Ghost in the Ising machine

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Coupled nonlinear systems have promise for parallel computing architectures. En route to realizing complex networks for Ising machines, we report an experimental and theoretical study of two coupled parametric resonators (parametrons). The coupling severely impacts the bifurcation topology and the number of available solutions of the system; in part of the stability diagram, we can access fewer solutions than expected. When applying noise to probe the stability of the states, we find that the switching rates and the phase-space trajectories of the system depend on the detuning in surprising ways. We present a theoretical framework that heralds the existence of ‘ghost bifurcations’. These bifurcations involve only unstable solutions and lead to avoided zones in phase space. The emergence of such ghost bifurcations is an important feature of parametron networks that can influence their application for parallel logic operations.

I. INTRODUCTION

Driven nonlinear systems were first considered as logic elements at the dawn of the digital era [1–3]. Their nonlinearity induces several stable oscillation states that can be used as elementary information units for computation and data storage. A prominent example of a nonlinear system is the parametric resonator, or ‘parametron’ [1, 2, 4, 5]. When driven close to twice its natural frequency, \(2f_d \approx 2f_0\), the parametron undergoes a time-translation symmetry breaking and locks into one of two ‘phase states’ at \(f_d\) that have the same oscillation amplitude but differ by \(\pi\) in phase [6–8], cf. Fig. 1(a) and (b). These phase states represent the binary information unit of the parametron. The parametron experienced a surge in interest with the recent development of nanomechanical, electrical and optical resonators that offer long-lived, error-resilient and tunable phase logic states [5, 9–19]. Several research fields are currently racing towards physical implementations of parametron networks for parallel computing, and their corresponding operation protocols [18, 20–24].

Parallel logic architectures have long been recognized as a way to overcome the limitations of standard (sequential) computers [25–29]. In a parallel network, a given task is encoded in the coupling between nodes, and the entire system evolves towards a stable ‘optimal’ configuration that represents the computational output, cf. Fig. 1(c). This behavior can be exploited to find answers to many optimization problems that are nearly intractable with sequential computers. Such tasks include the famous travelling salesman problem [30], the number partitioning problem [31], and the MAX-CUT problem [32, 33], but also fundamental questions in physics such as the ground state of the Ising spin model [34–36] or the structure of complex molecules [37].

In a parallel network of \(N\) parametrons, the solution of a computation corresponds to the optimal network configuration. The determination of this optimal configuration is achieved through a many-body state evolution from an initial setting to the final result [22–24, 32, 33, 38]. The evolution typically involves spon-
taneous or stochastic switching between the phase states of individual parametrons, while either the network connectivity or the strength of some type of fluctuation energy ("noise") is adiabatically adjusted. These aspects are intimately linked to the notion of phase transitions in out-of-equilibrium, in analogy to symmetry breaking, dissipation-induced phase transitions [39], or noise-induced Kramers’ instabilities [40].

A crucial subject in this context is the impact of the inter-parametron coupling strength on the very structure of the many-body states [41]. Current proposals for parametron-based computations are typically based on the notion of weak coupling that minimally modifies the available non-interacting many-body phase states at the expense of slow convergence times. Specifically, in the weak coupling limit, the entire overlap region in Fig. 1(d) can be mapped to an Ising system with $N$ nodes [24, 31, 32, 35, 42, 43]. In the opposite limit of strong coupling, the system is expected to split into hybridized normal modes whose parametric oscillation lobes ("Arnold tongues") only partially overlap in a phase diagram spanned by $\lambda$ and $f_d$ [41], cf. Fig. 1(d). Inside the overlap region, the system exhibits multiple stable oscillation states and is thus useful for parametron-based information encoding and computation. As a result, we seek an ideal configuration where the coupling strength is as large as possible for fast computation, without compromising the properties of the available many-body phase space.

In this paper, we test the validity of the Ising analogy in a two-parametron system. The inter-parametron coupling leads to the formation of hybridized normal modes. In contrast to earlier studies [41], we can access the intermediate coupling regime where the lobes only partially overlap. There, we find that the coupling qualitatively changes the bifurcation topology, erasing part of the Ising solution space in the region of overlapping lobes, cf. Fig. 1(d). We associate this effect to the existence of ‘ghost bifurcations’ that do not involve stable states and are therefore not directly observable. Probing the system with strong noise, we furthermore find that the ghost bifurcations strongly impact the dynamical behavior of the network in the remaining Ising space. For weak noise, the fluctuation spectra of the system around stable solutions highlight the intricate physics underpinning nonlinear networks, including the appearance of exceptional points. Our findings have important consequences for the fundamental understanding of coupled nonlinear systems and for the design and operation of future parametron-based parallel networks.

II. STEADY-STATE RESULTS

We built a setup of capacitively coupled parametrons using electrical lumped-element resonators. Each resonator (marked by index $i$) comprises a resistance $R = 47\, \text{M}\Omega$, an inductance $L = 87\, \mu\text{H}$, a tuning voltage $U_i = 2\, \text{V}$, and a nonlinear capacitance $C \approx 20\, \text{pF}$ in the form of a varicap diode, cf. Fig. 2(a). The resonators are driven and read out inductively through auxiliary coils. Using Kirchhoff’s laws, our electrical circuits are described by coupled equations of motion [15],

$$
\ddot{x}_i + \omega_i^2 \left[1 - \lambda \cos(2\omega_t t)\right] x_i + \alpha_i x_i^3 + \gamma_i \dot{x}_i - \sum_{j \neq i} J_{ij} x_j = 0. \quad (1)
$$

Here, dots indicate time derivatives, $x_i = u_i \cos(\omega t) - v_i \sin(\omega t)$ is the measured voltage with quadrature amplitudes $u_i$ and $v_i$, $\omega_i = 2\pi f_i$ is the angular resonance frequency, $\alpha_i$ the coefficient of the Duffing nonlinearity, $\gamma_i = \omega_i/Q_i$ the damping rate, and $Q_i$ the quality factor of the $i^{th}$ resonator. Our resonators are (nearly) identical in their bare characteristics and are tuned via $U_i$ to have (nearly) degenerate eigenfrequencies, $\omega_i \approx \omega_0 = 2\pi f_0$. They are linearly coupled with coefficients $J_{ij}$ ($i \neq j$).
and are all driven with the same parametric modulation depth $\lambda = 2U_d/(U_{th}Q)$ at an angular rate $2\omega_d = 4\pi f_d \approx 2\omega_0$, where $U_{th}$ is a threshold voltage for parametric oscillations. For further details on the individual resonators, cf. Ref. [15].

In our study, we measure the system with a lock-in amplifier, and are thus primarily interested in changes of the slow coordinates $(u_i, v_i)$ on timescales much longer than $1/\omega_0$. We therefore apply to lowest order the averaging method [41, 44, 45] to our model Eq. (1), yielding coupled “slow” equations of motion

$$
\dot{u}_i = -\frac{\gamma u_i}{2} - \left(\frac{3\alpha X_i^2}{8\omega_d} + \omega_0^2 - \omega_d^2 + \frac{\lambda\omega_0^2}{4\omega_d}\right) v_i + \frac{Jv_j}{2\omega_d},
$$

$$
\dot{v}_i = -\frac{\gamma v_i}{2} + \left(\frac{3\alpha X_i^2}{8\omega_d} - \omega_0^2 + \omega_d^2 - \frac{\lambda\omega_0^2}{4\omega_d}\right) u_i - \frac{Ju_j}{2\omega_d},
$$

where $X_i^2 = u_i^2 + v_i^2$. The averaging method is expected to yield valid results since $\lambda, \gamma/\omega_0, J/\omega_0^2$ and $(\alpha/\omega_0^2)X_i^2$ are all of order $\epsilon$ with $0 < \epsilon \ll 1$ [46]. Note that we assumed here homogeneous dissipation $\gamma_i = \gamma$, nonlinearities $\alpha_i = \alpha$, and coupling $J_{ij} = J$.

**Single parametron** – For vanishing $J$, each resonator can be driven into parametric resonance when $U_d \geq U_{th}$ [6, 7]. In Fig. 2(b), we show experimental sweeps with increasing and decreasing $f_d$ for constant $U_d$, exhibiting the standard nonlinear parametric response and hysteresis of the parametron labelled as 1 (similar results were obtained for parametron 2). Inside the region marked as (ii), the linear resonator becomes unstable, bifurcates and settles into one of the two phase states that are stabilized by $\alpha$ [7]. In region (iii), the phase states coexist with a stable solution at $X_1 = 0$. Repeating the upsweeps (increasing $f_d$) for different $U_d$, we recover the characteristic parametric instability lobe, cf. Fig. 2(c).

Using Eqs. (2), we can describe all of the measured results of the single parametron [13, 45]. We obtain the steady-state amplitude $X_1$ by applying the condition $(\dot{u}_1 = \dot{v}_1 = 0)$. This yields a quintic characteristic polynomial with up to three different stable solutions (attractors) in phase space, cf. Fig. 2(d). As a function of $f_d$, the number of stable solutions changes at specific ‘bifurcation points’. In the single parametron, we only observe pitchfork bifurcations, where the total number of solutions changes by 2, and the number of stable solutions changes by 1, cf. Fig. 2(e). Fitting the model to the measurement data, we determine the values $Q_1 = 295$, $f_0 = 2.6784$ MHz, $\alpha_1 = -9 \times 10^{-17}$ $V^{-2} s^{-2}$, and $U_{th} = 1.21$ V [13, 15]. In particular, from the fact that region (iii) appears at $f_d < f_0$, we infer that $\alpha < 0$ [47].

**Coupled parametrions** – Having characterized an individual parametron, we now couple two of them capacitively, cf. Fig. 3(a). We perform sweeps with increasing $f_d$ (upsweep) to probe the instability lobes, i.e., the equivalent of region (ii) in Fig. 2(b), cf. Fig. 1(d). Note that in Fig. 3(b), we plot $(u_1, v_2)$ instead of $(X_1, X_2)$ to visualize the relative signs of the oscillations in the two resonators; the phase states have a characteristic angle in phase space, such that $u_i \propto v_i \propto X_i$. For $U_d = 2.5$ V, the sweep yields two frequency segments with large responses. In one of them, the two resonator oscillations are in phase and virtually identical; in the other, the oscillations are out of phase and their amplitudes differ significantly. When increasing the parametric drive to $U_d = 3.7$ V, the response segments merge and the system directly jumps from the out-of-phase to the in-phase response slightly above 2.36 MHz.

To obtain the full measured stability diagram of the two-parametron system, we repeat frequency sweeps as in Fig. 3(b) for a wide range of $U_d$. Finite coupling $|J| > 0$ generally leads to the formation of hybridized normal modes with symmetric (in-phase) and antisymmetric (out-of-phase) combinations of the phase states [8, 41, 48, 49]. In order to appreciate these symmetries, we transform our data into corresponding combinations $|v_S| = |v_1 + v_2|/\sqrt{2}$ and $|v_A| = |v_1 - v_2|/\sqrt{2}$ (and analogous for $u_{S,A}$). White coloring in Fig. 3(c) indicates regions where the measured response was zero, while blue and red show the absolute values of the symmetric and antisymmetric combinations of $v_1, v_2$ (plotting $u_{S,A}$ yields similar images). As anticipated in Fig. 1(d), the overall shape is that of two normal modes with partially overlapping instability lobes. Surprisingly, however, the transition from antisymmetric ($A$) to symmetric ($S$) oscillations does not always occur at the boundaries of the individual lobes; rather, the symmetric region protrudes to the left at $U_d \approx 3.5$ V, and then proceeds along a diagonal line (brown line calculated from theory, see below). In addition, we observe a mixed state ($M$) in some parts of the lower left lobe. This mixed state comprises both symmetric and antisymmetric oscillation components, and manifests as different amplitudes of motion for each resonator, cf. left panel in Fig. 3(b). The mixed state and the reduced region of antisymmetric oscillations are unexpected from a naive weak coupling perspective [24, 35, 41–43]. As we show below, they stem from a dramatic change in the bifurcation topography of available stable solutions of the system, due to the bending direction of the negative nonlinearity.

**Stability analysis** – As for the single parametron, we apply Eq. (2) to model the coupled system. Here, the steady-state condition $(\dot{u}_i = \dot{v}_i = 0)$ results in a polynomial with 25 solutions. There are between 1 and 9 stable solutions that can be characterized by their symmetry in a space spanned by $v_S$ and $v_A$ (or alternatively by $u_S$ and $u_A$), cf. Fig. 3(d). We present calculated bifurcations and solutions in Fig. 3(e) and (f). The two cases correspond to the sweeps in Fig. 3(b) and use the experimentally determined parameters for $f_0$ and $J$, while $Q = 265$ was chosen for both resonators to achieve optimal agreement. Note that $\alpha$ does not impact the position of bifurcations.

The model reproduces all of the measured features. The mixed state observed for $U_d = 2.5$ V is caused by an additional bifurcation (brown dot) that causes one res-
A stable solution is involved, these 'ghost bifurcations' furcation points now connect only to unstable solutions, this bifurcation is obtained by a stability analysis against the antisymmetric solution (A) with the mixed solution (M) or the purely symmetric region (S). The position of this bifurcation is obtained by a stability analysis against small fluctuations (cf. Appendix A). This bifurcation represents the border for the stability of the antisymmetric solution (A) with the mixed solution (M) or the purely symmetric region (S). The position of this bifurcation is obtained by a stability analysis against small fluctuations (cf. Appendix A), which yields the expression

\[
\lambda_A = \frac{2\sqrt{\gamma^2\omega^2 + (2J - (\omega^2 - \omega_0^2))^2}}{\omega_0^2}.
\] (3)

As the coupling coefficient \( J \) is decreased (increased), the stability boundary of the antisymmetric solution \( \lambda_A \) approaches (departs) the right boundary of the antisymmetric instability lobe. This observation bears important consequence for the coupled system. The first of these consequences is that the antisymmetric solution connecting the right-hand side red bifurcation point to the brown one has become unstable, making the symmetric solution the only stable state across a large frequency range. This explains why the measured traces in Fig. 3(b) jumped to a symmetric state at \( f_d \approx 2.36 \text{ MHz} \).

In Fig. 3(g), we summarize the theoretical results for the phase diagram of stable solutions. The most prominent feature in this diagram is the pitchfork bifurcation shown as a solid brown line. It is caused by the combination of the inherent nonlinearity and the linear coupling between the resonators in Eq. (1). Together, these terms produce an effective parametric coupling between the symmetric and antisymmetric modes (cf. Appendix A).
implications for using parametron networks for Ising machines. Commonly, the weak coupling limit \((J \rightarrow 0)\) is explored, where the symmetric and antisymmetric solutions are stable in the entire overlap region \([41]\), and an Ising spin-based description correctly captures the symmetry (and behavior) of the stable solutions \([35, 43]\). However, as we find here, an increase in the coupling \((J)\) shifts the brown line to lower frequencies, and the region where the symmetric solution is the unique stable solution grows. Both the symmetric and antisymmetric solutions remain stable in the triangular overlap region delineated by the brown and blue dashed lines in Fig. 3(g), but not in the full overlap of the two instability lobes that is marked as a grey area in Fig. 1(d).

III. PROBING THE SYSTEM WITH NOISE

After charting the stable attractors of the system, we set out to explore its basic functionality as an Ising simulator. In many proposals, finding the optimal configuration of the network involves stochastic switching (‘annealing’) between the attractors \([36, 42]\). To test this procedure, we pump the system with a coherent drive tone \(U_d\) (as before), and additionally probe it with a fluctuating voltage \(U_\text{n}\) (with an approximately white power spectral density \(S_\text{n}\)) that mimics the effect of a thermal noise. We refer to this as the ‘pump-noisy-probe’ method. The noise enters Eq. (1), as an additional stochastic force \(\xi \propto U_\text{n,i,j}\), where \((\xi(t_1)\xi(t_2)) = \delta(t_1 - t_2)\) with a power spectral density calibrated to be \(\xi^2 = 4.93 \times 10^{-20} \text{Hz}^2 S_\text{n}\), see Appendix C. In the rotating picture, Eq. (2), this stochastic force \(\xi\) leads to additive uncorrelated noise terms \(\Xi_{\text{a},i}\), \(\Xi_{\text{v},i}\), with power spectral densities \(\sigma^2 = \xi^2 / 2\omega_d^2\) \([51, 52]\).

The combination of drives in the pump-noisy-probe method induces two main effects: (1) fluctuations of the parametron network around its stable oscillation state, and (2) noise-induced switching between stable oscillation states. The analysis of (1) is drawing much interest in recent years as a probe for dynamical responses in out-of-equilibrium systems \([39, 53, 54]\). We relegate our experimental and theoretical study of the rich fluctuation spectra in the coupled parametron network to Appendix B. We focus in the main text on (2) the switching between oscillation states, which is similar to noise-activated jumping over a barrier \(W\) in an equilibrium system \([40]\). In our rotating frame, \(W\) depends on the phase space zone that separates the stable attractors. Its value can be estimated using the Onsager-Machlup formalism for analyzing the underlying switching paths in phase space, with the Onsager-Machlup function \([55, 56]\)

\[
S_{\text{OM}}[\mathbf{Y}] = \int_{t_i}^{t_f} \frac{1}{4} \left( \dot{\mathbf{Y}} - f(\mathbf{Y}) \right)^2 dt,
\]

where \(t_i\) (\(t_f\)) is the initial (final) time of the trajectory of a system composed of \(N\) resonators, \(\mathbf{Y} = (u_1, v_1, ..., u_N, v_N)^T\), and \(f(\mathbf{Y})\) is the right hand side of Eq. (2). The switching probability density between two stable states along a path \(\mathbf{Y}(t)\) is given by \(e^{-2S_{\text{OM}}[\mathbf{Y}]/\sigma^2}\), and the total switching probability \(P_{ij}\) from \(\mathbf{Y}_i\) to \(\mathbf{Y}_j\) is obtained by integration over all trajectories connecting them. From this probability one can derive the switching rate \(\Gamma\) \([55]\), which in the weak-noise limit scales as \(\Gamma \propto \exp(-2W/\sigma^2)\) with barrier \(W\) \([57]\).

Single parametron – We test the pump-noisy-probe method with a single parametron whose properties are well known \([43, 50, 58, 59]\). In Fig. 4(a), we show an example of a measured time trace of \(u_1\) and \(v_1\) in the presence of applied noise. The resonator mostly dwells in the vicinity of a given phase state, and explores the local fluctuations of the specific attractor. Occasionally, the noise activates the system to switch to the other phase state. Collecting and plotting such a time trace in phase space yields a probability distribution which highlights both of these aspects, cf. Fig. 4(b). The weak fluctuations around the attractors activate the parametron fluctuation spectrum, cf. Appendix B. We concentrate here on the switching rate \(\Gamma\) between phase states, which we extract from the experimental data, cf. Appendix D and
Fig. 4(c). We find that $\Gamma$ decreases monotonically with increasing separation between the phase states, which we control here through $f_d$ [50]. Similar results have previously been measured in other parametron implementations [59].

The monotonic decrease of $\Gamma$ in Fig. 4(c) is readily derived using the Onsager-Machlup approach [50]. Specifically, at low noise, the switching rate $\Gamma$ is dominated by the path $\mathbf{Y}_{\text{min}}$ that minimizes $S_{\text{OM}}$ [55]. If $S_{\text{OM}}$ has more than one local minimum, one has to find the global minimum by comparison. Hence, we neglect the integration over all possible paths and the switching rate is approximately

$$\Gamma \approx \Gamma_{\text{min}} \equiv \Gamma_0 e^{-2S_{\text{OM}}[\mathbf{Y}_{\text{min}}]/\sigma^2}, \quad (5)$$

where $\Gamma_0$ is an overall prefactor and we identify the effective activation barrier $W_{\text{eff}} = S_{\text{OM}}[\mathbf{Y}_{\text{min}}]$ [55]. Typically, the minimal path $\mathbf{Y}_{\text{min}}$ connects between stable states via an unstable one [60, 61], although exceptions have been observed [62, 63]. We obtain this minimal path $\mathbf{Y}_{\text{min}}$ using the sgMAM method [64], see Fig. 4(b). It is an improved path optimization scheme using scaled time, leading to consistent converged results. We start with a guessed initial path that connects the phase states via the unstable (0-amplitude) attractor. Then, we perform numerical minimization of $S_{\text{OM}}$ by varying the path in phase space between the chosen end points. Repeating this estimation as a function of $f_d$ and calculating $\Gamma_{\text{min}}$ yields good agreement with the experimentally observed $\Gamma$, cf. Fig. 4(c). Note that the prefactor $\Gamma_0$ is not obtained by this method but reused from Ref. [50], leading to a slight overall shift towards larger $\Gamma$. The analytical formula derived in Ref. [50] produces a similarly good agreement, cf. Eq. (D1) in Appendix D.

**Coupled parametrons** – Turning to the two-parametron system, we obtain a noisy signal similar to that seen in Fig. 4(a). We collect time traces of the quadrature state vector $\mathbf{Y}$, cf. Appendix E, and plot it in the phase space spanned by $v_S$ and $v_A$, see Fig. 5(a). As for the single parametron, weak force noise probes the fluctuations around attractors of the system as depicted in Fig. 3, cf. our analysis in Appendix B, while large noise occasionally leads to switching events.

We start our investigation in that part of the symmetric lobe where only the symmetric normal mode shows a parametric instability, cf. Fig. 3(c) and blue dots in Fig. 3(d). Here, the oscillations of $x_1$ and $x_2$ are always in phase. As there is no stable antisymmetric state, we expect to find switching rates $\Gamma$ between the two symmetric states that obey the same functional form as for a single parametron. In Fig. 5(b), we plot the measured $\Gamma$ for switches between the two symmetric states as a function of $f_d$ for fixed parametric driving strength $U_d$. The investigated frequency range is divided into 4 regions separated by 3 bifurcations, out of which 2 are ghost bifurcations, cf. upper dashed line in Fig. 3(g). In regions I and II, $\Gamma$ decreases exponentially as the separation of the states in phase space grows larger with decreasing $f_d$, in agreement with results from single parametric resonators [50, 57, 58] and our results in Fig. 4(c).
trast, we observe a distinct kink at the border between regions II and III, coincident with a ghost bifurcation. Finally, in region IV the slope changes sign and $\Gamma$ increases as $f_\text{ad}$ is decreased, signalling the formation of additional stable Ising-type configurations.

A numerical simulation of the full noisy time evolution for the experimental parameters [cf. Eq. (2) with appropriate stochastic terms] reproduces the experimental results, see Fig. 5(b). Doing so, we first verify that our microscopic stochastic model indeed captures the observed physics. We then simulate the coupled parametron system at different noise strengths, and verify that the activation rate satisfies the weak noise scaling form of Eq. (5) [57], see Appendix D. In this limit, $\Gamma_0$ and $W$ are independent of the noise and purely depend on the properties of the non-stochastic system, and on the switching paths in phase space. This procedure allows us to extract $\Gamma_0$ and $W$ for the experimental and the numerical data [Fig. 5(c)] as well as calculate $\Gamma$ at different noise strengths. Here, we can also see a qualitative change in $W$ at the boundary between regions II and III.

The surprising deviation from a simple exponential law in Fig. 5(b) indicates that the emergence of new unstable states at the ghost bifurcation impacts $\Gamma$. Moreover, rather than acting as additional obstacles in the switching paths, these unstable states appear to favor an increase in $\Gamma$. In order to gain a better understanding of this remarkable effect, we apply the sgMAM method [64] to Eq. (4) of the coupled system with $Y = (u_1, v_1, u_2, v_2)^T$, where we choose one of the symmetric states as the initial point and the other one as the final point, and try different unstable states as intermediate points. We thus obtain the corresponding locally-minimizing switching paths $Y_{\text{min}}$, see Fig. 5(a).

In region I, we find only one optimal path, which is passing through the 0-amplitude state. Following the first ghost bifurcation [cf. Fig. 3], in region II there are three unstable states: two antisymmetric solutions that are unstable in one direction (singly unstable), and the solution at 0-amplitude that is unstable in two directions (doubly unstable). Correspondingly, we find 2 additional switching paths that avoid the 0-amplitude state and rather pass through the antisymmetric ones. This is in line with the experimental data that exhibits a broader distribution around the 0-amplitude state, extending to the two unstable antisymmetric states. Similar paths appear also in region III and IV, where the additional states that emerged from the ghost bifurcations provide unstable ‘ledges’ during switching events. These unstable states, which we dub ‘ghosts’ to emphasize their origin, slow down the decrease in $\Gamma$.

Using the sgMAM method, we also obtain the effective activation barrier $W_{\text{eff}}$ of each path, compare them with the experimental extracted barrier $W$, and identify the dominant transition path, see Fig. 5(c). Within regions I-III, we find good qualitative and approximate quantitative agreement (save for a small overall shift). Interestingly, in region II, the effective activation via the antisymmetric ghosts equals that of the original path via the center. In region III, the former overtakes the latter, which manifests as the observed kink in Fig. 5(b). This signals that the additional ghost states support the antisymmetric switching, and markedly participate and modify the expected stochastic annealing of the system. In region IV, we are within the full Ising domain, where both symmetric and antisymmetric phase states are stable. Here, the analytical calculation deviates from the experimental and numerical results. This deviation is likely due to the fact that the Onsager-Machlup method only considers switches that connect the two symmetrical states, while the counting algorithm that we used for the experimental and numerical data includes all possible switches between states. The latter includes repeated switches between a symmetric and an antisymmetric state as individual events.

To summarize, we have shown the response of the two-parametron network to stochastic force noise, which is a basic building block for simulated annealing [33, 36, 42, 65] in so-called ‘Ising machines’ [22, 23, 33]. The switching between a single pair of phase states is readily configurable and obeys a simple exponential scaling. Surprisingly, we see that the switching dynamics between Ising-type states of the system are severely impacted by the ghost bifurcations, as these dictate the path of least action that the system can take. Finally, within the full Ising domain the switching scaling involves additional activation paths, which require independent calibration beyond that performed here.

IV. DISCUSSION

Our experimental and theoretical results demonstrate unambiguously the existence of an unusual type of bifurcation in a two-parametron network. Arising from the coupling between the individual resonators, these ‘ghost bifurcations’ are responsible for the disappearance of stable antisymmetric solutions in a considerable part of the stability diagram, cf. Fig. 3(f). In addition, they impact the inter-state switching rate $\Gamma$ and the relative dwell times in the symmetric and antisymmetric states.

Generalizing our results to $N$ identical resonators with identical all-to-all coupling, we theoretically predict that the ghost bifurcations, as well as the altered stability of the symmetric/antisymmetric regions persist. Such a network has a single symmetric lobe and a $N-1$ fold degenerate antisymmetric lobe [41]. The boundaries delineating the overlapping region once more involve ghost bifurcations. However, one of the ghost bifurcations is now $N-1$ fold degenerate. As demonstrated above for two resonators, the antisymmetric states only become stable when they undergo another bifurcation, described by the general equation

$$\lambda_A = \sqrt{4\gamma^2\omega^2 + (J(N + 2) - 2(\omega^2 - \omega_0^2))^2}$$

(6)
resulting in an extended symmetric region. The ghost bifurcations (and the ghosts emerging from them) will also impact the switching behaviour of the system as new switching paths open up, in analogy to our observations in Fig. 5.

When the coupling coefficient \( J_{ij} \) is different for each resonator pair, the degeneracy is lifted. However, the instability lobes still overlap and generally form ghost bifurcations. We also expect to see mixed states, although they are more difficult to investigate for strongly varying coupling coefficients. More experimental and theoretical result are needed to find suitable prediction tools for the most general case. Our work provides the motivation for such future investigations.

All of our observations have direct and important consequences for logic networks built from parametrons, because they impact the solution that a network will choose after a finite transition time. In the most extreme case, a computation at a particular point in \( f_a - \lambda \) space might always produce the same state because all other states have unexpectedly become unstable. Of course, such a computation would be of no value.

Our results do not represent a ‘no-go’ theorem for computation with parametron networks. With proper modelling and calibration of the ghost bifurcations and the mixed states, a network can be operated at a position in \( f_a - \lambda \) space where the many-body character of the network is preserved; in our example, this would be the case in the small triangle between the solid brown and the dashed blue lines in Fig. 3(c). This area is expected to become larger for weaker coupling \( J \). It becomes clear from our work that a careful analysis of the bifurcation topology of a network is indispensable for performing any meaningful calculation.

V. SUMMARY AND OUTLOOK

In summary, we experimentally and theoretically investigated the dynamics of a system of two coupled nonlinear parametric resonators (parametrons). Our study implement the smallest form of parametron network, and puts to the test central assumptions about the solution space that are taken for granted in contemporary proposals for parallel computation. Surprisingly, we find that the coupling leads to an unusual type of ‘ghost bifurcation’ that reduces the number of available solutions in some regions of the stability diagram. The ghost bifurcations and the corresponding ‘ghosts’ that emerge from them have also important consequences for the switching dynamics of the system as it progresses towards its most stable configuration, affording a glimpse of the ‘ghost in the Ising machine’.

As coupled networks of parametric resonators are one of the main candidates for future parallel computation architectures, our study provides crucial input for a growing subcommunity working towards classical and quantum analog computation [18, 20–24]. Furthermore, it provides additional incentive for the fundamental exploration of complex driven-dissipative nonlinear networks in a multitude of fields [8].

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Appendix A: Nonlinear coupling between normal modes

A system of two, nearly identical, coupled linear resonators can be described in terms of uncoupled normal modes by moving to symmetric and antisymmetric coordinates. Nonlinearities in general couple the symmetric and antisymmetric modes. For our nonlinear parametric system, the slow-flow equations for \((u_S, v_S)\) are given by:

\[
\begin{pmatrix}
    \dot{u}_S \\
    \dot{v}_S
\end{pmatrix} = 
\begin{pmatrix}
    -\frac{v_S}{u_S} \left( J + \frac{\omega_d^2}{\omega_0^2} v_A^2 + v_X^2 - \omega_0^2 - J + \frac{\omega_0^2}{\omega_0^2} u_A^2 + u_X^2 \right) - \frac{v_S}{u_S} \left( 4 \lambda \omega_0^2 - 3 \alpha \left( u_A^2 - v_A^2 \right) \right) - \frac{3 \alpha u_A u_S v_A}{8 \omega_d} - \frac{3 \alpha v_S \left( u_A^2 + v_A^2 \right)}{8 \omega_d} - \frac{3 \alpha v_S \left( u_A^2 + v_A^2 \right)}{16 \omega_d} - \frac{7 u_S}{2} \\
    \frac{u_S}{v_S} \left( J + \frac{\omega_d^2}{\omega_0^2} v_A^2 + v_X^2 - \omega_0^2 - J + \frac{\omega_0^2}{\omega_0^2} u_A^2 + u_X^2 \right) - \frac{u_S}{v_S} \left( 4 \lambda \omega_0^2 - 3 \alpha \left( u_A^2 - v_A^2 \right) \right) + \frac{3 \alpha u_A u_S v_A}{8 \omega_d} + \frac{3 \alpha v_S \left( u_A^2 + v_A^2 \right)}{8 \omega_d} + \frac{3 \alpha v_S \left( u_A^2 + v_A^2 \right)}{16 \omega_d} - \frac{\gamma v_S}{2}
\end{pmatrix}.
\] (A1)

Note that because of the Duffing nonlinearity, the effective eigenfrequency of the symmetric mode, \( \omega_S^2 - J + \frac{\omega_0^2}{\omega_0^2} u_A^2 + u_X^2 \), as well as its parametric driving strength, \( 4 \lambda \omega_0^2 \rightarrow 4 \lambda \omega_0^2 - 3 \alpha \left( u_A^2 - v_A^2 \right) \) now explicitly depends on the coordinates \((u_A, v_A)\) of the antisymmetric mode change. In addition to the usual expected bifurcations for both normal modes, the aforementioned interplay triggers a further bifurcation, cf. brown point in Fig. 3(c), which heralds the mixed state (M). At this bifurcation, the oscillations of the antisymmetric mode are strong enough to drive parametric oscillations of the symmetric mode through the nonlinearity, leading to the mixed state. This effect takes place in the instability lobe with lower (higher) eigenfrequency for negative (positive)
where each PSD is given by the theoretically predicted form:

\[ S_y = \frac{1}{2 \pi} \text{Re} \left( \sum_i \lambda_i \langle u_i, v_i \rangle e^{-\lambda_i t} \right) + \text{Im} \left( \sum_i \lambda_i \langle u_i, v_i \rangle e^{-\lambda_i t} \right) \]

where \( \lambda_i \) are the eigenvalues, \( \langle u_i, v_i \rangle \) are the eigenvector components, and \( e^{-\lambda_i t} \) are the exponential decay factors. The real part of the PSD gives the mean square deviation, while the imaginary part gives the autocorrelation of the fluctuations.

Appendix B: Stability and probing the system with weak noise

In the following, we characterize the basins of attraction of the different solutions in the rotating-frame Hamiltonian landscape via a stability analysis. The corresponding frequency, decay rate, and symmetry of small fluctuations determines the dynamical regime of small fluctuations in the system. We provide here a general discussion for a network of \( N \) coupled resonators, whereas the experimental and theoretical results for the cases of a single resonator (\( N = 1 \)) and two coupled resonators (\( N = 2 \)) are summarized in Figs. 6 and 7, respectively.

Small fluctuations around a steady-state solution \( Y_s \)

\[
(\delta u_1, \delta v_1, \delta u_2, \delta v_2, \ldots, \delta u_N, \delta v_N)^T = \delta Y = Y - Y_s \quad (B1)
\]

are typically investigated by linearizing the equation of motion (2)

\[
\delta \dot{Y} = M_J \delta Y, \quad (B2)
\]

where \( M_J \) is the Jacobian matrix of the right side of Eq. (2) evaluated at the coordinates of the selected solution \( Y_s \) [39, 44, 53, 54]. The time evolution of the eigenvector components is then governed by \( e^{\mu t} \), where \( \mu_i \) is the \( i^{th} \) eigenvalue of the Jacobian matrix \( M_J \).

A state \( Y_s \) is stable if and only if the real parts of all eigenvalues are negative. The stability analysis further determines if a steady state is stable, singly unstable (one positive real part) or doubly unstable (two positive real parts), etc.. At a bifurcation, the real part of an eigenvalue switches sign and the corresponding state changes its stability. We used this criterion to calculate the stability boundary of the antisymmetric solution, cf. Eq. (3).

The regime of small fluctuations around a stable solution is naturally probed by weak noise. The imaginary (real) part of the eigenvalues describe the frequency (decay rate) of the fluctuations, while the eigenvectors determine the symmetry of the corresponding fluctuation component. We reiterate that these properties (symmetry, frequency, decay time) do not follow from those of the underlying stable solution. For instance, the frequency of the fluctuations are not equal to the drive frequency \( \omega_d \), but rather arise from the curvature of the rotating-frame Hamiltonian. Similarly, we always find a symmetric and an antisymmetric fluctuation component for every symmetric, antisymmetric, or \( 0 \)-amplitude solution in our system. A symmetric (antisymmetric) fluctuation component describes an oscillating motion in the symmetric (antisymmetric) subspace spanned by \( u_S \) and \( v_S \) \((u_A \) and \( v_A)\). For the mixed state, the fluctuation components also have a mixed symmetry.

Experimentally, we apply weak fluctuating forces \( \Xi_i \) with voltage PSD \( S_n \) to determine \( \mu_i \) for different stable solutions, see Figs. 6(a) and 7(a) and (b). From a timetrace measured with constant driving frequency \( f_d = \omega_d/2\pi \) and amplitude \( U_d \) and demodulated with a lock-in amplifier, we obtain the PSD of the \( u_i \) and \( v_i \), see Figs. 6(b) and 7(b). In the single-sided PSD of each coordinate measurement of \( N \) coupled resonators, one finds in general \( N \) resonance peaks, corresponding to the \( N \) normal mode fluctuations.

To extract \( \mu_i \) from the measured signal, the fluctuation modes can be decoupled from each other and investigated individually. Concretely, \( N \) identical oscillators in the symmetric or antisymmetric state with all-to-all coupling have a Jacobian matrix \( M_J \) with two symmetric and \( 2N - 2 \) antisymmetric eigenvectors \( w_j \otimes (1, e_j)^T \), and corresponding eigenvalues \( \mu_j \). The entries of the \( N \)-dimensional \( w_j \) take values of \( \pm 1 \) and describe the phase states of the individual oscillators. When applicable, transforming the data into subspaces spanned by the eigenvectors of specific symmetries (e.g. symmetric and antisymmetric for \( N = 2 \)) generates \( N \) PSDs with only one peak each that we index by \( j \). The value \( \mu_j \), whose real (imaginary) part corresponds to the width (position) of the peak. We fit the peaks of PSDs \( \text{PSD}_{u_j} + \text{PSD}_{v_j} \), where each PSD is given by the theoretically predicted form:
The fluctuations in these regions is partially locked to the
dependent parametric amplification. The frequency of
\( \mu \) and \( v \). We observe regions where the imaginary part of
fluctuate around the 0-amplitude state, see Figs. 6(d)
and 7(e). We assign regions to the imaginary part of
the PSD of the mixed state.

Hence, a linear combination of the antisymmetric and
symmetric PSD of equal weight is unsuitable to depict
fluctuations around the 0-amplitude, the symmetric, the anti-symmetric and the mixed states, respectively. Standard errors are
smaller than the point size. The experimental imaginary parts are symmetric around zero but only positive values are shown.

The eigenfrequencies of the experimental setup are slightly shifted compared to Fig. 3, which we assign to small changes in the
smaller than the point size. The experimental imaginary parts are symmetric around zero but only positive values are shown.

When the parametric driving strength \( \lambda \) is increased, the
parametric drive (flat theory lines). The eigenvalue dia-
gram is similar to that of a damped harmonic resonator
going from an underdamped to an overdamped motion,
establishing an ‘exceptional point’ [39]. We note that
multiple such scenarios emerge in our network, cf. Fig. 7.
When the parametric driving strength \( \lambda \) is increased, the
system also realizes large-amplitude solutions, which are
stable as long as all of their \( \mu_j \) have negative real parts,
see Figs. 6(e) and 7(f) and (g).

We successfully demonstrated the visualization of the
basins of attraction of stationary solutions of a nontriv-
ial driven-dissipative system. The agreement between
experiments and theory is excellent in all cases, confirm-
ing the usefulness of the ‘pump-noisy-probe’ technique
to study the system dynamical properties. The method
can straightforwardly be applied to investigate the many-
body physics of \( N \) coupled oscillators, and can be used
to observe noise squeezing prior to a bifurcation [53].
1000
100
10
1
\( \Gamma [\text{Hz}] \)
\( 1/\sigma^2 \text{[Hz}^{-1}\text{V}^{-2}] \)

FIG. 8: Switching rate \( \Gamma \) between the symmetric states of the two-parametron system as a function of inverse noise strength, \( 1/\sigma^2 \), obtained from simulation (black) for \( U_d = 3.7 \text{ V} \) and \( f_d = 2.3725 \text{ MHz} \). The optimal fit (gray) corresponds to Eq. (5) for the fitting parameters \( \Gamma_0 = 2 \times 10^4 \text{ Hz} \) and \( W = 0.024 \text{ Hz}^{-1}\text{V}^{-2} \).

Appendix C: Fluctuating versus Coherent Signal Amplitude

To obtain optimal agreement between the measured switching rates and the theoretical predictions, we consistently found that the noise power spectral density in the model had to be a factor \( \approx 4.2 \) smaller than the value applied in the experiment. This discrepancy is likely due to an additional attenuation of a factor 2 in the path of the fluctuating voltage, for instance a voltage division at a 50 \( \Omega \) matched input port. The fluctuating signal with power spectral density \( S_n \) was provided by a dedicated voltage source (two individual sources with the same output intensity in the case of two parametrons) and added to the coherent signal via the ADD channel of the Zurich Instruments HF2LI lock-in amplifier. The resulting noise process \( \xi_i \) acting on our system has power spectral density \( \xi^2 = C_{in} S_n \), where the coefficient for the signal in-coupling efficiency is \( C_{in} = 4.93 \times 10^{-20} \text{ Hz}^2 \) for the single-parametron experiment. For the two-parametron experiment, we find best agreement for a slightly lower value for \( C_{in} \), which is probably due to differences in the coil geometry between the devices or between the experimental runs.

Appendix D: Determination of the switching rate \( \Gamma \)

The experimental determination of the switching rate \( \Gamma \) in Fig. 5(b) was performed with a lock-in amplifier (Zurich Instruments HF2LI). We used a sampling rate of 450 Hz and a total measurement time of 300 s for \( f_d \leq 2.3696 \text{ Hz} \) and 60 s for \( f_d > 2.3696 \text{ Hz} \). Counting of the switching events was done with a numerical algorithm that compared the amplitudes and phases of successive measurement points for an entire time trace measurement. Concretely, the program increased the switching counter by 1 if the phase difference of two successive points was above a ‘phase threshold’ (130°) while at least one of the points was above an ‘amplitude threshold’ (0.5 mV), or if exactly one out of two successive points was above the amplitude threshold. The same algorithm was used to evaluate the switching rate in numerical simulations that emulated the measurements (including the effective sampling rate). Note that similar results were obtained by finding the maximal turning point of Allan deviations of the phase.

For a single parametric oscillator, the switching rate was calculated in [50] and is given by

\[
\Gamma = \frac{\left( \gamma \sqrt{\frac{\lambda^2 \omega_0^2}{\gamma^2 \omega_0^2} - 4 - 4\omega + 4\omega_0} \right)}{\sqrt{1 - \frac{\lambda^2 \omega_0^2}{4\gamma^2 \omega_0}}} \exp \left( -\frac{\gamma^2 \omega^2 \left( \frac{\lambda^2 \omega_0^2}{\gamma^2 \omega_0} - 4 - 4\omega + 4\omega_0 \right)^2}{3\alpha \lambda^2 \sigma^2 \omega_0^4} \right) \frac{2\sqrt{2\pi}}{\sqrt{1 - \frac{\lambda^2 \omega_0^2}{4\gamma^2 \omega_0}}} .
\] (D1)

In Fig 8 we verify that our analysis of two coupled resonators is in the low noise limit by showing that Eq. (5) is obeyed. The optimal fit (gray) yields \( \Gamma_0 = 2 \times 10^4 \text{ Hz} \) and \( W = 0.024 \text{ Hz}^{-1}\text{V}^{-2} \). For the numerical switching rate in Fig. 5(b), we used this scaling law to convert the numerical data, simulated at 1.3 times stronger noise, to the experimentally applied noise.

Appendix E: Switching via the antisymmetric state

In Fig. 9, we show examples of timetraces during noise-induced switching between symmetric states. Figure 9(a) corresponds to \( f_d = 2.37 \text{ MHz} \) in region I of Fig. 5, where switches occur via the unstable 0-amplitude state because both resonators switch synchronously. In Fig. 9(b), we show an example for \( f_d = 2.36 \text{ MHz} \) in region IV, where the two resonators switch with a finite delay. In the short time interval between the two switches, the system dwells in the antisymmetric state.
FIG. 9: Examples of switching events in the two-parametron system. (a) For $U_d = 3.7 \text{ V}$ and $f_d = 2.37 \text{ MHz}$ (marked as I in Fig. 5), all four measured coordinates $(u_1, v_1, u_2, v_2)$ switch simultaneously on our sampling timescale. The system switches from one symmetric (S) configuration to the opposite one via the 0-amplitude (0) state. (b) For $U_d = 3.7 \text{ V}$ and $f_d = 2.36 \text{ MHz}$ (marked as IV in Fig. 5), the coordinates $(u_1, v_1)$ switch first, followed by $(u_1, v_1)$ after a delay of roughly 0.5 ms. In the time span between the jumps, the system occupies the antisymmetric (A) state.

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