Robustness of unconventional $s$-wave superconducting states against disorder

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We investigate the robustness against disorder of superconductivity in multiband systems where the fermions have four internal degrees of freedom. This permits unconventional $s$-wave pairing states, which may transform nontrivially under crystal symmetries. Using the self-consistent Born approximation, we develop a general theory for the effect of impurities on the critical temperature, and find that the presence of these novel $s$-wave channels significantly modifies the conclusions of single-band theories. We apply our theory to two candidate topological superconductors, YPtBi and Cu$_4$Bi$_2$Se$_3$, and show that the novel $s$-wave states display an enhanced resilience against disorder, which extends to momentum-dependent pairing states with the same crystal symmetry. The robustness of the $s$-wave states can be quantified in terms of their superconducting fitness, which can be readily evaluated for model systems.

I. INTRODUCTION

It is a textbook result that the critical temperature $T_c$ of a conventional $s$-wave spin-singlet superconductor is insensitive to nonmagnetic disorder [1]. This is a consequence of Anderson’s theorem [2]: since this state has an isotropic gap and pairs electrons in time-reversed partner states, there is no depairing effect from the time-reversal-invariant scattering off the impurities. On the other hand, the sign-reversing gaps of unconventional superconductors are averaged to zero by the impurity-scattering across the Fermi surface, and these pairing states are suppressed by weak disorder with normal state elastic scattering rate (SR) $\tau^{-1} \sim k_BT_c$.

Recently there has been much interest in $s$-wave pairing states which do not pair time-reversed partner states [3-10]. This can occur in systems where the electrons have additional discrete degrees of freedom, such as orbital or sublattice indices. These permit novel ways to satisfy the fermionic antisymmetry of the Cooper pair wavefunction in a relative $s$-wave, e.g. a spin-triplet orbital-singlet state. Such pairing states typically belong to a nontrivial irreducible representation (irrep) of the point group. They have been proposed in a variety of materials [3,8], but here we focus on Cu$_4$Bi$_2$Se$_3$ [9] and YPtBi [10]. Experiments indicate a fully-gapped nematic superconducting state in Cu$_4$Bi$_2$Se$_3$ [11,13], which naturally arises from a time-reversal-invariant combination of the odd-parity $s$-wave states in the $E_u$ irrep [13]. In YPtBi there is evidence of a nodal superconducting gap [15], which could be explained by a time-reversal symmetry-breaking combination of even-parity $s$-wave states which support exotic Bogoliubov Fermi surfaces [10,16].

Since the novel $s$-wave states do not pair time-reversed partners, Anderson’s theorem does not apply and we may expect them to be highly sensitive to disorder. Indeed, expressed in a pseudospin band basis, the novel $s$-wave states show a sign-changing gap, which averages to zero across the Fermi surface [10,17]. However, since the impurity potential in the pseudospin band basis may be anisotropic, the impurity-averaged gap may not vanish, which can lead to unconventional impurity effects [18,19]. This anisotropy naturally appears when the states at the Fermi surface have a strong spin-orbital texture. Indeed, it was shown in Ref. [20] that the spin-orbital texture of the electronic states at the Fermi surface in Cu$_4$Bi$_2$Se$_3$ generates such an anisotropy, granting the novel $s$-wave $A_{1u}$ state enhanced robustness against disorder. It is nevertheless unclear if a general principle underlies this result, or if it applies to other pairing states.

In this paper we use the self-consistent Born approximation to study the effect of disorder on the critical temperature of a superconducting state in a system where the fermions have four degrees of freedom. In Sec. [11] we develop a general framework which explicitly keeps track of these degrees of freedom, naturally generalizing the familiar results of single-band theories with disorder [1]. Our computationally-straightforward approach generalizes and extends earlier works [20,22], and can be readily applied to new materials. As concrete examples, in Secs. [13] and [14] we apply our theory to YPtBi and Cu$_4$Bi$_2$Se$_3$, respectively. We find that nontrivial $s$-wave states proposed for these systems show a parametrically-enhanced robustness against disorder, which is shared with other states in the same irrep according to their similarity to the $s$-wave states at the Fermi surface. In the discussion of Sec. [15] we show that the robustness of the $s$-wave states is quantified in terms of the superconducting fitness [23,24], which can be readily evaluated for model Hamiltonians. Although robust unconventional states are generally possible, systems with nontrivial inversion operator are particularly favourable.
II. GENERAL THEORY

Our starting point is a generic model of a fermionic system with four internal degrees of freedom that is invariant under time reversal and inversion. The normal-state Hamiltonian is $H = \sum_k \epsilon_k \hat{c}^\dagger_k \hat{c}_k$, where $\epsilon_k$ is a four-component spinor encoding the internal degrees of freedom, and the matrix $\hat{H}_k$ has the general form [25].

$$\hat{H}_k = \epsilon_{k,0} \mathbb{1}_4 + \vec{\epsilon}_k \cdot \vec{\gamma},$$

where $\mathbb{1}_4$ is the $4 \times 4$ unit matrix and $\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$ is the vector of the five mutually-anticommuting Euclidean Dirac matrices. The real functions $\epsilon_{k,0}$ and $\vec{\epsilon}_k = (\epsilon_{k,1}, \epsilon_{k,2}, \epsilon_{k,3}, \epsilon_{k,4}, \epsilon_{k,5})$ are the coefficients of these matrices. The Hamiltonian in Eq. 1 has the doubly degenerate eigenvalues $E_{k,\pm} = \epsilon_{k,0} \pm |\vec{\epsilon}_k|$. The internal degrees of freedom can either transform trivially ($I = \mathbb{1}_4$) or nontrivially ($I = \gamma^1$) under inversion. The time-reversal operator is $\mathcal{T} = U \tau K$, where $K$ is complex conjugation and the unitary part can be expressed in terms of the Euclidean Dirac matrices without loss of generality as $U \tau = \gamma^3 \gamma^5$.

The pairing potential for a general superconducting state is $\Delta_k = \Delta_0 \Delta_k$, where $\Delta_0$ is the magnitude and

$$\Delta_k = f_k \gamma^0 \gamma^3 U_T. \tag{2}$$

Here $f_k$ is a normalized form factor, chosen such that fermionic antisymmetry $\Delta_k = -\Delta^\dagger_{-k}$ is satisfied. Because the pairing potential $\hat{\Delta}_k$ is a $4 \times 4$ matrix, there are six terms in Eq. 2 for which an s-wave form factor (i.e. $f_k = 1$) is permitted by fermionic antisymmetry. This is always possible for $\alpha = \beta = 0$ (where $\gamma^0 = \mathbb{1}_4$), which describes pairing between electrons in time-reversed partner states, and hence generalizes the usual s-wave spin-singlet state. The five other channels where an s-wave form-factor is allowed have a nontrivial dependence on the internal degrees of freedom, where $\alpha$ and $\beta$ in Eq. 2 are different and not both zero. These additional s-wave channels typically belong to nontrivial irreps.

The nontrivial s-wave channels do not generally pair electrons in time-reversed partner states, and hence typically involve both intraband and interband pairing. To quantify the degree of interband pairing for a pairing state $\Delta_k$ at wavevector $k$, Ref. 23 introduced the quantity $F_C(k) = \frac{1}{4} \text{Tr} \{ [\hat{H}_k \Delta_k - \Delta^\dagger_k \hat{H}_k^T] \}$, where $\Delta_k \hat{H}_k \Delta_k - \Delta^\dagger_k \hat{H}_k^T$ is referred to as the "superconducting fitness" [23] and is vanishing if there is no interband pairing. The superconducting fitness also controls the form of the superconducting gap in the low-energy spectrum. Specifically, the s-wave states (i.e. $\Delta_k = \Delta_0 \Delta$) open a gap of magnitude [26]

$$\Delta_0 \sqrt{1 - F_C(k)}, \tag{3}$$

where $\tilde{F}_C(k) = 4 F_C(k)/|\vec{\epsilon}_k|^2 \text{Tr}[\hat{\Delta} \hat{\Delta}^\dagger]$ is normalized such that $F_C(k) \leq 1$. If $F_C(k) = 1$, there is no intraband pairing, and so the s-wave states must necessarily display a gap node. Since the spin-singlet analogue state is perfectly fit (i.e. $F_C(k) = 0$), it hence opens a full gap and there is no interband pairing, as anticipated by the fact that it pairs time-reversed partners. In contrast, the nontrivial matrix structure of the anomalous s-wave states typically results in a nonzero fitness and possibly the formation of nodes.

We consider isotropic scattering off potential impurities distributed randomly at positions $r_j$, described by the Hamiltonian

$$H_{\text{imp}} = V \sum_{j,k,k'} \epsilon^{(k'-k)} \cdot r_j \hat{c}^\dagger_k \hat{c}_{k'} \tag{4}$$

where $V$ is the impurity potential and $\Omega$ is the volume. We restrict ourselves here to the use of a scattering potential that is isotropic in the spin and orbital indices, as is the standard approach for nonmagnetic impurities [1, 20, 21]. Although more complicated impurity potentials are possible in systems with orbital degrees of freedom [2, 27], our intention here is to understand the relationship between the spin-orbital texture of the normal-state bands and the robustness of the superconducting state to disorder. To this end, we focus on the simplest possible scattering potential in the spin-orbital basis. This simplification does not imply that intra- and interband scattering processes are equivalent, however, as such processes depend on matrix elements introduced by the transformation to the band basis. Within the self-consistent Born approximation, the Green’s functions of the disordered system are

$$\tilde{G}(k, i\omega_n) = \sum_{j,m} \frac{1}{i\omega_{nj} - E_{k,j}} \mathcal{P}_{k,j} \tag{5}$$

where $\mathcal{P}_{k,j} = \frac{1}{2} (\mathbb{1}_4 \pm \vec{\epsilon}_k \cdot \vec{\gamma})$ projects into the $\pm$ band at momentum $k$ and $\vec{\epsilon}_k = \vec{\epsilon}_k / |\vec{\epsilon}_k|$. The effect of impurities is accounted for in the renormalized Matsubara frequencies $\omega_{nj} = \omega_n + (2\tau_{k,j})^{-1} \text{sgn}(\omega_n)$, where the SR in band $j$ is

$$\frac{1}{\tau_{k,j}} = \pi n_{\text{imp}} V^2 \sum_{m=\pm} N_{m}(1 + jm \vec{\epsilon}_k \cdot \langle \vec{\epsilon}_k \rangle_{\text{FS},m}). \tag{6}$$

Here $n_{\text{imp}}$ is the concentration of impurities, $N_{m}$ is the density of states of band $m = \pm$ at the Fermi surface, and $\langle ... \rangle_{\text{FS},m}$ denotes the average over the Fermi surface of this band. The second term in the parentheses of Eq. 6 is an additional contribution to the scattering rate which arises from a net average polarization in the internal degrees of freedom on the (single band) Fermi surface. In the following we will assume a weak momentum-dependence of the SR and replace $\tau_{k,j}^{-1}$ by its Fermi surface average in Eq. 6.
The critical temperature in the presence of disorder can be determined from the lowest-order terms of the Ginzburg-Landau free energy in the Born approximation, expanded in powers of the gap $\Delta^*$. Cancelling an overall factor of the gap magnitude $\Delta_0$ gives an expression for the linearized gap equation

$$\frac{1}{g_\nu} = \frac{1}{2\beta} \sum_{i\omega_n} \int \frac{d^3 k}{(2\pi)^3} \text{Tr} \{ \Sigma_{\Delta_k} G(k, i\omega_n) (\Delta_k + \Delta_0 \Sigma_2) G_k(k, i\omega_n) \}$$

Figure 1. Diagrammatic form of the linearized gap equation, taking Cooperon ladder diagrams into account. The dotted line represents the interaction with the impurity, denoted by the star, and the double line is the Green’s function dressed by interactions with the impurity via the normal self-energy.

$$\Sigma_2 = -n_{\text{imp}} V^2 \int \frac{d^3 k}{(2\pi)^3} \bar{G}_k(k, i\omega_n) (\Delta_k + \Sigma_2) G_k(k, i\omega_n) \quad (9)$$

The anomalous self-energy vanishes unless the lowest-order contribution is nonzero:

$$\Sigma_2^{(0)} = -n_{\text{imp}} V^2 \int \frac{d^3 k}{(2\pi)^3} \bar{G}_k(k, i\omega_n) \Delta_k G_k(k, i\omega_n)$$

$$= \pi n_{\text{imp}} V^2 \sum_{j=\pm} \frac{N_j}{|\omega_{n,j}|} \langle P_{k,j} \bar{\Delta}_k P_{-k,j} \rangle_{\text{FS,j}} \quad (10)$$

where, in the final line, we have made the assumption that the bands are well separated and interband contributions to the self-energy are therefore small and can be neglected. Equation Eq. (10) is the central result of our analysis. Because of the nontrivial form of the projection operators, the Fermi-surface average will not necessarily vanish for an unconventional state. Although our theory has been developed for a two-band model, this result readily generalizes to an arbitrary number of bands. For the two-band system considered here, explicitly evaluating Eq. (10) for the general pairing state Eq. (2) yields nontrivial form factors. Moreover, we observe that since $\Sigma_2^{(0)}$ (and hence $\Sigma_2$) is independent of momentum, it must belong to one of the $s$-wave channels, and can thus

Due to the Fermi surface averages of the form-factor $f_k$ with the coefficients of the $\gamma$ matrices in the Hamiltonian Eq. [1], the self-energy may be nonzero even for
be nonzero for any state in the same irrep. This modifies the solution of Eq. (5), such that these states acquire some protection against the disorder. This represents the crucial difference to the single-band case, where the trivial form of the projection operators implies that the anomalous self-energy vanishes for any state with a sign-changing gap, and the critical temperature of these states is suppressed in a universal fashion [1].

Although our theory applies to a general two-band system, the analysis of systems with multiple Fermi surfaces is complicated. To more clearly reveal the universal physics due to the spin-orbital texture, therefore, in the following we study two examples of the simpler case where only one of the bands intersects the Fermi energy.

### III. APPLICATION TO YPtBi

YPtBi is a zero-band-gap semimetal, where the states close to the Fermi energy belong to the $T_5$ band. Ignoring a weak antisymmetric spin-orbit coupling due to the broken inversion symmetry [10], this is described by the Luttinger-Kohn model for the $j = \frac{3}{2}$ states in a cubic material

$$H = (\alpha|\mathbf{k}|^2 - \mu)(\mathbb{1} + \beta_1 \sum_i k_i^2 J_i^2 + \beta_2 \sum_{i \neq i'} k_i k_i' J_i J_i' ),$$

where $i$ and $i'$ enumerate the Cartesian coordinates. The $j = \frac{3}{2}$ internal angular momentum of the electrons constitutes the four degrees of freedom in our general model, and the $\gamma$ matrices in Eq. (11) can be parameterized as $\hat{\gamma} = \{J_1^2 - J_3^2, \frac{1}{2}(J_2^2 - J_3^2 - J_1^2), \frac{1}{4}(J_y, J_z), \frac{1}{\sqrt{3}}\{J_x, J_y\}, \frac{1}{\sqrt{6}}\{J_z, J_x\}\}$ with $\hat{\gamma}_k = (\sqrt{3} \beta_1 (k_x^2 - k_y^2) + \beta_2 (3k_z^2 - |\mathbf{k}|^2)/2, \sqrt{3} \beta_2 k_y k_z, \sqrt{3} \beta_2 k_z k_y, \sqrt{3} \beta_2 k_z k_z)$. The $j = \frac{3}{2}$ index transforms trivially under inversion. Experiments show hole-like carriers in YPtBi, and so we set the chemical potential to lie in the lower band.

The six $s$-wave pairing states in YPtBi are tabulated in Table I. Apart from the $A_{1g}$ singlet state, there are also five quintet states which pair electrons with total internal angular momentum $J = 2$, and which belong to the $E_g$ and $T_{2g}$ irreps. Evaluating Eq. (11), we find that the lowest-order contribution to the anomalous self-energy for the $s$-wave gaps in YPtBi is

$$\Sigma_2^{(0)} = \pi n_{\text{imp}} V^2 \frac{N}{4|\omega_n|} \left[1 + \sum_{l=1}^5 \lambda_l (\hat{\gamma}_k^2)_{\text{FS}}\right] \Delta,$$

where $\lambda_l = \pm 1$ is tabulated for each channel in Table I. Solving Eq. (6), we obtain the full self-energy

$$\Sigma_2 = \frac{\Sigma_2^{(0)}}{1 - \Sigma_2^{(0)}},$$

where $\Sigma_2^{(0)} = \Sigma_2^{(0)} \Delta$. Inserting this into the linearized gap equation, we find that the critical temperature $T_c$ of the $s$-wave state in channel $\nu$ is given by the solution of

$$\log \left( \frac{T_c}{T_{c0}} \right) = \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} + \frac{1}{4\pi k_B T_{c0}} \right)$$

where $T_{c0}$ is the critical temperature in the absence of disorder, $\psi(z)$ is the digamma function, and the effective SR is

$$\frac{1}{\tau_\nu} = \frac{1}{2\tau_0} \left(1 - \sum_{l=1}^5 \lambda_l (\hat{\gamma}_k^2)_{\text{FS}}\right)$$

with $\tau_0^{-1} = 2\pi n_{\text{imp}} V^2 N$. We see that $\lambda_l = +1$ decreases the effective SR, whereas $\lambda_l = -1$ brings it closer to the normal-state value $\tau = \tau_0$. Since all $\lambda_l = 1$ for the $A_{1g}$ $s$-wave state, we find that $\tau_{A_{1g}}^{-1} = 0$ and it is hence insensitive to disorder, consistent with Anderson’s theorem. The effective SR of the other $s$-wave states are reduced relative to the normal state value, and in each case there is one $l$ for which $\lambda_l = +1$. This gives a modest degree of protection against disorder, as shown in Fig. 3.

The enhanced stability of the nontrivial $s$-wave states extends to other pairing potentials: the critical temperature for an arbitrary state $\Delta_k$ satisfies

$$\log \left( \frac{T_c}{T_{c0}} \right) = \psi \left( \frac{1}{2} \right) - (1 - \alpha_\nu(\Delta_k)) \psi \left( \frac{1}{2} + \frac{1}{4\pi k_B T_{c0}} \right)$$

Table I. The six $s$-wave pairing states for YPtBi. The first line gives the irrep of $O_h$, the second line gives the form of the pairing potential in terms of the $\gamma$ matrices defined in the text, the third line gives the nodal structure, and the fourth and fifth lines give the values of $l$ corresponding to the $\gamma$ matrices for which $\lambda_l = 1$ and $\lambda_l = -1$, respectively.

- $A_{1g}$
- $E_g$
- $T_{2g}$
- $\Delta U_\nu$
- $\nu$
- $\lambda_l$
- $l$
- $l$
- $T_{c0}$

| $\Delta U_\nu$ | $\nu$ | $\lambda_l$ | $l$ | $l$
|-------------|------|------------|-----|-----|
| $E_g$       | $\gamma^1$ | $\lambda_l = 1$ | 1 | 2 |
| $E_g$ | $\gamma^1$ | $\lambda_l = -1$ | 2,3,4,5 | 1,3,4,5 |
| $A_{1g}$ | $\gamma$ | $\lambda_l = 1$ | 3 | 4 |
| $A_{1g}$ | $\gamma$ | $\lambda_l = -1$ | 5 | 6 |
| $T_{2g}$ | $\gamma^4$ | $\lambda_l = 1$ | 1,2,3,4,5 | 1,2,3,4,5 |
| $T_{2g}$ | $\gamma^5$ | $\lambda_l = -1$ | 1,2,3,4,5 | 1,2,3,4,5 |
\[-\alpha_v(\Delta k)\psi \left( \frac{1}{2} + \frac{1}{4\pi k_B T_c^\nu} \right), \tag{17}\]

where

\[\alpha_v(\Delta k) = \frac{\langle \text{Tr}[\Delta_k^\nu P_k \Delta_k^\nu P_k]^2 \rangle_{FS}}{\langle \text{Tr}[\Delta_k^\nu P_k \Delta_k^\nu P_k] \rangle_{FS} \langle \text{Tr}[\Delta_k^\nu P_k \Delta_k^\nu P_k] \rangle_{FS}}. \tag{18}\]

This parameter measures the similarity of \(\Delta_k\) to the s-wave state \(\Delta_s\) at the Fermi surface. The closer \(\alpha_v\) is to one, the more similar these states are to one another, and hence their response to disorder is also similar. In this way, a general state in an irrep with a nontrivial s-wave pairing potential can also acquire some robustness against disorder. Indeed, as shown in Fig. 3, the singlet d-wave \(E_g\) state \(\Delta_k = (k_x^2 - k_y^2)U_T\) is almost as stable as these states as the quintet s-wave \(E_g\) states, reflecting the nearly-identical form of these states at the Fermi surface. It is instructive to examine the lowest-order contribution to the anomalous self-energy for this state. In particular, the second term inside the brackets of Eq. (11) gives the overlap with the s-wave \(E_g\) state \(\gamma^1 U_T\):

\[\Sigma_2^{(2)} = -\pi n_imp V^2 \left[ \frac{N}{2|\omega_n|} \right] \langle \epsilon_{k,1}(k_x^2 - k_y^2) \rangle_{FS} \gamma^1 U_T, \tag{19}\]

This is nonzero since \(\epsilon_{k,1} = \sqrt{3}\beta_1(k_x^2 - k_y^2)/2\). The full anomalous self-energy will have the same form as Eq. (14), where \(\Sigma_2^{(0)}\) is the coefficient of \(\gamma^1 U_T\) in the expression above.

### IV. APPLICATION TO Cu₄Bi₂Se₃

The low-energy electron states in Cu₄Bi₂Se₃ derive from \(p_z\)-like orbitals which are located on opposite sides of each Bi₂Se₃ quintuple layer, implying a sublattice degree of freedom. The \(\mathbf{k} \cdot \mathbf{p}\) Hamiltonian for these states to lowest order in \(\mathbf{k}\) for each term is given by [20]

\[H = -\mu \sigma_0 \otimes \eta_0 + \sigma_0 \eta_0 + \nu k_x \sigma_0 \eta_y + \nu (k_x \sigma_y - k_y \sigma_x) \otimes \eta_2 + \lambda k_x (k_x^2 - 3k_y^2) \sigma_z \otimes \eta_2, \tag{20}\]

where \(\sigma_\nu\) and \(\eta_\nu\) are the Pauli matrices in spin and sublattice space, respectively. We choose the \(\gamma\) matrices to be \(\gamma = (\sigma_0 \otimes \eta_0, \sigma_0 \otimes \eta_y, \sigma_x \otimes \eta_2, \sigma_y \otimes \eta_2, \sigma_z \otimes \eta_2, \sigma_z \otimes \eta_2)\). The copper intercalation in Cu₄Bi₂Se₃ dopes electrons into the system, giving a Fermi surface in the upper band.

The six s-wave pairing channels in Cu₄Bi₂Se₃ are summarized in Table I. In addition to two \(A_{1g}\) states, there are four odd-parity states, which are permitted due to the swapping of the sublattice index under inversion. The \(A_{1g}\) states are insensitive to disorder [20], although the analysis is more involved than for YPtBi since the anomalous self-energy includes components from both pairing potentials. The critical temperatures of the odd-parity channel \(\nu\) is the solution of Eq. (13) where the effective SR is

\[\frac{1}{\tau_\nu} = \frac{1}{4\tau_0} \left( 1 + \frac{1}{4\tau_0} \right)^2 \sum_{l=1}^{5} \lambda_l \langle \epsilon_{k,l}^2 \rangle_{FS}, \tag{21}\]

and the \(\lambda_l\) are tabulated in Table I. The second term in the brackets is due to the nonzero sublattice polarization of the normal-state bands arising from the mass term. Our result is consistent with the analysis for the \(A_{1u}\) channel in [20]. We plot the critical temperature as a function of disorder strength for each channel in Fig. 4. Other odd-parity states in Cu₄Bi₂Se₃ also enjoy some degree of protection against disorder. In particular, a nontrivial dependence on the sublattice degrees of freedom is not required. For example, consider the two p-wave spin-triplet sublattice-trivial pairing states in \(A_{1u}\):

\[A_{1u}^{(2)}: \quad \Delta_k = \hat{k}_x \sigma_x \otimes \eta_0 \tag{22}\]

\[A_{1u}^{(6)}: \quad \Delta_k = -\hat{k}_x \sigma_x \otimes \eta_0 + \hat{k}_y \sigma_0 \otimes \eta_0 \tag{23}\]

As shown in Fig. 4 the robustness of these p-wave states is comparable to the \(A_{2u}\) and \(E_u\) s-wave states, because of their overlap with the significantly more stable \(A_{1u}\) s-wave state.

### V. DISCUSSION

Our analysis reveals a remarkable robustness of the nontrivial s-wave pairing states against disorder, which is manifested by an effective SR which can be greatly reduced from the normal-state SR. The s-wave states play a crucial role, as their robustness can be shared with, but not exceeded by, any other state in the same irrep.

Since they do not exclusively pair time-reversed partners, the nontrivial s-wave states may involve both pairing of electrons in the same (intraband pairing) and different (interband pairing) bands. Remarkably, the effective SR of the s-wave state in channel \(\nu\) can be expressed in terms of the Fermi surface average of the superconducting fitness function \(\hat{F}_C(k)\), which measures the degree of interband pairing:

\[\frac{1}{\tau_\nu} = \frac{1}{\tau} - \frac{1}{\tau_0} \left( 1 - \hat{F}_C \right), \tag{24}\]
Here $F_C = \langle \tilde{F}_C(k) \rangle_{FS} = 1 (0)$ implies completely interband (intraband) pairing across the Fermi surface. The effective SR is reduced, and hence the robustness against disorder is enhanced, according to the degree that the $s$-wave state involves intraband pairing. This result follows from the observation that $\lambda_1 = +1 (-1)$ when $\gamma^I \Delta_1 - \Delta_2 \gamma_1^L \tau = 0 \left(2 \gamma^I \Delta_2 \right)$. We emphasize that Eq. (24) only applies to the s-wave states: for other states, the value of $F_C$ does not supply any information about the robustness against disorder.

The extreme limit where a nontrivial $s$-wave potential $\tilde{\Delta}$ is perfectly fit (i.e. $F_C = 0$) is instructive. As shown in 9, the Bogoliubov-de Gennes Hamiltonian can then be mapped to that for the trivial $s$-wave state using $c_k \rightarrow \exp(i \gamma^I \Delta) c_k$. This global transformation leaves the impurity Hamiltonian Eq. (4) invariant, and so the nontrivial $s$-wave state is insensitive to nonmagnetic disorder, giving a generalization of Anderson’s theorem 22. The Hamiltonian will generally contain terms which violate the fitness condition (i.e. $F_C > 0$), however, which spoils this correspondence. Nevertheless, the nontrivial $s$-wave state will retain some robustness against disorder.

This effect is very sensitive to the material parameters. For example, it is known that the robustness of the odd-parity $s$-wave states in Cu$_2$Bi$_2$Se$_3$ is enhanced by reducing the mass term $m$ in Eq. (20) 20. This is immediately evident in our framework, where the effective scattering rate is always enhanced by a finite mass. For the odd-parity $s$-wave gaps, $\tilde{\Delta} = i \gamma^I \gamma^j \tau$, the enhancement is $\tau_0 / \tau_0 = \tilde{m}^2 / \langle \Delta^2 \rangle_{FS}$, and the $A_{1u}$ state is the most stable as $\langle \Delta^2 \rangle_{FS}$ is the smallest component of the Hamiltonian. This is a direct consequence of the fact that the mass term in the Hamiltonian is proportional to the nontrivial inversion symmetry operator $\mathcal{I} = \gamma^I$, and thus the odd-parity gaps must by definition have $\lambda_1 = -1$.

Equation (24) gives a simple diagnostic for the existence of a highly-robust nontrivial irrep in a general system: there must be an $s$-wave state in this irrep such that $F_C \ll 1$. A nontrivial inversion operator is highly desirable: in this case, the odd-parity $s$-wave states involve the product of two $\gamma$ matrices (one of which is the inversion operator), and hence commute with three $\gamma$ matrices in the general Hamiltonian Eq. (1). In contrast, the even-parity $s$-wave states commute with only one $\gamma$ matrix in the Hamiltonian when inversion is trivial. Assuming roughly equal values of all the coefficients $c_{k,l}$ at the Fermi surface, $F_C$ will typically be smaller for the $s$-wave states in the system with nontrivial inversion. This is exemplified by the greater robustness of the $s$-wave states in Cu$_2$Bi$_2$Se$_3$ compared to YPtBi.

The analysis presented above has focused entirely on understanding the role of the spin-orbital texture of the normal-state bands. The impurity physics of superconductors is a rich field 31, and although the self-consistent Born approximation utilized here can successfully account for the pair-breaking physics in the dilute impurity limit, effects beyond this approximation can be important. For example, it has recently been shown that the enhancement of the local density of states due to the presence of resonant levels at the impurity sites can increase the critical temperature in unconventional multiorbital superconductors above the clean-limit result 32. We nevertheless expect our results to remain qualitatively valid for more sophisticated treatments, as the spin-orbital texture and the superconducting fitness are properties of the clean-limit Bogoliubov-de Gennes equations. Indeed, the role of the mass term in controlling the robustness against disorder in Cu$_2$Bi$_2$Se$_3$ has been numerically confirmed using a self-consistent $T$-matrix theory 21. Extending our theory beyond the self-consistent Born approximation is a promising direction for future work.

During final preparation of our manuscript we became aware of a similar analysis in Ref. 25. However, our results for the effective SR disagree: whereas we find that this involves the superconducting fitness with respect to the normal-state Hamiltonian, in Ref. 25 the superconducting fitness with respect to the impurity Hamiltonian appears. This gives a complete insensitivity of the pairing state to disorder, in contrast to the parametric enhancement of the robustness found here, and disagrees with previous studies 20 22.

**VI. CONCLUSIONS**

In this manuscript we have shown that unconventional superconducting states in multiband systems are generally less sensitive to the presence of nonmagnetic disorder than unconventional states in single-band materials, due to the spin-orbital texture of the normal-state bands. The enhanced stability occurs for pairing states in irreps for which there is a nontrivial $s$-wave state. The degree to which an $s$-wave state is robust against disorder can be quantified in terms of the Fermi surface average of the
superconducting fitness parameter, and provides an upper bound for the stability of all other states in the same irrep. Our theory offers a straightforward way to assess the robustness against disorder of unconventional pairing states for any multiband system, and can thus guide the search for novel superconducting states.

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