Mechanism of decoherence-free coupling between giant atoms

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Giant atoms are a new paradigm of quantum optics going beyond the usual local coupling. Building on this, a new type of decoherence-free (DF) many-body Hamiltonians was shown in a broadband waveguide. Here, these are incorporated in a general framework (not relying on master equations) and contrasted to dispersive DF Hamiltonians with normal atoms: the two schemes are shown to correspond to qualitatively different ways to match the same general condition for suppressing decoherence. Next, we map the giant atoms dynamics into a cascaded collision model (CM), providing an intuitive interpretation of the connection between non-trivial DF Hamiltonians and coupling points topology. The braided configuration is shown to implement a scheme where a shuttling system subject to periodic phase kicks mediates a DF coupling between the atoms. From the viewpoint of CMs theory, this shows a collision model where ancillas effectively implement a dissipationless, maximally-entangling two-qubit gate on the system.

I. INTRODUCTION

Engineering decoherence-free (DF) Hamiltonians is a major task in the field of quantum technologies and many-body physics, with special regard to quantum optics implementations [1–7]. In particular, DF mediated Hamiltonians describe coherent interactions, typically between (pseudo) atoms or qubits which crosstalk via a quantum bus (usually some photonic environment) that yet does not introduce decoherence [8]. In terms of Lindblad master equation [9], this implies realizing a net second-order Hamiltonian that couple the atoms to one another, getting rid at once of the (usually present) dissipator term. Thereby one is left with an effective unitary dynamics of the atoms, where the environmental degrees of freedom are eliminated.

One of the typical strategies to achieve DF Hamiltonians is adiabatic elimination in the dispersive regime. In cavity QED, it is typically obtained by coupling a set of atoms far off-resonantly to cavity modes [10–12]. This gives rise to a separation of time scales such that incoherent second-order interactions average to zero, while coherent ones result in an effective Hamiltonian. An analogous working principle underpins DF Hamiltonians in structured photonic lattices [5, 6, 13], which are seeded by tuning the atomic frequency within a photonic bandgap entailing an off-resonant interaction with all the lattice modes (this results in short-range, potentially tunable, inter-atomic couplings).

Recently, a new class of DF Hamiltonians was predicted [14] and experimentally observed [15], which employs giant atoms [16] in broadband waveguides. Giant atoms are a new playground of quantum optics [17–23], where the usual pointlike model of an emitter (normal atom) breaks down. In contrast, as sketched in Fig. 1, a giant atom (typically an artificial two-level system) couples to the field at a discrete set of distinct coupling points (an alternative implementation is an atom in front of a mirror [24–26]). By appropriate engineering, the distance between coupling points can be made several wavelength long. This introduces tunable phase shifts, yielding interference effects unattainable with normal atoms. Notably, these can be harnessed in particular to suppress dissipative interactions, giving rise to DF Hamiltonians [14, 19]. As a distinctive feature of giant atoms, such Hamiltonians can happen to be trivial (i.e., identically zero or with null coupling terms) depending on the coupling points topology. For two giant atoms, for instance, only one out of the three possible topologies leads to non-trivial DF Hamiltonians [14, 15].

So far occurrence of giant atoms DF Hamiltonians was mostly investigated through the explicit master equation of the atoms. This is a rather involved object featuring both local and non-local dissipative terms, based on which the essential mechanism behind emergence of DF Hamiltonians is not straightforwardly interpreted. In the case of the aforementioned DF dispersive Hamiltonians, however, a derivation solely based...
on average Hamiltonian theory [27–31] (thus not relying on master equations) is possible [12]. One thus naturally asks whether a similar picture can be defined for giant atoms and used to study conditions for occurrence of DF Hamiltonians.

With the above motivations, in this work we consider a general framework for deriving DF Hamiltonians in the spirit of average Hamiltonian theory. The main condition (“DF condition”) is to arrange for an interaction Hamiltonian averaging to zero over a coarse-grained time scale (in the interaction picture). Dispersive schemes and giant atoms in a broadband waveguide are compared and shown to be different ways to match the DF condition. This occurs through destructive interference of a continuum of phase factors defining the interaction Hamiltonian in the former case and only a discrete, possibly small, number in the latter. We next focus on giant atoms schemes, reviewing their general description through a collision-model picture and connecting it to the DF-Hamiltonians framework. Is is then shown that each collective collision with all the atoms can be decomposed as a cascade of ordered subcollisions (see Fig. 1), each involving a single coupling point. Topologies yielding a zero effective Hamiltonian (such as serial and nested configurations) correspond to subcollisions combined with their time-reversed analogues so as to produce a zero net evolution. Instead, non-zero DF Hamiltonians are seeded for topologies (such as the braided) that feature combinations of subcollisions and their time-reversed versions not leading to an overall identity evolution.

Moreover, we show that the cascaded-collision picture allows to map giant atoms dynamics into a mediator (embodied by a field time bin) shuttling between the atoms and subject to periodic phase kicks. The equivalent quantum circuit, a sequence of parametric iSWAP and local phase gates, is presented.

This paper starts in Section II by developing a general framework for occurrence of DF Hamiltonians based on the Magnus expansion of the joint propagator in each time interval. We next review in Section III how the necessary requirements for having a DF Hamiltonian are fulfilled by standard dispersive schemes. In Section IV, giants atoms coupled to a broadband waveguide are introduced, showing how they provide a different way to match the DF condition. In Section V, we review how the joint dynamics of giant atoms and field can be described through an average-Hamiltonian approach, decomposing into a sequence of elementary units in each of which the atoms jointly collide with a field time bin. Section VI shows how each collision in turn can be decomposed into cascaded subcollisions, one for each coupling point. This highlights the physical origin of the effective Hamiltonian so as to link it to the coupling points topology, a task carried out in Section VII. In Section VIII, we show that the giant-atoms setup can be seen as an implementation of a scheme where a shuttling qubit, subject to periodic phase gates, mediates an indirect DF coupling between the atoms, and the equivalent quantum circuit is given. The theory developed in Sections VI and VII is extended to a bidirectional (generally chiral) waveguide in Section IX, and to more than two coupling points in Section X. Finally, we draw our conclusions in Section XI.

II. GENERAL SCHEME FOR DECOHERENCE-FREE HAMILTONIANS

Consider an unspecified quantum system $S$ coupled to a quantum environment $E$. The Hamiltonian reads

$$H = H_S + H_E + V,$$  \hspace{1cm} (1)

with $H_S$ ($H_E$) the free Hamiltonian of $S$ ($E$) and $V$ their interaction Hamiltonian. In the interaction picture with respect to $H_0 = H_S + H_E$, the joint state $\sigma_t$ evolves as

$$\dot{\sigma}_t = -i[V_t, \sigma_t]$$  \hspace{1cm} (2)

with $V_t = e^{iH_0 t} V e^{-iH_0 t}$. Thus at time $t$,

$$\sigma_t = U_t \sigma_0 U_t^\dagger$$  \hspace{1cm} (3)

with the propagator $U_t$ given by $U_t = \mathcal{T} \exp[-i \int_{t_0}^t ds V(s)]$, where $\mathcal{T}$ is the usual time ordering operator.

Consider now a mesh of the time axis defined by $t_n = n \Delta t$ with $n = 0, 1, ...$ and $\Delta t$ the time step, in terms of which the propagator can be decomposed as

$$U_t = \prod_{n=1}^{[t/\Delta t]} U_n \quad \text{with} \quad U_n = \mathcal{T} e^{-i \int_{t_{n-1}}^{t_n} ds V(s)}.$$  \hspace{1cm} (4)

Now, if $\Delta t$ is short enough compared to the characteristic time of interaction, applying the Magnus expansion [32] each unitary $U_n$ can be approximated to second order as

$$U_n \simeq 1 - i(\nabla_n + \mathcal{H}_n) \Delta t - \frac{1}{2} \nabla_n^2 \Delta t^2$$  \hspace{1cm} (5)

with $\mathbb{1}$ the identity operator and

$$\nabla_n = \frac{i}{{\Delta t}} \int_{t_{n-1}}^{t_n} ds V_s,$$  \hspace{1cm} (6)

$$\mathcal{H}_n = -\frac{i}{{\Delta t}} \int_{t_{n-1}}^{t_n} ds \int_{s_{n-1}}^{s} ds' |V_s, V_{s'}|.$$  \hspace{1cm} (7)

The averaged interaction $\nabla_n$ and Hamiltonian $\mathcal{H}_n$, respectively of first and second order in the coupling strength, are the two central quantities to consider for implementing DF Hamiltonians. In sketchy terms, one
seeks to fulfill $\nabla_n = 0$ (henceforth referred to as the “DF condition”) in a way that $\mathcal{H}_n$ yields (upon partial trace) a dissipationless effective Hamiltonian of $S$, $H_{\text{eff}}$. This is formalized in detail in the following.

Let $\sigma_n$ be the joint $S$-$E$ state at time $t_n$ and $\rho_n = \mathrm{Tr}_E (\sigma_n)$ the reduced state of the system at the same time. We will consider a coarse-grained time scale defined by $\Delta t$ short enough that (5) holds. In the corresponding continuous-time limit, $t_n \to t$, $\sigma_n \to \sigma$, $\dot{\sigma} \simeq \Delta \sigma_n / \Delta t$ where we set $\Delta \sigma_n = \sigma_n - \sigma_n - 1$ (analogously for $\rho_n$).

We also define
\[
\langle \mathcal{H}_n \rangle_{\rho_0} = \mathrm{Tr}_S \{ \mathcal{H}_n \rho_0 \},
\]
\[
H_{\text{eff}} = \mathrm{Tr}_E \{ \mathcal{H}_n \rho_E \}
\]
with $\mathrm{Tr}_S(E)\{\}$ the partial trace over $S$ ($E$). These are effective Hamiltonians on $E$ and $S$, respectively. When $S$ is multipartite, in particular, $H_{\text{eff}}$ will generally feature mutual couplings between subsystems of $S$.

The following property holds.

**Property.** Let the system and environment be initially in the uncorrelated state $\rho_0 \otimes \rho_E$ with $\rho_0$ ($\rho_E$) the initial state of the system (environment). If
\[
\nabla_n = 0
\]
in each time interval $[t_{n-1}, t_n]$, and
\[
[H_{\mathcal{H}_n}, 1_S \otimes \rho_E] = 0
\]
then in the continuous-time limit
\[
\dot{\rho} = -i [H_{\text{eff}}, \rho].
\]

This embodies a rather general working principle for realizing DF effective Hamiltonians: conditions (10) and (11) entail a unitary reduced dynamics of $S$ generated by the effective Hamiltonian $H_{\text{eff}}$. Among (10) and (11), the former (DF condition) is the most relevant: it means that the interaction Hamiltonian $V_t$ averages to zero over the coarse-grained time scale $\Delta t$.

The above property is easily shown (see Appendix A), from which in particular it turns out that $\sigma_n = \rho_n \otimes \rho_E$ namely $E$ remains in its initial state, uncorrelated with $S$.

A typical case where (11) occurs is when $\mathcal{H}_n$ acts trivially on $E$, then (11) is matched for any $\rho_E$ and $\mathcal{H}_n \equiv H_{\text{eff}}$ (this happens with giant atoms as we will see). Another instance is when $S$ is a two-level system and $E$ a harmonic oscillator with $\mathcal{H}_n \sim \sigma_z b^\dagger b$ (dispersive regime of the Jaynes-Cummings model [33]) (see next section). Then (11) holds when $\rho_E$ is any mixture of Fock states.

Note that the above framework, alongside related approaches [29, 30, 34], bypasses any direct use of master equations or the Born-Markov approximation, being instead mostly based on propagators and Hamiltonians.

### III. DISPERSIVE HAMILTONIANS

A longstanding way for matching condition (10) in quantum optics is coupling atoms to a single- or multi-mode photonic environment dispersively, i.e., off-resonantly. A standard model to illustrate this is a set of identical two-level atoms of frequency $\omega_0$ and ground (excited) state $|g\rangle$ ($|e\rangle$) weakly coupled to a bosonic field. The atoms play the role of system $S$ and the field of environment $E$. Their free Hamiltonians read
\[
H_S = \omega_0 \sum_j \sigma_j^z \sigma^j_j, \quad H_E = \sum_k \omega_k b_k^\dagger b_k,
\]
while the interaction Hamiltonian in the rotating-wave approximation is given by
\[
V = \sum_{j,k} g_{jk} \sigma_j^z b^\dagger_k + \mathrm{H.c.}
\]
with coupling strength $g_{jk}$ generally complex, $\sigma_j = |g\rangle_j / \langle e|$ and $b_k$ bosonic ladder operators of the field (here $k$ labels the field normal modes and in general could comprise both discrete and continuous indexes).

In the interaction picture, $V$ turns into
\[
V_t = \sum_j \sum_k g_{jk} \sigma_j^z b^\dagger_k e^{i\Delta_k t} + \mathrm{H.c.}
\]
with $\Delta_k = \omega_k - \omega_0$ the detuning between mode $k$ and the atomic transition frequency $\omega_0$.

Consider now the off-resonance regime such that the detunings are all much larger than the typical order of magnitude of the interaction, which is generally expressed as $\min_k |\Delta_k| \gg \max_{jk} |g_{jk}|$. Then one can choose a coarse-grained time scale $\Delta t$ such that
\[
\max_k |\Delta_k|^{-1} \ll \Delta t \ll \min_{jk} |g_{jk}|^{-1}.
\]

Accordingly,
\[
\int_{t_{n-1}}^{t_n} dt e^{\pm i\Delta_k t} \simeq 0 \quad \text{for any } k,
\]

hence (15) averages to zero in each time interval $[t_{n-1}, t_n]$ of length $\Delta t$ so as to fulfill the DF condition (10). The Hamiltonian term (7) is given by
\[
\mathcal{H}_n = -\sum_{j,k,k'} \frac{g_{jk} g^*_{jk'}}{2\Delta_k} \sigma_j^z \sigma^k_{j'} + \mathrm{H.c.} + \sum_{j,k,k'} \frac{g_{jk} g^*_{jk'}}{\Delta_k} \sigma_j^z b^\dagger_k b_{k'}
\]
with the primed sum running over all $k,k'$ such that $\omega_k = \omega_{k'}$. For a single atom and only one field mode, this reduces to the interaction Hamiltonian $\sim \sigma_z b^\dagger b$ arising in the dispersive regime of the Jaynes-Cummings model [35].
FIG. 2. Frequency regimes for realizing DF Hamiltonians. The photonic environment typically features frequency bands separated by bandgaps. The dashed line marks the atomic frequency \( \omega_0 \), which couples resonantly with a bandwidth of modes (in red) of the order of the atom-field coupling rate. Dispersive schemes (left panel) work off-resonance by tuning atoms far off-resonance from lattice bands. Instead, DF Hamiltonians with giant atoms (on the right) operate well within a photonic band, which can thus be modeled as infinite.

Choosing \( \rho_E = |0\rangle \), with \(|0\rangle\) the field vacuum state, condition (11) is fulfilled, hence we get the DF effective Hamiltonian [cf. Eq. (9)]

\[
H_{\text{eff}} = - \sum_{j\ell} \sum_k \frac{g_{j\ell k}}{2\Delta_k} \sigma_j^+ \sigma_j + \text{H.c.} \tag{18}
\]

featuring atom-atom couplings.

IV. GIANT ATOMS IN A BROADBAND WAVEGUIDE

A standard way to realize the scheme in the previous section is to couple the atoms to a photonic lattice and tune \( \omega_0 \) far from any band (see Fig. 2). The atoms then interact with the photonic environment far off-resonantly, which results in the separation of time scales (16).

Instead, decoherence-free Hamiltonians via giant atoms work in the regime in which the atomic frequency \( \omega_0 \) is well within a photonic band which can thus be approximated as infinite (see Fig. 2). This is possible due to non-local coupling (the hallmark of giant atoms) as will become clear later. Thus consider a set of giant two-level atoms weakly coupled to a one-dimensional waveguide [36–38] with \( \omega_0 \) well inside a band of the waveguide field. The free atomic Hamiltonians of \( S \) and \( E \) are still given by Eq. (13), where (compared to the general case in the previous section) \( k \) is now intended as the wavevector. The \( j \)th atom is coupled to the waveguide at \( N_j \) distinct coupling points [see Fig. 3(a)], the coordinate of each being \( x_{j\ell} \) with \( \ell = 1, \ldots, N_j \) (such that \( x_{j1} < x_{j2} < \ldots \)). Accordingly, the interaction Hamiltonian in the interaction picture now reads

\[
V_i = \sum_{j\ell} \sum_k g_{j\ell k} \sigma_j^+ \sigma_j \mathrm{e}^{ik\ell x_{j\ell}} + \text{H.c.} \tag{19}
\]

with \( g_{j\ell k} \) the coupling strength to mode \( k \) of the \( \ell \)th coupling point of atom \( j \). Unlike the previous section, resonant modes \( k \simeq \pm k_0 \) (with \( \omega_{k0} = \omega_0 \)) will now dominate, thus (17) does not hold.

The coupling strengths more explicitly read \( g_{j\ell k} = g_k \mathrm{e}^{-ik\ell x_{j\ell}} \), where taking advantage of weak coupling we can approximate \( g_k \simeq g_{k0} \) (which thus become \( k \)-independent). Accordingly, we can write

\[
g_{j\ell k} = g_{k0} \mathrm{e}^{-ik\ell x_{j\ell}} \mathrm{e}^{-i(k-k_0)x_{j\ell}}, \tag{20}
\]

where (for the sake of argument) we are assuming for now a unidirectional field. Plugging into \( V_i \), we get

\[
V_i = g_{k0} \sum_{j\ell} \sum_k \mathrm{e}^{-ik\ell x_{j\ell}} \sigma_j^+ \sigma_j \mathrm{e}^{-i(k-k_0)x_{j\ell}} b_k^+ + \text{H.c.}, \tag{21}
\]

where we defined the coupling point phases

\[
\varphi_{j\ell} = k_0 x_{j\ell} \tag{22}
\]

and performed the variable change \( k \to k - k_0 \) (wave vector measured from \( k_0 \)). Consistently with the weak-coupling regime, we can linearize the photonic dispersion law around the atomic frequency as \( \omega_k \simeq \omega_0 + \nu k \) with \( \nu \) the photon velocity. Using \( \nu \), the coupling points...
coordinates can be expressed in the time domain as 
\[ \tau_{jt} = v / x_{jt}. \] 
Thereby, (21) becomes
\[ V_t = g_{k_0} \sum_j e^{-i\varphi_j} \sigma_j \sum_k e^{-i\omega_k \left(t - \tau_{jt} \right)} b_k^\dagger + \text{H.c.} \]  
(23)
Averaging (23) over a time interval \([t_{n-1}, t_n]\) yields
\[ \nabla_n = g_{k_0} \sum_j \left( \sum \right) e^{-i\varphi_j} \sigma_j \int_{t_{n-1}}^{t_n} ds \sum_k e^{-i\omega_k (s - \tau_{jt})} b_k^\dagger + \text{H.c.} \]  
(24)
we have also split the sum over \(j\) and \(t\). Note that \(\tau_{jt} - \tau_{jt'}\) is the time delay taken by light to travel from the \(t\) coupling point of atom \(j\) to the \(t\) coupling point of \(j'\). If all these time delays are negligible compared to \(\Delta t\), then (24) can be approximated as
\[ \nabla_n \approx g_{k_0} \sum_j \left( \sum \right) e^{-i\varphi_j} \sigma_j \int_{t_{n-1}}^{t_n} ds \sum_k e^{-i\omega_k s} b_k^\dagger + \text{H.c.} \]  
(25)
Now, the key point is that each atomic operator \(\sigma_j\) comes with a pre-factor \(\sum_j e^{i\varphi_jj}\), which – due to non-local coupling – it can vanish for all atoms at the same time. This occurs when the coupling point phases are adjusted so as to match the condition
\[ \sum_{t=1}^{N_j} e^{-i\varphi_j} = 0 \]  
for any \(j\),
(26)
which is the DF condition (10) for giant atoms. Note that this cannot be satisfied by normal atoms: each atom must have at least two coupling points \((N_j \geq 2)\).

It is interesting to compare (26) with (17). Each can be seen as a destructive interference condition, involving a continuum of phase factors in the dispersive scheme but only a discrete, possibly small, number in the scheme with giant atoms.

V. GIANT ATOMS DYNAMICS: AVERAGE-HAMILTONIAN DESCRIPTION

The previous section showed how the DF condition \(\nabla_n = 0\) is realized with giant atoms in comparison with off-resonance schemes. When it comes to giant atoms, depending on the topology of coupling points, the DF condition can result in a vanishing (thus trivial) \(H_{\text{eff}}\) [39]. Clarifying the requisites for obtaining a non-trivial effective Hamiltonian, and especially the related physical interpretation, is a major goal of this work. Prior to this, however, we reformulate the microscopic model with giant atoms in terms of time mode operators of the field (generalizing at once to a bidirectional, generally chiral, waveguide), this being the theoretical basis for the mapping of the dynamics into a cascaded collision model that will be discussed in the next sections.

To begin with, as shown in Fig. 3(b), it is convenient to introduce a single index \(v = 1, \ldots, N\) labeling all the coupling points from left to right, where \(N = \prod_j N_j\) is the total number of coupling points. For each coupling point so indexed, we define atomic operators depending on the corresponding pair \((j, \ell)\) as
\[ S_v = \sigma_j e^{-i\varphi_jj}, S'_v = \sigma_j e^{i\varphi_jj}, \]  
(27)
For instance, in the case of Fig. 3, \(S_3 = \sigma_1 e^{-i\varphi_{12}}\) and \(S'_3 = \sigma_1 e^{i\varphi_{12}}\) with \(\varphi_{12} = k_0 \chi_{12}\). Note that \(S_v\) generally does not commute with \(S_v'\) (e.g., in Fig. 3, \([S_2, S'_2] \neq 0\)).

Next, we come back to (23) and extend it to a bidirectional waveguide using the newly introduced operators as (see Ref. [40] for more details)
\[ V_t = g_{k_0} \sum_v S_v \sum_k e^{-i\omega_k \left(t - \tau_{vt} \right)} b_k^\dagger + \text{H.c.} \]  
+ \(g_{k_0} \sum_v S_v \sum_k e^{i\omega_k \left(t + \tau_{vt} \right)} b_k^\dagger + \text{H.c.} \]  
(28)
with ladder operators \(b_k^\dagger\) \((b_k^\dagger)\) now corresponding to right-going (left-going) modes [in the first (second) sum \(k\) is measured from \(k_0 \) \((-k_0)\)].

In the limit in which the field becomes a continuum of modes, \(V_t\) can be expressed in the form [40]
\[ V_t = \sqrt{\gamma} \sum_v S_v b_{1-vt} + \sqrt{\gamma'} \sum_v S'_v b_{1+v} + \text{H.c.} \]  
(29)
with \(\gamma = g_{k_0}^2 / v\) and \(\gamma' = g_{-k_0}^2 / v\) (which we allow to be generally different). Here, \(b_t\) are right-going time modes fulfilling \([b_t, b_t^\dagger] = \delta(t - t'), [b_t, b_v] = [b_t^\dagger, b_v^\dagger] = 0\). Likewise, \(b_t^\dagger\) define left-going modes with analogous commutation rules. Before proceeding further, recalling Eq. (27), it is convenient to define the collective atomic operators
\[ S = \sum_v S_v = \sum_j \left( \sum \right) e^{-i\varphi_j} \sigma_j, \]  
(30)
\[ S' = \sum_v S'_v = \sum_j \left( \sum \right) e^{i\varphi_jj} \sigma_j. \]  
(31)

The regime of negligible time delays is defined by \(\tau_{N} - \tau_1 \ll \gamma^{-1}, \gamma'^{-1}\), allowing to coarse grain the dynamics over a characteristic time scale \(\Delta t\) such that
\[ \tau_{N} - \tau_1 \ll \Delta t \ll \gamma^{-1}, \gamma'^{-1}. \]  
(32)
In this regime, it can be shown [40] that (6) and (7) reduce to
\[ \nabla_n = \frac{1}{\sqrt{\Delta t}} \left( \sqrt{\gamma} S b_1^\dagger + \sqrt{\gamma'} S' b_1^\dagger + \text{H.c.} \right), \]  
(33)
\[ H_n = \frac{i}{2} \sum_{v \neq v'} \left( \gamma S_v^\dagger S_{v'} + \gamma' S_v^\dagger S'_v - \text{H.c.} \right), \]  
(34)
where
\[ b_n = \frac{1}{\sqrt{\Delta}} \int_{t_{n-1}}^{t_n} dt \, b_t, \quad b'_n = \frac{1}{\sqrt{\Delta}} \int_{t_{n-1}}^{t_n} dt \, b'_t. \] (35)

Eq. (35) define a discrete set of ladder operators of the environment fulfilling bosonic commutation rules [\( [b_n, b'_m] = \delta_{n,m}, [b_n, b_m] = [b'_n, b'_m] = 0 \) (and likewise for \( b'_n \)), as is easily checked using the commutation rules of time-mode operators \( b_t \) and \( b'_t \).

Based on (33), the DF condition [cf. Eq. (10)] for a bidirectional waveguide in terms of collective atomic operators (30) and (31) simply reads
\[ S = S' = 0. \] (36)

This is equivalent to (26) \( S = 0 \iff S' = 0 \) if Eq. (26) is matched so is the analogous equation for \( \varphi'_j \rightarrow -\varphi'_j \).

A. Effective Hamiltonian

When \( W_n = 0 \), atoms will evolve unitarily with effective Hamiltonian [recall Eq. (9)] \( H_{\text{eff}} = H_n (H_n \text{ acts trivially on the field}). \)

The effective Hamiltonian \( H_{\text{eff}} = H_n \) can be written more explicitly as [cf. Eqs. (27) and (34)]
\[ H_{\text{eff}} = \frac{i}{2} \sum_{v > v'} \left( \gamma e^{i(\varphi_{v'} - \varphi_v)} \sigma_j^+ \sigma_j - \gamma' e^{i(\varphi_{v'} - \varphi_v)} \sigma_j^+ \sigma'_j - \text{H.c.} \right), \] (37)

where \((j, \ell)\) are understood as the pair of indexes corresponding to \( v \) [and likewise \((j', \ell')\) with respect to \( v'\)]. This in turn can be expressed in the compact form
\[ H_{\text{eff}} = \sum_{jj'} \langle j|j\rangle \sigma_j^+ \sigma_{j'} + \text{H.c.} \] (38)

with
\[ \langle j|j\rangle = \sum_{v > v'} \left[ \frac{\gamma + \gamma'}{2} \sin(\varphi_{v'} - \varphi_v) 
+ i \frac{\gamma - \gamma'}{2} \cos(\varphi_{v'} - \varphi_v) \right], \] (39)

and where \( v_{j\ell} \) is the (previously introduced) discrete map returning the coupling point index for each pair \((j, \ell)\). Note that for isotropic coupling \( (\gamma = \gamma') \), each \( \langle j|j\rangle \) (for given \( j \) and \( j' \)) reduces to a sum of sines, where the argument of each sine is the phase shift associated to a pair of coupling points (one of atom \( j \) one of \( j' \)). Alternatively, \( \langle j|j\rangle \) can be expressed by separating the right- and left-going contributions as
\[ \langle j|j\rangle = \gamma K_{ij} + \gamma' K_{ij'}^* \] (40)

with
\[ K_{ij} = \frac{1}{\sqrt{2}} \sum_{v > v'} e^{i(\varphi_{v'} - \varphi_v)^2 + \frac{\gamma - \gamma'}{2}}. \] (41)

The issue is now raised as to whether or not \( H_{\text{eff}} \neq 0 \) when decoherence is inhibited [condition (26)]. It turns out that there generally exist patterns of coupling points such that \( H_{\text{eff}} = 0 \) and patterns for which \( H_{\text{eff}} \neq 0 \), where the former yield a trivial dynamics (the system just does not evolve) and are thus unwanted. The best instance for illustrating this is a pair of giant atoms 1 and 2, such that \( N_1 = N_2 = 2 \), with equally spaced coupling points
\[ k_{0j}x = (\nu - 1)\varphi \quad \text{with} \quad \nu = 1, 2, 3, 4, \] (42)

the field being unidirectional (\( \gamma' = 0 \)). Three different types of patterns are then possible: serial, nested and braided [see Fig. 4(a), (c), (e)]. For the serial and nested topology, we choose \( \varphi = \pi \) while in the braided case we take \( \varphi = \pi/2 \). Each of these settings ensures that there is \((2n+1)\pi\)-phase-shift between the two coupling points of each giant atom, thus matching the DF condition [recall Eq. (26)]. Using (38), in the serial and nested topologies we get \( H_{\text{eff}} = 0 \), while the braided yields \[ H_{\text{eff}} = \gamma (\sigma_1^+ \sigma_2^+ + \sigma_1^+ \sigma_2), \] (43)

(we absorbed a phase factor \( e^{-i\pi/2} \) in the definition of \( \varphi_2 \)). Analogous conclusions hold for isotropic coupling \( (\gamma' = \gamma = \Gamma/2) \), in which case (43) is generalized by replacing \( \gamma \) with \( \Gamma \).

Before concluding this section, we recall that, as discussed in Ref. [40], based on Eqs. (4), (5), (33) and (34) one can effectively see the joint dynamics as a collision model [41–45] [see Fig. 5]. According to this (we first consider the unidirectional case \( \gamma' = 0 \)), the field is decomposed into a discrete stream of right-going time bins, each being a bosonic mode with ladder operator \( b_n \) defined by Eq. (35). During the time interval \([t_{n-1}, t_n]\), the \( n \)th time bin undergoes a collision with all the atoms at once, which is described by unitary \( U_n \) [cf. Eq. (5)]. The atoms at the same time are effectively subject to a mutual coherent interaction described by \( \mathcal{H}_n \equiv H_{\text{eff}} \) (encompassed in the collision unitary \( U_n \)). The extension to the bidirectional case is natural: there is now an additional stream of left-going time bins (each with ladder operator \( b'_n \)). One can equivalently think of two-mode time bins \( (b_n, b'_n) \) such that in each collision the atoms collide with both time-bin subsystems \( b_n \) and \( b'_n \) (see Fig. 5).
with the coupling point”). Using Eqs. (30), (31), (33) and (d): nested topology. (e) and (f): braided topology. The phase of each coupling point is shown in (a), (c), (e). Each sub-collision is described by the unitary $e^{-iV_n\Delta t}$ with $V_n$ given by Eq. (44). In (f), we absorbed a phase factor $e^{-i\pi/2}$ in the definition of $\sigma_2$.

VI. MAPPING INTO A CASCaded COLLISION MODEL

While, as pointed out in the previous section, the collision $U_n$ formally describes a simultaneous collision with all the atoms, we show next that it can be effectively decomposed as a cascade of sub-collisions each involving only one coupling point. Cascaded collision models (for normal atoms) were introduced in Refs. [46, 47] (see also Ref. [48]). Here, a unidirectional waveguide is considered, the extension to the bidirectional case being postponed to Section IX.

For each coupling point $\nu$, let us define the interaction Hamiltonian

$$V_n = \sqrt{\frac{2}{\Delta t}} \left( S_\nu b_\nu^\dagger + \text{H.c.} \right)$$  \hspace{1cm} (44)

coupling the $n$th time bin to atom $j$ with phase $\varphi_{j\ell}$ [cf. Eq. (27)], where $(j, \ell)$ is the pair corresponding to coupling point $\nu$ (in the remainder we introduce a convenient terminology and say that the time bin “interacts with the coupling point”). Using Eqs. (30), (31), (33) and (44), it is easily immediately checked that the average interaction Hamiltonian is just the sum of the $V_n$’s

$$V_n = \sum_\nu V_{n\nu}.$$  \hspace{1cm} (45)

More importantly, as shown in the remainder, it turns out that, when the DF condition (26) is matched, the unitary collision $U_n$ can be decomposed as

$$U_n = e^{-iV_n\Delta t} \ldots e^{-iV_1\Delta t}.$$  \hspace{1cm} (46)

Thereby, one can think of each collision (see Fig. 6) as the result of $N$ cascaded sub-collisions in each of which the time bin “collides” with one of the coupling points according to unitary $e^{-iV_n\Delta t}$ with $V_n$ given by (44). Of course, this in particular entails that the same time bin collides with a given atom as many times as the number of respective coupling points $N_j$. Yet, the sub-collisions with the same atom occur with different cascaded sub-collisions are generally non-consecutive (i.e., between two sub-collisions with the same atom there may be sub-collisions with atoms $j' \neq j$), which is key to the occurrence of a non-trivial DF Hamiltonian as we will see shortly.

To prove (46), we expand to second order each sub-collision unitary on the right hand side as $e^{-iV_n\Delta t} \simeq 1 - iV_n\Delta t - \frac{1}{2} V_n^2\Delta t^2$. This yields (to leading order)

$$\prod_{\nu=1}^{N} e^{-iV_{n\nu}\Delta t} \simeq 1 - i \left( \bar{V}_n + \bar{H}_n \right) \Delta t - \frac{1}{2} \bar{V}_n^2 \Delta t^2 \hspace{1cm} (47)$$

with the order in the product understood as in (46) and

$$\bar{H}_n = i \frac{\Delta t}{2} \sum_{\nu \neq \nu'} \left[ V_{n\nu'}, V_{n\nu} \right],$$  \hspace{1cm} (48)

where we used (45). Using (26), it is easily shown that $\bar{H}_n = \bar{H}_n \equiv H_{\text{eff}}$ (see Appendix B). Upon comparison with (5), we thus conclude that (46) holds true.

The decomposition in terms of cascaded sub-collisions in particular highlights the physical origin of
effective Hamiltonian (44): if, instead of being sequential, the sub-collisions occurred simultaneously (corresponding to perfectly co-located coupling points) then the overall collision unitary would be $e^{-i\sum_n V_n \Delta t} \equiv e^{-iV_n \Delta t}$, the corresponding second-order expansion being just (47) without term $\tilde{H}_n \equiv H_{\text{eff}}$. Thus the effective Hamiltonian arises precisely because the time bin collides with the coupling points in a cascaded fashion. This is in fact the same mechanism underpinning emergence of effective Hamiltonians in chiral quantum optics with normal atoms [49], the difference yet being that decoherence cannot be suppressed in the latter case (because $V_n$ cannot vanish with normal atoms).

\[ \mathcal{V}_j = \sqrt{\frac{\pi}{\Delta t}} (\sigma_j b_{n}^\dagger + \text{H.c.}) \tag{51} \]

with $j = 1, 2$ (the dependence on $n$ is left implicit). No phase factor appears in this definition.

Consider first the serial scheme in Fig. 4(a), in which case we set $\varphi = \pi$. Then [see Fig. 4(b)],

\[ V_{n1} = \mathcal{V}_1, \quad V_{n2} = -\mathcal{V}_1, \quad V_{n3} = \mathcal{V}_2, \quad V_{n4} = -\mathcal{V}_2. \tag{52} \]

This results in the collision unitary [cf. (46)]

\[ U_n = e^{i\mathcal{V}_2 \Delta t} e^{-i\mathcal{V}_1 \Delta t} e^{i\mathcal{V}_1 \Delta t} e^{-i\mathcal{V}_2 \Delta t} = 1, \tag{53} \]

that is a trivial dynamics such that $H_{\text{eff}} = 0$. This case is in fact an extension the single giant atom considered above.

For the nested case in Fig. 4(c), we set $\varphi = \pi$. Then [see Fig. 4(d)],

\[ V_{n1} = \mathcal{V}_1, \quad V_{n2} = -\mathcal{V}_2, \quad V_{n3} = \mathcal{V}_2, \quad V_{n4} = -\mathcal{V}_1. \tag{54} \]

Thus the second pair of sub-collisions is the first pair time-reversed

\[ U_n = e^{i\mathcal{V}_1 \Delta t} e^{-i\mathcal{V}_2 \Delta t} e^{i\mathcal{V}_2 \Delta t} e^{-i\mathcal{V}_1 \Delta t} = 1, \tag{55} \]

ensuing again a trivial dynamics and $H_{\text{eff}} = 0$. Equivalently, the pair of central sub-collisions, both involving atom 2, are the time-reversed of one another. Thus atom 2 simply disappears from $U_n$, which reduces to $U_n = e^{i\mathcal{V}_1 \Delta t} e^{-i\mathcal{V}_1 \Delta t} = 1$. 

\[ S_1 = -S_2 = \sigma_1 \text{ and [cf. Eq. (44)]} \]

\[ V_{n1} = \sqrt{\frac{\pi}{\Delta t}} (\sigma_1 b_{n}^\dagger + \text{H.c.}), \quad V_{n2} = \sqrt{\frac{\pi}{\Delta t}} (-\sigma_1 b_{n}^\dagger + \text{H.c.}). \tag{49} \]

Thus $V_{n1} = -V_{n2}$ and [see Eq. (46)]

\[ U_n = e^{-i\mathcal{V}_{n2} \Delta t} e^{-i\mathcal{V}_{n1} \Delta t} = 1, \tag{50} \]

meaning that the collision has no effect overall. This, in particular, necessarily entails $H_{\text{eff}} = 0$ [recall Eq. (5)]. In other words, the two sub-collisions are the time-reversed of one another (so that the net effect is null). To sum up, in order to ensure the DF condition $V_n = 0$ for a single giant atom, one must adjust the phase shift so that $V_{n2} = -V_{n1}$. This yet brings about that one sub-collision is just the other one time-reversed, trivially yielding $U_n = 0$ hence $H_{\text{eff}} = 0$.

When it comes to a pair of giant atoms, instead, conditions $V_n = 0$ and $U_n \neq 0$ can be matched simultaneously. To see this, we reconsider uniformly-spaced atoms as in Eq. (42) and always set $\varphi$ so as to ensure a $2(2n+1)\pi$-phase-shift between the pair of coupling points of each atom, hence $V_n = 0$ (similarly to the single-atom instance just discussed).

For convenience, we define the coupling Hamiltonians

\[ \mathcal{V}_j = \sqrt{\frac{\pi}{\Delta t}} (\sigma_j b_{n}^\dagger + \text{H.c.}) \tag{51} \]
For the braided arrangement of Fig. 4(e), we set $\varphi = \pi/2$. Then [see Fig. 4(f)],

$$V_{n1} = V_1, \ V_{n2} = V_2, \ V_{n3} = -V_1, \ V_{n4} = -V_2 \quad (56)$$

[with $V_2$ now defined by (51) for $j = 2$ under the replacement $\sigma_2 \rightarrow -i\sigma_2$]. The collision unitary is given by

$$U_n = e^{i\gamma_2 \Delta t} e^{iV_1 \Delta t} e^{-iV_2 \Delta t} e^{-iV_1 \Delta t} = e^{-i\gamma (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) \Delta t} \neq 1,$$

(57)

Therefore, $\nabla_n = 0$ is fulfilled but now $H_{\text{eff}} \neq 0$.

The above shows that, while being irrelevant for realizing the DF condition $\nabla_n = 0$, the coupling points topology is crucial in order to have a non-vanishing effective Hamiltonian. In terms of propagators [cf. Eqs. (4) and (5)], this is ultimately due to the fact that the second-order term $K_n$ is affected by the time-ordering operator, while $\nabla_n$ and (of course) $\nabla_n^2$ are fully insensitive to it.

### A. Many atoms

The above arguments are naturally extended to more than two giant atoms. Again, the DF condition (26) is matched when for each atom the phase shift between its coupling points is a multiple integer of $\pi$, that is $\varphi_{j,2} - \varphi_{j,1} = (2n_j + 1) \pi$ for some integer $n_j$ and for any $j$. Based on the above discussion, we can state that if there exists an atom $j$ such that no coupling point of other atoms lies between its coupling points, i.e., $x_{j',\ell'} \notin [x_{j,1}, x_{j,2}]$ for all $j' \neq j$ and $\ell' = 1, 2$, then $H_{\text{eff}}$ simply does not contain $\sigma_j$ and $\sigma_j^\dagger$ (i.e., atom $j$ is fully decoupled from the field and other atoms). This is because the pair of sub-collision unitaries corresponding to coupling points $x_{j,1}$ and $x_{j,2}$ are the time-reversed of one another, hence atom $j$ is fully decoupled from the field and all other atoms. A braided configuration is thus generally defined [14] as one such that no atom exhibits the above phenomenon.

### VIII. EQUIVALENT SCHEME USING A MEDIATOR

Performing quantum information processing tasks [50] on two remote systems, say 1 and 2, is a longstanding problem. The challenge is due to the impossibility of acting jointly on the systems (owing to their distance). A standard strategy to get around this is employing a mobile mediator: a third quantum system $M$ which can shuttle between 1 and 2 so as to mediate an indirect coupling, which, e.g., can be exploited for generating entanglement.

The dynamics realizing the DF Hamiltonian for two giant atoms in a braided configuration [see Fig. 4(e)] in fact implements a two-qubit gate via a shuttling mediator. To see this, for simplicity and in view of the equivalent quantum circuit to be discussed later, we consider the field initially in the vacuum state, in which case each time bin behaves as an effective qubit $M$ [51]. Then $V_j$ [cf. Eq. (51)] reads

$$V_j = \sqrt{\frac{\pi}{M}} \left( \sigma_j \sigma_{M+} + \sigma_j^\dagger \sigma_{M-} \right) \quad (58)$$

with $\sigma_{M-} = \sigma_{M+}^\dagger$ usual spin-1/2 ladder operators of $M$. Note that, for convenience, we gauged away a phase factor $e^{-i\varphi/2}$ [due to $\varphi = \pi/2$, cf. Eq. (42)] by absorbing it in the definition of $\sigma_2$.

Unitary (57) (braided configuration) can be interpreted as follows (see Fig. 7). $M$ is first put close to 1 and the M-1 two-qubit gate $e^{-iV_1 \Delta t}$ applied on them [Fig. 7(a)], then $M$ is moved close to 2 and M-2 gate $e^{-iV_2 \Delta t}$ applied (b), next $M$ goes back to 1 and gate $e^{iV_1 \Delta t}$ is applied (c). Finally, $M$ lies close to 2 again and gate $e^{iV_2 \Delta t}$ is applied (d).

Consider now the more realistic case that only one two-qubit gate per atom can be implemented, say $e^{-iV_1 \Delta t}$ and $e^{-iV_2 \Delta t}$; we ask whether the other two can be obtained from these by adding extra single-qubit (local) gates. Noting that the local unitary transformation defined by $U_M = \sigma_{Mz}$ transforms the $M$'s ladder operators as $\sigma_{M\pm} \rightarrow -\sigma_{M\pm}$, we have

$$\sigma_{Mz} V_j \sigma_{Mz} = -V_j \Rightarrow e^{iV_j \Delta t} = \sigma_{Mz}^\dagger e^{-iV_j \Delta t} \sigma_{Mz} \quad (59)$$

with $j = 1, 2$. Plugging the last decomposition of $e^{iV_j \Delta t}$
in (57) thus yields

\[ U_n = \sigma_{Mz} e^{-i\hat{V}_2 \Delta t} e^{-i\hat{V}_1 \Delta t} \sigma_{Mz} e^{-i\hat{V}_2 \Delta t} e^{-i\hat{V}_1 \Delta t}, \]

(60)

where we used \( \sigma_{Mz}^2 = 1 \). Recalling that in the braided configuration, \( U_n = e^{-iH_{\text{eff}} \Delta t} \) with \( H_{\text{eff}} \) given by (43), we conclude

\[ e^{-i\hat{V}(\sigma_{Mz}^2 + \text{H.c.}) \Delta t} = \sigma_{Mz} e^{-i\hat{V}_2 \Delta t} e^{-i\hat{V}_1 \Delta t} \sigma_{Mz} e^{-i\hat{V}_2 \Delta t} e^{-i\hat{V}_1 \Delta t}. \]

(61)

The collision unitary in the braided configuration – hence the entire dynamics in fact – can thus be seen as alternate subcollisions (of the same type) of \( M \) with 1 and 2 where a local \( \pi \)-phase gate is applied on \( M \) at the end of each cycle. This effectively implements a DF interaction between 1 and 2. Note that replacing \( \sigma_{Mz} = \text{diag}(1, e^{i\pi}) \) (written in matrix form) with another phase gate \( U_M = \text{diag}(1, e^{i\phi}) \) with \( \phi \neq (2n + 1)\pi \) generally gives rise to an overall unitary \( U_n \) which does not act trivially on \( M \), thus introducing decoherence.

The above is somewhat reminiscent of dynamical decoupling schemes [52], where suitable local pulses are repeatedly applied in order to effectively decouple the system from the environment thus suppressing decoherence. Note however that in our case the environment (embodied by \( M \), that is the field in fact) has an active role since it allows 1 and 2 to effectively crosstalk.

**A. Quantum circuit**

To express (61) in the language of quantum circuits [50], we recall the definition of an XY gate, also known as piSWAP or parametric i-SWAP,

\[
    \text{XY}(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi \delta & -i \sin \pi \delta & 0 \\ 0 & -i \sin \pi \delta & \cos \pi \delta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

(62)

which coincides with each \( M \)-j unitary \( e^{-i\hat{V}_j \Delta t} \) for \( \delta = \frac{1}{\pi} \sqrt{\gamma \Delta t} \). Thereby we get the equivalent quantum circuit of the collision unitary (57) displayed in Fig. 8 (where \( Z = \sigma_\gamma \)).

Thus, remarkably, the waveguide setup with giant atoms can be seen as implementing iterated applications of the elementary quantum circuit in Fig. 8. A canonical (maximally entangling) i-SWAP two-qubit gate \( U_n = e^{-i(\pi \sigma_z^2 + \text{H.c.}) \Delta t} \) is obtained after \( N = \lfloor (\gamma \Delta t)^{-1} \rfloor \) iterations [recall Eq. (32)].

**IX. BIDIRECTIONAL CHIRAL WAVEGUIDE**

The considerations in Section VII naturally extend to a bidirectional waveguide. In Section VII A, we saw that, in the unidirectional case, if an atom \( j \) is such that \( x_{j,j'} \notin \{x_{j,1}, x_{j,2} \} \) for any \( j' \neq j \) and \( j' = 1, 2 \) (configuration not braided) then \( \sigma_j \) and \( \sigma_{j'} \) do not appear in \( H_{\text{eff}} \) under the DF condition. The same holds for a bidirectional waveguide since, according to Eqs. (38) and (40), if \( J_{j,j'} \) vanishes in the unidirectional case \( \gamma' = 0 \) then so does in the bidirectional one (when \( \gamma' \neq 0 \)).

For completeness, it is however worth showing that one can reach the same conclusion even through a purely collisional argument. To this aim, we note that, even for a bidirectional waveguide, under the DF condition (26) the collision unitary [cf. Eqs. (5), (33), (34)] can be decomposed into cascaded sub-collisions [cf. Eq. (46)] as

\[ U_n = e^{-i(V_{nN} + V'_{n}) \Delta t} \ldots e^{-i(V_{n1} + V'_{n}) \Delta t}. \]

(63)

with [cf. Eq. (44)]

\[ V'_{n} = \sqrt{\frac{\gamma}{\Delta t}} \left( S'_{j} b'^{+}_{j} + \text{H.c.} \right) \]

(64)

[recall Eqs. (27) and (35)]. This is because [cf. Eq. (5)] \( V'_{n} = \sum_{j} (V_{n,j} + V'_{n,j}) \) and, as shown in Appendix B, \( \hat{H}_n = \hat{H}_{\text{eff}} \). Here, \( \hat{H}_n \) [cf. Eq. (48)] is now generally defined as

\[ \hat{H}_n = i \frac{\Delta t}{2} \sum_{\nu > \nu'} \left[ V_{n,\nu} + V'_{n,\nu+1-\nu}, V_{n,\nu} + V'_{n,\nu+1-\nu} \right]. \]

(65)

Note that, as sketched in Fig. 9, in the first sub-collision, the right-going time bin interacts with coupling point \( \nu = 1 \) and the left-going time bin with coupling point \( \nu = N \). Then, in the second sub-collision, the right-(left-)going time bin interacts with coupling point \( \nu = 2 \) (\( \nu = N - 2 \)) and so on.

In Appendix C, we use (63) to show that a giant atom does not appear in \( H_{\text{eff}} \) whenever it is untangled from other atoms, i.e., when no coupling points of other atoms lie in between its two coupling points.
condition) fully decouples from the field, i.e., phase shift between its two coupling points (i.e., the DF lied on the property that a single giant atom with a \( \pi \) grows from top to bottom). (cf. Eq. (50)). For more than two coupling points, the \( \nu \) spectively interact with coupling points \( \phi \) set drop subscript for a bidirectional waveguide (cf. Eq. (63)). In

\[ \sum_{j} \sin \left( \frac{2\pi j}{3} \right) \sigma_z = 0 \]  

(67) (the sum of the three sines is \( \approx 0.87 \)).

More generally, for an atom \( j \) such that \( x_{j'} \neq j \), and fulfilling the DF condition \( \sum_{j} V_{n,j,j'} = 0 \), in general

\[ e^{-iV_{n,j,j'}} \Delta t \ldots e^{-iV_{n,j',j} \Delta t} \neq 1, \]  

(68) where \( N_j > 2 \) (if \( N_j = 2 \), the identity holds). However, (68) is anyway of the form \( e^{-i\delta_j \sigma_z \Delta t} \) (with \( \delta_j \) a frequency shift), hence all terms of \( H_{\text{eff}} \) coupling \( j \) to any other atom will vanish, i.e., in (38) \( I_{jj'} = 0 \) only for \( j' = j \). Thus, if the only focus is coupling the atoms, then the braided topology remains the only one yielding a non-trivial \( H_{\text{eff}} \). This remains true for a chiral waveguide (\( \gamma' \neq 0 \)) since (40) shows that if \( I_{jj'} = 0 \) for \( \gamma' = 0 \) then it vanishes also for \( \gamma' \neq 0 \).

XI. CONCLUSIONS

In this work, we investigated the physical mechanism underpinning implementation of DF Hamiltonians with giant atoms. We first introduced a general framework for obtaining DF Hamiltonians through second-order interactions mediated by an environment. The key “DF condition” is having an interaction Hamiltonian averaging to zero over a suitable coarse-grained time scale. The framework was first illustrated with standard dispersive Hamiltonians, in which case large detunings ensure a vanishing average interaction. We then considered giant atoms in a broadband waveguide and showed that, thanks to the non-local nature of the coupling, the DF condition can still be fulfilled, but in a qualitatively different way.

The above framework was then connected to a collisional picture of the joint dynamics of giant atoms and field in terms of elementary pairwise collisions between the atoms and field time bins. We showed that each collision can be decomposed as cascaded subcollisions, providing an intuitive understanding of the origin of the effective Hamiltonian. This was used to interpret the relationship between topology of the coupling points and occurrence of trivial/non-trivial DF Hamiltonians. In addition, we showed that the giant atoms dynamics can be mapped into a system shuttling between the atoms and subject to periodic phase kicks so as to effectively mediate a DF inter-atomic interaction, a mechanism in some respects reminiscent of dynamical decoupling schemes.

While here we did not consider lossy photonic environments [19, 34, 53], the considered framework could be naturally extended to accommodate these. Likewise, a generalization to giant atoms in gapped structured reservoirs [54] appears viable.

From the viewpoint of collision models and their quantum information processing applications (see, e.g., [55–57]), Sections VI and VIII in fact introduce a class of cascaded collision models implementing maximally-entangling multi-qubit gates free from decoherence. Notably, these correspond to second-order effective Hamiltonians (decoherence-free), at variance with schemes in Refs. [55, 56] which are of first order (see also Ref. [43]).
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Appendix A: Proof of property

The propagator in each time interval (5) for $\nabla_n = 0$ reduces to $U_n = 1 - i\mathcal{H}_n \Delta t$. Accordingly,

$$\sigma_n = \sigma_{n-1} - i [\mathcal{H}_n, \sigma_{n-1}] \Delta t . \quad (A1)$$

At the first step, $\sigma_{n-1} = \rho_0 \otimes \rho_E$, thus using condition (11)

$$\sigma_1 = \rho_0 \rho_E - i [\mathcal{H}_n, \rho_0 \rho_E] \Delta t = \rho_0 \rho_E - i [\mathcal{H}_n, \rho_0] \rho_E \Delta t = (1 - i [\mathcal{H}_n, \rho_0]) \rho_E \Delta t . \quad (A2)$$

This implies $\sigma_1 = \rho_1 \otimes \rho_E$ with

$$\frac{\Delta \rho_1}{\Delta t} = -i[H_{\text{eff}}, \rho_0] \quad (A3)$$

(recall Eq. (9)). By induction, we get that at each step $\sigma_n = \rho_n \otimes \rho_E$ with $\rho_n$ fulfilling

$$\frac{\Delta \rho_n}{\Delta t} = -i[H_{\text{eff}}, \rho_{n-1}] . \quad (A4)$$

Taking the continuous-time limit, $\rho_{n-1} \rightarrow \rho$ so that (A4) reduces to Eq. (12).

Appendix B: $\mathcal{H}_n' = \mathcal{H}_n$

Let us begin with the unidirectional case. Each commutator in (48) is explicitly worked out as

$$[V_{m'} V_m] = \frac{\gamma}{\Delta t} \left( [S_{\nu'}, S_{\nu}^\dagger] - \text{H.c.} \right) b_{m'}^\dagger b_n + \frac{\gamma}{\Delta t} \left( S_{\nu'}^\dagger S_{\nu} - \text{H.c.} \right) . \quad (B1)$$

Upon comparison with (34) for $\gamma' = 0$, the proof thus reduces to showing that the sum over $\nu > \nu'$ of terms $\propto b_{m'}^\dagger b_n$ vanish.

Each commutator $[S_{\nu'}, S_{\nu}^\dagger]$ is non-zero only when coupling points $\nu$ and $\nu'$ belong to the same atom. Thus, in light of (27),

$$\sum_{\nu > \nu'} \left[ S_{\nu'}, S_{\nu}^\dagger \right] - \text{H.c.} = \sum_j \sum_{\ell > \ell'} e^{i(\phi_{j\ell} - \phi_{j\ell'})} \left[ \sigma_j, \sigma_j^\dagger \right] - \text{H.c.} = \sum_j \left( \sum_{\ell > \ell'} e^{i(\phi_{j\ell} - \phi_{j\ell'}) - \text{c.c.}} \right) \sigma_j \quad (B2)$$

(recall that $x_1 < x_2 < ...$). When (26) holds, the coefficient of $\sigma_j$ vanishes for each $j$

$$\sum_{\ell > \ell'} e^{i(\phi_{j\ell} - \phi_{j\ell'})} - \text{c.c.} = \sum_{\ell > \ell'} e^{i(\phi_{j\ell} - \phi_{j\ell'})} - \text{c.c.} = \sum_{\ell = \ell'} e^{i\phi_{j\ell}} \sum_{\ell' = 1}^{N_j} e^{-i\phi_{j\ell'}} - \text{c.c.} = 0 . \quad (B3)$$

Thus

$$\sum_{\nu > \nu'} [V_{m'} V_m] = \frac{\gamma}{\Delta t} \sum_{\nu > \nu'} \left( S_{\nu'}^\dagger S_{\nu} - \text{H.c.} \right) , \quad (B4)$$

completing the proof.

In the bidirectional case, each commutator in (65) reads

$$\left[ V_{n',n} + V_{n,n+1-v'}, V_{n,v} + V_{n+1-v',n} \right] =$$

$$\left[ V_{n,n+1-v'}, V_{n,v} \right] + \left[ V_{n',n+1-v'}, V_{n,N+1-v} \right] + \left[ V_{n,v'}, V_{n,N+1-v} \right] + \left[ V_{n',n+1-v'}, V_{n,v} \right] . \quad (B5)$$

The last line features terms $\propto [b_{m'}^\dagger b_n]$ and $\propto [b_{n'}^\dagger b_n]$, which vanish because left- and right-going time-bin operators commute. Additionally, there are terms $\propto b_{m'}^\dagger b_n$ (or $\propto b_m^\dagger b_n$) featuring quantities like (B2) where however one of the two phases is primed: these vanish as well since (B3) holds even if $\phi_{j\ell'} \rightarrow \phi_{j\ell'}'$. We are thus only left with terms analogous to (B1) given by

$$\left[ V_{n,n'}, V_{n,v} \right] \left[ V_{n,n+1-v'}, V_{n,N+1-v} \right] =$$

$$\frac{\gamma}{\Delta t} \left( S_{\nu'}^\dagger S_{\nu} + S_{\nu'}^\dagger S_{\nu}^\dagger - \text{H.c.} \right) . \quad (B6)$$

Summing this over $\nu > \nu'$ yields

$$\frac{\gamma}{\Delta t} \left( S_{\nu'}^\dagger S_{\nu} + S_{\nu'}^\dagger S_{\nu}^\dagger - \text{H.c.} \right) \quad (\text{where we used that} \quad N+1-v' > N+1-v \quad \text{for} \quad \nu > \nu')$$

completing the proof.

Appendix C: Extension of Section VII A to a bidirectional waveguide

In order to extend the considerations in Section VII A to a bidirectional waveguide, we essentially need to show that, for an atom $j$ such that $x_{j,\ell'} \notin [x_{j,1}, x_{j,2}]$ for any $j' \neq j$ and $\ell' = 1, 2$ (recall Section VII A), $\sigma_j$ and $\sigma_j^\dagger$ do not appear in $H_{\text{eff}}$ under the DF condition. We first recall that $\nabla_n = 0 \Leftrightarrow \nabla_n' = 0$ (since $S = 0 \Leftrightarrow S' = 0$). In this configuration, the overall coupling point index runs over $v = 1, 2, ..., k_{j1}, k_{j1} + 1, ..., N$ with $k_j \leftrightarrow (j, 1)$ and $k_j + 1 \leftrightarrow (j, 2)$ labeling the left and right coupling
points of atom \( j \). Accordingly, in unitary (63), the sub-collision unitaries involving the \( j \)th atom are (to make notation lighter we drop subscript \( n \))

\[
\hat{U}_j = e^{-i(V_{j+1} + V_{N-j})\Delta t} e^{-i(V_{j} + V'_{N-j})\Delta t}
\]

(1)

and

\[
\hat{W}_j = e^{-i(V'_{N-j} + V'_{j})\Delta t} e^{-i(V_{N-j} + V_{j+1})\Delta t}.
\]

(C2)

Upon inspection, the pair of \( V \)'s in each exponent commute because one involves a coupling point of atom \( j \) and a right-going time bin, while the other one features a coupling point of an atom \( j' \neq j \) and a left-going time bin. Therefore, (C1) can be decomposed as

\[
\hat{U}_j = e^{-iV_{j+1}\Delta t} e^{-iV'_{N-j}\Delta t} e^{-iV_{j}\Delta t} e^{-iV'_{N-j}\Delta t} = e^{-iV_{j+1}\Delta t} e^{-iV'_{N-j}\Delta t} e^{-iV_{j}\Delta t} e^{-iV'_{N-j}\Delta t}.
\]

(C3)

Under the DF condition, \( V_{j+1} = -V_j \) so that the first two exponentials in the last line reduce to the identity. An analogous conclusion holds for (C2). We thus conclude that \( U_n \), hence \( H_{\text{eff}} \), does not contain atom \( j \), completing the proof.

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