A NOTE ON CONNECTED COFILTERED COALGEBRAS, CONILPOTENT COALGEBRAS AND HOPF ALGEBRAS

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ABSTRACT. This note gives a proof that a connected coaugmented cofiltered coalgebra is a conilpotent coalgebra and thus a connected coaugmented cofiltered bialgebra is a Hopf algebra. This applies in particular to a connected coaugmented cograded coalgebra and a connected coaugmented cograded bialgebra.

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1. Introduction

In this note we revisit the notion of a connected filtered coalgebra, and the fact that a connected filtered bialgebras is a Hopf algebra. These notion and result and their graded variation is fundamental in the study of a large number of Hopf algebras, including the Hopf algebra of Feynman graphs in the Connes-Kreimer approach to renormalization of perturbative quantum field theory \([\cite{1}].\) It is one of the most used methods to obtain Hopf algebras in combinatorics and algebra and is included in several references. There are variations in the precise meaning of the notion of connected filtered coalgebra (see for example \([\cite{2}]\)) and how the connectedness implies the Hopf property in the literature, such as \([\cite{3}, \cite{4}, \cite{5}, \cite{6}]\). As noted in \([\cite{7}]\), the graded and filtered properties of a coalgebra should include their compatibility with the counit, as well as the coproduct. Further, what works for connected graded bialgebras becomes quite subtle for connected filtered bialgebras. Moreover, there is a closely related concept called conilpotency of a bialgebra which also implies the Hopf algebra property \([\cite{8}, 1.2.4;1.3.4]\).

By elucidating these points in this note, we present a proof that a coaugmented connected filtered coalgebra is conilpotent. Hence a connected filtered bialgebra is a connected conilpotent bialgebra and hence is a Hopf algebra.

Convention Throughout this paper, algebras are associative unitary algebras, and algebras and coalgebras are taken to be over a commutative unitary ring \(k\), as are the linear maps and tensor products.

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2. FROM COFILTERED COALGEBRAS TO CONILPOTENT COALGEBRAS

In this section, we show that a connected coaugmented cofiltered coalgebra is a conilpotent coalgebra.

We start with some basic concepts on coalgebras. For further details, see standard references such as [1] [2] [3]. We include the notations $\beta_\ell, \beta_r$ [4] for later applications.

**Definition 2.1.** A coalgebra is a triple $(C, \Delta, \varepsilon)$ consisting of a $k$-module $C$, and linear maps $\Delta : C \to C \otimes C$, called the coproduct, and $\varepsilon : C \to k$, called the counit, that make the following diagrams commute:

(1) (Coassociativity) $\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ C \otimes C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes C \otimes C \end{array}$

(2) (Counicity) $\begin{array}{ccc} k \otimes C & \xrightarrow{\varepsilon \otimes \text{id}} & C \otimes C \\ \beta_\ell & \xleftarrow{\beta_r} & C \otimes k \\ \Delta \downarrow & & \Delta \downarrow \end{array}$

Here $\beta_\ell$ and $\beta_r$ are linear isomorphisms given by

$\beta_\ell : C \to k \otimes C, \quad c \mapsto 1_k \otimes c,$

$\beta_r : C \to C \otimes k, \quad c \mapsto c \otimes 1_k, \quad k \in k, c \in C.$

Using Sweedler's notations, we have

$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$

By iteration, we define

$\Delta^k(x) := (\text{id} \otimes \Delta^{k-1})\Delta(x), \quad k \geq 2$

and denote

$\Delta^k(x) = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(k+1)} \in C^{\otimes (k+1)}.$

**Definition 2.2.** A coalgebra $(C, \Delta, \varepsilon)$ is called coaugmented if there is a linear map $u : k \to C$, called the coaugmentation, such that $\varepsilon \cdot u = \text{id}_k$.

**Lemma 2.3.** Let $(C, \Delta, \varepsilon, u)$ be a coaugmented coalgebra. Then

$C = \text{im}(u \varepsilon) \oplus \ker(u \varepsilon) = \text{im} u \oplus \ker \varepsilon.$

**Proof.** It follows from $\varepsilon u = \text{id}_k$ that $\varepsilon$ is surjective, $u$ is injective and $u \varepsilon$ is idempotent. Then idempotency gives the linear decomposition

$C = \text{im}(u \varepsilon) \oplus \ker(u \varepsilon).$

Further $\text{im}(u \varepsilon) = \text{im} u$ since $\varepsilon$ is surjective and $\ker(u \varepsilon) = \ker \varepsilon$ since $u$ is injective. Therefore, $C = \text{im} u \oplus \ker \varepsilon.$
Lemma 2.4. Let \((C, \Delta, \varepsilon, u)\) be a coaugmented coalgebra. Then for \(x \in \ker \varepsilon\),
\[
\Delta(x) = x \otimes u(1_k) + u(1_k) \otimes x + \sum_{(x)} x' \otimes x'',
\]
where \(x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon\).

Proof. By Lemma 2.3, we have
\[
C \otimes C = (\im u \otimes \im u) \oplus (\im \varepsilon \otimes \im u) \oplus (\ker \varepsilon \otimes \im u) \oplus (\ker \varepsilon \otimes \ker \varepsilon).
\]
Now \(\im u = k \cdot u(1_k)\) and so we can write
\[
\Delta(x) = u(k) \otimes u(n) + y \otimes u(1_k) + u(1_k) \otimes z + \sum_{(x)} x' \otimes x'',
\]
where \(k, n \in k, y, z \in \ker \varepsilon\) and \(x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon\).

By the commutativity of the left triangle of the counicity property in Eq. (4), we obtain
\[
x = \beta^{-1}_E(\varepsilon \otimes \id_C)\Delta(x)
\]
\[
= \beta^{-1}_E(\varepsilon \otimes \id_C)((u(k) \otimes u(n) + y \otimes u(1_k) + u(1_k) \otimes z + \sum_{(x)} x' \otimes x''))
\]
\[
= \beta^{-1}_E(\varepsilon(u(k)) \otimes u(n) + 1_k \otimes z)
\]
\[
= \beta^{-1}_E(1_k \otimes ku(n) + 1_k \otimes z)
\]
\[
= ku(n) + z,
\]
whence \(x - z = ku(n) = u(kn)\). Since \(x, z \in \ker \varepsilon, u(kn) \in \im u\) and \(\im u \cap \ker \varepsilon = 0\), we have \(x - z = 0 = u(kn)\). Hence \(x = z\) and \(u(kn) = 0\). Then
\[
u(k) \otimes u(n) = u(k) \otimes nu(1_k) = u(k)n \otimes u(1_k) = u(kn) \otimes u(1_k) = 0.
\]
Similarly, by the commutativity of the right triangle in the counicity property, we get \(x = y\). Thus
\[
\Delta(x) = x \otimes u(1_k) + u(1_k) \otimes x + \sum_{(x)} x' \otimes x'',
\]
where \(x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon\).

Let \((C, \Delta, \varepsilon, u)\) be a coaugmented coalgebra. By Lemma 2.3, it makes sense to define the reduced coproduct
\[
\bar{\Delta} : \ker \varepsilon \to \ker \varepsilon \otimes \ker \varepsilon, \quad x \mapsto \Delta(x) - x \otimes u(1_k) - u(1_k) \otimes x \quad \text{for all } x \in C.
\]

For the remainder of the paper, we use the shorthand notations
\[
\bar{\Delta}(x) = \sum_{(x)} x' \otimes x'' = \sum_{(x)} x^{(1)} \otimes x^{(2)}.
\]

In general, define \(\bar{\Delta}^k := (\id \otimes \bar{\Delta}^{k-1})\bar{\Delta}\) for \(k \geq 2\). Then
\[
\bar{\Delta}^n(x) = \sum_{(x)} x^{(1)} \otimes \cdots \otimes x^{(n+1)}.
\]

Now we introduce two kinds of coalgebras, namely cograded coalgebras and cofiltered coalgebras by adapting the usual notions from [3, 4].
Definition 2.5. (a) A coalgebra \((C, \Delta, \varepsilon)\) is called **cograded** if there is a grading \(C = \bigoplus_{n \geq 0} C^{(n)}\) of \(k\)-modules that is compatible with the coproduct \(\Delta\) and the counit \(\varepsilon\) in the sense that
\[
\Delta(C^{(n)}) \subseteq \bigoplus_{p+q=n} C^{(p)} \otimes C^{(q)} \quad \text{for } n \geq 0,
\]
\[\ker \varepsilon = \bigoplus_{n \geq 1} C^{(n)}.\]

Elements \(x \in C^{(n)}\) are said to have degree \(n\), denoted by \(\deg(x) = n\).

(b) A coaugmented cograded coalgebra \((C, \Delta, \varepsilon, u)\) is called **connected** if \(C^{(0)} = \im u\) (hence \(C = \im u \oplus (\bigoplus_{n \geq 1} C^{(n)})\)).

Remark 2.6. As noted in [3], it is not true that \(C = C^{(0)} \oplus \ker \varepsilon\) and \(C = C^{(0)} \oplus (\bigoplus_{n \geq 1} C^{(n)})\) imply \(\ker \varepsilon = \bigoplus_{n \geq 1} C^{(n)}\), as was often taken for granted in the literature. This is simply because \(C = A \oplus B = A \oplus D\) does not imply \(B = D\). So the condition \(\bigoplus_{n \geq 1} C^{(n)} \subseteq \ker \varepsilon\) is added in the definition of a cograded coalgebra.

Lemma 2.7. For a connected coaugmented cograded coalgebra \((C, \Delta, \varepsilon, u)\), we have \(\ker \varepsilon = \bigoplus_{n \geq 1} C^{(n)}\).

Proof. Since \(\bigoplus_{n \geq 1} C^{(n)} \subseteq \ker \varepsilon\), by the modular law, we have
\[
\ker \varepsilon = \ker \varepsilon \cap (\im u \oplus (\bigoplus_{n \geq 1} C^{(n)})) = (\ker \varepsilon \cap \im u) \oplus (\bigoplus_{n \geq 1} C^{(n)}) = \bigoplus_{n \geq 1} C^{(n)}.
\]

As a more general concept, we have

Definition 2.8. (a) A coaugmented coalgebra \((C, \Delta, \varepsilon, u)\) is called **cofiltered** if there are \(k\)-submodules \(C^n, n \geq 0\), of \(C\) such that
(i) \(C = \bigcup_{n=0}^{\infty} C^n\);
(ii) \(C^n \subseteq C^{n+1}\) for \(n \geq 0\);
(iii) \(\Delta(C^n) \subseteq \bigoplus_{p+q=n} C^p \otimes C^q\) for all \(n \geq 0\);
(iv) \(C^n = \im u \oplus (C^n \cap \ker \varepsilon)\).

Elements \(x \in C^n \setminus C^{n-1}\) are said to have degree \(n\), denoted by \(\deg(x) = n\).

(b) A coaugmented cofiltered coalgebra \((C, \Delta, \varepsilon, u)\) is called **connected** if \(C^0 = \im u\).

Remark 2.9. A (connected) cograded coalgebra is a (connected) cofiltered coalgebra with the filtration defined by
\[
C^n = \bigoplus_{k \leq n} C^{(k)} \quad \text{for } n \geq 0.
\]

So we will mostly focus on cofiltered coalgebras.

Under the connected cofiltration condition, Lemma 2.4 can be strengthened:

Lemma 2.10. Let \((C, \Delta, \varepsilon, u)\) be a connected coaugmented cofiltered coalgebra. Then for \(x \in \ker \varepsilon\),
\[
\Delta(x) = x \otimes u(1_k) + u(1_k) \otimes x + \sum_{(x)} x' \otimes x'',
\]
where \(x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon\) and \(0 < \deg(x') < \deg(x), 0 < \deg(x'') < \deg(x)\).
Proof. By Lemma 2.4, for \( x \in C \) we have
\[
\Delta(x) = x \otimes u(1_k) + u(1_k) \otimes x + \sum_{(x)} x' \otimes x'',
\]
where \( x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon \).

Let \( x \in C \) with \( \deg(x) = n \geq 1 \). Since \( x', x'' \in \ker \varepsilon \), we have \( x', x'' \notin C^0 \) by \( C^0 \cap \ker \varepsilon = 0 \) and so \( \deg(x'), \deg(x'') > 0 \). By Definition 2.8 (4), we get
\[
x' \in C^p \text{ and } x'' \in C^q \text{ with } p + q = n = \deg(x).
\]
Using the definition of degree in Definition 2.8, we obtain
\[
\deg(x') + \deg(x'') \leq p + q = \deg(x) \text{ and so } \deg(x'), \deg(x'') < \deg(x)
\]
by \( \deg(x'), \deg(x'') > 0 \). This completes the proof. \( \square \)

We next relate these concepts to the conilpotency of coalgebras [7, Section 1.2.4]. Compare with the related notion in [3, § 5.2].

Definition 2.11. Let \((C, \Delta, \varepsilon, u)\) be a coaugmented coalgebra.

(a) The **coradical filtration** on \( C \) is defined by
\[
F_0 C := \text{im } u \text{ and } F_n C := \text{im } u \oplus \{ x \in \ker \varepsilon \mid \Delta^k(x) = 0 \text{ for } k \geq n \} \text{ for } n \geq 1.
\]

(b) \( C \) is said to be **conilpotent** if the filtration is exhaustive, that is \( C = \bigcup_{n \geq 0} F_n C \). Equivalently, \( \ker \varepsilon = \bigcup_{n \geq 1} \ker \Delta^n \).

Theorem 2.12. Let \((C, \Delta, \varepsilon, u)\) be a connected coaugmented cofiltered coalgebra. Then \( C \) is conilpotent.

Proof. Let the filtration of \( C \) be given by \( C = \bigcup_{n \geq 0} C^n \). We proceed to prove \( \Delta^k(C^n) = 0 \) for all \( k \geq n \geq 1 \) by induction on \( n \). When \( n = 1 \), we have \( \Delta(x) = 0 \) for all \( x \in C^1 \) by Lemma 2.4. When \( n \geq 2 \), let \( x \in C^n \). Then \( \deg(x) \leq n \). By Lemma 2.11,
\[
\Delta(x) = \sum_{(x)} x' \otimes x'',
\]
where \( 0 < \deg(x'), \deg(x'') < \deg(x) \leq n \). So \( x', x'' \in C^{n-1} \). By the induction hypothesis, we have
\[
\Delta^{k-1}(x'') = 0 \text{ for all } k - 1 \geq n - 1,
\]
and so by Eq. (6),
\[
\Delta^{k}(x) := (\text{id} \otimes \Delta^{k-1})\Delta(x) = \sum_{(x)} x' \otimes \Delta^{k-1}(x'') = 0 \text{ for all } k \geq n.
\]
This completes the proof. \( \square \)

Since every cograded coalgebra naturally gives a cofiltration, we obtain

Corollary 2.13. Let \((C, \Delta, \varepsilon, u)\) be a connected coaugmented cograded coalgebra. Then

(a) for \( x \in \ker \varepsilon \), we have
\[
\Delta(x) = x \otimes u(1_k) + u(1_k) \otimes x + \sum_{(x)} x' \otimes x'',
\]
where \( x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon \) and \( 0 < \deg(x'), \deg(x'') < \deg(x) \).

(b) \( C \) is conilpotent.

Proof. It follows from Lemma 2.11, Theorem 2.12 and Remark 2.9. \( \square \)
3. From connected bialgebras to Hopf algebras

In this section, we prove that connected cograded bialgebras and connected cofiltered bialgebras are Hopf algebras.

**Definition 3.1.** A bialgebra is a quintuple \((H, m, u, \Delta, \varepsilon)\) where \((H, m, u)\) is an algebra and \((H, \Delta, \varepsilon)\) is a coalgebra such that \(m : H \otimes H \rightarrow H\) and \(u : k \rightarrow H\) are morphisms of coalgebras.

**Remark 3.2.** The conditions that \(m\) and \(u\) are morphisms of coalgebras can be equivalent replaced by the conditions that \(\Delta : H \rightarrow H \otimes H\) and \(\varepsilon : H \rightarrow k\) are morphisms of algebras.

Since \(H\) is a bialgebra, we have \(\varepsilon u = \text{id}_k\). Hence the unit \(u : k \rightarrow H\) gives a coaugmentation of the coalgebra \((H, \Delta, \varepsilon)\). Thus Lemma 2.3 applies.

For an algebra \(A\) and a coalgebra \(C\), the convolution of two linear maps \(f, g\) in \(\text{Hom}(C, A)\) is defined to be the map \(f * g\) given by the composition

\[
C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A.
\]

In other words,

\[
(f * g)(a) = \sum_{(a)} f(a_{(1)}) g(a_{(2)}) \quad \text{for} \ a \in A.
\]

**Definition 3.3.** Let \((H, m, u, \Delta, \varepsilon)\) be a bialgebra. A linear endomorphism \(S\) of \(H\) is called an antipode for \(H\) if it is the inverse of \(\text{id}_H\) under the convolution product:

\[
S * \text{id}_H = \text{id}_H * S = u \varepsilon.
\]

A Hopf algebra is a bialgebra \(H\) with an antipode \(S\).

The following is the main theorem of this paper. Note that the grading and filtration are required to be compatible with the coproduct \(\Delta\) and the counit \(\varepsilon\), not with the product \(m\) or the unit \(u\). So the result can be applied to a bialgebra with a grading or filtration which is not necessarily compatible with the product.

**Theorem 3.4.** Let \((H, m, u, \Delta, \varepsilon)\) be a bialgebra such that \((H, \Delta, \varepsilon, u)\) is a connected coaugmented cofiltered (in particular cograded) coalgebra. Then \(H\) is a Hopf algebra and the antipode \(S\) is given by

\[
S(1_H) = 1_H \quad \text{and} \quad S(x) = -x + \sum_{n \geq 1} (-1)^{n+1} m^n \tilde{\Delta}^n(x) \quad \text{for} \ x \in \ker \varepsilon.
\]

**Proof.** Observe from [7, p. 20] that a conilpotent bialgebra automatically has an antipode \(S\) given by Eq. (8). Then the result follows from Theorem 2.12 and Corollary 2.13. □

Recall our aforementioned notations for \(\tilde{\Delta}^n(x)\) in Eqs. (5) and (6).

**Remark 3.5.** The formula for the antipode in Eq. (8) can be restated as

\[
S(1_H) = 1_H \quad \text{and} \quad S(x) = -x + \sum_{n \geq 1} \sum_{(x)} (-1)^{n+1} x^{(1)} \cdots x^{(n+1)} \quad \text{for} \ x \in \ker \varepsilon,
\]

which coincides with the following recursive formula on degree:

\[
S(1_H) = 1_H \quad \text{and} \quad S(x) = -x - \sum_{(x)} S(x') x'' = -x - \sum_{(x)} x' S(x'') \quad \text{for} \ x \in \ker \varepsilon.
\]
Here $x' \otimes x'' \in \ker \varepsilon \otimes \ker \varepsilon$ and $0 < \deg(x'), \deg(x'') < \deg(x)$ by Corollary 2.13 (resp. Lemma 2.10). This coincidence can be seen by induction on degree. Obviously, they agree on the initial step, that is $S(1_H) = 1_H$. For the induction step,

$$\begin{align*}
-x - \sum_{(x)} S(x')x'' &= -x - \sum_{(x)} \left( -x' + \sum_{n \geq 1} \sum_{(x')} (-1)^{n+1} x^{(1)} \cdots x^{(n+1)} \right) x'' \\
&= -x + \sum_{(x)} \left( (-1)^2 x' x'' + \sum_{n \geq 1} \sum_{(x')} (-1)^{n+2} x^{(1)} \cdots x^{(n+2)} \right) \\
&= -x + \sum_{(x)} \left( (-1)^2 x_{(1)} x_{(2)} + \sum_{n \geq 2} (-1)^{n+1} x^{(1)} \cdots x^{(n+1)} \right) \\
&= -x + \sum_{n \geq 1} \sum_{(x)} (-1)^{n+1} x^{(1)} \cdots x^{(n+1)} \\
&= -x + \sum_{n \geq 1} \sum_{(x)} (-1)^{n+1} x^{(1)} \cdots x^{(n+1)},
\end{align*}$$

as required. Similarly, we can show the case of $S(x) = -x - \sum_{(x)} x'S(x'')$.

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**References**

[1] E. Abe, Hopf Algebras, Cambridge University Press, 1980.

[2] A. Connes, D. Kreimer, Hopf algebras, renormalization and non-commutative geometry, *Comm. Math. Phys.* 199 (1998), 203-242.

[3] D. Grinberg, Errata to Dominique Manchon: Hopf algebras, from basics to applications to renormalization, version v2 (2006), http://www.cip.ifi.lmu.de/~grinberg/algebra/manchon-errata-update.pdf.

[4] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.

[5] L. Guo, S. Paycha and B. Zhang, Algebraic Birkhoff factorization and the Euler-Maclaurin formula on cones, *Duke Math. J.* 166 (2017), 537-571.

[6] M. Hazewinkel, N. M. Gubareni and V. V. Kirichenko, Algebras, Rings, and Modules: Lie Algebras and Hopf Algebras, AMS Press, 2010.

[7] J. L. Loday, B. Vallette, Algebraic Operads, *Grundlehren Math. Wiss.* 346, Springer, Heidelberg, 2012.

[8] D. Manchon, Hopf algebras in renormalisation, *Handbook of algebra*, Vol. 5, 365–427, Handb. Algebr., 5, Elsevier/North-Holland, Amsterdam, 2008.

[9] S. Montgomery, Hopf Algebras and Their Actions on Rings, *CBMS Regional Conference Series in Mathematics* 82 (1993) 238 pp.

[10] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[11] T. J. Zhang, X. Gao and L. Guo, Hopf algebras of rooted forests, cocyles, and free Rota-Baxter algebras, *J. Math. Phys.* 57 (2016).
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