TOTAL CURVATURE OF GRAPHS
AFTER MILNOR AND EULER

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We define a new notion of total curvature, called net total curvature, for finite graphs embedded in $\mathbb{R}^n$, and investigate its properties. Two guiding principles are given by Milnor’s way of measuring using a local Crofton-type formula, and by considering the double cover of a given graph as an Eulerian circuit. The strength of combining these ideas in defining the curvature functional is that it allows us to interpret the singular/noneuclidean behavior at the vertices of the graph as a superposition of vertices of a 1-dimensional manifold, so that one can compute the total curvature for a wide range of graphs by contrasting local and global properties of the graph utilizing the integral geometric representation of the curvature. A collection of results on upper/lower bounds of the total curvature on isotopy/homeomorphism classes of embeddings is presented, which in turn demonstrates the effectiveness of net total curvature as a new functional measuring complexity of spatial graphs in differential-geometric terms.

1. Introduction: curvature of a graph

The celebrated Fáry–Milnor theorem states that a curve in $\mathbb{R}^n$ of total curvature at most $4\pi$ is unknotted.

As a key step in his proof, John Milnor [1950] showed that for a smooth Jordan curve $\Gamma$ in $\mathbb{R}^3$, the total curvature equals half the integral over $e \in S^2$ of the number $\mu(e)$ of local maxima of the linear height function $\langle e, \cdot \rangle$ along $\Gamma$. This equality can be regarded as a Crofton-type representation formula of total curvature where the order of integrations over the curve and the unit tangent sphere (the space of directions) are reversed. The Fáry–Milnor theorem follows, since total curvature less than $4\pi$ implies there is a unit vector $e_0 \in S^2$ so that $\langle e_0, \cdot \rangle$ has a unique local maximum, and therefore that this linear function is increasing on an interval of $\Gamma$ and decreasing on the complement. Without changing the pointwise value of this height function, $\Gamma$ can be topologically untwisted to a standard embedding of $S^1$
into $\mathbb{R}^3$. The Fenchel theorem, that any curve in $\mathbb{R}^3$ has total curvature at least $2\pi$, also follows from Milnor’s key step, since for all $e \in S^2$, the linear function $\langle e, \cdot \rangle$ assumes its maximum somewhere along $\Gamma$, implying $\mu(e) \geq 1$. Milnor’s proof is independent of the proof of Istvan Fáry, published earlier [1949], which takes a different approach.

We would like to extend the methods of Milnor’s seminal paper, replacing the simple closed curve by a finite graph $\Gamma$ in $\mathbb{R}^3$. $\Gamma$ consists of a finite number of points, the vertices, and a finite number of simple arcs, the edges, each of which has as its endpoints one or two of the vertices. We shall assume $\Gamma$ is connected. The degree of a vertex $q$ is the number $d(q)$ of edges which have $q$ as an endpoint. (Another word for degree is “valence”.) We remark that it is technically not needed that the dimension $n$ of the ambient space equals three. All the arguments can be generalized to higher dimensions, although in higher dimensions ($n \geq 4$) there are no nontrivial knots, and any two homeomorphic graphs are isotopic.

The key idea in generalizing total curvature for curves to total curvature for graphs is to consider the Euler circuits, namely, parametrizations by $S^1$, of the double cover of the graph. We note that given a graph of even degree, there can be several Euler circuits, or ways to trace it without lifting the pen. A topological vertex of a graph of degree $d$ is a singularity, in that the graph is not locally Euclidean. However by considering an Euler circuit of the double of the graph, the vertex becomes locally the intersection point of $d$ paths. We will show (Corollary 3.7) that at the vertex, each path through it has a (signed) measure-valued curvature, and the absolute value of the sum of those measures is well-defined, independent of the choice of the Euler circuit of the double cover. We define (Definition 2.1) the net total curvature (NTC) of a piecewise $C^2$ graph to be the sum of the total curvature of the smooth arcs and the contributions from the vertices as described.

This notion of net total curvature is substantially different from the total curvature, denoted TC, as defined by Taniyama [1998]. (Taniyama writes $\tau$ for TC.) See Section 2 below.

This is consistent with known results for the vertices of degree $d = 2$; with vertices of degree three or more, this definition helps facilitate a new Crofton-type representation formula (Theorem 3.13) for total curvature of graphs, where the total curvature is represented as an integral over the unit sphere. Recall that the vertex is now seen as $d$ distinct points on an Euler circuit. The way we pick up the contribution of the total curvature at the vertices identifies the $d$ distinct points, and thus the $2d$ unit tangent spheres on a circuit. As Crofton’s formula in effect reverses the order of integrations — one over the circuit, the other over the space of tangent directions — the sum of the $d$ exterior angles at the vertex is incorporated in the integral over the unit sphere. On the other hand the integrand of the integral over the unit sphere counts the number of net local maxima of the height function.
along an axis, where net local maximum means the number of local maxima minus the number of local minima at these \( d \) points of the Euler circuit. This establishes a correspondence between the differential geometric quantity (net total curvature) and the differential topological quantity (average number of maxima) of the graph, as stated in Theorem 3.13 below.

In Section 2, we compare several definitions for total curvature of graphs which have appeared in the recent literature. In Section 3, we introduce the main tool (Lemma 3.5) which in a sense reduces the computation of NTC to counting intersections with planes.

Milnor’s treatment [1950] of total curvature also contained an important topological extension. Namely, in order to define total curvature, the curve needs only to be continuous. This makes the total curvature a geometric quantity defined on any homeomorphic image of \( S^1 \). In this article, we first define net total curvature (Definition 2.1) on piecewise \( C^2 \) graphs, and then extend the definition to continuous graphs (Definition 2.3). In analogy to Milnor, we approximate a given continuous graph by a sequence of polygonal graphs. In showing the monotonicity of the total curvature (Proposition 4.1) under the refining process of approximating graphs we use our representation formula (Theorem 3.13) applied to the polygonal graphs.

Consequently the Crofton-type representation formula is also extended to cover continuous graphs (Theorem 4.9). Additionally, we are able to show that continuous graphs with finite total curvature (NTC or TC) are tame. We say that a graph is tame when it is isotopic to an embedded polyhedral graph.

In sections 5 through 8, we characterize NTC with respect to the geometry and the topology of the graph. Proposition 5.5 shows the subadditivity of NTC under the union of graphs which meet in a finite set. In Section 6, the concept of bridge number is extended from curves to graphs, in terms of which the minimum of NTC can be explicitly computed, provided the graph has at most one vertex of degree \( > 3 \). In Section 7, Theorem 7.1 gives a lower bound for NTC in terms of the width of an isotopy class. The infimum of NTC is computed for specific graph types: the two-vertex graphs \( \theta_m \), the “ladder” \( L_m \), the “wheel” \( W_m \), the complete graph \( K_m \) on \( m \) vertices and the complete bipartite graph \( K_{m,n} \).

Finally we prove a result (Theorem 8.5) which gives a Fenchel type lower bound \(( \geq 3\pi \) for total curvature of a theta graph (an image of the graph consisting of a circle with an arc connecting a pair of antipodal points), and a Fáry–Milnor type upper bound \(( < 4\pi \) to imply the theta graph is isotopic to the standard embedding. A similar result was given by Taniyama [1998], referring to TC. In contrast, for graphs of the type of \( K_m \) \(( m \geq 4 \)), the infimum of NTC in the isotopy class of a polygon on \( m \) vertices is also the infimum for a sequence of distinct isotopy classes (Corollary 8.3).
Many of the results in our earlier preprint [Gulliver and Yamada 2008] have been incorporated into the present paper.

We thank Yuya Koda for his comments regarding Proposition 6.1, and Jaigyoung Choe and Rob Kusner for their comments about Theorem 8.5, especially about the sharp case $\text{NTC}(\Gamma) = 3\pi$ of the lower bound estimate.

2. Definitions of total curvature

The first difficulty, in extending the results of Milnor's classic paper, is to understand the contribution to total curvature at a vertex of degree $d(q) \geq 3$. We first consider the well-known case:

**Definition of total curvature for curves.** For a smooth closed curve $\Gamma$, the total curvature is

$$\mathcal{E}(\Gamma) = \int_{\Gamma} |\vec{k}| \, ds,$$

where $s$ denotes arc length along $\Gamma$ and $\vec{k}$ is the curvature vector. If $x(s) \in \mathbb{R}^3$ denotes the position of the point measured at arc length $s$ along the curve, then $\vec{k} = \frac{d^2x}{ds^2}$. For a piecewise smooth curve, that is, a graph with vertices $q_1, \ldots, q_N$ having always degree $d(q_i) = 2$, the total curvature is readily generalized to

$$\mathcal{E}(\Gamma) = \sum_{i=1}^{N} c(q_i) + \int_{\Gamma,\text{reg}} |\vec{k}| \, ds,$$

where the integral is taken over the separate $C^2$ edges of $\Gamma$ without their endpoints; and where $c(q_i) \in [0, \pi]$ is the exterior angle formed by the two edges of $\Gamma$ which meet at $q_i$. That is, $\cos(c(q_i)) = \langle T_1, -T_2 \rangle$, where $T_1 = \frac{dx}{ds}(q_i^+) \text{ and } T_2 = -\frac{dx}{ds}(q_i^-)$ are the unit tangent vectors at $q_i$ pointing into the two edges which meet at $q_i$. The exterior angle $c(q_i)$ is the correct contribution to total curvature, since any sequence of smooth curves converging to $\Gamma$ in $C^0$, with $C^1$ convergence on compact subsets of each open edge, includes a small arc near $q_i$ along which the tangent vector changes from near $\frac{dx}{ds}(q_i^-)$ to near $\frac{dx}{ds}(q_i^+)$. The greatest lower bound of the contribution to total curvature of this disappearing arc along the smooth approximating curves equals $c(q_i)$.

Note that $\mathcal{E}(\Gamma)$ is well defined for an immersed curve $\Gamma$.

**Definitions of total curvature for graphs.** When we turn our attention to a graph $\Gamma$, we find the above definition for curves (degree $d(q) = 2$) does not generalize in any obvious way to higher degree (see [Gulliver 2007]). The ambiguity of the general formula (2-1) is resolved if we specify the replacement for $c(0)$ when $\Gamma$ is the cone over a finite set $\{T_1, \ldots, T_d\}$ in the unit sphere $S^2$. 
The earliest notion of total curvature of a graph appears in the context of the first variation of length of a graph, which we call variational total curvature, and is called the mean curvature of the graph in [Allard and Almgren 1976]: we shall write VTC. The contribution to VTC at a vertex $q$ of degree 2, with unit tangent vectors $T_1$ and $T_2$, is $\text{vtc}(q) = |T_1 + T_2| = 2 \sin(c(q)/2)$. At a nonstraight vertex $q$ of degree 2, $\text{vtc}(q)$ is less than the exterior angle $c(q)$. For a vertex of degree $d$, the contribution is $\text{vtc}(q) = |T_1 + \cdots + T_d|$.

A rather natural definition of total curvature of graphs was given in [Taniyama 1998]. We have called this maximal total curvature $\text{TC}(0)$ in [Gulliver 2007]. The contribution to total curvature at a vertex $q$ of degree $d$ is

$$\text{tc}(q) := \sum_{1 \leq i < j \leq d} \arccos(T_i, -T_j).$$

In the case $d(q) = 2$, the sum above has only one term, the exterior angle $c(q)$ at $q$. Since the length of the Gauss image of a curve in $S^2$ is the total curvature of the curve, $\text{tc}(q)$ may be interpreted as adding to the Gauss image in $\mathbb{R}P^2$ of the edges, a complete great-circle graph on $T_1(q), \ldots, T_d(q)$, for each vertex $q$ of degree $d$. Note that the edge between two vertices does not measure the distance in $\mathbb{R}P^2$ but its supplement.

In [Gulliver and Yamada 2006], studying the density of an area-minimizing two-dimensional rectifiable set $\Sigma$ spanning $\Gamma$, we found that it was very useful to apply the Gauss–Bonnet formula to the cone over $\Gamma$ with a point $p$ of $\Sigma$ as vertex. The relevant notion of total curvature in that context is cone total curvature $\text{CTC}(\Gamma)$, defined using $\text{ctc}(q)$ as the replacement for $c(q)$ in (2-1):

$$\text{ctc}(q) := \sup_{e \in S^2} \left\{ \sum_{i=1}^{d} \left( \frac{\pi}{2} - \arccos(T_i, e) \right) \right\}.$$

Note that in the case $d(q) = 2$, the supremum above is assumed at vectors $e$ lying in the smaller angle between the tangent vectors $T_1$ and $T_2$ to $\Gamma$, so that $\text{ctc}(q)$ is then the exterior angle $c(q)$ at $q$. The main result of [Gulliver and Yamada 2006] is that $2\pi$ times the area density of $\Sigma$ at any of its points is at most equal to $\text{CTC}(\Gamma)$. The same result had been proven by Eckholm, White and Wienholtz for the case of a simple closed curve [Ekholm et al. 2002]. Taking $\Sigma$ to be the branched immersion of the disk given by Douglas [1931] and Radó [1933], it follows that if $\mathcal{E}(\Gamma) \leq 4\pi$, then $\Sigma$ is embedded, and therefore $\Gamma$ is unknotted. Thus [Ekholm et al. 2002] provided an independent proof of the Fáry–Milnor theorem. However, $\text{CTC}(\Gamma)$ may be small for graphs which are far from the simplest isotopy types of a graph $\Gamma$.

In this paper, we introduce the notion of net total curvature $\text{NTC}(\Gamma)$, which is the appropriate definition for generalizing — to graphs — Milnor’s approach to
isotopy and total curvature of curves. For each unit tangent vector $T_i$ at $q$, where $1 \leq i \leq d = d(q)$, let $\chi_i : S^2 \rightarrow \{-1, +1\}$ be equal to $-1$ on the hemisphere with center at $T_i$, and $+1$ on the opposite hemisphere (modulo sets of zero Lebesgue measure). We then define

$$(2-3) \quad \text{ntc}(q) := \frac{1}{4} \int_{S^2} \left[ \sum_{i=1}^{d} \chi_i(e) \right]^+ dA_{S^2}(e).$$

We note that the function $\sum_{i=1}^{d} \chi_i(e)$ is odd, hence the quantity above can be written as

$$\text{ntc}(q) := \frac{1}{8} \int_{S^2} \left| \sum_{i=1}^{d} \chi_i(e) \right| dA_{S^2}(e).$$

as well. In the case $d(q) = 2$, the integrand of (2-3) is positive (and equals 2) only on the set of unit vectors $e$ which have negative inner products with both $T_1$ and $T_2$, ignoring $e$ in sets of measure zero. This set is bounded by great semicircles orthogonal to $T_1$ and to $T_2$, and has spherical area equal to twice the exterior angle. So in this case, ntc($q$) is the exterior angle. Thus, in the special case where $\Gamma$ is a piecewise smooth curve, the following quantity NTC($\Gamma$) coincides with total curvature, as well as with TC($\Gamma$) and CTC($\Gamma$):

**Definition 2.1.** We define the net total curvature of a piecewise $C^2$ graph $\Gamma$ with vertices $\{q_1, \ldots, q_N\}$ as

$$(2-4) \quad \text{NTC}(\Gamma) := \sum_{i=1}^{N} \text{ntc}(q_i) + \int_{\Gamma_{\text{reg}} |\vec{k}|} ds.$$ 

For the sake of simplicity, elsewhere in this paper, we consider the ambient space to be $\mathbb{R}^3$. However the definition of the net total curvature can be generalized for a graph in $\mathbb{R}^n$ by defining the vertex contribution in terms of an average over $S^{n-1}$:

$$\text{ntc}(q) := \pi \int_{S^{n-1}} \left[ \sum_{i=1}^{d} \chi_i(e) \right]^+ dA_{S^{n-1}}(e),$$

which is consistent with the definition (2-3) of ntc when $n = 3$.

Recall that Milnor defines the total curvature of a continuous simple closed curve $C$ as the supremum of the total curvature of all polygons inscribed in $C$. By analogy, we define net total curvature of a continuous graph $\Gamma$ to be the supremum of the net total curvature of all polygonal graphs $P$ suitably inscribed in $\Gamma$ as follows.

**Definition 2.2.** For a given continuous graph $\Gamma$, we say a polygonal graph $P \subset \mathbb{R}^3$ is $\Gamma$-approximating, provided that its topological vertices (those of degree $\neq 2$) are
exactly the topological vertices of $\Gamma$, and having the same degrees; and that the
arcs of $P$ between two topological vertices correspond one-to-one to the edges of
$\Gamma$ between those two vertices.

Note that if $P$ is a $\Gamma$-approximating polygonal graph, then $P$ is homeomorphic
to $\Gamma$. According to the statement of Proposition 4.1, whose proof will be given
in the next section, if $P$ and $\tilde{P}$ are $\Gamma$-approximating polygonal graphs, and $\tilde{P}$ is
a refinement of $P$, then $\text{NTC}(\tilde{P}) \geq \text{NTC}(P)$. Here $\tilde{P}$ is said to be a refinement
of $P$ provided the set of vertices of $P$ is a subset of the vertices of $\tilde{P}$. Assuming
Proposition 4.1 for the moment, we can generalize the definition of the total
curvature to nonsmooth graphs.

**Definition 2.3.** Define the net total curvature of a continuous graph $\Gamma$ by

$$\text{NTC}(\Gamma) := \sup_P \text{NTC}(P)$$

where the supremum is taken over all $\Gamma$-approximating polygonal graphs $P$.

For a polygonal graph $P$, applying Definition 2.1,

$$\text{NTC}(P) := \sum_{i=1}^{N} \text{ntc}(q_i),$$

where $q_1, \ldots, q_N$ are the vertices of $P$.

Definition 2.3 is consistent with Definition 2.1 in the case of a piecewise $C^2$
graph $\Gamma$. Namely, as Milnor showed [1950, p. 251], the total curvature $\mathcal{C}(\Gamma_0)$ of
a smooth curve $\Gamma_0$ is the supremum of the total curvature of inscribed polygons,
which gives the required supremum for each edge. At a vertex $q$ of the piecewise-
$C^2$ graph $\Gamma$, as a sequence $P_k$ of $\Gamma$-approximating polygons become arbitrarily
fine, a vertex $q$ of $P_k$ (and of $\Gamma$) has unit tangent vectors converging in $S^2$ to the
unit tangent vectors to $\Gamma$ at $q$. It follows that for $1 \leq i \leq d(q)$, $\chi^P_k \to \chi^\Gamma_i$ in
measure on $S^2$, and therefore $\text{ntc}_{P_k}(q) \to \text{ntc}_\Gamma(q)$.

### 3. Crofton-type representation formula for total curvature

We would like to explain how the net total curvature $\text{NTC}(\Gamma)$ of a piecewise $C^2$
graph $\Gamma$ is related to more familiar notions of total curvature. Recall that $\Gamma$ has an
Euler circuit if and only if its vertices all have even degree, by a theorem of Euler.
An Euler circuit is a closed, connected path which traverses each edge of $\Gamma$ exactly
once. Of course, we do not have the hypothesis of even degree. We can attain that
hypothesis by passing to the double $\tilde{\Gamma}$ of $\Gamma$: $\tilde{\Gamma}$ is the graph with the same vertices
as $\Gamma$, but with two copies of each edge of $\Gamma$. Then at each vertex $q$, the degree
as a vertex of $\tilde{\Gamma}$ is $\tilde{d}(q) = 2d(q)$, which is even. By Euler’s theorem, there is an
Euler circuit $\Gamma'$ of $\tilde{\Gamma}$, which may be thought of as a closed path which traverses
each edge of $\Gamma$ exactly twice. Now at each of the points \{\(q_1, \ldots, q_d\)\} along $\Gamma'$ which are mapped to $q \in \Gamma$, we may consider the exterior angle $c(q_i)$. The sum of these exterior angles, however, depends on the choice of the Euler circuit $\Gamma'$. For example, if $\Gamma$ is the union of the $x$-axis and the $y$-axis in Euclidean space $\mathbb{R}^3$, then one might choose $\Gamma'$ to have four right angles, or to have four straight angles, or something in between, with completely different values of total curvature. In order to form a version of total curvature at a vertex $q$ which only depends on the original graph $\Gamma$ and not on the choice of Euler circuit $\Gamma'$, it is necessary to consider some of the exterior angles as partially balancing others. In the example just considered, where $\Gamma$ is the union of two orthogonal lines, two opposing right angles will be considered to balance each other completely, so that $ntc(q) = 0$, regardless of the choice of Euler circuit of the double.

It will become apparent that the connected character of an Euler circuit of $\tilde{\Gamma}$ is not required for what follows. Instead, we shall refer to a parametrization $\Gamma'$ of the double $\tilde{\Gamma}$, which is a mapping from a 1-dimensional manifold without boundary, not necessarily connected; the mapping is assumed to cover each edge of $\tilde{\Gamma}$ once.

The nature of $ntc(q)$ is clearer when it is localized on $S^2$, analogously to [Milnor 1950]. In the case $d(q) = 2$, Milnor observed that the exterior angle at the vertex $q$ equals half the area of those $e \in S^2$ such that the linear function $\langle e, \cdot \rangle$, restricted to $\Gamma$, has a local maximum at $q$. In our context, we may describe $ntc(q)$ as one-half the integral over the sphere of the number of net local maxima, which is half the difference of local maxima and local minima. Along the parametrization $\Gamma'$ of the double of $\Gamma$, the linear function $\langle e, \cdot \rangle$ may have a local maximum at some of the vertices $q_1, \ldots, q_d$ over $q$, and may have a local minimum at others. In our construction, each local minimum balances against one local maximum. If there are more local minima than local maxima, the number $nlm(e, q)$, the net number of local maxima, will be negative; however, our definition uses only the positive part $[nlm(e, q)]^+$. We need to show that

$$\int_{S^2} [nlm(e, q)]^+ dA_{S^2}(e)$$

is independent of the choice of parametrization, and in fact is equal to $2ntc(q)$; this will follow from another way of computing $nlm(e, q)$ (see Corollary 3.7).

**Definition 3.1.** Let a parametrization $\Gamma'$ of the double of $\Gamma$ be given. Then a vertex $q$ of $\Gamma$ corresponds to a number of vertices $q_1, \ldots, q_d$ of $\Gamma'$, where $d$ is the degree $d(q)$ of $q$ as a vertex of $\Gamma$. Choose $e \in S^2$. If $q \in \Gamma$ is a local extremum of $\langle e, \cdot \rangle$, then we consider $q$ as a vertex of degree $d(q) = 2$. Let $lmax(e, q)$ be the number of local maxima of $\langle e, \cdot \rangle$ along $\Gamma'$ at the points $q_1, \ldots, q_d$ over $q$, and similarly let $lmin(e, q)$ be the number of local minima. We define the number of
net local maxima of \( \langle e, \cdot \rangle \) at \( q \) to be
\[
\text{nlm}(e, q) = \frac{1}{2} [\text{lmax}(e, q) - \text{lmin}(e, q)].
\]

**Remark 3.2.** The definition of \( \text{nlm}(e, q) \) appears to depend not only on \( \Gamma \) but on a choice of the parametrization \( \Gamma' \) of the double of \( \Gamma \): \( \text{lmax}(e, q) \) and \( \text{lmin}(e, q) \) may depend on the choice of \( \Gamma' \). However, we shall see in Corollary 3.6 below that the number of net local maxima \( \text{nlm}(e, q) \) is in fact independent of \( \Gamma' \).

**Remark 3.3.** We have included the factor \( \frac{1}{2} \) in the definition of \( \text{nlm}(e, q) \) in order to agree with the difference of the numbers of local maxima and minima along a parametrization of \( \Gamma \) itself, if \( d(q) \) is even.

We shall assume for the rest of this section that a unit vector \( e \) has been chosen, and that the linear height function \( \langle e, \cdot \rangle \) has only a finite number of critical points along \( \Gamma \); this excludes \( e \) belonging to a subset of \( S^2 \) of measure zero. We shall also assume that the graph \( \Gamma \) is subdivided to include among the vertices all critical points of the linear function \( \langle e, \cdot \rangle \), with degree \( d(q) = 2 \) if \( q \) is an interior point of one of the topological edges of \( \Gamma \).

**Definition 3.4.** Choose a unit vector \( e \). At a point \( q \in \Gamma \) of degree \( d = d(q) \), let the **up-degree** \( d^+ = d^+(e, q) \) be the number of edges of \( \Gamma \) with endpoint \( q \) on which \( \langle e, \cdot \rangle \) exceeds \( \langle e, q \rangle \), the height of \( q \). Similarly, let the **down-degree** \( d^-(e, q) \) be the number of edges along which \( \langle e, \cdot \rangle \) is less than its value at \( q \). Note that \( d(q) = d^+(e, q) + d^-(e, q) \), for almost all \( e \in S^2 \).

**Lemma 3.5** (combinatorial lemma). *For all \( q \in \Gamma \) and for almost all \( e \in S^2 \),
\[
\text{nrm}(e, q) = \frac{1}{2} [d^-(e, q) - d^+(e, q)].
\]

**Proof.** Let a parametrization \( \Gamma' \) of the double of \( \Gamma \) be chosen, with respect to which \( \text{lmax}(e, q) \) and \( \text{lmin}(e, q) \) are defined. Recall the assumption above, that \( \Gamma \) has been subdivided so that along each edge, the linear function \( \langle e, \cdot \rangle \) is strictly monotone.

Consider a vertex \( q \) of \( \Gamma \), of degree \( d = d(q) \). Then \( \Gamma' \) has \( 2d \) edges with an endpoint among the points \( q_1, \ldots, q_d \) which are mapped to \( q \in \Gamma \). On \( 2d^+ \), resp. \( 2d^- \) of these edges, \( \langle e, \cdot \rangle \) is greater resp. less than \( \langle e, q \rangle \). But for each \( 1 \leq i \leq d \), the parametrization \( \Gamma' \) has exactly two edges which meet at \( q_i \). Depending on the up/down character of the two edges of \( \Gamma' \) which meet at \( q_i, 1 \leq i \leq d \), we can count:

- (+) If \( \langle e, \cdot \rangle \) is greater than \( \langle e, q \rangle \) on both edges, then \( q_i \) is a local minimum point; there are \( \text{lmin}(e, q) \) of these among \( q_1, \ldots, q_d \).
- (-) If \( \langle e, \cdot \rangle \) is less than \( \langle e, q \rangle \) on both edges, then \( q_i \) is a local maximum point; there are \( \text{lmax}(e, q) \) of these.
(0) In all remaining cases, the linear function $\langle e, \cdot \rangle$ is greater than $\langle e, q \rangle$ along one edge and less along the other, in which case $q_i$ is not counted in computing $l_{\text{max}}(e, q)$ nor $l_{\text{max}}(e, q)$; there are $d(q) - l_{\text{max}}(e, q) - l_{\text{min}}(e, q)$ of these.

Now count the individual edges of $\Gamma'$:

(+) There are $l_{\text{min}}(e, q)$ pairs of edges, each of which is part of a local minimum, both of which are counted among the $2d^+(e, q)$ edges of $\Gamma'$ with $\langle e, \cdot \rangle$ greater than $\langle e, q \rangle$.

(-) There are $l_{\text{max}}(e, q)$ pairs of edges, each of which is part of a local maximum; these are counted among the number $2d^-(e, q)$ of edges of $\Gamma'$ with $\langle e, \cdot \rangle$ less than $\langle e, q \rangle$. Finally,

(0) there are $d(q) - l_{\text{max}}(e, q) - l_{\text{min}}(e, q)$ edges of $\Gamma'$ which are not part of a local maximum or minimum, with $\langle e, \cdot \rangle$ greater than $\langle e, q \rangle$; and an equal number of edges with $\langle e, \cdot \rangle$ less than $\langle e, q \rangle$.

Thus, the total number of these edges of $\Gamma'$ with $\langle e, \cdot \rangle$ greater than $\langle e, q \rangle$ is

$$2d^+ = 2l_{\text{min}} + (d - l_{\text{max}} - l_{\text{min}}) = d + l_{\text{min}} - l_{\text{max}}.$$ Similarly,

$$2d^- = 2l_{\text{max}} + (d - l_{\text{max}} - l_{\text{min}}) = d + l_{\text{max}} - l_{\text{min}}.$$ Subtracting gives the conclusion:

$$n_{\text{lm}}(e, q) := \frac{l_{\text{max}}(e, q) - l_{\text{min}}(e, q)}{2} = \frac{d^-(e, q) - d^+(e, q)}{2} \square$$

**Corollary 3.6.** The number of net local maxima $n_{\text{lm}}(e, q)$ is independent of the choice of parametrization $\Gamma'$ of the double of $\Gamma$.

**Proof.** Given a direction $e \in S^2$, the up-degree and down-degree $d^\pm(e, q)$ at a vertex $q \in \Gamma$ are defined independently of the choice of $\Gamma'$. \square

**Corollary 3.7.** For any $q \in \Gamma$, we have $n_{\text{tc}}(q) = \frac{1}{2} \int_{S^2} [n_{\text{lm}}(e, q)]^+ dA_{S^2}$.

**Proof.** Consider $e \in S^2$. In the definition (2-3) of $n_{\text{tc}}(q)$, $\chi_i(e) = \pm 1$ whenever $\pm \langle e, T_i \rangle < 0$. But the number of $1 \leq i \leq d$ with $\pm \langle e, T_i \rangle < 0$ equals $d^+(e, q)$, so that

$$\sum_{i=1}^d \chi_i(e) = d^-(e, q) - d^+(e, q) = 2n_{\text{lm}}(e, q)$$

by Lemma 3.5, for almost all $e \in S^2$. \square

**Definition 3.8.** For a graph $\Gamma$ in $\mathbb{R}^3$ and $e \in S^2$, define the multiplicity at $e$ as

$$\mu(e) = \mu_\Gamma(e) = \sum n_{\text{lm}}^+(e, q) : q \text{ a vertex of } \Gamma \text{ or a critical point of } \langle e, \cdot \rangle.$$
Note that \( \mu(e) \) is a half-integer. Note also that in the case when \( \Gamma \) is a curve, or equivalently, when \( d(q) = 2 \), \( \mu(e) \) is exactly the integer \( \mu(\Gamma, e) \), the number of local maxima of \( \langle e, \cdot \rangle \) along \( \Gamma \) as defined in [Milnor 1950, p. 252].

**Corollary 3.9.** For almost all \( e \in S^2 \) and for any parametrization \( \Gamma' \) of the double of \( \Gamma \), \( \mu_\Gamma(e) \leq \frac{1}{2} \mu_{\Gamma'}(e) \).

**Proof.** We have

\[
\mu_\Gamma(e) = \frac{1}{2} \sum_q \left[ \text{lmax}_{\Gamma'}(e, q) - \text{lmin}_{\Gamma'}(e, q) \right] \leq \frac{1}{2} \sum_q \text{lmax}_{\Gamma'}(e, q) = \frac{1}{2} \mu_{\Gamma'}. \]

If, in place of the positive part, we sum \( nlm(e, q) \) itself over \( q \) located above a plane orthogonal to \( e \), we find a useful quantity:

**Corollary 3.10.** For almost all \( s_0 \in \mathbb{R} \) and almost all \( e \in S^2 \),

\[
2 \sum \{ nlm(e, q) : \langle e, q \rangle > s_0 \} = \#(e, s_0),
\]

the cardinality of the fiber \( \{ p \in \Gamma : \langle e, p \rangle = s_0 \} \).

**Proof.** If \( s_0 > \max_{p \in \Gamma} \langle e, p \rangle \), then \( \#(e, s_0) = 0 \). Now proceed downward, using Lemma 3.5 by induction. \( \square \)

Note that the fiber cardinality of Corollary 3.10 is also the value obtained for curves, where the more general \( nlm \) may be replaced by the number of local maxima [Milnor 1950].

**Remark 3.11.** In analogy with Corollary 3.10, we expect that an appropriate generalization of NTC to curved polyhedral complexes of dimension \( \geq 2 \) will in the future allow computation of the homology of level sets and sublevel sets of a (generalized) Morse function in terms of a generalization of \( nlm(e, q) \).

**Corollary 3.12.** The multiplicity of a graph in direction \( e \in S^2 \) may also be computed as \( \mu(e) = \frac{1}{2} \sum_{q \in \Gamma} |nlm(e, q)| \).

**Proof.** It follows from Corollary 3.10 with \( s_0 < \min_{p \in \Gamma} \langle e, p \rangle \) that \( \sum_{q \in \Gamma} nlm(e, q) = 0 \), which is the difference of positive and negative parts. The sum of these parts is \( \sum_{q \in \Gamma} |nlm(e, q)| = 2 \mu(e) \). \( \square \)

It was shown in Theorem 3.1 of [Milnor 1950] that, in the case of curves, \( \zeta(\Gamma) = \frac{1}{2} \int_{S^2} \mu(e) \, dA_{S^2} \), where Milnor refers to Crofton’s formula. We may now extend this result to graphs:

**Theorem 3.13.** For a (piecewise \( C^2 \)) graph \( \Gamma \) mapped into \( \mathbb{R}^3 \), the net total curvature has the representation

\[
\text{NTC}(\Gamma) = \frac{1}{2} \int_{S^2} \mu(e) \, dA_{S^2}(e). 
\]
Proof. We have $\text{NTC}(\Gamma) = \sum_{j=1}^{N} \text{ntc}(q_j) + \int_{\Gamma_{\text{reg}}} |\vec{k}| \, ds$, where $q_1, \ldots, q_N$ are the vertices of $\Gamma$, including local extrema as vertices of degree $d(q_j) = 2$, and where $\text{ntc}(q) := \frac{1}{4} \int_{S^2} \left[ \sum_{i=1}^{d} \chi_i(e) \right] \, dA_{S^2}(e)$ by the definition (2-3) of $\text{ntc}(q)$. Applying Milnor’s result to each $C^2$ edge, we have $\epsilon(\Gamma_{\text{reg}}) = \frac{1}{2} \int_{S^2} \mu_{\Gamma_{\text{reg}}}(e) \, dA_{S^2}$. But $\mu_{\Gamma}(e) = \mu_{\Gamma_{\text{reg}}}(e) + \sum_{j=1}^{N} \text{nlm}^+(e, q_j)$, and the theorem follows. \hfill \Box

**Corollary 3.14.** If $f : \Gamma \to \mathbb{R}^3$ is piecewise $C^2$ but is not an embedding, then the net total curvature $\text{NTC}(\Gamma)$ is well defined, using the right-hand side of the conclusion of Theorem 3.13. Moreover, $\text{NTC}(\Gamma)$ has the same value when some or all of the points of self-intersection of $\Gamma$ are redefined as vertices.

For $e \in S^2$, we use the notation $p_e : \mathbb{R}^3 \to e\mathbb{R}$ for the orthogonal projection $\langle e, \cdot \rangle$. We sometimes identify $\mathbb{R}$ with the one-dimensional subspace $e\mathbb{R}$ of $\mathbb{R}^3$.

**Corollary 3.15.** If $\{\Gamma\}$ is any homeomorphism type of graphs, then the infimum $\text{NTC}(\{\Gamma\})$ of net total curvature among mappings $f : \Gamma \to \mathbb{R}^n$ is assumed by a mapping $f_0 : \Gamma \to \mathbb{R}$.

For any isotopy class $[\Gamma]$ of embeddings $f : \Gamma \to \mathbb{R}^3$, the infimum $\text{NTC}(\{\Gamma\})$ of net total curvature is assumed by a mapping $f_0 : \Gamma \to \mathbb{R}$ in the closure of the given isotopy class.

Conversely, if $f_0 : \Gamma \to \mathbb{R}$ is in the closure of a given isotopy class $[\Gamma]$ of embeddings into $\mathbb{R}^3$, then for all $\delta > 0$ there is an embedding $f : \Gamma \to \mathbb{R}^3$ in that isotopy class with $\text{NTC}(f) \leq \text{NTC}(f_0) + \delta$.

**Proof.** Let $f : \Gamma \to \mathbb{R}^3$ be any piecewise smooth mapping. By Corollary 3.14 and Corollary 3.10, the net total curvature of the projection $p_e \circ f : \Gamma \to \mathbb{R}$ of $f$ onto the line in the direction of almost any $e \in S^2$ is given by $2\pi \mu(e) = \pi(\mu(e) + \mu(-e))$. It follows from Theorem 3.13 that $\text{NTC}(\Gamma)$ is the average of $2\pi \mu(e)$ over $e$ in $S^2$. But the half-integer-valued function $\mu(e)$ is lower semicontinuous almost everywhere, as may be seen using Definition 3.1. Let $e_0 \in S^2$ be a point where $\mu$ attains its essential infimum. Then $\text{NTC}(\Gamma) \geq \pi \mu(e_0) = \text{NTC}(p_{e_0} \circ f)$. But $(p_{e_0} \circ f)e_0$ is the limit as $e \to 0$ of the map $f_\varepsilon$ whose projection in the direction $e_0$ is the same as that of $f$ and is multiplied by $e$ in all orthogonal directions. Since $f_\varepsilon$ is isotopic to $f$, $(p_{e_0} \circ f)e_0$ is in the closure of the isotopy class of $f$.

Conversely, given $f_0 : \Gamma \to \mathbb{R}$ in the closure of a given isotopy class, let $f$ be an embedding in that isotopy class uniformly close to $f_0 e_0$; $f_\varepsilon$ as constructed above converges uniformly to $f_0$ as $e \to 0$, and $\text{NTC}(f_\varepsilon) \to \text{NTC}(f_0)$. \hfill \Box

**Definition 3.16.** We call a mapping $f : \Gamma \to \mathbb{R}^n$ flat (or NTC-flat) if $\text{NTC}(f) = \text{NTC}([\Gamma])$, the minimum value for the topological type of $\Gamma$, among all ambient dimensions $n$.

Corollary 3.15 above shows that for any $\Gamma$, there is a flat mapping $f : \Gamma \to \mathbb{R}$. 
Proposition 3.17. Consider a piecewise $C^2$ mapping $f_1 : \Gamma \to \mathbb{R}$. There is a mapping $f_0 : \Gamma \to \mathbb{R}$ which is monotonic along the topological edges of $\Gamma$, has values at topological vertices of $\Gamma$ arbitrarily close to those of $f_1$, and has $\text{NTC}(f_0) \leq \text{NTC}(f_1)$.

Proof. Any piecewise $C^2$ mapping $f_1 : \Gamma \to \mathbb{R}$ may be approximated uniformly by mappings with a finite set of local extreme points, using the compactness of $\Gamma$. Thus, we may assume without loss of generality that $f_1$ has only finitely many local extreme points. Note that for a mapping $f : \Gamma \to \mathbb{R} = \mathbb{R}^e$, $\text{NTC}(f) = 2\pi \mu(e)$: hence, we only need to compare $\mu_{f_0}(e)$ with $\mu_{f_1}(e)$.

If $f_1$ is not monotonic on a topological edge $E$, then it has a local extremum at a point $z$ in the interior of $E$. For concreteness, we shall assume $z$ is a local maximum point; the case of a local minimum is similar. Write $v, w$ for the endpoints of $E$. Let $v_1$ be the closest local minimum point to $z$ on the interval of $E$ from $z$ to $v$ (or $v_1 = v$ if there is no local minimum point between), and let $w_1$ be the closest local minimum point to $z$ on the interval from $z$ to $w$ (or $w_1 = w$). Let $E_1 \subset E$ denote the interval between $v_1$ and $w_1$. Then $E_1$ is an interval of a topological edge of $\Gamma$, having end points $v_1$ and $w_1$ and containing an interior point $z$, such that $f_1$ is monotone increasing on the interval from $v_1$ to $z$, and monotone decreasing on the interval from $z$ to $w_1$. By switching $v_1$ and $w_1$ if needed, we may assume that $f_1(v_1) < f_1(w_1) < f_1(z)$.

Let $f_0$ equal $f_1$ except on the interior of the interval $E_1$, and map $E_1$ monotonically to the interval of $\mathbb{R}$ between $f_1(v_1)$ and $f_1(w_1)$. Then for $f_1(w_1) < s < f_1(z)$, the cardinality $\#(s, f_1)_{f_0} = \#(s, f_1)_{f_1} - 2$. For $s$ in all other intervals of $\mathbb{R}$, this cardinality is unchanged. Therefore, $\text{nlm}_{f_1}(w_1) = \text{nlm}_{f_0}(w_1) - 1$, by Lemma 3.5. This implies that $\text{nlm}_{f_1}^+(w_1) \geq \text{nlm}_{f_0}^+(w_1) - 1$. Meanwhile, $\text{nlm}_{f_1}(z) = 1$, a term which does not appear in the formula for $\mu_{f_0}$ (see Definition 3.8). Thus $\mu_{f_0} \leq \mu_{f_1}$, and $\text{NTC}(f_0) \leq \text{NTC}(f_1)$.

Proceeding inductively, we remove each local extremum in the interior of any edge of $\Gamma$, without increasing NTC. \qed

4. Representation formula for nowhere-smooth graphs

Recall that, while defining the total curvature for continuous graphs in Section 2, we needed the monotonicity of $\text{NTC}(P)$ under refinement of polygonal graphs $P$. We are now ready to prove this.

Proposition 4.1. Let $P$ and $\tilde{P}$ be polygonal graphs in $\mathbb{R}^3$, having the same topological vertices, and homeomorphic to each other. Suppose that every vertex of $P$ is also a vertex of $\tilde{P}$: $\tilde{P}$ is a refinement of $P$. Then for almost all $e \in S^2$, the multiplicity $\mu_{\tilde{P}}(e) \geq \mu_P(e)$. As a consequence, $\text{NTC}(\tilde{P}) \geq \text{NTC}(P)$. 

Proof. We may assume, as an induction step, that $\tilde{P}$ is obtained from $P$ by replacing the edge having endpoints $q_0$, $q_2$ with two edges, one having endpoints $q_0$, $q_1$ and the other having endpoints $q_1$, $q_2$. Choose $e \in S^2$. We consider various cases:

If the new vertex $q_1$ satisfies $\langle e, q_0 \rangle < \langle e, q_1 \rangle < \langle e, q_2 \rangle$, then $\text{nlm}_P(e, q_1) = \text{nlm}_P(e, q_i) = i = 0, 2$ and $\text{nlm}_{\tilde{P}}(e, q_1) = 0$, hence $\mu_{\tilde{P}}(e) = \mu_P(e)$.

If $\langle e, q_0 \rangle < \langle e, q_2 \rangle < \langle e, q_1 \rangle$, then $\text{nlm}_{\tilde{P}}(e, q_0) = \text{nlm}_P(e, q_0)$ and $\text{nlm}_{\tilde{P}}(e, q_1) = 1$. The vertex $q_2$ requires more careful counting: the up- and down-degree satisfy $d_{\tilde{P}}^+(e, q_2) = d_{\tilde{P}}^+(e, q_2) = 1$, so that by Lemma 3.5, $\text{nlm}_{\tilde{P}}(e, q_2) = \text{nlm}_P(e, q_2) - 1$.

Meanwhile, for each of the polygonal graphs, $\mu(e)$ is the sum over $q$ of $\text{nlm}^+_P(e, q)$, so the change from $\mu_P(e)$ to $\mu_{\tilde{P}}(e)$ depends on the value of $\text{nlm}_P(e, q_2)$:

(a) If $\text{nlm}_P(e, q_2) \leq 0$, then $\text{nlm}^+_P(e, q_2) = \text{nlm}^+_P(e, q_2) = 0$.
(b) If $\text{nlm}_P(e, q_2) = \frac{1}{2}$, then $\text{nlm}^+_P(e, q_2) = \text{nlm}^+_P(e, q_2) - \frac{1}{2}$.
(c) If $\text{nlm}_P(e, q_2) \geq 1$, then $\text{nlm}^+_P(e, q_2) = \text{nlm}^+_P(e, q_2) - 1$.

Since the new vertex $q_1$ does not appear in $P$, recalling that $\text{nlm}_{\tilde{P}}(e, q_1) = 1$, we have $\mu_{\tilde{P}}(e) - \mu_P(e) = +1, +\frac{1}{2}$ or 0 in the respective cases (a), (b) or (c). In any case, $\mu_{\tilde{P}}(e) \geq \mu_P(e)$.

The reverse inequality $\langle e, q_1 \rangle < \langle e, q_2 \rangle < \langle e, q_0 \rangle$ may be reduced to the case just above by replacing $e \in S^2$ with $-e$, since $\mu_P(-e) = -\mu_P(e)$ for any polyhedral graph $P$. Then, depending whether $\text{nlm}_P(e, q_2)$ is $\leq -1, = -\frac{1}{2}$ or $\geq 0$, we find that $\mu_{\tilde{P}}(e) - \mu_P(e) = \text{nlm}^+_P(e, q_2) - \text{nlm}^+_P(e, q_2) = 0, \frac{1}{2}$, or 1. In any case, $\mu_{\tilde{P}}(e) \geq \mu_P(e)$.

These arguments are unchanged if $q_0$ is switched with $q_2$. This covers all cases except those in which equality occurs between $\langle e, q_i \rangle$ and $\langle e, q_j \rangle$ ($i \neq j$). The set of such unit vectors $e$ form a set of measure zero in $S^2$. The conclusion NTC($\tilde{P}$) $\geq$ NTC($P$) now follows from Theorem 3.13. $\square$

We remark here that this step of proving the monotonicity for the nowhere-smooth case differs from Milnor’s argument for the total curvature of curves, where it was shown by two applications of the triangle inequality for spherical triangles.

Milnor extended his results for piecewise smooth curves to continuous curves in [Milnor 1950]; we shall carry out an analogous extension to continuous graphs.

**Definition 4.2.** We say a point $q \in \Gamma$ is critical relative to $e \in S^2$ when $q$ is a topological vertex of $\Gamma$ or when $\langle e, \cdot \rangle$ is not monotone in any open interval of $\Gamma$ containing $q$.

Note that at some points of a differentiable curve, $\langle e, \cdot \rangle$ may have derivative zero but still not be considered a critical point relative to $e$ by our definition. This is appropriate to the $C^0$ category. For a continuous graph $\Gamma$, when NTC($\Gamma$) is finite, we shall show that the number of critical points is finite for almost all $e$ in $S^2$ (see Lemma 4.7 below).
Lemma 4.3. Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$, and choose a sequence $\hat{P}_k$ of $\Gamma$-approximating polygonal graphs with $\text{NTC}(\Gamma) = \lim_{k \to \infty} \text{NTC}(\hat{P}_k)$. Then for each $e \in S^2$, there is a refinement $P_k$ of $\hat{P}_k$ such that $\lim_{k \to \infty} \mu_{P_k}(e)$ exists in $[0, \infty]$.

Proof. First, for each $k$ in sequence, we refine $\hat{P}_k$ to include all vertices of $\hat{P}_{k-1}$. Then for all $e \in S^2$, $\mu_{\hat{P}_k}(e) \geq \mu_{\hat{P}_{k-1}}(e)$, by Proposition 4.1. Second, we refine $\hat{P}_k$ so that the arc of $\Gamma$ corresponding to each edge of $\hat{P}_k$ has diameter $\leq 1/k$. Third, given a particular $e \in S^2$, for each edge $E_k$ of $\hat{P}_k$, we add 0, 1 or 2 points from $\Gamma$ as vertices of $\hat{P}_k$ so that $\max E_k \langle e, \cdot \rangle = \max E \langle e, \cdot \rangle$ where $E$ is the closed arc of $\Gamma$ corresponding to $E_k$; and similarly so that $\min E_k \langle e, \cdot \rangle = \min E \langle e, \cdot \rangle$. Write $P_k$ for the result of this three-step refinement. Note that all vertices of $P_{k-1}$ appear among the vertices of $P_k$. Then by Proposition 4.1,

$$\text{NTC}(\hat{P}_k) \leq \text{NTC}(P_k) \leq \text{NTC}(\Gamma),$$

so we still have $\text{NTC}(\Gamma) = \lim_{k \to \infty} \text{NTC}(P_k)$.

Now compare the values of $\mu_{P_k}(e) = \sum_{q \in P_k} \text{nlm}_{P_k}(e, q)$ with the same sum for $P_{k-1}$. Since $P_k$ is a refinement of $P_{k-1}$, Proposition 4.1 gives $\mu_{P_k}(e) \geq \mu_{P_{k-1}}(e)$.

Therefore the values $\mu_{P_k}(e)$ are nondecreasing in $k$, which implies they are either convergent or properly divergent; in the latter case we write $\lim_{k \to \infty} \mu_{P_k}(e) = \infty$. \hfill $\Box$

Definition 4.4. For a continuous graph $\Gamma$, define the multiplicity at $e \in S^2$ as $\mu_\Gamma(e) := \lim_{k \to \infty} \mu_{P_k}(e) \in [0, \infty]$, where $P_k$ is a sequence of $\Gamma$-approximating polygonal graphs, refined with respect to $e$, as given in Lemma 4.3.

Remark 4.5. Note that any two $\Gamma$-approximating polygonal graphs have a common refinement. Hence, from the proof of Lemma 4.3, any two choices of sequences $\{\hat{P}_k\}$ of $\Gamma$-approximating polygonal graphs lead to the same value $\mu_\Gamma(e)$.

Lemma 4.6. Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$. Then $\mu_\Gamma : S^2 \to [0, \infty]$ takes its values in the half-integers, or $+\infty$. Now assume $\text{NTC}(\Gamma) < \infty$. Then $\mu_\Gamma$ is integrable, hence finite almost everywhere on $S^2$, and

$$\text{(4-1)} \quad \text{NTC}(\Gamma) = \frac{1}{2} \int_{S^2} \mu_\Gamma(e) \, dA_{S^2}(e).$$

For almost all $e \in S^2$, a sequence $P_k$ of $\Gamma$-approximating polygonal graphs, converging uniformly to $\Gamma$, may be chosen (depending on $e$) so that each local extreme point $q$ of $\langle e, \cdot \rangle$ along $\Gamma$ occurs as a vertex of $P_k$ for sufficiently large $k$.

Proof. Given $e \in S^2$, let $\{P_k\}$ be the sequence of $\Gamma$-approximating polygonal graphs from Lemma 4.3. If $\mu_\Gamma(e)$ is finite, then $\mu_{P_k}(e) = \mu_\Gamma(e)$ for $k$ sufficiently large, a half-integer.
Suppose $\text{NTC}(\Gamma) < \infty$. Then the half-integer-valued functions $\mu_{P_k}$ are nonnegative, integrable on $S^2$ with bounded integrals since $\text{NTC}(P_k) < \text{NTC}(\Gamma) < \infty$, and monotone increasing in $k$. Thus for almost all $e \in S^2$, $\mu_{P_k}(e) = \mu_{\Gamma}(e)$ for $k$ sufficiently large.

Since the functions $\mu_{P_k}$ are nonnegative and pointwise nondecreasing almost everywhere on $S^2$, it now follows from the monotone convergence theorem that

$$\int_{S^2} \mu_{\Gamma}(e) dA_{S^2}(e) = \lim_{k \to \infty} \int_{S^2} \mu_{P_k}(e) dA_{S^2}(e) = 2\text{NTC}(\Gamma).$$

Finally, the polygonal graphs $P_k$ have maximum edge length $\to 0$. For almost all $e \in S^2$, $\langle e, \cdot \rangle$ is not constant along any open arc of $\Gamma$, and $\mu_{\Gamma}(e)$ is finite. Given such an $e$, choose $l = l(e)$ sufficiently large that $\mu_{P_k}(e) = \mu_{\Gamma}(e)$ and $\mu_{P_k}(-e) = \mu_{\Gamma}(-e)$ for all $k \geq l$. Then for $k \geq l$, along any edge $E_k$ of $P_k$ with corresponding arc $E$ of $\Gamma$, the maximum and minimum values of $\langle e, \cdot \rangle$ along $E$ occur at the endpoints, which are also the endpoints of $E_k$. Otherwise, as $P_k$ is further refined, new interior local maximum and local minimum points of $E$ would each contribute a new, positive value to $\mu_{P_k}(e)$ or $\mu_{P_k}(-e)$, respectively, as $k$ increases. Since the diameter of the corresponding arc $E$ of $\Gamma$ tends to zero as $k \to \infty$, any local maximum or local minimum of $\langle e, \cdot \rangle$ must become an endpoint of some edge of $P_k$ for $k$ sufficiently large, and for $k \geq l$ in particular. 

Our next lemma focuses on the regularity of a graph $\Gamma$, originally only assumed continuous, provided it has finite net total curvature, or another notion of total curvature of a graph which includes the total curvature of the edges.

**Lemma 4.7.** Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$, with $\text{NTC}(\Gamma) < \infty$. Then $\Gamma$ has continuous one-sided unit tangent vectors $T_1(p)$ and $T_2(p)$ at each point $p$, not a topological vertex. If $p$ is a vertex of degree $d$, then each of the $d$ edges which meet at $p$ have well-defined unit tangent vectors at $p$: $T_1(p), \ldots, T_d(p)$.

For almost all $e \in S^2$,

$$\mu_{\Gamma}(e) = \sum_q \{\text{nlm}(e, q)\}^+, \tag{4-2}$$

where the sum is over the finite number of topological vertices of $\Gamma$ and critical points $q$ of $\langle e, \cdot \rangle$ along $\Gamma$. Further, for each $q$, $\text{nlm}(e, q) = \frac{1}{2}[d^-(e, q) - d^+(e, q)]$. All of these critical points which are not topological vertices are local extrema of $\langle e, \cdot \rangle$ along $\Gamma$.

**Proof.** We have seen in the proof of Lemma 4.6 that for almost all $e \in S^2$, the linear function $\langle e, \cdot \rangle$ is not constant along any open arc of $\Gamma$, and by Lemma 4.3 there is a sequence $\{P_k\}$ of $\Gamma$-approximating polygonal graphs with $\mu_{\Gamma}(e) = \mu_{P_k}(e)$ for $k$ sufficiently large. We have further shown that each local maximum point of $\langle e, \cdot \rangle$ is a vertex of $P_k$, possibly of degree two, for $k$ large enough. Recall that
\[ \mu_{P_k}(e) = \sum_q \operatorname{nlm}^{+}_{P_k}(e, q). \] Thus, each local maximum point \( q \) for \( \langle e, \cdot \rangle \) along \( \Gamma \) provides a nonnegative term \( \operatorname{nlm}^{+}_{P_k}(e, q) \) in the sum for \( \mu_{P_k}(e) \). Fix such an integer \( k \).

Consider a point \( q \in \Gamma \) which is not a topological vertex of \( \Gamma \) but is a critical point of \( \langle e, \cdot \rangle \). We shall show, by an argument similar to one used in [van Rooij 1965], that \( q \) must be a local extreme point. As a first step, we show that \( \langle e, \cdot \rangle \) is monotone on a sufficiently small interval on either side of \( q \). Choose an ordering of the closed edge \( E \) of \( \Gamma \) containing \( q \), and consider the interval \( E_+ \) of points \( \geq q \) with respect to this ordering. Suppose that \( \langle e, \cdot \rangle \) is not monotone on any subinterval of \( E_+ \) with \( q \) as endpoint. Then in any interval \( (q, r_1) \) there are points \( p_2 > q_2 > r_2 \) so that the numbers \( \langle e, p_2 \rangle, \langle e, q_2 \rangle, \langle e, r_2 \rangle \) are not monotone. It follows by an induction argument that there exist decreasing sequences \( p_n \to q, q_n \to q \), and \( r_n \to q \) of points of \( E_+ \) such that for each \( n \), \( r_{n-1} > p_n > q_n > r_n > q \), but the value \( \langle e, q_n \rangle \) lies outside of the closed interval between \( \langle e, p_n \rangle \) and \( \langle e, r_n \rangle \). As a consequence, there is a local extremum \( s_n \in (r_n, p_n) \). Since \( r_{n-1} > p_n \), the \( s_n \) are all distinct, \( 1 \leq n < \infty \).

But by Lemma 4.6, all local extreme points, specifically \( s_n \), of \( \langle e, \cdot \rangle \) along \( \Gamma \) occur among the \textit{finite} number of vertices of \( P_k \), a contradiction. This shows that \( \langle e, \cdot \rangle \) is monotone on an interval to the right of \( q \). An analogous argument shows that \( \langle e, \cdot \rangle \) is monotone on an interval to the left of \( q \).

Recall that for a critical point \( q \) relative to \( e \), \( \langle e, \cdot \rangle \) is not monotone on any neighborhood of \( q \). Since \( \langle e, \cdot \rangle \) is monotone on an interval on either side, the sense of monotonicity must be opposite on the two sides of \( q \). Therefore every critical point \( q \) along \( \Gamma \) for \( \langle e, \cdot \rangle \), which is not a topological vertex, is a local extremum.

We have chosen \( k \) large enough that \( \mu_{P_k}(e) = \mu_{P_k}(e) \). Then for any edge \( E \) of \( P_k \), the function \( \langle e, \cdot \rangle \) is monotone along the corresponding arc \( E \) of \( \Gamma \), as well as along \( E_k \). Also, \( E \) and \( E_k \) have common end points. It follows that for each \( \epsilon \in \mathbb{R} \), the cardinality \( \#(e, t) \) of the fiber \( \{ q \in \Gamma : \langle e, q \rangle = t \} \) is the same for \( P_k \) as for \( \Gamma \). We may see from Lemma 3.5 applied to \( P_k \) for each vertex or critical point \( q \), \( \operatorname{nlm}^{+}_{P_k}(e, q) = \frac{1}{2}[d^{+}_{P_k}(e, q) - d^{+}_{P_k}(e, q)] \); but \( \operatorname{nlm}(e, q) \) and \( d^{+}(e, q) \) have the same values for \( \Gamma \) as for \( P_k \). The formula \( \mu_{P_k}(e) = \sum_q \{ \operatorname{nlm}_{P_k}(e, q) \}^{+} \) now follows from the corresponding formula for \( P_k \), for almost all \( e \in S^2 \).

Consider an open interval \( E \) of \( \Gamma \) with endpoint \( q \). We have just shown that for almost all \( e \in S^2 \), \( \langle e, \cdot \rangle \) is monotone on a subinterval with endpoint \( q \). Choose a sequence \( p_l \) from \( E \), \( p_l \to q \), and write

\[ T_l := \frac{p_l - q}{|p_l - q|} \in S^2. \]

Then \( \lim_{l \to \infty} T_l \) exists. Otherwise, since \( S^2 \) is compact, there are subsequences \( \{ T_{m_n} \} \) and \( \{ T_{k_n} \} \) with \( T_{m_n} \to T' \) and \( T_{k_n} \to T'' \neq T' \). But for an open set of \( e \in S^2 \),
For a continuous graph

**Theorem 4.9.** A generalization of Theorem 3.13: for

0

and

χ

µ

has the representation

→∞

k

contradicting monotonicity on an interval starting at q for almost all e ∈ S².

This shows that Γ has one-sided tangent vectors \( T_1(q), \ldots, T_d(q) \) at each point \( q ∈ Γ \) of degree \( d = d(q) \) (\( d = 2 \) if q is not a topological vertex). Further, as \( k → ∞, T_i^{P_n}(q) → T_i^Γ(q), \) 1 ≤ i ≤ d(q), since edges of \( P_k \) have diameter ≤ \( \frac{1}{k} \).

The remaining conclusions follow readily.

**Corollary 4.8.** Let Γ be a continuous, finite graph in \( \mathbb{R}^3 \), with NTC(Γ) < ∞. Then for each point q of Γ, the contribution at q to net total curvature is given by (2-3), where for e ∈ S², \( \chi_i(e) = \text{the sign of } (-T_i(q), e), \) 1 ≤ i ≤ d(q). (Here, if q is not a topological vertex, we understand \( d = 2 \).)

**Proof.** According to Lemma 4.7, for 1 ≤ i ≤ d(q), \( T_i(q) \) is defined and tangent to an edge \( E_i \) of Γ, which is continuously differentiable at its end point q. If \( P_n \) is a sequence of Γ-approximating polygonal graphs with maximum edge length tending to 0, the corresponding unit tangent vectors \( T_i^{P_n}(q) → T_i^Γ(q) \) as \( n → ∞ \). For each \( P_n \), we have

\[
\text{ntc}^{P_n}(q) = \frac{1}{4} \int_{S^2} \left[ \sum_{i=1}^{d} \chi_i^{P_n}(e) \right]^+ dA_{S^2}(e),
\]

and \( \chi_i^{P_n} → \chi_i^{Γ} \) in measure on \( S^2 \). Hence, the integrals for \( P_n \) converge to those for Γ, which is (2-3).

We are ready to state the formula for net total curvature, by localization on \( S^2 \), a generalization of Theorem 3.13:

**Theorem 4.9.** For a continuous graph Γ, the net total curvature NTC(Γ) ∈ (0, ∞] has the representation

\[
\text{NTC}(Γ) = \frac{1}{4} \int_{S^2} \mu(e) dA_{S^2}(e),
\]

where, for almost all e ∈ S², the multiplicity \( \mu(e) \) is a positive half-integer or +∞, given as the finite sum (4-2).

**Proof.** If NTC(Γ) is finite, the theorem follows from Lemma 4.6 and Lemma 4.7.

Choose \( e ∈ S^2 \). Suppose NTC(Γ) = sup NTC(\( P_k \)) is infinite, where \( P_k \) is a refined sequence of polygonal graphs as in Lemma 4.3. Then \( \mu_Γ(e) \) is the nondecreasing limit of \( \mu_{P_k}(e) \) for all \( e ∈ S^2 \).

Thus \( \mu_Γ(e) ≥ \mu_{P_k}(e) \) for all \( e \) and \( k \), and \( \mu_Γ(e) = \mu_{P_k}(e) \) for \( k ≥ l(e) \). This implies that \( \mu_Γ(e) \) is a positive half-integer or ∞. Since NTC(Γ) is infinite, the integral

\[
\text{NTC}(P_k) = \frac{1}{2} \int_{S^2} \mu_{P_k}(e) dA_{S^2}(e)
\]
is arbitrarily large as $k \to \infty$, but for each $k$ is less than or equal to

$$\frac{1}{2} \int_{S^2} \mu_\Gamma(e) \, dA_{S^2}(e).$$

Therefore this latter integral equals $\infty$, and thus equals NTC($\Gamma$). \hfill \Box

We turn our attention next to the tameness of graphs of finite total curvature.

**Proposition 4.10.** Let $n$ be a positive integer, and write $Z$ for the set of $n$th roots of unity in $\mathbb{C} = \mathbb{R}^2$. Given a continuous one-parameter family $S_t$, $0 \leq t < 1$, of sets of $n$ points in $\mathbb{R}^2$, there exists a continuous one-parameter family $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ of homeomorphisms with compact support such that $\Phi_t(S_t) = Z$, $0 \leq t < 1$.

**Proof.** It is well known that there is an isotopy $\Phi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi_0(S_0) = Z$ and $\Phi_0 = \text{id}$ outside of a compact set. This completes the case $t_0 = 0$ of the following continuous induction argument.

Suppose that $[0, t_0] \subset [0, 1)$ is a subinterval such that there exists a continuous one-parameter family $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ of homeomorphisms with compact support, with $\Phi_t(S_t) = Z$ for all $0 \leq t \leq t_0$. We shall extend this property to an interval $[0, t_0+\delta]$. Write $B_\delta(Z)$ for the union of balls $B_\delta(\zeta_i)$ centered at the $n$ roots of unity $\zeta_1, \ldots, \zeta_n$. For $\epsilon < \sin \frac{\pi}{n}$, these balls are disjoint. We may choose $0 < \delta < 1 - t_0$ such that $\Phi_{t_0}(S_t) \subset B_\delta(Z)$ for all $t_0 \leq t \leq t_0 + \delta$. Write the points of $S_t$ as $x_i(t)$, $1 \leq i \leq n$, where $\Phi_{t_0}(x_i(t)) \in B_\delta(\zeta_i)$. For each $t \in [t_0, t_0 + \delta]$, each of the balls $B_\delta(\zeta_i)$ may be mapped onto itself by a homeomorphism $\psi_t$, varying continuously with $t$, such that $\psi_{t_0}$ is the identity, $\psi_t$ is the identity near the boundary of $B_\delta(\zeta_i)$ for all $t \in [t_0, t_0 + \delta]$, and $\psi_t(\Phi_{t_0}(x_i(t))) = \zeta_i$ for all such $t$. For example, we may construct $\psi_t$ so that for each $y \in B_\delta(\zeta_i)$, $y - \psi_t(y)$ is parallel to $\Phi_{t_0}(x_i(t)) - \zeta_i$. We now define $\Phi_t = \psi_t \circ \Phi_{t_0}$ for each $t \in [t_0, t_0 + \delta]$.

As a consequence, we see that there is no maximal interval $[0, t_0] \subset [0, 1)$ such that there is a continuous one-parameter family $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ of homeomorphisms with compact support with $\Phi_t(S_t) = Z$, for all $0 \leq t \leq t_0$. Thus, this property holds for the entire interval $0 \leq t < 1$. \hfill \Box

In the following theorem, the total curvature of a graph may be understood in terms of any definition which includes the total curvature of edges and which is continuous as a function of the unit tangent vectors at each vertex. This includes net total curvature, TC of [Taniyama 1998] and CTC of [Gulliver and Yamada 2006].

**Theorem 4.11.** Suppose $\Gamma \subset \mathbb{R}^3$ is a continuous graph with finite total curvature. Then for any $\epsilon > 0$, $\Gamma$ is isotopic to a $\Gamma$-approximating polygonal graph $P$ with edges of length at most $\epsilon$, whose total curvature is less than or equal to that of $\Gamma$. 
Proof. Since $\Gamma$ has finite total curvature, by Lemma 4.7, at each topological vertex of degree $d$ the edges have well-defined unit tangent vectors $T_1, \ldots, T_d$, which are each the limit as $\varepsilon \to 0$ of the unit tangent vectors to the corresponding edges of $P$. If at each vertex the unit tangent vectors $T_1, \ldots, T_d$ are distinct, then any sufficiently fine $\Gamma$-approximating polygonal graph will be isotopic to $\Gamma$; this easier case is proven.

We thus consider $n$ edges $E_1, \ldots, E_n$ ending at a vertex $q$, with common unit tangent vectors $T_1 = \cdots = T_n$. Choose orthogonal coordinates $(x, y, z)$ for $\mathbb{R}^3$ so that this common tangent vector $T_1 = \cdots = T_n = (0, 0, -1)$ and $q = (0, 0, 1)$. For some $\varepsilon > 0$, in the slab $1 - \varepsilon \leq z \leq 1$, the edges $E_1, \ldots, E_n$ project one-to-one onto the $z$-axis. After rescaling about $q$ by a factor $\geq 1/\varepsilon$, the edges $E_1, \ldots, E_n$ form a braid $B$ of $n$ strands in the slab $0 \leq z < 1$ of $\mathbb{R}^3$, plus the point $q = (0, 0, 1)$. Each strand $E_i$ has $q$ as an endpoint, and the coordinate $z$ is strictly monotone along $E_i$, $1 \leq i \leq n$. Write $S_t = B \cap \{z = t\}$. Then $S_t$ is a set of $n$ distinct points in the plane $\{z = t\}$ for each $0 \leq t < 1$. By Proposition 4.10, there are homeomorphisms $\Phi_t$ of the plane $\{z = t\}$ for each $0 \leq t < 1$, isotopic to the identity in that plane, continuous as a function of $t$, such that $\Phi_t(S_0) = Z \times \{t\}$, where $Z$ is the set of $n$-th roots of unity in the $(x, y)$-plane, and $\Phi_0$ is the identity outside of a compact set of the plane $\{z = t\}$.

We may suppose that $S_t$ lies in the open disk of radius $a(1 - t)$ of the plane $\{z = t\}$, for some (arbitrarily small) constant $a > 0$. We modify $\Phi_t$, first replacing its values with $(1 - t)\Phi_t$ inside the disk of radius $a(1 - t)$. We then modify $\Phi_t$ outside the disk of radius $a(1 - t)$, such that $\Phi_t$ is the identity outside the disk of radius $2a(1 - t)$.

Having thus modified the homeomorphisms $\Phi_t$ of the planes $\{z = t\}$, we may now define an isotopy $\Phi$ of $\mathbb{R}^3$ by mapping each plane $\{z = t\}$ to itself by the homeomorphism $\Phi_t^{-1} \circ \Phi_t$, $0 \leq t < 1$; and extend to the remaining planes $\{z = t\}$, $t \geq 1$ and $t < 0$, by the identity. Then the closure of the image of the braid $B$ is the union of line segments from $q = (0, 0, 1)$ to the $n$ points of $S_0$ in the plane $\{z = 0\}$. Since each $\Phi_t$ is isotopic to the identity in the plane $\{z = t\}$, $\Phi$ is isotopic to the identity of $\mathbb{R}^3$.

This procedure may be carried out in disjoint sets of $\mathbb{R}^3$ surrounding each unit vector which occurs as tangent vector to more than one edge at a vertex of $\Gamma$. Outside these sets, we inscribe a polygonal arc in each edge of $\Gamma$ to obtain a $\Gamma$-approximating polygonal graph $P$. By Definition 2.3, $P$ has total curvature less than or equal to the total curvature of $\Gamma$.

Artin and Fox [1948] introduced the notion of tame and wild knots in $\mathbb{R}^3$; the extension to graphs is the following:

**Definition 4.12.** We say that a graph in $\mathbb{R}^3$ is tame if it is isotopic to a polyhedral graph; otherwise, it is wild.
Milnor [1950] proved that curves of finite total curvature are tame. More generally, we have

**Corollary 4.13.** A continuous graph \( \Gamma \subset \mathbb{R}^3 \) of finite total curvature is tame.

**Proof.** This is an easy consequence of Theorem 4.11, since the \( \Gamma \)-approximating polygonal graph \( P \) is isotopic to \( \Gamma \). \( \square \)

**Observation 4.14.** Tameness does not imply finite total curvature.

For a well-known example, let \( \Gamma \subset \mathbb{R}^2 \) be the continuous curve

\[
\{(x, h(x)) : x \in [-1, 1]\},
\]

where

\[
h(x) = -\frac{x}{\pi} \sin \frac{\pi}{x} \quad \text{for} \quad x \neq 0
\]

and \( h(0) = 0 \). This function has a sequence of zeroes \( \pm 1/n \to 0 \) as \( n \to \infty \). The total curvature of \( \Gamma \) between \( (0, 1/n) \) and \( (0, 1/(n+1)) \) converges to \( \pi \) as \( n \to \infty \). Thus \( \mathcal{E}(\Gamma) = \infty \).

On the other hand, \( h(x) \) is continuous on \([-1, 1]\), from which it readily follows that \( \Gamma \) is tame.

### 5. On vertices of small degree

We will now illustrate some properties of net total curvature \( \text{NTC}(\Gamma) \) in a few relatively simple cases, and make some observations regarding \( \text{NTC}((\Gamma)) \), the minimum net total curvature for the homeomorphism type of a graph \( \Gamma \subset \mathbb{R}^n \) (see Definition 3.16 above).

**Minimum curvature for given degree.**

**Proposition 5.1.** If a vertex \( q \) has odd degree \( d(q) = 2m+1 \), then \( \text{ntc}(q) \geq \pi/2 \). If \( d(q) = 3 \), then equality holds if and only if the three tangent vectors \( T_1, T_2, T_3 \) at \( q \) are coplanar but do not lie in any open half-plane. If \( q \) has even degree \( 2m \), then the minimum value of \( \text{ntc}(q) \) is 0. Moreover, the equality \( \text{ntc}(q) = 0 \) only occurs when \( T_1(q), \ldots, T_{2m}(q) \) form \( m \) opposite pairs.

**Proof.** Let \( q \) have odd degree \( d(q) = 2m+1 \). Then from Lemma 3.5, for any \( e \in S^2 \), we see that \( \text{nlm}(e, q) \) is one of the half-integers \( \pm \frac{1}{2}, \ldots, \pm \frac{2m+1}{2} \). In particular, \( |\text{nlm}(e, q)| \geq \frac{1}{2} \). Corollary 3.7 and the proof of Corollary 3.12 show that

\[
\text{ntc}(q) = \frac{1}{4} \int_{S^2} |\text{nlm}(e, q)| \, dA_{S^2}.
\]

Therefore \( \text{ntc}(q) \geq \pi/2 \).

If the degree \( d(q) = 3 \), then \( |\text{nlm}(e, q)| = \frac{1}{2} \) if and only if both \( d^+(q) \) and \( d^-(q) \) are nonzero, that is, \( q \) is not a local extremum for \( \langle e, \cdot \rangle \). If \( \text{ntc}(q) = \pi/2 \), then
this must be true for almost every direction \( e \in S^2 \). Thus, the three tangent vectors must be coplanar, and may not lie in an open half-plane.

If \( d(q) = 2m \) is even and equality \( \text{ntc}(q) = 0 \) holds, then the formula above for \( \text{ntc}(q) \) in terms of \( |\text{nlm}(e, q)| \) would require \( \text{nlm}(e, q) \equiv 0 \), and hence \( d^+(e, q) = d^-(e, q) = m \) for almost all \( e \in S^2 \): whenever \( e \) rotates so that the plane orthogonal to \( e \) passes \( T_i \), another tangent vector \( T_j \) must cross the plane in the opposite direction, for almost all \( e \), which implies \( T_j = -T_i \).

**Observation 5.2.** If a vertex \( q \) of odd degree \( d(q) = 2p + 1 \), has the minimum value \( \text{ntc}(q) = \pi/2 \), and a hyperplane \( P \subset \mathbb{R}^n \) contains an even number of the tangent vectors at \( q \), and no others, then these tangent vectors form opposite pairs.

The proof is seen by fixing any \((n - 2)\)-dimensional subspace \( L \) of \( P \) and rotating \( P \) by a small positive or negative angle \( \delta \) to a hyperplane \( P_\delta \) containing \( L \). Since \( P_\delta \) must have \( k \) of the vectors \( T_1, \ldots, T_{2p+1} \) on one side and \( k + 1 \) on the other side, for some \( 0 \leq k \leq p \), by comparing \( \delta > 0 \) with \( \delta < 0 \) it follows that exactly half of the tangent vectors in \( P \) lie nonstrictly on each side of \( L \). The proof may be continued as in the last paragraph of the proof of Proposition 5.1. In particular, any two independent tangent vectors \( T_i \) and \( T_j \) share the 2-plane they span with a third, the three vectors not lying in any open half-plane: in fact, the third vector needs to lie in any hyperplane containing \( T_i \) and \( T_j \).

For example, a flat \( K_{5,1} \) in \( \mathbb{R}^3 \) must have five straight segments, two being opposite; and the remaining three being coplanar but not in any open half-plane. This includes the case of four coplanar line segments, since the four must be in opposite pairs, and either opposing pair may be considered as coplanar with the fifth segment.

**Nonmonotonicity of NTC for subgraphs.**

**Observation 5.3.** If \( \Gamma_0 \) is a subgraph of a graph \( \Gamma \), then \( \text{NTC}(\Gamma_0) \) might not be \( \leq \text{NTC}(\Gamma) \).

For a simple polyhedral example, we may consider the “butterfly” graph \( \Gamma \) in the plane with six vertices: \( q_0^\pm = (0, \pm 1) \), \( q_1^\pm = (1, \pm 3) \), and \( q_2^\pm = (-1, \pm 3) \). \( \Gamma \) has seven edges: three vertical edges \( L_0, L_1 \) and \( L_2 \) are the line segments \( L_i \) joining \( q_i^- \) to \( q_i^+ \). Four additional edges are the line segments from \( q_0^\pm \) to \( q_1^\pm \) and from \( q_0^\pm \) to \( q_2^\pm \), which form the smaller angle \( 2\alpha \) at \( q_0^\pm \), where \( \tan \alpha = 1/2 \), so that \( \alpha < \pi/4 \).

The subgraph \( \Gamma_0 \) will be \( \Gamma \) minus the interior of \( L_0 \). Then \( \text{NTC}(\Gamma_0) = \text{ntc}(\Gamma_0) = 6\pi - 8\alpha \). However, \( \text{NTC}(\Gamma) = 4(\pi - \alpha) + 2(\pi/2) = 5\pi - 4\alpha \), which is less than \( \text{NTC}(\Gamma_0) \). \( \square \)

The monotonicity property, which Observation 5.3 shows fails for \( \text{NTC}(\Gamma) \), is a virtue of Taniyama’s total curvature \( \text{TC}(\Gamma) \).
Net total curvature ≠ cone total curvature ≠ Taniyama’s total curvature. It is not difficult to construct three unit vectors $T_1, T_2, T_3$ in $\mathbb{R}^3$ such that the values of $ntc(q), ctc(q)$ and $tc(q)$, with these vectors as the $d(q) = 3$ tangent vectors to a graph at a vertex $q$, have different values. For example, we may take $T_1, T_2$ and $T_3$ to be three unit vectors in a plane, making equal angles $2\pi/3$. According to Proposition 5.1, we have the contribution to net total curvature $ntc(q) = \pi/2$. But the contribution to cone total curvature is $ctc(q) = 0$. Namely, 

$$ctc(q) := \sup_{e \in S^2} \sum_{i=1}^{3} \left( \frac{\pi}{2} - \arccos\langle T_i, e \rangle \right).$$

In this supremum, we may choose $e$ to be normal to the plane of $T_1, T_2$ and $T_3$, and $ctc(q) = 0$ follows. Meanwhile, $tc(q)$ is the sum of the exterior angles formed by the three pairs of vectors, each equal to $\pi/3$, so that $tc(q) = \pi$.

A similar computation for degree $d$ and coplanar vectors making equal angles gives $ctc(q) = 0$, and $tc(q) = \frac{d}{2} \left[ \frac{1}{2} (d-1)^2 \right]$ (floor function), while $ntc(q) = \pi/2$ for $d$ odd, $ntc(q) = 0$ for $d$ even. This example indicates that $tc(q)$ may be significantly larger than $ntc(q)$. In fact, we have

**Observation 5.4.** If a vertex $q$ of a graph $\Gamma$ has degree $d = d(q) \geq 2$, then

$$tc(q) \geq (d - 1) ntc(q).$$

This follows from the definition (2-3) of $ntc(q)$. Let $T_1, \ldots, T_d$ be the unit tangent vectors at $q$. The exterior angle between $T_i$ and $T_j$ is

$$\arccos(-T_i, T_j) = \frac{1}{4} \int_{S^2} (\chi_i + \chi_j)^+ dA_{S^2}.$$

The contribution $tc(q)$ at $q$ to total curvature $TC(\Gamma)$ equals the sum of these integrals over all $1 \leq i < j \leq d$. The sum of the integrands is

$$\sum_{1 \leq i < j \leq d} (\chi_i + \chi_j)^+ \geq \left[ \sum_{1 \leq i < j \leq d} (\chi_i + \chi_j) \right]^+ = (d - 1) \left[ \sum_{i=1}^{d} \chi_i \right]^+.$$

Integrating over $S^2$ and dividing by 4, we have $tc(q) \geq (d - 1) ntc(q)$. □

**Conditional additivity of net total curvature under taking union.** Observation 5.3 shows the failure of monotonicity of NTC for subgraphs due to the cancellation phenomena at each vertex. The following subadditivity statement specifies the necessary and sufficient condition for the additivity of net total curvature under taking union of graphs.
Proposition 5.5. Given two graphs $\Gamma_1$ and $\Gamma_2 \subseteq \mathbb{R}^n$ with $\Gamma_1 \cap \Gamma_2 = \{ p_1, \ldots, p_N \}$, the net total curvature of $\Gamma = \Gamma_1 \cup \Gamma_2$ obeys the subadditivity law

\[
\text{NTC}(\Gamma) = \text{NTC}(\Gamma_1) + \text{NTC}(\Gamma_2)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{N} \int_{S^2} [\text{nlm}_{\Gamma_1}^+(e, p_j) - \text{nlm}_{\Gamma_1}^+(e, p_j) - \text{nlm}_{\Gamma_2}^+(e, p_j)] dA_{S^2}
\]

\[
\leq \text{NTC}(\Gamma_1) + \text{NTC}(\Gamma_2).
\]

In particular, additivity holds if and only if

\[
\text{nlm}_{\Gamma_1}(e, p_j) \text{nlm}_{\Gamma_2}(e, p_j) \geq 0
\]

for all points $p_j$ of $\Gamma_1 \cap \Gamma_2$ and almost all $e \in S^2$.

Proof. The edges of $\Gamma$ and vertices other than $p_1, \ldots, p_N$ are edges and vertices of $\Gamma_1$ or of $\Gamma_2$, so we only need to consider the contribution at the vertices $p_1, \ldots, p_N$ to $\mu(e)$ for $e \in S^2$ (see Definition 3.8). The subadditivity follows from the general inequality $(a + b)^+ \leq a^+ + b^+$ for any real numbers $a$ and $b$. Namely, let $a := \text{nlm}_{\Gamma_1}(e, p_j)$ and $b := \text{nlm}_{\Gamma_2}(e, p_j)$, so that $\text{nlm}_\Gamma(e, p_j) = a + b$, as follows from Lemma 3.5. Now integrate both sides of the inequality over $S^2$, sum over $j = 1, \ldots, N$ and apply Theorem 3.13.

As for the equality case, suppose that $ab \geq 0$. We then note that either $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$, or $a = 0$, or $b = 0$. In all four cases, we have $a^+ + b^+ = (a + b)^+$. Applied with $a = \text{nlm}_{\Gamma_1}(e, p_j)$ and $b = \text{nlm}_{\Gamma_2}(e, p_j)$, assuming that $\text{nlm}_{\Gamma_1}(e, p_j)\text{nlm}_{\Gamma_2}(e, p_j) \geq 0$ holds for all $j = 1, \ldots, N$ and almost all $e \in S^2$, this implies that $\text{NTC}(\Gamma_1 \cup \Gamma_2) = \text{NTC}(\Gamma_1) + \text{NTC}(\Gamma_2)$.

To show that the equality $\text{NTC}(\Gamma_1 \cup \Gamma_2) = \text{NTC}(\Gamma_1) + \text{NTC}(\Gamma_2)$ implies the inequality $\text{nlm}_{\Gamma_1}(e, p_j)\text{nlm}_{\Gamma_2}(e, p_j) \geq 0$ for all $j = 1, \ldots, N$ and for almost all $e \in S^2$, we suppose, to the contrary, that there is a set $U$ of positive measure in $S^2$, such that for some vertex $p_j$ in $\Gamma_1 \cap \Gamma_2$, whenever $e$ is in $U$, the inequality $ab < 0$ is satisfied, where $a = \text{nlm}_{\Gamma_1}(e, p_j)$ and $b = \text{nlm}_{\Gamma_2}(e, p_j)$. Then for $e$ in $U$, $a$ and $b$ are of opposite signs. Let $U_1$ be the part of $U$ where $a < 0 < b$ holds: we may assume $U_1$ has positive measure, otherwise exchange $\Gamma_1$ with $\Gamma_2$. On $U_1$, we have

\[
(a + b)^+ < b^+ = a^+ + b^+.
\]

Recall that $a + b = \text{nlm}_\Gamma(e, p_j)$. Hence the inequality between half-integers

\[
\text{nlm}_1^+(e, p_j) < \text{nlm}_{\Gamma_1}^+(e, p_j) + \text{nlm}_{\Gamma_2}^+(e, p_j)
\]

is valid on the set $U_1$, which has positive measure. This, in turn, implies that $\text{NTC}(\Gamma_1 \cup \Gamma_2) < \text{NTC}(\Gamma_1) + \text{NTC}(\Gamma_2)$, contradicting the assumption of equality. \hfill \Box
One-point union of graphs.

Proposition 5.6. If the graph $\Gamma$ is the one-point union of graphs $\Gamma_1$ and $\Gamma_2$, where the points $p_1$ chosen in $\Gamma_1$ and $p_2$ chosen in $\Gamma_2$ are not topological vertices, then the minimum NTC among all mappings is subadditive, and the minimum NTC minus $2\pi$ is superadditive:

$$\text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2]) - 2\pi \leq \text{NTC}([\Gamma]) \leq \text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2]).$$

Further, if the points $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$ may appear as extreme points on mappings of minimum NTC, then the minimum net total curvature among all mappings, minus $2\pi$, is additive:

$$\text{NTC}([\Gamma]) = \text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2]) - 2\pi.$$

Proof. Write $p \in \Gamma$ for the identified points $p_1 = p_2 = p$.

Choose flat mappings $f_1 : \Gamma_1 \to \mathbb{R}$ and $f_2 : \Gamma_2 \to \mathbb{R}$, adding constants so that the chosen points $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$ have $f_1(p_1) = f_2(p_2) = 0$. Further, by Proposition 3.17, we may assume that $f_1$ and $f_2$ are strictly monotone on the edges of $\Gamma_1$ and $\Gamma_2$ containing $p_1$ and $p_2$, respectively. Let $f : \Gamma \to \mathbb{R}$ be defined as $f_1$ on $\Gamma_1$ and as $f_2$ on $\Gamma_2$. Then at the common point of $\Gamma_1$ and $\Gamma_2$, $f(p) = 0$, and $f$ is continuous. But since $f_1$ and $f_2$ are monotone on the edges containing $p_1$ and $p_2$, $\text{nlm}_{\Gamma_1}(p_1) = 0 = \text{nlm}_{\Gamma_2}(p_2)$, so we have $\text{NTC}([\Gamma]) \leq \text{NTC}(f) = \text{NTC}(f_1) + \text{NTC}(f_2) = \text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2])$ by Proposition 5.5.

Next, for all $g : \Gamma \to \mathbb{R}$, we show that $\text{NTC}(g) \geq \text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2]) - 2\pi$. Given $g$, write $g_1, g_2$ for the restriction of $g$ to $\Gamma_1, \Gamma_2$. Then

$$\mu_g(e) = \mu_{g_1}(e) - \text{nlm}_{g_1}^+(p_1) + \mu_{g_2}(e) - \text{nlm}_{g_2}^+(p_2) + \text{nlm}_g^-(p).$$

Now for any real numbers $a$ and $b$, the difference $(a + b)^+ - (a^+ + b^+)$ is equal to $\pm a, \pm b$ or 0, depending on the various signs. Let $a = \text{nlm}_{g_1}(p_1)$ and $b = \text{nlm}_{g_2}(p_2)$. Then since $p_1$ and $p_2$ are not topological vertices of $\Gamma_1$ and $\Gamma_2$, respectively, we have $a, b \in \{-1, 0, +1\}$ and $a + b = \text{nlm}_g(p)$ by Lemma 3.5. In any case, we have

$$\text{nlm}_g^+(p) - \text{nlm}_{g_1}^+(p_1) - \text{nlm}_{g_2}^+(p_2) \geq -1.$$

Thus, $\mu_g(e) \geq \mu_{g_1}(e) + \mu_{g_2}(e) - 1$, and multiplying by $2\pi$, $\text{NTC}(g) \geq \text{NTC}(g_1) + \text{NTC}(g_2) - 2\pi \geq \text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2]) - 2\pi$.

Finally, assume $p_1$ and $p_2$ are extreme points for flat mappings $f_1 : \Gamma_1 \to \mathbb{R}$ and $f_2 : \Gamma_2 \to \mathbb{R}$. We may assume that $f_1(p_1) = 0 = \min f_1(\Gamma_1)$ and $f_2(p_2) = 0 = \max f_2(\Gamma_2)$. Then $\text{nlm}_{f_1}(p_2) = 1$ and $\text{nlm}_{f_1}(p_1) = -1$, and hence using Lemma 3.5, $\text{nlm}_f(p) = 0$. So $\mu_f(e) = \mu_{f_1}(e) - \text{nlm}_{f_1}^+(p_1) + \mu_{f_2}(e) - \text{nlm}_{f_2}^+(p_2) + \text{nlm}_f^+(p) = \mu_{f_1}(e) + \mu_{f_2}(e) - 1$. Multiplying by $2\pi$, we have

$$\text{NTC}([\Gamma]) \leq \text{NTC}(f) = \text{NTC}([\Gamma_1]) + \text{NTC}([\Gamma_2]) - 2\pi.$$
6. Net total curvature for degree 3

Simple description of net total curvature.

Proposition 6.1. For any graph $\Gamma$ and any parametrization $\Gamma'$ of its double, $$\text{NTC}(\Gamma) \leq \frac{1}{2} \ell(\Gamma').$$

If $\Gamma$ is a trivalent graph, that is, having vertices of degree at most three, then $\text{NTC}(\Gamma) = \frac{1}{2} \ell(\Gamma')$ for any parametrization $\Gamma'$ that does not immediately repeat any edge of $\Gamma$.

Proof. The first conclusion follows from Corollary 3.9.

Now consider a trivalent graph $\Gamma$. Observe that $\Gamma'$ would be forced to immediately repeat any edge which ends in a vertex of degree 1; thus, we may assume that $\Gamma$ has only vertices of degree 2 or 3. Since $\Gamma'$ covers each edge of $\Gamma$ twice, we need only show, for every vertex $q$ of $\Gamma$, having degree $d = d(q) \in \{2, 3\}$, that

$$2 \text{ntc}_\Gamma(q) = \sum_{i=1}^{d} c_{\Gamma'}(q_i),$$

where $q_1, \ldots, q_d$ are the vertices of $\Gamma'$ over $q$. If $d = 2$, since $\Gamma'$ does not immediately repeat any edge of $\Gamma$, we have $\text{ntc}_\Gamma(q) = c_{\Gamma'}(q_1) = c_{\Gamma'}(q_2)$, so (6-1) clearly holds. For $d = 3$, write both sides of (6-1) as integrals over $S^2$, using the definition (2-3) of $\text{ntc}_\Gamma(q)$. Since $\Gamma'$ does not immediately repeat any edge, the three pairs of tangent vectors $\{T_{\Gamma'}^{1}(q), T_{\Gamma'}^{2}(q), T_{\Gamma'}^{3}(q)\}$, $1 \leq j \leq 3$, comprise all three pairs taken from the triple $\{T_{\Gamma}^{1}(q), T_{\Gamma}^{2}(q), T_{\Gamma}^{3}(q)\}$. We need to show that

$$2 \int_{S^2} [\chi_1 + \chi_2 + \chi_3]^+ dA_{S^2} = \int_{S^2} [\chi_1 + \chi_2]^+ dA_{S^2} + \int_{S^2} [\chi_2 + \chi_3]^+ dA_{S^2} + \int_{S^2} [\chi_3 + \chi_1]^+ dA_{S^2},$$

where at each direction $e \in S^2$, $\chi_j(e) = \pm 1$ is the sign of $\langle -e, T_{\Gamma}^j(q) \rangle$. But the integrands are equal at almost every point $e$ of $S^2$:

$$2[\chi_1 + \chi_2 + \chi_3]^+ = [\chi_1 + \chi_2]^+ + [\chi_2 + \chi_3]^+ + [\chi_3 + \chi_1]^+,$$

as may be confirmed by cases: $6 = 6$ if $\chi_1 = \chi_2 = \chi_3 = +1$; $2 = 2$ if exactly one of the $\chi_i$ equals $-1$, and $0 = 0$ in the remaining cases.

Simple description of net total curvature fails, $d \geq 4$.

Observation 6.2. We have seen in Proposition 6.1 that for graphs with vertices of degree $\leq 3$, if a parametrization $\Gamma'$ of the double $\tilde{\Gamma}$ of $\Gamma$ does not immediately repeat any edge of $\Gamma$, then $\text{NTC}(\Gamma) = \frac{1}{2} \ell(\Gamma')$, the total curvature in the usual
sense of the link $\Gamma'$. A natural suggestion would be that for general graphs $\Gamma \subset \mathbb{R}^3$, $\text{NTC}(\Gamma)$ might be half the infimum of total curvature of all such parametrizations $\Gamma'$ of the double. However, in some cases, we have the strict inequality $\text{NTC}(\Gamma) < \inf_{\Gamma'} \frac{1}{2} \text{NTC}(\Gamma')$.

In light of Proposition 6.1, we choose an example of a vertex $q$ of degree four, and consider the local contributions to $\text{NTC}$ for $\Gamma = K_{1,4}$ and for $\Gamma'$, which is the union of four arcs.

Suppose that for a small positive angle $\alpha$, ($\alpha \leq 1$ radian would suffice) the four unit tangent vectors at $q$ are $T_1 = (1, 0, 0)$; $T_2 = (0, 1, 0)$; $T_3 = (-\cos \alpha, 0, \sin \alpha)$; and $T_4 = (0, -\cos \alpha, -\sin \alpha)$. Write the exterior angles as $\theta_{ij} = \pi - \arccos \langle T_i, T_j \rangle$. Then $\inf_{\Gamma'} \frac{1}{2} \epsilon(\Gamma') = \theta_{13} + \theta_{14} = 2\alpha$. However, $\text{nfc}(q)$ is strictly less than $2\alpha$. This may be seen by writing $\text{nfc}(q)$ as an integral over $S^2$, according to the definition (2-3), and noting that cancellation occurs between two of the four lune-shaped sectors. □

**Minimum NTC for trivalent graphs.** Using the relation $\text{NTC}(\Gamma) = \frac{1}{2} \text{NTC}(\Gamma')$ between the net total curvature of a given trivalent graph $\Gamma$ and the total curvature for a nonreversing double cover $\Gamma'$ of the graph, we can determine the minimum net total curvature of a trivalent graph embedded in $\mathbb{R}^n$, whose value is then related to the Euler characteristic of the graph $\chi(\Gamma) = -k/2$.

First we introduce the following definition.

**Definition 6.3.** For a given graph $\Gamma$ and a mapping $f : \Gamma \to \mathbb{R}$, let the extended bridge number $B(f)$ be one-half the number of local extrema. Write $B([\Gamma])$ for the minimum of $B(f)$ among all mappings $f : \Gamma \to \mathbb{R}$. For a given isotopy type $[\Gamma]$ of embeddings into $\mathbb{R}^3$, let $B([\Gamma])$ be one-half the minimum number of local extrema for a mapping $f : \Gamma \to \mathbb{R}$ in the closure of the isotopy class $[\Gamma]$.

For an integer $m \geq 3$, let $\theta_m$ be the graph with two vertices $q^+$, $q^-$ and $m$ edges, each of which has $q^+$ and $q^-$ as its two endpoints. Then $\theta = \theta_3$ has the form of the lower-case Greek letter $\theta$.

**Remark 6.4.** For a curve, the number of local maxima equals the number of local minima. The minimum number of local maxima is called the bridge number, and equals the number of local minima. This is consistent with our Definition 6.3 of the extended bridge number. Of course, for curves, the minimum bridge number among all isotopy classes $B([S^1]) = 1$, and only $B([S^1])$ is of interest for a specific isotopy class $[S^1]$. For certain graphs, the minimum numbers of local maxima and local minima may not occur at the same time for any mapping: see the example of Observation 6.9 below. For isotopy classes of $\theta$-graphs, Goda [1997] has given a definition of an integer-valued bridge index which is similar in spirit to the definition above.
Theorem 6.5. If $\Gamma$ is a trivalent graph, and if $f_0 : \Gamma \to \mathbb{R}$ is monotone on topological edges and has the minimum number $2B([\Gamma])$ of local extrema, then

$$\text{NTC}(f_0) = \text{NTC}([\Gamma]) = \pi(2B([\Gamma]) + k/2),$$

where $k$ is the number of topological vertices of $\Gamma$. For a given isotopy class $[\Gamma]$, $\text{NTC}([\Gamma]) = \pi(2B([\Gamma]) + k/2)$.

Proof. Recall that $\text{NTC}([\Gamma])$ denotes the infimum of $\text{NTC}(f)$ among $f : \Gamma \to \mathbb{R}^3$ or among $f : \Gamma \to \mathbb{R}$, as may be seen from Corollary 3.15.

We first consider a mapping $f_1 : \Gamma \to \mathbb{R}$ with the property that any local maximum or local minimum points of $f_1$ are interior points of topological edges. Then all topological vertices $v$, since they have degree $d(v) = 3$ and $d^+(v) \neq 0$, have $\text{nlm}(v) = \pm 1/2$, by Proposition 5.1. Let $\Lambda$ be the number of local maximum points of $f_1$, $V$ the number of local minimum points, $\lambda^+$ the number of vertices with $\text{nlm} = +1/2$, and $\lambda^-$ the number of vertices with $\text{nlm} = -1/2$. Then $\lambda^+ + \lambda^- = k$, the total number of vertices, and $\Lambda + V \geq 2B([\Gamma])$. Hence, applying Corollary 3.12,

$$\mu = \frac{1}{2} \sum_v |\text{nlm}(v)| = \frac{1}{2} \left(\Lambda + V + \frac{\lambda^+ + \lambda^-}{2}\right) \geq B([\Gamma]) + k/4,$$

with equality if and only if $\Lambda + V = 2B([\Gamma])$.

We next consider any mapping $f_0 : \Gamma \to \mathbb{R}$ in general position: in particular, the critical values of $f_0$ are isolated. In a similar fashion to the proof of Proposition 3.17, we shall replace $f_0$ with a mapping whose local extrema are not topological vertices. Specifically, if $f_0$ assumes a local maximum at any topological vertex $v$, then, since $d(v) = 3$, $\text{nlm}_{f_0}(v) = 3/2$. $f_0$ may be isotoped in a small neighborhood of $v$ to $f_1 : \Gamma \to \mathbb{R}$ so that near $v$, the local maximum occurs at an interior point $q$ of one of the three edges with endpoint $v$, and thus $\text{nlm}_{f_1}(q) = 1$; while the up-degree $d^+_1(v) = 1$ and the down-degree $d^-_1(v) = 2$, so that $\text{nlm}_{f_1}(v)$ is now $1/2$. Thus, $\mu_{f_1}(e) = \mu_{f_0}(e)$. Similarly, if $f_0$ assumes a local minimum at a topological vertex $w$, then $f_0$ may be isotoped in a neighborhood of $w$ to $f_1 : \Gamma \to \mathbb{R}$ so that the local minimum of $f_1$ near $w$ occurs at an interior point of any of the three edges with endpoint $w$, and $\mu_{f_1}(e) = \mu_{f_0}(e)$. Then any local extreme points of $f_1$ are interior points of topological edges. Thus, we have shown that $\mu_{f_0}(e) \geq B([\Gamma]) + k/4$, with equality if $f_1$ has exactly $2B([\Gamma])$ as its number of local extrema, which holds if and only if $f_0$ has the minimum number $2B([\Gamma])$ of local extrema. Thus

$$\text{NTC}([\Gamma]) = 2\pi \mu_{f_0}(e) = 2\pi \left(B([\Gamma]) + k/4\right) = \pi(2B([\Gamma]) + k/2).$$

Similarly, for a given isotopy class $[\Gamma]$ of embeddings into $\mathbb{R}^3$, we may choose $f_0 : \Gamma \to \mathbb{R}$ in the closure of the isotopy class, deform $f_0$ to a mapping $f_1$ in the
closure of $[\Gamma]$ having no topological vertices as local extrema and count $\mu_{f_0}(e) = \mu_{f_1}(e) \geq B([\Gamma]) + k/4$, with equality if $f_0$ has the minimum number $2B([\Gamma])$ of local extrema. This shows that $\text{NTC}([\Gamma]) = \pi \left( 2B([\Gamma]) + k/2 \right)$. \qed

**Remark 6.6.** An example geometrically illustrating the lower bound is given by the dual graph $\Gamma^*$ of the one-skeleton $\Gamma$ of a triangulation of $S^2$, with the $\{\infty\}$ not coinciding with any of the vertices of $\Gamma^*$. The Koebe–Andreev–Thurston (see [Stephenson 2003]) theorem says that there is a circle packing that realizes the vertex set of $\Gamma^*$ as the set of centers of the circles. The so realized $\Gamma^*$, stereographically projected to $\mathbb{R}^2 \subset \mathbb{R}^3$, attains the lower bound of Theorem 6.5 with $B((\Gamma^*)) = 1$, namely $\text{NTC}([\Gamma]) = \pi(2 + \frac{k}{2}) = \pi(2 - \chi(\Gamma^*))$, where $k$ is the number of vertices.

**Corollary 6.7.** If $\Gamma$ is a trivalent graph with $k$ topological vertices, and $f_0 : \Gamma \to \mathbb{R}$ is a mapping in general position, having $\Lambda$ local maximum points and $V$ local minimum points, then

$$\mu_{f_0}(e) = \frac{1}{2}(\Lambda + V) + \frac{1}{4}k \geq B([\Gamma]) + \frac{1}{4}k.$$ 

**Proof.** Follows immediately from the proof of Theorem 6.5: $f_0$ and $f_1$ have the same number of local maximum or minimum points. \qed

An interesting trivalent graph is $L_m$, the “ladder of $m$ rungs” obtained from two unit circles in parallel planes by adding $m$ line segments (“rungs”) perpendicular to the planes, each joining one vertex on the first circle to another vertex on the second circle. For example, $L_4$ is the 1-skeleton of the cube in $\mathbb{R}^3$. Note that $L_m$ may be embedded in $\mathbb{R}^2$, and that the bridge number $B((L_m)) = 1$. Since $L_m$ has $2m$ trivalent vertices, we may apply Theorem 6.5 to compute the minimum NTC for the type of $L_m$:

**Corollary 6.8.** The minimum net total curvature $\text{NTC}((L_m))$ for graphs of the type of $L_m$ equals $\pi(2 + m)$.

**Observation 6.9.** For certain connected trivalent graphs $\Gamma$ containing cut points, the minimum extended bridge number $B([\Gamma])$ may be greater than 1.

**Example.** Let $\Gamma$ be the union of three disjoint circles $C_1, C_2, C_3$ with three edges $E_i$ connecting a point $p_i \in C_i$ with a fourth vertex $p_0$, which is not in any of the $C_i$, and which is a cut point of $\Gamma$: the number of connected components of $\Gamma \setminus p_0$ is greater than for $\Gamma$. Given $f : \Gamma \to \mathbb{R}$, after a permutation of $\{1, 2, 3\}$, we may assume there is a minimum point $q_1 \in C_1 \cup E_1$ and a maximum point $q_3 \in C_3 \cup E_3$. If $q_1$ and $q_3$ are both in $C_1 \cup E_1$, we may choose $C_2$ arbitrarily in what follows. Restricted to the closed set $C_2 \cup E_2$, $f$ assumes either a maximum or a minimum at a point $q_2 \neq p_0$. Since $q_2 \neq p_0$, $q_2$ is also a local maximum or a local minimum for $f$ on $\Gamma$. That is, $q_1, q_2, q_3$ are all local extrema. In the notation of the proof
of Theorem 6.5, we have the number of local extrema \( V + \Lambda \geq 3 \). Therefore
\[ B(\{\Gamma\}) \geq \frac{3}{2}, \quad \text{and} \quad \NTC(\{\Gamma\}) \geq \pi(3 + k/2) = 5\pi. \]

The reader will be able to construct similar trivalent examples with \( B(\{\Gamma\}) \) arbitrarily large.

In contrast to the results of Theorem 6.5 and of Theorem 6.11, below, for trivalent or nearly trivalent graphs, the minimum of NTC for a given graph type cannot be computed merely by counting vertices, but depends in a more subtle way on the topology of the graph:

**Observation 6.10.** When \( \Gamma \) is not trivalent, the minimum \( \NTC(\{\Gamma\}) \) of net total curvature for a connected graph \( \Gamma \) with \( B(\{\Gamma\}) = 1 \) is not determined by the number of vertices and their degrees.

**Example.** We shall construct two planar graphs \( S_m \) and \( R_m \) having the same number of vertices, all of degree 4.

Choose an integer \( m \geq 3 \) and take the image of the embedding \( f_\varepsilon \) of the “sine wave” \( S_m \) to be the union of the polar-coordinate graphs \( C_\pm \subset \mathbb{R}^2 \) of two functions: \( r = 1 \pm \varepsilon \sin(m\theta) \). \( S_m \) has \( 4m \) edges; and \( 2m \) vertices, all of degree 4, at \( r = 1 \) and \( \theta = \pi/m, 2\pi/m, \ldots, 2\pi \). For \( 0 < \varepsilon < 1 \), \( f_\varepsilon(S_m) = C_+ \cup C_- \) is the union of two smooth cycles. For small positive \( \varepsilon \), \( C_+ \) and \( C_- \) are convex. The \( 2m \) vertices all have \( \nlm(q) = 0 \), so

\[ \NTC(f_\varepsilon) = \NTC(C_+) + \NTC(C_-) = 2\pi + 2\pi. \]

Therefore \( \NTC(\{S_m\}) \leq \NTC(f_\varepsilon) = 4\pi \).

For the other graph type, let the “ring graph” \( R_m \subset \mathbb{R}^2 \) be constructed by adding \( m \) disjoint small circles \( C_i \), each crossing one large circle \( C \) at two points \( v_{2j-1}, v_{2j} \), \( 1 \leq i \leq m \). Then \( R_m \) has \( 4m \) edges. We construct \( R_m \) so that the \( 2m \) vertices \( v_1, v_2, \ldots, v_{2m} \) appear in cyclic order around \( C \). Then \( R_m \) has the same number \( 2m \) of vertices as does \( S_m \), all of degree 4. At each vertex \( v_j \), we have \( \nlm(v_j) = 0 \), so in this embedding, \( \NTC(R_m) = 2\pi(m + 1) \). We shall show that \( \NTC(f_1) \geq 2\pi m \) for any \( f_1 : R_m \to \mathbb{R}^3 \). According to Corollary 3.15, it is enough to show for every \( f : R_m \to \mathbb{R} \) that \( \mu_f \geq m \). We may assume \( f \) is monotone on each topological edge, according to Proposition 3.17. Depending on the order of \( f(v_{2i-2}), f(v_{2i-1}) \) and \( f(v_{2i}) \), \( \nlm(v_{2i-1}) \) might equal \( \pm 1 \) or \( \pm 2 \), but cannot be 0, as follows from Lemma 3.5, since the unordered pair \( \{d^-(v_{2i-1}), d^+(v_{2i-1})\} \) may only be \( \{1, 3\} \) or \( \{0, 4\} \). Similarly, \( v_{2i} \) is connected by three edges to \( v_{2i-1} \) and by one edge to \( v_{2i+1} \). For the same reasons, \( \nlm(v_{2i}) \) might equal \( \pm 1 \) or \( \pm 2 \), and cannot = 0. So \( |\nlm(v_j)| \geq 1, 1 \leq j \leq 2m \), and thus by Corollary 3.12, \( \mu = 1/2 \sum |\nlm(v_j)| \geq m \). Therefore the minimum of net total curvature \( \NTC(\{R_m\}) \geq 2m\pi \), which is greater than \( \NTC(\{S_m\}) \leq 4\pi \), since \( m \geq 3 \).
of the isotopy class. Choose $f$ mapping points are interior points of edges, with $nlm$ is odd, then $nlm$ is even, or for half of $m + 1$, if $m$ is odd.

By Corollary 3.12, we have

$$
\mu_g(e) = \frac{1}{2} \sum |nlm(v)| \geq \frac{1}{2} (\Lambda + V + k/2) \geq B(\Gamma) + k/4.
$$

This shows that

$$
NTC(\Gamma) \geq \pi(2B(\Gamma)) + k/2).
$$

Now let $f_0 : \Gamma \to \mathbb{R}$ be monotone on topological edges and have the minimum number $2B(\Gamma)$ of local extreme points (see Proposition 3.17). As in the proof of Theorem 6.5, $f_0$ may be modified without changing $NTC(f_0)$ so that all $2B(\Gamma)$ local extreme points are interior points of edges. $f_0$ may be further modified so that the distinct vertices $v_1, \ldots, v_m$ which share edges with $w$ are balanced: $f(v_j) < f(w)$ for half of the $j = 1, \ldots, m$, if $m$ is even, or for half of $m + 1$, if $m$ is odd. Having chosen $f(v_j)$, we define $f$ along the (unique) edge from $w$ to $v_j$ to be monotone, for $j = 1, \ldots, m$. Therefore if $m$ is even, then $nlm_f(w) = 0$; and if $m$ is odd, then $nlm_f(w) = \frac{1}{2}$, by Lemma 3.5. We compute

$$
\mu_f(e) = \frac{1}{2} \sum |nlm(v)| = \frac{1}{2} (\Lambda + V + k/2) = B(\Gamma) + k/4.
$$

We conclude that $NTC(\Gamma) = \pi(2B(\Gamma)) + k/2)$.

For a given isotopy class $\Gamma$, the proof is analogous to the above. Choose a mapping $g : \Gamma \to \mathbb{R}$ in the closure of $\Gamma$, and modify $g$ without leaving the closure of the isotopy class. Choose $f : \Gamma \to \mathbb{R}$ which has the minimum number $2B(\Gamma)$ of local extreme points, and modify it so that topological vertices are not local extreme points. In contrast to the proof of Theorem 6.5, a balanced arrangement of vertices may not be possible in the given isotopy class. In any case, if $m$ is even, then $|nlm_f(w)| \geq 0$; and if $m$ is odd, $|nlm_f(w)| \geq \frac{1}{2}$, by Proposition 5.1. Thus applying Corollary 3.12, we find $NTC(\Gamma) \geq \pi(2B(\Gamma)) + k/2)$.
Observation 6.12. When all vertices of \( \Gamma \) are trivalent except \( w \), \( d(w) \geq 4 \), and when \( w \) shares more than one edge with another vertex of \( \Gamma \), then in certain cases, \( \text{NTC}(\{\Gamma\}) > \pi \left( 2B(\{\Gamma\}) + \frac{k}{2} \right) \), where \( k \) is the number of vertices of odd degree.

Example. Choose \( \Gamma \) to be the one-point union of \( \Gamma_1 \), \( \Gamma_2 \) and \( \Gamma_3 \), where \( \Gamma_i = \theta = \theta_3 \), \( i = 1, 2, 3 \), and the point \( w_i \) chosen from \( \Gamma_i \) is one of its two vertices \( v_i \), \( w_i \). Then the identified point \( w = w_1 = w_2 = w_3 \) of \( \Gamma \) has \( d(w) = 9 \), and each of the other three vertices \( v_1, v_2, v_3 \) has degree 3.

Choose a flat map \( f : \Gamma \rightarrow \mathbb{R} \). We may assume that \( f \) is monotone on each edge, applying Proposition 3.17. If \( f(v_1) < f(v_2) < f(w) < f(v_3) \), then \( d^+(w) = 3 \), \( d^-(w) = 6 \), so \( \text{nlm}(w) = \frac{1}{2} \), while \( v_i \) is a local extreme point, so \( \text{nlm}(v_i) = \pm \frac{1}{2} \), \( 1 = 1, 2, 3 \). This gives \( \mu = 3 \). The case where \( f(v_1) < f(w) < f(v_2) < f(v_3) \) is similar. If \( w \) is an extreme point of \( f \), then \( \text{nlm}(w) = \pm \frac{9}{2} \) and \( \mu \geq \frac{9}{2} \geq 3 \), contradicting flatness of \( f \). This shows that \( \text{NTC}(\{\Gamma\}) = \text{NTC}(f) = 6\pi \).

On the other hand, we may show as in Observation 6.9 that \( B(\{\Gamma\}) = \frac{3}{2} \). All four vertices have odd degree, so \( k = 4 \), and \( \pi \left( 2B(\{\Gamma\}) + k/2 \right) = 5\pi \).

Let \( W_m \) denote the “wheel” of \( m \) spokes, consisting of a cycle \( C \) containing \( m \) vertices \( v_1, \ldots, v_m \) (the “rim”), a central vertex \( w \) (the “hub”) not on \( C \), and edges \( E_i \) (the “spokes”) connecting \( w \) to \( v_i \), \( 1 \leq i \leq m \).

Corollary 6.13. The minimum net total curvature \( \text{NTC}(\{W_m\}) \) for graphs in \( \mathbb{R}^3 \) homeomorphic to \( W_m \) equals \( \pi \left( 2 + \lfloor m/2 \rfloor \right) \).

Proof. We have one “hub” vertex \( w \) with \( d(w) = m \), and all other vertices have degree 3. Observe that the bridge number \( B(\{W_m\}) = 1 \). According to Theorem 6.11, we have \( \text{NTC}(\{W_m\}) = \pi \left( 2B(\{W_m\}) + k/2 \right) \), where \( k \) is the number of vertices of odd degree: \( k = m \) if \( m \) is even, or \( k = m + 1 \) if \( m \) is odd: \( k = 2[\lfloor m/2 \rfloor] \). Thus \( \text{NTC}(\{W_m\}) = \pi \left( 2 + \lfloor m/2 \rfloor \right) \).

7. Lower bounds of net total curvature

The width of an isotopy class \( [\Gamma] \) of embeddings of a graph \( \Gamma \) into \( \mathbb{R}^3 \) is the minimum among representatives of the class of the maximum number of points of the graph meeting a family of parallel planes. More precisely, we write

\[
\text{width}(\{\Gamma\}) := \min_{f : \Gamma \rightarrow \mathbb{R}^3} \min_{e \in S^2} \max_{s \in \mathbb{R}} \#(e, s).
\]

For any homeomorphism type \( \{\Gamma\} \) define \( \text{width}(\{\Gamma\}) \) to be the minimum over isotopy types.

Theorem 7.1. Let \( \Gamma \) be a graph, and consider an isotopy class \( [\Gamma] \) of embeddings \( f : \Gamma \rightarrow \mathbb{R}^3 \). Then

\[
\text{NTC}(\{\Gamma\}) \geq \pi \cdot \text{width}(\{\Gamma\}).
\]
As a consequence,

$$\text{NTC}([\Gamma]) \geq \pi \ \text{width}([\Gamma]).$$

Moreover, if for some $e \in S^2$, an embedding $f : \Gamma \to \mathbb{R}^3$ and $s_0 \in \mathbb{R}$, the integers $(e, s)$ are increasing in $s$ for $s < s_0$ and decreasing for $s > s_0$, then $\text{NTC}([\Gamma]) = \#(e, s_0) \pi$.

Proof. Choose an embedding $g : \Gamma \to \mathbb{R}^3$ in the given isotopy class, with

$$\min_{e \in S^2} \max_{s \in \mathbb{R}} \#(e, s) = \text{width}([\Gamma]).$$

There exist $e \in S^2$ and $s_0 \in \mathbb{R}$ with $(e, s_0) = \max_{s \in \mathbb{R}} \#(e, s) = \text{width}([\Gamma])$. Replace $e$ if necessary by a nearby point in $S^2$ so that the values $g(v_i), i = 1, \ldots, m$ are distinct. Next do cylindrical shrinking: without changing $\#(e, s)$ for $s \in \mathbb{R}$, shrink the image of $g$ in directions orthogonal to $e$ by a factor $\delta > 0$ to obtain a family $\{g_3\}$ from the same isotopy class $[\Gamma]$, with $\text{NTC}(g_3) \to \text{NTC}(g_0)$, where we may identify $g_0 : \Gamma \to \mathbb{R} e \subset \mathbb{R}^3$ with $p_e \circ g = p_e \circ g_3 : \Gamma \to \mathbb{R}$. But

$$\text{NTC}(p_e \circ g) = \frac{1}{2} \int_{S^2} \mu(u) \, dA_{S^2}(u) = 2\pi \mu(e),$$

since for $p_u \circ p_e \circ g$, the local maximum and minimum points are the same as for $p_e \circ g$ if $\langle e, u \rangle > 0$ and reversed if $\langle e, u \rangle < 0$ (recall that $\mu(-e) = \mu(e)$).

We write the topological vertices and the local extrema of $g_0$ as $v_1, \ldots, v_m$. Let the indexing be chosen so that $g_0(v_i) < g_0(v_{i+1}), i = 1, \ldots, m - 1$. Now estimate $\mu(e)$ from below: using Lemma 3.5 and Corollary 3.10,

$$\mu(e) = \sum_{i=1}^{m} \text{nlm}^+_{g_0}(e, v_i) \geq \sum_{i=k+1}^{m} \text{nlm}_{g_0}(e, v_i) = \frac{1}{2} \#(e, s)$$

for any $s, g_0(v_k) < s < g_0(v_{k+1})$. This shows that $\mu(e) \geq \frac{1}{2} \text{width}([\Gamma])$, and therefore

$$\text{NTC}(g) \geq \text{NTC}(g_0) = 2\pi \mu(e) \geq \pi \text{width}([\Gamma]).$$

Now suppose that the integers $(e, s)$ are increasing in $s$ for $s < s_0$ and decreasing for $s > s_0$. Then for $g_0(v_i) > s_0$, we have $\text{nlm}(e, g_0(v_i)) \geq 0$ by Lemma 3.5, and the inequality (7-1) becomes equality at $s = s_0$. \qed

**Lemma 7.2.** For an integer $l$, the minimum width of the complete graph $K_{2l}$ on $2l$ vertices is width($K_{2l}$) = $l^2$; for $2l + 1$ vertices, width($K_{2l+1}$) = $l(l + 1)$.

Proof. Write $E_{ij}$ for the edge of $K_m$ joining $v_i$ to $v_j$, $1 \leq i < j \leq m$, and suppose $g : K_m \to \mathbb{R}$ has distinct values at the vertices: $g(v_1) < g(v_2) < \cdots < g(v_m)$.

Then for any $g(v_k) < s < g(v_{k+1})$, there are $k(m-k)$ edges $E_{ij}$ with $i \leq k < j$; each of these edges has at least one interior point mapping to $s$, which shows that $\#(e, s) \geq k(m-k)$. If $m$ is even: $m = 2l$, these lower bounds have the maximum
value $l^2$ when $k = l$. If $m$ is odd: $m = 2l + 1$, these lower bounds have the maximum value $l(l + 1)$ when $k = l$ or $k = l + 1$. This shows that the width of $K_{2l} \geq l^2$ and the width of $K_{2l+1} \geq l(l + 1)$. On the other hand, equality holds for the piecewise linear embedding of $K_m$ into $\mathbb{R}$ with vertices in general position and straight edges $E_{ij}$, which shows that width($[K_{2l}]$) = $l^2$ and width($[K_{2l+1}]$) = $l(l + 1)$.

**Proposition 7.3.** For all $g : K_m \rightarrow \mathbb{R}$, NTC$(g) \geq \pi l^2$ if $m = 2l$ is even; and NTC$(g) \geq \pi l(l + 1)$ if $m = 2l + 1$ is odd. Equality holds for an embedding of $K_m$ into $\mathbb{R}$ with vertices in general position and monotone on each edge; therefore NTC($[K_{2l}]$) = $\pi l^2$, and NTC($[K_{2l+1}]$) = $\pi l(l + 1)$.

**Proof.** The lower bound on NTC($[K_m]$) follows from Theorem 7.1 and Lemma 7.2.

Now suppose $g : K_m \rightarrow \mathbb{R}$ is monotone on each edge, and number the vertices of $K_m$ so that for all $i$, $g(v_i) < g(v_{i+1})$. Then as in the proof of Lemma 7.2, $(e, s) = (k(m - k) \leq g(v_k) < s < g(v_{k+1})$. These cardinalities are increasing for $0 \leq k \leq l$ and decreasing for $l + 1 < k < m$. Thus, if $g(v_i) < s_0 < g(v_{i+1})$, then by Theorem 7.1, NTC($[K_i]$) = $(e, s_0) \pi = l(m - l) \pi$, as claimed.

Let $K_{m,n}$ be the complete bipartite graph with $m + n$ vertices divided into two sets: $v_i, 1 \leq i \leq m$ and $w_j, 1 \leq j \leq n$, having one edge $E_{ij}$ joining $v_i$ to $w_j$, for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

**Proposition 7.4.** NTC($[K_{m,n}]$) = $[mn/2] \pi$.

**Proof.** $K_{m,n}$ has vertices $v_1, \ldots, v_m$ of degree $d(v_i) = n$ and vertices $w_1, \ldots, w_n$ of degree $d(w_j) = m$. Consider a mapping $g : K_{m,n} \rightarrow \mathbb{R}$ in general position, so that the $m + n$ vertices of $K_{m,n}$ have distinct images. We wish to show $\mu(e) = \mu(g(e)) \geq mn/4$, if $m$ or $n$ is even, or $(mn + 1)/4$, if both $m$ and $n$ are odd.

For this purpose, according to Proposition 3.17, we may first reduce $\mu(e)$ or leave it unchanged by replacing $g$ with a mapping (also called $g$) which is monotone on each edge $E_{ij}$ of $K_{m,n}$. The values of $nlm(w_j)$ and of $nlm(v_i)$ are now determined by the order of the vertex images $g(v_1), \ldots, g(v_m), g(w_1), \ldots, g(w_n)$. Since $K_{m,n}$ is symmetric under permutations of $\{v_1, \ldots, v_m\}$ and permutations of $\{w_1, \ldots, w_n\}$, we shall assume that $g(v_i) < g(v_{i+1})$, $i = 1, \ldots, m - 1$ and $g(w_j) < g(w_{j+1})$, $j = 1, \ldots, n - 1$. For $i = 1, \ldots, m$ we write $k_i$ for the largest index $j$ such that $g(w_j) < g(v_i)$. Then $0 \leq k_1 \leq \cdots \leq k_m \leq n$, and these integers determine $\mu(e)$. According to Lemma 3.5, $nlm(v_i) = k_i - n/2$, $i = 1, \ldots, m$. For $j \leq k_1$ and for $j \geq k_m + 1$, we have $nlm(w_j) = \pm m/2$; for $k_1 < j \leq k_2$ and for $k_{m-1} < j \leq k_m$, we find $nlm(w_j) = \pm (m/2 - 1)$; and so on until we find $nlm(w_j) = 0$ on the middle interval $k_p < j \leq k_{p+1}$, if $m = 2p$ is even; or, if $m = 2p + 1$ is odd, $nlm(w_j) = -\frac{1}{2}$ for $k_p < j \leq k_{p+1}$ and $nlm(w_j) = +\frac{1}{2}$ for the other middle interval $k_{p+1} < j \leq k_{p+2}$. Thus according to Lemma 3.5 and Corollary 3.12, if $m = 2p$ is
even, (7-2)

\[ 2\mu(e) = \sum_{i=1}^{m} |\text{nlm}(u_i)| + \sum_{j=1}^{n} |\text{nlm}(w_j)| = \sum_{i=1}^{m} |k_i - \frac{n}{2}| + (k_1 + n - k_m)\frac{m}{2} \]

\[ + (k_2 - k_1 + k_m - k_{m-1})\left[\frac{m}{2} - 1\right] + \cdots \]

\[ + (k_p - k_{p-1} + k_{p+2} - k_{p+1})\left[\frac{m}{2} - (p - 1)\right] + (k_{p+1} - k_p)[0] \]

\[ = \sum_{i=1}^{m} |k_i - \frac{n}{2}| + mn/2 + \sum_{i=1}^{p} k_i - \sum_{i=p+1}^{m} k_i \]

\[ = \frac{mn}{2} + \sum_{i=1}^{p} \left[ |k_i - \frac{n}{2}| + (k_i - \frac{n}{2}) \right] + \sum_{i=p+1}^{m} \left[ |k_i - \frac{n}{2}| - (k_i - \frac{n}{2}) \right]. \]

Note that formula (7-2) assumes its minimum value \(2\mu(e) = mn/2\) when

\[ k_1 \leq \cdots \leq k_p \leq n/2 \leq k_{p+1} \leq \cdots \leq k_m. \]

If \(m = 2p + 1\) is odd, then

\[ 2\mu(e) = \sum_{i=1}^{m} |k_i - \frac{n}{2}| + (k_1 + n - k_m)\frac{m}{2} + (k_2 - k_1 + k_m - k_{m-1})\left[\frac{m}{2} - 1\right] + \cdots \]

\[ + (k_{p+3} - k_{p+2})\left[\frac{m}{2} - (p - 1)\right] + (k_{p+2} - k_p)\left[\frac{1}{2}\right] = \]

\[ = \sum_{i=1}^{m} |k_i - \frac{n}{2}| + mn/2 + \sum_{i=1}^{p} k_i - \sum_{i=p+2}^{m} k_i \]

\[ = \frac{mn}{2} + \sum_{i=1}^{p} \left[ |k_i - \frac{n}{2}| + (k_i - \frac{n}{2}) \right] \]

\[ + \sum_{i=p+2}^{m} \left[ |k_i - \frac{n}{2}| - (k_i - \frac{n}{2}) \right] + k_{p+1} - \frac{n}{2}. \]

Observe that formula (7-3) has the minimum value \(2\mu(e) = \frac{1}{2}mn\) when \(n\) is even and \(k_1 \leq \cdots \leq k_p \leq \frac{1}{2}n = k_{p+1} \leq \cdots \leq k_m\). If \(n\) as well as \(m\) is odd, then the last term \(|k_{p+1} - \frac{1}{2}n|\) is at least \(\frac{1}{2}\), and the minimum value of \(2\mu(e)\) is \(\frac{1}{2}(mn + 1)\), attained if and only if \(k_1 \leq \cdots \leq k_p \leq \frac{1}{2}n \leq k_{p+2} \leq \cdots \leq k_m\).

This shows that for either parity of \(m\) or of \(n\), \(\mu(e) \geq \frac{1}{4}mn\). If \(n\) and \(m\) are both odd, we have the stronger inequality \(\mu(e) \geq \frac{1}{4}(mn + 1)\). We may summarize these conclusions as \(2\mu(e) \geq \left\lceil \frac{1}{2}mn \right\rceil\), and therefore as in the proof of Corollary 3.15, NTC((\(K_m, n\)) \(\geq \left\lceil \frac{1}{2}mn \right\rceil\pi\), as we wished to show.

By abuse of notation, write the formula (7-2) or (7-3) as \(\mu(k_1, \ldots, k_m)\).
To show the inequality in the opposite direction, we need to find a mapping \( f : K_{m,n} \to \mathbb{R} \) with \( \text{NTC}(f) = \frac{1}{2} mn \pi \) (\( m \) or \( n \) even) or \( \text{NTC}(f) = \frac{1}{2} (mn + 1) \pi \) (\( m \) and \( n \) odd). The above computation suggests choosing \( f \) with \( f(v_1), \ldots, f(v_m) \) together in the middle of the images of the \( w_j \). Write \( n = 2l \) if \( n \) is even, or \( n = 2l + 1 \) if \( n \) is odd. Choose values \( f(w_1) < \cdots < f(w_l) < f(v_1) < \cdots < f(v_m) < f(w_{l+1}) < \cdots < f(w_n) \), and extend \( f \) monotonically to each of the \( mn \) edges \( E_{ij} \). From formulas (7-2) and (7-3), we have \( \mu(f(e)) = \mu(l, \ldots, l) = \frac{1}{4} mn \), if \( m \) or \( n \) is even; or \( \mu(f(e)) = \mu(l, \ldots, l) = \frac{1}{4} (mn + 1) \), if \( m \) and \( n \) are odd. \( \square \)

Recall that \( \theta_m \) is the graph with two vertices \( q^+, q^- \) and \( m \) edges.

**Corollary 7.5.**

\( \text{NTC}((\theta_m)) = m \pi. \)

**Proof.** \( \theta_m \) is homeomorphic to the complete bipartite graph \( K_{m,2} \), and by the proof of Proposition 7.4, we find \( \mu(e) \geq \frac{1}{2} m \) for almost all \( e \in S^2 \), and hence \( \text{NTC}((K_{m,2})) = m \pi \). \( \square \)

**8. Fáry–Milnor type isotopy classification**

Recall the Fáry–Milnor theorem, which states that if the total curvature of a Jordan curve \( \Gamma \) in \( \mathbb{R}^3 \) is less than or equal to \( 4\pi \), then \( \Gamma \) is unknotted. As we have demonstrated above, there are a collection of graphs whose values of the minimum total net curvatures are known. It is natural to hope when the net total curvature is small, in the sense of being in a specific interval to the right of the minimal value, that the isotopy type of the graph is restricted, as is the case for curves: \( \Gamma = S^1 \). The following proposition and corollaries, however, tell us that results of the Fáry–Milnor type cannot be expected to hold for more general graphs.

**Proposition 8.1.** If \( \Gamma \) is a graph in \( \mathbb{R}^3 \) and if \( C \subset \Gamma \) is a cycle, such that for some \( e \in S^2 \), \( p_e \circ C \) has at least two local maximum points, then for each positive integer \( q \), there is a nonisotopic embedding \( \tilde{\Gamma}_q \) of \( \Gamma \) in which \( C \) is replaced by a knot not isotopic to \( C \), with \( \text{NTC}(\tilde{\Gamma}_q) \) as close as desired to \( \text{NTC}(p_e \circ \Gamma) \).

**Proof.** It follows from Corollary 3.15 that the one-dimensional graph \( p_e \circ \Gamma \) may be replaced by an embedding \( \tilde{\Gamma} \) into a small neighborhood of the line \( \mathbb{R} e \) in \( \mathbb{R}^3 \), with arbitrarily small change in its net total curvature. Since \( p_e \circ C \) has at least two local maximum points, there is an interval of \( \mathbb{R} \) over which \( p_e \circ C \) contains an interval which is the image of four oriented intervals \( J_1, J_2, J_3, J_4 \) appearing in that cyclic order around the oriented cycle \( C \). Consider a plane presentation of \( \Gamma \) by orthogonal projection into a generic plane containing the line \( \mathbb{R} e \). Choose an integer \( q \in \mathbb{Z}, |q| \geq 3 \). We modify \( \tilde{\Gamma} \) by wrapping its interval \( J_1, q \) times around \( J_3 \) and returning, passing over any other edges of \( \Gamma \), including \( J_2 \) and \( J_4 \), which it encounters along the way. The new graph in \( \mathbb{R}^3 \) is called \( \tilde{\Gamma}_q \). Then, if \( C \) was the
unknot, the cycle $\tilde{C}_q$ which has replaced it is a $(2, q)$-torus knot (see [Lickorish 1997]). In any case, $\tilde{C}_q$ is not isotopic to $C$, and therefore $\tilde{\Gamma}_q$ is not isotopic to $\Gamma$.

As in the proof of Theorem 7.1, let $g_{\delta} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by cylindrical shrinking, so that $g_1$ is the identity and $g_0 = \tilde{p}$. Then $p_e \circ \tilde{\Gamma}_q = g_0(\tilde{\Gamma}_q)$, and for $\delta > 0$, $g_{\delta}(\tilde{\Gamma}_q)$ is isotopic to $\tilde{\Gamma}_q$. But $\text{NTC}(g_{\delta}) \to \text{NTC}(g_0)$ as $\delta \to 0$. □

**Corollary 8.2.** If $e = e_0 \in S^{n-1}$ minimizes $\text{NTC}(p_e \circ \Gamma)$, and there is a cycle $C \subset \Gamma$ so that $p_{e_0} \circ C$ has two (or more) local maximum points, then there is a sequence of nonisotopic embeddings $\tilde{\Gamma}_q$ of $\Gamma$ with $\text{NTC}(\tilde{\Gamma}_q)$ less than, or as close as desired, to $\text{NTC}(\Gamma)$, in which $C$ is replaced by its connected sum with a $(2, q)$-torus knot.

**Corollary 8.3.** If $\Gamma$ is an embedding of $K_m$ into $\mathbb{R}^3$, linear on each topological edge of $K_m$, $m \geq 4$, then there is a sequence of nonisotopic embeddings $\tilde{\Gamma}_q$ of $\Gamma$ with $\text{NTC}(\tilde{\Gamma}_q)$ as close as desired to $\text{NTC}(\Gamma)$, in which an unknotted cycle $C$ of $\Gamma$ is replaced by a $(2, q)$-torus knot.

**Proof.** According to Corollary 8.2, we only need to construct an isotopy of $K_m$ with the minimum value of NTC, such that there is a cycle $C$ so that $p_e \circ C$ has two local maximum points, where $\mu(e)$ is a minimum among $e \in S^2$.

Choose $g : K_m \to \mathbb{R}$ which is monotone on each edge of $K_m$, and has distinct values at vertices. Then according to Proposition 7.3, we have $\text{NTC}(g) = \text{NTC}([K_m])$. Number the vertices $v_1, \ldots, v_m$ so that $g(v_1) < g(v_2) < \cdots < g(v_m)$. Write $E_{ij}$ for the edge $E_{ij}$ with the reverse orientation, $i \neq j$. Then the cycle $C$ formed in sequence from $E_{13}$, $E_{32}$, $E_{24}$ and $E_{41}$ has local maximum points at $v_3$ and $v_4$, and covers the interval $(g(v_2), g(v_3)) \subset \mathbb{R}$ four times. Since $C$ is formed out of four straight edges, it is unknotted. The procedure of Corollary 8.2 replaces $C$ with a $(2, q)$-torus knot, with an arbitrarily small increase in NTC. □

Note that Corollary 8.2 gives a set of conditions for those graph types where a Fáry–Milnor type isotopy classification might hold. In particular, we consider one of the simpler homeomorphism types of graphs, the *theta graph* $\theta = \theta_3 = K_{3, 2}$ (cf. description following Definition 6.3). The *standard theta graph* is the isotopy class in $\mathbb{R}^3$ of a plane circle plus a diameter. We have seen in Corollary 7.5 that the minimum of net total curvature for a theta graph is $3\pi$. On the other hand note that in the range $3\pi \leq \text{NTC}(\Gamma) < 4\pi$, for $e$ in a set of positive measure of $S^2$, $p_e(\Gamma)$ cannot have two local maximum points. In Theorem 8.5 below, we shall show that a theta graph $\Gamma$ with $\text{NTC}(\Gamma) < 4\pi$ is isotopically standard.

We may observe that there are nonstandard theta graphs in $\mathbb{R}^3$. For example, the union of two edges might be knotted. Moreover, as S. Kinoshita has shown, there are $\theta$-graphs in $\mathbb{R}^3$, not isotopic to a planar graph, such that each of the three cycles formed by deleting one edge is unknotted [Kinoshita 1972].

We begin with a well-known property of curves, whose proof we give for the sake of completeness.
Lemma 8.4. Let \( C \subset \mathbb{R}^3 \) be homeomorphic to \( S^1 \), and not a convex planar curve. Then there is a nonempty open set of planes \( P \subset \mathbb{R}^3 \) which each meet \( C \) in at least four points.

Proof. For \( e \in S^2 \) and \( t \in \mathbb{R} \) write the plane \( P_t^e = \{ x \in \mathbb{R}^3 : \langle e, x \rangle = t \} \).

If \( C \) is not planar, then there exist four noncoplanar points \( p_1, p_2, p_3, p_4 \), numbered in order around \( C \). Note that no three of the points can be collinear. Let an oriented plane \( P_0 \) be chosen to contain \( p_1 \) and \( p_3 \) and rotated until both \( p_2 \) and \( p_4 \) are above \( P_0 \) strictly. Write \( e_1 \) for the unit normal vector to \( P_0 \) on the side where \( p_2 \) and \( p_3 \) lie, so that \( P_0 = P_{t_0=0}^{e_1} \). Then the set \( P_t \cap C \) contains at least four points, for \( t_0 = 0 < t < \delta_1 \), with some \( \delta_1 > 0 \), since each plane \( P_t = P_t^{e_1} \) meets each of the four open arcs between the points \( p_1, p_2, p_3, p_4 \). This conclusion remains true, for some \( 0 < \delta < \delta_1 \), when the normal vector \( e_1 \) to \( P_0 \) is replaced by any nearby \( e \in S^2 \), and \( t \) is replaced by any \( 0 < t < \delta \).

If \( C \) is planar but nonconvex, then there exists a plane \( P_0 = P_0^{e_1} \), transverse to the plane containing \( C \), which supports \( C \) and touches \( C \) at two distinct points, but does not include the arc of \( C \) between these two points. Consider disjoint open arcs of \( C \) on either side of these two points and including points not in \( P_0 \). Then for \( 0 < t < \delta \ll 1 \), the set \( P_t \cap C \) contains at least four points, since the planes \( P_t = P_t^{e_1} \) meet each of the four disjoint arcs. Here once again \( e_1 \) may be replaced by any nearby unit vector \( e \), and the plane \( P_t^e \) will meet \( C \) at least four points, for \( t \) in a nonempty open interval \( t_1 < t < t_1 + \delta \).

\( \square \)

Using the notion of net total curvature, we may extend the theorems of Fenchel [1929] as well as the Fáry–Milnor theorem, for curves homeomorphic to \( S^1 \), to graphs homeomorphic to the theta graph. An analogous result is given by Taniyama [1998], who showed that the minimum of TC for polygonal \( \theta \)-graphs is 4\( \pi \), and that any \( \theta \)-graph \( \Gamma \) with TC(\( \Gamma \)) < 5\( \pi \) is isotopically standard.

Theorem 8.5. Suppose \( f : \theta \to \mathbb{R}^3 \) is a continuous embedding, \( \Gamma = f(\theta) \). Then NTC(\( \Gamma \)) \geq 3\( \pi \). If NTC(\( \Gamma \)) < 4\( \pi \), then \( \Gamma \) is isotopic in \( \mathbb{R}^3 \) to the planar theta graph. Moreover, NTC(\( \Gamma \)) = 3\( \pi \) if and only if the graph is a planar convex curve plus a straight chord.

Proof. We consider first the case when \( f : \theta \to \mathbb{R}^3 \) is piecewise \( C^2 \).

1. We have shown the lower bound 3\( \pi \) for NTC(\( f \)), where \( f : \theta \to \mathbb{R}^n \) is any piecewise \( C^2 \) mapping, since \( \theta = \theta_3 \) is one case of Corollary 7.5, with \( m = 3 \).

2. We show next that if there is a cycle \( C \) in a graph \( \Gamma \) (a subgraph homeomorphic to \( S^1 \)) which satisfies the conclusion of Lemma 8.4, then \( \mu(e) \geq 2 \) for \( e \) in a nonempty open set of \( S^2 \). Namely, for \( t_0 < t < t_0 + \delta \), a family of planes \( P_t^e \) meets \( C \), and therefore meets \( \Gamma \), in at least four points. This is equivalent to saying that the cardinality \#(\( e, t \)) \geq 4 \). This implies, by Corollary 3.10, that \( \sum \{nlm(e, q) : \)}
\[ p_e(q) > t_0 \geq 2. \] Thus, since \( \text{nlm}^+(e, q) \geq \text{nlm}(e, q) \), using Definition 3.8, we have \( \mu(e) \geq 2 \).

Now consider the equality case of a theta graph \( \Gamma \) with \( \text{NTC}(\Gamma) = 3\pi \). As we have seen in the proof of Proposition 7.4 with \( m = 3 \) and \( n = 2 \), the multiplicity \( \mu(e) \geq \frac{3}{2} = \frac{1}{4}mn \) for almost all \( e \in S^2 \), while the integral of \( \mu(e) \) over \( S^2 \) equals \( 2 \text{NTC}(\Gamma) = 6\pi \) by Theorem 3.13, implying \( \mu(e) = 3/2 \) almost everywhere on \( S^2 \). Thus, the conclusion of Lemma 8.4 is impossible for any cycle \( C \) in \( \Gamma \). By Lemma 8.4, all cycles \( C \) of \( \Gamma \) must be planar and convex.

Now \( \Gamma \) consists of three arcs \( a_1, a_2 \) and \( a_3 \), with common endpoints \( q^+ \) and \( q^- \). As we have just shown, the three Jordan curves \( \Gamma_1 := a_2 \cup a_3, \Gamma_2 := a_3 \cup a_1 \) and \( \Gamma_3 := a_1 \cup a_2 \) are each planar and convex. It follows that \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) lie in a common plane. In terms of the topology of this plane, one of the three arcs \( a_1, a_2 \) and \( a_3 \) lies in the middle between the other two. But the middle arc, say \( a_2 \), must be a line segment, as it needs to be a shared piece of two curves \( \Gamma_1 \) and \( \Gamma_3 \) bounding disjoint convex open sets in the plane. The conclusion is that \( \Gamma \) is a planar, convex Jordan curve \( \Gamma_2 \), plus a straight chord \( a_2 \), whenever \( \text{NTC}(\Gamma) = 3\pi \).

(3) We next turn our attention to the upper bound of \( \text{NTC} \), to imply that a \( \theta \)-graph is isotopically standard: we shall assume that \( g : \theta \to \mathbb{R}^3 \) is an embedding in general position with \( \text{NTC}(g) < 4\pi \), and write \( \Gamma = g(\theta) \). By Theorem 3.13, since \( S^2 \) has area \( 4\pi \), the average of \( \mu(e) \) over \( S^2 \) is less than \( 2 \), and it follows that there exists a set of positive measure of \( e_0 \in S^2 \) with \( \mu(e_0) < 2 \). Since \( \mu(e_0) \) is a half-integer, and since \( \mu(e) \geq \frac{3}{2} \), as we have shown in part (1) of this proof, we have \( \mu(e_0) = \frac{3}{2} \) exactly.

From Corollary 6.7 applied to \( p_{e_0} \circ g : \theta \to \mathbb{R} \), we find \( \mu_g(e_0) = \frac{1}{2}(\Lambda + V) + \frac{k}{4} \), where \( \Lambda \) is the number of local maximum points, \( V \) is the number of local minimum points and \( k = 2 \) is the number of vertices, both of degree 3. Thus, \( \frac{3}{2} = \frac{1}{2}(\Lambda + V) + \frac{k}{4} \), so that \( \Lambda + V = 2 \). This implies that the local maximum/minimum points are unique, and must be the unique global maximum/minimum points \( p_{\text{max}} \) and \( p_{\text{min}} \) (which may be one of the two vertices \( q^\pm \)). Then \( p_{e_0} \circ g \) is monotone along edges except at the points \( p_{\text{max}}, p_{\text{min}} \) and \( q^\pm \).

Introduce Euclidean coordinates \((x, y, z)\) for \( \mathbb{R}^3 \) so that \( e_0 \) is in the increasing \( z \)-direction. Write \( t_{\text{max}} = p_{e_0} \circ g(p_{\text{max}}) = (e_0, p_{\text{max}}) \) and \( t_{\text{min}} = (e_0, p_{\text{min}}) \) for the maximum and minimum values of \( z \) along \( g(\theta) \). Write \( t^\pm \) for the value of \( z \) at \( g(q^\pm) \), where we may assume \( t_{\text{min}} \leq t^- < t^+ \leq t_{\text{max}} \).

We construct a “model” standard \( \theta \)-curve \( \hat{\Gamma} \) in the \((x, z)\)-plane, as follows. \( \hat{\Gamma} \) will consist of a circle \( C \) plus the straight chord of \( C \), joining \( \hat{q}^- \) to \( \hat{q}^+ \) (points to be chosen). Choose \( C \) so that the maximum and minimum values of \( z \) on \( C \) equal \( t_{\text{max}} \) and \( t_{\text{min}} \). Write \( \hat{p}_{\text{max}} \) and \( \hat{p}_{\text{min}} \) for the maximum and minimum points of \( z \) along \( C \). Choose \( \hat{q}^+ \) as a point on \( C \) where \( z = t^+ \). There may be two nonequivalent choices
for $\hat{q}^-$ as a point on $C$ where $z = t^-$. we choose so that $\hat{p}_{\text{max}}$ and $\hat{p}_{\text{min}}$ are in the same or different topological edge of $\hat{C}$, where $p_{\text{max}}$ and $p_{\text{min}}$ are in the same or different topological edge, respectively, of $\Gamma$. Note that there is a homeomorphism from $\Gamma$ to $\hat{C}$ which preserves $\hat{z}$.

We now proceed to extend this homeomorphism to an isotopy. For $t \in \mathbb{R}$, write $P_t$ for the plane $\{z = t\}$. As in the proof of Proposition 4.10, there is a continuous 1-parameter family of homeomorphisms $\Phi_t : P_t \to P_t$ such that $\Phi_t(\hat{C} \cap P_t) = \hat{C} \cap P_t$; $\Phi_t$ is the identity outside a compact subset of $P_t$; and $\Phi_t$ is isotopic to the identity of $P_t$, uniformly with respect to $t$. Defining $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi(x, y, z) := \Phi_z(x, y)$, we have an isotopy of $\Gamma$ with the model graph $\hat{C}$.

(4) Finally, consider an embedding $g : \theta \to \mathbb{R}^3$ which is only continuous, and write $\Gamma = g(\theta)$. It follows from Theorem 4.11 that for any $\theta$-graph $\Gamma$ of finite net total curvature, there is a $\Gamma$-approximating polygonal $\theta$-graph $P$ isotopic to $\Gamma$, with $\text{NTC}(P) \leq \text{NTC}(\Gamma)$ and as close as desired to $\text{NTC}(\Gamma)$.

If a $\theta$-graph $\Gamma$ would have $\text{NTC}(\Gamma) < 3\pi$, then the $\Gamma$-approximating polygonal graph $P$ would also have $\text{NTC}(P) < 3\pi$, in contradiction to what we have shown for piecewise $C^2$ theta graphs in part (1) above. This shows that $\text{NTC}(\Gamma) \geq 3\pi$.

If equality $\text{NTC}(\Gamma) = 3\pi$ holds, then $\text{NTC}(P) \leq \text{NTC}(\Gamma) = 3\pi$, so that by the equality case part (2) above, $\text{NTC}(P)$ must equal $3\pi$, and $P$ must be a convex planar curve plus a chord. But this holds for all $\Gamma$-approximating polygonal graphs $P$, implying that $\Gamma$ itself must be a convex planar curve plus a chord.

Finally, if $\text{NTC}(\Gamma) < 4\pi$, then $\text{NTC}(P) < 4\pi$, implying by part (3) above that $P$ is isotopic to the standard $\theta$-graph. But $\Gamma$ is isotopic to $P$, and hence is isotopically standard. \[\square\]

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Robert Gulliver and Sumio Yamada

Entire solutions of Donaldson’s equation

Weiyong He

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