INTRINSIC CONTRACTIVITY OF FEYNMAN-KAC SEMIGROUPS FOR SYMMETRIC JUMP PROCESSES WITH INFINITE RANGE JUMPS

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Abstract. Let \((X_t)_{t \geq 0}\) be a symmetric strong Markov process generated by non-local regular Dirichlet form \((D, \mathcal{D}(D))\) as follows
\[
D(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x, y) \, dx \, dy, \quad f \in \mathcal{D}(D)
\]
where \(J(x, y)\) is a strictly positive and symmetric measurable function on \(\mathbb{R}^d \times \mathbb{R}^d\).

We study the intrinsic hypercontractivity, intrinsic supercontractivity and intrinsic ultracontractivity for the Feynman-Kac semigroup
\[
T_t^V(f)(x) = \mathbb{E}^x \left( \exp \left( - \int_0^t V(X_s) \, ds \right) f(X_t) \right), \quad x \in \mathbb{R}^d, f \in L^2(\mathbb{R}^d; dx).
\]
In particular, we prove that for
\[
J(x, y) \asymp |x - y|^{-d-\alpha} \mathbf{1}_{\{|x-y| \leq 1\}} + e^{-|x-y|} \mathbf{1}_{\{|x-y| > 1\}}
\]
with \(\alpha \in (0, 2)\) and \(V(x) = |x|^\lambda\) with \(\lambda > 0\), \((T_t^V)_{t \geq 0}\) is intrinsically ultracontractive if and only if \(\lambda > 1\); and that for symmetric \(\alpha\)-stable process \((X_t)_{t \geq 0}\) with \(\alpha \in (0, 2)\) and \(V(x) = \log(1 + |x|)\) with some \(\lambda > 0\), \((T_t^V)_{t \geq 0}\) is intrinsically ultracontractive (or intrinsically supercontractive) if and only if \(\lambda > 1\), and \((T_t^V)_{t \geq 0}\) is intrinsically hypercontractive if and only if \(\lambda \geq 1\). Besides, we also investigate intrinsically contractive properties of \((T_t^V)_{t \geq 0}\) for the case that \(\lim_{|x| \to \infty} V(x) \neq \infty\).

Keywords: symmetric jump process; Lévy process; Dirichlet form; Feynman-Kac semigroup; intrinsic contractivity

MSC 2010: 60G51; 60G52; 60J25; 60J75.

1. Introduction and Main Results

1.1. Setting and Assumptions. Let \(((X_t)_{t \geq 0}, \mathbb{P}^x)\) be a symmetric strong Markov process on \(\mathbb{R}^d\) generated by the following non-local symmetric regular Dirichlet form
\[
D(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x, y) \, dx \, dy,
\]
\[
\mathcal{D}(D) = C^1_c(\mathbb{R}^d)^D_1,
\]
Here \(J(x, y)\) is a strictly positive and symmetric measurable function on \(\mathbb{R}^d \times \mathbb{R}^d\) satisfying that

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\begin{itemize}
  \item There exist constants $\alpha_1, \alpha_2 \in (0, 2)$ with $\alpha_1 \leq \alpha_2$ and positive constants $\kappa, c_1, c_2$ such that
  \begin{align}
  c_1 |x - y|^{-d - \alpha_1} & \leq J(x, y) \leq c_2 |x - y|^{-d - \alpha_2}, \quad 0 < |x - y| \leq \kappa, \\
  \int_{|x - y| > \kappa} J(x, y) \, dy & < \infty; \\
  \end{align}
  \end{itemize}

$L^\infty_c (\mathbb{R}^d)$ denotes the space of $C^1$ functions on $\mathbb{R}^d$ with compact support, and $L^\infty_c (\mathbb{R}^d)^{D_1}$ denotes the closure of $L^\infty_c (\mathbb{R}^d)$ under the norm $\|f\|_{D_1} := \sqrt{\int |f|^2 \, dx + \int f^2 \, dx}$. According to [1, Theorems 1.1 and 1.2], $(X_t)_{t \geq 0}$ has a symmetric, bounded and positive transition density function $p(t, x, y)$ defined on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, whence the associated strongly continuous Markov semigroup $(T_t)_{t \geq 0}$ is given by

$$T_t f(x) := \mathbb{E}^x (f(X_t)) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy, \quad x \in \mathbb{R}^d, \ t > 0, \ f \in B_b (\mathbb{R}^d),$$

where $\mathbb{E}^x$ denotes the expectation under the probability measure $\mathbb{P}^x$. Throughout this paper, we further assume that for every $t > 0$, the function $(x, y) \mapsto p(t, x, y)$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$, see [5, 6, 7, 1, 8] and the references therein. For symmetric Lévy process $(X_t)_{t \geq 0}$, the continuity of density function is equivalent to $e^{-t\Psi_0 (\cdot)} \in L^1 (\mathbb{R}^d; \, dx)$ for any $t > 0$, where $\Psi_0$ is the characteristic exponent or the symbol of Lévy process $(X_t)_{t \geq 0}$

$$\mathbb{E}^x (e^{i \xi \cdot (X_t - x)}) = e^{-t \Psi_0 (\xi)}, \quad \xi \in \mathbb{R}^d, \ t > 0.$$

Let $V$ be a non-negative measurable and locally bounded measurable (potential) function on $\mathbb{R}^d$. Define the Feynman-Kac semigroup $(T^V_t)_{t \geq 0}$

$$(T^V_t f)(x) = \mathbb{E}^x \left( \exp \left( - \int_0^t V(X_s) \, ds \right) f(X_t) \right), \quad x \in \mathbb{R}^d, \ f \in L^2 (\mathbb{R}^d; \, dx).$$

It is easy to check that $(T^V_t)_{t \geq 0}$ is a bounded symmetric semigroup on $L^2 (\mathbb{R}^d; \, dx)$. By assumptions of $(X_t)_{t \geq 0}$, for each $t > 0$, $T^V_t$ is also bounded from $L^1 (\mathbb{R}^d; \, dx)$ to $L^\infty (\mathbb{R}^d; \, dx)$, and there exists a symmetric, bounded and positive transition density function $p^V(t, x, y)$ such that for every $t > 0$, the function $(x, y) \mapsto p^V(t, x, y)$ is continuous, and for every $1 \leq p \leq \infty$,

$$T^V_t f(x) = \int_{\mathbb{R}^d} p^V(t, x, y) f(y) \, dy, \quad x \in \mathbb{R}^d, \ f \in L^p (\mathbb{R}^d; \, dx),$$

see e.g. [10, Section 3.2]. Suppose that for every $r > 0$,

$$|\{ x \in \mathbb{R}^d : \ V(x) \leq r \}| < \infty,$$

where $|A|$ denotes the Lebesgue measure of Borel set $A$. According to [4, Proposition 1.1] (which is essentially based on [20, Corollary 1.3]), the semigroup $(T^V_t)_{t \geq 0}$ is compact. By general theory of semigroups for compact operators, there exists an orthonormal basis in $L^2 (\mathbb{R}^d; \, dx)$ of eigenfunctions $\{ \phi_n \}_{n=1}^\infty$ associated with corresponding eigenvalues $\{ \lambda_n \}_{n=1}^\infty$ satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ and $\lim_{n \to \infty} \lambda_n = \infty$. That is, $L_V \phi_n = -\lambda_n \phi_n$ and $T^V_t \phi_n = e^{-\lambda_n t} \phi_n$, where $(L_V, \mathcal{D}(L_V))$ is the infinitesimal generator of the semigroup $(T^V_t)_{t \geq 0}$. The first eigenfunction $\phi_1$ is called ground
state in the literature. Indeed, in our setting there exists a version of \( \phi_1 \) which is bounded, continuous and strictly positive, e.g. see [4, Proposition 1.2].

In the following, we always assume that (1.4) holds, and that the ground state \( \phi_1 \) is bounded, continuous and strictly positive.

1.2. Previous Work and Motivation. We are concerned with the intrinsic contractivity for the semigroup \((T^V_t)_{t \geq 0}\). We first recall some definitions of intrinsic contractivity for Markov semigroups introduced in [11]. The semigroup \((T^V_t)_{t \geq 0}\) is intrinsically ultracontractive if and only if for any \( t > \) Lévy process \( \gamma \) is intrinsically supercontractive, which is in turn stronger than the intrinsic ultracontractivity of Feynman-Kac semigroups introduced in [11]. The semigroup \((1.2)\) is bounded, continuous and strictly positive, e.g. see [4, Proposition 1.2].

Define

\[
\tilde{T}^V_t f(x) := \frac{e^{\lambda^* t}}{\phi_1(x)} T^V_t ((\phi_1 f))(x), \quad t > 0,
\]

which is a Markov semigroup on \( L^2(\mathbb{R}^d; \phi_1^2(x) \, dx) \). Then, \((T^V_t)_{t \geq 0}\) is intrinsically ultracontractive if and only if \((\tilde{T}^V_t)_{t \geq 0}\) ultracontractive, i.e., for every \( t > 0 \), \( \tilde{T}^V_t \) is a bounded operator from \( L^2(\mathbb{R}^d; \phi_1^2(x) \, dx) \) to \( L^\infty(\mathbb{R}^d; \phi_1^2(x) \, dx) \). If for every \( 2 < p < \infty \), there exists a constant \( t_0(p) \geq 0 \) such that for all \( t > t_0(p) \), \( \tilde{T}^V_t \) is a bounded operator from \( L^p(\mathbb{R}^d; \phi_1^2(x) \, dx) \) to \( L^\infty(\mathbb{R}^d; \phi_1^2(x) \, dx) \), then we say \((\tilde{T}^V_t)_{t \geq 0}\) is hypercontractive, and equivalently, \((T^V_t)_{t \geq 0}\) is intrinsically hypercontractive. If one can take \( t_0(p) = 0 \), then we say \((\tilde{T}^V_t)_{t \geq 0}\) is supercontractive, and equivalently, \((T^V_t)_{t \geq 0}\) is intrinsically supercontractive. In particular, the intrinsic ultracontractivity is stronger than the intrinsic supercontractivity, which is in turn stronger than the intrinsic hypercontractivity.

The intrinsic ultracontractivity of \((T^V_t)_{t \geq 0}\) associated with pure jump symmetric Lévy process \((X_t)_{t \geq 0}\) has been investigated in [15, 13, 14]. The approach of all these cited papers is based on two-sided estimates for ground state \( \phi_1 \) corresponding to the semigroup \((T^V_t)_{t \geq 0}\). However, some restrictions on the density function \( \rho \) of Lévy measure and the potential function \( V \) are needed, see [4, Assumptions 2.1 and 2.5] or assumptions (H1)–(H3) below. Recently, the authors make use of super Poincaré inequalities with respect to infinite measure developed in [17] and functional inequalities for non-local Dirichlet forms recently studied in [19, 22, 3] to study the intrinsic ultracontractivity of Feynman-Kac semigroups for symmetric jump processes in [4]. The main result [4, Theorem 1.3] can deal with symmetric jump process such that associated jump kernel is given by

\[
J(x, y) \asymp |x - y|^{-d-\alpha} \mathbf{1}_{\{|x - y| \leq 1\}} + e^{-|x - y|\gamma} \mathbf{1}_{\{|x - y| > 1\}}
\]

with \( \alpha \in (0, 2) \) and \( \gamma \in (1, \infty) \), for which the approach of [15, 13, 14] does not work. In particular, when \( \gamma = \infty \),

\[
J(x, y) \asymp |x - y|^{-d-\alpha} \mathbf{1}_{\{|x - y| \leq 1\}},
\]

which is associated with the truncated symmetric \( \alpha \)-stable-like process.

As already mentioned in [4], in the model above finite range jumps play an essential role in the behavior of the associated process. In the present setting, the
argument of [4] could apply to obtain some sufficient conditions for intrinsic ultra-contractivity of \((T^V_t)_{t \geq 0}\). However, as we will see from examples below, the conclusions yielded by the approach of [4] are far from optimality, since now large range jumps make dominant roles. This explains the motivation of our present paper.

The main contribution of this paper is to derive explicit and sharp criterion for intrinsically contractive properties of Feynman-Kac semi groups for symmetric jump processes with infinite range jumps. We will see later that, even for symmetric Lévy process with infinite range jumps, our results can get rid of many technical restrictions used in [15, 13, 14]. Moreover, our method here efficiently applies to strong Markov process generated by non-local Dirichlet forms. In particular, the associated process is usually not space-homogeneous. We can also obtain some sufficient conditions for the intrinsic supercontractivity and intrinsic hypercontractivity of \((T^V_t)_{t \geq 0}\), and for the case that \(\lim_{|x| \to \infty} V(x) \neq \infty\). Both of them, to the best of our knowledge, do not appear in the literature.

1.3. Main Results. To state our main result, we need some necessary assumptions and notations. For \(x \in \mathbb{R}^d\), define

\[
J^*(x) = \begin{cases} 
\inf_{y-z \in B(x,3/2)} J(y,z), & |x| \geq 3, \\
1 & |x| < 3,
\end{cases}
\]

and

\[
V^*(x) = \sup_{z \in B(x,1)} V(z), \quad \varphi(x) = \frac{J^*(x)}{1 + V^*(x)}.
\]

For any \(s, r > 0\), set

\[
\alpha(r, s) := \inf \left\{ \frac{2}{|B(0,t)|} \inf_{x \in B(0,r+t)} \varphi^2(x) : t \leq r \text{ and } \frac{2 \sup_{0 < |x - y| \leq t} J(x,y)^{-1}}{|B(0,t)|} \leq s \right\},
\]

1.3.1. Regular Potential Function: \(\lim_{|x| \to \infty} V(x) = \infty\).

**Theorem 1.1.** Suppose that

\[
\lim_{|x| \to \infty} V(x) = \infty.
\]

For any \(s, \delta_1, \delta_2 > 0\), define

\[
\Phi(s) = \inf_{|x| \geq s} V(x)
\]

and

\[
(1.6) \quad \beta(s) := \beta(s; \delta_1, \delta_2) = \delta_1 \alpha \left( \Phi^{-1} \left( \frac{4}{s \wedge \delta_2} \right), \frac{s \wedge \delta_2}{4} \right).
\]

Then, we have the following three statements.

1. If for any constants \(\delta_1, \delta_2 > 0\),

\[
\int_t^\infty \frac{\beta^{-1}(s)}{s} ds < \infty, \quad t > \inf \beta,
\]

then the semigroup \((T^V_t)_{t \geq 0}\) is intrinsically ultracontractive.

2. If for any constants \(\delta_1, \delta_2 > 0\),

\[
\lim_{s \to 0} s \log \beta(s) = 0,
\]

then the semigroup \((T^V_t)_{t \geq 0}\) is intrinsically supercontractive.
(3) If for any constants \( \delta_1 \) and \( \delta_2 > 0 \),

\[
\limsup_{s \to 0} s \log \beta(s) < \infty
\]

then the semigroup \( (T_t^V)_{t \geq 0} \) is intrinsically hypercontractive.

To apply Theorem 1.1, we do not need the following extra restrictions (see [14, Assumptions 2.1, 2.3 and 2.5]) on \( \rho \) and \( V \):

(B1) There are constants \( c_3 \) and \( c_4 \geq 1 \) such that

\[
c_3^{-1} \sup_{B(x,1)} \rho(z) \leq \rho(x) \leq c_3 \inf_{z \in B(x,1)} \rho(z), \quad |x| > 2
\]  

(H1)

and

\[
\int_{\{|z-x|>1,|z-y|>1\}} \rho(x-z)\rho(z-y) \, dz \leq c_4 \rho(x-y), \quad |x-y| > 1.
\]  

(H2)

(B2) For all \( 0 < r_1 < r_2 < r \leq 1 \),

\[
\sup_{x \in B(0,r_1)} \sup_{y \in B(r_0,r_2)} G_{B(0,r)}(x,y) < \infty,
\]

where \( B(0,r) \) denotes the ball with center 0 and radius \( r \), and \( G_{B(0,r)}(x,y) \)

is the Green function for the killed process of \( (X_t)_{t \geq 0} \) on domain \( B(0,r) \).

(B3) There exists a constant \( c_5 \geq 1 \) such that

\[
\sup_{z \in B(x,1)} V(z) \leq c_5 V(x).
\]  

(H3)

In particular, even if assumption (B2) above is weak in applications, it heavily depends on time-space estimates for the transition density function, which are not available for general (symmetric) Lévy processes.

To see the power of Theorem 1.1, we consider the following example.

**Example 1.2.** Let

\[
J(x,y) \asymp |x-y|^{-d-\alpha} 1_{\{|x-y| \leq 1\}} + e^{-|x-y|^\gamma} 1_{\{|x-y| > 1\}},
\]

where \( \alpha \in (0,2) \) and \( \gamma \in (0,1] \). Let \( V(x) = |x|^\lambda \) for some \( \lambda > 0 \). Then, there is a constant \( C_1 > 0 \) such that for all \( x \in \mathbb{R}^d \),

\[
\phi_1(x) \geq \frac{C_1}{(1 + |x|)^{\lambda e^{|x|^\gamma}}},
\]

and the associated semigroup \( (T_t^V)_{t \geq 0} \) is intrinsically ultracontractive if and only if \( \lambda > \gamma \). Furthermore, if \( \lambda > \gamma \) and for every \( x \in \mathbb{R}^d \),

\[
\int \{ |z| |J(x,x+z) - J(x,x-z)| \, dz < \infty,
\]

then there is a constant \( C_2 > 0 \) such that for all \( x \in \mathbb{R}^d \),

\[
\phi_1(x) \leq \frac{C_2}{(1 + |x|)^{\lambda e^{|x|^\gamma}}}.
\]

**Remark 1.3.** (1) For symmetric Lévy process, (1.7) is automatically satisfied.

(2) For Example 1.2, one can follow the argument in [4] and obtain that \( (T_t^V)_{t \geq 0} \) is intrinsically ultracontractive if \( V(x) = |x|^\lambda \) with \( \lambda > 1 \), which however is not sharp. This indicates that Theorem 1.1 is more accurate for Example 1.2.
To further see that Theorem 1.1 yields optimal criteria for the intrinsic hypercontractivity and intrinsic supercontractivity besides the intrinsic ultracontractivity, we take the following example.

**Example 1.4.** Let \((X_t)_{t \geq 0}\) be a symmetric \(\alpha\)-stable process with some \(\alpha \in (0, 2)\), i.e.

\[
J(x, y) = \rho(x - y) := c(d, \alpha)|x - y|^{d - \alpha},
\]

where \(c(d, \alpha)\) is a constant only depending on \(d\) and \(\alpha\). Let \(V(x) = \log^\lambda(1 + |x|)\) for some \(\lambda > 0\). Then,

1. The semigroup \((T_t^V)_{t \geq 0}\) is intrinsically ultracontractive if and only if \(\lambda > 1\).
2. The semigroup \((T_t^V)_{t \geq 0}\) is intrinsically supercontractive if and only if \(\lambda > 1\).
3. The semigroup \((T_t^V)_{t \geq 0}\) is intrinsically hypercontractive if and only if \(\lambda \geq 1\).

### 1.3.2. Irregular Potential Function: \(\lim_{|x| \to \infty} V(x) \neq \infty\)

To consider the case that \(\lim_{|x| \to \infty} V(x) \neq \infty\), we make the following assumption on \(V\) similar to [4].

**(A)** There exists a constant \(K > 0\) such that

\[
\lim_{R \to \infty} \Phi_K(R) = \infty,
\]

where

\[
\Phi_K(R) := \inf_{|x| \geq R, V(x) \geq K} V(x), \quad R > 0.
\]

Let

\[
\Theta_K(R) := \left|\left\{x \in \mathbb{R}^d : |x| \geq R, V(x) \leq K\right\}\right|, \quad R > 0,
\]

where \(K\) is the constant given in (A). Then, by (1.4), \(\lim_{R \to \infty} \Theta_K(R) = 0\).

**Theorem 1.5.** Suppose that assumption (A) holds, and that \(d > \alpha_1\), where \(\alpha_1 \in (0, 2)\) is given in (1.1). For any \(s, \delta_i > 0\) with \(1 \leq i \leq 4\), define

\[
\hat{\beta}(s) := \hat{\beta}(s; \delta_1, \delta_2, \delta_3, \delta_4) = \delta_1 \alpha \left(\Psi_K^{-1}\left(\frac{8}{s \wedge \delta_2}\right) \wedge \delta_3, \frac{s \wedge \delta_2}{8}\right),
\]

where

\[
\Psi_K(R) := \left[\frac{1}{\Phi_K(R)} + \delta_4 \Theta_K(R)^{\alpha_1/d}\right]^{-1}.
\]

Then the three statements in Theorem 1.1 hold with \(\beta(s)\) replaced by \(\hat{\beta}(s)\).

Note that, when \(\lim_{|x| \to \infty} V(x) = \infty\), for any constant \(K > 0\) there exists \(R_0 > 0\) such that \(\Theta_K(R) = 0\) and \(\Psi_K(R) = \Phi_K(R)\) for \(R \geq R_0\). Therefore, by (1.8) and (1.6), in this case Theorem 1.5 essentially is the same as Theorem 1.1. To show that Theorem 1.5 is sharp, we reconsider symmetric \(\alpha\)-stable process both with irregular potential function.

**Example 1.6.** Let \((X_t)_{t \geq 0}\) be a symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\) with \(d > \alpha\), and let \(V\) be a nonnegative measurable function defined by

\[
V(x) = \begin{cases} 
\log^\lambda(1 + |x|), & x \notin A, \\
1, & x \in A,
\end{cases}
\]

where \(\lambda > 1\) and \(A\) is an unbounded set on \(\mathbb{R}^d\).
(1) Suppose that
\[ |A \cap B(0, R)^c| \leq \frac{c_1}{\log^\theta R}, \quad R > 1 \]
holds with some constants \(c_1, \theta > 0\). Then, the associated semigroup \((T_t^V)_{t \geq 0}\) is intrinsically ultracontractive (and also intrinsically supercontractive) if \(\theta > \frac{4}{\alpha}; (T_t^V)_{t \geq 0}\) is intrinsically hypercontractive if \(\theta \geq \frac{4}{\alpha}\).

(2) For any \(\varepsilon > 0\), let
\[ A = \bigcup_{m=1}^{\infty} B(x_m, r_m), \]
where \(x_m \in \mathbb{R}^d\) with \(|x_m| = e^{m^k_0}\), and \(r_m = m^{-\frac{k_0}{\alpha}} + \frac{\varepsilon}{8}\) for some \(k_0 > \frac{2}{\varepsilon}\). Then,
\[ |A \cap B(0, R)^c| \leq \frac{c_2}{\log^{\frac{\alpha}{\alpha - \varepsilon}} R}, \quad R > 1 \]
holds for some constant \(c_2 > 0\); however, the semigroup \((T_t^V)_{t \geq 0}\) is not intrinsically ultracontractive.

The reminder of this paper is arranged as follows. In the next section, we will present some preliminary results, including lower bound estimate for the ground state and intrinsic local super Poincaré inequalities for non-local Dirichlet forms with infinite range jumps. Section 3 is devoted to the proofs of all the theorems and examples.

2. Some Technical Estimates

2.1. Lower bound for the ground state. In this subsection, we consider lower bound estimate for the ground state \(\phi_1\). Recall that for \(x \in \mathbb{R}^d\)
\[ J^*(x) = \begin{cases} \inf_{y-z \in B(x, 3/2)} J(y, z), & |x| \geq 3, \\ 1, & |x| < 3, \end{cases} \]
and
\[ V^*(x) = \sup_{z \in B(x, 1)} V(z), \quad \varphi(x) = \frac{J^*(x)}{1 + V^*(x)}. \]

Proposition 2.1. Let \(\varphi\) be the function defined above. Then there exists a constant \(C_0 > 0\) such that for all \(x \in \mathbb{R}^d\),
\[ C_0 \phi_1(x) \geq \varphi(x). \]

The proof of Proposition 2.1 is mainly based on the argument of [15, Theorem 1.6] (in particular, see [15, pp. 5054-5055]). For the sake of completeness, we present the details here. First, for any Borel set \(D \subseteq \mathbb{R}^d\), let \(\tau_D := \inf\{t > 0 : X_t \notin D\}\) be the first exit time from \(D\) of the process \((X_t)_{t \geq 0}\). The following result is a consequence of [2, Theorem 2.1], and the reader can refer to [4, Lemma 3.1] for the proof of it.

Lemma 2.2. There exist constants \(c_0 := c_0(\kappa) > 0\) and \(r_0 := r_0(\kappa) \in (0, 1]\) such that for every \(r \in (0, r_0]\) and \(x \in \mathbb{R}^d\),
\[ P^x (\tau_{B(x,r)} \geq c_0 r^{\alpha_2\frac{(\alpha_2 - \alpha_1)\kappa}{\alpha_1}}) \geq \frac{1}{2}. \]
In the following, we will fix $r_0, c_0$ in Lemma 2.2 and set $t_0 = c_0 r_0^\frac{\alpha_2 - \alpha_1 d}{\alpha_1}$.

**Lemma 2.3.** Let $0 \leq t_1 < t_2 < t_0$, $x \in \mathbb{R}^d$ with $|x| \geq 3$, $D = B(0, r_0)$ and $B = B(x, r_0)$. We have

\begin{equation}
P^x(X_{\tau_B} \in D/2, t_1 \leq \tau_B < t_2) \geq c_1(t_2 - t_1) J^x(x)
\end{equation}

for some constant $c_1 > 0$.

**Proof.** Denote by $p_B(t, x, y)$ the density of the process $(X_t)_{t \geq 0}$ killed on exiting the set $B$, i.e.

\[ p_B(t, x, y) = p(t, x, y) - E^x(\tau_B \leq t; p(t - \tau_B, X(\tau_B), y)) \]

According to the Ikeda-Watanabe formula for $(X_t)_{t \geq 0}$ (see e.g. [15, Proposition 2.5]), we have

\[ P^x(X(\tau_B) \in D/2, t_1 \leq \tau_B < t_2) \]

\[ = \int_B \int_{t_1}^{t_2} p_B(s, x, y) \, ds \int_{D/2} J(y, z) \, dy \, dz \]

\[ \geq |D/2| \inf_{y - z \in B(x, 3r_0/2)} J(y, z) \int_{t_1}^{t_2} \int_B p_B(s, x, y) \, dy \, ds \]

\[ \geq c_2 \inf_{y - z \in B(x, 3r_0/2)} J(y, z) \int_{t_1}^{t_2} P(\tau_B \geq s) \, ds \]

\[ \geq c_2 \left[ \inf_{|z| \geq 3} \inf_{|y| \geq 3} J(y, z) \right] (t_2 - t_1) J(y, z) \]

\[ \geq \frac{c_2}{2} (t_2 - t_1) J^x(x), \]

which in the forth inequality we have used Lemma 2.2 and the fact that $r_0 \leq 1$. This completes the proof. \(\square\)

Now, we are in a position to present the

**Proof of Proposition 2.1.** We only need to consider $x \in \mathbb{R}^d$ with $|x| \geq 3$. Still let $B = B(x, r_0)$ and $D = B(0, r_0)$. First, we have

\[ \phi_1(x) = e^{\lambda t_0} T_{t_0}^V(\phi_1)(x) \geq e^{\lambda t_0} T_{t_0}^V(\mathbb{1}_D \phi_1)(x) \]

\[ \geq e^{\lambda t_0} (\inf_{x \in D} \phi_1(x)) T_{t_0}^V(\mathbb{1}_D)(x) \geq c_2 T_{t_0}^V(\mathbb{1}_D)(x), \]

where in the last inequality we have used the fact that $\phi_1$ is strictly positive and continuous.

Second, by the strong Markov property, it holds that

\[ T_{t_0}^V(\mathbb{1}_D)(x) \]

\[ = E^x(X_{t_0} \in D; e^{-\int_0^{t_0} V(X_s) \, ds}) \]

\[ \geq E^x(X_{\tau_B} \in D/2, \tau_B < t_0, X_s \in D \text{ for all } s \in [\tau_B, t_0]; e^{-\int_{\tau_B}^{t_0} V(X_s) \, ds - \int_{\tau_B}^{t_0} V(X_s) \, ds}) \]

\[ \geq e^{-\int_{t_0}^{\sup_{s \in D} V(z)} E^x(X_{\tau_B} \in D/2, \tau_B < t_0, X_s \in D \text{ for all } s \in [\tau_B, t_0]; e^{-\int_{\tau_B}^{t_0} V(X_s) \, ds}) \]

\[ \geq e^{-\int_{t_0}^{\sup_{s \in D} V(z)} E^x(X_{\tau_B} \in D/2, \tau_B < t_0; e^{-\int_{\tau_B}^{t_0} V(X_s) \, ds} \cdot P^{X_{\tau_B}}(\tau_D > t_0))} \]
\[ e^{-t_0 \sup_{z \in D} V(z)} \left( \inf_{|z| \leq r_0/2} P^z(\tau_{B(z,r_0/2)} > t_0) \right) E^x(X_{\tau_B} \in D/2, \tau_B < t_0; e^{-\int_0^{\tau_B} V(X_s) \, ds}) \geq c_3 E^x(X_{\tau_B} \in D/2, \tau_B < t_0; e^{-\int_0^{\tau_B} V(X_s) \, ds}), \]

where in the last inequality we have used Lemma 2.2.

Third, according to (2.12),

\[ E^x(X_{\tau_B} \in D/2, \tau_B < t_0; e^{-\int_0^{\tau_B} V(X_s) \, ds}) \geq \sum_{j=1}^\infty e^{-t_0/j} J^2(x) \sup_{z \in B(x,r_0)} V(z) \]

where the forth inequality follows from [15, Lemma 5.2], i.e.

\[ \sum_{j=1}^\infty e^{-r/j} \geq e^{-1} \frac{r}{r+1}, \quad r \geq 0. \]

Combining all the conclusions above, we prove the desired assertion.

\[ \square \]

2.2. Intrinsic local super Poincaré inequality. In this subsection, we are concerned with the local intrinsic super Poincaré inequality for \( D^V(f,f) \).

**Proposition 2.4.** Let \( \varphi \) be a strictly positive measurable function on \( \mathbb{R}^d \). Then for any \( s, r > 0 \) and any \( f \in C^2_c(\mathbb{R}^d) \),

\[ \int_{B(0,r)} f^2(x) \, dx \leq s D^V(f,f) + \alpha(r,s) \left( \int |f|(x) \varphi(x) \, dx \right)^2, \]

where

\[ \alpha(r,s) = \inf \left\{ \frac{2}{|B(0,t)| \inf_{x \in B(0,r+t)} \varphi^2(x)} : t \leq r \text{ and } \frac{2 \sup_{0 \leq |x-y| \leq t} J(x,y)^{-1}}{|B(0,t)|} \leq s \right\}. \]

**Proof.** Since \( V \geq 0 \),

\[ D(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x,y) \, dx \, dy \leq D^V(f,f), \quad f \in C^2_c(\mathbb{R}^d), \]

it suffices to prove (2.13) with \( D^V(f,f) \) replaced by \( D(f,f) \).
We can follow step (1) of the proof of [6, Theorem 3.1] or [22, Lemma 2.1] to verify that for any $0 < s \leq r$ and $f \in C^2_c(\mathbb{R}^d)$,

$$
\int_{B(0,r)} f^2(x) \, dx
\leq \left( \frac{2 \sup_{0 < |x-y| \leq s} J(x,y)^{-1}}{|B(0,s)|} \right) \int_{|x-y| \leq s} (f(x) - f(y))^2 J(x,y) \, dx \, dy 
+ \frac{2}{|B(0,s)|} \left( \int_{B(0,r+s)} |f(x)| \, dx \right)^2.
$$

(2.14)

Note that, if (2.14) holds, then for any $0 < s \leq r$ and $f \in C^2_c(\mathbb{R}^d)$

$$
\int_{B(0,r)} f^2(x) \, dx \leq \left( \frac{2 \sup_{0 < |x-y| \leq s} J(x,y)^{-1}}{|B(0,s)|} \right) D(f,f) 
+ \frac{2}{|B(0,s)|} \left( \inf_{x \in B(0,r+s)} \varphi^2(x) \int_{B(0,r+s)} |f(x)| \, dx \right)^2.
$$

This immediately yields (2.13) by the definition of $\alpha(s,r)$.

Next, we turn to the proof of (2.14). For any $0 < s \leq r$ and $f \in C^2_c(\mathbb{R}^d)$, define

$$
f_s(x) := \frac{1}{|B(0,s)|} \int_{B(x,s)} f(z) \, dz, \quad x \in B(0,r).
$$

We have

$$
\sup_{x \in B(0,r)} |f_s(x)| \leq \frac{1}{|B(0,s)|} \int_{B(0,r+s)} |f(z)| \, dz,
$$

and

$$
\int_{B(0,r)} |f_s(x)| \, dx \leq \int_{B(0,r)} \frac{1}{|B(0,s)|} \int_{B(x,s)} |f(z)| \, dz \, dx 
\leq \int_{B(0,r+s)} \left( \frac{1}{|B(0,s)|} \int_{B(z,s)} dx \right) |f(z)| \, dz \leq \int_{B(0,r+s)} |f(z)| \, dz.
$$

Thus,

$$
\int_{B(0,r)} f_s^2(x) \, dx \leq \left( \sup_{x \in B(0,r)} |f_s(x)| \right) \int_{B(0,r)} |f_s(x)| \, dx 
\leq \frac{1}{|B(0,s)|} \left( \int_{B(0,r+s)} |f(z)| \, dz \right)^2.
$$

Therefore, for any $f \in C^2_c(\mathbb{R}^d)$ and $0 < s \leq r$,

$$
\int_{B(0,r)} f^2(x) \, dx 
\leq 2 \int_{B(0,r)} (f(x) - f_s(x))^2 \, dx + 2 \int_{B(0,r)} f_s^2(x) \, dx
\leq 2 \int_{B(0,r)} \frac{1}{|B(0,s)|} \int_{B(x,s)} (f(x) - f(y))^2 \, dy \, dx + \frac{2}{|B(0,s)|} \left( \int_{B(0,r+s)} |f(z)| \, dz \right)^2
\leq \left( \frac{2 \sup_{0 < |x-y| \leq s} J(x,y)^{-1}}{|B(0,s)|} \right) \int_{|x-y| \leq s} (f(x) - f(y))^2 J(x,y) \, dx \, dy.
$$
and for any $f$

This, along with (2.13) and (2.11), gives us that for any $L$

Due to the fact that

This proves the desired assertion (2.14). \qed

3. Proofs of Theorems and Examples

We begin with proofs of Theorems 1.1 and 1.5.

Proof of Theorem 1.1. (1) Since for all $r > 0$ and $f \in C_c^2(\mathbb{R}^d)$,

This, along with (2.13) and (2.11), gives us that for any $r, \tilde{s} > 0$,

For any $s > 0$, taking $r = \Phi^{-1}(2/s)$ and $\tilde{s} = s/2$ in the inequality above, we arrive at

(2) Let $(\tilde{T}_t^V)_{t \geq 0}$ be the strongly continuous Markov semigroup defined by (1.5). Due to the fact that $L_V \phi_1 = -\lambda_1 \phi_1$, the (regular) Dirichlet form $(D_{\phi_1}, \mathcal{D}(D_{\phi_1}))$ associated with $(\tilde{T}_t^V)_{t \geq 0}$ enjoys the properties that, $C_c^2(\mathbb{R}^d)$ is a core for $(D_{\phi_1}, \mathcal{D}(D_{\phi_1}))$, and for any $f \in C_c^2(\mathbb{R}^d)$,

(3.16) $D_{\phi_1}(f, f) = D^V(f \phi_1, f \phi_1) - \lambda_1 \int_{\mathbb{R}^d} f^2(x) \phi_1^2(x) \, dx$.

Let $\mu_{\phi_1}(dx) = \phi_1^2(x) \, dx$. Combining (3.16) with (3.15) gives us the following intrinsic super Poincaré inequality

In particular, for any $s \in (0, 1/(2\lambda_1))$,

which implies that

$\mu_{\phi_1}(f^2) \leq s D_{\phi_1}(f, f) + 2 \lambda_1 \mu_{\phi_1}(f^2) + 2 C_0^2 \alpha \left( \Phi^{-1} \left( \frac{2}{s} \right), \frac{s}{2} \right) \mu_{\phi_1}(|f|) 2, \quad f \in C_c^2(\mathbb{R}^d)$,

where $\beta(s)$ is the rate function defined by (1.6) with some proper constants $\delta_1, \delta_2 > 0$.

Therefore, the desired assertions for the ultracontractivity, supercontractivity and hypercontractivity of the semigroup $(\tilde{T}_t^V)_{t \geq 0}$ (or, equivalently, the intrinsic ultracontractivity, intrinsic supercontractivity and intrinsic hypercontractivity of the semi-group $(\tilde{T}_t^V)_{t \geq 0}$) follow from [18, Theorem 3.3.13] and [16, Theorem 3.1]. \qed
Proof of Theorem 1.5. By (1.1) and \( d > \alpha_1 \), there is a constant \( c_1 := c_1(\kappa) > 0 \) such that the following Sobolev inequality holds

\[
\|f\|_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d;dx)}^2 \leq c_1 \left[ D(f, f) + \|f\|_{L^2(\mathbb{R}^d;dx)}^2 \right], \quad f \in C_c^\infty(\mathbb{R}^d),
\]

See [4, Proposition 3.7].

For the constant \( K \) in (A), let \( A_1 := \{ x \in \mathbb{R}^d : V(x) > K \} \) and \( A_2 := \mathbb{R}^d \setminus A_1 \). Then, for any \( R > 0 \) and \( f \in C_c^\infty(\mathbb{R}^d) \),

\[
\int_{B(0,R)^c} f^2(x) \, dx = \int_{B(0,R)^c \cap A_1} f^2(x) \, dx + \int_{B(0,R)^c \cap A_2} f^2(x) \, dx \\
\leq \frac{1}{\Phi_K(R)} \int_{B(0,R)^c \cap A_1} f^2(x) V(x) \, dx \\
+ |B(0,R)^c \cap A_2|^{\alpha_1/d} \|f\|_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d;dx)}^2 \\
\leq \frac{1}{\Phi_K(R)} D_V(f,f) + \Theta_K(R)^{\alpha_1/d} \|f\|_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d;dx)}^2.
\]

This, along with (2.11), (2.13) and (3.17), gives us that for any \( R, \bar{s} > 0 \),

\[
\int f^2(x) \, dx \leq \left( \frac{1}{\Phi_K(R)} + \bar{s} + c_1 \Theta_K(R)^{\alpha_1/d} \right) D_V(f,f) \\
+ C^2_0 \alpha(R,\bar{s}) \left( \int |f|(x) \phi_1(x) \, dx \right)^2 + c_1 \Theta_K(R)^{\alpha_1/d} \int f^2(x) \, dx \\
\leq \left( \Psi_K(R)^{-1} + \bar{s} \right) D_V(f,f) + C^2_0 \alpha(R,\bar{s}) \left( \int |f|(x) \phi_1(x) \, dx \right)^2 \\
+ \Psi_K(R)^{-1} \int f^2(x) \, dx,
\]

where \( \Psi_K \) is defined in the theorem with \( \delta_4 = c_1 \).

For any \( s > 0 \), taking \( R = \Psi^{-1}_K \left( \frac{4}{s} \right) \wedge \Psi^{-1}_K(2) \) and \( \bar{s} = \frac{4}{s} \) in the inequality above, we arrive at

\[
\int f^2(x) \, dx \leq s D_V(f,f) \\
+ 2C^2_0 \alpha \left( \Psi^{-1}_K \left( \frac{4}{s} \right) \wedge \Psi^{-1}_K(2), \frac{s}{4} \right) \times \left( \int |f|(x) \phi_1(x) \, dx \right)^2.
\]

According to the intrinsic super Poincaré inequality (3.18) and the argument of part (2) in Theorem 1.1, we can obtain the desired conclusions. \( \square \)

Finally, we present the proofs of Examples 1.2, 1.4 and 1.6.

Proof of Example 1.2. Let \( V(x) = (1 + |x|)^{\lambda} \) for some \( \lambda > 0 \). Then, according to Theorem 1.1, the rate function \( \beta \) given by (1.6) satisfies that

\[
\beta(s) = c_1 \exp \left( c_2 (1 + s^{-\gamma/\lambda}) \right).
\]

Therefore, by Theorem 1.1 (1), the semigroup \( (T_t^V)_{t \geq 0} \) is intrinsically ultracontractive for any \( \lambda > \gamma \). To verify that the semigroup \( (T_t^V)_{t \geq 0} \) is not intrinsically ultracontractive for \( \lambda \in (0, \gamma] \), we can follow the proof of Example 1.4 (1) below, by using [8, (1.18)] instead. We omit the details here.
The lower bound estimate for $\phi_1$ follows from Proposition 2.1. Now, we turn to the upper bound estimate. It is easy to check that for any $r > 0$ large enough,

\begin{equation}
3.19 \quad x \mapsto \mathbb{1}_{B(0,2r)} \int_{\{x+z \leq r\}} J(x, x + z) \, dz \in L^2(\mathbb{R}^d; dx),
\end{equation}

According to [21, Theorem 1.1], (1.7) and (3.19), $C^2_\infty(\mathbb{R}^d) \subset \mathcal{D}(L^V)$ and for any $f \in C^2_\infty(\mathbb{R}^d),$

\begin{align*}
L^V f(x) &= \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \right) J(x, x + z) \, dz \\
&\quad + \frac{1}{2} \int_{\{|z| \leq 1\}} \langle \nabla f(x), z \rangle (J(x, x + z) - J(x, x - z)) \, dz - V(x) f(x) \\
&= : Lf(x) - V(x) f(x).
\end{align*}

Let

$$
\psi(x) := \frac{e^{-(1+|x|^2)^{\gamma/2}}}{C_0 + (1+|x|^2)^{\lambda/2}},
$$

where $C_0 \geq 1$ is a constant to be determined by later. By the approximation argument, it is easy to verify that $\psi \in \mathcal{D}(L^V)$. Next, we set

$$
\rho(z) := |z|^{-d-\alpha} 1_{\{|z| \leq 1\}} + e^{-|z|^\gamma} 1_{\{|z| > 1\}}.
$$

Then, for any $x \in \mathbb{R}^d$ with $|x| > 3,$

\begin{align*}
L\psi(x) &= \int_{\{|z| \leq 1\}} \left( \psi(x + z) - \psi(x) - \langle \nabla \psi(x), z \rangle \right) J(x, x + z) \, dz \\
&\quad + \int_{\{|z| > 1\}} \left( \psi(x + z) - \psi(x) \right) J(x, x + z) \, dz \\
&\quad + \frac{1}{2} \int_{\{|z| \leq 1\}} \langle \nabla \psi(x), z \rangle (J(x, x + z) - J(x, x - z)) \, dz, \\
&\leq c_3 \psi(x) + \int_{\{|z| > 1\}} \frac{c_4 \rho(x + z)}{C_0 + V(x + z)} \rho(z) \, dz \\
&\leq c_3 \psi(x) + \frac{c_4}{C_0} \int_{\{|x+z| \leq 1\}} \rho(z) \, dz + \frac{c_4}{C_0} \int_{\{|x|,|z| > 1\}} \rho(z) \rho(x + z) \, dz \\
&\leq c_3 \psi(x) + c_5 \sup_{z \in B(x,1)} \rho(z) + \frac{c_6}{C_0} \rho(x) \\
&\leq c_3 \psi(x) + \frac{c_7}{C_0} \rho(x),
\end{align*}

where the constants $c_i$ ($i = 3, \ldots, 7$) are independent of the choice of $C_0$. Here, in the first inequality we have used (1.7) and the fact that there exists a constant $c_0 > 0$ such that for all $x \in \mathbb{R}^d$ with $|x| \geq 3,$

$$
\sup_{z \in B(x,1)} (|\nabla \psi(x)| + |\nabla^2 \psi(x)|) \leq c_0 \psi(x),
$$

and the third and the forth inequalities follow from (H1)–(H2) (they have been verified in [14, Example 4.1]). Thus, for any $x \in \mathbb{R}^d$ with $|x|$ large enough,

\begin{equation}
L^V \psi(x) \leq c_3 \psi(x) + \frac{c_7}{C_0} \rho(x) - \frac{V(x) e^{(1+|x|^2)^{\gamma/2}}}{C_0 + (1+|x|^2)^{\lambda/2}},
\end{equation}
In particular, taking $C_0 \geq 1 + 2c_7$ large enough in the inequality above, we get by the fact that $V(x) = |x|^\lambda \to \infty$ as $|x| \to \infty$,

$$L^V \psi(x) \leq 0 \quad \text{for } |x| \text{ large enough.}$$

On the other hand, since $\psi \in C_b^2(\mathbb{R}^d)$, it is easy to check that the function $x \mapsto L^V \psi(x)$ is locally bounded. Therefore, there exists $\lambda > 0$ such that for any $x \in \mathbb{R}^d$,

$$L^V \psi(x) \leq \lambda \psi(x),$$

which implies that

$$T_t^V \psi(x) \leq e^{\lambda t} \psi(x), \quad x \in \mathbb{R}^d, t > 0.$$ 

Furthermore, according to [11, Theorem 3.2], the intrinsic ultracontractivity of $(T_t^V)_{t \geq 0}$ implies that for every $t > 0$, there is a constant $c_t > 0$ such that

$$p^V(t, x, y) \geq c_t \phi_1(x) \phi_1(y), \quad x, y \in \mathbb{R}^d.$$ 

Therefore,

$$\psi(x) \geq e^{-\lambda t} T_t^V \psi(x) = e^{-\lambda} \int p^V(1, x, y) \psi(y) \, dy \geq c_8 e^{-\lambda} \int \psi(y) \phi_1(y) \, dy \phi_1(x) = c_9 \phi_1(x),$$

which yields the required upper bound for the ground state $\phi_1$.  

\textit{Proof of Example 1.4.} (1) Let $V(x) = \log^\lambda (1 + |x|)$ for some $\lambda > 0$. Then, according to Theorem 1.1, the rate function $\beta$ given by (1.6) satisfies that

$$\beta(s) = c_1 \exp\left(c_2 (1 + s^{-1/\lambda})\right).$$

Therefore, by Theorem 1.1 (1), the semigroup $(T_t^V)_{t \geq 0}$ is intrinsically ultracontractive for any $\lambda > 1$.

To prove that for any $\lambda \in (0, 1]$, the semigroup $(T_t^V)_{t \geq 0}$ is not intrinsically hypercontractive. We mainly follow the proof of [15, Theorem 1.6] (see [15, pp. 5055-5056]). Let $p(t, x, y)$ be the heat kernel for the symmetric $\alpha$-stable process $(X)_{t \geq 0}$. It is well known that for any fixed $t \in (0, 1]$ and $|x - y|$ large enough,

$$p(t, x, y) \leq \frac{c_4 t}{|x - y|^{d + \alpha}}.$$ 

Set $D = B(0, 1)$. For $|x|$ large enough,

$$T_t^V(\mathbb{1}_D)(x) \leq \int_D p(t, x, y) \, dy \leq \frac{c_4 t}{|x|^{d + \alpha}}.$$ 

On the other hand, since $\lambda \in (0, 1]$, for $|x|$ large enough and $t \in (0, 1]$,

$$T_t^V(\mathbb{1}_{B(1, 1)})(x) \geq \mathbb{E}^x\left(\tau_{B(1, 1)} > t; \exp\left(-\int_0^t V(X_s) \, ds\right)\right) \geq c_3 \mathbb{P}^x(\tau_{B(1, 1)} > t) e^{-t \log |x|} \geq c_4 \mathbb{P}^x(\tau_{B(1, 1)} > 1) e^{-t \log |x|} \geq \frac{c_7}{|x|^t}. $$
Combining both conclusions above, we get that for any fixed \( t \in (0, d + \alpha) \), there is not a constant \( C_t > 0 \) such that for \( |x| \) large enough,

\[
T^V_t (\mathbb{1}_D)(x) \geq C_t T^V_t (\mathbb{1}_{B(x, 1)})(x),
\]

which contradicts with [15, Condition 1.3, p. 5027]. Hence, according to the remark below [15, Condition 1.3, p. 5027], the semigroup \((T^V_t)_{t \geq 0}\) is not intrinsically ultracontractive.

(2) According to (3.20) and Theorem 1.1 (2), we know that if \( \lambda > 1 \), then the semigroup \((T^V_t)_{t \geq 0}\) is intrinsically supercontractive. Now, suppose that the semigroup \((T^V_t)_{t \geq 0}\) is intrinsically supercontractive for any \( \lambda \in (0, 1] \), which is equivalently saying that the semigroup \((\hat{T}^V_t)_{t \geq 0}\) defined by (1.5) is supercontractive for any \( \lambda \in (0, 1] \). Then, by [18, Theorem 3.3.13 (2)], we know that the following super Poincaré inequality

\[
\int f(x)^2 \phi^2_t(x) \, dx \leq r D_{\phi_t}(f, f)
\]

holds with some rate function \( \beta \) such that \( \lim_{r \to 0} r \log \beta(r) = 0 \), where the bilinear form \( D_{\phi_t} \) is given by (3.16).

For a fixed strictly positive \( \phi \in C_c^2(\mathbb{R}^d) \) and any \( \varepsilon > 0 \), define

\[
\hat{L}\varepsilon f(x) := \frac{1}{\phi(x)} \int_{|x-y|\geq \varepsilon} \left( f(y) - f(x) \right) \phi(y) \frac{c(d, \alpha)}{|x-y|^{d+\alpha}} \, dy,
\]

Then,

\[
\begin{align*}
L^V(\phi f)(x) &= c(d, \alpha) \text{ p.v.} \int \left( (\phi f)(y) - (\phi f)(x) \right) \frac{1}{|x-y|^{d+\alpha}} \, dy - V(x)(\phi f)(x) \\
&= \phi(x) \lim_{\varepsilon \to 0} \hat{L}\varepsilon f(x) \\
&\quad + f(x) \left[ c(d, \alpha) \text{ p.v.} \int \left( \phi(y) - \phi(x) \right) \frac{1}{|x-y|^{d+\alpha}} \, dy - V(x)\phi(x) \right] \\
&= \phi(x) \lim_{\varepsilon \to 0} \hat{L}\varepsilon f(x) + f(x) L^V \phi(x),
\end{align*}
\]

where p.v. denotes the principal value integral. Therefore, for the probability measure \( \mu(dx) := \phi^2(x) \, dx \), we get that

\[
D^V(\phi f, \phi f) = -\langle \phi f, L^V(\phi f) \rangle_{L^2(\mathbb{R}^d; dx)}
\]

\[
= -\left\langle f, \frac{1}{\phi} L^V(\phi f) \right\rangle_{L^2(\mathbb{R}^d; \mu)}
\]

\[
= -\lim_{\varepsilon \to 0} \left\langle f, \hat{L}\varepsilon f \right\rangle_{L^2(\mathbb{R}^d; \mu)} - \left\langle f, \frac{f L^V}{\phi} \right\rangle_{L^2(\mathbb{R}^d; \mu)}
\]

\[
= -\lim_{\varepsilon \to 0} \int_{|x-y|\geq \varepsilon} c(d, \alpha)(f(y) - f(x)) f(x) \frac{\phi(y)\phi(x)}{|x-y|^{d+\alpha}} \, dx \, dy \\
&\quad - \left\langle f, \frac{f L^V}{\phi} \right\rangle_{L^2(\mathbb{R}^d; \mu)}
\]

\[
= \frac{c(d, \alpha)}{2} \int\int \frac{(f(y) - f(x))^2}{|x-y|^{d+\alpha}} \phi(y)\phi(x) \, dx \, dy
\]
\[- \left\langle f, \frac{f}{\phi} L^R \phi \right\rangle_{L^2(\mathbb{R}^d, \mu)}\]

where in the third equality we have used the dominated convergence theorem, and the last equality follows from the symmetry of kernel \(\frac{c(d, \alpha)}{|x-y|^{d+\alpha}}\). Whence, for any \(\phi_1 \in C_c^1(\mathbb{R}^d)\),

\[
D_{\phi_1}(f, f) = \frac{c(d, \alpha)}{2} \iint \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \phi_1(x) \phi_1(y) \, dx \, dy
\]

(3.23)

\[
= \frac{c(d, \alpha)}{2} \iint \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \phi_1(x) \phi_1(y) \, dx \, dy.
\]

Since \(C_c^1(\mathbb{R}^d)\) is a core for \((L^R, \mathcal{D}(L^R))\) and \(\phi_1 \in \mathcal{D}(L^R)\), by the standard approximation argument we get that (3.23) is still true for ground state \(\phi_1\) without the assumption that \(\phi_1 \in C_c^1(\mathbb{R}^d)\).

Second, according to [14, Corollary 2.2] (in this case, [14, Assumption 2.3] holds true and so [14, Corollary 2.2] applies), there exists a constant \(c_1 > 1\) such that

(3.24)

\[
\frac{c_1^{-1}}{(1 + |x|)^{d+\alpha} \log^\lambda(1 + |x|)} \leq \phi_1(x) \leq \frac{c_1}{(1 + |x|)^{d+\alpha} \log^\lambda(1 + |x|)}.
\]

Third, we consider the following reference function \(g_n \in C_c^1(\mathbb{R}^d)\) for \(n \geq 1\) such that

\[
g_n(x) = \begin{cases} 
0, & |x| \leq n; \\
\in [0, 1], & n \leq |x| \leq 2n; \\
1, & |x| > 2n,
\end{cases}
\]

and \(|\nabla g_n(x)| \leq 2/n\) for all \(x \in \mathbb{R}^d\). It is easy to see that

\[
\int g_n(x)^2 \phi_1^2(x) \, dx \geq \frac{c_2}{n^{d+2\alpha} \log^{2\lambda}(1 + n)}
\]

and

\[
\left( \int |g_n(x)| \phi_1^2(x) \, dx \right)^2 \leq \frac{c_3}{n^{2d+4\alpha} \log^{4\lambda}(1 + n)}
\]

hold for some constants \(c_2, c_3 > 0\). On the other hand,

\[
D_{\phi_1}(g_n, g_n) = c(d, \alpha) \int_{\{|x| \leq n, |y| \geq n\}} \frac{(g_n(x) - g_n(y))^2}{|x - y|^{d+\alpha}} \phi_1(x) \phi_1(y) \, dx \, dy
\]

\[
+ c(d, \alpha) \int_{\{|x| > n\}} \frac{(g_n(x) - g_n(y))^2}{|x - y|^{d+\alpha}} \phi_1(x) \phi_1(y) \, dx \, dy
\]

\[= : I_1 + I_2.\]
Then, by (3.24),
\[
I_1 \leq \frac{c_1^*}{n^{d+\alpha} \log^\lambda (1+n)} \left[ \frac{1}{n^2} \iint_{\{|x-y| \leq n\}} \frac{|x-y|^2}{|x-y|^{d+\alpha}} \, dy \phi_1(x) \, dx \\
+ \iint_{\{|x-y| > n\}} \frac{1}{|x-y|^{d+\alpha}} \, dy \phi_1(x) \, dx \right]
\]
\[
\leq \frac{c_2^*}{n^{d+2\alpha} \log^\lambda (1+n)}.
\]
Similarly,
\[
I_2 \leq \frac{c_3^*}{n^{d+\alpha} \log^\lambda (1+n)} \left[ \frac{1}{n^2} \iint_{\{|x-y| \leq n\}} \frac{|x-y|^2}{|x-y|^{d+\alpha}} \, dx \phi_1(y) \, dy \\
+ \iint_{\{|x-y| > n\}} \frac{1}{|x-y|^{d+\alpha}} \, dx \phi_1(y) \, dy \right]
\]
\[
\leq \frac{c_4^*}{n^{d+2\alpha} \log^\lambda (1+n)}.
\]
Combining all the conclusions above, we obtain
\[
\frac{c_2}{\log^\lambda (1+n)} \leq c_4 r + \frac{c_3 \beta(r)}{n^{d+2\alpha} \log^\lambda (1+n)}
\]
for some constant $c_4 > 0$. Taking $r = r_n := \frac{c_2^*}{2 c_3 \log^\lambda (1+n)}$, we get that
\[
\beta(r_n) \geq \frac{c_4}{2 c_3^*} n^{d+2\alpha} \log^{2\lambda} (1+n).
\]
In particular, due to $\lambda \in (0,1)$,
\[
\limsup_{r \to 0} r \log \beta(r) \geq \limsup_{r \to 0} r^{1/\lambda} \log \beta(r) \geq \liminf_{n \to \infty} r_n^{1/\lambda} \log \beta(r_n) > 0,
\]
which contradicts with $\lim_{r \to 0} r \log \beta(r) = 0$. This proves the second desired assertion.

(3) By (3.20) and Theorem 1.1 (3), the semigroup $(T^V_t)_{t \geq 0}$ is intrinsically hypercontractive for $\lambda \geq 1$. Assume that the semigroup $(T^V_t)_{t \geq 0}$ is intrinsically hypercontractive for any $\lambda \in (0,1)$. Then, by [18, Theorem 3.3.13 (1)], the super Poincaré inequality (3.22) holds with
\[
\beta(r_n) \leq \exp(c(1+r^{-1})) \quad r > 0.
\]
Now, we can follow the proof of part (2) above, and obtain that
\[
\liminf_{n \to \infty} r_n^{1/\lambda} \log \beta(r_n) > 0,
\]
where $r_n$ is the same sequence as that in (2). In particular, $r_n \to 0$ as $n \to \infty$, and
\[
\beta(r_n) \geq \exp(c_1 r_n^{-1/\lambda})
\]
for $n$ large enough and some constant $c_1 > 0$. This is a contradiction with (3.25), also thanks to the fact that $\lambda \in (0,1)$. Hence, we complete the proof. \hfill \Box

Proof of Example 1.6. (1) Take $K = 1$ in assumption (A), and then
\[
\Phi_K(r) = \log^\lambda (1+r), \quad \Theta_K(r) = c_1 \log^{-\theta} (1+r)
\]
for \( r \geq 1 \) large enough. Thus, according to Theorem 1.5, the rate function \( \hat{\beta} \) given by (1.8) satisfies that
\[
\hat{\beta}(s) \leq c_2 \exp \left( c_3 \left( 1 + s^{-\max\left(1, \frac{\alpha}{d+\alpha}\right)} \right) \right).
\]
This, along with Theorem 1.5 again, yields the first desired assertion.

This proves (1.10).

(2) For any \( R > 0 \) with \( e^{m \kappa_0} \leq R \leq e^{(m+1) \kappa_0} \) for some \( m \geq 1 \),
\[
|A \cap B(0, R)^c| \leq \sum_{k=m}^{\infty} |B(x_k, r_k)| = c_0 \sum_{k=m}^{\infty} k^{-\frac{d \kappa_0}{\alpha} + 1}
\leq c_1 m^{-\frac{d \kappa_0}{\alpha} + 2} \leq c_2 (m + 1)^{\kappa_0(-\frac{d}{\alpha} + \frac{2}{\kappa_0})} \leq \frac{c_2}{\log^{d-\epsilon}_{\alpha} R}.
\]
This proves (1.10).

Let \( D = B(0, 1) \) and \( t = 1 \). According to (3.21), for all \( m \) large enough,
\[
T^V_1 (\mathbb{1}_D)(x_m) \leq \frac{c_3}{|x_m|^{d+\alpha}} = c_3 \exp \left( - (d + \alpha) m \kappa_0 \right).
\]
On the other hand, by the definition of \( V \) and the space-homogeneous property and scaling property of symmetric \( \alpha \)-stable process, for \( m \) large enough,
\[
T^V_1 (\mathbb{1}_B)(x_m) \geq T^V_1 (\mathbb{1}_B(x_m, r_m))(x_m)
\geq \mathbb{E}^{x_m} \left( \tau_{B(x_m, r_m)} > 1 \right) \exp \left( - \int_0^1 V(X_s) ds \right)
= e^{-1} \mathbb{P}^{x_m} \left( \tau_{B(x_m, r_m)} > 1 \right)
= e^{-1} \mathbb{P}^{0} \left( \tau_{B(0, r_m)} > 1 \right)
= e^{-1} \mathbb{P}^{0} \left( \tau_{B(0, 1)} > r_m^{-\alpha} \right).
\]
Let \( p_B(t, x, y) \) be the Dirichlet heat kernel of symmetric \( \alpha \)-stable process killed on exiting \( B \). We find that the right hand side of the inequality above is just
\[
\int_{B(0,1)} p_B(0,1) (r_m^{-\alpha}, 0, z) dz \geq c_4 e^{-\lambda r_m^{-\alpha}} = c_4 e^{-\lambda m \kappa_0 - \frac{2}{3}}
\]
for some positive constants \( c_4 \) and \( \lambda \), where the inequality above follows from [9, Theorem 1.1(ii)]. Hence, we have
\[
T^V_1 (\mathbb{1}_B(x_m, 1))(x_m) \geq c_4 e^{-\lambda m \kappa_0 - \frac{2}{3}}.
\]

According to (3.26) and (3.27) above, we know that for any constant \( C > 0 \), the following inequality
\[
T^V_1 (\mathbb{1}_B(x_m))(x) \leq C T^V_1 (\mathbb{1}_D)(x).
\]
does not hold for \( x = x_m \) with \( m \) large enough. In particular, [15, Condition 1.3, p. 5027] is not satisfied, and so the semigroup \((T^V_t)_{t \geq 0}\) is not intrinsically ultraccontractive.

**Acknowledgements.** The authors would like to thank Professor Mu-Fa Chen and Professor Feng-Yu Wang for introducing them the field of functional inequalities when they studied in Beijing Normal University, and for their continuous encouragement and great help in the past few years. Financial support through National
Natural Science Foundation of China (No. 11201073) and the Program for Non-linear Analysis and Its Applications (No. IRTL1206) (for Jian Wang) is gratefully acknowledged.

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