LELEK FAN FROM A PROJECTIVE FRA"ISSE LIMIT

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Abstract. We show that a natural quotient of the projective Fraïssé limit of a family that consists of finite rooted trees is the Lelek fan. Using this construction, we study properties of the Lelek fan and of its homeomorphism group. We show that the Lelek fan is projectively universal and projectively ultrahomogeneous in the class of smooth fans. We further show that the homeomorphism group of the Lelek fan is totally disconnected, generated by every neighbourhood of the identity, has a dense conjugacy class, and is simple.

1. Introduction

1.1. Lelek fan. A continuum is a compact and connected metric space. Let $C$ denote the Cantor set. The Cantor fan $F$ is the cone over the Cantor set, that is $C \times [0, 1]/\sim$, where $(a, b) \sim (c, d)$ if and only if $a = c$ and $b = d$ or $a = c = 0$. Recall that an arc is a homeomorphic image of the closed unit interval $[0, 1]$. If $X$ is a space and $h : [0, 1] \to X$ is a homeomorphism onto its image, we call $h(0) = a$ and $h(1) = b$ the endpoints of the arc given by $h$ and denote this arc as $ab$. An endpoint of a continuum $X$ is a point $e$ such that for every arc $ab$ in $X$, if $e \in ab$, then $e = a$ or $e = b$. Finally, a Lelek fan $L$ is a non-degenerate subcontinuum of the Cantor fan with a dense set of endpoints.

In the literature, a Lelek fan is often defined as a smooth fan with a dense set of endpoints. However, smooth fans are exactly fans that can be embedded into the Cantor fan (see [4], Proposition 4, the definition of a smooth fan is given there as well); and any subcontinuum of the Cantor fan is either a fan, or an arc, or a point.

A Lelek fan was constructed by Lelek in [13]. Several characterizations of a Lelek fan were collected in [5], Theorem 12.14. A remarkable property of a Lelek fan is its uniqueness (see [2] and [3]): any two non-degenerate subcontinua of the Cantor fan with a dense set of endpoints are homeomorphic.

A very interesting and well-studied by many people is the space of endpoints of the Lelek fan $L$. The set of endpoints of the Lelek fan is a dense $G_δ$ set in $L$, therefore it is separable and completely metrizable. Moreover, it is a 1-dimensional space. It is homeomorphic to the complete Erdős space, which is homeomorphic to the set of endpoints of the Julia set of the exponential map, the set of endpoints of the separable universal $\mathbb{R}$-tree; see [9] for more details.

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In [6], it is shown that the space of Lelek fans in the Cantor fan is homeomorphic to
the separable Hilbert space.

Here we introduce some notation that we will need later on. By \(v\) denote the \emph{top} of
the Cantor fan \(F\), that is, \(v = (0,0)/\sim\). For a point \(x \in F\), let \( [v, x] \) denote the
closed line segment joining \(v\) and \(x\). If \(x\) is in the line segment \([v, y]\), we denote by \([x, y]\)
the line segment \( ([v, y] \setminus [v, x]) \cup \{x\}\). Points in \(F\) will be denoted by \((c, y)\),
where \(c \in C\) and \(y \in [0, 1]\). Let \(\pi_1\) be the projection from \( F \setminus \{v\} \)
onto \(C\) that takes \((c, x)\) to \(c\). Let \(E\) be the set of endpoints of the Lelek fan \(L\), and let \(H(L)\) be
the group of all homeomorphisms of the Lelek fan.

1.2. \textbf{Projective Fra"ıssé limits.} Given a language \(L\) that consists of relation
symbols \(\{r_i\}_{i \in I}\), and function symbols \(\{f_j\}_{j \in J}\), a \emph{topological \(L\)-structure}
is a compact zero-dimensional second-countable space \(A\) equipped with closed relations \(r_i^A\)
and continuous functions \(f_j^A, i \in I, j \in J\). A continuous surjection \(\phi: B \to A\) is an
\emph{epimorphism} if it preserves the structure, more precisely, for a function symbol \(f\) of arity \(n\) and \(x_1, \ldots, x_n \in B\)
we require:

\[
    f^A(\phi(x_1), \ldots, \phi(x_n)) = \phi(f^B(x_1, \ldots, x_n));
\]

and for a relation symbol \(r\) of arity \(m\) and \(x_1, \ldots, x_m \in A\) we require:

\[
    r^A(x_1, \ldots, x_m)
    \iff \exists y_1, \ldots, y_m \in B \left( \phi(y_1) = x_1, \ldots, \phi(y_m) = x_m, \text{ and } r^B(y_1, \ldots, y_m) \right).
\]

By an \emph{isomorphism} we mean a bijective epimorphism.

For the rest of this section fix a language \(L\). Let \(G\) be a family of finite topological
\(L\)-structures. We say that \(G\) is a \emph{projective Fra"ıssé family} if the following two conditions hold:

\begin{itemize}
    \item [(JPP)] (the joint projection property) for any \(A, B \in G\) there are \(C \in G\) and epimorphisms
        from \(C\) onto \(A\) and from \(C\) onto \(B\);
    \item [(AP)] (the amalgamation property) for \(A, B_1, B_2 \in G\) and any epimorphisms \(\phi_1: B_1 \to A\) and
        \(\phi_2: B_2 \to A\), there exist \(C \in G\), \(\phi_3: C \to B_1\), and \(\phi_4: C \to B_2\) such that
        \(\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4\).
\end{itemize}

A topological \(L\)-structure \(L\) is a \emph{projective Fra"ıssé limit} of \(G\) if the following three
conditions hold:

\begin{itemize}
    \item [(L1)] (the projective universality) for any \(A \in G\) there is an epimorphism from \(L\) onto \(A\);
    \item [(L2)] for any finite discrete topological space \(X\) and any continuous function \(f: L \to X\)
        there are \(A \in G\), an epimorphism \(\phi: L \to A\), and a function \(f_0: A \to X\) such that
        \(f = f_0 \circ \phi\).
    \item [(L3)] (the projective ultrahomogeneity) for any \(A \in G\) and any epimorphisms \(\phi_1: L \to A\) and
        \(\phi_2: L \to A\) there exists an isomorphism \(\psi: L \to L\) such that \(\phi_2 = \phi_1 \circ \psi\);
\end{itemize}
**Theorem 1** (Irwin-Solecki, [8]). Let \( G \) be a countable projective Fraïssé family of finite topological \( L \)-structures. Then:

1. there exists a projective Fraïssé limit of \( G \);
2. any two topological \( L \)-structures that are projective Fraïssé limits of \( G \) are isomorphic.

We will frequently use the following property of the projective Fraïssé limit, often called the **extension property**.

**Proposition 2.** If \( L \) is the projective Fraïssé limit of \( G \) the following condition holds: Given \( A, B \in G \) and epimorphisms \( \phi_1 : B \to A \) and \( \phi_2 : L \to A \), there is an epimorphism \( \psi : L \to B \) such that \( \phi_2 = \phi_1 \circ \psi \).

### 1.3. Summary of results

In Section 2, we show how to construct the Lelek fan \( L \) as a natural quotient of the projective Fraïssé limit of a family \( F \) of finite reflexive fans. We then use this construction to show projective universality and projective ultrahomogeneity of the Lelek fan in the family of all smooth fans (see Theorem 16). In particular, we obtain that every smooth fan is a continuous image of the Lelek fan.

In Section 3, we prove that the homeomorphism group of the Lelek fan, \( H(L) \), satisfies the following properties.

1. The group \( H(L) \) is totally disconnected (Proposition 18).
2. The group \( H(L) \) is generated by every neighbourhood of the identity, that is, for every \( \varepsilon > 0 \) and \( h \in H(L) \), there are homeomorphisms \( f_0, f_1, \ldots, f_n \in H(L) \) such that for every \( i \), \( d_{\text{sup}}(f_i, \text{Id}) < \varepsilon \), and \( h = f_0 \circ \ldots \circ f_n \) (Theorem 19).
3. The group \( H(L) \) has a dense conjugacy class (Theorem 25).
4. The group \( H(L) \) is (algebraically) simple (Theorem 35).

To prove properties (2) and (3), we use our projective Fraïssé limit construction.

In [16] (Question 5) W. Lewis and Y.C. Zhou asked whether every homeomorphism group of a continuum, which is generated by every neighbourhood of the identity, has to be connected. As \( H(L) \) satisfies properties (1) and (2) above, the answer to this question is negative.

For a detailed discussion of motivation, connections to other known results, etc., of each of these four properties, we refer to Section 3.

### 2. Lelek fan as a quotient of a projective Fraïssé limit

#### 2.1. Construction of the Lelek fan

Let \( T \) be a finite tree, that is, an undirected simple graph which is connected and has no cycles. We will consider only rooted trees, i.e. trees with a distinguished element \( r \in T \). On a rooted tree \( T \) there is a natural partial order \( \leq_T \): for \( t, s \in T \) we let \( s \leq_T t \) if and only if \( s \) belongs to the path connecting \( t \) and the root. We say that \( t \) is a successor of \( s \) if \( s \leq_T t \). It is an immediate successor if additionally there is no \( p \in T \), \( p \neq s, t \), with \( s \leq_T p \leq_T t \). A chain is a rooted tree
Let $T$ be a binary relation symbol. Consider the language $L = \{ R \}$. For $s,t \in T$ we let $R^T(s,t)$ if and only if $s = t$ or $t$ is an immediate successor of $s$. Let $\mathcal{F}_0$ be the family of all finite rooted trees, viewed as topological $L$-structures, equipped with the discrete topology.

Notice that $\phi: (S, R^S) \to (T, R^T)$ is an epimorphism if it is a surjection satisfying: for every $t_1, t_2 \in T$, $R^T(t_1, t_2)$ if and only if there are $s_1, s_2 \in S$ with $\phi(s_1) = t_1, \phi(s_2) = t_2$, and $R^S(s_1, s_2)$.

Let $\mathcal{F}$ be a family that consists of trees $T \in \mathcal{F}_0$ such that for every $s, t \in T$ which are incomparable in $\leq_T$, if $p \neq s, t$ is such that $R^T(p, s)$ and $R^T(p, t)$, then $p$ is the root of $T$, and moreover all branches of $T$ have the same length.

**Remark 3.** The family $\mathcal{F}$ is coinitial in $\mathcal{F}_0$, that is, for every $T \in \mathcal{F}$ there are $S \in \mathcal{F}_0$ and an epimorphism $\phi: S \to T$.

**Proposition 4.** The family $\mathcal{F}_0$ is a projective Fraïssé family.

**Proof.** JPP: Take trees $S_1$ and $S_2$ in $\mathcal{F}_0$. Then the tree $T$ obtained as the disjoint union of $S_1$ and $S_2$ with their roots identified, together with the natural projections from $T$ onto $S_1$ and from $T$ onto $S_2$ witness the JPP.

AP: Take trees $P, Q, S$ together with epimorphisms $\phi_1: Q \to P$ and $\phi_2: S \to P$. Without loss of generality, as $\mathcal{F}$ is coinitial in $\mathcal{F}_0$, $Q$ and $S$ are in $\mathcal{F}$.

Take a branch $b$ in $Q$. Let $a = \phi_1(b)$. Note that $a$ is an initial segment of a branch of $P$. Take any branch $c$ in $S$ such that $a \subseteq \phi_2(c)$. Take a chain $d_b$ and $R$-preserving maps $\psi_1$ and $\psi_2$ defined on $d_b$ (we do not require them to be surjective) such that $\psi_1(d_b) = b$, $\psi_2(d_b) \subseteq c$, and for every $t \in d_b$, $\phi_1 \circ \psi_1(t) = \phi_2 \circ \psi_2(t)$.

We get $d_b$ for every branch $b$ in $Q$ and we get $d_b$ for every branch $b$ in $S$. Without loss of generality, all chains $d_b$ are of the same length. We get a disjoint union of chains $d_b$, where $B$ is a branch in $Q$ or in $S$. Identify roots of all chains $d_b$ and get a tree $T \in \mathcal{F}$. Functions $\psi_1$ and $\psi_2$ are well defined on $T$, $\psi_1$ is onto $Q$, $\psi_2$ is onto $S$, and $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$.

By Theorem 1 there exists a unique Fraïssé limit of $\mathcal{F}_0$, which we denote by $\mathbb{L} = (\mathbb{L}, R^L)$.

The following remark justifies that we can work only with the family $\mathcal{F}$.

**Remark 5.** By Remark 3, the family $\mathcal{F}$ is coinitial in $\mathcal{F}_0$. It easily follows that $\mathcal{F}$ is also a projective Fraïssé family and that the projective Fraïssé limit of $\mathcal{F}$ is isomorphic to the projective Fraïssé limit of $\mathcal{F}_0$. 


Symmetrize \( R^2 \), that is, take \( R_S^2 \) that satisfies \( R_S^2(s,t) \iff R^2(s,t) \) or \( R^2(t,s) \), for every \( s,t \in \mathbb{L} \).

**Theorem 6.** The relation \( R_S^2 \) is an equivalence relation which has only one and two element equivalence classes.

**Proof.** First, observe that since for every \( T \in \mathcal{F} \), \( R^T \) is reflexive, so is \( R_S^l \). Clearly, \( R_S^2 \) is symmetric.

Let \( p,q,r \) be pairwise different. Suppose towards a contradiction that \( R_S^2(p,q) \) and \( R_S^2(p,r) \). Note that, since each member of \( \mathcal{F} \) is a tree, it cannot happen that \( R_S^2(q,p) \) and \( R_S^2(r,p) \). There are two cases to consider.

**Case 1.** We have \( R_S^2(p,q) \) and \( R_S^2(p,r) \). Consider any clopen partition \( P \) of \( \mathbb{L} \) such that \( p,q,r \) are in different clopens of \( P \). Using (L2) in the definition of the projective Fraïssé limit, take \( T \in \mathcal{F} \) and an epimorphism \( \psi_1 : \mathbb{L} \to T \) refining \( P \) (i.e. for every \( t \in T \), \( \psi_1^{-1}(t) \) is contained in a clopen of \( P \)). Notice that \( p' = \psi_1(p) \), \( q' = \psi_1(q) \) and \( r' = \psi_1(r) \) are pairwise different and that \( R^T(p',q') \) and \( R^T(p',r') \). Take \( S \) which is equal to \( T \) with \( p' \) doubled. More precisely, we let \( R_S^S(p',p'), R_S^S(p',r') \), and for \( x,y \in T \), we let \( R_S^S(x,y) \) if and only if \( R^T(x,y) \) and \( (x,y) \neq (p',r') \), in particular \( \neg R_S^S(p',r') \). Then \( \phi : S \to T \) that sends \( p' \) to \( p' \), and other points to themselves, is an epimorphism. Using the extension property, we get an epimorphism \( \psi_2 : \mathbb{L} \to S \) such that \( \psi_1 = \phi \circ \psi_2 \). Then either \( \psi_2(p) = p' \) or \( \psi_2(p) = p' \). In each of these cases we derive a contradiction.

**Case 2.** We have \( R_S^2(q,p) \) and \( R_S^2(p,r) \). Consider again any clopen partition of \( \mathbb{L} \) such that \( p,q,r \) are in different clopens of it. Using (L2) take \( T \in \mathcal{F} \) and an epimorphism \( \psi_1 : \mathbb{L} \to T \) refining this partition. Notice that \( p' = \psi_1(p) \), \( q' = \psi_1(q) \) and \( r' = \psi_1(r) \) are pairwise different and satisfy \( R^T(q',p') \) and \( R^T(p',r') \). Take \( S \) which is equal to \( T \) with \( p' \) doubled. More precisely, we let \( R_S^S(p',p'), R_S^S(p',r') \), and for \( x,y \in T \), we let \( R_S^S(x,y) \) if and only if \( R^T(x,y) \) and \( (x,y) \neq (p',r') \). Then \( \phi : S \to T \) that sends \( p' \) to \( p' \), and other points to themselves, is an epimorphism. Using the extension property, we find an epimorphism \( \psi_2 : \mathbb{L} \to S \) such that \( \psi_1 = \phi \circ \psi_2 \). Then either \( \psi_2(p) = p' \) or \( \psi_2(p) = p' \). Either option leads to a contradiction.

It follows that \( R_S^2 \) is an equivalence relation and that every equivalence class has at most two elements.

\( \square \)

Take the quotient \( \mathbb{L}/R_S^2 \) and denote it by \( L \). Let \( \pi : \mathbb{L} \to L \) be the quotient map.

**Theorem 7.** The space \( L \) is the Lelek fan.

First we need a lemma.

**Lemma 8.** The space \( L \) is Hausdorff, compact, second countable, and connected.

**Proof.** Since \( \mathbb{L} \) and \( R_S^2 \) are compact and \( \pi \) is continuous, it follows that \( L \) is Hausdorff, compact, and second countable, since \( \mathbb{L} \) is such.
Suppose, towards a contradiction, that \( L \) is not connected. Let \( U \) be a clopen non-empty subset of \( L \) such that \( L \setminus U \) is also non-empty. Let \( V = \pi^{-1}(U) \). Let \( T \in \mathcal{F} \) and let \( \phi : \mathbb{L} \rightarrow T \) be an epimorphism refining the partition \( \{V, \mathbb{L} \setminus V \} \). It follows that there are \( x \in V \) and \( y \in \mathbb{L} \setminus V \) such that \( R^L(x, y) \). Since \( \pi(x) = \pi(y) \), we get a contradiction. 

\[ \square \]

We can write \( \mathbb{L} \) as the inverse limit of a sequence \( (T_n, f_n) \), where \( T_n \in \mathcal{F} \) and \( f_n : T_n \rightarrow T_{n-1} \) is an epimorphism that has the following properties (see [8], the proof of Theorem 2.4).

1. For any \( T \in \mathcal{F} \) there is an \( n \) and an epimorphism from \( T_n \) onto \( T \).
2. For any \( m, S, T \in \mathcal{F} \), and any epimorphisms \( \phi_1 : T_m \rightarrow T \) and \( \phi_2 : S \rightarrow T \) there exists \( m < n \) and an epimorphism \( \phi_3 : T_n \rightarrow S \) such that \( \phi_1 \circ f_m^n = \phi_2 \circ \phi_3 \).

We will write \( f_m^n \) to mean \( f_{m+1} \circ \ldots \circ f_n \), whenever \( m < n \) and let \( f_m^\infty = \text{Id}_{T_m} \). Denote by \( f_m^\infty : \mathbb{L} \rightarrow T_n \) the epimorphism such that \( f_m^n \circ f_m^\infty = f_m^\infty \).

Any sequence \( (T_n, f_n) \) that satisfies properties (1) and (2) above will be called a \textit{Fraïssé} sequence.

Our goal now is to show the following proposition.

**Proposition 9.** The continuum \( L \) can be embedded into the Cantor fan \( F \).

We describe a topological \( L \)-structure \( \mathbb{F} = (\mathbb{F}, R_\mathbb{F}) \) such that \( F \) is isomorphic to \( \mathbb{F}/R_\mathbb{F}^\mathbb{L} \), where \( R_\mathbb{F}^\mathbb{L} \) is the symmetrization of \( R_\mathbb{F} \). In order to do this, we first describe a topological \( L \)-structure \( \mathbb{I} = (\mathbb{I}, R^\mathbb{I}) \) such that \( I = [0,1] \) is isomorphic to \( \mathbb{I}/R_\mathbb{I}^\mathbb{L} \). As the underlying set of \( \mathbb{I} \), we take the Cantor set viewed as the middle third Cantor set. For every interval \( (a, b) \) we removed from \( [0,1] \) in its construction, we let \( R^\mathbb{I}(a, b) \), and for every \( a \in \mathbb{I} \), let \( R^\mathbb{I}(a, a) \). Then, \( \mathbb{I}/R_\mathbb{I}^\mathbb{L} \) is homeomorphic to \([0,1] \). We define a topological \( L \)-structure \( \mathbb{F} \) as follows. As the underlying set, we take \( \mathbb{I} \times \mathbb{I} \) mod out by \( \mathbb{I} \times \{0\} \). Let

\[ R^\mathbb{F}((x_1, y_1), (x_2, y_2)) \iff x_1 = x_2 \text{ and } R^\mathbb{I}(y_1, y_2). \]

Note that \( F = \mathbb{F}/R^\mathbb{F} \) is the Cantor fan.

We find an injective, \( R \)-preserving, continuous map from \( \mathbb{L} \) into \( \mathbb{F} \). This will induce a topological embedding from \( L \) into \( F \).

The following lemma provides a representation of \( (\mathbb{F}, R^\mathbb{F}) \) as a specific inverse limit of a sequence of trees in \( \mathcal{F} \).

**Lemma 10.** Let \( (S_n, g_n) \), where \( S_n \in \mathcal{F} \) and \( g_n : S_n \rightarrow S_{n-1} \) is an epimorphism, be an inverse sequence such that for every \( m \) there is \( n > m \) such that (i) \( m, n \), (ii) \( m, n \) hold, and for every \( m, (iii)_m \) holds.

(i) \( m, n \) For every branch \( b \) in \( S_m \) there are distinct branches \( b_1, b_2 \) in \( S_n \) which are mapped by \( g_n^m \) onto \( b \).

(ii) \( m, n \) For every branch \( b \) in \( S_n \) and \( x \in b \), there is \( x' \in b, x' \neq x \), such that \( g_m^n(x) = g_m^n(x') \).
(iii)$_m$ For every branch $b$ in $S_m$, for every branch $b'$ in $S_{m+1}$ such that $g_{m+1}(b') \subseteq b$, we have $g_{m+1}(b') = b$.

Then the inverse limit of $(S_n, g_n)$ is isomorphic to $(\mathbb{F}, R^\mathbb{F})$.

Proof. First, observe that the inverse limit of an inverse sequence $(I_n, h_n)$, where $I_n$ is a finite chain and $h_n : I_n \to I_{n-1}$ is an epimorphism, satisfying for every $m$ there is $n > m$ so that for every $x \in I_n$ there is $x' \in I_n$, $x' \neq x$ such that $h^n_m(x) = h^n_m(x')$, is isomorphic to $(\mathbb{I}, R^\mathbb{F})$. Therefore, for any sequence of branches $B = (b_n)$, $b_n$ is in $S_n$, $g_n(b_n) = b_{n-1}$, the inverse limit of $(b_n, g_n \upharpoonright b_n)$ is isomorphic to $(\mathbb{I}, R^\mathbb{F})$.

Since for every $m$ there is $n$ such that $(i)_{m,n}$ hold, we conclude that the inverse limit of $(S_n, g_n)$ is isomorphic to $(\mathbb{F}, R^\mathbb{F})$.

We point out that above we write $(i)_{m,n}$, rather than say $(i)^{(S_n, g_n)}$. It will always be clear which inverse sequence we are working with, so this will not cause ambiguities. We will write $(i)_n$ and $(ii)_n$ for $(i)_{n-1,n}$ and $(ii)_{n-1,n}$, respectively.

Recall that we represented $\mathbb{L}$ as the inverse limit of a Fraïssé sequence $(T_n, f_n)$, where $T_n \in \mathcal{F}$ and $f_n : T_n \to T_{n-1}$ is an epimorphism. We show that we may require the sequence to satisfy additional conditions.

**Lemma 11.** We can write $\mathbb{L}$ as the inverse limit of an inverse sequence $(\mathcal{T}_n, \mathcal{F}_n)$ such that for every $n$, $(i)_n$ and $(ii)_n$ hold.

Proof. Write $\mathbb{L}$ as the inverse limit of the Fraïssé sequence $(T_n, f_n)$. Using (2) in the properties of the Fraïssé sequence listed above, find $(k_n)$, $(\mathcal{T}_n)$, $\phi_n : \mathcal{T}_n \to T_{k_n}$ and $\psi_n : T_{k_{n+1}} \to \mathcal{T}_n$ such that:

1. $f_{k_n}^{k_{n+1}} = \phi_n \circ \psi_n$;
2. for every branch $b$ in $T_{k_n}$ there are branches $b_1, b_2$ in $\mathcal{T}_n$ which are mapped by $\phi_n$ onto $b$;
3. for every branch $b$ in $\mathcal{T}_n$ and $x \in b$, there is $x' \in b$, $x' \neq x$, such that $\phi_n(x) = \phi_n(x')$.

Let $\mathcal{F}_n = \psi_{n-1} \circ \phi_n$. Then $(\mathcal{T}_n, \mathcal{F}_n)$ works.

**Lemma 12.** Suppose that $(\mathcal{T}_n, \mathcal{F}_n)$, $\mathcal{T}_n \in \mathcal{F}$ and $\mathcal{F}_n : \mathcal{T}_n \to \mathcal{T}_{n-1}$ is an epimorphism, is an inverse sequence satisfying $(ii)_n$ for every $n$. Then there is an inverse sequence $(S_n, g_n)$, where $S_n \in \mathcal{F}$ and $g_n : S_n \to S_{n-1}$ is an epimorphism, satisfying $(i)_n, (ii)_n, (iii)_n$ for every $n$ (therefore, in particular, satisfying the assumptions of Lemma 11), such that $\mathcal{T}_n$ is a subtree of $S_n$ and $\mathcal{F}_n = g_n \upharpoonright \mathcal{T}_n$. In particular, the inverse limit of $(\mathcal{T}_n, \mathcal{F}_n)$ embeds into $\mathbb{F}$.

Proof. First, we find an inverse sequence satisfying $(ii)_n, (iii)_n$ for every $n$. We obtain the required inverse sequence via an inductive procedure.
Suppose that for some $N$ we have $(S'_n, g'_n)$ such that $T_n$ is a subtree of $S'_n$ and $\overline{T}_n = g'_n \upharpoonright \overline{T}_n$, for every $n$, (ii)$_n$ holds, and for every $n \leq N$, (iii)$_n$ holds. We find $(S''_n, g''_n)$ such that that $S'_n$ is a subtree of $S''_n$ and $g'_n = g''_n \upharpoonright S'_n$, for every $n$, (ii)$_n$ holds, and for every $n \leq N + 1$, (iii)$_n$ holds.

For $n \leq N$, let $S''_n = S'_n$, and let $g''_n = g'_n$. Pick a branch $b \subseteq S'_{N+1}$. If $g'_{N+1}(b)$ is properly contained in a branch $b_N \subseteq S'_N$, we extend $b$ to some $b_{N+1} \supseteq b$ and we extend $g'_{N+1}$ to $b_{N+1}$ in a way that $g'_{N+1}(b_{N+1}) = b_N$ and for every $x \in b_{N+1}$, there is $x' \in b_{N+1}$, $x' \neq x$, such that $g'_{N+1}(x) = g'_{N+1}(x')$. Next, in $S'_{N+2}$, we add a new branch $b_{N+2}$ and extend $g'_{N+2}$ to $b_{N+2}$ so that $g'_{N+2}(b_{N+2}) = b_{N+1}$ and for every $x \in b_{N+2}$, there is $x' \in b_{N+2}$, $x' \neq x$, such that $g'_{N+2}(x) = g'_{N+2}(x')$. Repeat this procedure with every branch $b$ in $S'_{N+1}$. The resulting $(S''_n, g''_n)$ is as required.

Denote by $(\overline{S}_n, \overline{g}_n)$ the inverse sequence satisfying (ii)$_n$, (iii)$_n$ for every $n$, we obtained from this inductive procedure.

Now, suppose that for some $N$ we have $(S'_n, g'_n)$ such that $\overline{S}_n$ is a subtree of $S'_n$ and $\overline{g}_n = g'_n \upharpoonright \overline{S}_n$, for every $n$, (ii)$_n$ and (iii)$_n$ hold, and for every $n \leq N$, (i)$_n$ holds. We find $(S''_n, g''_n)$ such that that $S'_n$ is a subtree of $S''_n$ and $g'_n = g''_n \upharpoonright S'_n$, for every $n$, (ii)$_n$ (iii)$_n$ hold, and for every $n \leq N + 1$, (i)$_n$ holds.

For $n \leq N$, let $S''_n = S'_n$, and let $g''_n = g'_n$. For each $n > N$ we take $S''_n$ to be two disjoint copies of $S'_n$: $\tilde{S}'_n$ and $\tilde{S}'_n$, with their roots identified, and let $g''_n$ be equal to $g_n$ on each copy and take $\tilde{S}'_n$ to $\tilde{S}'_{n-1}$, and take $\tilde{S}'_n$ to $\tilde{S}'_{n-1}$. The $(S''_n, g''_n)$ we obtained is as required.

Denote by $(S_n, g_n)$ the inverse sequence satisfying (i)$_n$, (ii)$_n$, (iii)$_n$ for every $n$, we obtained from this inductive procedure.

Proof of Proposition 13 It follows from Lemmas 10, 11 and 12.

Now let us focus on showing the density of endpoints of the Lelek fan. Let $A$ be a topological $L$-structure. We say that $K \subseteq A$ is $R$-connected if for every two non-empty, disjoint clopen subsets $K_1, K_2$ in $K$ such that $K_1 \cup K_2 = K$, there are $x \in K_1$ and $y \in K_2$ such that $R^A(x, y)$ or $R^A(y, x)$. Write $L$ as an inverse limit of a Fraïssé sequence $(T_n, f_n)$. Let $r_n$ be the root of $T_n$. The point $r = (r_n)$ will be called the root of $L$.

Proposition 13. The set of endpoints in $L$ is dense in $L$.

Proof. Let $U \subseteq L$ be open and non-empty. Let $V = \pi^{-1}(U)$. We find an endpoint in $U$. Take $n_1$ such that there is $e_{n_1} \in T_{n_1}$ with $(f_{n_1}^\infty)^{-1}(e_{n_1}) \subseteq V$. Let $T \in \mathcal{F}$, $\psi_1 : T \to T_{n_1}$, and $x \in T$ be such that $\psi_1(x) = e_{n_1}$ and $x$ is an endpoint of $T$ (i.e. $x$ is such that for no $y \in T$, $y \neq x$, we have $R^T(x, y)$). Using that $(T_n, f_n)$ is a Fraïssé sequence, find $n_2$ and $\psi_2 : T_{n_2} \to T$ such that $f_{n_2}^{n_1} = \psi_1 \circ \psi_2$. Pick any endpoint in $T_{n_2}$ in the preimage of
\[ x \text{ by } \psi_2. \] Denote it by \( e_{n_2} \). For \( n > n_2 \) inductively pick an endpoint \( e_n \) in \( T_n \) such that \( f_{n-1}^n(e_n) = e_{n-1} \) and for \( n < n_2 \) let \( e_n = f_{n+2}^n(e_{n_2}) = e_n \). Denote \( e = (e_n) \). Note that \( e \in V \) and therefore \( \pi(e) \in U \).

We show that \( \pi(e) \) is an endpoint of \( L \). Let \( i: I \to L \) be a homeomorphic embedding of the interval \( I = [0,1] \) such that \( \pi(e) \in i(I) \). Suppose towards a contradiction that \( \pi(e) \neq i(0) \) and \( \pi(e) \neq i(1) \). Denote \( K = \pi^{-1}(i(0,1)), M = \pi^{-1}(i(0,\pi^{-1}(\pi(e)]), \) and \( N = \pi^{-1}(i([\pi^{-1}(\pi(e)),1])) \). All three sets \( K, M, N \) are compact and \( R \)-connected in \( \mathbb{L} \).

We have either \( M \cap N = \{e\} \), or \( M \cap N = \{e,e'\} \), where \( e' \) is such that \( \pi(e') = \pi(e) \) and \( R^e(e,e') \) or \( R^e(e,e') \). Let \( K_n = f^\infty_n(K), M_n = f^\infty_n(M), \) and \( N_n = f^\infty_n(N) \). All sets \( K_n, M_n, N_n \) are \( R \)-connected in \( T_n \). Without loss of generality, \( \pi(r) \notin i(I) \), where \( r = (r_n) \) is the root of \( \mathbb{L} \). This implies that for some \( \mathbb{N} \), whenever \( n > \mathbb{N}, K_n \) and \( N_n \) is contained in a single branch.

Let \( x = (x_n) \in N \setminus M \) and let \( y = (y_n) \in M \setminus N \). Suppose that \( e \) is such that \( M \cap N = \{e,e'\} \), where \( e' \) is such that \( \pi(e') = \pi(e) \) and \( R^e(e',e) \) (other cases being similar). Then for \( n > \mathbb{N}, \) either \( r_n < x_n < y_n < e'_n < e_n \), or we have for \( n > \mathbb{N}, \) \( r_n < y_n < x_n < e'_n < e_n \). Without loss of generality, we may assume that the former holds. Since for every \( n, x_n, e_n \in N_n, \) \( R \)-connectivity of each \( N_n \) implies \( y_n \in N_n \) for \( n > \mathbb{N} \). Therefore \( y \in N \), which is a contradiction.

\[ \square \]

2.2. Properties of the Lelek fan: Projective universality and Projective ultrahomogeneity. The main goal of this subsection is to prove Theorem 16. This is an analog of Theorem 4.4. in [8].

\[ \text{Let } \text{Aut}(\mathbb{L}) \text{ be the group of all automorphisms of } \mathbb{L}, \text{ that is, the group of all homeomorphisms of } \mathbb{L} \text{ that preserve the relation } R^e. \text{ This is a topological group when equipped with the compact-open topology inherited from } H(\mathbb{L}), \text{ the group of all homeomorphisms of the Cantor set underlying the structure } \mathbb{L}. \text{ Since } R^e \text{ is closed in } \mathbb{L} \times \mathbb{L}, \text{ the group } \text{Aut}(\mathbb{L}) \text{ is closed in } H(\mathbb{L}). \]

\[ \text{Remark 14. Every } h \in \text{Aut}(\mathbb{L}) \text{ induces a homeomorphism } h^* \in H(L) \text{ satisfying } h^* \circ \pi(x) = \pi \circ h(x) \text{ for } x \in \mathbb{L}. \text{ We will frequently identify } \text{Aut}(\mathbb{L}) \text{ with the corresponding subgroup } \{h^* : h \in \text{Aut}(\mathbb{L})\} \text{ of } H(L). \text{ Observe that the compact-open topology on } \text{Aut}(\mathbb{L}) \text{ is finer than the topology on } \text{Aut}(\mathbb{L}) \text{ that is inherited from the compact-open topology on } H(L). \]

\[ \text{Smooth fan } X \text{ is a fan such that whenever } t_n \to t, t_n, t \in X \text{ then the sequence of arcs } t_nw \text{ converges to the arc } tv \text{ (in the Hausdorff metric), where } w \text{ is the top point of } X. \text{ Smooth fans are exactly fans that can be embedded into the Cantor fan (see [4], Proposition 4, the definition of a fan is given there as well). These are exactly non-degenerate subcontinua of the Cantor fan } F \text{ that contain the top point } v \in F \text{ and are not homeomorphic to the interval } [0,1]. \]
We will say that a continuous surjection \( f: L \to X \), where \( X \) is a smooth fan, is monotone on segments if \( f(v) = w \), where \( v \) is the top of \( L \) and \( w \) is the top of \( X \), and for every endpoint \( e \in L \) and every \( y \in X \), \( f^{-1}(y) \cap [v, e] \) is connected. Equivalently, a continuous surjection \( f: L \to X \) is monotone on segments if \( f(v) = w \) and for every line segment \([v, e]\) in \( L \), \( f \upharpoonright [v, e] \) is a non-decreasing map into some line segment \([w, f]\) in \( X \), where \( e \) is an endpoint of \( L \) and \( f \) is an endpoint of \( X \). To prove Theorem 16, we need the following lemma.

**Lemma 15.** Let \( \epsilon > 0 \). Let \( X \) be a smooth fan with the top \( w \). Then there is \( A \in \mathcal{F} \) and an open cover \( \{U_a\}_{a \in A} \) of \( X \) such that (1) for each \( a \in A \), \( \text{diam}(U_a) < \epsilon \), (2) for each \( a, a' \in A \), \( R^A(a, a') \) if and only if \( U_a \cap U_{a'} \neq \emptyset \) and whenever \( x \in U_a \setminus U_{a'} \) and \( y \in U_{a'} \) we have that \([w, y] \not\subset [w, x] \), (3) for every \( a \in A \) and every line segment \([x, y] \subset X \), the set \( U_a \cap [x, y] \) is connected, and (4) for every \( a \in A \) there is \( x \in X \) such that \( x \in U_a \setminus (\bigcup \{U_{a'}: a' \in A, a' \neq a\}) \).

**Proof.** We first show that the lemma holds for the Cantor fan \( F \). Let \( \{O_1, O_2, \ldots, O_n\} \) be an open \( \frac{\epsilon}{2} \)-cover of the unit interval \( I = [0, 1] \) such that \( O_i \cap O_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \) and for \( i, j \), \( x \in O_i \setminus O_j \) and \( y \in O_j \) we have \( x < y \). Let \( \{V_1, V_2, \ldots, V_m\} \) be a clopen \( \frac{\epsilon}{2} \)-cover of the Cantor set \( C \). Then \( \{O_i \times V_j: i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\} \) is an open \( \epsilon \)-cover of \( C \times I \). Let \( O \subset F \) be an open neighbourhood of the top \( w \) of \( F \) of the form \( O = \bigcup_{j=1}^m O_i \times V_j/\sim \), where \((a, b) \sim (c, d)\) if and only if either \( a = c \) and \( b = d \) or \( a = c = 0 \). The desired cover is then \( \mathcal{V} = \{O\} \cup \{O_i \times V_j: i = 2, \ldots, n, j = 1, \ldots, m\} \) with \( A = \{R \} \cup \{(i, j): i = 2, \ldots, n, j = 1, \ldots, m\} \), where \( R^A(r, (i, j)) \) if and only if \( i = 2 \) and for \((i, j), (i', j') \in \{2, \ldots, n\} \times \{1, \ldots, m\} \), \( R^A((i, j), (i', j')) \) if and only if \( j = j' \) and \( i' - i \leq 1 \).

If \( X \) is a smooth fan, we think of \( X \) as embedded in \( F \) and define the cover to be \( \{V \cap X: V \in \mathcal{V}\} \) and obtain the structure \( A \) in the same manner as for \( F \). We can also arrange that all branches of \( A \) have the same length.

**Theorem 16.**

1. Each smooth fan is a continuous image of the Lelek fan \( L \) via a map that is monotone on segments.
2. Let \( X \) be a smooth fan with a metric \( d \). If \( f_1, f_2: L \to X \) are two continuous surjections that are monotone on segments, then for any \( \epsilon > 0 \) there exists \( h \in \text{Aut}(\mathbb{L}) \) such that for all \( x \in L \), \( d(f_1(x), f_2 \circ h^*(x)) < \epsilon \).

**Proof.**

1. Let \( X \) be a smooth fan viewed as a subfan of the Cantor fan \( F \). We show how to obtain \( X \) in a canonical way as a quotient of a certain topological \( L \)-structure. While proving Proposition 9 we already described how to obtain the Cantor fan as a quotient of a topological \( L \)-structure. We generalize what we did there.

Let \( C \) be a Cantor set viewed as the middle third Cantor set. Let \( f: C \to [0, 1] \) be given by \( f(0.a_1a_2a_3 \ldots) = 0.a'_1a'_2a'_3 \ldots \), where \( a'_n = 0 \), when \( a_n = 0 \), and \( a'_n = 1 \), when \( a_n = 2 \). Consider \( \text{Id} \times f/\sim: C \times C/\sim \to F \), where \((a, b) \sim (c, d)\) if and only
if \(a = c\) and \(b = d\), or \(b = d = 0\). Let \(X = (\text{Id} \times f/\sim)^{-1}(X)\). Let \(R^X((a, b), (c, d))\) if and only if \(a = c\) and \(b = d\), or \(a = c\) and \((b, d)\) is an interval removed from \([0, 1]\) in the construction of \(C\). Then \(X = (X, R^X)\) is a topological \(L\)-structure. Observe that \(\mathbb{X}/R^S_S = X\) (recall that \(R_S\) denotes the symmetrization of \(R\)). By the proof of Lemma 15 and using the Lebesgue covering lemma, we get that every open cover of \(X\) can be refined by an epimorphism onto a structure in \(\mathcal{F}\). Therefore by Lemma 2.6 in [8], there is an epimorphism \(f : \mathbb{L} \to \mathbb{X}\). This epimorphism induces a continuous surjection \(\bar{f}\) from \(L = \mathbb{L}/R^S_S\) onto \(X = \mathbb{X}/R^S_S\). We show that \(\bar{f}\) is monotone on segments. Clearly, \(\bar{f}(v) = w\), where \(v\) and \(w\) are the tops of \(L\) and \(X\) respectively. Let \(x, y \in \mathbb{L}\) be such that \(\bar{f}(\pi(x)) \neq \bar{f}(\pi(y))\) and \(\pi(x) \in [v, \pi(y)]\). Then for some \(e\), an endpoint of \(X\), \(\bar{f}(\pi(x)), \bar{f}(\pi(y)) \in [w, e]\). We will show that \(\bar{f}(\pi(x)) \in [w, \bar{f}(\pi(y))]\). Let \(T \in \mathcal{F}\) and let \(\phi : X \to T\) be an epimorphism refining an open cover of \(X\) that separates \(f(x)\) and \(f(y)\). Using (L2), find \(S \in \mathcal{F}\) and \(\psi : \mathbb{L} \to S\), an epimorphism refining the cover \(\{f^{-1} \circ \phi^{-1}(t) : t \in T\}\). Let \(\mu : S \to T\) be such that \(\phi \circ f = \mu \circ \psi\). This \(\mu\) is an epimorphism. Let \(b\) be the branch in \(T\) such that \(\phi(f(x)), \phi(f(y)) \in b\) and let \(c\) be the branch in \(S\) such that \(\psi(x), \psi(y) \in c\). Note that \(\mu(c) \subseteq b\). Since \(\pi(x) \in [v, \pi(y)]\), there is a sequence \(\psi(x) = c_0, c_1, \ldots, c_n = \psi(y)\) such that \(R^S(c_i, c_{i+1})\) for \(i = 0, 1, \ldots, n-1\). We have that \(\phi \circ f(x) = \mu(c_0)\) and \(\phi \circ f(y) = \mu(c_n)\), and \(R^F(\mu(c_i), \mu(c_{i+1}))\) for \(i = 0, 1, \ldots, n-1\), as \(\mu\) is an epimorphism. If follows that \(\bar{f}(\pi(x)) \in [w, \bar{f}(\pi(y))]\).

(2) Take \(A \in \mathcal{F}\) and an open \(\epsilon\)-cover \(\{U_a\}_{a \in A}\) of \(X\) as in Lemma 15. Using the Lebesgue covering lemma, find \(\delta\) be such that for \(M \subseteq X\) with \(\text{diam}(M) < \delta\) there exists \(a \in A\) such that \(M \subseteq U_a\). Since \(f_1 \circ \pi\) and \(f_2 \circ \pi\) are uniformly continuous on \(\mathbb{L}\), there are \(B_1, B_2 \in \mathcal{F}\) and epimorphisms \(\phi_i : \mathbb{L} \to B_i\), \(i = 1, 2\) such that for \(b \in B_i\), \(\text{diam}(f_i \circ \pi \circ \phi_i^{-1}(b)) < \delta\). Let \(\psi_i : B_i \to A\) be defined as follows: \(\psi_i(b) = a\) if and only if \(f_i \circ \pi \circ \phi_i^{-1}(b) \subseteq U_a\) and whenever \(f_i \circ \pi \circ \phi_i^{-1}(b) \subseteq U_{a'}\), then \(R^A(a', a)\).

We show that \(\psi_i\) is an epimorphism for \(i = 1, 2\). Firstly, \(\psi_i\) is onto. That follows from the fact that \(\{U_a : a \in A\}\) and \(\{f_i \circ \pi \circ \phi_i^{-1}(b) : b \in B_i\}\) are covers of \(X\) and from Lemma 15 part (4). Secondly, let \(b, b' \in B_i\) be such that \(R^{B_i}(b, b')\). This implies that \(f_i \circ \pi \circ \phi_i^{-1}(b)\) and \(f_i \circ \pi \circ \phi_i^{-1}(b')\) intersect, and therefore also \(U_{\psi_i(b)}\) and \(U_{\psi_i(b')}\) intersect (they may be identical). This gives \(R^A(\psi_i(b), \psi_i(b'))\) or \(R^A(\psi_i(b'), \psi_i(b))\).

**Claim 1.** Let \(R^{B_i}(b, b')\) and let \(w\) be the top of \(X\). Then for every \(x \in f_i \circ \pi \circ \phi_i^{-1}(b')\) there exists \(y\) in the line segment \([w, x]\) belonging to \(f_i \circ \pi \circ \phi_i^{-1}(b)\).

**Proof of Claim 1.** Let \(x \in f_i \circ \pi \circ \phi_i^{-1}(b')\). Since \(f_i\) is monotone on segments, \(f_i^{-1}(x) \cap [v, e]\) is a connected line segment for some endpoint \(e\) of \(L\) and \(v\) the top of \(L\). Then \(\phi_i\) maps \(\pi^{-1}[v, e]\) into the branch \([r_{B_i}, \phi_i(e)]\). In particular, by the choice of \(x\), the image of \(\pi^{-1}[v, e]\) under \(\phi_i\) contains \(b'\), and consequently \(b\). Since \(R^{B_i}(b, b')\), it follows that \(f_i \circ \pi \circ \phi_i^{-1}(b) \cap [w, x] \neq \emptyset\), which finishes the proof.

Suppose towards the contradiction that \(R^A(\psi_i(b'), \psi_i(b))\) and that \(\psi_i(b') \neq \psi_i(b)\). It means that there exists \(x \in f_i \circ \pi \circ \phi_i^{-1}(b') \setminus U_{\psi_i(b)} \subseteq U_{\psi_i(b')} \setminus U_{\psi_i(b)}\). By
Claim 1 there exists \( y \in [w, x] \cap f_i \pi \phi_i^{-1}(b) \subseteq U_{\psi_i(b)} \). This contradicts the choice of the cover \( U_a \) in Lemma 15, part (2). Thirdly, let \( a, a' \in A \) be such that \( R^A(a, a') \), that is, \( U_a \cap U_{a'} \neq \emptyset \). Since by Lemma 15 part (3), \( U_a \) and \( U_{a'} \) are connected on line segments of \( X \), and since \( \{ f_i \pi \phi_i^{-1}(b) : b \in B_i \} \) is a finite closed cover of \( X \), there must be \( b, b' \in B_i \) such that \( \psi_i(b) = a \) and \( \psi_i(b') = a' \) and \( f_i \pi \phi_i^{-1}(b) \cap f_i \pi \phi_i^{-1}(b') \neq \emptyset \). We get that either \( R^B(b, b') \) or \( R^B(b', b) \). Using Claim 1 and the same reasoning as above for showing that \( \psi_i \) preserves the relation \( R \), we get \( R^B(b, b') \). This finishes the proof that \( \psi_i \) is an epimorphism for \( i = 1, 2 \).

Finally, by (L3), there exists \( h \in \text{Aut}(\mathbb{L}) \) such that \( \psi_1 \circ \phi_1 = \psi_2 \circ \phi_2 \circ h \). This gives that for each \( y \in \mathbb{L} \), there is \( a \in A \) such that \( f_1 \circ \pi(y), f_2 \circ \pi \circ h(y) \in U_a \).

Hence for all \( x \in \mathbb{L} \), \( d(f_1(x), f_2 \circ h^*(x)) < \epsilon \).

\[ \square \]

A metric space \( X \) is uniformly pathwise connected if

1. there exists a family \( P \) of paths in \( X \) such that for \( x, y \in X \) there is a path in \( P \) joining \( x \) and \( y \), and
2. for every \( \epsilon > 0 \) there is a positive integer \( n \) such that each path in \( P \) can be partitioned into \( n \) pieces of diameter at most \( \epsilon \).

As shown in [11], continuous images of the Cantor fan are precisely uniformly pathwise connected continua.

Since the Lelek fan is a continuous image of the Cantor fan (it is clearly uniformly pathwise connected), and since the Cantor fan is a continuous image of the Lelek fan (by the first part of Theorem 16), we obtain the following corollary.

**Corollary 17.** Continuous images of the Lelek fan are precisely uniformly pathwise connected continua.

### 3. The homeomorphism group of the Lelek fan

#### 3.1. Connectivity properties of \( H(L) \)

We show that \( H(L) \) – the homeomorphism group of the Lelek fan \( L \) – is totally disconnected and it is generated by every neighbourhood of the identity (i.e. every homeomorphism can be written as a finite composition of \( \epsilon \)-homeomorphisms, defined below). W. Lewis in [14] showed that the homeomorphism group of the pseudo-arc is generated by every neighbourhood of the identity. However, it is not known whether that group is totally disconnected (see [15]).

There are examples of totally disconnected Polish groups that are generated by every neighbourhood of the identity. (Recall that a Polish group is a separable and completely metrizable topological group.) The first such example, solving Problem 160 in the Scottish Book ([18]), posed by S. Mazur, asking whether a complete metric group that is generated by every neighbourhood of the identity must be connected, was given by Stevens [19]; another example was presented by Hjorth [7]. Our example is different. The group is non-abelian (which follows from Remark 24 and Theorem 25) and it is
explicitly given as a homeomorphism group of a compact space. W. Lewis and Y.C. Zhou ask in [16] (Question 5) whether every homeomorphism group of a continuum that is generated by every neighbourhood of the identity has to be connected. Our example shows that the answer is negative.

Recall that a topological space $X$ is totally disconnected if for any $x, y \in X$ there is a clopen set $C \subseteq X$ such that $x \in C$ and $y \notin (X \setminus C)$. Note that this implies that every subspace of $X$ containing more than one element is not connected (the latter property is in literature often used as a definition of being totally disconnected).

**Proposition 18.** The group $H(L)$ is totally disconnected.

*Proof.* Let $h_1, h_2 \in H(L)$. We show that there is a clopen set $A$ in $H(L)$ such that $h_1 \in A$ and $h_2 \notin A$.

First, we show that there is $e \in E$ such that $h_1(e) \neq h_2(e)$, where $E$ denotes the set of endpoints of $L$. Suppose that this is not the case and let $h_0 = h_1 \circ h_2^{-1}$. Then for every $e \in E$, $e = h_0(e)$, and consequently $h_0 \upharpoonright [v, e]$ is a homeomorphism onto $[v, e]$. We show that $h_0$ is the identity map. Suppose the contrary, i.e. for some $e \in E$, $h_0 \upharpoonright [v, e]$ is not the identity map. It means that for some $t \in [v, e]$, $h(t) \neq t$. Let $e_n$ be a sequence of endpoints that converges to $t$. We have that $h_0(e_n) = e_n$, for each $n$, and therefore the sequence $h_0(e_n)$ converges to $t \neq h_0(t)$. This contradicts continuity of $h_0$.

Let $e \in E$ be such that $h_1(e) \neq h_2(e)$. Let $X_0$ be a clopen set in $C$ such that $\pi_1(h_1(e)) \in X_0$ and $\pi_1(h_2(e)) \notin X_0$. Let $X = (\pi_1 \upharpoonright E)^{-1}(X_0)$. Then $X$ is a clopen set in $E$. Let $A = \{h \in H(L): h(e) \in X\}$. Since $H(L) \to E$, $h \mapsto h(e)$ is continuous, $A$ is a clopen set in $H(L)$. Finally, $h_1 \in A$, and $h_2 \notin A$. □

Fix a compatible metric $d$ on $L$. Denote the corresponding supremum metric on $H(L)$ by $d_{\text{sup}}$. A homeomorphism $h \in H(L)$ is called an $\epsilon$-homeomorphism if $d_{\text{sup}}(h, \text{Id}) < \epsilon$.

**Theorem 19.** For every $\epsilon > 0$ and $h \in H(L)$, there are $\epsilon$-homeomorphisms $f_0, f_1, \ldots, f_n \in H(L)$ such that $h = f_0 \circ \ldots \circ f_n$. Moreover, if $h \in \text{Aut}(\mathbb{L})$, we can choose $f_0, f_1, \ldots, f_n$ in Aut(\mathbb{L}).

*Proof.* Let us first roughly describe steps of the proof before we turn to technical details. First, we use the density of Aut(\mathbb{L}) in $H(L)$ to find $g \in \text{Aut}(\mathbb{L})$ within the distance less than $\epsilon$ from $h$. Then we look for $f_0, \ldots, f_n$ in Aut(\mathbb{L}). We start with an $\frac{\epsilon}{2}$-cover $\mathcal{C}$ of \mathbb{L} which corresponds to a finite tree in \mathcal{F}, or more precisely, to an epimorphism from \mathbb{L} onto a finite tree in \mathcal{F}. We refine $\mathcal{C}$ by a cover $\mathcal{D}$ such that $g(\mathcal{D})$ also refines $\mathcal{C}$. We pick a branch $d$ in $\mathcal{D}$ and branches $c_1$ and $c$ in $\mathcal{C}$ such that $d$ refines $c_1$ and $g(d)$ refines $c$. Then in $\epsilon$-steps we pull the branch $g(d)$ along the branch $c$ all the way down to the root and then pull it up along the branch $c_1$ to get the branch $d$. At each $\epsilon$-step, using (L3), we get an automorphism $h_i$. The required $\epsilon$-homeomorphisms are of the form $h_i^* \circ (h_{i+1})^{-1}$. We proceed in the same way with the remaining branches of $\mathcal{D}$.
Now, we proceed to the proof. Let \( g \in \text{Aut}(\mathbb{L}) \) be such that \( d_{\sup}(h, g^*) < \epsilon \). Let \( \mathcal{B}_0 \) be an open cover of \( L \) as in Lemma \([13]\) (for \( \epsilon' \) and \( X = L \)). Let \( \mathcal{B} = \{ \pi^{-1}(B) : B \in \mathcal{B}_0 \} \) be an open cover of \( \mathbb{L} \), where \( \pi : \mathbb{L} \to L \) is the natural quotient mapping. Let \( S \in \mathcal{F} \) and \( \alpha : \mathbb{L} \to S \) be an epimorphism that refines \( \mathcal{B} \). Note that then whenever \( s, s' \in S \) are such that \( R^S(s, s') \), then \( \text{diam}(\pi \circ \alpha^{-1}(s) \cup \pi \circ \alpha^{-1}(s')) < \epsilon \). Let \( T \in \mathcal{F} \), \( \beta : T \to S \), \( \gamma : \mathbb{L} \to T \) be epimorphisms such that \( \alpha = \beta \circ \gamma \) and \( g(D) = \{ g \circ \gamma^{-1}(t) : t \in T \} \), where \( D = \{ \gamma^{-1}(t) : t \in T \} \), refines \( \mathcal{C} = \{ \alpha^{-1}(s) : s \in S \} \). To find such a \( D \) we use the uniform continuity of \( g \) and the Lebesgue covering lemma. Let \( \beta_0 = \alpha \circ g \circ \gamma^{-1} \) and let \( \gamma_0 = \gamma \circ g^{-1} \).

Note that \( \beta_0 \) is an epimorphism and \( \alpha = \beta_0 \circ \gamma_0 \).

Enumerate all branches in \( S \) as \( c_1, \ldots, c_n \). Enumerate all branches in \( T \) as \( d_1, \ldots, d_l \). For a reason that will become clear later, we also require that for every \( 1 \leq i \leq k \) there are at least \( k + 1 \) branches in \( T \) such that for each such branch \( d_i \), \( \beta_0 \upharpoonright d_i \) is onto \( c_i \). If the original tree \( T \) does not have this property, we take \( T' \) and \( \phi : T' \to T \) such that for every branch \( b \) in \( T \) there are \( k + 1 \) branches in \( T' \) that are mapped by \( \phi \) onto \( b \). We apply the extension property to \( \phi \) and \( \gamma_0 \) and get \( \psi : \mathbb{L} \to T' \) such that \( \gamma_0 = \phi \circ \psi \). We replace \( T \) by \( T' \), \( \gamma_0 \) by \( \psi \), \( \beta_0 \) by \( \beta_0 \circ \phi \), \( \gamma_0 \) by \( \psi \circ g \), and \( \beta \) by \( \alpha \circ g^{-1} \circ \psi^{-1} \).

It is enough to show that there are epimorphisms \( \beta_1, \ldots, \beta_n = \beta \), for some \( n \), such that for every \( 0 \leq i < n \) and for every \( t \in T \), \( R^S(\beta_i(t), \beta_{i+1}(t)) \) or \( R^S(\beta_{i+1}(t), \beta_i(t)) \). Then using the extension property, we find \( \gamma_1, \ldots, \gamma_n = \gamma \) such that \( \alpha = \beta_i \circ \gamma_i, i = 1, 2, \ldots, n \). Using projective ultrahomogeneity, we find \( g = h_0, h_1, \ldots, h_{n-1}, h_n = \text{Id} \in \text{Aut}(\mathbb{L}) \) such that \( \gamma = \gamma_i \circ h_i \). For each automorphism \( h_i \), let \( h_i^* \) denote the corresponding homeomorphism of \( L \). Then \( f_0 = h \circ (h_0^*)^{-1}, f_1 = h_0 \circ (h_1^*)^{-1}, f_2 = h_1 \circ (h_2^*)^{-1}, \ldots, f_{n-1} = h_{n-1}^*, f_n = h_n^* \) are \( \epsilon \)-homeomorphisms: Indeed, for every \( x \in L \) and \( i = 0, 1, \ldots, n-1, R^S(\alpha \circ h_i(x), \alpha \circ h_{i+1}(x)) \) or \( R^S(\alpha \circ h_{i+1}(x), \alpha \circ h_i(x)) \), since \( \alpha \circ h_i(x) = \beta_i \circ \gamma_i \circ h_i(x) = \beta_i \circ \gamma_i \circ \alpha \circ h_i(x) = \beta_i \circ \gamma_i \circ \alpha \circ h_{i+1}(x) \) and \( R^S(\beta_i(t), \beta_{i+1}(t)) \) or \( R^S(\beta_{i+1}(t), \beta_i(t)) \) for every \( t \in T \). Consequently, by previous remarks, we get that for every \( x \in L \), \( \text{diam}(\pi \circ \alpha^{-1}(\alpha \circ h_i(x)) \cup \pi \circ \alpha^{-1}(\alpha \circ h_{i+1}(x))) < \epsilon \), and therefore, \( d_{\sup}(h_i^*, h_{i+1}^*) < \epsilon \). Clearly, the composition \( f_0 \circ \ldots \circ f_n \) is equal to \( h \).

Let us additionally assume that \( d_1, \ldots, d_l \) are enumerated in a way that for every \( 1 \leq i \leq k \), \( \beta \upharpoonright d_i \) is onto \( c_i \). Pick \( d_1 \). We construct \( \beta_1, \beta_2, \ldots, \beta_{n_1} \), for some \( n_1 \) determined later. Let \( c = (c(0), c(1), \ldots, c(m_1)) \) be such that \( \beta_0(d_1) \subseteq c \). We have \( \beta(d_1) = c_1 = (c_1(0), c_1(1), \ldots, c_1(m_2)) \). Let \( \beta_1(t) \) be equal to \( c(m_1 - 1) \) if \( t \in d_1 \) and \( \beta_0(t) = c(m_1) \), and be equal to \( \beta_0(t) \) otherwise. More generally, for \( i = 1, \ldots, m_1 \), let

\[
\beta_i(t) = \begin{cases} 
  c(m_1 - i) & \text{if } t \in d_1 \\
  \beta_{i-1}(t) & \text{otherwise.}
\end{cases}
\]

Let \( \beta_{m_1+1}(t) \) be equal to \( c_1(1) \) if \( t \in d_1 \) and \( \beta(t) \in \{ c_1(1), \ldots, c_1(m_2) \} \), and be equal to \( \beta_{m_1}(t) \) otherwise. More generally, for \( i = 1, \ldots, m_2 \), let

\[
\beta_{m_1+i}(t) = \begin{cases} 
  c_1(i) & \text{if } t \in d_1 \text{ and } \beta(t) \in \{ c_1(i), \ldots, c_1(m_2) \} \\
  \beta_{m_1+i-1}(t) & \text{otherwise.}
\end{cases}
\]
We constructed epimorphisms $\beta_1, \ldots, \beta_n$ for $n_1 = m_1 + m_2$ and we are done with the branch $d_1$. Take $d_2$ and by an analogous procedure construct $\beta_{n_1+1}, \ldots, \beta_{n_2}$, etc. Note that since we required that for every $1 \leq i \leq k$, there are at least $k+1$ branches in $T$ that are mapped onto $c_i$ by $\beta_0$, each of the $\beta_i$’s is onto. Each $\beta_i$ is an epimorphism and they satisfy the required condition: for every $0 \leq i < n$ and for every $t \in T$, $R^S(\beta_i(t), \beta_{i+1}(t))$ or $R^S(\beta_{i+1}(t), \beta_i(t))$.

The ‘moreover’ part of the theorem follows from the proof (we pick $g = h$).

**Corollary 20.** The group $H(L)$ has no proper open subgroup.

Corollary 20 will help us derive that $H(L)$ is neither locally compact, nor non-archimedean. Let us first recall the following classical theorem about locally compact groups.

**Theorem 21** (van Dantzig). A totally disconnected locally compact group admits a basis at the identity that consists of compact open subgroups.

**Corollary 22.** The group $H(L)$ is not locally compact.

**Proof.** By Proposition 18, $H(L)$ is totally disconnected. If $H(L)$ was also locally compact, then by a theorem of van Dantzig, it would contain a proper open subgroup. This however contradicts Corollary 20.

Recall that a Polish group is *non-archimedean* if it contains a basis of the identity that consists of open subgroups. This class of groups is equal to the class of automorphism groups of countable model-theoretic structures.

**Corollary 23.** The group $H(L)$ is not a non-archimedean group.

**Remark 24.** Theorems in this section are interesting only if we know that $H(L)$ is non-trivial, it means that there is $f \in H(L)$, $f \neq \text{Id}$. To see that this is the case, take any clopen $X \subseteq C$ such that $\pi_1^{-1}(X) \cap L$ and $\pi_1^{-1}(C \setminus X) \cap L$ are non-empty. Then each of $L_1 = \{v\} \cup (\pi_1^{-1}(X) \cap L)$ and $L_2 = \{v\} \cup (\pi_1^{-1}(C \setminus X) \cap L)$ is homeomorphic a Lelek fan. A Lelek fan is unique, so there is a homeomorphism $f_0 : L_1 \to L_2$. This homeomorphism sends $v$ to $v$. The map

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in L_1 \\ f_0^{-1}(x) & \text{if } x \in L_2 \end{cases}$$

is a non-trivial homeomorphism of the Lelek fan $L$.

**3.2. Conjugacy classes of $H(L)$.** In this subsection, we show the following theorem.

**Theorem 25.** The group of all homeomorphisms of the Lelek fan, $H(L)$, has a dense conjugacy class.
This will follow from Theorem 26.

**Theorem 26.** The group of all automorphisms of $\mathbb{L}$, $\text{Aut}(\mathbb{L})$, has a dense conjugacy class.

Let us first see how Theorem 26 implies Theorem 25.

**Proof of Theorem 25.** As observed in Remark 14, the group $\text{Aut}(\mathbb{L})$ is identified with a subgroup of $H(\mathbb{L})$ and its topology is finer than the one inherited from $H(\mathbb{L})$. Therefore it is enough to show that $\text{Aut}(\mathbb{L})$ is a dense subset of $H(\mathbb{L})$. However, this follows from the second part of Theorem 16 (take $X = \mathbb{L}$, an arbitrary $f_1 \in H(\mathbb{L})$, and $f_2 = \text{Id}$).

To show Theorem 26 we use the criterion stated in Proposition 27 below. The proof of this criterion is given in [12] in Theorem A1, and it is an analog of a theorem due to Kechris and Rosendal in [10] in the context of the (injective) Fraïssé theory.

Let $s$ be a binary relation symbol and let $L'$ be the language $\{R, s\}$. We will need a class $\mathcal{G}$ of finite $L'$-structure defined as follows:

$$\mathcal{G} = \{(A, s^A) : A \in F \text{ and there are } \phi : \mathbb{L} \to A \text{ and } f \in \text{Aut}(\mathbb{L}) \text{ such that } \phi : (\mathbb{L}, \text{graph}(f)) \to (A, s^A) \text{ is an epimorphism}\},$$

where $\text{graph}(f)$ is viewed as a closed relation on $\mathbb{L}$: $\text{graph}(f)(x, y)$ if and only if $f(x) = y$.

Recall from Introduction that $\mathcal{G}$ has the joint projection property (the JPP) if and only if for every $(A, s^A), (B, s^B) \in \mathcal{G}$ there is $(C, s^C) \in \mathcal{G}$ and epimorphisms from $(C, s^C)$ onto $(A, s^A)$ and from $(C, s^C)$ onto $(B, s^B)$.

**Proposition 27.** The group $\text{Aut}(\mathbb{L})$ has a dense conjugacy class if and only if $\mathcal{G}$ has the JPP.

In order to show that $\mathcal{G}$ has the JPP, we describe the family $\mathcal{G}$ in more concrete terms in Lemma 28 and Lemma 29.

**Lemma 28.** We have that $(T, s^T) \in \mathcal{G}$ if and only if there is $S \in F$ and there are epimorphisms $p_1 : S \to T$ and $p_2 : S \to T$ such that $s^T = \{ (p_1(s), p_2(s)) : s \in S \}$.

**Proof.** ($\Leftarrow$) Let $S, p_1, p_2$ be as in the hypothesis. Let $\phi : \mathbb{L} \to S$ be any epimorphism (it exists by the universality property). Let $\phi_1 = p_1 \circ \phi$ and let $\phi_2 = p_2 \circ \phi$. Using projective ultrahomogeneity (L3) get $f \in \text{Aut}(\mathbb{L})$ such that $\phi_1 \circ f = \phi_2$. Then $\phi_1 : (\mathbb{L}, \text{graph}(f)) \to (T, s^T)$ is an epimorphism. So $(T, s^T) \in \mathcal{G}$.

($\Rightarrow$) Let $(T, s^T) \in \mathcal{G}$. Let $\psi : (\mathbb{L}, f) \to (T, s^T)$ be an epimorphism. Denote $\phi_1 = \psi$ and $\phi_2 = \phi_1 \circ f$. Let $X$ be the common refinement of the partitions $\phi_1^{-1}(T)$ and $\phi_2^{-1}(T)$. Applying (L2) to $f : \mathbb{L} \to X$ such that $x \in f(x)$, we find a refinement $S \in F$ of $X$ witnessed by an epimorphism $\xi : \mathbb{L} \to S$. Then $p_1 : S \to T$ satisfying $\phi_1 = p_1 \circ \xi$ and $p_2 : S \to T$ satisfying $\phi_2 = p_2 \circ \xi$ are as required.
Every tree in \( \mathcal{F} \) is specified by its height \( k \) and its width \( n \). Recall that all branches of a tree in \( \mathcal{F} \) have the same length. The width of a tree is the number of its branches and, for us, the height of a tree is the number of elements in a branch minus one (we do not count the root). Let \( T \) be a tree of height \( k \) and width \( n \). Recall that if \( b \) is a branch in a tree \( T \) of height \( k \), we denote by \( b(j) \) the \( j \)-th element of \( b \), where \( j = 0, 1, 2, \ldots, k \) (\( b(0) \) is the root).

We say that a relation \( s^T \) on \( T \) is surjective if for every \( t \in T \) there are \( r, s \in T \) such that \( s^T(t, r) \) and \( s^T(s, t) \). Let \( s^T \) be a surjective relation on \( T \). Let \( b_1, b_2, \ldots, b_n \) be the list of all branches of \( T \) and let \( r_T \) be the root of \( T \). It means that \( b_1(0) = b_2(0) = \ldots = b_n(0) = r_T \).

We say that \((x_1, y_1) \in T^2 \) is \( s^T \)-adjacent to \((x_0, y_0) \in T^2 \) if and only if \( s^T(x_0, y_0) \), \( s^T(x_1, y_1) \), and there are \( i, i', j, j', k, k' \) such that \((x_0, y_0) = (b_i(j), b_{i'}(j')) \), \((x_1, y_1) = (b_i(k), b_{i'}(k')) \), and either \( k = j, k' = j' + 1 \), or \( k = j + 1, k' = j' + 1 \). (Note that this relation is not symmetric.) We say that \((c, d) \in T^2 \) is \( s^T \)-connected to \((a, b) \) if and only if there are \((a, b) = (x_0, y_0), (x_1, y_1), \ldots, (c, d) = (x_i, y_i) \) such that for each \( i \), \((x_{i+1}, y_{i+1}) \) is \( s^T \)-adjacent to \((x_i, y_i) \).

**Lemma 29.** \((T, s^T) \in \mathcal{G} \) if and only if \( s^T \) is surjective \( s^T(r_T, r_T) \) and for every \((x, y) \) such that \( s^T(x, y) \), \((x, y) \) is \( s^T \)-connected to \( (r_T, r_T) \).

**Proof.** (\( \Leftarrow \)) We define \( S, p_1, p_2 \) as in Lemma 28. Let \( k \) be the height of \( T \). For every \((x, y) \) such that \( s^T(x, y) \) we pick a chain of length \( 2k + 2 \) and denote it by \( b_{(x,y)} \). Let \( S \) be the disjoint union of all chains \( b_{(x,y)} \) with their roots identified. Now we define \( p_1 \) and \( p_2 \). Fix \((x, y) \) such that \( s^T(x, y) \). Fix a sequence \((r_T, r_T) = (x_0, y_0), (x_1, y_1), \ldots, (x_l, y_l) = (x, y) \) witnessing that \((x, y) \) is \( s^T \)-connected to \((r_T, r_T) \). We let \( p_1(b_{(x,y)}(i)) = x_i \) and \( p_2(b_{(x,y)}(i)) = y_i \), whenever \( i \leq l \). We let \( p_1(b_{(x,y)}(i)) = x \) and \( p_2(b_{(x,y)}(i)) = y \), whenever \( i > l \).

(\( \Rightarrow \)) Let \((T, s^T) \in \mathcal{G} \). Clearly \( s^T(r_T, r_T) \). Take \((x, y) \) such that \( s^T(x, y) \) and \( S, p_1, p_2 \) as in Lemma 28. Let \( S \) be such that \((x, y) = (p_1(s), p_2(s)) \). Let \( b \) be a branch in \( S \) connecting \( r_S \) to \( s \), i.e. \( r_S = s_0 = b(0), s_1 = b(1), \ldots, s_l = b(l) \). Then the sequence \((r_T, r_T) = (p_1(s_0), p_2(s_0)), (p_1(s_1), p_2(s_1)), \ldots, (p_1(s_l), p_2(s_l)) = (x, y) \) witnesses that \((x, y) \) is \( s^T \)-connected to \((r_T, r_T) \).

**Proposition 30.** The family \( \mathcal{G} \) has the JPP.

**Proof.** Let \((T_1, s^{T_1}), (T_2, s^{T_2}) \in \mathcal{G} \). For the JPP, take \( T \) to be the disjoint union of \( T_1 \) and \( T_2 \) with their respective roots identified. For \( x, y \in T \) we let \( s^T(x, y) \) if and only if either \( x, y \in T_1 \) and \( s^{T_1}(x, y) \), or \( x, y \in T_2 \) and \( s^{T_2}(x, y) \). Then, using Lemma 22 we conclude that \((T, s^T) \in \mathcal{G} \). Moreover, \( \phi_1 : (T, s^T) \rightarrow (T_1, s^{T_1}) \) such that \( \phi_1 \upharpoonright T_1 = \text{Id}_{T_1} \) and \( \phi_1 \upharpoonright T_2 \) is mapped to the root, and \( \phi_2 : (T, s^T) \rightarrow (T_2, s^{T_2}) \) such that \( \phi_2 \upharpoonright T_2 = \text{Id}_{T_2} \) and \( \phi_2 \upharpoonright T_1 \) is mapped to the root, are epimorphisms.

3.3. Simplicity of \( H(L) \). Recall that a group is simple if it has no non-trivial proper normal subgroups. Note that this is a stronger notion than being topologically simple, where we require the non-existence of a non-trivial proper closed normal subgroup.
In this subsection, we show that the homeomorphism group of the Lelek fan, \(H(L)\), is simple. In [1], Anderson gave a criterion for a group of homeomorphisms that implies its simplicity. Anderson’s criterion is satisfied for instance by the homeomorphism group of the Cantor set, the homeomorphism group of the universal curve, or by the group of all orientation-preserving homeomorphisms of \(S^2\). A modification of that criterion applies to \(H(L)\).

There are various recent results concerning simplicity of topological groups, see for example [17], for a general result on simplicity of automorphism groups. Recently, Tent and Ziegler [20] showed that the isometry group of the bounded Urysohn space is simple.

Recall that \(E\) denotes the set of endpoints of \(L\), \(v\) the top of \(L\), \(C\) the Cantor set, and \(F\) the Cantor fan. Recall that \(\pi_1\) is the natural projection from \(F \setminus \{v\}\) onto the Cantor set \(C\) and let \(\pi_2\) be the natural projection from \(F\) onto \([0,1]\). Define

\[
K = \{k \subseteq L: \text{ both } k \text{ and } (L \setminus k) \cup \{v\} \text{ are closed and non-empty}\}.
\]

The properties listed below follow immediately from the definition of \(K\).

**Remark 31.**
1. Let \(k \in K\). Then for any \(e \in E\), we have either \([v,e] \subseteq k\) or \([v,e] \subseteq (L \setminus k) \cup \{v\}\).
2. Whenever \(k \in K\) and \(g \in H(L)\), then also \(g(k) \in K\).
3. If \(k \in K\) then \(k \setminus \{v\}\) is an open set in \(L\).
4. If \(k \in K\), then \((L \setminus k) \cup \{v\} \in K\), and if \(k,k' \in K\) are such that \(k \cup k' \neq L\), then \(k \cup k' \in K\).
5. If \(X \subseteq C\) is a clopen set such that \(\pi_1^{-1}(X) \cap L\) and \(\pi_1^{-1}(C \setminus X) \cap L\) are non-empty, then \((\pi_1^{-1}(X) \cap L) \cup \{v\}\) is in \(K\).

Let \(G^0\) denote the subgroup of \(H(L)\) consisting of those \(g \in H(L)\) that are the identity when restricted to some \(k \in K\). We say that \(g \in G^0\) is supported on \(k \in K\) if \(g \upharpoonright (L \setminus k)\) is the identity. For \(k \in K\) let \(E(k)\) denote the set of endpoints of \(k\). Observe that by Remark 31 part (1), \(E \cap k = E(k)\).

**Lemma 32.** The family \(K\) satisfies the following properties:

1. the elements of \(K\) are homeomorphic to \(L\), in particular, they are non-degenerate and homeomorphic to each other;
2. for every \(h \in H(L)\) different from the identity, there is \(k \in K\) such that \(k \cap (h(k) \cup h^{-1}(k)) = \{v\}\);
3. for \(k \in K\), \(L \setminus k \in K\);

**Proof.** (1) Let \(k \in K\). To show that \(k\) is homeomorphic to \(L\), it is enough to show that \(E(k)\) is dense in \(k\). Let \(x \in k \setminus \{v\}\). There is a sequence \((e_i)\) of endpoints of \(L\) that converges to \(x\). By passing to a subsequence, we can assume that either every \(e_i\) is in \(k\), or every \(e_i\) is in \(L \setminus k\). Since \((L \setminus k) \cup \{v\}\) is closed and \(x \neq v\), the latter possibility cannot be true. Therefore, since \(E \cap k = E(k)\), the sequence \((e_i)\) is a sequence of endpoints of \(k\) and it converges to \(x\). The above argument
shows that $E(k)$ is dense in $k \setminus \{v\}$. However, since $k \setminus \{v\} = k$, $E(k)$ is also dense in $k$.

(2) Let $e \in E$ be such that $h([v, e]) \neq [v, e]$, and consequently $h([v, e]) \cap [v, e] = \{v\}$. For the proof that such $e \in E$ exists, see the first paragraph of the proof of Proposition \[18\].

Note that then also $h^{-1}([v, e]) \subseteq [v, e] = \{v\}$.

Let $l_1, l_2, l_3 \in K$ be such that $l_1 \cap l_2 = \{v\}$, $l_1 \cap l_3 = \{v\}$, $e \in l_1$, $h(e) \in l_2$, and $h^{-1}(e) \in l_3$. Let $k = h^{-1}(l_2) \cap l_1 \cap h(l_3)$. Then $k \in K$, $k \subseteq l_1$, $h(k) \subseteq l_2$, and $h^{-1}(k) \subseteq l_3$. Therefore $k \cap (h(k) \cup h^{-1}(k)) = \{v\}$.

(3) By the definition of $K$, $(L \setminus k) \cup \{v\}$ is closed. Since $k$ is closed and $L$ is connected, $L \setminus k$ cannot be closed. Therefore $L \setminus k = (L \setminus k) \cup \{v\} \subseteq K$.

\[\square\]

For $k \in K$, define the \textit{height} of $k$ to be $\max(\pi_2(k))$. We say that a sequence $(k_i)_{i \in \mathbb{Z}}$ of elements of $K$ is a \textit{$\beta$-sequence} if (1) $\bigcup_{i \in \mathbb{Z}} k_i \subseteq K$ and $k_i \cap k_j = \{v\}$ for $i \neq j$, and (2) $\lim_{i \to \infty} \text{ht}(k_i) = 0 = \lim_{i \to -\infty} \text{ht}(k_i)$.

**Lemma 33.** For every $k \in K$ there exist a $\beta$-sequence $(k_i)$ with $\bigcup k_i = k$ and $\rho_1, \rho_2 \in G^0$ supported on $k$ such that

1. $\rho_1(k_i) = k_{i+1}$ for each $i$;
2. $\rho_2 \upharpoonright k_0 = \rho_1 \upharpoonright k_0$, $\rho_2 \upharpoonright k_{2i} = \rho_2 \upharpoonright k_{2i}$ for $i > 0$, and $\rho_2 \upharpoonright k_{2i-1} = \rho_1 \upharpoonright k_{2i-1}$ for $i > 0$;
3. if, for each $i$, $\phi_i \in G^0$ is supported on $k_i$, then there exists $\phi \in G^0$ supported on $k$ such that $\phi \upharpoonright k_i = \phi_i \upharpoonright k_i$ for every $i$;
4. for any $k' \in K$, there exists $\eta \in H(L)$ such that $\eta(k') = k$.

Moreover, $K$ and $H(L)$ satisfy the following conditions

5. for any $k \in K$ and $g \in H(L)$ for which $g(k) \cap k = \{v\}$ while $g(k) \cup k \neq L$, there exists $\lambda \in G^0$, with $\lambda$ supported on $k' = k \cup g(k)$, such that $\lambda \upharpoonright k = g \upharpoonright k$;
6. for any $k_1, k_2, k_3, k_4 \in K$ with $k_1 \cap k_2 = k_3 \cap k_4 = \{v\}$ and $k_1 \cup k_2 \neq L \neq k_3 \cup k_4$, there exists $\mu \in H(L)$ such that $\mu(k_1) = k_3$ and $\mu(k_2) = k_4$.

**Proof.** Given $k \in K$, we first construct a sequence $(k'_i)_{i \in \mathbb{N}}$ of elements of $K$ such that $\bigcup_{i \in \mathbb{N}} k'_i = k$, $k'_i \cap k'_j = \{v\}$ for $i \neq j$, and $\lim_{i \to \infty} \text{ht}(k'_i) = 0$. We do the construction by induction. Fix a metric on the Cantor set $C$ such that $\text{diam}(C) \leq 1$.

To construct $k'_0$, pick $e \in E(k)$ such that $\pi_2(e) < 2^{-1}$. Let $X \subseteq C$ be a clopen such that $\pi_1(e) \in X$ and $\text{ht}(l) < 2^{-1}$, where $l = (\pi_1^{-1}(X) \cap L) \cup \{v\}$. Such $X$ exists by the compactness of $L$. Let $k'_0 = (k \setminus l) \cup \{v\}$. Then $k'_0 \in K$ and $k \setminus k'_0 \neq \emptyset$ since $e \in k \setminus k'_0$. Note that $\text{ht}((k \setminus k'_0) \cup \{v\}) \leq \text{ht}(l) < 2^{-1}$.

Suppose that we have already constructed $k'_0, k'_1, \ldots, k'_n$ such that (a) for every $i \neq j$, $i, j \leq n$, $k'_i \cap k'_j = \{v\}$ and $k \setminus \bigcup_{i \leq n} k'_i \neq \emptyset$, (b) for every $i \leq n$, $\text{ht}(k'_i) < 2^{-i}$ and $\text{ht}((k \setminus \bigcup_{i \leq n} k'_i) \cup \{v\}) < 2^{-(i+1)}$, and (c) for every $i \leq n$, $\text{diam}(\pi_1(k \setminus \bigcup_{j \leq i} k'_j)) < 2^{-(i-1)}$. Then $\text{ht}(k \setminus (k \cap L) \cup \{v\}) < 2^{-n}$ and $\text{diam}(\pi_1(k \setminus (k \cap L) \cup \{v\})) < 2^{-(n-1)}$. Therefore, using the properties (a) and (c), we can find $\rho_1(k) \in G^0$ supported on $k$, such that $\rho_1(k) \upharpoonright k_i = \rho_1 \upharpoonright k_i$ for each $i \leq n$.

Let $k'_0 \upharpoonright k_0 = \rho_1 \upharpoonright k_0$, and $k'_i \upharpoonright k_{2i} = k'_i \upharpoonright k_{2i}$ for $i > 0$, and $k'_i \upharpoonright k_{2i-1} = k'_i \upharpoonright k_{2i-1}$ for $i > 0$.
Now we construct $k'_{n+1}$ such that conditions (a), (b), and (c), with $n$ replaced by $n + 1$, are fulfilled: Pick $e \in E(k \setminus \bigcup_{j \leq n} k'_j)$ such that $\pi_2(e) < 2^{-(n+2)}$ (note that $(k \setminus \bigcup_{j \leq n} k'_j) \cup \{v\} \in K$). Let $X \subseteq C$ be a clopen such that $\pi_1(e) \in X$ and $\text{ht}(l) < 2^{-(n+2)}$, where $l = (\pi_1^{-1}(X) \cap L) \cup \{v\}$. By shrinking $l$ if necessary, we can assume that $(l \cap k) \cup \{k \setminus \bigcup_{j \leq n} k'_j \neq k$ and $\text{diam}(\pi_1(l \setminus \{v\})) < 2^{-n}$. Let $k'_{n+1} = (k \setminus (\bigcup_{j \leq n} k'_j \cup l)) \cup \{v\}$. Then $k'_{n+1} \in K$ is as required. In particular, $k \setminus \bigcup_{j \leq n+1} k'_j \neq \emptyset$, $\text{ht}((k \setminus \bigcup_{j \leq n+1} k'_j) \cup \{v\}) \leq \text{ht}(l) < 2^{-(n+2)}$ and $\text{diam}(\pi_1(k \setminus \bigcup_{j \leq n+1} k'_j)) \leq \text{diam}(\pi_1(l \setminus \{v\})) < 2^{-n}$.

The sequence $(k_i)$ such that $k_0 = k'_0$, $k_{-i} = k'_{2i}$ for $i = 1, 2, \ldots$, and $k_i = k'_{2i-1}$ for $i = 1, 2, \ldots$, is a $\beta$-sequence satisfying $\bigcup_{i \in \mathbb{N}} k_i = k$.

Let us first show that (3) holds. Let $\phi_i$ be as in the assumptions. Let $\phi$ be such that $\phi \upharpoonright k_i = \phi_i \upharpoonright k_i$ and let $\phi$ be equal to the identity outside $k$. We want to show that $\phi$ is a homeomorphism. Clearly $\phi$ is a bijection. Since $L$ is compact, it is enough to show that $\phi$ is continuous. Let $x \in L$. We show that $\phi$ is continuous at $x$. If $x \neq v$, then $x \in k_i \setminus \{v\}$ for some $i$ or $x \in L \setminus k$. Since each of $k_i \setminus \{v\}$ and $L \setminus k$ is open, whenever $(x_n)$ converges to $x$, then eventually $x_n \in k_i \setminus \{v\}$ for some $i$ or $x_n \in L \setminus k$, respectively. Therefore, eventually also $\phi(x_n) \in k_i \setminus \{v\}$ for some $i$ or $\phi(x_n) \in L \setminus k$, respectively. Since each $\phi_i$ is continuous, $\phi(x_n)$ converges to $\phi(x)$. Now let $x = v$ and let $(x_n)$ converge to $v$. We show that $\phi(x_n)$ converges to $v = \phi(v)$. Fix an open neighbourhood $U$ of $v$. Since $\text{ht}(k_i) \rightarrow 0$ both for $i \rightarrow \infty$ and for $i \rightarrow -\infty$, we can find $i_0 > 0$ be such that when $i > i_0$ or $i < -i_0$, then $k \subseteq U$. By continuity of $\phi_i$, find $n_0$ be such that whenever $n > n_0$ and $x_n$ is in one of $k_i$, $-i_0 \leq i \leq i_0$, or in $L \setminus k$, then $\phi(x_n) = \phi_i(x_n) \in U$, or $\phi(x_n) = x_n \in U$ respectively. Then in fact, whenever $n > n_0$, we have $\phi(x_n) \in U$. This shows continuity of $\phi$ at $v$.

To show (1) we let $\rho_i^1 : k_i \rightarrow k_{i+1}$ to be any homeomorphism. Such a homeomorphism exists by the uniqueness of the Lelek fan. Let $\rho_1$ be such that $\rho_1 \upharpoonright k_i = \rho_i^1$, $i \in \mathbb{Z}$, and $\rho_1$ is the identity outside $k$. Then reasoning similarly as in the proof of (3), we show that $\rho_1$ is a homeomorphism of $L$.

Having defined $\rho_1$, we define $\rho_2$ on each $k_i$, $i \geq 0$, as in (2), and let $\rho_2$ be the identity otherwise. Then reasoning similarly as in the proof of (3), we show that $\rho_2$ is a homeomorphism of $L$.

Parts (4), (5), and (6) are straightforward to show. In the proof we use the definition of $K$ and the uniqueness of the Lelek fan.

\[\square\]

Remark 34. Anderson in \cite{Anderson} showed that whenever $G$ is a group of homeomorphisms of a space $X$, and there exists a family of closed sets $K$ that satisfies conditions similar to those given in Lemmas \ref{lem32} and \ref{lem33}, then $G$ is a simple group. He assumes that sets $(k_i)$ in the definition of a $\beta$-sequence are disjoint, and that for every open non-empty set $U \subseteq X$ there exists $k \in K$ such that $k \subseteq U$. The latter condition is false in our situation. Nevertheless, it is enough to substitute it by the condition (2) of Lemma \ref{lem32}.
Other than that, our definition of a $\beta$-sequence and conditions (5) and (6) of Lemma 33 are stated in a slightly weaker form than needed in [1].

**Theorem 35.** The group of all homeomorphisms of the Lelek fan, $H(L)$, is simple.

Having Lemmas 32 and 33, the proof of Theorem 35 will go along the lines of the proof of simplicity of certain homeomorphism groups, presented in [1]. We sketch here the proof of Theorem 35 for the reader’s convenience, and for the details we refer the reader to [1].

We will need the following lemma (analogous to Theorem I in [1]).

**Lemma 36.** Let $h \in H(L)$ be different from the identity. Then every $g \in G^0$ is the product of four conjugates of $h$ and $h^{-1}$ (appearing alternately).

**Proof.** Since every two elements of $K$ are homeomorphic via a homeomorphism of $L$ (Lemma 33, condition (4)), it is enough to show that there exists $k_0 \in K$ such that for any $g_0 \in G^0$ supported on $k_0$, $g_0$ is the product of four conjugates of $h$ and $h^{-1}$.

Let $k \in K$ be such that $k \cap (h(k) \cup h^{-1}(k)) = \{v\}$. Let $(k_i)$ be a $\beta$-sequence such that $\bigcup_i k_i = k$ and let $\rho_1$ and $\rho_2$ be as in (1) and (2) of Lemma 33. We show that $k_0$ is as required. Let $g_0 \in G^0$ be supported on $k_0$. For $i \geq 0$, let $\phi_i = \rho_i^k g_0 \rho_i^{-1}$ and let $\phi_i$ be the identity on $k_i$, when $i < 0$. Take $\phi$ as in (3) of Lemma 33. Take $f = h^{-1} \phi^{-1} h \phi$. Note that $f$ is supported on $k \cup h^{-1}(k)$ and $f \upharpoonright k = \phi \upharpoonright k$, and $f \upharpoonright (h^{-1}(k)) = h^{-1} \phi^{-1} h \phi$. Take $\rho = h^{-1} \rho_2^k h \rho_1^{-1}$. Note that $\rho$ is supported on $k \cup h^{-1}(k)$ and $\rho \upharpoonright k = \rho_1^{-1} \upharpoonright k$, and $\rho \upharpoonright (h^{-1}(k)) = h^{-1} \rho_2 h \upharpoonright (h^{-1}(k))$. Let $w = \rho^{-1} f^{-1} \rho f$. Then $w = (\rho^{-1} f^{-1} \phi^{-1} h \phi)(\rho^{-1} h \rho)(h^{-1}) (\phi^{-1} h \phi)$, therefore it is a product of four conjugates of $h$ and $h^{-1}$. Unraveling definitions of $\phi$, $f$, $\rho$, and $w$, we can check that $w = g_0$.

**Proof of Theorem 35.** Let $g \in H(L)$ and let $h \in H(L)$, $h \neq \text{Id}$. We show that $g$ is the product of 8 conjugates of $h$ and $h^{-1}$. This will immediately imply that $H(L)$ is simple.

Let $k \in K$ be such that $g(k) \cap k = \{v\}$ and $g(k) \cup k \neq L$. Take $\alpha \in H(L)$ such that $\alpha \upharpoonright k = g \upharpoonright k$, $\alpha \upharpoonright g(k) : g(k) \to k$ is a homeomorphism, and $\alpha$ is equal to the identity outside $g(k) \cup k$. Notice that $\alpha, (\alpha^{-1} g) \in G^0$ and $g = \alpha (\alpha^{-1} g)$. By Lemma 36, $g$ is the product of 8 conjugates of $h$ and $h^{-1}$.

**Remark 37.** As in [1], one can modify the proof of Theorem 35 to show that whenever $g \in H(L)$ and $h \in H(L)$, $h \neq \text{Id}$, then $g$ is the product of 6 conjugates of $h$ and $h^{-1}$.

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