FULL GROUPS AND SOFICITY

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ABSTRACT. First, we answer a question of Giordano and Pestov by proving that the full group of a sofic equivalence relation is a sofic group. Then, we give a short proof of the theorem of Grigorchuk and Medynets that the topological full group of a minimal Cantor homeomorphism is LEF. Finally, we show that for certain non-amenable groups all the generalized lamplighter groups are sofic.

1. Introduction

1.1. Sofic groups and LEF groups. The notion of sofic groups was introduced by Weiss [12] and Gromov [5] (in a somewhat different form). A group $\Gamma$ is sofic if for any finite set $F \subset \Gamma$ and $\epsilon > 0$ there exists a finite set $A$ and a mapping $\Theta : \Gamma \to \text{Map}(A)$ such that

\begin{align*}
&\bullet \text{ If } f, g, fg \in F, \text{ then } d_H(\Theta(fg) - \Theta(f)\Theta(g)) \leq \epsilon, \text{ where } \\
&\quad d_H(\alpha, \beta) = \frac{|\{x \in A \mid \alpha(x) \neq \beta(x)\}|}{|A|}.
\end{align*}

\begin{itemize}
\item $\bullet$ If $1 \neq f \in F$, then $d_H(\Theta(f), 1) > 1 - \epsilon$.
\item $\Theta(1) = 1$.
\end{itemize}

All amenable and residually finite groups are sofic. It is an open question whether non-sofic groups exist. If we add the extra requirement that $\Theta(fg) = \Theta(f)\Theta(g)$, then we get the class of LEF-groups (locally embeddable into finite groups). This class of groups was introduced by Gordon and Vershik [11]. Clearly, all residually finite groups are LEF. However, simple, finitely presented groups are not LEF. Nevertheless, by a recent result of Juschenko and Monod [6] (and Theorem 2), there exist simple, finitely generated LEF-groups.

1.2. Sofic equivalence relations. Let $X = \{0, 1\}^\mathbb{N}$ be the standard Borel space with the natural product measure $\mu$. Let $\Phi : \mathbb{F}_\infty \acts X$ be a (not necessarily free) Borel action of the free group of countably infinite generators $\{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}, \ldots\}$ preserving $\mu$. Note that $\mathbb{F}_\infty = \bigcup_{r=1}^\infty \mathbb{F}_r$, where $\mathbb{F}_r$ is the free group of rank $r$. Hence, we also have probability measure preserving (p.m.p.) Borel actions $\Phi_r : \mathbb{F}_r \acts X$. We say that $x, y \in X$ are equivalent, $x \sim_\Phi y$ if there exists $w \in \mathbb{F}_\infty$, such that $w(x) = y$. Note that slightly abusing the notation we write $w(x)$ instead of $\Phi(w)(x)$. Thus, the action $\Phi$ represents a countable measured equivalence relation $E_\Phi$ on $X$. Similarly, each $\Phi_r$ represents a countable measured equivalence relation $E_{\Phi_r}$ on $X$.

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1943
and $E_\Phi = \bigcup_{i=1}^\infty E_{\Phi^i}$. Each equivalence relation $E_{\Phi^i}$ defines a graphing $[7] G_r$ on $X$:

- $V(G_r) = X$.
- $(x, y) \in E(G_r)$ if $\gamma_i x = y$ or $\gamma_i y = x$ for some $i$ (so, there may be loops in $G_r$).

Observe that each component of $G_r$ is a countable graph of bounded vertex degrees. We label each directed edge $(x, y)$ with all the generators mapping $x$ to $y$. Thus an edge, even a loop, may have multiple labels.

Now let us consider transitive actions of $F_r$ on countable sets. If $\alpha : F_r \curvearrowright Y$ is such an action, then we have a bounded degree graph structure on $Y$ with multiple labels on the edges from the set $\{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_r, \gamma_r^{-1}\}$. Let $T_r$ be the set of graphs of all countable $F_r$-actions with a distinguished vertex (the root) such that all the vertices are labeled by the elements of $\{0, 1\}^r$. Let $G \in T_r$. We define the the $k$-ball around the root $x$, $B_k(x)$, as the induced subgraph on vertices of $G$ in the form of $w(x)$, where $w \in F_r$ is a reduced word of length at most $k$. That is, $B_k(x)$ is the ball centered at $x$ of radius $k$ with respect to the shortest path metric of $G$. The ball $B_k(x)$ is a finite rooted graph with edge-colors from the set $\{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_r, \gamma_r^{-1}\}$ and vertex labels from the set $\{0, 1\}^r$. We denote the set of all possible $k$-balls arising from $F_r$-actions by $U^k_r$. We can define a compact metric structure on the set $T_r$ the following way. Let $d_r(G, H) = 1/k$ if $k$ is the maximal number such that the $k$-balls around the roots of $G$, resp. $H$, are isomorphic as rooted, labeled graphs.

Observe that if $\Theta : F_\infty \curvearrowright X$ is a p.m.p. action, then for each $r \geq 1$ and $x \in X$ one can associate an element $G(\Theta, x) \in T_r$. Namely, the orbit graph of $x$, where the vertex labels are given by the $X$-values, restricted on the first $r$ coordinates. Thus, we have a Borel map $\pi_\Theta : X \to T_r$. For $\kappa \in U^k_r$, let $\mu^{k,\kappa}_{\Theta_r}(\kappa) = (\pi_\Theta)_*(\mu(L_\kappa))$, where $L_\kappa \subset T_r$ is the set of elements $G$ such that the $k$-ball around the root of $G$ is isomorphic to $\kappa$. In other words, $\mu^{k,\kappa}_{\Theta_r}(\kappa)$ is the probability that the $k$-ball around a $\mu$-random element of $X$ is isomorphic to $\kappa$. Now let $\alpha : F_r \curvearrowright Y$ be an $F_r$-action on a finite set. Then for each element $y$ of $Y$, we can associate an element of $T_r$; namely, $Y$ itself with root $y$. Hence, we can define a probability distribution $\mu^{k,\kappa}_{\alpha_r}$ on $U^k_r$. Following [1] we say that the action $\Theta : F_\infty \curvearrowright X$ is sofic if for all $r \geq 1$, there exists a sequence of finite $F_r$-actions $\{\alpha_n\}_{n=1}^\infty$ such that for each $k \geq 1$ and $\kappa \in U^k_r$,

\[ \lim_{n \to \infty} \mu^{k,\kappa}_{\alpha_n} = \mu^k_{\Theta_r}(\kappa). \]

In [1] the authors proved that

- Soficity is a property of the underlying equivalence relations. That is, if an action $\Theta_1$ is orbit equivalent to a sofic action $\Theta_2$, then $\Theta_2$ is sofic as well.
- Treeable equivalence relations are sofic.
- Actions associated to Bernoulli shifts of sofic groups are sofic.

### 1.3. Full groups

Let $E(X, \mu)$ be a countable, measured equivalence relation on a Borel set $X$ with invariant measure $\mu$. The Borel full group of $E$ is the group $[E]_B$ of all Borel bijections $T : X \to X$ such that for any $x \in X$, $T(x) \sim_E x$. We call two such bijections $T_1, T_2$ equivalent if

\[ \mu(\{x \in X \mid T_1(x) = T_2(x)\}) = 1. \]
The measurable full group $[E]$ is the group formed by the equivalence classes. Obviously, $[E] = [E]_B/N$, where $N$ is the normal subgroup of elements in $[E]_B$ fixing almost all points of $X$.

Now, let $T : C \to C$ be a homeomorphism of the Cantor set $C$. The topological full group $[[T]]$ is the group of homeomorphisms $S : C \to C$ such that $C$ can be partitioned into finitely many clopen sets $C = \bigcup_{i=1}^n A_i$ such that $S|_{A_i} = T^{n_i}$ for some integer $n_i$.

1.4. Results. Answering a question of Giordano and Pestov, we prove the following theorem.

**Theorem 1.** The measurable full group of a sofic equivalence relation is sofic.

Then, we give a very short proof of a result of Grigorchuk and Medynets [4].

**Theorem 2.** The topological full group of a minimal Cantor homeomorphism is LEF.

Let $X$ be a countably infinite set and $\Gamma$ be a countable group acting faithfully and transitively on $X$. Then $\Gamma$ can be represented by automorphisms on the Abelian group $\bigoplus_{x \in X} \{0, 1\}$. The groups $\bigoplus_{x \in X} \{0, 1\} \rtimes \Gamma$ are called the lamplighter group of the $\Gamma$-action. If the action is the natural translation action on $\Gamma$, then we get the classical lamplighter group $\bigoplus_{\gamma \in \Gamma} \{0, 1\} \rtimes \Gamma$ is sofic. If $\Gamma$ is amenable, then all its generalized lamplighter groups are amenable, hence sofic. Nevertheless, we show that there exist non-amenable groups for which all the generalized lamplighter groups are sofic.

**Theorem 3.** Let $\Gamma^k$ be the $k$-fold free product of the cyclic group of two elements. Then, for any transitive, faithful action of $\Gamma^k$ on a countable set the associated lamplighter group is LEF.

2. Compressed sofic representations

Let $\Gamma$ be a countable sofic group with elements $\{\gamma_1, \gamma_2, \ldots\}$. A compressed sofic representation of $\Gamma$ is defined in the following way. For any $i \geq 1$ we have a constant $\epsilon_i > 0$, and for any $n \geq 1$ we have mappings $\Theta_n : \Gamma \to Map(A_n)$ such that $|A_n| < \infty$ satisfying the following condition: For all $r > 0$ and $\epsilon > 0$ there exists $K_{r, \epsilon} > 0$ such that if $n > K_{r, \epsilon}$, then

- $d_H(\Theta_n(\gamma_i \gamma_j) \Theta_n(\gamma_i) \Theta_n(\gamma_j)) < \epsilon$ if $1 \leq i, j \leq r$.
- $d_H(\Theta_n(\gamma_i), Id) > \epsilon_i$ if $1 \leq i \leq r$.

Thus, in a compressed sofic representation we allow a large amount of fixed points for each $\gamma \in \Gamma$.

**Lemma 2.1.** If $\Gamma$ has a compressed sofic representation, then $\Gamma$ is sofic.

**Proof.** Let $\tilde{\Theta}_n^k : \Gamma \to Map(A_n^k)$ be defined by

$$\tilde{\Theta}_n^k(\gamma)(x_1, x_2, \ldots, x_k) = (\Theta_n(\gamma)(x_1), \Theta_n(\gamma)(x_2), \ldots).$$

Observe that if $\gamma, \delta \in \Gamma$, then

- $d_H(\tilde{\Theta}_n^k(\gamma \delta), \tilde{\Theta}_n^k(\gamma) \tilde{\Theta}_n^k(\delta)) \leq (1 - d_H(\Theta_n(\gamma \delta), \Theta_n(\gamma) \Theta_n(\delta)))^k$.
- $d_H(\tilde{\Theta}_n^k(\gamma), Id) > 1 - (1 - d_H(\Theta_n(\gamma), Id))^k$.

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Hence, we can choose \( \epsilon, n \) and \( k \) appropriately to obtain for any \( F \subset \Gamma \) and \( \epsilon' > 0 \) a map \( \Theta \) as in the Introduction, proving the soficity of \( \Gamma \). \( \square \)

3. The proof of Theorem 1

Let \( \Phi : F_\infty \curvearrowright \{0,1\}^N \) be a sofic action preserving the product measure \( \mu \). Let \( \Gamma \subset [E] \) be a finitely generated group, where \([E]\) is the equivalence relation defined by \( \Phi \). So, we have an action \( \Phi_\Gamma : \Gamma \curvearrowright \{0,1\}^N \). Our goal is to construct a compressed sofic representation of \( \Gamma \). Let \( \{\gamma_n\}_{n=1}^\infty \) be an enumeration of the elements of \( \Gamma \). Let \( \epsilon_n = \mu(\text{Fix}(\Phi_\Gamma(\gamma_n)))/2 \). Since \( \Gamma \) is in the full group, \( \epsilon_n > 0 \).

Now, fix a subset \( F \subset \Gamma \) and \( \epsilon > 0 \). We need to construct a map \( \Theta : F \to \text{Map}(A) \) for some finite set \( A \) such that if \( \gamma_i, \gamma_j, \gamma_i\gamma_j \in F \), then

\[
\begin{align*}
(1) & \quad d_H(\Theta(\gamma_i)\Theta(\gamma_j)) < \epsilon, \\
(2) & \quad d_H(\Theta(\gamma_i), 1) > \epsilon_i.
\end{align*}
\]

Let \( \{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}\} \) be a symmetric generating set for \( \Gamma \). Observe that we have an action \( \Sigma_\Gamma : F_m \curvearrowright \{0,1\}^N \) preserving \( \mu \) such that \( \Sigma_\Gamma(\delta) = \Phi_\Gamma(\tau(\delta)) \), where \( \tau : F_m \to \Gamma \) is the natural quotient map. A dyadic \( E \)-map of depth \( k \) is a Borel map \( Q : X \to X \) defined in the following way. For each \( \rho \in \{0,1\}^k \) we pick \( \chi_\rho(\rho) \in F_k \subset F_\infty \) and define \( Q(x) = \Phi(\chi_\rho(\rho))(x) \) if the first \( k \)-coordinate of \( x \) is \( \rho \).

A dyadic approximation of \( \Gamma \) is a sequence of families \( \{Q_k(s_i)\}_{i=1}^m, \{Q_k(s_i^{-1})\}_{i=1}^m \), where for any \( 1 \leq i \leq m \):

- \( Q_k(s_i) : X \to X, Q_k(s_i^{-1}) : X \to X \) are dyadic \( E \)-maps of depth \( k \).
- \( \lim_{k \to \infty} \mu(\{x \in X \mid Q_k(s_i)(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0 \).
- \( \lim_{k \to \infty} \mu(\{x \in X \mid Q_k(s_i^{-1})(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0 \).

We do not require \( Q_k \) to be a bijection. Nevertheless, \( Q_k \) can be extended to a homomorphism from \( F_m \) to \( \text{Map}(X) \). Note that for simplicity we identified the generating set of \( F_m \) by the set \( \{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}\} \).

Since all the \( \Sigma_\Gamma(s_i) \)'s are Borel bijections, such dyadic approximations clearly exist. The following lemma is an immediate consequence of the definition of the dyadic approximation.

**Lemma 3.1.** For any \( \delta \in F_m \)

\[
\lim_{k \to \infty} \mu(\text{Fix}(Q_k(\delta))) = \mu(\text{Fix}(\Sigma_\Gamma(\delta))).
\]

**Proposition 3.1.** There exists a sequence of mappings \( \hat{\Theta}_k : F_m \to \text{Map}(B_k) \), where \( |B_k| < \infty \) such that for any \( \delta \in F_m \),

\[
\lim_{k \to \infty} \left( \mu(\text{Fix}(Q_k(\delta))) - \frac{|\text{Fix}(\hat{\Theta}_k(\delta))|}{|B_k|} \right) = 0.
\]

That is,

\[
\lim_{k \to \infty} \frac{|\text{Fix}(\hat{\Theta}_k(\delta))|}{|B_k|} = \mu(\text{Fix}(\Sigma_\Gamma(\delta))).
\]

**Proof.** Let \( \Phi_k : F_k \curvearrowright \{0,1\}^N \) be the restriction of \( \Phi \). Since \( \Phi \) is sofic, there exists a sequence of mappings \( \{\chi^n_k : F_k \to \text{Perm}(C_{k,n})\}_{n=1}^\infty \), where \( C_{k,n} \) is a finite \( \{0,1\}^k \)-vertex labeled graph such that for any \( t \geq 1 \) and \( \kappa \in U_k^t \),

\[
\lim_{n \to \infty} \mu_{\Phi_k}^{t,k}(\kappa) = \mu_{\Phi_k}^{t,k}(\kappa).
\]
Recall that $Q_k$ is not necessarily an action, only a homomorphism from $F_m$ to $\text{Map}(X)$. Hence, the local statistics of $Q_k$ cannot be described using the elements of $U_k^l$ as in the case of honest $F_m$-actions. So, let $W_k^l$ be the set of isomorphism classes of rooted $t$-balls of vertex degrees at most $2m$, where the vertices are labeled by elements of the set $\{0, 1\}^k$ and the edges (possibly loops) are labeled by subsets of $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}\}$. Note that $U_k^l \subset W_k^l$. Let $x, y \in X$ be points such that $B_{k^2}^Q(x)$ and $B_{k^2}^Q(y)$ represent the same element in $U_k^{k^2}$. Here $B_{k^2}^Q(x)$ denotes the $k$-ball with respect to the graphing associated to $\Phi_k$. Then, by the definition of the dyadic approximations $B_{k^2}^Q(x)$ and $B_{k^2}^Q(y)$ represent the same elements in $W_k^k$. Now we construct a sequence of maps $\hat{\Theta}_k^n : F_m \curvearrowright \text{Map}(C_{k, n})$ the following way:

$$\hat{\Theta}_k^n(s_i)(x) = \iota_k^n(w_{Q_k(s_i)}(\rho(x)))(x),$$

where $\rho(x)$ is the $\{0, 1\}^k$-label of $x$. By the previous observation, for any $\delta \in F_m$

$$\lim_{n \to \infty} \frac{|\text{Fix}(\hat{\Theta}_k^n(\delta))|}{|C_{k, n}|} = \mu(\text{Fix}(Q_k(\delta))).$$

This finishes the proof of the proposition. \hfill $\square$

Pick a section $\sigma : \Gamma \to F_m$, that is, a map such that $\tau \sigma = Id$. Let $\hat{\Theta}_k$ be as in Proposition 3.1. Define $\Theta_k : \Gamma \to \text{Map}(B_k)$ by

$$\Theta_k(\gamma) = \hat{\Theta}_k(\sigma(\gamma)).$$

Then $\{\Theta_k\}_{k = 1}^\infty$ is a compressed sofic representation of $\Gamma$. \hfill $\square$

4. The proof of Theorem 2

Let $T : C \to C$ be a minimal homeomorphism and $\Gamma \subset [[T]]$ be a finitely generated subgroup of the topological full group of $T$ with symmetric generating set $S = \{a_1, a_2, \ldots, a_k\}$. It is enough to prove that $\Gamma$ is LEF. Let $x \in C$ and consider the $T$-orbit $\{T^n(x)\}_{\infty}^{\infty}$. We define the map $\phi : \Gamma \to \text{Perm}(Z)$ of $\Gamma$ into the permutation group of the integers in the following way. Let $\phi(\gamma)(n) = m$, if $\gamma(T^n(x)) = T^m(x)$. Since $T$ acts freely on $C$, $\phi$ is well-defined.

**Lemma 4.1.** $\phi$ is an injective homomorphism.

**Proof.** If $\phi(\gamma) = Id$, then $\gamma$ fixes all the elements of the orbit of $x$. Since all the orbits are dense, this implies that $\gamma = 1$. The fact that $\phi$ is a homomorphism follows immediately, since $\phi$ is the restriction of the $T$-action onto the orbit of $x$. \hfill $\square$

Let $a = \max |n|$, where for some $p \in C$ and $a_i \in S$, $a_i(p) = T^n(p)$. We define a sequence

$$l : Z \to \{-a, -a + 1, \ldots, 0, 1, \ldots, a - 1, a\}^S$$

in the following way. Let $l(n) := (t_{a_1}, t_{a_2}, \ldots, t_{a_k})$, where $a_i(T^n(x)) = T^{n + l_i}(x)$. The following lemma is well-known; we prove it for the sake of completeness.

**Lemma 4.2.** $l$ is a repetitive sequence; that is, if we find a substring $\sigma$ in $l$, then there exists $m \geq 1$ such that for any interval of length $m$ we can find $\sigma$.

**Proof.** For a point $p \in C$, we can define its $n$-pattern

$$g_n(p) := \{-n, -n + 1, \ldots, 0, 1, \ldots, n - 1, n\} \to \{-a, -a + 1, \ldots, a - 1, a\}$$
by \( q_n(p)(j) := (a_{t_1}, t_{a_2}, \ldots, t_{a_k}) \), where \( a_i(T^j(x)) = T^{j+t_{a_i}}(x) \). Observe that the set of points with a given \( n \) pattern is closed. Now, let us suppose that for a sequence \( \{k_r\}_{r=1}^{\infty} \subset \mathbb{Z} \) the intervals \( (k_r, k_r+r) \) do not contain \( \sigma \) as a substring. Then, if \( z \) is a limit point of \( \{T^{k_r}(x)\}_{r=1}^{\infty} \), no translates of \( z \) have \( \sigma \) as a part of their \( n \)-patterns. Therefore the orbit closure of \( z \) does not contain \( x \), in contradiction with the minimality of \( T \).

\[
\text{Proof.}
\]

We define a lazy random walk on \( p \) transition probability for each pair \( \gamma \in \Gamma \) that is the product of at most \( r \) generators by \( \phi(\gamma)(i) - i \leq ar \). Pick \( n > 10a^r \) such that

- \( l_1{-ar,-ar+1,...,ar-1,ar} = \sigma_r \).
- for any \( \gamma \in \Gamma \) that is the product of at most \( r \) generators, there is \( 0 < j < n \) such that \( \gamma(j) \neq j \).

Now we define \( \phi_r : W^r \rightarrow \text{Perm}(\mathbb{Z}_n) \), where \( W^r \) is the set of elements in \( \Gamma \) that are products of at most \( r \) generators by \( \phi_r(i) = \phi(i)(\text{mod } n) \). Clearly, \( \phi_r \) is injective, and if \( x, y, xy \in W^r \), then \( \phi_r(x)\phi_r(y) = \phi_r(xy) \). This implies that \( \Gamma \) is LEF.

\[\square\]

5. THE PROOF OF THEOREM 5

Let \( \alpha : \Gamma^k \rightarrow X \) be a transitive and faithful action of the free product group. Consider the Schreier graph \( G_\alpha \) of the action with respect to the generators of the \( k \) cyclic groups \( \{a_1, a_2, \ldots, a_k\} \). Recall that \( V(G_\alpha) \) is \( X \) and \( (x, y) \in E(G) \) if \( y = a_ix \) for some \( i \geq 1 \). Hence \( G_\alpha \) is a connected graph of vertex degree bound \( k \).

**Proposition 5.1.** Let \( \alpha \) be as above. Then for any \( 1 \neq w \in \Gamma^k \), there exist infinitely many \( y \in X \) such that \( \alpha(w)(y) \neq y \).

**Proof.** We will need the following lemma.

**Lemma 5.1.** For any finite set \( S \subseteq X \), there exists \( g \in \Gamma^k \) such that \( gS \cap S = \emptyset \).

**Proof.** We define a lazy random walk on \( X \) in the following way. For \( y \in X \) the transition probability \( p(x, y) = 1/k \), where \( l \) is the number of generators \( a_i \) such that \( a_ix = y \). It is well-known (see e.g. [9], [8]) that the probabilities \( p_n(x, y) \) tend to zero for each pair \( x, y \in X \). Now consider the standard random walk on the Cayley graph of \( \Gamma^k \), the \( k \)-regular tree. Let \( P_n(g) \) be the probability being at \( g \) after taking \( n \) steps starting from the identity. Then,

\[
p_n(x, y) = \sum_{g \in \Gamma, gx = y} P_n(g).
\]

By the previous observation, if \( n \) is large enough, then

\[\sum P_n(g) < 1,\]

where the summation is taken for all \( g \in \Gamma^k \) such that \( gx \in S \), for some \( x \in S \). Hence, there exists \( g \in \Gamma^k \) such that \( gS \cap S = \emptyset \).

Now let us suppose that \( w \in \Gamma^k \) fixes all points of \( X \) outside a finite set \( S \). That is, \( \alpha(w)(S) = S \). Let \( gS \cap S = \emptyset \). Then \( gwg^{-1} \) fixes all the points of \( X \) outside \( gS \). Therefore the commutator \([w, gwg^{-1}] \) fixes all elements of \( X \), in contradiction with the assumption that the action is faithful.
Now fix a vertex $x \in X$ and consider the ball of radius $n$, $B_n(x)$, around $x$. We define an action $\alpha_n : \Gamma^k \looparrowright B_n(x)$ in the following way. Let $\partial B_n(x)$ be the boundary of the ball $B_n(x)$, that is, the set of all $y \in B_n(x)$ such that there exists $a_i$ for which $\alpha(a_i)y \notin B_n(x)$. If $y \notin \partial B_n(x)$, then let $\alpha_n(a_i)y = \alpha(a_i)y$. If $y \in \partial B_n(x)$ and $\alpha(a_i)y \notin B_n(x)$, then let $\alpha_n(a_i)(y) = y$. Finally, if $y \in \partial B_n(x)$ and $\alpha(a_i)y \in B_n(x)$, then let $\alpha_n(a_i)(y) = \alpha(a_i)(y)$. Now let $L^k_n = \{0, 1\}^{B_n(x)} \times \alpha_n \alpha_n(\Gamma^k)$ be the associated finite lamplighter group and $L^k = \bigoplus_{x \in X} \{0, 1\} \times \alpha \Gamma^k$. Our goal is to embed $L^k$ into $L^k_n$ locally. That is, for any finite set $F \subset L^k$ we construct an injective map $\Theta : F \to L^k_n$ such that $\Theta(fg) = \Theta(f)\Theta(g)$. Recall that each element of $L^k$ can be uniquely written in the form $a \cdot w$, where $a \in \bigoplus_{x \in X} \{0, 1\}$ and $w \in \Gamma^k$. We regard the elements of the lamplighter group as permutations of the set $\bigoplus_{x \in X} \{0, 1\}$. If $\kappa \in \bigoplus_{x \in X} \{0, 1\}$ and $p \in X$, then 
\[(a \cdot w)(\kappa) |_p = a(p) + \kappa(a(w^{-1})(p)).\]

We will also use the product formula 
\[(a_2 \cdot w_2)(a_1 \cdot w_1) = (a_2 + \alpha(w_2)(a_1), w_2w_1), \]
where $\alpha(w_2)(a_1)(q) = a_1(\alpha(w_2^{-1})(q))$. For $l \geq 1$, let $H_l$ be the set of elements of $L^k$ in the form of $a \cdot w$, where $w$ is a word of length at most $l$ and the support of $a$ is contained in $B_l(x)$. For $n \geq l$ we define the map $\tau^n_l : H_l \to L^k_n$ by $\tau^n_l(a \cdot w) := a \cdot \alpha_n(w)$.

**Lemma 5.2.** If $n$ is large enough, then $\tau^n_l$ is injective.

**Proof.** If $n$ is large enough, then $B_n(x)$ contains a point $y$ such that
- $\alpha(w)(y) \neq y$,
- $d(y, \partial B_n(x)) > l$,
- $d(y, B_l(x)) > l$,
where $d$ is the shortest path distance on the Schreier graph $G_{\alpha}$. Let $\kappa \in \bigoplus_{x \in X} \{0, 1\}$ be the element which is 1 at $y$ and zero otherwise. Then
\[\tau^n_l(a \cdot w)(\kappa) |_{\alpha_n(w)(y)} = 1,\]
hence $\tau^n_l(a \cdot w)$ is not trivial. \hfill $\square$

The following lemma finishes the proof of Theorem 3.

**Lemma 5.3.** Suppose that $(a_1 \cdot w_1), (a_2 \cdot w_2)$ and $(a_2 \cdot w_2)(a_1 \cdot w_1) \in H_l$ and $n$ is large enough. Then
\[\tau^n_l((a_2 \cdot w_2)(a_1 \cdot w_1)) = \tau^n_l((a_2 \cdot w_2)(a_1 \cdot w_1)).\]

**Proof.** We need to prove that
\[(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1)) = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2w_1)\]
holds in $L^k_n$. Fix an element $\kappa \in \{0, 1\}^{B_n(x)}$. Let $n > 10l$ and $d(p, \partial B_n(x)) > 5l$. Then
\[(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa) |_p = (a_2 \cdot w_2)(a_1 \cdot w_1)(\kappa) |_p\]
and
\[(a_2 + \alpha(w_2)(a_1) \cdot \alpha_n(w_2w_1))(\kappa) |_p = (a_2 + \alpha(w_2)(a_1) \cdot (w_2w_1))(\kappa) |_p,\]
where $\kappa$ is an extension of $\kappa$ onto $X$. On the other hand, if $d(p, \partial B_n(x)) \leq 5l$,
then
\[
(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)|_p = \alpha_n(w_2)\alpha_n(w_1)(\kappa)|_p
\]
\[
= \alpha_n(w_2w_1)(\kappa)|_p = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2w_1)(\kappa)|_p.
\]
\[\square\]

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