What kind of coordinate can keep the Hawking temperature invariant for the static spherically symmetric black hole?

Chikun Ding\textsuperscript{1,2} and Jiliang Jing\textsuperscript{1,2,3}

\textsuperscript{1} Institute of Physics and Department of Physics, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
\textsuperscript{2} Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
E-mail: jljing@hunnu.edu.cn

Received 9 February 2008, in final form 20 April 2008
Published 30 June 2008
Online at stacks.iop.org/CQG/25/145015

Abstract

By studying the Hawking radiation of the most general static spherically symmetric black hole arising from scalar and Dirac particles tunneling, we find the Hawking temperature is invariant in the general coordinate representation (2.3), which satisfies two conditions: (a) its radial coordinate transformation is regular at the event horizon; and (b) there is a time-like Killing vector.

PACS numbers: 04.70.Dy, 04.62.+v, 97.60.Lf

1. Introduction

In recent years, a semi-classical method of modeling Hawking radiation as a tunneling effect has been developed and has excited a lot of interest [1–22]. Tunneling provides not only a useful verification of the thermodynamic properties of black holes but also an alternate conceptual means for understanding the underlying physical process of black hole radiation. In the tunneling approach, the particles are allowed to follow classically forbidden trajectories, by starting just behind the horizon onward to infinity. The particles must then travel necessarily back in time, since the horizon is local to the future of the static or stationary external region. The classical, one-particle action becomes complex, signaling the classical impossibility of the motion, and gives the amplitude an imaginary part, providing a semi-classical approximation to free field propagators. In general the tunneling methods involve calculating the imaginary part of the action $I$ for the (classically forbidden) process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature, i.e.,

$$\Gamma \propto e^{-2\ln I} = e^{-E/T_\text{H}},$$

(1.1)

\textsuperscript{3} Author to whom all correspondence should be addressed.
where $T_H$ is the Hawking temperature of the black hole, $E$ is the energy of the tunneling particles.

There are two different approaches that are used to calculate the imaginary part of the action for the emitted particle. The first method developed was the null geodesic method used by Parikh and Wilczek [3]. The other approach is the Hamilton–Jacobi ansatz used by Agheben et al [4] which is an extension of the complex path analysis of Padmanabhan et al [5–7]. For the Hamilton–Jacobi ansatz it is assumed that the action of the emitted scalar particle satisfies the relativistic Hamilton–Jacobi equation. From the symmetries of the metric one picks an appropriate ansatz for the form of the action and plugs it into the relativistic Hamilton–Jacobi equation to solve.

Since a black hole has a well-defined temperature it should radiate all types of particles like a black body at that temperature. The emission spectrum therefore contains particles of all spins such as Dirac particles. In this paper, we will use the Hamilton–Jacobi ansatz method for calculating the Hawking temperature.

Can the Hawking temperature keep invariant under any coordinate transformation? At first glance, the Hawking temperature is invariant. However, this invariance has been lost in the following isotropic coordinate [4, 8] for the Schwarzschild black hole

$$t \rightarrow t, \quad r \rightarrow \rho, \quad \ln \rho = \int \frac{dr}{r \sqrt{1 - \frac{2M}{r}}} \quad (1.2)$$

And so the line element of the Schwarzschild black hole becomes

$$ds^2 = -\left(\frac{2\rho - M}{2\rho + M}\right)^2 dt^2 + \left(\frac{2\rho + M}{2\rho}ight)^4 d\rho^2 + \frac{(2\rho + M)^4}{16\rho^2} d\Omega^2, \quad (1.3)$$

and the horizon $\rho_H = M/2$. Substituting it and $\phi = e^{i(-Et + W(\rho) + J(\theta, \varphi))/\hbar}$ into the Klein–Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - \frac{m^2}{\hbar^2} \phi = 0, \quad (1.4)$$

we can obtain

$$\text{Im} W_\pm(\rho) = \pm \text{Im} \left[ \int \frac{(2\rho + M)^3 d\rho}{4\rho^2(2\rho - M)} \sqrt{E^2 - \left(\frac{2\rho - M}{2\rho + M}\right)^2 (m^2 + g^{ij} J_i J_j)} \right] = \pm 4\pi M E. \quad (1.5)$$

The probability is [5–7]

$$\Gamma = \frac{\Gamma_{\text{out}}}{\Gamma_{\text{in}}} \propto \exp[-4 \text{Im} W_+] = \exp[-16\pi M E] = \exp \left[-\frac{E}{T_H}\right], \quad (1.6)$$

since $W_- = -W_+$. Then the black hole’s temperature is [8]

$$T_H = \frac{1}{16\pi M}, \quad (1.7)$$

which is one-half of the standard Hawking temperature $T_H = 1/8\pi M$. The example tells us that the invariance is missing in the isotropic coordinate. The reason for the phenomenon comes from the coordinate transformation (1.2) itself. In the radial coordinate transformation

$$\ln \rho = \int \frac{dr}{r \sqrt{1 - \frac{2M}{r}}} = \int F(r) dr, \quad (1.8)$$

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the function \( F(r) = \frac{1}{\sqrt{r - 2M}} \) has singularity at the horizon \( r = 2M \). So it needs to discuss in which coordinates Hawking temperature can be invariant.

The purpose of this paper is to investigate the invariance of the Hawking temperature of the most general static spherically symmetric black hole from scalar and Dirac particles tunneling in a general coordinate representation. In order to do that, we introduce the metrics of the static spherically symmetric black in the two coordinates: Schwarzschild-like and a general coordinate. This general coordinate should satisfy two conditions: (a) its radial coordinate transformation is regular at the event horizon; (b) there exists a time-like Killing vector.

This paper is organized as follows. In section 2, the different coordinate representations for the general static spherically symmetric black hole are presented. In section 3, the Hawking temperature of the general static spherically symmetric black hole for scalar particle tunneling is investigated. In section 4, the Hawking temperature of the general static spherically symmetric black hole from Dirac particles tunneling is studied. The last section is devoted to a summary.

### 2. Coordinate representations for the general static spherically symmetric black hole

In this section we introduce two kinds of the coordinate representations for the general static spherically symmetric black hole, i.e., the Schwarzschild-like and a general coordinate.

#### 2.1. Schwarzschild-like coordinate representation

In the Schwarzschild-like coordinate the line element for the most general static spherically symmetric black hole in four dimensional spacetime is described by

\[
d s^2 = -f(r) \, dt^2 + \frac{1}{g(r)} \, dr^2 + R(r)(d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

where \( f(r), g(r) \) and \( R(r) \) are functions of \( r \), and \( t \) is the Schwarzschild-like time coordinate. Because the spacetime (2.1) is a static and spherically symmetric one, a time-like Killing vector field \( \xi^\mu = (1, 0, 0, 0) \) exists. An interesting feature of the black hole, worthy of note, is that the norm of the Killing field \( \xi^\mu \) is zero on the event horizon \( r_H \) since the horizon is a null surface and the vector \( \xi^\mu \) is normal to the horizon. Then, for the non-extreme case we have \( f(r) = f_1(r)(r - r_H) \) and \( g(r) = g_1(r)(r - r_H) \), where \( f_1(r) \) and \( g_1(r) \) are regular functions in the region \( r_H < r < \infty \) and their values are nonzero on the outermost event horizon.

#### 2.2. General coordinate representation

In order to ensure that there is a time-like Killing vector in the spacetime, the most general coordinate \((v, u, \theta, \phi)\) that transforms from the Schwarzschild-like coordinate (2.1) is

\[
v = \lambda t + \int dr \, G(r), \quad u = \int dr \, F(r),
\]

where \( v \) is the time coordinate, \( u \) is the radial one and the angular coordinates remain unchanged; \( \lambda \) is an arbitrary nonzero constant which re-scales the time; \( G \) is arbitrary functions of \( r \) and \( F \) is a regular function of \( r \). The line element (2.1) in the new coordinate becomes

\[
d s^2 = -\frac{f(r(u))}{\lambda^2} \, dv^2 + 2\frac{f(r(u))G(r(u))}{\lambda^2 F(r(u))} \, dv \, du + \frac{\lambda^2 - f(r(u))g(r(u))G^2(r(u))}{\lambda^2 g(r(u)) F^2(r(u))} \, du^2
\]

\[
+ R(r(u))(d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

We now show that the two well-known coordinates, the Painlevé and Lemaître coordinates, are the spacial cases of the metric (2.3).
2.2.1. Painlevé coordinate representation. In the transformation (2.2), one sets \( \lambda = 1, G(r) = \sqrt{\frac{1}{1 - g(r)}} \) and \( F(r) = 1 \), the line element (2.3) becomes the Painlevé coordinate representation [2, 23]

\[
d s^2 = -f(r) \, dt^2 + 2 \frac{f(r)(1 - g(r))}{g(r)} \, dt \, dr + \sqrt{f(r) \left( 1 - g(r) \right)} \, dr^2 + R(r)(d\theta^2 + \sin^2\theta \, d\phi^2),
\]

(2.4)

where \( t \) is the Painlevé time. The metric (2.4) has no singularity at \( g(r) = 0 \), so the metric is regular at the horizon of the black hole. That is to say, the coordinate complies with the perspective of a free-falling observer, who is expected to experience nothing out of the ordinary upon passing through the event horizon.

2.2.2. Lemaître coordinate representation. In the transformation (2.2), one sets \( \lambda = 1, G(r) = \frac{1}{2} \sqrt{\frac{g(r)}{f(r)}} \left( 1 - g(r) \right) + \sqrt{1 - g(r)} \cdot \sqrt{g(r)} \) and \( F(r) = \frac{1}{2} \sqrt{\frac{g(r)}{f(r)}} \), the line element (2.3) at present becomes the Lemaître coordinate representation [23, 24]

\[
d s^2 = -f(r)[dV^2 + dU^2] + 2 \frac{f(r)(2 - g(r))}{g(r)} \, dV \, dU + R(r)(d\theta^2 + \sin^2\theta \, d\phi^2),
\]

(2.5)

where \( U \) is the Lemaître radial coordinate and \( V \) is the Lemaître time one. We can see the fact that the Lemaître coordinate is a time-dependant system, suggests that there could be a genuine particle production.

3. Temperature of the general static spherically symmetric black hole from scalar particle tunneling

We now investigate scalar particle tunneling of the general static spherically symmetric black hole.

3.1. Scalar particles tunneling in the Schwarzschild-like coordinate

Applying the WKB approximation

\[
\phi(t, r, \theta, \phi) = \exp \left[ \frac{1}{\hbar} I(t, r, \theta, \phi) + I_1(t, r, \theta, \phi) + \mathcal{O}(\hbar) \right]
\]

(3.1)

to the Klein–Gordon equation (1.4), then, to leading order in \( \hbar \) we get the following relativistic Hamilton–Jacobi equation:

\[
g^{\mu\nu}(\partial_\mu I, \partial_\nu I) + m^2 = 0.
\]

(3.2)

As usual, due to the symmetries of the metric (2.1) and neglecting the effects of the self-gravitation of the particles, there exists a solution in the form

\[
I = -Et + W(r) + J(\theta, \phi).
\]

(3.3)

Inserting equation (3.3) and the metric (2.1) into the Hamilton–Jacobi equation (3.2), we find

\[
W_\pm(r) = \pm \int \frac{dr}{\sqrt{f(r)g(r)}} \sqrt{E^2 - f(r)(m^2 + g^{ij}J_iJ_j)},
\]

(3.4)

where \( J_i = \partial_\mu I, i = \theta, \phi \). One solution of equation (3.4) corresponds to the scalar particles moving away from the black hole (i.e. ‘+’ outgoing) and the other solution corresponds to particles moving toward the black hole (i.e. ‘−’ incoming). Imaginary parts of the action can
only come due to the pole at the horizon. The probability of a particle tunneling from inside to outside the horizon is [5–7]

\[
\Gamma = \frac{\Gamma_{\text{out}}}{\Gamma_{\text{in}}} \propto \exp[-4 \text{ Im } W_+] = \exp\left[-\frac{E}{T_H}\right],
\]

(3.5)
since \(W_- = -W_+\). Integrating around the pole at the horizon leads to

\[
\text{Im } W_+ = \frac{\pi E}{\sqrt{f'(r_H)g'(r_H)}}.
\]

(3.6)

Substituting (3.6) into (3.5), we obtain the Hawking temperature

\[
T_H = \frac{\sqrt{f'(r_H)g'(r_H)}}{4\pi},
\]

(3.7)

which shows that the temperature of the general static spherically symmetric black hole is the same as previous works [1, 2].

3.2. Scalar particles tunneling in a general coordinate

Here we study the scalar tunneling in a general coordinate (2.3). Employing the ansatz

\[
I = -Ev + W(u) + J(\theta, \phi)
\]

(3.8)

and substituting the metric (2.3) into the Hamilton–Jacobi equation (3.2), we obtain

\[
\left[g(r(u))G^2(r(u)) - \frac{\lambda^2}{f(r(u))}\right]E^2 - 2g(r(u))G(r(u))F(r(u))E W'(u) + g(r(u))F^2(r(u))[W'(u)]^2 + g^{ij}J_i J_j + m^2 = 0.
\]

(3.9)

Then \(W'(u)\) is

\[
W'_\pm(u) = \frac{G(r(u))}{F(r(u))} E \pm \frac{\sqrt{\lambda^2 E^2 - f(r(u))[g^{ij}J_i J_j + m^2]}}{F(r(u))\sqrt{f'(r(u))g'(r(u))}}.
\]

(3.10)

We will study the temperature for two cases: \(G(r(u))\) is a regular function and \(G(r(u))\) has a pole at the horizon.

3.2.1. \(G(r(u))\) is a regular function at the horizon. When \(G(r(u))\) is regular at the horizon, \(g_{uu}\) of metric (2.3) shows that there is still a coordinate singularity at the horizon \(r_H\). From equation (3.10) we get

\[
\text{Im } W_{\pm}(u) = \text{Im} \int du \left\{ \frac{G(r(u))}{F(r(u))} E \pm \frac{\sqrt{\lambda^2 E^2 - f(r(u))[g^{ij}J_i J_j + m^2]}}{F(r(u))\sqrt{f'(r(u))g'(r(u))}} \right\}
\]

\[
= \pm \frac{\lambda E\pi}{\sqrt{f'(r_H)g'(r_H)}}.
\]

(3.11)

We can see that the Im\(W_{\pm}(u)\) are like those in the Schwarzschild-like coordinate. Using

\[
\Gamma \propto \exp[-4 \text{ Im } W_+] = \exp\left[-\frac{\lambda E}{T_H}\right],
\]

(3.12)

we can recover the Hawking temperature (3.7).
3.2.2. G(r(u)) has a pole at the horizon. When G(r(u)) has a pole at the horizon, without loss of generality, it can be expressed as

\[ G(r(u)) = C(r(u)) \sqrt{f(r(u))g(r(u))} + D(r(u)), \]

where \( C(r(u)) \) and \( D(r(u)) \) are the regular functions at the horizon. From equation (3.10), we obtain

\[
\text{Im } W_{\pm}(u) = \int \text{Im} \left\{ \frac{D(r(u))}{F(r(u))} E + \frac{C(r(u)) E \pm \sqrt{\lambda^2 E^2 - f(r(u)) g(r(u))}}{F(r(u)) \sqrt{f(r(u))g(r(u))}} \right\} \frac{\text{d}u}{f'(r_H)g'(r_H)}. \tag{3.13}
\]

In the following, we will consider two cases, i.e., \( C(r_H) \neq \lambda \) and \( C(r_H) = \lambda \):

(i) If \( C(r_H) \neq \lambda \), after substituting \( G(r(u)) = C(r(u)) \sqrt{f(r(u))g(r(u))} + D(r(u)) \) into \( g_{uu} \) of metric (2.3), it is easy to see that there is still a coordinate singularity at the horizon \( r_H \), and the probabilities are

\[
\Gamma_{\text{out}} \propto \exp \left[ -2 \left( \frac{C(r_H)}{\lambda} + 1 \right) \frac{\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E \right], \quad \Gamma_{\text{in}} \propto \exp \left[ -2 \left( \frac{C(r_H)}{\lambda} - 1 \right) \frac{\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E \right]. \tag{3.14}
\]

It is interesting to note that \( \Gamma_{\text{out}}, \Gamma_{\text{in}} \) are different from that in the Schwarzschild-like coordinate, but the total probability is

\[
\Gamma = \frac{\Gamma_{\text{out}}}{\Gamma_{\text{in}}} \propto \exp \left[ -\frac{4\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E \right], \tag{3.15}
\]

and the Hawking temperature (3.7) is also recovered.

(ii) If \( C(r_H) = \lambda \), we can write \( C(r(u)) = \lambda + H(r(u)) \sqrt{f(r(u))g(r(u))} \), where \( H(r(u)) \) is a regular function at the horizon. Then we have \( G(r(u)) = \frac{\lambda}{\sqrt{f(r(u))g(r(u))}} + D(r(u)) \). Substituting it into \( g_{uu} \) of metric (2.3), we find that there is no coordinate singularity at the horizon \( r_H \) now. From equation (3.13), we obtain

\[
\text{Im } W_{+}(u) = \frac{2\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E, \quad \text{Im } W_{-}(u) = 0; \tag{3.16}
\]

this implies that \( \Gamma_{\text{in}} = 1 \). So the overall tunneling probability is

\[
\Gamma = \frac{\Gamma_{\text{out}}}{\Gamma_{\text{in}}} \propto \exp \left[ -2 \text{Im } W_{+} \right] = \exp \left[ -\frac{4\pi E}{\sqrt{f'(r_H)g'(r_H)}} \right]. \tag{3.17}
\]

It is obvious that the Hawking temperature (3.7) is recovered.

From the above discussions we know that the Hawking temperature of the general static spherically symmetric black hole arising from the scalar particles tunneling is invariant in the general coordinate (2.3).

4. Temperature of the general static spherically symmetric black hole from Dirac particle tunneling

In this section, we study the Dirac particle tunneling of the black hole in the coordinates (2.1) and (2.3).

4.1. Dirac particles tunneling in the Schwarzschild-like coordinate

For a general background spacetime, the Dirac equation is [25]

\[
\left[ \gamma^{\mu} \delta_{\alpha}^{\mu} (\partial_{\mu} + \Gamma_{\mu}) + \frac{m}{\hbar} \right] \psi = 0, \tag{4.1}
\]

\[\text{Im } W_{\pm}(u) = \int \text{Im} \left\{ \frac{D(r(u))}{F(r(u))} E + \frac{C(r(u)) E \pm \sqrt{\lambda^2 E^2 - f(r(u)) g(r(u))}}{F(r(u)) \sqrt{f(r(u))g(r(u))}} \right\} \frac{\text{d}u}{f'(r_H)g'(r_H)}. \tag{3.13}
\]

In the following, we will consider two cases, i.e., \( C(r_H) \neq \lambda \) and \( C(r_H) = \lambda \):

(i) If \( C(r_H) \neq \lambda \), after substituting \( G(r(u)) = C(r(u)) \sqrt{f(r(u))g(r(u))} + D(r(u)) \) into \( g_{uu} \) of metric (2.3), it is easy to see that there is still a coordinate singularity at the horizon \( r_H \), and the probabilities are

\[
\Gamma_{\text{out}} \propto \exp \left[ -2 \left( \frac{C(r_H)}{\lambda} + 1 \right) \frac{\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E \right], \quad \Gamma_{\text{in}} \propto \exp \left[ -2 \left( \frac{C(r_H)}{\lambda} - 1 \right) \frac{\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E \right]. \tag{3.14}
\]

It is interesting to note that \( \Gamma_{\text{out}}, \Gamma_{\text{in}} \) are different from that in the Schwarzschild-like coordinate, but the total probability is

\[
\Gamma = \frac{\Gamma_{\text{out}}}{\Gamma_{\text{in}}} \propto \exp \left[ -\frac{4\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E \right], \tag{3.15}
\]

and the Hawking temperature (3.7) is also recovered.

(ii) If \( C(r_H) = \lambda \), we can write \( C(r(u)) = \lambda + H(r(u)) \sqrt{f(r(u))g(r(u))} \), where \( H(r(u)) \) is a regular function at the horizon. Then we have \( G(r(u)) = \frac{\lambda}{\sqrt{f(r(u))g(r(u))}} + D(r(u)) \). Substituting it into \( g_{uu} \) of metric (2.3), we find that there is no coordinate singularity at the horizon \( r_H \) now. From equation (3.13), we obtain

\[
\text{Im } W_{+}(u) = \frac{2\pi}{\sqrt{f'(r_H)g'(r_H)}} \lambda E, \quad \text{Im } W_{-}(u) = 0; \tag{3.16}
\]

this implies that \( \Gamma_{\text{in}} = 1 \). So the overall tunneling probability is

\[
\Gamma = \frac{\Gamma_{\text{out}}}{\Gamma_{\text{in}}} \propto \exp \left[ -2 \text{Im } W_{+} \right] = \exp \left[ -\frac{4\pi E}{\sqrt{f'(r_H)g'(r_H)}} \right]. \tag{3.17}
\]

It is obvious that the Hawking temperature (3.7) is recovered.

From the above discussions we know that the Hawking temperature of the general static spherically symmetric black hole arising from the scalar particles tunneling is invariant in the general coordinate (2.3).
with
\[ \Gamma_\mu = \frac{1}{2} \{ \gamma^a, \gamma^b \} e_\mu^a e_\nu^b, \]
where \( \gamma^a \) are the Dirac matrices and \( e_\mu^a \) is the inverse tetrad defined by \( \{ e_\mu^a \gamma^a, e_\nu^b \gamma^b \} = 2g^{\mu\nu} \times 1 \). For the general static spherically symmetric black hole in the Schwarzschild-like metric (2.1) the tetrad can be taken as
\[ e_\mu^a = \text{diag} \left( \frac{1}{\sqrt{f(r)}}, \frac{1}{\sqrt{g(r)}}, \frac{1}{\sqrt{R(r)}}, \frac{1}{\sqrt{R(r) \sin \theta}} \right). \] (4.2)

We employ the following ansatz for the Dirac field,
\[ \psi^\uparrow = \begin{pmatrix} A(t, r, \theta, \phi) \xi^\uparrow \\ B(t, r, \theta, \phi) \xi^\uparrow \end{pmatrix} \exp \left( \frac{i}{\hbar} I^\uparrow (t, r, \theta, \phi) \right) = \begin{pmatrix} A(t, r, \theta, \phi) \\ B(t, r, \theta, \phi) \end{pmatrix} \exp \left( \frac{i}{\hbar} I^\uparrow (t, r, \theta, \phi) \right), \]
\[ \psi^\downarrow = \begin{pmatrix} C(t, r, \theta, \phi) \xi^\downarrow \\ D(t, r, \theta, \phi) \xi^\downarrow \end{pmatrix} \exp \left( \frac{i}{\hbar} I^\downarrow (t, r, \theta, \phi) \right) = \begin{pmatrix} 0 \\ C(t, r, \theta, \phi) \end{pmatrix} \exp \left( \frac{i}{\hbar} I^\downarrow (t, r, \theta, \phi) \right). \] (4.3)

where \( \uparrow \) and \( \downarrow \) represent the spin-up and spin-down cases, and \( \xi^\uparrow \) and \( \xi^\downarrow \) are the eigenvectors of \( \sigma^3 \). Inserting equations (4.2), (4.3) into the Dirac equation (4.1) and employing
\[ I^\uparrow = -E_t + W(r) + J(\theta, \phi), \]
(4.4)
to the lowest order in \( \hbar \) we obtain
\[ -\frac{A}{\sqrt{f(r)}} E + \sqrt{g(r)} B W'(r) + mA = 0, \] (4.5)
\[ \frac{B}{\sqrt{R(r)}} \left( J_0 + \frac{i}{\sin \theta} J_\psi \right) = 0, \] (4.6)
\[ \frac{B}{\sqrt{f(r)}} E - \sqrt{g(r)} A W'(r) + mB = 0, \] (4.7)
\[ -\frac{A}{\sqrt{R(r)}} \left( J_0 + \frac{i}{\sin \theta} J_\psi \right) = 0, \] (4.8)

where we consider only the positive frequency contributions without loss of generality. Equations (4.6) and (4.8) both yield \( \left( J_0 + \frac{i}{\sin \theta} J_\psi \right) = 0 \) regardless of \( A \) or \( B \), implying that \( J(\theta, \phi) \) must be a complex function. We can therefore ignore \( J \) from this point (or else pick the trivial \( J = 0 \) solution).

Consider first the massless case \( m = 0 \); equations (4.5) and (4.7) give
\[ W_\pm (r) = \pm \int \frac{E \, dr}{\sqrt{f(r) g(r)}}. \] (4.9)

We therefore recover the expected Hawking temperature (3.7) in the massless case.

In the massive case \( m \neq 0 \), equations (4.5) and (4.7) show
\[ \left( \frac{A}{B} \right)^2 = \frac{E}{\sqrt{f(r)}} + m \]
\[ \frac{E}{\sqrt{g(r)}} - m. \] (4.10)
and
\[ W_\pm(r) = \int \frac{E}{\sqrt{f(r)}g(r)} \frac{2 (\frac{\lambda}{\hbar})}{1 + (\frac{\lambda}{\hbar})^2} dr. \] (4.11)

Noting \( \lim_{r \to r_H} (\frac{\lambda}{\hbar})^2 = 1 \), we find that the result of integrating around the pole for \( W \) in the massive case is the same as the massless case and we recover the Hawking temperature (3.7).

For the spin-down case the calculation is very similar to the spin-up case discussed above. Other than some changes of sign, the equations are of the same form as the spin-up case. For both the massive and massless spin-down cases the Hawking temperature (3.7) is obtained, implying that both spin-up and spin-down particles are emitted at the same temperature.

### 4.2. Dirac particles tunneling in a general coordinate

We take
\[ \psi_\uparrow = \begin{pmatrix} A(v, u, \theta, \phi) \xi_\uparrow \\ B(v, u, \theta, \phi) \xi_\uparrow \end{pmatrix} \exp \left( \frac{i}{\hbar} I_\uparrow(v, u, \theta, \phi) \right) = \begin{pmatrix} A(v, u, \theta, \phi) \\ 0 \\ B(v, u, \theta, \phi) \\ 0 \end{pmatrix} \exp \left( \frac{i}{\hbar} I_\uparrow(v, u, \theta, \phi) \right), \]
\[ \psi_\downarrow = \begin{pmatrix} C(v, u, \theta, \phi) \xi_\downarrow \\ D(v, r, \theta, \phi) \xi_\downarrow \end{pmatrix} \exp \left( \frac{i}{\hbar} I_\downarrow(v, u, \theta, \phi) \right) = \begin{pmatrix} C(v, u, \theta, \phi) \\ 0 \\ D(v, u, \theta, \phi) \\ 0 \end{pmatrix} \exp \left( \frac{i}{\hbar} I_\downarrow(v, u, \theta, \phi) \right) \] (4.12)

where \( I_\uparrow = -Ev + W(u) + J(\theta, \phi) \). For the line element (2.3), we chose the tetrad
\[ e^\mu_{\alpha} = \begin{pmatrix} \frac{\lambda}{\sqrt{f(r(u))}} & \sqrt{g(r(u))}G(r(u)) & 0 & 0 \\ 0 & \sqrt{g(r(u))}F(r(u)) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{R(r(u))}} \\ 0 & 0 & \frac{1}{\sqrt{R(r(u))}} & \frac{1}{\sqrt{r(u) \sin \theta}} \end{pmatrix}. \] (4.13)

Then the Dirac equation (4.1) can be expressed as
\[ \begin{bmatrix} -\frac{\lambda}{\sqrt{f(r(u))}} A - \sqrt{g(r(u))}G(r(u))B \\ 0 \end{bmatrix} E + \sqrt{g(r(u))}F(r(u))BW'(u) + mA = 0 \] (4.14)
\[ \begin{bmatrix} \frac{\lambda}{\sqrt{f(r(u))}} B + \sqrt{g(r(u))}G(r(u))A \\ 0 \end{bmatrix} E - \sqrt{g(r(u))}F(r(u))ABW'(u) + mB = 0. \] (4.15)

For the case \( m = 0 \), we find
\[ W'(u) = \left[ \frac{G(r(u))}{F(r(u))} E \pm \frac{\lambda E}{\sqrt{f(r(u))}g(r(u))F(r(u))} \right], \] (4.16)
which is similar to equation (3.10). Taking the same method used in section 3.2, it is easy to get the Hawking temperature (3.7).

For the case \( m \neq 0 \), we find
\[ \left( \frac{A}{B} \right)^2 = \frac{E}{\frac{E}{\sqrt{f(r(u))}} + m} \] (4.17)
and \( \lim_{u \to 0} \frac{A}{B} = \pm 1 \). We have

\[
W'(u) = \left[ \frac{G(r(u))}{F(r(u))} \pm \frac{2\lambda}{E} \sqrt{f(r(u))g(r(u))F(r(u))} \right],
\]

which is similar to equation (3.10). Taking the same method used in section 3.2, we also find the same Hawking temperature (3.7).

From the above discussions we know that the Hawking temperature of the general static spherically symmetric black hole arising from the Dirac particles tunneling is also invariant in the general coordinate (2.3).

5. Summary

The Hawking temperature of the Schwarzschild black hole in the isotropic coordinate shows us that the temperature is not invariant. What kinds of coordinates can keep the Hawking temperature invariant for the general static spherically symmetric black hole? By studying the Hawking radiation of the most general static spherically symmetric black hole arising from scalar and Dirac particle tunneling, we find that it is invariant in the general coordinate representation (2.3), which satisfies two conditions: (a) its radial coordinate transformation is regular at the event horizon; and (b) there is a time-like Killing vector.

We also find some other interesting results: (1) for the non-singular coordinate representations, such as the general coordinate (2.3) with \( C(r_H) = \lambda \) (include the Painlevé (2.4) and Lemaître (2.5)), \( W_+ \) has a pole at the event horizon but \( W_- \) has a well-defined limit at the horizon. Then the imaginary part of \( W_- \) is zero since the imaginary parts of the action can only come from the pole and the probability of a particle tunneling from inside to outside the horizon is described by \( \Gamma = \Gamma_{\text{out}} \). (2) The mass of the particles and the angular quantum number do not affect the Hawking temperature for both scalar and Dirac particles. (3) When the time coordinate transforms from \( t_s \) to \( \lambda t_s \), i.e., we re-scale the time, the corresponding energy \( E \) of the total tunneling particles is increased by \( \lambda \) times, so a re-scaling of the time does not affect the Hawking temperature.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under grant no. 10675045; the FANEDD under grant no. 200317; the Hunan Provincial Natural Science Foundation of China under grant no. 08JJ3010; and the construct program of the key discipline in Hunan province.

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