QUERMASSINTEGRAL PRESERVING CURVATURE FLOW IN HYPERBOLIC SPACE

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Abstract. We consider the quermassintegral preserving flow of closed $h$-convex hypersurfaces in hyperbolic space with the speed given by any positive power of a smooth symmetric, strictly increasing, and homogeneous of degree one function $f$ of the principal curvatures which is inverse concave and has dual $f^*$ approaching zero on the boundary of the positive cone. We prove that if the initial hypersurface is $h$-convex, then the solution of the flow becomes strictly $h$-convex for $t > 0$, the flow exists for all time and converges to a geodesic sphere exponentially in the smooth topology.

1. Introduction

Let $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a closed smooth hypersurface in the hyperbolic space $\mathbb{H}^{n+1}$. We consider the smooth family of immersions $X : M^n \times [0,T) \to \mathbb{H}^{n+1}$ satisfying

$$\begin{align*}
\frac{\partial}{\partial t} X(x,t) &= (\phi(t) - \Psi(W(x,t)))\nu(x,t), \\
X(\cdot,0) &= X_0(\cdot),
\end{align*}$$

(1.1)

where $\nu(x,t)$ is the unit outward normal of $M_t = X(M,t)$, and $\Psi(W) = F^{\alpha}(W)$ where $F$ is a smooth invariant function of the Weingarten matrix $W = (h_{ij}^2)$ of $M_t$. The global term $\phi(t)$ is chosen to keep one of the Quermassintegrals of the hypersurface constant (we will explain this below). We assume that $\alpha > 0$ and $F$ satisfies the following conditions:

Assumption 1.1. \( F(W) = f(\kappa(W)) \), where $\kappa(W)$ gives the eigenvalues of $W$ and $f$ is a smooth symmetric function on

$$\Gamma_+ = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, \ i = 1, \cdots, n \}.$$

(i) $f$ is strictly increasing, i.e., $\dot{f}^i = \partial f / \partial \kappa_i > 0$ on $\Gamma_+$, $\forall i = 1, \cdots, n$.

(ii) $f$ is homogeneous of degree 1, i.e., $f(k\kappa) = kf(\kappa)$ for any $k > 0$.

(iii) $f$ is strictly positive on $\Gamma_+$ and is normalized such that $f(1, \cdots, 1) = 1$.

(iv) $f$ is inverse concave, i.e., the function

$$f_*(x_1, \cdots, x_n) = f(x_1^{-1}, \cdots, x_n^{-1})^{-1}$$

(1.2)

is concave.

(v) $f_*$ approaches zero on the boundary of $\Gamma_+$.

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To describe the global term $\phi(t)$ in (1.1), we first recall the (normalized) $k$-th mean curvature $E_k$ of a smooth closed hypersurface $M$ and the quermassintegrals $W_k(\Omega)$ of the bounded domain $\Omega$ enclosed by $M$: If $\Omega$ is a (geodesically) convex domain in $\mathbb{H}^{n+1}$, then the quermassintegrals of $\Omega$ are defined as follows (see \cite{30, 31, 33}):

$$W_k(\Omega) = \frac{(n+1-k)!\omega_{n-k}}{(n-k)!\omega_n} \int_{\mathcal{L}_k} \chi(L_k \cap \Omega) dL_k, \quad k = 1, \ldots, n,$$

(1.3)

where $\mathcal{L}_k$ is the space of $k$-dimensional affine subspaces $L_k$ in $\mathbb{H}^{n+1}$. The function $\chi$ is defined to be 1 if $L_k \cap \Omega \neq \emptyset$ and to be 0 otherwise. In particular, we have

$$W_0(\Omega) = |\Omega|, \quad W_{n+1}(\Omega) = \frac{\omega_n}{n+1}, \quad W_1(\Omega) = \frac{1}{n+1}|\partial \Omega|.$$

If the boundary $M = \partial \Omega$ is smooth (at least of class $C^2$), we can define the principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ as the eigenvalues of the Weingarten matrix $W$ of $M$. For each $k \in \{1, \ldots, n\}$ the $k$-th mean curvature $E_k$ of $M$ is then defined as the normalized $k$-th elementary symmetric functions of the principal curvatures of $M$:

$$E_k = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$ 

These include $E_1 = H/n = (\kappa_1 + \cdots + \kappa_n)/n$ (the normalized mean curvature) and $E_n = \kappa_1 \cdots \kappa_n$ (the Gauss curvature). The curvature integrals of $\Omega$ are then defined by

$$V_{n-k}(\Omega) = \int_{\partial \Omega} E_k, \quad k = 0, 1, \ldots, n.$$ 

(1.4)

The quermassintegrals and curvature integrals of smooth convex domain $\Omega$ in $\mathbb{H}^{n+1}$ are related as follows:

$$V_{n-k}(\Omega) = (n+1) \left( W_{k+1}(\Omega) + \frac{k}{n+2-k} W_{k-1}(\Omega) \right), \quad k = 1, \ldots, n$$ 

(1.5)

$$V_n(\Omega) = (n+1) W_1(\Omega) = |\partial \Omega|.$$ 

(1.6)

Besides the (geodesic) convexity, there is a stronger notion of convexity for regions in $\mathbb{H}^{n+1}$: horospherical convexity. A domain $\Omega$ in $\mathbb{H}^{n+1}$ is called $h$-convex (or, horospherically convex) if at every boundary point $p \in M = \partial \Omega$ there is a horoball $\mathcal{H}$ of $\mathbb{H}^{n+1}$ which contains $\Omega$ and touches at $p$ (i.e. $p$ is a boundary point of $\mathcal{H}$). Recall that a horoball in $\mathbb{H}^{n+1}$ is the union of those geodesics balls which have centre on a given geodesic ray from $p$ and which have $p$ as a boundary point. A smooth domain $\Omega$ is $h$-convex in $\mathbb{H}^{n+1}$ if and only if all the principal curvatures of $M = \partial \Omega$ are bounded from below by 1, and $\Omega$ is strictly $h$-convex if all the principal curvatures of $M = \partial \Omega$ are strictly bigger than 1. In this paper, when we say a hypersurface $M$ is (strictly) $h$-convex, we mean that the domain $\Omega$ enclosed by $M$ is (strictly) $h$-convex.

Fix an integer $k = 1, \ldots, n$. If we define the function $\phi(t)$ in (1.1) by

$$\phi(t) = \int_{M_t} E_k \mu dt \int_{M_t} E_k \mu dt,$$ 

(1.7)

then the quermassintegral $W_k(\Omega_t)$ of $\Omega_t$ remains constant along the flow (1.1) (see §2). We call flows of the form (1.1) with $\phi(t)$ given by (1.7) quermassintegral preserving curvature flows. In particular, in the case $k = 0$, these are volume preserving curvature flows. The main result of this paper is the following:
Theorem 1.2. Let $k \in \{1, \cdots, n\}$ and $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth embedding such that $M_0 = X_0(M)$ is a closed h-convex hypersurface in $\mathbb{H}^{n+1}$. Then for any power $\alpha > 0$, the flow (1.1) with $F$ satisfying Assumption (1.1) and the global term $\phi(t)$ given by (1.7) has a smooth h-convex solution $M_t$ for all time $t \in [0, \infty)$, and $M_t$ converges smoothly and exponentially to a geodesic sphere of radius $r_\infty$ determined by $W_k(B_{r_\infty}) = W_k(\Omega_0)$ as $t \to \infty$.

Remark 1.3. Some important examples of functions satisfying the Assumption 1.1 include:

1. $f = E_{k}^{1/k}$ for all $k = 1, \cdots, n$;
2. the power-means $H_r = (\frac{1}{n} \sum_i x_i^r)^{1/r}$ for all $r > 0$;
3. any function of the form $E_{\alpha}^\sigma G^{1-\sigma}$ where $0 < \sigma < 1$ and $G$ is homogeneous degree one, normalised, increasing in each argument, and inverse-concave. See §2 for further discussion.

Constrained curvature flows have been studied extensively in recent years. In 1987, Huisken [20] studied the volume preserving mean curvature flow in Euclidean space $\mathbb{R}^{n+1}$, and proved that starting from any strictly convex hypersurface a solution exists for all time $t \in [0, \infty)$ and converges smoothly to a round sphere. In 2001, the first named author [3] studied volume preserving anisotropic mean curvature flows in $\mathbb{R}^{n+1}$ and obtained a similar result. Later, McCoy [24, 25, 26, 27] studied some mixed volume preserving curvature flow driven by homogeneous of degree one curvature functions. For higher homogeneity, by imposing a strong pinching assumption on the initial hypersurface, Cabezas-Rivas and Sinestrari [14] proved convergence results for the flow (1.1) in $\mathbb{R}^{n+1}$ with $\Psi = E_{k}^{\alpha/k}$ where $k = 1, \cdots, n$ and $\alpha > 1$. Using the monotonicity of the isoperimetric ratio, Sinestrari [29] proved a convergence result for the flow (1.1) with $\Psi = H^\alpha$ in $\mathbb{R}^{n+1}$ and for any positive power $\alpha > 0$. This was generalized in [11] for volume (and area) preserving non-homogeneous mean curvature flow in $\mathbb{R}^{n+1}$. Very recently, the authors [8] removed the pinching assumption in [14] and proved that the flow (1.1) in $\mathbb{R}^{n+1}$ with $\Psi = E_{k}^{\alpha/k}$ for $k = 1, \cdots, n$ and any $\alpha > 0$, will deform any strictly convex hypersurface to a round sphere smoothly.

The volume preserving flow in hyperbolic space $\mathbb{H}^{n+1}$ was firstly studied by Cabezas-Rivas and Miquel [13] in 2007. By imposing $h$-convexity on the initial hypersurface, they proved that the flow (1.1) with $\Psi = H$ exists for all time and converges smoothly to a geodesic sphere. This result was generalized recently by Bertini and Pipoli [12] to volume preserving non-homogeneous mean curvature flow. In particular, their result includes the flow with velocity given by $\Psi = H^\alpha$ with $\alpha > 0$. Some mixed volume preserving flows were considered in [28, 33] with $\Psi$ given by some homogeneous of degree one curvature function $F$. By assuming $h$-convexity and strong pinching on the initial hypersurface, Guo-Li-Wu [17] proved the convergence of the flow (1.1) with $\Psi = E_{k}^{\alpha/k}$, $k = 1, \cdots, n$ and power $\alpha > 1$ by following the same procedure as the Euclidean case in [14].

The paper is organized as follows: In section 2, we collect some preliminaries on hyperbolic geometry, the evolution equations along the flow (1.1), and examples of the function satisfying the Assumption 1.1. In section 3, we prove that the $h$-convexity is preserved along the flow (1.1) for inverse concave $f$ and for all power $\alpha > 0$. To show this, we will apply the tensor maximum principle proved by the first named author in [4] (which generalized Hamilton’s [19] theorem). In section 4, the preservation of $W_k(\Omega_t)$ and the $h$-convexity will be used to estimate the inner radius and outer radius of $\Omega_t$. Then we adapt Tso’s [32] technique to prove a uniform upper bound on $F$. In section 5, by
assumption that $f_*$ approaches zero on the boundary of $\Gamma_+$ and the upper bound on $F$, we derive a uniform upper bound on the principal curvatures. The $h$-convexity together with the boundedness of principal curvatures makes the evolution equation uniformly parabolic. By projecting the flow solution into the unit ball in $\mathbb{R}^{n+1}$ and using the Gauss map parametrization, we write the flow (1.1) as a scalar parabolic partial differential equation for the support function which is concave with respect to the second spatial derivatives. Then the Hölder estimate of Krylov-Evans [21] and the parabolic Schauder theory [22] can be applied to derive the higher order derivative estimates of the solution $M_t$. From this we conclude that the solution of (1.1) exists for all time $t \in [0, \infty)$.

In previous work, the convergence of solutions as $t \to \infty$ was deduced using either the monotonicity of curvature pinching ratios [20, 24, 25, 26, 27, 13, 14, 17, 28] or of isoperimetric ratios [3, 29, 11, 12, 8]. In our situation for general $F$ and $\alpha$, neither of these arguments is available. Instead, we apply the Alexandrov reflection method to prove that the hypersurfaces approach a sphere, and a linearisation argument to prove exponential convergence.

Remark 1.4. In the case where $F = E_k^{1/k}$, we have as in [8] that the quermassintegral $W_k(\Omega_t)$ is monotone decreasing along the volume preserving flow for any $\alpha > 0$. This can also be used to deduce the smooth convergence to the geodesic sphere. We describe this alternative argument in the appendix.

Remark 1.5. As in [11, 12], a result similar to Theorem 1.2 is also true for non-homogeneous constrained flows (1.1) with $\Psi = \psi(f)$, where $f$ is a function satisfying Assumption 1.1, and $\psi : [0, +\infty) \to \mathbb{R}$ is in $C^0([0, \infty)) \cap C^2((0, \infty))$ and satisfies

(i) $\psi(r) > 0$, $\psi'(r) > 0$ for all $r > 0$;
(ii) $\lim_{r \to \infty} \psi(r) = \infty$;
(iii) $\lim_{r \to \infty} \frac{\psi'(r)^2}{\psi(r)} = \infty$;
(iv) $\psi''(r)r + 2\psi'(r) \geq 0$ for all $r > 0$

In fact, the flow is parabolic due to item (i); item (iv) is used to show that the $h$-convexity is preserved along the flow (see Remark 3.3); items (ii)–(iii) are used to estimate the upper bound of $f$. The remaining proof is similar.

2. Preliminaries

In this section, we collect some preliminary results concerning hyperbolic geometry, the properties of inverse concave functions, and examples of functions satisfying Assumption 1.1. We also collect the evolution equations for several geometric quantities of the solution $M_t$ of the flow (1.1).

2.1. Hyperbolic geometry. Let $M$ be a smooth closed hypersurface in $\mathbb{H}^{n+1}$. We denote by $g_{ij}, h_{ij}$ and $\nu$ the induced metric, the second fundamental form and unit outward normal vector of $M$. The Weingarten map is denoted by $W = (h^i_j)$, where $h^i_j = h_{ik}g^{kj}$. The principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ of $M$ are defined as the eigenvalues of $W$. As mentioned before, $M$ is $h$-convex if and only if $\kappa_i \geq 1$ for all $i = 1, \ldots, n$.

A remarkable property of an $h$-convex hypersurface $M = \partial \Omega$ in $\mathbb{H}^{n+1}$ is that its inner radius and outer radius are comparable. Recall that the inner radius $r_-$ and outer radius
\( \rho_+ \) of a bounded domain \( \Omega \) are defined by
\[
\rho_- = \sup \{ \rho : B_\rho(p) \subset \Omega \text{ for some } p \in \mathbb{H}^{n+1} \}
\]
and
\[
\rho_+ = \inf \{ \rho : \Omega \subset B_\rho(p) \text{ for some } p \in \mathbb{H}^{n+1} \},
\]
where \( B_\rho(p) \) denotes the geodesic ball of radius \( \rho \) about \( p \) in \( \mathbb{H}^{n+1} \). The following results can be found in \([9, 10, 13, 28]\).

**Theorem 2.1.** Let \( \Omega \) be a compact h-convex domain in \( \mathbb{H}^{n+1} \) and denote the center of an inball by \( o \) and its inner radius by \( \rho_- \). Then we have

1. The maximum of the distance \( d_\mathbb{H}(o,p) \) between \( o \) and the points on \( \partial \Omega \) satisfies
\[
\max_{p \in \partial \Omega} d_\mathbb{H}(o,p) \leq \rho_- + \ln \frac{(1 + \sqrt{\tanh \rho_- / 2})^2}{1 + \tanh \rho_- / 2} < \rho_- + \ln 2.
\]
Therefore there exists a constant \( c > 0 \) such that the outer radius
\[
\rho_+ \leq c(\rho_- + \rho_-^{1/2}).
\]

2. For any interior point \( p \) of \( \Omega \), and any boundary point \( q \in \partial \Omega \),
\[
Dr_p(\nu(q)) \geq \tanh(d_\mathbb{H}(p,\partial \Omega)),
\]
where \( r_p(x) = d_\mathbb{H}(p,x) \).

For smooth \( h \)-convex domains in \( \mathbb{H}^{n+1} \), inequalities of Alexandrov-Fenchel type for quermassintegrals were proved by Wang-Xia in \([33]\). See also \([18, 23]\) for related Alexandrov-Fenchel type inequalities for the curvature integrals \((1.4)\).

**Theorem 2.2** \(([33])\). For any smooth bounded domain \( \Omega \) in \( \mathbb{H}^{n+1} \) with \( h \)-convex boundary \( \partial \Omega \), and \( 0 \leq l < k \leq n \), we have
\[
W_k(\Omega) \geq f_k \circ f_l^{-1}(W_l(\Omega))
\]
with equality if and only if \( \Omega \) is a geodesic ball. Here the function \( f_k : [0, \infty) \rightarrow \mathbb{R}_+ \) is increasing and is defined by \( f_k(r) = W_k(B_r) \), with \( B_r \) a geodesic ball in \( \mathbb{H}^{n+1} \). \( f_l^{-1} \) is the inverse function of \( f_l \).

2.2. **Inverse concave functions.** For a smooth symmetric function \( F(A) = f(\kappa(A)) \), where \( A = (A_{ij}) \in \text{Sym}(n) \) is a symmetric matrix and \( \kappa(A) = (\kappa_1, \ldots, \kappa_n) \) gives the eigenvalues of \( A \), we denote by \( \dot{F}^{ij} \) and \( \ddot{F}^{ij,kl} \) the first and second derivatives of \( F \) with respect to the components of its argument, so that
\[
\frac{\partial}{\partial s} F(A + sB) \bigg|_{s=0} = \dot{F}^{ij}(A)B_{ij}
\]
and
\[
\frac{\partial^2}{\partial s^2} F(A + sB) \bigg|_{s=0} = \ddot{F}^{ij,kl}(A)B_{ij}B_{kl}
\]
for any two symmetric matrices \( A, B \). We also use the notation
\[
\dot{f}^i(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa), \quad \ddot{f}^{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\kappa).
\]
for the derivatives of $f$ with respect to $\kappa$. At any diagonal $A$ with distinct eigenvalues, the second derivative $\ddot{F}$ of $F$ in direction $B \in \text{Sym}(n)$ is given in terms of $\dot{f}$ and $\ddot{f}$ by (see [1, 4]):

$$
\dddot{F}_{ijkl} B_{ij} B_{kl} = \sum_{i,k} \dddot{f}_{ik} B_{ii} B_{kk} + 2 \sum_{i>k} \dddot{f}_{ik} B_{ik}^2. 
$$

(2.5)

This formula makes sense as a limit in the case of any repeated values of $\kappa_i$. Since $\Psi(A) = F^{\alpha}(A)$, we will use the same notations and formulas for the derivatives of $\Psi$ and $\psi = f^\alpha$.

For any positive definite symmetric matrix $A \in \text{Sym}(n)$ with eigenvalues $\kappa(A) \in \Gamma_+$, define $F_\star(A) = F(A^{-1})^{-1}$. Then $F_\star(A) = f_\star(\kappa(A))$, where $f_\star$ is defined in (1.2). Since $f$ is defined on the positive definite cone $\Gamma_+$, the following lemma characterizes the inverse concavity of $f$ and $F$.

**Lemma 2.3 ([4, 7]).**

(i) $f$ is inverse concave if and only if the following matrix

$$
\left( \dddot{f}_{kl} + 2 \frac{\dddot{f}_{kl}}{\kappa_k} \delta_{kl} \right) \geq 0. 
$$

(2.6)

(ii) $F_\star$ is concave if and only if $f_\star$ is concave;

(iii) $F_\star$ is inverse concave if and only if

$$
\left( \dddot{f}_{kl} + 2 \frac{\dddot{f}_{kl}}{\kappa_k} \delta_{kl} \right) \geq 0, \quad \text{and} \quad \frac{\dddot{f}_{kl}}{\kappa_k - \kappa_l} + \frac{\dddot{f}_{kl}}{\kappa_l} + \frac{\dddot{f}_{kl}}{\kappa_k} \geq 0, \quad k \neq l. 
$$

(2.7)

(i) If $f$ is inverse concave, then

$$
\sum_{i=1}^{n} \dddot{f}_{i} \kappa_i^2 \geq \dot{f}^2. 
$$

(2.8)

Since the function $f = E_k^{1/k}$ is inverse concave for all $k = 1, \cdots, n$ and has dual function

$$
f_\star(z) = \left( \frac{E_n(z)}{E_n^{-k}(z)} \right)^{1/k}, \quad z \in \Gamma_+
$$

which vanishes on the boundary of $\Gamma_+$, we have that $f = E_k^{1/k}$, $k = 1, \cdots, n$, satisfy the Assumption 1.1. We can also easily see that a convex function $f : \Gamma_+ \to \mathbb{R}$ satisfies the Assumption 1.1. Firstly, the inequality (2.6) is obviously true since $f$ is convex and strictly increasing. Secondly, the convexity of $f$ implies that

$$
f(x_1, \cdots, x_n) \geq \frac{1}{n} \sum_{i=1}^{n} x_i.
$$

Then the dual function $f_\star$ satisfies

$$
f_\star(z_1, \cdots, z_n) = f \left( \frac{1}{z_1}, \cdots, \frac{1}{z_n} \right)^{-1} \leq n \left( \frac{1}{z_1} + \cdots + \frac{1}{z_n} \right)^{-1} = \frac{E_n(z)}{E_n^{-1}(z)}.
$$

Thus $f_\star$ approaches zero on the boundary of $\Gamma_+$. Other important examples of functions satisfying Assumption 1.1 are the power means $H_r = (\frac{1}{n} \sum x_i^r)^{1/r}$, which are inverse-concave for $r \geq -1$, concave for $r \geq 1$, and have $f_\star$ approaching zero on $\partial \Gamma_+$ for $r \geq 0$, and so satisfy our requirements for all $r \geq 0$. More examples can be constructed as follows: If $G_1$ is homogeneous of degree one, increasing in each argument, and inverse-concave, and
G_2 satisfies Assumption 1.1, then F = G_1^n G_2^{-\sigma} satisfies Assumption 1.1 for any 0 < \sigma < 1 (see [4, 6] for more examples of inverse concave or convex functions).

2.3. Evolution equations. Along the flow
\[ \frac{\partial}{\partial t} X(x, t) = (\phi(t) - \Psi(W(x, t))) \nu(x, t) \]
in hyperbolic space \( \mathbb{H}^{n+1} \), we have the following evolution equations (see [23]) for the induced metric \( g_{ij} \), unit outward normal \( \nu \), induced area element \( d\mu_t \) and Weingarten matrix \( W = (h^j_i) \) of \( M_t = X(M^n, t) \):
\[ \frac{\partial}{\partial t} g_{ij} = 2(\phi(t) - \Psi) h_{ij} \quad (2.9) \]
\[ \frac{\partial}{\partial t} \nu = \nabla \Psi \quad (2.10) \]
\[ \frac{\partial}{\partial t} d\mu_t = nE_1(\phi(t) - \Psi) d\mu_t \quad (2.11) \]
\[ \frac{\partial}{\partial t} h^j_i = \nabla^j \nabla_i \Psi + (\Psi - \phi(t))(h^k_l h^i_k - \delta^i_k) \quad (2.12) \]
where \( \nabla \) denotes the Levi-Civita connection with respect to the induced metric \( g_{ij} \) on \( M_t \).

As an immediate consequence of (2.12), we have that the curvature function \( \Psi = \Psi(W) \) evolves by
\[ \frac{\partial}{\partial t} \Psi = \dot{\Psi}^{kl} \nabla^k \nabla_l \Psi + (\Psi - \phi(t))(\dot{\Psi}^{ij} h^k_l h^l_k - \dot{\Psi}^{ij} \delta^i_k), \quad (2.13) \]
where \( \dot{\Psi}^{kl} \) denotes the derivatives of \( \Psi \) with respect to the components of \( W = (h^j_i) \).

Throughout this paper we will always evaluate the derivatives of \( \Psi = F^\alpha \) at \( W = (h^j_i) \) and the derivatives of \( \psi = f^\alpha \) at \( \kappa(W) = (\kappa_1, \ldots, \kappa_n) \).

The following lemma gives a parabolic type equation of \( h^j_i \).

**Lemma 2.4.** Along the flow (1.1), the Weingarten matrix \( h^j_i \) of \( M_t \) evolves by
\[ \frac{\partial}{\partial t} h^j_i = \dot{\Psi}^{kl} \nabla^k \nabla_l h^j_i + \dot{\Psi}^{k, pq} \nabla_i h_{kl} \nabla^j h_{pq} + (\dot{\Psi}^{kl} h^k_l h_{rl} + \dot{\Psi}^{kl} g_{kl}) h^j_i - \dot{\Psi}^{kl} h_{kl}(h^k_p h^i_l + \delta^i_l) + (\Psi - \phi(t))(h^k_p h^i_l - \delta^i_l), \quad (2.14) \]
where \( \Psi = F^\alpha \) and \( \dot{\Psi}^{kl}, \dot{\Psi}^{k, pq} \) denote the derivatives of \( \Psi \) with respect to the components of \( W = (h^j_i) \).

**Proof.** Firstly, combining the Gauss and Codazzi equation in hyperbolic space gives the following generalized Simons’ identity (see [5]):
\[ \nabla^j \nabla_i \Psi = \nabla^k \nabla_l h_{kl} = (h^k_p h_{pl} + g_{kl}) h_{ij} - h_{kl}(h^p_l h_{pj} + g_{ij}), \quad (2.15) \]
where the brackets denote symmetrisation. Then
\[ \nabla^j \nabla_i \Psi = \dot{\Psi}^{kl} \nabla^j \nabla_i h_{kl} + \dot{\Psi}^{k, pq} \nabla_i h_{kl} \nabla^j h_{pq} \]
\[ = \dot{\Psi}^{kl} \nabla^j h^i_k + \dot{\Psi}^{k, pq} \nabla_i h_{kl} \nabla^j h_{pq} + \Psi^{kl}(h^p_l h_{pl} + g_{kl}) h^i_l - \dot{\Psi}^{kl} h_{kl}(h^p_l h^i_l + \delta^i_l). \quad (2.16) \]

The equation (2.14) follows from (2.12) and (2.16) immediately. \( \square \)
Using the evolution equations (2.11) and (2.12), we can also derive the evolution equation for the curvature integral defined in (1.4)

\[ \frac{d}{dt}V_{n-k}(\Omega_t) = \int_{M_t} \left( \frac{\partial}{\partial t} E_k d\mu_t + E_k \frac{\partial}{\partial t} d\mu_t \right) \]

\[ = \int_{M_t} \left( \frac{\partial E_k}{\partial h^j_i} \nabla_j \nabla_i F^\alpha + (F^\alpha - \phi(t))(\frac{\partial E_k}{\partial h^j_i} (h^k_i h^j_k - \delta^j_i) + nE_k E_k) \right) d\mu_t \]

\[ = \int_{M_t} ((\phi(t) - F^\alpha)((n-k)E_{k+1} + kE_{k-1})) d\mu_t, \]  

where we used the facts that

\[ \nabla_j (\frac{\partial E_k}{\partial h^j_i}) = 0 \]  

and

\[ \frac{\partial E_k}{\partial h^j_i} h^j_k = nE_k E_k - (n-k)E_{k+1}, \quad \frac{\partial E_k}{\partial h^j_i} \delta^j_i = kE_{k-1}. \]

By applying induction argument to (2.17) and (1.5), we have the following evolution equation for the quermassintegrals of \( \Omega_t \) along the flow (1.1),

\[ \frac{d}{dt}W_k(\Omega_t) = \frac{n+1-k}{n+1} \int_{M_t} E_k (\phi(t) - F^\alpha) d\mu_t, \quad k = 0, \ldots, n, \]  

(2.18)

which was also derived in [33]. Thus for the function \( \phi(t) \) defined in (1.7), the quermassintegral \( W_k(\Omega_t) \) remains constant along the flow (1.1).

**Remark 2.5.** If \( \phi(t) \) is defined as

\[ \phi(t) = \frac{1}{|M_t|} \int_{M_t} F^\alpha d\mu_t, \]  

(2.19)

then the volume \( |\Omega_t| \) remains constant. The flow (1.1) with \( \phi(t) \) given by (2.19) is called the volume preserving curvature flow.

### 3. Preserving of h-convexity

In this section, we will use the tensor maximum principle to prove that the \textit{h-convexity} is preserved along the flow (1.1) if \( f \) is inverse concave. For the convenience of readers, we include here the statement of the tensor maximum principle, which was first proved by Hamilton [19] and was generalized by Andrews [4].

**Theorem 3.1** ([4]). Let \( S_{ij} \) be a smooth time-varying symmetric tensor field on a compact manifold \( M \), satisfying

\[ \frac{\partial}{\partial t} S_{ij} = a^{kl} \nabla_k \nabla_l S_{ij} + u^k \nabla_k S_{ij} + N_{ij}, \]  

(3.1)

where \( a^{kl} \) and \( u \) are smooth, \( \nabla \) is a (possibly time-dependent) smooth symmetric connection, and \( a^{kl} \) is positive definite everywhere. Suppose that

\[ N_{ij} v^i v^j + \sup_\lambda 2a^{kl} \left( 2\Lambda_k^p \nabla_l S_{ij} v^i - \Lambda^q_k \Lambda^q_l S_{pq} \right) \geq 0 \]  

(3.2)

whenever \( S_{ij} \geq 0 \) and \( S_{ij} v^j = 0 \). If \( S_{ij} \) is positive definite everywhere on \( M \) at \( t = 0 \) and on \( \partial M \) for \( 0 \leq t \leq T \), then it is positive on \( M \times [0, T] \).

The main result of this section is the following.
**Theorem 3.2.** For any power \( \alpha > 0 \), if the initial hypersurface \( M_0 \) is \( h \)-convex and \( f \) is inverse concave, then along the flow (1.1) in \( \mathbb{H}^{n+1} \) the flow hypersurface \( M_t \) is strictly \( h \)-convex for \( t > 0 \).

**Proof.** Denote

\[ S_{ij} = h^j_i - \delta^j_i. \]

Then the \( h \)-convexity is equivalent to \( S_{ij} \geq 0 \). By (2.14), the tensor \( S_{ij} \) evolves by

\[
\frac{\partial}{\partial t} S_{ij} = \ddot{\psi}^{kl} \nabla_k \nabla_l S_{ij} + \ddot{\psi}^{kl,pq} \nabla_i h_{kl} \nabla_j h_{pq} + (\ddot{\psi}^{kl} h^r_l h_{rl} + \ddot{\psi}^{kl} g_{kl}) S_{ij} + \left( \Psi - \phi(t) - \dot{\psi}^{kl} h_{kl} \right)(S_{ik} S_{kj} + 2S_{ij}) + \ddot{\psi}^{kl} (h^r_k h_{rl} + g_{kl} - 2h_{kl}) \delta^j_i. \tag{3.3}
\]

To apply the tensor maximum principle in Theorem 3.1, we need to show the inequality (3.2) whenever \( S_{ij} \geq 0 \) and \( S_{ij} v^j = 0 \) (so that \( v \) is a null vector of \( S \)). Let \( (x_0, t_0) \) be the point where \( S_{ij} \) has a null vector \( v \). By continuity we can assume that \( h^j_i \) has all eigenvalues distinct and in increasing order at \( (x_0, t_0) \), that is \( \kappa_n > \kappa_{n-1} > \cdots > \kappa_1 \). The null eigenvector condition \( S_{ij} v^j = 0 \) implies that \( v = e_1 \) and \( S_{11} = \kappa_1 - 1 = 0 \) at \( (x_0, t_0) \).

The lower order terms in (3.3) involving \( S_{ij} \) and \( S_{ik} S_{kj} \) satisfy the null vector condition and can be ignored: In particular for the last term in (3.3), we have

\[
\ddot{\psi}^{kl} (h^r_k h_{rl} + g_{kl} - 2h_{kl}) = \sum_k \dot{\psi}^k (\kappa_k^2 + 1 - 2\kappa_k) = \sum_k \dot{\psi}^k (\kappa_k - 1)^2 \geq 0. \tag{3.4}
\]

Thus it remains to show that

\[
Q_1 := \ddot{\psi}^{kl,pq} \nabla_1 h_{kl} \nabla_1 h_{pq} + 2 \sup_{\Lambda} \ddot{\psi}^{kl} \left( 2\Lambda_k^p \nabla_k S_{1p} - \Lambda_k^p \Lambda^q S_{pq} \right) \geq 0. \tag{3.5}
\]

Note that \( S_{11} = 0 \) and \( \nabla_k S_{11} = 0 \) at \( (x_0, t_0) \), the supremum over \( \Lambda \) can be computed exactly as follows:

\[
2\ddot{\psi}^{kl} \left( 2\Lambda_k^p \nabla_k S_{1p} - \Lambda_k^p \Lambda^q S_{pq} \right)
= 2 \sum_{k=1}^n \sum_{p=2}^n \dot{\psi}^k \left( 2\Lambda_k^p \nabla_k S_{1p} - (\Lambda_k^p)^2 S_{pp} \right)
= 2 \sum_{k=1}^n \sum_{p=2}^n \dot{\psi}^k \left( \frac{(\nabla_k S_{1p})^2}{S_{pp}} - \left( \Lambda_k^p - \frac{\nabla_k S_{1p}}{S_{pp}} \right)^2 S_{pp} \right).
\]

It follows that the supremum is obtained by choosing \( \Lambda_k^p = \nabla_k S_{1p} \). The required inequality for \( Q_1 \) becomes:

\[
Q_1 = \ddot{\psi}^{kl,pq} \nabla_1 h_{kl} \nabla_1 h_{pq} + 2 \sum_{k=1}^n \sum_{p=2}^n \dot{\psi}^k \frac{(\nabla_k S_{1p})^2}{S_{pp}} \geq 0. \tag{3.6}
\]
Using (2.5) to express the second derivatives of $\Psi$ and noting that $\psi = f^\alpha$, $\nabla_1 S_{1p} = \nabla_1 h_{1p} = \nabla_p h_{11} = 0$ at $(x_0, t_0)$, we have
\begin{align*}
Q_1 = & \alpha f^{\alpha - 1} j^{kl} \nabla_1 h_{kk} \nabla_1 h_{ll} + \alpha (\alpha - 1) f^{\alpha - 2} (\nabla_1 F)^2 \\
+ & 2 \alpha f^{\alpha - 1} \sum_{k>l} \frac{j^k - j^l}{\kappa_k - \kappa_l} (\nabla_1 h_{kl})^2 + 2 \alpha f^{\alpha - 1} \sum_{k>l, l>1} \frac{j^k}{\kappa_l - 1} (\nabla_1 h_{kl})^2.
\end{align*}
(3.7)
To make use the inverse concavity of $f$, let $\tau_i = 1/\kappa_i$ and $f_s(\tau) = f(\kappa)^{-1}$. We compute that
\begin{align*}
j^k &= f_s^{-2} \frac{\partial f_s}{\partial \tau_k} \frac{1}{\kappa_k^2} \\
j^{kl} &= - f_s^{-2} \frac{\partial^2 f_s}{\partial \tau_k \partial \tau_l} \frac{1}{\kappa_k^2 \kappa_l^2} + 2 f_s^{-3} \frac{\partial f_s}{\partial \tau_k} \frac{1}{\kappa_k} \frac{\partial f_s}{\partial \tau_l} \frac{1}{\kappa_l} - 2 f_s^{-2} \frac{\partial f_s}{\partial \tau_k} \frac{1}{\kappa_k} \delta_{kl} \\
&= - f_s^{-2} \frac{\partial^2 f_s}{\partial \tau_k \partial \tau_l} \frac{1}{\kappa_k^2 \kappa_l^2} + 2 f^{-1} j^k j^l - 2 \frac{j^k}{\kappa_k} \delta_{kl}.
\end{align*}
By the concavity of $f_s$, the first term of (3.7) can be estimated as
\begin{align*}
\alpha f^{\alpha - 1} j^{kl, pq} \nabla_1 h_{kl} \nabla_1 h_{pq} & \geq 2 \alpha f^{\alpha - 1} \left( f^{-1} (\nabla_1 F)^2 - \sum_k \frac{j^k}{\kappa_k} (\nabla_1 h_{kk})^2 \right)
\end{align*}
Then
\begin{align*}
\frac{Q_1}{\alpha f^{\alpha - 1}} & \geq (\alpha + 1) f^{-1} (\nabla_1 F)^2 - 2 \sum_k \frac{j^k}{\kappa_k} (\nabla_1 h_{kk})^2 \\
& + 2 \sum_{k>l} \frac{j^k - j^l}{\kappa_k - \kappa_l} (\nabla_1 h_{kl})^2 + 2 \sum_{k>l, l>1} \frac{j^k}{\kappa_l - 1} (\nabla_1 h_{kl})^2 \\
& \geq (\alpha + 1) f^{-1} (\nabla_1 F)^2 - 2 \sum_k \frac{j^k}{\kappa_k} (\nabla_1 h_{kk})^2 \\
& - 2 \sum_{k\neq l>1} \frac{j^k}{\kappa_l} (\nabla_1 h_{kl})^2 + 2 \sum_{k>l, l>1} \frac{j^k}{\kappa_l - 1} (\nabla_1 h_{kl})^2 \\
& = (\alpha + 1) f^{-1} (\nabla_1 F)^2 + 2 \sum_{k>l, l>1} \left( \frac{j^k}{\kappa_l - 1} - \frac{j^k}{\kappa_l} \right) (\nabla_1 h_{kl})^2 \\
& \geq 0
\end{align*}
for all $\alpha > 0$, where we used the second inequality of (2.7) and the fact $\nabla_k h_{11} = 0$. The tensor maximum principle implies that the $h$-convexity is preserved along the flow (1.1).

Finally we show that $\Mt$ is strictly $h$-convex for $t > 0$. If this is not true, there exists some interior point $(x_0, t_0)$ such that the smallest principal curvature is $1$. By the strong maximum principle there exists a parallel vector field $v$ such that $S_{ij} v^i v^j = 0$ on $\Mt$. Then the smallest principal curvature is $1$ on $\Mt$ everywhere. On the other hand, a standard argument shows that on any closed hypersurface in $\mathbb{H}^{m+1}$, there exists at least one point where all the principal curvatures are strictly bigger than one. This contradiction completes the proof of Theorem 3.2. \qed
Remark 3.3. The above argument implies that $h$-convexity is also preserved along the flow

$$
\frac{\partial}{\partial t}X(x, t) = (\phi(t) - \Psi(W(x, t)))\nu(x, t)
$$

in hyperbolic space $\mathbb{H}^{n+1}$ with $\Psi = \psi(f)$, where $f$ is an inverse concave function and $\psi : [0, +\infty) \rightarrow \mathbb{R}_+$ satisfies $\psi'(r) > 0$ and $\psi''(r)r + 2\psi'(r) \geq 0$ for all $r > 0$.

4. Upper bound of $F$

In this section, we will prove that $F$ is uniformly bounded from above along the flow (1.1). Firstly, the preservation of $W_k(\Omega_t)$ and the $h$-convexity of $M_t = \partial \Omega_t$ imply uniform two-sided bounds on inner radius and outer radius of $\Omega_t$.

Lemma 4.1. Denote by $\rho_-(t)$ and $\rho_+(t)$ the inner and outer radii of the domain $\Omega_t$ enclosed by $M_t$. Then there exist positive constants $c_1, c_2$ depending only on $n, k, M_0$ such that

$$0 < c_1 \leq \rho_-(t) \leq \rho_+(t) \leq c_2$$

(4.1)

for all time $t \in [0, T)$.

Proof. On the one hand, since $W_k(\Omega_t) = W_k(\Omega_0)$, we have

$$W_k(B_{\rho_+(t)}) \geq W_k(\Omega_t) = W_k(\Omega_0),$$

where $B_{\rho_+(t)}$ is the geodesic ball of radius $\rho_+(t)$ that encloses $\Omega_t$. Thus

$$\rho_+(t) \geq f_k^{-1}(W_k(\Omega_0)) > 0,$$

where $f_k^{-1}$ is the inverse function of $f_k(r) = W_k(B_r)$. Similarly, $\rho_-(t) \leq f_k^{-1}(W_k(\Omega_0))$. Since each $M_t$ is $h$-convex, the estimate (4.1) follows by the inequality (2.2). \qed

By (4.1), the inner radius of $\Omega_t$ is bounded below by a positive constant $c_1$. This implies that there exists a geodesic ball of radius $c_1$ contained in $\Omega_t$ for each $t \in [0, T)$. The following lemma shows the existence of a geodesic ball with fixed center enclosed by the flow hypersurfaces on a suitable time interval.

Lemma 4.2. Let $M_t$ be a smooth $h$-convex solution of (1.1) on $[0, T)$ with global term $\phi(t)$ given by (1.7). For any $t_0 \in [0, T)$, let $B(p_0, \rho_0)$ be the inball of $\Omega_{t_0}$, where $\rho_0 = \rho_-(t_0)$. Then

$$B(p_0, \rho_0/2) \subset \Omega_t, \quad t \in [t_0, \min\{T, t_0 + \tau\})$$

(4.2)

for some $\tau$ depending only on $n, \alpha, k, \Omega_0$.

Proof. Given $p_0$, we denote by $r_{p_0}$ the distance function to $p_0$ in $\mathbb{H}^{n+1}$ and by $\partial_r = \partial_{r_{p_0}}$ the gradient vector of $r_{p_0}$. For any $x \in M_t$,

$$
\frac{\partial}{\partial t}\sinh^2 r_{p_0}(x) = 2\langle \sinh r_{p_0}(x)\partial_r, \frac{\partial}{\partial t}(\sinh r_{p_0}(x)\partial_r) \rangle
= 2\sinh r_{p_0}(x)\cosh r_{p_0}(x)(\phi(t) - F^\alpha(x, t))\langle \partial_r, \nu \rangle,
$$

(4.3)

where we used the conformal property of the vector field $\sinh r\partial_r$, i.e.,

$$\langle \nabla_X (\sinh r\partial_r), Y \rangle = \cosh r \langle X, Y \rangle$$

(4.4)
for any tangential vector fields $X, Y$ in $\mathbb{H}^{n+1}$ (see, e.g., [16]). It follows from (4.3) that
\[
\partial_t r_{p_0}(x) = (\phi(t) - F^\alpha(x, t))(\partial r, \nu) \geq -F^\alpha(x, t)(\partial r, \nu),
\]
since $\phi(t) > 0$ and $\langle \partial r, \nu \rangle > 0$ on $M_t$. Denote $r(t) = \min_{M_t} r_{p_0}(x)$. At the minimum point, we have $\langle \partial r, \nu \rangle = 1$ and $\kappa_i \leq \coth r(t)$. Then $F \leq \coth r(t)$ at the minimum point and
\[
\frac{d}{dt} r(t) \geq -\coth (r(t))\alpha. \tag{4.5}
\]
Note that $0 < c_1 \leq r(t) \leq 2\rho_+ \leq 2c_2$, where $c_1, c_2$ are the constants in (4.1). Then
\[
\coth^{\alpha - 1} r(t) \leq \max\{\coth^{\alpha - 1}(2c_2), \coth^{\alpha - 1}(c_1)\} =: c_3. \tag{4.6}
\]
We deduce from (4.5) and (4.6) that
\[
\frac{d}{dt} r(t) \geq -c_3 \coth r(t),
\]
from which we solve that
\[
\cosh r(t) \geq \cosh r(0) \exp\{-c_3 t\}.
\]
In particular,
\[
\begin{align*}
r(t) & \geq \frac{r(0)}{2} = \frac{\rho_0}{2},
\end{align*}
\]
provided that
\[
t - t_0 \leq \frac{1}{c_3} \ln \frac{\cosh r(0)}{\cosh \frac{r(0)}{2}} =: \tau
\]
which depends only on $n, \alpha, k, \Omega_0$. Then $B(p_0, \rho_0/2) \subset \Omega_t$ for $t \in [t_0, \min\{T, t_0 + \tau\})$. \hfill \Box

Consider the support function $u(x, t) = \sinh r_{p_0}(x)(\partial r_{p_0}, \nu)$ of $M_t$ with respect to the point $p_0$. Then by (2.3) and (4.2),
\[
\begin{align*}
u(x, t) & \geq \sinh\left(\frac{\rho_0}{2}\right) \tanh\left(\frac{\rho_0}{2}\right) =: 2c
\end{align*}
\]
on $M_t$ for any $t \in [t_0, \min\{T, t_0 + \tau\})$. On the other hand, the estimate (4.1) implies that $u(x, t) \leq \sinh(2c_2)$ on $M_t$ for all $t \in [0, T)$.

**Lemma 4.3.** The support function $u(x, t)$ evolves by
\[
\frac{\partial}{\partial t} u = \tilde{\Psi}^{kl} \nabla_k \nabla_i u + \cosh r_{p_0}(x) \left(\phi(t) - \Psi - \hat{\Psi}^{kl} h_{kl}\right) + \hat{\Psi}^{ij} h_i^k h_{kj} u. \tag{4.8}
\]

**Proof.** Firstly, by (4.4) and (2.10),
\[
\frac{\partial}{\partial t} u = \cosh r_{p_0}(x) (\phi(t) - \Psi) + \sinh r_{p_0}(x) (\partial r, \nabla \Psi). \tag{4.9}
\]
Secondly, the spatial derivatives of $u$ can also be computed using (4.4):
\[
\nabla_j u = \sinh r_{p_0}(x) (\partial_r, h_i^k \partial_k)
\]
\[
\nabla_i \nabla_j u = \cosh r_{p_0}(x) h_{ij} + \sinh r_{p_0}(x) (\partial_r, \nabla^k h_{ij} \partial_k - h_i^k h_{kj} \nu). \tag{4.11}
\]
Then the evolution equation (4.8) follows by a direct computation using (4.9) – (4.11). \hfill \Box

Now we can use the technique that was first introduced by Tso [32] to prove the upper bound of $F$ along the flow (1.1).
Theorem 4.4. Let $M_t$ be a smooth $h$-convex solution of (1.1) on $[0,T)$ with global term $\phi(t)$ given by (1.7). Then we have $\max_{M_t} F \leq C$ for any $t \in [0,T)$, where $C$ depends on $n, k, \alpha, M_0$ but not on $T$.

Proof. For any given $t_0 \in [0,T)$, define the auxiliary function

$$W(x,t) = \frac{\Psi(x,t)}{u(x,t) - c},$$

which is well-defined for all $t \in [t_0, \min\{T, t_0 + \tau\})$ by (4.7). Combining (2.13) and (4.8), we can compute the evolution equation of the function $W$

$$\frac{\partial}{\partial t} W = \Psi^{ij} \left( \nabla_j \nabla_i W + \frac{2}{u - c} \nabla_i u \nabla_j W \right) - \frac{\phi(t)}{u - c} \left( \Psi^{ij} (h^h k_i - \delta^j_i) + W \cosh r_{p_0}(x) \right) + \frac{\Psi}{(u - c)^2} \left( \Psi + \Psi^{ij} h_{kl} \right) \cosh r_{p_0}(x) - \frac{c \Psi}{(u - c)^2} \Psi^{ij} h_k^i k_l^j - W \Psi^{ij} \delta^j_i. \quad (4.12)$$

The $h$-convexity of $M_t$ implies that $\Psi^{ij} (h^h k_i - \delta^j_i) \geq 0$. So the terms involving $\phi(t)$ in (4.12) are non-positive. Since $\Psi = F^\alpha$ where $F$ is inverse concave, we have $\Psi + \Psi^{ij} h_{kl} = (1 + \alpha) \Psi$ and $\Psi^{ij} h_k^i k_l^j \geq \alpha F^{\alpha + 1}$. The last term of (4.12) is obviously non-positive. Therefore,

$$\frac{\partial}{\partial t} W \leq \Psi^{ij} \left( \nabla_j \nabla_i W + \frac{2}{u - c} \nabla_i u \nabla_j W \right) + (\alpha + 1) W^2 \cosh r_{p_0}(x) - \alpha c W^2. \quad (4.13)$$

Using (4.7) and the upper bound $r_{p_0}(x) \leq 2c_2$, we obtain the following estimate

$$\frac{\partial}{\partial t} W \leq \Psi^{ij} \left( \nabla_j \nabla_i W + \frac{2}{u - c} \nabla_i u \nabla_j W \right) + W^2 \left( (\alpha + 1) \cosh(2c_2) - \alpha c^{1+\frac{1}{\alpha}} W^{1/\alpha} \right) \quad (4.14)$$

holds on $[t_0, \min\{T, t_0 + \tau\})$. Let $\tilde{W}(t) = \sup_{M_t} W(\cdot, t)$. Then (4.14) implies that

$$\frac{d}{dt} \tilde{W}(t) \leq 2 \tilde{W}^2 \left( (\alpha + 1) \cosh(2c_2) - \alpha c^{1+\frac{1}{\alpha}} \tilde{W}^{1/\alpha} \right)$$

from which it follows using the maximum principle that

$$\tilde{W}(t) \leq \max \left\{ \left( \frac{2(1 + \alpha) \cosh(2c_2)}{\alpha} \right)^\alpha c^{-(\alpha + 1)}, \frac{2}{1 + \alpha} \left( \frac{\alpha}{1 + \alpha} \right)^{1+\alpha} c^{1-\alpha} (t - t_0)^{-\frac{\alpha}{1 + \alpha}} \right\}. \quad (4.15)$$

Then the upper bound on $F$ follows from (4.15) and the facts that

$$c = \frac{1}{2} \sinh\left( \frac{\rho_0}{2} \right) \tanh\left( \frac{\rho_0}{2} \right) \geq \frac{1}{2} \sinh\left( \frac{c_1}{2} \right) \tanh\left( \frac{c_1}{2} \right)$$

and $u - c \leq 2c_2$, where $c_1, c_2$ are constants in (4.1) depending only on $n, k, M_0$. \hfill $\square$

As an corollary of the upper bound of $F$ and the $h$-convexity of $M_t$, there exist constants $c_4, c_5$ depend only on $n, k, \alpha, M_0$ such that

$$c_4 \leq \phi(t) \leq c_5 \quad (4.16)$$

on $[0,T)$. 


5. LONG-TIME EXISTENCE

Until now, we only used the fact that \( f \) is inverse concave. The upper bound on \( F \) proved in §4 implies that the dual function \( f^* \) of \( f \) is bounded from below by a positive constant. Applying condition (vi) in Assumption 1.1 that \( f^* \) approaches zero on the boundary of \( \Gamma_+ \), there exists a positive constant \( C \) such that \( \frac{1}{\kappa_i} \geq C \) for all \( i = 1, \ldots, n \), which implies a uniform upper bound on the Weingarten matrix \( W = (h^i_j) \) along the flow (1.1) for all \( t \in [0, T) \).

**Lemma 5.1.** There exists a constant \( C > 0 \) depending only on \( n, k, \alpha, M_0 \) such that along the flow (1.1), the principal curvatures \( \kappa = (\kappa_1, \ldots, \kappa_i) \) of the solution \( M_t \) satisfy

\[
1 \leq \kappa_i \leq C, \quad i = 1, \ldots, n
\]

for all \( t \in [0, T) \).

In the following, we will derive higher order estimates on the solution \( M_t \) of the flow (1.1) and prove that the solution \( M_t \) exists for all time \( t \in [0, \infty) \).

Let us denote by \( \mathbb{R}^{1,n+1} \) the Minkowski spacetime, that is the vector space \( \mathbb{R}^{n+2} \) endowed with the Minkowski spacetime metric \( \langle \cdot, \cdot \rangle \) given by

\[
\langle X, X \rangle = -X_0^2 + \sum_{i=1}^{n} X_i^2
\]

for any vector \( X = (X_0, X_1, \ldots, X_n) \in \mathbb{R}^{n+2} \). The hyperbolic space \( \mathbb{H}^{n+1} \) is then

\[
\mathbb{H}^{n+1} = \{ X \in \mathbb{R}^{1,n+1}, \langle X, X \rangle = -1, \ X_0 > 0 \}
\]

An embedding \( X : M^n \to \mathbb{H}^{n+1} \) induces an embedding \( Y : M^n \to B_1(0) \subset \mathbb{R}^{n+1} \) by

\[
X = \frac{(1, Y)}{\sqrt{1 - |Y|^2}}
\]

where \( Y \in B_1(0) \subset \mathbb{R}^{n+1} \). Let \( \{x_i\}, i = 1, \ldots, n \) be a local coordinate system on \( M \) and \( \{\partial_i\} \) be the corresponding coordinate vectors. Then

\[
\partial_i X = X_s(\partial_i) = \frac{(0, \partial_i Y)}{\sqrt{1 - |Y|^2}} + \frac{X}{1 - |Y|^2} \langle Y, \partial_i Y \rangle
\]

and

\[
\partial_i \partial_j X = \frac{(0, \partial_i \partial_j Y)}{\sqrt{1 - |Y|^2}} + \frac{(0, \partial_j Y)}{(1 - |Y|^2)^{3/2}} \langle Y, \partial_i Y \rangle + \partial_i X \frac{\langle Y, \partial_j Y \rangle}{1 - |Y|^2} \frac{1}{1 - |Y|^2}
\]

\[
+ X \partial_i \left( \frac{\langle Y, \partial_j Y \rangle}{1 - |Y|^2} \right).
\]

Let \( \nu \in T\mathbb{H}^{n+1}, h^X_{ij} \) and \( N \in \mathbb{R}^{n+1}, h^Y_{ij} \) be the unit normal vectors and the second fundamental forms of \( X(M^n) \subset \mathbb{H}^{n+1} \) and \( Y(M^n) \subset \mathbb{R}^{n+1} \) respectively. Note that \( \langle \nu, X \rangle = \langle \nu, \partial_i X \rangle = 0 \). Taking the inner product of (5.2) with \( \nu \), we also have

\[
\langle \nu, (0, \partial_i Y) \rangle = 0
\]
Therefore,

\[ h^X_{ij} = -\langle \partial_i \partial_j X, \nu \rangle = - \frac{1}{\sqrt{1 - |Y|^2}} \langle (0, \partial_i \partial_j Y), \nu \rangle \]

\[ = - \frac{\langle \partial_i \partial_j Y, N \rangle}{\sqrt{1 - |Y|^2}} \langle (0, N), \nu \rangle = \frac{h^Y_{ij}}{\sqrt{1 - |Y|^2}} \langle (0, N), \nu \rangle \]  \hspace{1cm} (5.4)

By (5.3), we can write \( \nu = a_1(0, N) + a_2(1, 0) \) where \( a_1, a_2 \) can be computed as follows:

\[ 1 = \langle \nu, \nu \rangle = a_1^2 - a_2^2 \]

\[ 0 = \langle \nu, X \rangle = \frac{1}{\sqrt{1 - |Y|^2}} (a_1 \langle N, Y \rangle - a_2) \]

from which we deduce that

\[ \langle (0, N), \nu \rangle = a_1 = \frac{1}{\sqrt{1 - \langle N, Y \rangle^2}}. \]

Substituting this into (5.4) yields

\[ h^X_{ij} = \frac{h^Y_{ij}}{\sqrt{(1 - |Y|^2)(1 - \langle N, Y \rangle^2)}}. \]  \hspace{1cm} (5.5)

From (5.2), we also have that the induced metrics \( g^X_{ij} \) and \( g^Y_{ij} \) of \( X(M^n) \subset \mathbb{H}^{n+1} \) and \( Y(M^n) \subset \mathbb{R}^{n+1} \) are related by

\[ g^X_{ij} = \frac{\langle \partial_i Y, \partial_j Y \rangle}{1 - |Y|^2} + \frac{\langle Y, \partial_i Y \rangle \langle Y, \partial_j Y \rangle}{(1 - |Y|^2)^2} \]

\[ = \frac{1}{1 - |Y|^2} \left( g^Y_{ij} + \frac{\langle Y, \partial_i Y \rangle \langle Y, \partial_j Y \rangle}{(1 - |Y|^2)^2} \right) \]

Suppose \( X : M^n \times [0, T) \to \mathbb{H}^{n+1} \) is a solution to the flow (1.1). We next derive the evolution equation of the corresponding \( Y : M^n \times [0, T) \to \mathbb{R}^{n+1} \) related by (5.1). Denote by \( W^X = (h^X_{ik} \delta^X_{kj}) \) the Weingarten matrix of \( X(M^n) \subset \mathbb{H}^{n+1} \), where \( (g^X_{kj}) \) denotes the inverse matrix of the induce metric \( g^X_{kj} \). Then

\[ \phi(t) - F^\alpha(W^X) = \langle \partial_t X, \nu \rangle \]

\[ = \frac{\langle (0, \partial_t Y), \nu \rangle}{\sqrt{1 - |Y|^2}} + \frac{\langle X, \nu \rangle \langle Y, \partial_t Y \rangle}{1 - |Y|^2} \]

\[ = \langle \partial_t Y, N \rangle \frac{\langle (0, N), \nu \rangle}{\sqrt{1 - |Y|^2}} \]

\[ = \langle \partial_t Y, N \rangle \frac{1}{\sqrt{(1 - |Y|^2)(1 - \langle N, Y \rangle^2)}} \]

where we used the facts \( \langle X, \nu \rangle = 0 \) and \( \langle (0, \partial_t Y), \nu \rangle = 0 \) in the third equality. Thus up to a tangential diffeomorphism, \( Y : M^n \times [0, T) \to \mathbb{R}^{n+1} \) satisfies the following evolution equation:

\[ \partial_t Y = \sqrt{(1 - |Y|^2)(1 - \langle N, Y \rangle^2)} \left( \phi(t) - F^\alpha(W^X) \right) N. \]  \hspace{1cm} (5.6)
Here \( \mathcal{W}^X \) is the Weingarten matrix of \( X(M^n, t) \subset \mathbb{H}^{n+1} \), which we next relate to the geometry of \( Y \): In local coordinates, the inverse matrix \( \mathcal{W}^{-1}_X \) of \( \mathcal{W}^X \) satisfies

\[
(\mathcal{W}^{-1}_X)_{ij} = (h^{-1}_X)^{jk} g_{ki}^Y = (h^{-1}_Y)^{kj} \left( g_{ki}^Y + \frac{\langle Y, \partial_i Y \rangle \langle Y, \partial_k Y \rangle}{(1 - |Y|^2)} \right) \sqrt{\frac{1 - \langle N, Y \rangle^2}{1 - |Y|^2}}
\]

By the estimate (4.1), \( X \) stays in a bounded subset of \( \mathbb{H}^{n+1} \). Then there exists a positive constant \( c > 0 \) depending only on \( n, k, M_0 \) such that

\[
0 < c \leq 1 - |Y|^2 \leq 1, \quad 0 < c \leq 1 - \langle N, Y \rangle^2 \leq 1.
\]

Since each \( M_t \) is \( h \)-convex in \( \mathbb{H}^{n+1} \), the equation (5.5) implies that each \( Y_t = Y(M^n, t) \) is strictly convex in \( \mathbb{R}^{n+1} \). We can parametrise \( Y_t \) using the Gauss map and the support function \( s(z) := \langle Y(N^{-1}(z)), z \rangle \), where \( N^{-1} : \mathbb{S}^n \to M^n \) is the inverse of the Gauss map which exists due to the strict convexity of \( Y_t \). Then \( Y \) is given by the embedding \( Y : \mathbb{S}^n \to \mathbb{R}^{n+1} \) with (cf. [2, 7])

\[
Y(z) = s(z)z + \nabla s
\]

where \( \nabla \) is the gradient with respect to the standard round metric \( \bar{g} \) on \( \mathbb{S}^n \). The derivative of this map is given by

\[
\partial_i Y = \tau_{ik} \bar{g}^{kl} \partial_l z
\]

in local coordinates, where \( \tau_{ij} \) is given as follows

\[
\tau_{ij} = \nabla_i \nabla_j s + s \bar{g}_{ij}.
\]

The eigenvalues \( \tau_i \) of the matrix \( \tau_{ij} \) with respect to the metric \( \bar{g} \) are the inverse of the principal curvatures \( \kappa_i \), i.e., \( \tau_i = 1/\kappa_i \), and are called the principal radii of curvature. Then

\[
g_{ki}^Y = \langle \partial_k Y, \partial_l Y \rangle = \tau_{kp} \bar{g}^{pq} \tau_{qi}
\]

and

\[
\frac{\langle Y, \partial_i Y \rangle \langle Y, \partial_k Y \rangle}{(1 - |Y|^2)} = \tau_{kp} \frac{\langle \bar{g}^{pa} \nabla_a s, \bar{g}^{qb} \nabla_b s \rangle}{1 - s^2 - |\nabla s|^2} \tau_{qi}
\]

We can express (5.7) using \( s, \nabla s \) and the matrix \( \tau_{ij} \) as follows:

\[
(\mathcal{W}^{-1}_X)_{ij} = (h^{-1}_Y)^{jk} \left( g_{ki}^Y + \frac{\langle Y, \partial_i Y \rangle \langle Y, \partial_k Y \rangle}{(1 - |Y|^2)} \right) \sqrt{\frac{1 - \langle N, Y \rangle^2}{1 - |Y|^2}}
\]

\[
= \tau_{ij} (\tau^{-1})^{ls} \bar{g}_{st} (\tau^{-1})^{tk} \tau_{kp} \left( \bar{g}^{pq} + \frac{\langle \bar{g}^{pa} \nabla_a s, \bar{g}^{qb} \nabla_b s \rangle}{1 - s^2 - |\nabla s|^2} \right) \tau_{qi} \sqrt{\frac{1 - s^2}{1 - s^2 - |\nabla s|^2}}
\]

\[
= \left( \bar{g}^{ij} + \frac{i \bar{g}^{ia} \nabla_a s, \bar{g}^{b} \nabla_b s}{1 - s^2 - |\nabla s|^2} \right) \tau_{qi} \sqrt{\frac{1 - s^2}{1 - s^2 - |\nabla s|^2}}
\]

(5.8)

where \( \tau^{-1} \) denotes the inverse matrix of \( \tau_{ij} \). The solution of the evolution equation (5.6) is then given up to a tangential diffeomorphism by solving the following scalar parabolic equation

\[
\partial_t s = \sqrt{(1 - s^2 - |\nabla s|^2)(1 - s^2)} \left( \phi(t) - F^{-\alpha}_s ((\mathcal{W}^{-1}_X)_{ij}) \right)
\]

for the support function \( s(z, t) \), where \( (\mathcal{W}^{-1}_X)_{ij} \) is the matrix given in (5.8) in terms of \( s, \nabla s \) and the matrix \( \tau_{ij} = \nabla_i \nabla_j s + s \bar{g}_{ij} \).
By (4.1) and Lemma 5.1, we already have uniform \( C^2 \) estimates on the support function \( s(z,t) \). Denote the right hand side of (5.9) by \( G(\nabla^2 s, \nabla s, s, z, t) \). Then

\[
\dot{G}^{ij} = \frac{\partial G}{\partial (\nabla^2_{ij}s)} = \alpha F_*^{-\alpha - 1} \sqrt{(1 - s^2 - |\nabla s|^2)(1 - s^2)} \dot{F}_{pq} \frac{\partial (W_X^{-1})_{pq}}{\partial \nabla^2_{ij}s} \\
= \alpha F_*^{-\alpha - 1}(1 - s^2) \dot{F}_{pq} \left( \bar{g}^{pq} + \frac{(\bar{g}^{pa} \bar{\nabla}_a s, \bar{g}^{pb} \bar{\nabla}_b s)}{1 - s^2 - |\nabla s|^2} \right) \bar{g}_{ij},
\]

and

\[
\frac{1}{\sqrt{(1 - s^2 - |\nabla s|^2)(1 - s^2)}} \dot{G}^{ij,kl} = \alpha F_*^{-\alpha - 1} \dot{F}_{pq,rt} \frac{\partial (W_X^{-1})_{pq}}{\partial \nabla^2_{ij}s} \frac{\partial (W_X^{-1})_{rt}}{\partial \nabla^2_{kl}s} \\
- \alpha (\alpha + 1) F_*^{-\alpha - 2} \dot{F}_{pq} \frac{\partial (W_X^{-1})_{pq}}{\partial \nabla^2_{ij}s} \dot{F}_{rt} \frac{\partial (W_X^{-1})_{rt}}{\partial \nabla^2_{kl}s} \tag{5.10}
\]

The uniform bound on \( F \) and Lemma 5.1 imply that there exists a constant \( C > 0 \) such that \( 0 < C^{-1}I \leq (\dot{F}_{ij}^*) \leq CI \). The estimates on \( s, \nabla s \) and \( F \) then imply that

\[
\lambda I \leq (\dot{G}^{ij}) \leq \Lambda I
\]

for some constants \( \lambda, \Lambda > 0 \). By the concavity of \( F_* \) and \( \alpha > 0 \), from (5.10) the operator \( G \) is concave with respect to \( \nabla^2 s \). Since we have uniform \( C^2 \) estimates on \( s \) in space-time, we can apply the H"older estimate of [21] (as in [8, 14]) to obtain the \( C^{2,\alpha} \) estimate on \( s \) and \( C^\alpha \) estimate on \( \partial_t s \) for some \( \alpha \in (0,1) \) in space-time. By the parabolic Schauder theory [22], we can derive estimates on all higher order derivatives of \( s \). A standard continuation argument yields that the flow (1.1) exists for all time \( [0, \infty) \).

**Proposition 5.2.** Let \( M_t \) be a smooth \( h \)-convex solution to the flow (1.1) with \( \alpha > 0 \) and \( \phi(t) \) given by (1.7). If \( f \) is inverse concave and \( f_* \) approaches zero on the boundary of \( \Gamma_+ \), then the solution \( M_t \) exists for all time \( t \in [0, \infty) \).

### 6. Smooth convergence

In this section, we will use the Alexandrov reflection method to show that the solution of (1.1) converges smoothly to a geodesic sphere as \( t \to \infty \).

Let \( \gamma \) be a geodesic line in \( \mathbb{H}^{n+1} \), and let \( H_{\gamma(s)} \) be the totally geodesic hyperbolic \( n \)-plane in \( \mathbb{H}^{n+1} \) which is perpendicular to \( \gamma \) at \( \gamma(s) \), \( s \in \mathbb{R} \). We use the notation \( H^+_s \) and \( H^-_s \) for the half-spaces in \( \mathbb{H}^{n+1} \) determined by \( H_{\gamma(s)} \):

\[
H^+_s := \bigcup_{s' \geq s} H_{\gamma(s')}, \quad H^-_s := \bigcup_{s' \leq s} H_{\gamma(s')}
\]

For a bounded domain \( \Omega \) in \( \mathbb{H}^{n+1} \), denote

\[
\Omega^+(s) = \Omega \cap H^+_s, \quad \Omega^-(s) = \Omega \cap H^-_s.
\]

The reflection map across \( H_{\gamma(s)} \) is denoted by \( R_{\gamma,s} \). We define

\[
S^+_t(\Omega) := \inf \{ s \in \mathbb{R} \mid R_{\gamma,s}(\Omega^+_t) \subset \Omega^-(s) \}. \tag{6.1}
\]

**Lemma 6.1.** For any geodesic line \( \gamma \) in \( \mathbb{H}^{n+1} \), \( S^+_t(\Omega_t) \) is strictly decreasing along the flow (1.1) unless \( R_{\gamma,s}(\Omega_t) = \Omega_t \) for some \( s \in \mathbb{R} \) (in which case \( S^+_t(\Omega_t) = \bar{s} \) for all \( t \)).
Proof. Fix \( t_0 \in [0, \infty) \), and set \( \bar{s} = S_{\gamma}^+(\Omega_{t_0}) \). By definition we have \( R_{\gamma \bar{s}}(\Omega_{t_0}^+(\bar{s})) \subset \Omega_{t_0}^-(\bar{s}) \), and since \( s \) cannot be decreased below \( \bar{s} \) we must have one of two possibilities: Either (i) there is a point \( \bar{x} \in R_{\gamma \bar{s}}(\partial \Omega_{t_0}^+(\bar{s})) \cap \partial \Omega_{t_0}^-(\bar{s}) \) which is not in \( H_{\gamma}(s) \), or (ii) there is a point \( \bar{x} \) in \( \partial \Omega_{t_0} \cap H_{\gamma}(s) \) such that \( T_{\bar{x}} \partial \Omega_{t_0} \) is mapped to itself by \( R_{\gamma \bar{s}} \).

For case (i), since both \( M_t^-(s) \) and \( R_{\gamma}(s)(M_t^+(s)) \) are strictly h-convex, we locally express them as graphs of functions \( r^-(\theta,t) \) and \( r^+(\theta,t) \) over a domain \( U \) of a geodesic sphere for \( t \) sufficiently close to \( t_0 \). Define \( \omega(\theta,t) := r^-(\theta,t) - r^+(\theta,t) \). Then \( \omega(\theta,t_0) \) is nonnegative for \( \theta \in U \) and there exists a point \( \theta_0 \in U \) where the minimum \( \omega(\theta_0, t_0) = 0 \) is achieved. We will argue below using the strong maximum principle that \( \omega \) vanishes identically, and it follows from this that \( M_t^-(s) \) coincides with \( R_{\gamma}(s)(M_t^+(s)) \) and hence \( M_t^0 \) is reflection symmetric across the totally geodesic plane \( H_{\gamma}(s) \) at \( s = S_{\gamma}^+(\Omega_{t_0}) \).

In order to apply the strong maximum principle we first recall the graphical representation of hypersurfaces in \( \mathbb{H}^{n+1} \). Let \( M \subset \mathbb{H}^{n+1} \) be a hypersurface which can be written as a radial graph over a sphere \( S^n \), i.e., \( M = \{ (\theta, r(\theta)) : \theta \in S^n \} \) for a smooth function \( r \) on \( S^n \). Let \( \{ \theta^i \}, i = 1, \cdots, n \) be a local coordinate system on \( S^n \). The induced metric on \( M \) from \( \mathbb{H}^{n+1} \) takes the form

\[
g_{ij} = D_i r D_j r + \sinh^2 r \sigma_{ij},
\]

where \( \sigma_{ij} \) denotes the standard metric on \( S^n \). Denote

\[
v = \sqrt{1 + |D_r|^2 / \sinh^2 r}, \quad \text{and} \quad \tilde{\sigma}^{ij} = \sigma^{jk} - \frac{D_j r D_k r}{v^2 \sinh^2 r} \tag{6.2}
\]

where \( \sigma^{jk} \) is the inverse matrix of \( \sigma_{jk} \) and \( D^2 r = \sigma^{jk} D_k r \). Then the Weingarten matrix \( (h^i_j) \) can be expressed as (cf. [15])

\[
h^i_j = \frac{\coth r}{v} \delta^i_j + \frac{\coth r}{v^2 \sinh^2 r} D^i r D_j r - \frac{\tilde{\sigma}^{jk}}{v \sinh^2 r} D_k D_j r. \tag{6.3}
\]

Up to a tangential diffeomorphism, the flow equation (1.1) is equivalent to the following scalar parabolic PDE

\[
\frac{\partial r}{\partial t} = (\psi(t) - F^a(h^i_j)) \sqrt{1 + |D_r|^2 / \sinh^2 r}. \tag{6.4}
\]

Denote the right hand side of (6.4) by \( \Phi(D^2 r, Dr, r, \theta, t) \).

We now come back to the \( M_t^-(s) \) and \( R_{\gamma}(s)(M_t^+(s)) \) and \( \omega(\theta,t) = r^-(\theta,t) - r^+(\theta,t) \). Since the flow (1.1) is invariant under reflection, by (6.4) the function \( \omega(\theta,t) \) satisfies the following equation

\[
\frac{\partial \omega}{\partial t} = \Phi(D^2 r^-, Dr^-, r^-, \theta, t) - \Phi(D^2 r^+, Dr^+, r^+, \theta, t)
= \frac{\partial \Phi}{\partial D^2 r^+} (\chi, \xi, \eta) D^2 r \omega + \frac{\partial \Phi}{\partial D r^+} (\chi, \xi, \eta) D_r \omega + \frac{\partial \Phi}{\partial r} (\chi, \xi, \eta) \omega, \tag{6.5}
\]

where \( (\chi, \xi, \eta) = (\rho D^2 r^+ + (1 - \rho) D^2 r^-, \rho D r^- + (1 - \rho) D r^+, \rho r^- + (1 - \rho) r^+) \) for some \( \rho \in [0, 1] \). The uniform estimates in §5 implies that the equation (6.5) is uniformly parabolic, i.e., there exist constants \( \lambda, \Lambda > 0 \) such that

\[
\lambda I \leq \frac{\partial \Phi}{\partial D^2 r^+} \leq \Lambda I.
\]
The coefficients \( \frac{\partial \phi}{\partial M_j} \) and \( \frac{\partial \phi}{\partial M_k} \) are uniformly bounded and smooth. Since \( \omega(\theta, t_0) \) is non-negative and is positive somewhere in \( U \), the strong maximum principle applied to the equation (6.5) yields that \( \omega > 0 \) everywhere in \( U \) for \( t > t_0 \), unless it is identically zero. This in turn implies that \( S_\gamma^+(\Omega_t) < S_\gamma^+(\Omega_{t_0}) \) for \( t > t_0 \).

The discussion for case (ii) is similar. We again write \( M^+(s) \) and \( R_\gamma(s)(M^+(s)) \) locally as graphs of functions \( r^-(\theta, t) \) and \( r^+(\theta, t) \) over a domain \( U \) of a geodesic sphere for \( t \) sufficiently close to \( t_0 \), and then apply the boundary strong maximum principle (the Hopf Lemma). \( \square \)

Now we prove that the flow (1.1) converges smoothly to a geodesic sphere. For any fixed \( \tau \), we define the flow \( \Omega_\tau(t) \) by \( \Omega_\tau(t) := \Omega_{t+\tau} \). The uniform estimates in §5 imply that there exists a sequence of \( \tau_k \to \infty \) such that the families \( \Omega_{\tau_k}(t) \) converge smoothly to a limiting flow \( \Omega_\infty(t), t \in [0, \infty) \) which is again a solution of (1.1). By the monotonicity of \( S_\gamma^+(\Omega_t) \) proved in Lemma 6.1, we have

\[
S_\gamma^+(\Omega_\infty(t)) = \lim_{t' \to \infty} S_\gamma^+(\Omega_{t'})
\]  

which exists by the monotonicity. The right hand side of (6.6) is independent of \( t \) and is finite. We conclude that the limiting flow \( \Omega_\infty(t) \) is symmetric under reflection across a totally geodesic hyperplane \( H_\gamma \) which is perpendicular to the geodesic line \( \gamma \). Since \( \gamma \) is arbitrary, we conclude that \( \Omega_\infty(t) \) is a geodesic sphere. This implies the subsequential smooth convergence of \( \Omega_t \) to a geodesic sphere of radius \( r_\infty = f_k^{-1}(W_k(\Omega_t)) \).

The linearisation of the flow (1.1) about the geodesic sphere of radius \( r_\infty \) can be used to deduce the stronger convergence results. The hypersurface near the geodesic sphere can be written as a graph of a smooth function \( r \) over the geodesic sphere. Setting \( r = r_\infty(1+\epsilon \eta) \). The linearised equation of the flow (6.4) about the geodesic sphere of radius \( r_\infty \) is given by

\[
\frac{\partial \eta}{\partial t} = \frac{\alpha \coth \alpha - 1}{n \sinh^2 r_\infty} \left( \Delta \eta + n \eta - \frac{n}{|S^n|} \int_{S^n} \eta d\sigma \right)
\]  

This equation is the same as that for mixed volume preserving mean curvature flow in [25, Eqn.(21)] (see also [3, §12]). Thus the same argument as in [3] gives that the flow (1.1) converges exponentially in smooth topology to the geodesic sphere with radius \( r_\infty \). This completes the proof of Theorem 1.2.

**APPENDIX A. SMOOTH CONVERGENCE: \( f = E_k^{1/k} \)**

In this appendix, we provide an alternative approach to the proof of smooth convergence to a geodesic sphere for the volume preserving flow (1.1) with \( f = E_k^{1/k} \) and \( \alpha > 0 \). The key ingredient is the monotonicity of the quermassintegral \( W_k(\Omega_t) \).

**Lemma A.1.** For any integer \( k \in \{1, \cdots, n\} \), let \( M_t \) be a smooth convex solution of the volume preserving flow (1.1) with \( f = E_k^{1/k} \) and \( \alpha > 0 \). Denote \( \Omega_t \) the domain enclosed by \( M_t \). Then \( W_k(\Omega_t) \) is monotone decreasing in time \( t \).

**Proof.** The function \( \phi(t) \) in (1.1) for the volume preserving flow is defined as in (2.19). From the evolution equation (2.18) for the quermassintegral of \( \Omega_t \):

\[
\frac{d}{dt} W_k(\Omega_t) = \frac{n+1-k}{n+1} \int_{M_t} E_k(\phi(t) - F^\alpha) \, d\mu_t
\]  

(A.1)
and $F = E^{1/k}_{k}$, we have

$$\frac{d}{dt} W_k(\Omega_t) = \frac{n + 1 - k}{n + 1} \left( \frac{1}{|M_t|} \int_{M_t} E_k \int_{M_t} E^\alpha/k - \int_{M_t} E^{1+\alpha/k}_{k} d\mu_t \right).$$

Then the monotonicity of $W_k(\Omega_t)$ follows immediately from the Jensen’s inequality

$$\int_{M_t} E^{(\alpha+k)/k}_{k} d\mu_t \geq \frac{1}{|M_t|} \int_{M_t} E_k d\mu_t \int_{M_t} E^\alpha/k d\mu_t. \quad (A.2)$$

If the initial hypersurface $M_0$ is $h$-convex, then the long time existence of the flow (1.1) has been proved in §5. To show the smooth convergence to a geodesic sphere, we need the following lemma. Denote

$$\overline{E}_k = \frac{1}{|M_t|} \int_{M_t} E_k d\mu_t.$$

**Lemma A.2.** For any integer $k \in \{1, \cdots, n\}$, let $M_t$ be a smooth $h$-convex solution of the volume preserving flow (1.1) with $f = E^{1/k}_{k}$ and $\alpha > 0$. Then there exists a sequence of times $t_i \to \infty$ such that

$$\int_{M_{t_i}} (E_k - \overline{E}_k)^2 d\mu_{t_i} \to 0, \quad \text{as} \quad i \to \infty. \quad (A.3)$$

**Proof.** Since $M_t$ is $h$-convex, the Alexandrov-Fenchel inequality (2.4) implies that,

$$W_k(\Omega_t) \geq f_k \circ f^{-1}_0(W_0(\Omega_t)) = f_k \circ f^{-1}_0(W_0(\Omega_0)) > 0,$$

where we used the condition $|\Omega_t| = |\Omega_0|$. By Lemma A.1, $W_k(\Omega_t)$ is monotone decreasing. Then we can find a sequence of times $t_i \to \infty$ such that

$$\frac{d}{dt} \bigg|_{t_i} W_k(\Omega_t) \to 0, \quad \text{as} \quad i \to \infty,$$

Then from the proof of Lemma A.1, equality is attained in Jensen’s inequality (A.2) as $t_i \to \infty$, or equivalently

$$0 \leq \int_{M_{t_i}} E^{\alpha/k}_{k} (E_k - \overline{E}_k) d\mu_{t_i} = \int_{M_{t_i}} \left( E^{\alpha/k}_{k} - \overline{E}^{\alpha/k}_{k} \right) (E_k - \overline{E}_k) d\mu_{t_i} \to 0, \quad \text{as} \quad i \to \infty. \quad (A.4)$$

Since

$$\left( E^{\alpha/k}_{k} - \overline{E}^{\alpha/k}_{k} \right) (E_k - \overline{E}_k) \geq C \left( E_k - \overline{E}_k \right)^2, \quad (A.5)$$

as in the proof of Lemma 6.1 of [8], then (A.3) follows from (A.4) and (A.5) immediately. □

The uniform estimates on all the higher derivatives of the Weingarten map ($h^j_i$) imply that for any sequence of times $t_i \to \infty$, there exists a subsequence (still denoted by $t_i$) such that $M_{t_i}$ converges to a limit hypersurface $M_\infty$ (up to an ambient isometry). Thus by Lemma A.2, we can find a sequence of times $t_i \to \infty$ such that $M_{t_i}$ converges smoothly to a geodesic sphere up to an isometry. For any other sequence of times $t_j \to \infty$, we can also find a subsequence of time (labeled by the same $t_j$) such that $M_{t_j}$ converges to a
The limit $\hat{M}_\infty = \partial \hat{\Omega}_\infty$. The monotonicity of $W_k(\Omega_t)$ then yields $W_k(\hat{\Omega}_\infty) = W_k(B^{n+1})$, which implies that $\hat{\Omega}_\infty$ is a geodesic ball by the equality case of Theorem 2.2. Since the above argument works for any sequence, we conclude that the whole family of $M_t$ converges to a geodesic sphere as $t \to \infty$ up to an isometry. The exponential convergence is the same as in §6.

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