Induced superconductivity distinguishes chaotic from integrable billiards

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Random-matrix theory is used to show that the proximity to a superconductor opens a gap in the excitation spectrum of an electron gas confined to a billiard with a chaotic classical dynamics. In contrast, a gapless spectrum is obtained for a non-chaotic rectangular billiard, and it is argued that this is generic for integrable systems.

PACS numbers: 05.45.+b, 73.23.Ps, 74.50.+r, 74.80.Fp

The quantization of a system with a chaotic classical dynamics is the fundamental problem of the field of “quantum chaos” [1,2]. It is known that the statistics of the energy levels of a two-dimensional confined region (a “billiard”) is different if the dynamics is chaotic or integrable [3,4]: A chaotic billiard has Wigner-Dyson statistics, while an integrable billiard has Poisson statistics. The two types of statistics are entirely different as far as the level correlations are concerned [6]. However, the mean level spacing is essentially the same: Particles of mass \( m \) in a billiard of area \( A \) have density of states \( mA/2\pi\hbar^2 \), regardless of whether their dynamics is chaotic or not.

In the solid state, chaotic billiards have been realized in semiconductor microstructures known as “quantum dots” [7]. These are confined regions in a two-dimensional electron gas, of sufficiently small size that the electron motion remains ballistic and phase-coherent on long time scales. (Long compared to the mean dwell time \( t_{\text{dwell}} \) of an electron in the confined region, which itself is much longer than the ergodic time \( t_{\text{erg}} \) in which an electron explores the available phase space.) A tunneling experiment measures the density of states in the quantum dot, if its capacitance is large enough that the Coulomb blockade can be ignored. As mentioned above, this measurement does not distinguish chaotic from integrable dynamics.

In this paper we show that the density of states becomes a probe for quantum chaos if the electron gas is brought into contact with a superconductor. We first consider a chaotic billiard. Using random-matrix theory, we compute the density of states \( \rho(E) \) near the Fermi level \( (E = 0) \), and find that the coupling to a superconductor via a tunnel barrier induces an energy gap \( E_{\text{gap}} \) of the order of the Thouless energy \( E_T \approx \hbar/t_{\text{dwell}} \). More precisely,

\[
E_{\text{gap}} = cN\Gamma\delta/2\pi, \tag{1}
\]

where \( N \) is the number of transverse modes in the barrier, \( \Gamma \) is the tunnel probability per mode, \( 2\delta \) is the mean level spacing of the isolated billiard, and \( c \) is a number which is weakly dependent on \( \Gamma \) (\( c \) decreases from 1 to 0.6 as \( \Gamma \) increases from 0 to 1). Eq. (1) requires \( 1 \ll NT \ll \Delta/\delta \), where \( \Delta \) is the energy gap in the bulk of the superconductor. In this limit \( \rho(E) \) vanishes identically for \( E \leq E_{\text{gap}} \). In contrast, for a rectangular billiard we do not find an energy gap in which \( \rho = 0 \), but instead find that the density of states vanishes linearly with energy near the Fermi level. We present a general argument that in an integrable billiard \( \rho \) has a power-law dependence on \( E \) for small \( E \).

The system considered is shown schematically in the inset of Fig. 1. A confined region in a normal metal (N) is connected to a superconductor (S) by a narrow lead containing a tunnel barrier. The lead supports \( N \) propagating modes at the Fermi energy. Each mode may have a different tunnel probability \( \Gamma_n \), but later on we will take all \( \Gamma_n \)'s equal to \( \Gamma \) for simplicity. The proximity effects considered here require time-reversal symmetry, so we assume zero magnetic field. (The case of broken time-reversal symmetry has been studied previously [3–5].) The quasi-particle excitation spectrum of the system is discrete for energies below \( \Delta \). We are interested in the low-lying part of the spectrum, consisting of (positive) excitation energies \( E_n \ll \Delta \). We assume that the Thouless energy \( E_T \approx NT\delta/2\pi \) is also much smaller than \( \Delta \).

There are two methods to compute the spectrum in the regime \( E, E_T \ll \Delta \). The first method is a scattering approach, which leads to the determinant equation

\[
\det[1 + S_0(E)S_0^*(-E)] = 0. \tag{2}
\]

The \( N \times N \) unitary matrix \( S_0(E) \) is the scattering matrix of the quantum dot plus tunnel barrier at an energy \( E \) above the Fermi level. Eq. (2) is a convenient starting point for the case that the quantum dot is an integrable billiard. For the chaotic case, we will use an alternative — but equivalent — determinant equation involving an effective Hamiltonian [5,10],

\[
\det(E - H) = 0, \quad H = \begin{pmatrix}
H_0 & -\pi W W^T \\
-\pi W W^T & -H_0^T 
\end{pmatrix}. \tag{3}
\]

The \( M \times M \) Hermitian matrix \( H_0 \) is the Hamiltonian of the isolated quantum dot. (The finite dimension \( M \)}
where \( d \) is determined by the Hamiltonian \( H \) and structure of \( W \). To leading order in \( 1/M \), the Green function \( G(z) = \langle (z - H)^{-1} \rangle \) consists of four \( M \times M \) blocks \( G_{11}, G_{12}, G_{21}, G_{22} \). By taking the trace of each block separately, one arrives at a 2 \( \times \) 2 matrix Green function

\[
G = \frac{\lambda}{M} \left( \begin{array}{cc} \text{Tr} G_{11} & \text{Tr} G_{12} \\ \text{Tr} G_{21} & \text{Tr} G_{22} \end{array} \right). 
\]

(7)

We seek the density of states

\[
\rho(E) = -\pi^{-1} \text{Im} \text{Tr} \langle (E + i0^+ - H)^{-1} \rangle, 
\]

(6)

where \( \langle \cdots \rangle \) denotes an average over \( H \) for fixed \( W \) and \( H_0 \) distributed according to Eq. (3). The method we use to evaluate this average is a perturbation expansion in \( 1/M \), adapted from Refs. [3, 4]. Because of the block structure of \( H \) [see Eq. (3)], the Green function \( G(z) = \langle (z - H)^{-1} \rangle \) consists of four \( M \times M \) blocks \( G_{11}, G_{12}, G_{21}, G_{22} \). By taking the trace of each block separately, one arrives at a 2 \( \times \) 2 matrix Green function

\[
G = \frac{\lambda}{M} \left( \begin{array}{cc} \text{Tr} G_{11} & \text{Tr} G_{12} \\ \text{Tr} G_{21} & \text{Tr} G_{22} \end{array} \right). 
\]

We have multiplied by \( \lambda/M = \delta/\pi \) for later convenience.) One more trace yields the density of states,

\[
\rho(E) = -\frac{\delta}{\pi} \text{ImTr} G(E + i0^+). 
\]

(8)

To leading order in \( 1/M \), the matrix \( G \) satisfies

\[
G = \frac{\lambda}{M} \sum_{n=1}^{M} \left( \begin{array}{cc} z - \lambda G_{11} & \pi w_n^2 + \lambda G_{12} \\ \pi w_n^2 + \lambda G_{21} & z - \lambda G_{22} \end{array} \right)^{-1}, 
\]

(9)

where we have abbreviated \( w_n^2 = (WW^T)_{nn} \). Eq. (9) is a matrix-generalization of Pastur’s equation [5]. A unique solution is obtained by demanding that \( G \) goes to \( \lambda/z \) times the unit matrix as \( |z| \to \infty \).

We now restrict ourselves to identical tunnel probabilities, \( \Gamma_n = \Gamma \). For \( M \gg N \gg 1/\Gamma \) Eq. (9) simplifies to

\[
NG_{11} = \pi z G_{12} \left(-G_{12} + 1 - 2/\Gamma\right), 
\]

\[
G_{22} = G_{11}, \quad G_{21} = G_{12}, \quad G_{12}^2 = G_{11}^2 + 1 - G_{11}. 
\]

(10a, 10b)

This set of equations can be solved analytically [6]. The result is that \( \rho(E) = 0 \) for \( E \leq E_{\text{gap}} \), where \( E_{\text{gap}} \) is determined by

\[
G_{11} = 2/\Gamma, \quad G_{12} = 1/\Gamma, \quad G_{21} = G_{22} = 1. 
\]

(11)

FIG. 1. Density of states of a chaotic billiard coupled to a superconductor (inset), for various coupling strengths. The energy is in units of the Thouless energy \( E_T = N\Gamma \delta/2\pi \). The solid curves are computed from Eqs. (6) and (10), for \( \Gamma = 1, 0.5, 0.25, 0.1 \). The dashed curve is the asymptotic result (11) for \( \Gamma \ll 1 \). The data points are a numerical solution of Eq. (10), averaged over \( 10^5 \) matrices \( H_0 \) in the Gaussian orthogonal ensemble \( (M = 400, N = 80) \). The deviation from the analytical curves is mainly due to the finite dimensionality \( M \) of \( H_0 \) in the numerics.

\[
\frac{k^6 - k^4}{(1 - k)^6} x^6 - \frac{3k^4 - 20k^2 + 16}{(1 - k)^4} x^4 + \frac{3k^2 + 8}{(1 - k)^2} x^2 = 1, 
\]

\[
x = E_{\text{gap}}/E_T, \quad k = 1 - 2/\Gamma. 
\]

(11)

The solution of this gap equation is the result [6] announced in the introduction. The complete analytical solution of Eq. (10) is omitted here for lack of space. In Fig. 1 we plot the resulting density of states. In the limit \( \Gamma = 1 \) of ideal coupling it is given by

\[
\rho(E) = \frac{E_T\sqrt{3}}{6E\delta} [Q_+(E/E_T) - Q_-(E/E_T)], 
\]

(12a)

\[
Q_{\pm}(x) = \frac{8 - 36x^2 \pm 3x\sqrt{3x^2 + 132x^2 - 48}}{12} \sqrt{3/4 - \Gamma^2}, 
\]

(12b)

\[
E > E_{\text{gap}} = 2E_T \gamma^{5/2} \approx 0.6 E_T, 
\]

(12c)

where \( \gamma = \frac{1}{2}(\sqrt{5} - 1) \) is the golden number. In the opposite limit \( \Gamma \ll 1 \) of weak coupling we find

\[
\rho(E) = E^{-1}(E^2 - E_T^2)^{-1/2}, \quad E > E_{\text{gap}} = E_T. 
\]

(13)

To check the validity of the perturbation theory, we have computed \( \rho(E) \) numerically from Eq. (10) by generating a large number of random matrices \( H_0 \) in the Gaussian orthogonal ensemble. The numerical results (data
The density of states, which is the quantity following directly from the numerical computation. The energy $E$ because of the energy difference $2E$ between electron and hole, plus a phase shift of $-\pi$ from two Andreev reflections. For $s \to \infty$ we find $P(s) \to 8(A/W)^2 s^{-3}$, which implies a linear $E$-dependence of the density of states near the Fermi-level,

$$\rho_{\text{BS}}(E) \to \frac{4E}{N\delta^2} = \frac{2E}{\pi E_T \delta}, \quad E \to 0. \quad (15)$$

In Fig. 2 we see that the exact quantum mechanical density of states also has (approximately) a linear $E$-dependence near $E = 0$, but with a smaller slope than the semi-classical Bohr-Sommerfeld approximation.

We argue that the absence of an excitation gap found in the rectangular billiard is generic for the whole class of integrable billiards. Our argument is based on the Bohr-Sommerfeld approximation. It is known \[17,18\] that an integrable billiard has a power-law distribution of path lengths, $P(s) \sim s^{-p}$ for $s \to \infty$. Eq. (14) then implies a power-law density of states, $\rho(E) \propto E^{p-2}$ for $E \to 0$.

To conclude, we have shown that the presence of an excitation gap in a billiard connected to a superconductor is a signature of quantum chaos, which is special in two respects: It appears in the spectral density rather than in a spectral correlator, and it manifests itself on the macroscopic energy scale of the Thouless energy rather than on the microscopic scale of the level spacing. Both these characteristics are favorable for experimental observation. Our theoretical results are rigorous for a chaotic billiard and for an integrable rectangular billiard. We have presented an argument that the results for the rectangle are generic for the whole class of integrable billiards, based on the semi-classical Bohr-Sommerfeld approximation. It remains a challenge to develop a rigorous general theory for the integrable case.

This work was supported by the Dutch Science Foundation NWO/FOM and by the Human Capital and Mobility program of the European Community.

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