Statistical Recognition Methods Based on Complex HRRP

Xuan Zhao, Jianbin Lu*

School of Electronic Engineering Naval University of Engineering, Wuhan, China

*Corresponding author e-mail: xzhao415@163.com

Abstract. The radar high range resolution echo (radar target complex echo signal) is the sum of the projection vectors of the scattering points of the target detected by high resolution radar in the radar line of sight, and it contains the rich geometrical structure information of the target in that direction. At present, most of the statistical recognition work based on radar high-resolution range echo is using the amplitude information (i.e. High Range Resolution Profile, HRRP) obtained after high range resolution echo of radar target, which will result in limited recognition performance of the recognition system. In fact, the phase information of radar target high range resolution echo is same as amplitude information, and it also reflects some geometrical structure information of radar target, which is valuable to improve the recognition performance of radar target recognition system. In this paper, the complex HRRP was studied. Complex Probabilistic Principal Component Analysis and Complex Adaptive Gaussian Classifier models were used for statistical modeling, and the simulation results were given.

1. Real AGC and PPCA statistical models

1.1. Adaptive Gaussian Classifier (AGC) statistical model

The Adaptive Gaussian Classifier (AGC) assumes that each distance element in the radar HRRP sample is independent of each other and is subjected to a Gaussian distribution. Given a set of radar target data $Y = [y_c^{n_m}]$, where $y_c^{n_m}$ represents the $n_m^{th}$ ($n = 1, \cdots, N_c$) radar high range resolution echo sample, the dimension is $L$, and $c$ represents the class $c^a (c=1, \cdots, G)$ object, $m$ represents the $m^a (m=1, \cdots, M)$ frame of the class $c^a$ object. The estimated parameter set of AGC model is $\Theta_{AGC} = \{ \mu^{c^a}, \Sigma^{c^a} \}$, where, the estimation formula of the mean value $\mu^{c^a}$ and variance $\Sigma^{c^a}$ of the samples in the $m^a$ frame of the target $c^a$ class is:

$$\mu^{c^a} = \frac{1}{N_c^{m_a}} \sum_{j=1}^{N_c^{m_a}} y_j^{c^a}$$ (1)

$$\Sigma^{c^a} = \text{diag} \left[ \frac{1}{N_c^{m_a}} \sum_{j=1}^{N_c^{m_a}} (y_j^{c^a} - \mu^{c^a})(y_j^{c^a} - \mu^{c^a})^\top \right]$$ (2)
Where \( N^c \) represents the total number of samples for the \( c \)th class, \( \text{diag} [\cdot] \) represents the matrix diagonalization operator. The conditional likelihood function of the test sample \( y \) can be written as:

\[
p(y \mid \Theta) = (2\pi)^{-d/2} |\Sigma|^{-1/2} \cdot \exp \left( -\frac{1}{2} (y - \mu)^T \Sigma^{-1} (y - \mu) \right)
\]  

(3)

If the logarithm of both sides of equation (3) is taken, the following formula can be obtained. Just take the minimum value of \( D^c \) in the decision process:

\[
D^c = \ln |\Sigma^c| + (y^* - \mu^c)^T \Sigma^{-1} (y^* - \mu^c)
\]  

(4)

1.2. Probabilistic Principal Component Analysis (PPCA) statistical model

Probabilistic Principal Component analysis (PPCA) model is a target recognition model based on Gaussian distribution. PPCA model is the extension of Principal Component Analysis (PCA) model under probabilistic framework. A given real number sample \( y \) can be expressed in the following form according to the PPCA model:

\[
y = Az + \mu + \epsilon
\]  

(5)

Where, the matrix \( A \) is the \( q \times d \) dimensional loading matrix, and the vector \( \mu \) is the mean vector of the \( d \times 1 \) dimension. The vector \( z \) of the \( q \times 1 \) dimension is the corresponding hidden variable, and \( z \) obeys the Gaussian distribution of the identity matrix whose mean is 0 and whose covariance matrix is identity matrix \( I_q \), which can be written as \( z \sim G(0, I_q) \). \( \epsilon \) is the noise vector in \( d \times 1 \) dimension. \( \epsilon \) also obeys the Gaussian distribution, and it can be written as \( \epsilon \sim G(0, \sigma^2 I_d) \). Since the Gaussian distribution is linear, \( y \sim G(\mu, AA^T + \sigma^2 I_d) \) can be deduced. As can be seen from (5), PPCA model divides the space of HRRP training sample \( y \) into two parts, namely signal subspace and noise subspace respectively, and the characteristics of two subspaces can be considered separately. The probability density function of the real number sample \( y \) can be expressed as:

\[
p(y) = (2\pi)^{d/2} |AA^T + \sigma^2 I_d|^{-1/2} \cdot \exp \left( -\frac{1}{2} (y - \mu)^T (AA^T + \sigma^2 I_d)^{-1} (y - \mu) \right)
\]  

(6)

Where, \( |\cdot| \) represents the determinant operator for the matrix. PPCA model parameters are estimated using the maximum likelihood estimation method. The real sample set is represented by \( Y = \{y_i \mid i = 1, 2, \cdots, N\} \), then the logarithmic likelihood function of the sample can be expressed as:

\[
L(\Theta) = \sum_{i=1}^{N} \ln p(y_i) = -\frac{N}{2} \left[ d \ln (2\pi) + \ln |AA^T + \sigma^2 I_d| + \sum_{i=1}^{N} (y_i - \mu)^T (AA^T + \sigma^2 I_d)^{-1} (y_i - \mu) \right]
\]  

(7)

Where, the parameter set of PPCA is \( \Theta = \{\mu, A, \sigma^2\} \). Taking the partial derivative of equation (7) and making the derivative equal to zero, the maximum likelihood estimation of the mean value can be obtained:

\[
\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} y_i
\]  

(8)

It can be seen from equation (7) that the parameters \( A \) and \( \sigma^2 \) have a mutual coupling relationship, so the maximum likelihood estimation cannot be directly obtained as the parameter \( \mu \). Therefore, the
Expectation Maximization (EM) algorithm [1] is adopted to estimate these parameters. The steps of estimating PPCA model parameters using EM algorithm are described below.

First, the expected value of logarithmic joint likelihood function of PPCA model is defined as:
\[
Q(\Theta) = E\left\{ \sum_{i=1}^{N} \ln[p(y_i, z_i) | Y] \right\} = E\left\{ \sum_{i=1}^{N} \ln \left[ \frac{1}{(2\pi)^{d/2} \sigma^2 I} \exp \left[ -\frac{1}{2} (\bar{y}_i - A\mu)^T (\bar{y}_i - A\mu) - \frac{1}{2} z_i^T z_i \right] \right] \right\}
\]
\[
= -\frac{1}{2} \sum_{i=1}^{N} \left( y_i^T (\sigma^2 I)_{x} y_i - 2y_i^T (\sigma^2 I_{x})^{-1} A E(z_i | y_i) + Tr\left( A^T (\sigma^2 I_x)^{-1} A + I_x \right) E(z_i z_i^T | y_i) \right) - \frac{N(d+q)}{2} \ln(\pi) - \frac{N}{2} \ln(\sigma^2 I_x)
\]

Where \( \bar{y}_i = y_i - \bar{\mu} \).

Step 1: Set the number of iterations \( \tau = 1 \), give the threshold of iteration termination, and set the initial parameters \( \hat{\Theta}(1) = \{\hat{\lambda}(1), \hat{\sigma}^2(1)\} \) of the model (since the estimated mean value has been obtained from equation (8), the parameter set is updated to \( \hat{\Theta} = \{\hat{\lambda}, \hat{\sigma}^2\} \) here);

Step 2: assuming that \( \hat{\Theta}(\tau) = \{\hat{\lambda}(\tau), \hat{\sigma}^2(\tau)\} \) is the estimated parameter value obtained during the \( \tau \)th iteration, the conditional expectation and the expectation of conditional correlation matrix of \( z \) can be respectively calculated by the following formula:
\[
\begin{align*}
E(z_i | y_i; \hat{\Theta}(\tau)) &= \hat{\lambda}(\tau) \hat{C}^{-1}(\tau) y_i \\
E(z_i z_i^T | y_i; \hat{\Theta}(\tau)) &= I_x - \hat{\lambda}(\tau) \hat{C}^{-1}(\tau) + E(z_i | y_i; \hat{\Theta}(\tau)) E(z_i^T | y_i; \hat{\Theta}(\tau))
\end{align*}
\]

Where \( \bar{y}_i = y_i - \bar{\mu}, \hat{C}(\tau) = \hat{\lambda}(\tau) \hat{A}^T (\tau) + \hat{\sigma}^2(\tau) I_x, z_i \) is the implicit variable corresponding to \( y_i \), and \( E(\cdot) \) denotes the conditional expectation operator. The conditional expectation logarithm joint likelihood function of \( y \) and \( z \) is defined as:
\[
Q(\Theta, \hat{\Theta}(\tau)) = E\left\{ \sum_{i=1}^{N} \ln[p(y_i, z_i; \hat{\Theta}(\tau))] \right\} = E\left\{ \sum_{i=1}^{N} \ln \left[ \frac{1}{(2\pi)^{d/2} \sigma^2 I} \exp \left[ -\frac{1}{2} (\bar{y}_i - A\mu)^T (\bar{y}_i - A\mu) - \frac{1}{2} z_i^T z_i \right] \right] \right\}
\]
\[
= -\frac{1}{2} \sum_{i=1}^{N} \left( y_i^T (\sigma^2 I_x) y_i - 2y_i^T (\sigma^2 I_x)^{-1} A E(z_i | y_i) + Tr\left( A^T (\sigma^2 I_x)^{-1} A + I_x \right) E(z_i z_i^T | y_i) \right) - \frac{N(d+q)}{2} \ln(\pi) - \frac{N}{2} \ln(\sigma^2 I_x)
\]

Where \( Tr(\cdot) \) represents the matrix trace operator.

Step 3: Fix the conditional mean value and conditional correlation matrix expectation of the hidden variable \( z \), take the partial derivative of \( Q(\Theta, \hat{\Theta}(\tau)) \) with respect to parameters \( A \) and \( \sigma^2 \), and set the derivative to zero, so as to obtain the parameter value updated in the \( (\tau + 1) \)th iteration, that is:
\[
\begin{align*}
\hat{\lambda}(\tau + 1) &= \left[ \sum_{i=1}^{N} E(z_i | y_i; \hat{\Theta}(\tau)) \right]^{-1} \left[ \sum_{i=1}^{N} E(z_i z_i^T | y_i; \hat{\Theta}(\tau)) \right]^{-1} \\
\hat{\sigma}^2(\tau + 1) &= \frac{1}{Nd} Tr\left( \sum_{i=1}^{N} \bar{y}_i y_i^T - \hat{\lambda}(\tau + 1) E(z_i | y_i; \hat{\Theta}(\tau)) \right)
\end{align*}
\]
Step 4: Determine whether the convergence condition is reached. If \( |L(\hat{\theta}(r + 1)) - L(\hat{\theta}(r))|/|L(\hat{\theta}(r))| > \varepsilon \), then update the number of iterations, and at the same time continue step 2; Otherwise, the convergence condition is considered to be achieved. At this point, let \( \hat{\theta} = \hat{\theta}(r + 1) \) and end the EM algorithm.

Therefore, the estimated covariance matrix is \( \bar{C} = \bar{A}^T + \bar{\sigma}^2 I_d \). The inverse matrix formula can be used to obtain \( \bar{C}^{-1} = \bar{\sigma}^{-2} (I_d - \bar{\sigma}^{-2} \bar{M}^{-1} \bar{A}^T) \) and \( M = \bar{\sigma}^{-2} \bar{A}^T + I_d \), which can reduce the calculation of the inverse covariance matrix. Then the probability density function of the real number sample is:

\[
p(y) = (2\pi)^{d/2} |\bar{A}^T + \bar{\sigma}^2 I_d|^{-1/2} \exp\left[-\frac{1}{2} \bar{\sigma}^{-2} (y - \bar{\mu})^T (I_d - \bar{\sigma}^{-2} \bar{M}^{-1} \bar{A}^T)(y - \bar{\mu}) \right]
\]

It can be seen that we estimate the sample covariance matrix indirectly by estimating the model parameters \( A \) and \( \sigma^2 \) using the PPCA model, which can improve the accuracy of parameter estimation, avoid direct inversion of the high-dimensional matrix, and reduce the calculation quantity.

2. Modeling and analysis based on CAGC and CPPCA

2.1. Analysis of statistical characteristics of complex HRRP

Real HRRP have sensitivity problems during target recognition, such as aspect sensitivity, amplitude-scale sensitivity and time-shift sensitivity. Complex HRRP not only have these sensitivity problems, but also have the problem of initial phase sensitivity. This section will focus on the effects of the initial relative modeling of complex HRRP.

The HRRP samples are evenly divided into frames. According to the scattering center model [2,3], the radar HRRP echo in any frame can be expressed as (14):

\[
x(m) = \exp(j\theta_\lambda)[x_1(m), x_2(m), \cdots, x_n(m)]
\]

Where, \( \theta_\lambda = -4\pi R(m)/\lambda \) is the initial phase of HRRP, \( R(m) \) is the distance from the radar observation target center to the radar, \( \lambda \) is the wavelength, and \( x_i(m) \) represents the echo in the \( n^d \) resolution cell, which can be expressed in formula (15)

\[
x_i(m) = \sum_{l=1}^{V_e} \sigma_{il} \exp(j\phi_{il}(m)) = y_i(m) \exp(j\phi_{il}(m))
\]

Where, \( V_e \) represents the number of scattered points in the \( n^d \) resolution cell, and \( \sigma_{il} \) and \( \phi_{il}(m) \) are the subecho amplitude and phase of the \( l^i \) scattering point in the resolution cell respectively. \( y_i(m) \) and \( \phi_{il}(m) \) are the amplitude and phase of the echo of the resolution cell respectively.

By combining formulas (14) and (15), it can be found that the phase information in the complex HRRP can be divided into the following two parts: the initial phase \( \theta_\lambda \) and the resolution cell echo phase \( \phi_{il}(m) \). First, the influence of the initial phase \( \theta_\lambda \) is analyzed. \( \theta_\lambda \) is constrained by \( \lambda \) and \( R(m) \). If the wavelength \( \lambda \) has been determined, \( \theta_\lambda \) is only determined by \( R(m) \), while \( R(m) \) only represents the distance between the target and the radar, and does not contain the structural information of the target. Therefore the initial phase does not affect recognition.

The frequency of wideband radar is generally high, so the complex HRRP is prone to initial phase sensitivity. However, the accuracy of radar nowadays has not yet met the requirement of initial phase compensation, and \( \theta_\lambda \) is always coupled with \( \phi_{il}(m) \), which affects the direct use of \( \phi_{il}(m) \) for target
recognition. Therefore, we will analyze the influence of $\theta_{m0}$ on the statistical characteristics of complex HRRP.

Assuming that the target distance $R(m)$ is uniform change, so the initial phase $\theta_{m0}$ is a random variable uniformly distributed on the $[0, 2\pi]$ [4], which can be denoted by

$$\theta_{m0}\sim U(0, 2\pi)$$

(16)

The echo $x_{m}(m)$ of each resolution cell is made up of multiple scattered point echo with different amplitude-scale. According to the central limit theorem, it can be considered that $x_{m}(m)$ is subject to the complex Gaussian distribution of zero mean, which can be denoted as

$$x_{m}(m)\sim CG\left(x_{m}(m)\mid 0, \sigma_{x_{m}(m)}^{2}\right)$$

(17)

Where $\sigma_{x_{m}(m)}^{2}$ represents the variance of $x_{m}(m)$. For the sake of simplicity, we assume that $x_{m}(m)$ is subject to the circular symmetric complex Gaussian distribution [5], that is, the real and imaginary parts of $x_{m}(m)$ are independent of each other, and the real and imaginary parts are subject to the same Gaussian distribution (all the complex Gaussian distribution refer to this distribution if no special explanation is given below).

$$\begin{align*}
\gamma_{x}(m) & \sim \text{Rei}\left(\gamma_{x}(m)\mid \sigma_{x_{m}(m)}^{2}/2\right) \\
\varphi_{x}(m) & \sim U(\varphi_{x}(m)\mid 0, 2\pi)
\end{align*}$$

(18)

Considering the influence of initial phase to $x_{m}(m)$, let

$$\begin{align*}
\tilde{x}_{m}(m) & = \exp\left(j\theta_{m0}\right) x_{m}(m) = \gamma_{x}(m) \exp\left[j(\theta_{m0} + \varphi_{x}(m))\right] = \gamma_{x}(m) \exp\left(j\tilde{\theta}_{x}(m)\right)
\end{align*}$$

(19)

According to the property of uniform distribution and the periodicity of the complex exponential function (the period is $2\pi$), we can obtain

$$\tilde{\theta}_{x}(m) \sim U(\tilde{\theta}_{x}(m)\mid 0, 2\pi)$$

(20)

And finally obtain:

$$\tilde{x}_{m}(m)\sim CG\left(\tilde{x}_{m}(m)\mid 0, \sigma_{x_{m}(m)}^{2}\right)$$

(21)

By comparing formulas (17) and (21), it can be seen that the initial phase does not change the statistical distribution of the echo of each resolution cell, that is, it has no impact on the statistical characteristics of HRRP, so we can directly model the complex HRRP statistics. For cyclic symmetric complex Gaussian distribution, it is completely determined by its sufficient statistics, namely mean value and variance. Since all complex HRRP targets have the same zero mean, there is no discriminant information in the distribution of the echo of each resolution cell, that is, it has no impact on the statistical characteristics of HRRP, so we can directly model the complex HRRP statistics. For cyclic symmetric complex Gaussian distribution, it is completely determined by its sufficient statistics, namely mean value and variance. Since all complex HRRP targets have the same zero mean, there is no discriminant information in the mean of complex HRRP, so all discriminant information is contained in the covariance matrix of complex HRRP. This means that simple classifiers using only first-order statistics will no longer be suitable for complex HRRP recognition. Next, we will use the CPPCA and CAGC models to model complex HRRP, which can better describe the second-order statistics of complex HRRP.
2.2. CAGC statistical modeling

The Complex Adaptive Gaussian Classifier (CAGC) model and the real Adaptive Gaussian Classifier model all have the hypothesis that each range cell is independent of each other. Therefore, the HRRP complex echo sample $x$ obeys the complex Gaussian distribution, and the mean value is 0, denoted as $x \sim \text{CG}(x|0, \Lambda)$. Where $\Lambda \in \mathbb{R}^{L \times L}$ is a diagonal matrix, representing the covariance matrix of the complex echo sample. For a given HRRP complex echo sample $X = [x_1, \ldots, x_N]$, the covariance matrix $\Lambda$ can be estimated by ML algorithm:

$$
\Lambda = \frac{1}{N} \text{diag} \left( \sum_{i=1}^{N} x_i x_i^H \right)
$$

(22)

Where, $\text{diag}(\cdot)$ represents matrix diagonalization operator, superscript $H$ represents matrix conjugate transposition operator.

Using CAGC model to identify radar target is divided into two stages, namely, training stage and testing stage. In the training stage, the complex echoes are pre-processed, then the statistical models are used to model each frame and the estimated parameter values are obtained according to the ML estimation algorithm. In the testing phase, the category of test samples is judged according to the probability density function based on Bayes theory.

2.3. CPPCA statistical modeling

It can be seen from the literature [6] that correlation exists between various range cell of complex HRRP. Combined with the results discussed in section 1.2.1 of this chapter, we can conclude that complex HRRP is subject to the combined complex Gaussian distribution of zero mean. Therefore, this section adopts the Complex Probabilistic Principal Component Analysis (CPPCA) model for modeling. Given a complex HRRP sample $x$, the CPPCA model can be expressed as:

$$
x = Ay + \varepsilon
$$

(23)

In the formula, the complex matrix $A$ of $d \times q$ dimension is the projection matrix, the complex vector $y \sim \text{CG}(y|0, I_q)$ of $q \times 1$ dimension is the hidden variable corresponding to $x$, and the complex vector $\varepsilon \sim \text{CG}(\varepsilon|0, \sigma^2 I_d)$ of $d \times 1$ dimension is noise vector. As we can see, the CPPCA model extended the subspace and noise space of the PPCA model to the complex domain. According to the linear property of the complex Gaussian distribution, we can obtain $x \sim \text{CG}(x|0, AA^H + \sigma^2 I_d)$. The CPPCA model has the following advantages for the complex HRRP modeling:

1. The complex HRRP dimension is higher and the number of samples is limited. The covariance matrix error obtained by direct estimation is large and may even be singular. However, the CPPCA model constructed sample covariance matrix by using $A$ and $\sigma^2$, which greatly reduced the number of parameters and improved the accuracy of parameter estimation.

2. It is reasonable to think that complex HRRP is also concentrated distributed in some subspace, while CPPCA model can better describe the structure of complex HRRP's subspace.

Then, model parameter estimation is discussed below. Here, I directly extended the EM algorithm to the complex field to estimate the parameters of the CPPCA model. Here, the EM algorithm is directly extended to the complex domain to estimate the parameters of the CPPCA model. Given a set of complex HRRP samples $X = \{x_1, x_2, \ldots, x_N\}$, the expected logarithmic likelihood function of the model is:
Where, $\Theta = [A, \sigma^2]$ is the parameter set of the CPPCA model, superscript $H$ represents the conjugate transpose of the matrix or vector. The conditional mean value and conditional correlation matrix of hidden variables can be obtained through deduction:

$$
\begin{align*}
\mathbb{E}(y_i | x_i; \Theta) &= A^H (A A^H + \sigma^2 I_d) x_i = A^H C^{-1} x_i \\
\mathbb{E}(y_i y_i' | x_i; \Theta) &= I_q - A^H C^{-1} A + \mathbb{E}(y_i | x_i; \Theta) \mathbb{E}(y_i' | x_i; \Theta)
\end{align*}
$$

(25)

Where $C = A^H A + \sigma^2 I_d$. Accordingly, in the $(\tau+1)^{th}$ iteration of EM algorithm, the parameter updating formula of CPPCA model is:

$$
\begin{align*}
\hat{A}(\tau+1) &= \frac{1}{Nd} \text{Tr} \left[ \sum_{i=1}^{N} x_i x_i^H - \hat{A}(\tau+1) \mathbb{E}(y_i | x_i; \hat{\Theta}(\tau)) x_i^H \right] \\
\hat{\sigma}^2(\tau+1) &= \frac{1}{N} \text{Tr} \left[ \sum_{i=1}^{N} x_i x_i^H \right] - \hat{A}(\tau+1) \mathbb{E}(y_i | x_i; \hat{\Theta}(\tau)) x_i^H
\end{align*}
$$

(26)

Where, $\hat{\Theta}(\tau) = [\hat{A}(\tau), \hat{\sigma}^2(\tau)]$ is the model parameter value estimated by the $\tau^{th}$ iteration.

Next, we analyze the influence of initial phase on CPPCA model and EM algorithm. Multiply both sides of equation (23) by a random initial phase $\exp(j \theta_i)$, and we can obtain formula (27),

$$
\begin{align*}
x \exp(j \theta_0) &= A y \exp(j \theta_0) + \varepsilon \exp(j \theta_0)
\end{align*}
$$

(27)

Let $x' = x \exp(j \theta_0)$, $y' = y \exp(j \theta_0)$, $\varepsilon' = \varepsilon \exp(j \theta_0)$, the formula (27) can be rewritten as

$$
\begin{align*}
x' &= A y' + \varepsilon'
\end{align*}
$$

(28)

According to the properties of the complex Gaussian distribution, we can obtain:

$$
\begin{align*}
y' &\sim CG(y' | 0, I_q), \varepsilon' \sim CG(\varepsilon' | 0, \sigma^2 I_d), x' \sim CG(x' | 0, A A^H + \sigma^2 I_d)
\end{align*}
$$

(29)

That is, the statistical distribution of $y', \varepsilon', x'$ is identical to that of $y, \varepsilon, x$, which means that the initial phase has no effect on the statistical properties of the CPPCA model.

The influence of initial phase on EM algorithm is also discussed. The samples in $X$ are respectively multiplied by different random initial phase $\exp(j \theta_i), i=1,2,\cdots,N$, denoted as $X'=[x'_1, x'_2, \cdots, x'_N]$, where $x'_i = x_i \exp(j \theta_0)$. According to formula (25), the conditional mean value and conditional correlation matrix of these samples corresponding to implicit variables $\{y'_i, y''_i, \cdots, y''_i\}$ are changed to:

$$
\begin{align*}
\mathbb{E}(y'_i | x'_i; \Theta) &= A^H (A A^H + \sigma^2 I_d) x'_i = A^H C^{-1} x'_i = \exp(j \theta_0) \mathbb{E}(y_i | x_i; \Theta) \\
\mathbb{E}(y''_i | x'_i; \Theta) &= I_q - A^H C^{-1} A + \mathbb{E}(y'_i | x'_i; \Theta) \mathbb{E}(y''_i | x'_i; \Theta)
\end{align*}
$$

(30)
In the EM algorithm, the parameter update formula is correspondingly changed to:

\[
\begin{align*}
A^{(t+1)} &= \left[ \sum_{i=1}^{N} x_i^t E \left(y_i^t | x_i^t; \hat{\Theta}(t) \right) \right] \left[ \sum_{i=1}^{N} E \left(y_i^t | x_i^t; \hat{\Theta}(t) \right) \right]^{-1} \\
\mathcal{H}(t+1) &= \frac{1}{Nd} \text{Tr} \left( \sum_{i=1}^{N} x_i^t \left( x_i^t - A^{(t+1)} E \left(y_i^t | x_i^t; \hat{\Theta}(t) \right) \right) \right) \\
\Psi^{(t+1)} &= \frac{1}{Nd} \text{Tr} \left( \sum_{i=1}^{N} x_i^t \left( x_i^t - A^{(t+1)} E \left(y_i^t | x_i^t; \hat{\Theta}(t) \right) \right) \right)
\end{align*}
\]  

(31)

By comparing formulas (25) and (30), formulas (26) and (31), we can find that the initial phase affects the mean value of the hidden state condition, but does not change the parameter value estimated by EM algorithm.

3. Noise correction algorithm

The typical working mode of target recognition system is to use the template of high signal-to-noise ratio (SNR) sample training to identify low SNR test sample. The SNR mismatch between the template and the test sample can lead to a significant decrease in system recognition performance. In order to improve the recognition performance of the system under the condition of low SNR, a common method is to make corresponding corrections according to the SNR model parameters of the test samples. In the complex HRRP recognition, the noise is additive noise, i.e.

\[
\hat{x} = as + n
\]  

(32)

Where, \( \hat{x} \) is the noise-complex HRRP sample, \( s \) is the normalized target signal, \( a \) is the amplitude scaling factor, which is related to such factors as radar-target center distance and radar transmitter power. \( \eta \sim \text{CG}(\boldsymbol{n}, \sigma_n^2 I_d) \) \( s \) complex Gaussian white noise, and \( \sigma_n^2 \) is noise power. A noise correction algorithm for the CPPCA model is described below.

In order to overcome the amplitude sensitivity of complex HRRP, the energy of \( \hat{x} \) should be normalized before identification. Here, the energy of \( \hat{x} \) is recorded as \( \hat{x}_e \) after normalization, then

\[
\hat{x}_e = \frac{\hat{x}}{\sqrt{a^2 + d\sigma^2}} = \frac{a}{\sqrt{a^2 + d\sigma^2}} s + \frac{1}{\sqrt{a^2 + d\sigma^2}} n
\]  

(33)

For the normalized target signal \( s \) without noise, it can be seen from section 1.2.3 that \( s \sim \text{CG}(\boldsymbol{0}, \mathcal{A}a^2 + \sigma_s^2 I_d) \). Therefore, the mean value and covariance matrix of \( \hat{x}_e \) are:

\[
E(\hat{x}_e) = \frac{a}{\sqrt{a^2 + d\sigma^2}} E(s) + \frac{1}{\sqrt{a^2 + d\sigma^2}} E(n) = 0
\]  

(34)

\[
\text{Cov}(\hat{x}_e) = \frac{a^2}{a^2 + d\sigma^2} \text{Cov}(s) + \frac{1}{a^2 + d\sigma^2} \text{Cov}(n) = \frac{a^2}{a^2 + d\sigma^2} (\mathcal{A}a^2 + \sigma_s^2 I_d) + \frac{\sigma_s^2}{a^2 + d\sigma^2} I_d
\]  

(35)

The covariance matrix of normalized target signal \( s \) without noise is denoted as \( \Sigma \), then the covariance matrix of \( \hat{x}_e \) can be rewritten as:

\[
\text{Cov}(\hat{x}_e) = \frac{a^2}{a^2 + d\sigma^2} \Sigma + \frac{\sigma_s^2}{a^2 + d\sigma^2} I_d
\]  

(36)

In the formula (36), \( \text{Cov}(\cdot) \) represents the operator of covariance matrix. The SNR of \( \hat{x}_e \) is

\[
\eta = 10 \log_{10} \left( \frac{a^2}{d\sigma_s^2} \right)
\]

which is substituted into the formula (36). The covariance matrix of \( \hat{x}_e \) can be rewritten as:
\[
\text{Cov}(\hat{x}_r) = \frac{10^{\eta_0}}{1 + 10^{\eta_0}} \Sigma + \frac{1}{d(1 + 10^{\eta_0})} I_d
\]  

(37)

In the testing stage, based on the SNR \( \eta \) of the test samples, the covariance matrix of the CPPCA model is modified according to the formula (37). Then sound robust complex HRRP recognition can be realized. In the complex HRRP recognition, it is much simpler to make noise adaptive correction to the model, and the accuracy of CPPCA model is higher because the error caused by approximation is avoided.

The real Gaussian model also has a similar noise correction method, but the noise robust correction method based on the real Gaussian model approximates that the noise in the test sample containing noise is Gaussian additive noise when it modifies the model parameters, i.e.

\[
|\hat{x}| = |ax + n| \approx a|v| + |n|
\]

(38)

In fact, the approximation of formula (38) will cause error between the modified real Gaussian model and the test sample with noise, which will affect the recognition performance of the radar target recognition system.

Figure 1. The Recognition system flow chart using the noise adaptive correction algorithm

4. Experimental results and discussion

In this paper, target recognition experiments were conducted on three kinds of ships (named ship 1, ship 2 and ship 3) accurately modeled by CAD. Radar line of sight direction and the radial direction of warship overlap. The angle of the radar line of sight is 10° with the radial direction of the ship, and the pitch angle is constant. The initial angle of azimuth is 0°, and a set of ECHO data is calculated at every 0.01° azimuth, azimuth changes 30°, so that each ship obtains 3000 sets of echo data. 600 groups of learning samples and 300 sets of test samples were selected evenly from 0° to 30° in 3000 sets of data for each class of aircraft.

In this section, we respectively use CPPCA model and CAGC model to model complex HRRP and PPCA model and AGC model to model real HRRP, and compare their recognition performance. The recognition results of these models for different targets are given in table 1.

Table 1. Average recognition rate of the four models

| Target | AGC model | CAGC model | PPCA model | CPPCA model |
|--------|-----------|------------|------------|-------------|
| Ship 1 | 87.2%     | 88.7%      | 92.7%      | 93.4%       |
| Ship 2 | 86.5%     | 87.3%      | 90.8%      | 91.7%       |
| Ship 3 | 85.9%     | 87.0%      | 89.6%      | 90.3%       |
| Average | 86.53%    | 87.67%     | 91.03%     | 91.80%      |

According to table 1, the average accurate identification rate of the CAGC is 1.14% higher than that of the AGC, and the average accurate identification rate of the CPPCA model is 0.77% higher.
than that of the PPCA model. In general, the recognition performance of the complex Gaussian model (CAGC and CPPCA model) is better than that of the real Gaussian model (AGC and PPCA model), which indicates that the use of phase information of radar target's high resolution range echo can improve the recognition effect and verify the validity of the complex Gaussian model. In addition, the PPCA model has higher recognition rate than the AGC model, and the PPCA model has higher average correct recognition rate than the CAGC model. This is because both the AGC and the CAGC assume that the radar target data samples are statistically independent, but in fact this assumption is not very reasonable. The geometric structure and scattering characteristics of different parts of radar target may be similar, and the multiple scattering between radar target scattering points will also cause the statistical correlation between range cells of radar high range resolution echo samples. In addition, the rotation of radar target may also cause the echo amplitude fluctuation of some range cells of radar target's high range resolution echo sample to have a certain correlation. PPCA and CPPCA models take this correlation into account and assume that the radar target sample obeys the joint Gaussian distribution.

Figure 2 shows the variation curves of recognition rates of two models, namely, CPPCA model and CAGC model. Fig.2 shows that the recognition rate of all models increases with the increase of SNR. When the SNR is greater than 30dB, the recognition rate tends to be stable, and the recognition rate of PPCA model is higher than that of AGC model. When the SNR is between 10dB and 30dB, the recognition rate of the four models changes dramatically, and the noise has great influence on the statistical characteristics of target echo, and there is a serious problem of adaptation between the training sample and the test sample. When the SNR is low (less than 10dB), the adaptation between the training sample and the test sample becomes more serious. Therefore, it is of great significance to study noise correction research of the complex Gaussian model.

Figure 2. Recognition rate of CPPCA and CAGC before noise adaptive correction

Figure 3 shows the variation curves of recognition rate with SNR before and after the adaptive noise correction of four models, namely, PPCA model, AGC model, PPCA model and CAGC model. It can be seen from fig.3 that:

1. The adaptive noise correction algorithm significantly improves the recognition efficiency of the four models, especially in the case of low SNR. The accuracy rate of PPCA model after noise correction is increased by 11.32% on average compared with that before noise correction. The accuracy rate of AGC model after noise correction is increased by 4.37% on average compared with that before noise correction. The accuracy rate of CAGC model after noise correction is increased by 17.85% on average compared with that before noise correction. The accuracy rate of CAGC model after noise correction is increased by 13.76% on average compared with that before noise correction. The accuracy rate of CPPCA model after noise correction increased by 5.94% on average compared
with that of CAGC model after noise correction. It can be seen that the adaptive noise correction algorithm significantly improves the SNR mismatch between training samples and test samples.

2. Both the CPPCA model and the CAGC model with modified noise have higher recognition rate than PPCA model and AGC model with modified noise. There are mainly two factors: one is that both the CPPCA and CAGC models use the phase information including the structure information of the target for identification, while the two real number models are not utilized.

The first reason is that both CPPCA and CAGC models use phase information which include target structure information for identification, while PPCA model and AGC model are not utilized. The second reason is that the noise of PPCA model and AGC model approximate as Gaussian white noise in the process of noise correction, and this kind of approximation results in error.

1. When the SNR is less than 30dB, the recognition rate of the CPPCA model before the noise correction is lower than that of the PPCA model before the noise correction. The main reason is that there is a mismatch between the phase and amplitude of the training samples and the test samples during the target recognition process.

![Figure 3. Recognition rate of different models under different SNR before and after noise adaptive correction](image)

5. Summary
Firstly, based on real PPCA and AGC model, this paper analyzes the modeling method of radar automatic target recognition in real domain. Then the initial phase sensitivity of the complex model is analyzed, and the CPPCA and CAGC models are used for the statistical modeling of the radar target recognition in the complex domain. Finally, an adaptive noise correction algorithm is proposed, which improves the recognition performance under the condition of low SNR, and effectively solves the mismatch between the SNR of the test sample and the training sample. The simulation results show that the recognition performance of the complex model is better than that of the corresponding real model, and the rationality of the statistical modeling of the radar target recognition in the complex domain is proved by using CPPCA and CAGC model.

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