ON MICROLENSING EVENT RATES AND OPTICAL DEPTH TOWARD THE GALACTIC CENTER

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ABSTRACT

The dependence of microlensing timescale frequency distributions and optical depth toward the Galactic center on Galactic model parameters is explored in detail for a distribution of stars consisting of the Zhao bar and nucleus and the Bahcall & Soneira double exponential disk. The high sensitivity of these two microlensing measures to the circular velocity model, velocity dispersions, bulge mass, direction of the line of sight, bar axis orientation, star spatial distribution, and the stellar mass function means no single Galaxy property can be constrained very well without constraining most of the others. However, this same sensitivity will make microlensing a powerful member of the suite of observational techniques that will eventually define the Galaxy properties. The model timescale frequency distributions are compared throughout with that determined empirically by the MACHO group. Although the MACHO empirical data are matched quite well with a nominal velocity model and with a mass function only of hydrogen-burning stars that varies as \( m^{-2.2} \) in the M-star region, uncertainties in galactic structure, kinematics, and content, together with the paucity of published microlensing data, preclude any claim of the model representing the real world. A variation of the mass function \( \sim m^{-1} \) in the M-star region obtained from recent star counts, both local and in the Galactic bulge, fails to yield a sufficient number of short timescale events compared to the MACHO data. The high sensitivity of the microlensing measures to the direction of the line of sight may mean that sufficient microlensing data to constrain the bar distribution of stars is already in hand. The procedure developed here for determining the timescale frequency distribution is particularly convenient for rapidly incorporating model changes as data from all sources continue to accumulate.

Subject headings: Galaxy: center — Galaxy: structure — gravitational lensing

1. INTRODUCTION

The successful development of techniques to monitor microlensing events toward the center of the Galaxy has provided an additional observational probe of Galactic properties. An optical depth that exceeds predictions (Alcock et al. 1996; Udalski et al. 1994) already has confirmed other indications (Dwek et al. 1995) that the Galactic bulge must be bar-shaped with the long axis pointing toward us (Paczyński et al. 1994). More massive lenses tend to have longer timescales, so microlensing places statistical constraints on the mass function of the stellar lenses, although other variables entering into the timescale weaken these constraints. Attempts to define the mass function of the lenses from statistical arguments or from the event timescale frequency distribution have led to models with a large population of brown dwarfs in the bulge (Han & Gould 1996; Han & Lee 1998), but other models exist with few or no brown dwarfs (Zhao, Rich, & Spergel 1996; Zhao & deZeeuw 1998; Mera, Cabrier, & Schaeffer 1998). By measuring optical depth and the distribution of event timescales as a function of the line-of-sight direction, microlensing can also constrain the distribution of stars in the bar. So far there are insufficient data published (40 secure events from the MACHO group, Alcock et al. 1996; and 12 events from the Optical Gravitational Lensing Experiment [OGLE] group, Udalski et al. 1994) for such constraints to be very definitive. In fact, the lack of sufficient observations of all types that can constrain the properties of the Galaxy means that few of the potential constraints on Galactic structure, kinematics, or content that microlensing can contribute to can be yet realized—the major exception perhaps being the confirmation of the bar.

The microlensing optical depth is the probability that a microlensing event will be occurring at any instant in a particular direction for a single source. The timescale frequency is the number of events of a particular duration that occur per unit time, where the unit of time can be the entire observational period and the number of sources can be all of those monitored. These two measures are routinely obtained from a microlensing observational program aimed at monitoring stars toward the Galactic center, and they form the basis for constraining Galactic properties. It is important to appreciate the sensitivity of such data to the various Galactic parameters on which they depend, since they are related to the Galaxy through models with assumed values of the parameters. Most of the Galactic parameters such as stellar velocities as a function of position are poorly constrained observationally, which may allow several ways to match microlensing data with very different Galactic models. Given this uncertainty, we wish to investigate the sensitivity of timescale frequency distributions and optical depths to changes in the values of several Galactic parameters, and to compare the model microlensing measures at every step with the empirical measures deduced from the microlensing events followed by the MACHO group during the summer of 1993 (Alcock et al. 1996). We eliminated the two stars that are likely to be variable stars, two stars that might fail to pass the MACHO criteria for inclusion, and one binary star from the data set. This leaves a rather meager data set of only 40 events, but we chose not to add the 12 OGLE events in the interest of having a data set with all of whose entries are analyzed the same way. The small gain in the statistics from adding the OGLE events does not warrant the added complication.
The best model of the distribution of stars in the Galactic bulge is that of Zhao (1996). This model of the bar-shaped Galactic bulge is based on a model of Dwek et al. (1995) that reproduces the COBE IR surface brightness distributions, but which is also consistent with all other available observations of bulge kinematics. A highly peaked Galactic nucleus is added to the triaxial “boxy” Gaussian distribution of the bar to produce the observed central kinematics. We use the Zhao distribution of stellar mass (bar plus nucleus) and the double exponential disk distribution of stars of Bahcall & Soneira (1980) as a basis for determining microlensing event timescale frequency distributions and microlensing optical depths as functions of Galactic parameters and direction of the line of sight.

In § 2 we develop a general scheme for determining timescale frequency distributions, in which it is relatively easy to substitute a variety of stellar spatial distributions (although we limit these particular changes to different bar orientations), stellar velocities that depend on position in the Galaxy, mass functions, luminosity functions, and line-of-sight directions. In § 3 we discuss Galactic models in general and give details of the particular model that will be the basis for the calculations. A short § 4 describes the Monte Carlo integration scheme for two choices of independent variables in the six-dimensional integral defining the timescale frequency distribution. Section 5 is a series of example calculations, where the first example is a demonstration of the robustness of the timescale frequency distribution obtained with either choice of independent variables. The remaining examples show the effects of changing the visibility of the sources, the orientation of the long axis of the Galactic bar relative to the line from the observer to the Galactic center, the velocity model, the velocity dispersions within one model, the mass function, and the direction of the line of sight. The empirical timescale frequency distribution is superposed on most of the graphs for comparison. The last example in § 5 demonstrates an almost perfect match of the average of model timescale frequency distributions over the 24 MACHO field directions with the empirical distribution.

A brief derivation of the optical depth averaged over the source distribution along the line of sight begins § 6. This expression is used to demonstrate a relatively large variation in optical depth over a region that includes Baade’s window. The spatial distribution of optical depths along each MACHO line of sight is compared with the average optical depth determined empirically from the MACHO observations, where the same model giving such a good match to the empirical timescale frequency distribution gives an average optical depth that is about 2 σ below the empirical mean. The average of the model optical depths over the MACHO fields is raised almost to the empirical mean if the bulge mass is multiplied by 1.5. The variation of optical depth for a particular direction on source visibility and orientation of the bulge axis is also determined. In both §§ 5 and 6, we show the contributions of each separate part of the distribution of stars, which demonstrates how little the disk contributes to both the event rate and optical depth relative to the bulge. The discussion in § 7 points out virtues of and weaknesses in our particular formulation, emphasizes the sensitivity of results to assumptions within the models, and cautions the reader about taking our model match to the data too seriously given the large uncertainties in the real Galactic model, but also points out the eventual power of microlensing to play a major role in the entire suite of observations that will define the properties of the Galaxy in the future. Our conclusions are enumerated in a brief § 8.

2. EVENT TIMESCALE FREQUENCY DISTRIBUTION

A microlensing event is said to occur whenever a source star and lens star pass each other at an angular separation within that of the Einstein ring radius $R_E$ of the lens. The timescale for such an event is defined as $t_E = R_E/v$, where $v$ is the magnitude of the relative transverse velocity between source and lens projected onto the lens plane. This projected transverse relative velocity is given by

$$v = v_L - v_O - (v_S - v_O) \frac{D_{OL}}{D_O},$$

where $v_L$, $v_S$, and $v_O$ are the transverse velocities of the lens, source, and observer, respectively, relative to inertial space, and $D_{OL}$ and $D_O$ are the distances to the lens and source. The Einstein ring radius is

$$R_E = \frac{4Gm}{c^2} \frac{D_{OL}(D_O - D_{OL})}{D_O},$$

where $G$ is the gravitational constant, $m$ is the lens mass, and $c$ is the velocity of light.

A single source at distance $D_O$ will lead to microlensing events with lenses within mass range $dm$ of mass $m$, within $dD_{OL}$ of $D_{OL}$, and with Einstein ring radii $R_E$ at the rate of

$$2vR_E \frac{dn_l(m, D_{OL})}{dm} dD_{OL} dm$$

$$= \frac{8v^2}{c^2} \frac{Gm}{D_O} \frac{D_{OL}(D_O - D_{OL})}{D_O} \frac{d\rho_l(m, D_{OL})}{dm} dD_{OL} dm,$$

where $(dn_l/dm)dm$ is the number density of lenses with masses within $dm$ of $m$, so that $(dn_l/dm)dm$ is the areal density of lenses in the mass range $dm$ in the slab of thickness $dD_{OL}$ at $D_{OL}$. The rate is expressed in terms of mass density of lenses through $d\rho_l(m, D_{OL})/dm = m \frac{dn_l}{dm} / \frac{D_{OL}}{D_O}$, where the $m$ in the denominator cancels the $m$ in $R_E$ in the second form. The transverse velocity $v$ has components $v_b$ and $v_l$ in the directions of increasing Galactic latitude and longitude respectively, such that

$$v = (v_b^2 + v_l^2)^{1/2},$$

with

$$v_b = v_{lb} - v_{ob} - \frac{(v_{sb} - v_{ob})D_{OL}}{D_O},$$

$$v_l = v_{li} - v_{ol} - \frac{(v_{sl} - v_{ol})D_{OL}}{D_O},$$

$$v_{ob} = v_{ob}, \quad v_{ol} = v_{LSR} + v_{ol}, \quad v_{lb} = v_{lb},$$

$$v_{li} = v_{LO} + v_{li}, \quad v_{sb} = v_{sb}, \quad v_{si} = v_{so} + v_{si}.$$

In equation (5), $v_{LSR}$ is the circular velocity of the local standard of rest, $v_{ob,li}$ are the peculiar velocity components of the observer relative to the LSR, $v_{ol}(D_{OL})$ is the circular velocity of the Galaxy at $D_{OL}$, $v_{lb,li}$ are the random lens velocities relative to the mean Galactic rotation, $v_{so}(D_{OL})$ is the circular velocity of the Galaxy at $D_{OS}$, and $v_{sb,li}$ are the
random source velocities. We shall separate the distribution of stars in the Galaxy into disk and triaxial bulge distributions with their own circular velocity distributions and velocity dispersions, where the latter are Gaussian distributed about the mean circular velocities.

The number of source stars in a given field that are visible in range $dD_{OS}$ at $D_{OS}$ is proportional to $n_{s}(D_{OS})D_{OS}^{2.5}$, where $n_{s}(D_{OS})$ is the number density of sources at $D_{OS}$, and where the exponent follows from the increase in area of the cross section of the angular field with distance, and from the assumption that the fraction of stars with luminosities greater than some $L_{*}$ varies as $L_{*}^{6}$ (Kiraga & Paczyński 1994). Hence, the fraction of the visible sources in the field that are within the slab of thickness $dD_{OS}$ is given by

\[
\frac{n(D_{OS})d^{2.5}D_{OS}dD_{OS}}{\int_{0}^{\infty} n(D_{OS})D_{OS}^{2.5}dD_{OS}}.
\]

Equation (3) is constrained to lenses within $dD_{OL}$ at $D_{OL}$, and to lens masses within $dm$ of $m$. To further constrain the event rate to particular ranges of lens and source random velocities and to a range of source distances $dD_{OS}$, we must multiply by the respective fractions of lenses and sources contained within the small range of each parameter. Hence

\[
dF = \frac{8}{t_{E}} G D_{OL}(D_{OS} - D_{OL}) \frac{d\rho_{L}(m, D_{OL})}{dm} dD_{OL}
\]

\[
\times f_{v_{LSR}} f_{v_{SB}} f_{v_{LSR}} n(D_{OS}) D_{OS}^{2.5} dv_{LSR} dv_{SB} dv_{SI}
\]

\[
\times dD_{OL} dD_{OS} dm \int_{0}^{\infty} n(D_{OS}) D_{OS}^{2.5} dD_{OS}^{-1}.
\]

is the rate of events for a single source within $dD_{OS}$ of $D_{OS}$ that also has random velocities within $dv_{LSR}$, $dv_{SB}$ of $(v_{LSR}, v_{SB})$ for lenses within $dD_{OL}$ of $D_{OL}$, with random velocities within $dv_{LSR}$, $dv_{SB}$ of $(v_{LSR}, v_{SB})$ and with mass within $dm$ of $m$. The Gaussian velocity distributions are represented by

\[
f_{v_{LSR}} = \frac{1}{\sqrt{2\pi}\sigma_{v_{LSR}}} \exp \left( -\frac{v_{LSR}^2}{2\sigma_{v_{LSR}}^2} \right),
\]

where $\sigma_{v_{LSR}}$ is the rms value of $v_{LSR}$. The circular velocities of the Galaxy are assumed to be known from the Galactic model. The particular values of all seven parameters in equation (6) determine a particular value of the timescale $t_{E} = R_{E}/v_{c}$, which we have substituted.

To determine the total event rate per unit timescale interval for a particular timescale $t_{E}$, we multiply equation (6) by $d(t_{E} - t_{E}'$) and integrate it over the seven dimensional volume. Before doing this, it is expedient to normalize all distances by $D_{E} = 8$ kpc (the distance to the Galactic center), all velocity components by $v_{LSR}$, and the masses by the solar mass $M_{\odot}$. The timescale $t_{E}$ is normalized by

\[
t_{0} = \frac{4GM_{\odot}D_{E}}{v_{LSR}^2 c^2} = 66.72 \text{ days },
\]

which is the timescale for an event with a solar mass lens located at a distance of 4 kpc with the source at the Galactic center (8 kpc). The source is assumed at rest and the lens velocity and observer velocity are both assumed to be the circular velocity $v_{LSR}$, consistent with a flat Galaxy rotation curve. The numerical value for $t_{0}$ is obtained with $v_{LSR} = 210 \text{ km s}^{-1}$ (Alvarez, May, & Bronfmann 1990; Clemens 1985). These normalizations yield

\[
F = \frac{8 \times 10^{23}}{(3.0856 \times 10^{19})^3} \sqrt{\frac{GD_{E}^6 v_{LSR}^4}{M_{\odot} c^2}}
\]

\[
\times \int_{0.1}^{1.2} \int_{0}^{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{t_{E} - t_{E}'} d\rho_{L}(z, m)
\]

\[
\times f_{v_{LSR}} f_{v_{SB}} f_{v_{LSR}} n_{s}(\zeta) \delta(t_{E} - t_{E}')
\]

\[
\times d\zeta dz dm v_{LSR} dv_{SB} \int_{0.1}^{1.2} \frac{1}{\zeta^2 + 2\zeta n_{s}(\zeta)} d\zeta^{-1}.
\]

as the number of events s$^{-1}$ source$^{-1}$ (unit $t_{E}$)$^{-1}$ at timescale $t_{E} = t_{E}'$. The numerical coefficient means $\rho_{L}$ is expressed in $M_{\odot} \text{ pc}^{-3}$, $\zeta = D_{OL}/D_{E}$, $z = D_{OL}/D_{E}$, and it is understood that all velocities are normalized by $v_{LSR}$ and $t_{E}$ by $t_{0}$. The integral limits correspond to the variables in the same order as the differentials in the integrand. The integral over $\zeta$ starts at 0.1 instead of 0 to avoid the singularity in the integrand. There likely be no sources involved in microlensing events closer to us than 0.8 kpc. The upper limit on $\zeta$ extends the range beyond the Galactic center where source visibility begins to fade from either distance or blending. One can experiment with the effect of changing these limits, since the extent of blending of stellar images depends on observing site seeing.

The integral is to be evaluated approximately with a Monte Carlo technique, but first the dimension is reduced by 1 by changing variables from $m, z, L_{E}, v_{LSR}, v_{SB}, v_{SI}$ to $t_{E}, z, L_{E}, v_{LSR}, v_{SB}, v_{SI}$ and integrating over the $\delta$ function. The Jacobian determinant of the transformation is simply $\partial \zeta / \partial t_{E}$. In dimensionless units,

\[
t_{E} = \frac{m(z^{2} - z)}{\sqrt{v_{LSR}^2 + v_{SB}^2} / \zeta^{2} - z},
\]

such that

\[
\partial \zeta / \partial t_{E} = 2t_{E}(v_{LSR}^2 + v_{SB}^2) / \zeta(z^{2} - z),
\]

The change of variables in the integrand of equation (8) is effected by replacing $dm$ with $(\partial \zeta / \partial t_{E}) dt_{E}$. Integration over $t_{E}$ fixes $t_{E}$ at the value selected by the $\delta$ function, and the remaining six-dimensional integral is

\[
F = \frac{16 \times 10^{33}}{(3.0856 \times 10^{19})^3} \sqrt{\frac{GD_{E}^6 v_{LSR}^4}{M_{\odot} c^2}}
\]

\[
\times \int_{0.1}^{1.2} \int_{0}^{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{t_{E} - t_{E}'} d\rho_{L}(z, m)
\]

\[
\times f_{v_{LSR}} f_{v_{SB}} f_{v_{LSR}} n_{s}(\zeta) \delta(t_{E} - t_{E}')
\]

\[
\times \frac{1}{v_{LSR} dv_{SB} \int_{0.1}^{1.2} \frac{1}{\zeta^2 + 2\zeta n_{s}(\zeta)} d\zeta^{-1}}.
\]

where $t_{E}$ appears only in the expression for $m$ in $d\rho_{L}(z, m)/dm$.

As a check on the procedure, we eliminate $z$ instead of $m$ as an independent variable in equation (8); i.e., $m, L_{E}, \zeta, v_{LSR}, v_{SB}, v_{SI}$ go to $m, t_{E}, \zeta, v_{LSR}, v_{SB}, v_{SI}$, with $\partial \zeta / \partial t_{E}$ being the only term in the Jacobian determinant. Implicit differen-
tiation of the square of equation (9) yields

\[ \frac{\partial z}{\partial t_E} = \left[ \frac{t_{\text{L}}(\zeta - z)}{2\sigma(\zeta - z)} - \frac{t_{\text{L}}(\zeta - z)}{\text{mz}(\zeta - z)} \left( \frac{\partial v_b}{\partial z} + \frac{\partial v_l}{\partial z} \right) \right]^{-1}, \tag{12} \]

where

\[ \frac{\partial v_b}{\partial z} = \frac{v_{\text{sb}} - v_{\text{sb}}}{\zeta}, \quad \frac{\partial v_l}{\partial z} = -1 + \frac{v_{\text{ol}} - v_{\text{ol}}}{\zeta}. \]

Some important assumptions are used in arriving at the above forms of these partial derivatives that will be pointed out below. From equation (9), we can write

\[ a = \frac{m + t_{\text{L}}^2(1 - v_{\text{sb}})^2 + v_{\text{sb}}^2}{\zeta}, \]

\[ b = -m\zeta + 2t_{\text{L}}^2(v_{\text{L}}(1 - v_{\text{sb}}) - v_{\text{Lb}}v_{\text{sb}}), \]

\[ c = t_{\text{L}}^2(\zeta(v_{\text{L}} + v_{\text{LB}})), \]

where \( v_{\text{x,y}} = v'_{\text{x,y}} - v_{\text{L}} \) are transverse velocities relative to the observer. The solutions of the quadratic equation for \( z \) allow its expression in terms of the remaining variables where it occurs in \( \partial z/\partial t_E \) and elsewhere in the integrand. The change of variables in the integrand of equation (8) is effected by replacing \( dz \) with \((\partial z/\partial t_E) dt_E \). Integration over \( t_E \) fixes \( t_E \) at the value selected by the \( \delta \) function, and the remaining six-dimensional integral is

\[ F = \frac{8 \times 10^{-13}}{(3.0856 \times 10^{19})^3} \sqrt{G D_{\odot} v_{\text{LSR}}} \]

\[ \times \int_{0.1}^{1.2} \int_{0.08}^{1.2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{t_E} \frac{z(\zeta - z)}{\zeta} \frac{d\rho(\zeta, m)}{dm} \]

\[ \times \int f_{\text{ej}, f_{\text{ej}}, f_{\text{ej}}, f_{\text{ej}}, f_{\text{ej}}, f_{\text{ej}}}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\zeta}{t_E} \frac{\partial z}{\partial t_E} \frac{\partial z}{\partial t_E} \]

\[ \times \int_{\text{ej}} \int_{\text{ej}} \int_{\text{ej}} \int_{\text{ej}} \int_{\text{ej}} \int_{\text{ej}} \left[ \frac{1.2}{t_{\text{ej}}} \frac{1.2}{t_{\text{ej}}} \frac{2 + 2\eta_2(\zeta)}{t_{\text{ej}}} \right]^{-1}. \tag{13} \]

With fixed \( t_E \), the solutions of the quadratic equation for \( z \), the fractional distance of the lens to the center of the Galaxy, for random choices of the independent variables within their physical ranges must be real with the added condition that \( z < \zeta \). Those sets of variables for which there are no real solutions for \( z \) are simply incompatible with the particular value of \( t_E \). The two expressions for \( F \), equations (11) and (13), should yield identical timescale frequency distributions. We shall demonstrate this below, but we will use the simpler form of equation (11) in subsequent calculations, wherein it is easy to introduce additional \( z \) and \( \zeta \) dependence into the velocities.

3. MODELS

There is an arbitrary degree of complexity that one may incorporate into the Galactic models that specify the distributions of stars and their velocities, and many of the parameters are not well constrained by observations. The random velocities \( v_{\text{sb}, i} = v_{\text{sb}, i}(m, z, b, l) \) in general, with similar dependencies on parameters for \( v_{\text{sb}, i} \). The dependency of the velocity dispersion on stellar mass \( m \) is consistent with the increasing thickness of the spatial distribution in the disk with decreasing mass (see, e.g., Mihalas & Binney 1981, p. 252). The best model of the Galactic bar consistent with the COBE IR intensity distribution (Zhao 1996) explicitly shows the dependence of the dispersions on \( b \) and \( l \). Part of the dependence of the dispersions on \( z \) can be accounted for by separating the Galaxy into disk and bulge regions, with different velocity dispersions in each region but with the dispersions being otherwise independent of \( z \), respectively, for lenses and sources. This latter assumption was used explicitly in evaluating \( \partial v_{\text{sb}, i}/\partial z \) in equation (12), where the \( z \)-dependence of the velocities was ignored. We will relax this assumption in some of the examples below.

The mean circular velocity of the gas clouds in the Galactic disk appears to be a function of \( z \) or \( \zeta \) (Clemens 1985), but the rather complex rotation curve is usually approximated as flat with all mean circular velocities equal to \( v_{\text{lsr}} \) (Alvarez et al. 1990). Here we ignore any peculiar velocity of the LSR relative to the local mean circular velocity (Clemens 1985). The mean circular velocity of the stars is assumed to be the same as that of the gas clouds. A distribution of box-like orbits in the Zhao (1996) bar model matches the observed H I velocity distribution, and it predicts the velocity dispersions that should be expected as a function of \( b \) and \( l \), and presumably as a function of \( z \) or \( \zeta \) inside the bulge.

Microlensing should eventually prove to be a powerful constraint on the stellar mass function for the stellar spatial distribution that dominates the lens population, which is apparently the central bar. This is especially so for the lower mass stars, since there is no other way of observing these stars. Generally, one expects the mass function to depend on \( z \) (or \( \zeta \)), but that dependence is unknown. The frequency of event timescales is the microlensing constraint on the mass function, where the fraction of shorter timescales increases as the fraction of the mass density that is in the smallest stars or brown dwarfs is increased. However, increasing the velocity dispersions or otherwise increasing relative transverse velocities between lenses and sources and concentrating more of the lenses closer to the center of the Galaxy both increase the fraction of short timescale events. Therefore constraints on dispersions and spatial distributions of lenses and sources must be secure before definitive mass functions can be ascertained.

We have adopted above the assumption of Kiraga & Paczyński (1994) that the luminosity function of the sources is such that the fraction of stars with luminosities greater than some luminosity \( L_* \) is proportional to \( L_*^\beta \). This appears to be a fairly good approximation over the important range of luminosities in the K band (Tiede, Frogel, & Terndrup 1995), but less so in the visual band where the microlensing observations are made. More complex luminosity functions may affect conclusions based on microlensing data but are not warranted by observational constraints at this time. Data provided in Tiede et al. (1995) yield \( \beta \approx -1 \), a value that is assumed in most of the examples herein.

In principle, one could incorporate any of these complexities into the Galactic model, but there is little motivation for doing very much of this until observational constraints become more secure. Even without this security, one can determine the effect of various assumptions on the microlensing timescale rate and optical depth (Zhao & deZeeuw 1998) in anticipation of future selection. For our purposes here, we shall adopt a Galactic model for the integration of equation (11) that consists of a double exponential disk and
a bar-shaped bulge. The disk model is that of Bahcall & Soneira (1980):

$$\rho_m = \int_{\text{max}}^{\text{min}} \frac{d\rho}{dm} \propto \rho_{\text{m0}} \exp \left( -\left| z' \right| - \frac{r}{300 \text{ pc}} \right),$$  

where $z'$ is the coordinate perpendicular to the plane of the Galaxy, $r$ is the radial coordinate in the plane of the Galaxy, and $s_j$ is the scale length in the radial direction, which may be less than the 3.5 kpc used by Bahcall & Soneira (1980) (Zhao, Spergel, & Rich 1995; Kent, Dame, & Fazio 1991), and where $\rho_{\text{m0}}$ is chosen such that $\rho_m = 0.05 M_\odot \text{ pc}^{-3}$ at $z' = 0$ and $r = 8$ kpc. Like Zhao et al. (1996), we choose the triaxial bulge model of Zhao (1996) but keep the nucleus and do not terminate the bulge at 3.3 kpc:

$$\rho_m = \rho_0 \left[ \exp \left( -\frac{s_b^2}{2} \right) + s_a^{-1.85} \exp (-s_a) \right],$$  

where

$$s_b^2 = \left[ \left( \frac{x}{\sigma_x} \right)^2 + \left( \frac{y}{\sigma_y} \right)^2 \right]^2 + \left( \frac{z'}{\sigma_z} \right)^4,$$

$$s_a^2 = \frac{q_0^2 (x^2 + y^2) + z^2}{\sigma_z^2},$$  

with $q_0 = 0.6$, $\sigma_x = 1.49$ kpc, $\sigma_y = 0.58$ kpc, and $\sigma_z = 0.40$ kpc. The coefficient $\rho_0$ determines the mass of the bulge. The observations are consistent with the long axis of the bulge inclined about 13°–20° relative to the line of sight to the Galactic center with the near side of the bar lying in the first quadrant; however, here are large uncertainties in this inclination angle (Zhao 1998).

In the applications, the origin of coordinates for both disk and bulge distributions will be transferred to the position of the observer with the following transformation:

$$\begin{align*}
x &= \cos \theta - \zeta \cos b \cos (l - \theta), \\
y &= -\sin \theta - \zeta \cos b \sin (l - \theta), \\
z &= \zeta \sin b,
\end{align*}$$  

for a source in the bulge with $xyz'$ being Galactocentric coordinates along the bulge principal axes, and $\theta$ is the inclination of the bulge axis to the line between the observer and the Galactic center measured in the direction of increasing $l$. If a bulge star acts as a lens, $\zeta$ is replaced by $z$ in equation (17). For the disk distribution, the transformation is

$$\begin{align*}r^2 &= 1 + z^2 \cos^2 b - 2z \cos b \cos l, \\
z' &= z \sin b.\end{align*}$$  

Since $b$ and $l$ are always small angles, $\cos b \approx \cos l \approx 1$ is usually an adequate approximation, and the exponent in equation (14) can be replaced by $(1 - s_j D_8 \sin b/300 \text{ pc}) D_8 z/s_j$.

In some of the examples below, we shall assume the circular velocity of the disk to be 210 km s$^{-1}$, independent of radial position $r$, with the random velocities superposed on this base. The bar can be assumed uniformly rotating at an angular velocity of 63.6 km s$^{-1}$ kpc$^{-1}$, which is slightly higher than the 60 km s$^{-1}$ kpc$^{-1}$ pattern speed chosen by Zhao (1996) but is arbitrarily chosen to match the disk rotation at 3.3 kpc. A bulge velocity dispersion will be superposed on this bar rotation. We shall also consider the case where the bar is not rotating, but neither of these assumptions is consistent with the results of Alvarez et al. (1990) who find the flat rotation curve of the Galaxy gas clouds (210 km s$^{-1}$) to persist down to 2 kpc from the center, with velocities exceeding 210 km s$^{-1}$ at smaller Galactocentric distances. However, there are noncircular streaming motions in the bar that could boost the apparent rotation inside 3.3 kpc (Zhao 1996). Finally, in most of our examples we shall consider the case where both circular velocities and velocity dispersions for lenses and sources are functions of where they are located in the Galaxy; e.g., circular velocity = 210 km s$^{-1}$, and velocity dispersions are those appropriate to the disk for stars further than 3.3 kpc from the Galactic center but would be those appropriate to the bulge for smaller radial distances.

Because there is no compelling argument specifying the mass function variation with Galactic radius, we shall assume a single mass function applies throughout the Galaxy. Most of the mass functions will be limited to the range $0.08 \leq m \leq 2.0 M_\odot$. The upper limit is chosen because there are so few stars with larger mass, and the lower limit is chosen arbitrarily at the hydrogen-burning limit. The consequences of using several different mass functions, sometimes extended into the brown dwarf region, will be demonstrated in examples that illustrate the sensitivity of the microlensing technique to the eventual constraint of the mass function.

## 4. MONTE CARLO INTEGRATION

For the choice of variables $z$, $\zeta$, $v_{\text{l}}$, $v_{\text{L}}$, $v_{\text{g}}$, $v_{\text{g}}$ (eq. [11]), with no explicit $z$ or $\zeta$ dependence for the dispersions, the volume over which the integration is distributed randomly in the Monte Carlo integration is chosen to be $(1.1)(1.2)(6.4) \sigma_{\text{L}} \sigma_{\text{L}} \sigma_{\text{L}} \sigma_{\text{g}} \sigma_{\text{g}} \sigma_{\text{g}}$, so that the random velocities are truncated at $\pm 3\sigma_{xy}$. For the examples where the dispersions and circular velocities are functions of $z$ or $\zeta$, all of the $s_j$'s in the expression for the volume are replaced by the maximum among them. Those random values of $z > \zeta$ are outside the volume over which the integration is taken. For the variables $m$, $\zeta$, $v_{\text{l}}$, $v_{\text{L}}$, $v_{\text{g}}$, $v_{\text{g}}$ (eq. [13]), the volume is $(1.92)(1.1)(6.4) \sigma_{\text{L}} \sigma_{\text{L}} \sigma_{\text{L}} \sigma_{\text{g}} \sigma_{\text{g}} \sigma_{\text{g}}$. We choose $0.1 \leq z, \zeta \leq 1.2$ since there are not likely to be sources closer than 0.8 kpc, and faintness and blending will likely make sources beyond 10 kpc unusable, with the possible exception of the clump giants. The values of the six variables in the integrand are chosen randomly for each evaluation of the integrand, and values of $m$ in the first set and $z$ in the second set as functions of these variables, and the fixed $f_{\text{E}}$, are determined by the solutions of equation (9). All points for which $m$ is not within the specified range in the first form of $F$, or where there are no real solutions for $z$ (the particular choice of random variables not being consistent with the particular timescale) in the second, or for $z \geq \zeta$, do not contribute to the integral. In the second form, the integral is evaluated separately for each of two viable values of $z$, and the results are added to get the total contribution to the integral for a particular set of random variables. A sufficient number of points in the six-dimensional volume are used in evaluating the Monte Carlo integral (typically $10^6$) to yield a relatively smooth frequency distribution as a function of $f_{\text{E}}$.

## 5. EXAMPLES

First we show that both means of calculating the timescale frequency distribution (eqs. [11] and [13]) yield the same result. For this purpose, we assume $v_{\text{L}} = v_{\text{L}} = 210$
km s\(^{-1}\), \(v_{\alpha_0} = 0\), \((\sigma_{L_{\alpha}}, \sigma_{L_{\beta}}, \sigma_{\delta_{\alpha}}, \sigma_{\delta_{\beta}}) = (16, 20, 124, 102)\) km s\(^{-1}\), and \((b, l) = (-4, 4')\), where the bulge velocity dispersions correspond to those given for these coordinates by Zhao (1996), bulge mass \(= 2.2 \times 10^{10} M_{\odot}\), local mass density in the disk model \(= 0.05 M_{\odot} \text{pc}^{-3}\) with the Basu & Rana (1992) mass function (eq. [19]), and disk scale factors are 2.7 kpc and 300 pc. Figure 1 shows values for \(\theta = -20^\circ\) and \(-65^\circ\) that the two means of determining the distribution yield results that differ by amounts that are at most comparable to the error estimates in the Monte Carlo integrals. This gives some unknown confidence in the correctness of the procedure. The first form of the distribution has a much simpler integrand and requires about one-fourth as much computer time, so it will be used hereafter in all calculations. The 189 days and \(12.6 \times 10^6\) source stars used to scale the coordinate are the span of the MACHO bulge observations for the 1993 bulge season (Alcock et al. 1996) and the number of sources they monitored. All of the timescale frequency distribution curves in the figures are such that \(\int_{t_{E2}}^{t_{E1}} F(t) dt\) is the number of events with \(t_{E1} < t < t_{E2}\) for the \(12.6 \times 10^6\) source stars over the 189 day period.

Alcock et al. (1996) empirically determine a sampling efficiency of detection as a function of event timescale, and they further reduce this efficiency by a factor of 0.75 for main-sequence turnoff stars to account for the effects of blending. There is no reduction of the sampling efficiency assumed for the clump giant sources. We use the reciprocal of the overall efficiencies as the inferred number of events of a particular rate and optical depths as lines of sight pass near the otherwise singular density at the Galactic center.

Our choice of the \(z\) and \(\zeta\) dependence of the stellar velocities is somewhat arbitrary, although the rotation of the inner Galaxy is close to the rigid body rotation of the bar pattern adopted by Zhao (1996), and the flat velocity curve of the disk is that determined by Clemens (1985) and Alvarez et al. (1990). However, the latter two sets of authors find the flat rotation curve to apply for \(r > 2\) kpc, with gas cloud velocities even increasing at smaller distances; however, these high central velocities may not be circular. The \(z\) and \(\zeta\) dependence of the velocities used here is perhaps the simplest possibility that is semiconsistent with the observations, although more complicated (still unmotivated) dependences could be handled in principle.

Six mass functions that will be considered are shown below. Proper overall coefficients are introduced in the integrals to yield the Bahcall & Soneira (1980) local density of \(0.05 M_{\odot} \text{pc}^{-3}\) for the disk and a bulge mass of \(2.2 \times 10^{10} M_{\odot}\).

---

**Basu & Rana (1992):**

\[
\phi(m) = \begin{cases} 
0.7294 m^{-6.83}, & 1.205 \leq m \leq 2.0, \\
0.3527 m^{-2.5}, & 0.08 \leq m \leq 0.5327, \\
1.2588 m^{-1}, & 0.08 \leq m \leq 0.5327. 
\end{cases}
\] (19)

**Modified Basu & Rana (1992):**

\[
\phi(m) = \begin{cases} 
0.2861 m^{-6.83}, & 1.205 \leq m \leq 2.0, \\
0.1401 m^{-3}, & 0.08 \leq m \leq 0.5327, \\
0.1919 m^{-2.5}, & 0.08 \leq m \leq 0.5327. 
\end{cases}
\] (20)

**Holtzman et al. (1998):**

\[
\phi(m) = \begin{cases} 
0.5081 m^{-2.2}, & 0.7 \leq m \leq 2.0, \\
0.7795 m^{-1}, & 0.08 \leq m \leq 0.7, \\
0.2542 m^{-2.2}, & 0.08 \leq m \leq 2.0. 
\end{cases}
\] (22)

---

Only the 40 secure events for single stars are retained in the MACHO data set.

For the remainder of our examples, with the exception of Figure 4, we choose the velocities of source and lens to be determined by their position in the Galaxy. For \(r > 3.3\) kpc, we assume the stars have "disk" properties with a flat rotation curve at 210 km s\(^{-1}\) and velocity dispersions represented by \(\sigma_{b1} = 16\) km s\(^{-1}\) and \(\sigma_{b2} = 20\) km s\(^{-1}\) in the directions of Galactic latitude and longitude, respectively. For \(r < 3.3\) kpc, the stars have "bulge" properties with circular velocities decreasing linearly from 210 km s\(^{-1}\) at 3.3 kpc to 0 km s\(^{-1}\) at the center of the Galaxy, and with velocity dispersions \(\sigma_{b1} = 110\) km s\(^{-1}\). The dimensionless distances \(z\) and \(\zeta\) are chosen randomly from their respective intervals, and their values determine the velocity properties of the lens and source, respectively. For example, both lens and source could lie in the bulge with high-velocity dispersions for both. We shall also assume the Zhao (1996) bulge model described above \((M = 2.2 \times 10^{10} M_{\odot})\) along with a double exponential disk model with scale lengths of 2.7 kpc in the plane and 300 pc perpendicular to the plane for all cases below. We shall truncate the inner part of the otherwise singular nucleus in the Zhao (1996) model by requiring the stellar mass density to depend only on \(z\) for values of \(r = (x^2 + y^2)^{1/2} < 560\) pc. This avoids the very large event rates and optical depths as lines of sight pass near the otherwise singular density at the Galactic center.
Kroupa (1995):

\[
\phi(m) = \begin{cases} 
0.1038m^{-2.35}, & 0.075 \leq m \leq 0.35, \\
0.6529m^{-0.6}, & 0.35 \leq m \leq 0.6, \\
0.2674m^{-2.35}, & 0.6 \leq m \leq 10.0.
\end{cases}
\]

The mass functions are such that \( m \phi(m) \, dm \) represents the mass of stars pc\(^{-3}\) in mass range \( dm \) about \( m \) normalized such that the total mass density is \( 1 \, M_\odot \, pc^{-3} \). The first mass function (eq. [19]) is that of Basu & Rana (1992) for local stars, and the second (eq. [20]) is the same except the index on the low-mass region is changed from \(-1\) to \(-2.5\). The third mass function (eq. [21]) is determined by Holtzman et al. (1998), and the fourth (eq. [22]) is our modification, where we have changed the index from \(-1\) to \(-2.2\) in the low-mass region. The fifth and sixth mass functions (eqs. [23] and [24]) are those of Kroupa (1995) and Gould et al. (1997) as represented by Mera et al. (1998). The coefficients are those appropriate to the lower mass cutoff of 0.08 \( M_\odot \), except for the Kroupa (1995) mass function that is cut off at 0.075 \( M_\odot \) for later comparison with some results of Mera et al. (1998). We will extend the latter two mass functions into the brown dwarf region where the coefficients are altered to be consistent with the normalization of 1 \( M_\odot \) pc\(^{-3}\). The modification of the Gould et al. (1997) mass function by Mera et al. (1998) consists of the top line in equation [24] for \( m < 0.11 \). This modification is not supported by the Gould et al. (1997) data, but we will keep it for the purposes of illustration. Also, the index of \(-0.6\) in the second line of equation [24] is reduced to about \(-1.0\) if the unresolved binaries are accounted for (A. Gould 1998, private communication); however, we shall maintain the \(-0.6\) index again for illustration. The Kroupa (1995) and the Basu & Rana (1992) mass functions are based on measured parallaxes of the nearby stars, the Gould et al. (1997) mass function is derived from a luminosity function of more distant M stars (Mera et al. 1998), and the Holtzman et al. (1998) mass function is determined for bulge stars within Baade’s window. The Basu & Rana (1992) mass function, and more often its modified form, will be used in most of the following examples.

Figure 2 shows the effect of changing the luminosity index \( \beta \) from \(-0.5\) to \(-2.0\) superposed on the distribution obtained from the 1993 MACHO bulge data (Alcock et al. 1996). [Although we shall see that the modeled distributions should be averaged over the MACHO field positions for comparison to the empirical distribution of timescale frequencies, the choice of a line of sight toward Galactic coordinates \((b, l) = (-4°, 1°)\) in Baade’s window yields distributions very close to this average shown in Fig. 11.] Fewer sources are visible as this index gets more negative, and the expected decrease in the number of events observed as the index varies from \(-0.5\) to \(-2.0\) is displayed. The modified Basu & Rana (1992) mass function is used, and the curve for \( \beta = -1 \) is the same as that shown as the “complete model” in Figure 8. The other model parameters are specified in the figure caption.

Figure 3 shows the distribution of timescale frequencies for several bar orientations using the Basu & Rana (1992) mass function. The number of events decreases as the magnitude of the inclination of the bulge increases as expected, since fewer stars are located along the line of sight as the long axis of the bulge is pointed further away from the observer. Since the bulge distribution is so dominant over that of the disk, the timescale frequency distribution is almost proportional to the mass of the bulge. Hence, the
Fig. 4.—Effect of changing the velocity model. Galactic model is the same as Fig. 3 except $v_{L0} = 210$ km s$^{-1}$, and $\sigma_L = \sigma_S = \sigma_{Sl} = 110$ km s$^{-1}$. Short timescale event rate and total number of events is increased significantly.

curve for $\theta = -13^\circ$ is in some ways a closer match to the MACHO data if the bulge mass is increased by a factor of 1.5. Zhao et al. (1996) also pointed out the sensitivity of the event rate to the bulge mass, but notice that simply increasing the bulge mass to match the peak in the distribution produces too many events in the 8–40 day range. None of the model distributions seem to produce as many long timescale events as the MACHO group observed.

To show how a different velocity model can affect the model timescale frequencies, we repeat the exercise of where all parameters are identical to those used to Figure 3, construct that figure except the velocities. Instead of assigning circular velocities and velocity dispersions as a function of position in the Galaxy, we assume here that $v_{L0} = 210$ km s$^{-1}$ for all $z$, $v_{S0} = 0$, and $\sigma_L = \sigma_S = \sigma_{Sl} = 110$ km s$^{-1}$ for all $z$ and $\zeta$. Although this dispersion is not realistic, it might be a reasonable approximation if most of the lenses were in fact in the bulge. On the other hand, keeping the source circular velocities zero while assuming that all of the lenses maintain the flat velocity curve of the disk already introduces a significant relative transverse velocity between source and lens before the dispersions are applied. Then the large dispersions superposed onto this velocity difference result in a large increase in the relative number of short timescale events and in the total number of events. The results in Figure 4 show this large increase in the timescale frequency distribution for all of the bulge orientations. Even with the use of the Basu & Rana (1992) mass function, the MACHO data is now well matched by the distribution for $\theta = -13^\circ$ without increasing the bulge mass, but this match is an artifact of an unrealistic velocity model.

We return now to the velocity model of where both circular velocities and the dispersions depend on $z$ and $\zeta$ along with the other parameters assumed in the construction of that figure to show explicitly the dependence of the model timescale frequency distribution on velocity dispersion. Figure 5 displays the distributions for several choices of velocity dispersions, where more long timescale events are produced but with fewer total events, and the peak in the distribution moves to longer timescales as the velocity dispersion is decreased. Notice, however, that increasing the velocity dispersion of the disk part of the Galaxy to the values appropriate to the bulge results in only a modest increase in the distribution over that where only the bulge has the high dispersion. This is because in the latter case most of the lenses already have a large dispersion, since relatively few lenses are in the disk distribution.

The sensitivity of the microlensing timescale frequency distribution to the stellar mass function of the lenses is demonstrated in Figures 6 and 7, where consequences of assuming the several mass functions given in equations (19)–(24) for various extensions into the brown dwarf region

Fig. 5.—Effect of changing the velocity dispersions. The model is the same as that in Fig. 2 except for the changed velocity dispersions, the fixed value of $\beta$, and the use of the unmodified Basu & Rana (1992) mass function.

Fig. 6.—Comparison of the timescale frequency distributions obtained from the Kroupa (1995) mass functions extended into the brown dwarf region with that from the MACHO 1993 bulge data. All model parameters except the mass function are those of Fig. 2.
are compared with the MACHO data. In Figure 6 we show the timescale frequency distributions with $\theta = -13^\circ$ for three Kroupa (1995) mass functions with successively larger excursions into the brown dwarf region as indicated by the minimum mass. The indices shown in the figure apply only for the lowest mass range in the definitions of the mass functions. Except for the mass functions, all model parameters are those of Figure 2 with $\beta = -1$. Figure 7 shows the distributions of the Gould et al. (1997) mass functions, where the same model assumptions apply as in Figure 6.

First notice the degree of sensitivity of the model timescale frequency distributions to the mass functions, which is the only variation in the models. Herein lies the eventual power of the microlensing technique to constrain the Galactic mass function far from the Sun. The Kroupa (1995) mass function extended down to $m = 0.05$ yields a distribution that is a fair match to the MACHO data, although it is somewhat “thin” in the region around the peak. Improving the fit on the longer timescale side of the peak by increasing the Galactic bulge mass makes the fit worse for $t_E$ near 33 days. A similar problem is faced by the Basu & Rana (1992) mass function for the bulge mass increased by a factor of 1.5 in Figure 3, where there are too many events in the 8–40 day range of $t_E$. Perhaps the best fit to the MACHO data is for the modified Basu & Rana (1992) mass function in Figure 2 for $\beta = -1$ and $\theta = -13^\circ$, even though the peak is too high. This fit could even be improved somewhat by decreasing the mass of the bulge slightly (this would lower the peak) while making the $-2.5$ index in the M-star region slightly less negative (this would “fatten” the curve to bring it back into coincidence with the MACHO data). It is noteworthy that the best fit to the MACHO data is obtained with only hydrogen-burning stars, albeit with the M-star region enhanced with an index of $-2.5$.

Mera et al. (1998), with a different Galactic model, find a timescale frequency distribution that is consistent with the MACHO data using the Kroupa (1995) mass function with lower bound 0.075 $M_\odot$—slightly into the brown dwarf region. This same mass function yields a poor match to the MACHO data with the model used here. This illustrates the model and perhaps procedure dependence of conclusions about the mass functions when so many crucial Galactic parameters are so poorly constrained by observations.

Extending the Kroupa (1995) mass functions further into the brown dwarf region to enhance the timescale frequency distribution is only partially successful. If one believes the statistics of small numbers characterizing the MACHO data, increasing the fraction of the mass distribution in the brown dwarfs makes the distribution too narrow and shifts the peak too far toward shorter timescale events. The distributions so derived have (slightly) too few events of moderate timescales. The Gould et al. (1997) mass functions, (Fig. 7) are even worse matches to the MACHO data, which was also found by Mera et al. (1998) for their model. This mass function has many fewer stars between 0.11 and 0.35
and for $m > 0.08$ there is only a short range where the mass function appears to turn up steeply in the Mera et al. (1998) extension. The severe lack of small mass stars leads to the shift in the peak in the distribution to longer timescales, more moderately long period events, and a rather modest total number of events. Extending the very steep mass function into the brown dwarf region ($m > 0.04$) fails worse than the same exercise with the Kroupa (1995) mass function in matching the MACHO data since the peak of the distribution is moved even further toward very short timescales (less than 4.5 days), exceeding the MACHO numbers there while failing by a factor of at least 2 to produce a sufficient number of events of timescales between 5 and 20 days. Since the modified Basu & Rana (1992) mass function gives the best fit to the MACHO data, we shall use this mass function in subsequent examples, except in where the Figure 11 shows the relative importance of various contributions to the model mass distribution in determining a timescale frequency distribution for $\theta = -13^\circ$. Removing the disk stars has relatively little effect on the distribution, which confirms the suspicion that most of the lenses are in fact located in the bulge. Consistently, the number of events is drastically reduced when only the disk is present, although the peak in the distribution is not changed much because the velocities still have the $z$ and $\zeta$ dependence of Figures 2 and 3. Removing the nucleus decreases the number of events by a factor of about 2.5. The nucleus contributes about 21% of the total bulge mass, but it has such a large effect because the line of sight to $(b, l) = (-4^\circ, 1^\circ)$ for $\theta = -13^\circ$ encounters a comparable number of nucleus and bar stars. We should point out here that Zhao et al. (1996) were able to get a good match to the MACHO data with a comparable mass function and velocity dispersion as that used here without the nucleus, whereas we require the nucleus be kept for a good match. Like the Mera et al. (1998) match to the MACHO data with the Kroupa (1995) mass function that failed for our model, this again illustrates the model and procedure dependence of estimated timescale frequencies that is independent of and thereby weakens the constraints on the mass function. The Zhao (1996) model is also very sensitive to the Galactic coordinates of the line of sight. This is seen in Figures 9 and 10, which show the timescale frequency distributions along two perpendicular traces through Baade's window whose Galactic latitude ($-4^\circ$) bisects the more extended region where the MACHO data was obtained. The smaller scale lengths assumed in the $z$ direction for both the bar and nucleus results in the more rapid fall off in events at higher Galactic latitudes than for higher Galactic longitudes. Recall that we have eliminated the singular nature of the nucleus by truncating the nucleus density at the Galactic center by assuming that density uniform for constant $z'$ inside 560 pc. The nucleus thereby loses about 36% of its mass, but the whole bulge loses only 7%. If the nucleus is not truncated, both the rate of events and the optical depth become very large as the line of sight approaches the center of the Galaxy. Zhao et al. (1996) avoided this problem by eliminating the nucleus altogether, but we require the nucleus stars to match the MACHO data as discussed above. The large dependence of the event rate on coordinates of the line of sight means that microlensing will also eventually prove a powerful constraint on bar model star distributions as well as mass functions, if other parameters such as the velocity dispersions can be adequately defined. As the MACHO fields are spread over several degrees near Baade's window, (Fig. 1 of Alcock et al. 1996), it is appropriate that the model event frequency distributions be averaged over the MACHO fields for comparison with their results. This average is shown in Figure 11 for Basu & Rana (1992) and Holtzman et al. (1998) mass functions and their modifications by the more negative indices for low-mass stars. Both modified mass functions yield good agreement with the data with the best match probably obtained with an index between $-2.2$ and $-2.5$. However, given all of the uncertainties in the model parameters discussed above and
the simplicity of the model itself, this agreement must be regarded as fortuitous and should not be regarded as a definitive constraint on the mass function. Note, however, that there is as yet no need to assume that there is any significant fraction of the bulge mass in brown dwarfs—a conclusion also reached by Zhao et al. (1996). On the other hand, we do need more M stars in the mass functions than obtained from the star counts in order to match the MACHO data. There may be a not unreasonable Galaxy model that will yield a sufficient number of short timescale events with an $m^{-1}$ variation of the mass function in the M-star region, but the necessary changes in the parameters of the model adopted here seem too extreme. Yet the uncertainties in the most relevant mass function of the Galactic bulge stars obtained by Holtzman et al. (1998), the paucity of the microlensing data, and modest observational constraints on the Galactic model itself do not warrant a search for a drastically different Galactic model.

We now turn to a determination of the optical depth as a function of the line of sight. We shall see that a similar average over the MACHO lines of sight for our best matching model also gives reasonable agreement with the MACHO estimates. We shall eventually conclude that the Zhao (1996) model distribution of stars with our assumptions about the mass function and the velocity distribution yields a good match to both the estimated optical depth and the timescale frequency distribution of the 1993 MACHO data, but that the meagerness of this data and the uncertainties within the Galactic models means that the microlensing constraints on both the mass function and the distributions of lens and source stars are very weak indeed.

6. OPTICAL DEPTH

If we use the definition of a microlensing event as a source at distance $D_{os}$ passing within the Einstein ring radius $Re$ of a lens at distance $D_{ol}$, the cross section of the lens for such an event is just $\pi Re^2$, where $Re(m, D_{ol}, D_{os})$ is given by equation (2). For such lenses within a slab of thickness $dD_{ol}$ at $D_{ol}$, the probability $dP$ that a ray from a source will pass through an Einstein ring on its way to the observer is thus $\pi Re^2 dm dD_{ol}$, where $n_{\ell}$ is the number density of lenses and where we have selected only the lenses with masses within $dm$ of $m$. If $I$ is the intensity of rays from the source that have not passed through an Einstein ring in traveling distance $D_{os} - D_{ol}$, on their way to the observer, then the fractional change in traversing $dD_{ol}$ from lenses with masses within $dm$ of $m$ is $dI/I = -dP$, or

$$
\frac{dI}{I} = - \frac{4\pi G m}{c^2} \left[ \frac{D_{ol}(D_{os} - D_{ol})}{D_{os}} \right] \times \left[ \frac{1}{m} \frac{d\rho_\ell(m, D_{ol})}{dm} \right] dm dD_{ol}, \tag{25}
$$

where $\rho_\ell$ is the stellar mass density and equation (2) has been used. Integration of equation (25) leads to the optical depth for microlensing to distance $D_{os}$ of

$$
\tau = \frac{4\pi G}{c^2} \int_0^{D_{os}} \rho_\ell(D_{ol}) \frac{D_{ol}(D_{os} - D_{ol})}{D_{os}} dD_{ol}. \tag{26}
$$

where cancellation of the masses in equation (25) allows immediate integration over the lens mass function. Finally we introduce the dimensionless variables used earlier and average the optical depth over the distribution of visible sources along the line of sight to obtain the result of Kiraga & Paczyński (1994)

$$
\langle \tau \rangle = \frac{4\pi G D_\ell^2}{c^2} \int_0^{1.2} \rho_\ell(z) \frac{z(1 - z)}{\zeta} n_{\ell}(\zeta) \zeta^2 + 2\beta \times dz \frac{1}{\zeta^2} \left[ \int_0^{1.2} n_{\ell}(\zeta) \zeta^2 + 2\beta d\zeta \right]^{-1}, \tag{27}
$$

where $n_{\ell}(\zeta)$ is the source number density and where the coefficient is $3.86 \times 10^{-5}$ if $\rho_\ell$ is expressed in $M_\odot \text{pc}^{-3}$. We shall omit the brackets around $\tau$ hereinafter, but always understand that it is averaged over the source distribution.

The integral in the numerator of equation (27) is evaluated with a Monte Carlo technique like that used for the timescale frequency distribution, whereas the denominator is evaluated with a standard numerical technique. Our examples of optical depth will be mostly limited to the model of Figure 2, since that model best matches the timescale frequency distribution averaged over the MACHO lines of sight. This model is the Zhao (1996) bar and truncated nucleus with total mass of $2.2 \times 10^{10} M_\odot$ and the double exponential disk with scale lengths of 0.3 and 2.7 kpc, but the optical depth is independent of the mass function. The optical depth is quite sensitive to the coordinates of the line of sight for this model as indicated in Figure 12, where we show $\tau$ as a function of latitude and longitude around Baade’s window. The dots in Figure 12 mark the coordinates where we have evaluated the timescale frequency distribution in Figures 9 and 10, and they display the expected correlation between the optical depth and number of events. If we reduce the radius inside of which the mass density of the nucleus is constant, the optical depth, like the timescale frequency distribution, is increased for latitudes and longitudes less than about 4°, but is little affected for larger coordinates.

In Figure 13 optical depths for the MACHO line of sight directions are distributed across horizontal lines designating the estimated MACHO empirical optical depth aver-

![Fig. 12.—Dependence of optical depth on direction of the line of sight for the Zhao (1996) bar and truncated nucleus ($M = 2.2 \times 10^{10} M_\odot$) with double exponential disk with scale lengths of 0.3 and 2.7 kpc and local density of $0.05 M_\odot \text{pc}^{-3}$. The dots correspond to the directions for which the timescale frequency distribution was determined in Figs. 9 and 10.](image-url)
aged over all of the 24 fields and 1 and 2 \( \sigma \) error estimates. The points for a bulge mass of \( 2.2 \times 10^{10} M_{\odot} \), which gave such a good match for the timescale frequency distribution, fall somewhat low with about one-half falling within the 2 \( \sigma \) lines. In some sense, this can be regarded as not unreasonable agreement given the uncertainties in the Galactic model and the sparseness of the data. Recall in Figure 3 that we multiplied the timescale frequency distribution obtained for the Zhao (1996) model bulge together with our standard velocity model but with the Basu & Rana (1992) mass function by a factor of 1.5 to match the peak in the empirical distribution. Although this leads to too many mid-duration events, it could be regarded as a better fit. This multiplication simulates a similar increase in the bulge mass. In Figure 13, we see that the optical depths for the MACHO field coordinates are more compatible with the empirical estimate for the higher mass bulge. However, in addition to the poorer match of the Basu & Rana (1992) mass function to the moderate duration events obtained by MACHO in Figure 3, the higher mass bulge may be less compatible with the stellar kinematics as determined by Zhao (1996). The model optical depth averaged over the MACHO field directions is \( 1.54 \times 10^{-6} \) for a bulge mass of \( 2.2 \times 10^{10} M_{\odot} \) and \( 2.14 \times 10^{-6} \) for \( 3.3 \times 10^{10} M_{\odot} \), compared with \( (2.4 \pm 0.5) \times 10^{-6} \) for the MACHO empirical value.

In Figure 14 we show the effect of different positions of the bar axis along with luminosity function index \( \beta \) for the direction \( (b, l) = (-4^\circ, 1^\circ) \). The optical depth shows the expected decrease as the magnitude of the index increases—location of fewer visible sources at the larger distances pushes the mean position of the sources closer to the observer with a result of less optical depth. The optical depth also shows the expected decrease as the bar is rotated further from the line of sight. We show the optical depth for the near side of the bar in the fourth quadrant \( (\theta = +13^\circ) \) to demonstrate the expected decrease when the near side of the bar is not in the same quadrant as the line of sight. The dots indicate values of \( \beta \) and \( \theta \) for which we determined the timescale frequency distribution in Figure 3 where the optical depths and the number of events are again appropriately correlated.

As a companion to Figure 8 showing the relative contributions of various parts of the Galactic model to the timescale frequency distribution, we show in Figure 15 the effect of the same dissection of the model on optical depth. Again, the disk contributes relatively little to the optical depth—like its meager contribution to the timescale frequency distributions. The nucleus becomes more important when the line of sight passes very close to the Galactic center. The optical depth for the complete model would turn up sharply at small Galactic latitudes in Figure 15 without the truncation of the nucleus.
7. DISCUSSION

We have developed a procedure for determining model timescale frequency distributions for microlensing events toward the center of the Galaxy that can be easily modified for improvements in Galaxy models as they become available. Two methods of evaluating the integral used to accumulate all of the combinations of variables leading to the same timescale give the same distribution within the errors of the Monte Carlo integrations (Fig. 1). This robustness of the model distribution gives one some confidence that the computational procedure is sound. This confidence is increased by the fact that changes of parameters in a series of examples resulted in expected changes in the distributions.

The complete variable dependence of the event timescale is maintained within the integral used to accumulate all the combinations of variables leading to the same timescale. This contrasts to some work in the literature where event rates are determined for all of the lenses being $1 M_* \odot$ with a later convolution over the mass function (see, e.g., & Kiraga & Paczyński 1994; Zhao et al. 1996). However, it seems implicitly assumed in this latter procedure that Galactic models are secure enough that the mass function can be separated out as the major unknown and can be meaningfully constrained by substituting various forms into the convolution. There would also appear to be some danger of introducing some artifacts into the computation by such piecewise averaging. Within the Zhao (1996) bulge model (bar plus nucleus) and the double exponential Bahcall & Soneira (1980) disk model that we have adopted, the timescale frequency distributions are very sensitive to the total bulge mass, the circular velocity model, velocity dispersions, and the direction of the line of sight as well as the mass function. Other bulge and disk distributions of stars will introduce still other variations. So until the properties of the Galaxy are better constrained, it seems premature to assign the highest priority to the mass function in matching the model timescale frequency distributions to the observations.

With our nominal adornments to the Zhao (1996) bulge model, we obtain a good match to the MACHO timescale frequency distribution with an optical depth that is only slightly too low (Figs. 11 and 13). But we do this only by changing the index of either the & Basu & Rana (1992) or the Holtzman et al. (1998) mass function from $-1$ to $-2.5$ and $-2.2$, respectively, in the M-star region (Fig. 11). The Kroupa (1995) mass function has a slope of $-2.35$ for the latest M stars, but the Gould et al. (1997) mass function is even flatter than that of Basu & Rana (1992; $-0.6$ vs. $-1.0$), with the steep extension into the brown dwarf region by Mera et al. (1998) being completely speculative. From Figure 11 one is led to an index somewhere between $-2.2$ and $-2.5$ for an even better fit to the MACHO data within the overall model. But the Holtzman et al. (1998) mass function for bulge stars in the M-star region ($\sim m^{-1}$), where most of the lenses reside, is like that of Basu & Rana (1992) and also Gould et al. (1997) if unresolved binaries are accounted for in the latter. So the only observational support for the steeper mass function in the M-star region is consistency with a meager MACHO data set. Still, microlensing may be detecting M stars that are not accounted for in the star counts because of difficulties in estimating the completeness of the optical surveys (J. A. Holtzman 1998, private communication), or the meagerness of the microlensing data set may make the empirical event rates not entirely representative.

With a different velocity model, or with other than a Zhao (1996) bulge distribution of stars, our match may become less striking. Still, it is significant that we found a model consistent with the microlensing data without resorting to a large population of brown dwarfs as advocated by several authors (see, e.g., Han & Gould 1996; Han & Lee 1998). In fact, we find that extending the Kroupa (1995) and Gould et al. (1997) mass functions into the brown dwarf region produced too many very short timescale events and an insufficient number of moderate timescale events (Figs. 6 and 7), although at least the Kroupa (1995) mass function, which includes brown dwarfs, might have been a better fit if we had just decreased the velocity dispersions or changed the bulge mass.

Our use of the Kiraga & Paczyński (1994) luminosity function index $\beta$ as a measure of source visibility is useful for verifying consistent behavior of the distributions derived by this procedure (Fig. 2), but actually it is not a good representation. First, the luminosity function is not so simply behaved in the visible region, and second, blending is a much more important limitation on source visibility than apparent brightness. The representation might be useful in accounting for source visibility in highly obscured regions, but there are likely to be too few visible sources in these circumstances to motivate an extensive microlensing search.

Penetration of these regions by using infrared wavelengths would return us to blending as the most important limitation. Although there is a need for a more realistic model of source visibility, it is probably less important than other uncertainties in the Galactic model.

The change in the distribution function as we altered the inclination $\theta$ of the bar axis to the observer-Galactic center line also showed consistent behavior for the model (Fig. 3), as did the optical depth (Fig. 14). Although this inclination is uncertain to some extent, which in fact motivated our investigation of its effect, changing the inclination also requires changing the shape of the bar to remain consistent with the COBE observations (Zhao 1998). We left the bar shape unaltered during this exercise, but other uncertainties preclude a more complicated model.

It is not clear just how well our velocity model represents the real world, but a more complicated one is not warranted now. Certainly, the velocity dispersion must have a more complicated $z$ and $\zeta$ dependence than the one that we mostly used (e.g., Figs. 2 and 3), and the flat circular velocity of the disk and ramped circular velocity of the bulge can only be rather crude approximations. Still, it is useful to realize the extreme variation in the calculated timescale frequency distributions simply by changing the velocity dispersions within the model (Fig. 5) or by a change in the model itself (Fig. 4). A velocity model, both circular velocities and dispersions, or even more complicated models accounting for the boxlike orbits in the Zhao (1996) bulge model, that does not represent the real Galaxy very well may lead to, say, the adoption of a mass function in matching microlensing data that is far from reality.

We have so far in this discussion concentrated on the uncertainties in deducing Galactic properties from microlensing timescale frequency distributions and optical depths toward the Galactic center; however, the sensitivity of such microlensing data to changes in the model illustrated by our series of examples shows just how powerful the technique...
will eventually prove to be in the study of Galactic structure. Figures 9, 10, 12, and 13 show the very rapid variation in both optical depth and timescale frequency distributions with relatively small changes in the line-of-sight direction. If such variations do not emerge with the collection and analysis of many more events to obtain the necessary spatial resolution, one would have to drastically revise current ideas about the nature of the bar. Much better spatial resolution may already be possible from the more than 200 additional events that have been monitored but remain unanalyzed and unpublished since 1993. Recall also that all ways to improve the fit to the data are not equivalent. In Figure 3, the Basu & Rana (1992) mass function led to a timescale frequency distribution that produced far too few short timescale events for \( \theta = -13^\circ \), one of the preferred inclination angles for the bar axis. We increased the number of these events for this inclination by increasing the Galactic mass by a factor of 1.5, an increase that might be tolerated by other observational constraints. If the MACHO distribution is close to the correct one, the number of events predicted between about 15 and 40 days is now too large. A better way to match the data was to increase the magnitude of the index in the M-star range of the Basu & Rana (1992) or Holtzman et al. (1998) mass functions as shown in Figure 11.

Therefore microlensing will place constraints on the Galactic distribution of stars, their distributions of velocities, and the mass function, but other more conventional observations must also be accumulated to independently constrain the parameters for a gradual convergence to the true properties of the Galaxy. Certainly with only 40 data points, very uncertain velocities and still uncertain star distributions, in spite of the success of the Zhao (1996) model (we have not even discussed deviations from axial, radial, or bar symmetry), it is premature to assert any constraints on Galactic properties in general from the microlensing data (except for the existence of the bar) and in particular on the mass function. We found a mass function that works along with our other assumptions, but we are unable to assert that it is the right one.

8. CONCLUSIONS

We end by enumerating our conclusions.

1. The procedure developed here appears to be computationally sound (Fig. 1 and later consistencies) and provides a convenient framework for rapidly determining the predictions for microlensing surveys of the Galaxy. Position dependence of various Galactic properties such as stellar velocities are relatively easily incorporated when observations justify such detail.

2. The sensitivity of the predictions of our overall model to changes in various Galactic parameters used in the model means that microlensing will be a major player in the set of many types of observations that will ultimately constrain the structure, content, and dynamics of the Galaxy.

3. There is no motivation for invoking a large population of brown dwarfs to account for the number of short timescale events observed by the MACHO group. Brown dwarfs might be accommodated by reducing the velocity dispersion, for example, but there appears to be a sharp falloff in the number of brown dwarfs as companions to local solar type stars (Mazeh, Goldberg, & Latham 1998), and there is no apparent theoretical or observational reason for expecting many brown dwarfs in the bulge.

4. The sensitivity of the microlensing event rates and optical depth to the direction of the line of sight for bar models means it will be a sensitive probe of the star distribution in such models.

5. The disk stars contribute to a minor fraction of the microlensing events (Figs. 8 and 15).

6. We obtain a better fit to the MACHO data by including the nucleus in the Zhao (1996) bulge model, albeit we truncated the singular peak in the density to avoid excessively large optical depths and event rates for lines of sight near the Galactic center. This truncation is probably academic in any case, as obscuration increases drastically for lines of sight very close to the Galactic center and no data have been taken there.

7. Our model timescale frequency distributions dance around the empirical distribution from the MACHO observations as various parameters are varied, but we are able to get a very good match to this data from the distributions from our nominal model averaged over the MACHO field directions (Fig. 11). About half of the optical depths for these directions fall within the 2 \( \sigma \) error bars of the empirical average optical depth (Fig. 13), and the mean optical depth is about 2 \( \sigma \) away from the MACHO mean. It is important to remember that although no brown dwarfs are necessary to match the MACHO data, the number of M stars had to be increased rather drastically above those obtained from recent star counts both locally as in the Basu & Rana (1992) or Gould et al. (1997) mass functions and in the Galactic bulge itself from the Holtzman et al. (1998) mass function.

8. For our particular model, the addition of brown dwarfs to the mass function seems to be a detriment rather than an asset in matching the MACHO data (Figs. 6 and 7).

9. In spite of the success of our nominal model in matching the MACHO data, uncertainties in all of the Galactic parameters and the meagerness of the published data preclude definitive microlensing constraints on any of them—including the mass function. This frustration should be diminished as the microlensing data set grows. In fact, additional existing data may already yield sufficient spatial resolution to constrain the bar shape and orientation—if the set were analyzed and published.

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