STAR DISCREPANCY FOR NEW STRATIFIED RANDOM SAMPLING I: OPTIMAL EXPECTED STAR DISCREPANCY

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ABSTRACT. We introduce a class of convex equivolume partitions. Expected star discrepancy results are compared for stratified samples under these partitions, including simple random samples. There are four main parts of our results. First, among these newly designed partitions, there is one that minimizes the expected star discrepancy, thus we partly answer an open question in [F. Pausinger, S. Steinerberger, J. Complex. 2016]. Second, there are an infinite number of such class of partitions, which generate point sets with smaller expected discrepancy than classical jittered sampling for large sampling number, leading to an open question in [M. Kiderlen, F. Pausinger, Monatsh. Math. 2021] being solved. Third, we prove a strong partition principle and generalize the expected star discrepancy under these partition models from $L_2$—discrepancy to star discrepancy, hence an open question in [M. Kiderlen, F. Pausinger, J. Complex. 2021] is answered. In the end, optimal expected star discrepancy upper bound under this class of partitions is given, which is better than using jittered sampling.

1. INTRODUCTION

In real life sampling processes, it is necessary to know how well-spread these sampling points are. One can select the sampling set randomly which has achieved successful applications in the field of Monte Carlo sampling, compressed sensing, image processing and learning theory [8,10,12,25]. The concept of discrepancy can be introduced to quantify these point distributions. There is a list of interesting discrepancy measures, such as extreme discrepancy, $G$—discrepancy, isotrope discrepancy, lattice discrepancy, (see e.g., [21,22]). Among them, star discrepancy is the most widely studied.

Star discrepancy. The star discrepancy of a sampling set $P_{N,d} = \{t_1, t_2, \ldots, t_N\}$ is defined by:
\[ D_N^* (t_1, t_2, \ldots, t_N) := \sup_{x \in [0,1]^d} \left| \lambda([0,x]) - \frac{\sum_{n=1}^{N} I_{[0,x]}(t_n)}{N} \right|, \]

where \( \lambda \) denotes the \( d \)-dimensional Lebesgue measure and \( I_{[0,x]} \) denotes the characteristic function defined on the \( d \)-dimensional rectangle \([0,x]\).

The significance of the star discrepancy comes from the Koksma-Hlawka inequality, which is given by:

\[ \left| \int_{[0,1]^d} f(x) dx - \frac{1}{N} \sum_{t \in P_{N,d}} f(t) \right| \leq D_N^* (t_1, t_2, \ldots, t_N) V(f), \]

where \( D_N^* (t_1, t_2, \ldots, t_N) \) is the star discrepancy of \( P_{N,d} \) defined in (1.1) and \( V(f) \) is the total variation of \( f \) in the sense of Hardy and Krause. For discussion on \( V(f) \), a new concept of variation of multivariate functions is introduced to obtain Koksma-Hlawka type inequalities, see [35].

For fixed \( d \), the best known asymptotic upper bounds for star discrepancy are of the form

\[ O\left(\frac{(\ln N)^{\alpha_d}}{N}\right), \]

where \( \alpha_d \geq 0 \) are constants depending on dimension \( d \). These involve special deterministic point set constructions, which are low discrepancy point sets. Examples of such point sets can be found in [14,34]. For applications arising in computer graphics, quantitative finance and learning theory, see e.g., [1,9,31,32] and the literature mentioned therein.

Although low discrepancy (deterministic) point sets are widely used in numerical integration, the simulation of many phenomena in the real world requires the introduction of random factors. Recently, a large amount of research investigating random sampling for different function spaces has emerged in [3,4,24], due to the simplicity, flexibility and effectiveness of the subject. Besides, in the field of discrepancy, probabilistic star discrepancy bounds for Monte Carlo point sets (simple random sampling sets) are considered in [2,28], while centered discrepancy of random sampling and Latin hypercube random sampling are investigated in [23]. Motivated by these developments, we incorporate a random viewpoint into our study of star discrepancy to consider a special random sampling method, which is stratified sampling. Its special case is called jittered sampling that is formed by grid-based equivolume partition.

Some random sampling strategies, for example, simple random sampling, stratified sampling, Latin hypercube sampling, etc. are commonly used in the real sampling process, see [11,33,37]. Formers have made sufficient research on estimating the
expected star discrepancy with random samples. For researches on expected star discrepancy of jittered sampling, we refer to [20, 36]. Both the upper and the lower bounds for the discrepancy of jittered sampling are given in [36], while the bounds in [20] improve them and remove the asymptotic requirement that $m$ is sufficiently large compared to dimensions $d$ (where $N = m^d$ means the number of subcubes of grid-based equivolume partition). Expected bound

$$E(D_N^*(T)) \leq c(d) \frac{\ln^{\frac{1}{2}} N}{N^{\frac{1}{2} + \frac{d}{2}}}$$

of the star discrepancy for stratified sampling set $T$ under equivolume partition is derived in [6], while existence result is obtained in [5], where $c(d)$ is an implicit constant depending only on dimension $d$. Starting from the star discrepancy itself, rather than estimating its bound, in [30], it is shown that jittered sampling construction gives rise to a set whose expected $L_p$-discrepancy is smaller than that of purely random points. Further, a theoretical conclusion that the jittered sampling does not have the minimal expected $L_2$-discrepancy among all stratified samples from convex equivolume partitions with the same number of points is presented in [29].

Our research will be carried out on the $d$-dimensional unit cube, which can be easily extended to a more general compact convex set. Studies on convex bodies are extensive, see [7, 26]. In the following, we shall construct a class of convex body partitions to analyze expected star discrepancy, which turns out to provide better results than jittered sampling.

Throughout this paper, we adopt the idea of stratified random sampling to study star discrepancy. First, we design an infinite family of partitions with partition parameter $0 \leq \theta \leq \frac{\pi}{2}$ that generates point sets with a smaller expected star discrepancy than classical stratified sampling for sampling number $N = m^d$, which is,

$$E(D_N^*(Z)) \leq E(D_N^*(Y)),$$

where $Z$ and $Y$ denote stratified samples generated by the infinite family of partitions and grid-based equivolume partition respectively. The equal sign holds if and only if stratified sampling set $Z$ is selected for the jittered sampling set $Y$. This means, among the family of partitions, parameter $\theta = \frac{\pi}{2}$ comes back to the case of grid-based equivolume partition. An optimal expected star discrepancy under newly designed class of partitions is obtained when we choose $\theta = \arctan \frac{1}{2}$. Secondly, we prove the expected star discrepancy of a point set generated from jittered sampling is always strictly smaller than that of a set of $N = m^d$ uniform random samples, which is,

$$E(D_N^*(Y)) < E(D_N^*(X)),$$

where $Y$ and $X$ denote jittered samples and uniform random samples respectively. Thirdly, optimal expected star discrepancy bound is also provided under this class
of partitions, that is better than the employment of jittered sampling, which gives

$$\mathbb{E}[D^*_N(Z)] \leq \sqrt{2d + \frac{2p(\theta)}{3d+2-N^2} + 1} \frac{N^{2d-\frac{1}{2}}}{N^{\frac{d}{2} + \frac{1}{2}}}$$

where $P(\theta) = -\frac{2}{45}$ and $P(\theta) = 0$ are chosen in the upper bounds for optimal partition and grid-based equivolume partition respectively. All of these answer some open questions in [29,30,36].

The rest of this paper is organized as follows. In Section 2 we present some preliminaries on stratified sampling, $\delta-$covers and newly designed partition models. In Section 3 we provide comparisons of the expected star discrepancy for simple random sampling and stratified sampling under a series of convex equivolume partitions, expected star discrepancy bounds for these newly designed stratified random sampling models are also obtained. In Section 4 we include the proofs of all theorems and lemmas. Finally, in section 5 we conclude the paper with a short summary.

2. Preliminaries on stratified sampling, $\delta-$covers and new partition models

Before introducing the main result, we list preliminaries used in this paper.

2.1. Stratified sampling. Stratified sampling is a special random sampling, that is different from simple random sampling, see Figure 1. The original sampling area is divided, and a uniformly distributed random sample point is selected in each subset of partitions. Jittered sampling is a special case of stratified sampling, involving grid-based equivolume partition. Explicitly, $[0,1]^d$ is divided into $m^d$ axis parallel boxes $Q_i, 1 \leq i \leq N$, each with sides $\frac{1}{m}$, see Figure 2. Research on the jittered sampling are extensive, see [11,20,29,30,36].

We now consider a rectangle $R = [0, x]$ (we shall call it the test set in the following) in $[0,1]^d$ anchored at 0. For an isometric grid partition $\Omega = \{Q_1, Q_2, \ldots, Q_N\}$ of $[0,1]^d$, we put

$$I_N := \{j : \partial R \cap Q_j \neq \emptyset\},$$

and

$$C_N := |I_N|,$$

which means the cardinality of the index set $I_N$. For $C_N$, it is easy to obtain

$$C_N \leq d \cdot N^{1-\frac{1}{d}}.$$
2.2. $\delta$-covers. Secondly, to discretize the star discrepancy, we use the definition of $\delta$-covers as in [19], which is well known in empirical process theory, see, e.g., [38].

**Definition 2.1.** For any $\delta \in (0, 1]$, a finite set $\Gamma$ of points in $[0, 1)^d$ is called a $\delta$-cover of $[0, 1)^d$, if for every $y \in [0, 1)^d$, there exist $x, z \in \Gamma \cup \{0\}$ such that $x \leq y \leq z$ and $\lambda([0, z]) - \lambda([0, x]) \leq \delta$. The number $\mathcal{N}(d, \delta)$ denotes the smallest cardinality of a $\delta$-cover of $[0, 1)^d$.

From [19] [27], combining with Stirling's formula, the following estimation for $\mathcal{N}(d, \delta)$ holds, that is, for any $d \geq 1$ and $\delta \in (0, 1]$,
\[(2.2) \quad \mathcal{N}(d, \delta) \leq 2^d \cdot \frac{e^d}{\sqrt{2\pi d}} \cdot (\delta^{-1} + 1)^d.\]

Let \( P = \{p_1, p_2, \ldots, p_N\} \subset [0, 1]^d \) and \( \Gamma \) be \( \delta - \)covers, then
\[ D^*_N(P) \leq D_\Gamma(P) + \delta, \]
where
\[(2.3) \quad D_\Gamma(P) := \max_{x \in \Gamma} |\lambda([0, x]) - \frac{\sum_{n=1}^{N} I_{[0,x]}(p_n)}{N}|.\]

Formula (2.3) provides convenience for estimating the star discrepancy.

2.3. New partition models. In the end of this section, we design a class of partitions that generates point sets with a smaller star discrepancy than those by jittered sampling for sampling number \( N \). We will construct it step by step. First, we consider the two-dimensional case.

**Step one: a class of partitions design for two dimension.**

Our designed equivolume partition is actually a special case of general equivolume partition (see Figure 3 for illustration in two and three dimensional cases). For a grid-based equivolume partition in two dimension, we merge the two squares in the upper right corner to form a rectangle, then we use a series of straight line partitions to divide the rectangle into two equal-volume parts, which will be converted to a one-parameter model if we set the angle between the dividing line and the long side of the rectangle \( \theta \), where we suppose \( 0 \leq \theta \leq \frac{\pi}{2} \). From simple calculations, we can conclude the arbitrary straight line must pass through the center of the rectangle. For convenience of notation, we set this partition model \( \Omega \sim = (\Omega_1, \sim, \Omega_2, \sim, Q_3, \ldots, Q_N) \) in two dimension case.

In the above one-parameter model, the case will be grid-based equivolume partition if we choose \( \theta = \frac{\pi}{2} \). The case \( \theta = \text{arctan} \frac{1}{2} \) is introduced in [29], see Figure 4 for two and three dimensional cases. For notation convenience, we set this partition model \( \Omega\setminus = (\Omega_{1\setminus}, \Omega_{2\setminus}, Q_3, \ldots, Q_N) \) in two dimension case.

The only difference between the new partition model and grid-based equivolume partition is to change two closed hypercubes into two special convex bodies, see illustration in Figure 5.

**Step two:** Suppose the original rectangle is \( I \), for the convenience of calculation, we set the lower left corner of the rectangle at the origin \((0, 0)\) and the side length of the small square to 1. Now, consider \( I = [0, 2] \times [0, 1] \) and its two equivolume partitions \((\Omega_{1\setminus}, \Omega_{2\setminus})\) into two closed squares and \((\Omega_{1\setminus}, \Omega_{2\setminus})\) into two convex bodies with
In order to measure the two different partition methods in the subsequent proof and calculation, we introduce the concept of $L_2$--discrepancy.

$L_2$--discrepancy. $L_2$--discrepancy of a sampling set $P_{N,d} = \{t_1, t_2, \ldots, t_N\}$ is defined by

$$
\Omega_1,| = [0,1] \times [0,1], \quad \Omega_{1,\sim} = \text{conv}\{(0,0), (1 + \frac{\cot\theta}{2}, 0), (0,1), (1 - \frac{\cot\theta}{2}, 1)\},
$$

where $\text{conv}$ denotes the convex hull.
\[(2.4) \quad L_2(D_N, P, N) = \left( \int_{[0,1]^d} |z_1 z_2 \ldots z_d - \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{[0,z_i]}(t_i)|^2 dz \right)^{1/2}, \]

where \( \mathbf{1}_A \) denotes the characteristic function on set \( A \). For the applications of \( L_2 \)-discrepancy, see [15–18].

In the definition of \( L_2 \)-discrepancy, if we introduce the counting measure \( # \), (2.4) can also be expressed as

\[(2.5) \quad L_2(D_N, P, N) = \left( \int_{[0,1]^d} |\lambda([0,z]) - \frac{1}{N} #(P, N \cap [0,z])|^2 dz \right)^{1/2}, \]

where \( #(P, N \cap [0,z]) \) denotes the number of points falling into the set \([0,z]\).

To simplify the expression of \( L_2 \)-discrepancy, we employ the discrepancy function \( \Delta(P, N, z) \) via:

\[(2.6) \quad \Delta(P, N, z) = \lambda([0,z]) - \frac{1}{N} #(P, N \cap [0,z]). \]

Accordingly, the \( L_2 \)-discrepancy can be extended to a fixed compact convex set \( K \subset \mathbb{R}^d \) with \( \lambda(K) > 0 \), see [29]. Discrepancy function in (2.6) of a finite set of points \( P = \{x_1, x_2, \ldots, x_n\} \subset K \) is now given by

\[(2.7) \quad \Delta(P, x) = \frac{\lambda((-\infty, x] \cap K)}{\lambda(K)} - \frac{1}{N} #(P \cap (-\infty, x]). \]

**Step three:** We consider the translation and stretch of the rectangle \( I = [0,2] \times [0,1] \) into

\[ I' = [a_1, a_1 + 2b] \times [a_2, a_2 + b], \]

\[ \Omega_1, \Omega_2, \Omega_1, \Omega_2, \Omega_1, \Omega_2, \Omega_1, \Omega_2. \]
the above two dimensional case in Step one can then be extended to $d$-dimension as \[29\]. Consider $d$-dimensional cuboid

\[ I_d = I' \times \prod_{i=3}^{d} [a_i, a_i + b] \]

and its three equivolume partitions $\Omega'_1 = (\Omega'_1, \Omega'_2)$ into two closed hypercubes, $\Omega'_\parallel = (\Omega'_1, \Omega'_2, \parallel)$ into two closed, regular triangular hyperprisms and $\Omega'_\sim = (\Omega'_1, \sim, \Omega'_2, \sim)$ into two closed, trapezoidal superconvex bodies with

\[ \Omega'_1 = \prod_{i=1}^{d} [a_i, a_i + b], \]

\[ \Omega'_\parallel = \text{conv}\{(a_1, a_2), (a_1 + 2b, a_2), (a_1, a_2 + b)\} \times \prod_{i=3}^{d} [a_i, a_i + b], \]

and

\[ \Omega'_\sim = \text{conv}\{(a_1, a_2), (a_1 + b + \frac{b \cdot \cot \theta}{2}, a_2), (a_1, a_2 + b), (a_1 + b - \frac{b \cdot \cot \theta}{2}, a_2 + b)\} \]

\[ \times \prod_{i=3}^{d} [a_i, a_i + b], \]

where \text{conv} denotes the convex hull.

Just as grid-based partition $N = m^d$, where $m$ represents the number of partitions in each dimension and $d$ denotes the dimensions. If we choose $a_1 = \frac{m-2}{m}, a_2 = \frac{m-1}{m}, b = \frac{1}{m}$, then, through the construction method from step one to step three, we get a series of partitions (where we set $0 \leq \theta \leq \frac{\pi}{2}$) that we call local convex partition, denoted by

\[ \Omega^* = (\Omega^*_1, \sim, \Omega^*_2, \sim, Q_3, \ldots, Q_N). \]

Among the above local convex partition $\Omega^*_\sim$, if we choose the partition parameter $\theta = \frac{\pi}{2}$, isometric grid with partition number $m$ in each dimension is obtained, which we set

\[ \Omega^*_\parallel = (\Omega^*_1, \parallel, \Omega^*_2, \parallel, Q_3, \ldots, Q_N). \]

Likewise, if we choose the partition parameter $\theta = \arctan \frac{1}{2}$, partition model in two dimension case introduced above can then be extended to $d$ dimension, and we
choose $a_1 = \frac{m-2}{m}, a_2 = \frac{m-1}{m}, b = \frac{1}{m}$ in (2.10), then this partition model is denoted by

$$\Omega_\setminus = (\Omega_{1\setminus}, \Omega_{2\setminus}, Q_3, \ldots, Q_N).$$

3. Expected star discrepancy for stratified and simple random sampling

In this section, comparisons of expected star discrepancy under different partition models are obtained, comparisons for simple random sampling are also included. Furthermore, we study expected star discrepancy and several bounds are given under different newly designed partition models.

3.1. Expected star discrepancy under different partition models.

**Theorem 3.1.** Let $m, d \in \mathbb{N}$ with $m \geq d \geq 2, 0 \leq \theta \leq \frac{\pi}{2}$. Let $N = m^d$, $X = \{X_1, X_2, \ldots, X_N\}$ is a simple random sampling set. Stratified random $d$-dimension point sets $Y = \{Y_1, Y_2, Y_3, \ldots, Y_N\}$, $Z = \{Z_1, Z_2, Z_3, \ldots, Z_N\}$ are uniformly distributed in the grid-based stratified subsets of $\Omega_\setminus$ and stratified subsets of $\Omega_\sim$ respectively, then

$$E(D_N^*(Z)) \leq E(D_N^*(Y)) < E(D_N^*(X)),$$

where $\Omega_\sim$ and $\Omega_\setminus$ are defined in (2.11) and (2.12) respectively, $\theta$ is the partition parameter related to $\Omega_\sim$.

**Remark 3.2.** Theorem 3.1 means Strong Partition Principle [30] holds, thus an open question in [29] is answered, where the authors thought a proof of a Strong Partition Principle for star discrepancy out of reach. It is mentioned in [29, 36] that the strong partition principle has not been proved for star discrepancy case, while the $L_2$—discrepancy and $L_p$—discrepancy cases have been provided in [36] and [30] respectively.

**Remark 3.3.** Furthermore, in Theorem 3.1, as an infinite family of partitions is designed to generate point sets with a smaller expected star discrepancy than classical stratified sampling (jittered sampling) for same large sampling number $N$, open question 3 in [30] is solved(The authors of [30] asked whether an infinite family of partitions that generates point sets with a smaller expected discrepancy than classical jittered sampling for large $N$ exists.) Among this class of partitions, there is a constructed convex equivolume partition that minimizes star discrepancy on average. Thus we partly answer an open question in [36], where it asks which partition will minimize the $D_N^*—$discrepancy on average and raises the possibility that jittered sampling is not optimal. The partition model introduced in [29] is also generalized from $L_2—$discrepancy to star discrepancy.
3.2. Expected star discrepancy bounds under the new partition models. In this subsection, expected star discrepancy bounds under new partition models are given. Optimal result is also obtained under this class of partitions.

**Theorem 3.4.** Let \( m, d \in \mathbb{N} \) with \( m \geq d \geq 2, 0 \leq \theta \leq \frac{\pi}{2} \). Let \( N = m^d \), the stratified random \( d \)-dimension point set \( Z = \{Z_1, Z_2, Z_3, \ldots, Z_N\} \) distributed in subsets of \( \Omega^*_\theta \) defined in (2.11), then

\[
E[D^*_N(Z)] \leq \sqrt{2d + \frac{2P(\theta)}{3d^2 - N^2 - \frac{1}{4}}} + 1
\]

where

\[
P(\theta) = \begin{cases} 
\frac{2}{45} \tan^2 \theta + \frac{2}{15} \tan^2 \theta - \frac{\tan \theta}{6}, & 0 \leq \theta < \arctan \frac{1}{2}, \\
-\frac{2}{45}, & \theta = \arctan \frac{1}{2}, \\
-\frac{1}{24\tan \theta} + \frac{1}{120\tan^2 \theta} + \frac{1}{1440\tan^3 \theta}, & \arctan \frac{1}{2} < \theta \leq \frac{\pi}{2}.
\end{cases}
\]

**Remark 3.5.** Noticing that in Theorem 3.4, \( P(\theta) \) is a continuous function, decreases monotonically between 0 and \( \arctan \frac{1}{2} \) and increases monotonically between \( \arctan \frac{1}{2} \) and \( \frac{\pi}{2} \). The optimal expected star discrepancy bound under this class of partitions is obtained at \( \arctan \frac{1}{2} \). Choose parameter \( \theta = \frac{\pi}{2} \) in Theorem 3.4, then we are back to the case of classical jittered sampling. Furthermore, when \( \theta = 0, \frac{\pi}{2} \), \( P(\theta) \) attain their maximum values 0; in \((0, \frac{\pi}{2})\), all \( P(\theta) \) take values less than 0, this implies all of these local convex partitions with parameter \( \theta \in (0, \frac{\pi}{2}) \) obtain better upper bounds of expected star discrepancy than the jittered sampling. See illustration in Figure 6.
Corollary 3.6. Let $m, d \in \mathbb{N}$ with $m \geq d \geq 2$. Let $N = m^d$, $d$-dimensional simple random sampling point set $X = \{X_1, X_2, X_3, \ldots, X_N\}$ distributed in $[0, 1]^d$, then

$$
\mathbb{E}[D_{X}^*(X)] \leq \frac{\sqrt{2d} + 1}{N^{\frac{1}{2}}}.
$$

Corollary 3.7. Let $N = m^d$ and $m, d \in \mathbb{N}$ with $m \geq d \geq 2$. For $\Omega_\ast$, there exists partition $\Omega = \Omega_\ast \in \Omega_\ast^\ast$ such that

$$
\min_{\Omega \in \Omega_\ast} \mathbb{E}(D_{X}^*(P_\Omega)) = \mathbb{E}(D_{X}^*(P_{\Omega_\ast}^\ast)),
$$

where $\Omega_\ast^\ast$ defined in (2.13) denotes the optimal partition.

Remark 3.8. Actually, for local convex partition as Figure 2, optimal expected star discrepancy can be given if we choose parameter $\theta = \arctan \frac{1}{2}$ in Theorem 3.1. In other words, Corollary 3.7 gives the existence of the optimal expected star discrepancy.
under the local convex partition, and such an optimal partition manner is presented explicitly. The optimal partition manner is considered in [29] too. But the conclusion that this case produces optimal expected star discrepancy is not mentioned, because $L_2$-discrepancy measure instead of star discrepancy measure is employed.

3.3. **Some Examples.** This subsection presents some examples of expected star discrepancy bounds under different sampling models for $N = m^d$. The case of $\theta = \arctan \frac{1}{2}$ acquires better result than that of jittered sampling; both of them are better than that of simple random sampling.

**Example 1.** Expected bound of stratified sampling set for $\theta = 0$

$$
\mathbb{E}[D_N^*] \leq \frac{\sqrt{2d} + 1}{N^{\frac{1}{2} + \frac{d}{2}}}. 
$$

**Figure 7.** Stratified sampling for $\theta = 0$

**Example 2.** Expected bound of stratified sampling set for $\theta = \frac{\pi}{2}$

$$
\mathbb{E}[D_N^*] \leq \frac{\sqrt{2d} + 1}{N^{\frac{1}{2} + \frac{d}{2}}}. 
$$
Example 3. Expected bound of stratified sampling set for $\theta = \arctan \frac{1}{2}$

$$\mathbb{E}[D_N^*] \leq \sqrt{2d - \frac{4}{45.3^{d-2}N^{2-\frac{1}{d}}} + 1} \frac{N^{\frac{1}{2}}}{N^2 + \frac{1}{2}}.$$ 

Example 4. Expected bound of simple random sampling set

$$\mathbb{E}[D_N^*] \leq \frac{\sqrt{2d} + 1}{N^{\frac{1}{2}}}.$$
4. Proofs

In this section, we present the proofs of Theorem 3.1 and 3.4. We first introduce some key lemmas. The following Lemma 4.1 is a standard result which can be found in textbooks on probability, see, e.g., [13] and no proof will be given here.

**Lemma 4.1 (Bernstein’s inequality).** Let \( Z_1, \ldots, Z_N \) be independent random variables with expected values \( \mathbb{E}(Z_j) = \mu_j \) and variances \( \sigma_j^2 \) for \( j = 1, \ldots, N \). Assume \( |Z_j - \mu_j| \leq C \) (\( C \) is a constant) for each \( j \) and set \( \Sigma^2 := \sum_{j=1}^{N} \sigma_j^2 \), then for any \( \lambda \geq 0 \),

\[
\mathbb{P} \left\{ \left| \frac{1}{N} \sum_{j=1}^{N} [Z_j - \mu_j] \right| \geq \lambda \right\} \leq 2 \exp \left( -\frac{\lambda^2}{2\Sigma^2 + \frac{2}{3}C\lambda} \right).
\]

The following lemma reveals the quantitative relationship between the two partition models \( \Omega_{\sim}^* \) and \( \Omega_{\sim}^* \), and the \( L_2 \)-discrepancy is employed to measure it.

**Lemma 4.2.** For two equivolume partitions \( \Omega_{\sim}^* = (\Omega_{1, \sim}^*, \Omega_{2, \sim}^*, Q_3, \ldots, Q_N) \) and \( \Omega_{\sim}^* = \{Q_1, Q_2, Q_3, \ldots, Q_N\} \) defined in (2.11) and (2.12) respectively, we have

\[
\mathbb{E} L_2^2(D_N, P_{\Omega_{\sim}^*}) - \mathbb{E} L_2^2(D_N, P_{\Omega_{\sim}^*}) = \begin{cases} 
\frac{1}{N^3} \cdot \frac{1}{3^{d-2}} \cdot P_1(\theta), & 0 \leq \theta < \arctan \left( \frac{1}{2} \right), \\
- \frac{2}{45} \cdot \frac{1}{N^3} \cdot \frac{1}{3^{d-2}}, & \theta = \arctan \left( \frac{1}{2} \right), \\
\frac{1}{N^3} \cdot \frac{1}{3^{d-2}} \cdot P_2(\theta), & \arctan \left( \frac{1}{2} \right) < \theta \leq \frac{\pi}{2},
\end{cases}
\]

where

\[
P_1(\theta) = \frac{2}{45} \tan^3 \theta + \frac{2}{15} \tan^2 \theta - \frac{\tan \theta}{6}.
\]
and

\[ P_2(\theta) = -\frac{1}{24\tan\theta} + \frac{1}{120\tan^2\theta} + \frac{1}{1440\tan^3\theta}. \]

**Remark 4.3.** When \( \theta = \arctan\frac{1}{2} \), equivolume partition manner is consistent with that in [29], which will also return to the result of remark 6 in it.

### 4.1. Proof of Lemma 4.2

For equivolume partition \( \Omega_0,\sim = (\Omega_1,\sim, \Omega_2,\sim) \) of \( I \) (the same argument if we replace \( \Omega_0,\sim \) with \( \Omega_{0,1} \)), from [Proposition 2] in [29], which is, for an equivolume partition \( \Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_N\} \) of a compact convex set \( K \subset \mathbb{R}^d \) with \( \lambda(K) > 0 \), \( P_\Omega \) is the corresponding stratified sampling set, then

\[ (4.2) \quad \mathbb{E}L_2^2(D_N, P_\Omega) = \frac{1}{N^2\lambda(K)} \sum_{i=1}^{N} \int_K q_i(x)(1 - q_i(x))dx, \]

where

\[ (4.3) \quad q_i(x) = \frac{\lambda(\Omega_i \cap [0, x])}{\lambda(\Omega_i)}. \]

Through simple derivation, it follows that

\[ (4.4) \quad \mathbb{E}L_2^2(D_N, P_{\Omega_0,\sim}) = \frac{1}{8} \sum_{i=1}^{2} \int_I q_i(x)(1 - q_i(x))dx, \]

and

\[ (4.5) \quad q_i(x) = \frac{\lambda(\Omega_{i,\sim} \cap [0, x])}{\lambda(\Omega_{i,\sim})} = \lambda(\Omega_{i,\sim} \cap [0, x]). \]

Conclusion (4.4) is equivalent to the following

\[ 8\mathbb{E}L_2^2(D_N, P_{\Omega_0,\sim}) = 1 - \sum_{i=1}^{2} \int_I q_i^2(x)dx. \]

We adapt the line of proof of [Lemma 4] in [29]. However, the geometric structure of trapezoid is different for a test set \([0, x]\). We first consider parameter \( \arctan\frac{1}{2} \leq \theta \leq \frac{\pi}{2} \), then we define the following two functions for simplicity of the expression.

\[ F(x) = \frac{1}{2} \cdot [(x_1 - 1)\tan\theta + x_2 - \frac{1}{2}] \cdot [(x_1 - 1) + (x_2 - \frac{1}{2}) \cdot \cot\theta], \]

and
\[
G(x) = x_1x_2 - x_2 - \frac{\cot \theta}{2} x_2 + \frac{1}{2} x_2^2 \cdot \cot \theta,
\]
where \( x = (x_1, x_2) \).

Furthermore, for \( \Omega_{0,\|} = (\Omega_{1,\|}, \Omega_{2,\|}) \) defined in Step two of subsection 2.3, (4.5) implies
\[
q_{1,\|}(x) = \begin{cases} 
  x_1 x_2, & x \in \Omega_{1,\|} \\
  x_2, & x \in \Omega_{2,\|},
\end{cases}
\]
and
\[
q_{2,\|}(x) = \begin{cases} 
  0, & x \in \Omega_{1,\|} \\
  (x_1 - 1)x_2, & x \in \Omega_{2,\|}.
\end{cases}
\]
Besides,
\[
q_{1,\sim}(x) = \begin{cases} 
  x_1 x_2, & x \in \Omega_{1,\sim}, \\
  x_1 x_2 - F(x), & x \in \Omega_{2,\sim,1}, \\
  x_1 x_2 - G(x), & x \in \Omega_{2,\sim,2},
\end{cases}
\]
and
\[
q_{2,\sim}(x) = \begin{cases} 
  0, & x \in \Omega_{1,\sim}, \\
  F(x), & x \in \Omega_{2,\sim,1}, \\
  G(x), & x \in \Omega_{2,\sim,2},
\end{cases}
\]
where \( \Omega_{1,\sim}, \Omega_{2,\sim} \) denote subsets of partition \( \Omega_{0,\sim} \). In the following, we shall continue to divide subsets \( \Omega_{1,\sim} = \{\Omega_{1,\sim,1}, \Omega_{1,\sim,2}\} \) and \( \Omega_{2,\sim} = \{\Omega_{2,\sim,1}, \Omega_{2,\sim,2}\} \) to facilitate calculation. See figures 11 to 12.

Therefore, for \( \theta = \frac{\pi}{2} \), we introduce two symbols \( B_{1,\|}, B_{2,\|} \) and have
\[
(4.6) \quad B_{1,\|} = \int_I q_{1,\|}^2(x)dx = \int_{\Omega_{1,\|}} x_1^2 x_2^2 dx + \int_{\Omega_{2,\|}} x_2^2 dx = \frac{1}{9} + \frac{1}{3} = \frac{4}{9},
\]
and
\[
(4.7) \quad B_{2,\|} = \int_I q_{2,\|}^2(x)dx = \int_{\Omega_{2,\|}} (x_1 - 1)^2 x_2^2 dx = \frac{1}{9}.
\]
Thus,
\[
(4.8) \quad 8\mathbb{E}(L_2^2(P_{\Omega_{0,\|}})) = 1 - (B_{1,\|} + B_{2,\|}) = \frac{4}{9}.
\]
Furthermore, we introduce \( B_{1,\sim} \) and \( B_{2,\sim} \), then
\( B_{1,\sim} = \int_I q_{1,\sim}^2(x)dx = \int_{\Omega_{1,\sim}} x_1^2x_2^2dx + \int_{\Omega_{2,\sim,1}} (x_1x_2 - F(x))^2dx \\
+ \int_{\Omega_{2,\sim,2}} (x_1x_2 - G(x))^2dx, \)

and

\( B_{2,\sim} = \int_I q_{2,\sim}^2(x)dx = \int_{\Omega_{2,\sim,1}} F^2(x)dx + \int_{\Omega_{2,\sim,2}} G^2(x)dx. \)

We divide our calculation in three steps. First, we compute \( \int_{\Omega_{1,\sim}} x_1^2x_2^2dx \), see Figure 11 for illustration.

\[ \int_{\Omega_{1,\sim,1}} x_1^2x_2^2dx = \int_0^{1-\cot\theta} x_1^2dx_1 \cdot \int_0^1 x_2^2dx_2 = \frac{(2 - \cot\theta)^3}{72}. \]

\[ \int_{\Omega_{1,\sim,2}} x_1^2x_2^2dx = \int_{1-\cot\theta}^{1+\cot\theta} x_1^2dx_1 \cdot \int_0^{(1-x_1)\tan\theta + \frac{1}{2}} x_2^2dx_2 \\
= \frac{60\tan^2\theta - 36\tan\theta + 7}{720\tan^3\theta}. \]

Therefore, (4.10) and (4.11) imply
\[
\int_{\Omega_{1,\sim}} x_1^2 x_2^2 \, dx = \int_{\Omega_{1,\sim,1}} x_1^2 x_2^2 \, dx + \int_{\Omega_{1,\sim,2}} x_1^2 x_2^2 \, dx
\]
\[
= -\frac{1}{12 \tan \theta} + \frac{1}{30 \tan^2 \theta} - \frac{1}{240 \tan^3 \theta} + \frac{1}{9}.
\]

Second, we compute \( \int_{\Omega_{2,\sim,1}} (x_1 x_2 - F(x))^2 \, dx \) and \( \int_{\Omega_{2,\sim,2}} (x_1 x_2 - G(x))^2 \, dx \).

\[
\int_{\Omega_{2,\sim,1}} (x_1 x_2 - F(x))^2 \, dx = \int_{1 + \frac{\cot \theta}{2}}^{1} \int_{0}^{1} (x_1 x_2 - F(x))^2 \, dx_2 \, dx_1
\]
\[
= \frac{180 \tan^2 \theta - 12 \tan \theta + 5}{720 \tan^3 \theta},
\]

\[
\int_{\Omega_{2,\sim,2}} (x_1 x_2 - G(x))^2 \, dx = \int_{1 + \frac{\cot \theta}{2}}^{2} \int_{0}^{1} (x_1 x_2 - G(x))^2 \, dx_2 \, dx_1
\]
\[
= -\frac{\cot^3 \theta}{240} - \frac{\cot^2 \theta}{30} - \frac{\cot \theta}{12} + \frac{1}{3}.
\]

Thus, (4.13) and (4.14) imply

\[
\int_{\Omega_{2,\sim,1}} (x_1 x_2 - F(x))^2 \, dx + \int_{\Omega_{2,\sim,2}} (x_1 x_2 - G(x))^2 \, dx
\]
\[
= \frac{1}{3} + \frac{1}{6 \tan \theta} - \frac{1}{20 \tan^2 \theta} + \frac{1}{360 \tan^3 \theta}.
\]

Combining (4.9), (4.12), (4.15), we have
\[
B_{1,\sim} = \int_{\Omega_{1,\sim}} x_1^2 x_2^2 dx + \int_{\Omega_{2,\sim,1}} (x_1 x_2 - F(x))^2 dx + \int_{\Omega_{2,\sim,2}} (x_1 x_2 - G(x))^2 dx
\]
\[
= \frac{1}{12\tan\theta} - \frac{1}{60\tan^2\theta} - \frac{1}{720\tan^3\theta} + \frac{4}{9}.
\]

Third, we compute \(\int_{\Omega_{2,\sim,1}} F^2(x) dx\) and \(\int_{\Omega_{2,\sim,2}} G^2(x) dx\).

\[
\int_{\Omega_{2,\sim,1}} F^2(x) dx = \int_{1+\cot\theta}^{1-\cot\theta} \int_{(1-x_1)\tan\theta + \frac{1}{2}}^1 F^2(x) dx_2 dx_1
\]
\[
= \frac{1}{120\tan^3\theta}.
\]

\[
\int_{\Omega_{2,\sim,2}} G^2(x) dx = \int_2^{1+\cot\theta} \int_0^1 G^2(x) dx_2 dx_1
\]
\[
= \frac{1}{9} - \frac{1}{24\tan\theta} + \frac{1}{120\tan^2\theta} - \frac{11}{1440\tan^3\theta}.
\]

Combining (4.17) and (4.18), we have

\[
B_{2,\sim} = \int_{\Omega_{2,\sim,1}} F^2(x) dx + \int_{\Omega_{2,\sim,2}} G^2(x) dx
\]
\[
= \frac{1}{9} - \frac{1}{24\tan\theta} + \frac{1}{120\tan^2\theta} + \frac{1}{1440\tan^3\theta}.
\]

Thus,

\[
B_{1,\sim} + B_{2,\sim} = \frac{1}{24\tan\theta} - \frac{1}{120\tan^2\theta} - \frac{1}{1440\tan^3\theta} + \frac{5}{9}.
\]

Therefore,

\[
8\mathbb{E}(L_2^2(P_{\Omega_0,\sim})) = 1 - (B_{1,\sim} + B_{2,\sim})
\]
\[
= -\frac{\cot\theta}{24} + \frac{\cot^2\theta}{120} + \frac{\cot^3\theta}{1440} + \frac{4}{9},
\]

where \(\arctan\frac{1}{2} \leq \theta < \frac{\pi}{2}\). Easy to show that when \(\tan\theta = \frac{1}{2}\),
Expected star discrepancy for stratified sampling

Minimum value of $\mathbb{E}(L^2(P_{\Omega_0,\sim}))$ is obtained, for $\theta = \frac{\pi}{2}$. From (4.8), we have

$$8\mathbb{E}(L^2(P_{\Omega_0,\sim})) = \frac{4}{9}.$$  

Figure 13. Division of the integral region.

Considering the case $0 \leq \theta < \arctan\frac{1}{2}$, we denote the partition by $\Omega'_\sim = \{\Omega'_{1,\sim}, \Omega'_{2,\sim}\}$, see Figure 13. Let

$$q'_{1,\sim}(x) = \begin{cases} x_1x_2, & x \in \Omega'_{1,\sim}, \\ x_1x_2 - H(x), & x \in \Omega'_{2,\sim,1}, \\ x_1x_2 - J(x), & x \in \Omega'_{2,\sim,2}. \end{cases}$$

and

$$q'_{2,\sim}(x) = \begin{cases} 0, & x \in \Omega'_{1,\sim}, \\ H(x), & x \in \Omega'_{2,\sim,1}, \\ J(x), & x \in \Omega'_{2,\sim,2}. \end{cases}$$

Where

$$(4.22) \quad H(x) = \frac{1}{2} \cdot [x_2 - (1 - x_1)\tan \theta - \frac{1}{2}] \cdot [\cot \theta \cdot x_2 - 1 + x_1 - \frac{1}{2}\cot \theta],$$

and

$$(4.23) \quad J(x) = [x_2 - \tan \theta - \frac{1}{2}] \cdot x_1 + \frac{1}{2}x_1^2 \cdot \tan \theta,$$

and we divide subsets $\Omega'_{1,\sim} = \{\Omega'_{1,\sim,1}, \Omega'_{1,\sim,2}\}$ and $\Omega'_{2,\sim} = \{\Omega'_{2,\sim,1}, \Omega'_{2,\sim,2}\}$ to facilitate calculation. See Figure 13.

Hence, let
\[ B'_{1,\sim} = \int_I q^2_{1,\sim}(x)dx = \int_{\Omega_{1,\sim}} x_1^2 x_2^2 dx + \int_{\Omega_{2,\sim,1}} (x_1 x_2 - H(x))^2 dx \]
\[ + \int_{\Omega_{2,\sim,2}} (x_1 x_2 - J(x))^2 dx, \]
and
\[ B'_{2,\sim} = \int_I q^2_{2,\sim}(x)dx = \int_{\Omega_{2,\sim,1}} H^2(x)dx + \int_{\Omega_{2,\sim,2}} J^2(x)dx. \]

If we follow the calculation process of (4.10)-(4.20), then we obtain
\[ B'_{1,\sim} = -\frac{4}{45} \tan^3 \theta - \frac{4}{15} \tan^2 \theta + \frac{\tan \theta}{3} + \frac{4}{9}, \]
and
\[ B'_{2,\sim} = \frac{2}{45} \tan^3 \theta + \frac{2}{15} \tan^2 \theta - \frac{\tan \theta}{6} + \frac{1}{9}. \]

Thus,
\[ B'_{1,\sim} + B'_{2,\sim} = -\frac{2}{45} \tan^3 \theta - \frac{2}{15} \tan^2 \theta + \frac{\tan \theta}{6} + \frac{5}{9}. \]

Hence,
\[ 8\mathbb{E}(L^2(P_{\Omega_{\sim}})) = 1 - (B'_{1,\sim} + B'_{2,\sim}) \]
\[ = \frac{4}{9} + \frac{2}{45} \tan^3 \theta + \frac{2}{15} \tan^2 \theta - \frac{\tan \theta}{6}, \]
where \(0 \leq \theta < \arctan \frac{1}{2} \).

Combining with (4.21) and considering the translation and stretch of the rectangle \( I = [0, 2] \times [0, 1] \) into \( I' = [a_1, a_1 + 2b] \times [a_2, a_2 + b], \)
where \(a_1 = \frac{m-2}{m}, a_2 = \frac{m-1}{m}, b = \frac{1}{m} \), we obtain
\[ \mathbb{E}(L^2(P_{\Omega_{\sim}})) \leq \mathbb{E}(L^2(P_{\Omega_{\sim}})), \]
where \(\Omega_{\sim}^* \) is the infinite family of equivolume partitions defined in (2.11), \(\Omega_{\sim}^* \) is grid-based equivolume partition defined in (2.12). The equal sign of (4.29) is established if and only if partition parameter \(\theta = 0, \frac{\pi}{2} \).
We now give a proof of (4.29). For $d$-dimensional case, we first prove the case $b = 1$ and $(a_1, a_2, \ldots, a_d) = (0, 0, \ldots, 0)$. Let $I'_d = [0, 2] \times [0, 1] \times [0, 1]^{d-2}$ and we denote partition manner of this special case $\Omega''_\sim = \{\Omega''_{1, \sim}, \Omega''_{2, \sim}\}$.

For $i = 1, 2$, we have

$$q'_i, \sim(x) = q_i, \sim(x_1, x_2) \cdot \prod_{j=3}^{d} x_j,$$

where $q'_i, \sim(x)$ is defined as (4.5) for $\Omega''_\sim$.

Thus,

$$\int_{I'_d} q'^2_{i, \sim}(x) dx = B_{i, \sim} \cdot \int_{[0,1]^{d-2}} \prod_{j=3}^{d} x^2_j dx_3 dx_4 \ldots dx_d = \frac{1}{3^{d-2}} \cdot B_{i, \sim},$$

where $B_{i, \sim}, i = 1, 2$ have been calculated in (4.16) and (4.19) respectively.

As we have

$$\int_{I'_d} \lambda([0, x]) dx = \int_{[0,1]^{d-2}} \prod_{j=3}^{d} x_j dx_3 dx_4 \ldots dx_d = \frac{1}{2^{d-2}}.$$  

Then we obtain,

$$8\mathbb{E}(L_2^2(P_{\Omega''_\sim})) = \frac{1}{2^{d-2}} - \frac{1}{3^{d-2}} \cdot (B_{1, \sim} + B_{2, \sim}).$$  

The following equation is a conclusion in [29],

$$\mathbb{E}_d^2(P_{\Omega''_\sim}) = \mathbb{E}_d^2(P_{\Omega''_\sim}) = \frac{1}{N^3} \cdot \mathbb{E}(L_2^2(P_{\Omega''_\sim})) = \frac{1}{N^3} \cdot \mathbb{E}(L_2^2(P_{\Omega''_\sim})) - \mathbb{E}(L_2^2(P_{\Omega''_\sim}))).$$  

We now give a detailed calculation process of (4.31). Firstly, for $I_d$ in (2.8), we define a vector

$$\mathbf{a} = \{a_1, a_2, \ldots, a_d\}.$$

We then prove (4.2) is independent of $\mathbf{a}$. In $I_d$, we choose $\mathbf{a} = 0$, set

$$I^0_d = [0, 2b] \times [0, b]^{d-1},$$

and

$$I^0_{d,m} = [0, \frac{2}{m}] \times [0, \frac{1}{m}]^{d-1}.$$

It suffices to show that
\begin{equation}
\frac{1}{N^2 \lambda(I_d)} \sum_{i=1}^{N} \int_{I_d} q_i(x)(1 - q_i(x))dx = \frac{1}{N^2 \lambda(I_0^d)} \sum_{i=1}^{N} \int_{I_0^d} q_i(x)(1 - q_i(x))dx.
\end{equation}

We only consider \( N = 2 \) in (4.35), this is because we choose \( K = I_d \) and \( K = I_0^d \) in (4.2) respectively. This means \( I_d, I_0^d \) are divided into two equal volume parts respectively.

Let
\begin{equation}
x_i - a_i = t_i, \quad 1 \leq i \leq d.
\end{equation}

According to (4.3) and plugging (4.36) into the left side of (4.35), the desired result is obtained.

Secondly, we prove (4.31). From (4.2) and let \( K = [0, 1]^d \), we have

\begin{equation}
\mathbb{E}L_2^2(P_{\Omega \sim}) - \mathbb{E}L_2^2(P_{\Omega \mid})
= \frac{1}{N^2} \sum_{i=1}^{N} \int_{[0,1]^d} \tilde{q}_i(x)(1 - \tilde{q}_i(x))dx - \frac{1}{N^2} \sum_{i=1}^{N} \int_{[0,1]^d} \bar{q}_i(x)(1 - \bar{q}_i(x))dx,
\end{equation}

where
\begin{align*}
\tilde{q}_i(x) &= \frac{\lambda(\Omega^*_{i,\sim} \cap [0, x])}{\lambda(\Omega^*_{i,\sim})}, \quad \bar{q}_i(x) = \frac{\lambda(\Omega^*_{i,\mid} \cap [0, x])}{\lambda(\Omega^*_{i,\mid})}, \quad i = 1, 2, \\
\end{align*}

and
\begin{align*}
\tilde{q}_i(x) &= \tilde{q}_i(x) = \frac{\lambda(Q_i \cap [0, x])}{\lambda(Q_i)}, \quad i = 3, 4, \ldots, N.
\end{align*}

Let \( I_{d,m}^0 = \{\Omega^*_{1,\sim}, \Omega^*_{2,\sim}\} \), \( I_{d,m}^0 = \{\Omega^*_{1,\mid}, \Omega^*_{2,\mid}\} \) denote two different partitions of \( I_{d,m}^0 \).

It can easily be seen only \( I_{d,m}^0 \) contributes to the difference between two expected \( L_2 \)-discrepancies, thus
\[
\mathbb{E}L_2^2(P_{\Omega}) - \mathbb{E}L_2^2(P_{\Omega'}) \\
= \frac{1}{N^2} \sum_{i=1}^{2} \int_{I_d} (\tilde{q}_i(x) - \tilde{q}_i(x))dx + \frac{1}{N^2} \sum_{i=1}^{2} \int_{I_d} (\tilde{q}_i^2(x) - \tilde{q}_i^2(x))dx \\
= \frac{1}{N} \sum_{i=1}^{2} \int_{I_d} (\lambda(\Omega_i \cap [0, x]) - \lambda(\Omega_i \cap [0, x]))dx \\
\quad + \sum_{i=1}^{2} \int_{I_d} (\lambda^2(\Omega_i \cap [0, x]) - \lambda^2(\Omega_i \cap [0, x]))dx \\
\quad + \frac{1}{N^3} \sum_{i=1}^{2} \int_{I_d^2} (\lambda(\Omega_i'' \cap [0, x]) - \lambda(\Omega_i'' \cap [0, x]))dx \\
\quad + \frac{1}{N^3} \sum_{i=1}^{2} \int_{I_d^2} (\lambda^2(\Omega_i'' \cap [0, x]) - \lambda^2(\Omega_i'' \cap [0, x]))dx.
\] (4.38)

Furthermore, employing (4.2) again, we have

\[
\mathbb{E}(L_2^2(P_{\Omega'})) - \mathbb{E}(L_2^2(P_{\Omega''})) \\
= \frac{1}{8} \sum_{i=1}^{2} \int_{I_d} q_i'(x)(1 - q_i'(x))dx - \frac{1}{8} \sum_{i=1}^{2} \int_{I_d} q_i'(x)(1 - q_i'(x))dx \\
= \frac{1}{8} \sum_{i=1}^{2} \int_{I_d} (\lambda(\Omega_i'' \cap [0, x]) - \lambda(\Omega_i'' \cap [0, x]))dx \\
\quad + \frac{1}{8} \sum_{i=1}^{2} \int_{I_d^2} (\lambda^2(\Omega_i'' \cap [0, x]) - \lambda^2(\Omega_i'' \cap [0, x]))dx.
\] (4.39)

An easy induction from (4.38) to (4.39) gives (4.31).

In the end, combining with (4.6), (4.7), (4.8), (4.20), (4.21), (4.27) and (4.30), we obtain the conclusion (4.29). Simultaneously, (4.31) combining with the above calculations prove the lemma.

4.2. Proof of Theorem 3.1. For arbitrary test set \(R = [0, x]\), we unify a label \(W = \{W_1, W_2, \ldots, W_N\}\) for the sampling point sets formed by different equivolume partition models, and we consider the following discrepancy function,
\begin{align}
\Delta_\varphi(x) &= \frac{1}{N} \sum_{n=1}^{N} 1_{R}(W_n) - \lambda(R). 
\end{align}

For an equivolume partition \( \Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_N\} \), we divide the test set \( R \) into two parts, one is the disjoint union of \( \Omega_i \) entirely contained by \( R \) and another is the union of remaining pieces which are the intersections of some \( \Omega_j \) and \( R \), i.e.,

\begin{align}
R &= \bigcup_{i \in I_0} \Omega_i \cup \bigcup_{j \in J_0} (\Omega_j \cap R),
\end{align}

where \( I_0, J_0 \) are two index-sets.

Set

\( T = \bigcup_{j \in J_0} (\Omega_j \cap R) \),

from (4.40), we have

\begin{align}
\Delta_\varphi(x) &= \frac{1}{N} \sum_{n=1}^{N} 1_{R}(W_n) - \lambda(R) = \frac{1}{N} \sum_{n=1}^{N} 1_{T}(W_n) - \lambda(T),
\end{align}

equation (4.42) is based on the fact discrepancy function equals 0 on \( \bigcup_{i \in I_0} \Omega_i \).

According to the definition of \( L_2 \)-discrepancy and (4.42), it follows that

\begin{align}
\mathbb{E}(L_2^2(D_N, W)) &= \mathbb{E}(\int_{[0,1]^d} |\frac{1}{N} \sum_{n=1}^{N} 1_{T}(W_n) - \lambda(T)|^2 dx).
\end{align}

Consider the whole sum in (4.43) as a random variable which is defined on a region \( P_\Omega \). Besides we set the probability measure be \( w \), then we have

\begin{align}
\mathbb{E}(L_2^2(D_N, W)) &= \int_{P_\Omega} \int_{[0,1]^d} |\frac{1}{N} \sum_{n=1}^{N} 1_{T}(W_n) - \lambda(T)|^2 dx dw \\
&= \int_{[0,1]^d} \int_{P_\Omega} |\frac{1}{N} \sum_{n=1}^{N} 1_{T}(W_n) - \lambda(T)|^2 dw dx.
\end{align}

It can easily be checked that,

\( \mathbb{E}(\frac{1}{N} \sum_{n=1}^{N} 1_{T}(W_n)) = \int_{P_\Omega} \frac{1}{N} \sum_{n=1}^{N} 1_{T}(W_n) dw = \lambda(T) \).
Hence,

\[
\int_{R_0} \left( \frac{1}{N} \sum_{n=1}^{N} 1_T(W_n) - \lambda(T) \right)^2 d\nu = \text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} 1_T(W_n) \right).
\]

(4.45)

Now, for sampling sets \( Z = \{Z_1, Z_2, \ldots, Z_N\} \) and \( Y = \{Y_1, Y_2, \ldots, Y_N\} \) defined in Theorem 3.1, from (4.29), we have

\[
\mathbb{E}(L_2^2(D_N, Z)) \leq \mathbb{E}(L_2^2(D_N, Y)).
\]

(4.46)

Hence,

\[
\text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} 1_T(Z_n) \right) \leq \text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} 1_T(Y_n) \right).
\]

(4.47)

Otherwise, combining with (4.44) and (4.45), a contradiction with equation (4.46).

Theorem 3.1 will be proved if we could employ (4.47) to establish the relationship between \( L_2 \)-discrepancy and star discrepancy of two sampling sets \( Y \) and \( Z \). Next, we shall finish this work.

For test set \( R = [0, x) \), we choose \( R_0 = [0, y) \) and \( R_1 = [0, z) \) such that \( y \leq x \leq z \) and \( \lambda(R_1) - \lambda(R_0) \leq \frac{1}{N} \), then \( (R_0, R_1) \) constitute the \( \frac{1}{N} \)-covers. For \( R_0 \) and \( R_1 \), we can divide them into two parts as we did for (4.41) respectively. Let

\[
T_0 = \bigcup_{j \in J_0} (\Omega_j \cap R_0),
\]

and

\[
T_1 = \bigcup_{j \in J_0} (\Omega_j \cap R_1).
\]

From (4.47), we have the same conclusion for \( T_0 \) and \( T_1 \). In order to unify the two cases \( T_0 \) and \( T_1 \) (Because \( T_0 \) and \( T_1 \) are generated from two test sets with the same cardinality, and the cardinality is the covering numbers), we consider a set \( R' \) which can be divided into two parts

\[
R' = \bigcup_{k \in K} \Omega_k \cup \bigcup_{l \in L} (\Omega_l \cap R'),
\]

(4.48)

where \( K, L \) are two index sets. Moreover, we set the cardinality of \( R' \subset [0, 1)^d \) at most \( 2^{d-1} \frac{e^d}{\sqrt{2\pi d}} (N + 1)^d \) (the \( \delta \)-covering numbers, where we choose \( \delta = \frac{1}{N} \)), and we let
\[ T' = \bigcup_{i \in L} (\Omega_i \cap R'). \]

We define new random variables \( \chi_j, 1 \leq j \leq |L|, \) as follows
\[ \chi_j = \begin{cases} 
1, & W_j \in \Omega_j \cap R', \\
0, & \text{otherwise}.
\end{cases} \]

Then,
\[
N \cdot D_N^* (W_1, W_2, \ldots, W_N; R') = N \cdot D_N^* (W_1, W_2, \ldots, W_N; T') \\
= | \sum_{j=1}^{|L|} \chi_j - N(\sum_{j=1}^{|L|} \lambda(\Omega_j \cap T')) |.
\]

Since
\[
P(\chi_j = 1) = \frac{\lambda(\Omega_j \cap T')}{\lambda(\Omega_j)} = N \cdot \lambda(\Omega_j \cap T'),
\]
we get
\[
E(\chi_j) = N \cdot \lambda(\Omega_j \cap T').
\]

Thus, from (4.49) and (4.50), we obtain
\[
N \cdot D_N^* (W_1, W_2, \ldots, W_N; R') = | \sum_{j=1}^{|L|} (\chi_j - E(\chi_j)) |.
\]

Let
\[
\sigma_j^2 = E(\chi_j - E(\chi_j))^2, \Sigma = (\sum_{j=1}^{|L|} \sigma_j^2)^{\frac{1}{2}}.
\]

Therefore, from Lemma 4.1, for every \( R' \), we have,
\[
P \left( \left| \sum_{j=1}^{|L|} (\chi_j - E(\chi_j)) \right| > \lambda \right) \leq 2 \cdot \exp\left( - \frac{\lambda^2}{2\Sigma^2 + \frac{6\lambda}{3}} \right).
\]

Let \( \mathcal{R} = \bigcup_{R'} \left( \left| \sum_{j=1}^{|L|} (\chi_j - E(\chi_j)) \right| > \lambda \right) \), then using \( \delta \)-covering numbers, we have
\[(4.52) \quad \Pr(\mathcal{B}) \leq (2e)^d \cdot \frac{1}{\sqrt{2\pi d}} \cdot (N + 1)^d \cdot \exp(-\frac{\lambda^2}{2\Sigma^2 + \frac{2\lambda}{3}}).\]

Combining with (4.51), we get

\[(4.53) \quad \Pr \left( \bigcup_{R'} (N \cdot D_N^* (W_1, W_2, \ldots, W_N'; R') > \lambda) \right) \leq (2e)^d \cdot \frac{1}{\sqrt{2\pi d}} \cdot (N + 1)^d \cdot \exp(-\frac{\lambda^2}{2\Sigma^2 + \frac{2\lambda}{3}}).\]

For partitions \(\Omega^*_\sim\) and \(\Omega'\), point sets \(Z\) and \(Y\), if we let

\[
\Sigma_0^2 = \text{Var} \left( \sum_{n=1}^{N} 1_{T'}(Z_n) \right), \quad \Sigma_1^2 = \text{Var} \left( \sum_{n=1}^{N} 1_{T'}(Y_n) \right).
\]

Then (4.47) implies

\[\Sigma_0^2 \leq \Sigma_1^2.\]

Besides, as (4.53), we have

\[(4.54) \quad \Pr \left( \bigcup_{R'} (N \cdot D_N^* (Z_1, Z_2, \ldots, Z_N; R') > \lambda) \right) \leq (2e)^d \cdot \frac{1}{\sqrt{2\pi d}} \cdot (N + 1)^d \cdot \exp(-\frac{\lambda^2}{2\Sigma_0^2 + \frac{2\lambda}{3}}),\]

and

\[\Pr \left( \bigcup_{R'} (N \cdot D_N^* (Y_1, Y_2, \ldots, Y_N; R') > \lambda) \right) \leq (2e)^d \cdot \frac{1}{\sqrt{2\pi d}} \cdot (N + 1)^d \cdot \exp(-\frac{\lambda^2}{2\Sigma_1^2 + \frac{2\lambda}{3}}),\]

respectively.

Suppose \(A(d, q, N) = d \ln(2e) + d \ln(N + 1) - \frac{\ln(2\pi d)}{2} - \ln(1 - q),\) and we choose

\[\lambda = \sqrt{2\Sigma_0^2 \cdot A(d, q, N) + \frac{A^2(d, q, N)}{9} + \frac{A(d, q, N)}{3}}\]

in (4.54), then we have
\[ (4.55) \quad \mathbb{P}\left( \bigcup_{R'} (N \cdot D^*_N(Z_1, Z_2, \ldots, Z_N; R') > \lambda) \right) \leq 1 - q. \]

Hence, from (4.55), it can easily be verified

\[ (4.56) \quad \max_{R_{i, t=0,1}} D^*_N(Z_1, Z_2, \ldots, Z_N; R_i) \leq \frac{\sqrt{2\Sigma^2_0} \cdot A(d, q, N) + \frac{A^2(d, q, N)}{9}}{N} + \frac{A(d, q, N)}{3N} \]

holds with probability at least \( q \).

From (2.3), combining with \( \delta \)-covering numbers (where \( \delta = \frac{1}{N} \)), we get,

\[ (4.57) \quad D^*_N(Z) \leq \frac{\sqrt{2\Sigma^2_0} \cdot A(d, q, N) + \frac{A^2(d, q, N)}{9}}{N} + \frac{A(d, q, N) + 3}{3N} \]

holds with probability at least \( q \), the last inequality in (4.57) holds because \( A(d, q, N) \geq 3 \) holds for all \( q \in (0, 1) \).

Same analysis with point set \( Y \), we have

\[ (4.58) \quad D^*_N(Y) \leq \frac{\sqrt{2\Sigma^2_1} \cdot A(d, q, N) + \frac{A^2(d, q, N)}{9}}{N} + \frac{A(d, q, N) + 3}{3N} \]

holds with probability at least \( q \).

Now, we fix a probability value \( q_0 \in (0, 1) \) in (4.57), i.e., we suppose (4.57) holds with probability exactly \( q_0 \), where \( q_0 \in [q, 1) \). Choose this \( q_0 \) in (4.58), we have

\[ D^*_N(Y) \leq (\sqrt{2} \cdot \Sigma_1 + 1) \frac{A(d, q_0, N)}{N}, \]

holds with probability \( q_0 \).

Therefore from \( \Sigma_0 \leq \Sigma_1 \), we obtain,

\[ (4.59) \quad D^*_N(Y) \leq (\sqrt{2} \cdot \Sigma_0 + 1) \frac{A(d, q_0, N)}{N} \]

holds with probability \( q_1 \), where \( 0 < q_1 \leq q_0 \).

We use the following fact to estimate expected star discrepancy.
where $D^*_N(W)$ denotes the star discrepancy of point set $W$.

Plugging $q_0$ into (4.57), we have

\[(4.61)\]

$$D^*_N(Z) \leq (\sqrt{2} \cdot \Sigma_0 + 1) \frac{A(d, q_0, N)}{N}$$

holds with probability $q_0$. Then (4.61) is equivalent to

$$\mathbb{P}(D^*_N(Z) \geq (\sqrt{2} \cdot \Sigma_0 + 1) \frac{A(d, q_0, N)}{N}) = 1 - q_0.$$

Now releasing $q_0$ and let

\[(4.62)\]

$$t = (\sqrt{2} \cdot \Sigma_0 + 1) \frac{A(d, q_0, N)}{N},$$

\[(4.63)\]

$$C_0(\Sigma_0, N) = \frac{\sqrt{2} \cdot \Sigma_0 + 1}{N},$$

and

\[(4.64)\]

$$C_1(d, \Sigma_0, N) = \frac{\sqrt{2} \cdot \Sigma_0 + 1}{N} \cdot (d \ln(2e) + d \ln(N + 1) - \frac{\ln(2\pi d)}{2}).$$

Then

\[(4.65)\]

$$t = C_1(d, \Sigma_0, N) - C_0(\Sigma_0, N) \ln(1 - q_0).$$

Thus from (4.60) and $q_0 \in [q, 1)$, we have

\[(4.66)\]

$$\mathbb{E}[D^*_N(Z)] = \int_0^1 \mathbb{P}(D^*_N(Z) \geq t)dt$$

$$= \int_{1-e^{-C_1(d, \Sigma_0, N) \ln(1 - q_0)}}^{1-e^{-C_0(\Sigma_0, N) \ln(1 - q_0)}} \mathbb{P}(D^*_N(Z) \geq (\sqrt{2} \cdot \Sigma_0 + 1) \frac{A(d, q_0, N)}{N}) \cdot C_0(\Sigma_0, N) \cdot \frac{1}{1 - q_0} dq_0$$

$$= \int_q^{1-e^{-C_1(d, \Sigma_0, N) \ln(1 - q_0)}} C_0(\Sigma_0, N) \cdot \frac{1 - q_0}{1 - q_0} dq_0.$$
\[ P(D_N^*(Y) \geq (\sqrt{2} \cdot \Sigma_0 + 1) \frac{A(d, q_0, N)}{N}) = 1 - q_1. \]

Following the steps from (4.62) to (4.65), we obtain,

\[ \mathbb{E}[D_N^*(Y)] = \int_0^1 P(D_N^*(Y) \geq t)dt = \int_q^{1 - e^{-C_1(d, \Sigma_0, N)/C_0(\Sigma_0, N)}} C_0(\Sigma_0, N) \cdot \frac{1 - q_1}{1 - q_0} dq_0. \]

From \( q_1 \leq q_0 \), we obtain

\[ \frac{1 - q_1}{1 - q_0} \geq \frac{1 - q_0}{1 - q_0}. \]

Hence,

(4.67) \[ \mathbb{E}(D_N^*(Z)) \leq \mathbb{E}(D_N^*(Y)). \]

In the end, let

\[ \Sigma_2^2 = \text{Var}\left(\sum_{n=1}^{N} 1_{T^n}(X_n)\right). \]

From the conclusion of [Theorem 1] in [30], if we choose \( p = 2 \), then we get

(4.68) \[ \mathbb{E}(L_2^2(D_N, Z)) < \mathbb{E}(L_2^2(D_N, X)), \]

and

(4.69) \[ \mathbb{E}(L_2^2(D_N, Y)) < \mathbb{E}(L_2^2(D_N, X)), \]

respectively.

Following the derivation from (4.43) to (4.47), we get

(4.70) \[ \Sigma_0^2 < \Sigma_2^2, \Sigma_1^2 < \Sigma_2^2. \]

Following the derivation (4.54)-(4.67), the proof is completed.
4.3. **Proof of Theorem 3.4.** We only consider the case $\arctan \frac{1}{2} \leq \theta \leq \frac{\pi}{2}$, the calculation of case $0 \leq \theta < \arctan \frac{1}{2}$ is similar to it.

First, we have

$$P_2(\theta) = -\frac{1}{24 \tan \theta} + \frac{1}{120 \tan^2 \theta} + \frac{1}{1440 \tan^3 \theta}.$$ 

Then from Lemma 4.2, we obtain

\[(4.71)\quad \mathbb{E}L_2^2(P_{\Omega^*}) - \mathbb{E}L_2^2(P_{\Omega^*}) = \frac{1}{N^3} \cdot \frac{1}{3^d-2} \cdot P_2(\theta).\]

We set

$$P_{\Omega^*} = \{U_1, U_2, \ldots, U_N\},$$
and

$$P_{\Omega^*} = \{W_1, W_2, \ldots, W_N\},$$

which denote stratified samples under different partition models $\Omega^*$ and $\Omega^*$ respectively.

As \[(4.41)\], after dividing the test set $R$, we set

$$T = \bigcup_{j \in J_0} (\Omega_j \cap R).$$

Therefore,

\[(4.72)\quad \mathbb{E}L_2^2(P_{\Omega^*}) - \mathbb{E}L_2^2(P_{\Omega^*}) = \int_{[0,1]^d} \text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} 1_T(U_n) \right) dx - \int_{[0,1]^d} \text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} 1_T(W_n) \right) dx = \frac{1}{N^2} \cdot \int_{[0,1]^d} \text{Var} \left( \sum_{n=1}^{N} 1_T(U_n) \right) dx - \frac{1}{N^2} \cdot \int_{[0,1]^d} \text{Var} \left( \sum_{n=1}^{N} 1_T(W_n) \right) dx.\]

Hence, we have

\[(4.73)\quad \text{Var} \left( \sum_{n=1}^{N} 1_T(U_n) \right) - \text{Var} \left( \sum_{n=1}^{N} 1_T(W_n) \right) \leq \frac{1}{N} \cdot \frac{1}{3^d-2} \cdot P_2(\theta).\]

Otherwise, combining with \[(4.72)\], a contradiction with equation \[(4.71)\].

For $T = \bigcup_{j \in J_0} (\Omega_j \cap R)$, the same proceeding as \[(4.49)\] and \[(4.51)\], we have
\[ N \cdot D_N^*(W_1, W_2, \ldots, W_N; T) = | \sum_{j \in J_0} [1_T(W_j) - \mathbb{E}(1_T(W_j))] |, \]

where \(1_T(W_j)\) denotes the characteristic function defined on set \(T\) for samples \(W_j, 1 \leq j \leq N\).

Let \(\sigma_j^2 = \text{Var}(1_T(W_j))\), then we have

\[
\sum_{j \in J_0} \sigma_j^2 = \text{Var}\left( \sum_{n=1}^{N} 1_T(W_n) \right) = \sum_{n=1}^{N} \text{Var}(1_T(W_n)) = \sum_{j \in J_0} \sigma_j^2.
\]

Hence, from (2.1), we get

\[
\Sigma^2 \leq d \cdot N^{1 - \frac{1}{d}}.
\]

Combining with (4.73), we obtain

\[
\text{Var}\left( \sum_{n=1}^{N} 1_T(U_n) \right) \leq d \cdot N^{1 - \frac{1}{d}} + \frac{1}{N} \cdot \frac{1}{3^{d-2}} \cdot P_2(\theta).
\]

Therefore, for any test set \(R\), we could find \(R_0, R_1\), which form a \(\delta\)–cover, such that

\[ R_0 \subseteq R \subset R_1, \]

and

\[
\lambda(R_1) - \lambda(R_0) \leq \frac{1}{N}.
\]

Similarly, if we let

\[ T_0 = \bigcup_{j \in J_1} (\Omega_j \cap R_0), T_1 = \bigcup_{j \in J_2} (\Omega_j \cap R_1). \]

Employing the Bernstein inequality, we have

\[
\mathbb{P} \left( \left| N \cdot D_N^*(U_1, U_2, \ldots, U_N; R_0) \right| > \lambda \right) \leq 2 \cdot \exp\left( -\frac{\lambda^2}{2 \text{Var}(\sum_{n=1}^{N} 1_{T_0}(U_n)) + \frac{2\lambda}{3}} \right),
\]

and
\[
\mathbb{P}\left( \left| N \cdot D_N^*(U_1, U_2, \ldots, U_N; R_1) \right| > \lambda \right) \leq 2 \cdot \exp\left(-\frac{\lambda^2}{2\text{Var}(\sum_{n=1}^{N} 1_{T_1}(U_n)) + \frac{2\lambda}{3}}\right).
\]

For \( T_0 \) and \( T_1 \), same derivation process with above, we get

\[
\text{Var}\left( \sum_{n=1}^{N} 1_{T_0}(U_n) \right) \leq d \cdot N^{1-\frac{1}{d}} + \frac{1}{N} \cdot \frac{1}{3d-2} \cdot P_2(\theta),
\]

and

\[
\text{Var}\left( \sum_{n=1}^{N} 1_{T_1}(U_n) \right) \leq d \cdot N^{1-\frac{1}{d}} + \frac{1}{N} \cdot \frac{1}{3d-2} \cdot P_2(\theta).
\]

We unify notations \( R' \) and \( T' \) for all of sets as \( R_0, R_1, T_0, T_1 \), and we let

\[
B = \bigcup_{R'} \left( \left| N \cdot D_N^*(U_1, U_2, \ldots, U_N; R') \right| > \lambda \right).
\]

Then using \( \delta \)–cover numbers, we have

\[
\mathbb{P}(B) \leq (2e)^d \cdot \frac{1}{\sqrt{2\pi d}} \cdot (N + 1)^d \cdot \exp\left(-\frac{\lambda^2}{2\text{Var}(\sum_{n=1}^{N} 1_{T'}(U_n)) + \frac{2\lambda}{3}}\right),
\]

where

\[
\text{Var}\left( \sum_{n=1}^{N} 1_{T'}(U_n) \right) \leq d \cdot N^{1-\frac{1}{d}} + \frac{1}{N} \cdot \frac{1}{3d-2} \cdot P_2(\theta).
\]

Following the derivation process from (4.54) to (4.56), combining with (4.84), we have

\[
\max_{R_{i_1, i_2}, i=0,1} \ D_N^*(U_1, U_2, \ldots, U_N; R_{i_1, i_2}) \leq \sqrt{2d \cdot A(d, q, N) + \frac{2P_2(\theta) \cdot A(d, q, N)}{3d - 2} N^{2-\frac{1}{2}}} + \frac{2A(d, q, N)}{3N}.
\]

Thus, from (2.3), we obtain
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\[ (4.86) \quad D_N^*(U) \leq \frac{\sqrt{2d} \cdot A(d, q, N) + \frac{2p_2(\theta) \cdot A(d, q, N)}{3^d - 2 \cdot N^2 - \frac{1}{3}}}{N^{\frac{1}{2} + \frac{1}{2d}}} + \frac{2A(d, q, N)}{3N} + \frac{1}{N} \]

holds with probability at least \( q \).

According to (4.63) and (4.66), it follows that

\[ (4.87) \quad \mathbb{E}[D_N^*(U)] \leq \frac{\sqrt{2} \cdot \sqrt{\text{Var}(\sum_{n=1}^{N} 1_{T^*(U_n)})} + 1}{N}. \]

Plugging (4.84) into (4.87), we obtain

\[ (4.88) \quad \mathbb{E}[D_N^*(U)] \leq \frac{\sqrt{2d} + \frac{2p_2(\theta)}{3^d - 2 \cdot N^2 - \frac{1}{3}} + 1}{N^{\frac{1}{2} + \frac{1}{2d}}}, \]

noting that in (4.88), we use the fact

\[ \frac{1}{N} \leq \frac{1}{N^{\frac{1}{2} + \frac{1}{2d}}}, \]

holds for all \( d, N \in \mathbb{N} \) to simplify it.

The conclusion of the case \( 0 \leq \theta < \arctan \frac{1}{2} \) can be added as long as we employ the result from (4.24) to (4.28), we have thus proved the theorem.

5. Conclusion

We study optimal expected star discrepancy under a class of newly designed convex equivolume partitions. First, the expected star discrepancy under different partition models are compared. Second, the expected star discrepancy upper bound under the new partition models is obtained. Third, an optimal partition model that minimizes expected star discrepancy is designed and an optimal expected star discrepancy upper bound is given explicitly. In next paper, a similar technique can be used to prove the optimal probabilistic star discrepancy under a class of convex equal volume partitions, which will have more corresponding applications.

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