Optimal investment and consumption for Ornstein-Uhlenbeck spread financial markets with logarithmic utility *

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Abstract
We consider a spread financial market defined by the multidimensional Ornstein–Uhlenbeck (OU) process. We study the optimal consumption/investment problem for logarithmic utility functions in the base of stochastic dynamical programming method. We show a special Verification Theorem for this case. We find the solution to the Hamilton–Jacobi–Bellman (HJB) equation in explicit form and as a consequence we construct the optimal financial strategies. Moreover, we study the constructed strategy by numerical simulations.

keywords Optimality, Feynman–Kac mapping, Hamilton–Jacobi–Bellman equation, Itô formula, Brownian motion, Ornstein–Uhlenbeck process, Stochastic processes, Financial market, Spread market.

AMS subject classification primary 62P05, secondary 60G05

1 Introduction
This paper deals with an optimal investment/consumption problem during a fixed time interval [0, T] for a financial market generated by risky spread assets defined through the general multidimensional Ornstein–Uhlenbeck (OU)

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processes (see, for example, [3] and [1]). The idea of spread market goes back to the 1980’s where the team of Nunzio Tartaglia at Morgan Stanley proposed the pairs trading idea to take advantage of market mispricing to gain profit [6]. Several studies have been using the notion of spread to examine the behaviour of financial market. For example for the precious metals spread, it has been examined the spread of gold future market and the Treasury bill future market by [12]. In addition, it has been studied for oil markets such as [7] have been investigated the long term price relationship between futures prices of crude oil and heating oil. The idea of pairs trading is widely used however the academic research about it is still small [11]. In this paper we are concerned on the time-series approach of pairs-trading. It is been proposed in [6] the mean-reverting Gaussian Markov chain model and [13] has discussed the classical study of pairs trading of Royal Dutch and Shell stocks. Also in other sectors like the microstructure level within the airline industry (see, for example, [14]) as well it is known in many hedge funds [4]. Moreover, these problems for Black-Scholes (Bl-Sch) market and stochastic utility market are considered in many papers (see for example [8], [9], [5] and [2]). The affine processes proposed in [5] and [10] to be used in the financial market in the general framework, however, unfortunately we can not use these methods due to the additional variable in the HJB equation corresponding to the risky asset. So in this paper we investigate the optimal investment/consumption problem for logarithm utility functions with no constraints or transaction fees over the whole investment interval $[0,T]$. Using the stochastic dynamical programming method in solving this type of problems, we obtain all optimal solutions in explicit form. To this end we studied the Hamilton–Jacobi–Bellman (HJB) equation and we found its solution in an explicit form. We shown a special new verification theorem for this case and making use of this theorem we construct the optimal strategy. The main difference between this model and Black - Scholes model is that in this model we obtained in the HJB equation the additional multivariate spread variables corresponding to the O-U process. By these reasons, we need to develop a new analytical tool method for this optimisation problem.

The rest of the paper is organized as follows. In section 2 we formulate the problem and we define the price process for the Ornstein-Uhlenbeck model. In section 3 we write the HJB equation. In section 4 we state the main results of the paper. Numerical simulations are given in section 5. The corresponding verification theorem is stated in section 6. Some auxiliary results are stated in the Appendix.
2 Market model

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a standard filtered probability space with \((\mathcal{F}_t)_{0 \leq t \leq T}\) adapted Wiener processes \(W = (W_t)_{0 \leq t \leq T} \in \mathbb{R}^m\). Our financial market consists of one riskless bond \((\hat{S}_t)_{0 \leq t \leq T}\) and risky spread stocks \((S_t)_{0 \leq t \leq T}\) governed by the following equations:

\[
\begin{align*}
    d\hat{S}_t &= r\hat{S}_tdt, & \hat{S}_0 &= 1, \\
    dS_t &= AS_tdt + \sigma dW_t, & S_0 &> 0,
\end{align*}
\]

where \(r \geq 0\) is the interest rate for riskless asset, the \(d\) vector risky assets \(S_t = (S_1(t), S_2(t), \ldots, S_d(t))\), the standard Brownian motion \((W_t)_{0 \leq t \leq T}\) with values in \(\mathbb{R}^m\), the volatility \(\sigma\) is a \(d \times m\) matrix such that \((\sigma \sigma')^{-1}\) exists, and the \(d \times d\) mean reverting matrix \(A\) is given by

\[
    A = \begin{pmatrix}
        a_{11} & a_{12} & \cdots & a_{1d} \\
        a_{21} & a_{22} & \cdots & a_{2d} \\
        \vdots & \vdots & \ddots & \vdots \\
        a_{d1} & a_{d2} & \cdots & a_{dd}
    \end{pmatrix},
\]

with negative real eigenvalues i.e. \(\text{Re} \lambda_i(A) < 0\). Let now \(\alpha_t\) be the number of riskless assets \(\hat{S}\) and \(\alpha_t = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_d(t)) \in \mathbb{R}^d\) be the number of risky assets at the moment \(0 \leq t \leq T\), and the consumption rate is given by a non negative integrated function \((c_t)_{0 \leq t \leq T}\) \[8\]. Thus the wealth process for \(X_t = \alpha_t\hat{S}_t + \alpha'_tS_t\) is given by

\[
dX_t = \alpha_t d\hat{S} + \alpha'_tdS_t - c_t dt,
\]

which can be written as

\[
dX_t = (rX_t - \alpha'_t\hat{S} - c_t)dt + \alpha'_t\sigma dW_t,
\]

where \(\hat{S} = A_1S = (\hat{S}_1, \ldots, \hat{S}_d)' \in \mathbb{R}^d\) and \(A_1 = rI_d - A\), the prime \('\) denotes the transposition. Note that in this case the matrix \(A_1\) is invertible, i.e. there exists \(A_1^{-1}\). In this paper we use the logarithmic utility functions, i.e., we need the following definition for the admissible strategies.

**Definition 2.1.** The strategy \(\nu = (\nu_t)_{0 \leq t \leq T}\) is called admissible if it is adapted, equation (2.3) has a unique positive strong solution and the following conditions hold

\[
    \mathbb{E}\left(\int_0^T (\ln c_t)_- dt\right) < +\infty \quad \text{and} \quad \mathbb{E} \sup_{0 \leq t \leq T} \left(\ln(X^\nu_t)_-\right) < +\infty.
\]

We denote by \(\mathcal{V}\) the set of all admissible strategies.
Now for any \( \upsilon \in \mathcal{V} \) and \( \varsigma = (X, S) \in \mathbb{R}^N \), where \( N = d + 1 \), we define the objective function as

\[
J(\varsigma, \upsilon) := \mathbb{E}_\varsigma \left( \int_0^T (\ln c_u) du + \varpi \ln (X^u_T) \right),
\]

where \( \mathbb{E}_\varsigma \) is the expectation under condition \( \varsigma_0 = \varsigma = (x, s) \). Our goal in this paper is to maximize this function, i.e.

\[
J^*(\varsigma) := \sup_{\upsilon \in \mathcal{V}} J(\varsigma, \upsilon).
\] (2.4)

To study this problem we use the stochastic dynamic programming method. To this end we need to study the value functions \( J^*(\varsigma, t) \) defined as

\[
J^*(\varsigma, t) = \sup_{\upsilon \in \mathcal{V}} \mathbb{E}_{\varsigma, t} \left( \int_t^T (\ln c_u) du + \varpi \ln (X^u_T) \right),
\]

where \( \varpi > 0 \) and \( \mathbb{E}_{\varsigma, t} \) is the expectation under condition \( \varsigma_0 = \varsigma = (x, s) \). Thus we need to study the HJB equation which is given in the following section.

3 Hamilton–Jacobi–Bellman equation

Denoting by \( \varsigma_t = (X_t, S_t)' \in \mathbb{R}^N \) where \( N = d + 1 \), we can rewrite the wealth and stock equations given in Eq. (2.1) and (2.3) respectively in the following form

\[
d\varsigma_t = \tilde{a}(\varsigma_t, \upsilon_t) dt + \tilde{b}(\varsigma_t, \upsilon_t) dW_t, \quad \varsigma_0 = \varsigma,
\] (3.1)

where \( \tilde{a} \in \mathbb{R}^N \) and \( \tilde{b} \) is the matrix of \( N \times m \) functions such that for any \( \varsigma = (x, s) \in \mathbb{R}^N \)

\[
\tilde{a}(\varsigma, u) = \begin{pmatrix} r x - \alpha' A_1 s - c \\ x'S \end{pmatrix} \quad \text{and} \quad \tilde{b}(\varsigma, u) = \begin{pmatrix} \alpha' \\ \sigma \\ s \end{pmatrix},
\]

with the control variable \( u = (\alpha, c) \) with \( \alpha \in \mathbb{R}^d \) and \( c > 0 \). Now, for any \( q = (q_1, \ldots, q_N)' \in \mathbb{R}^N \) and \( N \times N \) symmetric matrix \( M = (M_{ij})_{1 \leq i, j \leq N} \), we set the Hamilton function as

\[
H(\varsigma, q, M) := \sup_{u \in \Theta} H_0(\varsigma, q, M, u), \quad \Theta \in \mathbb{R}^d \times \mathbb{R}_+,
\] (3.2)

where

\[
H_0(\varsigma, q, M, u) := \tilde{a}'(\varsigma, u) q + \frac{1}{2} \text{tr} [\tilde{b}'(\varsigma, u) M] + \ln c.
\]
In order to study problem (2.4), we need to solve the HJB equation which is given by

$$\begin{cases}
z_t(\varsigma, t) + H(\varsigma, \partial z(\varsigma, t), \partial^2 z(\varsigma, t)) = 0, & t \in [0, T], \\
z(\varsigma, T) = \varpi \ln x, & \varsigma \in \mathbb{R}^N,
\end{cases}$$

(3.3)

where \( \partial z(\varsigma, t) = (z_x, z_{s_1}, \ldots, z_{s_d})' \in \mathbb{R}^N \) and

$$\partial^2 z(\varsigma, t) = \begin{pmatrix}
z_{xx} & z_{xs_1} & z_{xs_2} & \cdots & z_{xs_d} \\
z_{s_1x} & z_{s_1s_1} & z_{s_1s_2} & \cdots & z_{s_1s_d} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
z_{s_dx} & z_{s_ds_1} & z_{s_ds_2} & \cdots & z_{s_ds_d}
\end{pmatrix}_{N \times N}.$$

To calculate the Hamilton function (3.2), note that

$$H_0(\varsigma, q, M, \nu) = (rx - \alpha'\tilde{s} - c)q_1 + \sum_{i=1}^d \tilde{s}_i q_{1+i}$$

$$+ \frac{1}{2} \left( \alpha'\sigma\sigma'\alpha M_{11} + 2 \sum_{i=1}^d <\sigma\sigma'\alpha >_{i} M_{1,1+i} + \sum_{k,i=1}^d <\sigma\sigma' >_{ki} M_{1+k,1+i} \right) + \ln c,$$

where \( \tilde{s} = As = (\tilde{s}_1, \ldots, \tilde{s}_d)' \in \mathbb{R}^d \). The symbol \(<X>_i\) denotes the \(i\)th element of the vector \(X\) and \(<Y>_{ij}\) denotes the \((i,j)\)th element of the matrix \(Y\). Note that due to (3.2), if \(M_{11} \geq 0, q_1 \leq 0\) then the Hamilton function \(H(\varsigma, q, M) = +\infty\). So, we maximize the function \(H_0(\varsigma, q, M, \nu)\) over \(\alpha\) and \(c\) under condition that \(M_{11} < 0\) and \(q_1 > 0\). We obtain that optimal values for this maximization problem are given by

$$\alpha^0(s, q, M) = \frac{(\sigma\sigma')^{-1}\tau}{M_{11}} \quad \text{and} \quad c^0(s, q, M) = \frac{1}{q_1},$$

(3.4)

where \(\tau = q_1 S - \sigma\sigma'\mu\) and \(\mu = (M_{1,1+1}, \ldots, M_{1,1+d})'\). Now we replace \(\alpha^0\) and \(c^0\) into \(H_0\) to obtain the Hamilton function, so we get

$$H(\varsigma, q, M) = rxq_1 - \ln q_1 + \frac{\tau'(\sigma\sigma')^{-1}\tau}{2|M_{11}|} + \sum_{i=1}^d \tilde{s}_i q_{1+i}$$

$$+ \sum_{k,i=1}^d <\sigma\sigma' >_{ki} M_{1+i,1+k} - 1.$$
From the preceding Hamilton function and the HJB equation (3.3), we obtain

\[
    z_t + rxz_x + \frac{r'(\sigma\sigma')^{-1}r}{2|z_{xx}|} - 1 - \ln z_x + \sum_{i=1}^{d} \tilde{s}_i z_{s_i} + \sum_{k,i=1}^{d} <\sigma\sigma'>_{ki} z_{s_ks_k} = 0,
\]

where \( \sigma(\zeta, T) = \ln x \) for any \( \zeta \in \mathbb{R}_+ \times \mathbb{R}^d \). To write the solution for this equation, we need to introduce the \( d \times d \) matrix \( g = (g_{ij})_{1 \leq i,j \leq d} \) which is the solution of the following differentiable equation

\[
    \dot{g} + \frac{1}{2} \rho(t) A_1'(\sigma\sigma')^{-1} A_1 - A'(g + g') = 0, \quad g(T) = 0. \tag{3.6}
\]

Here, the dot “.” denotes the derivative. Moreover, we set

\[
    f(t) = \sum_{k,i=1}^{d} <\sigma\sigma'>_{ki} (\tilde{g}_{ki}(v) + \tilde{g}_{ik}(v)) + f_0(t), \tag{3.7}
\]

where \( \tilde{g}(t) = \int_t^T g(v)dv, \)

\[
    f_0(t) = \frac{1}{2} r \left( t^2 - 2t(T+1) + T(T+2) \right) + \rho(t) \ln \rho(t) \quad \text{and} \quad \rho(t) = T - t + 1.
\]

We show that the equation (3.5) has the following solution

\[
    z(x, s, t) = \rho(t) \ln x + s'g(t)s + f(t). \tag{3.8}
\]

**Remark 3.1.** As we see in the HJB equation, the additional variable \( s \in \mathbb{R}^d \) is the main difference from the Bl-Sch market.

**4 Main results**

First of all we have to study the HJB equation (3.5) to calculate the value function (2.4).

**Theorem 4.1.** The function (3.8) satisfies the HJB equation (3.3).

Furthermore, to construct the optimal strategies we set

\[
    \tilde{\alpha}(\zeta, t) = \alpha^0(\zeta, \partial z, \partial^2 z) = -(\sigma\sigma')^{-1} \tilde{s} x \quad \text{and} \quad \tilde{c}(\zeta, t) = c^0(\zeta, \partial z, \partial^2 z) = \frac{x}{\rho(t)}.
\]
Recall that \( \hat{s} = A_1 s = (\hat{s}_1, \ldots, \hat{s}_d)' \in \mathbb{R}^d \). Using these functions we define the optimal strategies \( v^* = (\alpha^*, c^*) \) as

\[
\alpha^*(t) = \hat{\alpha}(\varsigma^*, t) = (\sigma^\prime)^{-1} \hat{S}_t X^*_t \quad \text{and} \quad c^*(t) = \hat{c}(\varsigma^*, t) = \frac{X^*_t}{\rho(t)} .
\] (4.1)

Here \( \varsigma^*_t = (X^*_t, S_t) \) and \( X^*_t \) is the optimal wealth process defined by the following stochastic differential equation

\[
dX^*_t = X^*_t a^*(t) dt + X^*_t (b^*(t))' dW_t , \quad X^*_0 = x ,
\] (4.2)

where

\[
a^*(t) = r - \hat{s}_t'(\sigma^\prime)^{-1} \hat{s}_t - \frac{1}{\rho(t)} \quad \text{and} \quad b^*(t) = \sigma^{-1} \hat{s}_t .
\]

Now we show that these processes are optimal solutions for the problem (2.4).

**Theorem 4.2.** The processes (4.1) and (4.2) are the optimal strategies for the problem (2.4) and

\[
J^*(x, s, t) = z(x, s, t) = \rho(t) \ln x + s' g(t) s + f(t) ,
\] (4.3)

where \( \rho, g \) and \( f \) are given in (3.6).

**Example 1.** For one dimensional case where a riskless and risky assets are given respectively by

\[
\begin{align*}
\begin{cases}
  d\tilde{S}_t = r \tilde{S}_t dt , \\
  dS_t = -\kappa S_t dt + \sigma dW_t ,
\end{cases}
\end{align*}
\] (4.4)

where \( r \geq 0 \) is the interest rate of the riskless asset, \( \kappa > 0 \) and \( \sigma \) are respectively the mean reverting speed and the volatility for risky assets. Therefore, for \( \kappa_1 = \kappa + r > 0 \), the optimal strategies and the HJB equation are given by

\[
\alpha^*(t) = \hat{\alpha}^0(\varsigma^*_t, t) = -\frac{\kappa_1 S_t X^*_t}{\sigma^2} \quad \text{and} \quad c^*(t) = \hat{c}^0(\varsigma^*_t, t) = \frac{X^*_t}{\rho(t)} ,
\]

Moreover, the differential wealth process for this example is given by

\[
dX^*_t = X^*_t a^*(t) dt + X^*_t b^*(t) dW_t ,
\]

where

\[
a^*(t) = \hat{a}(S_t, t) = r + \kappa^2 S_t^2 / \sigma^2 - 1 / \rho(t) \quad \text{and} \quad b^*(t) = \hat{b}(S_t, t) = \kappa_1 S_t / \sigma .
\]
Example 2. For multidimensional case where the market assets are given by

\[
\begin{cases}
    d\tilde{S}_t = r\tilde{S}_t dt, & \tilde{S}_0 = 1, \\
    dS_t = AS_t dt + \sigma dW_t, & S_0 > 0,
\end{cases}
\] (4.5)

where \( r \) is the interest rate for riskless asset \( \tilde{S} \), \( S_t \) is a \( d \)-dimensional vector of risky assets \( S_t = (S_1(t), S_2(t), \ldots, S_d(t)) \in \mathbb{R}^d \), \((W_t)\) is a standard Brownian motion with values in \( \mathbb{R}^d \), the market volatility matrix \( \sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_d) \), and the mean reverting matrix \( A \) is given by

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1d} \\
    a_{21} & a_{22} & \cdots & a_{2d} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{d1} & a_{d2} & \cdots & a_{dd}
\end{pmatrix},
\]

with negative real eigenvalues i.e. \( \text{Re} \lambda_i(A) < 0 \). The optimal wealth process \((X^*_t)_{0 \leq t \leq T}\) is defined by the following stochastic equation

\[
dX^*_t = X^*_t a^*(t) dt + X^*_t (b^*(t))' dW_t, \quad X^*_0 = x,
\]

where

\[
a^*(t) = r + \sum_{i=1}^d \tilde{S}_i^0(t) - \frac{1}{\rho(t)}, \quad b^*(t) = (b^*_1(t), \ldots, b^*_d(t))' \quad \text{and} \quad b^*_i(t) = \frac{\tilde{S}_i(t)}{\sigma_i}.
\]

Using the preceding stochastic differential equation, the optimal strategies \( \nu^* = (\alpha^*, c^*) \) for all \( 0 \leq t \leq T \) is of the form:

\[
\alpha^*_i(t) = \hat{\alpha}_i(\zeta^*_i, t) = \frac{\tilde{S}_i(t) X^*_t}{\sigma_i^2} \quad \text{and} \quad c^*(t) = \hat{c}_i(\zeta^*_i, t) = \frac{X^*_t}{\rho(t)}, \quad (4.6)
\]

where \( \hat{\alpha}_i \) is the number of riskless assets \( \hat{S} \) and \( \alpha_i = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_d(t)) \in \mathbb{R}^d \) be the number of risky assets \( S \) at the moment \( 0 \leq t \leq T \).

Remark 4.1. It should be noted that the behaviour of theses optimal strategies are described by the transformed spread process \( \tilde{S}_t = A_t S_t' \). In the scalar case this is the same as \( S_t \). However, in the general multidimensional case we need to take into account all components of the spread processes.
5 Numerical Simulation

For 1–dimensional case. Fig. 1 shows the value function \( z(\varsigma, t) \) given by Eq. (3.8). The following parameters have been used: \( T = 1, r = 0.01, \kappa = 0.1, \sigma = 0.5 \) and the initial endowment \( x = 100 \).

Now, we simulate the optimal strategies \( \alpha_t^* \) and \( c_t^* \) given in Eq. (4.1) with the optimal wealth process \( x_t^* \). In the following figures, we used different parameters to show the behaviour of the strategies with different values of \( r, \kappa, \) and \( \sigma \). As seen in the figures below, we see that the behaviour of the wealth process is increasing constantly when \( \kappa \) has large values (see Fig. 3a and Fig. 5a). However, it is clear that the wealth process is decreasing when \( \kappa \) has a quite small value as seen in Fig. 2a and Fig. 4a. In addition we see that the volatility in the investment process increases and decreases depending on the fraction \( \kappa / \sigma^2 \). Thus the range of volatility in figures (Fig. 2b, Fig. 3b and Fig. 4b) is less than Fig. 5b which jumps to 4000 points. This is due to the higher number we get from the fraction which is nearly 50.

Figure 2: The wealth process with the parameters \( \alpha \) and \( c \) when \( \sigma = 1, r = 0.01 \) and \( \kappa = 0.5 \).
Figure 3: The wealth process with the parameters $\alpha$ and $c$ when $\sigma = 5$, $r = 4$ and $\kappa = 5$.

Figure 4: The wealth process with the parameters $\alpha$ and $c$ when $\sigma = 20$, $r = 0.01$ and $\kappa = 0.5$ with $n = 1000$.

Figure 5: The wealth process with the parameters $\alpha$ and $c$ when $\sigma = 0.1$, $r = 0.01$ and $\kappa = 5$ with $n = 1000$. 
6 Verification theorem

Now we give some modifications for the verification theorem from [2]. Consider on the interval \([0, T]\), the stochastic control process given by \(N\)-dimensional Itô process

\[
\begin{aligned}
dς^\nu_t &= \hat{a}(ς^\nu_t, t, \nu)dt + \hat{b}(t, ς^\nu_t, ν)dW_t, \quad t \geq 0, \\
ς_0^\nu &= x \in \mathbb{R}^N,
\end{aligned}
\]

where \((W_t)_{0 \leq t \leq T}\) is a standard \(m\)-dimensional Brownian motion. We assume that the control process \(\nu\) takes values in some set \(Θ\). Moreover, we assume that the coefficients \(\hat{a}\) and \(\hat{b}\) satisfy the following conditions:

\(V_1\) For all \(t \in \[0, T]\), the functions \(\hat{a}(\cdot, t, \cdot)\) and \(\hat{b}(\cdot, t, \cdot)\) are continuous on \(\mathbb{R}^N \times Θ\); where \(Θ \in \mathbb{R} \times \mathbb{R}_+\).

\(V_2\) For every deterministic vector \(ν \in Θ\), the stochastic differential equation

\[
dς^\nu_t = \hat{a}(ς^\nu_t, t, ν)dt + \hat{b}(ς^\nu_t, t, ν)dW_t,
\]

with an \(N \times m\) matrix \(\hat{b}\), has a unique strong solution.

Now we introduce an admissible control process for equation Eq. (6.1).

**Definition 6.1.** We set

\[\mathcal{F}_t = \sigma\{W_u, 0 \leq u \leq t\}, \quad \text{for any} \quad 0 < t \leq T,\]

where a stochastic control process \(v = (v_t)_{t \geq 0} = (α_t, c_t)_{t \geq 0}\) is called admissible on \([0, T]\) with respect to equation Eq. (6.1) if it is \((\mathcal{F}_t)_{0 \leq t \leq T}\) progressively measurable with values in \(Θ\), and equation Eq. (6.1) has a unique strong a.s. continuous solution \((ς^\nu_t)_{0 \leq t \leq T}\) such that

\[
E \int_0^T (f(ς^u_t, u, v_u))_+ dt < +\infty, \quad E \sup_{0 \leq t \leq T} (h(ς^u_T))_+ < +\infty,
\]

and

\[
\int_0^T (|\hat{a}(ς^u_t, u, v_u)| + |\hat{b}(ς^u_t, u, v_u)|^2) dt + \int_0^T |f(ς^u_t, u, v_u)| du \leq \infty \quad \text{a.s.}
\]

We denote by \(V\) the set of all admissible control processes with respect to equation Eq. (6.1).
Moreover, let $f : \mathbb{R}^m \times [0, T] \times \Theta \to [0, \infty)$ and $h : \mathbb{R}^m \to [0, \infty)$ be continuous utility functions. We define the cost function by

$$J(x, t, v) = \mathbb{E}_{x,t} \left( \int_t^T f(\varsigma, u, v_u) du + h(\varsigma_u^t) \right), \quad 0 \leq t \leq T,$$

where $\mathbb{E}_{x,t}$ is the expectation operator conditional on $\varsigma^v_t = x$. Our goal is to solve the optimization problem (2.4) given by

$$J^*(x, t) := \sup_{v \in \mathcal{V}} J(x, t, v).$$

To this end we introduce the Hamilton function, i.e. for any $\varsigma$ and $0 \leq t \leq T$, with $q \in \mathbb{R}^N$ and symmetric $N \times N$ matrix $M$ we set

$$H(\varsigma, t, q, M) := \sup_{\theta \in \Theta} H_0(\varsigma, t, q, M, \theta), \quad (6.5)$$

where

$$H_0(\varsigma, t, q, M, \theta) := \tilde{a}'(\varsigma, t, \theta)q + \frac{1}{2} tr[\tilde{b}\tilde{b}'(\varsigma, t, \theta)M] + f(\varsigma, t, \theta).$$

In order to find the solution to Eq. (2.4), we investigate the HJB equation

$$\begin{cases}
    z_t(\varsigma, t) + H(\varsigma, t, z_\varsigma(\varsigma, t), z_{\varsigma\varsigma}(\varsigma, t)) = 0, & t \in [0, T], \\
    z(\varsigma, T) = h(\varsigma), & \varsigma \in \mathbb{R}^N.
\end{cases} \quad (6.6)$$

Here, $z_t$ denotes the partial derivative of $z$ with respect to $t$, $z_\varsigma(\varsigma, t)$ the gradient vector with respect to $\varsigma$ in $\mathbb{R}^N$ and $z_{\varsigma\varsigma}(\varsigma, t)$ denotes the symmetric hessian matrix, that is the matrix of the second order partial derivatives with respect to $\varsigma$.

We assume the following conditions hold:

$\mathbf{H}_1$) There exists a function $z(\varsigma$ from $C^{2,1}(\mathbb{R}^N \times [0, T]), t)$ from $\mathbb{R}^N \times [0, T] \to (0, \infty)$ which satisfies the HJB equation.

$\mathbf{H}_2$) There exists a measurable function $\theta^* : \mathbb{R}^N \times [0, T] \to \Theta$, such that for all $\varsigma \in \mathbb{R}^N$ and $0 \leq t \leq T$,

$$H(\varsigma, t, z_\varsigma(\varsigma, t), z_{\varsigma\varsigma}(\varsigma, t)) = H_0(\varsigma, t, z_\varsigma(\varsigma, t), z_{\varsigma\varsigma}(\varsigma, t), \theta^*(\varsigma^0, t)).$$

$\mathbf{H}_3$) Assume that for any $v \in \mathcal{V}$, any $0 \leq t \leq T$ and $x$,

$$\mathbb{E}_{x,t} \sup_{t \leq u \leq T} \left( z(X^v_u, u) \right) < +\infty.$$
There exists a unique strong solution to the Itô equation

\[ d\varsigma^*_t = \tilde{a}(\varsigma^*_t, t)dt + \tilde{b}(\varsigma^*_t, t)dW_t, \quad \varsigma^*_0 = x, \quad t \geq 0, \]

where \( \tilde{a}(., t) = \tilde{a}(., t, \theta^*(., t)) \) and \( \tilde{b}(., t) = \tilde{b}(., t, \theta^*(., t)) \). Moreover, the optimal control process \( v^*_t = \theta^*(v^*_t, t) \) for \( 0 \leq t \leq T \) belongs to \( \mathcal{V} \), and

\[ \mathbb{E} \sup_{t \leq u \leq T} |z(x^*_u, u)| < +\infty. \]

**Theorem 6.2.** Assume that conditions \( H_1 \)–\( H_4 \) hold

\[ \Rightarrow v^*_t = (v^*_t)_{0 \leq t \leq T}, \]

is a solution to this problem.

**Proof.** For \( v \in \mathcal{V} \), let \( X^v \) be the associated wealth process with initial value \( X^v_0 = x \). Define a stopping time

\[ \tau_n = \inf \left\{ s \geq t : \int_t^s |\tilde{b}'(\varsigma^v, u) \partial_\varsigma z(\varsigma^v, u)|^2 du \geq n \right\} \wedge T. \]

Note that condition (6.4) implies that \( \tau_n \to T \) as \( n \to \infty \) a.s.. By continuity of \( z(., .) \) and of \( (\varsigma^v_t)_{0 \leq t \leq T} \) we obtain

\[ \lim_{n \to \infty} z(\varsigma^v_{\tau_n}, \tau_n) = z(\varsigma^v_T, T) = h(\varsigma^v_T) \quad \text{a.s..} \quad (6.7) \]

To simplify we use the notation \( \tilde{a}_t = \tilde{a}(\varsigma_t, v_t, t) \) and \( \tilde{b}_t = \tilde{b}(\varsigma_t, v_t, t) \). Then by Itô formula

\[ dz(\varsigma_t, t) = z_t(\varsigma_t, t)dt + \sum_{i=1}^N \frac{\partial}{\partial \varsigma_i} z(\varsigma_t, t)d\varsigma_i + \frac{1}{2} \sum_{i,j=1}^N \frac{\partial^2}{\partial \varsigma_i \partial \varsigma_j} z(\varsigma_t, t)d<\varsigma_i, \varsigma_j> \quad (6.8) \]

By using the definition of \( d\varsigma \), this equation becomes

\[ dz(\varsigma_t, t) = z_t(\varsigma_t, t) + (\partial z(\varsigma_t, t))' \tilde{a}_t dt + \frac{1}{2} \text{tr}(\tilde{b}_t(\tilde{b}_t)' \partial^2 z(\varsigma_t, t)) dt \\
+ (\partial z(\varsigma_t, t))' \tilde{b}_t dW_t. \]

Taking the integration for both sides we get

\[ z(\varsigma_T, T) - z(\varsigma_t, t) = \left[ z_u(\varsigma_u, t) + (\partial z(\varsigma_u, u))' \tilde{a}_u + \frac{1}{2} \text{tr}(\tilde{b}_u(\tilde{b}_u)' \partial^2 z(\varsigma_u, u)) \right]_t^T \\
+ \int_t^T (\partial z(\varsigma_u, u))' \tilde{b}_u dW_u. \]
Add and subtract \( \int_t^T f(\varsigma, u) \, du \) and let 
\[ z(\varsigma_T, T) = h(\varsigma) \]
we get
\[ z(\varsigma_t, t) = h(\varsigma) - \int_t^T (\partial z(\varsigma_u, u)') \dot{b}^u \, dW_u + \int_t^T f(\varsigma_u, u) \, du \]
\[ - \int_t^T \left( z_u(\varsigma_u, u) + (\partial z(\varsigma_u, u)') \dot{a}^u + \frac{1}{2} \text{tr} \left( \ddot{b}^u \right) \partial^2 z(\varsigma_u, u) + f(\varsigma_u, u) \right). \]  
(6.9)

Take the expectation on both sides under \( \varsigma \) and \( t \) noting that 
\( \mathbb{E}_{\varsigma, t} z(\varsigma_t, t) = z(\varsigma_t, t) \)
\[ z(\varsigma_t, t) = \mathbb{E}_{\varsigma, t} h(\varsigma_T) - \mathbb{E}_{\varsigma, t} \int_t^T |(\dot{b}(\varsigma_u, v_u, u))' \partial z(\varsigma_u, u)|^2 \, du + \mathbb{E}_{\varsigma, t} \int_t^T f(\varsigma, u) \, du \]
\[ - \mathbb{E}_{\varsigma, t} \int_t^T \left( z_u(\varsigma_u, u) + (\partial z(\varsigma_u, u)') \dot{a}^u + \frac{1}{2} \text{tr} \left( \ddot{b}^u \right) \partial^2 z(\varsigma_u, u) + f(\varsigma_u, u) \right) \, du. \]

From the condition that 
\( J(\varsigma, v, t) = \mathbb{E}_{\varsigma, t} \left( \int_t^T f(\varsigma_u, v_u, u) \, du + h(\varsigma_T) \right), \)  
(6.10)
and 
\[ H_0(t, \varsigma, q, M) = a'(\varsigma_t, v_t, t)q + \frac{1}{2} \text{tr} [bb'(\varsigma_t, v_t, t)M] + f(\varsigma, v, t), \]  
(6.11)
where \( q = (q_1, \ldots, q_N)' \in \mathbb{R}^N \) and a symmetric \( N \times N \) matrix \( M = (M_{ij})_{1 \leq i, j \leq N} \), 
then we have 
\[ z_t(\varsigma, t) = J(\varsigma_t, v_t, t) - \mathbb{E}_{\varsigma, t} \left( \int_t^T (z_u(\varsigma_u, u) + H_0(\varsigma_u, u, q, M, v)) \, du \right). \]

We present the first term as 
\[ \int_t^\tau f(\varsigma_u, u, v_u) \, du = \int_t^\tau (f(\varsigma_u, u, v_u))_+ \, du - \int_t^\tau (f(\varsigma_u, u, v_u))_- \, du. \]

Taking into account that 
\( \mathbb{E} \int_0^T (f(\varsigma_u, u, v_u))_- \, du < +\infty. \)

We obtain by the Monotone Convergence Theorem that 
\[ \lim_{n \to \infty} \mathbb{E} \int_0^\tau f(\varsigma_u, u, v_u) \, du = \mathbb{E} \int_t^T f(\varsigma_u, u, v_u) \, du. \]
From the following conditions:

\[ z_t(\varsigma, t) + H(\varsigma, t, q, M) = 0 , \quad (6.12) \]

and the Hamilton function

\[ H(\varsigma, t, q, M) = \sup_{v \in \Theta} H_0(\varsigma, t, q, M, v) . \quad (6.13) \]

Then \( z(., .) \) becomes

\[ z(\varsigma, t) = J(\varsigma, u, q, t) + E_{\varsigma,t} \left( \int_{t}^{T} (H(\varsigma, u, q, M) - H_0(\varsigma, u, q, M, v)) du \right) . \]

Moreover, taking into account that

\[ E_{\varsigma,t} \sup_{n \geq 1} (z(\varsigma^n, \tau_n))_+ \leq E_{\varsigma,t} \sup_{0 \leq t \leq T} (z(\varsigma^*, t))_+ \leq +\infty . \]

We thereby Fatou’s Lemma obtain that

\[ \lim_{n \to \infty} E_{\varsigma,t} z(\varsigma^*, \tau_n) \geq E_{\varsigma,t} \lim_{n \to \infty} z(\varsigma^*, \tau_n) = E_{\varsigma,t} z(\varsigma^*, T) = E_{\varsigma,t} h(\varsigma_T) . \]

Finally, we obtain that

\[ z(\varsigma, t) \geq E_{\varsigma,t} \left( \int_{t}^{T} f(\varsigma^*, u, v_u) du + h(\varsigma_T) \right) = J(\varsigma, v, t) . \]

Therefore, \( z(\varsigma, t) \geq J^*(\varsigma, t) \) for all \( 0 \leq t \leq T \). Similarly, replacing \( v \) in (6.9) by \( v^* \) as defined by \( H_2 - H_3 \) we obtain

\[ z(\varsigma, t) = E_{\varsigma,t} \int_{t}^{T} f(\varsigma^*, u, v_u^*) du + E_{\varsigma,t} z(\varsigma^*, \tau_n) . \]

Condition \( H_4 \) implies that the sequence \((z(\varsigma^*_n, \tau_n))_{n \in \mathbb{N}}\) is uniformly integrable. Therefore, by (6.7)

\[ \lim_{n \to \infty} E_{\varsigma,t} z(\varsigma^*_n, \tau_n) = E_{\varsigma,t} \lim_{n \to \infty} z(\varsigma^*_n, \tau_n) = E z(\varsigma^*, T) = E_{\varsigma,t} h(\varsigma^*_T) , \]

and we obtain

\[ z(\varsigma, t) = \lim_{n \to \infty} E_{\varsigma,t} \int_{t}^{T} f(\varsigma^*_n, u, v_u^*) du + \lim_{n \to \infty} E_{\varsigma,t} z(\varsigma^*_n, \tau_n) \]

\[ = E_{\varsigma,t} \left( \int_{t}^{T} f(\varsigma^*, u, v_u^*) du + h(\varsigma^*_T) \right) = J(\varsigma, t, v^*) . \]

We arrive at \( z(\varsigma, t) = J^*(\varsigma, t) \). This proves Theorem 6.2. \( \square \)
Remark 6.1. The difference in Theorem 6.2 from the verification theorem from [2] is that the functions $f$ and $h$ are positive but from the logarithmic utilities these functions are negative. So, we can not use directly the verification theorem in [2].
7 Proofs

7.1 Proof of Theorem 4.1

Now, by taking the derivatives of \( z(\varsigma, t) \) defined in (3.8) with respect to \( t \) and \( s \) and apply them into equation (3.5) we obtain

\[
s'\dot{g}(t)s + \dot{f}(t) + r\rho(t) + \sum_{k,i=1}^d (<\sigma\sigma'>_{ki} (g_{ki} + g_{ik}))-\ln \rho(t) - 1
\]

\[
+ \sum_{j=1}^d \sum_{l=1}^d A_{jl}s_l <(g + g')s>_j + \frac{\rho(t)(\sigma\sigma')^{-1}s}{2} = 0,
\]

where the dot "\( \cdot \)" denotes the first derivative and \( g \) is a \( d \times d \) matrix defined in (3.6). Then this can be written as

\[
s'(\dot{g}(t) + \frac{1}{2}\rho(t)A'_1(\sigma\sigma')^{-1}A_1 - A'(g + g'))s + \dot{f}(t) + \sum_{k,i=1}^d <\sigma\sigma'>_{ki} (g_{ki} + g_{ik}) - 1
\]

\[-\ln \rho(t) + r\rho(t) = 0.
\]

After calculation we get that for \( s \in \mathbb{R}^d \),

\[
f(t) = \sum_{k,i=1}^d <\sigma\sigma'>_{ki} (\tilde{g}_{ki}(v) + \tilde{g}_{ik}(v)) + f_0(t),
\]

and

\[
s'(\dot{g}(t) + \frac{1}{2}\rho(t)A'_1(\sigma\sigma')^{-1}A_1 - A'(g + g'))s = 0, \quad g(T) = 0,
\]

where

\[
\tilde{g}(t) = \int_t^T g(v)dv \quad \text{and} \quad f_0(t) = \frac{1}{2}r\left(t^2 - 2t(T+1) + T(T+2)\right) + \rho(t)\ln \rho(t).
\]

The last term in the preceding equation can be written as

\[
<A'(g + g')>_{ij} = \sum_{l=1}^d<A'>_{il} (g_{lj} + g_{jl}),
\]

\[
= \sum_{l=1}^d (<A>_{li} g_{lj} + <A>_{lj} g_{ij}).
\]
Let we denote by \( H = (h_{ij})_{1 \leq i,j \leq d} \), where \( h = \text{vect}(H) \) a vector in \( \mathbb{R}^m \) such that \( h = (h_1, h_2, \ldots, h_m) \), with \( h_{(j-1)d+i} = \langle H \rangle_{i,j} \) and \( Z(t) = \text{vect}(g(t)) \) where \( Z_{(j-1)d+i} = g_{ij} \). Therefore, the last equation becomes

\[
< A'(g + g') >_{ij} = \sum_{l=1}^{d} < A >_{li} \left( Z_{(j-1)d+l} + Z_{(l-1)d+j} \right),
\]

\[
= \sum_{l=1}^{d} < A >_{li} Z_{(j-1)d+l} + \sum_{l=1}^{d} < A >_{li} Z_{(l-1)d+j},
\]

\[
= \sum_{l=1}^{d} \sum_{k=1}^{d} \left( < A >_{li} 1_{\{k=j\}} Z_{(k-1)d+l} + < A >_{ki} 1_{\{l=j\}} Z_{(k-1)d+l} \right).
\]

This can be written in the following form

\[
\text{Vect}(A'(g + g')) = \Gamma Z,
\]

where \( \Gamma = (\hat{\gamma}_{s,t})_{1 \leq s,t \leq m} \) and \( \hat{\gamma}_{s,t} = \langle A >_{li} 1_{\{k=j\}} + < A >_{ki} 1_{\{l=j\}} \), with \( s = (j-1)d+i \) and \( t = (k-1)d+l \). Therefore, for all \( m \times m \) matrix \( \Gamma \), with \( m = d^2 \),

\[
< A'(g + g') >_{ij} = \langle \Gamma Z \rangle_{ij},
\]

where \( \Gamma = (\hat{\gamma}_{s,t})_{1 \leq s,t \leq m} \). Thus equation (3.6) can be written as

\[
\dot{Z} - \Gamma Z + \frac{1}{2} \rho(t) \tilde{b} = 0, \quad Z(T) = 0,
\]

where \( \tilde{b} = \text{vect}(A'_1 (\sigma \sigma')^{-1} A_1) \in \mathbb{R}^m \). Therefore, the solution of \( Z(t) \) is given by

\[
Z(t) = b^{-1}_2 \int_t^T \rho(v) e^{\Gamma(v-t)} dv.
\]

This proves Theorem 4.1. \( \square \)

### 7.2 Proof of Theorem 4.2

We apply the Verification Theorem 6.2 to Problem (2.4) for the stochastic control differential equation (2.3). For fixed \( u = (\alpha, c) \), where \( \alpha \in \mathbb{R}^d \) and \( c \in [0, \infty) \), the coefficients in model (6.1) are defined as

\[
\tilde{a}(\varsigma, u) = \begin{pmatrix} \frac{r x - \alpha' \hat{s} - c}{A} \\ A \hat{s} \end{pmatrix}, \quad \hat{b}(\varsigma, u) = \begin{pmatrix} \alpha' \sigma \\ \sigma \end{pmatrix}, \quad \text{and} \quad h(x) = \ln x.
\]
This implies immediately condition $H_1)$. Moreover, by Definition (6.1), the coefficients are continuous, hence (6.4) holds for every $u \in V$. To check $H_1) - H_3)$ we calculate the Hamilton function (6.4) for Problem (2.4). We have

$$H(\varsigma, q, M) = \sup_{u \in \mathbb{R}^d \times \mathbb{R}_+} H_0(\varsigma, q, M, u),$$

where

$$H_0(\varsigma, q, M, u) = \hat{a}'(t, \varsigma, u)q + \frac{1}{2} \text{tr}[\hat{b}b'(t, \varsigma, u)M] + U(c).$$

As

$$<\hat{b}b'>_{1+k,1+j} = \sum_{l=1}^m \hat{b}_{1+k,l} \hat{b}_{1+j,l} = \sum_{l=1}^m \sigma_{kl} \sigma_{jl} = <\sigma\sigma' >_{kj},$$

and

$$\text{tr} (\hat{b}b'M) = \alpha' \sigma\sigma' \alpha M_{11} + 2 \sum_{j=1}^d <\sigma\sigma' >_{j} M_{1,1+j} + \sum_{k,j=1}^d <\sigma\sigma' >_{kj} M_{1+k,1+j}.$$

Therefore, $H_0$ can be written as

$$H_0(\varsigma, q, M, u) = rxq_1 + \sum_{j=1}^d \tilde{s}_j q_{1+j} + \sum_{k,j=1}^d <\sigma\sigma' >_{kj} M_{1+k,1+j} - cq_1 + \ln c + J(\alpha),$$

where

$$J(\alpha) = \frac{\alpha' \sigma\sigma' \alpha}{2} M_{11} + \sum_{j=1}^d <\sigma\sigma' >_{j} M_{1,1+j} - \alpha' \tilde{s} q_1.$$

Now in order to find the Hamilton function we have to maximize $H_0$,

$$\max_{\alpha} J(\alpha) = \frac{\tau'(\sigma\sigma')^{-1} \tau}{2|M_{11}|},$$

where $\tau = \sigma\sigma' \mu - q_1 \hat{s}$ and $\mu = (M_{1,1+1}, \ldots, M_{1,1+d})'$. Therefore,

$$H(\varsigma, q, M) = rxq_1 + \sum_{j=1}^d \tilde{s}_j q_{1+j} + \sum_{k,j=1}^d <\sigma\sigma' >_{kj} M_{1+j,1+k} - \ln q_1 + \frac{\tau'(\sigma\sigma')^{-1} \tau}{2|M_{11}|}.$$

Therefore, the HJB equation can be written as

$$z_t + r x z_x + \sum_{j=1}^d \tilde{s}_j z_{s_j} + \sum_{k,j=1}^d <\sigma\sigma' >_{kj} z_{s_j s_k} - 1 - \ln z_x + \frac{\tau'(\sigma\sigma')^{-1} \tau}{2|z_{xx}|} = 0,$$
where \( \tau = \tau(\varsigma, t) \) and \( \tau_j = \sum_{i=1}^d <\sigma'_i z_{x_i} - z_x \tilde{s}_j > \). By taking
\[
z(x, s, t) = (T - t + 1) \ln x + s' g(t) s + f(t),
\]
with \( g \) and \( f \) are given in (3.6), we obtain
\[
s' \dot{g}(t) s + \dot{f}(t) + r \rho(t) + s' A'(g + g') s + \text{tr}(\sigma' \sigma' (g + g')) - \ln \rho(t) + \frac{\rho(t) \delta^{2} (\sigma' \sigma') - \delta}{2} = 0.
\]
So, \( z(\varsigma, t) \) given in (3.8) is the solution to the HJB equation (3.3). One can check directly that the strategy \( \nu \) in with optimal strategy in (4.1) satisfies the conditions \( H_1 \) - \( H_3 \). Now to check condition \( H_4 \) we have to verify that
\[
\sup_{0 \leq t \leq T} E_{\varsigma} |z(\varsigma^*_t, t)| < +\infty.
\]
Thus, as
\[
z(\varsigma, t) = (T - t + 1) \ln x + s' g(t) s + f(t),
\]
and as \( g(t) \) and \( f(t) \) are bounded functions and \( X_t^* \) is given by
\[
X_t^* = x \exp \left\{ \int_0^t \left( \tilde{a}^*(u) - (\tilde{b}^*(u))^2 / 2 \right) du + \int_0^t \tilde{b}^*(u) dW_u \right\},
\]
then we have to show that
\[
\sup_{\tau \in M_t} E \left( \left| \ln(X_t^*) \right| + S_t^2 \right) |X_t = x, S_t = s) < +\infty.
\]
Moreover, note that \( S_t = e^{-\kappa(T-t)} s + \xi_{t, \tau} \) and \( \xi_{t, \tau} = \sigma e^{-\kappa \tau} \int_t^T e^{\kappa u} dW_u \).

Since \( |S_t| \leq |s| + |\xi_{t, \tau}| \), one needs to check that
\[
\sup_{0 \leq t \leq T} E(\ln(X_t^*) + \xi_t^2) < +\infty.
\]
From equation (4.2), we have that \( |\tilde{a}^*(t)| \leq c_1(1 + S_t^2) \) and \( |\tilde{b}^*(t)| \leq c_2 |S_t| \).
Thus by the OU process, \( E \int_0^T |\tilde{b}^*(t)|^m du \leq +\infty \), which implies that
\[
E \left( \int_0^T \tilde{b}^*(u) dW_u \right)^2 = E \int_0^T |\tilde{b}^*(u)|^2 du \leq c.
\]
Therefore,
\[
E|\ln X_t^*| \leq E \int_0^T \left( |\tilde{a}^*(u)| + \frac{1}{2} (\tilde{b}^*(u))^2 \right) du + \sqrt{E \left( \int_0^T \tilde{b}^*(u) du \right)^2} < +\infty.
\]
This proves Theorem 4.2. \( \square \)
8 Appendix

The R simulation codes:

```r
## defining the variables
T=1
r=0.01
kappa=0.1
kappa1=kappa+r
sigma=0.5
x=100
s<-seq(-10,10, length=30)
t<-seq(0,1, by=.1)

## The function g
g<- function(t) (-kappa1^2/2*sigma^2 )*((-2*kappa*exp(2*kappa*(t-T))
+ t-T-1 + exp(2*kappa*(t-T)) - 1 )/(4* kappa^2))

## The function f
f<-function(t) sigma^2*g(t) + T-t-(T-t+1)* log(T-t+1)

## The function Z
Zvarsigma<-function(s,t) log(T-t+1) +s^2*g(t)+f(t)

## The plot ###
z<-outer(s,t, Zvarsigma)
jet.colors <- colorRampPalette( c("blue", "green") )
nbcol <- 100
color <- jet.colors(nbcol)
nrz <- nrow(z)
ncz <- ncol(z)
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
facetcol <- cut(zfacet, nbcol)

persp(s, t, z, col = color[facetcol], phi = 30, theta = -30, xlab = "S",
ylab = "t", zlab ="Z", ticktype = "detailed")
# ticktype -- to give details in the numbers or values of each variable

## The strategies ##
```

\[
\begin{align*}
\text{rho} & \leftarrow T - t + 1 \\
\text{astar} & \leftarrow r + (\kappa \cdot \sigma^2) - \frac{1}{\rho} \\
\text{bstar} & \leftarrow \kappa \cdot \sigma \\
\text{xstar} & \leftarrow \text{seq}(0, 100) \\
\text{alphastar} & \leftarrow s \cdot \text{xstar} / \sigma^2 \\
\text{persp}(s, \text{xstar}, \text{alphastar}, \text{xlab} = "S", \text{ylab} = "xstar", \text{zlab} = "alpha^*"), \\
\text{cstar} & \leftarrow \text{xstar} / (T - t + 1) \\
\text{persp}(\text{xstar}, t, \text{cstar}, \text{xlab} = "xstar", \text{ylab} = "t", \text{zlab} = "c^*"
\end{align*}
\]
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