ON THE ANTIDERIVATIVE OF INVERSE FUNCTIONS

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Abstract. One of the basics of calculus is the following proposition: If $F$ and $G$ are antiderivatives of two real functions $f$ and $g$ resp., then an antiderivative of $\lambda f + g$ is $\lambda F + G$. It may be surprising that such systematic an integration formula exists for the inverse of a function $f$, a fact that seems to have been discovered for the first time by Laisant in 1905 ([2]), and seems to be still insufficiently known. More precisely, if $f$ is an invertible real function, and if $F$ is an antiderivative of $f$, then the antiderivative of $f^{-1}$ is $x f^{-1}(x) - F \circ f^{-1}(x) + C$. Laisant, and other authors after him (e.g. [3,4]), assumes that $f^{-1}$ is differentiable, in which case the proof of this formula is immediate. Recently, it has been shown by Key that this additional assumption is unnecessary ([1]). In this paper, we give two different proofs of this result. The first proof, of geometrical spirit, relies to Fubini’s theorem, while the second proof, purely analytic, is based on the Stieltjes integral.

Key Words: integral of inverse functions, antiderivative of inverse functions, integration formula for inverse functions.

Apparently, the aforementioned result has been first discovered by Laisant in 1905 ([2]), but seems to be unconsciously used by mathematicians, each time they need to compute the integral of the inverse of a function $f$. This elementary theorem was rediscovered several time after Laisant (see e.g. [3,4]), assuming furthermore that $f$ is differentiable, in which case the proof of the formula is immediate. It seems that Key was the first who proved the exactness of this formula, without this additional assumption ([1]). In his article, he uses it as an intermediate tool to prove that the shell and disk methods for computing the volume of a solid of revolution agree.

The author of the present article, which was unaware of the papers of Laisant and Key, was lead to the same result by simple geometric considerations in 1999. He then asked several mathematicians (some of renown) if this theorem was not unknown to them. They replied that they are unaware of any previous statement of this theorem, and that they are rather surprised that such elementary a result is not included in any introducing book of calculus.

Actually, if this theorem has already been discovered in the far past, it would be interesting to understand why it is not widely known at the elementary level. Also, it may be equally interesting to understand why this result was not sufficiently spread since it was published in 1994. The answer to this question may be that it is relatively recent (only twenty years). Another answer may be the
way Key presented this theorem: The main topic, in his paper, is the proof of
the agreement of the shell and disk methods. It is not entirely clear, from his
article, that the intermediate proposition is a result comparable, in usefulness,
to the integration by part formula.

The proof of Key is based on the very definition of the Riemann-Darboux in-
tegral, that is, the upper or lower limit of the Riemann-Darboux sums. His
argument is very clear and elegant, but it is based on the theory of Darboux
sums, which is not always taught at the undergraduate level anymore. In con-
trast, the first of the two alternative proofs offered in this article uses more
advanced tools of integral calculus (Fubini’s theorem), but it does not assume
any knowledge of the Riemann-Darboux integral, which makes it independent
of the kind of integral taught to students. The second proof, essentially based
on the change of variable theorem in the Stieltjes integral, has the advantage
to be generalizable to more involved formulae. Also, it should be observed that
the argument needed to establish the change of variable formula in the Stieltjes
integral is very similar, in spirit, to the argument used in the proof of Key.

Here is the precise statement of the aforementioned proposition.

**Theorem 0.1.** Let \( f : [a, b] \to [c, d] \subseteq \mathbb{R} \) be a continuous and invertible function,
with \( a \geq -\infty \) and \( b \leq +\infty \). Then \( f \) and \( f^{-1} \) have antiderivatives, and if \( F \) is
an antiderivative of \( f \), then the antiderivative of \( f^{-1} \) is
\[
G(x) = xf^{-1}(x) - F \circ f^{-1}(x) + C.
\]

**Example:** Let \( f(x) = e^x \); then by the formula above, one gets immediately
\[
\int \log x \, dx = x \log(x) - x + C.
\]

**Proof:** Since \( f \) is continuous and invertible \( [a, b] \to [c, d] \), it is an elementary
result that \( f^{-1} \) must be continuous. Therefore, the fundamental theorem of
calculus implies that both \( f \) and \( f^{-1} \) have antiderivatives in \( [a, b] \) and \( [c, d] \)
respectively.

Notice also that \( f \) and \( f^{-1} \) must be either both increasing, or both decreasing,
as can be seen easily.

Because of the fundamental theorem of calculus, it suffices to show that
\[
G(y) = \int_{c}^{y} f^{-1}(t) \, dt
\]
is equal to \( yf^{-1}(y) - F \circ f^{-1}(y) + C \), with \( C \in \mathbb{R} \) and \( F(x) = \int_{f^{-1}(c)}^{x} f(x) \, dx \).

Observe first that
\[
f^{-1}(t) = \int_{f^{-1}(c)}^{f^{-1}(t)} \, dx + f^{-1}(c),
\]
therefore
\[
G(y) = \int_{c}^{y} \int_{f^{-1}(c)}^{f^{-1}(t)} \, dx \, dt + f^{-1}(c)y + C, \quad \text{with} \quad C = -f^{-1}(c)c. \tag{1}
\]
In a system of axes \((t, x)\), let \(D\) denote the domain between the horizontal straight \(x = f^{-1}(c)\), the curve \(x = f^{-1}(t)\), and the vertical straight \(t = y\). More formally, if \(f^{-1}\) is increasing,
\[
D = \{(x, t) \in \mathbb{R}^2 : c \leq t \leq y \text{ and } f^{-1}(c) \leq x \leq f^{-1}(t)\},
\]
and if \(f^{-1}\) is decreasing,
\[
D = \{(x, t) \in \mathbb{R}^2 : c \leq t \leq y \text{ and } f^{-1}(t) \leq x \leq f^{-1}(c)\}.
\]
Let us put \(\varepsilon = 1\) if \(f^{-1}\) is increasing, and \(\varepsilon = -1\) if \(f^{-1}\) is decreasing. Hence \(\varepsilon = 1\) if \(f^{-1}(c) < f^{-1}(y)\) and \(\varepsilon = -1\) if \(f^{-1}(c) > f^{-1}(y)\).

By Fubini’s theorem,
\[
\int_{c}^{y} \int_{f^{-1}(c)}^{f^{-1}(y)} dx \, dt = \varepsilon \int \int_{D} dt \otimes dx.
\tag{2}
\]

For every fixed \(x\) between \(f^{-1}(c)\) and \(f^{-1}(y)\), the function \(\varphi(t) \mapsto 1_D(t, x)\) is equal to 1 between \(f(x)\) and \(y\), and to 0 otherwise. Indeed, assuming for example that \(f\) is increasing (hence \(f^{-1}\) is increasing), an element \((t, x)\) belongs to \(D\) if and only if \(c \leq t \leq y\) and \(f^{-1}(c) \leq x \leq f^{-1}(t)\). Equivalently, \(c \leq t \leq y\) and \(c = f(a) \leq f(x) \leq t\). This can be put into the more compact form: \(c \leq f(x) \leq t \leq y\).

Similarly, if \(f\) is decreasing (hence so is \(f^{-1}\)), then \((t, x)\) belongs to \(D\) if and only if \(c \leq t \leq y\) and \(f^{-1}(t) \leq x \leq f^{-1}(c)\). Equivalently, \(c \leq t \leq y\) and \(c \leq f(x) \leq t\) (applying \(f\) to the two sides of an inequality reverses the inequality since \(f\) is decreasing). In compact form: \(c \leq f(x) \leq t \leq y\). Thus, the above contention is established.

Fubini’s theorem can be applied again:
\[
\int \int_{D} dt \otimes dx = \int \int \varphi(t) dt \, dx = \varepsilon \int \int_{f^{-1}(c)}^{f^{-1}(y)} \int_{f(x)}^{y} dt \, dx
\]
\[
= \varepsilon \int_{f^{-1}(c)}^{f^{-1}(y)} (y - f(x)) \, dx = \varepsilon \left[yf^{-1}(y) - yf^{-1}(c) - F(f^{-1}(y)) + F(f^{-1}(c))\right]
\]
\[
= \varepsilon \left[yf^{-1}(y) - yf^{-1}(c) - F(f^{-1}(y))\right] + C.
\]

Taking eq. (1) and (2) into account, it follows that
\[
G(y) = yf^{-1}(y) - F(f^{-1}(y)) + C,
\]
as was to be shown. \(\square\)

**Second proof:** As above, we calculate
\[
G(y) = \int_{c}^{y} f^{-1}(y) \, dy.
\]

In what follows, the classic Stieltjes integral, as well as its famous integration by part and change of variable theorems, will be used.
Since \( f \) is an isomorphism \([a, b] \rightarrow [c, d]\), the change of variable formula is licit, that is, with \( y = f(x) \),

\[
G(y) = \int_{f^{-1}(c)}^{x} f^{-1}(f(x)) \, df(x) = \int_{f^{-1}(c)}^{x} x \, df(x).
\]

For the sake of clarity, let us put \( g(x) = x \) and \( \alpha = f^{-1}(c) \) (necessarily, \( \alpha = a \) or \( \alpha = b \)). The integration by part formula for the Stieltjes integral is

\[
\int_{\alpha}^{x} g(x) \, df(x) + \int_{\alpha}^{x} f(x) \, dg(x) = f(x)g(x) - f(\alpha)g(\alpha).
\]

In other words,

\[
\int_{\alpha}^{x} x \, df(x) = f(x)x - \int_{\alpha}^{x} f(x) \, dx + C' \quad \text{with} \quad C' = -f(\alpha)g(\alpha).
\]

Setting \( F(x) = \int_{\alpha}^{x} f(x) \, dx \), and substituting \( x = f^{-1}(y) \) inside this equation, there holds

\[
G(y) = yf^{-1}(y) - F(f^{-1}(y)) + C, \quad \text{with} \quad C = C' - F(\alpha).
\]

This ends the second proof of the theorem.

Remarks:

1) The same approach can be used to obtain more involved formulae containing inverse functions, such as

\[
\int H(y, f^{-1}(y)) \, dy = H(y, f^{-1}(y))y - \int_{\alpha}^{f^{-1}(y)} f(x) \, dH(f(x), x) + C.
\]

2) If it can be supposed that \( f^{-1} \) is absolutely continuous, then a simpler proof can be given, based on known theorems about absolute continuity: Indeed, \( f^{-1} \) is invertible and monotonic by hypothesis, hence, if \( f^{-1} \) is also absolutely continuous, the function

\[
G(x) = xf^{-1}(x) - F \circ f^{-1}(x) + C
\]

is also absolutely continuous, because it is the sum, product, and composition of absolutely continuous and bounded functions. By a known theorem due to Lebesgue, \( f^{-1} \) and \( G \) are differentiable almost everywhere, and are the integral of their derivative. But at every point where \( f^{-1} \) is differentiable, \( G(x) \) is differentiable with derivative equal to \( f^{-1} \). This prove that the integral of \( f^{-1} \) is precisely \( G \).

References

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