Improved Lower Bounds for the Critical Probability of
Oriented-Bond Percolation in Two Dimensions

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Abstract
We present a coupled decreasing sequence of random walks on \(\mathbb{Z}(\mathcal{X}(i), i \in \mathbb{N})\) that dominates the edge process of oriented-bond percolation in two dimensions. Using the concept of “random walk in a strip”, we construct an algorithm that generates an increasing sequence of lower bounds that converges to the critical probability of oriented-bond percolation. Numerical calculations of the first ten lower bounds thereby generated lead to an improved, ie higher, rigorous lower bound to this critical probability, viz. \(p_c \geq 0.63328\). Finally a computer simulation technique is presented; the use thereof establishes 0.64450 as a non-rigorous five-digit-precision (lower) estimate for \(p_c\).

Key Words: oriented percolation, discrete time contact processes, critical probability, edge process, Markov chain in a strip, coupling, simulation.

1 Introduction
Oriented percolation in two dimensions or discrete time contact process on \(\mathbb{Z}\) is a one-parameter family of discrete time stochastic processes defined on \(\{0, 1\}^2\); the parameter \(p\), the infection rate, taking values on \([0, 1]\). It is a well established fact that this family exhibits a phase transition, as the value of \(p\) increases from 0 to 1: if \(p < p_c\), the process dies out almost surely; whereas if \(p > p_c\), the process survives with positive probability (see [1] for instance).

As usual, in critical phenomena theory, an analytical expression for \(p_c\) is unknown and its value has been estimated both in mathematical and physical literatures [1 Sec.6].

Up to the present moment, the best (rigorous) lower and upper bounds for \(p_c\) are respectively 0.6298 and 2/3 [4].

In this paper, we present an algorithm that generates an increasing sequence of lower bounds that converges to the critical probability of oriented bond percolation in two dimensions and, calculating the first ten lower bounds thereby generated, we were able to improve the best lower bound known up to now from 0.6298 to 0.63328.

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1 “\(p_c\)” is known as the critical probability of the family.
More specifically:

1. a numerical sequence of lower bounds for \( p_c, \{p_c^{(i)}\}_{i \in \mathbb{N}} \), was constructed in the following steps:

   (a) Associated to a decreasing\(^2\) sequence \((\text{in } i)\) of random walks on \( \mathbb{Z} \), \( \{X_n^{(i)}\}_{n \in \mathbb{N}} \), that dominate the edge process of oriented percolation, we constructed a sequence \((\text{in } i)\) of finite, discrete time Markov chains, \( \{Y_n^{(i)}\}_{n \in \mathbb{N}} \), whose transition probabilities are specific\(^3\) functions of \( p \), the infection probability; and

   (b) calculated the mathematical expectation of the random variable 
   \[ E\left((X_1^{(i)} - X_0^{(i)} \mid Y_0^{(i)})\right), \]
   the mean jump on the \( Y \)-configuration, with respect to \( \pi^{(i)} \), the stationary measure for \( Y^{(i)} \).

   (c) \( p_c^{(i)} \) was then defined to be the (only) value of \( p \) that nullifies the above expectation. In other words, when \( p = p_c^{(i)} \), the random walk \( \{X_n^{(i)}\}_{n \in \mathbb{N}} \) has zero mean drift.

2. the numerical sequence \( \{p_c^{(i)}\}_{i \in \mathbb{N}} \) was shown to converge in a non-decreasing fashion to \( p_c \), ie \( p_c^{(i)} \nearrow p_c \). Moreover,

3. the first ten lower bounds \( \{p_c^{(0)}, p_c^{(1)}, ..., p_c^{(9)}\} \) were numerically calculated, thereby improving the best rigorous lower bound known up to the moment from 0.6298 to 0.63328.

In the last section we present a simulation technique which in our opinion has important advantages in comparison with the usual so called Monte Carlo simulation techniques, in the sense that it exhibits a clear cut off between the subcritical and supercritical phases; thereby enabling a precise estimation of the critical probability of Oriented Bond Percolation without the aid of scaling techniques. By means thereof a lower bound for \( p_c \) was obtained within a precision of 5 digits, viz \( p_c^{1000} = 0.64451 \), so that we can (non-rigorously) state that \( p_c = (0.64451 \pm 0.00001) \).

2 Definitions and Constructions

2.1 The Environment

Let \( \mathcal{G} \equiv (\mathcal{V}, \mathcal{E}) \) be the oriented graph, having \( \mathcal{V} = \{(n, m): n \in \mathbb{N} \text{ and } (n + m) \text{ is even}\} \) as its set of vertices/sites, and \( \mathcal{E} = \{e_{n,m}^r, e_{n,m}^l : (n, m) \in \mathcal{V}\} \) as its set of bonds.

Bond \( e_{n,m}^r \) points from site \((n, m)\) to site \((n + 1, m - 1)\), whereas bond \( e_{n,m}^l \) points from site \((n, m)\) to site \((n + 1, m + 1)\). Sometimes the natural association: \( l \leftrightarrow -1 \quad r \leftrightarrow +1 \) will be assumed through out this text.

It is useful to think of \( n \) as a (discrete) time coordinate and of \( m \) as a (discrete) space coordinate of the graph \( \mathcal{G} \).

\( \mathcal{V}_n \) denotes the \( n \)-th slice of \( \mathcal{V} \), ie \( \mathcal{V}_n = \{(n, m) \in \mathcal{V} : n \text{ is fixed}\} \), and \( \mathbb{Z}_n \) the set of integers \( m \) such that \((m + n)\) is even.

It will be often useful in the forthcoming definitions to identify \( \mathcal{V}_n \) as \( \mathbb{Z}_n \), and to think of \( \{0, 1\}^{\mathbb{Z}_n} \) as being \( \{0, 1\}^{\mathcal{V}_n} \).

Therefore arises the loose, yet natural, notation: 
\[ \eta_n(m) \equiv \eta_n(n, m), \quad \eta_n \in \{0, 1\}^{\mathcal{V}_n}. \]

\(^2\)In a sense to be specified later by means of coupling
\(^3\)polynomial
2.2 The Probability Structure

Let \( \{ \xi_{nm}^j : (n, m) \in \mathcal{V}, j \in \{l, r\} \} \) be a family of independent and uniformly distributed (onto \([0, 1]\)) random variables, defined on the same abstract probability space \( \Omega \). Starting from this three-indexed countable family, we construct the four-indexed uncountable family of iid Bernoulli random variables on the same probability space \( \Omega \):

\[
\{ p \xi_{nm}^j : (n, m) \in \mathcal{V}, j \in \{l, r\}, p \in \[0, 1]\} \tag{1}
\]

such that

\[
p \xi_{nm}^j = 1_{\{ \xi_{nm}^j \leq p \}} \tag{2}
\]

It follows straightforwardly from (2) that

\[
\begin{align*}
\mathbb{P}(p \xi_{nm}^j = 1) &= p \tag{a} \\
\mathbb{P}(p \xi_{nm}^j = 0) &= 1 - p := q \tag{b}
\end{align*}
\]

and the Fundamental Coupling:

\[
p_1 \leq p_2 \Rightarrow p_1 \xi_{nm}^j(\omega) \leq p_2 \xi_{nm}^j(\omega), \ \forall \omega \in \Omega, \ (n, m) \in \mathcal{V}, \ j \in \{l, r\} \tag{3}
\]

We observe that \( p \xi_{nm}^l = 1 \) is interpreted as a channel open to infection propagation from site \((n, m)\) to site \((n + 1, m - 1)\); \( p \xi_{nm}^r = 0 \), as a channel obstructed to infection propagation from site \((n, m)\) to site \((n + 1, m + 1)\); and so forth ...

2.3 The Processes

2.3.1 The Strengthened Discrete Time Contact Processes (SDTCP)

\( \eta_n \in \{0, 1\}^{\mathcal{V}_n} \), is interpreted as a infection state on slice \( \mathcal{V}_n \):

\[
\begin{cases}
\eta_n(m) = 1: \text{site } m \text{ is infected at time } n \\
\eta_n(m) = 0: \text{site } m \text{ is healthy at time } n
\end{cases} \tag{5}
\]
Particularly $\eta_0 \in \{0,1\}^{V_0}$ will denote an initial state of infection over $V_0$; the set of even integers, according to Section 2.4 above.

At this point, the following notation (to be used throughout this paper) should be kept into account: $\mathbb{N} = \{0, 1, 2, \cdots \}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Now for each $i \in \overline{\mathbb{N}}$, we define (by induction on $n$) the sequence of $\{0,1\}^{V_n}$-valued random variables $\{pX_n^{(i)}\}_{n \in \mathbb{N}}$ as

**Definition 2.1.**

(a) $pX_0^{(i)} := \eta_0$, so that $pX_0^{(i)}(m) = \eta_0(m), \forall m \in \mathbb{Z}_0$

(b) $pX_{n+1}^{(i)}(m) = \sup \left\{ pX_n^{(i)}(m-1), p\in \mathbb{Z}_n, m-1; pX_n^{(i)}(m+1), p\in \mathbb{Z}_n, m+1; 1_{\{pX_n^{(i)}(m) = 0, \forall m \geq m\}} \right\}$,

where $pX_{n+1}^{(i)} = \sup_{m \in \mathbb{Z}_n, j \in \{-1,1\}} \left\{ (m+j) : \{\omega \in \Omega : pX_n^{(i)}(m), \xi^j \} = 1 \right\}$

The role of the indicator function in Definition 2.1 (b) above is to infect by force all the sites lying farther than $2i$ on the left side of $pX_{n+1}^{(i)}$, the utmost right infected site at time $n+1$. It is natural, therefore, to call the stochastic process $\{pX_n^{(i)}\}_{n \in \mathbb{N}}$, defined above, as the **Strengthened Discrete Time Contact Process** (of $i$-th order and infection parameter $p$).

It follows directly from Definition 2.1 that

(a) when $i = \infty$, the indicator function does not act anymore, and we recover the ordinary **Discrete Time Contact Process**, also called **Oriented Percolation in Two Dimensions**, described in [1, Sec. 2]. So forth in this paper we shall refer to it as $\{pX_n^{(\infty)}\}_{n \in \mathbb{N}}$.

(b) the sequence of SDTCPs $\{pX_n^{(i)}\}_{n \in \mathbb{N}}$ is decreasing in $i$, in the following sense:

$$i \leq j; i, j \in \overline{\mathbb{N}} \Rightarrow pX_n^{(i)}(m)[\omega] \geq pX_n^{(j)}(m)[\omega], V \omega \in \Omega, (n, m) \in V, p \in [0, 1]$$

In particular, the **Oriented Percolation Process** is the weakest of them all.

Inequality 6 above is called **Coupling of First Kind**

(c) the family of SDTCPs $\{pX_n^{(i)}\}_{n \in \mathbb{N}}$ is decreasing in $p$ in the following sense

$$p_1 \leq p_2 \Rightarrow p_1X_n^{(i)}(m)[\omega] \leq p_2X_n^{(i)}(m)[\omega], \forall \omega \in \Omega, (n, m) \in V, i \in \overline{\mathbb{N}}$$

Inequality 7 above is called **Coupling of Second Kind**

In this paper, unless otherwise stated, $\eta_0$ will be chosen to be $1_{\bullet \leq 0} \in \{0,1\}^{V_0}$, where

$$1_{\bullet \leq 0}(m) = \begin{cases} 1, & m \leq 0 \\ 0, & m > 0 \end{cases}$$

We shall write $\{pX_n^{(i)}\}_{n \in \mathbb{N}}$ to emphasize that $\eta_0 = \eta$, when $\eta$ is some specific element of $\{0,1\}^{V_0}$ (probably different from $1_{\bullet \leq 0}$). Therefore the symbols $\{pX_n^{(i)}\}_{n \in \mathbb{N}}$, $\{1_{\bullet \leq 0}pX_n^{(i)}\}_{n \in \mathbb{N}}$, $1_{\bullet \leq 0}pX_\bullet^{(i)}$, $pX_\bullet^{(i)}$ have all the same meaning and, for simplicity’s sake, the last will be chosen, when no confusion may arise.

As usual in Particle System’s notation we write $\eta_n(\theta) \geq \eta_n(m)$, whenever $\eta_n(m) \geq \eta_n(\theta)$ for every $m \in \mathbb{Z}_n$.

Definition 2.1 also implies that, for $\eta, \theta \in \{0,1\}^{V_0}$

$$\eta \geq \theta \Rightarrow \forall p \in [0, 1], (n, m) \in V, \omega \in \Omega, i \in \overline{\mathbb{N}}$$

Inequality 8 is called **Coupling of Third Kind**.
2.3.2 The Right Edge Processes (REP)

Given a particular SDTCP \( pX^{(i)} \) and assuming that \( \sigma_{X^{(i)}} = 0 \), the second line of Definition 2.1(b) above defines a (non-markovian) random process on \( \mathbb{Z} \) denoted by \( \{ pX^{(i)} \}_{n \in \mathbb{N}} \) (or \( p\overline{X}^{(i)} \) in abbreviated fashion). In Figure 2 above \( pX^{(2)} = 0, pX^{(1)} = -3, pX^{(2)} = -2, pX^{(3)} = -3, pX^{(4)} = -2, \ldots \)

Again, in case of \( i = \infty \), the Edge Process of Oriented Percolation cited in [11] is recovered. Throughout this text, this term will be employed in a generalized way \((i \in \mathbb{N})\).

It is useful to think of \( pX^{(i)} \) as random walks on \( \mathbb{Z} \).

2.3.3 The Induced Markov Chains

Definition 2.2 The Markov chain \( \{ pY^{(i)} \}_{n \in \mathbb{N}}, i \in \mathbb{N}, \) with state space \( S^{(i)} = \{0, 1\}^{\{2^i, \ldots, 2^{i+1}\}} \) defined by

\[
pY_n^{(i)}(2j) = pX_n^{(i)} \left( \frac{pX_n^{(i)} - 2j}{2} \right), 1 \leq j \leq i
\]

is called Induced Markov Chain (IMC) for the SDTCP \( pX^{(i)} \).

In what follows, a generic element \( \sigma \in S^{(i)} \), will be labeled by \( n \in \{0, 1, \ldots, 2^i - 1\} \) according to

\[
\sigma = \sigma_n \Leftrightarrow n = \sum_{j=1}^{i} 2^{i-j} \sigma(2j)
\]  

(10)

Accordingly, in Figure 2 above \( pY^{(2)} = \{0, 1\}, pY^{(3)} = \{0, 1\}, pY^{(4)} = \{0, 1\}, \ldots \)

The transition probabilities for the IMC \( pY^{(i)} \)

\[
q_{im}^{(i)} = P \left[ Y_{n+1}^{(i)} = \sigma_m \mid Y_n^{(i)} = \sigma_l \right], 0 \leq l, m \leq 2^i - 1
\]  

(11)
are all strictly positive polynomial functions of $p$ (provided $0 < p < 1$). Therefore $\pi^{(i)}$, its stationary measure on $S^{(i)}$, is well defined.

The notation

$$\pi^{(i)}_l \overset{\text{def}}{=} \pi^{(i)}(\sigma_l), l = 0, 1, \ldots, 2^i - 1$$

is self explanatory.

At this point, in accordance with [3, Sec.3.1], we state

**Definition 2.3**

$$P^{(i)}_{(l,m,k)}(p) \overset{\text{def}}{=} \mathbb{P}\left[pX_{n+1}^{(i)} = pX_n^{(i)} + (1 - 2k), pY_{n+1}^{(i)} = \sigma_m \mid pY_n^{(i)} = \sigma_l\right], \quad k \in \mathbb{N}, 0 \leq l, m \leq 2^i - 1$$

the transition probabilities from state $\sigma_l$ to state $\sigma_m$ with a jump of magnitude $(1 - 2k)$

**Definition 2.4**

$$M^{(i)}_l(p) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \sum_{m=0}^{2^i-1} (1 - 2k) P^{(i)}_{(l,m,k)}(p), \quad 0 \leq l \leq 2^i - 1$$

the mean jump/drift of the REP on configuration $\sigma_l$

and

**Definition 2.5**

$$M^{(i)}(p) \overset{\text{def}}{=} \sum_{l=0}^{2^i-1} M^{(i)}_l(p) \pi^{(i)}_l$$

the mean jump/drift of the REP.

$M^{(i)}(p)$ is also called the (right)edge speed in accordance with Section 2.4 below. As it will soon become clear, Definition 2.5 is of fundamental importance in this paper.

Under this framework, the SDTCPs $pX^{(i)}$, described above, can be regarded, in case of $i \in \mathbb{N}$, as *Markov Chains/Random Walks in a 2$^i$-rowed Strip*:

$$pX_n^{(i)} = pX_{n-1}^{(i)} \left( pX_n^{(i)}, pY_n^{(i)} \right)$$

(12)

a slightly different idea of *Markov Chains in a Half Strip*, developed in [3, Sec.3.1].

### 2.4 Critical Probabilities

In Section 3 below it will be shown that, for $i \in \mathbb{N}$ and $p \in [0, 1]$:

(i) $M^{(i)}(p)$ is a strictly increasing function of $p$;

(ii) $M^{(i)}(p)$ has only one real root (in $[0, 1]$);

(iii) $\frac{pX_n^{(i)}}{n} \xrightarrow{\text{a.s.}} M^{(i)}(p)$
In case of \( i = \infty \), \( \alpha(p) \), the (right) edge speed of oriented percolation, plays precisely the same role of \( M^{(i)}(p) \) in the finite case depicted above\(^4\). (For details, see \([1]\), for instance). So the notation
\[
M^{(\infty)}(p) \overset{\text{def}}{=} \alpha(p)
\]
suggests itself and we state the following

**Definition 2.6** The critical (infection) probability \( p_c^{(i)} \) for the family of stochastic processes \( pX^{(i)}_n \), \( i \in \mathbb{N} \), is the only real root of \( M^{(i)}(p) \) (in \((0,1)\)) , the edge speed. Hence
\[
M^{(i)}(p_c^{(i)}) = 0, \forall i \in \mathbb{N}
\]
The heuristic meaning of Definition 2.6 above is:

(i) for \( p < p_c^{(i)} \), \( \lim_{n \to \infty} pX^{(i)}_n = -\infty \) \( a.s \), ie the infection dies out with probability one;

(ii) for \( p > p_c^{(i)} \), \( \lim_{n \to \infty} pX^{(i)}_n = +\infty \) \( a.s \), ie the infection spreads out over all \( \mathbb{Z} \).

In the sequel, we prove an important relation concerning the \( p_c^{(i)} \) s, \( i \in \mathbb{N} \), just defined, viz. \( p_c^{(i)} \overset{\text{ref}}{=} p_c^{(\infty)} \) \(^5\).

This non-decreasing convergence to the critical probability of Oriented Percolation (to be called The Convergence Theorem), besides the possibility of calculating the \( p_c^{(i)} \) s ( \( i \in \mathbb{N} \) ) by algebraical means (Section 4 below), is the cornerstone of this work.

3 The Convergence Theorem and Preliminary Results

**Lemma 3.1** \( \frac{pX^{(i)}_n}{n} \overset{(n)}{\to} M^{(i)}(p) \) \( a.s \), \( \forall i \in \mathbb{N}, p \in (0,1] \)

**Proof:**

**First case** \((i \in \mathbb{N})\): Let \( n_1^{(i)} \) be the (random) number of visits that the IMC \( pY^{(i)}_n \) makes to state \( \sigma_i \) up to time \( n \) (so that \( \sum_{i=0}^{2^n-1} n_1^{(i)} = n \)) and \( J_{l,k}^{(i)} \) the \( k \)th jump of the REP \( pX^{(i)}_n \) on state/row \( \sigma_i \).

The (strong) Markov property of SDTCP \( pX^{(i)}_n \) makes \( J_{l,k}^{(i)}, k \in \{1, 2, 3, \ldots\} \) iid rvs with \( E[J_{l,k}^{(i)}] = M_1^{(i)} \).

Now
\[
\frac{pX^{(i)}_n}{n} = \sum_{i=0}^{2^n-1} \sum_{k=1}^{n_1^{(i)}} \frac{J_{l,k}^{(i)}}{n} = \sum_{i=0}^{2^n-1} \sum_{k=1}^{n_1^{(i)}} \frac{J_{l,k}^{(i)} n_1^{(i)}}{n} = \sum_{i=0}^{2^n-1} \frac{n_1^{(i)}}{n} \sum_{k=1}^{n_1^{(i)}} J_{l,k}^{(i)} ,
\]
\[
\frac{n_1^{(i)}}{n} \overset{(n)}{\to} \pi_1^{(i)} \text{ a.s and } \sum_{k=1}^{n_1^{(i)}} J_{l,k}^{(i)} \overset{(n)}{\to} M_1^{(i)}(p) \text{ a.s}
\]

Taking the limit \( n \to \infty \) and bearing definition 2.6 in mind, we get the desired result.

**Second case** \((i = \infty)\): See \([1]\) pag.1004

\[^4\alpha(p) = -\infty, \text{when } p < p_c; \text{ so that the strict increasing behaviour does not apply to } \alpha(p) \text{ precisely.}\]

\[^5p_c^{(\infty)} \overset{\text{def}}{=} p_c, \text{ putting the notations of Sections 1 and 2 into agreement.}\]
Lemma 3.2 For $i \in \mathbb{N}$, each function $M^{(i)} : (0, 1] \to (-\infty, 1]$ is strictly increasing (in $p$). Moreover, $M^{(i)}$ is a surjection from $(0, 1]$ onto $(-\infty, 1]$.

Idea of Proof: The non-decreasing behaviour of $M^{(i)}(p)$ follows from the Second Kind of Coupling (inequality 7 above) and Lemma 3.1 just established. Definitions 2.2, 2.3, 2.4 and 2.5 make $M^{(i)}(p)$ a rational function of $p$ such that $M^{(i)}(0_+) = -\infty$ and $M^{(i)}(1) = 1$; so $M^{(i)}(p)$ cannot be constant on any interval $[p_1, p_2] \subset (0, 1]$ and the strict behaviour follows.

Commentary on Lemma 3.2:
At this point, it is worth observing that the function $\alpha(p) \overset{\text{def}}{=} M^{(\infty)}(p)$ is non-decreasing; strictly positive, when $p > p_c$; null, when $p = p_c$ and infinitely negative, when $p < p_c$. Again, the reference is [1].

Lemma 3.2 above yields

Corollary 3.3 For each $i \in \mathbb{N}$, $M^{(i)}$ has only one real root in $(0, 1]$, denoted by $p_{c}^{(i)}$, in accordance with Section 2.4.

Lemma 3.4 The sequence of functions $\{M^{(i)}(p)\}_{i \in \mathbb{N}}$ is non-increasing, ie $i \leq j$; $i, j \in \mathbb{N} \Rightarrow M^{(i)}(p) \geq M^{(j)}(p), \forall p \in (0, 1]$. In particular $\alpha(p) \leq M^{(i)}(p); \forall i \in \mathbb{N}, \forall p \in (0, 1]$.

Idea of Proof:
The non-increasing behaviour (in $i$) of $\{M^{(i)}(p)\}$ follows from the First Kind of Coupling (inequality 6 above) and again from Lemma 3.1.

Lemma 3.4 yields

Corollary 3.5 The numerical sequence $\left\{p_{c}^{(i)}\right\}_{i \in \mathbb{N}}$ is non-decreasing. Hence $\lim_{i \to \infty} p_{c}^{(i)}$ is well defined and $p_{c}^{(i)} \nearrow \lim_{i \to \infty} p_{c}^{(i)}$. Moreover $p_{c}^{\overset{\text{def}}{=} p_{c}^{(\infty)}} \geq p_{c}^{(i)}, \forall i \in \mathbb{N}$ and

Corollary 3.6 $p_{c}^{(\infty)} \geq \lim_{i \to \infty} p_{c}^{(i)}$.

Now we turn our attention to the reverse (and more difficult) inequality, viz. $p_{c} \leq \lim_{i \to \infty} p_{c}^{(i)}$. For that consider the following probability spaces:

- $(\Omega, \mathbb{P})$: the abstract probability space, where the iid rv $\xi_{n,m}^{j}$ were defined;
- $(\mathcal{S}^{(i)}, \pi^{(i)})$: the finite probability space of the IMC $Y_{*}^{(i)}$, endowed with its stationary measure $\pi^{(i)}$;
- $(\Omega \times \mathcal{S}^{(i)}, \mathbb{P} \times \pi^{(i)}):$ the product space.
In the product space, we define the stochastic processes:

- \( \left\{ pX_n^{(i)} \right\}_{n \in \mathbb{N}} \), by \( pX_n^{(i)}(\omega, \sigma) \overset{\text{def}}{=} pX_n^{(i)}(\omega), \forall (\omega, \sigma) \in \Omega \times S^{(i)} \)

- \( \left\{ \pi_n^{(i)} \right\}_{n \in \mathbb{N}} \), by \( \pi_n^{(i)} \overset{\text{def}}{=} \eta \sigma \pi_n^{(i)}(\omega), \forall (\omega, \sigma) \in \Omega \times S^{(i)} \)

where \( \eta \sigma \in \{0, 1\}^\mathbb{N} \) is such that \( \eta \sigma(2m) \overset{\text{def}}{=} \begin{cases} 0 & m > 0 \\ \sigma(-2m) & -1 \leq m \leq -i \\ 1 & m < -i \end{cases} \)

so that the IMC \( pY_n^{(i)} \) starts on state \( \sigma \in S^{(i)} \).

Now we state

**Lemma 3.7** \( \mathbb{E} \left[ \frac{X_n^{(i)}}{n} \right] \geq M^{(i)}(p), \forall n, i \in \mathbb{N}, p \in (0, 1] \).

**Proof:**

Inequality \( \overset{\text{□}}{\text{□}} \) implies that \( \forall (\omega, \sigma) \in \Omega \times S^{(i)}, pX_n^{(i)}(\omega, \sigma) = pX_n^{(i)}(\omega) \geq \pi_n^{(i)}(\omega) = \pi_n^{(i)}(\omega, \sigma) \).

Integrating (with respect to \( \mathbb{P} \times \pi_n^{(i)} \)) both sides of the inequality above, yields:

\[
\mathbb{E} \left[ pX_n^{(i)} \right] = \mathbb{E} \left[ \pi_n^{(i)} \right] \geq \mathbb{E} \left[ \pi_n^{(i)}(\omega) \right] = nM^{(i)}(p),
\]

where the last equality above comes from the fact that the process \( \left\{ \pi_n^{(i)} \right\}_{n \in \mathbb{N}} \) has stationary increments of mean \( M^{(i)}(p) \).

**Figure 3** below refers to the next lemma:

**Lemma 3.8** \( \lim_{i \to \infty} \mathbb{E} \left[ pX_n^{(i)} \right] = \mathbb{E} \left[ pX_n^{(\infty)} \right] \) for all (fixed) \( n \in \mathbb{N} \) and \( p \in (0, 1] \).
Proof:

As the jumps to the right are bounded by +1, it follows that \( \forall \omega \in \Omega, \ pX_n^{(i)} \leq n \implies pX_n^{(i)} - m \leq n - m \). However, for a site \((n, m)\) to be infected by force, we must have \( pX_n^{(i)} - m > 2i \) (Def. 24 b) above). Hence, if \( n - m \leq 2i \), then \( pX_n^{(i)} - m \leq 2i, \forall \omega \in \Omega \) and site \((n, m)\) will not be infected by force, for any \( \omega \in \Omega \). Accordingly, it can be proved by induction on \( n \) that \( n - m \leq 2i \implies X_n^{(\infty)}(\omega) = X_n^{(i)}(m, \omega), \forall \omega \in \Omega \), i.e all sites \((n, m)\) such that \( n - m \leq 2i \) have the same infection state regarding the processes \( pX_n^{(\infty)} \) and \( pX_n^{(i)} \). (See Figure 5 above for the case: \( i = 4 \)).

Hence, for any (fixed) time \( n \),

\[
pX_n^{(\infty)} < pX_n^{(i)} \implies pX_n^{(\infty)}(m) = 0, \forall m \geq n - 2i
\]  

(13)

The reasoning behind (13) is as follows:

\[
pX_n^{(i)} \geq n - 2i \implies pX_n^{(i)} \left( pX_n^{(i)} \right) = pX_n^{(i)} \left( pX_n^{(i)} \right), \quad \text{def} \quad \tag{10}
\]

\[
pX_n^{(i)} \equiv \sup \{ m \in \mathbb{Z}_n : pX_n^{(i)}(m) = 1 \} = \sup \{ m \in \mathbb{Z}_n : pX_n^{(i)}(m) = 1 \} \equiv pX_n^{(\infty)} \implies \{ pX_n^{(i)} \geq n - 2i \} \subset \{ pX_n^{(i)} = pX_n^{(i)} \} \implies \{ pX_n^{(i)} \neq pX_n^{(i)} \} \subset \{ pX_n^{(i)} < n - 2i \} \subset \{ pX_n^{(i)} < n - 2i \}
\]

(10) follows identifying \( \{ pX_n^{(i)} \neq pX_n^{(i)} \} \) with \( \{ pX_n^{(i)} < pX_n^{(i)} \} \) and \( \{ pX_n^{(i)} < n - 2i \} \) with \( \{ pX_n^{(i)}(m) = 0, \forall m \geq n - 2i \} \).

Now, observe that, if the infection is not present on the set \( \{ (n, m) \in \mathbb{V}_0 : m \geq n - 2i \} \), all the paths joining its sites to slice \( \mathbb{V}_0 \) must be obstructed somewhere. In particular all the \((i+1)\) straight lines joining site \((0, -2j)\) to site \((n, n - 2j)\), \(0 \leq j \leq i\) must be interrupted at some point (Figure 5 above). As these lines are made of different, independent bonds, the probability of this event equals \((1 - p^n)^{i+1}\), and we have

\[
\mathbb{P} \left( pX_n^{(i)} \neq pX_n^{(i)} \right) \leq \mathbb{P} \left( pX_n^{(\infty)}(m) = 0, \forall m \geq n - 2i \right) \leq (1 - p^n)^{i+1}
\]

So that \( pX_n^{(i)} \to pX_n^{(\infty)} \) in probability, and there is a sub-sequence \( (i_k)_{k \in \mathbb{N}} \) such that \( pX_n^{(i_k)} \to pX_n^{(\infty)} \) as \( \text{Theorem 7.6} \). As the whole sequence \( \{ pX_n^{(i)} \}_{i \in \mathbb{N}} \) is non increasing (in \( i \)), we must have \( pX_n^{(i)} \to pX_n^{(\infty)} \) as \( i \) well, and we can apply the Monotone Convergence Theorem to conclude that \( \mathbb{E} \left[ pX_n^{(i)} \right] \to \mathbb{E} \left[ pX_n^{(\infty)} \right], \forall n \in \mathbb{N} \).

Now we can prove the Convergence Theorem:

**Theorem 3.9** \( p_c^{(i)} \to p_c \)

**Proof:** Suppose that \( \lim_{i \to \infty} p_c^{(i)} < p_c \), so that we can choose \( p \) such that \( \lim_{i \to \infty} p_c^{(i)} < p < p_c \). Then Lemma 62 and Corollary 3 ensure that \( \forall i \in \mathbb{N}, \ M(i)(p) > M(i)(p_i^{(i)}) \) def 0 and Lemma 7 ensure that \( \mathbb{E} \left[ pX_n^{(i)} \right] \geq nM(i)(p) > 0, \forall i \in \mathbb{N}, \forall n \in \mathbb{N} \). Thus

\[
\lim_{i \to \infty} \mathbb{E} \left[ pX_n^{(i)} \right] \geq 0, \forall n \in \mathbb{N}
\]  

(14)

(inequality ensures that this limit is well defined)

By the other side,

\[
p < p_c \implies pX_n^{(\infty)} \to -\infty \text{ a s} \quad \text{Lemma 3.1 and commentary on Lemma 62}
\]

\[
\implies \mathbb{E} \left[ pX_n^{(\infty)} \right] \to -\infty \quad \text{Fatou’s Lemma}
\]

\[
\implies \exists \mathbb{F} : \mathbb{E} \left[ pX_n^{(\infty)} \right] < 0
\]

10
Applying Lemma 3.8 for $n$ yields
\[
\lim_{i \to \infty} \mathbb{E} \left[ pX_i \right] = \mathbb{E} \left[ pX_\infty \right] < 0
\]
which contradicts inequality 14 above. Hence, $\lim_{i \to \infty} p_c(i) \geq p_c$ and the theorem follows from Corollary 3.6.

\[\square\]

# 4 Numerical Calculations

## 4.1 Algebraical Determination of the Critical Probabilities

Theorem 3.9 (The Convergence Theorem) is of theoretical interest by itself. However weaker results such as Corollaries 3.5 and 3.6 already indicate that each $p_c(i), i \in \mathbb{N}$, is an improved lower bound (regarding its predecessors) for the critical probability of Oriented Percolation.

Although tacitly present in Sections 2.3.3 and 2.4 above, we present below the algorithm for calculating the critical probabilities $p_c(i), i \in \mathbb{N}$, exactly.

According to Definitions 2.2, 2.3, 2.4, 2.5 and 2.6, the critical probabilities $p_c(i), i \in \mathbb{N}$, may be determined in the following steps:

(i) **Determination of the Probabilities** $P^{(i)}_{(l,m,k)}$ (Def. 2.3) in terms of $q \overset{\text{def}}{=} 1 - p$:

Elementary combinatorics show that the probabilities $P^{(i)}_{(l,m,k)}, 0 \leq l, m \leq 2^i - 1, k \in \mathbb{N}$, may be expressed as polynomial functions of $q$, ie

\[
P^{(i)}_{(l,m,k)} = P^{(i)}_{(l,m,k)}(q) : \text{polynomial in } q
\]

(ii) **Determination of the Transition Matrix** $\left( q^{(i)}_{lm} \right)_{0 \leq l, m \leq 2^i - 1}$ of the IMC $pY_\bullet^{(i)}$:

The transition probabilities defined in (11) may be expressed as

\[
q^{(i)}_{lm} = \sum_{k=0}^{\infty} P^{(i)}_{(l,m,k)}(q)
\]

As the numerical sequence $\left( P^{(i)}_{(l,m,k)}(q) \right)_{k \in \mathbb{N}}$ is a geometric progression (ratio $q^2$) starting from the $(i + 2)^{th}$ term, the transition probabilities $q^{(i)}_{lm}$ may be expressed as rational functions of $q$. As a matter of fact, these probabilities are always polynomials in $q$:

\[
q^{(i)}_{lm} = q^{(i)}_{lm}(q) : \text{polynomial in } q
\]

(iii) **Determination of the Stationary Measure** $(\pi^i)$ of the IMC $pY_\bullet^{(i)}$:

The combinatorial calculus that lead to equation (15) above show that the transition probabilities $q^{(i)}_{lm}(q), 0 \leq l, m \leq 2^i - 1$, are strictly positive for $q \in (0, 1)$, so that the transition matrix $\left( q^{(i)}_{lm} \right)$ is irreducible and aperiodic. Hence $\pi^i$ is a well defined probability measure on $S^i$ and can be algebraically determined from $\left( q^{(i)}_{lm} \right)$: resulting that

\[
\pi^i_l(q) : \text{a strictly positive rational function of } q, q \in (0, 1), 0 \leq l \leq 2^i - 1
\]
(iv) **Determination of** $M_{l}^{(i)}$, the Mean Jump of the SDTCP $p_{X_{l}^{(i)}}$ on State $\sigma_{l}$:

Definition 2.4 and the fact that the numerical sequence $\left( P_{lmk}^{(i)}(q) \right)_{k\in\mathbb{N}}$ is essentially a geometric progression yield that

$$M_{l}^{(i)}(q) : \text{ a rational function of } q, \ 0 \leq l \leq 2^{i} - 1$$

(v) **Determination of** $M^{(i)}$, the Mean Jump of the SDTCP $p_{X_{l}^{(i)}}$:

Definition 2.5 and steps (iii) and (iv) above ensure that

$$M^{(i)}(q) : \text{ a rational function of } q$$

(vi) **Determination of the Critical Probability** $p_{c}^{(i)}$:

according to Corollary 3.3 and Definition 2.6, $p_{c}^{(i)}$ is the only real root of $M^{(i)}(p)$ on the interval $(0, 1)$; so that it is the only real root of the polynomial in the numerator of $M^{(i)}(p)$. Its calculation thus can be done numerically.

In the sequel, we employ the algorithm described above for calculating the first critical probability in algebraical terms:

**4.1.1 The Critical Probability of Zeroth Order, $p_{c}^{(0)}$:**

In this case $|S^{(i)}| = 1$, so $p_{Y_{l}^{(i)}}$ is trivial:

$$M^{(0)}(q) = M_{0}^{(0)}(q) = \sum_{k=0}^{\infty} (1 - 2k)P_{0,0,k}^{(0)}(q) = 1.p - 1.q(1 - q^{2}) - 3.q^{3}(1 - q^{2}) - 5.q^{5}(1 - q^{2}) - \ldots$$

$$= (1 - q) - q(1 - q^{2}).[1 + 3q^{2} + 5q^{4} + 7q^{6} + \ldots] = (1 - q) - q(1 - q^{2})\frac{1 + q^{2}}{(1 - q^{2})^{2}} =$$

$$= (1 - q) - \frac{q + q^{3}}{1 - q^{2}} = \frac{1 - 2q - q^{2}}{1 - q^{2}}$$

Hence,

$$M^{(0)}(q) = 0 \Leftrightarrow 1 - 2q - q^{2} = 0 \Rightarrow p_{c}^{(0)} = 2 - \sqrt{2} = 0.58579\ldots$$

**4.1.2 The Critical Probability of First Order, $p_{c}^{(1)}$:**

$$\left(q_{lm}^{(1)}(q) = \begin{bmatrix} q - q^{3} + q^{4} & 1 - q + q^{3} - q^{4} \\ q^{2} & 1 - q^{2} \end{bmatrix} \right), \quad \pi_{l}^{(1)}(q) = \frac{q^{2}, 1 - q + q^{3} - q^{4}}{1 - q^{2} + q^{3} - q^{4}}$$

$$M_{0}^{(1)}(q) = \frac{1 - 2q - 3q^{2} + 2q^{4}}{1 - q^{2}}, \quad M_{1}^{(1)}(q) = \frac{1 - 2q - q^{2}}{1 - q^{2}}$$

$$M^{(1)}(q) = \frac{1 - 3q + 2q^{2} - 6q^{4} + q^{5} + 3q^{6}}{1 - q + 2q^{3} - 2q^{4} - q^{5} + q^{6}}$$

Hence,

$$M^{(1)}(q) = 0 \Rightarrow p_{c}^{(1)} = 0.604233\ldots$$
4.1.3 The Critical Probability of Second Order, $p_c^{(2)}$:

$$\left(q_m^{(2)}(q)\right) = \begin{bmatrix}
q - q^3 + q^6 \\
q^2 - q^3 + q^4 + q^6 - q^6 \\
2q^2 - q^3 + q^4 + q^6 \quad q - q^3 + q^4 - q^6 \\
q^2 - q^3 + q^4 + q^6 \quad q - q^3 + q^4 - q^6 \\
2q^2 - q^3 + q^4 + q^6 \quad q - q^3 + q^4 - q^6 \\
1 - 2q + q^2 + q^4 - q^6 \\
1 - 2q + 2q^2 - q^3 + q^5 + q^6
\end{bmatrix}$$

$$\pi_0^{(2)}(q) = \frac{-2q^4 + 2q^3 - 4q^2 + 6q - 14q^3 + 15q^4 - 9q^{10} + 2q^{11}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 + 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$\pi_1^{(2)}(q) = \frac{-q^2 + 2q^3 - 2q^4 + 9q^5 - 5q^6 - 6q^7 + 12q^8 - 8q^9 + 2q^{10}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 + 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$\pi_2^{(2)}(q) = \frac{-q^2 + q^3 + 2q^4 - 9q^5 + 15q^6 - 14q^7 + 8q^8 - 2q^9}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 + 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$\pi_3^{(2)}(q) = \frac{-1 + 3q - 4q^2 + 5q^3 - 13q^4 + 25q^5 - 27q^6 + 13q^7 + 13q^8 - 34q^9 + 37q^{10} - 26q^{11} + 11q^{12} - 2q^{13}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 + 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$M^{(2)}(q) = \frac{1}{1 - q^2} \left[ 1 - 2q - 5q^2 + 4q^4, 1 - 2q - 3q^2 + 2q^3, 1 - 2q - q^2 + 2q^4, 1 - 2q - q^2 \right]$$

$$M^{(2)}(q) = \frac{-1 + 5q - 11q^2 + 16q^3 - 25q^4 + 52q^5 - 75q^6 + 96q^7 - 58q^8 - 60q^9 + 152q^{10} - 114q^{11} - 5q^{12} + 74q^{13} - 49q^{14} + 10q^{15}}{(1 - q^2)(-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 + 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13})}$$

Hence,

$$M^{(2)}(q) = 0 \Rightarrow p_c^{(2)}(q) = 0.614187$$

4.2 Numerical Determination of the Critical Probabilities:

The combinatorial calculations leading to the transition matrix $(q^{(i)})_{lm}$ and to the mean-drift vector $M^{(i)}_l$ described in Section 4.1.1 above, yet fastidious for humans, are tailor-made for computers: while it took us a whole afternoon for calculating $(q^{(2)})_{lm}$ algebraically (16 polynomial entries), a FORTRAN 77 program, running at 600 MHz, calculated $(q^{(9)})_{lm}$ (40 polynomial entries) in less than one minute time!

The greatest problem in writing a computer program version for the algorithm described in Section 4.1.1 arises precisely in step(iii), viz. the determination of the stationary measure $\pi^{(i)}$ (in algebraical terms), starting from the transition matrix $(q^{(i)})_{lm}(q)$. True, it can be done straightforwardly, acting on the polynomials as if they were numbers. However, if we don’t take into account the eventual and probable simplifications that may take place, the degree of the resulting polynomials soon get unmanageable. And finding out these simplifications is far from obvious.

Therefore we have adopted a different approach:

(i) After having algebraically calculated the transition matrix $(q^{(i)})_{lm}(q)$ and the mean-drift vector $M^{(i)}_l(q)$, the program calculated a numerical transition matrix and a numerical mean-drift vector for a decreasing sequence of numerical values of $q$.

(ii) From each numerical transition matrix, a numerical stationary measure was obtained by solving a system of linear equations. It is worth-while mentioning that partial pivoting was of fundamental importance in keeping numerical errors under control.

(iii) The numerical mean-drift was then obtained, performing the inner-product of Definition 2.6.

(iv) According to Definition 2.6, $q^c$ ($= 1 - p^{(i)}_c$) lies between the last value of $q$ for which $M^{(i)}_l(q)$ is negative and the first value of $q$ for which $M^{(i)}_l(q)$ is positive.
This approach has the onus of introducing numerical errors (basically in step ii above), from which we hitherto have got rid of. So as to produce reliable numerical data for the critical probabilities, rigorous upper bounds for the numerical errors must be provided. In doing so, we followed the set-up of forward analysis described in \[5\], which consists fundamentally in providing partial upper bounds after each arithmetic operation performed in the flow of numerical calculations leading to the final numerical result. This rather crude approach has the advantage of providing a secure upper bound for numerical errors, regardless of particular features the linear system may present, as it is often the case in the so called backward analysis (\[5\]).

In the sequel, we present the numerical data produced following steps i-iv above\(^6\):

| i=1 | p | \(M^{(1)}_{\text{Num}}(p)\) | \(\Delta M^{(1)}(p)\) |
|-----|---|----------------|-----------------|
| i=2 | p | \(M^{(2)}_{\text{Num}}(p)\) | \(\Delta M^{(2)}(p)\) |
| i=3 | p | \(M^{(3)}_{\text{Num}}(p)\) | \(\Delta M^{(3)}(p)\) |
| i=4 | p | \(M^{(4)}_{\text{Num}}(p)\) | \(\Delta M^{(4)}(p)\) |
| i=5 | p | \(M^{(5)}_{\text{Num}}(p)\) | \(\Delta M^{(5)}(p)\) |
| i=6 | p | \(M^{(6)}_{\text{Num}}(p)\) | \(\Delta M^{(6)}(p)\) |
| i=7 | p | \(M^{(7)}_{\text{Num}}(p)\) | \(\Delta M^{(7)}(p)\) |
| i=8 | p | \(M^{(8)}_{\text{Num}}(p)\) | \(\Delta M^{(8)}(p)\) |
| i=9 | p | \(M^{(9)}_{\text{Num}}(p)\) | \(\Delta M^{(9)}(p)\) |

Based upon the data displayed above, the first ten critical probabilities may be determined:

| i  | \(p_{c}^{(i)}\) |
|----|----------------|
| 0  | 0.585787       |
| 1  | 0.604233       |
| 2  | 0.614187       |
| 3  | 0.620205       |
| 4  | 0.624211       |
| 5  | 0.627066       |
| 6  | 0.629203       |
| 7  | 0.630864       |
| 8  | 0.632193       |
| 9  | 0.63328        |

So that we can establish the inequality:

\[ p_{c} \geq 0.63328 \]

\(^6\)the calculations were carried out on double precision.

\(^7\)|\(M^{(i)}(p) - M^{(i)}_{\text{Num}}(p)| \leq \Delta M^{(i)}(p)\); where \(M^{(i)}_{\text{Num}}(p)\) denotes the numerically calculated mean drift of \(i^{th}\) order.
5 Simulations

The boolean and inductive features of definition 2.1 make it extremely suitable for computer simulations: at each step a FORTRAN77-written computer program determines the value of \( p \frac{X_n^{(i)}}{n} \), the right edge mean speed at moment \( n \). In accordance with lemma 3.1, this sequence converges (a.s. in \( n \)) to \( M^{(i)}(p) \), the right edge mean speed. Therefore running the program for different values of \( p \), one can estimate the value of \( p_c^{(i)} \) observing the height of the plateau established with the increase of \( n \), according with the rule:

(a) height of plateau < 0 \( \Rightarrow \) \( p < p_c^{(i)} \),

(b) height of plateau > 0 \( \Rightarrow \) \( p > p_c^{(i)} \).

The two figures displayed below illustrate the use of this technique when \( i = 1000 \):

Figure 4: Independent (not coupled) trajectories of the processes \( \frac{X_n^{(1000)}}{n} \) e \( \frac{X_n^{(1000)}}{n} \).

Figure 5: Enlarged detail of the figure above.
Table 1 below summarizes the numerical results obtained by means of the technique for different values of $i$:

| $i$ | $p_{c}^{(i)}$ |
|-----|---------------|
| 5   | 0.627         |
| 6   | 0.629         |
| 9   | 0.6332        |
| 20  | 0.638         |
| 40  | 0.641         |
| 100 | 0.643         |
| 200 | 0.6438        |
| 1000| 0.64451       |

Table 1: Critical probabilities obtained by means of simulation.

It is interesting to observe that the plateau pattern exhibited in figure 4 is present even in the vicinity of criticality, i.e. $p \approx p_{c}^{(i)}$ and $i \rightarrow \infty$. Therefore the critical probabilities $p_{c}^{(i)}$ (even for large values of $i$ and so very close to criticality) can be estimated within the desired precision simply by increasing the value of $n$, i.e. running the program until the plateau pattern becomes clear. We believe that this feature distinguishes our technique from the usual Monte Carlo simulation methods, wherein instability appears near criticality and the use of scaling techniques is called for.

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