On the pion decay constant

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Abstract

The pion decay constant $f_\pi$ plays a crucial role in many areas of low energy particle physics. Its value may e.g. be deduced from experimental data on leptonic pion decays. Here, we provide comments on several aspects of this evaluation. In particular, we point out that at the present level of experimental accuracy, the value of $f_\pi$ is sensitive to the value of the pion mass chosen in its chiral expansion.

Key words: Chiral symmetries, Chiral perturbation theory, Chiral Lagrangians, Meson decay constants

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1 Introduction

In the framework of QCD, the pion decay constant $f_\pi$ may be defined through the coupling of the axial current to the pion [1],

$$\langle 0| A_\mu(0)|\pi^{-}(p)\rangle = ip_\mu f_\pi; \quad A_\mu = \bar{u}\gamma_\mu\gamma_5d,$$

(1)

where $|\pi^{-}(p)\rangle$ denotes a one-pion state with normalization $\langle \pi^{-}(p')|\pi^{-}(p)\rangle = 2(2\pi)^3p^0\delta^{(3)}(p'-p)$.

The pion decay constant plays a crucial role in many areas of low energy particle physics. First of all, it dictates the strength of leptonic pion decays,

$$\pi^{-} \rightarrow \ell^{-}\bar{\nu}_\ell; \quad \ell = e, \mu,$$

(2)
with rate:\footnote{To ease notation, we often write in the following $\pi \to \ell \nu$, which stands for $\pi^- \to \mu^- \bar{\nu}_\mu(\gamma)$ and lifetime, and for $\pi^+ \to \ell^+ \bar{\nu}_\ell$.}

$$\Gamma^{(0)}(\pi \to \ell \nu) = \frac{G_F^2|V_{ud}|^2 f_\pi^2 m_\pi m_\ell^2}{8\pi} \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$ \hspace{1cm} (3)

in the absence of electromagnetic interactions. It also enters the Goldberger-Treiman relation \cite{2},

$$f_\pi g_{\pi N} = \sqrt{2} m_p g_A ,$$ \hspace{1cm} (4)

which relates the weak and strong coupling constants $f_\pi, g_A$ and $g_{\pi N}$ with the proton mass $m_p$. This relation is exact in the chiral limit $m_u = m_d = 0$ \cite{3} and may well be considered the starting point of precise low energy hadron physics. It has been and still is a crucial test of our understanding of low energy hadron dynamics. Further, $f_\pi$ happens to determine the strength of $\pi\pi$ interactions and thus acts as an (inverse) expansion parameter in Chiral Perturbation Theory (ChPT) \cite{4,5}. Last but not least, $f_\pi$ is now amenable to lattice calculations, see e.g. Ref. [6] for an evaluation with $N_f = 2 + 1 + 1$, and for further references.

The most precise determination of $f_\pi$ is presently obtained from leptonic decays of the pion. In fact, experiments have reached a level of precision which makes it mandatory to include radiative corrections, and to consider the rate for

$$\pi \to \ell \nu(\gamma) .$$ \hspace{1cm} (5)

According to PDG \cite{1},

$$f_\pi = (130.4 \pm 0.04 \pm 0.2) \text{ MeV} .$$ \hspace{1cm} (6)

This value is based on data [branching fraction for $\pi^- \to \mu^- \bar{\nu}_\mu(\gamma)$ and lifetime], and on theoretical work performed in Refs. [7,8,9,10,11,12,13,14,15]. The first (second) uncertainty is due to the uncertainty in the value of $V_{ud}$ (to the uncertainties in the higher order corrections in the evaluation of the matrix element for the decay (5)). It is the main purpose of this Letter to comment on recent calculations \cite{7,9,12,15} of the transition matrix element for this process, and on the value of $f_\pi$ reported in Eq. (6), see Section 4 for details on the questions investigated here.

\section{The effective Lagrangian}

A very elegant and convenient tool to perform the calculation is the effective field theory framework set up in Refs. \cite{8,9,12,15}. We adhere here to this
method, and come back to its relation to the underlying theory below. The pertinent lowest-order effective Lagrangian for three flavours can be found in Ref. [12]. Here, we consider its two-flavour version,

\[
\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle + e^2 F^4 Z \langle u^\dagger Qu^2 Q u^\dagger \rangle - \frac{1}{4} F_{\mu
u} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \sum_\ell \left[ \bar{\ell} (i\gamma^\mu + e A^\mu - m_\ell) \ell + \bar{\nu}_L i\gamma^\mu \nu_L \right],
\]

(7)

where the flavour-trace is indicated by \(\langle\rangle\), and

\[
u = e^{i\phi/2F}, \quad \phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}. \]

(8)

Here, \(F\) denotes the pion decay constant in the chiral limit, in a normalization which is standard in ChPT: Let \(f = f_\pi|_{m_u=m_d=0}\). Then \(f = \sqrt{2}F\). In the absence of pseudoscalar densities, one has

\[
\chi_+ = u^\dagger \chi u^\dagger + u \chi u, \quad \chi = 2B \text{diag}(m_u, m_d). \]

(9)

The external vector and axial vector external sources \(v_\mu\) and \(a_\mu\) contain also the lepton and photon fields,

\[
u_\mu = i \left[ u^\dagger (\partial_\mu - i r_\mu) u - u (\partial_\mu - i l_\mu) u^\dagger \right],
\]

\[
l_\mu = v_\mu - a_\mu - eQ A_\mu + \sum_{\ell=e,\mu} \left( \bar{\ell} \gamma_\mu \nu_L Q^W + \text{h.c.} \right),
\]

\[
r_\mu = v_\mu + a_\mu - eQ A_\mu,
\]

(10)

with

\[
Q = \frac{1}{3} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q^W = -2\sqrt{2}G_F \begin{pmatrix} 0 & V_{ud} \\ 0 & 0 \end{pmatrix}. \]

(11)

For further notation, and for the terms at next-to-leading order, see Ref. [12].

3 \(\pi \to \ell\nu\) without electromagnetic interactions

We first consider the non-radiative decay \(\pi \to \ell\nu\) in the absence of electromagnetic corrections, and set \(e = 0\) in the effective Lagrangian. It is convenient to use the axial current as an interpolating field for the pion,

\[
\langle \bar{\nu}_\ell(q_2) \ell^-(q_1); \text{out} | A_\mu(0)^\dagger | 0 \rangle = \frac{-i f_\pi p_\mu}{m_\pi^2 - p^2} T(q_1, q_2); p = q_1 + q_2. \]

(12)
At $p^2 = m^2_\pi$, the quantity $T(q_1, q_2)$ is the transition amplitude for $\pi^- \to \ell^- \bar{\nu}_\ell$,
\begin{equation}
T = i \sqrt{2} G_F V_{ud} m_\ell \bar{u}(q_1) v_L(q_2). \tag{13}
\end{equation}

As has been pointed out by the authors of Ref.[15], the chiral expansion of the transition amplitude, evaluated in the effective field theory framework mentioned, amounts to the chiral expansion of $f_\pi$. A proof of the statement is provided in the Appendix. It relies on the fact that the pion decay constant also occurs in the correlator of two axial currents,
\begin{equation}
A^{\mu\nu} = i \int d^4x e^{ipx} \langle 0 | T A^\mu(x) A^{\nu\dagger}(0) | 0 \rangle = B(p^2) p^\mu p^\nu + C(p^2) g^{\mu\nu}, \tag{14}
\end{equation}
\begin{equation}
B(p^2) = \frac{f^2_\pi}{m^2_\pi - p^2} + R(p^2). \tag{15}
\end{equation}

The quantity $R$ is holomorphic in the complex $p^2$-plane, cut along the real axis for $p^2 \geq 9m^2_\pi$. [The ambiguities inherent in the definition of $A^{\mu\nu}$, generated by the short distance singularities in $\langle 0 | T A^\mu(x) A^{\nu\dagger}(0) | 0 \rangle$, affect $R$ and $C$ only.] One could as well use the relation (15) to define $f_\pi$ in pure QCD.

4 Two questions

Because the effects of real and virtual photons have to be included for the confrontation of theory with experiment, infrared (IR) singularities occur in intermediate steps of the calculation. One source of these singularities is the fact that the correlator $A^{\mu\nu}$, evaluated at $e \neq 0$, develops a branch point in the form factor $B(p^2)$ at $p^2 = m^2_\pi$, as a result of which there is no isolated pole-contribution – the decomposition Eq. (15) does not hold anymore in the presence of electromagnetic interactions. In addition, the formula Eq. (12) cannot be used without further ado when $e \neq 0$. So, one may wonder about the role of the pion decay constant that is determined in pion decays:

i) What is its relation to the correlator $A^{\mu\nu}$ at $e \neq 0$?
ii) What is its relation to the pion decay constant $f_\pi$ in pure QCD, as it occurs in the decomposition Eq. (15)?

To the best of our knowledge, these two questions were never discussed in full detail in the literature, and we find it instructive to shed additional light on the issue. On the other hand, we do not question the final algebraic result for the rate as provided in the works mentioned – we have nothing to add here.

5 Photons generate a branch point

5.1 Infrared regularization

To perform the calculations, one may tame the IR singularities by providing the photon with a small mass $m_\gamma$, such that the decomposition Eq. (15) still
a) Fig. 1. Diagrams that contribute to the correlator $A_{\mu\nu}^e$ at one-loop order. Double lines denote axial currents, solid (wavy) lines pions (photons). Contributions from counterterms are not shown, nor do we indicate graphs which generate local terms only.

holds, with $f_\pi \rightarrow \bar{f}_\pi$, where the constant $\bar{f}_\pi$ now also includes contributions from virtual photons. The remainder $R(p^2)$ generates a branch point at $p^2 = (m_\pi + m_\gamma)^2$. The quantity $\bar{f}_\pi$ is gauge dependent and diverges logarithmically as the photon mass is sent to zero. Further, the LSZ formula Eq. (12) remains true, with $f_\pi \rightarrow \bar{f}_\pi$. The IR singularities cancel at the end when adding the rates for $\pi \rightarrow \ell \nu$ and for $\pi \rightarrow \ell \nu (n\gamma)$, and sending $m_\gamma$ to zero at the very end of the calculation provides the desired result. This is the method used in Refs. [9,12,15].

Because dimensional regularization is a very useful ultraviolet and infrared regulator for our purpose, we adhere in the following to this alternative regularization [16,17,18,19], where the photon mass is set to zero from the very beginning. We start the discussion with the evaluation of the correlator

$$A_{\mu\nu}^e = i \int d^4xe^{ipx} \langle 0|TA^\mu(x)A^{\nu(\mu)}|0\rangle_e$$

in $d$ space-time dimensions, in the presence of electromagnetic interactions, in the framework of the effective theory defined by the Lagrangian Eq. (7). The index $e$ indicates that virtual photons are included.

5.2 Loop contributions at $d \neq 4$: explicit expressions

To keep everything as simple as possible, we restrict ourselves to a one-loop calculation. Some of the pertinent graphs which contribute to $A_{\mu\nu}^e$ are displayed in Fig. 1. A typical contribution generated e.g. by diagrams Figs. 1a+1e reads

$$A_{\mu\nu}^e = \frac{2p_\mu p_\nu F^2}{M^2 - p^2} \left( 1 + \frac{e^2 M^2 J(p^2)}{M^2 - p^2} \right) + \cdots ,$$

(17)
where \( J(p^2) \) denotes the one-loop integral

\[
J(p^2) = \frac{1}{i} \int \frac{d^4l}{(2\pi)^4} \frac{1}{(M^2 - (p-l)^2)(-l^2)}. \tag{18}
\]

As for the notation, we recall that the chiral expansion of the charged pion mass starts out with \( M_{\pi^+}^2 = M^2 + O(e^2, m_0^2) \), where \( M^2 = (m_u + m_d)B \), see Refs. \([8,9]\). The ellipsis in Eq. (17) stands for additional contributions. The double pole at \( p^2 = M^2 \) is removed by mass renormalization in the standard manner. We split off the part which contributes to the pion mass,

\[
J(p^2) = J(M^2) + (1 - z)J(p^2); \quad z = \frac{p^2}{M^2}, \tag{19}
\]

such that

\[
A_{\mu\nu} = \frac{2p_{\mu}p_{\nu}F^2}{M_1^2 - p^2}(1 + e^2J(p^2)) + \cdots; \quad M_1^2 = M^2(1 - e^2J(M^2)). \tag{20}
\]

If the photon were massive, the photon propagator in Eq. (18) would be replaced by \( 1/(-l^2) \rightarrow 1/(m_0^2 - l^2) \), and the corresponding loop function \( \mathcal{J}(p^2, m_0) \) would be of the form \( \mathcal{J}(p^2, m_0) = C + O(p^2 - M^2) \), with \( C \) a finite constant at \( d = 4 \), and the decomposition Eq. (15) would hold for \( A_{\mu\nu} \) as well. However, massless photons render loop contributions singular at threshold. To investigate the structure of \( A_{\mu\nu} \), we evaluate loop integrals at \( d \neq 4, p^2 \neq M^2 \). In particular, the quantity \( J(p^2) \) can be expressed in terms of hypergeometric functions \([20]\). Expanding the parts which are regular at \( p^2 = M^2 \), we find

\[
J(p^2) = M^{2w} f(w) \left\{ 1 - \frac{\Gamma(-2w)}{\Gamma(-w)}(1 - z) \left[ (1 - z)^{2w}g_1(w, z) + g_2(w, z) \right] \right\},
\]

\[
f(w) = \frac{\Gamma(-w)}{1 + 2w(4\pi)^{2+w}}; \quad g_1 = \frac{\Gamma(1 + w)}{z^{1+w}},
\]

\[
g_2 = \sum_{n \geq 1} \frac{\Gamma(n - w)}{\Gamma(n - 2w)}(1 - z)^{n-1} = -g_1 + w^2h(d, z), \quad w = \frac{d}{2} - 2. \tag{21}
\]

The function \( h(d, z) \) is regular at \( d = 4, z = 1 \), and \( \mathcal{J}(p^2) \) behaves as

\[
\mathcal{J}(p^2) = (L + k) \left[ (1 - z)^{d-4} - 1 \right] z^{-1} + \cdots,
\]

\[
L = \frac{\mu^{d-4}}{N} \left[ \frac{1}{d - 4} - \frac{1}{2}\left( \ln 4\pi + \Gamma'(1) + 1 \right) \right]; \quad k = \frac{1}{2N} \left[ \ln \frac{M^2}{\mu^2} - 1 \right],
\]

\[
N = 16\pi^2 \tag{22}
\]

in the vicinity of \( p^2 = M^2, d = 4 \). The renormalization scale is denoted by \( \mu \), and the ellipsis stands for terms that are irrelevant in the following. The region where this representation is valid is indicated in Fig. 2a with the shaded circle [use \( M_{\pi^+}^2 = M^2 \)].
Fig. 2. The one-loop integral $J$ and the correlator $A^\mu_\nu$ in the $d, p^2$ plane. In Fig. a), the shaded circle indicates the region where the expression Eq. (22) is valid. In Fig. b), we indicate 3 different limiting procedures. The limit 1 (2) is relevant for renormalization (for the LSZ formalism). In the limit 3, $J$ and $A^\mu_\nu$ do not exist. For $J$ use $M_{\pi^+}^2 = M^2$.

### 5.3 Three limiting procedures

We now consider three different limiting procedures [21]. First, we let $d \to 4$ off the mass shell $p^2 = M^2$. This limit is relevant for renormalization and is indicated by the path $(\text{1})$ in Fig. 2b. Second, for the LSZ formalism, one goes to the mass shell first and then considers $d \to 4^+$ [path $(\text{2})$ in the figure], see Refs. [18,19,22]. The result is

$$J(p^2) = \begin{cases} (16\pi^2 z)^{-1} \ln (1 - z) & \text{[limit (1)]} \\ -(L + k) & \text{[limit (2)]}. \end{cases}$$

(23)

In the limit $(\text{3})$, $J$ does not exist.

As a result of Eq. (23), the correlator generates a branch point at $p^2 = M_1^2$,

$$A^\mu_\nu = \frac{2p_\mu p_\nu F^2}{M_1^2(1 - \frac{p^2}{M_1^2})^{1 - \frac{2}{d+1}}} + \cdots [\text{limit (1)}].$$

(24)

On the other hand, going on the mass shell $p^2 = M_1^2$ at $d > 4$ results in

$$A^\mu_\nu = \frac{2p_\mu p_\nu F^2}{M_1^2 - p^2}(1 - e^2(L + k) + \cdots) [\text{limit (2)}].$$

(25)

In other words, the standard renormalized contribution generates a branch point in the propagator. Going to the mass shell at $d > 4$ results in the standard pole behaviour of the propagator.
6 The correlator $A_{\xi}^{\mu\nu}$ at one loop

Including all the graphs at one-loop order results in the following expression for the correlator in the first limit,

$$A_{\xi}^{\mu\nu} = \frac{p^{\mu} p^{\nu} \kappa_1^2}{M_{\pi^+}^2 (1 - \frac{p^2}{M_{\pi^+}^2})^{1 + e^2 g_1}} + \cdots \text{ [limit (1)]} ,$$

$$\kappa_1^2 = 2 F^2 \left( 1 + a_1 + e^2 b_1 + \mathcal{O}[M^4, e^4, M^2 e^2] \right) ,$$

$$a_1 = -\frac{1}{NF^2} \left( M^2 \ln \frac{M^2}{\mu^2} + M_{\pi^+}^2 \ln \frac{M_{\pi^+}^2}{\mu^2} \right) + \frac{2 M^2}{F^2} r_4 ,$$

$$b_1 = \frac{1}{N} \left[ -3 + \xi (1 + \ln \frac{M^2}{\mu^2}) \right] + K^r (\mu) ,$$

$$g_1 = \frac{1}{N} (6 - 2 \xi) , \quad K^r (\mu) = \frac{20}{9} (k_1^r + k_2^r) + 4 k_3^r . \quad (26)$$

Here, $l_4^r (k_i^r)$ are LECs in the effective Lagrangian at order $p^4 \ (e^2 p^2)$, see Refs. [5] ([23,24]). We have used the result Ref. [25, Eq. (C.10)] for the renormalization of $k_9$ in any gauge. The above expression reveals the branch point at $p^2 = M_{\pi^+}^2$, with strengths $g_1, \kappa_1$ which are gauge dependent. The ellipsis indicates terms that are less singular at $p^2 = M_{\pi^+}^2$, and terms proportional to $g_1^{\mu\nu}$. The branch point is generated by Fig. 1e alone – this explains the fact that a branch point with identical strength $g_1$ is also present in the electron propagator in the framework of QED [22], or in the two-point function of two charged pion fields in scalar QED [8]. In Ref. [22], it is shown that for the electron propagator, the leading infrared singularity at any order in $e$ is given by the coefficient $g_1$.

Performing the second limit gives

$$A_{\xi}^{\mu\nu} = \frac{p^{\mu} p^{\nu} \kappa^2}{M_{\pi^+}^2 - p^2} + \cdots \text{ [limit (2)]} ,$$

$$\kappa^2 = 2 F^2 \left( 1 + a_1 + e^2 b + \mathcal{O}[M^4, e^4, M^2 e^2] \right) ,$$

$$b = (6 - 2 \xi) L + \frac{1}{N} (3 \ln \frac{M^2}{\mu^2} - 4) + K^r (\mu) . \quad (27)$$

The ellipsis has the same meaning as before. It is seen that in this case, the correlator does have – aside from less singular terms – a pole contribution at $p^2 = M_{\pi^+}^2$, with a residue that is divergent at $d = 4$, and gauge dependent. At $e = 0$, the quantity $\kappa = \kappa_1$ coincides with the pion decay constant $f_\pi$, evaluated at one-loop order in ChPT [5],

$$f_\pi = \sqrt{2} F \left( 1 - \frac{M^2}{16 \pi^2 F^2} \ln \frac{M^2}{\mu^2} + \frac{M_{\pi^+}^2}{F^2} r_4^l + \mathcal{O}(M^4) \right) . \quad (28)$$

Neufeld and Rupertsberger [9] have evaluated the pion decay constant with the photon mass $m_\gamma$ as an infrared regulator. With the identification
the quantity $\kappa/\sqrt{2}$, evaluated at $\xi = 1$, agrees with their $F_{\pi \pm}$, translated to the $SU(2) \times SU(2)$ case by use of the matching relations for the LECs $l_i^4$, $k_i^r_{1,2,9}$ worked out in Refs. [26,27].

In Refs. [11,28], it is shown that the LEC $k_i^r$ depends logarithmically on the scale of the underlying theory [we use the matching of $k_r^9$ to $K_{r12}^r$ as worked out in Ref. [27]], see also Ref. [14]. In other words, the residues $\kappa^2, \kappa_1^2$ are not uniquely defined in the framework of QCD+QED. An analogous scale dependence of $k_i^r$ occurs in the effective theory of the linear sigma model, coupled to electromagnetism [25].

7 Width for $\pi \to \ell \nu(\gamma)$

We come back to the evaluation of the width for leptonic pion decays (5). The reduction formula reads

$$\langle \bar{\nu}_\ell(q_2)\ell^{-}(q_1) ; \text{out}|A_\mu(0)^\dagger|0\rangle_c = \frac{-i\kappa p_\mu}{M_{\pi^+}^2 - p^2} T_\nu(q_1, q_2) ; p = q_1 + q_2$$

(30)

for the non-radiative part, and similarly for the case when a photon is emitted in addition. Here, it is understood that the limit $p^2 \to M_{\pi^+}^2$ is taken at $d > 4$. Let us denote by class I (class II) the set of graphs which do not (which do) contain a virtual charged lepton, and disregard the leptonic counter term contributions for a moment. The graphs in class I are obtained by replacing one of the axial currents in Fig. 1 by $\ell^-\gamma_\mu\bar{\nu}_\ell L$. From the previous discussion and from Eq. (13), it is clear that the graphs in class I generate the amplitude

$$i\sqrt{2}G_F V_{ud} k m_\ell \bar{u}(q_1) v_L(q_2) \equiv C_{I\kappa}$$

(31)

see also the Appendix. The graphs from class II are displayed in Fig. 3. These contributions are of order $F$ in this order of the momentum expansion. This remains true including the leptonic counterterms, and we denote the sum by $C_{II\ell e^2 F}$. Finally, one has to add the effect from real photon emission – the pertinent amplitude is denoted by $C_{IIIeF}$. All in all, the width at one loop is obtained from

$$\Gamma(\pi \to \ell \nu(\gamma)) = \langle |C_{I\kappa} + C_{II\ell e^2 F}|^2 \rangle + \langle |C_{IIIeF}|^2 \rangle ,$$

(32)

where the symbol $\langle | \rangle$ denotes phase space integrations, including all kinematic factors. The IR singularities and the gauge dependence cancel out in $\Gamma(\pi \to \ell \nu(\gamma))$, and one ends up with the expression first given in Eq. (5.1) of Ref. [12], matched to the $SU(2) \times SU(2)$ case considered here. Further, as pointed out in Ref. [14], the above mentioned scale dependence of $k_i^r$ is cancelled by the scale dependence of the leptonic LEC $X_6^r$ introduced in Ref. [12].
8 The answers to the questions in Section 4

We are now prepared to answer the questions raised in Section 4.

Answer to question i):

The relation of the pion decay constant to the correlator of two axial currents is evident from Eq. (32): the first term on the right-hand side in Eq. (32) contains the residue of $A_{\mu \nu}^e$ at $d > 4$, which itself contains – among other contributions – the pion decay constant $f_\pi$ in pure QCD. At this order in the low energy expansion, one may therefore factor out $f_\pi^2$ in the expression for the width [12]. The final result may be written [12] in the form given by Marciano and Sirlin (Eq. (7a) of Ref. [7]),

$$
\Gamma(\pi \to \ell \nu(\gamma)) = \frac{G^2|V_{ud}|^2 f_\pi^2 M_{\pi^+} m_\ell^2}{8\pi} (1 - z_\ell)^2 \times \left\{ 1 + \frac{\alpha}{\pi} \left[ \log \frac{M_{\rho}^2}{m_\rho^2} - \frac{3}{2} \log \frac{m_\rho}{M_{\pi^+}} + F(\sqrt{z_\ell}) - C_1 \right] \right\},
$$

(33)

with $z_\ell = m_\ell^2/M_{\pi^+}^2$. The function $F(x)$ is given in Eq. (7b) of Ref. [7], and the parameters $C_{2,3}$ introduced there do not occur at this order in the low energy expansion [12]. The constant $C_1$ can be expressed in terms of LECs and mass logarithms, see Eq. (5.11) of Ref. [12] [its adaption to the $SU(2) \times SU(2)$ case considered here is straightforward]. It is now clear that, once $G_\mu, V_{ud}$ and $C_1$ are known, the pion decay constant is fixed through data on $\pi \to \ell \nu(\gamma)$. This is how the numerical result Eq. (6) was obtained by PDG [where higher order corrections in the width, as worked out in Ref. [15], were taken into account as well].

Answer to question ii):

Here arise two subtle points.

a) In factoring out $f_\pi$ in Eq. (33), one makes use of the chiral expansion of $f_\pi$ as given in Eq. (28). The choice of the mass $M$ in that expansion amounts to a convention: one may use either the neutral or the charged pion mass. Scrutinizing the calculations performed in Refs. [12,15], we find that the value Eq. (6) corresponds to the choice $M = M_{\pi^0} \simeq 135$ MeV [physical value of the neutral pion mass]. Factoring out $f_\pi$ evaluated at $M = M_{\pi^+} = 139.57$ MeV
[physical value of the charged pion mass] amounts to a renormalization of the constant \( C_1 \). The formula (28) allows one to determine the difference between the two cases,

\[
f_\pi(M_{\pi^+}) = f_\pi(M_{\pi^0}) + \frac{M_{\pi^0}^2 - M_{\pi^0}^2}{8\pi^2f} \left( \tilde{l}_4 - 1 \right) + \mathcal{O}[p^4, (M_{\pi^+}^2 - M_{\pi^0}^2)^2]
\]

\( \tilde{l}_4 = 16\pi^2 l_4^r - \ln \frac{M_{\pi^+}^2}{\mu^2} \). \quad (34)

Using the value \( \tilde{l}_4 = 4.4 \) [29], we find

\[
f_\pi(M_{\pi^+}) = f_\pi(M_{\pi^0}) + 0.4 \text{ MeV} . \quad (35)
\]

The difference is thus quite significant – about twice the uncertainty reported in the PDG-value Eq. (6). The induced change in \( C_1 \) is

\[
C_1 \rightarrow C_1 + 2.8 . \quad (36)
\]

b) Concerning the second point, we note that, extracting a value for \( f_\pi \) [defined in pure QCD, e.g. through Eq. (15)] from leptonic pion decays, where real and virtual photons are included, requires that one performs a splitting between strong and electromagnetic effects. This splitting is ambiguous [30] - the result depends on the procedure chosen. It is at this stage that the matching of the effective theory to the underlying theory matters. For a detailed analysis of this fact in a case which can be analyzed in a perturbative manner [linear sigma model, coupled to electromagnetism], we refer the reader to Ref. [25]. Here we note that, in the language of the frameworks used in Refs. [11,12,13,14,15], the quantity \( f_\pi \) depends on the scale of the underlying theory [31,25]: \( \mu_0 \frac{df}{d\mu} = \mathcal{O}[e^2m_q] \). A different method to perform the matching consists in evaluating quantities in QCD with values of the parameters relevant in QCD+QED at a scale \( \mu_1 \). In this case, strong quantities are scale independent, but do depend on the matching scale \( \mu_1 \). This scenario is discussed in detail in Refs. [25,32], see also Refs. [33,34]. For the pion decay constant in the chiral limit, the dependence on \( \mu_1 \) can be worked out in the framework of QCD, using the observation that \( f \) is proportional to the renormalization group invariant scale of QCD, and applying the formula Eq. (11.6) in Ref. [30][2]. It turns out that, if \( \mu_1 \) is changed by a factor of 2, \( f \) changes by a negligible amount of about 8 keV. The effect is so small, because the electromagnetic renormalization of the strong coupling constant \( g \) starts out at two-loop order and is \( \mathcal{O}(e^2g^3) \) [30]-the one-loop contribution, which would be of order \( e^2g \), vanishes. In the linear sigma model, the scale dependence of \( f \) is more than one order of magnitude larger [25].

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2 We are indebted to H. Leutwyler for pointing this out to us.
In our opinion, it would be very useful to perform the matching of the effective theory constructed in Refs. [8,9,12,15] to the underlying theory [the Standard Model] in this setting, which is used in [34].

9 Effects from $m_d \neq m_u$

Finally, we comment on $m_d - m_u$ effects in pure QCD, and note that the difference $m_d - m_u$ can occur only with even powers in $f_\pi$ [35]. One-loop contributions in ChPT are linear in the quark masses, and thus cannot contain isospin breaking terms, whereas they do occur at and beyond two-loop order. Indeed, the latter have been evaluated in Ref. [36] and found to be tiny, $f_{\pi^+}/f_{\pi^0} - 1 \simeq 0.7 \times 10^{-4}$. Barring unexpectedly large higher order contributions, one concludes [36] that $f_{\pi^+} \simeq f_{\pi^0}$ in pure QCD, to a very good approximation. A numerical estimate of the contributions at order $e^2$ in $f_{\pi^+}/f_{\pi^0} - 1$ can be found in Ref. [37].

10 Summary

We have evaluated the correlator of two charged axial currents at one loop in ChPT, including virtual photons, for any value of the gauge fixing parameter $\xi$. As is seen from the result Eq. (26), photon loops modify the holomorphic properties of the correlator in a fundamental manner: the pole at the charged pion mass becomes a branch point, with strengths $\kappa_1, g_1$ that are gauge dependent. The residue $\kappa_2$ furthermore depends on the scale of the underlying theory [QCD+QED], through the LEC $k^9_{9}$ [11,14]. For the evaluation of leptonic pion decays in the framework of the effective field theory framework set up in Refs. [8,9,12,15], one may use the standard LSZ formalism also at vanishing photon mass, provided that the graphs are worked out in $d$ space-time dimensions, and provided that the residue is evaluated at $d > 4$ [18,19]. We have discussed the manner in which the pion decay constant shows up in the final formula for the decay width, and have pointed out that the value reported in Eq. (6) corresponds to the case where the pion mass is identified with the neutral one. Evaluating $f_\pi$ at the charged pion mass increases its value by about 0.4 MeV. Furthermore, we note that the value Eq. (6) is based on a matching procedure which differs from the one used in Ref. [34]. In particular, $f_\pi$ depends on the scale of the underlying theory.

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Appendix

Here we prove Eq. (13), which is true in the absence of electromagnetic inter-
actions. In this case, there are no virtual leptons in the corresponding Feynman
diagrams, and one may evaluate the transition amplitude by considering the
lepton current as a classical external field which we denote by \( X_\mu \), such that

\[
\begin{align*}
  l_\mu &= v_\mu - a_\mu - X_\mu, \\
r_\mu &= v_\mu + a_\mu,
\end{align*}
\]

and

\[
\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \tag{A2}
\]
is the pertinent leading order Lagrangian, which has the structure

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi\pi} + \langle \ell_\mu L^\mu \rangle + \langle r_\mu R^\mu \rangle + \mathcal{O}(r^2, l^2, rl); \quad \mathcal{L}_{\pi\pi} = \mathcal{L}_{\text{eff}}|_{l=r=0},
\]

(A3)

with operator-valued currents \( L_\mu, R_\mu \). Let \( L^\mu \pm R^\mu = O^\mu_{\pm} \), with \( O^\mu_{\pm}(\phi) = \pm O^\mu_{\pm}(-\phi) \). At \( v_\mu = 0 \), the terms quadratic in the external fields in the \( S \)-operator are

\[
S = \int dx dy Te^{i\int \mathcal{L}_{\pi\pi}(z)dz} \langle a_\mu O^\mu_{\pm} \rangle_0 \langle (a_\rho + X_\rho)O^\rho_{\pm} \rangle_0, \tag{A4}
\]

up to contact terms, which do not contribute to the matrix element in ques-
tion. The symbol \( T \) denotes time ordering. From Eq. (A4), it is seen that the
coefficient of axial current \( \times \) axial current is identical to the coefficient of axial current \( \times \) lepton current. It is furthermore clear from the derivation that
this statement remains true in the presence of additional terms in the La-
grangian which contain lepton fields exclusively through their presence in the
left current \( l_\mu \). In particular, the coefficients are still the same in the presence
of strong counterterms. From these observations and from Eqs. (12,14,15)
follows Eq. (13). Eq. (31) is true, because the above statements also hold
in the presence of an external electromagnetic field and of the mass term
\( e^2 F^4 Z \langle u^\dagger Qu^2 Qu \rangle \).
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