Superconducting zero temperature phase transition in two dimensions and in the magnetic field

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We derive the Ginzburg-Landau-Wilson theory for the superconducting phase transition in two dimensions and in the magnetic field. Without disorder the theory describes a fluctuation induced first-order quantum phase transition into the Abrikosov lattice. We propose a phenomenological criterion for determining the transition field and discuss the qualitative effects of disorder. Comparison with recent experiments on MoGe films is discussed.

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I. INTRODUCTION

In the last decade it became well appreciated that the superconducting transition can be tuned not only by changing temperature, but also by varying some other parameter, like the level of disorder, magnetic field, film thickness, or doping of a high-parameter, like the level of disorder, magnetic field, film thickness, or doping of a high-

A particularly interesting example is the superconductor-insulator transition in two dimensions (2D), where the superconducting state at $T = 0$ is destroyed by increasing the level of static disorder in the system. For a while it has been thought that the magnetic field would have a similar (but not exactly the same) disordering effect on the superconducting ground state (2D). Recently, however, the experiments of Mason and Kapitulnik have suggested that by increasing the magnetic field at $T = 0$ the 2D system undergoes a superconductor-metal phase transition, and, moreover, that this transition may actually be discontinuous. The metallic state they found seems to be quite unusual, in that its resistivity is anomalously low, and it exhibits a very large magnetoresistance. Only at a higher magnetic field the expected 2D insulating state is recovered.

Motivated by these intriguing results, in this paper we study theoretically the $T = 0$ superconducting phase transition in 2D and in the perpendicular magnetic field. We begin by demonstrating that, due to the pair-breaking nature of the magnetic field, in 2D (and without disorder) there exists a regular Ginzburg-Landau-Wilson (GLW) theory for the fluctuating superconducting order parameter even at $T = 0$. Long time ago, it was shown by Maki and Tsuzuki that the GLW theory in the magnetic field and in 3D is almost regular at $T = 0$; the quartic term coefficient was singular only weakly as $T$ approaches zero. We show that in 2D this singularity is removed by the absence of the third direction along which the motion would be unaffected by the field. The resulting GLW theory describes the quantum fluctuations of the lowest Landau level (LLL) superconducting order parameter (OP), and bears close resemblance to the finite-$T$ GLW theory for a 3D superconductor. Based on the existing understanding of the finite-$T$ Abrikosov transition in 3D, we then argue that the quantum Abrikosov transition in 2D and without disorder is generically first-order, and propose a phenomenological criterion for determining the first-order superconductor-metal transition field $H_{sm}$. In particular, we show that the true transition field $H_{sm}$ at $T = 0$ is always below the mean-field critical field $H_0$, and that the quantum phase transition is from the superconducting into a zero-temperature equivalent of the vortex liquid phase: a metallic phase, strongly renormalized by the superconducting fluctuations, which we propose may be related to the anomalous metal observed by Mason and Kapitulnik.

In a real experiment, of course, disorder is always present, and may actually be quite strong, so it can be expected to play an important role in determining the nature of the transition. In principle, quenched disorder lifts the degeneracy of the LLL for the order parameter, and, we argue, if strong enough could bring back the continuous phase transition to a glassy superconducting phase. We provide a qualitative discussion of this phase transition at strong disorder within the framework of a self-consistent Hartree approach. The superconducting transition in this approach would correspond to the condensation into the lowest extended Hartree eigenstate, that in principle needs to be computed numerically. At weak disorder, on the other hand, based on the existing renormalization group studies of the thermal Abrikosov transition, we expect that the superconducting transition at $T = 0$ remains discontinuous.

The organization of the paper is as follows. In Sec. II, we derive the effective GLW action appropriate for the superconductor-metal quantum phase transition in 2D in the magnetic field and without disorder. In Sec. III, we show how this effective action leads to a fluctuation induced first-order transition, and calculate the critical field. In Sec. IV, we discuss the qualitative effects of quenched disorder on the phase transition, and in Sec. V, we relate our results to experiment and other theoretical studies and provide a brief summary. Technical details
are relegated to the Appendices.

II. QUANTUM ACTION AT $T = 0$

We begin by considering the $T = 0$ action for a system of 2D fermions in a magnetic field ($\hbar = c = 1$)

\[ S = \sum_\sigma \int d^2 r d\tau \left[ \bar{\psi}_\sigma \partial_\tau \psi_\sigma + \frac{1}{2m}(|\nabla - ieA|)^2 \psi_\sigma |^2 - \mu \bar{\psi}_\sigma \psi_\sigma 
- \frac{V}{2} \bar{\psi}_\sigma \psi_{\sigma^-} \psi_{\sigma^-} \psi_\sigma \right] , \]

where $\psi \equiv \psi(r, \tau)$, $\sigma$, $e$, and $m$ denote the spin, charge, and mass of fermions, $\mu$ is the chemical potential, and $H = \nabla \times A$ is the external magnetic field. The integration over the imaginary time $\tau$ is over the whole real axis ($T = 0$), and we neglect the Zeeman coupling of the magnetic field to spin. We assume the interaction $V > 0$ which corresponds to the $s$-wave attraction. The partition function is the functional integral over Grassman fields $\psi$ with the action $S$ as the Boltzmann weight. Following the standard Hubbard-Stratonovich decoupling procedure, in the Cooper channel, the action can be expanded in powers of the OP, terms in the action can be represented diagrammatically, and the higher Landau levels are not critical, and their effect only the lowest Landau level (LLL) configurations of the OP need to be retained in the effective action, since the OP feels the magnetic field, one may expand it in terms of the Landau levels, which are the solutions of the single-particle Schrödinger equation in the magnetic field for charge $2e$. Near the superconducting transition only the lowest Landau level (LLL) configurations of the OP need to be retained in the effective action, since the higher Landau levels are not critical, and their effect in principle could be only to renormalize the coefficients in the GLW expansion for the LLL modes. Choosing the Landau gauge $A = H(0, x, 0)$ to represent the magnetic field $H = H\hat{z}$ perpendicular to the 2D plane, the OP may then be expanded in terms of the LLL functions

\[ \Delta(r, \tau) \equiv \sum_{k \in \text{LLL}} C_n(\tau) e^{\phi_k n y} \psi_n(x); \quad \bar{\psi}_n(x) = e^{-eH(x - \frac{2\pi}{L})} , \]

where $k = 2\pi/L$ ($L$ is the linear size of the system) and $n$ is an arbitrary integer.

Going to Matsubara frequency space, the quadratic part of the action can be written as $S_2 = \int d\omega \tilde{S}_2(\omega)$, and to the lowest order in small parameter $\omega \tau_H$ we find (see Appendix)

\[ \tilde{S}_2(\omega) = [1 - g \ln(2\Omega\tau_H)] - \sqrt{\pi g \tau_H} |\omega|/|\Delta(\omega)|^2 , \]
Similarly, the quartic part of the action $g_N/2$ is the dimensionless coupling ($N = m/2\pi$ is the constant density in 2D), $\Delta_\omega(r)$ is the Fourier transform of $\Delta(r, \tau)$, and $\Omega$ is the usual ultraviolet cutoff. Also

$$\langle |\Delta_\omega(r)|^2 \rangle = \int dr |\Delta_\omega(r)|^2 = \frac{2\pi}{k} \sum_n |C_n(\omega)|^2 \sqrt{\frac{\pi}{2eH}}.$$  \hspace{1cm} (7)

Similarly, the quartic part of the action $S_4 = \int d\omega d\omega_1 d\omega_2 S_4(\omega_1, \omega_2)$, where at zero frequencies \[ \text{(Appendix B),} \]

$$\bar{S}_4(\omega_1 = \omega_2 = 0) = a g^2 \frac{2\pi}{N} \langle |\Delta_0(r)|^4 \rangle$$

with the constant $a = 4\pi [\ln(1 + \sqrt{2})]$. The average

$$\langle |\Delta_0(r)|^4 \rangle = \int dr |\Delta_0 = 0(r)|^4 = \frac{2\pi}{k} \sum_{n, m, p, q} \delta(n - m + p - q)C_n(0)C_m(0)C_p(0)C_q(0) \times \sqrt{\frac{\pi}{4eH}} e^{-\frac{\pi^2}{4eH}|(n^2 + m^2 + p^2 + q^2) - (n + m + p + q)^2/4|}.$$ \hspace{1cm} (9)

Note that the coefficient before $\langle |\Delta_0(r)|^4 \rangle$ is finite in a finite magnetic field. At zero field it diverges, as well-known. In 3D the same coefficient is $\alpha - \ln(T/H)$, and logarithmically divergent as $T \to 0$. This is a consequence of the fact that the motion in the $z$ direction remains unaffected by the magnetic field. In 2D, however, the singular behavior of the Landau expansion around the normal state $\Delta = 0$ for the $T = 0$ action is completely cured.

The effective GLW action for the superconductormetal phase transition in 2D at $T = 0$ can thus be written as

$$S_{GLW}[\Phi] = \int d\tau d\Phi \left[ \Phi^\dagger \partial_\tau \Phi + \alpha |\Phi|^2 + \beta |\Phi|^4 \right]$$ \hspace{1cm} (10)

where $\Phi \equiv \Phi(r, \tau)$, and the partition function is $Z = \int D[\Phi^\dagger, \Phi] \exp(-S_{GLW}[\Phi])$. In principle the action contains higher powers of the field and the higher order time derivatives, but we make the usual approximation in retaining only the most relevant terms. The operator $\partial_\tau$ corresponds to $\omega$ in Matsubara space and we have rescaled the field $\Phi = (\sqrt{\pi g_{LH}})^{1/2} \Delta$ to bring the coefficient of the $|\omega|^2$ term in (10) to unity. Thus $\alpha = \frac{1}{1 + \ln(2\Omega H)} \ln(\sqrt{\pi \tau H})$ and $\beta = \frac{\alpha}{\pi N}$.

**III. FLUCTUATION INDUCED FIRST-ORDER TRANSITION**

When the fluctuations of the OP are ignored (the mean-field approximation), the system described by the action (10) at $T = 0$ undergoes a continuous phase transition from normal to the Abrikosov superconducting phase as the magnetic field is decreased, at the point where $\alpha = 0$. The mean-field critical field is thus $H_0 = 2\Omega^2 |(1/\sqrt{\pi g_{LH}})| e^{-2g}$, and exponentially small for a weak coupling $g$. One can easily verify that this satisfies the standard relation $H_0 \xi^2 \approx 1$ where $\xi$ is the superconducting coherence length. This mean-field critical field is simply the end ($T = 0$) point of the standard $H_c2(T)$ Abrikosov line in the $H$-$T$ phase diagram of a 2D type-II superconductor. However, today it is well established both theoretically \[ \text{(11),} \] and experimentally \[ \text{(12),} \] that at finite $T$ Abrikosov transition is one of those rare cases where fluctuations even *qualitatively* alter the nature of the transition, turning it into a first-order in the clean case. This may be understood as a consequence of the macroscopic degeneracy of the LLL manifold, which is a unique feature of the Abrikosov transition, and which greatly enhances the effect of fluctuations. Any OP configuration within the LLL can be parametrized in terms of its zeroes (vortices), and the Abrikosov transition is equivalent to a freezing transition of a complicated classical many body system, which is then typically first order \[ \text{(13).} \]

Our quantum action at $T = 0$ is almost identical to the finite-$T$ GLW theory for a 3D superconductor in the magnetic field (except for a linear instead of a quadratic derivative with respect to the third coordinate), and one may expect that, in this case, quantum fluctuations of the OP may also turn the $T = 0$ transition into first order. To approximately obtain the critical magnetic field $H_{sm}$ of the first-order transition, it is convenient to work with dimensionless quantities and first rescale the fields, length, and time as $\left(4\pi^2 \tau H^2 \right)^{1/2} \delta \Phi \to \Phi$, $r/(\sqrt{\pi \ell}) \to r$, and $\tau/\tau_H \to \tau$. The action in (10) then becomes

$$S_{GLW}[\Phi] = \int d\tau d\Phi \left[ g_{\tau,\alpha} \partial_\tau |\Phi|^2 + g_{\alpha} |\Phi|^4 + \frac{1}{4} |\Phi|^4 \right],$$ \hspace{1cm} (11)

where the two dimensionless couplings are $\alpha = (1/\tau_H, \alpha)(\pi^2 \tau H \beta^{1/2})$. First, consider only the OP configurations that have no $\tau$ dependence ($\omega = 0$ modes). For those the first term of (11) vanishes, and the thermodynamics of the system would depend exclusively on the single dimensionless coupling $g_{\alpha}$. The theory would then look zero-dimensional, however, this is deceiving: the macroscopic degeneracy of the LLL still remains. In fact, without the first term the partition function with the action (11) would be identical to the finite-$T$ partition function that describes the vortex-liquid-to-solid transition, and which is known to have a weak first-order transition at $g_{\alpha} = g_{LH} \approx -6.7$. This has been established in detailed Monte Carlo simulations \[ \text{(14),} \] as well as in the density-functional theory \[ \text{(15).} \] To find the transition field with the full $\tau$-dependence of the OP included we assume that the transition is driven by the same mechanism of growing positional correlations between vortices, the only difference now being that the vortices are “straight” in $\tau$-direction over an imaginary time scale $\tau$, instead of $\tau_H$. 


We identify this time scale with the correlation “length” in \( \tau \) direction. This ansatz works remarkably well in describing the finite-\( T \) first-order transition line in clean YBCO \([\text{19}]\) and on that basis we expect it to be a good approximation here as well. The transition field is then determined by the condition

\[
g_M = g_\alpha \tilde{\tau}^{\frac{1}{2}}, \tag{12}
\]

for \( \tilde{\tau} > 1 \) in units of \( \tau_H \). To approximately determine \( \tilde{\tau} \) we will use the self-consistent Hartree approximation \([\text{19}]\) to the action in \([\text{11}]\). The correlation time is then defined as

\[
\tilde{\tau} = \frac{g_\alpha}{g_\alpha + \langle |\Phi(r)|^2 \rangle / 4}, \tag{13}
\]

where the quantum mechanical average appearing in \([\text{14}]\) is determined self-consistently as

\[
\langle |\Phi(r)|^2 \rangle = \frac{1}{2\pi} \int g_\alpha |\omega| + g_\alpha + \langle |\Phi(r)|^2 \rangle / 4. \tag{14}
\]

Solving Eqs. \([\text{12}]-[\text{14}]\), one obtains the equation for the first-order transition field

\[
H_{\text{sm}} = \frac{2\Omega^2}{eV_0} e^{-2/g_{\text{eff}}}, \tag{15}
\]

where the effective coupling satisfies

\[
g_{\text{eff}}^{-1} = g^{-1} + \frac{4a|g_M| \sqrt{\ln \Omega_1} \sqrt{2\tilde{\tau} H_{\text{sm}}}}{\pi k_F}, \tag{16}
\]

in the \( \Omega_1 \to \infty \) limit. The parameter \( \Omega_1 \) is a dimensionless upper cutoff we introduced to regularize the integral in \([\text{14}]\). It may be chosen to yield the correct answer when the next order in “frequency” (\( i.e. \), \( \omega^2 \)) term is kept in the effective action, for example. Since \( \Omega_1 \) appears only under a log the result is little sensitive to its precise value. More importantly, it is evident that \( g_{\text{eff}} \) is smaller than \( g \) which in turn implies that the true transition field \( H_{\text{sm}} \) is below the mean-field \( H_0 \), which would correspond to the \( g_M = 0 \) case.

In Fig. \([\text{2}]\) we present the \( H-g \) phase diagram for the 2D system in the magnetic field and at \( T = 0 \). The superconductor-metal phase transition induced by the order parameter fluctuations is first-order. The mean-field result represented by the dashed line is included for comparison. Her we have assumed \( \Omega = \epsilon F, \Omega_1 = 10^4 \), and \( g_M = -7.0 \). The mean-field and the true transition field are indistinguishably close at very small coupling, but they start to differ significantly at larger couplings. In principle, one may also expect a somewhat renormalized value of \( g_M \) from the static value (\( \approx -7 \)), but the difference does not qualitatively alter our results.

![FIG. 2. \( H-g \) phase diagram of superconductor-to-metal QPT in 2D in a magnetic field. The dashed line corresponds to the to the mean-field second-order transition. The solid line is determined by solving Eqs. \([\text{15}]\) and \([\text{16}]\) and corresponds to the first-order transition induced by the quantum fluctuations of the superconducting order parameter.]

**IV. EFFECTS OF DISORDER**

So far we have completely neglected the effects of static disorder on the superconducting transition. Since the inclusion of disorder complicates the problem significantly, we will discuss its effects only qualitatively. First, it should be possible to model disorder by including a random potential term \( V(r) |\Phi(r)|^2 \) into the OP effective action in \([\text{11}]\). That such a term indeed arises in the OP theory was demonstrated by one of us \([\text{9}]\) in a related problem of disordered \( d \)-wave superconductor. For simplicity one may assume that the random potential is uncorrelated in space. With this term included one faces an interacting theory with quenched disorder (and in the LLL) which is notoriously difficult to analyze. So in what follows we assume that disorder is strong (compared to the quartic term), so that the self-consistent Hartree treatment of the interaction may be a reasonable starting point \([\text{21}]\). In this spirit we replace the quartic term in \([\text{11}]\) as

\[
\Phi^4 \to \langle \Phi^2 \rangle \Phi^2, \tag{17}
\]

where the average is self-consistently determined as

\[
\langle |\Phi(r)|^2 \rangle = \frac{1}{2\pi} \sum_n \int d\omega \frac{|\phi_n(r)|^2}{g_r |\omega| + \epsilon_n}, \tag{18}
\]

where \( \phi_n \) are the LLL eigenfunctions, and \( \epsilon_n \) eigenvalues of the random potential, Hartree ”screened” by the interactions

\[
\hat{V}(r) = g_\alpha + V(r) + \frac{1}{4} |\Phi(r)|^2. \tag{19}
\]

The superconducting phase transition in this approach would correspond to the vanishing of the lowest eigen-
value $\varepsilon_n$ and condensation into the corresponding extended random eigenstate \[2\]. This transition is expected to be continuous, but to verify this scenario one needs to implement the self-consistent procedure numerically, which we leave for future work. The reader should note however, that the main effect of disorder is to lift the degeneracy of the LLL manifold, and thus restore the possibility of a continuous transition.

In case of weak disorder the above Hartree approximation becomes inadequate and we may only speculate what happens with the transition. Our prejudice, which we discuss more shortly, is that the transition remains first order for weak disorder, and turns continuous only at some critical disorder strength.

V. SUMMARY AND DISCUSSION

Our conclusion of a first order transition in the clean case is in agreement with the renormalization group studies of Brézin, Nelson, and Thiaville \[22\] and more recently of Moore and Newman \[22\]. These authors have shown that due to macroscopic degeneracy of the LLL the renormalization group flow in dimensions less than six is always unstable, which is usually interpreted as a sign of a first-order transition. Although they studied a thermal, and not quantum, Abrikosov transition, their methods and conclusions can be readily translated to our case. Furthermore, our conclusion is also in agreement with the large-$N$ treatment of the Abrikosov transition \[23,24\] at finite temperature. As for the disorder case, Moore and Newman \[22\] also demonstrated that with weak disorder the renormalization group flow remains unstable, which would indicate that the transition is still discontinuous. The case of strong disorder is outside the domain of validity of their perturbative approach, and we feel that the self-consistent Hartree treatment of the interaction term we discussed is more appropriate.

Our theory offers a natural explanation for the hysteretic behavior observed at the $T = 0$ superconductor-metal transition by Mason and Kapitulnik. We argue that it is a consequence of the fluctuation induced first-order transition. Also, that the non-superconducting state appears metallic, and not insulating, may be related to the fact that a weak magnetic field cuts off the weak-localization effects, and thus relegates the localization effects in 2D to lower temperatures \[25\]. The observed anomalously small resistivity and the large magnetoresistance of the Mason and Kapitulnik metal could be related to the strong superconducting fluctuations near the critical field. Additional support for this picture comes from the experimental observation that the resistivity away from the critical field $H_{sm}$ vanishes as $R \sim (H - H_0)$, where $H_0 > H_{sm}$. This power-law follows naturally from our theory as follows. Away from the critical field (outside the critical region dominated by the OP fluctuations) the transition appears continuous, and the resistivity should vanish according to the simple (Aslamazov-Larkin like) scaling \[25\]

$$R \sim \xi^{d-2},$$

where $\xi$ is the superconducting correlation length. The GLW action (10) for the OP at the mean-field level implies $d = 0$ (since the OP is confined to the LLL which completely eliminates the gradient terms in 2D). At the mean-field level $\xi \sim 1/(H - H_0)^{1/2}$, so the power counting implies

$$R \sim (H - H_0),$$

in the crossover region, precisely as seen in the experiment \[6\]. This, of course, ceases to be valid upon entering the critical region, in which, as we argued, fluctuations eventually drive the transition first order at a lower critical field $H_{sm}$.

Finally, we note that the GLW theory in Eq. (10) has the frequency dependence characteristic of a dissipative system. This is analogous to what was found in the theory of disordered $d$-wave superconducting phase transition at $T = 0$ \[3\], and is related to the pair-breaking nature of the magnetic field. This may provide the theoretical basis for the Mason and Kapitulnik interpretation of the superconductor-metal transition \[6\].

To summarize, we have studied the $T = 0$ superconducting phase transition for a 2D system in a magnetic field. We derived an effective Ginzburg-Landau-Wilson action for the fluctuating superconducting order parameter which enables one to investigate the effects of quantum fluctuations and the order of the phase transition. It is argued that without, or with weak disorder, the quantum superconductor-metal phase transition is of first order. For the case of strong disorder we expect that the possibility of a continuous phase transition is restored.

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APPENDIX A: DERIVATION OF $S_2$

In this Appendix we give the explicit derivation of quadratic $S_2$ in Eq. (8). If we write $S_2 = \int d\omega \overline{S_2(\omega)}$, based on the diagram in Fig. 1, one has at $T = 0$ (apart from the constant term)

$$\overline{S_2(\omega)} = -\frac{V}{2} \times$$

$$\int \frac{d\nu}{2\pi} dr_1 dr_2 G_{\nu + \omega}(r_1, r_2) G_{-\nu}(r_1, r_2) \Delta^\dagger(\nu)(r_1) \Delta(\omega)(r_2),$$

(A1)
where $\Delta_\omega$ is given by (1) with $C_n(\tau) \rightarrow C_n(\omega)$ in terms of LLL. In the semi-classical approximation, the 2D single-particle Green’s function $G$ is given by (2) in which

$$G^0_{\nu}(\mathbf{r}_1, \mathbf{r}_2) = \int \frac{d\nu}{2\pi} e^{i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} \frac{1}{i\nu - \epsilon_p}$$

(A2)

where $\epsilon_p = \mathbf{p}^2/2m - \epsilon_F$ and the phase $\phi(\mathbf{r}_1, \mathbf{r}_2) = eH(x_1 + x_2)(y_1 - y_2)/2$. We have assumed that the field $\mathbf{A} = H(0, x, 0)$, which leads to $\mathbf{H} = H \hat{z}$. After substitution of the above into (A1) and carrying out the integrations over $y_1$ and $y_2$, one obtains

$$S_2(\omega) = -\frac{V}{2} \sum_{n,m} (2\pi)^2 \int \frac{d\nu}{2\pi} dx_1 dx_2 \frac{d\mathbf{p}}{(2\pi)^2} \frac{dt}{(2\pi)^2} C_m^*(\omega) \times$$

$$C_m(\omega) \delta(ty + eH(x_1 + x_2) - km ) \delta(kn - km) \times$$

$$e^{itx_1(x_2 - x_3)} (i[\nu + \epsilon_F - \epsilon_p][\nu - (\epsilon_F - \nu) \cdot \mathbf{t}]) \psi_n(x_1) \psi_m(x_2).$$

(A3)

Here we have set $\mathbf{t} \equiv \mathbf{q} + \mathbf{p}$ and $\epsilon_q \approx \epsilon_p - \nu \cdot \mathbf{v}_F$ with $\mathbf{v}_F$ the Fermi velocity.

Using $\int \frac{d\nu}{2\pi} = N \int d\nu \frac{d\nu}{2\pi}$ (with $N = m/2\pi$ as the 2D density of states) and the integral ($\Omega$ is the usual ultraviolet cutoff)

$$\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{\Omega}^{\infty} \frac{d\nu}{2\pi} \frac{1}{i[\nu + \epsilon_F - \epsilon_p][\nu - (\epsilon_F - \nu) \cdot \mathbf{t}]}$$

$$= \ln \left[ \frac{4\pi^2}{(\epsilon_F - \mathbf{t} \cdot \mathbf{t})^2 + \omega^2} \right],$$

(A4)

one obtains after some calculation

$$\tilde{S}_2(\omega) = K_2(\omega) |\Delta_\omega(\mathbf{r})|^2,$$

(A5)

where $|\Delta_\omega(\mathbf{r})|^2$ is defined in (2) and

$$K_2(\omega) = -\frac{g}{4\pi eH} \int d\mathbf{t} e^{-\frac{\mathbf{t} \cdot \mathbf{t}}{2\omega^2}} \ln \left[ \frac{4\pi^2}{(\epsilon_F - \mathbf{t} \cdot \mathbf{t})^2 + \omega^2} \right].$$

(A6)

Here $g = N\nu/2$. Expanding $K_2(\omega) = K_2(0) + A|\omega| + B\omega^2 + ...$, we found the zero-frequency term

$$K_2(0) = -g \ln(2\Omega \tau_H)$$

(A7)

with $\tau_H = \ell/\nu_F$ and $\ell = 1/\sqrt{2eH}$, and the first-order coefficient

$$A = \frac{\partial K_2(\omega)}{\partial \omega} \bigg|_{\omega \rightarrow 0} = \sqrt{\pi} g \tau_H.$$  

(A8)

The combination of (A7) and (A8) gives the result in (3).

**APPENDIX B: DERIVATION OF $S_4$**

The quartic action $S_4$ is expressed diagrammatically in Fig. 1. The similar diagram has been calculated by Maki and Tsuzuki [9] for 3D and finite-$T$. Here we need $S_4$ in 2D and at $T = 0$. One can write the quartic term as $S_4 = \int d\omega d\omega d\omega \tilde{S}_4(\omega_1, \omega_1, \omega_2, \omega_2)$, where at zero frequencies $(\omega_1 = \omega_1 = \omega_2 = 0)$

$$\tilde{S}_4(0) = \frac{V^2}{4} \int \frac{d\nu}{2\pi} d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4 \times$$

$$G_{\nu}(\mathbf{r}_3, \mathbf{r}_4)G_{-\nu}(\mathbf{r}_1, \mathbf{r}_2) \Delta_{\nu}(\mathbf{r}_1) \Delta_{0}(\mathbf{r}_2) \Delta_{0}(\mathbf{r}_3) \Delta_{0}(\mathbf{r}_4).$$

(B1)

In analogy to Appendix A, after a lengthy calculation we find

$$\tilde{S}_4(0) = \frac{1}{4} gV\nu_F^{-2} \frac{2\pi}{k} \sum_{n,m,p,q} C_n^* C_m^* C_q R(n, m, p, q),$$

(B2)

where for brevity, $C_n \equiv C_n(0)$, and

$$R(n, m, p, q) = \delta(n - m + p - q) \int_0^{2\pi} \frac{d\theta}{2\pi} \left[ \frac{\Theta(1, 3; 2, 4)}{\tauH e^{\eta}} \right] \times$$

$$\int dx_1 dx_2 dx_3 dx_4 |x_1 + x_3 - x_2 - x_4| e^{-i\tan \theta eH \tauH} \psi_n(x_1) \psi_m(x_2) \psi_q(x_3) \psi_q(x_4).$$

(B3)

Here $\theta$ is the azimuthal angle in 2D,

$$\Theta(1, 3; 2, 4) = 1, \quad \text{if } x_1, x_3 > x_2, x_4 \text{ or } x_1, x_3 < x_2, x_4$$

0, otherwise,

(B4)

and

$$T = -(x_1 - \frac{kn}{2eH})^2 + (x_2 - \frac{km}{2eH})^2 - (x_3 - \frac{kp}{2eH})^2$$

$$+(x_4 - \frac{kq}{2eH})^2 + \left( \frac{k}{2eH} \right)^2 (n^2 - m^2 + p^2 - q^2).$$

(B5)

One can further solve

$$\tilde{S}_4(0) = \frac{1}{4} gV\nu_F^{-2} \frac{2\pi}{k} \sqrt{\frac{\pi}{4eH}} \sum R(n, m, p, q) \times$$

$$e^{-Z} C_n^* C_m^* C_q f,$$

(B6)

where

$$Z = \frac{k^2}{4eH} \left[ (n^2 + m^2 + p^2 + q^2) - \frac{1}{4} (n^2 + m^2 + p + q) \right]^2$$

$$= \frac{k^2}{16eH} \left[ (n^2 + m^2 + p^2 + q^2) + (n - m)^2 + (m - p)^2 + (p - q)^2 \right]$$

(B7)

and

$$f = \int_0^{2\pi} \frac{d\theta}{2\pi} \left[ \frac{1}{\cos^2 \theta} \int_0^{\infty} dt \int_0^{\infty} du du e^{-\eta t^2} \right] \cos(\sqrt{\tauH} \Sigma).$$

(B8)

Here $(\alpha = \tan \theta)$

$$\Lambda = \frac{1}{2} (1 - \alpha^2)(1 - u^2) + \frac{1}{2} (1 + \alpha^2) + \frac{1}{4} (1 + \alpha^2)^2.$$
and
\[ \Sigma = \frac{k}{2 \sqrt{eH}} \times \]
\[ (1 - i \alpha)(1 - u)(n - p) + (1 + i \alpha)u(m - q). \]  
(B10)

As pointed out by Maki and Tsuzuki, due to the factor of \( e^{-Z} \) in (B6), there is a dominance of \( n = m = p = q \) when \( k^2 / 4eH \gg 1 \) (i.e., in the weak-field limit). In this limit \( \Sigma \approx 0 \), and thus drops out of \( f \) in (B8). As a consequence, \( f \) becomes independent of \( n, m, p, q \) and Eq. (B6) is simplified to
\[ \bar{S}_4(0) = \frac{1}{N V_F^2} f (\langle |\Delta_0(r)|^4 \rangle), \]  
(B11)

where \( \langle |\Delta_0(r)|^4 \rangle \) is given in (B4). To the leading order in field, we found
\[ f = f(H) = \frac{4 \pi}{eH} \ln(1 + \sqrt{2})^2. \]  
(B12)

It is interesting to note that for 3D and finite-\( T \), the same term will be \( f \propto -\ln(T/H) \), and divergent at \( T \to 0 \).

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