The quantum dilogarithm and representations of quantum cluster
to varieties

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To David Kazhdan for his 60th birthday

"Лошадь состоит из трёх неравных половин".
Конский лечебник,
Сочинение г. А. де Барра. Москва 1868.

1 "A horse consists of three unequal halves". cf. A. de Barr, Horse doctor. Moscow 1868.
Cluster varieties are relatives of cluster algebras [FZI]. They are schemes over $\mathbb{Z}$ glued from algebraic tori $\mathbb{G}^N_m$. Cluster modular groups act by automorphisms of cluster varieties. They include classical modular groups of punctured surfaces, also known as the mapping class/Teichmüller groups. We employ certain extensions of these groups, called saturated cluster modular groups.

Our main result is a construction of series of $\ast$-representations of quantum cluster $\mathcal{X}$-varieties. By this we mean the following. A cluster $\mathcal{X}$-variety $\mathcal{X}$ is equipped with a natural Poisson structure. It admits a canonical non-commutative $q$-deformation. The latter gives rise to a non-commutative $\ast$-algebra $L(\mathcal{X}_q)$, which should be thought of as the algebra of regular functions on the $q$-deformation. Given a complex number $\hbar$, we consider its Langlands modular double, which is a $\ast$-algebra $L_{\mathcal{X}} := L(\mathcal{X}_q) \otimes_{\mathbb{Z}} L(\mathcal{X}^\vee_q)$, $q = e^{i\pi\hbar}$, $q^\vee := e^{i\pi/\hbar}$.

Here $\mathcal{X}^\vee$ is the Langlands dual cluster $\mathcal{X}$-variety\footnote{When $\mathcal{X}$ is related to a split reductive group $G$, $\mathcal{X}^\vee$ is related to the Langlands dual group $G^\vee$ [FC4].} for $\mathcal{X}$. The (saturated) cluster mapping class group $\hat{\Gamma}$ of $\mathcal{X}$ acts by automorphisms of the $\ast$-algebra $L_{\mathcal{X}}$.

Assume now that $\hbar$ is a positive real number, so that $|q| = 1$. We define a Freschet linear space Schwartz space of $L_{\mathcal{X}}$, and a $\ast$-representation of $L_{\mathcal{X}}$ in $\mathcal{S}_X$. Using the quantum dilogarithm function we construct a projective representation of the group $\hat{\Gamma}$ in the Schwartz space $\mathcal{S}_X$, intertwining the action of $\hat{\Gamma}$ on $L_{\mathcal{X}}$. The space $\mathcal{S}_X$ is a dense subspace of a Hilbert space $V_{\mathcal{X}}$. The group $\hat{\Gamma}$ acts by unitary operators in $V_{\mathcal{X}}$. Summarizing:

A $\ast$-representation of quantum cluster $\mathcal{X}$-variety is a triple $(\hat{\Gamma}, L_{\mathcal{X}}, \mathcal{S}_X \subset V_{\mathcal{X}})$. \hfill (2)

This representation is decomposed according to the unitary characters of the center of algebra $L_{\mathcal{X}}$.

Equivalently, $\mathcal{S}_X$ is a $\hat{\Gamma}$-equivariant module over the $\ast$-algebra $L_{\mathcal{X}}$, equipped with a $\hat{\Gamma}$-invariant unitary scalar product – the Hilbert space $V_{\mathcal{X}}$ is the completion of $\mathcal{S}_X$ for the scalar product.

There is a space of generalized functions $\mathcal{S}^*_X$, defined as the topological dual of $\mathcal{S}_X$.

Cluster modular groups are automorphism groups of objects of cluster modular groupoids. We construct representation (2) for groupoids rather then groups – this generalisation in fact greatly simplifies the construction.

The unitary projective representation of the group $\hat{\Gamma}$ in the Hilbert space $V_{\mathcal{X}}$ can be viewed as a rather sophisticated analog of the Weil representation of the metaplectic cover of the group $Sp(2n, \mathbb{Z})$.

In both cases representations are given by integral operators, intertwining representations of certain

\footnote{This just means that $\mathcal{S}_X$ is a (projective) module over the semidirect product of the group algebra $\mathbb{Z}[\hat{\Gamma}]$ and $L_{\mathcal{X}}$.}
Heisenberg-type algebras. These intertwiners in our case are of two types: the simpler ones are Weil’s intertwiners; the kernels of the new ones are given by the quantum dilogarithm function.

The program of quantization of cluster $\mathcal{X}$-varieties, including a construction of intertwiners, was initiated in [FG2] (the final version appeared in [FG2II]). However it lacked some ingredients, including a proof of crucial relations for the intertwiners, so representations of $\hat{\Gamma}$ were not available.

The main new features of the present approach are the following. We give another construction of the intertwiners, clarifying their structure. We introduce the Schwartz space $S_X$. Since the algebra $L_X$ actually acts in $S_X$, the claim that the intertwiners indeed intertwine this action makes sense, and we prove it. We show that this implies the relations for the intertwiners. In the quasiclassical limit they give functional equations for the classical dilogarithm. The simplest instance of this program, quantization of the moduli space $\mathcal{M}_{0,5}$, was implemented in [Go2]. Not overshadowed by issues of algebraic nature, it may serve as an introduction to the proof of our main result.

One of applications of our construction is quantum higher Teichmüller theory. Let $\hat{S}$ be a surface $S$ with holes and a finite collection of marked points at the boundary, considered modulo isotopy. Let $G$ be a split reductive group. The pair $(G,\hat{S})$ gives rise to a moduli space $X_{G,\hat{S}}$ [FG1], related to the moduli space of $G$-local systems on $S$. The modular group $\Gamma_S$ of $S$ acts on $X_{G,\hat{S}}$.

The moduli space $X_{G,\hat{S}}$, in the case when $G$ has trivial center, has a natural a cluster $\mathcal{X}$-variety structure (see loc. cit. for the case of $G = PGL_n$ [5]). This means, in particular, that the moduli space $X_{G,\hat{S}}$ carries a $\Gamma_S$-equivariant collection of rational coordinate systems (atlas) with the following properties:

(i) The natural Poisson structure $\{*, *\}$ on $X_{G,\hat{S}}$ in each of the coordinate system $\{X_i\}$ has a standard quadratic form: $\{X_i, X_j\} = \varepsilon_{ij}X_iX_j$ for certain $\varepsilon_{ij} \in \mathbb{Z}$ depending on the coordinate system.

(ii) The transition transformations between different coordinate systems are compositions of certain standard transformations, cluster mutations.

The set parametrising coordinate systems of the cluster atlas on $X_{G,\hat{S}}$ includes a subset provided by ideal triangulations of $\hat{S}$ equipped with an additional data at each triangle.

Therefore our general construction provides a family of infinite dimensional unitary projective representations of the saturated cluster modular group $\hat{\Gamma}_{G,\hat{S}}$ related to the pair $(G,\hat{S})$. The group $\hat{\Gamma}_{G,\hat{S}}$ includes, as a subquotient, the classical modular group $\Gamma_S$ of $S$, but can be bigger if $G \neq PGL_2$.

Here is the story for $G = GL_1$. The moduli space $\mathcal{L}_{GL_1,S}$ of $G$-local systems on $G$ is an algebraic torus $\text{Hom}(H_1(S),GL_1)$. Assume that $S$ has no holes. Then $H_1(S)$ is a lattice with a symplectic structure, so $\mathcal{L}_{GL_1,S}$ is a symplectic algebraic torus. The group $\Gamma_S$ acts by its automorphisms through the quotient $\text{Sp}(H_1(S)) \cong \text{Sp}(2g,\mathbb{Z})$. Our projective representation of $\Gamma_S$ in this case is nothing else but the projection $\Gamma_S \to \text{Sp}(2g,\mathbb{Z})$ composed with the Weil representation of $\text{Sp}(2g,\mathbb{Z})$. If $S$ has holes, we get a family of representations parametrised by unitary characters of copies of $\mathbb{R}_+^*$ assigned to the holes.

We conjecture that these representations for a given $G$ and different $S$ form an infinite dimensional modular functor – understanding representations of quantum cluster varieties as data [2] allows to state this precisely – see Section 6.2. The algebra [1] in this case contains the Langlands modular double of the $q$-deformed algebra of functions on the moduli space of $G$-local systems on $S$.

Let $K_G$ be the maximal compact group of the complex group $G(\mathbb{C})$. Quantization of the moduli space of $K_G$-local systems on $S$ given by Witten [W1], [W2], Hitchin [H] and others leads to a

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4$L_X$ acts by unbounded operators, and thus does not act on $V_X$ – the space $S_X$ is its maximal domain of definition.

5A similar result for an arbitrary $G$ will appear in elsewhere.
construction of certain local systems on the moduli spaces $M_{g,n}$, the local systems of conformal blocks for the Wess-Zumino-Witten theory. One can view them as finite-dimensional projective representations of the modular group of $S$.

Let us take now instead of the maximal compact subgroup $K_G$ a non-compact real form of the complex Lie group $G(\mathbb{C})$, namely the split real Lie group $G(\mathbb{R})$. The quantum higher Teichmüller theory can be viewed as a quantization of the moduli space of $G(\mathbb{R})$-local systems on $S$, and provides infinite-dimensional unitary projective representations of the modular group of $S$.

The relationship between the two stories is similar to the relationship between finite dimensional representations of $G$ and principal series representations of the real Lie group $G(\mathbb{R})$.

To prove relations for the intertwiners we introduce and study a geometric object encapsulating their properties: the symplectic double of a cluster $\mathcal{X}$-variety and its non-commutative $q$-deformation, the quantum double. The quantum double is determined by the holonomic system of difference relations for the intertwiners. The symplectic double has a structure of the symplectic groupoid related to the Poisson variety $\mathcal{X}$. The relation between the intertwiner and the symplectic double is similar to the one between the Fourier transform on a vector space $V$ and the cotangent bundle $T^*V$.

We define a canonical representation of the quantum double. It is given by a data similar to [2], and includes a canonical unitary projective representation of $\hat{\Gamma}$. This representation is realized in the Hilbert space $L^2(\mathcal{A}^+)$, where $\mathcal{A}^+$ is the manifold of real positive points of the cluster $\mathcal{A}$-variety assigned to $\mathcal{X}$. In the situation related to a pair $(G, \hat{S})$, there is a dual moduli space $\mathcal{A}_{G,\hat{S}}$ introduced in [FG1], closely related to the moduli space of unipotent $G$-local systems on $S$. It has a cluster $\mathcal{A}$-variety structure, and the canonical representation is realized in the Hilbert space $L^2(\mathcal{A}_{G,\hat{S}}^+)$.

The symplectic/quantum double is an object of independent interest. To support this, we show in [FG5] how it appears in the higher Teichmüller theory. This is new even in the classical set-up: we construct a collection of canonical coordinates on a certain modification and completion of the Teichmüller space for the double of surface $S$. A partial tropicalisation of the symplectic double delivers the cluster algebra with principal coefficients of Fomin and Zelevinsky [FZIV].

This paper provides a more transparent treatment of a part of [FG2]: Using the quantum dilogarithm we simplify and clarify construction of cluster $\mathcal{X}$-varieties and their $q$-deformations.

The quantum double and the intertwiner  Here is an outline of this relationship. We use the standard cluster terminology, recalled in Section 2.1. The saturated cluster modular groupoid $\hat{\mathcal{G}}$ has a combinatorial description. Its objects are feeds $i = (\hat{I}, \hat{\varepsilon}_{ij}, d_i)$, where $\hat{I}$ is a finite set, $d_i \in \mathbb{Q}_{>0}$, and $\hat{\varepsilon}_{ij}, i, j \in \hat{I}$, is an integral valued matrix such that $d_i \hat{\varepsilon}_{ij}$ is skew-symmetric. The morphisms are feed cluster transformations modulo certain equivalence relations.

Denote by $\mathbb{G}_m$ the multiplicative algebraic group – one has $\mathbb{G}_m(K) = K^*$ for a field $K$. Each feed $i$ gives rise to a split algebraic torus $\mathcal{A}_i := \mathbb{G}_m^I$ equipped with canonical coordinates $A_i, i \in I$. A feed cluster transformation $c : i \to i'$ gives rise to a positive birational map $\mathcal{A}_i \to \mathcal{A}_{i'}$, which induces an isomorphism of their sets of real positive points.

To define the canonical representation of the modular groupoid $\hat{\mathcal{G}}$, we assign to a feed $i$ a Hilbert space $L^2(\mathcal{A}_i^+)$, where $\mathcal{A}_i^+ = \mathbb{R}^I$ is a vector space with coordinates $a_i := \log A_i, A_i \geq 0$. The quantum torus algebra $\mathcal{D}_i = \mathbb{D}_i$ is represented as the algebra of all $h$-difference operators in this space ($q = e^{\pi i \hbar}$). It is generated by operators of multiplications by $\exp(a_i)$, and shift by $2\pi i \hbar$ of a given coordinate $a_j$. The intertwiner corresponding to a feed cluster transformation $c : i \to i'$ is a

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6The reader might benefit from taking a brief look at Section 2.1.
unitary map of Hilbert spaces

\[ \mathbf{K}_{\epsilon^0} : L^2(A^+_v) \to L^2(A^+_v). \]  

(3)

Its Schwarz kernel satisfies a system of difference relations, consisting of \( h \)-difference relations, and their “Langlands dual” counterparts. If \( h \) is irrational, it is characterised uniquely up to a constant by these relations. The \( h \)-difference relations alone do not characterize it.

The \( h \)-difference relations form an ideal in \( \text{D}^q_{v} \otimes \text{D}^q_{i} \). It gives rise to an isomorphism of the fraction fields of algebras \( \text{D}^q_{v} \) and \( \text{D}^q_{i} \). The quantum double is obtained by gluing the corresponding quantum tori along these maps. Setting \( q = 1 \), we get the symplectic double.

To show that the intertwiners provide a representation of a modular groupoid we need to prove that relations between cluster transformations give rise to the identity maps up to constants. One of the issues is that in many important cases, e.g. in quantization of higher Teichmüller spaces, we do not know an explicit form of all trivial (i.e. equivalent to the identity) cluster transformations.

In the rest of the Introduction we give a more detailed account on the results of the paper and relevant connections. Reversing the logic, we discuss the double first, and the quantum dilogarithm and the intertwiners after that.

1.2 The symplectic double and its properties

Cluster \( X \)-varieties are equipped with a positive atlas and a Poisson structure. They admit a canonical non-commutative \( q \)-deformation. Given a Poisson variety \( X \), we denote by \( X^{\text{op}} \) the same variety equipped with the opposite Poisson structure \( \{*,*\}^{\text{op}} := -\{*,*\} \).

The symplectic double \( D \) of a cluster \( X \)-variety is a symplectic variety equipped with a positive atlas and a Poisson map

\[ \pi : D \to X \times X^{\text{op}}. \]  

(4)

There is an embedding \( j : X \hookrightarrow D \). Its image is a Lagrangian subvariety. The following diagram, where \( \Delta_X \) is the diagonal in \( X \times X^{\text{op}} \), is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{j} & D \\
\downarrow & & \downarrow \pi \\
\Delta_X & \hookrightarrow & X \times X^{\text{op}}
\end{array}
\]  

(5)

There is a canonical involution \( i \) of \( D \) interchanging the two projections in (4). The pair \((X, D)\) together with the maps \( i, j, \pi \) has a structure of the symplectic groupoid \([?]\) related to the Poisson space \( X \), where \( D \) serves as the space of morphisms, and \( X \) as the space of objects.

A cluster \( X \)-variety \( X \) comes together with a dual object \( A \), called a \textit{cluster A-variety}. The algebra of regular functions on \( A \) coincides with the upper cluster algebra from \([BFZ]\). There is a map \( p : A \to X \); the triple \((A,X,p)\) is called a \textit{cluster ensemble}. There is a 2-form \( \Omega_A \) on \( A \); the map \( p \) is the quotient along the null space of this form. The form \( \Omega_A \) comes from a class in \( K_2(A) \).

The double \( D \) sits in a commutative diagram:

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\varphi} & D \\
p \times p & \downarrow & \\
X \times X^{\text{op}} & \xrightarrow{\pi}
\end{array}
\]  

(6)

The modular group \( \hat{\Gamma} \) acts by automorphisms of the triple \((A,X,p)\), and of diagrams (4) and (6).
The very existence of the positive $\hat{\Gamma}$-equivariant space $\mathcal{D}$ implies that there is a symplectic space $\mathcal{D}(\mathbb{R}_{>0})$ of the positive points of $\mathcal{D}$, equipped with an action of $\hat{\Gamma}$. It is isomorphic to $\mathbb{R}^N$. We found another interesting $\hat{\Gamma}$-equivariant real subspace in $\mathcal{D}(\mathbb{C})$, the unitary part $\mathcal{D}^U$ of $\mathcal{D}(\mathbb{C})$.

The quantum double. We define in Section 3 a non-commutative $\hat{\Gamma}$-equivariant $q$-deformation $\mathcal{D}_q$ of the double. There is a map of quantum spaces $\pi: \mathcal{D}_q \to \mathcal{X}_q \times \mathcal{X}_q^{\text{op}}$, and an involution $i: \mathcal{D}_q \to \mathcal{D}_q^{\text{op}}$ interchanging the two components of $\pi$. The two real subspaces $\mathcal{D}(\mathbb{R}_{>0})$ and $\mathcal{D}^U$ of $\mathcal{D}(\mathbb{C})$ match two $\ast$-algebra structures on $\mathcal{D}_q$.

Duality conjectures for the double. We conjecture that there exists a canonical basis in the space $\mathbb{L}(\mathcal{D})$ of regular positive functions on the positive space $\mathcal{D}$, as well as in its $q$-analog. Its vectors should be parametrised by the set of integral tropical points of the Langlands dual double $\mathcal{D}^\text{dual}$. This is closely related to the duality conjectures from Section 4 of [FG2].

Relation with the work of Fomin and Zelevinsky. After we found the symplectic double, we learned about a very interesting paper [FZIV] were, among the other things, “cluster algebras with partial tropicalization of the formulas which we use in the definition of the double $\mathcal{D}$.

1.3 Quantum dilogarithm, quantum double, and quantization

The dilogarithm plays a crucial role in our construction and understanding of cluster $\mathcal{X}$-varieties and their doubles, both on the classical and quantum levels. The story goes as follows.

The dilogarithm function and its two quantum versions

(i) Recall the classical dilogarithm function $\text{Li}_2(x) := -\int_0^x \log(1-t)dt \log t$. We employ its version

$$L_2(x) := \int_0^x \log(1+t)\frac{dt}{t} = -\text{Li}_2(-x).$$

(ii) The $q$-exponential as a “quantum dilogarithm”. Consider the following formal power series

$$\Psi^q(x) := \prod_{a=1}^\infty \left(1 + q^{2a-1}x\right)^{-1} = \frac{1}{(1 + qx)(1 + q^3x)(1 + q^5x)(1 + q^7x)\ldots}.$$ 

It is characterized by the difference relation $\Psi^q(q^2x) = (1 + qx)\Psi^q(x)$. If $|q| < 1$ the power series $\Psi^q(x)^{-1}$ converge, providing an analytic function in $x \in \mathbb{C}$. The dilogarithm appears in the quasiclassical limit: $\Psi^q(x)$ admits an asymptotic expansion when $q \to 1^-$:

$$\Psi^q(x) \sim \exp\left(\frac{L_2(x)}{\log q^2}\right).$$

(7)

Indeed, setting $x = e^t$, $q = e^{-s}$ we have

$$\log \Psi^q(x) = -\sum_{k=1}^\infty \log(1 + e^{t-(2k-1)s}) \sim_{s \to 0} -\int_0^{e^t} \log(1+u)\frac{du}{u}/2s = L_2(e^t)/-2s.$$

The function $\Psi^q(x)$ appeared in XIX-th century under the names $q$-exponential, infinite Pochhammer symbol etc. The name $q$-exponential is justified by the following power series expansion:

$$\Psi^q(x) = \sum_{n=0}^\infty \frac{x^n}{(q - q^{-1})(q^2 - q^{-2})\ldots(q^n - q^{-n})}.$$
Schützenberger [Sch] found a remarkable relation for the function $\Psi^q(X)$ involving the 2-dimensional quantum torus algebra. Its version was rediscovered and interpreted as a quantum analog of the Abel’s pentagon relation for the dilogarithm by Faddeev and Kashaev [FK].

(iii) The quantum dilogarithm for $|q| = 1$. Let $\text{sh}(u) := (e^u - e^{-u})/2$. Then

$$\Phi^h(z) := \exp \left( -\frac{1}{4} \int_{\Omega} \frac{e^{-ipz}}{\text{sh}(\pi p)\text{sh}(\pi hp)} \frac{dp}{p} \right), \quad h \in \mathbb{R}_{>0},$$

where the contour $\Omega$ goes along the real axes from $-\infty$ to $\infty$ bypassing the origin from above. Let

$$q = e^{\pi ih}, \quad q^v = e^{\pi i/h}, \quad h \in \mathbb{R}_{>0}.$$

The function $\Phi^h(z)$ is characterized by the two difference relations (Property B5 in Section 4.2):

$$\Phi^h(z + 2\pi ih) = (1 + q^v e^z)\Phi^h(z), \quad \Phi^h(z + 2\pi i) = (1 + q^v e^{z/h})\Phi^h(z).$$

It is related in several ways to the dilogarithm (Section 4.2), e.g. has an asymptotic expansion

$$\Phi^h(z) \sim \exp \left( \frac{L_2(e^z)}{2\pi i h} \right) \text{ as } h \to 0.$$

The functions $\Phi^h(z)$ and $\Psi^q(e^z)$ are close relatives: both deliver the classical dilogarithm in the quasiclassical limit, both are characterized by difference relations. When the Planck constant $h$ is a complex number with $\text{Im} \ h > 0$, they are related by an infinite product presentation

$$\Phi^h(z) = \frac{\Psi^q(e^z)}{\Psi^{1/q^v}(e^{z/h})}.$$ 

In this case $|q| < 1$ as well as $|1/q^v| < 1$. (The map $q \mapsto 1/q^v$ is a modular transformation).

The function $\Phi^h(z)$ goes back to Barnes [Ba], and reappeared in second half of XX-th century in the works [Sh], [Bax], [Fad], [K1], [CF] and many others. The quantum pentagon relation for the function $\Phi^h(z)$ was suggested in [Fad] and proved in different ways/forms in [Wo], [FKV] and [Go2].

**The dilogarithm and cluster varieties** Every feed $i$ provides a torus $X_i$, called the feed $X$-torus. It is equipped with canonical coordinates $X_i$, $i \in I$, providing an isomorphism $X_i \to \mathbb{C}^I_m$. There is a Poisson structure defined by

$$\{X_i, X_j\} = \bar{e}_{ij} X_i X_j, \quad \bar{e}_{ij} := \varepsilon_{ij} d_j^{-1}. \quad (8)$$

A cluster $\mathcal{X}$-variety is obtained from the feed $X$-tori by the following gluing procedure. Every element $k \in I$ determines a feed $i'$, called the mutation of the feed $i$ in the direction $k$, and a Poisson birational isomorphism $\mu_k : X_i \to X_{i'}$. A cluster $\mathcal{X}$-variety is obtained by gluing the cluster feed tori $X_i$ parametrised by the feeds related to an initial feed by mutations, and taking the affine closure.

The cluster $\mathcal{A}$- and $\mathcal{D}$-varieties are constructed similarly from the feed $\mathcal{A}$- and $\mathcal{D}$-tori. In each case the combinatorial set-up is the same: the groupoid $\hat{\mathcal{G}}$. However the gluing patterns are different. The obtained objects carry different structures: $\mathcal{X}$-varieties are Poisson, $\mathcal{D}$-varieties are symplectic with the symplectic structure coming from a class in $K_2$, and $\mathcal{A}$-varieties carry a class in $K_2$.\[7\text{One should rather call by quantum dilogarithms the logarithms of the functions } \Psi^q \text{ and } \Phi^h. \text{ The function } \Phi^h \text{ is also known as the "non-compact quantum dilogarithm".}
One of the ideas advocated in this paper is that one should decompose the mutation birational isomorphism $\mu_k$ into a composition of two maps, called “the automorphism part” and “the monomial part” of the mutation. We introduce such decompositions for the $\mathcal{A}$-, $\mathcal{D}$- and $\mathcal{X}$-varieties. They are compatible with the relating them maps (Theorem 2.3). The “the automorphism part” of the $\mathcal{X}$-mutation is a Poisson birational automorphism of the feed torus $X_i$ and the “monomial part” is a Poisson isomorphism $X_i \rightarrow X'_i$. The situation for the $\mathcal{D}$- and $\mathcal{A}$-spaces is similar – the $K_2$-classes are preserved. The “monomial part” is a very simple isomorphism. The “automorphism part” is governed by the quantum dilogarithm.

Here is how we use the classical and quantum dilogarithm functions in the gluing process. The cluster coordinate $X_i$ on a feed torus $X_i$, via the projection $\pi$, see (4), lifts to the functions $X_i$ and $X'_i$ on the feed $\mathcal{D}_i$-torus.

(i) Recall that the Hamiltonian flow for a Hamiltonian function $H$ on a Poisson space with coordinates $y_i$ is given by $dy_i(t)/dt = \{H, y_i\}$. The “automorphism part” of the mutation map $\mu_k$ is the Hamiltonian flow for the time 1 for the Hamiltonian functions given by:

- The dilogarithm function $L_2(X_k)$ for the cluster $\mathcal{X}$-variety.
- The difference of the dilogarithm functions $L_2(X_k) - L_2(X'_k)$ for the double $\mathcal{D}$.

(ii) Similarly, the “quantum dilogarithm” $\Psi^q(X)$ is used to describe the quantum mutation maps: The “automorphism part” of the quantum mutation in the direction $k$ is given by conjugation by

- The “quantum dilogarithm” function $\Psi^q(X_k)$ for the quantum cluster $\mathcal{X}$-variety.
- The ratio of the “quantum dilogarithms” $\Psi^q(X_k)/\Psi^q(X'_k)$ for the quantum double $\mathcal{D}$.

So although both $L_2(x)$ and $\Psi^q(X)$ are transcendental functions, they provide birational transformations, “the automorphism parts” of the mutation maps, and make all properties of the quantum $\mathcal{X}$- and $\mathcal{D}$-spaces very transparent.

**Representations of quantum cluster $\mathcal{X}$-varieties** A lattice (= free abelian group) $\Lambda$ with a skew-symmetric form $\bar{\varepsilon} : \Lambda \times \Lambda \rightarrow \frac{1}{\hbar}\mathbb{Z}$ provides a quantum torus algebra $T$. It is an associative $*$-algebra over $\mathbb{Z}[q^{1/N}, q^{-1/N}]$ with generators $e_\lambda, \lambda \in \Lambda$, and relations

$$e_\lambda e_\mu = q^{-(\lambda, \mu)}e_{\lambda+\mu}, \quad *e_\lambda = e_\lambda, \quad *q = q^{-1}. \quad (9)$$

A basis $\{e_i\}, i \in I$ in $\Lambda$ provides a set of generators $\{X_i^{\pm1}\}$ of the algebra $T$, satisfying the relations

$$q^{-\bar{\varepsilon}_{ij}}X_iX_j = q^{-\bar{\varepsilon}_{ij}}X_jX_i, \quad \bar{\varepsilon}_{ij} := \bar{\varepsilon}(e_i, e_j). \quad (10)$$

Its quasiclassical limit delivers the Poisson structure (8). The feed algebras for the quantum $\mathcal{X}$- and $\mathcal{D}$-varieties are quantum tori algebras related to the corresponding Poisson structures.

Given a quantum torus $T$, we associate to it a Heisenberg algebra $\mathcal{H}_T$. It is a topological $*$-algebra over $\mathbb{C}$ generated by elements $x_i$ such that

$$[x_p, x_q] = 2\pi i\hbar \bar{\varepsilon}_{pq}; \quad *x_p = x_p, \quad q = e^{\pi i\hbar}.$$ 

Setting $X_p := \exp(x_p)$ we get an embedding $T \hookrightarrow \mathcal{H}_T$.

**Representations.** The Heisenberg $*$-algebra $\mathcal{H}_T$ for $|q| = 1$ has a family of $*$-representations by unbounded operators in a Hilbert space, parametrized by the central characters $\lambda$ of $T$. 
**Example.** The simplest quantum torus algebra is generated by \(X, Y\) with the relation \(qXY = q^{-1}YX\). The corresponding Heisenberg algebra is generated by \(x, y\) with the relation \([x, y] = -2\pi i\hbar\). It has a \(*\)-representation in \(L^2(\mathbb{R})\): \(x \mapsto x, \ y \mapsto 2\pi i\hbar \frac{\partial}{\partial x}\).

Denote by \(V_{\lambda, i}\) a \(*\)-representation of the quantum feed \(\mathcal{X}\)-torus algebra assigned to a feed \(i\) and a central character \(\lambda\). Given a mutation of feeds \(i \to i'\), we define a unitary operator

\[
K_{i' \to i} : V_{\lambda, i'} \to V_{\lambda, i}, \quad K_{i' \to i} = K^\sharp \circ K'.
\]

Here \(K' : V_{\lambda, i'} \to V_{\lambda, i}\) is a very simple unitary operator corresponding to the “monomial part” of the mutation map. The main hero is the automorphism \(K^\sharp : V_{\lambda, i} \to V_{\lambda, i}\), given by

\[
K^\sharp := \Phi^h(\mathbb{x}_k),
\]

where \(\mathbb{x}_k\) is a self-adjoint operator provided by the image \(x_k\) in the representation \(V_{\lambda, i}\). The Schwartz kernels of the operators \(K'\) and \(K^\sharp\) are characterised by a Langlands dual pairs of systems of difference equations. They describe the relevant (“monomial”/“automorphism”) parts of the mutation map of the feed \(\mathcal{X}\)-torus. Compositions of elementary intertwiners \([\Pi]\) give us the intertwiners \(K_{\mathcal{X}, \psi}\) assigned to cluster transformations \(\mathcal{c}\).

We define the Schwartz space \(S_{\lambda, i} \subset V_{\lambda, i}\) as the maximal domain of the algebra \(L\) in \(V_{\lambda, i}\). Further, let \(\hat{\mathcal{G}}\) be the saturated cluster modular groupoid of \(\mathcal{X}\). So \(\hat{\Gamma}\) is the automorphism group of its objects.

**Theorem 1.1** The datum \((\hat{\mathcal{G}}, L, \{S_{\lambda, i} \subset V_{\lambda, i}\}; \{K_{\mathcal{X}, \psi}\})\) provides a \(*\)-representation of quantum cluster \(\mathcal{X}\)-variety. In particular operators \([\Pi]\) provide a unitary projective representation of the groupoid \(\hat{\mathcal{G}}\), and hence of the group \(\hat{\Gamma}\).

**Remark.** The mutation maps are described in (ii) via the formal power series \(\Psi^h(X)\). There is a similar description using the function \(\Phi^h(x)\) and \([\Pi]\). The advantage of the description via \(\Phi^h(x)\) is that it works in representations, while \(\Psi^h(X)\) at \(|q| = 1\) makes sense only as a power series, and thus does not act in a representation.

**Quantization of higher Teichmüller spaces** Let \(S\) be a surface with punctures. According to \([\text{FG}]\), Section 10, the pair of moduli spaces \((\mathcal{A}_{G, S}, \mathcal{X}_{G, S})\) for \(G = SL_m\) has a cluster ensemble structure. Therefore thanks to Theorem \([\Pi]\) we arrive at quantization of \(\mathcal{X}_{G, S}\). It includes a series of its \(*\)-representations, and thus a family of unitary projective representations of the saturated cluster modular group \(\hat{\Gamma}_{G, S}\). They are parametrised by unitary characters of the group \(H(\mathbb{R}_{>0})\{\text{punctures of } S\}\), where \(H\) is the Cartan group of \(G\). By the results of loc. cit. the classical modular group \(\Gamma_S\) is a subgroup of the group \(\hat{\Gamma}_{G, S}\). The latter is a quotient of \(\hat{\Gamma}_{G, S}\). So \(\Gamma_S\) is a sub-quotient of \(\hat{\Gamma}_{G, S}\). We show that the obtained representations of the group \(\hat{\Gamma}_{G, S}\) descend to projective representations of \(\Gamma_S\). Here is our second main result:

**Theorem 1.2** The cluster structure of \((\mathcal{A}_{G, S}, \mathcal{X}_{G, S})\) gives rise to a series of \(*\)-representations of quantum moduli space \(\mathcal{X}_{G, S}\). They provide a series of infinite dimensional unitary projective representations of the classical mapping class group \(\Gamma_S\).

The \(G = SL_2\) case was the subject of works \([K]\) (which deals with a single representation with trivial central character) and \([\text{CF}]\). Unfortunately the argument presented in \([\text{CF}]\) as a proof of the pentagon relation has a serious problem. As a result, the approach to quantization of Teichmüller spaces advocated in loc. cit. was put on hold. The proof of the pentagon relation for the quantum dilogarithm given in \([\text{Go2}]\) serves as a model for the proofs in Section 5.

We conjecture that the family of these representations for different surfaces \(S\) form an infinite dimensional modular functor. For \(G = SL_2\) the proof is claimed by J. Teschner in \([T]\).
A canonical representation of the quantum double

It is defined similarly, by using the quantum double $\mathcal{D}$ instead of $\mathcal{X}$. Recall the quantum torus algebra $D_i$ for the quantum double assigned to a feed $i$. Its Heisenberg algebra $\mathcal{H}_{D_i}$ has a unique $*$-representation. It has a natural realization in $L^2(\mathcal{A}_i^+)$ where $D_i$ acts as the algebra of $\hbar$-difference operators described in Section 1.1.

Take the inverse images of $X_i \otimes 1$ and $1 \otimes X_i$ under the quantum map $\pi^+$. Let $x_i, \tilde{x}_i$ be their “logarithms”, i.e. the elements of the Heisenberg algebra $\mathcal{H}_{D_i}$. Given a mutation $\mu_k : i \to i'$, we define a unitary operator

$$K_{i' \to i} : L^2(\mathcal{A}_i^+) \to L^2(\mathcal{A}_i^+)$$

as a composition $K_{i' \to i} = K^{\sharp} \circ K'$, where $K'$ is the unitary operator induced on functions by a linear map of spaces $A_i^+ \to A_{i'}^+$, the “monomial part” of the $\mathcal{D}$-mutation map, and

$$K^\sharp := \Phi(\tilde{x}_k)\Phi(x_k)^{-1}.$$ (14)

The operator (13) coincides with the one defined in [FG2II] in a different way. The Schwartz kernels of the operators $K'$ and $K^\sharp$ are characterized by Langlands dual pairs of systems of difference equations. They describes the relevant (“monomial”/“automorphism”) parts of the coordinate transformations for the mutation $i \to i'$, employed in the definition of the quantum double. In particular the symplectic double describes the quasiclassical limit of intertwiners (13). Compositions of elementary intertwiners (13) give us the intertwiners $K_{c \circ}$, see (3).

The quantum double $\mathcal{D}_q$ is a functor: we assign to a feed $i$ the quantum torus algebras $D_i$, and to a feed cluster transformation $c : i \to i'$ an isomorphism $\gamma_{c \circ}$ of the fraction fields of algebras $D_{i'}$ and $D_i$. The algebra of regular functions $L(D_q)$ on $\mathcal{D}_q$ consists of collections of elements $F_i \in D_i$ identified by these maps: $\gamma_{c \circ}(F_i) = F_i$. The crucial role plays the algebra $L := L(D_q) \otimes L(D_{i'})$ of regular functions on the quantum Langlands modular double $\mathcal{D}_q \times \mathcal{D}_{i'}$. For each feed $i$ it is identified with a subalgebra $L_i \subset D_{i-q} \times D_{i',q'}$. The map $\gamma_{c \circ}$ induces an isomorphism $L_{i'} \simrightarrow L_i$. The algebra $L_i$ acts by unbounded operators in the Schwartz space $S_i \subset L^2(\mathcal{A}_i^+)$, defined as the maximal common domain of operators from $L_i$. We prove that intertwiners restrict to isomorphisms between the Schwartz spaces. This allows to introduce distribution spaces respected by the intertwiners. Summarizing, we get

**Theorem 1.3** The datum $\left(\tilde{\mathcal{G}}, L_i, S_i \subset L^2(\mathcal{A}_i^+), K_{c \circ}\right)$ provides a canonical $*$-representation of the quantum cluster double. **This means that:**

- We have a unitary projective representation of the groupoid $\tilde{\mathcal{G}}$ given by the Hilbert spaces $L^2(\mathcal{A}_i^+)$ and the unitary maps $K_{c \circ}$ between them;
- The maps $K_{c \circ}$ preserve the Schwartz spaces $S_i$ and intertwine on them the action of the groupoid $\mathcal{G}$ on the algebras $L_i$: for any $s \in S_{i'}$ and $A \in L_{i'}$ one has

$$K_{c \circ}(s) = \gamma_{c \circ}(A)K_{c \circ}.$$ (15)

So for every cluster transformation $c : i \to i'$ there are commutative diagrams

$$\begin{array}{ccc}
L_i & \text{acts on} & S_i \leftrightarrow L^2(\mathcal{A}_i^+) & \text{acts on} & S^*_i \\
\gamma_{c \circ} \downarrow & & K_{c \circ} \downarrow & & K_{c \circ} \downarrow \\
L_{i'} & \text{acts on} & S_{i'} \leftrightarrow L^2(\mathcal{A}_{i'}^+) & \text{acts on} & S^*_i \\
\end{array}$$

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The quantum map $\pi$ provides a subalgebra $\pi_q^*(L_X \otimes L_{X}^{\text{op}}) \subset L$. It is isomorphic to the quotient of $L_X \otimes L_{X}^{\text{op}}$ by a central subalgebra isomorphic to $\text{Center}(L_X)$. Restricting the canonical representation to this subalgebra and decomposing it according to its central characters, identified with the characters of $\text{Center}(L_X)$, we get, for every feed $i$, a decomposition into an integral of Hilbert spaces:

$$L^2(A_i^+) = \int_{\lambda} \text{End}(V_{\lambda,i})d\lambda.$$

These decompositions are respected by the intertwiners. Thus the canonical representation is an integral of the endomorphisms of the principal series representations of the quantum cluster variety $X_q$. So the canonical representation is a regular representation for the $*$-algebra of regular functions on $X_q$. Using this one can easily deduce Theorem 1.1 from Theorem 1.3.

### 1.4 The symplectic double and higher Teichmüller theory [FG5]

Let $S$ be an oriented topological surface with boundary, and $G$ a split semi-simple group over $\mathbb{Q}$ with trivial center. Cluster ensembles provide a framework and tool for study of the dual pair $(\mathcal{A}_{G,S}, X_{G,S})$ of moduli spaces related to $S$ and $G$ [FG1]. In [FG5] we show that the same happens with the double.

A moduli space $D_{G,S}$. Let $S_D$ be the double of $S$. It is a topological surface obtained by gluing the surface $S$ with its “mirror”, given by the same surface with the opposite orientation, along the corresponding boundary components. We introduce a moduli space $D_{G,S}$. It is a rather non-trivial relative of the moduli space of $G$-local systems on the double $S_D$. In particular it contains a divisor whose points do not correspond to any kind of local systems on $S_D$. Its irreducible components match components of the boundary of $S$.

The modular group $\Gamma_S$ of $S$ acts on $D_{G,S}$. The moduli space $D_{G,S}$ has a natural $\Gamma_S$-invariant symplectic structure and a class in $K_2$. There are commutative diagrams similar to (5) and (6). We prove that, unlike the classical moduli space of $G$-local systems on $S_D$, the moduli space $D_{G,S}$ is rational, and equip it with a positive $\Gamma_S$-equivariant atlas. It is described as the symplectic double of the cluster $X$-variety structure on the moduli space $X_{G,S}$. This was not known even for $G = \text{PGL}_2$.

The symplectic double and quasifuchsian representations. The space $D_{G,S}(\mathbb{R}_{>0})$ of positive real points of $D_{G,S}$ is a symplectic space isomorphic to $\mathbb{R}^{-2\chi(S)\dim G}$. We believe that it is closely related to the space of framed quasifuchsian representations $\pi_1(S) \to G(\mathbb{C})$ modulo conjugation. For $G = \text{PGL}_2$ and closed $S$ this boils down to the Bers double uniformization theorem.

Relation with the work of Bondal. We will show elsewhere that the symplectic groupoid introduced in [B] is the symplectic double in our sense of a certain twisted moduli space $X_{G,S}$.

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2 The symplectic double of a cluster $\mathcal{X}$-variety

2.1 Cluster $\mathcal{X}$- and $\mathcal{A}$-varieties

For the convenience of the reader we recall below the language of cluster $\mathcal{X}$- and $\mathcal{A}$-varieties in the form it was introduced in [FG2], borrowing from Section 2 of loc. cit., presenting a simplified version, without frozen variables.

We reprove all the results mentioned below in Theorems 2.3 and 3.3 and Lemma 2.12. Doing the quantum case first and using the decomposition of mutations we simplify a lot the original proofs given in loc. cit.

A feed $\mathfrak{f}i$ is a triple $(I, \varepsilon, d)$, where $I$ is a finite set, $\varepsilon$ is a matrix $\varepsilon_{ij}$, where $i, j \in I$, such that $\varepsilon_{ij} \in \mathbb{Z}$, and $d = \{d_i\}$, where $i \in I$, are positive integers, such that the matrix $\hat{\varepsilon}_{ij} = \varepsilon_{ij}d_j^{-1}$ is skew-symmetric.

For a feed $\mathfrak{i}$ we associate a torus $\mathcal{X}_\mathfrak{i} = (\mathbb{G}_m)^I$ with a Poisson structure given by

\[\{X_i, X_j\} = \hat{\varepsilon}_{ij}X_iX_j,\]  \hspace{1cm} (15)

where \{\!\{X_i\}i \in I\\} are the standard coordinates on the factors. It is called the cluster $\mathcal{X}$-torus.

Let $\mathfrak{i} = (I, \varepsilon, d)$ and $\mathfrak{i}' = (I', \varepsilon', d')$ be two feeds, and $k \in I$. A mutation in the direction $k \in I$ is an isomorphism $\mu_k : I \rightarrow I'$ satisfying the following conditions: $d'_{\mu_k(i)} = d_i$, and

\[\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \\ \varepsilon_{ij} + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0. \end{cases}\]  \hspace{1cm} (16)

An automorphism $\sigma$ of a feed $\mathfrak{i} = (I, \varepsilon, d)$ is an automorphism of the set $I$ preserving the matrix $\varepsilon$ and the numbers $d_i$. Automorphisms and mutations induce rational maps between the corresponding cluster $\mathcal{X}$-tori, denoted by the same symbols $\mu_k$ and $\sigma$ and acting on the coordinate functions by the formulae $\sigma^*X_\sigma(i) = X_i$ and

\[\mu_k^*X_{\mu_k(i)} = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})-\varepsilon_{ik}}) & \text{if } i \neq k. \end{cases}\]  \hspace{1cm} (17)

A cluster transformation between two feeds (and between two cluster $\mathcal{X}$-tori) is a composition of automorphisms and mutations. Two feeds are called equivalent if they are related by a cluster transformation. The equivalence class of a feed $\mathfrak{i}$ is denoted by $|\mathfrak{i}|$.

Recall that the affine closure of a scheme $Y$ is the Spectrum of the ring of regular functions on $Y$. For instance the affine closure of $\mathbb{C}^2 - \{0\}$ is $\mathbb{C}^2$. A cluster $\mathcal{X}$-variety is a scheme over $\mathbb{Z}$ obtained by gluing the feed $\mathcal{X}$-tori for the feeds equivalent to a given feed $\mathfrak{i}$ via the above birational isomorphisms, and taking the affine closure of the obtained scheme. It is denoted by $\mathcal{X}_{|\mathfrak{i}|}$, or simply by $\mathcal{X}$ if the equivalence class of feeds is apparent. Every feed provides our cluster $\mathcal{X}$-variety with a rational coordinate system. The corresponding rational functions are called cluster coordinates.

Cluster transformations preserve the Poisson structure. In particular a cluster $\mathcal{X}$-variety has a canonical Poisson structure. The mutation formulas (16) are recovered from the ones (17) and the condition that the latter preserve the Poisson structure.

Recall now the definition of the cluster $\mathcal{A}$-variety. Given a feed $\mathfrak{i}$, we define a torus $\mathcal{A}_\mathfrak{i} = (\mathbb{G}_m)^I$ with the standard standard coordinates \{\!\{A_i\}i \in I\\} on the factors. We call it the feed $\mathcal{A}$-torus.

\[\text{A feed is a combinatorial data obtained by excluding cluster coordinates from a seed of } \mathbb{F}ZI.\]
Automorphisms and mutations give rise to birational maps between the corresponding feed $\mathcal{A}$-tori, which are given by $\sigma^* A_{\sigma(i)} = A_i$ and

$$
\mu_k^* A_{\mu_k(i)} = \begin{cases} 
A_i & \text{if } i \neq k, \\
A_k^{-1} \left( \prod_{i\mid \varepsilon_{ki} > 0} A_i^{\varepsilon_{ki}} + \prod_{i\mid \varepsilon_{ki} < 0} A_i^{-\varepsilon_{ki}} \right) & \text{if } i = k.
\end{cases}
$$

The cluster $\mathcal{A}$-variety corresponding to a feed $i$ is a scheme over $\mathbb{Z}$ obtained by gluing all feed $\mathcal{A}$-tori for the feeds equivalent to a given feed $i$ using the above birational isomorphisms, and taking the affine closure. It is denoted by $\mathcal{A}_{[i]}$, or simply by $\mathcal{A}$.

There is a 2-form $\Omega$ on the cluster $\mathcal{A}$-variety ($\mathcal{G}$V$^2$, $\mathcal{F}$G$^2$), given in every cluster coordinate system by

$$
\Omega_i = \sum_{i, j \in I} \varepsilon_{ij} d \log A_i \wedge d \log A_j, \quad \varepsilon_{ij} = d_i \varepsilon_{ij}.
$$

There is a class in $\mathcal{W}_A \in K_2(\mathcal{A})$ given in every cluster coordinate system by

$$
\sum_{i, j \in I} \varepsilon_{ij} \{ A_i, A_j \} \in K_2(\mathcal{Q}(\mathcal{A}_i)).
$$

It is a $K_2$-avatar of the 2-form $\Omega_A$, in the sense that $\Omega_A = d \log(\mathcal{W}_A)$, see Section 2.3 below.

There is a map $p : \mathcal{A} \rightarrow \mathcal{X}$, given in every cluster coordinate system by $p^* X_k = \prod_{i \in I} A_i^{\varepsilon_{ki}}$. It is the quotient of the space $\mathcal{A}$ along the null-foliation of the 2-form $\Omega_A$.

The cluster modular groupoid. The inverse of a mutation is a mutation: $\mu_k \mu_k = \text{id}$. Feed cluster transformations inducing isomorphisms of the feed $\mathcal{A}$-tori as well as the feed $\mathcal{X}$-tori are called trivial feed cluster transformations. The cluster modular groupoid $\mathcal{G} = \mathcal{G}_{[i]}$ is a category whose objects are feeds equivalent to a given feed $i$, and $\text{Hom}(i, i')$ is the set of all feed cluster transformations from $i$ to $i'$ modulo the trivial ones. In particular, given a feed $i$, the cluster modular group $\Gamma_i$ is the automorphism group of the object $i$ of $\mathcal{G}$. By the very definition, it acts by automorphisms of the cluster $\mathcal{A}$-variety. It preserves the class in $K_2$.

The $(h + 2)$-gon relations. They are crucial examples of trivial cluster transformations. Denote by $\sigma_{ij}$ the map of feeds induced by interchanging $i$ and $j$. Let $h = 2, 3, 4, 6$ when $p = 0, 1, 2, 3$ respectively. Then if $\varepsilon_{ij} = -p \varepsilon_{ji} = -p$, then $(\sigma_{ij} \circ \mu_i)^{h+2} = \text{Id}$ on feeds, and

$$
(\sigma_{ij} \circ \mu_i)^{h+2} = \text{a trivial cluster transformation}.
$$

Relations (20) are affiliated with the rank two Dynkin diagrams, i.e. $A_1 \times A_1, A_2, B_2, G_2$. The number $h = 2, 3, 4, 6$ is the Coxeter number of the diagram. We do not know any other general procedure to generate trivial cluster transformations.

The saturated coordinate groupoid $\hat{\mathcal{G}}$. Its objects are the isomorphism classes of feeds equivalent to a given one. The morphisms admit an explicit description as compositions of the cluster transformations (20). Its fundamental group $\hat{\Gamma}$ is the saturated cluster modular group. There is a canonical surjective map $\hat{\Gamma} \rightarrow \Gamma$.

The quantum space $\mathcal{X}_q$. It is a canonical non-commutative $q$-deformation of the cluster $\mathcal{X}$-variety. We start from the feed quantum torus algebra $\mathcal{T}_1^q$, defined as an associative $*$-algebra with generators $X_i^\pm, i \in I$ and $q^\pm$ and relations

$$
q^{-\varepsilon_{ij}} X_i X_j = q^{-\varepsilon_{ji}} X_j X_i, \quad *X_i = X_i, \quad *q = q^{-1}.
$$

$^9$Below we skip the subscripts $[i]$ whenever possible.
Let QTor\(^*\) be a category whose objects are quantum torus algebras and morphisms are subtraction free \(*\)-homomorphisms of their fraction fields over \(\mathbb{Q}\). The quantum space \(X_q\) is understood as a contravariant functor
\[
\eta^q : \text{The modular groupoid } \hat{G} \rightarrow \text{QTor}\(^*\).
\]
It assigns to a feed \(i\) the quantum torus \(*\)-algebra \(T_1\), and to a mutation \(i \rightarrow i'\) a map of the fraction fields \(\text{Frac}(T_1) \rightarrow \text{Frac}(T_i)\), given by a \(q\)-deformation of formulas \([17]\) (\(\text{[FG2, Section 3]}\))\(^\text{[10]}\) The group \(\hat{G}_i\) acts by automorphisms of the quantum space \(X_q\).

One can understand cluster \(A\)- and \(X\)-varieties as similar functors: this is equivalent to their definition as schemes. For the quantum spaces this is the only way to do it.

There are tori \(H_X\) and \(H_A\) related to a cluster ensemble. They play the following role. There is a surjective projection \(\theta : X \rightarrow H_X\), as well as its quantum version \(\theta_q : X_q \rightarrow H_X\). The torus \(H_A\) acts on \(A\), and the map \(p : A \rightarrow X\) provides an embedding \(A/H_A \hookrightarrow X\). There is a canonical isomorphism \(H_A \otimes \mathbb{Q} = H_X \otimes \mathbb{Q}\). In particular \(H_A(\mathbb{R}_{>0}) = H_X(\mathbb{R}_{>0})\).

The chiral dual to a feed \(i = (I, \varepsilon, d)\) is the feed \(i^\circ := (I, -\varepsilon, d)\). Mutations commute with the chiral duality on feeds. Therefore given a cluster \(\mathcal{X}\)-variety \(X\) (respectively \(A\)-variety \(A\)), there is the chiral dual cluster \(\mathcal{X}\)-variety (respectively \(A\)-variety) denoted by \(\mathcal{X}^\circ\) (respectively \(A^\circ\)).

The cluster \(\mathcal{X}\)-varieties \(X\) and \(\mathcal{X}^\circ\) are canonically isomorphic as schemes: for every feed, the isomorphism is given by inversion of the coordinates: \(X_i \rightarrow X_i^{-1}\). However this isomorphism changes the Poisson bracket to the opposite one: the Poisson structure on \(\mathcal{X}^\circ\) differs from the one on \(\mathcal{X}\) by the sign. The chiral dual cluster \(A\)-variety \(A^\circ\) is canonically isomorphic to \(A\). The isomorphism is given by the identity maps on each feed torus. However \(W_{A^\circ} = - W_A\).

The Langlands dual to a feed \(i = (I, \varepsilon_{ij}, d)\) is the feed \(i^\vee = (I, \varepsilon_{ij}^\vee, d^\vee)\), where \(\varepsilon_{ij}^\vee := -\varepsilon_{ji}^\vee\) and \(d^\vee = d^{-1}\). The Langlands duality on feeds commutes with mutations. Therefore it gives rise to the Langlands dual cluster \(A\)-, and \(\mathcal{X}\)-varieties, denoted \(A^\vee\) and \(\mathcal{X}^\vee\).

### 2.2 The symplectic double

We follow the same pattern as in the definition of cluster \(\mathcal{X}\)- and \(A\)-varieties: define feed tori, introduce the relevant structures on them, glue the feed tori in a certain specific way, and show that the gluing respects the structures. We assign to each feed \(i\) a split torus \(D_i\), equipped with canonical coordinates \((B_i, X_i), i \in I\). There is a Poisson structure on the torus \(D_i\), given in coordinates by
\[
\{B_i, B_j\} = 0, \quad \{X_i, B_j\} = d_i^{-1} \delta_{ij} X_i B_j, \quad \{X_i, X_j\} = \tilde{\varepsilon}_{ij} X_i X_j, \quad \tilde{\varepsilon}_{ij} := \varepsilon_{ij} d_j^{-1}.
\]

There is a symplectic 2-form \(\Omega_i\) on the torus \(D_i\):
\[
\Omega_i := -\frac{1}{2} \sum_{i,j \in I} d_i \cdot \varepsilon_{ij} d \log B_i \wedge d \log B_j - \sum_{i \in I} d_i \cdot d \log B_i \wedge d \log X_i.
\]

So \(\Omega_i(\partial B_i \wedge \partial B_i) = \tilde{\varepsilon}_{ij}\) thanks to the coefficient \(1/2\) in front of the first term. The following lemma is equivalent to Lemma \([24]\) which we prove below.

**Lemma 2.1** The Poisson and symplectic structures on \(D_i\) are compatible: The Poisson structure \([21]\) coincides with the one defined by the 2-form \(\Omega_i\).

\(^{10}\)A more transparent and less computational definition is given in Section 3.
Mutations. Set

$$\mathbb{B}_k^+ := \prod_{i | \varepsilon_{ki} > 0} B_i^{\varepsilon_{ki}}, \quad \mathbb{B}_k^- := \prod_{i | \varepsilon_{ki} < 0} B_i^{-\varepsilon_{ki}}.$$  

Given an element \( k \in I \), let us define a birational transformation \( \mu_k : D \to D' \). Abusing notation, we identify \( I' \) and \( I \) via the map \( \mu_k : I \to I' \). Let \( \{ X'_i, B'_i \} \) be the coordinates on the torus \( D' \). Set

$$\mu_k^* : X'_i \mapsto \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k. \end{cases}$$  

$$\mu_k^* : B'_i \mapsto \begin{cases} B_i & \text{if } i \neq k, \\ \frac{B_i + X_k B'_k}{B_k(1+X_k)} & \text{if } i = k. \end{cases}$$

**Definition 2.2** The symplectic double \( D \) is a scheme over \( \mathbb{Z} \) obtained by gluing the feed tori \( D_i \) using formulas (23)-(24), and then taking the affine closure.

Given a Poisson variety \( Y \), we denote by \( Y^{\text{op}} \) the same variety with the opposite Poisson structure. Let us introduce a notation

$$\tilde{X}_i := X_i \frac{B_k^+}{B_k^-} = X_i \prod_{j \in I} B_j^{\varepsilon_{ij}}.$$  

We denote by \( (A_i, A_i^o) \) the coordinates on the cluster torus \( A_i \times A_i \). Let \( p_-, p_+ \) be the projections of \( A \times A \) onto the two factors. The key properties of the symplectic double are the following:

**Theorem 2.3**

a) There are \( \hat{\Gamma} \)-equivariant positive symplectic spaces \( A, D \) and \( X \).

b) There is a map \( \varphi : A \times A \to D \), given in any cluster coordinate system by the formulas

$$\varphi^*(X_i) = \prod_j A_j^{\varepsilon_{ij}}, \quad \varphi^*(B_i) = \frac{A_i^o}{A_i}.$$  

It respects the canonical 2-forms: \( \varphi^* \Omega_D = p^+ \Omega_A - p^+_A \Omega_A \).

c) There is a Poisson map \( \pi : D \to X \times X^{\text{op}} \), given in any cluster coordinate system by

$$\pi^*(X_i \otimes 1) = X_i, \quad \pi^*(1 \otimes X_j) = \tilde{X}_j.$$  

d) There are commutative diagrams

$$\begin{array}{ccc}
A \times A & \xrightarrow{\varphi} & D \\
\downarrow p \times p & \nearrow \varphi & \searrow \pi \\
X \times X^{\text{op}} & \xleftarrow{j} & D \\
\downarrow \Delta_X & & \downarrow \pi \\
\Delta_X & \leftarrow X \times X^{\text{op}}
\end{array}$$

\( e \) The map \( j \) is a Lagrangian embedding. The intersection of its image with each cluster torus is given by equations \( B_i = 1 \), \( i \in I \).

\( f \) There is an involutive isomorphism \( i : D \to D^{\text{op}} \) interchanging the two components of the map \( \pi \). It is given in any cluster coordinate system by \( i^*(B_i) = B_i^{-1}, \quad i^*(X_i) = \tilde{X}_i \).

g) The map \( \varphi \) is the quotient by the diagonal action of the torus \( H_A \). The map \( \pi \) is the quotient by a free Hamiltonian action of the torus \( H_X \) on \( D \). Its commuting Hamiltonians are given by the Poisson composition map \( D \xrightarrow{\pi} X \times X^{\text{op}} \xrightarrow{\vartheta} H_X \) followed by characters of \( H_X \).
The proof is postponed till Section 3. It uses a decomposition of mutations from Section 2.4.

The symplectic groupoid structure. The data \((\mathcal{X}, D, \pi, i, j)\) describes a symplectic groupoid related to the Poisson space \(\mathcal{X}\). A definition of a symplectic groupoid see in [?]. Namely, \(\mathcal{X}\) is the space of objects, \(D\) is the space of morphisms. The two components \(\pi_-\) and \(\pi_+\) of \(\pi\) provide the source and target maps. There is an involution \(i\) on \(D\), reversing the sign of the symplectic structure and interchanging the source and the target maps. It is the inversion map. There is a partial composition \(D \times D \to D\), defined if \(\pi_+ y_1 = \pi_- y_2\), so that of \(\pi_-(y_1 \circ y_2) = \pi_-(y_1)\) and \(\pi_+(y_1 \circ y_2) = \pi_+(y_2)\). The subvariety of \(D \times D^{\text{op}} \times D^{\text{op}}\) consisting of triples \((y_1, y_2, y_3)\) such that \(y_3\) is the composition of \(y_1\) and \(y_2\) is isotropic. The Lagrangian subspace \(j(\mathcal{X})\) is the subspace of the identity morphisms.

2.3 Cluster linear algebra

Let \(\Lambda^*_\mathcal{X}\) be a finite rank lattice (a free abelian group) with a skew-symmetric bilinear form \(\langle *, * \rangle_\mathcal{X}\). Let \(\{e_i\}, i \in I\) be a basis in \(\Lambda^*_\mathcal{X}\). Then there is a dictionary translating the notion of a feed into the linear algebra language:

\[
\text{Feed} = (\text{lattice, skew-symmetric form, basis, multiplier}) = (\Lambda^*_\mathcal{X}, \langle *, * \rangle_\mathcal{X}, \{e_i\}, \{d_i\}).
\]  

Indeed, \(\varepsilon_{ij} := d_j(e_i, e_j)_\mathcal{X}\). Set \(\tilde{\varepsilon}_{ij} := (e_i, e_j)_\mathcal{X}\).

Consider the following two lattices associated with \(\Lambda^*_\mathcal{X}\):

\[
\Lambda^*_\mathcal{X} := \text{Hom}_\mathbb{Z}(\Lambda^*_\mathcal{X}, \mathbb{Z}), \quad \Lambda_D := \Lambda^*_\mathcal{X} \oplus \Lambda^*_\mathcal{X}.
\]

The basis \(\{e_i\}\) in \(\Lambda^*_\mathcal{X}\) provides the dual basis \(\{e^\vee_j\}\) in \(\Lambda^*_\mathcal{X}\). We define a quasidual basis \(\{f_j\}\) in \(\Lambda^*_\mathcal{X}\) by setting \(f_i := d^{-1}_i e^\vee_i\). There is a basis \(\{e_i, f_j\}\) in \(\Lambda_D\).

Canonical maps. (i) The bilinear form on the lattice \(\Lambda^*_\mathcal{X}\) provides a canonical, i.e. defined without using a basis, homomorphism

\[
p^* : \Lambda^*_\mathcal{X} \longrightarrow \Lambda^*_\mathcal{X}, \quad e_i \longmapsto \sum_{j \in I} (e_i, e_j)_\mathcal{X} e^\vee_j = \sum_{j \in I} \varepsilon_{ij} f_j.
\]  

(ii) There is a canonical homomorphism

\[
\pi^* := (\text{Id}, \text{Id} + p^*) : \Lambda^*_\mathcal{X} \oplus \Lambda^*_\mathcal{X} \longrightarrow \Lambda_D, \quad e_l \oplus e_r \longmapsto e_l + e_r + p^*(e_r).
\]

(iii) Denote by \(\{f_i, f^o_i\}\) the basis in \(\Lambda^*_\mathcal{X} \oplus \Lambda^*_\mathcal{X}\). There is a canonical homomorphism

\[
\varphi^* := \begin{pmatrix} p^* & -\text{Id}_- \\ 0 & \text{Id}_+ \end{pmatrix} : \Lambda_D \longrightarrow \Lambda^*_\mathcal{X} \oplus \Lambda^*_\mathcal{X}, \quad e_i \longmapsto p^*(e_i), f_j \longmapsto f^o_j - f_j.
\]

(iv) There is a canonical involution

\[
i^* : \Lambda_D \longrightarrow \Lambda_D, \quad f_j \longmapsto -f_j, \quad e_i \longmapsto e_i + p^*(e_i).
\]

Canonical bilinear forms/ bivectors. The bilinear form \(\langle *, * \rangle_\mathcal{X}\) can be viewed as an element \(\omega^*_\mathcal{X} \in \Lambda^2 \Lambda^*_\mathcal{X}\). It is written in the basis as

\[
\omega^*_\mathcal{X} = \frac{1}{2} \sum_{i,j \in I} \tilde{\varepsilon}_{ij} \cdot f_i \wedge f_j, \quad \tilde{\varepsilon}_{ij} = d_i \varepsilon_{ij}.
\]  

\[
16
\]
We define a canonical element \( \omega_D \in \Lambda^2 \Lambda_D \) by setting
\[
\omega_D = - \sum_{i \in I} e_i^\vee \wedge e_i - \frac{1}{2} \cdot e_i^\vee \wedge p^*(e_i) = - \frac{1}{2} \sum_{i,j \in I} \varepsilon_{ij} \cdot f_i \wedge f_j - \sum_{i \in I} d_i \cdot f_i \wedge e_i. \tag{29}
\]

It is evidently non-degenerate. Thus it determines a dual element \( \omega_D^* \in \Lambda^2 \Lambda_{D}^* \), which we view as a skew-symmetric bilinear form \((\ast, \ast)_D\) on the lattice \( \Lambda_D \). Clearly \( i^* \omega_D^* = - \omega_D^* \).

**Lemma 2.4** The form \((\ast, \ast)_D\) is given explicitly by
\[
(e_i, e_j)_D := (e_i, e_j)_X, \quad (e_i, f_j)_D = d_i^{-1} \delta_{ij}, \quad (f_i, f_j)_D = 0. \tag{30}
\]

**Proof.** The symplectic structure \( \omega_D \) on \( \Lambda_D^* \) provides an isomorphism \( I : \Lambda_D^* \to \Lambda_D \), given by \( \langle a, I(b) \rangle = \omega_D(a, b) \). One easily checks that
\[
I(f_i^*) = d_i e_i + \sum_{j \in I} \varepsilon_{ij} e_j, \quad I(e_i^*) = d_i f_i, \quad I^{-1}(f_i) = -d_i^{-1} e_i^*, \quad I^{-1}(e_i) = d_i^{-1} f_i^* + \sum_{j \in I} \varepsilon_{ij} e_j^*.
\]

Since by definition \( \omega_D^*(a, b) = \omega_D(I^{-1}a, I^{-1}b) \), we have
\[
\omega_D^*(e_i, e_j) = \omega_D(d_i^{-1} f_i^* + \sum_k \varepsilon_{ik} e_k^*, d_j^{-1} f_j^* + \sum_s \varepsilon_{js} e_s^*) = -d_i^{-1} d_j^{-1} \varepsilon_{ij} + \varepsilon_{ij} + \varepsilon_{ij} = \varepsilon_{ij}.
\]

and \( \omega_D^*(e_i, f_j) = d_j^{-1} \delta_{ij} \). The lemma is proved.

Denote the bivector related to the second summand in \( \Lambda_X^* \oplus \Lambda_X^* \) by \( \omega_X^* \). Denote by \( \Lambda_X^{\text{op}} \) the lattice \( \Lambda_X \) equipped with the opposite bilinear form \((\ast, \ast)_X^{\text{op}} := -((\ast, \ast)_X)\).

Let \( j^* : \Lambda_D = \Lambda_X^* \oplus \Lambda_X \to \Lambda_X \) be the canonical projection.

**Proposition 2.5** (i) There are commutative diagrams
\[
\begin{array}{ccc}
\Lambda_X & \xleftarrow{j^*} & \Lambda_D \\
\uparrow (\text{id}, \text{id}) & & \uparrow \pi^* \\
\Lambda_X & \xrightarrow{(\text{id}, \text{id})} & \Lambda_X \times \Lambda_X^{\text{op}}
\end{array}
\] \quad \begin{array}{ccc}
\Lambda_X^* \oplus \Lambda_X^* & \xleftarrow{\varphi^*} & \Lambda_D \\
\uparrow p^* \times p^* & & \uparrow \pi^* \\
\Lambda_X^* \oplus \Lambda_X^{\text{op}} & \xrightarrow{\pi^*} & \Lambda_X \times \Lambda_X^{\text{op}}
\end{array} \tag{31}
\]

The map \( \pi^* \) respects the bilinear forms in \( \Lambda_X^* \oplus \Lambda_X^{\text{op}} \) and \( \Lambda_D \). One has
\[
\varphi^*(\omega_D) = \omega_X - \omega_X^*. \tag{32}
\]

(ii) The involution \( i \) reverses the sign of the symplectic \((\ast, \ast)_D\) form in \( \Lambda_D \).

**Proof.** (i) The commutativity is clear. The claim about \( \pi^* \) is clear for the left component of \( \pi^* \). For the right one it follows from \( (32) \). To check the claim about \( \varphi^* \), notice that
\[
- \frac{1}{2} \sum_{i,j \in I} \varepsilon_{ij} \cdot (f_i^o - f_i) \wedge (f_j^o - f_j) - \sum_{i,j \in I} d_i \cdot (f_i^o - f_i) \wedge \varepsilon_{ij} \cdot f_j = \frac{1}{2} \sum_{i,j \in I} \varepsilon_{ij} \cdot (f_i \wedge f_j - f_i^o \wedge f_j^o).
\]

(ii) Indeed, \( (i^* e_i, i^* f_j)_D = (e_i + p^*(e_i), -f_j)_D = -(e_i, f_j)_D \). and
\[
(i^* e_i, i^* e_j)_D = (e_i + \sum_{s \in I} \varepsilon_{is} f_s, e_j + \sum_{t \in I} \varepsilon_{jt} f_t)_D = \varepsilon_{ij} - \varepsilon_{ij} + \varepsilon_{ji} = -\varepsilon_{ij}. \tag{33}
\]
Mutated bases. Set $[\alpha]_+ = \alpha$ if $\alpha \geq 0$ and $[\alpha]_+ = 0$ otherwise. Given $k \in I$, choose another basis \( \{e'_i\} \) in \( \Lambda_X \) given by

\[
e'_i := \begin{cases} 
e_i + [\varepsilon_{ik}] + e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k. \end{cases}
\]

(34)

Lemma 2.6 The quasidual basis \( \{f'_i\} \) in \( \Lambda^*_X \) is given by

\[
f'_i := \begin{cases} -f_k + \sum_{j \in J} [-\varepsilon_{kj}] + f_j & \text{if } i = k \\ f_i & \text{if } i \neq k. \end{cases}
\]

(35)

Proof. Let \( t_k \) be the automorphism of \( \Lambda_X \) given by \( e_i \mapsto e'_i \). The matrix of the dual map \( t_k^* \) in the dual basis \( e'_i \) is obtained by the transposition of the matrix of the map \( t_k \). Thus the map \( t_k^* \) acts as \( e'_i \mapsto -e^*_k + \sum_j [\varepsilon_{jk}] + e'_j \) and \( e^*_i \mapsto e^*_i \) if \( i \neq k \). Changing to the basis \( f_j \) we get the lemma.

Lemma 2.7 Set \( \varepsilon'_{ij} := (e'_i, e'_j)_X \). Then one has

\[
\varepsilon'_{ij} = \begin{cases} \varepsilon_{ij} & \text{if } i = k \text{ or } j = k \\ \varepsilon_{ij} + |\varepsilon_{ik}| \varepsilon_{kj} & \text{if } i \neq k. \end{cases}
\]

(36)

Proof. Clearly \( (e'_i, e'_k) = (e_i + [\varepsilon_{ki}] + e_k, -e_k) = -\varepsilon_{ik} = \varepsilon'_{ik} \). Assume that \( k \notin \{i, j\} \). Then

\[
(e'_i, e'_j) = (e_i + [\varepsilon_{ik}] + e_k, e_j + [\varepsilon_{jk}] + e_k) = \varepsilon_{ij} + [\varepsilon_{ik}] + \varepsilon_{kj} + \varepsilon_{ik}[\varepsilon_{jk}] +
\]

\[
= \varepsilon_{ij} + [\varepsilon_{ik}] + \varepsilon_{kj} + \varepsilon_{ik}[\varepsilon_{kj}] = \varepsilon'_{ij}.
\]

The lemma is proved.

Remarks. 1. Since the multipliers \( d_i \) do not change under mutations, the mutation formula for \( \varepsilon_{ij} \) is equivalent to formula (36).

2. The map \( t_k \) is an automorphism of the lattice \( \Lambda_X \). However it does not preserve the bilinear form \((*, *)_X \) on \( \Lambda_X \). Its square \( t_k^2 \) is not the identity map in general. However it preserves the bilinear form \((*, *)_X \) on \( \Lambda_X \).

We define a category of algebraic tori (viewed as group schemes) as the the category opposite to the category of abelian groups of finite rank: A finite rank abelian group \( A \) gives rise to the torus \( \text{Hom}(A, \mathbb{G}_m) \). So if \( A \) is finite, it is a torsion group, a product of finite multiplicative groups, and if \( A \) is torsion free, it is a product of copies of \( \mathbb{G}_m \). It is an abelian category.

Having in mind dictionary (26), denote by \( \Lambda_{D,i} \) the lattice \( \Lambda_D \) equipped with the basis corresponding to the feed \( i \), and similarly for the lattices \( \Lambda_X \) and \( \Lambda^*_X \). Then for the feed tori we have:

\[
D_i = \text{Hom}(\Lambda_{D,i}, \mathbb{G}_m), \quad \Lambda_i = \text{Hom}(\Lambda_{X,i}, \mathbb{G}_m), \quad A_i = \text{Hom}(\Lambda^*_X, \mathbb{G}_m).
\]

(37)

The cluster coordinates for these tori are provided by the corresponding lattice bases.

We define the cluster tori \( H_X \) and \( H_A \) by setting

\[
H_X := \text{Hom}(\ker p^*, \mathbb{G}_m), \quad H_A := \text{Hom}(\text{coker} p^*, \mathbb{G}_m).
\]

So \( H_X \) is torsion free, while \( H_A \) may have torsion. The exact sequence of abelian groups \( 0 \to \ker p^* \to \Lambda_X \xrightarrow{p^*} \Lambda^*_X \to \text{coker} p^* \to 0 \) gives rise to an exact exact sequence of tori

\[
0 \to H_A \to A_i \to \Lambda_i \to H_X \to 0.
\]

(38)
2.4 Decomposition of mutations

In this subsection we decompose mutations of the $\mathcal{A}$-, $\mathcal{D}$- and $\mathcal{X}$-spaces into a composition of an automorphism and a monomial transformation, so that

Monomial part of the mutation = basis change from Section 2.3.

Given $k \in I$, the bases changes in the lattices can be interpreted as isomorphisms

$$
\tau_k : \Lambda_{\mathcal{D},i} \longrightarrow \Lambda_{\mathcal{D},i}', \quad \tau_k : \Lambda_{\mathcal{X},i} \longrightarrow \Lambda_{\mathcal{X},i}', \quad \tau_k : \Lambda_{\mathcal{X}',i}^* \longrightarrow \Lambda_{\mathcal{X},i}^*.
$$

(39)

Definition 2.8 The mutation isomorphisms

$$
\mu_k' : \mathcal{D}_i \longrightarrow \mathcal{D}_i', \quad \mu_k' : \mathcal{X}_i \longrightarrow \mathcal{X}_i', \quad \mu_k' : \mathcal{A}_i \longrightarrow \mathcal{A}_i
$$

are the isomorphisms determined by the maps of the lattices (39) and (37).

Lemmas 2.6 and 2.7 imply

Corollary 2.9 The maps $\mu_k' : \mathcal{D}_i \longrightarrow \mathcal{D}_i'$ and $\mu_k' : \mathcal{X}_i \longrightarrow \mathcal{X}_i'$ are Poisson maps.

Let us proceed now to decomposition of mutations.

The $\mathcal{D}$-space. Let us define a birational automorphism $\mu_k^\sharp$ of the feed torus $\mathcal{D}_1$ which acts on the coordinates as follows:

$$
B_i \mapsto B_i^\sharp := \begin{cases} B_i & \text{if } i \neq k, \\ B_k(1 + X_k)(1 + ̅X_k)^{-1} & \text{if } i = k. \end{cases}
$$

(40)

$$
X_i \mapsto X_i^\sharp := X_i(1 + X_k)^{-\epsilon_{ik}}
$$

(41)

Denote by $(B_i^\sharp, X_i^\sharp)$ the coordinates of the feed torus $\mathcal{D}_1'$. The isomorphism $\mu_k' : \mathcal{D}_1 \longrightarrow \mathcal{D}_1'$ acts on the coordinates as follows (observe that $B_k^\sharp = B_k^\sharp'$):

$$
B_i' \mapsto \begin{cases} B_i & \text{if } i \neq k, \\ B_k/B_k^\sharp & \text{if } i = k. \end{cases} \quad X_i' \mapsto \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k)^{\epsilon_{ik}} & \text{if } i \neq k. \end{cases}
$$

(42)

Lemma 2.10 The mutation $\mu_k : \mathcal{D}_1 \longrightarrow \mathcal{D}_1'$ is decomposed into a composition $\mu_k = \mu_k' \circ \mu_k^\sharp$.

Proof. We should check that on the level of functions we have $\mu_k = (\mu_k')^* \circ (\mu_k^\sharp)^*$. Indeed,

$$
B_k' \mapsto \frac{1 + X_k B_k^+ / B_k^-}{B_k(1 + X_k)^{-\epsilon_{ik}}} = \frac{B_k^- + X_k B_k^+}{B_k(1 + X_k)}.
$$

The rest is an obvious computation. The lemma is proved.

The $\mathcal{X}$-space. The decomposition is given by formulas (41) and the right formula in (42).

Lemma 2.11 The mutation $\mu_k : \mathcal{X}_i \longrightarrow \mathcal{X}_i'$ is decomposed into a composition $\mu_k = \mu_k' \circ \mu_k^\sharp$. 

Proof. We have $X'_k \mapsto X^{-1}_k \mapsto X^{-1}_k$, and

$$X'_i \mapsto X_i(X_k)^{[\varepsilon_{ik}]} \mapsto X_i(1 + X_k)^{-\varepsilon_{ik}}(X_k)^{[\varepsilon_{ik}]} = X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}}.$$ 

The $\mathcal{A}$-space. Set

$$\mathcal{A}_k^+ := \prod_{j \in I} A_j^{[\varepsilon_{kj}]} \quad \text{and} \quad \mathcal{A}_k^- := \prod_{j \in I} A_j^{-[\varepsilon_{kj}]}.$$ 

We define a birational automorphism $\mu_k^\sharp$ of the feed torus $\mathcal{A}_i$ which acts on the coordinates as follows:

$$A_i \mapsto \begin{cases} A_i & \text{if } i \neq k, \\ A_k(1 + p^*X_k)^{-1} = A_k(1 + \mathcal{A}_k^+/\mathcal{A}_k^-)^{-1} & \text{if } i = k. \end{cases} \tag{43}$$

Denote by $A'_i$ the coordinates of the feed torus $\mathcal{A}_i'$. The isomorphism $\mu_k' : \mathcal{A}_i \mapsto \mathcal{A}_i'$ acts on them as follows:

$$A'_i \mapsto \begin{cases} A_i & \text{if } i \neq k, \\ A_k^-/A_k & \text{if } i = k. \end{cases} \tag{44}$$

Lemma 2.12 The mutation $\mu_k : \mathcal{A}_i \mapsto \mathcal{A}_i'$ is decomposed into a composition $\mu_k = \mu_k' \circ \mu_k^\sharp$.

Proof. We need to check only how the composition acts on $A'_k$. It is given by

$$A'_k \mapsto \mathcal{A}_k^-/A_k = \mathcal{A}_k^- (1 + \mathcal{A}_k^+/\mathcal{A}_k^-)^{-1} = \frac{\mathcal{A}_k^+ + \mathcal{A}_k^-}{A_k}.$$ 

Decomposition of mutations is compatible with the map $p : \mathcal{A}_i \mapsto \mathfrak{X}_i$:

Lemma 2.13 The map $p : \mathcal{A}_i \mapsto \mathfrak{X}_i$ intertwines each of the components $\mu_k', \mu_k^\sharp$ of the mutation map.

Proof. We have to show that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{A}_i & \xrightarrow{\mu_k^\sharp} & \mathcal{A}_i \\
\downarrow p & & \downarrow p \\
\mathfrak{X}_i & \xrightarrow{\mu_k'} & \mathfrak{X}_i
\end{array} \tag{45}$$

In the left square, going up and to the left we get $X_i \mapsto \prod_{j \in I} A_j^{[\varepsilon_{ij}]} \mapsto \prod_{j \in I} A_j^{[\varepsilon_{ij}]} (A_k^+/A_k)^{[\varepsilon_{ik}]}$. Going to the left and up we get the same:

$$X_i \mapsto X_i(1 + X_k)^{-[\varepsilon_{ik}]} \mapsto \prod_{j \in I} A_j^{[\varepsilon_{ij}]}(1 + \frac{\mathcal{A}_k^+}{\mathcal{A}_k^-})^{-[\varepsilon_{ik}]}.$$ 

The claim for the right square is equivalent to the existence of the map $p^*$ of lattices, see (27).
2.5 The motivic dilogarithm structure on the symplectic double

Recall that for a field $F$ the abelian group $K_2(F)$ is the quotient of the abelian group $\Lambda^2 F^*$, the wedge square of the multiplicative group $F^*$ of $F$, by the subgroup generated by the so-called Steinberg relations $(1 - x) \land x, x \in F^* - \{1\}$. Further, if $F$ is the field of rational functions $\mathbb{Q}(X)$ on a variety $X$, there is a canonical map

$$d \log : \Lambda^2 \mathbb{Q}(X)^* \longrightarrow \Omega^2_{\log}(X), \quad f \land g \longmapsto d \log(f) \land d \log(g).$$

where $\Omega^2_{\log}(X)$ is the space of 2-forms with logarithmic singularities on $X$. It apparently kills the Steinberg relations, inducing a map $K_2(\mathbb{Q}(X)) \longrightarrow \Omega^2_{\log}(X)$.

Working modulo 2-torsion, one can replace the Steinberg relation by the following one:

$$(1 + x) \land x = (1 - (-x)) \land x = (1 - (-x)) \land (-x) + (1 - (-x)) \land -1. \quad (46)$$

The symplectic structure on $D$ comes from a class in $K_2$ as follows. For each feed i, the symplectic form is the image under the $d \log$ map of an element $W_i \in \Lambda^2 F_i^*$, where $F_i$ is the function field of the feed torus. For a mutation $i \to i'$ we present the difference $W_i - W_{i'}$ as a difference of Steinberg relations (46). So the class of the element $W_i$ in $K_2$ is mutation invariant. The difference of Steinberg relations gives rise to an element of the Bloch group, which is the motivic avatar of generating function of the symplectic map $D_i \to D_{i'}$, a mutation of the feed tori. Let us implement this program.

We use $\cdot$ for the product of an element of the wedge product by an integer. The element $\omega_D$, see (29), provides an element

$$W_1 = -\frac{1}{2} \sum_{i,j \in I} \varepsilon_{ij} \cdot B_i \land B_j - \sum_{i \in I} d_i \cdot B_i \land X_i \in \Lambda^2 \mathbb{Q}(D_1)^*, \quad \varepsilon_{ij} := d_i \varepsilon_{ij}. \quad (47)$$

Applying to $W_1$ the map $d \log$ we get the 2-form $\Omega_1$ on the torus $D_1$, see (22).

Proposition 2.14 Given a mutation $i \to i'$ in the direction $k$, one has

$$\mu^*_k W_{i'} - W_i = d_k \cdot \left( (1 + \bar{X}_k) \land \bar{X}_k - (1 + X_k) \land X_k \right).$$

Proof. It follows from the two claims, describing behavior of the elements $W_1$ under the automorphism $\mu^*_k$ and the map $\mu'_k$. It is convenient to set $W_i^k := (\mu^*_k)^* W_i$.

Lemma 2.15 One has $W^*_k - W_i = d_k \cdot \left( (1 + \bar{X}_k) \land \bar{X}_k - (1 + X_k) \land X_k \right)$.

Proof. We have

$$-2 \cdot (W^*_k - W_i) = \sum_{i \neq k, j \neq k} d_i \varepsilon_{ij} \cdot B_i^k \land B_j^k + d_k \sum_j \varepsilon_{k} \cdot B_k^j \land B_j^k + d_k \sum_i \varepsilon_{ik} \cdot B_i^k \land B_k^i \quad (48)$$

$$- \left( \sum_{i \neq k, j \neq k} d_i \cdot \varepsilon_{ij} \cdot B_i \land B_j + d_k \sum_j \varepsilon_{k} \cdot B_k \land B_j + d_k \sum_i \varepsilon_{ik} \cdot B_i \land B_k \right) \quad (49)$$

$$+ 2 \sum_i d_i \cdot B_i^k \land X_i^k - 2 \sum_i d_i \cdot B_i \land X_i. \quad (50)$$

Since $B_i^k = B_i$ for $i \neq k$, and $X_k^k = X_k$, we have

$$\sum_{i \neq k, j \neq k} d_i \varepsilon_{ij} \cdot B_i^k \land B_j^k + d_k \sum_j \varepsilon_{k} \cdot B_k^j \land B_j^k = 2 d_k \cdot B_k^i \land \frac{B_i}{B_k} \land \frac{B_k}{B_k}. \quad (51)$$
Further, formula (50) equals
\[
2 \sum_{i \neq k} d_i \cdot B_i \wedge (X_i^2 / X_k) + 2d_k \frac{B_k^2}{B_k} \wedge X_k = -2 \sum_{i \neq k} d_i \varepsilon_{ik} \cdot B_i \wedge (1 + X_k) + 2d_k \cdot \frac{B_k^2}{B_k} \wedge X_k
\]
(52)
\[
= 2d_k \cdot \left( \frac{B_k^2}{B_k} \wedge (1 + X_k) + \frac{B_k^2}{B_k} \wedge X_k \right).
\]
(53)
Here we used \(-d_i \varepsilon_{ik} = d_k \varepsilon_{ki}\) to get the third equality. Therefore we get
\[
W_i^\sharp - W_i = -(51) - (53) = 2d_k \cdot \left( \frac{B_k^2}{B_k} \wedge (1 + X_k) + \frac{B_k^2}{B_k} \wedge (1 + X_k) \right) =
\]
\[-d_k \cdot \left( (1 + X_k)(1 + \tilde{X}_k)^{-1} \wedge X_k \frac{B_k^+}{B_k} + \frac{B_k^+}{B_k} \wedge (1 + X_k) \right) = d_k \cdot \left( 1 + \tilde{X}_k \wedge \tilde{X}_k - (1 + X_k) \wedge X_k \right).
\]
The lemma is proved. The next Lemma is an immediate corollary of Corollary 2.19.

**Lemma 2.16** One has \((\mu_k')^* W_{i'} = W_i^\sharp\).

Lemmas 2.15 and 2.16 imply the proposition.

The element \(W_i\) gives rise to an element of \(K_2\) of the field \(\mathbb{Q}(\mathcal{D}_i)\), which evidently lies in \(K_2(\mathcal{D}_i)\). Proposition 2.14 implies that mutations preserve these elements. So they give rise to a \(\Gamma\)-invariant element \(W \in K_2(\mathcal{D})\). Thus the double \(\mathcal{D}\) has a \(\Gamma\)-invariant symplectic structure \(\Omega = d \log(W)\).

*The case of the \(A\)-space.* The element \(\omega_A^\times\), see (28), provides an element
\[
W_{A,i} := \frac{1}{2} \cdot \sum_{i,j \in I} \bar{\varepsilon}_{ij} A_i \wedge A_j \in \Lambda^2 \mathbb{Q}(A_i)^*.
\]
(54)

**Lemma 2.17** Given a mutation \(i \to i'\) in the direction \(k\), one has
\[
(\mu_k^\sharp)^* W_{A,i'} - W_{A,i} = d_k \cdot (1 + X_k) \wedge X_k, \quad (\mu_k')^* W_{A,i} = W_{A,i}.
\]

**Proof.** The second follows from Lemma 2.6. For the first, one has
\[
(\mu_k^\sharp)^* W_{A,i} - W_{A,i} = \frac{1}{2} \sum_{i,j \in I} \bar{\varepsilon}_{ij} \cdot \left( A_i^2 \wedge A_j^2 - A_i \wedge A_j \right) = \sum_{i \in I} \bar{\varepsilon}_{ik} \left( A_i \wedge A_k^2 / A_k \right) = (1 + \kappa_k^+ / \kappa_k^-) \wedge \kappa_k^+ / \kappa_k^-.
\]

**Corollary 2.18** There is a \(\Gamma\)-invariant element \(W_A \in K_2(A)\), and thus a \(\Gamma\)-invariant presymplectic structure \(\Omega = d \log(W_A)\) on the space \(A\).
2.6 The unitary part of the symplectic double.

Since $D$ is a positive space, the set of its positive real points is well defined. It is a real symplectic subspace of $D(\mathbb{C})$, with the symplectic form $\text{Re}\Omega$.

It turns out that there is another real subspace $D^U$ of $D(\mathbb{C})$, with a real symplectic structure given by $\text{Im}\Omega$, which we call the unitary part of the complex double. It is defined by setting the $A$-coordinates unitary, and the phases of the $X$-coordinates expressed via the $A$-coordinates:

**Definition 2.19** The unitary part $D^U$ of $D(\mathbb{C})$ is the real subspace obtained by gluing the real subvarieties of the complex cluster tori defined by the following equations:

$$|B_i| = 1, \quad \frac{X_i}{\bar{X}_i} = \prod_{j \in I} B_j^{\varepsilon_{ij}}, \quad i \in I. \quad (55)$$

Let $\Sigma_X$ be the “antiholomorphic diagonal” in $\mathcal{X}(\mathbb{C}) \times \mathcal{X}(\mathbb{C})$, defined as the set of stable points of the composition of the antiholomorphic involution on $\mathcal{X}(\mathbb{C}) \times \mathcal{X}(\mathbb{C})$ with the one $(x_1, x_2) \mapsto (x_2, x_1)$:

$$\Sigma_X = \{(x_1, x_2) \in \mathcal{X}(\mathbb{C}) \times \mathcal{X}(\mathbb{C}) \mid (x_1, x_2) = (\bar{x}_2, \bar{x}_1)\}.$$  

**Proposition 2.20** a) There is a commutative square

$$D^U \xrightarrow{j^U} D(\mathbb{C}) \xrightarrow{\rho} \xrightarrow{\pi} \Sigma_X \xrightarrow{\pi} \mathcal{X}(\mathbb{C}) \times \mathcal{X}(\mathbb{C})$$

The modular group acts on $D^U$ by real birational transformations.

b) The space $D^U$ has a symplectic structure given by $\text{Im}\Omega$.

c) The space $D^U$ is a real Lagrangian subspace for the real symplectic structure $\text{Re}\Omega$ on $D(\mathbb{C})$.

**Proof.** a) The commutativity just means that $\pi^* \bar{X}_i = \pi_+^* X_i$, that is $\bar{X}_i = \bar{X}_i$. This is equivalent to the second equation in (55). Let us check that the gluing respects the equations (55):

$$|B_k B'_k| = \left| \frac{B_k^- + X_k B_k^+}{1 + X_k} \right| = \left| \frac{1 + X_k B_k^+ / B_k^-}{1 + X_k} \right| = \left| \frac{1 + \bar{X}_k}{1 + X_k} \right| = 1.$$  

Since the second condition in (55) just means that the image of the map $\pi$ lies in the antiholomorphic diagonal, it is also invariant.

b, c) Set $B_j = |B_j| e^{\sqrt{-1}b_j}$ and $X_j = r_j e^{\sqrt{-1}x_j}$. Then the restriction of the holomorphic 2-form $\Omega$ to the unitary part $D^U$ is given by

$$\Omega|_{D^U} = -2 \sum_{i,j \in I} d_i \varepsilon_{ij} db_j \wedge db_i + \sum_{i \in I} \sum_{j \in I} d_i \varepsilon_{ij} db_i \wedge db_j + \sqrt{-1} \sum_{i \in I} da_i \wedge dr_i = \sqrt{-1} \sum_{i \in I} db_i \wedge dr_i. \quad (56)$$

The proposition is proved.

If $\varepsilon_{ij}$ is non-degenerate, there exists a complex structure on $D^U$ for which the maps $\pi_-, \pi_+ : D^U \rightarrow \mathcal{X}(\mathbb{C})$ are, respectively, holomorphic and antiholomorphic.
2.7 Appendix

Relating the quantum spaces $\mathcal{X}_q$ and $\mathcal{A}_q$ for nondegenerate $\varepsilon_{ij}$  
Recall the quantum cluster algebras of Berenstein and Zelevinsky \[BZq\]. They are defined using some additional data (the companion matrix). However if $\det(\varepsilon_{ij}) \neq 0$, there is a canonical non-commutative $q$-deformation of the cluster algebra related to a feed with the cluster function $\varepsilon_{ij}$. It is the one which we will use. It is easily translated into the language of quantum spaces. Given a feed $\mathfrak{i}$, a quantum torus $\mathcal{A}_{q,i}$ is defined by the generators $A_i$, $i \in I$, and relations $q^{-\lambda_{kl}} A_k A_l = q^{-\lambda_{lk}} A_l A_k$, where matrix $\lambda_{ij}$ is the inverse to the one $\varepsilon_{ij}$. (The cluster matrix $b_{ij}$ from \[BZq\] is the transposed to our $\varepsilon_{ij}$). These quantum tori are glued into a quantum space $\mathcal{A}_q$ via the rule given in formula 4.23 in \[BZq\].

Lemma 2.21 Given a feed $\mathfrak{i}$, there is an injective map of noncommutative algebras

$$p^*_i : \mathbb{Z}[\mathcal{X}_{q,i}] \rightarrow \mathbb{Z}[\mathcal{A}_{q,i}], \quad p^*_i X_i = q^{-\sum_{k < l} \lambda_{kl} \varepsilon_{ik} \varepsilon_{jl}} A_k^\varepsilon_{ik}$$  

(57) commuting with mutations.

Proof. It is an easy calculation using the definitions of our Section 4 and Chapter 4 of \[BZq\], which we left to the reader.

So we get a map of quantum spaces $p : \mathcal{A}_q \rightarrow \mathcal{X}_q$.

Relations between different types of cluster transformations  
A feed cluster transformation $\mathfrak{c}$ gives rise to the corresponding cluster transformations in several set-ups: the classical $\mathcal{X}$- and $\mathcal{A}$-cluster transformation $\mathfrak{c}^x$ and $\mathfrak{c}^a$, and the quantum $\mathcal{X}$- and $\mathcal{A}$-cluster transformation $\mathfrak{c}^x_q$ and $\mathfrak{c}^a_q$. Quantum mutations, and hence cluster transformations, admit the $q = 1$ specialization. So $\mathfrak{c}^x_q = \text{Id}$ implies $\mathfrak{c}^a = \text{Id}$, and similarly in the $a$-version.

Lemma 2.22 Assume that $\det \varepsilon_{ij} \neq 0$. Let $\mathfrak{c}$ be a feed cluster transformation. Then $\mathfrak{c}^a = \text{Id}$ implies $\mathfrak{c}^x_q = \text{Id}$.

Proof. By \[BZq\], Theorem 6.1, $\mathfrak{c}^a = \text{Id}$ implies $\mathfrak{c}^a_q = \text{Id}$. Since $\det \varepsilon_{ij} \neq 0$, the algebra map (57) is injective. Since $\mathfrak{c}_q^a = \text{Id}$, this implies the claim.

3 The quantum double

3.1 Mutation maps for the quantum double

Recall the quantum torus algebra (9) provided by a lattice $\Lambda$ with a bilinear skew-symmetric form $(*,*) \in \frac{1}{N} \mathbb{Z}$. Morphisms between the quantum torus algebras related to the lattices $\Lambda_1$ and $\Lambda_2$ with skew-symmetric bilinear forms are in bijection with homomorphisms of lattices $\Lambda_2 \rightarrow \Lambda_1$ respecting the forms.

Let $\mathfrak{i}$ be a feed. Set $q_i := q^{1/d_i}$. The lattice $\Lambda_D$ with the form $(*,*)_D$ related to the feed $\mathfrak{i}$ gives rise to the quantum torus algebra $D_1$. A cluster basis $(e_i, f_i), i \in I$, of the latter provides the generators $B_i, X_i$, satisfying the relations

$$q_i^{-1} X_i B_i = q_i B_i X_i, \quad B_i X_j = X_j B_i \quad \text{if } i \neq j, \quad q^{-\varepsilon_{ij}} X_i X_j = q^{-\varepsilon_{ji}} X_j X_i.$$  

(58)

Denote by $D^q_1$ the (non-commutative) fraction field of $D_1$. Let $\mu_k : \mathfrak{i} \rightarrow \mathfrak{i}'$ be a mutation. Our goal is to define a quantum mutation map, understood as a homomorphism of non-commutative fields $\mu^q_k : D^q_1 \rightarrow D^q_1$. Recall the quantum dilogarithm $\Psi^q(x) = \prod_{k=1}^{\infty} (1 + q^{2k-1} x)^{-1}$. The generators $X_i$ and $\tilde{X}_j$ commute. Thus $\Psi^q(X_k)$ commutes with $\Psi^q(\tilde{X}_k)$. 

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Definition 3.1  (i) The automorphism $\mu^\sharp_k : \mathbb{D}_i \to \mathbb{D}_i$ is the conjugation by $\Psi^{q_k}(X_k)/\Psi^{q_k}(\bar{X}_k)$:

$$
\mu^\sharp_k := \Ad_{\Psi^{q_k}(X_k)/\Psi^{q_k}(\bar{X}_k)}.
$$

(ii) The isomorphism $\mu'_k : \mathbb{D}_V \to \mathbb{D}_1$ is induced by the lattice map $t'_k : \Lambda_{\mathbb{D}, V} \to \Lambda_{\mathbb{D}, 1}$.

(iii) The mutation map $\mu'^\sharp_k : \mathbb{D}_V \to \mathbb{D}_1$ is the composition $\mu'^\sharp_k = \mu^\sharp_k \circ \mu'_k$.

Remarks. 1. The map $\mu^\sharp_k$ acts on the $X$-coordinates conjugating them by $\Psi^{q_k}(X_k)$.

2. The map $\mu'_k : \mathbb{D}_V \to \mathbb{D}_1$ is a homomorphism of algebras thanks to Corollary 2.9.

3. The definition of the automorphism $\mu^\sharp_k$ via adjoint action of the ratio of quantum dilogarithms $a\text{ priori}$ produces a formal power series in the generators. However Lemma 3.2 below shows that the range of the automorphism $\mu^\sharp_k$ is as already anticipated in Definition 3.1.

Here is an explicit computation of the automorphism $\mu^\sharp_k$. It is a bit involved. However what is really used is not this formula but transparent Definition 3.1.

Lemma 3.2 The automorphism $\mu^\sharp_k$ is given on the generators by the formulas

$$
B_i \mapsto B^\sharp_i := \begin{cases} 
B_i & \text{if } i \neq k, \\
B_k(1 + q_k X_k)(1 + q_k \bar{X}_k)^{-1} & \text{if } i = k.
\end{cases}
$$

$$
X_i \mapsto X^\sharp_i := \begin{cases} 
X_i(1 + q_k X_k)(1 + q_k^3 X_k) \cdots (1 + q_k^{2|\varepsilon_{ik}|-1} X_k) & \text{if } \varepsilon_{ik} \leq 0, \\
X_i \left((1 + q_k^{-1} X_k)(1 + q_k^{-3} X_k) \cdots (1 + q_k^{1-2|\varepsilon_{ik}|} X_k)\right)^{-1} & \text{if } \varepsilon_{ik} \geq 0.
\end{cases}
$$

Proof. For any formal power series $\varphi(x)$ the relation $q^{-\varepsilon_{ki}} X_k X_i = q^{-\varepsilon_{ik}} X_i X_k$ implies

$$
\varphi(X_k)X_i = X_i \varphi(q^{-2\varepsilon_{ik}} X_k).
$$

Recall the difference equation characterising $\Psi^q(x)$ up to a constant:

$$
\Psi^q(q^2 x) = (1 + qx)\Psi^q(x), \quad \text{or, equivalently,} \quad \Psi^q(q^{-2} x) = (1 + q^{-1} x)^{-1}\Psi^q(x).
$$

It implies that formulas (59) and (60) can be rewritten as

$$
B^\sharp_i = B_k \cdot \Psi^{q_k}(q_k^{\varepsilon_{ik}} X_k)\Psi^{q_k}(X_k)^{-1} \cdot \Psi^{q_k}(q_k^{-2\varepsilon_{ik}} \bar{X}_k)^{-1}\Psi^{q_k}(\bar{X}_k),
$$

$$
X^\sharp_i = X_i \cdot \Psi^{q_k}(q_k^{-2\varepsilon_{ik}} X_k)\Psi^{q_k}(X_k)^{-1}.
$$

Using (61) and $q_k^{-2\varepsilon_{ik}} = q^{-2\varepsilon_{ik}}$, we get

$$
\Psi^{q_k}(X_k)X_i \Psi^{q_k}(X_k)^{-1} = X_i \Psi^{q_k}(q^{-2\varepsilon_{ik}} X_k)\Psi^{q_k}(X_k)^{-1} = X^\sharp_i.
$$

Formula (63) is proved similarly. The lemma is proved.
3.2 The quantum double and its properties

Recall that quantum spaces are functors from the modular groupoid $\hat{\mathcal{G}}^o$ (the opposite to $\hat{\mathcal{G}}$) to the category $\mathrm{QTor}^*$ (Section 2.1). Maps between quantum spaces are monomial morphisms of the functors.

To relate with a geometric language, recall that the category of affine schemes is dual to the category of commutative algebras: A map $Y_1 \to Y_2$ of affine schemes is the same as a map of the corresponding commutative algebras $Y_2 \to Y_1$, where $Y_i$ is the algebra corresponding to the space $Y_i$.

To define a quantum space $D_q^o$ we use gluing isomorphisms $\mu^q_k : D_i' \to D_i$ and show that they satisfy relations (20). We can talk about them geometrically, saying that gluing isomorphisms correspond to birational maps of non-commutative space $D_i^o,q \to D_i',q$, and that the quantum scheme $D_q^o$ is obtained by gluing them via these maps. However the only meaning we put into this is that there is a functor $\hat{\mathcal{G}}^o \to \mathrm{QTor}^*$.

The main properties of the quantum double are summarized in the following theorem:

**Theorem 3.3**

a) There is a $\hat{\Gamma}$-equivariant quantum space $D_q$. It is equipped with an involution $\ast_R : D_q \to D_q^{op}$, given in any cluster coordinate system by

$$
\ast_R(q) = q^{-1}, \quad \ast_R(X_i) = X_i, \quad \ast_R(B_i) = B_i.
$$

a') There is a $\hat{\Gamma}$-equivariant quantum space $X_q$. It is equipped with an involution $\ast_R : X_q \to X_q^{op}$, given in any cluster coordinate system by

$$
\ast_R(q) = q^{-1}, \quad \ast_R(X_i) = X_i.
$$

b) There is a map of quantum spaces $\pi : D_q \to X_q \times X_q^{op}$ given in a cluster coordinate system by

$$
\pi^*(X_i \otimes 1) = X_i, \quad \pi^*(1 \otimes X_i) = \tilde{X}_i := X_i \prod_j B_j^{\varepsilon_{ij}}.
$$

(65)

$c$) There is an involutive isomorphism of quantum spaces $i : D_q \to D_q^{op}$ interchanging the two components of the projection $\pi$, given in any cluster coordinate system by

$$
i^*B_i = B_i^{-1}, \quad i^*X_i = \tilde{X}_i.
$$

When $q = 1$, the unitary part $D_U$ of the symplectic double is given by $\overline{B}_j = i^*(B_j), \overline{X}_j = i^*(X_j)$.

d) There is a canonical map of quantum spaces $\theta_q : X_q \to H_X$.

**Proof.** a), a'). Conjugation is an automorphism, so this and Remark 2 in Section 3.1 imply that the mutation map $\mu^q_k : D_i \to D_i$ is a $\ast_R$-algebra homomorphism. Similarly the mutation map $\mu^q_k : X_i \to X_i$ is a $\ast_R$-algebra homomorphism. So to prove that we get a $\Gamma$-equivariant quantum spaces $D_q$ and $X_q$ it remains to prove the following Lemma.

**Lemma 3.4** Cluster transformation maps for the quantum double satisfy relations (20).

**Proof.** This can be done by explicit calculations. Here is a trick which allows to skip involved computations. Let us do first the commutative case. One can embed a feed $\mathbf{i} = (I, \varepsilon_{ij}, d_i)$ to a bigger feed $\mathbf{i} = (\tilde{I}, \tilde{\varepsilon}_{ij}, \tilde{d}_i)$ so that $I \subset \tilde{I}$, and $\det \tilde{\varepsilon}_{ij} = 1$. We put tilde’s over the spaces related to the feed $\mathbf{i}$.
Then the map \( \varphi : \tilde{A} \times \tilde{A} \to \tilde{D} \) is an isomorphism. We claim that it is enough to check the relations \([20]\) for the space \( \tilde{A} \times \tilde{A} \). Indeed, if we put then \( B_s = 1 \) for every \( s \in \tilde{I} \) we recover the mutation for the feed \( i \). Since for the \( A \)-space these identities are known, and much easier to check, we are done.

In the quantum case we use the quantum cluster algebras of \([BZq]\). If \( \det(e_{ij}) = 1 \), there is a canonical non-commutative \( q \)-deformation of the cluster algebra. Namely, given a feed \( i \), a quantum torus \( \mathcal{A}_{i,q} \) corresponds to the lattice \( \Lambda_i^* \) with the bilinear form dual to the one on \( \Lambda_i \). These quantum tori are glued into a quantum space \( \mathcal{A}_q \) via the rule given in formula 4.23 in loc. cit.. Since \( \det(e_{ij}) = 1 \), there is a canonical isomorphism of quantum spaces \( p : \mathcal{A}_q \to \mathcal{X}_q \) \([King] \) or Appendix in Section 2, which for every feed \( i \) is induced by the lattice isomorphism \( p^* \) from Section 2.3. We embed a feed \( i \) to a feed \( \tilde{i} \) as above. Then there is an isomorphism \( \mathcal{A}_q \times \mathcal{A}_{q_{\tilde{i}}}^\op \sim \mathcal{D}_q \), given by a composition \( \tilde{A}_q \times \tilde{A}_{q_{\tilde{i}}}^\op \to \tilde{X}_q \times \tilde{X}_{q_{\tilde{i}}}^\op \to \mathcal{D}_q \). Thus we can reduce checking the quantum relations \([20]\) to the case of rank 2 quantum cluster algebras, where they are known \([BZq]\). The lemma is proved.

b). For a feed \( i \), thanks to Lemma \([25]\) the lattice map \( \pi^* \) in Section 2.3 (ii) provides a homomorphism of algebras \( \pi^*_i : X_i \otimes X_i^\op \to D_i \). It is given by formulas \([65]\).

Let us show that the quantum map \( \pi \) commutes with each component of the mutation map.
For \( \mu^*_k \) this is clear: Since \( X_i \) and \( X_j \) commute, conjugation by \( \Psi_q(X_k)/\Psi_q(X_k) \) acts on \( \pi^*(X_i \otimes 1) \) as \( \text{Ad}_{\Psi_q(X_k)} \), and on \( \pi^*(1 \otimes X_i) \) as \( \text{Ad}_{\Psi_q(X_k)}^{-1} \).

The maps \( \mu^*_k \) and \( \pi \) are monomial by definition, and Poisson by Corollary \([29]\) and Lemma \([25]\). On the classical level the map \( \pi \) intertwines the monomial parts of the mutation maps for the spaces \( D \) and \( X \) since the map of lattices \( \pi^* \) in Section 2.3 (ii) is defined without using a basis. This implies the claim in the quantum case.

c). Since \( i^* \) is a monomial map, and its classical version is Poisson by Section 2.3 (iv), the quantum map \( i^* : D_i^\op \to D_i \) is an involutive automorphism. It commutes with the automorphism \( \mu^*_k \) since

\[
i^* \circ \text{Ad}_{\Psi_q(X_k)/\Psi_q(X_k)} \circ (i^*)^{-1} = \text{Ad}_{\Psi_q(X_k)/\Psi_q(X_k)}^{-1}.
\]

The claim that it commutes with the monomial part of the mutation follows from the fact that the map of lattices \( i^* \) in Section 2.3 (iv) is defined without using a basis.

d) For a given feed \( i \) the map of tori \( \chi_q \to H_X \) is dual to the canonical embedding \( \text{Ker } p^* \to \Lambda_X^\vee \). Since \( \text{Ker } p^* \) is the kernel of the form \( (*,*)_X^\vee \), the automorphism part of the mutation map acts trivially on \( H_X \). The part d) is proved. The Theorem is proved.

**Remark.** The obtained results include a construction of the quantum \( X \)-variety. It is independent of the one given in \([FG2]\). It is simpler and more transparent thanks to the decomposition of mutations and the fact that the automorphism part of mutation is given by conjugation by \( \Psi_q(X_k) \).

**Connections between quantum \( X \)-varieties.** There are three ways to alter the space \( X_q^\op \):

(i) change \( q \) to \( q^{-1} \),

(ii) change the quantum space \( X_q^\op \) to its chiral dual \( X_q^\ast \),

(iii) change the quantum space \( X_q^\ast \) to the opposite quantum space \( X_q^\op \).

The resulting three quantum spaces are canonically isomorphic \([FG21]\), Lemma 2.1:

**Lemma 3.5** There are canonical isomorphisms of quantum spaces

\[
\alpha^q_X : X_q \to X^\op_{q^{-1}}, \quad (\alpha^q_X)^* : X_i \to X_i.
\]

\[
i^q_X : X_q \to X^\ast_{q^{-1}}, \quad (i^q_X)^* : X_i \to X_i^{-1}.
\]
The duality conjectures from \cite{FG2}, Section 4 should have the following version related to the double.

\[
\beta^q_k := \alpha^q_k \circ i^q_k : \mathcal{X}_q^o \rightarrow \mathcal{X}_q^{op}, \quad X_i \mapsto X_i^{o-1}.
\]

There are similar three ways to alter the space \( \mathcal{D}_q \), and three similar isomorphisms, acting the same way on the \( X \)-coordinates, and identically on the \( B \)-coordinates. They are compatible with the projection \( \pi \), while the involution \( i \) from Theorem \[2.3\] interchanges the two components of \( \pi \).

### 3.3 Proof of Theorem \[2.3\]

a), c), d), f). The claim in the part a) regarding the \( \mathcal{A} \)-space is known from the basic properties of the cluster algebras, see \cite{FZ1}. Its only non-trivial part is a proof of the \((h + 2)\)-gon relations in the \( \mathcal{A} \)-case. The rest of a) plus c), d), f) follow from the similar quantum properties proved in Theorem 3.3.

b) For a feed \( i \), the projection \( \varphi \) is given by a homomorphism \( \varphi_i : \mathcal{A}_i \times \mathcal{A}_i \rightarrow \mathcal{D}_i \), determined by the lattice map \( \varphi^* \). We have to show that for a mutation \( \mu_k : i \rightarrow i' \) the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{A}_i \times \mathcal{A}_i & \xrightarrow{\mu_k} & \mathcal{A}_{i'} \times \mathcal{A}_{i'} \\
\varphi_i \downarrow & & \downarrow \varphi_{i'} \\
\mathcal{D}_i & \xrightarrow{\mu_k} & \mathcal{D}_{i'}
\end{array}
\]

We need to compare pull backs of the coordinates \( B'_k, X'_i \) obtained in two different ways. The claim for the coordinates \( X'_i \) follows from the existence of the map \( p : \mathcal{A} \rightarrow \mathcal{X} \). For the \( B' \)-coordinates it is non-obvious only for the coordinate \( B'_k \), and the map \( \mu_k^2 \). Going up and left in the diagram we get

\[
B'_k \rightarrow \frac{A'_k}{A_k} \rightarrow \frac{A^o_k (1 + \Lambda_k^+/\Lambda_k^-)}{A_k (1 + \Lambda_k^{op}+/\Lambda_k^{-})}, \quad \Lambda_k^{op} = \prod_i (A_i^o)^{[\pm\epsilon_k]_+}.
\]

Going left and up in the diagram we get the same:

\[
B'_k \rightarrow \frac{1 + X_k}{1 + X'_k} \rightarrow \frac{A^o_k (1 + \Lambda_k^+/\Lambda_k^-)}{A_k (1 + \Lambda_k^{op}+/\Lambda_k^-)}.
\]

The claim that \( \varphi^* \pi^* = p^* \times p^* \) and the claim about the 2-forms follows from Proposition \[2.5\](i).

e) Write \( \pi = (\pi_-, \pi_+) \). Setting \( B_i = 1 \) for all \( i \in I \) we get subvariety of \( \mathcal{D} \) identified with \( \mathcal{X} \) by \( \pi_- \). The restriction of the 2-form \( \Omega \) to this subvariety is clearly zero.


g) It is straightforward linear algebra statement valid in every feed torus. We leave it to the reader. The Theorem is proved.

### 3.4 Duality conjectures for the double

The duality conjectures from \cite{FG2}. Section 4 should have the following version related to the double.

The semiring \( \mathbb{L}_+(\mathcal{D}) \) of regular positive functions on the positive scheme \( \mathcal{D} \) consists of all rational functions which are Laurent polynomials with positive integral coefficients in every cluster coordinate system on \( \mathcal{D} \). We conjecture that there exists a canonical basis in the space of regular positive functions on the Langlands dual space \( \mathcal{D}' \), parametrised by the set \( \mathcal{D}(\mathbb{Z}) \) of the integral tropical points of \( \mathcal{D} \). Moreover, it should admit a \( q \)-deformation to a basis in the space \( \mathbb{L}_+(\mathcal{D}_q) \) of regular positive functions on the space \( \mathcal{D}_q \). The latter, by definition, consists of all (non-commutative) rational functions which in every cluster coordinate system are Laurent polynomials with positive integral coefficients in the cluster variables and \( q \).

The canonical basis should be compatible with the canonical bases on the cluster \( \mathcal{A} \)- and \( \mathcal{X} \)-varieties from Section 4 of \cite{FG2}. Here is a precise conjecture. Recall that for a set \( S \) we denote by \( \mathbb{Z}_+\{S\} \) the semigroup of all \( \mathbb{Z}_+ \)-valued function on the set \( S \) with finite support.
**Conjecture 3.6**
a) There exists a canonical isomorphism $I_D : Z_+\{D(Z^t)\} \sim \to L_+(D^\vee)$, which fits into the following commutative diagram:

$$
\begin{array}{c}
Z_+\{(A \times A)(Z^t)\} \\
\downarrow \\
Z_+\{D(Z^t)\} \\
\downarrow \\
Z_+\{(A^\vee \times A^\vee)(Z^t)\} \\
\end{array}
\quad
\begin{array}{c}
L_+(A^\vee \times A^\vee) \\
\downarrow \\
L_+(D^\vee) \\
\downarrow \\
L_+(A^\vee \times A^\vee) \\
\end{array}
$$

Here the vertical maps are induced by the maps in the diagram (6). The top and the bottom horizontal maps are the isomorphisms predicted by the duality conjecture in loc. cit.

b) The isomorphism $\mathbb{I}_D$ admits a $q$-deformation, given by an isomorphism $\mathbb{I}^q_D : D(Z^t) \sim \to L_+(D^\vee_q)$.

The maps $\mathbb{I}_D$ and $\mathbb{I}^q_D$ should satisfy additional properties just like the ones listed in loc. cit.

4 The quantum dilogarithm and its properties

4.1 The quantum logarithm function and its properties

We assume throughout this Section that $\hbar > 0$. The quantum logarithm is the following function:

$$
\phi^\hbar(z) := -2\pi \hbar \int_{\Omega} \frac{e^{-ipz}}{(e^{\pi p} - e^{-\pi p})(e^{\pi \hbar p} - e^{-\pi \hbar p})} dp;
$$

where the contour $\Omega$ goes along the real axes from $-\infty$ to $\infty$ bypassing the origin from above.

**Proposition 4.1** The function $\phi^\hbar(x)$ enjoys the following properties.

(A1) $\lim_{\hbar \to 0} \phi^\hbar(z) = \log(e^z + 1)$.

(A2) $\phi^\hbar(z) - \phi^\hbar(-z) = z$.

(A3) $\overline{\phi^\hbar(z)} = \phi^\hbar(\overline{z})$.

(A4) $\phi^\hbar(z)/\hbar = \phi^1/(z/\hbar)$.

(A5) $\phi^\hbar(z + i\pi \hbar) - \phi^\hbar(z - i\pi \hbar) = \frac{2\pi i \hbar}{e^{-z} + 1}$, $\phi^\hbar(z + i\pi) - \phi^\hbar(z - i\pi) = \frac{2\pi i}{e^{-z/\hbar} + 1}$.

(A6) $\phi^1(z) = \frac{z}{1 - e^{-z}}$.

(A7) The form $\phi^\hbar(z)dz$ is meromorphic with simple poles at the points of the upper half plane

$$
\{\pi i \left((2m - 1) + (2n - 1)\hbar\right) | m, n \in \mathbb{N}\}
$$

with residues $2\pi i \hbar$,

and at the points of the lower half plane

$$
\{-\pi i \left((2m - 1) + (2n - 1)\hbar\right) | m, n \in \mathbb{N}\}
$$

with residues $-2\pi i \hbar$.
\[ \phi^h(z) = \frac{\hbar}{(\hbar + 1)} \left( \phi^{h+1}(z + i\pi) + \phi^{\frac{h+1}{2}}(z/h - i\pi) \right) = \frac{\hbar}{(\hbar + 1)} \left( \phi^{h+1}(z - i\pi) + \phi^{\frac{h+1}{2}}(z/h + i\pi) \right). \]

(A9) \[ \sum_{l=\frac{1}{2}}^{\frac{c-1}{2}} \sum_{m=\frac{1}{2}}^{\frac{c-1}{2}} \phi^h(z + \frac{2\pi i}{r} l + \frac{2\pi i \hbar}{s} m) = s\phi^h(rz), \]

where the sum is taken over half-integer if the summation limits are half-integers.

**Proof. A1.** Since the contour \( \Omega \) bypasses the origin from above, \( \lim_{z \to -\infty} \phi^h(z) = \lim_{z \to -\infty} \ln(e^{z} + 1) = 0. \) Thus it is sufficient to prove that \( \lim_{\hbar \to 0} \frac{\partial}{\partial z} \phi^h(z) = \frac{1}{e^{z} + 1}. \) The l.h.s. of the latter is computed using residues:

\[ \lim_{\hbar \to 0} \frac{\partial}{\partial z} \phi^h(z) = \lim_{\hbar \to 0} -\frac{\pi \hbar}{2} \int_{\Omega} -i \rho e^{-i\rho z} d\rho = \frac{i}{2} \int_{\Omega} e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = \frac{1}{2} \int_{\Omega} e^{-i\rho z} \frac{1 + e^{\rho}}{\rho} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = \frac{\rho h}{2} 2 \pi e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} = \frac{1}{e^{z} + 1}. \]

**A2.** It is verified by computing using residues:

\[ \phi^h(z) - \phi^h(-z) = -\frac{\pi \hbar}{2} \int_{\Omega} e^{-i\rho z} - e^{i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = -\frac{\pi \hbar}{2} \left( \int_{\Omega} e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho + \int_{\Omega} e^{i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho \right) = \frac{\pi \hbar}{2} 2 \pi e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} = z. \]

**A3.** It is obtained by the change of the integration variable \( q = -\bar{\rho}: \)

\[ \phi^h(z) = -\frac{\pi \hbar}{2} \int_{\bar{\Omega}} e^{i\bar{\rho} z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = \frac{\pi \hbar}{2} \int_{-\Omega} e^{i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = -\frac{\pi \hbar}{2} \int_{\Omega} e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = \phi^h(z). \]

**A4.** It is obtained by the change of the integration variable \( q = p/\hbar. \)

**A5** The proof is similar to the proof of A1. We give only the proof of the first identity:

\[ \phi^h(z + i\pi h) - \phi^h(z - i\pi h) = -\frac{\pi \hbar}{2} \int_{\Omega} \frac{e^{-i\rho z}(e^{\pi \hbar \rho} - e^{-\pi \hbar \rho})}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = -\frac{\pi \hbar}{2} \int_{\Omega} e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho - \frac{\pi \hbar}{2} \int_{\Omega} e^{i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = \frac{2\pi \hbar}{e^{z} + 1}. \]

**A6.** It is done by explicit residue calculations:

\[ \phi^h(z) = -\frac{\pi \hbar}{2} \int_{\Omega} e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = -\frac{\pi \hbar}{2} \int_{\Omega} e^{-i\rho z} \frac{1}{\sin(\pi \rho) \sin(\pi \hbar \rho)} d\rho = \frac{2\pi \hbar}{e^{z} + 1}. \]

**A7.** The integral (66) converges for \( |\Re z| < \pi(1 + \hbar). \) Using A5, the function \( \phi^h \) can be continued to the whole complex plane. Then A7 is obvious. Property A9 is an easy consequence of the identity

\[ \sum_{l=\frac{1}{2}}^{\frac{c-1}{2}} e^{lx} = \frac{\sin((l + 1/2)x)}{\sin((1/2)x)}. \]
4.2 The quantum dilogarithm and its properties

Recall the quantum dilogarithm function:

\[ \Phi^\hbar(z) := \exp\left(-\frac{1}{4} \int_\Omega \frac{e^{-ipz} \text{sh}(\pi p) \text{sh}(\pi \hbar p)}{p} \, dp \right). \]

**Proposition 4.2** The function \( \Phi^\hbar(x) \) enjoys the following properties.

1. \( 2\pi i \hbar d\log \Phi^\hbar(z) = \phi^\hbar(z) \, dz. \)
2. \( \lim_{\Re z \to -\infty} \Phi^\hbar(z) = 1. \)
   
   Here the limit is taken along a line parallel to the real axis.
3. \( \lim_{\hbar \to 0} \Phi^\hbar(z)/\exp L z^2 e(z^2/2\pi i \hbar) = 1. \)
4. \( \Phi^\hbar(z)\Phi^\hbar(-z) = \exp\left(\frac{z^2}{4\pi i \hbar} \right) e^{-\pi i (\hbar^2 + 1)}. \)
5. \( \Phi^\hbar(z) = (\Phi^\hbar(z))^\frac{1}{h}. \)
6. \( \Phi^\hbar(z + 2\pi i \hbar) = \Phi^\hbar(z)(1 + q^z), \quad \Phi^\hbar(z + 2\pi i) = \Phi^\hbar(z)(1 + q e^{z/\hbar}). \)
7. \( \Phi^1(z) = e^{(\pi^2/6 - Li_2(1-e^z))/2\pi i}. \)

where the r.h.s. should be understood as the analytic continuation from the origin.

**Proposition 4.2** The function \( \Phi^\hbar(z) \) is meromorphic. Its poles are simple poles located at the upper half plane at the points

\[ \{-\pi i \left( (2m - 1) - (2n - 1)\hbar \right) \mid m, n \in \mathbb{N} \}, \]

and its zeroes are located at the lower half plane at the points

\[ \{\pi i \left( (2m - 1) + (2n - 1)\hbar \right) \mid m, n \in \mathbb{N} \}. \]

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Proof. Property B0 follows from the Cauchy lemma for \( \text{Im} z \leq 1 + \hbar \) and can be extended to any line parallel to the real line using the relation B5. Properties B3-B5 and B7-B9 follow immediately from the corresponding properties of the function \( \phi^h \) and property B0. Indeed, since their logarithmic derivatives give exactly the relations for the functions \( \phi \) one needs only to verify that they are valid in the limit \( z \to -\infty \), what is obvious in these cases.

The proof of Property B1 is also analogous to the proof of A1. According to the latter we have:

\[
\int_\Omega e^{-ipz} \frac{1}{\text{sh}(\pi p)} = \frac{-2i}{e^{-z} + 1}.
\]

By integrating both sides twice one has

\[
\int_\Omega e^{-ipz} \frac{1}{p^2 \text{sh}(\pi p)} = 2i \text{Li}_2(e^z).
\]

It implies that

\[
\lim_{\hbar \to 0} -\frac{1}{4} \int_\Omega e^{-ipz} \frac{1}{\text{sh}(\pi p) \text{sh}(\pi hp)} dp \frac{\text{Li}_2(e^z)}{2\pi i} = 0
\]

since this difference is an odd function of \( \hbar \) and the poles of the both terms cancel each other.

Property B2 can be proved by residue computations:

\[
\Phi^h(z)\Phi^h(-z) = \exp\left(-\frac{1}{4} \int_\Omega \left( \frac{e^{-ipz}}{\text{sh}(\pi p) \text{sh}(\pi hp)} - \frac{e^{ipz}}{\text{sh}(\pi p) \text{sh}(\pi hp)} \right) dp \right) =
\]

\[
= \exp\left(\pi i \text{Res}_{p=0} \frac{e^{-ipz}}{\text{sh}(\pi p) \text{sh}(\pi hp)} \right) =
\]

\[
= \exp\left(-\frac{1}{2\pi i} \text{Res}_{p=0} \frac{e^{-ipz}}{p^3(1 + (\pi p)^2/6)(1 + (\pi hp)^2/6)} \right) \text{Res}_{p=0} \frac{e^{-ipz}}{p^3(1 + (\pi p)^2/6)(1 + (\pi hp)^2/6)} \right) = \exp\left(\frac{z^2}{4\pi i h} e^{-\frac{\pi i}{12}(h + h^{-1})}\right).
\]

Property B6 can be proven by a direct calculation:

\[
\Phi^1(z) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^z \phi^1(t) dt \right) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^z zd(1 - e^z) \right) = \exp\left(\frac{\text{Li}_2(1) - \text{Li}_2(1 - e^z)}{2\pi i}\right).
\]

5 Representations of quantum cluster varieties

5.1 The intertwiner

The logarithmic coordinates These are the coordinates \( a_i := \log A_i \) on the set \( A_i(\mathbb{R} > 0) \) of positive real points of the feed torus \( A_i \). Here is a relevant algebraic notion: The logarithmic feed \( A \)-space \( A_{\log,i} \) is the affine space with the coordinates \( a_i \). So there is a canonical isomorphism

\[
A_{\log,i}(\mathbb{R}) = A_i(\mathbb{R} > 0).
\]

Below we use the notation \( A_i^+ \) for \( A_{\log,i}(\mathbb{R}) \), and \( L^2(A_i^+) \) for \( L^2(A_{\log,i}(\mathbb{R})) \).

It is easy to see that the volume form \( da_1 \ldots da_n \) on \( A^+ \) changes the sign under the mutations. Thus mutations provide isomorphisms of the Hilbert spaces \( L^2(A_i^+) \) for different feeds.
The set-up  Let us set

\[ h_k := 2 \pi i h_k, \quad q_k := e^{\pi i h_k}, \quad h_k^\vee := 1/\hbar, \quad h_k^\vee := 1/\hbar, \quad q_k^\vee := e^{\pi i/\hbar} = e^{\pi i h_k^\vee}. \]

It is handy to introduce the following notation:

\[ \alpha_k^+ := \sum_{j \in I} [\varepsilon_{kj}]_+ a_j, \quad \alpha_k^- := \sum_{j \in I} [-\varepsilon_{kj}]_+ a_j; \quad \text{so} \quad \alpha_k^+ - \alpha_k^- = \sum_{j \in I} \varepsilon_{kj} a_j. \quad (67) \]

Consider the following differential operators in \( A_{\log,1} \):

\[ \hat{x}_p := 2 \pi i h_p \frac{\partial}{\partial a_p} - \alpha_p^+, \quad \hat{b}_p := a_p. \quad (68) \]

They satisfy the commutation relations of the Heisenberg \(*\)-algebra \( \mathcal{H}_1^h \):

\[ [\hat{x}_p, \hat{x}_q] = 2 \pi i h \delta_{pq}, \quad [\hat{x}_p, \hat{b}_q] = 2 \pi i h_p \delta_{pq}, \quad \ast \hat{x}_p = \hat{x}_p, \quad \ast \hat{b}_p = \hat{b}_p. \]

Furthermore, consider another collection of first order differential operators in \( A_{\log,1} \):

\[ \tilde{x}_p = \hat{x}_p + \sum_q \varepsilon_{pq} \hat{b}_q = 2 \pi i h_p \frac{\partial}{\partial a_p} - \alpha_p^- . \quad (69) \]

They commute with the operators (68), and, together with the operators \( \hat{b}_p \), satisfy the commutation relations of the opposite Heisenberg \(*\)-algebra \( \mathcal{H}_1^{op} \):

\[ [\tilde{x}_p, \tilde{x}_q] = -2 \pi i h \delta_{pq}, \quad [\tilde{x}_p, \tilde{b}_q] = 2 \pi i h_p \delta_{pq}, \quad [\tilde{x}_p, \tilde{x}_q] = 0. \quad \ast \tilde{x}_p = \tilde{x}_p. \]

The exponentials \( \tilde{X}_p := \exp(\tilde{x}_p) \) and \( \tilde{B}_p := \exp(\tilde{b}_p) \) are difference operators:

\[ \tilde{X}_p f(A_1, ..., A_n) = (A_p^+)^{-1} \cdot f(A_1, ..., A_p A_p^2, ..., A_n), \quad \tilde{B}_p f = A_p f. \quad (70) \]

They satisfy the commutation relations of the quantum torus \(*\)-algebra \( \mathbf{D}_1^q \).

There is a canonical isomorphism \( \mathcal{H}_1^{op} \to \mathcal{H}_1^{\ast} \) acting as the identity on the generators.

Finally, let us introduce the Langlands modular dual collection of operators

\[ \tilde{x}_p := \hat{x}_p/h_p, \quad \hat{b}_p := \hat{b}_p/h_p, \quad \tilde{X}_p := \exp(\tilde{x}_p), \quad \tilde{B}_p := \exp(\tilde{b}_p). \quad (71) \]

The operators \{\( \tilde{x}_p, \tilde{b}_p \)\} satisfy the commutation relations of the Heisenberg \(*\)-algebra \( \mathcal{H}_1^{\ast} \). The operators \{\( \tilde{X}_p, \tilde{B}_p \)\} satisfy the commutation relations of the \(*\)-algebra \( \mathcal{D}_1^{\ast} \).

**Remark.** One can say that operators (68) (respectively (69) and (70)) provide a \(*\)-representation by unbounded operators of the Heisenberg \(*\)-algebra \( \mathcal{H}_1 \) (respectively \( \mathcal{H}_1^{op} \) and \( \mathbf{D}_1^q \)) in the Hilbert space \( L^2(A_1^+) \).

Since the operators \( \hat{x}_k \) and \( \hat{x}_k \) are self-adjoint in \( L^2(A_1^+) \), one can apply to them any continuous function on the real line. If the function is unitary, i.e. takes the values at the unit circle, we get a unitary operator. We employ below the quantum dilogarithm function \( \Phi^h(x) \), which is unitary by the property \textbf{B3}. 

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Definition 5.1 Given a mutation $\mu_k : i \to i'$, the intertwining operator

$$K_{i', i} : L^2(A_i^+) \to L^2(A_{i'}^+)$$

is defined as a composition \[K_{i', i} := K^2 \circ K',\] where:

- The operator $K^2$ is the ratio of quantum dilogarithms of the operators $\tilde{x}_k$ and $\tilde{x}_k'$:

$$K^2 := \Phi^{h_k}(\tilde{x}_k)\Phi^{h_{k'}}(\tilde{x}_k)^{-1}.$$  

- The operator $K'$ is induced by the logarithmic version $\mu_{\log,k} : A_i^+ \to A_{i'}^+$ of the cluster $A$-transformation $\mu'_k$, see [44], which is a linear map acting on the coordinates as follows:

$$a_i' \mapsto \begin{cases} a_i & \text{if } i \neq k, \\ \alpha_k - a_k & \text{if } i = k. \end{cases}$$

Let us write the operator $K^2$ by using the Fourier transform $F_{a_k}$ along the $a_k$-coordinate:

$$F_{a_k}(f)(c) = \hat{f}(c) = \int e^{a_k c/2\pi i h} f(a_k) da_k; \quad f(a_k) = \frac{1}{(2\pi i)^{2h}} \int e^{-a_k c k/2\pi i h} \hat{f}(c) dc.$$  

Here $c$ is the variable dual to $a_k$. We omit the variables $a_1, \ldots, a_k, \ldots, a_n$ in both $f$ and $\hat{f}$. Then

$$-2\pi i h \frac{\partial \hat{f}}{\partial a_k} = c \hat{f}, \quad a_k \hat{f} = 2\pi i h \frac{\partial \hat{f}}{\partial c}.$$  

Equivalently:

$$a_k = F_{a_k}^{-1} \circ (2\pi i h \frac{\partial}{\partial c}) \circ F_{a_k}, \quad -2\pi i h \frac{\partial}{\partial a_k} = F_{a_k}^{-1} \circ c \circ F_{a_k}.$$  

Then

$$K^2 = F_{a_k}^{-1} \circ \Phi^{-a_k} \left(-d_k^{-1} c - a_k^+\right) \Phi^{a_k} \left(-d_k^{-1} c - a_k^-\right)^{-1} \circ F_{a_k}.$$  

The inverse $K_{i', i}^{-1}$ of the intertwiner has a bit simpler presentation as an integral operator:

$$(K_{i', i}^{-1} f)(a_1, \ldots, a_k', \ldots, a_n) := \int G(a_1, \ldots, a_k', a_k, \ldots, a_n f(a_1, \ldots, a_k, \ldots, a_n) da_k,$$

where

$$G(a_1, \ldots, a_n) := \frac{1}{(2\pi i)^{2h}} \int \Phi^{a_k} \left(-d_k^{-1} c - a_k^+\right)^{-1} \Phi^{a_k} \left(-d_k^{-1} c - a_k^-\right)^{-1} \exp \left(c \frac{a_k - a_k^+}{2\pi i h}\right) dc.$$  

The kernel of the integral operator $K^2$ depends on a choice of a representation of the Heisenberg algebra $H_1$ in $L^2(A_i^+)$. Here is a different realization, employed in [FG21]:

$$\tilde{x}_p^{old} := \pi i h_p \frac{\partial}{\partial a_p} - \sum_q \epsilon_{pq} a_q, \quad \tilde{b}_p^{old} = 2a_p.$$  

\[\text{11}\] We suppress the indices when this does not lead to a confusion.
5.2 Main results

A space $W_1$. Let $W_1 \subset L^2(A_1^+)$ be the space of finite linear combinations of functions

\[ P\exp(-\alpha \sum_{i \in I} (a_i^2/2 + b_i a_i)), \quad \text{where } \alpha > 0, \ b_i \in \mathbb{C}, \text{ and } P \text{ is a polynomial in } a_i, \ i \in I. \quad (79) \]

It is invariant under the Fourier transform. The operators $\hat{B}_i, \hat{B}_i^\vee, \hat{X}_i, \hat{X}_i^\vee$ are symmetric unbounded operators defined on $W_i$. They preserve $W_i$, and satisfy, on $W_i$, the standard commutation relations.

The *-algebra $L$ and its Schwartz space. Denote by $\mathbb{L}_q$ the space of universally Laurent polynomials for the cluster double $D_q$, and by $L$ the algebra $\mathbb{L}_q \otimes \mathbb{L}_q^\vee$. The algebra $L$ is *-invariant.

**Definition 5.2** The Schwartz space $S_i = S_{L,i}$ is a subspace of $L^2(A_1^+)$ consisting of vectors $f$ such that the functional $w \mapsto (f, \hat{A}w)$ on $W_i$ is continuous for the $L^2$-norm, for all $A \in L$.

Denote by $(\ast, \ast)$ the scalar product in $L^2(A_1^+)$. The Schwartz space $S_i$ is the common domain of definition of operators from $L$ in $L^2(A_1^+)$. Indeed, since $W_i$ is dense in $L^2(A_1^+)$, the Riesz theorem implies that for any $f \in S_i$ there exists a unique $g \in L^2(A_1^+)$ such that $(g, w) = (f, \hat{A}w)$. We set $\hat{A}^* f := g$. Equivalently, let $W_i^*$ be the algebraic linear dual to $W_i$. So $L^2(A_1^+) \subset W_i^*$. Then

\[ S_i = \{ v \in W_i^* \mid \hat{A}^* v \in L^2(A_1^+) \text{ for any } A \in L \} \cap L^2(A_1^+). \]

The Schwartz space $S_i$ has a natural topology given by seminorms

\[ \rho_B(f) := ||Bf||_{L^2}, \quad B \text{ runs through a basis in } L. \]

**Definition 5.3** The distribution space $S_i^\ast$ is the topological dual of the space $S_i$.

**Intertwiners for cluster transformations.** A feed cluster transformation $c : i \rightarrow i'$ provides a unitary operator

\[ K_{c^\ast} : L^2(A_1^+) \rightarrow L^2(A_i^+). \]

Indeed, we assigned to a feed mutation $i \rightarrow i'$ an intertwiner $K_{i' \rightarrow i} : L^2(A_i^+) \rightarrow L^2(A_i^+)$. Further, an automorphism $\sigma$ of a feed $i$ gives rise to a unitary operator given by a permutation of coordinates in the space $A_i^+$. The feed cluster transformation $c$ is a composition of mutations and automorphisms. Taking the reverse composition of the corresponding intertwiners, we get the map $K_{c^\ast}$.

A feed cluster transformation $c$ gives rise to cluster transformations $c_q^x$ and $c_q^d$ of the quantum spaces $X_q$ and $D_q$. Denote by $\gamma_{c^\ast}$ the *-algebra automorphism of $L$ corresponding to $c$.

The following theorem is one of the main results of this paper.

**Theorem 5.4** (i) The operator $K_{c^\ast}$ provides a map of Schwartz spaces

\[ K_{c^\ast} : S_i' \rightarrow S_i. \]

It intertwines the automorphism $\gamma_{c^\ast}$ of $L$, i.e. for any $A \in L$ and $s \in S_i$ one has

\[ K_{c^\ast} \hat{A} K_{c^\ast}^{-1} s = \gamma_{c^\ast}(A)s. \quad (80) \]

(ii) Suppose that the cluster transformations $c_q^x$ and $c_q^d$ of the quantum spaces $X_q$ and $D_q$ are identity maps. Then the operator $K_{c^\ast}$ is proportional to the identity: $K_{c^\ast} = \lambda_{c^\ast} \text{Id}$, where $|\lambda_{c^\ast}| = 1$.  

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It is sufficient to prove the part (i) of the theorem for mutations. The relation between the mutations and the intertwiners is summarized in the following diagram:

\[
\begin{align*}
L & \xrightarrow{\mu'_k} L & S_i' & \xrightarrow{K'} S_i \\
L & \xrightarrow{\mu'_k} L & S_i & \xrightarrow{K^2} S_i
\end{align*}
\]

Here the algebra \(L\) acts on the Schwartz spaces \(S_i\) at the bottom. The operators \(K'\) and \(K^2\) intertwine the isomorphisms at the top line.

Here are main issues and a strategy of the proof.

The part (i) implies the part (ii) if we can show that the algebra \(L\) is big enough, e.g. of maximal functional dimension. So we start from the part (ii), and explain why it requires the part (i).

First, we do not know explicitly neither all trivial feed cluster transformations, nor feed cluster transformations satisfying the condition of the part (ii) of the theorem.\(^{12}\) We suspect that in many cases relations \((20)\) generate all of them, but this is not always the case, and is not known at the moment even for the cluster varieties \(X_{G,S}\) with \(G = SL_m, m > 3\). This eliminates a hope for a proof by an explicit calculation of the integrals providing the intertwiner.

To prove the part (ii), we would like to show that the operator \(K_c\) commutes with the operators from algebra \(D^q_i \otimes D^q_i\) of the \(q\)-difference operators acting on the space \(W_i\). This, for \(\hbar \not\in \mathbb{Q}\), easily implies that \(K_c = \lambda c \text{Id.}\) The algebra \(D^q_i\) alone would not do the job, since there are many operators commuting with it, e.g. \(D^q_i\). Since \(K_c\) depends continuously on \(\hbar\), the claim follows then for all \(\hbar\).

Since the operator \(K_c\) is a composition of elementary intertwiners \(K\), we would like to have commutation relations with them. The problem is that we deal with unbounded operators, so \(AK = KA\) means that \(AKf = KAf\) for \(f\)'s from a certain dense subspace \(D_A \subset L^2(A^+)\). However \textit{a priori} it is not clear what the intertwiner \(K\) does with the domain \(D_A\), so if both \(A\) and \(B\) commute with \(K\), \textit{a priori} it is not clear why \(AB\) should commute with \(K\), or even whether \(BKf\) is defined.

To make sense out of this we restrict to the subalgebra \(L\) – otherwise meaningless denominators would appear in the commutation formulas, and work with the corresponding Schwartz spaces – the intertwiners do not respect the spaces \(W_i\). However then we have to show that the subalgebra \(L\) is big enough. This creates some issues of purely algebraic nature.

Here is our scheme of the proof:

1) First, we prove in Theorem 5.6 that the operator \(K^2\) intertwines the mutation automorphism of the algebra \(L\) on the subspace \(W_i\). Here we use the pair of difference relations for the function \(\Phi^h\), and in addition to this the following remarkable property of the function \(\Phi^h(z)\): it is analytic in the upper half plane, and its inverse is analytic in the lower half plane (!) – see Property B7 in Section 4 – plus at most exponential growth of \(\Phi^h(z)\) when \(|\text{Re} z| \to \infty\).

We would like to stress that the pair of difference relations for the function \(\Phi^h\) is not sufficient for the proof. Indeed, it is clear from the proof, that any pole of the function \(\Phi^h(z)\) in the upper half plane, or any zero in the lower half plane would be an obstruction to the intertwining property for general \(h\).

2) Second, we prove that the space \(W_i\) is dense in the Freschet space \(S_i\). This eventually allows us to prove that the intertwiner respects the Schwartz spaces.

3) Finally, we show that the subalgebra \(L\) is big enough. This allows to use 2) to deduce (i) from (ii) in Theorem 5.4.

\(^{12}\) We conjecture that these two types of cluster transformations coincide.
Recall the dual groupoid $\tilde{G}^o$ (Section 2.1). Its objects are feeds equivalent to a given feed $i$. The set of morphisms Hom$(i, i')$ consists of feed cluster transformations $i' \to i$ modulo the ones \([20]\). The group $\Gamma$ acts by automorphisms of the quantum double (Theorem \(3.3a\)).

The (h + 2)-gon relations. The automorphism $\sigma_{ij} : I \to I$ permuting $i$ and $j$ induces an automorphism of $L^2(A_i^+)$, also denoted by $\sigma_{ij}$. Denote by $K_{\mu^o_i}$ the intertwiner for the mutation of a feed $i$ at the direction $i$. Set

$$K_{i,j} := \sigma_{ij} \circ K_{\mu^o_i}.$$

**Theorem 5.5**

(i) Given a pair \(\{i, j\} \subset I\) such that $\varepsilon_{ij} = -p\varepsilon_{ji} = p \in \{0, 1, 2, 3\}$, one has

$$K_{i,j}^{h+2} = \lambda \text{Id}, \quad |\lambda| = 1,$$

where $h = 2, 3, 4, 6$ for $p = 0, 1, 2, 3$ respectively.

(ii) Assigning to a feed $i$ the Hilbert space $L^2(A_i^+)$, and to a feed cluster transformation $c : i \to i'$ the operator $K_c$, we get a unitary projective representation of the saturated cluster modular groupoid $\tilde{G}^o$, and hence a unitary projective representation of the saturated cluster modular group $\tilde{\Gamma}_1$ in the Hilbert space $L^2(A_i^+)$.

(iii) If $\det \varepsilon_{ij} \neq 0$, we get a unitary projective representation of the cluster modular groupoid $\tilde{G}^o$, and hence a unitary projective representation of the cluster modular group $\tilde{\Gamma}_1$ in $L^2(A_i^+)$.  

**Proof.** The part (ii) follows immediately from (i). Since relations \([20]\) are valid for the quantum cluster spaces $A_q$ and $D_q$ (Theorem \(3.3a\)), the part (i) follows from Theorem \(5.4\).

If $\det \varepsilon_{ij} \neq 0$, by Lemma \(2.22\), $c^o = \text{Id}$ implies $c^x_q = \text{Id}$. Thanks to the quantum Laurent Phenomenon Theorem \([BZq]\), $c^o = \text{Id}$ implies $c^x_q = \text{Id}$. There is a canonical projection $A_q \times A_q \to D_q$, defined for every feed by the canonical $q$-deformation of the monomial map $\varphi$. It is surjective thanks to $\det \varepsilon_{ij} \neq 0$. Using this, we deduce that $c^q = \text{Id}$. So the part (iii) follows from Theorem \(5.4\).

The theorem is proved.

### 5.3 Commutation relations for the intertwiner

Recall that a mutation $\mu_k : i \to i'$ provides automorphisms $\mu^o_k$ and $\mu^x_k$ of $L$. They are tensor products of the corresponding automorphisms of $\mathbb{L}_q$ and $\mathbb{L}_q^\vee$.

**Theorem 5.6** For any $w \in W'$, $A \in L$ one has

$$K^x \hat{A}w = \mu^x_k(A)K^x w, \quad K' \hat{A}w = \mu^o_k(A)K^x w, \quad (82)$$

Thus both $\mu^x_k(A)K^x w$ and $\mu^o_k(A)K' w$ lie in $L^2(A_i^+)$.

**Proof.** The proof of the right identity in \(82\) is straightforward. Let us prove the left identity in \(82\). We do it for $A \in \mathbb{L}_q$. The proof for $A \in \mathbb{L}_q^\vee$ is completely similar, using the second difference relations for the function $\Phi^A$. So from now on $A \in \mathbb{L}_q$.

Let $\mathbb{L}_q'$ be the space of Laurent $q$-polynomials $F$ in $B_i, X_i$ such that $\mu^x_k(F)$ is again a Laurent $q$-polynomial. It is easy to check that the following elements belong to $\mathbb{L}_q'$:

\[
\begin{align*}
[1] \quad & B_i \pm^1, \ i \neq k, & [2] \quad & B_k(1 + q_k \bar{X}_k), \quad [3] \quad & (1 + q_k X_k)B_k^{-1}, \\
[4] \quad & X_k \pm^1, & [5] \quad & X_k \frac{\Psi q_k(X_k)}{\Psi q_k(q_k^{-2\varepsilon_{ik}} X_k)} \quad \text{if } \varepsilon_{ik} \geq 0, & [6] \quad & X_k^{-1} \quad \text{if } \varepsilon_{ik} \geq 0,
\end{align*}
\]

(83)

(84)
requires an assumption that the function \( \hbar \) Here we used the following facts, guaranteed by Lemma 5.7

\[ i \alpha \]

commute, they are obvious for [4]. The commutation relation for [6] and [7] are deduced formally to the one

\[ \langle \rangle \]

Therefore, we should move the integration contour \( \int \)... 

\[ \text{Im}(\cdot) \]

Looking into the pairs \((i, q)\) and \((\bar{i}, q^{-1})\), where \( \epsilon_{ij} \) = \( \epsilon_{ij} \). Finally, [3] is reduced to [2] via a similar trick using the involution \( i : D_\hbar \to D_\hbar^0 \) from Theorem 3.3. So one needs to prove relations [2], [5], [8].

[2]. Recall one of the two difference relations for the function \( \Phi^{\hbar k}(z) \), see B5 in Section 4:

\[ \Phi^{\hbar k}(z + 2\pi i\hbar) = \Phi^{\hbar k}(z)(1 + q_k e^z) \quad \langle \rangle \quad \Phi^{\hbar k}(z - 2\pi i\hbar) = \Phi^{\hbar k}(z)(1 + q_k^{-1} e^z)^{-1}. \]  

Recall the notation \( \alpha_k^\pm \), see [37], and \( B_k^2 = B_k(1 + q_k X_k)(1 + q_k \bar{X}_k)^{-1} \). Set

\[ T_{2\pi i\hbar}(c) = T_{2\pi i\hbar}^0(c) := \varphi(c + 2\pi i\hbar). \]

Here is an important point. For the shift in the imaginary direction, the usual identity

\[ \hat{A}_k \Phi^{-1}_{\alpha_k} \varphi(c) = \Phi_{\alpha_k}^{-1} T_{2\pi i\hbar} \varphi(c) \]

requires an assumption that the function \( \varphi(c) \) admits an analytic continuation to the strip \( 0 \leq \text{Im}(c) \leq 2\pi \hbar \), and decays sufficiently fast in this strip when \( |\text{Re}(c)| \to \infty \). Indeed,

\[ \Phi_{\alpha_k}^{-1} T_{2\pi i\hbar} \varphi(c) = \int_{C_0} e^{-a_k c/2\pi i\hbar} \varphi(c + 2\pi \hbar) dc = e^{a_k} \int_{C_0} e^{-a_k(c + 2\pi i\hbar)/2\pi i\hbar} \varphi(c + 2\pi \hbar) dc = e^{a_k} \int_{C_0} e^{-a_k(c/2\pi i\hbar) \varphi(c) dc = \hat{A}_k \Phi^{-1}_{\alpha_k} \varphi(c). \]

To justify the marked by \( ? \) equality, we should move the integration countour \( C_{2\pi i\hbar} := \{ c | \text{Im} c = 2\pi \hbar \} \) to the one \( C_0 := \{ c | \text{Im} c = 0 \} \). Below we use this in the situation when \( \varphi(c) \) is analytic in the strip, decays as \( e^{-c^2} \) at infinity, so one indeed can move the countour.

Using [74]-[75], we have

\[ K^2 \left( \hat{A}_k(1 + q_k \bar{X}_k)^w \right) = \]

\[ F_{\alpha_k}^{-1} \circ \Phi_{\hbar k} \left( -d_k^{-1} c - \alpha_k^+ \right) \Phi_{\hbar k} \left( -d_k^{-1} c - \alpha_k^- \right)^{-1} \circ T_{2\pi i\hbar} \circ F_{\alpha_k} \circ (1 + q_k \bar{X}_k)^w = \]

\[ F_{\alpha_k}^{-1} \circ T_{2\pi i\hbar} \circ T_{-2\pi i\hbar} \circ \Phi_{\hbar k} \left( -d_k^{-1} c - \alpha_k^+ \right) \Phi_{\hbar k} \left( -d_k^{-1} c - \alpha_k^- \right)^{-1} \left( 1 + q_k^{-1} e^{-d_k^{-1} c - \alpha_k^+} \right) \circ T_{2\pi i\hbar} \circ F_{\alpha_k} \circ \phi(c) = \]

\[ \hat{A}_k \circ F_{\alpha_k}^{-1} \circ T_{-2\pi i\hbar} \circ \Phi_{\hbar k} \left( -d_k^{-1} c - \alpha_k^+ \right) \Phi_{\hbar k} \left( -d_k^{-1} (c + 2\pi i\hbar) - \alpha_k^- \right)^{-1} \circ T_{2\pi i\hbar} \circ F_{\alpha_k} \circ \phi(c). \]

Here we used the following facts, guaranteed by \( h > 0 \) and property B7 from Section 4:

(i) The function \( \Phi_{\hbar k}(c) \) is analytic in the upper half plane.
(ii) The function $\Phi^h_k(c - 2\pi h - \alpha_k^+)^{-1}$ is analytic in the strip $0 \leq \text{Im}(c) \leq 2\pi h$ – indeed, $\Phi^h_k(c)$ is analytic in the lower half plane.

(iii) The functions $\Phi^h_k(c)^{\pm 1}$ grow at most exponentially when $|\text{Re}(c)| \to \infty$, while the function $\omega(a)$ and its Fourier transform decay as $e^{-a^2}$.

Thus we can use (87) and the fact that the function $\Phi$ is analytic in the lower half plane.

We proved the claim for the expression $[2]$.

Denote by $(a; a_i, a_k)$ the arguments of $\omega$, and by $(a; a_i, c)$ the ones of $\tilde{\omega}$. Here is the idea. The only variables participating non-trivially in the transformations are the ones $a_i$ and $a_k$ or $c$. So if we pay attention to them only, and do a formal computation, the operator $\frac{\tilde{\omega}}{\tilde{\omega}}$ acts on them as shift

$$c \mapsto c - [\varepsilon_{ik}] + 2\pi h, \quad a_i \mapsto a_i + 2\pi h.$$

To justify this formal computation, we use (87) and the fact that the function $\Phi^h_k$ is analytic in the upper half plane, just like in the proof of [2]. Finally, since $\varepsilon_{ik} h_k = -\varepsilon_{ki} h_i$ and $[\varepsilon_{ik}]_+ - [-\varepsilon_{ik}]_+ = \varepsilon_{ik}$

$$\Phi^h_k \left( -d_k^{-1} (c - [\varepsilon_{ik}] + 2\pi h) - \alpha_k^+ - [\varepsilon_{ki}] + 2\pi h \right) \equiv \Phi^h_k \left( -d_k^{-1} (c - \alpha_k^+ + \varepsilon_{ik} 2\pi h k) \right).$$

The statement is proved.

We have to compare two expressions:

$$\mathcal{F}_a \mathcal{F}_k \tilde{X}_i \mathcal{F}_k^{-1} \Phi^h_k \tilde{\omega} \quad \text{and} \quad \mathcal{F}_a \Phi^h_k \mathcal{F}_k \tilde{X}_i \frac{\Psi_k(\tilde{X}_k)}{\Psi_k(q_{ik} X_k)} \mathcal{F}_a^{-1} \tilde{\omega}.$$

The computation is very similar to the one for [8], and thus omitted. The Proposition and hence Theorem [5.6] are proved.
5.4 The Schwartz spaces $S_i$ and the intertwiners

The key property of the Schwartz space $S_i$ which we use below is the following.

**Proposition 5.8** The space $W_i$ is dense in the Schwartz space $S_i$.

One interprets Proposition 5.8 by saying that the $*$-algebra $L$ is essentially self-adjoint in $L^2(A_i^+)$. 

**Proof.** Here is the scheme of the proof.

- **Lemma 5.9** For any $w \in W_i, s \in S_i$, the convolution $s * w$ lies in $S_i$.

  Let $w_\varepsilon := (2\pi)^{-\frac{n}{2}} \varepsilon^{-\frac{n}{2}} e^{-\frac{i}{2} |x|^2} \in W_i$ be a sequence converging as $\varepsilon \to 0$ to the $\delta$-function at 0. By Lemma 5.9 $w_\varepsilon * s$ lies in $S_i$. Clearly one has in the topology of $S_i$

$$
\lim_{\varepsilon \to 0} w_\varepsilon * s = s(x). \quad (91)
$$

- **Lemma 5.10** For any $w \in W, s \in S_i$, the Riemann sums for the integral

$$
s * w(x) = \int_V s(v) \omega(x - v) dv = \int_V s(v) T_v w(x) dv \quad (92)
$$

converges in the topology of $S_i$ to the convolution $s * w$.

- For every $\varepsilon > 0$, we approximate the function $(s * w_\varepsilon)(x) \in S_i$ by the finite Riemann sums of the integral (92). Each of these Riemann sums is an element of $W$. Then, letting $\varepsilon \to 0$ and using Lemma 5.9 together with the triangle inequality, we get the Proposition.

**Proof of Lemma 5.9** Let $V$ be a finite dimensional vector space with a norm $|v|$ and the corresponding Lebesgue measure $dv$, and $v, x \in V$, set $T_v f(x) := f(x - v)$. Write

$$
s * w(x) = \int_V w(v) (T_v s)(x) dv.
$$

Below $V \cong \mathbb{R}^n \cong A_i^+$. For any seminorm $\rho_B$ on $S_i$ the operator $T_v : (S_i, \rho_B) \to (S_i, \rho_B)$ is a bounded operator with the norm bounded by $e^{|v|}$ for some $c$ depending on $B$. Thus the operator $\int_V w(v) T_v dv$ is a bounded operator on $(S_i, \rho_B)$. This implies the Lemma.

**Proof of Lemma 5.10** Let us show first that (92) is convergent in $L^2(A_i^+)$. The key fact is that a shift of $w \in W_i$ quickly becomes essentially orthogonal to $w$. More precisely, in the important for us case when $w(x) = e(-\alpha(x \cdot x) / (2 + b \cdot x))$, $\alpha > 0$, (this includes any $w \in W_0$) we have

$$
(w(x), T_v w(x)) < C_w e^{-\alpha(x \cdot x) / (2 + b \cdot v)}. \quad (93)
$$

Therefore in this case

$$
\left( \int_V s(v) T_v w(x) dv, \int_V s(v) T_v w(x) dv \right) \leq C_w \int_V \int_V e^{-\alpha(v_1 - v_2)^2 / (2 + b \cdot v_1 - v_2)} |s(v_1) s(v_2)| dv_1 dv_2 =
$$

$$
C_w \int_V e^{-\alpha(v \cdot v) / 2 + (b \cdot v)} \int_V |s(t)s(t + v)| dt dv \leq C_w \|s\|_{L^2}^2 \int_{-\infty}^{\infty} e^{-\alpha(v \cdot v) / 2 + (b \cdot v)} dv.
$$
We leave the case of an arbitrary $w$ to the reader: it is not used in the proof of the theorem. The convergence with respect to the seminorm $\|Bf\|$ is proved by the same argument. The Lemma is proved. The proof of the Proposition is finished.

**Remark.** The same arguments show that the space $W_1$ is dense in the space $S_{L_1}$ defined for any subalgebra $L'$ of $L$.

**Proof Theorem 5.4(i).** The case of $K'$ is obvious. To prove it for $K^2$ we use Theorem 5.6.

We use shorthand $K := K^2$ and $\gamma := \gamma^2$. Let us show that $K^{-1}s \in S_{L_1}$ for an $s \in S_1$. For this we need to check that for any $B \in L$ the functional $w' \mapsto (K^{-1}s, B^*w')$ is continuous on $W_{L_1}$.

Since $W_1$ is dense in $S_1$ by Proposition 5.8, there is a sequence $v_i \in W_1$ converging to $s$ in $S_1$. This means that

$$\lim_{i \to \infty} (\hat{B}v_i, w) = (\hat{B}s, w) \quad \text{for any } B \in L, w \in W_1. \quad (94)$$

In particular, let us put for the record the fact that

The sequences $v_i$ and $\gamma(B)v_i$ converge in $L^2(A_i^\ast)$ to, respectively, $s$ and $\gamma(B)s$. \quad (95)

One has

$$(K^{-1}s, \hat{B}^*w') = (s, K\hat{B}^*w') \overset{\text{Th. 5.6}}{=} (s, \gamma(\hat{B}^*)Kw') \overset{(95)}{=} \lim_{i \to \infty} (v_i, \gamma(B)Kw') \overset{\text{def}}{=} \lim_{i \to \infty} (\gamma(B)v_i, Kw') \overset{(95)}{=} (\gamma(B)s, Kw') = (K^{-1}\gamma(B)s, w').$$

Since the functional on the right is continuous, $K^{-1}s \in S_{L_1}$, and we have (80). The theorem is proved.

**Definition 5.11** The space of distributions $S_1^\ast$ is the topological dual to the Schwartz space $S_1$.

**Corollary 5.12** The operator $K_{c^o}$ gives rise to an isomorphism of topological spaces $K^o_s : S_1^\ast \to S_{L_1}$ intertwining the automorphism $\gamma_{c^o}$ of $L$.

### 5.5 Relations for the intertwiners

**Proof of part (ii) of Theorem 5.4** Step 1.

**Proposition 5.13** Assume that $\det(\varepsilon_{ij}) = 1$. The intertwiner $K_{c^o}$ corresponding to a trivial cluster transformation $c$ is proportional to the identity.

**Proof.** Let us show first that the algebra $L$ is big enough. Recall the quantum cluster algebras $\text{[BZq]}$. Thanks to the quantum Laurent Phenomenon from loc. cit., for every feed $i$ each monomial $\prod_{i \in I} A_i^{n_i}$ where $n_i \geq 0$ is a universally Laurent polynomial. Since $\det(\varepsilon_{ij}) = 1$, there are canonical isomorphisms of feed tori

$$A_i \times A_i^\ast \overset{\sim}{\longrightarrow} D_i \overset{\sim}{\longrightarrow} X_i \times X_i^\ast \quad (96)$$

as well as the corresponding quantum feed tori algebras. Thanks to this there is a cone (i.e. a semigroup) $C_+$ of full rank $\dim D$ in the group of characters $X^*(D_i)$ of the torus $D_i$ such that the quantum monomials assigned to its vectors are belong to $L$. This cone is the preimage of the square of the positive octant cone under the map $X^*(D_i) \to X^*(A_i) \times X^*(A_i^\ast)$ provided by isomorphism (96). This way we know that $L$ is big.

**Lemma 5.14** Let $A \in L$ be a monomial. Then $\hat{A}^{-1}K_{c^o}w = K_{c^o}\hat{A}^{-1}w$ for any $w \in W_1$. 41
Proof. By Theorem 5.6 we have $\hat{A}K_{w}w = K_{w}\hat{A}w$. Thus

$$K_{w}\hat{A}^{-1}\hat{A}w = K_{w}w = \hat{A}^{-1}\hat{A}K_{w}w = \hat{A}^{-1}K_{w}\hat{A}w.$$  

Since for a monomial $A$ the operator $\hat{A}$ is an automorphism of the space $W$, the lemma follows.

Corollary 5.15 For any $A \in D_1$ and $w \in W_1$ one has $K_{w}\hat{A}w = \hat{A}K_{w}w$.

Proof. Thanks to Theorem 5.6 and Lemma 5.14 we know the claim for monomials corresponding to the vectors of the cones $C_+$ and $-C_+$. The semigroup generated by these cones is the group of characters of the torus $D_1$. Finally, if $A_1, A_2 \in D_1$ commute with $K_{w}$ on $W_1$, then for any $w \in W_1$ one has $K_{w}\hat{A}_1\hat{A}_2w = \hat{A}_1\hat{A}_2K_{w}w$. Indeed, the operators $\hat{A}_i$ preserve $W_1$. The corollary is proved.

Now we can finish the proof of the theorem. Let

$$E = \{ f \in L^2(A_1^+) | \exp(\sum_i n_i a_i)f \in L^2(A_1^+) \text{ for any } n_i \geq 0 \}.$$  

We define a topology in $E$ by using the seminorms related to the operators of multiplication by $\exp(\sum_i n_i a_i)$ just as for the Schwartz spaces. The same argument as in the proof of Proposition 5.8 shows that the space $W$ is dense in $E$, see the Remark after the proof of the Proposition 5.8.

Lemma 5.16 $K_{w}(E) \subseteq E$.

Proof. The elements $B_i \in D_1$ act as the operators of multiplication by $\exp(a_i)$. Thus by Corollary 5.15 one has $K_{w}e^{na_i}w = e^{na_i}K_{w}w$ for any $n > 0, w \in W_1$. Since $W_1$ is dense in $E$, we get the Lemma.

Lemma 5.17 $K_{w}$ is the operator of multiplication by a function $F(a)$.

Proof. For a given point $a_0$, the value $(K_{w}f)(a_0)$ depends only on the value $f(a_0)$. Indeed, for any $f_0 \subseteq E$ with $f_0(a_0) = f(a_0)$ we have $f(a) = (e^{a-a_0})f_0(a) + f_0(a)$, where $\phi = (f - f_0)/(e^{a} - e^{a_0}) \in E$. Thus by Corollary 5.15 $K_{w}f = (e^{a} - e^{a_0})K_{w}\phi(a) + K_{w}f_0(a)$. So $K_{w}f(a_0) = K_{w}f_0(a_0)$. Now define $F(a_0)$ from $K_{w}f_0(a_0) = F(a_0)f_0(a_0)$. The lemma is proved.

Lemma 5.18 The function $F(a)$ is a constant.

Proof. Let us show that $F(a)$ extends to an analytic function in the product of the upper half planes $\text{Im } a_i > 0$. The operator $X_k$ acts as a shift by $2\pi i h_k$ followed by multiplication by $p^k X_k$. The latter is an exponential in the logarithmic coordinates $a_k$. By Corollary 5.15 it commutes with $K_{w}$. It follows that $K_{w}$ commutes with the operator of shift by $2\pi i h_k$ along the variable $a_i$.

Lemma 5.19 Suppose that positive powers of the shift operator $T_{2\pi i h}$, $h > 0$, commute on the subspace $W \subseteq L^2(\mathbb{R})$ with the operator of multiplication by a function $F(a)$. Then $F(a)$ is extended to an analytic function in the upper half space $\text{Im } a > 0$, and is invariant by the shift by $2\pi i h$.

Proof. Since $T^n(Fw) \in L^2(\mathbb{R})$, its Fourier transform $F(Fw)$ is in $L^2(\mathbb{R})$. Thus $e^{ax}F(Fw) \in L^2(\mathbb{R})$. Making the inverse Fourier transforms and using the Payley-Wiener argument we see that $Fw$ is analytic in the upper half plane for any $w \in W_1$. The Lemma follows.

This implies that $F(a)$ is analytic in the product of the upper half planes $\text{Im } a_k > 0$, and is invariant by the shift by $2\pi i h_k$ along the variable $a_k$. Employing the Langlands dual family of the
operators $\hat{X}_k^\epsilon$ we conclude that $F(a)$ is invariant under the shift by $2\pi i$ along the variable $a_k$. So if $h$ is irrational then $F$ is a constant. Since $K$ evidently depends continuously on $h$, $F$ is a constant for all $h$. Proposition 5.13 is proved.

**Step 2.**

**Proposition 5.20** One has

$$K_{c^0} \hat{X}_i w = \hat{X}_i K_{c^0} w, \quad K_{c^0} \hat{B}_i w = \hat{B}_i K_{c^0} w, \quad w \in W, i \in I. \quad (97)$$

**Proof.** Let us prove the first claim.

**Reduction to the case when $\det(\varepsilon_{ij}) = 1$.** Take a set $\bar{I}$ containing $I$ and a skew-symmetric function $\varpi$ on $\bar{T} \times \bar{T}$ extending the function $\varepsilon$, with $\det(\varpi) = 1$. Denote by $\bar{I} = (\bar{I}, \varepsilon_{ij}, \varpi_i)$ the obtained feed. There is a canonical projection:

$$\varpi : A_{\bar{I}} \rightarrow X_{\bar{I}}, \quad X_i \mapsto \varpi^* X_i = \prod_{j \in I} A_{\bar{I}}^{\varepsilon_{ij}}, \quad i \in I, \quad (98)$$

as well as its $q$-deformed version $\bar{\varpi}_q : A_{q,\bar{I}} \rightarrow X_{q,\bar{I}}$. Write $\bar{\varpi}_q^* X_i = \hat{\lambda}_i^+ / \hat{\lambda}_i^-$. Here $\hat{\lambda}_i^\pm$ are monomials (with non-negative exponents), so thanks to the quantum Laurent Phenomenon theorem $L_i^\pm := \varpi^{\pm}(\hat{\lambda}_i^\pm)$ are quantum Laurent polynomials. By the assumption of Theorem 5.21 we have $c^q(X_i) = X_i$. Therefore $\varpi_q(X_i) = X_i$. Thus

$$L_i^+(L_i^-)^{-1} = \varpi_q(\hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1}) = \varpi_q(\bar{\varpi}_q^* X_i) = \bar{\varpi}_q^* X_i = \hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} \implies L_i^+ = \hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} L_i^- \quad (99)$$

We interpret below $\hat{\lambda}_i^\pm$ as monomials in the quantum feed torus algebra for $D_{\bar{T}}$ via the quantum version of $[96]$. Thus they act on the space $W_{\bar{T}}$ by the operators $\hat{\lambda}_i^\pm$. Denote by $\hat{K}_{c^0}$, or simply by $\hat{K}$, the intertwiner related to the cluster transformation $c$ for the feed $\bar{I}$. Since the operator $(\hat{\lambda}_i^-)^{-1}$ is an isomorphism on $W_{\bar{T}}$, we have

$$\hat{K}_{c^0} \hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} w \overset{\text{Prop. } 5.13}{=} L_i^+ \hat{K}_{c^0} (\hat{\lambda}_i^-)^{-1} w \overset{99}{=} \hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} L_i^- \hat{K}_{c^0} (\hat{\lambda}_i^-)^{-1} w \overset{\text{Prop. } 5.13}{=} (100)$$

$$\hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} \hat{K}_{c^0} \hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} w = \hat{\lambda}_i^+(\hat{\lambda}_i^-)^{-1} \hat{K}_{c^0} w. \quad (101)$$

In particular the right hand side is in $L^2(A_{\bar{T}})$. We are going to deduce from this the left relation in $[97]$ by using the restriction to the feed $\bar{i}$. Observe that setting $A_l = 1$ for $l \notin \bar{T} - I$, the space $W_{\bar{T}}$ restricts to $W_{\bar{i}}$. There are two issues:

**Issue (i).** Extending the feed $\bar{i}$, we change the action of a mutation $\mu_k$ on $B_j$-coordinates, $j \in I$, although we do not change its action on the $X_i$-coordinates, $j \in I$.

**Issue (ii).** Extending the feed $\bar{i}$, we change the intertwiner $K_i$.

To handle them, we employ the following facts:

**Fact (i).** The $D$-cluster transformation corresponding to the mutation $\bar{\pi}_k$ of the feed $\bar{T}$ in the direction $k \in I$ preserves the functions $B_l$, $l \in \bar{T} - I$. So we can restrict it to the subvariety of the double given by the equations $B_l = 1$, $l \in \bar{T} - I$. Then restriction of the function $B_i$, $i \in I$, to this subvariety mutates the same way as under the $D$-cluster transformation corresponding to $\mu_k$.

**Fact (ii).** The natural embedding $A_i^+ \hookrightarrow A_{\bar{T}}^+$ gives rise to an injective map of linear spaces

$$i_{\bar{i} \rightarrow i} : L^2(A_i^+) \hookrightarrow \text{Dist}(A_{\bar{T}}^+), \quad f \mapsto f \prod_{l \in \bar{T} - I} \delta(a_l), \quad (102)$$
where on the right stands the dual to the classical Schwarz space on $A_1^+$. The intertwiner $\overline{K}$ provides a map of the classical Schwarz spaces: indeed, it is a composition of Fourier transforms and operators of multiplication by a smooth function of absolute value 1. Therefore it gives rise to a map of the distribution spaces. We claim that it restricts to an operator between the images of map (102), which coincides with the intertwiner $K$. In other words, there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}^2(A_1^+) & \longrightarrow & \operatorname{Dist}(A_1^+) \\
\overline{K} & \downarrow & \overline{K} \\
\mathcal{L}^2(A_1^+) & \longrightarrow & \operatorname{Dist}(A_1^+) \\
\end{array}
\]

Fact (i) is obvious. Fact (ii) is clear from the definition of the intertwiner: Restricting the differential operator $\tilde{x}_k$, $k \in I$, acting on $A_1^+$ to the distributions annihilated by operators of multiplication by $a_l$, $l \in I - I$, we get the operators $\tilde{x}_k$ acting on $A_1^+$. Thus the operator $K'$ restricts to $K^2$. The operator $\overline{K}'$ obviously restricts to $K'$. Thus (100)-(101) imply the left identity in (97).

The second claim in (97) is proved similarly. Although $\mathcal{C}_q(B_i)$ may differ from $B_i$, we run the same argument as in (100)-(101) with the following modification: we apply the operators to the distribution $i_{e^{-i}}(w) \in \operatorname{Dist}(A^-_1)$ corresponding to $w \in W_1$, and use the fact that the restriction of $\mathcal{C}_q(B_i)$ to the subspace given by the equations $B_l = 1, l \in I - I$ coincides with $B_l$. Proposition 5.20 is proved.

Theorem 5.4(ii) follows from Proposition 5.20. Indeed, since $K_{c_0}$ commutes with the operators $\tilde{B}_i$, $i \in I$, we conclude, just like in the proof of Lemma 5.17, that $K_{c_0}$ is an operator of multiplication by a function. Then, since $K_{c_0}$ commutes with the operators $\tilde{X}_i$ and $\tilde{X}_i^\vee$, arguing as in the end of the proof of Proposition 5.13 we conclude that this function is a constant. Theorem 5.4 is proved.

**A result providing assumptions of Theorem 5.4(ii).** Let $\tilde{1} := (I, I_0, \tilde{c}_{ij}, \tilde{d}_i)$ be a feed. Restricting to $I - I_0$, we get a feed $\iota := (I - I_0, c_{ij}, d_i)$. There is a canonical projection, given as a composition

\[
A_{\tilde{1}} \longrightarrow X_{\tilde{1}} \longrightarrow X_{\iota}, \quad X_i \longmapsto \prod_{j \in I} A_j^{c_{ij}}, i \in I - I_0.
\] (103)

The precursor of the following result is Lemma 2.22.

**Proposition 5.21** Assume that a feed cluster transformation $c : \iota \rightarrow \iota$, considered as a cluster transformation $\overline{c} : \tilde{1} \rightarrow \tilde{1}$, is trivial. Assume that the composition (103) is surjective. Then the induced by $c$ cluster maps $c_q^+$ and $c_q^d$ of the $q$-deformed spaces $X_{q,[\iota]}$ and $D_{q,[\iota]}$ are trivial.

**Proof.** Since $\overline{c}$ is trivial, by [BZ1], the cluster transformation $c_q^d$ of the $q$-deformed cluster algebra related to the feed $\tilde{1}$ is trivial. Recall the surjective monomial map $\overline{\rho} : A_{q,[\iota]} \longrightarrow X_{q,[\iota]}$. It commutes with mutations (Appendix to Section 2). This implies the proposition for the $X$-space.

**The $D$-space.** A feed torus $D_{\tilde{1}}$ has cluster coordinates $(B_i, X_i)$, $i \in I$. Consider a torus $D_{\tilde{1}}^X$ with coordinates $(B_i, X_j)$, where $i \in I, j \in I - I_0$. There is a canonical projection $\pi_I : D_{\tilde{1}}^X \longrightarrow D_{\tilde{1}}$, given by dropping the $\mathbb{G}_m$-components of $D_{\tilde{1}}$ with the coordinates $X_j$, $j \in I - I_0$. Mutations in the directions of the set $I - I_0$ of the $X$-coordinates of a $D$-space parametrised by $I - I_0$ do not involve the $X$-coordinates parametrised by $I_0$. Thus, using the same formulas as in the definition of the $D$-space, we define a new positive space $D_{\tilde{1}}^X$. There is a canonical projection $\pi : D_{\tilde{1}}^X \longrightarrow D_{\iota}^X$, and a unique Poisson structure on $D_{\tilde{1}}^X$ for which the map $\pi$ is Poisson. It is given by the same formulas as for $D_{\iota}^X$. Similarly there is a $q$-deformation $D_{q,[\iota]}^X$ of the space $D_{\tilde{1}}^X$, as well as of the projection $\pi$. 

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There is a projection
\[ \tilde{\varphi} : \mathcal{A}_{q,[i]} \times \mathcal{A}_{q,[i]}^0 \longrightarrow D_{q,[i]}^\times, \quad \tilde{\varphi} := \pi \circ \varphi. \] (104)

**Lemma 5.22** Assume that the rectangular matrix \( \varepsilon_{ij} \), where \( i \in I, j \in I - I_0 \), is non-degenerate. For any \( q \)-deformation \( A_{q,[i]} \) of the \( A \)-space as in [BZq], there is a map of quantum spaces
\[ \tilde{\varphi}^q : A_{q,[i]} \times A_{q,[i]}^0 \longrightarrow D_{q,[i]}^\times. \]

**Proof.** The map \( \tilde{\varphi}^q \) is a monomial map, which for the feed tori is determined by its \( q = 1 \) counterpart \( \tilde{\varphi}_1^q \). One checks that it commutes with the mutations, see also [Ki ng].

**Lemma 5.23** If the matrix \( \varepsilon_{ij} \), \( i \in I, j \in I - I_0 \), is non-degenerate, then the map \( (104) \) is surjective.

**Proof.** Assume that \( \tilde{\varphi}^* \prod_{i,j} X_i^j B_j^h = 1 \). Then \( b_j = 0 \) since \( \tilde{\varphi}^* B_j = A_j^0 / A_j \), and \( \tilde{\varphi}^* X_j \) does not involve \( A_j^0 \)'s. Thus the assumption of the lemma implies \( x_i = 0 \). The lemma is proved.

There is an embedding \( D_{q,[i]} \hookrightarrow D_{q,[i]}^\times \) given in any cluster coordinate system by setting \( B_i = 1 \) for \( i \in I - I_0 \). Now \( \tilde{\varphi} \equiv \text{Id} \) implies \( \tilde{\varphi}^q = \text{Id} \) by [BZq], the latter implies by Lemma 5.23 that the cluster transformation \( \tilde{c}_q^d \) of the space \( D_{q,[i]}^\times \) is the identity. Thus the restriction \( \tilde{c}_q^d \) to \( D_{q,[i]} \) is the identity map. The proposition is proved.

### 5.6 A proof of Theorem 1.1

**Canonical representation.** Let \( \Lambda \) be a lattice with a symplectic form \((*,*)\) with values in \( \frac{1}{\sqrt{N}} \mathbb{Z} \). Denote by \( T_\Lambda \) the corresponding quantum torus \(*\)-algebra with the parameter \( q = e^{\pi i h} \), and by \( T_\Lambda^\vee \) the algebra with the dual parameter \( q^\vee = e^{\pi i / h} \). So \( T_\Lambda \otimes T_\Lambda^\vee \) is the modular double. Set \( \Lambda_Q := \Lambda \otimes \mathbb{Q} \). Choose a decomposition \( \Lambda_Q = P \oplus \mathbb{Q} \) into a direct sum of two Lagrangian subspaces. Set \( P_R := P \otimes \mathbb{R} \). One defines, similarly to (79), a space \( W(P_R) \) with a canonical representation of the \(*\)-algebra \( T_\Lambda \otimes T_\Lambda^\vee \); elements \( p \in P_Q \) and \( q \in Q_Q \) provide operators
\[ \tilde{p} \longmapsto T_{2\pi i h p}, \quad \tilde{q} \longmapsto e^{(q,s)}, \quad \tilde{p}^* \longmapsto T_{2\pi i p}, \quad \tilde{q}^* \longmapsto e^{(q/h,s)}. \]

Here \( T_\delta f(p) := f(p - a) \), and \( e^{(q,s)} \) is the operator of multiplication by the function \( e^{(q,p)} \). The Schwartz space \( \mathcal{S}_{P,Q} \) is the maximal domain of the \(*\)-algebra \( T_\Lambda \otimes T_\Lambda^\vee \) in \( L^2(P_R) \). It has a natural Fréchet topology; \( W(P_R) \) is dense in this topology – the proof is the same as of Proposition 5.8.

It is easy to see that, given another decomposition \( \Lambda_Q := P' \oplus Q' \), there is a canonical up to a constant unitary operator \( \mathcal{S}_{P,Q} \rightarrow \mathcal{S}_{P',Q'} \) intertwining the action of the algebra \( T_\Lambda \otimes T_\Lambda^\vee \). Its kernel is the exponential of an imaginary quadratic expression of the coordinates. If \( h \notin \mathbb{Q} \), it is determined uniquely by the difference equations expressing the commutation relations. Therefore the representation of the \(*\)-algebra \( T_\Lambda \otimes T_\Lambda^\vee \) is defined up to a canonical modulo constant unitary isomorphism. Having in mind this, we can say that there is a canonical representation \((V,S_V)\) of \( T_\Lambda \otimes T_\Lambda^\vee \), where \( V \) is a Hilbert space, and \( S_V \) its subspace where the modular double \( T_\Lambda \otimes T_\Lambda^\vee \) acts.

If the form \((*,*)\) on \( \Lambda \) is degenerate, denote by \( Z \subset \Lambda \) its kernel. A model for the canonical representation is defined then by choosing a pair of subspaces \( P,Q \) containing \( Z \), which project to a pair of transversal Lagrangian subspaces in \( \Lambda_Q / Z_Q \). One can decompose it further via the characters \( \lambda \) of \( Z_R \) using the Fourier transform, getting a family of representations \( V_\lambda \) and a canonical decomposition into an integral of Hilbert spaces \( V = \int V_\lambda d\lambda \).

The representation of \( T_{\Lambda_D} \otimes T_{\Lambda_D^\vee} \) in \( L^2(A^+) \) from Section 5.1 is the canonical representation. Take the sublattice \( \pi^*(\Lambda_X \oplus \Lambda_X^{opp}) \) of \( \Lambda_D \). The following lemma is an easy linear algebra exercise.
Lemma 5.24 The canonical representation of \( T_{\pi^*(\Lambda_X \oplus \Lambda_X^\op)} \otimes T_{\pi^*(\Lambda_X \oplus \Lambda_X^\op)}^\vee \) is identified with restriction of the canonical representation of \( T_{\Lambda_D} \otimes T_{\Lambda_D}^\vee \).

The group \( H_A(\mathbb{R}_{>0}) \) acts on \( \mathcal{A}^+ \) and hence on \( L^2(\mathcal{A}^+) \). On the other hand, let \( Z \) be the kernel of the restriction of the symplectic form \((*,*)_D\) to the lattice \( \pi^*(\Lambda_X \oplus \Lambda_X^\op) \otimes \mathbb{Q} \). In realization (78) it gives rise to an action of the real torus with the group of cocharacters \( Z \otimes \mathbb{R} \). Indeed, given a cluster basis \( \{e_i\} \) of \( \Lambda_X \), the sublattice \( Z \) consists of the vectors \( (\sum c_i e_i, \sum c_i e_i) \) where \( \sum c_i e_i = 0 \) for every \( j \in I \). So in realization (78) the corresponding operator in the representation is just \( \sum_i c_i \partial/\partial e_i \) - the linear term is gone thanks to the condition on \( c_i \)'s. These two real tori are canonically isomorphic, and the two actions coincide.

Therefore the Fourier decomposition of \( L^2(\mathcal{A}^+) \) according to the characters of \( H_A(\mathbb{R}_{>0}) \) is the same as the Fourier decomposition according to the center of \( T_{\pi^*(\Lambda_X \oplus \Lambda_X^\op)} \otimes T_{\pi^*(\Lambda_X \oplus \Lambda_X^\op)}^\vee \). Thanks to theorem 5.4 these two natural decompositions are respected by the intertwiners. Finally, a module over \( T_{\pi^*(\Lambda_X \oplus \Lambda_X^\op)} \otimes T_{\pi^*(\Lambda_X \oplus \Lambda_X^\op)}^\vee \) with the central character \((\lambda,\lambda)\) is identified with the representation \( \text{End}V^\lambda_A \). The theorem is proved.

5.7 The intertwiner and the quantum double

Our goal is to show how the intertwiner \( K_{V,i} \) determines the coordinate transformations which we used to define the quantum double. The idea is similar to the \( D \)-module approach to integral geometry from [Go1], used here for the difference equations.

Recall the (non-commutative) algebra \( D_i \) of regular functions on the quantum double feed torus for the feed \( i \). We realize it as the algebra of all \( q \)-difference operators on the torus \( \mathcal{A}_i \), see (70). Its fraction field \( D_i \) is a non-commutative field.

The ring \( D_i \otimes D_i^\op \) as a bimodule: a left \( D_i \)-module and a right \( D_i \)-module. Denote by \( \mathcal{R}_{i,i'} \subset D_i \otimes D_i^\op \) be the ideal for this bimodule structure generated by all \( q \)-difference equations satisfied by the generalized function \( K_{V,i}(a,a') \) representing the kernel of the intertwiner \( K_{V,i} \). Consider the corresponding bimodule

\[
M_{i,i'} := \frac{D_i \otimes D_i^\op}{\mathcal{R}_{i,i'}}.
\]

Denote by \( M_{i,i'} \) its localization. It is a module over the tensor product \( D_i \otimes D_i^\op \).

Proposition 5.25 The restriction of the bimodule \( M_{i,i'} \) to each of the factors in \( D_i \otimes D_i^\op \) is a free rank one module with a canonical generator \( K = K_{V,i} \).

Proof. This is equivalent to the similar property for the kernel \( K^\sharp \).

Corollary 5.26 There is a homomorphism of fields \( \kappa: D_i \rightarrow D_i \), \( AK = \kappa(A) \).

This non-commutative field homomorphism is the coordinate transformation employed in the definition of the quantum double.

6 Quantization of higher Teichmüller spaces

6.1 Principal series representations of quantum moduli spaces \( \mathcal{X}_{G,S} \)

In this Section we denote by \( S \) an oriented surface with holes and a finite collection of marked points on the boundary, considered modulo isotopy. We reserve the notation \( S \) to the surface with the
marked points omitted. This differs from the notation we used in [FG1] and the Introduction, where the surface with marked points was denoted by $\widetilde{S}$.

In Chapter 10 of [FG1] we defined a cluster ensemble structure for the pair of moduli spaces $(\mathcal{A}_{G,S}, \mathcal{X}_{G,S})$ where $G = \text{SL}_m$. Namely, let us shrink all holes without marked points at the boundary to punctures. Let us pick a point, called a special point, in each connected component of $\{\text{the boundary of } \widetilde{S} \text{ (minus punctures)} \} - \{\text{the marked points}\}$.

An ideal triangulation $T$ of $S$ (loc. cit.) is a triangulation of $\widetilde{S}$ with vertices at the special points and punctures. An ideal triangulation $T$ produces a feed $i_T$. To each edge $E$ of $T$ we assign a flip of the triangulation, producing a new ideal triangulation $T'$. The feed $i_T'$ is obtained from $i_T$ by a sequence of $(m - 1)^2$ mutations, described in loc. cit.. Their composition is a feed cluster transformation $c_E : i_T \to i_T'$. The classical modular groupoid $\mathbb{G}_S$ of $S$ is a groupoid whose objects are ideal triangulations of $S$ up to an isomorphism. Its morphisms are generated by the flips. The relations are generated by “rectangles” and “pentagons”. The former relation tells that flips at disjoint edges commute. The pentagon relation tells that the composition of five flips of the diagonal of an ideal pentagon, presented on Fig. 1, is an automorphism.

Figure 1: The composition of five flips is the identity.

The cluster structure of the pair of moduli spaces $(\mathcal{A}_{G,S}, \mathcal{X}_{G,S})$ provides a saturated cluster modular group $\hat{\Gamma}_{G,S}$, and a cluster modular group $\Gamma_{G,S}$. Let $\hat{\Gamma}_{G,S} \to \Gamma_{G,S}$ be the canonical epimorphism, We conjecture that it is an isomorphism. Since a flip is decomposed into a composition of mutations, there is a canonical embedding $\Gamma_S \hookrightarrow \Gamma_{G,S}$ (loc. cit.). The group $\hat{\Gamma}_S$ is the preimage of $\Gamma_S$ in $\hat{\Gamma}_{G,S}$. There are similar maps of modular groupoids. So we are getting the diagrams

$$
\begin{array}{c}
\hat{\Gamma}_S \hookrightarrow \hat{\Gamma}_{G,S} \\
\downarrow \quad \quad \downarrow \\
\Gamma_S \hookrightarrow \Gamma_{G,S}
\end{array}
\quad
\begin{array}{c}
\hat{\mathbb{G}}_S \hookrightarrow \hat{\mathbb{G}}_{G,S} \\
\downarrow \quad \quad \downarrow \\
\mathbb{G}_S \hookrightarrow \mathbb{G}_{G,S}
\end{array}
$$

Applying the main construction of this paper to the cluster ensemble assigned to the pair $(\mathcal{A}_{G,S}, \mathcal{X}_{G,S})$, we get a unitary projective representation of the modular groupoid $\hat{\mathbb{G}}_{G,S}$, and thus a representation of the groupoid $\mathbb{G}_S$. Our goal is to show that the latter descends to representations of the classical modular groupoid $\mathbb{G}_S$. This is equivalent to the part (iii) of following:

**Theorem 6.1** Let $S$ be an open surface, and $G = \text{SL}_m$. Then

(i) The modular group $\Gamma_S$ acts by automorphisms of the $q$-deformed spaces $\mathcal{X}_{G,S}^q$ and $\mathcal{D}_{G,S}^q$.

(ii) The composition of the intertwiners corresponding to the pentagon relation in the classical modular groupoid is a multiple of the identity transformation.

(iii) The group $\Gamma_S$ acts in the canonical unitary projective representation related to the pair of moduli spaces $(\mathcal{A}_{G,S}, \mathcal{X}_{G,S})$.

**Proof.** Theorem 5.4 plus (i) implies (ii). Clearly (ii) implies (iii).

(i) Recall the moduli space $\text{Conf}_m \mathcal{B}$ (respectively $\text{Conf}_m \mathcal{A}$) of configurations of $m$ flags (respectively affine flags) in $G$. There is a canonical surjective projection

$$
p : \text{Conf}_m \mathcal{A} \longrightarrow \text{Conf}_m \mathcal{B}.
$$
We have to prove the quantum pentagon relation, asserting that the cluster transformation \( c_P \), corresponding to a pentagon \( P \) of an ideal triangulation of \( S \) is the identity map for the quantum spaces \( X_{G,S}^q \) and \( D_{G,S}^q \). The map \( c_P \) is the composition of quantum mutations corresponding to the sequence of 5 flips of the pentagon \( P \). Since we work with a single ideal pentagon, it is enough to handle the case of the quantum configuration space \( \text{Conf}_5^q(B) \) and the corresponding \( D \)-space.

Let \( \tilde{i} := (I, I_0, \varepsilon_{ij}) \) be a feed corresponding to a triangulated pentagon which describes the cluster structure of the moduli space \( \text{Conf}_5(A) \). The set \( I \) (respectively the frozen set \( I_0 \)) corresponds to the vertices of the \( m \)-triangulation assigned to the vertices (respectively external vertices) of the pentagon. The case \( m = 3 \) is presented on Fig. 2. Denote by \( i \) the restriction of the feed \( \tilde{i} \) to \( I - I_0 \). Then the configuration space \( \text{Conf}_5^q(B) \) (respectively \( \text{Conf}_5^q(A) \)) has a cluster \( X \)- (respectively Figure 2: A feed for a configuration of 5 affine flags in \( SL_3 \).)

\( A \)-) variety structure described by the feed \( i \) (respectively \( \tilde{i} \)). The cluster structure of projection \( (105) \) is described as a composition \( (103) \).

The feed cluster transformation \( c_P : i \rightarrow \tilde{i} \) satisfies conditions of Proposition \( 5.21 \). Indeed, the corresponding cluster transformation \( \tilde{c}_P \) of the cluster variety \( A_{[i]} \) is trivial, since it is obviously so for the corresponding moduli space \( \text{Conf}_5(A) \), and projection \( (105) \) is surjective. The theorem is proved.

6.2 The modular functor conjecture

Let us cut a surface \( S \) along a simple loop \( \gamma \), getting a surface \( S' \). It has two new boundary components \( \gamma_{\pm} \). So a principal series representation assigned to \( S \) is parametrised by a central character \( \lambda \), a one assigned to \( S' \) is parametrised by a triple \( (\lambda, \chi, \chi') \) where \( \chi_{\pm} \) are character of the torus \( H(\mathbb{R}_{>0}) \). The Weyl group \( W \) acts on the latter characters, and the dominant ones form a set of representatives.

**Conjecture 6.2** Given a surface \( S \) with holes, a principal series representation \( V_{G,S,\lambda} \) with a central character \( \lambda \) has a natural \( \Gamma_{G,S'} \)-equivariant decomposition into an integral of Hilbert spaces

\[
V_{G,S,\lambda} \xrightarrow{\sim} \int_{\chi} V_{G,S',\lambda,\chi,\chi^{-1}} d\mu_{\chi}
\]

(106)

where \( V_{G,S',\lambda,\chi,\chi^{-1}} \) is the principal series representation assigned to the surface \( \tilde{S}' \) and the central character \( (\lambda, \chi, \chi^{-1}) \), where \( \chi \) is a dominant character of the group \( H(\mathbb{R}_{>0}) \).

Let us put an extra data on the Hilbert spaces which rigidifies isomorphism \( (106) \). Denote by \( L_S \) and \( L_{S'} \) the \( L \)-algebras for the surfaces \( S \) and \( S' \). For every boundary component \( \alpha \) of \( S \), the algebra \( L_S \) has a central subalgebra \( L_\alpha \) identified with the algebra of regular functions on the Cartan group \( H \). Thus the algebra \( L_{S'} \) contains a subalgebra \( \bar{L}_{S'} \) given by the condition “the monodromies around the boundary components \( \gamma_- \) and \( \gamma_+ \) are opposite to each other”, i.e. by the (anti)diagonal subalgebra in \( L_{\gamma_-} \otimes L_{\gamma_+} \). There should be (see the comments below) a \( \Gamma_{S'} \)-equivariant embedding

\[
i_{S',S} : \bar{L}_{S'} \hookrightarrow L_S.
\]

The word “natural” in the formulation of Conjecture 6.2 means that
For (almost every) character $\chi$, map (106) leads to a map of the corresponding Schwartz spaces

$$\alpha_\chi : S(V_S) \rightarrow S(V_{S',\chi}),$$

commuting with the action of the algebra $\tilde{L}_{S'}$ – the latter acts on the left via the map $i_{S',S}$.

Comments. The algebra generated by the monodromies $M_\alpha$ along loops $\alpha$ on $S$ generate the algebra of regular functions on the moduli space $L_{G,S}$ of $G$-local systems on $S$. The monodromy lies in $G/\text{Ad}G$, so any $W$-invariant function on the Cartan group gives rise to a regular function on $L_{G,S}$.

It was proved in [FG1] that the monodromy $M_\alpha$ provides regular functions on the cluster $\mathcal{X}$-variety $\mathcal{X}_{G,S}$. Let us assume a similar claim for the $q$-deformed space. Then the algebra $L_S$ is of the “right size”, and evidently contains $\tilde{L}_{S'}$ as a subalgebra: the latter is generated by the quantum monodromies along the loops on $S$ which do not intersect the loop $\gamma$.

If two loops on $S$ do not intersect, the corresponding monodromy functions on $L_{G,S}$ commute for the Poisson bracket. Let us assume the corresponding quantum statement. Then the subalgebra $\tilde{L}_{S'}$ centralizes the one $L_\gamma$ generated by the monodromy along the loop $\gamma$. Thus the isomorphism (106) is the spectral decomposition for the algebra $L_\gamma$, understood as a $\Gamma_{S'}$-equivariant $\tilde{L}_{S'}$-module.

Finally, to justify the name “modular functor” one should add to Conjecture 6.2 relations between the spaces $V_{G,S,\lambda}$ associated to surfaces $S$ with varying number of holes which correspond geometrically to closing a hole (i.e. filling the hole by a disc by gluing a disc to the boundary component of $S$ corresponding to the hole), sometimes called ‘‘propagation of vacua” in the context of the conformal field theory.

Representation of the modular groupoid as a combinatorial connection. A rational conformal field theory provides a bundle with a flat connections on the moduli space $\mathcal{M}_{g,n}$ – the bundle of holomorphic conformal blocks.

In our case we get a projective unitary representation of the cluster modular groupoid $G_{G,S}$. Let us discuss first the case of $G = \text{PGL}_2$. Then the vertices of the modular groupoid $G_S$ are ideal triangulations $T$ of $S$. Our construction assigns to $T$ a Hilbert vector space $V_T$. The edges of $G_S$ correspond to flips $T \rightarrow T'$. We assign to them unitary maps $V_T \rightarrow V_{T'}$, and interpret this as a connection on the polyhedral complex $G_S$. The pentagon equation just means that this connection is flat. In general we get a flat connection on the polyhedral complex $G_{G,S}$.

The universal cover of the polyhedral complex $G_{G,S}$ can be viewed as sitting at the projectivisation of the Thurston boundary of the higher Teichmüller space $\mathcal{X}_{G,S}(\mathbb{R}_+)$). Indeed, assume first that $G = \text{PGL}_2$. Then an ideal triangulation $T$ of $S$ provides an integral $X$-lamination in the terminology of [FG1], Section 12, by assigning the weight 1 to every edge of $T$. Flips provide paths connecting these boundary points, etc. In general a feed $i$ provides a point $\mathcal{X}_{G,S}(\mathbb{R}_+)$ with all the tropical coordinates equal to 1.

It would be interesting to get this connection as a limit of a flat $\Gamma_{G,S}$-equivariant connection on the higher Teichmüller space $\mathcal{X}_{G,S}(\mathbb{R}_+)$.

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