Supersymmetric solutions for non-relativistic holography

Aristomenis Donos\textsuperscript{1} and Jerome P. Gauntlett\textsuperscript{2}

\textsuperscript{1}DESY Theory Group, DESY Hamburg
Notkestrasse 85, D 22603 Hamburg, Germany

\textsuperscript{2}Theoretical Physics Group, Blackett Laboratory,
Imperial College, London SW7 2AZ, U.K.

\textsuperscript{2}The Institute for Mathematical Sciences,
Imperial College, London SW7 2PE, U.K.

Abstract

We construct families of supersymmetric solutions of type IIB and $D = 11$
supergravity that are invariant under the non-relativistic conformal algebra for various values of dynamical exponent $z \geq 4$ and $z \geq 3$, respectively. The solutions are based on five- and seven-dimensional Sasaki-Einstein manifolds and generalise the known solutions with dynamical exponent $z = 4$ for the type IIB case and $z = 3$ for the $D = 11$ case, respectively.
1 Introduction

There has recently been much interest in finding holographic realisations of systems invariant under the non-relativistic conformal algebra starting with the work [1], [2] and discussed further in related work [3]-[32]. Such systems are invariant under Galilean transformations, generated by time and spatial translations, spatial rotations, Galilean boosts and a mass operator, which is a central element of the algebra, combined with scale transformations. If $x^+$ is the time coordinate, and $x$ denotes $d$ spatial coordinates, the scaling symmetry acts as

$$x \rightarrow \mu x, \quad x^+ \rightarrow \mu^{\frac{z}{2}} x^+, \quad (1.1)$$

where $z$ is called the dynamical exponent. When $z = 2$ this non-relativistic conformal symmetry can be enlarged to an invariance under the Schrödinger algebra which includes an additional special conformal generator.

The solutions found in [1], [2] with $d = 2$ and $z = 2$ were subsequently embedded into type IIB string theory in [8],[9],[10] and were based on an arbitrary five-dimensional Sasaki-Einstein manifold, $SE_5$. The work of [9] also constructed type IIB solutions with $d = 2$ and $z = 4$ and again these were constructed using an arbitrary $SE_5$. It was also shown in [9] that the solutions with $z = 2$ and $z = 4$ can be obtained from a five dimensional theory with a massive vector field after a Kaluza-Klein reduction on the $SE_5$ space [9]. This procedure was generalised to solutions of $D = 11$ supergravity in [31]: using a similar KK reduction on an arbitrary seven-dimensional Sasaki-Einstein space, $SE_7$, solutions with non relativistic conformal symmetry with $d = 1$ and $z = 3$ were found.

The type IIB solution of [8],[9],[10] with $z = 2$ do not preserve any supersymmetry [9]. One aim of this note is to show that, by contrast, the type IIB solutions of [9] with $z = 4$ and the $D = 11$ solutions of [31] with $z = 3$ are both supersymmetric and generically preserve two supersymmetries. A second aim is to generalise both of these supersymmetric solutions to different values of $z$. We will construct new supersymmetric solutions using eigenmodes of the Laplacian acting on one-forms on the $SE_5$ or $SE_7$ space. If the eigenvalue is $\mu$ then we obtain type IIB solutions with $z = 1 + \sqrt{1+\mu}$ and $D = 11$ solutions with $z = 1 + \frac{1}{2}\sqrt{4+\mu}$. This gives rise to type IIB solutions with $z \geq 4$ and $D = 11$ solutions with $z \geq 3$, respectively. For the case of $S^5$ we get solutions with $z = 4, 5, \ldots$ while for the case of $S^7$ we get solutions with $z = 3, 3\frac{1}{2}, 4, \ldots$ and both of these preserve 8 supersymmetries.

Our constructions have some similarities with the construction of type IIB solutions in [24] that were based on eigenmodes of the Laplacian acting on scalar functions
on the $SE_5$ space. Our IIB solutions preserve the same supersymmetry and we show how our solutions can be superposed with those of [24] while maintaining a scaling symmetry. An analogous superposition is possible for the $D = 11$ solutions, which we shall also describe.

2 The type IIB solutions

The ansatz for the type IIB solutions we shall consider is given by

$$
\begin{align*}
  ds^2 &= \frac{dr^2}{r^2} + r^2 \left[ 2dx^+ dx^- + dx_1^2 + dx_2^2 \right] + ds^2(SE_5) + 2r^2C dx^+ \\
  F_5 &= 4r^3 dx^+ \wedge dx^- \wedge dr \wedge dx_1 \wedge dx_2 + 4Vol(SE_5) \\
       &- dx^+ \wedge \left[ *_{CY_3} dC + d(r^4C) \wedge dx_1 \wedge dx_2 \right] 
\end{align*}
$$

(2.1)

where $SE_5$ is an arbitrary five-dimensional Sasaki-Einstein space and the metric $ds^2(SE_5)$ is normalised so that the Ricci tensor is equal to four times the metric (i.e. the same normalisation as that of a unit radius five-sphere). Recall that the metric cone over the $SE_5$,

$$
  ds^2(CY_3) = dr^2 + r^2 ds^2(SE_5) , 
$$

(2.2)

is Calabi-Yau. The Kähler form on the $CY_3$ is denoted $\omega_{ij}$ and the complex structure is defined by $J_{ij} = \omega_{ik} g^{kj}$, where $g_{ij}$ is the Calabi-Yau cone metric. We will define the one-form $\eta$, which is dual to the Reeb vector on $SE_5$ by

$$
  \eta_i = -J^j_i \left( d \log r \right)_j .
$$

(2.3)

The one-form $C$ is a one-form on the $CY_3$ cone. When $C = 0$ we have the standard $AdS_5 \times SE_5$ solution of type IIB which, in general, preserves eight supersymmetries (four Poincaré and four superconformal), corresponding to an $N = 1$ SCFT in $d = 4$. More generally, we can deform this solution by choosing $C \neq 0$ provided that $dC$ is co-closed on $CY_3$:

$$
  d *_{CY} dC = 0 .
$$

(2.4)

With this condition, $F_5$ is closed and in fact it is also sufficient for the type IIB Einstein equations to be satisfied. As we will show these solutions preserve one

\footnote{While this is standard in the physics literature, often in the maths literature $J_{ij} = -\omega_{ik} g^{kj}$.}
half of the Poincaré supersymmetries. Note that the solution is invariant under the transformation
\[ x^- \rightarrow x^- - \Lambda, \quad C \rightarrow C + d\Lambda \]  
for some function \( \Lambda \) on the CY cone. Thus, if \( dC = 0 \), we can remove \( C \), at least locally, by such a transformation.

We will look for solutions where the one-form \( C \) has weight \( \lambda \) under the action of \( r \partial_r \). Then it is straightforward to check, following [1] and [2] that our solution is invariant under non-relativistic conformal transformations with two spatial dimensions \( x^1, x^2 \) and dynamical exponent \( z = 2 + \lambda \). For example the scaling symmetry is acting as in (1.1) combined with \( r \rightarrow \mu^{-1} r, \ x^- \rightarrow \mu^{2-z} x^- \). Following the analysis of closed and co-closed two forms on cones (such as \( dC \)) in appendix A of [33] we consider solutions constructed from a co-closed one-form \( \beta \) on the \( SE_5 \) space that is an eigenmode of the Laplacian \( \Delta_{SE} = (d^\dagger d + dd^\dagger)_{SE} \):
\[ C = r^\lambda \beta, \quad \Delta_{SE} \beta = \mu \beta, \quad d^\dagger \beta = 0. \]  
(2.6)

It is straightforward to check that \( dC \) is co-closed providing that \( \mu = \lambda (\lambda + 2) \). For our applications we choose the branch \( \lambda = -1 + \sqrt{1 + \mu} \) leading to solutions with
\[ z = 1 + \sqrt{1 + \mu}. \]  
(2.7)

A general result valid for any five-dimensional Einstein space, normalised as we have, is that for co-closed 1-forms \( \mu \geq 8 \) and \( \mu = 8 \) holds iff the 1-form is dual to a Killing vector (see section 4.3 of [34]). Thus in general our construction leads to solutions with
\[ z \geq 4. \]  
(2.8)

Since all \( SE_5 \) manifolds have at least the Reeb Killing vector, dual to the one-form \( \eta \), this bound is always saturated. Indeed the solution of [9] with \( z = 4 \) is in our class. Specifically it can be obtained by setting \( C = \sigma r^2 \eta \) (and redefining \( x^- \rightarrow -x^-/2 \)): one can explicitly check that \( \eta \) is co-closed on \( SE_5 \) and is an eigenmode of \( \Delta_{SE} \) with eigenvalue \( \mu = 8 \). Note that for this solution the two-form \( dC \) is proportional to the Kähler-form of the Calabi-Yau cone: \( dC = 2\sigma \omega \).

On \( S^5 \) the spectrum of \( \Delta_{S^5} \) acting on one-forms is well known and we have \( \mu = (s+1)(s+3) \) for \( s = 1, 2, 3 \ldots \) (see for example [35] eq (2.20)) leading to \( \lambda = s+1 \) and hence new classes of solutions with \( z = 4, 5, 6 \ldots \). Note that these solutions come in families, transforming in the \( SO(6) \) irreps \( 15, 64, 175, \ldots \). To obtain similar results for \( T^{1,1} \) one can consult [36].
We now discuss a construction that can be used when the spectrum of the Laplacian acting on functions is known, but not acting on one-forms. For example, the scalar Laplacian was studied in [40] for the $Y^{p,q}$ metrics [41], but as far as we know it has not been discussed acting on one-forms. Specifically we construct $(1,1)$ forms $dC$ on the CY cone using scalar functions $\Phi$ on the cone as follows. We write

$$C_i = J_i^j \partial_j \Phi \quad (2.9)$$

for some function $\Phi$ on $CY_3$. A short calculation shows that if

$$\nabla^2_{CY} \Phi = \alpha \quad (2.10)$$

for some constant $\alpha$ then $dC$ is co-closed. The two-form $dC$ is a $(1,1)$ form on $CY_3$ and it is primitive, $J^i_j dC_{ij} = 0$, if and only if $\alpha = 0$. Observe that the solution of [9] with $z = 4$ fits into this class by taking $\Phi = -\sigma r^2/2$ and $\alpha = -6\sigma$, leading to $C = \sigma r^2 \eta$.

We now consider solutions with $\alpha = 0$, corresponding to harmonic functions on the CY cone with $dC (1,1)$ and primitive. We next write

$$\Phi = r^\lambda f \quad (2.11)$$

where $f$ is a function on the $SE_5$ space satisfying

$$- \nabla^2_{SE_5} f = kf \quad (2.12)$$

with $k = \lambda(\lambda + 4)$ (see e.g. [37]). For the solutions of interest we choose the branch $\lambda = -2 + \sqrt{4+k}$ leading to $z = \sqrt{4+k}$. For the special case of the five-sphere we can check with the results that we obtained above. The eigenfunctions $f$ on the five-sphere are given by spherical harmonics with $k = l(l+4)$, $l = 1, 2, \ldots$ and hence $z = l + 2$. The $l = 1$ harmonic appears to violate the bound (2.8). However, it is straightforward to see that the construction for $l = 1$ leads to $dC = 0$ for which $C$ can be removed by a transformation of the form (2.5). Thus for $S^5$ we should consider $l \geq 2$ leading to solutions with $z = 4, 5, \ldots$, as above. It is worth pointing out that for higher values of $l$ some of the eigenfunctions will also lead to closed $C$: if we consider the harmonic function on $\mathbb{R}^6$ given by $x^{i_1} \ldots x^{i_l} c_{i_1 \ldots i_l}$ where $c$ is symmetric and traceless then, with $J = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6$ we see that $dC = 0$ if $J_{[i} c_{j]i_2 \ldots i_l} = 0$.

Note that in general the one-form $C$ defined in (2.9) has a component in the $dr$ direction, unlike in (2.6). However, locally we can remove it by a transformation of the form (2.5). Also, one can directly show that the resulting one-form $\beta$ is co-closed on the $SE_5$ space.
2.1 Supersymmetry

We introduce the frame

\[ e^+ = rdx^+ \]
\[ e^- = r(dx^- + C) \]
\[ e^2 = rdx_1 \]
\[ e^3 = rdx_2 \]
\[ e^4 = \frac{dr}{r} \]
\[ e^m = e^m_{SE}, \quad m = 5, \ldots, 9 \] (2.13)

where \( e^m_{SE} \) is an orthonormal frame for the \( SE_5 \) space. We can write

\[ F_5 = B_5 + *_{10}B_5 \] (2.14)
\[ B_5 = 4e^+ \wedge e^- \wedge e^2 \wedge e^3 \wedge e^4 - re^+ \wedge dC \wedge e^2 \wedge e^3 \] (2.15)

where we have chosen \( \epsilon_{+-23456789} = +1 \). The Killing spinor equation can be written

\[ D_M \epsilon + \frac{i}{2} \Gamma_M \epsilon = D_M \epsilon + \frac{1}{2} \bar{\epsilon} \] (2.16)

We are using the conventions for type IIB supergravity \([42][43]\) as in \([44]\) and in particular, \( \Gamma_{11} = \Gamma_{+ -23456789} \) with the chiral IIB spinors satisfying \( \Gamma_{11} \epsilon = -\epsilon \).

If \( \epsilon \) are the Killing spinors for the \( AdS_5 \times SE_5 \) solution, then we find that we must also impose that

\[ \Gamma^{+-23} \epsilon = i\epsilon \]
\[ \Gamma^+ \epsilon = 0 \] (2.17)

The first condition maintains the Poincaré supersymmetries but breaks all of the superconformal supersymmetries (this can be explicitly checked using, for example, the results of \([45]\)). The second condition breaks a further half of these\(^3\). Thus when \( dC \neq 0 \), we preserve two Poincaré supersymmetries for a generic \( SE_5 \) and this is increased to eight Poincaré supersymmetries for \( S^5 \).

\(^3\)That we preserve the Poincaré supersymmetries suggests that we can extend our solutions away from the near horizon limit of the D3-branes. This is indeed the case but we won’t expand upon that here.
3 The $D = 11$ solutions

The construction of the $D = 11$ solutions is very similar. We consider the ansatz for $D=11$ supergravity solutions:

$$
\begin{align*}
 ds^2 &= \frac{d\rho^2}{4\rho^2} + \rho^2 [2dx^+ dx^- + dx^2] + ds^2(SE_7) + 2\rho^2 C dx^+ \\
 G &= -3\rho^2 dx^+ \wedge dx^- \wedge d\rho \wedge dx + dx^+ \wedge dx \wedge d(\rho^3 C)
\end{align*}
$$

(3.1)

where $SE_7$ is a seven-dimensional Sasaki-Einstein space and $ds^2(SE_7)$ is normalised so that the Ricci tensor is equal to six times the metric (this is the normalisation of a unit radius seven-sphere). It is convenient to change coordinates via $\rho = r^2$ to bring the solution to the form

$$
\begin{align*}
 ds^2 &= \frac{dr^2}{r^2} + r^4 [2dx^+ dx^- + dx^2] + ds^2(SE_7) + 2r^4 C dx^+ \\
 G &= -6r^5 dx^+ \wedge dx^- \wedge dr \wedge dx + dx^+ \wedge dx \wedge d(r^6 C).
\end{align*}
$$

(3.2)

In these coordinates the cone metric

$$
 ds^2_{CY} = dr^2 + r^2 ds^2(SE_7)
$$

(3.3)

is a metric on Calabi-Yau four-fold. We will use the same notation for the $CY$ space as in the previous section.

When the one-form $C$ is zero we have the standard $AdS_4 \times SE_7$ solution of $D = 11$ supergravity that, in general, preserves eight supersymmetries. We again find that all the equations of motion are solved if $C$ is a one-form on $CY_4$ and the two-form $dC$ is co-closed

$$
 d*_{CY} dC = 0.
$$

(3.4)

The solutions are again invariant under the transformation (2.5). We will consider solutions where the one-form $C$ has weight $\lambda$ under the action of $r \partial_r$, corresponding to dynamical exponent $z = 2 + \lambda/2$. As before, using the results in appendix A of [33], we consider solutions constructed from a co-closed one-form $\beta$ on the $SE_7$ space that is an eigenmode of the Laplacian $\Delta_{SE}$:

$$
 C = r^\lambda \beta, \quad \Delta_{SE} \beta = \mu \beta, \quad d^\dagger \beta = 0.
$$

(3.5)

One can check that $dC$ is co-closed providing that $\mu = \lambda(\lambda + 4)$. For our applications we choose the branch $\lambda = -2 + \sqrt{4 + \mu}$ leading to solutions with

$$
 z = 1 + \frac{1}{2} \sqrt{4 + \mu}.
$$

(3.6)
A general result valid for any seven-dimensional Einstein space, normalised as we have, is that for co-closed 1-forms \( \mu \geq 12 \) and \( \mu = 12 \) holds iff the 1-form is dual to a Killing vector (see section 4.3 of [34]). Thus in general our construction leads to solutions with

\[
z \geq 3
\]  

(3.7)

and the bound is again saturated for all \( SE_7 \) spaces. Observe that the solutions of [31] with \( z = 3 \) fit into this class. Specifically they are obtained by setting \( C = \sigma r^2 \eta \) (after redefining \( x \to x/2 \) and \( x^- \to -x^-/8 \)). On \( S^7 \) the spectrum of \( \Delta_{S^7} \) is well known and we have \( \mu = s(s+6) + 5 \) for \( s = 1, 2, 3 \ldots \) (see for example [31] eq (7.2.5)) leading to \( \lambda = 1 + s \) and hence new classes of solutions with \( z = 3, 3\frac{1}{2}, 4, \ldots \). These solutions come in families transforming in the \( SO(8) \) irreps \( 28, 160_v, 567_v, \ldots \). Results on the spectrum of the Laplacian on some homogeneous \( SE_7 \) spaces can be found in [46],[47],[48].

As before we can construct \((1,1)\) co-closed two-forms \( dC \) using scalar functions \( \Phi \) on \( CY_4 \). We write

\[
C_i = J^i_j \partial_j \Phi, \quad \nabla^2_{CY} \Phi = \alpha.
\]  

(3.8)

and \( dC \) is again primitive if and only if \( \alpha = 0 \). The solutions of [31] with \( z = 3 \) arise by taking \( \Phi = \sigma r^2 \) and \( \alpha = -8\sigma \) leading to \( C = \sigma r^2 \eta \). We now focus on solutions with \( \alpha = 0 \), corresponding to harmonic functions on the CY cone. We take

\[
\Phi = r^\lambda f
\]  

(3.9)

where \( f \) is a function on the \( SE_7 \) space satisfying

\[
- \nabla^2_{SE_7} f = kf
\]  

(3.10)

with \( k = \lambda(\lambda + 6) \). For our applications we choose the branch \( \lambda = -3 + \sqrt{9+k} \) leading to solutions with \( z = \frac{1}{2} + \frac{1}{2}\sqrt{9+k} \). For example, on the seven-sphere the eigenfunctions \( f \) are given by spherical harmonics with \( k = l(l+6) \) with \( l = 1, 2, \ldots \) and hence \( z = 2+l/2 \). Excluding the \( l = 1 \) harmonic, as it can be removed by a transformation of the form (2.5), for \( S^7 \) we are left with solutions with \( z = 3, 7/2, 4, \ldots \), as above.
3.1 Supersymmetry

We introduce a frame

\[ e^+ = r^2 dx^+ \]
\[ e^- = r^2 (dx^- + C') \]
\[ e^2 = r^2 dx \]
\[ e^3 = \frac{dr}{r} \]
\[ e^m = e^m_{SE}, \quad m = 4, \ldots, 10. \]  \hspace{1cm} (3.11)

We thus have

\[ G = 6 e^+ \wedge e^- \wedge e^2 \wedge e^3 + r^2 e^+ \wedge e^2 \wedge dC \]
\[ \ast_{11} G = -6 Vol(SE_7) + dx^+ \ast_{CY} dC \]  \hspace{1cm} (3.12)

where we have chosen the orientation \( \epsilon_{+23, \ldots, 10} = +1 \).

The Killing spinor equation can be written as

\[ \nabla_M \epsilon + \frac{1}{288} \left[ \Gamma_M^{N_1 N_2 N_3 N_4} - 8 \delta_M^{N_1} \Gamma_{N_2 N_3 N_4} \right] G_{N_1 N_2 N_3 N_4} \epsilon = 0. \]  \hspace{1cm} (3.13)

We are using the conventions for \( D = 11 \) supergravity [49] as in [50] and in particular \( \Gamma_{+2345678910} = +1 \).

If \( \epsilon \) are the Killing spinors arising for the \( AdS_4 \times SE_7 \) solution, then we find that we must also impose that

\[ \Gamma^{+2} \epsilon = -\epsilon \]
\[ \Gamma^+ \epsilon = 0. \]  \hspace{1cm} (3.14)

The first condition maintains the Poincaré supersymmetries but breaks all of the superconformal supersymmetries. The second condition breaks a further half of these. Thus when \( dC \neq 0 \), we preserve two Poincaré supersymmetries for a generic \( SE_7 \) and this is increased to eight Poincaré supersymmetries for \( S^7 \).

3.2 Skew-Whiffed Solutions

If \( AdS_4 \times SE_7 \) is a supersymmetric solution of \( D = 11 \) supergravity, then if we “skew-whiff” by reversing the sign of the flux (or equivalently changing the orientation of \( SE_7 \)) then apart from the special case when the \( SE_7 \) space is the round \( S^7 \), all supersymmetry is broken [51]. Despite the lack of supersymmetry, such solutions are known to be perturbatively stable [51]. Similarly, if we reverse the sign of the flux in our new solutions (3.2), we will obtain solutions of \( D = 11 \) supergravity that will generically not preserve any supersymmetry.
4 Further Generalisation

We now discuss a further generalisation of the solutions that we have considered so far, preserving the same amount of supersymmetry, which incorporate the construction of [24]. For type IIB the metric is now given by

\[
ds^2 = \frac{dr^2}{r^2} + r^2 \left[ 2dx^+dx^- + dx_1^2 + dx_2^2 \right] + ds^2(\text{SE}_5) + r^2 \left[ 2Cdx^+ + h(dx^+)^2 \right]
\]

(4.1)

with the five-form unchanged from (2.1). The conditions on the one-form \(C\) are as before and we demand that \(h\) is a harmonic function on the \(CY_3\) cone:

\[
\nabla^2_{CY}h = 0 .
\]

(4.2)

Choosing \(h\) to have weight \(\lambda'\) under \(r\partial_r\) we take

\[
h = r^{\lambda'} f' ,
\]

(4.3)

where \(f'\) is an eigenfunction of the Laplacian on \(SE_5\) with eigenvalue \(k'\)

\[
- \nabla^2_{SE_5} f' = k' f'
\]

(4.4)

with \(k' = \lambda' (\lambda' + 4)\). If we set \(C = 0\) and choose the branch \(\lambda' = -2 + \sqrt{4 + k'}\) then these are the solutions constructed in section 5 of [24] and have dynamical exponent \(z = \frac{1}{2} \sqrt{4 + k'}\). As noted in [24] an application of Lichnerowicz’s theorem [52],[53] implies that these solutions have \(z \geq 3/2\) with \(z = 3/2\) only possible for \(S^5\). Now if there is a scalar eigenfunction with eigenvalue \(k'\) and a one-form eigenmode of the Laplacian on \(SE_5\) with eigenvalue \(\mu\) that satisfy \(z = \frac{1}{2} \sqrt{4 + k'} = 1 + \sqrt{1 + \mu}\) then we can superpose the solution with \(h\) as in (4.3) and the one-form \(C\) as in (2.6) and have a solution with scaling symmetry with this value of \(z\). For example on \(S^5\), using the notation as before, we have \(k' = l'(l' + 4)\), \(l' = 1, 2, \ldots\) and \(\mu = (s + 1)(s + 3)\), \(s = 1, 2, \ldots\) and hence we must demand that \(l' = 2(s + 2)\), \(s = 1, 2, \ldots\), giving solutions with \(z = 3 + s\).

The story for \(D = 11\) is very similar. The metric is now given by

\[
ds^2 = \frac{dr^2}{r^2} + r^4 \left[ 2dx^+dx^- + dx_1^2 + dx_2^2 \right] + ds^2(\text{SE}_7) + r^4 \left[ 2Cdx^+ + h(dx^+)^2 \right]
\]

(4.5)

with the four-form unchanged from (3.2). The conditions on the one-form \(C\) are as before and we demand that \(h\) is a harmonic function on the \(CY_4\) cone:

\[
\nabla^2_{CY}h = 0 .
\]

(4.6)
Choosing \( h \) to have weight \( \lambda' \) under \( r \partial_r \), we take

\[
h = r^\lambda f',
\]

(4.7)

where \( f' \) is an eigenfunction of the Laplacian on \( SE_7 \) with eigenvalue \( k' \)

\[
-\nabla^2_{SE_7} f' = k' f'
\]

(4.8)

with \( k' = \lambda'(\lambda' + 6) \). If we set \( C = 0 \) and chose the branch \( \lambda' = -3 + \sqrt{9 + k'} \) then these solutions have dynamical exponent \( z = \frac{1}{4}(1 + \sqrt{9 + k'}) \). Lichnerowicz’s theorem [52], [53] implies that these solutions have \( z \geq 5/4 \) with \( z = 5/4 \) only possible for \( S^7 \).

If there is a scalar eigenfunction with eigenvalue \( k' \) and a one-form eigenmode of the Laplacian on \( SE_7 \) with eigenvalue \( \mu \) that satisfy

\[
z = \frac{1}{4}(1 + \sqrt{9 + k'}) = 1 + \frac{1}{2}\sqrt{4 + \mu}
\]

then we can superpose the solution with \( h \) as in (4.7) and the one-form \( C \) as in (3.5) and have a solution with scaling symmetry with this value of \( z \). For example on \( S^7 \), using the notation as before, we have \( k' = l'(l' + 6), l' = 1, 2, \ldots \) and \( \mu = s(s + 6) + 5, s = 1, 2, \ldots \) and hence we must demand that \( l' = 2(s + 3), s = 1, 2, \ldots \), giving solutions with \( z = \frac{1}{2}(5 + s) \).

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