An optimized Rayleigh-Schrödinger expansion scheme of solving the functional Schrödinger equation with an external source is proposed to calculate the effective potential beyond the Gaussian approximation. For a scalar field theory whose potential function has a Fourier representation in a sense of tempered distributions, we obtain the effective potential up to the second order, and show that the first-order result is just the Gaussian effective potential. Its application to the $\lambda\phi^4$ field theory yields the same post-Gaussian effective potential as obtained in the functional integral formalism.

I. INTRODUCTION

Since the mid 1980s, the effective potential (EP) in the quantum field theory (QFT) beyond the Gaussian approximation (GA) has received much attention, because it collects merits and removes weakpoints of the conventional perturbation theory and the GA as shown by the so-called variational perturbation theory in other fields. Some schemes have proposed to calculate such an EP, for example, designing some non-Gaussian trial wavefunctionals (i), using the Brueckner-Goldstone formula with a variational basis (ii), rearranging loop diagrams in the functional integral formalism with the background field method (iii), optimized expansions in the functional integral formalism with the steepest-descent method (iv) and with the background field method (v), and so on. Noting that the well-developed Rayleigh-Schrödinger (RS) expansion in quantum mechanics was generalized to QFT, recently we have developed a variational perturbation scheme with the RS expansion to calculate the EP as per the variational minimum definition. In the scheme, a free-field Hamiltonian with a mass fixed from the GA was adopted as a solvable part so as to perform the RS expansion. Obviously, this scheme amounts to a series expansion around the Gaussian EP, and so do those schemes in Ref. (i)—(iii). In the present paper, taking a free-field Hamiltonian instead with an arbitrary mass parameter $\mu$ as the solvable part, we will apply the RS expansion to solve the functional Schrödinger equation in presence of an external source so that the vacuum energy functional of the source can be obtained. Then, adopting the alternative, but equivalent definition of the EP in the functional Schrödinger picture, we extract the EP through a Legendre transform of the energy functional. In the new scheme, $\mu$ will be determined according to the principle of minimal sensitivity and, consequently, the value of $\mu$ in an approximation up to one order is different from those up to other orders of the expansion. This way of fixing $\mu$ makes the expansion give the Gaussian EP with its lowest orders, but be not an expansion around the Gaussian EP. Furthermore, unlike Ref. (i), the vacuum expectation value $\varphi$ of the field operator will naturally be given in the present scheme. Hereafter, we will call this scheme the optimized RS expansion (ORSE). Since, for the case of scalar field theory, the EP beyond the GA was given only for the $\lambda\phi^4$ model except for the $\phi^6$ model (1985) up to now, we use the ORSE in this paper to give the EP for a generic class of scalar field models (see Sect. III) up to the second order. The resultant formula can easily be used to give the EP beyond the GA for a number of concrete scalar field models, for instance, models with polynomial or/and exponential interactions. We also show that its application to $\lambda\phi^4$ field theory gives the same post-Gaussian EP as Ref. (iv,v), and yields the result in Ref. (ii,iii) if one chooses to use the same constraint on $\varphi$ as in Ref. (ii).
Next, for the sake of convenience, we will first give the complete basis set for a free field theory with an external source, which will be employed in the ORSE. In Sect. III, the ORSE will be proposed. In Sect. IV, we will perform the ORSE to obtain the EP for a class of scalar field theories up to the second order, and its application to the $\lambda\phi^4$ field theory will be given in Sect. V. Conclusions will be made in Sect. VI with discussions on some possible extensions and developments of the present work.

II. A FREE FIELD THEORY WITH AN EXTERNAL SOURCE

In this section, we discuss the free-field theory with an external source $J_x \equiv J(\vec{x})$ in a time-fixed functional Schrödinger picture. The Hamiltonian is given by

$$H_0^{I,\mu} = \int_x \left[ \frac{1}{2} \Pi_x^2 + \frac{1}{2} (\partial_x \phi_x)^2 + \frac{1}{2} \mu^2 \phi_x^2 - J_x \phi_x - \frac{1}{2} \int_y J_y h^{-1}_{xy} J_y \right],$$

where $x = (x^1, x^2, \cdots, x^D)$ represents a position in $D$-dimensional space, $\int_x \equiv \int d^Dx$, $\mu$ an arbitrary mass parameter and $\phi_x \equiv \phi(\vec{x})$ the field at $x$. $\Pi_x \equiv -i \frac{\partial}{\partial \phi_x}$ is canonically conjugate to $\phi_x$ with the commutation relation, $[\phi_x, \Pi_y] = i\delta(x-y)$. In Eq.(1), $f_{xy} \equiv \left(\sqrt{\frac{1}{2} \delta_2 + \mu^2}\right)\delta(x-y)$ with $\int_x f_{xy} f_{yx}^{-1} = \delta(x-y)$, and $h_{xy} \equiv (-\partial_x^2 + \mu^2)\delta(x-y)$ with $\int_x h_{xx}^{-1} = \delta(x-y)$.

The functional Schrödinger equation for Eq.(1), $H_0^{I,\mu} |n; J > = E_n^{0|} |J| n; J >$, is easily solved [3] (Here, the subscript $n$ in $E_n^{0|} |J|$ is the index of eigenstates, and the superscript “(k)” means “at the $k$th order of $\delta$”. See the next section.). First, the eigenenergy of the vacuum state vanishes, and the corresponding wavefunctional is a Gaussian-type functional

$$|0; J > = \mathcal{N} \exp\left\{ -\frac{1}{2} \int_{x,y} (\phi_x - \int_z h_{xz}^{-1} J_z f_{xy}(\phi_y - \int_z h_{yz}^{-1} J_z) \right\},$$

where $\mathcal{N}$ is the normalization constant (i.e., $< J ; 0 | 0 ; J > = 1$). One can show that $\langle J ; 0 | \phi_x | 0; J > = \int_x h_{xx}^{-1} J_x$ which, unlike Eq.(3) in Ref. [3], is dependent on $x$. Then, for the above vacuum, the annihilation and creation operators can respectively be constructed as

$$A_f(p;J) = \left(\frac{1}{2(2\pi)^D f(p)}\right)^{1/2} \int_x e^{-ipx} [f(p)(\phi_x - \int_z h_{xz}^{-1} J_z) + i\Pi_x]$$

and

$$A_f^\dagger(p;J) = \left(\frac{1}{2(2\pi)^D f(p)}\right)^{1/2} \int_x e^{ipx} [f(p)(\phi_x - \int_z h_{xz}^{-1} J_z) - i\Pi_x]$$

with $[A_f(p;J), A_f^\dagger(p';J)] = \delta(p' - p)$ and $A_f(p)|0; J > = 0$. It is not difficult to verify that $H_0^{J,\mu} = \int d^Dp f(p) A_f^\dagger(p;J) A_f(p;J)$, where $f(p) = \sqrt{p^2 + \mu^2}$ arises from $f_{xy} = \int d^Dp f(p)e^{ip(x-y)}$ with $p = (p^1, p^2, \cdots, p^D)$. Consequently, the eigenwavefunctionals for excited states can be easily written as

$$|n; J > = \frac{1}{\sqrt{n!}} \prod_{i=1}^n A_f^\dagger(p_i;J) |0; J >, \quad n = 1, 2, \cdots, \infty$$

and the corresponding eigenenergies are

$$E_n^{0|} |J| = \sum_{i=1}^n f(p_i).$$

Evidently, the eigenwavefunctionals $|n; J >$ and $|0; J >$ are orthogonal and normalized, $< J ; m | n; J > = \delta_{mn} \frac{1}{\sqrt{n!}} \prod_{k=1}^n \delta(p'_k - p_k)$. Here, $P_t(n)$ represents a permutation of the set $\{i_k\} = \{1, 2, \cdots, n\}$ and the summation is over all $P_t(n)s$. $|n; J >$ describes a $n$-particle state with the continuous momenta $p_1, p_2, \cdots, p_n$, $|0; J >$ and $|n; J >$ with $n = 1, 2, \cdots, \infty$ constitute the complete set for $H_0^{J,\mu}$, and satisfy the closure $|0; J > < J; 0 > + \sum_{n=1}^\infty \int d^Dp_1 d^Dp_2 \cdots d^Dp_n |n; J > < J; n > = 1$. 

2
III. OPTIMIZED RAYLEIGH-SCHRÖDINGER EXPANSION FOR THE EFFECTIVE POTENTIAL

The EP for a field system is equivalently defined through the Feynman graphs, the operator representation, the path integral \[1\], or the minimum expected energy in a set of normalized states \[2,3\]. They were used to give the loop or Gaussian EP and propose those schemes in Refs. \[2,3\]. Yet another equivalent definition of the EP is given through the vacuum energy functional of an external source obtained by solving the relevant functional Schrödinger equation \[3\]. Based on it, we will construct the ORSE in this section.

In this and next sections, we work with a scalar field model whose Lagrangian density is \[3\]

\[
\mathcal{L} = \frac{1}{2} \partial_x \phi_x \partial^\mu \phi_x - V(\phi_x).
\] (7)

In Eq.(7), the model potential is assumed to be written as \(V(\phi_x) = \int \frac{d\Omega}{\sqrt{2\pi}} \tilde{V}(\Omega) e^{i\Omega \phi_x}\), at least, in a sense of tempered distributions \[10\]. It represents several scalar-field models, such as \(\lambda \phi^4\) model \[2,11\], general and special \(\phi^4\) models \[12\], sine-Gordon and sinh-Gordon models \[13\], massive and double sine-Gordon model \[14\], Liouville model \[15\], as well as two generic models investigated in Ref. \[16\].

For the system, Eq.(7), the time-independent functional Schrödinger equation in the presence of an external source \(J_x\) is

\[
(H - \int_x J_x \phi_x)\Psi_n > = E_n[J] \Psi_n >
\] (8)

with the Hamiltonian \(H = \int_x \left[\frac{1}{2} \partial_x^2 + \frac{1}{2} (\partial_x \phi_x)^2 + V(\phi_x)\right]\). Here, the eigenvalue \(E_n[J]\) is a functional of \(J_x\).

For our purpose, Eq.(8) will be modified. We make a shift \(\phi_x \rightarrow \phi_x + \Phi\) (\(\Phi\) is a constant), and Eq.(8) can equivalently be rewritten as

\[
\left[H(\phi_x + \Phi) - \int_x J_x (\phi_x + \Phi)\right]\Psi_n [\phi_x + \Phi, J] = E_n[J, \Phi] \Psi_n [\phi_x + \Phi, J]
\] (9)

with \(H(\phi_x + \Phi) = \int_x \left[\frac{1}{2} \partial_x^2 + \frac{1}{2} (\partial_x \phi_x)^2 + V(\phi_x + \Phi)\right]\). This shift is really in the spirit of the background-field method \[17,2\](iii). Further, normal-ordering the Hamiltonian in Eq.(9) with respect to a normal-ordering mass \(M\) \[18\], and inserting a vanishing term \(\int_x \left[\frac{1}{2} \mu^2 \phi_x^2 - \frac{1}{2} \mu^2 \phi_x^2\right]\) with \(\mu\) an arbitrary mass parameter into the Hamiltonian \[13\], one can have

\[
\mathcal{N}_M[H(\phi_x + \Phi) - \int_x J_x (\phi_x + \Phi)] = H_0^L + H_1^\mu \Phi - C
\] (10)

with

\[
H_1^\mu \Phi = \int_x \left\{-\frac{1}{2} \mu^2 \phi_x^2 + \mathcal{N}_M[V(\phi_x + \Phi)]\right\}
\] (11)

and

\[
C = \int_x \left[-\frac{1}{2} I_{xx} + \frac{1}{2} \int_y J_x h_{xy}^{-1} J_y + \frac{1}{2} I_0(M^2) - \frac{M^2}{4} I_1(M^2) + J_x \Phi\right].
\] (12)

Here, the notation \(\mathcal{N}_M[\cdots]\) represents normal-ordered form with respect to \(M\), \(I_n(Q^2) \equiv \int \frac{d^D p}{(2\pi)^D} \sqrt{p^2 + Q^2} f(p)\), and \(\mathcal{N}_M[V(\phi_x + \Phi)] = \int \frac{d^D \Omega}{\sqrt{2\pi}} \tilde{V}(\Omega) e^{i\Omega (\phi_x + \Phi)} + \frac{d^D p}{(2\pi)^D} I_n(M^2)\) \[3\]. From now on, we will use the normal-ordered Hamiltonian in Eq.(10) instead of the original one, which will naturally make the EP in \((1+1)\) dimensions free of explicit ultraviolet divergences \[3,18\]. Noting that \(C\) is a constant independent of \(\phi_x\) and \(H_0^L\) is an exactly-solved Hamiltonian, we can formally treat \(H_1^\mu \Phi\) as a “perturbed” interaction in the RS expansion \[4\]. To mark the order of the RS expansion, an index factor \(\delta\) will be inserted in front of \(H_1^\mu \Phi\) in Eq.(10). Consequently, Eq.(9) is modified as

\[
[H_0^L + \delta H_1^\mu \Phi] \Psi_n [\phi_x + \Phi, J; \delta] = \left(E_n[J, \Phi, \delta] + C\right) \Psi_n [\phi_x + \Phi, J; \delta].
\] (13)

Now, applying the RS expansion, one can solve Eq.(13) to get energy eigenvalues, \(E_n[J, \Phi, \delta]\), and eigenwavefunctionals, \(\Psi_n [\phi_x + \Phi, J; \delta]\). Here, we are interested only in the eigenenergy functional for vacuum state, \(E_0[J, \Phi, \delta]\). Obviously, the zeroth-order approximation to \(E_0[J, \Phi, \delta]\), \(E_n^{(0)}[J, \Phi]\), satisfies
\[ E_0^{(0)}[J; \phi] + C = E_0^{(0)}[J] = 0 \]  

and at the \( n \)-th order of \( \delta \), the correction to \( E_0^{(0)}[J; \phi] \) is

\[ E_0^{(n)}[J; \phi] = \langle 0 | H^{\mu} \Phi [Q_0 \frac{1}{H_0^{\mu} - E_0^{(0)}[J]} (E_0^{(1)}[J; \phi] - H_0^{\mu} \Phi)]^{n-1} | 0 \rangle \]

with \( Q_n = \sum_{j=1}^{\infty} \int d^Dp_1 d^Dp_2 \cdots d^Dp_j (\langle j | > < j |) \). Thus, \( E_0[J; \Phi, \delta] = E_0^{(0)}[J; \phi] + \sum_{n=1}^{\infty} \delta^n E_0^{(n)}[J; \phi] \).

With the vacuum energy functional \( E_0[J; \Phi, \delta] \), one can have

\[ \frac{\delta E_0[J; \Phi, \delta]}{\delta J_x} = - \int D\phi \Phi^\dagger (\phi_x + \Phi, J; \delta)(\phi_x + \Phi) \psi_0 | \phi_x + \Phi, J; \delta \rangle \equiv - \varphi_x \]

where we used the Feynman-Hellmann theorem [20]. Evidently, for \( \delta = 1, \varphi_x = \langle \psi_0 | \phi_x | \psi_0 \rangle \). Then, a Legendre transformation of \( E_0[J; \Phi, \delta] \) yields the static effective action [3]

\[ \Gamma_s[\varphi; \Phi, \delta] = -E_0[J; \Phi, \delta] - \int J_x \varphi_x . \]

To calculate EP, one can conveniently take \( \varphi_x = \Phi \) in Eq.(16) to fix the arbitrary shifted parameter \( \Phi \). In analogy to Appendix A of Ref. [2] (1990) but with the Feynman-Hellmann theorem [20], one can show that other choices of \( \Phi \) will give rise to the same EP (when the wavefunction renormalization procedure is not needed). Finally, one can have the EP [3]

\[ V(\Phi) \equiv - \frac{\Gamma_s[\Phi, \delta; \varphi]}{\int x} \bigg|_{\varphi_x = \text{constant} = \Phi, \delta = 1} . \]

If the EP is truncated at some order of \( \delta \), the extrapolation of \( \delta = 1 \) should be made after renormalizing the approximated EP.

When truncated at a given order of \( \delta \), the right side of Eq.(18) will depend upon \( \mu \). To obtain an approximated EP up to the same order, one can determine \( \mu \) according to the principle of minimal sensitivity [3]. That is, \( \mu \) should be such a value that the approximated EP up to the given order is optimized to be as insensitive to variations in \( \mu \) as possible. This can be realized by analyzing the vanishing first- or higher-order derivatives of the truncated result with respect to \( \mu \) [3](v). From the next section, one will see that the EP up to the first order is just the Gaussian EP. Thus, the ORSE is a systematic tool of improving the Gaussian EP.

**IV. OPTIMIZED EFFECTIVE POTENTIAL FOR A CLASS OF MODELS UP TO THE SECOND ORDER**

In this section, we carry out the ORSE for the system, Eq.(7), to calculate the EP up to the second order.

The matrix elements which appear in Eq.(15) involve only Gaussian integrals except commutators of creation and annihilation operators and, thus, can be readily calculated as follows,

\[ \langle n | V_M[\phi_x + \Phi])m \rangle = \frac{1}{\sqrt{n!m!}} \sum_{i=0}^{n} C_n^i C_m^{n-i}(n-i)!(2(2\pi)^D)^{-\frac{m-n+2i}{2}} \]

\[ \cdot \left( \prod_{j=n-i+1}^{m} f(p_j) \right) \prod_{k=n-i+1}^{n} f(p'_k)^{-\frac{1}{2}} e^{i \left( \sum_{k=n-i+1}^{n} p'_k - \sum_{j=n-i+1}^{n} p_j \right)x} \prod_{l=1}^{n-i} \delta(p'_l - p_l) \]

\[ \cdot \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-\alpha^2 V(m-n-2i)} (\alpha \sqrt{f_{xx}^{-1}} I_1(M^2) + \Phi + \int J_x h_{xx}^{-1} J_x) \]

with \( n \leq m \). In Eq.(19), \( V^k(z) \equiv \frac{d^k V(z)}{(dz)^k} \). For simplicity, in getting the above results, we have employed the permutation symmetry of momenta in Eq.(15) for various products of \( \delta \) functions. Note that matrix elements of \( \phi_x^2 \) are special cases of Eq.(19).

Substituting the above matrix elements into Eq.(15), one can obtain the first- and the second-order corrections to \( E_0^{(0)}[J; \phi] \) as
\[ E_0^{(1)}[J; \phi] = \int_x \left\{ -\frac{\mu^2}{2} \left( \int_z h_{xz}^{-1} J_z \right)^2 + \frac{1}{2} f_{xx}^{-1} \right\} dx + \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi + \int_z h_{xz}^{-1} J_z) \right\} \] 

and

\[ E_0^{(2)}[J; \phi] = -\frac{\mu^2}{2} \int_x \frac{d^2 p}{(2\pi)^D} \frac{1}{f^2(p)} \int_{x_1} e^{ipx} h_{xz}^{-1} J_z \int_{x_2} e^{ipx} h_{xz}^{-1} J_z \int_{x_3} e^{ipx} \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V^{(1)}(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi + \int_z h_{xz}^{-1} J_z) \right\} \]

respectively. Here, \(\cdots\) represents the absolute value.

Next, we extract the approximated EP for the system order by order.

At the zeroth order, \(E_0^{(0)}[J; \phi] = -C\) from Eq.(14), and so, taking \(-\frac{\delta E_0^{(0)}[J; \phi]}{\delta J_x} = \int_y h_{xy}^{-1} J_y + \Phi = \varphi_x^{(0)}\) as \(\Phi\), one has \(J^{(0)} = 0\). Consequently, the EP at the zeroth-order of \(\delta\) is

\[ \mathcal{V}^{(0)}(\Phi) = -\frac{\Gamma_x^{(0)}[\varphi; \Phi, \delta]}{\varphi_x = \Phi} = \left\{ \frac{1}{2} f_{xx}^{-1} - \frac{1}{2} J_0(M^2) + \frac{M^2}{4} I_1(M^2) \right\} \]

Up to the first order (hereafter, any Greek-number superscript, such as \(^{\Gamma}\), \(^{II}\), means “up to the order whose number is consistent with the Greek number”),

\[ E_0'[\Phi, \delta; J] = E_0^{(0)}[J; \phi] + \delta E_0^{(1)}[J; \phi] \]

and \(-\frac{\delta E_0'[\Phi, \delta; J]}{\delta J_x} = \varphi_x' = \Phi\) yields

\[ \int_y h_{xy}^{-1} J_y + \delta \int_y h_{xy}^{-1} \left( \mu^2 \int_z h_{yz}^{-1} J_z \right) - \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V^{(1)}(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi + \int_z h_{xz}^{-1} J_z) \right\} = 0 \] 

When extracting the EP up to first order, Eqs.(17), (18) and (23) imply that only the \(J_x\) up to the first order, \(J^I\), is necessary. Owing to \(J^{(0)} = 0\), it is enough to take \(J_x = 0\) for the last term in the left hand of Eq.(24). Thus, \(J^I\) can be solved from Eq.(24) as

\[ J^I = \delta \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V^{(1)}(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi) \]

Even \(J^I\) will not be needed to get the EP up to the first order of \(\delta\), because there exists no linear, but the quadratic term of \(J_x\) in the zeroth-order term of \(E_0'[J; \Phi, \delta] - \int_x J_x \Phi\) as shown in Eq.(12). In fact, to obtain the EP up to the \(n\)th order, one need the approximated \(J\) only up to the \((n - 1)\)th order. Now one can write down the EP up to the first order
\[ V^I(\Phi, \delta) = \frac{1}{2} [f_{xx} - I_0(M^2)] + \frac{1}{4} M^2 I_1(M^2) - \delta \frac{1}{4} \mu^2 f_{xx}^{-1} + \delta \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi). \] (26)

Obviously, this result will yield nothing but the Gaussian EP [11] (1995, 2002).

Finally, we consider the second order. \( \varphi^l_x = \varphi^{lI}_x = -\frac{\delta E^{\mu}_l[f_{x}, \Phi, \delta]}{\delta f_{x}} = \Phi \) can be solved for \( J^{II} \). In the present case, however, it is enough to use only \( J^I \) for the EP. Substituting \( J^I \) into Eq. (18), we obtain the EP for the system, Eq. (7), up to the second order as

\[ V^{II}(\Phi, \delta) = \frac{1}{2} [f_{xx} - I_0(M^2)] + \frac{1}{4} M^2 I_1(M^2) - \delta \frac{1}{4} \mu^2 f_{xx}^{-1} + \delta \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi) \]

\[ -\delta^2 \mu^2 \int_{16} \frac{d^Dp}{f(p)} [\mu^2 - 2 \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V^{(2)}(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi)] \]

\[ -\delta^2 \sum_{j=2}^{2D} \int \frac{d^Dp_k}{f^{(j-1)}(p_k)} \frac{1}{f(\sum_{k=1}^{j-1} p_k) \prod_{k=1}^{j-1} f(p_k)} \]

\[ \cdot \frac{1}{(f(\sum_{k=1}^{j-1} p_k) + \sum_{k=1}^{j-1} f(p_k))} \left[ \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} V^{(j)}(\alpha \sqrt{f_{xx}^{-1} - I_1(M^2)} + \Phi) \right]^2, \] (27)

where, one should take \( \delta = 1 \) after renormalizing \( V^{II}(\Phi, \delta) \), and \( \mu \) is determined from the stationary condition

\[ \frac{\partial V^{II}(\Phi)}{\partial \mu} = 0. \] (28)

Here, \( V^{II}(\Phi) \) is the EP after \( V^{II}(\Phi, \delta) \) is renormalized. If Eq. (28) has no real solutions, \( \mu \) can be fixed by \( \frac{\partial^2 V^{II}(\Phi)}{\partial (\mu^2)} = 0 \). Note that in (1+1) dimensions, \( \{ \frac{1}{4} [f_{xx} - I_0(M^2)] + \frac{1}{4} M^2 I_1(M^2) - \frac{1}{4} \mu^2 f_{xx}^{-1} \} \) and \( \{ f_{xx}^{-1} - I_1(M^2) \} \) in Eq. (27) with \( \delta = 1 \) is finite and, thus, for any (1+1)-dimensional theories which make the series in Eq. (30) finite, no renormalization procedure is needed.

Similarly, employing Eq. (19), one can obtain higher order corrections to the Gaussian EP from Eq. (15). To conclude this section, we emphasize that Eq. (27) can easily be used to give the EPs for a number of scalar field theories including those discussed in Refs. [11, 13, 15].

V. APPLICATION TO \( \lambda \phi^4 \) FIELD THEORY

In this section, we consider the potential,

\[ V(\phi_x) = \frac{1}{2} m^2 \phi_x^2 + \lambda \phi_x^4, \] (29)

which was widely studied with several variational perturbation techniques [40].

Substituting Eq. (29) into Eq. (27), and noting that \( \int_{-\infty}^{\infty} \alpha^{2n+1} e^{-\alpha^2} d\alpha = 0 \) for \( n = 1, 2, \ldots \) and \( \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha = \sqrt{\pi} \), one can easily obtain the EP for the system, Eq. (29), up to the second order

\[ V^{II}(\Phi, \delta) = \frac{1}{2} [f_{xx} - I_0(M^2)] + \frac{1}{4} M^2 I_1(M^2) + \delta \frac{1}{2} m^2 \Phi^2 + \lambda \Phi^4 - \frac{1}{4} \mu^2 f_{xx}^{-1} \]

\[ + \frac{1}{4} \int \frac{d^Dp}{f(p)} [m^2 + 12 \Phi^2 + 3 \lambda (f_{xx}^{-1} - I_1(M^2))] \]

\[ - \frac{\delta^2}{16} \mu^2 \int \frac{d^Dp}{f(p)} [m^2 - \mu^2 + 12 \Phi^2 + 6 \lambda (f_{xx}^{-1} - I_1(M^2))] \]

\[ - \delta^2 2 \lambda^2 \Phi^2 \int \frac{d^Dp_1 d^Dp_2}{f(p_1) + f(p_2) + f(p_1 + p_2)} \]

\[ - \delta^2 2 \lambda^2 \Phi^2 \int \frac{d^Dp_1 d^Dp_2 d^Dp_3}{f(p_1) + f(p_2) + f(p_1 + p_2) + f(p_1 + p_2 + p_3)} \]

\[ - \delta^2 2 \lambda^2 \Phi^2 \int \frac{d^Dp_1 d^Dp_2 d^Dp_3}{f(\sum_{k=1}^{3} p_k) + f(\sum_{k=1}^{3} p_k) + f(\sum_{k=1}^{3} p_k) + f(\sum_{k=1}^{3} p_k) \prod_{k=1}^{3} f(p_k)}. \] (30)
Discarding terms with \( I_n(M^2)(n = 0, 1) \), the above result becomes identical to Eq.(2.36) in Ref. \[2\] (1990). This can be verified by carrying out integrations of \( I_n(\Omega)(n = 0, 1) \) and \( I^{(n)}(\Omega)(n = 2, 3, 4) \) in Ref. \[2\] (v) over one component of each Euclidean momentum. In \((1 + 1)\) dimensions, taking \( \delta = 1 \), one finds that Eq.(30) is finite. Because the \((1+1)\)-dimensional Gaussian EP in the Coleman’s normal-ordering prescription is consistent \[9\] (2002) with that in the Stevenson’s reparametrization scheme \[11\] (1985), Eq.(30) with \( \delta = 1 \) and \( D = 1 \) is consistent with the \( \Delta m^2 \) version of Sect. IV in Ref. \[2\] (1991). As for the case of \((2 + 1)\) dimensions, Stancu has performed a renormalization procedure to make \( V^{I(\Phi)} \) finite \[2\] (1991).

Using \( \mu \) fixed at the GA result for each order of the ORSE will simultaneously imply that up to each order, the vacuum expectation value of the field operator \( \phi_\mu \) is identical to that in the GA. If one chooses to do so, Eq.(30) and also the second-order result in our former work Ref. \[6\] will yield the result in Ref. \[2\] (ii,iii).

Finally, we point out that taking \( \mu = m \) will lead to the conventional perturbation result on the EP.

VI. DISCUSSION AND CONCLUSION

In this paper, an optimized RS expansion scheme of solving the functional Schrödinger equation with an external source is proposed to calculate the EP beyond the GA. For the class of scalar field theories, Eq.(7), we obtain a general expression on the EP up to the second order which can be used easily to obtain the EP for several models. Since the RS expansion is a basic tool in quantum physics and the Schrödinger picture can give us some quantum-mechanical intuition in QFT, we believe our investigation is interesting and useful.

Some investigations relevant to the present work can be envisioned. To our knowledge, scalar field models in Refs. \[2\] \[3\] were seldom investigated beyond the GA, and so it is worth applying Eq.(27) to those concrete models as well as to some bosonized models in condensed matter physics. This paper discussed just ground state solution of Eq.(13), and actually, one can consider excited state solutions of Eq.(13) to give the EP for the excited states. Our work here also implies that it is possible to introduce the optimized procedure to those schemes proposed in Ref. \[2\] (i,ii,iii). Furthermore, since Ref. \[6\] has generalized the RS perturbation theory to the spinor theory, QED and Yang-Mills theory, it should be viable to generalize the ORSE here to those higher-spin field theories. Besides, since an effective action contains complete information of a field system, it will be useful to develop the ORSE here to calculate the effective action \[2\] (iii). Finally, Cornwall, Jackiw and Tomboulis developed a generalized EP for composite operators and calculated it with Rayleigh-Ritz procedure \[21\], and it will be interesting to develop the ORSE here for calculating the generalized EP which will go beyond the variational result.

ACKNOWLEDGMENTS

This project was supported by the Korea Science and Engineering Foundation through the Center for Strongly correlated materials Research (SNU). Lu’s work was also supported in part by the National Natural Science Foundation of China under the grant No. 19875034.

[1] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127 (1962) 965; G. Jona-Lasinio, Nuovo Cimento 34 (1964) 1700; S. Coleman and E. Weinberg, Phys. Rev. D 7 (1973) 1888; R. Jackiw, ibid. D 9 (1974) 1686.

[2] i. M. Funke and H. Kümmer, Phys. Rev. D 32 (1985) 1435; ii. I. Yotsuyanagi, Z. Phys. C 35 (1987) 453; iii. I. Yotsuyanagi, Z. Phys. C 35 (1987) 453; R. Ibañez-Meier, A. Mattingly, U. Ritschel and P. M. Stevenson, Phys. Rev. D 45, 2893 (1992); ii. P. Cea and L. Tedesco, ibid. D 55 (1997) 4967; iii. G. H. Lee and J. H. Yee, ibid. D 56 (1997) 6573; iv. A. Okopińska, ibid. D 35 (1987) 1835; v. I. Stancu and P. M. Stevenson, ibid. D 42 (1990) 2710; I. Stancu, ibid. D 43 (1991) 1283.

[3] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics, World Scientific, Singapore, 2nd Ed., Chapter 5, 1995; H. Kleinert and V. Schulte-Frohlinde, Critical Properties of \( \phi^4 \)-Theories, World Scientific, Singapore, Chapter 19, 2001; S. K. You, K. J. Jeon, C. K. Kim and K. Nahm, Eur. J. Phys. 19, 179 (1998); W. F. Lu, S. K. You, J. Bak, C. K. Kim and K. Nahm, J. Phys. A 35 (2002) 21.

[4] E. Schrödinger, Ann. Phys. (Leipzig) 80 (1926) 437; A. Z. Capri, Nonrelativistic Quantum Mechanics, The Benjamin/Cummings, Menlo Park, California, 1985.

[5] B. Hatfield, Quantum Field Theory of Point Particles and Strings, Addison-Wesley, Redwood, California, Chapters 11 & 18, 1992.
[6] W. F. Lu, C. K. Kim, J. H. Yee and K. Nahm, Phys. Rev. D 64 (2001) 025006.
[7] P. M. Stevenson, Phys. Rev. D 23 (1981) 2916; Phys. Lett. B 100 (1981) 61; Phys. Rev. D 24 (1981) 1622; S. K. Kaufmann and S. M. Perez, J. Phys. A 17, 2027 (1984); P. M. Stevenson, Nucl. Phys. B 231 (1984) 65.
[8] P. M. Stevenson, Phys. Rev. D 30 (1984) 1712.
[9] W. F. Lu, S. Q. Chen and G. J. Ni, J. Phys. A 28 (1995) 7233; W. F. Lu, ibid. A 32 (1999) 739; W. F. Lu and C. K. Kim, ibid. A 35 (2002) 393.
[10] N. Boccara, Functional Analysis — An Introduction for Physicists, Academic Press, New York, Chapter 4, 1990.
[11] P. M. Stevenson, Phys. Rev. D 32 (1985) 1389; W. F. Lu, B. W. Xu and Y. M. Zhang, ibid. D 49 (1994) 5625.
[12] P. M. Stevenson and I. Roditi, Phys. Rev. D 33 (1986) 2305; W. F. Lu, G. J. Ni and Z. G. Wang, J. Phys. G 24 (1998) 673.
[13] R. Ingermanson, Nucl. Phys. B 266 (1986) 620; W. F. Lu, B. W. Xu and Y. M. Zhang, Phys. Lett. B 309 (1993) 109.
[14] J. Fröhlich, Phys. Rev. Lett. 34 (1975) 833; W. F. Lu, Phys. Rev. D 59 (1999) 105021; S. Y. Lou and G. J. Ni, Commun. Theor. Phys. 11 (1988) 87; W. F. Lu, B. W. Xu and Y. M. Zhang, ibid. 27 (1997) 485.
[15] R.P. Ignatius, V.C. Kuriakose and K. Babu Joseph, Phys.Lett.B 220 (1989) 181.
[16] G. Rosen, Phys. Rev. Lett. 16 (1966) 704; Bo-Sture K. Skagerstam, Phys. Rev. D 13 (1976) 2827.
[17] B. S. DeWitt, Phys. Rev. 162 (1967) 1195; 162 (1967) 1293; F. Abbott, Nucl. Phys. B 185 (1981) 189; K. Zarembo, Mod. Phys. Lett. A 13 (1998) 1709 or hep-th/9803233.
[18] S. Coleman, Phys. Rev. D 11 (1975) 2088; S. J. Chang, 1976 ibid. D 13 (1976) 2778.
[19] Y. Nambu, Phys. Rev. 117 (1960) 648; Y. Nambu and G. Jona-Lasinio, ibid. 122 (1961) 345; T. R. Koehler, ibid. 165 (1968) 942; R. Seznec and J. Zinn-Justin, J. Math. Phys. 20 (1979) 1398.
[20] R. P. Feynman, Phys. Rev. 56 (1939) 340; D. B. Lichtenberg, ibid. D 40 (1989) 4196; W. Namgung, J. Korean. Phys. Soc. 32 (1998) 649.
[21] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D 10 (1974) 2428.