Ergodic and strong Feller properties of affine processes

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Abstract. For general (1+1)-affine Markov processes, we prove the ergodicity and exponential ergodicity in total variation distances. Our methods follow the arguments of ergodic properties for Lévy-driven OU-processes and a coupling of CBI-processes constructed by stochastic equations driven by time-space noises. Then the strong Feller property is considered.

Key words: Affine Markov processes; Ergodicity; Strong Feller property; Total variation distances

1 Introduction

Let $m \geq 0$ and $n \geq 0$ be integers. A time-homogeneous Markov processes $\{X_t : t \geq 0\} = \{(Y_t, Z_t) : t \geq 0\}$ taking values in $G := \mathbb{R}_+^m \times \mathbb{R}^n$ is called an affine Markov process if its characteristic function satisfies

$$
E_x[e^{\langle X_t, u \rangle}] = \exp \left\{ \langle x, V(t, u) \rangle + \int_0^t \psi(V(s, u)) \, ds \right\}, \quad x \in G, u \in i\mathbb{R}^{m+n}, \quad (1.1)
$$

where $V$ and $\psi$ are two complex-valued functions and $V$ satisfies certain generalized Riccati equations. The affine property means roughly that the logarithm of the characteristic function is affine with respect to the initial state. The concept of affine Markov processes enables a unified treatment of two important Markov classes including continuous state branching processes with immigration (CBI-processes) and Ornstein-Uhlenbeck type processes (OU-type processes), where the OU-type processes also include Lévy processes as a particular case. Roughly speaking, affine processes with state space $\mathbb{R}_+^m$ are $m$-dimensional CBI-processes, and those with state space $\mathbb{R}^n$ are $n$-dimensional OU-type processes. The processes involve rich common mathematical structures and have found interesting connections and applications in several areas. The general theory of finite-dimensional affine Markov processes including several equivalent characterizations and common financial applications was given by Duffie et al. (2003) under a regularity assumption, which requires the functions

$$
t \mapsto V(t, u) \quad \text{and} \quad t \mapsto \int_0^t \psi(V(s, u)) \, ds
$$

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are differentiable at $t = 0$ and continuous at $u = 0$. The regularity problem asks whether this property holds automatically for stochastically continuous affine processes. This property was established in Dawson and Li (2006) under the first moment condition. The problem was finally settled in Keller-Ressel et al. (2011), where it was proved that any stochastically continuous affine Markov process is regular. The connection of the regularity problem with Hilbert’s fifth problem was also explained in Keller-Ressel et al. (2011).

The ergodicity and strong Feller property of CBI- and OU-type processes have been studied by a number of authors. In particular, a sufficient and necessary integrability condition for the ergodicity of a one-dimensional subcritical or critical CBI-process was announced in Pinsky (1972); see Li (2011) for a proof. It was proved in Sato and Yamazato (1984) that a finite-dimensional OU-type process is ergodic if and only if the eigenvalues of its coefficient matrix have strictly negative real parts. The coupling property and strong Feller property of finite-dimensional OU-type processes was studied in Priola and Zabczyk (2009) and Wang (2011). The ergodicity and exponential ergodicity of such processes in total variation distances were proved in Schilling and Wang (2012) and Wang (2012). The strong Feller property and exponential ergodicity in the total variation distance of one-dimensional CBI-processes were shown in Li and Ma (2015) by a coupling method; see also Li (2020a). In the recent work of Li (2020b), the ergodicities and exponential ergodicities in Wasserstein and total variation distances of Dawson-Watanabe superprocesses with or without immigration were proved, which clearly includes the finite-dimensional CBI-processes.

For general finite-dimensional affine Markov processes with the strictly negative real parts of eigenvalues of its coefficient matrix, a sufficient condition for ergodicity in weak convergence was given in Jin et al. (2020). The necessity of the condition was not established in Jin et al. (2020), so their result partially covers the those in Pinsky (1972), Li (2011) and Sato and Yamazato (1984). The exponential ergodicity of finite-dimensional affine Markov processes in the Wasserstein distance was established in Friesen et al. (2020). Zhang and Glynn (2018) provided sufficient conditions for ergodicity and exponential ergodicity of such processes in the total variation distance.

The main purpose of this paper is to study the ergodicity of the finite-dimensional affine processes. For simplicity, we focus on the (1+1)-dimensional affine processes. We prove some results on the ergodicity and exponential ergodicity of the processes in total variation distances under natural conditions. Instead of Zhang and Glynn (2018), our approach is based on coupling methods developed in Schilling and Wang (2012) and Wang (2012); see also Li and Ma (2015), which answers the question appeared on Jin et al. (2017), pp.1145; see also Friesen and Jin (2020), pp.646.

The remainder of this paper is organized as follows. In section 2, we prove the ergodicity of (1+1)-dimensional affine processes in total variance distances. The exponential ergodic property in total variance distances is established under stronger conditions in section 3. Finally, the strong Feller property is studied in section 4.
2 Ergodicity in total variance distances

In this and next section, we mainly prove the ergodicity and exponential ergodicity of (1+1)-affine Markov processes in total variance distances by dividing the progress of two processes issued from different points into two parts by a natural coupling of CBI-processes. It is well known that the coupling property along with the existence of a stationary measure can yield the ergodicity for the process; see, e.g., Wang (2012), which motivates the basic proof in this section, as well as for section 3. More precisely, we prove the coupling property for the process in Proposition 2.3 under proper conditions, and show the existence of a stationary distribution in Lemma 2.4. Based on those results, we prove the ergodic property in Theorem 2.5.

We first give some notations. Write $\mathbb{G} = \mathbb{R}_+ \times \mathbb{R}$. Define $U = \mathbb{C}_- \times i\mathbb{R}$, where $\mathbb{C}_- = \{a + ib : a \in \mathbb{R}_-, b \in \mathbb{R}\}$ and $i\mathbb{R} = \{ia : a \in \mathbb{R}\}$. We further define two functions on $U$ as follows:

$$
\phi(u) = -a_1u_1 - b_1u_2 + (\alpha_{11} + \alpha_{12})u_1^2 + 2(\sqrt{\alpha_{11}\alpha_{21}} + \sqrt{\alpha_{12}\alpha_{22}})u_1u_2 + (\alpha_{21} + \alpha_{22})u_2^2 + \int_G (e^{(u,z)} - 1 - \langle u, z \rangle) m(dz),
$$
$$
\psi(u) = a_2u_1 - b_0u_2 + \frac{1}{2}\sigma^2u_2^2 + \int_G (e^{(u,z)} - 1 - z_2u_2) n(dz),
$$

where $a_1, b_i \in \mathbb{R}$ ($i = 0, 1, 2$), $a_2, \sigma \in \mathbb{R}_+$, $(\alpha_{ij})_{2 \times 2}$ is a nonnegative matrix, $m$ and $n$ are two Lévy measures supported on $\mathbb{G} \setminus \{0\}$ satisfying

$$
\int_\mathbb{G} (z_1 \wedge z_2^2 + |z_2|^2) m(dz) + \int_\mathbb{G} (1 \wedge z_1 + |z_2| \wedge |z_2|^2) n(dz) < \infty. \tag{2.1}
$$

Denote by $\{P_t : t \geq 0\}$ the transition semigroup of (1+1)-affine Markov process $\{X_t : t \geq 0\}$. It is well known that $\{P_t : t \geq 0\}$ can be uniquely determined by

$$
\int_G e^{(u, \xi)} P_t(x, d\xi) = \exp \left\{ \langle x, V(t, u) \rangle + \int_0^t \psi(V(s, u)) ds \right\},
$$

where

$$
\begin{cases}
\frac{\partial V_1}{\partial t}(t, u) = \phi(V(t, u)), & V_1(0, u) = u_1. \\
V_2(t, u) = e^{-b_2t}u_2.
\end{cases}
$$

We can also obtain the process as a unique strong solution to a stochastic integral equation system. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. Denote by $W_i(ds, du), i = 0, 1, 2$ the $(\mathcal{F}_t)$-Gaussian white noises on $(0, \infty)^2$ with intensity $dsdu$, $M(ds, du, dz)$ be an $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty)^2 \times \mathbb{G}$ with intensity $dsdum(dz)$ and $\bar{N}(ds, dz)$ an $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty) \times \mathbb{G}$ with intensity $dsmn(dz)$, the corresponding compensated measures are defined by $\bar{M}(ds, du, dz)$ and $\bar{N}(ds, dz)$. Let $P^M_i(t) = \int_0^t \int_\mathbb{G} z_i M(ds, du, dz)$ and $P^N_i(t) = \int_0^t \int_\mathbb{G} z_i N(ds, dz)$ for $i = 1, 2$. We assume those
random elements are independent of each other. Let $Y_0, Z_0$ be $\mathcal{F}_0$-measurable random variables and $Y_0 \geq 0$. Let us consider the following stochastic integral equation system:

$$
Y_t = Y_0 + \int_0^t (a_2 - a_1 Y_s) \, ds + \sqrt{2\alpha_{11}} \int_0^t \int_0^{Y_s} W_1(ds, du) + \sqrt{2\alpha_{12}} \int_0^t \int_0^{Y_s} W_2(ds, du) + \int_0^t \int_0^{Y_s} z_1 \, N(ds, dz) + \int_0^t \int_0^{Y_s} z_2 \, \tilde{M}(ds, du, dz),
$$

(2.2)

$$
Z_t = Z_0 - \int_0^t (b_0 + b_1 Y_s + b_2 Z_s) \, ds + \sigma \int_0^t \int_0^{Y_s} W_0(ds, du) + \sqrt{2\alpha_{21}} \int_0^t \int_0^{Y_s} W_1(ds, du) + \sqrt{2\alpha_{22}} \int_0^t \int_0^{Y_s} W_2(ds, du) + \int_0^t \int_0^{Y_s} z_2 \, \tilde{N}(ds, dz) + \int_0^t \int_0^{Y_s} z_2 \, \tilde{M}(ds, du, dz).
$$

(2.3)

Here and in the sequel, we understand that for any $a \leq b \in \mathbb{R}$

$$
\int_a^b = \int_{[a,b]} \quad \text{and} \quad \int_{\infty} = \int_{(a,\infty)}.
$$

The existence of the solution to (2.2)–(2.3) is a consequence of Theorem 6.2 in Dawson and Li (2006), where a weakly equivalent stochastic equation system was studied. The pathwise uniqueness for (2.2)–(2.3) follows by modifications of the proofs in Dawson and Li (2006, 2012). Denote by \{X_t : t \geq 0\} = \{Y_t, Z_t : t \geq 0\} the unique strong solution to (2.2)–(2.3). Then \{Y_t : t \geq 0\} is a one-dimensional CBI-process and \{X_t : t \geq 0\} is an (1+1)-dimensional affine Markov process.

For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{G}$, let \{X_t(x) : t \geq 0\} and \{X_t(y) : t \geq 0\} be the affine processes defined by (2.2)–(2.3) starting from $x$ and $y$, respectively. Let \(\varsigma = \inf\{t \geq 0 : Y_t(x_1) = Y_t(y_1)\}\) be the coalescence time of the coupling \{(Y_t(x_1), Y_t(y_1)) : t \geq 0\}. Given $f \in B_b(\mathbb{G})$, one can see that

$$
|P_t f(x) - P_t f(y)| \leq \mathbb{P}\{|f(X_t(x)) - f(X_t(y))|1_{\{t < \varsigma\}}\}
$$

$$
+ \mathbb{P}\{|f(X_t(x)) - f(X_t(y))|1_{\{t \geq \varsigma\}}\}
$$

$$
\leq 2\|f\| \mathbb{P}\{Y_t(x_1) \neq Y_t(y_1)\}
$$

$$
+ \mathbb{P}\{|f(X_t(x)) - f(X_t(y))|1_{\{t \geq \varsigma\}}\}. \quad (2.4)
$$

For any $\varepsilon > 0$, we define a finite measure $n_\varepsilon$ on $\mathbb{R}$ such that

$$
n_\varepsilon(B) = \begin{cases} 
n(\mathbb{R}_+ \times B), & \text{if } n(\mathbb{R}_+ \times \mathbb{R}) < \infty; \\
n(\mathbb{R}_+ \times B_\varepsilon), & \text{if } n(\mathbb{R}_+ \times \mathbb{R}) = \infty,
\end{cases}
$$

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where $B \in \mathcal{B}(\mathbb{R})$ and $B_\varepsilon := B \setminus \{z_2 : |z_2| < \varepsilon\}$.

In the following we give some key estimates. For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{G}$, let \{\(X_t(x) : t \geq 0\)\} and \{\(X_t(y) : t \geq 0\)\} be the affine processes defined by (2.2)–(2.3) starting from $x$ and $y$, respectively. In view of (2.2)–(2.3) we have

\[
Z_t(x) = T_t x_2 - \int_0^t T_{t-s} [b_0 + b_1 Y_s(x_1)] \, ds + \sqrt{2\alpha_2} \int_0^t \int_0^{Y_s(x_1)} T_{t-s} W_1(ds, du) \\
+ \sqrt{2\alpha_2} \int_0^t \int_0^{Y_s(x_1)} T_{t-s} W_2(ds, du) + \sigma \int_0^t \int_0^{Y_s(x_1)} T_{t-s} W_0(ds, du) \\
+ \int_0^t \int_0^{Y_s(x_1)} \int_G T_{t-s} z_2 M(ds, du, dz) + \int_0^t \int_G T_{t-s} z_2 \tilde{N}(ds, dz),
\]

(2.5)

and

\[
Z_t(x) - Z_t(y) = T_t (x_2 - y_2) - b_1 \int_0^t T_{t-s} \left(Y_s(x_1) - Y_s(y_1)\right) ds \\
+ \sqrt{2\alpha_2} \int_0^t Y_s(x_1) \, T_{t-s} W_1(ds, du) \\
+ \sqrt{2\alpha_2} \int_0^t Y_s(x_1) \, T_{t-s} W_2(ds, du) \\
+ \int_0^t Y_s(x_1) \int_G T_{t-s} z_2 M(ds, du, dz),
\]

(2.6)

where $T_t = e^{-bt_2}$ for $t \geq 0$.

**Lemma 2.1** Suppose $0 < 2b_2 < a_1$. Then there exist strictly positive constants $C_1$ and $C_2$ such that for any $x, y \in \mathbb{G}$

\[
\mathbb{E}|Z_t(x) - Z_t(y)| \leq T_t \left(|x_2 - y_2| + C_2(x_1 - y_1) + \sqrt{C_1(x_1 - y_1)}\right),
\]

(2.7)

\[
\mathbb{P}\{|Z_t(x) - Z_t(y)| > T_t \eta\} \leq \frac{1}{\eta} \left(|x_2 - y_2| + C_2(x_1 - y_1) + \sqrt{C_1(x_1 - y_1)}\right)
\]

(2.8)

for $\eta > 0$.

**Proof.** Note that $\mathbb{E}[Y_s(x_1) - Y_s(y_1)] = (x_1 - y_1)e^{-a_1s}$. By Martingale inequality and Cauchy-Schwartz inequality we see that

\[
\mathbb{E} \left| \int_0^t \int_{Y_s(x_1)} \int_G e^{bs} z_2 M(ds, du, dz) \right| \leq \left[ \mathbb{E} \left| \int_0^t \int_{Y_s(x_1)} \int_G e^{bs} z_2 M(ds, du, dz) \right|^2 \right]^{\frac{1}{2}} \\
\leq \left[ \frac{e^{(2b_2-a_1)t} - 1}{2b_2 - a_1} (x_1 - y_1) \int_G |z_2|^2 m(dz) \right]^{\frac{1}{2}}
\]

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where \( C \) is not hard to see that

\[
\sqrt{2 \alpha_1} \int_0^t \int_{Y_s(y)} e^{2bs} W_1(ds, du) + \sqrt{2 \alpha_2} \int_0^t \int_{Y_s(y)} e^{2bs} W_2(ds, du) \leq C_{11}(x_1 - y_1),
\]

similarly,

\[
\begin{align*}
&\mathbb{E}\left|\sqrt{2 \alpha_1} \int_0^t \int_{Y_s(x)} e^{2bs} W_1(ds, du) + \sqrt{2 \alpha_2} \int_0^t \int_{Y_s(x)} e^{2bs} W_2(ds, du)\right| \\
&\leq \left[2 \alpha_1 \frac{e^{(2b_2-a_1)t} - 1}{2b_2 - a_1} (x_1 - y_1) \right]^{\frac{1}{2}} + \left[2 \alpha_2 \frac{e^{(2b_2-a_1)t} - 1}{2b_2 - a_1} (x_1 - y_1) \right]^{\frac{1}{2}} \\
&\leq \sqrt{C_{12}(x_1 - y_1)},
\end{align*}
\]

and

\[
\mathbb{E} \left| b_1 \int_0^t e^{bs} (Y_{s-}(x) - Y_{s-}(y)) ds \right| = \left| \frac{b_1 (e^{(b_2-a_1)t} - 1)}{b_2 - a_1} (x_1 - y_1) \right| \leq C_2(x_1 - y_1),
\]

where \( C_{11} = \frac{1}{a_1 - 2b_2} \int_{\mathbb{R}} |z|^2 m(dz), C_{12} = 8 \max \left\{ \frac{\alpha_{21}}{a_1 - 2b_2}, \frac{\alpha_{22}}{a_1 - 2b_2} \right\} \) and \( C_2 = \frac{|b_1|}{a_1 - b_2} \). By (2.6) it is not hard to see that

\[
\mathbb{E}|Z_t(x) - Z_t(y)| \leq T_t \left( |x_2 - y_2| + C_2(x_1 - y_1) + \sqrt{C_1(x_1 - y_1)} \right),
\]

where \( C_1 = 4 \max \{C_{11}, C_{12}\} \). The second inequality is an immediate result following from Markov inequality and (2.7). \( \square \)

For a bounded measurable function \( f \) on \( \mathbb{R} \), define the supremum norm \( \|f\| = \sup_x |f(x)| \). Given two bounded measures \( \mu \) and \( \nu \) on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \), let \( \mu \wedge \nu = \mu - (\mu - \nu)^+ = \nu - (\nu - \mu)^+ \), where the superscript "+" refers to the positive part in the Jordan-Hahn decomposition. It is easy to see that \( \mu \wedge \nu = \nu \wedge \mu = 2^{-1}(\mu + \nu - |\mu - \nu|) \), where \( |\mu - \nu| = (\mu - \nu)^+ + (\nu - \mu)^+ \) is the total variation measure. Let \( \| \cdot \|_{\text{var}} \) denote the total variation norm defined by

\[
\|f\|_{\text{var}} = \sup_{\|f\| \leq 1} |\mu(f) - \nu(f)|,
\]

where \( \mu(f) = \int f \, d\mu \). For the convenience, we formulate the following conditions:

(A) Denote the branching mechanism of the CBI-processes \( \{Y_t : t \geq 0\} \) by

\[
\phi_0(x) = a_1 x + (\alpha_{11} + \alpha_{12}) x^2 + \int_G (e^{-xz_1} - 1 + xz_1) m(dz), \quad x \geq 0.
\]

There exists \( \theta > 0 \) such that for any \( z \geq \theta, \phi_0(z) > 0 \), and

\[
\int_\theta^\infty \phi_0^{-1}(z) \, dz < \infty.
\]

(B) There exist two constants \( \varepsilon, \eta > 0 \) such that

\[
\inf_{|a| \leq \eta} n_\varepsilon \wedge (\delta_a * n_\varepsilon)(\mathbb{R}) > 0, \quad \int_{\{|z| > \varepsilon\}} |z| n(dz) < \infty.
\]
Remark 2.2

(1). Condition (A) is called Grey’s condition; see, e.g., Grey (1974). This condition has been used to study the exponentially ergodic property in total variance distances and strong Feller property of one-dimensional subcritical CBI-processes; see, e.g., Li and Ma (2015) and Li (2020a).

(2). Condition (B) is sharp to study the coupling property for the Lévy process $(L_t)_{t\geq0}$ with Lévy measure $\kappa$ and the Ornstein-Uhlenbeck process driven by $(L_t)_{t\geq0}$. Intuitively, it is one possibility to guarantee the sufficient jump activity such that the process admits a successful coupling; see, e.g., Schilling and Wang (2011, 2012) and Wang (2012).

Proposition 2.3 Suppose that Conditions (A,B) hold and $0 < 2b_2 < a_1$. Then there exist constants $\bar{C}, \kappa > 0$, such that for any $t > 0$, $x, y \in \mathbb{R}$, we have

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq \bar{C} \left(1 + (\bar{b}_{\kappa} + 1)|x_1 - y_1| + \sqrt{|x_1 - y_1|} + |x_2 - y_2| \right) \frac{1}{\sqrt{t}},$$

where $\bar{v}_t$ is the unique solution of the following equation:

$$\frac{d}{dt} \bar{v}_t = -\phi_0(\bar{v}_t), \quad t > 0$$

with initial condition $\bar{v}_{0+} = \infty$. Moreover, the mapping $t \mapsto \bar{v}_t$ is decreasing.

Proof. Under Condition (A), the solution $\bar{v}_t$ to (2.9) is unique, and the mapping $t \mapsto \bar{v}_t$ is decreasing; see, e.g., Theorem 3.6, Theorem 3.7 and Corollary 3.14 in Li (2020a). Under Condition (B), for simplicity, we let

$$L_t^\varepsilon = \int_0^t \int 1_{\{|z| > \varepsilon\}} z_2 N(ds, dz)$$

and define a sequence of stopping times $\Upsilon^\varepsilon = \inf\{t > \Upsilon^\varepsilon_{t-1} : L_t^\varepsilon \neq L_{t-}\}$ with convention $\Upsilon^\varepsilon_0 = 0$. For $i \geq 1$ let $\tau_i^\varepsilon = \Upsilon_i^\varepsilon - \Upsilon_{i-1}^\varepsilon$ and $U_i^\varepsilon = \int_{\{|z| > \varepsilon\}} z_2 N(ds, dz)$. Then $(\tau_i^\varepsilon)_{i \geq 1}$ are i.i.d. random variables which are exponentially distributed with intensity $C_\varepsilon = n_\varepsilon(\mathbb{R})$ and $(U_i^\varepsilon)_{i \geq 1}$ of i.i.d. random variables on $\mathbb{R}$ with distribution $\tilde{n}_\varepsilon := n_\varepsilon/C_\varepsilon$. Moreover, the two sequences $(U_i^\varepsilon)_{i \geq 1}$ and $(\tau_i^\varepsilon)_{i \geq 1}$ are independent of each other. Let $N_t^\varepsilon = \sup\{k \geq 1 : \sum_{i=1}^k \tau_i^\varepsilon \leq t\}$. Then $(N_t^\varepsilon)_{t \geq 0}$ is a Poisson process of intensity $C_\varepsilon$. Now we can rewrite

$$\int_0^t \int 1_{\{|z| > \varepsilon\}} z_2 N(ds, dz) = \sum_{i=1}^{N_t^\varepsilon} U_i^\varepsilon$$

with $\sum_{i=1}^0 = 0$ by convention. It is not hard to check that

$$\int_0^t \int 1_{\{|z| \geq \varepsilon\}} T_{t-s} z_2 N(ds, dz) = 0 \cdot 1_{\{|\Upsilon_1^\varepsilon| > t\}} + \sum_{k=1}^{\infty} 1_{\{\tau_k^\varepsilon \leq t < \Upsilon_{k+1}^\varepsilon\}} \sum_{i=1}^k T_{t-\Upsilon_i^\varepsilon} U_i^\varepsilon.$$
Let us use the following notations for the convenience,

\[
\zeta_{t,\varepsilon} = T_t \left\{ (b_0 - \int_{\{ |z| > \varepsilon \}} z_2 n(dz)) \int_0^t e^{b_2 s} \, ds \\
+ \sigma \int_0^t \int_0^1 e^{b_2 s} W_0(ds, du) + \int_0^t \int_{\{ |z| \leq \varepsilon \}} e^{b_2 s} z_2 \, \tilde{N}(ds, dz) \right\},
\]

\[
\theta_t(x_1) = -b_1 \int_0^t e^{b_2 s} Y_s(x_1) \, ds + \int_0^t \int_0^t \int_G e^{b_2 s} z_2 \, \tilde{M}(ds, du, dz),
\]

Moreover, for any \( t \geq 0 \), we denote the distribution of \( \theta_t(x_1) - \theta_t(y_1) \) by \( \Gamma_{t,x_1-y_1} \). In view of (2.3), for any \( f \in B_b(\mathbb{G}) \),

\[
\mathbb{E} f(Y_t(x_1), Z_t(x)) = \mathbb{E} f \left( Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} + \sum_{k=1}^{N_{\varepsilon}^t} T_{t-Y_{k}^t} U_{k}^t \right)
\]

\[
= \mathbb{E} \left[ f \left( Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} \right) \mathbf{1}_{\{N_{\varepsilon}^t=0\}} \right] + \mathbb{E} f \left( Y_t(x_1), \sum_{k=1}^{\infty} \mathbf{1}_{\{ T_{k} \leq t < T_{k+1} \}} \left( T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} + \sum_{j=1}^{k} T_{Y_{j}^t} U_{j}^t \right) \right)
\]

\[
= \mathbb{E} \left[ f \left( Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} \right) \mathbf{1}_{\{N_{\varepsilon}^t=0\}} \right] + \sum_{k=1}^{\infty} \int_{\mathbb{R}^k} \ldots \int_{\sum_{i=1}^{k} t_i \leq \sum_{i=1}^{k+1} t_i} C_{\varepsilon}^{k+1} e^{-C_{\varepsilon} \sum_{i=1}^{k+1} t_i} \, dt_1 \ldots dt_{k+1}
\]

\[
\times \int_{\mathbb{R}^k} \mathbb{E} f \left( Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} + \sum_{i=1}^{k} T_{\sum_{j=1}^{i} T_j r_j} \right) \tilde{n}_{\varepsilon}(dr_1) \ldots \tilde{n}_{\varepsilon}(dr_k)
\]

\[
= \mathbb{E} \left[ f \left( Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} \right) \mathbf{1}_{\{N_{\varepsilon}^t=0\}} \right] + \sum_{k=1}^{\infty} \int_{\mathbb{R}^k} \ldots \int_{\sum_{i=1}^{k} t_i \leq \sum_{i=1}^{k+1} t_i} C_{\varepsilon}^{k+1} e^{-C_{\varepsilon} \sum_{i=1}^{k+1} t_i} \, dt_1 \ldots dt_{k+1}
\]

\[
\times \int_{\mathbb{R}^k} \mathbb{E} f \left( Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,\varepsilon} + z \right) n_{t_1,\ldots,t_k}(dz),
\]

where the second equality partly follows from formula (2.10) in Schilling and Wang (2012); see also Lemma 2.2 in Schilling and Wang (2012). Here, \( n_{t_1,\ldots,t_k}(dz) \) is the probability measure on \( \mathbb{R} \), which is the image of the \( k \)-fold product measure \( \tilde{n}_{\varepsilon} \times \ldots \times \tilde{n}_{\varepsilon} \) under the linear transformation \( J_{t_1,\ldots,t_k} : \mathbb{R}^{k+1} \rightarrow \mathbb{R} \) given by \( J_{t_1,\ldots,t_k}(r_1, \ldots, r_k) = T_{t_1} r_1 + \ldots + T_{t_1+\ldots+t_k} r_k \). Given \((x_1, x_2), (y_1, y_2) \in \mathbb{G}\), without loss of generality, we can assume that \( x_1 \geq y_1 \), and we have

\[
\sup_{\|f\| \leq 1} \left| \mathbb{E} \left[ (f(X_t(x)) - f(X_t(y))) \mathbf{1}_{\{t \geq \varepsilon\}} \right] \right|
\]
≤ 2e^{−C_ε t} \sum_{k=1}^{∞} \int \cdots \int_{\sum_{i=1}^{k} t_i \leq t < \sum_{i=1}^{k+1} t_i} C_{ε}^{k+1} e^{−C_ε \sum_{i=1}^{k+1} t_i} dt_1 \cdots dt_{k+1}
\times \sup_{\|f\| \leq 1} \left| \int_{\mathbb{R}} \mathbb{E} f\left(Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,ε} + z\right) n_{t_1,\ldots,t_k}(dz) \right|
\times \delta_{T_t(x_2-y_2+z)}(dz)\right| \Gamma_{t,x_1-y_1}(dz_2)
\leq 2e^{−C_ε t} \sum_{k=1}^{∞} \int \cdots \int_{\sum_{i=1}^{k} t_i \leq t < \sum_{i=1}^{k+1} t_i} C_{ε}^{k+1} e^{−C_ε \sum_{i=1}^{k+1} t_i} dt_1 \cdots dt_{k+1}
\times \int_{W} \mathbb{E} f(Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta_{t,ε} + z) n_{t_1,\ldots,t_k}(dz)\bigg| \Gamma_{t,x_1-y_1}(dz_2)
+ 2 \sum_{k=1}^{∞} \int \cdots \int_{\sum_{i=1}^{k} t_i \leq t < \sum_{i=1}^{k+1} t_i} C_{ε}^{k+1} e^{−C_ε \sum_{i=1}^{k+1} t_i} dt_1 \cdots dt_{k+1} \int_{\mathbb{R}\setminus W} \Gamma_{t,x_1-y_1}(dz_2),

where W = \{z_2 : |z_2 + x_2 - y_2| < \eta\} for some \(\eta > 0\) and

If we set \(p(t, x, y) := \int_{\mathbb{R}\setminus W} \Gamma_{t,x_1-y_1}(dz_2)\), one can see that

\[\sum_{k=1}^{∞} \int \cdots \int_{\sum_{i=1}^{k} t_i \leq t < \sum_{i=1}^{k+1} t_i} C_{ε}^{k+1} e^{−C_ε \sum_{i=1}^{k+1} t_i} dt_1 \cdots dt_{k+1} \Gamma_{t,x_1-y_1}(dz_2)\]
\[\leq p(t, x, y)e^{−C_ε t} \sum_{k=1}^{∞} C_{ε}^{k+1} \int \cdots \int_{t \geq \sum_{i=1}^{k} t_i} dt_1 \cdots dt_k\]
\[= p(t, x, y)C_{ε}(1 − e^{−C_ε t}).\]

By (2.6), we have \(T_t^{-1}\left|Z_t(x) - Z_t(y)\right| = (x_2 - y_2) + \theta_t(x_1) - \theta_t(y_1)\). It follows that \(T_t^{-1|Z_t(x) - Z_t(y)| = |x_2 - y_2 + \theta_t(x_1) - \theta_t(y_1)|}\). Then

\[p(t, x, y) \leq \mathbb{P}\{\left|Z_t(x) - Z_t(y)\right| > T_t\eta\}\]
\[\leq \frac{1}{\eta} \left(\left|x_2 - y_2\right| + C_2(x_1 - y_1) + \sqrt{C_3(x_1 - y_1)}\right), \tag{2.10}\]

where the last inequality follows from (2.8). On the other hand, following the proof of Theorem 1.1 in Schilling and Wang (2012), for \(|x_2 - y_2 + z_2| \leq \eta, t \geq t_1 + \ldots + t_k\) and \(k \geq 1\) we can
find some constants $C_{1,\eta}, C_3 > 0$ such that

$$w(z_2, t, t_1, ..., t_k) \leq \frac{C_{1,\eta}}{\sqrt{k}},$$

and

$$\frac{C_3}{\sqrt{t}} \geq 2e^{-C_\varepsilon t} + \sum_{k=1}^{\infty} \int ... \int_{\sum_{i=1}^{k-1} t_i \leq \sum_{i=1}^{k+1} t_i} C_{e}^{k+1} e^{-C_\varepsilon \sum_{i=1}^{k+1} t_i} dt_1 ... dt_{k+1}$$

$$\times \int_W w(z_2, t, t_1, ..., t_k) \Gamma_{t,x_1-y_1}(dz_2).$$

In conclusion, we have

$$\|E[(f(X_t(x)) - f(X_t(y)))1_{\{t \geq c\}}]\| \leq \frac{\|f\|C_3}{\sqrt{t}} + \frac{2\|f\|C_\varepsilon}{\eta}$$

$$\times \left( |x_2 - y_2| + C_2(x_1 - y_1) + \sqrt{C_1(x_1 - y_1)} \right).$$

In view of (2.4), for $f \in B_b(\mathbb{G})$

$$|P_t f(x) - P_t f(y)| \leq 2\|f\|\|\bar{v}_t(x_1 - y_1)\| + \frac{\|f\|C_3}{\sqrt{t}}$$

$$+ \frac{2\|f\|C_\varepsilon}{\eta} \left( |x_2 - y_2| + C_2(x_1 - y_1) + \sqrt{C_1(x_1 - y_1)} \right).$$

For any $s \in (0, t)$, by using the Markov property and (2.7),

$$|P_t f(x) - P_t f(y)|$$

$$= \left| E\left\{ P_s f(Y_{t-s}(x_1), Z_{t-s}(x)) - P_s f(Y_{t-s}(y_1), Z_{t-s}(y)) \right\} \right|$$

$$\leq 2\|f\|\bar{v}_s E(Y_{t-s}(x_1) - Y_{t-s}(y_1)) + \frac{\|f\|C_3}{\sqrt{s}}$$

$$+ \frac{2\|f\|C_\varepsilon}{\eta} \left( E|Z_{t-s}(x) - Z_{t-s}(y)| + \left[ C_1 E(Y_{t-s}(x_1) - Y_{t-s}(y_1)) \right]^{\frac{1}{2}}$$

$$+ C_2 E(Y_{t-s}(x_1) - Y_{t-s}(y_1)) \right)$$

$$\leq 2\|f\|\bar{v}_s(x_1 - y_1)e^{-a_1(t-s)} + \frac{\|f\|C_3}{\sqrt{s}}$$

$$+ \frac{2\|f\|C_\varepsilon}{\eta} \left\{ \sqrt{C_1(x_1 - y_1)e^{-a_1(t-s)} + C_2(x_1 - y_1)e^{-a_1(t-s)}}$$

$$+ e^{-b_2(t-s)}(|x_2 - y_2| + C_2(x_1 - y_1) + \sqrt{C_1(x_1 - y_1)}) \right\},$$

which implies that

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}}$$
\[
\leq C_4 \left[ 1 + (\bar{v}_s + 1)(x_1 - y_1) + \sqrt{x_1 - y_1} + |x_2 - y_2| \right] \left( e^{-b_2(t-s)} \vee \frac{1}{\sqrt{t-s}} \right)
\leq C_4 \left[ 1 + (\bar{v}_s + 1)(x_1 - y_1) + \sqrt{x_1 - y_1} + |x_2 - y_2| \right] \left( \frac{\kappa}{\sqrt{t-s}} \vee \frac{1}{\sqrt{t-s}} \right)
\]

for some constants \(C_4 > 0\) and \(\kappa = 1 \vee \frac{1}{e b_2}\). We finally obtain the required assertion by setting \(s = \frac{1}{\kappa^2+1} t, \hat{C} = \sqrt{\kappa^2 + 1} C_4\). \(\square\)

Note that the initial assumptions of Lévy measures \(m\) and \(n\) in Jin et al. (2020) are weaker than (2.1). Before we establish the main result in this section, we need the following lemma, which is a consequence of Theorem 2.7 in Jin et al. (2020).

**Lemma 2.4** Assume that \(a_1 > 0, b_2 > 0, (2.1)\) and

\[\int_{\{x_1 \geq 1\}} \log z_1 n(dz) < \infty \tag{2.11}\]

hold. Then the law of \(X_t\) converges weakly to a limiting distribution \(\pi\) given by

\[\int_G e^{\langle u, y \rangle} \pi(dy) = \exp \left\{ \int_0^\infty \psi(V(s,u)) \, ds \right\}, \quad u \in U,\]

Moreover, \(\pi\) is the unique stationary distribution for \(X\).

Based on the Lemma 2.4 and Proposition 2.3 above, we can prove the ergodicity of the transition semigroup \((P_t)_{t \geq 0}\). More explicitly, we have the following result:

**Theorem 2.5** Suppose \(\int_{\{x_1 \geq 1\}} \log z_1 n(dz) < \infty\) and the conditions of Proposition 2.3 are satisfied. Then the affine process \((X_t)_{t \geq 0}\) is ergodic in the total variation distance. Namely, there exists a unique invariant measure \(\pi\) for the process such that for any \(x \in G\),

\[\lim_{t \to \infty} \|P_t(x, \cdot) - \pi(\cdot)\|_{\text{var}} = 0.\]

**Proof.** By (2.1) and Lemma 2.4 there exists a unique invariant measure \(\pi\) for \((P_t)_{t \geq 0}\). Fix \(x \in G\), one can see that

\[\|P_t(x, \cdot) - \pi(\cdot)\|_{\text{var}} \leq \int_G \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \pi(dy).
\]

For any \(\epsilon > 0\), we choose \(\delta > 0\) such that \(\pi\{|x_1 - y_1| + |x_2 - y_2| > \delta\} \leq \epsilon\). By Proposition 2.3 we have

\[
\|P_t(x, \cdot) - \pi(\cdot)\|_{\text{var}} \leq \int_{G \setminus \Xi} \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \pi(dy) + \int_{\Xi} \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \pi(dy)
\]

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\[
\leq \frac{\hat{C}}{\sqrt{t}} \int_{G \setminus \Xi} \left( 1 + \left( \bar{v} \frac{1}{\kappa + 1} + 1 \right) \delta + \sqrt{\delta} + \delta \right) \pi(dy) + 2\pi(\Xi)
\]

\[
\leq \frac{\hat{C}}{\sqrt{t}} \int_{G \setminus \Xi} \left( 1 + \left( \bar{v} \frac{1}{\kappa + 1} + 1 \right) \delta + \sqrt{\delta} + \delta \right) \pi(dy) + 2\epsilon,
\]

where \( \Xi = \{ y : |x_1 - y_1| + |x_2 - y_2| > \delta \} \). Letting \( t \to \infty \) first and then \( \epsilon \to 0 \), we prove the desired result. \( \square \)

### 3 Exponential ergodicity in total variance distances

In this section, we prove the exponential ergodicity of (1+1)-affine Markov processes defined by \( (2.2) - (2.3) \) with \( (2.1) \) in total variance distance under proper conditions. Since the exponential ergodicity implies the ergodicity, it's reasonable to strengthen the assumptions to guarantee the exponential ergodicity. We also give the following conditions before moving forward:

(C) There exists \( \varepsilon > 0 \), such that

\[
\limsup_{\rho \to 0} \left[ \sup_{|a| \leq \rho} \frac{\|n_{\varepsilon} - \delta_a * n_{\varepsilon}\|_{\text{var}}}{\rho} \right] < \infty, \quad \int_{\{|z_2| > \varepsilon\}} |z_2| n(dz) < \infty.
\]

(C') There exists \( \varepsilon > 0 \), such that

\[
\limsup_{\rho \to 0} \left[ \sup_{|a| \leq \rho} \frac{\|n_{\varepsilon} - \delta_a * n_{\varepsilon}\|_{\text{var}}}{\rho} \right] < \infty, \quad \int_{\{|z_2| > 1\}} |z_2|^2 n(dz) < \infty.
\]

**Remark 3.1** Condition (C') plays an important role in characterizing the exponential ergodicity. This condition implies \( \int_G |z_2|^2 n(dz) < \infty \), so it is stronger than (C). Condition (B) is weaker than (C); see, e.g., Remark 1 in Wang (2012) for a proof.

**Proposition 3.2** Suppose that Conditions (A, C) hold and \( 0 < 2b_2 < a_1 \). Then there exist constants \( \tilde{C}, \tilde{\kappa} > 0 \), such that for any \( t > 0, x, y \in \mathbb{G} \), we have

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq \tilde{C} \left( 1 + \left( \bar{v} \frac{2t}{c_\varepsilon} + 1 \right)|x_1 - y_1| + \sqrt{|x_1 - y_1| + |x_2 - y_2|} \right) e^{-\tilde{\kappa} t},
\]

where \( \bar{v}_t \) is defined as that in Proposition 2.3.

**Proof.** Following the arguments in the proof of Proposition 2.3, it implies that on \( \{N_\varepsilon^t \geq 1\} \)

\[
\int_0^t \int_{\{|z_2| \geq \varepsilon\}} T_{t-s} z_2 N(ds, dz) = \sum_{k=1}^{N_\varepsilon^t} T_{t-\tau_k} U_{\varepsilon k}.
\]

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For given $f \in B_0(\mathcal{G})$, $x \in \mathcal{G}$ we have the following decomposition
\[
\mathbb{E}[f(Y_t(x_1), Z_t(x))] = \mathbb{E}[f(Y_t(x_1), Z_t(x))1_{\{N_t^\varepsilon \geq 1\}}] + P_t^1 f(x),
\]
where $P_t^1 f(x) = \mathbb{E}[f(Y_t(x_1), Z_t(x))1_{\{N_t^\varepsilon \geq 1\}}]$ and
\[
P_t^1 f(x) = \mathbb{E}\left\{1_{\{N_t^\varepsilon \geq 1\}} \int f(Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta^1_{t, \varepsilon} + T_{t-T^\varepsilon_{N_t^\varepsilon}} U_{N_t^\varepsilon}^\varepsilon \right\}
\]
\[
= \frac{1}{C_\varepsilon} \mathbb{E}\left\{1_{\{N_t^\varepsilon \geq 1\}} \int f(Y_t(x_1), T_t(x_2 + \theta_t(x_1)) + \zeta^1_{t, \varepsilon} + T_{t-T^\varepsilon_{N_t^\varepsilon}} z) n_\varepsilon(dz) \right\},
\]
where
\[
\zeta^1_{t, \varepsilon} = T_t \left\{ (b_0 - \int_{\{|z| > \varepsilon}\} z_2 n(dz)) \int_0^t e^{b_2 s} ds + \sigma \int_0^t \int_0^t e^{b_2 s} W_0(ds, du)
\]
\[
+ \sum_{k=1}^{N_t^\varepsilon - 1} T_{-T^\varepsilon_{k}} U_k^\varepsilon + \int_0^t \int_{|z| \leq \varepsilon} e^{b_2 s} z_2 \tilde{N}(ds, dz) \right\}.
\]
We recall again that the distributions of $\theta_t(x_1) - \theta_t(y_1)$ is $\Gamma_{t,x_1-y_1}$. Given $x, y \in \mathcal{G}$, without loss of generality, it suffices to consider the case of $x_1 \geq y_1$.
\[
\mathbb{E}[|f(Y_t(x_1) - f(Y_t(y_1)))1_{\{N_t^\varepsilon \geq 1\}}1_{\{t \geq \varsigma\}}|]
\]
\[
\leq \frac{1}{C_\varepsilon} \mathbb{E}\left\{1_{\{N_t^\varepsilon \geq 1\}} \left| \int_\mathbb{R} f(z_1, t, y_1, y_2) n_\varepsilon(dz_1) - T_{T_{N_t^\varepsilon}}^\varepsilon (x_2 - y_2 + z_2) \right| \times \Gamma_{t,x_1-y_1}(dz_2) \right\}
\]
\[
\leq \frac{(1-e^{-C_\varepsilon t})\Lambda}{C_\varepsilon} \|f\| \int_\mathbb{R} \left| x_2 - y_2 \right| \Gamma_{t,x_1-y_1}(dz_2),
\]
where
\[
\Lambda = \sup_{\rho > 0} \left\{ \frac{\sup_{|a| \leq \rho} \|n_\varepsilon - \delta_\alpha * n_\varepsilon\|_{var}}{\|a\|} \right\},
\]
\[
f(z_1, t, y_1, y_2) = f(Y_t(y_1), T_t(y_2 + \theta_t(y_1)) + \zeta^1_{t, \varepsilon} + T_{t-T^\varepsilon_{N_t^\varepsilon}} z_1).
\]
We have $\Lambda < \infty$ due to the Condition (C) and the fact that $\sup_{|a| \leq \rho} \|n_\varepsilon - \delta_\alpha * n_\varepsilon\|_{var} \leq 2C_\varepsilon$. It together with Lemma 2.1 follow that
\[
\mathbb{E}[|f(Y_t(x_1) - f(Y_t(y_1)))1_{\{N_t^\varepsilon \geq 1\}}1_{\{t \geq \varsigma\}}|] \leq \frac{\Lambda \|f\|}{C_\varepsilon} \left\{ \left| x_2 - y_2 \right| + \sqrt{C_1 (x_1 - y_1) + C_2 (x_1 - y_1)} \right\},
\]
On the other hand,
\[
\mathbb{E}[f(Y_t(x_1), Z_t(x))1_{\{N_t^\varepsilon = 0\}}1_{\{t \geq \varsigma\}}] \leq \|f\| e^{-C_\varepsilon t}, \quad t \geq 0, f \in B_0(\mathcal{G}),
\]
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it follows that
\[
\left| \mathbb{E}[(f(X_t(x)) - f(X_t(y)))1_{\{t\geq s\}}] \right| \\
\leq 2 \|f\|e^{-C_xt} + \frac{A}{C_x}\|f\| \left\{ \|x_2 - y_2\| + \sqrt{C_1(x_1 - y_1)} + C_2(x_1 - y_1) \right\}.
\]

Therefore, it together with (2.4) and Theorem 10.3 in Li (2020a) imply that
\[
|P_tf(x) - P_tf(y)| \leq 2 \|f\| (\bar{v}_t(x_1 - y_1) + e^{-C_xt}) \\
+ \|f\| \left\{ \frac{A}{C_x}(\|x_2 - y_2\| + \sqrt{C_1(x_1 - y_1)} + C_2(x_1 - y_1)) \right\}. 
\]

Similarly as that in Proposition 2.3 for any \(s \in (0, t)\), by using the Markov property and (2.7) we have
\[
|P_tf(x) - P_tf(y)| \\
= \left| \mathbb{E}\left[ P_s f(Y_{t-s}(x), Z_{t-s}(x)) - P_s f(Y_{t-s}(y), Z_{t-s}(y)) \right] \right| \\
\leq 2 \|f\| \bar{v}_s \mathbb{E}[Y_{t-s}(x_1) - Y_{t-s}(y_1)] + 2 \|f\| e^{-Cxs} \\
+ \|f\| \frac{A}{C_x} \left\{ \mathbb{E}[Z_{t-s}(x) - Z_{t-s}(y)] + \left\{ C_1 \mathbb{E}[Y_{t-s}(x_1) - Y_{t-s}(y_1)] \right\}^{\frac{1}{2}} \\
+ C_2 \mathbb{E}[Y_{t-s}(x_1) - Y_{t-s}(y_1)] \right\} \\
\leq 2 \|f\| \bar{v}_s (x_1 - y_1)e^{-a_1(t-s)} + 2 \|f\| e^{-C_xt} \\
+ \|f\| \frac{A}{C_x} \left\{ e^{-b_2(t-s)} \left( |x_2 - y_2| + C_2(x_1 - y_1) \\
+ \sqrt{C_1(x_1 - y_1)} + [C_1(x_1 - y_1)e^{-a_1(t-s)}]^{\frac{1}{2}} + C_2(x_1 - y_1)e^{-a_1(t-s)} \right) \right\},
\]

it follows that
\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq \tilde{C} \left( 1 + (\bar{v}_s + 1)(x_1 - y_1) + \sqrt{x_1 - y_1} + |x_2 - y_2| \right) \left\{ e^{-C_xt} + e^{-b_2(t-s)} \right\},
\]

where \(\tilde{C} = \max\{2, \frac{A}{C_x}, \frac{C_2A}{C_x}, \frac{\sqrt{C_1A}}{C_x}\}\). Setting \(s = \frac{b_4t}{c_x + b_2}\), we obtain
\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq \tilde{C} \left( 1 + (\bar{v}_s + 1)(x_1 - y_1) + \sqrt{x_1 - y_1} + |x_2 - y_2| \right) e^{-\tilde{\kappa}t},
\]

where \(\tilde{\kappa} = \frac{b_4C_x}{c_x + b_2}\). Then the required assertion holds. \(\square\)

Based on Proposition 3.2 we have the following result:

**Theorem 3.3** Suppose that Conditions (\(A', C'\)) hold and \(0 < 2b_2 < a_1\). Moreover, assume that
\[
\int_{\{z_1 > 1\}} z_1 n(\text{d}z) < \infty
\]
holds. Then the affine process \((X_t)_{t \geq 0}\) is exponentially ergodic in the total variation distance.
Proof. Given \( x \in \mathbb{G} \), it follows from Proposition 3.2 that

\[
\| P_t(x, \cdot) - \pi(\cdot) \|_{\text{var}} \leq \int_M \| P_t(x, \cdot) - P_t(y, \cdot) \|_{\text{var}} \pi(dy)
\]

\[
\leq C e^{-\kappa t} \left( 1 + (\bar{v}, \frac{e_t}{e_c} + 1)(x_1 + \Delta_1) + \sqrt{x_1 + \Delta_1 + |x_2| + \Delta_2} \right),
\]

where

\[
\Delta_1 = \int_{\mathbb{G}} y_1 \pi(dy), \quad \Delta_2 = \int_{\mathbb{G}} |y_2| \pi(dy).
\]

In order to prove the exponential ergodic property, it suffices to prove \( \Delta_1 \) and \( \Delta_2 \) are finite. For \( u \in U \), setting \( u = (u_1, 0) \), \( u_1 \leq 0 \), then

\[
\int_{\mathbb{G}} e^{u_1 y_1} \pi(dy) = \exp \left\{ \int_0^\infty P(V_1(s, u)) ds \right\}, \quad \frac{\partial V_1(s, u)}{\partial s} = \tilde{\phi}_0(V_1(s, u)),
\]

where \( P \) and \( \tilde{\phi}_0 \) are two functions defined on \( \mathbb{R}^- \) such that

\[
P(x) = a_2 x + \int_{\mathbb{G}} (e^{x z_1} - 1) n(dz), \quad \tilde{\phi}_0(x) = -a_1 x + (\alpha_{11} + \alpha_{12}) x^2 + \int_{\mathbb{G}} (e^{x z_1} - 1 - x z_1) m(dz).
\]

Following the proof of Theorem 10.4 in Li (2020a), in the case of \( a_1 > 0 \), we can see that for \( u \in U \), \( \lim_{s \to \infty} V_1(s, u) = 0 \) and

\[
\int_0^t P(V_1(s, u)) ds = -\int_{V_1(t, u)}^u \frac{P(z)}{\phi_0(z)} \, dz.
\]

This implies that

\[
\int_{\mathbb{G}} e^{u_1 y_1} \pi(dy) = \exp \left\{ -\int_0^{u_1} \frac{P(z)}{\phi_0(z)} \, dz \right\}.
\]

By differentiating both sides of above at \( \lambda_1 = 0 \) we get \( \Delta_1 = a_1^{-1} [a_2 + \int_{\mathbb{G}} z_1 n(dz)] < \infty \). On the other hand, \( \{Z_t\}_{t \geq 0} \) can be constructed by

\[
Z_t = e^{-b_2 t} Z_0 + \int_0^t e^{-b_2 (t-s)} dL_s^Y, \quad t \geq 0,
\]

where

\[
dL_s^Y = -(b_0 + b_1 Y_s) dt + \sigma \int_0^1 W_0(dt, du) + \sqrt{2\alpha_{21}} \int_0^{Y_s} W_1(dt, du)
\]

\[
+ \sqrt{2\alpha_{22}} \int_0^{Y_s} W_2(dt, du) + \int_{\mathbb{G}} z_2 \tilde{N}(dt, dz) + \int_0^{Y_s-} \int_{\mathbb{G}} z_2 \tilde{M}(dt, du, dz).
\]

We assume that \( Z_0 = y_2, Y_0 = y_1 \). By Cauchy-Schwartz inequality and the result for the first moment of CBI-processes; see, e.g., Li (2020a), pp.33, one can see that

\[
\mathbb{E} \left| \int_0^t e^{b_2 s} dL_s^Y \right| \leq |b_0| \frac{e^{b_2 t} - 1}{b_2} + |b_1| \int_0^t e^{b_2 s} \left( y_1 e^{-a_1 s} + \frac{\gamma}{a_1} (1 - e^{-a_1 s}) \right) \, ds
\]

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\[ + \int_G z^2 m(dz) \int_0^t e^{2b_2 s} \left( y_1 e^{-a_1 s} + \gamma \frac{a_1}{a_1} (1 - e^{-a_1 s}) \right) ds \]\n
\[ + \int_G z^2 n(dz) (e^{b_2 t} - 1) b_2^{-1} \],

where \( \gamma = a_2 + \int_G z_1 n(dz) \). Then

\[ \sup_{t>0} \mathbb{E} \left| \int_0^t e^{-b_2(t-s) \partial} d L_s \right| \leq 2 \left| b_1 \right| y_1 + C_5 + \left[ \frac{2 \int_G z^2 m(dz) y_1 + C_6}{2b_2 - a_1} \right] \]

for some constants \( C_5 \) and \( C_6 \). Further, for any \( k \geq 1 \) and \( t > 0 \),

\[ \mathbb{E}(\mid Z_t(y) \mid \wedge k) \leq e^{-b_2 t} \mid y_2 \mid \wedge k + \sup_{t>0} \mathbb{E} \left| \int_0^t e^{-b_2(t-s) \partial} d L_s \right|, \]

and so

\[ \int_{\mathbb{R}} (\mid y_2 \mid \wedge k) \pi(dy) \leq \int_{\mathbb{R}} (e^{-b_2 t} \mid y_2 \mid \wedge k) \pi(dy) + C_7, \quad t > 0, \quad k \geq 1 \]

for some constant \( C_7 \) since \( \Delta_1 < \infty \). Letting first \( t \to \infty \) and then \( k \to \infty \), it follows from dominated convergence theorem and \( b_2 > 0 \) that

\[ \int_{\mathbb{R}} (\mid y_2 \mid \wedge k) \pi(dy) < \infty. \]

That completes the proof. \( \square \)

### 4 Strong Feller property

In this section, we study the strong Feller property of (1+1)-dimensional affine Markov processes constructed by (2.2)–(2.3). We first formulate the following condition:

**D** There exists a nonnegative measurable function \( \rho_0 \) on \( \mathbb{R} \) such that

\[ n(\mathbb{R}_+ \times dz_2) \geq \sigma_0(dz_2) := \rho_0(z_2) dz_2, \quad \sigma_0(\mathbb{R}) > 0. \]

For a positive integrally function \( g \) defined on \( \mathbb{R} \), for \( k \geq 1 \), let \( \sigma_k := (kg \wedge \rho_0)(z_2) dz_2, \)

\[ \sigma_k(\mathbb{R}) := \int_{\mathbb{R}} (kg \wedge \rho_0)(z_2) dz_2. \]

There exists \( K \geq 1 \) such that for all \( k \geq K \), the measure \( \sigma_k \) satisfies

\[ \limsup_{\rho \to 0} \left[ \sup_{|z| \leq \rho} \left\| \sigma_k - \delta_{\alpha} * \sigma_k \right\|_{\text{var}} \right] < \infty, \quad \int_{\mathbb{R}} |z_2| \sigma_k(dz_2) < \infty. \]

Under Condition **D**, \( \sigma_k(\mathbb{R}) < \infty \) for \( k \geq 1 \). Then for \( k \geq K \) we can define a compound Poisson process as follows:

\[ L^k_t := \int_0^t \int_G z_2 N_k(ds, dz), \]
Lemma 4.1 Suppose Condition (D) holds. For $k \geq K$ let $P^k_t$ be defined by

$$P^k_t f(x) := \mathbb{E} \left[ f(Y_t(x_1), Z_t(x)) \mathbf{1}_{\{x^+_i \leq t\}} \right], \quad f \in B_b(\mathbb{G}), \quad x \in \mathbb{G}.$$ 

Then there exist a constant $C_{k,8} > 0$ and non-negative functions $t \mapsto C_1(t)$ and $t \mapsto C_2(t)$ such that for any given $x, y \in \mathbb{G},$

$$\sup_{\|f\| \leq 1} |P^k_t f(x) - P^k_t f(y)| \leq 2e^{-\sigma_k(\mathbb{R})t} + C_{k,8} \left\{ |x_2 - y_2| + \sqrt{C_1(t)|x_1 - y_1|} + C_2(t)|x_1 - y_1| \right\}.$$ 

Proof. By a modified proof in Proposition 3.2, we have

$$\sup_{\|f\| \leq 1} |P^k_t f(x) - P^k_t f(y)| \leq 2e^{-\sigma_k(\mathbb{R})t} + \frac{\Lambda_k}{\sigma_k(\mathbb{R})} \left\{ |x_2 - y_2| + \sqrt{C_1(t)|x_1 - y_1|} + C_2(t)|x_1 - y_1| \right\},$$

where

$$C_1(t) = \bar{c} \cdot \frac{\exp(2b_2 - a_1)t - 1}{2b_2 - a_1}, \quad C_2(t) = \frac{|b_1| \exp(2b_2 - a_1)t - 1}{b_2 - a_1},$$

$$\bar{c} = 4 \max \left\{ 8(\alpha_{21} \vee \alpha_{22}), \int_{\mathbb{G}} |z_2|^2 m(dz) \right\}, \quad \Lambda_k = \sup_{\rho > 0} \left[ \frac{\sup_{|a| \leq \rho} \|\sigma_k - \delta_a \ast \sigma_k\|_{\text{var}}}{\rho} \right]$$

by conventions $C_1(t) = \bar{c}t$ for $2b_2 = a_1$ and $C_2(t) = |b_1|t$ for $b_2 = a_1$. And we prove the required assertion by setting $C_{k,8} = \frac{\Lambda_k}{\bar{c} \sigma_k(\mathbb{R})}$.

Theorem 4.2 Suppose Conditions (A, D) hold with $\sigma_0(\mathbb{R}) = \infty$. Then the affine Markov process $(X_t)_{t \geq 0}$ satisfies the strong Feller property.
Proof. It follows from (2.4) that for \( x, y \in G \) we have
\[
\sup_{\|f\| \leq 1} |P_t f(x) - P_t f(y)| \leq 2 \bar{v}_t |x_1 - y_1| + \sup_{\|f\| \leq 1} |P_t^k f(x) - P_t^k f(y)|, \quad k \geq K,
\]
which together with Lemma 4.1 imply that
\[
\lim_{y \to x} \sup_{\|f\| \leq 1} |P_t f(x) - P_t f(y)| \leq 2 e^{-\sigma_k(R)t}, \quad k \geq K, \quad t > 0.
\]
Since \( \sigma_k(R) \uparrow \infty \) as \( k \uparrow \infty \), we conclude the strong Feller property of \((X_t)_{t \geq 0}\). \qed

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