A GLOBAL TORELLI THEOREM FOR CERTAIN CALABI-YAU THREEFOLDS

MAO SHENG AND JINXING XU

To the memory of Yi Zhang

Abstract. We establish a global Torelli theorem for the complete family of Calabi-Yau threefolds arising from cyclic triple covers of \( \mathbb{P}^3 \) branched along stable hyperplane arrangements.

1. Introduction

Classical Hodge theory attaches complex algebraic varieties with (mixed) Hodge structures and global Torelli theorem asserts that the attached Hodge structures determine complex algebraic varieties up to isomorphism. There are very rare cases of classes of complex algebraic varieties for which global Torelli theorem holds. The renowned examples include the smooth projective curves [2], Abelian varieties, polarized K3 surfaces [9, 3, 8] (see also birational global Torelli for hyperkähler manifolds [13]) and cubic fourfolds [14, 6, 7]. In this paper, we add one more example into the above list.

Our example stems from our early studies [11, 12] on Calabi-Yau varieties arising from cyclic covers of projective spaces branched along hyperplane arrangements. They are cyclic triple covers of \( \mathbb{P}^3 \) branched along six hyperplanes which are stable in the sense of GIT [5] (for hyperplane arrangements in general position (non-unique) crepant resolutions of singular CY varieties exist [11]). We show that the attached weight 3 Hodge structures of these CY varieties are pure (Propositions 2.1, 5.2). The main result of the paper is the following:

Theorem. Let \( X \) and \( Y \) be two Calabi-Yau threefolds which are cyclic triple covers of \( \mathbb{P}^3 \) branched along stable hyperplane arrangements. Then \( X \) and \( Y \) are isomorphic if and only if \( H^3(X, \mathbb{Z}) \) and \( H^3(Y, \mathbb{Z}) \) are isomorphic as polarized Hodge structure.

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The theorem is a direct consequence of Theorems 4.4, 5.3 in the text. The method to establish the results is pretty standard, namely using the variation of Hodge structure attached to a universal family. The analysis of the corresponding period map (and its extension to stable locus) is based on the celebrated work of Deligne-Mostow [4].

Inspired by a discussion with Chin-Lung Wang on a possible classification of complete Calabi-Yau families with Yukawa coupling length one, we were led to compare our example with other known families with Yukawa coupling length one. Surprisingly, it turns out our example is essentially not new. Indeed, we show in Appendix that our family over the smooth locus formed by hyperplane arrangements in general position is birationally equivalent to the one constructed by J. Rohde [10] via Borcea-Voisin construction (Propositions A.3, A.4). To the authors’ best knowledge, our example (which is essentially the one of Rohde) is the only example in literature of complete families of CY varieties with the global Torelli property.

2. FAMILIES FROM HYPERPLANE ARRANGEMENTS

Given a hyperplane arrangement $\mathfrak{A}$ in $\mathbb{P}^n$ in general position, the cyclic cover of $\mathbb{P}^n$ branched along $\mathfrak{A}$ is an interesting algebraic variety. When the hyperplane arrangement $\mathfrak{A}$ moves in the coarse moduli space of hyperplane arrangements, we get a family of projective variety. In this section, we collect some known facts about Hodge structures of these cyclic covers.

We say an ordered arrangement $\mathfrak{A} = (H_1, \cdots, H_m)$ of hyperplanes in $\mathbb{P}^n$ is in general position if no $n+1$ of the hyperplanes intersect in a point, or equivalently, if the divisor $\sum_{i=1}^m H_i$ has simple normal crossings.

Given an odd number $n$, we set $r = \frac{n+3}{2}$. Then for each ordered hyperplane arrangement $(H_1, \cdots, H_{n+3})$ in $\mathbb{P}^n$ in general position, we can define a (unique up to isomorphism) degree $r$ cyclic cover of $\mathbb{P}^n$ branched along the divisor $\sum_{i=1}^{n+3} H_i$. In this way, if we denote the coarse moduli space of ordered $n+3$ hyperplane arrangements in $\mathbb{P}^n$ in general position by $\mathcal{M}_{n,n+3}$, then we obtain a universal family $f_n : \mathcal{X}_{AR} \to \mathcal{M}_{n,n+3}$ of degree $r$ cyclic covers of $\mathbb{P}^n$ branched along $n+3$ hyperplane arrangements in general position. In [11], we constructed a simultaneous crepant resolution $\pi : \tilde{\mathcal{X}}_{AR} \to \mathcal{X}_{AR}$ for the family $f$ without changing the middle cohomology of fibers. Moreover, this simultaneous crepant resolution gives an $n$-dimensional projective Calabi-Yau family which is
maximal in the sense that its Kodaira-Spencer map is an isomorphism at each point of \( \mathcal{M}_{n,n+3} \). We denote this smooth projective Calabi-Yau family by \( \mathcal{f}_n : \mathcal{X}_{AR} \rightarrow \mathcal{M}_{n,n+3} \).

Now we recall the relation between a cyclic cover of \( \mathbb{P}^1 \) branched along points and that of \( \mathbb{P}^n \) branched along hyperplane arrangements.

Suppose \((p_1, \ldots, p_{n+3})\) is a collection of \( n + 3 \) distinct points on \( \mathbb{P}^1 \), and put \( H_i = \{p_i\} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \). By the natural identification between \( \mathbb{P}^n \) and the symmetric power \( \text{Sym}^n(\mathbb{P}^1) \) of \( \mathbb{P}^1 \), we can view each \( H_i \) as a hyperplane in \( \mathbb{P}^n \). Then it can be shown that \((H_1, \ldots, H_{n+3})\) is a hyperplane arrangement in \( \mathbb{P}^n \) in general position ([11], Lemma 3.4). A direct computation shows that this construction gives an isomorphism between the moduli space \( \mathcal{M}_{1,n+3} \) and \( \mathcal{M}_{n,n+3} \) ([11], Lemma 3.5). Moreover, for \( r = \frac{n+3}{2} \), if we denote \( C \) as the \( r \)-fold cyclic cover of \( \mathbb{P}^1 \) branched along the \( n + 3 \) points, and \( X \) as the \( r \)-fold cyclic cover of \( \mathbb{P}^n \) branched along the corresponding hyperplane arrangement \((H_1, \ldots, H_{n+3})\), then we have an isomorphism

\[
X \cong C^n / N \rtimes S_n.
\]

Here \( N \) is the kernel of the summation homomorphism \((\mathbb{Z}/r\mathbb{Z})^n \rightarrow \mathbb{Z}/r\mathbb{Z})\). The action of \( \mathbb{Z}/r\mathbb{Z} \) on \( C \) is induced from the cyclic cover structure, and \( S_n \) acts on \( C^n \) by permutating the \( n \) factors.

We summarize the properties of the Hodge structures on \( X \) and \( C \) as the following proposition ([11], Lemma 2.7, Proposition 3.7):

**Proposition 2.1.** Suppose \( n \) is an odd number, and \((p_1, \ldots, p_{n+3})\) is a collection of \( n + 3 \) distinct points on \( \mathbb{P}^1 \). For each \( 1 \leq i \leq n + 3 \), put \( H_i = \gamma(\{p_i\} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1) \), viewed as a hyperplane in \( \mathbb{P}^n \). Let \( r = \frac{n+3}{2} \), and \( C \) be the \( r \)-fold cyclic cover of \( \mathbb{P}^1 \) branched along \( \sum_{i=1}^{n+3} p_i \). Suppose \( X \) is the \( r \)-fold cyclic cover of \( \mathbb{P}^n \) branched along \( \sum_{i=1}^{n+3} H_i \). Then we have:

1. The natural \( \mathbb{Q} \)-mixed Hodge structure on the middle cohomology group \( H^n(X, \mathbb{Q}) \) is pure.
2. \( H^n(X, \mathbb{C})_{(i)} \cong \wedge^n H^1(C, \mathbb{C})_{(i)} \), for each \( 1 \leq i \leq r - 1 \).

Here and from now on, we fix a primitive \( r \)-th root of unity \( \zeta \), a generator \( \sigma \) of the cyclic group \( \mathbb{Z}/r\mathbb{Z} \), and we use \( H^n(X, \mathbb{C})_{(i)} \) to denote the \( i \)-eigenspace \( \{ \alpha \in H^n(X, \mathbb{C}) | \sigma \alpha = \zeta^i \alpha \} \) of \( H^n(X, \mathbb{C}) \). The notation \( H^1(C, \mathbb{C})_{(i)} \) has the similar meaning.
Remark 2.2. Since the simultaneous crepant resolution $\tilde{\mathcal{X}}_{AR} \to \mathcal{X}_{AR}$ of the universal family $\mathcal{X}_{AR} \xrightarrow{f} \mathcal{M}_{n,n+3}$ does not change the middle cohomologies of the fibers, the two $\mathbb{Q}$-PVHS (rational polarized variation of Hodge structures) $R^n f_* C_{\mathcal{X}_{AR}}$ and $R^n f_* C_{\tilde{\mathcal{X}}_{AR}}$ are isomorphic.

3. The monodromy group and period map: curve case

In this section, we first determine the monodromy group of the universal family of cyclic triple covers of $\mathbb{P}^1$ branched along six distinct points. Then we recall Deligne-Mostow’ result about period maps of this family.

Take five distinct points $a_1, \cdots, a_5 \in \mathbb{C}$. Let $C$ be the smooth projective curve whose affine model is defined by the equation

$$\{(x, y) \in \mathbb{C}^2 | y^3 = \prod_{i=1}^{5}(x - a_i)\}.$$ 

The cyclic triple covering structure induces a natural automorphism of $C$:

$$\sigma : C \to C$$

$$(x, y) \mapsto (x, \omega y)$$

where $\omega = \exp\left(\frac{2\pi \sqrt{-1}}{3}\right)$ is a primitive cubic root of unity.

We have the decomposition of $\mathbb{Z}[\omega]$-modules:

$$H^1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega] = H^1(C, \mathbb{Z}[\omega])_{\omega} \oplus H^1(C, \mathbb{Z}[\omega])_{\bar{\omega}}$$

where

$$H^1(C, \mathbb{Z}[\omega])_{\omega} := \{\alpha \in H^1(C, \mathbb{Z}[\omega]) | \sigma^* \alpha = \omega^i \alpha \}; \ i = 1, 2.$$ 

The formula $h(\alpha, \beta) := -iQ(\alpha, \bar{\beta})$ defines an Hermitian form

$$h : H^1(C, \mathbb{Z}[\omega])_{\omega} \times H^1(C, \mathbb{Z}[\omega])_{\bar{\omega}} \to \mathbb{Z}[\omega],$$

where $Q$ means the intersection pairing on $H^1(C, \mathbb{Z})$. We can verify that $H^1(C, \mathbb{Z}[\omega])_{\omega}$ is a rank four free $\mathbb{Z}[\omega]$-module, and the Hermitian form $h$ is unimodular with signature $(3, 1)$.

Now we can describe the monodromy group in the curve case. Let $(\Lambda, h)$ be a fixed $\mathbb{Z}[\omega]$-lattice of signature $(3, 1)$, and let $\mathcal{M}_{1,6} := \{(z_1, \cdots, z_6) \in (\mathbb{P}^1)^6 | z_i \neq z_i, \forall i \neq j)/\text{PGL}(2, \mathbb{C})$ be the moduli space of ordered six distinct points on $\mathbb{P}^1$. Let $f : C \to \mathcal{M}_{1,6}$ be the universal family of cyclic triple covers of $\mathbb{P}^1$ branched along six distinct points. Fix a base point $s \in \mathcal{M}_{1,6}$, and let $C := f^{-1}(s)$ be the fiber over $s$. Then we
have the monodromy representation \( \rho : \pi_1(\mathfrak{M}_{1,6}, s) \to Aut(H^1(C, \mathbb{Z}[\omega])_\omega, h) \). Since 
\( (H^1(C, \mathbb{Z}[\omega])_\omega, h) \simeq (\Lambda, h) \), we can view the monodromy group \( \Gamma := \rho(\pi_1(\mathfrak{M}_{1,6}, s)) \) as a subgroup of \( Aut(\Lambda, h) \).

In order to describe \( \Gamma \), we first introduce some notations. Let \( \theta := \omega - \bar{\omega} = \sqrt{3}i. \) Then \( V := \Lambda/\theta \Lambda \) is a four dimensional vector space over the finite field \( \mathbb{F}_3 \simeq \mathbb{Z}[\omega]/\theta \mathbb{Z}[\omega] \), and \( h \) reduces to a quadratic form \( q \) on \( V \). Let \( \psi : Aut(\Lambda, h) \to Aut(V, q) \) be the natural reduction map, and let \( v : Aut(V, q) \to \mathbb{F}_3^*/\mathbb{F}_3^{*2} \simeq \mathbb{Z}/2\mathbb{Z} \) be the spinor norm. Define the following notations:

\[
\begin{align*}
Aut^+(V, q) &:= kerv \\
Aut^+(\Lambda, h) &:= ker(v \circ \psi) \\
\Gamma_\theta &:= ker\psi \\
S &:= \{ \pm 1, \pm \omega, \pm \omega^2 \} \subset Aut(\Lambda, h) \\
S_0 &:= \{ 1, \omega, \omega^2 \} \subset S \\
P\Aut(\Lambda, h) &:= Aut(\Lambda, h)/S \\
P\Gamma_\theta &:= \Gamma_\theta/S_0
\end{align*}
\]

**Proposition 3.1.** \( \Gamma \) is a subgroup of \( \Gamma_\theta \), and \( \Gamma_\theta \) is generated by \( \Gamma \) and \( \omega \).

As a direct consequence, we have the following corollary.

**Corollary 3.2.** The projectified monodromy representation

\[
P\rho : \pi_1(\mathfrak{M}_{1,6}, s) \to P\Aut(\Lambda, h)
\]

has image \( P\Gamma_\theta \).

Before starting the proof of Proposition 3.1, we first introduce some auxiliary spaces and families of curves over them. Define \( M' := \{ (z_1, \cdots, z_6) \in \mathbb{C}^6 | \forall i \neq j, \ z_i \neq z_j. \} \). Obviously the permutation group \( S_6 \) acts on \( M' \). Let \( \tilde{M}' := M'/S_6 \) be the quotient space. Similarly as the universal family \( \mathcal{C} \overset{\pi}{\to} \mathfrak{M}_{1,6} \), we have a universal family \( \mathcal{C}' \to M' \), such that for each \( s = (z_1, \cdots, z_6) \in M' \), the fiber \( \mathcal{C}'_s \) is the cyclic triple cover of \( \mathbb{P}^1 \) branched along the six points \( z_1, \cdots, z_6 \). Obviously this family \( \mathcal{C}' \to M' \) descends to a family \( \tilde{\mathcal{C}}' \to \tilde{M}' \) over \( \tilde{M}' \). Fixing base points \( s \in M' \) and \( \bar{s} \in \tilde{M}' \), we also have the monodromy representations \( \rho : \pi_1(M', s) \to Aut(\Lambda, h) \) and \( \rho : \pi_1(\tilde{M}', \bar{s}) \to Aut(\Lambda, h) \).

**Lemma 3.3.** The monodromy groups can be determined as follows:
(1) The monodromy group of the family \( \mathcal{C}' \to \bar{M}' \) over \( \bar{M}' \) is \( \rho(\pi_1(\bar{M}', \bar{s})) = \text{Aut}^+(\Lambda, h) \).

(2) The monodromy group of the family \( \mathcal{C}' \to M' \) over \( M' \) is \( \rho(\pi_1(M', s)) = \Gamma_\theta \).

**Proof.** (1) It is well known that the fundamental group \( \pi_1(\bar{M}', \bar{s}) \) is isomorphic to the braid group \( B_6 \). In order to describe it, we suppose the base point \( \bar{s} \) represents the unordered subset \( B = \{1, 2, \cdots, 6\} \) of \( \mathbb{C} \). It is a standard fact that \( B_6 \) is isomorphic to \( \text{Mod}_c(\mathbb{C}, B) \), the compactly-supported mapping-class group of the pair \((\mathbb{C}, B)\). Moreover, \( B_6 \) admits standard generators \( \tau_1, \cdots, \tau_5 \), where \( \tau_i \) denotes the right Dehn half-twist around a loop enclosing the interval \([i, i+1]\) in \( \mathbb{C} \). These five generators satisfy the braid relation

\[
\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}
\]

for \( i = 1, \cdots, 5 \), as well as the commutation relation \( \tau_i \tau_j = \tau_j \tau_i \) for \( |i - j| > 1 \); and these generators and relations give a presentation for \( B_6 \).

As for the monodromy action of \( B_6 \) on \( \Lambda \), we can see \( \tau_i \) acts on \( \Lambda \) as a \(-\omega\)-reflection. Then an application of Lemma (7.12) in [1] shows that \( \rho(\pi_1(\bar{M}', \bar{s})) = \text{Aut}^+(\Lambda, h) \).

(2) We know that \( \pi_1(M', s) \simeq PB_6 \), the pure braid group of six points, and \( PB_6 \) is generated by \( \tau_1^2, \tau_2^2, \cdots, \tau_5^2 \). Moreover, \( PB_6 \) is normal subgroup of \( B_6 \), and the quotient group \( B_6/PB_6 \) is isomorphic to \( S_6 \), the permutation group of six elements.

By the arguments in (1), we see \( \rho(\tau_i) \) is a \(-\omega\)-reflection in \( \Lambda \), for \( i = 1, 2, \cdots, 5 \). So \( \rho(\tau_i^2) \) is a \( \omega\)-reflection in \( \Lambda \), for \( i = 1, 2, \cdots, 5 \), and hence by the definition of \( \Gamma_\theta \), the monodromy group \( \rho(\pi_1(M', s)) \) is contained in \( \Gamma_\theta \). By Lemma (4.5) in [1], we have a short exact sequence of group \( 1 \to \Gamma_\theta \to \text{Aut}^+(\Lambda, h) \to \text{Aut}^+(V, q) \to 1 \). Then we get

\[
[\text{Aut}^+(\Lambda, h) : \rho(\pi_1(M', s))] \geq [\text{Aut}^+(\Lambda, h) : \Gamma_\theta] = |\text{Aut}^+(V, q)| = 720.
\]

On the other hand,

\[
[\text{Aut}^+(\Lambda, h) : \rho(\pi_1(M', s))] = [\rho(\pi_1(\bar{M}', \bar{s})) : \rho(\pi_1(M', s))] \leq [B_6 : PB_6] = |S_6| = 720.
\]

So we obtain \( \rho(\pi_1(M', s)) = \Gamma_\theta \).

□
Remark 3.4. As a byproduct of the proof of Lemma 3.3 we get the commutative diagram:

\begin{equation}
\begin{array}{cccccc}
1 & \longrightarrow & PB_6 & \longrightarrow & B_6 & \longrightarrow & S_6 & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \Gamma_\theta & \longrightarrow & Aut^+(\Lambda, h) & \longrightarrow & Aut^+(V, q) & \longrightarrow & 1
\end{array}
\end{equation}

(3.4.1)

where the rows are short exact sequences and the homomorphism $S_6 \to Aut^+(V, q)$ is an isomorphism.

Proof of Proposition 3.1:

Let $M'':=\{(z_1, \cdots, z_5) \in \mathbb{C}^5|z_i \neq z_j, \forall i \neq j\}$ be the moduli space of five distinct ordered points in $\mathbb{C}$. Let $\mathcal{C}'' \to M''$ be the family of smooth projective curves whose affine model is defined by $y^3 = \prod_{i=1}^{5}(x - z_i)$.

We have the following inclusion of moduli spaces:

$$\mathfrak{M}_{1,6} \hookrightarrow M''$$

Here we identify $\mathfrak{M}_{1,6}$ as the space $\{(z_1, z_2, z_3) \in (\mathbb{C}\backslash\{0, 1\})^3|z_i \neq z_j, \forall i \neq j\}$, and $i$ maps a point $(z_1, z_2, z_3)$ to the point $(0, 1, z_1, z_2, z_3)$ in $M''$.

It is easy to see that, through the map $i$, the space $M''$ is homeomorphism to the product space $\mathfrak{M}_{1,6} \times \mathbb{C} \times \mathbb{C}^*$. Moreover, the restriction $\mathcal{C}''|_{\mathfrak{M}_{1,6}}$ is isomorphic to the family $\mathcal{C}$ over $\mathfrak{M}_{1,6}$. So the monodromy group of $\mathcal{C}''|_{\mathfrak{M}_{1,6}} \to \mathfrak{M}_{1,6}$ is also $\Gamma$. On the other hand, it can be seen directly from the construction that, the monodromy groups of the two families $\mathcal{C}'' \to M''$ and $\mathcal{C} \to M'$ are isomorphic. We then identify these two monodromy groups, and by Lemma 3.3 (2), we know the monodromy group of $\mathcal{C}'' \to M''$ is $\Gamma_\theta$.

Now we compare the monodromy groups of $\mathcal{C}'' \to M''$ and its restriction $\mathcal{C}''|_{\mathfrak{M}_{1,6}} \to \mathfrak{M}_{1,6}$. Since $M'' \simeq \mathfrak{M}_{1,6} \times \mathbb{C} \times \mathbb{C}^*$, by fixing a base point $s = (z_1, \cdots, z_5) \in \mathfrak{M}_{1,6}$, the fundamental group $\pi_1(M'', s)$ is generated by $\pi_1(\mathfrak{M}_{1,6}, s)$ and a loop $\mu : \theta \mapsto (e^{i\theta}z_1, \cdots, e^{i\theta}z_5)$, $0 \leq \theta \leq 2\pi$. It can be verified directly that the monodromy action induced by $\mu$ is $\rho(\mu) = \omega \in \Gamma_\theta$. So we obtain that the monodromy group of the family $\mathcal{C}'' \to M''$ is generated by $\omega$ and the monodromy group of $\mathcal{C}''|_{\mathfrak{M}_{1,6}} \to \mathfrak{M}_{1,6}$. This in turn implies that $\Gamma_\theta$ is generated by $\Gamma$ and $\omega$. 
Now we describe the period map of the family \( f : C \to \mathcal{M}_{1,6} \). Recall \((\Lambda, h)\) is a \(\mathbb{Z}[\omega]\)-lattice of signature \((3, 1)\). Let \(B_3 := \{v \in \mathbb{P}(\Lambda \otimes \mathbb{Z}[\omega] \otimes \mathbb{C}) | h(v, v) < 0\}\), which is isomorphic to the three dimensional unit ball \(B_3\). For any point \(s \in \mathcal{M}_{1,6}\), the space \(H^{1,0}(C_s, \mathbb{C}) := \{\alpha \in H^1(C, \mathbb{C}) | \sigma^*\alpha = \bar{\omega}\alpha\}\) is a one-dimensional linear space over \(\mathbb{C}\). By associating to \(s\) the line in \(H^1(C_s, \mathbb{C})\) generated by \(H^{1,0}(C_s, \mathbb{C})\), we get a well defined holomorphic map (called the period map):

\[
P_C : \mathcal{M}_{1,6}^{uni} \to B_3
\]

where \(\mathcal{M}_{1,6}^{uni}\) is the universal cover of \(\mathcal{M}_{1,6}\).

By definition, \(P_C\) is equivariant under the projectified monodromy representation:

\[
P\rho : \pi_1(\mathcal{M}_{1,6}, s) \to P\text{Aut}(\Lambda, h).
\]

Let \(K\) be the kernel of \(P\rho\), then \(P_C\) descends to the following holomorphic map, still denoted by \(P_C\)

\[
\tilde{\mathcal{M}}_{1,6} := \mathcal{M}_{1,6}^{uni}/K \to B_3.
\]

Let \(\mathcal{M}_{1,6}^s := \{f : \{1, 2, \ldots, 6\} \to (\mathbb{P}^1)^6 | \forall a \in \mathbb{P}^1, \#f^{-1}(a, a, \ldots, a) \leq 2\}/\text{PGL}(2)\) be the moduli space of stable six ordered points on \(\mathbb{P}^1\). Then \(\mathcal{M}_{1,6}^s\) is a smooth complex manifold and \(\tilde{\mathcal{M}}_{1,6}^s \setminus \mathcal{M}_{1,6}^s\) is a normal crossing divisor. If we denote \(\tilde{\mathcal{M}}_{1,6} \to \tilde{\mathcal{M}}_{1,6}^s\) as the Fox completion of \(\tilde{\mathcal{M}}_{1,6} \to \mathcal{M}_{1,6}\), then By [4], the period map \(P_C : \tilde{\mathcal{M}}_{1,6} \to B_3\) extends to an isomorphism \(\tilde{\mathcal{M}}_{1,6} \to B_3\). The following proposition is the starting point of our global Torelli theorem.

**Proposition 3.5.** The period mapping induces a bijective map: \(\mathcal{M}_{1,6}^s/S_6 \to B_3/\text{Aut}(\Lambda, h)\).

**Proof.** By Corollary 3.2, the covering \(\tilde{\mathcal{M}}_{1,6} \to \mathcal{M}_{1,6}\) is a Galois cover with deck transformation group \(P\Gamma_\theta\), so we have \(\mathcal{M}_{1,6}^s \simeq \tilde{\mathcal{M}}_{1,6}^s/P\Gamma_\theta \simeq B_3/P\Gamma_\theta\). We have seen from the diagram \(3.4.1\) that \(P\text{Aut}(\Lambda, h)/P\Gamma_\theta \simeq \text{Aut}^+(V, q) \simeq S_6\), so we get \(\mathcal{M}_{1,6}^s/S_6 \simeq B_3/P\text{Aut}(\Lambda, h) = B_3/\text{Aut}(\Lambda, h)\). \(\square\)

4. The monodromy group and period map: Calabi-Yau threefold case

In this section, we analyze the monodromy group and period map of the universal family \(f_3 : \mathcal{X}_{AR} \to \mathcal{M}_{3,6}\), which is the family of cyclic triple covers of \(\mathbb{P}^3\) branched along six hyperplane arrangements in general position. Our strategy is to use the correspondence between this family and the family of curves considered in the previous section.
Let $H_1, \ldots, H_6$ be six hyperplanes in general position in $\mathbb{P}^3$. Let $X$ be the cyclic triple cover of $\mathbb{P}^3$ branched along the divisor $\sum_{i=1}^{6} H_i$. Similarly with the curve case, we have a natural $\mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle$ action on $X$, and we have the eigen-subspace decomposition

$$H^3(X, \mathbb{Z}[\omega]) = H^3(X, \mathbb{Z}[\omega])_\omega \oplus H^3(X, \mathbb{Z}[\omega])_{\bar{\omega}}$$

where

$$H^3(X, \mathbb{Z}[\omega])_\omega := \{ \alpha \in H^3(X, \mathbb{Z}[\omega]) | \sigma^* \alpha = \omega^i \alpha \}; \quad i = 1, 2.$$

By results in Section 2, we know that there exist six distinct points $p_1, \ldots, p_6$ on $\mathbb{P}^1$, such that $H_i$ can be identified with $\{ p_i \} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Moreover, let $C$ be the cyclic triple cover of $\mathbb{P}^1$ branched along $p_1, \ldots, p_6$, then the correspondence between $C$ and $X$ shows that the lattice $(H^3(X, \mathbb{Z}[\omega]), h)$ and $(H^1(C, \mathbb{Z}[\omega]), h)$ are isomorphic, both of which are rank four $\mathbb{Z}[\omega]$-lattice with signature $(3, 1)$. Here the Hermitian form $h$ on $H^3(X, \mathbb{Z}[\omega])$ is defined by $h(\alpha, \beta) = -i Q(\alpha, \bar{\beta})$, in the same way as the curve case.

Let $(\Lambda, h)$ be a $\mathbb{Z}[\omega]$-lattice of signature $(3, 1)$, and let $\rho : \pi_1(\mathcal{M}_{3,6}, s) \to Aut(\Lambda, h)$ be the monodromy representation of the family $f_3$. A family version of the correspondence in Section 2 shows that under the association isomorphism $\phi : \mathcal{M}_{1,6} \to \mathcal{M}_{3,6}$, if $s_2 = \phi(s_1)$, then

$$X_{AR,s_2} \simeq \frac{C_{s_3} \times C_{s_1} \times C_{s_1}}{N \rtimes S_3}.$$

The isometry $\phi_\Omega : (\Lambda_{X_{s_2}}, h) \xrightarrow{\sim} (\Lambda_{C_{s_1}}, h)$ implies the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(\mathcal{M}_{1,6}, s) & \xrightarrow{\phi_*} & \pi_1(\mathcal{M}_{3,6}, \phi(s)) \\
\downarrow{\phi} & & \downarrow{\phi} \\
P Aut(\Lambda, h) & & \end{array}$$

(4.0.1)

Keeping the same notations as in Section 3, the commutative diagram (4.0.1) and Corollary (3.2) give the following proposition.

**Proposition 4.1.** The projectified monodromy representation

$$P \rho : \pi_1(\mathcal{M}_{3,6}, s) \to P Aut(\Lambda, h)$$

has image $P \Gamma_\theta$. □
Moreover, by Proposition (2.1), we have the decompositions
\[
H^3(X, \mathbb{C})_\omega = H^{3,0}(X, \mathbb{C}) \oplus H^{2,1}(X, \mathbb{C})
\]
(4.1.1)
\[
H^3(X, \mathbb{C})_{\bar{\omega}} = H^{3,0}(X, \mathbb{C}) \oplus H^{2,1}(X, \mathbb{C}).
\]
By associating the isomorphic class of $X$ with the point $[H^{3,0}(X, \mathbb{C})]$ in $\mathbb{B}_3 \simeq \{ v \in \mathbb{P}(\Lambda \otimes_{\mathbb{Z}[\omega]} \mathbb{C}| h(v,v) < 0) \}$, we get the period map $P_X : \mathcal{M}^{uni}_{3,6} \to \mathbb{B}_3$, where $\mathcal{M}^{uni}_{3,6}$ is the universal cover of $\mathcal{M}_{3,6}$. As in the curve case, the period map $P_X$ is equivariant under the projectified monodromy representation $P\rho : \pi_1(\mathcal{M}_{3,6}, s) \to P\text{Aut}(\Lambda, h)$. Let $K$ be the kernel of $P\rho$, then $P_X$ descends to the holomorphic map $\tilde{\mathcal{M}}_{3,6} := \mathcal{M}^{uni}_{3,6}/K \to \mathbb{B}_3$, still denoted by $P_X$.

The association isomorphism $\phi : \mathcal{M}_{1,6} \to \mathcal{M}_{3,6}$ gives the commutative diagram relating period maps:

\[
\begin{array}{ccc}
\mathcal{M}_{1,6}^{uni} & \xrightarrow{\sim} & \mathcal{M}_{3,6}^{uni} \\
\text{P}_X & \searrow & \text{P}_X \\
& \mathbb{B}_3 & \\
\end{array}
\]

This diagram descends to the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{\mathcal{M}}_{1,6} & \xrightarrow{\sim} & \tilde{\mathcal{M}}_{3,6} \\
\text{P}_{\tilde{X}} & \searrow & \text{P}_{\tilde{X}} \\
& \mathbb{B}_3 & \\
\end{array}
\]

(4.1.2)

Recall $\mathcal{M}_{1,6}^s$ is the moduli space of stable six ordered points on $\mathbb{P}^1$. Let $\mathcal{M}_{3,6}^s$ be the moduli space of stable six ordered hyperplanes in $\mathbb{P}^3$, which consists six ordered hyperplanes in $\mathbb{P}^3$ with at worst four-fold intersection point. Then the association isomorphism $\mathcal{M}_{1,6} \xrightarrow{\sim} \mathcal{M}_{3,6}$ extends to an isomorphism $\mathcal{M}_{1,6}^s \xrightarrow{\sim} \mathcal{M}_{3,6}^s$, and further extends to an isomorphism between Fox completions:

\[
\begin{array}{ccc}
\tilde{\mathcal{M}}_{1,6}^s & \xrightarrow{\phi_{ass}} & \tilde{\mathcal{M}}_{3,6}^s \\
& \mathcal{M}_{1,6}^s & \xrightarrow{\sim} \mathcal{M}_{3,6}^s \\
\end{array}
\]

We have the following proposition.
Proposition 4.2. There exists a unique isomorphism $\tilde{M}_{3,6}^{s} \xrightarrow{P\chi} \mathbb{B}_{3}$ extending the period map $\tilde{M}_{3,6}^{s} \xrightarrow{P\chi} \mathbb{B}_{3}$. Moreover, the following diagram is commutative:

Proof. By [4], the period map $P_{\chi} : M_{1,6} \rightarrow \mathbb{B}_{3}$ extends to an isomorphism $\tilde{M}_{1,6}^{s} \xrightarrow{\sim} \mathbb{B}_{3}$. Since $\tilde{M}_{3,6}^{s} \setminus \tilde{M}_{3,6}^{s}$ is a normal crossing divisor, the extendability and uniqueness follow from Riemann’s extension theorem. The commutative diagram follows from the diagram (4.1.2).

Similarly as Proposition 3.5, we have the following proposition.

Proposition 4.3. The period mapping induces a bijective map: $M_{3,6}^{s} / S_{6} \xrightarrow{\sim} \mathbb{B}_{3} / \text{Aut}(\Lambda, h)$.

As a corollary, we have the following global Torelli type theorem.

Theorem 4.4. Suppose $a = (H_{1}, \cdots, H_{6})$ and $b = (H'_{1}, \cdots, H'_{6})$ are two hyperplane arrangements in general position in $\mathbb{P}^{3}$. Let $X_{a}$ (resp. $X_{b}$) be the cyclic triple cover of $\mathbb{P}^{3}$ branched along $a$ (resp. $b$). Then the polarized $\mathbb{Z}$-Hodge structures $H^{3}(X_{a}, \mathbb{Z})$ and $H^{3}(X_{b}, \mathbb{Z})$ are isomorphic if and only if after a permutation, the hyperplane arrangements $\{H_{1}, \cdots, H_{6}\}$ and $\{H'_{1}, \cdots, H'_{6}\}$ are projectively equivalent.

Proof. Let $\phi : H^{3}(X_{a}, \mathbb{Z}) \xrightarrow{\sim} H^{3}(X_{b}, \mathbb{Z})$ be an isomorphism of polarized $\mathbb{Z}$-Hodge structures. By the decomposition (4.1.1), we see that $\phi$ is compatible with the $\mathbb{Z}/3\mathbb{Z}$-actions. Then $\phi$ induces an isomorphism $H^{3}(X_{a}, \mathbb{Z}[\omega])_{\omega} \xrightarrow{\sim} H^{3}(X_{b}, \mathbb{Z}[\omega])_{\omega}$. From this we see $a$ and $b$ have the same image under the period map $M_{3,6}^{s} \rightarrow \mathbb{B}_{3}/\text{Aut}(\Lambda, h)$. Then Proposition 4.3 implies $a$ and $b$ represent the same point in $M_{3,6}^{s} / S_{6}$, which means exactly that after a permutation, the hyperplane arrangements $a$ and $b$ are projectively equivalent.

5. Analysis of stable degenerations

In this section, we want to extend the global Torelli type Theorem 4.4 to the stable hyperplane arrangement case.
We first analyze boundary correspondence under the period mapping. Recall \((\Lambda, h)\) is a rank four \(\mathbb{Z}[\omega]\) lattice with signature \((3, 1)\), and we realize \(\mathcal{B}_3\) as the open subset of \(\mathbb{P}(\Lambda \otimes \mathbb{Z}[\omega] \mathbb{C})\) consisting of negative lines. We call a vector \(r \in \Lambda\) a short root, if \(h(r, r) = 1\). Denote \(R\) for the set of short roots in \(\Lambda\). For any \(r \in R\), define the hyperplane orthogonal to \(r\):

\[
H_r := \{ [v] \in \mathcal{B}_3 \subset \mathbb{P}(\Lambda \otimes \mathbb{Z}[\omega] \mathbb{C}) | v \in \Lambda, \ h(v, r) = 0 \}.
\]

We write \(\mathcal{H} := \cup_{r \in R} H_r\). Proposition 4.2 gives an isomorphism \(P_X : \widehat{\mathcal{M}}_{3,6} \rightarrow \mathcal{B}_3\).

**Proposition 5.1.** \(P_X\) induces isomorphisms \(\widehat{\mathcal{M}}_{3,6} \rightarrow \mathcal{B}_3 \setminus \mathcal{H}\) and \(\widehat{\mathcal{M}}_{3,6} \setminus \widehat{\mathcal{M}}_{3,6} \rightarrow \mathcal{H}\).

**Proof.** Note first that \(\widehat{\mathcal{M}}_{3,6} \setminus \widehat{\mathcal{M}}_{3,6}\) is a union of 15 irreducible hypersurfaces in the complex manifold \(\widehat{\mathcal{M}}_{3,6}\). Since \(P_X : \widehat{\mathcal{M}}_{3,6} \rightarrow \mathcal{B}_3\) is a isomorphism, it suffices to show \(P_X(\mathcal{M}_{3,6}) \subset \mathcal{B}_3 \setminus \mathcal{H}\). If \(x \in \mathcal{M}_{3,6}\) and \(P_X(x) \in H_r\) for some \(r \in R\), then we can choose an \(\omega\)-reflection \(\alpha_r\) along \(r\) in \(P \Gamma_\theta\). In particular, \(r\) is a fixed point of \(\alpha_r\). Next we choose \(\gamma_r \in \pi_1(\mathcal{M}_{3,6}, s)\) satisfying \(P \rho(\gamma_r) = \alpha_r\). Since the period mapping \(\widehat{\mathcal{M}}_{3,6} \xrightarrow{P_X} \mathcal{B}_3\) is \(\pi_1(\mathcal{M}_{3,6}, s)\)-equivariant, we see \(P_X(\gamma_r(x)) = \alpha_r(P_X(x)) = P_X(x)\). Since \(P_X\) is injective, we see \(\gamma_r(x) = x\). Note the cover \(\mathcal{M}_{3,6} \rightarrow \mathcal{M}_{3,6}\) is Galois, so any nontrivial deck transformation has no fixed points. This implies \(\gamma_r\) belongs to the kernel of the monodromy representation \(P \rho : \pi_1(\mathcal{M}_{3,6}, s) \rightarrow P \Gamma_\theta\), and hence \(\alpha_r = P \rho(\gamma_r) = id\) is the identity element in \(P \Gamma_\theta\). This contradicts with the chosen of \(\alpha_r\). So we get \(P_X(\mathcal{M}_{3,6} \setminus \mathcal{M}_{3,6}) \subset \mathcal{H}\). \(\square\)

In order to give a geometric interpretation of the period mapping \(P_X\) on the boundary \(\mathcal{M}_{3,6} \setminus \mathcal{M}_{3,6}\), we study the Hodge structure on cyclic triple covers of \(\mathbb{P}^3\) branched along stable six hyperplanes. For \(a = (H_1, \cdots, H_6) \in \mathcal{M}_{3,6} \setminus \mathcal{M}_{3,6}\), we can still define \(X_a\) to be the cyclic triple cover of \(\mathbb{P}^3\) branched along the divisor \(\sum_{i=1}^6 H_i\).

**Proposition 5.2.** For each \(a \in \mathcal{M}_{3,6}\), Deligne’s mixed Hodge structure on \(H^3(X_a, \mathbb{Q})\) is pure.

**Proof.** Denote the homogeneous coordinates on \(\mathbb{P}^3\) by \([X_0 : \cdots : X_3]\), and for \(1 \leq i \leq 6\), let \(\ell_i\) be the defining homogeneous linear equation of the hyperplane \(H_i\). Over the rational function field \(K(\mathbb{P}^3)\) of \(\mathbb{P}^3\), there exists the finite Galois extension \(L := K(\mathbb{P}^3)(\sqrt[12]{\ell_1}, \cdots, \sqrt[6]{\ell_6})\). Define \(Y_a\) to be the normalization of \(\mathbb{P}^3\) in the Galois extension field \(L\). It is not hard to see that \(Y_a\) is a complete intersection of two degree three
hypersurfaces in \( \mathbb{P}^5 \), and \( Y_a \) is smooth if \( a \in \mathcal{M}_{3,6} \). Moreover, the finite abelian group \( N_1 \), which is defined as the kernel of the summation homomorphism \( \sum_{i=0}^{5} \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \), acts on \( Y_a \), and \( X_a \) is isomorphic to the quotient variety \( Y_a/N_1 \). For details of these claims, one can see [12], section 2.2.

Since \( X_a \cong Y_a/N_1 \), we obtain that the (mixed) Hodge structure \( H^3(X_a, \mathbb{Q}) \) is isomorphic to \( H^3(Y_a, \mathbb{Q})^{-N_1} \), the \( N_1 \)-invariant part of \( H^3(Y_a, \mathbb{Q}) \).

If \( a \in \mathcal{M}_{3,6} \), then the mixed Hodge structure on \( H^3(Y_a, \mathbb{Q}) \) is pure, since in this case \( Y_a \) is a smooth projective variety. So the mixed Hodge structure on \( H^3(X_a, \mathbb{Q}) \cong H^3(Y_a, \mathbb{Q})^{-N_1} \) is also pure.

If \( a = (H_1, \cdots, H_6) \in \mathcal{M}_{3,6} \setminus \mathcal{M}_{3,6} \), we can see that \( Y_a \) has only isolated singularities. In order to show the mixed Hodge structure on \( H^3(X_a, \mathbb{Q}) \cong H^3(Y_a, \mathbb{Q})^{-N_1} \) is pure, we can assume, without loss of generality, that the first four hyperplanes \( H_1, H_2, H_3, H_4 \) pass through a common point \( p = [1:0:0:0] \in \mathbb{P}^3 \), and \( \sum_{i=1}^{6} H_i \) is a normal crossing divisor on \( \mathbb{P}^3 \setminus \{p\} \). By an automorphism of \( \mathbb{P}^3 \), we can assume the defining equations of \( H_i \) (\( 1 \leq i \leq 6 \)) are the following:

\[
\ell_1 = X_0; \quad \ell_2 = X_1; \quad \ell_3 = X_2; \quad \ell_4 = X_3; \quad \ell_5 = X_1 + X_2 + X_3; \quad \ell_6 = X_0 + b_1 X_1 + b_2 X_2 + b_3 X_3.
\]

Here \( b_i \) (\( 1 \leq i \leq 3 \)) are complex numbers. Then we can see \( Y_a \) is the complete intersection in \( \mathbb{P}^5 \) defined by the following two homogeneous equations:

\[
Y_4^3 = Y_1^3 + Y_2^3 + Y_3^3; \\
Y_5^3 = Y_0^3 + b_1 Y_1^3 + b_2 Y_2^3 + b_3 Y_3^3
\]

where \( [Y_0: \cdots : Y_5] \) are the homogeneous coordinates on \( \mathbb{P}^5 \). The quotient morphism from \( Y_a \) to \( X_a \) is

\[
\pi : Y_a \to X_a \\
[Y_0: \cdots : Y_5] \mapsto [Y_0^3 : Y_1^3 : Y_2^3 : Y_3^3].
\]

The finite abelian group \( N_1 = ker(\sum_{i=0}^{5} \mathbb{Z}/3\mathbb{Z} \to \sum_{i=0}^{5} \mathbb{Z}/3\mathbb{Z}) \) acts on \( Y_a \) by the following way:

\[
N_1 \times Y_a \to Y_a \\
((a_0, \cdots, a_5), [Y_0: \cdots : Y_5]) \mapsto [\omega^{a_0} Y_0 : \cdots : \omega^{a_5} Y_5]
\]

where \( \omega \) is a primitive cubic root of unity.
It is easy to see the singular subset of $Y_a$ is $\pi^{-1}([1 : 0 : 0 : 0]) = \{[1 : 0 : 0 : 0 : \omega^i]| i = 0, 1, 2\}$. Blowing up $Y_a$ along these singular points, we get a smooth projective variety $\tilde{Y}_a$, and we can see the exceptional divisor $E$ on $\tilde{Y}_a$ is a disjoint union of smooth cubic surfaces.

Now we take a sufficient small open neighborhood $V$ of $[1 : 0 : 0 : 0]$ in $\mathbb{P}^3$, such that $V$ is biholomorphic to an open ball. Let $U_1$ be the inverse image of $V$ in $\tilde{Y}_a$, and let $U_2 = \tilde{Y}_a \setminus E$. Then $\tilde{Y}_a = U_1 \cup U_2$ and we get the following exact sequence from the Meyer-Vietoris sequence:

$$H^2(U_1 \cap U_2, \mathbb{Q})^{N_1} \rightarrow H^3(\tilde{Y}_a, \mathbb{Q})^{N_1} \rightarrow H^3(U_1, \mathbb{Q})^{N_1} \oplus H^3(U_2, \mathbb{Q})^{N_1} \rightarrow H^3(U_1 \cap U_2, \mathbb{Q})^{N_1}.$$ 

It is not difficult to see that the exceptional divisor $E$ is a deformation retract of $U_1$. Since $E$ is a disjoint union of smooth cubic surfaces, we get $H^3(U_1, \mathbb{Q}) \simeq H^3(E, \mathbb{Q}) = 0$. On the other hand, we can see $U_2/N_1 \simeq X_a \setminus \{p\}$, where $p$ is the inverse image of $[1 : 0 : 0 : 0] \in \mathbb{P}^3$. This implies the isomorphism $H^3(U_2, \mathbb{Q})^{N_1} \simeq H^3(X_a \setminus \{p\}, \mathbb{Q})$.

Next we consider $H^2(U_1 \cap U_2, \mathbb{Q})^{N_1}$ and $H^3(U_1 \cap U_2, \mathbb{Q})^{N_1}$. Let $Z$ by the hypersurface in $\mathbb{C}^4$ defined by the equation $x_1^3 = x_1x_2x_3(x_1 + x_2 + x_3)$, then we can see the quotient space $U_1 \cap U_2/N_1$ is homotopic to $Z \setminus \{(0, 0, 0, 0)\}$. Since a direct computation shows that both the cohomology groups $H^2(Z \setminus \{(0, 0, 0, 0)\}, \mathbb{Q})$ and $H^3(Z \setminus \{(0, 0, 0, 0)\}, \mathbb{Q})$ are vanishing, we see $H^2(U_1 \cap U_2, \mathbb{Q})^{N_1} = H^3(U_1 \cap U_2, \mathbb{Q})^{N_1} = 0$. Moreover, a similar Meyer-Vietoris sequence analysis on $X_a$ shows that $H^3(X_a \setminus \{p\}, \mathbb{Q}) \simeq H^3(X_a, \mathbb{Q})$.

Putting the above analysis together, we conclude $H^3(X_a, \mathbb{Q}) \simeq H^3(\tilde{Y}_a, \mathbb{Q})^{N_1}$. Since $\tilde{Y}_a$ is a smooth projective variety, we see the mixed Hodge structure on $H^3(X_a, \mathbb{Q})$ is pure.

□

For a stable hyperplane arrangement $a = (H_1, \cdots, H_6) \in \mathcal{M}_{3,6}^6$, the divisor $\sum_{i=1}^6 H_i$ has at worst four-fold intersection point, and the number of four-fold intersection points is less than or equal to 3. Furthermore, if the number of four-fold intersection points is $k$, then $\dim_{\mathbb{Q}} H^3(X_a, \mathbb{Q}) = 8 - 2k$, and the $(H^3(X_a, \mathbb{Z}[\omega])_{\mathbb{Z}[\omega]}, h)$ is a rank $4 - k$ lattice with signature $(3 - k, 1)$. All these facts can be checked by a careful degeneration analysis as that in the proof of Proposition 5.2. Since it is routine and a little tedious, we omit it here.

Now we can extend Theorem 4.4 to stable hyperplane arrangements.
Theorem 5.3. Suppose \( a = (H_1, \cdots, H_6) \) and \( b = (H'_1, \cdots, H'_6) \) are two stable hyperplane arrangements in \( \mathbb{P}^3 \). Let \( X_a (\text{resp. } X_b) \) be the cyclic triple cover of \( \mathbb{P}^3 \) branched along \( a \) (resp. \( b \)). Then the polarized \( \mathbb{Z} \)-Hodge structures \( H^3(X_a, \mathbb{Z}) \) and \( H^3(X_b, \mathbb{Z}) \) are isomorphic if and only if after a permutation, the hyperplane arrangements \( \{ H_1, \cdots, H_6 \} \) and \( \{ H'_1, \cdots, H'_6 \} \) are projectively equivalent.

Proof. Suppose \( \dim_{\mathbb{Q}} H^3(X_a, \mathbb{Q}) = \dim_{\mathbb{Q}} H^3(X_b, \mathbb{Q}) = 8 - 2k \), with \( k = 0, 1, 2, 3 \). Then we discuss the four cases according to \( k \).

(1) \( k = 0 \): In this case, both \( a \) and \( b \) are in general position, so it follows from Theorem 4.4.

(2) \( k = 1 \): Since \( \mathcal{M}_{3,6}^a \backslash \mathcal{M}_{3,6} \simeq \mathcal{H}/Aut(\Lambda, h) \) is irreducible, we can choose an irreducible component \( H \) of \( \mathcal{M}_{3,6}^a \backslash \mathcal{M}_{3,6} \) and \( \tilde{a}, \tilde{b} \) in \( H \) over \( a \) and \( b \) respectively. Suppose the isomorphism \( P_X : \mathcal{M}_{3,6}^a \rightarrow \mathbb{B}_3 \) maps \( H \) to \( \mathbb{B}_2^r := \{ [v] \in \mathbb{P}(\Lambda^r \otimes_{\mathbb{Z}[\omega]} \mathbb{C}) | h(v, v) < 0 \} \). Here \( r \in R \) is a short root, and \( \Lambda^r := \{ \alpha \in \Lambda | h(\alpha, r) = 0 \} \) is a free \( \mathbb{Z}[\omega] \)-module of rank three. We can see the isomorphism \( P_X : H \rightarrow \mathbb{B}_2^r \) coincides with the period map \( H \rightarrow \mathbb{B}_2^r \) which associates with a point \( c \) in the smooth locus of \( H \) the line \( [H^{3,0}(X_c, \mathbb{C})] \) in \( \mathbb{P}(H^3(X_c, \mathbb{Z}[\omega]) \otimes_{\mathbb{Z}[\omega]} \mathbb{C}) \), where \( X_c \) is the cyclic triple cover of \( \mathbb{P}^3 \) branched along \( c \). Then since the polarized \( \mathbb{Z} \)-Hodge structures \( H^3(X_a, \mathbb{Z}) \) and \( H^3(X_b, \mathbb{Z}) \) are isomorphic, we get \( P_X(\tilde{a}) = P_X(\tilde{b}) \). By Proposition 4.3, this in turn implies \( a \) and \( b \) has the same image in \( \mathcal{M}_{3,6}^a/S_6 \), so after a permutation, the hyperplane arrangements \( a \) and \( b \) are projectively equivalent.

(3) \( k = 2 \): In this case, we can choose two irreducible components \( H_1, H_2 \) of \( \widetilde{\mathcal{M}}_{3,6}^a \backslash \widetilde{\mathcal{M}}_{3,6} \) and points \( \tilde{a}, \tilde{b} \) in \( H_1 \cap H_2 \) over \( a \) and \( b \) respectively. Moreover, if the isomorphism \( P_X : \widetilde{\mathcal{M}}_{3,6}^a \rightarrow \mathbb{B}_3 \) maps \( H_i \) to \( \mathbb{B}_2^{r_i} \), \( i = 1, 2 \), then \( r_1 \) and \( r_2 \) are two perpendicular short roots. Similar as the \( k = 1 \) case, the restriction \( P_X : H_1 \cap H_2 \rightarrow \mathbb{B}_2^{r_1} \cap \mathbb{B}_2^{r_2} \) coincides with the period map by associating a point \( c \) in the smooth locus of \( H_1 \cap H_2 \) the line \( [H^{3,0}(X_c, \mathbb{C})] \) in \( \mathbb{P}(H^3(X_c, \mathbb{Z}[\omega]) \otimes_{\mathbb{Z}[\omega]} \mathbb{C}) \).

Then this case follows from the same argument as in the \( k = 1 \) case.

(4) \( k = 3 \): In this case, both \( a \) and \( b \) have three fourfold intersection points, then an elementary analysis on the configuration of hyperplane arrangements shows that after a permutation, the hyperplane arrangements \( a \) and \( b \) are projectively equivalent.
Appendix A. Comparison with Rohde’s example

Given distinct $a, b, c \in \mathbb{C}\{0, 1\}$, Rohde [10] constructed a singular Calabi-Yau threefold $X'$ in the following way:

Let $W$ be the surface in the weighted projective space $\mathbb{P}(2, 2, 1, 1)$ defined by the equation $y_1^3 + y_2^3 + x_0 x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)(x_1 - cx_0) = 0$. Let $F$ be the Fermat curve in $\mathbb{P}^2$ defined by the homogeneous equation $z_0^3 + z_1^3 + z_2^3 = 0$. Then the cyclic group $G = \mathbb{Z}/3\mathbb{Z}$ acts on $W$ and $F$. Fixing a generator $\sigma$ of $G$ and let $\omega$ be a fixed primitive cubic root of unity. We define these actions explicitly:

$$\sigma : W \to W \quad [x_0 : x_1 : y_1 : y_2] \mapsto [x_0 : x_1 : \omega y_1 : \omega y_2]$$

$$\sigma : F \to F \quad [z_0 : z_1 : z_2] \mapsto [\omega z_0 : z_1 : z_2]$$

Rohde [10] constructs the Calabi-Yau threefold as a crepant resolution of the quotient threefold $X' := W \times F / G$, where $G$ acts on $W \times F$ in the diagonal way. Moreover, varying the parameters $a, b, c \in \mathbb{C}\{0, 1\}$, Rohde obtains a family of Calabi-Yau threefolds $X' \to \mathcal{M}_{1,6}$, where we recall that $\mathcal{M}_{1,6}$ is the moduli space of ordered six distinct points in $\mathbb{P}^1$. The main goal of this section is to show Rohde’s family is birationally equivalent to the family $X'_{AR} \to \mathcal{M}_{3,6}$, which is the universal family of cyclic triple covers of $\mathbb{P}^3$ branched along six hyperplanes in general position.

We first analyze the structure of the singular surface $W$. In general, if $X_1$ and $X_2$ are two varieties with $G$-action, we say $X_1$ and $X_2$ are $G$-birationally equivalent if there exists a birational map from $X_1$ to $X_2$ compatible with the $G$-action.

Let $C$ be the cyclic triple cover of $\mathbb{P}^1$ branched along the six points $\{0, 1, \infty, a, b, c\}$ whose affine model is the curve in $\mathbb{C}^2$ defined by the equation $y^3 - x(x - 1)(x - a)(x - b)(x - c) = 0$. Then $G$ acts on $C$ in the following way:

$$\sigma : C \to C \quad (x, y) \mapsto (x, \omega y)$$

Let $G$ act on the product $C \times F$ diagonally, then $G$ acts on the quotient $C \times F / G$ through the $G$-action on the first factor $C$. 
Lemma A.1. $W$ is $G$-birationally equivalent to the quotient $C \times F/G$.

Proof. Let $F_0$ be the affine surface $\{(z_1, z_2) \in \mathbb{C}^2 | 1 + z_1^3 + z_2^3 = 0\}$, and define a $G$-action on $F_0$ by $\sigma(z_1, z_2) = (\omega z_1, \omega z_2)$. Then $F_0$ is $G$-birational to the Fermat curve $F$. Define the following morphism:

$$C \times F_0 \to W$$

$$(x, y, z_1, z_2) \mapsto (x, z_1y, z_2y)$$

It is easy to see this morphism induces a $G$-birational equivalent between $C \times F_0/G$ and $W$. So $W$ is $G$-birationally equivalent to $C \times F/G$. □

Now we consider the following six hyperplanes in $\mathbb{P}^3$ which are in general position

$$H_i : X_i = 0 \ (0 \leq i \leq 3), \ H_4 : \sum_{i=0}^3 X_i = 0, \ H_5 : X_0 + aX_1 + bX_2 + cX_3 = 0,$$

where $[X_0 : \cdots : X_3]$ is the homogeneous coordinates on $\mathbb{P}^3$. Let $X$ be the cyclic triple cover of $\mathbb{P}^3$ branched along $\sum_{i=0}^5 H_i$.

In order to analyze the structure of $X$, we define some auxiliary varieties.

Let $u_1, v_1$ be linear functions of $u, v$ defined by the following relations:

(A.1.1) \[
\begin{align*}
  u_1 &= 1 + u + v \\
  v_1 &= a + bu + cv.
\end{align*}
\]

We define $Y$ as the following affine variety:

$$Y = \{(t_1, u, v, y_1) \in \mathbb{C}^4 | y_1^3 = \frac{uvt_1(t_1 + 1)}{u_1v_1(v_1 - u_1)}\}.$$

Let $S$ be the following affine surface:

$$S = \{(w, u, v) \in \mathbb{C}^3 | w^3 = \frac{uv}{u_1v_1(v_1 - u_1)}\}.$$

Let $G$ acts on $S$ by $\sigma(w, u, v) = (\omega^2 w, u, v)$.

Lemma A.2. We have the following birational isomorphisms:

1. $X$ is birationally equivalent to $Y$.
2. $Y$ is birationally equivalent to $S \times F/G$, where $G$ acts on $S \times F$ diagonally.
3. $S$ is $G$-birationally equivalent to $C \times F/G$.

Proof. (1) We take the following affine model of $X$:

$$X_1 = \{(x_1, x_2, x_3, y) \in \mathbb{C}^4 | y^3 = x_1x_2x_3(1 + x_1 + x_2 + x_3)(1 + ax_1 + bx_2 + cx_3)\}.$$
Under the coordinate transformation
\[
\begin{align*}
x_1 &= t \\
x_2 &= tu \\
x_3 &= tv \\
y &= y
\end{align*}
\]
the hypersurface \(X_1\) is birationally equivalent to the following hypersurface in \(\mathbb{C}^4\):
\[
X_2 = \{(t, u, v, y) \in \mathbb{C}^4 | y^3 = t^3 uv(1 + u_1 t)(1 + v_1 t)\},
\]
where \(u_1, v_1\) are defined by the equations (A.1.1). Then we can see \(X_2\) is birational to \(Y\) under the following coordinate transformation:
\[
\begin{align*}
u &= u \\
v &= v \\
t &= \left(\frac{1}{v_1} - \frac{1}{u_1}\right)t_1 - \frac{1}{u_1} \\
y &= tu_1v_1\left(\frac{1}{v_1} - \frac{1}{u_1}\right)y_1.
\end{align*}
\]

(2) It is direct to see the following affine curve \(F_1\) is \(G\)-birationally equivalent to the Fermat curve \(F\):
\[
F_1 = \{(t_1, x) \in \mathbb{C}^2 | x^3 = t_1(t_1 + 1)\},
\]
where \(G\) acts on \(F_1\) by \(\sigma(t_1, x) = (t_1, \omega x)\).

Then the following rational map induces the desired \(G\)-birationally equivalence between \(S \times F/G\) and \(Y\):
\[
S \times F_1 \rightarrow Y
\]
\[
(w, u, v, t_1, x) \mapsto (t_1, u, v, wx).
\]

(3) From the equations (A.1.1), we can view \((w, u_1, v_1)\) as coordinate system on \(\mathbb{C}^3\), and we make the following coordinate transformation:
\[
\begin{align*}
w &= w \\
u_1 &= t_2 \\
v_1 &= t_2z.
\end{align*}
\]
Under this coordinate transformation, we see \(S\) is \(G\)-birationally equivalent to the following surface:
\[
S_1 = \{(w, t_2, z) \in \mathbb{C}^3 | w^3 = \frac{(A_1 t_2 + a_1)(B_1 t_2 + b_1)}{t_2^2 z (z - 1)}\},
\]
where
\[
\begin{align*}
A_1 &= \frac{b-c}{b-c} \\
B_1 &= \frac{c-b}{c-b} \\
a_1 &= \frac{a-b}{b-c} \\
b_1 &= \frac{a-c}{c-b},
\end{align*}
\]
and \(G\) acts on \(S_1\) by \(\sigma(w, t_2, z) = (\omega^2 w, t_2, z)\).

A further coordinate transformation:
\[
\begin{align*}
z &= z \\
t_3 &= t_3 + \frac{a_1}{A_1 B_1} \\
w_1 &= \frac{w t_2}{A_1 B_1 (\frac{a_1}{A_1} - \frac{a_1}{B_1})}
\end{align*}
\]

shows that \(S_1\) is \(G\)-birational to the surface:
\[
S_2 = \{(z, t_3, w_1) \in \mathbb{C}^3 | w_1^3 = \frac{t_3(t_3 + 1)}{z(z - 1)A_1B_1(A_1b_1 - B_1a_1)}\},
\]
where \(G\) acts on \(S_2\) by \(\sigma(z, t_3, w_1) = (z, t_3, \omega^2 w_1)\).

Now let \(C_1\) be the following affine curve:
\[
C_1 := \{(z, w_2) \in \mathbb{C}^2 | w_2^3 = z(z - 1)A_1B_1(A_1b_1 - B_1a_1)\},
\]
and let \(G\) act on \(C_1\) by \(\sigma(z, w_2) = (z, \omega^2 w_2)\). We consider the affine model
\[
F_1 = \{(t_3, x) \in \mathbb{C}^2 | x^3 = t_3(t_3 + 1)\}
\]
of the Fermat curve \(F\) as before. The rational map
\[
C_1 \times F_1 \to S_2 \\
(z, w_2, t_3, x) \mapsto (z, t_3, \frac{x}{w_2}).
\]
gives a \(G\)-birationally equivalence between \(C_1 \times F_1/G\) and \(S_2\). Moreover, we see the smooth projective model of \(C_1\) is isomorphic to \(C\), the cyclic triple cover of \(\mathbb{P}^1\) branched along the six points \(\{0, 1, \infty, a, b, c\}\). By combining all of the birational equivalences above, we obtain \(S\) is \(G\)-birationally equivalent to \(C \times F/G\). \(\square\)

Now it is direct to see that, by combining Lemma A.1 and Lemma A.2 we obtain the following birational equivalence.
Proposition A.3. Given distinct \( a, b, c \in \mathbb{C} \setminus \{0, 1\} \), Rohde’s singular Calabi-Yau threefold \( X' = W \times F/G \) is birational to \( X \), which is the cyclic triple cover of \( \mathbb{P}^3 \) branched along \( \sum_{i=0}^{5} H_i \), with \( H_i \) defined by

\[
H_i : X_i = 0 \quad (0 \leq i \leq 3), \quad H_4 : \sum_{i=0}^{3} X_i = 0, \quad H_5 : X_0 + aX_1 + bX_2 + cX_3 = 0.
\]

Note that to give distinct \( a, b, c \in \mathbb{C} \setminus \{0, 1\} \) is equivalent to give six distinct points \( \{0, 1, \infty, a, b, c\} \) in \( \mathbb{P}^1 \). Since the moduli space \( \mathcal{M}_{1,6} \) of ordered six distinct points in \( \mathbb{P}^1 \) is isomorphic to the moduli space \( \mathcal{M}_{3,6} \) of ordered six hyperplane arrangements in general in \( \mathbb{P}^3 \), the following birational equivalence is a direct consequence of Proposition A.3.

Proposition A.4. Rohde’s Calabi-Yau family \( X' \rightarrow \mathcal{M}_{1,6} \) is birationally equivalent to the universal family \( X_{AR} \rightarrow \mathcal{M}_{3,6} \) of cyclic triple covers of \( \mathbb{P}^3 \) branched along six hyperplanes in general position.

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E-mail address: msheng@ustc.edu.cn
E-mail address: xujx02@ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China