SMOOTHLY NON-ISOTOPIC LAGRANGIAN DISK FILLINGS OF LEGENDRIAN KNOTS

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Abstract. In this paper, we construct the first families of distinct Lagrangian ribbon disks in the standard symplectic 4-ball which have the same boundary Legendrian knots, and are not smoothly isotopic or have non-homeomorphic exteriors.

1. Introduction

Given a ribbon knot $K$ in $S^3$, it may bound distinct ribbon disks. For instance, in [1], Akbulut constructed a pair of ribbon disks which have the same boundary knots and diffeomorphic exteriors, and are not smoothly isotopic relative to the boundary.

The standard contact 3-sphere $(S^3, \xi_{st})$ has a canonical symplectic filling $(B^4, \omega_{st})$, the standard symplectic 4-ball. Given a Legendrian knot $L$ in $(S^3, \xi_{st})$, if there is a Lagrangian surface $S$ in $(B^4, \omega_{st})$ whose boundary is $L$, then we say that $S$ is a Lagrangian filling of $L$. There are some restrictions for a Legendrian knot admitting a Lagrangian filling. For instance, the sum of the Thurston-Bennequin invariant of a Lagrangian fillable Legendendrian knot and the Euler characteristic of its bounding Lagrangian surface is zero. See for example [6], [8] and [10]. If a Legendrian knot $L$ is Lagrangian fillable, then one can naturally ask how many distinct Lagrangian surfaces filling $L$. People have found examples of Legendrian knots each of which bounds distinct exact Lagrangian surfaces up to Hamiltonian isotopy [12], [15], [11]. In particular, in [11] Section 3], Ekholm found two Lagrangian disks in $(B^4, \omega_{st})$ filled by a Legendrian knot $\tilde{9}_{46}$ which are not Hamiltonian isotopic to each other. On the other hand, Auroux exhibited a Legendrian knot in the boundary of a Stein domain other than $(B^4, \omega_{st})$ which fills two different Lagrangian disks [4, Corollary 3.4]. Those two disks are distinguished by the first homology group of their exteriors.

In this paper, we construct the first families of distinct Lagrangian disks in $(B^4, \omega_{st})$ which fill the same Legendrian knots in $(S^3, \xi_{st})$, and are not smoothly isotopic or have non-homeomorphic exteriors.

Theorem 1.1. There exist Legendrian knots in $(S^3, \xi_{st})$ filling two Lagrangian disks which are not smoothly isotopic relative to the boundary, and have diffeomorphic exteriors.

Our first example is a Legendrian knot $\tilde{9}_{46}$ shown in Figure 1. It bounds two Lagrangian disks which are not smoothly isotopic relative to the boundary. Their exteriors are diffeomorphic. These two Lagrangian disks are exactly the same as those in [11]. So Theorem 1.1 implies Ekholm’s result. Our second example is a Legendrian knot $\tilde{9}_{36} \# \tilde{9}_{46}$ shown in Figure 2. It bounds four Lagrangian disks which
are pairwisely not smoothly isotopic relative to the boundary. In particular, some pairs of these four Lagrangian disks have non-homeomorphic exteriors in $B^4$.

**Theorem 1.2.** There exists a Legendrian knot in $(S^3, \xi_{st})$ filling two Lagrangian disks whose exteriors are not homeomorphic.

In [7], Conway, Etnyre and Tosun proved that the contact (+1)-surgery along a Legendrian knot $L$ in $(S^3, \xi_{st})$ is strongly symplectically fillable if and only if $L$ bounds a Lagrangian disk $D$ in $(B^4, \omega_{st})$, and the symplectic filling of the contact (+1)-surgery along $L$ can be constructed by removing a neighborhood of $D$ from $B^4$. They asked the following question.

**Question 1.3 ([7]).** Let $L$ be a Legendrian knot in $(S^3, \xi_{st})$ with two (or more) distinct Lagrangian disk fillings in $(B^4, \omega_{st})$. Does contact (+1)-surgery on $L$ have more than one Stein (or symplectic) filling up to symplectomorphism?

Theorem 1.2 give an evidence for Question 1.3.

**Corollary 1.4.** Let $L$ be the Legendrian knot $9_{46} \# 9_{46}$ depicted in Figure 2. Then the contact (+1)-surgery on $L$ has two Stein fillings up to homeomorphism.

![Figure 1. A Legendrian knot of knot type $9_{46}$.](image1)

![Figure 2. A Legendrian knot of knot type $9_{46} \# 9_{46}$. It is a Legendrian connected sum of $L$ and itself.](image2)

Taking connected sums of multiple copies of Legendrian $9_{46}$ in Figure 1, we can find arbitrarily many distinct smooth isotopy classes of Lagrangian disks that fill a Legendrian knot.

**Theorem 1.5.** For any positive integer $N$ there exists a Legendrian knot $L$ in $(S^3, \xi_{st})$ such that the number of smooth isotopy classes of Lagrangian disks that fill $L$ in $(B^4, \omega_{st})$ relative to the boundary is greater than $N$.

Here we raise a conjecture.
Conjecture 1.6. Let \( L \) be a Legendrian ribbon knot in \((S^3, \xi_{st})\). Then \( L \) bounds only finitely many Lagrangian isotopy types of Lagrangian ribbon disks in \( B^4 \).

Notice that it is not known whether this conjecture is true or not even in the case of smooth ribbon disks.

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2. Preliminaries

2.1. A smooth knot \( L \) in a contact 3-manifold \((M, \xi)\) is Legendrian if \( L \) is everywhere tangent to \( \xi \). A properly embedded smooth surface \( S \) in a symplectic 4-manifold \((W, \omega)\) is Lagrangian if \( i^* \omega = 0 \), where \( i : S \to W \) is the inclusion map. In this paper, we consider Lagrangian surfaces in \((B^4, \omega_{st})\) which is bounded by a Legendrian knot in \((S^3, \xi_{st})\). The Legendrian unknot with Thurston-Bennequin invariant \(-1\) bounds a Lagrangian disk. In [12], Ekholm, Honda and Kálmán proved that a sequence of moves of three types below gives a Lagrangian surface in \((B^4, \omega_{st})\) whose boundary is a Legendrian link in \((S^3, \xi_{st})\).

- Adding a maximal Thurston-Bennequin unknot,
- Pinch move,
- Legendrian isotopy.

\[
\begin{array}{c}
\text{Legendrian} \\
\text{link}
\end{array}
\quad \xrightarrow{\text{Legendrian}} \\
\begin{array}{c}
\text{Legendrian} \\
\text{link}
\end{array}
\]

Figure 3. The top arrow is an adding a maximal Thurston-Bennequin unknot. The bottom arrow is a pinch move.

Hence, if a Legendrian knot can be changed into a maximal Thurston-Bennequin Legendrian unlink by a sequence of the (inverse) pinch moves and Legendrian isotopies, then the moves give a Lagrangian surface in \((B^4, \omega_{st})\). Such Lagrangian surfaces are said to be decomposable [2].

We use the notation \( \simeq \) to represent that the two surfaces in \( B^4 \) are smoothly isotopic relative to boundary.

2.2. Let \( L \) be a Legendrian knot in a contact 3-manifold \((Y, \xi)\). Perform \( \frac{p}{q} \)-surgery with respect to the contact framing, \( pq \neq 0 \), we obtain a closed 3-manifold, and denoted it by \( Y_{\xi}^L \). Let \( \nu(L) \) be the standard tubular neighborhood of \( L \). Extend the contact structure \( \xi \) on \( Y \setminus \nu(L) \) to \( Y_{\xi}^L \) by a tight contact structure on the new glued-up solid torus, we obtain a contact structure on \( Y_{\xi}^L \), and denote it by
This operation is called a contact $\xi_q$-surgery. In [4], Ding and Geiges proved that every closed contact 3-manifold can be obtained by contact $(\pm 1)$-surgery along a Legendrian link in $(S^3, \xi_{st})$. So the contact $(\pm 1)$-surgery plays a fundamental role in the constructions of contact 3-manifolds. The contact $(-1)$-surgery, a.k.a Legendrian surgery, along a Legendrian knot in $(S^3, \xi_{st})$ yields a Stein fillable contact 3-manifold. However, the contact $(+1)$-surgery does not yield a symplectically fillable contact 3-manifold in general. Conway, Etnyre and Tosun gave a necessary and sufficient condition for a contact $(+1)$-surgery to be strongly symplectically fillable.

**Theorem 2.1** ([7]). Let $L$ be a Legendrian knot in $(S^3, \xi_{st})$. Then contact $(+1)$-surgery on $L$ is strongly symplectically fillable if and only if $L$ bounds a Lagrangian disk in $(B^4, \omega_{st})$. In particular, if $L$ bounds a decomposable Lagrangian disk in $(B^4, \omega_{st})$, then the filling can be taken to be Stein.

2.3. We recall a topological characterization of Stein 4-manifolds given by Eliashberg and Gompf.

**Theorem 2.2** ([13]). A smooth, compact, connected, oriented 4-manifold $X$ admits a Stein structure (inducing the given orientation) if and only if it can be presented as a handlebody by attaching 2-handles to a framed link in $\partial(D^3 \cup 1$-handles) $= \sharp m S^1 \times S^2$, where the link is drawn in a Legendrian standard form and the framing coefficient on each link component $K$ is given by $tb(K) - 1$.

Any Stein 4-manifold has a special property about smooth embedded surfaces by using the adjunction inequality of the Seiberg-Witten theory. Here, we state the following result.

**Theorem 2.3** ([3]). If $X$ is a 4-dimensional Stein manifold and $\Sigma \subset X$ is a closed, connected embedded surface of genus $g$ in it, then

$$|\Sigma|^2 + |\langle c_1(X), [\Sigma] \rangle| \leq 2g(\Sigma) - 2,$$

unless $\Sigma$ is a sphere with $[\Sigma] = 0$ in $H_2(X, \mathbb{Z})$.

This theorem implies that there is no smoothly embedded square $-1$ sphere in any Stein 4-manifold.

3. **Non-isotopic Lagrangian disks**

The Legendrian knot of knot type $9_46$ in Figure 1 has two Lagrangian disk fillings as shown in Figure 4. Let $D_1$ and $D_2$ denote the two Lagrangian disks. In [11], Ekholm shows that $D_1$ and $D_2$ are not Hamiltonian isotopic to each other using a DGA. Our following result follows his result.

**Proposition 3.1.** $D_1 \not\cong D_2$. Their exteriors are diffeomorphic.

**Proof.** Let $W_1$ and $W_2$ denote the exteriors of $D_1$ and $D_2$ respectively. By symmetry, $W_1$ is diffeomorphic to $W_2$.

According to [2] Section 1.4, $W_1$ and $W_2$ have handle decompositions as shown in the top left and the bottom left of Figure 5 respectively. By assumption, $\partial D_1$ and $\partial D_2$ are the same knot in $S^3$, so their exteriors in $S^3$ coincide pointwisely. Both $\partial W_1$ and $\partial W_2$ can be seen as the 0-surgery along $\partial D_1 = \partial D_2$. Extending the identity of the knot exteriors to the 0-surgeries, we get an identity map $Id : \partial W_1 \to \partial W_2$.

Suppose that $D_1 \simeq D_2$. Then the map $Id : \partial W_1 \to \partial W_2$ can be extended to a diffeomorphism $f : W_1 \to W_2$. We attach a $(-1)$-framed 2-handle $h$ on $W_1$ as
in the top second diagram in Figure 5. Then $W_1 \cup h$ is diffeomorphic to $W_2 \cup h$ via $f$. By handle sliding, we obtain the top third diagram. We replace the handle decomposition of $W_1$ in the top third diagram by that of $W_2$, and obtain the bottom second diagram in Figure 5 which represents $W_2 \cup h$.

Figure 5. The top first diagram is $W_1$. The top second diagram is $W_1 \cup h$. The $(-1)$-framed unknots stand for the attached 2-handles $h$. The top horizontal arrow is a diffeomorphism realized by a handle sliding. The bottom first diagram is $W_2$. The bottom second diagram is $W_2 \cup h$. The bottom horizontal arrow is a cancellation of 1/2-canceling pair. The bottom last diagram is a Stein handlebody. The coefficients $-1$ in the last diagram, and all other Stein handlebody diagrams in this paper, stand for Legendrian surgery.

Canceling a 1/2-canceling pair in the handle decomposition of $W_2 \cup h$ in the bottom second diagram of Figure 5, we obtain a simplified handle decomposition of $W_2 \cup h$ which is shown in the bottom third diagram of Figure 5. Furthermore, by Gompf’s result in [13], we can transform it to a Stein handlebody as in the second fourth diagram. This means $W_2 \cup h$ admits a Stein structure.

On the other hand, in $W_1 \cup h$, we can find an embedded square $-1$ sphere by taking the union of the core disk of $h$ and the slice disk of the attaching sphere.
obtained by pushing the spanning disk in the top second diagram into $B^4$. In particular $W_1 \cup h$ never admit Stein structure by Theorem 2.3. As a result, $W_1 \cup h$ and $W_2 \cup h$ are not diffeomorphic. This implies $D_1 \not\simeq D_2$. □

According to [12], the Legendrian knot of knot type $9_{46} \# 9_{46}$ shown in Figure 2 bounds four Lagrangian disks $D_{11}$, $D_{12}$, $D_{21}$ and $D_{22}$ depicted in Figure 6.

![Figure 6. Four Lagrangian disks $D_{ij}$, where $1 \leq i, j \leq 2$. The dotted lines of each picture present the positions of saddle points of the slice disks.](image)

The boundary connected sum of two properly embedded surfaces $S_1$ and $S_2$ in $B^4 (\partial S_1 \neq \emptyset)$ is removing two disk neighborhoods in $\partial B^4$ of two points in $\partial S_1$ and $\partial S_2$, respectively, and connecting the two arcs of $\partial S_1$ and $\partial S_2$ in the removed disks by an embedded band $I \times I$. Here $I \times \partial I$ is attached to the two arcs and $\partial I \times I$ is embedded in $\partial B^4$. In general, if orientations of $\partial S_1$ and $\partial S_2$ is given, then the boundary connected sum of $S_1$ and $S_2$ has two isotopy types. One is the case where the orientation of the boundary of the resulted surface is consistent with the given orientation of $\partial S_1$ and $\partial S_2$, and the other is the inconsistent case. The result of boundary connected sum of two surfaces $S_1$ and $S_2$ via any band is denoted by $S_1 \natural S_2$. See Figure 7.

![Figure 7. Two types of motion pictures of boundary connected sum of surfaces. Here, the straight band surgery gives a consistent orientation, while the twisted band surgery gives an inconsistent orientation.](image)

The boundary connected sums of the two Lagrangian disks $D_1$ and $D_2$ have four possibilities depending on the choices of $D_i$ and two ways of boundary connected.
sums. We remark that $D_{ij}$ in Figure 6 is the boundary connected sum of the Lagrangian disks $D_i$ and $D_j$ for $i, j \in \{1, 2\}$.

Here we prove the following:

**Proposition 3.2.** $D_{11} \not\cong D_{22}$, $D_{12} \not\cong D_{21}$. Both of these two pairs of disks have diffeomorphic exteriors.

**Proof.** Let $W_{ij}$ denote the exteriors of disks $D_{ij}$ in $B^4$. According to [2] Section 1.4, the handle decompositions of $W_{22}$ and $W_{11}$ are shown as in Figure 8. By symmetry of the diagrams, $W_{11}$ is diffeomorphic to $W_{22}$. Similarly, we know that $W_{12}$ is diffeomorphic to $W_{21}$.

We prove that $D_{11} \not\cong D_{22}$. The way of proof is essentially the same as that in the proof of Proposition 3.1. We attach two $(-1)$-framed 2-handles $k_1$ and $k_2$ on $W_{22}$. See the middle first diagram in Figure 8. The two attaching circles can be moved to the position shown in the middle second diagram by handle sliding. If $D_{11} \cong D_{22}$, then by the same argument as in the proof of Proposition 3.1, there is a diffeomorphism between the exteriors $W_{11}$ and $W_{22}$ which fixes the boundary. Replacing the handle decomposition of $W_{22}$ in the middle right diagram by that of $W_{11}$, we get the bottom left diagram. By handle cancellation, we obtain the last
diagram which can be deformed into a Stein handlebody in a similar way in the proof of Proposition 3.1. Hence, $W_{11} \cup k_1 \cup k_2$ admits a Stein structure. On the other hand, by Theorem 2.3 again, $W_{22} \cup k_1 \cup k_2$ never admit any Stein structure, since it has two square $-1$ spheres constructed by capping the core disks over the slice disk of the attaching sphere. This means $D_{11} \not\cong D_{22}$.

We can also prove $D_{12} \not\cong D_{21}$ in the similar way. □

In fact, the disks $D_{1i} \not\cong D_{i2}$ and $D_{1j} \not\cong D_{2j}$ for $i, j \in \{1, 2\}$. We will prove that the fundamental groups of the exteriors of these pairs are not isomorphic in the next section.

**Proof of Theorem 1.1** It follows from Proposition 3.1 and Proposition 3.2 □

4. Non-homeomorphic Lagrangian disk exteriors

The handle decompositions of $W_{22}$ and $W_{12}$ are depicted in Figure 9. We prove that $W_{22}$ and $W_{12}$ have distinct topological types by showing that they have non-isomorphic fundamental groups.

**Figure 9.** The handle decompositions of $W_{22}$ and $W_{12}$.

**Figure 10.** The handle decompositions of $W_{22}$ and $W_{12}$ and the generators and relators of the fundamental groups.

**Lemma 4.1.** The fundamental groups of $W_{22}$ and $W_{12}$ are computed as follows:

$$\pi_1(W_{22}) = \langle x_1, x_2, x_3, x_1x_2x_3^{-1}x_2^{-1}x_1x_2, x_3x_2x_3^{-1}x_2^{-1}x_1x_2 \rangle,$$

$$\pi_1(W_{12}) = \langle x_1, x_2, x_3, x_1x_2x_3^{-1}x_2^{-1}x_1x_2, x_3x_2^{-1}x_3^{-1}x_2x_3x_2^{-1} \rangle.$$
Proof. Deforming the handle decomposition in Figure 9, we get the diagrams of Figure 10. Then we have
\[ r_1 = x_1 x_2 x_1^{-1} x_2^{-1} x_1 x_2, \]
\[ r_2 = x_3 x_2 x_3^{-1} x_2^{-1} x_3 x_2, \]
\[ r_3 = x_3 x_2^{-1} x_3^{-1} x_2 x_3 x_2^{-1}. \]

The presentations of \( \pi_1(W_{22}) \) and \( \pi_1(W_{12}) \) are \( \langle x_1, x_2, x_3 | r_1, r_2 \rangle \) and \( \langle x_1, x_2, x_3 | r_1, r_3 \rangle \), respectively.

Lemma 4.2. \( \pi_1(W_{22}) \) and \( \pi_1(W_{12}) \) are not isomorphic.

Proof. The Fox derivatives
\[ \frac{\partial r_1}{\partial x_1} = 1 + x_1 x_2 (-x_1^{-1} + x_1^{-1} x_2^{-1}), \]
\[ \frac{\partial r_1}{\partial x_2} = x_1 (1 + x_2 x_1^{-1} (-x_2^{-1} + x_2^{-1} x_1)), \]
\[ \frac{\partial r_2}{\partial x_2} = x_3 (1 + x_2 x_3^{-1} (-x_2^{-1} + x_2^{-1} x_3)), \]
\[ \frac{\partial r_3}{\partial x_2} = 1 + x_3 x_2 (-x_3^{-1} + x_3^{-1} x_2^{-1}), \]
\[ \frac{\partial r_3}{\partial x_3} = x_3 (-x_2^{-1} + x_2^{-1} x_3^{-1} (1 - x_2 x_3 x_2^{-1})), \]
\[ \frac{\partial r_3}{\partial x_3} = 1 + x_3 x_2^{-1} (-x_3^{-1} + x_3^{-1} x_2), \]
and, obviously, \( \frac{\partial r_1}{\partial x_3} = \frac{\partial r_2}{\partial x_1} = \frac{\partial r_3}{\partial x_1} = 0. \)

Both the abelianizations of \( \pi_1(W_{22}) \) and \( \pi_1(W_{12}) \) are \( \mathbb{Z} \). Let \( t \) be a generator of \( \mathbb{Z} \).

In the abelianization of \( \pi_1(W_{22}) \),
\[ x_1 \mapsto t, \]
\[ x_2 \mapsto t^{-1}, \]
\[ x_3 \mapsto t. \]

So the presentation matrix of \( \pi_1(W_{22}) \)
\[ \left[ \begin{array}{ccc} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial x_3} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial x_3} \\ \frac{\partial r_3}{\partial x_1} & \frac{\partial r_3}{\partial x_2} & \frac{\partial r_3}{\partial x_3} \end{array} \right] \mapsto \left[ \begin{array}{ccc} 2 - t^{-1} & 2t - 1 & 0 \\ 0 & 2t - 1 & 2 - t \end{array} \right], \]
and the Alexander polynomial of \( \pi_1(W_{22}) \) is \((2 - t^{-1})^2\).

In the abelianization of \( \pi_1(W_{12}) \),
\[ x_1 \mapsto t, \]
\[ x_2 \mapsto t^{-1}, \]
\[ x_3 \mapsto t^{-1}. \]

So the presentation matrix of \( \pi_1(W_{12}) \)
\[ \left[ \begin{array}{ccc} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial x_3} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial x_3} \\ \frac{\partial r_3}{\partial x_1} & \frac{\partial r_3}{\partial x_2} & \frac{\partial r_3}{\partial x_3} \end{array} \right] \mapsto \left[ \begin{array}{ccc} 2 - t^{-1} & 2t - 1 & 0 \\ 0 & -2 + t & 2 - t \end{array} \right], \]
and the Alexander polynomial of \( \pi_1(W_{12}) \) is \((2 - t^{-1})(2 - t)\).

Since \( \pi_1(W_{22}) \) and \( \pi_1(W_{12}) \) have different Alexander polynomials, they are not isomorphic. \( \square \)
Proof of Theorem 1.2. It follows from Lemma 4.1 and Lemma 4.2. □

5. Arbitrarily many non-isotopic Lagrangian disks

In this section we construct arbitrarily many smoothly non-isotopic Lagrangian fillings for Legendrian knots. At first, we give a Stein handlebody decomposition of $W_2$, the exterior of the disk $D_2$ in $B^4$.

Lemma 5.1. There is a Stein handle decomposition of $W_2$ shown as in the bottom last diagram in Figure 11.

Proof. We isotope the handle decomposition of $W_2$ shown as in Figure 5 to the top two diagrams in Figure 11. Then we change the dotted circle presentation of the 1-handles to the ordinary presentation, and transform the smooth handle decomposition in the bottom second diagram to a Stein handlebody in the last diagram in Figure 11. Note that the Legendrian knot in the bottom last diagram has Thurston-Bennequin invariant 1. □

![Diagram of Stein structure on $W_2$, the exterior of the Lagrangian disk $D_2$ in $B^4$. All these arrows are diffeomorphisms.](image)

Figure 11. A Stein structure on $W_2$, the exterior of the Lagrangian disk $D_2$ in $B^4$. All these arrows are diffeomorphisms.

Now we give a proof of Theorem 1.5.

Proof of Theorem 1.5. Taking Legendrian connected sum of $n$ copies of the Legendrian $9_{46}$ in Figure 1, we get a Legendrian knot as in the first row of Figure 12. We denote it by $L_n$.

Choosing one of the two pinch positions per a connected sum summand, we can get $2^n$ Lagrangian disks $D_{i_1i_2\ldots i_n}$, where $i_j \in \{1, 2\}$. In fact, $D_{i_1i_2\ldots i_n}$ is the boundary connected sum $D_{i_1} \natural D_{i_2} \natural \ldots \natural D_{i_n}$ of $D_{i_1}$, $D_{i_2}$, $\ldots$, and $D_{i_n}$. The first row in Figure 12 represents one example of Lagrangian disks by indicating $n$ positions to pinch. By Lemma 5.1, the exterior of this disk has a Stein structure which can be drawn by piling vertically $n$ Stein handlebody diagrams of the four patterns in the first row of Figure 14 according to the $n$-tuple $(i_1, i_2, \ldots, i_n)$, and put $n - 1$ 0-framed 2-handle to connect adjacent summands as in Figure 13.
Figure 12. The top diagram is $D_{i_1 i_2 \cdots i_n}$. The middle and the bottom diagrams are $W \cup h$ and $W' \cup h$, respectively.

Figure 13. A 0-framed 2-handle connecting 1-handles and its Stein handlebody.

Figure 14. The diagrams of the first row are rotations of Legendrian knot in Figure[11] The diagram in the second row is a handle decomposition of $W_2$ or $W_1$ union with a 2-handle $h$.

Suppose that $D_{i_1 i_2 \cdots i_n} \simeq D_{j_1 j_2 \cdots j_n}$, where the two $n$-tuples $(i_1, i_2, \cdots, i_n)$ and $(j_1, j_2, \cdots, j_n)$ in $\{1, 2\}^n$ are distinct. Then there exists $r \in \{1, \cdots, n\}$ such that $i_r \neq j_r$, say $i_r = 1$ and $j_r = 2$. Let $W$ and $W'$ denote the exteriors of two disks $D_{i_1 i_2 \cdots i_n}$ and $D_{j_1 j_2 \cdots j_n}$ respectively. On the $r$-th summand we put a $(-1)$-framed 2-handle $h$ (see the second row in Figure[12]) in a similar way depicted in Figure[5]
After doing a similar isotopy in Figure 5 we replace $W$ by $W'$. Then, $W \cup h$ and $W' \cup h$ should be diffeomorphic by the same reason as in the proof of Proposition 3.1.

By the same argument as in the proofs of Proposition 3.1 and Theorem 2.3, $W \cup h$ does not admit any Stein structure. We claim that $W' \cup h$ admits a Stein structure. We can perform a cancellation of 1/2-canceling pair as in the proof of Proposition 3.1. The Stein handlebody diagram of $W' \cup h$ can be constructed by piling $n$ choices of the five patterns in Figure 14 vertically, taking the last Stein diagram exactly once, and connecting them by $n-1$ 0-framed 2-handles. See Figure 15 for an example of Stein handlebody decomposition of $W' \cup h$. Thus, $W \cup h$ and $W' \cup h$ are not diffeomorphic and this means that $D_{i_1 i_2 \cdots i_n} \not\approx D_{j_1 j_2 \cdots j_n}$.

Hence, for any number $N$, the Legendrian knot $L_n$ where $n$ is an integer with $n > \log_2 N$, has at least $N + 1$ mutually smoothly non-isotopic Lagrangian ribbon disks in $B^4$.

\section{Two remarks}

Here we give two remarks.

\textbf{Remark 6.1.} Let $C$ be a contractible 4-manifold depicted in Figure 16, which is called Akbulut cork. It is well known that the diffeomorphism on $\partial C$, which is the $180^\circ$ rotation about the horizontal axis, cannot extend to any diffeomorphism on $C$. Akbulut gave smoothly non-isotopic slice disks $d_1$ and $d_2$ of a common knot $K_{Ak}$ Figure 7 in [1] in $B^4$. The slice disk was constructed by the cork twist of $C$. In fact, the exterior of $d_i$ ($i = 1, 2$) in $B^4$ is diffeomorphic to the exterior of a slice disk in $C$. The property of the cork yields $d_1 \not\approx d_2$.

Our non-isotopy property of $D_1$ and $D_2$ can also be reinterpreted by a cork. The exterior of the ribbon disk $D_i$ ($i = 1, 2$) in $B^4$ is diffeomorphic to the exterior of a slice disk $D$ in the Akbulut cork $C$. The diffeomorphism can be seen in Figure 17. The spanning disk of right dotted circle in the last picture in the figure is $D$. This
The diffeomorphism relates the symmetries of the ribbon disks $D_1$ and $D_2$ of $\bar{9}_{46}$ and the cork twist of $C$. Let $\gamma$ and $\delta$ be the meridians of a dotted 1-handle and a 2-handle of the handle decomposition of $C$. See the last picture of Figure 17. The two meridians are isotopic to the positions in the first picture in Figure 17 by the diffeomorphism. Furthermore, the meridians are moved to a symmetric position in $W_1$ and $W_2$ by an isotopy. In Figure 18 we draw the positions of $\gamma$ and $\delta$ in $S^3 \setminus \bar{9}_{46}$. Thus, we can understand that the non-isotopiness of two disks is caused by the non-extendability property of the cork twist. This idea is due to Akbulut [1].

![Figure 16. Akbulut cork C.](image)

![Figure 17. The slice disk complement of $\bar{9}_{46}$ is diffeomorphic to a slice disk complement of the Akbulut cork.](image)

![Figure 18. The symmetric positions of $\gamma$ and $\delta$ in $S^3 \setminus \bar{9}_{46}$.](image)

**Remark 6.2.** Both of Akbulut’s slice disks $d_1$, $d_2$ are not isotopic to any Lagrangian disk even if taking the mirror image. In fact, any ribbon disk in $B^3$ of both of $K_Ak$ and the mirror image $\bar{K}_Ak$ is not isotopic to a Lagrangian disk. If $K_Ak$ has a Legendrian knot $L$ that fills a Lagrangian disk, then $tb(L) = -1$ due to [5]. By [14], $tb(L)$ has an upper bound $tb(L) \leq \min \deg_{sa}(F_{K_Ak}) - 1$. Here $\min \deg_{sa}(F_{K_Ak})$ is the
minimal $a$-degree of the Kauffman polynomial $F_{K_{Ak}}(a,z)$. By an easy calculation using the skein relation, $\text{min deg}_a(F_{K_{Ak}}) = -1$. So $tb(L) \leq -2$, and there is no Lagrangian disk filling for $K$. Since $\text{min deg}_a(F_{\overline{K_{Ak}}}) = -8$, we have no Lagrangian disk filling $\overline{K_{Ak}}$ for the same reason.

Hence, since whether the exteriors of $d_1$ and $d_2$ admit a Stein structure or not is not known, our argument cannot be immediately applied to this case.

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