lemma A.4 (Restatement of (Lyu and Li, 2019, Lemma B.1)). Suppose the setting of eq. (A.1) and also $\ell \in \{\ell_{\exp}, \ell_{\log}\}$. If $\mathcal{R}(u_\tau) < \ell(0)/n$, then, for every $t \geq \tau$,
\[
\frac{d}{dt} \|u_t\| > 0 \quad \text{and} \quad \left\langle u_t, \dot{u}_t \right\rangle > 0,
\]
and moreover $\lim_t \|u_t\| = \infty$.

Proof. Since $\tilde{R}$ is nonincreasing during gradient flow, it suffices to consider any $u_s$ with $\tilde{R}(u_s) < \ell(0)/n$. To apply (Lyu and Li, 2019, Lemma B.1), first note that both the exponential and logistic losses can be handled, e.g., via the discussion of the assumptions at the beginning of (Lyu and Li, 2019, Appendix A.1). Next, the statement of that lemma is
\[
\frac{d}{ds} \ln \|u_s\| > 0,
\]
but note that $\|u_s\| > 0$ (otherwise $\tilde{R}(u_s) < \ell(0)/n$ is impossible), and also that
\[
\frac{d}{ds} \|u_s\| = \frac{\langle u_s, \dot{u}_s \rangle}{\|u_s\|}, \quad \text{and} \quad \frac{d}{ds} \ln \|u_s\| = \frac{\langle u_s, \dot{u}_s \rangle}{\|u_s\|^2},
\]
which together with $(d/\mathrm{ds}) \ln \|u_s\| > 0$ from (Lyu and Li, 2019, Lemma B.1) imply the main part of the statement; all that remains to show is $\|u_s\| \to \infty$, but this is given by (Lyu and Li, 2019, Lemma B.6).
Next, even without the assumption $\hat{R}(u_s) < \ell(0)/n$ (which at a minimum requires a two-phase proof, and certain other annoyances), note that once $\|u_s\|$ is large, then the gradient can be related to margins, even if they are negative, which will be useful in circumventing the need for dual convergence and other assumptions present in prior work (e.g., as in [Chizat and Bach, 2020]).

**Lemma A.5.** (See also [Ji and Telgarsky, 2020a, Proof of Lemma C.5]). Suppose the setting of eq. (A.1) and also $\ell = \ell_{\text{exp}}$. Then, for any $u$ and any $((x_i, y_i))_{i=1}^n$ (and corresponding $\hat{R}$),

$$\frac{\langle u, -n\hat{\partial}_u \hat{R}(u) \rangle}{L\|u\|^L} \leq Q\left[\gamma(u) + \frac{\ln n}{\|u\|^L}\right] \leq Q\left[\gamma(u) + \frac{\ln n}{\|u\|^L}\right].$$

**Proof.** Define $\pi(p) = -\ln \sum \exp(-p) = \bar{\gamma}(u)$, whereby $q = \nabla_p \bar{\pi}(p)$. Since $\pi$ is concave in $p$,

$$\left\langle u, -n\hat{\partial}_u \hat{R}(u) \right\rangle = \sum_i |\ell_i| \left\langle u, \hat{\partial}_u \hat{R}(u) \right\rangle = LQ \sum_i q_i p_i = LQ \left\langle \nabla_p \bar{\pi}(p), p \right\rangle$$

$$= LQ \left\langle \nabla_p \bar{\pi}(p), p - 0 \right\rangle \leq LQ \left[ \bar{\pi}(p) - \bar{\pi}(0) \right] = LQ \left[ \bar{\gamma} + \ln n \right].$$

Moreover, by standard properties of $\pi$, letting $k$ be the index of any example with $p_k(u) = \min_i p_i(u)$,

$$\bar{\gamma} = -\ln \sum \exp(-p) \leq -\ln \exp(-p_k) = p_k = \gamma(u)\|u\|^L.$$

Combining these inequalities and dividing by $L\|u\|^L$ gives the desired bounds.

Lastly, a key abstract potential function lemma: this potential function is a proxy for mass accumulating on certain weights with good margin, and once it satisfies a few conditions, large margins are implied directly. This is the second component needed to remove dual convergence from [Chizat and Bach, 2020].

**Lemma A.6.** Suppose the setup of eq. (A.1) with $L = 2$, and additionally that there exists a constant $\bar{\gamma} > 0$, a time $\tau$, and a potential function $\Phi(u)$ so that $\Phi(u_\tau) > -\infty$, and for all $t \geq \tau$,

$$\Phi(u) \leq \frac{1}{L} \ln \|u\|,$$

$$\frac{d}{dt} \Phi(u) \geq Q(u)\bar{\gamma}.$$ 

Then it follows that $\hat{R}(u) \to \infty$, and $\|u\| \to \infty$, and $\int_{\tau}^t Q(u_s) \, ds = \infty$, and $\lim \inf_t \gamma(u_t) \geq \bar{\gamma}$.

**Proof.** First it is shown that if $\inf_s \hat{R}(u_s) > 0$ (which is well-defined since since $\hat{R}$ is nonincreasing and bounded below by 0), then $\int_{\tau}^{\infty} Q_s \, ds = \infty$ and $\|u\| \to \infty$. Since $\hat{R}(u_s) = \frac{1}{n} Q_s$, this implies $\inf_s Q_s > 0$, and consequently

$$\int_{\tau}^{\infty} Q_s \, ds = \infty,$$

which also implies

$$\lim \inf_t \frac{1}{L} \ln \|u_t\| \geq \lim \inf_t \Phi(u_t) - \Phi(u_\tau) + \Phi(u_\tau) = \Phi(u_\tau) + \lim \inf_t \int_{\tau}^{t} \frac{d}{ds} \Phi(u_s) \, ds$$

$$\geq \Phi(u_\tau) + \lim \inf_t \int_{\tau}^{t} \bar{\gamma} Q_s \, ds = \infty,$$
thus \(\|u_s\| \to \infty\). On the other hand, if \(\inf_s \hat{R}(u_s) = 0\), then there exists \(t_1\) so that for all \(t \geq t_1\), then \(\gamma_t > 0\) (also making use of non-decreasing margins (Lyu and Li [2019])), which is only possible if \(\|u_s\| \to \infty\), and thus moreover we can take \(t_2 \geq t_1\) so that additionally \(\|u_t\| \geq \ln(n)/\gamma_t\); (which will hold for all \(t' > t_2\) by Lemma [A.4], and by Lemma [A.5] and the restriction \(L = 2\) means

\[
\frac{d}{dt} \ln \|u\| = \sum_i \frac{\|u\|^2}{\|u\|^2} \langle u, \delta p_i(u) \rangle = \frac{QL \sum_i q_i p_i(u)}{\|u\|^2} \leq QL(\gamma_t + \gamma_t^\prime) \leq 2LQ\gamma_t,
\]

which after integrating and upper bounding \(\gamma_t \leq 1\) means \(\int_{t_2}^{\infty} Q_s ds \geq \lim_t [\ln \|u_t\| - \ln \|u_{t_2}\|] = \infty\). As such, independent of whether \(\inf_s \hat{R}(u_s) = 0\), then still \(\|u_s\| \to \infty\) and \(\int_{t_2}^{\infty} Q_s ds = \infty\).

This now suffices to complete the proof. If \(\lim sup_t \gamma_t \geq \hat{\gamma}\), then in fact \(\lim_t \gamma_t\) is well-defined (by non-decreasing margins and \(\|u\| \to \infty\)) and \(\lim_t \gamma_t \geq \hat{\gamma} > 0\), whereby \(\lim sup_t \hat{R}(u_t) \leq \lim sup_t \ell(-\gamma_t\|w_t\|^2) = 0\). Alternatively, suppose contradictorily that \(\lim inf_t \gamma_t < \hat{\gamma}\); and choose any \(\epsilon \in (0, \hat{\gamma}/4)\) so that \(\lim inf_t \gamma_t < \hat{\gamma} - 3\epsilon\). Noting that \(\gamma_t\) is monotone once there exists some \(\gamma_s > 0\), then, choosing \(t_3\) large enough so that \(\|u_t\|^2 \geq \ln(n)/\epsilon\) for all \(t \geq t_3\) and \(\gamma_t < \hat{\gamma} - 2\epsilon\) for all \(t \geq t_3\), it follows by Lemma [A.5] that

\[
0 \leq \lim inf_t \left[ \frac{1}{L} \ln \|u_t\| - \Phi(u_t) \right] \\
\leq \frac{1}{L} \ln \|u_{t_3}\| - \Phi(u_{t_3}) + \lim inf_t \int_{t_3}^{t} \frac{1}{L} \ln \|u_s\| - \Phi(u_s) \right] ds \\
\leq \frac{1}{L} \ln \|u_{t_3}\| - \Phi(u_{t_3}) + \lim inf_t \int_{t_3}^{t} (\mathcal{Q}(\hat{\gamma} - \epsilon) - 2\mathcal{Q}) ds \\
\leq \frac{1}{L} \ln \|u_{t_3}\| - \Phi(u_{t_3}) + \lim inf_t \int_{t_3}^{t} (-\epsilon) ds \\
= -\infty,
\]

a contradiction, and since \(\epsilon \in (0, \hat{\gamma}/4)\) was arbitrary, it follows that \(\lim inf \gamma_t \geq \hat{\gamma}\). \(\square\)

B Proofs for Section 2

This section contains proofs with a dependence on \(\gamma_{\text{init}}\).

B.1 SGD proofs

The following application of Freedman’s inequality is used to obtain the test error bound.

**Lemma B.1** (Nearly identical to (Ji and Telgarsky 2020b, Lemma 4.3)). Define \(\mathcal{Q}(W) := \mathbb{E}_{x,y}[\ell'(p(x, y; W))]\) and \(\mathcal{Q}_i(W) := |\ell'(p(x_i, y_i; W))|\). Then \(\sum_{i < t} [\mathcal{Q}(W_i) - \mathcal{Q}_i(W_i)]\) is a martingale difference sequence, and with probability at least \(1 - \delta\),

\[
\sum_{i < t} \mathcal{Q}(W_i) \leq 4 \sum_{i < t} \mathcal{Q}_i(W_i) + 4 \ln(1/\delta),
\]

**Proof.** This proof is essentially a copy of one due to (Ji and Telgarsky 2020b, Lemma 4.3); that one is stated for the analog of \(p_t\) used there, and thus need to be re-checked.

Let \(\mathcal{F}_i := \{(x_j, y_j) : j < i\}\) denote the \(\sigma\)-field of all information until time \(i\), whereby \(x_i\) is independent of \(\mathcal{F}_i\), whereas \(w_i\) deterministic after conditioning on \(\mathcal{F}_i\). Consequently, \(\mathbb{E} [\mathcal{Q}(W_i) - \mathcal{Q}_i(W_i)|\mathcal{F}_i] = 0\), whereby \(\sum_{i < t} [\mathcal{Q}(W_i) - \mathcal{Q}_i(W_i)]\) is a martingale difference sequence.
The high probability bound will now follow via a version of Freedman’s inequality \cite{Agarwal2014} Lemma 9. To apply this bound, the conditional variances must be controlled: noting that $|\ell'(z)| \in [0,1]$, then $Q(W_i) - Q_i(W_i) \leq 1$, and since $Q_i(W_i) \in [0,1]$, then $Q_i(W_i)^2 \leq Q(W_i)$, and thus

$$
\mathbb{E} \left[ (Q(W_i) - Q_i(W_i))^2 \mid F_i \right] = \mathbb{E} \left[ Q_i(W_i)^2 \mid F_i \right] - Q(W_i)^2 \\
\leq \mathbb{E} \left[ Q_i(W_i) \mid F_i \right] - 0 \\
= Q(W_i).
$$

As such, by the aforementioned version of Freedman’s inequality \cite{Agarwal2014} Lemma 9,

$$
\sum_{i<t} [Q(W_i) - Q_i(W_i)] \leq (e - 2) \sum_{i<t} \mathbb{E} \left[ (Q(W_i) - Q_i(W_i))^2 \mid F_i \right] + \ln(1/\delta)
\leq (e - 2) \sum_{i<t} Q(W_i) + \ln(1/\delta),
$$

which rearranges to give

$$
(3 - e) \sum_{i<t} Q(W_i) \leq \sum_{i<t} Q_i(W_i) + \ln(1/\delta),
$$

which gives the result after multiplying by 4 and noting $4(3 - e) \geq 1$.

With Lemma \ref{lem:tau} and the Gaussian concentration inequalities from Appendix \ref{app:gaussian} in hand, a proof of a generalized form of Theorem 2.1 is as follows.

Proof of Theorem 2.1. Let $(w_j)_{j=1}^m$ be given with corresponding $(\bar{a}_j, \bar{v}_j) := \bar{\theta}_j := \theta(w_j)/\sqrt{m}$ (whereby $\|\bar{\theta}_j\| \leq 2$ by construction), and define

$$
r := \frac{10\eta\sqrt{m}}{\gamma} \leq \frac{\gamma\sqrt{m}}{640}, \quad R := 8r = \frac{80\eta\sqrt{m}}{\gamma} \leq \frac{\gamma\sqrt{m}}{80}, \quad \bar{W} := r\bar{\theta} + W,
$$

which implies $r \geq 1$, and $R \geq 1$, and $\eta \leq R/16$. Since Assumption 1.4 implies that Assumption 1.3 holds for $((x_i, y_i))_{i \leq t}$ with probability 1, for the remainder of the proof, rule out the $7\delta$ failure probability associated with the second part of Lemma A.3 (which is stated in terms of Assumption 1.3 not Assumption 1.4), whereby simultaneously for every $\|W' - W_0\| \leq R$

$$
\min_i \left\langle \bar{W}, \partial p_i(W') \right\rangle \geq \frac{r\gamma\sqrt{m}}{2} - 160r^2 \geq \frac{r\gamma\sqrt{m}}{4} \geq \frac{\gamma^2 m}{2560} \geq \ln(t), \quad (B.1)
$$

$$
\min_i \left\langle \bar{\theta}, \partial p_i(W') \right\rangle \geq \gamma\sqrt{m} - \sqrt{32\ln(\nu/\delta)} - 4R - 4 \geq \gamma\sqrt{m} - \frac{\gamma\sqrt{m}}{8} - \frac{\gamma\sqrt{m}}{10} \geq \frac{\gamma\sqrt{m}}{2}, \quad (B.2)
$$

and also $\|a_0\| \leq 2$ and $\|V_0\| \leq 2\sqrt{m}$.

The proof now proceeds as follows. Let $\tau$ denote the first iteration where $\|W_{\tau} - W_0\| \geq R$, whereby $\tau > 0$ and max $\leq t$ $\|W_s - W_0\| \leq R$. Assume contrarily that $\tau \leq t$; it will be shown that this implies $\|W_{\tau} - W_0\| \leq R$.

Consider any iteration $s < \tau$. Expanding the square,

$$
\|W_{s+1} - \bar{W}\|^2 = \|W_s - \eta\partial \ell_s(W_s) - \bar{W}\|^2 \\
= \|W_s - \bar{W}\|^2 - 2\eta \left\langle \partial \ell_s(W_s), W_s - \bar{W} \right\rangle + \eta^2 \left\| \partial \ell_s(W_s) \right\|^2 \\
= \|W_s - \bar{W}\|^2 + 2\eta \ell_s'(W_s) \left\langle \partial p_s(W_s), \bar{W} - W_s \right\rangle + \eta^2 \ell_s'(W_s)^2 \left\| \partial p_s(W_s) \right\|^2.
$$

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By convexity, \( \|W_s - W_0\| \leq R \), and eq. \([B.1]\),

\[
\ell_s'(W_s) \langle \hat{\delta} p_s(W_s), \overline{W} - W_s \rangle = \ell_s'(W_s) \left( \left[ \langle \hat{\delta} p_s(W_s), \overline{W} \rangle - p_s(W_s) \right] - p_s(W_s) \right)
\]

\[
\leq \ell_s \left( \langle \hat{\delta} p_s(W_s), \overline{W} \rangle - p_s(W_s) \right) - \ell_s(W_s)
\]

\[
\leq \ln(1 + \exp(-\ln(t))) - \ell_s(W_s),
\]

\[
\leq \frac{1}{t} - \ell_s(W_s),
\]

which combined with the preceding display gives

\[
\|W_{s+1} - \overline{W}\|^2 \leq \|W_s - \overline{W}\|^2 + 2\eta \left( \frac{1}{t} - \ell_s(W_s) \right) + \eta^2 \ell_s'(W_s)^2 \|\hat{\delta} p_s(W_s)\|^2.
\]

Since this inequality holds for any \( s < \tau \), then applying the summation \( \sum_{s<\tau} \) and rearranging gives

\[
\|W_\tau - \overline{W}\|^2 + 2\eta \sum_{s<\tau} \ell_s(W_s) \leq \|W_0 - \overline{W}\|^2 + 2\eta + \sum_{s<\tau} \eta^2 \ell_s'(W_s)^2 \|\hat{\delta} p_s(W_s)\|^2.
\]

To simplify the last term, using \( \|V_0\| \leq 2\sqrt{m} \) and \( \|a_0\| \leq 2 \) and \( \|W_s - W_0\| \leq R \) gives

\[
\|\hat{\delta} p_s(W)\|^2 = \|\sigma(V_s x_s)\|^2 + \left\| \sum_j e_j a_{i,j} \sigma'(v_{i,j}^T x_s) x_s \right\|^2
\]

\[
\leq \|\sigma(V_s x_s)\|^2 + \|a_s\|^2
\]

\[
\leq 2\|V_s - V_0\|^2 + 2\|V_0\|^2 + 2\|a_s - a_0\|^2 + 2\|a_0\|^2
\]

\[
\leq 2R^2 + 8m + 8,
\]

\[
\leq 10m,
\]

and moreover the first term can be simplified via

\[
\|W_\tau - \overline{W}\|^2 = \|W_\tau - W_0\|^2 - 2 \langle W_\tau - W_0, \overline{W} - W_0 \rangle + \|\overline{W} - W_0\|^2
\]

\[
\geq \|W_\tau - W_0\|^2 - 2r\|W_\tau - W_0\| + \|\overline{W} - W_0\|^2;
\]

whereby combining these all gives

\[
\|W_\tau - W_0\|^2 - 2r\|W_\tau - W_0\| + \|\overline{W} - W_0\|^2 + 2\eta \sum_{s<\tau} \ell_s(W_s)
\]

\[
\leq \|W_\tau - \overline{W}\|^2 + 2\eta \sum_{s<\tau} \ell_s(W_s)
\]

\[
\leq \|W_0 - \overline{W}\|^2 + 2\eta + \sum_{s<\tau} \eta^2 \ell_s'(W_s)^2 \|\hat{\delta} p_s(W_s)\|^2
\]

\[
\leq \|W_0 - \overline{W}\|^2 + 2\eta + 10\eta^2 m \sum_{s<\tau} |\ell_s'(W_s)|,
\]

which after canceling and rearranging gives

\[
\|W_\tau - W_0\|^2 + 2\eta \sum_{s<\tau} \ell_s(W_s) \leq 2r\|W_\tau - W_0\| + 2\eta + 10\eta^2 m \sum_{s<\tau} |\ell_s'(W_s)|.
\]
To simplify the last term, note by eq. (B.2) that
\[
\|W_\tau - W_0\| = \sup_{\|W\| \leq 1} \langle W, W_\tau - W_0 \rangle
\]
\[
\geq \frac{1}{2} \langle -\bar{\theta}, W_\tau - W_0 \rangle
\]
\[
= \frac{\eta}{2} \sum_{s < \tau} \langle -\bar{\theta}, \hat{\partial} \ell_s(W_s) \rangle
\]
\[
= \frac{\eta}{2} \sum_{s < \tau} |\ell'_s(W_s)| \langle \bar{\theta}, \hat{\partial} p_i(W_s) \rangle
\]
\[
\geq \frac{\eta}{2} \sum_{s < \tau} |\ell'_s(W_s)| \frac{\gamma \sqrt{m}}{2}, \tag{B.3}
\]
and thus, by the choice of \( R \), and since \( \|W_\tau - W_0\| \geq 1 \) and \( \eta \leq R/16 \),
\[
\|W_\tau - W_0\|^2 + 2\eta \sum_{s < t} \ell_s(W_s) \leq 2r\|W_\tau - W_0\| + 2\eta + \frac{40\eta \sqrt{m}\|W_\tau - W_0\|}{\gamma}
\]
\[
\leq \left( \frac{R}{4} + \frac{R}{8} + \frac{R}{2} \right) \|W_\tau - W_0\|.
\]
Dropping the term \( 2\eta \sum_{s < t} \ell_s(W_s) \geq 0 \) and dividing both sides by \( \|W_\tau - W_0\| \geq R > 0 \) gives
\[
\|W_\tau - W_0\| \leq \frac{R}{4} + \frac{R}{8} + \frac{R}{2} < R,
\]
the desired contradiction, thus \( \tau > t \) and all above derivations hold for all \( s \leq t \).

To finish the proof, combining eq. (B.3) with \( \|W_t - W_0\| \leq R = 80\eta \sqrt{m}/\gamma \) gives
\[
\sum_{s < t} |\ell'_s(W_s)| \leq \frac{4\|W_t - W_0\|}{\eta \gamma \sqrt{m}} \leq \frac{320}{\gamma^2}.
\]
Lastly, for the generalization bound, defining \( Q(W) := \mathbb{E}_{x,y}|\ell'(p(x,y;W))| \), discarding an additional \( \delta \) failure probability, by Lemma B.1
\[
\sum_{i < t} Q(W_s) \leq 4 \ln(1/\delta) + 4 \sum_{s < t} |\ell'_s(W_s)| \leq 4 \ln(1/\delta) + \frac{1280}{\gamma^2}.
\]
Since \( 1[p_s(W_s) \leq 0] \leq 2|\ell'_s(W_s)| \), the result follows.

### B.2 GF proofs

This section culminates in the proof of Theorem 2.2 which is immediate once Lemmas 2.3 and 2.4 are established.

Before proceeding with the main proofs, the following technical lemma is used to convert a bound on \( \ell' \) to a bound on \( \ell \).

**Lemma B.2.** For \( \ell \in \{\ell_{\text{log}}, \ell_{\exp}\} \), then \( |\ell'(z)| \leq 1/8 \) implies \( \ell(z) \leq 2|\ell'(z)| \).

**Proof.** If \( \ell = \ell_{\exp} \), then \( \ell' = -\ell \), and thus \( \ell(z) \leq 2|\ell'(z)| \) automatically. If \( \ell(z) = \ell_{\text{log}} \), the logistic loss, then \( |\ell'(z)| \leq 1/8 \) implies \( z \geq 2 \). By the concavity of \( \ln(\cdot) \), for any \( z \geq 2 \), since \( 1 + e^{-z} \leq 7/6 \), then
\[
\ell(z) = \ln(1 + e^{-z}) \leq e^{-z} \leq \frac{(7/6)e^{-z}}{1 + e^{-z}} \leq 2|\ell'(z)|,
\]
thus completing the proof. \( \square \)
Next comes the proof of Lemma 2.3 which follows the same proof plan as Theorem 2.1.

**Proof of Lemma 2.3.** This proof is basically identical to the SGD in Theorem 2.1. Despite this, proceeding with amnesia, let rows \((w_j)_{j=1}^m\) of \(W_0\) be given with corresponding \((a_j, \tau_j) := \theta_j := \theta(w_j)/\sqrt{m}\) (whereby \(\|\theta_j\| \leq 2\) by construction), and define

\[
r := \frac{\gamma\|W\|}{640}, \quad R := 8r = \frac{\gamma\|W\|}{20}, \quad \overline{W} := r\theta + W,
\]

with immediate consequences that \(r \geq 1\) and \(R \geq 8\). For the remainder of the proof, rule out the 7\(\delta\) failure probability associated with the second part of Lemma A.3 whereby simultaneously for every \(\|W' - W_0\| \leq R\),

\[
\min_i \left\langle \overline{W}, \partial p_i(W') \right\rangle \geq \frac{r^2\gamma\|W\|}{80} - 160r^2 \geq \frac{\gamma\|W\|}{2560} \geq \ln(t), \tag{B.4}
\]

\[
\min_i \left\langle \theta, \partial p_i(W') \right\rangle \geq \frac{\gamma\|W\|}{2} - \sqrt{32\ln(n/\delta)} - 4R - 4 \geq \frac{\gamma\|W\|}{8} - \frac{\gamma\|W\|}{10} \geq \frac{\gamma\|W\|}{2}. \tag{B.5}
\]

The proof now proceeds as follows. Let \(\tau\) denote the earliest time such that \(\|W_\tau - W_0\| = R\); since \(W_s\) traces out a continuous curve and since \(R > 0 = \|W_0 - W_0\|\), this quantity is well-defined. As a consequence of the definition, \(\sup_{s < \tau} \|W_s - W_0\| \leq R\). Assume contradictorily that \(\tau \leq t\); it will be shown that this implies \(\|W_\tau - W_0\| < R\).

By the fundamental theorem of calculus (and the chain rule for Clarke differentials), convexity of \(\ell\), and since \(\|W_s - W_0\| \leq R\) holds for \(s \in [0, \tau]\), which implies eq. (B.4) holds,

\[
\|W_\tau - \overline{W}\|^2 - \|W_0 - \overline{W}\|^2 = \int_0^\tau \|W_s - \overline{W}\|^2 \, ds
= \int_0^\tau 2 \left\langle W_s, W_s - \overline{W} \right\rangle \, ds
= \frac{2}{n} \int_0^\tau \sum_i \ell'_i(W_s) \left\langle \partial p_i(W_s), W_s - \overline{W} \right\rangle \, ds
= \frac{2}{n} \int_0^\tau \sum_i \ell'_i(W_s) \left( \left\langle \partial p_i(W_s), \overline{W} \right\rangle - p_i(W_s) \right) \, ds
\leq \frac{2}{n} \int_0^\tau \sum_i \left( \ell_i \left( \left\langle \partial p_i(W_s), \overline{W} \right\rangle - p_i(W_s) \right) - \ell_i(W_s) \right) \, ds
\leq \frac{2}{n} \int_0^\tau \sum_i \left( \frac{1}{\ell} - \ell_i(W_s) \right) \, ds
\leq 2 - 2 \int_0^\tau \mathcal{R}(W_s) \, ds.
\]

To simplify the left hand side,

\[
\|W_\tau - \overline{W}\|^2 - \|W_0 - \overline{W}\|^2 = \|W_\tau - W_0\|^2 - 2 \left\langle W_\tau - W_0, \overline{W} - W_0 \right\rangle \geq \|W_\tau - W_0\|^2 - 2r\|W_\tau - W_0\|,
\]

which after combining, rearranging, and using \(r \geq 1\) and \(\|W_\tau - W_0\| \geq R \geq 1\) gives

\[
\|W_\tau - W_0\|^2 + 2 \int_0^\tau \mathcal{R}(W_s) \, ds \leq 2 + 2r\|W_\tau - W_0\| \leq 4r\|W_\tau - W_0\|,
\]

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which implies
\[ \|W_\tau - W_0\| \leq 2r = \frac{R}{2} < R, \]
a contradiction since \( W_\tau \) is well-defined as the earliest time with \( \|W_\tau - W_0\| = R \), which thus contradicts \( \tau \leq t \). As such, \( \tau \geq t \), and all of the preceding inequalities follows with \( \tau \) replaced by \( t \).

To obtain an error bound, similarly to the key perceptron argument before, using eq. (B.5),
\[ \|W_t - W_0\| = \sup_{\|W\| \leq 1} \langle W, W_t - W_0 \rangle \]
\[ \geq \frac{1}{2} \lbrack -\bar{\sigma}, W_t - W_0 \rbrack \]
\[ = \frac{1}{2} \lbrack -\bar{\sigma}, \int_0^t \dot{a}_s \, ds \rbrack \]
\[ = \frac{1}{2n} \int_0^t \sum_i |\ell'_i(W_s)| \lbrack -\bar{\sigma}, \dot{\hat{p}}_i(W_s) \rbrack \, ds \]
\[ \geq \frac{\gamma_{\text{atk}} \sqrt{m}}{4n} \int_0^t \sum_i |\ell'_i(W_s)| \, ds, \]
which implies
\[ \frac{1}{n} \int_0^t \sum_i |\ell'_i(W_s)| \, ds \leq \frac{4\|W_t - W_0\|}{\gamma_{\text{atk}} \sqrt{m}} \leq \frac{1}{20}, \]
and in particular
\[ \inf_{s \in [0,t]} \frac{1}{n} \sum_i |\ell'_i(W_s)| \leq \frac{1}{10}, \]
and in particular
\[ \inf_{s \in [0,t]} \frac{1}{n} \sum_i |\ell'_i(W_s)| \leq \frac{1}{10n} \int_0^t \sum_i |\ell'_i(W_s)| \, ds \leq \frac{4\|W_t - W_0\|}{t\gamma_{\text{atk}} \sqrt{m}} \leq \frac{1}{20t}, \]
and so there exists \( k \in [0,t] \) with
\[ \frac{1}{n} \sum_i |\ell'_i(W_k)| \leq \frac{1}{10t}. \]
Since this also implies \( \max_i |\ell'_i(W_k)| \leq n/(10t) \leq 1/10 \), it follows by Lemma [B.2] that \( \hat{R}(W_k) \leq 1/(5t) \), and the claim also holds for \( t' \geq t \) since the empirical risk is nonincreasing with gradient flow.

Next, the proof of the Rademacher complexity bound used for all GF sample complexities.

**Proof of Lemma 2.5.** For any \((a, v) \in \mathbb{R}^{d+1}\) and any \(x\), recalling the notation defining \( \bar{a} := \text{sgn}(a) \) and \( \bar{v} := v/\|v\| \),
\[ a\sigma(v^Tx) = \|av\| \bar{a}\sigma(\bar{v}^Tx) \leq \frac{\|(a,v)\|^2}{2} \bar{a}\sigma(\bar{v}^Tx), \]
and therefore, letting
\[ \text{sconv}(S) := \left\{ \sum_{j=1}^m p_j u_j : m \geq 0, p \in \mathbb{R}^m, \|p\|_1 \leq 1, u_j \in S \right\} \]
denote the symmetrized convex hull as used throughout Rademacher complexity (Shalev-Shwartz and Ben-David, 2014), then
\[
\mathcal{F} := \left\{ x \mapsto \frac{1}{\|W\|^2} \sum_j a_j \sigma(v_j^T x) : m \geq 0, W \in \mathbb{R}^{m \times (d+1)} \right\}
\]
\[
= \left\{ x \mapsto \frac{1}{\|W\|^2} \sum_j \|a_j v_j\| \tilde{a} \sigma(v_j^T x) : m \geq 0, W \in \mathbb{R}^{m \times (d+1)} \right\}
\]
\[
\subseteq \left\{ x \mapsto \sum_j \frac{(a_j, v_j)^T}{2\|W\|^2} \tilde{a} \sigma(v_j^T x) : m \geq 0, W \in \mathbb{R}^{m \times (d+1)} \right\}
\]
\[
\subseteq \left\{ x \mapsto \sum_j p_j \sigma(u_j^T x) : m \geq 0, p \in \mathbb{R}^m, \|p\|_1 \leq \frac{1}{2}, \|u_j\|_2 = 1 \right\}
\]
\[
= \frac{1}{2} \text{scov} \left\{ x \mapsto \sigma(v^T x) : \|v\|_2 = 1 \right\}.
\]

As such, by standard rules of Rademacher complexity (Shalev-Shwartz and Ben-David, 2014),
\[
\text{Rad}(\mathcal{F}) \leq \frac{1}{\sqrt{2}} \text{Rad} \left( \left\{ x \mapsto \sigma(v^T x) : \|v\|_2 = 1 \right\} \right) \leq \frac{1}{\sqrt{2}\sqrt{n}},
\]
and thus, by a refined margin bound for Rademacher complexity (Srebro et al., 2010, Theorem 5), with probability at least 1 − δ, simultaneously for all γ̃gl and all m, every W ∈ R^{m \times (d+1)} with γ̃(W) ≥ γ̃gl satisfies
\[
\Pr[p(x, y; W) \leq 0] = O \left( \frac{\ln(n)^3 \text{Rad}(\mathcal{F})^2 + \ln \ln \frac{1}{\gamma_{\tilde{g}l}} + \ln \frac{1}{\delta}}{n} \right) = O \left( \frac{\ln(n)^3}{n \gamma_{\tilde{g}l}^2} + \frac{\ln 1}{n} \right),
\]
and to finish, instantiating this bound with γ̃(W) gives the desired form. □

The proof of Lemma 2.4 now follows.

**Proof of Lemma 2.4.** By the second part of Lemma A.3 with probability at least 1 − 7δ, simultaneously \(\|\alpha\| \leq 2\), and \(\|V\| \leq 2\sqrt{m}\), and for any \(\|W' - W_0\| \leq R\), then
\[
\min_i \left\langle \tilde{\theta}_j, \hat{\partial} p_i(W') \right\rangle \geq \gamma_{tak} \sqrt{m} - \left[ \sqrt{32 \ln(n/\delta)} + 8R + 4 \right] \geq \gamma_{tak} \sqrt{m},
\]
where \(\tilde{\theta}_j := \theta(w_j)/\sqrt{m}\) as usual, and \(\|\tilde{\theta}\| \leq 2\); for the remainder of the proof, suppose these bounds, and discard the corresponding 7δ failure probability. Moreover, for any W' with \(\hat{\mathcal{R}}(W') < \ell(0)/n\) and \(\|W' - W_0\| \leq R\), as a consequence of the preceding lower bound and also the property \(\sum_i q_i(W') \geq 1\)
Ji and Telgarsky (2019, Lemma 5.4, first part, which does not depend on linear predictors),

\[ \| \tilde{\gamma}(W') \| = \sup_{\| W \| \leq 1} \langle W, \tilde{\gamma}(W') \rangle \]

\[ \geq \frac{1}{2} \left\langle \bar{\theta}, \sum_i q_i \bar{\partial} p_i(W') \right\rangle \]

\[ = \frac{1}{2} \sum_i q_i \left\langle \bar{\theta}, \bar{\partial} p_i(W') \right\rangle \]

\[ \geq \frac{\gamma_{atk} \sqrt{m}}{4} \sum_i q_i \]

\[ \geq \frac{\gamma_{atk} \sqrt{m}}{4}. \]

Now consider the given \( W_\tau \) with \( \hat{\mathcal{R}}(W_\tau) < \ell(0)/n \) and \( \| W_\tau - W_0 \| \leq R/2 \). Since \( s \mapsto W_s \) traces out a continuous curve and since norms grow monotonically and unboundedly after time \( \tau \) (cf. Lemma A.4), then there exists a unique time \( r \) with \( \| W_t - W_0 \| = R \). Furthermore, since \( \hat{\mathcal{R}} \) is nonincreasing throughout gradient flow, then \( \hat{\mathcal{R}}(W_s) < \ell(0)/n \) holds for all \( s \in [\tau, t] \). Then

\[ \tilde{\gamma}(W_t) - \tilde{\gamma}(W_\tau) = \int_\tau^t \left\langle \bar{\partial} \tilde{\gamma}(W_s), \dot{W}_s \right\rangle \, ds \]

\[ = \int_\tau^t \| \bar{\partial} \tilde{\gamma}(W_s) \| \cdot \| \dot{W}_s \| \, ds \]

\[ \geq \frac{\gamma_{atk} \sqrt{m}}{4} \int_\tau^t \| \dot{W}_s \| \, ds \]

\[ \geq \frac{\gamma_{atk} \sqrt{m}}{4} \int_\tau^t \| \dot{W}_s \| \, ds \]

\[ = \frac{\gamma_{atk} \sqrt{m}}{4} \| W_t - W_\tau \| \, ds \]

\[ \geq \frac{\gamma_{atk} R \sqrt{m}}{8} \]

\[ \geq \frac{\gamma_{atk}^2 m \sqrt{m}}{256}. \]

Since \( \| W_0 \| \leq 3 \sqrt{m} \), thus \( \| W_t \| \leq 3 \sqrt{m} + \gamma_{atk} \sqrt{m}/32 \leq 4 \sqrt{m} \), and the normalized margin satisfies

\[ \tilde{\gamma}(W_t) \geq \frac{\tilde{\gamma}(W_\tau)}{\| W_t \|^2} + \frac{1}{\| W_t \|^2} \int_\tau^t \frac{d}{ds} \tilde{\gamma}(W_s) \, ds \geq 0 + \frac{\gamma_{atk}^2 m / 256}{16m} = \frac{\gamma_{atk}^2}{4096}. \]

Furthermore, it holds that \( \tilde{\gamma}(W_s) \geq \tilde{\gamma}(W_t) \) for all \( s \geq t \) (Lyu and Li, 2019), which completes the proof for \( W_t \) under the standard parameterization.

Now consider the rebalanced parameters \( \hat{W}_t := (a_t/\sqrt{\gamma_{atk}}, V_t/\sqrt{\gamma_{atk}}) \); since \( m \geq 256/\gamma_{atk}^2 \), which means \( 16 \leq \gamma_{atk} \sqrt{m} \), then

\[ \| a_t \| \leq \| a_0 \| + \| a_t - a_0 \| \leq 2 + R \leq \frac{\gamma_{atk} \sqrt{m}}{8} + \frac{\gamma_{atk} \sqrt{m}}{32} \leq \frac{\gamma_{atk} \sqrt{m}}{4}, \]

\[ \| V_t \| \leq \| V_0 \| + \| V_t - V_0 \| \leq 2 \sqrt{m} + R \leq 3 \sqrt{m}, \]
then the rebalanced parameters satisfy
\[ \|\tilde{W}_t\| \leq \|a_t/\sqrt{\gamma_{\text{atk}}}\| + \|V_t\sqrt{\gamma_{\text{atk}}}\| \leq \frac{\sqrt{\gamma_{\text{atk}}}}{4} + 3\sqrt{\gamma_{\text{atk}}^2 m} \leq 4\sqrt{\gamma_{\text{atk}}^2 m}, \]
and thus, for any \((x, y)\), since
\[ p(x, y; w) = \sum_j a_j \sigma(v_j^T x) = \sum_j \frac{a_j}{\sqrt{\gamma_{\text{atk}}}} \sigma(\sqrt{\gamma_{\text{atk}}} v_j^T x) = p(x, y; \tilde{W}), \]
then
\[ \hat{\gamma}(\tilde{W}) = \min_i \frac{p_i(\tilde{W})}{\|W\|^2} = \min_i \frac{p_i(w)}{\|W\|^2} \geq \frac{\gamma^2 m / 256}{16 \gamma_{\text{atk}} m} \geq \frac{\gamma_{\text{atk}}}{4096}, \]
which completes the proof.

Thanks to Lemmas 2.3 and 2.4, the proof of Theorem 2.2 is now immediate.

**Proof of Theorem 2.4** As in the statement, define \( R := \gamma_{\text{atk}} \sqrt{m}/32 \), and note that Assumption 1.3 holds almost surely for any finite sample due to Assumption 1.4. The analysis now uses two stages. The first stage is handled by Lemma 2.3 run until time \( \tau := n \), whereby, with probability at least \( 1 - 7\delta \),
\[ \hat{R}(W) \leq \frac{1}{5n} < \frac{\ell(0)}{n}, \quad \|W_t - W_0\| \leq \frac{\gamma_{\text{atk}} \sqrt{m}}{80} \leq \frac{R}{2}. \]
The second stage now follows from Lemma 2.4 since \( W_t \) as above satisfies all the conditions of Lemma 2.4 there exists \( W_t \) with \( \|W_t - W_0\| = R \), and \( \hat{\gamma}(W_t) \geq \frac{\gamma_{\text{atk}}^2}{256} \) for all \( s \geq t \), and \( \hat{\gamma}(\tilde{W}) \geq \frac{\gamma_{\text{atk}}}{4096} \) for the rebalanced iterates \( \tilde{W}_t = (a_t/\sqrt{\gamma_{\text{atk}}}, V_t\sqrt{\gamma_{\text{atk}}}) \). The first generalization bound now follows by Lemma 2.5. The second generalization bound uses \( p(x, y; W_t) = p(x, y; \tilde{W}_t) \) for all \((x, y)\), whereby \( \Pr[p(x, y; W_t) \leq 0] = \Pr[p(x, y; \tilde{W}_t) \leq 0] \), and the earlier margin-based generalization bound can be invoked with the improved margin of \( \tilde{W}_t \).

Lastly, here are the details from the construction leading to Proposition 2.6 which give a setting where GF converges to parameters with higher margin than the maximum margin linear predictor.

**Proof of Proposition 2.6** Let \( u \) be the maximum margin linear separator for \(((x_i, y_i))_{i=1}^n\); necessarily, there exists at least one support vector in each of \( S_1 \) and \( S_2 \), since otherwise \( u \) is also the maximum margin linear predictor over just one of the sets, but then the condition
\[ \min\{\langle x_i, x_j \rangle : x_i \in S_1, x_j \in S_2\} \leq -\frac{1}{\sqrt{2}} \tag{B.6} \]
implies that \( u \) points away from and is incorrect on the set with no support vectors.

As such, let \( v_1 \in S_1 \) and \( v_2 \in S_2 \) be support vectors, and consider the addition of a single data point \((x', +1)\) from a parameterized family of points \( \{z_\alpha : \alpha \in [0, 1]\} \), defined as follows. Define \( v_0 := (I - v_1 v_1^T)u/\|I - v_1 v_1^T\|u\| \), which is orthogonal to \(-v_1\) by construction, and consider the geodesic between \( v_0 \) and \(-v_1\):
\[ z_\alpha := \alpha v_0 - \sqrt{1 - \alpha^2} v_1, \]
which satisfies \( \|z_\alpha\| = 1 \) by construction by orthogonality, meaning
\[ \|z_\alpha\|^2 = \alpha^2 - 2\alpha \sqrt{1 - \alpha^2} \langle v_0, v_1 \rangle + (1 - \alpha^2) = 1. \]

In order to pick a specific point along the geodesic, here are a few observations.
1. First note that $-v_2$ is a good predictor for $S_1$: \[ \min_{x \in S_1} \langle -v_2, x \rangle = -\max_{x \in S_1} \langle v_2, x \rangle \geq \frac{1}{\sqrt{2}}. \]

It follows similarly that $-v_1$ is a good predictor for $S_2$: analogously, \[ \min_{x \in S_2} \langle -v_1, x \rangle \geq \frac{1}{\sqrt{2}}. \]

2. It is also the case that $-v_1$ is a good predictor for every $z_\alpha$ with $\alpha \leq \frac{1}{\sqrt{2}}$: \[ \langle -v_1, z_\alpha \rangle = \sqrt{1 - \alpha^2} \geq \frac{1}{\sqrt{2}}. \]

3. Now define a 2-ReLU predictor \[ f(x) := (\sigma(-v_1^T x) + \sigma(-v_2^T x))/2. \] As a consequence of the two preceding points, for any $x \in S_1 \cup S_2 \cup \{z_\alpha\}$, then $f(x) \geq 1/\sqrt{2}$ so long as $\alpha \leq 1/\sqrt{2}$. As such, by Lemma A.1, Assumption 1.3 is satisfied with $\gamma_1 \geq 1/(64\sqrt{d})$.

4. Lastly, as $\alpha \to 0$, then $z_\alpha \to -v_1$, but the resulting set of points $S_1 \cup S_2 \cup \{z_0\}$ is linearly separable only with margin at most zero (if it is linearly separable at all). Consequently, there exist choices of $\alpha_0 \in [0, 1]$ so that the resulting maximum margin linear predictor over $S_1 \cup S_2 \cup \{z_0\}$ has arbitrarily small yet still positive margins, and for concreteness choose some $\alpha_0$ so that the resulting linear separability margin $\gamma$ satisfies $\gamma \in \left(0, c/(2^{15}d)\right)$, where $c$ is the positive constant in Lemma 2.3.

As a consequence of the last two bullets, using label +1 and inputs $S_1 \cup S_2 \cup \{z_\alpha\}$, the maximum margin linear predictor has margin $\gamma$, Assumption 1.3 is satisfied with margin at least $\gamma_1$, but most importantly, by Lemma 2.3 (with $m_0$ given by the width lower bound required there), GF will achieve a margin $\gamma_2 \geq c\gamma_1/4 \geq 2\gamma$. It only remains to argue that the linear predictor itself is a KKT point (for the margin objective), but this is direct and essentially from prior work (Lyu and Li, 2019): e.g., taking all outer weights to be equal and positive, and all inner weights to be equal to $u$ and of equal magnitude and equal to the outer layer magnitude, and then taking all these balanced norms to infinity, it can be checked that the gradient and parameter alignment conditions are asymptotically satisfied (indeed, the ReLU can be ignored since all nodes have positive inner product with all examples), which implies convergence to a KKT point (Lyu and Li, 2019, Appendix C).

C Proofs for Section 3

This section develops the proofs of Theorem 3.2 and Theorem 3.3. Before proceeding, here is a quick sampling bound which implies there exist ReLUs pointing in good directions at initialization, which is the source of the exponentially large widths in the two statements.

**Lemma C.1.** Let $(\beta_1, \ldots, \beta_r)$ be given with $\|\beta_k\| = 1$, and suppose $(v_j)_{j=1}^m$ are sampled with $v_j \sim \mathcal{N}_d/\sqrt{d}$, with corresponding normalized weights $\tilde{v}_j$. If \[ m \geq 2 \left( \frac{2}{\epsilon} \right)^d \ln \frac{r}{\delta}, \]
then \[ \max_k \min_j \|\tilde{v}_j - \beta_k\| \leq \epsilon, \] alternatively \[ \min_k \max_j \tilde{v}_j^T \beta_k \geq 1 - \frac{c^2}{2}. \]
Proof. By standard sampling estimates \cite{Ball1997} Lemma 2.3, for any fixed \( k \) and \( j \), then

\[
\Pr[\|\tilde{v}_j - \beta_k\| \leq \epsilon] \geq \frac{1}{2} \left( \frac{\epsilon}{2} \right)^{d-1},
\]

and since all \((v_j)_{j=1}^m\) are iid,

\[
\Pr[\exists j \cdot \|\tilde{v}_j - \beta_k\| \leq \epsilon] = 1 - \Pr[\forall j \cdot \|\tilde{v}_j - \beta_k\| > \epsilon] = 1 - \left( 1 - \Pr[\|\tilde{v}_1 - \beta_k\| \leq \epsilon] \right)^m \\
\geq 1 - \exp \left( - \frac{m}{2} \left( \frac{\epsilon}{2} \right)^{d-1} \right) \geq 1 - \frac{\delta}{r},
\]

and union bounding over all \((\beta_k)_{k=1}^r\) gives \( \max_k \min_j \|\tilde{v}_j - \beta_k\| \leq \epsilon \). To finish, the inner product form comes by noting \( \|\tilde{v}_j - \beta_k\|^2 = 2 - 2\tilde{v}_j^T\beta_k \) and rearranging.

First comes the proof of Theorem \ref{thm:3.2}, whose entirety is the construction of a potential \( \Phi \) and a verification that it satisfies the conditions in Lemma \ref{lem:a6}.

Proof of Theorem \ref{thm:3.2} To start the construction of \( \phi \), first define per-weight potentials \( \phi_{k,j} \) which track rotation of mass towards each \( \beta_k \):

\[
\phi_{k,j} := \phi_k(w_j) := \phi \left( \tilde{\alpha}_k a_j \sigma(v_j^T \beta_k) \geq (1 - \epsilon)\|a_j v_j\| \right).
\]

The goal will be to show that \( \frac{d}{dt}\phi_{k,j} \) is monotone nondecreasing, which shows that weights get trapped pointing towards each \( \beta_k \) once sufficiently close. As such, to develop \( \frac{d}{dt}\phi_{k,j} \), note (using \( \tilde{a_j} := \text{sgn}(a_j) \) and \( \tilde{v}_j := v_j/\|v_j\| \)),

\[
\frac{d}{dt}\tilde{\alpha}_k a_j \sigma(v_j^T \beta_k) \quad = \quad \tilde{\alpha}_k \left[ \tilde{a}_j \sigma(v_j^T \beta_k) + a_j \frac{d}{dt} \sigma(v_j^T \beta_k) \right] \\
= \quad -\tilde{\alpha}_k \sum_i \ell_i' y_i \left[ \sigma(v_j^T x_i) \sigma(v_j^T \beta_k) + a_j^2 \sigma'(v_j^T \beta_k) x_i^T \beta_k \right],
\]

\[
= \quad -\tilde{\alpha}_k \sum_i \ell_i' y_i \left[ \|v_j\|^2 \sigma(\tilde{v}_j^T x_i) \sigma(\tilde{v}_j^T \beta_k) + a_j^2 \sigma'(v_j^T \beta_k) x_i^T \beta_k \right],
\]

\[
\frac{d}{dt}\|a_j v_j\| = \frac{d}{dt} \left( a_j v_j, a_j v_j \right)^{1/2} \\
= \quad 2 \left( a_j v_j, \tilde{a}_j v_j + a_j \tilde{v}_j \right) \\
= \quad 2 \left( a_j v_j, a_j v_j \right)^{1/2} \\
= \quad \left( a_j v_j, - \sum_i \ell_i' y_i \left[ v_j^T \sigma(v_j^T x_i) + a_j^2 \sigma'(v_j^T x_i) x_i \right] \right) \\
= \quad \|a_j v_j\| \\
= \quad - \sum_i \ell_i' y_i \|v_j\| \|w_j\| \\
= \quad - \sum_i \ell_i' y_i \|w_j\| \|a_j v_j\| \\
= \quad - \sum_i \ell_i' y_i \|\tilde{v}_j^T x_i\| \|a_j v_j\|,
\]

\[
\left( \frac{d}{dt} \phi_j \right) = \frac{d}{dt} \left[ \tilde{\alpha}_k a_j \sigma(v_j^T \beta_k) - (1 - \epsilon)\|a_j v_j\| \right] \\
= \quad - \sum_i \ell_i' y_i \left[ \|v_j\|^2 \tilde{a}_j \sigma(\tilde{v}_j^T x_i) \left( \tilde{\alpha}_k \tilde{a}_j \sigma(\tilde{v}_j^T \beta_k) \right) - (1 - \epsilon) \right] \\
= \quad a_j^2 \left( \tilde{\alpha}_k \sigma'(v_j^T \beta_k) \tilde{a}_j x_i - (1 - \epsilon) \tilde{a}_j \sigma(\tilde{v}_j^T x_i) \right).
\]
Analyzing the last two terms separately, the first (the coefficient to \( \|v\|^2 \)) is nonnegative since the term in parentheses is a rescaling of \( \phi_j \):

\[
(\tilde{\alpha}_k \tilde{\alpha}_j \sigma(v_j^\top \beta_k) - (1 - \epsilon)) = \frac{\phi_j}{\|a_j v_j\|} > 0,
\]

and moreover this holds for any choice of \( \epsilon > 0 \). The second term (the coefficient of \( a_j^2 \)) is more complicated; to start, fixing any example \((x_i, y_i)\) and writing \( z_i := c\beta_k + z_\perp \), where necessarily \( c \geq \gamma_{nc} \), note (using \( \tilde{u}_j := \tilde{a}_j \tilde{v}_j \))

\[
\left( \frac{z_\perp}{\|z_\perp\|}, \tilde{u}_j \right)^2 \leq \| (I - \beta_k \beta_k^\top) \tilde{u}_j \|^2 = 1 - \langle \beta_k, \tilde{u}_j \rangle^2 \leq 1 - (1 - \epsilon)^2 = 2\epsilon - \epsilon^2 \leq 2\epsilon,
\]

and thus, since \( \phi_j \in (0, 1) \) whereby \( \sigma(v_j^\top x_i) = v_j^\top x_i \), the nonnegativity of the second term follows from the narrowness of the around \( \beta_k \) (cf. Assumption 3.1):

\[
\tilde{\alpha}_k \beta_k^\top x_i y_i - (1 - \epsilon) \tilde{a}_j \tilde{v}_j^\top x_i y_i \geq c - (1 - \epsilon) \langle \tilde{a}_j \tilde{v}_j, c\beta_k + z_\perp \rangle \\
\geq c\epsilon - (1 - \epsilon) \|z_\perp\| \sqrt{2\epsilon} \\
\geq c\epsilon - (1 - \epsilon) \gamma_{nc} \epsilon \\
\geq 0,
\]

meaning \( d\phi_{k,j}/dt \geq 0 \) for every pair \((k, j)\).

With this in hand, define the overall potential as

\[
\Phi(w) := \frac{1}{4} \sum_k |\alpha_k| \ln \sum_j \phi_{k,j} \|a_j v_j\|,
\]

whereby

\[
\frac{d}{dt} \Phi(w) = \frac{1}{4} \sum_k |\alpha_k| \left[ \sum_j \left( \phi_{k,j} \frac{d}{dt} \|a_j v_j\| + \|a_j v_j\| \frac{d}{dt} \phi_{k,j} \right) \right] \\
\geq \frac{1}{4} \sum_k |\alpha_k| \left[ - \sum_i \epsilon_i y_i \sum_j \phi_{k,j} \tilde{a}_j \sigma(v_j^\top x_i) \|w_j\|^2 \right] \\
\geq \frac{1}{4} Q \sum_k |\alpha_k| \sum_{i \in S_k} \frac{y_i \sum_j \phi_{k,j} \tilde{a}_j \sigma((\tilde{v}_j - \beta_k + \beta_k^\top) x_i) \|w_j\|^2}{\sum_j \phi_{k,j} \|a_j v_j\|} \\
\geq \frac{1}{4} Q \sum_k |\alpha_k| \sum_{i \in S_k} \frac{Q y_i \sum_j \phi_{k,j} (\gamma_{nc} - \epsilon) 2\|a_j v_j\|}{\sum_j \phi_{k,j} \|a_j v_j\|} \\
\geq Q \left( \frac{\gamma_{nc} - \epsilon}{r} \right).
\]

Written another way, this establishes \( d\Phi(W_t)/dt \geq Q \tilde{\gamma} \) with \( \tilde{\gamma} = (\gamma_{nc} - \epsilon)/k \), which is one of the conditions needed in Lemma A.6. The other properties meanwhile are direct:

\[
\Phi(W_t) \leq \frac{1}{4} \sum_k |\alpha_k| \ln \sum_j \phi_{j,k} \|w_j\|^2 \leq \frac{1}{4} \sum_k |\alpha_k| \ln \|W_t\|^2 = \frac{1}{2} \ln \|W_t\|,
\]

and \( \Phi(W_0) > -\infty \) due to random initialization (cf. Lemma C.1), which allows the application of Lemma A.6 and Lemma 2.5 and completes the proof. \(\square\)
To close, the proof of Theorem 3.3.

Proof of Theorem 3.3. Throughout the proof, use $W = ((a_j, b_k))_{j=1}^m$ to denote the full collection of parameters, even in this scalar parameter setting.

To start, note that $a_k^2 = b_k^2$ for all times $t$; this follows directly, from the initial condition $a_k(0)^2 = b_k(0)^2 = 1/\sqrt{m}$, since

$$a_k(t)^2 - b_k(t)^2 = a_k(t)^2 - a_k(0)^2 - b_k(t)^2 + b_k(0)^2$$

and

$$= \int_0^t (a_k \dot{a}_k - b_k \dot{b}_k) \, ds$$

$$= \int_0^t \sum_i |\ell'_i| (a_k \sigma(b_k v_k^T x_i) - a_k \sigma'(b_k v_k^T x_i) v_k^T x_i) \, ds$$

$$= 0.$$ (This also implies that $a_k^2 + b_k^2 = 2a_k^2 = 2|a_k| \cdot |b_k|$.)

For each $\beta_k$, choose $j$ so that $||b_j v_j - \beta_k|| \leq \epsilon = \gamma_{\epsilon|}/2$ and $\bar{a}_j = \text{sgn}(\alpha_k)$; this holds with probability at least $1 - 3\delta$ over the draw of $W$ due to the choice of $m$, first by noting that with probability at least $1 - \delta$, there are at least $m/4$ positive $a_j$ and $m/4$ negative $a_j$, and then with probability $1 - 2\delta$ by applying Lemma C.1 to $(b_j v_j)_{j=1}^m$ (which are equivalent in distribution to $(v_j)_{j=1}^m$ for each choice of output sign. For the rest of the proof, reorder the weights $((a_j, b_j, v_j))_{j=1}^m$ so that each $(\alpha_k, \beta_k)$ is associated with $(a_k, b_k, v_k)$.

Now consider the potential function

$$\Phi(W) := \frac{1}{4} \sum_{k=1}^r |\alpha_k| \ln \left( \frac{a_k^2 + b_k^2}{a_k^2 + b_k^2} \right).$$

The time derivative of this potential can be lower bounded in terms of the reference margin:

$$\frac{d}{dt} \Phi(W) = \sum_k \sum_i |\ell'_i| y_i |\alpha_k| \frac{a_k \sigma(b_k v_k^T x_i) \bar{\sigma}(b_k v_k^T x_i)}{a_k^2 + b_k^2}$$

$$= \sum_k \sum_i q_i y_i |\alpha_k| \frac{|b_k v_k||\sigma(b_k v_k^T x_i)|}{2|a_k| \cdot |b_k|}$$

$$= \sum_k \sum_i q_i y_i |\alpha_k| \frac{(b_k v_k - \beta_k)^T x_i}{|b_k - \beta_k|}$$

$$\geq \sum_k \sum_i q_i y_i |\alpha_k| \frac{(b_k v_k - \beta_k)^T x_i}{|b_k - \beta_k|}$$

$$\geq \sum_k \sum_i q_i y_i |\alpha_k| \frac{(b_k v_k - \beta_k)^T x_i}{|b_k - \beta_k|}$$

$$\geq \sum_k \sum_i q_i y_i |\alpha_k| \frac{(b_k v_k - \beta_k)^T x_i}{|b_k - \beta_k|}$$

As an immediate consequence, the signs of all $((a_k, b_k))_{k=1}^r$ never flip (since this would require their values to pass through 0, which would cause $\Phi(W) = -\infty < \Phi(W(0))$). This implies the preceding lower bound always holds, and since $\Phi(W_0) > -\infty$ thanks to random initialization, and

$$\Phi(W) = \frac{1}{4} \sum_{k=1}^r |\alpha_k| \ln \left( \frac{a_k^2 + b_k^2}{a_k^2 + b_k^2} \right) \leq \frac{1}{4} \sum_{k=1}^r |\alpha_k| \ln \|W\|^2 = \frac{1}{2} \ln \|W\|,$$

all conditions of Lemma A.6 are satisfied, and the proof is complete after applying Lemma 2.5.