VISCOUS SHOCK PROFILE AND SINGULAR LIMIT FOR HYPERBOLIC SYSTEMS WITH CATTANEO’S LAW

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ABSTRACT. In the current paper, we consider large time behavior of solutions to scalar conservation laws with an artificial heat flux term. In the case where the heat flux is governed by Fourier’s law, the equation is scalar viscous conservation laws. In this case, existence and asymptotic stability of one-dimensional viscous shock waves have been studied in several papers. The main concern in the current paper is a $2 \times 2$ system of hyperbolic equations with relaxation which is derived by prescribing Cattaneo’s law for the heat flux. We consider the one-dimensional Cauchy problem for the system of Cattaneo-type and show existence and asymptotic stability of viscous shock waves. We also obtain the convergence rate by utilizing the weighted energy method. By letting the relaxation time zero in the system of Cattaneo-type, the system is formally deduced to scalar viscous conservation laws of Fourier-type. This is a singular limit problem which occurs an initial layer. We also consider the singular limit problem associated with viscous shock waves.

1. Introduction. We consider large time behavior of solutions to a scalar conservation laws with an artificial heat flux

$$u_t + f(u)_x + q_x = 0$$

over a one-dimensional full space $\mathbb{R} := (-\infty, \infty)$. Here $u = u(t,x) \in \mathbb{R}$ is an unknown function; $f(u) \in \mathbb{R}$ is a flux function which is a smooth given function of $u$; $q = q(t,x) \in \mathbb{R}$ is an artificial heat flux. We assume that $f(u)$ is strictly convex, that is, there exists a positive constant $c$ such that

$$f''(u) \geq c > 0$$

holds for an arbitrary $u$. For the heat flux $q$, we prescribe the following two types of relations:

- Fourier’s law : $\mu u_x + q = 0,$
- Cattaneo’s law : $\varepsilon q_t + \mu u_x + q = 0,$

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where \( \varepsilon > 0 \) is a relaxation time and \( \mu > 0 \) is a viscosity coefficient.

The main concern of the current paper is a viscous shock wave, which is a smooth traveling wave solution, for the equation (1) prescribed with Fourier’s law or Cattaneo’s law. The initial value problem for (1) with Fourier’s law is given by

\[
\begin{align*}
    u_t + f(u)_x + q_x &= 0, \\ \\
    \mu u_x + q &= 0, \\ \\
    u(0, x) &= u_0(x) \to u_\pm \quad (x \to \pm \infty),
\end{align*}
\]

(3a)-(3c) where \( u_\pm \) are constants satisfying

\[
    u_+ < u_-.
\]

From this condition as well as the convexity condition (2), we have

\[
f'(u_+) < f'(u_-).
\]

We call the system (3a) and (3b) a system of Fourier-type. Notice that the system of Fourier-type is deduced to a scalar viscous conservation laws for \( u \) as

\[
u_t + f(u)_x = \mu u_{xx}.
\]

(4)

For the scalar viscous conservation laws (4), asymptotic stability of a viscous shock wave has been studied. The pioneering work was done by Il’in and Oleinik [5]. It was shown in [5] that the viscous shock wave is asymptotically stable with exponential decay if the initial disturbance decays exponentially as \( |x| \to \infty \). The proof is based on the maximum principle. For the isentropic model of compressible viscous fluid, Matsumura and Nishihara [9] proved asymptotic stability of the viscous shock wave by using the \( L^2 \) energy method for the integrated system. Goodman [3] also used the \( L^2 \) energy method for the uniformly parabolic system and showed asymptotic stability of the viscous shock wave. The \( L^2 \) energy method for the integrated system was generalized to the full system of an ideal polytropic gases and the Broadwell model of the discrete Boltzmann equation by Kawashima and Matsumura [6]. The case where the flux function \( f(u) \) is non-convex was handled in [7, 8, 10, 11]. Especially in [10], the technical weight function with using the viscous shock wave was developed in order to obtain the convergence rate.

In place of Fourier’s law, Cattaneo’s law has been widely used for describing the finite speed of heat conduction. As for the model systems with Cattaneo’s law, see [2, 14] for the thermoelasticity and [4] for the compressible viscous fluid. See also the references in these papers.

The first aim of the current paper is to show existence and asymptotic stability of the viscous shock wave for the equation (1) with Cattaneo’s law, which is a \( 2 \times 2 \) system of hyperbolic equations and called a system of Cattaneo-type through the current paper. Since the system of Cattaneo-type has the relaxation time \( \varepsilon \) as a small parameter, the solution to the system is denoted by \((u^\varepsilon, q^\varepsilon)\). Thus the initial value problem for Cattaneo-type is given by

\[
\begin{align*}
    u^\varepsilon_t + f(u^\varepsilon)_x + q^\varepsilon_x &= 0, \\ \\
    \varepsilon q^\varepsilon_t + \mu u^\varepsilon_x + q^\varepsilon &= 0, \\ \\
    (u^\varepsilon, q^\varepsilon)(0, x) &= (u_0, q_0)(x) \to (u_\pm, q_\pm) \quad (x \to \pm \infty),
\end{align*}
\]

(5a)-(5c) where \( q_\pm \) is a constant and it is shown in Section 2 that \( q_+ = q_- = 0 \) holds for existence of the viscous shock wave.
By letting $\varepsilon \to 0$ in Cattaneo-type (5), we formally obtain Fourier-type (3). This is a relaxation limit from a $2 \times 2$ hyperbolic system to a scalar parabolic equation. Since the initial data in (5c) does not necessarily satisfy the relation $q_0 = -\mu u_{0x}$, the difference $q_0 + \mu u_{0x}$ remains as an initial layer. Thus this problem is a singular limit problem. The second aim of the current paper is to consider the singular limit associated with the viscous shock wave.

The outline of the current paper is as follows. In Section 2 we consider the Fourier-type (3). Since this case is deduced to a scalar viscous conservation laws (4), existence and asymptotic stability of the viscous shock wave are already proved. However, we apply the method for the Fourier-type to the problem of Cattaneo-type in Section 3, we show the proofs of existence and asymptotic stability of the viscous shock wave for the Fourier-type by introducing the method in [6, 9, 10]. In Section 3, we consider the system of Cattaneo-type (5) and show existence of the viscous shock wave in Theorem 3.1 and its asymptotic stability in Theorem 3.2. To show asymptotic stability, we use the $L^2$ energy method for the integrated system under suitably determined shift, and obtain the uniform a priori estimate by assuming that the initial perturbation and the shock strength $\delta := |u_+ - u_-|$ are sufficiently small. Then we obtain the convergence rate by using the weighted energy method with employing the technical weight function developed in [10]. Namely, we show in Theorem 3.8 that if the initial perturbation belongs to the exponentially weighted Sobolev space, then the solution to (5) tends to the corresponding viscous shock wave exponentially fast as $t \to \infty$. In Section 4, we consider the relaxation limit $\varepsilon \to 0$. We firstly show in Theorem 4.1 that the viscous shock wave $\tilde{u}^{\varepsilon}$ for Cattaneo-type tends to $\tilde{u}$ for Fourier-type as $\varepsilon \to 0$ in $L^p$ norm. We secondly discuss the singular limit problem. Namely we show in Theorem 4.3 that the solution $(u^{\varepsilon}, q^{\varepsilon})$ to (5) tends to the solution $(u, q)$ to (3) as $\varepsilon \to 0$ uniformly in $t$. It is also verified that the initial layer decays exponentially as $t \to \infty$ or $\varepsilon \to 0$. The proof is based on the Gronwall inequality with using the exponential decay of the perturbation.

**Notations.** For $p \in [1, \infty]$, $L^p = L^p(\mathbb{R})$ denotes a standard Lebesgue space over $\mathbb{R}$ equipped with a norm $\| \cdot \|_{L^p}$. For a non-negative integer $s$, $H^s = H^s(\mathbb{R})$ denotes an $s$-th order Sobolev space over $\mathbb{R}$ in the $L^2$ sense with a norm $\| \cdot \|_{H^s}$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted $L^2$ space by $L^2_\alpha := L^2(e^{\alpha |x|})$ of which norm is given by

$$\|u\|_{L^2_\alpha} := \left( \int_{\mathbb{R}} e^{\alpha |x|} |u(x)|^2 \, dx \right)^{1/2}.$$

We define the exponentially weighted $H^s$ space by $H^s_\alpha := H^s(e^{\alpha |x|})$ of which norm is given by

$$\|u\|_{H^s_\alpha} := \left( \sum_{k=0}^{s} \| \partial_x^k u \|_{L^2_\alpha}^2 \right)^{1/2}.$$

Through the paper, $c$ and $C$ denote several generic positive constants.

2. **Fourier-type: Scalar viscous conservation laws.** In this section, we consider existence and asymptotic stability of the viscous shock wave for Fourier-type (3) by introducing the results in [6, 9, 10].

2.1. **Existence of viscous shock wave.** We firstly show the existence of the viscous shock wave. Let $(\tilde{u}, \tilde{q})(\xi)$ be a smooth traveling wave solution to (3) satisfying
\( \hat{u}(\xi) \to u_{\pm} (\xi \to \pm\infty) \), where \( \xi := x - st \) and \( s \) is a shock speed. Thus the equations for \( (\hat{u}, \hat{q}) \) are given by

\[
- s\hat{u}_\xi + f(\hat{u})_\xi + \hat{q}_\xi = 0, \\
\mu\hat{u}_x + \hat{q} = 0.
\]

From (6b), we see that \( \hat{q}(\xi) \to 0 \) as \( \xi \to \pm\infty \) and hence \( q_+ = q_- = 0 \). Substituting (6b) in (6a), we get a single equation for \( \hat{u} \) as

\[
\hat{u}(\xi) \to u_{\pm} (\xi \to \pm\infty).
\]

Integrating (7a) over \( \mathbb{R} \), we have

\[-s(u_+ - u_-) + f(u_+) - f(u_-) = 0,
\]
which gives the Rankine–Hugoniot condition

\[s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.
\]

Integrating (7a) over \( (\pm\infty, \xi) \), we get the ordinary differential equation of first order for \( \hat{u} \) as

\[
\mu\hat{u}_\xi = h(\hat{u}) := -s\hat{u} + f(\hat{u}) - (-s u_+ + f(u_+)), \\
\hat{u}(0) = u_*, \quad \hat{u}(\xi) \to u_{\pm} (\xi \to \pm\infty),
\]

where \( u_* \in (u_+, u_-) \) is a constant satisfying

\[h'(u_*) = 0.
\]

Due to the uniform convexity (2) of \( f(u) \), we have the Lax shock condition

\[f'(u_+) < s < f'(u_-)
\]
and hence \( h'(u_+) < 0 \) and \( h'(u_-) > 0 \). Therefore we obtain the existence of the non-degenerate viscous shock wave which converges to \( u_{\pm} \) exponentially fast as \( \xi \to \pm\infty \). Notice that (2) and (11) give the unique existence of \( u_* \) satisfying (10).

**Theorem 2.1 ([6, 9]).** The problem (9) has a unique smooth solution \( \hat{u}(\xi) \) satisfying

\[|\partial^k_\xi (\hat{u}(\xi) - u_-)| \leq C\delta e^{c\delta \xi} (\xi \leq 0), \quad |\partial^k_\xi (\hat{u}(\xi) - u_+)| \leq C\delta e^{-c\delta \xi} (\xi \geq 0)
\]
for \( k = 0, 1, \ldots \), where \( \delta := |u_+ - u_-| \).

In deriving (12), we have used the fact that

\[-c_1\delta \leq h'(u_+) \leq -c_0\delta, \quad c_0\delta \leq h'(u_-) \leq c_1\delta
\]
for a sufficiently small \( \delta \), where \( c_0 \) and \( c_1 \) are positive constants independent of \( \delta \). The estimate (13) follows from the asymptotic expansion

\[h'(u_{\pm}) = \frac{1}{2}f''(u_{\pm})\delta + O(\delta^2) \quad (\delta \to 0)
\]
as well as the uniform convexity (2).
2.2. **Asymptotic stability.** We next consider asymptotic stability of $\tilde{u}$ obtained in Theorem 2.1 by introducing the $L^2$ energy method for the integrated equation developed in [6, 9]. Define a perturbation $\varphi$ of the solution $u$ to Fourier-type (3) from $\tilde{u}$ as

$$\varphi(t, \xi) = u(t, \xi + st) - \tilde{u}(\xi + x_0),$$

where $x_0 \in \mathbb{R}$ is a shift to be determined later. Thus the equation for $\varphi$ is given by

$$\varphi_t - s\varphi_{\xi} + (f(\tilde{u} + \varphi) - f(\tilde{u}))_{\xi} - \mu \varphi_{\xi\xi} = 0. \quad (15)$$

Integrating (15) over $(0, t) \times \mathbb{R}$, we formally get

$$\int_{\mathbb{R}} \varphi(t, \xi) \, d\xi = \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}(\xi + x_0)) \, d\xi.$$  

We determine the shift $x_0$ to satisfy

$$I(x_0) := \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}(\xi + x_0)) \, d\xi = 0 \quad (16)$$

provided that $u_0 - \tilde{u} \in L^1$. Since we have $I'(x_0) = -(u_+ - u_-)$, it holds that $I(x_0) = I(0) = (u_+ - u_-)x_0$. Therefore, by determining $x_0$ as

$$x_0 = \frac{1}{u_+ - u_-} I(0) = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}(\xi)) \, d\xi, \quad (17)$$

we get (16) and hence $\int_{\mathbb{R}} \varphi(t, \xi) \, d\xi = 0$. Then we define an anti-derivative of $\varphi$ by

$$\Phi(t, \xi) = \int_{-\infty}^{\xi} (u(t, \xi + st) - \tilde{u}(\xi + x_0)) \, d\xi.$$  

Notice that $\Phi_t = \varphi$. The initial value problem for $\Phi$ is derived by integrating (15) as

$$\Phi_t - s\Phi_{\xi} + f(\tilde{u} + \Phi_{\xi}) - f(\tilde{u}) - \mu \Phi_{\xi\xi} = 0, \quad (18a)$$

$$\Phi(0, \xi) = \phi_0(\xi) := \int_{-\infty}^{\xi} (u_0(\xi) - \tilde{u}(\xi + x_0)) \, d\xi. \quad (18b)$$

The asymptotic stability of the viscous shock wave $\tilde{u}$ is shown in the next theorem by deriving the a priori estimate in the function space

$$X(0, T) := \bigcap_{k=0}^{1} C^k([0, T]; H^{3-2k}).$$

**Theorem 2.2 ([6, 9]).** Let $u_0 - \tilde{u} \in L^1$ and $\phi_0 \in H^3$. Then there exists a positive constant $\eta$ such that if $\|\phi_0\|_{H^3} \leq \eta$, the problem (18) has a unique solution $\Phi \in X(0, \infty)$. Moreover, the solution $u(t, x)$ to (3) converges to the viscous shock wave $\tilde{u}(x - st + x_0)$ as $t \to \infty$:

$$\sup_{x \in \mathbb{R}}|u(t, x) - \tilde{u}(x - st + x_0)| \to 0 \quad (t \to \infty). \quad (19)$$

**Remark 1.** (i) Notice that the asymptotic stability in [6, 9] is proved in the $H^2$ framework. We show Theorem 2.2 in the $H^3$ framework since we need $H^3$ regularity in considering a singular limit problem in Section 4.

(ii) It is known that Theorem 2.2 is proved under the more mild assumption, that is, the Oleinik shock condition, than the convexity of $f(u)$ in (2). If we do not assume the convexity (2), the constants $c$ and $C$ in Lemma 2.7 may depend on the size of shock strength $\delta$. This dependency yields some difficulty in deriving the
weighted energy estimate for the system of Cattaneo-type in Section 3 since we need a smallness assumption on δ for Cattaneo-type. To avoid this difficulty, we assume the convexity (2) in stead of the Oleinik shock condition.

Theorem 2.2 is proved by combining the uniform a priori estimate of Φ with the existence of the solution locally in time summarized in the following lemma.

**Lemma 2.3.** For Φ₀ ∈ Hᵖ, there exists a positive constant T₀ depending on ‖Φ₀‖ₚ such that the problem (18) has a unique solution Φ ∈ X(0,T₀).

Lemma 2.3 is probed by a standard iteration method so that we omit the details of proof. We next show the a priori estimate of the solution Φ. To do this, we define the energy norm defined by

\[ E(t) := \sup_{τ ∈ [0,t]} ‖Φ(τ)‖ₚ. \]

**Proposition 1.** Let Φ ∈ X(0,T) be a solution to (18) for a certain T > 0. Then there exists a positive constant η such that if E(T) ≤ η, the solution satisfies

\[ ‖Φ(t)‖ₚ² + \int_0^t ‖Φₓ(t)‖ₚ² dτ ≤ C ‖Φ₀‖ₚ² \]

for t ∈ [0,T].

From the uniform estimate (20) as well as the standard continuity argument, we obtain the existence of the solution Φ globally in time. Moreover, the dissipative estimate in (20) gives the convergence ‖Φₓ(t)‖ₚ → 0 (t → ∞) which yields the asymptotic stability (19). Therefore it suffices to show Proposition 1.

To show Proposition 1, by following the method in [6, 9], we firstly obtain the basic L² estimate for Φ in the following lemma.

**Lemma 2.4.** Suppose that the same assumption as in Proposition 1 holds. Then we have

\[ ‖Φ(t)‖ₐ² + \int_0^t ‖(\sqrt{−uₓΦ, Φₓ)(τ)}‖ₐ² dτ ≤ C ‖Φ₀‖ₐ². \]

**Proof.** Multiplying (18a) by Φ, we have

\[ \frac{1}{2}Φₓ² + \frac{1}{2}Φₓ(Φₓ) − \frac{1}{2}f'(u)Φₓ² + μΦₓ² = R₀, \]

\[ R₀ := −f(Φₚ) − f(Φₚ) − f'(Φₚ)Φₚ. \]

Integrating (22) over (0,T) × ℝ and using

\[ \int_ℝ |R₀| dξ ≤ C \int ℝ |Φₓ²| dξ ≤ C ‖Φ‖ₚₚ ‖Φₓ‖ₐ² ≤ CE(T) ‖Φₓ‖ₐ² \]

as well as the Sobolev inequality ‖Φ‖ₚₚ ≤ C ‖Φ‖ₚ, we obtain the desired inequality (21). □

We next show the estimate for higher order derivatives.

**Lemma 2.5.** Let k = 1, 2, 3. Suppose that the same assumption as in Proposition 1 holds. Then we have

\[ ‖Φₓ̅̅̅ₖ(t)‖ₐ² + \int_0^t ‖Φₓ̅̅̅ₖ₊¹(t)‖ₐ² dτ ≤ C ‖Φₓ̅̅̅ₖΦ₀‖ₐ² + C \int_0^t ‖Φₓ(t)‖ₐ² dτ. \]
Proof. Applying \( \partial_{\xi}^{k-1} \) to (18a) and multiplying the resultant equality by \(-\partial_{\xi}^{k+1}\Phi\), we get
\[
\frac{1}{2}(\partial_{\xi}^{k}\Phi^2)_{t} + \left(-\partial_{\xi}^{k}\Phi \frac{\partial_{\xi}^{k-1}\Phi_{t}}{2} + \frac{s}{2}(\partial_{\xi}^{k}\Phi)^2\right)_{\xi} + \mu(\partial_{\xi}^{k+1}\Phi)^2 = \partial_{\xi}^{k+1}\Phi \partial_{\xi}^{k-1}(f(\tilde{u} + \Phi_{\xi}) - f(\tilde{u})).
\] (24)

Integrating (24) and substituting the inequality
\[
\int_{R} |\partial_{\xi}^{k+1}\Phi \partial_{\xi}^{k-1}(f(\tilde{u} + \Phi_{\xi}) - f(\tilde{u}))| d\xi \leq \frac{1}{2} \|\partial_{\xi}^{k+1}\Phi\|_{L^2}^2 + C\|\Phi_{\xi}\|_{H^{k-1}},
\]
we obtain the desired inequality (23). \( \square \)

Using (21) and applying an induction to (23) with respect to \( k \), we obtain the desired estimate (20). Thus we complete the proofs of Proposition 1 and Theorem 2.2.

We also show the estimate for derivatives with respect to \( t \) which is used for the singular limit problem in Section 4.

**Lemma 2.6.** Suppose that the same assumption as in Theorem 2.2 holds. Then the solution \( \Phi \in X(0, \infty) \) to (18) satisfies
\[
\|\Phi_{t}(t)\|_{H^1} + \int_{0}^{t} \|\Phi_{t}(\tau)\|_{H^1} d\tau \leq C\|\Phi_{0}\|_{H^3}.\] (25)

Proof. Applying \( \partial_{t} \) to (18a) and multiplying the resultant equality by \( \Phi_{t} \), we have
\[
\left(\frac{1}{2}\Phi_{t}^2\right)_{t} - \left(\frac{s}{2}\Phi_{t}^2 + \mu \Phi_{t}\Phi_{\xi}\right)_{\xi} + \mu \Phi_{t}^2 = -\Phi_{t}(f(\tilde{u} + \Phi_{\xi}) - f(\tilde{u}))_{t}.\] (26)

Integrating (26) and using the estimate
\[
\int_{R} |\Phi_{t}(f(\tilde{u} + \Phi_{\xi}) - f(\tilde{u}))_{t}| d\xi \leq \frac{1}{2} \|\Phi_{\xi}\|_{L^2}^2 + C\|\Phi_{t}\|_{L^2}^2,
\]
we obtain
\[
\|\Phi_{t}(t)\|_{L^2}^2 + \int_{0}^{t} \|\Phi_{t}(\tau)\|_{L^2}^2 d\tau \leq C\|\Phi_{0}\|_{H^2}^2 + C\int_{0}^{t} \|\Phi_{t}(\tau)\|_{L^2}^2 d\tau \leq C\|\Phi_{0}\|_{H^2}.\] (27)

Here we have used
\[
\int_{0}^{t} \|\Phi_{t}\|_{L^2}^2 d\tau \leq C\int_{0}^{t} \|\Phi_{t}, \Phi_{\xi}\|_{L^2}^2 d\tau \leq C\|\Phi_{0}\|_{H^1}^2,
\]
which is derived from the equation (18a) and the estimate (20). In the similar way, we obtain the estimate for \( \Phi_{\xi t} \) as
\[
\|\Phi_{\xi t}(t)\|_{L^2}^2 + \int_{0}^{t} \|\Phi_{\xi t}(\tau)\|_{L^2}^2 d\tau \leq C\|\Phi_{0}\|_{H^3}.\] (28)

Combining the estimates (27) and (28), we obtain the desired estimate (25). \( \square \)
2.3. Convergence rate. We next obtain the convergence rate for asymptotic stability in Theorem 2.2 by introducing the results in [6, 10]. To obtain the convergence rate, we derive the weighted energy estimate with employing an weight function in terms of \( \tilde{u} \) developed in [10] defined by

\[
\omega(\tilde{u}) := \frac{-(g(\tilde{u}))^{1-\beta \delta^2}}{-h(\tilde{u})} \quad (0 \leq \beta \leq 1),
\]

where \( g(\tilde{u}) := (\tilde{u} - u_+)(\tilde{u} - u_-) \) and \( h(\tilde{u}) \) is defined in (9a).

**Lemma 2.7.** Let \( \tilde{u} \) be a viscous shock wave obtained in Theorem 2.1. Then we have

(i) \( c \leq \frac{h(\tilde{u})}{g(\tilde{u})} \leq C \),

(ii) \( 0 < \omega(\tilde{u}) \leq Ce^{c|\xi|} \quad (\xi \in \mathbb{R}) \),

(iii) \( -(\omega(\tilde{u}) h(\tilde{u}))'' \tilde{u}_\xi \geq c(\beta \delta^4 - \tilde{u}_\xi)\omega(\tilde{u}), \) and

(iv) \( |\omega(\tilde{u})| \leq C(\beta \delta^4 - \tilde{u}_\xi)\omega(\tilde{u}), \)

where \( c \) and \( C \) are positive constants independent of \( \delta \).

**Proof.** (i) Since \( g(u) < 0 \) and \( h(u) < 0 \) for \( u \in (u_+, u_-) \), we see \( \frac{h(u)}{g(u)} > 0 \). Moreover, since we see

\[
\lim_{u \to u_\pm} \frac{h(u)}{g(u)} = h'(u_\pm) \frac{1}{g'(u_\pm)} = \frac{1}{2} f''(u_\pm) \delta + O(\delta^2) = \frac{1}{2} f''(u_\pm) + O(\delta) \quad (\delta \to 0),
\]

the uniform convexity (2) yields that the function \( \frac{h(u)}{g(u)} \) is positive and bounded uniformly in \( u \in [u_+, u_-] \) and \( \delta \).

(ii) Let \( \xi_1 \in \mathbb{R} \) be a constant satisfying \( \tilde{u}(\xi_1) = (u_+ + u_-)/2 \). In the case of \( \xi \geq \xi_1 \), since \( \tilde{u} \) is monotonically decreasing and satisfies \( cde^{-C\delta \xi} \leq \tilde{u} - u_+ \leq C\delta e^{-\delta \xi} \), we see

\[
g(\tilde{u}) \geq \frac{\delta}{2}(\tilde{u} - u_+) \geq c\delta^2 e^{-C\delta \xi}.
\]

Therefore, we have

\[
\omega(\tilde{u}) \leq C(-g(\tilde{u}))^{-\beta \delta^2} \leq C\delta^{-2\beta \delta^2} e^{C\beta \delta^2 \xi} \leq C e^{C\beta \xi}
\]

for \( \xi \geq \xi_1 \), where we have used the uniform bound of \( \delta^{-\delta^2} \) with respect to \( \delta \). The case of \( \xi \leq \xi_1 \) is handled in the same way.

(iii) From a straightforward computation, we have

\[
(\omega(u)h(u))'' = (1 - \beta \delta^2)(-g(u))^{-\beta \delta^2 - 1} A(u),
\]

\[
A(u) := \beta \delta^2 (g'(u))^2 - g(u)g''(u).
\]

Since \( A(u) \geq -g(u)g''(u) = -2g(u) \), we see

\[
-(\omega(\tilde{u}) h(\tilde{u}))'' \tilde{u}_\xi \geq -2\tilde{u}_\xi(-g(\tilde{u}))^{-\beta \delta^2} \geq -2\tilde{u}_\xi \frac{-(g(\tilde{u}))^{1-\beta \delta^2}}{-h(\tilde{u})} \frac{h(\tilde{u})}{g(\tilde{u})} \geq -c\tilde{u}_\xi \omega(\tilde{u}).
\]

Also, since the function \( A(u) = (4\beta \delta^2 - 2)(u - (u_+ + u_-)/2)^2 + \delta^2/2 \) takes a minimal value \( \beta \delta^4 \) at \( u = u_\pm \), we see

\[
-(\omega(\tilde{u}) h(\tilde{u}))'' \tilde{u}_\xi \geq -\beta \delta^4 (-g(\tilde{u}))^{-\beta \delta^2 - 1} \tilde{u}_\xi = \beta \delta^4 \frac{-(g(\tilde{u}))^{1-\beta \delta^2}}{-h(\tilde{u})} \frac{h(\tilde{u})^2}{g(\tilde{u})^2} \geq c\beta \delta^4 \omega(\tilde{u}).
\]
The inequalities (30) and (31) give the desired inequality. 

(iv) A straight forward computation gives

$$\omega'(u) = B(u) \omega(u) - \beta \delta^2 \omega(u) \frac{g'(u)}{g(u)}, \quad B(u) := \frac{g'(u)}{g(u)} - \frac{h'(u)}{h(u)}.$$ 

We see

$$\lim_{u \to u_\pm} B(u) = \frac{1}{2} \left( \frac{g''(u_\pm)}{g'(u_\pm)} - \frac{h''(u_\pm)}{h'(u_\pm)} \right) = \frac{1}{1 + \delta} - \frac{f''(u_\pm)}{f'(u_\pm) \delta + O(\delta^2)}$$

which yields that $B(u)$ is bounded uniformly in $u$ and $\delta$. Therefore, using the boundedness of $\frac{g'(u)}{g(u)} \hat{u}_\xi = \frac{g'(u)}{g(u)} h(\hat{u})$, we obtain

$$|\omega(\hat{u})\xi| = \left| B(\hat{u}) \omega(\hat{u}) \hat{u}_\xi - \beta \delta^2 \omega(\hat{u}) \frac{g'(\hat{u})}{g(\hat{u})} \hat{u}_\xi \right| \leq -C \hat{u}_\xi \omega(\hat{u}) + C \beta \delta^2 \omega(\hat{u}).$$

Therefore we complete the proof. \( \square \)

By using the weighted energy method, we obtain the weighted energy estimate which yields the convergence rate. To do this, we define a weighted $L^2$ norm with weight function $\omega$ as

$$|\Phi|_{\omega(\hat{u})} := \left( \int_{\mathbb{R}} \omega(\hat{u})|\Phi(\xi)|^2 \, d\xi \right)^{1/2}.$$ 

**Theorem 2.8** [6, 10]. Let $u_0 - \hat{u} \in L^1$ and $\Phi_0 \in H^{3} \cap L^2_\alpha$ for a certain $\alpha > 0$. Then there exist positive constants $\eta$ and $\gamma$ such that if $||\Phi_0||_{H^3} \leq \eta$, the solution $\Phi$ to (18) verifies

$$||\Phi(t)||_{L^2} \leq C e^{-\gamma \delta^3 t} \quad (t \geq 0).$$

**Proof:** Multiplying (22) by the weight function $\omega = \omega(\hat{u})$ with using the computation

$$\omega \left( \frac{1}{2} \omega' \Phi^2 - \mu \omega \Phi \xi \xi \right) - \frac{1}{2} \omega h'' \hat{u}_\xi \Phi^2 = \frac{1}{2} \omega h' \Phi^2 - \mu \omega \Phi \xi \xi - \frac{1}{2} (\omega h')' \hat{u}_\xi \Phi^2 + (\mu \omega \xi - \omega h') \Phi \xi$$

we get

$$\left( \frac{1}{2} \omega \Phi^2 \right)_t + \left( \frac{1}{2} (\omega h') \Phi^2 - \mu \omega \Phi \xi \xi \right)_\xi - \frac{1}{2} (\omega h')' \hat{u}_\xi \Phi^2 + (\mu \omega \xi - \omega h') \Phi \xi = \omega R_0. \quad (34)$$

Integrating (34) in $\xi \in \mathbb{R}$ and using Lemma 2.7-(iii) and

$$\int_{\mathbb{R}} |\omega R_0| \, d\xi \leq C ||\Phi||_{L^\infty} |\Phi_{\xi,2}| \leq C ||\Phi_0||_{H^3} |\Phi_{\xi,2}|,$$

we obtain

$$\frac{d}{dt} |\Phi|_{\omega}^2 + c \beta \delta^2 |\Phi|_{\omega}^2 + c |\Phi_{\xi}|_{\omega}^2 \leq 0, \quad (35)$$

for a certain $c$. The proof is complete. \( \square \)
where we have assumed that $\|\Phi_0\|_{H^2}$ is sufficiently small. Multiplying (35) by a time-weight function $e^{\gamma \delta t}$ for a positive constant $\gamma$ and integrating the resultant inequality in $t$ give

$$e^{\gamma \delta t} |\Phi|^2_\omega + \int_0^t e^{\gamma \delta \tau} \left( \beta \delta |\Phi|^2_\omega + |\Phi|^2_k \right) d\tau$$

$$\leq C |\Phi_0|^2 + C \gamma \delta \int_0^t e^{\gamma \delta \tau} |\Phi|^2_k d\tau. \quad (36)$$

By using Lemma 2.7-(ii), we take $\beta$ depending on $\alpha$ to satisfy $|\Phi_0|_{\omega(u)} \leq C |\Phi_0|_{L^2}$. Moreover, we take $\gamma$ to satisfy $C \gamma < \beta$. These computations yield from (36) that

$$e^{\gamma \delta t} |\Phi|^2_\omega + \int_0^t e^{\gamma \delta \tau} |\Phi|^2_k d\tau \leq C |\Phi_0|^2_{L^2},$$

which yields the desired estimate (32).

\[\Box\]

**Remark 2.** From (20) and (32) with the aid of the interpolation inequality

$$\| \partial^k \Phi \|_{L^2} \leq C \| \partial^k \phi \|^p_{\Omega} \| \Phi \|^{1-p}_{L^2}, \quad \theta = \frac{k}{3}, \quad k = 1, 2,$$

we have the convergence rate for the higher derivatives. Namely, there exists a positive constant $\gamma$ such that we have

$$\| \Phi(t) \|_{H^2} \leq C e^{-\gamma t} \quad (t \geq 0). \quad (37)$$

3. **Cattaneo-type: System of hyperbolic equations.** In this section, we consider the system of Cattaneo-type (5) and show existence and asymptotic stability of the viscous shock wave.

3.1. **Existence of viscous shock wave.** Let $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi)$ be a viscous shock wave, where $\xi = x - st$. Thus $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi)$ is a smooth traveling wave solution to (5) and satisfies

$$-s \tilde{u}^\varepsilon + f(\tilde{u}^\varepsilon) \xi + \tilde{q}^\varepsilon = 0, \quad (38a)$$

$$-\varepsilon \tilde{q}^\varepsilon + \mu \tilde{u}^\varepsilon + \tilde{q}^\varepsilon = 0, \quad (38b)$$

$$(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(\xi) \rightarrow (u_\pm, 0) \quad (\xi \rightarrow \pm \infty). \quad (38c)$$

Integrating (38a) over $\mathbb{R}$, we have

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0.$$ 

Therefore the shock speed $s$ is determined by the same condition (8) as the case of Fourier-type. Integrating (38a) over $(\pm \infty, \xi)$ gives

$$\tilde{q}^\varepsilon = s \tilde{u}^\varepsilon - f(\tilde{u}^\varepsilon) - (su_+ - f(u_+)) = -h(\tilde{u}^\varepsilon). \quad (39)$$

Substituting (39) in (38b), we have the differential equation of first order for $\tilde{u}^\varepsilon$ as

$$\mu \tilde{u}^\varepsilon = h_\pm(\tilde{u}^\varepsilon) := \frac{\mu h(\tilde{u}^\varepsilon)}{\mu + \varepsilon sh'(\tilde{u}^\varepsilon)}, \quad (40a)$$

$$\tilde{u}^\varepsilon(0) = u_+, \quad \tilde{u}^\varepsilon(\xi) \rightarrow u_\pm \quad (\xi \rightarrow \pm \infty), \quad (40b)$$

where $u_\pm \in (u_+, u_-)$ is a constant satisfying (10). Note that $u_\pm$ are equilibrium points of (40) since $h^\varepsilon(u_\pm) = 0$. Under an assumption

$$\varepsilon < \inf_{u \in [u_+, u_-]} \frac{\mu}{|sh'(u)|}, \quad (41)$$
we see that $\mu + \varepsilon sh'(u) > 0$ for $u \in [u_+, u_-]$. Moreover, since we have
\[
h'_{\varepsilon}(u_+|_{u_+} = \frac{\mu h'(u_+)h'(u_+) - \mu \varepsilon sh(u_+)h''(u_+)}{(\mu + \varepsilon sh'(u_+))^2} = \frac{\mu h'(u_+)}{\mu + \varepsilon sh'(u_+)},
\]
the Lax condition (11) yields $h'_{\varepsilon}(u_+) < 0$ and $h'_{\varepsilon}(u_-) > 0$. Moreover, from (14), we have the asymptotic expansion of $h'_{\varepsilon}(u_+)$ as
\[
h'_{\varepsilon}(u_+) = \frac{1}{2} f''(u_+) \delta + O(\delta^2) \quad (\delta \to 0).
\]
Therefore we get the existence of the non-degenerate viscous shock wave for (38) summarized in Theorem 3.1.

**Theorem 3.1.** For a small $\varepsilon$ satisfying (41), the problem (40) has a unique smooth solution $\tilde{u}^\varepsilon(\xi)$. Moreover, if $\varepsilon$ is sufficiently small, the solution $\tilde{u}^\varepsilon(\xi)$ satisfies
\[
|\partial^k_x (\tilde{u}^\varepsilon(\xi) - u_+)| \leq C_{\delta} e^{-\alpha \delta^k} \quad (\varepsilon < 0),
\]
\[
|\partial^k_x (\tilde{u}^\varepsilon(\xi) - u_-)| \leq C_{\delta} e^{-\alpha \delta^k} \quad (\xi > 0)
\]
for $k = 0, 1, \ldots$, where $c_0$ is a positive constant independent of $\delta$ and $\varepsilon$.

**3.2. Asymptotic stability.** We next show the asymptotic stability of the viscous shock wave $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)$ to (38) of Cattaneo-type. Define a perturbation $(\varphi^\varepsilon, \psi^\varepsilon)$ of the solution $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)$ to (5) from the viscous shock wave $(\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)$ as
\[
(\varphi^\varepsilon, \psi^\varepsilon)(t, \xi) = (u^\varepsilon, q^\varepsilon)(t, \xi + \delta t) - (u^\varepsilon, q^\varepsilon)(\xi + x_0^\varepsilon),
\]
where $x_0^\varepsilon$ is a shift to be determined later. From (5) and (38), the equations for $(\varphi^\varepsilon, \psi^\varepsilon)$ are given by
\[
\varphi^\varepsilon - s \varphi^\varepsilon + (f(\tilde{u}^\varepsilon + \varphi^\varepsilon) - f(\tilde{u}^\varepsilon))\xi + \psi^\varepsilon = 0,
\]
\[
\varepsilon \psi^\varepsilon - \varepsilon s \psi^\varepsilon + \psi^\varepsilon + \mu \varphi^\varepsilon = 0.
\]
We determine the shift $x_0^\varepsilon$ in the similar way to (17). Namely, under an assumption that $u_0 - \tilde{u}^\varepsilon \in L^1(\mathbb{R})$, we integrate (43a) over $(0, t) \times \mathbb{R}$ to get
\[
\int_{\mathbb{R}} \varphi^\varepsilon(t, \xi) d\xi = \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon)) d\xi.
\]
Then we determine $x_0^\varepsilon$ to satisfy
\[
I^\varepsilon(x_0^\varepsilon) := \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon)) d\xi = 0.
\]
Since we see $I^\varepsilon(x_0^\varepsilon) = I^\varepsilon(0) - (u_+ - u_-)x_0^\varepsilon$, letting $x_0^\varepsilon$ as
\[
x_0^\varepsilon = \frac{1}{u_+ - u_-} I^\varepsilon(0) = \frac{1}{u_+ - u_-} \int_{\mathbb{R}} (u_0(\xi) - \tilde{u}^\varepsilon(\xi)) d\xi
\]
yields (44) and $\int_{\mathbb{R}} \varphi^\varepsilon(t, \xi) d\xi = 0$. Then we employ an anti-derivative of $\varphi^\varepsilon$ by
\[
\Phi^\varepsilon(t, \xi) = \int_{-\infty}^{\xi} (u^\varepsilon(t, \xi + st) - u^\varepsilon(\xi + x_0^\varepsilon)) d\xi,
\]
and deduce the problem (43) to that for $(\Phi^\varepsilon, \psi^\varepsilon)$ as
\[
\Phi^\varepsilon - s \Phi^\varepsilon + f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon) + \psi^\varepsilon = 0,
\]
\[
\varepsilon \psi^\varepsilon - \varepsilon s \psi^\varepsilon + \psi^\varepsilon + \mu \Phi^\varepsilon = 0,
\]
\[
(\Phi^\varepsilon, \psi^\varepsilon)(0, \xi) = (\Phi^\varepsilon_0, \psi^\varepsilon_0)(\xi),
\]
where $\Phi^\varepsilon_0, \psi^\varepsilon_0$ are determined by (38).
where \((\Phi_0^\varepsilon, \psi_0^\varepsilon)\) is an initial perturbation defined by
\[
\Phi_0^\varepsilon(\xi) := \int_{-\infty}^{\xi} (u_0(\xi) - \tilde{u}^\varepsilon(\xi + x_0^\varepsilon)) \, d\xi, \quad \psi_0^\varepsilon(\xi) := q_0(\xi) - \tilde{q}^\varepsilon(\xi + x_0^\varepsilon).
\]

The main result in the present section is the asymptotic stability of the viscous shock wave \((\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)\) summarized in the next theorem. To this end, we define a function space
\[
Y(0, T) := \bigcap_{k=0}^{2} C^k([0, T]; H^{3-k} \times H^{2-k})
\]
and a norm of the initial perturbation
\[
E_0^\varepsilon := \sqrt{\|\Phi_0^\varepsilon\|_{H^3}^2 + \|\psi_0^\varepsilon\|_{H^2}^2}.
\]

**Theorem 3.2.** Let \(u_0 - \tilde{u}^\varepsilon \in L^1\) and \((\Phi_0^\varepsilon, \psi_0^\varepsilon) \in H^3 \times H^2\). Then there exists a positive constant \(\eta\) such that if \(E_0^\varepsilon + \delta \leq \eta\), the problem (46) has a unique solution \((\Phi^\varepsilon, \psi^\varepsilon) \in Y(0, \infty)\). Moreover, the solution \((u^\varepsilon, q^\varepsilon)\) to (3) converges to the viscous shock wave \((\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)\) as \(t \to \infty\):
\[
\sup_{x \in \mathbb{R}} |(u^\varepsilon, q^\varepsilon)(t, x) - (\tilde{u}^\varepsilon, \tilde{q}^\varepsilon)(x - st + x_0^\varepsilon)| \to 0 \quad (t \to \infty).
\]

To show **Theorem 3.2**, we combine the uniform a priori estimate of \((\Phi^\varepsilon, \psi^\varepsilon)\) with the existence of the solution locally in time summarized in the next lemma.

**Lemma 3.3.** For \((\Phi_0^\varepsilon, \psi_0^\varepsilon) \in H^3 \times H^2\), there exists a positive constant \(T_0\) depending on \(E_0^\varepsilon\) such that the problem (46) has a unique solution \((\Phi^\varepsilon, \psi^\varepsilon) \in Y(0, T_0)\).

**Proof.** Applying the differential operator \(\varepsilon(\partial_t - s\partial_\xi)\) to (46a) and substituting (46b) in the resultant equation, we have
\[
\varepsilon(\partial_t - s\partial_\xi)^2 \Phi^\varepsilon + \varepsilon(\partial_t - s\partial_\xi)(f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon)) - \psi^\varepsilon - \mu \Phi^\varepsilon_{\xi\xi} = 0.
\]

Dropping out \(\psi^\varepsilon\) from the above equation by using (46a), we obtain the single equation of second order for \(\Phi^\varepsilon\) as
\[
\varepsilon \Phi^\varepsilon_{tt} + (-\varepsilon s + \varepsilon f'(\tilde{u}^\varepsilon + \Phi^\varepsilon]\Phi^\varepsilon_{\xi\xi} = \varepsilon s^2 - \varepsilon f'(\tilde{u}^\varepsilon + \Phi^\varepsilon) - \mu) \Phi^\varepsilon_{\xi\xi} = 0.
\]

where the right-hand side consists of lower order terms. Since the discriminant of the equation (47) is \(\varepsilon^2 f'(\cdot)^2 + 4\varepsilon \mu > 0\), the equation (47) is a quasi-linear and strictly hyperbolic equation. Thus the standard iteration argument gives the existence of a solution to (47) in \(H^3\). Namely, under the initial conditions
\[
\Phi^\varepsilon\big|_{t=0} = \Phi_0^\varepsilon \in H^3, \quad \Phi^\varepsilon\big|_{t=0} = s\Phi_0^\varepsilon - f(\tilde{u}^\varepsilon + \Phi_0^\varepsilon) + f(\tilde{u}^\varepsilon) - \psi_0^\varepsilon \in H^2,
\]
there exists a positive constant \(T_0\) depending on \(E_0^\varepsilon\) such that the equation (47) has a unique solution \(\Phi^\varepsilon \in C^k([0, T_0]; H^{3-k})\) \((k = 0, 1, 2)\). Also, \(\psi^\varepsilon\) is given by using \(\Phi^\varepsilon\) from (46a) and satisfies \(\psi^\varepsilon \in C^k([0, T_0]; H^{2-k})\) \((k = 0, 1, 2)\). Thus we obtain a solution \((\Phi^\varepsilon, \psi^\varepsilon) \in Y(0, T_0)\) to (46).

To show **Theorem 3.2**, it suffices to obtain the uniform a priori estimate for \((\Phi^\varepsilon, \psi^\varepsilon)\). To this end, we employ an energy norm defined by
\[
E^\varepsilon(t) := \sup_{\tau \in [0, t]} ||\Phi^\varepsilon(\tau)||_{H^3}.
\]
Proposition 2. Let \((\Phi^\varepsilon, \psi^\varepsilon) \in Y(0,T)\) be a solution to (46) for a certain \(T > 0\). Then there exists a positive constant \(\eta\) such that if \(E^\varepsilon(T) + \delta \leq \eta\), the solution satisfies
\[
\|\Phi^\varepsilon(t)\|_2^2 + \varepsilon\|\psi^\varepsilon(t)\|_2^2 + \int_0^t \|(\Phi^\varepsilon_x, \psi^\varepsilon)(\tau)\|_2^2 d\tau \leq C\|\Phi^\varepsilon_0\|^2_2 + C\|\psi^\varepsilon_0\|^2_2
\]
for \(t \in [0,T]\).

The a priori estimate (48) and the local existence as well as the continuity argument give the global existence. Moreover, the estimate (48) gives a convergence \(\|(\Phi^\varepsilon_x, \psi^\varepsilon)(t)\|_{L^\infty} \to 0\) \((t \to \infty)\) which yields the asymptotic stability in Theorem 3.2. To show Proposition 2, we firstly obtain the \(L^2\) estimate of \(\Phi^\varepsilon\).

Lemma 3.4. Suppose that the same assumption as in Proposition 2 holds. Then we have
\[
\|\Phi^\varepsilon(t)\|_2^2 + \int_0^t \|(\Phi^\varepsilon_x, \Phi^\varepsilon)\|_2^2 d\tau \leq C\|\Phi^\varepsilon_0\|^2_2 + C\|\psi^\varepsilon(t)\|^2_2 + C\int_0^t \|\psi^\varepsilon(\tau)\|^2_2 d\tau.
\]

Proof. Subtracting (46b) from (46a), we have
\[
\Phi^\varepsilon_t - s\Phi^\varepsilon_x + f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon) - \mu\Phi^\varepsilon_{xx} = \varepsilon\psi^\varepsilon_t - \varepsilon s\psi^\varepsilon_x.
\]

Multiplying (50) by \(\Phi^\varepsilon\) and using the equalities
\[
(f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon))\Phi^\varepsilon = f'(\tilde{u}^\varepsilon)\Phi^\varepsilon\Phi^\varepsilon_x + (f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon) - f'(\tilde{u}^\varepsilon)\Phi^\varepsilon)\Phi^\varepsilon
\]
\[
= \left(\frac{1}{2} f'(	ilde{u}^\varepsilon)(\Phi^\varepsilon)^2\right) - \frac{1}{2} f'(	ilde{u}^\varepsilon)\Phi^\varepsilon_x(\Phi^\varepsilon)^2 + (f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon) - f'(\tilde{u}^\varepsilon)\Phi^\varepsilon)\Phi^\varepsilon_x,
\]
and \(h'(\tilde{u}^\varepsilon) = -s + f'(\tilde{u}^\varepsilon)\), we have
\[
\left(\frac{1}{2} (\Phi^\varepsilon)^2\right)_t + \left(\frac{1}{2} h'(\tilde{u}^\varepsilon)(\Phi^\varepsilon)^2 - \mu\Phi^\varepsilon\Phi^\varepsilon_x\right)_x - \frac{1}{2} f''(\tilde{u}^\varepsilon)\Phi^\varepsilon_x(\Phi^\varepsilon)^2 + \mu(\Phi^\varepsilon)^2
\]
\[
= (\varepsilon\Phi^\varepsilon\psi^\varepsilon)_t - (\varepsilon s\Phi^\varepsilon\psi^\varepsilon)_x + \varepsilon(f(\tilde{u}^\varepsilon + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon))\psi^\varepsilon + \varepsilon(\psi^\varepsilon)^2
\]
\[
=: R^\varepsilon_0 + R^\varepsilon_1.
\]

Integrating (51) over \((0,t) \times \mathbb{R}\), we obtain
\[
\|\Phi^\varepsilon\|_2^2 + \int_0^t \|(\Phi^\varepsilon_x, \Phi^\varepsilon)\|_2^2 d\tau \leq C\|\Phi^\varepsilon_0\|^2_2 + C\varepsilon \int_{\mathbb{R}} |\Phi^\varepsilon\psi^\varepsilon| d\xi + \int_0^t \int_{\mathbb{R}} |R^\varepsilon_0 + R^\varepsilon_1| d\xi d\tau.
\]

The second term in the right-hand side of (52) is estimated as
\[
C\varepsilon \int_{\mathbb{R}} |\Phi^\varepsilon\psi^\varepsilon| d\xi \leq \frac{1}{2} \|\Phi^\varepsilon\|^2_2 + C\varepsilon\|\psi^\varepsilon\|^2_2.
\]
The third term is estimated as
\[ \int_{\mathbb{R}} |R_0^\varepsilon| \, d\xi \leq C \int_{\mathbb{R}} |\Phi^\varepsilon(\Phi^\varepsilon_\xi)| \, d\xi \leq C \varepsilon^2(T) \|\Phi^\varepsilon\|_{L^2}, \] (53)
and
\[ \int_{\mathbb{R}} |R_1^\varepsilon| \, d\xi \leq \frac{1}{2} \|\Phi^\varepsilon\|_{L^2}^2 + C \|\psi^\varepsilon\|_{L^2}^2, \] (54)
where we have used \( \|\Phi^\varepsilon\|_{L^\infty} \leq C \varepsilon^2(T) \). Substituting these estimates in (52) and letting \( E^\varepsilon(T) \) suitably small, we obtain the desired inequality (49).

We next obtain the estimate of \( (\Phi^\varepsilon_\xi, \psi^\varepsilon) \). Notice that this estimate corresponds to the basic \( L^2 \) estimate for the hyperbolic system (43) since \( \Phi^\varepsilon_\xi = \varphi^\varepsilon \).

**Lemma 3.5.** Suppose that the same assumption as in Proposition 2 holds. Then we have
\[ \|\Phi^\varepsilon_\xi\|_{L^2}^2 + \varepsilon \|\psi^\varepsilon(t)\|_{L^2}^2 + \int_0^t \|\psi^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq C \|\Phi^\varepsilon_{0\xi} + \psi^\varepsilon_0\|_{L^2}^2 + C \varepsilon^2(T) + \delta \int_0^t \|\Phi^\varepsilon_\xi(\tau)\|_{L^2}^2 \, d\tau. \] (55)

**Proof.** Differentiating (46a) in \( \xi \) and multiplying the resultant equality by \( \mu \Phi^\varepsilon_\xi \), we get
\[ \left( \frac{\mu}{2} (\Phi^\varepsilon_\xi)^2 \right)_t - \left( \frac{\mu s}{2} (\Phi^\varepsilon_\xi)^2 \right)_\xi + \mu \Phi^\varepsilon_\xi \psi^\varepsilon = -\mu(f(\tilde{u}^\varepsilon + \Phi^\varepsilon_\xi) - f(\tilde{u}^\varepsilon)) \xi \Phi^\varepsilon_\xi. \] (56)
Multiply (46b) by \( \psi^\varepsilon \) to get
\[ \left( \frac{\varepsilon}{2} (\psi^\varepsilon)^2 \right)_t - \left( \frac{\varepsilon s}{2} (\Phi^\varepsilon_\xi)^2 \right)_\xi + (\psi^\varepsilon)^2 + \mu \Phi^\varepsilon_\xi \psi^\varepsilon = 0. \] (57)
Adding (56) to (57) and using the equality
\[ (f(\tilde{u}^\varepsilon + \Phi^\varepsilon_\xi) - f(\tilde{u}^\varepsilon)) \xi \Phi^\varepsilon_\xi = f'(\tilde{u}^\varepsilon + \Phi^\varepsilon_\xi) \Phi^\varepsilon_\xi \Phi^\varepsilon_\xi - f'(\tilde{u}^\varepsilon) \tilde{u}^\varepsilon \Phi^\varepsilon_\xi \]
\[ = \left( \frac{1}{2} f'(\tilde{u}^\varepsilon + \Phi^\varepsilon_\xi)(\Phi^\varepsilon_\xi)^2 \right)_\xi - R^\varepsilon_2, \] (58)
we obtain
\[ \left( \frac{\mu}{2} (\Phi^\varepsilon_\xi)^2 + \frac{\varepsilon}{2} (\psi^\varepsilon)^2 \right)_t - \left( \frac{\mu s}{2} (\Phi^\varepsilon_\xi)^2 + \frac{\varepsilon s}{2} (\psi^\varepsilon)^2 - \mu \Phi^\varepsilon_\xi \psi^\varepsilon - \frac{\mu}{2} f'(\tilde{u}^\varepsilon + \Phi^\varepsilon_\xi)(\Phi^\varepsilon_\xi)^2 \right)_\xi \]
\[ + (\psi^\varepsilon)^2 = \mu R^\varepsilon_2. \] (59)
Integrating (59) over \((0, t) \times \mathbb{R}\) and using the estimate for \( R^\varepsilon_2 \) as
\[ \int_{\mathbb{R}} |R^\varepsilon_2| \, d\xi \leq C \int_{\mathbb{R}} (|\tilde{u}^\varepsilon_\xi| + |\Phi^\varepsilon_\xi|)(\Phi^\varepsilon_\xi)^2 \, d\xi \leq C(\varepsilon^2(T) + \delta) \|\Phi^\varepsilon\|_{L^2}^2, \] (60)
where we have used \( \|\Phi^\varepsilon_\xi\|_{L^\infty} \leq C \varepsilon^2(T) \), we obtain the desired inequality (55).

Combining (49) and (55) and letting \( E^\varepsilon(T) + \delta \) suitably small, we obtain
\[ \|\Phi^\varepsilon(t)\|_{H^1}^2 + \varepsilon \|\psi^\varepsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{-\tilde{u}^\varepsilon \Phi^\varepsilon_\xi, \Phi^\varepsilon_\xi, \psi^\varepsilon}(\tau)\|_{L^2}^2 \, d\tau \]
\[ \leq C \|\Phi^\varepsilon_0\|_{H^1}^2 + C \|\psi^\varepsilon_0\|_{L^2}^2. \] (61)
We next obtain the estimate of higher order derivatives.
Lemma 3.6. Let \( k = 1, 2 \). Suppose that the same assumption as in Proposition 2 holds. Then we have

\[
\left\| \partial_x^{k+1} \Phi^\epsilon(t) \right\|_{L^2}^2 + \epsilon \left\| \partial_x^{k} \psi^\epsilon(t) \right\|_{L^2}^2 + \int_0^t \left\| (\partial_x^{k+1} \Phi^\epsilon, \partial_x^{k} \psi^\epsilon)(\tau) \right\|_{L^2}^2 d\tau \\
\leq C \left\| \Phi_0^\epsilon \right\|_{H^{k+1}}^2 + C \left\| \psi_0^\epsilon \right\|_{H^k}^2 + C(E^\epsilon(T) + \delta) \int_0^t \left\| \Phi^\epsilon(\tau) \right\|_{H^k}^2 d\tau. \tag{62}
\]

Proof. Applying \( \partial_x^{k+1} \) to (46a) and multiplying the resultant equality by \( \mu \partial_x^{k+1} \Phi^\epsilon \), we have

\[
\left( \frac{\mu}{2} (\partial_x^{k+1} \Phi^\epsilon)^2 \right)_t - \left( \frac{\mu s}{2} (\partial_x^{k+1} \Phi^\epsilon)^2 \right)_x + \mu \partial_x^{k+1} \Phi^\epsilon \partial_x^{k+1} \psi^\epsilon \\
= -\mu \partial_x^{k+1}(f(\tilde{u}^\epsilon + \Phi^\epsilon) - f(\tilde{u}^\epsilon)) \partial_x^{k+1} \Phi^\epsilon. \tag{63}
\]

Apply \( \partial_x^k \) to (46b) and multiply the resultant equality by \( \partial_x^k \psi^\epsilon \) to get

\[
\left( \frac{\epsilon}{2} (\partial_x^k \psi^\epsilon)^2 \right)_t - \left( \frac{\epsilon s}{2} (\partial_x^k \psi^\epsilon)^2 \right)_x + (\partial_x^k \psi^\epsilon)^2 + \left( \mu \partial_x^{k+1} \Phi^\epsilon \partial_x^k \psi^\epsilon \right)_x \\
- \mu \partial_x^{k+1} \Phi^\epsilon \partial_x^{k+1} \psi^\epsilon = 0. \tag{64}
\]

Adding (63) to (64), we have

\[
\left( \frac{\mu}{2} (\partial_x^{k+1} \Phi^\epsilon)^2 + \frac{\epsilon}{2} (\partial_x^k \psi^\epsilon)^2 \right)_t - \left( \frac{\mu s}{2} (\partial_x^{k+1} \Phi^\epsilon)^2 + \frac{\epsilon s}{2} (\partial_x^k \psi^\epsilon)^2 - \mu \partial_x^{k+1} \Phi^\epsilon \partial_x^k \psi^\epsilon \right)_x \\
+ (\partial_x^k \psi^\epsilon)^2 = -\mu \partial_x^{k+1}(f(\tilde{u}^\epsilon + \Phi^\epsilon) - f(\tilde{u}^\epsilon)) \partial_x^{k+1} \Phi^\epsilon. \tag{65}
\]

By using the estimate

\[
\left| \int_R \partial_x^{k+1}(f(\tilde{u}^\epsilon + \Phi^\epsilon) - f(\tilde{u}^\epsilon)) \partial_x^{k+1} \Phi^\epsilon d\xi \right| \leq C(E^\epsilon(T) + \delta) \left\| \Phi^\epsilon \right\|_{H^k}^2, \tag{66}
\]

which is obtained in the similar way to (58) and (60), integrating (65) over \((0, t) \times \mathbb{R}\) gives

\[
\left\| \partial_x^{k+1} \Phi^\epsilon \right\|_{L^2}^2 + \epsilon \left\| \partial_x^k \psi^\epsilon \right\|_{L^2}^2 + \int_0^t \left\| \partial_x^k \psi^\epsilon \right\|_{L^2}^2 d\tau \\
\leq C \left\| \partial_x^{k+1} \Phi_0^\epsilon \right\|_{L^2}^2 + C \left\| \partial_x^k \psi_0^\epsilon \right\|_{L^2}^2 + C(E^\epsilon(T) + \delta) \int_0^t \left\| \Phi^\epsilon \right\|_{H^k}^2 d\tau. \tag{67}
\]

We next obtain the dissipative estimate of \( \partial_x^k \Phi^\epsilon \). Applying \( \partial_x^k \) to (50) and multiplying the resultant equality by \( \partial_x^k \Phi^\epsilon \), we get

\[
\left( \frac{1}{2} (\partial_x^k \Phi^\epsilon)^2 \right)_t - \left( \frac{s}{2} (\partial_x^k \Phi^\epsilon)^2 + \mu \partial_x^k \Phi^\epsilon \partial_x^k \psi^\epsilon \right)_x + \mu (\partial_x^{k+1} \Phi^\epsilon)^2 \\
= (\epsilon \partial_x^k \Phi^\epsilon \partial_x^k \psi^\epsilon)_t - (\epsilon s \partial_x^k \Phi^\epsilon \partial_x^k \psi^\epsilon)_x + R^3_3 + R^3_4, \tag{68}
\]

where

\[
R^3_3 := -\partial_x^k(f(\tilde{u}^\epsilon + \Phi^\epsilon) - f(\tilde{u}^\epsilon)) \partial_x^k \Phi^\epsilon, \\
R^3_4 := \epsilon \partial_x^k(f(\tilde{u}^\epsilon + \Phi^\epsilon) - f(\tilde{u}^\epsilon)) \partial_x^k \psi^\epsilon + \epsilon (\partial_x^k \psi^\epsilon)^2.
\]

In the same way as (54) and (66), the integrals of \( R^3_3 \) and \( R^3_4 \) are estimated as

\[
\int_\mathbb{R} | R^3_3 | d\xi \leq C(E^\epsilon(T) + \delta) \left\| \Phi^\epsilon \right\|_{H^{k-1}}^2, \\
\int_\mathbb{R} | R^3_4 | d\xi \leq \frac{1}{2} \left\| \partial_x^{k+1} \Phi^\epsilon \right\|_{L^2}^2 + C \left\| \Phi^\epsilon \right\|_{H^{k-1}}^2 + C \left\| \partial_x^k \psi^\epsilon \right\|_{L^2}^2.
\]
Lemma 3.7. Suppose that the same assumption as in Theorem 3.2 holds. Denoting the solution \((\Phi^\varepsilon, \psi^\varepsilon) \in \mathcal{Y}(0, \infty)\) satisfies

\[
\varepsilon \epsilon \left( \frac{\partial t}{\partial T} \right) \| \Phi^\varepsilon \|_{L^2}^2 + \varepsilon \epsilon \| \psi^\varepsilon \|_{L^2}^2 + \varepsilon \int_0^t \| (\Phi^\varepsilon, \psi^\varepsilon) (\tau) \|_{H^1}^2 d\tau \leq C \| \Phi^\varepsilon_0 \|_{H^1}^2 + C \| \psi^\varepsilon_0 \|_{H^2}^2 + \varepsilon \epsilon \left( \frac{\partial t}{\partial T} \right) \| \Phi^\varepsilon \|_{L^2}^2 + \varepsilon \epsilon \| \psi^\varepsilon \|_{L^2}^2 \quad (70)
\]

Proof. Applying \( \partial_t \partial_t \) to (46a) and multiplying the resultant equation by \( \mu \Phi^\varepsilon \), we get

\[
\left( \frac{\mu}{2} (\Phi^\varepsilon)^2 \right)_t - \left( \frac{\mu s}{2} (\Phi^\varepsilon)^2 \right)_{\xi} + \mu \Phi^\varepsilon \psi^\varepsilon = -\mu (f(\tilde{u}^\varepsilon) + \Phi^\varepsilon) - f(\tilde{u}^\varepsilon)_{\xi} \Phi^\varepsilon. \quad (71)
\]

Differentiate (46b) in \( t \) and multiply the resultant equation by \( \psi^\varepsilon \) to get

\[
\left( \frac{\varepsilon}{2} (\psi^\varepsilon)^2 \right)_t - \left( \frac{\varepsilon s}{2} (\psi^\varepsilon)^2 \right)_{\xi} + (\psi^\varepsilon)^2 + (\mu \Phi^\varepsilon \psi^\varepsilon)_{\xi} = \mu \Phi^\varepsilon \psi^\varepsilon = 0. \quad (72)
\]

Adding (71) to (72), integrating the resultant equality and estimating the nonlinear terms in the similar way to (66), we obtain

\[
\| \Phi^\varepsilon \|_{L^2}^2 + \varepsilon \| \psi^\varepsilon \|_{L^2}^2 + \int_0^t \| \psi^\varepsilon \|_{L^2}^2 d\tau \leq C \varepsilon \epsilon \| \Phi^\varepsilon_0 \|_{H^2}^2 + C \epsilon \| \psi^\varepsilon_0 \|_{H^1}^2 \quad \text{and} \| \Phi^\varepsilon \|_{L^\infty} \leq C \| \Phi^\varepsilon \|_{H^1} \leq C E_0. \quad (73)
\]

We next show the estimate of derivatives in \( t \).

Lemma 3.7. Suppose that the same assumption as in Theorem 3.2 holds. Denoting the solution \((\Phi^\varepsilon, \psi^\varepsilon) \in \mathcal{Y}(0, \infty)\) satisfies

\[
\| \partial_t^k \Phi^\varepsilon \|_{L^2}^2 + \int_0^t \| \partial_t^{k+1} \Phi^\varepsilon \|_{L^2}^2 d\tau \leq C \| \Phi^\varepsilon_0 \|_{H^1}^2 + C \| \psi^\varepsilon_0 \|_{H^2}^2 + C \int_0^t \| \Phi^\varepsilon \|_{H^k}^2 d\tau + C(E^\varepsilon(T) + \delta) \int_0^t \| \Phi^\varepsilon \|_{H^k}^2 d\tau \quad (69)
\]

by using an induction with the aid of (61). Combining (67) and (69) gives the desired inequality (62).

Proof of Proposition 2. Summing up (62) for \( k = 1, 2 \) as well as (61) and letting \( E^\varepsilon(T) + \delta \) suitably small, we obtain the desired a priori estimate (48).
differentiating (50) in \( t \), multiplying by \( \Phi_t^\varepsilon \) and integrating the resultant equality with using (73) yield

\[
\|\Phi_t^\varepsilon\|_{L^2}^2 + \int_0^t \|\Phi_{\xi t}^\varepsilon\|_{L^2}^2 \, d\tau \\
\leq C\|\Phi_0^\varepsilon\|_{H^2}^2 + C\|\psi_0^\varepsilon\|_{H^1}^2 + C(E_0^\varepsilon + \delta) \int_0^t \|\Phi_t^\varepsilon, \Phi_{\xi t}^\varepsilon\|_{L^2}^2 \, d\tau.
\]

Combining (73) and (74) and letting \( E_0^\varepsilon + \delta \) suitably small with using \( \|\Phi_t^\varepsilon\|_{L^2}^2 \leq C((\Phi_t^\varepsilon, \psi^\varepsilon)\|_{L^2}^2 \), we obtain

\[
\varepsilon\|\Phi_t^\varepsilon\|_{H^2}^2 + \varepsilon^2\|\psi_t^\varepsilon\|_{L^2}^2 + \varepsilon \int_0^t \|\Phi_{\xi t}^\varepsilon, \psi_{\xi t}^\varepsilon\|_{L^2}^2 \, d\tau \leq C\|\Phi_0^\varepsilon\|_{H^2}^2 + C\|\psi_0^\varepsilon\|_{H^1}^2.
\]

Similarly, we obtain the estimate of \((\Phi_{\xi t}^\varepsilon, \psi_{\xi t}^\varepsilon)\). Thus we complete the proof.

3.3. Convergence rate. We next show the convergence rate associated with the asymptotic stability in Theorem 3.2. To do this, we employ the weight function \( \omega(\tilde{u}^\varepsilon) \) defined in (29). Namely, \( \omega(\tilde{u}^\varepsilon) \) is given by

\[
\omega(\tilde{u}^\varepsilon) = \frac{(-g(\tilde{u}^\varepsilon))^{1-\beta_2}}{-h(\tilde{u}^\varepsilon)}.
\]

In the same way as Lemma 2.7, we see that \( \omega(\tilde{u}^\varepsilon) \) satisfies

\[
0 < \omega(\tilde{u}^\varepsilon) \leq Ce^{C|\beta|\xi} \quad (\xi \in \mathbb{R}),
\]

\[
-(\omega(\tilde{u}^\varepsilon)h(\tilde{u}^\varepsilon))''\tilde{u}^\varepsilon \geq c(\beta^4 - \tilde{u}^\varepsilon)\omega(\tilde{u}^\varepsilon),
\]

\[
|\omega(\tilde{u}^\varepsilon)| \leq C(\beta^4 - \tilde{u}^\varepsilon)\omega(\tilde{u}^\varepsilon),
\]

where the positive constants \( c \) and \( C \) in the above estimates are independent of \( \delta \) and \( \varepsilon \).

**Theorem 3.8.** Let \( u_0 - \tilde{u}^\varepsilon \in L^1 \) and \((\Phi_0^\varepsilon, \psi_0^\varepsilon) \in (H^3 \times H^2) \cap (H^1_0 \times L_0^2) \) for a certain \( \alpha > 0 \). Then there exist positive constants \( \eta \) and \( \gamma \) such that if \( E_0^\varepsilon + \delta \leq \eta \), the solution \((\Phi^\varepsilon, \psi^\varepsilon)\) to (46) verifies

\[
\|\Phi^\varepsilon(t)\|_{H^1}^2 + \varepsilon\|\psi^\varepsilon(t)\|_{L^2}^2 \leq Ce^{-\gamma\delta t} \quad (t \geq 0).
\]

**Remark 3.** In the same way as the case of Fourier-type, the convergence (79) and the interpolation inequality give

\[
\|\Phi^\varepsilon(t)\|_{H^2}^2 + \varepsilon\|\psi^\varepsilon(t)\|_{H^1}^2 \leq Ce^{-\gamma\delta t} \quad (t \geq 0).
\]

To show Theorem 3.8, we first obtain the weighted \( L^2 \) estimate of \( \Phi^\varepsilon \).

**Lemma 3.9.** Suppose that the same assumption as in Theorem 3.8 holds and let \( \gamma \) be a positive constant. Then we have

\[
e^{\gamma \delta t}\|\Phi^\varepsilon(t)\|_{\omega(\tilde{u}^\varepsilon)}^2 + \int_0^t e^{\gamma \delta \tau}(\beta^4|\Phi^\varepsilon(\tau)|_{\omega(\tilde{u}^\varepsilon)}^2 + |\Phi_{\xi t}^\varepsilon(\tau)|_{\omega(\tilde{u}^\varepsilon)}^2) \, d\tau \\
\leq C|\Phi_0^\varepsilon, \psi_0^\varepsilon|_{\omega(\tilde{u}^\varepsilon)}^2 + C\varepsilon e^{\gamma \delta t}|\psi^\varepsilon(\tau)|_{\omega(\tilde{u}^\varepsilon)}^2 + C\int_0^t e^{\gamma \delta \tau}|\psi^\varepsilon(\tau)|_{\omega(\tilde{u}^\varepsilon)}^2 \, d\tau \\
+ C\gamma \delta t \int_0^t e^{\gamma \delta \tau}|\Phi^\varepsilon(\tau)|_{\omega(\tilde{u}^\varepsilon)}^2 \, d\tau + C(E_0^\varepsilon + \delta + \beta) \int_0^t e^{\gamma \delta \tau}|\Phi_{\xi t}^\varepsilon(\tau)|_{\omega(\tilde{u}^\varepsilon)}^2 \, d\tau.
\]
Proof. Multiplying (51) by the weight function $\omega = \omega(\tilde{u}^c)$ with using

$$\omega \left( \frac{1}{2} h'(\Phi^c)^2 - \mu \Phi^c \Phi_{\xi}^c \right)_{\xi} = \frac{1}{2} \omega h'' \tilde{u}_{\xi}^c (\Phi^c)^2$$

we have

$$\frac{1}{2} (\omega h)' (\Phi^c)^2 - \mu \omega \Phi^c \Phi_{\xi}^c \right)_{\xi} = \frac{1}{2} (\omega h)' (\Phi^c)^2 + (\mu \omega - \omega') \Phi^c \Phi_{\xi}^c$$

which is obtained in the same way as (33), we have

$$\left( \frac{1}{2} \omega (\Phi^c)^2 \right)_t + \frac{1}{2} (\omega h)' (\Phi^c)^2 - \mu \omega \Phi^c \Phi_{\xi}^c \right)_{\xi} = \frac{1}{2} (\omega h)' (\Phi^c)^2 + \omega (\Phi_{\xi}^c)^2$$

In the similar way to derivation of (53) and (54), we have

$$\left( \frac{1}{2} \omega (\Phi^c)^2 \right)_t + \left( \frac{1}{2} (\omega h)' (\Phi^c)^2 - \mu \omega \Phi^c \Phi_{\xi}^c \right)_{\xi} - \frac{1}{2} (\omega h)' (\Phi^c)^2 + \omega (\Phi_{\xi}^c)^2$$

Thus, integrating (82) in $\xi$ with using (77) and substituting (83)–(85), we get

$$\frac{d}{dt} |\Phi_{\xi}^c|^2_{\omega(u^r)} + c \int \beta \delta^2 \omega |\Phi^c|^2 d\xi + C |\psi|^2_{\omega(u^r)} \leq \frac{1}{2} \int \beta \delta^4 \omega (\Phi^c)^2 d\xi + C (\delta + \beta) |(\Phi_{\xi}^c, \psi)|^2_{\omega(u^r)}.$$ (86)

Multiplying (86) by $e^{\gamma \delta^2 t}$ and integrating in $t$ with using

$$\int e^{\gamma \delta^2 t} |\omega \Phi^c \psi^c| d\xi \leq \frac{1}{2} |\Phi_{\xi}^c|^2_{\omega(u^r)} + C |\psi|^2_{\omega(u^r)}$$

we obtain the desired inequality (81).}

Lemma 3.10. Suppose that the same assumption as in Theorem 3.8 holds and let $\gamma$ be a positive constant. Then we have

$$e^{\gamma \delta^2 t} \left[ (|\Phi_{\xi}^c|^2_{\omega(u^r)} + \epsilon |\psi|^2_{\omega(u^r)}) \right] + \int_0^t e^{\gamma \delta^4 t} |\psi^c|^2_{\omega(u^r)} d\tau 

\leq C (|\Phi_{\xi}^c|^2_{\omega(u^r)} + C (E_0^\infty + \delta) \int_0^t e^{\gamma \delta^4 t} |(\Phi_{\xi}^c, \psi^c)|^2_{\omega(u^r)} d\tau).$$ (87)
Proof. Multiplying (59) by the weight function $\omega = \omega(\tilde{u}^\varepsilon)$, we have
\[
\left(\frac{\mu}{2} \omega(\tilde{u}^\varepsilon)^2 + \frac{\varepsilon}{2}(\omega \psi^\varepsilon)^2\right)_t - (\omega B)_\xi + \omega(\psi^\varepsilon)^2 = \mu \omega R_2^\varepsilon - \omega \xi B, \tag{88}
\]
from which we get
\[
B := \frac{\mu s}{2}(\tilde{u}^\varepsilon)^2 + \frac{\varepsilon s}{2}(\psi^\varepsilon)^2 - \frac{s}{2} f'(\tilde{u}^\varepsilon)\xi (\tilde{u}^\varepsilon)^2.
\]
From (78), we have the estimate of $\omega \xi B$ as
\[
\int_\mathbb{R} |\omega \xi B| \, d\xi \leq C \int_\mathbb{R} (|\beta \delta^2 - \tilde{u}^\varepsilon|^2 + (\psi^\varepsilon)^2) \, d\xi \leq C \delta |(\tilde{u}^\varepsilon, \psi^\varepsilon)|^2_\omega.
\]
Using these estimates as well as
\[
\frac{\gamma}{2} \int_0^t e^{\gamma \delta^4 \tau} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon)^2 + \varepsilon|\psi^\varepsilon|^2 d\xi\right| d\tau = C \delta \int_0^t e^{\gamma \delta^4 \tau} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 d\tau, \tag{89}
\]
we obtain the desired inequality (87) by multiplying (88) by $e^{\gamma \delta^4 t}$ and integrating the resultant equality.

Proof of Theorem 3.8. Substituting (81) in (87) and adding the resultant inequality to (87), we get
\[
e^{\gamma \delta^4 t} \left(\left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 \right| + \varepsilon|\psi^\varepsilon|^2 \right) + \int_0^t e^{\gamma \delta^4 \tau} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 d\tau + C \delta \int_0^t e^{\gamma \delta^4 \tau} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 d\tau, \tag{90}
\]
where we have assumed that $E_0^\varepsilon + \delta$ is sufficiently small. Taking $\beta$ small to satisfy $|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)| \leq C \int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2$ which follows from (76), and taking $\gamma$ small to satisfy $C \gamma < \beta$, we obtain
\[
e^{\gamma \delta^4 t} \left(\left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 \right| + \varepsilon|\psi^\varepsilon|^2 \right) + \int_0^t e^{\gamma \delta^4 \tau} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 d\tau + C \delta \int_0^t e^{\gamma \delta^4 \tau} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon, \psi^\varepsilon)^2 d\tau
\]
which yields the desired convergence (79). Thus we complete the proof.

We conclude this section by showing the better estimate of $\psi^\varepsilon$ in terms of $\varepsilon$.

Lemma 3.11. Suppose that the same assumption as in Theorem 3.8 holds. Then there exists a positive constant $\gamma$ such that
\[
\|\psi^\varepsilon(t)\|_{L^2}^2 \leq Ce^{-\gamma \delta^4 t} \quad (t \geq 0). \tag{90}
\]
Proof. Multiply (46b) by $\psi^\varepsilon$ and integrate the resultant equality to get
\[
e^{\gamma \delta^4 t} \left|\int_\mathbb{R} (\tilde{u}^\varepsilon)^2 + \varepsilon|\psi^\varepsilon|^2 d\xi\right| d\tau \leq C \|\tilde{u}^\varepsilon\|_{L^2}^2, \tag{91}
\]
Multiplying (91) by $e^{t/\varepsilon}$ and integrate in $t$ with using the exponential decay of $\Phi_{\xi \xi}$ in (80), we have
\[
e^{t/\varepsilon} \|\psi^\varepsilon\|_{L^2}^2 \leq \|\psi_0\|_{L^2}^2 + C \int_0^t e^{t/\varepsilon} e^{-\gamma \delta^4 \tau} d\tau \leq \|\psi_0\|_{L^2}^2 + Ce^{t/\varepsilon} e^{-\gamma \delta^4 t},
\]
which gives the desired estimate (90).
4. Relaxation limit. In this section, we consider the relaxation limit \( \varepsilon \to 0 \). We firstly show in Section 4.1 that the viscous shock wave \( \tilde{u}^\varepsilon \) of Cattaneo-type tends to \( \tilde{u} \) of Fourier-type as \( \varepsilon \to 0 \) by obtaining the estimate of \( \tilde{u}^\varepsilon - \tilde{u} \) in terms of \( \varepsilon \). In Section 4.2, we consider the singular limit \( \varepsilon \to 0 \) for the solutions to the initial value problem. Namely, we show in Theorem 4.3 that the solution \((u^\varepsilon, q^\varepsilon)\) to (5) tends to \((u, q)\) to (3) as \( \varepsilon \to 0 \) uniformly in \( t \).

4.1. Relaxation limit for viscous shock wave. In the paper [1], Caffisch obtained the estimate of shock profiles between the Boltzmann equation and the Navier–Stokes equations. Related to this result, we obtain the estimate of the difference of viscous shock waves \( \tilde{u}^\varepsilon - \tilde{u} \) in \( L^p \) norm in terms of \( \varepsilon \), which is summarized in the following theorem.

**Theorem 4.1.** Under the same assumptions as in Theorems 2.1 and 3.1, the viscous shock waves \( \tilde{u}^\varepsilon \) and \( \tilde{u} \) satisfy

\[
||\tilde{u}^\varepsilon - \tilde{u}||_{L^p} \leq C\varepsilon \delta^{2-1/p} \quad \text{for} \quad p \in [1, \infty].
\]  
(92)

To show Theorem 4.1, we define

\[
\tilde{v}(\xi) := \tilde{u}^\varepsilon(\xi) - \tilde{u}(\xi).
\]

From (9) and (40), we have the equation for \( \tilde{v} \) as

\[
\mu \tilde{v}_\xi = h_\varepsilon(\tilde{u}^\varepsilon) - h(\tilde{u}),
\]  
(93a)

\[
\tilde{v}(0) = 0, \quad \tilde{v}(\xi) \to 0 \quad (\xi \to \pm \infty).
\]  
(93b)

**Lemma 4.2.** There exists a positive constant \( \xi_\delta \) independent of \( \varepsilon \) such that

\[
\tilde{v}(\xi)^2 \leq C(\tilde{v}(\xi_\delta)^2 + \delta^4 \varepsilon^2) e^{-c_0 \delta \xi} \quad (|\xi| \geq \xi_\delta).
\]  
(94)

**Proof.** We show (94) for a positive \( \xi > 0 \). The case for \( \xi < 0 \) is handled in the same way. Notice that \( \tilde{u}(\xi), \tilde{u}^\varepsilon(\xi) \in [u_+, u_-] \) for \( \xi > 0 \). Since \( h'(u) \) is monotonically increasing, we have from (13) that \( h'(u_+) \leq h'(u) \leq h'(u_-) = 0 \) and hence \( |h'(u)| \leq |h'(u_+)| \leq C \delta \) for \( u \in [u_+, u_-] \). Thus, from (13) and (42), we have

\[
|h_\varepsilon(\tilde{u}^\varepsilon) - h(\tilde{u})| = |\frac{\varepsilon \delta}{1 + \varepsilon s h'(\tilde{u}^\varepsilon)} h'(\tilde{u}^\varepsilon) h(\tilde{u})| \leq C \delta^3 \varepsilon e^{-c_0 \delta \xi},
\]  
(95)

where we have used

\[
|h(\tilde{u}^\varepsilon)| = |h(\tilde{u}^\varepsilon) - h(u_+)| \leq \int_0^1 |h'(u_+ + \theta(\tilde{u}^\varepsilon - u_+))(\tilde{u}^\varepsilon - u_+)| d\theta \leq C \delta^2 e^{-c_0 \delta \xi}.
\]

Also, we have

\[
h(\tilde{u}^\varepsilon) - h(\tilde{u}) \approx h'(\tilde{u}_+) \tilde{v} + (h'(\tilde{u}) - h'(\tilde{u}_+)) \tilde{v} + h(\tilde{u} + \tilde{v}) - h(\tilde{u}) - h'(\tilde{u}) \tilde{v},
\]  
(96)

\[
\left| |R| \leq C|\tilde{u} - u_+||\tilde{v}| + C|\tilde{v}|^2 \leq C \delta e^{-c_0 \delta \xi} |\tilde{v}|.
\]

Thus multiplying (93a) by \( \tilde{v} \) and using (95) and (96) as well as (13), we get

\[
\left( \frac{\mu}{2} \tilde{v}^2 \right)_\xi + c_0 \delta \tilde{v}^2 \leq C \delta e^{-c_0 \delta \xi} |\tilde{v}|^2 + C \delta^3 \varepsilon e^{-c_0 \delta \xi} |\tilde{v}|
\]

\[
\leq C_1 \delta e^{-c_0 \delta \xi} |\tilde{v}|^2 + C \delta^5 \varepsilon^2 e^{-2c_0 \delta \xi}.
\]  
(97)

We define \( \xi_\delta \) by

\[
\xi_\delta := \frac{1}{c_0 \delta} \log \frac{2 C_1}{c_0}.
\]
Then we see that \( C_1 e^{-c_0 \xi} \leq C_1 e^{-c_0 \xi} = c_0 / 2 \) for \( \xi \geq \xi_0 \). Substituting this estimate in (97), we obtain

\[
\left( \frac{\mu}{2} v^2 \right)_\xi + c_0 \delta \tilde{v}^2 \leq C \delta^5 \varepsilon e^{-2c_0 \xi} \quad (\xi \geq \xi_0).
\]

Multiplying (98) by \( e^{c_0 \xi} \) and integrating the resultant inequality in \( \xi \in (\xi_0, \infty) \), we arrive at

\[
e^{c_0 \xi} \tilde{v}(\xi)^2 \leq e^{c_0 \xi} \tilde{v}(\xi_0)^2 + C \delta^4 \varepsilon^2 \varepsilon^{-c_0 \xi},
\]

which gives the desired estimate.

**Proof of Theorem 4.1.** From (97), we have

\[
\left( \frac{\mu}{2} v^2 \right)_\xi \leq C \delta \tilde{v}^2 + C \delta^5 \varepsilon e^{-2c_0 \xi},
\]

which gives

\[
\tilde{v}(\xi)^2 \leq C \delta^4 \varepsilon^2 e^{c_0 \xi} \quad (\xi \in [0, \xi_0])
\]

by the Gronwall inequality. Substituting (99) in (94) and combining with (99), we obtain

\[
\tilde{v}(\xi)^2 \leq C \delta^4 \varepsilon^2 \quad (\xi \geq 0).
\]

Notice that we can also obtain (100) for \( \xi \leq 0 \). Therefore, we get the \( L^\infty \) estimate in (92). We next show the \( L^1 \) estimate. Dividing the integration interval into \((0, \xi_0)\) and \((\xi_0, \infty)\) we have

\[
\int_0^{\xi_0} |\tilde{v}(\xi)| d\xi \leq C \delta^2 \varepsilon \xi_0 \leq C \delta \varepsilon,
\]

\[
\int_{\xi_0}^\infty |\tilde{v}(\xi)| d\xi \leq C \delta^2 \varepsilon \int_{\xi_0}^\infty e^{-c_0 \xi} d\xi \leq C \delta \varepsilon.
\]

By obtaining the same estimate over \((-\infty, 0)\), we get the \( L^1 \) estimate in (92). The interpolation inequality gives the \( L^p \) estimate in (92). Thus we complete the proof.

4.2. **Singular limit.** In the papers [13, 15], singular limit problem with initial layer is considered between the Boltzmann equation and the compressible Euler equations obtained as the first approximation of the Chapman-Enskog expansion. For model systems of semi-conductors, the singular limit problem from hydrodynamic model to drift-diffusion model associated with stationary waves is considered in the paper [12].

In the present section we show that the solution \((u^\varepsilon, q^\varepsilon)\) to (5) tends to the solution \((u, q)\) to (3) as \( \varepsilon \to 0 \). Since the relation \( q = -\mu u_x \) does not holds for the system of Cattaneo-type, the initial data \( q_0 \) is not necessarily equal to \(-\mu u_{0x}\). Thus the difference \( q_0 + \mu u_{0x} \) appears as the initial layer and hence the relaxation limit is a singular limit. We also show that the initial layer decays as \( t \to \infty \) or \( \varepsilon \to 0 \).

Notice that we do not use the smallness of \( \delta \) directly in the computations of the singular limit. Thus we write constants which may depend on \( \delta \) as \( C \) in the present section.

**Theorem 4.3.** Suppose that the same assumptions as in Theorems 2.8 and 3.8 hold. Then the solutions \((u, q)\) to (3) and \((u^\varepsilon, q^\varepsilon)\) to (5) satisfy the following.
(i) The estimates
\[
\|(u^\varepsilon - u)(t)\|_{L^1}^2 \leq C\varepsilon e^{C_1 t},
\]
\[
\|(q^\varepsilon - q)(t)\|_{L^2}^2 \leq \|q_0 + \mu u_0\|_{L^2}^2 e^{-t/\varepsilon} + C\varepsilon e^{C_1 t}
\]
hold for \( t \geq 0 \), where \( C \) and \( C_1 \) are independent of \( \varepsilon \) and \( t \).

(ii) There exists a positive constant \( \lambda_0 \in (0, 1) \) such that
\[
\|(u^\varepsilon - u)(t)\|_{L^1}^2 \leq C\varepsilon^{\lambda_0},
\]
\[
\|(q^\varepsilon - q)(t)\|_{L^2}^2 \leq \|q_0 + \mu u_0\|_{L^2}^2 e^{-t/\varepsilon} + C\varepsilon^{\lambda_0}
\]
hold for \( t \geq 0 \), where \( \lambda_0 \) and \( C \) are independent of \( \varepsilon \) and \( t \).

Before showing the proof of Theorem 4.3, we show the estimate of \( |x^\varepsilon_0 - x_0| \) in terms of \( \varepsilon \) summarized in the next lemma.

Lemma 4.4. We have

(i) \( \int_\mathbb{R} |\tilde{u}(\xi + y) - \hat{u}(\xi)|^2 d\xi \leq Cy \) for \( y \in (0, 1) \), and

(ii) \( |x^\varepsilon_0 - x_0| \leq C\varepsilon \).

Proof. (i) Noticing that \( |\tilde{u}(\xi + y) - \hat{u}(\xi)| \leq |\tilde{u}(\xi + y) - u_\pm| + |\hat{u}(\xi) - u_\pm| \leq C e^{-\alpha |\xi|} \)
where \( \alpha \) is a positive constant, we define \( R := -\frac{1}{\alpha} \log y > 0 \), that is, \( e^{-\alpha R} = y \). For the case of \( |\xi| \leq R \), by using \( |\tilde{u}(\xi + y) - \hat{u}(\xi)| \leq \int_{\xi}^{\xi + y} |\tilde{u}(\xi)| d\xi \leq Cy \), we have
\[
\int_{|\xi| \leq R} |\tilde{u}(\xi + y) - \hat{u}(\xi)|^2 d\xi \leq Cy^2 R = \frac{C}{\alpha} y^2 \log y \leq Cy,
\]
where we have used the fact that the function \(-y \log y\) is bounded for \( y \in (0, 1) \).

On the other hand, for the case of \( |\xi| \geq R \), we have
\[
\int_{|\xi| \geq R} |\tilde{u}(\xi + y) - \hat{u}(\xi)|^2 d\xi \leq C \int_{R}^{\infty} e^{-2\alpha \xi} d\xi \leq Ce^{-2\alpha R} = Cy^2.
\]
Thus we obtain the desired estimate.

(ii) From (17) and (45) with using the \( L^1 \) estimate in (92), we have
\[
|x^\varepsilon_0 - x_0| \leq \frac{1}{\delta} \int_{\mathbb{R}} |\tilde{u}^\varepsilon(\xi) - \hat{u}(\xi)| d\xi \leq C\varepsilon,
\]
which completes the proof.

We next show the proof of Theorem 4.3. To this end, we define \( v \) and \( w \) as difference of the solutions between Cattaneo-type and Fourier-type as
\[
v(t, x) := u^\varepsilon(t, x) - u(t, x),
\]
\[
w(t, x) := q^\varepsilon(t, x) - q(t, x) = q^\varepsilon(t, x) + \mu u_x(t, x).
\]

From (3) and (5), we see that \( v \) and \( w \) satisfy the equations
\[
v_t + (f(u + v) - f(u))_x + w_x = 0,
\]
\[
\varepsilon w_t + \mu w_x + u = -\varepsilon q_v,
\]
\[
(v, w)(0, x) = (0, q_0 + \mu u_0).
\]

Proof of Theorem 4.3. (i) We firstly show (101). Multiplying (105a) by \( v \) gives
\[
\left( \frac{1}{2} v^2 \right)_t + (v(f(u + v) - f(u)) - \varepsilon v q_v - \mu v w_x)_x + \mu v^2 = v_x (f(u + v) - f(u) - \varepsilon q_v^2). \tag{106}
\]
By using the estimate of the right-hand side of (106) as
\[
\int_{\mathbb{R}} |v_x(f(u + v) - f(u)) - \varepsilon v_x q_t^2| \, dx \leq \frac{\mu}{2} \|v_x\|_{L^2}^2 + C \|v\|_{L^2}^2 + C \varepsilon^2 \|q_t\|_{L^2}^2,
\]
we integrate (106) in \(x\) to get
\[
\frac{d}{dt} \|v\|_{L^2}^2 + c \|v_x\|_{L^2}^2 \leq C \|v\|_{L^2}^2 + C \varepsilon^2 \|q_t\|_{L^2}^2.
\]
(107)

Next, we multiply (105a) by \(-v_{xx}\) to get
\[
\left(\frac{1}{2} v_{x}^2\right)_t - (v_t v_x)_x + \mu v_{x x} = v_{xx} (f'(u + v) - f(u)) u_x + f'(u + v) v_x - \varepsilon q_x^2.
\]
(108)

Using the estimate of the right-hand side of (108) as
\[
\int_{\mathbb{R}} |v_{xx} ((f'(u + v) - f(u)) u_x + f'(u + v) v_x - \varepsilon q_x)| \, dx
\leq \frac{\mu}{2} \|v_{xx}\|_{L^2}^2 + C \|v\|_{H^1}^2 + C \varepsilon^2 \|q_{xx}\|_{L^2}^2,
\]
where we have used the uniform bound of \(u_x = \Phi \xi + \tilde{u}_\xi\), we integrate (108) in \(x\) to obtain
\[
\frac{d}{dt} \|v_x\|_{L^2}^2 + c \|v_{xx}\|_{L^2}^2 \leq C \|v\|_{H^1}^2 + C \varepsilon^2 \|q_x\|_{L^2}^2.
\]
(109)

Therefore, summing up (107) and (109) yields
\[
\frac{d}{dt} \|v\|_{H^1}^2 \leq C \|v\|_{H^1}^2 + C \varepsilon^2 \|q_t\|_{H^1}^2,
\]
which gives the desired inequality (101) with the aid of the Gronwall inequality and
\[
\varepsilon \int_0^t \|q_t\|_{H^1}^2 \, d\tau \leq C \varepsilon \int_0^t \|\psi\|_{H^1}^2 \, d\tau + C \varepsilon \int_0^t \|\psi\|_{H^1}^2 \, d\tau + C \varepsilon \int_0^t \|\tilde{q}_t\|_{H^1}^2 \, d\tau
\leq C (1 + t) \leq C e^{C t}
\]
which follows from (48) and (70).

We secondly show (102). Multiplying (105b) by \(w\) and integrating the resultant equality, we have
\[
\varepsilon \frac{d}{dt} \|w\|_{L^2}^2 + \|w\|_{L^2}^2 \leq C \|v_x\|_{L^2}^2 + C \varepsilon^2 \|q_t\|_{L^2}^2,
\]
(110)

where we have used
\[
\int_{\mathbb{R}} |w(\mu v_x + \varepsilon q_t)| \, dx \leq \frac{1}{2} \|w\|_{L^2}^2 + C \|v_x\|_{L^2}^2 + C \varepsilon^2 \|q_t\|_{L^2}^2.
\]

Multiply (110) by \(\frac{1}{\varepsilon} e^{t/\varepsilon}\) and integrate the resultant inequality in \(t\) to get
\[
e^{t/\varepsilon} \|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 + C \varepsilon \int_0^t e^{\tau/\varepsilon} \|v_x\|_{L^2}^2 \, d\tau + C \varepsilon \int_0^t e^{\tau/\varepsilon} \|q_t\|_{L^2}^2 \, d\tau.
\]
(111)

From (101), we have
\[
\frac{1}{\varepsilon} \int_0^t e^{\tau/\varepsilon} \|v_x\|_{L^2}^2 \, d\tau \leq C e^{C_1 t} \int_0^t e^{\tau/\varepsilon} \, d\tau \leq C \varepsilon e^{C_1 t} e^{t/\varepsilon}.
\]
(112)
Also, using \( \eta_t = -\mu u_t x - \mu (\partial_t - s \partial_x)(\Phi_{xx} + \tilde{u}_x) \), we see
\[
\varepsilon \int_0^t e^{\tau/\varepsilon} \| \eta_t \|^2_{L^2} \, d\tau \leq C \int_0^t e^{\tau/\varepsilon} \left( \| \Phi_{xx} \|^2_{L^2} + \| \Phi_{xx} \|^2_{L^2} + \| \tilde{u}_x \|^2_{L^2} \right) d\tau
\]
\[
\leq C \varepsilon e^{t/\varepsilon} \int_0^t \left( \| \Phi_{xx} \|^2_{L^2} + \| \Phi_{xx} \|^2_{L^2} \right) d\tau + C \varepsilon \int_0^t e^{\tau/\varepsilon} \, d\tau \leq C \varepsilon e^{t/\varepsilon},
\] (113)
where we have used (20) and (25). Substituting (112) and (113) in (111) yields the desired inequality (102).

(ii) Define \( T \) := \( -\frac{1}{C_1} \log \varepsilon^\lambda \) for an arbitrary constant \( \lambda \in (0, 1) \), where \( C_1 \) is a positive constant in (101) and (102). We see from (101) and (102) that
\[
\| \varphi(t) \|^2_{H^1} \leq C \varepsilon e^{C_1 T} = C \varepsilon^{1-\lambda},
\] (114)
\[
\| w(t) \|^2_{L^2} \leq \| w(0) \|^2_{L^2} e^{-t/\varepsilon} + C \varepsilon^{1-\lambda}
\] (115)
for \( t \in [0, T] \). On the other hand, we see from (37) and (80) that
\[
\| \Phi_x(t) \|^2_{L^2} + \| \Phi_x(t) \|^2_{L^2} \leq C \varepsilon^{-\gamma \delta^4 T} = C \varepsilon^\frac{1}{\gamma \delta^4} \lambda
\]
for \( t \geq T \). Also, from Lemma 4.4, we see
\[
\int_{\mathbb{R}} \| \tilde{u}(\xi + x_0^2) - \tilde{u}(\xi + x_0) \|^2 \, d\xi \leq C |x_0^2 - x_0| \leq C \varepsilon.
\]

Thus, from Theorem 4.1 and Lemma 4.4, we obtain
\[
\| v(t) \|^2_{L^2} \leq C \left( \| \Phi_x(t) \|^2_{L^2} + \| \Phi_x(t) \|^2_{L^2} + \| \tilde{u}(\cdot + x_0^2) - \tilde{u}(\cdot + x_0) \|^2_{L^2} + \| \tilde{u} - \tilde{u} \|^2_{L^2} \right)
\]
\[
\leq C \varepsilon^{\frac{1}{\gamma \delta^4}} \lambda
\] (116)
for \( t \geq T \). We fix \( \lambda := \lambda = \frac{C_1}{C_1 + \gamma \delta^4} \) and define \( \lambda_1 := 1 - \lambda = \frac{\gamma \delta^4}{C_1} \lambda \). Then (114) and (116) yield
\[
\| v(t) \|^2_{L^2} \leq C \varepsilon^{\lambda_1}
\]
for \( t \geq 0 \). The interpolation inequality and the uniform bound of \( v_{xx} \) \( \| v_{xx} \|_{L^2} \leq C(\| \Phi_{xx} \|_{L^2} + \| \Phi_{xx} \|_{L^2} + \| \tilde{u}_x \|_{L^2} + \| \tilde{u}_x \|_{L^2}) \leq C \) give the estimate of \( v_x \) as
\[
\| v_x(t) \|^2_{L^2} \leq C \| v(t) \|^2_{L^2} \| v_{xx}(t) \|^2_{L^2} \leq C \varepsilon^{\lambda_1/2}
\]
for \( t \geq 0 \). Therefore we obtain (103) with \( \lambda_0 := \lambda_1/2 \).

We next obtain the estimate (104) of \( w \). In the same way as the derivation of (116), we have
\[
\| w(t) \|^2_{L^2} \leq C \left( \| \psi_x(t) \|^2_{L^2} + \| \Phi_x(t) \|^2_{L^2} + \| \tilde{u}(\cdot + x_0^2) - \tilde{u}(\cdot + x_0) \|^2_{L^2} + \| \tilde{u} - \tilde{u} \|^2_{L^2} \right)
\]
\[
\leq C \varepsilon^{\lambda_1}
\]
for \( t \geq T \), where we have used (37) and (90). Therefore, combining this estimate with (115), we obtain the desired estimate (104). Thus we complete the proof.

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