Axiomatic, Parameterized, Off-Shell Quantum Field Theory

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Axiomatic QFT attempts to provide a rigorous mathematical foundation for QFT, and it is the basis for proving some important general results, such as the well-known spin-statistics theorem. Free-field QFT meets the axioms of axiomatic QFT, showing they are consistent. Nevertheless, even after more than 50 years, there is still no known non-trivial theory of quantum fields with interactions in four-dimensional Minkowski spacetime that meets the same axioms.

This paper provides a similar axiomatic basis for parameterized QFT, in which an invariant, fifth path parameter is added to the usual four spacetime position arguments of quantum fields. Dynamic evolution is in terms of the path parameter rather than the frame-dependent time coordinate. Further, the states of the theory are allowed to be off shell. Particles are therefore fundamentally “virtual” during interaction but, in the appropriate non-interacting, large-time limit, they dynamically tend towards “physical”, on-shell states.

Unlike traditional QFT, it is possible to define a mathematically consistent interaction picture in parameterized QFT. This may be used to construct interacting fields that meet the same axioms as the corresponding free fields. One can then re-derive the Dyson series for scattering amplitudes, but without the mathematical inconsistency of traditional, perturbative QFT. The present work is limited to the case of scalar fields, and it does not address remaining issues of gauge symmetry and renormalization. Nevertheless, it still demonstrates that the parameterized formalism can provide a consistent foundation for the interpretation of QFT as used in practice and, perhaps, for better dealing with its further mathematical issues.

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I. INTRODUCTION

A. Why Off Shell?

The momentum mass-shell condition, which is \( p^2 = -m^2 \) using a spacetime metric signature of \( (-++++) \), is the fundamental kinematic relationship in special relativity. It therefore seems obvious that it should be carried over to relativistic quantum mechanics. Indeed, the operator form of this condition is just the Klein-Gordon equation in quantum field theory (QFT) and it also determines the spectrum of the relativistic momentum operator.

However, the state space of QFT still naturally includes both on-shell states that meet the mass-shell condition and off-shell states that do not. The “physical,” on-shell states need to be picked out as those satisfying the mass-shell constraint. Nevertheless, off-shell states cannot be completely eliminated from the theory, since propagation is off shell along the internal segments of Feynman diagrams generated using perturbative QFT.

In the usual derivation (as given in any traditional QFT textbook, such as [1] or [2]), off-shell particles appear in the formalism because the end points of an internal segment may have either time ordering. However, physical particles are considered to propagate forward in time, with the on-shell propagator formalism because the end points of an internal segment may have either time ordering. Nevertheless, off-shell states cannot be completely eliminated from the theory, since propagation is off shell along the internal segments of Feynman diagrams generated using perturbative QFT.

In the usual derivation (as given in any traditional QFT textbook, such as [1] or [2]), off-shell particles appear in the formalism because the end points of an internal segment may have either time ordering. However, physical particles are considered to propagate forward in time, with the on-shell propagator

\[
\Delta^+(x-x_0) \equiv (2\pi)^{-3} \int d^3p \frac{e^{i(-\omega_p(x^0-x^0_0)+p(x-x_0))}}{2\omega_p}.
\]

for \( x^0 > x^0_0 \) and \( \omega_p = \sqrt{p^2 + m^2} \). When “backward” time propagation is then re-interpreted as being for an (on-shell) antiparticle propagating forward in time, and the particle and antiparticle propagators are combined, the result is the relativistically invariant Feynman propagator

\[
\Delta(x-x_0) = \theta(x^0 - x^0_0)\Delta^+(x-x_0) + \theta(x^0_0 - x^0)\Delta^+(x-x_0)^* = -i(2\pi)^{-4} \int d^4p \frac{e^{i(p(x-x_0))}}{p^2 + m^2 - i\varepsilon}.
\]

And this allows for the transfer of any possible four-momentum, not just a momentum that meets the mass-shell condition.

Such internal segments are said to represent virtual particles because, of course, “real,” physical particles must be on shell. However, as Feynman noted [3], “real” particles actually always have a finite lifetime (at least if they are ever to be detected), and so effectively propagate on finite-length internal segments of Feynman diagrams. As a result, they are all technically virtual particles!

Of course, a particle does not have to propagate very far before the interaction by which it is observed is essentially in the future light cone of all classical observers of interest. In this case, the particle is unambiguously in the future of all such observers and can be considered to be on shell. But still, fundamentally, a particle is only truly, exactly on shell in the limit of propagation into the infinite future without interaction, at which point it can be considered to be unambiguously in the future of all potential observers (except, of course, that without interaction it can never really be observed).

Now, considerations such as the above may seem purely philosophical. However, they do suggest the possibility that, perhaps, one could formulate an interaction theory that starts by considering (so-called) virtual, off-shell particles to be the “true” particles. On-shell particles would then be considered a limiting case of particles with an infinite lifetime and a useful approximation for particles with finite but “long” lifetimes.

The important benefit of this alternative viewpoint is that it simplifies the treatment of the state space and allows for state dynamics to be formulated in a way that is closely analogous to non-relativistic quantum mechanics. However, dynamic state evolution could no longer be in terms of time, as it is in non-relativistic quantum mechanics, because this would be relativistically frame dependent. Instead, evolution must be in terms of some additional, invariant parameter. But what, then, is this parameter, and how might relativistic quantum mechanics be formulated in terms of it?

B. Why Parameterized?

In non-relativistic quantum mechanics, the Hamiltonian operator \( \hat{H} \) determines the time evolution of a (Schrödinger-picture) wave function \( \psi(t) \) via the Schrödinger equation

\[
\frac{i}{\hbar} \frac{d\psi(t)}{dt} = \hat{H}\psi(t)
\]
(here and in the following I take $\hbar = 1$). Importantly, while $\hat{H}$ also represents the observable energy of the system, there is no “time operator” – time is not an observable in non-relativistic quantum mechanics, it is just an evolution parameter.

In relativistic quantum mechanics, the Hamiltonian can also be taken to be the energy operator and the generator of time translations. However, this operator is now just the time component $\hat{P}^0$ of the relativistic four-vector momentum operator $\hat{P}$. And, of course, such a single component is not Lorentz invariant, so there can be no relativistically invariant analog of the Schrödinger equation using this definition of the Hamiltonian. Further, the position operator $X^0$ conjugate to $\hat{P}^0$ is a (frame-dependent) observable for time, which, therefore, can no longer be considered to be just an evolution parameter.

On the other hand, one can instead define a truly relativistic Hamiltonian which, for a free particle, is $\hat{H} = \hat{P}^2 + m^2$. However, applying this operator to an on-shell particle state $\psi$ then gives $\hat{H}\psi = 0$. That is, the Hamiltonian vanishes identically when applied to on-shell states. Particularly when interactions and gravitation are included in $\hat{H}$, this equation is known as the Wheeler-DeWitt equation, which is fundamental for quantum gravity and cosmology [4, 5].

Thus, rather than being a generator of system evolution, the Hamiltonian acts to impose a constraint on all physical states. A Hamiltonian-based mechanics for on-shell relativistic quantum mechanics therefore requires the use of Dirac’s theory of constraints [6, 7], a powerful but complicating generalization of standard Hamiltonian mechanics. And, in general, the Wheeler-DeWitt equation is not easy to solve. (For example, see the extensive discussion in [8] on Hamiltonian constraint mechanics and solving the Wheeler-DeWitt equation in the context of quantum gravity.)

But now suppose we did not require $\psi$ to be on shell. In this case we would have $\hat{H}\psi \neq 0$. Indeed, by analogy with the non-relativistic case, we could then consider $\hat{H}$ to generate evolution in a relativistically invariant parameter $\lambda$, such that

$$-\frac{i}{\hbar}\frac{d\psi(\lambda)}{d\lambda} = \hat{H}\psi(\lambda),$$

(3)
a clear analog of the non-relativistic Schrödinger equation. A parameterized formalism for relativistic quantum mechanics is one that includes an invariant evolution parameter such as $\lambda$.

But what exactly is this evolution parameter?

To see, consider that, for wave functions in the position representation, such a parameter acts as a fifth argument, in addition to the usual four position arguments of Minkowski space. Using the relativistic Hamiltonian operator in the position representation,

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + m^2,$$

the solution to Eq. (3) is

$$\psi(x; \lambda) = \int d^4x_0 \Delta(x - x_0; \lambda - \lambda_0)\psi(x_0; \lambda_0).$$

(4)

with the propagation kernel

$$\Delta(x - x_0; \lambda - \lambda_0) = (2\pi)^{-4} \int d^4p e^{ip(x-x_0)}e^{-i(p^2+m^2)(\lambda-\lambda_0)}.$$  

(5)

The wave function $\psi(x; \lambda)$ is just the parameterized probability amplitude function defined by Stueckelberg [8]. If we take the path of a particle to be a curve in spacetime, $x^\mu = q^\mu(\lambda)$, with $\lambda$ as its path parameter, then the $\psi(x; \lambda)$ represent the probability amplitude for a particle to reach position $x$ at the point along its path with parameter value $\lambda$.

This viewpoint of $\lambda$ as a path parameter is reinforced by the observation, first made by Feynman [10], that the propagation kernel given in Eq. (5) can be written in the form of a spacetime path integral

$$\Delta(x - x_0; \lambda_1 - \lambda_0) = \eta \int D^4q \delta^4(q(\lambda_1) - x)\delta^4(q(\lambda_0) - x_0) \exp \left(i \int_{\lambda_0}^{\lambda_1} d\lambda L(q^2(\lambda)) \right)$$

(6)

for an appropriate normalization constant $\eta$ and the Lagrangian function

$$L(q^2) = \frac{1}{4}q^2 - m^2.$$
In the path integral above, the notation $D^4q$ indicates that the integral is over the four functions $q^\mu(\lambda)$ and the delta functions constrain the starting and ending points of the paths integrated over. Further, if we integrate this over paths from $x_0$ to $x$ with all possible intrinsic lengths $\lambda_1 - \lambda_0$, the result is just the Feynman propagator \cite{10–13},

$$\Delta(x - x_0) = \int_{\lambda_0}^{\lambda_1} d\lambda_1 \Delta(x - x_0; \lambda_1 - \lambda_0). \quad (7)$$

Further, consider a state $\psi$ that maintains a fixed three-momentum $p$, and integrate it over all paths that end at some future time $t$. Then, as shown in \cite{13}, as $t \to \infty$, the energy $p^0 \to \sqrt{p^2 + m^2}$. That is, in this limit, we dynamically recover the mass-shell constraint $p^2 = -(p^0)^2 + p^2 = -m^2$, as desired.

Thus, we can think of the $\lambda$ as parameterizing the position of a particle along its path in spacetime. Note, however, that such a parameter is not necessarily the proper time for the particle, as has sometimes suggested. Indeed, constraining $\lambda$ to be the proper time in the paths integrated over in Eq. \textcolor{red}{(6)} would mean constraining the particle four-velocity to be timelike, which is equivalent to re-introducing the mass-shell constraint. The result would not then be the relativistic propagation kernel, and the integral in Eq. \textcolor{red}{(7)} would not produce the Feynman propagator. Thus, for an off-shell theory, it is necessary to consider all possible paths, not just ones with timelike four-velocities.

There is actually a long history of approaches using such a fifth parameter for relativistic quantum mechanics, going back to proposals in the late thirties and early forties of Fock \cite{14} as well as Stueckelberg \cite{9,15}. The idea appeared subsequently in the work of a number of well-known authors, including Nambu \cite{16}, Feynman \cite{10,17}, Schwinger \cite{18}, DeWitt-Morette \cite{19} and Cooke \cite{20}. However, it was not until the seventies and eighties that the theory was more fully developed, particularly by Horwitz and Piron \cite{21,22} and Fanchi and Collins \cite{23–26}, into what has come to be called relativistic dynamics. The approach is particularly applicable to the study of quantum gravity and cosmology, in which the fundamental equations (such as the Wheeler-DeWitt equation) make no explicit distinction for the time coordinate (see, e.g., \cite{3,11,12,27–30}).

Extension of relativistic dynamics to a second-quantized QFT has been somewhat more limited, focusing largely on application to quantum electrodynamics \cite{31–37}. I have previously proposed a foundational parameterized formalism for QFT and scattering based on spacetime paths \cite{13,38–40}, along the lines that I outlined above. The purpose of the present paper is to provide a more formal grounding for parameterized QFT, using an axiomatic approach. While the concept of paths will occasionally still be helpful for intuitive motivation, the theory is presented here entirely in field-theoretic mathematical language, without the use of spacetime path integrals.

### C. Why Axiomatic?

Beginning in the 1950s and 1960s, a number of researchers attempted to provide rigorous foundations for QFT. Gårding and Wightman introduced a set of precise axioms for QFT on Minkowski space \cite{41,42}, and Haag and Kastler developed a related approach to local functions of the field \cite{43}. These approaches had some early important successes, including modeling free-field theory, proving that a field theory can be reconstructed from its vacuum expectation values and rigorously establishing the link between spin and statistics (see, for example, \cite{14} and \cite{15}). This early work also has led to considerable subsequent mathematical work on algebraic, Euclidean and constructive QFT.

In addition, axiomatic QFT provided the basis for the first rigorous proof of Haag’s theorem. This theorem states that, under the Wightman axioms for QFT, any field that is unitarily equivalent to a free field must itself be a free field \cite{44,46,47}. This is troublesome, because the usual Dyson perturbation expansion of the scattering matrix is based on the interaction picture, in which the interacting field is presumed to be related to the free field by a unitary transformation. And, as Streeter and Wightman note \cite{14}, Haag’s theorem means that such a picture should not exist in the presence of actual interaction.

Indeed, after more than 50 years, there is still no known, non-trivial, interacting QFT in four-dimensional Minkowski space that satisfies all of the Wightman axioms. While the existence of such a theory has not been ruled out (and there has been progress for simplified cases, such as in fewer dimensions), this situation does suggest that perhaps the Wightman axioms do not provide the most useful formal grounding for interacting QFT after all.

In this paper, I propose a set of axioms for parameterized QFT that are inspired by the Wightman axioms, but differ in some important ways. In particular, the underlying states of the theory are not required to be on shell, which means that the spacetime symmetry group of a particle field is no longer restricted to just a single irreducible representation of the Poincaré group. This also requires a different treatment of Hamiltonian operators, which act as generators of parameter evolution (as foreshadowed in Sec. \textcolor{red}{13}), and for which, also, a vacuum state is only unique relative to a given Hamiltonian.

Section \textcolor{red}{14} presents the formal axiomatic theory of parameterized QFT, starting with the axioms for the Hilbert space of states and for the field operators defined on those states. Not surprisingly, it is straightforward to construct
a theory of free fields that meets the given axioms. However, it turns out that, in the parameterized theory, it is, in fact, possible to also straightforwardly construct interacting fields that meet the axioms. The critical result is that the parameterized theory has no analog of Haag’s theorem (for a more in-depth look at why this is so, see also [48]). This means that parameterized QFT has a mathematically consistent interaction picture, allowing for the construction of an interacting field using a unitary transformation from a free field, so that it naturally meets the field axioms.

Section III then takes up the task of applying the theory to the modeling of interactions and computing scattering amplitudes. This development is not complete, because it includes only cursory consideration, at this point, of regularization and renormalization, which are still required to obtain finite results for scattering amplitudes. Nevertheless, it is shown that the parameterized formulation can formally reproduce, term for term, the usual Dyson expansion for a scattering amplitude obtained using traditional perturbative QFT. And, since the use of the interaction picture is mathematically consistent in parameterized QFT, this actually explains how the perturbative expansions in traditional QFT can produce such empirically accurate results, even in the face of Haag’s theorem.

For simplicity, only massive, scalar fields will be considered in this paper. However, the parameterized formalism has been extended to non-scalar fields in other work (see, for example, [39]), and there is no reason to believe this could not be incorporated into the axiomatic formalism, much as for traditional QFT. On the other hand, including massless and gauge fields in the parameterized axiomatic theory is more difficult and is an important avenue for future work (see also [32] and [34] for some earlier work addressing gauge theory in the context of parameterized quantum electrodynamics).

II. THEORY

This section describes the basic theory of parameterized QFT. Sections II A and II B present the axioms of parameterized QFT, generally paralleling the axioms for traditional QFT as presented in [44]. Section II C then demonstrates the consistency of the axioms by constructing a theory of free fields that satisfies them, and Sec. II D discusses general results related to the vacuum expectation values of fields. Section II E shows how the theory can also cover the case of interacting fields, a topic which will be addressed further in Sec. III. Section II F develops the formalism for integrating fields over the path parameter and Sec. II G addresses the related issue of the modeling of antiparticles.

A. Hilbert Space

As in traditional QFT, the states in the theory, over which fields will be defined, are given by the unit rays in a separable Hilbert space \( H \). For any vectors \( \Psi, \Phi \in H \), the inner product defined on \( H \) is denoted \( (\Psi, \Phi) \). For all \( \Psi \in H \), \( \Psi^2 \equiv (\Psi, \Psi) \) must be finite and non-negative (i.e., vectors are normalizable), and \( \Psi^2 = 0 \) if and only if \( \Psi = 0 \).

We can introduce the usual Dirac bra and ket notation by first extending \( H \) to a rigged Hilbert space, or Gel’fand triple \[ K \subset H \subset K' \]

Here, \( K \) is a nuclear space that is a dense subset of \( H \) [45]. This is the space of kets, \( |\Phi\rangle \in K \). \( K' \) is the space of continuous, linear, complex-valued functionals \( F[\Phi] \) on \( K \), known as distributions. This is the space of bras, \( \langle F| \in K' \). (For a complete treatment of bras, kets and Gel’fand triples, see [51].)

For any \( |F\rangle \in K' \), the action of \( \langle F| \) on a ket \( |\Phi\rangle \) is denoted by the bracket

\[
\langle F| \Phi \rangle \equiv F[\Phi].
\]

For any \( \Psi \in H \), there is a dual bra \( \langle \Psi| \) whose action is defined by

\[
\langle \Psi| \Phi \rangle = (\Psi, \Phi).
\]

It is under this duality mapping that \( H \) may be considered a subset of \( K' \).

The following axiom establishes the basic assumptions about \( H \), including its relativistic nature. The assumed relativistic transformation law is the same as in traditional relativistic quantum mechanics, but the axiom does not include the usual spectral conditions on the spacetime translation operator.

**Axiom 0 (Relativistic States).** The states of the theory are the unit rays in a separable Hilbert space \( H \) on which is defined a unitary representation \( \{\Delta x, \Lambda\} \to \hat{U}(\Delta x, \Lambda) \) of the Poincaré group.

Since the spacetime translation operator \( \hat{U}(\Delta x, 1) \) is unitary, it can be written as

\[
\hat{U}(\Delta x, 1) = e^{iP \Delta x},
\]
where $\hat{P}$ is a Hermitian operator (i.e., $\hat{P} = \hat{P}^\dagger$). As usual, we will interpret $\hat{P}$ as the relativistic energy-momentum operator. However, as noted above, we will not require that the eigenvalues of $\hat{P}$ be on-shell. Section III will address the emergence of physical on-shell states in the context of scattering processes.

We will also not take $\hat{P}^0$, the (frame-dependent) energy operator and generator of time translations, to be the Hamiltonian operator. Instead, we allow the separate identification of one or more Hamiltonian operators on $\mathcal{H}$, as long as they meet the following definition.

**Definition** (Hamiltonian operator). An operator $\hat{H}$ on $\mathcal{H}$ with the following properties:

1. $\hat{H}$ is Hermitian.
2. $\hat{H}$ commutes with all spacetime transformations $\hat{U}(\Delta x, \Lambda)$ (and, hence, with $\hat{P}$).
3. $\hat{H}$ has an eigenstate $|0\rangle \in \mathcal{K} \subset \mathcal{H}$ such that $\hat{H}|0\rangle = 0$, and this is the unique null eigenstate for $\hat{H}$, up to a constant phase.

Note that this definition essentially introduces the concept of a vacuum state, but only relative to a choice of Hamiltonian operator. Each Hamiltonian operator must have a unique vacuum state, but different Hamiltonian operators defined on the same Hilbert space may have different vacuum states. On the other hand, the Hamiltonian operator is not required to be positive definite, since it is no longer considered to represent the energy observable, so its vacuum state is not a “ground state” in the traditional sense. However, when $\mathcal{H}$ is a Fock space with a particle interpretation, the vacuum state for an identified Hamiltonian can be considered to be the “no particle” state (as will be the case for the free-field theory defined in Sec. III).

Rather than generating time translation, a Hamiltonian operator $\hat{H}$ is taken instead to generate evolution in the parameter $\lambda$, as given by the exponential operator $e^{-i\hat{H}\lambda}$. Since $\hat{H}$ commutes with $\hat{P}$, momentum is conserved under evolution in $\lambda$. The parameter $\lambda$ itself is not considered to be an observable in the theory, but is, rather, treated as just an evolution parameter, similarly to time in non-relativistic quantum mechanics.

By further analogy with non-relativistic theory, we can define both Schrödinger and Heisenberg pictures for evolution in $\lambda$. That is, a Schrödinger-picture state $\Psi_S(\lambda)$ evolves in $\lambda$ and is related to the corresponding Heisenberg-picture state $\Psi_H(\lambda) = e^{-i\hat{H}\lambda}\Psi_H$, while a Heisenberg-picture operator $A_H(\lambda)$ evolves in $\lambda$ and is related to the corresponding Schrödinger-picture operator $\hat{A}_S$ by $\hat{A}_H(\lambda) = e^{i\hat{H}\lambda}\hat{A}_Se^{-i\hat{H}\lambda}$, so that $\langle \Psi_H|\hat{A}_H(\lambda)|\Phi_H\rangle = \langle \Psi_S(\lambda)|\hat{A}_S|\Phi_S(\lambda)\rangle$, for all $\Psi, \Phi$ and $\lambda$.

The evolution of Schrödinger-picture states can also be given by the Stueckelberg-Schrödinger equation

$$i\frac{d\Psi_S}{d\lambda} = \hat{H}\Psi_S.$$  
(8)

Similarly, Heisenberg-picture operators evolve according to

$$i\frac{d\hat{A}_H}{d\lambda} = [\hat{A}, \hat{H}].$$  
(9)

The single null eigenstate of a Hamiltonian $\hat{H}$ is a vacuum state for $\mathcal{H}$, invariant in the $\lambda$ evolution generated by $\hat{H}$. Such a state must also be invariant under Poincaré transformations, since $\hat{U}(\Delta x, \Lambda)|0\rangle = \hat{U}(\Delta x, \Lambda)|\hat{H}|0\rangle = 0$, so $\hat{U}(\Delta x, \Lambda)|0\rangle$ must equal $|0\rangle$ (up to a phase, which can be absorbed into the definition of $|0\rangle$). Note, however, that $\hat{H}$ is not prohibited from having other eigenstates that are invariant under Poincaré transformations, but evolve in $\lambda$.

**B. Fields**

As in traditional QFT, we can define a field as an operator $\hat{\psi}(x)$ on $\mathcal{H}$, which is a function of spacetime position $x$ (the present paper only covers the case of scalar fields). For the parameterized theory, though, we can also consider both Schrödinger-picture and Heisenberg-picture versions of the field operators, relative to parameter evolution according to a given Hamiltonian $\hat{H}$. Taking $\hat{\psi}(x)$ to be a Schrödinger-picture operator, the corresponding Heisenberg-picture operator is

$$\hat{\psi}(x; \lambda) \equiv e^{i\hat{H}\lambda}\hat{\psi}(x)e^{-i\hat{H}\lambda}.$$  

That is, in the (parameterized) Heisenberg picture, the field operators $\hat{\psi}(x; \lambda)$ are functions of both the spacetime position and the parameter $\lambda$. 
However, fields that are functions of position are generally too singular to be well-defined operators on $\mathcal{H}$. Instead, to obtain mathematically well-defined entities, one must smear the fields with a test function $f(x)$ from the space $\mathcal{D}$ of smooth, parameterized, complex-valued functions with compact support in spacetime and a norm
\[ |f|^2 \equiv \int d^4x f^*(x)f(x) \]
that is finite.

So, for any $f \in \mathcal{D}$, formally define the operator-valued functional
\[ \hat{\psi}[f] = \int d^4x f^*(x)\hat{\psi}(x) = \int d^4p f^*(p)\hat{\psi}(p) \]
and its adjoint
\[ \hat{\psi}^\dagger[f] = \int d^4x f(x)\hat{\psi}^\dagger(x) = \int d^4p f(p)\hat{\psi}^\dagger(p). \]
The position and momentum representations of the functions $f$ and fields $\hat{\psi}$ are related by four-dimensional Fourier transforms:
\[ f(p) = (2\pi)^{-2}\int d^4x e^{-ip\cdot x}f(x) \]
and
\[ \hat{\psi}(p) = (2\pi)^{-2}\int d^4x e^{-ip\cdot x}\hat{\psi}(x). \]
It is the smeared fields $\hat{\psi}[f]$ and $\hat{\psi}^\dagger[f]$ that are actually well-defined operators in the theory.

The Heisenberg-picture versions of the smeared fields are, similarly,
\[ \hat{\psi}[f; \lambda] = \int d^4x f^*(x)\hat{\psi}(x; \lambda) = \int d^4p f^*(p)\hat{\psi}(p; \lambda) \]
and its adjoint
\[ \hat{\psi}^\dagger[f; \lambda] = \int d^4x f(x)\hat{\psi}^\dagger(x; \lambda) = \int d^4p f(p)\hat{\psi}^\dagger(p; \lambda), \]
for any value of the path parameter $\lambda$. Note that the Heisenberg-picture fields are smeared over the spacetime argument, but not the parameter argument.

**Axiom I** (Domain and Continuity of Fields). For all $f \in \mathcal{D}$, the operators $\hat{\psi}[f]$ and their adjoints $\hat{\psi}^\dagger[f]$ are defined on a domain $D$ of states dense in $\mathcal{H}$. The $\hat{U}(\Delta x, \Lambda)$, any Hamiltonian $\hat{H}$, $\hat{\psi}[f]$ and $\hat{\psi}^\dagger[f]$ all carry vectors in $D$ into vectors in $D$.

The domain $D$ always contains the domain $D_0$ of states obtained by applying polynomials in the fields and their adjoints to the vacuum state of a given Hamiltonian $\hat{H}$. Typically, we will be able to assume that $D_0 = D$.

**Axiom II** (Field Transformation Law). For any Poincaré transformation $\{\Delta x, \Lambda\}$, for any $\Psi \in D$,
\[ \hat{U}(\Delta x, \Lambda)\hat{\psi}[f]\hat{U}^{-1}(\Delta x, \Lambda)\Psi = \hat{\psi}[\{\Delta x, \Lambda\} f]\Psi, \]
where
\[ (\{\Delta x, \Lambda\} f)(x) \equiv f(\Lambda^{-1}(x - \Delta x)). \]
This field transformation law is consistent with the spacetime transformation law for states given in Axiom 0 (note again that only scalar particles are being considered in the present paper). While the axiom is given in terms of the Schrödinger-picture field, the commutivity of $\hat{U}(\Delta x, \Lambda)$ and $\hat{H}$ implies that it also applies to the Heisenberg-picture fields.

For the unsmeared field $\hat{\psi}(x)$, the above transformation law for a Poincaré transformation $\{\Delta x, \Lambda\}$ gives
\[ \hat{U}(\Delta x, \Lambda)\hat{\psi}(x)\hat{U}^{-1}(\Delta x, \Lambda) = \hat{\psi}(\Delta x + \Delta x). \]
(13)
Taking the limit of infinitesimal $\Delta x$ with $\Lambda = 1$ then gives
\[ i\frac{\partial}{\partial x}\hat{\psi}(x) = [\hat{\psi}(x), \hat{P}]. \]
Axiom III (Commutation Relations). The field $\hat{\psi}$ and its adjoint satisfy the (bosonic) commutation relations, for all $f', f \in \mathcal{D}$ and any $\Psi \in \mathcal{D}$,

$$[\hat{\psi}[f'], \hat{\psi}[f]]\Psi = [\hat{\psi}^\dagger[f'], \hat{\psi}^\dagger[f]]\Psi = 0$$

and

$$[\hat{\psi}[f'], \hat{\psi}^\dagger[f]]\Psi = (f', f)\Psi,$$

where

$$(f', f) \equiv \int d^4 x f'*(x)f(x).$$

Note that $(f', f) = (|f' + f|^2 - |f'|^2 - |f|^2)/2$, and so this is well-defined if the norm given in Eq. (10) is.

In terms of the unsmeared field, Axiom III gives the four-dimensional commutation relation:

$$[\hat{\psi}(x'), \hat{\psi}^\dagger(x)] = \delta^4(x' - x).$$

or, for the Heisenberg-picture fields, the equal-$\lambda$ commutation relation:

$$[\hat{\psi}(x'; \lambda), \hat{\psi}^\dagger(x; \lambda)] = \delta^4(x' - x).$$

As a four-dimensional commutation relation, this is stronger than the usual “local commutivity” axiom for traditional fields, which only requires that fields commute when the positions $x'$ and $x$ are spacelike. In contrast, Eq. (14) requires that fields commute for all $x'$ and $x$ other than $x' = x$.

Further, since the Heisenberg-picture field operators are smeared over the four-position $x$, but not the parameter $\lambda$, there is no mathematical issue with going from the unsmeared form of the commutation relations to the more rigorously defined smeared form. This is in contrast to the equal-time commutation relations commonly imposed in traditional QFT, which would require fields to make sense as operators when smeared over only the three-position, an additional strong, and possibly questionable, assumption (see the discussion in [44], p. 101).

Axiom IV (Cyclicity of the Vacuum). If $\hat{H}$ is a Hamiltonian operator, then its vacuum state $|0\rangle$ is in the domain $\mathcal{D}$ of the field operators, and polynomials in the fields $\hat{\psi}[f]$ and their adjoints $\hat{\psi}^\dagger[f]$, when applied to $|0\rangle$, yield a set $\mathcal{D}_0$ of states dense in $\mathcal{H}$.

An operator $\hat{A}$ constructed as a polynomial in the smeared fields is a functional

$$\hat{A}[[f_i], \{g_j\}] = \hat{A}(\{\hat{\psi}_{n_i}[f_i]\}, \{\hat{\psi}^\dagger_{m_j}[g_j]\}),$$

where the functions $f_i$ are arguments of the field operators and the functions $g_j$ are arguments of their adjoints. Such polynomial operators clearly form an algebra $\mathcal{A}$ under addition, multiplication and scalar multiplication. Further, $\mathcal{A}$ is a $\ast$-algebra, with involution given by the adjoint operation

$$\hat{A}^\dagger[[g_j], \{f_i\}] \equiv (\hat{A}[[f_i], \{g_j\}])^\dagger.$$

C. Free Fields

This section presents a theory of free fields that satisfy the axioms presented in Sec. III. Free fields represent particles that do not interact. Therefore, as in traditional QFT, we can develop a theory of parameterized free fields that act on a Fock space constructed as the direct sum

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)},$$

where $\mathcal{H}^{(N)}$ is the subspace of those states with exactly $N$ particles.

Let $\mathcal{H}^{(N)}$ be the Hilbert space of functions $\Psi(x_1, \ldots, x_N)$, symmetric in the interchange of any two arguments (for the case of bosonic scalar particles considered in this paper), taken from the space $L^2(R^{4N}, C)$ of complex-valued, square-integrable functions on $R^{4N}$. The inner product is given by

$$(\Psi_1, \Psi_2) \equiv \int d^4 x_1 \cdots \int d^4 x_N \Psi_1(x_1, \ldots, x_N)^* \Psi_2(x_1, \ldots, x_N).$$
\[ (\Psi, \Psi) = \int d^4x_1 \cdots \int d^4x_N \Psi(x_1, \ldots, x_N)^* \Psi(x_1, \ldots, x_N) = 1. \] (17)

The spacetime transformation law for these states is
\[ \hat{U}(\Delta x, \Lambda) \Psi(x_1, \ldots, x_N) = \{ \Delta x, \Lambda \} \Psi(x_1, \ldots, x_N) = \Psi(\Lambda^{-1}(x_1 - \Delta x), \ldots, \Lambda^{-1}(x_N - \Delta x)). \]

The Fock space \( \mathcal{H} \) of states of any number of particles then, by construction, satisfies the assumptions of Axiom [IV]

Next, consider \( \mathcal{H}^{(N)} \) as being in the Gel’fand triple \( \mathcal{K}^{(N)} \subset \mathcal{H}^{(N)} \subset \mathcal{K}^{(N)'} \). Here, \( \mathcal{K}^{(N)} \) is a dense subset of \( \mathcal{H}^{(N)} \) consisting of smooth functions \( f(x_1, \ldots, x_N) \) with compact support, and \( \mathcal{K}^{(N)'} \) is the space of continuous, linear, complex-valued functionals \( F[f] \) on \( \mathcal{K}^{(N)} \). \( \mathcal{H} \) is then in the Gel’fand triple \( \mathcal{K} \subset \mathcal{H} \subset \mathcal{K}' \), where \( \mathcal{K} = \bigoplus_{N=0}^{\infty} \mathcal{K}^{(N)} \) and \( \mathcal{K}' = \bigoplus_{N=0}^{\infty} \mathcal{K}^{(N)'} \).

The subspace \( \mathcal{H}^{(0)} \subset \mathcal{H} \) contains the single zero-particle state \( |0\rangle \), and we require that any free Hamiltonian have this state as its vacuum state. As for traditional QFT, we can then define a parameterized field theory of free particles that satisfies the axioms given in Sec. [II] by having the fields build up the Fock space \( \mathcal{H} \) from the vacuum state \( |0\rangle \).

Specifically, let \( |\Psi\rangle \in \mathcal{K}^{(N)} \), for \( N > 0 \). Then free fields are defined such that
\[ \hat{\psi}[f] |0\rangle = 0, \]
\[ \hat{\psi}[f] |\Psi\rangle = \sum_{i=1}^{N} \hat{f}_i |\Psi\rangle \in \mathcal{K}^{(N-1)}, \] (18)

where the operators \( \hat{f}_i \) act as
\[ (\hat{f}_i |\Psi\rangle)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) = \int d^2x_i f^*(x_i) \Psi(x_1, \ldots, x_i, \ldots, x_N), \]

and
\[ \hat{\psi}^\dagger[f] |0\rangle = |f\rangle \in \mathcal{K}^{(1)}, \]
\[ \hat{\psi}^\dagger[f] |\Psi\rangle = \text{Sym}( |f\rangle \otimes |\Psi\rangle ) \in \mathcal{K}^{(N+1)}, \] (19)

where Sym(…) indicates symmetrization on function arguments.

In this way, we can build up any element of \( \mathcal{K}^{(N)} \) for any \( N > 0 \), and thus \( \mathcal{K} \), by applying a polynomial in \( \hat{\psi}^\dagger[f] \). We can thus take \( D = D_0 = \mathcal{K} \) and satisfy Axioms [II] and [IV]. Further, inserting fields transformed under \( \hat{U}(\Delta x, \Lambda) \), it is clear that these fields also satisfy Axiom [III]. Finally,
\[ \hat{\psi}[f'][\hat{\psi}^\dagger[f] |\Psi\rangle = \sum_{i=1}^{N} \hat{f}'_i \text{Sym}( |f\rangle \otimes |\Psi\rangle ) \]
\[ = (f', f) |\Psi\rangle + \text{Sym}( |f\rangle \otimes \sum_{i=1}^{N-1} \hat{f}'_i |\Psi\rangle ) \]
\[ = (f', f) |\Psi\rangle + \hat{\psi}^\dagger[f] \hat{\psi}[f'] |\Psi\rangle, \]

so
\[ \{ \hat{\psi}[f'], \hat{\psi}^\dagger[f] \}|\psi\rangle = (f', f) |\psi\rangle, \]

satisfying Axiom [III].

### D. Vacuum Expectation Values

One of the central results of traditional axiomatic QFT is that a field theory can be reconstructed from its vacuum expectation values, and this result can largely be carried over to parametrized QFT. To do so, let \( \psi[f] \) be a field
then there exists a unitary transformation $\hat{H}$ be a Hamiltonian with vacuum state $|0\rangle \in \mathcal{H}$. Then the vacuum expectation values for the field are given by the Wightman distributions

$$\mathcal{W}_{(m,n)}(f_1, \ldots, f_m; g_1, \ldots, g_n) = \langle 0 | \hat{\psi}^{\dagger}[f_1] \cdots \hat{\psi}^{\dagger}[f_m] \hat{\psi}[g_1] \cdots \hat{\psi}[g_n] | 0 \rangle \mathcal{H},$$

for any value of $\lambda$. Note that the definitions here have been adapted for non-Hermitian, scalar field operators. These distributions have the following properties.

1. **Relativistic Invariance.** For any Poincaré transformation $\{\Delta x, \Lambda\}$,

$$\mathcal{W}_{(m,n)}[f_1, \ldots, f_m; g_1, \ldots, g_n] = \mathcal{W}_{(m,n)}[\{\Delta x, \Lambda\}f_1, \ldots, \{\Delta x, \Lambda\}f_m; \{\Delta x, \Lambda\}g_1, \ldots, \{\Delta x, \Lambda\}g_n].$$

2. **Hermiticity.**

$$\mathcal{W}_{(m,n)}[f_1, \ldots, f_m; g_1, \ldots, g_n] = \mathcal{W}_{(n,m)}^*[g_n, \ldots, g_1; f_m, \ldots, f_1].$$

3. **Commutativity.** For any $j$ or $k$,

$$\mathcal{W}_{(m,n)}[f_1, \ldots, f_m; g_1, \ldots, g_n] = \mathcal{W}_{(m,n)}[f_1, \ldots, f_{j+1}, f_j, \ldots, f_m; g_1, \ldots, g_n]$$

and, if $(f_m, g_1) = 0$,

$$\mathcal{W}_{(m,n)}[f_1, \ldots, f_m; g_1, \ldots, g_n] = \mathcal{W}_{(m,n)}[f_1, \ldots, f_{m-1}, g_1; f_m, g_2, \ldots, g_n].$$

4. **Positive Definiteness.** For any sequence of test functions $\{f_{11}, f_{21}, f_{22}, f_{31}, \ldots\}$ and each integer $n \geq 0$,

$$\sum_{j=0, k=0}^{n} \mathcal{W}_{(j,k)}^*[f_j^{\dagger}, \ldots, f_{j+1}^{\dagger}; f_k, \ldots, f_{kk}] \geq 0.$$
The proof of this theorem is essentially the same as for traditional QFT. However, the spectral conditions required on traditional Wightman functions are necessary for proving the cluster decomposition property for those functions, which is in turn used to prove the uniqueness of the constructed vacuum state. Since the Wightman functions for the parameterized theory do not have spectral restrictions, the Reconstruction Theorem as stated above does not claim that $|0\rangle$ is necessarily unique.

Nevertheless, suppose that $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces related by a unitary transformation $\hat{G}$, with corresponding vacuum states $|0\rangle$ and $|0\rangle'$, and fields $\hat{\psi}$ and $\hat{\psi}'$, as in the second part of the Reconstruction Theorem. Suppose further that $\hat{H}$ is a Hamiltonian defined on $\mathcal{H}$ with $|0\rangle$ as its unique vacuum state. The corresponding Hamiltonian operator in $\mathcal{H}'$ is $\hat{H}' = \hat{G}\hat{H}\hat{G}^{-1}$. Then $\hat{H}'|0\rangle' = \hat{G}\hat{H}|0\rangle = \hat{G}|0\rangle = 0$, and, since $\hat{H}'$ has the same spectrum as $\hat{H}$, $|0\rangle'$ must be the unique null eigenstate of $\hat{H}'$.

In addition, if $\hat{\psi}[f;\lambda]$ is the Heisenberg-picture version of $\hat{\psi}[f]$, using the Hamiltonian $\hat{H}$, then the Heisenberg-picture for $\hat{\psi}'[f]$, using $\hat{H}'$, is

$$\hat{\psi}'[f;\lambda] = e^{i\hat{H}'\lambda}\hat{\psi}'[f]e^{-i\hat{H}'\lambda} = e^{i\hat{H}'\lambda}\hat{G}e^{-i\hat{H}\lambda}\hat{\psi}'[f;\lambda]e^{i\hat{H}\lambda}\hat{G}^{-1}e^{-i\hat{H}'\lambda} = \hat{G}\hat{\psi}[f;\lambda]\hat{G}^{-1},$$

where the last equality follows from $\hat{H}'\hat{G} = \hat{G}\hat{H}$. So the Heisenberg-picture forms of the fields are also unitarily related by $\hat{G}$.

### E. Interacting Fields

Let the field $\hat{\psi}$ and Hamiltonian $\hat{H}$ be unitarily related to the field $\hat{\psi}'$ and Hamiltonian $\hat{H}'$, as discussed in Sec. [11]. Now, however, suppose that the fields are both defined on the same Hilbert space $\mathcal{H}$. Then the unitary transformation $\hat{G}$ maps $\mathcal{H}$ onto $\mathcal{H}$ and, for all $\Psi \in \mathcal{H}$, it is also the case that $\Psi' = \hat{G}\Psi \in \mathcal{H}$. In the Schrödinger picture, $\Psi_S(\lambda)$ evolves according to $\hat{H}$, while $\Psi'_S(\lambda)$ evolves according to $\hat{H}'$. Nevertheless,

$$\Psi'_S(\lambda) = e^{-i\hat{H}'\lambda}\Psi'_H = e^{-i\hat{H}'\lambda}\hat{G}\Psi_H = \hat{G}e^{-i\hat{H}\lambda}\Psi_H = \hat{G}\Psi_S(\lambda),$$

since $\hat{H}'\hat{G} = \hat{G}\hat{H}$.

Under the unitary transformation $\hat{G}$, an operator $\hat{A}$ maps to $\hat{A}' = \hat{G}\hat{A}\hat{G}^{-1}$. Now, in the Heisenberg picture, $\hat{A}$ evolves according to the Hamiltonian $\hat{H}$, $\hat{A}_H(\lambda) = e^{i\hat{H}\lambda}\hat{A}e^{-i\hat{H}\lambda}$, while $\hat{A}'$ evolves according to $\hat{H}'$, $\hat{A}'_H(\lambda) = e^{i\hat{H}'\lambda}\hat{A}'e^{-i\hat{H}'\lambda}$. However, since $\hat{A}'$ is also an operator on $\mathcal{H}$, it can equally well be evolved using $\hat{H}$. This gives the interaction picture form for $\hat{A}'$:

$$\hat{A}'_H(\lambda) = e^{i\hat{H}\lambda}\hat{A}'_S e^{-i\hat{H}\lambda} = \hat{G}(\lambda)\hat{A}_H(\lambda)\hat{G}^{-1}(\lambda),$$

where $\hat{G}(\lambda) = e^{i\hat{H}\lambda}\hat{G}e^{-i\hat{H}\lambda}$.

The corresponding interaction-picture form for a state $\Psi'$ must then be related to its Schrödinger-picture form by

$$\Psi'_I(\lambda) = e^{i\hat{H}\lambda}\Psi'_S(\lambda) = \hat{G}(\lambda)\Psi_H,$$

so that

$$\langle \Psi'_I(\lambda)|\hat{A}'_I(\lambda)|\Phi'_I(\lambda)\rangle = \langle \Psi'_S(\lambda)|\hat{A}'_S(\lambda)|\Phi'_S(\lambda)\rangle = \langle \Psi'_H|\hat{A}'_H(\lambda)|\Phi'_H\rangle,$$

for all $\Psi'$, $\Phi'$ and $\lambda$. This also allows Heisenberg-picture states constructed using the field $\hat{\psi}$ to be compared to interaction-picture states constructed using the field $\hat{\psi}'$, such that

$$\langle \Psi_H|\hat{\psi}'_I(\lambda)\rangle = \langle \Psi_H|\hat{G}(\lambda)|\Phi_H\rangle = \langle \Psi_S(\lambda)|\hat{G}|\Phi_S(\lambda)\rangle = \langle \Psi_S(\lambda)|\Phi'_S(\lambda)\rangle,$$

for any $\lambda$.

To see why the name “interaction picture” is applicable here, consider the interaction-picture form of the field $\hat{\psi}'[f]$:

$$\hat{\psi}'[f;\lambda] = e^{i\hat{H}\lambda}\hat{\psi}'[f]e^{-i\hat{H}\lambda} = \hat{G}(\lambda)\hat{\psi}[f;\lambda]\hat{G}^{-1}(\lambda).$$
Then
\[
\frac{\partial \hat{\psi}'[f; \lambda]}{\partial \lambda} = \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{\psi}[f; \lambda] \hat{G}^{-1}(\lambda) + \hat{G}(\lambda) \frac{\partial \hat{\psi}[f; \lambda]}{\partial \lambda} \hat{G}^{-1}(\lambda) - \hat{G}(\lambda) \hat{\psi}[f; \lambda] \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda)
\]
\[
= \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda) \hat{\psi}'[f; \lambda] + i \hat{G}(\lambda) [\hat{H}, \hat{\psi}[f; \lambda]] \hat{G}^{-1}(\lambda) - \hat{G}(\lambda) \hat{\psi}'[f; \lambda] \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda)
\]
\[
= i[\hat{H}'(\lambda), \hat{\psi}'[f; \lambda]] + \left[ \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda), \hat{\psi}'[f; \lambda] \right],
\]
where
\[
\hat{H}'(\lambda) = e^{i \hat{H} \lambda} \hat{H} e^{-i \hat{H} \lambda} = \hat{G}(\lambda) \hat{H} \hat{G}^{-1}(\lambda).
\]

But, since \(\hat{\psi}'[f; \lambda]\) evolves according to \(\hat{H}\),
\[
\frac{\partial \hat{\psi}'[f; \lambda]}{\partial \lambda} = i[\hat{H}, \hat{\psi}'[f; \lambda]],
\]
so we can take
\[
\hat{H}'(\lambda) = \hat{H} + \Delta \hat{H}(\lambda),
\]
where
\[
\Delta \hat{H}(\lambda) = i \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda).
\]

Now, suppose that \(\hat{\psi}[f]\) is a free field, as in Sec. IIIC and that \(\hat{H}\) is the free, relativistic Hamiltonian such that
\[
[\hat{\psi}[f; \lambda], \hat{H}] = (\hat{P}^2 + m^2) \hat{\psi}[f; \lambda],
\]
where \(m\) is the free-particle mass. Then the field equation for \(\hat{\psi}[f; \lambda]\) is
\[
i \frac{\partial \hat{\psi}[f; \lambda]}{\partial \lambda} = (\hat{P}^2 + m^2) \hat{\psi}[f; \lambda].
\]
However, since \(\hat{G}\) and \(\hat{H}\) commute with \(\hat{P}\), it is also the case that
\[
[\hat{\psi}'[f; \lambda], \hat{H}'(\lambda)] = (\hat{P}^2 + m^2) \hat{\psi}'[f; \lambda].
\]
Substituting this into Eq. (20) then gives
\[
i \frac{\partial \hat{\psi}'[f; \lambda]}{\partial \lambda} = (\hat{P}^2 + m^2) \hat{\psi}'[f; \lambda] + [\Delta \hat{H}(\lambda), \hat{\psi}'[f; \lambda]].
\]
If we take this as the field equation for \(\hat{\psi}'[f; \lambda]\), then we can consider \(\hat{\psi}'\) to be an interacting field, with the interaction Hamiltonian \(\Delta \hat{H}(\lambda)\).

In traditional QFT, Haag’s Theorem essentially disallows the use of the interaction picture to relate an interacting field to a free field, stating that any field unitarily related to a free field must itself be a free field \[44, 16, 47\]. The traditional interaction picture is relative to evolution in time, and any two fields unitarily equivalent at any one time will necessarily have the same vacuum expectation values at that time. Haag’s Theorem is then proved by showing that Lorentz invariance requires that fields with equal same-time vacuum expectation values will also have equal different-time vacuum expectation values and that this implies that if one of the fields is free the other one must be too.

For parameterized QFT, however, the evolution parameter is not one of the spacetime coordinates and, thus, parameter evolution is not related to spacetime transformations. It is therefore possible to have a field such as \(\hat{\psi}'[f; \lambda]\) that is unitarily related to a free field \(\hat{\psi}[f; \lambda]\) for each value of \(\lambda\) but with a different transformation \(\hat{G}(\lambda)\) for each \(\lambda\). In traditional QFT, Lorentz covariance prohibits having an analogous unitary transformation that depends on time.
Note also that the $\lambda$-dependent transformation $\hat{G}(\lambda)$ essentially induces a different effective vacuum state $|0'; \lambda\rangle = \hat{G}(\lambda)|0\rangle$ for each value of $\lambda$. The two-point equal-$\lambda$ vacuum expectation value for $\hat{\psi}^\dagger[f; \lambda]$, for example, is thus

$$\langle 0'; \lambda|\hat{\psi}^\dagger[f; \lambda]\hat{\psi}^\dagger[g; \lambda]|0'; \lambda\rangle = \langle 0|\hat{\psi}[f; \lambda]\hat{\psi}^\dagger[g; \lambda]|0\rangle .$$

When constructing the vacuum expectation values for $\hat{\psi}^\dagger[f; \lambda]$ across different $\lambda$ values, though, one must be careful to use the appropriate effective vacuum for each value. For example, the two point different-$\lambda$ expectation value is

$$\langle 0'; \lambda'|\hat{\psi}^\dagger[f; \lambda']\hat{\psi}^\dagger[g; \lambda]|0'; \lambda\rangle = \langle 0|\hat{\psi}[f; \lambda']\hat{G}^{-1}(\lambda')\hat{G}(\lambda)\hat{\psi}^\dagger[g; \lambda]|0\rangle \neq \langle 0|\hat{\psi}[f; \lambda']\hat{\psi}^\dagger[g; \lambda]|0\rangle .$$

The effective vacuum $|0'; \lambda\rangle$ is a null eigenstate of the effective Hamiltonian $\hat{H}'(\lambda)$ for each specific value of $\lambda$. However, it is not an eigenstate of $\hat{H}$ for any $\lambda$. Considered as a function of $\lambda$, $|0'; \lambda\rangle$ is therefore an example of a state that is invariant under Poincaré transformations, but evolves in $\lambda$, as permitted in the parameterized theory.

F. Integrated Fields

In general, the two-point vacuum expectation value

$$\Delta[f, g; \lambda - \lambda_0] \equiv \langle 0|\hat{\psi}[f; \lambda]\hat{\psi}^\dagger[g; \lambda_0]|0\rangle$$

represents the propagation of a particle from parameter value $\lambda_0$ to $\lambda$. The quantity $\Delta[f, g; \lambda - \lambda_0]$ clearly depends only on the difference $\lambda - \lambda_0$, since applying the same unitary parameter translation to the field operator at each parameter value leaves the vacuum expectation value unchanged. Intuitively, such an expectation value represents the propagation of a particle over any path of intrinsic length $\lambda - \lambda_0$ (as similarly described in the introductory discussion in Sec. [13]). Integrating over all intrinsic lengths then gives the amplitude for propagation over all possible paths:

$$\Delta[f, g] = \int_{0}^{\infty} d\lambda \Delta[f, g; \lambda] .$$

For this purpose, it is convenient to introduce the integrated particle fields

$$\hat{\psi}_{\lambda_0}[f] \equiv \int_{\lambda_0}^{\infty} d\lambda \hat{\psi}[f; \lambda] ,$$

such that

$$\langle 0|\hat{\psi}_{\lambda_0}[f]\hat{\psi}^\dagger[g; \lambda_0]|0\rangle = \langle 0|\int_{\lambda_0}^{\infty} d\lambda \hat{\psi}[f; \lambda]\hat{\psi}^\dagger[g; \lambda_0]|0\rangle = \Delta[f, g] .$$

for all $\lambda_0$. Note that the integrated field operator still depends on a parameter value $\lambda_0$, even though $\Delta[f, g]$ is independent of it. This reflects the fact that, even when integrating over intrinsic path lengths, the freedom remains to arbitrarily choose the parameter value at the start of the paths being integrated along.

Parameter translation is, in fact, a global symmetry of the integrated field, so such a field has the eleven-dimensional symmetry group $P \otimes U(1)$, where $P$ is the ten-dimensional (inhomogeneous) Poincaré group. This is just the Four-Space Formalism (FSF) group described in [31]. (However, it is different than the SO(3, 2) and SO(4, 1) groups discussed in [32] and [34].)

The following theorems follow directly from the definition of the integrated field and the axioms for the corresponding unintegrated field.

**Theorem 2.** For each parameter value $\lambda_0$ and all $f \in \mathcal{D}$, the operator $\hat{\psi}_{\lambda_0}[f]$ is defined on a domain $\mathcal{K}'$ of states dense in $\mathcal{K}'$ containing the vacuum state $\langle 0 |$. It carries vectors in $\mathcal{K}'$ into vectors in $\mathcal{K}'$. The domain $\mathcal{K}'$ here is a subset of the generalized state space $\mathcal{K}'$, rather than $\mathcal{H}$, because the result of applying $\hat{\psi}_{\lambda_0}[f]$ to a state in $\mathcal{H}$ generally results in a vector that is not normalizable. As a result, an expression such as $\langle \Psi | \hat{\psi}_{\lambda_0}[f] | \Phi \rangle$ should strictly be understood as the application of $\langle \Psi | \hat{\psi}_{\lambda_0}[f] \rangle \in \mathcal{K}'$ to $| \Phi \rangle \in \mathcal{K}$. However, with a further common abuse of ket notation, we can formally use the notation $A | \Phi \rangle$ for any $A$ constructed as a polynomial in the integrated fields, with the understanding that this generally only has meaning in the context of an expression like $\langle \Psi | A | \Phi \rangle$, in which case it means $\langle \Psi | A$ applied to $| \Phi \rangle$. 


Theorem 3. For any Poincaré transformation \( \{ \Delta x, \Lambda \} \),

\[
\hat{U}(\Delta x, \Lambda) \hat{\psi}_{\lambda_0}[f] \hat{U}^{-1}(\Delta x, \Lambda) = \hat{\psi}_{\lambda_0}(\{ \Delta x, \Lambda \}f).
\]

Given the Hamiltonian \( \hat{H} \), for any parameter translation \( \Delta \lambda \),

\[
e^{i\hat{H}\Delta \lambda} \hat{\psi}_{\lambda_0}[f] e^{-i\hat{H}\Delta \lambda} = \hat{\psi}_{\lambda_0+\Delta \lambda}[f].
\]

These equations are valid when applied to any element of \( K' \).

The use of the integrated fields can be generalized to cover any \( \hat{A} \) in the algebra \( A \) of operators formed as polynomials in the (unintegrated) smeared field operators. Given the functional form for \( \hat{A} \) in Eq. (16), define

\[
\hat{A}(\{ f_i \}, \{ g_j \}) = A(\{ \hat{\psi}_{\lambda_0}[f_i], \{ \hat{\psi}[g_j; \lambda_0] \} \}) = \hat{A}(\{ f_i \}, \{ g_j \}).
\]

That is, all instances of the \( \hat{\psi} \) operators in the construction of \( \hat{A} \) are replaced with their integrated versions, but not instances of the adjoint \( \hat{\psi}^\dagger \) operators (for which the Heisenberg-picture versions are used here, with a parameter argument consistent with the integrated fields). To simplify the notation, an overbar will also be placed on a functional argument to indicate that it is to be used as the argument of an integrated field, as in \( f_i \) in the second equality above.

Define the special adjoint \( \hat{A}^\ddagger \), such that

\[
\hat{A}^\ddagger(\{ g_j \}, \{ f_i \}) = \overline{\hat{A}^\dagger(\{ g_j \}, \{ f_i \})} = \hat{A}^\dagger(\{ \hat{\psi}_{\lambda_0}[f_i], \{ f_j \} \}) = \overline{(\hat{A}(\{ f_i \}, \{ g_j \}))}^\dagger,
\]

where \( \hat{A}^\ddagger \) is defined as in Eq. (16). Note that the special adjoints differs from the regular adjoint of \( \hat{A} \) in that

\[
\hat{A}^\ddagger(\{ g_j \}, \{ f_i \}) = (\hat{A}(\{ f_i \}, \{ g_j \})) = \hat{A}^\dagger(\{ g_j \}, \{ f_i \}) \neq \hat{A}^\dagger(\{ g_j \}, \{ f_i \}).
\]

In particular, the special adjoint of the field operator \( \hat{\psi}_{\lambda_0}[f] \) is

\[
\hat{\psi}_{\lambda_0}^\ddagger[f] = \overline{\hat{\psi}_{\lambda_0}^\dagger[f]} = (\hat{\psi}[\{ f \}, \{ \} \}; \lambda_0) = (\hat{\psi}[f; \lambda_0]^\dagger = \hat{\psi}_{\lambda_0}^\ddagger[f]
\]

and

\[
\hat{\psi}^\ddagger[f; \lambda_0] = \overline{\hat{\psi}_{\lambda_0}^\dagger[f]} = \hat{\psi}_{\lambda_0}^\ddagger[f].
\]

This special adjoint has all the properties required of an involution on the \(*\)-algebra of operators of the form \( \hat{A} \). First,

\[
\hat{A}^\ddagger \hat{A}^\ddagger = (\overline{\hat{A}^\dagger})^\dagger = \overline{\hat{A}^\dagger} = \hat{A}.
\]

Further, clearly

\[
(\hat{A} + \hat{B})^\ddagger = \hat{A}^\ddagger + \hat{B}^\ddagger.
\]

and, for any complex number \( a \),

\[
(a\hat{A})^\ddagger = (\overline{a\hat{A}})^\dagger = a^*\hat{A}^\ddagger = a^*\overline{\hat{A}^\dagger} = a^*\hat{A}^\ddagger.
\]

Finally, note that,

\[
\hat{A}(\{ f_i \}, \{ g_j \}) \hat{B}(\{ f'_i \}, \{ g'_j \}) = \hat{A}(\{ f_i \}, \{ g_j \}) \hat{B}(\{ f'_i \}, \{ g'_j \}) = (\hat{A}\hat{B})(\{ f_i \}, \{ f'_i \}; \{ g_j \}, \{ g'_j \}) = (\overline{\hat{A}\hat{B}})(\{ f_i \}, \{ f'_i \}; \{ g_j \}, \{ g'_j \}) = (\hat{A}\hat{B})^\ddagger(\{ f_i \}, \{ f'_i \}; \{ g_j \}, \{ g'_j \}),
\]

so \( \hat{A}\hat{B} = AB \). Then

\[
(\hat{A}\hat{B})^\ddagger = (\overline{\hat{A}\hat{B}})^\dagger = (\overline{\hat{A}})^\dagger (\overline{\hat{B}})^\dagger = (\overline{\hat{A}})^\dagger (\overline{\hat{B}})^\dagger = (\overline{\hat{A}})^\dagger (\overline{\hat{B}})^\dagger = \hat{A}^\ddagger \hat{B}^\ddagger.
\]

With the above definitions, the following theorem holds.
\textbf{Theorem 4.} Operators constructed as polynomials in the field \( \hat{\psi}_{\lambda_0}[f] \) and its adjoint \( \hat{\psi}_{\lambda_0}^{\dagger}[f] \) form a \( \ast \)-algebra \( \hat{\mathcal{A}} \), with involution given by the special adjoint \( \hat{\mathcal{A}}^{\dagger} \) for \( \hat{A} \in \hat{\mathcal{A}} \), isomorphic to the algebra \( \mathcal{A} \) defined using the unintegrated fields.

It is also possible to define unsmeared versions of the integrated fields:

\[
\hat{\psi}_{\lambda_0}[f] = \int d^4 x \ f^*(x) \hat{\psi}_{\lambda_0}(x),
\]

where the unsmeared integrated fields are

\[
\hat{\psi}_{\lambda_0}(x) \equiv \int_{\lambda_0}^{\infty} d\lambda \hat{\psi}(x; \lambda).
\]

The special adjoint operates on these fields as \( \hat{\psi}^\dagger_{\lambda_0}(x) = \hat{\psi}^\dagger(x; \lambda_0) \) and \( \hat{\psi}^{\dagger\dagger}_{\lambda_0}(x) = \hat{\psi}_{\lambda_0}(x) \).

\[\text{G. Antiparticles}\]

As mentioned in Sec. [1A] in perturbative QFT, backward-time particle propagation is traditionally re-interpreted as forward-time antiparticle propagation. Alternatively, one can also view antiparticles as particles that formally propagate backwards in time [22]. Indeed, Stueckelberg considers segments of particle paths with reversed-time propagation to represent antiparticles [9, 15]. In [13], I presented a related but importantly different formulation, in which only forward-time antiparticle propagation is allowed. Alternatively, one can also view antiparticles as particles that formally propagate backwards in time [52]. Indeed, Stueckelberg considers segments of particle paths with reversed-time propagation.

However, in [40], I found that, in order to produce complete Feynman diagrams for scattering using the spacetime path formalism, it was also necessary to introduce particle propagation that was reversed in the sense of the path parameter \( \lambda \), as well as propagation reversed in the sense of time. It turns out, though, that such reverse-propagating particles can also be used directly to represent antiparticles, under the condition in which forward propagation in \( \lambda \) is aligned with forward propagation in time. This will be discussed further in Sec. [113]. In the present section, I will present the formal model of antiparticles as reverse-propagating particles, which will be used in the modeling of interactions in Sec. [111].

To do this, first note that, in Eq. (23) defining \( \hat{\psi}_{\lambda_0} \), the integral is only over parameter values \( \lambda > \lambda_0 \). This reflects the conception that \( \lambda - \lambda_0 \) is a path length and, therefore, must be positive. However, it was actually an arbitrary choice to presume that particle propagation along a path was for values of \( \lambda \) increasing from \( \lambda_0 \). One could just as well take \( \lambda \) to decrease from \( \lambda_0 \) and consider the (positive) intrinsic path lengths \( \lambda_0 - \lambda \).

Such reverse propagation is reflected in the field vacuum expectation values, such that

\[
\langle 0 | \hat{\psi}_{\lambda_0}^\dagger[f; \lambda] \hat{\psi}_{\lambda_0}^\dagger[g; \lambda_0] | 0 \rangle = \Delta[f, g; \lambda_0 - \lambda],
\]

where \( \lambda_0 - \lambda \) is positive for \( \lambda < \lambda_0 \). Note the difference with Eq. (22), even assuming an equivalent underlying form for the functional \( \Delta[f, g; \Delta \lambda] \). For a forward-propagating field, if \( \lambda < \lambda_0 \), the \( \Delta \lambda \) argument in \( \Delta[f, g; \Delta \lambda] \) is negative, while, for a reverse-propagating field, it is positive.

Therefore, in terms of propagation, a reverse-propagating field really is a different sort of field than a forward-propagating one. Nevertheless, for fields that otherwise have vacuum expectation values given by the same functional \( \Delta[f, g; \Delta \lambda] \) (e.g., particles of the same mass and spin), we will take the reverse-propagating field \( \hat{\psi}_{\lambda_0} \) to represent the antiparticle of the particle represented by the corresponding forward-propagating field \( \hat{\psi}_{\lambda_0} \) (as motivated earlier). That is, antiparticles propagate in the \textit{opposite direction} in \( \lambda \) from their corresponding particles.

This also means that, when defining integrated fields for antiparticles, for a given \( \lambda_0 \), propagation is downwards from \( \lambda_0 \) to lower values of \( \lambda \). Therefore, the integration must be over all \( \lambda < \lambda_0 \):

\[
\hat{\psi}_{-\lambda_0}[f] \equiv \int_{-\infty}^{\lambda_0} d\lambda \hat{\psi}_{-\lambda_0}[f; \lambda].
\]

However, the integrated vacuum expectation value is then

\[
\langle 0 | \hat{\psi}_{-\lambda_0}[f] \hat{\psi}_{-\lambda_0}^{\dagger}[0] | 0 \rangle = \langle 0 | \int_{-\infty}^{\lambda_0} d\lambda \hat{\psi}_{-\lambda_0}[f; \lambda] \hat{\psi}_{-\lambda_0}^{\dagger}[g; \lambda_0] | 0 \rangle = \int_{-\infty}^{\lambda_0} d\lambda \Delta[f, g; \lambda_0 - \lambda]
\]

\[
= \int_{-\infty}^{\lambda_0} d\lambda \Delta[f, g; -\lambda] = \int_{0}^{\lambda_0} d\lambda \Delta[f, g; \lambda] = \Delta[f, g].
\]
So, it would seem that, for an on-shell state with four-position \(x\) such that \(|x\rangle \equiv \langle 0|\psi(x)\in \mathcal{K}'\). These bras act as a continuous basis for \(\mathcal{K}\), such that, for any \(|f\rangle \in \mathcal{K}\), \(|x|f\rangle = f(x)\). Informally, we can also define the dual kets \(|x\rangle\), such that \langle x'|x\rangle = \delta^4(x' - x), so

\[
\langle x'|x\rangle = \int d^4x f(x)|x\rangle.
\] (27)

Of course, with the above normalization, the \(|x\rangle\) are not actually in \(\mathcal{K}\) (or \(\mathcal{H}\), but, similarly to the smeared field operators, they may be used with care as long as it is understood that it is really only expressions such as Eq. (27) that are properly defined, for \(f\in \mathcal{D}\).

For example, consider the momentum version of such single particle states, \(|p\rangle = \hat{\psi}^\dagger(p)|0\rangle\), where \(\hat{\psi}(x)\) is the four-dimensional Fourier transform of \(\psi(x)\). The free Hamiltonian acts on these momentum states as \(\hat{H}|p\rangle = (p^2 + m^2)|p\rangle\). So, it would seem that, for an on-shell state with \(p^2 = -m^2\), one would have \(\hat{H}|p\rangle = 0\), which would be a non-vacuum null eigenstate of \(\hat{H}\), in violation of the requirements for a Hamiltonian operator.

There is, of course, no actual violation here, since \(|p\rangle \notin \mathcal{H}\). On the other hand, \(|p\rangle \in \mathcal{K}'\) is properly defined, for all \(p\), such that \langle p|f\rangle = f(p), where \(f(p)\) is the Fourier transform of \(f(x)\). One can thus define

\[
\langle m| \equiv \int d^4p \delta^4(p^2 + m^2)\theta(p^0)|p\rangle = \int \frac{d^3p}{2\omega_p} \langle \omega_p, p| 
\]

such that \langle m|f\rangle as the probability amplitude for an arbitrary state \(|f\rangle\) to be on shell. (A similar approach is effectively used in Sec. III D to ensure the on-shell nature of external legs of a scattering process.)

With the above caveat, then, we can take \(|x\rangle = \hat{\psi}^\dagger(x)|0\rangle\). Similarly, for the Heisenberg-picture fields, we can define \(|x; \lambda\rangle \equiv \hat{\psi}^\dagger(x; \lambda)|0\rangle\). Now, despite their apparent dependence on \(\lambda\), the \(|x; \lambda\rangle\) are actually Heisenberg-picture states, since the \(\hat{\psi}^\dagger(x; \lambda)\) are Heisenberg-picture operators. They represent the state of a particle localized at a specific four-position \(x\) and at a specific parameter value \(\lambda\).
Multiple applications of $\hat{\psi}^\dagger(x; \lambda)$ result in the multi-particle position states

$$|x_1, \lambda_1; \ldots; x_N, \lambda_N\rangle \equiv \hat{\psi}^\dagger(x_1; \lambda_1) \cdots \hat{\psi}^\dagger(x_N; \lambda_N) |0\rangle,$$

which represent multiple particles localized at the four-positions $x_i$, each with their own specific parameter values $\lambda_i$. It will also be convenient to have a shorthand notation for the case of multiple particles at different positions, but all with the same parameter value $\lambda_0$:

$$|x_1, \ldots, x_N; \lambda_0\rangle \equiv |x_1, \lambda_0; \ldots; x_N, \lambda_0\rangle.$$

Clearly, for any one value of $\lambda$,

$$\langle x; \lambda|x_0; \lambda\rangle = \langle 0|\hat{\psi}(x; \lambda)\hat{\psi}^\dagger(x_0; \lambda)|0\rangle = \delta^4(x - x_0),$$

by Axiom III and the normalization of $|0\rangle$. However, for different $\lambda$ values, using Eq. (26) gives

$$\langle x; \lambda|x_0; \lambda_0\rangle = \langle 0|\hat{\psi}(x; \lambda)\hat{\psi}^\dagger(x_0; \lambda_0)|0\rangle = \Delta(x - x_0; \lambda - \lambda_0).$$

The two-point different-$\lambda$ expectation value for $\hat{\psi}(x; \lambda)$ is the probability amplitude for the propagation of a particle from position $x_0$ at parameter value $\lambda_0$ to position $x$ at parameter value $\lambda$. As discussed in Sec. III, the full propagation amplitude is then given by integrating over $\lambda$, which is equal to the vacuum expectation value of the integrated field:

$$\langle 0|\hat{\psi}_{\lambda_0}(x)\hat{\psi}^\dagger_{\lambda_0}(x_0)|0\rangle = \int_{\lambda_0}^\infty \lambda \Delta(x - x_0; \lambda - \lambda_0) = \Delta(x - x_0). \quad (28)$$

(Note the use here of the special adjoint defined in Sec. III)

Next, following on the idea introduced at the end of Sec. II, consider an interacting-field operator $\hat{\psi}_{\text{int}}^\dagger(x; \lambda)$ that, in the interaction picture, is related to the free-field operator by a $\lambda$-dependent unitary transformation:

$$\hat{\psi}_{\text{int}}^\dagger(x; \lambda) = \hat{G}(\lambda)\hat{\psi}(x; \lambda)\hat{G}^{-1}(\lambda),$$

where $\hat{G}(\lambda)$ is constructed as a functional of the field operator. Actually, it will be more convenient to work primarily with integrated field operators. However, rather than directly integrating $\hat{\psi}_{\text{int}}^\dagger(x; \lambda)$ (which would not be proper in the interaction picture), instead take $\hat{\psi}_{\lambda_0}^\dagger(x)$ to be the operator in the algebra $\mathcal{A}$ corresponding to $\hat{\psi}_{\text{int}}^\dagger(x; \lambda)$ in the algebra $\mathcal{A}$ (as discussed at the end of Sec. III). This is

$$\hat{\psi}_{\lambda_0}^\dagger(x) = \hat{G}(\lambda_0)\hat{\psi}_{\lambda_0}(x)\hat{G}^{-1}(\lambda_0),$$

where the operator $\hat{G}$ has the same functional form as $\hat{G}$, but with the unintegrated field operator and its adjoint replaced by the integrated field operator and its special adjoint. With this notation, it is easy to create interacting-particle position states:

$$|x_1, \ldots, x_N; \lambda_0\rangle_{\text{int}} \equiv \hat{\psi}_{\lambda_0}^\dagger(x_1) \cdots \hat{\psi}_{\lambda_0}^\dagger(x_N)|0; \lambda_0\rangle_{\text{int}}$$

$$= \hat{G}(\lambda_0)\hat{\psi}_{\lambda_0}^\dagger(x_1) \cdots \hat{\psi}_{\lambda_0}^\dagger(x_N)|0\rangle$$

$$= \hat{G}(\lambda_0)\hat{\psi}(x_1; \lambda_0) \cdots \hat{\psi}(x_N; \lambda_0)|0\rangle$$

$$= \hat{G}(\lambda_0)|x_1, \ldots, x_N; \lambda_0\rangle,$$

where $|0; \lambda_0\rangle_{\text{int}} \equiv \hat{G}(\lambda_0)|0\rangle$ is the interacting vacuum.

B. Interactions

The goal now is to construct the interaction operator $\hat{G}(\lambda_0)$ to represent interactions between different types of particles. In order to do this, introduce a set of (integrated) free fields $\hat{\psi}_{n, \lambda_0}(x)$, indexed by the particle type $n$. Each of these fields act on the same Hilbert space $\mathcal{H}$, with the same domain $D$, they each individually satisfy the axioms
from Sec. II.B and each otherwise commutes with all fields of other particle types (and their adjoints). They also each have corresponding interacting fields given by

$$\hat{\psi}_{n,\lambda_0}^{\text{int}}(x) = \hat{G}(\lambda_0)\hat{\psi}_{n,\lambda_0}(x)\hat{G}^{-1}(\lambda_0).$$

While the free field $\hat{\psi}_{n,\lambda_0}$ of a specific particle type $n$ is independent of the fields for all other particle types, the interacting field $\hat{\psi}_{n,\lambda_0}^{\text{int}}$ will depend on the fields for other particle types through the construction of $\hat{G}$.

Note that, if $\hat{G}$ is unitary, then, from the definition of the special adjoint, Eq. (24),

$$\hat{G}^\dagger = \overline{\hat{G}}^\dagger = \overline{\hat{G}}^{-1} = \hat{G}^{-1}.$$

That is, $\hat{G}$ is unitary with respect to the special adjoint. From now forward, we will drop the overbar from $\hat{G}$ and just consider the operator $\hat{G}$ as constructed from the integrated field operators and their adjoints. Since $\hat{G}$ is unitary with respect to the special adjoint, we can take it to have the form

$$\hat{G}(\lambda_0) = e^{-i\hat{V}(\lambda_0)},$$

where $\hat{V}(\lambda_0)$ is self-adjoint with respect to the special adjoint (that is, $\hat{V}(\lambda_0) = \hat{V}(\lambda_0)$).

From Theorem 3 an integrated field $\hat{\psi}_{\lambda_0}(x)$ evolves in $\lambda_0$ according to the Hamiltonian of the corresponding unintegrated field $\hat{\psi}(x; \lambda)$. Therefore, we can carry the discussion of interacting fields in Sec. II.E over to the case of interacting fields $\hat{\psi}_{\lambda_0}^{\text{int}}(x)$ to have the form

$$\hat{H}(\lambda_0) = \hat{H}_0 + \hat{H}^{\text{int}}(\lambda_0),$$

where $\hat{H}^{\text{int}}(\lambda_0)$ is the interaction Hamiltonian in the interaction picture. Then, using Eqs. (21) and (30),

$$\hat{H}^{\text{int}}(\lambda_0) = -i\frac{d\hat{G}(\lambda_0)}{d\lambda_0}\hat{G}^{-1}(\lambda_0) = -\frac{d\hat{V}(\lambda_0)}{d\lambda_0}.$$

Further, taking the series expansion for $\hat{G}$,

$$\hat{G}(\lambda_0) = \sum_m \frac{(-i)^m}{m!} \hat{V}^m(\lambda_0),$$

the $m$th term in the series represents the case of exactly $m$ interactions. The factor $1/m!$ accounts for the $m!$ possible permutations of the $m$ factors of $\hat{V}$ in the term. Thus, $\hat{V}$ is the vertex operator that determines the effect of the individual interaction vertices collected within $\hat{G}$. Take

$$\hat{V}(\lambda_0) = \int d^4x \hat{V}_{\lambda_0}(x),$$

for some $\hat{V}_{\lambda_0}(x)$ that is self-adjoint. $\hat{V}_{\lambda_0}(x)$ represents an interaction at the specific four-position $x$ and is to be constructed from the integrated field operators.

Now, we need to account for both particles and antiparticles when constructing the interaction at a vertex. Let $n_+$ denote a particle type and $n_-$ denote the corresponding antiparticle type. Then, as discussed in Sec. II.C, we take the particle field $\hat{\psi}_{n_+}(x; \lambda)$ to be forward propagating, while the antiparticle field $\hat{\psi}_{n_-}(x; \lambda)$ is backward propagating. Therefore,

$$\langle 0 | \hat{\psi}_{n_+}^{\dagger}(x; \lambda) \hat{\psi}_{n_+}^{\dagger}(x_0; \lambda_0) | 0 \rangle = \Delta_n(x - x_0; \pm(\lambda - \lambda_0)),$$

where $\Delta_n(x - x_0; \lambda - \lambda_0)$ is $\Delta(x - x_0; \lambda - \lambda_0)$ from Eq. (30), using the mass $m_n$, which is the mass of both particles of type $n_+$ and the corresponding antiparticles of type $n_-$. Note also that a particle field is taken to commute with the corresponding antiparticle field (and its adjoint), as for fields of completely different particle types. The integrated fields $\hat{\psi}_{n_\pm,\lambda_0}(x)$ are constructed as described in Sec. II.C such that

$$\langle 0 | \hat{\psi}_{n_\pm,\lambda_0}(x) \hat{\psi}_{n_\pm,\lambda_0}^{\dagger}(x_0) | 0 \rangle = \Delta(x - x_0).$$
Then, at a specific interaction position $x$, there may be either the destruction of an “incoming” particle by $\hat{\psi}_{n+,\lambda_0}(x)$ or the creation of an “outgoing” antiparticle by $\hat{\psi}_{n-,\lambda_0}^\dagger(x)$. This can be reflected in the combined field

$$\hat{\psi}_{n,\lambda_0}(x) \equiv \hat{\psi}_{n+,\lambda_0}(x) + \hat{\psi}_{n-,\lambda_0}^\dagger(x).$$  \hspace{1cm} (34)$$

Since the special adjoint has the properties of an involution, we also have

$$\hat{\psi}_{n,\lambda_0}^\dagger(x) \equiv \hat{\psi}_{n+,\lambda_0}^\dagger(x) + \hat{\psi}_{n-,\lambda_0}(x),$$  \hspace{1cm} (35)$$

which represents the creation of an “outgoing” particle by $\hat{\psi}_{n+,\lambda_0}^\dagger(x)$ or the destruction of an “incoming” antiparticle by $\hat{\psi}_{n-,\lambda_0}(x)$.

An interaction vertex can be represented in terms of these operators as

$$\hat{V}_{\lambda_0}(x) = g : \prod_i \hat{\psi}_{n_i,\lambda_0}^\dagger(x) \hat{\psi}_{n_i,\lambda_0}(x) \prod_j \hat{\psi}_{n_j}^\dagger(x) :,$$  \hspace{1cm} (36)$$

where $g$ is a coupling constant, $: \cdots :$ represents normal ordering (that is, placing all $\hat{\psi}^\dagger$ operators to the left of all $\hat{\psi}$ operators in any product, for particles or antiparticles), and the $\hat{\psi}^\dagger$ are self-adjoint fields given by

$$\hat{\psi}_{n_j}^\dagger(x) \equiv \hat{\psi}_{n_j,\lambda_0}(x) + \hat{\psi}_{n_j,\lambda_0}^\dagger(x).$$

A $\hat{V}_{\lambda_0}(x)$ constructed in this way is clearly self-adjoint, as required. Further, it has the important commutivity property given in the following theorem (proved in the appendix).

**Theorem 5** (Commutivity of the Vertex Operator). Let $\hat{V}_\lambda(x)$ be defined as in Eq. (36). Then

$$[\hat{V}_{\lambda_1}(x_1), \hat{V}_{\lambda_2}(x_2)] = 0,$$

for all values of the $x_i$ and $\lambda_i$.

**Corollary.** Let

$$\hat{V}(\lambda) = \int d^4x \hat{V}_\lambda(x).$$

Then

$$[\hat{V}(\lambda_1), \hat{V}(\lambda_2)] = 0,$$

for all values of the $\lambda_i$.

Because the $\hat{V}(\lambda)$ commute for different $\lambda$, so will $\hat{H}^{\text{int}}(\lambda) = -d\hat{V}(\lambda)/d\lambda$. Therefore, Eq. (34) may be integrated to obtain

$$\hat{V}(\lambda_0) = \int_{\lambda_0}^{\infty} d\lambda \hat{H}^{\text{int}}(\lambda).$$  \hspace{1cm} (37)$$

Note that, for the chosen integration bounds and any non-trivial $\hat{H}^{\text{int}}(\lambda_0)$, $\hat{V}(\lambda_0)$ will be non-zero for all finite $\lambda_0$, and so $G(\lambda_0)$ will not be the identity for any $\lambda_0$. Thus, unlike the traditional interaction picture, in which the free and interacting fields coincide at one point in time, there is no $\lambda_0$ for which the free and interacting parameterized fields are “the same.”

**C. Regularization**

The operator $\hat{V}_{\lambda_0}(x)$, as defined using normal ordering in Eq. (36), is actually a proper distribution. However, this is no longer the case when it is integrated over all spacetime, as in Eq. (37). As a result, the series expansion of $\hat{G}$ in Eq. (32) gives terms with integrals (over spacetime or momentum space) that are not well defined. These integrals require regularization, as in the case of perturbation expansions in traditional QFT.
As an example, consider the use of Pauli-Villars regularization \[52\]. In this approach, the Feynman propagator for a particle of mass \(m\),

\[
\Delta(x - x_0; m) = -i(2\pi)^{-4} \int d^4p \frac{e^{ip(x-x_0)}}{p^2 + m^2 - i\varepsilon},
\]
is replaced with the effective propagator \(\Delta(x - x_0; m) - \Delta(x - x_0; M)\), for \(M > m\). The integrals in the expansion are regulated by the extra term in \(M\) and can be evaluated. Then, at the end of the calculation, \(M\) is taken to infinity, such that \(\Delta(x - x_0; M) \to 0\).

Now, consider that (see Eq. (28))

\[
\Delta(x; m) = \int_{\lambda_0}^{\infty} d\lambda \int d^4p e^{ip(x-x_0)} e^{-i(p^2 + m^2)(\lambda - \lambda_0)} = \int_{\lambda_0}^{\infty} d\lambda [\hat{\psi}(x; \lambda), \hat{\psi}^\dagger(x_0; \lambda_0)].
\]

Therefore,

\[
\Delta(x - x_0; m) - \Delta(x - x_0; M) = \int_{\lambda_0}^{\infty} d\lambda \int d^4p e^{ip(x-x_0)} \left[ e^{-i(p^2 + m^2)(\lambda - \lambda_0)} - e^{-i(p^2 + M^2)(\lambda - \lambda_0)} \right]
= \int_{\lambda_0}^{\infty} d\lambda \left[ 1 - e^{-i(M^2 - m^2)(\lambda - \lambda_0)} \right] \int d^4p e^{ip(x-x_0)} e^{-i(p^2 + m^2)(\lambda - \lambda_0)}
= \int_{\lambda_0}^{\infty} d\lambda \left[ 1 - e^{-i(M^2 - m^2)(\lambda - \lambda_0)} \right] [\hat{\psi}(x; \lambda), \hat{\psi}^\dagger(x_0; \lambda_0)].
\]

Define

\[
\hat{\psi}_f(x) \equiv \int_{\lambda_0}^{\infty} d\lambda f(\lambda - \lambda_0)\hat{\psi}(x; \lambda).
\]

This is a generalization of the definition of the integrated field operator, which reduces to \(\hat{\psi}_{\lambda_0}(x)\) for \(f(\lambda - \lambda_0) = 1\). Taking, instead,

\[
f(\lambda - \lambda_0) = 1 - e^{-i(M^2 - m^2)(\lambda - \lambda_0)}
\]

then gives

\[
[\hat{\psi}_f(x), \hat{\psi}^\dagger_{\lambda_0}(x_0)] = \Delta(x - x_0; m) - \Delta(x - x_0; M).
\]

Thus, we can achieve Pauli-Villars regularization simply by replacing \(\hat{\psi}_{n,\lambda_0}(x)\) with \(\hat{\psi}_{n,f_n}(x)\) in the vertex operator \(\hat{V}\), for each particle type \(n\) (with \(f_n\) defined as given above, but with \(m\) replaced by \(m_n\)), but leaving the adjoints \(\hat{\psi}^\dagger_{n,\lambda_0}(x)\) unchanged. This is similar to the idea discussed in Sect. IID of \[13\], based on earlier proposals in \[54, 55\], to regularize by making the interaction coupling dependent on the intrinsic path length (though in those cases with a more physical function for correlation in the path parameter).

The topic of regularization in parameterized QFT, and its relation to renormalization of the expansion of \(\hat{G}\), will be explored in future work. For the present paper, we will simply consider Eq. (32) to be a formal series expansion and see, in the next section, how this expansion can be used to reproduce the similarly formal series expansion that results from perturbative scattering theory in traditional QFT.

### D. Scattering

This section shows how the usual scattering amplitudes can be reproduced by the parameterized formalism, given a vertex operator of the form defined in Eq. (63). We will be using only integrated fields in this section. Presume that a specific, fixed value of \(\lambda_0\) has been chosen, so the notation may be simplified by omitting explicit references to \(\lambda_0\).

Consider, first, scattering that takes place limited to just a specific four volume \(\mathcal{V}\). A vertex operator \(\hat{V}_\mathcal{V}\) restricted to \(\mathcal{V}\) may be defined as in Eq. (33) but with the integral over all spacetime replaced by an integral over only the four-volume \(\mathcal{V}\). That is,

\[
\hat{V}_\mathcal{V} \equiv \int_\mathcal{V} d^4x \hat{V}(x).
\]
The corresponding interaction operator restricted to $\mathcal{V}$ is then

$$\hat{G}_V \equiv e^{-i\hat{V}_V}.$$  

It is another immediate corollary of Theorem 1 that the $\hat{V}(x)$ commute for different $x$, so the $\hat{V}_V$ commute for different $\mathcal{V}$. Therefore, the restricted interaction operator has the property

$$\hat{G}_{V_1 \cup V_2} = \hat{G}_{V_1} \hat{G}_{V_2},$$

which allows for easy separation of interactions within a system in a certain four-volume from interactions that occur in the environment of the system $[10]$.

Now consider states $|\Phi_{in}\rangle$ that are superpositions of position states with positions outside of $\mathcal{V}$, that is, they are constructed from applications of the field operators $\hat{\psi}_{n+}^\dagger(x)$ and $\hat{\psi}_{n-}^\dagger(x)$ that have $x$ outside of $\mathcal{V}$. Using Eq. (29), but with the restricted interaction operator, then gives $\hat{G}_V|\Phi_{in}\rangle$ as the state representing the particles entering $\mathcal{V}$ from outside and interacting there (or not interacting at all). Since the free and interacting Hilbert spaces are the same, we can expand the interacting state in the free position state basis:

$$\hat{G}_V|\Phi_{in}\rangle = \sum_{N=0}^{\infty} \left( \prod_{i=1}^{N} \int d^4x_i \right) |x_1, \ldots, x_N; \lambda_0\rangle \langle x_1, \ldots, x_N; \lambda_0|\hat{G}_V|\Phi_{in}\rangle.$$  

The coefficients $(x_1, \ldots, x_N|\hat{G}_V|\Phi_{in}\rangle$ are the probability amplitudes that the incoming particles, after interacting in $\mathcal{V}$, result in $N$ particles at the given positions. Thus, if we construct states $\langle \Phi_{out}|$ using the integrated fields $\hat{\psi}_{n+}(x)$ and $\hat{\psi}_{n-}(x)$, with $x$ also outside of $\mathcal{V}$, then

$$\langle \Phi_{out}|\hat{G}_V|\Phi_{in}\rangle = \sum_{N=0}^{\infty} \left( \prod_{i=1}^{N} \int d^4x_i \right) \langle \Phi_{out}|x_1, \ldots, x_N; \lambda_0\rangle \langle x_1, \ldots, x_N; \lambda_0|\hat{G}_V|\Phi_{in}\rangle$$

is the amplitude for the incoming particles to scatter only in $\mathcal{V}$ (or not interact at all) and then propagate out of $\mathcal{V}$ into the outgoing state $\langle \Phi_{out}|$.

If we take the series expansion for $\hat{G}_V$,

$$\hat{G}_V = \sum_m \frac{(-i)^m}{m!} \hat{V}_V^m,$$

then the $m$th term in the series represents the case of exactly $m$ interactions taking place within $\mathcal{V}$. Using this expansion, any scattering amplitude $\langle \Phi_{out}|\hat{G}_V|\Phi_{in}\rangle$ can be written as a (weighted) sum of terms of the form

$$\langle 0|\hat{\psi}_{n_1}^\dagger(x_1') \cdots \hat{\psi}_{n_M}^\dagger(x_M') \frac{(-i)^m}{m!} \hat{V}_V^m \hat{\psi}_{n_1}(x_1) \cdots \hat{\psi}_{n_M}(x_M)|0\rangle,$$

where the $x_i$ and $x_i'$ are all outside $\mathcal{V}$ and each $n_i$ and $n_i'$ may be a particle or antiparticle type.

As in the traditional derivation of Feynman diagrams, we can now move all annihilation operators to the right in each such term, generating commutators with intermediate field applications of the same particle type $[1, 2, 57]$. Thus, the pairing of a factor $\hat{\psi}_{n}(x')$ in $\hat{V}_V$ with an “in” particle factor $\hat{\psi}_{n+}^\dagger(x)$ gives the commutator

$$[\hat{\psi}_{n}(x'), \hat{\psi}_{n+}^\dagger(x)] = [\hat{\psi}_{n+}(x'), \hat{\psi}_{n+}^\dagger(x)] = \Delta_n(x' - x),$$

and the pairing of a factor $\hat{\psi}_{n+}^\dagger(x')$ in $\hat{V}_V$ with an “in” antiparticle factor $\hat{\psi}_{n-}^\dagger(x)$ gives the commutator

$$[\hat{\psi}_{n+}^\dagger(x'), \hat{\psi}_{n-}^\dagger(x)] = [\hat{\psi}_{n-}(x'), \hat{\psi}_{n-}^\dagger(x)] = \Delta_n(x' - x).$$

Pairings of “out” particle and antiparticle annihilation operators with corresponding creation operators within $\hat{V}$ (or directly with those for incoming particles) give similar factors, as do the pairings of annihilation and creation operators within $\hat{V}$.

Next, consider a region $\mathcal{V}(t_F, t_l)$ bounded by hyperplanes at $t = t_l$ and $t = t_F > t_l$, but unbounded in space. Further, let $|\Phi_{in}\rangle = |\Phi_1\rangle$, a superposition of position states with $t < t_l$, and $\langle \Phi_{out}| = \langle \Phi_2|$, a superposition of
positions states with $t > t_F$. Then, in the commutators of Eqs. (38) and (39) for incoming particles and antiparticles, $t' = x'^0 \geq t_1 > x_0$, while, in the commutators for outgoing particles and antiparticles, $t' > t_F \geq t$.

Using Eq. (2), it is clear that, for $t' > t$, $\Delta_n(x' - x) = \Delta_n(t' - t, x' - x) = \Delta_n^\dagger(t' - t, x' - x)$, which represents the propagation of an on-shell, positive-energy particle. That is, particles propagate into or out of $\mathcal{V}(t_F, t_1)$ on-shell. Within $\mathcal{V}(t_F, t_1)$, $x'$ and $x$ are not time-ordered, so propagation is still off-shell and virtual, as given by $\Delta_n(x' - x)$.

Thus, rather than constructing virtual particle propagation from on-shell propagation, as in traditional QFT, we have recovered on-shell propagation as a special case. The basic $\hat{\psi}_{n\pm}(x)$ fields themselves represent off-shell virtual particles and do not in general satisfy the Klein-Gordon equation. On the other hand, it is straightforward to also construct fields that are positive-energy solutions of the Klein-Gordon equation:

$$\hat{\psi}_{n\pm}^\dagger(x) \equiv \int d^4x_0 \Delta_n^\dagger(x - x_0) \hat{\psi}_{n\pm}(x_0; \lambda_0),$$

such fields directly represent on-shell particles and antiparticles, such that

$$\{\hat{\psi}_{n\pm}^\dagger(x'), \hat{\psi}_{n\pm}^\dagger(x)\} = \Delta_n^\dagger(x' - x).$$

Now, suppose that, in each term in the expansion of $\hat{G}_{\mathcal{V}(t_F, t_1)}$, we replace all occurrences of $\hat{\psi}_{n\pm}(x)$ with $\hat{\psi}_{n\pm}^\dagger(x)$ (but do not replace the $\hat{\psi}_{n\pm}^\dagger(x)$). Call the resulting operator $\hat{S}(t_F, t_1)$.

To determine the effect of this replacement, consider that the vertex operator $\hat{V}_{\mathcal{V}(t_F, t_1)}$ may be written

$$\hat{V}_{\mathcal{V}(t_F, t_1)} = \int_{t_1}^{t_F} dt \int d^3x \hat{V}(t, x).$$

Because the $\hat{V}(t, x)$ commute for different $t$ (per Theorem 3), each term in the series expansion of $\hat{G}_{\mathcal{V}(t_F, t_1)}$ may then be rewritten to take order the vertex operators:

$$\frac{1}{m!} \hat{V}_{\mathcal{V}(t_F, t_1)}^m = \int_{t_1}^{t_F} dt \int_{t_1}^{t_F} dt_2 \cdots \int_{t_1}^{t_F} dt_2 T[\hat{V}(t_m) \cdots \hat{V}(t_1)],$$

where

$$\hat{V}(t) \equiv \int d^3x \hat{V}(t, x)$$

and $T[\cdots]$ indicates the sum of all time-ordered permutations of the bracketed factors.

Make the $\hat{\psi}_{n\pm}$ to $\hat{\psi}_{n\pm}^\dagger$ replacement in $\hat{V}(t)$ and call the resulting operator $\hat{v}(t)$. The series expansion for $\hat{S}(t_F, t_1)$ is then

$$\hat{S}(t_F, t_1) = \sum_m \frac{(-i)^m}{m!} \int_{t_1}^{t_F} dt \int_{t_1}^{t_F} dt_2 \cdots \int_{t_1}^{t_F} dt_2 T[\hat{v}(t_m) \cdots \hat{v}(t_1)] = e^{T[-i \int_{t_1}^{t_F} dt \hat{v}(t)]},$$

where time ordering is now not optional, since the $\hat{v}(t)$ do not commute. This is essentially just a Dyson expansion.

Make similar field operator replacements in $\langle \Phi_F |$ to get $\langle \Phi_F^+ |$. Then the amplitude $\langle \Phi_F^+ | \hat{S}(t_F, t_1) | \Phi_1 \rangle$ has an expansion that is term-for-term parallel to the expansion of $\langle \Phi_F | \hat{G}_{\mathcal{V}(t_F, t_1)} | \Phi_1 \rangle$, but with the appropriate operator replacements in each term. For incoming and outgoing particles, this will clearly result in commutators that generate the same $\Delta_n^\dagger(x' - x)$ factors as before the replacement. For internal propagations, the result will be as in the usual derivation for Feynman diagrams from the time-ordered Dyson series 11 12 15 23:

$$\theta(t' - t)[\hat{\psi}_{n\pm}^\dagger(t', x'), \hat{\psi}_{n\pm}^\dagger(t, x)] + \theta(t - t')[\hat{\psi}_{n\pm}^\dagger(t, x), \hat{\psi}_{n\pm}^\dagger(t', x')]
= \theta(t' - t)\Delta_n^\dagger(t' - t, x' - x) + \theta(t - t')\Delta_n^\dagger(t - t', x - x')
= \theta(t' - t)\Delta_n^\dagger(t' - t, x' - x) + \theta(t - t')\Delta_n^\dagger(t' - t, x' - x)^*$$

which is virtual particle propagation, as before. The expansions before and after the field replacements therefore produce the same propagation factors, so

$$\langle \Phi_F^+ | \hat{S}(t_F, t_1) | \Phi_1 \rangle = \langle \Phi_F | \hat{G}_{\mathcal{V}(t_F, t_1)} | \Phi_1 \rangle.$$
Now let \( t_F \to +\infty \) and \( t_1 \to -\infty \). Then \( \hat{S}(t_F, t_1) \to \hat{S} \), the traditional scattering operator, and \( V(t_F, t_1) \) goes to all spacetime, so \( \hat{G}_{V(t_F, t_1)} \to \hat{G} \). In taking the time limits in \( \langle \Phi_F \mid \Phi_1 \rangle \), we would like to hold the three-momenta of the particles to fixed, given values. To do this, construct these states from the time-dependent three-momentum field operators

\[
\hat{\psi}_{n \pm}(t, \mathbf{p}) \equiv (2\pi)^{-3/2} \int d^3x e^{-i(-\omega_p t + \mathbf{p} \cdot \mathbf{x})} \hat{\psi}_{n \pm}(t, \mathbf{x})
\]

with their respective adjoints. The corresponding three-momentum field operators to use in \( \langle \Phi_F^\dagger \mid \hat{S} \mid \Phi_1 \rangle \) are

\[
\hat{\psi}_{n \pm}^+(t, \mathbf{p}) = (2\pi)^{-3/2} \int d^3x e^{-i(-\omega_p t + \mathbf{p} \cdot \mathbf{x})} \hat{\psi}_{n \pm}^+(t, \mathbf{x})
\]

For an incoming particle, the propagation factor in \( \langle \Phi_F^\dagger \mid \hat{S} \mid \Phi_1 \rangle \) is then given by

\[
[\hat{\psi}_{n \pm}^+(t', \mathbf{x}'), \hat{\psi}_{n \pm}^+(t, \mathbf{p})] = (2\pi)^{-3/2} \int d^3x' e^{i(-\omega_p t' + \mathbf{p} \cdot \mathbf{x}')} \Delta^+_n (t' - t, \mathbf{x}' - \mathbf{x})
\]

\[
= (2\pi)^{-3/2} (2\omega_p)^{-1} e^{i(-\omega_p t' + \mathbf{p} \cdot \mathbf{x}')}.
\]

This is the proper factor for an incoming, on-shell particle of three-momentum \( \mathbf{p} \) and is independent of \( t \) as \( t \to -\infty \). For an outgoing particle, the factor is

\[
[\hat{\psi}_{n \pm}^+(t', \mathbf{p'}), \hat{\psi}_{n \pm}^+(t, \mathbf{x})] = (2\pi)^{-3/2} \int d^3x' e^{-i(-\omega_p t' + \mathbf{p'} \cdot \mathbf{x}')} \Delta^+_n (t' - t, \mathbf{x}' - \mathbf{x})
\]

\[
= (2\pi)^{-3/2} (2\omega_p)^{-1} e^{-i(-\omega_p t' + \mathbf{p'} \cdot \mathbf{x})}.
\]

Again, this is the proper factor for an outgoing, on-shell particle of three-momentum \( \mathbf{p}' \) and is independent of \( t' \) as \( t' \to +\infty \).

Therefore, in the \( t_F \to +\infty, t_1 \to -\infty \) limit taken as above, \( \langle \Phi_F^\dagger \mid \hat{S} \mid \Phi_1 \rangle \) is just the traditional scattering amplitude between on-shell, three-momentum states. And, since, in this limit, \( \langle \Phi_F^\dagger \mid \hat{S} \mid \Phi_1 \rangle = \langle \Phi_F \mid \hat{G} \mid \Phi_1 \rangle \), \( \langle \Phi_F \mid \hat{G} \mid \Phi_1 \rangle \) is the same scattering amplitude.

**IV. CONCLUSION**

Mathematically, the parameterized approach is appealing. It makes the formalism for relativistic quantum theory much more parallel to that of the non-relativistic theory, and it does not treat time in a privileged way that is in conflict with the nature of relativistic spacetime. Nevertheless, it is reasonable to ask whether this benefit is worth the cost of introducing another parameter that is (at least as presented here), in the end, physically unobservable.

Recall, though, that all the empirically tested results of traditional QFT come from canonical, perturbative QFT. And, even before issues of regularization and renormalization are addressed, Haag’s theorem already renders the axiomatic theory of QFT with interactions.

The parameterized formulation presented here resolves this problem. The key to doing so is basing the theory on a Hilbert space of off-shell states, as given in Sec. II A. A Hamiltonian operator on such states as then defined as a generator of propagation in the path parameter, rather than time. And a vacuum state is a null eigenstate unique relative to a chosen Hamiltonian operator, rather than being a priori unique in the Hilbert space.

The corresponding axioms for fields given in Sec. II B admit a free-field theory equivalent to that of traditional QFT, as shown in Sec. III C. However, unlike traditional, canonical QFT, Haag’s theorem is absent from parameterized QFT, crucially because the Hamiltonian generates evolution in a parameter separate from the spacetime coordinates that are constrained by Poincaré invariance. It is then relatively straightforward to construct an interacting theory that also meets the parameterized QFT axioms, as shown in Sec. III D.

Further, by allowing the Fock representation of a free field to be extended to the corresponding interacting field (as discussed in Sec. III A), the intuitive quanta interpretation of the free theory can be carried over to the interacting theory. Indeed, the approach can also provide for a fuller interpretation in terms of spacetime paths, decoherence and consistent histories over spacetime, as addressed in [38, 40].

And, as shown in Sec. III D, the parameterized formalism can be used to derive series expansions for scattering amplitudes that formally match those derived using traditional perturbative QFT, term for term. Admittedly, this
shows no more than that the key empirically tested results of traditional QFT can be duplicated by parameterized QFT. But it does explain the otherwise surprising fact that perturbative QFT produces excellent empirical results even though, in the face of Haag’s theorem, it is mathematically inconsistent as usually formulated.

Moreover, as argued in [58], different formulations of QFT may lead to different interpretations, even while being empirically equivalent. Clearly, one would like to base any interpretation on a formulation that is rigorously defined mathematically. But this is problematic for traditional canonical QFT, since models of realistic interactions using the canonical formulation run afoul of Haag’s Theorem. Parameterized QFT does not have this problem.

Of course, this does not resolve all the mathematical issues with traditional QFT, such as those involved in renormalization. And the axiomatic approach discussed here has not yet been extended to cover gauge theories, as required to model real interactions. Nevertheless, such issues all revolve around the mathematical treatment of interactions in QFT, and having a consistent foundation for modeling interactions therefore seems to be a prerequisite to resolving them.

In that regard, the work presented here is offered as a step toward building a firmer foundation, both mathematically and interpretationally, for QFT in general.

Appendix: Commutivity of the Vertex Operator

**Theorem.** Let

\[ \hat{V}_\lambda(x) = g : \prod_i \hat{\psi}_{n_i,\lambda}(x) \hat{\psi}_{n_i,\lambda}(x) \prod_j \hat{\psi}_{n_j,\lambda}(x) : , \]

where

\[ \hat{\psi}_{n,\lambda}(x) = \hat{\psi}_{n+\lambda}(x) + \hat{\psi}^\dagger_{n-\lambda}(x) \]

and

\[ \hat{\psi}'_{n,\lambda}(x) = \hat{\psi}_{n,\lambda}(x) + \hat{\psi}^\dagger_{n,\lambda}(x) . \]

Then

\[ [\hat{V}_{\lambda_1}(x_1), \hat{V}_{\lambda_2}(x_2)] = 0 , \]

for all values of the \( x_i \) and \( \lambda_i \).

**Proof.** First, consider that

\[ [\hat{\psi}_{n+\lambda_1}(x_1), \hat{\psi}_{n+\lambda_2}(x_2)] = \int_{-\infty}^{\infty} d\lambda \Delta (x_1 - x_2, \lambda - \lambda_2) \]

\[ = \int_{\lambda_1 - \lambda_2}^{\infty} d\lambda \Delta (x_1 - x_2, \lambda) \]

and

\[ [\hat{\psi}_{n-\lambda_2}(x_2), \hat{\psi}'_{n-\lambda_1}(x_1)] = \int_{-\infty}^{\lambda_2} d\lambda \Delta (x_2 - x_1, \lambda_1 - \lambda) \]

\[ = \int_{\lambda_1 - \lambda_2}^{\lambda_2} d\lambda \Delta (x_2 - x_1, -\lambda) \]

\[ = \int_{\lambda_2}^{\infty} d\lambda \Delta (x_1 - x_2, \lambda) \]

\[ = [\hat{\psi}_{n+\lambda_1}(x_1), \hat{\psi}_{n+\lambda_2}(x_2)]^{-1} , \]

so

\[ [\hat{\psi}_{n,\lambda_1}(x_1), \hat{\psi}'_{n,\lambda_2}(x_2)] = [\hat{\psi}_{n+\lambda_1}(x_1) + \hat{\psi}'_{n-\lambda_1}(x_1), \hat{\psi}'_{n+\lambda_2}(x_2) + \hat{\psi}_{n-\lambda_2}(x_2)] \]

\[ = [\hat{\psi}_{n+\lambda_1}(x_1), \hat{\psi}'_{n+\lambda_2}(x_2)] - [\hat{\psi}_{n-\lambda_2}(x_2), \hat{\psi}'_{n-\lambda_1}(x_1)] \]

\[ = 0 , \]
and, similarly, $[\hat{\psi}^+_{n,\lambda_1}(x_1), \hat{\psi}_{n,\lambda_2}(x_2)] = 0$. Next, formally defined the normal-ordered products by the limiting process

$$
: \hat{\psi}^+_{n,\lambda_1}(x_1) \hat{\psi}_{n,\lambda_2}(x_1) : = \lim_{x'_1, x''_1 \to x_1} \{ \hat{\psi}^+_{n,\lambda_1}(x'_1) \hat{\psi}_{n,\lambda_2}(x''_1) - [\hat{\psi}_{n,\lambda_1}(x''_1), \hat{\psi}^+_{n,\lambda_2}(x'_1)] \}
$$

$$
= \lim_{x'_1, x''_1 \to x_1} \{ \hat{\psi}^+_{n,\lambda_1}(x'_1) \hat{\psi}_{n,\lambda_2}(x''_1) - \Delta(x''_1 - x'_1) \}.
$$

Then

$$
[ : \hat{\psi}^+_{n,\lambda_1}(x_1) \hat{\psi}_{n,\lambda_2}(x_1) : ; : \hat{\psi}^+_{n,\lambda_2}(x_2) \hat{\psi}_{n,\lambda_2}(x_2) : ] = \lim_{x'_1, x''_1 \to x_1, x'_2, x''_2 \to x_2} \{ \hat{\psi}^+_{n,\lambda_1}(x'_1) \hat{\psi}_{n,\lambda_2}(x''_1), \hat{\psi}^+_{n,\lambda_2}(x'_2) \hat{\psi}_{n,\lambda_2}(x''_2) \}
$$

$$
= 0.
$$

Further,

$$
[\hat{\psi}_{n,\lambda_1}(x_1), \hat{\psi}_{n,\lambda_2}(x_2)] = [\hat{\psi}_{n,\lambda_1}(x_1), \hat{\psi}^+_{n,\lambda_1}(x_1), \hat{\psi}^+_{n,\lambda_2}(x_2) + \hat{\psi}_{n,\lambda_2}(x_2)]
$$

$$
= [\hat{\psi}_{n,\lambda_1}(x_1), \hat{\psi}^+_{n,\lambda_2}(x_2)] + [\hat{\psi}^+_{n,\lambda_1}(x_1), \hat{\psi}_{n,\lambda_2}(x_2)] = 0.
$$

Thus, the factor for each particle type in the definition of $\hat{V}_{\lambda}(x_1)$ commutes with the similar factor in the definition of $\hat{V}_{\lambda_2}(x_2)$. Since fields of different particle types all commute with each other, $\hat{V}_{\lambda_1}(x_1)$ as a whole commutes with $\hat{V}_{\lambda_2}(x_2)$ as a whole.

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