Integrals over $\text{SU}(N)$

Jesse Carlsson

E-mail: jesse.carlsson@gmail.com

Abstract. In this paper we calculate a number of integrals over $\text{SU}(N)$ of interest in Hamiltonian Lattice Gauge Theory calculations.
1. Introduction: file preparation and submission

This paper is concerned with the analytic calculation of certain group integrals over SU(N) which are of interest in Hamiltonian Lattice Gauge Theory (LGT) [1]. In particular, the integrals in question arise in calculations of ground state energies and mass gaps using the plaquette expansion method [2]. While these integrals have traditionally been handled with numerical Monte Carlo integration, in this paper we demonstrate how these integrals may be calculated analytically.

Traditionally Hamiltonian LGT techniques have been restricted to simple “trial” states (most commonly the strong coupling ground state which is annihilated by the chromo-electric field). The techniques presented in this paper are of use in the extension of standard Hamiltonian LGT techniques beyond trivial “trial” states, at least in 2+1 dimensions, to include more complicated trial states than the strong coupling ground state. For example, the techniques presented in Section 2 have been applied in variational estimates of glueball masses in 2+1 dimensions [3, 4]. It is thought that similar techniques may be of use in plaquette expansion method calculations of glueball masses in 2+1 dimensions.

Much work has been carried out on the topic of integration over the classical compact groups. The subject has been studied in great depth in the context of random matrices and combinatorics. Many analytic results in terms of determinants are available for integrals of various functions over unitary, orthogonal and symplectic groups [5]. Unfortunately similar results for SU(N) are not to our knowledge available. Their primary use has been in the study of Ulam’s problem concerning the distribution of the length of the longest increasing subsequence in permutation groups [6, 7]. Connections between random permutations and Young tableaux [8] allow an interesting approach to combinatorial problems involving Young tableaux. A problem of particular interest is the counting of Young tableaux of bounded height [9] which is closely related to the problem of counting singlets in product representations. Group integrals similar to those examined in this paper have also appeared in studies of the distributions of the eigenvalues of random matrices [10, 11].

In the last 20 years it appears that not much effort has been devoted to the subject of group integration in the context of LGT. The last significant development was due to Creutz who developed a diagrammatic technique for calculating specific SU(N) integrals [12] using link variables. This technique allows strong coupling matrix elements to be calculated for SU(N) [13] and has more recently been used in the loop formulation of quantum gravity where spin networks are of interest [14, 15, 16].

This paper is structured as follows. Section 2 contains calculations of some general SU(N) integrals which are used in later sections. Section 3 contains integrals of SU(N)
Integrals over $SU(N)$

group elements which are then applied in Section 4 to the calculation of a number of integrals that arise in the plaquette expansion method in 2+1 dimensional Hamiltonian LGT using a one plaquette trial state.

2. Preliminaries

In an earlier paper [4] a useful technique for performing $SU(N)$ integrals was described. In this paper the following results in terms of Toeplitz determinants were derived,

$$G_{U(N)}(c, d) = \int_{U(N)} dU e^{c Tr U + d Tr U^\dagger} = \det \left[ I_{j-i} \left( 2\sqrt{cd} \right) \right]_{1 \leq i,j \leq N}$$

(1)

$$G_{SU(N)}(c, d) = \int_{SU(N)} dU e^{c Tr U + d Tr U^\dagger} = \sum_{l=-\infty}^{\infty} \left( \frac{d}{c} \right)^{lN/2} \det \left[ I_{l+j-i} \left( 2\sqrt{cd} \right) \right]_{i \leq i,j \leq N}.$$  

(2)

Here $c$ is any number, $dU$ is the Haar measure normalised so that $\int dU = 1$, the quantities inside the determinant are to be interpreted as the $(i,j)$-th entry of an $N \times N$ matrix, and $I_n$ is the $n$-th order modified Bessel function of the first kind defined, for integers $n$, by

$$I_n(2x) = \sum_{k=0}^{\infty} \frac{x^{2k+n}}{k!(k+n)!}.$$  

(3)

Eq. (2) can be used to as a generating function to calculate integrals of the form

$$\int_{SU(N)} (Tr U)^n (Tr U^\dagger)^m dU e^{c Tr U + d Tr U^\dagger},$$

by direct differentiation. This process allows easy calculation of integrals of polynomials in $Tr U$ and $Tr U^\dagger$. It does not allow the simple calculation of integrals involving the more complicated trace variables, $(Tr U^n)^m$ for integers $n$ and $m$.

Making use of the same technique described in [4], the following more generals integrals can be calculated,

$$\int_{U(N)} dU \chi_{n_1 n_2 \ldots n_N} (U) e^{c Tr U + d Tr U^\dagger} = \left( \frac{d}{c} \right)^{\sum_i n_i/2} \det I_{j-i+n_i}(2\sqrt{cd})$$

and

$$\int_{SU(N)} dU \chi_{n_1 n_2 \ldots n_N} (U) e^{c Tr U + d Tr U^\dagger} = \left( \frac{d}{c} \right)^{\sum_k n_k/2} \sum_{l=-\infty}^{\infty} \left( \frac{d}{c} \right)^{lN/2} \det I_{j-i+l+n_i}(2\sqrt{cd}).$$

(5)

(6)

To arrive at these results, instead of Eq. (15) in [4], we make use of the following result due to Weyl [17] (with implicit sums over repeated indices understood) and follow the same procedure,

$$\Delta(\phi_1, \ldots, \phi_N) \chi_{n_1 \ldots n_N} = \frac{1}{\sqrt{N!}} \epsilon_{i_1 \ldots i_n} e^{i\phi_1 (N-i_1+n_1)} e^{i\phi_2 (N-i_2+n_2)} \ldots e^{i\phi_N (N-i_N+n_N)}.$$  

(7)

Here $\epsilon_{i_1 \ldots i_n}$ is the totally antisymmetric Levi-Civita tensor defined to be 1 if $\{i_1, \ldots, i_n\}$ is an even permutation of $\{1, 2, \ldots, n\}$, $-1$ if it is an odd permutation and 0 otherwise (i.e. if an index is repeated).
Integrals over SU(\(N\))

Eqs. (5) and (6) allow the simple calculation of many integrals involving \((\text{Tr} U^n)^m\) for integers \(n\) and \(m\). To perform these calculations expressions of the form \((\text{Tr} U^n)^m\) must be written in terms of characters. To do this the following results are useful [17, 18],

\[
\chi_{r_1 r_2 \ldots r_{N-1}}(U) = \det \left[ \chi_{i j} \right]_{1 \leq i, j \leq N-1}, \quad \text{where}
\]

\[
\chi_n(U) = \sum_{k_1, \ldots, k_n} \delta \left( \sum_{i=1}^n i k_i - n \right) \prod_{j=1}^n k_j^{(\text{Tr} U^j)^{k_j}}. \quad \text{(8)}
\]

Using these results, expressions for characters in terms of trace variables can be calculated. These results can be rearranged to give expressions for trace variables in terms of characters. As an example, for SU(\(N\)) it can be shown that

\[
(\text{Tr} U)^2 = \chi_2(U) + \chi_{11}(U) \quad \text{and} \quad \text{Tr}(U^2) = \chi_2(U) - \chi_{11}(U). \quad \text{(10)}
\]

Making use of Eqs. (10), (11) and (6) the following integrals can be easily calculated

\[
\int_{\text{SU}(N)} dU \text{Tr}(U^2) e^{c \text{Tr} U + d \text{Tr} U^\dagger} \quad \text{and} \quad \int_{\text{SU}(N)} dU (\text{Tr} U)^2 e^{c \text{Tr} U + d \text{Tr} U^\dagger}. \quad \text{(12)}
\]

3. Creutz’s Method

In this section we make use of Creutz’s method for calculating strong coupling integrals over SU(\(N\)) to calculate more complicated integrals. In the terminology of Hamiltonian LGT, we use Creutz’s method to calculate one-plaquette trial state integrals. The integrals we consider in this section are of use in the calculation of matrix elements arising in Hamiltonian LGT, particularly in calculations involving Hamiltonian moments.

The first two integrals we consider calculate in this section are,

\[
I_{a b a' b'}(c) = \int dU e^{c \text{Tr}(U + U^\dagger)} U_{a b} U_{a' b'} \quad \text{and} \quad J_{a b a' b'}(c) = \int dU e^{c \text{Tr}(U + U^\dagger)} U_{a b} (U^\dagger)_{a' b'}, \quad \text{(13)}
\]

where \(c\) is a complex valued variable.

Let us start with \(I_{a b a' b'}(c)\). Expanding the exponential in a Taylor series and also expanding the resulting binomial terms gives,

\[
I_{a b a' b'}(c) = \sum_{m=0}^{\infty} \sum_{i=0}^{m} \binom{m}{i} \frac{c^m}{m!} \int dU U_{a_1 a_1} \cdots U_{a_{m-1} a_{m-1}} (U^\dagger)_{b_1 b_1} \cdots (U^\dagger)_{b_i b_i} U_{a b} U_{a' b'}. \quad \text{(15)}
\]

Following the diagrammatic procedure first described by Creutz [12], the contributing integrals within the sum reduce to products of paired Levi-Civita symbols. The indices of each pair of Levi-Civita symbols are completely contracted with one another with the exception of those that contain an uncontracted \(U\) index \(a, b, a'\) or \(b'\). Following
Integrals over SU($N$)

Creutz’s procedure, each non-zero integral under the sum in Eq. (15) can be reduced to linear combinations of the following products of Levi-Civita symbols,

$$
e_{a_1A_1\cdots A_{N-1}}\delta_{a'A_1}\cdots\delta_{a'A_{N-1}} \propto \delta_{ab}\delta_{a'b'},$$

as the paired Levi-Civita symbols can contract with uncontracted operations result in the following two equations:

$$I_{ab;ab'} = \text{det} \left( \delta_{a_i'b_j} \right)_{1\leq i,j\leq n}. \tag{18}$$

Hence, $I_{ab;ab'}$ may be expressed as a linear combination of delta functions,

$$I_{ab;ab'}(c) = b_2(c)\delta_{ab}\delta_{a'b'} + b_1(c)\delta_{ab}\delta_{ba'}, \tag{19}$$

where $b_1$ and $b_2$ are functions of $c$ to be determined. To find these functions we contract over indices in Eq. (19) to construct equations for $b_1$ and $b_2$ in terms of known integrals.

For this particular integral we construct two equations for $b_1$ and $b_2$ by firstly, multiplying both sides of Eq. (19) by $\delta_{a'b'a}$ and summing over repeated indices and secondly, multiplying both sides of Eq. (19) by $\delta_{a'a'}\delta_{b'a}$ and summing over repeated indices. These operations result in the following two equations:

$$I_{ab;ab}(c) = N^2b_1(c) + Nb_2(c) \tag{20}$$

$$I_{ab;ba}(c) = Nb_1(c) + N^2b_2(c), \tag{21}$$

which can be solved for $b_1$ and $b_2$ and substituted in Eq. (19) to give our final result for $I_{ab;ab'}$ in terms of known integrals,

$$I_{ab;ab'}(c) = \frac{\delta_{ab}\delta_{a'b'}}{N^2 - 1} \int dU e^{c\text{Tr}(U+U^\dagger)} \left[ (\text{Tr}U)^2 - \frac{\text{Tr}(U^2)}{N} \right] + \frac{\delta_{a'b'}\delta_{ba'}}{N^2 - 1} \int dU e^{c\text{Tr}(U+U^\dagger)} \left[ \text{Tr}(U^2) - \frac{(\text{Tr}U)^2}{N} \right]. \tag{22}$$

Our approach to calculating $J_{ab;ab'}$, defined in Eq. (14), follows the approach for $I_{ab;ab'}$ described above. As was done for $I_{ab;ab'}$, it can be shown that $J_{ab;ab'}$ may be expressed as a linear combination of delta functions using the method due to Creutz,

$$J_{ab;ab'}(c) = d_1(c)\delta_{ab}\delta_{a'b'} + d_2(c)\delta_{ab}\delta_{ba'}. \tag{24}$$

By contracting with appropriate delta functions we arrive at simultaneous equations for $d_1$ and $d_2$, which are solved to give an expression for $J_{ab;ab'}$ in terms of known integrals

$$J_{ab;ab'}(c) = \frac{\delta_{ab}\delta_{a'b'}}{N^2 - 1} \int dU e^{c\text{Tr}(U+U^\dagger)} (\text{Tr}U \text{Tr}U^\dagger - 1) \tag{25}$$

$$+ \frac{\delta_{a'b'}\delta_{ba'}}{N^2 - 1} \int dU e^{c\text{Tr}(U+U^\dagger)} \left( N - \frac{1}{N} \text{Tr}U \text{Tr}U^\dagger \right). \tag{26}$$
This result agrees with the strong coupling result \( J_{ab'a'b'}(0) = \delta_{ab'}\delta_{ba'}/N \) since
\[
\int dUT\text{Tr}U\text{Tr}U^\dagger = \int dU = 1. \tag{27}
\]

This procedure can be used to evaluate more complicated integrals. For example, following Creutz’s method again,
\[
I_{ab'a'b':a''b''}(c) = \int dU e^{c\text{Tr}(U+U^\dagger)}U_{ab'}U_{a''b''}, \tag{28}
\]
can be written as linear combinations of the following products of Levi-Civita symbols,
\[
\epsilon_{aa'\cdots A_{N-3}}\epsilon_{bb'\cdots A_{N-3}}\epsilon_{cc'\cdots A_{N-3}}\epsilon_{dd'\cdots A_{N-3}}\epsilon_{ee'\cdots A_{N-3}}\epsilon_{ff'\cdots A_{N-3}} \propto \delta_{ab}\delta_{a'b'}\delta_{a''b''}, \tag{29}
\]
\[
\epsilon_{aa'\cdots A_{N-3}}\epsilon_{bb'\cdots A_{N-3}}\epsilon_{cc'\cdots A_{N-3}}\epsilon_{dd'\cdots A_{N-3}}\epsilon_{ee'\cdots A_{N-3}} \propto (\delta_{ab}\delta_{a'b'} - \delta_{ab}\delta_{ba'})\delta_{a''b''}, \tag{30}
\]
\[
\epsilon_{aa'\cdots A_{N-3}}\epsilon_{bb'\cdots A_{N-3}}\epsilon_{cc'\cdots A_{N-3}}\epsilon_{dd'\cdots A_{N-3}} \propto (\delta_{ab}\delta_{a'b'} - \delta_{ab}\delta_{ba'})\delta_{a''b''}, \tag{31}
\]
\[
\epsilon_{aa'\cdots A_{N-3}}\epsilon_{bb'\cdots A_{N-3}}\epsilon_{cc'\cdots A_{N-3}} \propto (\delta_{a'b'}\delta_{a''b''} - \delta_{a'b'}\delta_{b'a''})\delta_{ab}, \quad \text{and} \quad \epsilon_{aa'\cdots A_{N-3}}\epsilon_{bb'\cdots A_{N-3}} \propto (\delta_{a'b'}\delta_{a''b''} - \delta_{a'b'}\delta_{b'a''})\delta_{ab}, \tag{32}
\]
\[
\epsilon_{aa'\cdots A_{N-3}}\epsilon_{bb'\cdots A_{N-3}} \propto \delta_{a'b'}(\delta_{a''b''} - \delta_{a'b'}\delta_{a''b''}). \tag{33}
\]
These results are easily derived using Eq. (18). Hence, \( I_{ab'a'b':a''b''} \) may be written as the following linear combination
\[
I_{ab'a'b':a''b''}(c) = x_1\delta_{ab}\delta_{a'b'}\delta_{a''b''} + x_2\delta_{ab}\delta_{a'b'}\delta_{a''b''} + x_3\delta_{ab}\delta_{a'b'}\delta_{a''b''} + x_4\delta_{ab}\delta_{a'b'}\delta_{a''b''} + x_5\delta_{ab}\delta_{a'b'}\delta_{a''b''} + x_6\delta_{ab}\delta_{a'b'}\delta_{a''b''}. \tag{34}
\]

A collection of six equations for the \( x_i \) coefficients may be constructed by contracting appropriate indices as follows:
\[
\begin{pmatrix}
N^2 & N & N & N^2 & N^2 & N^3 \\
N & N^2 & N^2 & N & N^3 & N^2 \\
N^3 & N^2 & N^2 & N & N & N^2 \\
N & N^2 & N^2 & N & N^2 & N \\
N^2 & N^3 & N^2 & N^2 & N \\
N^2 & N^3 & N^2 & N^2 & N \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{pmatrix}
= \begin{pmatrix}
I_{aa'bb'cc} \\
I_{aa'bb'cc} \\
I_{ab'cc'ba} \\
I_{ab'cc'ba} \\
I_{ab'cc'ba} \\
I_{ab'cc'ba} \\
\end{pmatrix}
\equiv \begin{pmatrix}
i_1 \\
i_2 \\
i_3 \\
i_4 \\
i_5 \\
i_6 \\
\end{pmatrix} \tag{35}
\]

This system may be solved to give,
\[
x_1 = \frac{2i_2 + 2i_4 - N(i_1 + i_5 + i_6) + (N^2 - 2)i_3}{(N - 2)(N - 1)N(N + 1)(N + 2)} \tag{36}
\]
\[
x_2 = \frac{2i_1 + 2i_6 - N(i_2 + i_3 + i_4) + (N^2 - 2)i_5}{(N - 2)(N - 1)N(N + 1)(N + 2)} \tag{37}
\]
\[
x_3 = \frac{2i_1 + 2i_5 - N(i_2 + i_3 + i_4) + (N^2 - 2)i_6}{(N - 2)(N - 1)N(N + 1)(N + 2)} \tag{38}
\]
\[
x_4 = \frac{2i_2 + 2i_3 - N(i_1 + i_5 + i_6) + (N^2 - 2)i_4}{(N - 2)(N - 1)N(N + 1)(N + 2)} \tag{39}
\]
\[
x_5 = \frac{2i_3 + 2i_4 - N(i_1 + i_5 + i_6) + (N^2 - 2)i_2}{(N - 2)(N - 1)N(N + 1)(N + 2)} \tag{40}
\]
\[
x_6 = \frac{2i_5 + 2i_6 - N(i_1 + i_5 + i_6) + (N^2 - 2)i_1}{(N - 2)(N - 1)N(N + 1)(N + 2)} \tag{41}
\]
Making use of the definitions in Eq. (33), we see that \( i_2 = i_3 = i_4 \) and \( i_5 = i_6 \). Making these replacements in Eqs. (36—41) and substituting the results in Eq. (34) gives,

\[
I_{ab'a''b''}(c) = \frac{-N(i_1 + 2i_5) + (N^2 + 2)i_2}{N(N^2 - 1)(N^2 - 4)}(\delta_{ab'}\delta_{a'b'}\delta_{a''b} + \delta_{ab'}\delta_{a'b'}\delta_{a''b} + \delta_{ab}\delta_{a'b'}\delta_{a''b}).
\]

\[
+ \frac{2i_1 - 3Ni_2 + N^2i_5}{N(N^2 - 1)(N^2 - 4)}(\delta_{ab'}\delta_{a'b'}\delta_{a''b} + \delta_{ab'}\delta_{a'b'}\delta_{a''b})
\]

\[
+ \frac{4i_5 - 3Ni_2 + (N^2 - 2)i_1}{N(N^2 - 1)(N^2 - 4)}\delta_{ab}\delta_{a'b'}\delta_{a''b'}.
\]  

(42)

The integrals, \( i_1, i_2 \) and \( i_5 \) can all be calculated using the procedure set out in Section 2.

In order to calculate all integrals of three \( U_{ij} \) or \( (U^\dagger)_{kl} \) factors only one more integral is required,

\[
J_{ab'a''b''}(c) = \int dU e^{i\text{Tr}(U + U^\dagger)}U_{ab}(U^\dagger)_{a'b'}U_{a''b''}.
\]  

(43)

Other integrals of three group elements can be calculated using results for \( I_{ab'a''b''} \) and \( J_{ab'a''b''} \) by making appropriate changes of variables.

\( J_{ab'a''b''} \) can be calculated in the same way as \( I_{ab'a''b''} \), in fact \( J_{ab'a''b''} \) is a linear combination of the same delta functions as \( I_{ab'a''b''} \) with coefficients defined by Eqs. (36—41) but with the definitions of \( i_1, \ldots, i_6 \) modified to reflect the integral in question (i.e. \( i_1 = J_{aa'bb'cc'}, \ldots, i_6 = J_{ab'cc'ba} \)). The final result can be shown to be

\[
J_{ab'a''b''}(c) = \frac{2Ni_5 - Ni_1 + (N^2 - 2)i_3}{N(N^2 - 1)(N^2 - 4)}\delta_{ab'}\delta_{a'b'}\delta_{a''b}
\]

\[
+ \frac{2i_1 - Ni_3 - N^2i_5}{N(N^2 - 1)(N^2 - 4)}(\delta_{ab'}\delta_{a'b'}\delta_{a''b} + \delta_{ab'}\delta_{a'b'}\delta_{a''b})
\]

\[
+ \frac{2i_3 - Ni_1 + N(N^2 - 2)i_5}{N(N^2 - 1)(N^2 - 4)}(\delta_{ab'}\delta_{a'b'}\delta_{a''b} + \delta_{ab'}\delta_{a'b'}\delta_{a''b})
\]

\[
+ \frac{2(2 - N^2)i_5 - Ni_3 + (N^2 - 2)i_1}{N(N^2 - 1)(N^2 - 4)}\delta_{ab}\delta_{a'b'}\delta_{a''b'}.
\]  

(44)

Using a software package capable of symbolic manipulation, such as Mathematica, this technique can be easily extended to more complicated integrals, involving four or more \( U \) and / or \( U^\dagger \) group elements in the integrand. In order to accelerate the calculation of such integrals, \( I \), the following procedure is used:

(i) Express \( I_{a_1b_1\ldots a_mb_m} \) as a linear combination of appropriate delta function products, with unknown coefficients, \( x_i \).

(ii) Simplify this expression, taking into account symmetries of the integral with respect to \( \{a_1, \ldots, a_m\} \) and \( \{b_1, \ldots, b_m\} \). Some \( x_i \) will be found to be equal to others, reducing the number of independent \( x_i \). For example, from Eq. (28), we see that \( I_{ab'a''b''} = I_{a'b'b'a''ab} \). Making use of this and Eq. (31) we find that \( x_2 = x_3 \) and \( x_4 = x_5 \).

(iii) Construct a system of equations for \( x_i \) in terms of known integrals by contracting appropriate combinations of indices.
(iv) Solve this system for $x_i$ and substitute into the original expression for $I_{a_1 b_1; \ldots; a_m b_m}$ from step (i).

4. Integrals Required for Moment Calculations

In this section we calculate a number of integrals useful in Hamiltonian LGT, especially for those techniques requiring the calculation of Hamiltonian moments. The calculations make use of the results of Section 3.

The Hamiltonian methods for LGT generally require the calculation of various integrals over $SU(N)$. The $t$-expansion [19] and plaquette expansion [2] methods are two approaches to Hamiltonian LGT which require the calculation of Hamiltonian moments. These moments can be simplified so that all electric field operators are removed, and thus reduced to a set of group integrals over $SU(N)$. For simple one-plaquette trial states in 2+1 dimensions these integrals can be calculated analytically. More complicated scenarios (3+1 dimensions or more complicated trial states) typically require numerical evaluation.

In the calculation of Hamiltonian moments in 2+1 dimensions, the following integrals require calculation at the fourth order for diagrams on two-plaquette skeletons (in the following $U$ and $V$ represent the neighbouring plaquettes of the two-plaquette connected skeleton):

\begin{align*}
Z_1(c, N) &= \langle \phi'_0 | \text{Tr}(U V) \text{Tr}(U V) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (45) \\
Z_2(c, N) &= \langle \phi'_0 | \text{Tr}(U V) \text{Tr}(U V^\dagger) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (46) \\
Z_3(c, N) &= \langle \phi'_0 | \text{Tr}(U V^\dagger) \text{Tr}(U V^\dagger) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (47) \\
Z_4(c, N) &= \langle \phi'_0 | \text{Tr}(U V) \text{Tr}(U^\dagger V^\dagger) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (48) \\
Z_5(c, N) &= \langle \phi'_0 | \text{Tr}(U U V V) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (49) \\
Z_6(c, N) &= \langle \phi'_0 | \text{Tr}(U U V V^\dagger) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (50) \\
Z_7(c, N) &= \langle \phi'_0 | \text{Tr}(U V U V) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (51) \\
Z_8(c, N) &= \langle \phi'_0 | \text{Tr}(U V^\dagger U V) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (52) \\
Z_9(c, N) &= \langle \phi'_0 | \text{Tr}(U V^\dagger U V^\dagger) | \phi'_0 \rangle \equiv \left\langle \begin{array}{c}
\begin{array}{c}
\text{rectangle}
\end{array}
\end{array} \right| \langle \phi_0 | \phi_0 \rangle \quad (53)
\end{align*}
Integrals over \( SU(N) \)

Here each integral is calculated over \( SU(N) \) and \( |\phi_0\rangle \) is the one plaquette trial state:

\[
|\phi_0\rangle = \exp \left[ \frac{c}{2} \sum_p \text{Tr}(U_p + U_p^\dagger) \right] |0\rangle,
\]

where \( |0\rangle \) is the strong coupling vacuum and the sum is over all plaquettes on the lattice. \( |\phi'_0\rangle \) is the reduced trial state incorporating only the neighbouring plaquettes \( U \) and \( V \),

\[
|\phi'_0\rangle = \exp \left[ \frac{c}{2} \text{Tr}(U + U^\dagger + V + V^\dagger) \right] |0\rangle.
\]

Eqs. (45), (47), (48), (51) and (54) can be calculated using standard character expansion techniques [1]. Here we will calculate these integrals using a different method which is able to be applied to the calculation of the other integrals presented above. We start with \( Z_1(c, N) \), which can be expressed as (assuming summation over repeated indices):

\[
Z_1(c, N) = \int dU dV e^{c \text{Tr}(U+U^\dagger)} e^{c \text{Tr}(V+V^\dagger)} U_{ab} V_{ba} U^\prime_{a'b'} V^\prime_{b'a'}
\]

\[
= I_{ab,a'b'}(c) I_{ba,b'a'}(c).
\]

Making use of Eqs. (19) and (23), \( Z_1 \) can be reduced to an expression involving known single plaquette integrals as follows:

\[
Z_1(c, N) = (b_1 \delta_{ab} \delta_{a'b'} + b_2 \delta_{ab} \delta_{ba'}) (b_1 \delta_{ba} \delta_{b'ba'} + b_2 \delta_{ba} \delta_{ab'})
\]

\[
= N^2 b_1^2 + 2N b_1 b_2 + N^2 b_2^2
\]

\[
= \frac{N I_{aa;bb}^2 - 2I_{aa;bb} I_{ab;ba} + N I_{ab;ba}^2}{N(N^2 - 1)}
\]

Next we calculate \( Z_2 \). Using the same technique we have,

\[
Z_2(c, N) = \int dU dV e^{c \text{Tr}(U+U^\dagger)} e^{c \text{Tr}(V+V^\dagger)} U_{ab} V_{ba} U_{a'b'} (V^\dagger)_{b'a'}
\]

\[
= I_{ab,a'b'}(c) J_{ba,b'a'}(c).
\]

Making use of Eqs. (19) and (23) and also Eqs. (24) and (26), \( Z_2 \) can be reduced to an expression involving known integrals as follows,

\[
Z_2(c, N) = I_{ab,a'b'}(c) J_{ba,b'a'}(c)
\]

\[
= N^2 b_1 d_1 + Nb_1 d_2 + N b_2 d_1 + N^2 b_2 d_2
\]

\[
= \frac{NI_{aa;bb} J_{aa;bb} - N I_{aa;bb} J_{ab;ba} - I_{ab;ba} J_{aa;bb} + NI_{ab;ba} J_{ab;ba}}{N(N^2 - 1)}.
\]

\( Z_3 \) can be shown to be equal to \( Z_1 \) as follows,

\[
Z_3(c, N) = \int dU dV e^{c \text{Tr}(U+U^\dagger)} e^{c \text{Tr}(V+V^\dagger)} U_{ab} U_{cd} (V^\dagger)_{ba} (V^\dagger)_{dc}
\]

\[
= I_{ab,cd}(c) I_{ba,dc}(c) = Z_1(c, N).
\]

Here the last line follows from the fact that an integral of a single plaquette loop does not depend on its direction.
The remainder of the integrals can be calculated similarly. The results are as follows:

\[ Z_4(c, N) = \frac{1}{N^2 - 1} \left( J_{aa;bb}^2 + J_{ab;ba}^2 \right) - \frac{2}{N(N^2 - 1)} J_{ab;ba} J_{aa;bb} \]  

(67)

\[ Z_5(c, N) = \frac{1}{N} I_{ab;ba}^2 \]  

(68)

\[ Z_6(c, N) = Z_5(c, N) \]  

(69)

\[ Z_7(c, N) = \frac{2NI_{aa;bb} I_{ab;ba} - I_{aa;bb}^2 - I_{ab;ba}^2}{N(N^2 - 1)} \]  

(70)

\[ Z_8(c, N) = \frac{NI_{aa;bb} J_{ab;ba} - J_{aa;bb} J_{ab;ba} - I_{ab;ba} J_{ab;ba} + N I_{ab;ba} J_{aa;bb}}{N(N^2 - 1)} \]  

(71)

\[ Z_9(c, N) = \frac{2NJ_{aa;bb} J_{ab;ba} - J_{aa;bb}^2 - J_{ab;ba}^2}{N(N^2 - 1)} \]  

(72)

\[ Z_{10}(c, N) = Z_6(c, N) \]  

(73)

The techniques presented in this section can be extended to the calculation of loop integrals on three or more neighbouring plaquettes using Eqs. (42) and (44) and their extensions.

5. Conclusion

In this paper a number of integrals over SU(N) have been calculated. Further work could incorporate these calculations into plaquette expansion calculations of glueball masses in 2+1 dimensions. Other calculations involving Hamiltonian moments may be able to be extended to more complicated trial states, at least in 2+1 dimensions, using the techniques presented in this paper.

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