Carleson and sampling measures on Bernstein spaces on Siegel CR manifolds

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Abstract
In this paper, we introduce and study Carleson and sampling measures on Bernstein spaces on a class of quadratic CR manifold called Siegel CR manifolds. These are spaces of entire functions of exponential type whose restrictions to the given Siegel CR manifold are $L^p$-integrable with respect to a natural measure. For these spaces, we prove necessary and sufficient conditions for a Radon measure to be a Carleson or a sampling measure. We also provide sufficient conditions for sampling sequences.

KEYWORDS
Bernstein spaces, Carleson measures, entire functions of exponential type, Paley–Wiener spaces, quadratic CR manifolds, sampling measures

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1 | INTRODUCTION

For $\kappa > 0$, let $\mathcal{E}_\kappa(\mathbb{C})$ be the space of an entire function of exponential type at most $\kappa$, that is,

$$\mathcal{E}_\kappa(\mathbb{C}) = \left\{ f \in \text{Hol}(\mathbb{C}) : \limsup_{w \to \infty} \frac{\log |f(w)|}{1 + |w|} \leq \kappa \right\}.$$

For $p \in (0, \infty]$, the classical Bernstein spaces are defined as the spaces of functions in $\mathcal{E}_\kappa(\mathbb{C})$ whose restrictions to the real line are in $L^p(\mathbb{R})$, that is, writing $w = u + iv$ and letting $f_v : u \mapsto f(u + iv)$, $\mathcal{B}_p^\kappa = \{ f \in \mathcal{E}_\kappa(\mathbb{C}) : f_0 \in L^p(\mathbb{R}) \},$ (1)

endowed with the quasi-norm $\|f\|_{\mathcal{B}_p^\kappa} := \|f_0\|_{L^p}$. If $f \in \mathcal{B}_p^\kappa$, then the Phragmén–Lindelöf principle easily implies that in fact $|f(w)| \leq Ce^{\kappa|w|}$ for all $w \in \mathbb{C}$. The case $p = 2$ corresponds to the classical Paley–Wiener space $PW_\kappa$, and in fact also the Bernstein spaces are sometimes referred to as the Paley–Wiener spaces. See [21, 39, 42] for more information on classical Bernstein spaces.

In [10], an analog of Bernstein spaces in several variables is considered, where the role of $\mathbb{R}$ is played by a “Siegel CR submanifold.” More precisely, given a complex Hilbert space $E$ of dimension $n$, a real Hilbert space $F$ of dimension $m$, and a Hermitian map $\Phi : E \times E \to F_C$, we consider the quadratic (or quadric) CR manifold (cf. [3, 38])

$$Q := \{ (\zeta, x + i\Phi(x)) : \zeta \in E, x \in F \} = \{ (\zeta, z) \in E \times F_C : \rho(\zeta, z) = 0 \},$$
where

$$\Phi(\zeta) := \Phi(\zeta, \zeta) \quad \text{and} \quad \rho(\zeta, z) := \text{Im } z - \Phi(\zeta)$$

for every $(\zeta, z) \in E \times F_C$. Then, $Q$ is a CR manifold of CR dimension $n$ and real codimension $m$. The manifold $Q$ can be canonically identified with a two-step nilpotent Lie group $\mathcal{N} := E \times F$, endowed with the product

$$(\zeta, x)(\zeta', x') := (\zeta + \zeta', x + x' + 2 \text{Im } \Phi(\zeta, \zeta'))$$

for every $(\zeta, x), (\zeta', x') \in E \times F$. Notice that the mapping $\mathcal{N} \ni (\zeta, x) \mapsto (\zeta, x + i\Phi(\zeta)) \in E \times F_C$ is a canonical isomorphism of $\mathcal{N}$ onto the subgroup $Q := \rho^{-1}(0)$ of $E \times F_C$, endowed with the product

$$(\zeta, z) \cdot (\zeta', z') := (\zeta + \zeta', z + z' + 2i\Phi(\zeta', \zeta)).$$

Consequently, $\mathcal{N}$ acts freely and affinely on the complex space $E \times F_C$. Then, given a compact convex subset $K$ of $F^t$, we consider the Bernstein spaces

$$B^p_K(\mathcal{N}) := \left\{ f \in \text{Hol}(E \times F_C) : \forall h \in F \quad \| f_h \|_{L^p(\mathcal{N})} \leq e^{H_K(\rho(\zeta, z))}, f_0 \in L^p(\mathcal{N}) \right\},$$

endowed with the quasi-norm $\| f \|_{B^p_K(\mathcal{N})} := \| f_0 \|_{L^p(\mathcal{N})}$, where

$$f_h : \mathcal{N} \ni (\zeta, x) \mapsto f(\zeta, x + i\Phi(\zeta) + ih)$$

for every function $f$ on $E \times F_C$ and for every $h \in F$, while

$$H_K : F \ni h \mapsto \sup_{\lambda \in K} \langle \lambda, h \rangle \in [-\infty, \infty)$$

is the support function associated with $K$ (cf. [17, section 4.3] and [5, Exercise 9 of section 2]). We observe that, if $n = 0$, $m = 1$, and $K = [-\tau, \tau]$, then $H_K(h) = \tau |h|$ for every $h \in \mathbb{R}$, and $\mathcal{N} = \mathbb{R}$. Thus, the space $B^p_K(\mathcal{N})$ reduces to the classical Bernstein space $B^p_K$ defined in (1). Furthermore, if $n = 0$ and $K$ is a convex symmetric compact set containing the origin (a convex body), then $\mathcal{N}$ is (again abelian and coincides with) $\mathbb{R}^m$ and the spaces $B^p_K(\mathcal{N})$ are the Paley–Wiener spaces implicitly considered in [41, Chapter III]. Finally, we observe explicitly that the case $n \geq 1$, $m = 1$, and $K = [0, a]$ was first introduced and studied in [30]. In this case, $\mathcal{N}$ coincides with the classical Heisenberg group $H_n$. We discuss these cases in more detail in Examples 3.1–3.3 in Section 3.

We refer the reader to [10] for various characterizations and other basic properties of $B^p_K(\mathcal{N})$. It is worth mentioning that, as a consequence of [10, Theorem 1.10], $B^p_K(\mathcal{N}) = B^p_{K \cap \Lambda_+}(\mathcal{N})$ when $p < \infty$, where

$$\Lambda_+ := \left\{ \lambda \in F' : \forall \zeta \in E \setminus \{ 0 \} \langle \lambda, \Phi(\zeta) \rangle > 0 \right\}.$$ 

Notice that, if $E = \{ 0 \}$, then $\Lambda_+ = F'$. It is therefore natural to restrict our attention to the case in which the open convex cone $\Lambda_+$ is nonempty, in which case, $Q$ is said to be a “Siegel” CR submanifold of $E \times F_C$, while $\mathcal{N}$, endowed with the CR structure induced by $Q$, is said to be a “Siegel” CR manifold.

Therefore, the spaces $B^p_K(\mathcal{N})$ constitute a natural, highly nontrivial multidimensional extension of the classical Bernstein spaces. In Section 3, we discuss their main properties and features, connections with the classical spaces in one variable, with other extensions in several variables, and present some examples.

A classical research topic in the study of Bernstein spaces and other spaces of holomorphic functions is the determination of Carleson measures and sampling measures and sequences, which, more generally, may actually be regarded as fundamental objects in function theory. In this paper, we propose to investigate Carleson and sampling measures for the spaces $B^p_K(\mathcal{N})$. Let us now recall the definition of these objects. Given a Hausdorff space $X$, a quasi-Banach space $Y$ of functions on $X$, and $p \in (0, \infty)$, a $p$-Carleson measure for $Y$ is a positive Radon measure $\mu$ on $X$ such that $Y \subseteq L^p(\mu)$.
continuously. In other words, each element of $Y$ is $\mu$-measurable and there is a constant $C > 0$ such that

$$\|f\|_{L^p(\mu)} \leq C \|f\|_Y$$

for every $f \in Y$. If, in addition, the canonical mapping $Y \rightarrow L^p(\mu)$ is compact, then $\mu$ is called a compact $p$-Carleson measure for $Y$. If, in addition, the canonical mapping $Y \rightarrow L^p(\mu)$ is an isomorphism onto its image, then $\mu$ is called a $p$-sampling measure for $Y$. This means that we may find $C \geq 1$ such that

$$\frac{1}{C} \|f\|_Y \leq \|f\|_{L^p(\mu)} \leq C \|f\|_Y$$

for every $f \in Y$.

We also recall that, classically, if $Y$ is a reproducing kernel Hilbert space, a locally finite sequence of distinct points $(x_j)_{j \in J}$ in $X$ is called a 2-sampling sequence for $Y$ (or simply a sampling sequence, since the parameter $p = 2$ is understood from the context) if the measure $\mu := \sum_{j \in J} c_j^{-1} \delta_{x_j}$ is a 2-sampling measure for $Y$. Here, $c_j = k_{x_j}(x_j)$, where $k_{x_j}$ is the reproducing kernel of $Y$ at the point $x_j$, and $\delta_x$ denotes the Dirac delta at $x \in X$. In other words, $(x_j)_{j \in J}$ is a sampling sequence if the mapping $Y \ni f \mapsto (c_j^{-1/2}f(x_j)) \in \ell^2(J)$ is an isomorphism onto its image. Notice that $c_j^{1/2}$ may also be interpreted as the norm of the continuous linear mapping $Y \ni f \mapsto f(x_j) \in \mathbb{C}$. If $Y$ is only a quasi-Banach space, but the mappings $Y \ni f \mapsto f(x) \in \mathbb{C}$ are continuous (with norm $c_j'$), then a sequence of distinct points $(x_j)_{j \in J}$ in $X$ is called a $p$-sampling sequence for $Y$ if the mapping $Y \ni f \mapsto (c_j^{-1}f(x_j)) \in \ell^p(J)$ is an isomorphism onto its image, equivalently, if $\sum_{j \in J} c_j'^{-p} \delta_{x_j}$ is a $p$-sampling measure for $Y$. Clearly, the notion of $p$-sampling measures for $Y$ is a generalization of the notion of $p$-sampling sequences for $Y$.

Carleson measures were introduced by L. Carleson in [11, 12] in order to study the corona problem in the classical Hardy spaces on the unit disc. The study of these measures has flourished since then, and has been generalized to several different settings, such as weighted Bergman spaces, the Dirichlet space, Fock spaces, model spaces, and Bernstein spaces. Sampling measures arose as extensions of sampling sequences. See [40] and the references therein for more information on sampling (and interpolating) sequences for various function spaces.

In the case of the classical Bernstein spaces $B^p_k$, $p \in (0, \infty]$, $p$-sampling sequences on the real line were studied by Plancherel and Pólya in [39, Nos. 40, 44]. They proved that for every $p \in (0, \infty]$ and $\kappa' > \kappa$, there exist two constants $C_{p,\kappa,\kappa'}$, $C'_{p,\kappa,\kappa'} > 0$ such that, for every $f \in B^p_k$,

$$C_{p,\kappa,\kappa'} \|f\|_{B^p_k} \leq \left( \sum_{n \in \mathbb{Z}} |f(n\pi/\kappa')|^p \right)^{1/p} \leq C'_{p,\kappa,\kappa'} \|f\|_{B^p_k}$$

(modification if $p = \infty$). If $p \in (1, \infty)$, then one may take $\kappa' = \kappa$, while, if $p = 2$, the classical Whittaker–Kotelnikov–Shannon theorem shows that $C_{2,\kappa,\kappa} = C'_{2,\kappa,\kappa} = \sqrt{\kappa/\pi}$. General sampling sequences for $B^p_k$ have been studied by Beurling [2] for sampling sequences in $\mathbb{R}$, and by Seip in [40, Theorem 10 of Chapter 6] for sampling sequences in $\mathbb{C}$, see also [35].

We shall now review our main results. Concerning Carleson measures for the spaces $B^p_k(\mathcal{N})$, we are able to provide a complete characterization for those measures, which are supported in a band

$$\rho^{-1}(\overline{B}(0, R)) = \{ (\zeta, z) \in E \times F_C : |\text{Im } z - \Phi(\zeta)| \leq R \}, \quad (3)$$

for some $R > 0$. In the one-dimensional situation, this corresponds to restricting our attention to those measures that are supported on a horizontal strip $\{ z \in \mathbb{C} : |\text{Im } z| \leq R \}$ for some $R > 0$. The case of measures supported in a band is more general than that of measures supported in $Q$ (which correspond to the measures supported on the real line in the one-dimensional situation), but may still be treated with the same techniques. The general case is much more complicated except in the one-dimensional situation, where it can be settled essentially as for Hardy spaces on the upper half-plane. The proof of the one-dimensional result (cf. Proposition 2.1) suggests that, at least when $E = \{ 0 \}$ and $K$ is a polyhedron, the study of Carleson measures for $B^p_k(\mathcal{N})$ should be somewhat related to the study of Carleson measures for the Hardy spaces $H^p(\rho^{-1}(\Omega))$, where $\Omega$ runs through the set of tangent cones of $K$ at its vertices (cf. the discussion after the proof of Corollary 4.9). Nonetheless, the study of Carleson measures for these Hardy spaces is far from being settled, even in the simplest settings (with the notable exception of the one-dimensional case).
For these reasons, we shall content ourselves with providing some sufficient and some necessary conditions for general Carleson measures without finding a complete characterization. For what concerns Carleson measures which are supported in a band, we have the following result (cf. Theorem 4.10).

We denote by $M_+(E \times F_C)$ the set of positive Radon measures on $E \times F_C$. Furthermore, we denote by $M_H(\mu)$ the function

$$M_H(\mu) : E \times F_C \ni (\zeta, z) \mapsto \mu((\zeta, z) \cdot H) \in \mathbb{R}_+.$$  

(4)

**Theorem A.** Take a compact subset $K$ of $\Lambda_+$ with a nonempty interior, $p, q \in (0, \infty]$ with $q < \infty$, and $\mu \in M_+(E \times F_C)$ such that $\rho(\text{Supp} \ \mu)$ is bounded. Fix a compact neighborhood $H$ of $(0, 0)$ in $E \times F_C$. If $p \leq q$, then the following conditions are equivalent:

1. $M_H(\mu)$ is bounded (resp. and vanishes at $\infty$);
2. $\mu$ is a (resp. compact) $q$-Carleson measure for $B^p_K(\mathcal{N})$.

If, otherwise, $p > q$, then the following conditions are equivalent:

1. $M_H(\mu) \in L^{p/q}(E \times F_C)$;
2. $\mu$ is a $q$-Carleson measure for $B^p_K(\mathcal{N})$;
3. $\mu$ is a compact $q$-Carleson measure for $B^p_K(\mathcal{N})$.

Concerning sampling measures, we shall again only consider the case of measures supported in a band. In the one-dimensional case, sampling measures and sampling sequences in the real line may be (almost, if $p > 1$) characterized by means of Beurling densities, as we shall review in Section 2. Sampling sequences for $B^2_{[-\tau, \tau]}(\mathbb{R})$ may be even fully characterized in this setting. Unfortunately, already when $n = 0$ and $K$ is a square, Beurling densities may not be used to provide sufficient conditions for sampling sequences, as shown in [40, p. 122]. Nonetheless, there are some necessary conditions that apply to a large class of reproducing kernel Hilbert spaces (essentially based on Landau’s necessary conditions for Paley–Wiener spaces in one and several variables), cf. [15], and also some sufficient conditions essentially based on an argument by Beurling, cf. [2, 32].

The general necessary conditions in our context appear as follows (cf. Proposition 5.18). See Section 3 for a definition of $B^d$ and $|Pf|$. Here and in the sequel, we denote by $H^d$ the $d$-dimensional Hausdorff measure.

**Theorem B.** Take a compact subset $K$ of $\Lambda_+$, and let $(\xi_j, x_j + i\Phi(\xi_j)) \in \mathbb{N}$ be a (locally finite) sampling sequence for $B^2_K(\mathcal{N})$ in $Q$. Then,

$$\liminf_{R \to +\infty} \inf_{(\xi, x) \in \mathcal{N}} \text{Card} \left\{ \xi_j, x_j \in B^{\infty}_K((\xi, x), R) \right\} \geq \frac{2^{n-m}}{\pi^{n+m}} \int_K |Pf(\lambda)| \, d\lambda.$$

We may also derive somewhat analogous, yet less precise, necessary conditions for sampling measures using standard techniques (cf. e.g., [8, 26]). See Proposition 5.6.

**Proposition C.** Take a compact subset $K$ of $\Lambda_+$ with a nonempty interior, $p \in (0, \infty)$, and a $p$-sampling measure $\mu$ for $B^p_K(\mathcal{N})$ such that $\rho(\text{Supp} \ \mu)$ is bounded. Then, there are $C > 0$ and a compact neighborhood $H$ of $(0, 0)$ in $E \times F_C$ such that (cf. formula 4)

$$M_H(\mu)(\xi, z) \geq C$$

for every $(\xi, z) \in Q$.

Concerning the sufficient conditions, a procedure to characterize sampling measures for weighted Bergman spaces in the unit disc (or, equivalently, in the upper half-plane) was introduced in [29]. This technique has later proved fruitful to characterize sampling sequences for Fock spaces in $\mathbb{C}$ (cf. [43, Lemma 4.25]) and to provide sufficient conditions for sampling measures for weighted Bergman spaces in homogeneous Siegel domains (cf. [8]). We shall employ this procedure to
prove Theorem 5.8, which may be considered our main result. Because of its technical nature, though, in this Introduction we simply indicate some of its corollaries, which are of interest in their own right. First of all, we shall characterize the so-called “dominating” sets. In the classical situation where this term seems to have originated, namely, the setting of weighted Bergman spaces in the upper half-plane (or, equivalently, in the unit disc)

\[ A^p_s(\mathbb{C}+) = \left\{ f \in \text{Hol}(\mathbb{C}+) : \int_{\mathbb{C}+} |f(z)|^p |\text{Im } z|^{s-1} \, dz < \infty \right\}, \]

there is a canonical \( p \)-Carleson measure \( \mu_s \), namely,

\[ d\mu_s(z) = (\text{Im } z)^{s-1} \, dz \quad (s > 0), \]

and a \( \mu_s \)-measurable subset \( D \) of \( \mathbb{C}+ \) is said to be dominating for \( A^p_s(\mathbb{C}+) \) if the measure \( \chi_D \cdot \mu_s \) is \( p \)-sampling for \( A^p_s(\mathbb{C}+) \). In other words, \( D \) is dominating for \( A^p_s(\mathbb{C}+) \) if and only if there is a constant \( C > 0 \) such that

\[ \int_{\mathbb{C}+} |f(z)|^p |\text{Im } z|^{s-1} \, dz \leq C \int_D |f(z)|^p |\text{Im } z|^{s-1} \, dz \]

for every \( f \in A^p_s(\mathbb{C}+) \). The notion of dominating sets then extends to other function spaces endowed with a distinguished “base” (Carleson) measure. In our setting, a reasonable “base” measure could be the Haar measure on \( \mathbb{C} \), since this is the measure which was used to define the quasi-norm of \( \mathbb{B}_K^p(\mathbb{N}) \). Nonetheless, since \( \mathbb{B}_K^p(\mathbb{N}) \) is naturally a space of functions defined on \( E \times F \), on one may argue that the Lebesgue measure \( \mathcal{H}^{2n+2m} \) — or rather the measures \( (\chi_{B(0,R)} \circ \rho) \cdot \mathcal{H}^{2n+2m} \) for \( R > 0 \), if we want the distinguished measure to be a Carleson measure — should be preferred. Because of this possible ambiguity, we shall characterize dominating sets with respect to any \( \mathbb{N} \)-invariant Carleson “base” measure supported in a band. See Corollary 5.12 for a more precise version of the following statement.

**Theorem D.** Take a compact subset \( K \) of \( \Lambda_+ \) with a nonempty interior, \( p \in (0, \infty) \), an \( \mathbb{N} \)-invariant \( p \)-Carleson measure \( \mu \) on \( E \times F \), such that \( \varphi(\text{Supp } \mu) \) is bounded, and a \( \mu \)-measurable subset \( D \) of \( E \times F \). Then, \( D \) is a dominating set (with “base” measure \( \mu \)) for \( \mathbb{B}_K^p(\mathbb{N}) \) if and only if there are \( C > 0 \) and a compact neighborhood \( H \) of \( (0,0) \) such that

\[ M_H(\chi_D \cdot \mu) \geq C \]

on \( \mathbb{Q} \).

We shall then provide some sufficient conditions for “lattice-like” sampling sequences in \( \mathbb{Q} \). This kind of sampling sequences is particularly interesting as it may be considered as a reasonable analog of the classical lattices (i.e., discrete subgroups) in \( \mathbb{R} \), which were the first sequences for which sampling was investigated in [39]. This result extends the sufficient conditions provided in [32, Theorem 4.1] (on which the proof is partially based), and (almost) extends the sufficient conditions found in [30] when \( m = 1 \) and \( p = 2 \). See Corollary 5.14 for a more precise statement.

**Theorem E.** Take a compact subset \( K \) of \( \Lambda_+ \) with a nonempty interior, \( p \in (0, \infty) \), \( r < \frac{\pi}{2} \), and \( \lambda \in F' \). Assume that

\[ \langle \lambda', \Phi(\zeta) \rangle < \frac{\pi}{2} |\zeta|^2 \]

for every nonzero \( \zeta \in E \) and identify \( E \) with \( \mathbb{C}^n \) by means of some orthonormal basis. For every \( \zeta \in \mathbb{Z}[i]^n \subseteq \mathbb{C}^n \), fix a sequence \( (x_{\zeta,j})_{j \in \mathbb{N}} \) in \( F \) such that

\[ \bigcup_{j \in \mathbb{N}} (x_{\zeta,j} + r[(K - \lambda) \cup (\lambda - K)])^c = F \tag{5} \]

and such that \( \inf_{j \neq j'} |x_{\zeta,j} - x_{\zeta,j'}| > 0 \). Then, \( (\zeta, x_{\zeta,j} + i\Phi(\zeta))_{\zeta \in \mathbb{Z}[i]^n, j \in \mathbb{N}} \) is a sampling sequence for \( \mathbb{B}_K^p(\mathbb{N}) \).

Some comments are in order. First of all, \( [(K - \lambda) \cup (\lambda - K)]^c \) denotes the polar of \( (K - \lambda) \cup (\lambda - K) \). In other words, if we endow \( F' \) with the norm for which the symmetric convex envelope of \( K - \lambda \) is the (closed) unit ball, then \( [(K - \lambda) \cup (\lambda - K)]^c \) is the unit ball \( \mathcal{B}_K,\lambda(0,1) \) with respect to the dual norm, namely, \( H_{(K-\lambda),\lambda-K} \) (see Example 3.2 in Section 3).
Thus, the covering condition (5) may be rephrased requiring that the balls $\overline{B}_{K,\lambda}(x_{\zeta,j},r)$ cover $F$. Since $K$ is assumed to be contained in the cone $\Lambda_+$, and is therefore far from being symmetric (unless $n = 0$), translating $K$ by means of $\lambda$ may shrink the symmetric convex envelope of $K - \lambda$ and thus enlarge the dual unit ball $\overline{B}_{K,\lambda}(0, 1)$. Consequently, a convenient choice of $\lambda$ may weaken the covering condition (5): This is the reason why $\lambda$ has been added in the statement.

Second, the condition that $\inf_{j''} x_{\zeta,j''} \neq x_{\zeta,j} - x_{\zeta,j'}$ is positive is essentially related to the condition that $\sum \delta(\zeta, x_{\zeta,j} + i \Phi(\zeta))$ be a $p$-Carleson measure for $B^p_L(N)$. Equivalence is achieved if $(x_{\zeta,j})$ are simply assumed to be finite unions of families satisfying the preceding conditions.

Finally, we observe explicitly that the covering condition (5) may be substantially weakened, simply requiring that the balls $\overline{B}_{K,\lambda}(x_{\zeta,j},r)$ cover $F \setminus \overline{B}(x_{\zeta,j}, R)$ for some $x_{\zeta,j} \in F$ and for some fixed $R > 0$.

This latter reduction is essentially related to the following extension of [22, Lemma 4], which shows that we may always remove a measure with compact support from a sampling measure for $B^p_L(N)$ and still get a sampling measure for a slightly smaller space $B^p_L(N)$. See Proposition 5.5 for a more precise statement.

**Proposition F.** Take two compact subsets $K, K'$ of $\Lambda_+$ such that $K'$ is contained in the interior of $K$, $p \in (0, \infty)$, and two positive measures $\mu, \mu'$ on $E \times F_C$ such that $\mu + \mu'$ is $p$-sampling for $B^p_L(N)$, $\mu'$ has compact support, and $\rho(\text{Supp}(\mu + \mu'))$ is bounded. Then, $\mu$ is $p$-sampling for $B^p_L(N)$.

To conclude this review of our main results, we would like to emphasize that the majority of our sufficient conditions for sampling measures (and sequences) actually imply that the given measure $\mu$ satisfies a stronger condition, namely, that the condition $f \in L^p(\mu)$ is sufficient to determine if a holomorphic function $f \in E \times F_C$ satisfying certain natural growth conditions (e.g., $f \in B^p_L(N)$) belong to $B^p_L(N)$. Notice that this property is not satisfied by the “limit” sampling lattices $(j \pi / \kappa)$ for the one-dimensional Bernstein space $B^p_{\text{loc}}$, $p \in (1, \infty)$. For example, $\sin(\pi \cdot \kappa)$ belongs to $B^p_{\text{loc}}$ and vanishes on the aforementioned lattice, but does not belong to $B^p_{\text{loc}}$ for any $p \in (0, \infty)$.

The paper is structured as follows. For the reader’s convenience, in Section 2, we review some of the known results for Carleson measures and sampling sequences for Bernstein spaces in the one-dimensional setting. In Section 3, we recall some basic definitions and facts, which will be needed in the following sections. We introduce the Bernstein spaces in our setting and discuss the known results in the classical one-dimensional case and the extensions to several variables present in the literature. In Section 4, we shall prove our main results on Carleson measures. After providing some general sufficient (cf. Proposition 4.7) and necessary (cf. Proposition 4.8) conditions, we characterize the Carleson measures, which are supported in a band, that is, in $\rho^{-1}(\overline{B}_E(0, R))$ for some $R > 0$ (cf. Theorem 4.10).

In Section 5, we consider only measures $\mu$ supported in a band, that is, in $\rho^{-1}(\overline{B}_E(0, R))$ for some $R > 0$, and we provide general necessary (cf. Proposition 5.6) and sufficient (cf. Theorem 5.8) conditions for $\mu$ to be $p$-sampling for $B^p_L(N)$. In addition, we provide a number of sufficient criteria for sampling measures (cf. Corollaries 5.12 and 5.13) and for sampling sequences (cf. Corollaries 5.14 and 5.16). We then extend to this setting the known relation between sampling sequences for the various $B^p_L(N)$ (cf. [32, Theorem 2.1] and Proposition 5.17), and we specialize to our setting the general Beurling-type necessary conditions for sampling sequences proved in a very general context in [15] (cf. Proposition 5.18).

2 | THE ONE-DIMENSIONAL CASE

We keep the notation of the Introduction section.

We begin with the discussion of Carleson measures. The sequences $(z_j)$ in $C$ such that $\sum \{1 + |\text{Im } z_j|\} e^{-2|\text{Im } z_j|} \delta_{z_j}$ is a 2-Carleson measure for $B^2_L$ are essentially characterized in [40, Chapter VI, section 2]. Using a similar argument, one may then prove the following characterization of $q$-Carleson measures $B^q_L$ when $p \leq q$.

**Proposition 2.1.** Take $p, q \in (0, \infty)$, $p \leq q$, $\kappa > 0$, and $\mu \in \mathcal{M}_+(C)$. Then, $\mu$ is $q$-Carleson for $B^q_L(\mathbb{R})$ if and only if there is a constant $C > 0$ such that

$$\left( \int_{B(x,R)} e^{\kappa |\text{Im } z|} \, d\mu(z) \right)^{1/q} \leq CR^{1/p}$$

for every $x \in \mathbb{R}$ and for every $R \geq 1$. 
As the proof shows, this characterization is essentially a consequence of the characterization of $q$-Carleson measures for the Hardy space $H^p(C_+)$. One might also consider the available characterization of $q$-Carleson measures for the Hardy space $H^p(C_+)$ (cf. [27, Theorem C]) for $q < p$ and verify if a corresponding characterization of $q$-Carleson measures for $B^p_q$ holds. Since the characterization of $q$-Carleson measures for the Hardy space $H^p(C_+)$ for $q < p$ is considerably more involved, and since a complete treatment of the one-dimensional case is beyond the scope of this paper, we shall not pursue this task.

**Proof.** In order to avoid repetitions, we shall only provide the proof under the assumption that the result is known for measures supported in a band. In other words, we shall refer freely to Theorem 4.10.

The main idea in the proof is then the following: The mapping $f \mapsto (e^{i\omega(\cdot)} f, e^{-i\omega(\cdot)} f)$ induces an isomorphism of $B^p_q$ onto a closed subspace of $H^p(C_+) \times H^p(C_-)$, thanks to the Plancherel–Pólya inequalities (cf., e.g., [10, Theorem 1.7]). Thus, any measure that induces Carleson measures on $e^{-i\omega(\cdot)} H^p(C_+)$ and $e^{i\omega(\cdot)} H^p(C_-)$ is a Carleson measure for $B^p_q$, whence sufficiency. Necessity is proved refining the techniques employed for the Hardy spaces.

Let us prove that the condition is sufficient. Observe that, by Theorem 4.10 below, we may assume that $\mu$ is supported in $\{ z \in \mathbb{C} : |\Im z| \geq 1 \}$. Then, observe that $e^{i\omega(\cdot)} B^p_q \subseteq H^p(C_+)$ continuously while $e^{-i\omega(\cdot)} B^p_q \subseteq H^p(C_-)$ continuously. Now, the assumptions and [27, VII.2] show that the restrictions of $e^{i\Im(\cdot)} \mu$ and $e^{-i\Im(\cdot)} \mu$ to $C_+$ and $C_-$, respectively, are $q$-Carleson measures for $H^p(C_+)$ and $H^p(C_-)$, respectively. Hence, $\mu$ is a $q$-Carleson measure for $B^p_q$.

Let us now prove that the condition is necessary. By assumption, there is a constant $C_1 > 0$ such that

$$
\|f\|_{L^q(\mu)} \leq C_1 \|f_0\|_{L^p(\mathbb{R})}
$$

for every $f \in B^p_q(\mathbb{R})$. Then, take $k \in \mathbb{N}$ so that $kp > 1$, and fix $\varphi : \mathbb{R} \to \mathbb{R}$ so that $\varphi(\lambda) = \frac{1}{(k-1)!} (\lambda + x)^{k-1}$ for every $\lambda \in (-\infty, 0]$, and so that $\varphi$ is of class $C^\infty$ on $(-\infty, +\infty)$ and vanishes on $[\infty, +\infty)$. Define

$$
f(z) : \mathbb{C} \ni w \mapsto \int_{-\kappa}^{\kappa} \varphi(\lambda) e^{i\lambda(w-z)} d\lambda
$$

for every $z \in \mathbb{C}$. Observe that, integrating by parts,

$$
f(z)(w) = (-1)^k e^{-i\omega(w-z)} \left[ 1 + \int_0^{2\kappa} \varphi^{(k)}(\lambda-x) e^{i\lambda(w-z)} d\lambda \right]
$$

for every $w, z \in \mathbb{C}$ with $w \neq z$. Observe that, for every $w, z \in \mathbb{C}$,

$$
\left| \int_0^{2\kappa} \varphi^{(k)}(\lambda-x) e^{i\lambda(w-z)} d\lambda \right| \leq \|\varphi^{(k)}\|_{L^q((\kappa, \kappa))} \frac{1 - e^{-2x|\Im(w+z)} \Im(w+z)}{\Im(w+z)},
$$

which we may assume to be $\leq \frac{1}{2}$ if $|\Im(w+z)| \geq M$ for some $M > 0$. Then,

$$
\|f(z)\|_{L^p(\mathbb{R})} \leq \frac{C_2}{(\Im z)^{k-1}} e^{p\Im z}
$$

for every $z \in \mathbb{C}$ with $\Im z \geq M$, where $C_2 := \frac{3p}{2p} \int_{\mathbb{R}} \frac{1}{|x+1|^p} dx$. Analogously,

$$
\|f(z)\|_{L^q(\mu)} \geq \frac{e^{p\Im z}}{2^q} \int_{B(x,R+R_1)} e^{p\Im w} \frac{e^{p\Im x}}{2^{kq/2}(R+\Im z)^{kq}} d\mu(w)
$$

for every $z \in \mathbb{C}$ with $\Im z \geq M$, and for every $R > 0$. Choosing $z = x + iR$, this implies that

$$
\int_{B(x,R+y) \cap \mathbb{C}_+} e^{p\Im w} d\mu(w) \leq 2^{q(1+3k/2)} R^{kq} \left( \|f(z)\|_{L^q(\mu)} \right)^q \leq C_1^q C_2^{q/p} 2^{q(1+3k/2)} R^{q/p}
$$
for every $x \in \mathbb{R}$ and for every $R \geq M$. Thus,
\[
\left( \int_{B(x,R) \cap \mathbb{C}_+} e^{q|\text{Im } w|} \, \text{d}\mu(w) \right)^{1/q} \leq C_1 C_2^{1/p} 2^{1+3k/2} R^{1/p}
\]
for every $x \in \mathbb{R}$, for every $R \geq M$. In a similar way, one shows that there is a constant $C_3 > 0$ such that
\[
\left( \int_{B(x,R) \cap \mathbb{C}_+} e^{q|\text{Im } w|} \, \text{d}\mu(w) \right)^{1/q} \leq C_3 R^{1/p}
\]
for every $x \in \mathbb{R}$, and for $R$ sufficiently large. By means of Theorem 4.10 below, this completes the proof.

We now turn to sampling sequences. Given a subset $S$ of $\mathbb{R}$, its lower Beurling density is
\[
D^-(S) := \liminf_{R \to +\infty} \inf_{x \in \mathbb{R}} \frac{\text{Card}(S \cap [x, x+R))}{R}.
\]
Then, we have the following result (cf., e.g., [40, pp. 82 and 118]).

**Theorem 2.2.** Take $p \in (0, \infty)$, $\kappa > 0$, and a subset $S$ of $\mathbb{R}$, which is a finite union of uniformly separated sequences (i.e., $\mu = \sum_{x \in S} \delta_x$ is $p$-Carleson for $\mathbb{B}^p_\kappa$). Then, the following hold:

1. If $D^-(S) > \frac{\kappa}{\pi}$, then $S$ is $p$-sampling (i.e., $\mu$ is a $p$-sampling measure) for $\mathbb{B}^p_\kappa$.
2. If $S$ is $p$-sampling for $\mathbb{B}^p_\kappa$, then $D^-(S) \geq \frac{\kappa}{\pi}$.
3. If $p \leq 1$, then $S$ is $p$-sampling for $\mathbb{B}^p_\kappa$ if and only if $D^-(S) > \frac{\kappa}{\pi}$.

Using a general procedure introduced in [33], one may then transform the previous description of sampling sequences into an analogous description of sampling measures.

The characterization of general 2-sampling sequences (in $\mathbb{C}$) for $\mathbb{B}^2_\kappa$ is far more complicated. Here, we content ourselves with a brief description of [40, Theorem 10 (ii)], for the sake of completeness.

**Theorem 2.3.** Take $\kappa, \eta > 0$ and a sequence $(z_j)$ of elements of $\mathbb{C}$ such that
\[
\inf_{j \neq j'} \frac{|z_j - z_{j'}|}{1 + |z_j - z_{j'}|} > 0.
\]
Then, $(z_j)$ is 2-sampling for $\mathbb{B}^2_\kappa$ if and only if there are $u, v \in L^\infty(\mathbb{R})$ with $\|v\|_{L^\infty(\mathbb{R})} < \frac{\pi}{4}$, and a Blaschke sequence (possibly empty or finite) $(w_k)$ in $\mathbb{C}_+$ with no accumulation points in $\mathbb{R}$ such that
\[
\int_0^x \left( \sum_j \frac{|\text{Im } z_j| + \eta}{(t - \text{Re } z_j)^2 + (|z_j| + \eta)^2} - \sum_k \frac{|\text{Im } w_k| + \eta}{(t - \text{Re } w_k)^2 + (|w_k| + \eta)^2} \right) \, dt = xx + \bar{u}(x) + v(x)
\]
for almost every $x \in \mathbb{R}$.

Recall that, according to our definition of 2-sampling sequences for $\mathbb{B}^2_\kappa$, $(z_j)$ is sampling if and only if there is a constant $C > 0$ such that
\[
\frac{1}{C} \|f_0\|_{L^2(\mathbb{R})} \leq \sum_j \left( 1 + |\text{Im } z_j| \right) e^{-2\kappa|\text{Im } z_j|} \left| f(z_j) \right|^2 \leq C \|f_0\|_{L^2(\mathbb{R})}
\]
for every $f \in \mathbb{B}^2_\kappa$ (cf. footnote 3). In addition, recall that $(w_j)$ is a Blaschke sequence in $\mathbb{C}_+$ if $\sum_j |\text{Im } w_j^{-1}| < \infty$. 
Notice that the separation condition (6) is essentially a necessary condition for the sequence \((z_j)\) to be 2-sampling for \(B^2_\mathbb{C}(\mathbb{C}, F^0)\), in the sense that every 2-sampling sequence for \(B^2_\mathbb{C}(\mathbb{C}, F^0)\) is a finite union of subsequences satisfying (6).

3 BERNSTEIN SPACES ON SIEGEL CR MANIFOLDS

In this section, we introduce our main notation and collect some definitions and basic results that will be needed in the sequel. For the ease of the reader, we shall repeat all the definitions stated in the Introduction section that we shall use in the sequel.

We shall denote by \(E\) a complex Hilbert space of finite dimension \(n\), by \(F\) a real Hilbert space of finite dimension \(m\), and by \(\Phi : E \times E \to F_\mathbb{C}\) a Hermitian mapping such that the open convex cone

\[ \Lambda_+ := \{ \lambda \in F' : \forall \zeta \in E \setminus \{0\} \langle \lambda, \Phi(\zeta) \rangle > 0 \} \]

is not empty. Then, \(\Phi\) is nondegenerate and \(\Lambda_+\) is the interior of the polar of \(\Phi(E)\) (cf. [7, Proposition 2.5]). By the polar of a subset \(A\) of \(F\), we mean

\[ A^\circ := \{ \lambda \in F' : \forall h \in A \langle \lambda, h \rangle \geq -1 \}. \]

We define the polar of the subsets of \(F'\) (identifying \(F\) with \(F''\)) analogously, so that \(A^{\infty}\) is the closed convex envelope of \(A \cup \{0\}\) (cf. [5, Theorem 1 of Chapter II, section 6, No. 3]). In particular, if \(A \subseteq B \subseteq A^\circ\), then \(A^\circ = B^\circ\).

We define

\[ \rho : E \times F_\mathbb{C} \ni (\zeta, z) \mapsto \text{Im} z - \Phi(\zeta) \in F \]

and identify \(\mathcal{N} := E \times F\) with the CR submanifold \(\rho^{-1}(0)\) of \(E \times F_\mathbb{C}\) by means of the canonical mapping \((\zeta, x) \mapsto (\zeta, x + i\Phi(\zeta))\) (cf. [3] for more information on (quadratic or quadric) CR manifolds). If we endow \(\mathcal{N}\) with the two-step nilpotent Lie group structure induced by the product

\[ (\zeta, x)(\zeta', x') := (\zeta + \zeta', x + x' + 2\text{Im}\Phi(\zeta, \zeta')) \]

for every \((\zeta, x), (\zeta', x') \in \mathcal{N}\), then the CR structure of \(\mathcal{N}\) is left-invariant and generated by the left-invariant vector fields \(Z_v\) which induce the Wirtinger derivative \(\frac{i}{2}(\partial_v - i\partial_{\bar{v}})\) at \((0,0), v \in E\). Explicitly,

\[ Z_v = \frac{1}{2}(\partial_v - i\partial_{\bar{v}}) + i\Phi(v, \cdot)\partial_F. \]

Thus, by a CR function on \(\mathcal{N}\), we shall mean a function \(f\) of class \(C^1\) such that \(Z_v f = 0\) for every \(v \in E\).

We may also endow \(E \times F_\mathbb{C}\) with a two-step nilpotent Lie group structure induced by the product

\[ (\zeta, z) \cdot (\zeta', z') := (\zeta + \zeta', z + z' + 2\text{Im}\Phi(\zeta, \zeta')), \]

so that \(\rho^{-1}(0)\) becomes a subgroup of \(E \times F_\mathbb{C}\) and the mapping \(\mathcal{N} \ni (\zeta, x) \mapsto (\zeta, x + i\Phi(\zeta)) \in \rho^{-1}(0)\) an isomorphism. Given a function \(f\) on \(E \times F_\mathbb{C}\), we define

\[ f_h : \mathcal{N} \ni (\zeta, x) \mapsto f(\zeta, x + i\Phi(\zeta) + ih) \]

for every \(h \in F\). Given \((\zeta, z) \in E \times F_\mathbb{C}\), we define \(L_{(\zeta, z)} f := f((\zeta, z)^{-1} \cdot)\). Given \((\zeta, x) \in \mathcal{N}\) and a function \(g\) on \(\mathcal{N}\), we define \(L_{(\zeta, x)} g = g((\zeta, x)^{-1} \cdot)\) analogously.

Given a compact subset \(K\) of \(F\), we define \(\mathcal{O}_K(\mathcal{N})\) as the space of CR functions \(f\) of class \(C^\infty\) on \(\mathcal{N}\), which grow polynomially with every left- (or right-)invariant derivative, and such that \(F_F[f(\zeta, \cdot)]\) is supported in \(K\) for every \(\zeta \in E\), where \(F_F\) denotes the Fourier transform on \(F\) (cf. [7] for more information and Paley–Wiener–Schwartz theorems...
associated with this space). We shall denote by $H^d$ the (suitably normalized) $d$-dimensional Hausdorff measure on the relevant metric space, for every $d \in \mathbb{N}$. In particular, $H^{2n+m}$ and $H^{2n+2m}$ are left and right Haar measures on $\mathcal{N}$ and $E \times F_\mathbb{C}$, respectively.

Given a compact subset $K$ of $F'$, we define

$$H_K : F \ni h \mapsto \sup_{\lambda \in -K} \langle \lambda, h \rangle \in [-\infty, \infty),$$

so that $H_K$ is the support function of the convex envelope of $K$ (cf. [17, Section 4.3] or [5, Exercise 9 of section 2]). In particular, $H_K = -\infty$ if and only if $K = \emptyset$, while $H_K(h) > -\infty$ for every $h \in F$ when $K \neq \emptyset$. In addition, $H_K$ is continuous and subadditive, and may be identified with the Minkowski functional (or gauge) associated with $K^\circ$ when $0 \in K$. The case $K = \emptyset$ is considered for notational convenience.

If $K$ is a compact convex subset of $F'$, then the mapping $f \mapsto f_0$ induces a bijection of the set of $f \in \text{Hol}(E \times F_\mathbb{C})$ for which there are $N, C > 0$ such that

$$||f(\xi, z)|| \leq C (1 + ||\xi|| + |z|)^N e^{H_K(\rho(\xi, z))}$$

for every $(\xi, z) \in E \times F_\mathbb{C}$, onto $\mathcal{O}_K(\mathcal{N})$ (cf. [7, Theorem 3.3]). For this reason, given a (not necessarily convex) compact subset $K$ of $F'$, we define $\text{Hol}_K(E \times F_\mathbb{C})$ as the set of $f \in \text{Hol}(E \times F_\mathbb{C})$ satisfying the above estimate and such that $f_0 \in \mathcal{O}_K(\mathcal{N})$.

Notice that $\mathcal{O}_K(\mathcal{N}) = \mathcal{O}_{K \cap \mathbb{R}^+}(\mathcal{N})$ for every compact subset $K$ of $F'$, thanks to [7, Proposition 5.7], so that we may reduce to considering only $K \subseteq \mathbb{R}_+$. For every $p \in (0, \infty]$ and for every compact subset $K$ of $F'$, we then define

$$B^p_K(\mathcal{N}) := \{ f \in \text{Hol}_K(E \times F_\mathbb{C}) : f_0 \in L^p(\mathcal{N}) \},$$

endowed with the quasi-norm $f \mapsto ||f_0||_{L^p(\mathcal{N})}$. This definition agrees with the one given in [10] when $K$ is convex (which is the only case considered therein). As before, $B^p_K(\mathcal{N}) = B^p_{K \cap \mathbb{R}^+}(\mathcal{N})$, so that we may always assume that $K \subseteq \mathbb{R}_+$.

We now illustrate a few examples of our setting.

**Examples 3.1.** First of all, the classical Bernstein spaces $B_k$ considered in the Introduction section correspond to the case $n = 0$ and $m = 1$, so that $E = \{ 0 \}$ and $\mathcal{N} = \mathbb{R}$, and $K = [-\kappa, \kappa]$. If $K$ is a convex compact subset of $\mathbb{R}$, then the space $B_k(\mathbb{R})$ is isomorphic to $B_k$ via the multiplication by a suitable character $e^{i\alpha z}$. For this classical case, see, for example, [21, 42]. The case of a general compact subset $K$ was considered in [20], where necessary conditions for a sequence to be sampling were established.

For the reader’s convenience, we recall that when $K$ is an interval, $K = [a, b]$ for some $a \leq b$, then the support function $H_K$ is given by

$$H_K(h) = \begin{cases} -ah & \text{if } h \geq 0 \\ -bh & \text{if } h \leq 0 \end{cases}$$

for every $h \in \mathbb{R}$. In particular, $H_{[-\kappa, \kappa]}(h) = \kappa |h|$ for every $\kappa \geq 0$ and for every $h \in \mathbb{R}$.

**Examples 3.2.** The case in which $n = 0$, $m > 1$, and $K$ is a compact parallelotope was studied in [39]. More generally, let $K \subseteq F$ be a convex symmetric compact set containing the origin. Its polar set

$$K^\circ := \{ \lambda \in F' : \forall x \in K \ | \langle x, \lambda \rangle | \leq 1 \}$$

defines a norm (the Minkowski functional) given by

$$|x|_{K^\circ} := \sup_{x \in K} |\langle x, \lambda \rangle|,$$
and this norm coincides with the support function of $K$, defined in (2). It is worthwhile mentioning that the space $B_K^2(F)$ is precisely the space of the holomorphic extensions of the Euclidean Fourier transforms of $L^2(K)$. More precisely, we have the following:

1. If $g \in L^2(K)$, then the function $f$ on $F_C$ defined by
   
   \[ f(z) = \frac{1}{(2\pi)^m} \int_K e^{i(z,\xi)} g(\xi) \, d\xi, \]

   for every $z \in F_C$, belongs to $B_K^2(F)$ and satisfies $\|f\|_{B_K^2} = \frac{1}{(2\pi)^{m/2}} \|g\|_{L^2}$.

2. If $f \in B_K^2(F)$, then $\text{Supp}(\hat{f}_0) \subseteq K$ and $\|f\|_{B_K^2} = \frac{1}{(2\pi)^{m/2}} \|\hat{f}_0\|_{L^2}$;

see [41, Chapter III.4]).

In all these cases, $\mathcal{N}$ is abelian and the Fourier transform is the classical Euclidean Fourier transform.

Examples 3.3. If $n \geq 1$ and $m = 1$, then $\rho^{-1}(0)$ is the topological boundary of the Siegel upper half-space \( \{(\zeta, z) \in \mathbb{C}^n \times \mathbb{C} : \text{Im} z > |\zeta|^2\} \). In this case, $\mathcal{N}$ is the $n$-dimensional Heisenberg group $H_n$. The spaces $B_K^p(H_n)$, with $K = [0, \tau]$ were introduced and studied in [30], where the authors established a sharp sampling theorem for a class of sequences on $\rho^{-1}(0)$.

In the general case, the spaces $B^p_K(\mathcal{N})$, when $K$ is convex, were introduced and studied in [10].

We observe explicitly that $\rho^{-1}(0)$ is totally real if and only if $n = 0$, and a hypersurface if and only if $m = 1$. Further, $\mathcal{N}$ is abelian if and only if $n = 0$.

4 | CARLESON MEASURES

In this section, we consider the $q$-Carleson measures for the Bernstein spaces $B^p_K(\mathcal{N})$, that is, the (positive) Radon measures $\mu$ on $\mathcal{N}$ such that $B^p_K(\mathcal{N})$ embeds as a closed subspace of $L^q(\mu)$.

When the (positive Radon) measure $\mu$ is supported in a band, that is, when $\rho(\text{Supp} \mu)$ is bounded, it turns out that $\mu$ is $p$-Carleson for $B^p_K(\mathcal{N})$ if (and only if) it is controlled “in mean” by the Lebesgue measure on $E \times F_C$. In other words, once a left-invariant distance $d$ on $E \times F_C$ is chosen, the ratio $\mu(B((\zeta, z), R))/H^{2n+2m}(B((\zeta, z), R))$ should be uniformly bounded as $(\zeta, z)$ runs through $E \times F_C$ for some (hence every) $R > 0$. Since $H^{2n+2m}(B((\zeta, z), R))$ does not depend on $(\zeta, z)$ by left invariance, it is then sufficient to require the uniform boundedness of $\mu(B((\zeta, z), R))$.

The reason why this happens lies in the fact that the Plancherel–Pólya inequalities (cf. [10, Theorem 1.7]) and the subharmonicity properties of the powers of the moduli of holomorphic functions imply that the quasi-norm on $B^p_K(\mathcal{N})$ is equivalent to the quasi-norm

\[ f \mapsto \left( \int_{\rho^{-1}(B_2(0,R_2))} \max_{B((\zeta, z), R_1)} |f|^p \, d(\zeta, z) \right)^{1/p} \]

for every $R_1, R_2 > 0$. Choosing $R_2$ sufficiently large and applying a suitable discretization by means of a convenient notion of lattices, sufficiency is proved. The same argument also provides sufficient conditions for $q$-Carleson measures for $B^p_K(\mathcal{N})$. Necessity is then proved by means of suitable left-invariant families of functions, and the argument works also when $p \leq q$. The case $p > q$ needs a more delicate treatment by means of Khintchine’s inequalities.

The same methods may be then used to deal with general positive Radon measures on $E \times F_C$, but the sufficient and the necessary conditions we get are different, and neither of them agrees with the known characterization of Carleson measures in the one-dimensional case (cf. Proposition 2.1).

Let us then proceed with a convenient definition of the left-invariant distances in $\mathcal{N}$ and $E \times F_C$.

Definition 4.1. Define $\vartheta := \frac{1}{2}$ if $n > 0$, and $\vartheta := 1$ if $n = 0$. 


We denote by $d_{\mathcal{N}}$ a left-invariant $\theta$-homogeneous distance on $\mathcal{N}$, with respect to the dilations given by $t \cdot (\zeta, x) := (t^{1/2}\zeta, tx)$, so that $d_{\mathcal{N}}(t \cdot (\zeta, x), t \cdot (\zeta', x')) = t^\theta d_{\mathcal{N}}((0, 0), (\zeta, x)^{-1}(\zeta', x'))$ for every $(\zeta, x), (\zeta', x') \in \mathcal{N}$ and for every $t > 0$.

We endow $E \times F_C$ with the distance
\[
d : ((\zeta, z), (\zeta', z')) \mapsto \max(d_{\mathcal{N}}((\zeta, \Re z), (\zeta', \Re z')), |\rho(\zeta, z) - \rho(\zeta', z')|^{\theta}),
\]
which is left-invariant and $\theta$-homogeneous with respect to the dilations given by $t \cdot (\zeta, z) := (t^{1/2}\zeta, tz)$. We denote by $\mathcal{M}_+(E \times F_C)$ the set of positive Radon measures on $E \times F_C$.

When $n = 0$, one may define $d_{\mathcal{N}}$ as the Euclidean distance on $F$, so that $d((z, z')) = \max(|\Re (z - z')|, |\Im (z - z')|)$ for every $(z, z') \in F$. In general, $d_{\mathcal{N}}((0, 0), (\zeta, z))$ is controlled from above and below by $|\zeta|^\theta + |z|^\theta$. The reason why we introduced $\theta$ lies in the fact that, when $n = 0$, it seems reasonable to allow the Euclidean distance (at least on $\mathcal{N}$), whereas, when $n > 0$, no 1-homogeneous left-invariant distance (with respect to the above dilations) exists, since it would induce a nonzero subadditive 2-homogeneous function on $C(\zeta, 0)$ for every $\zeta \in E$ (and such a function cannot exist unless $\zeta = 0$).

**Definition 4.2.** Given $\delta > 0$ and $R > 1$, by a $(\delta, R)$-lattice on a metric space $X$, we shall mean a family $(x_j)$ of elements of $X$ such that the balls $B_X(\delta)$ are pairwise disjoint, while the balls $B_X(R\delta)$ cover $X$.

By a restricted $(\delta, R)$-lattice on $E \times F_C$ we shall mean a family $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ of elements of $E \times F_C$ such that the balls $B((\zeta_{j,k}, z_{j,k}), \delta)$ are pairwise disjoint, such that the balls $B((\zeta_{j,k}, z_{j,k}), R\delta)$ cover $E \times F_C$, and such that $\rho(\zeta_{j,k}, z_{j,k})$ does not depend on $j \in J$ for every $k \in K$.

If we define $h_k := \rho(\zeta_{j,k}, z_{j,k})$, then the balls $B_F(h_k, \delta^{1/\theta})$ are pairwise disjoint and the balls $\overline{B}_F(h_k, (R\delta)^{1/\theta})$ cover $F$ by our choice of $d$. In other words, $(h_k)$ is a $(\delta^{1/\theta}, R^{1/\theta})$-lattice on $F$, endowed with the Euclidean distance.

Arguing as in the proof of [9, Lemma 2.55], one may prove that there are restricted $(\delta, 2)$-lattices on $E \times F_C$ for every $\delta > 0$. Indeed, one may start with a $(\delta^{1/\theta}, 2)$-lattice $(h_k)$ in $F$, that is, a sequence with $|h_k - h_{k'}| \leq 2\delta^{1/\theta}$ for every $k \neq k'$, and with a $(\delta, 2)$-lattice $(\zeta_j, x_j)$ on $\mathcal{N}$ (constructed analogously), and then define $(\zeta_{j,k}, z_{j,k}) := (\zeta_j, x_j + i\Phi(\zeta_j) + i h_k)$.

**Definition 4.3.** For every $\mu \in \mathcal{M}_+(E \times F_C)$ and for every $R > 0$, we define
\[
M_R(\mu) : E \times F_C \ni (\zeta, z) \mapsto \mu(B((\zeta, z), R)) \in \mathbb{R}_+.
\]

For notational convenience, we also define $M_{K,R}(\mu) = M_R(e^{H_K \circ \rho} \cdot \mu)$ for every compact subset $K$ of $F'$, so that $M_R(\mu) = M_{\{0\}, R}$.

We define
\[
L^{p,q}(E \times F_C) := \left\{ f : E \times F_C \rightarrow \mathbb{C} : f \text{ is measurable, } \left\| h \mapsto \|f_h\|_{L^p(\mathcal{N})}\right\|_{L^q(F)} < \infty \right\},
\]
and we define $L^{p,q}_0(E \times F_C)$ as the closure of the set of measurable step functions in $L^{p,q}(E \times F_C)$. We define $\ell^{p,q}(J, K)$ and $\ell^{p,q}_0(J, K)$ analogously, for any two sets $J$ and $K$.

**Lemma 4.4.** Take a compact subset $K$ of $F'$, $R > 0$ and $\mu \in \mathcal{M}_+(\mathcal{N})$. Then, $M_{K,R}(\mu)$ is upper semicontinuous.

**Proof.** Just observe that there is a sequence $(\varphi_j)$ of elements of $C_c(E \times F_C)$ such that $\chi_{B((0,0), R)} \leq \varphi_j \leq \chi_{B((0,0), (1+2^{-j})R)}$ for every $j \in \mathbb{N}$, so that $M_{K,R}(\mu)$ is the pointwise infimum of the continuous functions
\[
(\zeta, z) \mapsto \int_{E \times F_C} \varphi_j((\zeta, z)^{-1}(\zeta', z')) e^{H_K(\rho(\zeta', z'))} \, d\mu(\zeta', z')
\]
as $j$ runs through $\mathbb{N}$. \qed

In the following technical lemma, we show that we may discretize some integral conditions on $M_{K,R}$ (and that these conditions do not depend on $R > 0$). This is the first step to prove Proposition 4.7.
Lemma 4.5. Fix a compact subset $K$ of $F'$ and $p, q \in (0, \infty)$. Then, for every $\delta, R' > 0$ and for every $R > 1$ there is a constant $C > 0$ such that

$$
\frac{1}{C} \left\| M_{K, R'}(\mu) \right\|_{L^p_q(\mathbb{E} \times F_C)} \leq \left\| M_{K, R\delta}(\mu(\zeta_{j,k}, z_{j,k})) \right\|_{L^p_q(J, K)} \leq C \left\| M_{K, R'}(\mu) \right\|_{L^p_q(\mathbb{E} \times F_C)}
$$

for every $\mu \in \mathcal{M}_+(E \times F_C)$ and for every restricted $(\delta, R)$-lattice $(\xi_{j,k}, z_{j,k})_{j,k \in I, K}$ on $E \times F_C$.

In addition, $M_{K, R'}(\mu) \in L^p_q(E \times F_C)$ if and only if $M_{K, R\delta}(\mu(\zeta_{j,k}, z_{j,k})) \in \ell^p_q(J, K)$.

The proof is analogous to those of [31, Lemmas 2.9 and 2.12] and [8, Lemma 5.1].

Proof. Since $M_{K, R''}(\mu) = M_{R''}(e^{H_K \circ \rho \mu})$ for every $R'' > 0$, we may assume that $K = \{0\}$.

Step I. Let us first prove that for every $R'' > R'$, there is a constant $C_1 > 0$ such that

$$
\left\| M_{R'}(\mu) \right\|_{L^p_q(E \times F_C)} \leq \left\| M_{R''}(\mu) \right\|_{L^p_q(E \times F_C)} \leq C_1 \left\| M_{R'}(\mu) \right\|_{L^p_q(E \times F_C)}
$$

for every $\mu \in \mathcal{M}_+(E \times F_C)$. The first inequality is obvious. Then, observe that, since $d$ is homogeneous and left-invariant, $B((\zeta, z), R'')$ is compact for every $(\zeta, z) \in E \times F_C$. Then, there are $(\zeta_1, z_1), \ldots, (\zeta_k, z_k) \in E \times F_C$ such that $B((0, 0), R'' \subseteq \bigcup_{j=1}^k B((\zeta_j, z_j), R')$, so that, by left invariance,

$$
M_{R''}(\mu) \leq \sum_{j=1}^k M_{R'}(\mu(\cdot (\zeta_j, z_j)) \in L^p_q(E \times F_C)
$$

for every $\mu \in \mathcal{M}_+(E \times F_C)$. Hence,

$$
\left\| M_{R''}(\mu) \right\|_{L^p_q(E \times F_C)} \leq k^{\max(1/p, 1/q, 1)} \left\| M_{R'}(\mu) \right\|_{L^p_q(E \times F_C)}
$$

for every $\mu \in \mathcal{M}_+(E \times F_C)$.

Step II. Let $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ be a restricted $(\delta, R)$-lattice on $E \times F_C$. Observe that

$$
M_{R\delta}(\mu, \xi_{j,k}, z_{j,k}) \leq M_{R+1\delta}(\mu)(\zeta, z)
$$

for every $(\zeta, z) \in B((\zeta_{j,k}, z_{j,k}), \delta)$, for every $j \in J$, and for every $k \in K$, so that

$$
\left\| M_{R\delta}(\mu(\xi_{j,k}, z_{j,k})) \right\|_{L^p_q(J, K)} \leq H^{2n+m}(B_{F}(h_k, \delta))^{-1/p} \left\| [M_{R+1\delta}(\mu)]_h \right\|_{L^p_q(\mathbb{E})}
$$

for every $k \in K$ and for every $h \in B_{F}(h_k, \delta)$, since the balls $B((\zeta_{j,k}, z_{j,k}), \delta)$ are pairwise disjoint. Then,

$$
\left\| M_{R\delta}(\mu(\xi_{j,k}, z_{j,k})) \right\|_{L^p_q(J, K)} \leq H^{2n+m}(B_{F}(h_k, \delta))^{-1/p} H^{m}(B_{F}(0, \delta))^{-1/q} \left\| M_{R+1\delta}(\mu) \right\|_{L^p_q(E \times F_C)}.
$$

By Step I, this proves that there is a constant $C_2 > 0$ such that

$$
\left\| M_{R\delta}(\mu(\xi_{j,k}, z_{j,k})) \right\|_{L^p_q(J, K)} \leq C_2 \left\| M_{R'}(\mu) \right\|_{L^p_q(E \times F_C)}
$$

for every $\mu \in \mathcal{M}_+(E \times F_C)$.

Step III. Take a restricted $(\delta, R)$-lattice $(\xi_{j,k}, z_{j,k})_{j \in J, k \in K}$ on $E \times F_C$. Let us first prove that there is a constant $C_3 > 0$ such that

$$
\left\| M_{R+1\delta}(\mu(\xi_{j,k}, z_{j,k})) \right\|_{L^p_q(J, K)} \leq C_3 \left\| M_{R\delta}(\mu(\xi_{j,k}, z_{j,k})) \right\|_{L^p_q(J, K)}
$$

for every $\mu \in \mathcal{M}_+(E \times F_C)$. Indeed, for every $(j, k) \in J \times K$, define

$$
J_{j,k} := \left\{ (j', k') \in J \times K : \overline{B}_{F}(\zeta_{j',k'}, z_{j',k'}), R\delta) \cap \overline{B}(\zeta_{j,k}, z_{j,k}), (R + 1)\delta) \neq \emptyset \right\},
$$
and observe that there is \( N \in \mathbb{N} \), depending only on \( \delta \) and \( R \), such that \( \text{Card}(J_{j,k}) \leq N \) for every \((j,k) \in J \times K\), and such that each \((j',k') \in J \times K\) is contained in at most \( N \) of the sets \( J_{j,k} \) (cf., e.g., [9, Proposition 2.56 and its proof]). For every \( k \in K \), define \( h_k := \rho(\zeta_{j,k}, z_{j,k}) \) for some/every \( j \in J \), and set \( K_k := \left\{ k' \in K : \overline{B}_F(h_k', R) \cap \overline{B}_F(h_k, (R + 1)\delta) \neq \emptyset \right\} \). Observe that we may assume that \( \text{Card}(K_k) \leq N \) for every \( k \in K \), and that each \( k' \in K \) is contained in at most \( N \) of the sets \( K_k \) (cf., again, [9, Proposition 2.56 and its proof]).

Then,

\[
M_{(R+1)\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \leq \sum_{(j',k') \in J_{j,k}} M_{R\delta}(\mu)(\zeta_{j',k'}, z_{j',k'})
\]

so that

\[
\left\| M_{(R+1)\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^p(J)} \leq N^{\max(1,1/p)} \left\| M_{R\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^p(J)},
\]

whence

\[
\left\| M_{(R+1)\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^p q(J,K)} \leq N^{\max(1,1/p)+\max(1,1/q)} \left\| M_{R\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^p q(J,K)}
\]

and our assertion.

Then, observe that

\[
M_\delta(\mu)(\zeta, z) \leq M_{(R+1)\delta}(\mu)(\zeta_{j,k}, z_{j,k})
\]

for every \((\zeta, x) \in \overline{B}((\zeta_{j,k}, z_{j,k}), R\delta)\) and for every \((j,k) \in J \times K\), so that

\[
\left\| M_\delta(\mu) \right\|_{L^p(\mu)} \leq H^{2m+(m(B_{\mathcal{K}}((0,0), R\delta)))^{1/p} H^m(B_F(0, R\delta))^{1/q}} \left\| M_{(R+1)\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^p q(J,K)},
\]

By Step I, this proves that there is a constant \( C_4 > 0 \) such that

\[
\left\| M_{R^1}(\mu) \right\|_{L^p(\mu)} \leq C_4 \left\| M_{R\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^p q(J,K)}
\]

for every \( \mu \in \mathcal{M}_+(E \times F_C) \). This completes the proof of the first assertion.

Step IV. The second assertion follows from the first one, approximating \( M_{R_1}(\mu) \) with \( M_{R_1}(\chi_{B((0,0), R_2)} \cdot \mu) \), for \( R_2 \to +\infty \), for every \( R_1 > 0 \).

\[\square\]

**Definition 4.6.** For every \( p \in (0, \infty] \), we define \( p' := \max(1, p) \), so that \( p' = \infty \) if \( p \leq 1 \), while \( \frac{1}{p} + \frac{1}{p'} = 1 \) otherwise.

The following result provides some quantitative sufficient conditions for a measure to be \( q \)-Carleson for \( \mathcal{M}_p^q(\mathcal{K}) \). The statement is somewhat technical since we shall need some rather precise estimates in some subsequent proofs (cf. Proposition 5.6 and Lemma 5.9). Notice that these sufficient conditions cannot be necessary (in general) if \( \mu \) is not supported in a band, as Proposition 2.1 shows.

**Proposition 4.7.** Take a compact subset \( K \) of \( \Lambda^0 \), \( p, q \in (0, \infty] \), with \( q < \infty \), and \( R > 0 \). Then, there is a constant \( C > 0 \) such that

\[
\left\| f \right\|_{L^q(\mu)} \leq C \left\| M_{qK,1}(\mu) \right\|_{L^{p/q'}(\mathcal{K}(\mathcal{M}_p^q(\mathcal{F}_C)))} \sup_{h \in F} e^{-H_{\mathcal{K}}(h)} \left\| (\chi_{B((\operatorname{Supp} \mu), R)} f)_{\mathcal{R}}(h) \right\|_{L^p(\mathcal{N})}
\]

for every \( \mu \in \mathcal{M}_+(E \times F_C) \).

In particular, if \( M_{qK,1}(\mu) \in L^{p/q'}(E \times F_C) \) (resp. \( M_{qK,1}(\mu) \in L^{p/q'}(E \times F_C) \)), then \( \mu \) is a (resp. compact) \( q \)-Carleson measure for \( \mathcal{B}_K(\mathcal{N}) \).

Here, we set \( B(\operatorname{Supp} \mu, R) := \bigcup_{(\zeta, x) \in \operatorname{Supp} \mu} \overline{B}((\zeta, x), R) = (\operatorname{Supp} \mu) B((0,0), R) \).
The proof is based on standard techniques, cf., for example, [8, 25, 31].

**Proof.** We may assume that $K$ is convex and nonempty.

**Step I.** Let $(ξ_{j,k}, z_{j,k})_{j \in J, k \in K}$ be a restricted $(1/4, 4)$-lattice on $E \times F_C$, and define

$$(S_+ f)_{j,k} := \max_{\mathcal{B}(ξ_{j,k}, x_{j,k}, 1)} e^{-H_{K_0, \rho} X_{\text{Supp} \mu} |f|} \in \mathbb{R}_+$$

for every $f \in B^p_K(\mathcal{N})$. Then,

$$\|f\|_{L^q(\mu)} \leq \left\| (S_+ f)_{j,k} M_{qK, 1}(\mu) (ξ_{j,k}, z_{j,k})^{1/q} \right\|_{\ell^q(J \times K)}$$

$$\leq \left\| S_+ f \right\|_{\ell^{p, \infty}(J, K)} \left\| M_{qK, 1}(\mu) (ξ_{j,k}, z_{j,k}) \right\|_{\ell^q(J)}^{1/q}$$

where $s := q \left( \frac{p}{q} \right)' = \frac{pq}{(p-q)+}$. Thanks to Lemma 4.5, in order to prove the first assertion, it will suffice to prove that there is a constant $C_1 > 0$ such that

$$\left\| (S_+ f)_{j,k} M_{qK, 1}(\mu) (ξ_{j,k}, z_{j,k}) \right\|_{\ell^q(J)} \leq C_1 \left\| \chi_{\text{Supp} \mu} f \right\|_{L^p(\mathcal{N})}$$

for every $f \in B^p_K(\mathcal{N})$.

Observe first that

$$\frac{1}{C_2} e^{-H_K(h')} \leq e^{-H_K(h)} \leq C_2 e^{-H_K(h')}$$

for every $h, h' \in F$ such that $|h - h'| \leq 1 + R^{1/\theta}$, where $C_2 := \sup_{|h| \leq 1 + R^{1/\theta}} e^{H_K(h)}$. Then, observe that, by the subharmonicity of $|f|^{\min(1, p)}$,

$$|f(0, 0)|^{\min(1, p)} \leq \frac{1}{H^{2n+2m}(B_{E \times F_C}((0, 0), R'))} \int_{B_{E \times F_C}((0, 0), R')} |f|^{\min(1, p)} \, dH^{2n+2m}$$

for every $R' > 0$ and for every $f \in \text{Hol}(E \times F_C)$, where $B_{E \times F_C}((0, 0), R')$ denotes the Euclidean ball of center $(0, 0)$ and radius $R'$ in $E \times F_C$ (cf., e.g., [19, Theorem 2.14 and Corollary 2.15]). Hence, choosing $R' > 0$ so that $B_{E \times F_C}((0, 0), R') \subseteq B((0, 0), R)$, and using the left invariance of $H^{2n+2m}$, we infer that there is a constant $C_3 > 0$ such that

$$|f(ξ, z)|^{\min(1, p)} \leq C_3 \int_{B_0((ξ, z), R)} |f|^{\min(1, p)} \, dH^{2n+2m} = C_3 \int_{B_0((ξ, z), R) / \mathcal{N}} \int_{B(ξ, Re z)} |f|^{\min(1, p)} \, dH^{2n+m} \, d\mathcal{H}$$

for every $(ξ, z) \in E \times F_C$ and for every $f \in \text{Hol}(E \times F_C)$. Hence, there is a constant $C_4 > 0$ such that

$$(S_+ f)_{j,k}^{\min(1, p)} \leq C_4 e^{-H_K(h_k)} \int_{B_{E \times F_C}((ξ_{j,k}, z_{j,k}), 1 + R)} \left\| \chi_{\text{Supp} \mu} f \right\|_{L^p(\mathcal{N})}^{\min(1, p)} \, dH^{2n+m}$$

for every $(j, k) \in J \times K$. Using the finite intersection property of the balls $B_{\mathcal{N}}((ξ_{j,k}, z_{j,k}), 1 + R)$ (cf., e.g., [9, Proposition 2.56 and its proof]), we then infer that there is a constant $C_5 > 0$ such that

$$\left\| (S_+ f)_{j,k} \right\|_{\ell^p(U)} \leq C_5 e^{-H_K(h_k)} \int_{B_{E \times F_C}((ξ_{j,k}, z_{j,k}), 1 + R)} \left\| \chi_{\text{Supp} \mu} f \right\|_{L^p(\mathcal{N})} \, d\mathcal{H}$$

for every $k \in K$, whence the first assertion.
Step II. Now, assume that $M_{qK,1}(\mu) \in L^{p/q}(\mathcal{N})$. By means of Step I and [10, Theorem 1.7], we see that
\[ \|f\|_{L^q(\mu)} \leq C \left\| M_{qK,1}(\mu) \right\|^{1/q}_{L^{p/q}(\mathcal{N})} \sup_{h \in F} e^{-H_K(h)} \|f_h\|_{L^p(\mathcal{N})} = C \left\| M_{qK,1}(\mu) \right\|^{1/q}_{L^{p/q}(\mathcal{N})} \|f_0\|_{L^p(\mathcal{N})} \]
for every $f \in B^p_K(\mathcal{N})$, so that $\mu$ is a $q$-Carleson measure for $B^p_K(\mathcal{N})$.

Step III. Finally, assume that $M_{qK,1}(\mu) \in L^{p/q}(\mathcal{N})$. Then, Step II shows that the mappings
\[ t_k : B^p_K(\mathcal{N}) \ni f \mapsto \chi_{B((0,0),k+1)} f \in L^q(\mu) \]
converge to the inclusion mapping $B^p_K(\mathcal{N}) \to L^q(\mu)$ in $L^q(B^p_K(\mathcal{N}) ; L^q(\mu))$, so that it will suffice to show that the $t_k$ are compact. Now, observe that $B^p_K(\mathcal{N})$ embeds continuously into $\text{Hol}(E \times F_C)$ (cf. [10, Corollary 3.3]), and that the mapping $\text{Hol}(E \times F_C) \ni f \mapsto \chi_{B((0,0),k+1)} f$ is clearly compact for every $k \in \mathbb{N}$. The assertion follows. 

We now pass to the necessary conditions and the characterization of Carleson measures supported in a band. First, we provide some general necessary conditions. Again, these conditions cannot be sufficient in general, as the one-dimensional cases shows (cf. Proposition 2.1).

**Proposition 4.8.** Take a compact subset $K$ of $\Lambda_+$, $p, q \in (0, \infty)$, with $q < \infty$, and $\mu \in \mathcal{M}_+(E \times F_C)$. Assume that $B^p_K(\mathcal{N}) \neq \{0\}$. Then, there is $R > 0$ such that, if $\mu$ is a $q$-Carleson measure for $B^p_K(\mathcal{N})$, then the mapping
\[ (\zeta, z) \mapsto e^{-qH_K(-\rho(\zeta,z))} M_R(\mu)(\zeta, z) \]
is bounded.

If, in addition, $B^p_K(\mathcal{N}) \neq \{0\}$ for some $r < \infty$ and $\mu$ is a compact $q$-Carleson measure for $B^p_K(\mathcal{N})$, then the above function vanishes at the point at infinity.

Notice that, by [10, Proposition 3.4], if $K$ is convex and $p < \infty$, then $B^p_K(\mathcal{N}) \neq \{0\}$ if and only if $K$ has a nonempty interior. In addition, $B^p_K(\mathcal{N}) \neq \{0\}$ if and only if $K \neq \emptyset$. Thus, when $K$ is convex, the assumption for the second part of the statement means that $K$ has a nonempty interior; this assumption is always empty unless $p = \infty$.

The proof is analogous to those of [31, Theorem 3.5] and [8, Proposition 5.3].

**Proof.** Step I. By assumption, there is a constant $C > 0$ such that
\[ \|f\|_{L^q(\mu)} \leq C \|f\|_{B^p_K(\mathcal{N})} \]
for every $f \in B^p_K(\mathcal{N})$. Take $f \in B^p_K(\mathcal{N})$ with $\|f_0\|_{L^p(\mathcal{N})} = 1$, and observe that, by translation invariance, we may assume that there is $R > 0$ such that $f(\zeta, z) \neq 0$ for every $(\zeta, z) \in B((0,0), R)$. Since $B((0,0), R)$ is compact, there is a constant $C' > 1$ such that
\[ \frac{1}{C'} \leq |f(\zeta, z)| \leq C' \]
for every $(\zeta, z) \in \overline{B}((0,0), R)$. Define $f^{(\zeta,z)} := f((\zeta,z)^{-1})$ for every $(\zeta, z) \in E \times F_C$, so that $f^{(\zeta,z)} \in B^p_K(\mathcal{N})$, $\|f^{(\zeta,z)}\|_{B^p_K(\mathcal{N})} = \|f^{(\zeta,z)}\|_{L^p(\mathcal{N})} \leq e^{H_K(-\rho(\zeta,z))}$ (cf. [10, Theorem 1.7]), and
\[ \frac{1}{C'} \leq |f^{(\zeta,z)}(\zeta', z')| \leq C' \]
for every $(\zeta', z') \in B((\zeta, z), R)$, then
\[ e^{H_K(-\rho(\zeta,z))} C \geq \|f^{(\zeta,z)}\|_{L^q(\mu)} \geq \frac{1}{C'} M_R(\mu)(\zeta, z)^{1/q} \]
for every $(\zeta, z) \in E \times F_C$, whence the first assertion.
Step II. Assume, now, that \( B^p_K(\mathcal{N}) \neq \{0\} \) for some \( r < \infty \) and that \( \mu \) is a compact \( q \)-Carleson measure for \( B^p_K(\mathcal{N}) \). Notice that we may take \( r = p \) if \( p < \infty \). Take a nonzero \( f \in B^p_K(\mathcal{N}) \subseteq B^p_K(\mathcal{N}) \). We may then define \( R, C' \) and \( f(\zeta, z) \), for every \((\zeta, z) \in E \times F_C\), as in Step I. Then, fix \( h \in F \) and let \( \mathcal{U} \) be an ultralimit on \( E \times F_C \), which is finer than the filter \( \{ (\zeta, z) \to \infty \} \), and observe that, by the compactness of the inclusions \( B^p_K(\mathcal{N}) \subseteq L^q(\mu) \) and \( B^p_K(\mathcal{N}) \subseteq \text{Hol}(E \times F_C) \), \( f(\zeta, z) \) has limits \( g_0 \) and \( g_1 \) in \( L^q(\mu) \) and in \( \text{Hol}(E \times F_C) \) along \( \mathcal{U} \), respectively. Since convergence in \( L^q(\mu) \) implies convergence in measure, it is clear that \( g_0 = g_1 \) \( \mu \)-almost everywhere. In addition, observe that

\[
\lim_{(\zeta, z) \to \infty} e^{-H^p_K(\rho(\zeta, z))} f(\zeta, z) = 0
\]

pointwise, by [10, Theorem 3.2], so that \( g_1 = 0 \). Hence, the arguments of Step I imply that

\[
\lim_{(\zeta, z) \to \infty} e^{-qH^p_K(\rho(\zeta, z))} M_R(\mu)(\zeta, z) = 0.
\]

By the arbitrariness of \( \mathcal{U} \), this implies that \( e^{-qH^p_K(\rho(\cdot))} M_R(\mu) \in L_0^\infty(E \times F_C) \). \( \square \)

**Corollary 4.9.** Take a compact subset \( K \) of \( \overline{\mathcal{N}}_+ \), \( p, q \in (0, \infty) \) with \( p \leq q \), and \( \mu \in \mathcal{M}_+(E \times F_C) \) such that \( \rho(\text{Supp} \mu) \) is bounded. Assume that \( B^p_K(\mathcal{N}) \neq \{0\} \). Then, \( \mu \) is a (resp. compact) \( q \)-Carleson measure for \( B^p_K(\mathcal{N}) \) if and only if \( M_1(\mu) \in L^\infty(E \times F_C) \) (resp. \( M_1(\mu) \in L_{0}^\infty(E \times F_C) \)).

**Proof.** The assertion follows from Lemma 4.5 and Propositions 4.7 and 4.8. \( \square \)

From now on, we shall restrict ourselves to measures supported in a set of the form \( \rho^{-1}(B^p(K, R)) \) for some \( R > 0 \), since there are no simple criteria to determine general Carleson measures as in the one-dimensional case. More precisely, assume that \( \mathcal{N} = F \) and that \( K \) is polyhedral for simplicity. We may then assume that \( 1 \) is an extreme point of \( K \). Let \( C \) be the corresponding tangent cone (i.e., \( \mathbb{R}_+ K \)), and \( \Omega \) the interior of its polar (which is the set where \( H_K = 1 \)). Observe that, if \( \mu \) is the vague limit on \( F + i \Omega \) of a sequence of measures of the form \( r_k^{q/m}(p(r_k \cdot), \mu_k) \), with \( r_k \to 0^+ \) and the \( \mu_k \) uniformly \( q \)-Carleson for \( B^p_K(F) \), then \( \mu \) is \( q \)-Carleson for \( H^p(F + i \Omega) \). Conversely, if \( \mu \) is concentrated in \( F + i \Omega \) and \( q \)-Carleson for \( H^p(F + i \Omega) \), then it is \( q \)-Carleson for \( B^p_K(F) \). Thus, the problem of the determining Carleson measures for \( B^p_K(F) \) is closely related to the problem of determining the Carleson measures for \( H^p(F + i \Omega) \), where \( \Omega \) runs through the set of the interiors of the polars of the tangent cones to \( K \) at its extremal points.

Consequently, already when \( K \) is a parallelogram, determining \( p \)-Carleson measures for \( B^p_K(F) \) would require imposing a condition of the form \( \int_{\gamma(U)} e^{pH^m(\gamma(x))} \mu(x) \leq C \mathrm{H}^m(U) \) for every connected open subset \( U \) of \( F \), where \( \gamma(U) \) is a suitable “tent” on \( U \), adapted to \( K \). The situation for general polyhedral cones is even less clear, whereas for general cones even this loose connection with Hardy spaces is no longer applicable.

In the following result, we shall therefore content ourselves with a complete characterization of the \( q \)-Carleson measures for \( B^p_K(\mathcal{N}) \), which are supported in a band.

**Theorem 4.10.** Take a compact subset \( K \) of \( \overline{\mathcal{N}}_+ \) with a nonempty interior, \( p, q \in (0, \infty) \) with \( q < \infty \), and \( \mu \in \mathcal{M}_+(E \times F_C) \) such that \( \rho(\text{Supp} \mu) \) is bounded. Then, \( \mu \) is a (resp. compact) \( q \)-Carleson measure for \( B^p_K(\mathcal{N}) \) if and only if \( M_1(\mu) \in L^{(p/q)'}(E \times F_C) \) (resp. \( M_1(\mu) \in L_0^{(p/q)'}(E \times F_C) \)).

The proof is based on a technique developed in [28], and then applied also in [8, 31].

**Proof.** One implication follows from Proposition 4.7. In addition, by Corollary 4.9, we may reduce to the case in which \( q < p \), in which case \( L_0^{(p/q)'}(E \times F_C) = L^{(p/q)'}(E \times F_C) \). Then, assume that \( \mu \) is a \( q \)-Carleson measure for \( B^p_K(\mathcal{N}) \), and let us prove that \( M_1(\mu) \in L^{(p/q)'}(E \times F_C) \).

Take a compact convex subset \( K' \) with a nonempty interior contained in \( K \cap \Lambda_+ \), and fix a nonzero \( \varphi \in B^{(1,p)}_{\min}(F) \) (cf. [10, Proposition 3.4]). Observe that the Plancherel–Pólya inequalities (cf., e.g., [10, Theorem 1.7]) show that

\[
\|\varphi_h\|_{L^{p/(1,p)}(F)} \leq e^{H^p_K(h)} \|\varphi_0\|_{L^{p/(1,p)}(F)}.
\]
In particular, if we define \( f : E \times F \rightarrow \mathbb{C} \) by \( (\zeta, z) \mapsto \varphi(z) \in \mathbb{C} \), then \( f \in \mathcal{B}_{K}^{\min(1,p)}(\mathcal{N}) \) (cf., e.g., the proof of [10, Proposition 5.3]). Indeed, \( f_{h}(\zeta, x) = \varphi(x + i\Phi(\zeta) + ih) = \varphi_{h+i\zeta}(x) \) for every \( (\zeta, x) \in \mathcal{N} \), so that

\[
\|f_{h}\|_{L^{\min(1,p)}(\mathcal{N})} \leq \|\varphi_{0}\|_{L^{\min(1,p)}(F)} e^{H_{K}(h)} \|e^{H_{K}(\varphi)}\|_{L^{\min(1,p)}(E)},
\]

for every \( h \in F \). Notice that \( \|e^{H_{K}(\varphi)}\|_{L^{\min(1,p)}(E)} \) is finite since \( K' \subseteq \Lambda^+ \), so that there are \( c, c' > 0 \) such that

\[
H_{K'}(\Phi(\zeta)) \leq -c|||\Phi(\zeta)||| \leq -c' |||\zeta|||^{2}
\]

for every \( \zeta \in \Lambda \). Now, identify \( \mathcal{N} \) with \( E \) and \( \mathcal{N} \) with \( F \) by means of some fixed orthonormal bases, and observe that, by [10, Lemma 6.1], there is a constant \( c'' > 1 \) such that, if \( (\zeta_{j})_{j \in J} \) is a \((\delta, R)\)-lattice in \( E \) and \( (x_{j'})_{j' \in J'} \) is a \( (\delta'/\sqrt{m}, \sqrt{m})\)-lattice in \( F \), then \((\zeta_{j}, x_{j'})_{j, j'}\) is a \((\delta, R/\sqrt{m})\)-lattice in \( \mathcal{N} \).

Observe that, up to a translation of \( \varphi \), we may assume that there is \( R > 0 \) such that \( f(\zeta, z) \neq 0 \) for every \( (\zeta, z) \in B((0, 0), R) \). Then, fix \( \delta > 0 \) so that

\[
\frac{c'' \delta}{\sqrt{n}} \leq \frac{1}{2} R, \quad J = \mathbb{Z}[i]^{n}, \quad J' = \mathbb{Z}^{m}, \quad \zeta_{j} = 2c'' \delta j, \quad x_{j'} = 2(\delta/\sqrt{m})^{1/2} j',
\]

so that \( (\zeta_{j}, x_{j'}) \) is a \((\delta/\sqrt{m}, \sqrt{m})\)-lattice in \( \mathcal{N} \). Notice that we may assume that \( \delta > 0 \) is so small that there is a constant \( C_{1} > 0 \) such that

\[
\frac{1}{C_{1}} \sum_{j'} g(x_{j'}) \leq \|g\|_{L^{1}(F)} \leq C_{1} \sum_{j'} |g(x_{j'})|
\]

for every \( g \in B_{K}^{1}(F) \), thanks to [10, Theorem 1.18].

Then, consider the mapping

\[
\Psi : \lambda \mapsto \sum_{j, j'} \lambda_{j, j'} L(\zeta_{j}, x_{j'} + i\Phi(\zeta_{j})),
\]

and let us prove that \( \Psi \) maps \( \ell^{p}(J \times J') \) into \( B_{K}^{p}(\mathcal{N}) \) continuously. Let us first prove that \( \Psi \) maps \( c_{0}(J \times J') \) (actually, \( \ell^{\infty}(J \times J') \)) into \( B_{K}^{\infty}(\mathcal{N}) \) continuously. Indeed, let \( Q \) be the fundamental cube of the lattice \( (\zeta_{j}) \) (so that \( 0 \in Q \) and the \( \zeta_{j} + Q \) form a partition of \( E \)), and observe that, for every \( h \in F \) and for every \( \lambda \in \mathbb{C}(J \times J') \),

\[
\|\Psi(\lambda)_{h}\|_{L^{\infty}(F)} \leq \|\lambda\|_{\ell^{\infty}(J \times J')} \sup_{j, j'} \|\varphi_{h+i\Phi(\zeta_{j} - \zeta_{j})} + i(h + \Phi(\zeta_{j} - \zeta_{j}))\|
\]

\[
\leq C_{1} \|\lambda\|_{\ell^{\infty}(J \times J')} \sup_{j} \|\varphi_{h+i\Phi(\zeta_{j} - \zeta_{j})}\|_{L^{1}(F)}
\]

\[
\leq C_{1} \|\lambda\|_{\ell^{\infty}(J \times J')} \|\varphi_{0}\|_{L^{1}(F)} \sup_{j} e^{H_{K}(h)}(h + \Phi(\zeta_{j} - \zeta_{j}))
\]

\[
\leq C_{1} \|\lambda\|_{\ell^{\infty}(J \times J')} \|\varphi_{0}\|_{L^{1}(F)} e^{H_{K}(h)} \sum_{\zeta \in Q} \sum_{j} e^{H_{K}(\Phi(\zeta - \zeta_{j}))}.
\]

To conclude the proof of our assertion, observe that there is a constant \( c'' > 0 \) such that

\[
H_{K'}(\Phi(\zeta - \zeta_{j})) \leq -c|||\zeta - \zeta_{j}\||| \leq c'' - (c'/2)|||\zeta|||^{2}
\]

for every \( \zeta \in Q \) and for every \( j \in J \). Then, there is a constant \( C_{2} > 0 \) such that

\[
\|\Psi(\lambda)_{h}\|_{L^{\infty}(\mathcal{N})} \leq C_{2} \|\lambda\|_{\ell^{\infty}(J \times J')} e^{H_{K}(h)}
\]

for every \( \lambda \in \mathbb{C}(J \times J') \), and for every \( h \in F \). This is sufficient to prove that \( \Psi \) maps \( c_{0}(J \times J') \) in \( B_{K}^{\infty}(\mathcal{N}) \). In particular, \( \Psi(\ell^{p}(J \times J')) \subseteq B_{K}^{\infty}(\mathcal{N}) \) so that, in order to prove that \( \Psi \) maps \( \ell^{p}(J \times J') \) continuously into \( B_{K}^{p}(\mathcal{N}) \), it will suffice to prove that the mapping \( \lambda \mapsto \Psi(\lambda)_{0} \) maps \( \ell^{p}(J \times J') \) continuously into \( L^{p}(\mathcal{N}) \).
If $p \leq 1$, then clearly
\[
\|\Psi(\lambda)\|_{L^p(\mathcal{N})} \leq \left\| \lambda_{j,j'} \left\| L(\mathcal{N}, \mathcal{N}) \right\| L^p(\mathcal{N}) \right\|_{L^p(J \times J')} = \|\lambda\|_{L^p(J \times J')} \|f_0\|_{L^p(\mathcal{N})},
\]
whence our claim in this case. The case $p > 1$ then follows by interpolation, since $(L^1(\mathcal{N}), L^\infty(\mathcal{N}))^{1/p} = L^{p/(p-1)}(\mathcal{N})$, where $(\cdot, \cdot)_{[\cdot]}$ denotes the complex interpolation functor (cf., e.g., [1, Theorem 5.1.2]).

Now, take a probability space $(X, \nu)$ and a countable family $(r_{j,j'})_{j \in J, j' \in J'}$ of $\nu$-measurable functions on $X$ such that
\[
\left( \bigotimes_{(j,j') \in J''} r_{j,j'}(x) \right)(\nu) = \frac{1}{2\text{Card}(J'')} \sum_{\varepsilon \in \{-1,1\}^{J''}} \delta_{\varepsilon}
\]
for every finite subset $J''$ of $J \times J'$ (cf. [16, C.1]). By Khintchine’s inequality, there is a constant $C_3 > 0$ such that
\[
\frac{1}{C_3} \left( \sum_{j,j'} |a_{j,j'}|^2 \right)^{1/2} \leq \left\| \sum_{j,j'} a_{j,j'} r_{j,j'} \right\|_{L^q(\nu)} \leq C_3 \left( \sum_{j,j'} |a_{j,j'}|^2 \right)^{1/2}
\]
for every $(a_{j,j'}) \in C(J \times J')$ (cf. [16, C.2]). By the assumptions and the continuity of $\Psi$, there is a constant $C_4 > 0$ such that, for every $\lambda \in C(J \times J')$,\[
\left\| \Psi\left( r_{j,j'}(x) \lambda_{j,j'}(x) \right) \right\|_{L^q(\mu)} \leq C_4 \left\| r_{j,j'}(x) \lambda_{j,j'}(x) \right\|_{L^q(J \times J')} = C_4 \|\lambda\|_{L^q(J \times J')}
\]
for $\nu$-almost every $x \in X$. Therefore, by means of Tonelli’s theorem, we see that\[
\left\| \left( \sum_{j,j'} |\lambda_{j,j'} L(\mathcal{N}, \mathcal{N}) \right| f_{j,j'} \right\|_{L^q(\mu)} \leq C_3 \left\| \Psi\left( r_{j,j'}(x) \lambda_{j,j'} \right) \right\|_{L^q(\mu)} \leq C_3 C_4 \|\lambda\|_{L^p(J \times J')}
\]
for every $\lambda \in C(J \times J')$. Now, observe that there is $N \in \mathbb{N}$ such that $\sum_{j,j'} \chi_{B((\mathcal{N}, \mathcal{N} + i \Phi(\mathcal{N})))} \leq N$ on $E \times F_C$. Then, setting $C_5 := \min_{B(0,0), R} |f| > 0,$\[
\left\| \left( \sum_{j,j'} |\lambda_{j,j'} L(\mathcal{N}, \mathcal{N}) f_{j,j'} \right| \right\|_{L^q(\mu)} \geq C_5 \left\| \left( \sum_{j,j'} |\lambda_{j,j'}|^2 \chi_{B((\mathcal{N}, \mathcal{N} + i \Phi(\mathcal{N)))}) \right) \right\|_{L^q(\mu)} \geq C_5 N^{-(1/q-1/2)} \left( \sum_{j,j'} |\lambda_{j,j'}|^q M_R(\mu)(\mathcal{N}, \mathcal{N} + i \Phi(\mathcal{N}))) \right)^{1/q}
\]
for every $\lambda \in C(J \times J')$. Using the natural duality between $\ell^p_0(J \times J')$ and $\ell^{p/(p-1)}(J \times J')$, we then see that\[
M_R(\mu)(\mathcal{N}, \mathcal{N} + i \Phi(\mathcal{N})) \in \ell^{p/q}(J \times J').
\]
Therefore, by means of Lemma 4.5, we see that $M_1((\chi_{B(0,0), R}) \cdot \mu) \in L^{(p/q)'(J \times J')}(E \times F_C)$. Applying the preceding arguments to (a finite number of) the translates $L_{(0,jh)} \mu$ of $\mu$ (which are still necessarily $q$-Carleson for $B^p_K(\mathcal{N})$, $h \in F$, the assertion follows.

5 | SAMPLING MEASURES

In this section, we consider $p$-sampling measures for $B^p_K(\mathcal{N})$. We keep the notation of Section 4. Unlike the preceding section, we shall limit ourselves to the measures supported in a band from the beginning. As a consequence, we shall no longer consider $M_K, R(\mu)$, but only $M_R(\mu)$. 
As anticipated in the Introduction section, our methods heavily rely on the analysis of a slightly larger Bernstein space \( B^p_K(\mathcal{N}) \), with \( K \) contained in the interior of \( K' \) (actually, less is needed when \( n > 0 \)), so that our sufficient conditions actually imply stronger sampling properties. Namely, under our sufficient conditions, it is possible to determine the \( f \in \text{Hol}_K(E \times F_C) \), which belong to \( B^p_K(\mathcal{N}) \) simply checking the finiteness of the \( L^p(\mu) \) quasi-norm of \( f \) (cf. Lemma 5.3). We call the measures satisfying this property “strongly \( p \)-sampling.”

We shall then show that one may always remove a “sparse” measure (cf. Definition 5.4) from a \( p \)-sampling measure for \( B^p_K(\mathcal{N}) \) and still get a (strongly) \( p \)-sampling measure for \( B^p_K(\mathcal{N}) \). Here, a sparse measure is a measure such that the quasi-norms of the Carleson embeddings \( B^p_K(\mathcal{N}) \rightarrow L^p(K(E \times F_C) \setminus B(0, R)) \cdot \mu \) go to 0 for \( R \rightarrow +\infty \). This notion is independent of \( K \) and \( p \), thanks to Theorem 4.10. This result is quite useful and comes from an extension of [22, Lemma 4].

We shall then provide some necessary conditions for \( p \)-sampling measures (cf. 5.6). Even though these conditions are far from being sufficient in general, they will allow us to characterize dominating sets with respect to \( \mathcal{N} \)-invariant measures in Corollary 5.12. More precise necessary conditions for 2-sampling sequences in \( \rho^{-1}(0) \) for \( B^p_K(\mathcal{N}) \), which specialize [15] in our context, will be presented in Proposition 5.18.

Our main result concerning sufficient conditions for \( p \)-sampling measures for \( B^p_K(\mathcal{N}) \) is Theorem 5.8. It is inspired by [29, Theorem 5], where a characterization of the sampling measures for the classical weighted Bergman spaces in the unit disc is provided. In this case, the (strong) \( p \)-sampling properties of a measure \( \mu \) for the space \( B^p_K(\mathcal{N}) \) are related to the properties of the supports of the vague limits of its \( \mathcal{N} \)-translates; namely, these supports have to be sets of uniqueness of \( B^p_K(\mathcal{N}) \). Even though this result may seem somewhat technical, it still allows us to draw several consequences.

**Definition 5.1.** Take a compact subset \( K \) of \( \overline{\mathbb{A}}_+ \), \( p \in (0, \infty) \), and \( \mu \in \mathcal{M}_+(E \times F_C) \). We say that \( \mu \) is \( p \)-sampling for \( B^p_K(\mathcal{N}) \) if \( B^p_K(\mathcal{N}) \subseteq L^p(\mu) \) continuously and the canonical mapping \( B^p_K(\mathcal{N}) \rightarrow L^p(\mu) \) is an isomorphism onto its image.

We say that \( \mu \) is strongly \( p \)-sampling for \( B^p_K(\mathcal{N}) \) if it is \( p \)-sampling for \( B^p_K(\mathcal{N}) \) and

\[
B^p_K(\mathcal{N}) = \{ f \in \text{Hol}_K(E \times F_C) : f \in L^p(\mu) \}.
\]

We say that a locally finite \( 6 \) family \( (\zeta_j, z_j) \) of elements of \( E \times F_C \) is (strongly) sampling for \( B^p_K(\mathcal{N}) \) if the measure \( \sum_j \delta(\zeta_j, z_j) \) is (strongly) \( p \)-sampling for \( B^p_K(\mathcal{N}) \).

Let us briefly comment on the notion of a strongly \( p \)-sampling measure. On the one hand, \( p \)-sampling measures allow to reconstruct (up to constants) the quasi-norm of a holomorphic function which is known to belong to the Bernstein space \( B^p_K(\mathcal{N}) \). On the other hand, strongly \( p \)-sampling measures allow to verify if a holomorphic function \( f \in \text{Hol}_K(E \times F_C) \) belongs to the Bernstein space \( B^p_K(\mathcal{N}) \). Obviously, some (growth) conditions on \( f \) have to be imposed, unless the measure \( \mu \) satisfies very strong properties (in particular, \( \rho(\mu) \) must be unbounded, a case that is not considered below).

Notice that the sampling results proved in [39] concern actually strongly sampling sequences, and hold uniformly for every \( p \in (0, \infty] \). Sharp sampling results are then obtained by a limiting procedure for \( p \in (1, \infty) \).

**Definition 5.2.** Take a compact subset \( K \) of \( \overline{\mathbb{A}}_+ \) and \( \varepsilon > 0 \). Then, we set \( K_\varepsilon := K + (\overline{B^p_F}(0, \varepsilon) \cap \overline{\mathbb{A}}_+) \).

In particular, \( \{ 0 \} \subseteq \overline{B^p_F}(0, \varepsilon) \cap \overline{\mathbb{A}}_+ \), so that \( K_\varepsilon = K + \{ 0 \} \). The sets \( K_\varepsilon \) may be considered as a fundamental system of neighborhoods of \( K \) “in the directions of \( \overline{\mathbb{A}}_+ \).”

**Lemma 5.3.** Take a compact subset \( K \) of \( \overline{\mathbb{A}}_+ \), \( p \in (0, \infty) \), and \( \mu \in \mathcal{M}_+(E \times F_C) \) such that \( \rho(\text{Supp} \mu) \) is bounded. If \( \mu \) is \( p \)-sampling for \( B^p_K(\mathcal{N}) \), then \( \mu \) is strongly \( p \)-sampling for \( B^p_K(\mathcal{N}) \).

**Proof.** Fix \( g \in B^\infty_{\{0\}+}(\mathcal{N}) \) so that \( g_0 \in S(\mathcal{N}) \) and \( g(0, 0, \zeta) = \|g_0\|_{{L^\infty}(\mathcal{N})} = 1 \) (cf. [7, Theorem 4.2 and Proposition 5.2]). Take \( f \in \text{Hol}_K(E \times F_C) \), so that \( f g^{(j)} \in \text{Hol}_K(E \times F_C) \) for every \( j \in \mathbb{N} \), where \( g^{(j)} := g(2^{-j} \cdot) \). In addition, \( f_0 g_0^{(j)} \in S(\mathcal{N}) \), so that \( f g^{(j)} \in B^p_K(\mathcal{N}) \). Then, there is a constant \( C > 0 \) such that

\[
\left\| f_0 g_0^{(j)} \right\|_{L^p(\mu)} \leq C \left\| f g^{(j)} \right\|_{L^p(\mu)} \leq C \left\| f \right\|_{L^p(\mu)} \sup_{h \in \rho(\text{Supp} \mu)} e^{C|h|}
\]

for every \( j \in \mathbb{N} \) (cf. [10, Theorem 1.7]). Passing to the limit for \( j \rightarrow \infty \), this proves that \( f_0 \in L^p(\mathcal{N}) \). The assertion follows. □
**Definition 5.4.** We say that $\mu \in M_+(E \times F_C)$ is sparse if there is $R > 0$ such that for every $\varepsilon > 0$, there is $R' > 0$ such that $M_R(\mu) \leq \varepsilon$ on $(E \times F_C) \setminus B((0,0),R')$.

Notice that, by Propositions 4.7 and 4.8, a positive measure $\mu$ supported in a band is sparse if and only if the quasi-norm of the canonical map $B^p_{K^c}(\mathcal{N}) \to L^p(\chi_{(E \times F_C)\setminus B((0,0),R)})$ goes to 0 as $R \to +\infty$, for every compact subset $K$ of $\Lambda_\varepsilon$ (with a nonempty interior) and for every $p \in (0,\infty)$. In particular, if $\mu$ is sparse and supported in a band, then for every $R > 0$ and for every $\varepsilon > 0$ there is $R' > 0$ such that $M_R(\mu) \leq \varepsilon$ on $(E \times F_C) \setminus B((0,0),R)$.

**Proposition 5.5.** Take a compact subset $K$ of $\Lambda_\varepsilon$, $\varepsilon > 0$, $p \in (0,\infty)$, and $\mu, \mu' \in M_+(E \times F_C)$ such that $\rho(\text{Supp}(\mu + \mu'))$ is bounded. If $\mu + \mu'$ is p-sampling for $B^p_{K^c}(\mathcal{N})$ and $\mu'$ is sparse, then $\mu$ is strongly p-sampling for $B^p_{K^c}(\mathcal{N})$.

This result is based on [22, Lemma 4].

**Proof.** Since $\mu + \mu'$ is $p$-sampling for $B^p_{K^c}(\mathcal{N})$, there is a constant $C_1 > 0$ such that

$$
\|f_0\|_{L^p(\mathcal{N})}^p \leq C_1 \|f\|_{L^p(\mu + \mu')}^p = C_1 \|f\|_{L^p(\mu)}^p + \|f\|_{L^p(\mu')}^p
$$

for every $f \in B^p_{K^c}(\mathcal{N})$. Choose $R > 0$ so that for every $\varepsilon > 0$, there is $R' > 0$ such that $M_R(\mu) \leq \varepsilon$ on $(E \times F_C) \setminus B((0,0),R')$. In addition, notice that by Proposition 4.7, there is a constant $C_2 > 0$ such that

$$
\|f\|_{L^p(\mu')} \leq C_2 \|M_R(\mu')\|_{L^\infty(\mathcal{N})}^{1/p} \|f_0\|_{L^p(\mathcal{N})}
$$

for every $f \in B^p_{K^c}(\mathcal{N})$ and for every $\mu'' \in M_+(E \times F_C)$ supported in $\rho^{-1}(\rho(\text{Supp}(\mu + \mu')))$. Take $\varepsilon > 0$ so that $C_1 C_2 p \varepsilon < \frac{1}{2}$, and observe that

$$
\|f\|_{L^p(\mu')} \leq \int_{B((0,0),R+R')} |f|^p \, d\mu' + C_2 p \varepsilon \|f_0\|_{L^p(\mathcal{N})}
$$

for every $f \in B^p_{K^c}(\mathcal{N})$, so that

$$
\|f_0\|_{L^p(\mathcal{N})} \leq 2C_1 \left( \int_{B((0,0),R+R')} |f|^p \, d\mu' + \|f\|_{L^p(\mu')} \right)
$$

for every $f \in B^p_{K^c}(\mathcal{N})$. Now, let us prove that the canonical mapping $B^p_{K^c}(\mathcal{N}) \to L^p(\mu)$ is one-to-one. Assume, by contradiction, that there is $f \in B^p_{K^c}(\mathcal{N})$ with $\|f_0\|_{L^p(\mathcal{N})} = 1$ and $\|f\|_{L^p(\mu)} = 0$. Observe that the mapping $B^\infty_{\{0\}_c}(\mathcal{N}) \ni g \mapsto fg \in B^p_{K^c}(\mathcal{N})$ is continuous and one-to-one by holomorphy. In particular, $f B^\infty_{\{0\}_c}(\mathcal{N})$ is an infinite-dimensional subspace of $B^p_{K^c}(\mathcal{N})$. By means of Riesz’s lemma we may then find, by induction, a sequence $(g(k))$ of elements of $B^\infty_{\{0\}_c}(\mathcal{N})$ such that $\|f_0 g_0(k)\|_{L^p(\mathcal{N})} = 1$ and $\|f_0 g_0(k) - g_0(k')\|_{L^p(\mathcal{N})} \geq \frac{1}{2}$ for every $k \in \mathbb{N}$ and for every $k' < k$. Since $f g_0(k) \in B^p_{K^c}(\mathcal{N})$ for every $k \in \mathbb{N}$ and since $B^p_{K^c}(\mathcal{N})$ embeds continuously in $\text{Hol}(E \times F_C)$ (cf. [10, Corollary 3.3]), we may assume that $(f g_0(k))$ converges locally uniformly on $E \times F_C$, so that $\int_{B((0,0),R+R')} |f g_0(k) - f g_0(k+1)|^p \, d\mu' \to 0$ for $k \to \infty$. In addition, $\|f g_0(k)\|_{L^p(\mu)} = 0$ for every $k \in \mathbb{N}$. Thus,

$$
\frac{1}{2^p} \leq \left\| f_0 (g_0(k) - g_0(k+1)) + f_0 g_0(k)+1 \right\|_{L^p(\mathcal{N})} \leq 2C_1 \left( \int_{B((0,0),R+R')} |f g_0(k) - f(g_0(k+1))|^p \, d\mu' \right) \to 0
$$

for $k \to \infty$: contradiction. Thus, the canonical mapping $B^p_{K^c}(\mathcal{N}) \to L^p(\mu)$ is one-to-one.

Now, assume by contradiction that the canonical mapping $B^p_{K^c}(\mathcal{N}) \to L^p(\mu)$ is not an isomorphism onto its image. Then, there is a sequence $(f(j))$ of elements of $B^p_{K^c}(\mathcal{N})$ such that $\left\| f(j) \right\|_{L^p(\mathcal{N})} = 1$ for every $j \in \mathbb{N}$, while $\left\| f(j) \right\|_{L^p(\mu)} \to 0$.

As before, we may assume that $f(j) \to f$ locally uniformly for some $f \in \text{Hol}(E \times F_C)$, so that $\|f_0\|_{L^p(\mathcal{N})} \leq 1$, $f \in B^p_{K^c}(\mathcal{N})$, 

and observe that
and \( \|f\|_{L^p(\mu)} = 0 \). Then, \( f = 0 \) by the preceding remarks. Therefore,

\[
1 = \lim_{j \to \infty} \|f^{(j)}\|_{L^p(\mathcal{N})}^p \leq 2C_1 \lim_{j \to \infty} \left( \int_{B((0,0),R+R')} |f^{(j)}|^p \, d\mu' + \|f^{(j)}\|_{L^p(\mu)}^p \right) = 0,
\]

which is absurd. The proof is then completed by means of Lemma 5.3.

We now prove a necessary condition for sampling measures, which is based on [26, Theorem 4.3] (cf. also [8, Proposition 7.2]).

**Proposition 5.6.** Take a compact subset \( K \) of \( \overline{\mathcal{N}}_+ \), \( p \in (0, \infty) \), and \( \mu \in \mathcal{M}_+(E \times F_C) \) such that \( \rho(\text{Supp} \, \mu) \) is bounded. Assume that \( B^p_K(\mathcal{N}) \neq \{0\} \) and that \( \mu \) is a \( p \)-sampling measure for \( B^p_K(\mathcal{N}) \). Then, \( M_1(\mu) \) is bounded and there are \( R, C > 0 \) such that

\[
[M_R(\mu)]_0 \geq C
\]
on \( \mathcal{N} \).

**Proof.** The first assertion follows from Corollary 4.9. Then, take \( R' > 0 \) such that \( \rho(\text{Supp} \, \mu') \subseteq B_F(0, R') \). By Proposition 4.7, there is a constant \( C_1 > 0 \) such that

\[
\|f\|_{L^p(\mu')} \leq C_1 \|M_1(\mu')\|_{L^\infty(E \times F_C)} \sup_{|h| \leq R'+1} \|X_{B(\text{Supp} \, \mu', 1)} f|_h\|_{L^p(\mathcal{N})}
\]
for every \( \mu' \in \mathcal{M}_+(E \times F_C) \) with \( \rho(\text{Supp} \, \mu') \subseteq B_F(0, R') \), and for every \( f \in B^p_K(\mathcal{N}) \). Then,

\[
\|\left(1 - X_{B((\xi, z), R''+1)} f\right|_h\|_{L^p(\mu)} \leq C_1 \|M_1(\mu)\|_{L^\infty(E \times F_C)} \sup_{|h| \leq R''+1} \|\left((1 - X_{B((\xi, z), R''+1)} f|_h\right\|_{L^p(\mathcal{N})}
\]
for every \( R'' > 0 \), for every \( (\xi, z) \in E \times F_C \), and for every \( f \in B^p_K(\mathcal{N}) \). Furthermore, by assumption, there is a constant \( C' > 0 \) such that

\[
\|f\|_{B^p_K(\mathcal{N})} \leq C' \|f\|_{L^p(\mu)}
\]
for every \( f \in B^p_K(\mathcal{N}) \).

Then, take \( f \in B^p_K(\mathcal{N}) \) so that \( \|f\|_{B^p_K(\mathcal{N})} = 1 \), and define \( f^{(\xi, x)} := L_{((\xi, x) + i\Phi(\xi)), R''+1)} f \) for every \( (\xi, x) \in \mathcal{N} \). Define \( C'' := \sup_{|h| \leq R'} \|h\|_{L^\infty(\mathcal{N})} \), so that \( C'' < \infty \) by [10, Theorem 3.2]. Therefore,

\[
C'' [M_{R''+1}(\mu)]_0(\xi, x) \geq \int_{B((\xi, x) + i\Phi(\xi), R''+1)} |f^{(\xi, x)}|^p \, d\mu
\]
\[
= \|f^{(\xi, x)}\|_{L^p(\mu)}^p - \|\left(1 - X_{B((\xi, x) + i\Phi(\xi), R''+1)} f^{(\xi, x)}\right|_h\|_{L^p(\mu)}^p
\]
\[
\geq \frac{1}{C'p} - C_1 \|M_1(\mu)\|_{L^\infty(\mathcal{N})} \sup_{|h| \leq R''+1} \|\left((1 - X_{B((\xi, x) + i\Phi(\xi), R''+1)} f|_h\right\|_{L^p(\mathcal{N})}
\]
\[
= \frac{1}{C'p} - C_1 \|M_1(\mu)\|_{L^\infty(\mathcal{N})} \sup_{|h| \leq R''+1} \|\left((1 - X_{B((0,0), R''+1)} f|_h\right\|_{L^p(\mathcal{N})}
\]
for every \( R'' > 0 \) and for every \( (\xi, x) \in \mathcal{N} \). Then, choose \( R'' > 0 \) so that

\[
\sup_{|h| \leq R''+1} \|\left((1 - X_{B((0,0), R''+1)} f|_h\right\|_{L^p(\mathcal{N})} < \frac{1}{C'c_1 \|M_1(\mu)\|_{L^\infty(\mathcal{N})}}.
\]

The assertion follows.
We now provide an abstract, yet quite powerful, sufficient condition for sampling measures, based on [29, Theorem 5] (cf. also [8, Theorem 7.9]). We shall then draw several corollaries.

**Definition 5.7.** Take \( \mu \in \mathcal{M}_+(E \times F_C) \). Then, we define \( W(\mu) \) as the closure of \( \left\{ L(\zeta, x + i \Phi(\zeta)) \mu : (\zeta, x) \in \mathcal{N} \right\} \) in the vague topology, that is, the weak dual topology of the space of Radon measures \( \mathcal{M}(E \times F_C) \) on \( E \times F_C \), interpreted as the dual of the space of continuous compactly supported functions on \( E \times F_C \).

**Theorem 5.8.** Take a compact subset \( K \) of \( \Lambda_+ \), \( p, q \in (0, \infty) \) with \( q \leq p \), \( \varepsilon > 0 \), and a \( p \)-Carleson measure \( \mu \) for \( B^q_{K}(\mathcal{N}) \) such that \( \rho(\text{Supp} \mu) \) is bounded. Assume that the support of every element of \( W(\mu) \) is a set of uniqueness for \( B^q_{K+K'}(\mathcal{N}) \). Then, \( \mu \) is a strongly \( p \)-sampling measure for \( B^q_{K}(\mathcal{N}) \).

Here, we say that a subset \( U \) of \( E \times F_C \) is a set of uniqueness for \( B^q_{K+K'}(\mathcal{N}) \) if the canonical mapping \( f \mapsto \chi_U f \) is one-to-one on \( B^q_{K+K'}(\mathcal{N}) \). In other words, if the only element of \( B^q_{K+K'}(\mathcal{N}) \) that vanishes on \( U \) is the zero function.

Before we pass to the proof, we need some lemmas.

**Lemma 5.9.** Let \( \mathcal{M} \) be a subset of \( \mathcal{M}_+(E \times F_C) \) such that
\[
\bigcup_{\mu \in \mathcal{M}} \rho(\text{Supp} \mu)
\]
is bounded, and
\[
\sup_{\mu \in \mathcal{M}} \| M_1(\mu) \|_{L^\infty(E \times F_C)} < \infty.
\]
Then,
\[
\lim_{\mu \to \mathbf{\emptyset}} \| f \|_{L^p(\mu)} = \| f \|_{L^p(\mu_0)}
\]
for every filter \( \mathbf{\emptyset} \) on \( \mathcal{M} \) which converges vaguely to some Radon measure \( \mu_0 \) on \( E \times F_C \), for every \( f \in B^q_{K}(\mathcal{N}) \), for every compact subset \( K \) of \( \Lambda_+ \), and for every \( p \in (0, \infty) \).

The proof is based on [29, Theorem 1] (cf. also [8, Lemma 7.10]).

**Proof.** Fix \( p \in (0, \infty) \) and \( R' > 0 \) so that \( \rho(\text{Supp} \mu) \subseteq \overline{B}_F(0, R') \) for every \( \mu \in \mathcal{M} \cup \{ \mu_0 \} \). By Lemma 4.5 and Proposition 4.7, there is a constant \( C > 0 \) such that
\[
\| (1 - \chi_{B((0,0),R+1)})f \|_{L^p(\mu)} \leq C \sup_{|h| \leq R+1} \| (1 - \chi_{B((0,0),R)})f \|_{L^p(\mu)}
\]
for every \( \mu \in \mathcal{M} \cup \{ \mu_0 \} \), for every \( f \in B^p_{K}(\mathcal{N}) \), and for every \( R > 0 \) (cf. the proof of Proposition 5.6). Then, fix \( f \in B^p_{K}(\mathcal{N}) \), and take \( \varepsilon > 0 \). Fix \( R > 1 \) so that \( \| (1 - \chi_{B((0,0),R)})f \|_{L^p(\mathcal{N})} \leq \varepsilon \) for every \( h \in \mathcal{N}_R \) (cf. the proof of Proposition 5.6), and choose \( \tau \in C^\infty(E \times F_C) \) so that \( \chi_{B((0,0),R+1)} \leq \tau \leq \chi_{B((0,0),2R+2)} \). Then,
\[
\limsup_{\mu \to \mathbf{\emptyset}} \left| \| f \|_{L^p(\mu)} - \| f \|_{L^p(\mu_0)} \right| = (C\varepsilon)^p
\]
for every \( \varepsilon > 0 \), whence the result. \( \Box \)

**Lemma 5.10.** Take \( p \in (0, \infty) \), two compact subsets \( K, K' \) of \( \Lambda_+ \), \( g \in \text{Hol}_{K'}(E \times F_C) \) with \( g(0,0) \neq 0 \), and \( \mu \in \mathcal{M}_+(E \times F_C) \) such that \( \rho(\text{Supp} \mu) \) is bounded in \( F \), such that \( M_1(\mu) \) is bounded, and such that the support of every element of \( W(\mu) \) is a set of uniqueness for \( B^p_{K+K'}(\mathcal{N}) \). Take \( \varepsilon > 0 \), and define, for every \( (\zeta, x) \in \mathcal{N} \),
\[
U_\varepsilon(\zeta, x) := \left\{ f \in \text{Hol}_K(E \times F_C) : |f_0(\zeta, x)| \geq \varepsilon \| f_0 L(\zeta, x) \|_{L^p(\mathcal{N})} \right\}.
\]
Then, there is a constant $C > 0$ such that 
\[ \left\| f L(\zeta, x) g \right\|_{L^p(\rho')} \geq C \left\| f \right\|_{L^p(\rho)} \left\| g \right\|_{L^q(\rho')} \]
for every $(\zeta, x) \in \mathcal{N}$, for every $f \in U_\varepsilon(\zeta, x)$, and for every $\mu' \in W(\mu)$.

The proof is based on [29, Lemma 4] (cf. also [8, Lemma 7.11]).

Proof. By the $\mathcal{N}$-invariance of $W(\mu)$, we may reduce to proving the assertion for $(\zeta, x) = (0, 0)$. Define
\[ \mu(\zeta, x) := L(\zeta, x) \mu \]
for every $(\zeta, x) \in \mathcal{N}$. Observe that Lemma 5.9 implies that it will suffice to prove the assertion with $W(\mu)$ replaced by \( \{ \mu(\zeta, x) : (\zeta, x) \in \mathcal{N} \} \). Then assume, by contradiction, that there are a sequence $(f(j))$ of elements of $U_\varepsilon(0, 0)$, a sequence $(\zeta_j, x_j)$ of elements of $\mathcal{N}$ such that 
\[ \left\| f(j) g \right\|_{L^p(\mu(\zeta_j, x_j))} = 1 \]
for every $j \in \mathbb{N}$, while
\[ \lim_{j \to \infty} \left\| f(j) g \right\|_{L^p(\mu')} = 0. \]
Observe that
\[ \left\| M_1(\mu(\zeta, x)) \right\|_{L^\infty(\mathcal{N})} = \left\| M_1(\mu) \right\|_{L^\infty(\mathcal{N})} \]
for every $(\zeta, x) \in \mathcal{N}$, so that $W(\mu)$ is bounded, hence compact and metrizable, in the vague topology. Therefore, we may assume that $(\mu(\zeta_j, x_j))$ converges vaguely to some (positive Radon) measure $\mu'$ on $E \times F_C$. Analogously, since $f(j) g$ is bounded in $B_{K+K'}^p(\mathcal{N})$, and since $B_{K+K'}^p(\mathcal{N})$ embeds continuously into $\text{Hol}(E \times F_C)$ (cf. [10, Corollary 3.3]), we may assume that $(f(j) g)$ converges locally uniformly to some $h \in B_{K+K'}^p(\mathcal{N})$, so that $|h(0, 0)| \geq \varepsilon |g(0, 0)| > 0$. Let $(\psi_k)_{k \in K}$ be a partition of the unity on $E \times F_C$ whose elements belong to $C_c(\mathcal{N})$, and observe that
\[ \lim_{j \to \infty} \left\| \psi_k \right\|^p_{L^p(\mu(\zeta_j, x_j))} = \left\| \psi \right\|^p_{L^p(\mu')} \]
by the previous remarks. Therefore, by Fatou's lemma,
\[ 0 = \lim_{j \to \infty} \left\| f(j) g \right\|^p_{L^p(\mu(\zeta_j, x_j))} \]
\[ = \lim_{j \to \infty} \sum_{k \in K} \left\| \psi_k \right\|^p_{L^p(\mu(\zeta_j, x_j))} \left\| f(j) g \right\|^p_{L^p(\mu(\zeta_j, x_j))} \]
\[ \geq \sum_{k \in K} \left\| \psi_k \right\|^p_{L^p(\mu')} \left\| h \right\|^p_{L^p(\mu')} \]
\[ = \left\| h \right\|^p_{L^p(\mu')} \]
Since the support of $\mu' \in W(\mu)$ is a set of uniqueness for $B_{K+K'}^p(\mathcal{N})$, this implies that $h = 0$, which is absurd, since $|h(0, 0)| \geq \varepsilon |g(0, 0)| > 0$. \( \square \)

Lemma 5.11. Take $p, q \in (0, \infty)$, with $q \leq p$, two compact subsets $K, K'$ of $\overline{\Lambda_+}$, and $g \in \mathcal{O}_K(\mathcal{N}) \cap L^q(\mathcal{N})$. For every $\varepsilon > 0$ and for every $f \in \mathcal{O}_{K'}(\mathcal{N})$, define
\[ B_{\varepsilon, f}^{p, q} := \left\{ (\zeta, x) \in \mathcal{N} : |f(\zeta, x)| \leq \varepsilon \left\| f L(\zeta, x) g \right\|_{L^q(\mathcal{N})} \right\} . \]
Then, there is a constant $C > 0$ such that

$$\left\| X_{B^q_{t, f}} \right\|_{L^p(N')} \leq C \varepsilon \left\| f \right\|_{L^p(N')}$$

for every $\varepsilon > 0$ and for every $f \in \mathcal{O}_K(N) \cap L^p(N')$.

The proof is based on [29, Lemma 2] (cf. also [8, Lemma 7.13]).

**Proof.** It will suffice to observe that the operator

$$T : f \mapsto \left[ (\zeta, x) \mapsto \int_{E \times F_C} f(\zeta', x') \left| L_{(\zeta', x')} g(\zeta', x') \right|^q d(\zeta', x') \right] = f * |g|^q$$

induces a continuous linear mapping of $L^{p/q}(\mathcal{N})$ into itself, which is a consequence of Young’s inequality.

**Proof of Theorem 5.8.** Notice first that, because of Lemma 5.3, it will suffice to prove that $\mu$ is $p$-sampling for $B^q_{t, f}(\mathcal{N})$.

Fix $g \in B^q_{t, f}(E \times F_C)$ such that $g(0, 0) \neq 0$ (cf. [10, Proposition 3.4]). Observe that Lemma 5.10 implies that for every $\varepsilon' > 0$, there is a constant $C_{\varepsilon'} > 0$ such that

$$\left\| f L_{(\zeta, x + i\Phi(\zeta))} g \right\|_{L^q(\mathcal{N})} \geq C_{\varepsilon'} \left\| f_0 L_{(\zeta, x)} g_0 \right\|_{L^q(\mathcal{N})}$$

for every $(\zeta, x) \in \mathcal{N}$ and for every $f \in \text{Hol}_K(E \times F_C)$ such that

$$|f(\zeta, x)| > \varepsilon' \left\| f L_{(\zeta, x)} g \right\|_{L^q(\mathcal{N})},$$

that is, for every $f \in \text{Hol}_K(E \times F_C)$ and for every $(\zeta, x) \in \mathcal{N} \setminus B^q_{t, f}(0)$, with the notation of Lemma 5.11. In addition, by [10, Theorem 3.2], there is a constant $C'' > 0$ such that

$$\left\| h \right\|_{L^\infty(\mathcal{N})} \leq C'' \left\| h \right\|_{L^q(\mathcal{N})}$$

for every $h \in \mathcal{O}_K(N)$, so that

$$\left\| f L_{(\zeta, x + i\Phi(\zeta))} g \right\|_{L^q(\mathcal{N})} \geq \frac{C_{\varepsilon'}}{C''} \left| g(0, 0) \right| \left\| f(\zeta, x) \right\|_{L^q(\mathcal{N})}$$

for every $f \in \text{Hol}_K(E \times F_C)$ and for every $(\zeta, x) \in \mathcal{N} \setminus B^q_{t, f}(0)$. By Lemma 5.11, we may choose $\varepsilon'$ so small that

$$\left\| h \right\|_{L^p(\mathcal{N})} \leq 2 \left\| (1 - X_{B^q_{t, f}}) h \right\|_{L^p(\mathcal{N})}$$

for every $h \in \mathcal{O}_K(N) \cap L^p(N)$. Therefore,

$$\left\| f_0 \right\|_{L^p(\mathcal{N})} \leq \frac{2C''}{C_{\varepsilon'}} \left\| (\zeta, x) \mapsto \left\| f L_{(\zeta, x + i\Phi(\zeta))} g \right\|_{L^q(\mathcal{N})} \right\|_{L^p(\mathcal{N})}$$

for every $f \in B^p_{t, f}(\mathcal{N})$. Hence, it will suffice to show that the linear mapping

$$T : f \mapsto \left[ (\zeta, x) \mapsto \int_{E \times F_C} f \left| L_{(\zeta, x + i\Phi(\zeta))} g \right|^q d\mu \right]$$

maps $L^{p/q}(\mu)$ continuously into $L^{p/q}(\mathcal{N})$. To see this, observe that, by Jensen’s inequality,

$$\left\| \int_{E \times F_C} f \left| L_{(\zeta, x + i\Phi(\zeta))} g \right|^q d\mu \right\|_{L^{p/q}(\mathcal{N})}^{p/q} \leq \left\| L_{(\zeta, x + i\Phi(\zeta))} g \right\|_{L^q(\mathcal{N})}^{p-q} \int_{E \times F_C} \left| f \right|^{p/q} \left| L_{(\zeta, x + i\Phi(\zeta))} g \right|^q d\mu$$
for every \((\zeta, x) \in \mathcal{N}\) and for every \(f \in L^{p/q}(\mu)\). Observe that, by Proposition 4.7, there is a constant \(C'''' > 0\) such that
\[
\|h\|_{L^q(\mu)} \leq C'''' \|h_0\|_{L^q(\mathcal{N})}
\]
for every \(h \in B^q_{\{0\}}(\mathcal{N})\), so that the preceding remarks show that
\[
\left\| \int_{E \times F} f L_{(\zeta, x + i\Phi(\zeta))} g \, d\nu \right\|_{L^{p/q}(\mu)} \leq C'''' \|g_0\|_{L^{p/q}(\mathcal{N})} \sup_{h \in \rho (\text{Supp } \mu)} \|g_h\|_{L^q(\mathcal{N})} \|f\|_{L^{p/q}(\mu)}
\]
for every \(f \in L^{p/q}(\mu)\). The proof is complete. \(\square\)

We now present several consequences of Theorem 5.8. We begin with the characterization of dominating sets. As mentioned in the Introduction section, in this context one may find several natural “base” measures, all having the common property of being \(\mathcal{N}\)-invariant. Therefore, we shall characterize dominating sets with respect to any \(\mathcal{N}\)-invariant measure supported in a band. We shall actually consider bounded functions, instead of the characteristic function of the given set, thus providing some information on Toeplitz operators (cf. [24] for more information on this kind of connections).

**Corollary 5.12.** Take \(p \in (0, \infty)\), a compact subset \(K\) of \(\Lambda_+\), \(\nu \in M_+(F)\) with compact support and a bounded Borel measurable function \(g\) on \(E \times F \mathbb{C}\). Assume that \(B^p_K(\mathcal{N}) \neq \{0\}\) and define a Radon measure \(\mu_{\nu, g}\) so that
\[
\int_{E \times F} \varphi \, d\mu_{\nu, g} = \int_{F} \int_{\mathcal{N}} g_0 \varphi_h \, dH^{2n+m}(\nu)(h)
\]
for every \(\varphi \in C_c(E \times F \mathbb{C})\). Then, \(\mu_{\nu, g}\) is a (actually, strongly) \(p\)-sampling measure for \(B^p_K(\mathcal{N})\) if and only if there are \(R, C > 0\) such that \(\|M_R(\mu_{\nu, g})\|_0 \geq C\) on \(\mathcal{N}\).

Notice that every measure on \(E \times F \mathbb{C}\), which is supported in a band and has a bounded density with respect to an \(\mathcal{N}\)-invariant measure may be written in the above form, by disintegration (cf., e.g., [6, Theorem 2 of Chapter VI, section 3, No. 3]). We stated our result in this form since it simplifies the proof.

**Proof.** One implication follows from Proposition 5.6, since \(\rho (\text{Supp } \mu_{\nu, g}) \subseteq \text{Supp } \nu\). Conversely, assume that there are \(R, C > 0\) such that \(\|M_R(\mu_{\nu, g})\|_0 \geq C\) on \(\mathcal{N}\). Observe that
\[
M_1(\mu_{\nu, g}) \leq \|g\|_{L^{\infty}(E \times F \mathbb{C})} \nu(F)H^{2n+m}(B_{\mathcal{N}}((0, 0), 1)),
\]
so that \(M_1(\mu_{\nu, g})\) is bounded. Then, take \(\mu' \in W(\mu_{\nu, g})\), and observe that \(\mu' \leq \|g\|_{L^{\infty}(E \times F \mathbb{C})} \nu_{\nu, 1}\), so that \(\mu'\) is absolutely continuous with respect to \(\mu_{\nu, 1}\). In particular, either \(\mu' = 0\) or \(\mu_{\nu, 1}(\text{Supp } \mu') > 0\). In the latter case, \(H^{2n+m}(\rho^{-1}(h) \cap \text{Supp } \mu') > 0\) for some \(h \in F\), so that \(\text{Supp } \mu'\) is a set of uniqueness for \(\text{Hol}(E \times F \mathbb{C})\). By Theorem 5.8, it will then suffice to prove that \(\mu' \neq 0\). To this aim, observe that, if \(\varphi \in C_c(E \times F \mathbb{C})\) and \(\varphi \geq \chi_B((0,0),K)\), then
\[
\int_{E \times F} \varphi \, dL_{(\zeta, x)}(\mu_{\nu, g}) \geq M_R(\mu_{\nu, g})(\zeta, x) \geq C
\]
for every \((\zeta, x) \in \mathcal{N}\), so that \(\int_{E \times F} \varphi \, d\mu' \geq C\). The proof is complete. \(\square\)

In the following result, we provide some more specific sufficient conditions for sampling measures. In this case, we require that the set where the means \(M_R(\mu)\) are small has a sufficiently well-distributed complement. This may be considered as an analog of Corollary 5.12 for general measures. Again, since there does not seem to be a canonical “base”
measure on $E \times F_C$, we allow the reader to choose a reasonable one ("$
u$"), which may be, for instance, the Haar measure on $E \times F_C$ or on $\rho^{-1}(0)$.

**Corollary 5.13.** Take $p \in (0, \infty)$, $\varepsilon, C > 0$, and a compact subset $K$ of $\bar{N}$. In addition, fix $\nu \in M_+(E \times F_C)$. For every $\mu \in M_+(E \times F_C)$ and for every $R > 0$, define

$$G_{\mu, R} := \{ (\zeta, z) \in E \times F_C : M_R(\mu)(\zeta, z) \geq \varepsilon \}.$$  

Then, there is $R > 0$ such that for every $R' > 0$ and for every $\mu \in M_+(E \times F_C)$ such that $\rho(\text{Supp}(\mu))$ is bounded,

$$M_1(\mu) \in L^\infty(E \times F_C),$$

and

$$\nu(G_{\mu, R} \cap B((\zeta, x + i\Phi(\zeta)), R)) \geq C$$

for every $(\zeta, x) \in N \setminus B_{\rho'}((0,0), R')$, the measure $\mu$ is strongly $p$-sampling for $B^p_K(N)$.

This result is inspired by [26, Theorem 4.2] (cf. also [8, Theorem 7.6]). The proof is different, though. See also [22, Theorem 5].

**Proof.** We first prove the assertion for $R' = 0$.

For every $R > 0$, fix $\varphi_R \in C_c(E \times F_C)$ such that $\chi_{B((0,0), R)} \leq \varphi_R \leq \chi_{B((0,0), 2R)}$. Define, for every $\mu \in M_+(E \times F_C)$ and for every $R > 0$,

$$M'_R(\mu) : E \times F_C \ni (\zeta, z) \mapsto \int_{E \times F_C} L(\zeta, z) \varphi_R \, d\mu \in \mathbb{C}$$

and

$$G'_{\mu, R} := \{ (\zeta, z) \in E \times F_C : M'_R(\mu)(\zeta, z) \geq \varepsilon \},$$

so that

$$M_R(\mu) \leq M'_R(\mu) \leq M_{2R}(\mu) \quad \text{and} \quad G_{\mu, R} \subseteq G'_{\mu, R} \subseteq G_{\mu, 2R}.$$  

By Proposition 4.7 and [10, Theorem 1.16], we may find $R > 0$ such that, for every $(R, 6)$-lattice $(\zeta_j, z_j)_{j \in J}$ on $\rho^{-1}(B_F((0, 2R)^{1/2}))$ (cf. Definition 4.1), the mapping

$$f \mapsto (f(\zeta_j, z_j))_j$$

induces an isomorphism of $B^p_K(N)$ (cf. Definition 5.2) onto a closed subspace of $\ell^p(J)$.  

Then, take $\mu \in M_+(E \times F_C)$ so that $\rho(\text{Supp}(\mu))$ is bounded and $M_1(\mu) \in L^\infty(E \times F_C)$, and assume that

$$C \leq \nu(G_{\mu, R} \cap B((\zeta, x + i\Phi(\zeta)), R)) \leq \nu(G'_{\mu, R} \cap B((\zeta, x + i\Phi(\zeta)), R))$$

for every $(\zeta, x) \in N$. Observe that, since $M_1(\mu) \in L^\infty(N)$, the set of $L(\zeta, x + i\Phi(\zeta))\mu$, as $(\zeta, x)$ runs through $N$, is bounded in $M_+(E \times F_C)$, hence relatively compact and metrizable in the vague topology. In particular, if $\mu' \in W(\mu)$, then there is a sequence $(\zeta_j, x_j)_{j \in J}$ of elements of $N$ such that $L(\zeta_j, x_j + i\Phi(\zeta_j))\mu$ converges vaguely to $\mu'$. Hence, $M'_R(L(\zeta_j, x_j + i\Phi(\zeta_j))\mu)$ converges locally uniformly to $M'_R(\mu')$, so that

$$\bigcap_{k \in \mathbb{N}} \bigcup_{j \geq k} G'_{L(\zeta_j, x_j + i\Phi(\zeta_j))\mu, R} \subseteq G'_{\mu', R}.$$
whence
\[ \nu(G'_\mu, R \cap \mathcal{B}(\zeta, x + i\Phi(\zeta), R)) \geq \lim_{k \to \infty} \nu \left( \bigcup_{j \geq k} G'_{L(\zeta_j, x_j), R} \cap \mathcal{B}(\zeta, x + i\Phi(\zeta), R) \right) \geq C' \]

for every \((\zeta, x) \in \mathcal{N}\). Therefore, \(\text{Supp } \mu' \cap \mathcal{B}(\zeta, x + i\Phi(\zeta), 2R) \neq \emptyset\) for every \((\zeta, x) \in \mathcal{N}\), so that we may find an \((R, 6)\)-lattice on \(\mathcal{B}_F((0, (2R)^{1/2}))\) whose elements belong to \(\text{Supp } \mu'\). Our choice of \(R\) then implies that every element of \(B_{K, 1}(\mathcal{N})\), which vanishes on \(\text{Supp } \mu'\), must vanish identically. Therefore, Theorem 5.8 implies that \(\mu\) is a strongly \(p\)-sampling measure for \(\mathcal{M}_p(K)\). The assertion for \(R' > 0\) follows from Proposition 5.5, applying the preceding arguments to \(\mu + \chi_{B_F((0,0),2R')}\)\(\cdot12\). □

In the next result, we shall provide some conditions for a class of sequences in \(\rho^{-1}(0)\) to be sampling. Even though these conditions are sharp in some contexts, they are not necessary.

**Corollary 5.14.** Take \(p \in (0, \infty), r < \frac{\pi}{2}, R, C > 0, \lambda \in F', \) and a compact subset \(K\) of \(\Lambda_+\). Fix a basis \((e_k)\) of \(E\) with dual basis \((e'_k)\), and assume that
\[ H_K(-\Phi(\zeta)) < \frac{\pi}{2} \sum_k \left| \langle e'_k, \zeta \rangle \right|^2 \]
for every \(\zeta \in E \setminus \{0\}\). Define \(J := \mathbb{Z}[i]^n\) and \(\zeta_j := \sum_k j_k e_k\) for every \(j \in J\). For every \(j \in J\), take a locally finite sequence \((x_j, j', j'')\) of elements of \(F\) such that \(M_1(\sum_{j''} \delta_{x_j, j''}) \leq C\) and such that
\[ \bigcup_{j'' \in \mathbb{N}} \overline{B_K(x_j, j'')} \supseteq F \setminus \overline{B_F(x_j, R)} \quad (8) \]
for some \(x_j \in F\), where \(\overline{B_K(x, R')}\) denotes the closed ball of center \(x \in F\) and radius \(R' > 0\) relative to the distance induced by the norm \(H_{(K-\lambda)\lambda, (\lambda - K)}\), namely, \(x + R'[(K - \lambda) \cup (\lambda - K)]^n\). Then, \((\zeta + \zeta_j, x_j, j' + i\Phi(\zeta + \zeta_j))_{j \in J, j' \in \mathbb{N}}\) is a strongly sampling family for \(B_{K, 1}(\mathcal{N})\) for every \(\zeta \in E\).

Observe that this result implies [32, Theorem 4.1] (of which is a consequence and to which it essentially reduces when \(n = 0\)) by means of [32, Theorem 2.1] (or of the following Proposition 5.17). Notice that [30, Theorem 1.6] gives a slightly stronger result, corresponding to the case in which \(m = 1, n = 1, p = 2, r = \frac{\pi}{2}, K = [0, a], \lambda = \frac{a}{2}, (e_k)\) is orthogonal with respect to \(\Phi\), and \(\{ x_{j, j'} : j' \in J' \} = \frac{2}{\pi} \mathbb{Z}\). Notice that this latter result only gives rise to sampling families (and not strongly sampling families). Actually, the above families cannot be (in general) strongly sampling if \(r = \frac{\pi}{2}\), at least when \(n = 0\) and \(K\) is a paralleletope (cf. [39, Nos. 34, 44]).

Using more general results on the sets of uniqueness in Fock spaces, one may replace \((\zeta_j + \zeta)\) with a more general family. Here, we content ourselves with affine lattices (i.e., translates of discrete subgroups) for the sake of simplicity.

For the sake of simplicity, we shall prove separately the following elementary lemma.

**Lemma 5.15.** Let \(X\) be a locally compact space with a countable base, and denote by \(\mathcal{M}_d(X)\) the set of discrete Radon measures \(\mu\) on \(X\) such that \(\mu(\{x\}) \in \mathbb{N}\) for every \(x \in X\). Then, \(\mathcal{M}_d(X)\) is vaguely closed. In addition, if \((\mu_j)\) is a sequence in \(\mathcal{M}_d(X)\), which converges vaguely to some \(\mu\), then
\[ \text{Supp } \mu = \left\{ x \in X : \exists (x_j) \in \prod_j \text{Supp } \mu_j, \lim_{j \to \infty} x_j = x \right\}. \]

**Proof.** Let us first prove that \(\mathcal{M}_d(X)\) is the set of Radon measures \(\mu\) on \(X\) such that \(\mu(U) \in \mathbb{N}\) for every relatively compact open subset \(U\) of \(X\). One implication is obvious. Conversely, take \(\mu\) as above, and observe that \(\mu(E) \in \mathbb{N}\) for every \(\mu\)-integrable subset of \(X\). Observe that \(X\) is a Polish space, thanks to [4, Corollary to Proposition 16 of Chapter IX, section 2,
No. 10, and Corollary to Proposition 2 of Chapter IX, section 6, No. 1], so that we may endow $X$ with a complete metric $d$. Take a compact subset $K$ of $X$ such that $\mu(K) > 0$, and observe that for every $k \in \mathbb{N}$, there is a finite family $(x_{k,j})$ of elements of $X$ such that $K \subseteq B_X(x_{k,j}, 2^{-k})$, so that $\mu(B_X(x_{k,j}, 2^{-k})) \geq 1$ for some $j_k$. By induction, we may assume that $x_{k,j_k} \in B_X(x_{k',j'}, 2^{-k'})$ for every $k' < k$. Thus, $(x_{k,j_k})$ is a Cauchy sequence, so that it converges to some $x \in K$ with $\mu(\{x\}) = \lim_{k \to \infty} \mu(B_X(x_{k,j_k}, 2^{-k})) \geq 1$. In particular, there is a $k \in \mathbb{N}$ such that $\mu(B_X(x_{k,j_k}, 2^{-k})) = \mu(\{x\})$ for every $k' \geq k$.

Since $K$ was arbitrary, applying the same argument to $K \setminus B_X(x_{k,j,k}, 2^{-k})$ with $k$ as before, we infer that the restriction of $\mu$ to $K$ belongs to $\mathcal{M}_d(K)$. By the arbitrariness of $K$, this implies that $\mu \in \mathcal{M}_d(X)$.

Now, let $\mathcal{F}$ be a filter on $\mathcal{M}_d(X)$ converging vaguely to some $\mu$. Let $U$ be a relatively compact open subset of $X$, and take a positive $\varphi \in C_c(X)$ so that $\varphi = 1$ on $U$. Then, [6, Proposition 22 of Chapter IV, section 5, No. 12], applied to $\varphi \cdot \mathcal{F}$ and $\varphi \cdot \mu$, shows that $\mu(U) \to \mu(U) \in \mathbb{N}$, as $\mu'$ runs along $\mathcal{F}$, provided that $\mu(\partial U) = 0$. In particular, observe that there is only a countable number of $\delta > 0$ such that $\mu(\partial U_{\delta}) > 0$, where $U_{\delta} := \{x \in U : d(x, X \setminus U) > \delta\}$, so that $\mu(U) = \lim_{\delta \to 0+} \mu(U_{\delta}) \in \mathbb{N}$. Thus, the preceding remarks imply that $\mu \in \mathcal{M}_d(X)$.

Now, let $(\mu_j)$ be a sequence in $\mathcal{M}_d(X)$, which converges vaguely to some $\mu$. Take $x \in \text{Supp } \mu$, and choose $r > 0$ so that $B_X(x, 2r)$ is relatively compact and $\mu(\{x\}) \delta_x$ on $B_X(x, 2r)$. Observe that, for every $k \in \mathbb{N}$, $\mu_j(B_X(x, 2^{-k}r)) \to \mu(B_X(x, 2^{-k}r)) = \mu(\{x\}) > 0$ by [6, Proposition 22 of Chapter IV, section 5, No. 12] again, so that there is $j_k \in \mathbb{N}$ such that $x_{j,k} \in B_X(x, 2^{-k}) \cap \text{Supp } \mu_j$ if $j \geq j_k$. Thus, we may find a sequence $(x_j') \subseteq \bigcap_j \text{Supp } \mu_j$, which converges to $x$. Conversely, if there is a sequence $(x_j) \subseteq \bigcap_j \text{Supp } \mu_j$, which converges to some $x$ in $X$, then clearly $\mu_j(B_X(x, \varepsilon)) \geq 1$ for every $\varepsilon > 0$, provided that $j$ is sufficiently large, so that $\mu(B_X(x, \varepsilon)) \geq 1$ for every $\varepsilon > 0$, whence $\mu(\{x\}) \geq 1$ and $x \in \text{Supp } \mu$. \hfill $\Box$

**Proof of Corollary 5.14.** Notice that the assumptions are weaker (while the conclusion is stronger) if we replace $K$ with its convex envelope. Therefore, we may assume that $K$ is convex. We may further assume that $K$ has a nonempty interior. In addition, observe that we may find $\varepsilon \in (0, 1)$ so that

$$H_{K_{\varepsilon}}(-\Phi(\zeta)) \leq H_K(-\Phi(\zeta)) + \varepsilon|\Phi(\zeta)| \leq \frac{\pi - \varepsilon}{2} \sum_k \langle e_{k', \zeta} \rangle^n$$

for every $\zeta \in E$, and such that

$$\overline{B}_{K_{\varepsilon}}(0, r/(1 - \varepsilon)) \supseteq \overline{B}_K(0, r/((1 - \varepsilon)(1 + \varepsilon C')) \supseteq \overline{B}_K(0, r),$$

where $C' := \sup_{x \in \overline{B}_K(0, 1)} |x|$.

**Step I.** Assume first that $E = \{0\}$, and let us prove that $\{x_{0,j'} : j' \in \mathbb{N}\}$ is a set of uniqueness of $B_{\infty}^K(F)$. Observe that, since $B_{\infty}^K(\mathbb{C}) = e^{-i\langle \lambda, \cdot \rangle} B_{\infty}^K(F)$, we may assume that $\lambda = 0$, that is, that the convex envelope of $K \cup (-K)$ is a neighborhood of $0$. Then, the assertion follows from [32, Theorem 4.1] and Proposition 5.5.

**Step II.** Let us prove that $\{\langle \zeta' + \zeta , j, j' + i \Phi(\zeta , j') \rangle : j \in J , j' \in \mathbb{N}\}$ is a set of uniqueness for $B_{\infty}^K(\mathcal{N})$. Notice that, up to a translation (and observing that the $(x_{j,j}, -2im \Phi(\zeta , j')$ satisfy the same assumptions as the $(x_{j,j})$, with $x_j$ replaced by $x_j - 2im \Phi(\zeta , j')$, we may assume that $\zeta = 0$. Take $f \in B_{\infty}^C(\mathcal{N})$, and assume that $f(\zeta_j, x_{j,j} + i \Phi(\zeta_j)) = 0$ for every $j \in J$ and for every $j' \in \mathbb{N}$. Observe first that, if we define, for every $j \in J$, $f^{(j)} := f(\zeta_j, \cdot + i \Phi(\zeta_j))$, then $f^{(j)} \in B_{\infty}^K(F)$, so that Step I implies that $f^{(j)} = 0$. Therefore, $f(\zeta , z) = 0$ for every $z \in F_C$ and for every $j \in J$. Observe that, since $f \in B_{\infty}^K(\mathcal{N})$, the function $f(\cdot , z)$ belongs to the Fock space

$$\left\{ g \in \text{Hol}(E) : \int_E |g(\zeta)|^2e^{(\pi - \varepsilon)\sum_k |e_{k', \zeta}|^2} d\zeta < \infty \right\}$$

for every $z \in F_C$, with $\varepsilon$ as above. Then, [30, Corollary 5.4] shows that $f(\cdot , z) = 0$ for every $z \in F_C$, whence $f = 0$.

**Step III.** As in Step II, we may reduce to the case $\zeta = 0$. Then, define $\mu_{(0,0)} := \sum_j \delta_{(j, x_{j,j'} + i \Phi(\zeta_j))}$ and

$$\mu_{(\zeta, x)} := L_{(x_{j,j} + i \Phi(\zeta_j))} \mu_{(0,0)}$$

for every $\zeta, x \in \mathcal{N}$. Observe that $\mu_{(0,0)}$ is a well-defined Radon measure on $E \times F_C$, and that $M_1(\mu_{(0,0)})$ is bounded. Take $\mu \in \mathcal{W}(\mu_{(0,0)})$, and observe that there is a sequence $(\zeta'_q, x_q')_{q \in \mathbb{N}}$ of elements of $\mathcal{N}$ such that $\mu_{(\zeta'_q, x'_q)}$ converges vaguely to $\mu$ (cf. the proof of Lemma 5.10). By Lemma 5.15, $\mu$ is a locally finite sum of Dirac deltas, and $\text{Supp } \mu$ is a suitable limit of
Supp \( \mu_{(\zeta_q', x_q')}(\zeta_q', x_q') \) = \( (\zeta_q' \cdot x_q' + i \Phi(\zeta_q')) \cdot \text{Supp} \mu_{(0,0)} \). For every \( R' > 0 \), define \( U_{R'} := \{ (\zeta, z) : \text{Re} z \in \mathbb{B}_F(0, R') \} \). If \( \mu(\partial U_{R'}) = 0 \) (which happens except for a countable number of \( R' \)), then [6, Proposition 22 of Chapter IV, section 5, No. 12], applied as in Lemma 5.15, shows that \( \chi_{U_{R'}} \cdot \mu_{(\zeta_q', x_q')} \) converges vaguely to \( \chi_{U_{R'}} \cdot \mu \). Observe that, if \( \text{pr}_E : E \times F \) denotes the canonical projection, then the restriction of \( \text{pr}_E \) to \( U_{R'} \cap \rho^{-1}(0) \) is proper, so that the measures \( (\text{pr}_E)_* \mu_{(\zeta_q', x_q')} \) belong to \( \mathcal{M}_F(E) \) (with the notation of Lemma 5.15) and converge vaguely to \( (\text{pr}_E)_* \mu \). If, in addition, \( R' > R + C' r \), with \( C' \) defined as above, then the (almost) covering condition (8) implies that

\[
\text{Supp}((\text{pr}_E)_* \mu_{(\zeta_q', x_q')}) = \text{pr}_E(\text{Supp} \mu_{(\zeta_q', x_q')}) = \zeta_q' + L
\]

for every \( q \in \mathbb{N} \), where \( L := \{ \zeta_j : j \in J \} \). Therefore, the support of \( (\text{pr}_E)_* \mu_{(\zeta_q', x_q')} \) is a suitable limit of the affine lattices \( \zeta_q' + L \) by Lemma 5.15. It then follows that there is \( \zeta'' \in E \) such that

\[
\text{pr}_E(U_{R'} \cap \text{Supp} \mu) = \text{Supp}((\text{pr}_E)_* \mu_{(\zeta_q', x_q')}) = \zeta'' + L
\]

for every \( R' \) as above, where the first equality follows from the fact that we chose \( R' \) so that \( \mu(\partial U_{R'}) = 0 \). Letting \( R' \to +\infty \), we see that \( \text{pr}_E(\text{Supp} \mu) = \zeta'' + L \).

Now, take a sequence \((j_q)\) in \( J \) such that \( \zeta_q' + \zeta_{j_q} \) converges to \( \zeta'' \), and define \( v_{j,q} := \sum_{j' \in \mathbb{N}} \delta_{x_{j,q} + j' \cdot x_{j,q} + j, j'} + 2 \text{Im} \Phi(\zeta_q' + \zeta_{j,q}) \), so that

\[
\mu_{(\zeta_q', x_q')} = \sum_{j \in J} \delta_{\zeta_q' + \zeta_{j,q}} \otimes L_0(\Phi(\zeta_q' + \zeta_{j,q})) v_{j,q},
\]

where \( L_0(\Phi(\zeta_q' + \zeta_{j,q})) v_{j,q} \) denotes the (left) translate of \( v_{j,q} \) by \( i \Phi(\zeta_q' + \zeta_{j,q}) \). In particular, for every \( j \in J \), the sequence \( v_{j,q} \) converges vaguely to a measure \( v_j \), which is a locally finite sum of Dirac deltas by Lemma 5.15, and

\[
\mu = \sum_{j \in J} \delta_{\zeta'' + \zeta_j} \otimes L_0(\Phi(\zeta'' + \zeta_j)) v_j.
\]

Now, assume that there is \( x \in F \) such that \( v_j(\mathbb{B}_F(x, r)) = 0 \) for some \( j \in J \). Then, [6, Proposition 22 of Chapter IV, section 5, No. 12] (applied as in Lemma 5.15) shows that \( v_{j,q}(\mathbb{B}_F(x, r)) \to 0 \) for \( q \to \infty \), so that \( v_{j,q}(\mathbb{B}_F(x, r)) = 0 \) for \( q \) sufficiently large. By the (almost) covering condition (8), this shows that \( x \in \mathbb{B}(x_q' + x_{j,q} + j + 2 \text{Im} \Phi(\zeta_{q,j}), R) \), that is, \( x_q' + x_{j,q} + j + 2 \text{Im} \Phi(\zeta_{q,j}) \in \mathbb{B}(x, R) \), for \( q \) sufficiently large. In particular, for every \( x' \in F \setminus \mathbb{B}(x, R + C'r) \), the (almost) covering condition (8) shows that

\[
v_{j,q}(\mathbb{B}_F(x', r)) \geq 1,
\]

so that, by upper semicontinuity,

\[
v_j(\mathbb{B}_F(x', r)) \geq 1.
\]

Hence,

\[
\text{Supp} v_j + \mathbb{B}_F(x', r) \supseteq F \setminus \mathbb{B}(x, R + C'r).
\]

If there is no \( x \) as above, then a similar argument shows that \( \text{Supp} v_j + \mathbb{B}_F(x', r) = F \). We have thus proved that there is an enumeration \((x_{j,q}')_{j \in \mathbb{N}} \) of \( \text{Supp} v_j \), which satisfies an analog of the (almost) covering condition (8) with \( R \) replaced by \( R + C'r \) and \( x_j \) replaced by a suitable element of \( F \). In addition, \( M_1(v_{j,q}) \leq C \) for every \( j \in J \) and for every \( q \in \mathbb{N} \), so that \( M_{1/2}(v_j) \leq C \) for every \( j \in J \). Then, Step II and Lemma 4.5 show that

\[
\text{Supp} \mu = \{ (\zeta'' + \zeta_j, x'' + i \Phi(\zeta'' + \zeta_j)) : j \in J, x'' \in \text{Supp} v_j \}.
\]
is a set of uniqueness for $B_{K^c}^p(N)$, hence also for $B_{K^c}^p(N)$. By the arbitrariness of $\mu$, Theorem 5.8 leads to the conclusion.

With a similar (but simpler) argument, one may also prove the following result. We formulate it for the space $L^p(N) \cap \Theta_K(N)$ instead of $B_{K}^p(N)$ in order to keep the notation as simple as possible, but this result is actually about sampling sequences (in $\rho^{-1}(0)$) for $B_{K}^p(N)$.

**Corollary 5.16.** Take $p, q \in (0, \infty)$ with $q \leq p$, a compact subset $K$ of $\overline{N}$, $\delta > 0$, $R > 1$, and $\varepsilon > 0$. Assume that $d$ satisfies the following “convexity” assumption: If $(\zeta, x) \in N$ and $d_N((0,0), (\zeta, x)) < 1$, then there is $(\zeta', x') \in N$ such that $d_N((0,0), (\zeta', x')), d_N((\zeta', x'), (\zeta, x)) < \frac{1}{2}$. If the support of every $(\delta, R)$-lattice on $N$ is a set of uniqueness for $L^p(N) \cap \Theta_K(N)$, then every $(\delta, R)$-lattice on $N$ is strongly sampling for $L^p(N) \cap \Theta_K(N)$.

**Proof.** The assertion follows from Theorem 5.8 and Lemma 5.15, since the condition imposed on $d_N$ guarantees that $\rho^{-1}(0)$ satisfies the following “convexity” assumption: If $(\zeta', x') \in N$ and $d_N((\zeta', x'), (\zeta, x)) < \frac{1}{2}$, then there is $(\zeta', x') \in N$ such that $d_N((0,0), (\zeta', x')), d_N((\zeta', x'), (\zeta, x)) < \frac{1}{2}$. If the support of every $(\delta, R)$-lattice on $N$ is a set of uniqueness for $L^p(N) \cap \Theta_K(N)$, then every $(\delta, R)$-lattice on $N$ is strongly sampling for $L^p(N) \cap \Theta_K(N)$.

In the following result, we extend to this context the comparison between sampling sequences in $B_{K}^p(N)$ and $B_{K^c}^\infty(N)$. As in the usual context $n = 0$, this correspondence requires a change in the compact set $K$. We shall state this result for general sampling measures, even though one of the two implications seems to be essentially tied to sampling sequences.

**Proposition 5.17.** Take $p \in (0, \infty)$, a compact subset $K$ of $F'$, $\varepsilon > 0$, and $\mu \in \mathcal{M}_+(E \times F_C)$ with $\rho(\text{Supp } \mu)$ bounded. Then, the following hold:

1. If $\mu$ is a $p$-sampling measure for $B_{K}^p(N)$, then the canonical inclusion $B_{K}^\infty(N) \to L^\infty(\mu)$ is an isomorphism onto its image and $B_{K}^\infty(N) = \{ f \in H_{K}(E \times F_C) : f \in L^\infty(\mu) \}$. 
2. If $M_k(\mu)$ is bounded, $\mu$ is discrete, $\inf_{x \in \{ 0 \}} \mu(\{ (\zeta, x) \}) > 0$, and the canonical mapping $B_{K^c}^\infty(N) \to L^\infty(\mu)$ is an isomorphism onto its image, then $\mu$ is a strongly $p$-sampling measure for $B_{K}^p(N)$.

This result extends (i) and (ii) of [32, Theorem 2.1].

**Proof.**

1. Assume by contradiction that the canonical mapping $B_{K}^\infty(N) \to L^\infty(\mu)$ is not an isomorphism onto its image. Then, there is a sequence $(f_j)$ of elements of $B_{K}^\infty(N)$ such that $\|f_0\|_{L^\infty(\mu)} = 1$ and $\|f_j\|_{L^\infty(\mu)} \leq 2^{-j}$ for every $j \in \mathbb{N}$. In particular, for every $j \in \mathbb{N}$, there is $(\zeta_j, x_j) \in N$ such that $\|f_j\|_{L^\infty(\zeta_j, x_j)} \geq \frac{1}{2}$. Fix $\varphi \in B_{K^c}^p(N)$ (cf. Definition 5.2) so that $\varphi(0,0) = 1$, and define $g_j := f_j L_{(\zeta_j, x_j + k(\zeta_j))} \varphi$ for every $j \in \mathbb{N}$. Then, $g_j \in B_{K^c}^p(N)$ and,

$$\|g_j\|_{L^{p}(\mu)} \leq 2^{-j} \|L_{(\zeta_j, x_j + k(\zeta_j))} \varphi\|_{L^{p}(\mu)} \leq C 2^{-j} \|\varphi_0\|_{L^{p}(N)},$$

where $C$ is the quasi-norm of the continuous inclusion $B_{K^c}^\infty(N) \subseteq L^p(\mu)$, for every $j \in \mathbb{N}$. Furthermore, since $B_{K^c}^p(N)$ embeds continuously into $B_{K^c}^\infty(N)$ (cf. [10, Theorem 3.2]), there is a constant $C' > 0$ such that

$$\|g_0\|_{L^{p}(\mu)} \geq C' \|g_0\|_{L^{\infty}(\mu)} \geq C' \|f_0(\zeta_j, x_j)\| \geq \frac{C'}{2}$$

for every $j \in \mathbb{N}$; contradiction. The proof of the second part is analogous to that of Lemma 5.3.

2. By Proposition 4.7, we know that $\mu$ is a $p$-Carleson measure for $B_{K}^p(N)$. Take $g \in B_{K^c}^p(N)$ so that $g(0,0) = 1$. Notice that, by assumption, there is a constant $C'' > 0$ such that

$$\|f_0\|_{L^{\infty}(\mu)} \leq C'' \|f\|_{L^{\infty}(\mu)}$$
for every $f \in B^p_{K_t}(\mathcal{N})$. Define $C''' := \sup_{\mu(\{(\zeta, z)\}) > 0} \mu(\{(\zeta, z)\})^{-1/p}$. Then, for every $f \in B^p_K(\mathcal{N})$,

$$
\|f_0\|_{L^p(\mathcal{N})} \leq \left\| (\zeta, x) \mapsto \|f_0 L(\zeta, x) g_0\|_{L^\infty(\mathcal{N})} \|f\|_{L^p(\mathcal{N})}
\right. \\
\leq C'' \left\| (\zeta, x) \mapsto \|f L(\zeta, x) \Phi(\zeta)\|_{L^\infty(\mu)} \|f\|_{L^p(\mathcal{N})}
\right. \\
\leq C'' C''' \left\| (\zeta', z') \mapsto f(\zeta', z') \|g(\zeta', z')\|_{L^p(\mathcal{N})} \|f\|_{L^p(\mu)}
\right. \\
\leq C'' C''' \left\| g_0\|_{L^p(\mathcal{N})} \sup_{\eta \in \eta(\text{Supp}\, \mu)} e^{\epsilon |\eta|} \|f\|_{L^p(\mu)}
\right. ,
$$

where the second inequality follows from the fact that $f L(\zeta, x) \Phi(\zeta) \in B^\infty_{K_t}(\mathcal{N})$ for every $(\zeta, x) \in \mathcal{N}$, while the last inequality follows from [10, Theorem 1.7]. Thus, $\mu$ is a $p$-sampling measure for $B^p_K(\mathcal{N})$. The conclusion follows by means of Lemma 5.3.

We now show how the general Beurling-type necessary conditions for sampling sequences proved in [15] look like in this setting. Here, for every $\lambda \in \Lambda^+$ we denote by $|\text{Pf}(\lambda)|$ the (complex) determinant of the positive Hermitian form $\langle \lambda, \Phi \rangle$ with respect to the scalar product of $E$.

**Proposition 5.18.** Take a compact subset $K$ of $\Lambda^+$, and let $S$ be a locally finite subset of $\mathcal{N}$ such that $\mu := \sum_{(\zeta, x) \in S} \delta(\zeta, x + i\Phi(\zeta))$ is a sampling (Radon) measure for $B^2_K(\mathcal{N})$. Then,

$$
\liminf_{R \to +\infty} \inf_{(\zeta, x) \in \mathcal{N}} \frac{\mu(B^\infty_N((\zeta, x), R))}{H^{2n+m}(B^\infty_N((\zeta, x), R))} \geq \frac{2^{n-m}}{\pi^{n+m}} \int_K |\text{Pf}(\lambda)| \, d\lambda.
$$

**Proof.** We may assume that $H^m(K) > 0$, that is, $B^2_K(\mathcal{N}) \neq \{0\}$. Then, the result will follow from [15, Theorem 2.2] and [10, Proposition 5.1] (extended to the case in which $K$ is not necessarily convex), the former applied with $X = \mathcal{N}$, $\mu = H^{2n+m}$, and $H = L^2(\mathcal{N}) \cap \mathcal{O}_K(\mathcal{N})$ (with the norm induced by $L^2(\mathcal{N})$), once we show that the assumptions of [15, section 2.1] are satisfied. The assumption of [15, section 2.1 (A)] are clearly satisfied, since the distance $d_{\mathcal{N}}$ is continuous and $H^{2n+m}(B^\infty_N((\zeta, x), R)) = CR^{(n+m)/2}$ for every $(\zeta, x) \in \mathcal{N}$ and for every $R > 0$, where $C = H^{2n+m}(B^\infty_N((0, 0), 1))$. Since the reproducing kernel $k$ of $H$ satisfies $k(\zeta, x) = L(\zeta, x) k_0$, also the first two assumptions of [15, section 2.1 (B)] are clear. Finally, the third assumption of [15, section 2.1 (B)] follows from the second one, Proposition 4.7, and Corollary 4.9.

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**ENDNOTES**

1. We warn the reader that the notation $M_\delta(\mu)$ will only be used within the Introduction section, and should not be confused with the notation $M_\delta(\mu)$ defined in Definition 4.3, which will be used in the rest of the paper.

2. For $p \in (0, 1]$, $(j\pi/k)$ is simply not a sampling sequence for $B^p_k$, cf., for example, [40, p. 82].
Notice that the reproducing kernel $k$ of $B^2_{\ell}$ is $\frac{\sin(\pi(z-w))}{\pi(z-w)}$, so that $k_z(z) = \frac{\sinh(2\pi x)}{2\pi x}$, which is comparable to $\frac{e^{2\pi |x|}}{|x|}$ for every $z \in \mathbb{C}$. Thus, these sequences are those for which $\sum \frac{1}{k_z(z)} \delta_z$ is a 2-Carleson measure for $B^2_{\ell}$.

A sequence is said to be uniformly separated if the mutual distances between its elements are bounded from below by a strictly positive constant. Here, $\tilde{u}$ denotes the image of $u$ under the conjugation operator, that is the operator $L^\infty(\mathbb{R}) \to BMO(\mathbb{R})$ defined by the Fourier multiplier $\overline{\chi'(-x)}$. In other words, if $U$ denotes the bounded harmonic functions on $C_+$ having $u$ as its nontangential boundary values, and $\tilde{U}$ is its harmonic conjugate, then $\tilde{u}$ is the nontangential boundary values of $\tilde{U}$, at least when $u \in L^p(\mathbb{R})$, $p \in (1, \infty)$. Observe that, since $\tilde{u}$ is defined only up to a constant, the equality (7) should be understood in the sense that $\tilde{u}$ has a representative such that the stated equality holds for almost every $x \in \mathbb{R}$.

This condition is only added to ensure that the corresponding measure be Radon.

Notice that, since we shall consider only sequences with elements in $\mathcal{P}(0)$, this notion of a sampling sequence agrees with the one stated in the Introduction section. Indeed, by $\mathcal{N}$-invariance, the norms of the evaluations at the points of $\mathcal{P}(0)$ are all equal.

This lemma is usually stated for normed spaces, but holds for quasi-normed spaces as well with the same proof. Namely, given a quasi-normed space $X$ and a nondense vector subspace $Y$ of $X$, there is $x \in X$ such that $\|x\|_X = 1$ and $\|x - y\|_X \geq \frac{1}{2}$ for every $y \in Y$.

Notice that this is possible since the set of $f \in B_p, h \in B_p(0, R' + 1)$ is compact in $L^p(\mathcal{N})$, as the mapping $F \ni h \mapsto f \in L^p(\mathcal{N})$ is continuous. To see this fact, observe that $B^2_{\ell}(\mathcal{N})$ is contained in the closure, in $L^p(\mathcal{N})$, of the elements $\{ f \circ \text{Hol}_\ell(E \times F) \mid f_0 \in S(\mathcal{N}) \}$ (argue as in the proof of Lemma 5.9). Since the assertion is clear for the elements of $\{ f \in \text{Hol}_\ell(E \times F) \mid f_0 \in S(\mathcal{N}) \}$, thanks to [7, Theorem 4.2], our claim follows.

Here, $g$ denotes the reflection of $g$, that is, $g(\zeta, x) = g((\zeta, x)) = g(-\zeta, -x)$ for every $(\zeta, x) \in \mathcal{N}$.

Let $(\zeta, x) \in \mathcal{N}$ be real. Then, we may define $i : \ell \to J$ so that $d((z_0, z_0), (z_0', x_0 + i\Phi(z_0))) \leq 6R$ for every $\ell \in \mathcal{L}$. Clearly, there is $N \in \mathbb{N}$ such that the fibers of $i$ have at most $N$ elements, provided that (say) $R \leq 1$. So, we may use [9, Proposition 4.7] and its proof. Therefore, using [9, Lemma 3.25] and Proposition 4.7, we see that there is $C > 0$ such that $\|f(z_0, x_0')\|_{\ell_2} \leq C \|f(z_0, x_0)\|_{\ell_2} + C \|f_0\|_{L^2(\mathcal{N})}$ for every $f \in B^2_{\ell}(\mathcal{N})$, provided that (say) $R \leq 1$. The assertion then follows by means of Proposition 4.7 and [10, Theorem 1.16].

It suffices to take a family $(\zeta, x)$, of elements of $\text{Supp } \mu$, which is maximal for the relation $d((\zeta, x), (\zeta, x)) \geq 2R$.

This is the case if $d_{\ell}$ is defined as a left-invariant homogeneous control distance. In addition, if $n = 0$, then $d_{\mathcal{N}}$ is the distance induced by a norm on $F$, so that this condition is automatically satisfied for every choice of $d_{\mathcal{N}}$.

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