SCALING GROUP FLOW AND LEFSCHETZ TRACE FORMULA FOR LAMINATED SPACES WITH $p$–ADIC TRANSVERSAL

ERIC LEICHTNAM

Abstract. In his approach to analytic number theory C. Deninger has suggested that to the Riemann zeta function $\hat{\zeta}(s)$ (resp. the zeta function $\zeta_Y(s)$ of a smooth projective curve $Y$ over a finite field $\mathbb{F}_q$, $q = p^f$) one could possibly associate a foliated Riemannian laminated space $(S_Q, \mathcal{F}, g, \phi^t)$ (resp. $(S_Y, \mathcal{F}, g, \phi^t)$) endowed with an action of a flow $\phi^t$ whose primitive compact orbits should correspond to the primes of $\mathbb{Q}$ (resp. $Y$). Precise conjectures were stated in our report [Lei03] on Deninger’s work. The existence of such a foliated space and flow $\phi^t$ is still unknown except when $Y$ is an elliptic curve (see Deninger [De02]). Being motivated by this latter case, we introduce a class of foliated laminated spaces $(S = L \times \mathbb{R}_q, \mathcal{F}, g, \phi^t)$ where $L$ is locally $D \times \mathbb{Z}_p^m$, $D$ being an open disk of $\mathbb{C}$.

Assuming that the leafwise harmonic forms on $L$ are locally constant transversally, we prove a Lefschetz trace formula for the flow $\phi^t$ acting on the leafwise Hodge cohomology $H^j_\tau(0 \leq j \leq 2)$ of $(S, \mathcal{F})$ that is very similar to the explicit formula for the zeta function of a (general) smooth curve over $\mathbb{F}_q$. We also prove that the eigenvalues of the infinitesimal generator of the action of $\phi^t$ on $H^1_\tau$ have real part equal to $\frac{1}{2}$.

Moreover, we suggest in a precise way that the flow $\phi^t$ should be induced by a renormalization group flow “à la K. Wilson”. We show that when $Y$ is an elliptic curve over $\mathbb{F}_q$ this is indeed the case. It would be very interesting to establish a precise connection between our results and those of Connes (page 553 [Co00], page 90 [Co02]) and Connes-Marcolli [Co-Ma04a, Co-Ma04b] on the Galois interpretation of the renormalization group.

Contents

1. Introduction 2
2. The case of an elliptic curve $E_0$ over $\mathbb{F}_q$ 5
2.1. The zeta function $\zeta_{E_0}(s)$ and the explicit formula 5
2.2. The Riemannian laminated foliated space $(S(E_0), \mathcal{F}, g, \phi^t)$ 5
2.3. Interpretation of $(S(E_0), \mathcal{F}, g, \phi^t)$ as a renormalization group flow 8
2.4. Further remarks and motivation of Section 3 and [Lei06] 9
3. The zeta function of a compact Riemannian foliated transversally $p$–adic laminated space $(S = L \times \mathbb{R}_q, \mathcal{F}, g, \phi^t)$ 12
4. Analytic results on $L$ 18
4.1. Sobolev spaces on $\mathbb{Z}_p^m$ 18
4.2. Sobolev spaces and harmonic forms on $L$ 18
5. Proof of Theorem 2 20
5.1. Leafwise Hodge Decomposition and Heat operator 20
5.2. Proof of Theorem 2.1 22
5.3. Proof of Theorem 2.3 23
6. Appendix: Renormalization group flow 25

Date: October 31, 2018.
1. Introduction

Several papers (see [De98], [De99], [De02], [De01b], [De01]) of Deninger lead to suggest that to the Riemann zeta function \( \zeta(s) \) one could possibly associate the following two data:

1) A Riemannian foliated space of the form

\[
(S_Q = \mathcal{L} \times \mathbb{R}^{*+} \cup \mathcal{Q}_Q^{*+}, \mathcal{F}, g)
\]

where \( \mathcal{L} \) is a \( \sigma \)-compact complex 1-dimensional laminated space on which \( \mathbb{Q}^{*+} \) acts. The path connected components of \( \mathcal{L} \) induce a foliation of \( \mathcal{L} \times \mathbb{R}^{*+} \) by Riemann surfaces and \( g \) is a leafwise kaehler metric.

2) A flow \( \phi^t \) acting on \( (S_Q = \mathcal{L} \times \mathbb{R}^{*+} \cup \mathcal{Q}_Q^{*+}, \mathcal{F}) \) whose primitive closed orbits correspond to the primes of \( \mathbb{Q} \) and admitting a fixed point in \( \mathcal{L} \). The action of \( \phi^t \) on the \( \mathcal{L} \)-leaf space \( \mathcal{L}^{*+} \cup \{pt\} \) should be given by \( \phi^t([x]) = [e^{-t}x] \) and \( \phi^t(pt) = pt \). Moreover \( (\phi^t)^*[\lambda_g] = e^t[\lambda_g] \) where [\( \lambda_g \)] denotes the reduced leafwise cohomology class of the leafwise kaehler form associated to \( g \).

The quotient \( \mathcal{L}^{*+} \) allows to compactify the space \( \mathcal{L} \times \mathbb{R}^{*+} \). For more precise informations and axioms see [Lei03] or even Section 3. Notice that the existence of such a quadruple \( (S_Q, \mathcal{F}, g, \phi^t) \) is still unknown. Recall that Alain Connes [Co99] has reduced the validity of the Riemann hypothesis to a trace formula (of Lefschetz type) on the quotient space \( \frac{\mathcal{L} \times \mathbb{R}}{\mathbb{Q}} \).

Notice nevertheless that \( \frac{\mathcal{L} \times \mathbb{R}}{\mathbb{Q}} \) does not satisfy the properties required for \( (S_Q, \mathcal{F}, g, \phi^t) \).

It is natural to guess that the action of \( \phi^t \) on \( S_Q \) should be induced by the action on \( \mathcal{L} \times \mathbb{R}^{*+} \) of a flow still denoted \( \phi^t \). Given the previous property 2) above, one is lead to search a flow of the form

\[
\phi^t(l, x) = (\psi^t(l), xe^{-t}), \ \forall (l, x) \in \mathcal{L} \times \mathbb{R}^{*+}.
\]

But if we write \( \psi^t = R_{x,e^{-t}} \) then we recognize the general scheme of the method of renormalization group "à la K. Wilson" ([page 554 QFT]), where \( R_{x,e^{-t}} \) should act on a space \( \mathcal{L} \) of lagrangians. We noticed this point in 2002 when we were preparing our lectures (on which [Lei03] is based) on Deninger’s work for Bar Ilan University.

Then we are lead to guess that a similar picture should exist for the zeta function of a smooth projective curve \( Y \) over \( \mathbb{Q} \). Namely that there should exists a Riemannian foliated laminated space \( (S_Y, \mathcal{F}, g, \phi^t) \) where \( \phi^t \) should be a renormalization group flow whose primitive closed orbits should correspond to the closed points of \( Y \).

In Section 2 we consider the case of an elliptic curve \( E_0 \) and the Riemannian foliated laminated space \( (S(E_0), \mathcal{F}, g, \phi^t) \) constructed by Deninger. Then we show that \( (S(E_0), \mathcal{F}, g, \phi^t) \) can be interpreted as a renormalization group flow. Now we motivate our main result and
explain its main context. Recall that the following explicit formula is satisfied for the zeta function \( \zeta_Y(s) \) of \( Y \). Let \( \alpha \in C_0^\infty(\mathbb{R}, \mathbb{R}) \). Then:

\[
\sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t) e^{t \left( \frac{2\pi i \nu}{\log q} \right)} dt - 2g \sum_{j=1}^{2g} \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t) e^{t (\rho_j + \frac{2\pi i \nu}{\log q})} dt + \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t) e^{t \left( \frac{1 + \frac{2\pi i \nu}{\log q}}{\log q} \right)} dt = (2 - 2g)\alpha(0) \log q + \sum_{\gamma} \sum_{k \geq 1} l(\gamma) \left( e^{-kl(\gamma)} \alpha(-kl(\gamma)) + \alpha(kl(\gamma)) \right)
\]

where the \( (\rho_j + \frac{2\pi i \nu}{\log q}) \) run over the zeroes of \( \zeta_Y \) and the \( \gamma \) (with norm \( e^{l(\gamma)} \)) run over the closed points of \( Y \). As pointed out by Deninger, the dissymmetry of the coefficients of \( \alpha(-kl(\gamma)) \) and \( \alpha(kl(\gamma)) \) in the right hand side is of arithmetic nature. It arises when one tries to intertwine the functional equation (addition) and the Eulerian product (multiplication) in the proof of the explicit formula \( (1) \).

Deninger has suggested that it might be possible to interpret the above formula as a Lefschetz trace formula. Having in mind in fact the case of number fields, he made interesting remarks on dynamical Lefschetz trace formulas on laminated foliated spaces see [Section5] [De01].

Consider briefly the case of a compact connected three dimensional manifold \( X \) endowed with a codimension one foliation \((X, \mathcal{F})\). Assume that \((X, \mathcal{F})\) is endowed with a flow \( \phi^t \) which acts transversally and whose closed orbits of \( \phi^t \) are simple, thus \((X, \mathcal{F})\) is Riemannian.

Let \( \pi^j \) denote the projection onto the leafwise Hodge cohomology \( H^j_\pi \). Then Alvarez-Lopez and Kordyukov [A-K00] have proved the following Lefschetz trace formula:

\[
\sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s) \pi^j \circ (\phi^s)^* \circ \pi^j ds = \chi_Y(0) + \sum_{\gamma} \sum_{k \geq 1} l(\gamma) \left( \epsilon_{-k} \alpha(-kl(\gamma)) + \epsilon_k \alpha(kl(\gamma)) \right)
\]

where \( \gamma \) runs over the primitive closed orbits of \( \phi^t \), \( \epsilon_{\pm k} = \text{sign det}(\text{id} - D\!\!\phi^{|\pm kl(\gamma)}_{T_y \mathcal{F}}) \), \( y \in \gamma \) and TR denotes the (usual) trace of trace class operators. Notice that here there is no dissymmetry for the coefficients of \( \alpha(-kl(\gamma)) \) and \( \alpha(kl(\gamma)) \). The reason for this absence of dissymmetry is due to the Guillemin-Sternberg formula [G-S77] which states that the geometric contribution of a closed orbit \( \pm k \gamma \) should be:

\[
\frac{l(\gamma) \alpha(\pm kl(\gamma))}{\sum_{j=0}^{2} (-1)^j \text{TR} \left( (D\!\!\phi^{|\pm kl(\gamma)}_{T_y \mathcal{F}})^* : \wedge^j T^*_y \mathcal{F} \mapsto \wedge^j T^*_y \mathcal{F} \right) / |\text{det}(\text{id} - D\!\!\phi^{|\pm kl(\gamma)}_{T_y \mathcal{F}})|}
\]

where \( y \) is any point on \( \gamma \). In Lemma 5 of Section 2.4 we shall provide in the case of an elliptic curve \( E_0 \) over \( \mathbb{F}_q \) a dynamical explanation of this dissymmetry pointing out the role of the \( p \)-adic transversal in \( S(E_0) \). This will lead us to propose a list of four Assumptions in Section 3 for a Riemannian foliated laminated space

\[
(S = \mathcal{L} \times \mathbb{R}^{+\ast} / q_\mathbb{Z}, \mathcal{F}, g, \phi^t)
\]

where \( \mathcal{L} \) is locally of the form \( D \times \mathbb{Z}_p^n \), \( D \) is a disk of \( \mathbb{C} \). The leaves of \( S \) are induced by \( \mathcal{C} \times \{x\} \) where \( \mathcal{C} \) is a path connected component of \( \mathcal{L} \). Assumption (iv) states that the
elements of the vector space $\mathcal{H}_L^1$ of continuous and leafwise harmonic forms on $L$ are locally constant along the $p$–adic transversal $\mathbb{Z}_p^m$. It implies that the vector space $\mathcal{H}_L^1$ is of finite even dimension $2g$. This Assumption is of course satisfied by the foliated space $(S(E_0), F)$ of Section 2 but we do not know if there are any example which satisfies this Assumption with $g \geq 2$, see the remark before Lemma \[\text{[6]}\] Anyway, we are forced to use this Assumption for the following two reasons. First if we want for $(S = E \times \mathbb{R}^+_{q^m}, F, g, \phi^t)$ a Lefschetz trace formula of the type $[\text{1}]$ then it seems that we need $\mathcal{H}_L^1$ to be of finite dimension $2g$. Second, we shall need the following operator

$$\int_\mathbb{R} \alpha(s)(\phi^t)^*ds \circ \pi^t_x$$

to be trace class. Morally to $\pi^t_x$ (resp. $\int_\mathbb{R} \alpha(s)(\phi^t)^*ds$) corresponds a regularizing process along the leaves $[C \times \{x\}]$ of $S$ (resp. the integral curves of $\phi^t$). But $\int_\mathbb{R} \alpha(s)(\phi^t)^*ds \circ \pi^t_x$ does not involve any regularizing process along the $p$–adic transversal $\mathbb{Z}_p^m$. Therefore, unless Assumption (iv) is satisfied we see no reason why $\int_\mathbb{R} \alpha(s)(\phi^t)^*ds \circ \pi^t_x$ should be trace class.

Then, our main result is Theorem \[\text{[2]}\] It proves a Lefschetz trace formula similar to $[\text{1}]$ and shows that the eigenvalues of the infinitesimal generator of $\pi^t_x \circ (\phi^t)^*$ acting on the leafwise Hodge cohomology $H^1_L$ have real part equal to $\frac{1}{2}$. Our main new ingredient is the transversal $p$–adic Laplacian $\Delta_{p,T}$ on $L$ (see Definition \[\text{[3]}\]). The intrinsic meaning of Assumption (iv) is the inclusion $\mathcal{H}_L^1 \subset \ker \Delta_{p,T}$. Moreover, Assumption (iv) allows to use a ”contraction process” along the $p$–adic transversal of $S$ (see Definition \[\text{[4]}\]). Then we use in essential way results of Alvarez-Lopez and Kordyukov since in some sense our foliated laminated spaces are closed to Riemannian foliations. Even in the case of an elliptic curve $E_0$, it seems of some interest to provide a proof of $[\text{1}]$ à la Atiyah-Bott-Lefschetz and the new ingredients introduced here should be useful in other contexts.

We also show (Proposition \[\text{[3]}\]) that the von Neumann algebra $W(S, F) \rtimes_{\phi^t} \mathbb{R}$, which describes the non commutative space of closed points, is of type $\text{III}_{\frac{1}{2}}$. This matches with Connes’s approach (see \[\text{[Co99]}, \text{Lei03}\]).

In a future paper \[\text{[Lei06]}\] we shall try to propose a list of axioms that a Riemannian foliated laminated space $(S_q, F, g, \phi^t)$ should satisfy in order to get a Lefschetz trace formula that should be analogous to the explicit formula for the Riemann Zeta function.

Now we try to explain why there should exist a connection with the work of Connes-Marcolli. Let $K$ be a local field with residue class field the finite field $\mathbb{F}_q$, let $f : z \to z^q$ be the canonical generator of the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_q$. Then local class field theory shows that the Galois group of the maximal unramified extension of $K$ admits a dense subgroup which is naturally isomorphic to $\{f^k, \ k \in \mathbb{Z}\}$. Now let $Y$ be a smooth projective variety over $\mathbb{F}_q$, then (see [page 292] \[\text{[Mil80]}\]) the automorphism $\text{Id} \otimes f$ of $Y = Y \times_{\mathbb{F}_q} \mathbb{F}_q$ is called the arithmetic Frobenius. On the other hand, the geometric Frobenius $F : Y \to Y$ is the $\mathbb{F}_q$–morphism sending the point $P$ with coordinates $(a_i), a_i \in \mathbb{F}_q$, to the point $F(P)$ with coordinates $(a_i^q)$. The action of $(\text{Id} \otimes f)^{-1}$ on $l$–adic cohomology coincides with the action of the geometric Frobenius $F$ (see [page 292] \[\text{[Mil80]}\]).

On the other hand, A. Connes (page 553 \[\text{[Co00]}\], page 90 \[\text{[Co02]}\]) has suggested that $\mathbb{R}^{++}$, as part of the renormalization group, should play the role of the missing (unramified) Galois group at the archimedean place of $Q$. A. Connes is motivated by an unramified local
class field analogy and his classification of type III factors. Moreover, Connes-Marcolli [Co-Ma04b] have shown that the renormalization group flow is an ambiguity Galois group acting on Quantum Field Theories (QFT’s). This seems reminiscent of a continuous version of $\text{Id} \otimes f$.

We suggest that the scaling group flow $\phi^t$, in $(S, F, g, \phi^t)$ as above, corresponds to a continuous version of $F$ and that Connes’s suggestion corresponds to a continuous version of $\text{Id} \otimes f$. It would be interesting to establish a precise connection between these two continuous notions of Frobenius by providing an ambiguity Galois group interpretation of $\phi^t$.

The results of [Co-Ma04a], [Co-Ma04b] should be very helpful with this respect. Moreover, this should allow to decide if to a smooth curve $Y$ over $\mathbb{F}_q$ one can (or cannot) associate a foliated space $(S_Y, F, g, \phi^t)$ satisfying the four Assumptions of Section 3 and such that the primitive closed orbits of $\phi^t$ should correspond to the closed points of $Y$. It is also tempting to try to establish a connection with Haran’s recent approach [Har05].

2. The case of an elliptic curve $E_0$ over $\mathbb{F}_q$

2.1. The zeta function $\zeta_{E_0}(s)$ and the explicit formula.

Let $E_0$ be an elliptic curve over a finite field $\mathbb{F}_q$. Recall that the zeta function $\zeta_{E_0}(s)$ of $E_0$ is given by:

$$\zeta_{E_0}(s) = \prod_{w \in |E_0|} \frac{1}{1 - (Nw)^{-s}} = \frac{(1 - \xi q^{-s})(1 - \overline{\xi} q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

(2)

where $|E_0|$ denotes the set of closed points of $E_0$ and $\xi$ is a complex number which by Hasse’s theorem satisfies $|\xi| = \sqrt{q}$. The explicit formula for $\zeta_{E_0}(s)$ takes the following form. Let $\alpha \in C^\infty_c(\mathbb{R}, \mathbb{R})$ and set for any real $s$, $\Phi(s) = \int_\mathbb{R} e^{st} \alpha(t) \, dt$. Then, one has:

$$\sum_{\nu \in \mathbb{Z}} \Phi \left( \frac{2\pi \nu i}{\log q} \right) - \sum_{\rho \in \zeta_{E_0}^{-1}(0)} \Phi(\rho) + \sum_{\nu \in \mathbb{Z}} \Phi \left( 1 + \frac{2\pi \nu i}{\log q} \right) = \sum_{w \in |E_0|} \log Nw \left( \sum_{k \geq 1} \alpha(k \log Nw) + \sum_{k \leq -1} (Nw)^k \alpha(k \log Nw) \right).$$

(3)

The idea of the proof is to apply the residue theorem to

$$s \to \left( \int_0^{+\infty} \sqrt{t} \alpha(\log t) t^s \, \frac{dt}{t} \right) \frac{\zeta_{E_0}'}{\zeta_{E_0}}(s)$$

and to use the functional equation $\zeta_{E_0}(s) = \zeta_{E_0}(1 - s)$. At the end of this Section we shall explain briefly how Deninger managed (see [De02] for the details) to interpret this explicit formula (3) as a Lefschetz trace formula.

2.2. The Riemannian laminated foliated space $(S(E_0), F, g, \phi^t)$.

Let $\phi_0 : E_0 \to E_0$ be the $q$–th power Frobenius endomorphism of $E_0$ over $\mathbb{F}_q$. Deninger has used (see [De02]) the following result due to Oort [Oor73]:

[Oor73] Oort, J. (1973). On the classification of abelian varieties. In Algebraic geometry, Academic Press, pp. 73–79.
Lemma 1. There exists:
1) a complete local integral domain \( R \) with field of fractions \( L \) a finite extension of \( \mathbb{Q}_p \) \((q = p^r)\) such that \( R/\mathfrak{M} = \mathbb{F}_q \) where \( \mathfrak{M} \) is the maximal ideal of \( R \).
2) an elliptic curve \( \mathcal{E} \) over \( \text{spec } R \) together with an endomorphism \( \phi : \mathcal{E} \to \mathcal{E} \) such that:
\[
(\mathcal{E}, \phi) \otimes \mathbb{F}_q = (E_0, \phi_0).
\]
So \((\mathcal{E}, \phi)\) is a lift of \((E_0, \phi_0)\) in characteristic zero.

Remark 1.
1) If the elliptic curve \( E_0 \) is ordinary, then one may take for \( R \) the ring of Witt vectors of \( \mathbb{F}_q \), \( W(\mathbb{F}_q) \), and then there is a canonical choice of the lifting \((\mathcal{E}, \phi)\). On the contrary, if \( E_0 \) is supersingular \([\text{page 137}] [\text{Si92}] \), then there is no canonical choice of \((\mathcal{E}, \phi)\).
2) It is possible to lift a curve of genus \( \geq 2 \) (over \( \mathbb{F}_q \)) in characteristic zero, but Hurwitz’s formula \([\text{page 41}] [\text{Si92}] \) shows that one cannot lift its Frobenius morphism.

Now (still following \([\text{De02}] \)), we denote by \( E = \mathcal{E} \otimes_R L \) the generic fibre. Then \( \text{End}_L(E) \otimes \mathbb{Q} = K \) is a field \( K \) which is either \( \mathbb{Q} \) or an imaginary quadratic extension of \( \mathbb{Q} \). We fix an embedding \( L \subset \mathbb{C} \) and consider the complex analytic elliptic curve \( E(\mathbb{C}) \). Let \( \omega \) be a non-zero holomorphic one form on \( E(\mathbb{C}) \) and let \( \Gamma \) be its period lattice. Then the Abel-Jacobi map:
\[
E(\mathbb{C}) \to \mathbb{C}/\Gamma, \ p \to \int_0^p \omega \bmod \Gamma
\]
induces an isomorphism. Next we choose the embedding \( K \subset \mathbb{C} \) such that for any \( \alpha \in K \), \( \Theta(\alpha) \) induces the multiplication by \( \alpha \) on the Lie algebra \( \mathbb{C}/\Gamma \) where \( \Theta \) is the natural homomorphism:
\[
\Theta : K = \text{End}_L(E) \otimes \mathbb{Q} \to \text{End}(\mathbb{C}/\Gamma) \otimes \mathbb{Q}.
\]
Next we consider the unique element \( \xi \in \Theta^{-1}(\text{End}_L(E)) \subset K \) such that \( \Theta(\xi) = \phi \otimes L \).
By construction one has \( \xi \Gamma \subset \Gamma \) and the complex elliptic curve \( \mathbb{C}/\Gamma \) endowed with the multiplication by \( \xi \) represents a lift of \((E_0, \phi_0)\). Now, we set
\[
V = \bigcup_{n \in \mathbb{N}} \xi^{-n}\Gamma, \ \Gamma \xi = \lim_{\longrightarrow} \frac{\Gamma}{\xi^n \Gamma}, \ \text{and } V_\xi \Gamma = \Gamma \xi \otimes_{\mathbb{Z}} \mathbb{Q}.
\]
The set \( \Gamma \xi \) is a Tate module defined by a projective limit and \( V_\xi \Gamma \) is a \( \mathbb{Q}_p \)-vector space of dimension 1 (resp. 2) if \( \xi \notin \mathbb{Z} \) (resp \( \xi \in \mathbb{Z} \)).

Any element \( v \) of \( V \) acts on \( \mathbb{C} \times V_\xi \Gamma \) by \( v.(z, \hat{v}) = (z + v, \hat{v} - v) \), we denote by \( \frac{\mathbb{C} \times V_\xi \Gamma}{\Gamma} \) the quotient space.

Lemma 2.
1) Let \( F \) be a finite set of representatives in \( \Gamma \) of the quotient group \( \frac{\Gamma}{\xi \Gamma} \). Then any element of \( V_\xi \Gamma \) is of the form \( \sum_{l \geq k} a_l \xi^l \) where \( k \in \mathbb{N} \) and the \( a_l \in F \). Moreover the multiplication by \( \xi \) defines an automorphism of \( V_\xi \Gamma \).
2) The natural homomorphism:
\[
\frac{\mathbb{C} \times \Gamma}{\Gamma} \to \frac{\mathbb{C} \times V_\xi \Gamma}{V}
\]
defines a \( \{\xi^l, \ l \in \mathbb{Z}\} \)-equivariant isomorphism where the action of \( \xi \) is induced by the diagonal action on \( \mathbb{C} \times \Gamma \) and \( \mathbb{C} \times V_\xi \Gamma \) respectively.
Proof. 1) Observe that any element of $T\Gamma$ is of the form $\sum_{l \in N} a_l \xi^l$ where the $a_l \in F$. Using Bezout’s theorem one checks that a prime number $\hat{p}$ not dividing $q$ (ie $\hat{p} \neq p$) induces by multiplication an automorphism of $T\Gamma$. If $\xi \notin \mathbb{Z}$ then the elliptic curve $\mathbb{C}/\Gamma$ has complex multiplication and its endomorphism ring is invariant under complex multiplication. So, in all cases, we have $\xi \Gamma \subset \Gamma$. Then using the equality $\xi \xi = q$ one gets the results of 1). Part 2) is now easy and left to the reader. □

Now, any element $q^\nu \in q \mathbb{Z}$ acts on $C \times V \xi \Gamma \times \mathbb{R}^+$ by

$$q^\nu.(z,\hat{v},x) = ([\xi^\nu z,\xi^\nu \hat{v}],xq^\nu).$$

In [De02], Deninger has introduced the (compact) laminated Riemannian foliated space $(S(E_0),F)$ where

$$S(E_0) = \frac{\mathbb{C} \times V \xi \Gamma}{V} \times q\mathbb{R}^+,$$

and the leaves of $F$ are the images of the sets $\mathbb{C} \times \{\hat{v}\} \times \{x\}$ by the natural map $\pi : C \times V \xi \Gamma \times \mathbb{R}^+ \to S(E_0)$. Observe that the domain of a typical foliation chart is locally isomorphic to $D \times \Omega \times [1,2]$ where $D$ is an open disk of $C$, $\Omega$ is an open subset of $T\Gamma$ so that the leaves are given by $D \times \{\omega\} \times \{x\}$ for $(\omega, x) \in \Omega \times [1,2]$; the term ”laminated” refers to the fact that the local transversal to the foliation $F$ is the disconnected space $\Omega \times [1,2]$.

Remark 2. Using the fact that $V$ (resp. $q\mathbb{Z}$) acts freely on $V \xi \Gamma$ (resp. $\mathbb{R}^+$), the reader will check that $(S(E_0),F)$ has trivial holonomy.

One defines a flow $\phi^t$ acting on $(S(E_0),F)$ and sending each leaf into another leaf by: $\phi^t(z,\hat{v},x) = (z,\hat{v},xe^{-t})$. Let $\mu_\xi$ denote a Haar measure on the group $V \xi \Gamma$ then, one has the following

Lemma 3. ([De02])

1) The measure $dx_1dx_2 \otimes \mu_\xi \otimes \frac{dx}{x}$ on $\mathbb{C} \times V \xi \Gamma \times \mathbb{R}^+$ induces a measure $\mu$ on $S(E_0)$.

2) The measure $\mu$ is invariant under the action of $\phi^t$.

Proof. 1) We just have to check that for any $\nu \in \mathbb{N}^*$ and any borel subset $A$ of $V \xi \Gamma$, one has

$$\mu_\xi(\xi^\nu A) = |\xi|^{-2\nu} \mu_\xi(A) = q^{-\nu} \mu_\xi(A).$$

Since $(\xi^\nu)_* \mu_\xi$ is also a Haar measure on $V \xi \Gamma$ it suffices to check this equality for $A = T\Gamma$. But this is an immediate consequence of the fact that

$$T\Gamma/(\xi^\nu T\Gamma) \simeq \Gamma/(\xi^\nu \Gamma)$$

has $|\xi|^{2\nu} = q^\nu$ elements.

2) This is obvious. □

Using the fact that $|\xi| = \sqrt{q}$, one checks that the Riemannian metric on the bundle $T\mathbb{C} \times V \xi \Gamma \times \mathbb{R}^+$ given by:

$$g_{z,\hat{v},x}(\eta_1,\eta_2) = x^{-1} \text{Re}(\eta_1 \overline{\eta_2})$$
induces a Riemannian metric $g$ along the leaves of $(S(E_0), \mathcal{F})$ so that the following property is satisfied:

$$\forall \eta \in T_{[z,v,x]}\mathcal{F}, \ g(T_{[z,v,x]}\phi_t(\eta), T_{[z,v,x]}\phi_t(\eta)) = e^t g(\eta, \eta).$$  \hspace{1cm} (4)

2.3. Interpretation of $(S(E_0), \mathcal{F}, g, \phi_t)$ as a renormalization group flow.

Now, recall that the additive group $(\mathbb{C}, +)$ is identified to its dual $(\widehat{\mathbb{C}}, \times)$ in the following way. For each character $\chi \in \widehat{\mathbb{C}}$ one can find one and only one complex number $\alpha$ such that:

$$z \in \mathbb{C} \rightarrow <\chi; z> = e^{\sqrt{-1}(R \alpha R z + 3\alpha \Im z)} \in S^1.$$  \hspace{1cm} (5)

We set $G = \frac{\mathbb{C} \times \Gamma}{\Gamma}$, since $G$ is a quotient of $\mathbb{C} \times \Gamma$, one checks that $\hat{G}$ can be identified to a subgroup of $\mathbb{C} \times \widehat{\Gamma}$. Let $\Gamma^*$ denote the dual lattice of $\Gamma$:

$$\Gamma^* = \{z \in \mathbb{C}/ \forall \gamma \in \Gamma, (z; \gamma) = R z R \gamma + 3z \Im \gamma \in 2\pi \mathbb{Z}\}.$$  

Lemma 4.  

1] One has a natural group isomorphism $\widehat{\Gamma} \simeq \cup_{n \in \mathbb{N}} (\xi^n \Gamma)^*$.  

2] Let $\pi$ denote the projection map $\pi : \cup_{n \in \mathbb{N}} (\xi^n \Gamma)^* \rightarrow \cup_{n \in \mathbb{N}} (\xi^n \Gamma)^*$. Then one has $\hat{G} = \{(z, \pi(z)) \in \cup_{n \in \mathbb{N}} (\xi^n \Gamma)^* \times \widehat{\Gamma}\}$.  

Proof. 1] We fix a set $F$ of representatives in $\Gamma$ of the quotient group $\frac{\Gamma}{\Gamma}$ which has exactly $q = |\xi|^2$ elements. Notice that for any $(a_k)_{k \in \mathbb{N}} \in F^\mathbb{N}$ the series $\sum_{k \in \mathbb{N}} a_k \xi^k$ converges in $\Gamma$ and that each element of $\Gamma$ is of the form $\sum_{k \geq n} a_k \xi^k$. Recall that $\Gamma$ is dense in the compact abelian group $\Gamma$ and that a fundamental system of open neighborhoods of $0 \in \Gamma$ is provided by the subsets $O_n = \{\sum_{k \geq n} a_k \xi^k/ \forall k \geq n, a_k \in F\}$. Now, let $\chi$ be any character of $\Gamma$ then there exists $\alpha \in \mathbb{C}$ such that

$$\forall \gamma \in \Gamma, <\chi; \gamma> = e^{\sqrt{-1}(R \alpha R \gamma + 3\alpha \Im \gamma)} \in S^1.$$  

In fact $\alpha$ is defined in $\frac{\mathbb{C}}{\Gamma}$. Notice that $\chi$ extends to a character of $\Gamma$ if and only if $\chi \equiv 1$ on $O_n$ for a suitable $n > > 1$. Therefore $\chi$ defines a character of $\Gamma$ if and only if $\alpha \in (\xi^n \Gamma)^*$ for a suitable $n > > 1$ and the result follows.

2] An element of $\hat{G}$ is defined by a couple $(\chi_1, \chi_2) \in \widehat{\mathbb{C}} \times \widehat{\Gamma}$ such that $(\chi_1)|_\Gamma = (\chi_2)|_\Gamma$ and $\chi_1$ is defined by $\alpha \in \mathbb{C}$ as in [1]. According to part 1], $(\chi_1)|_\Gamma$ extends to a character of $\Gamma$ if and only if $\alpha \in \cup_{n \in \mathbb{N}} (\xi^n \Gamma)^*$ and the result follows.  \hspace{1cm} $\square$

Corollary 1. One has the following group isomorphism:

$$\frac{\mathbb{C} \times \widehat{\Gamma}}{\hat{G}} \simeq \frac{\mathbb{C}}{\Gamma^*}.$$  

Proof. We fix a set $\mathcal{R} \subset \cup_{n \in \mathbb{N}} (\xi^n \Gamma)^*$ of representatives of $\frac{\cup_{n \in \mathbb{N}} (\xi^n \Gamma)^*}{\Gamma}$. Let $(u, v) \in \mathbb{C} \times \widehat{\Gamma}$, so we can find $z \in \mathcal{R}$ such that $\pi(z) = -v$. According to Lemma [1] 2], $(z, \pi(z)) \in \hat{G}$ and $(z, \pi(z)) \cdot (u, v) = (u + z, 0)$. Then for any $\gamma \in \Gamma$ one has: $(\gamma, \pi(\gamma)) \cdot (u + z, 0) = (u + z + \gamma, 0)$. Now the result follows immediately.  \hspace{1cm} $\square$
Now recall that
\[ S(E_0) = \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\Gamma} \times \mathbb{R}^+ \]
and the action of \( \phi^t \) on \( S(E_0) \) is given by \( \phi^t([z, \hat{v}, x]) = [z, \hat{v}, xe^{-t}] \). The following Proposition provides an interpretation of \((S(E_0), F, g, \phi^t)\) as a renormalization group flow for suitable quantum field theories defined over \( \mathbb{C} \).

**Proposition 1.** The set \( G = \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\Gamma} \) defines in a natural way a set of free lagrangians on \( \mathbb{C} \) (ie on which the renormalization semi-group \( R_{x, xe^{-t}} \) acts trivially, see the Appendix).

**Proof.** Observe that \( \hat{G} \) is a discrete subgroup of \( \mathbb{C} \times \mathbb{T}_{\Gamma} \) and thus acts on \( \mathbb{C} \times \mathbb{T}_{\Gamma} \) by translation. According to Corollary [1], \( \mathbb{C} \times \mathbb{T}_{\Gamma} \) appears as a \( \hat{G} - \)principal bundle over the space \( \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}} \simeq \mathbb{C} \Gamma \). Let \( h \in \hat{G} \simeq G \), then consider the fiber product \( \mathcal{E}_h = \left( \mathbb{C} \times \mathbb{T}_{\Gamma} \right) \times_{\hat{G}} \mathbb{C} \) where for any \( \chi_1 \in \hat{G} \), and \( ((z, \hat{v}), u) \in \left( \mathbb{C} \times \mathbb{T}_{\Gamma} \right) \times \mathbb{C} \), the point \( ((z, \hat{v}), u) \) is identified with the point \( (\chi_1 \cdot (z, \hat{v}), \chi_1(h)^{-1}u) \). The space \( \mathcal{E}_h \) defines a complex line bundle over \( \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}} \) and is endowed with a natural flat connection. Let \( S_{\pm} \) denote the spinor bundle on \( \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}} \) associated with the real 1-dimensional representation of \( \text{Spin}(1, 1) \) with spin \( \pm \frac{1}{2} \). Then we get (as in [page 587][QFT]) a natural Dirac type operator \( D_h \) acting on the sections of \((S_+ \oplus S_-) \otimes \mathcal{E}_h\) and the map

\[ s \in C^\infty(\frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}}; (S_+ \oplus S_-) \otimes \mathcal{E}_h) \rightarrow \int_{\frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}}} (\overline{s}; D_h(s))(y)dy = L_h(s) \]

defines a free Lagrangian \( L_h \) on \( \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}} \).

**Open Question 1.** In [page 277][Po84] (see also [page 558][QFT]) it is proved that the space of perturbative renormalizable theories on \( \mathbb{R}^4 \) is an attractor for the renormalization group flow acting on an infinite dimensional space of lagrangians. Is it possible to develop a (non pertubative) quantum field theory on the space \( \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}} \) in which there should exist a natural infinite dimensional space \( \mathcal{I} \) of lagrangians such that \( \frac{\mathbb{C} \times \mathbb{T}_{\Gamma}}{\hat{G}} \) should be an attractor of \( \mathcal{I} \) for the renormalization group flow?

### 2.4. Further remarks and motivation of Section 3 and [Lei00].

Denote by \( A^j_{\phi}(S(E_0)) \) \((0 \leq j \leq 2)\) the set of sections of the real vector bundle \( \wedge^j T^* F \rightarrow S(E_0) \) which are smooth along the leaves and continuous on \( S(E_0) \). The metric \( g \) of \([1]\) induces a metric \( h_j(\cdot, \cdot) \) on the bundle \( \wedge^j T^* F \) \((0 \leq j \leq 2)\), we then denote by \( L^2(S(E_0); \wedge^j T^* F) \) the Hilbert completion of \( A^j_{\phi}(S(E_0)) \) with respect to the real scalar product:

\[ \forall \omega, \omega' \in A^j_{\phi}(S(E_0)), \quad <\omega; \omega'> = \int_{S(E_0)} h_j(\omega, \omega')(\theta) d\mu(\theta). \quad (6) \]

Let \( d^j_{\phi} \) denote the formal adjoint of the leafwise exterior derivative \( d_{\phi} \). Then one defines the reduced \( L^2 \)-leafwise real cohomology groups \( H^j_{L^2, \tau}(S(E_0), \mathbb{R}) \), \((0 \leq j \leq 2)\) by

\[ H^j_{L^2, \tau}(S(E_0), \mathbb{R}) = \ker((d^j_{\phi})^* \cap (d^j_{\phi})^{**}), \quad 0 \leq j \leq 2 \]

where the unbounded operators \((d^j_{\phi})^*, (d^j_{\phi})^{**}\) both act on \( L^2(S(E_0); \wedge^j T^* F)\).
Theorem 1.
1) There is a natural bijection between the set of valuations $w$ of the function field $K(E_0)$ of $E_0$ and the set of primitive compact $\mathbb{R}$–orbits of $\phi^t$ on $S(E_0)$. It has the following property. If $w$ corresponds to $\gamma = \gamma_w$, then

$$l(\gamma_w) = \log N(w).$$

2) Denote by $(\phi^t)^*_j$ the operator $(\phi^t)^*$ acting on $H^2_{L^2,\tau}(S(E_0), \mathbb{R})$ for $0 \leq j \leq 2$. Then for any $\alpha \in C^\infty_c(\mathbb{R}; \mathbb{R})$ the following equality holds:

$$\sum_{j=0}^{j=2} (-1)^j \int_{\mathbb{R}} \text{TR}(\phi^t)^*_j \alpha(t) \, dt = \sum_{\gamma_w} l(\gamma_w) \sum_{k \geq 1} \alpha(kl(\gamma_w)) + \sum_{\gamma_w} \sum_{k \leq -1} e^{kl(\gamma_w)} \alpha(kl(\gamma_w))$$

(7)

where the right hand side coincides with the one of the explicit formula (4).

About part 2. Deninger has identified the left handside of (3) with the (spectral) left handside of (p) and has invoked (3) to get (7).

Now we make remarks about the structure of $(S(E_0), \mathcal{F})$ which will motivate the constructions and definitions of the Sections 3 and [Lei06].

First we introduce carefully a natural transverse measure on $(S(E_0), \mathcal{F})$ and point out its important role.

Set $\mathcal{L}_{E_0} = \mathcal{E} \times \mathcal{T}$, this is a compact laminated space which is foliated by its path-connected components. Any element $q^\tau \in q^\mathbb{R}$ acts on $[z, \tilde{v}] \in \mathcal{L}_{E_0}$ by $q^\tau \cdot [z, \tilde{v}] = [\xi^\tau z, \xi^\tau \tilde{v}]$. The Haar measure $\mu_\xi$ of $\mathcal{T}$ induces a transverse measure, still denoted $\mu_\xi$, of $\mathcal{L}_{E_0}$. For any Borel transversal $T$ of $\mathcal{L}_{E_0}$ one has $\mu_\xi(q \cdot T) = q^{-1} \mu_\xi(T)$.

Moreover, the metric $\tilde{g} = (dx_1)^2 + (dx_2)^2$ (where $z = x_1 + ix_2$) defines a leafwise metric on $\mathcal{L}_{E_0}$, let $\lambda_\tilde{g}$ be the associated leafwise volume form. Then $\lambda_\tilde{g} \mu_\xi$ defines a $q^\mathbb{R}$–invariant measure of $\mathcal{L}_{E_0}$.

The leafwise metric $g$ in (4) of $(S(E_0), \mathcal{F})$ is defined by $g = x^{-1}\tilde{g}$ and its associated leafwise volume form is given by $\lambda_g = x^{-1}dx_1 \wedge dx_2$.

Now it is clear that $\frac{\mu}{\lambda} = \mu_\xi dx$ defines a transverse measure with associated Ruelle-Sullivan current $C(\frac{\mu}{\lambda})$. We can pair sections of $\mathcal{A}_F^2(\mathcal{L}_{E_0})$ with $C(\frac{\mu}{\lambda})$, for instance the measure $\mu$ may be recovered by the formula:

$$\forall f \in C^0(S(E_0)), \quad (f \lambda; C(\frac{\mu}{\lambda})) = \int_{S(E_0)} f \, d\mu.$$  

Recall that $\mu$ is $\phi^t$–invariant. Moreover we observe that the scalar product (6) may be recovered by the formula:

$$\forall \omega, \omega' \in \mathcal{A}_F^4(S(E_0)), \quad \langle \omega; \omega' \rangle = (\omega \cup \ast \omega'; C(\frac{\mu}{\lambda}))$$

where $\ast$ denotes the leafwise Hodge star operator associated to $g$.

Now we come to formula (7). As explained by Deninger, the dissymmetry of the coefficients of $\alpha(kl(\gamma))$ for $k \geq -1$ and $k \geq 1$ is due to property (p) (see the remark following Corollary 1 of [Lei06]). We are going to propose a dynamical explanation, à la Guillemin-Sternberg, of this dissymmetry. Consider a point $(z_0, \tilde{v}_0, 1) \in S(E_0)$, with $\tilde{v}_0 \in \mathcal{T}$, such that $\phi^{-\log q}[z_0, \tilde{v}_0, 1] = [z_0, \tilde{v}_0, q] = [z_0, \tilde{v}_0, 1]$. Recall that by definition
Then a computation using (8) and (9) shows (see also [Section IV] [Co99]), that for
\[ \xi^{-1}z_0 = z_0 + \gamma, \quad \xi^{-1}\hat{v}_0 = \hat{v}_0 - \gamma. \] (8)
The operator \((\phi^t)^*\) acting on \(A^j_T(S(E_0))\) admits a Schwartz kernel defined by the formula:
\[ \forall \omega \in A^j_T(S(E_0)), \quad (\phi^t)^*(\omega)(y) = \int_{S(E_0)} (D\phi^t)^* \delta_{\phi^t(y)=y'} \omega(y') \, d\mu(y'). \]
Consider a point \(y = [z, \hat{v}, x] \) belonging to a small neighborhood of \(\{\phi^t[z_0, \hat{v}_0, 1], -\log q \leq t \leq 0\} \). Then, with the previous notations, one has:
\[ \phi^t(y) = (\xi^{-1}z - \gamma, \xi^{-1}\hat{v} + \gamma, q^{-1}xe^{-t}). \] (9)
In the following Lemma we show basically that the graph of the flow \(\phi^t\) is transverse to the diagonal and compute \(\delta_{\phi^t(y)=y}\).

**Lemma 5.**
1) \(z \in C \rightarrow \xi^{-1}z - \gamma - z\) and \(\hat{v} \in V\xi T \rightarrow \xi^{-1}\hat{v} + \gamma - \hat{v}\) are invertible and their jacobians are respectively given by:
\[ \text{Jac}(\xi^{-1}z - \gamma - z) = |\xi^{-1} - 1|^2, \quad \text{Jac}(\xi^{-1}\hat{v} + \gamma - \hat{v}) = q. \]
2) Let \(V\) be an open neighborhood of \((z_0, \hat{v}_0)\), set:
\[ U = \{(z, \hat{v}, e^{-s})/ s \in [-\log q, 0], (s, \hat{v}) \in V\}. \]
Consider \(\varepsilon > 0\) and \(V\) small enough so that \(t \in [-\log q, 0] \rightarrow (z_0, \hat{v}_0, e^{-t})\) is the only closed orbit of \(\phi^t\) contained in \(U\) with length in \([-\varepsilon - \log q, \varepsilon - \log q]\). Then one has the following equality as a distribution on \(U \times [-\varepsilon - \log q, \varepsilon - \log q]\):
\[ \delta_{\phi^t(y)=y} = \frac{1}{|\xi^{-1} - 1|^2} \delta_{z-z_0} \otimes \frac{1}{q} \delta_{\hat{v}-\hat{v}_0} \otimes \delta_{t+\log q}. \]

**Proof.**
1) We prove only the second equality. Recall that \(T \Gamma\) is an open compact subset of \(V\xi T\). Then, since \(\hat{v} \rightarrow \hat{v} - \xi \hat{v}\) defines an automorphism of \(T \Gamma\) whose inverse is \(\hat{v} \rightarrow \sum_{n \in \mathbb{N}} \xi^n \hat{v}, \) one has \(\text{Jac} (\hat{v} - \xi \hat{v}) = 1. \) Now recall that the proof of Lemma 3 shows that \(\mu_\xi(T \Gamma) = \frac{1}{q} \mu_\xi(T \Gamma)\) so that \(\text{Jac} (\xi \hat{v}) = \frac{1}{q}. \) By combining the last two equalities for \(\text{Jac}, \) one gets:
\[ \text{Jac}(\xi^{-1}\hat{v} + \gamma - \hat{v}) = q. \]
2) Using the change of variable formula for \(\int d\mu_\xi\) and the equality \(\xi^{-1}\hat{v}_0 + \gamma - \hat{v}_0 = 0, \) one sees that for \(\hat{v}\) close to \(\hat{v}_0\) one has
\[ \delta_{\xi^{-1}\hat{v} + \gamma - \hat{v}} = \frac{1}{\text{Jac}(\xi^{-1}\hat{v} + \gamma - \hat{v})} \delta_{\hat{v}-\hat{v}_0}. \]
Then a computation using (3) and (9) shows (see also [Section IV] [Co99]), that for \(y = [z, \hat{v}, x] \in U\) and \(t \in [-\varepsilon - \log q, \varepsilon - \log q]\) one has:
\[ \delta_{\phi^t(y)=y} = \frac{1}{\text{Jac}(\xi^{-1}z - \gamma - z)} \delta_{z-z_0} \otimes \frac{1}{\text{Jac}(\xi^{-1}\hat{v} + \gamma - \hat{v})} \delta_{\hat{v}-\hat{v}_0} \otimes \delta_{t+\log q}. \]
By combining 1) with this equality one gets the result. \(\Box\)
Recall now that $d\mu(y) = dx_1 dx_2 \otimes \mu_\xi \otimes dx$. The formula $\int_0^{-\log q} \frac{de^{-s}}{e^{-s}} = \log q$ and Lemma 2 show that for $t$ close to $-\log q$ the distributional trace

$$\int_{S(E_0)} \text{Tr} (D\phi_t)^* \delta_{\phi_t(y) = y} \, d\mu(y)$$

is well defined (near $-\log q$) and equal to:

$$\log q \sum_{\gamma_w, l(\gamma_w) = \log q} \frac{1}{q} \delta_{-l(\gamma_w)}$$

where $\gamma_w$ runs over the set of closed orbits of $\phi_t$ of length $l(\gamma_w) = \log q$. Since $\text{Jac}(\xi \hat{v} + \gamma - \hat{v}) = 1$ a similar argument shows that for $t$ close to $\log q$ the distributional trace

$$\int_{S(E_0)} \text{Tr} (D\phi_t)^* \delta_{\phi_t(y) = y} \, d\mu(y)$$

is well defined (near $\log q$) and equal to:

$$\log q \sum_{\gamma_w, l(\gamma_w) = \log q} \delta_{l(\gamma_w)}.$$

These observations motivate the definition of a transversally $p$-adic complex laminated space that we shall introduce in Section 3. Deninger pointed out to us that the condition (1) $(\phi_t)^* g = e^t g$ was probably too strong for being generalized. That is why in Proposition 2.2 we shall replace it by $(\phi_t)^* [\lambda_g] = e^t [\lambda_g]$ where $[\lambda_g]$ denotes the reduced leafwise cohomology class of the leafwise kaehler form associated to $g$ (see Serre [Se60] and Deninger-Singham [De-Si02]).

Now let $Y$ be a smooth projective absolutely irreducible curve over $\mathbb{F}_q$ admitting a rational point. As pointed out in [Lei03], one would like to associate to $Y$ a Riemannian laminated foliated space $(\mathcal{L} \times \mathbb{R}^{++}, \mathcal{F}, g, \phi_t)$ for which the analogous version of Theorem 1 should hold. Recall now Hurwitz’s formula for a non constant morphism $\psi : C_1 \to C_1$ where $C_1$ is a smooth projective curve of genus $g_1$ over a field of characteristic zero. One has:

$$2g_1 - 2 = (\deg \psi)(2g_1 - 2) + \sum_{P \in C_1} (e_\psi(P) - 1)$$

where the strictly positive integers $e_\psi(P)$ are all equal to one except for a finite number of them. If it could be possible to lift the Frobenius morphism of $Y$ in characteristic zero, one should get a morphism $\psi$ of degree $> 1$ which is not possible. Thus, unlike the case of an elliptic curve, we do not consider for $Y$ a flow of the simple form $\phi_t([l, x]) = [l, xe^{-t}]$ but a priori a (more general) renormalization group flow. We shall carry out the detailed constructions in Section 3.

3. The zeta function of a compact Riemannian foliated transversally $p$-adic laminated space $(S = \mathcal{L} \times \mathbb{R}^{++}, \mathcal{F}, g, \phi_t)$

We introduce the following definition which is a particular case of a notion due to Sullivan [Su93]:
Definition 1. A compact complex $1$–dimensional transversally $p$–adic laminated space $\mathcal{L}$ is a compact topological space satisfying the following property. There exists $m \in \mathbb{N}^*$ such that $\mathcal{L}$ admits a finite open cover $\bigcup_{i=1}^{N} A_i$ by local charts: $f_i : A_i \rightarrow U_i \times \mathbb{Z}_p^m$ where $U_i$ is an open disc of $\mathbb{C}$ such that each transition map $f_i \circ f_j^{-1}$ is of the form

\begin{align*}
  f_j(A_i \cap A_j) &\rightarrow U_i \times \mathbb{Z}_p^m \\
  (z, \theta) &\rightarrow (H(z), G(\theta)) = f_i \circ f_j^{-1}(z, \theta)
\end{align*}

(10) where $f_j(A_i \cap A_j) = \Omega_{i,j} \times \mathbb{Z}_p^m$ and

1) $z \mapsto H(z)$ is holomorphic on its domain of definition $\Omega_{i,j}$
2) There exists $M \in \text{GL}_m(\mathbb{Z}_p)$ and $B \in \mathbb{Z}_p^m$ such that for any $\theta \in \mathbb{Z}_p^m$ one has $G(\theta) = M\theta + B$.

Then the path connected components $\tilde{\mathcal{L}}$ of $\mathcal{L}$ carry a structure of $1$–dimensional complex manifold and constitute the leaves of a foliation. For any $l \in \mathcal{L}$ we denote by $T_l\mathcal{L}$ the tangent space at $l$ of the path connected component of $\mathcal{L}$ containing $l$.

We may and shall assume in the sequel that the charts $(A_j, f_j)$ are defined on open subsets $A'_j$ such that $\overline{A}_j \subset A'_j$.

The space $\mathcal{L}^{\mathrm{cx,\mathrm{GL}}}$ of Section 2 is an example of such an $\mathcal{L}$.

Definition 2. Let $\mathcal{L}$ be a laminated space as in Definition \[\text{II}\] and $j \in \{0, 1, 2\}$. Denote by $\mathcal{A}'_j(\mathcal{L})$ the set of sections $\omega$ of the complex vector bundle $\bigwedge^p T^* \mathcal{L} \otimes \mathbb{C} \rightarrow \mathcal{L}$ which are smooth along the leaves and continuous globally in the following sense. In any chart $(A_i, f_i)$ as above, $(f_i^{-1})^* \omega$ is a finite sum of terms $a(z, \theta)(dx_1)^\alpha \wedge (dx_2)^\beta$ where $z = x_1 + \sqrt{-1}x_2, \alpha + \beta = j$,

$$\forall \theta \in \mathbb{Z}_p^m, z = x_1 + \sqrt{-1}x_2 \rightarrow a(z, \theta)$$

belongs to $C^\infty(U_i, \mathbb{C})$ and for any $r_1, r_2 \in \mathbb{N}$, $(z, \theta) \rightarrow \partial_{x_1}^{r_1} \partial_{x_2}^{r_2} a(z, \theta)$ is continuous on $U_i \times \mathbb{Z}_p^m$.

Now we consider a $1$–dimensional complex compact transversally $p$–adic laminated space $\mathcal{L}$. In the sequel, we shall assume that $\mathcal{L}$ satisfies the following four properties which are stated with the notations of Definition \[\text{II}\] and motivated by subsection \[\text{II.4}\].

(i) The group $q^{-\mathbb{Z}}$ acts leafwise holomorphically on $\mathcal{L}$ in the following sense, $q^m \cdot l$ denoting the action of $q^m \in q^{-\mathbb{Z}}$ on $l \in \mathcal{L}$. Let $y \in \mathcal{L}$, assume that $y \in A_j$ and $q \cdot y \in A_i$ (cf Definition \[\text{II}\]). Then:

$$\forall(z, \theta) \in f_j(q^{-1}(A_i) \cap A_j), f_i \circ q \circ f_j^{-1}(z, \theta) = (Z(z), L(\theta) = M_q \theta + b_q)$$

on its domain of definition, where $Z(\cdot)$ is holomorphic, $b_q \in \mathbb{Z}_p^m$, and $M_q \in \text{M}_m(\mathbb{Z}_p) \cap \text{GL}_m(\mathbb{Q}_p)$ is such that $\text{Jac} M_q = \frac{1}{q}$.

(ii) There exists a smooth leafwise kaehler metric $\tilde{g}$ along the leaves of $\mathcal{L}$ with the following two properties. First, for each foliation chart $f_i : A_i \rightarrow U_i \times \mathbb{Z}_p^m$ as in Definition \[\text{II}\] $(f_i^{-1})^*(\tilde{g})$ does not depend on $\theta$. Second, for any $l \in \mathcal{L}$ one has:

$$\forall u \in T_l \mathcal{L}, \tilde{g}(q_u(u)) = q \tilde{g}(u).$$

We denote by $\lambda_{\tilde{g}}$ the leafwise Kaehler form associated with $\tilde{g}$: $\tilde{g}(u, v) = \lambda_{\tilde{g}}(u, Jv)$.

Since the leaves of $\mathcal{L}$ are oriented by their complex structure there is a leafwise Hodge star associated to $\tilde{g}$.

(iii) There exists a smooth family $(\psi^t_x)_{t \in \mathbb{R}, x > 0}$ of diffeomorphisms of $\mathcal{L}$ acting leafwise holomorphically in the following sense. Let $(t_0, x_0, y) \in \mathbb{R} \times \mathbb{R}^+ \times \mathcal{L}$, assume that $y \in A_j$
Remark. The assumption (iv) is satisfied in the case of the space \( C_p \) of Definition 1 of Definition 1 holds. Moreover, for any \((t, t', x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, t \in \mathcal{L}\) one assumes that:

\[
\psi^t_{qa}(q \cdot l) = q \cdot \psi^t_x(l), \quad \psi^t_{x+q} \left( \psi^t_x(l) \right) = \psi^t_{x+q}(l),
\]

Notice that the equation (12) implies that \( \psi^0_x = \text{Id} \), that each \( \psi^t_x \) preserves any path connected component of \( \mathcal{L} \) and induces a diffeomorphism of it.

Moreover, let \( k \in \mathbb{N}^* \) and \( l_0 \in A_j \) be such that \( (\psi^A_q \circ q)^k(l_0) = l_0 \). Then one can write:

\[
\forall (z, \theta) \in U' \times \mathbb{Z}_p^m, \quad f_j \circ (\psi^A_q \circ q)^k \circ f_j^{-1}(z, \theta) = (Z_k(z), Q_k \theta + b_k)
\]

where \( Z_k \) is holomorphic on an open neighborhood \( U'_j \subset U_j \) of \( f_j(l_0) \), \( b_k \in \mathbb{Z}_p^m \), and \( Q_k \in \text{M}_m(\mathbb{Z}_p) \) satisfies:

\[
\text{Jac}(Q_k) = q^{-k}, \quad \text{Jac}(\text{Id} - Q_k) = 1.
\]

(iv) Let \( \omega \in C^0(\mathcal{L}, \wedge^1 \mathbb{T}^* \mathcal{L}) \) be a leafwise \( C^\infty \) harmonic 1–form. Then for each chart \((A_j, f_j)\) of Definition 1 \( (f_j^{-1})^* \omega(z, \theta) \) does not depend on \( \theta \). For an intrinsic equivalent statement see Theorem 3.

**Remark** The assumption (iv) is satisfied in the case of the space \( \mathbb{C} \times \mathbb{V}_\Gamma \) of Section 2. Louis Boutet de Monvel has showed us an example (of the form \( \mathbb{H} \times \mathbb{Z}^q_{\mathbb{F}} \)) with hyperbolic leaves where it is not satisfied. Dennis Sullivan has told us that, in some sense, most of the examples of \( \mathcal{L} \) with hyperbolic leaves of Definition 1 do not satisfy this assumption. Bertrand Deroin has showed [Der04] that for a space \( \mathcal{L} \) with hyperbolic leaves, the space of leafwise holomorphic quadratic differentials is infinite dimensional. But his proof does not show that Assumption (iv) is not satisfied. Notice that there are examples of spaces \( \mathcal{L} \) which have hyperbolic and parabolic leaves and do not seem to be defined by an inverse limit of Riemann surfaces (e.g., the one described by E. Ghys [Section 6.4] [Ghys99] following an idea of R. Kenyon).

**Lemma 6.**

1] The data of the Haar measure \( \mu_{\mathbb{Z}_p^m} \) in each local chart \((A_j, f_j)\) of Definition 1 induce a transverse measure \( \mu_{\mathcal{L}} \) on \( \mathcal{L} \). Then \( \lambda_{A_j} \mu_{\mathcal{L}} \) defines a measure on \( \mathcal{L} \). Moreover, for any Borel transversal \( T \) of \((\mathcal{L}, \mathcal{F})\) one has \( \mu_{\mathcal{L}}(q \cdot T) = \frac{1}{q} \mu_{\mathcal{L}} \).

2] \( \forall (t, x) \in \mathbb{R} \times \mathbb{R}^+, (\psi^t_x)^*(\mu_{\mathcal{L}}) = \mu_{\mathcal{L}} \).

**Proof.** 1]. The fact that \( \mu_{\mathcal{L}} \) is well defined as a transverse measure is a consequence of Definition 1. The leaves are two dimensional so \( \lambda_{\mathcal{L}} \) defines the leafwise Riemannian volume form associated to \( \tilde{g} \) on \((\mathcal{L}, \mathcal{F})\). Then by definition, \( \lambda_{A_j} \mu_{\mathcal{L}} \) defines a measure on \( \mathcal{L} \). The equality \( \mu_{\mathcal{L}}(q \cdot T) = \frac{1}{q} \mu_{\mathcal{L}} \) is a consequence of Assumption (i) (\( \text{Jac} M_q = \frac{1}{q} \)).

2]. This is a consequence of the equality \( \text{Jac}(M_1) = 1 \) in Property (iii). \( \square \)
Consider the foliated laminated space \((S = \mathcal{L} \times \mathbb{R}^+, \mathcal{F})\) whose leaves are the sets \(\Pi(\mathcal{C} \times \{x\})\) where \(\Pi : \mathcal{L} \times \mathbb{R}^+ \to S\) denote the projection and \(\mathcal{C}\) runs over the set of path connected components of \(\mathcal{L}\). Let \([l, x_0] \in S\), assume that \(l \in A_i\) with the notations of Definition II. There exists \(\epsilon > 0\) such that \(A_i \times |x_0 - \epsilon, x_0 + \epsilon|\) may be identified with an open subset of \(S\). Then
\[
f_i \times \text{Id}_{|x_0-\epsilon, x_0+\epsilon|} : A_i \times |x_0 - \epsilon, x_0 + \epsilon| \to U_i \times \mathbb{Z}_p^m \times |x_0 - \epsilon, x_0 + \epsilon|
\]
defines a foliated chart of \(S\) centered at \([l, x_0]\).

**Definition 3.**
1] Denote by \(A_j^i(S) (0 \leq j \leq 2)\) the set of sections \(\omega\) of the complex vector bundle \(\wedge^j \mathcal{T}^* \mathcal{F} \otimes \mathbb{C} \to S\) which are smooth along the leaves and continuous globally in the following sense. In any foliation chart \(f_i \times \text{Id}_{|x_0-\epsilon, x_0+\epsilon|}\) as above, \(((f_i \times \text{Id}_{|x_0-\epsilon, x_0+\epsilon|})^{-1})\omega\) is a finite sum of terms \(a(z, \theta, x)(dx_1)^\alpha \wedge (dx_2)^\beta\) where \(z = x_1 + \sqrt{-1}x_2\), \(\alpha + \beta = j\) such that:
\[
\forall \theta \in \mathbb{Z}_p^m, (z = x_1 + \sqrt{-1}x_2) \to a(z, \theta, x) \in C^\infty(U_i \times |x_0 - \epsilon, x_0 + \epsilon|, \mathbb{R}),
\]
and for any \(r_1, r_2, r_3 \in \mathbb{N}\), \((z, \theta, x) \to \partial_{x_1}^r \partial_{x_2}^r \partial_{\theta}^r a(z, \theta, x)\) is continuous.
2] Denote, for \(j \in \{0, 1, 2\}\), by \(\mathcal{H}_j^\omega(S)\) the reduced leafwise cohomology \(\ker \frac{\text{Id}_x}{\text{Id}_x}\).

One then gets the following:

**Proposition 2.** Denote by \(\Pi\) the projection map \(\Pi : \mathcal{L} \times \mathbb{R}^+ \to \mathcal{L} \times \mathbb{R}^+ = S\) where \(q^Z\) acts diagonally. Consider the foliated laminated space \((S = \mathcal{L} \times \mathbb{R}^+, \mathcal{F})\) whose leaves are the sets \(\Pi(\mathcal{C} \times \{x\})\) where \(\mathcal{C}\) runs over the set of path connected components of \(\mathcal{L}\).
1] One defines a flow \(\phi^i\) acting on \(S\) by setting for any \((l, x) \in \mathcal{L} \times \mathbb{R}^+\)
\[
\phi^i([l, x]) = ([\varphi^i_x(l), xe^{-t}]),
\]
where \([l, x]\) denotes the class of \((l, x)\) in \(S\).
2] The metrics \(x^{-1}g\) on \(T_{[l, x]}\mathcal{F}\), define a leafwise kaehler metric \(g = (x^{-1}g)_{x \in \mathbb{R}^+}\) on \((S, \mathcal{F})\). Moreover, \(\lambda_g = x^{-1} \lambda_g\) defines a leafwise kaehler form on \((S, \mathcal{F})\) \(\lambda_g\) is defined in Assumption (ii) of this Section. For any \(t \in \mathbb{R}\), \((\phi^i)^* [\lambda_g] = e^t [\lambda_g]\) where \([\lambda_g]\) denotes the induced class in \(\mathcal{H}^2(S)\).
3] \(\lambda_g\) is also the leafwise Riemannian volume form associated to \(g\) on \((S, \mathcal{F})\). Moreover, \(\mu_{\mathcal{L}} dx\) defines a transverse measure denoted \(\Lambda\) on \((S, \mathcal{F})\), and \(\mu = \lambda_g \mu_{\mathcal{L}} dx\) induces a measure on \(S\).
4] Let \(W(\mathcal{L}, \mathcal{F})\) (resp. \(W(S, \mathcal{F})\)) denotes the von Neumann algebra of the foliation \((\mathcal{L}, \mathcal{F})\) (resp. \((S, \mathcal{F})\)) associated with the measure \(\lambda_g \mu_{\mathcal{L}}\) (resp. \(\lambda_g \mu_{\mathcal{L}} dx\)). Assume that \(W(\mathcal{L}, \mathcal{F})\) is a factor. Then the flow \(\phi^i\) induces an action, denoted \((\phi^i)^*\), on \((W(S, \mathcal{F}), \mathcal{F})\) by
\[
A \to (\phi^i)^* \circ A \circ (\phi^{-1})^*
\]
where \(A = (A_t)_{t \in S/\mathcal{F}} \in W(S, \mathcal{F})\) is a random operator. The cross-product von Neumann algebra \(W(S, \mathcal{F}) \rtimes (\phi^i)\) is a type II\(\frac{1}{4}\) factor.

**Remark** The algebra \(W(S, \mathcal{F}) \rtimes (\phi^i)\) \(\mathbb{R}\) represents morally the set of measurable functions on the noncommutative space of the orbits of \(\phi^i\) (ie the closed points). As explained in [Section 3] [Lei03] this matches with Connes’s approach to the zeta function of a function field.
Proof. 1] This assertion follows from [12].
2] The fact that $x^{-1}\tilde{g}$ defines a leafwise metric on $T_{l,x}\mathcal{F}$ is a consequence of Assumption (ii) above. Since $\psi^0_x = \text{Id}$ and $q^*\lambda_{\tilde{g}} = q\lambda_{\tilde{g}}$, one checks that one can write

$$\left(\psi_t^*\right)^*(\lambda_{\tilde{g}}) = \lambda_{\tilde{g}} + xe^{-t} \lim_{k \to +\infty} d_{\mathcal{F}} B^{t,x}_k$$

(14)

where $(B^{t,x})_{t \in \mathbb{R}, x > 0}$ is a smooth family of elements of $\mathcal{A}_\mathcal{F}^1(\mathcal{L})$ such that $B^{t,x} = q^* B^{t,qx}$. See the proof of Lemma 8 for a similar argument. The fact that $(\phi^j)^*[\lambda_g] = e^j[\lambda_g]$ is then a direct consequence of [13].

3] The fact that

$$\mu = \lambda_g \mu_\mathcal{L} \, dx = \lambda_{\tilde{g}} \mu_\mathcal{L} \frac{dx}{x}$$

induces a measure on $(S, \mathcal{F})$ is a consequence of Lemma [61] and of Assumption (ii) $(q^*\tilde{g} = q\tilde{g})$. It is clear that the fraction

$$\frac{\lambda_g \mu_\mathcal{L} \, dx}{\lambda_{\tilde{g}}} = \mu_\mathcal{L} \, dx$$

defines a transverse measure on $(S, \mathcal{F})$.

4] Let us first check that $W(\mathcal{L}, \mathcal{F})$ is a type $\Pi_{\infty}$-factor. Denote by $\tau$ the trace on $W(\mathcal{L}, \mathcal{F})$ induced by the transverse measure $\mu_\mathcal{L}$. Consider a chart $(A_i, f_i)$ as in Definition [11]. Consider a finite orthonormal family $h_j \in C_0^\infty(U_i, \mathbb{R})$ ($1 \leq j \leq j_0$). So:

$$\forall j, l \in \{0, \ldots, j_0\} \text{ with } j \neq l, \quad \int_{U_i} h_j^2 \lambda_{\tilde{g}} = 1, \quad \int_{U_i} h_j h_l \lambda_{\tilde{g}} = 0.$$  

For $k \in \mathbb{N}$, denote by $\pi(k, j_0) \in W(\mathcal{L}, \mathcal{F})$ the projection defined by

$$f_i^* \left( \sum_{j=1}^{j_0} h_j \langle \cdot, h_j \rangle 1_{p^j \mathbb{Z}_p} \right).$$

It is clear, as $k$ and $j_0$ vary, that the $\tau(\pi(k, j_0))$ do not belong to a ladder of $\mathbb{R}^+$ and can be arbitrarily large. Therefore $W(\mathcal{L}, \mathcal{F})$ is a type $\Pi_{\infty}$-factor. This implies easily that $W(S, \mathcal{F})$ is a type $\Pi_{\infty}$-von Neumann algebra whose center is $L^\infty(\mathbb{R}^{n+}, q^*)$. Now denote by $\tau_\Lambda$ the trace on $W(S, \mathcal{F})$ induced by $\Lambda$ (See [Section 3] [Lei03] for details). Then, Lemma 6 and the definition of $\phi^j$ allow to see that $\tau_\Lambda \circ (\phi^j)^* = e^{-t} \tau_\Lambda$. Using Connes’s results [pages 494 and 495] [Co94], one then checks that $W(S, \mathcal{F}) \rtimes (\phi^j)_*, \mathbb{R}$ is of type III. Moreover, one has:

$$S(\mathcal{W}(S, \mathcal{F}) \rtimes (\phi^j)_*, \mathbb{R}) \cap \mathbb{R}^{++} = \{\lambda > 0/ (\phi^j)^*(\lambda) = \text{Id} \} = q^\mathbb{Z}.$$  

Then [page 473] [Co94] implies that $W(S, \mathcal{F}) \rtimes (\phi^j)_*, \mathbb{R}$ is of type $\Pi_{\infty}$.

We think of $\mathcal{L}$ as a set of renormalizable quantum field theories, if we write $\psi^t_x = R_{x, x} e^{-t}$ then we recognize the general scheme of the method of the renormalization group flow à la Wilson ([page 554] [QFT]). The condition $\psi^t_{q^*} (q \cdot l) = q \cdot \psi^t_x (l)$ in [12] means that $q$ induces an action on $\mathcal{L}$ which commutes with the renormalization group flow up to rescaling: $R_{q^*} e^{-t} (q \cdot l) = q \cdot R_{x} e^{-t} (l)$ for any $l \in \mathcal{L}$.

Let $j \in \{0, 1, 2\}$. Denote by $\mathcal{H}_j \subset \mathcal{A}_\mathcal{F}^j(S)$ (see Definition [3]) the subspace of complex leafwise harmonic forms and by $\pi^j_\mathcal{L}$ the orthogonal projection onto $\mathcal{H}_j \subset \mathcal{A}_\mathcal{F}^j(S)$, see Theorem [4] and Proposition [4] for more on $\pi^j_\mathcal{L}$. Now, denote by $C(\Lambda)$ the Ruelle-Sullivan current
associated to the transverse measure $\Lambda = \mu_c dx$ of $(S, F)$. Then, one defines a scalar product on $H^2_\alpha$ by the formula $<\omega; \omega^* > = \langle \omega \cup \overline{\omega}; C(\Lambda) \rangle$ where $\overline{\omega}$ denotes the leafwise Hodge star of the metric $g$. We denote by $H^2_\alpha$ the $L^2$-completion of $H^2_\alpha$. The operator $\pi^j_\alpha \circ (\phi^j)^*$ defines a one parameter group acting on $H^2_\alpha$ (see Proposition 4 for details). We write $e^{t \Theta_j} = \pi^j_\alpha \circ (\phi^j)^*$ where the infinitesimal generator $\Theta_j$ defines an unbounded operator on the Hilbert space $H^2_\alpha$.

**Theorem 2.** Assume that the closed orbits $\gamma$ of the flow $\phi^j$ acting on $S = \mathcal{C} \times \mathbb{R}^{+,*}$ are non degenerate. Assume the four properties (i) to (iv), that $W(\mathcal{L}, F)$ is a factor and that $\mathcal{L}$ has a dense leaf. Let $\alpha \in C^\infty_c(\mathbb{R}, \mathbb{R})$.

1) For each $j \in \{0, 1, 2\}$, $\int_{\mathbb{R}} \alpha(t)e^{t \Theta_j} dt$ acting on $H^2_\alpha$ is trace-class and one has:

$$\text{TR} \int_{\mathbb{R}} \alpha(t)e^{t \Theta_j} dt = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t)e^{\frac{2i\pi\nu t}{\log q}} dt, \quad \text{TR} \int_{\mathbb{R}} \alpha(t)e^{t \Theta_2} dt = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t)e^{t + \frac{2i\pi\nu}{\log q}} dt.$$  

Moreover, there exists a finite subset $\{\rho_1, \ldots, \rho_{2g}\} \subset \mathbb{C}$ such that

$$\text{TR} \int_{\mathbb{R}} \alpha(t)e^{t \Theta_1} dt = \sum_{j=1}^{2g} \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t)e^{t(\rho_j + \frac{2i\pi\nu}{\log q})} dt.$$  

2) One has

$$\forall j \in \{1, \ldots, 2g\}, \quad \Re \rho_j = \frac{1}{2}.$$  

3) One has

$$\sum_{j=0}^{2g} \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}} \alpha(t)e^{t \Theta_j} dt = \chi_\Lambda(F)\alpha(0) + \sum_{\gamma} \sum_{k \geq 1} l(\gamma) \left( e^{-k\ell(\gamma)}\alpha(-k\ell(\gamma)) + \alpha(k\ell(\gamma)) \right)$$

where $\chi_\Lambda(F)$ denotes Connes’s $\Lambda$–Euler characteristic of $(S, F)$ (Co94) and $\gamma$ runs over the set of primitive closed orbits of $\phi^j$ and $l(\gamma)$ denotes the length of $\gamma$.

**Remark.** Define the Ruelle zeta function

$$\zeta_S(s) = \Pi_{\gamma} \frac{1}{1 - \exp(-s l(\gamma))}$$

where $\gamma$ runs over the set of primitive compact orbits of $\phi^j$ and $l(\gamma)$ denote its length. Deninger told us that Illies’s result [Illies99] and Theorem 2 should imply that $\zeta_S(s)$ is an alternate product of regularized determinants. Then Theorem 2 should imply (along the lines of Deninger’s formalism): meromorphic extension of $\zeta_S(s)$ to $\mathbb{C}$, functional equation $s \leftrightarrow 1 - s$, Riemann hypothesis (or Weil’s Theorem type) for $\zeta_S(s)$.

**Open Question 2.**

1) Let $Y$ be a smooth projective absolutely irreducible curve over $\mathbb{F}_q$ admitting a rational point. Does there exist a laminated foliated space $(S_Y = \mathcal{C} \times \mathbb{R}^{+,*}, F, g, \phi^j)$ satisfying all the assumptions of Proposition 4 and Theorem 2 and the following assumption:

(A) One has a natural bijection $w \mapsto \gamma_w$ between the set of closed points of $Y$ and the set of primitive closed orbits of $(S_Y, \phi^j)$ satisfying $\log Nw = l(\gamma_w)$.

If the answer is yes, one should obtain, via Theorem 2 a new proof of Weil’s Theorem.
2] Is it possible to interpret $L_Y$ as an attractor of the renormalization group (semi-)flow acting on a suitable infinite dimensional space of lagrangians? (compare with the end of [De01] and [pages 30 and 559][QFT]).

Of course, when $Y$ is an elliptic curve over $\mathbb{F}_q$, Deninger has shown ([De02]) that the answer to part 1] of the previous open question is yes.

4. Analytic results on $\mathcal{L}$

4.1. Sobolev spaces on $\mathbb{Z}_p^m$.

Recall that a character $\chi$ of $\mathbb{Z}_p^m = \bigoplus_{j=1}^m \mathbb{Z}_p$ may be written as:

$$\chi = \bigoplus_{j=1}^m \sum_{l=0}^{n_j} \frac{a_{l,j}}{p^l}$$

where the $a_{l,j}$ belong to $\{0,\ldots,p-1\} \subset \mathbb{Z}_p$ and $a_{n_j,j} \neq 0$ if $n_j \geq 1$. Notice that the $a_{l,j}$ are unique for $l \geq 1$.

If $\chi \neq 0$, we set

$$|\chi| = \max_{1 \leq j \leq m, n_j \geq 1} |a_{n_j,j}|_p.$$ 

If $\chi = 0$, we set $|0| = 0$.

**Definition 4.**
1] Let $k \in \mathbb{N}$ and $u \in L^1(\mathbb{Z}_p^m, d\mu_{\mathbb{Z}_p^m})$. We say that $u$ belongs to the Sobolev space $H^k(\mathbb{Z}_p^m)$ if the function

$$\theta \rightarrow \sum_{\chi \in \hat{\mathbb{Z}}_p^m} |\chi|^k \hat{u}(\chi) \langle \chi, -\theta \rangle$$

belongs to $L^2(\mathbb{Z}_p^m, d\mu_{\mathbb{Z}_p^m})$. Here, $\hat{u}(\chi)$ denotes the Fourier transform with the convention $\hat{1}_{\mathbb{Z}_p^m}(0) = 1$.

2] The operator $\Delta_p$ defined by

$$\Delta_p(u)(\theta) = \sum_{\chi \in \hat{\mathbb{Z}}_p^m} |\chi|^2 \hat{u}(\chi) \langle \chi, -\theta \rangle$$

induces a bounded operator from $H^{k+2}(\mathbb{Z}_p^m)$ to $H^k(\mathbb{Z}_p^m)$ for any $k \in \mathbb{N}$.

4.2. Sobolev spaces and harmonic forms on $\mathcal{L}$.

The measure $\lambda_{\tilde{g}}\mu_{\mathcal{L}}$ (see Lemma[3] and the leafwise metric $\tilde{g}$ allow to define a scalar product on $C^0(\mathcal{L}, \wedge^j T^*\mathcal{L})$ ($0 \leq j \leq 2$). The axioms of Definition[1] (eg the fact that $M \in \text{GL}_m(\mathbb{Z}_p)$) show that the following definition makes sense.

**Definition 5.** Let $(k, l) \in \mathbb{N}^2$. We say that a function $u : \mathcal{L} \to \mathbb{C}$ belongs to the Sobolev space $H^{k,l}(\mathcal{L}, \mathbb{C})$ if for any chart $(A_j, f_j)$ as in Definition[1] and any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq l$ the function:

$$(z, \theta) \rightarrow \sum_{\chi \in \hat{\mathbb{Z}}_p^m} (1 + |\chi|^k) \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} u \circ f_j^{-1}(z, \chi) \langle \chi, -\theta \rangle$$
belongs to $L^2(U_j \times \mathbb{Z}_p^m, dx_1 dx_2 \otimes d\mu_{\mathbb{Z}_p^m})$. Here, $\widetilde{\omega}$ denotes the Fourier tranform with respect to the second variable (ie $\theta$). Similarly one defines Sobolev spaces $H^{k,l}(\mathcal{L}, \Lambda^* T^* \mathcal{L})$ of differential forms.

**Lemma 7.** One defines a transverse $p-$adic Laplacian $\Delta_{p,T}$ acting on $\mathcal{A}_{\mathcal{T}}^*(\mathcal{L})$ (see Definition 3) by setting in each chart $(A_j, f_j)$ as in Definition 4

$$\forall u \in \mathcal{A}_{\mathcal{T}}^*(\mathcal{L}), \; \Delta_{p,T}(u|_{A_j}) = (f_j)^* \Delta_p((f_j^{-1})^* u|_{A_j}).$$

**Proof.** We prove the result for leafwise differential forms of degree one. Consider $u \in \mathcal{A}_{\mathcal{T}}^1(\mathcal{L})$ having compact support in $A_i \cap A_j$ (with the notations of Definition 4). We are going to show that

$$(f_i \circ f_j^{-1})^* \Delta_p((f_j^{-1})^* u) = \Delta_p((f_j^{-1})^* u).$$

This will prove that the operator $\Delta_{p,T}$ is intrinsically defined on $\mathcal{L}$. From the assumptions of Definition 4 we deduce that $f_i(A_i \cap A_j)$ is of the form $\Omega_{i,j} \times \mathbb{Z}_p^m$. Recall that

$$f_i \circ f_j^{-1}(z, \theta) = (H(z), G(\theta) = M\theta + B)$$

with $M \in \text{GL}_m(\mathbb{Z}_p)$.

For any tangent vector $v \in T_z \mathcal{L}$, we then have

$$(f_i \circ f_j^{-1})^* \Delta_p((f_j^{-1})^* u)(z, \theta) = \sum_{\chi \in \mathbb{Z}_p^m} |\chi|^2 \int_{\mathbb{Z}_p^m} ((f_j^{-1})^* u)(H(z), \xi)(D_z H(v)) \langle \chi, \xi - G(\theta) \rangle d\mu_{\mathbb{Z}_p^m}(\xi).$$

In this integral we make the change of variable $\xi = G(\theta')$. We then have

$$< \chi, G(\theta') - G(\theta) >= < \chi, G(\theta' - \theta) >= < ^t G(\chi), \theta' - \theta >.$$

Since $M \in \text{GL}_m(\mathbb{Z}_p)$ we observe that $\chi \rightarrow ^t G(\chi)$ defines a bijection of $\mathbb{Z}_p^m$ satisfying $| ^t G(\chi)| = |\chi|$. One then gets immediately that

$$(f_i \circ f_j^{-1})^* \Delta_p((f_j^{-1})^* u) = \Delta_p((f_j^{-1})^* u).$$

\[\square\]

**Theorem 3.**

1] Assumption (iv) of Section 3 is equivalent to the following (intrinsic) condition. Any leafwise $C^\infty$ harmonic $1-$form $\omega \in C^0(\mathcal{L}, \Lambda^1 T^* \mathcal{L})$ satisfies $\Delta_{p,T} \omega = 0$.

2] Assume Assumptions from (i) to (iv) of Section 3. Then the vector space $\mathcal{H}^1_\mathcal{L}$ of real harmonic $1-$forms $\omega \in C^0(\mathcal{L}, \Lambda^1 T^* \mathcal{L})$ is of finite dimension $2g$ where $g \in \mathbb{N}$.

**Proof.** 1] Left to the reader.

2] Denote by $\Delta_{\mathcal{L}}$ the (essentially self-adjoint) leafwise signature laplacian associated with the metric $\bar{g}$ as in Assumption (ii) of Section 3. Notice that $\Delta_{\mathcal{L}}$ is constant in $\theta$ in each chart $(A_j, f_j)$. Since $\mathcal{L}$ is compact, an ellipticity argument shows that the operator $\Delta_{\mathcal{L}} + \Delta_{p,T}$ is Fredholm from $\bigwedge_{k+l=2}^k H^{k+l}(\mathcal{L}, \Lambda^* T^* \mathcal{L})$ to $L^2(\mathcal{L}, \Lambda^* T^* \mathcal{L})$. Indeed, one constructs a parametrix by considering in each chart $(A_j, f_j)$ the operator:

$$\omega(z, \theta) \rightarrow \sum_{\chi \in \mathbb{Z}_p^m} (1 + |\chi|^2 + \Delta_{\mathcal{L}})^{-1} \widetilde{\omega}(z, \chi) \langle \chi, -\theta \rangle.$$
Now, observe that Assumption (iv) implies that $\mathcal{H}_L^1 \subset \ker(\Delta_L + \Delta_p \tau)$. Then one gets that $\mathcal{H}_L^1$ is finite dimensional. Moreover, the Hodge star $\star$ induces a complex structure on $\mathcal{H}_L^1$ so this dimension is an even integer $2g$.

**Proposition 3.** The scalar product defined at the beginning of this Subsection induces a hermitian scalar product $\langle ; , \rangle$ on $\mathcal{H}_L^1 \otimes \mathbb{C}$. There exists an orthonormal basis $(\omega_1, \ldots, \omega_{2g})$ of $\mathcal{H}_L^1 \otimes \mathbb{C}$ of eigenvectors for the action of $q$ on $\mathcal{H}_L^1 \otimes \mathbb{C}$. More precisely, for each $j \in \{1, \ldots, 2g\}$, there exists $\rho_j \in \mathbb{C}$ such that $\Re \rho_j = \frac{1}{2}$ and $q^*(\omega_j) = q^{\rho_j} \omega_j$.

**Proof.** Lemma 1 and Assumption (ii) of Section (eg $q^*(\tilde{g}) = q\tilde{g}$) allow to see that $q^*$ induces an operator acting on $\mathcal{H}_L^1 \otimes \mathbb{C}$ of the form $\sqrt{q}U$ where $U$ is unitary. This proves the result.

5. **Proof of Theorem 2**

5.1. **Leafwise Hodge Decomposition and Heat operator.**

We begin with describing a finite system of local foliated charts of $(S = \mathcal{L} \times \mathbb{R}^{+}, \mathcal{F})$ with the notations of Definition 1.

Set $I_1 = \{q^{\frac{1}{2}}, q\}$ and $I_2 = \{q^{-\frac{1}{2}}, q\}$. Set $\tilde{A}_{j,2} = \{(q^k \cdot l, q^k x)\}_{k \in \mathbb{Z}}$, $(l, x) \in A_j \times I_2$ and $F_{j,2}(\{q^k \cdot l, q^k x\}_{k \in \mathbb{Z}}) = (f_j(l), x) \in U_j \times \mathbb{Z}_p \times I_2$.

Set also $\tilde{A}_{i,1} = \{(q^k \cdot l, q^k x)\}_{k \in \mathbb{Z}}$, $(l, x) \in A_i \times I_1$ and $F_{i,1}(\{q^k \cdot l, q^k x\}_{k \in \mathbb{Z}}) = (f_i(l), x) \in U_i \times \mathbb{Z}_p \times I_1$.

Then, the $(\tilde{A}_{i,1}, F_{i,1}), (\tilde{A}_{j,2}, F_{j,2})$ $(1 \leq i, j \leq N)$ define a finite open cover of foliated charts of $(S, \mathcal{F})$. The transition maps are given in the following way. If $(z, \theta, x) \in f_j(A_j \cap A_i) \times \{q^{\frac{1}{2}}, q\}^\bot$, one has

$$F_{i,1} \circ F_{j,2}^{-1}(z, \theta, x) = (f_i \circ f_j^{-1}(z, \theta, x)).$$

If $(z, \theta, x) \in f_j(A_j \cap q^{-1}(A_i)) \times \{q^{-\frac{1}{2}}, 1\}$, one has

$$F_{i,1} \circ F_{j,2}^{-1}(z, \theta, x) = (f_i \circ q \circ f_j^{-1}(z, \theta, qx)).$$

We can now state the following:

**Definition 6.** Let $k \in \mathbb{N}$ and $j \in \{0, 1, 2\}$. We say that an $\mathcal{L}^2$–leafwise differential form $\omega \in \mathcal{L}^2(S, \wedge^j T^* \mathcal{F})$ belongs to the space $H_{0,k}(S, \wedge^j T^* \mathcal{F})$ if in any chart of the type $(\tilde{A}_{i,1}, F_{i,1}), (\tilde{A}_{j,2}, F_{j,2})$ as above, $P \omega(z, \theta, x) \in L^2$ for any differential operator of degree $\leq k$ $P$ in the variables $(\mathbb{R}^2, \mathbb{S}^2, x)$. We set:

$$H_{0,+\infty}(S, \wedge^j T^* \mathcal{F}) = \cap_{k \in \mathbb{N}} H_{0,k}(S, \wedge^j T^* \mathcal{F}).$$

**Theorem 4.** Let $j \in \{0, 1, 2\}$.

1) We have the following leafwise Hodge decomposition:

$$H_{0,+\infty}(S, \wedge^j T^* \mathcal{F}) = \mathcal{H}_{0,+\infty}^j \oplus \mathcal{H}_{0,+\infty}^j \oplus \mathcal{H}_{0,+\infty}^j \oplus \mathcal{H}_{0,+\infty}^j \oplus \mathcal{H}_{0,+\infty}^j \oplus \mathcal{H}_{0,+\infty}^j$$

where $\mathcal{H}_{0,+\infty}^j$ denotes the set of leafwise harmonic forms, $\delta$ denotes the adjoint of the leafwise exterior derivative $d_{\mathcal{F}}$ of $S$. The orthogonality is with respect to the $\mathcal{L}^2$–scalar product. Moreover, one has

$$\mathcal{H}_{0,+\infty}^1 = \mathcal{H}_{0,+\infty}^0 \oplus \mathcal{H}_{0,+\infty}^0.$$
where \( \mathcal{H}^{1,0}_{r,0,+\infty} \) and \( \mathcal{H}^{0,1}_{r,0,+\infty} \) denote respectively the harmonic forms of type \((1,0)\) and \((0,1)\).

2] Denote by \( \pi^j \) the orthogonal projection from \( H_{0,+\infty}(S, \wedge^j T^*F) \) onto \( \mathcal{H}^j_{r,0,+\infty} \). Then \( \pi^j \circ (\phi^t)^* \) defines a one parameter group acting on \( \mathcal{H}^j_{r,0,+\infty} \).

Proof.
1] The foliated space \((S, F)\) is morally closed to a Riemannian foliation. One has just to adapt to our context the proof of Theorem 1.1 of [A-K01]. We leave the details to the reader.

2] Denote, for \( j \in \{0,1,2\} \), by \( H^j_{0,+\infty} \) the reduced leafwise cohomology group associated with \( H_{0,+\infty}(S, \wedge^j T^*F) \). Part 1 allows to identify \( \mathcal{H}^j_{r,0,+\infty} \) with \( \mathcal{H}^j_{0,+\infty} \) on which \((\phi^t)^*\) acts naturally. One then gets easily the result. \( \square \)

Now we introduce a particular partition of unity of \( S \). Consider \( \beta_1 \in C^\infty_c([q^{3/2}, q], \mathbb{R}) \) and \( \beta_2 \in C^\infty_c([q^{3/2}, q], \mathbb{R}) \) such that:

\[
\forall x \in [q^{3/2}, q], \quad \beta_1(x) + \beta_2(x) = 1, \quad \forall x \in [q^{-3/2}, q], \quad \beta_1(x) + \beta_2(q^{-1}x) = 1,
\]

\[
\forall x \in [1, q^{3/2}], \quad \beta_2(x) = 1, \quad \forall x \in [q^{3/2}, q^{3/2}], \quad \beta_1(x) = 1.
\]

Next we set \( \tilde{V}_j = \{([q^k \cdot l, q^k x])_{k \in \mathbb{Z}}, \ l \in \mathcal{L}, x \in I_j \} \) for \( j = 1, 2 \). If \([([q^k \cdot l, q^k x])_{k \in \mathbb{Z}}] \in \tilde{V}_j \) with \( x \in I_j \), we set

\[
\tilde{F}_j([([q^k \cdot l, q^k x])_{k \in \mathbb{Z}}]) = (l(x), \tilde{\beta}_j([([q^k \cdot l, q^k x])_{k \in \mathbb{Z}}]) = \beta_j(x).
\]

By construction one has \( \tilde{\beta}_1(y) + \tilde{\beta}_2(y) \equiv 1 \) on \( S \). Now we may state the:

**Definition 7.** If \( \omega \in H_{0,k}(S, \wedge^j T^*F) \) and \( t \in \mathbb{R}^{+*} \), we set:

\[
\mathcal{R}_t(\omega) = \sum_{j=1}^{2} \left( \tilde{F}_j^{-1} e^{-t \Delta_{p,T}} \tilde{F}_j \right) (\tilde{\beta}_j \omega)
\]

where \( \Delta_{p,T} \) is defined in Lemma 7.

**Remark.** Since the operator \( \Delta_p \) of Definition 4 does not commute nicely with the multiplication by \( p \) one cannot define a \( p \)-adic transversal Laplacian on \((S, F)\). But the operator \( \Delta_{p,T} \) allows to define such a \( p \)-adic transversal Laplacian in each open subset \( \tilde{V}_1, \tilde{V}_2 \) of \( S \).

**Proposition 4.**
1] Let \( \omega \in \mathcal{H}^j_{r,0,+\infty} \). Then for any real \( t > 0 \), \( \mathcal{R}_t(\omega) = \omega \) and, \( \omega \) is continuous on \( S \) and constant in \( \theta \) in the above local charts \((\tilde{A}_{i,1}, F_{i,1}),(\tilde{A}_{j,2}, F_{j,2})\). Thus \( \mathcal{H}^j_{r,0,+\infty} = \mathcal{H}^j_r \) (see notation before Theorem 1) and

\[
\mathcal{H}^{1,0}_{r,0,+\infty} = \mathcal{H}^{1,0}_r, \quad \mathcal{H}^{0,1}_{r,0,+\infty} = \mathcal{H}^{0,1}_r.
\]

2] The operators \( \pi^j \circ (\phi^t)^* \) define a one parameter group acting on the Hilbert space \( \mathcal{H}^j_r \) of Theorem 2.

Proof.
Lemma 8. Let $\tilde{g}$ be an orthonormal basis of $H^i_\tau(x)$. Then, for each $t > 0$, one checks that $R_t(\omega)$ is a smooth leafwise harmonic form which is continuous on $S$. Assumption (iv) of Section 3 allows to see that $R_t(\omega)$ is constant in $\theta$ in the charts $(\tilde{A_i}, (F_i))$, $(A_j, (F_j))$. Letting $t$ go to $0^+$ one gets that $\omega$ is constant in $\theta$ in these local charts and is equal to $R_t(\omega)$ for any $t > 0$.

One checks that $(\phi^t)^*$ acts continuously on $H_{0, +\infty}(S, \wedge^* T^*F)$ with respect to the $L^2$-scalar product. The result then follows from part 1] and Theorem 4.2]. \hfill \Box

5.2. Proof of Theorem 2.1].

Recall that $e^{i\Theta_j} = \pi^j_+ \circ (\phi^t)^*$. Actually the results of this Subsection allow to (re)prove, using explicit computations, that $\pi^j_+ \circ (\phi^t)^*$ defines a one parameter group acting on $H^2_\tau$.

Recall that $\mathcal{L}$ has a dense leaf. Then a continuous function on $\mathcal{L}$ which is constant along the leaves is constant. Then one checks easily that $(x^{2\pi i\nu\theta} \omega_j)_{\nu \in \mathbb{Z}}$ is an orthonormal basis of $H^0_\tau \otimes_\mathbb{R} \mathbb{C}$ and that:

$$\text{TR} \int_\mathbb{R} \alpha(t)e^{i\Theta_0} dt = \sum_{\nu \in \mathbb{Z}} \int_\mathbb{R} \alpha(t) e^{2\pi i \nu \theta / \log q} dt.$$

In the same way, one checks that $(x^{2\pi i\nu \theta} - \nu \omega_j)_{\nu \in \mathbb{Z}}$ is an orthonormal basis of $H^2_\tau \otimes_\mathbb{R} \mathbb{C}$ (recall that $\lambda_\tilde{g}$ is defined in Assumption (ii) of Section 3). Then one easily checks that:

$$\text{TR} \int_\mathbb{R} \alpha(t)e^{i\Theta_2} dt = \sum_{\nu \in \mathbb{Z}} \int_\mathbb{R} \alpha(t) e^{t + 2\pi i \nu \theta / \log q} dt.$$

Next, using Proposition 3 and its notations, one checks that

$$(x^{2\pi i\nu \theta} - \nu \omega_j)_{\nu \in \mathbb{Z}}$$

is an orthonormal basis of $H^1_\tau \otimes_\mathbb{R} \mathbb{C}$.

**Lemma 8.** Let $j \in \{1, \ldots, 2g\}$ and $\nu \in \mathbb{Z}$. Then one has

$$(\phi^t)^* (x^{-\rho_j} x^{2\pi i\nu \theta} \omega_j) = e^{t(\rho_j + 2\pi i\nu \theta / \log q)} x^{-\rho_j - 2\pi i\nu \theta / \log q} \omega_j + d_x B^t$$

where $(B^t)_{t \in \mathbb{R}}$ is a smooth family of smooth leafwise functions on $S$ continuous on $S$.

**Proof.** One considers only the case $\nu = 0$ in order to simplify the notations. Using the proof of the Poincaré lemma and the fact that $\psi_x^0 = \text{Id}$, one can write along the leaves of $\mathcal{L}$:

$$\forall x \in \mathbb{R}^+, (\psi_x^t)^* (\omega_j x^{-\rho_j}) = \omega_j x^{-\rho_j} + d_x A^t x$$

where $(A^t x)$ is a smooth family of smooth leafwise functions on $S$. Now we use the $\beta_l$ and $l_i$ ($l = 1, 2$) introduced in Subsection 6.1. For $x \in \mathbb{R}^+$ such that $q^{-k} x \in I_i$ with $k \in \mathbb{Z}$, we set $B^t_l x = e^{\rho_j} \beta_l(q^{-k} x)(q^{-k} x)^* A^{t,q^{-k} x}$. Otherwise we set $B^t_l x = 0$. Then, using the formula $q^* \omega_j = q^{\rho_j} \omega_j$, one gets the result by setting $B^t = B^t_1 + B^t_2$. \hfill \Box

Using Proposition 3 and the previous Lemma one then checks that

$$\text{TR} \int_\mathbb{R} \alpha(t)e^{i\Theta_1} dt = \sum_{j = 1}^{2g} \sum_{\nu \in \mathbb{Z}} \int_\mathbb{R} \alpha(t) e^{t(\rho_j + 2\pi i\nu \theta / \log q)} dt.$$
5.3. Proof of Theorem 2.3.

Notice that in the previous subsection we have shown that for each \( j \in \{0, 1, 2\} \), the operator \( \int_{\mathbb{R}} \alpha(s) \pi^j_\tau \circ \phi^*_s \, ds \circ \pi^j_\tau \) is trace class.

Jesus Alvarez-Lopez pointed out to us the following lemma:

**Lemma 9.** (Alvarez-Lopez) Set \( \Theta(l, x) = \phi^{-\log x}(l, 1) \) and \( q(l, x) = (q \cdot l, qx) \) where \((l, x) \in \mathcal{L} \times \mathbb{R}^{+*} \). Then one has:

\[
\Theta^{-1} \circ \phi^j \circ \Theta(l, x) = (l, xe^{-t}).
\]

Moreover, for any \((l, x) \in \mathcal{L} \times \mathbb{R}^{+*} \), one has \( \Theta^{-1} \circ q \circ \Theta(l, x) = (\psi^{-\log q} \circ q(l), qx) \).

This Lemma shows that it is enough to prove Theorem 2.3 when the flow \( \phi^t \) is of the form \( \phi^t(l, x) = (l, xe^{-t}) \), by abuse of notation we shall still write \( \mathcal{R}_t, \pi^j_\tau \) instead of \( \Theta^* \circ \mathcal{R}_t \circ (\Theta^{-1})^*, \Theta^* \circ \pi^j_\tau \circ (\Theta^{-1})^* \) ....etc.

**Proposition 5.** For each \( j \in \{0, 1, 2\} \), denote by \( \Delta^j_\mathcal{F} \) the leafwise signature Laplacian of \((S = \mathcal{L} \times \mathbb{R}^{+*}, \mathcal{F})\) acting on leafwise differential forms of degree \( j \).

1) For any \((t, t') \in \mathbb{R}^{+*} \times \mathbb{R}^{+*} \) and any \( \alpha \in C^\infty_c(\mathbb{R}, \mathbb{R}) \), the operator

\[
\int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \mathcal{R}_t \circ e^{-t' \Delta^j_\mathcal{F}}
\]

is trace class.

2) For any \((t, t') \in \mathbb{R}^{+*} \times \mathbb{R}^{+*} \), one has:

\[
\sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \mathcal{R}_t \circ e^{-t' \Delta^j_\mathcal{F}} = \sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \pi^j_\tau.
\]

**Proof.** 1) In the case of a standard Riemannian foliation, Alvarez-Lopez and Kordyukov have proved (see \cite{AK03, AK00}) that \( \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ e^{-t' \Delta^j_\mathcal{F}} \) is trace class. Their proof can be adapted in our situation (with a \( p \)-adic transversal).

2) Proceeding as in the proof of Lemma 3.3 of \cite{AK00}, one checks easily that

\[
\sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \mathcal{R}_t \circ e^{-t' \Delta^j_\mathcal{F}}
\]

does not depend on \( t' > 0 \). Moreover, proceeding as in the proof of Lemma 3.2 of \cite{AK00}, one checks that for fixed \( t > 0 \)

\[
\lim_{t' \to +\infty} \sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \mathcal{R}_t \circ e^{-t' \Delta^j_\mathcal{F}} = \sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \mathcal{R}_t \circ \pi^j_\tau.
\]

Since, by Proposition 4, \( \mathcal{R}_t \circ \pi^j_\tau = \pi^j_\tau \), one gets the result. \( \square \)

**Proposition 6.** Assume that the support of \( \alpha \) is contained in \( [-\log \frac{q}{2}, \log \frac{q}{2}] \), then:

\[
\sum_{j=0}^{2} (-1)^j \text{TR} \int_{\mathbb{R}} \alpha(s)(\phi^*_s)^* \, ds \circ \pi^j_\tau = \chi_\Lambda(\mathcal{F}) \alpha(0).
\]

(See the notations of Theorem 2.3.).
Proof. Observe that the length of a closed orbit of $\phi^t$ is at least $\log q$. One has just to adapt the arguments of the proof of Theorem 1.2 and Proposition 4.3 of [A-K00]. □

**Proposition 7.** Assume that the support of $\alpha$ does not meet $[\frac{-\log q}{4}, \frac{\log q}{4}]$, then:

$$
\sum_{j=0}^{2} (-1)^j \text{Tr} \int_{\mathbb{R}} \alpha(s)(\phi^s)^* ds \circ \pi^j = \sum_{j=0}^{2} (-1)^j \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha(s)\text{Tr}_j(D\phi^s)^* \delta_{\phi^s(y)=y} \, d\mu(y) \, ds
$$

where $\text{Tr}_j$ denotes the bundle endomorphism trace of $(D\phi^t)^*$ acting on leafwise exterior forms of degree $j$. The measure $\mu$ is defined in Proposition [23].

Proof. In the left hand side of the equality of Proposition 5 one lets $t$ and $t'$ go to $0^+$ and one gets the desired result. □

From the previous results we deduce that Theorem [23] follows from the:

**Proposition 8.** Assume that the support of $\alpha$ does not meet $[\frac{-\log q}{4}, \frac{\log q}{4}]$, then:

$$
\sum_{j=0}^{2} (-1)^j \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha(s)\text{Tr}_j(D\phi^s)^* \delta_{\phi^s(y)=y} \, d\mu(y) \, ds = \sum_{\gamma} \sum_{k \geq 1} l(\gamma) \left( e^{-kl(\gamma)} \alpha(-kl(\gamma)) + \alpha(kl(\gamma)) \right)
$$

where $\gamma$ runs over the set of primitive closed orbits of $\phi^t$ and $l(\gamma)$ denotes the length of $\gamma$.

Proof. We recall Lemma [8] and the fact that we are reduced to the case $\phi^t(l, x) = (l, xe^{-t})$. Assumptions (i) and (iii) of Section [3] allows to see that $y \in \gamma$, $\det D\phi^t(\gamma)(y)_{T_yF} > 0$. Moreover, using these Assumptions (i) and (iii) (especially [13]) and proceeding exactly as in the proof of Lemma [5] one gets the proposition. □

One has proved the assertion $\Re \rho_j = \frac{1}{2}$ in Proposition 3 as a consequence of Assumption (ii) of Section [4]. Nevertheless, following an argument of Serre [Se60] and of Deninger-Singhof [Prop 4.6] [De-S02] we are going to explain how the assertion $\Re \rho_j = \frac{1}{2}$ follows formally from the equality $(\phi^t)^*[\lambda_g] = e^t [\lambda_g]$ of Proposition [22].

We replace the hermitian scalar product on $H^1_{\tau}$ introduced before Theorem 2 by the following (equivalent) one.

$$
\forall \alpha_1, \alpha_2 \in H^1_{\tau}, \quad <\alpha_1; \alpha_2> = -(\alpha_1 \cup J\alpha_2; C(\Lambda))
$$

where the operator $J$ is multiplication by $\sqrt{-1}$ [resp. $-\sqrt{-1}$] on $H^{1,0}_{\tau}$ [resp. $H^{0,1}_{\tau}$].

**Lemma 10.**

1) For any $\nu \in \mathbb{Z} \setminus \{0\}$, $(x^{\frac{\nu}{\log q}} \lambda_g; C(\Lambda)) = 0$. But $(\lambda_g; C(\Lambda)) \neq 0$.

2) For any $\alpha_1, \alpha_2 \in H^1_{\tau}$ one has:

$$
\forall t \in \mathbb{R}, \quad <e^{t\Theta_1} \alpha_1; e^{t\Theta_1} \alpha_2> = ((\phi^t)^*(\alpha_1 \cup J\alpha_2); C(\Lambda)).
$$

3) For any $\alpha_1, \alpha_2 \in H^1_{\tau}$ one has:

$$
\forall t \in \mathbb{R}, \quad <e^{t\Theta_1} \alpha_1; e^{t\Theta_1} \alpha_2> = e^t <\alpha_1; \alpha_2>.
$$

**Proof.**

1) Easy computation.
2] Recall that we have defined two notions of leafwise reduced cohomology (see Definition 3 and the proof of Theorem 4). We have a natural map between them:
\[ \overline{H}^j_F(S) \to \overline{H}^j_{0,+\infty}. \]
Observe that \( e^{i\Theta_1} \alpha = (\phi^t)^* \alpha \) modulo \( \text{Im} d_F \) and that the Ruelle-Sullivan current is closed. One then checks that
\[ < e^{i\Theta_1} \alpha_1; e^{i\Theta_1} \alpha_2 > = ( (\phi^t)^* \alpha_1 \cup J (\phi^t)^* \alpha_2; C(\Lambda)) . \]
Since \( (\phi^t)^* \) commutes with \( J \) and the complex conjugation, one gets the result.

3] One can write
\[ \pi^2_\tau (\alpha_1 \cup J \alpha_2) = \sum_{\nu \in \mathbb{Z}} c_{\nu} e^{2i\pi \nu} \lambda_g \in H^2_\tau. \]
Then, using parts 1] and 2] one checks that \( < e^{i\Theta_1} \alpha_1; e^{i\Theta_1} \alpha_2 > = ( (\phi^t)^* (c_0 \lambda_g); C(\Lambda)) \) and
\[ < \alpha_1; \alpha_2 >= (c_0 \lambda_g; C(\Lambda)). \]
Since \((\phi^t)^*[\lambda_g] = e^t[\lambda_g]\) by Proposition 2.2 one gets the result.

Now part 3] of the previous Lemma implies that
\[ \frac{d}{dt} ( < e^{i\Theta_1} \alpha_1; e^{i\Theta_1} \alpha_1 > )_{t=0} = < \alpha_1; \alpha_1 >. \]
Therefore one has:
\[ < (\Theta_1 - \frac{1}{2}) (\alpha_1); \alpha_1 > + < \alpha_1; (\Theta_1 - \frac{1}{2}) (\alpha_1) > = 0. \]
Now if \( \alpha_1 \in H^1_\tau \setminus \{0\} \) is such that \( \Theta_1 (\alpha_1) = \rho \alpha_1 \) with \( \rho \in \mathbb{C} \) then one gets \( \rho - \frac{1}{2} = -(\overline{\rho} - \frac{1}{2}) \).
Thus one gets \( \Re \rho = \frac{1}{2} \) which from the equality \( (\phi^t)^*[\lambda_g] = e^t[\lambda_g] \) as desired.

6. Appendix: Renormalization group flow

We first briefly and informally recall Wilson’s view point following [pages 554 and 557][QFT]. We consider a set \( \mathcal{S} \) of QFT defined by lagrangians. Let \( 0 < \Lambda_0 < \Lambda_1 \) be two scales of momenta. For each theory \( L \in \mathcal{S} \), one finds another theory \( R_{\Lambda_1,\Lambda_0}(L) \) which is the effective theory at the scale \( \Lambda_1 \) of the original theory \( L \) at the scale \( \Lambda_0 \).

In terms of Feynman integrals for QFT defined on \( \mathbb{R}^n \) one can write:
\[ \int_{\mathcal{B}(\Lambda_0)} A(\phi) \left( \int_{C(\Lambda_0,\Lambda_1)} e^{-L(\phi+\eta)} \, D\eta \right) \, D\phi = \int_{\mathcal{B}(\Lambda_0)} A(\phi) e^{-R_{\Lambda_1,\Lambda_0}(L)(\phi)} \, D\phi \]
where \( A \) is a function of the field \( \phi \), and \( C(\Lambda_0,\Lambda_1) \) (resp. \( \mathcal{B}(\Lambda_0) \)) denotes the set of fields whose Fourier transform has support in the corona \( \{ \xi \in \mathbb{R}^n, \Lambda_0 \leq |\xi| \leq \Lambda_1 \} \) (resp. the ball \( \{ \xi \in \mathbb{R}^n, |\xi| \leq \Lambda_0 \} \)).

Then the renormalization (semi-)group flow is defined by:
\[ \forall t \in \mathbb{R}^+, \forall (L, \Lambda_1) \in \mathcal{S} \times \mathbb{R}^{+\ast}, \phi^t(L, \Lambda_1) = (R_{\Lambda_1,\Lambda_0}(L), \Lambda_1 e^{-t}). \]
Notice that if \( L \) is a free lagrangian then \( \phi^t(L, \Lambda_1) = (L, \Lambda_1 e^{-t}) \) for any \( t \geq 0 \).

In [CK00] and [CK01] Connes and Kreimer have developed a mathematical theory of renormalization of perturbative QFT. Let \( G \) be the group of characters of the dual Hopf algebra of the enveloping algebra associated with the 1—particle irreducible Feynman graphs.
The unrenormalized theory gives rise to a meromorphic loop \( \gamma(z) \in G, \ z \in \mathbb{C} \). Connes and Kreimer have shown that the renormalized theory is the evaluation at the integer dimension \( z_0 \) of space-time of the holomorphic part \( \gamma_+ \) of the Birkhoff decomposition of \( \gamma \) for the Riemann-Hilbert problem. Moreover they view the renormalization group as a subgroup of \( G \). Then, for massless QFT they recover the action of the renormalization group on lagrangians.

7. Acknowledgements

I thank C. Deninger whose helpful comments on the first version have allowed to improve this paper. I thank A. Abdesselam, J. Alvarez-Lopez, A. Besser, U. Buenke, L. Boutet de Monvel, C. Deninger, D. Harari, S. Haran, M. Marcolli, G. Skandalis and, D. Sullivan for fruitful discussions. Part of this work was done while the author was visiting the universities of Bar Ilan, Beer Sheva, Technion, Tel Aviv and Savoie: he would like to thank these institutions for their very warm hospitality. I thank Etienne Blanchard for his interest in this work and having given me the opportunity to present these results in the \( C^* \)-algebras seminar of Paris.

References

[A-K00] J. A. Alvarez Lopez and Y. Kordyukov: Distributional Betti numbers of transitive foliations of codimension one. Foliations: geometry and dynamics (Warsaw, 2000), World Sci. Publishing, River Edge, NJ, (2002), pages 159-183.

[A-K01] J. A. Alvarez Lopez and Y. Kordyukov: Long time behaviour of leafwise heat flow for Riemannian foliations. Compositio Math. 125 (2001), pages 129-153.

[A-K05] J. A. Alvarez Lopez and Y. Kordyukov: Distributional Betti numbers for Lie foliations. Preprint.

[Co94] A. Connes: Noncommutative Geometry. Academic Press.

[Co99] A. Connes: Trace formula in noncommutative geometry and the zeroes of the Riemann zeta function. Selecta Math. (N.S.) 5 (1999), no. 1, 29-106.

[Co00] A. Connes: Noncommutative Geometry Year 2000, Special Volume GAFA 2000 Part II, pages 481-559.

[Co02] A. Connes: Symétries Galoisiennes et Renormalisation, Séminaire Bourbaphy, Octobre 2002, pages 75-91.

[CK00] A. Connes, and D. Kreimer: Renormalization in quantum field theory and the Riemann Hilbert problem I, Comm. Math. Phys. 210, 2000, No 1, pages 249-273.

[CK01] A. Connes, and D. Kreimer: Renormalization in quantum field theory and the Riemann Hilbert problem II, the \( \beta \) function, diffeomorphisms and the renormalization group, Comm. Math. Phys. 216, 2001, No 1, pages 249-273.

[Co-Ma04a] A. Connes, and M. Marcolli: \( \mathbb{Q} \)-lattices: quantum statistical mechanics and Galois theory, to appear in Journal of Geometry and Physics.

[Co-Ma04b] A. Connes, and M. Marcolli: From Physics to Number theory via Noncommutative Geometry. Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory, to appear in the volume "Frontiers in Number Theory, Physics, and Geometry".

[QFT] P. Deligne and Al: Quantum Fields and Strings: A Course for Mathematicians, vol I. Amer. Math. Soc. Institute for Advanced Study (1996).

[De94] C. Deninger: Motivic \( L\)-functions and regularized determinants . Proc. Symp. Pure Math. 55, 1 (1994), pages 707-743.

[De98] C. Deninger: Some analogies between number theory and dynamical systems on foliated spaces. Doc. Math. J. DMV. Extra volume ICM I, (1998), pages 23-46.
[De99] C. Deninger: *On dynamical systems and their possible significance for Arithmetic Geometry*. In: A. Reznikov, N. Schappacher (eds.), Regulators in Analysis, Geometry and Number Theory. Progress in Mathematics 171, (1999), Birkhauser, pages 29-87.

[De-Si00] C. Deninger and W. Singhof: *A note on dynamical trace formulas*, Dynamical, spectral and arithmetic zeta functions (San Antonio, TX, 1999), Contemp. Math., 290, Amer. Math. Soc., Providence, RI, (2001), pages 41-55.

[De-Si02] C. Deninger and W. Singhof: *Real polarizable Hodge structures arising from foliations*. Annals of Global Analysis and Geometry 21, 2002, pages 377-399.

[De01] C. Deninger: *Number theory and dynamical systems on foliated spaces*. Jahresberichte der DMV. 103, (2001), No 3, pages 79-100.

[De01b] C. Deninger: *A note on arithmetic topology and dynamical systems*. Algebraic number theory and algebraic geometry, Contemp. Math., 300, Amer. Math. Soc., Providence, RI, (2002), pages 99-114.

[De02] C. Deninger: *On the nature of explicit formulas in analytic number theory, a simple example*. Number theoretic methods (Iizuka, 2001), Dev. Math., 8, Kluwer Acad. Publ., Dordrecht, (2002), pages 97-118.

[Der04] B. Deroin: *Non rigidity of hyperbolic Riemann surfaces lamination*. Preprint 2004.

[Ghys99] E. Ghys: *Laminations par surfaces de Riemann*. Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthes, 8, Soc. Math. France, Paris, (1999), pages 49-95.

[G-S77] V. Guillemin and S. Sternberg: *Geometric Asymptotics*. Mathematical surveys and monographs. Number 14. Published by the A.M.S.

[Har05] S. Haran: *Arithmetic as Geometry I. The language of non-additive geometry*. Preprint 2005.

[Ih73] Y. Ihara: *On $(\infty \times p)-$adic coverings of curves (the simplest example)*. Trudy Mat. Inst. Steklov 132 (1973), pages 133-148.

[Illies99] G. Illies: *Cramer Functions and Guinand Equations*. Preprint IHES (1999).

[Lei03] E. Leichtnam: *An invitation to Deninger’s work on arithmetic zeta functions*. Geometry, spectral theory, groups, and dynamics, Contemp. Math., 387, Amer. Math. Soc., Providence, RI, (2005), pages 201-236.

[Lei06] E. Leichtnam: *Renormalization group flow and arithmetic Zeta functions*. In preparation.

[Meyer03] R. Meyer: *On a representation of the Idele class group related to primes and zeros of $L-$functions*. Preprint 2003.

[Mil80] J. Milne: *Étale Cohomology*, Princeton Mathematical Series, 33, (1980).

[Oor73] F. Oort: *Lifting an endomorphism of an elliptic curve to characteristic zero*. Indag. Math. 35, (1973), pages 466-470.

[Po84] J. Polchinski: *Renormalization and effective lagrangians*. Nuclear Physics B231, (1984), pages 269-295.

[Se] J-P. Serre: *Groupes algébriques et corps de classe*. Editions Hermann (1959).

[Se60] J-P. Serre: *Analogues Kachleriens de certaines conjectures de Weil*. Annals of Math. 65, (1960), pages 392-394.

[Si92] J. Silverman: *The arithmetic of elliptic curves*. Graduate text in Math. 106 (1992).

[Sul93] D. Sullivan: *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers*. Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, (1993), pages 543-564.