Surface counterterms and boundary stress-energy tensors for asymptotically non-anti-de Sitter spaces

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Abstract

For spaces which are not asymptotically anti-de Sitter where the asymptotic behavior is deformed by replacing the cosmological constant by a dilaton scalar potential, we show that it is possible to have well-defined boundary stress-energy tensors and finite Euclidean actions by adding appropriate surface counterterms. We illustrate the method by the examples of domain-wall black holes in gauged supergravities, three-dimensional dilaton black holes and topological dilaton black holes in four dimensions. We calculate the boundary stress-energy tensor and Euclidean action of these black configurations and discuss their thermodynamics. We find new features of topological black hole thermodynamics.

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1 Introduction

In the traditional Euclidean path integration approach to black hole thermodynamics [1, 2], except for the usual Gibbons-Hawking surface term which makes the variation principle well defined, one has to choose a suitable reference background and make subtraction in order to get a finite Euclidean action of black holes. However, the background subtraction procedure makes the action of black holes depend on the choice of reference background. Furthermore sometimes one may encounter the situations in which there are no appropriate reference backgrounds, as observed for the Taub-NUT-AdS and Taub-Bolt-AdS spaces [3, 4]. On the other hand, in the quasilocal formulation of gravity [5, 6], one can define the so-called quasilocal stress-energy tensor and conserved charges on the boundary of a given spacetime region. Unfortunately, such quantities often diverge as the boundary is taken to infinity. A suitable background subtraction must then be made for getting a finite result.

In the asymptotically anti-de Sitter spacetimes, the above difficulty has been solved recently. The proposal is that by adding suitable surface counterterms to the gravitational action, one can obtain a well-defined boundary stress-energy tensor and a finite Euclidean action for the black hole spacetimes [7]. A remarkable feature of this procedure is that the boundary stress-energy tensor and Euclidean action thus defined are independent of the reference background and the results are physically unique. Recently a lot of works have been devoted to this proposal and related topics [8]-[21]. In \((n + 1)\)-dimensional Einstein gravity with a negative cosmological constant \(\Lambda = -n(n-1)/2l^2\), the action can be written as

\[
S = \frac{1}{16\pi G} \int_M d^{n+1}x \sqrt{-g} \left( R + \frac{n(n - 1)}{l^2} \right) - \frac{1}{8\pi G} \int_{\partial M} d^n x \sqrt{-h} K, \tag{1.1}
\]

where the first term is called the bulk action, and the second term is just the Gibbons-Hawking surface term. Here \(h\) denotes the reduced metric of a timelike boundary \(\partial M\) and \(K\) represents the trace of its extrinsic curvature to be defined below. In ref. [8], an expression of surface counterterms has been given, which can cancel divergences up to
\[ n \leq 6: \]
\[
S_{ct} = -\frac{1}{8\pi G} \int_{\partial M} d^n x \sqrt{-h} \left[ \frac{n-1}{l} + \frac{l}{2(n-2)} \mathcal{R} + \frac{l^3}{2(n-4)(n-2)} \left( \mathcal{R}_{ab} \mathcal{R}^{ab} - \frac{n}{4(n-1)} \mathcal{R}^2 \right) \right],
\]

where \( \mathcal{R} \) and \( \mathcal{R}_{ab} \) are the Ricci scalar and Ricci tensor for the boundary metric \( h_{ab} \). The authors of [20] claimed that they have given the surface counterterms up to \( n \leq 8 \). From (1.2), one may see that the cosmological constant \( l \) plays a crucial role in this surface counterterm method. Once given the surface counterterms, one may define a quasilocal stress-energy tensor by
\[
T_{ab} = \frac{1}{8\pi G} \left[ K_{ab} - K h_{ab} + \frac{2}{\sqrt{-h}} \frac{\delta S_{ct}}{\delta h_{ab}} \right], \tag{1.3}
\]

where the extrinsic curvature is \( K_{ab} = -\frac{1}{2}(\Delta_a n_b + \Delta_b n_a) \), and \( n_a \) denotes the outward pointing normal vector to the boundary \( \partial M \).

Decomposing the boundary metric \( h_{ab} \) in the ADM form with a spacelike surface \( \mathcal{B} \) in \( \partial M \) with metric \( \sigma_{ij} \):
\[
h_{ab} dx^a dx^b = -N_B^2 dt^2 + \sigma_{ij} (dx^i + N^i dt)(dx^j + N^j dt), \tag{1.4}
\]

one can define a conserved charge
\[
Q_\xi = \int_{\mathcal{B}} d^{n-1} x \sqrt{\sigma} (u^a T_{ab} \xi^b), \tag{1.5}
\]

associated with a Killing vector \( \xi^a \), where \( u^a \) is a timelike unit normal to \( \mathcal{B} \). In this way one can have the definition of the mass of gravitational field as [7]
\[
M = \int_{\mathcal{B}} d^{n-1} x \sqrt{\sigma} N_B u^a u^b T_{ab}. \tag{1.6}
\]

Using the above prescription, Balasubramanian and Kraus [7] have obtained the boundary stress tensor associated with a gravitational system in asymptotically anti-de Sitter space. Via the AdS/CFT correspondence [22, 23, 24], the result is interpreted as the expectation value of the stress tensor of the boundary quantum conformal field theory. In particular, they have found a nonvanishing ground state energy for a global \( AdS_5 \), and have matched this energy with the Casimir energy of the dual \( \mathcal{N}=4 \) super Yang-Mills theory on \( R \times S^3 \).

So far, however, most of these works are restricted to the asymptotically anti-de Sitter space and its asymptotically flat limit. It is true that these two kinds of spacetimes are
much of physical interest, but there are also other interesting spacetimes which are neither asymptotically anti-de Sitter nor asymptotically flat. For instance, the geometry in the decoupling limit of the black D3-brane with NS $B$ field [25, 26] has been proposed as the gravity dual of the $\mathcal{N}=4$ super Yang-Mills theory in non-commutative spacetime. It is thus interesting to try to extend this approach to a more general class of spacetimes. In this paper we consider the kind of spacetimes which are not asymptotically anti-de Sitter, in which the asymptotically anti-de Sitter behavior is deformed by the presence of a dilaton potential in the bulk action. In this class of spacetimes, we find that it is also possible to have a well-defined boundary stress-energy tensor and a finite Euclidean action by slightly modifying the above prescription. We give a general form of the surface counterterms necessary to cancel the divergences and provide a formula for the coefficient in terms of the asymptotic behaviors of the metrics and potential in the solution.

The organization of this paper is as follows. In the next section we first consider domain-wall spacetimes in gauged supergravities which come from the sphere reduction of $Dp$-branes in type II supergravities, since in this case we can have a consistency check of our result. We will also consider a kind of charged domain-wall spacetimes. Our results can be regarded as a part of the realization of the so-called domain wall/QFT correspondence. In Sec. 3 we will discuss a three-dimensional dilaton black hole, where the BTZ black hole is deformed by a dilaton potential. In Sec. 4 we extend this discussion to the topological dilaton black holes in four dimensions. A brief summary is given in Sec. 5. The general formula for the surface counterterms and their coefficients are summarized in the appendix.

2 Domain-wall black holes

The AdS/CFT correspondence asserts that there is an equivalence between the bulk supergravity (string/M theories) and a boundary conformal field theory. This correspondence nicely illustrates the holographic principle [27, 28] which is widely believed to be a feature of any consistent theory of quantum gravity. Thus the AdS/CFT correspondence is just a special case of a more general correspondence between supergravity and quantum field theory (QFT) in one lower dimensions. On the basis of the observation that the AdS
metric in horoshperical coordinates is a special case of a domain-wall metric, Boonstra, Skenderis, and Townsend [29] have extended the AdS/CFT correspondence to the so-called domain-wall/QFT correspondence between the gauged supergravity and quantum field theory on domain walls. This correspondence has been discussed further in ref. [30] in various dimensions. It is straightforward to extend this to the correspondence between the domain-wall black holes and corresponding quantum field theory at finite temperature. In this section we will extract the stress-energy tensor of quantum field theory on the domain-walls in the spirit of domain-wall/QFT correspondence. We first discuss the case, in which the domain-wall black holes come from sphere reduction of Dp-branes in the “dual” frame. We will then consider the charged domain-wall black holes which come from singular sphere reductions of eleven-dimensional supergravity and ten-dimensional type IIB supergravity.

2.1 Neutral domain-wall black holes

Let us consider the black Dp-brane solution with “magnetic” charge in type II supergravity. In the string frame, the action is

\[ S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\partial\phi)^2 \right) - \frac{1}{2(8-p)!} F_{8-p}^2 \right], \]  

(2.1)

where \( G_{10} = 8\pi^6 \alpha'^4 \) is the gravitational constant in ten dimensions. The black Dp-brane solution is

\[ ds_{\text{string}}^2 = H^{-1/2}(-f^2 dt^2 + dx_p^2) + H^{1/2}(f^{-1}dr^2 + r^2d\Omega_{8-p}^2), \]

\[ e^\phi = g_s H^{(3-p)/4}, \]

\[ F_{8-p} = Q\epsilon_{8-p}, \]  

(2.2)

where \( g_s \) is the string coupling constant, \( \epsilon_{8-p} \) the volume form of \( S^{8-p} \) and

\[ H = 1 + \frac{r_0^{7-p} \sinh^2 \alpha}{r^{7-p}}, \quad f = 1 - \left( \frac{r_0}{r} \right)^{7-p}. \]  

(2.3)

In the decoupling limit: \( \alpha' \to 0 \), but keeping fixed \( U = r/\alpha' \), \( U_0 = r_0/\alpha' \) and the ’t Hooft coupling constant \( g_{\text{YM}}^2 N \), with \( g_{\text{YM}}^2 = g_s (\alpha')^{(p-3)/2} \), the harmonic function tends to

\[ H = 1 + g_{\text{YM}}^2 N \left( \alpha' \right)^2 U^{7-p} \implies g_{\text{YM}}^2 N (\alpha')^{-2} U^{p-7}. \]  

(2.4)
where we have absorbed an unimportant coefficient into $g_{YM}$ [29]. Except for the case of $p = 3$, the radius of angle part of the string metric (2.2) depends on $U$ and the metric is singular at $U = 0$ even in the case of $U_0 = 0$. To circumvent this problem, the so-called “dual frame” metric has been considered in [29]:

$$ds^2_{\text{dual}} = (Ne^\phi)^{2/(p-7)}ds^2_{\text{string}}.$$ (2.5)

In this frame, the action (2.1) becomes

$$S = \frac{N^2}{16\pi G_{10}} \int d^{10}x \sqrt{-g(Ne^\phi)^\lambda} \left[ R + \frac{4(p-1)(p-4)}{(7-p)^2}(\partial\phi)^2 - \frac{1}{2N^2(8-p)!} F^2_{8-p} \right],$$ (2.6)

where

$$\lambda = 2(p-3)/(7-p).$$ (2.7)

The decoupling limit solution in the “dual frame” is

$$ds^2_{\text{dual}} = \alpha' \left[ \left( g^2_{YM}N \right)^{-1} U^{5-p}(-f dt^2 + dx_p^2) + U^{-2} f^{-1} dU^2 + d\Omega_{8-p}^2 \right],$$

$$e^\phi = \frac{1}{N} \left[ \left( g^2_{YM}N \right)^{U^p-3} \right]^{(7-p)/4},$$

$$F_{8-p} = (7-p)N(\alpha')^{(7-p)/2} e_{8-p},$$ (2.8)

where $f = 1 - (U_0/U)^{7-p}$. The near-horizon “dual frame” metric is $AdS_{p+2} \times S^{8-p}$ for $p \neq 5$ and $E^{(1,6)} \times S^3$ for $p = 5$. An important feature of this frame is that the radius of the angle part of the metric becomes a constant.

Because $\alpha'$ is eventually canceled at the end of the calculations, we set $\alpha' = 1$ in what follows. In addition, by the transformation ($p \neq 5$)

$$u^2 = \mathcal{R}^2 \left( g^2_{YM}N \right)^{-1} U^{5-p}, \quad \mathcal{R} = 2/(5-p),$$ (2.9)

the above “dual frame” metric can be put in a standard form

$$ds^2_{\text{dual}} = \frac{u^2}{\mathcal{R}^2} \left( -f dt^2 + dx_p^2 \right) + \frac{\mathcal{R}^2}{u^2 f} du^2 + d\Omega_{8-p}^2,$$

$$e^\phi = \frac{1}{N} \left( g^2_{YM}N \right)^{(7-p)/2(5-p)} (u/\mathcal{R})^{(p-7)(p-3)/2(p-5)},$$

$$F_{8-p} = (7-p)N e_{8-p}.$$ (2.10)

It was found that the scale $u$ introduced above is just the holographic energy scale of the boundary QFT. Thus the “dual frame” was argued as the holographic frame describing supergravity probes [29].
Due to the fact that the radius of angle part of the metric is a constant, one may consistently reduce the angle part to get an effective gauged \((p+2)\)-dimensional supergravity.

In the Einstein frame, the resulting action is

\[
S = \frac{N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+2}x \sqrt{-\gamma} \left[ R - \frac{1}{2} (\partial \Phi)^2 + V(\Phi) \right] - \frac{2N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{-h} K, \tag{2.11}
\]

where for the later use, we have added the Gibbons-Hawking term to the bulk action, \(\Omega_{8-p}\) is the volume of a unit \((8-p)\)-sphere, and

\[
V(\Phi) = \frac{1}{2} (9-p)(7-p)N^{-2\lambda/p}e^{\alpha \Phi},
\]

\[
\Phi = \frac{2\sqrt{2(9-p)}}{\sqrt{p(7-p)}} \phi, \quad \alpha = -\frac{\sqrt{2}(p-3)}{\sqrt{p(9-p)}}, \tag{2.12}
\]

In the effective gauged supergravity action, its equations of motion have the following domain-wall black hole solutions:

\[
ds_{p+2}^2 = (Ne^\phi)^{2\lambda/p} \left[ \frac{u^2}{R^2} \left( -\tilde{f} dt^2 + dx_p^2 \right) + \frac{R^2}{u^2} du^2 \right],
\]

\[
\tilde{f} = 1 - \left( \frac{u_0}{u} \right)^{2(7-p)/(5-p)}, \tag{2.13}
\]

where \(\lambda, R\) and \(e^\phi\) are given in Eqs. (2.7), (2.9) and (2.10), respectively, and \(u_0\) is defined as \(u_0^2 = \frac{R^2 (g_{YM} N)^{-1}}{U_0^{5-p}}\).

Now we are interested in extracting the stress-energy tensor of quantum field which lives in the domain wall (2.13), according to the domain-wall/QFT correspondence. We find that the scalar potential occurring in the action (2.11) can play the same role as a cosmological constant does in the asymptotically anti-de Sitter spaces. Writing the scalar potential \(V(\Phi)\) as

\[
V(\Phi) \equiv \frac{n(n-1)}{l^2_{\text{eff}}} = \frac{p(p+1)}{l^2_{\text{eff}}}, \tag{2.14}
\]

one may introduce an “effective cosmological constant” \(1/l_{\text{eff}}\) defined by

\[
\frac{1}{l_{\text{eff}}} = \sqrt{\frac{V(\Phi)}{p(p+1)}}, \tag{2.15}
\]

According to the formulae (A.3), (A.7) and (A.9) in the appendix, by adding the following surface counterterm to (2.11):

\[
S_{\text{ct}} = -\frac{2N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{-h} \frac{c_0}{l_{\text{eff}}}, \quad c_0 = \sqrt{\frac{(9-p)p(p+1)}{2(7-p)}}, \tag{2.16}
\]
it is possible to cancel divergences in physical quantities such as stress-energy tensor and Euclidean action associated with the domain-wall black holes. Note that the surface term is similar to the first term in (1.2) but with different coefficient.

Using (1.3) we have

\[ T_{ab} = \frac{2N^2\Omega_{8-p}}{(2\pi)^7} \left[ K_{ab} - K h_{ab} - \frac{c_0}{l_{\text{eff}}} h_{ab} \right], \tag{2.17} \]

where the labels \( a, b \) run over the domain-wall directions. Substituting the solution (2.13) into (2.17) and using the “effective cosmological constant” (2.15), we obtain

\[ \frac{(2\pi)^7}{2N^2\Omega_{8-p}} T_{tt} = \frac{9 - p}{4} \left( g_{\text{YM}}^2 \right)^{(p-3)/p(5-p)} \left( \frac{u}{\Re} \right)^{(p^2-4p-9)/p(p-5)} \left( \frac{u_0}{u} \right)^{2(7-p)/(5-p)} + \cdots, \]

\[ \frac{(2\pi)^7}{2N^2\Omega_{8-p}} T_{ij} = \delta_{ij} \frac{5 - p}{4} \left( g_{\text{YM}}^2 \right)^{(p-3)/p(5-p)} \left( \frac{u}{\Re} \right)^{(p^2-4p-9)/p(p-5)} \left( \frac{u_0}{u} \right)^{2(7-p)/(5-p)} + \cdots, \tag{2.18} \]

where dots denote higher order terms, which will vanish when we take the boundary to the spatial infinity. Using (1.0) we get the mass of the domain-wall black hole,

\[ M = \int_{u \to \infty} d^p x \sqrt{-\gamma} N_B u^t u^t T_{tt} = \frac{\Omega_{8-p}}{(2\pi)^7 g_{\text{YM}}^4} \frac{9 - p}{2} U_0^{7-p} V_p, \tag{2.19} \]

where \( V_p \) is the spatial volume of the domain wall.

The surface metric \( \gamma_{ab} \) of the spacetime, in which the boundary quantum field lives, can be obtained as \[8\]

\[ \gamma_{ab} = \lim_{u \to \infty} \frac{9 \Re^2}{u^2} (N e^{\alpha})^{-2\lambda/p} h_{ab} = \eta_{ab}, \tag{2.20} \]

which means that the boundary quantum field theory lives in a flat domain wall. The stress-energy tensor of boundary quantum field theory can be obtained as \[8\]

\[ \sqrt{-\gamma} \gamma^{ab} \tau_{bc} = \lim_{u \to \infty} \sqrt{-h} h^{ab} T_{bc}. \tag{2.21} \]

Substituting (2.18) into the above formula, we finally arrive at

\[ \tau_{ab} = \frac{U_0^{7-p} \Omega_{8-p}}{(2\pi)^7 g_{\text{YM}}^4} \text{diag} \left[ \begin{array}{c} 9 - p \ 2 \\ 5 - p \ 2 \ \cdots \ 5 - p \end{array} \right], \tag{2.22} \]
which can be interpreted as the vacuum expectation value of the quantum field theory on the domain wall (2.13).

On the other hand, we can calculate the Euclidean action of the domain-wall black hole

\[
I = \frac{N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{g} \left[ R - \frac{1}{2} (\partial \Phi)^2 + V(\Phi) \right] + 2 \frac{N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{\hbar} K + 2 \frac{N^2 \Omega_{8-p}}{(2\pi)^7} \int d^{p+1}x \sqrt{\hbar} \frac{c_0}{l_{\text{eff}}},
\]

where the first line is the bulk contribution, the second term is the usual Gibbons-Hawking surface term, and the last is just the surface counterterm. Putting the solution into (2.23) yields a finite result

\[
I = - \frac{\Omega_{8-p} V_p U_0^{7-p}}{(2\pi)^7 g_{\text{YM}}^4 T} \frac{5 - p}{2},
\]

from which we obtain the free energy \( F \)

\[
F \equiv T I = - \frac{\Omega_{8-p} V_p U_0^{7-p}}{(2\pi)^7 g_{\text{YM}}^4 T} \frac{5 - p}{2} V_p U_0^{7-p},
\]

where \( T \) is the Hawking temperature of the domain-wall black hole (2.13). For \( p < 5 \), the free energy is negative, which implies that the system is thermodynamically stable, while for \( p > 5 \), the free energy becomes positive. In this case, the thermal excitations are thermodynamically unstable, and they will be suppressed in canonical ensemble. From (2.13) it might appear that the results derived above are applicable only to \( p < 5 \). In fact, the above results hold for \( p \geq 5 \) as well, for we may use the coordinate \( U \) in (2.8) instead of \( u \) in (2.13) and then the same results are obtained (the apparent singularities for \( p = 5 \) as in Eqs. (2.18) are absent in terms of \( U \) and \( U_0 \)). We also see from (2.22) and (2.23) that the case of \( p = 5 \) is a bit peculiar: the free energy and the pressure of thermal excitations on the domain wall vanish.

2.2 A consistency check

Note that the domain-wall black hole solution (2.13) comes from the sphere reduction of Dp-brane solution (2.2). It makes possible to calculate the stress-energy tensor of boundary quantum field theory and the free energy directly from the Dp-brane solution
and to compare with the results from the counterterm method. The formula to extract the stress-energy tensor of excitations of D\(_{p}\)-branes has been given in \([8]\):

\[
T_{ab} = \frac{1}{16\pi G_{10} g_s^2} \int_{r \to \infty} d\Omega_{8-p} r^{8-p} n^i \eta_{ab} \left( \partial_i h^c_c + \partial_i h^j_j - \partial_j h^i_i - \partial_i h_{ab} \right),
\]

(2.26)

where \(n^i\) is a radial unit in the transverse subspace, while \(h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}\) is the deviation of the (Einstein frame) metric from that for flat space. The labels \(a, b = 0, 1, \cdots, p\) run over the world-volume directions, while \(i, j = 1, 2, \cdots, 9 - p\) denote the transverse directions.

In addition, it should be reminded that the calculations in (2.26) must be done using asymptotically Cartesian coordinates.

Rewriting the D\(_{p}\)-brane solution (2.2) in the isotropic coordinates of the Einstein frame, we have

\[
d s_E^2 = H^{-\left(7-p\right)/8} \left(-f dt^2 + dx_p^2\right) + H^{\left(p+1\right)/8} r^2 \rho^{-2} \left(d\rho^2 + \rho^2 d\Omega_{8-p}^2\right),
\]

(2.27)

where \(\rho\) is the radial coordinate having the relation with \(r\) as

\[
r^{7-p} = \rho^{7-p} \left(1 + \frac{r_0^{7-p}}{4\rho^{7-p}}\right)^2.
\]

(2.28)

Substituting the solution into (2.26), one finds

\[
T_{ab} = \frac{(7 - p)r_0^{7-p} \Omega_{8-p}}{16\pi G_{10} g_s^2} \text{ diag } \left[\frac{8 - p}{7 - p} + \sinh^2 \alpha, \frac{1}{7 - p} - \sinh^2 \alpha, \cdots, \frac{1}{7 - p} - \sinh^2 \alpha\right].
\]

(2.29)

This stress-energy tensor includes the contribution from the extremal background, which can be obtained from (2.29) by taking \(r_0 \to 0\), but keeping \(\tilde{R}^{7-p} = r_0^{7-p} \sinh \alpha \cosh \alpha\) constant:

\[
(T_{ab})_{\text{ext}} = \frac{(7 - p)\Omega_{8-p}}{16\pi G_{10} g_s^2} \text{ diag } [\tilde{R}^{7-p}, -\tilde{R}^{7-p}, \cdots, -\tilde{R}^{7-p}].
\]

(2.30)

Subtracting the contribution of extremal background from (2.29) and taking the near-extremal limit: \(r_0^{7-p} \sinh^2 \alpha \approx \tilde{R}^{7-p} - r_0^{7-p} / 2\), we reach

\[
(\Delta T)_{ab} = \frac{\Omega_{8-p} r_0^{7-p}}{16\pi G_{10} g_s^2} \frac{1}{2} \text{ diag } [9 - p, 5 - p, \cdots, 5 - p],
\]

(2.31)

and its trace

\[
\Delta T = -\frac{(p - 3)^2 \Omega_{8-p} r_0^{7-p}}{2} \frac{1}{16\pi G_{10} g_s^2}.
\]

(2.32)
In the decoupling limit, (2.31) reduces to

\[(\Delta T)_{ab} = \frac{\Omega_{8-p} U_0^{7-p}}{(2\pi)^7 g_{YM}^4} \frac{1}{2} \text{diag} [9 - p, 5 - p, \cdots, 5 - p], \quad (2.33)\]

which precisely agrees with (2.22) obtained by the counterterm method. In addition, from the 00-component of \((\Delta T)_{ab}\) we can read off directly the energy of thermal excitations on the Dp-branes:

\[E = \frac{\Omega_{8-p}}{(2\pi)^7 g_{YM}^4} \frac{9 - p}{2} U_0^{7-p} V_p. \quad (2.34)\]

Obviously it is again the same as the mass (2.19) of the domain-wall black holes.

For the black Dp-brane (2.2), the Hawking temperature and entropy are

\[T = \frac{1}{4\pi r_0} \frac{7 - p}{\cosh \alpha}, \quad S = \frac{4\pi \Omega_{8-p} V_p}{(2\pi)^7 g_{YM}^2} r_0^{8-p} \cosh \alpha. \quad (2.35)\]

The free energy of the thermal excitations defined as \(\mathcal{F} = E - TS\) is

\[\mathcal{F} = -\frac{\Omega_{8-p}}{(2\pi)^7 g_{YM}^4} \frac{5 - p}{2} V_p U_0^{7-p}, \quad (2.36)\]

in the decoupling limit. Once again, this reproduces the result (2.25) by the surface counterterm method.

### 2.3 Charged domain-wall black holes

It is now clear that one can make consistent reductions of eleven-dimensional supergravity on \(S^4\) or \(S^7\), and ten-dimensional type IIB supergravity on \(S^5\). The Kaluza-Klein sphere reduction results in gauged supergravities. The anti-de Sitter spaces are vacuum solutions of these gauged supergravities. More recently an evidence has been provided that some singular limits of sphere reduction are also consistent and resulting gauged supergravities have domain-wall vacuum solutions, instead of the AdS spaces \([31]\).

The so-called domain-wall supergravities can be consistently truncated to the following bosonic Lagrangian \([31]\):

\[S = \frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{\alpha \phi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} b^2 e^{-\alpha \phi} \right], \quad (2.37)\]
where \( a^2 = 2/p \), \( b \) is a constant and \( F_{\mu\nu} \) denotes the Maxwell field strength. The action (2.37) for \( p = 5 \) comes from the reduction on \( S^3 \times R \) of eleven-dimensional supergravity, while \( p = 2 \) from the reduction on \( S^3 \times R^4 \) and \( p = 3 \) from the \( S^3 \times R^2 \) reduction of type IIB supergravity. But we consider an arbitrary \( p \) in what follows.

The equations of motion from the action (2.37) have domain-wall black hole solutions

\[
\begin{align*}
 ds^2 &= -f(r)dt^2 + f^{-1}(r)dr^2 + rdx_p^2, \\
 F_{tr} &= \frac{pq}{2}r^{-(p+1)/2}, \quad e^{\alpha\phi} = r, \\
 f(r) &= 2r \left( \frac{b^2}{p^2} + \frac{q^2}{4r^p} - \frac{m}{r^{p/2}} \right),
\end{align*}
\]

(2.38)

where \( m \) and \( q \) represent two integration constants. Choosing the surface counterterm as

\[
 S_{ct} = -\frac{1}{8\pi G_{p+2}} \int d^{p+1}x \sqrt{-h} c_0, \quad c_0 = \sqrt{p(p+1)}, \quad \frac{1}{l_{eff}} = be^{-a\phi/2} \sqrt{\frac{1}{2p(p+1)}},
\]

(2.39)

as prescribed in the appendix, we have the quasilocal stress-energy tensor

\[
\begin{align*}
 8\pi G_{p+2} T_{tt} &= \frac{bn}{\sqrt{2}} r^{-(p+1)/2} + \cdots, \\
 8\pi G_{p+2} T_{ij} &= 0 + O(r^{-(p+1)/2}) + \cdots.
\end{align*}
\]

(2.40)

The mass of the black hole can be obtained as follows:

\[
 M = \int_{r\to\infty} d^p x r^{p/2} f^{-1/2} T_{tt} = \frac{pmV_p}{16\pi G_{p+2}}.
\]

(2.41)

In this case, the surface metric \( \gamma_{ab} \) is

\[
 \gamma_{ab} dx^a dx^b = \lim_{r\to\infty} \frac{1}{r} h_{ab} dx^a dx^b = -\frac{2b^2}{p^2} dt^2 + dx^2_p,
\]

(2.42)

and the boundary stress-energy tensor \( \tau_{ab} \) is found to be

\[
 \tau_{ab} = \frac{bn}{8\pi G_{p+2} \sqrt{2}} [1, 0, \cdots, 0].
\]

(2.43)

Its pressure vanishes identically. This is reminiscent of the case of the \( p = 5 \) neutral domain-wall black holes in the previous subsection. Calculating the Euclidean action of the charged domain-wall black holes,

\[
 I = -\frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{a\phi} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} b^2 e^{-a\phi} \right] \\
 + \frac{1}{8\pi G_{p+2}} \int d^{p+1}x \sqrt{h} K + \frac{1}{8\pi G_{p+2}} \int d^{p+1}x \sqrt{h} c_0, \\
\]

(2.44)
we find that the Euclidean action $I$ vanishes identically. This is again the same as the case of $p = 5$ neutral domain-wall black holes. Here it should be reminded that the calculation \[2.44\] has been done implicitly in grand canonical ensemble, in which the electric potential of the charge of black holes is fixed at the boundary. The vanishing of the Euclidean action in grand canonical ensemble means that the Gibbs free energy $G = 0$ for the charged domain-wall black holes.

### 3 Three-dimensional dilaton black holes

The three-dimensional BTZ black hole plays an important role in understanding statistical entropy of black holes. The degrees of freedom of the BTZ black hole can be accounted for by a two-dimensional boundary conformal field theory, which is a special case of the AdS/CFT correspondence. In this section we consider a deformed BTZ black hole by a dilaton field and an exponential potential. Its action is

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R - 4(\partial \phi)^2 + 2\Lambda e^{b\phi} \right) - \frac{1}{8\pi G} \int d^2x \sqrt{-h} K, \quad (3.1)$$

where $b$ and $\Lambda$ are two constants. The action has the following black hole solution \[32, 33\]:

$$\begin{align*}
ds^2 &= -A(r) dt^2 + A(r)^{-1} dr^2 + \beta^2 r^N d\theta^2, \\
\phi &= k \ln r, \\
A(r) &= -\frac{16Gm}{N} r^{1-N/2} + \frac{8\Lambda \beta^2}{N(3N-2)} r^N, \\
k &= \pm \frac{1}{4} \sqrt{N(2-N)}, \quad bk = N - 2, \quad (3.2)
\end{align*}$$

where $m$ is the quasilocal mass identified at spatial infinity by using background subtraction. The positive mass ($m > 0$) black holes exist only for $2 \geq N > 2/3$ and $\Lambda > 0$. When $N = 2$, the solution reduces to the BTZ black hole. In addition, note that the radial coordinate $r$ is chosen to be dimensionless and $\beta$ is a length scale with dimension of length. (This should not be confused with the inverse Hawking temperature $1/T$. In this paper we do not use $\beta$ for the inverse Hawking temperature.)

The dilaton black hole solution \[3.2\] has the horizon at $r = r_+$ with

$$r_+^{(3N-2)/2} = \frac{2Gm(3N-2)}{\Lambda \beta^2}. \quad (3.3)$$
The Hawking temperature and entropy of the solution are

\[
T = \frac{2Gm(3N - 2)}{N\pi\beta} r_+^{-N/2}, \\
S = \frac{\pi\beta}{2G} r_+^{N/2}.
\]  

(3.4)

Hence we have the free energy of the solution

\[
F = m - TS = -2m(N - 1)/N.
\]  

(3.5)

For \(2 > N > 1\), the free energy is always negative and the dilaton black hole is thermodynamically stable as the BTZ black hole. For \(1 > N > 2/3\), however, the free energy becomes positive and the dilaton black hole is thermodynamically unstable.

Obviously the dilaton black hole solution (3.2) is not asymptotically anti-de Sitter, unless \(N = 2\). In what follows we will extract the boundary stress-energy tensor and its quantum expectation value of the corresponding boundary quantum field, by adding an appropriate surface counterterm to the action (3.1). As in the previous section, the occurrence of the dilaton potential makes possible to choose a suitable surface counterterm as

\[
S_{ct} = -\frac{1}{8\pi G} \int d^2x \sqrt{-h} c_0, \quad c_0 = \sqrt{\frac{2N}{3N - 2}}, \quad \frac{1}{l_{\text{eff}}} = \sqrt{\Lambda e^{b\phi/2}},
\]  

(3.6)

as given in the appendix. Using the quasilocal stress-energy tensor formula (1.3), in this case, we have

\[
8\pi GT_{tt} = \frac{4Gm}{\beta} c + \cdots, \\
8\pi GT_{\theta\theta} = 8Gm\beta \frac{N - 1}{Ne} + \cdots,
\]  

(3.7)

where \(c^2 = 8\Lambda\beta^2/N(3N - 2)\). The mass of the black hole is found to be

\[
M = \int_{r\to\infty} d\theta \beta r^{N/2} A^{1/2}(r)u^t u^t T_{tt} = m,
\]  

(3.8)

which means that the mass of the black hole from the counterterm method is the same as the quasilocal mass identified at the spatial infinity. Note that the latter is obtained by using background subtraction method [5, 6].

The surface metric is derived as

\[
\gamma_{ab} dx^a dx^b = \lim_{r\to\infty} \frac{1}{r^N} h_{ab} dx^a dx^b = -c^2 dt^2 + \beta^2 d\theta^2.
\]  

(3.9)
In this spacetime, the boundary stress-energy tensor of the corresponding quantum field can be calculated as in the previous examples and we get

$$\tau_{ab} = \frac{mc}{2\pi \beta} \left[ 1, \frac{2(N - 1)\beta^2}{Nc^2} \right].$$  \hspace{1cm} (3.10)

Furthermore, calculating the Euclidean action of the black hole,

$$I = -\frac{1}{16\pi G} \int d^3x \sqrt{g} (R - 4(\partial \phi)^2 + 2\Lambda e^{b\phi}) + \frac{1}{8\pi G} \int d^2x \sqrt{h} K + \frac{1}{8\pi G} \int d^2x \sqrt{h} \frac{c_0}{l_{\text{eff}}},$$ \hspace{1cm} (3.11)

yields a finite result

$$I = \frac{F}{T} = -2m \frac{N - 1}{NT},$$ \hspace{1cm} (3.12)

which gives us the same free energy as (3.5). The example of the three-dimensional dilaton black hole shows that the surface counterterm method works well as in the case of domain-wall black holes. We expect that this method is also applicable to other three-dimensional black holes with a nonvanishing scalar field.

4 Topological dilaton black holes

Recently it has been found that in the asymptotically anti-de Sitter spaces, except for the black holes whose horizon hypersurface has the topology of positive curvature sphere, there are other black hole solutions with horizon hypersurfaces of zero or negative constant curvature. The latter are called topological black holes. These topological black holes have been studied extensively in the AdS/CFT correspondence (for example, see [13] and references therein). In this section, we consider those topological black holes in dilaton gravities. That is, as in the case of three-dimensional dilaton black holes, the negative cosmological constant is replaced by a dilaton potential, which changes drastically the asymptotic behavior of black hole solutions.

The action we will consider is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - 2(\partial \phi)^2 + 2\Lambda e^{2b\phi} - e^{-2a\phi} F_{\mu\nu} F^{\mu\nu} \right] - \frac{1}{8\pi G} \int d^3x \sqrt{-h} K,$$  \hspace{1cm} (4.1)

where $a$ and $b$ are two constants. Assuming the solution has the following metric:

$$ds^2 = -A(r)dt^2 + A^{-1}(r)dr^2 + R^2(r)d\Sigma_k^2,$$ \hspace{1cm} (4.2)
where $d\Sigma_k^2$ is the line element of a two-dimensional hypersurface with constant curvature $2k$. Without loss of the generality, we may set $k = 1, 0$ and $-1$, respectively. When $k = 1$, the hypersurface $\Sigma$ has a positive constant curvature. This is the case of spherically symmetric black holes. The horizon surface is of the topology of two-sphere $S^2$. When $k = 0$, the hypersurface $\Sigma$ is a Ricci flat surface. In this case, we may have the two-torus topology $T^2$, or its infinite area limit $R^2$, or a cylinder topology $S^1 \times R$. Finally when $k = -1$, $\Sigma$ is a hyperbolic hypersurface. By an appropriate identification, in this case, one may get an arbitrary higher genus hypersurface. In the asymptotically anti-de Sitter spaces, black holes with these three horizon hypersurfaces exist. Now we discuss these so-called topological black holes in the dilaton gravity described by (4.1). Because of the dilaton potential, we will see that the topological dilaton black holes will not approach asymptotically the anti-de Sitter space. Let us consider the case $k = 0$ first.

### 4.1 $k = 0$ solutions

In the case of $k = 0$, we have the black hole solutions [34, 35]:

\[
\begin{align*}
A(r) &= -\frac{8\pi G m}{V N \beta^2} r^{1-2N} + \frac{\Lambda e^{2b\phi_0}}{N(4N-1)} r^{2N} + \frac{16\pi^2 Q^2 e^{2a\phi_0}}{NV^2 \beta^4} r^{-2N}, \\
R(r) &= \beta r^N, \\
\phi &= \phi_0 - \sqrt{N(1 - N)} \ln r, \\
F_{tr} &= \frac{4\pi Q}{VR^2} e^{2a\phi}, \\
a = b &= \sqrt{N(1 - N)/N},
\end{align*}
\]

(4.3)

where $\phi_0$ is an integration constant, $Q$ is the charge of the hole and $m$ is the quasilocal mass identified at spatial infinity. $N$ and $\beta$ are two parameters and $V$ is the area of the hypersurface $\Sigma$. In order for the solution (4.3) to have a black hole structure, it must be satisfied that $\Lambda > 0$ and $1/4 < N < 1$ [34].

For the solution (4.3), the results in the appendix tell us that the suitable surface counterterm is

\[
S_{ct} = -\frac{1}{8\pi G} \int d^3x \sqrt{-h} \frac{c_0}{l_{\text{eff}}} e^{b\phi}, \quad c_0 = 2 \sqrt{\frac{3N}{4N - 1}}, \quad \frac{1}{l_{\text{eff}}} = e^{b\phi} \sqrt{\frac{\Lambda}{3}},
\]

(4.4)
Using this surface counterterm, we obtain the following quasilocal stress-energy tensor:

\[
T_{tt} = \frac{m}{V \beta^2} c_1 r^{-N} + \cdots,
\]

\[
T_{xx} = \frac{m}{2Vc_1} \frac{2N - 1}{N} r^{-N} + \cdots,
\]

\[
T_{yy} = T_{xx},
\]

where \(c_1 = \Lambda e^{2b\phi_0} / N(4N - 1)\). According to the mass formula (1.6), the mass of the black hole is

\[
M = \int_{r \rightarrow \infty} d^2x \sqrt{\sigma} R^2 A^{-1/2} T_{tt} = m,
\]

where \(\sigma\) is the determinant of the metric of the hypersurface \(\Sigma\). The mass is the same as the quasilocal mass at the spatial infinity. The surface metric, in which the boundary quantum field lives, is

\[
\gamma_{ab} = \lim_{r \rightarrow \infty} \frac{1}{R^2} h_{ab} = -\beta^{-2} c_1^2 dt^2 + dx^2 + dy^2.
\]

Using (2.21), we then obtain the boundary stress-energy tensor of the corresponding boundary quantum field,

\[
\tau_{ab} = \frac{M}{Vc_1} \frac{2N - 1}{2N} \text{diag} \left[ \frac{2N}{2N - 1} \beta^2, 1, 1 \right].
\]

After a straightforward calculation using the counterterm (4.4), we are able to get a finite Gibbs free energy of the black hole:

\[
\mathcal{G} = \frac{(2N - 1) V\beta^2}{16\pi GN} \left[ \frac{\Lambda e^{2b\phi_0}}{4N - 1} r_+^{4N - 1} + \frac{16\pi^2 Q^2 e^{2a\phi_0}}{V^2 \beta^4} \frac{1}{r_+} \right],
\]

where \(r_+\) is the horizon of the black hole, which satisfies the equation \(A(r_+) = 0\). The Gibbs free energy (4.9) is consistent with the definition

\[
\mathcal{G} = M - TS - \mu Q,
\]

where \(T\) and \(S\) are the Hawking temperature and entropy of black holes, and \(\mu\) is the chemical potential corresponding to the charge. For the solution (4.3), we have

\[
T = \frac{1}{4\pi r_+} \left[ \frac{\Lambda e^{2b\phi_0}}{N} r_+^{2N} - \frac{16\pi^2 Q^2 e^{2a\phi_0}}{NV^2 \beta^4} r_+^{-2N} \right],
\]

\[
S = \frac{V \beta^2}{4G} r_+^{2N}, \quad \mu = \frac{4\pi Q e^{2a\phi_0}}{GV^2 r_+}.
\]
It is easy to verify that (4.10) reproduces the result (4.9). Because of $1/4 < N < 1$, we see from (4.9) that the free energy is always negative for $1/2 < N < 1$, but changes its sign at $N = 1/2$, becoming positive for $1/4 < N < 1/2$. In the latter case, the black hole is thermodynamically unstable. Note that in the Einstein [36] and Einstein-Maxwell [37, 13] gravities with a negative cosmological constant, the black holes with $k = 0$ are always thermodynamically stable. Therefore the change of asymptotic behavior of black hole solutions may change the thermodynamic stability. It is worth pointing out here that the sign change of the Gibbs free energy does not mean the occurrence of the Hawking-Page phase transition [38, 39] as in the Einstein gravity, because once given a black hole, the value $N$ is fixed and then the sign of the free energy is fixed as well and will not change due to the change of the size of black holes.

4.2 $k = -1$ solutions

In this case, we consider the following black hole solution [34]:

$$A(r) = -\frac{8\pi G m}{VN\beta^2} r^{1-2N} + \frac{\Lambda e^{2\phi_0}}{1-N} r^{2-2N} + \frac{16\pi^2 Q^2 e^{2\phi_0}}{NV^2 \beta^4} r^{-2N},$$

$$\phi = \phi_0 - \sqrt{N(1-N)} \ln r,$$

$$\Lambda = \frac{1-N}{1-2N} e^{-2\phi_0} \beta^2,$$

$$b = 1/a = N/\sqrt{N(1-N)}$$

(4.12)

where we use the same notations as in the previous subsection and in particular $R$ and $F_{tr}$ are the same. To have a black hole structure, we must have $0 < N < 1/2$.

For the hyperbolic black hole (4.12), we find that the appropriate surface counterterm is

$$S_{ct} = -\frac{1}{8\pi G} \int d^3x \sqrt{-h} c_0 \frac{c_0}{l_{eff}}, \quad c_0 = 2N \sqrt{\frac{3}{1-N}}, \quad \frac{1}{l_{eff}} = e^{\phi_0} \sqrt{\frac{\Lambda}{3}}.$$  (4.13)

The quasilocal stress-energy tensor then is

$$T_{tt} = \frac{me^{b\phi_0}}{V \beta^2} \sqrt{\frac{\Lambda}{1-N}} r^{1-3N} + \cdots.$$  (4.14)

Unfortunately, the black hole solution (4.12) has no well-defined surface metric $\gamma_{ab}$ in this case and the other components of the quasilocal stress-energy tensor are not well defined,
either. Nonetheless, as a consistency check of the surface counterterm (4.13), we may compute the mass of the black hole using (1.6). Once again, in this way, the mass of the black hole is found to coincide with the quasilocal mass $m$ at the spatial infinity:

$$M = \int_{r \to \infty} d^2 x \sqrt{\sigma} R^2 A^{-1/2} T_{tt} = m. \quad (4.15)$$

Note that due to the different asymptotic behavior, the so-called “negative mass” black holes [36, 37, 13] do not appear in this dilaton gravity. The Euclidean action of the hole is

$$I = -\frac{1}{16\pi G} \int d^4 x \sqrt{g} [R - 2(\partial \phi)^2 + 2\Lambda e^{2b\phi} - e^{-2a\phi} F_{\mu \nu} F^{\mu \nu}] + \frac{1}{8\pi G} \int d^3 x \sqrt{h} K + \frac{1}{8\pi G} \int d^3 x \sqrt{h} \frac{c_0}{l_{\text{eff}}}. \quad (4.16)$$

Substituting the solution and the “effective cosmological constant” into the above, we find

$$I \equiv G/T = \frac{V}{16\pi GT} \left[ -r_+ + \frac{1 - 2N}{N} \frac{16\pi^2 Q^2 e^{2a\phi_0}}{V^2 \beta^2 r_+} \right], \quad (4.17)$$

where $T$ is the Hawking temperature of the solution,

$$T = \frac{1}{4\pi r_+} \left[ \frac{\Lambda}{1 - N} e^{2b\phi_0} r_+^{2-2N} - \frac{16\pi^2 Q^2 e^{2a\phi_0}}{N V^2 \beta^2} r_+^{-2N} \right], \quad (4.18)$$

and $r_+$ is the horizon radius obeying the equation

$$\frac{1}{1 - 2N} r_+^2 - \frac{8\pi G m}{V N} r_+ + \frac{16\pi^2 Q^2 e^{2a\phi_0}}{N V^2 \beta^2} = 0. \quad (4.19)$$

From (4.17) we see that the first term is negative, while the second is positive. Therefore the Gibbs free energy may be negative for large black holes, while positive for small black holes, which is reminiscent of the Schwarzschild-anti-de Sitter black holes [38, 39], where the free energy is also negative for large black holes and positive for small black holes. Therefore the Hawking-Page phase transition takes place for the hyperbolic black holes (4.12), which occurs at $G = 0$, that is, at $r = r_+$ with

$$r_+^2 = \frac{1 - 2N}{N} \frac{16\pi^2 Q^2 e^{2a\phi_0}}{V^2 \beta^2}. \quad (4.20)$$

From the free energy (4.17) and the critical point (4.20), one may see that the charge plays a central role in the Hawking-Page phase transition: if $Q = 0$, the phase transition
will disappear. It would be interesting to note that the Hawking-Page phase transition does not appear in the hyperbolic black holes of the Einstein and Einstein-Maxwell AdS gravities \[36, 37, 13\]. In addition, as a consistency check, one may reproduce the Gibbs free energy \((4.17)\) by the definition \((4.10)\) with the same expressions in \((4.11)\) of black hole entropy \(S\) and the chemical potential \(\mu\).

### 4.3 \(k = 1\) solutions

Of course, the action \((1.1)\) has also spherically symmetric black hole solutions. Three sets of black hole solutions have been given in \([10]\). We consider here the first set of the solutions found there:

\[
A(r) = r^{\frac{2\alpha^2}{\beta}} \left( \frac{1 + a^2}{(1 - a^2)\beta^2} - \frac{2(1 + a^2)Gm}{\beta^2 r} + \frac{Q^2(1 + a^2)e^{2\alpha \phi}}{\beta^4 r^2} \right),
\]

\[
R^2(r) = \beta^2 r^{2N}, \quad N = 1/(1 + a^2),
\]

\[
\phi = \phi_0 - \frac{a}{1 + a^2} \ln r,
\]

\[
F_{tr} = \frac{Q e^{2\alpha \phi}}{R^2},
\]

\[
b = 1/a, \quad \Lambda = \frac{a^2}{(1 - a^2)\beta^2} e^{-2\phi_0/a}.
\]

For this solution, \(a^2 < 1\) must be satisfied in order for the solution to describe a black hole. In addition, one may notice that when \(a^2 \to 0\), the solution has a well-defined asymptotically flat limit: Reissner-Nordström (RN) black hole solution.

As described in the appendix, the appropriate surface counterterm is

\[
S = -\frac{1}{8\pi G} \int d^3x \sqrt{-h} \frac{c_0}{l_{\text{eff}}}, \quad c_0 = 2 \sqrt{\frac{3}{a^2(1 + a^2)}}, \quad \frac{1}{l_{\text{eff}}} = e^{b\phi} \sqrt{\frac{\Lambda}{3}}.
\]

Using this surface term, we have

\[
T_{tt} = \frac{m}{4\pi \beta^2} \sqrt{\frac{1 + a^2}{(1 - a^2)\beta^2}} r^{\frac{2\alpha^2}{\beta} + \cdots}.
\]

A well-defined surface metric \(\gamma_{ab}\) requires \(a^2 = 1\), but which is excluded by the existence of black hole solutions from \((4.21)\). Just as the case of \(k = -1\), we cannot obtain a well-defined stress-energy tensor of the corresponding quantum field here. Instead we can
calculate the mass of the black hole as before. The result is again in agreement with the quasilocal mass \( m \) identified at the spatial infinity,

\[
M = \int_{r \to \infty} d\theta d\varphi \sin^2 \theta R^2 A^{-1/2} T_{tt} = m. \tag{4.24}
\]

The Euclidean action of the black hole in the grand canonical ensemble is found to be

\[
I \equiv \mathcal{G}/T = \frac{1}{4GT} \left[ r_+ - (1 - a^2) \frac{Q^2 e^{2a\phi_0}}{\beta^2 r_+^2} \right]. \tag{4.25}
\]

Here \( T \) is the Hawking temperature of the black hole

\[
T = \frac{1}{4\pi} \frac{a^2}{r_+^{2 + \frac{1}{2}}} \left[ \frac{1 + a^2}{(1 - a^2)\beta^2} - \frac{Q^2 (1 + a^2)e^{2a\phi_0}}{\beta^2 r_+^2} \right], \tag{4.26}
\]

and \( r_+ \) is the horizon radius satisfying the equation

\[
\frac{1}{(1 - a^2)} - \frac{2Gm}{r_+} + \frac{Q^2 e^{2a\phi_0}}{\beta^2 r_+^2} = 0. \tag{4.27}
\]

It can be seen clearly from (4.25) that the free energy is negative for small black holes, while it becomes positive for large black holes, which changes its sign at

\[
r_+^2 = (1 - a^2) \frac{Q^2 e^{2a\phi_0}}{\beta^2}. \tag{4.28}
\]

Thus the small black holes are thermodynamically stable and large black holes will become thermodynamically unstable. This property is the same as that of the RN black holes in asymptotically flat spaces. As is well known, the heat capacity is positive for near-extremal RN black holes (small \( r_+ \)) and becomes negative beyond a certain critical point (\( r_+ \) gets larger) from extremal RN black holes. This thermodynamic behavior is completely opposite to that of the black holes in the anti-de Sitter spaces [38]. Therefore although an “effective negative cosmological constant” occurs here, the thermodynamic properties are similar to those of RN black holes in asymptotically flat spaces. Furthermore, we can check that the Gibbs free energy (4.25) is reproduced by the definition with the black hole entropy \( S \) and the chemical potential \( \mu \):

\[
S = \frac{\pi \beta^2}{G} r_+^{1 + \frac{1}{2}}; \quad \mu = \frac{Q e^{2a\phi_0}}{G\beta^2 r_+}. \tag{4.29}
\]

In addition, it is worth noting that because the solution (4.21) has a well-defined asymptotically flat limit as \( a \to 0 \), the surface counterterm (4.22) has also a well-defined asymptotically flat limit. In this limit, we again reproduce the thermodynamics of RN and Schwarzschild black holes.
5 Conclusions

In the dilaton gravities with a dilaton potential, in general, the black hole solutions do not approach asymptotically anti-de Sitter spaces due to the dilaton field, but we have found that for such black holes, it is also possible to extract a well-defined surface stress-energy tensor and to get finite Euclidean action by adding appropriate surface counterterms to the bulk action, in which the dilaton potential plays a similar role as the cosmological constant does in the Einstein(-Maxwell) gravity. In this paper using this prescription we studied some examples including domain-wall black holes in gauged supergravities, three-dimensional dilaton black holes and topological dilaton black holes in four dimensions. In these examples, this prescription works well.

Using the surface counterterm method, we have also obtained boundary stress-energy tensors and Euclidean actions of domain-wall black holes. These results have been checked to be consistent with those coming from direct calculations in the original Dp-brane configurations of type II supergravity. For a kind of charged domain-wall black holes in the domain-wall gauged supergravities, which result from singular limit of the sphere reduction of eleven-dimensional supergravity and ten-dimensional type IIB supergravity, we have found that the Gibbs free energy and the pressure of the thermal excitations on the domain-wall always vanish. This is the same as the situation for the D5-brane case.

We have also studied thermodynamics of these black configurations by calculating Euclidean action within this surface counterterm method. Some new features have been found in the topological dilaton black holes, which are not present in the Einstein(-Maxwell) gravities with a negative cosmological constant. For example, $k = 0$ dilaton black holes may be thermodynamically unstable; in the hyperbolic charged dilaton black holes ($k = -1$), the Hawking-Page phase transition may take place; in the case of $k = 1$ we have a well-defined asymptotically flat limit of the surface counterterm. Using it, we can reproduce the thermodynamics of Schwarzschild and RN black holes.
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A Formula for surface counterterm

Here we present our formula for the surface counterterm, which is applicable to all our cases.

Let us write our metric as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\sigma^2, \quad (A.1)$$

where the last term represents a $p$-dimensional Ricci-flat space, and the “effective cosmological constant” $l_{\text{eff}}$ is defined as

$$\frac{1}{l_{\text{eff}}} = \sqrt{\frac{V(\Phi)}{p(p+1)}}, \quad (A.2)$$

in the Einstein frame. By analogy with the surface counterterms in Eq. (1.2), we introduce the counterterm

$$-2 \int_{\Sigma} d^{p+1}x \sqrt{-h} c_0 l_{\text{eff}}, \quad (A.3)$$

where only the relative normalization with the Einstein term is written explicitly. In fact, we find that appropriate choice of the coefficient removes divergences from physical quantities. The asymptotic behaviors of the metrics $B, C$ and $V$ govern the coefficient $c_0$, whose formula is derived below.

Let the asymptotic behaviors of the fields be

$$A(r) = A_0 r^\alpha + \ldots,$$

$$B(r) = B_0 r^\beta + \ldots,$$

$$C(r) = C_0 r^\gamma + \ldots,$$

$$V(\Phi) = V_0 r^\delta + \ldots \quad (A.4)$$

It can be easily checked that in order to satisfy the Einstein equations, one must have

$$\beta + \delta = -2. \quad (A.5)$$
We then compute the boundary stress-energy tensor to obtain
\[
T_{tt} = \frac{-pA}{2\sqrt{BC}} + \frac{c_0}{l_{\text{eff}}} A,
\]
\[
\simeq \frac{-pA_0\gamma}{2\sqrt{B_0}} r^{\alpha-\beta/2-1} + \frac{cA_0\sqrt{V_0}}{\sqrt{p(p+1)}} r^{\alpha+\delta/2} + \ldots, (A.6)
\]
where prime indicates differentiation with respect to $r$. This leading behavior governs the finiteness of the physical quantities. Note that thanks to the relation (A.5), the two terms in Eq. (A.6) match with each other and allows to cancel the divergences.

Imposing the condition that the leading terms be absent in Eq. (A.6), we obtain the formula
\[
c_0 = \frac{\gamma p\sqrt{p(p+1)}}{2\sqrt{B_0V_0}}. (A.7)
\]
In fact, for the asymptotically AdS space
\[
\gamma = 2, \quad B_0 = l^2, \quad V_0 = \frac{p(p+1)}{l^2}, (A.8)
\]
which reproduces the first term in Eq. (1.2).

For the neutral domain-wall black holes, we find
\[
\gamma = \frac{2(p-9)}{p(p-5)}, \quad B_0V_0 = \frac{2(9-p)(7-p)}{(5-p)^2}, (A.9)
\]
giving (2.16). For the charged domain-wall, we have
\[
\gamma = 1, \quad B_0V_0 = \frac{p^2}{4}, (A.10)
\]
leading to (2.39). For the dilaton black holes,
\[
\gamma = N, \quad p = 1, \quad B_0V_0 = \frac{N(3N-2)}{4}, (A.11)
\]
yielding (3.6). For topological black holes, we find
\[
\gamma = 2N, \quad p = 2, (A.12)
\]
and
\[
B_0V_0 = 2N(4N-1), \quad \text{for } k = 0,
\]
\[
B_0V_0 = 2(1-N), \quad \text{for } k = -1,
\]
\[
B_0V_0 = \frac{2a^2}{1+a^2}, \quad \text{for } k = 1, (A.13)
\]
giving (4.4), (4.13) and (4.22), respectively.
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