The Large N Random Phase sine-Gordon Model.

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Abstract.
At large distances and in the low temperature phase, the quenched correlation functions in the 2d random phase sine-Gordon model have been argued to be of the form: \( \langle [\varphi(x) - \varphi(0)]^2 \rangle_* = A(\log |x|) + B\epsilon^2(\log |x|)^2 \), with \( \epsilon = (T - T_c) \). However, renormalization group computations predict \( B \neq 0 \) while variational approaches (which are supposed to be exact for models with a large number of components) give \( B = 0 \).

We introduce a large \( N \) version of the random phase sine-Gordon model. Using non-Abelian bosonization and renormalization group techniques, we show that the correlation functions of our models have the above form but with a coefficient \( B \) suppressed by a factor \( 1/N^3 \) compared to \( A \).

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The 2d random phase sine-Gordon model has been introduced to describe many disordered systems including the 2d XY model in a random field [1], interfacial roughening transition [2], randomly pinned flux lines in superconductors [3], etc.... Its action is:

$$S(\varphi|A_\mu;\xi) = \int \frac{d^2x}{4\pi} \left( K_2 (\partial_\nu \varphi)^2 - A_\nu(x) \partial_\nu \varphi - \xi(x)e^{i\varphi} - \xi^*(x)e^{-i\varphi} \right)$$ (1)

The coupling $K$ is proportional to the inverse temperature, $K \propto 1/T$. In addition to the random phase, we also introduced a random potential since this is required by one-loop renormalization [1]. The quenched random variables $A_\nu(x)$ and $\xi(x)$ are gaussian with law

$$P[A] = \exp \left[ -\frac{1}{2g} \int \frac{d^2x}{4\pi} A_\mu A_\mu \right] \quad \text{and} \quad P[\xi] = \exp \left[ -\frac{1}{2\sigma} \int \frac{d^2x}{4\pi} \xi \xi^* \right]$$ (2)

There are two different phases: a low temperature phase $K > K_c = 1$ in which the disorder is relevant, and a high temperature phase $K < K_c = 1$ in which the disorder is irrelevant. At $K = K_c$, the disorder is marginally irrelevant and only induces logarithmic corrections to the pure system. Of particular interest is the large distance behavior of the quenched average of the correlation functions of $\varphi(x)$. The first proposal for this correlation in the low temperature phase was found using renormalization group (RG) techniques to be of the following form [1, 4, 5]:

$$\left\langle |\varphi(x) - \varphi(0)|^2 \right\rangle_s = A(\log |x|) + B\epsilon^2(\log |x|)^2$$ (3)

with $B \neq 0$ and $\epsilon = \left( \frac{K - K_c}{K_c} \right) \ll 1$. There is a crossover from a $(\log |x|)$ to a $(\log |x|)^2$ behavior. The crossover length $R_{cross}$ is such that $(\log R_{cross}) \sim \frac{A}{B\epsilon^2}$, for $\epsilon = \left( \frac{K - K_c}{K_c} \right) \ll 1$. It is exponentially large close to the phase transition.

However the formula (3) is still controversial, theoretically as well as numerically. In fact variational approaches predict $B = 0$ and a $(\log |x|)$ behavior [3, 4, 5]. The RG flows was also found to be unstable again asymmetric replica perturbations [6]. The numerical verifications of (3) are also not settled: a $(\log |x|)$ behavior was found in ref. [10, 11], while more recent simulations [12] seem to indicate a $(\log |x|)^2$ behavior. In view of this conflict, and since the variational approaches are argued to be exact for systems with a large number of components [13], we studied a large $N$ version of the model [1]. As explained below, using RG computations based on the (a priori symmetric) replica trick, we find that our large $N$ model possesses a non-trivial infrared fixed point in which the correlation functions have the form (3) but with a coefficient $B$ suppressed by a factor $(1/N^3)$ compared to $A$. The occurrence of this factor could explain why the $(\log |x|)^2$ term does not manifest itself in the variational approaches.

\begin{itemize}
  \item *The large $N$ bosonic and fermionic actions.*
\end{itemize}

To introduce the large $N$ version of (1), it is convenient to fermionize it. The fermionic form of the random phase sine-Gordon model is a massless Thirring model coupled to a quenched potential $A_\mu$ and a random phase $\xi$. To define its large $N$ version, we need to introduce $N$ Dirac fermions with components $\psi^k_\pm$ and $\bar{\psi}^k_\pm$ with $k = 1, \ldots, N$. Let $z = x + iy$ and $\overline{z} = x - iy$ be the complex coordinates on the plane. The action is:

$$S^{(N)} = \int \frac{d^2x}{\pi} \left( \sum_{k=1}^{N} (\psi_{-k}\overline{\partial}_+ \psi^k_+ + \overline{\psi}_{-k}\partial_+ \overline{\psi}^k_+) - \frac{a}{N} (\sum_{k=1}^{N} \psi_{-k}\psi^k_+) (\sum_{k=1}^{N} \overline{\psi}_{-k}\overline{\psi}^k_+) \right)$$ (4)

$$- \int \frac{d^2x}{\pi} \left( A_z (\sum_{k=1}^{N} \psi_{-k}\psi^k_+) + A_z (\sum_{k=1}^{N} \overline{\psi}_{-k}\overline{\psi}^k_+) + \xi (\sum_{k=1}^{N} \psi_{-k}\overline{\psi}^k_+) + \xi^* (\sum_{k=1}^{N} \psi_{-k}\psi^k_+) \right)$$
In absence of disorder, it is conformally invariant. The random potential \( A_\mu = (A_\sigma, A_\tau) \) is coupled to the \( U(1) \) currents \( J_\tau = \sum_{k=1}^N \psi_{-k} \psi_k^* \) and \( \overline{J}_\tau = \sum_{k=1}^N \overline{\psi}_{-k} \overline{\psi}_k^* \) of the unperturbed theory. At \( \xi = 0 \), the random potential does not break conformal invariance.

There are a priori many ways to generalize the action (1) to a large \( N \) version. The action (1) has been designed in such way as (i) to keep the number of disordered variables fixed, (ii) to preserve the exact conformal invariance in absence of disorder, and (iii) to parallel as much as possible standard properties of large \( N \) models. In particular, we can implement a Hubbard-Stratonovitch transformation in order to disentangle the quartic interaction in (1). The action is then quadratic in the \( N \) fermions and we can integrate over them. This leads to the following representation of the partition function at fixed disorder:

\[
\int D\psi \exp \left( -S^{(N)} \right) = \int DQ \exp \left[ -N \left( \frac{1}{a} \int \frac{d^2 x}{\pi} Q_\sigma Q_\tau - Tr (\log H) \right) \right]
\]

with \( H = \left( \partial_\tau - A_\tau - Q_\tau, \xi^* - \partial_\tau - A_\tau - Q_\tau \right) \). Therefore, at large \( N \) the path integral (1) is dominated by the saddle point as usual in large \( N \) technique. However, this is not the way we will pursue.

The fermionic action (1) can be bosonized back using non-Abelian bosonization [14]. As usual, since the pure system describes \( N \) Dirac fermions, the pure bosonized theory will be described by a \( su(N) \) Wess-Zumino-Witten (WZW) model at level one plus a massless free field. The \( su(N) \) WZW model at level one possesses primary fields taking values in the \((N - 1)\) fundamental representations of \( su(N) \). Let \( \phi^k_\sigma \) and \( \phi^k_\tau \) be the chiral WZW primary fields which takes values in the defining representation of \( su(N) \) and in its complex conjugate. Their conformal weights are both equal to \( \frac{(N-1)}{2N} \). Let us denote by \( \varphi \) the gaussian free field. The original fermions \( \psi_{\pm}^k \) can be written as the product of these WZW primary fields by a vertex operator of the gaussian model. Namely, \( \psi^k_{+} = \phi^k_\sigma e^{\sqrt{N} \varphi} \) and \( \psi_{-k} = \phi^k_\tau e^{-\sqrt{N} \varphi} \), and similarly for the other chiral components \( \overline{\psi}^k_{+} \) and \( \overline{\psi}_{-k} \). The bosonic form of the action (1) is:

\[
S^{(N)} = S_{wzw} + \frac{K}{2} \int \frac{d^2 x}{4\pi} (\partial_\nu \varphi)^2 \\
- \int \frac{d^2 x}{4\pi} \left( A_\nu(x) \partial_\nu \varphi + \xi(x) \Phi e^{\sqrt{N} \varphi} + \xi^*(x) \Phi^* e^{-\sqrt{N} \varphi} \right)
\]

where \( S_{wzw} \) refers to the \( su(N)_1 \) WZW action. We have introduced the composite fields \( \Phi(z, \overline{z}) = \sum_{k=1}^N \phi^k_\sigma(z) \phi^k_\tau(\overline{z}) \) and \( \Phi^* = \sum_{k=1}^N \phi^k_\tau(z) \phi^k_\sigma(\overline{z}) \). Equivalently, \( \Phi = tr_{\phi}(G) \) with \( G \) the group valued field of the WZW model. For \( N = 1 \) the WZW terms are absent and we recover the action (1) of the random phase sine-Gordon model. We used the action (1) as definition of the model.

The effect of the Thirring interaction, specified by the coupling constant \( a \), is summarized in the normalization of the gaussian action for \( \varphi \). That is, \( K \) is a known function of \( a \) whose explicit form is not needed. The coefficient \( K \) fixes the scale of the dimension of the vertex operators: \( \dim \left( e^{\sqrt{N} \varphi} \right) = \frac{1}{K N} \). Since the dimension of \( \Phi \) is \( \left( \frac{N-1}{N} \right) \), the dimension of the field coupled to the random variable \( \xi \) is:

\[
h \equiv \dim \left( \Phi \right) \left( e^{\sqrt{N} \varphi} \right) = \frac{N - 1}{N} + \frac{1}{K N} = 1 + \left( \frac{1 - K}{K N} \right).
\]
From the Harris criterion we know that the disordered variable $\xi$ will be relevant if this dimension is less than one. Thus, the critical temperature is $K_c = 1$ : with our convention the critical temperature is independent of $N$. For $K > K_c$ the disorder is relevant while for $K < K_c$ it is irrelevant.

The WZW model possesses chiral conserved currents, which we denote by $J^a_z(z)$ and $\overline{J}^a_z(\overline{z})$ with $a = 1, \ldots, \dim su(N)$. In order to fixe the normalization in the WZW sector for later convenience, we give the operator product expansions of the primary fields $\phi$ and $\phi^k$ and of the currents:

$$J^a_z(z)J^a_0(0) = \frac{I^{ab}}{z^2} + \frac{if^{abc}}{z} J^c_0(0) + \cdots$$  \hspace{1cm} (8)

$$J^a_z(z)\phi^k_0(0) = \frac{t^a}{z^2} \phi^k_0(0) + \cdots$$  \hspace{1cm} (9)

$$\phi^k_z(z)\phi^j_0(0) = \frac{1}{z^2} \left( \delta^k_j + z I^{-1} t^a \delta^a_{jk} J^a_0(0) \right) + \cdots$$  \hspace{1cm} (10)

where $[t^a, t^b] = -i f^{abc} t^c$ with $f^{abc}$ the structure constants of $su(N)$ and $tr(t^a t^b) = I^{ab}$. Eq.(8) encodes the fact that the level of the representation of the $su(N)$ affine algebra is one. We will also need the Casimir operator whose value in the adjoint representation is denoted by $C_G$ and in the defining representation by $C_a$. We have $NC_a = I_d C_G$ with $d_G = \dim (su(N)) = N^2 - 1$. A convenient normalization is $C_G = 2N$, $C_a = (N^2 - 1)/N$ and thus $I = 1$.

Similarly as the random phase sine-Gordon model [5], the large $N$ model possesses a $U(1)$ symmetry which amounts to absorb any translation of $\varphi$ into the random variables $A_\mu$ and $\xi$. Its Noether current is represented by insertions of $(\partial_\mu \varphi)$ in the quenched connected correlation functions. This symmetry implies that all the quenched connected correlation functions of $(\partial_\mu \varphi)$ are unaffected by the disorder. But this symmetry has another consequence : the $g$-dependence of the quenched correlation functions can be factorized. The simplest way to visualize it consists in noticing that the $\xi$-dependence of the action can be absorbed into a shift of $\varphi$. Indeed, let us decompose $A_\mu$ as $A_\mu = \partial_\mu \Lambda + \epsilon_\mu \partial_\nu \zeta$, (this is always possible on the plane). The field $\zeta$ decouples from the action and from the measure. It is therefore irrelevant and we can set it to zero. Denoting by $S^{(N)}(\varphi|\Lambda; \xi)$ the action (3) with $A_\mu = \partial_\mu \Lambda$, we have:

$$S^{(N)}(\varphi|\Lambda, \xi) = S^{(N)}(\varphi - \Lambda/K|\Lambda = 0, \hat{\xi}) - \frac{1}{2K} \int \frac{d^2x}{4\pi} (\partial_\mu (\Lambda))^2,$$

with $\hat{\xi} = \xi e^{i\Lambda/K}$. For the correlation functions involving the vertex operators $e^{i\alpha_\nu \varphi(x_p)}$, this implies:

$$\langle e^{i\alpha_1 \varphi(x_1)} \cdots \rangle_{\Lambda, \xi} = \left( \prod_p e^{i\alpha_\nu A(p, x_p)} \right) \langle e^{i\alpha_1 \varphi(x_1)} \cdots \rangle_{\Lambda = 0, \hat{\xi}}$$  \hspace{1cm} (11)

Let $G_{\alpha_1, \ldots, (x_1, \ldots |g, \sigma)$ be their quenched averages. Using the fact that $\hat{\xi}$ and $\xi$ have the same measure, and integrating [11] over $\Lambda$ using the free field gaussian measure [2], $P[A_\mu] = \exp \left[-\frac{1}{2g} \int \frac{d^2x}{4\pi} (\partial_\mu \Lambda)^2 \right]$ with $A_\mu = \partial_\mu \Lambda$, we deduce that:

$$G_{\alpha_1, \ldots, (x_1, \ldots |g, \sigma) = \prod_{p < q} |x_p - x_q|^{2g\alpha_p \alpha_q / K^2} \hspace{1cm} G_{\alpha_1, \ldots, (x_1, \ldots |g = 0, \sigma)$$  \hspace{1cm} (12)
Equivalently,
\[
\partial_y G_{\alpha_1 \cdots \alpha_n | g, \sigma} = \left( \sum_{p < q} \frac{\alpha_p \alpha_q}{K^2} \log(|x_p - x_q|^2) \right) G_{\alpha_1 \cdots \alpha_n | g, \sigma}
\]  
(13)

This Ward identity will be useful for analyzing the renormalization group equations.

- **The effective action and the beta functions.**

To compute the renormalization group equations we used the (a priori symmetric) replica trick. Introducing \( n \) copies of the system with the same disorder and then averaging over the disorder gives the following effective action:

\[
S^{(N)}_{\text{eff}} = \sum_r S^{(r)}_{\text{wzw}} + \frac{1}{2} \int \frac{d^2 x}{4\pi} \left( \partial_\mu \varphi^r \right) G_{rs} \left( \partial_\mu \varphi^s \right)
\]

\[-2\sigma \int \frac{d^2 x}{4\pi} \sum_{r \neq s} \left( \Phi^r \Phi^s e^{i\sqrt{N}(\varphi^r - \varphi^s)} \right) - \lambda \int \frac{d^2 x}{4\pi} \sum_r \left( \sum_a J^a \tilde{T}^a \right)
\]

(14)

with \( G_{rs} = K\delta_{rs} - g \). In eq. (14), the indices \( r, s, \cdots \) refer to the replica index and therefore run from 1 to \( n \). The fields with a replica index refer to the copies of the original fields in the \( r \)th replicated system. The extra \( \lambda \)-term arises from the regularization of the \( \sigma \)-term for \( r = s \). Even if we would not have introduced it at this point, it would have been generated at one-loop. Since it does not couple the replica, we could have introduced it in eq. (6). There it represents a current-current interaction in the WZW sector.

Since the interaction only involves the difference of the fields \( \varphi^r \), it is convenient to decompose their kinetic term as follows:

\[
\frac{1}{2} \left( \partial_\mu \varphi^r \right) G_{rs} \left( \partial_\mu \varphi^s \right) = \frac{(K - ng)}{2n} \left( \partial_\mu \left( \sum_r \varphi^r \right) \right)^2 + \frac{K}{4n} \sum_{r \neq s} \left( \partial_\mu (\varphi^r - \varphi^s) \right)^2
\]

(15)

The field \( \sum_r \partial_\mu \varphi^r \) decouples. Since its correlation functions represent the averages of the connected correlation functions of the \( U(1) \) current, we recover that these correlations are unaffected by the disorder. The Ward identity (13) can also be recovered using this decomposition.

The renormalization group allows us to perturbatively analyse the behavior of the system in the low temperature phase. We present a one-loop computation, which is valid close to the critical temperature, i.e. \( \left( \frac{K - K_c}{K_c} \right) \ll 1 \). We recall that if the partition function and correlation functions are computed with the measure \( \int D\phi \exp(-S) \) with \( S = S_\ast + \sum_i g_i \int d^2 x O_i(x) \), where \( S_\ast \) is the unperturbed fixed point action and \( O_i(x) \) are a set of relevant primary operators of dimension \( h_i \), the beta functions at one-loop are:

\[
\dot{g}^i = \beta^i(g) = (2 - h_i) g^i - \pi \sum_{jk} C^i_{jk} g^j g^k + \cdots
\]

(16)

The summation in the second term is over all the relevant fields generated by the operator product expansions, the coefficients \( C_{jk} \) are determined by the relation:

\[
O_i(x) O_j(0) = \sum_k |x|^{h_k - h_i - h_j} C^k_{ij} O_k(0) + \cdots
\]
Let us introduce the following notation for the perturbing fields in the action (14):

\[ O_1 = \sum_{r \neq s} \phi^r \Phi^s e^{\sqrt{N} (\varphi^r - \varphi^s)} \]  
(17)

\[ O_2 = \frac{1}{2n} \sum_{r,s} i \partial_z (\varphi^r - \varphi^s) i \partial_z (\varphi^r - \varphi^s) \]  
(18)

\[ O_3 = \sum_r \left( \sum_a J^a_z J^a_z \right) \]  
(19)

The fields \( O_1 \) and \( O_3 \) are the perturbing fields associated to the coupling constants \( \sigma \) and \( \lambda \), respectively. The field \( O_2 \) is one of the kinetic field for the \( \varphi^r \)'s. We have the following operator product expansions:

\[ O_1(z)O_1(0) = \frac{2N(n - 2)}{|z|^{2h}} O_1(0) + \frac{2Nn}{|z|^{2h-2}} O_2(0) - \frac{2N(n - 1)}{|z|^{2h-2}} O_3(0) + \cdots \]

\[ O_3(z)O_3(0) = - C_G |z|^2 O_3(0) + \cdots \]  
(20)

\[ O_3(z)O_1(0) = - \frac{2C_G}{|z|^2} O_1(0) + \cdots \]

\[ O_2(z)O_1(0) = \frac{2/NK^2}{|z|^2} O_1(0) + \cdots \]

where the dots refer to irrelevant terms. The operator product expansions \( O_2(z)O_2(0) \) and \( O_3(z)O_2(0) \) only contain irrelevant terms. Using the formula (16) we get the beta functions at finite \( n \). Since the field (\( \sum_r \varphi^r \)) decouples, the coupling (\( K - ng \)) is unrenormalized at any order in perturbation theory, and \( \beta_K = n \beta_g \). Setting \( n = 0 \) as required by the replica trick, we get for \( \frac{K-K_c}{K_c} \ll 1 \):

\[ \beta_\sigma = \frac{2}{N} \left( \frac{K-K_c}{K_c} \right) \sigma - 2N\sigma^2 - C_G \lambda \sigma + \cdots \]

\[ \beta_\lambda = - \left( \frac{C_G}{4} \lambda^2 - \frac{2N}{I_a} \sigma^2 \right) + \cdots \]  
(21)

\[ \beta_g = \frac{N}{2} \sigma^2 + \cdots \]

and \( \beta_K = 0 \). Here the dots refer to higher loop contributions. So, \( K \) is unrenormalized at \( n = 0 \). These equations show that the couplings to \( A_\mu \) and to \( O_3 \) are generated at one-loop even if one does not include them at tree level. Notice the opposite signs in the \( \lambda^2 \) and \( \sigma^2 \) contribution to \( \beta_\lambda \).

From equation (21), we immediately see that in the low temperature phase \( (K > K_c) \), the beta functions \( \beta_\sigma \) and \( \beta_\lambda \) possess a non-trivial infrared zero at \( \sigma_* \) and \( \lambda_* \) given by:

\[ \lambda_*^2 = 8 \left( \frac{d_G}{G^2 C_G} \right) \sigma_*^2 \]

\[ \frac{2}{N} \left( \frac{K-K_c}{K_c} \right) = 2N\sigma_* + C_G \lambda_* \sim_{N \to \infty} 4N \sigma_* \]  
(22)

The last beta function \( \beta_g \) does not vanish at \( \sigma_* \), \( \lambda_* \), but it behaves like:

\[ \beta_g^* = \frac{1}{8N^2} \left( \frac{K-K_c}{K_c} \right)^2 + \cdots, \quad \text{for} \quad N \gg 1, \quad \frac{K-K_c}{K_c} \ll 1 \]  
(23)
Hence, even at $\sigma = \sigma_*$ and $\lambda = \lambda_*$, the coupling $g$ will continue to flow. We may characterize such pseudo-fixed point as a “run away fixed point”.

• The RG equations and the quenched correlation functions.

The non-vanishing of the beta function $\beta_g$ at the infrared pseudo-fixed point has direct consequences. Consider the quenched correlation functions $G_{\alpha_1,\ldots}(x_1, \cdots)$ involving the vertex operators $e^{i\alpha \varphi(x_p)}$. They satisfy the renormalization group equations:

$$\left[ \sum_p x_p^\nu \frac{\partial}{\partial x_p^\nu} + \sum_p \gamma_p - \sum_{c=\sigma,\lambda} \beta_c \frac{\partial}{\partial c} \right] G_{\alpha_1,\ldots}(x_1, \cdots) = 0$$

Here $\gamma_p$ are the anomalous dimensions. Since the $g$-dependence is explicitly known from eq.(13), we can separate the function $\beta_g$ from $\beta_\sigma$ and $\beta_\lambda$ to get:

$$\left[ \sum_p x_p^\nu \frac{\partial}{\partial x_p^\nu} + \sum_p \gamma_p - \sum_{c=\sigma,\lambda} \beta_c \frac{\partial}{\partial c} - \beta_g \sum_{p<q} \frac{\alpha_p \alpha_q}{K^2} \log(|x_p - x_q|^2) \right] G_{\alpha_1,\ldots}(x_1, \cdots) = 0$$

In particular at the infrared fixed point $\sigma_*, \lambda_*$, in which $\beta_\sigma^* = \beta_\lambda^* = 0$, we get:

$$\left[ \sum_p x_p^\nu \frac{\partial}{\partial x_p^\nu} + \sum_p \gamma_p^* - \beta_g^* \sum_{p<q} \frac{\alpha_p \alpha_q}{K^2} \log(|x_p - x_q|^2) \right] G_{\alpha_1,\ldots}^*(x_1, \cdots) = 0$$

where $\gamma_p^*$ are the values of the anomalous dimensions at the infrared fixed point and $\beta_g^*$ is given in eq.(23) for large $N$ and for $(K - K_c) \ll 1$. The net effect of the $g$-flow is to add the extra logarithmic term in the renormalization group equations (24).

Eq.(24) can be used to compute two-point functions at the infrared fixed point. Consider:

$$G_1(x) = \langle \exp(i\alpha (\varphi(x) - \varphi(0))) \rangle_*$$

$$G_2(x) = \langle \exp(i\alpha \varphi(x)) \rangle \langle \exp(-i\alpha \varphi(0)) \rangle_*$$

Let $\gamma_{1,2}^*$ be their anomalous dimensions at the infrared fixed point. The renormalization group equation (24) gives:

$$G_{1,2}(x) = |x|^{-2\gamma_{1,2}^*} \exp \left( -\frac{\alpha^2 \beta_g^*}{2K^2} (\log |x|)^2 \right)$$

Notice the $(\log |x|)^2$ correction which arises from the renormalization of $g$. The anomalous dimensions are even in the charge $\alpha$ and vanishes at $\alpha = 0$, therefore:

$$\gamma_{1,2}^* = \frac{\alpha^2}{K} \rho_{1,2}^* + O(\alpha^4), \quad \text{with} \quad \rho_1^* = 1 + O \left( \frac{K - K_c}{K_c} \right)$$

Expanding eq.(25) in power of $\alpha^2$ gives the two-point functions of $\varphi$:

$$\langle (\varphi(x) - \varphi(0))^2 \rangle_* = \frac{2\rho_1^*}{K} \log |x| + \frac{\beta_g^*}{2K^2} (\log |x|)^2$$

$$\left[ \langle \varphi(x) - \varphi(0) \rangle \right]^2_* = \frac{2\rho_2^*}{K} \log |x| + \frac{\beta_g^*}{2K^2} (\log |x|)^2$$

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Comparing the formula for $\rho_1^*$ and $\beta_g^*$ close to the phase transition, eq. (26,23), we see that at large $N$ the $(\log |x|)^2$ term is suppressed by a factor of $1/N^3$ compared to the $(\log |x|)$ term. Hence there is no a priori contradiction between the replica symmetric RG computations and the variational approaches: they coincide in the infinite $N$ limit, i.e. in the regime where the latter are expected to be exact. It would be interesting to compute the $1/N$ corrections in the variational approaches for comparison.

Note that the $(\log |x|)^2$ cancel in the connected correlation function $\langle [\varphi(x) - \varphi(0)]^2\rangle_{\text{conn}}$ as it should be, since this connected correlation function is unaffected by the disorder. Eq. (27, 28) are exact to all orders in perturbation theory provided we may trust the renormalization group using symmetric replica. Only the exact value of the beta function $\beta_g^*$ depends on the order of the perturbation theory.

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