STATISTICAL ESTIMATES FOR CHANNEL FLOWS DRIVEN BY A PRESSURE GRADIENT

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\textbf{Abstract.} We present rigorous estimates for some physical quantities related to turbulent and non-turbulent channel flows driven by a uniform pressure gradient. Such results are based on the concept of stationary statistical solution, which is related to the notion of ensemble average for flows in statistical equilibrium. We provide a lower bound estimate for the mean skin friction coefficient and improve on a previous upper bound estimate for the same quantity; both estimates in terms of the Reynolds number. We also present lower and upper bound estimates for the mean rate of energy dissipation, the mean longitudinal velocity (in the direction of the pressure gradient), and the mean kinetic energy. In particular, we obtain an upper bound related to the energy dissipation law, namely that the mean rate of energy dissipation is essentially bounded by a non-dimensional universal constant times the cube of the mean longitudinal velocity over a characteristic macro-scale length. Finally, we investigate the scale-by-scale energy injection due to the pressure gradient, proving an upper bound estimate for the decrease of this energy injection as the scale length decreases.

\section{Introduction}

While the existence of exact solutions of the Navier-Stokes Equations are not available in general, most of the classical research on turbulence theory consist of approximate methods based on a few exact deductions, supplemented with intuitive hypotheses about the nature of the phenomenon, such as scaling assumptions and moment truncation models; see for example \cite{1, 28}.

Recently, part of the theoretical research on turbulence has been concentrated on deriving rigorous bounds on characteristic quantities of turbulent flows directly from the equations of motion. These results are important to substantiate the ones obtained via the classical approximation methods.

Decomposing the turbulent flow into a stationary background flow and a fluctuation component, and using variational methods, Constantin and Doering derived
rigorous results for the long-time averaged rate of energy dissipation of flows in some geometries, in particular for the channel flow driven by a pressure gradient, in which case the estimate also yields an estimate for the mean skin friction coefficient; see [3,4].

Meanwhile, rigorous results were recently established for the three-dimensional theory of homogeneous stationary statistical turbulence in [13,14,15] using the concepts of stationary statistical solutions of the Navier-Stokes equations and generalized time average measures, and using energy-type methods.

This paper presents a combination of those results in the specific case of channel flows driven by a uniform pressure gradient. More specifically, we extend the upper bound estimate for the long-time averaged rate skin friction coefficient, obtained in [4], to general stationary statistical solutions, slightly simplifying their proof and slightly improving their estimates. We also obtain a lower bound estimate for the skin friction coefficient, which cannot be obtained by the variational principle method of [4]. More precisely, we show that for every stationary statistical solution the corresponding skin friction coefficient $C_f$ satisfies

$$\frac{10.88}{Re} \leq C_f \leq \frac{13.5}{Re},$$

for low Reynolds number flows, and

$$\frac{10.88}{Re} \leq C_f \leq 0.484 + O\left(\frac{1}{Re}\right),$$

for high Reynolds number flows, where the Reynolds number is defined by $Re = hU/\nu$, with $h$ being the height of the channel and $U$, the mean longitudinal velocity.

The lower-bound estimate for $C_f$ is nearly optimal in the sense that the stationary statistical solution is arbitrary and may be concentrated on the plane Poiseuille flow (which is unstable for high Reynolds number flows, but anyway exists in a mathematical sense), for which $C_f = 12/Re$. The upper-bound estimate might not be optimal since heuristic arguments suggest that $C_f \sim (\ln Re)^{-2}$ for high-Reynolds number turbulent flows. Nevertheless, it represents a nearly 19% improvement over the estimate obtained in [4] on the leading order constant term (from 0.597 to 0.484.)

We also give upper and lower bound estimates for some other physical quantities, such as the mean energy dissipation rate, the mean kinetic energy, and the mean longitudinal velocity. In particular, we prove an upper bound estimate related to the energy dissipation law, namely that for large Reynolds number flows the mean rate of energy dissipation is essentially bounded by a non-dimensional universal constant times the cube of the mean longitudinal velocity $U$ over the height $h$ of the channel:

$$\epsilon \leq \left(0.054 + O\left(\frac{1}{Re^2}\right)\right) \frac{U^3}{h}.$$

The leading order constant term obtained in [4] was approximately 0.0884, and it was remarked in that work that this term is much lower than 1, hinting that this result
is substantially more than a formalized dimensional analysis argument. The same applies here.

Finally, we study the scale-by-scale energy injection term due to the pressure gradient. We show that the energy injected into the modes larger than or equal to $\kappa$ is bounded by a term proportional to $\kappa^{-3/2}$. The motivation for the study of the decrease of energy injection comes from the Kolmogorov theory of turbulence. This theory argues that for turbulent flows there is a certain range of scales much lower than the energy injection scales and greater than the energy dissipative scales in which the kinetic energy is transferred to the small scales at a nearly constant rate equal to the energy dissipation rate. This theory was proposed in the idealized case of locally homogeneous turbulence, away from the boundaries, under the assumption that the energy injection is concentrated on the large scales. However, it is known from experiments that for wall bounded turbulence this hypothesis needs to be corrected [4, 28, 23]. In particular, the energy injection occurs at arbitrarily small scales. The estimates presented yield an upper bound on the rate of decrease of energy injection as the scale length decreases.

The remaining of the paper is organized as follows. In the next section we introduce the convenient mathematical setting used throughout the work. In Section 3, we define the notion of stationary statistical solution of the Navier-Stokes equations and present some related results. In Section 4, we rigorously define the characteristic quantities that will be estimated, such as the mean energy dissipation rate, mean kinetic energy, mean longitudinal velocity, and mean skin friction coefficient. In Section 5, we establish a relation between stationary statistical solutions and time averages. In Sections 6 and 7, we explicitly derive rigorous bounds for the mentioned physical quantities, utilizing both methods of [13, 14, 15] and of [4]. In Section 8, we list the explicit values of the mentioned physical quantities for the specific case of the laminar Poiseuille flow, verifying that some of the results obtained in Section 6 and Section 7 are optimal in some sense. In Section 9, we conclude the work with the discussion about the scale-by-scale energy injection.

2. Mathematical framework of the Navier-Stokes Equations

We consider an incompressible Newtonian flow confined to a rectangular periodic channel and driven by a uniform pressure gradient. More precisely, the velocity vector field $\mathbf{u} = (u_1, u_2, u_3)$ of the fluid satisfies the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \frac{P}{L_x} \mathbf{e}_1, \quad \nabla \cdot \mathbf{u} = 0, \quad \Omega = (0, L_x) \times (0, L_y) \times (0, h),$$

in the domain $\Omega = (0, L_x) \times (0, L_y) \times (0, h)$. The scalar $p$ is the kinematic pressure. We denote by $\mathbf{x} = (x, y, z)$ the space variable. The boundary conditions are no-slip on the planes $z = 0$ and $z = h$ and periodic in the $x$ and $y$ directions, with periods $L_x$ and $L_y$, respectively, for both $\mathbf{u}$ and $p$. The parameter $P/L_x$ denotes the magnitude of the applied pressure gradient. The parameter $\nu > 0$ is the kinematic viscosity, $\mathbf{e}_1$. 

is the unit vector in the $x$ direction, and $L_x$, $L_y$, $h$, $P > 0$. We sometimes refer to the
direction $x$ of the pressure gradient as the longitudinal direction.

The mathematical formulation of the Navier-Stokes equations in this geometry can
be easily adapted from the no-slip or fully-periodic case developed in [5,13,19,25,27].

The formulation yields a functional equation for the time-dependent velocity field
$u = u(t)$ of the form:

$$\frac{du}{dt} = F(u) = f_P - \nu A u - B(u, u), \quad (2.2)$$

where

$$f_P = \frac{P}{L_x} e_1. \quad (2.3)$$

Two fundamental spaces are defined by

$$H = \left\{ u = w |_{\Omega}; \begin{array}{l}
w \in (L_{loc}^2(\mathbb{R}^2 \times (0, h)))^3, \nabla \cdot w = 0, \\
w(x + L_x, y, z) = w(x, y, z), \\
w(x + L_y, y, z) = w(x, y, z), \text{ a.e. } (x, y, z) \in \mathbb{R}^2 \times (0, h). \\
w_3(x, y, 0) = w_3(x, y, h) = 0, \text{ a.e. } (x, y) \in \mathbb{R}^2. \end{array} \right\}$$

and

$$V = \left\{ u = w |_{\Omega}; \begin{array}{l}
w \in (H^1_{loc}(\mathbb{R}^2 \times (0, h)))^3, \nabla \cdot w = 0, \\
w(x + L_x, y, z) = w(x, y, z), \\
w(x + L_y, y, z) = w(x, y, z), \text{ a.e. } (x, y, z) \in \mathbb{R}^2 \times (0, h). \\
w(x, y, 0) = w(x, y, h) = 0, \text{ a.e. } (x, y) \in \mathbb{R}^2. \end{array} \right\}$$

The inner products in $H$ and $V$ are denoted respectively by

$$(u, v) = \int_{\Omega} u(x) \cdot v(x) \, dx, \quad (u, v) = \int_{\Omega} \sum_{i=1,3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx,$$

and the associated norms by $|u|_0 = (u, u)^{1/2}, \|u\| = (u, u)^{1/2}.$

We identify $H$ with its dual and consider the dual space $V'$ of $V$, so that $V \subseteq H \subseteq V'$, with the injections being continuous, each space dense in the following one. We also denote by $H_w$ the space $H$ endowed with its weak topology.

We denote by $P_{\text{LH}}$ the (Leray-Helmhotz) orthogonal projector in $L^2(\Omega)^3$ onto the
subspace $H$. The operator $A$ in (2.2) is the Stokes operator given by $Au = -P_{\text{LH}} \Delta u.$
The term $B(u, v) = P_{\text{LH}}((u \cdot \nabla)v)$ is a bilinear term associated with the inertial
term. Moreover, since the Stokes operator is a positive self-adjoint operator on $H$,
we consider its powers $A^s, s \in \mathbb{R}$, with domain $D(A^s)$. We have $V = D(A^{1/2})$ and its
dual $V'' = D(A^{-1/2}).$
The Stokes operator possesses a complete orthonormal basis of eigenvectors in $H$, \( \{ w_{j,l,k} \}_{j,l,k} \), of the form

\[
 w_{j,l,k}(x, y, z) = \exp \left( i\pi \left( \frac{jx}{L_x} + \frac{ly}{L_y} \right) \right) \hat{w}_{j,l,k}(z),
\]

where \( (j, l, k) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \), \( A w_{j,l,k} = \lambda_{j,l,k} w_{j,l,k} \), and each \( \hat{w}_{j,l,k}(z) \) satisfies a one-dimensional eigenvalue problem, with \( 0 < \lambda_{j,l,k} \to \infty \), when \( j, l, k \to \infty \). We write the spectral expansion of \( u \) in this basis as

\[
 u(x, y, z) = \sum_{j,l,k} \hat{u}_{j,l,k} w_{j,l,k}(x, y, z).
\]

To each eigenvalue \( \lambda_{j,l,k} \) we associate a wavenumber \( \kappa = \kappa_{j,l,k} = \lambda_{j,l,k}^{1/2} \). Since \( u \in V \) vanishes on the top and bottom walls, Poincaré inequality applies, yielding a bound on \( |u|_0 \) in terms of \( \|u\| \). In fact, we have precisely

\[
 |u|^2_0 \leq \lambda_1^{-1} \|u\|^2,
\]

where \( \lambda_1 = \pi^2/h^2 \) is the smallest positive eigenvalue of the Stokes operator on this geometry. The smallest positive wavenumber is \( \kappa_1 = \lambda_1^{1/2} = \pi/h \).

We define the component \( u_\kappa \) of the vector field \( u \), for a single wavenumber \( \kappa \), by

\[
 u_\kappa = \sum_{j,l,k=\kappa} \hat{u}_{j,l,k} w_{j,l,k},
\]

and the component \( u_{\kappa'\kappa''} \) with a range of wave numbers \( [\kappa', \kappa''] \) by

\[
 u_{\kappa'\kappa''} = \sum_{\kappa' \leq \kappa < \kappa''} u_\kappa.
\]

We then write the Navier-Stokes equations projected on those components in the form

\[
 \frac{du_{\kappa'\kappa''}}{dt} + \nu A u_{\kappa'\kappa''} + B(u, u)_{\kappa'\kappa''} = (f_P)_{\kappa'\kappa''},
\]

where

\[
 (f_P)_\kappa = \sum_{j,l,k=\kappa} (f_P, w_{j,l,k}) w_{j,l,k}.
\]

Now, taking the inner product in $H$ of the bilinear term with a third variable yields a trilinear term

\[
 b(u, v, w) = (B(u, v), w),
\]

which is defined for \( u, v, w \in V \). An important relation for the trilinear term is the orthogonality property

\[
 b(u, v, v) = 0,
\]

for \( u, v \in V \). It follows from this relation the anti-symmetry property

\[
 b(u, v, w) = -b(u, w, v),
\]

for \( u, v, w \in V \).
3. Statistical solutions and the Reynolds Equations

A mathematical framework for the conventional theory of turbulence is based on the concept of stationary statistical solution of the Navier-Stokes equations. This amounts to considering the space $H$ as a probability space with the $\sigma$-algebra of the Borel sets of $H$ and endowed with a Borel probability measure. The ensemble averages are then regarded as averages with respect to this Borel probability measure. In our three-dimensional case we work mostly with the weak topology. Fortunately the Borel $\sigma$-algebra generated by the weakly open sets coincides with that for the open sets in the strong topology. Since $H$ is a separable Hilbert space every Borel measure is automatically regular. An important consequence of the regularity of a Borel probability measure is the density of the continuous functions (or just weakly continuous functions) in the space of integrable functions.

We say that a measure $\mu$ in $H$ is carried by a measurable set $E$ when $E$ has full measure in $H$, i.e. $\mu (H \setminus E) = 0$. The support of a Borel probability measure $\mu$ is the smallest closed set which carries $\mu$. The ensemble averages are regarded as averages with respect to a Borel probability measure $\mu$ on $H$. If $\varphi : H \to \mathbb{R}$ is a Borel function representing some physical information $\varphi (u)$ extracted from a velocity field $u$, such as kinetic energy, velocity, enstrophy, etc., then its mean value is

$$ \langle \varphi \rangle = \int_H \varphi (u) d\mu (u). \quad (3.1) $$

The reader is referred to [13] for more details.

Now, we define a class of Borel functions that are particularly useful in order to make a rigorous definition of a stationary statistical solution of the Navier-Stokes equations.

**Definition 3.1.** We define the class $\mathcal{T}$ of test functions to be the set of real-valued functionals $\Psi = \Psi (u)$ on $H$ that are bounded on bounded subsets of $H$ and such that the following conditions hold:

1. For any $u \in V$, the Fréchet derivative $\Psi'(u)$ taken in $H$ along $V$ exists. More precisely, for each $u \in V$, there exists an element in $H$ denoted $\Psi'(u)$ such that

   $$ \frac{|\Psi(u + v) - \Psi(u) - (\Psi'(u), v)|}{|v|_0} \to 0 \quad \text{as} \quad |v|_0 \to 0, \; v \in V. \quad (3.2) $$

2. $\Psi'(u) \in V$ for all $u \in V$, and $u \to \Psi'(u)$ is continuous and bounded as a function from $V$ into $V$.

For example, we can take the cylindrical test functions $\Psi : H \to \mathbb{R}$ of the form $\Psi(u) = \psi ((u, g_1), \ldots, (u, g_m))$, where $\psi$ is a $C^1$ scalar function on $\mathbb{R}^m$, $m \in \mathbb{N}$, with
compact support, and \( g_1, \ldots, g_m \) belong to \( V \). For this case we have
\[
\Psi'(u) = \sum_{j=1}^{m} \partial_j \psi((u, g_j), \ldots, (u, g_j)) g_j,
\]
where \( \partial_j \psi \) denotes the derivative of \( \psi \) with respect to the \( j \)-th variable. It follows that \( \Psi'(u) \in V \) since it is a linear combination of the \( g_j \).

Now, we define the notion of a stationary statistical solution of the Navier-Stokes equations.

**Definition 3.2.** A stationary statistical solution of the Navier-Stokes equation is a Borel probability measure \( \mu \) on \( H \) such that
\[
\begin{align*}
(1) \quad & \int_{H} \|u\|^2 d\mu(u) < \infty; \\
(2) \quad & \int_{H} (F(u), \Psi'(u)) d\mu(u) = 0, \text{ for any } \Psi \in \mathcal{T}, \text{ where } F(u) \text{ is as in } (2.2); \\
(3) \quad & \int_{e_1 \leq \|u\|_{0}/2 < e_2} \{ \nu \|u\|^2 - (f_P, u) \} d\mu(u) \leq 0, \text{ for all } 0 \leq e_1 < e_2 \leq +\infty.
\end{align*}
\]

The first condition means that an arbitrary stationary statistical solution has finite mean enstrophy. This is natural when we compare with individual solutions whose time average is bounded uniformly with respect to the time interval. It is also needed to make sense out of the second condition.

The last condition on the definition above is an energy-type inequality, and one can deduce from it that the support of a stationary statistical solution is included in the weak attractor \( A_w \), see \([13, 17]\), which is bounded in \( H \) according to
\[
|u|_{0} \leq \frac{\nu G^*}{\kappa_1^{1/2}} = \frac{\nu h^{1/2}}{\pi^{1/2}} G^*, \quad \forall u \in A_w, \quad (3.3)
\]
where \( G^* \) is a nondimensional number called Grashof number.

The concept of stationary statistical solution is regarded as a generalization of the notion of invariant measure. It is relevant to our three-dimensional case, in which a semigroup is not well-defined.

Due to these regularity properties obtained for stationary statistical solutions (finite mean enstrophy and with support bounded in \( H \)), the mean value \( \langle \varphi(u) \rangle \) can be defined not only for weakly continuous functions bounded in \( H \) but for any real-valued function \( \varphi \) which is continuous in \( V \) and satisfies the estimate
\[
|\varphi|_0 \leq C(|u|_{0})(1 + \nu^{-2} \lambda_1^{1/2} \|u\|^2), \quad (3.4)
\]
where \( C(|u|_0) \) is bounded on bounded subsets of \( H \). Important examples of such \( \varphi \) are \( |u|_{0}^2, \|u\|^2, b(u_{\kappa_0, \kappa'}, u_{\kappa_0, \kappa'}, u_{\kappa, \kappa'}, u_{\kappa, \kappa'}, u_{\kappa, \kappa'}), \) and \( b(u_{\kappa, \kappa'}, u_{\kappa, \kappa'}, u_{\kappa_0, \kappa'}) \).
By a duality argument we can extend the ensemble averages to functions with value in some function space. More precisely, we define the velocity field \( \langle u \rangle \) and the mean value \( \langle B(u, u) \rangle \) of the inertial term by

\[
(\langle u \rangle, v) = \int_H (u, v) d\mu(u), \quad \forall v \in V',
\]

\[
(\langle B(u, u) \rangle, v) = \int_H (B(u, u), v) d\mu(u), \quad \forall v \in D(A^{3/8}).
\]

The mean flow \( \langle u \rangle \) is a vector field on \( \Omega \) with \( \langle u \rangle \in V \), while \( \langle B(u, u) \rangle \in D(A^{-3/8}) \).

Since we assume statistical equilibrium, the stationary form of the Reynolds equations can be recovered within this framework; see also [24]:

**Proposition 3.3.** Given a stationary statistical solution in the sense of Definition 3.2 the following functional form of the Reynolds equations hold in \( V' \):

\[
\nu A \langle u \rangle + \langle B(u, u) \rangle = f_P. \tag{3.5}
\]

**Proof.** Let \( \psi \) be a \( C^1 \) real-valued function with compact support on \( \mathbb{R} \). For any \( v \in V \) and any wavenumber \( \kappa \), the function \( \Phi(u) = \psi((u, v_{\kappa_1, \kappa})) \) is a cylindrical test function. Thus,

\[
\int_H \psi'((u, v_{\kappa_1, \kappa})) \{ (f_P, v_{\kappa_1, \kappa}) - \nu(A u, v_{\kappa_1, \kappa}) - b(u, u, v_{\kappa_1, \kappa}) \} d\mu(u) = 0.
\]

Let \( \psi' \) converge pointwise to 1 while being uniformly bounded, so that at the limit we find

\[
\int_H \{ (f_P, v_{\kappa_1, \kappa}) - \nu(A u, v_{\kappa_1, \kappa}) - b(u, u, v_{\kappa_1, \kappa}) \} d\mu(u) = 0.
\]

For each fixed \( v \in V \), we may let \( \kappa \) go to infinity to find (since \( \mu \) has finite enstrophy and with support bounded in \( H \))

\[
\int_H \{ (f_P, v) - \nu(A u, v) - b(u, u, v) \} d\mu(u) = 0.
\]

which gives us the result. \( \square \)

We end this section with a result concerning the Grashof number \( G^* \), which yields a bound in \( H \) on the weak attractor \( A_w \) in terms of \( P \):

**Lemma 3.1.** We have, more explicitly,

\[
G^* = \frac{\sqrt{3}L_y^{1/2}h^2}{6\pi^{1/2}L_x^{1/2}L_z^{1/2}} P. \tag{3.6}
\]

**Proof.** Since \( f_P = (P/L_x)e_1 \), we have that

\[
A^{-1}f_P = \left( \frac{P}{2L_x} z(h - z), 0, 0 \right).
\]
Hence,
\[ |A^{-1/2}f_P|_0^2 = (f_P, A^{-1}f_P) = \int_\Omega \frac{P}{L_x} \frac{P}{2L_x} z (h - z) dx = \frac{L_y h^3}{12L_x} P^2. \quad (3.7) \]
Taking the square root of the equality above and substituting in the definition of the Grashof number give us the result. \[ \square \]

**Remark 3.1.** The vector-field \( A^{-1}f_P \) is directly related to the plane Poiseuille flow. In fact, the plane Poiseuille flow is precisely \( u = A^{-1}f_P/\nu = (Pz(h - z)/2\nu L_x, 0, 0); \) see Section 7.

## 4. Characteristic dimensions and nondimensional numbers

The macroscopic characteristic length is considered to be \( h \) and the macroscale characteristic wavenumber is \( \kappa_0 = 1/h \). The total mass of the fluid in the channel is \( \rho_0 L_x L_y h \), where \( \rho_0 \) denotes the uniform mass density of the fluid. Then, for a given stationary statistical solution \( \mu \), the corresponding mean kinetic energy per unit mass and the mean energy dissipation rate per unit time and unit mass are given respectively by
\[
e = \frac{1}{2L_x L_y h} \langle |u|^2 \rangle, \quad \epsilon = \frac{\nu}{L_x L_y h} \langle \|u\|^2 \rangle.
\]

The mean longitudinal velocity is defined by
\[
U = \frac{1}{L_y h} \int_H \left( \int_0^h \int_0^{L_y} u_1(x, y, z) dy dz \right) d\mu(u).
\quad (4.1)
\]
Note that this definition makes sense and the expression does not depend on \( x \) due to the incompressibility and boundary conditions.

With this velocity scale we may define the following Reynolds number
\[
Re = \frac{U h}{\nu}.
\quad (4.2)
\]
A dimensionless ratio of the applied pressure gradient to the square of the flow velocity scale is provided by the skin friction coefficient given by
\[
C_f = \frac{P h}{L_x U^2}.
\quad (4.3)
\]

Thanks to condition (3) of Definition \[3.2\] and to the divergence-free and boundary conditions, the mean longitudinal velocity and the mean energy dissipation rate are related by
\[
\epsilon \leq \frac{U P}{L_x}.
\quad (4.4)
\]

Now, suppose that \( u(x, t) \) is a weak solution of the Navier-Stokes equations \[2.1\] with initial condition \( u_0(x) \). We define the finite-time average longitudinal velocity
by

\[ U_T = \frac{1}{L_y h T} \int_0^T \left( \int_0^h \int_0^{L_y} u_1(x, y, z, t) dy dz \right) dt. \]  

(4.5)

This expression is also well defined and independent of \( x \) due to the incompressibility and boundary conditions.

In the sequel, we will also consider the similarly defined time-averaged dissipation rate

\[ \frac{\nu}{L_x L_y h} \langle \|u\|^2 \rangle_T = \frac{\nu}{L_x L_y h T} \int_0^T \|u(t)\|^2 dt, \]

and time-averaged kinetic energy

\[ \frac{1}{2 L_x L_y h} \langle |u|^2 \rangle_T = \frac{1}{2 L_x L_y h T} \int_0^T |u(t)|^2 dt. \]

Notice that even if finite-time averages are bounded, their long-time limits may not exist.

5. TIME AVERAGES AND STATIONARY STATISTICAL SOLUTIONS

Since the usual limit of long-time averaged quantities may not exist, we aim to obtain eventual bounds for these time averaged quantities. In this section, we will establish, via generalized limits, a rigorous relationship between certain limits of these quantities and the stationary statistical solutions, see [13].

For example, suppose we are interested in estimating the upper limit of the time averaged velocity of a weak solution \( u(x, t) \) of (2.1)

\[ U = \frac{1}{L_y h} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \int_0^{L_y} \int_0^h u_1(x, y, z, t) dy dz \right) dt. \]  

(5.1)

Since an upper bound for the mean longitudinal velocity \( U \), associated with an arbitrary stationary statistical solution \( \mu \), is derived in (6.1), namely

\[ U_{\mu} \leq \frac{\sqrt{3} h^2}{6 \nu L_x} P, \]

we may establish a direct relation between the time average longitudinal velocity (5.1) and the mean longitudinal velocity associated with a specific stationary statistical solution \( \mu_0 \), such as

\[ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \int_0^{L_y} \int_0^h u_1(x, y, z, t) dy dz \right) dt = \int_H \left( \int_0^{L_y} \int_0^h u_1(x, y, z) dy dz \right) d\mu_0, \]  

(5.2)

in such a way that we can give an upper bound to (5.1) using (6.1), obtaining

\[ \frac{1}{L_y h} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \int_0^{L_y} \int_0^h u_1(x, y, z, t) dy dz \right) dt = U_{\mu_0} \leq \frac{\sqrt{3} h^2}{6 \nu L_x} P. \]  

(5.3)
This relation between long-time averages and stationary statistical solutions is realized via the notion of generalized limit, which is defined as follows.

**Definition 5.1.** A generalized limit is any linear functional, denoted \( \text{Lim}_{T \to \infty} \), defined on the space \( \mathcal{B}([0, \infty)) \) of all bounded real-valued functions on \( [0, \infty) \) and satisfying

1. \( \text{Lim}_{T \to \infty} g(T) \geq 0, \forall g \in \mathcal{B}([0, \infty)) \) with \( g(s) \geq 0, \forall s \geq 0; \)
2. \( \text{Lim}_{T \to \infty} g(T) = \lim_{T \to \infty} g(T), \forall g \in \mathcal{B}([0, \infty)) \)

such that the classical limit, denoted \( \text{lim}_{T \to \infty} \), exists.

**Remark 5.1.** It can be shown that given a particular \( g_0 \in \mathcal{B}([0, \infty)) \) and a sequence \( t_j \to \infty \) for which \( g_0 \) converges to a number \( l \), there exists a generalized limit \( \text{Lim}_{T \to \infty} \) satisfying \( \text{Lim}_{T \to \infty} g_0 = l \); see [2, 13].

**Proposition 5.1.** Let \( \varphi \in C(H_w) \). Suppose that for every stationary statistical solution \( \mu \), the associated average of \( \varphi \) satisfies \( \langle \varphi \rangle \leq C_1 \), for some constant \( C_1 \). Then, given a weak solution \( w(x, t) \) defined on \( [0, \infty) \), we have

\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \varphi(w(t))dt \leq C_1. \tag{5.4}
\]

Similarly, if for some constant \( C_2 \) we have \( \langle \varphi \rangle \geq C_2 \) for every stationary statistical solution, then

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \varphi(w(t))dt \geq C_2. \tag{5.5}
\]

**Proof.** We will prove inequality (5.4). Inequality (5.5) follows by a similar argument.

Let \( w_0 = w(0) \). Consider the set

\[
K_w = \left\{ v \in H; |v|^2 \leq |w_0|^2 + |f_P|^2 / \nu^2 \lambda_1^2 \right\},
\]

endowed with the weak topology of \( H \). \( K_w \) is compact in \( H_w \) and is such that \( w(t) \in K_w \), for all \( t \geq 0 \); see [5, 25].

Let \( \psi \in C(K_w) \). Since \( K_w \) is compact, the function \( t \mapsto \psi(w(t)) \) is continuous and bounded. Thus,

\[
g_0(t) = \frac{1}{t} \int_0^t \psi(w(s))ds
\]

makes sense, and is continuous and bounded for \( t \geq 0 \). Therefore, its generalized limit is well defined, and by Remark 5.1 if we choose a subsequence \( t_j \to \infty \) for which \( g_0(t_j) \) converges to \( \limsup_{t \to \infty} g_0(t) \), there exists a generalized limit \( \text{Lim}_{T \to \infty} \) satisfying

\[
\text{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \psi(w(t))dt = \limsup_{t \to \infty} \frac{1}{t} \int_0^t \psi(w(s))ds.
\]
Now, we relate this generalized time average with stationary statistical solutions.

Since the weak solution $t \mapsto w(t)$ belongs to the compact set $K_w$ in $H_w$, see [13], and since

$$
\psi \mapsto \lim_{T \to \infty} \frac{1}{T} \int_0^T \psi(w(t))dt
$$

is a positive linear functional on $C(K_w)$, we use the Kakutani-Riesz representation theorem, see [29], and conclude that there exists a measure $\mu_0$ on $H$ such that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \psi(w(x,t))dt = \int_H \psi(u) d\mu_0(u),
$$

(5.6)

for all $\psi \in C(K_w)$. It is shown in [13] that $\mu_0$ defined above is a stationary statistical solution.

Therefore, since $\varphi|_{K_w} \in C(K_w)$, and $\mu(H \setminus K_w) = 0$, for every stationary statistical solution, see [13], and in particular for $\mu_0$, we conclude that

$$
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \varphi(w(x,t))dt = \int_H \varphi(u) d\mu_0(u).
$$

Thus, since $\langle \varphi \rangle \leq C_1$ for all stationary statistical solution $\mu$, and in particular $\mu_0$, we have

$$
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \varphi(w(t))dt = \int_H \varphi(u) d\mu_0(u) \leq C_1.
$$

□

Now, returning to the mean longitudinal velocity example, since

$$
u \mapsto \int_0^L_y \int_0^h u_1(x,y,z)dydz
$$

belongs to $C(K_w)$, there exists a stationary statistical solution $\mu_0$ satisfying (5.2), which together with (6.1), yields the upper bound (5.3).

**Remark 5.2.** Proposition 5.1 shows that every estimate involving the average of a continuous quantity on $C(K_w)$ can be stated as a superior or inferior limit of its time average.

This is true for the energy injection term, $(f_P, u)$, and also for the mean longitudinal velocity $U$. However, we are also interested in estimating quantities involving $|u|^0$ and $\|u\|$, which are not weakly continuous. Fortunately, we are still able to estimate these quantities by approximating via Galerkin projections as shown in the next proposition.

**Proposition 5.2.** Let $w(x,t)$ be a weak solution of the NSE defined on $[0, \infty)$, and suppose that for every stationary statistical solution $\mu$, we have the following bounds

$$
C_1 \leq \int_H |u|^2_0 d\mu_0(u) \leq C_2,
$$
and
\[ C_3 \leq \int_H \|u\|^2 \, d\mu_0(u). \]

Then, we also have
\[ \liminf_{T \to \infty} \frac{1}{T} \int_0^T |w(t)|_0^2 \, dt \geq C_1, \quad (5.7) \]
\[ \limsup_{T \to \infty} \frac{1}{T} \int_0^T |w(t)|_0^2 \, dt \leq C_2, \quad (5.8) \]
and
\[ \liminf_{T \to \infty} \frac{1}{T} \int_0^T \|w(t)\|^2 \, dt \geq C_3. \quad (5.9) \]

Proof. Since \(|P_\kappa u|_0, \|P_\kappa u\| \in C(K_w)|,
where \(P_\kappa\) are the usual Galerkin projectors, we have by (5.6) that given a stationary statistical solution \(\mu_0\) generated by a generalized time average, the following equations are valid
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |P_\kappa w(t)|_0^2 \, dt = \int_H |P_\kappa u|_0^2 \, d\mu_0(u), \quad (5.10) \]
and
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \|P_\kappa w(t)\|^2 \, dt = \int_H \|P_\kappa u\|^2 \, d\mu_0(u). \quad (5.11) \]
Now, since
\[ |w(t)|_0^2 - |P_\kappa w(t)|_0^2 = |Q_\kappa w(t)|_0^2 \leq \kappa^{-2} \|w(t)\|^2, \]
and
\[ \frac{1}{T} \int_0^T \|w(t)\|^2 \, dt \leq C < \infty, \]
where \(C\) is independent of \(T\) \([5, 25]\), we have by the usual properties of the generalized limits that
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T |w(t)|^2 \, dt - \lim_{T \to \infty} \frac{1}{T} \int_0^T \|P_\kappa w(t)\|^2 \, dt \leq \kappa^{-2} \lim_{T \to \infty} \frac{1}{T} \int_0^T \|w(t)\|^2 \, dt \]
\[ \leq \kappa^{-2} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|w(t)\|^2 \, dt \leq C\kappa^{-2} \to 0, \quad \kappa \to \infty. \quad (5.12) \]
Thus, considering the generalized limit, \(\lim_{T \to \infty}\), that extends the left hand side of (5.7), we have
\[ \liminf_{T \to \infty} \frac{1}{T} \int_0^T |w(t)|_0^2 \, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T |w(t)|_0^2 \, dt = \lim_{\kappa \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T \|P_\kappa w(t)\|^2 \, dt \]
\[ = \lim_{\kappa \to \infty} \int_H \|P_\kappa u\|^2 \, d\mu_0(u) = \int_H \|u\|^2 \, d\mu_0(u) \geq C_1, \quad (5.13) \]
where the last equality of the expression above follows from the Monotone Convergence Theorem. The bound (5.8) follows in a similar way.
Now, we will prove (5.9). Consider the generalized limit, \( \lim_{T \to \infty} \), that extends the l.h.s. of (5.9), and notice that
\[
\| w(t) \| \geq \| P_\kappa w(t) \|. 
\]
Then, we have
\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \| w(t) \|^2 dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \| w(t) \|^2 dt \geq \lim_{\kappa \to \infty} \frac{1}{T} \int_0^T \| P_\kappa w(t) \|^2 dt = \lim_{\kappa \to \infty} \int_H \| P_\kappa u \|^2 d\mu_0(u) = \int_H \| u \|^2 d\mu_0(u) \geq C_3, \quad (5.14)
\]
where, again, the last equality in the expression above follows from the Monotone Convergence Theorem.

**Remark 5.3.** Propositions 5.1 and 5.2 show that, except for Theorem 7.8 and Proposition 7.9, every estimate in the sequel can be stated as a superior or inferior limit of their time averages. The reason why these results do not apply to Theorem 7.8 and Proposition 7.9 is that they involve an upper bound to \( \| u \| \), which is not considered by the propositions above. However, they can still be stated in terms of their time averages as seen in Remark 7.1.

6. **Estimates on the mean longitudinal velocity and on the skin friction coefficient**

We start by deriving an upper bound on the mean longitudinal velocity.

**Theorem 6.1.** For every stationary statistical solution, the mean longitudinal velocity \( U \) satisfies
\[
U \leq \frac{\sqrt{3} h^2}{6\pi \nu L_x} P. \quad (6.1)
\]

**Proof.** It follows directly from the definition of \( U \) and from the Cauchy-Schwarz and H"older inequalities that
\[
U \leq \frac{1}{L_x^{1/2} L_y^{1/2} h^{1/2}} \langle |u|^2 \rangle^{1/2}. \quad (6.2)
\]

Now, by (3.3) and (3.6), we can estimate the term \( \langle |u|^2 \rangle \) as follows
\[
\langle |u|^2 \rangle \leq \frac{\nu^2 h}{\pi} G^{*2} = \frac{L_y h^5}{12\pi^2 \nu^2 L_x} P^2. \quad (6.3)
\]

Substituting (6.3) into (6.2), we obtain the result. □

A lower bound for the skin friction coefficient, \( C_f \), follows directly from the theorem above.
Corollary 6.2. For every stationary statistical solution, the skin friction coefficient, $C_f$, satisfies

$$C_f \geq \frac{2\pi \sqrt{3}}{Re}.$$  \hfill (6.4)

Proof. It follows immediately from Theorem 6.1 that

$$C_f = \frac{Ph}{L_x U^2} \geq \frac{Ph}{L_x U} \frac{6\pi \nu L_x}{\sqrt{3h^2}P} = \frac{2\pi \sqrt{3}}{3hU},$$

and the result follows from the definition of the Reynolds number \((4.2)\). \hfill \square

Now, we give a lower bound estimate for the mean longitudinal velocity $U$ following the calculations of [4], but avoiding using an equation for the fluctuation $v$.

Proposition 6.3. For every stationary statistical solution, the mean longitudinal velocity $U$ satisfies

$$U \geq \sup \left\{ \frac{h^2}{12\nu L_x} P - \frac{\nu L_x}{Ph} \int_0^h \left( U_1'(z) - \frac{P}{2\nu L_x} (h - 2z) \right)^2 dz; \ U \in \mathcal{U} \right\},$$

where

$$\mathcal{U} = \left\{ \mathbf{U} \in \mathbf{V}; \ \mathbf{U}(x, y, z) = (U_1(z), 0, 0), \ U_1 \in H^1_0(0, h), \ (H_U(u - \mathbf{U})) \geq 0 \right\},$$

and

$$H_U(u - \mathbf{U}) = \nu \frac{\|u - \mathbf{U}\|^2}{2} + b(u - \mathbf{U}, \mathbf{U}).$$

Proof. Let $\mathbf{U} \in \mathbf{V}$ be of the form $\mathbf{U}(x, y, z) = (U_1(z), 0, 0)$. We have

$$\langle \|u - \mathbf{U}\|^2 \rangle = \langle \|u\|^2 \rangle - 2\langle \langle u, \mathbf{U} \rangle \rangle + \|\mathbf{U}\|^2. \hfill (6.5)$$

Now, since $\mathbf{U} \in \mathbf{V}$ is fixed, we can multiply it with the Reynolds equations and obtain

$$\nu\langle \langle u, \mathbf{U} \rangle \rangle = (f_p, \mathbf{U}) - \langle b(u, \mathbf{U}, \mathbf{U}) \rangle. \hfill (6.6)$$

Substituting (6.6) into (6.5), we obtain

$$\nu\langle \|u\|^2 \rangle = \nu\langle \|u - \mathbf{U}\|^2 \rangle + 2\langle (f_p, \mathbf{U}) - \langle b(u, \mathbf{U}, \mathbf{U}) \rangle \rangle - \nu\langle \|\mathbf{U}\|^2 \rangle. \hfill (6.7)$$

Since $\mathbf{U}(x, y, z) = (U_1(z), 0, 0)$, by the anti-symmetry property of the trilinear term and by (4.4), we have

$$L_xL_yhU \geq \nu \frac{L_x}{P} \langle \|u\|^2 \rangle \hfill (6.8)$$

$$= \frac{2L_x}{P} \langle \nu \frac{\|u - \mathbf{U}\|^2}{2} + b(u, \mathbf{U}, \mathbf{U}) \rangle + 2\frac{L_x}{P}(f_p, \mathbf{U}) - \frac{\nu L_x}{P} \|\mathbf{U}\|^2.$$

Due to the form of $\mathbf{U}$, we have $b(u, \mathbf{U}, \mathbf{U}) = 0$. The orthogonality property implies $b(u - \mathbf{U}, \mathbf{U}) = 0$. 

Thus,

\[ b(u, U, u) = b(u - U, U, u - U), \forall u \in V. \]

Since \( \mu \) is carried by \( V \), we find

\[
(L_x L_y h) U = \frac{2L_x}{P} \langle \nu \| u - U \|^2 \rangle + b(u - U, U, u - U) + \frac{2L_x}{P} (f_P, U) - \frac{\nu L_x}{P} \| U \|^2. \tag{6.9}
\]

Thus, considering only background flows \( U \) such that \( \langle H_U(v) \rangle \geq 0 \), we have

\[
(L_x L_y h) U \geq \frac{2L_x}{P} (f_P, U) - \frac{\nu L_x}{P} \| U \|^2. \tag{6.10}
\]

We obtain the result by completing the squares.

**Remark 6.1.** This theorem was shown in [4], in the context of long time averages. It was obtained from a derivation of an energy equation for the fluctuation variable \( v = u - U \):

\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \nu \| v \|^2 + \nu \langle (v, U) \rangle + b(U, U, v) + b(v, U, v) = (f_P, v) \tag{6.11}
\]

and by considering an energy equation for \( u \):

\[
\frac{d}{dt} \frac{1}{2} |u|^2 + \nu \| u \|^2 = (L_y h) PU. \tag{6.12}
\]

Taking the long time average in both sides of (6.12), considering the same hypothesis for \( U \), and substituting it into (6.11), they have obtained the corresponding result for long time averages.

However, since we want to consider any stationary statistical solutions and general weak solutions of the Navier-Stokes equations, we treat carefully the fluctuation component and avoid the energy equation (6.11). The slightly modified and simpler proof presented in Theorem 6.3 achieves this aim.

**Theorem 6.4.** For every stationary statistical solution, the mean longitudinal velocity and the skin friction coefficient satisfy

\[
U \geq \begin{cases} 
\frac{h^2}{2 \nu L_x^2} P, & \text{if } 0 < P \leq \frac{27 \sqrt{2} \pi^2 \nu^2 L_x}{4h^3}; \\
\frac{2^{5/4} \pi^{1/2} h^{1/2}}{3^{3/2}} P^{1/2} - \frac{\sqrt{2} \pi^2 \nu}{2h}, & \text{if } P > \frac{27 \sqrt{2} \pi^2 \nu^2 L_x}{4h^3}.
\end{cases}
\tag{6.13}
\]

and
\[ C_f \leq \begin{cases} \frac{27}{2} \frac{1}{\text{Re}} & \text{if } 0 < P \leq \frac{27\sqrt{2\pi^2}\nu^2 L_x}{4h^3}; \\ \frac{27\sqrt{2}}{8\pi^2} \left(1 + \frac{\sqrt{2\pi^2}}{2} \frac{1}{\text{Re}}\right)^2 & \text{if } P \geq \frac{27\sqrt{2\pi^2}\nu^2 L_x}{4h^3}. \end{cases} \] (6.14)

**Proof.** Following Constantin and Doering, consider the background-flow of the form
\[ U(x, y, z) = (U_1(z), 0, 0) \]
with
\[ U_1(z) = \begin{cases} \frac{V}{\delta} z, & 0 \leq z \leq \delta; \\ \frac{V}{\delta} z, & \delta \leq z \leq h - \delta; \\ \frac{V}{\delta} (h - z), & h - \delta \leq z \leq h. \end{cases} \]

We will verify that this flow satisfies the spectral constraint \( H_U(v) \geq 0 \), for appropriate choices of \( V \) and \( \delta \). For that purpose, we bound the integral of \( U'_1(z)v_1v_3 \) in terms of \( \delta \) and \( \|v\|^2 \). First, divide this integral into two parts, one from 0 to \( \delta \), and the other from \( h - \delta \) to \( h \).

In order to bound the first integral, consider the spaces \( \tilde{H} = L^2(0, \delta) \), with the usual \( L^2 \) inner product, and \( \tilde{V} = \{ u \in H^1(0, \delta); u(0) = 0 \} \), with the inner product \( (u, v) = \int_0^\delta u'(z)v'(z)dz \). Consider the operator \( \tilde{A} : \tilde{V} \rightarrow \tilde{H} \) defined by
\[ (\tilde{A}u, v) = ((u, v)), \quad \forall v \in \tilde{V}, \]
and \( D(\tilde{A}) = \{ u \in \tilde{V}; \tilde{A}u \in \tilde{H} \} \). One can show that \( \tilde{A} \) is self-adjoint and invertible, with compact inverse, and that the smallest associated eigenvalue is \( \tilde{\lambda}_1 = \pi^2/4\delta^2 \). Therefore,
\[ \tilde{\lambda}_1 \int_0^\delta |u(z)|^2 dz \leq \int_0^\delta \left| \frac{\partial u(z)}{\partial z} \right|^2 dz. \] (6.15)

A similar statement can be made for the integral between \( h - \delta \) and \( h \). Thus, the integral of \( U'(z)v_1v_3 \) can be estimated in the following way
\[ \left| \int_0^{L_x} \int_0^{L_y} \int_0^h U'(z)v_1v_3 dxdydz \right|. \]
\[
\begin{align*}
&\quad \leq \frac{V}{\delta} \left| \int_0^{L_x} \int_0^{L_y} \int_0^{\delta} v_1 v_3 \, dx \, dy \, dz - \int_0^{L_x} \int_0^{L_y} \int_{h-\delta}^h v_1 v_3 \, dx \, dy \, dz \right| \\
&\quad \leq \frac{V}{\delta} \int_0^{L_x} \int_0^{L_y} \int_0^{\delta} \alpha \left| \frac{v_1}{2} \right|^2 + \frac{1}{2\alpha} \left| \frac{v_3}{2\alpha} \right|^2 \, dx \, dy \, dz \\
&\quad \quad + \frac{V}{\delta} \int_0^{L_x} \int_0^{L_y} \int_{h-\delta}^h \alpha \left| \frac{v_1}{2} \right|^2 + \frac{1}{2\alpha} \left| \frac{v_3}{2\alpha} \right|^2 \, dx \, dy \, dz \\
&\quad \leq \frac{2V\delta}{\pi^2} \left( \alpha \left| \frac{\partial v_1}{\partial z} \right|_0 + \frac{1}{\alpha} \left| \frac{\partial v_3}{\partial z} \right|_0 \right) \\
&\quad \leq \frac{2V\delta}{\pi^2} \left( \alpha \left| \frac{\partial v_1}{\partial z} \right|_0 + \frac{1}{2\alpha} \left( \left| \frac{\partial v_3}{\partial z} \right|_0 + \left| \frac{\partial v_1}{\partial x} \right|_0 + \left| \frac{\partial v_2}{\partial y} \right|_0 + \left| \frac{\partial v_1}{\partial y} \right|_0 + \left| \frac{\partial v_2}{\partial x} \right|_0 \right) \right).
\end{align*}
\]

The last step above follows from the following inequality stated in [3]:

\[
\left| \frac{\partial v_3}{\partial z} \right|_0 \leq \frac{1}{2} \left( \left| \frac{\partial v_3}{\partial z} \right|_0 + \left| \frac{\partial v_1}{\partial x} \right|_0 + \left| \frac{\partial v_2}{\partial y} \right|_0 + \left| \frac{\partial v_1}{\partial y} \right|_0 + \left| \frac{\partial v_2}{\partial x} \right|_0 \right),
\]

which is valid for divergence-free vector fields.

Thus, choosing \( \alpha = \sqrt{2}/2 \), we have

\[
\left| \int_0^{L_x} \int_0^{L_y} \int_{h-\delta}^h U_1'(z) v_1 v_3 \, dx \, dy \, dz \right| \leq \frac{\sqrt{2}}{\pi^2} V\delta \|v\|^2.
\]

Hence, \( H_U(v) \) is bounded from below by

\[
H_U(v) \geq \left( \frac{\nu}{2} - \frac{\sqrt{2}}{\pi^2} V\delta \right) \|v\|^2.
\]

Therefore, \( H_U \) is non-negative if \( \delta \leq \nu \pi^2 / 2\sqrt{2}V \), with \( V \) sufficiently large to fulfill the compatibility hypothesis \( \delta < h/2 \). Now, by substituting \( U \) in (6.10), we give a lower bound for \( U \):

\[
U \geq 2L_x L_y \left( hV - \delta V - \frac{\nu L_x V^2}{\delta P} \right).
\]  

(6.16)

We maximize the lower bound above, respecting the compatibility hypotheses, with the following choices of \( V \) and \( \delta \):

\[
V = \frac{\pi h^{1/2}}{3^{1/2} 2^{3/4} L_x^{1/2}} P^{1/2}, \quad \delta = \frac{3^{1/2} \nu L_x^{1/2}}{2^{3/4} h^{1/2} P^{1/2}} \quad \text{if} \quad P > 27\frac{\sqrt{2}\pi^2 \nu^2 L_x}{4h^3},
\]

(6.17)

and

\[
V = \frac{h^2 P}{9\nu L_x}, \quad \delta = \frac{h}{3} \quad \text{if} \quad P \leq 27\frac{\sqrt{2}\pi^2 \nu^2 L_x}{4h^3}.
\]  

(6.18)
The result follows immediately from the substitution of (6.17) and (6.18) into (6.16).

Remark 6.2. Theorem 6.4 gives a uniform upper bound estimate for the skin friction coefficient for high Reynolds numbers. Even though this constant upper bound estimate is predicted by the Kolmogorov theory of homogeneous turbulence, it is known from experiments that corrections are necessary for turbulence in the presence of walls, see [3, 4, 23, 28]. Actually, closure approximation theories establish the following logarithmic friction law which has been confirmed by high Reynolds number pipe flow experiments:

\[ C_f \sim \frac{1}{(\ln R)^2}. \] (6.19)

Thus, we conclude that while empirical arguments and experimental data predicts a logarithmic friction law, our rigorous mathematical bounds can only assert that

\[ \frac{2\pi \sqrt{3}}{Re} \leq C_f \leq \frac{27\sqrt{2}}{8\pi^2} + O\left(\frac{1}{Re^2}\right). \] (6.20)

The lower bound for \( C_f \) is of the order of the skin friction coefficient for the plane Poiseuille flow; see Section 8.

Remark 6.3. Note also that for high Reynolds number flows, the characteristic background velocity which leads to the estimate above is of the order of

\[ V \sim \left(\frac{h}{L}\right)^{1/2}P^{1/2}, \]

while the corresponding “boundary layer” length is of the order of

\[ \delta \sim \left(\frac{\nu L^1}{h^{1/2}}\right)P^{-1/2}. \]

7. Other Estimates

We start by deriving a lower bound for the energy dissipation rate \( \epsilon \).

**Theorem 7.1.** For every stationary statistical solution, the energy dissipation rate satisfies

\[ \epsilon \geq \begin{cases} 
\frac{2}{27\nu^2 L_x^2} h^2 P^2, & \text{if } 0 < P \leq \frac{27\sqrt{2\pi^2 \nu^2 L_x}}{4h^3}, \\
\frac{2^{5/4}}{3^{3/2}} \frac{h^{1/2}}{L_x^{3/2}} P^{3/2} - \frac{\sqrt{2\pi^2 \nu}}{2h}, & \text{if } P > \frac{27\sqrt{2\pi^2 \nu^2 L_x}}{4h^3}.
\end{cases} \] (7.1)

**Proof.** The result follows from noticing that the estimate (6.8) obtained in Proposition 6.3 is actually a lower bound for \( \langle \|u\|^2 \rangle \), and, therefore, we can follow the subsequent calculations in the exact same way with this term instead of \( U \). □

Now, we give a lower bound on the mean kinetic energy.
Theorem 7.2. For every stationary statistical solution, the mean kinetic energy $e$ satisfies

$$e \geq \frac{h}{6L_x} P - \frac{4\nu}{L_y h^3} \left( \frac{\nu^2 L_y^2 h^2}{P} + \frac{L_y^2 h^5}{12L_x} \right)^{1/2} P^{1/2} + \frac{4\nu^2}{h^2}.$$  \hspace{1cm} (7.2)

Proof. Taking the inner product with $A^{-1}f_P$ in the Reynolds equation yields

$$|A^{-1/2}f_P|_0^2 = \nu \langle (u,f_P) \rangle + b \langle (u,u,A^{-1}f_P) \rangle \leq \nu \langle |u|_0 \rangle |f_P|_0 + \langle |b (u,A^{-1}f_P,u)| \rangle,$$  \hspace{1cm} (7.3)

and since

$$\langle |b (u,A^{-1}f_P,u)| \rangle = \langle \int_\Omega u_3 \left( \frac{\partial P}{\partial z} 2L_x z(h-z) \right) u_1 dx \rangle$$

$$\leq \frac{P}{2L_x} \langle \int_\Omega |h-2z| |u_3| |u_1| dx \rangle$$

$$\leq \frac{Ph}{4L_x} \langle \int_\Omega |u_3|^2 + |u_1|^2 dx \rangle$$

$$\leq \frac{Ph}{4L_x} \langle |u|_0^2 \rangle,$$  \hspace{1cm} (7.4)

we find from (7.3) that

$$|A^{-1/2}f_P|_0^2 \leq \nu |f_P| \langle |u|_0^2 \rangle^{1/2} + \langle |b (u,A^{-1}f_P,u)|_0 \rangle$$

$$\leq \nu \frac{PL_y^{1/2} h^{1/2}}{L_x^{1/2}} \langle |u|_0^2 \rangle^{1/2} + P \frac{h}{4L_x} \langle |u|_0^2 \rangle.$$  \hspace{1cm} (7.5)

Then, using (3.7), we have

$$\frac{L_y h^3}{12L_x} P \leq \frac{\nu}{L_x} \langle |u|_0^2 \rangle^{1/2} + \frac{h}{4L_x} \langle |u|_0^2 \rangle,$$  \hspace{1cm} (7.6)

which is of the form $ar^2 + br + c \geq 0$ for $r = \langle |u|_0^2 \rangle^{1/2}$, $a = h/4L_x$, $b = \nu/L_x$, $c = (L_y h^3)/12L_x$. It gives us $r^2 \geq b^2/2a^2 - b(b^2 + 4ac)^{1/2} / a + c/a$, which implies

$$\langle |u|_0^2 \rangle \geq \frac{h^2 L_y}{3} P - \frac{8\nu L_x}{h^2} \left( \frac{\nu^2 L_y^2 h^2}{P} + \frac{L_y^2 h^5}{12L_x} \right)^{1/2} P^{1/2} + \frac{8\nu^2 L_x L_y}{h}.$$  \hspace{1cm} \square

Proposition 7.3. For every stationary statistical solution, the mean kinetic energy $e$ satisfies

$$e \leq \frac{h^4}{24\pi^2 \nu^2 L_x^2} P^2.$$  \hspace{1cm} (7.7)

Proof. This follows directly from inequality (6.3).  \hspace{1cm} \square
Theorem 7.4. For every stationary statistical solution, the energy dissipation rate $\epsilon$ satisfies

$$\epsilon \leq \frac{\sqrt{3}h^2}{6\pi\nu L_x^2} P^2. \quad (7.8)$$

Proof. This follows directly from inequalities (4.4) and (6.1).

Now, we state a partial rigorous confirmation of the Kolmogorov dissipation law in terms of $U$.

Proposition 7.5. For every stationary statistical solution, and sufficiently large pressure drop $P$, namely $P \geq 27\sqrt{2\pi^2\nu^2 L_x/4h^3}$, the associated energy dissipation rate satisfies

$$\epsilon \leq \left(\frac{3}{25/2\pi^2} + \frac{27\pi}{4} \frac{1}{\Re} + \frac{27\pi^2}{2^{7/2}} \frac{1}{\Re^2}\right) \frac{U^3}{h}. \quad (7.9)$$

Proof. Taking the square of both sides of the second inequality in (6.13), we have

$$P \leq \frac{3L_x}{2^{5/2}\pi h^2} U^2 + \frac{27L_x\pi\nu}{4h^2} U + \frac{27\pi^2 L_x\nu^2}{2^{7/2}h^3}. \quad (7.10)$$

Substituting (7.10) into (4.4), we obtain the result.

Remark 7.1. Note that we cannot invoke Proposition 5.1 neither Proposition 5.2 to state the results (7.9) and (7.8) in terms of their time averages. However, we can improve these results as follows.

Let $u(x,t)$ be a weak solution of the Navier-Stokes equations. It follows from the classical energy inequality for weak solutions of the NSE defined on $[0,\infty)$, see [5, 25], that

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \nu \|u(s)\|^2 \, ds \leq \liminf_{T \to \infty} \frac{1}{T} \int_0^T (\mathbf{f}_P, u(s)) \, ds. \quad (7.11)$$

Then, an inequality similar to (4.4) can be stated

$$\frac{\nu}{L_x L_y h} \limsup_{T \to \infty} (\|u(s)\|^2)_T \leq \frac{P}{L_x} \liminf_{T \to \infty} U_T. \quad (7.12)$$

Hence, if we consider inequality (6.1) for the stationary statistical solution $\mu_0$ that extends the inferior limit of the time-averaged longitudinal velocity of $u(x,t)$, we have

$$\frac{\nu}{L_x L_y h} \limsup_{T \to \infty} (\|u(s)\|^2)_T \leq \frac{P}{L_x} \liminf_{T \to \infty} U_T = \frac{PU_{\mu_0}}{L_x} \leq \frac{\sqrt{3}h^2}{6\nu L_x^2} P^2.$$

Similarly, considering inequality (6.13) for the same $\mu_0$ above, we have

$$P \leq \frac{3L_x}{2^{5/2}\pi h^2} U_{\mu_0}^2 + \frac{27L_x\pi\nu}{4h^2} U_{\mu_0} + \frac{27\pi^2 L_x\nu^2}{2^{7/2}h^3} \frac{1}{\mu_0}$$

$$= \frac{3L_x}{2^{5/2}\pi h^2} \left(\liminf_{T \to \infty} U_T\right)^2 + \frac{27L_x\pi\nu}{4h^2} \left(\liminf_{T \to \infty} U_T\right) + \frac{27\pi^2 L_x\nu^2}{2^{7/2}h^3}. \quad (7.13)$$
Substituting (7.13) into (7.12), we obtain

\[
\frac{\nu}{L_x L_y h} \limsup_{T \to \infty} \langle \|u(s)\|^2 \rangle_T \leq \left( \frac{3}{25/2 \pi^2 h} + \frac{27\pi}{4 \text{ Re}} + \frac{27\pi^2}{27/2 \text{ Re}^2} \right) \liminf_{T \to \infty} \frac{U_3^3}{h}.
\]

(7.14)

8. The plane Poiseuille flow

In order to see how sharp our estimates are, we calculate the characteristic quantities for a specific explicit flow. It can be easily shown that there exists an explicit solution for the stationary version of the channel flow problem in this geometry, known as the plane Poiseuille flow:

\[
u_{\text{Poiseuille}}(x, y, z) = \frac{P}{2\nu L_x} z(h - z)e_1.
\]

(8.1)

A straightforward calculation gives us the following estimates:

**Proposition 8.1.** We have, for the plane Poiseuille flow,

\[
\epsilon_{\text{Poiseuille}} = \frac{1}{2L_x L_y h} \int_\Omega \frac{P^2}{4\nu^2 L_x^2} z^2(h - z)^2 dx dy dz = \frac{h^4}{240\nu^2 L_x^2} P^2; \quad (8.2)
\]

\[
\epsilon_{\text{Poiseuille}} = \frac{\nu}{L_x L_y h} \int_\Omega \frac{P^2}{4\nu^2 L_x^2} (h - 2z)^2 dx dy dz = \frac{h^2}{6\nu L_x^2} P^2; \quad (8.3)
\]

\[
C_{f\text{Poiseuille}} = \frac{12}{\text{Re}}; \quad (8.4)
\]

\[
U_{\text{Poiseuille}} = \frac{P h^2}{12\nu L_x}; \quad (8.5)
\]

and

\[
\text{Re}_{\text{Poiseuille}} = \frac{U_{\text{u}P} h}{\nu} = \frac{P h^3}{12\nu^2 L_x}. \quad (8.6)
\]

**Remark 8.1.** Notice that the upper bound estimates for the mean kinetic energy, and mean energy dissipation obtained in the previous sections are sharp in the sense that they are of the same order (up to a multiplicative constant) as those just presented for the plane Poiseuille flow. They are sharp independently of the value of the applied pressure. As far as we know, these estimates were known to be sharp only when the applied pressure is low, since in this case the plane Poiseuille flow is globally asymptotically stable.
9. The rate of decrease of energy injection with respect to the scales of the flow

In the classical theory of homogeneous turbulence, it is argued that for turbulent flows, the energy injection is concentrated on the large scale motions, whereas the energy dissipated into heat due to the molecular viscosity occurs on scales that are much smaller than those.

In 1941, Kolmogorov [18] proposed that within a certain range of scales much lower than the energy injection scales and greater than the energy dissipative scales, the energy is transferred to the small scales at a nearly constant rate equal to the energy dissipation rate. This mechanism is called the energy cascade, and sufficient conditions were rigorously derived in [13, 14, 15] for the existence of this phenomenon. This theory was proposed in the idealized case of locally homogeneous turbulence, away from the boundaries, with the injection of energy restricted to the large scales. However, it is well known from experiments that for wall bounded turbulence, this hypothesis needs to be corrected; see [4, 28, 23]. In particular, the energy injection occurs at arbitrarily small scales. Our next result gives an upper bound for the rate of decrease of mean energy injection at progressively small scales.

By taking the scalar product of the Navier-Stokes equations with the component \( u_{\kappa',\kappa''} \) of the flow we find the energy equation for the scales of motion in the range \([\kappa', \kappa'']\):

\[
\frac{1}{2} \frac{d}{dt} |u_{\kappa',\kappa''}|^2 + \nu \|u_{\kappa',\kappa''}\|^2 + b(u, u, u_{\kappa',\kappa''}) = ((f_P)_{\kappa',\kappa''}, u_{\kappa',\kappa''}),
\]

where \( f_P \) is the pressure force. If \( \kappa'' < \infty \), and

\[
\frac{1}{2} \frac{d}{dt} |u_{\kappa',\infty}|^2 + \nu \|u_{\kappa',\infty}\|^2 + b(u, u, u_{\kappa',\infty}) \leq ((f_P)_{\kappa',\infty}, u_{\kappa',\infty}),
\]

if \( \kappa'' = \infty \).

The energy injection at each scale associated with a wavenumber \( \kappa \) due to the pressure gradient is thus given by

\[
\mathcal{E}_\kappa(u) = \frac{1}{L_x L_y h} ((f_P)_\kappa, u_\kappa).
\]

The energy injection in a range of wavenumbers \([\kappa', \kappa'']\) is given by

\[
\mathcal{E}_{\kappa',\kappa''}(u) = \frac{1}{L_x L_y h} ((f_P)_{\kappa',\kappa''}, u_{\kappa',\kappa''}).
\]

In particular, the energy injection into the wavenumbers larger than or equal to a given wavenumber \( \kappa \) is given by \( \mathcal{E}_{\kappa,\infty}(u) \).

The mean energy injection is given by the average value of those quantities with respect to a given stationary statistical solution. In order to estimate the mean energy injection at different length scales let us prove the following lemma.
Lemma 9.1. The forcing term component \((f_P)_\kappa\), for a given wavenumber \(\kappa\), satisfies

\[ |(f_P)_\kappa|_0 = \begin{cases} \frac{2L_y^{1/2}}{L_x^{1/2}h^{1/2}} \frac{P}{\kappa}, & \text{if } \kappa = \frac{k\pi}{h}, \ k \in \mathbb{N}, \ k \text{ odd;} \\ 0, & \text{otherwise.} \end{cases} \]  

(9.3)

Proof. First, notice that we can write

\[ (f_P)_\kappa = \sum_{\lambda_{j,l,k}=\kappa^2} (f_P, w_{j,l,k}) w_{j,l,k}, \]  

(9.4)

so that by the Parseval identity we have

\[ |(f_P)_\kappa|^2 = \sum_{\lambda_{j,l,k}=\kappa^2} |(f_P, w_{j,l,k})|^2. \]  

(9.5)

We also notice that each projection satisfies

\[ (f_P, w_{j,l,k}) = \frac{P}{L_x} \int_\Omega w_{1,j,l,k}^1 dx, \]  

(9.6)

where \(w_{j,l,k}^1\) denotes the first component of the eigenvector \(w_{j,l,k}\). Now, by inspecting the expression (2.4) for \(w_{j,l,k}\), we notice that the integral (9.6) vanishes for all \((j,l) \neq (0,0)\). Thus, (9.5) reduces to

\[ |(f_P)_\kappa|^2 = P^2 L_y^2 \sum_{\lambda_{0,0,k}=\kappa^2} \left| \int_0^h \hat{w}_{0,0,k}^1(z) dz \right|^2. \]  

(9.7)

Furthermore, one can deduce from the Stokes problem and the expansion (2.4) that the component \(\hat{w}_{0,0,k}^1(z)\) satisfies following one-dimensional eigenvalue problem:

\[ \begin{cases} -\frac{\partial^2 \hat{w}_{0,0,k}^1(z)}{\partial z^2} = \lambda_{0,0,k} \hat{w}_{0,0,k}^1(z), \\ -\frac{\partial^2 \hat{w}_{0,0,k}^2(z)}{\partial z^2} = \lambda_{0,0,k} \hat{w}_{0,0,k}^2(z), \\ \hat{w}_{0,0,k}^2(z) = 0, \end{cases} \]  

And the normalized solution to this equation is

\[ \hat{w}_{0,0,k}(0,0,z) = \frac{1}{L_x^{1/2}L_y^{1/2}h^{1/2}} \left( \sin\left(\frac{k\pi}{h}z, \sin\left(\frac{k\pi}{h}z\right), 0 \right), \right) \]  

with \(\lambda_{0,0,k} = (k\pi/h)^2\). Then, we can once more reduce (9.5) to

\[ |(f_P)_\kappa|^2 = \begin{cases} P^2 L_y^2 \left| \int_0^h \hat{w}_{0,0,k}^1(z) dz \right|^2 & \text{if } \kappa = \frac{k\pi}{h}, \ \text{for some } k \in \mathbb{N}, \\ 0, & \text{if } \kappa \neq \frac{k\pi}{h}, \ \text{for every } k \in \mathbb{N} \end{cases} \]  

(9.8)
Then the result follows directly from the following calculation:

\[
\int_0^h \dot{w}_{0,0,k}(z)dz = \frac{1}{L_x^{1/2}L_y^{1/2} h^{1/2}} \int_0^h \sin\left(\frac{k\pi}{h} z\right)dz = \begin{cases} 
\left(\frac{2h^{1/2}}{\pi L_x^{1/2}L_y^{1/2}}\right) \frac{1}{k}, & \text{if } k \text{ is odd,} \\
0, & \text{if } k \text{ is even.} \quad (9.9)
\end{cases}
\]

We now estimate the energy injection at a given wavenumber \( \kappa \).

**Proposition 9.1.** The mean energy injection at a given wavenumber \( \kappa \) with respect to an arbitrary stationary statistical solution satisfies

\[
\langle \dot{f}_\kappa(u) \rangle \leq \begin{cases} 
\frac{2}{L_x^{3/2}L_y^{1/2} h^{3/2} \kappa^2} \|u_\kappa\|, & \text{if } \kappa = \frac{k\pi}{h}, \ k \in \mathbb{N}, \ k \text{ odd,} \\
0, & \text{otherwise.} \quad (9.10)
\end{cases}
\]

**Proof.** We have

\[
\langle \dot{f}_\kappa(u) \rangle = \frac{1}{L_x L_y h} \langle (f_P)_\kappa, u_\kappa \rangle \leq \frac{1}{L_x L_y h} |A^{-1/2}(f_P)_\kappa|_0 \|u_\kappa\| = \frac{1}{L_x L_y h \kappa} |(f_P)_\kappa|_0 \|u_\kappa\|,
\]

and the result follows from using Lemma 9.1. \( \square \)

**Proposition 9.1.** The mean energy injection on the modes larger than or equal to \( \kappa \) with respect to an arbitrary stationary statistical solution satisfies

\[
\langle \dot{f}_{\kappa,\infty}(u) \rangle \leq \frac{1}{\kappa^{3/2}} \left(\frac{2P}{\pi^{1/2} L_x^{3/2} L_y^{1/2} h}\right) \|u_{\kappa,\infty}\| \leq \frac{1}{\kappa^{3/2}} \left(\frac{2P}{\pi^{1/2} L_x^{1/2} L_y^{1/2} h}\right) \epsilon^{1/2}. \quad (9.11)
\]

**Proof.** We have:

\[
\frac{1}{L_x L_y h} \langle (f_P, u_{\kappa,\infty}) \rangle \leq \frac{1}{L_x L_y h} |A^{1/2}(f_P)_{\kappa,\infty}|_0 \|u_{\kappa,\infty}\|.
\]

Estimating the term \( |A^{-1/2}(f_P)_{\kappa,\infty}|_0 \), we have by the Parseval identity that

\[
|A^{-1/2}(f_P)_{\kappa,\infty}|_0^2 = \sum_{\kappa' = \kappa}^{\infty} |A^{-1/2}(f_P)_{\kappa'}|_0^2 = \sum_{\kappa' = \kappa}^{\infty} \frac{1}{\kappa'^2} |(f_P)'_{\kappa'}|_0^2.
\]

Then, if \( k \) is the smallest odd number such that \( \kappa \leq k\pi/h \), and using Lemma 9.1, the following estimate holds

\[
|A^{-1/2}(f_P)_{\kappa,\infty}|_0^2 = \frac{4L_y^3 P^2}{\pi^4 L_x} \sum_{j = k, j \text{ odd}}^{\infty} \frac{1}{j^4} \leq \frac{4L_y^3 P^2}{\pi^4 L_x} \int_k^{\infty} \frac{1}{s^4} ds = \frac{4L_y^3 P^2}{\pi^4 L_x} \frac{1}{3k^3} \leq \frac{4L_y P^2}{\pi L_x} \frac{1}{k^3}.
\]
Thus,
\[
\frac{1}{L_x L_y h} \langle (f_P, u_{\kappa, \infty}) \rangle \leq \frac{2P}{\pi^{1/2} L_x^{3/2} L_y h^{1/2} h \kappa^{3/2}} \| u_{\kappa, \infty} \|,
\]
which completes the proof. \qed

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