Integrability of the Pentagram Map

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Abstract

The pentagram map was introduced by R. Schwartz in 1992 for convex planar polygons. Recently, V. Ovsienko, R. Schwartz, and S. Tabachnikov proved Liouville integrability of the pentagram map for generic monodromies by providing a Poisson structure and the sufficient number of integrals in involution on the space of twisted polygons.

In this paper we prove algebraic-geometric integrability for any monodromy, i.e., for both twisted and closed polygons. For that purpose we show that the pentagram map can be written as a discrete zero-curvature equation with a spectral parameter, study the corresponding spectral curve, and the dynamics on its Jacobian. We also prove that on the symplectic leaves Poisson brackets discovered for twisted polygons coincide with the symplectic structure obtained from Krichever-Phong’s universal formula.

Introduction

The pentagram map was introduced by R. Schwartz in [1] as a map defined on convex polygons understood up to projective equivalence on a real projective plane. Here is a picture of this map for a pentagon and a hexagon:

Figure 1: The pentagram map defined on a pentagon and a hexagon

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This map sends an \( i \)-th vertex to the intersection of 2 diagonals: \((i-1, i+1)\) and \((i, i+2)\). The definition implies that this map is invariant under projective transformations.

Surprisingly, this simple map stands at the intersection of many branches of mathematics: dynamical systems, integrable systems, projective geometry, and cluster algebras. In this paper we focus on integrability of the pentagram map.

Its integrability was thoroughly studied in the paper \[3\], where the authors considered the pentagram map on a more general space \( \mathcal{P}_n \) of the so-called twisted polygons. A twisted polygon is a piecewise linear curve, which is not necessarily closed, but has a monodromy relating its vertices after \( n \) steps (we state its precise definition in the next section). They proved the Arnold-Liouville integrability for the pentagram map on this space:

**Theorem 0.1** \([3]\). There exists a Poisson structure on the space \( \mathcal{P}_n \) invariant under the pentagram map. The number of vertices of the polygons is \( n \), and \( n \geq 4 \). When \( n \) is even, the Poisson brackets have 4 independent Casimirs, and \( n-2 \) invariant functions in involution. When \( n \) is odd, there are only 2 Casimirs, and \( 2q \) (where \( q = \lfloor n/2 \rfloor \)) invariant functions in involution.

The total dimension of \( \mathcal{P}_n \) for all monodromies together is \( 2n \), and this theorem implies the Arnold-Liouville complete integrability on \( \mathcal{P}_n \). I.e., a Zariski open subset of \( \mathcal{P}_n \) is foliated into tori, and the time evolution is a quasiperiodic motion on these tori. The authors of \[3\] posed an open question about integrability for regular closed polygons. Closed polygons form a submanifold \( \mathcal{C}_n \) of codimension 8 in \( \mathcal{P}_n \), but it is difficult to find out what happens with Poisson brackets and integrability on this submanifold. One of the main results of the present study is a solution of this problem (see Theorem C below) in the complexified case.

Note that R.Schwartz conjectured that the pentagram map is a quasi-periodic motion in \[1\], introduced the integrals of motion and proved their algebraic independence in \[2\].

The central component of the algebraic-geometric integrability is a Lax representation with a spectral parameter, which is introduced for the pentagram map in Theorem 2.2. This Lax representation implies our main results, which can be formulated in the following 4 theorems.

**Theorem A.** Consider a spectral curve \( \Gamma_0 \subset \mathbb{CP}^2 \) defined by the equation:

\[
R(z, k) = k^3 - k^2 \left( \sum_{j=0}^{q} J_j z^{q-j} \right) + k \left( \sum_{j=0}^{q} I_j z^{q-j} \right) z^{-n} - z^{-n} = 0,
\]

where \( I_j, J_j, 0 \leq j \leq q \) are complex parameters. Let the normalization of \( \Gamma_0 \) be \( \Gamma \). The genus \( \Gamma \) is \( g = n - 2 \) for even \( n \), and \( g = n - 1 \) for odd \( n \). The Jacobian of \( \Gamma \) is \( J(\Gamma) \).

Then there exists the spectral map \( S : \mathcal{P}_n \to (\Gamma, J(\Gamma)) \), which has a non-degenerate Jacobian matrix at a generic point, i.e., locally, \( S \) is one-to-one.

Notice that we consider polygons on a complex projective plane instead of a real projective plane, which does not change any formulas for the pentagram map.

Next theorem, along with the previous one, establishes the algebraic-geometric integrability:
Theorem B. Let \([D_{0,0}] \in J(\Gamma)\) be the point that corresponds to a twisted polygon at time \(t = 0\) under the map \(S\), and \([D_{0,t}]\) be the point describing the twisted polygon at time \(t\). Then \([D_{0,t}]\) is related to \([D_{0,0}]\) by the formulas:

- when \(n\) is odd,
  \[[D_{0,t}] = [D_{0,0} - tO_1 + tW_2] \in J(\Gamma),\]
- when \(n\) is even,
  \[[D_{0,t}] = \left[D_{0,0} - tO_1 + \left[\frac{1 + t}{2}\right] W_2 + \left[\frac{t}{2}\right] W_3\right].\]

For odd \(n\) the time evolution in \(J(\Gamma)\) is along a straight line, whereas for even \(n\) the evolution resembles a “staircase.”

The point \(O_1 \in \Gamma\) corresponds to \((z = 0, k\) is finite), and the points \(W_2, W_3 \in \Gamma\) correspond to \((z = \infty, k = 0)\).

Theorem C. Closed polygons are singled out by the condition that \((z, k) = (1, 1)\) is a triple point of \(\Gamma\). The latter is equivalent to 5 linear relations on \(I_j, J_j\):

\[
\sum_{j=0}^{q} I_j = \sum_{j=0}^{q} J_j = 3, \quad \sum_{j=0}^{q} jI_j = \sum_{j=0}^{q} jJ_j = 3q - n, \quad \sum_{j=0}^{q} j^2I_j = \sum_{j=0}^{q} j^2J_j.
\]

The genus of \(\Gamma\) drops to \(g = n - 5\) when \(n\) is even, and to \(g = n - 4\) when \(n\) is odd. The dimension of the Jacobian \(J(\Gamma)\) drops by 3 for closed polygons. Theorem [A] holds with this genus adjustment, and Theorem [B] holds verbatim for closed polygons.

The relations on \(I_j, J_j\) found in Theorem 4 in [3] are equivalent to those in Theorem [C]

Corollary. The dimension of the phase space \(C_n\) in the periodic case is \(2n - 8\). In the complexified case, \(C_n\) is fibred over the base of dimension \(2q - 3\). The coordinates on the base are \(I_j, J_j, 0 \leq j \leq n - 1\), subject to the constraints from Theorem [C]. The fibres are Jacobians (complex tori) of dimension \(2q - 3\) for odd \(n\), and of dimension \(2q - 5\) for even \(n\). Note that the restriction of the symplectic form (which corresponds to the Poisson brackets on the symplectic leaves) to the space \(C_n\) is always degenerate, therefore the Arnold-Liouville theorem is not directly applicable for closed polygons. Nevertheless, the algebraic-geometric methods guarantee that the pentagram map exhibits quasi-periodic motion on a Jacobian.

Finally, we prove that:

Theorem D. Krichever-Phong’s universal formula (defined in [5, 6]) provides a pre-symplectic 2-form on the space \(P_n\). This 2-form becomes a symplectic form of rank \(2g\) after the restriction to the leaves: \(\delta I_q = \delta J_q = 0\) for odd \(n\), and \(\delta I_q = \delta J_q = \delta J_q = 0\) for even \(n\). These leaves coincide with the symplectic leaves of the Poisson structure found in [3]. The symplectic form is invariant under the pentagram map and coincides with the inverse of the Poisson structure restricted to the symplectic leaves.

We would also like to point out that there is some similarity between the pentagram map and the integrable model [7] which corresponds to the \(N = 2\) SUSY SU(\(N\)) Yang-Mills theory with a hypermultiplet in the antisymmetric representation.
1 Definition of the pentagram map

In this section, we give a definition of a twisted polygon, following [3], introduce coordinates on the space of such polygons, and give formulas of the map in terms of these coordinates.

Definition 1.1. A twisted $n$-gon is a map $\phi : \mathbb{Z} \to \mathbb{CP}^2$, such that $\phi(k + n) = M \circ \phi(k)$ for any $k$, and $M \in PSL(3, \mathbb{C})$ is a projective transformation of the plane $\mathbb{CP}^2$. $M$ is called the monodromy of $\phi$. Two twisted $n$-gons are equivalent if there is a transformation $g \in PSL(3, \mathbb{C})$, such that $g \circ \phi_1 = \phi_2$. The space of $n$-gons considered up to $PSL(3, \mathbb{C})$ transformations is called $\mathcal{P}_n$.

Notice that the monodromy is transformed as $M \to gMg^{-1}$ under transformations $g \in PSL(3, \mathbb{C})$. The dimension of $\mathcal{P}_n$ is $2n$, because a twisted $n$-gon depends on $2n$ variables representing coordinates of $\phi(k), 0 \leq k \leq n - 1$, on a monodromy matrix $M$ (8 additional parameters), and the equivalence relation reduces the dimension by 8.

The coordinates on $\mathcal{P}_n$ are introduced in the following way. If we assume that $n$ is not divisible by 3, then there exists the unique lift of the points $\phi(k) \in \mathbb{P}^2$ to the vectors $V_k \in \mathbb{C}^3$ provided that $\det(V_j, V_{j+1}, V_{j+2}) = 1$ for all $j$. We associate a difference equation to the sequence of vectors $V_k$:

$$V_{j+3} = a_j V_{j+2} + b_j V_{j+1} + V_j \text{ for all } j.$$  

The sequences $(a_j)$ and $(b_j)$ are $n$-periodic, i.e., $a_{j+n} = a_j, b_{j+n} = b_j$ for all $j$. The monodromy is a matrix $M \in SL(3, \mathbb{C})$, such that $V_{j+n} = MV_j$ for all $j$. The variables $a_i, b_i, 0 \leq i \leq n - 1$ are coordinates on the space $\mathcal{P}_n$.

Notice, that sometimes this map is not defined. For example, it happens when 3 consecutive points lie on one line. When it is defined, and $n = 3m + 1$ or $n = 3m + 2$, the map is given by the formulas:

$$T^*(a_i) = a_{i+2} \prod_{l=1}^{m} \frac{1 + a_{i+3l} + 2b_{i+3l+1}}{1 + a_{i-3l+2} + 2b_{i-3l+1}}, \quad T^*(b_i) = b_{i-1} \prod_{l=1}^{m} \frac{1 + a_{i-3l}b_{i-3l-1}}{1 + a_{i+3l}b_{i+3l-1}}. \quad (1.1)$$

The proof of these formulas is a direct calculation, which has been performed in [3].

2 A Lax representation and the geometry of the spectral curve

The key ingredient of the algebraic-geometric integrability is a Lax representation with a spectral parameter. First, we show that the map (1.1) has such a representation. It implies the conservation of all invariant functions from Theorem 0.1. The Lax representation organizes these invariant functions in the form of the so-called spectral curve. We investigate some properties of the spectral curve, which are important for our purposes.

\footnote{There is a typo in the formula (4.14) for $T^*(b_i)$ in [3].}
In the continuous case, a zero-curvature equation is a compatibility condition for an over-determined system of linear differential equations, hence the name (for example, see [9] for details). In the discrete case, a system of differential equations becomes a system of linear difference equations on functions $\Psi_{i,t}, i, t \geq 0$ of an auxiliary variable $z$ (called a spectral parameter):

$$\begin{align*}
L_{i,t}(z)\Psi_{i,t}(z) &= \Psi_{i+1,t}(z) \\
P_{i,t}(z)\Psi_{i,t}(z) &= \Psi_{i,t+1}(z).
\end{align*}$$

(2.1)

The indices $i$ and $t$ are integers and represent discrete space and time variables. The initial polygon corresponds to $t = 0$. It is convenient to represent several functions $\Psi_{i,t}, i, t \geq 0$ and their relationship on a diagram:

Equations (2.1) form an over-determined system, whose compatibility condition imposes a relation on the functions $L_{i,t}$ and $P_{i,t}$. This relation is called a discrete zero-curvature equation.

**Definition 2.1.** A discrete zero-curvature equation is the compatibility condition for system (2.1), which reads explicitly as:

$$L_{i,t+1}(z) = P_{i+1,t}(z)L_{i,t}(z)P_{i,t}^{-1}(z),$$

(2.2)

where $L_{i,t}$ is called a Lax function.

**Theorem 2.2.** A Lax function for the pentagram map is

$$L_{i,t}(z) = \begin{pmatrix}
-b_i & 1 & 0 \\
-a_i/z & 1/z & 0 \\
1 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & b_i \\
0 & z & a_i
\end{pmatrix}^{-1},$$

where $a_i, b_i, 0 \leq i \leq n - 1$, depend on time $t$. Their dependence on $t$ is not indicated in the notation, but it will always be clear from the context which moment of time they correspond to.

**Proof.** The proof is to check that formulas (2.1) are equivalent to equation (2.2) for an appropriate choice of the function $P_{i,t}$. The matrix function $P_{i,t}$ is different for $n = 3m + 1$ and $n = 3m + 2$. When $n = 3m + 1$, the function $P_{i,t}(z)$ is

$$P_{i,t} = \begin{pmatrix}
-a_i\lambda_{i-1} & 0 & \lambda_{i-1} \\
\lambda_{i-3} & -a_{i+1}\lambda_i & b_{i-1}\lambda_{i-3} \\
0 & z\lambda_{i-2} & 0
\end{pmatrix},$$

where $\lambda_i = \prod_{l=1}^{m}(1 + a_{i+3l+1}b_{i+3l}).$
If $n = 3m + 2$, then $P_{t,t}(z)$ is

$$P_{t,t} = \begin{pmatrix} -a_i\lambda_i\lambda_{i-1}(1 + a_{i+1}b_i) & 0 & \lambda_i\lambda_{i-1}(1 + a_{i+1}b_i) \\ \lambda_i\lambda_{i-2}(1 + a_{i+1}b_i) & -a_i+1\lambda_i\lambda_{i+1}(1 + a_{i+2}b_{i+1}) & \lambda_{i+1}\lambda_{i-1}(1 + a_{i+2}b_{i+1}) \\ 0 & \lambda_i\lambda_{i-2}(1 + a_{i+1}b_i) & 0 \end{pmatrix}.$$  

Notice that $P_{i+n,t} = P_{t,t}$ and $L_{i+n,t} = L_{i,t}$ for all $i$. The rest of the proof is a straightforward calculation using the formulas:

- for $n = 3m + 1$: $T^*(a_i) = a_{i+2} \frac{\lambda_{i+1}}{\lambda_{i-1}}$, $T^*(b_i) = b_{i-1} \frac{\lambda_{i-3}}{\lambda_{i-1}}$, $1 + a_{i+1}b_i \lambda_i = \lambda_{i-3}$,
- for $n = 3m + 2$: $T^*(a_i) = a_{i+2} \frac{\lambda_{i+1}}{\lambda_{i}}$, $T^*(b_i) = b_{i-1} \frac{\lambda_{i-2}}{\lambda_{i-1}}$, $1 + a_{i+3}b_{i+2} \lambda_{i+2} = \lambda_{i-1}$.

\[\square\]

A discrete analogue of the monodromy matrix is a monodromy operator:

**Definition 2.3.** Monodromy operators $T_{0,t}, T_{1,t}, \ldots, T_{n-1,t}$ are defined as the following ordered products of the Lax functions:

$$T_{0,t} = L_{n-1,t}L_{n-2,t} \ldots L_{0,t},$$

$$T_{1,t} = L_{0,t}L_{n-1,t}L_{n-2,t} \ldots L_{1,t},$$

$$T_{2,t} = L_{1,t}L_{0,t}L_{n-1,t}L_{n-2,t} \ldots L_{2,t},$$

$$\ldots$$

$$T_{n-1,t} = L_{n-2,t}L_{n-3,t} \ldots L_{0,t}L_{n-1,t}.$$

Similarly to the continuous case, one can define Floquet-Bloch solutions:

**Definition 2.4.** A Floquet-Bloch solution $\psi_{i,t}$ of a difference equation $\psi_{i+1,t} = L_{i,t}\psi_{i,t}$ is an eigenvector of the monodromy operator: $T_{i,t}\psi_{i,t} = k\psi_{i,t}$.  

**Definition 2.5.** A spectral curve of the monodromy operator $T_{i,t}(z)$ is

$$R(k, z) = \det (T_{i,t}(z) - kI) = 0.$$  

The Floquet-Bloch solutions are parameterized by the points $(k, z)$ of the spectral curve.

**Theorem 2.6.** The spectral curve for the pentagram map is

$$R(k, z) = k^3 - k^2trT_{i,t} + ktr(T_{i,t}^{-1})z^{-n} - z^{-n} = 0, \quad (2.3)$$

where the functions $tr(T_{i,t}^{-1})$ and $trT_{i,t}$ are equal to

$$tr(T_{i,t}^{-1}) = \sum_{j=0}^{q} I_j z^{q-j}, \quad trT_{i,t} = \sum_{j=0}^{q} J_j z^{j-q}. \quad (2.4)$$

Here $q$ is an integer part of $n/2$, i.e., $q = \lfloor n/2 \rfloor$. The coefficients $I_j, J_j$ are polynomials in $a_i, b_i, 0 \leq i \leq n - 1$, and they coincide with the invariants introduced in \[3\].

The spectral curve is independent on $i$ and $t$.  

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We observe that $L$ implies that formulas (2.4) and (2.5) are identical.

Since $\det L_{i,t} = 1/z$, equation (2.3) follows from Vieta’s formula. Equation (2.2) implies that the monodromy operators satisfy the discrete-time Lax equation:

$$T_{i,t+1}(z) = P_{i,t}(z)T_{i,t}(z)P_{i,t}^{-1}(z),$$

i.e., monodromies $T_{i,t}$ are conjugated to each other for different $t$. Consequently, the function $\det (T_{i,t}(z) - kI)$ is independent on $t$. The monodromy operators $T_{i,t}(z)$ with a fixed $t$ and different $i$’s are also conjugated to each other, therefore $R(k, z)$ is independent on $i$.

The definition of $I_j, J_j$ in [3] is:

$$\text{tr } (N_0N_1\ldots N_{n-1}) = \sum_{j=0}^{q} I_j s^{w(j)}, \quad \text{tr } (N_{n-1}^{-1}\ldots N_1^{-1}N_0^{-1}) = \sum_{j=0}^{q} J_j s^{-w(j)}, \quad (2.5)$$

where $N_j = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_j/s \\ 0 & 1 & a_js \end{pmatrix}$, $w(j) = n + 3j - 3q$.

We observe that $L_j^{-1} = (gN_jg^{-1})/s$, where $g = \text{diag } (s, s^2, 1)$, if we identify $z = s^{-3}$. It implies that formulas (2.4) and (2.5) are identical.

We will need the explicit expressions for some of the invariant functions (Proposition 5.3 in [3]):

for any $n$, $I_n = \prod_{j=0}^{n-1} a_j$, $J_n = (-1)^n \prod_{j=0}^{n-1} b_j$,

for even $n$, $I_0 = \prod_{j=0}^{q-1} b_{2j} + \prod_{j=0}^{q-1} b_{2j+1}$, $J_0 = (-1)^q \prod_{j=0}^{q-1} a_{2j} + (-1)^q \prod_{j=0}^{q-1} a_{2j+1}$.

**Theorem 2.7.** A homogenous polynomial $R(k, z, w) = 0$ corresponding to (2.3) defines an algebraic curve $\Gamma_0$ in $\mathbb{C}P^2$. For generic values of the parameters $I_i, J_i$, this curve is singular only at 2 points: $(1:0:0), (0:1:0) \in \mathbb{C}P^2$. Its normalization $\Gamma$ is a Riemann surface of genus $g = 2(n - q - 1)$.

**Proof.** A homogenous polynomial that corresponds to equation (2.3) is

$$R(k, z, w) = k^3z^n - \sum_{j=0}^{q} J_j k^2z^{n+j-q}w^{1-j+q} + \sum_{j=0}^{q} I_j k z^{q-j}w^{n+2+j-q} - w^{n+3}.$$  

The equation $R(k, z, w) = 0$ defines an algebraic curve in $\mathbb{C}P^2$, which we denote by $\Gamma_0$. Singular points are the points where $\partial_k R = \partial_z R = \partial_w R = R = 0$. One can check that the only singular points with $w = 0$ are the points $(1:0:0), (0:1:0) \in \mathbb{C}P^2$. Let us
show that there are no singular points in the affine chart \((k : z : 1)\). By Euler’s theorem on homogenous functions, we have \(k \partial_k R + z \partial_z R + w \partial_w R = (n + 3) R\). Therefore, we have a system of 3 equations for the singular points:

\[
\begin{align*}
\partial_k R &= 3k^2 z^n - \sum_{j=0}^q 2J_j k z^{n+j-q} + \sum_{j=0}^q I_j z^{q-j} = 0 \\
\partial_z R &= n k^3 z^{n-1} - \sum_{j=0}^q (n+j-q) J_j k^2 z^{n+j-q-1} + \sum_{j=0}^q (q-j) I_j k z^{q-j-1} = 0 \\
R &= k^3 z^n - \sum_{j=0}^q J_j k^2 z^{n+j-q} + \sum_{j=0}^q I_j k z^{q-j} - 1 = 0.
\end{align*}
\]

These polynomials may have a solution in common only if \(I_j, J_j\) satisfy some non-trivial polynomial equation as follows, for example, from Sylvester’s resultant formula. This polynomial cannot vanish identically, because the system of equations has no solutions for \(I_0 = J_0 = \ldots = I_q = J_q = 0\). Therefore, for generic values of the parameters \(I_j, J_j\) there are no singular points in the chart \((k : z : 1)\). For the same reason, one may assume that all branch points of \(\Gamma_0\) on \(z\)-plane are simple, since the branch points of index 3 are given by 3 equations: \(R = \partial_k R = \partial^2_k R = 0\).

According to the normalization theorem, there always exists the unique Riemann surface \(\Gamma\) with a map \(\sigma : \Gamma \to \Gamma_0\) biholomorphic away from the singular points. We will always work with the normalized curve \(\Gamma\). The genus \(g\) of \(\Gamma\) is called the geometric genus of the algebraic curve \(\Gamma_0\). To find it, we have to analyze the type of singularities of \(\Gamma_0\), i.e., find the formal series solutions at the singular points.

**Lemma 2.8.** The singularities of the curve \(\Gamma_0\) are as follows:

- If \(n\) is even, the equation \(R(k, z, 1) = 0\) has 3 distinct formal series solutions at \(z = 0\):
  
  \[
  \begin{align*}
  O_1 : \quad &k_1 = \frac{1}{I_q} \left( \frac{I_{q-1}}{I_2^2} - z + O(z^2) \right), \\
  O_2 : \quad &k_2 = (-1)^q \left( \prod_{j=0}^{q-1} a_{2j} \right) \frac{1}{z^q} + O\left( \frac{1}{z^{q-1}} \right), \\
  O_3 : \quad &k_3 = (-1)^q \left( \prod_{j=0}^{q-1} a_{2j+1} \right) \frac{1}{z^q} + O\left( \frac{1}{z^{q-1}} \right),
  \end{align*}
  \]

  and also 3 solutions at \(z = \infty\):

  \[
  \begin{align*}
  W_1 : \quad &k_1 = J_q + \frac{J_{q-1}}{z} + O\left( \frac{1}{z^2} \right), \\
  W_2 : \quad &k_2 = \left( \prod_{j=0}^{q-1} b_{2j} \right)^{-1} \frac{1}{z^q} + O\left( \frac{1}{z^{q+1}} \right), \\
  W_3 : \quad &k_3 = \left( \prod_{j=0}^{q-1} b_{2j+1} \right)^{-1} \frac{1}{z^q} + O\left( \frac{1}{z^{q+1}} \right).
  \end{align*}
  \]
• if $n$ is odd, the equation $R(k, z, 1) = 0$ has 3 distinct Puiseux series solutions at $z = 0$:

$$O_1 : k_1 = \frac{1}{I_q} - \frac{J_{q-1}}{I_q^2} z + O(z^2),$$

$$O_2 : k_2 = \frac{\sqrt{-I_q}}{z^{n/2}} + \frac{J_0}{2z^{(n-1)/2}} + O\left(\frac{1}{z^{(n-2)/2}}\right),$$

$$k_3 = -\frac{\sqrt{-I_q}}{z^{n/2}} + \frac{J_0}{2z^{(n-1)/2}} + O\left(\frac{1}{z^{(n-2)/2}}\right),$$

and 3 solutions at $z = \infty$:

$$W_1 : k_1 = J_q + \frac{J_{q-1}}{z} + O\left(\frac{1}{z^2}\right),$$

$$W_2 : k_2 = \frac{1}{\sqrt{-J_q}} z^{n/2} + \frac{I_0}{2J_q z^{(n+1)/2}} + O\left(\frac{1}{z^{(n+2)/2}}\right),$$

$$k_3 = -\frac{1}{\sqrt{-J_q}} z^{n/2} + \frac{I_0}{2J_q z^{(n+1)/2}} + O\left(\frac{1}{z^{(n+2)/2}}\right).$$

If $\sigma : \Gamma \to \Gamma_0$ is a normalization of $\Gamma_0$, the singularities of $\Gamma_0$ correspond to several points on $\Gamma$:

• for odd $n$, $\sigma^{-1}(1 : 0 : 0) = O_2$, $\sigma^{-1}(0 : 1 : 0) = \{W_1, W_2\}$,

• for even $n$, $\sigma^{-1}(1 : 0 : 0) = \{O_2, O_3\}$, $\sigma^{-1}(0 : 1 : 0) = \{W_1, W_2, W_3\}$.

The point $O_1 \in \Gamma$ is non-singular.

Proof. The proof is a computation using equation (2.3). \qed

Now we can complete the proof of Theorem 2.7. First, we find the number of branch points of $\Gamma$, and then we use the Riemann-Hurwitz formula to find the genus of $\Gamma$.

The number of branch points of $\Gamma$ on $z$-plane equals the number of zeroes of the function:

$$\partial_k R(z, k) = 3k^2 - 2k \left( \frac{J_0}{z^q} + \frac{J_1}{z^{q-1}} + \ldots + \frac{J_{q-1}}{z} + J_q \right) + \left( \frac{I_0}{z^{n-q}} + \frac{I_1}{z^{n-q+1}} + \ldots + \frac{I_{q-1}}{z^{n-1}} + \frac{I_q}{z^n} \right)$$

with an exception of the singular points. The function $\partial_k R(z, k)$ is meromorphic on $\Gamma$, therefore the number of its zeroes equals the number of its poles. For any $n$, $\partial_k R$ has poles of total order $3n$ at $z = 0$, and $\partial_k R$ has zeroes of total order $n$ at $z = \infty$. For even $n$ the Riemann-Hurwitz formula implies that $2 - 2g = 6 - (3n - n)$, thus the genus of $\Gamma$ is $g = n - 2$. For odd $n$ we have $2 - 2g = 6 - (3n - n + 2)$, and $g = n - 1$. The difference between odd and even values of $n$ occurs because $O_2, W_2$ are branch points for odd $n$. \qed
3 Direct and inverse spectral transforms

In this section we prove Theorems \(A\) and \(B\). Recall that Theorem \(A\) reads as follows:

**Theorem \(A\).** At a generic point, the spectral map \(S : \mathcal{P}_n = (a_i, b_i, 0 \leq i \leq n-1) \rightarrow (\Gamma, [D])\) has a non-degenerate Jacobian matrix, i.e., locally, it is one-to-one. The spectral curve \(\Gamma\) is defined in Theorem 2.7 and \([D]\) is a point in the Jacobian \(J(\Gamma)\).

**Remark 3.1.** \(\Gamma\) is determined by \(2q+2\) parameters: \(I_j, J_j, 0 \leq j \leq q\), and a point \([D]\) in the Jacobian of \(\Gamma\) is determined by \(g\) parameters, therefore the dimensions of the spaces on the left and the right hand sides of the map \(S\) match. Since the map \(S\) is locally biholomorphic, it implies the functional (and algebraic) independence of the invariants \(I_j, J_j, 0 \leq j \leq q\). Their independence was proved in [3] by a different method.

The proof of Theorem \(A\) consists of two parts: the direct spectral transform (the construction of the map \(S\) itself) and the inverse spectral transform (the construction of the map \(S^{-1}\)). Combined together, they imply the functional independence of the parameters \(I_j, J_j, 0 \leq j \leq q\) and coordinates in \(J(\Gamma)\).

### 3.1 Direct spectral transform.

Given a set of parameters \((a_i, b_i, 0 \leq i \leq n-1)\), we construct the spectral curve and the Floquet-Bloch solution \(\tilde{\psi}_{0,0}\). The vector function \(\tilde{\psi}_{0,0}\) is defined up to a multiplication by a scalar function. To get rid of this ambiguity, we normalize \(\tilde{\psi}_{0,0}\) by dividing it by the sum of its components. As a result, the vector function \(\tilde{\psi}_{0,0}\) always satisfies the identity: \(\sum_{i=1}^{3} \psi_{0,0,i} \equiv 1\). The Abel map assigns a point in the Jacobian \(J(\Gamma)\) of the curve \(\Gamma\) to each divisor on \(\Gamma\). We denote the pole divisor of \(\tilde{\psi}_{0,0}\) by \(D_{0,0}\), and the corresponding point in \(J(\Gamma)\) by \([D_{0,0}]\). A pair \(\Gamma\) and \([D_{0,0}]\) \(\in J(\Gamma)\) is called the spectral data, and it is used to define the map \(S\).

Notice that once we define the function \(\tilde{\psi}_{0,0}\), all other functions \(\tilde{\psi}_{i,t}\) with \(i, t \geq 0\) are uniquely determined using equations (2.1). However, in Theorem \(B\) below we need to normalize each vector \(\tilde{\psi}_{i,t}\), and we denote the normalized vectors by \(\tilde{\psi}_{i,t}\). The vectors \(\tilde{\psi}_{0,0}\) and \(\tilde{\psi}_{0,0}\) are identical in this notation. The following proposition establishes the number of poles of the normalized Floquet-Bloch solution with any values of \(i, t\).

**Proposition 3.2.** A Floquet-Bloch solution \(\tilde{\psi}_{i,t}\) is a meromorphic vector function on \(\Gamma\). It is uniquely defined if we require \(\sum_{j=1}^{3} \tilde{\psi}_{i,t,j} \equiv 1\). Its pole divisor \(D_{i,t}\) has degree \(g+2\).

**Proof.** Firstly, we show that \(\tilde{\psi}_{i,t}\) is a meromorphic function. By definition, it is a solution to the linear equation: \((T_{i,t} - k)u = 0\). By Cramer’s rule, the components of the vector \(u\) are rational functions in the entries of the matrix \(T_{i,t} - k\) and, consequently, they are rational functions in \(k\) and \(z\). The normalized solution \((u\) divided by the sum of its components \(u_1 + u_2 + u_3\)) is also a rational function in \(k\) and \(z\), i.e., a meromorphic function on \(\Gamma\).

Secondly, we find the behavior of \(\tilde{\psi}_{i,t}\) at the branch points. Let the expansion of \(k(z)\) at the branch point \((k_0, z_0) \in \Gamma\) be \(k(z) = k_0 + k_1 \sqrt{z - z_0} + O(z - z_0)\). If \(k_1 = 0\), then the
The divisors of the functions \( \psi \) determined as follows:

\[
\psi(v, w) = \frac{1}{n} \sum_{i=1}^{3} v_i, \quad \sum_{i=1}^{3} v_i = 1, \quad \sum_{i=1}^{3} w_i = 0.
\]

The latter equations determine \( v, w \) uniquely, and they imply that \( k_0 \) corresponds to a Jordan block of the matrix \( T_{i,t}(z_0) \).

Thirdly, we find the number of the poles of \( \tilde{\psi}_{i,t} \). If \( u_1 + u_2 + u_3 = 0 \), then the function \( \tilde{\psi}_{i,t} \) may develop a pole. For generic values of the parameters \( a_i, b_i \), we may assume that these poles are distinct from the branch points of \( \Gamma \). Let \( k_i, 1 \leq i \leq 3 \) be the solutions of equation (2.3) for a fixed value of \( z \). Then \( Q_i = (k_i, z), 1 \leq i \leq 3 \), correspond to 3 points on \( \Gamma \), and we can form a matrix \( \tilde{\Psi}_{i,t}(z) = \{\tilde{\psi}_{i,t}(Q_1), \tilde{\psi}_{i,t}(Q_2), \tilde{\psi}_{i,t}(Q_3)\} \). Obviously, this matrix depends on the ordering of the roots \( k_1, k_2, k_3 \). However, an auxiliary function \( F(z) = \det \tilde{\Psi}_{i,t}(z) \) is independent on that ordering. Consequently, \( F(z) \) is a well-defined meromorphic function on \( \Gamma \). Generically, it is not singular at the points \( z = 0 \) and \( z = \infty \), which follows from Proposition 3.3 below. One can check using the above series expansion of \( \tilde{\psi}_{i,t} \) that \( F(z) \) has zeroes precisely at the branch points of \( \Gamma \), and that these zeroes are simple. In Theorem 2.7 we found that the number of the branch points of \( \Gamma \) is \( \nu = 2g + 4 \). The pole divisor of \( F(z) \) equals \( 2\pi(D_{i,t}) \). Consequently, we have \( \deg D_{i,t} = \nu/2 = g + 2 \).

### 3.2 Inverse spectral transform.

The construction of the map \( S^{-1} \) consists of 3 parts (which we describe in detail below):

- **Proposition 3.3** establishes analytic properties of the Floquet-Bloch solution \( \psi_i \). These properties allow us to reconstruct the components \( \psi_{i,j}, 1 \leq j \leq 3 \), up to a multiplication by constants. Since the construction of \( S^{-1} \) doesn’t depend on time \( t \), we drop the index \( t \) in Propositions 3.3, 3.4, 3.7.

- Given a spectral curve \( \Gamma \) with marked points \( O_i, W_i, 1 \leq i \leq l \), where \( l = 2 \) or 3 depending on whether \( n \) is odd or even, and a point \([D] \in J(\Gamma)\), Proposition 3.4 allows us to reconstruct Lax matrices \( L_j', 0 \leq j \leq n - 1 \):

\[
L_j'(z) = \begin{pmatrix} 0 & 0 & c_j' \\ d_j' & 0 & b_j' \\ 0 & e_j'z & a_j' \end{pmatrix}^{-1}, 0 \leq j \leq n - 1, \quad \prod_{i=0}^{n-1} c_id_i' = 1.
\]

- If \( n \) is not divisible by 3, Proposition 3.7 allows us to perform the unique reduction from \( L_j' \) to \( L_j \), which completes the construction of \( S^{-1} \). It will be evident from the construction that \( S \circ S^{-1} = Id \), which concludes the proof of Theorem A.

**Proposition 3.3.** The divisors of the functions \( \psi_{i,j}, 0 \leq i \leq n - 1, 1 \leq j \leq 3 \) have the following properties:
• when $n$ is odd,

\[(\psi_{1,1}) \geq -D + (1 - i)O_2 + (1 + i)W_2, \quad (\psi_{1,2}) \geq -D - iO_2 + W_1 + (1 + i)W_2, \quad (\psi_{1,3}) \geq -D + (2 - i)O_2 + iW_2;\]

• when $n$ is even,

\[(\psi_{2k,1}) \geq -D - kO_2 + (1 - k)O_3 + kW_2 + (1 + k)W_3, \quad (\psi_{2k,2}) \geq -D - kO_2 - kO_3 + W_1 + kW_2 + (1 + k)W_3, \quad (\psi_{2k,3}) \geq -D + (1 - k)O_2 + (1 - k)O_3 + kW_2 + kW_3, \quad (\psi_{2k+1,1}) \geq -D - kO_2 - kO_3 + (1 + k)W_2 + (1 + k)W_3, \quad (\psi_{2k+1,2}) \geq -D - (1 + k)O_2 - kO_3 + W_1 + (1 + k)W_2 + (1 + k)W_3, \quad (\psi_{2k+1,3}) \geq -D - kO_2 + (1 - k)O_3 + kW_2 + (1 + k)W_3.\]

**Proof.** First we establish the necessary properties of $\psi_0$. They are different for even and odd $n$. When $n$ is even, the expansion of $T_{0,t}(z)$ at $z = 0$ is:

\[
T_{0,t}(z) = \left( (-1)^q \prod_{i=0}^{q-1} a_{2i} \right) C_1 \left( (-1)^q \prod_{i=0}^{q-1} a_{2i+1} \right) C_2 \left( \frac{1}{z^q} + O \left( \frac{1}{z^{q-1}} \right) \right),
\]

where $C_1, C_2$ are some non-trivial polynomials in $a_i, b_i$.

Using Lemma 2.8 the definition of the Floquet-Bloch solution, and the identity $\psi_{0,1} + \psi_{0,2} + \psi_{0,3} \equiv 1$, one can check that $\psi_0$ is holomorphic at the points $O_1, O_2, O_3$ and that

\[
\psi_0(O_3) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_0,3(O_2) = 0, \quad a_0 = \lim_{Q \to O_1} \frac{\psi_{0,3}(Q)}{\psi_{0,1}(Q)}.
\]

Similarly, the expansion of $T_{0,t}^{-1}(z)$ at $z = \infty$ is:

\[
T_{0,t}^{-1}(z) = \left( \prod_{i=1}^{q-1} b_{2i} \right) \left( \prod_{i=0}^{q-1} b_{2i+1} \right) \left( \frac{z^q}{C_3} \prod_{i=1}^{q} b_{2i-1} \right) z^q + O(z^{q-1}),
\]

which, along with the identity $T_0^{-1} \psi_0 = k^{-1} \psi_0$, implies that $\psi(Q)$ is holomorphic at the points $W_1, W_2, W_3$ and that

\[
\psi_0(W_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_{0,2}(W_1) = 0, \quad b_0 = \lim_{Q \to W_1} \frac{\psi_{0,2}(Q)}{\psi_{0,1}(Q)}, \quad b_{n-1} = -\lim_{Q \to W_1} \frac{\psi_{0,1}(Q)}{\psi_{0,3}(Q)}.
\]
We provide a similar analysis for odd \( n \):

\[
T_{0,t}(z) = \begin{pmatrix}
O(z^{-q}) & \frac{\prod_{i=1}^q (-a_{2i-1})}{z^q} + O(z^{1-q}) & O(z^{-q}) \\
\prod_{i=0}^q (-a_{2i}) + O(z^{-q}) & O(z^{-q}) & \frac{\prod_{i=1}^q (-a_{2i})}{z^q+1} + O(z^{-q}) \\
O(z^{-q}) & \frac{\prod_{i=1}^q (-a_{2i})}{z^q+1} + O(z^{-q}) & O(z^{-q})
\end{pmatrix},
\]

which implies that:

\[
\psi_0(O_2) = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad \psi_0(W_2) = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}, \quad \psi_{0,2}(W_1) = 0, \quad \psi_{0,3}(O_2) = 0, \quad (3.1)
\]

\[
a_0 = \lim_{Q \to O_1} \frac{\psi_{0,3}(Q)}{\psi_{0,1}(Q)}, \quad b_0 = \lim_{Q \to W_2} \frac{\psi_{0,2}(Q)}{\psi_{0,1}(Q)}, \quad b_{n-1} = \lim_{Q \to W_1} \frac{\psi_{0,1}(Q)}{\psi_{0,3}(Q)}. \quad (3.2)
\]

Notice that a cyclic permutation of indices \( (n-1, n-2, \ldots, 1, 0) \) changes \( T_1 \to T_{i+1} \) and \( \tilde{\psi}_i \to \tilde{\psi}_{i+1} \). For even \( n \), it also permutes \( \tilde{\psi}_i(O_2) \leftrightarrow \tilde{\psi}_i(O_3) \) and \( \tilde{\psi}_i(W_2) \leftrightarrow \tilde{\psi}_i(W_3) \). This observation allows us to write formulas similar to (3.2) for \( a_i, b_i, i > 0 \):

\[
a_i = \lim_{Q \to O_1} \frac{\psi_{i,3}(Q)}{\psi_{i,1}(Q)}, \quad b_{2k} = \lim_{Q \to W_2} \frac{\psi_{2k,2}(Q)}{\psi_{2k,1}(Q)}, \quad b_{2k+1} = \lim_{Q \to W_3} \frac{\psi_{2k+1,2}(Q)}{\psi_{2k+1,1}(Q)}. \quad (3.3)
\]

We can use the vectors \( \psi_i, i > 0 \) instead of \( \tilde{\psi}_i, i > 0 \) in formulas (3.3), because they do not depend on the normalization. Formulas for \( b_{2k} \) and \( b_{2k+1} \) coincide for odd \( n \).

Now using formulas (3.3) and the equation \( \psi_{i+1} = L_i \psi_i \), one can check that the components of \( \psi_i \) have the required properties. 

**Proposition 3.4.** Given the spectral curve \( \Gamma \) with marked points \( O_i, W_i, 1 \leq i \leq l \), where \( l = 2 \) for odd \( n \), and \( l = 3 \) for even \( n \), and a point \([D] \) in the Jacobian \( J(\Gamma) \), one can recover a sequence of \( n \) matrices:

\[
L_j'(z) = \begin{pmatrix}
0 & 0 & c_j' \\
d_j' & 0 & b_j' \\
0 & e_j' & a_j'
\end{pmatrix}^{-1}, \quad 0 \leq j \leq n-1, \quad \prod_{i=0}^{n-1} c_i' d_i' e_i' = 1.
\]

This sequence is unique up to gauge transformations: \( L_j' \to g_{j+1} L_j' g_j^{-1} \), where \( g_j, 0 \leq j \leq n-1 \), are non-degenerate diagonal matrices.

**Proof.** The procedure to reconstruct the matrices \( L_j' \), \( 0 \leq j \leq n-1 \), consists of 3 steps:

1. We pick an arbitrary divisor \( D \) of degree \( g+2 \) in the equivalence class \([D] \in J(\Gamma)\).
2. We observe that the degree of all divisors in Proposition 3.3 is \(-g\). According to the Riemann-Roch theorem, it means that each function \(\psi_{i,j}\) is determined up to a multiplication by a constant. We pick arbitrary non-zero constants, and thus obtain a sequence of vectors \(\psi_i\).

3. For each \(i\), we find the matrix \(L'_i\) from the equation \(\psi_{i+1} = L'_i\psi_i\). One can check using Proposition 3.7 that the matrices \(L'_i\) are uniquely determined for all \(i\) at this step.

The remaining part is to prove that \(\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1\) and that the matrices \(L'_j\) are uniquely up to gauge transformations.

Since \(\psi_j' = k\psi_j'\), the determinant of the product \(T'_0 = L'_{n-1} L'_{n-2} \ldots L'_0\) equals \(k_1 k_2 k_3 = z^{-n}\). On the other hand, we have \(\det T'_0 = (z^n \prod_{i=0}^{n-1} c'_i d'_i e'_i)^{-1}\). Consequently, \(\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1\).

Assume that we have a divisor \(D'\) of degree \(g + 2\) equivalent to \(D\). Two divisors are equivalent if and only if there is a meromorphic function \(f\) on \(\Gamma\) with zeroes at \(D\) and with poles at \(D'\). Therefore, a choice of the divisor \(D'\) instead of \(D\) at step 1 is equivalent to multiplying all functions \(\psi_i, 0 \leq i \leq n,\) by the function \(f\). Clearly, such multiplication does not change the matrices \(L'_i\), which we obtain at step 3.

A different choice of constants at step 2 is equivalent to a transformation \(\psi_i \rightarrow g_i \psi_i\), where \(g_i\) is a non-degenerate diagonal matrix. As a result, the matrix \(L'_i\), which we obtain at step 3, is transformed to \(g_{i+1} L'_i g_i^{-1}\), i.e., we obtain a gauge-equivalent sequence of matrices \(L'_i\).

Remark 3.5. The reason to introduce the marked points \(O_i, W_i\) in the statement of Proposition 3.4 is the following. When \(n\) is even, there is no natural way to distinguish the points \(O_2, O_3\) and \(W_2, W_3\). Since the divisors in Proposition 3.3 are not symmetric with respect to swaps \(O_2 \leftrightarrow O_3, W_2 \leftrightarrow W_3\), different markings of the curve \(\Gamma\) may result in 4 non-equivalent sequences of matrices \(L'_j, 0 \leq j \leq n - 1\).

Remark 3.6. Note that one can define the spectral transform \(S'\) for the matrices \(L'_{j}, 0 \leq j \leq n - 1\), in the same way as the transform \(S\) in Section 3.1. The space \(\mathcal{P}_n'\) of the matrices \(L'_j\) is of dimension \(5n - 1\) and it is parameterized by the variables

\[(a'_i, b'_i, c'_i, d'_i, e'_i, 0 \leq i \leq n - 1)\]

subject to the constraint \(\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1\). Then at generic points there is a bijection: \(\mathcal{P}_n'/G \leftrightarrow (\Gamma, [D], O_i, W_i, 1 \leq i \leq l)\), where \(G\) denotes the action by gauge transformations defined in Proposition 3.4. The last statement is a particular case of the general construction proposed in [8].

Proposition 3.7. If \(n\) is not divisible by 3, any sequence of \(n\) matrices:

\[L'_j(z) = \begin{pmatrix} 0 & 0 & c'_j \\ d'_j & 0 & b'_j \\ 0 & c'_j z & a'_j \end{pmatrix}^{-1}, 0 \leq j \leq n - 1, \prod_{i=0}^{n-1} c'_i d'_i e'_i = 1\]

may be transformed to a unique sequence of matrices \(L_j(z)\) (defined in Theorem 2.2) with help of gauge transformations: \(L_j = g_{j+1} L'_j g_j^{-1}\), where \(g_j = \text{diag}(\alpha_j, \beta_j, \gamma_j)\) (\(0 \leq j \leq n - 1, g_n = g_0\)) are diagonal matrices.
Proof. The equation \( L_j = g_{j+1} L'_j g_j^{-1} \) reads as:

\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & b_j \\
0 & z & a_j
\end{pmatrix} = g_{j+1} \begin{pmatrix}
d'_j & 0 & b'_j \\
d_j' & 0 & d'_j \\
e_j' & a_j & e_j'
\end{pmatrix} g_j^{-1},
\]

and it implies a system of equations for \( \alpha_j, \beta_j, \gamma_j, 0 \leq j \leq n - 1 \):

\[
c_j' \frac{\alpha_j}{\gamma_j+1} = d_j' \frac{\beta_j}{\alpha_j+1} = e_j' \frac{\gamma_j}{\beta_j+1} = 1.
\]

The latter system of equations has a one-parameter family of solutions provided that

\[
\prod_{i=0}^{n-1} c_i' d_i' e_i' = 1.
\]

The parameter appears because a multiplication of all matrices \( g_j \) by an arbitrary constant: \( g_j \rightarrow \mu g_j \) leaves the above equations invariant. The variables \( a_j, b_j \) are independent on \( \mu \) due to their defining equations:

\[
a_j = a_j' \frac{\gamma_j}{\gamma_j+1}, \quad b_j = b_j' \frac{\beta_j}{\gamma_j+1}.
\]

\( \square \)

Remark 3.8. Another set of coordinates was proposed in \([3]\). It is related to \( a_i, b_i \) via the formulas:

\[
x_i = \frac{a_{i-2}}{b_{i-2} b_{i-1}}, \quad y_i = -\frac{b_{i-1}}{a_{i-2} a_{i-1}}. \tag{3.4}
\]

Notice, however, that the coordinates \( x_i, y_i \) may be defined independently of \( a_i, b_i \). As opposed to \( a_i, b_i \), they are well-defined for all \( n \), and there is no “non-divisibility by 3” requirement. A Lax representation also exists for the variables \( x_i, y_i \). It is related to the function \( L_j(z) \) that we are using in the following way:

\[
\tilde{L}_j = -\frac{b_j+1}{a_j} (g_j^{-1} L_j g_j) = \begin{pmatrix}
1/z & -1/x_{j+2} & 0 \\
1/z & 0 & 1/z \\
-y_{j+2} & 0 & 0
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & 0 & -1/y_{j+2} \\
-x_{j+2} & 0 & -1/y_{j+2} \\
0 & z & 1/y_{j+2}
\end{pmatrix},
\]

where \( g_j = \text{diag}(1, b_j, -a_j) \) is a gauge matrix. The matrix \( P_{i,t} \), which determines the time evolution, becomes independent on \( n \):

\[
\tilde{P}_{i,t}(z) = \begin{pmatrix}
1 - x_{i+2} y_{i+2} & 0 & 1 - x_{i+2} y_{i+2} \\
x_{i+1} y_{i+1} (1 - x_{i+2} y_{i+2}) & 1 - x_{i+1} y_{i+1} & 1 - x_{i+2} y_{i+2} \\
0 & -z y_{i+2} (1 - x_{i+3} y_{i+3}) & 0
\end{pmatrix}.
\]

All theorems of this paper hold with minor changes in the coordinates \( x_i, y_i \) without the “non-divisibility by 3” requirement. Note that Proposition 3.7 is an obstacle to integrability in the coordinates \( a_i, b_i \), when \( n \) is a multiple of 3.
### 3.3 Time evolution.

The remaining part of this section is to describe the time evolution of the pentagram map and to prove:

**Theorem B.** The equivalence class of the pole divisor $D_{i,t}$ of $\tilde{\psi}_{i,t}$ changes as:

- when $n$ is odd,
  \[ [D_{i,t}] = [D_{0,0} - tO_1 + iO_2 + (t - i)W_2] \in J(\Gamma), \]

- when $n$ is even,
  \[ [D_{i,t}] = \left[ D_{0,0} - tO_1 + \left( \frac{1 + i}{2} \right) O_2 + \left( \frac{i}{2} \right) O_3 + \left( \frac{1 + t - i}{2} \right) W_2 + \left( \frac{t - i}{2} \right) W_3 \right], \]

where $\deg D_{i,t} = g + 2$, and $D_{0,0} \equiv D$ determines the point in $J(\Gamma)$ at $t = 0$. For odd $n$ the time evolution in $J(\Gamma)$ takes place along a straight line, whereas for even $n$ the evolution goes along a “staircase” (i.e., its square goes along a straight line).

The time evolution of the pentagram map is described by the equation: $\psi_{i,t+1} = P_{i,t}\psi_{i,t}$, where $t$ is an integer parameter. The value $t = 0$ corresponds to an initial $n$-gon. Proposition 3.9 describes the time evolution at the level of divisors:

**Proposition 3.9.** The divisors of the functions $\psi_{i,t,j}, 0 \leq i \leq n-1, 1 \leq j \leq 3$ have the following properties:

- when $n$ is odd,
  \[
  (\psi_{i,t,1}) \geq -D + tO_1 + (1 - i)O_2 + (1 + i - t)W_2,
  \]
  \[
  (\psi_{i,t,2}) \geq -D + tO_1 - iO_2 + W_1 + (1 + i - t)W_2,
  \]
  \[
  (\psi_{i,t,3}) \geq -D + tO_1 + (2 - i)O_2 + (i - t)W_2,
  \]

- when $n$ is even,
  \[
  (\psi_{i,t,1}) \geq -D + tO_1 + \left( \frac{1 - i}{2} \right) O_2 + \left( \frac{2 - i}{2} \right) O_3 + \left( \frac{1 + i - t}{2} \right) W_2 + \left( \frac{2 + i - t}{2} \right) W_3,
  \]
  \[
  (\psi_{i,t,2}) \geq -D + tO_1 + \left( \frac{-i}{2} \right) O_2 + \left( \frac{1 - i}{2} \right) O_3 + W_1 + \left( \frac{1 + i - t}{2} \right) W_2 + \left( \frac{2 + i - t}{2} \right) W_3,
  \]
  \[
  (\psi_{i,t,3}) \geq -D + tO_1 + \left( \frac{2 - i}{2} \right) O_2 + \left( \frac{3 - i}{2} \right) O_3 + \left( \frac{i - t}{2} \right) W_2 + \left( \frac{1 + i - t}{2} \right) W_3,
  \]

where $[x]$ is the greatest integer less than or equal to $x$. 
Proof. The proof is a direct verification using Proposition 3.3, formulas for $P_{i,t}$, and 2 additional formulas:

$$\psi_j(O_1) \propto \left(1, \frac{1}{a_{j+1}} + b_j, a_j\right)^T,$$

$$b_{j-1} = -\lim_{Q \to W_1} \frac{\psi_{j,1}(Q)}{\psi_{j,3}(Q)}.$$

where “$\propto$” means “proportional to.”

Propositions 3.4, 3.7 and 3.9 allow us to reconstruct the time evolution of an $n$-gon completely.

Now we are in a position to prove Theorem B itself:

Proof. The vector functions $\psi_{i,t}$ with $i, t \neq 0$ are not normalized. The normalized vectors are equal to $\tilde{\psi}_{i,t} = \psi_{i,t}/f_{i,t}$, where $f_{i,t} = \sum_{j=1}^{3} \psi_{i,t,j}$. According to Proposition 3.9 the divisor of each function $f_{i,t}$ is:

- for odd $n$, $(f_{i,t}) = D_{i,t} - D_{0,0} + tO_1 - iO_2 + (i - t)W_2,$

- for even $n$, $(f_{i,t}) = D_{i,t} - D_{0,0} + tO_1 + \left[\frac{-i}{2}\right]O_2 + \left[\frac{1-i}{2}\right]O_3 + \left[\frac{i-t}{2}\right]W_2 + \left[\frac{1+i-t}{2}\right]W_3.$

Since the divisor of any meromorphic function is equivalent to $[0]$, the result of the theorem follows.

Remark 3.10. For even $n$, it is not sufficient to know $[D_{i,t}]$ to recover the polygon uniquely for each $t$. Proposition 3.4 additionally requires the marking of the spectral curve $\Gamma$ for each $t$. Only one marking is possible for odd $n$, but 4 are possible for even $n$. Notice that Proposition 3.9 contains more information about the time evolution. If one specifies the marking at $t = 0$, Proposition 3.9 determines the time evolution completely.

Remark 3.11. If we use a different normalization $\psi_{0,0,0} \equiv 1$ (i.e., if we divide the vector function $\psi_{0,0}$ by the first component instead of the sum of all components), the divisor $D$ becomes:

- $D = D_g + O_2 + W_2$ for odd $n$,

- $D = D_g + O_3 + W_3$ for even $n$,

where $D_g$ is a generic divisor of degree $g$ on $\Gamma$. 

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4 Periodic case - closed polygons

In this section we prove:

**Theorem C.** Closed polygons are singled out by the condition that \((z,k) = (1,1)\) is a triple point of \(\Gamma\). The latter is equivalent to 5 linear relations on \(I_j, J_j\):

\[
\sum_{j=0}^{q} I_j = \sum_{j=0}^{q} J_j = 3, \quad \sum_{j=0}^{q} jI_j = \sum_{j=0}^{q} jJ_j = 3q - n, \quad \sum_{j=0}^{q} j^2I_j = \sum_{j=0}^{q} j^2J_j. \tag{4.1}
\]

The genus of \(\Gamma\) drops to \(g = n - 5\) when \(n\) is even, and to \(g = n - 4\) when \(n\) is odd. The dimension of the Jacobian \(J(\Gamma)\) drops by 3 for closed polygons. Theorem A holds with this genus adjustment, and Theorem B holds verbatim for closed polygons.

**Proof.** The monodromy matrix from the definition of the twisted \(n\)-gon equals \(T_{0,t}(1)\). Clearly, an \(n\)-gon is closed if and only if \(T_{0,t}(1) = I\). The latter condition implies that \((1,1)\) is a self-intersection point for \(\Gamma_0\). The algebraic conditions implying that \((1,1)\) is a triple point are:

- \(R(1,1) = 0\),
- \(\partial_k R(1,1) = \partial_z R(1,1) = 0\),
- \(\partial^2_k R(1,1) = \partial^2_z R(1,1) = \partial^2_{kz} R(1,1) = 0\).

They are equivalent to 5 linear relations among \(I_j, J_j\):

\[
\sum_{j=0}^{q} I_j = \sum_{j=0}^{q} J_j = 3, \quad \sum_{j=0}^{q} jI_j = \sum_{j=0}^{q} jJ_j = 3q - n, \quad \sum_{j=0}^{q} j^2I_j = \sum_{j=0}^{q} j^2J_j.
\]

Equivalent relations were found in Theorem 4 in [3].

The proofs of Theorems A and B apply, mutatis mutandis, to the periodic case with one change: a count of the number of branch points \(\nu\) of \(\Gamma\) and the corresponding calculation for the genus \(g\) of \(\Gamma\).

As before, the function \(\partial_k R\) has poles of total order \(3n\) above \(z = 0\), and zeroes of total order \(n\) about \(z = \infty\). Now since \(R(z,k)\) has a triple point \((1,1)\), \(\partial_k R\) has a double zero at \((1,1)\). But \(z = 1\) is not a branch point of the normalization \(\Gamma\). Consequently, \(\partial_k R\) has double zeroes on 3 sheets of \(\Gamma\) above \(z = 1\). The Riemann-Hurwitz formula for even \(n\) becomes: \(2-2g = 6-\nu, \ \nu = 3n-n-6 = 2n-6\), and for odd \(n\): \(2-2g = 6-\nu, \ \nu = 3n-n-6+2 = 2n-4\). Therefore, we have \(g = n - 5\) for even \(n\), and \(g = n - 4\) for odd \(n\).

Remark 3.1 implies that there are no other relations among \(I_i, J_i, 0 \leq i \leq q\), except for (4.1) in the periodic case. The dimension of the Jacobian \(J(\Gamma)\) is 3 less than for twisted polygons. Therefore, closed polygons form a subspace of codimension 8 in \(P_n\).
Corollary 4.1. The dimension of the phase space $\mathcal{C}_n$ in the periodic case is $2n - 8$. In the complexified case, $\mathcal{C}_n$ is fibred over the base of dimension $2q - 3$. The coordinates on the base are $I_j, J_j, 0 \leq j \leq n - 1$, subject to the constraints from Theorem C. The fibres are Jacobians (complex tori) of dimension $2q - 3$ for odd $n$, and of dimension $2q - 5$ for even $n$. Note that the restriction of the symplectic form (which corresponds to the Poisson brackets on the symplectic leaves) to the space $\mathcal{C}_n$ is always degenerate, therefore the Arnold-Liouville theorem is not directly applicable for closed polygons. Nevertheless, the algebraic-geometric methods guarantee that the pentagram map exhibits quasi-periodic motion on a Jacobian.

5 The symplectic form

Definition 5.1 ([5, 6]). Krichever-Phong's universal formula defines a pre-symplectic form on the space of Lax operators, i.e., on the space $\mathcal{P}_n$. It is given by the expression:

$$\omega = -\frac{1}{2} \sum_{z=0,\infty} \text{res} \text{Tr} \left( \Psi_0^{-1}T_0^{-1}\delta T_0 \wedge \delta \Psi_0 \right) \frac{dz}{z}.$$  

The matrix $\Psi_{0,t}$ is defined in Proposition 3.2. In this section we drop the index $t$, because all variables correspond to the same moment of time.

The leaves of the 2-form $\omega$ are defined as submanifolds of $\mathcal{P}_n$, where the expression $\delta \ln kdz/z$ is holomorphic. The latter expression is considered as a one-form on the spectral curve $\Gamma$.

Remark 5.2. A heuristic principle justified by many examples is that when $\omega$ is restricted to these leaves, it becomes a symplectic form of rank $2g$, where $g$ is the genus of $\Gamma$. Moreover, one can prove ([8]) that $\omega$ does not depend on the normalization of the eigenvectors used to construct the matrix $\Psi_{0,t}$, and on gauge transformations $L_j \rightarrow g_j + 1 L_j g_j^{-1}$, $g_j \in GL(3, \mathbb{C})$, when restricted to the leaves.

Remark 5.3. There exist different variations of the universal formula, which provide 2 or even more compatible Hamiltonian structures for some integrable systems. However, it seems likely that other modifications of the universal formula lead to degenerate 2-forms for the pentagram map.

Theorem [D]. The 2-form $\omega$ defined above equals:

$$\omega = \sum_{(i,j) \in \Lambda} (\delta \ln a_i \wedge \delta \ln a_j - \delta \ln b_i \wedge \delta \ln b_j),$$

where the set $\Lambda$ consists of pairs $(i, j), 0 \leq i \leq n - 1, i < j \leq n - 1$, such that either both $i$ and $j$ are even, or $i$ is odd and $j$ is arbitrary.

The 2-form $\omega$ is a symplectic form of rank $2g$, where $g$ is the genus of the spectral curve $\Gamma$, when restricted to the leaves: $\delta I_q = \delta J_q = 0$ for odd $n$, and $\delta I_0 = \delta I_q = \delta J_0 = \delta J_q = 0$ for even $n$. These leaves coincide with the symplectic leaves of the Poisson bracket found in Proposition 4.12 in [3]. The inverse of $\omega$ coincides with the Poisson bracket on the symplectic leaves.
Proof. First we find the equations which define the leaves of the 2-form $\omega$.

**Lemma 5.4.** The one-form $\delta \ln kdz/z$ is holomorphic on the spectral curve $\Gamma$ when restricted to the leaves: $\delta I_q = \delta J_q = 0$ for odd $n$, and $\delta I_0 = \delta J_0 = \delta I_q = \delta J_q = 0$ for even $n$.

Proof. Using Lemma 2.8 one can calculate the principal parts of the one-form $\delta \ln kdz/z$ at the points $O_2, W_2$ for odd $n$, and at the points $O_2, O_3, W_2, W_3$ for even $n$. The equations defining the leaves follow after equating these principal parts to zero. 

Now we proceed to the computation of $\omega$. Note that

$$\text{Tr} \left( \Psi_0^{-1} T_0^{-1} \delta T_0 \wedge \delta \Psi_0 \right) = \sum_{k=0}^{n-1} \text{Tr} \left( \Psi_0^{-1} L_k^{-1} \delta L_k \wedge \delta \Psi_0 \right) =$$

$$= \sum_{k=0}^{n-1} \text{Tr} \left( L_k^{-1} L_k \wedge \delta \Psi_0 \right) - \sum_{k=0}^{n-2} \text{Tr} \left( L_k^{-1} L_k \wedge \delta (L_k-1 \wedge L_0) \right),$$

where $\Psi_k = L_k^{-1} \Psi_0$ (this transformation is similar to the one used in [8]). Notice that the last sum does not have any poles except at the points $z = 0$ and $z = \infty$ and vanishes after the summation over both residues. Therefore,

$$\omega = -\frac{1}{2} \sum_{j=0}^{n-1} \text{res}_{0, \infty} \text{Tr} \left( \Psi_j^{-1} L_j^{-1} \delta L_j \wedge \delta \Psi_j \right) \frac{dz}{z}.$$

To compute $\omega$, we use a normalization of $\psi_0$ in which $\psi_{0,1} \equiv 1$. It corresponds to the case when the first line of $\Psi_0$ is $(1, 1, 1)$. Note that a different normalization is used in Proposition 3.3. Therefore, when we use Proposition 3.3 in this proof, we have to keep in mind Remark 3.11. In particular, this remark implies that the vector $\psi_i$ may acquire additional poles at the points $O_2, W_2$ or $O_3, W_3$.

The matrices $\Psi_j, j > 0$, are not normalized. A normalized matrix $\tilde{\Psi}_j, j > 0$, is related to $\Psi_j$ by a diagonal matrix $F_j$: $\tilde{\Psi}_j = \Psi_j F_j$. The matrices $F_j, j > 0$, may have poles or zeroes at $z = 0, \infty$. We have the formula:

$$\text{Tr} \left( \Psi_j^{-1} L_j^{-1} \delta L_j \wedge \delta \Psi_j \right) = \text{Tr} \left( \tilde{\Psi}_j^{-1} L_j^{-1} \delta L_j \wedge \delta \tilde{\Psi}_j \right) - \text{Tr} \left( \tilde{\Psi}_j^{-1} L_j^{-1} \delta L_j \tilde{\Psi}_j \wedge \delta \ln F_j \right)$$

Notice that the product $L_j^{-1} \delta L_j$ is

$$L_j^{-1} \delta L_j = \begin{pmatrix} 0 & 0 & 0 \\ -\delta b_j & 0 & 0 \\ -\delta a_j & 0 & 0 \end{pmatrix},$$

and the first line of $\delta \tilde{\Psi}_j$ is always zero due to the normalization. Consequently, we obtain the formula:

$$\omega = \frac{1}{2} \sum_{j=0}^{n-1} \text{res}_{0, \infty} \text{Tr} \left( \tilde{\Psi}_j^{-1} L_j^{-1} \delta L_j \tilde{\Psi}_j \wedge \delta \ln F_j \right) \frac{dz}{z}.$$
Proof. First, we prove 2 formulas:

\[ \psi_0 = \frac{1}{2} \sum_{j=0}^{n-1} \sum_i \text{res}_{O_i,W_i} \text{Tr} \left( \psi_j^* L_j^{-1} \delta L_j \tilde{\psi}_j \wedge \delta \ln f_j \right) \frac{dz}{z}, \]

where \( \psi_j^* \) is an eigen-covector: \( \psi_j^* T_j = k \psi_j^* \). Covectors are normalized by \( \psi_j^* \tilde{\psi}_j = 1 \), and \( \tilde{\psi}_{j,1} \equiv 1 \). One can check that \( \psi_j^* L_j^{-1} \delta L_j \tilde{\psi}_j = -\psi_{j,3}^* \delta a_j - \psi_{j,2}^* \delta b_j \). The formula for \( \omega \) becomes:

\[ \omega = -\frac{1}{2} \sum_{j=0}^{n-1} \sum_i \text{res}_{O_i,W_i} \left( \psi_{j,3}^* \delta a_j + \psi_{j,2}^* \delta b_j \right) \wedge \delta \ln f_j \frac{dz}{z} = \sum_i \omega_{O_i} + \sum_i \omega_{W_i}. \quad (5.1) \]

We use formula (5.1) to compute \( \omega \). We compute the terms \( \omega_{O_i} \) and \( \omega_{W_i} \) with different \( i \) separately, and then sum them up.

**Lemma 5.5.** The contribution from the point \( O_1 \) is independent on the parity of \( n \) and is given by:

\[ \omega_{O_1} = -\frac{1}{2} \sum_{j=2}^{n-1} \delta \ln a_j \wedge \delta \ln \left( \prod_{k=1}^{j} a_k \right). \]

**Proof.** First, we prove 2 formulas:

\[ \tilde{\psi}_j(O_1) = \left( 1, \frac{1}{a_{j+1}} + b_j, a_j \right)^T, \quad \psi_j^*(O_1) = (0, 0, 1/a_j), \quad (5.2) \]

then we find \( f_j(O_1) \), and compute \( \omega_{O_1} \) using formula (5.1).

The vectors \( \psi_0, \psi_0^* \) and the matrix \( T_0 \) are related to \( \tilde{\psi}_j, \psi_j^*, T_j \) by a permutation of the variables \( a_0, \ldots, a_{n-1} \) and \( b_0, \ldots, b_{n-1} \). Therefore, formulas (5.2) are equivalent to 2 formulas (which we prove below):

\[ \psi_0(O_1) = \left( 1, \frac{1}{a_1} + b_0, a_0 \right)^T, \quad \psi_0^*(O_1) = (0, 0, 1/a_0). \]

**Proposition 3.3** and formulas (3.2) imply that \( \psi_0(O_1) = (1, x, a_0)^T \) for some constant \( x \). Using the value of \( T_0^{-1} \) at \( z = 0 \):

\[ T_0^{-1}(0) = \begin{pmatrix} 0 & 0 & I_q/a_0 \\ 0 & 0 & (1 + a_1 b_0)I_q/(a_0 a_1) \\ 0 & 0 & I_q \end{pmatrix}, \]

and the formula \( T_0^{-1}(0) \psi_0(O_1) = I_q \psi_0(O_1) \), we find that \( x = (1/a_1) + b_0 \).

One can check that the equation \( \psi_0 T_0 = k \psi_0 \) implies that \( \psi_0^*(O_1) = (0, 0, y) \) for some constant \( y \). Since \( \psi_0^* \psi_0 = 1 \), we find that \( y = 1/a_0 \).

To find \( f_j(O_1) \), we have to compare \( \tilde{\psi}_j \) and \( L_{j-1} \ldots L_0 \psi_0 \) at the point \( O_1 \). One can check that \( L_0 \psi_0(O_1) = (1/a_1, *, 1)^T \). Therefore, \( f_1(O_1) = a_1 \). When \( i > 0 \), we have \( L_i \tilde{\psi}_i = (1/a_{i+1}, *, 1)^T \). Consequently, we find that \( f_i(O_1)/f_{i-1}(O_1) = a_i \). Multiplying the latter equations with \( 2 \leq i \leq j \) by each other, we obtain that \( f_j(O_1) = \prod_{k=1}^{j} a_k \).

Substituting \( f_j(O_1) \) and \( \psi_j^*(O_1) \) into formula (5.1), we obtain \( \omega_{O_1} \).

\[ \square \]
Similarly, the contribution from the point $W_1$ is given by:

**Lemma 5.6.** For both even and odd $n$,

$$\omega_{W_1} = \frac{1}{2} \sum_{j=1}^{n-1} \delta \ln b_j \land \delta \ln \left( \prod_{k=0}^{j-1} b_k \right).$$

**Proof.** In the same way as in Lemma 5.5, we find that

$$\tilde{\psi}_j(W_1) = (1, 0, -1/b_{j-1})^T, \quad \tilde{\psi}^*_j(W_1) = (1, -1/b_j, 0), \quad f_j(W_1) = (-1)^j \prod_{k=0}^{j-1} b_k^{-1},$$

which implies the formula for $\omega_{W_1}$.

The computation at the points $O_2, O_3, W_2, W_3$ is trickier, because it differs for even and odd $n$.

**Lemma 5.7.** If $n$ is odd, then

$$\omega_{O_2} = -\frac{1}{2} \sum_{j=1}^{n-1} \delta \ln a_j \land \delta \ln \left( \prod_{k=0}^{j-1} \prod_{i=0}^{q} a_{k+2i} \right).$$

**Proof.** First, we need to prove 2 formulas:

$$\tilde{\psi}_j = \left( 1, \frac{(-1)^{q+1} \prod_{i=0}^{q} a_{j+2i}}{\sqrt{-I_q}} \frac{1}{\sqrt{z}} + O(1), \frac{(-1)^{q} \prod_{i=0}^{q-1} a_{j+2i}}{\sqrt{-I_q}} \sqrt{z} + O(z) \right)^T \text{ at } O_2,$$  \hspace{1cm} (5.3)

$$\tilde{\psi}^*_j(O_2) = \left( \frac{1}{2}, 0, -\frac{1}{2a_j} \right).$$  \hspace{1cm} (5.4)

Note that a cyclic permutation of the variables $a_j \rightarrow a_{j+1}, b_j \rightarrow b_{j+1}$ (for all $j$) permutes the eigenvectors and covectors as follows: $\tilde{\psi}_j \rightarrow \tilde{\psi}_{j+1}, \tilde{\psi}^*_j \rightarrow \tilde{\psi}^*_{j+1}$. Therefore, we only need to find $\tilde{\psi}_0$ at $O_2$ and $\tilde{\psi}^*_0(O_2)$ to prove formulas (5.3) and (5.4).

Proposition 3.3 implies that $\psi_0 = (1, \alpha/\sqrt{z} + O(1), \beta \sqrt{z} + O(z))^T$ around the point $O_2$. Since $T_0 \psi_0 = (\sqrt{-I_q} z^{-n/2} + O(z^{-q})) \psi$, we find that

$$\alpha = \frac{(-1)^{q+1} \prod_{i=0}^{q} a_{2i}}{\sqrt{-I_q}}.$$

One can check that $(T_0^{-1})_{32} = I_q z/a_{n-1} + O(z^2)$, and since $T_0^{-1} \psi_0 = \psi_0 O(z^{n/2})$ in the neighborhood of $O_2$, we deduce that $\beta = -\alpha/a_{n-1}$. Formula (5.3) with $j = 0$ is proven.

The equation $\psi^*_0 \psi_0 = 1$ implies that $\psi^*_0 = (\alpha' + O(\sqrt{z}), \beta' \sqrt{z} + O(z), \gamma' + O(\sqrt{z}))$ at the point $O_2$. Using the identity $\psi_0 T_0 = (\sqrt{-I_q} z^{-n/2} + O(z^{-q})) \psi_0$, we find that

$$\beta' \prod_{i=0}^{q} (-a_{2i}) = \alpha' \sqrt{-I_q}, \quad \beta' \prod_{i=1}^{q} (-a_{2i}) = \gamma' \sqrt{-I_q}, \quad \alpha' + \beta' \frac{(-1)^{q+1} \prod_{i=0}^{q} a_{2i}}{\sqrt{-I_q}} = 1.$$
Solving these equations for $\alpha, \beta', \gamma'$, we obtain that $\psi^*_0(O_2) = (1/2, 0, -1/(2a_0))$.

Now we find the value of $\delta \ln f_j(O_2)$. Since $(L_0\psi_0)_1 = \psi_{0,2} - b_0\psi_{0,1}$, we obtain that $\delta \ln f_1(O_2) = -\delta \ln \alpha$. The argument similar to the one used in the proof of Lemma 5.5 along with the condition $\delta I_q = 0$, implies that

$$\delta \ln f_j(O_2) = -\delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=0}^q a_{k+2i}\right).$$

Finally, using formula (5.4), we obtain that

$$\omega_{O_2} = \frac{1}{2} \sum_{j=1}^{n-1} 2 \cdot \frac{1}{2} \delta \ln a_j \land \delta \ln f_j(O_2) = -\frac{1}{2} \sum_{j=1}^{n-1} \delta \ln a_j \land \delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=0}^q a_{k+2i}\right).$$

The coefficient “2” in the last formula appears because $O_2$ is a branch point. The local parameter around the point $O_2$ is $\sqrt{z}$, and one has to use the formula $2(d\sqrt{z})/\sqrt{z}$ instead of $dz/z$ to compute the residue at $O_2$.

\textbf{Lemma 5.8.} If $n$ is odd, then

$$\omega_{W_2} = -\frac{1}{2} \sum_{j=1}^{n-1} \delta \ln b_j \land \delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=1}^q b_{k+2i+1}\right).$$

\textit{Proof}. The computation of $\omega_{W_2}$ is very similar to that of $\omega_{O_2}$ in Lemma 5.7. By computing $\psi_0, \psi^*_0(W_2), \delta \ln f_1(W_2)$, we find the expressions for $\delta \ln f_j(W_2)$ and $\psi^*_j(W_2)$ with arbitrary $j$:

$$\delta \ln f_j(W_2) = -\delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=1}^q b_{k+2i+1}\right), \quad \psi^*_j(W_2) = \left(0, \frac{1}{2b_j}, 0\right). \quad (5.5)$$

From Proposition 3.3 and formula (3.2) it follows that $\psi_0 = (1, b_0 + \beta/\sqrt{z} + O(1/z), \alpha \sqrt{z} + O(1))$ near the point $W_2$. From the identity $(T_0^{-1}\psi_0)_1 = k^{-1}\psi_{0,1}$ we find that $\alpha \prod_{i=1}^q b_{2i} = \sqrt{-J_q}$. The identity $(T_0\psi_0)_1 = k\psi_{0,1}$, along with the formulas:

$$T_0(W_2) = \begin{pmatrix} J_q & -J_q/b_0 & 0 \\ 0 & 0 & 0 \\ -J_q/b_{n-1} & J_q/(b_0b_{n-1}) & 0 \end{pmatrix}, \quad T_0(z)_{13} = J_q/(b_0b_1z) + O(z^{-2}) \text{ near } W_2,$$

implies that $\beta b_1 = \alpha$. Solving the above equations for $\beta$, we find that $\beta = \left(\prod_{i=1}^q b_{2i+1}\right)/\sqrt{-J_q}$.

Since $(L_0\psi_0)_1 = \beta/\sqrt{z} + O(1/z)$, we obtain that $\delta \ln f_1(W_2) = -\delta \ln \beta$. On the symplectic leaf we have $\delta J_q = 0$, therefore $\delta \ln f_1(W_2) = -\delta \ln \left(\prod_{i=1}^q b_{2i+1}\right)$.

Now we find the covector $\psi^*_0$ at the point $W_2$. The identity $\psi^*_0\psi_0 \equiv 1$ implies that $\psi^*_0 = (A + O(1/\sqrt{z}), B + O(1/\sqrt{z}), C/\sqrt{z} + O(1/z))$, and that $A + b_0B + \alpha C = 1$. The identity $(\psi^*_0 T_0^{-1})_2 = k^{-1}\psi^*_{0,2}$ implies that $C \prod_{i=1}^q b_{2i-1} = B \sqrt{-J_q}$. One can check that since
Lemma 5.9. We prove that the following identities hold:

Substituting formulas (5.5) into formula (5.1), we obtain:

\[ \omega_{W_2} = \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{2} \cdot \frac{1}{2} \delta \ln b_j \land \delta \ln f_j(W_2) = -\frac{1}{2} \sum_{j=1}^{n-1} \delta \ln b_j \land \delta \ln \left( \prod_{k=0}^{j-1} \prod_{i=1}^{q} b_{k+2i+1} \right). \]

The coefficient “-2” appears in the last formula because the local parameter at \( W_2 \) is \( z^{-1/2} \), and the formula \(-2d(z^{-1/2})/z^{-1/2}\) should be used instead of \( dz/z \) to compute the residue. \( \square \)

Now we find the contribution to \( \omega \) from the points \( O_2, O_3, W_2, W_3 \) for even \( n \).

**Lemma 5.9.** If \( n \) is even, then

\[ \omega_{O_2} = -\frac{1}{2} \sum_{j=1}^{q-1} \delta \ln a_{2j} \land \delta \ln \prod_{k=0}^{j-1} a_{2k}, \quad \omega_{O_3} = -\frac{1}{2} \sum_{j=1}^{q-1} \delta \ln a_{2j+1} \land \delta \ln \prod_{k=0}^{j-1} a_{2k+1}. \]

**Proof.** We prove that the following identities hold:

\[ \psi_{2j}^*(O_2) = (1, 0, -1/a_{2j}), \quad \delta \ln f_{2j}(O_2) = -\delta \ln \prod_{k=0}^{j-1} a_{2k}, \quad \psi_{2j+1}^*(O_2) = (0, 0, 0), \]

\[ \psi_{2j}^*(O_3) = (1, 0, -1/a_{2j+1}), \quad \delta \ln f_1(O_3) = -\delta \ln \eta, \]

\[ \delta \ln f_{2j+1}(O_3) = -\delta \ln \left( \eta \prod_{k=0}^{j-1} a_{2k+1} \right), \quad \psi_{2j}^*(O_3) = (0, 0, 0). \]

These identities and formula (5.1) imply the lemma. The parameter \( \eta \) vanishes from the final formulas on the symplectic leaf \( \delta \ln (a_1a_3...a_{2q-1}) = 0 \).

Note that a cyclic permutation of the variables \( a_j \rightarrow a_{j+1}, b_j \rightarrow b_{j+1} \) (for all \( j \)) permutes the eigenvectors and covectors as follows: \( \psi_{2j}(O_2) \rightarrow \psi_{2j+1}(O_3), \psi_{2j}(O_3) \rightarrow \psi_{2j+1}(O_2), \psi_{2j}^*(O_2) \rightarrow \psi_{2j}^*(O_3), \psi_{2j}^*(O_3) \rightarrow \psi_{2j}^*(O_2) \). The use of these permutations and the usual argument to find the functions \( f_j, j > 0 \), imply that the identities above are equivalent to the following 4 formulas (which we prove below):

\[ \delta \ln f_2(O_2) = -\delta \ln a_0, \quad \psi_0^*(O_2) = (1, 0, -1/a_0), \quad \delta \ln f_1(O_3) = -\delta \ln \eta, \quad \psi_0^*(O_3) = (0, 0, 0). \]

Proposition 3.3 implies that \( \psi_0 = (1, O(1), O(z))^T \) at the point \( O_2 \). One can check that the principal part of \( (L_1 L_0 \psi_0)_1 \) at \( O_2 \) is \(-a_0/z\), which implies that \( \delta \ln f_2(O_2) = -\delta \ln a_0 \).

Let the covector \( \psi_0^*(O_2) \) be \((\alpha, \beta, \gamma)\). The equation \((\psi_0^* T_0)_1 = k \psi_{0,1}^* \) implies that \( \beta = 0 \). Since \( \psi_0^*(O_2) \psi_0(O_2) = 1 \), we find that \( \alpha = 1 \). One can check that since the product \( \psi_0^* T_0^{-1} \) has zero of order \( q \) at \( O_2 \), it must be that \( \gamma = -1/a_0 \). Therefore, we obtain that \( \psi_0^*(O_2) = (1, 0, -1/a_0) \).
Proposition 3.3 implies that \( \psi_0 = (1, \eta/z + O(1), O(1))^T \) at the point \( O_3 \). The principal part of \( (L_0 \psi_0)_1 \) at \( O_3 \) is \( \eta/z \), therefore \( \delta \ln f_1(O_3) = -\delta \ln \eta \). Since the product \( \tilde{\psi}_0 \psi_0 \) is holomorphic at \( O_3 \), it must be that \( \psi_0^*(O_3) = 0 \) and \( \psi_0^*(O_3) = (\alpha, 0, \beta) \) for some \( \alpha, \beta \). One can check that the equation \( \psi_0^* T_0 = k \psi_0^* \) implies that \( \alpha = \beta = 0 \), thus \( \psi_0^*(O_3) = (0, 0, 0) \).

Lemma 5.10. If \( n \) is even, then

\[
\omega_{W_2} = \frac{1}{2} \sum_{j=1}^{q-1} \delta \ln b_{2j} \wedge \delta \ln \prod_{k=0}^{j} b_{2k}, \quad \omega_{W_3} = \frac{1}{2} \sum_{j=1}^{q-1} \delta \ln b_{2j+1} \wedge \delta \ln \prod_{k=0}^{j} b_{2k+1}.
\]

Proof. The proof of this lemma is very similar to the proof of Lemma 5.9. We prove that:

\[
\psi_{2j}(W_2) = (0, 1/b_{2j}, 0), \quad \psi_{2j+1}(W_2) = (0, 0, 0), \quad \delta \ln f_2(W_2) = \delta \ln \prod_{k=1}^{j} b_{2k},
\]

\[
\psi_{2j}^*(W_3) = (0, 1/b_{2j+1}, 0), \quad \psi_{2j}^*(W_3) = (0, 0, 0), \quad \delta \ln f_1(W_3) = \delta \ln \prod_{k=1}^{j} b_{2k+1}.
\]

These identities, along with formula (5.1) and the equations for the symplectic leaves:

\[
\delta \ln (b_0 b_2 \cdots b_{2q-2}) = \delta \ln (b_1 b_3 \cdots b_{2q-1}) = 0,
\]

imply the statement of the lemma. Due to a cyclic permutation \( a_j \rightarrow a_{j+1}, b_j \rightarrow b_{j+1}, \tilde{\psi}_{2j}(W_2) \rightarrow \tilde{\psi}_{2j+1}(W_3), \psi_{2j}(W_3) \rightarrow \psi_{2j+1}(W_2), \psi_{2j}^*(W_2) \rightarrow \psi_{2j+1}^*(W_3), \psi_{2j}^*(W_3) \rightarrow \psi_{2j+1}^*(W_2) \) (for all \( j \)), we only need to prove 4 formulas:

\[
\delta \ln f_2(W_2) = \delta \ln b_2, \quad \psi_0^*(W_2) = (0, 1/b_0, 0), \quad \delta \ln f_1(W_3) = \delta \ln \xi, \quad \psi_0^*(W_3) = (0, 0, 0).
\]

Proposition 3.3 implies that \( \psi_0 = (1, b_0 + O(1/z), O(1))^T \) at the point \( W_2 \). By definition of \( f_2 \), we have \( \psi_2 = f_2 L_1 \psi_0 \). We deduce that \( f_2 \psi_0 = (L_0^{-1} L_1^{-1} \psi_0)_1 = b_2 z + O(z^2) \) at \( W_2 \). Since \( \psi_0 = 1 \), we find that \( \delta \ln f_2(W_2) = \delta \ln b_2 \).

Let the covector \( \psi_0^*(W_2) \) be \( (\alpha, \beta, \gamma) \). One can check that the highest order terms of the equation \( \psi_0^* T_0^{-1} = k^{-1} \psi_0^* \) imply that \( \alpha = \gamma = 0 \). Since \( \psi_0^* \psi_0 = 1 \), we find that \( \beta = 1/b_0 \), and \( \psi_0^*(W_2) = (0, 1/b_0, 0) \).

Proposition 3.3 implies that \( \psi_0 = (1, O(1), O(z))^T \) at the point \( W_3 \). Therefore, \( (L_0 \psi_0)_1 \) is a constant, which we denote by \( 1/\xi \). Hence, \( \delta \ln f_1(W_3) = \delta \ln \xi \). Since \( \psi_0^* \psi_0 \) is holomorphic at \( W_3 \), it must be that \( \psi_0^*(W_3) = (\alpha, 0, \beta) \) for some \( \alpha, \beta \). One can check that \( \psi_0^* T_0 = k \psi_0^* \) implies \( \alpha = \beta = 0 \). Therefore, \( \psi_0^*(W_3) = (0, 0, 0) \). \( \square \)

Proof of Theorem D (continued).

Finally, by using the contributions at different points that we have found in Lemmas 5.5, 5.10, one can show that their sum equals the expression in the statement of the theorem for both even and odd \( n \).

The remaining part of the proof is to show that the inverse of \( \omega \) coincides with the Poisson structure found in 3. It is easier to do using the variables \( x_i, y_i \) defined by (3.4).
As is shown in Proposition 4.12 in [3], the Poisson bracket in the variables \( x_i, y_i \) is:

\[
\{ x_i, x_j \} = (\delta_{i,j-1} - \delta_{i,j+1}) x_i x_j, \quad \{ y_i, y_j \} = (\delta_{i,j+1} - \delta_{i,j-1}) y_i y_j,
\]

and all other brackets vanish. The symplectic leaves for these brackets have positive codimension, therefore the corresponding 2-form is not unique. One of the possible 2-forms is

\[
\omega_0 = \sum_{j=0}^{q-1} \delta \ln x_{2j+1} \wedge \delta \ln \left( \prod_{k=0}^{j} x_{2k} \right) - \sum_{j=0}^{q-1} \delta \ln y_{2j+1} \wedge \delta \ln \left( \prod_{k=0}^{j} y_{2k} \right).
\]

Substituting formulas (3.4) into \( \omega_0 \) and using the equations for the symplectic leaves, one can show that \( \omega_0 \) equals \( \omega \). Consequently, \( \omega \) has the same rank as \( \omega_0 \) when restricted to the symplectic leaves, i.e., its rank is \( 2g \).

\[ \square \]

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