CONTROLLABILITY OF A SYSTEM OF DEGENERATE PARABOLIC EQUATIONS WITH NON-DIAGONALIZABLE DIFFUSION MATRIX

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Abstract. In this paper we study the null controllability of some non-diagonalizable degenerate parabolic systems of PDEs, we assume that the diffusion, coupling and controls matrices are constant and we characterize the null controllability by an algebraic condition so called Kalman’s rank condition.

1. Introduction and main result. In this paper we focus on the controllability properties of the following non-diagonalizable parabolic degenerate system

\[
\begin{cases}
\partial_t Y = (D M + A)Y + B v 1_\omega & \text{in } Q, \\
C Y = 0 & \text{on } \Sigma, \\
Y(0, x) = Y_0(x) & \text{in } (0, 1),
\end{cases}
\]

(1)

where \( Q := (0, T) \times (0, 1), \) \( \Sigma := (0, T) \times \{0, 1\}, \) for \( T > 0 \) and \( \omega \subset (0, 1) \) is a (small) nonempty open control region, \( 1_\omega \) denotes the characteristic function of \( \omega. \) The diffusion matrix \( D \) is a non-diagonalizable \( n \times n \) matrix that satisfies the following assumptions:

- there exists \( \alpha_0 > 0 \) such that

\[
D\xi \cdot \xi \geq \alpha_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n
\]

(2)

- there exists a non-singular matrix

\[
P \in \mathcal{L}(\mathbb{C}^n) \text{ such that } D = PJP^{-1}
\]

(3)

for some \( J \in \mathcal{L}(\mathbb{C}^n) \) of the form

\[
J = \text{diag}(J_1, \cdots, J_p),
\]

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where the $J_i$ are the Jordan blocks associated to the eigenvalues $d_i$ of $D$.

$$J_i = \begin{bmatrix} d_i & 1 \\ & \ddots \\ & & 1 \\ & \ddots & \\ & & & d_i \end{bmatrix}$$  \hspace{1cm} (4)

with $\text{Re} \ d_i > 0$.

The coupling matrix $A$ is a $n \times n$ constant matrix and the control matrix $B$ is a $n \times m$ constant matrix. The operator $\mathcal{M}$ is defined by $\mathcal{M}y = (ay_y)_x$ for $y \in D(\mathcal{M}) \subset L^2(0,1)$. For $Y = (y_1, \cdots, y_n)^*$, $\mathcal{M}Y$ denotes $(\mathcal{M}y_1, \cdots, \mathcal{M}y_n)^*$. The function $a$ is a diffusion coefficient which degenerates at 0 (i.e., $a(0) = 0$) and which can be either weak degenerate (WD), i.e.,

$$ \begin{cases} 
(\text{WD}) \\
(i) \ a \in C((0,1]) \cup C^1((0,1]), \ a > 0 \text{ in } (0,1], \ a(0) = 0, \\
(ii) \ \exists K \in [0,1) \text{ such that } xa'(x) \leq Ka(x), \ \forall x \in \mathbb{R},
\end{cases}$$

or strong degenerate (SD), i.e.,

$$ \begin{cases} 
(\text{SD}) \\
(i) \ a \in C^1([0,1]), \ a > 0 \text{ in } (0,1], \ a(0) = 0, \\
(ii) \ \exists K \in [1,2) \text{ such that } xa'(x) \leq Ka(x) \forall x \in [0,1], \\
(iii) \ \exists \theta \in (1,K]x \mapsto \frac{a(x)}{x^{\theta}} \text{ is nondecreasing near 0, if } K > 1, \\
\end{cases}$$

The boundary condition $CY = 0$ is either $Y(0) = Y(1) = 0$ in the weak degenerate case (WD) or $Y(1) = (aY_x)(0) = 0$ in the strongly degenerate case (SD).

It will be said that the system (1) is \textit{null-controllable} at time $T$ if, for any $Y_0 \in (L^2(0,1))^n$, there exists $v \in L^2((0,T) \times \omega)^m$ such that the associated solution satisfies

$$y(T, x) = 0 \text{ in } (0,1).$$

The system (1) is said to be \textit{approximately controllable} at time $T$ if, for any $Y_0, Y_1 \in L^2((0,1))^n$ and $\varepsilon > 0$, there exists $v \in L^2((0,T) \times \omega)^m$ such that the solution of (1) corresponding to the initial condition $Y_0$ satisfies

$$\|y(T, x) - Y_1\|_{(L^2(0,1))^n} \leq \varepsilon.$$  

Controlling coupling systems of partial differential equations attracted growing interest during the last decade, the main question is whether it is possible to control such systems with fewer controls (i.e the number of controls is less than the number of equations). For finite dimensional linear systems, the controllability can be characterized by algebraic rank condition on the matrices generating the dynamics and taking account of the control action. The theory has been adapted and extended to more general systems including infinite dimensional systems. At our knowledge, in the nondegenerate case, M. Gonzalez-Burgos, L. de Teresa [18] provided a null controllability result for a cascade parabolic system by one control force under a condition on the sub-diagonal of the coupling matrix. F. Ammar Khodja et al. [5, 6] obtained several results characterizing the null controllability of fully coupled systems with $m$-control forces by a generalized Kalman rank condition. In [16], the authors gave controllability results for a system in the case where the diffusion matrix is non diagonalizable.
For degenerate systems, the case of two coupled equations \( n = 2 \), cascade systems are considered in \([13, 14]\) and in \([1, 2]\) the authors have studied the null controllability of degenerate noncascade parabolic systems.

In \([15]\), we have extended the null controllability results obtained by Ammar Khodja et al. \([6]\) to a class of parabolic degenerate systems of PDEs in the two following cases

1. the coupling matrix \( A \) is a cascade one and the diffusion matrix \( D = \text{diag}(d_1, \ldots, d_n) \) where \( d_i > 0, \, i = 1, \ldots, n \),
2. the coupling matrix \( A \) is a full matrix (noncascade) and the diffusion matrix \( D = dI_n \) with \( d > 0 \).

On the other hand, in \([3]\) we studied the null controllability of system (1) in the case where the diffusion matrix \( D \) is diagonalizable \( n \times n \) matrix with positive real eigenvalues, i.e.,

\[
D = P^{-1}JP, \quad P \in \mathcal{L}(\mathbb{R}^n), \quad \text{det}(P) \neq 0, \tag{7}
\]

where \( J = \text{diag}(d_1, \ldots, d_n), \, d_i > 0, \, 1 \leq i \leq n \).

In the current paper, we assume that diffusion matrix \( D \) is non-diagonalizable. We use the same approach as \([16]\) without imposing that Jordan’s block sizes are bounded by 4. Thus our proof is also an improvement of the one given in \([16]\) for the nondegenerate case.

Notice that, if the condition (2) is satisfied, for every \( v \in L^2((0, T) \times \omega; \mathbb{R}^m) \) and every \( y_0 \in L^2([0, T]; \mathbb{R}^n) \), the system (1) possesses a unique weak solution \( y \), with \( y \in C([0, T], (L^2(0, 1))^n) \cap L^2(0, T; (H^1_0)^n) \) see Section 2.

In order to study the null controllability of system (1), we will consider the following corresponding adjoint problem

\[
\begin{cases}
-z_t - D^*Mz = A^*z & \text{in } Q, \\
Cz = 0 & \text{on } \Sigma, \\
z(T, x) = z_T(x) & \text{in } (0, 1).
\end{cases} \tag{8}
\]

Since the null controllability of system (1) is equivalent to the existence of a positive constant \( C \) such that, for every \( z_T \in (L^2(0, 1))^n \), the solution \( z \in C^0([0, T]; (L^2(0, 1))^n) \) to the adjoint system (8) satisfies the observability inequality:

\[
\|z(\cdot, 0)\|_{L^2(0, 1)^n}^2 \leq C \int_{[0, T] \times \omega} |B^*z(x, t)|^2, \tag{9}
\]

The strategy used in this case is slightly different from the one used in \([3]\), although in both cases it is necessary to show Carleman estimates for a scalar PDE of order \( 2n \) in space.

Note that this technique failed in the case where \( M \) is a compact operator, since its spectrum admits zero as a point of accumulation, so the Kalman condition is no longer verified, which does not ensure a perfect coupling of the equations.

All along the article, we use generic constants for the estimates, whose values may change from line to line.

Let us remark that when \( A \in \mathcal{L}(\mathbb{R}^n) \) and \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \) are constant matrices, \([A|B] \in \mathcal{L}(\mathbb{R}^{nm}, \mathbb{R}^n)\) is the matrix given by

\[
[A|B] = (A^{n-1}B|A^{n-2}B| \cdots |AB|B)
\]

With this notation, we have the following main result.
Theorem 1.1. Let assume that $D, A \in \mathcal{L}(\mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ such that $D$ satisfies the assumptions (2)-(4). Then system (1) is null controllable at time $T$ if and only if
\[
\text{rank}[\lambda_i D - A|B] = n \quad \forall i \geq 1. \tag{10}
\]

The rest of this paper is organized as follows. In Section 2, we prove the wellposedness of the problem (1). Section 3 is devoted to some controllability results for one parabolic equation. In Section 4, we prove some useful estimates under the assumption (10) and we establish Carleman estimates for a scalar PDE of order $2n$ in space. In Section 5, we give the proof of the main result. And finally, in Section 6 we study the null controllability of semilinear systems.

2. Wellposedness of the problem. The semigroup generated by the operator $(\mathcal{M}, D(\mathcal{M}))$ is analytic with angle $\frac{\pi}{2}$ (see [9, Theorem 2.8]). In order to prove the wellposedness of the problem (1), it suffices to show that the operator $D\mathcal{M}$ generates a $c_0$-semigroup. In fact, similarly like in [22], under the assumption that all eigenvalues of the diffusion matrix $D$ have positive real part, we prove that $D\mathcal{M}$ is the generator of an analytic semigroup. Let us first introduce the following weighted spaces:
\[
H^1_\alpha = \{ u \in L^2(0,1)/u \text{ absolutely continuous in } [0,1], \sqrt{\alpha}u_x \in L^2(0,1) \text{ and } u(1) = u(0) = 0 \}\n\]
and
\[
H^2_\alpha = \{ u \in H^1_\alpha(0,1)/au_x \text{ in } H^1(0,1) \}.
\]
in the (WD) case and
\[
H^1_\alpha = \{ u \in L^2(0,1)/u \text{ absolutely continuous in } [0,1], \sqrt{\alpha}u_x \in L^2(0,1) \text{ and } u(1) = 0 \}
\]
and
\[
H^2_\alpha = \{ u \in H^1_\alpha(0,1)/au_x \text{ in } H^1(0,1) \}
\]
\[
= \{ u \in L^2(0,1)/u \text{ absolutely continuous in } [0,1], au \in H^1_\alpha(0,1), au_x \in H^1(0,1) \text{ and } (au_x)(0) = 0 \},
\]
in the (SD) case with the norms
\[
\| u \|_{H^1_\alpha}^2 = \| u \|_{L^2(0,1)}^2 + \| \sqrt{\alpha}u_x \|_{L^2(0,1)}^2, \quad \| u \|_{H^2_\alpha}^2 = \| u \|_{H^1_\alpha}^2 + \| (au_x)_x \|_{L^2(0,1)}^2.
\]

We recall also some results about the spectrum of the operator $-\mathcal{M}$ already used in [3], indeed, the operator $-\mathcal{M}$ is a definite positive operator. Knowing that $D(\mathcal{M}) = H^2_\alpha(0,1)$ is compactly embedded in $L^2(0,1)$ see [10]. Thus, the spectrum of $-\mathcal{M}$ consists of eigenvalues

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \quad \text{with} \quad \lambda_j \to +\infty \tag{11}
\]

Therefore: There exists a complete orthonormal set $\{w_j\}$ of eigenvectors of $-\mathcal{M}$.

For all $z \in D(-\mathcal{M})$ we have
\[
-\mathcal{M}z = \sum_{j=1}^{+\infty} \lambda_j \langle z, w_j \rangle w_j = \sum_{j=1}^{+\infty} \lambda_j E_j z, \tag{12}
\]
where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0,1)$ and
\[
E_j z = \langle z, w_j \rangle w_j.
\]
So, \( \{E_j\} \) is a family of complete orthogonal projections in \( L^2(0, 1) \) and
\[
z = \sum_{j=1}^{+\infty} E_j z, \quad z \in L^2(0, 1).
\]

\(-\mathcal{M}\) generates a strongly continuous semigroup \( \{e^{-\mathcal{M}t}\} \) given by
\[
e^{-\mathcal{M}t} z = \sum_{j=1}^{+\infty} e^{\lambda_j t} E_j z.
\]

In the Hilbert space \( \mathbb{H} := (L^2(0, 1))^n \) we define the following linear operator:
\[
\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H} \quad \text{given by} \quad D(\mathcal{A}) = (H^2_\alpha)^n \quad \text{and} \quad \mathcal{A}u = -\mathcal{D}\mathcal{M}u.
\]

We have the following result.

**Theorem 2.1.** Assume that all eigenvalues of \( \mathcal{D} \) have positive real part. Then \( \mathcal{A} \) is sectorial and therefore, \( -\mathcal{A} \) is the generator of an analytic semigroup \( \{e^{-\mathcal{A}t} : t \geq 0\} \) in \( \mathbb{H} \).

**Proof.** Let \( \theta \in (0, \frac{\pi}{2}) \) such that \( |\arg \lambda_D| < \theta \) for any eigenvalue \( \lambda_D \) of \( D \). We prove that the sector
\[
S = \{z \in \mathbb{C} : \theta \leq |\arg z| \leq \pi, \quad z \neq 0\}
\]
is in the resolvent set of \( \mathcal{A} \) and there exists a constant \( C \) such that for any \( z \in S \)
\[
\| (z - \mathcal{A})^{-1} \| \leq \frac{C}{|z|}
\]
For \( z \in S \) and \( f \in \mathbb{H} \), let \( u \) be given by
\[
u = \sum_{j=1}^{\infty} (z - \lambda_j D)^{-1} f_j w_j
\]
where \( f_j = \int_{(0, 1)} f w_j dx \). Since \( z \in S \) implies \( \frac{z}{\lambda_j} \) is not an eigenvalue of \( D \), the matrix \( z - \lambda_j D \) is invertible and there exists a constant \( C > 0 \) such that \( \|(z - \lambda_j D)^{-1}\| \leq \frac{C}{|z|} \) for all \( j \geq 1 \). It follows that the above series is convergent in \( \mathbb{H} \), so \( u \) is well defined and \( \|u\| \leq \frac{C}{|z|} \|f\| \). Also,
\[
z u + D\mathcal{M}u = \sum_{j=1}^{\infty} \left[ (z - \lambda_j D)^{-1} f_j - \lambda_j D f_j \right] w_j
\]
\[
= \sum_{j=1}^{\infty} (z - \lambda_j D)^{-1} [z - \lambda_j D] f_j w_j
\]
\[
= \sum_{j=1}^{\infty} f_j w_j = f,
\]
so, \( u = (z - \mathcal{A})^{-1} f \). Therefore, \( z \) is in the resolvent set of \( \mathcal{A} \),
\[
\| (z - \mathcal{A})^{-1} \| \leq \frac{C}{|z|}
\]
and the proof is complete. \( \square \)

Thus, system (1) is well posed in the sense of semigroup theory and the following global existence result holds.
Theorem 2.2. Under the Hypothesis (3)-(4), for all \((y_1^0, \cdots, y_n^0) \in (L^2(0, 1))^n\) and \(v \in (L^2(Q))^m\) there exists a unique mild solution \((y_1(t), \cdots, y_n(t))\) of system (1) which belongs to
\[
X_T := C \left([0, T], (L^2(0, 1))^n\right) \cap L^2 \left(0, T; (H^1_\delta)^n\right)
\]
and satisfies
\[
\sup_{t \in [0, T]} \|(y_1, \cdots, y_n)(t)\|_{(L^2(0, 1))^n}^2 + \int_0^T \|\sqrt{a}y_{1x}, \cdots, \sqrt{a}y_{nx}\|_{L_2}^2 dt \\
\leq C_T \left(\|(y_1^0, \cdots, y_n^0)\|_{(L^2(0, 1))^n}^2 + \|v\|_{(L^2(Q))^m}^2\right),
\]
for a constant \(C_T > 0\).

Moreover, if \((y_1^0, \cdots, y_n^0) \in (H^1_\delta)^n\), then
\[
(y_1, \cdots, y_n) \in C \left([0, T], (H^1_\delta)^n\right) \cap H^1 \left(0, T; (L^2(0, 1))^n\right) \cap L^2 \left(0, T; (H^2_\delta)^n\right)
\]
and
\[
\sup_{t \in [0, T]} \|(y_1, \cdots, y_n)(t)\|_{(H^1_\delta)^n}^2 + \int_0^T \|(y_{1t}, \cdots, y_{nt})\|_{L_2}^2 + \|(ay_{1x})_x, \cdots, (ay_{nx})_x\|_{L_2}^2 dt \\
\leq C_T \left(\|(y_1^0, \cdots, y_n^0)\|_{(H^1_\delta)^n}^2 + \|v\|_{(L^2(Q))^m}^2\right)
\]
for a constant \(C_T > 0\).

3. Carleman estimates for one equation. In order to establish a Carleman estimate for the adjoint system (8), we are led to see Carleman estimates already established in the case of one single parabolic degenerate equation of order 2 in space \([3, 15]\).

\[
\begin{cases}
\begin{aligned}
  u_t - (a(x)u_x)_x + bu &= f, &\text{in } Q, \\
  Cu &= 0 &\text{on } \Sigma, \\
  u(T, x) &= u_0(x) &\text{in } (0, 1).
\end{aligned}
\end{cases}
\]

For this purpose, let us consider the following time and space weight functions
\[
\left\{
\begin{aligned}
  \theta(t) &= \frac{1}{t^4(T-t)^2}, \\
  \psi(x) &= \lambda \left(\int_0^x \frac{y}{a(y)} dy - c\right) \quad \text{and} \quad \varphi(t, x) = \theta(t)\psi(x), \\
  \Phi(t, x) &= \psi(x) \quad \text{and} \quad \Psi(x) = e^{\rho\sigma(x)} - e^{2\rho\nu\sigma(x)}.
\end{aligned}
\right.
\]

where \(\sigma\) is a \(C^2([0, 1])\) function such that \(\sigma(x) > 0\) in \((0, 1)\), \(\sigma(0) = \sigma(1) = 0\) and \(\sigma_x(x) \neq 0\) in \(0, 1\) \(\omega_0\), \(\omega_0\) is an open subset of \(\omega\), and the parameters \(c, \rho\) and \(\lambda\) are chosen as in \([15]\) such that
\[
c > 4^n c_0, \quad \rho > \frac{\ln \left(\frac{4^n(c - c_0)}{c - 4^n c_0}\right)}{\|\sigma\|_\infty},
\]
\[
e^{2\rho\nu\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty} < \lambda < \frac{4^n}{(4^n - 1)c} \left(e^{2\rho\nu\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}\right),
\]
where 
\[
c_0 = \int_0^1 \frac{x}{a(x)} dx.
\]
We recall first the following useful properties of the weighted functions for which the proofs are given in \([15]\).
Lemma 3.1. Under assumptions (16)-(17), we have
\[ \frac{4^n}{4^n - 1} \Phi < \varphi \leq \Phi, \] (18)
and
\[ \frac{4}{3} \Phi < \frac{4^2}{4^2 - 1} \Phi < \cdots < \frac{4^{(n-1)}}{4^{(n-1)} - 1} \Phi < \frac{4^n}{4^n - 1} \Phi. \] (19)

Let \( \omega' \) a subset of \( \omega \) and set \( \omega'' := \{ x_1', x_2'' \} \subset \subset \omega' \) and \( \xi \in C^\infty((0,1]) \) such that \( 0 \leq \xi(x) \leq 1 \) for \( x \in (0,1) \), \( \xi(x) = 1 \) for \( x \in (0,x_1') \) and \( \xi(x) = 0 \) for \( x \in (x_2'', 1) \).

The two following results have been proved in [3].

Proposition 1. Let \( T > 0 \) and \( \tau \in \mathbb{R} \). Then there exist two positive constants \( C \) and \( s_0 \) such that, for all \( u_0 \in L^2(0,1) \), the solution \( u \) of equation (14) satisfies
\[ \int_Q \left( (s\theta)^{\tau-1} \xi^2 u_t^2 + (s\theta)^{\tau-1} \xi^2 (Mu)^2 + (s\theta)^{1+\tau} a \xi^2 u_t^2 + (s\theta)^{3+\tau} \frac{x^2}{a} \xi^2 u^2 \right) e^{2\varphi(t,x)} dx dt \]
\[ \leq C \left( \int_Q \xi^2 (s\theta)^{\tau} f^2(t,x) e^{2\varphi(t,x)} dx dt + \int_{Q''} (s\theta)^{3+\tau} u_2^2 e^{2\varphi(t,x)} dx dt \right) \] (20)
for all \( s \geq s_0 \), with \( Q_{\omega'} = (0, T) \times \omega' \).

Proposition 1 gives a Carleman estimate in \((0, x_1')\). The following proposition is a non-degenerate Carleman estimate to the equation (14) on the interval \((x_1', 1)\).

Proposition 2. Let \( T > 0 \) and \( \tau \in \mathbb{R} \). Then, there exist two positive constants \( C \) and \( s_0 \) such that for every \( u_0 \in L^2(0,1) \), the solution \( u \) of equation (14) satisfies
\[ \int_Q \left( (s\theta)^{\tau-1} \xi^2 u_t^2 + (s\theta)^{\tau-1} \xi^2 (Mu)^2 + (s\theta)^{1+\tau} a \xi^2 u_t^2 + (s\theta)^{3+\tau} \frac{x^2}{a} \xi^2 u^2 \right) e^{2\varphi(t,x)} dx dt \]
\[ \leq C \left( \int_Q \xi^2 (s\theta)^{\tau} f^2(t,x) e^{2\varphi(t,x)} dx dt + \int_{Q''} (s\theta)^{3+\tau} u_2^2 e^{2\varphi(t,x)} dx dt \right) \] (21)
for all \( s \geq s_0 \), with \( \varsigma = 1 - \xi \) and \( Q_{\omega'} = (0, T) \times \omega' \).

4. Some useful results. Since the number of control forces is less than the number of equations, we need to highlight the equation coupling tools. Indeed, the equations are coupled by means of this algebraic condition \( \text{rank} \{ \lambda_i D - A[B] \} = n \forall i \geq 1 \). Let us introduce the following operators \( K : D(K) \subset L^2([0,1]; \mathbb{R}^n) \to L^2([0,1]; \mathbb{R}^n) \) and \( K_r : D(K_r) \subset L^2([0,1]; \mathbb{R}^n) \to L^2([0,1]; \mathbb{R}^nm) \) with
\[ D(K) = \{ v \in L^2([0,1]; \mathbb{R}^nm) : [-D]\mathcal{M} - A[B]v \in L^2([0,1]; \mathbb{R}^n) \} \]
\[ D(K_r) = \{ \varphi \in L^2([0,1]; \mathbb{R}^n) : [-D]\mathcal{M} - A[B]^{\tau}\varphi \in L^2([0,1]; \mathbb{R}^nm) \} \]
and
\[ K_r := [-D]\mathcal{M} - A[B]v, \quad K_r \varphi := [-D]\mathcal{M} - A[B]^{\tau}\varphi \] (22)
\( K \) and \( K_r \) are densely defined unbounded operators. We have the following estimate

Proposition 3. Let us assume that \( \text{rank} \{ \lambda_i D - A[B] \} = n \forall i \geq 1 \), then for every \( \varphi \) such that \( (-\mathcal{M})^k(K_r\varphi) \) in \( L^2(0,1) \) we have
\[ \int_0^1 |\varphi(x,t)|^2 dx \leq R \int_0^1 |(-\mathcal{M})^k(K_r\varphi)(x,t)|^2 dx \] (23)
for any \( t \in [0, T] \) and any \( k \geq (n-1)^2 \), where \( R \) depends only on \( n, D \) and \( A \).
Proof. We adapt the same argument as in [16] to our degenerate case. Let denote by $K_i$ the matrices $K_i = [\lambda_iD - A]B \in \mathcal{L}(\mathbb{R}^{nm}; \mathbb{R}^n)$ for $i \geq 1$. Let $f \in L^2([0, 1]; \mathbb{R}^n)$ be given

$$f = \sum_{i=1}^{p} f_i w_i$$

where $f_i \in \mathbb{R}^n$ for some $p \geq 1$, then

$$K_* f = \sum_{i=1}^{p} (K_i^* f_i) w_i, \quad (-\mathcal{M})^k K_* f = \sum_{i=1}^{p} \lambda_i^k (K_i^* f_i) w_i,$$

hence

$$\|(-\mathcal{M})^k K_* f\|_{L^2}^2 = \sum_{i=1}^{p} \lambda_i^{2k} |K_i^* f_i|^2.$$

Let us denote by $\eta_i^j$, for $1 \leq j \leq n$, the real and nonnegative eigenvalues of $K_i^* K_i^*$. Then we have

$$|K_i^* f_i|^2 = (K_i^* K_i^* f_i, f_i) \geq \eta_i^1 |f_i|^2.$$  

There exists $c_1$ such that

$$\det K_i K_i^* \geq c_1 \quad \forall i \geq 1.$$ 

Indeed, let us set $p(\lambda) = \det \tilde{K}(\lambda)\tilde{K}(\lambda)^*$ for all $\lambda$, with

$$\tilde{K}(\lambda) := [\lambda D - A]B.$$ 

Thus, $p(\lambda)$ is a polynomial function of degree $2n(n-1)$, $p(\lambda) \geq 0$ for all $\lambda$ and $p(\lambda_i) \neq 0$ for all $i$. Since the roots of $p(\lambda) = 0$ are in a disk of radius $R$ for some $R > 0$, then, there exists $C_2 > 0$ such that $p(\lambda) \geq C_2$ for $|\lambda| \geq R$. Moreover, for some $\ell$, one has $\lambda_i > R$. Hence,

- Either $i \leq \ell - 1$ and then $\det K_i K_i^* \geq C_3 := \min_{\ell \leq i} \det K_j K_j^*$,

- Or $i \geq \ell$ and then $\lambda_i \geq \lambda_\ell > R$ and $\det K_i K_i^* \geq C_2$.

Thus, $\det K_i K_i^* \geq c_1 = \min(C_2, C_3)$.

Furthermore, for each $i \geq 1$ and each $\ell = 1, \cdots, n$ there exists $\tilde{f}_i \in \mathbb{R}^n \setminus \{0\}$ such that

$$\eta_i^\ell = \frac{(K_i^* K_i^* \tilde{f}_i, \tilde{f}_i)}{\|\tilde{f}_i\|^2} \leq \|K_i K_i^*\|_2 \leq C_4(1 + \lambda_i^{2(n-1)}),$$

where $\|\cdot\|_2$ in the usual Euclidean norm in $\mathcal{L}(\mathbb{R}^n)$. Then we infer

$$\eta_i^\ell \geq \left( \frac{\det K_i K_i^*}{\prod_{\ell \geq 2} \eta_i^\ell} \right) \geq \frac{c_1}{C_4^{n-1}(1 + \lambda_i^{2(n-1)})^{n-1}} \geq C_5 \lambda_i^{-2(n-1)^2}.$$ 

Coming back to (26) we get

$$|K_i^* f_i|^2 = (K_i^* K_i^* f_i, f_i) \geq \eta_i^1 |f_i|^2 \geq C_5 \lambda_i^{-2(n-1)^2} |f_i|^2.$$ 

Therefore

$$\|(-\mathcal{M})^k K_* f\|_{L^2}^2 \geq \sum_{i=1}^{p} \lambda_i^{2k} |K_i^* f_i|^2 \geq C_5 \sum_{i=1}^{p} \lambda_i^{2(k-(n-1))^2} |f_i|^2 \geq C\|f\|_{L^2}^2.$$ 

As this is true for all $f$ spanned by a finite amount of the $w_i$, then we infer that this must also hold for all $f \in L^2([0, 1]; \mathbb{R}^n)$ such that $(-\mathcal{M})^k K_* f \in L^2([0, 1]; \mathbb{R}^n)$. In particular, we obtain the estimate (23).
From now on, we consider \( \phi \) with the monomial derivative \( \mathcal{M}^i \partial_t^j \phi \in L^2(0, T; H^2_0(0, 1)) \) for every \( i, j \in \mathbb{N} \), a solution of the following scalar degenerate parabolic equation of order \( 2n \) in space

\[
\begin{cases}
P(\partial_t, \mathcal{M}) \phi = 0 & \text{in } Q, \\
\mathcal{C}M^k \phi = 0, & k \geq 0, \text{ on } \Sigma,
\end{cases}
\]

(28)

where \( P(\partial_t, \mathcal{M}) \) is the operator defined by \( P(\partial_t, \mathcal{M}) = \text{det}(\partial_t I_d + \mathcal{D}^* \mathcal{M} + \mathcal{A}^*) \).

In fact, we prove that all components of every solution of the adjoint system (8) are solutions of the scalar PDE (28). So, it will be necessary to establish Carleman estimate for the scalar PDE (28). We recall first the following result given in [3, 7, 16].

**Proposition 4.** Let \( z_0 \in \mathbb{D}^n \) and let \( z = (z_1, \cdots, z_n)^* \) be the corresponding solution of problem (8). Then, \( z \in C^k([0, T]; D(\mathcal{M}^p)^n) \) for every \( k, p \geq 0 \), and for every \( i, z_i \) solves equation (28).

The following proposition is the crucial result in this paper, since it generalizes [16, Lemma 4.1] to the case where the Jordan block size exceeds 4.

**Proposition 5.** For any \( k \geq 0 \) and \( j \geq 0 \), we can find an integer \( m(k, j) \geq 0 \), a constant \( C(k, j) > 0 \) and an open set \( \omega(k, j) \) satisfying \( \omega \in \omega(k, j) \subseteq \omega_1 \), such that

\[
I(\tau, (-\mathcal{M})^k \partial_t^j \phi) \leq C(k, j) \int_{(0, T) \times \omega(k, j)} (s\theta)^{m(k, j)} |\phi|^2 e^{2s\Phi} dx dt
\]

(29)

where \( \phi \) satisfies (28) and

\[
I(\tau, z) = \int_Q \left( (s\theta)^{-1} z_t^2 + (s\theta)^{-1} (\mathcal{M}z)^2 + (s\theta)^{-r} a(x) z_x^2 + (s\theta)^{r+3} \frac{x^2}{a(x)} z^2 \right) e^{2s\Phi} dx dt.
\]

**Proof.** We will prove (29) by induction on \( k \) and \( j \) in two steps.

**Step 1.** Proof of (29) for \( k = j = 0 \)

Let us see that, for \( s \) large enough, one has

\[
I(\tau, \phi) \leq C(0, 0) \int_{(0, T) \times \omega(0, 0)} (s\theta)^{m(0, 0)} |\phi|^2 e^{2s\Phi} dx dt
\]

(30)

for some integer \( m(0, 0) \), a positive constant \( C(0, 0) \) and an open subset \( \omega(0, 0) \) satisfying \( \omega \subseteq \omega(0, 0) \subseteq \omega_1 \).

We assume that \( \mathcal{D} \) satisfies the assumptions (2)-(4), then we have for some \( p \geq 1 \)

\[
I_d \partial_t + \mathcal{D}^* \mathcal{M} + \mathcal{A}^* = \begin{bmatrix}
H_1(\partial_t, \mathcal{M}) & A_{12}^* & \cdots & A_{1p}^* \\
A_{21}^* & H_2(\partial_t, \mathcal{M}) & \cdots & A_{2p}^* \\
\vdots & \vdots & \ddots & \vdots \\
A_{p1}^* & A_{p2}^* & \cdots & H_p(\partial_t, \mathcal{M})
\end{bmatrix}
\]

(31)

where \( H_i(\partial_t, \mathcal{M}) \) is the non-scalar operator \( H_i(\partial_t, \mathcal{M}) := I_d \partial_t + J_i^* \mathcal{M} + A_{ii}^* \), the \( J_i^* \) are Jordan blocks, i.e. each of them is of the form (4) for some \( d_i \in \mathbb{C} \) and the \( A_{ij} \) provide the corresponding block decomposition of \( A \). Thus we can write (28) as follow

\[
\prod_{i=1}^p \det H_i(\partial_t, \mathcal{M}) \phi = F(\phi)
\]

(32)
in the term $F(\phi)$ we find the composition of at most $p - 2$ operators of kind \( \det H_1(\partial_t, M) \) applied to $\phi$. Let us define the functions $\psi_i$ by

$$
\psi_1 = \phi, \quad \psi_2 = \det H_1(\partial_t, M)\psi_1, \ldots, \psi_p = \det H_{p-1}(\partial_t, M)\psi_{p-1}.
$$

Thus, the equation (32) can be written as

$$
\begin{cases}
\det H_p(\partial_t, M)\psi_p = F(\phi), \\
\det H_{p-1}(\partial_t, M)\psi_{p-1} = \psi_p, \\
\ldots \\
\det H_1(\partial_t, M)\psi_1 = \psi_2.
\end{cases}
$$

By hypothesis, we have

$$
C_\phi = C_\psi_2 = \cdots = C_\psi_p = 0 \text{ on } \Sigma.
$$

Let us consider the first PDE of (33), assume that $J_p$ is the Jordan block of dimension $r$ associated to the complex eigenvalue $\alpha$ with $\text{Re}(\alpha) > 0$ and let denote by $\eta_1, \ldots, \eta_r$ the diagonal components of $A_{pp}$. Then this PDE can be rewritten as

$$
\prod_{i=1}^{r} (\partial_t + \alpha M + \eta_i)\psi_p = F(\phi) - G(\psi_p),
$$

where $G(\psi_p)$ is a linear combination of partial derivatives of $\psi_p$.

Again, let us introduce the new variables

$$
\zeta_r = \psi_p, \quad \zeta_{r-1} = (\partial_t + \alpha M + \eta_r)\zeta_r, \ldots, \zeta_1 = (\partial_t + \alpha M + \eta_2)\zeta_2.
$$

Therefore, we can rewrite (34) as a first-order system for the $\zeta_i$:

$$
\begin{cases}
(\partial_t + \alpha M + \eta_1)\zeta_1 = F(\phi) - G(\psi_p), \\
(\partial_t + \alpha M + \eta_2)\zeta_2 = \zeta_1, \\
\ldots \ldots \ldots \\
(\partial_t + \alpha M + \eta_r)\zeta_r = \zeta_{r-1}.
\end{cases}
$$

with

$$
C_\zeta_1 = C_\zeta_2 = \cdots = C_\zeta_r = 0 \text{ on } \Sigma.
$$

Notice that $|G(\psi_p)|^2$ is bounded by a sum of squares of derivatives of $\psi_p$. More precisely, we have $|G(\psi_p)|^2 \leq CI_G(\psi_p)$, with

$$
I_G(\psi_p) := \sum_{l=0}^{r-1} \sum_{j_1, \ldots, j_r = 0}^{r} \sum_{b=0}^{r-(l+1)} |(-M)^b \prod_{i=1}^{l} (\partial_t + \alpha M + \eta_{j_i})\psi_p|^2.
$$

The following $I_{\alpha, \mathbf{A}}(\tau, z)$ term already used in [3] is defined by

$$
I_{\alpha, \mathbf{A}}(\tau, z) = \iint_{Q} \left( (s\theta)^{r-1} a^2 z_t^2 + (s\theta)^{r-1} a^2 (Mz)^2 + (s\theta)^{r+1} a(x) a^2 z^2 + (s\theta)^{r+3} \frac{x^2}{a(x)} a^2 z^2 \right) e^{2\mathbf{A} dx dt}
$$

where $\mathbf{A} \in \{ \varphi, \Phi \}$ and $\alpha \in \{ \zeta, \zeta \}$ used in Proposition 1 and Proposition 2.

Applying Carleman estimates (20) established in Proposition 1 to the first PDE in system (35), we get

$$
I_{\xi, \varphi}(\tau, \zeta_1) \leq C \left( \iint_{Q} \xi^2 (s\theta)^r (|F(\phi)|^2 (t, x) + I_G(\psi_p)) e^{2\mathbf{A} dx dt} + \int_{Q_{\omega'}} (s\theta)^{2+\tau} \xi_1 e^{2\mathbf{A} dx dt} \right).
$$
And for the \( j \text{th} \) PDE in system (35) where \( j = 2, \cdots, r \), we have
\[
I_{\xi, \varphi}(\tau, \zeta_j) \leq C \left( \int_Q \xi^2(s\theta)^r |\zeta_{j-1}|^2 e^{2s\varphi} \, dx \, dt + \int_{Q_{\omega}} (s\theta)^{r+2} |\zeta_j|^2 e^{2s\varphi} \, dx \, dt \right).
\]

The first term in the right hand-side can be bounded as follow
\[
\int_Q \xi^2(s\theta)^r |\zeta_{j-1}|^2 e^{2s\varphi} \, dx \, dt
= \int_Q (\xi(s\theta)^{r/2} \sqrt{\frac{a}{x^2}} |\zeta_{j-1}| |e^{s\varphi}|) (\xi(s\theta)^{r/2} \sqrt{\frac{x^2}{a}} |\zeta_{j-1}| |e^{s\varphi}|) \, dx \, dt
\leq \int_Q \xi^2(s\theta)^{r} \frac{a}{x^2} |\zeta_{j-1}|^2 e^{2s\varphi} \, dx \, dt + \int_Q \xi^2(s\theta)^{r} \frac{x^2}{a} |\zeta_{j-1}|^2 e^{2s\varphi} \, dx \, dt.
\]

Knowing that \( (\xi \zeta_{j-1} e^{s\varphi})_x = \xi_x \zeta_{j-1} e^{s\varphi} + \xi \zeta_{j-1} e^{s\varphi} + \xi_s \zeta_x e^{s\varphi} \) and using Hardy-Poincaré inequality and \( \text{supp} \xi_x \subset Q_{\omega} \) we get
\[
\int_Q \xi^2(s\theta)^{r} \frac{a}{x^2} |\zeta_{j-1}|^2 e^{2s\varphi} \, dx \, dt
\leq C \int_Q (s\theta)^{r} a (\xi \zeta_{j-1} e^{s\varphi})^2 \, dx \, dt
\leq C \int_Q (s\theta)^{r} a (\xi_x \zeta_{j-1} e^{s\varphi} + \xi \zeta_{j-1} e^{s\varphi} + \xi_s \zeta_x e^{s\varphi})^2 \, dx \, dt
\leq C \int_{Q_{\omega}} (s\theta)^{r} \xi^2 \zeta_{j-1}^2 e^{2s\varphi} \, dx \, dt + C \int_Q (s\theta)^{r} \xi^2 a \zeta_{j-1}^2 e^{2s\varphi} \, dx \, dt
+ C \int_Q (s\theta)^{r+2} \xi^2 \frac{x^2}{a} \zeta_{j-1}^2 e^{2s\varphi} \, dx \, dt.
\]

Consequently, an appropriate linear combination of the terms in the left hand sides absorbs the global weighted integrals of \( |\zeta_j|^2 \) for \( j = 2, \cdots, r \).
\[
\sum_{j=1}^r I_{\xi, \varphi}(\tau, \zeta_j) \leq C \left( \sum_{j=1}^r \int_{Q_{\omega}} (s\theta)^{r+2} |\zeta_j|^2 e^{2s\varphi} \, dx \, dt \right)
+ C \left( \int_Q \xi^2(s\theta)^r (|F(\phi)|^2(t, x) + I_G(\psi_p)) e^{2s\varphi} \, dx \, dt \right).
\]  
(39)

Now let us apply the operators \((-M)^k\) to the \( j \text{th} \) PDE in system (35) where \( k = 1, \cdots, r-1 \) and \( j = k + 1, \cdots, r \), we have
\[
(\partial_t + \alpha M + \eta_j)(-M)^k \zeta_j = (-M)^k \zeta_{j-1}.
\]

By induction and using Proposition 1, we prove the following estimates
\[
I_{\xi, \varphi}(\tau, (-M)^k \zeta_j) \leq C \left( \sum_{i=0}^{k} I_{\xi, \varphi}(\tau - 2i, \zeta_{j-i}) \right).
\]
(40)
Terms in the right-hand side can be bounded by \( \sum_{l=0}^{k} I_{\xi,\tau}(\tau, \zeta_{j-l}) \) for \( s \) enough large.

Let us denote by \( \mathcal{J}_{\xi,\phi}(\tau, \zeta) \) the following sum

\[
\mathcal{J}_{\xi,\phi}(\tau, \zeta) = \sum_{k=0}^{r-1} \sum_{j=k+1}^{r} I_{\xi,\phi}(\tau, (-\mathcal{M})^{k} \zeta_{j})
\]

where \( \zeta = (\zeta_{1}, \cdots, \zeta_{r}) \). Then, we can add all these new terms (40) to the left hand side of (39) and obtain

\[
\mathcal{J}_{\xi,\phi}(\tau, \zeta) \leq C \left( \sum_{j=1}^{r} \int_{Q_{\omega}} (s)_{\tau+2} |\zeta|^{2} e^{2s} dxdt \right) + C \left( \int_{Q_{\omega}} \xi^{2} (s)_{\tau} (|F(\phi)|^{2})^{(t, x) + I_{G}(\psi_{p})) e^{2s} dxdt \right)
\]

where \( C \) is a new positive constant, and \( s \) sufficiently large. From now on, we fix \( s \) sufficiently large and we try to replace the local terms in (42) corresponding to \( \zeta_{1}, \cdots, \zeta_{r-1} \) by a term of the form \( (\psi_{p} = \zeta_{r}) \) using the same computation [15, Lemma 3.7], we can show the existence of a constant \( C > 0 \) and an integer \( \ell_{1} \) such that:

\[
\mathcal{J}_{\xi,\phi}(\tau, \zeta) \leq C \int_{Q_{\omega'}} s^{\ell_{1}} t^{1} \psi_{p}^{2} e^{2s} dxdt + C \int_{Q_{\omega'}} \xi^{2} s^{r} \theta^{r} (|F(\phi)|^{2})^{(t, x) + I_{G}(\psi_{p})) e^{2s} dxdt.
\]

Since the operators \( (\partial_{t} + \alpha M + \eta_{j}) \), \( j = 1, \cdots, r \) commute, then the formula (34) can be rewritten equivalently in the form

\[
\prod_{i=1}^{r} (\partial_{t} + \alpha M + \eta_{\sigma(i)}) \psi_{p} = F(\phi) - G(\psi_{p}),
\]

where \( \sigma \) is any permutation in \( P_{r} \). Hence, we can introduce the new variables

\[
\zeta_{1}^{\sigma} = \psi_{p}, \zeta_{r-1}^{\sigma} = (\partial_{t} + \alpha M + \eta_{\sigma(r)}) \zeta_{r}^{\sigma}, \cdots, \zeta_{2}^{\sigma} = (\partial_{t} + \alpha M + \eta_{\sigma(2)}) \zeta_{2}^{\sigma}
\]

and we can also rewrite (34) as a first-order system for the \( \zeta_{i}^{\sigma} \):

\[
\begin{align*}
(\partial_{t} + \alpha M + \eta_{\sigma(1)}) \zeta_{1}^{\sigma} &= F(\phi) - G(\psi_{p}), \\
(\partial_{t} + \alpha M + \eta_{\sigma(2)}) \zeta_{2}^{\sigma} &= \zeta_{1}^{\sigma}, \\
&\quad \cdots \quad \cdots \\
(\partial_{t} + \alpha M + \eta_{\sigma(r)}) \zeta_{r}^{\sigma} &= \zeta_{r-1}^{\sigma}.
\end{align*}
\]

Again, with

\[
C \zeta_{1}^{\sigma} = C \zeta_{2}^{\sigma} = \cdots = C \zeta_{r}^{\sigma} = 0 \quad \text{on} \quad \Sigma.
\]

Similarly, we obtain an estimate like (43)

\[
\mathcal{J}_{\xi,\phi}(\tau, \zeta^{\sigma}) \leq C \int_{Q_{\omega'}} s^{\ell_{1}} t^{1} \psi_{p}^{2} e^{2s} dxdt + C \int_{Q_{\omega'}} \xi^{2} s^{r} \theta^{r} (|F(\phi)|^{2})^{(t, x) + I_{G}(\psi_{p})) e^{2s} dxdt.
\]

In this inequality we have used the notation \( \zeta^{\sigma} = (\zeta_{1}^{\sigma}, \cdots, \zeta_{r}^{\sigma}) \). Now, let us define \( \mathcal{I}_{\xi,\phi}(\tau, \zeta) \) by

\[
\mathcal{I}_{\xi,\phi}(\tau, \psi_{p}) = \sum_{\sigma \in P_{r}} \mathcal{J}_{\xi,\phi}(\tau, \zeta^{\sigma}).
\]

We have

\[
\mathcal{I}_{\xi,\phi}(\tau, \psi_{p}) \leq C \int_{Q_{\omega'}} s^{\ell_{1}} t^{1} \psi_{p}^{2} e^{2s} dxdt + C \int_{Q_{\omega'}} \xi^{2} s^{r} \theta^{r} (|F(\phi)|^{2})^{(t, x) + I_{G}(\psi_{p})) e^{2s} dxdt.
\]
Observe that all the terms in $I_G(\psi_p)$ are also in the left multiplied by weights of the form $s^2 \theta^1 e^{2s\varphi}$. Consequently, for sufficiently large $s$, these terms are absorbed and we find

$$I_{\xi,\varphi}(\tau, \psi_p) \leq C \int_{Q_{\omega'}} s^{t_1} \theta^1 \psi_p^2 e^{2s\varphi} \, dx \, dt + C \int_Q \xi^2 s^2 \theta^1 |F(\phi)|^2 e^{2s\varphi} \, dx \, dt. \quad (45)$$

Let us now consider the second PDE in (33). Arguing in the same way, we deduce the following estimate for all the $\psi$:

$$I_{\xi,\varphi}(\tau, \psi_{p-1}) \leq C \int_{Q_{\omega'}} s^{t_2} \theta^2 |\psi_{p-1}|^2 e^{2s\varphi} \, dx \, dt + C \int_Q \xi^2 s^2 \theta^2 |\psi_{p-1}|^2 e^{2s\varphi} \, dx \, dt. \quad (46)$$

The corresponding similar estimate also holds for $\psi_{p-2}, \ldots, \psi_1$. Thus, adding the previous terms and taking into account that $\psi_1 = \phi$ and the global integrals of $\psi_p, \ldots, \psi_2$ in the right hand side are smaller than the terms in the left, we get an estimate for all the $\psi_1$:

$$\sum_{i=1}^p I_{\xi,\varphi}(\tau, \psi_i) \leq C \left( \int_Q \xi^2 s^2 \theta^1 |F(\phi)|^2 e^{2s\varphi} \, dx \, dt \right. \left. + \int_{Q_{\omega'}} (s\theta)^{t_2} |\phi|^2 e^{2s\varphi} \, dx \, dt + \sum_{i=2}^p \int_{Q_{\omega'}} (s\theta)^{t_2} |\psi_i|^2 e^{2s\varphi} \, dx \, dt \right). \quad (47)$$

Again, using the cascade structure of system (33), all the local integrals in the right can be absorbed by the left hand side, with the exception of the local weighted integral of $|\phi|^2$. All we have to do is to enlarge the open set $\omega'$ and argue like in the passage from (42) to (43). Therefore, the following is obtained:

$$\sum_{i=1}^p I_{\xi,\varphi}(\tau, \psi_i) \leq C \left( \int_Q \xi^2 s^2 \theta^1 |F(\phi)|^2 e^{2s\varphi} \, dx \, dt \right. \left. + \int_{Q_{\omega'}} (s\theta)^{t_2} |\phi|^2 e^{2s\varphi} \, dx \, dt \right). \quad (48)$$

Taking into account that the operators $\det H_i(\partial_1, M)$, $i = 1, \ldots, p$, commute we can rewrite (32) in the form

$$\prod_{i=1}^p \det H_{\sigma(i)}(\partial_1, M) \phi = F(\phi)$$

where $\sigma$ is any permutation in $P_n$. This means that another equivalent formulation of (28) is

$$\begin{align*}
\det H_{\sigma(p)}(\partial_1, M) \psi_p^\sigma &= F(\phi), \\
\det H_{\sigma(p-1)}(\partial_1, M) \psi_{p-1}^\sigma &= \psi_p^\sigma, \\
& \quad \ldots \\
\det H_{\sigma(1)}(\partial_1, M) \psi_1^\sigma &= \psi_2^\sigma
\end{align*}$$

and we obtain an estimate like (48) where the terms in the left hand sides are a global weighted integrals of $\phi$, $\psi_2^\sigma$, $\psi_3^\sigma$, $\psi_p^\sigma$ and in the right hand sides we have terms concerning $F(\phi)$ and $\phi$. Recall that $F(\phi)$ is a sum of terms where, at most, $p-2$ operators of the kind $\det H_j(\partial_1, M)$ are applied to $\phi$. Since $\sigma$ is arbitrary in $P_n$, using all these possible estimates together and arguing as above, it becomes also clear that the terms containing $|F(\phi)|^2$ can be absorbed by the terms in the left. This gives

$$\sum_{\sigma \in P_n} \sum_{i=1}^p I_{\xi,\varphi}(\tau, \psi_i^\sigma) \leq C \int_{Q_{\omega'}} (s\theta)^{t_2} |\phi|^2 e^{2s\varphi} \, dx \, dt. \quad (49)$$
Likewise, by applying Proposition 2 we infer
\[
\sum_{\sigma \in \mathcal{P}_n} \sum_{i=1}^{\mu} I_{\xi, \phi}(\tau, \psi^n_i) \leq C \int_{Q_{\omega'}} (s\theta)^{m(k,j)}|\phi|^2 e^{2s\Phi} \, dx dt.
\] (50)
From (49) and (51) we deduce
\[
\sum_{\sigma \in \mathcal{P}_n} \sum_{i=1}^{\mu} I(\tau, \psi^n_i) \leq C \int_{Q_{\tilde{\omega}}} (s\theta)^{\tilde{\ell}}|\phi|^2 e^{2s\Phi} \, dx dt.
\] (51)
where \( \tilde{\ell} = \max(\ell_2, \ell_3) \) and \( I(\tau, \psi^n_i) = I_{\xi, \phi}(\tau, \psi^n_i) + I_{\xi, \phi}(\tau, \psi^n_i) \). This proves (30).

**Step 2.** Induction on \( k \) and \( j \).
Assume that (29) is true for any \( k' = 0, 1, \ldots, k \) and \( j' = 0, 1, \ldots, j \) and any \( \phi \) solution to (28) satisfying the assumptions of the Proposition 5. Let us prove (29) (for instance) with \( k \) replaced by \( k + 1 \); the proof with the same \( k \) and \( j \) replaced by \( j + 1 \) is essentially the same. Since \( \tilde{\phi} := (-M)\phi \) also satisfies (28), we have by hypothesis
\[
I(\tau, (-M)^{k+1} \partial_t^j \tilde{\phi}) = I(\tau, (-M)^{k+1} \partial_t^j \tilde{\phi})
\]
\[
\leq C(k, j) \int_{(0, T) \times \omega(k,j)} (s\theta)^{m(k,j)}|\tilde{\phi}|^2 e^{2s\Phi} \, dx dt
\]
\[
\leq C(k, j) \int_{(0, T) \times \omega(k,j)} (s\theta)^{m(k,j)}|M\phi|^2 e^{2s\Phi} \, dx dt
\]
\[
\leq C(k, j) C' \int_{(0, T) \times \omega(k,j)} (s\theta)^{m(k,j)}|M\phi|^2 e^{2s\Phi} \, dx dt
\]
\[
\leq C(k, j) C' I(m(k, j) + 1, \phi).
\]
Applying the formula (30), there exist a positive constant \( C'' \) an integer \( m' \) and an open subset \( \mathcal{O}' \) satisfying \( \omega \subset \mathcal{O}' \subset \omega_1 \) such that
\[
I(m(k, j) + 1, \phi) \leq C'' \int_{(0, T) \times \mathcal{O}'} (s\theta)^{m'}|\phi|^2 e^{2s\Phi} \, dx dt.
\]
Thus, we have
\[
I(\tau, (-M)^{k+1} \partial_t^j \phi) \leq C(k + 1, j) \int_{(0, T) \times \omega(k+1,j)} (s\theta)^{m(k+1,j)}|\phi|^2 e^{2s\Phi} \, dx dt
\]
with \( m(k+1,j) = m' \), \( C(k+1, j) = C(k, j) C' C'' \) and \( \omega(k+1, j) = \mathcal{O}' \). This ends the proof.

5. **Proof of the main result.**

**Proof of Theorem 1.1.** Let us first assume that the system (1) is null-controllable. If we have \( \text{rank}\{\lambda_i, D + A|B\} \leq n - 1 \) for some \( i \), then the associated ordinary differential system is not null-controllable. This means there exists \( y_T \in \mathbb{R}^n \setminus \{0\} \) such that the solution to the Cauchy problem
\[
\begin{cases}
-\partial_t y + (\lambda_i D^* + A^*)y = 0 & \text{in } Q, \\
y(T) = y_T & \text{on } (0, T),
\end{cases}
\] (52)
satisfies
\[
B^* y(t) = 0.
\]

end.
Taking \( z_T = y_T w_i \) in the adjoint system (8), where \( w_i \) is an eigenfunction associated to \( \lambda_i \), we can see as in [3] that the corresponding solution cannot satisfy the observability inequality (9). Consequently, the condition (10) must hold.

Conversely, let us assume that (10) is satisfied, and let us prove that the system (1) is null-controllable.

Let \( z \) be the solution to the adjoint system (8) corresponding to a final data \( z_T \), by Proposition 3, for \( k \geq (n - 1)^2 \) there exists a positive constant \( C \) depends on \( n, D \) and \( A \) such that

\[
\int_0^1 |z(x,t)|^2 \, dx \leq C \int_0^1 |(-\mathcal{M})^k(K_* z)(x,t)|^2 \, dx
\]

for any \( t \in [0, T] \). From (22) the components of \( K_* z \) are appropriate linear combinations of the components of \( z \) and their second-order in space derivatives. Notice again that, for all \( t \in [0, T] \), \( z(\cdot, t) \) is regular enough to give a sense to \((-\mathcal{M})^k(K_* z)\), which belongs to \( L^2(0, 1) \).

By Proposition 4 \( \mathcal{Z} \in \mathcal{C}^k([0, T]; D(M^p)^n) \) for every \( k, p \geq 0 \) and for every \( i \) the component \( z_i \) of \( z \) solves equation (28). Thus, we can write (29) for any component of \( B^* z \). This gives the following inequality for all \( j, k \geq 0 \) and all \( \ell = 1, \cdots, m \)

\[
\int_Q |(B^* ((-\mathcal{M})^k \partial_\ell z))_i|^2 e^{2\beta_\ell} dx \, dt \leq C(k, j) \int_{(0, T) \times \omega(k, j)} |(B^* z)_i|^2 e^{2\beta_\Phi} dx \, dt.
\]

Let \( M_0 = \max_{x \in (0, 1)} |\psi(x)| \), thus

\[
\int_Q |(-\mathcal{M})^k K_* z|^2 e^{-2\alpha M_0} dx \, dt \leq C \sum_{\ell=1}^m \int_Q |(B^* ((-\mathcal{M})^k \partial_\ell z))_i|^2 e^{2\beta_\ell} dx \, dt
\]

\[
\leq C \sum_{\ell=1}^m C(k, j) \int_{(0, T) \times \omega} |(B^* z)_i|^2 e^{2\beta_\Phi} dx \, dt
\]

\[
\leq C \int_{(0, T) \times \omega} |B^* z|^2 e^{2\beta_\Phi} dx \, dt. \tag{54}
\]

From the estimates (53) and (54), we will easily deduce the observability inequality (9) and, therefore, the null controllability of system (1).

\[ \square \]

6. **Null controllability for semilinear systems.** Now we consider the following semi-linear non-diagonalizable parabolic degenerate systems.

\[
\begin{cases}
\partial_t Y = D M Y + F(Y) + B v 1_{\omega} & \text{in } Q, \\
C Y = 0 & \text{on } \Sigma, \\
Y(0, x) = Y_0(x) & \text{in } (0, 1),
\end{cases} \tag{55}
\]

where \( F \in \mathcal{C}^1(\mathbb{R}^n) \) is a globally Lipschitz function depending only on \( Y \) and \( F(0) = 0 \). Our goal is to study the null controllability of the system (55). To this end, we adopt a standard strategy, as in [24, 8, 10, 1], which consists in using the linearization technique, the approximate null controllability, the variational approach and the Schauder fixed point theorem. The system (55) can be written as follow

\[
\begin{cases}
\partial_t Y = (D M + A_Y) Y + B v 1_{\omega} & \text{in } Q, \\
C Y = 0 & \text{on } \Sigma, \\
Y(0, x) = Y_0(x) & \text{in } (0, 1),
\end{cases}
\]
where \( A_Y \) is the matrix defined by 
\[
\alpha_{i,j}^Y = \int_0^1 \partial_i F_t(\tau Y) d\tau.
\]

In the sequel, we assume
\[
\text{rank}[\lambda_i, D - A_Y] = n \ \forall i \geq 1 \text{ for all } Y \in (L^2(0,1))^n. \tag{56}
\]

Let us recall the set \( X_T \) (see Theorem 2.2) induced with the norm
\[
\|Y\|_{\tilde{X}_T}^2 = \sup_{t \in [0,T]} \|Y(t)\|_{L^2(0,1)}^2 + \int_0^T \|\sqrt{a}Y(t)\|_{(L^2(0,1))^n}^2 dt.
\]

For a fixed \( \tilde{Y} \) in \( X_T \), consider the associated linear system
\[
\begin{aligned}
\frac{\partial}{\partial t} Y &= (D \mathcal{M} + A_Y^\varepsilon)Y + Bv1_\omega \quad \text{in } Q, \\
CY &= 0 \quad \text{on } \Sigma, \\
Y(0, x) &= Y_0(x) \quad \text{in } (0, 1),
\end{aligned}
\tag{57}
\]

and its adjoint system
\[
\begin{aligned}
\frac{\partial}{\partial t} Z &= (D^* \mathcal{M} + A_Y^\varepsilon)Z \quad \text{in } Q, \\
CZ &= 0 \quad \text{on } \Sigma, \\
Z(0, x) &= Z_0(x) \quad \text{in } (0, 1). 
\end{aligned}
\tag{58}
\]

Thus, from the assumption (56), it follows that the pair of matrices \((A_Y^\varepsilon, B)\) satisfies the algebraic condition (10), which means that the linearized system (57) is null controllable. In order to construct a suitable fixed point operator, we start at first by proving the uniqueness of the control with minimal norm. For a given \( \varepsilon > 0 \) and \( Y_0 \in (L^2(0,1))^n \) we consider the following functional
\[
J_{\varepsilon, \tilde{Y}}(v) = \frac{1}{2} \int_0^T \|v\|_{(L^2(0,1))^n}^2 dt + \frac{1}{2\varepsilon} \|Y(T)\|_{(L^2(0,1))^n}^2, 
\]
\[
J^*_{\varepsilon, \tilde{Y}}(Z_0) = \frac{1}{2} \int_{Q_\omega} |B^* Z|^2 + \frac{\varepsilon}{2} \|Z_0\|_{(L^2(0,1))^n}^2 + \int_0^1 \left( \sum_{i=1}^n Z_i(T)Y_0 \right) dx 
\tag{59}
\]
where \( Y \) is the solution of system (57) with initial data \( Y_0 \) and \( Z \) is the solution of (58) with initial data \( Z_0 \). By a classical arguments, minimization problems
\[
\min \{ J_{\varepsilon, \tilde{Y}}(v), Bv \in L^2(Q)^n \} \quad \text{and} \quad \min \{ J^*_{\varepsilon, \tilde{Y}}(Z_0), Z_0 \in (L^2(0,1))^n \}
\]

admit unique solutions \( v_{\varepsilon, \tilde{Y}} \) and \( Z^*_{\varepsilon, \tilde{Y}} \) such that
\[
v = B^* Z 1_\omega, \tag{60}
\]
\[
Z^*_{\varepsilon, \tilde{Y}} = -\frac{1}{\varepsilon} Y^{\varepsilon, \tilde{Y}}(T) \tag{61}
\]
where \( Y^{\varepsilon, \tilde{Y}} \) is the solution of (57) associated to the control \( v \) and \( Z^{\varepsilon, \tilde{Y}} \) is the solution of the adjoint problem (58) with the initial data \( Z^*_{\varepsilon, \tilde{Y}} \). Since \( J^*_{\varepsilon, \tilde{Y}}(Z^*_{\varepsilon, \tilde{Y}}) \leq 0 \), then from (59) and (61) we infer
\[
\frac{1}{2} \int_{Q_\omega} |B^* Z|^2 + \frac{\varepsilon}{2} \|Y^{\varepsilon, \tilde{Y}}(T)\|_{(L^2(0,1))^n}^2 \leq \|Z(T, \cdot)\|_{(L^2(0,1))^n} \|Y(0, \cdot)\|_{(L^2(0,1))^n}.
\tag{62}
\]

On the other hand, by the observability inequality (9)
\[
\|Z(T, \cdot)\|_{(L^2(0,1))^n}^2 \leq C \int_{Q_\omega} |B^* Z(t, x)|^2. \tag{63}
\]
From (60) and the estimates (62) – (63) we infer
\[ \frac{1}{2} \int_0^T \| v^\varepsilon, \tilde{Y} \|^2_{(L^2(0,1))^m} dt + \frac{1}{2\varepsilon} \| Y^\varepsilon, \tilde{Y} (T) \|^2_{(L^2(0,1))^n} \leq C \| Y(0, \cdot) \|^2_{(L^2(0,1))^n}. \]  
(64)

The uniqueness of the control $v^\varepsilon, \tilde{Y}$ allows to define the operator
\[ K_\varepsilon : X_T \rightarrow X_T \]
\[ \tilde{Y} \mapsto Y^\varepsilon, \tilde{Y}. \]

Any fixed point $Y^\varepsilon$ of $K_\varepsilon$ is a solution of the semilinear system (55) associated to $v^\varepsilon, \tilde{Y}$ and it satisfies
\[ \| Y^\varepsilon \|^2_{(L^2(0,1))^n} \leq \varepsilon C. \]

Indeed, suppose first that $Y_0 \in (H^1_0)^n$. From (13) and since the matrix $A$ is constant, we have the following estimate
\[ \sup_{t \in [0,T]} \| Y^\varepsilon, \tilde{Y} (t) \|^2_{(H^1_0)^n} + \int_0^T \left( |\partial_t Y^\varepsilon, \tilde{Y} |^2_{L^2} + |\mathcal{M} Y^\varepsilon, \tilde{Y} |^2_{L^2} \right) dt \]
\[ \leq C T \left( \| Y_0^\varepsilon, \tilde{Y} \|^2_{(H^1_0)^n} + \| v^\varepsilon, \tilde{Y} \|^2_{(L^2(Q))^n} \right). \]
(65)

Thus, by the estimate (64) we infer
\[ \| Y^\varepsilon, \tilde{Y} \|_{X_T} \leq C \| Y_0 \|_{(H^1_0)^n}, \]
(66)
\[ \| Y^\varepsilon, \tilde{Y} \|_{Y_T} \leq C \| Y_0 \|_{(H^1_0)^n}, \]
(67)

where $Y_T := H^1 (0, T; (L^2(0,1))^n) \cap L^2 (0, T; (H^1_0)^n)$ with the norm
\[ \| Y \|_{Y_T} = \int_0^T \left( \| Y(t) \|^2_{H^1_0} + |\partial_t Y |^2_{L^2} + |\mathcal{M} Y |^2_{L^2} \right) dt. \]

Thus, the range of $K_\varepsilon$ is include in the ball $B(0, R)$ of $X_T$ with the radius $R = C \| Y_0 \|_{(H^1_0)^n}$ where $C$ is the constant used in (66). Then $K_\varepsilon(B(0, R)) \subset B(0, R)$.

Now let us prove that the operator $K_\varepsilon$ is continuous and compact. The compactness of $K_\varepsilon$ results from the compactness of the embedding
\[ Y_T \hookrightarrow X_T \]
(68)

see [10, Theorem 4.4]. For the continuity, let us consider the sequence $\tilde{Y}_n$ that converge to $\tilde{Y}$ in $X_T$. To simplify, let denote $Y^\varepsilon, \tilde{Y}_n$ and $v^\varepsilon, \tilde{Y}_n$ respectively by $Y_n$ and $v_n$ (for a fixed $\varepsilon$). From (67) we deduce that the sequence $Y_n$ is bounded in the space $Y_T$. Thus we can extract a subsequence that converges weakly in $Y_T$ to $\tilde{Y}$ and strongly in $X_T$ by dint of (68). Likewise, thanks to (64) we can assume that $v_n$ converges weakly to $v$. So $\tilde{Y}$ is then the solution of (57) associated to $\tilde{Y}$ and $\tilde{v}$. Therefore, in order to show that $K_\varepsilon(\tilde{Y}) = \tilde{Y}$ it suffices to prove that $\tilde{v} = v^\varepsilon, \tilde{Y}$.

From the definition of $v_n$, we have for all $v$ in $L^2(Q)^m$
\[ \frac{1}{2} \int_0^T \| v_n \|^2_{(L^2(0,1))^m} dt + \frac{1}{2\varepsilon} \| Y_n(T) \|^2_{(L^2(0,1))^n} \]
\[ \leq \frac{1}{2} \int_0^T \| v \|^2_{(L^2(0,1))^m} dt + \frac{1}{2\varepsilon} \| Y_n, v(T) \|^2_{(L^2(0,1))^n}, \]
(69)
where \( Y_{0,v} \) is the solution of (57) associated to \( \tilde{Y}_n \) and \( v \). Passing to the limit in the inequality (69), one has for all \( v \) in \( L^2(Q)^m \)

\[
\frac{1}{2} \int_0^T \| v \|^2_{(L^2(0,1))^m} dt + \frac{1}{2\varepsilon} \| \mathbf{Y}(T) \|^2_{(L^2(0,1))^n} \leq \frac{1}{2} \int_0^T \| v \|^2_{(L^2(0,1))^m} dt + \frac{1}{2\varepsilon} \| Y_{0,v}(T) \|^2_{(L^2(0,1))^n}.
\]

This means that \( v \) minimizes \( J_{\varepsilon,Y} \). Consequently \( K_{\varepsilon}(\mathbf{Y}) = \mathbf{Y} \), Hence the continuity of \( K_{\varepsilon} \). Thus, the following result is then proved.

**Theorem 6.1.** Assume the condition (56) is fulfilled. For all \( Y_0 \) in \( (H^1_{0a})^n \) the semi-linear parabolic degenerate system (55) is approximatively null controllable. i.e. for all \( \varepsilon > 0 \) there exists a control \( v_\varepsilon \in L^2(Q)^m \) for which the associated solution \( Y_{\varepsilon} \) satisfies

\[
\| Y_{\varepsilon}(T) \|_{(L^2(0,1))^n} \leq \varepsilon.
\]

Moreover, there exists a positive constant \( C > 0 \) such that

\[
\| v_\varepsilon \|^2_{L^2(Q)^m} \leq C |Y_0|^2_{(L^2(0,1))^n}.
\]

From this theorem, we deduce following result

**Theorem 6.2.** Assume the condition (56) is fulfilled. For all \( Y_0 \) in \( (H^1_{0a})^n \) the semi-linear parabolic degenerate system (55) is null controllable. i.e. There exists a control \( v \) in \( L^2(Q)^m \) for which the associated solution \( Y_{\varepsilon} \) satisfies

\[
Y_{\varepsilon}(T,x) = 0, \quad \forall x \in (0,1).
\]

Moreover, there exists a positive constant \( C > 0 \) such that

\[
\| v \|^2_{L^2(Q)^m} \leq C |Y_0|^2_{(L^2(0,1))^n}.
\]

**Proof.** From Theorem 6.1 the set \( \{ v_\varepsilon, \varepsilon > 0 \} \) is bounded in \( L^2(Q)^m \), thus it contains a sequence \( (v_{\varepsilon^n}) \) that converges (weakly) in \( L^2(Q)^m \) to a limit \( v_0 \) that satisfies

\[
\| v_0 \|^2_{L^2(Q)^m} \leq C |Y_0|^2_{(L^2(0,1))^n}.
\]

The sequence \( (Y_{\varepsilon^n}) \) converges strongly to \( Y_{\varepsilon^0} \) in \( X_T \). Moreover \( (Y_{\varepsilon^0}) \) is the solution to (55) with \( v = v_0 \). So according to (70), for all \( x \in (0,1) \) we have

\[
Y_{\varepsilon^0}(T,x) = 0.
\]

Thus, the semi-linear parabolic degenerate system (55) with regular initial data is null controllable. \( \square \)

Now we are able to give the proof of the null controllability of the semi-linear parabolic degenerate system (55) with general initial data. As in [10, 4, 1], we can show also the following well posedness of degenerate parabolic semi-linear systems which is of great utility.

**Proposition 6.** For all \( Y_0 \) in \( L^2(0,1) \) the semi-linear system

\[
\begin{align*}
\partial_t U &= D \mathbf{M} U + F(U) \quad \text{in} \quad Q, \\
C U &= 0 \quad \text{on} \quad \Sigma, \\
U(0,x) &= Y_0(x) \quad \text{in} \quad (0,1),
\end{align*}
\]

admits a solution \( U \) in \( X_T \).
Theorem 6.3. For all $Y_0$ in $(L^2(0,1))^n$ the semi-linear system parabolic degenerate system (55) is null controllable.

Proof. By Proposition 6 the system (71) in the set $(0,T/2) \times (0,1)$ with initial data $Y_0$ in $(L^2(0,1))^n$ admits a solution $\bar{Y}$ in $X_{T/2}$. Thus, for $t_0 \in (0,T/2)$ we have $\bar{Y}(t_0) \in (H^1)^n$. Now let us consider

$$
\begin{cases}
\partial_t \bar{U} = D M \bar{U} + F(\bar{U}) + B v_1 \mathbb{1}_{\omega} & \text{in } Q_{t_0}, \\
C \bar{U} = 0 & \text{on } \Sigma, \\
\bar{U}(t_0,x) = \bar{Y}(t_0)(x) & \text{in } (0,1),
\end{cases}
$$

(72)

where $Q_{t_0} = (t_0,T) \times (0,1)$. Due to Theorem 6.2, there exists a control $v_1 \in L^2(Q_{t_0})$ for which the system (72) admits a solution $\bar{U}$ that satisfies $\bar{U}(T,x) = 0$ for all $x \in (0,1)$. Now let us define $(Y,v)$ as follow:

$$
Y = \begin{cases}
U \text{ in } [0,t_0], \\
\bar{U} \text{ in } [t_0,T].
\end{cases}
$$

$$
v = \begin{cases}
0 \text{ in } [0,t_0], \\
v_1 \text{ in } [t_0,T].
\end{cases}
$$

$Y$ is then a solution of the system (55) that satisfies $Y(T,x) = 0$ for all $x \in (0,1)$ which completes the proof.

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