ORBITAL STABILITY OF SMOOTH SOLITARY WAVES FOR THE DEGASPERIS-PROCESI EQUATION

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Abstract. We study localized smooth solitary waves in the Desgasperis-Procesi (DP) equation on the real line. This work extends our previous work on spectral stability of these solitons [21] by establishing their orbital stability. The main difficulty stems from the fact that the translation symmetry for the DP equation gives rise to a conserved quantity equivalent to $L^2$-norm, which by itself can not bound the higher-order nonlinear terms in the Lagrangian. The remedy is to observe that, given a sufficiently smooth initial condition satisfying a measurable constraint, the $L^\infty$ orbital norm of the perturbation is bounded above by a function of its $L^2$ orbital norm, yielding the orbital stability in the $L^2 \cap L^\infty$ space.

1. Introduction

The Degasperis-Procesi (DP) equation
\begin{equation}
    m_t + 2ku_x + 3mu_x + um_x = 0, \quad x \in \mathbb{R}, \ t > 0,
\end{equation}
with momentum density $m \equiv u - u_{xx}$ and $k > 0$ as a parameter related to the critical shallow water speed, was originally derived by Degasperis and Procesi [13] using the method of asymptotic integrability up to the third order as one of three equations in the family of third-order dispersive PDE conservation laws of the form
\begin{equation}
    u_t - \alpha_2^2 u_{xxt} + \alpha_2 u_{xxx} + c_0 u_x = \partial_x(c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx}).
\end{equation}
The other two integrable equations in the family, after rescaling and applying a Galilean transformation, are the Korteweg-de Vries (KdV) equation [18],
\begin{equation}
    u_t + u_{xxx} + uu_x = 0,
\end{equation}
and the Camassa-Holm(CH) shallow-water equation [1, 15] (see also [5] for a rigorous justification in shallow water approximation),
\begin{equation}
    m_t + 2ku_x + 2mu_x + um_x = 0, \quad m = u - u_{xx}.
\end{equation}
The DP equation is also an approximation to the incompressible Euler equations for shallow water and its asymptotic accuracy is the same as that of the CH shallow-water equation [5] in the CH scaling, where the solution \( u(t, x) \) of (1.6) represents the horizontal velocity field at height \( z_0 = \sqrt{\frac{23}{36}} \) after the re-scaling within \( 0 \leq z_0 \leq 1 \) at time \( t \) in the spatial \( x \)-direction with momentum density \( m \).

The DP equation (1.6) has an apparent similarity to the CH equation (1.3), and both of them are important model equations for shallow water waves with breaking phenomena, i.e., the wave remains bounded but its slope becomes unbounded [6, 7, 26, 29]. However, there is not much known about qualitative properties and long-time dynamics of the DP equation, and what is known about the CH equation cannot be directly applied to the DP equation, due to major structural differences between the DP equation and the CH equation. For instance, the isospectral problems in the Lax pair for the DP equation (1.6) and the CH equation are respectively a third-order equation [12]

\[
\psi_x - \psi_{xxx} - \lambda m \psi = 0,
\]

and a second-order equation [1]

\[
\psi_{xx} - \frac{1}{4} \psi - \lambda m \psi = 0,
\]

where \( m = u - u_{xx} \) in both cases. Moreover, the CH equation is a re-expression of geodesic flow on the diffeomorphism group [8] and on the Bott-Virasoro group [28], while no such geometric derivation of the DP equation is available.

When it comes to solitons, the main focus of this work, the DP equation also distinguishes from the CH equation although they share some similarity. There are two scenarios, depending on the value of \( k \). In the limiting case of vanishing linear dispersion (\( k = 0 \)), smooth solitary waves become peaked solitons, called peakons. More specifically, when \( k = 0 \), the CH equation can be written as

(1.4)  
\[ u_t + \partial_x \left( \frac{1}{2} u^2 + p \ast (\frac{1}{2} u_x^2 + u^2) \right) = 0, \quad t > 0, \quad x \in \mathbb{R}, \]

and the DP equation as

(1.5)  
\[ u_t + \partial_x \left( \frac{1}{2} u^2 + p \ast (\frac{3}{2} u^2) \right) = 0, \quad t > 0, \quad x \in \mathbb{R}, \]

where \( p(x) = \frac{1}{2} e^{-|x|} \) and “ \( \ast \) ” stands for convolution with respect to the spatial variable \( x \in \mathbb{R} \). Peakons are weak solutions of these conservation laws and are true solitons that interact via elastic collisions respectively under the CH dynamics and the DP dynamics. Moreover, as a fundamental qualitative property in nonlinear dynamics, the orbital stability of peakons of the CH and DP equation has been verified [10, 23]. Relevant stability results for waves approximating peakons are also available [9]. The novel feature of the DP equation is that for \( k = 0 \), not only does it have peakon solitons [1, 12] of the form \( u(t, x) = ce^{-|x-\alpha|}, \) \( c \in \mathbb{R} \), it also admits shock peakons [14, 27] of the form

\[ u(t, x) = -\frac{1}{t+a} \text{sgn}(x)e^{-|x|}, \quad a > 0. \]

It is not clear if such a discontinuous solution is stable or not in proper settings.
In the case of non-vanishing linear dispersion \((k \neq 0)\), the existence and stability of localized smooth solitary waves of the CH equation \((1.3)\) are well understood by now \([2, 11]\), while the DP equation case has been barely explored so far except in our former work \([21]\) for existence and spectral stability. The goal of this paper is to establish orbital stability results of smooth solitons for the DP equation \((1.6)\).

We start with a rigorous definition of solitary waves, i.e., solitons. Firstly, a solution of the DP equation \(u(t, x)\) is a traveling wave if there exist a real number \(c\) and a scalar function \(\phi : \mathbb{R} \to \mathbb{R}\) such that

\[
u(t, x) = \phi(x - ct).
\]

Moreover, a traveling wave of the DP equation \(\phi(x - ct)\) is a solitary wave if there exists \(\xi_0 \in \mathbb{R}\) such that

- \(\max_{\xi \in \mathbb{R}} \phi(\xi) = \phi(\xi_0)\) and \(\lim_{\xi \to \pm \infty} \phi(\xi) = 0\).
- \(\phi\) is strictly increasing on \((-\infty, \xi_0)\) and strictly decreasing on \((\xi_0, \infty)\).

We have the following existence result

**Theorem 1.1** (existence). \([21]\) Under the physical condition \(c > 2k\), there exists a unique \(c\)-speed solitary wave \(\phi(\xi; c)\) with its maximum height

\[
\frac{c - 2k}{4} < \phi_{\text{max}} \triangleq \max_{\xi \in \mathbb{R}} \{\phi\} < c - 2k.
\]

In addition, the function \(\phi(\xi; c)\) is even and strictly monotonically increases from 0 to \(\phi_{\text{max}}\) for negative values of \(\xi\).

Thanks to the translation invariance of the equation, for any given solitary wave \(\phi(\xi; c)\), its spatial translation generates a family of solutions, called the orbit of the solitary wave, denoted as

\[
O_c = \{\phi(\cdot + x_0; c) \mid x_0 \in \mathbb{R}\}.
\]

Moreover, a solitary-wave solution \(\phi\) of the DP equation is called orbitally stable if a wave starting close to the solitary wave \(\phi\) remains close to the orbit of the solitary wave up to the existence time. A generic feature of nonlinear dispersive equation is that their solutions usually tend to be oscillations that, as time evolves, spread out spatially in a quite nonlinear and complicated way. When it comes to solitary waves, one would naively expect that a small perturbation of a solitary wave would at least yield another one with a different speed and phase shift, if not more complicated, which makes the stability of solitary waves counter-intuitive and thus fascinating. We have the following stability theorem as the main result of the present paper.

**Theorem 1.2.** Let \(\phi(x - ct) = \phi^c(x - ct)\) be the solitary-wave solution of \((1.6)\) traveling with speed \(c > 2k\). Such a solitary-wave \(\phi^c\) is orbitally stable. More specifically, for every \(\epsilon > 0\), there is \(\delta > 0\) such that if \(u \in C((0, T); H^s), s > \frac{3}{2}\) for some \(0 < T \leq \infty\) is a solution of \((1.6)\) with

\[
\|u_0 - \phi^c\|_X < \delta,
\]
(see (1.10) for definitions), and with \( m_0 + \frac{2k}{3} = u_0 - u_{0xx} + \frac{2k}{3} \) being a positive Radon measure, then the solution \( u(t, x) \) is global, and for every \( t \geq 0 \),

\[
\inf_{x_0 \in \mathbb{R}} \| u(t, \cdot) - \phi(\cdot - x_0) \|_X < \epsilon, \quad \inf_{x_0 \in \mathbb{R}} \| u(t, \cdot) - \phi(\cdot - x_0) \|_{L^\infty} < O(\epsilon^{\frac{2}{3}}).
\]

Particularly, if \( u \in C([0, T); H^s) \), \( s \geq 3 \) for some \( 0 < T \leq \infty \) with

\[
\| u_0 - \phi^c \|_X < \delta, \quad \| u(0, \cdot) - \phi(\cdot) \|_{H^3} \leq \frac{2\sqrt{2}}{3} k,
\]

then \( m_0 + \frac{2k}{3} = u_0 - u_{0xx} + \frac{2k}{3} \) is a positive Radon measure and for every \( t \geq 0 \),

\[
\inf_{x_0 \in \mathbb{R}} \| u(t, \cdot) - \phi(\cdot - x_0) \|_X < \epsilon, \quad \inf_{x_0 \in \mathbb{R}} \| u(t, \cdot) - \phi(\cdot - x_0) \|_{L^\infty} < O(\epsilon^{\frac{2}{3}}).
\]

The orbital stability proof is essentially based on the Hamiltonian structure of the DP equation. Actually, the DP equation (1.1) in terms of \( u \), that is,

\[
(1.6) \quad u_t - u_{xx} + 2ku_x - 3u_xu_{xx} - uu_{xxx} + 4uu_x = 0,
\]

can be written as an infinite dimensional Hamiltonian PDE, that is,

\[
(1.7) \quad u_t = J\frac{\delta H}{\delta u}(u),
\]

where

\[
J \triangleq \partial_x(4 - \partial_x^2)(1 - \partial_x^2)^{-1}, \quad H(u) \triangleq \frac{1}{6} \int \left( u^3 + 6k \left( (4 - \partial_x^2)^{-\frac{1}{2}} u \right)^2 \right) dx.
\]

If these solitary waves under the study are local minima of the Hamiltonian, the proof is relatively straightforward. But this is not the case, as typical for nonlinear dispersive PDEs with extra conserved quantities. In fact, our solitary waves here are not even critical points of the Hamiltonian. Instead, they are critical points, but, unfortunately, not local minima, of the Lagrangian, which is a linear combination of the Hamiltonian and some conserved quantity to be specified later. The remedy is based on a framework seminally developed by Grillakis, et.al. [16, 20], which turns the characterization of soliton stability into the verification of the coercivity of the bilinear form of the second variational derivative of the Hamiltonian on a restrained space. The idea is to check whether the unstable directions are prohibited by constraints arising from symmetries, kernel of the skew-symmetric operator and foliation decomposition of the solution nearby the orbit of the solitary wave, so that the nonlinear wave under study becomes a constrained minimizer and thus orbitally stable. Typically, this framework requires verification of several conditions, listed below.

- Bounded invertibility of the skew symmetric operator in the Hamiltonian PDE;
- The linear operator, denoted as \( \mathcal{L} \), corresponding to the second variational derivative of the Hamiltonian, admits certain spectral properties. Typically, the kernel should be finite-dimensional. The intersection of the spectrum and the negative axis consists of finite many negative eigenvalues and the intersection with the positive axis admits a positive distance from the origin.
• convexity of a scalar function, the derivative of which is typically a map from the group
velocity of the nonlinear wave to the value of the Momentum evaluated at the nonlinear wave
profile with the parameter vector taking the value of the group velocity.
• higher order terms be of higher order under the right norm.

We point out major difficulties to directly follow this method through a comparison between the
DP equation and the CH equation. We start with conserved quantities. It is observed that some
relevant conservation laws of the DP equation (1.6) are generically weaker than those of the CH
equation (1.3). More specifically, there are at least three relevant conservation laws of (1.6) in study
of stability—the conservation of momentum \( M(u) \), the Hamiltonian \( H(u) \), the conserved quantity
\( S(u) \) arising from the translation symmetry, respectively taking the following forms.

\[
M(u) = \int_{\mathbb{R}} (1 - \partial_x^2)u \, dx, \quad H(u) = -\frac{1}{6} \int_{\mathbb{R}} (u^3 + 6ku \cdot (4 - \partial_x^2)^{-1}u) \, dx, \\
S(u) = \frac{1}{2} \int_{\mathbb{R}} (1 - \partial_x^2)(4 - \partial_x^2)^{-1}u \cdot u \, dx,
\]

(1.8)

while the corresponding ones of the CH equation (1.3) are the following,

\[
M(u) = \int_{\mathbb{R}} (1 - \partial_x^2)u \, dx, \quad \tilde{H}(u) = \int_{\mathbb{R}} (u^3 + uu_x^2 + 2ku^2) \, dx, \\
\tilde{S}(u) = \int_{\mathbb{R}} (u^2 + uu_x^2) \, dx.
\]

(1.9)

**Remark 1.1.** While it is straightforward to verify that \( M \) and \( H \) are conserved quantities, the
verification of the conservation of \( S(u) \) under the flow is relatively nontrivial. In fact, the conservation
of \( S(u) \) holds as long as the Hamiltonian density at spatial infinity equal to zero. More specifically,
for any solution \( u(t,x) \) to the DP equation with initial condition \( u(0, \cdot) \in H^s(\mathbb{R}) \) with \( s > 3/2 \), the
solution \( u(t,x) \) is continuous in \( x \) with \( \lim_{x \to \pm \infty} u(t,x) = 0 \) and

\[
\frac{dS}{dt} = ((1 - \partial_x^2)(4 - \partial_x^2)^{-1}u, u_t) = ((1 - \partial_x^2)(4 - \partial_x^2)^{-1}u, J\frac{\delta H}{\delta u}(u)) \\
= -(\partial_x u, \frac{\delta H}{\delta u}(u)) = \int_{\mathbb{R}} \partial_x h(u(t,x)) \, dx \\
= h(u(t, \infty)) - h(u(t, -\infty)) = h(0) - h(0) = 0,
\]

where \( h(u) = -\frac{1}{6} u^3 + 6k \left( (4 - \partial_x^2)^{-1}u \right)^2 \) is the Hamiltonian density.

In particular, one can see that the conservation law \( S \) for the DP equation is equivalent to \( \|u\|_{L^2}^2 \).
In fact, by the Fourier transform, we have

\[
S(u) = \int_{\mathbb{R}} m(4 - \partial_x^2)u \, dx = \int_{\mathbb{R}} \frac{1 + \xi^2}{4 + \xi^2} |\hat{u}(\xi)|^2 \, d\xi \sim \|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2.
\]
We shall use for convenience the equivalent norm
\[(1.10) \quad \| \cdot \|_X \triangleq \sqrt{S(\cdot)} \quad \text{satisfying} \quad \frac{1}{2}\|u\|_{L^2} \leq \|u\|_X \leq \|u\|_{L^2}.
\]

Due to such a weaker conservation law $S$ for the DP equation, compared with $\tilde{S}$ of the CH case, we can only expect orbital stability of solitons in the sense of the $L^2$ norm (the norm $\| \cdot \|_X$), which makes the study of the stability of smooth DP solitons more subtle.

In fact, taking advantage of the fact that the conserved energy $\tilde{S}$ in (1.9) of the CH equation is $H^1$ norm of the solution and fixed sign of the momentum density, the variational framework by Grillakis, et.al. \cite{16} can be successfully applied without too much trouble to obtain orbital stability of smooth CH solitons \cite{9}. More specifically,

- According to the conservation law of momentum $M$, the CH skew symmetric operator
  \[ J_{CH} = -\partial_x(1 - \partial_x^2)^{-1} \]
  is bounded and invertible when restricted to the zero-average co-dimensional one subspace.
- By the Liouville substitution, the linearized operator
  \[ L_{CH} = -\partial_x((2c - 2\phi)\partial_x) - 6\phi + 2\phi'' + 2(c - k) \]
  with respect to the soliton $\phi$, defined on the space $H^2(\mathbb{R})$, is transformed into a regular self-adjoint Sturm-Liouville operator, which is, as one readily sees, a relatively compact perturbation of a second order differential operator with constant coefficients. The required spectral properties of $L_{CH}$ then follows directly from the Sturm-Louville theory.
- The corresponding convexity condition is easily verified in the CH soliton case which takes big advantage of the simple form of the conservation law $E_3$.

As for the orbital stability of smooth DP solitons, there are several obstacles to tackle.

- The corresponding DP skew symmetric operator
  \[ J_{DP} = -\partial_x(4 - \partial_x^2)(1 - \partial_x^2)^{-1} \]
  is not bounded invertible. This obstacle is mild, since the generator of the translation symmetry is $\partial_x$, annihilating the unbounded part $\partial_x^{-1}$ in the pseudo inversion $J_{DP}^{-1}$, and making $J_{DP}^{-1}\partial_x$ bounded invertible, as in the KdV case.
- The corresponding linearized operator
  \[ L_{DP} = (c - 2k - c\partial_x^2)(4 - \partial_x^2)^{-1} - \phi \]
  fails to directly transform into a regular self-adjoint Sturm-Liouville type operator, so the study of its spectral properties becomes highly nontrivial.
- The verification of the convexity of the Lagrangian evaluated at solitary wave profiles with respect to the wave speed $c$ is also nontrivial and relies substantially on the special structure of the DP equation.

We had succeeded in overcome all the above in \cite{21}, yet not in establishing the orbital stability of DP solitons but only spectral stability. This is due to the main difference between DP and CH, say the weaker conservation law of DP and more complicated form of the Hamiltonian structure:
• The $L^2$ coercivity of the $CH$ equation is readily lifted to the $H^1$ coercivity so that the higher order term is of higher order in the right $H^1$ norm.

• The $L^2$ coercivity of the DP equation can not be lifted to the $H^1$ coercivity for the reason that the operator $L_{DP}$ is not of differential. As a result, the higher order term, say $\int h^3 dx$ is not of higher order in the right form.

Fortunately, we are inspired from the new work of Khorbatly and Molinet [19], which introduces an intelligent method to obtain $L^\infty$ control in terms of $L^2$, and are able to express our above mentioned $H^1$-higher order term $\int h^3 dx$ in terms of $L^2$-higher order term and to prove the orbital stability in the present paper.

We also remark that for the case of null-linear dispersion, the uniform $L^\infty$ control is not need. Instead, the control of a point distance is enough, say the difference between the peak of the peakon and that of the perturbation.

The remainder of the paper is organized as follows. In Section 2 we recall the local well-posedness of the Cauchy problem of the DP equation and several useful results that are crucial in the proof of the stability theorem. In Section 3, we recall the existence and uniqueness of soliton for fixed speed as well as some key spectrum properties. In Section 4 we give the proof of the orbital stability result (Theorem 1.2).

2. Well-posedness and A priori estimates

The local existence theory of the initial value problem is informative for our study of nonlinear stability. We briefly collect the needed results in this section.

Denote $p(x) = \frac{1}{2} e^{|x|}, x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * (u - u_{xx}) = u$. Using this identity, we rewrite the DP equation (1.6) as follows:

$$ u_t + \partial_x \left( \frac{1}{2} u^2 + p * \left( \frac{3}{2} u^2 + 2ku \right) \right) = 0, \quad t > 0, \quad x \in \mathbb{R}. $$

The local well-posedness of the Cauchy problem of equation (1.5), which does not include linear dispersion, with initial data $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$, is obtained in [30] applying Kato’s theorem [17]. With exactly the same argument, we have the following local well-posedness result for (2.1):

**Lemma 2.1.** Given $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique solution $u$ to equation (2.1) such that

$$ u = u(\cdot, u_0) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})). $$

Moreover, the solution depends continuously on the initial data; i.e., the mapping 

$$ u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{R}) \to C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})) $$

is continuous and the maximal time of existence $T > 0$ can be chosen to be independent of $s$.

The following two lemmas show that the only way that a classical solution to (2.1) may fail to exist for all time is that the wave may break. Corresponding results were established in [30] and [26] for the case of vanishing linear dispersion. With minor modification, their argument leads to the following two lemmas.
Lemma 2.2. Given \( u_0 \in H^s(\mathbb{R}), s > \frac{3}{2} \), blowup of the solution \( u = u(\cdot, u_0) \) in finite time \( T < +\infty \) occurs if and only if

\[
\liminf_{t \to T^-} \inf_{x \in \mathbb{R}} [u_x(t, x)] = -\infty.
\]

Lemma 2.3. Assume \( u_0 \in H^s(\mathbb{R}), s > \frac{3}{2} \). Let \( T \) be the maximal existence time of the solution \( u \) to equation (2.1). Then we have

\[
\|u(t, x)\|_{L^\infty} \leq (3\|u_0\|_{L^2}^2 + 4k\|u_0\|_{L^2})t + \|u_0(x)\|_{L^\infty}, \quad \forall t \in [0, T].
\]

Consider the following differential equation

\[
(2.2) \quad \begin{cases}
q_t = u(t, q), & t \in [0, T), \\
qu(0, x) = x, & x \in \mathbb{R}.
\end{cases}
\]

The following two results on \( q \) are useful. They are proved in [31] for the case of vanishing linear dispersion. For the current case, the proof follows from their argument with minor modification.

Lemma 2.4. Let \( u_0 \in H^s(\mathbb{R}), s \geq 3 \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( u \) to equation (2.1). Then equation (2.2) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}, \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
q_x(t, x) = \exp \int_0^t u_x(s, q(s, x))ds > 0 \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]

Lemma 2.5. Let \( u_0 \in H^s(\mathbb{R}), s \geq 3 \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( u \) to equation (2.1). Setting \( m = u - u_{xx} \), we have

\[
[m(t, q(t, x))] + \frac{2}{3}k|q_x|^3 \equiv m_0(x) + \frac{2}{3}k \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]

This lemma implies that for smooth enough solution, \( m(t, x) + \frac{2}{3}k \) is of fixed sign if \( m_0 + \frac{2}{3}k \) does. Based on this information, we prove the following global existence result.

Proposition 2.1. Assume \( u_0 \in H^s(\mathbb{R}), s > \frac{3}{2} \). If \( m_0 + \frac{2}{3}k = u_0 - u_0_{xx} + \frac{2}{3}k \) is a Radon measure of fixed sign, then (2.1) has a global strong solution

\[
u = u(\cdot, u_0) \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R})).
\]

Moreover,

(a) \( |u_x(t, x)| \leq |u(t, x)| + \frac{2}{3}k \) on \( \mathbb{R} \).

(b) \[
\|u(t)\|_{L^\infty} \leq 2(1 + \sqrt{2})\|u_0\|_{L^\infty} + \frac{4}{3}k \quad \forall t \in [0, T].
\]

Proof. We only assume \( s = 3 \) to prove the above lemma.

(a) Let \( T \) be the maximal time of existence of the solution \( u \) to equation (2.1) with initial data \( u_0 \in H^3(\mathbb{R}) \). We consider the case where \( m_0 + \frac{2}{3}k \geq 0 \) on \( \mathbb{R} \). The other case is exactly the same.
If $m_0 + \frac{2}{3}k \geq 0$, then Lemma 2.5 ensure that $m(t,x) + \frac{2}{3}k \geq 0$ for all $t \in [0,T)$ and $x \in \mathbb{R}$. From $u = p \ast m$, $u + \frac{2}{3}k = p \ast (m + \frac{2}{3}k)$, we infer that $u(t,x) + \frac{2}{3}k \geq 0$. We have

$$u(t,x) + \frac{2}{3}k = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^\eta [m(t,\eta) + \frac{2}{3}k] d\eta + \frac{e^x}{2} \int_{x}^{\infty} e^{-\eta}[m(t,\eta) + \frac{2}{3}k] d\eta,$$

and

$$u_x(t,x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^\eta [m(t,\eta) + \frac{2}{3}k] d\eta + \frac{e^x}{2} \int_{x}^{\infty} e^{-\eta}[m(t,\eta) + \frac{2}{3}k] d\eta,$$

from which we deduce that

$$u(t,x) + \frac{2}{3}k + u_x(t,x) = e^x \int_{x}^{\infty} e^{-\eta}[m(t,\eta) + \frac{2}{3}k] d\eta,$$

$$u(t,x) + \frac{2}{3}k - u_x(t,x) = e^{-x} \int_{-\infty}^{x} e^\eta [m(t,\eta) + \frac{2}{3}k] d\eta.$$

From the above and the fact that $m(t,x) + \frac{2}{3}k \geq 0$, we obtain for all $t \in [0,T)$, $x \in \mathbb{R}$,

$$|u_x(t,x)| \leq |u(t,x) + \frac{2}{3}k|.$$

(b). Fix any $t \in [0,T)$, $x \in \mathbb{R}$ and denote by $[x]$ the integer part of $x$. Since $u(t,\cdot) \in H^s \subset C(\mathbb{R})$, the Mean-Value theorem ensures that there exists $\bar{\xi} \in [[x] - 1,[x]]$ such that

$$u^2(t,\bar{\xi}) = \int_{[x]}^{[x]+1} u^2(t,\eta) d\eta \leq \|u(t,\cdot)\|_{L^2(\mathbb{R})}^2 \leq 4S(u(t,\cdot)) = 4S(u_0).$$

Therefore, since $0 \leq x - \bar{\xi} \leq 2$, we have

$$u(t,x) = u(x,\bar{\xi}) + \int_{\bar{\xi}}^{x} u_\eta(t,\eta) d\eta \geq -2\|u_0\|_X - \int_{\bar{\xi}}^{x} (|u(\eta)| + \frac{2}{3}k) d\eta$$

$$\geq -2\|u_0\|_X - \frac{4}{3}k - \sqrt{2}\|u(t,\cdot)\|_{L^2(\mathbb{R})}$$

$$\geq -2(1 + \sqrt{2})\|u_0\|_X - \frac{4}{3}k.$$

To prove (b), we suppose otherwise that there exists $x_*$ such that $u(t,x_*) > 2(1 + \sqrt{2})\|u_0\|_X + \frac{4}{3}k$. Then, on one side the Mean-Value theorem ensures that there exists $\bar{\xi}_* \in [[x_*] + 1,[x_*] + 2]$ such that

$$u^2(t,\bar{\xi}_*) = \int_{[x_*]}^{[x_*]+2} u^2(t,\eta) d\eta \leq \|u(t,\cdot)\|_{L^2(\mathbb{R})}^2 \leq 4S(u(t,\cdot)) = 4S(u_0).$$

On the other side, part (a) leads to

$$u(t,\bar{\xi}_*) = u(t,x_*) + \int_{x_*}^{\bar{\xi}_*} u_0(t,\eta) d\eta \geq 2(1 + \sqrt{2})\|u_0\|_X + \frac{4}{3}k - (\int_{x_*}^{\bar{\xi}_*} (|u(t,\eta)| + \frac{2}{3}k))$$

$$\geq 2(1 + \sqrt{2})\|u_0\|_X + \frac{4}{3}k - \left(\frac{4}{3}k + \sqrt{2}\|u(t,\cdot)\|_{L^2(\mathbb{R})}\right)$$

$$\geq 2\|u_0\|_X.$$
The fact that the above two estimates are not compatible completes the proof of part (b).

(a) and (b) then imply \( u_x \) is uniformly bounded for all \( t \in [0, T), x \in \mathbb{R} \), which together with Lemma 2.2 implies non-blowup and global existence of strong solution.

\[ \square \]

The following proposition is of pivotal importance in establishing the orbital stability of the DP solitons. We follow the argument of Khorbatly and Molinet [19] with little modification.

**Proposition 2.2.** Assume \( u_0 \in H^s(\mathbb{R}), s > \frac{2}{3} \). If \( m_0 + \frac{2}{3} k = u_0 - u_{0,x} + \frac{2}{3} k \) is a Radon measure of fixed sign, then

\[
(2.3) \quad \|u(t, \cdot) - \psi^c(\cdot)\|_{L^\infty(\mathbb{R})} \leq 2\|u(t, \cdot) - \psi^c(\cdot)\|_{L^\infty(\mathbb{R})} + 2\|\psi\|_{L^\infty(\mathbb{R})}.
\]

**Proof.** Denote \( \alpha = \|u(t, \cdot) - \psi^c(\cdot)\|_{L^\infty(\mathbb{R})}^2 \). Fixing \( x \in \mathbb{R} \), there exists \( k \in \mathbb{Z} \) such that \( x \in [k\alpha, (k+1)\alpha) \).

By the Mean-Value theorem, there exists \( \bar{x} \in [(k-1)\alpha, k\alpha] \) such that

\[
[u(t,\bar{x}) - \psi(\bar{x})]^2 = \frac{1}{\alpha} \int_{(k-1)\alpha}^{k\alpha} [u(t,\eta) - \psi(\eta)]^2 d\eta \leq \frac{4}{\alpha}\|u(t,\cdot) - \psi^c(\cdot)\|_{L^\infty(\mathbb{R})}^2 = 4\alpha^2.
\]

In view of Proposition 2.1 part (a) and the fact that \( 0 \leq x - \bar{x} \leq 2\alpha \), we get

\[
(2.4) \quad u(t,x) - \psi(x) = u(t,\bar{x}) - \psi(\bar{x}) + \int_{\bar{x}}^{x} [u(\eta) - \psi(\eta)] d\eta \\
\geq -2\alpha - \frac{4\alpha}{3}k - \sqrt{2\alpha}\|(|u(t,\cdot)| + |\psi'|)\|_{L^2([k\alpha,(k+1)\alpha])} \\
\geq -2\alpha - \frac{4\alpha}{3}k - \sqrt{2\alpha}\|2|u(t,\cdot)| - \psi| + |\psi'|\|_{L^2([k\alpha,(k+1)\alpha])} \\
\geq -2\alpha - \frac{4\alpha}{3}k - \sqrt{2\alpha}\|2|u(t,\cdot)| - \psi|\|_{L^\infty(\mathbb{R})} + \sqrt{2\alpha}\|\psi'\|_{L^\infty(\mathbb{R})} + \|\psi\|_{L^\infty(\mathbb{R})} \\
= -2\alpha(1 + \frac{2}{3}k + \sqrt{2\alpha} + \|\psi\|_{L^\infty(\mathbb{R})} + \|\psi'\|_{L^\infty(\mathbb{R})}).
\]

Now, suppose that there exists \( x_* \in \mathbb{R} \) such that

\[
u(t, x_*) - \psi(x_*) > 2\alpha(1 + \frac{2}{3}k + \sqrt{2\alpha} + \|\psi\|_{L^\infty(\mathbb{R})} + \|\psi'\|_{L^\infty(\mathbb{R})}).
\]

Similarly, there exists \( k_* \in \mathbb{R} \) with \( x_* \in [k_*\alpha,(k_*+1)\alpha) \) such that by the Mean-Value theorem, there exists \( \bar{x}_* \in [(k_*+1)\alpha, (k_*+2)\alpha] \) such that, on one hand,

\[
[u(t, \bar{x}_*) - \psi(\bar{x}_*)]^2 = \frac{1}{\alpha} \int_{(k_*+1)\alpha}^{(k_*+2)\alpha} [u(t,\eta) - \psi(\eta)]^2 d\eta \leq 4\alpha^2.
\]
On the other hand, proceeding as in (2.4):

\[ u(t, \bar{x}_*) - \psi(\bar{x}_*) = u(t, x_*) - \psi(x_*) + \int_{x_*}^{\bar{x}_*} [u_\eta(t, \eta) - \psi'(\eta)] d\eta \]

\[ > 2\alpha \left( 1 + \frac{2}{3} k + \sqrt{2\alpha} + \|\psi\|_{L^\infty(\mathbb{R})} + \|\psi'\|_{L^\infty(\mathbb{R})} \right) \]

\[ - \frac{4\alpha}{3} k - \frac{\sqrt{2\alpha}}{2} \left( \|u(t, \cdot)\| + \|\psi'(\cdot)\|_{L^2([k\alpha, (k+2)\alpha])} \right) \]

\[ \geq 2\alpha \left( 1 + \frac{2}{3} k + \sqrt{2\alpha} + \|\psi\|_{L^\infty(\mathbb{R})} + \|\psi'\|_{L^\infty(\mathbb{R})} \right) \]

\[ - \frac{4\alpha}{3} k - 2\alpha \left( \sqrt{2\alpha} + \|\psi\|_{L^\infty(\mathbb{R})} + \|\psi'\|_{L^\infty(\mathbb{R})} \right) \]

\[ = 2\alpha. \]

Again, the incompatibility of the above two estimates completes the proof of the proposition. \(\square\)

3. Useful Soliton Properties

We recall in this section useful properties about smooth solitary wave of (1.7):

\[ u_t = J\delta H \delta u(u), \]

where

\[ J = \partial_x(4 - \partial_x^2)(1 - \partial_x^2)^{-1}, \quad H(u) = -\frac{1}{6} \int \left( u^3 + 6k \left( (4 - \partial_x^2)^{-\frac{1}{2}} u \right)^2 \right) dx. \]

Changing the \((t, x)\) coordinates into the traveling frame \((t, \xi)\) with \(\xi \triangleq x - ct\) and slightly abusing the notation by denoting \(u(t, \xi) \triangleq u(t, x - ct)\), the equation (1.7) is now written as

\[ u_t = J \delta H \delta u(\phi) + cu_{\xi} = J \frac{\delta H}{\delta u}(u) + c \frac{\delta S}{\delta u}(u), \]

where we recall

\[ S(u) = \frac{1}{2} \int (1 - \partial_\xi^2)(4 - \partial_\xi^2)^{-1} u \cdot u d\xi. \]

Introducing the Lagrangian

\[ Q_c(u) \triangleq H(u) + cS(u), \]

the solitary wave with speed \(c > 0\), denoted as \(\phi(\xi; c)\) or \(\phi'(\xi)\) or \(\phi\), is a steady state of (3.1) and a critical point of the Lagrangian, namely,

\[ \frac{\delta Q_c}{\delta u}(\phi) = \frac{\delta H}{\delta u}(\phi) + c \frac{\delta S}{\delta u}(\phi) = 0, \]
which is equivalent to
\[
-\left[ \frac{1}{2} \phi^2 + (4 - \partial^2_{\xi})^{-1} 2k\phi \right] + c(1 - \partial^2_{\xi})(4 - \partial^2_{\xi})^{-1}\phi = 0,
\]
and by applying \( 4 - \partial^2_{\xi} \) is equivalent to
\[
-2\phi^2 + \phi \phi_{\xi} + \phi_{\xi}^2 - 2k\phi + c(\phi - \phi_{\xi}) = 0,
\]
possessing a first integral
\[
\Phi(\phi, \psi) = \phi^2 \left( \frac{1}{2} \phi^2 - c\phi + \frac{2}{3} k\phi + \frac{1}{2} c^2 - kc \right) - \frac{1}{2} (c - \phi)^2 \psi^2.
\]
A solitary wave of (3.1) corresponds to the \( \phi \) entry of the connected component of the level curve
\[
\Phi(\phi, \psi) = \Phi(0, 0) = 0,
\]
which connects to the origin. Any point \((\phi, \psi)\) on the level curve \(\Phi(\phi, \psi) = 0\) satisfies
\[
\phi^2 \left( \frac{1}{2} \phi^2 - c\phi + \frac{2}{3} k\phi + \frac{1}{2} c^2 - kc \right) = \frac{1}{2} (c - \phi)^2 \phi_{\xi}^2.
\]
A more careful study \([21]\) about the above curve leads to Theorem 1.1.

We next present spectrum properties of the corresponding linear operator of the second-order variational derivative of the Lagrangian, which is critical to the stability of smooth solitary waves. From now on, we simply write \(\phi\) for the solitary wave profile \(\phi(\xi; c)\) unless specified.

Consider under the traveling frame \((t, \xi)\) the linearization of (3.1) along the soliton \(\phi\),
\[
v_t = JL_{c}v,
\]
where \(v \in L^2(\mathbb{R})\), and
\[
L_{c} = \frac{\delta^2 Q_{c}}{\delta u^2}(\phi) = -\phi - 2k(4 - \partial^2_{\xi})^{-1} + c(1 - \partial^2_{\xi})(4 - \partial^2_{\xi})^{-1} = c - \phi - (3c + 2k)(4 - \partial^2_{\xi})^{-1}.
\]
It is straightforward to see that \(L_{c} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\) is a well-defined, self-adjoint, bounded linear operator. Moreover, we proved in \([21]\) the following spectral theorem about the operator \(L_{c}\):

**Theorem 3.1.** \([21]\) The spectrum set of the operator \(L_{c}\), denoted as \(\sigma(L_{c})\), admits the following properties.

1. The spectrum set \(\sigma(L_{c})\) lies on the real line; that is, \(\sigma(L_{c}) \subset \mathbb{R}\).
2. 0 is a simple eigenvalue of \(L_{c}\) with \(\partial_{\xi}\phi(\xi)\) as its eigenfunction.
3. On the negative axis \((-\infty, 0)\), the spectrum set \(\sigma(L_{c})\) admits nothing but only one simple eigenvalue, denoted as \(\lambda_\ast\), with its corresponding normalized eigenfunction, denoted as \(\phi_\ast\).
4. The set of essential spectrum \(\sigma_{ess}(L_{c})\) lies on the positive real axis, admitting a positive distance to the origin.

The next result is crucial:
Lemma 3.1. [21] The momentum $S$ is an increasing function about the wave speed when evaluated at the solitary wave:

$$\frac{d}{dc} S(\phi(\cdot, c)) = \frac{3c^2(c + k)}{(3c + 2k)^2} \sqrt{\frac{c - 2k}{c}} > 0, \text{ for any } c > 2k > 0.$$ 

Lemma 3.1 and Theorem 3.1 enable us to establish the desired $L^2$ coercivity property, i.e. Proposition 4.1.

4. Orbital Stability of Degasperis-Procesi Solitons

In this section, we give a proof of the Theorem 1.2, which is based on the framework of Grillakis, et.al. [16, 20] with major modification on nonlinear estimates. More specifically, we recall that the DP equation is a Hamiltonian PDE with the spatial translation symmetry, yielding the conserved quantity,

$$S(u) = \frac{1}{2} \int (1 - \partial_x^2)(4 - \partial_x^2)^{-1} u \cdot u \, dx.$$ 

We now consider the orbital stability of solitary waves $\phi$ under the conservation constraint $S(u) = S(\phi)$.

We note that this constraint is dropped in Theorem 1.2 since the value of $S$ evaluated at $u$ is equivalent to the $L^2$-norm of $u$.

Introducing the translation operator $T(r)u(\cdot) = u(\cdot + r)$ and assuming that the solution $u(t, x)$ is sufficiently close to the solitary wave orbit $O_c$ for a time $T_0 > 0$, then there exist $r(t) \in \mathbb{R}$ and $h(t, \cdot) \in L^2$ with $(h, \partial_x \phi) = 0$ such that $u(t, x)$ admits the foliation decomposition

$$u(t, x) = T(r(t)) \left( \phi(x) + h(t, x) \right).$$

We now recall the definition of the Lagrangian

$$Q_c(u) = H(u) + cS(u) = \frac{1}{6} \int \left( u^3 + 6k \left( (4 - \partial_\xi^2)^{-\frac{1}{2}} u \right)^2 \right) \, d\xi + \frac{c}{2} \int (1 - \partial_\xi^2)(4 - \partial_\xi^2)^{-1} u \cdot u \, d\xi$$

and introduce the time-invariant quantity

$$\overline{Q_c} \triangleq Q_c(u) - Q_c(\phi),$$

which admits

$$\bar{Q}_c = Q_c(\phi(x) + h(t, x)) - Q_c(\phi) = \frac{1}{2} (L_c h, h) - \frac{1}{6} \int h^3 \, d\xi,$$

where $h$ lies in the nonlinear admissible set

$$\mathcal{A} \triangleq \{ h \in H^1(\mathbb{R}) \mid S(h + \phi) = S(\phi), (h, \partial_\xi \phi) = 0 \}.$$ 

For the sake of narrative coherency, we give the following proposition and leave the proof to the end of the section.
Proposition 4.1. For small $h \in \mathcal{A}$, there exist $\alpha, \beta > 0$ such that

$$\frac{1}{2}(L_{c}h, h) \geq \alpha \|h\|_{X}^{2} - \beta \|h\|_{X}^{3}. \tag{4.2}$$

We now take advantage of the expansion of $Q_{c}$ and the above proposition to prove Theorem 1.2.

Proof of Theorem 1.2. Combining (4.1,4.2) and setting in Proposition 2.2 appropriate $\psi(\cdot) \triangleq \phi(\cdot + r(t))$, we readily see that there exist some $\beta_{1} > 0$, which depends on $k,c$ in definite way, such that

$$|Q_{c}| \geq \alpha \|h\|_{X}^{2} - \beta \|h\|_{X}^{3} - \frac{1}{6} \|h\|_{L^{\infty}} \cdot \|h\|_{L^{2}}^{2}$$

$$\geq \alpha \|h\|_{X}^{2} - \beta \|h\|_{X}^{3} - \frac{4}{3} \left(\|h\|_{X}^{2} \cdot \left(1 + \frac{2}{3} k + \|\phi\|_{L^{\infty}} + \|\phi'\|_{L^{\infty}} + \sqrt{2}\|h\|_{X}^{2}\right)\right) \cdot \|h\|_{X}^{2}$$

$$= \alpha \|h\|_{X}^{2} - \beta_{1}(c,k) \|h\|_{X}^{8} - \beta \|h\|_{X}^{3} - \frac{4\sqrt{2}}{3} \|h\|_{X}^{10}.$$ \hspace{1cm} (4.3)

By noting that the function $f(r) \triangleq |Q_{c}| - \alpha r^{2} + \beta_{1}(c,k)r^{8} + \beta r^{3} + \frac{4\sqrt{2}}{3} r^{10}$ admits for small $|Q_{c}|$ two positive roots

$$0 < r_{1} = O(|Q_{c}|^{1/2}) < r_{2} = O(1),$$

we conclude from the inequality (4.3) and the continuity of $h(t)$ that if $\|h_{0}\|_{X} \in (0,r_{1})$, then $\|h(t)\|_{X} \in (0,r_{1})$ for the time interval of existence. Therefore, for any $\varepsilon > 0$, we can choose $\delta \in (0,\alpha/(2\beta_{1}))$ such that for any

$$\|h_{0}\|_{X} = \|u_{0} - T(r(0))\phi\|_{X} = \inf_{r \in \mathbb{R}} \|u_{0} - T(r)\phi\|_{X} \leq \delta,$$

Note that $|S(u_{0}) - S(\phi)| = O(\delta^{2})$ and

$$|H(u_{0}) - H(\phi)| \leq |Q_{c}(u_{0}) - Q_{c}(\phi)| + O(\delta^{2})$$

$$= \frac{1}{2}(L_{c}h_{0}, h_{0}) - \frac{1}{6} \int h_{0}^{3}d\xi + O(\delta^{2})$$

$$= O(\|h_{0}\|_{X}^{2}) + O(\|h_{0}\|_{X}^{2}) \cdot O(\|h_{0}\|_{X}^{3}) + O(\delta^{2})$$

$$= o(\delta) \text{ for } \delta \ll 1.$$ \hspace{1cm} (4.4)

It follows that $|Q_{c}|$ is sufficiently small (which is $o(\delta)$) so that $r_{1} < \min\{\varepsilon, \alpha/(2\beta_{1})\}$ and $r_{2} > \alpha/(2\beta_{1})$. As a result,

$$\inf_{r} \|u(t, \cdot) - T(r)\phi\|_{X} = \|h(t)\|_{X} < r_{1} < \varepsilon,$$

for any $t$ in the existence interval. This establishes the desired $L^{2}$ (or $\|\cdot\|_{X}$) stability estimate under the assumption that $m_{0} + \frac{2k}{3} = u_{0} - u_{0xx} + \frac{2k}{3}$ is a positive Radon measure. The $L^{\infty}$ estimate follows from the $L^{2}$ estimate and Proposition 2.2.

In particular, it is straightforward to compute from (3.4) the following:

$$\phi - \phi'' = \frac{2k}{3} \left(\frac{c^{3}}{(c - \phi)^{3}} - 1\right) > 0.$$
It follows that if \( \|u(0, \cdot) - \phi(\cdot)\|_{H^3} \leq \frac{2\sqrt{2}}{3} k \), then
\[
|\langle u_0 - u_{0,xx}, \phi'' \rangle| \leq \frac{1}{\sqrt{2}}\|u(0, \cdot) - \phi(\cdot)\|_{H^3} \leq \frac{2k}{3},
\]
and \( m_0 + \frac{2k}{3} = u_0 - u_{0,xx} + \frac{2k}{3} \) is a positive Radon measure.

This completes the proof of Theorem 1.2. \( \square \)

We are now left to prove Proposition 4.1. In order to do so, we denote \( \tilde{\psi} \equiv (1 - \partial^2_\xi)(4 - \partial^2_\xi)^{-1} \phi \) and introduce the linear admissible space
\[
A' \equiv \{ h \in L^2 \mid \langle h, \tilde{\psi} \rangle = \langle h, \partial_\xi \phi \rangle = 0 \}.
\]
It is then straightforward to see that any \( h \in A \) with \( \|h\|_X \) small admits the decomposition
\[
h = \tilde{h} + a\phi,
\]
where \( \tilde{h} \in A' \) and \( |a| = O(\|h\|_{L^2}^2) \). As a result, to prove Proposition 4.1, it suffices to prove the following lemma.

**Lemma 4.1.** For \( h \in A' \), there exists \( \alpha > 0 \) such that
\[
(L_c h, h) \geq \alpha\|h\|_{L^2}^2.
\]

**Proof.** Introduce the self-adjoint projection
\[
\Pi u = u - \frac{(u, \tilde{\psi})}{(\psi, \psi)} \tilde{\psi}.
\]

The constrained operator \( L^\Pi_c \equiv \Pi L_c : A' \to A' \) is self-adjoint and bounded invertible, and thus admits only real spectra. Moreover, the essential spectrum of \( L^\Pi_c \) and \( L_c \) are the same, due to the fact that \( L^\Pi_c - L_c \) restricted to \( A' \) is rank one and thus compact. We now just need to show that the smallest eigenvalue, called the ground-state eigenvalue and denoted as \( \tilde{\lambda}_s \), if there is any, is strictly positive. We first recall that \( \lambda_\ast \) is the ground-state eigenvalue of the full linear operator \( L_c \) with its eigenfunction \( \phi_\ast \), and note that
\[
\tilde{\lambda}_s = \inf_{u \in A'} \frac{(L_c u, u)}{(u, u)} \geq \inf_{u \in L^2} \frac{(L_c u, u)}{(u, u)} \geq \lambda_\ast.
\]

As a matter of fact, we have an improved estimate
\[
\tilde{\lambda}_s > \lambda_\ast,
\]
due to the fact that \( \phi_\ast \notin A' \), which, in turn, is a natural consequence of the fact that
\[
(\phi_\ast, \tilde{\psi}) \neq 0.
\]

To prove this inequality, we only need to show that both \( \phi_\ast \) and \( \tilde{\phi} \) are functions of one sign. According to Theorem 3.1, \( L_c \phi_\ast = \lambda_\ast \phi_\ast \) can be rewritten as
\[
\partial^2_\xi w_\ast - A(\xi, \lambda_\ast) w_\ast = 0
\]
where \( w_* \triangleq (4 - \partial^2_\xi)^{-1} \phi_* \) is even and everywhere positive, yielding, for all \( \xi > 0 \),
\[
\phi(\xi) = (4 - \partial^2_\xi) w_*(\xi) = (4 - A(\xi, \lambda_*)) w_*(\xi) = \frac{3c + 2k}{c - \phi(\xi) - \lambda_*} w_*(\xi) > 0.
\]

On the other hand, we have \( \tilde{\psi} = (1 - \partial^2_\xi)(4 - \partial^2_\xi)^{-1} \phi = \phi - 3(4 - \partial^2_\xi)^{-1} \phi \), where the profile \( w \triangleq (4 - \partial^2_\xi)^{-1} \phi \) can be expressed in terms of \( \phi \). More specifically, from the traveling wave equation (3.4),
\[
c(1 - \partial^2_\xi)(4 - \partial^2_\xi)^{-1} \phi - \frac{1}{2}(\phi)^2 + (4 - \partial^2_\xi)^{-1} 2k \phi = 0,
\]
we have
\[
c(\phi - 3w) = \frac{1}{2} \phi^2 + 2kw,
\]
which, after simple rearrangements, yields
\[
w = \frac{2c\phi - \phi^2}{6c + 4k}.
\]
Thus, we have
\[
(4.4) \quad \tilde{\psi} = (1 - \partial^2_\xi)(4 - \partial^2_\xi)^{-1} \phi = \frac{3\phi + 4k}{2(3c + 2k)} \phi > 0,
\]
which concludes the proof that \( \tilde{\lambda}_* > \lambda_* \). Denoting the \( L^2 \)-normalized eigenfunction of \( L_c^{\Pi} \) with respect to \( \tilde{\lambda}_* \) as \( \tilde{\phi}_* \), there exists \( b \in \mathbb{R} \) such that
\[
L_c \tilde{\phi}_* = \tilde{\lambda}_* \tilde{\phi}_* + b\tilde{\phi}, \quad \Rightarrow \quad \tilde{\phi}_* = b(L_c - \tilde{\lambda}_*)^{-1} \phi.
\]
Noting that \( \tilde{\phi}_* \in \mathcal{A}' \) imposes the constraint \( (\tilde{\phi}, \tilde{\phi}_*) = 0 \), we introduce the scalar function
\[
g(\lambda) \triangleq ((L_c - \lambda)^{-1} \phi, \phi),
\]
which is real analytic for \( \lambda \notin \sigma(L_c) \setminus \{0\} \); strictly increasing on connected smooth intervals; and most importantly, \( g(\lambda) = 0 \) if and only if \( \lambda \) is an eigenvalue of \( L_c^{\Pi} \). Moreover, \( g(\lambda) \) is strictly increasing on \( (\lambda_*, \lambda_0) \), where \( \lambda_0 > 0 \) is given in the proof of Theorem 3.1, and admits \( \lambda_* \) as a pole. Consequently, \( \tilde{\lambda}_* > 0 \iff g(0) < 0 \).

To see that \( g(0) < 0 \), we recall that \( \phi \) is a critical point of the Lagrangian \( Q_c \), that is,
\[
\frac{\delta Q_c}{\delta u}(\phi) = \frac{\delta H}{\delta u}(\phi) + c \frac{\delta S}{\delta u}(\phi) = 0,
\]
which, taken derivative with respect to \( c \) to both sides, yields,
\[
L_c \partial_c \phi = -\frac{\delta S}{\delta u}(\phi) = (1 - \partial^2_\xi)(4 - \partial^2_\xi)^{-1} \phi = -\tilde{\phi}.
\]
Consequently, we rewrite \( g(0) \) as
\[
g(0) = (L_c^{-1} \phi, \phi) = -(\partial_c \phi, \frac{\delta S}{\delta u}(\phi)) = -\frac{d}{dc} S(\phi).
\]
We then conclude from Lemma 3.1 that \( g(0) < 0 \) and \( \tilde{\lambda}_* > 0 \).
Proof of Proposition 4.1. The proof has been given above, based on Lemma 4.1 and Lemma 3.1.

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