Curvature-induced superconductivity enhancement for ultra-thin superconducting films

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We derive the linearized Ginzburg-Landau (GL) equation for an ultra-thin superconducting film with curvature in a magnetic field. By introducing a novel transverse order parameter that varies slowly along the film, and applying the superconducting/vacuum boundary condition, we decouple the linearized GL equation into a transverse part and a surface part that includes the superconducting geometric potential (GP). The nucleation of the superconducting state in curved thin superconducting films can be equivalently described by the surface part equation. In the equivalent GL free energy of a curved superconducting film, the superconducting GP enables the film to remain in the superconducting state even when the superconducting parameter $\alpha$ turns positive by further reducing the quadratic term of the order parameter. Furthermore, we numerically investigate the phase transition of a rectangle thin superconducting film bent around a cylindrical surface. Our numerical results show that the superconducting GP enhances the superconductivity of the curved film by weakening the effect of the magnetic field, and the increase of the critical temperature, in units of the bulk critical temperature, is equal to the product of the negative superconducting GP and the square of the zero-temperature coherence length, which agrees with our theoretical predictions.

I. INTRODUCTION

With the progress in micro- and nanofabrication techniques, quantization effects associated with the confinement of superconducting condensate at length scales of the order of the superconducting coherence length, $\xi$, have been intensively studied for more than four decades\textsuperscript{1–3}. In these mesoscopic superconducting systems, both the superconducting critical parameters and the vortex matter change substantially with respect to the reference of the bulk materials\textsuperscript{2,4,5}. Good examples here are the normal-superconducting phase boundary $T_c(H)$ of a submicron cylinder that shows a cusplike behavior superimposed with a linear background\textsuperscript{1} and the phase transitions accompanied by bistable region\textsuperscript{4}. Close to $T_c$, the vortices do not order themselves in a triangular vortex lattice. Instead, they form vortex patterns consistent with the geometry of the sample, so that superconductivity nucleates in form of a giant vortex at the center of mesoscopic disks\textsuperscript{7,8} and vortex-antivortex pairs can be spontaneously generated in structures with discrete symmetry\textsuperscript{5,9–12}. These symmetry-induced effects will disappear and the triangular vortex pattern will be recovered by decreasing the temperature\textsuperscript{13}.

In addition to the effects caused by the scale confinement, the geometric curvatures of quasi-two-dimensional (quasi-2D) mesoscopic superconductors can also act on the Cooper-pairs inside. Generally, the quasi-2D structures with radius of curvature much larger than $\xi$, can be treated as fully 2D\textsuperscript{14}. While the studies for systems with radii of curvature comparable to the coherence length become interesting but still at its infancy stage. A much rigorous and physically sound candidate for studying low-dimensional curved structures embedded in three-dimensional (3D) Euclidean space is the thin-layer quantization scheme (TLQS), which was proposed by Jensen, Koppe, and da Costa\textsuperscript{15,16}, and has been constantly developing until now\textsuperscript{17–20}. In this approach, the extrinsic curvature leads to an alteration of the kinetic term in the tangential Hamiltonian with an additional geometric potential (GP)\textsuperscript{21}. This geometric effect is not only a theoretical result, but has been observed through the Tomonaga-Luttinger liquid exponent in a one-dimensional (1D) metallic peanut-shaped $C_{60}$ polymer\textsuperscript{22}, and experimentally realized by an optical analogue in a photonic topological crystal\textsuperscript{14}. In addition, such GP can lead to a topological band structure for electrons confined to periodic minimal surfaces\textsuperscript{23}, an oscillatory behavior of the transmission coefficient of carriers bound to bent cylindrical surface\textsuperscript{24}, and atomic-like bound states in rolled-up nanotubes\textsuperscript{25}. Over the last decade, detecting the micro-stresses in graphene through measuring nanoscale Aharonov-Bohm interferences modulated by elastic deformation fields has been proposed\textsuperscript{26}, and the topological charge pump driven by an periodic magnetic field cooperating with the curvature modulated Rashba spin-orbit interaction has been realized\textsuperscript{27}.

Recent studies have shown that varying the curved geometric profile of Josephson junctions and superconducting ferromagnetic hybrid nanowires can tuned their critical current and temperature, respectively\textsuperscript{28,29}. Moreover, the topological superconductivity can also be induced by
controlling the surface curvature and size of the topological insulator surface$^{30}$. Inspired by the TLQS, in this paper, we aim to study the effect of geometric bending on superconductivity by analyzing on the linearized Ginzburg-Landau (GL) equation confined in a curved superconducting film. It is noteworthy that the strain effect can significantly enhance the superconductivity in superconducting films$^{32,33}$. Generally, it is impossible to distinguish between curvature-induced quantum effects and curvature-induced strain effects in deformed films$^{19,34}$. For the sake of clarity, we solely focus on the curvature-induced effect on the GL equation. This paper is organized as follows: In Sec. II, we derive the linearized GL equation in an ultra-thin superconducting film with curvature, and decouple it into a surface part and a transverse part, which demonstrates that decoupling the linearized GL equation under the superconducting/vacuum $(S/V)$ boundary condition is feasible. In Section III, we analyze the mechanism of geometric potential on enhancing the critical temperature of superconducting thin films based on the Ginzburg-Landau free energy theory. In Sec. IV, we numerically study the nucleation of a rectangular thin superconducting film bent around the surface of a cylinder, and discuss the phase boundary modified by the superconducting GP. Finally, the conclusions are given in Sec. V.

II. LINEARIZED GL EQUATION ON A CURVED THIN SUPERCONDUCTING FILM

We start the derivation with writing the linearized GL equation, which can well describe the nucleation of the superconducting state close to the phase transitions. In the presence of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, with $\mathbf{A}$ representing the vector potential, the stationary linearized GL equation is defined as$^{35,36}$

$$
\frac{1}{2\mu} \left( -i\hbar \nabla - \frac{Q}{c} \mathbf{A} \right)^2 \psi = -\alpha \psi, \tag{1}
$$

where $\mu$ and $Q$ are the mass and charge of a Cooper-pair, respectively, $\psi$ is the superconducting order parameter with its modulus squared proportional to the Cooper-pair density, and the GL parameter

$$
-\alpha = \frac{\hbar^2}{2\mu \xi_0^2} \left( 1 - \frac{T}{T_c} \right), \tag{2}
$$

where $\xi_0$ is the zero-temperature coherent length, $T$ is the actual temperature of the superconductor, and $T_c$ represents the bulk critical temperature at zero magnetic field. Eq. (1) is invariant under the following gauge transformations$^{37}$:

$$(\psi, \mathbf{A}) \rightarrow (\psi', \mathbf{A}') = (\psi \exp(iQ\gamma/\hbar), \mathbf{A} + \nabla \gamma), \tag{3}$$

with $\gamma$ being an arbitrary scalar function.

Let us consider a curved thin superconducting film with uniform thickness $d$, sketched in Fig. 1, immersed in an insulating medium or vacuum. The space occupied by the superconductor, denoted by $\Omega$, is bounded by two parallel surfaces, $S_1$ and $S_2$, and the side surfaces, denoted by $S_0$, around. For clarity, we introduce the central surface $S$ which is equidistant from both $S_1$ and $S_2$, and suppose that $S_0$ is perpendicular to $S$ at their intersections. The boundary condition for this sample is

$$
\left. \{ \mathbf{n} \cdot [i\hbar \nabla + (Q/c) \mathbf{A}] \psi \} \right|_{\partial \Omega} = 0, \tag{4}
$$

where $\partial \Omega \equiv S_0 \cup S_1 \cup S_2$, and $\mathbf{n}$ indicates the outward unit vector normal to $\partial \Omega$.

When $d$ goes to zero, the Cooper-pairs in $\Omega$ are constrained in the immediate neighborhood of the surface $S$. Following the parametrization of da Costa$^{16}$, the surface $S$ can be parametrized by the position vector $\mathbf{r}(q^1, q^2)$ with $q^1, q^2$ the surface coordinates on $S$, and the portion of $\Omega$ can be parametrized as

$$
\mathbf{R}(q^1, q^2, q^3) = \mathbf{r}(q^1, q^2) + q^3 \mathbf{e}_3(q^1, q^2), \tag{5}
$$

where $\mathbf{e}_3(q^1, q^2)$ is the normal basis vector depending only on $q^1$ and $q^2$, and $|q|^2$ gives the distance from the surface $S$. In what follows, the indices $i, j, k = 1, 2, 3$ stand the spatial coordinates and the spatial components of vector fields in $\Omega$, whereas the indices $a, b, c = 1, 2, 3$ enumerate the surface coordinates and the surface components for vector field on $S$. Einstein summation is also used in the following. The total derivative of $\mathbf{R}(q^1, q^2, q^3)$ is

$$
\frac{d\mathbf{R}(q^1, q^2, q^3)}{dq^a} = \epsilon_a dq^a + e_3 dq^3, \tag{6}
$$

where

$$
\epsilon_a = \partial_a \mathbf{R} = \partial_a \mathbf{r} + q^3 \partial_a \mathbf{e}_3 \tag{7}
$$

with $\partial_a$ being the derivative with respect to the spatial variable $q^a$, and $\epsilon_1, \epsilon_2$ and $\epsilon_3$ composing the basis vectors of the da Costa’s 3D curvilinear coordinates in $\Omega$. The basis vectors for surface coordinates $q^a$ are $\epsilon_a = \partial_a \mathbf{r}$, and the normal basis vector $\epsilon_3$ is defined as $\epsilon_3 = \epsilon_1 \times \epsilon_2 / |\epsilon_1 \times \epsilon_2|$. The derivatives of $\epsilon_3(q^1, q^2)$ lie in the tangent plane and satisfy

$$
\partial_a \epsilon_3 = w^b_a \partial_a \mathbf{r} \tag{8}
$$

with $w^b_a$ the components of the Weingarten curvature matrix on $S$. The covariant components of the metric tensor in $\Omega$ and on $S$ are, respectively, defined by $G_{ij} = \partial_i \mathbf{R} \cdot \partial_j \mathbf{R}$ and $g_{ab} = \partial_a \mathbf{r} \cdot \partial_b \mathbf{r}$. From Eq. (5) we obtain

$$
G_{ab} = g_{ab} + (\hat{w} \hat{g} + \hat{g} \hat{w}^T \hat{w}^T)_{ab} q^3 + \left( \hat{w} \hat{g} \hat{w}^T \right)_{ab} (q^3)^2, \tag{9}
$$

where $\hat{w}$ and $\hat{g}$ are the matrices of $w_a^b$ and $g_{ab}$, respectively, and the superscript $T$ denotes the matrix transpose. From Eq. (9), we have

$$
G = f^2 g, \tag{10}
$$
where $G$ and $g$ are the determinants with respect to the metric tensors $G_{ij}$ and $g_{ab}$, respectively, and $f = 1 + 2Mq^3 + K(q^3)^2$ with $M = \text{Tr}(\tilde{\omega})/2$ and $K = \det(\tilde{\omega})$ being the mean curvature and the Gaussian curvature, respectively.

In da Costa’s 3D curvilinear coordinates, the divergence of a vector field $\mathbf{v}$ reads

$$\nabla \mathbf{v} = \nabla_i v^i = \partial_i v^i + \delta^i_j \Gamma^k_{ji} v^k,$$

where $\nabla_i$ are the covariant derivatives, $v^i$ are the contravariant components of $\mathbf{v}$, $\Gamma^k_{ji}$ are the Christoffel symbols, and $\delta^i_j \Gamma^k_{ji} = (1/\sqrt{G}) \partial_i \sqrt{G}$. Therefore, the linearized GL equation and the boundary conditions can be rewritten as

$$\begin{align}
\frac{1}{2\mu} \left[ -\frac{\hbar^2}{\sqrt{G}} \partial_i \left( \sqrt{G} G^{ij} \partial_j \psi \right) + \frac{2i \hbar Q}{c} G^{ij} A_i \partial_j \psi \right. \\
+ \frac{i \hbar Q}{c \sqrt{G}} \partial_i \left( \sqrt{G} G^{ij} A_j \right) \psi + \frac{Q^2}{c^2} G^{ij} A_i A_j \psi \right] = -\alpha \psi \tag{12}
\end{align}$$

and

$$\left\{ n \cdot \left[ G^{ij} (i \hbar \partial_i + (Q/c) A_i) \psi \mathbf{e}_j \right] \right\} |_{\partial \Omega} = 0, \tag{13}$$

where $G^{ij}$ are the reciprocals of $G_{ij}$, and $A_i$ denote the covariant components of the vector potential $\mathbf{A}$. The structure of $G^{ij}$ implies that the linearized GL equation can be separated into surface and transverse parts. Meanwhile, the order parameter $\psi$ from Eq. (12) can take the factorizable form $\psi = f^{-1/2} \chi$, where $f^{-1/2}$ is employed for the conservation of the Cooper-pair density\textsuperscript{16,38}. Then, we take into account the compressing procedure: in the limit of confinement, the order parameter $\psi$ is constrained on $S$ with its value being different from zero only for an extremely small range of $|q^3| \leq d/2$.

With the limit $d \to 0$, i.e., $q^3 \to 0$, the corresponding relations between the original order parameter $\psi$ and the new order parameter $\chi$ and their derivatives are

$$\begin{align}
\lim_{q^3 \to 0} \psi = \chi, & \quad \lim_{q^3 \to 0} \partial_3 \psi = \partial_3 \chi - M \chi, \\
\lim_{q^3 \to 0} \partial_3^2 \psi = \partial_3^2 \chi - 2M \partial_3 \chi + (3M^2 - K) \chi. \tag{14}
\end{align}$$

From both Eqs. (12) and (13), we have the linearized GL equation

$$\begin{align}
\frac{\hbar^2}{2\mu} \left[ -\frac{1}{\sqrt{g}} \partial_a \left( \sqrt{g} g^{ab} \partial_b \chi \right) - \partial_3^2 \chi \right] + \frac{i \hbar Q}{2\mu c} \left[ \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} A_b) \chi + \partial_3 A_3 \chi + 2(g^{ab} A_a \partial_b \chi + A_3 \partial_3 \chi) \right] \\
+ \frac{Q^2}{2\mu c^2} (g^{ab} A_a A_b + A_3^2) \chi + V_s \chi = -\alpha \chi, \tag{15}
\end{align}$$

and the boundary conditions (13) separated into the surface part and the transverse part as follows,

$$\begin{align}
&\left\{ n \cdot \left[ g^{ab} (i \hbar \partial_a + (Q/c) A_a) \chi \mathbf{e}_b \right] \right\} |_{S_0} = 0, \tag{16} \\
&\left[ i \hbar (\partial_3 \chi - M \chi) + (Q/c) A_3 \chi \right] |_{S_1 \cup S_2} = 0. \tag{17}
\end{align}$$

where $V_s = -\hbar^2/(2\mu)(M^2 - K)$ is the well-known GP\textsuperscript{16,18,39}, $A_1$ and its derivative are calculated at $q^3 = 0$. In Eq. (15), the term $A_3 \partial_3 \chi$ couples the vector potential along $q^3$ with the order parameter on $S$. Applying the gauge transformations (3), we can cancel the coupling term by choosing

$$\gamma (q^1, q^2, q^3) = -\int_0^{q^3} A_3 (q^1, q^2, z) \, dz \tag{18}$$

\textsuperscript{18}. In the limit $q^3 \to 0$, we have $A'_3 = 0$ and $\partial_3 A'_3 = 0$ while $A_1$ and $A_2$ unchanged. In the conventional TLQS, due to the restriction that transverse fluctuations depend only on $q^3$\textsuperscript{31}, a transverse Neumann-type constraint generally makes Schrödinger-type equations of motion on surfaces with non-constant mean curvature unseparable\textsuperscript{40}. Here, to obtain a surface linearized GL equation accommodating the transverse part of the boundary conditions (17), we introduce a new order parameter

$$\chi = \chi_s (q^1, q^2) \chi_t (q^1, q^2, q^3), \tag{19}$$

with $\chi_s$ and $\chi_t$ the surface part and the transverse part of the order parameter, respectively. It is easy to verify that the separability requirement in TLQS is still met when $\chi_t$ is such a slowly varying function along $S$ that $\lim_{q^3 \to 0} \partial_3 \chi_t = 0$\textsuperscript{40}.

After substituting the gauge transformation (18) into Eq. (15) and assuming $\alpha = \alpha_s + \alpha_t$, where $\alpha_s$ and $\alpha_t$ are the eigenvalues corresponding to the surface and the transverse eigenstates, respectively, Eq. (15) is separated into two independent equations:

$$-\frac{\hbar^2}{2\mu} \partial_3^2 \chi_t = -\alpha_t \chi_t, \tag{20}$$

$$\frac{1}{2\mu} \left( -i \hbar \nabla_s - \frac{Q}{c} A_s \right)^2 \chi_s + V_s \chi_s = -\alpha_s \chi_s, \tag{21}$$

where $\nabla_s = g^{ab} e_a \partial_b$ and $A_s = g^{ab} e_a A_b$. Eq. (20) represents the 1D linearized GL equation along $q^3$, while Eq. (21) describes the order parameter attached to the surface $S$ in the magnetic field, presented in shorthand form.
The transverse part of the boundary condition from Eq. (17) is given by

\[ [(\partial_t - M) \chi_t] \big|_{S_1 \cup S_2} = 0. \]  

(22)

From Eq. (20), the permissible eigenstate subjected to the boundary condition (22) is \( \chi_t \sim \exp(Mq^2) \), with the corresponding eigenvalue \(-\alpha_s = -h^2 M^2/(2\mu)\). Under the surface part of the boundary condition (16), \(-\alpha_s\) can be determined by computing the lowest eigenvalue of the following equation:

\[ \frac{1}{2\mu} \left( -i\hbar \nabla_s - \frac{Q}{c} A_s \right)^2 \chi_s + V^s_g \chi_s = -\alpha_s \chi_s, \]  

(23)

where

\[ V^s_g = -\frac{\hbar^2}{2\mu} (2M^2 - K) \]  

(24)

is the GP for the thin superconducting film, which is always negative on curved surfaces. At this point, we can state one of the most fundamental conclusions in this paper: By introducing a novel transverse order parameter, depending both on the surface and transverse coordinates, that varies very slowly along the surface, the linearized GL equation can be analytically decoupled into the transverse part and the surface part under the S/V boundary condition (16), and the nucleation of the superconducting state can be described by Eq. (23).

III. GL FREE ENERGY

In this section, we explore the effect of the superconducting GP on the nucleation of the superconducting film from the viewpoint of the GL free energy. Within the framework of the GL theory, the free energy of the superconducting film sketched in Fig. 1 reads

\[ F = F_n + \int_{\Omega} \left( \bar{\psi} \hat{L} \psi + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \varepsilon_{\text{mag}} \right) \, dv, \]  

(25)

where \( F_n \) represents the free energy in the normal state, and \( \beta \) is the GL parameter, \( \hat{L} = (2\mu)^{-1} \left(-i\hbar \nabla - (Q/c) A_s \right)^2 \) denotes the GL kinetic term, and \( \varepsilon_{\text{mag}} \) corresponds to the energy density of the magnetic field. Employing da Costa’s 3D curvilinear coordinates and considering both Eqs. (18) and (19), along with the solution for \( \chi_t \), we can equivalently reduced Eq. (25) to a surface integral over the GL free energy density as follows:

\[ F = F_n + A \int_{S} \left( \bar{\chi}_s \hat{L}_s \chi_s + \alpha |\chi_s|^2 + V^s_g |\chi_s|^2 + \frac{\beta}{2} |\chi_s|^4 + \varepsilon_{\text{mag}} \right) \, ds, \]  

(26)

where the coefficient \( A \) depends on the integrating over the film thickness, \( \hat{L}_s = (2\mu)^{-1} \left(-i\hbar \nabla_s - (Q/c) A_s \right)^2 \) is the surface part of \( \hat{L} \), and \( ds = \sqrt{g} dq^1 dq^2 \). The superconducting GP incorporates the effects resulting from both the curvature and the transverse constraint on the order parameter. This potential facilitates a further reduction in the superconducting free energy through the quadratic term of the order parameter. Consequently, the film is able to retain its superconducting state even when the parameter \( \alpha \) turns positive.

For a flat superconducting film, where \( V^s_g \) vanishes, the critical temperature \( T^* \) is determined by the equation

\[ -\alpha_0 = \frac{h^2}{2\mu \xi_0^2} \left( 1 - \frac{T^*}{T_c} \right), \]  

(27)

with \(-\alpha_0\) being the lowest positive eigenvalue of the kinetic energy term from Eq. (23). When the superconducting film is bent into a curved film with uniform curvature, the critical temperature \( \tilde{T}^* \) is modified by \( V^s_g \) and satisfies the relation

\[ (\tilde{T}^* - T^*)/T_c = -V^s_g c_0^2. \]  

(28)

Therefore, we can state the other fundamental conclusion of this paper at the end of this section: The increase in the critical temperature of a superconducting film with uniform curvature relative to its flat shape, measured in units of the bulk critical temperature, is equal to the product of the negative superconducting GP and the square of the zero-temperature coherence length.

IV. CURVATURE-INDUCED NUCLEATION OF A SUPERCONDUCTING FILM

To present the geometric effects on the normal-superconducting phase transition, in the following, we numerically study the nucleation of a rectangular thin superconducting film bent around the surface of a cylinder. First of all, we introduce the dimensionless variables for convenience,

\[ \xi = \xi_0 \xi', \quad q^a = \xi_0 q^a, \quad \partial_a = \xi_0^{-1} \partial_a', \quad M = M'/\xi_0, \]

\[ K = K'/\xi_0^2, \quad A = \frac{hc}{\xi_0 Q} A', \quad \alpha = \frac{h^2}{2\mu \xi_0^2} \alpha', \]  

(29)

with \( \partial_a' = \partial/\partial q^a \). Using the algebraic transform (29) and omitting the primes in the dimensionless variables, we have the scaled Eqs. (23) and (16) in the following form:

\[ \frac{1}{\sqrt{g}} \partial_a \left( \sqrt{g} g^{ab} \partial_b \chi_s \right) + \frac{i}{\sqrt{g}} \partial_a \left( \sqrt{g} g^{ab} A_b \right) \chi_s + 2i g^{ab} A_a \partial_b \chi_s + g^{ab} A_a A_b \chi_s + V^s \chi_s = -\alpha \chi_s, \]

(30)

\[ \{ n \cdot [g^{ab} (i \partial_a + A_a) \chi_s e_b] \} \big|_{S_0} = 0, \]  

(31)

where the scaled superconducting GP \( V^s \) is \(-(2M^2 - K)\), and the scaled GL parameter \(-\alpha = 1 - T/T_c\). As shown in Fig. 2, we consider a rectangular superconducting film
with width \( U \) and height \( V \), bent around the surface of a cylinder with radius \( R \) along the width direction. This superconducting film can be parameterized by two dimensionless surface coordinates \((u, v)\) where \( u \) and \( v \) correspond to the coordinates along the width and height directions, respectively. To purely study the impact of geometric bending on the superconductivity of the film, we apply a magnetic field perpendicular to the film with uniform intensity, ensuring that the magnetic field-induced free energy remains constant. Such a magnetic field can be achieved by attaching the superconducting film to a magnetic substrate with perpendicular magnetization\(^{41,42}\). The diamagnetic susceptibility of a superconducting film in the normal state, due to the fluctuation, is many orders of magnitude smaller than the full diamagnetic susceptibility in the Meissner state\(^{43}\). In other words, the magnetic field generated by the diamagnetism is usually negligible.

By choosing the local Landau gauge with zero transverse component, one of the most suitable vector potentials on the superconducting film, determined by Eq. (18), is \((A_u, A_v) = (0, Bu)\), which corresponds to \( \nabla \times A = B \). From Eq. (30), the linearized GL equation on the film reads

\[
-\partial_u^2 \chi_s - (\partial_v - iBu)^2 \chi_s - \frac{1}{2R^2} \chi_s = -\alpha \chi_s, \tag{32}
\]

where \(-1/(2R^2)\) is the dimensionless superconducting GP. With the origin of the coordinates \((u, v)\) chosen at the center of the film, the boundary conditions (16) for Eq. (32) become

\[
\partial_u \chi_s \big|_{u=\pm U/2} = 0, \quad (\partial_v - iBu) \chi_s \big|_{v=\pm V/2} = 0. \tag{33}
\]

Although Eq. (32) bears resemblance to the well-known 2D free electron equation in a magnetic field, the boundary conditions (33) pose challenges in analytically deriving its eigenstates. Therefore, we employ finite element simulations to determine the critical temperature \( T^* \), corresponding to the lowest positive eigenvalue of \(-\alpha\), for the model depicted in Fig. 2. In panel (a) of Fig. 3, we present the phase diagram of the critical temperature \( T^* \) (measured in units of \( T_c \)), with \( U = 8 \xi_0 \) and \( V = 6 \xi_0 \), as a function of the mean curvature \( M \) (measured in units of \( \xi_0^{-1} \)) and the magnetic field intensity \( B \) (measured in units of \( \hbar c \xi_0^{-2} Q^{-1} \)). In panel (b) of Fig. 3, the slices of \( T^* \) at different intensities of magnetic field as a function of the mean curvature are presented. The geometric effect can weaken the action of the magnetic field, leading to an increase of \( T^* \) with the mean curvature. The numerical results indicate that \( T^* \) depends quadratically on \( M \). Fig. 3 (c) shows the slices of \( T^* \) at different mean curvatures as a function of the intensity of magnetic field \( B \). The critical temperature \( T^* \), decreases monotonically with increasing the magnetic field intensity and exhibits Little-Parks oscillations, with each cusp corresponding to a change in fluxoid quantum number \( L \) by one\(^2\). By connecting the corresponding cusps on all \( \tilde{T}^*(B) \) curves at different mean curvatures in Fig. 3 (a) (black-and-white dashed lines), we can subdivided the \( M-B \) plane into areas characterized by distinct \( L \) values. Compared to the model without curvature, all \( T^*(B) \) curves for finite \( M \) exhibit upward shifts along the \( T^* \) axis. Furthermore, numerical results reveal that these shifts amount to \( 2M^2 \), which precisely corresponds to the modifications induced by \( V_s^2 \) on the lowest \(-\alpha\) of the flat film. These findings corroborate that the geometric curvature can effectively enhance the critical temperature of the film, in agreement with the prediction in Eq. (28).

V. CONCLUSIONS

In this paper, we have developed a general linearized GL equation valid for any curved thin superconducting film surrounded by an insulating medium or vacuum in the presence of a magnetic field. By introducing the novel order parameter \( \chi \), given in Eq. (19), whose transverse component varies extremely slowly along the surface, and taking into account Ferrari’s gauge transformation to eliminate the terms involving \( A_3 \) and \( \partial_3 A_3 \), we have decoupled the linearized GL equation into its surface and transverse components. Furthermore, we have provided an analytical solution for the transverse component of \( \chi \), and have derived the superconducting GP, \( V_s^2 = -\hbar^2 / (2\mu) \left(2M^2 - K^2\right)\), under the S/V boundary condition.

Using the thin-layer quantization scheme, we have derived the equivalent 2D reduced GL free energy for curved superconducting films. Based on the equivalent free energy, we have clarified the mechanism behind the curvature-induced superconductivity enhancement. That is, the superconducting GP facilitates an additional reduction in the free energy through the quadratic term of the order parameter, which allows the film to maintain
Its superconducting state even when the superconducting parameter $\alpha$ turns positive. Moreover, the increase in critical temperature of a planar superconducting film bent into a uniformly curved film, measured in unit of the bulk critical temperature, equals the product of the negative superconducting GP and the zero-temperature coherence length squared.

To present the effects of geometric curvature on the superconductivity, we have numerically studied the normal-superconducting phase transition of a rectangular thin superconducting film bent around the surface of a cylinder. With applying the magnetic field perpendicular to the film with uniform intensity over the film, we have numerically computed the critical temperature $T^*$ depending both on the intensity of magnetic field $B$ and the mean curvature $M$, and have charted the phase diagram. Our numerical calculations have demonstrated that the geometric curvature can enhance the critical temperature of the superconducting film by compensating the impact resulting from the magnetic field. In particular, we have found that the critical temperature depends quadratically on the mean curvature, and have observed the Little-Parkers oscillations in the $T^*-B$ phase boundaries. These numerical findings have supported our theoretical estimates of the critical temperature for superconducting films with uniform curvature.

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