We consider spherical collapse in the Randall-Sundrum type II model and estimate the critical over density for black hole formation in the radiation dominated era. It is found that when (density)$^2$-term is dominant in the modified Friedmann equation the critical density is smaller than in the standard cosmology, which implies that PBHs are more easily produced in the Randall-Sundrum model.

I. INTRODUCTION

Primordial black holes (PBHs) are produced in the early Universe as a result of initial large density fluctuations. PBHs are stable against evaporation if their masses are small ($\lesssim 10^{15}$ g), and hence such PBH is a good candidate for dark matter of the Universe. PBHs with smaller masses evaporate by now and can give significant contributions to background $\gamma$-rays and cosmic rays. In particular, evaporation of PBHs may account for the excess of low energy anti-protons observed by the BESS experiment.

Since regions with large over density can collapse into black holes, the number density of PBHs is very sensitive to the spectrum of the initial density fluctuations and the critical over density $\delta_c$ above which the black hole is formed. In the standard cosmology, the critical density was estimated by Carr for a radiation dominated universe and it was applied to a number of PBH scenarios (e.g. [4, 5, 6], for review see [7]).

However, the evolution of the early universe may be drastically changed in the presence of extra-dimensions and branes which are predicted by string theories. In braneworld cosmology, the standard particles including ourselves are confined in one of branes and only gravity can travel in a bulk space. Randall and Sundrum proposed a simple and attractive model (called Randall-Sundrum Type II model) for the braneworld, where a brane with positive tension is embedded in a 5D anti-de Sitter space. In this model the Friedmann equation is modified and hence cosmological evolutions of the scale factor and density are different from those in the standard cosmology, which leads to different evolution of the PBHs after their formation.

In this letter, we consider spherical collapse of a region with large over density in the radiation dominated era of the Randall-Sundrum model and determine the critical over density $\delta_c$. It is shown that the critical density is small compared with result of the standard cosmology, which suggests that more abundance of PBHs is expected in the Randall-Sundrum cosmology.

II. SPHERICAL COLLAPSE MODEL IN THE STANDARD COSMOLOGY

First we consider the collapse in the standard cosmology in order to clarify the difference between the standard and the Randall-Sundrum models. Let us consider a spherical region with radius $R$ and over density $\delta$ in a radiation-dominated universe. If the radius is larger than the horizon, the region behaves like a Friedmann universe. Thus, the evolution of $R$ is described by

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3M_P^2} \rho R^2 - K$$

This equation describes the evolution of the radius of a spherical region with over density $\delta$ in a radiation dominated universe.
where $t$ is the cosmic time, $M_4$ is the 4D Planck mass, $\rho$ is the radiation density and $K$ is a positive constant. The solution of this equation is written as

$$R = A \sin \theta,$$

$$t = B(1 - \cos \theta),$$

where $\theta$ is a parameter. From Eq. (1) the constants $A$ and $B$ is related by

$$\frac{A^4}{B^2} = \frac{8\pi}{3M_4^2}\rho R^4,$$

$$\frac{A^2}{B^2} = K.$$  

Please notice that the radiation density $\rho$ changes as $\sim R^{-4}$ and hence $\rho R^4$ is constant.

The spherical region stops expanding when $\theta = \pi/2$, which gives the critical time $t_c$ and radius $R_c$,

$$R_c = A,$$

$$t_c = B.$$  

For $\theta \ll 1$ $R$ and $t$ are expanded as

$$R \simeq A(\theta - \theta^3/6 + \ldots),$$

$$t \simeq B(\theta^2/2 - \theta^4/24 + \ldots).$$

Then $\theta$ and $R$ can be expressed as a function of $t$ as

$$\theta^2 \simeq \frac{2t}{B} + \frac{\theta^4}{12} \simeq \frac{2t}{B} + \frac{t^2}{6B},$$

$$R \simeq A \left(\frac{2t}{B}\right)^{1/2} \left(1 + \frac{t}{12B} - \frac{t}{3B}\right) \simeq A \left(\frac{2t}{B}\right)^{1/2} \left(1 - \frac{t}{4B}\right).$$

On the other hand, the homogeneous background universe expands as

$$R_b = A_b \left(\frac{2t_b}{B_b}\right)^{1/2},$$

where the subscript "b" represents the homogeneous background and

$$\frac{A_b^4}{B_b^2} = \frac{8\pi}{3M_4^2}\rho_b R_b^4.$$  

In order to calculate the over density $\delta = (\rho - \rho_b)/\rho_b$ we must fix the coordinate system or gauge. Here we take “synchronous gauge”, i.e., $t = t_b$, $A = A_b$ and $B = B_b$. Then, the over density given by

$$1 + \delta = \frac{R^4}{R_b^4} \simeq 1 + \frac{t}{B}.$$  

Therefore, at the initial time $t_i$ the over-density $\delta_i$ is given by

$$\delta_i = \frac{t_i}{B}.$$
Using Eqs. (6) and (7) we obtain
\[ \frac{t_c}{t_i} = \frac{1}{\delta_i}, \]  
(16)
\[ \frac{R_c}{R_i} = \frac{A}{A\theta_i} = \left( \frac{B}{2t_i} \right)^{1/2} = \left( \frac{t_c}{2t_i} \right)^{1/2} = \frac{1}{(2\delta_i)^{1/2}}. \]  
(17)

For the region to collapse and become a black hole, the critical radius must be larger than the Jeans radius \( R_J \) which is given by [see Eq. (37)]
\[ R_J = \frac{\ell_H(t_c)}{\sqrt{6}}, \]  
(18)
where \( \ell_H(t) \) is the particle horizon at \( t \) and \( \alpha(\sim O(1)) \) is introduced to take account of ambiguity in defining \( R_J \). By requiring \( R_c > R_J \) we obtain the condition for the black hole formation,
\[ \delta_i > \delta_c \equiv \frac{\alpha^2}{3}. \]  
(19)

Thus, when \( R_i = \ell_H(t_i) = 2t_i \), the critical over-density \( \delta_c \) is given by
\[ \delta_c = \frac{\alpha^2}{3}. \]  
(20)

### III. SPHERICAL COLLAPSE IN RANDALL-SUNDRUM MODEL

Now we study the spherical collapse in Randall-Sundrum model. In this model the evolution of the spherical region with radius \( R \) is described by the modified Friedmann equation as
\[ \left( \frac{dR}{dt} \right)^2 = \frac{8\pi}{3M_5^2} \rho R^2 + \frac{16\pi^2}{9M_5^5} \rho^2 R^2 + \frac{\mu}{R^2} - K, \]  
(21)
where \( M_5 \) is the 5D Planck mass and \( \mu \) represents the dark radiation. When the density is small \( (\rho < \frac{3M_5^6}{(2\pi M_5^4)}) \) and the dark radiation is absent, Eq. (21) is the same as the standard one. However, for the case of large density, \( \rho^2 \)-term dominates the RHS of Eq. (21) and the cosmological evolution is quite different from that in the standard cosmology. Therefore, we only consider the latter case. Then, the modified Friedmann equation is given by
\[ \left( \frac{dR}{dt} \right)^2 = \frac{16\pi^2}{9M_5^4} \rho^2 R^2 - K, \]  
(22)
In similar way to the standard case, we obtain the solution of Eq. (22) in the radiation dominated era,
\[ R = A \sin^{1/3} \theta, \]  
(23)
\[ t = B \int_0^\theta d\theta' \sin^{1/3} \theta', \]  
(24)

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1 This critical over density is the same as the result in Ref. 3. However, this is accidental because the different gauge and Jeans radius are used. If we use “uniform Hubble gauge” adopted in Ref. 3 the critical density is given by \( \delta_c = 2\alpha^2/3 \).

2 Here we presume that the spherical collapse is described by the modified Friedmann equation. However, the bulk contribution (5D Wyle tensor) might spoil this picture 13. Since the evaluation of the bulk contribution is very difficult and is beyond the scope of the present letter, we simply assume that the bulk contribution is small.
where the constants $A$ and $B$ is related by

$$\frac{A^8}{9B^2} = \frac{16\pi^2}{9M_5^5}\rho^2 R^8, \quad (25)$$

$$\frac{A^2}{9B^2} = K. \quad (26)$$

The critical time $t_c$ and radius $R_c$ are given by

$$R_c = A, \quad (27)$$

$$t_c = -B\sqrt{\frac{\pi}{1}} (-1/3) \frac{1}{1} \approx 1.29B \equiv \beta B. \quad (28)$$

For small $\theta$, $R(\theta)$ and $t(\theta)$ are expanded as

$$R \approx A\theta^{1/3}(1 - \theta^2/18 + \ldots), \quad (29)$$

$$t \approx \frac{3}{4}B\theta^{4/3}(1 - \theta^2/45 + \ldots). \quad (30)$$

Then, $R$ is expressed in term of $t$ as

$$R \approx A \left(\frac{4t}{3B}\right)^{1/4} \left(1 - \frac{1}{20} \left(\frac{4t}{3B}\right)^{3/2} + \ldots\right) \quad (31)$$

Choosing the same gauge as before, the background universe evolve as

$$R_b = A \left(\frac{4t}{3B}\right)^{1/4}, \quad (32)$$

which leads to the initial over density

$$\delta_i = \frac{1}{5} \left(\frac{4t_i}{3B}\right)^{3/2}. \quad (33)$$

From Eq. (33) the critical time $t_i$ and the critical radius $R_i$ are

$$\frac{t_c}{t_i} = \frac{4\beta}{3(5\delta_i)^{2/3}}, \quad (34)$$

$$\frac{R_c}{R_i} = \frac{1}{(5\delta_i)^{1/6}}. \quad (35)$$

Next, let us consider the condition that the region collapses into a black hole, $R_c > R_J$. The Jeans length can be read from the mode equation of the density perturbations $\delta_k$ for the modified Friedmann model \cite{10,14},

$$\ddot{\delta}_k + H\dot{\delta}_k + \left[-\frac{16\pi}{3M_4^5}\rho - \frac{32\pi^2}{M_5^5}\rho^2 + \frac{1}{3} \left(\frac{k}{a}\right)^2\right]\delta_k = 0, \quad (36)$$

where $a$ is the scale factor and $H$ is the Hubble. From Eq. (36) the Jeans length is given by

$$\frac{a}{k_J} = \begin{cases} \frac{4\sqrt{\pi\rho}}{M_4} = \frac{1}{\sqrt{6}} \ell_H \text{ (low } \rho) \\ \frac{4\sqrt{\pi\rho}}{M_5} = \frac{1}{\sqrt{6}} \ell_H \text{ (high } \rho) \end{cases} \quad (37)$$
Therefore, taking $R_J = \alpha/\sqrt{6}\ell_H$, the condition for the black hole formation is written as

$$\delta_i > \frac{8\alpha^2\beta^2}{135} \left( \frac{4t_i}{3R_i} \right)^2 \simeq 0.10 \alpha^2 \left( \frac{4t_i}{3R_i} \right)^2 .$$  \hspace{1cm} (38)

Then, the critical over density at $R_i = \ell_H(t_i) = 4t_i/3$ is given by

$$\delta_c \simeq 0.1 \alpha^2 .$$  \hspace{1cm} (39)

IV. CONCLUSION

From Eqs. (39) and (20), the critical value in Randall-Sundrum model is about 3 times smaller than in the standard cosmology. Thus, PBHs are more easily produced when the $\rho^2$-term controls the cosmic expansion in the Randall-Sundrum model. This can be explained by the growth rate of the density fluctuations. As is seen in Eq. (33) the density perturbations grows as $\sim t^{3/2}$ which is faster than $\sim t$ for the standard case.

When the distribution for the density fluctuations is Gaussian, the mass fraction $f(M) = \rho_{\text{BH}}(M)/\rho$ of PBHs with mass $M$ is given by

$$f(M) = \int_{\delta_c}^{\infty} \frac{d\delta}{\sqrt{2\pi}\sigma(M)} \exp\left( -\frac{\delta^2}{2\sigma(M)^2} \right) \simeq \sigma(M) \exp\left( -\frac{\delta_c^2}{2\sigma(M)^2} \right) ,$$  \hspace{1cm} (40)

where $\sigma(M)$ is the mass variance at horizon crossing and we assume $\sigma(M) \ll \delta_c$. This shows that the abundance of PBHs is very sensitive to $\delta_c$. Therefore, the smaller critical value (39) gives a significant effect on the mass spectrum of PBHs. Suppose that the $\rho^2$-dominated era ends at $t = t_\star$. The horizon mass $M_\star$ at $t_\star$ is $\sim \rho_\star t_\star^3$. Since the mass of the produced PBH is $\sim$ horizon mass, PBHs with mass smaller than $M_\star$ are produced in $\rho^2$-dominated era and have much larger abundance than more massive PBHs. This fact may be useful when one searches the extra-dimension by observing $\gamma$-ray and anti-protons from evaporation of PBHs.

Finally we make a comment on the choice of $R_i$ in estimating the critical density. In this letter we take $R_i = \ell_H(t_i)$. Another reasonable choice is $R_i = H^{-1}(t_i)$. In this case, the difference between $\delta_i$(standard) and $\delta_i$(Randall-Sundrum) becomes larger. So the conclusion that the PBHs formation is easier in Randall-Sundrum cosmology does not change.

In summary, we have considered the spherical collapse in Randall-Sundrum model and estimated the critical over density for the black hole formation in the radiation dominated era. It has been found that when the Hubble expansion is dominated by $\rho^2$-term in the modified Friedmann equation the critical density is smaller than in the standard cosmology, which implies that PBHs are more easily produced in the Randall-Sundrum model.

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