A Characterization Theorem for the $L^2$-Discrepancy of Integer Points in Dilated Polygons

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Abstract Let $C$ be a convex $d$-dimensional body. If $\rho$ is a large positive number, then the dilated body $\rho C$ contains $\rho^d |C| + O(\rho^{d-1})$ integer points, where $|C|$ denotes the volume of $C$. The above error estimate $O(\rho^{d-1})$ can be improved in several cases. We are interested in the $L^2$-discrepancy $D_C(\rho)$ of a copy of $\rho C$ thrown at random in $\mathbb{R}^d$. More precisely, we consider

$$D_C(\rho) := \left\{ \int_{T^d} \int_{SO(d)} \left| \text{card} \left( (\rho \sigma(C) + t) \cap \mathbb{Z}^d \right) - \rho^d |C| \right|^2 \, d\sigma \, dt \right\}^{1/2},$$

where $T^d = \mathbb{R}^d / \mathbb{Z}^d$ is the $d$-dimensional flat torus and $SO(d)$ is the special orthogonal group of real orthogonal matrices of determinant 1. An argument of Kendall shows that $D_C(\rho) \leq c \rho^{(d-1)/2}$. If $C$ also satisfies the reverse inequality $D_C(\rho) \geq c_1 \rho^{(d-1)/2}$, we say that $C$ is $L^2$-regular. Parnovski and Sobolev proved that, if $d > 1$, a $d$-dimensional unit ball is $L^2$-regular if and only if $d \not\equiv 1 \pmod{4}$. In this paper we characterize the $L^2$-regular convex polygons. More precisely, we prove that a convex
polygon is not \( L^2 \)-regular if and only if it can be inscribed in a circle and it is symmetric about the centre.

**Keywords** Discrepancy · Integer points in polygons · Fourier analysis

**Mathematics Subject Classification** Primary 11K38 · Secondary 11P21

1 Introduction

We identify the \( d \)-dimensional flat torus \( T^d = \mathbb{R}^d / \mathbb{Z}^d \) with the unit cube \([-\frac{1}{2}, \frac{1}{2})^d \) and we recall that a sequence \( \{t_j\}_{j=1}^{+\infty} \subset T^d \) is uniformly distributed if one of the following three equivalent conditions is satisfied: (i) for every \( d \)-dimensional box \( I \subset [-\frac{1}{2}, \frac{1}{2})^d \) with volume \( |I| \),

\[
\lim_{N \to +\infty} \frac{1}{N} \text{card}\{t_j \in I : 1 \leq j \leq N\} = |I|;
\]

(ii) for every continuous function \( f \) on \( T^d \),

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} f(t_j) = \int_{T^d} f(t) \, dt;
\]

and (iii) for every \( 0 \neq k \in \mathbb{Z}^d \),

\[
\lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i k \cdot t_j} = 0,
\]

where “\( \cdot \)” denotes the \( d \)-dimensional inner product.

The concept of uniform distribution and the defining properties given above go back to a fundamental paper written one hundred years ago by Weyl [35]; see [25] for the basic reference on uniformly distributed sequences. Observe that the above definition does not show the quality of a uniformly distributed sequence. In the late thirties J. van der Corput coined the term discrepancy: let \( \mathcal{D}_N := \{t_j\}_{j=1}^{N} \) be a sequence of \( N \) points in \( T^d \), henceforth called a distribution (of \( N \) points), and let

\[
D(\mathcal{D}_N) := \sup_{I \subset T^d} |\text{card}(\mathcal{D}_N \cap I) - N| |I||
\]

be the (non normalized) discrepancy associated with \( \mathcal{D}_N \) with respect to the \( d \)-dimensional boxes \( I \) in \( T^d \). There are different approaches to define a discrepancy that measures the quality of a distribution of points; see e.g. [3,11,18,19,25,26] for an introduction of discrepancy theory. See e.g. [10,14–16] for the connections of discrepancy to energy and numerical integration.
Throughout this paper we shall denote by $c, c_1, \ldots$ positive constants that may change from step to step.

Roth [31] proved the following lower estimate: for every distribution $\mathcal{D}_N$ of $N$ points in $\mathbb{T}^2$, we have

$$\int_{\mathbb{T}^2} \left| \text{card}(\mathcal{D}_N \cap I_{x,y}) - Nxy \right|^2 \, dx \, dy \geq c \log N,$$

where $I_{x,y} := [0, x] \times [0, y]$ and $0 \leq x, y < 1$. This yields $D(\mathcal{D}_N) \geq c \log^{1/2} N$. Davenport [17] proved that the estimate (1) is sharp.

Schmidt [32] investigated the discrepancy with respect to discs. His results were improved and extended, independently, by Beck [1] and Montgomery [27]: for every convex body $C \subset [-\frac{1}{2}, \frac{1}{2}]^d$ of diameter less than one and for every distribution $\mathcal{D}_N$ of $N$ points in $\mathbb{T}^d$, one has

$$\int_0^1 \int_{SO(d)} \int_{\mathbb{T}^d} \left| \text{card}(\mathcal{D}_N \cap (\lambda \sigma(C) + t)) - \lambda^d N |C| \right|^2 \, dt \, d\sigma \, d\lambda \geq c N^{(d-1)/d}. \tag{2}$$

This relation implies that for every distribution $\mathcal{D}_N$ there exists a translated, rotated, and dilated copy $\mathcal{C}$ of a given convex body $C \subset [-\frac{1}{2}, \frac{1}{2}]^d$ having diameter less than one, such that

$$\left| \text{card}(\mathcal{D}_N \cap \mathcal{C}) - N |\mathcal{C}| \right| \geq c \, N^{(d-1)/(2d)}. \tag{4}$$

Beck and Chen [2] proved that (2) is sharp. Indeed, they showed that for every positive integer $N$ there exists a distribution $\mathcal{D}_N \subset \mathbb{T}^d$ satisfying

$$\int_{SO(d)} \int_{\mathbb{T}^d} \left| \text{card}(\mathcal{D}_N \cap C) - N |C| \right|^2 \, dt \, d\sigma \leq c \, N^{(d-1)/d}. \tag{3}$$

This distribution $\mathcal{D}_N$ can be obtained either by applying a probabilistic argument or by reduction to a lattice point problem; see [5, 12, 13, 33] for a comparison of probabilistic and deterministic results.

In the following, we shall consider bounds for the integral in (3) for distributions of $N$ points that are restrictions of a shrunk integer lattice to the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Due to an argument in [9, p. 3533] that also extends to higher dimensions, we may assume that $N$ is a $d$th power $N = M^d$ for a positive integer $M$. More precisely, we consider distributions

$$\mathcal{D}_N = \left( \frac{1}{N^{1/d}} \mathbb{Z}^d \right) \cap \left[ -\frac{1}{2}, \frac{1}{2} \right]^d.$$

Given a convex body $C \subset [-\frac{1}{2}, \frac{1}{2}]^d$ of diameter less than one, we then have

$$\text{card} (\mathcal{D}_N \cap C) - N |C| = \text{card} \left( \mathbb{Z}^d \cap N^{1/d} C \right) - N |C|. \tag{4}$$
Estimation of the RHS in (4) is a classical lattice point problem. Results concerning lattice points are extensively used in different areas of pure and applied mathematics; see, for example, [20, 21, 24].

For the definition of a suitable discrepancy function, we change the discrete dilation \( N^{1/d} \) in (4) to an arbitrary dilation \( \rho \geq 1 \) and replace the convex body \( C \) in (4) with a translated, rotated and then dilated copy \( \rho \sigma (C) + t \), where \( \sigma \in SO(d) \) and \( t \in \mathbb{T}^d \). Thus the discrepancy

\[
D^\rho_C(\sigma, t) := \text{card}(\mathbb{Z}^d \cap (\rho \sigma (C) + t)) - \rho^d |C| = \sum_{k \in \mathbb{Z}^d} \chi_{\rho \sigma(C)+t}(k) - \rho^d |C|
\]

is defined as the difference between the number of integer lattice points in the set \( \rho \sigma (C) + t \) and its volume \( \rho^d |C| \) (here, \( \chi_A \) denotes the characteristic function for the set \( A \)). It is easy to see (e.g., [5]) that the periodic function \( t \mapsto D^\rho_C(\sigma, t) \) has the Fourier series expansion

\[
\rho^d \sum_{0 \neq m \in \mathbb{Z}^d} \hat{\chi}_{\sigma(C)}(\rho m) e^{2\pi im \cdot t}.
\]

(5)

Kendall [22] seems to have been the first to realize that multiple Fourier series expansions can be helpful in certain lattice point problems. Using our notation, he proved that for every convex body \( C \subset \mathbb{R}^d \) and \( \rho \geq 1 \)

\[
\|D^\rho_C\|_{L^2(SO(d) \times \mathbb{T}^d)} \leq c \rho^{(d-1)/2}.
\]

(6)

This also follows from more recent results in [30] and [8] as demonstrated next. Given a convex body \( C \subset \mathbb{R}^d \), we define the (spherical) average decay of \( \hat{\chi}_C \) as

\[
\|\hat{\chi}_C( \cdot \rho \cdot)\|_{L^2(\Sigma_{d-1})} := \left\{ \int_{\Sigma_{d-1}} |\hat{\chi}_C(\rho \tau)|^2 \ d\tau \right\}^{1/2},
\]

where \( \Sigma_{d-1} := \{ t \in \mathbb{R}^d : |t| = 1 \} \) and \( \tau \) is the rotation invariant normalized measure on \( \Sigma_{d-1} \). Extending an earlier result of Podkorytov [30], Brandolini, Hofmann, and Iosevich [8] proved that

\[
\|\hat{\chi}_C( \cdot \rho \cdot)\|_{L^2(\Sigma_{d-1})} \leq c \rho^{-(d+1)/2}.
\]

(7)

By applying the Parseval identity to the Fourier series (5) of the discrepancy function, we obtain Kendall’s result (6); i.e.,

\[
\|D^\rho_C\|_{L^2(SO(d) \times \mathbb{T}^d)}^2 = \rho^{2d} \sum_{0 \neq k \in \mathbb{Z}^d} \int_{SO(d)} |\hat{\chi}_{\sigma(C)}(\rho k)|^2 \ d\sigma
\]

\[
\leq c \rho^{2d} \sum_{0 \neq k \in \mathbb{Z}^d} |\rho k|^{-(d+1)} \leq c_1 \rho^{d-1}.
\]

(8)
We are interested in the reversed inequality
\[ \| D_\rho^C \|_{L^2(SO(d) \times T^d)}^2 \geq c_1 \rho^{d-1}, \] (9)
which, as we shall see, may or may not hold. To understand this, let us assume that (7) can be reversed
\[ \| \hat{\chi}_C(\rho \cdot \cdot) \|_{L^2(\Sigma_{d-1})} \geq c_1 \rho^{-(d+1)/2}. \] (10)
This relation (10) is true for a simplex (see [6, Theorem 2.3]) but it is not true for every convex body (see Appendix).

Moreover, relations (9) and (10) are connected in the following way (see [6, Proof of Theorem 3.7]).

**Proposition 1** Let \( C \) be a convex body in \( \mathbb{R}^d \). If (10) holds, then also (9).

**Proof** Indeed,
\[
\| D_\rho^C \|_{L^2(SO(d) \times T^d)}^2 = \rho^{2d} \sum_{0 \neq k \in \mathbb{Z}^d} \int_{SO(d)} |\hat{\chi}_C(\rho k)|^2 \ d\sigma \\
\geq c \rho^{2d} \int_{SO(d)} |\hat{\chi}_C(\rho k')|^2 \ d\sigma \geq c_1 \rho^{d-1},
\] (11)
where \( k' \) is any non-zero element in \( \mathbb{Z}^d \).

We shall see that (9) does not always imply (10).

### 2 \( L^2 \)-Regularity of Convex Bodies

We say that a convex body \( C \subset \mathbb{R}^d \) is \( L^2 \)-regular if there exists a positive constant \( c_1 \) such that
\[ c_1 \rho^{(d-1)/2} \leq \| D_\rho^C \|_{L^2(SO(d) \times T^d)} \] (12)
(by (6) we already know that \( \| D_\rho^P \|_{L^2(SO(d) \times T^d)} \leq c_2 \rho^{(d-1)/2} \) for some \( c_2 > 0 \). If (12) fails we say that \( C \) is \( L^2 \)-irregular.

Let \( d > 1 \). Parnovski and Sobolev [29] proved that the \( d \)-dimensional ball \( B_d := \{ t \in \mathbb{R}^d : |t| \leq 1 \} \) is \( L^2 \)-regular if and only if \( d \equiv 1 \) (mod 4).

More generally, it was proved [4] that if \( C \subset \mathbb{R}^d \) is a convex body with smooth boundary, having everywhere positive Gaussian curvature, then \( C \) is \( L^2 \)-irregular if and only if it is symmetric about a point and \( d \equiv 1 \) (mod 4).

Parnovski and Sidorova [28] studied the above problem for the non-convex case of a \( d \)-dimensional annulus. They provided a complete answer in terms of the width of the annulus.

In the case of a polyhedron \( P \), inequality (6) was extended to \( L^p \) norms in [6]: for any \( p > 1 \) and \( \rho \geq 1 \) we have
\[
\| D_\rho^P \|_{L^p(SO(d) \times T^d)} \leq c_p \rho^{(d-1)(1-1/p)}
\]
and, specifically for simplices $S$, one has
\[ c_p' \rho^{(d-1)(1-1/p)} \leq \| D_S^p \|_{L^p(SO(d) \times T^d)} \leq c_p \rho^{(d-1)(1-1/p)}. \]

In particular, this implies that the $d$-dimensional simplices are $L^2$-regular.

For the planar case it was proved in [7, Theorem 6.2] that every convex body with piecewise $C^\infty$ boundary that is not a polygon is $L^2$-regular. Related results can be found in [6,13,23]. Until now no example of a $L^2$-irregular polyhedron has been found. We are interested in identifying the $L^2$-regular convex polyhedrons. In this paper we give a complete answer for the planar case.

Let us first compare the $L^2$-regularity for a disc $B \subset \mathbb{R}^2$ and a square $Q \subset \mathbb{R}^2$. Note that both their characteristic functions $\chi_B$ and $\chi_Q$ do not satisfy (10), see the Appendix. So Proposition 1 cannot be applied and $B$ and $Q$ may be $L^2$-irregular.

It is well-known that the disc is $L^2$-regular (see [29] or [7, Theorem 6.2]), so that (9) does not imply (10). We shall prove in this paper that $Q$ is $L^2$-irregular.

The $L^2$-irregularity of the square $Q$ is shared by each member of the family of polygons described in the following definition.

**Definition 2** Let $\mathcal{P}$ be the family of all convex polygons in $\mathbb{R}^2$ that can be inscribed in a circle and are symmetric about the centre.

### 3 Statements of the Results

We now state our main result.

**Theorem 3** A convex polygon $P$ is $L^2$-regular if and only if $P \notin \mathcal{P}$.

The “only if” part is a consequence of the following more precise result.

**Proposition 4** If $P \in \mathcal{P}$, then for every $\varepsilon > 0$ there is a divergent increasing sequence $\{\rho_u\}_{u=1}^{\infty}$ such that
\[ \| D_P^{\rho_u} \|_{L^2(SO(2) \times T^2)} \leq c_\varepsilon \rho_u^{1/2} \log^{-1/(32+\varepsilon)}(\rho_u). \]

Theorem 3 above and [7, Theorem 6.2] yield the following more general result.

**Corollary 5** Let $C$ be a convex body in $\mathbb{R}^2$ having piecewise smooth boundary. Then $C$ is not $L^2$-regular if and only if it belongs to $\mathcal{P}$.

The following result shows that Theorem 3 is essentially sharp.

**Proposition 6** For every $P \in \mathcal{P}$, for $\varepsilon > 0$ arbitrary small, and for any $\rho$ large enough,
\[ \| D_P^{\rho} \|_{L^2(SO(2) \times T^2)} \geq c_\varepsilon \rho^{1/2-\varepsilon}, \]
where $c_\varepsilon$ is independent of $\rho$.  

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The “if” part of Theorem 3 is a consequence of the following three lemmas.

**Lemma 7** Let \( P \) in \( \mathbb{R}^2 \) be a polygon having a side not parallel to any other side. Then \( P \) is \( L^2 \)-regular.

**Lemma 8** Let \( P \) in \( \mathbb{R}^2 \) be a convex polygon with a pair of parallel sides having different lengths. Then \( P \) is \( L^2 \)-regular.

**Lemma 9** Let \( P \) in \( \mathbb{R}^2 \) be a convex polygon that cannot be inscribed in a circle. Then \( P \) is \( L^2 \)-regular.

### 4 Notation and Preliminary Arguments

In the remainder of the paper, a polygon \( P \) is given by its vertex set \( \{ P_h \}_{h=1}^s \), where it is assumed that the numbering indicates counterclockwise ordering of the vertices; we write \( P \sim \{ P_h \}_{h=1}^s \). For convenience we use periodic labeling; i.e., \( P_{h+s}, P_{h+2s}, \ldots \) refer to the same point \( P_h \) for \( 1 \leq h \leq s \). For every \( h \) let

\[ \tau_h := \frac{P_{h+1} - P_h}{|P_{h+1} - P_h|} \]

be the direction of the oriented side \( P_h P_{h+1} \) and \( \ell_h := |P_{h+1} - P_h| \) its length.

For every \( h \) let \( v_h \) be the outward unit normal vector corresponding to the side \( P_h P_{h+1} \). Let

\[ \mathcal{L}_h := |P_h + P_{h+1}| \]

be the length of the vector \( P_h + P_{h+1} \). Observe that if \( |P_h| = |P_{h+1}| \) (in particular if the polygon \( P \) is inscribed in a circle centred at the origin) then,

\[ P_h + P_{h+1} = \mathcal{L}_h v_h. \]

We shall always assume \( \ell_h \geq 1 \) and \( \mathcal{L}_h \geq 1 \).

Let \( v(s) \) be the outward unit normal vector at a point \( s \in \partial P \) that is not a vertex of \( P \). By applying Green’s formula we see that, for any \( \rho \geq 1 \), we have

\[ \widehat{\chi}_P(\rho \Theta) = \int_P e^{-2\pi i \rho \Theta \cdot t} \, dt \]

\[ = -\frac{1}{2\pi i \rho} \int_{\partial P} e^{-2\pi i \rho \Theta \cdot s} (\Theta \cdot v(s)) \, ds \]

\[ = -\frac{1}{2\pi i \rho} \sum_{h=1}^s \ell_h (\Theta \cdot v_h) \int_0^1 e^{-2\pi i \rho \Theta \cdot (P_{h+1} - P_h)} \, d\lambda \]

\[ = -\frac{1}{4\pi^2 \rho^2} \sum_{h=1}^s \frac{\Theta \cdot v_h}{\Theta \cdot \tau_h} \left[ e^{-2\pi i \rho \Theta \cdot P_{h+1}} - e^{-2\pi i \rho \Theta \cdot P_h} \right] \]
\[= - \frac{1}{4\pi^2 \rho^2} \sum_{h=1}^{s} \frac{\Theta \cdot \nu_h}{\Theta \cdot \tau_h} e^{-\pi i \rho \Theta \cdot (P_{h+1} - P_h)} \left[ e^{-\pi i \rho \Theta \cdot (P_{h+1} - P_h)} - e^{\pi i \rho \Theta \cdot (P_{h+1} - P_h)} \right] \]

\[= \frac{i}{2\pi^2 \rho^2} \sum_{h=1}^{s} \frac{\Theta \cdot \nu_h}{\Theta \cdot \tau_h} e^{-\pi i \rho \mathcal{L}_h \Theta \cdot \nu_h} \sin(\pi \rho \mathcal{L}_h \Theta \cdot \tau_h). \tag{13} \]

For any \(1 \leq h \leq s\), let \(\theta_h \in [0, 2\pi)\) be the angle defined by

\[\tau_h =: (\cos \theta_h, \sin \theta_h). \tag{14}\]

Hence

\[\nu_h = (\sin \theta_h, -\cos \theta_h) \tag{15}\]

and, if \(\Theta =: (\cos \theta, \sin \theta)\), then

\[\Theta \cdot \tau_h = \cos(\theta - \theta_h), \quad \Theta \cdot \nu_h = -\sin(\theta - \theta_h). \]

Thus, (13) can be written as

\[\hat{\chi}_P(\rho \Theta) = - \frac{i}{2\pi^2 \rho^2} \sum_{h=1}^{s} \sin(\theta - \theta_h) \cos(\theta - \theta_h) e^{\pi i \rho \mathcal{L}_h \sin(\theta - \theta_h)} \sin(\pi \rho \mathcal{L}_h \cos(\theta - \theta_h)) \]

and the equality in (8) yields

\[\| D^\rho_P \|_{L^2(SO(2) \times \mathbb{T}^2)}^2 = \rho^4 \sum_{0 \neq k \in \mathbb{Z}^2} \int_0^{2\pi} |\hat{\chi}_P(\rho |k| \Theta)|^2 d\theta = \sum_{0 \neq k \in \mathbb{Z}^2} \frac{1}{|k|^4} \times \int_0^{2\pi} \left| \sum_{h=1}^{s} \frac{\sin(\theta - \theta_h)}{\cos(\theta - \theta_h)} e^{-\pi i \rho |k| \mathcal{L}_h \sin(\theta - \theta_h)} \sin(\pi \rho |k| \ell_h \cos(\theta - \theta_h)) \right|^2 d\theta. \tag{16}\]

For \(P \in \mathcal{P}\), relation (16) can be further simplified. Let \(P \in \mathcal{P}\) have \(s = 2n\) sides (i.e. \(P \sim \{P_h\}_{h=1}^{2n}\)) and be inscribed in a circle centred at the origin. Then \(P_h P_{h+1} = -P_{n+h} P_{n+h+1}\) for any \(1 \leq h \leq n\) and \(P_{h+1} + P_h = \mathcal{L}_h \nu_h\). Therefore, for every \(1 \leq h \leq n\),

\[\tau_h = -\tau_{n+h}, \quad \nu_h = -\nu_{n+h}, \quad \ell_h = \ell_{n+h}, \quad \mathcal{L}_h = \mathcal{L}_{n+h}. \]

Then relation (13) becomes

\[\hat{\chi}_P(\rho \Theta) = \frac{1}{\pi^2 \rho^2} \sum_{h=1}^{n} \frac{\sin(\theta - \theta_h)}{\cos(\theta - \theta_h)} \sin(\pi \rho \mathcal{L}_h \sin(\theta - \theta_h)) \sin(\pi \rho \ell_h \cos(\theta - \theta_h)) \]

\[\tag{17}\]
and the equality in (8) yields

\[ \|D_P^\rho\|_{L^2(SO(2) \times \mathbb{T}^2)}^2 = c \sum_{0 \neq k \in \mathbb{Z}^2} \frac{1}{|k|^4} \left| \sum_{h=1}^n \frac{\sin(\theta - \theta_h)}{\cos(\theta - \theta_h)} \right|^2 \]

\[ \times (\pi \rho |k| \ell_h \cos(\theta - \theta_h)) \sin(\pi \rho |k| \mathcal{L}_h \sin(\theta - \theta_h)) \right| d\theta \]

\[ \leq c \sum_{0 \neq k \in \mathbb{Z}^2} \frac{1}{|k|^4} \sum_{h=1}^n \int_0^{\pi/2} \left| \frac{\sin(\pi \rho |k| \ell_h \sin \theta)}{\sin \theta} \sin(\pi \rho |k| \mathcal{L}_h \cos \theta) \right|^2 d\theta. \]

(18)

The last relation holds for every \( P \in \mathfrak{P} \) with 2n sides.

5 Proofs

Proof of Lemma 7 The proof of Lemma 7 is essentially the proof of [6, Theorem 3.7], which is stated for a simplex but the argument also works for every polyhedron having a face not parallel to any other face. \( \square \)

Proof of Lemma 8 By Lemma 7, we can assume that \( P \sim \{ P_h \}_{h=1}^{2n} \) is a convex polygon with an even number of sides, and that for every \( h = 1, \ldots, n \) the sides \( P_h \) and \( P_{h+1} \) are parallel. Assume the existence of an index \( 1 \leq j \leq n \) such that the length \( \ell_j \) of the \( j \)th side \( P_j P_{j+1} \) is longer than the length \( \ell_{j+n} \) of the opposite side \( P_{j+n} P_{j+n+1} \). Then there exist \( 0 < \varepsilon < 1 \) and \( 0 < \alpha < 1 \) such that

\[ (1 + \varepsilon) \frac{\ell_{j+n}}{\ell_j} < \alpha. \] (19)

Let \( H > 1 \) be a large constant satisfying

\[ \sin(\theta - \theta_j) \geq \sqrt{\alpha} (\theta - \theta_j) \quad \text{if} \quad 0 \leq \theta - \theta_j \leq \frac{1 + \varepsilon}{H}. \] (20)

We further assume (recall \( \rho \geq 1 \))

\[ \frac{1}{H \pi \rho \ell_j} \leq \theta - \theta_j \leq \frac{1 + \varepsilon}{H \pi \rho \ell_j}. \]

Observe that (19) and (20) yield

\[ |\sin(\pi \rho \ell_j \sin(\theta - \theta_j))| - |\sin(\pi \rho \ell_{j+n} \sin(\theta - \theta_{j+n}))| \]

\[ \geq \sin(\pi \rho \ell_j \sqrt{\alpha} (\theta - \theta_j)) - \sin(\pi \rho \ell_{j+n} (\theta - \theta_{j+n})) \]

\[ \geq \frac{\alpha}{H} - \frac{1 + \varepsilon}{H} \frac{\ell_{j+n}}{\ell_j} =: a_j > 0. \]
Hence

\[
\frac{\sin(\pi \rho \ell_j \sin(\theta - \theta_j))}{\sin(\theta - \theta_j)} \cos(\theta - \theta_j)e^{-\pi i \rho \Theta_h(P_{j+1}+P_j)} + \frac{\sin(\pi \rho \ell_{j+n} \sin(\theta - \theta_j))}{\sin(\theta - \theta_j)} \cos(\theta - \theta_{j+n})e^{-\pi i \rho \Theta_h(P_{j+n+1}+P_{j+n})} \\
\geq \frac{|\cos(\theta - \theta_j)|}{|\sin(\theta - \theta_j)|} \left( |\sin(\pi \rho \ell_j \sin(\theta - \theta_j))| - |\sin(\pi \rho \ell_{j+n} \sin(\theta - \theta_{j+n}))| \right) \\
\geq a_j \frac{|\cos(\theta - \theta_j)|}{|\sin(\theta - \theta_j)|}.
\]

(21)

We use the previous estimates to evaluate the last integral in (16) in a neighborhood of \( \theta \), and therefore obtain an estimate from below of \( \|D^\rho_P\|_{L^2(SO(2) \times \mathbb{T}^2)} \). By the arguments in [6, Theorem 2.3] or [33, Lemma 10.6], the contribution of all the sides \( P_h P_{h+1} \) (with \( h \neq j \) and \( h \neq j+n \)) to the term \( \|D^\rho_P\|_{L^2(SO(2) \times \mathbb{T}^2)} \) is \( O(1) \). Then (11), (17) and (21) yield

\[
\|D^\rho_P\|_{L^2(SO(2) \times \mathbb{T}^2)}^2 \geq c \int \frac{1}{1 - \frac{\pi \rho(\ell_j)}{2}} \frac{\cos^2 \theta}{\sin^2 \theta} d\theta + c_1 \geq c \int \frac{1}{1 - \frac{\pi \rho(\ell_j)}{2}} \frac{d\theta}{\theta^2} + c_1 \geq c_2 \rho.
\]

\[\square\]

Proof of Lemma 9 We can assume that \( P \sim \{P_h\}_{h=1}^{2n} \) is a convex polygon such that for every \( h = 1, \ldots, n \) the sides \( P_h P_{h+1} \) and \( P_{h+n} P_{h+n+1} \) are parallel and of the same length (that is, \( \ell_h = \ell_{h+n}, \tau_h = -\tau_{h+n}, v_h = -v_{h+n} \)). Then we may assume that \( P \) is symmetric about the origin. As \( P \) cannot be inscribed in a circle, there exists an index \( 1 \leq j \leq n \) such that the two opposite equal and parallel sides \( P_j P_{j+1} \) and \( P_{j+n} P_{j+n+1} \) are not the sides of a rectangle. Then \( P_j + P_{j+1} \) is not orthogonal to \( P_{j+1} - P_j \). Let \( \phi_j \in [\theta_j - \pi, \theta_j] \) be defined by

\[
P_{j+1} + P_j = L_j(\cos \phi_j, \sin \phi_j).
\]

Since \( \tau_j = (\cos \theta_j, \sin \theta_j) \) and \( \nu(j) = (\cos(\theta_j - \frac{\pi}{2}), \sin(\theta_j - \frac{\pi}{2})) \), see (14) and (15), we have \( \phi_j - \theta_j \neq -\frac{\pi}{2} \). We put \( \varphi_j := \phi_j - \theta_j \). Then

\[
\varphi_j \in [-\pi, 0] \setminus \left[ -\frac{\pi}{2} \right].
\]

Again we need to find a lower bound for the last integral in (16). As in the previous proof it is enough to consider

\[
F_j(\theta) := \sum_{h \in \{j, j+n\}} \frac{\sin(\theta - \theta_h)}{\cos(\theta - \theta_h)} \sin(\pi \rho \ell_j \cos(\theta - \theta_j))e^{-\pi i \rho \Theta_h(P_{j+1}+P_h)} \\
= \frac{\sin(\theta - \theta_j)}{\cos(\theta - \theta_j)} \sin(\pi \rho \ell_j \cos(\theta - \theta_j)) \left[ e^{-\pi i \rho L_j \cos(\theta - \phi_j)} - e^{\pi i \rho L_j \cos(\theta - \phi_j)} \right] \\
= -2i \frac{\sin(\theta - \theta_j)}{\cos(\theta - \theta_j)} \sin(\pi \rho \ell_j \cos(\theta - \theta_j)) \sin(\pi \rho L_j \cos(\theta - \phi_j)).
\]
We write
\[
\int_0^{2\pi} \left| \frac{\sin(\theta - \theta_j)}{\cos(\theta - \theta_j)} \sin(\pi \rho L_j \cos(\theta - \phi_j)) \right|^2 d\theta
\]
\[
= \int_0^{2\pi} \left| \frac{\sin(\pi \rho L_j \sin \theta)}{\sin \theta} \cos \theta \sin(\pi \rho L_j \sin(\theta - \varphi_j)) \right|^2 d\theta.
\]

We shall integrate \( \theta \) in a neighborhood of 0 (actually \( 0 \leq \theta \leq 1 \) suffices). As for \( \varphi_j \) we first assume \( \varphi_j \in (-\frac{\pi}{2}, 0] \). Then \( \cos \varphi_j > 0 \) and \( \sin \varphi_j \leq 0 \). Let \( 0 < \gamma < 1 \) satisfy \( \cos \varphi_j > \gamma \). In order to prove that \( \left| \sin(\pi \rho L_j \sin(\theta - \varphi_j)) \right| \geq c \) we consider two cases.

Case 1: \( |\sin(\pi \rho L_j \sin \varphi_j)| > \gamma/2 \).
We need to bound \( \sin(\theta - \varphi_j) - |\sin \varphi_j| \). Since \( \sin \varphi_j \leq 0 \) one has
\[
\frac{\theta}{2} \cos \varphi_j + \left[ 1 - \frac{\theta^2}{2} \right] |\sin \varphi_j| \leq \sin \theta \cos \varphi_j - \cos \theta \sin \varphi_j \leq \theta \cos \varphi_j + |\sin \varphi_j|.
\]
Therefore
\[
\frac{\theta}{2} \gamma - \frac{\theta^2}{2} \leq \sin(\theta - \varphi_j) - |\sin \varphi_j| \leq \theta. \tag{22}
\]
Let \( \rho \geq 1 \) and assume
\[
\frac{\gamma}{8\pi \rho L_j} \leq \theta \leq \frac{\gamma}{4\pi \rho L_j}.
\]
We recall that \( L_j \geq 1 \). Again we have to estimate \( \sin(\theta - \varphi_j) - |\sin \varphi_j| \). By (22) we have
\[
0 < \frac{\gamma}{16\pi \rho L_j} - \frac{\gamma^2}{32(\pi \rho L_j)^2} < \sin(\theta - \varphi_j) - |\sin \varphi_j| \leq \frac{\gamma}{4\pi \rho L_j}.
\]
Therefore
\[
0 < \pi \rho L_j \sin(\theta - \varphi_j) - \pi \rho L_j |\sin \varphi_j| \leq \frac{\gamma}{4}. \tag{23}
\]
Hence the assumption of Case 1 and (23) yield
\[
|\sin(\pi \rho L_j \sin(\theta - \varphi_j))|
\]
\[
= |\sin(\pi \rho L_j |\sin(\theta - \varphi_j) + \sin \varphi_j|) - \pi \rho L_j |\sin \varphi_j|)
\]
\[
= |\sin(\pi \rho L_j |\sin(\theta - \varphi_j) - |\sin \varphi_j|) \cos(\pi \rho L_j \sin \varphi_j)
- \cos(\pi \rho L_j |\sin(\theta - \varphi_j) - |\sin \varphi_j|) \sin(\pi \rho L_j \sin \varphi_j)|
\]
\[
\geq |\sin(\pi \rho L_j \sin \varphi_j)| |\cos(\pi \rho L_j \sin(\theta - \varphi_j) - \pi \rho L_j |\sin \varphi_j|)|
- |\sin(\pi \rho L_j \sin(\theta - \varphi_j) - \pi \rho L_j |\sin \varphi_j|)|
\]
Case 2: $|\sin(\pi \rho L_j \sin \varphi_j)| \leq \gamma / 2$.

Let $\rho$ be large so that $0 \leq \theta \leq \frac{3}{2\pi \rho L_j}$ implies $\sin \theta \geq (1 - \delta) \theta$, with $\delta < 1/20$. Then for

$$\frac{1}{\pi \rho L_j} \leq \theta \leq \frac{3}{2 \pi \rho L_j}$$

(24)

we have

$$\theta (1 - \delta) \gamma + \left[ 1 - \frac{\theta^2}{2} \right] |\sin \varphi_j| \leq \sin \theta \cos \varphi_j - \cos \theta \sin \varphi_j \leq \theta + |\sin \varphi_j|$$

and

$$\theta \gamma (1 - \delta) - \frac{\theta^2}{2} \leq \sin(\theta - \varphi_j) - |\sin \varphi_j| \leq \theta.$$  

(25)

For $\rho$ large enough we have $\frac{9}{8(\pi \rho L_j)^2} < \gamma^2 \pi \rho L_j$. Then (24) and (25) yield

$$\frac{\gamma (1 - 2 \delta)}{\pi \rho L_j} < \frac{\gamma (1 - \delta)}{\pi \rho L_j} - \frac{9}{8(\pi \rho L_j)^2} \leq \theta \gamma (1 - \delta) - \frac{\theta^2}{2}$$

$$< \sin(\theta - \varphi_j) - |\sin \varphi_j| \leq \theta \leq \frac{3}{2 \pi \rho L_j}$$

and

$$\gamma (1 - 2 \delta) < \pi \rho L_j \sin(\theta - \varphi_j) - \pi \rho L_j |\sin \varphi_j| \leq \frac{3}{2}.$$  

(26)

We choose $\gamma$ small enough so that

$$\sin(\gamma (1 - 2 \delta)) \geq (1 - 2 \delta)^2 \gamma \quad \text{and} \quad \gamma^2 / 4 < 2 \delta.$$  

Then (26) and the assumption of Case 2 yield

$$|\sin(\pi \rho L_j \sin(\theta - \varphi_j))|$$

$$= |\sin(\pi \rho L_j \left[ \sin(\theta - \varphi_j) - \sin \varphi_j \right] + \pi \rho L_j \sin \varphi_j)|$$

$$\geq |\cos(\pi \rho L_j \sin \varphi_j)| \left| \sin(\pi \rho L_j \left[ \sin(\theta - \varphi_j) - \pi \rho L_j |\sin \varphi_j| \right]) \right|$$

$$- |\sin(\pi \rho L_j \sin \varphi_j)|$$

$$\geq \gamma (1 - 2 \delta)^2 \sqrt{1 - \frac{\gamma^2}{4}} - \frac{\gamma}{2} > \gamma \left[ (1 - 2 \delta)^{5/2} - \frac{1}{2} \right] > \frac{\gamma}{4}.$$

\(\text{Birkhäuser}\)
Cases 1 and 2 prove that for a suitable choice of $0 < \gamma < 1$ such that $\cos \varphi_j > \gamma$, there exist $0 < \alpha < \beta$ such that for $\frac{\alpha}{\pi \rho L_j} \leq \theta \leq \frac{\beta}{\pi \rho L_j}$ and $\rho$ large enough we have

$$\left|\sin(\pi \rho L_j \sin(\theta - \varphi_j))\right| > \frac{\gamma}{5}. \quad (27)$$

If $\varphi_j \in [-\pi, -\frac{\pi}{2})$ we have $\cos \varphi_j < 0$ and $\sin \varphi_j \leq 0$. Then for $0 \leq \theta < 1$ we have

$$-\theta + \left[1 - \frac{\rho^2}{2}\right] |\sin \varphi_j| \leq \sin \theta \cos \varphi_j - \cos \theta \sin \varphi_j \leq -\sin |\cos \varphi_j| + |\sin \varphi_j|.$$

Hence, for a positive constant $K$,

$$|\sin \theta| \cos \varphi_j \leq |\sin \varphi_j| - \sin(\theta - \varphi_j) \leq K \theta.$$

If we choose a suitable constant $\gamma > 0$ such that $|\cos \varphi_j| > \gamma$, we can prove as for the case $\varphi_j \in (-\frac{\pi}{2}, 0]$ that (27) still holds for $\frac{\alpha}{\pi \rho L_j} \leq \theta \leq \frac{\beta}{\pi \rho L_j}$, with $0 < \alpha < \beta$ and $\rho$ large enough. Then (27) yields

$$\int_0^{2\pi} \left|F_j(\theta)\right|^2 d\theta \geq \int_{\frac{\alpha}{\pi \rho L_j}}^{\frac{\beta}{\pi \rho L_j}} \left|\frac{\sin(\pi \rho L_j \sin(\theta - \varphi_j))}{\sin \theta} \cos \theta \sin(\pi \rho L_j \sin(\theta - \varphi_j))\right|^2 d\theta$$

$$\geq c \gamma^2 \int_{\frac{\alpha}{\pi \rho L_j}}^{\frac{\beta}{\pi \rho L_j}} \left|\frac{\sin(\pi \rho L_j \sin(\theta))}{\sin \theta}\right|^2 d\theta$$

$$\geq c_1 \rho \int_{c_1}^{c_2} \frac{|\sin(t)|^2}{t} dt \geq c_2 \rho.$$

This ends the proof. \(\square\)

The proof of Theorem 3 will be complete after the proof of Proposition 4. We need a simultaneous approximation lemma, see [29, Lemma 3.4].

**Lemma 10** Let $r_1, r_2, \ldots, r_n \in \mathbb{R}$. For every positive integer $j$ there exists $j \leq q \leq j^{n+1}$ such that $\|r_s q\| < j^{-1}$ for any $1 \leq s \leq n$, where $\|x\|$ denotes the distance between the real $x$ and the nearest integer.

**Proof of Proposition 4** Let $P \sim \{P_j\}_{j=1}^{2n}$ be a polygon in $\mathcal{P}$. For every positive integer $u$ let

$$A_u^j := \{k \in \mathbb{Z}^2 : 0 < L_j |k| \leq u^2\} \quad \text{for} \quad j = 1, \ldots, n, \\
A_u := \bigcup_{j=1}^{n} A_u^j.$$

Observe that $\text{card}(A_u^j) \leq 4u^4$ and therefore $\text{card}(A_u) \leq 4nu^4$. By Lemma 10 there exists a sequence $\{\rho_u\}_{u=1}^{+\infty}$ of positive integers such that, for every $k \in A_u$ and every $j = 1, \ldots, n$,
\[ u \leq \rho_u \leq u^{4n u^4 + 1}, \quad |\sin(\pi \rho_u |k|L_j)| < 1/u. \] (28)

Observe that (28) implies
\[ u \geq c_{\varepsilon} \log \frac{1}{\varepsilon} (\rho_u) \] (29)

for every \( \varepsilon > 0 \). For any \( 1 \leq j \leq n \) and \( k \in A^j_u \) we split the integral in (18) into several parts.

\[ E^{\rho}_{1,j,|k|} := \int_0^{(8\rho_u |k|)^{-1}} \left| \sin(\pi \rho_u |k|L_j \sin \theta) \frac{\sin(\pi \rho_u |k|L_j \cos \theta)}{\sin \theta} \right|^2 d\theta. \]

For \( 0 \leq \theta \leq (8\rho_u |k|)^{-1} \) we have \( 0 \leq 1 - \cos \theta \leq (128\rho_u^2 |k|^2)^{-1} \). Then (28) yield

\[
\left| \sin(\pi \rho_u |k|L_j \cos \theta) \right| = \left| \sin(\pi \rho_u |k|L_j (\cos \theta - 1 + 1)) \right| \\
\leq \left| \sin(\pi \rho_u |k|L_j (\cos \theta - 1)) \cos(\pi \rho_u |k|L_j) \right| \\
+ \left| \sin(\pi \rho_u |k|L_j) \cos(\pi \rho_u |k|L_j (\cos \theta - 1)) \right| \\
\leq \left| \sin(\pi \rho_u |k|L_j (1 - \cos \theta)) \right| + \left| \sin(\pi \rho_u |k|L_j) \right| \\
\leq \frac{\pi L_j}{128\rho_u |k|} + \left| \sin(\pi \rho_u |k|L_j) \right| \\
\leq c \frac{1}{u}. \] (30)

By (30) we obtain

\[
E^{\rho}_{1,j,|k|} \leq \frac{c}{u^2} \int_0^{(8\rho_u |k|)^{-1}} \left| \sin(\pi \rho_u |k|L_j \sin \theta) \frac{\sin(\pi \rho_u |k|L_j \cos \theta)}{\sin \theta} \right|^2 d\theta \\
\leq c_1 \frac{|k|\rho_u}{u^2} \int_0^1 \left| \frac{\sin(t)}{t} \right|^2 dt \leq c_2 \frac{|k|\rho_u}{u^2}.
\]

Let

\[ E^{\rho}_{2,j,|k|} := \int_{(8\rho_u |k|)^{-1}}^{\left(8\rho_u^{1/4} |k|^{1/2}\right)^{-1}} \left| \sin(\pi \rho_u |k|L_j \sin \theta) \frac{\sin(\pi \rho_u |k|L_j \cos \theta)}{\sin \theta} \right|^2 d\theta. \]

For \( (8\rho_u |k|)^{-1} \leq \theta \leq \left(8\rho_u^{1/4} |k|^{1/2}\right)^{-1} \) we have

\[
\frac{1}{2000\rho_u^2 |k|^2} \leq 2 \sin^2 (\theta/2) = 1 - \cos \theta \leq \frac{1}{128\rho_u^{1/2} |k|}. \]
As in (30) we obtain

\[
\left| \sin(\pi \rho_u |k| \mathcal{L}_j \cos \theta) \right| \leq \left| \sin(\pi \rho_u |k| \mathcal{L}_j (1 - \cos \theta)) \right| + \left| \sin(\pi \rho_u |k| \mathcal{L}_j) \right| \\
\leq \frac{\pi \mathcal{L}_j}{128u^{1/2}} + \frac{1}{u} \leq c u^{-1/2}
\]

and then

\[
E^\rho_{2,j,|k|} \leq c \frac{1}{u} \int \left( \frac{8u^{1/4} \rho_u^{1/2} |k|^{1/2}}{|8 \rho_u|^{-1}} \right) \sin(\pi \rho_u |k| \ell_j \sin \theta) \left| \frac{\sin(\pi \rho_u |k| \mathcal{L}_j \cos \theta)}{\sin \theta} \right| d\theta \leq c_1 \frac{1}{u} \int \left( \frac{8u^{1/4} \rho_u^{1/2} |k|^{1/2}}{|8 \rho_u|^{-1}} \right) \frac{d\theta}{\theta^2} \leq c_2 \frac{\rho_u |k|}{u}.
\]

Let \(1/4 < \lambda < 1/2\) and let

\[
E^\rho_{3,j,|k|} := \int_0^\lambda \left( \frac{8u^{1/4} \rho_u^{1/2} |k|^{1/2}}{|8 \rho_u|^{-1}} \right) \sin(\pi \rho_u |k| \ell_j \sin \theta) \left| \frac{\sin(\pi \rho_u |k| \mathcal{L}_j \cos \theta)}{\sin \theta} \right| d\theta.
\]

We have

\[
E^\rho_{3,j,|k|} \leq \int_0^\lambda \left( \frac{8u^{1/4} \rho_u^{1/2} |k|^{1/2}}{|8 \rho_u|^{-1}} \right) \frac{d\theta}{\theta^2} \leq 8u^{1/4} \rho_u^{1/2} |k|^{1/2}.
\]

Finally we have

\[
E^\rho_{4,j,|k|} := \int_0^{\pi/2} \left| \frac{\sin(\pi \rho_u |k| \ell_j \sin \theta)}{\sin \theta} \right| \sin(\pi \rho_u |k| \mathcal{L}_j \cos \theta) \right| \right| d\theta \leq c.
\]

By the above estimates, (18), (28) and (29) we have

\[
\| D^\rho_{p_2} \|_{L^2(SO(2) \times \mathbb{R}^2)}^2 \leq c \rho_u \sum_{k \in A_u} \frac{1}{|k|^3} \left( \frac{1}{u^2} + \frac{1}{u} + u^{1/4} \rho_u^{-1/2} |k|^{-1/2} + \rho_u^{-1} |k|^{-1} \right)
\]

\[
+ c_1 \sum_{k \notin A_u} \frac{1}{|k|^4} \int_0^{\pi/2} \left| \frac{\sin(\pi \rho_u |k| \ell_j \sin \theta)}{\sin \theta} \right|^2 d\theta \leq c \rho_u \sum_{0 \neq k \in A_u} \frac{1}{|k|^3} u^{-1/4}
\]

\[
+ c_1 \sum_{|k| > c_1 u^2} \frac{1}{|k|^4} \left( \int_0^{(\rho_u |k|)^{-1/2}} (\rho_u |k|) d\theta + \int_0^{(\rho_u |k|)^{-1/2}} \frac{1}{\theta^2} d\theta \right) \leq c \rho_u \sum_{0 \neq k \in \mathbb{Z}^2} \frac{1}{|k|^3} \log^{-1/\pi \pi} (\rho_u) + c \sum_{|k| > c_1 u^2} \frac{1}{|k|^4} (\rho_u |k|)^{1/2}.
\]
\[
\leq c_\varepsilon \rho_u \log^{-\frac{1}{16\varepsilon^2}} (\rho_u) + c \rho_u^{1/2} \int_{|t| = 1/|t| > c_1 u^2} \frac{1}{|t|^{1/2}} \, dt
\]
\[
\leq c_\varepsilon \rho_u \log^{-\frac{1}{16\varepsilon^2}} (\rho_u).
\]

We now turn to the proof of Proposition 6, which depends on the following lemma proved by Parnovski and Sobolev, see [29, Lemma 3.3].

**Lemma 11** For any \( \varepsilon > 0 \) there exist \( \rho_0 \geq 1 \) and \( 0 < \alpha < 1/2 \) such that for every \( \rho \geq \rho_0 \) there exists \( k \in \mathbb{Z}^d \) such that \( |k| \leq \rho \varepsilon \) and \( \| \rho k \| \geq \alpha \).

**Proof of Proposition 6** Let \( P \sim \{ P_j \}_{j=1}^n \) be a polygon in \( \mathbb{P} \). Let \( \varepsilon > 0 \) and let \( j \in \{ 1, 2, \ldots, n \} \). By Lemma 11 there exist \( \rho_0 \geq 1 \) and \( 0 < \alpha < 1/2 \) such that for any \( \rho \geq \rho_0 \) there is \( \tilde{k} \in \mathbb{Z}^2 \) such that \( |\tilde{k}| \leq \rho \varepsilon \) and \( |\sin(\pi \rho |\tilde{k}| \ell_j)| > \alpha \). Then we consider the interval

\[
\theta_j \leq \theta \leq \theta_j + \frac{1}{\pi \rho |k|}.
\]

We have

\[
0 \leq 1 - \cos(\theta - \theta_j) \leq \frac{1}{2(\pi \rho |k|)^2}.
\]

Then for large \( \rho \) we have

\[
|\sin(\pi \rho |\tilde{k}| \ell_j \cos(\theta - \theta_j))| \geq |\sin(\pi \rho |\tilde{k}| \ell_j)||\cos(\pi \rho |\tilde{k}| \ell_j(1 - \cos(\theta - \theta_j)))|
\]
\[
- |\sin(\pi \rho |\tilde{k}| \ell_j(1 - \cos(\theta - \theta_j)))|
\]
\[
\geq c \left[ 1 - \frac{\ell_j^2}{8(\pi \rho |k|)^2} \right] - \frac{\ell_j}{2\pi \rho |k|} > c_1.
\]

As before the sides non parallel to \( P_j P_{j+1} \) give a bounded contribution to the integration of \( |D_{P_n}^\rho|^2 \) over the interval in (31). Finally (32) yields

\[
\| D_{P_n}^\rho \|_{L^2(SO(2) \times \mathbb{T}^2)}^2 \geq c + c_1 \frac{1}{|k|^4} \int_{\theta_j}^{\theta_j + 1/(\pi \rho |k|)} \left| \frac{\sin(\pi \rho |\tilde{k}| \ell_j \sin(\theta - \theta_j))}{\sin(\theta - \theta_j)} \right| d\theta
\]
\[
\times |\sin(\pi \rho |\tilde{k}| \ell_j \cos(\theta - \theta_j))|^2 |\cos(\theta - \theta_j)|^2 \, d\theta
\]
\[
\geq c + c_2 \frac{1}{|k|^4} \int_0^{1/(\pi \rho |k|)} \left| \frac{\sin(\pi \rho |\tilde{k}| \ell_j \sin \theta)}{\sin \theta} \right|^2 \, d\theta
\]
\[
\geq c + c_3 \rho \frac{1}{|k|^3}
\]
\[
\geq c_4 \rho^{1-\varepsilon}.
\]

\[\square\]
The proofs of Lemmas 7, 8, and 9 actually show that \( \| \hat{\chi}_{\mathcal{P}}(\rho \cdot) \|_{L^2(\Sigma_1)} \geq c \rho^{-3/2} \) whenever \( P \notin \mathfrak{P} \). Hence, Theorem 3 and Proposition 1 readily yield the following result.

**Corollary 12** Let \( P \) be a polygon in \( \mathbb{R}^2 \). Then \( P \) satisfies

\[
\| \hat{\chi}_{\mathcal{P}}(\rho \cdot) \|_{L^2(\Sigma_1)} \geq c \rho^{-3/2}
\]

if and only if \( P \notin \mathfrak{P} \).

The arguments in this paper (apart from Lemma 7) seem to work only in the planar setting. For the multi-dimensional case we have only partial results. We hope to be able to address this problem in the future.

**Appendix: On the Average Decay of \( \hat{\chi}_C \) When \( C \) is a Disc or a Square**

We consider a disc \( B \subset \mathbb{R}^2 \) and a square \( Q \subset \mathbb{R}^2 \). We observe that both their characteristic functions \( \chi_B \) and \( \chi_Q \) do not satisfy (10).

Indeed, \( \hat{\chi}_B(\xi) = |\xi|^{-1} J_1(2\pi |\xi|) \), where \( J_1 \) is the Bessel function of the first kind of order 1 (see e.g. [34]). Then the zeros of \( J_1 \) form a divergent increasing sequence \( \{\rho_u\}_{u=1}^{\infty} \) such that

\[
\| \hat{\chi}_B \left((2\pi)^{-1} \rho_u \cdot\right) \|_{L^2(\Sigma_1)} = 0.
\]

As for a square \( Q \) it was observed in [6] that

\[
\| \hat{\chi}_Q(\nu \cdot) \|_{L^2(\Sigma_1)} \leq cn^{-7/4}.
\]

for all positive integers \( n \), where \( c \) is a positive constant. Indeed, let \( Q = \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \) and let \( \Theta := (\cos \theta, \sin \theta) \). Then an explicit computation of \( \hat{\chi}_Q \) yields

\[
\int_0^{2\pi} \left| \hat{\chi}_Q(n\Theta) \right|^2 d\theta = 8 \int_0^{\pi/4} \left| \frac{\sin(n \pi \cos \theta) \sin(n \pi \sin \theta)}{\pi n \cos \theta} \right|^2 \frac{\pi n \cos \theta}{\pi n \sin \theta} d\theta \leq c \frac{1}{n^4} \int_0^{\pi/4} \left| \sin(n \pi \cos \theta) \right|^2 \frac{\pi n \cos \theta}{\sin \theta} d\theta = c \frac{1}{n^4} \int_0^{\pi/4} \left| \sin(n \pi (1 - 2 \sin^2(\theta/2))) \right|^2 \frac{\pi n \cos \theta}{\sin \theta} d\theta \leq c' \frac{1}{n^4} \int_0^{\pi/4} \left| \sin(2\pi n \sin^2(\theta/2)) \right|^2 \theta^{-2} d\theta \leq c'' \frac{1}{n^4} \int_0^{\pi/4} \left| \sin(\pi n (1 - 2 \sin^2(\theta/2))) \right|^2 \frac{\pi n \cos \theta}{\sin \theta} d\theta \leq c''' \frac{1}{n^4} \int_0^{\pi/4} \left| \sin(2\pi n \sin^2(\theta/2)) \right|^2 \theta^{-2} d\theta \leq c'''' n^{-7/2}.
\]
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