Plethystic formulas for permutation enumeration

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Abstract

We prove several general formulas for the distributions of various permutation statistics over any set of permutations whose quasisymmetric generating function is a symmetric function. Our formulas involve certain kinds of plethystic substitutions on quasisymmetric generating functions, and the permutation statistics we consider include the descent number, peak number, left peak number, and the number of up-down runs. We apply these results to cyclic permutations, involutions, and derangements, and more generally, to derive formulas for counting all permutations by the above statistics jointly with the number of fixed points and jointly with cycle type. A number of known formulas are recovered as special cases of our results, including formulas of Désarménien–Foata, Gessel–Reutenauer, Stembridge, Fulman, Petersen, Diaconis–Fulman–Holmes, Zhuang, and Athanasiadis.

Keywords: permutation statistics, descents, peaks, cycle type, symmetric functions, plethysm

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Let \( \pi = \pi(1)\pi(2) \cdots \pi(n) \) be an element of the symmetric group \( \mathfrak{S}_n \) of permutations of the set \( [n] := \{1, 2, \ldots, n\} \). We say that \( i \in [n - 1] \) is a descent of \( \pi \) if \( \pi(i) > \pi(i + 1) \). Let \( \text{des}(\pi) \) denote the number of descents of \( \pi \) and let \( \text{maj}(\pi) \) denote the sum of all descents of \( \pi \). The descent number \( \text{des} \) and major index \( \text{maj} \) are classical permutation statistics whose study dates back to Percy MacMahon \([27]\).

The distribution of the descent number over \( \mathfrak{S}_n \) for \( n \geq 1 \) is encoded by the \( n \)th Eulerian polynomial

\[
A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)+1},
\]

and the joint distribution of the descent number and major index by the \( n \)th \( q \)-Eulerian polynomial

\[
A_n(q, t) := \sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi)}t^{\text{des}(\pi)+1},
\]

In some works, \( A_n(t) \) is instead defined to be \( \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} \).
with $A_0(t) = A_0(q, t) = 1$. MacMahon [27, Vol. 2, Section IX] proved a formula of which a special case is

$$
\frac{A_n(q, t)}{(1 - t)(1 - tq) \cdots (1 - tq^n)} = \sum_{k=0}^{\infty} [k]_q^n t^k,
$$

(1.1)

where

$$[k]_q := 1 + q + q^2 + \cdots + q^{k-1}.
$$

(This formula is often attributed to Carlitz [5].) Note that (1.1) reduces to the classical identity

$$\frac{A_n(t)}{(1 - t)^{n+1}} = \sum_{k=0}^{\infty} k^n t^k
$$

(1.2)

(which is sometimes used as the definition for Eulerian polynomials).

Equation (1.1) allows one to compute the joint distribution of the descent number and major index over $S_n$, but one may also want to study the joint distribution of these statistics, and others, over certain interesting subsets of $S_n$.

Our work is concerned with descent statistics: permutation statistics that are determined by the descent set. Examples of descent statistics include the descent number and major index, as well as the peak number, the left peak number, and the number of up-down runs, which we will study in this paper. Quasisymmetric functions—which are certain power series in infinitely many variables that generalize symmetric functions—encode descent sets, and in many cases, generating functions for descent statistics can be extracted in a useful way from quasisymmetric functions. For example, if $\Pi$ is a subset of $S_n$ with quasisymmetric generating function $Q(\Pi)$ (see Section 2.3 for relevant definitions), then

$$\sum_{\pi \in \Pi} t^{\text{des}(\pi)+1} q^{\text{maj}(\pi)} = \sum_{k=0}^{\infty} \phi_k(Q(\Pi)) t^k
$$

(1.3)

where $\phi_k(f) := f(q^{k-1}, q^{k-2}, \ldots, 1)$. (See [15, Section 4].) If $Q(\Pi)$ is symmetric, then we can use operations on symmetric functions to extract generating function formulas for permutation statistics.

In 1993, the first author and Christophe Reutenauer [18] proved that the set of permutations with a prescribed cycle type has a symmetric quasisymmetric generating function, and used this fact and (1.3) to derive formulas for the joint distribution of des and maj over cyclic permutations, involutions, and derangements. Later, Jason Fulman [12, Theorem 1] used the work of Gessel and Reutenauer to derive a formula for the joint distribution of des, maj, and cycle type over $S_n$. These results are notable because the descent number and major index are statistics which encode properties of a permutation in one-line representation, and one would not expect such statistics to be compatible with the cycle structure. See [10, 11, 24, 30, 31, 34, 41] for related work.

Given $\pi \in S_n$, we say that $i \in \{2, \ldots, n-1\}$ is a peak of $\pi$ if $\pi(i-1) < \pi(i) > \pi(i+1)$, and that $i \in [n-1]$ is a left peak of $\pi$ if $i$ is a peak or if $i = 1$ and $\pi(1) > \pi(2)$. A birun of $\pi$ is a maximal monotone consecutive subsequence, and an up-down run of $\pi$ is a birun of

\footnote{See [10] Section 3 for more on MacMahon’s work.}

\footnote{Biruns are also commonly called “alternating runs”.
}
Let \( \pi(1) \leq \pi(2) \) denote the number of peaks of \( \pi \), \( \text{lpk}(\pi) \) the number of left peaks of \( \pi \), and \( \text{udr}(\pi) \) the number of up-down runs of \( \pi \). For example, if \( \pi = 71462853 \), the peaks of \( \pi \) are 4 and 6; the left peaks of \( \pi \) are 1, 4, and 6; and the up-down runs of \( \pi \) are 7, 71, 146, 62, 28, and 853. Thus, \( \text{pk}(\pi) = 2 \), \( \text{lpk}(\pi) = 3 \), and \( \text{udr}(\pi) = 6 \).

The **peak number** \( \text{pk} \), the **left peak number** \( \text{lpk} \), and the **number of up-down runs** \( \text{udr} \) are—like the descent number and major index—statistics which deal with permutations in one-line representation. All of these statistics have been well-studied (see, e.g., [19, 42, 43] by the present authors, and the references therein), but to our knowledge the only known result concerning a distribution of any of these statistics while refining by cycle structure is due to Diaconis, Fulman, and Holmes, who derived a formula [8, Corollary 3.8] for the joint distribution of the peak number and cycle type over \( S_n \) in their analysis of casino shelf shuffling machines.

The main results of our paper are general formulas, analogous to (1.3), which can be used to study the joint distribution of \( \text{pk} \) and \( \text{des} \), the joint distribution of \( \text{lpk} \) and \( \text{des} \), and the distribution of \( \text{udr} \) over any subset of \( S_n \) whose quasisymmetric generating function is symmetric. These formulas involve plethysm, an operation on symmetric functions originating in the representation theory of the general linear groups \( GL_n(\mathbb{C}) \) and the symmetric groups \( S_n \). In recent decades, plethysm has been extended to more general formal power series rings and has found numerous applications within algebraic combinatorics, e.g., in the theory of Macdonald polynomials. See Macdonald [26, Chapter I, Section 8], Stanley [37, Chapter 7, Appendix 2], Haglund [22, Chapter 1], and Loehr–Remmel [25] for introductory references on plethysm.

The structure of our paper is as follows. In Section 2, we review some relevant results from the basic theory of symmetric and quasisymmetric functions, give an introductory account of plethystic calculus, prove some preliminary lemmas involving plethystic substitutions, and give ribbon and power sum expansions of several symmetric function identities which are relevant for the study of peaks, left peaks, and up-down runs. We prove our main results in Section 3. The rest of the paper presents applications of our main results: Section 4 concerns cyclic permutations, Section 5 concerns involutions, Section 6 concerns derangements and (more generally) counting permutations by fixed points, and Section 7 concerns counting permutations by cycle type. We recover as special cases the formulas of Fulman and Diaconis–Fulman–Holmes mentioned above—as well as various formulas by Désarménien–Foata, Gessel–Reutenauer, Stembridge, Petersen, Zhong, and Athanasiadis—but most of our results are new and pertain to permutation statistic distributions that have been previously unstudied. We conclude in Section 8 with a brief discussion of future work.

## 2. Preliminaries

### 2.1. Review of basic symmetric function theory

We assume familiarity with basic definitions from the theory of symmetric functions at the level of Stanley [37, Chapter 7]. In this section we establish notation and review some
elementary facts which will be needed for our work. See also Macdonald [26, Chapter 1], Sagan [33, Chapter 4], and Grinberg–Reiner [20, Section 2] for other treatments of basic symmetric function theory.

We use the notations $\lambda \vdash n$ and $|\lambda| = n$ to indicate that $\lambda$ is a partition of $n$, and we let $l(\lambda)$ denote the number of parts of $\lambda$. We write $\lambda = (1^{m_1}2^{m_2}\cdots)$ to mean that $\lambda$ has $m_1$ parts of size $1$, $m_2$ parts of size $2$, and so on; alternatively, we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ to mean that $\lambda$ has parts $\lambda_1, \lambda_2, \ldots, \lambda_r$. For example, $\lambda = (1^23^41)$ and $\lambda = (4, 3, 3, 3, 1, 1)$ denote the same partition, and here we have $|\lambda| = 18$ and $l(\lambda) = 7$.

Let $\Lambda$ denote the $\mathbb{Q}$-algebra of symmetric functions in the variables $x_1, x_2, \ldots$. We recall the important bases for $\Lambda$: the monomial symmetric functions $m_\lambda$, the complete symmetric functions $h_\lambda$, the elementary symmetric functions $e_\lambda$, the power sum symmetric functions $p_\lambda$, and the Schur functions $s_\lambda$. (As usual, we write $h_n$ as $h_n$, $e_n$ as $e_n$, and $p_n$ as $p_n$.)

We will also work with symmetric functions with coefficients involving additional variables such as $t$, $y$, $z$, and $\alpha$, and symmetric functions of unbounded degree like $H(z)$, defined below.

Define $H(z) := \sum_{n=0}^\infty h_n z^n$ and $E(z) := \sum_{n=0}^\infty e_n z^n$ to be the ordinary generating functions for the $h_n$ and $e_n$, respectively. It is well known [37, Equations (7.11) and (7.12)] that

$$H(z) = \prod_{n=1}^\infty (1 - x_n z)^{-1} \quad \text{and} \quad E(z) = \prod_{n=1}^\infty (1 + x_n z),$$

from which the identity

$$H(z) = E(-z)^{-1} \quad (2.1)$$

follows. It is then a consequence of [37, Proposition 7.7.4] that

$$H(z) = \exp \left( \sum_{k=1}^\infty \frac{p_k}{k} z^k \right) \quad (2.2)$$

and

$$E(z) = \exp \left( \sum_{k=1}^\infty (-1)^{k-1} \frac{p_k}{k} z^k \right). \quad (2.3)$$

We adopt the notation $H := H(1) = \sum_{n=0}^\infty h_n$ and $E := E(1) = \sum_{n=0}^\infty e_n$.

For a partition $\lambda = (1^{m_1}2^{m_2}\cdots)$, let $z_\lambda := 1^{m_1}m_1!2^{m_2}m_2!\cdots$. An important property of the numbers $z_\lambda$ is that if $\lambda$ is a partition of $n$ then $n! / z_\lambda$ is the number of permutations in $\mathfrak{S}_n$ of cycle type $\lambda$, i.e., permutations in which the lengths of the cycles correspond to the parts of $\lambda$.

The following lemma is helpful in working with expressions like (2.2). We omit the proof, which is a straightforward computation (cf. Macdonald [26, p. 25]).

**Lemma 2.1.** For any sequence $a_1, a_2, \ldots$ we have

$$\exp \left( \sum_{k=1}^\infty \frac{a_k}{k} z^k \right) = \sum_\lambda \frac{x^{\lambda}}{z_\lambda} \prod_{k=1}^{l(\lambda)} a_{\lambda_k},$$

where the sum on the right is over all partitions $\lambda$. 

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It follows from (2.2) and Lemma 2.1 that

\[ H = \sum_{\lambda} p_{\lambda}/z_{\lambda}. \]

Let \( \langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbb{Q} \) denote the usual scalar product on symmetric functions defined by

\[ \langle m_\lambda, h_\tau \rangle = \begin{cases} 1, & \text{if } \lambda = \tau, \\ 0, & \text{otherwise,} \end{cases} \]

for all partitions \( \lambda \) and \( \tau \) and extending bilinearly, that is, by requiring that \( \{ m_\lambda \} \) and \( \{ h_\tau \} \) be dual bases. Then we have

\[ \langle p_\lambda, p_\tau \rangle = \begin{cases} z_{\lambda}, & \text{if } \lambda = \tau, \\ 0, & \text{otherwise.} \end{cases} \]

for all \( \lambda \) and \( \tau \) [36, Proposition 7.9.3].

We extend the scalar product in the obvious way to symmetric functions of unbounded degree and symmetric functions with coefficients that involve other variables such as \( t, z, \) and \( \alpha \). Note that the scalar product is not always defined for unbounded symmetric functions: for example, \( \langle H(z), H \rangle = \sum_{n=0}^{\infty} z^n = (1 - z)^{-1} \) but \( \langle H, H \rangle \) is undefined.

2.2. Plethysm

The aim of this section is to give a brief, self-contained introduction to plethysm and to develop a few preliminary lemmas involving plethystic substitutions that we will need later in this paper. None of the results in this section are new, but we have chosen to provide proofs in order to help guide readers who are unfamiliar with plethysm.

Let \( R \) be a commutative ring containing \( \mathbb{Q} \). We define a lambda ring over \( R \) to be a commutative \( R \)-algebra \( A \) together with an \( R \)-algebra endomorphism \( \psi_i : A \to A \) for every \( i \in \mathbb{P} \) (the set of positive integers) such that \( \psi_1 \) is the identity map, and \( \psi_i \circ \psi_j = \psi_{ij} \) for every \( i, j \in \mathbb{P} \). These homomorphisms \( \psi_i \) are called Adams operations. We define an operation \( \Lambda \times A \to A \), where the image of \( (f, a) \in \Lambda \times A \) is denoted \( f[a] \), by these two properties:

1. For any \( i \geq 1 \), \( p_i[a] = \psi_i(a) \).

2. For any fixed \( a \in A \), the map \( f \mapsto f[a] \) is an \( R \)-algebra homomorphism from \( \Lambda \) to \( A \).

Throughout this paper, we will take our ring \( R \) to be \( \mathbb{Q} \) and our lambda ring \( A \) to be a \( \mathbb{Q} \)-algebra of formal power series containing \( \Lambda \) as a subalgebra. Moreover, from this point on we shall take the \( i \)th Adams operation \( \psi_i \) to be the result of replacing each variable with its \( i \)th power. For a symmetric function \( f \in \Lambda \), this means that \( p_i[f] = \psi_i(f(x_1, x_2, \ldots)) = f(x_1^i, x_2^i, \ldots) \) but more generally, if \( f \) contains other variables, then in \( p_i[f] \) they are all raised to the \( i \)th power as well. For example, \( p_i[q t^2 p_m] = \psi_i(q t^2 p_m) = q^i t^{2i} p_m \). The map \( (f, a) \mapsto f[a] \) is called plethysm. The terms “composition” and “plethystic substitution” are also used for this operation. (Sometimes the term “plethysm” is restricted to the case in which \( A = \Lambda \), but we will use it for the more general operation.)

As with the scalar product, we extend plethysm in the obvious way to symmetric functions of unbounded degree with coefficients involving other variables. If \( f \) is a symmetric function of unbounded degree and if \( a \) has a nonzero constant term, then \( f[a] \) need not be defined. For
example, \( H(x)[1] = (1 - x)^{-1} \) but \( H[1] \) is undefined. In some of our formulas (e.g., Lemma 2.3), we assume implicitly that any infinite sums involved converge as formal power series.

Note that plethysm does not commute with evaluation of variables: if \( \alpha \) is a variable then \( p_n[\alpha] = \alpha^n \), but if \( y \) is a rational number then \( p_n[y] = y \), so \( p_n[\alpha]|_{\alpha=y} \neq p_n[y] \).

We call \( m \in A \) a monic term if it is a monomial with coefficient 1. The next theorem gives a straightforward method for evaluating the plethystic substitution \( f[a] \) if \( a \) is expressed as a sum of monic terms.

**Theorem 2.2.** Suppose that \( a \in A \) can be expressed as a (possibly infinite) sum of monic terms \( m_1 + m_2 + \cdots \). Then for any \( f = f(x_1, x_2, \ldots) \in \Lambda \), we have \( f[a] = f(m_1, m_2, \ldots) \).

**Proof.** It is sufficient to prove this result for \( f = p_i \), and this is straightforward:

\[
p_i[a] = \psi_i(m_1 + m_2 + \cdots) = m_1^i + m_2^i + \cdots = p_i(m_1, m_2, \ldots).
\]

For example, 1 is a monic term, so if \( k \) is a positive integer and \( f \in \Lambda \) then

\[
f[k] = f(1, 1, \ldots, 1).
\]

In what follows, we let \( X := x_1 + x_2 + \cdots = p_1 \), so that \( f[X] = f \). The next several lemmas concern plethystic substitutions of the form \( H[a] \) for certain kinds of elements \( a \in A \).

**Lemma 2.3.** Let \( f, g \in A \), let \( m \in A \) be a monic term, and let \( k \in \mathbb{Z} \). Then

(a) \( H[f + g] = H[f]H[g] \),

(b) \( H[mf] = H(m)[f] \), and

(c) \( H[kf] = H[f]^k \).

**Proof.** First, we have

\[
H[f + g] = \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n} [f + g] \right)
\]

\[
= \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n} [f + g] \right)
\]

\[
= \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} (p_n[f] + p_n[g]) \right)
\]

\[
= \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n} [f] \cdot \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n} \right)[g] \right)
\]

\[
= H[f]H[g]
\]
which proves (a). For (b), we have

\[ H[mf] = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\right)[mf] = \exp\left(\sum_{n=1}^{\infty} m^n p_n[f]\right) = \exp\left(\sum_{n=1}^{\infty} m^n \frac{p_n}{n}\right)[f] = H(m)[f]. \]

We omit the proof of (c), which is similar to that of (a).

\[ \square \]

**Lemma 2.4.** Let \( m \in A \) be a monic term, let \( \alpha \in A \) be a variable, and let \( k \in \mathbb{Z} \). Then

(a) \( H[mX] = H(m) \),

(b) \( H[-mX] = E(-m) \),

(c) \( H[m] = 1/(1 - m) \), and

(d) \( H(m)[k(1 - \alpha)] = (1 - m\alpha)^k/(1 - m)^k \).

**Proof.** Part (a) is a special case of Lemma 2.3 (b), and we have

\[ H[-mX] = H[mX]^{-1} \quad \text{(by Lemma 2.3 (c))} \]

\[ = H(m)^{-1} \quad \text{(by Lemma 2.4 (a))} \]

\[ = E(-m) \quad \text{(by (2.1))} \]

which proves part (b). Next, by Theorem 2.2 we have

\[ H[m] = \sum_{n=0}^{\infty} h_n(m, 0, 0, \ldots) = \sum_{n=0}^{\infty} m^n = \frac{1}{1 - m}; \]

this proves part (c). Finally, to prove part (d), observe that by Lemma 2.3 (c) it suffices to prove the case \( k = 1 \). We have

\[ H(m)[1 - \alpha] = H(m(1 - \alpha)] \quad \text{(by Lemma 2.3 (b))} \]

\[ = H[m]H[-m\alpha] \quad \text{(by Lemma 2.3 (a))} \]

\[ = \frac{H[m]}{H[m] \alpha} \quad \text{(by Lemma 2.3 (c))} \]

\[ = \frac{1 - m\alpha}{1 - m} \quad \text{(by Lemma 2.3 (c))}, \]

thus completing the proof. \[ \square \]

**Lemma 2.5.** Suppose that \( g \in A \) does not contain any of the variables \( x_1, x_2, \ldots \). Then for any symmetric function \( f \), we have \( f[g] = \langle f[X], H[gX]\rangle \).

**Proof.** We prove this result for \( f = p_\lambda \), and then the result follows by linearity. First, it is easy to show that for any partition \( \tau \), we have \( p_\tau[gX] = p_\tau[g]p_\tau[X] \). Thus

\[ H[gX] = \sum_\tau \frac{1}{z_\tau} p_\tau[gX] = \sum_\tau \frac{1}{z_\tau} p_\tau[g]p_\tau[X] \]

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and so

\[ \langle p_\lambda[X], H[gX] \rangle = \left\langle p_\lambda, \sum_\tau \frac{1}{z_\tau} p_\tau[g] p_\tau \right\rangle. \]

Since \( g \) does not contain any of the variables \( x_1, x_2, \ldots \), the same is true for \( p_\tau[g] \) for any \( \tau \). Thus we can pull out \( p_\tau[g] \) from the scalar product expression to obtain

\[ \langle p_\lambda[X], H[gX] \rangle = \sum_\tau \left( p_\lambda, \frac{p_\tau}{z_\tau} \right) p_\tau[g] = p_\lambda[g] \]

and we are done. \( \square \)

2.3. Descent compositions, cycle type, and quasisymmetric generating functions

We use the notation \( L \models n \) to indicate that \( L \) is a composition of \( n \). Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences—or equivalently, maximal consecutive subsequences containing no descents—which we call increasing runs. The descent composition of \( \pi \), denoted \( \text{Comp}(\pi) \), is the composition whose parts are the lengths of the increasing runs of \( \pi \) in the order that they appear. For example, the increasing runs of \( \pi = 85712643 \) are 8, 57, 126, 4, and 3, so the descent composition of \( \pi \) is \( \text{Comp}(\pi) = (1, 2, 3, 1, 1) \).

For a composition \( L = (L_1, L_2, \ldots, L_k) \), let \( \text{Des}(L) := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\} \). It is easy to see that if \( L \) is the descent composition of \( \pi \), then \( \text{Des}(L) \) is the set of descents (i.e., descent set) of \( \pi \). Recall that the fundamental quasisymmetric function \( F_L \) is defined by

\[ F_L := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \]

where the sum is over all \( i_1, \ldots, i_n \) satisfying

\[ L_1 \leq \cdots \leq L_3 > L_1 + L_2 \leq \cdots \leq L_{1+L_2} > \cdots > L_{1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n. \]

Given a set \( \Pi \) of permutations, its quasisymmetric generating function \( Q(\Pi) \) is defined by

\[ Q(\Pi) := \sum_{\pi \in \Pi} F_{\text{Comp}(\pi)}. \]

Moreover, given a composition \( L \), let \( r_L \) denote the skew Schur function of ribbon shape \( L \). Thus \( r_L \) is defined by

\[ r_L = \sum_{i_1, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \]

where the sum is over all \( i_1, \ldots, i_n \) satisfying

\[ L_1 \leq \cdots \leq L_3 > L_1 + L_2 \leq \cdots \leq L_{1+L_2} > \cdots > L_{1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n. \]

The following is [15 Corollary 4].

\(^5\)Ribbon shapes are also called “skew-hooks” (e.g., by Gessel–Reutenauer [16]) or “border strips” (e.g., by Macdonald [26] and Stanley [34, 37]).
Theorem 2.6. Suppose that \( Q(\Pi) \) is a symmetric function. Then the number of permutations in \( \Pi \) with descent composition \( L \) is equal to \( \langle Q(\Pi), r_L \rangle \).

Recall that a permutation \( \pi \) has cycle type \( \lambda = (1^{m_1}2^{m_2} \cdots) \) if \( \pi \) has exactly \( m_1 \) cycles of length 1, \( m_2 \) cycles of length 2, and so on. Henceforth, cycles of length \( i \) are called \( i \)-cycles, 1-cycles in particular are called fixed points, and the number of fixed points of a permutation \( \pi \) is denoted \( \text{fix}(\pi) \). For \( n \in \mathbb{P} \), define the symmetric function \( L_n \) by

\[
L_n := \frac{1}{n} \sum_{d|n} \mu(d)p_d^{n/d}
\]

where \( \mu \) is the number-theoretic Möbius function. Then, given a partition \( \lambda = (1^{m_1}2^{m_2} \cdots) \), define \( L_\lambda \) by

\[
L_\lambda := h_{m_1}[L_1]h_{m_2}[L_2] \cdots
\]

The symmetric functions \( L_\lambda \) are called Lyndon symmetric functions. Gessel and Reutenauer [18, Theorem 2.1] showed that \( L_\lambda \) is the quasisymmetric generating function for the set of permutations with cycle type \( \lambda \).

Corollary 2.7. The number of permutations \( \pi \) with cycle type \( \lambda \) and descent composition \( M \) is equal to \( \langle L_\lambda, r_M \rangle \).

2.4. Ribbon expansions

In the work that will follow, we will need to expand the symmetric function expressions \((1 - tE(yx)H(x))^{-1}, H(x)/(1 - tE(yx)H(x))\), and \((1 + tH(x))/(1 - t^2E(x)H(x))\) in terms of the ribbon Schur functions \( r_L \) and in terms of the power sums \( p_\lambda \). First we give the ribbon expansions, which reveal connections between these expressions and permutation statistics.

We let \( \text{des}(L) \), \( \text{pk}(L) \), \( \text{lpk}(L) \), and \( \text{udr}(L) \) be equal to the values \( \text{des}(\pi) \), \( \text{pk}(\pi) \), \( \text{lpk}(\pi) \), and \( \text{udr}(\pi) \), respectively, of any permutation \( \pi \) with descent composition \( L \). These are well-defined because the statistics \( \text{des} \), \( \text{pk} \), \( \text{lpk} \), and \( \text{udr} \) depend only on the descent composition; in other words, \( \text{Comp}(\pi) = \text{Comp}(\sigma) \) implies \( \text{st}(\pi) = \text{st}(\sigma) \) for \( \text{st} = \text{des}, \text{pk}, \text{lpk}, \text{udr} \). Recall that permutation statistics with this property are called descent statistics. In general, if \( \text{st} \) is a descent statistic, then we let \( \text{st}(L) \) denote the value of \( \text{st} \) on any permutation \( \pi \) with descent composition \( L \).

Lemma 2.8. We have the formulas

(a) \[
\frac{1}{1 - tE(yx)H(x)} = \frac{1}{1 - t} + \frac{1}{1 + y} \sum_{n=1}^{\infty} \sum_{L \sqsubseteq n} \left( \frac{1 + yt}{1 - t} \right)^{n+1} \left( \frac{(1 + y)^2t}{(y + t)(1 + yt)} \right)^{\text{pk}(L)+1} \left( \frac{y + t}{1 + yt} \right)^{\text{des}(L) + 1} x^{n r_L},
\]

(b) \[
\frac{H(x)}{1 - tE(yx)H(x)} = \frac{1}{1 - t} + \frac{1}{1 + y} \sum_{n=1}^{\infty} \sum_{L \sqsubseteq n} \left( \frac{1 + yt}{1 - t} \right)^{n+1} \left( \frac{(1 + y)^2t}{(y + t)(1 + yt)} \right)^{\text{lpk}(L)} \left( \frac{y + t}{1 + yt} \right)^{\text{des}(L)} x^{n r_L},
\]
and
\[
\frac{1 + tH(x)}{1 - t^2 E(x) H(x)} = \frac{1}{1 - t} + \frac{1}{2(1 - t)^2} \sum_{n=1}^{\infty} \sum_{L \vdash n} \frac{(1 + t^2)^n}{(1 - t^2)^{n-1}} \left( \frac{2t}{1 + t^2} \right)^{\text{udr}(L)} x^n r_L.
\]

**Proof.** These formulas are obtained directly from Lemma 4.1, Lemma 4.6, and Corollary 4.12 of [3]—which are noncommutative versions of these formulas—via the canonical projection map from noncommutative symmetric functions to symmetric functions. 

We note that part (a) of Lemma 2.8 can be rewritten as
\[
\frac{1}{1 - vE(u x)H(x)} = \frac{1}{1 - v} + \frac{1}{1 + u} \sum_{n=1}^{\infty} \sum_{L \vdash n} \left( \frac{1 + uv}{1 - v} \right)^n y^{\text{pk}(L) + \text{ld}(L) + 1} x^n r_L
\]
where
\[
u = \frac{1 + t^2 - 2yt - (1 - t)\sqrt{(1 + t)^2 - 4yt}}{2(1 - y)t}
\]
and
\[
u = \frac{(1 + t)^2 - 2yt - (1 + t)\sqrt{(1 + t)^2 - 4yt}}{2yt}.
\]
To see this, we replace \( y \) with \( u \) and replace \( t \) with \( v \) in part (a). Then we set \( y = \frac{(1 + u)\nu}{(u + v)(1 + uv)} \) and \( t = \frac{u + v}{1 + uv} \); solving these two equations yields the above expressions for \( u \) and \( v \). By the same reasoning, part (b) of Lemma 2.8 is equivalent to
\[
\frac{H(x)}{1 - vE(u x)H(x)} = \frac{1}{1 - v} + \sum_{n=1}^{\infty} \sum_{L \vdash n} \frac{(1 + uv)^n}{(1 - v)^{n+1}} y^{\text{pk}(L) + \text{ld}(L)} x^n r_L
\]
where \( u \) and \( v \) are the same as above, and part (c) can be “inverted” in a similar way.

There are explicit formulas for expressing powers of \( u \) and \( v \) in terms of \( y \) and \( t \). It is easy to check that \( v = ytP^2 \) and \( u = (1 - y)tQ^2 \), where
\[
P = \frac{1 + t - \sqrt{(1 + t)^2 - 4yt}}{2yt} = \frac{2}{1 + t + \sqrt{(1 + t)^2 - 4yt}}
\]
and
\[
Q = \frac{1 - t - \sqrt{(1 + t)^2 - 4yt}}{2(1 - y)t} = \frac{2}{1 - t + \sqrt{(1 + t)^2 - 4yt}}.
\]
Using the fact that \( P = 1 - tP + ytP^2 \) and \( Q = P/(1 - tP) \), we can apply Lagrange inversion (see, e.g., [17, Equation (2.4.5)]) to find an explicit formula for the expansion of \( P^aQ^b \) in powers of \( y \) and \( t \),
\[
P^aQ^b = \sum_{i,j=0}^{\infty} (-1)^{j-i} \left[ \binom{a + b + i + j - 1}{i} \binom{a + j - 1}{j - i} - \binom{a + b + i + j - 1}{i - 1} \binom{a + j}{j - i} \right] y^i t^j,
\]
which yields explicit formulas for expanding powers of \( u \) and \( v \).
Furthermore, it is interesting to note that part (c) of Lemma 2.8 can be derived from parts (a) and (b). First, we can split the left side of (c) into even and odd powers of $t$. Using the fact that the valley number, val, has the same distribution as pk, we may apply the case $y = 1$ of (a) and (b) to express these two sums in terms of val and lpk. Then we may use the identities \( \text{val}(L) + 1 = \lfloor \text{udr}(L)/2 \rfloor \) and \( \text{lpk}(L) = \lfloor \text{udr}(L)/2 \rfloor \) (see [19, Lemma 2.2]) to express these two sums in terms of udr. Finally we split these two sums into even and odd values of \( \text{udr}(L) \) and rearrange to obtain the right side of (c).

Before continuing, we present here symmetric function formulas analogous to those in Lemma 2.8 for two more descent statistics: the double descent number \( \text{ddesc} \) and the number of biruns \( \text{br} \). We will not discuss them further, but they could be used to derive formulas analogous to those we will give later on for des, pk, lpk, and udr.

A double descent of a permutation \( \pi \in \mathcal{S}_n \) is an index \( i \) with \( 2 \leq i \leq n - 1 \) such that \( \pi(i - 1) > \pi(i) > \pi(i + 1) \). We denote by \( \text{ddesc}(\pi) \) the number of double descents of \( \pi \). (See, for example, [42, Theorem 12] for the generating function for permutations by double descents.) Using the fact that each descent of a permutation \( \pi \) either occurs in position 1 or is preceded by a descent or an ascent, we see that \( \text{ddesc}(\pi) + \text{lpk}(\pi) = \text{des}(\pi) \), so \( \text{ddesc}(\pi) = \text{des}(\pi) - \text{lpk}(\pi) \). Thus, we might expect that a symmetric function formula for double descents can be derived from Lemma 2.8 (b). To obtain the simplest formula for double descents, we first replace \( t \) with \( t^2 \) in Lemma 2.8 (b), then set \( y = 1/t \), and then replace \( x \) with \( tx \). This gives

\[
\frac{H(tx)}{1 - t^2 E(x)H(tx)} = \frac{1}{1 - t^2} + \frac{1}{1 - t^2} \sum_{L \leq n} \sum_{n=1}^{\infty} \left( \frac{t}{1 - t} \right)^n (t - 1 + t^{-1})^{\text{ddesc}(L)} x^n r_L. \tag{2.4}
\]

Recall that a birun of a permutation is a maximal monotone consecutive subsequence. We denote by \( \text{br}(\pi) \) the number of biruns of \( \pi \). Since biruns are closely related to up-down runs, we might expect a formula similar to Lemma 2.8 (c) for biruns, and in fact there is such a formula:

\[
\frac{2 + tH(x) + tE(x)}{1 - t^2 E(x)H(x)} = \frac{2}{1 - t} + \frac{2t}{(1 - t)^2} \sum_{h_1} + \frac{(1 + t)\beta}{2(1 - t)} \sum_{L \leq n} \sum_{n=2}^{\infty} \left( \frac{1 + t^2}{1 - t^2} \right)^n \left( \frac{2t}{1 + t^2} \right)^{\text{br}(L)} x^n r_L. \tag{2.5}
\]

Formula (2.5) can be proved using the approach of [42].

### 2.5. Power sum expansions and Eulerian polynomials

Next, we give the power sum expansions of \( (1 - tE(yx)H(x))^{-1}, H(x)/(1 - tE(yx)H(x)), \) and \( (1 + tH(x))/(1 - t^2 E(x)H(x)) \), which—perhaps surprisingly—involve Eulerian polynomials and type B Eulerian polynomials. The \( n \)th type B Eulerian polynomial \( B_n(t) \) may be defined by the formula

\[
\frac{B_n(t)}{(1 - t)^{n+1}} = \sum_{k=0}^{\infty} (2k + 1)^n t^k
\]

\[\text{Given } \pi \in \mathcal{S}_n, \text{ we say that } i \in \{2, \ldots, n\} \text{ is a valley of } \pi \text{ if } \pi(i - 1) > \pi(i) < \pi(i + 1). \text{ Then val}(\pi) \text{ is the number of valleys of } \pi.\]
and gives the distribution of the type B descent number over the nth hyperoctahedral group; see [13, Section 2.3] for details. Recall that \( l(\lambda) \) is the number of parts of the partition \( \lambda \). We define \( o(\lambda) \) to be the number of odd parts of \( \lambda \) and we write \( \sum_{\lambda \text{ odd}} \) to denote a sum over partitions \( \lambda \) in which every part is odd.

**Lemma 2.9.** We have the formulas

(a) \[
\frac{1}{1 - tE(yx)H(x)} = \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} A_{l(\lambda)}(t) \frac{\lambda^{|\lambda|}}{(1 - t)^{|\lambda| + 1}} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k}),
\]

(b) \[
\frac{H(x)}{1 - tE(x)H(x)} = \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} B_{o(\lambda)}(t) x^{|\lambda|},
\]

and

(c) \[
\frac{1 + tH(x)}{1 - t^2E(x)H(x)} = \sum_{\lambda \text{ odd}} \frac{p_{\lambda}^2 o(\lambda)}{z_{\lambda}} A_{l(\lambda)}(t^2) \frac{\lambda^{|\lambda|}}{(1 - t^2)^{|\lambda| + 1}} x^{|\lambda|} + t \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} B_{o(\lambda)}(t^2) \frac{x^{|\lambda|}}{(1 - t^2)^{o(\lambda) + 1}}.
\]

**Proof.** First, we have

\[
\frac{1}{1 - tE(yx)H(x)} = \sum_{n=0}^{\infty} t^n \exp \left( n \sum_{k=1}^{\infty} \frac{p_k}{k} (-1)^{k-1} y^k x^k \right) \exp \left( n \sum_{k=1}^{\infty} \frac{p_k}{k} x^k \right)
= \sum_{n=0}^{\infty} t^n \exp \left( n \sum_{k=1}^{\infty} \frac{p_k}{k} (1 - (-y)^k) x^k \right).
\]

By Lemma 2.1, we have

\[
\exp \left( n \sum_{k=1}^{\infty} \frac{p_k}{k} (1 - (-y)^k) x^k \right) = \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} x^{|\lambda|} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k}).
\]

It follows that

\[
\frac{1}{1 - tE(yx)H(x)} = \sum_{n=0}^{\infty} t^n \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} x^{|\lambda|} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k})
= \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} \sum_{n=0}^{\infty} x^{|\lambda|} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k})
= \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} A_{l(\lambda)}(t) \frac{\lambda^{|\lambda|}}{(1 - t)^{|\lambda| + 1}} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k}),
\]

where in the last step we are using (1.2). This proves part (a). Note that setting \( y = 1 \) in (a) gives

\[
\frac{1}{1 - tE(x)H(x)} = \sum_{\lambda \text{ odd}} \frac{p_{\lambda}^2 o(\lambda)}{z_{\lambda}} A_{l(\lambda)}(t) \frac{\lambda^{|\lambda|}}{(1 - t)^{|\lambda| + 1}} x^{|\lambda|}.
\]

(2.6)
To prove part (b), we begin with
\[
\frac{H(x)}{1 - tE(yx)H(x)} = \exp \left( \sum_{k=1}^{\infty} \frac{p_k}{k} x^k \right) \sum_{n=0}^{\infty} t^n \exp \left( n \sum_{k=1}^{\infty} \frac{p_k}{k} (-1)^{k-1} y^k x^k \right) \exp \left( n \sum_{k=1}^{\infty} \frac{p_k}{k} x^k \right)
\]
\[
= \sum_{n=0}^{\infty} t^n \exp \left( \sum_{k=1}^{\infty} \frac{p_k}{k} (n(1 - (-y)^k) + 1) x^k \right)
\]
\[
= \sum_{n=0}^{\infty} t^n \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} x^{[\lambda]} \prod_{k=1}^{\ell(\lambda)} (n(1 - (-y)^k) + 1).
\]

Setting \( y = 1 \), we obtain
\[
\frac{H(x)}{1 - tE(x)H(x)} = \sum_{n=0}^{\infty} t^n \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} x^{[\lambda]} \prod_{k=1}^{\ell(\lambda)} (n(1 - (-1)^k) + 1)
\]
\[
= \sum_{n=0}^{\infty} t^n \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} x^{[\lambda]} (2n + 1)^{\omega(\lambda)}
\]
\[
= \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} \frac{B_{\omega(\lambda)}(t)}{(1 - t)^{\omega(\lambda)+1}} x^{[\lambda]}.
\]

Finally, using parts (a) and (b), we have
\[
\frac{1 + tH(x)}{1 - t^2 E(x)H(x)} = \frac{1}{1 - tE(x)H(x) + t} \frac{H(x)}{1 - t^2 E(x)H(x)}
\]
\[
= \sum_{\lambda \text{ odd}} \frac{p_{\lambda}}{z_{\lambda}} x^{[\lambda]} \frac{A_{\ell(\lambda)}(t^2)}{(1 - t^2)^{\ell(\lambda)+1}} + t \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} \frac{B_{\omega(\lambda)}(t^2)}{(1 - t^2)^{\omega(\lambda)+1}} x^{[\lambda]},
\]
thus proving part (c).

3. General plethystic formulas

In this section, we derive general formulas—analogous to (1.3)—that will allow us to compute the joint distribution of \( p_k \) and \( d_{es} \), the joint distribution of \( l_{pk} \) and \( d_{es} \), and the distribution of \( u_{dr} \) over any set \( \Pi \subseteq S_n \) of permutations whose quasisymmetric generating function \( Q(\Pi) \) is symmetric. Our formulas will involve the following map: Given \( y \in A \) and an integer \( k \in \mathbb{Z} \), define the homomorphism \( \Theta_{y,k} : \Lambda \to \mathbb{Q}[[y]] \) by
\[
\Theta_{y,k}(f) := f[k(1 - \alpha)]|_{\alpha = -y}.
\]
That is, \( \Theta_{y,k} \) first sends a symmetric function \( f \) to the plethystic substitution \( f[k(1 - \alpha)] \), where \( \alpha \) is a variable, and then evaluates this expression at \( \alpha = -y \).
3.1. General formula for peaks and descents

Given a set $\Pi$ of permutations, define

$$P^{(pk,des)}(\Pi; y, t) := \sum_{\pi \in \Pi} y^{pk(\pi)+1} t^{des(\pi)+1},$$

$$P^{pk}(\Pi; t) := \sum_{\pi \in \Pi} t^{pk(\pi)+1},$$

and

$$A(\Pi; t) := \sum_{\pi \in \Pi} t^{des(\pi)+1}.$$

These encode the joint distribution of $pk$ and $des$, the distribution of $pk$, and the distribution of $des$, respectively, over $\Pi$. If $\Pi$ has a symmetric quasisymmetric function $Q(\Pi)$, then the following theorem allows one to describe these polynomials in terms of $\Theta_{y,k}(Q(\Pi))$. Moreover, if we know the power sum expansion of $Q(\Pi)$, then this theorem allows us to describe these polynomials in terms of Eulerian polynomials.

**Theorem 3.1.** Let $\Pi \subseteq \mathfrak{S}_n$, with $n \geq 1$, and suppose that the quasisymmetric generating function $Q(\Pi)$ is a symmetric function with power sum expansion $Q(\Pi) = q(p_1, p_2, p_3, \ldots) = \sum_{\lambda \vdash n} c_\lambda p_\lambda$. Then

(a) \[ \frac{1}{1+y} \left( \frac{1+yt}{1-t} \right)^{n+1} P^{(pk,des)}(\Pi; \frac{(1+y)^2t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}) = \sum_{k=0}^{\infty} \Theta_{y,k}(Q(\Pi)) t^k = \sum_{\lambda \vdash n} c_\lambda \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}} \prod_{k=1}^{l(\lambda)} (1 - (y)^{\lambda_k}), \]

(b) \[ \frac{1}{2} \left( \frac{1+t}{1-t} \right)^{n+1} P^{pk}(\Pi; \frac{4t}{(1+t)^2}) = \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)) t^k = \sum_{\lambda \vdash n, \text{odd}} c_\lambda 2^{l(\lambda)} \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}} = \sum_{k=1}^{n} a_k 2^{k} \frac{A_k(t)}{(1-t)^{k+1}}, \]

where \[ \sum_{k=1}^{n} a_k w^k = q(w, 0, w, 0, w, 0, \ldots), \]

and

(c) \[ \frac{A(\Pi; t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} \Theta_{0,k}(Q(\Pi)) t^k = \sum_{\lambda \vdash n} c_\lambda \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}} = \sum_{k=1}^{n} b_k \frac{A_k(t)}{(1-t)^{k+1}}, \]

where \[ \sum_{k=1}^{n} b_k w^k = q(w, w, w, \ldots). \]

The proof of Theorem 3.1 requires the following technical lemma.
Lemma 3.2. Let $f \in A$, let $\alpha \in A$ be a variable, and let $k \in \mathbb{Z}$. Then $f[k(1 - \alpha)] = \langle f, H^k E(-\alpha)^k \rangle$.

Proof. We have

\[
    f[k(1 - \alpha)] = \langle f[X], H[k(1 - \alpha)X] \rangle = \langle f[X], H[(1 - \alpha)X]^k \rangle = \langle f[X], H[X]^k H[-\alpha X]^k \rangle = \langle f, H^k E(-\alpha)^k \rangle
\]

(by Lemma 2.3 (a))

thus completing the proof.

Proof of Theorem 3.1. We first prove part (a). Consider the following three expressions for the scalar product $\langle Q(\Pi), (1 - tE(y)H)^{-1} \rangle$. First, from Lemma 2.8 (a) we have

\[
    \langle Q(\Pi), \frac{1}{1 - tE(y)H} \rangle = \frac{1}{1 + y} \sum_{L \in n} \left( \frac{1 + yt}{1 - t} \right)^n \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \right)^{pk(\Pi) + 1} \frac{y + t}{1 + yt} \langle Q(\Pi), r_L \rangle
\]

which by Theorem 2.6 simplifies to

\[
    \langle Q(\Pi), \frac{1}{1 - tE(y)H} \rangle = \frac{1}{1 + y} \left( \frac{1 + yt}{1 - t} \right)^n \sum_{\pi \in H} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \right)^{pk(\pi) + 1} \frac{y + t}{1 + yt} \langle Q(\Pi), r_L \rangle
\]

Next, from Lemma 3.2 we have

\[
    \langle Q(\Pi), \frac{1}{1 - tE(y)H} \rangle = \sum_{k=0}^{\infty} \langle Q(\Pi), E(y)^k H^k t^{k^2} \rangle = \sum_{k=0}^{\infty} \Theta_{y,k}(Q(\Pi)) t^k.
\]

Finally, from Lemma 2.9 (a) we have

\[
    \langle Q(\Pi), \frac{1}{1 - tE(y)H} \rangle = \sum_{\lambda \in n} c_{\lambda} p_{\lambda, \tau} \langle A_{l(\tau)}(t), \frac{1}{z_\tau} \prod_{k=1}^{l(\tau)} (1 - (-y)^{\tau_k}) \rangle
\]

\[
    = \sum_{\lambda \in n} c_{\lambda} \frac{A_{l(\lambda)}(t)}{(1 - t)^{l(\lambda) + 1}} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k})
\]

The desired result follows from equating these three expressions for $\langle Q(\Pi), (1 - tE(y)H)^{-1} \rangle$.

Setting $y = 1$ in part (a) gives the first two equalities in part (b). The last equality in (b) comes from the fact that $\sum_{\lambda \in n \text{ odd}} c_{\lambda} w^{l(\lambda)}$ is obtained from $Q(\Pi)$ by setting $p_i = w$ for $i$ odd and $p_i = 0$ for $i$ even. Part (c) is obtained similarly.
We note that since \( \Theta_{0,k}(Q(\Pi)) = Q(\Pi)[k] \), the first equality of part (c) is the \( q = 1 \) specialization of (1.3).

We may also refine Theorem 3.1 by additional permutation statistics. Given an integer-valued statistic \( \text{st} \), define
\[
Q_{\text{st}}(\Pi) := \sum_{\pi \in \Pi} F_{\text{Comp}(\pi)} z^{\text{st}(\pi)}
\]
and
\[
P^{(\text{lpk},\text{des},\text{st})}(\Pi; y, t, z) := \sum_{\pi \in \Pi} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)} z^{\text{st}(\pi)}.
\]
Then the next theorem is proved in a similar way to Theorem 3.1 (a). We will also use analogous generalizations of parts (b) and (c) of Theorem 3.1 but we will not state them here.

**Theorem 3.3.** Let \( \Pi \subseteq S_n \), with \( n \geq 1 \), and let \( \text{st} \) be a permutation statistic. Suppose that \( Q_{\text{st}}(\Pi) \) is a symmetric function (in the variables \( x_1, x_2, \ldots \)). Then we have
\[
\frac{1}{1 + y} \left( \frac{1 + yt}{1 - t} \right)^{n+1} P^{(\text{lpk},\text{des},\text{st})}(\Pi; (1 + yt)^2 t, \frac{y + t}{1 + yt}, z) = \sum_{k=0}^{\infty} \Theta_{1,k}(Q_{\text{st}}(\Pi))[X + 1] t^k.
\]

### 3.2. General formula for left peaks and descents

We now prove an analogue of Theorem 3.1 for the joint distribution of \( \text{lpk} \) and \( \text{des} \). Given a set of permutations \( \Pi \), define
\[
P^{(\text{lpk},\text{des})}(\Pi; y, t) := \sum_{\pi \in \Pi} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)} \quad \text{and} \quad P^{\text{lpk}}(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{lpk}(\pi)}.
\]

**Theorem 3.4.** Let \( \Pi \subseteq S_n \) and suppose that the quasisymmetric generating function \( Q(\Pi) \) is a symmetric function with power sum expansion \( Q(\Pi) = q(p_1, p_2, p_3, \ldots) = \sum_{\lambda \vdash n} c_\lambda \rho_\lambda \). Then

(a) \[
\frac{(1 + yt)^n}{(1 - t)^{n+1}} P^{(\text{lpk},\text{des})}(\Pi; (1 + yt)^2 t, \frac{y + t}{1 + yt}) = \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)[X + 1]) t^k
\]

and

(b) \[
\frac{(1 + t)^n}{(1 - t)^{n+1}} P^{\text{lpk}}(\Pi; 4t, \frac{4t}{1 + t}) = \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)[X + 1]) t^k
\]

where
\[
\sum_{k=0}^{n} d_k w^k = q(w, 1, w, 1, w, 1, \ldots).
\]

We will need a lemma involving plethysm in order to prove this theorem.
Lemma 3.5. Let $f, g \in A$, and let $m \in A$ be a monic term. Then $\langle f[X + m], g \rangle = \langle f, H[mX]g \rangle$.

In the proof of this lemma, we use the notation $\tau \cup \{n\}$ to denote the partition obtained from $\tau$ by adding a part of size $n$, and we write $\tau - \tau_i$ to denote the partition obtained by removing the part $\tau_i$ from $\tau$.

Proof. By linearity, it suffices to prove the result for $f = m_\lambda$ and $g = h_\tau$. First, because $X + m$ is a sum of monic terms, we have
\[
m_\lambda[X + m] = m_\lambda(m, x_1, x_2, \ldots) = m_\lambda + \sum_{\lambda_i} m_{\lambda - \lambda_i} m^{\lambda_i},
\]
where the sum is over all distinct parts $\lambda_i$ of $\lambda$. For example, if $\lambda = (2, 2, 1)$, then we have
\[
m_{(2,2,1)}[X + m] = m_{(2,2,1)} + m_{(2,2)} m + m_{(2,1)} m^2.
\]
Thus,
\[
\langle m_\lambda[X + m], h_\tau \rangle = \sum_{\lambda_i} \langle m_{\lambda - \lambda_i}, h_\tau \rangle m^{\lambda_i}
\]
\[
= \begin{cases} m^n, & \text{if } \lambda = \tau \cup \{n\} \\ 0, & \text{otherwise} \end{cases}
\]
\[
= \sum_{n=0}^{\infty} \langle m_\lambda, h_{\tau \cup \{n\}} \rangle m^n
\]
\[
= \sum_{n=0}^{\infty} \langle m_\lambda, h_n h_\tau \rangle m^n
\]
\[
= \langle m_\lambda, H(m) h_\tau \rangle
\]
\[
= \langle m_\lambda, H[mX] h_\tau \rangle
\]
and we are done. \qed

Proof of Theorem 3.4. We first derive two expressions for $\langle Q(\Pi), H/(1 - tE(y)H) \rangle$. From Lemma 2.8 (b) and Theorem 2.6, we obtain
\[
\left\langle Q(\Pi), \frac{H}{1 - tE(y)H} \right\rangle = \sum_{L \in n} \frac{(1 + yt)^n}{(1 - t)^{n+1}} \left( \frac{(1 + y)^2t}{(y + t)(1 + yt)} \right)^{lpk(L)} \left( \frac{y + t}{1 + yt} \right)^{des(L)} \langle Q(\Pi), r_L \rangle
\]
\[
= \frac{(1 + yt)^n}{(1 - t)^{n+1}} \sum_{\pi \in \Pi} \left( \frac{(1 + y)^2t}{(y + t)(1 + yt)} \right)^{lpk(\pi)} \left( \frac{y + t}{1 + yt} \right)^{des(\pi)}
\]
\[
= \frac{(1 + yt)^n}{(1 - t)^{n+1}} P^{lpk, des}(\Pi; \frac{(1 + y)^2t}{(y + t)(1 + yt)} \frac{y + t}{1 + yt}).
\]
In addition, we have
\[
\left\langle Q(\Pi), \frac{H}{1 - tE(y)H} \right\rangle = \left\langle Q(\Pi)[X + 1], \frac{1}{1 - tE(y)H} \right\rangle \quad \text{(by Lemma 3.5)}
\]
\[
= \sum_{k=0}^{\infty} \left\langle Q(\Pi)[X + 1], E(y)^k H^k \right\rangle t^k
\]
\[
= \sum_{k=0}^{\infty} \Theta_{y,k} \langle Q(\Pi)[X + 1] \rangle t^k \quad \text{(by Lemma 3.2)}.
\]

Equating these two expressions yields part (a). The first two equalities in part (b) follow from equating the \(y = 1\) evaluation of these two expressions with
\[
\sum_{\lambda \vdash n} c_{\lambda} w^{\omega(\lambda)}(1 - t)^{\omega(\lambda) + 1},
\]
which is obtained using Lemma 2.9 (b). The last equality in (b) comes from the fact that \(\sum_{\lambda \vdash n} c_{\lambda} w^{\omega(\lambda)}\) is obtained from \(Q(\Pi)\) by setting \(p_i = w\) for \(i\) odd and \(p_i = 1\) for \(i\) even.

We state a refinement of Theorem 3.4 (a) with an additional statistic \(st\), which is proved in a similar way. Define
\[
P^{(lpk,\text{des},st)}(\Pi; y, t, z) := \sum_{\pi \in \Pi} y^{lpk(\pi)} t^{\text{des}(\pi)} z^{\text{st}(\pi)}.
\]

**Theorem 3.6.** Let \(\Pi \subseteq S_n\), with \(n \geq 1\), and let \(st\) be a permutation statistic. Suppose that \(Q^{st}(\Pi)\) is a symmetric function (in the variables \(x_1, x_2, \ldots\)). Then
\[
\frac{(1 + yt)^n}{(1 - t)^{n+1}} P^{(lpk,\text{des},st)}(\Pi; \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}, z) = \sum_{k=0}^{\infty} \Theta_{y,k} (Q^{st}(\Pi)[X + 1]) t^k.
\]

### 3.3. General formula for up-down runs

We can also provide analogous results for the udr statistic. Given a set of permutations \(\Pi\) and a permutation statistic \(st\), define
\[
P^{\text{udr}}(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{udr}(\pi)} \quad \text{and} \quad P^{(\text{udr},st)}(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{udr}(\pi)} z^{\text{st}(\pi)}.
\]
Theorem 3.7. Let \( \Pi \subseteq \mathfrak{S}_n \), with \( n \geq 1 \), and suppose that the quasisymmetric generating function \( Q(\Pi) \) is a symmetric function with power sum expansion \( Q(\Pi) = \sum_{\lambda \vdash n} c_\lambda p_\lambda \). Then we have

\[
\frac{(1 + t^2)^n}{2(1 - t)^2(1 - t^2)^{n-1}} P_{\text{udr}}(\Pi; \frac{2t}{1 + t^2}) = \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)t^{2k} + \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)[X + 1])t^{2k+1}
\]

\[
= \sum_{\lambda \vdash n \text{ odd}} c_\lambda 2^{l(\lambda)} \frac{A_{l(\lambda)}(t^2)}{(1 - t^2)(1 - t^2)_{\lambda} + t \sum_{\lambda \vdash n \text{ odd}} c_\lambda \frac{B_{o(\lambda)}(t^2)}{(1 - t^2)(1 - t^2)_{o(\lambda) + 1}}
\]

\[
= \sum_{k=1}^{n} a_k 2^{k} \frac{A_{k}(t^2)}{(1 - t^2)_{k + 1}} + t \sum_{k=0}^{n} d_k \frac{B_{k}(t^2)}{(1 - t^2)^{k + 1}}
\]

with \( a_k \) and \( d_k \) as in Theorems 3.1 and 3.4.

Proof. We derive three expressions for \( \langle Q(\Pi), (1 + tH)/(1 - t^2EH) \rangle \). First, from Lemma 2.8 (c) and Theorem 2.6 we have

\[
\langle Q(\Pi), \frac{1 + tH}{1 - t^2EH} \rangle = \frac{1}{2(1 - t)^2 \sum_{L \vdash n} (1 + t^2)^2 \frac{2t}{1 + t^2} \text{udr}(L)} \langle Q(\Pi), r_L \rangle
\]

\[
= \frac{(1 + t^2)^2}{2(1 - t)^2(1 - t^2)^{n-1}} \sum_{\pi \in \Pi} \frac{2t}{1 + t^2} \text{udr}(\pi)
\]

\[
= \frac{(1 + t^2)^n}{2(1 - t)^2(1 - t^2)^{n-1}} P_{\text{udr}}(\Pi; \frac{2t}{1 + t^2}).
\]

Moreover, following the reasoning of the proof of Theorem 3.1 (a), we obtain

\[
\langle Q(\Pi), \frac{1 + tH}{1 - t^2EH} \rangle = \langle Q(\Pi), \frac{1}{1 - t^2EH} \rangle + \langle Q(\Pi), \frac{tH}{1 - t^2EH} \rangle
\]

\[
= \langle Q(\Pi), \frac{1}{1 - t^2EH} \rangle + \langle Q(\Pi)[X + 1], \frac{t}{1 - t^2EH} \rangle
\]

\[
= \sum_{k=0}^{\infty} \langle Q(\Pi), E^k H^k \rangle t^{2k} + \sum_{k=0}^{\infty} \langle Q(\Pi)[X + 1], E^k H^k \rangle t^{2k+1}
\]

\[
= \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)) t^{2k} + \sum_{k=0}^{\infty} \Theta_{1,k}(Q(\Pi)[X + 1]) t^{2k+1}.
\]

Finally, Lemma 2.9 (c) implies

\[
\langle Q(\Pi), \frac{1 + tH}{1 - t^2EH} \rangle = \sum_{\lambda \vdash n} c_\lambda p_\lambda \sum_{\tau \text{ odd}} \frac{p_{\tau} 2^{l(\tau)} A_{l(\tau)}(t^2)}{(1 - t^2)^{l(\tau) + 1}} + \langle \sum_{\lambda \vdash n} c_\lambda p_\lambda, t \sum_{\tau} \frac{B_{o(\tau)}(t^2)}{(1 - t^2)^{o(\tau) + 1}} \rangle
\]
\[
\sum_{\lambda \vdash n, \tau \text{ odd}} c_{\lambda} \sum_{\mu} c_{\mu} \frac{A_{\mu}(t^2)}{(1-t^2)^{\lambda + \mu + 1}} + t \sum_{\lambda \vdash n, \tau} \frac{B_\mu(t^2)}{(1-t^2)^{\lambda + \mu + 1}}
\]

The fourth expression follows from the third as in Theorems 3.1 and 3.4.

Theorem 3.8. Let \( \Pi \subseteq S_n \) and let \( st \) be a permutation statistic. Suppose that \( Q^{st}(\Pi) \) is a symmetric function (in the variables \( x_1, x_2, \ldots \)). Then

\[
\left(1 + t^2\right)^n \frac{2}{2(1-t)^2(1-t^2)^{n-1}} P_{(u,d,r)} \left( \Pi; \frac{2t}{1 + t^2}, z \right)
\]

\[
= \sum_{k=0}^\infty \Theta_{1,k}(Q^{st}(\Pi)) t^{2k+1} + \sum_{k=0}^\infty \Theta_{1,k}(Q^{st}(\Pi)[X+1]) t^{2k+1}.
\]

4. Cyclic permutations

We say that a permutation of length \( n \) is a cyclic permutation (or a cycle) if it has cycle type \( (n) \). We denote the set of cyclic permutations of length \( n \) by \( C_n \).

Recall that the Lyndon symmetric function

\[
L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}
\]  

(4.1)

is the quasisymmetric generating function for \( C_n \).

4.1. Counting cyclic permutations by peaks and descents

Define

\[
C_{n}^{(pk,des)}(y,t) := P_{(pk,des)}(C_n; y,t) = \sum_{\pi \in C_n} y^{pk(\pi)+1} t^{des(\pi)+1},
\]

\[
C_{n}^{pk}(t) := P_{pk}(C_n; t) = \sum_{\pi \in C_n} t^{pk(\pi)+1}, \quad \text{and} \quad C_{n}(t) := A(C_n; t) = \sum_{\pi \in C_n} t^{des(\pi)+1}.
\]

Our first result from this section is a formula for the polynomials \( C_{n}^{(pk,des)}(y,t) \) in terms of Eulerian polynomials.

Theorem 4.1. Let \( n \geq 1 \). Then

\[
C_{n}^{(pk,des)} \left( \frac{(1+y)^2}{(y+t)(1+yt)} \frac{y+t}{1+yt} \right) = \frac{(1+y)}{n(1+yt)^{n+1}} \sum_{d|n} \mu(d)(1+(-y)^d)^{n/d}(1-t)^{n-n/d} A_{n/d}(t).
\]
Proof. The result follows immediately from Theorem 3.1 (a) and Equation (4.1).

Although this formula may seem complicated, it allows for easy computation of the polynomials \( C_n^{(pk, \text{des})}(y, t) \). By inverting this formula (cf. the discussion after Lemma 2.8), we obtain

\[
C_n^{(pk, \text{des})}(y, t) = \frac{(1 + u)}{n(1 + uv)^{n+1}} \sum_{d|n} \mu(d)(1 - (-u)^{d/n}(1 - v)^{n-d}A_{n/d}(v))
\]

where

\[
u = \frac{1 + t^2 - 2yt - (1 - t)\sqrt{(1 + t)^2 - 4yt}}{2(1 - y)t}
\]

and

\[
v = \frac{(1 + t)^2 - 2yt - (1 + t)\sqrt{(1 + t)^2 - 4yt}}{2yt}.
\]

For example, with \( n = 7 \), we have

\[
C_7^{(pk, \text{des})}(y, t) = \frac{(1 + u)}{7(1 + uv)^8}((1 + u)^7A_7(v) - (1 + u^7)(1 - v)^6A_1(v))
\]

\[
= \frac{(1 + u)}{7(1 + uv)^8}((1 + u)^7(v + 120v^2 + 1191v^3 + 2416v^4 + 1191v^5 + 120v^6 + v^7) - (1 + u^7)(1 - v)^6v)
\]

\[
= (y + 17y^2)t^2 + (2y + 64y^2 + 102y^3)t^3 + (3y + 99y^2 + 207y^3 + 39y^4)t^4
\]

\[
+ (2y + 64y^2 + 102y^3)t^5 + (y + 17y^2)t^6,
\]

where the last equality was obtained by substituting in the above expressions for \( u \) and \( v \), and then simplifying using Maple.

Observe that in \( C_7^{(pk, \text{des})}(y, t) \), the coefficient of \( t^{k+1} \) is equal to the coefficient of \( t^{n-k} \) for each \( k \), indicating that the number of cyclic permutations of length 7 with \( j \) peaks and \( k \) descents is equal to the number of cyclic permutations of length 7 with \( j \) peaks and \( n - 1 - k \) descents. In fact, this is true for any \( n \in \mathbb{P} \) not congruent to 2 modulo 4. This is not readily apparent from Theorem 4.1, but is a special case of the following result.

**Theorem 4.2.** Suppose that \( \lambda \) is a partition of \( n \) with no parts congruent to 2 modulo 4 and that every odd part of \( \lambda \) occurs only once. Then the number of permutations of cycle type \( \lambda \) with \( j \) peaks and \( k \) descents is equal to the number of permutations of cycle type \( \lambda \) with \( j \) peaks and \( n - 1 - k \) descents.

**Corollary 4.3.** If \( n \) is not congruent to 2 modulo 4, then the number of cyclic permutations of length \( n \) with \( j \) peaks and \( k \) descents is equal to the number of cyclic permutations of length \( n \) with \( j \) peaks and \( n - 1 - k \) descents.

To prove Theorem 4.2, we need the following lemma. Given a descent set \( D \subseteq [n - 1] \), the complementary descent set \( D^c \) of \( D \) is defined by \( D^c := [n - 1] \setminus D \).
Lemma 4.4. Suppose that $\lambda$ is a partition of $n$ with no parts congruent to 2 modulo 4 and that every odd part of $\lambda$ occurs only once. Then the number of permutations with cycle type $\lambda$ and descent set $D \subseteq [n-1]$ is equal to the number of permutations with cycle type $\lambda$ and the complementary descent set $D^c$.

Lemma 4.4 is due to Gessel and Reutenauer [18, Theorem 4.1]; a bijective proof was later given by Steinhardt [38].

Proof of Theorem 4.2. Given $\pi \in S_n$, we say that $i \in \{2, \ldots, n\}$ is a valley of $\pi$ if $\pi(i-1) > \pi(i) < \pi(i+1)$. It is not difficult to verify that conjugation by the decreasing permutation $n(n-1) \cdots 1$ (i.e., the reverse-complement operation) preserves both the descent number and cycle type but toggles between peaks and valleys. In other words, the number of permutations of cycle type $\lambda$ with $j$ peaks and $k$ descents is equal to the number of permutations of cycle type $\lambda$ with $j$ valleys and $k$ descents. The descent number, peak number, and valley number are all descent statistics, and it is easy to see that a permutation with descent set $D$ has $j$ valleys and $k$ descents if and only if a permutation with the complementary descent set $D^c$ has $j$ peaks and $n-1-k$ descents. Thus, if $\lambda$ satisfies the conditions of Lemma 4.4, then the number of permutations of cycle type $\lambda$ with $j$ peaks and $k$ descents is equal to the number of permutations of cycle type $\lambda$ with $j$ peaks and $n-1-k$ descents.

Let us return to Theorem 4.1. Setting $y = 1$ and $y = 0$ yields simpler formulas for counting cyclic permutations by the peak number and by the descent number, respectively. Note that the latter result is the $q = 1$ evaluation of [18, Corollary 6.2].

Corollary 4.5. Let $n \geq 1$. Then

\[
\begin{align*}
(a) \quad C_n^{\text{pk}} \left( \frac{4t}{(1+t)^2} \right) &= \frac{1}{n(1+t)^{n+1}} \sum_{d|n, d \text{ odd}} \mu(d) 2^{n/d+1}(1-t)^{n-n/d} A_{n/d}(t) \\
(b) \quad C_n(t) &= \frac{1}{n} \sum_{d|n} \mu(d)(1-t)^{n-n/d} A_{n/d}(t).
\end{align*}
\]

Now, consider the polynomial

\[
P_n^{\text{pk}}(t) := P_n^{\text{pk}}(\mathfrak{S}_n; t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{pk}(\pi)+1}.
\]

giving the distribution of the peak number over all of $\mathfrak{S}_n$. Just as Corollary 4.5 (b) allows one to express the polynomials $C_n(t)$ counting cyclic permutations by descents in terms of the polynomials $A_n(t)$ counting all permutations by descents, we can use Corollary 4.5 (a) to express $C_n^{\text{pk}}(t)$ in terms of the $P_n^{\text{pk}}(t)$.

Corollary 4.6. Let $n \geq 1$. Then

\[
C_n^{\text{pk}}(t) = \frac{1}{n} \sum_{d|n, d \text{ odd}} \mu(d)(1-t)^{(n-n/d)/2} P_{n/d}^{\text{pk}}(t).
\]
Proof. It is known that

\[ A_k(t) = \left( \frac{1 + t}{2} \right)^{k+1} P_k^p \left( \frac{4t}{(1+t)^2} \right) \]  \hspace{1cm} (4.2)

for every \( k \geq 1 \); see [39] \footnote{We will later recover this formula as a special case of Theorem 6.3 (a).} Combining (4.2) with Corollary 4.5 (a), we obtain

\[ C_n^p \left( \frac{4t}{(1+t)^2} \right) = \frac{1}{n} \sum_{\substack{d|n \\ d \text{ even}}} \mu(d) \left( \frac{1 - t}{1 + t} \right)^{n - n/d} P^p_{n/d} \left( \frac{4t}{(1+t)^2} \right) \]

Inverting this formula and simplifying completes the proof. \qed

4.2. Counting cyclic permutations by left peaks

Let

\[ C_{n,p}^l := P_{n,p}^l = \sum_{\pi \in \mathcal{C}_n} l_{n,p}(\pi). \]

We now state a result analogous to Corollary 4.5 (a) but for the polynomials \( C_{n,p}^l(t) \).

**Theorem 4.7.** Let \( n \geq 2 \). Then

\[ C_{n,p}^l \left( \frac{4t}{(1+t)^2} \right) = \frac{B_n(t) - (1 - t)^n}{n(1 + t)^n} \]

if \( n \) is a power of 2; otherwise,

\[ C_{n,p}^l \left( \frac{4t}{(1+t)^2} \right) = \frac{1}{n(1+t)^n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) (1 - t)^{n - n/d} B_{n/d}(t). \]

The proof of this theorem requires a Möbius function identity.

**Lemma 4.8.**

\[ \sum_{\substack{d|n \\ d \text{ even}}} \mu(d) = \begin{cases} -1, & \text{if } n = 2^j \text{ with } j > 0, \\ 0, & \text{otherwise}. \end{cases} \]

Proof. The result is true for \( n = 1 \), so suppose that \( n > 1 \). Then by a fundamental property of the Möbius function, \( \sum_{d|n} \mu(d) = 0 \), so

\[ \sum_{\substack{d|n \\ d \text{ even}}} \mu(d) = - \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d). \]

Now let \( m \) be the largest odd divisor of \( n \). Then

\[ \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d) = \sum_{\substack{d|m \\ d \text{ even}}} \mu(d) = \begin{cases} 1, & \text{if } m = 1, \\ 0, & \text{otherwise}, \end{cases} \]

and the result follows. \qed
Proof of Theorem 4.7. By Theorem 3.4 (b), we have

\[ C_{n}^{lpk}\left(\frac{4t}{(1+t)^2}\right) = \frac{1}{n(1+t)^n} \sum_{d|n \atop d \text{ odd}} \mu(d) (1-t)^{n-n/d} B_{n/d}(t) + \frac{1}{n} \left(\frac{1-t}{1+t}\right)^{n} \sum_{d|n \atop d \text{ even}} \mu(d). \]

Applying Lemma 4.8 and noting that 1 is the only odd divisor of any power of 2, we obtain the desired result.

Now, let

\[ P_{n}^{lpk}(t) := P_{lpk}(\mathfrak{S}_n; t) = \sum_{\pi \in \mathfrak{S}_n} t^{lpk(\pi)}. \]

Our next result expresses the polynomials \( C_{n}^{lpk}(t) \) in terms of the \( P_{n}^{lpk}(t) \).

**Corollary 4.9.** Let \( n \geq 2 \). Then

\[ C_{n}^{lpk}(t) = \frac{1}{n} \left( P_{n}^{lpk}(t) - (1-t)^{n/2} \right) \]

if \( n \) is a power of 2; otherwise,

\[ C_{n}^{lpk}(t) = \frac{1}{n} \sum_{d|n \atop d \text{ odd}} \mu(d) (1-t)^{(n-n/d)/2} P_{n/d}^{lpk}(t). \]

**Proof.** It is known that

\[ B_k(t) = (1+t)^k P_k^{lpk}\left(\frac{4t}{(1+t)^2}\right) \]

for every \( k \geq 0 \); see [28, Proposition 4.15]. Combining this formula with Theorem 4.7 yields

\[ C_{n}^{lpk}\left(\frac{4t}{(1+t)^2}\right) = \frac{1}{n} \left( P_{n}^{lpk}\left(\frac{4t}{(1+t)^2}\right) - \left(\frac{1-t}{1+t}\right)^{n} \right) \]

if \( n \) is a power of 2, and

\[ C_{n}^{lpk}\left(\frac{4t}{(1+t)^2}\right) = \frac{1}{n} \sum_{d|n \atop d \text{ odd}} \mu(d) \left(\frac{1-t}{1+t}\right)^{n-n/d} P_{n/d}^{lpk}\left(\frac{4t}{(1+t)^2}\right) \]

otherwise. Inverting these formulas and simplifying yields the desired result.

4.3. Counting cyclic permutations by up-down runs

Finally, we state the analogous result for the polynomials

\[ C_{n}^{udr}(t) := P_{udr}(\mathfrak{C}_n; t) = \sum_{\pi \in \mathfrak{C}_n} t^{udr(\pi)}, \]

which is proved in the same way but using Theorem 3.7.

\[ \text{We will later recover this formula as a special case of Theorem 6.8.} \]
Theorem 4.10. Let \( n \geq 2 \). Then
\[
C_n^{\text{adr}} \left( \frac{2t}{1 + t^2} \right) = \frac{2}{n(1 + t^2)^n(1 + t)^2} (2^n A_n(t^2) + tB_n(t^2) - t(1 - t^2)^n)
\]
if \( n \) is a power of 2; otherwise,
\[
C_n^{\text{adr}} \left( \frac{2t}{1 + t^2} \right) = \frac{2}{n(1 + t^2)^n(1 + t)^2} \sum_{d|n, d \text{ odd}} \mu(d) (1 - t^2)^{n-n/d} (2^{n/d} A_{n/d}(t^2) + tB_{n/d}(t^2)).
\]

5. Involutions

A permutation \( \pi \) is called an involution if \( \pi^2 \) is the identity permutation, or equivalently, if \( \pi \) has no cycles of length greater than 2. We denote the set of involutions of length \( n \) by \( \mathcal{I}_n \).

The quasisymmetric generating function for all involutions weighted by length and number of fixed points is known to be
\[
Q_I(x) = \sum_{n=0}^{\infty} Q^{\text{fix}}(\mathcal{I}_n) x^n = \prod_i \frac{1}{1 - z x x_i} \prod_{i < j} \frac{1}{1 - x^2 x_i x_j};
\]
see [18, Equation (7.1)].

5.1. Counting involutions by peaks, descents, and fixed points

Define
\[
I_n^{(\text{pk, des, fix})}(y, t, z) := P^{(\text{pk, des, fix})}(\mathcal{I}_n; y, t, z) = \sum_{\pi \in \mathcal{I}_n} y^{\text{pk}(\pi) + 1} t^{\text{des}(\pi) + 1} z^{\text{fix}(\pi)},
\]
this polynomial encodes the joint distribution of the peak number, descent number, and number of fixed points over involutions of length \( n \). Our first result of this section is a formula for the generating function of these polynomials.

Theorem 5.1.
\[
\frac{1}{1-t} + \frac{1}{1+y} \sum_{n=1}^{\infty} \left( \frac{1 + yt}{1 - t} \right)^{n+1} I_n^{(\text{pk, des, fix})} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}, z \right) x^n
= \sum_{k=0}^{\infty} \frac{(1 + z x y)^k (1 + x^2 y)^{k^2} t^k}{(1 - z x)^k (1 - x^2)^{(k^2)} (1 - x^2 y)^{(k+1)}}.
\]

To prove Theorem 5.1, we will need to evaluate the expression \( Q_I[k(1 - \alpha)] \). In doing so, we shall first give an alternate description of the quasisymmetric generating function \( Q_I \) as a plethystic substitution of \( H \).

Lemma 5.2. \( Q_I = H[z x e_1 + x^2 e_2] \)

Proof. This follows directly from Theorem 2.2. \( \square \)
Lemma 5.3.

\[ Q_I[k(1-\alpha)] = \frac{(1-zx\alpha)^k(1-x^2\alpha)^k}{(1-zx)^k(1-x^2)\binom{k}{2}(1-x^2\alpha^2)\binom{k+1}{2}}. \]

Proof. First, observe that

\[ (zxe_1 + x^2e_2)[k(1-\alpha)] = \left(zxp_1 + \frac{x^2}{2}(p_1^2 - p_2)\right)[k(1-\alpha)] \]
\[ = kzx(1-\alpha) + \frac{x^2}{2}(k^2(1-\alpha)^2 - k(1-\alpha^2)) \]
\[ = kzx - kzx\alpha + \left(\frac{k}{2}\right)x^2 + \left(\frac{k+1}{2}\right)x^2\alpha^2 - k^2x^2\alpha. \quad (5.1) \]

Thus, we have

\[ Q_I[k(1-\alpha)] = H[zxe_1 + x^2e_2][k(1-\alpha)] \]
\[ = H\left[kzx - kzx\alpha + \left(\frac{k}{2}\right)x^2 + \left(\frac{k+1}{2}\right)x^2\alpha^2 - k^2x^2\alpha\right] \quad (\text{by Lemma 5.2}) \]
\[ = \frac{H[zx]^kH[x^2]^\left(\frac{k}{2}\right)H[x^2\alpha^2]^\left(\frac{k+1}{2}\right)}{H[zx\alpha]^kH[x^2\alpha]^k} \quad (\text{by Lemma 2.3}) \]
\[ = \frac{(1-zx\alpha)^k(1-x^2\alpha)^k}{(1-zx)^k(1-x^2)\binom{k}{2}(1-x^2\alpha^2)\binom{k+1}{2}}, \quad (\text{by Lemma 2.4 (c)}) \]

thus completing the proof. \(\square\)

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.3 we have

\[ \sum_{k=0}^{\infty} \frac{(1+zx)(1+x^2y^2)^k t^k}{(1-zx)^k(1-x^2)\binom{k}{2}(1-x^2y^2)\binom{k+1}{2}} = \sum_{k=0}^{\infty} Q_I[k(1-\alpha)]_{\alpha=-y} t^k \]
\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Theta_{y,k}(Q_{\text{fix}}(J_n)) t^k x^n \]
\[ = \frac{1}{1-t} + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \Theta_{y,k}(Q_{\text{fix}}(J_n)) t^k x^n. \quad (5.2) \]

Then we apply Theorem 3.3 to obtain

\[ \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \Theta_{y,k}(Q_{\text{fix}}(J_n)) t^k x^n \]
\[ = \frac{1}{1+y} \sum_{n=1}^{\infty} \left(\frac{1+yt}{1-t}\right)^n I_n^{(p_k,\text{des,fix})} \left(\frac{(1+y)^2}{y+t}(1+y), \frac{y+t}{1+y}, z\right) x^n, \]

which combined with (5.2) proves the result. \(\square\)
Define
\[ I_{n}^{(pk,\text{fix})}(t, z) := \sum_{\pi \in \mathcal{J}_n} t^{pk(\pi)+1} z^{\text{fix}(\pi)} \quad \text{and} \quad I_{n}^{(\text{des,fix})}(t, z) := \sum_{\pi \in \mathcal{J}_n} t^{\text{des}(\pi)+1} z^{\text{fix}(\pi)}. \]

Specializing Theorem 5.1 at \( y = 1 \) yields the following formula for the joint distribution of \( pk \) and \( \text{fix} \) over \( \mathcal{J}_n \).

**Corollary 5.4.**
\[ \frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(1+t)}{1-t} I_{n}^{(pk,\text{fix})} \left( \frac{4t}{(1+t)^2}, z \right) x^n = \sum_{k=0}^{\infty} \frac{(1+zx)^k (1+x^2)^{k^2} t^k}{(1-zx)^k (1-x^2)^{k+1}}. \]

Specializing at \( y = 0 \), on the other hand, yields a formula for the joint distribution of \( \text{des} \) and \( \text{fix} \) over \( \mathcal{J}_n \), which is equivalent to Equation (5.5) of Désarménien and Foata [7] and Equation (7.3) of Gessel–Reutenauer [18]. See also Strehl [40] and Guo–Zeng [21].

**Corollary 5.5.**
\[ \frac{1}{1-t} + \sum_{n=1}^{\infty} I_{n}^{(\text{des,fix})}(t, z) x^n = \sum_{k=0}^{\infty} \frac{t^k}{(1-zx)^k (1-x^2)^{k+1}}. \]

Our next theorem gives formulas relating \( I_{n}^{(pk,\text{fix})}(t, z) \) and \( I_{n}^{(\text{des,fix})}(t, z) \) to Eulerian polynomials.

**Theorem 5.6.** Let \( n \geq 1 \). Then
\[ (a) \quad \frac{1}{2} \left( \frac{1+t}{1-t} \right)^{n+1} I_{n}^{(pk,\text{fix})} \left( \frac{4t}{(1+t)^2}, z \right) = \sum_{k=1}^{n} a_{n,k}(z) 2^k \frac{A_k(t)}{(1-t)^{k+1}}, \]
where
\[ 1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} a_{n,k}(z) w^k = \left( \frac{1+zx}{1-zx} \right)^{w/2} \left( \frac{1+x^2}{1-x^2} \right)^{w^2/4}, \]
and
\[ (b) \quad I_{n}^{(\text{des,fix})}(t, z) = \sum_{k=1}^{n} b_{n,k}(z) \frac{A_k(t)}{(1-t)^{k+1}}, \]
where
\[ 1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} b_{n,k}(z) w^k = \frac{1}{(1-zx)^w (1-x^2)^{\left( \frac{w^2}{4} \right)}}. \]

**Proof.** We first expand \( Q_I \) into power sum symmetric functions. By Lemma 5.2 we have
\[ Q_I = H[ x e_1 + x^2 e_2 ] = H[ x p_1 + x^2 (p_1^2 - p_2)/2 ] = \exp \left( \sum_{n=1}^{\infty} z^n x^n p_n \frac{n}{n} + \sum_{n=1}^{\infty} x^{2n} \left( \frac{p_n^2 - p_{2n}}{2n} \right) \right) = \exp \left( \sum_{n \text{ odd}} z^n x^n p_n \frac{n}{n} + \sum_{n \text{ even}} (z^n - 1) x^n p_n \frac{n}{n} + \sum_{n=1}^{\infty} x^{2n} p_n^2 \frac{2n}{2n} \right). \]
Thus, if we set $a$ appropriately, we obtain the following formulas for the polynomials.

\[
1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} a_{n,k}(z) w^k = \exp \left( \sum_{n=1}^{\infty} z^n x^n \frac{w}{n} + \sum_{n=1}^{\infty} x^{2n} \frac{w^2}{2n} \right) = \exp \left( \frac{w}{2} \log \left( \frac{1 + zx}{1 - zx} \right) + \frac{w^2}{4} \log \left( \frac{1 + x^2}{1 - x^2} \right) \right) = \left( \frac{1 + zx}{1 - zx} \right)^{w/2} \left( \frac{1 + x^2}{1 - x^2} \right)^{w^2/4}.
\]

Similarly, by applying the analogous generalization of Theorem 3.1 (c) gives the formula for $I_n^{(\text{des,fix})}(t, z)$ with

\[
1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} b_{n,k}(z) w^k = \exp \left( \sum_{n=1}^{\infty} z^n x^n \frac{w}{n} + \sum_{n=1}^{\infty} x^{2n} \frac{w^2 - w}{2n} \right) = \exp \left( -w \log(1 - zx) - \frac{w(w - 1)}{2} \log(1 - x^2) \right) = \frac{1}{(1 - zx)w(1 - x^2)^{w/2}}.
\]

The polynomials $a_{n,k}(z)$ and the numbers $b_{n,k}(1)$ have simple combinatorial interpretations. By the exponential formula [37, Corollary 5.1.9, p. 7], the exponential generating function for permutations with cycles of length $n$ weighted $u_n$ is $\exp(\sum_{n=1}^{\infty} u_n x^n/n)$. Thus, from (5.4), we see that $n! \sum_{j=1}^{\infty} a_{n,k}(z) w^k$ counts permutations in $\mathfrak{S}_n$ with no cycle lengths divisible by 4 in which odd cycles of length $m$ are weighted $z^m w$ and even cycles (which must have lengths congruent to 2 modulo 4) are weighted $w^2$. For $z = 1$, we can restate this in a more elegant way. Note that the square of an odd cycle of length $m$ is an odd cycle of length $m$ and the square of an even cycle of length $m$ is a product of two cycles, each of length $m/2$. Thus, if we set $a_{n,k} := a_{n,k}(1)$, then $n!a_{n,k}$ is the number of permutations $\pi$ in $\mathfrak{S}_n$ with no cycles having length divisible by 4 for which $\pi^2$ has $k$ cycles. Similarly—as can be seen most easily be setting $z = 1$ and $p_n = w$ in (5.3)—if we set $b_{n,k} := b_{n,k}(1)$, we see that $n!b_{n,k}$ is the number of permutations $\pi$ in $\mathfrak{S}_n$ for which $\pi^2$ has $k$ cycles. Therefore, by specializing Theorem 5.6 appropriately, we obtain the following formulas for the polynomials

\[
I_n^{pk}(t) := P^{pk}(\mathfrak{J}_n; t) = \sum_{\pi \in \mathfrak{J}_n} t^{\text{des}^\text{fix}(\pi)+1} \quad \text{and} \quad I_n(t) := A(\mathfrak{J}_n; t) = \sum_{\pi \in \mathfrak{J}_n} t^{\text{des}(\pi)+1}.
\]

**Corollary 5.7.** Let $n \geq 1$. Then

\[
(a) \quad \frac{1}{2} \left( \frac{1 + t}{1 - t} \right)^{n+1} I_n^{pk} \left( \frac{4t}{(1 + t)^2} \right) = \sum_{k=1}^{n} a_{n,k} 2^k \frac{A_k(t)}{(1 - t)^{k+1}}.
\]

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9 We do not have a combinatorial interpretation for the polynomials $b_{n,k}(z)$, which have some negative coefficients.
where \( n! a_{n,k} \) is the number of permutations \( \pi \in \mathfrak{S}_n \) with no cycles having length divisible by 4 and for which \( \pi^2 \) has \( k \) cycles, and

\[
\frac{I_n(t)}{(1 - t)^{n+1}} = \sum_{k=1}^{n} b_{n,k} \frac{A_k(t)}{(1 - t)^{k+1}}
\]

where \( n! b_{n,k} \) is the number of permutations \( \pi \in \mathfrak{S}_n \) for which \( \pi^2 \) has \( k \) cycles.

A formula equivalent to Corollary 5.7 (b) was given by Athanasiadis [2, Proposition 2.22]. Athanasiadis’s proof involves computations with irreducible characters of symmetric groups.

We can use Corollary 5.7 to obtain an analogue of Corollary 4.6 for involutions. The proof is omitted as it is essentially the same as the proof of Corollary 4.6.

**Corollary 5.8.** Let \( n \geq 1 \). Then

\[
I_n^{pk}(t) = \sum_{k=1}^{n} a_{n,k}(1 - t)^{(n-k)/2} P_k^{pk}(t)
\]

with \( n! a_{n,k} \) as in Corollary 5.7.

We note that for \( a_{n,k} \) to be nonzero, \( n \) and \( k \) must have the same parity, i.e., \( (n - k)/2 \) must be an integer. Thus the above formula does not contain any square roots.

Next, we derive a formula for the polynomials

\[
I_n^{(pk, des)}(y, t) := P^{(pk, des)}(\mathfrak{I}_n; y, t) = \sum_{\pi \in \mathfrak{I}_n} y^{pk(\pi)+1} t^{\text{des}(\pi)+1}.
\]

Below, we let \( \lambda^2 = (\lambda_1^2, \lambda_2^2, \ldots) \) be the cycle type of \( \pi^2 \) for any permutation \( \pi \) of cycle type \( \lambda \). We denote the parts of the partition \( \lambda^2 \) by \( \lambda_1^2, \lambda_2^2, \ldots \) (so \( \lambda_k^2 \) is not the same as \( (\lambda_k)^2 \)).

**Theorem 5.9.** For \( n \geq 1 \), we have

\[
\frac{1}{1 + y \left( \frac{1 + yt}{1 - t} \right)^{n+1}} I_n^{(pk, des)} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt} \right) = \sum_{\lambda \vdash n} \frac{A_l(\lambda^2)(t)}{z_{\lambda}(1 - t)^{l(\lambda^2)+1}} \prod_{k=1}^{l(\lambda^2)} (1 - (-y)^{\lambda_k^2}).
\]

**Proof.** First we set \( z = 1 \) in (5.3), which yields

\[
Q_I|_{z=1} = \exp \left( \sum_{n \text{ odd}} x^n \frac{P_n}{n} + \sum_{n=1}^{\infty} x^n \frac{P_n^2}{2n} \right)
= \exp \left( \sum_{n \text{ odd}} x^n \frac{P_n}{n} + \sum_{n \text{ even}} x^n \frac{P_n^2}{n} \right).
\]

By Lemma 2.4 we have

\[
Q(\mathfrak{I}_n) = \sum_{\lambda \vdash n} \frac{q_{\lambda}}{z_{\lambda}} \tag{5.5}
\]
where \( q_\lambda = \prod_{k=1}^{l(\lambda)} q_{\lambda_k} \) with \( q_n = p_n \) if \( n \) is odd and \( q_n = p_n^2 \) if \( n \) is even. Since the square of a cycle of odd length \( n \) is a cycle of length \( n \), and the square of a cycle of even length \( n \) is a product of two cycles of length \( n/2 \), it follows that \( q_\lambda = p_{\lambda^2} \). Thus we may rewrite (5.5) as

\[
Q(I_n) = \sum_{\lambda \vdash n} p_{\lambda^2} z_\lambda
\]  (5.6)

and the desired formula follows from Theorem 3.1 (a) and (5.6).

Theorem 5.9 can also be proven using the approach of Athanasiadis (see the proof of [2, Proposition 2.22]). Moreover, we note that by setting \( y = 1 \) and \( y = 0 \), Theorem 5.9 can be used to obtain Corollary 5.7.

Finally, we note the following symmetry result on the joint distribution of \( pk \) and \( des \) over \( I_n \), similar to Theorem 4.2.

**Theorem 5.10.** The number of involutions of length \( n \) with \( j \) peaks and \( k \) descents is equal to the number of involutions of length \( n \) with \( j \) peaks and \( n - 1 - k \) descents.

The proof of Theorem 5.10 is the same as that of Theorem 4.2 but uses the next lemma in place of Lemma 4.4.

**Lemma 5.11.** The number of involutions of length \( n \) with descent set \( D \subseteq [n - 1] \) is equal to the number of involutions of length \( n \) with the complementary descent set \( D^c \).

Lemma 5.11, like Lemma 4.4, was first proven by Gessel and Reutenauer [18, Theorem 4.2] and later proven bijectively by Steinhardt [38].

**5.2. Counting involutions by left peaks, descents, and fixed points**

We now give an analogue of Theorem 5.2 for the polynomials

\[
I_n^{(pk, des, fix)}(y, t, z) := P^{(pk, des, fix)}(I_n; y, t, z) = \sum_{\pi \in I_n} y^{lpk(\pi)} t^{des(\pi)} z^{fix(\pi)}.
\]

**Theorem 5.12.**

\[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + yt)^n}{(1 - t)^{n+1}} I_n^{(pk, des, fix)}(y, t, z) = \sum_{k=0}^{\infty} \frac{(1 + zxy)^k(1 + x^2y)^{k+2} + k + k}{(1 - zx)^{k+1}(1 - x^2)(1 - x^2y^2)(1 - x^2)^{k+1}}.
\]

The proof of Theorem 5.12 proceeds in the same way as the proof for Theorem 5.1, but we use Theorem 3.6 in place of Theorem 3.3 and we use the next lemma in place of Lemma 3.3; the details are omitted.

**Lemma 5.13.**

\[
Q_I[X + 1][k(1 - \alpha)] = \frac{(1 - zx\alpha)^k(1 - x^2\alpha)^{k+2} + k}{(1 - zx)^{k+1}(1 - x^2)(1 - x^2\alpha^2)^{k+1}}.
\]
Proof. First, observe that
\[
(zxe_1 + x^2e_2)[X + 1] = zxp_1[X + 1] + \frac{x^2}{2}(p_1^2 - p_2)[X + 1]
\]
\[
= zx(p_1 + 1) + \frac{x^2}{2}((p_1 + 1)^2 - (p_2 + 1))
\]
\[
= zx(p_1 + 1) + \frac{x^2}{2}(p_1^2 + 2p_1 - p_2).
\]
Then, following the proof of Lemma 5.3, we have
\[
(zxe_1 + x^2e_2)[X + 1][k(1 - \alpha)] = \left(zx(p_1 + 1) + \frac{x^2}{2}(p_1^2 + 2p_1 - p_2)\right)[k(1 - \alpha)]
\]
\[
= zx(k(1 - \alpha) + 1) + \frac{x^2}{2}(k(1 - \alpha)^2 + 2k(1 - \alpha) - k(1 - \alpha^2))
\]
\[
= (k + 1)zx - kzx\alpha + \left(\frac{k + 1}{2}\right)x^2 + \left(\frac{k + 1}{2}\right)x^2\alpha^2 - (k^2 + k)x^2\alpha
\]
and thus
\[
Q_l[X + 1][k(1 - \alpha)] = H[\phi zxe_1 + x^2e_2][X + 1][k(1 - \alpha)]
\]
\[
= H\left[(k + 1)zx - kzx\alpha + \left(\frac{k + 1}{2}\right)x^2 + \left(\frac{k + 1}{2}\right)x^2\alpha^2 - (k^2 + k)x^2\alpha\right]
\]
\[
= \frac{H[\phi zxe_1 + x^2e_2][X + 1][k(1 - \alpha)]}{H[\phi zxe_1 + x^2e_2][X + 1][k(1 - \alpha)]}
\]
\[
= \frac{(1 - z\alpha)k(1 - x^2\alpha)^{k^2 + k}}{(1 - z\alpha)k(1 - x^2\alpha)^{k^2 + k}}.
\]
Define
\[
I_{n}(t, z) := \sum_{\pi \in \mathcal{P}_n} t^{\lambda k(\pi)} z^{\lambda(\pi)}.
\]
Specializing Theorem 5.12 at \(y = 1\) yields the following corollary.

Corollary 5.14.
\[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t)^n}{(1 - t)^{n+1}} I_{n}(t, z) = \sum_{k=0}^{\infty} \frac{(1 + z\alpha)^k(1 + x^2)^{k^2 + k}}{(1 - z\alpha)^k(1 - x^2)^{k^2 + k}}.
\]

We also state an analogue of Theorem 5.6 for left peaks. We omit the proof as it is similar to that of Theorem 5.6.

Theorem 5.15. Let \(n \geq 1\). Then
\[
\frac{(1 + t)^n}{(1 - t)^{n+1}} I_{n}(t, z) = \sum_{k=0}^{n} d_{n,k}(z) B_k(t) \frac{1}{(1 - t)^{k+1}}
\]

where
\[
1 + \sum_{n=1}^{\infty} x^n \sum_{k=0}^{n} d_{n,k}(z) w^k = \frac{1}{(1 - x^4)^{1/4}} \frac{1 + x^2}{1 - x^2} w^{2/4} \left(\frac{1 + z\alpha w}{1 - z\alpha w}\right)^{1/2} \left(\frac{1 - x^2}{1 - z^2 x^2}\right)^{1/2}.
\]
As with Theorem 5.6, we have a simple combinatorial interpretation when \( z = 1 \): if we set \( d_{n,k} := d_{n,k}(1) \), then \( n! d_{n,k} \) is the number of permutations \( \pi \in S_n \) for which \( \pi^2 \) has \( k \) odd cycles (i.e., cycles of odd length). Thus we have the following formula for the polynomials

\[
I_{lpk}^n(t) := P_{lpk}^n(J_n; t) = \sum_{\pi \in \mathcal{J}_n} \ell_{lpk}(\pi).
\]

**Corollary 5.16.** Let \( n \geq 1 \). Then

\[
\frac{(1 + t)^n}{(1 - t)^{n+1}} I_{lpk}^n \left( \frac{4t}{(1 + t)^2} \right) = \sum_{k=0}^{n} d_{n,k} \frac{B_k(t)}{(1 - t)^{k+1}}
\]

where \( n! d_{n,k} \) is the number of permutations \( \pi \in S_n \) for which \( \pi^2 \) has \( k \) odd cycles.

Corollary 5.16 leads to an expression for \( I_{lpk}^n(t) \) in terms of the ordinary left peak polynomials \( P_{lpk}^n(t) \). The proof is similar to that of Corollary 4.9.

**Corollary 5.17.** Let \( n \geq 1 \). Then

\[
I_{lpk}^n(t) = \sum_{k=0}^{n} d_{n,k} (1 - t)^{(n-k)/2} P_{k}^n(t)
\]

with \( d_{n,k} \) as in Corollary 5.16.

For \( d_{n,k} \) to be nonzero, \( n \) and \( k \) must have the same parity, so that \( (n - k)/2 \) must be an integer. Therefore, like the formula in Corollary 5.8, the above formula does not contain any square roots.

### 5.3. Counting involutions by up-down runs and fixed points

We conclude this section with analogous results for counting involutions by the number of up-down runs. Define

\[
I_{n}^{(udr, fix)}(t, z) := P_{n}^{(udr, fix)}\left(J_n; t, z\right) = \sum_{\pi \in \mathcal{J}_n} t^{udr(\pi)} z^{fix(\pi)}
\]

and

\[
I_{n}^{udr}(t) := P_{n}^{udr}\left(J_n; t\right) = \sum_{\pi \in \mathcal{J}_n} t^{udr(\pi)}.
\]

**Theorem 5.18.**

\[
\frac{1}{1 - t} + \frac{1}{2(1 - t)^2} \sum_{n=1}^{\infty} \frac{(1 + t^2)^n}{(1 - t^2)^{n-1}} I_{n}^{(udr, fix)}(\frac{2t}{1 + t^2}, z) x^n
\]

\[
= \sum_{k=0}^{\infty} (1 + zx)^k (1 + x^2)^{k^2} t^{2k} (1 + (1 + x^2)^k t) \left( \frac{1}{1 - zx} \right) \left( \frac{1}{1 - x^2} \right).
\]
Theorem 5.19. Let \( n \geq 1 \). Then
\[
\frac{(1 + t^2)^n}{2(1 - t)^2(1 - t^2)^{n-1}} I_{\text{udr}} \left( \frac{2t}{1 + t^2} \right) = \sum_{k=1}^{n} a_{n,k} 2^k A_k(t^2) + t \sum_{k=0}^{n} d_{n,k} B_k(t^2)
\]
with \( a_{n,k} \) and \( d_{n,k} \) as in Corollaries 5.7 and 5.16.

Theorem 5.18 is proved using Theorem 3.8, Lemma 5.3, and Lemma 5.13, whereas the proof of Theorem 5.19 uses Theorem 3.7. We omit the details.

6. Fixed points and derangements

Let \( D_n \) denote the set of derangements—permutations with no fixed points—of length \( n \). In this section, we will prove formulas for the joint distribution of \( pk, \text{des}, \) and \( \text{fix} \), the joint distribution of \( lpk, \text{des}, \) and \( \text{fix} \), and the joint distribution of \( \text{udr} \) and \( \text{fix} \) over all permutations and then specialize these results to the case of derangements.

The quasisymmetric generating function for all permutations weighted by length and number of fixed points is known to be
\[
Q := \sum_{n=0}^{\infty} Q_{\text{fix}}(\mathfrak{S}_n)x^n = \frac{H(zx)}{H(x)(1 - p_1x)};
\]
see the proof of \([18, \text{Theorem 8.4}]\).

6.1. Counting permutations by peaks, descents, and fixed points

Let us define
\[
P_n^{(pk,\text{des},\text{fix})}(y, t, z) := P_n^{(pk,\text{des},\text{fix})}(\mathfrak{S}_n; y, t, z) = \sum_{\pi \in \mathfrak{S}_n} y^{pk(\pi) + 1} t^{\text{des}(\pi) + 1} z^{\text{fix}(\pi)}
\]
and
\[
D_n^{(pk,\text{des})}(y, t) := P_n^{(pk,\text{des})}(D_n; y, t) = \sum_{\pi \in D_n} y^{pk(\pi) + 1} t^{\text{des}(\pi) + 1}.
\]

Our first theorem of this section provides generating function formulas for these polynomials.

Theorem 6.1. We have

(a)
\[
\frac{1}{1 - t} + \frac{1}{1 + y} \sum_{n=1}^{\infty} \left( \frac{1 + yt}{1 - t} \right)^{n+1} P_n^{(pk,\text{des},\text{fix})} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \cdot \frac{y + t}{1 + yt} \cdot z \right) x^n
\]
\[
= \sum_{k=0}^{\infty} \frac{(1 + zxy)^k}{(1 - zx)^k} \frac{(1 - x)^k}{(1 + xy)^k} \frac{t^k}{1 - k(1 + y)x}
\]
and

(b)
\[
\frac{1}{1 - t} + \frac{1}{1 + y} \sum_{n=1}^{\infty} \left( \frac{1 + yt}{1 - t} \right)^{n+1} D_n^{(pk,\text{des})} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \cdot \frac{y + t}{1 + yt} \right) x^n
\]
\[
= \sum_{k=0}^{\infty} \frac{(1 - x)^k}{(1 + xy)^k} \frac{t^k}{1 - k(1 + y)x}.
\]
Proof. First, note that

\[
\sum_{k=0}^{\infty} \Theta_{y,k}(Q) t^k = \sum_{k=0}^{\infty} \frac{\Theta_{y,k}(H(zx))}{\Theta_{y,k}(H(x)) \Theta_{y,k}(1-p_1 x)} t^k \\
= \sum_{k=0}^{\infty} \frac{(1+zx)^k (1-x)^k}{(1-zx)^k (1+xy)^k} \frac{t^k}{1-k(1+y)x}
\]

(6.1)

by Lemma [2.4] (d). Thus, we have

\[
\sum_{k=0}^{\infty} \frac{(1+zx)^k (1-x)^k}{(1-zx)^k (1+xy)^k} \frac{t^k}{1-k(1+y)x} = \sum_{k=0}^{\infty} \frac{\Theta_{y,k}(Q)}{\Theta_{y,k}(H(zx))} t^k
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Theta_{y,k}(Q^{\text{fix}}(S_n)) t^k x^n
\]

\[
= \frac{1}{1-t} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \Theta_{y,k}(Q^{\text{fix}}(S_n)) t^k x^n.
\]

(6.2)

Applying Theorem 3.3, we obtain

\[
\frac{1}{1+y} \left(1 \frac{1+yt}{1-t}\right)^{n+1} P_{\text{pk,fix}}(t,z) = \sum_{\pi \in \Theta_n} t^{\text{pk}(\pi)+1} z^{\text{fix}(\pi)}, \quad \frac{1}{1+y} \left(1 \frac{1+yt}{1-t}\right)^{n+1} A_{\text{fix}}(t,z) = \sum_{\pi \in \Theta_n} t^{\text{des}(\pi)+1} z^{\text{fix}(\pi)}
\]

combining this with (6.2) yields part (a). Part (b) is simply the \(z = 0\) specialization of part (a), so we are done.

Now, define

\[
P_{\text{pk,fix}}(t, z) := \sum_{\pi \in \Theta_n} t^{\text{pk}(\pi)+1} z^{\text{fix}(\pi)}, \quad A_{\text{fix}}(t, z) := \sum_{\pi \in \Theta_n} t^{\text{des}(\pi)+1} z^{\text{fix}(\pi)}
\]

\[
D_{\text{pk}}(t) := P_{\text{pk}}(D_n; t, z) = \sum_{\pi \in D_n} t^{\text{pk}(\pi)+1}, \quad D_n(t) := A(D_n; t) = \sum_{\pi \in D_n} t^{\text{des}(\pi)+1}.
\]

We easily obtain the following generating function formulas for these polynomials by specializing Theorem 6.1 appropriately.

**Corollary 6.2.** We have

(a) \(\frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \left(1 \frac{1+t}{1-t}\right)^{n+1} P_{\text{pk,fix}}(t,z)^{n+1} x^n = \sum_{k=0}^{\infty} \frac{(1+zx)^k (1-x)^k}{(1-zx)^k (1+x)^k} \frac{t^k}{1-k(1+y)x}\)

(b) \(\frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \left(1 \frac{1+t}{1-t}\right)^{n+1} D_{\text{pk}}(t)^{n+1} x^n = \sum_{k=0}^{\infty} \frac{(1-x)^k}{(1+x)^k} \frac{t^k}{1-k(1+y)x}\)
\[
\frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{A_{n}^{\text{fix}}(t, z)}{(1-t)^{n+1}} x^n = \sum_{k=0}^{\infty} \frac{(1-x)^k}{1-\theta z^k} t^k,
\]

and

\[
\frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{D_n(t)}{(1-t)^{n+1}} x^n = \sum_{k=0}^{\infty} \frac{(1-x)^k t^k}{1-\theta}. \tag{d}
\]

Note that part (c) of Corollary 6.2 is the \(q = 1\) evaluation of a formula by Gessel and Reutenauer [18, Equation (8.3)].

We now give formulas which express \(P_{n}(p_k, \text{fix})(t, z)\) and \(A_{n}^{\text{fix}}(t, z)\) in terms of Eulerian polynomials. We omit the proof as it is similar to the proof of Theorem 5.6.

**Theorem 6.3.** Let \(n \geq 1\). Then

(a) \[
\frac{1}{2} \left( \frac{1+t}{1-t} \right)^{n+1} P_n^{(p_k, \text{fix})} \left( \frac{4t}{(1-t)^2}; z \right) = \sum_{k=1}^{n} a_{n,k}(z) \frac{A_k(t)}{(1-t)^{k+1}} \]

where \[
1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} a_{n,k}(z) w^k = \frac{1}{1-wx} \left( \frac{1+zx}{1-zx} \right)^{w/2} \left( \frac{1-x}{1+x} \right)^{w/2},
\]

and

(b) \[
\frac{A_{n}^{\text{fix}}(t, z)}{(1-t)^{n+1}} = \sum_{k=1}^{n} b_{n,k}(z) \frac{A_k(t)}{(1-t)^{k+1}}
\]

where \[
1 + \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} b_{n,k}(z) w^k = \frac{1}{1-wx} \left( \frac{1-x}{1-zx} \right)^{w}.
\]

Theorem 6.3 can be used to obtain the next corollary; the proof is very similar to that of Corollary 4.6.

**Corollary 6.4.** Let \(n \geq 1\). Then

\[
P_n^{(p_k, \text{fix})}(t, z) = \sum_{k=1}^{n} a_{n,k}(z) (1-t)^{(n-k)/2} P_k^{p_k}(t)
\]

with \(a_{n,k}(z)\) as in Theorem 6.3.

Like the formulas in Corollaries 5.8 and 5.17, the above formula does not contain any square roots.

Specializing Theorem 6.3 and Corollary 6.4 at \(z = 0\) yields analogous results for derangements; we omit these formulas here. In addition, by specializing Theorem 6.3 (a) at \(z = 1\) and simplifying, we recover Stembridge’s [39] formula

\[
A_n(t) = \left( \frac{1+t}{2} \right)^{n+1} P_n^{p_k} \left( \frac{4t}{(1+t)^2} \right)
\]

relating the Eulerian and peak polynomials.
6.2. Counting permutations by left peaks, descents, and fixed points

Next, we prove analogous formulas for the polynomials

\[ P_{n}^{(\text{lpk,des,fix})}(y, t, z) := P_{n}^{(\text{lpk,des,fix})}(\mathfrak{S}_{n}; y, t, z) = \sum_{\pi \in \mathfrak{S}_{n}} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)} z^{\text{fix}(\pi)} \]

and

\[ D_{n}^{(\text{lpk,des})}(y, t) := P_{n}^{(\text{lpk,des})}(\mathfrak{D}_{n}; y, t) = \sum_{\pi \in \mathfrak{D}_{n}} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)}. \]

**Theorem 6.5.** We have

(a) \[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + yt)^{n}}{(1 - t)^{n+1}} P_{n}^{(\text{lpk,des,fix})} \left( \frac{(1 + y)^{2}t}{(y + t)(1 + yt)}, y + t, z \right) x^{n} = \sum_{k=0}^{\infty} \frac{(1 + zxy)^{k} (1 - x)^{k+1}}{(1 - zx)^{k+1} (1 + xy)^{k} 1 - (k(1 + y) + 1)x} t^{k} \]

and

(b) \[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + yt)^{n}}{(1 - t)^{n+1}} D_{n}^{(\text{lpk,des})} \left( \frac{(1 + y)^{2}t}{(y + t)(1 + yt)}, y + t \right) x^{n} = \sum_{k=0}^{\infty} \frac{(1 - x)^{k+1}}{(1 + xy)^{k} 1 - (k(1 + y) + 1)x} t^{k}. \]

The following lemma is needed for the proof of Theorem 6.5.

**Lemma 6.6.** Let \( m \) be a monic term. Then

(a) \( H(m)[X + 1] = H(m)/(1 - m) \),

(b) \( E(m)[X + 1] = (1 + m)E(m) \).

**Proof.** We have

\[
H(m)[X + 1] = H[mX + m] \quad \text{(by Lemma 2.3 (b))}
\]

\[
= H[mX]H[m] \quad \text{(by Lemma 2.3 (a))}
\]

\[
= \frac{H(m)}{1 - m} \quad \text{(by Lemma 2.4 (a) and (c))}
\]

which proves (a). To prove (b), first observe that (a) implies

\[
(H(-m)[X + 1])^{-1} = (1 + m)H(-m)^{-1}.
\]

Therefore

\[
E(m)[X + 1] = H(-m)^{-1}[X + 1]
\]

\[
= (H(-m)[X + 1])^{-1}
\]

\[
= (1 + m)H(-m)^{-1}
\]

\[
= (1 + m)E(m)
\]

and we are done. \( \square \)
Proof of Theorem 6.5. By Lemmas 6.6 (a) and 2.4 (d), we have
\[
\sum_{k=0}^{\infty} \Theta_{y,k}(Q[X + 1]) t^k = \sum_{k=0}^{\infty} \frac{\Theta_{y,k}(H(x)[X + 1])}{\Theta_{y,k}(H(x)[X + 1])} t^k
\]
\[
= \sum_{k=0}^{\infty} \frac{1 - x}{1 - z x} \Theta_{y,k}(H(x)) \Theta_{y,k}(1 - (p_1 + 1) x) t^k
\]
\[
= \sum_{k=0}^{\infty} \frac{(1 + z x y)^k (1 - x)^{k+1}}{(1 - z x)^{k+1} (1 + x y)^k} \frac{t^k}{1 - (k(1 + y) + 1) x}.
\]
(6.3)

Then
\[
\sum_{k=0}^{\infty} \frac{(1 + z x y)^k (1 - x)^{k+1}}{(1 - z x)^{k+1} (1 + x y)^k} \frac{t^k}{1 - (k(1 + y) + 1) x} = \sum_{k=0}^{\infty} \Theta_{y,k}(Q[X + 1]) t^k
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Theta_{y,k}(Q^n_{\text{fix}}(S)[X + 1]) t^k x^n
\]
\[
= \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \Theta_{y,k}(Q^n_{\text{fix}}(S)[X + 1]) t^k x^n,
\]
which by Theorem 3.6 is equal to
\[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + y t)^{n}}{(1 - t)^{n+1}} P_{n}^{(l_{p,k}, \text{des}, \text{fix})}
\left( \frac{(1 + y t)^2}{(y + t)(1 + y t)}; \frac{y + t}{1 + y t}; z \right) x^n,
\]
thus establishing part (a). Part (b) follows from setting \(z = 0\) in part (a).

Specializing Theorem 6.5 appropriately, we obtain generating function formulas for the polynomials
\[
P_{n}^{(l_{p,k}, \text{fix})}(t, z) := \sum_{\pi \in S_{n}} t^{l_{p,k}(\pi)} z^{\text{fix}(\pi)} \quad \text{and} \quad D_{n}^{(l_{p,k})}(t) := P_{n}^{(l_{p,k}, S)}(t) = \sum_{\pi \in S_{n}} t^{l_{p,k}(\pi)}.
\]

Corollary 6.7. We have
\[
(a) \quad \frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t)^{n}}{(1 - t)^{n+1}} P_{n}^{(l_{p,k}, \text{fix})}
\left( \frac{4t}{(1 + t)^2}; z \right) x^n
\]
\[
= \sum_{k=0}^{\infty} \frac{(1 + z x)^k (1 - x)^{k+1}}{(1 - z x)^{k+1} (1 + x)^k} \frac{t^k}{1 - (2k + 1) x}
\]
and
\[
(b) \quad \frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t)^{n}}{(1 - t)^{n+1}} D_{n}^{(l_{p,k})}
\left( \frac{4t}{1 + t} \right) x^n = \sum_{k=0}^{\infty} \frac{(1 - x)^{k+1}}{(1 + x)^k} \frac{t^k}{1 - (2k + 1) x}.
\]

We omit the proofs of the next theorem and its corollary, as they are very similar to the proofs of analogous results from previous sections.
Theorem 6.8. Let \( n \geq 1 \). Then

\[
\frac{(1 + t)^n}{(1 - t)^{n+1}} P_n^{(lpk, \text{fix})} \left( \frac{4t}{(1 + t)^2}, z \right) = \sum_{k=0}^{n} d_{n,k}(z) \frac{B_k(t)}{(1 - t)^{k+1}}
\]

where

\[
1 + \sum_{n=1}^{\infty} x^n \sum_{k=0}^{n} d_{n,k}(z) w^k = \frac{1}{1 - x w} \left( \frac{1 - x^2}{1 - z^2 x^2} \right)^{1/2} \left( \frac{1 + z x}{1 - z x} \right)^{w/2} \left( \frac{1 - x}{1 + x} \right)^{w/2}.
\]

Corollary 6.9. Let \( n \geq 1 \). Then

\[
P_n^{(lpk, \text{fix})} (t, z) = \sum_{k=0}^{n} d_{n,k}(z) (1 - t)^{(n-k)/2} P_k^{lpk}(t)
\]

with \( d_{n,k}(z) \) as in Theorem 6.8.

Like the formulas in Corollaries 5.8, 5.17, and 6.4, the above formula does not contain any square roots.

We note that setting \( z = 1 \) in Theorem 6.8 recovers Petersen’s formula \cite{Petersen} Proposition 4.15]

\[
B_n(t) = (1 + t)^n P_n^{lpk} \left( \frac{4t}{(1 + t)^2} \right)
\]

relating the type B Eulerian and left peak polynomials.

### 6.3. Counting permutations by up-down runs and fixed points

Finally, let us present analogous formulas for the polynomials

\[
P_n^{(udr, \text{fix})} (t, z) := \sum_{\pi \in S_n} t^{\text{udr}(\pi)} z^{\text{fix}(\pi)} \quad \text{and} \quad D_n^{\text{udr}} (t) := P_n^{\text{udr}} (D_n; t) = \sum_{\pi \in D_n} t^{\text{udr}(\pi)}.
\]

The proofs are omitted as they are very similar to that of analogous results from previous sections.

**Theorem 6.10.** We have

(a) \[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t^2)^n}{2(1 - t)^2(1 - t^2)^{n-1}} P_n^{(udr, \text{fix})} \left( \frac{2t}{1 + t^2}, z \right)
= \sum_{k=0}^{\infty} \frac{(1 + z x)^k (1 - x)^{k} t^{2k}}{(1 - z x)(1 + x)^k} \left( \frac{1}{1 - 2k x} + \frac{1 - x}{1 - z x} \frac{t}{1 - (2k + 1)x} \right)
\]

and

(b) \[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t^2)^n}{2(1 - t)^2(1 - t^2)^{n-1}} D_n^{\text{udr}} \left( \frac{2t}{1 + t^2} \right)
= \sum_{k=0}^{\infty} \frac{(1 - x)^{k} t^{2k}}{(1 + x)^k} \left( \frac{1}{1 - 2k x} + \frac{(1 - x)t}{1 - (2k + 1)x} \right).
\]
Theorem 6.11. Let \( n \geq 1 \). Then
\[
\frac{(1 + t^2)^n}{2(1 - t)^2(1 - t^2)^{n-1}} P_n^{(\text{udr,fix})}(\frac{2t}{1 + t^2}, z) = \sum_{k=1}^{n} a_{n,k}(z) 2^k \frac{A_k(t^2)}{(1 - t^2)^{k+1}} + t \sum_{k=0}^{n} d_{n,k}(z) \frac{B_k(t^2)}{(1 - t^2)^{k+1}}
\]
with \( a_{n,k}(z) \) and \( d_{n,k}(z) \) as in Theorems 6.3 and 6.8.

Let
\[
P_n^{\text{udr}}(t) := P_n^{\text{udr}}(\mathfrak{S}_n; t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{udr}(\pi)}.
\]

We note that setting \( z = 1 \) in Theorem 6.11 yields
\[
2^n A_n(t^2) + tB_n(t^2) = \frac{(1 + t)^2(1 + t^2)^n}{2} P_n^{\text{udr}}(\frac{2t}{1 + t^2}),
\]
which is equivalent to the formula
\[
\hat{B}_n(t) = \frac{(1 + t)(1 + t^2)^n}{2t} P_n^{\text{udr}}(\frac{2t}{1 + t^2})
\]
previously obtained by the second author [43, Corollary 4.18]. Here, \( \hat{B}_n(t) \) is the nth flag descent polynomial, which encodes the distribution of the flag descent number over the nth hyperoctahedral group; see [43, Section 2.3] for details.

7. Cycle type

In Section 6, we derived formulas for counting permutations by the number of fixed points jointly with the peak number and descent number, with the left peak number and descent number, and with the number of up-down runs. We will now extend these results by deriving formulas for counting permutations by these statistics along with cycle type.

Given a permutation \( \pi \), let \( N_i(\pi) \) denote the number of \( i \)-cycles in \( \pi \). Similarly, given a partition \( \lambda \), let \( N_i(\lambda) \) denote the number of parts of size \( i \) in \( \pi \). Recall that the Lyndon symmetric function
\[
L_\lambda = h_{m_1}[L_1] h_{m_2}[L_2] \cdots
\]
is the quasisymmetric generating function for the set of permutations with cycle type \( \lambda = (1^{m_1}2^{m_2} \cdots) \). Then
\[
P := 1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash n} L_\lambda x^n \prod_{i=1}^{\infty} z_i^{N_i(\lambda)}
\]
\[
= \sum_{m_1,m_2,\ldots} \prod_{i=1}^{\infty} h_{m_i}[L_i](z_i x^i)^{m_i}
\]
\[
= \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} h_{m_i}[L_i](z_i x^i)^{m_i}
\]
is the quasisymmetric generating function for all permutations refined by cycle type and length.
7.1. Counting permutations by peaks, descents, and cycle type

Define
\[ F_{n}^{(pk,\text{des})}(y, t, z_1, z_2, \ldots) := \sum_{\pi \in S_n} y^{pk(\pi)+1} t^{\text{des}(\pi)+1} \prod_{i=1}^{\infty} z_i^{N_i(\pi)}, \]
\[ F_{n}^{pk}(t, z_1, z_2, \ldots) := \sum_{\pi \in S_n} t^{pk(\pi)+1} \prod_{i=1}^{\infty} z_i^{N_i(\pi)}, \]
and
\[ F_{n}^{\text{des}}(t, z_1, z_2, \ldots) := \sum_{\pi \in S_n} t^{\text{des}(\pi)+1} \prod_{i=1}^{\infty} z_i^{N_i(\pi)}. \]

Let us also define the numbers
\[ f_{i,k} := \frac{1}{i} \sum_{d \mid i} \mu(d) k^{i/d} \quad \text{and} \quad g_{i,k} := \frac{1}{2i} \sum_{d \mid i \text{ odd}} \mu(d) (2k)^{i/d}, \]
which will appear in our formulas for these cycle type polynomials. It is known that \( f_{i,k} \) is the number of primitive necklaces of length \( i \) from the alphabet \([k]\) (see [32]) and that \( g_{i,k} \) is the number of nowhere-zero primitive twisted necklaces of length \( i \) from the alphabet \( \pm[k] := \{\pm 1, \pm 2, \ldots, \pm k\} \) (see [14, Remark 3.2]).

**Theorem 7.1.** We have

(a) \[ \frac{1}{1-t} + \frac{1}{1+y} \sum_{n=1}^{\infty} \left( \frac{1+y t}{1-t} \right)^{n+1} F_{n}^{(pk,\text{des})} \left( \frac{(1+y)^2 t^2}{(y+t)(1+y t)}, y + t, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} t^k \prod_{i=1}^{\infty} \left( \frac{1 + z_i x^i}{1 - z_i x^i} \right)^{g_{i,k}}, \]

(b) \[ \frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1+t}{1-t} \right)^{n+1} F_{n}^{pk} \left( \frac{4t}{(1+t)^2}, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} t^k \prod_{i=1}^{\infty} \left( \frac{1 + z_i x^i}{1 - z_i x^i} \right)^{f_{i,k}}, \]

and

(c) \[ \frac{1}{1-t} + \sum_{n=1}^{\infty} F_{n}^{\text{des}}(t, z_1, z_2, \ldots) x^n = \sum_{k=0}^{\infty} t^k \prod_{i=1}^{\infty} \left( \frac{1}{1 - z_i x^i} \right)^{f_{i,k}}. \]

Part (b) is due to Diaconis, Fulman, and Holmes [8, Corollary 3.8] and part (c) to Fulman [12, Theorem 1].

**Proof.** It is easy to show that

\[ \frac{1}{1-t} + \frac{1}{1+y} \sum_{n=1}^{\infty} \left( \frac{1+y t}{1-t} \right)^{n+1} F_{n}^{(pk,\text{des})} \left( \frac{(1+y)^2 t^2}{(y+t)(1+y t)}, y + t, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} \Theta_{y,k}(P) t^k \quad (7.1) \]
(cf. the proofs of Theorems 3.3 and 6.1 (a)), so we shall need to compute \( \Theta_{y,k}(P) \). Since

\[
L_i[k(1-\alpha)] = \frac{1}{i} \sum_{d|i} \mu(d)(k(1-\alpha^d))^{i/d},
\]

we have

\[
P[k(1-\alpha)] = \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} h_{m_i} [L_i][k(1-\alpha)](z_i x^i)^{m_i}
\]

\[
= \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} h_{m_i} (z_i x^i)^{m_i} \left[ \frac{1}{i} \sum_{d|i} \mu(d)(k(1-\alpha^d))^{i/d} \right]
\]

\[
= \prod_{i=1}^{\infty} \exp \left( \sum_{m_i=1}^{\infty} \frac{1}{m_i} \sum_{d|i} \mu(d)(k(1-\alpha^d))^{i/d} \right) (z_i x^i)^{m_i}
\]

\[
= \prod_{i=1}^{\infty} \exp \left( \sum_{m_i=1}^{\infty} \frac{(z_i x^i)^{m_i}}{i m_i} \sum_{d|i} \mu(d)(k(1-\alpha^{d m_i}))^{i/d} \right). \quad (7.2)
\]

Combining (7.1) with (7.2) yields part (a).

To prove part (b), we set \( y = 1 \) in (7.1) to obtain

\[
\frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1+t}{1-t} \right)^{n+1} F_n^{p_k} \left( \frac{4t}{(1+t)^2}, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} \Theta_{1,k}(P) t^k. \quad (7.3)
\]

Setting \( \alpha = -1 \) in (7.2), we obtain \( 1 - \alpha^{d m_i} = 2 \) if both \( d \) and \( m_i \) are odd and \( 1 - \alpha^{d m_i} = 0 \) otherwise, so

\[
\Theta_{1,k}(P) = \prod_{i=1}^{\infty} \exp \left( \sum_{m_i=1}^{\infty} \frac{(z_i x^i)^{m_i}}{i m_i} \sum_{d|i} \mu(d)(2k)^{i/d} \right)
\]

\[
= \prod_{i=1}^{\infty} \exp \left( \sum_{m_i=1}^{\infty} \frac{2(z_i x^i)^{m_i}}{m_i} g_{i,k} \right)
\]

\[
= \prod_{i=1}^{\infty} \exp \left( \sum_{m_i=1}^{\infty} \frac{(z_i x^i)^{m_i}}{i m_i} - \sum_{m_i=1}^{\infty} \frac{(-z_i x^i)^{m_i}}{i m_i} g_{i,k} \right)
\]

\[
= \prod_{i=1}^{\infty} \exp \left( \log(1 - z_i x^i) - \log(1 + z_i x^i) \right) g_{i,k}
\]

\[
= \prod_{i=1}^{\infty} \left( \frac{1 + z_i x^i}{1 - z_i x^i} \right) g_{i,k}. \quad (7.4)
\]

Combining (7.3) and (7.4) yields part (b). Part (c) can be proved similarly to part (b), but we set \( y = 0 \) and \( \alpha = 0 \) instead of \( y = 1 \) and \( \alpha = -1 \).
7.2. Counting permutations by left peaks, descents, and cycle type

We now prove an analogue of Theorem 7.1 for left peaks. Let

\[ F_n^{\text{lpk,des}}(y, t, z_1, z_2, \ldots) := \sum_{\pi \in S_n} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)} \prod_{i=1}^{\infty} z_i^{N_i(\pi)} \]

and

\[ F_n^{\text{lpk}}(t, z_1, z_2, \ldots) := \sum_{\pi \in S_n} t^{\text{lpk}(\pi)} \prod_{i=1}^{\infty} z_i^{N_i(\pi)}. \]

Also, let

\[ h_{i,k} := \begin{cases} (1 + 2k)^i - 1, & \text{if } i = 2^j \text{ with } j \geq 0 \\ \frac{1}{2i} \sum_{d \mid i, d \text{ odd}} \mu(d)(1 + 2k)^{i/d}, & \text{otherwise.} \end{cases} \]

The quantity \( h_{i,k} \) has a nice combinatorial interpretation as the number of primitive blinking necklaces of length \( i \) from the alphabet \( \{0\} \cup \pm[k] = \{0, \pm 1, \ldots, \pm k\}; \) these were first studied by Reiner [31] and have also appeared in work by Fulman [13].

**Theorem 7.2.** We have

(a) \[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + yt)^n}{(1 - t)^{n+1}} F_n^{\text{lpk,des}} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}, z_1, z_2, \ldots \right) x^n
= \sum_{k=0}^{\infty} t^k \prod_{i=1}^{\infty} \exp \left( \sum_{m_i=1}^{\infty} \frac{(z_i x^i)^{m_i}}{im_i} \sum_{d \mid i, d \text{ odd}} \mu(d)(1 + k(1 - (-y)^{dm_i}))^{i/d} \right)
\]

and

(b) \[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t)^n}{(1 - t)^{n+1}} F_n^{\text{lpk}} \left( \frac{4t}{(1 + t)^2}, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} \frac{t^k}{1 - z_1 x} \prod_{i=1}^{\infty} \left( 1 - \frac{z_i x^i}{1 - z_i x^i} \right)^{h_{i,k}}.
\]

**Proof.** First, it is easy to show that

\[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + yt)^n}{(1 - t)^{n+1}} F_n^{\text{lpk,des}} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}, z_1, z_2, \ldots \right) x^n
= \sum_{k=0}^{\infty} \Theta_{y,k}(P[X + 1]) t^k \quad (7.5)
\]

so we proceed by computing \( \Theta_{y,k}(P[X + 1]) \). We have

\[
L_i[X + 1][k(1 - \alpha)] = \frac{1}{i} \sum_{d \mid i} \mu(d)(1 + p_d)^{i/d} [k(1 - \alpha)]
= \frac{1}{i} \sum_{d \mid i} \mu(d)(1 + k(1 - \alpha^d))^{i/d};
\]
thus

\[
P[X + 1][k(1 - \alpha)] = \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} h_{m_i}[L_i][X + 1][k(1 - \alpha)][(z_i x^i)^{m_i}]
\]

\[
= \prod_{i=1}^{\infty} \sum_{m_i=0}^{\infty} h_{m_i}(z_i x^i)^{m_i}\left[\frac{1}{i} \sum_{d|i} \mu(d) (1 + k(1 - \alpha^d))^{i/d}\right]
\]

\[
= \prod_{i=1}^{\infty} \exp\left(\sum_{m_i=1}^{\infty} \frac{p_{m_i}}{m_i} \left[\frac{1}{i} \sum_{d|i} \mu(d) (1 + k(1 - \alpha^d))^{i/d}\right] (z_i x^i)^{m_i}\right)
\]

\[
= \prod_{i=1}^{\infty} \exp\left(\sum_{m_i=1}^{\infty} \left(\frac{z_i x^i}{i m_i} \sum_{d|i} \mu(d)(1 + k(1 - \alpha^{d m_i}))^{i/d}\right)\right).
\]

Then part (a) follows from (7.5) and (7.6).

To prove part (b), we first set \(y = 1\) in (7.5) to obtain

\[
\frac{1}{1 - t} + \sum_{n=1}^{\infty} \frac{(1 + t)^n}{(1 - t)^{n+1}} F_n^{l_{n,k}}\left(\frac{4t}{(1 + t)^2}; z_1, z_2, \ldots\right) x^n = \sum_{k=0}^{\infty} \Theta_{1,k}(P[X + 1]) t^k.
\]

(7.7)

Setting \(\alpha = -1\) in (7.6), we have \(1 + k(1 - \alpha^{d m_i}) = 1 + 2k\) if both \(d\) and \(m_i\) are odd and \(1 + k(1 - \alpha^{d m_i}) = 1\) otherwise. Hence, we have

\[
\Theta_{1,k}(P[X + 1]) = \prod_{i=1}^{\infty} \exp\left(\sum_{m_i \geq 2 \text{ even}} \frac{(z_i x^i)^{m_i}}{i m_i} \sum_{d|i} \mu(d)\right)
\]

\[
\times \exp\left(\sum_{m_i \geq 1 \text{ odd}} \frac{(z_i x^i)^{m_i}}{i m_i} \left(\sum_{d|i \text{ even}} \mu(d) + \sum_{d|i \text{ odd}} \mu(d)(1 + 2k)^{i/d}\right)\right).
\]

To simplify this expression, let us define

\[
h'_{i,k} := \begin{cases} 1/2 + h_{i,k}, & \text{if } i = 1, \\ h_{i,k}, & \text{otherwise,} \end{cases}
\]

and recall that \(\sum_{d|i} \mu(d) = 1\) if \(i = 1\) and \(\sum_{d|i} \mu(d) = 0\) otherwise. Applying this fact and Lemma 4.8, we obtain

\[
\Theta_{1,k}(P[X + 1]) = \exp\left(\sum_{m_i \geq 2 \text{ even}} \frac{(z_i x^i)^{m_i}}{m_i}\right) \prod_{i=1}^{\infty} \exp\left(\sum_{m_i \geq 1 \text{ odd}} \frac{2(z_i x^i)^{m_i}}{m_i} h'_{i,k}\right)
\]

\[
= \exp\left(\frac{1}{2} \log \frac{1}{1 - z_1^2 x^2}\right) \prod_{i=1}^{\infty} \exp\left(\sum_{m_i \geq 1 \text{ odd}} \frac{2(z_i x^i)^{m_i}}{m_i} h'_{i,k}\right).
\]
\[
\begin{align*}
&= \left( \frac{1}{1-z_1^2x^2} \right)^{\frac{1}{2}} \prod_{i=1}^{\infty} \left( \frac{1+z_i x^i}{1-z_i x^i} \right)^{h_{i,k}} \\
&= \left( \frac{1+z_1 x}{(1-z_1^2x^2)(1-z_1 x)} \right)^{\frac{1}{2}} \prod_{i=1}^{\infty} \left( \frac{1+z_i x^i}{1-z_i x^i} \right)^{h_{i,k}} \\
&= \frac{1}{1-z_1 x} \prod_{i=1}^{\infty} \left( \frac{1+z_i x^i}{1-z_i x^i} \right)^{h_{i,k}}.
\end{align*}
\]

Then combining (7.7) and (7.8) yields part (b). \hfill \square

### 7.3. Counting permutations by up-down runs and cycle type

We end with the analogous formula for the polynomials

\[
F_{\text{udr}}^n(t, z_1, z_2, \ldots) := \sum_{\pi \in S_n} t^{\text{udr}(\pi)} \prod_{i=1}^{\infty} z_i^{N_i(\pi)}.
\]

**Theorem 7.3.**

\[
\frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{(1+t^2)^n}{2(1-t)^2(1-t^2)^n-1} F_{\text{udr}}^n \left( \frac{2t}{1+t^2}, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} t^{2k+1} \prod_{i=1}^{\infty} \left( \frac{1+z_i x^i}{1-z_i x^i} \right)^{h_{i,k}}.
\]

**Proof.** It is easy to show that

\[
\frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{(1+t^2)^n}{2(1-t)^2(1-t^2)^n-1} F_{\text{udr}}^n \left( \frac{2t}{1+t^2}, z_1, z_2, \ldots \right) x^n = \sum_{k=0}^{\infty} \Theta_{1,k}(P)t^{2k} + \sum_{k=0}^{\infty} \Theta_{1,k}(P[X+1])t^{2k+1}.
\]

Then combining this equation with (7.4) and (7.8) yields the result. \hfill \square

### 8. Conclusion

In summary, we have derived general formulas for studying the joint distribution of the peak number and descent number, the joint distribution of the left peak number and the descent number, and the distribution of the number of up-down runs over any subset \(\Pi \subseteq S_n\) whose quasisymmetric function \(Q(\Pi)\) is symmetric. Furthermore, we have applied these results to produce more concrete formulas for distributions of these statistics over cyclic permutations, involutions, derangements, as well as jointly with the number of fixed points and with cycle type over all permutations.
There is a sizable literature on families of permutations whose quasisymmetric generating functions are symmetric. In addition to conjugacy classes of the symmetric group (i.e., permutations with fixed cycle type), these are known to include inverse descent classes and Knuth classes [18], sets of permutations with fixed inversion number [11], and more recently, certain classes of pattern-avoiding permutations [3, 4, 10, 23]. In ongoing work, the present authors are applying this property for inverse descent classes to study various distributions involving “inverse descent statistics”. One of our preliminary results is a rederivation of the formula

$$\sum_{\pi \in S_n} s^{\text{des}(\pi)} t^{\text{des}(\pi^{-1})} \frac{1}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{j,k=0}^{\infty} \binom{jk+n-1}{n} s^j t^k$$

originally due to Carlitz, Roselle, and Scoville [6] (see also [29]) for the joint distribution of the descent number and “inverse descent number” over $S_n$, and we have proved new formulas for similar “two-sided” bidistributions such that of the peak number and “inverse peak number” over $S_n$. It is also worth using the methods developed in this paper to study other families of permutations with symmetric quasisymmetric generating functions, and the case of pattern avoidance classes appears particularly interesting.

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