SOME DEGENERATE PARABOLIC PROBLEMS: EXISTENCE AND DECAY PROPERTIES

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ABSTRACT. We study the existence of solutions \( u \) belonging to \( L^1(0,T; W^{1,1}_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \) of a class of nonlinear problems whose prototype is the following

\[
\begin{align*}
    u_t - \text{div} \left( \frac{\nabla u}{(1+|u|)^\gamma} \right) &= 0, & \text{in } \Omega_T; \\
    u &= 0, & \text{on } \partial\Omega \times (0,T); \\
    u(x,0) &= u_0(x) \in L^2(\Omega), & \text{in } \Omega.
\end{align*}
\] (1)

We investigate also the asymptotic estimates satisfied by distributional solutions that we find and the uniqueness.

1. Introduction. In this paper we study the following class of nonlinear parabolic problems

\[
\begin{align*}
    u_t - \text{div} (a(x,t,u)\nabla u) &= 0, & \text{in } \Omega_T; \\
    u &= 0, & \text{on } \partial\Omega \times (0,T); \\
    u(x,0) &= u_0(x), & \text{in } \Omega;
\end{align*}
\] (2)

where \( u_0(x) \) belongs to some Lebesgue space \( L^m(\Omega) \), \( \Omega \) is a bounded open set of \( \mathbb{R}^N \), \( N \geq 2 \) and \( \Omega_T = \Omega \times (0,T), T > 0 \). Here \( a(x,t,\rho) : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R} \) is a Caratheodory function satisfying the following structure assumptions

\[
\frac{\alpha}{(1+|\rho|)^\gamma} \leq a(x,t,\rho) \leq \beta,
\] (3)

where \( \alpha, \beta \) and \( \gamma \) are positive constants. The simplest example is given by the parabolic problem (1).

The feature of our existence results is that the solutions only belong to the non reflexive space \( L^1(0,T; W^{1,1}_0(\Omega)) \) and the main difficulty of the problem is due to the principal part of the operator: it can degenerate for \( \rho \) large; that is when the solution is unbounded. Hence a slow diffusion can appear for large value of the solution.

In the elliptic case, this type of problems, under the assumption

\[
\frac{\alpha}{(1+|\rho|)^\gamma} \leq a(x,\rho) \leq \beta,
\] (4)

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was introduced in [3] and developed in [1]. In the last paper, the impact of a lower order term of order zero is considered

\[
\begin{aligned}
\text{div}(a(x, u)\nabla u) + u &= f \in L^m(\Omega), \quad \text{in } \Omega; \\
u &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

In these papers the authors study the existence of weak or distributional or entropy solutions (defined using the truncature, useful if the gradient of the solution does not belong to \(L^1\), see [4]); the paper [16] deals with uniqueness. In particular in [1], thanks to the presence of the lower order term (regularizing effect), some results of [3] are improved: it is proved for the boundary value problem (5)

- the existence of finite energy solutions, if \(m \geq \gamma + 2\);
- the existence of bounded weak solutions, if \(m > \gamma \frac{N}{2}\) and \(\gamma > 1\).

The paper [2] covers the borderline case \(m = \gamma = 2\) of (5), where the distributional solutions belong \(W^{1,1}_0(\Omega)\). Further existence and regularity results can be found in [10], [11] and [23].

In the parabolic case, the problem (2) was studied in [20] with \(u_0(x) \in L^m(\Omega), m \geq 1\). In particular the existence of solutions (which become distributional solutions only for suitable values of \(\gamma\) and \(m\)) is proved for every value of \(\gamma > 0\) and \(m \geq 1\). Moreover existence results when \(u_0\) is a Radon measure and \(\gamma > 1\) are proved in [21] and in [22] where a new notion of solution (measure-valued solutions) is given to study the cases of measure data. Notice that in all these last cases the solutions constructed are very weak and no information on the summability of the gradients of these solutions is given.

In this paper we prove the existence of distributional solutions of the problem (2) in the borderline case

\[
\begin{align}
\gamma &= 2. \\
u_0(x) &\in L^2(\Omega),
\end{align}
\]

The distinctive aspect of our existence results is that the gradient of the solutions only belongs to \(L^1\).

Moreover we study the behavior of the solution constructed for \(t\) large and we prove that decay estimates hold. In detail we show that if \(N \geq 3\), for every \(p < m\) the \(L^p(\Omega)\)-norm of such a solution decays in time according to the estimate (25). If \(N = 2\) a faster (exponential) decay occurs (see Theorem 3.2).

Finally we prove that if the function \(a\) in (2) is Lipschitz continuous with respect to \(\rho\), then the solution obtained by approximation is unique.

The plan of the paper is the following: in section 2 we enounce and prove the existence results. In section 3 we investigate on decay estimates. Finally in the last section we study the uniqueness of the solutions.

2. Existence.

**Theorem 2.1.** Assume (3), (6) and (7). Then there exists a distributional solution \(u \in L^1(0, T; W^{1,1}_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega))\) of problem (2).

**Proof.** The proof of the results will be achieved in several steps.

**Step 1 - Approximate problems.** Let us consider the following problems

\[
\begin{aligned}
\begin{cases}
(u_n)_t - \text{div} (a(x, t, T_n(u_n))\nabla u_n) = 0 & \text{in } \Omega_T; \\
u_n &= 0, & \text{on } \partial \Omega \times (0, T); \\
u_n(x, 0) &= u_{0,n}(x), & \text{in } \Omega.
\end{cases}
\end{aligned}
\]

\(25\).

\(3.2\).
where the functions $u_{0,n}(x)$ belong to $L^\infty(\Omega)$ and satisfy
\begin{equation}
\begin{cases}
  u_{0,n} \to u_0, \quad \text{strongly in } L^2(\Omega) \quad \text{and a.e. in } \Omega \\
  |u_{0,n}(x)| \leq |u_0(x)|, \quad \text{almost every } x \in \Omega,
\end{cases}
\end{equation}

like

$$T_n(u_0(x)) = \frac{u_0(x)}{1 + \frac{1}{n}|u_0(x)|},$$

where

$$T_k(s) = \min \{k, |s| \} \text{sign}(s), \quad G_k(s) = s - T_k(s), \quad \forall \ s \in \mathbb{R}. \quad (10)$$

The existence of $u_n \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^\infty(\Omega_T) \cap C^{\delta,\frac{\delta}{2}}(\Omega \times (0,T)) \ (\delta \in (0,1))$ is proved in [15]. We notice explicitly that all the test functions used below can be made rigorous by means of the Steklov averaging process (see [14], Chapter II).

**Step 2 - Estimates.** The following estimate holds true, $\forall \ 0 < t \leq T$, $k \geq 0$, taking $G_k(u_n)$ as test function in (8). Indeed

$$\frac{1}{2} \int_\Omega |G_k(u_n(t))|^2 \, dx + \alpha \int_0^T \int_{\Omega_T} \frac{|\nabla G_k(u_n)|^2}{(1 + |u_n|)^2} \, dx \, dt \leq \frac{1}{2} \int_\Omega |G_k(u_0)|^2 \, dx. \quad (11)$$

In particular we have (with $k = 0$)

$$\int_\Omega |u_n(t)|^2 \, dx \leq \int_\Omega |u_0|^2 \, dx, \quad \forall \ 0 < t \leq T. \quad (12)$$

Moreover (11) implies

$$\int_0^T \int_{\Omega_T} \frac{|\nabla G_k(u_n)|^2}{(1 + |u_n|)^2} \, dx \, dt \leq \frac{1}{2\alpha} \int_\Omega |G_k(u_0)|^2, \quad \forall \ k \geq 0. \quad (13)$$

Then, using estimates (12), (13) we deduce

$$\int_\Omega |\nabla u_n| \leq \left[ \int_{\Omega_T} \frac{|\nabla u_n|}{1 + |u_n|} (1 + |u_n|) \right]^{\frac{1}{2}} \left( \int_{\Omega_T} \frac{|\nabla G_k(u_0)|^2}{(1 + |u_n|)^2} \right)^{\frac{1}{2}} \leq \left( \frac{1}{2\alpha} \int_\Omega |G_k(u_0)|^2 \right)^{\frac{1}{2}} \left[ 2T \left( |\Omega| + \int_\Omega |u_0|^2 \right) \right]^{\frac{1}{2}},$$

that is the following inequality

$$\int_{\Omega_T} |\nabla u_n| \leq \left[ \frac{1}{2\alpha} \int_\Omega |G_k(u_0)|^2 \right]^{\frac{1}{2}} \left[ 2T \left( |\Omega| + \int_\Omega |u_0|^2 \right) \right]^{\frac{1}{2}}. \quad (14)$$

In particular we have ($k = 0$)

$$\int_{\Omega_T} |\nabla u_n| \leq \left[ \frac{1}{2\alpha} \int_\Omega |u_0|^2 \right]^{\frac{1}{2}} \left[ 2T \left( |\Omega| + \int_\Omega |u_0|^2 \right) \right]^{\frac{1}{2}}. \quad (15)$$

Now we prove that, for every measurable subset $A \subset \Omega_T$, we have

$$\int_A |\nabla T_k(u_n)| \leq \left( \frac{k(1 + k)^2 \|u_0\|_{L^2(\Omega)}}{\alpha} \right)^{\frac{1}{2}} \text{meas}(A)^{\frac{1}{2}}. \quad (16)$$
Indeed, we choose \( T_k(u_n) \) as test function in (8) and we use that
\[
\frac{1}{2}|T_k(s)|^2 \leq \Psi_k(s) \leq k|s|, \quad k > 0, \quad \forall \ s \in \mathbb{R},
\]
where
\[
\Psi_k(s) = \int_0^s T_k(\sigma)d\sigma.
\]
Then thanks to assumption (3) we obtain
\[
\alpha \iint_{\Omega_T} |\nabla T_k(u_n)|^2 \leq k\|u_0\|_{L^1(\Omega)},
\]
which implies
\[
\iint_{\Omega_T} |\nabla T_k(u_n)|^2 \leq \frac{k(1 + k^2)\|u_0\|_{L^2(\Omega)}|\Omega|}{\alpha}.
\]
By the previous inequality we deduce that, for every measurable subset \( A \subset \Omega_T \), we have
\[
\iint_{A} |\nabla T_k(u_n)| \leq \left( \iint_{\Omega_T} |\nabla T_k(u_n)|^2 \right)^{\frac{1}{2}} \text{meas}(A)^{\frac{1}{2}} \leq \left( \frac{k(1 + k^2)\|u_0\|_{L^2(\Omega)}|\Omega|}{\alpha} \right)^{\frac{1}{2}} \text{meas}(A)^{\frac{1}{2}},
\]
that is (16).

**Step 3 - Convergences.** Now we prove that the approximating solutions \( u_n \) converge to a solution \( u \) of (2).

In (15), we proved the boundedness of \( \{\nabla u_n\}_{n \in \mathbb{N}} \) in \( [L^1(\Omega_T)]^N \). The assumption (3) implies that \( \{(u_n)_t\}_{n \in \mathbb{N}} \) is bounded in \( L^1(0, T; W^{-1,1}(\Omega)) \). Hence we can apply Corollary 4 of [24], obtaining that there exists a measurable function \( u \) such that (up to subsequences)
\[
u_n \text{ converges to } u \text{ in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{17}
\]

Now we prove that the following convergence holds true
\[
\nabla u_n \rightharpoonup \nabla u, \text{ weakly in } (L^1(\Omega_T))^N. \tag{18}
\]
We follow a technique of [2]. We start proving that there exists a function \( Y \in (L^1(\Omega_T))^N \) such that the following convergence holds true
\[
\nabla u_n \rightharpoonup Y, \text{ weakly in } (L^1(\Omega_T))^N. \tag{19}
\]
The sequence \( \{\nabla u_n\}_{n \in \mathbb{N}} \) is bounded in \( (L^1(\Omega_T))^N \). Hence by Dunford-Pettis’ Theorem the assertion (19) follows if we show that for every positive \( \varepsilon \) there exists a positive constant \( \delta \) such that for every measurable \( A \subset \Omega_T \) satisfying \( \text{meas}(A) < \delta \) it results
\[
\int_A |\nabla u_n| \leq \varepsilon, \quad \forall \ n \in \mathbb{N}. \tag{20}
\]

We have, using (14) and (16),
\[
\int_A |\nabla u_n| \leq \int_{\Omega_T \cap \{|u_n| \geq k\}} |\nabla u_n| + \int_{A \cap \{|u_n| \leq k\}} |\nabla u_n| \\
\leq \left( \frac{1}{2\alpha} \int_\Omega |G_k(u_0)|^2 \right)^{\frac{1}{2}} \left[ 2T \left( |\Omega| + \int_\Omega |u_0|^2 \right) \right]^{\frac{1}{2}} + \left( \frac{k(1+k)^2 \|u_0\|_{L^2(\Omega)} \| \Omega \|^{\frac{1}{2}}}{\alpha} \right)^{\frac{1}{2}} \text{meas}(A)^{\frac{1}{2}},
\]
which implies (20).

Hence, to conclude the proof, it remains to prove that \( Y \) is equal to \( \nabla u \); this is a consequence of the definition of weak gradient and the convergences (17) and (18).

**Step 4 - Pass to the limit.** Note that, thanks to (17), (18), (20) and to the inequality
\[
0 \leq a(x, t, T_n(u_n))|\nabla \phi| \leq \beta|\nabla \phi|,
\]
it is easy to pass to the limit in the weak formulation of (8), that is
\[
\int_\Omega u_n \phi_t + \int_{\Omega_T} a(x, t, T_n(u_n))\nabla u_n \nabla \phi = 0.
\]
It follows that \( u \in L^1(0, T; W^{1,1}_0(\Omega)) \) and it solves (2). Notice that by (12) and again by (17) we obtain (thanks to Fatou’s Lemma) that
\[
\int_\Omega |u(t)|^2 \, dx \leq \int_\Omega |u_0|^2 \, dx, \quad 0 < t \leq T.
\]
That is \( u \) belongs also to \( L^\infty(0, T; L^2(\Omega)) \). \( \square \)

**Remark 1.** Our problems, in particular the model case (1), thanks to the change \( z = \frac{u}{1+|u|} \), are formally related to the singular problem
\[
\begin{cases}
\left( \frac{z}{1-|z|} \right)_t - \Delta z = 0, & \text{in } \Omega_T; \\
z = 0, & \text{on } \partial \Omega \times (0, T); \\
z(x, 0) = z_0(x), & \text{in } \Omega.
\end{cases}
\]

**Remark 2.** Notice that under the assumption (3) with \( \gamma > 0 \), if the summability coefficient \( m \) of the initial datum \( u_0 \) satisfies the following condition
\[
m > m_0 \equiv \min \left\{ \frac{\gamma + 2}{2}, \frac{N(\gamma + 1)}{N + 1} \right\},
\]
then there exists a weak solution of (2) in \( L^\infty(0, T; L^m(\Omega)) \cap L^h(0, T; W^{1,h}_0(\Omega)) \), where \( h > 1 \) depends on \( \gamma, m \) and \( N \) (see formula (2.7) and Theorem 2.9 in [20]). Notice that if \( m = \gamma = 2 \), that is the case considered here, the previous condition becomes
\[
m > m_0 \equiv \min\{2, \frac{3N}{N+1}\} = 2,
\]
since the assumption \( N > 2 \) is equivalent to require \( \frac{3N}{N+1} > 2 \), and hence it is not satisfied. Thus, as said before, we study here a limit case not considered in [20] and we prove, as stated in Theorem 2.1 above, the existence of a solution with summable gradient.
3. Decay. We show the following decay estimate.

**Theorem 3.1.** Assume (3), $N \geq 3$, $u_0 \in L^m(\Omega)$ and $m \geq 2$. Then the solution $u$ of problem (2) constructed in Theorem 2.1 satisfies the following decay estimate

$$\|u(t)\|_{L^p(\Omega)}^p \leq \frac{\|u_0\|_{L^p(\Omega)}^p}{(1 + C_0 t)^{\frac{N(m-p)}{m(m-2)}}}, \quad t \in (0, T), \quad (25)$$

where

$$1 < p < m, \quad (26)$$

and the constant $C_0$, given in (45), is determined in dependence on $\alpha, |\Omega|, u_0, m, N$ and $p$.

**Remark 3.** In particular, it results $m = 2 > p > 1$ implies $\|u(t)\|_{L^p(\Omega)} \leq \frac{\|u_0\|_{L^p(\Omega)}}{(1 + C_0 t)^{\frac{N(m-p)}{m(m-2)}}}$, if $u_0 \in L^2(\Omega), \quad m > 2 = p$ implies $\|u(t)\|_{L^2(\Omega)}^2 \leq \frac{\|u_0\|_{L^2(\Omega)}^2}{(1 + C_0 t)^{\frac{N(m-p)}{m(m-2)}}}$, if $u_0 \in L^m(\Omega), \quad p = 2$ and $m = N$ imply $\|u(t)\|_{L^2(\Omega)}^2 \leq \frac{\|u_0\|_{L^2(\Omega)}^2}{1 + C_0 t}$, if $u_0 \in L^N(\Omega)$.

**Remark 4.** Notice that by (25) it follows that also the $L^1(\Omega)$ norm of $u(t)$ decays since, $\Omega$ being a bounded open set, we have that

$$\|u(t)\|_{L^1(\Omega)} \leq \frac{\|u_0\|_{L^p(\Omega)} \|\Omega\|^{1-\frac{1}{p}}}{(1 + C_0 t)^{\frac{N(m-p)}{m(m-2)}}}, \quad \forall \ p > 1. \quad (27)$$

**Remark 5.** We recall that if the initial datum satisfies

$$u_0 \in L^m(\Omega), \quad m > \frac{\gamma N}{2} \quad (27)$$

then there exists a solution of problem (2) that becomes immediately bounded and satisfies the following decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \leq K_0 \frac{\|u_0\|_{L^m(\Omega)} \|\Omega\|^{1-\frac{1}{p}}}{(1 + C_0 t)^{\frac{N(m-p)}{m(m-2)}}}, \quad t > 0, \quad (28)$$

where $K_0$, $K$ and $\nu$ are positive constants depending only on the data (see Theorem 2.15 in [20]).

The above kind of estimates (28) are known in literature also as ultracontractive estimates or decay estimates to underline that they imply that the solutions go to zero (i.e. decay) when $t$ goes to infinity. There is a wide literature on this interesting phenomenon which is known to appear for a large class of parabolic problems (see for example [25], [7], [6], [8], [20], [17], [18] and the references cited therein).

Notice that the assumption (27) is optimal to obtain instantaneous boundedness (see section 6 in [20]).

We observe explicitly that in the limit case we study here such assumption is not satisfied, since being $\gamma = 2$ it becomes

$$m > N,$$

and hence the instantaneous boundedness phenomenon does not take place although there exists a solution that continue to decay in time.
We notice that decay estimates for solutions that do not become immediately bounded are less known in literature; some interesting examples can be found for other parabolic problems in [12], [13], [19] and [18].

Finally, in the particular case \( N = 2 \) we have the following faster decay result.

**Theorem 3.2.** Under the assumptions (3) with \( \gamma = 2 \), if \( N = 2 \) and \( u_0 \in L^2(\Omega) \cap L^p(\Omega) \), where \( p > 1 \), the solution \( u \) of problem (2) constructed in Theorem 2.1 satisfies the following decay estimate

\[
\|u(t)\|_{L^p(\Omega)}^p \leq \|u_0\|_{L^p(\Omega)}^p e^{-C_1 t},
\]

where

\[
C_1 = \frac{(p-1)\alpha S}{\rho(\|u_0\|^2_{L^2(\Omega)})},
\]

and \( S \) is the Sobolev constant (hence a constant depending only on \( N \)).

**Proof of Theorem 3.1.** We start proving that for every \( p > 1 \) arbitrarily fixed it results

\[
\frac{\partial}{\partial t} \int_\Omega |u_n(x,t)|^p dx + p(p-1)\alpha \int_\Omega \frac{|u_n|^{p-2}|\nabla u_n|^2}{(1 + |u_n|)^2} dx \leq 0,
\]

where all the terms in (30) are finite for a.e. \( t \in (0,T) \).

To this aim take \((|u_n| + \varepsilon)^{p-1} - \varepsilon^{p-1}|\text{sign}(u_n)\) as test function in (8), where \( \varepsilon \in (0,1) \) is arbitrarily fixed. We obtain

\[
\frac{1}{p} \int_\Omega (|u_n(x,t)| + \varepsilon)^p - \frac{1}{p} \int_\Omega (|u_0| + \varepsilon)^p + \varepsilon^{p-1} \left( \int_\Omega |u_0| - \int_\Omega |u_n(x,t)| \right) + (p-1) \int_0^t \int_\Omega a(x,t,T_n(u_n))|\nabla u_n|^2(|u_n| + \varepsilon)^{p-2} = 0.
\]

If \( p \geq 2 \), passing to the limit as \( \varepsilon \) goes to zero and using the assumption (3) we obtain (30).

If otherwise \( 1 < p < 2 \), notice that by (31) we have

\[
\int_0^t \int_\Omega a(x,t,T_n(u_n))|\nabla u_n|^2(|u_n| + \varepsilon)^{p-2} \leq C_{0,n},
\]

where we have set \( C_{0,n} = \frac{1}{p} \int_\Omega (|u_0| + 1)^p + \int_\Omega |u_n(x,t)| \). Hence applying Fatou’s Lemma we obtain

\[
\lim_{\varepsilon \to 0} \int_0^t \int_\Omega a(x,t,T_n(u_n))|\nabla u_n|^2(|u_n| + \varepsilon)^{p-2} \leq C_{0,n}.
\]

In particular it results \( a(x,t,T_n(u_n))|\nabla u_n|^2(|u_n| + \varepsilon)^{p-2} \in L^1(\Omega \times (0,t)) \). Since we have

\[
a(x,t,T_n(u_n))|\nabla u_n|^2(|u_n| + \varepsilon)^{p-2} \leq a(x,t,T_n(u_n))|\nabla u_n|^2|u_n|^{p-2},
\]

we can apply the Lebesgue Theorem obtaining

\[
\lim_{\varepsilon \to 0} \int_0^t \int_\Omega a(x,t,T_n(u_n))|\nabla u_n|^2(|u_n| + \varepsilon)^{p-2} = \int_0^t \int_\Omega a(x,t,T_n(u_n))|\nabla u_n|^2|u_n|^{p-2}.
\]
Hence we can pass to the limit in (31) getting
\[
\frac{1}{p} \int_{\Omega} |u_n(x, t)|^p - \frac{1}{p} \int_{\Omega} |u_{0,n}|^p + (p-1) \int_{0}^{\infty} \int_{\Omega} a(x, t, T_n(u_n))|\nabla u_n|^2 |u_n|^{p-2} = 0,
\]
from which the assertion (30) follows thanks to assumption (3).

Now notice that it results
\[
\int_{\Omega} \frac{|u_n|^{p-2} |\nabla u_n|^2}{(1 + |u_n|)^2} \geq C_0 S \left( \int_{\Omega} |u_n|^{\frac{pN}{N-1}} \, dx \right)^{\frac{2(N-1)}{N}},
\]
where $C_0 = p^{-2}(|\Omega| + \|u_0\|_{L^2(\Omega)})^{-1}$ and $S$ is the Sobolev constant (hence a constant depending only on $N$). As a matter of fact we have
\[
\int_{\Omega} \left[ ((|u_n| + \varepsilon)^{\frac{p}{2}} - \varepsilon) \text{sign}(u_n) \right] \, dx = \frac{p}{2} \int_{\Omega} \left( |u_n| + \varepsilon \right)^{\frac{p}{2}-1} |\nabla u_n| = \frac{p}{2} \int_{\Omega} \left( |u_n| + \varepsilon \right)^{\frac{p}{2}-1} \frac{|\nabla u_n|}{1 + |u_n|} \leq \frac{p}{2} \left( \int_{\Omega} \frac{(|u_n| + \varepsilon)^{p-2} |\nabla u_n|^2}{(1 + |u_n|)^2} \right)^{\frac{1}{2}} \left( \int_{\Omega} (1 + |u_n|)^2 \right)^{\frac{1}{2}} \leq p \left( \int_{\Omega} \frac{(|u_n| + \varepsilon)^{p-2} |\nabla u_n|^2}{(1 + |u_n|)^2} \right)^{\frac{1}{2}} (|\Omega| + \|u_0\|_{L^2(\Omega)})^{\frac{1}{2}},
\]
which implies
\[
\int_{\Omega} \frac{|u_n|^{p-2} |\nabla u_n|^2}{(1 + |u_n|)^2} \geq C_0 \left( \int_{\Omega} \left[ ((|u_n| + \varepsilon)^{\frac{p}{2}} - \varepsilon) \text{sign}(u_n) \right] \, dx \right)^2,
\]
where $C_0$ is the constant defined above. By the previous inequality and the Sobolev inequality in $W^{1,1}_0(\Omega)$ we deduce
\[
\int_{\Omega} \frac{(|u_n| + \varepsilon)^{p-2} |\nabla u_n|^2}{(1 + |u_n|)^2} \geq C_0 S \left( \int_{\Omega} \left[ ((|u_n| + \varepsilon)^{\frac{p}{2}} - \varepsilon) \text{sign}(u_n) \right] \, dx \right)^{\frac{2(N-1)}{N}}.
\]
Now (33) follows by the previous inequality passing to the limit as $\varepsilon$ that goes to zero. By (30) and (33) we deduce that
\[
\frac{\partial}{\partial t} \int_{\Omega} |u_n(x, t)|^p \, dx + C_1 \left( \int_{\Omega} |u_n|^{\frac{pN}{N-1}} \, dx \right)^{\frac{2(N-1)}{N}} \leq 0,
\]
where $C_1 = p(p-1)\alpha C_0 S$. Notice that it results
\[
\frac{pN}{2(N-1)} < p \iff N > 2,
\]
and hence assuming $p < m$ we have
\[
\frac{pN}{2(N-1)} < p < m.
\]
Thus the following interpolation inequality holds true
\[
\|u_n(t)\|_{L^p(\Omega)} \leq \|u_n(t)\|^{\lambda} \|u_n(t)\|^{1-\lambda}_{L^{\frac{pN}{2(N-1)}}(\Omega)} ,
\]
(37)
where $\lambda$ satisfies
\[
\frac{1}{p} = \frac{\lambda(N - 1)}{pN} + \frac{1 - \lambda}{m},
\]
that is
\[
\lambda = \frac{N(m - p)}{2m(N - 1) - pN}.
\tag{38}
\]
Recalling that (31) holds true for every $p > 1$, hence also for $p = m$, we obtain
\[
\|u_n(t)\|_{L^m} \leq \|u_{0,n}\|_{L^m}.
\tag{39}
\]
Using the previous inequality and (9) in (37) we deduce
\[
\|u_n(t)\|_{L^p(\Omega)} \leq \|u_n(t)\|_{L^\lambda(\Omega)}^{\frac{p}{m(N - 1)}} \|u_{0,n}\|_{L^m(\Omega)}^{\frac{m(N - 1)}{pN}} \|u_0\|_{L^m(\Omega)}^{\frac{m(N - 1)}{pN}},
\tag{40}
\]
where $\lambda$ is as before. By (40) we have (if $\|u_0\|_{L^m(\Omega)} \neq 0$, otherwise the assertion is evidently true) that
\[
\|u_n(t)\|_{L^p(\Omega)} \geq \|u_0\|_{L^m(\Omega)}^{\frac{p(1 - \lambda)}{m(N - 1)}}.
\tag{41}
\]
By (41) and (35) we obtain
\[
y'(t) + C_2 y(t)^\frac{\lambda}{1 - \lambda} \leq 0,
\tag{42}
\]
where we have set
\[
y(t) = \|u_n(t)\|_{L^p(\Omega)}, \quad C_2 = C_1 \|u_0\|_{L^m(\Omega)}^{\frac{p(1 - \lambda)}{m(N - 1)}}.
\tag{43}
\]
Integrating (42) we deduce
\[
y(t)^{1 - \frac{\lambda}{1 - \frac{\lambda}} - 1} = y(0)^{1 - \frac{\lambda}{1 - \frac{\lambda}} - 1} + C_2 t \leq 0,
\]
and hence, being $\lambda \in (0, 1)$, we have
\[
\frac{1}{(\frac{\lambda}{1 - \lambda}) - 1} y(0)^{\frac{\lambda}{1 - \frac{\lambda}} - 1} + C_2 t \leq 0,
\]
i.e.
\[
y(t) \leq \frac{y(0)}{\left[1 + C_2 (\frac{\lambda}{1 - \lambda}) y(0)^{\frac{\lambda}{1 - \frac{\lambda}} - 1} t\right]^\frac{1}{\frac{\lambda}{1 - \frac{\lambda}}}}.
\tag{44}
\]
Notice that by (9) and the previous inequality we obtain
\[
y(t) \leq \frac{\|u_0\|_{L^p}^p}{\left[1 + C_2 (\frac{\lambda}{1 - \lambda}) y(0)^{\frac{\lambda}{1 - \frac{\lambda}} - 1} t\right]^\frac{p}{\lambda}}.
\]
Hence the assertion follows by passing to the limit as $n \to +\infty$ (using Fatou’s Lemma) and setting
\[
C_0 = C_2 \left(\frac{1}{\lambda} - 1\right) \|u_0\|_{L^p}^{p(\frac{\lambda}{1 - \lambda} - 1)} = \tag{45}
\]
\[
\alpha \left(|\Omega| + \|u_0\|_{L^2(\Omega)}^2\right)^{-1} S(p - 1) \frac{m(N - 2)}{pN(m - p)} \left(\frac{\|u_0\|_p}{\|u_0\|_m}\right)^{\frac{m(N - 2)}{N(m - p)}},
\]
where $\lambda$ satisfies
\[
\frac{1}{p} = \frac{\lambda(N - 1)}{pN} + \frac{1 - \lambda}{m},
\]
that is
\[
\lambda = \frac{N(m - p)}{2m(N - 1) - pN}.
\tag{38}
\]
since by definition of $\lambda$ it results

$$\frac{\lambda}{1 - \lambda} = \frac{N(m - p)}{m(N - 2)}.$$  

Proof of Theorem 3.2. Proceeding exactly as in the proof of Theorem 3.1 we deduce that (35) holds true, which, under the assumption $N = 2$, reads

$$\frac{\partial}{\partial t} \int_{\Omega} |u_n(x, t)|^p \, dx + C_1 \int_{\Omega} |u_n|^p \, dx \leq 0,$$

where, as before, $C_1 = (p - 1)\alpha S[p(\|\Omega\| + \|u_0\|_{L^2(\Omega)})]^{-1}$. Hence the assertion follows by integrating the previous inequality and passing again to the limit on $n$.  

4. Uniqueness. We prove here the following uniqueness result.

**Theorem 4.1.** Assume (3) and that there exists a positive constant $L$ such that

$$|a(x, t, s) - a(x, t, z)| \leq L|s - z|, \quad a.e. \ (x, t) \in \Omega_T, \forall s, z \in \mathbb{R}. \quad (47)$$

Then if $u_n$ and $z_n$ solve the following problems

$$\begin{cases} (u_n)_t - \text{div} \ (a(x, t, u_n) \nabla u_n) = 0, & \text{in } \Omega_T; \\ u_n = 0, & \text{on } \partial \Omega \times (0, T); \\ u_n(x, 0) = u_{0,n}(x), & \text{in } \Omega; \end{cases} \quad (48)$$

$$\begin{cases} (z_n)_t - \text{div} \ (a(x, t, z_n) \nabla z_n) = 0, & \text{in } \Omega_T; \\ z_n = 0, & \text{on } \partial \Omega \times (0, T); \\ z_n(x, 0) = z_{0,n}(x), & \text{in } \Omega; \end{cases} \quad (49)$$

where $u_{0,n}(x)$ and $z_{0,n}(x)$ belong to $L^\infty(\Omega)$, then the following estimate holds true

$$\int_{\Omega} |u_n - z_n|(t) \, dx \leq \int_{\Omega} |u_{0,n}(x) - z_{0,n}(x)| \, dx \quad \forall \ t \in (0, T). \quad (50)$$

**Remark 6.** Notice also that if we choose $u_{0,n}(x) = T_n(u_0)$, then the solution of the approximating problem (8) (constructed in Section 2 to prove the existence of a solution of our problem) satisfies

$$\|u_n\|_{L^\infty(\Omega_T)} \leq \|u_{0,n}\|_{L^\infty(\Omega)} \leq n, \quad \Rightarrow \quad T_n(u_n) = u_n,$$

and hence $u_n$ solution of (8) coincides with the solution of (48).

**Remark 7.** Indeed the proof of Theorem 4.1 (and hence also that of Corollary 1 below) works for more general problems where the structure assumption (3) is replaced by the following weaker assumption

$$0 \leq a(x, t, s) \leq \beta, \quad (x, t) \in \Omega_T, \forall s \in \mathbb{R}, \quad (51)$$

under the assumption that there exist the solutions $u_n$ and $z_n$ of (48) and (49), respectively, in $L^\infty(\Omega_T) \cap L^2(0, T; H^1_0(\Omega))$.

An immediate consequence of Theorem 4.1 is that the solution of problem (2) obtained by approximation is unique. We notice that in the “classical case $\gamma = 0$” the uniqueness of solutions obtained by approximations can be found in [9].

**Corollary 1.** Let the structure conditions (3) and (47) hold true. Assume that $u$ and $v$ are obtained, respectively, as limit a.e. in $\Omega_T$ of the sequences $\{u_n\}_{n \in N}$ and $\{v_n\}_{n \in N}$ solutions of (48) and (49). Finally assume that the initial data $u_{0,n}(x)$ and $z_{0,n}(x)$ converge both to $u_0$ in $L^1(\Omega)$. Then we have that $u = v$ a.e. in $\Omega_T$.  

Proof of Theorem 4.1. As in [2] and [5] take $T_h(u_n - z_n)$ as test function in (48) and in (49) where $h > 0$ is arbitrarily fixed. Subtracting the obtained equations we deduce that

$$
\frac{1}{2} \int_0^t \int_\Omega [a(x,t,u_n) \nabla u_n - a(x,t,z_n) \nabla z_n] \nabla T_h(u_n - z_n) = \int_\Omega \Psi_h(u_0 - z_0, n) dx,
$$

where we have set

$$
\Psi_h(s) = \int_0^s T_h(\sigma) d\sigma.
$$

We estimate now the integrals in (52). It results

$$
\int_0^t \int_\Omega [a(x,t,u_n) \nabla u_n - a(x,t,z_n) \nabla z_n] \nabla T_h(u_n - z_n) =
$$

$$
\int_0^t \int_\Omega a(x,t,u_n) \nabla (u_n - z_n) \nabla T_h(u_n - z_n) +
$$

$$
\int_0^t \int_\Omega [a(x,t,u_n) - a(x,t,z_n)] \nabla z_n \nabla T_h(u_n - z_n).
$$

Notice that we have, since by (3) $a(x,t,s) \geq 0$,

$$
\int_0^t \int_\Omega a(x,t,u_n) \nabla (u_n - z_n) \nabla T_h(u_n - z_n) \geq 0,
$$

while for the second integral in the right-hand side of (53) we deduce, thanks to assumption (47),

$$
\left| \int_0^t \int_\Omega [a(x,t,u_n) - a(x,t,z_n)] \nabla z_n \nabla T_h(u_n - z_n) \right| \leq
$$

$$
\int_0^t \int_\Omega |\nabla z_n| |\nabla T_h(u_n - z_n)| \leq L h \int_0^t |\nabla z_n| |\nabla T_h(u_n - z_n)|.
$$

Moreover, noticing that it results

$$
sT_h(s) - \frac{h^2}{2} \leq \Psi_h(s) \leq h |s|,
$$

we can estimate the first and the last integral in (52), as follows

$$
\int_\Omega \Psi_h(u_n - z_n)(t) dx \geq \int_\Omega (u_n - z_n) T_h(u_n - z_n)(t) - \frac{h^2}{2} |\Omega|,
$$

$$
\int_\Omega \Psi_h(u_0,n - z_0,n) dx \leq h \int_\Omega |u_0,n - z_0,n|.
$$

Hence, putting together the previous inequalities we deduce

$$
\int_\Omega (u_n - z_n) T_h(u_n - z_n)(t) - \frac{h^2}{2} |\Omega| \leq L h \int_0^t |\nabla z_n| |\nabla T_h(u_n - z_n)| +
$$

$$
h \int_\Omega |u_0,n - z_0,n|,
$$

as follows.
from which it follows
\[
\frac{1}{h} \int_{\Omega} (u_n - z_n) T_h (u_n - z_n) (t) - \frac{h}{2} |\Omega| \leq L \int_0^t \int_{\Omega} |\nabla z_n| |\nabla T_h (u_n - z_n)| + \int_{\Omega} |u_{0,n} - z_{0,n}|
\]
Hence letting $h$ go to zero we deduce (50).

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