FLAT TRACES FOR A RANDOM PARTIALLY HYPERBOLIC MAP

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Abstract. We consider a $\mathbb{R}/\mathbb{Z}$ extension of an Anosov diffeomorphism of a compact Riemannian manifold by a random function $\tau$ and show that the flat traces of the transfer operator, reduced with respect to frequency in the fibers, converge in law towards Gaussians, up to an Ehrenfest time that decreases with the regularity of $\tau$. 

Date: June 29, 2020.

2010 Mathematics Subject Classification. 37D30 Partially hyperbolic systems and dominated splittings, 37E10 Maps of the circle, 60F05 Central limit and other weak theorems, 37C30 Zeta functions, (Ruelle-Frobenius) transfer operators, and other functional analytic techniques in dynamical systems.
1. Introduction
   1.1. Model
   1.2. Ruelle spectrum
   1.3. Eigenfunction Gaussian random fields
   1.4. Result
2. Proof
   2.1. Sketch of proof
   2.2. Construction of the fields $\delta r_j$

Appendix A. Upper bound for the distance between periodic points of a given period
Appendix B. Proof of Lemma 2.15
   B.1. Topological pressure
   B.2. Variationnal principle
   B.3. End of proof of Lemma 2.15
Appendix C. Proof of Proposition 1.4
Appendix D. Ruelle spectrum and flat trace

References
1. Introduction

This paper follows [Gos20] and extends its results to the case of an Anosov diffeomorphism on a compact Riemannian manifold.

The main object of study are the flat traces of the transfer operator. This operator acts by pulling back functions and its spectral properties are linked to the dynamical correlations. For Anosov diffeomorphisms, the statistical properties have been studied since the late 1960’s, with help of Markov partitions and symbolic dynamics [Bow75]. The construction of spaces, in the Anosov framework, in which the transfer operator is quasicompact was first achieved later by Blank, Keller and Liverani [BKL02].

For Anosov flows or partially hyperbolic diffeomorphisms, the neutral direction adds a substantial difficulty to the study. In this setting, Dolgopyat [Dol98] showed exponential decay of correlations for the geodesic flow on negatively curved surfaces, and Liverani [Liv04] generalized this result to all $\mathcal{C}^4$ contact Anosov flows, by constructing anisotropic Banach spaces in which the generating vector field has a spectral gap and resolvent. Tsujii [Tsu10] extended this method and showed quasicompactness of the transfer operator itself, with an explicit bound on the essential spectral radius, for contact Anosov flows, in some Hilbert spaces. Butterley and Liverani [BL07] then constructed Banach spaces to study the spectrum of general Anosov flows. Weich and Bonthoneau [BW17] on their side constructed outside the scope of compact manifolds appropriate spaces for geodesic flow on negatively curved manifolds with a finite number of cusps. Dyatlov and Guillarmou [DG16] did it for open hyperbolic systems.

A simple example of Anosov flow is the suspension of an Anosov diffeomorphism, or the suspension semi-flow of an expanding map. Pollicott [Pol85] showed exponential decay of correlations in this setting and Tsujii constructed suitable spaces for the transfer operator and gave an upper bound on its essential spectral radius in [Tsu08].

Here we consider a close model, namely a $\mathbb{R}$-extension of an Anosov diffeomorphism on a compact Riemannian manifold, for which dolgopyat [Dol02] has shown generic rapid decay of correlations. In a series of papers, de Simoi, Liverani, Poquet and Volk [DSLPV17] and de Simoi and Liverani [DSL16, DSL18] studied statistical properties of fast-slow dynamical systems. Their model generalizes $T$ extensions of circle expanding maps.

We investigate a small random perturbation of the roof function and show that the flat traces (1.5) of the iterates of the transfer operator (restricted to a given frequency $\xi$ in the fiber direction) satisfy a central limit theorem in a semiclassical regime linking time $n$ and frequency $\xi$ (theorem 1.5). We obtain convergence towards a Gaussian law up to a constant times the Ehrenfest time, this constant being a decreasing function of the regularity of the random function. The principle is the same as in [Gos20]: We show pointwise convergence of the characteristic function, by decomposing for each time $n$ the roof function as the sum of a random function that decorrelates at a scale corresponding to the minimal distance between periodic points of period $n$, and an other function that plays no role if the frequency is large enough.

Naud [Nau16] found in the case of circle extensions of some analytic Anosov maps of the torus lower bounds on the first eigenvalue, both in the deterministic and random settings. He makes use of the fact that with positive probability,
there is a lower bound on the modulus of the trace (of same order as the scaling $A_n$ from (1.13)), and takes advantage of the fact that the operator is trace class. This is not the case in our setting and we don’t know whether information on the Ruelle-Pollicott spectrum can be recovered from our estimation of the flat traces.

1.1. Model. Let $M$ be a smooth Riemannian manifold of dimension $d$ and $T : M \to M$ be a transitive Anosov diffeomorphism. This means that the tangent bundle admits a splitting $T^*_M = E^u \oplus E^s$ such that

1. $dT_x (E^i(x)) = E^i(f(x)), \ i \in \{u, s\},$

2. $\exists 0 < \lambda < 1, \exists C > 0, \forall n \in \mathbb{N}, \forall v \in E^u, \|dT^{-n} \cdot v\| \leq C \lambda^n \|v\|, \forall v \in E^s, \|dT^n \cdot v\| \leq C \lambda^n \|v\|$

and that $T$ has a dense orbit. We will be interested, given $k \geq 0$ and a $C^k$ function $\tau$, in the skew-product

$$F : \begin{cases} \mathbb{S}^* \bigtimes \mathbb{R} & \to \mathbb{S}^* \bigtimes \mathbb{R} \\ (x, y) & \mapsto (T(x), y + \tau(x)) \end{cases}.$$ 

1.2. Ruelle spectrum. To the map $F$ can be associated a transfer operator $L_\tau$ acting on $C^k(\mathbb{S}^* \bigtimes \mathbb{R})$ by composition:

$$L_\tau v = v \circ F.$$ 

Fourier analysis with respect to $y$ leads to the introduction of the family of operators on $C^k(\mathbb{S}^* \bigtimes \mathbb{R})$ indexed by $\xi \in \mathbb{R}$

$$L_{\xi,\tau} u = e^{i\xi \tau} u \circ T.$$ 

Indeed, if $v$ is a Fourier mode with respect to $y$, that is $v(x, y) = u(x) e^{i\xi y}$ for some $u \in C^k(\mathbb{S}^* \bigtimes \mathbb{R})$, $\in \mathbb{R}$,

$$L_{\xi,\tau} v(x, y) = L_{\xi,\tau} u(x) e^{i\xi y}.$$ 

These operators can be extended to distributions by duality. They have their essential spectral radius bounded by explicit constants in appropriate spaces. The operators are not trace class, but we can define a generalization of their trace, called flat trace, which has a connexion with their spectrum. See Appendix D for a brief discussion about this. The flat trace is the main object studied in this paper, we express a central limit theorem for a small random perturbation of a given function $\tau$ in the limit of large times $n$ and frequencies $\xi$ in Theorem (1.5) Its expression involves periodic points and is given by

$$\text{Tr}^\flat(L_{\xi,\tau}^n) = \sum_{x, T^n(x) = x} \frac{e^{i\xi \tau^n_x}}{\det(1 - d(T^n)_x)},$$

where $\tau^n_x$ denotes the Birkhoff sum

$$\tau^n_x := \tau(x) + \tau(T(x)) + \cdots + \tau(T^{n-1}(x)).$$
1.3. Eigenfunction Gaussian random fields.

**Definition 1.1.** We will call centered Gaussian field on $M$ a random distribution of the form

$$f = \sum_{j \geq 0} c_j \zeta_j \phi_j,$$

where the $c_j \geq 0$ grow at most polynomially with $j$, $\zeta_j$ are i.i.d centered Gaussian random variables of variance 1 and $(\phi_j)_j$ is a Hilbert basis of eigenfunctions of the Laplace-Beltrami operator:

$$\Delta \phi_j = \lambda_j \phi_j, \quad 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots.$$

This sum is in general understood in the sense of distributions, Proposition 1.4 thereafter expresses a link between the growth of $(c_j)_{j \geq 0}$ and the regularity of the field. We will only be interested in at least continuous fields in what follows.

**Example 1.2.** If $c_j = 1$ for all $j$, the random field $W$ is called white noise:

$$W = \sum_{j \geq 0} \zeta_j \phi_j.$$

It is a random distribution, almost surely not in $L^2(M)$.

On $M$, we have a notion of Sobolev spaces:

**Definition 1.3.** Let $s \in \mathbb{R}$. The Sobolev space $H^s(M)$ is defined by

$$H^s(M) = (1 + \Delta)^{-s/2} L^2(M).$$

**Proposition 1.4.** Assume that

$$c_j = O(j^{-\alpha})$$

for some $\alpha \in \mathbb{R}$. Then, almost surely, the centered Gaussian field

$$f := \sum_{j \geq 0} c_j \zeta_j \phi_j \in H^s(M)$$

for every $s < d(\alpha - \frac{1}{2})$. Thus,

$$f \in C^k(M)$$

for every $k < d(\alpha - 1)$ (where $C^k(M)$ is understood as $(C^{-k}(M))^\prime$ for negative $k$).

**Proof.** See appendix C. □
1.4. **Result.** If $x$ is a periodic point of $T$, we write its primitive period $m_x$. Let us define the amplitudes $A_n$ by

\begin{equation}
A_n := \left( \sum_{T^n(x) = x} \frac{m_x}{|\det(1 - dT^n_x)|^2} \right)^{-\frac{1}{2}}.
\end{equation}

Let also

\begin{equation}
\Lambda^\pm := \lim_{n \to \infty} \max_{x \in M} \|dT_x^{\pm n}\|^{\pm \frac{1}{2}}
\end{equation}

and

\begin{equation}
\Lambda := \max(\Lambda^+) = \Lambda^-. \tag{1.16}
\end{equation}

Let $h_{\text{top}} \leq \frac{2}{d^2} \log \Lambda$ be the topological entropy of the map $T$ (see Definition 3.1.3 in [KH97]).

**Theorem 1.5.** Let us fix any $\tau_0 \in C^0(M)$ and $\varepsilon > 0$. Let

\begin{equation}
\delta \tau = \sum_{j \geq 0} c_j \zeta_j \phi_j
\end{equation}

be a centered Gaussian field with $\zeta_j$ i.i.d. $N(0,1)$ such that

\begin{equation}
\frac{1}{C} j^{-\alpha} \leq c_j \leq C j^{-\beta}
\end{equation}

for some constants $C > 0, 0 < \beta > \alpha > 1$. By Proposition 1.4, the condition involving $\beta$ ensures that $\delta \tau$ is almost surely continuous, and that we can define the flat trace of $L_{x,\tau}$ for $\tau := \tau_0 + \varepsilon \delta \tau$. Then, for any $0 < c < 1$, we have the following convergence in law

\begin{equation}
A_n Tr^\flat (L_{x,\tau}^n) \rightsquigarrow \mathcal{N}(0,1)
\end{equation}

as $n$ and $\xi$ go to infinity under the relation

\begin{equation}
n \leq c \frac{\log \xi}{h_{\text{top}} + \frac{d}{2}(\alpha - \frac{1}{2}) \log \Lambda}. \tag{1.20}
\end{equation}

**Remark 1.6.** $\beta$ plays no other role than to make sure that the flat trace is well defined.

**Remark 1.7.** If for some reason we want to impose a certain regularity on the function $\delta \tau$ using Proposition 1.4, we need to take $\alpha$ large enough, and the larger it is, the more restrictive condition 1.20 imposed on the time $n$ is.

**Remark 1.8.** In [Gos20], we obtained instead of 1.20 the condition

\begin{equation}
n \leq c \frac{\log \xi}{\log l + (k + \frac{1}{2} + \frac{1}{2}) \log M}. \tag{1.21}
\end{equation}

In this setting, $k + \frac{1}{2} + \frac{1}{2}$ was the analog of $\alpha$, and $\log l$ the topological entropy. $M$ was analogous to $\Lambda^\#$. The dimension was 1 and the exponent $\frac{1}{2}$ comes from the fact that $T$ is invertible, while $E$ was not and had therefore possibly denser
periodic points. $M = \sup E'$ could have been refined as $\lim_n (\sup(E')^n)^{\frac{1}{n}}$ to match the definition of $\Lambda$. With this in mind, the bound of [Gos20] translates to

$$n \leq c \frac{\log \xi}{h_{\text{top}} + \frac{1}{2} \alpha \log \Lambda}.$$  

We obtain here a slightly better bound, with $\alpha - \frac{1}{2}$ instead of $\alpha$, due to the fact that our proof directly deals with the multivariate Gaussian probability density in a space of dimension approximately $e^{nh_{\text{top}}}$ instead of reducing to the unidimensional variables.

2. Proof

Remark 2.1. Let us first remark that the convergence in law stated in theorem 1.5 only involves the law of $\delta \tau$. Therefore, we will abusively name $\delta \tau$ another field that has the same law.

2.1. Sketch of proof. Let us choose $\alpha > 1$, $\eta > 0$ and $\delta \tau$ as in the statement of Theorem 1.5. We will show that condition (1.18) allows us to construct a centered Gaussian field with the same law as $\delta \tau$, as a sum of independent Gaussian fields

$$\delta \tau = \delta \tau_0 + \sum_{j \geq 1} \delta \tau_j,$$

so that the covariances $\mathbb{E}[\delta \tau_j(x)\delta \tau_j(y)]$ become very small at a distance greater than $\Lambda' - \frac{1}{2}$, for some $\Lambda' > \Lambda$ that we will choose small enough (i.e. close enough to $\Lambda$), which is smaller than the minimal distance between two periodic points of large period $j$, as we know from Lemma A.1. Here we have used the abusive notation described in the preliminary remark 2.1. Therefore, the phases appearing in the trace formula (1.5) will behave as independent random variables on $S^1$, almost uniform when $\xi$ is large enough, that is, under the condition (1.20).

2.2. Construction of the fields $\delta \tau_j$. Let $\chi \in \mathcal{S}(\mathbb{R})$ be a positive Schwartz function such that $\chi(0) = 0$, normalized so that

$$\int \chi^2 \, dx = (2\pi)^d.$$

Let us define the family of operators

$$P_h := \chi(h^2 \Delta).$$

Proposition 2.2 ([Zwo12] Theorem 14.9 p.358 and Theorem 9.6 p.209). $P_h$ is a $h$-pseudodifferential operator and its Schwartz kernel $K_h$ satisfies

$$\forall N > 0, \exists C_N > 0, \forall x \neq y, |K_h(x, y)| \leq \frac{C_N h^N}{d(x, y)^N}.$$  

Let $W_j$ be a family of independent white noises (defined in [L20]) independent of $\delta \tau$. Let $\gamma > 0$ and $\Lambda > \Lambda$ to be chosen small enough later and

$$h_j = \frac{\Lambda^{-\frac{1}{2}}}{\Lambda}.$$
\( P_{h_j} \) is a positive selfadjoint operator, so we can define
\[
\delta \tau_j := h_j^{d_{\alpha} + \gamma} \sqrt{P_{h_j} W_j}.
\]
In other terms, there exist i.i.d. random variables \( \zeta_{j,k} \) of law \( \mathcal{N}(0, 1) \) such that
\[
\delta \tau_j = h_j^{d_{\alpha} + \gamma} \sum_{k \geq 1} \sqrt{\lambda_k} \zeta_{j,k} \phi_k.
\]

**Lemma 2.3.** Let \( K_j \) be the Schwartz kernel of the operator \( h_j^{2(d_{\alpha} + \gamma)} P_{h_j} \). Then,
\[
E[\delta \tau_j(x) \delta \tau_j(y)] = K_j(x, y).
\]
Moreover, on the diagonal, \([\text{Zwo12, Theorem 14.10 p.361}]\)
\[
K_j(x, x) = h_j^{d(2\alpha - 1) + 2\gamma}(1 + O(h_j)).
\]

**Proof.** Indeed, on one hand
\[
E[\delta \tau_j(x) \delta \tau_j(y)] = h_j^{2(d_{\alpha} + \gamma)} \sum_{k, k' \geq 0} \sqrt{\lambda_k} \sqrt{\lambda_{k'}} E[\zeta_{j,k} \zeta_{j,k'} \phi_k(x) \phi_{k'}(y)]
\]
and on the other hand, since
\[
P_{h_j} \phi_k = \chi(h_j^2 \lambda_k) \phi_k,
\]
we have for any \( u, v \in L^2(M) \)
\[
\langle v, P_{h_j} u \rangle_{L^2(M)} = \int v(x) \sum_{k \geq 0} (P_{h_j} \phi_k)(x) \int u(y) \phi_k(y) \, dy \, dx
\]
\[
= \int \left( \sum_{k \geq 0} \chi(h_j^2 \lambda_k) \phi_k(x) \phi_k(y) \right) v(x) u(y) \, dx \, dy.
\]

Let us choose \( \Lambda < \Lambda' < \tilde{\Lambda} \) (recall that \( \tilde{\Lambda} \) is involved in the definition of \( h_j \) in \( (2.5) \)). As a consequence of Lemma [A.1] Proposition [2.2] and the definition of \( h_j \) we have the following decay:
\[
\forall N > 0, \exists C_N > 0, \forall x \neq y \in M, \forall j \text{ large enough,}
\]
\[
(T^j x = x \text{ and } T^j y = y) \implies |K_j(x, y)| \leq C_N \frac{h_j^{N+2(d_{\alpha} + \gamma)}}{d(x, y)^N}
\]
\[
\leq C_N \frac{h_j^N}{d(x, y)^N}
\]
\[
\leq C_N \frac{\Lambda'}{\Lambda} \right)^{\frac{N}{2j}}
\]
for some \( C > 0 \). Thus,
\( \forall k > 0, \exists C_k > 0, \forall x \neq y \in M, (T^j x = x \text{ and } T^j y = y) \implies |K_j(x, y)| \leq C_k e^{-kj}. \)

**Lemma 2.4.** Let us write as in Theorem 1.5

\[
\delta \tau = \sum_{k \geq 0} c_k \zeta_k \phi_k.
\]

We have

\[
\sum_{j \geq 1} \delta \tau_j = \sum_{k \geq 0} c'_k \zeta'_k \phi_k,
\]

where \((\zeta'_k)_k\) is a family of i.i.d. random variables of law \(N(0, 1)\) independent of the variables \(\zeta_k\), and

\[
c'_k = O(c_k).
\]

**Proof.**

\[
\sum_{j} \delta \tau_j = \sum_{j \geq 1} \left( \sum_{k \geq 1} h^{d\alpha + \gamma}_j \sqrt{\chi(h^2_j \lambda_k)} \zeta_{j,k} \right) \phi_k.
\]

Since every variables are independent from each other, the

\[
\left( \sum_{j \geq 1} h^{d\alpha + \gamma}_j \sqrt{\chi(h^2_j \lambda_k)} \zeta_{j,k} \right)_k
\]

are independent Gaussian variables of variances

\[
c'_k^2 := \sum_{j \geq 1} h^{2d\alpha + 2\gamma}_j \chi(h^2_j \lambda_k).
\]

Now, since \(\chi \in \mathcal{S}(\mathbb{R})\),

\[
\exists C > 0, \forall j, k, \chi(h^2_j \lambda_k) \leq \frac{C}{h^{2d\alpha}_j \lambda_k^{d\alpha}}.
\]

Also,

\[
\sum_{j \geq 1} h^{2\gamma}_j < \infty
\]

so the variances satisfy

\[
c_k'^2 = \sum_{j \geq 1} h^{2d\alpha + 2\gamma}_j \chi(h^2_j \lambda_k) = O(\lambda_k^{-d\alpha})
\]

By Weyl’s law \cite[(14.3.21) p.362]{Zwo12} there exists a constant \(C\) depending on \(M\) such that

\[
\lambda_k^d \sim C k^2
\]
Corollary 2.5. Consequently, up to the multiplication of each \( \delta \tau_j \) by the same constant, condition (1.18) allows us to define the field

\[
\delta \tau_0 := \sum_{k \geq 1} \sqrt{c_k^2 - c_k'^2} \zeta_k'' \phi_k,
\]

for i.i.d. random variables \( \zeta_k'' \) of law \( \mathcal{N}(0,1) \) independent of the variables \( \zeta_k \) and \( \zeta_k' \). According to Remark 2.1, we will abusively write

\[
\delta \tau = \sum_{j \geq 0} \delta \tau_j.
\]

Definition 2.6. We will use the following notations for periodic orbits in this paper: \( \text{Per}(n) \) will be the set of periodic orbits of period \( n \), while \( \mathcal{P}_m \) will be the set of periodic orbits of primitive period \( m \). This way, we have a disjoint union

\[
\text{Per}(n) = \coprod_{m \mid n} \mathcal{P}_m.
\]

If \( O \in \text{Per}(n) \), then the Birkhoff sums \( f_n^x = \sum_{k=0}^{n-1} f(T^k x) \) do not depend on the point \( x \in O \) and will be written \( f_n^O \). Similarly, \( \det(1 - d(T^n)_x) \) will denote the Jacobian \( \det(1 - d(T^n))_x \) for any \( x \in O \).

Proposition 2.7 ([KH97, Theorem 18.5.5 p.585]). There exists \( C > 0 \) such that

\[
\frac{1}{C} e^{nh_{\text{top}}} \leq \# \text{Per}(n) \leq Ce^{nh_{\text{top}}}.
\]

Remark 2.8. Note that, since we assume the map \( T \) to be transitive, by the Closing Lemma [KH97, Theorem 6.4.15 p.269] it has periodic orbits of arbitrary large period. Therefore, necessarily, \( h_{\text{top}} > 0 \). Thus,

\[
\# \text{Per}(n) \rightarrow \infty.
\]

Remark 2.9. Let us notice that Lemma [A.1] implies

\[
h_{\text{top}} \leq \frac{d}{2} \log \Lambda.
\]

Proof. Let indeed \( \Lambda' > \Lambda \). \( M \) can be covered by \( O(\Lambda'^{-d/2}) \) balls of radius \( \Lambda'^{-d/2} \). The constraint of Lemma [A.1] implies that each ball of radius \( \Lambda'^{-d/2} \) contains a bounded number of points of \( \text{Per}(n) \). (2.29) then implies

\[
e^{nh_{\text{top}}} \leq C \Lambda'^{-d/2}
\]

for every \( \Lambda' > \Lambda \). \( \Box \)

Recall that

\[
\tau = \tau_0 + \varepsilon \delta \tau = \tau_0 + \varepsilon \sum_{j \geq 0} \delta \tau_j.
\]
Lemma 2.10. We can write

\[ \text{Tr}^h(L^n_{\xi,r}) = \sum_{m|n} \sum_{O \in \mathcal{P}_m} e^{i\xi(X^n_O + Y^n_O)} \frac{\det(1 - dT^n_O)}{\left|\det(1 - dT^n_O)\right|^4} \]

where

1. For all \( n \in \mathbb{N} \), \( (X^n_O)_{O \in \text{Per}(n)} \) is a Gaussian random vector such that

\[ \exists C > 0, \forall n \in \mathbb{N}, \forall O \in \text{Per}(n), \quad \frac{1}{C} m h_n d(2\alpha - 1) + 2\gamma \leq \sigma^n_O := \mathbb{E}[(X^n_O)^2] \leq C n^2 h_n d(2\alpha - 1) + 2\gamma \]

and

2. For every integer \( n \) the random variable \( (X^n_O)_{O \in \text{Per}(n)} \) is independent of \( (X^n_O)_{O \in \text{Per}(n)} \).

Proof of lemma 2.10. Using (1.5), and the fact that the Birkhoff sums and differentials \( dT^n_O \) only depend on the orbit, one can pack the terms

\[ \text{Tr}^h(L^n_{\xi,r}) = \sum_{m|n} \sum_{O \in \mathcal{P}_m} e^{i\xi X^n_O} \frac{\det(1 - dT^n_O)}{\left|\det(1 - dT^n_O)\right|^4} \]

Now, we isolate the term \( \delta \tau_n \) in (2.33) (where \( n \) is the time appearing in the expression \( \text{Tr}^h(L^n_{\xi,r}) \)) and set

\[ X^n_O := (\delta \tau_n)_O \]

and

\[ Y^n_O := \left( \frac{\tau_0}{\varepsilon} + \sum_{j \neq n} \tau_j \right)^n_O. \]

The independence of the family \( (\zeta_{j,n})_{n \geq 0} \cup (\zeta'_{k})_{k \geq 0} \) involved in (2.7) and Corollary 2.5 gives the independence between the families \( (X^n_O)_{O \in \text{Per}(n)} \) and \( (Y^n_O)_{O \in \text{Per}(n)} \). Then, for an orbit \( O \in \mathcal{P}_m \subset \text{Per}(n) \),

\[ X^n_O = (\delta \tau_n)_O = \frac{m}{n} \sum_{x \in O} \delta \tau_n(x). \]

Thus \( X^n_O \) is a Gaussian random variable and

\[ (X^n_O)^2 = \frac{n^2}{m^2} \left( \sum_{x \in O} \delta \tau_n(x)^2 + \sum_{x \neq y \in O} \delta \tau_n(x) \delta \tau_n(y) \right). \]
So
\[
\mathbb{E}[(X^n_O)^2] = \frac{n^2}{m^2} \left( \sum_{x \in O} K_n(x, x) + \sum_{x, y \in O \atop x \neq y} K_n(x, y) \right)
\]
for every \(\alpha > 0\). Similarly
\[
\mathbb{E}[X^n_O X^n_{O'}] = \mathbb{E} \left[ \sum_{x \in O} \delta \tau_n(x) \sum_{y \in O'} \delta \tau_n(y) \right] = \sum_{x \in O, y \in O'} K_n(x, y) = O(e^{-\alpha n})
\]
for every \(\alpha > 0\).

This gives the expressions (2.35) and (2.36). □

Proposition 2.11. The characteristic function of the rescaled flat traces
\[
E(\mu, \nu) := \mathbb{E} \left[ \exp \left( i \langle \mu, A_n \text{Tr}^b(\mathcal{L}^n_{\xi, \tau}) \rangle \right) \right]
\]
converges pointwise towards \(e^{-\mu^2 + \nu^2} \).

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E(\mu, \nu) := \mathbb{E} \left[ \exp \left( i \langle \mu, A_n \text{Tr}^b(\mathcal{L}^n_{\xi, \tau}) \rangle \right) \right]
\]
satisfies
\[
E(\mu, \nu) \sim \prod_{m \mid n} \prod_{O \in \mathcal{P}_m} \int \exp \left( i \frac{m A_n}{\det(1 - d T^n_O)} \left( \mu \cos x + \nu \sin x \right) \right) dx
\]
under condition (1.20).

Proof. Let \(n \in \mathbb{N}\) and \(m \mid n\). Let us write
\[
N := \# \text{Per}(n) \sim e^{n h_{top}} = O \left( \Lambda^\frac{m^2}{n} \right).
\]
Let us write for \(O \in \mathcal{P}_m\) and \(t \in \mathbb{R}\)
\[
f_O(t) := \exp \left( i \frac{m A_n}{\det(1 - d T^n_O)} \left( \mu \cos(t) + \nu \sin(t) \right) \right)
\]
and for \(x \in \mathbb{R}^N\)
\[
g_n(x) := \frac{e^{\frac{i}{2} \langle x, \Sigma^{-1} x \rangle}}{\sqrt{(2\pi)^N \det(\Sigma_n)}},
\]
where \(\Sigma_n\) is the covariance matrix of \((X^n_O)_{O \in \text{Per}(n)}\): From Lemma 2.10.
\[\Sigma_n = \text{Diag}(\sigma_O^2)_{O \in \text{Per}(n)} + R_n,\]

where \(R_n\) is a matrix with entries uniformly \(O(e^{-kn})\) for every \(k > 0\).

The proposition will quickly lead to the pointwise convergence of the characteristic function, and its proof is a consequence of the following technical lemmas:

**Lemma 2.12.** For \(k \in \mathbb{Z}^N\), let
\[
H_k := \prod_{O \in \text{Per}(n)} \left[ \frac{2\pi}{\xi} k_O, \frac{2\pi}{\xi} (k_O + 1) \right].
\]

\[
\frac{E(\mu, \nu)}{\prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_0(t)dt} - 1 \leq 4\pi \sqrt{N} \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\|
\]

**Lemma 2.13.** \(\sqrt{N} \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\| \to 0\) under condition (1.20).

**Proof of Lemma 2.12.** Using Lemma 2.10, we see that
\[
E(\mu, \nu) = \int_{\mathbb{R}^{\text{Per}(n)}} \left( \int_{\mathbb{R}^{\text{Per}(n)}} \left( \prod_{O \in \text{Per}(n)} f_0(\xi(x_O + y_O)) \right) g_n(x)dx \right) d\mathbb{P}_Y(y)
\]

Let us write for a given \(y = (y_O)_{O \in \text{Per}(n)}\)
\[
E_y(\mu, \nu) = \int_{\mathbb{R}^{\text{Per}(n)}} \left( \prod_{O \in \text{Per}(n)} f_0(\xi(x_O + y_O)) \right) g_n(x)dx
\]

Since, for fixed \(y\) the functions \(x_O \mapsto f_0(\xi(x_O + y_O))\) are fast oscillating periodic functions, of period \(\frac{2\pi}{\xi}\), while the Gaussian factor is almost constant at this scale, we approximate the integral by splitting the space into hypercubes \(H_k\) of side length \(\frac{2\pi}{\xi}\).

Each \(H_k\) has diameter \(2\pi \sqrt{N}\). Let us notice that \(\int g_n = 1\) and that for every \(k \in \mathbb{Z}^N\) and \(y = (y_O)_{O \in \text{Per}(n)}\),
\[
\int_{H_k} \left( \prod_{O \in \text{Per}(n)} f_0(\xi(x_O + y_O)) \right) dx = \frac{1}{\xi^N} \sum_{O \in \text{Per}(n)} \int_0^{2\pi} f_0(t)dt.
\]

For any fixed \(y = (y_O)_{O \in \text{Per}(n)} \in \mathbb{R}^{\text{Per}(n)}\), by periodicity,
\[
\left| E_y(\mu, \nu) - \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_0(t)dt \right| = \left| E_y(\mu, \nu) - \left( \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_0(t)dt \right) \int g_n \right|
\]
\[
\leq \left| E_y(\mu, \nu) - \left( \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_0(t)dt \right) \left( \frac{2\pi}{\xi} \right)^N \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} g_n \left( \frac{2\pi k}{\xi} \right) \right|
\]
\[
+ \left| \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_0(t)dt \right| \left( \frac{2\pi}{\xi} \right)^N \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} g_n \left( \frac{2\pi k}{\xi} \right) - \int g_n \right|
\]
Splitting the integral in (2.54) into a sum of integrals over the $H_k$, and using (2.55) allows us to bound the first term of the right hand-side of (2.56) by

\[
(2.57) \quad \left| \sum_{k \in \mathbb{Z}^{\perp(n)}} \int_{H_k} \left( \prod_{O \in \text{Per}(n)} f_O(x) \right) \left( g_n(x) - g_n \left( 2\pi \frac{k}{\xi} \right) \right) \, dx \right| 
\leq \left| \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) \, dt \right| 2\pi \frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^N} \sup_{H_k} \| \nabla g_n \| 
\]

by mean-value inequality. Likewise, in the second term,

\[
(2.58) \quad \left| \left( \frac{2\pi}{\xi} \right)^N \sum_{k \in \mathbb{Z}^{\perp(n)}} g_n \left( 2\pi \frac{k}{\xi} \right) - \int g_n \right| = \left| \sum_{k \in \mathbb{Z}^{\perp(n)}} \int_{H_k} \left( g_n \left( 2\pi \frac{k}{\xi} \right) - g_n(x) \right) \, dx \right| 
\leq 2\pi \frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^{\perp(n)}} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \| \nabla g_n \|.
\]

This ends the proof of Lemma 2.12. \hfill \Box

**Proof of Lemma 2.13**

\[
(2.59) \quad \| \nabla g_n(x) \| = \frac{\| \Sigma_n^{-1} x \| e^{-\frac{1}{2} \| \sqrt{\Sigma_n^{-1}} x \| ^2} \} \sqrt{2\pi}^N \det(\Sigma_n)^{\frac{1}{2}}}{\sqrt{(2\pi)^N \det(\Sigma_n)^{\frac{1}{2}}}} \leq \| \sqrt{\Sigma_n^{-1}} x \| e^{-\frac{1}{2} \| \sqrt{\Sigma_n^{-1}} x \| ^2} \sqrt{(2\pi)^N \det(\Sigma_n)^{\frac{1}{2}}}
\]

Since $\| \nabla g_n \|$ is a function of $\| \sqrt{\Sigma_n^{-1}} x \|$, we will pack the terms of the sum

\[
(2.60) \quad \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \| \nabla g_n \|
\]

by level sets of $\| \Sigma_n^{-1} \|$. Let us first observe that, by (2.50)

\[
(2.61) \quad \sqrt{\Sigma_n^{-1}} = \text{Diag} \left( \frac{1}{\sigma_O} \right)_{O \in \text{Per}(n)} + O(e^{-kn})
\]

for every $k$ so, by Lemma 2.10.

\[
(2.62) \quad \| \sqrt{\Sigma_n^{-1}} \| = O(h_n^{-d(\alpha - \frac{1}{2}) - \frac{1}{2}})
\]

Provided that $\Lambda > \Lambda$ and $\gamma > 0$ are chosen small enough with respect to $\frac{1}{2}$ in (1.20), writing $1 + \delta := \frac{1}{2}$, we have

\[
(2.63) \quad \xi \geq e^{\frac{1}{2}h_{top}} \Lambda^{\frac{d}{2}}(\alpha - \frac{1}{2}) \geq e^{(1+\delta)h_{top} \Lambda^{\frac{d}{2}}(\alpha - \frac{1}{2}) + \gamma) \geq C N^{1+\delta} \| \sqrt{\Sigma_n^{-1}} \|.
\]
So

\( \exists \delta > 0, \| \sqrt{\Sigma_n^{-1}} \| = o \left( \frac{\xi}{N^{1+\delta}} \right) \).

Let us now fix such a \( \delta \) and write for \( j \geq 0 \)

\( \tag{2.65} C_j := \left\{ x \in \mathbb{R}^N, \frac{j}{N^{1+\delta}} < \| \sqrt{\Sigma_n^{-1}} x \| \leq \frac{j+1}{N^{1+\delta}} \right\}. \)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\linewidth]{fig1.png}
\caption{The hypercubes \( H_k \), pictured as a grid, have small diameter, relatively to the distance between the annuli \( C_j, C_{j+2} \) (or \( C_j \) and \( C_{j-2} \)). Consequently, the supremum \( \sup_{H_k} \| \nabla g_n \| \) in (2.60) can be replaced by a supremum over \( C_{j-1} \cap C_j \cap C_{j+1} \) if \( H_k \) intersects \( C_j \).

The diameter of the cubes \( H_k \) is then very small compared to the distance between \( C_j \) and \( C_{j+2} \) for every \( j \) (see Figure 1): Indeed,

\( \tag{2.66} x \in C_j \implies \| \sqrt{\Sigma_n^{-1}} x \| \leq \frac{j+1}{N^{1+\delta}} \)

and

\( \tag{2.67} y \in C_{j+2} \implies \| \sqrt{\Sigma_n^{-1}} y \| \geq \frac{j+2}{N^{1+\delta}}. \)

So by triangular inequality

\( \tag{2.68} \frac{1}{N^{1+\delta}} \leq \| \sqrt{\Sigma_n^{-1}} y \| - \| \sqrt{\Sigma_n^{-1}} x \| \leq \| \sqrt{\Sigma_n^{-1}} (x-y) \| \leq \| \sqrt{\Sigma_n^{-1}} \| \| x-y \|. \)
Thus by \((2.64)\), for all \(j, k\),

\[
(2.69) \quad d(C_{j-2}, C_j) \geq \frac{1}{N^{\frac{1}{1+\delta}} \|\sqrt{\Sigma_n}^{-1} x\|} \geq C \frac{N^{\frac{1}{2} + \delta}}{\xi} \gg 2\sqrt{\frac{N}{\xi}} = \text{Diam}(H_k).
\]

Consequently, we have

\[
(2.70) \quad \{H_k, H_k \cap C_j \neq \emptyset\} \subset C_{j-1} \cup C_j \cup C_{j+1}.
\]

Thus,

\[
(2.71) \quad \sum_{k \in \mathbb{Z}^N} \left(\frac{2\pi}{\xi}\right)^N \sup_{H_k} \|\nabla g_n\| \leq \sum_{j \in \mathbb{Z}^N} \sum_{k, H_k \cap C_j \neq \emptyset} \left(\frac{2\pi}{\xi}\right)^N \sup_{H_k} \|\nabla g_n\|
\]

\[
\leq \sum_{j \in \mathbb{Z}^N} \#\{k, H_k \cap C_j \neq \emptyset\} \left(\frac{2\pi}{\xi}\right)^N \sup_{C_{j-1} \cup C_j \cup C_{j+1}} \|\nabla g_n\|.
\]

We deduce from \((2.70)\) that

\[
\#\{k, H_k \cap C_j \neq \emptyset\} \left(\frac{2\pi}{\xi}\right)^N = \#\{k, H_k \cap C_j \neq \emptyset\} \cdot \text{Vol}(H_k)
\]

\[
\leq \text{Vol} \left( \bigcup_{l \leq j+1} C_l \right)
\]

\[
= \text{Vol}\{x, \|\sqrt{\Sigma_n}^{-1} x\| \leq \frac{j + 2}{N^{1+\delta}}\}
\]

\[
= \sqrt{\det \Sigma_n} \left(\frac{j + 2}{N^{1+\delta}}\right)^N \cdot \text{Vol}(B(0, 1)).
\]

And we also have

\[
(2.72) \quad \sup_{C_{j-1} \cup C_j \cup C_{j+1}} \|\nabla g_n\| \leq \left(\frac{2\pi}{\xi}\right)^N \text{Vol}(B(0, 1))
\]

\[
\leq \text{Vol}(B(0, 1)) = \frac{(2\pi)^N}{\sqrt{\pi}} \text{Vol}(B(0, 1)).
\]

We know moreover (see for instance \([B+97]\) p.5) that the unit ball of dimension \(N\) has a volume equivalent to

\[
(2.74) \quad \frac{1}{\sqrt{\pi}} \frac{(2\pi e)^{N/2}}{N^{N/2}}.
\]
Finally, putting together (2.71), (2.72), (2.73), and (2.74), we have the following upperbound for the sum

\[
\frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}} \left( \frac{2\pi}{\xi} \right)^N \max_{H_k} \| \nabla g_n \| \leq \left( 2.71 \right) \leq \frac{\sqrt{N}}{\xi} \text{Vol} B(0, 1) \sqrt{\det \Sigma_n} \sum_{j \geq 0} \left( j + \frac{2}{N^{1/2}} \right)^{N+1} e^{-\frac{(j-1)^2}{2 \left( \frac{1}{N^{1/2}} + \delta \right)}} \leq \left( 2.73 \right) \leq C \frac{e^{N/2}}{N^{(N+1)(1+\frac{2}{N})}} \frac{1}{N^{1/2}} \sum_{j \geq 0} \left( j + \frac{2}{N^{1/2}} \right)^{N+1} e^{-\frac{(j-1)^2}{2 \left( \frac{1}{N^{1/2}} + \delta \right)}} = C \frac{e^{N/2}}{N^{(N+1)(1+\frac{2}{N})}} \frac{1}{N^{1/2}} \sum_{j \geq 0} \left( j + \frac{2}{N^{1/2}} \right)^{N+1} e^{-\frac{(j-1)^2}{2 \left( \frac{1}{N^{1/2}} + \delta \right)}}.
\]

To conclude, we will now show that

\[
\left( 2.75 \right) \leq \frac{e^{N/2}}{N^{(N+1)(1+\frac{2}{N})}} \sum_{j \geq 0} \left( j + \frac{2}{N^{1/2}} \right)^{N+1} e^{-\frac{(j-1)^2}{2 \left( \frac{1}{N^{1/2}} + \delta \right)}} = O \left( N^{1/2+\frac{2}{N}} \right): \]

**Lemma 2.14.** Let

\[
\left( 2.76 \right) \frac{e^{N/2}}{N^{(N+1)(1+\frac{2}{N})}} \sum_{j \geq 0} \left( j + \frac{2}{N^{1/2}} \right)^{N+1} e^{-\frac{(j-1)^2}{2 \left( \frac{1}{N^{1/2}} + \delta \right)}} = O \left( N^{1/2+\frac{2}{N}} \right):
\]

It is first increasing then decreasing and admits a maximum \(O(1)\) at

\[
\left( 2.77 \right) f_N : \begin{cases} \mathbb{R} & \mapsto \mathbb{R} \\ x & \mapsto \frac{e^{N/2}}{N^{(N+1)(1+\frac{2}{N})}} \left( x + \frac{2}{N^{1/2}} \right)^{N+1} e^{-\frac{(x-1)^2}{2 \left( \frac{1}{N^{1/2}} + \delta \right)}}. \end{cases}
\]

It is first increasing then decreasing and admits a maximum \(O(1)\) at

\[
\left( 2.78 \right) x_0 \sim N^{1+\frac{2}{N}}
\]

and decays at a scale \(N^{1+\frac{2}{N}+\delta}\): If we write for \(k \geq -1\),

\[
\left( 2.79 \right) x_k = x_0 + k N^{1+\frac{2}{N}+\delta},
\]

we have

\[
\left( 2.80 \right) f_N(x_k) = O(e^{k-\frac{4}{N}k^2N^{\frac{2}{N}}})
\]
as \(N\) goes to infinity.

**Proof.** Differentiating the logarithm of \(f_N\) tells us that the function is increasing, then decreasing, and that the maximum is attained for \(x_0\) such that

\[
\left( 2.81 \right) x_0^2 + x_0 - 2 = N^{2+\delta} + N^{1+\delta}.
\]

This shows that

\[
\left( 2.82 \right) x_0 = N^{1+\frac{2}{N}} + \frac{1}{2} N^{\frac{2}{N}} (1 + o(1)),
\]

so

\[
\left( 2.83 \right) x_0 = N^{1+\frac{2}{N}} + O(N^{\frac{2}{N}}) = N^{1+\frac{2}{N}} (1 + O(N^{-1}))
\]
and hence

\[
\begin{align*}
(2.84) \quad \max \log f_N &= \frac{N}{2} - (N + 1)(1 + \frac{\delta}{2}) \log N \\
+ (N + 1) \log \left( N^{1 + \frac{\delta}{2}} (1 + O(N^{-1})) \right) - \frac{1}{2} \left( N^{1 + \frac{\delta}{2}} + O(N^{\frac{\delta}{2}}) \right)^2 N^{1 + \frac{\delta}{2}} = O(1)
\end{align*}
\]

\[
(2.85)
\]

\[x_k = x_0 + kN^{\frac{\delta}{2} + \frac{\delta}{6}} = N^{1 + \frac{\delta}{2}} + kN^{\frac{\delta}{2} + \frac{\delta}{6}} + O(N^{\frac{\delta}{2}}) = N^{1 + \frac{\delta}{2}} (1 + kN^{-\frac{\delta}{2}} + O(N^{-1})).\]

The remainder does not depend on \(k\). Therefore, for \(N \geq 1\)

\[
(2.86) \quad \log f_N(x_k) = \frac{N}{2} - (N + 1)(1 + \frac{\delta}{2}) \log N \\
+ (N + 1) \log \left( N^{1 + \frac{\delta}{2}} (1 + kN^{-\frac{\delta}{2}} + O(N^{-1})) \right) - \frac{1}{2} \left( N^{1 + \frac{\delta}{2}} + kN^{\frac{\delta}{2} + \frac{\delta}{6}} + O(N^{\frac{\delta}{2}}) \right)^2 N^{1 + \frac{\delta}{2}} \leq k(N + 1)N^{-\frac{\delta}{2}} + O(1) - kN^{1 + \frac{\delta}{2}} + \frac{1}{2} k^2 N^{\frac{\delta}{2}} + O(1) = -\frac{1}{2} k^2 N^{\frac{\delta}{2}} + k + O(1).
\]

Consequently,

\[
(2.87) \quad \sum_{j \geq 0} f_N(j) = \sum_{j < x_{-1}} f_N(j) + \sum_{x_{-1} \leq j < x_1} f_N(j) + \sum_{k=1}^{+\infty} \sum_{x_k \leq j < x_{k+1}} f_N(j) \leq f_N(x_{-1}) O(N^{1 + \delta}) + O(N^{\frac{\delta}{2} + \frac{\delta}{6}}) + O(N^{\frac{\delta}{2} + \frac{\delta}{6}}) = O(N^{\frac{\delta}{2} + \frac{\delta}{6}}).
\]

Together with (2.76), this concludes the proof of Lemma 2.13 and hence the proof of Proposition 2.11.

In order to conclude the proof, we need the following lemma, whose proof is postponed to Appendix B.

**Lemma 2.15.**

\[
(2.88) \quad \frac{1}{nA_n \sup_{O \in \operatorname{Per}(n)} \left| \frac{1}{\det(1 - dT^O)} \right|} \to 0. \quad n \to \infty
\]

Since

\[
(2.89) \quad \int_{\mathbb{R}} e^{i(\mu \cos u + \nu \sin u)} du = 1 - \frac{\mu^2 + \nu^2}{4} + o(\mu^2 + \nu^2),
\]

we obtain for a given \((\mu, \nu) \in \mathbb{R}^2\)
(2.90)  
\[
\prod_{O \in \text{Per}(n)} \int_{\mathbb{R}} f_O(u) du = \prod_{m|n} \prod_{O \in P_m} \left( 1 - \frac{\mu^2 + \nu^2}{4} \frac{m^2 A^2_n}{|\det(1 - dT^O_n)|^2} + o\left( \frac{m^2 A^2_n}{|\det(1 - dT^O_n)|^2} \right) \right)
\]
\[
= e^{\sum_{m|n} \sum_{O \in P_m} \log \left( 1 - \frac{\mu^2 + \nu^2}{4} \frac{m^2 A^2_n}{|\det(1 - dT^O_n)|^2} \right) + o\left( \frac{m^2 A^2_n}{|\det(1 - dT^O_n)|^2} \right)}
\]
\[
= e^{-\frac{\mu^2 + \nu^2}{4} \sum_{m|n} \sum_{O \in P_m} \frac{m^2 A^2_n}{|\det(1 - dT^O_n)|^2} + o\left( \sum_{m|n} \sum_{O \in P_m} \frac{m^2 A^2_n}{|\det(1 - dT^O_n)|^2} \right)}
\]
\[
(1.14)
\]

Levy's theorem then implies the convergence in law of \( A_n \text{Tr}^\flat(L_n^\xi, \tau) \) towards a complex Gaussian law \( N_{\mathbb{C}}(0, 1) \).

APPENDIX A. UPPER BOUND FOR THE DISTANCE BETWEEN PERIODIC POINTS OF A GIVEN PERIOD

Note that for a given period, there is a finite number of periodic points, whose mutual distance is bounded below. Then,

Lemma A.1.
(A.1) \( \forall \Lambda' > \Lambda, \exists C > 0, \forall n \in \mathbb{N}, \forall x \neq y \in M, (T^n x = x, T^n y = y) \implies d(x, y) \geq \frac{C}{\Lambda^{\frac{n}{2}}} \).

Anosov diffeomorphisms are expansive (see [KH97], p.125):

(A.2) \( \exists \delta > 0, \forall x, y \in M, (\forall k \in \mathbb{Z}, d(T^k(x), T^k(y)) \leq \delta) \implies x = y. \)

Let \( \Lambda' > \Lambda \), and let \( n_0 \in \mathbb{N} \) such that

(A.3) \( \forall |n| \geq n_0, \max_{x \in M} \|dT^\mu_x\| \leq \Lambda^{|n|}. \)

Let \( L = \max_{x \in M} \|dT^{\pm 1}_x\| \geq \Lambda \). For any \( x, y \in M \),

(A.4) \( d(T^k x, T^k y) \leq \left\{ \begin{array}{ll} \Lambda^{|k|} & \text{if } |k| \geq n_0 \\ L^{|k|} & \text{if } |k| < n_0 \end{array} \right. \)

Writing \( C = \left( \frac{L}{\Lambda} \right)^{n_0} \), we get for all \( n \in \mathbb{N} \), for all periodic points \( x, y \) of period \( n \), for all \( -n/2 \leq k \leq n/2 \),

(A.5) \( d(T^k x, T^k y) \leq CA^{n/2} d(x, y). \)

Therefore, if \( d(x, y) \leq \frac{\delta}{C \Lambda^{n/2}} \), then for all \( -\frac{n}{2} \leq k \leq \frac{n}{2} \), (thus for all \( k \in \mathbb{Z} \)) \( d(T^k x, T^k y) \leq \delta \) and \( x = y. \).
Appendix B. Proof of Lemma 2.15

B.1. Topological pressure.

Proposition B.1 ([KH97 Proposition 20.3.3]). Let $\phi : M \to \mathbb{R}$ be a Hölder function. The topological pressure of $\phi$ is given by

$$\text{Pr}(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{x \in T^n(x)} e^{\phi(x)} \right).$$

B.2. Variational principle. The topological pressure of a Hölder function is given by a variational principle:

Theorem B.2 ([Bow75 Theorem 2.17]). Let $\phi : M \to M$ be a Hölder function.

$$\text{Pr}(\phi) = \sup_{\mu} (h_{\mu}(T) + \int_M \phi \, d\mu)$$

where $\mu$ goes through the $T$-invariant probability measures. This supremum is attained at a unique measure [KH97 Proposition 20.3.7] called equilibrium measure of $\phi$.

Let us write

$$J_u(x) := \log |\det dT|_{E^u(x)}|.$$

$J_u$ is said to be cohomologous to a constant if

$$\exists c \in \mathbb{R}, \exists h \in C(M), \quad J_u(x) = c + h \circ T - h.$$

Lemma B.3 ([KH97 Proposition 20.3.10]). Let $\mu_{-\beta}$ be the equilibrium measure of $-\beta J_u$, $\beta > 0$. If $J_u$ is not cohomologous to a constant, the map $\beta \mapsto \mu_{-\beta}$ is injective.

Corollary B.4. As a consequence, the function

$$F : \left\{ \begin{array}{ccc} \mathbb{R}_+^* & \to & \mathbb{R} \\ \beta & \mapsto & \frac{1}{\beta} \text{Pr}(-\beta J_u) \end{array} \right.$$

is strictly decreasing.

Proof. If $J_u$ is cohomologous to a constant $c$, then by Proposition 2.7

$$\sum_{T^n(x) = x} e^{-\beta(J_u(x))} \leq e^{n(h_{\text{top}} - c\beta)}.$$

Hence,

$$\text{Pr}(-\beta J_u) = h_{\text{top}} - c\beta,$$

with $h_{\text{top}} > 0$, which gives the wanted result. If $J_u$ is not cohomologous to a constant, then the previous Lemma applies. Let $\beta' > \beta > 0$.

$$\int -\beta J_u \, d\mu_{\beta'} = \text{Pr}(-\beta J_u) > \int -\beta' J_u \, d\mu_{\beta'}.$$

Consequently,

$$F(\beta) = \int -J_u \, d\beta + \frac{h(\mu_{\beta})}{\beta} > \int -J_u \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta} \geq \int -J_u \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta'} = F(\beta').$$

□
$F$ is moreover bounded below, and admits consequently a limit as $\beta$ goes to infinity.

B.3. End of proof of Lemma 2.15. The quantity $\inf_{x \in M} (J_u)_x^n$ is subadditive. We can consequently define

(B.10) $J_u^{\min} := \lim_{n \to \infty} \inf_{x \in M} \frac{1}{n} (J_u)_x^n$.

**Lemma B.5.** The infimum can be taken over periodic points of period $n$:

(B.11) $J_u^{\min} = \lim_{n \to \infty} \inf_{\tau^n(x) = x} \frac{1}{n} (J_u)_x^n$.

**Proof.** For $n \in \mathbb{N}$, let $x_n \in M$ be the point such that

(B.12) $(J_u)_x^n = \inf_{x \in M} (J_u)_x^n$.

Let $\epsilon > 0$. By the specification property [KH97, Theorem 18.3.9], there exists $s \in \mathbb{N}$ and a periodic point $x'_n$ of period $n + s$ such that

(B.13) $\forall 0 \leq j \leq n - 1, \ d(T^j(x_n), T^j(x'_n)) \leq \epsilon$.

Thus, for some $C > 0$,

(B.14) $\frac{1}{n} \inf_{x \in M} (J_u)_x^n \leq \frac{1}{n} \inf_{\tau^s(x) = x} (J_u)_x^n \leq \frac{1}{n} \inf_{x \in M} (J_u)_x^n + C\epsilon$.

Then, we have

(B.15) $\frac{1}{n + s} \inf_{\tau^n(x) = x} (J_u)_x^n \geq \frac{1}{n} \inf_{x \in M} (J_u)_x^n + C\epsilon - \frac{C s}{n}$.

As a consequence,

(B.16) $\inf_{\tau^n(x) = x} \frac{1}{n} (J_u)_x^n \to J_u^{\min}$.

**Lemma B.6.**

(B.17) $F(\beta) \xrightarrow{\beta \to \infty} J_u^{\min}$

**Proof.** Let

(B.18) $F_n(\beta) := \frac{1}{n\beta} \log \sum_{\tau^n(x) = x} e^{-\beta(J_u)_x^n} \xrightarrow{n \to \infty} F(\beta)$.

Let $\epsilon > 0$. If $n$ is large enough, for every periodic point $x$ of period $n$

(B.19) $(J_u)_x^n \geq n(J_u)^{\min} - \epsilon$.

Moreover, there exists a periodic point of period $n$ such that

(B.20) $(J_u)_x^n \leq n(J_u)^{\min} + \epsilon$.

So

(B.21) $e^{-\beta n(J_u)^{\min} + \epsilon} \leq \sum_{\tau^n(x) = x} e^{-\beta(J_u)_x^n} \leq e^{n h_{top}} e^{-\beta n(J_u)^{\min} + \epsilon}$.
Taking the logarithm and taking $\varepsilon \to 0$ gives for every $\beta > 0$
\begin{equation}
-J_u^\text{min} \leq F(\beta) \leq \frac{h_{\text{top}}}{\beta} - J_u^\text{min},
\end{equation}
hence the result. \boxed{}

The proof of Lemma 2.15 then derives from the following facts:
\begin{equation}
\sup_{O \in \text{Per}(n)} |\det (1 - dT^n_{O})| = e^{-nJ_u^\text{min}} (1 + o(1)) = e^{n \lim_{\beta \to \infty} F(\beta) + o(n)}
\end{equation}
and
\begin{equation}
A_n = e^{nF(2) + o(n)}.
\end{equation}
(B.23) is clear, and the definition (1.14) implies that
\begin{equation}
\frac{1}{\sqrt{n}} \left( \sum_{T^n(x)=x} e^{-n(J_u)^n} (1 + o(1)) \right)^{-\frac{1}{2}} \leq A_n \leq \left( \sum_{T^n(x)=x} e^{-n(J_u)^n} (1 + o(1)) \right)^{-\frac{1}{2}}.
\end{equation}

Appendix C. Proof of Proposition 1.4

These results are based on the fact that if $\zeta_j$ are i.i.d. centered Gaussian variables of variance 1, then, almost surely, for every $\varepsilon > 0$,
\begin{equation}
\zeta_j = o(j^{\varepsilon}).
\end{equation}
It is a consequence of Borel-Cantelli Lemma:
\begin{equation}
P(|\zeta_j| > j^{\varepsilon}) \leq \frac{2}{\sqrt{2\pi}} e^{-j^{2\varepsilon}/2},
\end{equation}
it is thus summable. The second ingredient is Weyl’s law (2.24): the asymptotics of the sequence of eigenvalues $(\lambda_j)$ of the Laplace-Beltrami operator is given by
\begin{equation}
\lambda_j \sim Cj^{\frac{d}{2}}
\end{equation}
for some constant $C$ depending only on $M$ (and the metric).

Proof of Proposition 1.4. If $c_j = O(j^{-\alpha})$, then for $s < d(\alpha - \frac{1}{2})$, or equivalently $\alpha > \frac{s}{d} + \frac{1}{2}$, almost surely,
\begin{equation}
|c_j \zeta_j|^2 = O(j^{-2s/d - 1 - \delta})
\end{equation}
for some $\delta > 0$. Thus,
\begin{equation}
\sum_j |c_j \zeta_j|^2 j^{2\frac{d}{2}}
\end{equation}
converges, and so does
\begin{equation}
\sum_j |c_j \zeta_j|^2 (1 + \lambda_j)^s
\end{equation}
by Weyls law (C.3). Consequently, almost surely,
\begin{equation}
(1 + \Delta)^s f = \sum_j c_j \zeta_j (1 + \lambda_j)^{\frac{s}{2}} \phi_j \in L^2.
\end{equation}
In other terms, \( f \) belongs almost surely to \( H^s \). The statement about \( C^k(M) \) follows from the classical Sobolev embedding theorem:

**Theorem C.1** ([Tay96 Proposition 3.3 p.282]). For \( s > k + d/2 \),

\[
H^s(M) \subset C^k(M).
\]

\( \square \)

**Appendix D. Ruelle spectrum and flat trace**

**Theorem D.1** ([Bal18 Theorem 5.1]). In our setting, if \( \tau \) is \( C^k \), there exists a Banach space \( B^k \) such that

- \( \mathcal{L}_{\xi,\tau} : B^k \to B^k \) is a bounded operator,
- For any \( \alpha > k/2 \),

\[
C^\alpha(M) \subset B^k \subset C^\alpha(M)',
\]

- The spectral radius \( r(\mathcal{L}_{\xi,\tau}) \) of \( \mathcal{L}_{\xi,\tau} \) in \( B^k \) is

\[
r(\mathcal{L}_{\xi,\tau}) \leq 1,
\]

- The essential spectral radius \( r_{ess}(\mathcal{L}_{\xi,\tau}) \) of \( \mathcal{L}_{\xi,\tau} \) in \( B^k \) satisfies

\[
r_{ess}(\mathcal{L}_{\xi,\tau}) \leq \lambda m^k,
\]

where \( \lambda \) is the smallest constant satisfying the definition of Anosov diffeomorphism from Section [1.1].

This implies that the resolvent of the transfer operator admits a meromorphic extension as an operator \( C^\infty(M) \to \mathcal{D}'(M) \) to the region

\[
\{ |z| > m^k \}
\]

and that its poles in this domain are the eigenvalues of \( \mathcal{L}_{\xi,\tau} \) with same multiplicities and spectral projectors. These poles are called Ruelle-Pollicott resonances. For analytic Anosov maps, there are spaces in which the transfer operator is trace-class. There is however no hope for this to be true with less regularity. Yet, an analog of the trace can be defined, that coincides with the usual trace in the analytic case:

**Lemma D.2** ([Bal18 Section 6.2]). The operators \( \mathcal{L}_{\xi,\tau}^n, n \geq 1 \) have integral kernel

\[
K_{\xi,\tau}^n(x,y) = e^{i\xi\tau^n_x} \delta(y - T^n(x)),
\]

where

\[
\tau^n_x := \sum_{j=0}^{n-1} \tau(T^j(x)).
\]

Introducing a mollification, the distribution \( K_{\xi,\tau}^n \) can be integrated along the diagonal \( \{(x,x), x \in M\} \). The resulting quantity is called flat trace of \( \mathcal{L}_{\xi,\tau}^n \) and can be expressed as a sum over periodic points:

\[
\text{Tr}^\flat \mathcal{L}_{\xi,\tau}^n := \int_M K_{\xi,\tau}^n(x,x)dx = \sum_{x, T^n(x) = x} \frac{e^{i\xi\tau^n_x}}{|\det(1 - d(T^n)_x)|}.
\]

This notion of trace is linked to the eigenvalues of the operator in the following way ([Bal18 Theorem 6.2], [Jéz17 Theorem 2.4]):
Theorem D.3. Let $\xi \in \mathbb{R}$, let $\varepsilon > 0$ be such that $L_{\xi,\tau}$ has no resonance of modulus $\varepsilon$, then

\begin{equation}
\exists C_\xi > 0, \forall n \in \mathbb{N}, \left| \text{Tr}^n L_{\xi,\tau} - \sum_{\lambda \in \text{Res}(L_{\xi,\tau}) \atop |\lambda| > \varepsilon} \lambda^n \right| \leq C_\xi \varepsilon^n.
\end{equation}
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