Numerical solution through finite elements for dielectric materials in space

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Abstract. The study of physical models is generally modeled by means of partial differential equations which is still an open field in current research, such is the case of Laplace's differential equation which is surely one of the most important representations of the telecommunications industry. In the following article we will use techniques based on finite elements, through this technique we will find the behavior of the electrostatic potential in each part of the conductor: dielectric conductive material and armored material. The configuration of the coaxial cable allows to find the distribution of the dielectric displacement in the entire composition of the same in such a way that it guarantees the maximum data transfer as the signal transmission frequency increases and the attenuation of the external interference signals increases. In the first section we propose to obtain the linear terms that will calculate the weights of the values of a linear combination that numerically approximates the variable of interest through Galerkin's residual theory applying the finite element method, which for the two-dimensional case with elements Triangular linear lines that discretize the cross section of the coaxial cable, in such a way that it is easier to find these constants including the initial conditions and contour, thus allowing to guarantee the existence of a numerical solution.

1. Introduction
The finite element method simplifies the problems defined in a geometric domain, finding a solution in a finite number of points through the construction of a mesh, nodes [1]. In the case of two dimensions, when approximating the solution of a partial differential equation, these elements (subdomains) may be triangular or quadrilateral, tetrahedron or hexahedron in shape. On each element the unknown variable is approximated using a function (linear combination); usually, these functions are polynomials that depend on the nodes used to define the form of the finite element.

The governing equations in this method are integrated over each finite element and the contributions are assembled over the entire domain of the problem, thus obtaining a set of linear finite equations in terms of the unknown parameters in the nodal elements Equations to be solved by techniques of linear algebra [2,3].

2. Methodology
This method allows us to approximate the solution by means of the division of known geometrical figures that complete the original domain including the boundary conditions as integrals by means of Galerkin's residual theory, guaranteeing that the construction procedure is independent of the border conditions of each problem. Let us examine the following Equation (1) [4].
\[ \nabla^2 U(x, y) = 0, \quad (1) \]

with \((x, y) \in K\), and boundary conditions \(u(x_0, y_0) = u_0\) \(y u(x, y) = g\).

For domains \(K \subset \mathbb{R}^2\) by means of triangular elements as in Figure 1(a), we will build a mesh (Figure 1(b)). The boundary conditions are evaluated on each element Figure 1, then, \(K\) is fragmented in subdomains \(K_1, K_2, K_3, \ldots, K_q\) so that each element is consecutive with each element. Suppose that \(X^h\) is a space of polynomials of degree 1 defined in pieces \((K_1, K_2, K_3, \ldots, K_q)\), so it is required that \(X_q\) is a space of \(P_1(K_q)\) and for this to be fulfilled a test function is created \(\varphi(x, y) = a + bx + cy\). As we know the location of each node, we’ll know the values of each vertex in the test function. The position of the different nodes ensures the continuity of the functions defined in pieces, since, by taking the nodes of an element adjacent to the nodes of the next element, it will always be possible to find an element of the linear envelope of the elements of the domain [5].

\[ \text{Figure 1.} \quad \text{(a) Triangular elements; (b) Domain divided into triangular elements. Domain } K \text{ divided into triangular elements.} \]

The functions of the linear Figure are the interpolation polynomials that allow knowing the value of the variable of interest and are given in Equation (2).

\[ \begin{bmatrix} \phi_1 \\ \phi_j \\ \phi_k \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \quad (2) \]

where \(\phi_i, \phi_j, \phi_k\) are the values of the variable of interest in each node, \((i \leftrightarrow 1, j \leftrightarrow 2, k \leftrightarrow 3)\). From the system of equations proposed in Equation (2) one can obtain the values of \(\alpha_1, \alpha_2, \alpha_3\) as a function of the values of \(\phi_i, \phi_j, \phi_k\).

To solve the differential Equation (1) it is possible to apply the Galerkin method which consists in finding a functional that represents the numerical solution of the equation in each node (element) product of the discretization. According to the proposal in [6-9], it is possible to write a solution as a linear combination (see Equation (3)).

\[ V_h(x, y) = \sum_{i=1}^{N} \gamma_i \phi_i(x, y) \in K, \quad (3) \]

with the approximation of the Galerkin method, a linear system of equations of the form \([\alpha]^T \vec{\beta} = \vec{\beta}\) is obtained and the internal product between the functions of interpolation (functional) is defined as the integral (see Equation (4)).

\[ I(u(x, y)) = \iint \left\{ \frac{1}{2} \left( \frac{\partial u(x, y)}{\partial x} \right)^2 + \left( \frac{\partial u(x, y)}{\partial y} \right)^2 \right\} + f(x, y)u(x, y) \right\} dx dy, \quad (4) \]
3. Results and discussion

For dielectric materials, which are polarized when subjected to scalar or vector fields, the electric displacement vector $\vec{D} = \varepsilon \vec{E} = \varepsilon r \varepsilon_0 \vec{E}$ can be related, where $\varepsilon r$ represents the relative permittivity. Now, the Maxwell equation that models the phenomenon of dielectrics in space by means of a partial differential Equation is defined by [10] (see Equation (5) and Equation (6)).

$$\nabla^2 \vec{D} = \rho_v,$$

$$\frac{\partial^2 \vec{D}(xy)}{\partial x^2} + \frac{\partial^2 \vec{D}(xy)}{\partial y^2} = \rho_v,$$  

where $\rho_v$ represents the density of the volumetric charge, which for the case of the dielectric model is zero due to the fact that no polarization is generated (free charges). Therefore, Equation (1) and Equation (2) are rewritten in terms of a potential variable such as in Equation (7).

$$\nabla^2 \vec{D} = \nabla^2 \varepsilon \vec{E} = \nabla(-\varepsilon \nabla U) = 0.$$  

Therefore, with boundary conditions are defined as follows: $u = 1$ in the conductive material (Figure 2, internal circumference) and $u = 0$ in the outer material and the edge (Figure 3, outside material and edge) [11]. In this paper the model obtained in Equation (7) will be solved by means of partial differential equations for a coaxial cable. This phenomenon consists in the interaction of an electric field with objects encapsulated in a metallic shield (isolated), which is known as the outer limit. It is assumed that they are sufficiently long (long) and parallel in such a way that the field can be calculated in a cross section in the middle of the cable [12]. The geometry of the coaxial cable is simple, it consists of a metallic coating and a conductor cable, these two materials are separated by a polymer tube (insulator) $\varepsilon = 2\varepsilon_0 = 2.3 \times 8.85 \times 10^{-12}$.

The coaxial cable was fragmented in triangular elements as shown in Figure 2 (cross section of the coaxial cable), then the initial conditions of the problem are established as indicated in Equation (6), and Equation (7). So that the functional ($u$) is equal to the current flowing through the conductor (inner material Figure 4, Figure 5, Figure 6, and Figure 7) and equal to zero in the insulating materials and in the exterior (Figure 6, Figure 7, and Figure 8).

**Figure 2.** Cross section discretized domain.

**Figure 3.** Value of the potential $u_{max} = 1$ (internal conductor), $u_{min} = 0$ (insulator and external).
Figure 4. Dielectric displacement distribution in the x direction.

Figure 5. Dielectric displacement distribution in the y direction.

Figure 6. Gradient field of the dielectric displacement.

Figure 7. Electric field distribution. Perpendicular to the current circulation in the conductor.

Figure 8. Magnitude of the dielectric displacement along the domain.
4. Conclusions

Coaxial cables can be used in installations where it is necessary to transmit signals without external interference, in such a way that the materials fulfil the following functions: the external conductor separated from the internal conductor by means of a dielectric material (insulator), the external conductor, as well as a return conductor, it must serve as a shield to attenuate the interference signals coming from outside and consequently behaves as a stabilizer of the electrical parameters, in such a way that a signal can be confined and the losses limited as the frequency of the signal that is transmitted.

The distribution of the dielectric displacement vector makes it possible to observe the behavior of the polarized charges in the materials and the direction they take due to a vector source such as the electric field due to the circulation of current by the conductive material. The electrostatic field does not change along the conductive material except in regions where there is presence of dielectric materials (insulator) and in very remote regions (beyond the border).

By means of the numerical approximation and Galerkin's residual theorem, it is possible to find a linear combination that models the behavior of variables of interest in discretized domains, such as those obtained by the finite element method. The finite element method allows to calculate the weight (participation) of each of the interpolation functions defined in pieces in regular (or irregular) material or domains, guaranteeing the continuity of the potential that one wishes to calculate, that for the case treated in this document it meant finding the numerical solution of the Laplace differential equation in isotropic materials with null boundary conditions (coaxial cable) guaranteeing the attenuation of interference signals as shown in the results section.

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