ON THE BLAZHKO EFFECT IN RR LYRAE STARS
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ABSTRACT

The Blazhko effect is a long-term, generally irregular modulation of the light curves that occurs in a sizeable number of RR Lyrae stars. The physical origin of the effect has been a puzzle ever since its discovery over a hundred years ago. We build here upon the recent observational and theoretical work of Szabó et al. on RRab stars who found with hydrodynamical simulations that the fundamental pulsation mode can get destabilized by a 9 : 2 resonant interaction with the 9th overtone. Alternating pulsation cycles arise, although these remain periodic, i.e., not modulated as in the observations. Here we use the amplitude equation formalism to study this nonlinear, resonant interaction between the two modes. We show that not only does the fundamental pulsation mode break up into a period-two cycle through the nonlinear, resonant interaction with the overtone, but that the amplitudes are modulated, and that in a broad range of parameters the modulations are irregular as in the observations. This irregular behavior is in fact chaotic and arises from a strange attractor in the dynamics.

Key words: instabilities – stars: oscillations – stars: variables: RR Lyrae

Online-only material: color figure

1. INTRODUCTION

A large subclass of RR Lyrae stars undergo light curve modulations, typically on the timescale of some 60 periods, although the range extends from some tens to some hundreds of periods. The fundamental pulsation cycle itself lasts ~0.5 days. The effect was discovered by Blazhko (1907) over a hundred years ago, and a number of explanations have been proposed, such as closely spaced pulsation modes, a modal 1 : 2 resonance, an oblique rotator model, a nonradial modal interaction, and convective cycles (Stothers 2010; Molnár & Kolláth 2010). However, none of these mechanisms is without fault.

The giant step toward the explanation of the Blazhko effect has come from the unprecedentedly precise and continuous Kepler space telescope observations and their analysis (Szabó et al. 2010). Because we are going to build on this work, we first present a summary of these findings. Fourteen RR Lyr stars in their sample undergo Blazhko modulations. Unexpectedly, three of these stars also display period doublings, i.e., the shapes of the light curves show cycle-to-cycle alternations. The depths of these alternations change during the Blazhko cycle. Another recent observational finding is that the Blazhko cycle does not repeat regularly; see, e.g., Chadid et al. (2010), Kolenberg et al. (2011), and Sódor et al. (2011). This behavior poses another important constraint for the physical explanation of the Blazhko effect, as do the observed variations of the mean physical parameters of the stars during the Blazhko cycle (Jurcsik et al. 2009a, 2009b).

On the theoretical side, Kolláth et al. (2011) performed a systematic numerical hydrodynamical modeling survey of RR Lyr models. They found that over a relatively broad region of astrophysical model parameters the fundamental pulsation is unstable and develops into a pulsation with alternating cycles. Guided by the earlier work of Moskalik & Buchler (1990) that had shown that half-integer resonances can cause a bifurcation to alternating cycles, Kolláth et al. (2011) did some sleuthing. By computing the Floquet stability coefficients (Hartman 1992; Buchler et al. 1991) of the fundamental pulsation and searching for resonances of the fundamental mode with successive overtones, they showed that it is the 9th overtone that is in a 9 : 2 resonance and destabilizes the fundamental pulsation cycle. (It is of course a coincidence that it should be the 9th overtone that is in a 9 : 2 resonance.) The 9th overtone in these RR Lyr models turns out to be egregious in that it is a surface mode (dubbed “strange mode”) when it was first encountered by Buchler et al. (1997 and Buchler & Kolláth 2001). The 9 : 2 resonance appears in a relatively narrow, winding band in a log $L$–log $T$ diagram (Kolláth et al. 2011).

The numerical hydrodynamical simulations of Kolláth et al. (2011) were able to produce alternating cycles, and the ancillary analyses established that the 9 : 2 resonance is the cause of the symmetry breaking bifurcation. However, these hydrodynamical simulations were unable to produce either regular or irregular Blazhko-like modulations.

When a surface mode plays a dynamical role, the numerical hydrodynamical simulations become particularly sensitive to the mixing length parameters, to the zoning in the outer regions as well as to the surface boundary condition. It is therefore extremely difficult to make completely trustworthy and robust simulations. In fact, it is an open question whether hydrodynamics with a time-dependent mixing length treatment of convection is able to produce Blazhko modulations. For these reasons it is of importance to resort to an alternative, complementary approach as we do in this paper.

2. MODELING

In the 1980s, several groups developed the amplitude equation formalism in an effort to understand the systematics of stellar pulsations throughout the H–R diagram (for a review, see, e.g., Buchler 1993). The validity of this powerful formalism rests on the fact that for mildly nonadiabatic stars, such as the RR Lyrae and the classical Cepheids, one can decouple the “fast” pulsation and obtain equations for the slow secular behavior of the amplitudes and phases, akin to the Poincaré–Lindstedt approach in celestial mechanics.

The amplitude equation formalism is complementary to numerical hydrodynamical simulations. It allows one to understand the mathematical structure of the dynamics of the stellar models. As such it is therefore ideally suited for making
surveys of the possible pulsational behavior of a given type of star. However, when light curves or radial velocity curves for specific models are desired, one needs to resort to numerical hydrodynamics.

The form of these amplitude equations is generic and depends only on the resonances that exist between the excited linear modes of pulsation (e.g., Buchler 1993; Spiegel 1985). Amplitude equations have a solid mathematical foundation and are also known as “normal forms” in the mathematical literature (Guckenheimer & Holmes 1983).

We note in passing that the first success of the amplitude equation formalism in stellar pulsation theory was to prove the conjecture of Simon & Schmidt (1976) that the Hertzsprung equation formalism in stellar pulsation theory was to prove the fundamental instability strip. Amplitude equations have a solid mathematical foundation and are also known as “normal forms” in the mathematical literature (Guckenheimer & Holmes 1983).

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Mode pulsations (RRd) that we addressed in Kollát et al. (2002) are also known as “normal forms” in the mathematical literature (Guckenheimer & Holmes 1983). We also note that Szabó et al. (2004) were able to map out efficiently the pulsational behavior of RR Lyr stars throughout the whole instability strip by using a judicious combination of numerical hydrodynamical simulations and amplitude equations.

In this paper, we ignore the first overtone (RRc) and double mode pulsations (RRd) that we addressed in Kollát et al. (2002) and Szabó et al. (2004). We therefore do not need to include an amplitude equation corresponding to the first overtone in what follows. Hereafter, we label the fundamental mode with “a” and the 9th overtone with “b”.

The amplitude equations that are appropriate for the modal interaction between mode a that is in a 9 : 2 resonance with mode b (9\(\omega_a \sim 2\omega_b\)) have already been given in Moskalik & Buchler (1990):

\[
\begin{align*}
\frac{da}{dt} &= (i\omega_a + \kappa_a - Q_a |a|^2 - T_a |b|^2)a + c_a a^* b^2 \\
\frac{db}{dt} &= (i\omega_b + \kappa_b - Q_b |b|^2 - T_b |a|^2)b + c_b b^* a^2.
\end{align*}
\] (1)

By introducing real amplitudes A and B and phases \(\phi_a\) and \(\phi_b\), and by defining the relative phase

\[\Gamma = 2\phi_b - 9\phi_a,\]

we can cast Equations (1) into a set of three real equations

\[
\begin{align*}
\frac{dA}{dt} &= (\kappa_a - Q_a A^2 - T_a B^2)A + C_a A^2 B^2 \cos(\Gamma + \delta_a) \\
\frac{dB}{dt} &= (\kappa_b - Q_b B^2 - T_b A^2)B + C_b B^2 A^2 \cos(\Gamma - \delta_b) \\
\frac{d\Gamma}{dt} &= 2\Delta - 9C_a A^2 B^2 \sin(\Gamma + \delta_a) - 2C_b B^2 A^2 \sin(\Gamma - \delta_b).
\end{align*}
\] (2)

Here, \(\kappa_a\) and \(\kappa_b\) denote the growth rates of the two modes a and b. The quadratic terms \(Q_a\), \(Q_b\) and \(T_a\) and \(T_b\) appear in the amplitude equations even in the absence of any resonances. Their imaginary parts do not have a significant effect and for convenience we have assumed that they are all real. The quantities \(C_a\) and \(C_b\), and their phases \(\delta_a\) and \(\delta_b\), describe the resonant coupling, and

\[\Delta = \omega_b - \frac{9}{2} \omega_a\]

is the off-resonance parameter.

Two types of solutions exist depending on the parameters of the amplitude equations:

1. The fixed points \((dA/dt = 0, dB/dt = 0, d\Gamma/dt = 0)\).

Equations (2) have two possible fixed points:

(a) \(A = 0\) and \(B = 0\). This fixed point corresponds to the trivial, nonpulsating static model. It is unstable when mode a is self-excited, \(\kappa_a > 0\), i.e., inside the fundamental instability strip.

(b) A single-mode fixed point with \(A = A_0 = \sqrt{\kappa_a/Q_a}\) and \(B_0 = 0\). This fixed point, when stable, corresponds to steady periodic pulsations in the fundamental mode with constant amplitude. In some regime of parameters this solution is unstable to an excitation of mode b, in which case we have

(c) A two-mode fixed point with \(A = A_0\) and \(B = B_0\) (and \(\Gamma = \Gamma_0\)). Note that because of the 9 : 2 resonance condition, it takes two fundamental periods for the oscillation to repeat. This is the fixed point that Kollát et al. (2011) found in their numerical hydrodynamical simulations and that gives rise to the period alternations.

Generally, Equations (2) are complicated enough so that the values of \(A_0\) and \(B_0\) must be obtained numerically.

In some parameter regime this fixed point can also be unstable, as we describe now.

2. The oscillatory solutions for the amplitudes.

These solutions come in two types:

(a) Purely periodic oscillations in the amplitudes and phases, corresponding to periodic amplitude and phase modulations of the stellar pulsations.

(b) Irregular oscillations (irregular amplitude and phase modulations). These oscillations arise because of the existence of a strange attractor (chaos). It is these solutions that correspond to irregular Blazhko modulations.

Depending on the values of the parameters in the amplitude equations, all these solutions show up in the numerical integration of the amplitude equations.

The amplitude equations have too many parameters to make an exhaustive study of their solutions possible. However, we can impose some constraints. By placing ourselves inside the fundamental mode instability strip, we can assume that the growth rate \(\kappa_a > 0\). The quadratic terms in Equations (2), viz., \(Q_a\), \(Q_b\) and \(T_a\), and \(T_b\), are generally positive. We can scale the time by setting \(\kappa_a = 1\) and scale the amplitudes A and B by respectively setting \(Q_a = 1\) and \(Q_b = 1\).

In the following, we show an example of an irregular amplitude modulation. The values of the remaining parameters are chosen to be \(\kappa_b = -0.1, T_a = 10.0, T_b = 1.0, C_a = 0.75, C_b = 11.0, \delta_a = 0.5, \delta_b = 1.2,\) and \(\Delta = -2\).

The single-mode fixed point is located at \(A_0 = 1\) (by construction). The Floquet stability coefficients corresponding to an excitation in mode b (Moskalik & Buchler 1990),

\[F_b = \kappa_b - T_b A_0^2 \pm \sqrt{C_b A_0^{18} - \Delta^2},\]

have the values \(-12.034\) and 9.8343, and the single mode is therefore unstable to an excitation of mode b.

The two-mode fixed point is found to be located at \(A_1 = 0.7979, B_1 = 0.1918, \Gamma_1 = 0.1955,\) and its stability eigenvalues are calculated to be \(\sigma_1 = -3.3135\) and \(\sigma_2 = 0.2057 \pm i 2.7913\). This fixed point is therefore unstable as well, and an oscillation arises in the amplitudes.

The temporal variation of the amplitudes \(A(t)\) and \(B(t)\) is displayed in Figure 1, with a blowup in the right panel.
One notes that, in this example, the modulations in the amplitudes are irregular, akin to the observed Blazhko cycles. In RR Lyr stars, the relative growth rates $\kappa_0 \times P_0$ are typically of the order of a percent. The amplitude modulations occur on a timescale that is set by $\kappa_a$ that is therefore much longer than the fundamental pulsation period. In our example in which we have scaled $\kappa$ to be unity, the scaled “Blazhko period” comes out to be $\sim 2.5$ as the figure shows, corresponding thus to $\sim 250$ periods, in the ball park of the observed Blazhko cycle periods. We note, however, that this period can easily be shortened or lengthened with a fine-tuning of the parameters in the amplitude equations and one can get the whole range of Blazhko “periods.”

Admittedly, we have chosen the parameters in our amplitude equations in a somewhat ad hoc way, but we find that the range of parameters over which irregular modulations occur is quite broad.

3. NATURE OF THE IRREGULAR MODULATIONS

The reader may wonder how such irregular modulations arise. Because of the spiral-node nature of the fixed point, the trajectories in the complex amplitude $(a, b)$ space spiral away from the fixed point in the plane of the spiral roots ($\sigma_{\pm}$), get bent around in the nonlinear regime to be attracted back to the fixed point along the direction of the real, stable root ($\sigma_0$). Under broad parameter conditions this results in irregular oscillations. This behavior is displayed in a $B(t)-A(t)$ phase plot in Figure 2 for our example.

At first sight one might think that the dynamics of this strange attractor is of a stretch-and-fold type, similar to the well-known Rössler band. As a further test, we have constructed a first return map with the maxima of amplitude $A$, i.e., a plot of the successive pairs of the $A$ amplitude, $A_{n+1}^{\text{max}}$ versus $A_{n}^{\text{max}}$, shown in Figure 3. It exhibits a tent-like structure more akin to that of the Lorentz attractor (e.g., Thompson & Stewart 1986), although somewhat more complex because of the split. For reference we have drawn a diagonal line that intersects the map at two very closely spaced points. At both points the slope of the map is greater than unity ($|\text{slope}| > 1$), which establishes the presence of a strange attractor (Thompson & Stewart 1986).

Chaotic behavior had previously been detected in the pulsations of the RV Tauri class star R Scuti (Buchler et al. 1995, 1996). It had also been predicted to be the cause of irregular pulsations in long period W Vir stellar models (Buchler & Kovács 1987; Kovács & Buchler 1988; Moskalik & Buchler 1990). It is interesting that the irregular modulations in the

Figure 1. Temporal modulation of the two amplitudes of the irregularly modulated period-two pulsation. (A color version of this figure is available in the online journal.)

Figure 2. Phase plot: amplitude $B(t)$ vs. amplitude $A(t)$.

Figure 3. First return map on the successive amplitude maxima $A_{n}^{\text{max}}$ of amplitude $A$, namely $A_{n+1}^{\text{max}}$ vs. $A_{n}^{\text{max}}$. 

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Blazhko RR Lyr stars would also be caused by a chaotic dynamics. There is, however, a big physical difference between the two cases. In RR Lyr the amplitude modulations occur on a timescale that is long compared to the “period” of the pulsation, because this timescale is set by the pulsational growth rates that are small in these nonadiabatic stars. In contrast, the W Vir and RV Tau stars have growth rates that are comparable to the pulsation frequencies and therefore allow the modulations to occur over a cycle, resulting in irregular pulsations (usually called semi-regular in the astronomical literature).

4. OTHER RESONANCES

In this paper, we have only considered the modal coupling between the 9th overtone and the fundamental mode through the 9 : 2 resonance.

An inspection of the period ratio diagram in Figure 3 of Kolláth et al. (2011) shows that the spacings between the curves are close to 0.5, and that therefore there is a plenitude of additional near resonances, e.g.,

\[ \omega_9 - \omega_7 = \omega_0 \quad \text{and} \quad \omega_{10} - \omega_8 = \omega_0. \]

Furthermore, there are also several integer resonances, such as

\[ \omega_7 = 4\omega_0 \quad \text{and} \quad \omega_5 = 3\omega_0, \]

although this latter resonance may not be important because of the strong damping of mode O5. The mode couplings that arise from these additional resonances obviously can cause additional features and complexity in the light curves. For the time being, we have ignored them. Observations of RR Lyr stars indicate that the Blazhko effect can be more complicated in some RR Lyr stars than those explored in this paper (e.g., multi-periodic modulation as found by Södör et al. 2011). In these cases, it may be necessary to include these additional resonances in the description.

5. CONCLUSIONS

Kolláth et al. (2011) and Szabó et al. (2010) discovered with the help of numerical hydrodynamical simulations that the Blazhko effect is most likely associated with the half-integer (9 : 2) resonance between the fundamental pulsation mode and an overtone that destabilizes the fundamental RR Lyr full amplitude pulsation. Because of the half-integer nature of the resonance (9P2 = 2P0) it therefore takes two fundamental periods for the pulsation to repeat, hence the occurrence of alternating cycles. The hydrodynamical modeling indeed found this symmetry breaking to give rise to cycle-to-cycle alternations, i.e., short-term variations. The Blazhko effect, however, is a long-term irregular amplitude modulation of the pulsations which hydrodynamical simulations have not produced so far.

In this paper, using an entirely different approach, namely the amplitude equation formalism, we demonstrate that irregular amplitude modulations can occur quite naturally as a result of the nonlinear, resonant mode coupling between the 9th overtone and the fundamental mode. The phenomenon occurs over a broad range of physical parameters and is therefore quite robust. Furthermore, we find that the range of “periods” of the Blazhko-like amplitude modulations are in concordance with the observed ones.

It is important to emphasize that the same half-integer resonance, responsible for the destabilization of fundamental RR Lyr pulsations, is also capable of producing period doubling or amplitude modulations, depending on the coefficients in the equations.

However, observations may have other surprises in store and the Blazhko effect may turn out to be more complicated, so that this simple two-mode coupling may not account for all of its complexity. This is the case for CZ Lacertae (Södör et al. 2011), where the Blazhko modulation is multiperiodic. There exist, in effect, additional modal resonances or near resonances that may need to be added in the amplitude equation description, and that may then lead to more complicated light curves.

Finally, now that the astronomical community has accepted the fact that period alternations can occur in classical variable stars such as RR Lyr stars, the time might be ripe to also target BL Her stars. Theory has actually predicted such alternations more than 15 years ago (Buchler & Moskalik 1992; Moskalik & Buchler 1993; Buchler & Buchler 1994), but at the time this had been received very skeptically at best. It is true that a recent unpublished Fourier analysis by Buchler and Moskalik of the OGLE data of BL Her stars does not reveal any conclusive evidence for alternations, but then, that was certainly also the case for the pre-Kepler RR Lyr.

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