A control on quantum fluctuations in 2+1 dimensions

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Abstract

A functional method is discussed, where the quantum fluctuations of a theory are controlled by a mass parameter and the evolution of the theory with this parameter is connected to its renormalization. It is found, in the framework of the gradient expansion, that the coupling constant of a $\mathcal{N} = 1$ Wess-Zumino theory in 2+1 dimensions does not get quantum corrections.

1 Introduction

The understanding of the notion of renormalization had done a great step forward when the connection with the blocking procedure was made [1]. In this framework, the control of the quantum fluctuations can be performed by the progressive elimination of the Fourier components of the fields in a given theory [2]. The parameters of the theory are then functions of the running cut-off and the corresponding renormalization flows describe explicitly the dependence of the theory on the energy scale. There are different approaches here, due to the freedom in defining the coarse-graining procedure.

So as to avoid the dependence on a specific blocking procedure, an alternative approach to renormalization was proposed [3]. There the quantum fluctuations
are controlled by the mass of a scalar field: a large mass freezes the quantum corrections, the system is classical, and as the mass decreases the fluctuations appear and generate the quantum effects. The evolution of the parameters with this mass can thus be seen as renormalization flows and it was shown that these correspond to the usual flows at the one-loop level. Beyond one-loop, the "mass-controlled" flows do not coincide anymore with the loop expansion since they are generated in the framework of the gradient expansion, which is based on an expansion of the effective action (proper graphs generating functional) in derivatives of the field, rather than based on an expansion in powers of $\hbar$. An example of the difference with a loop expansion is given in [4], where the same method is applied to QED and the running mass is the fermions one. There the famous Landau pole is recovered at one-loop, but disappears if all the quantum corrections are taken into account. It should not be forgotten that this result holds at the lowest order in the gradient expansion and higher order derivatives were not taken into account. Finally, this functional method was also applied to the study of the Coulomb interaction in an electron gas [5] where the one-loop Lindhard function and screening are reproduced.

In the present paper, we propose to make use of this functional method in 2+1 dimensions, motivated by relativistic-like effective theories for the description of high-temperature planar superconductors [6]. The use of renormalization procedures here is essential: starting from lattice models dealing with the initial strongly correlated electron system, the corresponding theory in the continuum should give to the fore the relevant operators and interactions, and the present method is believed to be well appropriate for this.

Section 2 presents the method in the simple case of a self-interacting scalar field, so as to concentrate on the physical meaning of the flows that are obtained. The connection with the usual renormalization flows is made and the technical details, as well as the derivation of the evolution equation, are given in the Appendix A.

Section 3 deals with the supersymmetric generalization of the previous case. $N = 1$ supersymmetry in 2+1 dimensions does not have non-renormalization theorems, since the integrations over superspace have all the same measure $d^3x d^2\theta$ and no chiral superfield can be defined. It is thus interesting to make use of this functional method to learn about the behaviour of the parameters under the influence of quantum corrections. It will be found that, to the order of the gradient expansion and the truncation of the potential which are considered, the coupling constant does not get quantum corrections, at any order in $\hbar$. This absence of renormalization of the parameters, though, should not be confused with a non-renormalization theorem: higher powers of the superfield or higher order terms in the gradient expansion (higher derivatives, derivative interactions) would lead to a renormalization of the coupling constant. For the sake of clarity in the presentation, the derivations are given in the Appendix B.

Finally, the conclusion contains a discussion on the gradient expansion approximation.
2 Self interacting scalar field

We present here the functional method in the case of a self interacting scalar field. The difference with 3+1 dimensions is that the integrals do not need regularization, as will be seen.

The starting point is the Euclidean action

\[ S = \int d^3x \left\{ -\frac{1}{2} \phi \Box \phi + \frac{\lambda}{2} m_0^2 \phi^2 + \frac{e_0}{24} \phi^4 \right\}, \tag{1} \]

where the mass dimension of the coupling is \([e_0] = 1\). The classical system will correspond to \(\lambda \to \infty\), where the mass is infinite and thus the quantum effects are not present. These will gradually appear as \(\lambda\) decreases, and when \(\lambda \to 1\) the full quantum corrections will be present.

As usual, we introduce the effective action \(\Gamma\) as the Legendre transform of the connected graphs generator functional. After some manipulations (see Appendix A), we find then the following exact evolution equation with \(\lambda\):

\[ \partial_\lambda \Gamma = \frac{m_0^2}{2} \int d^3x \left\{ \phi^2(x) + \left( \frac{\delta^2 \Gamma}{\delta \phi^2(x)} \right)^{-1} \right\} \tag{2} \]

To obtain informations on the physical processes, we express the functional \(\Gamma\) in terms of its argument \(\phi\) via a gradient expansion and a truncation of the potential, to assume the following functional form:

\[ \Gamma = \int d^3x \left\{ -\frac{1}{2} \phi \Box \phi + \frac{\lambda}{2} m^2 \phi^2 + \frac{e}{24} \phi^4 \right\}. \tag{3} \]

In this approximation, we obtain the following evolution of the parameters (see Appendix A):

\[ \begin{align*}
\partial_\lambda (\lambda m^2) &= m_0^2 \left( 1 - \frac{e}{16\pi \lambda^{1/2} m} \right) \\
\partial_\lambda e &= \frac{3e^2 m_0^2}{32\pi \lambda^{3/2} m^3}. \tag{4}
\end{align*} \]

One should bare in mind that the equations (4) contain all the quantum corrections, in this approximation of the gradient expansion. The connection with the one-loop results is obtained by making the replacements \(m^2 \to m_0^2\) and \(e \to e_0\) in the right hand sides of Eqs.(4), since restoring the factors \(\hbar\) leads to \(e \to \hbar e\):

\[ \begin{align*}
\partial_\lambda (\lambda m^2) &= m_0^2 - \hbar \frac{e_0 m_0}{16\pi \lambda^{1/2}} + \mathcal{O}(\hbar^2) \\
\partial_\lambda e &= \hbar \frac{3e_0^2 m_0^2}{32\pi \lambda^{3/2} m_0} + \mathcal{O}(\hbar^2). \tag{5}
\end{align*} \]
The solutions are found by integrating from $\lambda = \infty$ to $\lambda = 1$:

$$m^2 - m_0^2 = -M^2 + h \frac{e_0 m_0}{8\pi} + \mathcal{O}(h^2)$$

$$e - E = -h \frac{3e_0^2}{16\pi m_0} + \mathcal{O}(h^2),$$

where $M^2$ and $E$ are constant of integration. The solutions (6) coincide with the usual one-loop results obtained when computing the appropriate Feynman graphs (see Appendix A), where the scale $M$ is linked to the cut-off of the theory and $E = e_0$. Therefore it has been shown that the evolution with the parameter $\lambda$ controls the quantum fluctuations and is consistent, at one loop, with the usual renormalization scheme.

3 $N = 1$ Wess-Zumino model

We deal here with the supersymmetric generalization of the previous example. In terms of the real scalar superfield $Q$, the supersymmetric generalization of the previous section is the Wess-Zumino model, with the following classical Euclidean action

$$S[Q] = \int d^5 z \left\{ \frac{1}{2} Q D^2 Q + \frac{\lambda}{2} m_0 Q^2 + \frac{g_0}{6} Q^3 \right\},$$

where $z = (x, \theta)$ is the superspace coordinate and the conventions are those taken in [7]. The coupling $g_0$ has mass dimension $[g_0] = 1/2$ and the on-shell Lagrangian contains, among other terms, the interaction $g_0 \phi^4$, where $\phi$ is the scalar component of $Q$.

The evolution equation of the proper graphs generator functional $\Gamma$ has a similar form as the one obtained in the non-supersymmetric case and is

$$\partial_\lambda \Gamma = \frac{m_0}{2} \int d^5 z \left\{ Q^2(z) + \left( \frac{\delta^2 \Gamma}{\delta Q^2(z)} \right)^{-1} \right\}.$$ (8)

In the framework of the gradient expansion, and truncating the potential to the order $Q^3$, we consider the following ansatz for the functional dependence of $\Gamma$:

$$\Gamma[Q] = \int d^5 z \left\{ \frac{1}{2} Q D^2 Q + \frac{\lambda}{2} m Q^2 + \frac{g}{6} Q^3 \right\},$$ (9)

In this approximation, the trace of the operator $(\delta^2 \Gamma)^{-1}$ is computed in the Appendix B and it is shown that the relevant quadratic term, as far as the evolution of the mass is concerned, is
\[ g^2 \int \frac{d^3k}{(2\pi)^3} \frac{d^2\theta}{2\pi} \, Q(k, \theta)Q(-k, \theta) \]
\[ \times \int \frac{d^3p}{(2\pi)^3} \frac{[p^2 - 3(\lambda m)^2]}{[p^2 + (\lambda m)^2]^2[(k - p)^2 + (\lambda m)^2]^2}. \quad (10) \]

In the approximation (9), the quantum corrections to the mass are obtained when \( k = 0 \) in the integrand. The identification of the non-derivative quadratic terms in both sides of the evolution equation (8) gives then
\[ \partial_\lambda (\lambda m) = m_0 + g^2 m_0 \int \frac{d^3p}{(2\pi)^3} \frac{p^2 - 3(\lambda m)^2}{[p^2 + (\lambda m)^2]^3}. \quad (11) \]

The integration over \( p \) is easy and gives a vanishing integral, such that
\[ \partial_\lambda (\lambda m) = m_0, \quad (12) \]
i.e. the evolution of \((\lambda m)\) is classical and does not have quantum corrections. The evolution of \( m \) with \( \lambda \) is then
\[ m = m_0 \left(1 + \frac{c}{\lambda}\right), \quad (13) \]
where \( c \) is a dimensionless constant, which is computed at the one-loop level in the Appendix B, using the Feynman graph technique. We can note that in a numerical resolution of the differential equation for the mass, the large but finite initial value for \( \lambda \) would not allow us to set exactly \( m = m_0 \) since quantum fluctuations, even tiny, are present for finite \( \lambda \). This is the reason why the constant of integration \( c \) would not vanish.

Let us look at the evolution of the coupling constant \( g(\lambda) \). It is seen in the Appendix B that the relevant cubic term, as far as the evolution of the coupling is concerned, is
\[ g^3 \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^2\theta}{2\pi} \, Q(k, \theta)Q(q, \theta)Q(-k - q, \theta) \]
\[ \times \int \frac{d^4p}{(2\pi)^4} \frac{4(\lambda m)(\lambda m)^2 - p^2}{[p^2 + (\lambda m)^2]^2[(p + q)^2 + (\lambda m)^2][(p - k)^2 + (\lambda m)^2]^4}. \quad (14) \]

In the framework of the approximation (9), the quantum corrections to the coupling are obtained when \( k = q = 0 \) in the integrand. The identification of the non-derivative cubic terms in both sides of the evolution equation (8) gives then
\[ \partial_\lambda g = 12g^3(\lambda m)m_0 \int \frac{d^4p}{(2\pi)^4} \frac{(\lambda m)^2 - p^2}{[p^2 + (\lambda m)^2]^4}. \quad (15) \]
Once again, the integration over $p$ is easy and gives a vanishing integral, such that

$$\partial_\lambda g = 0,$$

i.e. the coupling constant does not evolve with $\lambda$. This is consistent with the one-loop result (see Appendix B) and provides a generalization to all orders in $\bar{h}$, in the approximation (9).

Finally, the quantities $\lambda m$ and $g$ are frozen to their initial value given when $\lambda \to \infty$, i.e. when the quantum fluctuations can be neglected. The coupling constant $g$, in the present approximation (9), does not get any quantum correction.

## 4 Conclusion

To conclude, let us stress again the non-perturbative nature of the flows that are obtained with this method. The evolution equations of the coupling constants contain all the quantum corrections, independently of any perturbative expansion in $\bar{h}$, in the approximation of the gradient expansion and the polynomial truncation of the potential. This feature is interesting in the context of low-energy effective theories, where only low powers of the momentum are kept into account in the functional dependence of the effective action, as well as relevant operators only, in a renormalization group sense.

Let us be more precise concerning this gradient expansion approximation. The evolution equation of the effective action, giving $\partial_\lambda \Gamma$, is exact and contains all the quantum corrections. However, it looks like a one-loop equation, since it has the form (when the factor $\bar{h}$ is restored)

$$\partial_\lambda \Gamma = \bar{h} \mathcal{F}[\Gamma],$$

where $\mathcal{F}$ is some operator applied on $\Gamma$. This is the reason why the usual one-loop results are obtained, when $\Gamma$ is replaced by the classical action $S$ in the right-hand side of Eq.(17). The next step forward, beyond one-loop, is to assume a functional dependence of $\Gamma$ and to plug it inside the evolution equation (17). The resulting couplings’ evolution consist then in a partial resumation of the graphs. In the present example, where no new vertices were considered compared to the classical ones, the resumation takes into account the one-loop-like graphs, with the internal lines being the full propagators (full in the present approximation of the gradient expansion and polynomial truncation of the potential).

Note that more general gradient expansions could be considered, besides including higher order powers of the field: one could add higher order derivatives in the quadratic kinetic term, as well as derivative interactions. These would lead to new vertices and therefore to a more complete resumation of the graphs.
Finally, let us make a comment on $N = 2$ supersymmetry. The latter can be obtained by dimensional reduction of $N = 1$ supersymmetry in 3+1 dimensions and thus has non-renormalization theorems. The present method can then still be useful for the study of the kinetic operators, since these do get renormalized. The evolution equation with the parameter $\lambda$ would be quite different though, due to the existence of two kinds of integrals: $\int d^3 x d^2 \theta$ and $\int d^3 x d^2 \theta d^2 \bar{\theta}$. Indeed, the functional derivatives have to be taken with respect to the full superspace dependence, such that\[ałáááááááááááé\]

\begin{equation}
\frac{\delta Q(x, \theta, \bar{\theta})}{\delta Q(x', \theta', \bar{\theta}')} = \mathcal{T}^2 \delta^2(\theta - \theta')\delta^2(\bar{\theta} - \bar{\theta}')\delta^3(x - x') \tag{18}
\end{equation}

when $Q$ is a chiral superfield. As a consequence, the final evolution equation is more involved and is planned for a future work.

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**Appendix A: Evolution equation for the scalar field**

The Euclidean action, functional of the field $\hat{\phi}$, is

\begin{equation}
S[\hat{\phi}] = \int d^3 x \left\{ -\frac{1}{2} \hat{\phi} \Box \hat{\phi} + \frac{\lambda}{2} m_0^2 \hat{\phi}^2 + \frac{\epsilon_0}{24} \hat{\phi}^4 \right\}, \tag{19}
\end{equation}

and the connected graphs generator functional, function of the source $j$, is defined by $W[j] = -\ln Z[j]$ where

\begin{equation}
Z[j] = \int \mathcal{D}[\hat{\phi}] \exp \left\{ -S[\hat{\phi}] - \int d^3 x j(x) \hat{\phi}(x) \right\}. \tag{20}
\end{equation}

The functional derivative of $W$ defines the expectation value field $\phi$:

\begin{equation}
\frac{\delta W}{\delta j(x)} = \frac{1}{Z} \langle \hat{\phi}(x) \rangle = \phi(x), \tag{21}
\end{equation}

where

\begin{equation}
\langle \hat{\phi}(x) \rangle = \int \mathcal{D}[\hat{\phi}] \hat{\phi}(x) \exp \left\{ -S[\hat{\phi}] - \int d^3 y j(y) \hat{\phi}(y) \right\}. \tag{22}
\end{equation}

We also have
\[
\frac{\delta^2 W}{\delta j(x) \delta j(y)} = \phi(x) \phi(y) - \frac{1}{Z} < \phi(x) \phi(y) >. \tag{23}
\]

Inverting the relation (21) which gives \( \phi(x) \) as a function of \( j(x) \), we define the Legendre transform \( \Gamma \) (functional of \( \phi \)) of \( W \) by

\[
\Gamma[\phi] = W[j] - \int d^3 x j(x) \phi(x). \tag{24}
\]

From this definition we extract the following functional derivatives:

\[
\frac{\delta \Gamma}{\delta \phi(x)} = -j(x) \tag{25}
\]

\[
\frac{\delta^2 \Gamma}{\delta \phi(x) \delta \phi(y)} = - \left( \frac{\delta^2 W}{\delta j(x) \delta j(y)} \right)^{-1} \]

The evolution of \( W \) with the parameter \( \lambda \) is, according to (23),

\[
\partial_\lambda W = - \frac{m_0^2}{2Z} \int d^3 x < \phi^2(x) >
\]

\[
= \frac{m_0^2}{2} \int d^3 x \phi^2(x) - \frac{m_0^2}{2} \int d^3 x \frac{\delta^2 W}{\delta j^2(x)}. \tag{26}
\]

To compute the evolution of \( \Gamma \) with \( \lambda \), we have to keep in mind that its independent variables are \( \phi \) and \( \lambda \), such that

\[
\partial_\lambda \Gamma = \partial_\lambda W + \int d^3 x \frac{\delta W}{\delta j(x)} \partial_\lambda j(x) - \int d^3 x \partial_\lambda j(x) \phi(x) = \partial_\lambda W. \tag{27}
\]

Combining these different results, we finally obtain the exact evolution equation for the proper graphs generator functional \( \Gamma \):

\[
\partial_\lambda \Gamma = \frac{m_0^2}{2} \int d^3 x \left\{ \phi^2(x) + \left( \frac{\delta^2 \Gamma}{\delta \phi^2(x)} \right)^{-1} \right\}
\]

\[
= \frac{m_0^2}{2} \int \frac{d^3 p}{(2\pi)^3} \left\{ \phi(p) \phi(\bar{p}) + \left( \frac{\delta^2 \Gamma}{\delta \phi(p) \delta \phi(\bar{p})} \right)^{-1} \right\}, \tag{28}
\]

where we used the fact that \( \phi(-p) = \bar{\phi}(p) \) since \( \phi(x) \) is real. Note that, in Eq.(28), the integration of the operator \( (\delta^2 \Gamma)^{-1} \) has to be understood as

\[
\int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \left( \frac{\delta^2 \Gamma}{\delta \phi(p) \delta \phi(q)} \right)^{-1} (2\pi)^3 \delta^3 (p + q). \tag{29}
\]
In the approximation where we assume that

$$\Gamma = \int d^3x \left\{ -\frac{1}{2} \phi \Box \phi + \frac{\lambda}{2} m^2 \phi^2 + \frac{e}{24} \phi^4 \right\},$$

(30)

it is enough to consider a constant configuration $\phi_0$ of $\phi(x)$, which leads to

$$\Gamma[\phi_0] = \mathcal{V} \left( \frac{\lambda}{2} m^2 \phi_0^2 + \frac{e}{24} \phi_0^4 \right)$$

(31)

where $\mathcal{V} = \delta^3(0)$ is the space-time volume. The evolution equations for $m^2$ and $e$ are obtained after expanding the right-hand side of Eq.(31) in powers of $\phi_0$, which leads to:

$$\int \frac{d^3p}{(2\pi)^3} \left( \frac{\delta^2 \Gamma}{\delta \phi(p) \delta \phi(q)} \right)^{-1}_{\phi_0} = \frac{\delta^3(p + q)}{p^2 + \lambda m^2 + \frac{e}{2} \phi_0^2},$$

(32)

such that the identification of the coefficients of $\phi_0^2$ and $\phi_0^4$ in the evolution equation (28) leads to

$$\partial_\lambda (\lambda m^2) = m_0^2 \left( 1 - \frac{e}{16\pi \lambda^{1/2} m} \right)$$

$$\partial_\lambda e = \frac{3e^2 m_0^2}{32\pi \lambda^{3/2} m^3}$$

(33)

We finally compute the one-loop renormalization of the potential of this model, using the usual Feynman rules, and starting with the action for $\lambda = 1$:

$$S_{\lambda=1} = \int d^3x \left\{ -\frac{1}{2} \phi \Box \phi + \frac{m_0^2}{2} \phi^2 + \frac{e_0}{24} \phi^4 \right\}.$$  

(34)

The one-loop correction to the mass is given by the tadpole diagram. Taking into account the symmetry factor $(1/2)$, we have
\[
m^2 = m_0^2 - \hbar \frac{e_0}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m_0^2} = m_0^2 - \hbar \frac{e_0 A}{4\pi^2} + \hbar \frac{e_0 m_0}{8\pi} + O\left(\frac{1}{\Lambda}\right) \quad (35)
\]

where \(\Lambda\) is the cut-off. This last result coincides with Eq.\((6)\) if we make the identification \(M^2 \to \hbar e_0 A/4\pi^2\).

The one-loop correction to the charge is given by, at the limit of zero incoming momenta and taking into account the symmetry factor \((1/2)\) as well as the 3 different possibilities for the incoming momenta:

\[
e = e_0 + \hbar \frac{3}{2}(ie)^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2 + m_0^2)^2} = e_0 - \hbar \frac{3e_0^2}{16\pi m_0}, \quad (36)
\]

which is consistent with the result \((6)\).

**Appendix B: Evolution equation for the Wess-Zumino model**

The reader can find a complete presentation of superspace in 2+1 dimensions in \([7, 8]\) and we give here only the basic properties necessary for the present derivation. The conventions are those given in \([7]\).

The classical action we are interested in, functional of the (real) \(N = 1\) superfield \(Q\), is given by

\[
S[\hat{Q}] = \int d^5z \left\{ \frac{1}{2} \hat{Q} D^2 \hat{Q} + \frac{\lambda}{2} m_0 \hat{Q}^2 + \frac{g_0}{6} \hat{Q}^3 \right\}, \quad (37)
\]

where \(d^5z = d^3x d^2\theta\) and the spinorial derivative is

\[
D^\alpha = \frac{\partial}{\partial \theta_\alpha} + i(\gamma^\mu \theta)^\alpha \partial_\mu, \quad (38)
\]

where the gamma matrices are given by \(\gamma^0 = \sigma^2, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^3\), and \(\sigma^1, \sigma^2, \sigma^3\) are the Pauli matrices.

The construction of the expectation value superfield \(Q\) follows the one given in the Appendix A: if \(W\) is the connected graphs generator functional, function of the supersource \(P\), we have

\[
\frac{\partial W}{\partial P(z)} = Q(z). \quad (39)
\]
Following the steps of Appendix A, it is straightforward to see that the evolution equation of the proper graphs generator functional $\Gamma$ is

$$
\partial_\lambda \Gamma = \frac{m_0}{2} \int d^5 z \left\{ Q^2(z) + \left( \frac{\delta^2 \Gamma}{\delta Q^2(z)} \right)^{-1} \right\}
$$

(40)

$$
= \frac{m_0}{2} \int \frac{d^3 p}{(2\pi)^3} d^2 \theta \left\{ Q(p, \theta)\overline{Q}(p, \theta) + \left( \frac{\delta^2 \Gamma}{\delta Q(p, \theta)\delta Q(p, \theta)} \right)^{-1} \right\},
$$

where we used $Q(-p, \theta) = \overline{Q}(p, \theta)$ since $Q(z)$ is real. In Eq.(40), the integration of the operator $(\delta \Gamma)^{-1}$ has to be understood as

$$
\int \frac{d^3 p}{(2\pi)^3} d^3 q (2\pi)^3 d^2 \theta d^2 \theta' \left( \frac{\delta^2 \Gamma}{\delta Q(p, \theta)\delta Q(q, \theta')} \right)^{-1} (2\pi)^3 \delta^3(p + q) \delta^2(\theta - \theta').
$$

(41)

With the following functional dependence of $\Gamma$:

$$
\Gamma[Q] = \int d^5 z \left\{ \frac{1}{2} QD^2 Q + \frac{\lambda}{2} m Q^2 + \frac{g}{6} Q^3 \right\},
$$

(42)

we obtain

$$
\frac{\delta^2 \Gamma}{\delta Q(p, \theta)\delta Q(q, \theta')} = \left( \lambda m + D^2_{\mu\theta} \right) (2\pi)^3 \delta^3(p + q) \delta^2(\theta - \theta') + gQ(p + q, \theta) \delta^2(\theta - \theta'),
$$

(43)

where $D^\alpha_{\mu\theta} = \partial^\alpha - p_\mu (\gamma^\alpha \theta)$. We now take the inverse of $(\delta^2 \Gamma)$, using the expansion

$$
(A + B)^{-1} = A^{-1} - A^{-1} BA^{-1} + A^{-1} BA^{-1} BA^{-1} - A^{-1} BA^{-1} BA^{-1} BA^{-1} + ...
$$

(44)

where

$$
A = \left( \lambda m + D^2_{\mu\theta} \right) (2\pi)^3 \delta^3(p + q) \delta^2(\theta - \theta')
$$

$$
B = gQ(p + q, \theta) \delta^2(\theta - \theta'),
$$

(45)

such that $A$ is diagonal in Fourier space. We will also use the following properties [7]:

11
\[ \delta^2(\theta_1 - \theta_2)\delta^2(\theta_2 - \theta_1) = 0 \]
\[ \delta^2(\theta_1 - \theta_2)D^\alpha_\rho\delta^2(\theta_2 - \theta_1) = 0 \]
\[ \delta^2(\theta_1 - \theta_2)D^\beta_\rho\delta^2(\theta_2 - \theta_1) = \delta^2(\theta_1 - \theta_2) \]
\[ (D^2_\rho) = -p^2, \quad (46) \]

such that
\[ A^{-1} = \frac{\lambda m - D^2_\rho}{p^2 + (\lambda m)^2} (2\pi)^3 \delta^3(p + q)\delta^2(\theta - \theta'). \quad (47) \]

In the expansion (44), the constant and linear terms in \( B \) lead to constants after taking the trace (41). The trace of the quadratic term is
\[ g^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} d^2\theta d^2\theta_1 d^2\theta_2 \]
\[ \times Q(-p + k, \theta_1)Q(-k + p, \theta_2) \]
\[ \times \frac{\lambda m - D^2_\rho}{p^2 + (\lambda m)^2} \delta^2(\theta - \theta_1) \frac{\lambda m - D^2_\rho}{k^2 + (\lambda m)^2} \delta^2(\theta_1 - \theta_2) \]
\[ \times \frac{\lambda m - D^2_\rho}{p^2 + (\lambda m)^2} \delta^2(\theta_2 - \theta). \quad (48) \]

Using the properties (46), we obtain then for the quadratic term
\[ g^2 \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} d^2\theta [p^2 + (\lambda m)^2] \]
\[ \times \frac{1}{[(k - p)^2 + (\lambda m)^2]} \]
\[ \times (3\lambda m Q(k, \theta)D^2_\rho Q(-k - p, \theta) + [p^2 - 3(\lambda m)^2]Q(k, \theta)Q(-k, \theta). \quad (49) \]

In the previous integral, both terms give a kinetic contribution, when \( k \neq 0 \). In the present approximation (42), we neglect these kinetic terms and the remaining contribution is obtained for \( k = 0 \) in the second integrand. This gives the quantum corrections to the mass, obtained from Eq.(10).

The trace of the cubic term gives
\[ -g^3 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} d^2\theta d^2\theta_1 d^2\theta_2 d^2\theta_3 \]
\[ \times Q(-p + q, \theta_1)Q(-q + k, \theta_2)Q(-k + p, \theta_3) \]
\[
\times \frac{\lambda m - D_{p\theta}^2}{p^2 + (\lambda m)^2} \delta^2(\theta - \theta_1) \lambda m - D_{q\theta}^2 \delta^2(\theta_1 - \theta_2) \\
\times \frac{\lambda m - D_{k\theta_2}^2}{k^2 + (\lambda m)^2} \delta^2(\theta_2 - \theta_3) \lambda m - D_{p\theta_3}^2 \delta^2(\theta_3 - \theta) \\
\lambda m - D_{p\theta_4}^2 \delta^2(\theta_4 - \theta)
\]

Using again the properties (46), we obtain

\[
\frac{g^3}{(2\pi)^9} \int \frac{d^3p d^3q d^3k d^2\theta}{[p^2 + (\lambda m)^2][p^2 + (\lambda m)^2]}
\times \left( 4(\lambda m)^3 Q(k, \theta) Q(q, \theta) Q(-k - q, \theta) \right.
+ 6(\lambda m)^2 Q(q, \theta) Q(-k - q, \theta) D_{-p}^2 Q(k, \theta)
+ 4\lambda m Q(-k - q, \theta) D_{-p}^2 Q(k, \theta) D_{-p}^2 Q(k, \theta)
\left. + p^2 Q(q, \theta) Q(k, \theta) D_{-p}^2 Q(-k - q, \theta) \right)
\]

The evolution of the coupling constant \( g \) is obtained in the limit of zero incoming momenta, since in the approximation (42) the derivative interactions are neglected. Considering the fact that

\[
D_{-p-q}^2 D_{-p}^2 = -p^2 + D_{-q}^2 D_{-p}^2 D_{-p}^2 + 2pq\theta^2 D_{-p}^2,
\]

the non-derivative interaction terms coming from Eq.(51) are then

\[
g^3 \int \frac{d^3q d^3k}{(2\pi)^3} d^2\theta Q(k, \theta) Q(q, \theta) Q(-k - q, \theta)
\times \int \frac{d^4p}{(2\pi)^4} \frac{4(\lambda m)^3}{[p^2 + (\lambda m)^2]^2[(p + q)^2 + (\lambda m)^2][(p - k)^2 + (\lambda m)^2]}.
\]

for \( k = q = 0 \) in the integrand. The identification of both sides of the evolution equation (40) gives then Eq.(15).

We finally compute the one-loop renormalization of this model, using the Feynman graphs technique, and starting with the bare action for \( \lambda = 1 \):

\[
S_{\lambda=1} = \int d^5z \left\{ \frac{1}{2} Q D^2 Q + \frac{m_0}{2} Q^2 + \frac{g_0}{6} Q^3 \right\}.
\]

The one-loop correction to the quadratic term in \( Q \) is given by the self-energy graph

\[
\Gamma_2[Q] = g_0^2 \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} d^2\theta d^2\theta' Q(-p, \theta) Q(p, \theta')
\]

13
Using the properties (46), we obtain

\[
\Gamma_2[Q] = g^2_0 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^2\theta d^2\theta'}{(m_0^2 + q^2)(m_0^2 + (q + p)^2)} \left\{ -2\delta(\theta - \theta')m_0Q(-p, \theta)Q(p, \theta) + \delta(\theta - \theta')Q(p, \theta')D_{q\theta}^2[Q(-p, \theta)D_{q\theta}^2\delta^2(\theta' - \theta)] \right\}
\]

\[
= g^2_0 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^2\theta}{(m_0^2 + q^2)(m_0^2 + (q + p)^2)} \left[ Q(p, \theta)D_{q\theta}^2Q(-p, \theta) - 2m_0Q(p, \theta)Q(-p, \theta) \right].
\]

The correction to the mass is obtained for \( p = 0 \) in the non-derivative integrand, such that the one-loop mass is

\[
m^{(1)} = m_0 - \hbar g^2_0 \int \frac{d^3q}{(2\pi)^3} \frac{2m_0}{(m_0^2 + q^2)^2} = m_0 - \hbar \frac{g^2_0}{4\pi}.
\]

and the constant \( c \) appearing in Eq.(13) is, at one loop, \( c = -\hbar g^2_0/(4\pi m_0) \).

The one-loop correction to the cubic term in \( Q \) is given by

\[
\Gamma_3[Q] = g^3 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^2\theta d^2\theta_1 d^2\theta_2}{m_0^2 + k^2} \left\{ Q(p, \theta)Q(q, \theta_1)Q(-p - q, \theta_2) \times m_0 - D_{k\theta}^2\delta^2(\theta - \theta_1) \times m_0 - D_{k+p\theta\theta}^2\delta^2(\theta_1 - \theta_2) \times \delta^2(\theta_2 - \theta) \right\}
\]

The properties (46) give then

\[
\Gamma_3[Q] = g^3 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^2\theta}{(m_0^2 + k^2)(m_0^2 + (k + p)^2)} \frac{d^2\theta}{(m_0^2 + (k + p + q)^2)} \left\{ 3m_0Q(-p - q, \theta)D_{k+p\theta\theta}^2[Q(p, \theta)Q(q, \theta)] - 3m^2_0Q(p, \theta)Q(q, \theta)Q(-p - q, \theta) - D_{k+p\theta\theta}^2[Q(q, \theta)Q(-p - q, \theta)D_{k\theta}^2Q(p, \theta)] \right\}
\]

\[14\]
\[
\begin{align*}
&= g_0^3 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} d^2 \theta \\
&\times \frac{(k^2 - 3m_0^2)Q(p, \theta)Q(q, \theta)Q(-p - q, \theta)}{(m_0^2 + k^2)[m_0^2 + (k + p)^2][m_0^2 + (k + p + q)^2]} \\
&\quad + \text{derivative terms}
\end{align*}
\]

The correction to the coupling constant is obtained in the limit \( p, q \to 0 \) in the non-derivative integrand, i.e. it is given by the integral

\[
g_0^3 \int \frac{d^3 k}{(2\pi)^3} \frac{k^2 - 3m_0^2}{(m_0^2 + k^2)^3} = 0,
\]

such that the one-loop correction to the coupling constant vanishes, in accordance with the the result (16).

References

[1] K.Wislon, J.Kogut, Phys.Rep.C12 (1974) 75; K.Wislon, Rev.Mod.Phys.47 (1975) 773.
[2] F.Wegner, A.Houghton, Phys.Rev.A8 (1973) 40; J.Polchinsky, Nucl.Phys.B231 (1984) 269; C.Wetterich, Phys.Lett.B301 (1993) 90; M.Reuter, C.Wetterich, Nucl.Phys.B391 (1993) 147; T.Morris, Int.J.Mod.Phys.A9 (1994) 2411; U.Ellwanger, Phys.Lett.B335 (1994) 364; N.Tetradis, D.Litim, Nucl.Phys.B464 (1996) 492; J.Alexandre, V.Branchina, J.Polonyi, Phys.Lett.B445 (1999) 351. See also references therein.
[3] J.Alexandre, J.Polonyi, Annals Phys.288 (2001): 37.
[4] J.Alexandre, J.Polonyi, K.Seiler, Phys.Lett.B531 (2002): 316.
[5] S.Correia, J.Polonyi, J.Richert, Annals Phys.296 (2002): 214.
[6] N.Dorey, N.Mavromatos, Nucl.Phys.B386 (1992): 614; X.G.Wen, P.Lee, Phys.Rev.Lett.76 (1996) 503; L.Balent, M.Fisher, C.Nayak, Phys.Rev.B60 (1999) 1654; M.Franz, Z.Tesanovic, Phys.Rev.Lett.87 (2001) 257003; I.Herbut, Phys.Rev.Lett.88 (2002) 047006;
J. Alexandre, N. Mavromatos, S. Sarkar, Int. J. Mod. Phys. B17 (2003): 2359; See also references therein.

[7] S. J. Gates, M. Rocek, W. Siegel "Superspace", Benjamin/Cummings, Reading (1983).

[8] N. J. Hitchin, A. Karlhelde, U. Lindström, M. Rocek, Commun. Math. Phys. 108 (1987).

[9] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, M. J. Strassler, Nucl. Phys. B499 (1997) 67;
    J. de Boer, K. Hori, Y. Oz, Nucl. Phys. B500 (1997) 63;
    M. J. Strassler, hep-th/9912142.