A universal regularization method for ill-posed Cauchy problems for quasilinear partial differential equations

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Abstract
For the first time, a globally convergent numerical method is presented for ill-posed Cauchy problems for quasilinear PDEs. The key idea is to use Carleman Weight Functions to construct globally strictly convex Tikhonov-like cost functionals.

Keywords: Carleman estimates; Ill-Posed Cauchy problems; quasilinear PDEs

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1 Introduction
This is the first publication where a globally convergent numerical method is presented for ill-posed Cauchy problems for a broad class of quasilinear Partial Differential Equations (PDEs). All previous works were concerned only with the linear case. First, we present the general framework of this method. Next, we specify that framework for ill-posed Cauchy problems for quasilinear elliptic, parabolic and hyperbolic PDEs.

Let $G \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial G$. Let $\Gamma \subseteq \partial G$ be a part of the boundary. Let $A$ be a quasilinear Partial Differential Operator (PDO) of the second order acting on functions $u$ defined in the domain $G$ (details are given in section 2). Suppose that $\Gamma$ is not a characteristic hypersurface of $A$. Consider the Cauchy problem for the quasilinear PDE generated by the operator $A$ with the Cauchy data at $\Gamma$. Suppose that this problem is ill-posed in the classical sense, e.g. the Cauchy problem for a quasilinear elliptic equation. In this paper we construct a universal globally convergent regularization method for solving such problems. This method works for those PDOs, for which Carleman estimates are valid for linearized versions of those operators. Since these estimates are valid for three main types of linear PDOs of the second order, elliptic, parabolic and hyperbolic ones, then our method is a quite general one.

The key idea of this paper is to construct a weighted Tikhonov-like functional. The weight is the Carleman Weight Function (CWF), i.e. the function which is involved in the Carleman estimate for the principal part of the corresponding PDO. Given a ball $B(R)$ of an arbitrary radius $R$ in a certain Sobolev space, one can choose the parameter
\( \lambda(R) \) of this CWF so large that the weighted Tikhonov-like functional is strictly convex on \( B(R) \) for all \( \lambda \geq \lambda(R) \). The strict convexity, in turn guarantees the convergence of the gradient method of the minimization of this functional if starting from an arbitrary point of the ball \( B(R) \). Since restrictions of the radius \( R \) are not imposed, then this is global convergence. On the other hand, the major problem with conventional Tikhonov functionals for nonlinear ill-posed problems is that they usually have many local minima and ravines. The latter implies convergence of the gradient method only if it starts in a sufficiently small neighborhood of the solution, i.e. the local convergence.

First globally strictly convex cost functionals for nonlinear ill-posed problems were constructed by Klibanov \cite{23,24} for Coefficient Inverse Problems (CIPs), using CWFs. Based on a modification of this idea, some numerical studies for 1-d CIPs were performed in the book of Klibanov and Timonov \cite{25}. Recently there is a renewed interest to this topic, see Baudoin, DeBuhan and Ervedoza \cite{11}, Beilina and Klibanov \cite{3}, Klibanov and Thanh \cite{28} and Klibanov and Kamburg \cite{30}. In particular, the paper \cite{28} contains some numerical results. However, globally strictly convex cost functionals were not constructed for ill-posed Cauchy problems for quasilinear PDEs in the past.

Concerning applications of Carleman estimates to inverse problems, we refer to the method, which was first proposed by Bukhgeim and Klibanov \cite{9,10,19}. The method of \cite{9,10,19} was originally designed for proofs of uniqueness theorems for CIPs with single measurement data, see, e.g. some follow up works of Bukhgeim \cite{11}, Klibanov \cite{20}, Klibanov and Timonov \cite{25}, surveys of Klibanov \cite{27} and Yamamoto \cite{36}, as well as sections 1.10 and 1.11 of the book of Beilina and Klibanov \cite{2}.

There is a huge number of publications about ill-posed Cauchy problems for linear PDEs. Hence, we now refer only to a few of them. We note first that in the conventional case of a linear ill-posed problem, Tikhonov regularization functional is generated by a bounded linear operator, see, e.g. books of Ivanov, Vasin and Tanana \cite{16} and Lavrentiev, Romanov and Shishatskii \cite{33}. Unlike this, the idea, which is the closest one to this paper, is to use unbounded linear PDOs as generators of Tikhonov functionals for ill-posed Cauchy problems. This idea was first proposed in the book of Lattes and Lions \cite{32}. In \cite{32} Tikhonov functionals were written in the variational form. Since Carleman estimates were not used in \cite{32}, then convergence rates for minimizers were not established. The idea of using Carleman estimates for establishing convergence rates of minimizers of those Tikhonov functionals was first proposed in works of Klibanov and Santosa \cite{21} and of Klibanov and Malinsky \cite{22}. Next, it was explored in works of Klibanov with coauthors \cite{12,13,26}, where accurate numerical results were obtained; also see a recent survey \cite{29}. In addition, Bourgeois \cite{7,8} and Bourgeois and Dardé have used this idea for the case of the Cauchy problem for the Laplace equation, see, e.g. \cite{7,8}. We also refer to Berntsson, Kozlov, Mpinganzima and Turesson \cite{4}, Hao and Lesnic \cite{13}, Kabanikhin \cite{17}, Kabanikhin and Karchevsky \cite{18}, Kozlov, Maz’ya and Fomin \cite{31} and Li, Xie and Zou \cite{34} for some other numerical methods for ill-posed Cauchy problems for linear PDEs.

All functions considered below are real valued ones. In section 2 we present the general framework of our method. Next, in sections 3, 4 and 5 we specify this framework for quasilinear elliptic, parabolic and hyperbolic PDEs respectively.
2 The method

Let $A$ be a quasilinear PDO of the second order in $G$ with its principal part $A_0$,
\begin{align}
A(u) &= \sum_{|\alpha|=2} a_{\alpha}(x) D^\alpha u + A_1(x, \nabla u, u), \tag{2.1} \\
A_0 u &= \sum_{|\alpha|=2} a_{\alpha}(x) D^\alpha u, \tag{2.2}
\end{align}

where functions $a_{\alpha} \in C^1(G)$, $A_1 \in C(G) \times C^3(\mathbb{R}^{n+1})$. \tag{2.3}

Let $k_n = \lfloor n/2 \rfloor + 2$, where $\lfloor n/2 \rfloor$ is the largest integer which does not exceed the number $n/2$. By the embedding theorem
\[ H^{k_n}(G) \subset C^1(G) \quad \text{and} \quad \|f\|_{C^1(G)} \leq C \|f\|_{H^{k_n}(G)}, \forall f \in H^{k_n}(G), \tag{2.4} \]
where the constant $C = C(n, G) > 0$ depends only on listed parameters.

**Cauchy Problem.** Let the hypersurface $\Gamma \subseteq \partial G$ and assume that $\Gamma$ is not a characteristic hypersurface of the operator $A_0$. Consider the following Cauchy problem for the operator $A$ with the Cauchy data $g_0(x), g_1(x)$,
\begin{align}
A(u) &= 0, \tag{2.5} \\
\text{u |}_{\Gamma} &= g_0(x), \partial_n \text{u |}_{\Gamma} = g_1(x), \text{r} \in \Gamma. \tag{2.6}
\end{align}

Determine the solution $u \in H^{k_n}(G)$ of the problem (2.5), (2.6) either in the entire domain $G$ or at least in its subdomain.

2.1 The Carleman estimate

Let the function $\xi \in C^2(G)$ and $|\nabla \xi| \neq 0$ in $G$. For a number $c \geq 0$ denote
\[ \xi_c = \{ x \in G : \xi(x) = c \}, G_c = \{ x \in G : \xi(x) > c \}. \tag{2.7} \]

We assume that $G_c \neq \emptyset$. Choose a sufficiently small $\varepsilon > 0$ such that $G_{c+2\varepsilon} \neq \emptyset$. Obviously, $G_{c+2\varepsilon} \subset G_{c+\varepsilon} \subset G_c$. Let $\Gamma_c = \{ x \in \Gamma : \xi(x) > c \} \neq \emptyset$. Hence, the boundary of the domain $G_c$ consists of two parts,
\[ \partial G_c = \partial_1 G_c \cup \partial_2 G_c, \partial_1 G_c = \xi_c, \partial_2 G_c = \Gamma_c. \tag{2.8} \]

Let $\lambda > 1$ be a large parameter. Consider the function $\varphi_\lambda(x)$,
\[ \varphi_\lambda(x) = \exp(\lambda \xi(x)). \tag{2.9} \]

It follows from (2.8) and (2.9) that
\[ \min_{\Gamma_c} \varphi_\lambda(x) = \varphi_\lambda(x) |_{\xi_c} = e^{\lambda c}. \tag{2.10} \]

**Definition 2.1.** We say that the operator $A_0$ admits the pointwise Carleman estimate in the domain $G_c$ with the CWF $\varphi_\lambda(x)$ if there exist constants $\lambda_0(G_c, A_0) >$
In (2.11) the vector function $U(x)$ satisfies the following estimate
\[
(A_0u)^2 \varphi_\lambda^2 (x) \geq C_1 \lambda (\nabla u)^2 \varphi_\lambda^2 (x) + C_1 \lambda^3 u^2 \varphi_\lambda^2 (x) + \text{div} \, U, \quad \forall \lambda \geq \lambda_0, \forall u \in C^2 (\overline{G}), \forall x \in G_c. \tag{2.11}
\]

In (2.11) the vector function $U(x)$ satisfies the following estimate
\[
|U(x)| \leq C_1 \lambda^3 \left[(\nabla u)^2 + u^2\right] \varphi_\lambda^2 (x), \forall x \in G_c. \tag{2.13}
\]

### 2.2 The main result and the gradient method

Let $R > 0$ be an arbitrary number. Denote
\[
B(R) = \left\{ u \in H^{k_h} (G_c) : \|u\|_{H^{k_h}(G_c)} < R, u |_{\Gamma = g_0} = g_0 (x), \partial_n u |_{\Gamma = g_1} = g_1 (x) \right\}, \tag{2.14}
\]
\[
H_0^{k_h} (G_c) = \left\{ u \in H^{k_h} (G_c) : u |_{\Gamma = \partial_n u |_{\Gamma = \partial}} = 0 \right\}. \tag{2.15}
\]

To solve the above Cauchy problem, we consider the following minimization problem.

**Minimization Problem.** Assume that the operator $A_0$ satisfies the Carleman estimate (2.11), (2.12) for a certain number $c \geq 0$. Minimize the functional $J_{\lambda,a} (u)$ in (2.16) on the set $B(R)$, where
\[
J_{\lambda,\beta} (u) = e^{-2\lambda(c+\epsilon)} \int_{G_c} [A(u)]^2 \varphi_\lambda^2 dx + \beta \|u\|_{H^{k_h}(G_c)}^2, \tag{2.16}
\]

where $\beta \in (0,1)$ is the regularization parameter.

Thus, by solving this problem we approximate the function $u$ in the subdomain $G_c \subset G$. The multiplier $e^{-2\lambda(c+\epsilon)}$ is introduced to balance two terms in the right hand side (2.16): to allow to have $\beta \in (0,1)$. Theorem 2.1 is the main result of this paper. Note that since $e^{-\lambda x} << 1$ for sufficiently large $\lambda$, then the requirement of this theorem $\beta \in (e^{-\lambda x}, 1)$ means that the regularization parameter $\beta$ can change from being very small and up to unity.

**Theorem 2.1.** Let $R > 0$ be an arbitrary number. Let $B(R)$ and $H_0^{k_h} (G_c)$ be the sets defined in (2.14) and (2.15) respectively. Then for every point $u \in B(R)$ there exists the Fréchet derivative $J'_{\lambda,\alpha} (u) \in H_0^{k_h} (G_c)$. Assume that the operator $A_0$ admits the pointwise Carleman estimate in the domain $G_c$ of Definition 2.1 and let $\lambda_0 (G_c, A_0) > 1$ be the constant of this definition. Then there exists a sufficiently large number $\lambda_1 = \lambda_1 (G_c, A, R) > \lambda_0$ such that for all $\lambda \geq \lambda_1$ and for every $\beta \in (e^{-\lambda x}, 1)$ the functional $J_{\lambda,\beta} (u)$ is strictly convex on the set $B(R)$,
\[
J_{\lambda,\beta} (u_2) - J_{\lambda,\beta} (u_1) - J'_{\lambda,\beta} (u_1) (u_2 - u_1) \geq C_2 e^{2\lambda_x} \|u_2 - u_1\|_{H^1(G_c+2\epsilon)}^2 + \frac{\beta}{2} \|u_2 - u_1\|_{H^{k_h}(G_c)}^2, \forall u_1, u_2 \in B(R).
\]

Here the number $C_2 = C_2 (A, R, c)$ depends only on listed parameters.

We now show that this theorem implies the global convergence of the gradient method of the minimization of the functional (2.16) on the set $B(R)$. Consider an arbitrary
function \( u_1 \in B(R), \) which is our starting point for iterations of this method. Let the step size of the gradient method be \( \gamma > 0. \) For brevity, we do not indicate the dependence of functions \( u_n \) on parameters \( \lambda, \beta, \gamma. \) Consider the sequence \( \{u_n\}_{n=1}^{\infty} \) of the gradient method,

\[
    u_{n+1} = u_n - \gamma J'_{\lambda,\alpha}(u_n), \quad n = 1, 2, \ldots
\]  

(2.17)

**Theorem 2.2.** Let \( \lambda_1 \) be the parameter of Theorem 2.1. Choose a number \( \lambda \geq \lambda_0. \) Let \( \beta \in (e^{-\lambda\epsilon},1). \) Assume that the functional \( J_{\lambda,\beta} \) achieves its minimal value on the set \( B(R) \) at a point \( u_{\text{min}} \in B(R). \) Then such a point \( u_{\text{min}} \) is unique. Consider the sequence \( (2.17), \) where \( u_1 \in B(R) \) is an arbitrary point. Assume that \( \{u_n\}_{n=1}^{\infty} \subset B(R). \) Then there exists a sufficiently small number \( \gamma = \gamma(\lambda, \beta, B(R)) \in (0,1) \) and a number \( q = q(\gamma) \in (0,1) \) such that the sequence \( (2.17) \) converges to the point \( u_{\text{min}}. \)

\[
    \|u_{n+1} - u_{\text{min}}\|_{H^k_n(G_c)} \leq q^n \|u_0 - u_{\text{min}}\|_{H^k_n(G_c)}.
\]

Since the starting point \( u_1 \) of the sequence \( (2.17) \) is an arbitrary point of the set \( B(R) \) and a restriction on \( R \) is not imposed, then Theorem 2.2 claims the global convergence of the gradient method. This is the opposite to its local convergence for non-convex functionals. Since it was shown in [3] how a direct analog of Theorem 2.2 follows from an analog of Theorem 2.1, although for a different cost functional, we do not prove Theorem 2.2 here for brevity. We note that it is likely that similar global convergence results can be proven for other versions of the gradient method, e.g. the conjugate gradient method. However, we are not doing this here for brevity.

### 2.3 Proof of Theorem 2.1

In this proof \( C_2 = C_2(A, R, c) > 0 \) denotes different numbers depending only on listed parameters. Let \( u_1, u_2 \in B(R) \) be two arbitrary functions and let \( h = u_2 - u_1. \) Then (2.18) implies that

\[
    h \in H^k_0(G_c).
\]

(2.18)

Let

\[
    D = (A(u_2))^2 - (A(u_1))^2.
\]

(2.19)

We now put the expression for \( D \) in a form, which is convenient for us. First, we recall that the Lagrange formula implies

\[
    f(y + z) = f(y) + f'(y)z + \frac{z^2}{2}f''(y), \quad \forall y, z \in \mathbb{R}, \forall f \in C^2(\mathbb{R}),
\]

where \( \eta = \eta(y, z) \) is a number located between numbers \( y \) and \( z. \) By (2.2)

\[
    \|h\|_{C^1(\mathbb{R}_c)} = \|u_2 - u_1\|_{C^1(\mathbb{R}_c)} \leq 2CR.
\]

(2.20)

Hence, using this formula, (2.2) and (2.20), we obtain for the operator \( A_1 \)

\[
    A_1(x, \nabla (u_1 + h), u_1 + h) = A_1(x, \nabla u_1, u_1)
\]

\[
    + \sum_{i=1}^{n} \partial_{u_i}A_1(x, \nabla u_1, u_1) h_i + \partial_uA_1(x, \nabla u_1, u_1) h + F(x, \nabla u_1, u_1, h),
\]

where \( F(x, \nabla u_1, u_1, h) \) is defined by

\[
    F(x, \nabla u_1, u_1, h) =
\]

\[
    F(x, \nabla u_1, u_1, h) = f(y + z) - f(y) - f'(y)z.
\]

(2.21)

\[
    f(y + z) = f(y) + f'(y)z + \frac{z^2}{2}f''(y), \quad \forall y, z \in \mathbb{R}, \forall f \in C^2(\mathbb{R}),
\]

where \( \eta = \eta(y, z) \) is a number located between numbers \( y \) and \( z. \) By (2.2)

\[
    \|h\|_{C^1(\mathbb{R}_c)} = \|u_2 - u_1\|_{C^1(\mathbb{R}_c)} \leq 2CR.
\]

(2.20)

Hence, using this formula, (2.2) and (2.20), we obtain for the operator \( A_1 \)

\[
    A_1(x, \nabla (u_1 + h), u_1 + h) = A_1(x, \nabla u_1, u_1)
\]

\[
    + \sum_{i=1}^{n} \partial_{u_i}A_1(x, \nabla u_1, u_1) h_i + \partial_uA_1(x, \nabla u_1, u_1) h + F(x, \nabla u_1, u_1, h),
\]

where \( F(x, \nabla u_1, u_1, h) \) is defined by

\[
    F(x, \nabla u_1, u_1, h) =
\]

\[
    F(x, \nabla u_1, u_1, h) = f(y + z) - f(y) - f'(y)z.
\]
where the function $F$ satisfies the following estimate
\[ |F(x, \nabla u_1, u_1, h)| \leq C_2 \left( (\nabla h)^2 + h^2 \right), \forall x \in G_c, \forall u_1 \in B(R). \] (2.21)

Hence,
\[ A(u_1 + h) = A_0(u_1 + h) + A_1(x, \nabla (u_1 + h), u_1 + h) = \]
\[ A(u_1) + \left[ A_0(h) + \sum_{i=1}^n \partial_{u_i} A_1(x, \nabla u_1, u_1) h_{x_i} + \partial_u A_1(x, \nabla u_1, u_1) h \right] + F(x, \nabla u_1, u_1, h). \]
Hence, by (2.19)
\[ D = 2A(u_1) \left[ A_0(h) + \sum_{i=1}^n \partial_{u_i} A_1(x, \nabla u_1, u_1) h_{x_i} + \partial_u A_1(x, \nabla u_1, u_1) h \right] \]
\[ + \left[ A_0(h) + \sum_{i=1}^n \partial_{u_i} A_1(x, \nabla u_1, u_1) h_{x_i} + \partial_u A_1(x, \nabla u_1, u_1) h \right]^2 + F^2 \] (2.22)
\[- 2 \left[ A_0(h) + \sum_{i=1}^n \partial_{u_i} A_1(x, \nabla u_1, u_1) h_{x_i} + \partial_u A_1(x, \nabla u_1, u_1) h \right] F. \]

The expression in the first line of (2.22) is linear with respect to $h$. We denote it as $Q(u_1)(h)$. Consider the linear functional
\[ \tilde{J}_{u_1}(h) = \int_{G_c} Q(u_1)(h) \varphi^2 dx + 2 \beta [u_1, h], \] (2.23)
where $[,]$ denotes the scalar product in $H^{k_n}(G_c)$. Clearly, $\tilde{J}_{u_1}(h) : H^{k_n}(G_c) \to \mathbb{R}$ is a bounded linear functional. Hence, by the Riesz theorem, there exists a single element $P(u_1) \in H^{k_n}(G_c)$ such that $\tilde{J}_{u_1}(h) = [P(u_1), h], \forall h \in H^{k_n}(G_c)$. Furthermore, $\|P(u_1)\|_{H^{k_n}(G_c)} = \|\tilde{J}_{u_1}\|$. This proves the existence of the Frechét derivative
\[ J'_{\lambda, \beta}(u_1) = P(u_1) \in H^{k_n}(G_c). \] (2.24)

Let
\[ S(x, u_1, h) = D - 2A(u_1) \left[ A_0(h) + \sum_{i=1}^n \partial_{u_i} A_1(x, \nabla u_1, u_1) h_{x_i} + \partial_u A_1(x, \nabla u_1, u_1) h \right]. \]

Then, using (2.20), (2.22) and the Cauchy-Schwarz inequality, we obtain
\[ S \geq \frac{1}{2} (A_0 h)^2 - C_2 ((\nabla h)^2 + h^2), \forall x \in G_c, \forall u_1 \in B(R). \]
Hence, using (2.23) and (2.24), we obtain
\[ J_{\lambda, \beta}(u_1 + h) - J_{\lambda, \beta}(u_1) - J'_{\lambda, \beta}(u_1)(h) \]
\[ \geq \frac{1}{2} e^{-2\lambda(c+e)} \int_{G_c} (A_0 h)^2 \varphi^2 dx - C_2 e^{-2\lambda(c+e)} \int_{G_c} ((\nabla h)^2 + h^2) \varphi^2 dx + \beta \|h\|^2_{H^{k_n}(G_c)}. \] (2.25)
Next, integrate (2.11) over the domain $G_c$, using the Gauss’ formula, (2.12) and (2.13). Next, replace $u$ with $h$ in the resulting formula. Even though there is no guarantee that $h \in C^2(\overline{G}_c)$, still density arguments ensure that the resulting inequality remains true. Hence, using (2.13), we obtain

$$\frac{1}{2}e^{-2\lambda(c+\epsilon)} \int_{G_c} (A_0 h)^2 \varphi^2 dx \geq \frac{C_1}{2} e^{-2\lambda(c+\epsilon)} \int_{G_c} (\lambda (\nabla h)^2 + \lambda^3 h^2) \varphi^2 dx$$

$$- \frac{C_1}{2} \lambda^3 e^{-2\lambda} \int_{\xi_c} ((\nabla h)^2 + h^2) \varphi^2 dx.$$

Substituting this in (2.25), using again (2.20) and $\beta > e^{-\lambda \epsilon}$, we obtain for sufficiently large $\lambda$

$$J_{\lambda,\beta} (u_1 + h) - J_{\lambda,\beta} (u_1) (h) \geq C_2 e^{-2\lambda(c+\epsilon)} \int_{G_c} (\lambda (\nabla h)^2 + \lambda^3 h^2) \varphi^2 dx - C_2 e^{-2\lambda \epsilon} \lambda^3 \|h\|^2_{H^{2n}(G_c)} + \beta \|h\|^2_{H^{2n}(G_c)}$$

$$\geq C_2 e^{2\lambda \epsilon} \|h\|^2_{H^1(G_{c+2\epsilon})} - C_2 e^{-2\lambda \epsilon} \lambda^3 \|h\|^2_{H^{2n}(G_c)} + \frac{e^{-\lambda \epsilon}}{2} \|h\|^2_{H^{2n}(G_c)} + \frac{\beta}{2} \|h\|^2_{H^{2n}(G_c)}$$

$$\geq C_2 e^{2\lambda \epsilon} \|h\|^2_{H^1(G_{c+2\epsilon})} + \frac{\beta}{2} \|h\|^2_{H^{2n}(G_c)} \quad \Box$$

3 Cauchy problem for the quasilinear elliptic equation

In this section we apply Theorem 2.1 to the Cauchy problem for the quasilinear elliptic equation. We now rewrite the operator $A$ in (2.1) as

$$Au := L_{ell} (u) = \sum_{i,j=1}^n a_{i,j} (x) u_{x_i x_j} + A_1 (x, \nabla u, u), \quad x \in G,$$

(3.1)

$$A_0 u := L_0 u = \sum_{i,j=1}^n a_{i,j} (x) u_{x_i x_j},$$

(3.2)

where $a_{i,j} (x) = a_{j,i} (x), \forall i, j$ and $L_0$ is the principal part of the operator $L$. We impose assumption (2.3). The ellipticity of the operator $L_0$ means that there exist two constants $\mu_1, \mu_2 > 0, \mu_1 \leq \mu_2$ such that

$$\mu_1 |\eta|^2 \leq \sum_{i,j=1}^n a_{i,j} (x) \eta_i \eta_j \leq \mu_2 |\eta|^2, \forall x \in \overline{G}, \forall \eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n.$$  

(3.3)

As above, let $\Gamma \subset \partial G$ be the part of the boundary $\partial G$, where the Cauchy data are given. Assume that the equation of $\Gamma$ is $\Gamma = \{ x \in \mathbb{R}^n : x_1 = p(\overline{x}), \overline{x} = (x_2, ..., x_n) \in \Gamma' \subset \mathbb{R}^{n-1} \}$ and that the function $g \in C^2(\overline{\Gamma'})$. Here $\Gamma' \subset \mathbb{R}^{n-1}$ is a bounded domain. Changing variables as $x = (x_1, \overline{x}) \leftrightarrow (x'_1, \overline{x})$, where $x'_1 = x_1 - p(\overline{x})$, we obtain that in new variables
Γ = \{ x \in \mathbb{R}^n : x_1 = 0, \overline{x} \in \Gamma' \}. Here we keep the same notation for x_1 as before: for the simplicity of notations. This change of variables does not affect the property of the ellipticity of the operator L. Let X > 0 be a certain number. Thus, without any loss of generality, we assume that

\[ G \subset \{ x_1 > 0 \}, \quad \Gamma = \{ x \in \mathbb{R}^n : x_1 = 0, |\overline{x}| < X \} \subset \partial G. \]

**Cauchy Problem for the Quasilinear Elliptic Equation.** Suppose that conditions (3.3) hold. Find such a function \( u \in H^{k_0}(G) \) that satisfies the equation

\[ L_{\text{ell}}(u) = 0 \]  

and has the following Cauchy data \( g_0, g_1 \) at \( \Gamma \)

\[ u |_{\Gamma} = g_0(\overline{x}), \quad u_{x_1} |_{\Gamma} = g_1(\overline{x}). \]  

Let \( \lambda > 1 \) and \( \nu > 1 \) be two large parameters, which we define later. Consider two arbitrary numbers \( a, c = \text{const.} \in (0, 1/2) \), where \( a < c \). To introduce the Carleman estimate, consider functions \( \psi(x), \varphi_\lambda(x) \) defined as

\[ \psi(x) = x_1 + \frac{|\overline{x}|^2}{X^2} + a, \quad \varphi_\lambda(x) = \exp(\lambda|\psi|^\nu). \]

Then the analogs of sets (2.7) and \( \Gamma_\varepsilon \) are

\[ G_\varepsilon = \left\{ x : x_1 > 0, \left( x_1 + \frac{|\overline{x}|^2}{X^2} + a \right)^{-\nu} > c^{-\nu} \right\}, \]  

\[ \xi_\varepsilon = \left\{ x : x_1 > 0, \left( x_1 + \frac{|\overline{x}|^2}{X^2} + a \right)^{-\nu} = c^{-\nu} \right\}, \]  

\[ \Gamma_\varepsilon = \left\{ x : x_1 = 0, \left( \frac{|\overline{x}|^2}{X^2} + a \right)^{-\nu} > c^{-\nu} \right\}. \]

Hence, \( \partial G_\varepsilon = \xi_\varepsilon \cup \Gamma_\varepsilon \). Below in this sections we keep notations (3.8)-(3.10). We assume that \( G_\varepsilon \neq \emptyset \) and \( \overline{G_\varepsilon} \subset G \). By (3.3) and (3.10) \( \Gamma_\varepsilon \subset \Gamma \). For a sufficiently small number \( \varepsilon > 0 \) and for \( k = 1, 2 \) define the subdomain \( G_{\varepsilon+2\varepsilon} \) as

\[ G_{\varepsilon+2\varepsilon} = \left\{ x : x_1 > 0, \left( x_1 + \frac{|\overline{x}|^2}{X^2} + a \right)^{-\nu} > c^{-\nu} + 2\varepsilon \right\}. \]

Hence, \( G_{\varepsilon+2\varepsilon} \subset G_\varepsilon \). Lemma 3.1 follows immediately from Lemma 3 of §1 of Chapter 4 of the book [3].

**Lemma 3.1** (Carleman estimate). There exist a sufficiently large number \( \nu_0 = \nu_0 \left( a, \mu_1, \mu_2, \max_{i,j} \| a_{i,j} \|_{C^1(\overline{G_\varepsilon})}, X, n \right) > 1 \) and a sufficiently large absolute constant \( \lambda_0 > 1 \) such that for all \( \nu \geq \nu_0, \lambda \geq \lambda_0 \) and for all functions \( u \in C^2(\overline{G_{1/2}}) \) the following pointwise Carleman estimate is valid for all \( x \in G_{1/2} \) with a constant \( C = C \left( n, \max_{i,j} \| a_{i,j} \|_{C^1(\overline{G_{1/2}})} \right) > 0 \) and with the function \( \varphi_\lambda \) from (3.7)

\[ (L_0u)^2 \varphi_\lambda^2 \geq C\lambda |\nabla u|^2 \varphi_\lambda^2 + C\lambda^3 u^2 \varphi_\lambda^2 + \text{div } U, \]

\[ |U| \leq C\lambda^3 \left[ (|\nabla u|^2 + u^2) \right] \varphi_\lambda^2. \]
This Carleman estimate allows us to construct the weighted Tikhonov functional to solve the Cauchy problem (3.5), (3.6). Similarly with (2.16), we minimize the functional 
\[ J_{\lambda,\beta,\ell} (u) \] on the set \( B (R) \) defined in (2.14), where
\[
J_{\lambda,\beta,\ell} (u) = e^{-2 \lambda (\cdot - \epsilon + \tau)} \int_{G_c} [L_{\ell} (u)]^2 \varphi_\lambda^2 dx + \beta \| u \|^2_{H^{\nu} (G_c)},
\]
where the CWF \( \varphi_\lambda (x) \) is the one in (3.7) and functions \( g_0 \) and \( g_1 \) are ones in (3.6). Hence, Lemma 3.1 and Theorem 2.1 immediately imply Theorem 3.1.

**Theorem 3.1.** Let \( R > 0 \) be an arbitrary number. Let \( B (R) \) and \( H^{\nu} (G_c) \) be the sets defined in (2.14) and (2.15) respectively. Then for every point \( u \in B (R) \) there exists the Fréchet derivative \( J'_{\lambda,\beta,\ell} (u) \in H^{\nu} (G_c) \). Choose the numbers \( \nu = \nu_0, \lambda_0 \) as in Lemma 3.1. There exists a sufficiently large number \( \lambda_1 = \lambda_1 (R, L) > \lambda_0 > 1 \) such that for all \( \lambda \geq \lambda_1 \) and for every \( \beta \in (e^{-\lambda \epsilon}, 1) \) the functional \( J_{\lambda,\beta,\ell} (u) \) is strictly convex on the set \( B (R) \),
\[
J_{\lambda,\beta,\ell} (u_2) - J_{\lambda,\beta,\ell} (u_1) - J'_{\lambda,\beta,\ell} (u_1) (u_2 - u_1) \geq C_2 e^{2 \lambda \epsilon} \| u_2 - u_1 \|^2_{H^1 (G_{c+\epsilon})} + \frac{\beta}{2} \| u_2 - u_1 \|^2_{H^\nu (G_c)}, \forall u_1, u_2 \in B (R).
\]
Here the number \( C_2 = C_2 (L, R, c) > 0 \) depends only on listed parameters.

# 4 Quasilinear parabolic equation with the lateral Cauchy data

Choose an arbitrary \( T = \text{const.} > 0 \) and denote \( G_T = G \times (-T, T) \). Let \( L_{\text{par}} \) be the quasilinear elliptic operator of the second order in \( G_T \), which we define the same way as the operator \( L_{\ell} \) in (3.1)-(3.3) with the only difference that now its coefficients depend on both \( x \) and \( t \) and the domain \( G \) is replaced with the domain \( G_T \). Let \( L_{0,\text{par}} \) be the similarly defined principal part of the operator \( L_{\text{par}} \), see (3.2). Next, we define the quasilinear parabolic operator as \( P = \partial_t - L_{\text{par}} \). The principal part of \( P \) is \( P_0 = \partial_t - L_{0,\text{par}} \). Thus,
\[
L_{\text{par}} (u) = \sum_{i,j=1}^n a_{i,j} (x, t) u_{x_i x_j} + A_1 (x, t, \nabla u, u),
\]
\[
A u := Pu = u_t - L_{\text{par}} (u), (x, t) \in G_T,
\]
\[
P_0 u = u_t - L_{0,\text{par}} u = u_t - \sum_{i,j=1}^n a_{i,j} (x, t) u_{x_i x_j},
\]
\[
a_{i,j} \in C^1 (\overline{G_T}), A_1 \in C (\overline{G_T}) \times C^3 (\mathbb{R}^{n+1}),
\]
\[
\mu_1 |\eta|^2 \leq \sum_{i,j=1}^n a_{i,j} (x, t) \eta_i \eta_j \leq \mu_2 |\eta|^2, \forall (x, t) \in \overline{G_T}, \forall \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n.
\]
Let \( \Gamma \subseteq \partial G, \Gamma \in C^2 \) be the subsurface of the boundary \( \partial G \) with the same properties as ones in section 3. Denote \( \Gamma_T = \Gamma \times (-T, T) \). Without loss of generality we assume that \( \Gamma \) is the same as in (3.4). Consider the parabolic equation
\[
P (u) = u_t - L_{\text{par}} (u) = 0 \text{ in } G_T.
\]
Cauchy Problem with the Lateral Data for Quasilinear Parabolic Equation (4.6). Assume that conditions (3.4) hold. Find such a function \( u \in H^{k+1}(G_T) \) that satisfies equation (4.6) and has the following lateral Cauchy data \( g_0, g_1 \) at \( \Gamma_T \)

\[
  u \big|_{\Gamma_T} = g_0(x, t), \quad u_{x_1} \big|_{\Gamma_T} = g_1(x, t).
\]

We now introduce the Carleman estimate which is similar with the one of section 3. Let \( \lambda > 1 \) and \( \nu > 1 \) be two large parameters, which we define later. Consider two arbitrary numbers \( a, c = \text{const.} \in (0, 1/2) \), where \( a < c \). Consider functions \( \psi(x, t) \), \( \varphi_\lambda(x, t) \) defined as

\[
  \psi(x, t) = x_1 + \frac{|x|^2}{X^2} + \frac{t^2}{T^2} + a, \quad \varphi_\lambda(x, t) = \exp(\lambda \psi^{-\nu}).
\]

Analog of conditions (3.8)-(3.11) are

\[
  G_{T,c} = \left\{(x, t) : x_1 > 0, \left(x_1 + \frac{|x|^2}{X^2} + \frac{t^2}{T^2} + a\right)^{-\nu} > c^{-\nu}\right\},
\]

\[
  \xi_c = \left\{(x, t) : x_1 > 0, \left(x_1 + \frac{|x|^2}{X^2} + \frac{t^2}{T^2} + a\right)^{-\nu} = c^{-\nu}\right\},
\]

\[
  \Gamma_c = \left\{(x, t) : x_1 = 0, \left(x_1 + \frac{|x|^2}{X^2} + \frac{t^2}{T^2} + a\right)^{-\nu} > c^{-\nu}\right\},
\]

\[
  \partial G_{T,c} = \xi_c \cup \Gamma_c,
\]

\[
  G_{c+2\varepsilon,T} = \left\{(x, t) : x_1 > 0, \left(x_1 + \frac{|x|^2}{X^2} + \frac{t^2}{T^2} + a\right)^{-\nu} > c^{-\nu} + 2\varepsilon\right\}.
\]

We assume that

\[
  G_{T,c} \neq \emptyset, G_{T,c} \subset G_T \quad \text{and} \quad \overline{G_{T,c}} \cap \{t = \pm T\} = \emptyset.
\]

In (4.13) \( \varepsilon > 0 \) is so small that \( G_{c+2\varepsilon,T} \neq \emptyset \). Below in this section we use notations (4.8)-(4.13). By (4.11) and (4.14) \( \Gamma_c \subset \Theta_T \). Lemma 4.1 follows immediately from Lemma 3 of §1 of chapter 4 of the book [33].

**Lemma 4.1** (Carleman estimate). Let \( P_0 \) be the parabolic operator defined via (4.1)-

(4.3). There exist a sufficiently large number \( \nu_0 = \nu_0(a, c, \mu_1, \mu_2, \max_{ij} \|a_{ij}\|_{C^1(G_T)}; X, T) > 1 \) and a sufficiently large absolute constant \( \lambda_0 > 1 \) such that for all \( \nu \geq \nu_0, \lambda \geq \lambda_0 \) and for all functions \( u \in C^{2,1}(G_{T,1/2}) \) the following pointwise Carleman estimate is valid for all \( (x, t) \in G_{T,1/2} \) with a constant \( C = C(n, \max_{ij} \|a_{ij}\|_{C^1(\overline{G_T})}) > 0 \) and with the function \( \varphi_\lambda \) defined in (4.8)

\[
  (P_0 u)^2 \varphi_\lambda^2 \geq C \lambda |\nabla u|^2 \varphi_\lambda^2 + C \lambda^3 u^2 \varphi_\lambda^2 + \text{div } U + V,
\]

\[
  |U|, |V| \leq C \lambda^3 \left[ |\nabla u|^2 + u_t^2 + u_x^2 \right] \varphi_\lambda.
\]

Let \( R > 0 \) be an arbitrary number. Similarly with (2.14) and (2.15) let

\[
  B(R) = \left\{ u \in H^{k+1}(G_{T,c}) : \|u\|_{H^{k+1}(G_{T,c})} < R, u \big|_{\Gamma_c} = g_0(x, t), \partial_{x_1} u \big|_{\Gamma_c} = g_1(x, t) \right\}.
\]

\[
  (4.15)
\]
\[ H_{0}^{k+1}(G_{T,c}) = \{ u \in H_{0}^{k+1}(G_{T,c}) : u|_{\Gamma_{c}} = \partial_{u}|_{\Gamma_{c}} = 0 \}. \]  

(4.16)

To solve the problem (4.16), (4.17), we minimize the functional \( J_{\lambda,\beta,\text{par}}(u) \) in (4.17) on the set \( B(R) \) defined (4.15), where

\[
J_{\lambda,\beta,\text{par}}(u) = e^{-2\lambda(c-v)} \int_{G_{T,c}} |P(u)|^{2} \varphi^{2}dx + \beta \| u \|_{H_{0}^{k+1}(G_{T,c})}^{2},
\]

(4.17)

where the operator \( P \) is defined via (4.1)-(4.5). Hence, Lemma 4.1 and Theorem 2.1 imply Theorem 4.1.

**Theorem 4.1.** Let \( R > 0 \) be an arbitrary number. Let \( B(R) \) and \( H_{0}^{k+1}(G_{T,c}) \) be the sets defined in (4.15) and (4.16) respectively. Then for every point \( u \in B(R) \) there exists the Fréchet derivative \( J_{\lambda,\beta,\text{par}}'(u) \in H_{0}^{k+1}(G_{T,c}) \). Choose the numbers \( v = \nu_{0}, \lambda_{0} \) as in Lemma 4.1. There exists a sufficiently large number \( \lambda_{1} = \lambda_{1}(R, P) > \lambda_{0} > 1 \) such that for all \( \lambda \geq \lambda_{1} \) and for every \( \beta \in (e^{-\lambda}, 1) \) the functional \( J_{\lambda,\beta,\text{par}}(u) \) is strictly convex on the set \( B(R) \),

\[
J_{\lambda,\beta,\text{par}}(u_{2}) - J_{\lambda,\beta,\text{par}}(u_{1}) - J_{\lambda,\beta,\text{par}}'(u_{1})(u_{2} - u_{1}) \geq C_{3}e^{2\lambda} \| u_{2} - u_{1} \|_{H_{0}^{k+1}(G_{T,c})}^{2} + \beta \| u_{2} - u_{1} \|_{H_{0}^{k+1}(G_{T,c})}^{2}, \forall u_{1}, u_{2} \in B(R).
\]

Here the number \( C_{3} = C_{3}(L_{\text{par}}, R, c) > 0 \) depends only on listed parameters.

## 5 Quasilinear hyperbolic equation with lateral Cauchy data

In this section, notations for the domain \( G \subset \mathbb{R}^{n} \) and the time cylinder \( G_{T} \) are the same as ones in section 4. Denote \( S_{T} = \partial G \times (-T, T) \). The Carleman estimate of Lemma 5.1 of this section for was proved in Theorem 1.10.2 of the book of Belina and Klibanov [2]. Other forms of Carleman estimates for the hyperbolic case can be found in, e.g. Theorem 3.4.1 of the book of Isakov [15], Theorem 2.2.4 of the book of Klibanov and Timonov [25], Lemma 2 of §4 of chapter 4 of the book of Lavrentiev, Romanov and Shishatskii [33] and in Lemma 3.1 of Triggiani and Yao [35].

Let numbers \( a_{1}, a_{u} > 0 \) and \( a_{1} < a_{u} \). For \( x \in G \), let the function \( a(x) \) satisfy the following conditions in \( G \)

\[
a(x) \in [a_{1}, a_{u}], a \in C^{1}(\overline{G}).
\]

(5.1)

In addition, we assume that there exists a point \( x_{0} \in G \) such that

\[
(\nabla a(x), x - x_{0}) \geq 0, \forall x \in \overline{G},
\]

(5.2)

where \((\cdot, \cdot)\) denotes the scalar product in \( \mathbb{R}^{n} \). In particular, if \( a(x) \equiv 1 \), then (5.2) holds for any \( x_{0} \in G \). We need inequality (5.2) for the validity of the Carleman estimate of Lemma 5.1. Assume that the function \( A_{1} \) satisfies condition (1.1). Consider the quasilinear hyperbolic equation in the time cylinder \( G_{T} \) with the lateral Cauchy data \( g_{0}(x, t), g_{1}(x, t) \),

\[
L_{\text{hyp}}u = a(x) u_{tt} - \Delta u - A_{1}(x, t, \nabla u, u) = 0 \text{ in } G_{T},
\]

(5.3)

\[
u \big| s_{x} = g_{0}(x, t), \partial_{n}u \big|_{S_{T}} = g_{1}(x, t).
\]

(5.4)
Denote \( L_{0,\text{hyp}}u = a(x)u_t - \Delta u \).

**Cauchy Problem with the Lateral Data for the Hyperbolic Equation (5.3).**

Find the function \( u \in H^{k+1}_0(G_T) \) satisfying conditions (5.3), (5.4).

Let the number \( \eta \in (0, 1) \). Let \( \lambda > 1 \) be a large parameter. Define functions \( \xi(x,t) \) and \( \varphi_\lambda(x,t) \) as

\[
\xi(x,t) = |x - x_0|^2 - \eta t^2, \varphi_\lambda(x,t) = \exp[\lambda \psi(x,t)].
\]

(5.5)

Similarly with (2.7), for a number \( c > 0 \) define the hypersurface \( \xi_c \) and the domain \( G_{T,c} \) as

\[
\xi_c = \{(x,t) \in G_T : \xi(x,t) = c\}, \quad G_{T,c} = \{(x,t) \in G_T : \xi(x,t) > c\}.
\]

(5.6)

**Lemma 5.1** (Carleman estimate). Let \( n \geq 2 \) and conditions (5.1) be satisfied. Also, assume that there exists a point \( x_0 \in G \) such that (2.2) holds. Denote \( M = M(x_0,G) = \max_{x \in G} |x - x_0| \). Choose such a number \( c > 0 \) that \( G_{T,c} \neq \emptyset \). Let \( \varphi_\lambda(x,t) \) be the function defined in (5.5), sets \( \xi_c, G_{T,c} \) are the ones defined in (5.7). Theorem 5.1 and the Fréchet derivative (4.14) are valid for the case of the domain \( G \) of this section. Then there exists a number \( \eta_0 = \eta_0(G, M, a_t, a_u, \|\nabla a\|_{C(\overline{G})}) \in (0, 1) \) such that for any \( \eta \in (0, \eta_0) \) one can choose a sufficiently large number \( \lambda_0 = \lambda_0(G, M, a_t, a_u, \|\nabla a\|_{C(\overline{G})}, \eta_0, c) > 1 \) and the number \( C_4 = C_4(G, M, a_t, a_u, \|\nabla a\|_{C(\overline{G})}, \eta_0, c) > 0 \), such that for all \( u \in C^2(\overline{G}_{T,c}) \) and for all \( \lambda \geq \lambda_0 \) the following pointwise Carleman estimate is valid

\[
(L_{0,\text{hyp}}u)^2 \varphi_\lambda^2 \geq C_4 \lambda \left( |\nabla u|^2 + u_t^2 \right) \varphi_\lambda^2 + C_4 \lambda^3 u^2 \varphi_\lambda^2 + \text{div } U + V_t \text{ in } G_{T,c},
\]

\[
|U|, |V| \leq C_4 \lambda^3 \left( |\nabla u|^2 + u_t^2 + u^2 \right) \varphi_\lambda^2.
\]

In the case \( a(x) \equiv 1 \) one \( \eta_0 = 1 \).

Again, let \( R > 0 \) be an arbitrary number. Similarly with (4.15) and (4.16) let

\[
B(R) = \left\{ u \in H^{k+1}_0(G_{T,c}) : \|u\|_{H^{k+1}_0(G_{T,c})} < R, u|_{S_T} = g_0(x,t), \partial_n u|_{S_T} = g_1(x,t) \right\}, \quad (5.7)
\]

\[
H^{k+1}_0(G_{T,c}) = \left\{ u \in H^{k+1}_0(G_{T,c}) : u|_{S_T} = \partial_n u|_{S_T} = 0 \right\}. \quad (5.8)
\]

To solve the Cauchy problem posed in this section, we minimize the functional \( J_{\lambda,\beta,\text{hyp}}(u) \) in (5.9) on the set \( B(R) \) defined in (5.7), where

\[
J_{\lambda,\beta,\text{hyp}}(u) = e^{-2\lambda(\epsilon+c)} \int_{G_{T,c}} [L_{\text{hyp}}(u)]^2 \varphi_\lambda^2 dx + \beta \|u\|_{H^{k+1}_0(G_{T,c})}^2.
\]

(5.9)

Hence, Lemma 5.1 and Theorem 2.1 imply Theorem 5.1.

**Theorem 5.1.** Let \( R > 0 \) be an arbitrary number. Let \( B(R) \) and \( H^{k+1}_0(G_{T,c}) \) be the sets defined in (5.7) and (5.8) respectively. Then for every point \( u \in B(R) \) there exists the Fréchet derivative \( J'_{\lambda,\beta,\text{hyp}}(u) \in H_{0}^{k+1}(G_{T,c}) \). Let \( \lambda_0 = \lambda_0(G, M, a_t, a_u, \|\nabla a\|_{C(\overline{G})}, \eta_0, c) > 1 \) be the number of Lemma 5.1. There exists a sufficiently large number

\[
\lambda_1 = \lambda_1(R, L_{\text{hyp}}, G, M, a_t, a_u, \|\nabla a\|_{C(\overline{G})}, \eta_0, c) > \lambda_0 > 1 \text{ such that for all } \lambda \geq \lambda_1 \text{ and for every } \beta \in (e^{-\lambda_1}, 1) \text{ the functional } J_{\lambda,\beta,\text{hyp}}(u) \text{ is strictly convex on the set } B(R),
\]

\[
J_{\lambda,\beta,\text{par}}(u_2) - J_{\lambda,\beta,\text{hyp}}(u_1) - J'_{\lambda,\beta,\text{hyp}}(u_1)(u_2 - u_1)
\]
\[ \geq C_5 e^{2\lambda c} \| u_2 - u_1 \|^2_{H^1(G_{T,c+\varepsilon})} + \frac{\beta}{2} \| u_2 - u_1 \|^2_{H^{n+1}(G_{T,c})}, \forall u_1, u_2 \in B(R). \]

Here the number \( C_5 = C_3 (L_{hyp}, R, c) > 0. \)

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