A class of narrow-sense BCH codes over $\mathbb{F}_q$ of length $\frac{q^m-1}{2}$

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Abstract BCH codes with efficient encoding and decoding algorithms have many applications in communications, cryptography and combinatorics design. This paper studies a class of linear codes of length $\frac{q^m-1}{2}$ over $\mathbb{F}_q$ with special trace representation, where $q$ is an odd prime power. With the help of the inner distributions of some subsets of association schemes from bilinear forms associated with quadratic forms, we determine the weight enumerators of these codes. From determining some cyclotomic coset leaders $\delta_i$ of cyclotomic cosets modulo $\frac{q^m-1}{2}$, we prove that narrow-sense BCH codes of length $\frac{q^m-1}{2}$ with designed distance $\delta_i = \frac{q^m}{2} - \frac{q^m-1}{2} - 1 - \frac{q^m-1}{2} = 1 - i$, have the corresponding trace representation, and have the minimal distance $d = \delta_i$ and the Bose distance $d_B = \delta_i$, where $1 \leq i \leq \lfloor \frac{m+3}{4} \rfloor$.

Keywords Linear codes · BCH codes · association schemes · the weight distribution · quadratic forms

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1 Introduction

As an important class of cyclic codes, BCH codes with efficient encoding and decoding algorithms have many applications in communications, cryptography and combinatorics de-
sign. They were independently discovered by Hocquenghem [17] and Bose, Ray-Chaudhuri [3], and were generalized from binary fields to finite fields by Gorenstein and Zierler [15]. The determination of parameters of BCH codes is an interesting and difficult problem. There are many results on BCH codes [1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 19, 22, 23, 24, 26, 28, 30, 31, 33, 34, 38, 39, 40, 41, 42].

Let \( m \) be a positive integer and \( F_q = \mathbb{F}_{q^m} \) be a finite field. Let \( \beta \) be an element in \( F_q \) with order \( n \), where \( n \mid (q^n - 1) \). A BCH code \( C \) over \( F_q \) with designed distance \( d \) is a cyclic code with a generator polynomial \( g(x) \), where \( g(x) \) is determined by its \( \delta - 1 \) consecutive roots \( \beta, \beta^q, ..., \beta^{q^{\delta-1}} \). The minimal distance of the BCH code \( C \) is greater than \( \delta \). When \( n = q^n - 1 \), the code \( C \) is called primitive. When \( l = 1 \), this code \( C \) is called narrow-sense.

Two narrow-sense BCH codes with different designed distances may be the same. The Bose distance \( d_B \) of a BCH code is the largest designed distance. The Bose distance satisfies that \( d_B \geq q^m - q^m - d - 1 \).

The minimal distance of the BCH code is the minimum distance. The determination of the minimum distance \( d \) and the Bose distance of a narrow-sense BCH code attracts much interest [1, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 19, 22, 23, 24, 26, 28, 30, 31, 33, 34, 38, 39, 40, 41, 42].

Some results on \([n, k, d]\) narrow-sense BCH codes over \( F_q \) such that \( d = d_B \) are listed:

- the primitive case: \( n = q^n - 1 \)
  - \( d = d_B = q^m - 1 \), where \( 1 \leq i \leq m - 1 \) [59];
  - \( q = 2 \), \( d = d_B = 2^{m-1} - 2 \), where \( \frac{m - 2}{2} \leq i \leq m - \left\lfloor \frac{m}{2} \right\rfloor - 2 \) [2];
  - \( q = 2 \), \( d = d_B = 2^{m-1-i} - 2^{m-1-i} \), where \( 1 \leq i \leq m - s - 2 \) and \( 0 \leq s \leq 2i - 2 \) [20];
  - \( d = d_B = q^m + 1 \), where \( (m - i) \mid i \) or \( 2i \mid m \) [9];
  - \( d = d_B = (q - h_0)q^{m-h_0} - 1 \), where \( 0 \leq h_0 \leq q - 2 \) and \( 0 \leq h_0 \leq m - 1 \) [7];
  - \( d = d_B = q^m - q^m - q - 1 \), where \( \frac{m - 2}{2} \leq i \leq m - \left\lfloor \frac{m}{2} \right\rfloor - 1 \) [Theorem 1.2, [25]].
- \( n = \frac{q^n - 1}{2} \)
  - \( q = 3 \), \( d = d_B = \frac{2^{m-1} - 1}{2} - 1 - 2^{\frac{m - 2}{2} - i} \) for \( i = 1, 2 \) [27];
  - \( d = d_B = \frac{2^{m-1} - 1}{2} - 1 - 2^{\frac{m - 2}{2} - i} \) for \( i = 1, 2 \) [43];
  - \( d = d_B = \frac{q^m - q^m - 1}{2} - 1 - 2^{\frac{m - 2}{2} - i} \), where \( 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \) [this paper].

To determine the minimal distance \( d \) and the Bose distance \( d_B \) of narrow-sense primitive BCH codes with designed distance \( q^m - q^m - q - 1 \), Li [25] employed the framework of association schemes from bilinear forms associated with quadratic forms and presented the weight enumerators of these BCH codes, where these quadratic forms are corresponding to the trace representation of BCH codes. In [26], Tang et al. studied a class of ternary linear codes with trace representation, determined the weight distributions of some shortened codes and punctured codes of these three-weight subcodes, and presented some 2-designs from these codes.

Motivated by these papers and their ideas, this paper studies the following linear codes \( C_{1,h} \) (resp. \( C_{2,h} \)) of length \( n = \frac{q^n - 1}{2} \) with special trace representation over finite field \( F_q \) for \( m \) odd (resp. even), where \( q \) is odd prime power and

\[
C_{1,h} = \left\{ \left( \text{Tr}_q^n \left( \sum_{j=0}^{\frac{m-1}{2}} a_j \alpha^{q^{j+1}} \right) + a \right)^{n-1} : a_h, \ldots, a_{m-1} \in F_{q^m}, a \in F_q \right\}
\]

and

\[
C_{2,h} = \left\{ \left( \text{Tr}_q^n \left( \sum_{j=0}^{\frac{m-1}{2}} a_j \alpha^{q^{j+1}} \right) + a \right)^{n-1} : a_h, \ldots, a_{m-1} \in F_{q^m}, a \in F_q \right\}.
\]
Then $d$ of $C_{0,\alpha}$ is called a cyclic code with designed distance $d$ such that $d = d_\beta = \delta_i$, where $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$.

The rest of the paper is organized as follows. Section 2 introduces some basic results on BCH codes, quadratic forms and association schemes. Section 3 studies some linear codes with trace representation, uses association schemes to determine the weight enumerators of these codes, and presents the minimum distance $d$ and the Bose distance of the narrow-sense BCH codes of length $n = \frac{q^m-1}{q-1}$ with designed distance $\delta_i$. Section 4 makes a conclusion.

2 Preliminaries

In this section, some results on BCH codes, quadratic forms and association schemes are introduced.

2.1 BCH codes

Let $q$ be a prime power. An $[n,k,d]$ linear code $C$ over the finite field $\mathbb{F}_q$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$, where $d$ is the minimal distance of $C$. A linear code is called a cyclic code if $(c_0,c_1,\ldots,c_{n-1}) \in C$ implies that $(c_{n-1},c_0,\ldots,c_{n-2}) \in C$. We also write a cyclic code as a principal ideal of the ring $\mathbb{F}_q[x]/(x^n - 1)$, since there is a map

$$
\psi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q[x]/(x^n - 1)
$$

$$(c_0,c_1,\ldots,c_{n-1}) \mapsto c_1 + c_1 x + \cdots + c_{n-1} x^{n-1}.
$$

Then $C = \langle g(x) \rangle$, where $g(x)$ is a monic polynomial and is called the generator polynomial of $C$. Note that $g(0)$ is the parity-check polynomial of $C$. The code $C$ is called a cyclic code with $s$ zeros (resp. $s$ nonzeros) if $g(x)$ (resp. $h(x)$) can be factored into a product of $s$ irreducible polynomials over $\mathbb{F}_q$.

Let $n$ be a positive integer and $m$ be the smallest positive integer such that $n \mid (q^m - 1)$. Let $\alpha$ be a primitive element of $\mathbb{F}_{q^m}$ and $\beta = \alpha^{\frac{q^m - 1}{n}}$. Let $m_i(x)$ be the minimal polynomial of $\beta^i$ over $\mathbb{F}_q$, where $0 \leq i \leq n-1$. For each $2 \leq \delta \leq n$, define two polynomials

$$
g_{(\alpha,q,m,\delta)}(x) = \text{lcm}(m_1(x),\ldots,m_{\delta-1}(x)) \quad \text{and} \quad \tilde{g}_{(\alpha,q,m,\delta)}(x) = (x-1)g_{(\alpha,q,m,\delta)}(x),
$$

where $\text{lcm}$ denotes the least common multiple of the polynomials. Let $C_{\alpha,q,m,\delta} = \langle g_{(\alpha,q,m,\delta)}(x) \rangle$ and $\tilde{C}_{\alpha,q,m,\delta} = \langle \tilde{g}_{(\alpha,q,m,\delta)}(x) \rangle$. Then $C_{\alpha,q,m,\delta}$ is called a narrow-sense BCH code with designed distance $\delta$ and $C_{\alpha,q,m,\delta}$ is the even-like subcode of $C_{\alpha,q,m,\delta}$. Note that $\text{dim}(C_{\alpha,q,m,\delta}) = \text{dim}(C_{\alpha,q,m,\delta}) - 1$ and the minimal distances of $C_{\alpha,q,m,\delta}$ and $\tilde{C}_{\alpha,q,m,\delta}$ are at least $\delta$ and $\delta + 1$, respectively. If $n = \frac{q^m - 1}{n}$, the code $C_{\alpha,q,m,\delta}$ is called a narrow-sense primitive BCH code. Note that two BCH codes with different designed distances may be the same. For a narrow-sense BCH code $C_{\alpha,q,m,\delta}$, the Bose distance $[31]$ of this code is defined by the largest designed distance $d_\beta = \max(\delta')$, where $C_{\alpha,q,m,\delta} = C_{\alpha,q,m,\delta}$.
The narrow-sense BCH code $C_{(n, q, m, \delta)}$ has close relation with $q$-cyclotomic cosets modulo $n$. The $q$-cyclotomic coset of $s$ modulo $n$ is defined by

$$C_s = \{s, sq, sq^2, \ldots, sq^{l_s - 1}\} \mod n \subseteq \mathbb{Z}_n,$$

where $0 \leq s \leq n - 1$ and $l_s$ called the size of the $q$-cyclotomic coset is the smallest positive integer such that $s \equiv sq^{l_s} \mod n$. Note that $l_s \mid m$. The coset leader of $C_s$ is the smallest integer in $C_s$. Let $\Gamma_{(n, q)}$ be the set of all the coset leaders. Then $C_s \cap C_t = \emptyset$ and $\mathbb{Z}_n = \bigcup_{s \in \Gamma_{(n, q)}} C_s$, where $s, t \in \Gamma_{(n, q)}$ and $s \neq t$.

Let $m_s(x)$ be the minimal polynomial of $\beta^s$ over $\mathbb{F}_q$. Then

$$m_s(x) = \prod_{i \in C_s} (x - \beta^i) \in \mathbb{F}_q[x],$$

deg($m_s(x)$) = $l_s$, and

$$x^n - 1 = \prod_{s \in \Gamma_{(n, q)}} m_s(x).$$

Hence, the generator polynomial $g_{(n, q, m, \delta)}(x) = \prod_{s \in \Gamma_{(n, q)}} m_s(x)$, the parity-check polynomial of $C_{(n, q, m, \delta)}$ is

$$h(x) = (x - 1) \prod_{s \in \delta, s \in \Gamma_{(n, q)}} m_s(x),$$

and the dimension of $C_{(n, q, m, \delta)}$ is

$$\sum_{s \in \delta, s \in \Gamma_{(n, q)}} l_s + 1.$$
2.2 Quadratic forms

This subsection introduces some results on quadratic forms [29]. Let $q$ be an odd prime power and $V$ be an $n$-dimensional vector space over $\mathbb{F}_q$. A quadratic form $Q$ on $V$ is a function from $V$ to $\mathbb{F}_q$ satisfying $Q(\lambda x) = \lambda^2 Q(x)$, where $x \in V$ and $\lambda \in \mathbb{F}_q$. A symmetric bilinear form $B_Q(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$ on $V$ is associated with $Q$. The radical of $Q$ is $\text{Rad}(Q) = Q^{-1}(0) \cap \text{Rad}(B_Q)$, where $\text{Rad}(B_Q) = \{ y \in V : B(x, y) = 0, \forall x \in V \}$ is the radical of the symmetric bilinear form $B_Q$. The radical of $Q$ is a vector space over $\mathbb{F}_q$. The rank of $Q$ is $\text{rank}(Q) = n - \dim(\text{Rad}(Q))$ and the rank of $B_Q$ is $\text{rank}(B_Q) = n - \dim(\text{Rad}(B_Q))$.

**Lemma 2 (Lemma 3.6, [25])** Let $q$ be an odd prime power, $Q$ be a quadratic form on the $n$-dimensional vector space $V$ over $\mathbb{F}_q$ and $B_Q$ be its associated bilinear form. Then $\text{rank}(Q) = \text{rank}(B_Q)$.

Let $V = \mathbb{F}_q^n$ and $Q$ be a quadratic form on $V$. Then $Q$ has the following expression $Q(x) = \sum_{i,j=1}^n c_{ij}x_ix_j$. Two quadratic forms $Q$ and $Q'$ are equivalent if there is an $n \times n$ nonsingular matrix $A$ such that $Q(x_1, \ldots , x_n) = Q'(x_1, \ldots , x_n)$. Every quadratic form $Q$ is equivalent to $\sum_{i=1}^n a_i x_i^2$, where $a_i \in \mathbb{F}_q$ and $r = \text{rank}(Q)$. The type of $Q$ is $\tau = \eta(\prod_{i=1}^n a_i)$, where $\eta$ is the quadratic character of $\mathbb{F}_q$.

**Lemma 3 (Lemma 5.1, [35])** Let $q$ be an odd prime power and $Q$ be quadratic form of rank $r$ and type $\tau$. Let $N(b)$ be the number of solutions $Q(x) = b$, where $b \in \mathbb{F}_q$. Then

$$N(b) = \begin{cases} q^{n-1} + \tau \eta(-1)^v \eta(b)q^r & \text{if } r \text{ is odd;} \\ q^{n-1} + \tau \eta(-1)^v \eta(b)q^{r/2} & \text{if } r \text{ is even.} \end{cases}$$

where $\eta(0) = 0$, $\nu(x) = -1$ for $x \in \mathbb{F}_q^n$ and $\nu(0) = q - 1$.

Let $V = \mathbb{F}_q^n$. Choose a basis $a_1, \ldots , a_n$ of $V$ over $\mathbb{F}_q$. We have a bijection between $\mathbb{F}_q^n$ and $\mathbb{F}_q^n$. Note that the rank $r$ and the type $\tau$ of a quadratic form $Q$ on $V$ are independent of the choice of the basis.

2.3 Association schemes

This subsection introduces some results on association schemes [31, 18, 37]. Let $X$ be a finite set and $R_0, R_1, \ldots , R_k$ be a partition of $X \times X$. Then a pair $(X, (R_i))$ is called an association scheme with $n$ classes if it satisfies

- $R_0 = \{ (x, x) : x \in X \}$;
- for each $i$, there exists $j$ such that the inverse of $R_i$ equals $R_j$;
- if $(x, y) \in R_k$, the number of the set $\{ z \in X : (x, z) \in R_i, (z, y) \in R_j \}$ is a constant $p_{ij}^k$ depending only on $i$, $j$, and $k$ but not on the particular choice of $x$ and $y$.

An association scheme $(X, (R_i))$ is symmetric if for all $i$ the inverse of $R_i$ equals $R_i$. It is commutative if for all $i$, $j$, and $k$, $p_{ij}^k = p_{ji}^k$.

Let $(X, (R_i))$ be a commutative association scheme with $n$ classes and $A_i$ be the adjacency matrix of the digraph $(X, R_i)$. The matrices $A_0, A_1, \ldots , A_n$ span a vector space over the complex numbers called the Bose-Mesner algebra of $(X, (R_i))$ with dimension $n + 1$. This vector space has another uniquely defined basis consisting of minimal idempotent matrices $E_0, E_1, \ldots , E_n$. Then

$$A_i = \sum_{k=0}^n p_{ij}(k)E_k \text{ and } E_k = \frac{1}{|X|} \sum_{i=0}^n Q_i(i)A_i.$$
The uniquely defined numbers \( P(k) \) and \( Q_k(i) \) are called the \( P \)-numbers and the \( Q \)-numbers of \((X, (R_i))\), respectively.

Let \( V \) be a vector space over the finite field \( \mathbb{F}_q \) of dimension \( m \) and \( X(m,q) \) be the set of symmetric bilinear forms on \( V \), where \( q \) is an odd prime power. Let \( \alpha_1, \ldots, \alpha_m \) be a basis of \( V \). Then a symmetric bilinear form \( B \in X(m,q) \) has the \( m \times m \) symmetric matrix

\[
(B(\alpha_i, \alpha_j))_{1 \leq i, j \leq m}.
\]

The rank of \( B \) is the rank of this matrix, which is independent of the choice of the basis. This matrix is congruent to a diagonal matrix, whose diagonal is either zero or \( p \) for some nonzero \( p \in \mathbb{F}_q \). The type of \( B \) is \( \eta(z) \), where \( \eta \) is the quadratic character of \( \mathbb{F}_q \).

Let \( X_{\tau} \) be the set of all the symmetric bilinear forms with rank \( r \) and type \( \tau \). Define

\[
R_{\tau} = \{(A,B) \in X(m,q) \times X(m,q) : A-B \in X_{\tau}\}.
\]

Then \((X(m,q), (R_\tau))\) is an association scheme with \( 2m \) classes.

Let \( Y \) be a subset of \( X(m,q) \) and the inner distribution of \( Y \) be the sequence of numbers \((a_{\tau})\), where

\[
a_{\tau} = \frac{|(Y \times Y) \cap R_{\tau}|}{|Y|}.
\]

Let \( Q_{k}(r,\tau) \) be the \( Q \)-numbers of \((X(m,q), (R_\tau))\). The dual inner distribution of \( Y \) is the sequence of \((a'_{\tau})\), where

\[
a'_{\tau} = \sum_{\tau} Q_{k}(r,\tau)a_{\tau}.
\]

Note that \( \tau \in \{1, -1\} \). When \( r = 0 \), we write \( a_0 \).

**Definition 1** Let \( Y \) be a subset of \( X(m,q) \). The set \( Y \) is a \( d \)-code if \( a_{\tau} = a_{\tau+1} = 0 \) for each \( \tau \in \{1, 2, \ldots, d-1\} \). The set \( Y \) is a proper \( d \)-code if it is a \( d \)-code and it is not a \((d+1)\)-code. The set \( Y \) is a \( \tau \)-design if \( a_{\tau} = a_{\tau+1} = 0 \) for each \( \tau \in \{1, 2, \ldots, \tau\} \). The set \( Y \) is a \((2t+1, \tau)\)-design if it is a \((2t+1)\)-design and \( a_{2t+2,\tau} = 0 \).

Note that the designs involved in associate schemes are not the usual \( t \)-designs studied in combinatorial design theory. Define \( q^2 \)-analogs of binomial coefficients

\[
\binom{n}{k} = \frac{1}{k} \sum_{j=0}^{k} \binom{n}{k} (-1)^{j} q^{j(j-1)} \left( \frac{|Y|}{q^{2(m+1)(n+1+j)}} - 1 \right)
\]

for integers \( n \) and \( k \geq 0 \). Note that \( \binom{n}{0} = 1 \). The inner distribution of \( Y \) can be given by \( q^2 \)-analogs of binomial coefficients in the following theorem and proposition.

**Theorem 1** (Theorem 3.9, [13]) If \( Y \) is a \((2n-2\tau-1)\)-code and a \((2n-2\tau+3)\)-design in \( X(2n+1,q) \), then the inner distribution \((a_{\tau})\) of \( Y \) satisfies

\[
a_{2r-1,\tau} = \frac{1}{2} \binom{n}{i} \left( \frac{|Y|}{q^{2(m+1)(n+1+j)}} - 1 \right),
\]

\[
a_{2r,\tau} = \frac{1}{2} \binom{q^2 + \tau\eta(-1)q^3}{i} \binom{n}{i} \sum_{j=0}^{\delta} (-1)^{j} q^{j(j-1)} \left( \frac{|Y|}{q^{2(m+1)(n+1+j)}} - 1 \right).
\]
for $i > 0$. If $Y$ is a $(2\delta - 1)$-code and a $(2n - 2\delta + 1)\text{-design}$ in $X(2n,q)$, then the inner distribution $(a_{2\iota})$ of $Y$ satisfies

$$a_{2\iota - 1} = \frac{1}{2} (q^2 - 1) \left[ \frac{n}{i} \sum_{j=0}^{i-\delta} (-1)^j q^{j(i-1)} \left( \frac{|Y| q^{j}}{q^{2n+1}(n+1+j-i)} - 1 \right) \right]$$

$$+ \frac{1}{2} \eta(-1)^i q^i \left[ n \sum_{j=0}^{i-\delta} (-1)^j q^{j(i-1)} \left( \frac{|Y|}{q^{2n+1}(n+1+j-i)} - 1 \right) \right]$$

for $i > 0$.

**Proposition 2 (Proposition 3.10, [35])** If $Y$ is a $(2\delta)$-code and a $(2n - 2\delta + 1)$-design in $X(2n,q)$, then the inner distribution $(a_{\iota})$ of $Y$ satisfies

$$a_{2\iota - 1} = \frac{1}{2} (q^2 - 1) \left[ \frac{n}{i} \sum_{j=0}^{i-\delta} (-1)^j q^{j(i-1)} \left( \frac{|Y| q^{j}}{q^{2n+1}(n+1+j-i)} - 1 \right) \right]$$

$$+ \frac{1}{2} \eta(-1)^i q^i \left[ n \sum_{j=0}^{i-\delta} (-1)^j q^{j(i-1)} \left( \frac{|Y|}{q^{2n+1}(n+1+j-i)} - 1 \right) \right]$$

for $i > 0$. If $Y$ is a $(2\delta)$-code and a $(2n - 2\delta + 1, \eta(-1)^{\delta+1})\text{-design}$ in $X(2n+1,q)$, then the inner distribution $(a_{\iota})$ of $Y$ satisfies

$$a_{2\iota - 1} = \frac{1}{2} \left[ \frac{n}{i-1} \sum_{j=0}^{i-\delta} (-1)^j q^{j(i-1)} \left( \frac{|Y|}{q^{2n+1}(n+1+j-i)} - 1 \right) \right]$$

$$+ \frac{1}{2} \left( \frac{-1}{} \right) \left( \frac{n}{\delta - 1} \left( \frac{|Y|}{q^{2n+1}(n+1+j-i)} - 1 \right) \left( \frac{n-\delta}{n-1} \left( q^{n-\delta+1} + 1 \right) - \left( \frac{n\delta}{n-1} \right) \right) \right)$$

$$a_{2\iota} = \frac{1}{2} (q^2 + \eta(-1)^i q^i) \left[ \frac{n}{i} \sum_{j=0}^{i-\delta} (-1)^j q^{j(i-1)} \left( \frac{|Y|}{q^{2n+1}(n+1+j-i)} - 1 \right) \right]$$

$$+ \frac{1}{2} \left( \frac{-1}{} \right) \left( \frac{n}{\delta - 1} \left( q^{n-\delta+1} + 1 \right) \right) \left( \frac{|Y|}{q^{2n+1}(n+1+j-i)} - 1 \right)$$

for $i > 0$.

### 3 A class of linear codes of length $n = \frac{2^m-1}{2}$

In this section, let $q$ be an odd prime power, $m$ be a positive integer, $\alpha$ be a generator of $\mathbb{F}_{q^m}$, and $\beta = \alpha^2$. Then $\beta$ is a primitive $n$-th root of unity in $\mathbb{F}_{q^m}$, where $n = \frac{2^m-1}{2}$.

Let $h$ be a positive integer. When $m$ is odd, define the codes

$$C_{1,h} = \left\{ \left( \text{Tr}_{1}^{m} \left( \sum_{j=0}^{h-1} a_j \alpha^{(q^h+1)j} \right) + a \right)_{l=0}^{n-1} : a_0, \ldots, a_{h-1} \in \mathbb{F}_{q^m}, a \in \mathbb{F}_q \right\}$$
and
\[ \tilde{C}_{1,h} = \left\{ \left( \text{Tr}_q^n \left( \sum_{j=0}^{q^n-1} a_j x^{(q^n+1)j} \right) \right)^{n-1} : a_h, \ldots, a_{q^n-1} \in \mathbb{F}_{q^n}, a \in \mathbb{F}_q \right\}. \]

When \( m \) is even, define the codes
\[ C_{2,h} = \left\{ \left( \text{Tr}_q^n \left( a_2 x^{(q^n+1)1} + \sum_{j=0}^{q^n-1} a_j x^{(q^n+1)j} \right) + a \right)^{n-1} : a_h, \ldots, a_{q^n-1} \in \mathbb{F}_{q^n}, a \in \mathbb{F}_q \right\} \]
and
\[ \tilde{C}_{2,h} = \left\{ \left( \text{Tr}_q^n \left( a_2 x^{(q^n+1)1} + \sum_{j=0}^{q^n-1} a_j x^{(q^n+1)j} \right) + a \right)^{n-1} : a_h, \ldots, a_{q^n-1} \in \mathbb{F}_{q^n}, a \in \mathbb{F}_q \right\}. \]

Let \( Q(x) \) be a quadratic form defined by
\[ Q(x) = \text{Tr}_q^n \left( \sum_{j=1}^{t} a_j x^{q^j} \right), \]
where \( k_j \geq 0 \) and \( a_j \in \mathbb{F}_{q^n} \). The symmetric bilinear form associated with \( Q(x) \) is
\[ B_Q(x,y) = \frac{1}{2} (Q(x+y) - Q(x) - Q(y)) \]
\[ = \frac{1}{2} \sum_{j=1}^{t} \text{Tr}_q^n \left( a_j (x^{q^j} + y^{q^j})(x+y) - a_j x^{q^j+1} - a_j y^{q^j+1} \right) \]
\[ = \frac{1}{2} \sum_{j=1}^{t} \text{Tr}_q^n (a_j x^{q^j} y + a_j x y^{q^j}) \]
\[ = \frac{1}{2} \sum_{j=1}^{t} \text{Tr}_q^n (a_j x^{q^j} y) + \frac{1}{2} \sum_{j=1}^{t} \text{Tr}_q^n (a_j y^{q^j}) \]
\[ = \sum_{j=1}^{t} \text{Tr}_q^n \left( \left( a_j x^{q^j} \right) y \right) + \sum_{j=1}^{t} \text{Tr}_q^n \left( \left( a_j y^{q^j} \right) x \right) \]
\[ = \sum_{j=1}^{t} \text{Tr}_q^n \left( \left( \frac{a_j}{2} x^{q^j} \right) y + \left( \frac{a_j}{2} y^{q^j} \right) x \right). \]

Let \( h \geq 1 \). Define the set of quadratic forms:
\[ Q_1 = \left\{ \text{Tr}_q^n \left( \sum_{j=0}^{m-1} a_j x^{q^j} \right) : a_j \in \mathbb{F}_{q^n} \text{ for } h \leq j \leq \frac{m-1}{2} \right\}, \]
where \( m \) is odd. Define the set of quadratic forms:
\[ Q_2 = \left\{ \text{Tr}_q^n \left( a_{q^n} x^{q^n+1} + \sum_{j=0}^{m-3} a_j x^{q^j} \right) : a_{q^n} \in \mathbb{F}_{q^n}, a_j \in \mathbb{F}_{q^n} \text{ for } h \leq j \leq \frac{m-2}{2} \right\}. \]
where \( m \) is even. Then we have the following two sets of bilinear forms associated with \( Q_1 \) and \( Q_2 \) respectively.

\[
S_1 = \left\{ \text{Tr}_m \left( \sum_{j=0}^{m-1} \left( \frac{a_j}{2} x_j^{q^j} + \frac{a_j}{2} x_j^{q^{-j}} \right) \right) : a_j \in \mathbb{F}_{q^m} \text{ for } h \leq \frac{m-1}{2} \right\} \\
= \left\{ \text{Tr}_m \left( \sum_{j=0}^{m-1} (a_j x_j^{q^j} + a_j x_j^{q^{-j}}) \right) : a_j \in \mathbb{F}_{q^m} \text{ for } h \leq \frac{m-1}{2} \right\}
\]

and

\[
S_2 = \left\{ \text{Tr}_m \left( \sum_{j=0}^{m-1} \left( \frac{a_j}{2} x_j^{q^j} + \frac{a_j}{2} x_j^{q^{-j}} \right) \right) : a_j \in \mathbb{F}_{q^m} \text{ for } h \leq \frac{m-2}{2} \right\} \\
= \left\{ \text{Tr}_m \left( \sum_{j=0}^{m-1} (a_j x_j^{q^j} + a_j x_j^{q^{-j}}) \right) : a_j \in \mathbb{F}_{q^m} \text{ for } h \leq \frac{m-2}{2} \right\}.
\]

Then \( |S_1| = |S_2| = q^{m(\frac{m+1}{2} - h)}. \) The inner distributions of \( S_1 \) and \( S_2 \) are given in the following proposition.

**Proposition 3** Let \( (a_0, a_1, a_{1-1}, \ldots, a_{m_1}, a_{m_{-1}}) \) be the inner distribution of \( S_1 \). If \( m \) is odd, then \( S_1 \) is a proper \( (2h+1) \)-code and \( (m+1-2h) \)-design in \( X(m, q) \), and the inner distribution of \( S_1 \) is given in Theorem\( \)\( \text{[2]} \) If \( m \) is even, then \( S_1 \) is a proper \( (2h) \)-code and \( (m+1-2h) \)-design in \( X(m, q) \), and the inner distribution of \( S_2 \) is given in Proposition\( \)\( \text{[2]} \).

**Proof** We first prove that \( S_1 \) is a \( (m+1-2h) \)-design in \( X(m, q) \), where \( m \) is odd.

Let \( U \) be a \( t \)-dimensional subspace of \( \mathbb{F}_{q^m} \) and \( A \) be a symmetric bilinear form on \( U \), where \( t = m + 1 - 2h \). For \( a = (a_0, a_1, \ldots, a_{m-1}) \in \mathbb{F}_{q^m}^t \), define a bilinear form \( A_a \) on \( \mathbb{F}_{q^m}^t \)

\[
A_a(x, y) = \text{Tr}_m \left( \sum_{j=0}^{m-1} a_j x_j^{q^j+y} \right) = \text{Tr}_m \left( \sum_{j=0}^{m-1} a_j x_j^{q^j+y} \right).
\]

From Lemma 4.6 in \( \text{[33]} \), the set \( \{ A_a : a \in \mathbb{F}_{q^m}^t \} \) is a multiset in which each bilinear form on \( U \) occurs a constant number (depending only on \( t \)) of times. From Lemma 4.3 in \( \text{[25]} \), the number of elements in the multiset \( \{ D + D' : D \in \{ A_a : a \in \mathbb{F}_{q^m}^t \} \} \) that are an extension of \( A \) is a constant independent of \( U \) and \( A \).
For $\alpha = (a_0, a_1, \ldots, a_{t-1}) \in \mathbb{F}_q^t$, we have

$$B_{\alpha}(x, y) + B'_{\alpha}(x, y) = \text{Tr}^m_q \left( \sum_{j=h}^{m-h} a_{j-h}(x^{q^j}y + xy^{q^j}) \right).$$

Then

$$\{D + D' : D \in \{B_\alpha : \alpha \in \mathbb{F}_q^t\} \} = \left\{ \text{Tr}^m_q \left( \sum_{j=h}^{m-h} (a_{j-h} + a_{m-j-h})^{q^j}(x^{q^j}y + xy^{q^j}) \right) : \alpha \in \mathbb{F}_q^t \right\}.$$
Since the degree of the linearized polynomial \( \sum_{j=0}^{m} \left( a_j x^{q^j} + a_j^{m+1} x^{q^j q^m} \right) \) \( y \) over \( \mathbb{F}_{q^m} \) is at most \( m - 2h \), the dimension of the vector space \( \text{Rad}(B(x, y)) \) is at most \( m - 2h \). Hence, \( \text{rank}(B(x, y)) = m - \text{dim}(\text{Rad}(B(x, y))) \geq 2h \). Hence, \( S_1 \) is a \((2h)\)-code in \( X(m, q) \). Since \( S_1 \) is a \((2m + 1 - 2h)\)-design, then \( S_1 \) is a \((2m - 2h, \tau)\)-design. From Proposition 3 with \( \delta = h \), we have \( a_{2h+1, \tau} = 0 \) for \( \tau = \{1, -1\} \). Hence, \( S_1 \) is a \((2h + 1)\)-code. From Theorem 1 we have \( a_{2h+1, \tau} > 0 \). Hence, \( S_1 \) is a proper \((2h + 1)\)-code.

From a similar discussion, we have the corresponding results for \( S_2 \) when \( m \) is even.

**Lemma 4** Let \( q \) be odd and \( Q \) be a quadratic form of rank \( r \geq 1 \) and type \( \tau \) on \( \mathbb{F}_{q^m} \). Then the weight enumerator \( W_{r, \tau} \) of \( (Q(a^r) + a)_{j=0}^{m-1} \) is

\[
Z^{q^{m-1}} + \frac{q - 1}{2} Z^q(q^{m-1} - q^{\eta(-1)} q^m - \frac{m}{2}) + \frac{q - 1}{2} Z^{q^2}(q^{m-1} + q^{\eta(-1)} q^m - 1)
\]

if \( r \) is odd, or

\[
Z^{q^{m-1} - q^{\eta(-1)} q^m - \frac{m}{2}} + (q - 1) Z^{q^2}(q^{m-1} + q^{\eta(-1)} q^m - 1)
\]

if \( r \) is even.

**Proof** Let \( f(x) = Q(x) + a \). We just compute the weight \( \text{wt}(f) \). Let \( N(f = 0) \) be the number of solutions of \( f(x) = 0 \).

When \( a = 0 \), from Lemma 3 we have

\[
N(f = 0) = \begin{cases} q^{m-1}, & \text{if } r \text{ is odd;} \\ q^{m-1} + \eta(-1) q^{\frac{m}{2}}, & \text{if } r \text{ is even.} \end{cases}
\]

Then

\[
\text{wt}(f) = \frac{q^m - N(f = 0)}{2} = \begin{cases} \frac{q^{m-1}}{2}, & \text{if } r \text{ is odd;} \\ \frac{q^{m-1} + \eta(-1) q^m - \frac{m}{2}}{2}, & \text{if } r \text{ is even.} \end{cases}
\]

When \( a \neq 0 \) and \( \eta(-a) = 1 \), from Lemma 3 we have

\[
N(f = 0) = \begin{cases} q^{m-1} + \eta(-1) q^{\frac{m}{2}}, & \text{if } r \text{ is odd;} \\ q^{m-1} - \eta(-1) q^{\frac{m}{2}}, & \text{if } r \text{ is even.} \end{cases}
\]

Then

\[
\text{wt}(f) = \frac{q^m - 1 - N(f = 0)}{2} = \begin{cases} \frac{1}{2} \left( q^m - q^{m-1} - \eta(-1) q^{\frac{m}{2}} - 1 \right), & \text{if } r \text{ is odd;} \\ \frac{1}{2} \left( q^m - q^{m-1} + \eta(-1) q^{\frac{m}{2}} - 1 \right), & \text{if } r \text{ is even.} \end{cases}
\]

When \( a \neq 0 \) and \( \eta(-a) = -1 \), from Lemma 3 we have

\[
N(f = 0) = \begin{cases} q^{m-1} - \eta(-1) q^{\frac{m}{2}}, & \text{if } r \text{ is odd;} \\ q^{m-1} + \eta(-1) q^{\frac{m}{2}}, & \text{if } r \text{ is even.} \end{cases}
\]

Then

\[
\text{wt}(f) = \frac{q^m - 1 - N(f = 0)}{2} = \begin{cases} \frac{1}{2} \left( q^m - q^{m-1} + \eta(-1) q^{\frac{m}{2}} - 1 \right), & \text{if } r \text{ is odd;} \\ \frac{1}{2} \left( q^m - q^{m-1} - \eta(-1) q^{\frac{m}{2}} - 1 \right), & \text{if } r \text{ is even.} \end{cases}
\]

Hence, this lemma follows.
Lemma 5 Let \( q \) be odd and \( Q \) be a quadratic form of rank \( r \geq 1 \) and type \( \tau \) on \( \mathbb{F}_q \). Then the weight enumerator \( W_{r,\tau} \) of \( (Q(x'))_{i=0}^{n-1} \) is \( Z^{\frac{r+1}{2}} \) (resp. \( Z^{\frac{r+1}{2}} + Z^{\frac{r+1}{2}} \)) if \( r \) is odd (resp. even).

Proof From results of \( a = 0 \) in Lemma this lemma follows.

We determine the weight enumerators of \( C_{1,h}, C_{1,h}, C_{1,h}, \) and \( C_{1,h} \) in the following theorem.

Theorem 2 When \( m \) is odd, the weight enumerator of the code \( C_{1,h} \) is

\[
1 + (q - 1)Z^{\frac{r-1}{2}} + \sum_{r=2h+1} \sum_{\tau \in \{1, -1\}} a_{r,\tau}W_{r,\tau},
\]

where \( a_{r,\tau} \) is given in Theorem and \( W_{r,\tau} \) is given in Lemma. The weight enumerator of the code \( C_{1,h} \) is

\[
1 + \sum_{r=2h+1} \sum_{\tau \in \{1, -1\}} a_{r,\tau}W_{r,\tau},
\]

where \( a_{r,\tau} \) is given in Theorem and \( W_{r,\tau} \) is given in Lemma. When \( m \) is even, the weight enumerator of the code \( C_{2,h} \) is

\[
1 + (q - 1)Z^{\frac{r-1}{2}} + \sum_{r=2h} \sum_{\tau \in \{1, -1\}} a_{r,\tau}W_{r,\tau},
\]

where \( a_{r,\tau} \) is given in Proposition and \( W_{r,\tau} \) is given in Lemma. The weight enumerator of the code \( C_{2,h} \) is

\[
1 + \sum_{r=2h} \sum_{\tau \in \{1, -1\}} a_{r,\tau}W_{r,\tau},
\]

where \( a_{r,\tau} \) is given in Proposition and \( W_{r,\tau} \) is given in Lemma.

Proof We first give the weight enumerator of \( C_{1,h} \). When \( m \) is odd, we have

\[
C_{1,h} = \bigcup_{Q \in Q_1} (Q(x') + a)_{i=0}^{n-1}.
\]

The weight enumerator of \( \{ (a)_{i=0}^{n-1} : a \in \mathbb{F}_q \} \) is \( 1 + (q - 1)Z^{\frac{r-1}{2}} \). From Proposition \( S_1 \) is a proper \( (2h + 1) \)-code. For any nonzero \( Q(x) \in Q_1 \), we have \( \text{rank}(Q(x)) = r \geq 2h + 1 \). The weight enumerator \( W_{r,\tau} \) of \( (Q(x') + a)_{i=0}^{n-1} \) with rank \( r > 0 \) is given in Lemma. The inner distribution of \( Q_1 \) is just the inner distribution of \( S_1 \), which is given in Theorem. Hence, we have the weight enumerator of \( C_{1,h} \)

\[
1 + (q - 1)Z^{\frac{r-1}{2}} + \sum_{r=2h+1} \sum_{\tau \in \{1, -1\}} a_{r,\tau}W_{r,\tau}.
\]

From the similar method, we have the other results. Hence, this theorem follows.

Lemma 6 When \( m \geq 2 \) and \( 1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor \), then \( \delta_i = \frac{a-m^{r-1}}{2} - \frac{2^i}{2} \) is the \( i \)-th largest coset leader module \( n \).
Proof When $q = 3$, from [27], this lemma holds. Suppose that $q > 3$. When $m$ is odd, then the $q$-adic expansion of $\delta_i$ is
\[
\delta_i = \left(\frac{q-3}{2}\right)q^{m-1} + (q-1) \sum_{j=\frac{m-1}{2}+1}^{m-2} q^j + \left(\frac{q-1}{2}\right) \sum_{j=0}^{\frac{m-1}{2}-1} q^j.
\]
and its cyclotomic coset $C_i$ is
\[
\left\{ \frac{q^n - 1 - q^{l-1} - q^{l+\frac{m-1}{2}}}{2} : 1 \leq l \leq \frac{m+1}{2} - i \right\} \bigcup \left\{ \frac{q^n - 1 - q^{l-1} - q^{l+\frac{m-1}{2}}}{2} : m+3 - i \leq l \leq m \right\}.
\]
From Lemma 9 in [43], $\delta_i$ is the largest coset leader module $n$. Then we just prove that for $i > 1$, there does not exist a coset leader $s$ satisfying $\delta_i < s < \delta_{i-1}$.

Suppose such a coset leader $s$ satisfying $\delta_i < s < \delta_{i-1}$ exists. From Lemma 8 in [43], then the $q$-adic expansion of $s$ is
\[
s = \left(\frac{q-3}{2}\right)q^{m-1} + (q-1) \sum_{j=\frac{m-1}{2}+1}^{m-2} q^j + k \sum_{j=0}^{\frac{m-1}{2}-1} q^j,
\]
where $k = \frac{m-1}{2} + i - 1$, $\frac{m-1}{2} \leq s_j \leq q - 1$ and $(\frac{q-1}{2}) \sum_{j=0}^{k} q^j < \sum_{j=0}^{\frac{m-1}{2}-1} s_jq^j < (q-1)q^k + (\frac{q-1}{2}) \sum_{j=0}^{\frac{m-1}{2}-1} q^j$.

When $s_i = \frac{q-1}{2}$, from $t > \delta_i$, we have $(q^{m-1-2s} \mod n) < s$, where $0 \leq t < k$ and $i$ is the largest index such that $s_i > \frac{q-1}{2}$. It is a contradiction to the coset leader $s$.

When $s_i < \frac{q-1}{2} < q-1$, then $(q^{m-1+i} \mod n) < s$, which is a contradiction to the coset leader $s$.

Hence, $\delta_i$ is the $i$-th largest coset leader module $n$ when $m$ is odd. From the similar method, it holds when $m$ is even.

The following theorem gives parameters of the BCH code $C_{(a,q,m,\delta)}$.

**Theorem 3** Let $m \geq 2$, $1 \leq i \leq \left\lfloor \frac{m+1}{2} \right\rfloor$, and $h = \left\lfloor \frac{m}{2} \right\rfloor - i + 1$. Then the BCH code $C_{(a,q,m,\delta)}$ is $C_{1,h}$ (resp. $C_{2,h}$) for $m$ odd (resp. even) and it has parameters $[n, i+1 + \frac{m-1}{2} - \left\lfloor \frac{m}{2} \right\rfloor, m, \delta_i]$ and the Bose distance $d_B = \delta_i$. The code $\tilde{C}_{(a,q,m,\delta)}$ is $\tilde{C}_{1,h}$ (resp. $\tilde{C}_{2,h}$) for $m$ odd (resp. even) and it has dimension $(i + \frac{m-1}{2} - \left\lfloor \frac{m}{2} \right\rfloor)m$.

Proof Note that $\delta_i$ is the $i$-th largest coset leader module $n$. When $m$ is odd, we have $l_i = |C_{\delta_i}| = m$ and
\[
\text{Tr}_m \left( b \beta^{\delta_i} \right) = \text{Tr}_m \left( b \alpha^{-q^{n+1} + q^{n+1} + q^{n-1} + q^{n-1}} \right) = \text{Tr}_m \left( b \alpha^{q^{n+1} + q^{n-1}} \right) = \text{Tr}_m \left( b \alpha^{q^{n-1} + q^{n-1}} \right).
\]
Hence the code $C_{(a,q,m,\delta)} = C_{1,h}$ and the dimension of $C_{(a,q,m,\delta)}$ is $(i + 1 + \frac{m-1}{2} - \left\lfloor \frac{m}{2} \right\rfloor)m$.

When $r = 2h + 1$, the $(q(\alpha^j) + a)_{j=0}^{n-1}$ has a codeword of weight $\delta_i$. From Lemma 1, the code
has Bose distance $\delta_i$. Hence, the code $C_{n,q,m,\delta_i}$ has parameters $[r, \lfloor m + 1 + \frac{m - 1}{2} \rfloor, \delta_i]$ and Bose distance $\delta_i$. From the similar discussion, we have the other results of this proposition.

**Example 1** Let $q = 3$, $h = \lfloor \frac{m}{2} \rfloor$, and $i = 1$. When $m$ is odd (resp. even), the code $C_{n,3,m,\delta_1}$ is $C_1 \boxplus q$ (resp. $C_2 \boxplus \varphi$), and it has parameters $[\frac{q^m - 1}{2}, m + 1, \delta_1]$ (resp. $[\frac{q^{m+1} - 1}{2}, m, \delta_1]$) and the weight enumerator in Table II (resp. Table I) in Theorem 19 \cite{27}. When $m$ is odd (resp. even), the code $C_{n,3,m,\delta_1}$ is $C_1 \boxplus q$ (resp. $C_2 \boxplus \varphi$), and it has parameters $[\frac{q^m - 1}{2}, m, 3^{m-1}]$ (resp. $[\frac{q^{m+1} - 1}{2}, \frac{3}{2}m, 3^{m-1} + 3 \frac{3^m}{2}$) and the weight enumerator $1 + (3^m - 1)Z^{3^{m-1}}$ (resp. $1 + (3^m - 1)Z^{3^{m-1} + \frac{3^m}{2}$) in Theorem 22 \cite{27}.

**Example 2** Let $q = 3$, $h = \lfloor \frac{m}{2} \rfloor - 1$, and $i = 2$. When $m$ is odd (resp. even), the code $C_{n,3,m,\delta_2}$ is $C_1 \boxplus q$ (resp. $C_2 \boxplus \varphi$), and it has parameters $[\frac{q^m - 1}{2}, 2m + 1, \delta_2]$ (resp. $[\frac{q^{m+1} - 1}{2}, \frac{3}{2}m + 1, \frac{3^m}{2}$) and the weight enumerator in Table VI (resp. Table V) in Theorem 29 \cite{27}. When $m$ is odd (resp. even), the code $C_{n,3,m,\delta_2}$ is $C_1 \boxplus q$ (resp. $C_2 \boxplus \varphi$), and it has parameters $[\frac{q^m - 1}{2}, 2m, 3^{m-1} - 3^{\frac{m}{2}}]$ (resp. $[\frac{q^{m+1} - 1}{2}, \frac{3}{2}m, 3^{m-1} - 3^{\frac{3m}{2}}$) and the weight distribution in Table IV (resp. Table III) in Theorem 26 \cite{27}.

**Example 3** When $h = \lfloor \frac{m}{2} \rfloor$ and $i = 1$, the code $C_{n,q,m,\delta_i}$ (resp. $C_{n,q,m,\delta_i}$) is given in Theorem 5 (resp. Theorem 4) in \cite{43}. When $h = \lfloor \frac{m}{2} \rfloor - 1$ and $i = 2$, the code $C_{n,q,m,\delta_2}$ (resp. $C_{n,q,m,\delta_2}$) is given in Theorem 7 (resp. Theorem 6) in \cite{43}.

**Remark 1** Let $q = 3$, $h = \lfloor \frac{m}{2} \rfloor - 1$, and $i = 2$. The code $C_{n,3,m,\delta_2}$ is a three-weight code with the weight distribution in Table 1 of \cite{56}. Then these codes can be used to construct 2-designs.

### 4 Conclusion

This paper studies a class of linear codes of length $\frac{q^m - 1}{2}$ over $\mathbb{F}_q$ with special trace representation, uses association schemes from bilinear forms associated with quadratic forms, and determines the weight enumerators of these codes. From determining some cyclotomic coset leaders $\delta_i$ of cyclotomic cosets modulo $\frac{q^m - 1}{2}$, we prove that narrow-sense BCH codes of length $\frac{q^m - 1}{2}$ with designed distance $\delta_i = \frac{q^m - 1}{2} - 1 - \frac{q^m - 1}{2} = \delta_i$ have the corresponding trace representation and have the minimal distance $d = \delta_i$ and the Bose distance $d_B = \delta_i$, where $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$. It is interesting to determine parameters of more BCH codes.

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