Variational solutions to nonlinear stochastic differential equations in Hilbert spaces

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Abstract

One introduces a new variational concept of solution for the stochastic differential equation \(dX + A(t)X \, dt + \lambda X \, dt = X \, dW, \, t \in (0, T); \)
\(X(0) = x\) in a real Hilbert space where \(A(t) = \partial \varphi(t), \, t \in (0, T)\), is a maximal monotone subpotential operator in \(H\) while \(W\) is a Wiener process in \(H\) on a probability space \(\{\Omega, \mathcal{F}, \mathbb{P}\}\). In this new context, the solution \(X = X(t, x)\) exists for each \(x \in H\), is unique, and depends continuously on \(x\). This functional scheme applies to a general class of stochastic PDE not covered by the classical variational existence theory [15], [16], [17] and, in particular, to stochastic variational inequalities and parabolic stochastic equations with general monotone nonlinearities with low or superfast growth to \(+\infty\).

Keywords: Brownian motion, maximal monotone operator, subdifferential, random differential equation, minimization problem.

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1 Introduction

Here, for $\lambda \in (0, \infty)$, we consider the stochastic differential equation

\[dX(t) + A(t)X(t)dt + \lambda X(t)dt \ni X(t)dW_t, \quad t \in (0, T),\]
\[X(0) = x \in H,\]

in a real Hilbert space $H$ whose elements are generalized functions on a bounded domain $O \subset \mathbb{R}^d$ with a smooth boundary $\partial O$. In examples, we have in mind that $H$ is e.g. $L^2(O)$ or $H^1_0(O)$, $H^1(O)$, $H^{-1}(O)$.

The norm of $H$ is denoted by $| \cdot |_H$, its scalar product by $(\cdot, \cdot)$ and its Borel $\sigma$-algebra by $\mathcal{B}(H)$.

$W$ is a Wiener process of the form

\[W(t, \xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi) \beta_j(t), \quad \xi \in O, \quad t \geq 0,\]

where $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of real $(\mathcal{F}_t)$-Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$ and $\{e_j\}$ is an orthonormal basis in $H$ such that both $c_j$ and $e_j^2$, $j \in \mathbb{N}$, are multipliers in $H$, while $\mu_j \in \mathbb{R}$, $j = 1, 2, \ldots$, satisfy (1.9) below.

As regards the nonlinear (multivalued) operator $A = A(t, \omega) : H \to H$, the following hypotheses will be assumed below.

(i) Let $\varphi : [0, T] \times H \times \Omega \to \mathbb{R} = (-\infty, +\infty]$ be convex lower semicontinuous in $y \in H$ and progressively measurable, i.e., for each $t \in [0, T]$ the function $\varphi$ restricted to $[0, t] \times H \times \Omega$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(H) \otimes \mathcal{F}_t$ measurable, and let

\[A(t, \omega) = \partial \varphi(t, \omega), \quad \forall (t, \omega) \in [0, T] \times \Omega.\]

In particular, $y \to A(t, \omega, y)$ is maximal monotone in $H \times H$ for all $(t, \omega) \in [0, T] \times \Omega$. Furthermore, $\varphi$ is such that there exists $\alpha \in L^2([0, T] \times \Omega, H)$ and $\beta \in L^2([0, T] \times \Omega)$ such that

$\varphi(t, y, \omega) \geq (\alpha(t, \omega), y) - \beta(t, \omega)$ for $dt \otimes \mathbb{P} - a.e., (t, \omega) \in [0, T] \times \Omega$.

(ii) $e^{\pm W(t)}$ is a multiplier in $H$ such that there is an $(\mathcal{F}_t)_{t \geq 0}$-adapted $\mathbb{R}_+$-valued process $Z(t)$, $t \in [0, T]$, with

\[\sup_{t \in [0, T]} |Z(t)| < \infty, \quad \mathbb{P}\text{-a.s.,}\]

\[|e^{\pm W(t)} y|_H \leq Z(t)|y|_H, \quad \forall t \in [0, T], \quad y \in H,\]

$t \to e^{\pm W(t)} \in L(H, H)$ is continuous.
Recall that a multivalued mapping \( A : D(A) \subset H \rightarrow H \) is said to be maximal monotone if it is monotone, that is, for \( u_1, u_2 \in D(A) \),
\[
(z_1 - z_2, u_1 - u_2) \geq 0, \ \forall z_i \in Au_i, \ i = 1, 2,
\]
and the range \( R(\lambda I + A) \) is all of \( H \) for each \( \lambda > 0 \).

If \( \varphi : H \rightarrow \mathbb{R} \) is a convex, lower semicontinuous function, then its subdifferential \( \partial \varphi : H \rightarrow H \)
\[
\partial \varphi(u) = \{ v \in H; \ \varphi(u) \leq \varphi(\bar{u}) + (v, u - \bar{u}), \ \forall \bar{u} \in H \} \quad (1.6)
\]
is maximal monotone (see, e.g., [1]).

The conjugate \( \varphi^* \) of \( H \) defined by
\[
\varphi^*(v) = \sup\{(u, v) - \varphi(u); \ u \in H\} \quad (1.7)
\]
satisfies
\[
\varphi(u) + \varphi^*(v) \geq (u, v), \ \forall u, v \in H,
\]
\[
\varphi(u) + \varphi^*(v) = (u, v), \ \text{iff} \ v \in \partial \varphi(u). \quad (1.8)
\]

As regards the basis \( \{ e_j \} \) arising in the definition of the Wiener process \( W \), we assume also that, for the multipliers \( e_j^2 \), we have

(iii) For \( \gamma_j = \max\{\sup\{|ue_j|_H; \ |u|_H = 1\}, (\sup\{|ue_j^2|_H; \ |u|_H = 1\})^{\frac{1}{2}}, 1\} \), we assume
\[
\nu = \sum_{j=1}^{\infty} \mu_j^2 \gamma_j^2 < \infty, \quad (1.9)
\]
and that \( \lambda > \nu \).

Clearly, then
\[
\mu = \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 e_j^2 \quad (1.10)
\]
is a multiplier in \( H \).

It should be noted that the condition \( \lambda > \nu \) in (1.9) is made only for convenience. In fact, by the substitution \( X \rightarrow \exp(-\lambda t)X \) and replacing \( A(t) \) by \( u \rightarrow e^{-\lambda t}A(t)(e^{\lambda t}u) \) we can always change \( \lambda \) in (1.11) to a big enough \( \lambda \) which satisfies \( \lambda > \nu \). It should be emphasized that a general existence and uniqueness result for equation (1.11) is known only for the special case where \( A(t) \) are monotone and demicontinuous operators from \( V \) to \( V' \), where
$(V,V')$ is a pair of reflexive Banach spaces in duality with the Hilbert space $H$ as pivot space, that is, $V \subset H(\equiv H') \subset V'$ densely and continuously. If, in addition, for $\alpha_1 \in (0, \infty)$, $\alpha_2, \alpha_3 \in \mathbb{R}$,

$$\begin{align*}
V':(A(t)u,u)_V &\geq \alpha_1\|u\|_V^p + \alpha_2|u|_H^2, \forall u \in V, \\
\|A(t)u\|_{V'} &\leq \alpha_3\|u\|_{V'}^{p-1}, \forall u \in V,
\end{align*}$$

where $1 < p < \infty$, then equation (1.11) has under assumptions (i)–(iii) a unique strong solution $X \in L^p((0,T) \times \Omega;V)$ (see [15], [16], [17], [18]). We noted before that assumption (i) implies that $A(t,\omega)$ is maximal monotone in $H$ for all $t \in [0,T]$, though not every maximal monotone operator $A(t) : D(A(t)) \subset H \rightarrow H$ has a realization in a convenient pair of spaces $(V,V')$ such that (1.8)–(1.9) hold. Though assumptions (1.11)–(1.12) hold for a large class of stochastic parabolic equations in Sobolev spaces $W^{1,p}(\Omega)$, $1 \leq p < \infty$ (see [6]), some other important stochastic PDEs are not covered by this functional scheme. For instance, the variational stochastic differential equations, nonlinear parabolic stochastic equations in $W^{1,1}(\Omega)$, in Orlicz-Sobolev spaces on $\Omega$ or in $BV(\Omega)$ (bounded variation stochastic flows) cannot be treated in this functional setting. As a matter of fact, contrary to what happens for deterministic infinite differential equations, there is no general existence theory for equation (1.1) under assumption (i)–(iii). The definition of a convenient concept of a weak solution to be unique and continuous with respect to data is a challenging objective of the existence theory of the infinite dimensional SDE. In this paper, we introduce such a solution $X$ for (1.1) which is defined as a minimum point of a certain convex functional defined on a suitable space of $H$-valued processes on $(0,T)$. This idea was developed in [11] for nonlinear operators $A(t) : V \rightarrow V'$ satisfying condition (1.11)–(1.12) and is based on the so-called Brezis–Ekeland variational principle [11]. Such a solution in the sequel will be called the variational solution to (1.1). (Along these lines see also [2], [3], [5], [6].)

2 The variational solution to equation (1.1)

First, we transform equation (1.1) into a random differential equation via the substitution

$$X(t) = e^{W(t)}(y(t) + x), \ t \in [0,T],$$

which, by Itô’s product rule,
\[ dX = e^W dy + e^W (y + x) dW + \mu e^W (y + x) dt, \]

leads to

\[
\frac{dy(t)}{dt} + e^{-W(t)} A(t) (e^{W(t)} (y(t) + x)) + (\mu + \lambda) (y(t) + x) \geq 0, \quad t \in (0, T), \tag{2.2} \]

\[ y(0) = 0. \]

(In the following, we shall omit \( \omega \) from the notation \( A(t, \omega) \).)

As a matter of fact, the equivalence between (1.1) and (2.2) is true only for a smooth solution \( y \) to (2.2), that is, for pathwise absolutely continuous strong solutions to (2.2) (see [9], [10]). In the sequel, we shall define a generalized (variational) solution for the random Cauchy problem (2.2) and will call the corresponding process \( X \) defined by (2.1) the variational solution to (1.1).

We shall treat equation (2.2) by the operator method developed in [10]. Namely, consider the space \( \mathcal{H} \) of all \( H \)-valued processes \( y : [0, T] \rightarrow H \) such that

\[
|y|_{\mathcal{H}} = \left( \mathbb{E} \int_0^T |e^{W(t)} y(t)|^2_H dt \right)^{\frac{1}{2}} < \infty,
\]

which have an \( (\mathcal{F}_t)_{t \geq 0} \)-adapted version. Here \( \mathbb{E} \) denotes the expectation with respect to \( \mathbb{P} \). The space \( \mathcal{H} \) is a Hilbert space with the scalar product

\[
\langle y, z \rangle = \mathbb{E} \int_0^T (e^{W(t)} y(t), e^{W(t)} z(t)) dt, \quad y, z \in \mathcal{H}.
\]

We set \( \delta = \frac{1}{2} (\lambda - \nu) \). Now, consider the operators \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) and \( B : D(B) \subset \mathcal{H} \rightarrow \mathcal{H} \) defined by

\[
(A y)(t) = e^{-W(t)} A(t) (e^{W(t)} (y(t) + x)) + \delta (y + x), \quad y \in D(A), \quad t \in [0, T],
\]

\[
D(A) = \{ y \in \mathcal{H} ; \ e^{W(t)} (y(t) + x) \in D(A(t)), \ \forall t \in [0, T] \text{ and } e^{-W(t)} A(e^{W(t)} (y + x)) \in \mathcal{H} \}, \tag{2.3}
\]

\[
(B y)(t) = \frac{dy(t)}{dt} (t) + (\mu + \nu + \delta) (y + x), \quad \text{a.e.} \ t \in (0, T), \quad y \in D(B),
\]

\[
D(B) = \left\{ y \in \mathcal{H} ; \ y \in W^{1,2}_0 ([0, T]; H), \ \mathbb{P}-\text{a.s., } \frac{dy}{dt} \in \mathcal{H} \right\}. \tag{2.4}
\]
Here, $W^{1,2}_0([0, T]; H)$ denotes the space \( \{ y \in W^{1,2}([0, T]; H); y(0) = 0 \} \), where $W^{1,2}([0, T]; H)$ is the Sobolev space $\{ y \in L^2(0, T; H), \frac{dy}{dt} \in L^2(0, T; H) \}$.

We recall that $W^{1,2}([0, T]; H) \subset AC([0, T]; H)$, the space of all $H$-valued absolutely continuous functions on $[0, T]$.

Then we may rewrite equation (2.2) as

$$By + Ay \ni 0.$$  \hspace{1cm} (2.5)

(If $A(t)$ is multivalued, we replace $A(t)(e^W(y + x))$ in (2.3) by $\{ \eta(t); \eta(t) \in A(t)(e^W(y(t) + x)), \text{ a.e. } (t, \omega) \in (0, T) \times \Omega \}$.)

Consider the functions $\Phi : H \rightarrow \mathbb{R}$ defined by

$$\Phi(y) = \mathbb{E} \int_0^T (\varphi(t, e^W(y(t) + x)) + \frac{\delta}{2} |e^W(y(t) + x)|^2_H) dt, \ \forall y \in H. \quad (2.6)$$

It is easily seen that $\Phi$ is convex, lower-semicontinuous and

$$\partial \Phi = A. \quad (2.7)$$

As regards the operator $\mathcal{B}$, we have

**Lemma 2.1** For each $y \in D(\mathcal{B})$ we have

$$\langle \mathcal{B}y, y \rangle = \frac{1}{2} \mathbb{E}|e^W(T)y(T)|^2_H + (\nu + \delta)|y|^2_H - \frac{1}{2} \mathbb{E} \int_0^T \sum_{j=1}^\infty |e^W ye_j|^2_H \mu_j^2 dt \geq \frac{1}{2} \mathbb{E}|e^W(T)y(T)|^2_H + \frac{\lambda}{2} |y|^2_H. \quad (2.8)$$

**Proof.** We have

$$\langle \mathcal{B}y, y \rangle = \mathbb{E} \int_0^T \left( e^{W(t)} \frac{dy}{dt}(t), e^{W(t)} y(t) \right) dt + \mathbb{E} \int_0^T ((\mu + \nu + \delta)e^W y, e^W y) dt. \quad (2.9)$$

Taking into account that

$$d(e^W y) = e^W dy + e^W y dW + \mu e^W y dt, \ \forall y \in D(\mathcal{B}),$$
we get via Itô’s formula that (see [6])
\[
\frac{1}{2} d|e^W y|^2_H = \left( e^W \frac{dy}{dt}, e^W y \right) dt + (e^W y, e^W y dW) + (\mu e^W y, e^W y) dt \\
+ \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 |e^W ye_j|^2_H dt.
\]

Hence
\[
\mathbb{E} \int_0^T \left( e^W \frac{dy}{dt}, e^W y \right) dt = \frac{1}{2} \mathbb{E} |e^{W(T)} y(T)|^2_H - \mathbb{E} \int_0^T (\mu e^W y, e^W y) dt \\
- \frac{1}{2} \mathbb{E} \int_0^T \sum_{j=1}^{\infty} |e^W ye_j|^2_H \mu_j^2 dt,
\]
and so, because \( \lambda > \nu \), by (1.9), (2.9), we get (2.8), as claimed. \( \square \)

Consider now the conjugate \( \Phi^* : \mathcal{H} \to \mathbb{R} \) of functions \( \Phi \), that is,
\[
\Phi^*(z) = \sup \{ \langle z, y \rangle_H - \Phi(y); \ y \in \mathcal{H} \}.
\]
By (2.6), we see that (see [19])
\[
\Phi^*(z) = \mathbb{E} \int_0^T (\psi^*(t, e^{W(t)} z(t)) - (e^{W(t)} z(t), e^{W(t)} x)) dt,
\]
where \( \psi^* \) is the conjugate of the function
\[
\psi(t, y) = \varphi(t, y) + \frac{\delta}{2} |y|^2_H,
\]
that is,
\[
\psi^*(t, v) = \sup \{ \langle v, y \rangle - \varphi(t, y) - \frac{\delta}{2} |y|^2_H; \ y \in H \}.
\]
We recall (see [1.8]) that
\[
\Phi(y) + \Phi^*(u) \geq \langle y, u \rangle_H, \ \forall y, u \in \mathcal{H},
\]
with equality if and only if \( u \in \partial \Phi(y) \). We infer that \( y^* \) is a solution to equation (2.5) if and only if
\[
y^* = \arg \min_{(y, u) \in D(B) \times \mathcal{H}} \{ \Phi(y) + \Phi^*(u) - \langle y, u \rangle_H; \ B y + u = 0 \} = \arg \min_{(y, u) \in D(B) \times \mathcal{H}} \{ \Phi(y) + \Phi^*(u) + \langle B y, y \rangle_H; \ B y + u = 0 \}.
\]
and
\[
\Phi(y^*) + \Phi^*(u^*) + \langle B y^*, y^* \rangle = 0. \tag{2.15}
\]

Taking into account (2.10) and recalling (2.6), (2.8), we have

\[
y^* = \arg \min_{(y,u) \in D(B) \times H} \left\{ \mathbb{E} \int_0^T \left( \varphi(t, e^{W(t)}(y(t) + x)) + \frac{\delta}{2} |e^{W(t)}(y(t) + x)|^2_H
\right.
\]
\[
+ \psi^*(t, e^{W(t)}u(t)) - (e^{W(t)}u(t), e^{W(t)}x)
\]
\[
+ \eta(e^{W(t)}y(t)) \right) dt + \frac{1}{2} \mathbb{E}|e^{W(T)}y(T)|^2_H; \quad By + u = 0 \right\},
\]

where

\[
\eta(z) = (\nu + \delta)|z|^2_H - \frac{1}{2} \sum_{j=1}^{\infty} |ze_j|^2_H \mu_j^2. \tag{2.17}
\]

We note also that, by Itô’s product rule, we have, for \( u \in H, \ y \in D(B), \)

\[-\mathbb{E} \int_0^T (e^{W(t)}u(t), e^{W(t)}x) dt
\]
\[= \mathbb{E} \int_0^T \left( e^{W}x, e^{W} \left( \frac{dy}{dt} + (\mu + \nu + \delta)(y + x) \right) \right) dt
\]
\[= \mathbb{E} \int_0^T \left( e^{W}x, d(e^{W}y) - \int_0^T (e^{W}x, \mu e^{W}y - (\mu + \nu + \delta)(y + x)e^{W}) dt
\]
\[= \mathbb{E} \int_0^T \left( e^{W}x, (\mu + \delta + \nu)(y + x)e^{W} \right) dt
\]
\[+ \mathbb{E} \int_0^T d(e^{W}x, e^{W}y) - \int_0^T \left( (e^{W}y, e^{W}(1 + \mu)x) - (\mu e^{W}y, e^{W}x) \right) dt
\]
\[= \mathbb{E}(e^{W(T)}x, e^{W(T)}y(T)) - \int_0^T \left( (e^{W}y, e^{W}(1 + \mu)x) - (\mu e^{W}y, e^{W}x) \right) dt
\]
\[+ \mathbb{E} \int_0^T (e^{W}x, (\mu + \nu + \delta)(y + x)e^{W}) dt
\]
\[= \mathbb{E}(e^{W(T)}x, e^{W(T)}y(T)) + \mathbb{E} \int_0^T (e^{W}x, ((\mu + \nu + \delta)(y + x) - \mu y)e^{W}) dt
\]
\[- \int_0^T (e^{W}y, e^{W}(1 + \mu)x) dt
\]
\[= \mathbb{E} \int_0^T (e^{W}((\nu + \delta)(y + x) + \mu x), e^{W}x) dt + \mathbb{E}(e^{W(T)}y(T), e^{W(T)}x)
\]
\[- \int_0^T (e^{W}y, e^{W}(1 + \mu)x) dt.\]
Let $\mathcal{H}_0$ denote the set of all $u \in L^2([0, T] \times \Omega; H)$ which have an $(\mathcal{F}_t)_{t \geq 0}$-adapted version. We set, for $y \in \mathcal{H}$, $u \in \mathcal{H}_0$,

$$G_1(y) = \mathbb{E} \int_0^T \varphi(t, e^{W(t)}(y(t) + x))dt$$

$$+ \mathbb{E} \int_0^T ((e^{W(t)}((\nu + \delta)(y(t) + x) + \mu x), e^{W(t)}x)

+ \frac{\delta}{2} |e^{W(t)}(y(t) + x)|_H^2 + \eta(e^{W(t)}y(t)))dt

+ \frac{1}{2} \mathbb{E}|e^{W(T)}y(T)|_H^2 + \mathbb{E}(e^{W(T)}y(T), e^{W(T)}x)

- \mathbb{E} \int_0^T (e^{W(t)}y(t), e^{W(t)}(1 + \mu)x)dt,$$

$$G_2(u) = \mathbb{E} \int_0^T \psi^*(t, u(t))dt,$$

where $\psi^*$ is given by (2.12).

By (2.16) it follows that $y^*$ is a solution to equation (2.5) if and only if

$$y^* = \arg \min_{(y,u) \in \mathcal{D}(\mathcal{B}) \times \mathcal{H}_0} \{G_1(y) + G_2(u); \ e^{W}B_y + u = 0\}$$

(2.20)

and

$$G_1(y^*) + G_2(u^*) = 0.$$ (2.21)

It should be said, however, that under our assumptions the convex minimization problem (2.20) might have no solution $(y^*, u^*)$ because, in general, $G_2$ is not coercive on the space $\mathcal{H}$. ($G_2$ is, however, coercive if $\varphi$ is bounded on bounded sets of $H$. But such a condition is too restrictive for applications to PDEs.) So, we are led to replace (2.20) by a relaxed optimization problem to be defined below.

Let

$$\mathcal{X} = L^2(\Omega; (W^{1,2}([0, T]; H))'),$$

where $(W^{1,2}([0, T]; H))'$ is the dual space of $W^{1,2}([0, T]; H)$.

Define the operator $\widetilde{\mathcal{B}} : \mathcal{H} \times L^2(\Omega; H) \to \mathcal{X}$ by
\[ \mathcal{B}(y, y_1)(\theta) = \mathbb{E}(e^{W(T)}y_1, \theta(T)) + \mathbb{E} \int_0^T ((\nu + \delta)(y(t) + x) + \mu x)e^{W(t)}, \theta(t))dt - \mathbb{E} \int_0^T (e^{W(t)}y(t), \frac{d\theta}{dt}(t))dt, \quad \forall \theta \in L^2(\Omega; W^{1,2}([0, T]; H)). \] (2.23)

We note that \( y_1(\omega) \in H \) can be viewed as the trace of \( y(\omega) \) at \( t = T \).

Indeed, if \( y \in \mathcal{D}(\mathcal{B}) \), we have via Itô’s formula

\[
\mathbb{E} \int_0^T (e^{W} \mathcal{B}y, \theta)dt = \mathbb{E} \left( \int_0^T (d(e^{W} y), \theta) - \int_0^T (e^{W} \mu y, \theta)dt \right)
+ \mathbb{E} \int_0^T (e^{W}(\mu + \nu + \delta)(y + x), \theta)dt
= \mathbb{E}(e^{W(T)}y(T), \theta(T))
+ \mathbb{E} \int_0^T (e^{W}((y + x)(\nu + \delta) + \mu x), \theta)dt
- \mathbb{E} \int_0^T \left( e^{W} y, \frac{d\theta}{dt} \right)dt, \quad \forall \theta \in L^2(\Omega; W^{1,2}([0, T]; H)).
\]

This means that \( \tilde{\mathcal{B}}(y, y(T)) = e^{W} \mathcal{B}y, \forall y \in \mathcal{D}(\mathcal{B}) \). We set

\[
\tilde{G}_1(y, y_1) = \mathbb{E} \int_0^T \varphi(t, e^{W(t)}(y(t) + x))dt
+ \mathbb{E} \int_0^T \left( (e^{W(t)}((\nu + \delta)(y(t) + x) + \mu x), e^{W(t)}x \right)dt
+ \mathbb{E} \int_0^T \left( \frac{\delta}{2} |e^{W(t)}(y(t) + x)|^2_H + \eta(e^{W(t)}y(t)) \right)dt
- \mathbb{E} \int_0^T (e^{W(t)}y(t), e^{W(t)}(1 + \mu)x)dt + \frac{1}{2} \mathbb{E}|e^{W(T)}y_1|^2_H
+ \mathbb{E}(e^{W(T)}y_1, e^{W(T)}x), \quad \forall (y, y_1) \in \mathcal{H} \times L^2(\Omega; H) \]

and note that \( \tilde{G}_1(y; y(T)) = G_1(y), \forall y \in \mathcal{D}(\mathcal{B}) \).

We note also that, if \( y_n \in \mathcal{D}(\mathcal{B}) \) such that \( y_n \to y \) weakly in \( \mathcal{H} \) and \( y_n(T) \to y_1 \) weakly in \( L^2(\Omega; H) \), then

\[
e^{W} \mathcal{B}y_n \to \tilde{\mathcal{B}}(y, y_1) \text{ weakly in } \mathcal{X}. \quad (2.25)
\]
Let \( \overline{G} : \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X} \to \overline{\mathbb{R}} \) be the lower semicontinuous closure of the function \( G(y, y_1, u) = \tilde{G}_1(y, y_1) + G_2(u) \) in \( \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X} \), on the set \( \{(y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}; e^W \mathcal{B} y + u = 0\} \), that is,

\[
\overline{G}(y, y_1, u) = \liminf \{G(z, z(T), u); z(T) \to y_1 \text{ in } L^2(\Omega; H), \quad z \in \mathcal{D}(\mathcal{B}), (z, v) \to (y, u) \text{ in } \mathcal{H} \times \mathcal{X}; e^W \mathcal{B} z + v = 0\}.
\] (2.26)

(Here and everywhere in the following, by \( \to \) we mean weak convergence.)

Taking into account that the function \( \tilde{G}_1 \) is convex and lower semicontinuous in \( \mathcal{H} \times L^2(\Omega; H) \), we have by (2.26)

\[
\overline{G}(y, y_1, u) = \tilde{G}_1(y, y_1) + \liminf \{G_2(v); (z, v) \to (y, u) \text{ in } \mathcal{H} \times \mathcal{X}; e^W \mathcal{B} z + v = 0\}.
\] (2.27)

Now, we relax (2.20) to the convex minimization problem

\[
(P) \quad \text{Min} \{\overline{G}(y, y_1, u); \tilde{\mathcal{B}}(y, y_1) + u = 0; (y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}\}.
\]

We have

**Theorem 2.2** Let \( x \in \mathcal{H} \). Then problem (P) has a unique solution \((y^*, y_1^*, u^*)\) in \( \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X} \), with \( u^* = -\tilde{\mathcal{B}}(y^*, y_1^*) \). Moreover, \( \varphi(\cdot, e^W(y^* + x)) \in L^1((0, T) \times \Omega) \).

**Proof.** Let \( m \) be the infimum in (P) and let \((y_n, u_n) \in \mathcal{D}(\mathcal{B}) \times \mathcal{H}\) be such that

\[
m \leq G(y_n, y_n(T), u_n) \leq m + \frac{1}{n}, \quad \forall n \in \mathbb{N},
\] (2.28)

\[
e^W \mathcal{B} y_n + u_n = 0.
\] (2.29)

Since, by assumption (iii), for some \( C_1, C_2 \in ]0, \infty[\),

\[
\tilde{G}_1(y_n, y_n(T)) \geq C_1(|y_n|^2_H + \mathbb{E}|e^W(T)y_n(T)|^2_H) - C_2,
\]

we have along a subsequence

\[
y_n \to y^* \text{ weakly in } \mathcal{H}, \quad y_n(T) \to y_1^* \text{ weakly in } L^2(\Omega; H),
\]

and so, by (2.25), we have

\[
u_n \to u^* = -\tilde{\mathcal{B}}(y^*, y_1^*) \text{ weakly in } \mathcal{X}.
\]
As $\overline{G}$ is weakly lower semicontinuous on $\mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}$, we see by (2.28) that
$$\overline{G}(y^*, y_1^*, u^*) = m,$$
as claimed. The uniqueness of $(y^*, y_1^*, u^*)$ is immediate because the function $\overline{G}(.\cdot, u)$ is strictly convex on $\mathcal{H} \times L^2(\Omega; H)$ for all $u \in \mathcal{X}$.

**Definition 2.3** A pair $(y^*, y_1^*)$ such that $(y^*, y_1^*, u^*) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}$, $u^* = -\overline{B}(y^*, y_1^*)$, is a solution to problem (P), is called the variational solution to equation (2.2), and $X^* = e^W(y^* + x)$ is called the variational solution to equation (1.1).

The variational solution $X^* : (0, T) \to H$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted process.

**Theorem 2.4** Under hypotheses (i)–(iii), equation (1.1) has a unique variational solution $X^* \in L^2((0, T) \times \Omega; H)$ with $\varphi(t, X^*) \in L^1((0, T) \times \Omega)$.

It should be noted that $y^*$ and $X^*$, as well, are not pathwise continuous on $[0, T]$. As seen later on, this happens, however, in some specific cases with respect to a weaker topology.

In the next section, we shall see how problem (P) looks like in a few important examples of stochastic PDEs.

**Remark 2.5** The above formulation of the variational solution $X^*$ is strongly dependent on the subdifferential form (1.3) of the operator $A(t)$. The extension of the above technique to a general maximal monotone function $A(t) : H \to H$ remains to be done using the Fitzpatrick formalism (see [20]).

## 3 Nonlinear parabolic stochastic differential equations

We consider here the stochastic differential equation
$$
\begin{align*}
&dX - \text{div}_\xi(a(t, \nabla X))dt + \lambda X \, dt = X \, dW \text{ in } (0, T) \times \mathcal{O}, \\
&X = 0 \text{ on } (0, T) \times \partial \mathcal{O}, \\
&X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O} \subset \mathbb{R}^d,
\end{align*}
$$

(3.1)
where \( x \in H \), \( W \) is the Wiener process \((1.2)\) in \( H = L^2(\Omega) \), \( \Omega \) is a bounded and open subset of \( \mathbb{R}^d \) with smooth boundary \( \partial \Omega \), and \( a : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) is a nonlinear mapping of the form

\[
a(t, z) = \partial_z j(t, z), \quad \forall z \in \mathbb{R}^d, \quad t \in [0, T],
\]

(3.2)

where \( j : (0, T) \times \mathbb{R}^d \to \mathbb{R} \) is measurable, convex, lower semicontinuous in \( z \) and

\[
\lim_{|z| \to \infty} \frac{j(t, z)}{|z|} = +\infty, \quad t \in [0, T],
\]

(3.3)

\[
\lim_{|v| \to \infty} \frac{j^*(t, v)}{|v|} = +\infty, \quad t \in [0, T],
\]

(3.4)

uniformly with respect to \( t \in [0, T] \).

We note that, if the function \((t, y) \to j(t, y)\) is bounded on bounded subsets of \((0, T] \times \mathbb{R}^d\), then (3.4) automatically holds by the conjugacy formula \((1.8)\), that is,

\[
j^*(t, v) \geq v \cdot z - j(t, z), \quad \forall v, z \in \mathbb{R}^d, \quad t \in [0, T].
\]

It should be noted that equation (3.1) cannot be treated in the functional setting \((1.11)-(1.12)\) which require polynomial growth and boundedness for \( j(t, \cdot) \), while assumptions (3.3)–(3.4) allow nonlinear diffusions \( a \) with slow growth to \( +\infty \) as well as superlinear growth of the form

\[
a(t, z) = a_0 \exp(a_1 |z|^p \text{sgn } z).
\]

We note also that assumptions (3.2)–(3.4) do not preclude multivalued mappings \( a \). Such an example is

\[
j(t, z) \equiv |z|(\log(|z| + 1)),
\]

\[
a(t, z) = \left( \log(|z| + 1) + \frac{1}{|z| + 1} \right) \text{sgn } z, \quad \forall z \in \mathbb{R}^d.
\]

By \((2.1)\), one reduces equation (3.1) to the random parabolic differential equation

\[
\frac{\partial y}{\partial t} - e^{-W} \text{div}_x a(t, \nabla(e^W(y + x)) + (\lambda + \mu)(y + x)) = 0
\]

in \((0, T) \times \mathbb{O})

(3.5)

\[
y = 0 \text{ on } (0, T) \times \partial \mathbb{O},
\]

\[
y(0, \xi) = 0, \quad \xi \in \mathbb{O}.
\]
We are under the conditions of Section 2, where

\[ H = L^2(\mathcal{O}), \]
\[ A(t)y = -\text{div}_\xi a(t, \nabla y), \]
\[ D(A(t)) = \{y \in W^{1,1}_0(\mathcal{O}); \text{div}_\xi a(t, \nabla y) \in L^2(\mathcal{O})\} \]
\[ \varphi(t, y) = \int_\mathcal{O} j(t, \nabla y(\xi))d\xi. \]

By (2.12), we have

\[ \psi^*(t, v) = \int_\mathcal{O} (a(t, \nabla z) \cdot \nabla z - j(t, \nabla z) + \frac{\delta}{2} z^2) d\xi, \quad \forall v \in L^2(\mathcal{O}), \]

where \( z \) is the solution to the equation

\[ -\text{div} a(t, \nabla z) + \delta z = v \quad \text{in} \; \mathcal{O}, \]
\[ z = 0 \quad \text{on} \; \partial \mathcal{O}, \]

or, equivalently,

\[ z = \arg \min_{\tilde{z} \in W^{1,1}_0(\mathcal{O})} \left\{ \int_\mathcal{O} j(t, \nabla \tilde{z}) d\xi - \int_\mathcal{O} v \tilde{z} d\xi + \frac{\delta}{2} \int_\mathcal{O} \tilde{z}^2 d\xi \right\}. \]

By (3.3), it follows that (3.8) has, for each \( v \in L^2(\mathcal{O}) \) and \( t \in [0, T] \), a unique solution \( z \in W^{1,1}_0(\mathcal{O}) \). In fact, as easily seen, by condition (3.3) it follows that the functional arising in the right side part of (3.8) is convex, lower semicontinuous and coercive on \( W^{1,1}_0(\mathcal{O}) \). By (2.24), we have

\[ \tilde{G}_1(y, y_1) = \mathbb{E} \int_0^T \int_\mathcal{O} (a(t, \nabla (e^{W(t)}(y(t) + x))) + \frac{\delta}{2} |e^{W(t)}(y(t) + x)|_H^2 + e^{W(t)}((\nu + \delta)(y(t) + x) + \mu x)e^{W(t)}x) d\xi dt \]
\[ -\mathbb{E} \int_0^T (e^{W(t)}y(t), e^{W(t)}(1 + \mu)x) dt \]
\[ + \mathbb{E} \int_0^T \eta(e^{W(t)}y(t)) dt + \frac{1}{2} \mathbb{E} \int_\mathcal{O} |e^{W(T)}y_1(\xi)|^2 d\xi \]
\[ + \mathbb{E} (e^{W(T)}y_1, e^{W(T)}x), (y, y_1) \in \mathcal{H} \times L^2(\Omega; H), \]

where (see (2.17))

\[ \eta(z) = (\nu + \delta) \int_\mathcal{O} |z|^2 d\xi - \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 \int_\mathcal{O} |ze_j|^2 d\xi. \]
By (2.19) and (3.6)–(3.7), we also have
\[ G_2(u) = \mathbb{E} \int_0^T \int_{\mathcal{O}} (a(t, \nabla z(t, \xi)) \cdot \nabla z(t, \xi)) \]
\[ - j(t, \nabla z(t, \xi)) + \frac{\delta}{2} z^2(t, \xi) dt, \quad u \in \mathcal{H}, \] (3.11)
where \( z(t, \omega) \in W^{1,1}_0(\mathcal{O}) \) for \( dt \otimes \mathbb{P}\text{-a.e.}, \quad (t, \omega) \in (0, T) \times \Omega \), is given by (see (3.7))
\[ - \text{div} a(t, \nabla z) + \delta z = u \quad \text{in} \quad \mathcal{O}, \]
\[ z = 0 \quad \text{on} \quad \partial \mathcal{O}. \] (3.12)
Taking into account that \( a(t, \nabla z) \cdot \nabla z \geq j(t, \nabla z) - j(t, 0) \), we see by (3.3) and (3.12) that
\[ z \in L^1((0, T) \times \Omega; W^{1,1}(\mathcal{O})) \cap L^2((0, T) \times \mathcal{O} \times \Omega). \]
Recalling (1.7)–(1.8), we have
\[ a(t, \nabla z) \cdot \nabla z - j(t, \nabla z) = j^*(t, a(t, \nabla z)) \text{ a.e. in } (0, T) \times \mathcal{O}, \]
and this yields
\[ G_2(u) = \mathbb{E} \int_0^T \int_{\mathcal{O}} (j^*(t, a(t, \nabla z(t, \xi)))) + \frac{\delta}{2} z^2(t, \xi) d\xi dt. \] (3.13)
By (3.4), it follows via the Dunford–Pettis weak compactness theorem in \( L^1 \) that every level set
\[ \left\{ v; \mathbb{E} \int_0^T \int_{\mathcal{O}} j^*(t, v(t, \xi)) dx d\xi \leq M \right\}, \quad M > 0, \]
is weakly compact in the space \( L^1((0, T) \times \mathcal{O} \times \Omega) \). By (3.12) and (3.13), we see that, if \( G_2(u_n) \leq M \), where \( \{u_n\} \subset L^2((0, T) \times \mathcal{O} \times \Omega) \) and \( z_n \) is the solution to (3.12) with \( u_n \) replacing \( u \), then, by the Dunford–Pettis theorem, the sequence \( \{a(t, \nabla z_n)\} \) is weakly compact in \( L^1((0, T) \times \mathcal{O} \times \Omega) \). Hence \( \{u_n\} \) is weakly compact in \( L^1((0, T) \times \mathcal{O}; W^{-1,\infty}(\mathcal{O})) \).
By (3.13), it follows also that \( \{z_n\} \) is weakly compact in \( L^2((0, T) \times \mathcal{O} \times \Omega) \).
By (2.26), this means that, if \( x \in L^2(\mathcal{O}) \), then, for \( (y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}, \)
\[ \overline{G}(y, y_1, u) \]
\[ = \overline{G}_1(y, y_1) + \mathbb{E} \int_0^T \int_{\mathcal{O}} \left( j^*(t, a(t, \nabla z(t, \xi))) + \frac{\delta}{2} z^2(t, \xi) \right) d\xi dt, \] (3.14)
where $z \in L^1((0, T) \times \Omega; W^{1,-1}_0(\mathcal{O})) \cap L^2((0, T) \times \mathcal{O} \times \Omega)$ is the solution to (3.12).

Let $(y_n, u_n) \in \mathcal{H} \times \mathcal{H}$ be such that $e^W_B y_n + u_n = 0$ and $(y_n, u_n) \to (y, u)$ in $\mathcal{H} \times \mathcal{X}; y_n(T) \to y_1$ in $L^2(\Omega; H)$. Since $\sup\{G_1(y_n)\} < \infty$, by (3.3) and (3.9), it follows also that $\{\nabla (e^W(y_n + x))\}$ is weakly compact in $L^1((0, T) \times \mathcal{O} \times \Omega)$, and so $e^W (y + x) \in L^1((0, T) \times \mathcal{O}; W^{1,1}_0(\mathcal{O}))$. Moreover, it follows that $\{\frac{dy_n}{dt}\}$ is weakly compact in $L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O}))$, and so $\frac{dy}{dt} \in L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O}))$. This implies that the equation $\tilde{B}(y^*, y_1^*) + u^* = 0$ reduces to

$$e^W \frac{dy^*}{dt} + e^W (\mu + \nu + \delta)(y^* + x) + u^* = 0 \text{ in } D'(0, T), \text{ } \mathbb{P}\text{-a.s.,}
$$

$$y^*(0) = 0, \quad y^*(T) = y_1^*.$$  

Hence, if $D(G_1) = \{(y, y_1, u); \overline{G}_1(y, y_1, u) < \infty\}$, then we have

$$D(G_1) \subset \{(y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{H}; e^W y \in L^1((0, T) \times \Omega; W^{1,1}_0(\mathcal{O})); \frac{dy}{dt} \in L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O})); u \in L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O})), y_1 = y(T)\}.$$  

This means that, in this case, problem (P) can be rewritten as

$$\begin{align*}
\text{Min} \left\{ \overline{G}(y, y(T), u); y \in L^2((0, T) \times \mathcal{O} \times \Omega) \cap \mathcal{H}, \\
e^W (y + x) \in L^1((0, T) \times \Omega; W^{1,1}_0(\mathcal{O})), \\
\frac{dy}{dt} \in L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O})), \\
u \in L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O})) \cap \mathcal{X}; \\
\text{subject to} \\
\frac{dy}{dt} + (\mu + \nu + \delta)(y + x) + e^{-W} u = 0 \text{ on } (0, T); y(0) = 0 \right\},
\end{align*}$$

(3.15)

where $\overline{G}_1$ is defined by (3.14). By Theorem 2.2 there is a unique solution $(y^*, u^*)$ to (3.15). Taking into account that $u^* \in L^1((0, T) \times \Omega; W^{-1,\infty}(\mathcal{O}))$ and that

$$y^*(t) = -\int_0^t e^{-W} u^*(s)ds - \int_0^t (\mu + \nu + \delta)(y^*(s) + x)ds, \forall t \in (0, T),$$

we infer that the process $t \to y^*(t)$ in pathwise $W^{-1,\infty}(\mathcal{O})$ continuous on $(0, T)$. By Theorem 2.4 we have, therefore,
Theorem 3.1 Assume that \( x \in L^2(\mathcal{O}) \) and that conditions (3.2)–(3.4) hold. Then, equation (3.1) has a unique variational solution

\[
X^* \in L^2((0,T) \times \mathcal{O} \times \Omega), \quad e^W X^* \in L^1((0,T) \times \Omega; W^{1,1}_0(\mathcal{O})).
\]

Moreover, the process \( t \to X^*(t) \) is \( \mathcal{F}_t \) \( \mathcal{F}_t \geq 0 \)-adapted and pathwise \( W^{-1,\infty}(\mathcal{O}) \)-valued continuous on \((0,T)\).

The total variation flow

The stochastic differential equation

\[
dX - \text{div} \left( \frac{\nabla X}{|\nabla X|_d} \right) dt + \lambda X \, dt = X \, dW \text{ in } (0,T) \times \mathcal{O},
\]

\[
X(0) = x \text{ in } \mathcal{O},
\]

\[
X = 0 \text{ on } (0,T) \times \partial \mathcal{O}
\]

with \( x \in L^2(\mathcal{O}) \) is the equation of stochastic variational flow in \( \mathcal{O} \subset \mathbb{R}^d, \) \( 1 \leq d \leq 3. \) The existence and uniqueness of a generalized solution to (3.17) \( X : [0,T] \to BV(\mathcal{O}) \) was established in [9] by using some specific approximation techniques. We shall treat now equation (3.17) in the framework of variational solution developed above in the space \( H = L^2(\mathcal{O}) \) with the norm \( | \cdot |_H = | \cdot |_2 \) and the scalar product \( (\cdot, \cdot) \), and \( \varphi : L^2(\mathcal{O}) \to \mathbb{R} \) defined by

\[
\varphi(y) = \begin{cases} 
\|Dy\| + \int_{\partial \mathcal{O}} |\gamma_0(y)| d\mathcal{H}^{d-1}, & y \in BV(\mathcal{O}) \setminus L^2(\mathcal{O}), \\
+\infty & \text{otherwise.}
\end{cases}
\]

Here, \( BV(\mathcal{O}) \) is the space of functions with bounded variation and \( \|Dy\| \) is the total variation of \( y \in BV(\mathcal{O}). \) (See, e.g., [9].) Then, with the notations of Section 2, we have \( Ay = \partial \varphi(y) \), where \( \partial \varphi : L^2(\mathcal{O}) \to L^2(\mathcal{O}) \) is the subdifferential of \( \varphi \) and (see (2.2), (2.18))

\[
\frac{\partial y}{\partial t} + e^{-W} A(e^W (y + x)) + \mu(y + x) = 0 \text{ in } (0,T) \times \mathcal{O},
\]

\[
y(0,\xi) = 0, \quad \xi \in \mathcal{O},
\]

\[
y = 0 \text{ on } (0,T) \times \partial \mathcal{O}.
\]

The function \( \tilde{G}_1 \) is given, in this case, by
\[ G_1(y, y_1) = E \int_0^T \left( \varphi(e^{W(t)}(y(t) + x)) + \frac{\delta}{2} |e^{W(t)}(y(t) + x)|^2 \right. \]
\[ + (e^{W(t)}(\nu + \delta y(t) + \mu x), e^{W(t)}x) \right) dt \]
\[-E \int_0^T (e^{W(t)}y(t), e^{W(t)}(1 + \mu)x) dt \]
\[ + E \int_0^T \eta(e^{W(t)}y(t)) dt + \frac{1}{2} E|e^{W(T)}y_1|^2 \]
\[ + E(e^{W(T)}y_1, e^{W(T)}x), (y, y_1) \in \mathcal{H} \times L^2(\Omega; H), \]
where \( \eta \) is given by (3.10). We have also (see (2.11), (2.12), (3.6))
\[
\psi(y) = \varphi(y) + \frac{\delta}{2} |y|^2, \quad \forall y \in D(\varphi),
\]
\[
\psi^*(v) = (v, \theta) - \varphi(\theta) - \frac{\delta}{2} |\theta|^2, \quad v \in \partial \varphi(\theta) + \delta \theta.
\]
Hence,
\[
\psi^*(v) = \frac{\delta}{2} |\theta|^2 + (\partial \varphi(z), \theta) - \varphi(\theta)
\]
\[
= \frac{\delta}{2} |(\delta I + \partial \varphi)^{-1}(u)|^2 + \varphi^*(u - (\delta I + \partial \varphi)^{-1}v)
\]
and, therefore, by (2.19),
\[
G_2(u) = E \int_0^T \left( \frac{\delta}{2} |(\delta I + \partial \varphi)^{-1}(u)|^2 + \varphi^*(u - (\delta I + \partial \varphi)^{-1}u) \right) dt,
\]
where \( \varphi^*: L^2(\mathcal{O}) \rightarrow \mathbb{R} \) is the conjugate of the function \( \varphi \). This yields
\[
\mathcal{G}(y, y_1, u) = G_1(y, y_1) + \lim \inf_{(z,v) \to (y,u)} \left\{ E \int_0^T \left( \frac{\delta}{2} |(I + \partial \varphi)^{-1}(v(t))| \right. \right. \]
\[ + \left. \varphi^*(v - (\delta I + \partial \varphi)^{-1}(v(t))) \right) dt, e^Wz + v = 0 \right\},
\]
where the space \( \mathcal{X} \) is defined by (2.22).

By definition, the solution \((y^*, y_1^*)\) to the minimization problem
\[
\text{Min}\{\mathcal{G}(y, y_1, u); \mathcal{B}(y, y_1) + u = 0, (y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X} \}\]
is the \textit{variational solution} to the random differential equation (3.18).
Denote by $V^*$ the dual of the space $V = BV(\mathcal{O}) \cap L^2(\mathcal{O})$. We note that $\varphi^*$ can be extended as a convex lower semicontinuous convex function on $F^*$, and we also have
\[
\frac{\varphi^*(u)}{\|u\|_{V^*}} \to +\infty \text{ as } \|u\|_{V^*} \to +\infty.
\]

Then, if $(z_n, y_n) \in H \times H$ is convergent to $(y, u) \in H \times X$, it follows by the Dunford-Pettis compactness criterium (see [12]) that \{v_n\} is weakly compact in $L^1((0, T) \times \Omega; V^*)$. This implies that
\[
D(G) \subset L^1((0, T) \times \Omega; BV(\mathcal{O})) \times L^2(\Omega; H) \times L^1((0, T) \times \Omega; V^*),
\]
and so, in particular, it follows that
\[
y \in W^{1,1}([0, T]; V^*), \text{ } \mathbb{P}\text{-a.s.}
\]
We have, therefore,

**Theorem 3.2** Let $x \in BV(\mathcal{O}) \cap L^2(\mathcal{O})$. Then equation (3.17) has a unique variational solution $X = e^{W}(y + x)$ which is $V^*$-valued pathwise continuous and satisfies
\[
\varphi(X) \in L^1((0, T) \times \Omega), \quad \text{(3.22)}
\]
\[
X \in L^2((0, T) \times \mathcal{O} \times \Omega), \quad AX \in L^1((0, T) \times \Omega; V^*), \quad \text{(3.23)}
\]
\[
e^{-W}X \in W^{1,1}([0, T]; V^*), \quad \mathbb{P}\text{-a.s.} \quad \text{(3.24)}
\]

In [9], it was proved the existence and uniqueness of a generalized solution $X$, also called the variational solution, which was obtained as limit $X^* = \lim_{\varepsilon \to 0} X_\varepsilon$ in $L^2(\Omega; C((0, T); L^2(\mathcal{O})))$, where $X_\varepsilon$ is the solution to the approximating equation
\[
\begin{align*}
dX_\varepsilon - \text{div } a_\varepsilon(\nabla X_\varepsilon)dt + \lambda X_\varepsilon &= X_\varepsilon dW \text{ in } (0, T) \times \mathcal{O}, \\
X_\varepsilon(0) &= x, \quad X_\varepsilon = 0 \text{ on } (0, T) \times \mathcal{O}, \quad \text{(3.25)}
\end{align*}
\]
where $a_\varepsilon = \nabla j_\varepsilon$ and $j_\varepsilon$ is the Moreau–Yosida approximation of the function $r \to |r|_d$. Since, as strong solution to (3.25), $X_\varepsilon$ is also a variational solution to this equation in sense of Definition 2.3, it is clear by the structural stability of convex minimization problems that, for $\varepsilon \to 0$, we have also $X_\varepsilon \to X$, where $X$ is the variational solution given by Theorem 3.2.
We may infer, therefore, that the function $X$ given by Theorem 3.2 is just the generalized solution of (3.17) given by Theorem 3.1 in [9]. In particular, this implies that $X$ is $L^2(O)$-valued pathwise continuous.

In [4], it is developed a direct variational approach to (3.17), which leads via first order conditions of optimality to sharper results. (On these lines, see also [13].)

**Stochastic porous media equations**

Consider the equation

\[
\begin{aligned}
dX - \Delta \beta(X) dt + \lambda X dt &= X dW \text{ in } (0, T) \times O, \\
X &= 0 \text{ on } (0, T) \times \partial O, \\
X(0, \xi) &= x(\xi), \quad \xi \in O, 
\end{aligned}
\]  

(3.26)

where $O$ is a bounded and open domain of $\mathbb{R}^d$, $d \geq 1$, $\lambda > 0$, $W$ is a Wiener process in $H = H^{-1}(O)$ of the form (1.2) and $\beta$ is a continuous and monotonically nondecreasing function such that $\beta(0) = 0$ and

\[
\lim_{|r| \to \infty} \frac{f(r)}{|r|} = +\infty.
\]  

(3.27)

In this case,

\[
\begin{aligned}
H &= H^{-1}(O), \\
Ay &= -\Delta \beta(y), \\
D(A) &= \{y \in H^{-1}(O) \cap L^1(O), \beta(y) \in H^1_0(O)\} \text{ and} \\
A &= \partial \varphi, \text{ where } \varphi(y) = \int_O j(y(\xi))d\xi.
\end{aligned}
\]

By (2.11), we have also

\[
\psi^*(v) = \int_O j^*(\beta(\theta))d\xi + \frac{\delta}{2} |\theta|_{-1}^2, \quad v \in L^2(O),
\]

where $\theta \in H^{-1}(O) \cap L^1(O)$,

\[
\begin{aligned}
2\delta \theta - \Delta \beta(\theta) &= v \quad \text{in } O, \\
\theta &= 0 \quad \text{on } \partial O,
\end{aligned}
\]

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and $| \cdot |_{-1}$ is the norm of $H^{-1}(\mathcal{O})$. Then we have

$$
\tilde{G}_1(y, y_1) = \mathbb{E} \int_0^T \left( \int_{\mathcal{O}} \tilde{j}(e^{W(t)}(y(t) + x))d\xi + \frac{\delta}{2} |e^{W(t)}(y(t) + x)|_{-1}^2 \right) d\xi \, dt
$$

$$
+ \mathbb{E} \int_0^T \int_{\mathcal{O}} e^{W(t)}((\nu + \delta)(y(t) + x) + \mu x)e^{W(t)}x \, d\xi \, dt
$$

$$
- \mathbb{E} \int_0^T (e^{W(t)}y(t), e^{W(t)}(1 + \mu)x)dt
$$

$$
+ \mathbb{E} \int_0^T \eta(e^{W})dt + \frac{1}{2} \mathbb{E}|e^{W(T)}y(T)|_{-1}^2
$$

$$
+(e^{W(T)}y_1, e^{W(T)}x)_{-1},
$$

while

$$
G_2(u) = \mathbb{E} \int_0^T \left( \int_{\mathcal{O}} j^*(\beta(\tilde{z}))d\xi + \frac{\delta}{2} |\tilde{z}(t)|_{-1}^2 \right) \, dt,
$$

where

$$
\delta \tilde{z} - \Delta \beta(\tilde{z}) = u \quad \text{in } \mathcal{O},
$$

$$
\tilde{z} = 0 \quad \text{on } \partial \mathcal{O}. \tag{3.28}
$$

(Here, $(\cdot, \cdot)_{-1}$ is the scalar product of $H^{-1}(\mathcal{O})$.)

Taking into account that $\frac{j^*(r)}{|r|} \to +\infty$ as $|r| \to \infty$, it follows, as in the previous case, for each $M > 0$, the set

$$
\left\{ \beta(\tilde{z}); \mathbb{E} \int_0^T \int_{\mathcal{O}} j^*(\beta(\tilde{z}))d\xi dt \leq M \right\}
$$

is weakly compact in $L^1((0, T) \times \mathcal{O} \times \Omega)$, we infer that

$$
\overline{G}(y, y_1, u) = \tilde{G}_1(y, y_1) + \mathbb{E} \int_0^T \left( \int_{\mathcal{O}} j^*(\beta(\tilde{z}))d\xi + \frac{\delta}{2} |\tilde{z}(t)|_{-1}^2 \right) dt, \tag{3.29}
$$

where $\tilde{z}$ is the solution to (3.28). This implies that

$$
D(\overline{G}) \subset \{(y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X}; u \in L^1((0, T) \times \Omega; \mathcal{Z})\}.
$$

Here $\mathcal{Z} = (-\Delta)^{-1}(L^1(\mathcal{O})) \subset W^{1,p}_0(\mathcal{O}), \ 1 \leq p < \frac{d}{d-1}$, where $\Delta$ is the Laplace operator with homogeneous Dirichlet conditions and

$$
D(\overline{G}) = \{(y, y_1, u); \overline{G}(y, y_1, u) < \infty\}.
$$
We define, as above, the solution to (3.26) as \( X^* = e^W y^* \), where \((y^*, y_1^*, u^*)\) is the solution to the minimization problem

\[
\text{Min} \left\{ G(y, y_1, u); \frac{dy}{dt} + (\mu + \nu + \delta)(y + x) + e^{-W} u = 0, \; y(0) = 0, \; y(T) = y_1, \; (y, y_1, u) \in \mathcal{H} \times L^2(\Omega; H) \times \mathcal{X} \right\}
\]

(3.30)

(Here, \( \frac{dy}{dt} \) is taken in sense of distributions, i.e., in \( \mathcal{D}'(0, T; H) \).) We have, therefore,

**Theorem 3.3** Assume that \( x \in L^2(\mathcal{O}) \). Then equation (3.26) has a unique variational solution \( X^* \),

\[
X^* \in L^2((0, T) \times \mathcal{O} \times \Omega), \; \varphi(X^*) \in L^1((0, T) \times \mathcal{O} \times \Omega), \; e^{-W} X \in W^{1,1}([0, T]; W^{1,1}_0(\mathcal{O})), \; \mathbb{P}\text{-a.s.}
\]

Moreover, the process \( t \to X^*(t) \) is pathwise \( W^{1,1}_0(\mathcal{O}) \)-valued continuous on \((0, T)\).

**Remark 3.4** A different treatment of equation (3.26) under the general assumptions (3.27) was developed in [7] (see also [8], Ch. 5).

### 4 Stochastic variational inequalities

Consider the stochastic differential equation

\[
dX + A_0 X \, dt + N_K(X) \, dt + \lambda X \, dt \ni X \, dW, \; t \in (0, T),
\]

\[
X(0) = x,
\]

in a real Hilbert space \( H \) with the scalar product \((\cdot, \cdot)\) and the norm \(|\cdot|\). Assume that \( x \in H \) and

(j) \( A_0 : D(A_0) \subset H \to H \) is a linear self-adjoint, positive definite operator in \( H \).

(jj) \( W \) is the Wiener process (1.2) and \( \lambda > \nu \).

(jjj) \( K \) is a closed, convex subset of \( H \) such that \( 0 \in K, \; (I + \lambda A_0)^{-1} K \subset K, \; \forall \lambda > 0 \).
Here, $N_K : H \rightarrow 2^H$ is the normal cone to $K$, that is,

$$N_K(u) = \{ \eta \in H; \langle \eta, u - v \rangle \geq 0, \forall v \in K \}. \quad (4.2)$$

By the transformation (2.1), equation (4.1) reduces to the nonlinear random differential equation

$$\frac{dy}{dt} + e^{-W} A_0(e^W(y + x)) + e^{-W} N_K(e^W(y + x)) + \mu(y + x) = 0, \quad t \in (0, T), \quad (4.3)$$

$$y(0) = 0.$$  

(We note that, if $W(t) = \sum_{j=1}^{N} \mu_j \beta_j(t)$, then (4.3) reduces to a deterministic variational inequality.)

To represent this problem as an optimization problem of the form (P), we set

$$\phi(u) = \frac{1}{2} (A_0 u, u) + I_K(u), \quad \forall u \in H,$$

where $I_K$ is the indicator function

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise}. \end{cases}$$

The function $\phi : H \rightarrow \mathbb{R}$ is convex and lower semicontinuous. Then, by (2.6), (2.18), (2.19), we have

$$\tilde{G}_1(y, y_1) = \mathbb{E} \int_0^T \left( \frac{1}{2} (A_0(e^W(t)(y(t) + x)), e^W(t)(y(t) + x)) + e^{2W}((\nu + \delta)(y + x) + \mu y)x + \frac{\delta}{2} |e^W(t)(y(t) + x)|_H^2 \\
+ I_K(e^W(t)(y(t) + x)) + \eta(e^W(t)y(t)) \right) dt$$

$$- \mathbb{E} \int_0^T (e^W y, e^W(1 + \mu)y) dt \quad (4.4),$$

$$G_2(u) = \mathbb{E} \int_0^T \psi^*(u(t)) dt, \quad (4.6)$$

where, by (2.11)-(2.12), we have

$$\tilde{G}_1(y, y_1) = \mathbb{E} \int_0^T \left( \frac{1}{2} (A_0(e^W(t)(y(t) + x)), e^W(t)(y(t) + x)) + e^{2W}((\nu + \delta)(y + x) + \mu y)x + \frac{\delta}{2} |e^W(t)(y(t) + x)|_H^2 \\
+ I_K(e^W(t)(y(t) + x)) + \eta(e^W(t)y(t)) \right) dt$$

$$- \mathbb{E} \int_0^T (e^W y, e^W(1 + \mu)y) dt \quad (4.4),$$

$$G_2(u) = \mathbb{E} \int_0^T \psi^*(u(t)) dt, \quad (4.6)$$

where, by (2.11)-(2.12), we have
\[
\psi^*(e^W u) = \sup \left\{ (e^W u, v) - \frac{1}{2} (A_0 v, v) - \frac{\delta}{2} |v|^2 ; \, v \in K \right\},
\]
where \( A_0 z + \delta z + N_K(z) \ni e^W u \). (We note that, by (iii), \( z \) is uniquely defined.)

By (4.4)-(4.6), we see that
\[
G(y, y_1, u) = \tilde{G}_1(y, y_1) + \frac{1}{2} \liminf_{n \to \infty} \mathbb{E} \int_0^T \left( (A_0 z_n, z_n) + \frac{\delta}{2} |z_n|^2 \right) dt,
\]
where \( A_0 z_n + \delta z_n + N_K(z_n) \ni u_n, \ e^W B y_n + u_n = 0, \ y_n \to y \) in \( \mathcal{H} \), \( y_n(T) \to y_1 \) in \( L^2(\Omega; H) \), \( u_n \to u \) in \( \mathcal{X} \).

This yields
\[
\mathbb{E} \int_0^T |A_0^{1/2} z_n|^2 dt \leq C < \infty, \ \forall n \in \mathbb{N},
\]
and, therefore, we have
\[
\tilde{G}(y, y_1, u) = \tilde{G}_1(y, y_1) + \frac{1}{2} \mathbb{E} \int_0^T (|A_0^{1/2} z|^2 + \delta |z|^2) dt,
\]
where \( V = D(A_0^{1/2}) \). We note that \( D(\tilde{G}_1) \subset L^2((0, T) \times \Omega; V) \).

We may conclude, therefore, by Theorem (2.4) that

**Theorem 4.1** Under hypotheses (i)–(iij), there is a unique variational solution \( X^*(t) \in K \), a.e. \( t \in (0, T) \), \( X^* \in L^2((0, T) \times \Omega) \) to equation (4.1).

More insight into the problem can be gained in the following two special cases.

**Stochastic parabolic variational inequalities**

The stochastic differential equation
\[
dX - \Delta X dt + \lambda X dt + N_K(X) dt \ni X dW \text{ in } (0, T) \times \mathcal{O},
\]
\[X(0) = x \text{ in } \mathcal{O},\]
\[X = 0 \text{ on } (0, T) \times \partial \mathcal{O},\]
where \( N_K(X) \subset L^2(\mathcal{O}) \) is the normal cone to the closed convex set \( K \) of \( L^2(\mathcal{O}) \),
\[ K = \{ z \in L^2(\mathcal{O}); \ z \geq 0, \ \text{a.e. in } \mathcal{O} \}, \ \alpha \in \mathbb{R}, \]

can be treated following the above infinite-dimensional scheme in the space \( H = L^2(\mathcal{O}) \), where \( A_0 u = -\Delta u, \ u \in D(A_0) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}). \)

Then the variational solution to (4.11) is defined by \( X = e^W y \), where \( y \) is given by (4.11) and \( G \) is given by

\[
G(y, y_1, u_1) = \tilde{G}_1(y, y_1) + \frac{1}{2} E \int_0^T \int_\Omega (|\nabla z|^2 + \delta |z|^2) d\xi \ dt,
\]

where \( G_1 \) is defined by (4.4) and \( z = w - \lim_{n \to \infty} z_n \) in \( L^2((0, T) \times \mathcal{O}; H^1_0(\mathcal{O})) \).

Since \( u_n \to u \) in \( D'(0, T; L^2(\mathcal{O})) \) and \( \eta_n(t, \xi) \leq 0 \) a.e. \( (t, \xi) \in (0, T) \times \mathcal{O} \), by (4.12), we infer that

\[ -\Delta z + \delta z + \eta = u \text{ in } D'((0, T) \times \mathcal{O}), \]

where \( \eta, u \) are in \( \mathcal{M}((0, T) \times \mathcal{O}) \) the space of bounded measures on \( (0, T) \times \mathcal{O} \). If we denote by \( \eta_a, u_a \in L^1((0, T) \times \mathcal{O}) \) the absolutely continuous parts of \( \eta \) and \( u \), we get

\[ -\Delta z + \delta z + \eta_a = u_a \text{ in } L^1(\mathcal{O}), \]

\( z \in H^1_0(\mathcal{O}) \) and \( \eta_a(t, \xi) = 0, \ \text{a.e. on } [z(t, \xi) > 0] \)

\[ \eta_a(t, \xi) \geq 0, \ \text{a.e. on } [z(t, \xi) = 0]. \]

Then the process \( X = e^W(y + x) \) is the variational solution to (4.11) and so, by Theorem 4.1, we have

**Corollary 4.2** There is a unique variational solution \( X \in L^2((0, T) \times \Omega; H^1_0(\mathcal{O})) \), \( X \geq 0, \ \text{a.e. on } (0, T) \times \Omega. \)

**Finite dimensional stochastic variational inequalities**

Consider equation (1.11) in the special case \( K \subset \mathbb{R}^d, \ \text{int } K \neq \emptyset, \ 0 \in K, \)

\( W = \sum_{i=1}^N \mu_i \beta_i \) and \( A_0 \in L(\mathbb{R}^d, \mathbb{R}^d), \ A_0 = A_0^* \). Then, as easily seen by (2.13), we have

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\[
\psi^*(u) \geq \alpha_1 |u| - \alpha_2, \quad \forall u \in \mathbb{R}^d. \tag{4.13}
\]

Let \( z_n \) be the solution to (see (4.8))

\[
A_0 z_n + \delta z_n + N_K(z_n) \ni u_n. \tag{4.14}
\]

Since, by (4.13)-(4.14), the sequence \( \{u_n\} \) is bounded in \( L^1((0,T) \times \Omega, \mathbb{R}^d) \), it follows that it is weak-star compact in \( \mathcal{M}(0,T; \mathbb{R}^d) \), \( \forall \varepsilon > 0 \), and so \( u \in \mathcal{M}(0,T; \mathbb{R}^d) \). (Here, \( \mathcal{M}(0,T; \mathbb{R}^d) \) is the space of \( \mathbb{R}^d \)-valued bounded measures on \( (0,T) \). Letting \( n \to \infty \) in (4.14), we get

\[
A_0 z + \delta z + \zeta = u, \tag{4.15}
\]

where \( u \in \mathcal{M}(0,T; \mathbb{R}^d) \), \( \forall \varepsilon > 0 \), and \( \zeta \in \mathcal{M}((0,T); \mathbb{R}^d), \mathbb{P}\text{-a.s.} \) By the Lebesgue decomposition theorem, we have

\[
z = (A_0 + \delta I + N_K)^{-1}(u_a), \quad \zeta = u_a.
\]

As a matter of fact, the singular measure \( \zeta \) belongs to the normal cone \( N_K(z) \subset \mathcal{M}(0,T; \mathbb{R}^d) \) to the set \( K = \{ \tilde{z} \in C([0,T]; \mathbb{R}^d); \tilde{z}(t) \in K, \forall t \in [0,T] \} \) and it is concentrated on the set of \( t \)-values for which \( z(t) \) defined by (4.16) lies on the boundary \( \partial K \) of \( K \).

By (2.22)-(2.23), we have

\[
\overline{G}(y, y_1, u) = \tilde{G}_1(y, y_1) + \mathbb{E} \int_0^T \left( \frac{1}{2} \langle A_0 F(u_a), F(u_a) \rangle + \frac{\delta}{2} |F(u_a)|^2 \right) dt, \tag{4.17}
\]

where \( \tilde{G}_1 \) is given by (4.4) and \( y \in \mathcal{H} \) is solution to the equation

\[
\begin{align*}
y &= y_a + y_s, \quad y_a \in AC([0,T]; \mathbb{R}^d), \quad y_s \in BV([0,T]; \mathbb{R}^d), \quad \mathbb{P}\text{-a.s.}, \\
\frac{dy_a}{dt} + (\mu + \nu + \delta)(y_a + x) + e^{-W}u_a &= 0, \text{ a.e. on } (0,T), \\
y_a(0) &= 0, \\
\frac{dy_s}{dt} + e^{-W}u_s &= 0 \text{ in } \mathcal{D}'(0,T; \mathbb{R}^d),
\end{align*}
\]

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where $BV([0,T];\mathbb{R}^d)$ is the space of functions with founded variations on $[0,T]$. We note that, by (4.17), it follows also that

$$D(G) \subset \{(y, y_1, u) \in H \times L^2(\Omega; H) \times X; y \in BV([0,T]; \mathbb{R}^d), \; \mathbb{P}\text{-a.s.,}$$

$$F(u_a) \in L^2((0,T) \times \Omega \times \mathbb{R}^d)\}$$

where $D(G) = \{(y, y_1, u); G(y, y_1, u) < \infty\}$. We have, therefore,

**Theorem 4.3** The minimization problem

$$\text{Min}\{G(y, y_1, u); \; (y, y_1, u) \in H \times L^2(\Omega; \mathbb{R}^d) \times X, \text{ subject to (4.18)}\} \tag{4.19}$$

has a unique solution $(y^*, y_1^*) \in H \times L^2(\Omega; \mathbb{R}^d)$ satisfying (4.18). The process $X^* = e^{Wy^*}$ is the solution to the variational solution to (4.17).

**Remark 4.4** Since $y^* \in BV([0,T]; \mathbb{R}^d)$ and, as seen by (4.18), the singular measure $\zeta_s = u_s \neq 0$, it follows that the process $X^*$ is not pathwise continuous on $[0,T]$. However, by the Lebesgue decomposition, we have, $\mathbb{P}$-a.s., $X^*(t) = X_a^*(t) + X_1^*(t) + X_2^*(t), \; \forall t \in [0,T]$, where $t \rightarrow X_a^*(t)e^{-W(t)}$ is absolutely continuous, $X_1^*$ is a jump function and $X_2^*$ is a singular function, that is, $X_2^* = e^{Wy_2}$, where $\frac{dy_2}{dt} = 0.$ a.e.

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