Generation of Source Terms in General Relativity by differential structures

T. Asselmeyer*
Institut of Physics, Humboldt University
Berlin, Germany

October 9, 1996

Abstract

In this paper the relation between the choice of a differential structure and a smooth connection in the tangential bundle is discussed. For the case of an exotic $S^7$ one obtains corrections to the curvature after the change of the differential structure, which can not be neglected by a gauge transformation. In the more interesting case of four dimensions we obtain a correction of the connection constructed by intersections of embedded surfaces. This correction produce a source term in the equation of the general relativity theory which can be interpreted as the energy-momentum tensor of a embedded surface.

Pacs: 04.20.Gz, 02.40.Re, 02.40Vh

submitted to Classical and Quantum Gravity

1 Introduction

One of the outstanding problems in physics is the unification of the quantum and general relativity theory. As we learned from gauge field theory and from the mathematical part known as geometric quantization, the procedure of quantization leads to the introduction of topological invariants defined on the phase space of the systems. In the case of the quantization of gravity one works on the space of (pseudo-)Riemannian metrics or better on the space of connections (Levi-Civita or more general). The diffeomorphism group of the corresponding manifold is the gauge group of this theory. Knowing more about the structure of this group means knowing more about quantum gravity. In a first approximation we are interested in the mathematical question: If one smooth manifold is homeomorphic to another one, need it be diffeomorphic? In 1956 J. Milnor [1] gave a negative answer to this question. He constructed the first exotic manifold, the exotic $S^7$, as the total space of

*torsten@summa.physik.hu-berlin.de
a fiber bundle. After this remarkable result, the next important step was the proof of the generalized Poincaré conjecture for dimensions greater than 4, also known as h-cobordism theorem [2]. Together with the calculation of the h-cobordism group \( \Theta_n \) of homotopy \( n \)-spheres by Kervaire and Milnor [3] one obtains the classification of differential structures on topological \( n \)-spheres. Later Kirby and Siebenman [4] proved that every simply connected manifold of dimension greater than 4 can be classified by \( \Theta_n \). By obstruction theory one obtains an element of the cohomology group \( H^{n+1}(M, \Theta_n) \) of the manifold \( M \) which measures the existence of the differential structures. The rank of \( H^n(M, \Theta_n) \) gives the number of such structures. Because of the failure of Whitney’s trick, only the special case of dimension 4 remained unsolved for a long time. With the help of the technique of Casson handles, Freedman [5] proved in 1982, that the classification of topological 4-manifolds is given by the classification of the so-called intersection form. Soon afterwards Donaldson [6] published his famous theorem, which shows the existence of an exotic \( \mathbb{R}^4 \), which is impossible for the \( \mathbb{R}^n \) with \( n \neq 4 \). In contrast to the compact case there are uncountable many exotic \( \mathbb{R}^4 \) [7, 8]. So the most interesting dimension for physics has the richest structure. Now the question is: What is the physical relevance of the differential structure? Which physical observable will be modified after a change of the differential structure? In the case of the exotic \( \mathbb{R}^4 \) the first question was discussed by Brans and Randall [9] and later by Brans [10, 11] alone to guess, that exotic smoothness can be a source of non-standard solutions of Einstein’s equation. In connection with the topological quantum field theory other physical models are developed to relate the topological invariants to vacuum states of the quantum field theory. In this paper we try to give an answer to the second question. To this end, we introduce a toy model given by the exotic \( S^7 \) and study the change of the connection in the tangent bundle with respect to the change of the differential structure. The fact behind this construction is the localization of exotic smoothness which means that the “exoticness” of the structure is induced by a countable set of points on the manifold. In the compact case this can be described by certain characteristic classes as was done in the first paper of Milnor [1] or by generalized characteristic classes given by the bigraded version of the Weil-algebra [12]. In the non-compact case, as in \( \mathbb{R}^4 \), such powerful algebraic invariants are missing. Otherwise the compact case has physically more sense and we will investigate this case in detail.

In the next section we construct explicitly an exotic \( S^7 \) following Milnor [11] and choose a connection in the tangent bundle. Then we define a map between the standard and the exotic \( S^7 \) which is a differentiable map except for one point. The associated map between the tangent bundles possesses a singularity originating from the fact, that the exotic \( S^7 \) has non-trivial first Pontrjagin class. From the theory of Harvey and Lawson [13] we know that this singularity is related to the occurrence of a nontrivial Pontrjagin class and is explicitly realized as current. So we obtain the nice result that a change of the differential structure induces a change in the connection which can not be neglected by a diffeomorphism. The four-dimensional case is related to the Donaldson polynomial [14, 15] as topological invariant to classify the differential structures for the compact 4-manifolds \( M \). Because of the realization as
polynomial in $H_2(M)$, an associated change in the connection is discussed. The new invariants of Seiberg and Witten [13, 17] use also the relation between the elements in $H_2(M)$. So we construct with the help of these classes in $H_2(M)$ the change of the connection after the change of the differential structure. To this end we use the results of Kronheimer and Mrowka [18] for manifolds of simple type that there is $d$ classes $e_1, \ldots, e_d \in H^2(M, \mathbb{Z})$ and $d$ polynomials $f_1, \ldots, f_d \in \mathbb{Q}[z]$ such that the Donaldson's invariant is uniquely defined by these pairs. Furthermore the set of these classes $\{e_i\}$ with $i = 1, \ldots, k$ is a diffeomorphism invariant. So one can conclude that two different differential structures differ by the number of classes in $H^2(M, \mathbb{Z})$. From this knowledge we construct the change of the connection by changing the differential structure. A physical interpretation of this mathematical construction is given by the general relativity theory as the energy-momentum tensor of a embedded surface in the 4-manifold which can be interpreted as a world surface of a string after the choice of one coordinate as time coordinate.

2 The simplest example - the exotic $S^7$

The change of the differential structure will be understood in the following sense. We choose a topological, smoothable manifold $M$ with the micro tangential bundle $\tau(M)$ [19]. From this bundle we get back two different tangential bundles $TM$ and $TM'$ associated to different differential structures. The change of differential structure means that the following diagram of homeomorphisms commutes

$$
\begin{array}{ccc}
TM & \xrightarrow{d\alpha} & \tau(M) \\
\downarrow & & \downarrow \\
TM' & \xleftarrow{\tau(M)} & 
\end{array}
$$

The map $\alpha$ is a diffeomorphism except on a countable set of points. In the case of $M = S^7$ we know that a smooth structure exists which is not unique. To construct and detect such structure we refer to [1].

Let $S^7$ be the standard $S^7$ and $M^7_k$ one of the exotic $S^7$. Furthermore we assume that a Riemannian metric is chosen on both manifolds. Let $D_{TS^7}$ be a connection in the tangential bundle $TS^7$ and $D_{TM^7_k}$ in $TM^7_k$. We are interested in the construction of the difference connection $D_{dc}$ given by

$$
D_{TM^7_k} = D_{TS^7} + D_{dc} \quad \text{in } S^7 \quad .
$$

(1)

The reason for the introduction is the effect, that the transport of a section in $M^7_k$ is different from the transport of the same section in $S^7$ because the section smooth in $M^7_k$ is non-smooth in some points in $S^7$. The difference connection $D_{dc}$ compensate this effect. To construct this connection we investigate the map $h : M^7_k \rightarrow S^7$. This map is up to one point $x_0 \in M^7_k$ a diffeomorphism. So the map $dh : TM^7_k \rightarrow TS^7$ is singular with respect to this point $x_0$. Next we fix a frame $e$ in $TM^7_k$ and in $TS^7$
denoted by \( f \). Together with the map we obtain \( dh(e) = a \cdot f \) where \( a \) is a \( SO(7) \)-valued function with a singularity on the same place. By the standard procedure of gauging the connection, the difference connection is given by

\[
D_{dc} = a^{-1} da
\]  

with the property

\[
d(a^{-1} da) + a^{-1} da \wedge a^{-1} da = 0 \quad \text{in} \quad T(M_k^7 \setminus \{x_0\}) .
\]  

Loosely speaking the curvature induced by changing the differential structure is concentrated on the point \( x_0 \) and one can write

\[
d(a^{-1} da) + a^{-1} da \wedge a^{-1} da = a(x_0) \cdot \delta(x_0) \quad \text{in} \quad TM_k^7
\]

where \( \delta(x_0) \) is the delta distribution and \( a(x_0) \in SO(7) \) is the group element corresponding to the point \( x_0 \). To clarify this important point in the argumentation, we choose without loss of generality a coordinate system with \( x_0 = 0 \). Together with the parallelization of the topological \( S^7 \) we can split the differential \( dh : TM_k^7 \to TS^7 \) to get \( dh = (h, a) : M_k^7 \times IR^7 \to S^7 \times IR^7 \) with the same map \( a \) defined above. The group element \( a \in SO(7) \) is considered with respect to the map \( h \) which means that the \( a(0) \) corresponding to the point 0 is singular. Milnor [1] constructed the homeomorphism \( h \) with a Morse function \( f : M_k^7 \to IR \) having two non-degenerated, critical points \( x_0, x_1 \). Next we calculate the tangential vector field corresponding to the Morse function:

\[
\frac{d\vec{x}(t)}{dt} = \nabla_{\vec{x}} f
\]

According to Milnor we normalize the function to be \( f(x_1) = 0 \) and \( f(x_0) = 1 \). The solution of (3) induces the homeomorphism \( h \) which is a diffeomorphism except at one point \( x_0 \). Of course at the critical points the tangential vector field vanish and the group element \( a(x_0) \) is defined to be the transformation \( a(x_0)y = 0 \) where \( y \in IR^7 \) is an element of the tangential space over \( x_0 \). The group element \( a \) depends of the point in \( M_k^7 \) and can be considered as a element of the gauge group \( \mathcal{G}(M_k^7) \). This group is described via the set of sections of the automorphism bundle \( Aut(TM_k^7) \) and act on every connection one-form \( \omega \in T^*(TM_k^7) \otimes so(7) \) on \( TM_k^7 \), where \( so(7) \) is the Lie algebra of \( SO(7) \). The bundle \( Aut(TM_k^7) \) is a bundle of \( SO(7) \) groups and we construct form this bundle a associated real vector bundle \( V(TM_k^7) \) of rank 7. The difference connection \( D_{dc} \) is a homogeneous form over \( V(TM_k^7) \) which means the invariance of the form after left multiplication with a scalar. We denote the homogeneous one-form over \( V(TM_k^7) \) associated to the difference connection by \( \omega \in T^*M_k^7 \otimes T^*IR^7 \) and the coordinate system on every fiber by \( y \in IR^7 \). Next we use a theorem in [13] (chapter III) to get the formula for the exterior derivation of \( \omega \):

\[
(d\omega) = (d\omega)_{\text{regular}} + \left( \int_{S^1} \omega \right) \frac{1}{\gamma_7} (d * d(\frac{1}{|y|^5})) \quad \text{in} \quad V(TM_k^7)
\]

1The function has in general values in \( Gl(7, IR) \), but this group can be reduced to the \( SO(7) \) group by bundle reduction. Furthermore we assume without loss of generality that the space has a euclidian metric.
where $\gamma_7$ is the volume of the $S^6$ unit sphere. From the standard result

$$*d * d\left(\frac{1}{|y|^5}\right) = \Delta_y \frac{1}{|y|^5} = \gamma_7 \delta(y)$$

in functional analysis we get the outstanding delta function. Now we pull-back this formula to $TM^7_k$ via the element $a$ of the gauge group with $a(x_0)y = 0$ to get

$$(d a^* \omega) = (d a^* \omega)_{regular} + \left(\int_{S^1} a^* \omega\right) \frac{1}{\gamma_7} (d * d(a^*(\frac{1}{|y|^5}))) \text{ in } TM^7_k$$  \hspace{1cm} (7)

Because of the property $a(x_0)y = 0$ the support of the form $(d * d(a^*(\frac{1}{|y|^5})))$ is the single point $x_0$ leading to the final result (4).

According to [4] (Essay IV, Theorem 10.1) one can consider the smoothing problem as lifting problem described by obstruction theory [20, 21]. In our case $M = S^7$ the obstruction to smooth the $S^7$ is an element of $H^8(S^7, \mathbb{Z}_28)$ (see [3]) which vanish in this case. The number of smoothings are the number of elements in $H^7(M^7_k, \mathbb{Z}_{28}) = \mathbb{Z}_{28}$. In equation (7) the one-form $a^* \omega$ represents a non-trivial cohomology class. The 7-form $(a^* \omega)^7$ can be interpreted as the generator of the group $H^7(M^7_k, \mathbb{Z})$. Using the relation

$$\int_{S^7} \delta(x_0) * 1 = 1$$  \hspace{1cm} (8)

where $*$ is the Hodge operator, we obtain for the curvature

$$\Omega_{dc} = \frac{1}{28} \delta(x_0)a(x_0) \sum_{i<j} \frac{1}{21} dx^i \wedge dx^j$$  \hspace{1cm} (9)

with the normalization factor $1/28$. This relates the obstruction to the change in the curvature after a change in the differential structure.

In the interesting case of a 4-manifold we have to use a more sophisticated technique developed by Casson, Freedman and Donaldson to get the same results. This will be done in the following section.

3 Discussion of the four-dimensional case

3.1 Introducing Remarks

The four-dimensional case requires completely different methods than the higher-dimensional one. At first we consider a compact 4-manifold $M$. Now one may ask: What is the characterization of the differential structures in this manifold? Fortunately the invariants of Donaldson and Seiberg-Witten answer this question particularly. The main components of the 4-manifold are formed by the basis elements of the second homology group $H_2(M, \mathbb{Z})$ except the generators of the torsion...
subgroup. If we choose a simple-connected 4-manifold then the group admits no torsion. Now one can define a pairing
\[
\omega : H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \rightarrow \mathbb{Z} \\
(\alpha, \beta) \mapsto <\alpha, \beta>
\]
where \(<\alpha, \beta>\) denotes the intersection number of the two cycles. This pairing is known as the intersection form and following Freedman it is the main topological invariant to distinguish two simple-connected, topological 4-manifolds. With the help of the Poincaré dual we can switch to the picture of cohomology classes. Fortunately there is a unique characterization of the classes in \(H^2(M, \mathbb{Z})\) by complex line bundles over \(M\). The position of the cycles to each other, expressed by the topological properties of the moduli space of an appropriated (non-linear) gauge theory, creates diffeomorphism invariants to distinguish different differential structures. Because of the failure of the h-cobordism theorem in four dimensions we have to change the description from the Morse function picture to methods which use the complex line bundle.

Let \(P\) be a \(SU(2)\)-principal bundle over \(M\) with second Chern number \(k\) and the moduli space of irreducible, anti-self-dual, gauge-equivalent connections is denoted by \(\mathcal{M}\). Let \(\mathcal{G}\) be the gauge group of \(P\) and \(\mathcal{A}\) the space of irreducible connections. From the bundle \(P\) one can induce the tautological bundle \(\mathcal{P} = \mathcal{A} \times P\). But in the case of irreducible connections the gauge group \(\mathcal{G}\) does not freely act on \(\mathcal{A}\) and one has to factorize out the center of the \(SU(2)\) which is given by the group \(\mathbb{Z}_2\). This procedure results in a universal \(SO(3)\) bundle \(\mathcal{P}^{ad} = \mathcal{P}/\mathcal{G}\) over \(\mathcal{M} \times \mathcal{M}\) with the first Pontrjagin class \(p_1(\mathcal{P}^{ad})\). Next we define the map \(\mu : H_2(M, \mathbb{Q}) \rightarrow H^2(\mathcal{M}, \mathbb{Q})\) by
\[
\int_T \mu(\Sigma) = -\frac{1}{4} \int_{T \times [\Sigma]} p_1(\mathcal{P}^{ad})
\]
where \(\Sigma\) is a 2-dimensional submanifold of \(M\) and \(T \subset \mathcal{M}\) is a compact 2-dimensional submanifold of the moduli space. Now we consider the situation, in which the moduli space is even-dimensional \(\dim \mathcal{M}_k = 2d(k)\). This generalization superimposes a restriction on the manifold \(M\) but it does include all interesting cases. We now choose \(d\) classes \([\Sigma_1], \ldots, [\Sigma_d]\) in \(H_2(M, \mathbb{Z})\), corresponding to 2-dimensional submanifolds in \(M\). With the help of the \(\mu\)-map we get the corresponding cohomology elements \(\mu(\Sigma_i) \in H^2(\mathcal{M}_k, \mathbb{Z})\) and can now define the invariant as the pairing
\[
q_k = \int_{[\mathcal{M}]} \mu(\Sigma_1) \cup \ldots \cup \mu(\Sigma_d) \quad .
\]  

(10)

Because of the non-compactness of the moduli space this definition is wrong, but as Donaldson showed one can “repair” this defect by evaluating the cup-product \(\mu(\Sigma_1) \cup \ldots \cup \mu(\Sigma_d)\) on a regular subset of \(\mathcal{M}_k\). As a result of analysis of the structure of the Donaldsons invariant of a smooth simply connected 4-manifold, Kronheimer and Mrowka [18] proved the existence of \(d\) classes \(e_1, \ldots, e_d \in H^2(M, \mathbb{Z})\) and \(d\) polynomials \(f_1, \ldots, f_d \in \mathbb{Q}[z]\) such that the Donaldsons invariant is uniquely
defined by these pairs. Furthermore the set of these classes \( \{e_i\} \) with \( i = 1, \ldots, k \) is a diffeomorphism invariant. Suppose a smooth simply connected 4-manifold \( M \) with \( d \) basic classes is given. A useful realization of the \( \mu \)-map is given by the following construction. We choose a Dirac operator \( \bar{\partial}_\Sigma \) on \( \Sigma \) and the trivial \( SU(2) \) bundle \( E \) over \( \Sigma \). Next we twist the Dirac operator with a connection \( A \) of \( E \) to get \( \bar{\partial}_\Sigma, A \). By \( B \) we denote the set of all (framed) connections and by \( E \) the bundle over \((B/\text{SO}(3)) \times \Sigma \) induced by \( E \). The family index \( \text{ind}(\bar{\partial}_\Sigma, E) \) of the Dirac operator can be considered as a vector bundle over \( B/\text{SO}(3) \). Donaldson defined a line bundle by

\[
\mathcal{L}_\Sigma = \bigcup_{A \in B/\text{SO}(3)} (\Lambda^{\max} \ker (\bar{\partial}_\Sigma, A))^* \otimes \Lambda^{\max} \ker (\bar{\partial}_\Sigma, A)
\]

i.e. as determinant bundle of the index bundle \( \text{ind}(\bar{\partial}_\Sigma, E) \). This line bundle \( \mathcal{L}_\Sigma \) has the following property

\[
c_1(\mathcal{L}_\Sigma) = \mu([\Sigma]) \quad .
\]  

(11)

For more details see [15]. Together with this representation of the \( \mu \)-map we can rewrite the Donaldson invariant as

\[
q_k = \int_{[M]} c_1(\mathcal{L}_{\Sigma_1}) \cup \cdots \cup c_1(\mathcal{L}_{\Sigma_d}) .
\]

Another construction, which is probably connected with Donaldson's invariant, was introduced by Seiberg and Witten [16, 17]. To fix the notation we describe the procedure to get the Seiberg-Witten invariant (for details see [22]). Because of the fact [23] that every compact, connected, orientable, differentiable four-manifold admits a \( \text{Spin}^c \)-structure one choose such a structure together with a section, known as spinor. Then one write down a system of non-linear PDE describing the coupling between a self-dual \( U(1) \)-field and a harmonic spinor. We shall refer to a \( \text{Spin}^c \)-structure on \( M \) by specifying the Spinor Bundle \( S \otimes L \) where \( L \) is the square root of a complex line bundle. The equations of the gauge theory are given in terms of a pair \((A, \psi)\) of indeterminates, of which \( A \) is a connection on \( L \) and \( \psi \) is a smooth section of \( S^+ \otimes L \).

The equations are

\[
D_A \psi = 0 \quad , 
\]

(12)

\[
(F_A^+)_{ij} = \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j ,
\]

(13)

where \( D_A \) is the Dirac operator twisted by the connection \( A \), and \( F_A^+ \) is the self-dual part of the curvature associated to \( A \). Here \( \{e_i\} \) is a local basis of \( TX \) that acts on \( \psi \) by Clifford multiplication (see [22]), \( \{e^i\} \) is the dual basis of \( T^*X \), and \( <,> \) is the inner product on the fibres of \( S^+ \otimes L \). We denote the moduli space of non-trivial solution of the Seiberg-Witten equations modulo the gauge group \( \mathcal{G} = \text{Map}(M, S^1) \) by \( \mathcal{SW} \). Consider the group of all gauge transformations that fix a base point, \( \mathcal{G}_0 \subset \mathcal{G} \). Take the moduli space \( \mathcal{SW}_0 \) of solutions of Seiberg-Witten equations, modulo the action of \( \mathcal{G}_0 \). This space \( \mathcal{SW}_0 \) fibres as a principal \( U(1) \) bundle over the
moduli space $\mathcal{SW}$. Let $\mathcal{L}$ denote the line bundle over $\mathcal{SW}$ associated to this principal $U(1)$ bundle via the standard representation. The Seiberg–Witten invariant, relative to a choice of $\mathcal{L}$ such that the dimension of $\mathcal{SW}$ is positive and even,

$$2d = c_1(\mathcal{L})^2 - \frac{2\chi + 3\sigma}{4} > 0,$$

is given by the pairing of the $d$th-power of the Chern class of the line bundle $\mathcal{L}$ with the moduli space $\mathcal{SW}$,

$$N_L \equiv \int_{\mathcal{SW}} c_1(\mathcal{L})^d = \int_{\mathcal{SW}} c_d(\mathcal{L}^\otimes d).$$

If the dimension of $\mathcal{SW}$ is odd, the invariant is set to be zero.

There is a conjecture of Witten that the Donaldson invariants and the Seiberg-Witten invariants are related for manifolds of simple type. Afterwards Fintushel and Stern [24] proved this conjecture in the case of simple connected elliptic surfaces to be true. As one expected from the topological classification of four-manifolds, embedded surfaces and the intersection with another surface characterize also the differential structure of the manifold.

### 3.2 Changing the Differential Structure

According to the section 2 we introduce two manifolds $M$ and $M'$ with different differential structures and define a map $\alpha: M \to M'$. Let $TM$ be the tangential bundle $TM$ of the manifold and $TM'$ for the other case. In [18] the 4-manifold of simple type was introduced including the best-known examples like elliptic surfaces (see [23]). Later Fintushel and Stern [24, 26] determine the structure of Donaldson invariants of such 4-manifolds and its relation to the Seiberg-Witten invariant. Following the discussion in [18] we choose these two differential structures in such a manner that the number of classes in $H_2(M, \mathbb{Z})$, characterizing the Donaldson invariant, differs by 1. That $M$ has $d$ classes $\Sigma_1, \ldots, \Sigma_d \in H_2(M, \mathbb{Z})$ and $M'$ has $d + 1$ classes with a further $\Sigma_{d+1} \in H_2(M, \mathbb{Z})$. The interesting point of the map $\alpha: M \to M'$ is the mapping to the further class $\Sigma_{d+1}$ of $M'$. At first we map $d$ classes of $M$ to $M'$. The further class $\Sigma_{d+1}$ produce a singularity in the following sense. In [27, 28] Kronheimer and Mrowka investigate the case of an embedded surface in the 4-manifold and its influence on the Donaldson invariant. As a main result they proved that except for the case of a sphere the self-interactions $\Sigma \cdot \Sigma$ of the embedded surface $\Sigma$ is given by: $2g - 2 \leq \Sigma \cdot \Sigma$ where $g$ is the genus of $\Sigma$. For simplicity we neglect self-interactions and choose $g = 0$. According to [27] the tangential bundle $T\Sigma$ must be non-trivial. We choose suitable embeddings $i_1: \Sigma_1 \subset M'$ and $i_{d+1}: \Sigma_{d+1} \subset M'$. The extension of the bundles $\mathcal{L}_{\Sigma_1}$ and $\mathcal{L}_{\Sigma_{d+1}}$ defined in section 3.1 to the whole manifold $M'$ leads to

$$c_1(\mathcal{L}_{\Sigma_1}) \wedge c_1(\mathcal{L}_{\Sigma_{d+1}}) = c_2((i_1)_*\mathcal{L}_{\Sigma_1} \oplus (i_{d+1})_*\mathcal{L}_{\Sigma_{d+1}}).$$

\[2\]This is the simplest case which one can consider as the generator of the other cases.
and with the definition of the $\mu$-map we get

$$\int_{T_d} \mu([\Sigma_1]) \wedge \mu([\Sigma_{d+1}]) = \frac{1}{16} \int_{T_d \times N} p_2(\mathcal{P}^{ad})$$

(15)

where $N = (i_1)(\Sigma_1) \times (i_{d+1})(\Sigma_{d+1})$ is a submanifold of $M'$ and $T_d$ is a 4-dimensional submanifold of the moduli space $\mathcal{M}'$ of anti-selfdual connections. If the integral given above is non-trivial then both surfaces $\Sigma_1$ and $\Sigma_{d+1}$ intersect in $M'$. So the existence of the further class in $M'$ produce further intersections which does not exist in $M$. So the interesting points of the map $\alpha$ and the induced map in the tangential bundle $d\alpha$ are the further intersection points. For simplicity we assume that only one further intersection point between $\Sigma_1$ and $\Sigma_{d+1}$ exists. We denote this point by $x_0$. The tangential bundle $TM'$ in the neighborhood $U(x_0)$ of this point splits like $TM'|_{U(x_0)} = ((i_1)_*T\Sigma_1 \oplus (i_{d+1})_*T\Sigma_{d+1})|_{U(x_0)}$, because near the intersection point we have a natural bundle reduction of the rank-4-bundle into two rank-2-bundles. The connection of $TM'$ in the neighborhood of this point is the direct sum of the connections in $T\Sigma_1$ and $T\Sigma_{d+1}$. Because of the non-triviality of the bundles $T\Sigma_1$ and $T\Sigma_{d+1}$ we obtain also non-trivial connections in a neighborhood of the intersection point $x_0$. Because of the relation

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i = \int_{K} d\left(\frac{dz}{z}\right) = 2\pi i \int_{K} \delta(x)\delta(y) \, dx \wedge dy$$

(16)

with $z = x + iy \in \mathbb{C}$ and $\partial K = \{z \in \mathbb{C} : |z| = 1\}$, we can construct a connection on a nontrivial complex line bundle $T\Sigma_1$ by the map $\beta : \Sigma_1 \times \mathbb{C} \to T\Sigma_1$ with $\beta(x, 1) = a(x)$ and $a(x_0) = 0$. This connection is given by the push-forward of the form $z^{-1} \, dz$ by the map $\beta$ leading to $\beta_*(z^{-1} \, dz) = a_*(z^{-1} \, dz) = a^{-1} \, da$. In [13] this case is studied extensively and one obtains for a connection $\omega$ in $T\Sigma_1$

$$\omega = \frac{da}{a} = a^{-1} \, da$$

(17)

induced from the trivial connection on $\Sigma_1 \times \mathbb{C}$ with the property $d(a^{-1} \, da) \neq 0$. The map $d\alpha : TM \to TM'$ is in the neighborhood of the point $y_0$ with $\alpha(y_0) = x_0$ given by the map of the trivial bundle $(i_1)_*(\Sigma_1 \times \mathbb{C}) \times V \times \mathbb{C}$ with $V$ a two-dimensional subset of $M$ homeomorphic to a subset of the $\mathbb{R}^2$, to the nontrivial bundle $((i_1)_*T\Sigma_1 \oplus (i_{d+1})_*T\Sigma_{d+1})|_{U(y_0)}$. If we define the splitting of the map $d\alpha$ in the neighborhood $U(y_0)$ of $y_0$ by the expression $d\alpha|_{U(y_0)} = (b_1, b_{d+1})$ then the connection induced by the bundle map is given by $(b_1^{-1} \, db_1) \oplus (b_{d+1}^{-1} \, db_{d+1})$. From the summable property of the connections near the intersection point we obtain for the change of the connection the expression

$$\nabla' = \nabla + (b_1^{-1} \, db_1) \oplus (b_{d+1}^{-1} \, db_{d+1})$$

(18)

This is the final result of the connection change. The additional term disappears if the manifolds have the same differential structure. Next we will interpret the results given above in a physical context.
4 Interpretation

In this section we try to give an answer to the following questions: What is the physical relevance of the differential structure? Which physical observable will be modified after a change of the differential structure? In principle the second question can be answered by the results of the previous section but to learn more about this topic one has to investigate all invariants of the differential structure. In most cases these invariants are topological invariants which no relation to the analytical properties of manifolds, i.e. the homotopy type or the combinatorical construction of an invariant. But in most physical models defined by the equation of motion the geometry of the underlying manifold plays an essential role in the description and one is faced with the problem to transport a vector along a curve, to form derivatives of physical observables and so on. Usually one implicitly fixes the differential structure for the whole space and all problems are gone. But in a future version of quantum gravity all geometrical and topological states of the space are included in a space of states of the system.

In the general relativity theory (GRT), first of all one fixes the differential structure of the manifold to be able to write down the field equation for a description of the gravitational field. This field equation is a second order non-linear PDE explicitly given on a chart. From this fact it follows, that in the GRT only the infinitesimally generated diffeomorphisms in the chart (or in more mathematical terms, the diffeomorphisms connected to the identity component) are used. In physics this means that only the local gravitational field in a local system (for instance the famous elevator of Einstein) represented by one chart can be neglected by acceleration. To follow this effect, Einstein developed a field equation where the gravitational field is connected with the geometry of the space. The problem of this theory is the source of the gravitation given by the energy of the system which is not determined by the theory itself. So let us consider two different differential structures leading to different gravitational theories. As in the previous section we choose two 4-manifolds $M$ and $M'$ with different differential structures and non-diffeomorphic tangential bundles $TM$ and $TM'$. Next we consider a connection $\nabla$ of the tangential bundle $TM$ which produces the Riemannian curvature $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z$, where $X,Y,Z \in \Gamma(TM)$ are vector fields and $\Gamma(TM)$ denotes the set of all vector fields of $M$. Let $Ric(X,Y)$ be the Ricci tensor, $R$ the curvature scalar and $g(X,Y)$ the metric defined for all $X,Y \in \Gamma(TM)$. Then Einstein’s vacuum field equation is given by

$$Ric(X,Y) - \frac{1}{2} g(X,Y) R = 0 \quad \text{in} \quad M. \quad (19)$$

From the last section we know that the change of the differential structure produces a correction (18) to the connection leading to a different curvature. Here we remark that the Lie-algebra of the Lorentz group $SO(3,1)$ is isomorphic to $sl(2,\mathbb{C})$ which is the Lie-algebra of the $SL(2,\mathbb{C})$ or spin group. So the connection of $TM$ can be described by complex $2 \times 2$ matrices and the (real) connection 1-form is denoted by $\omega^i_j$ with $i,j = 1,2,3,4$. Together with the two complex functions $b_1 = b_1(z)$ and $b_{d+1} = b_{d+1}(z) \quad (z \in \mathbb{C})$, the change of the connection by changing the differential
structure according to the previous section can be written as
\[
\omega' = \omega + \begin{pmatrix}
\frac{db_1}{b_1} & 0 \\
0 & \frac{db_{d+1}}{b_{d+1}}
\end{pmatrix}.
\] 
(20)
If we denote the connection change by \(\Delta \omega\) then one obtains
\[
(\omega')_i^j = \omega_i^j + (\Delta \omega)_i^j = \omega_i^j + \delta_i^j \frac{db_i^j}{b_i^j}
\]
(21)
where \(\delta_i^j\) is Kroneckers \(\delta\)-function and \(b_1^1 = \text{Re}(b_1(z)), \ b_2^2 = \text{Im}(b_1(z)), \ b_3^3 = \text{Re}(b_{d+1}(z))\) etc. are abbreviations. According to the standard calculus with \(\omega_i^j = \Gamma_{ki}^j \, dx^k\) and the fact according to the theory of \(U(1)\) connections that \(\Delta \omega \wedge \Delta \omega = 0\), we obtain the correction to the Ricci tensor \(\Delta \text{Ric}\) and the scalar curvature \(\Delta \text{R}\)
\[
(\Delta \text{Ric})_{ik} = \partial_k (\Delta \Gamma)_j^i - \partial_j (\Delta \Gamma)_k^i
\]
\[
\Delta \text{R} = g^{ik} (\Delta \text{Ric})_{ik}
\]
(22)
(23)
with \((\Delta \omega)_i^j = (\Delta \Gamma)_k^i dx^k = (b_i^j)^{-1} db_i^j\). Here we remark that in the case of the Levi-Civita connection the metric also change after the change of the differential structure.

From the general relation between the metric tensor and the Levi-Civita connection one obtains the correction of the metric by solving the system of differential equations:
\[
(b^{-1} \partial_i b)_j^i = \frac{1}{2} g^{im} (\partial_j g_{mi} + \partial_i g_{jm} - \partial_m g_{ij})
\]
Finally we obtain the correction of the vacuum equation (19) by the change of the differential structure
\[
(\text{Ric})_{ik} - \frac{1}{2} g_{ik} R = \delta_i^j (\partial_j (b^{-1} \partial_i b)_l^j - \partial_k (b^{-1} \partial_j b)_l^j)
\]
\[
+ \frac{1}{2} g_{ik} (g^{lm} (\partial_j (b^{-1} \partial_l b)_m^j - \partial_l (b^{-1} \partial_j b)_m^j)))
\]
(24)
But from the relation
\[
\oint_{|b|=1} \frac{db}{b} = 2i\pi w = 2i\pi w \int_{\partial K} \delta(b) * 1
\]
(25)
where \(w\) is the winding number or multiplicity of the zero \(b(z) = 0, \partial K = \{b \in \mathbb{C}; |b| = 1\}\) and \(*1\) is the volume form of \(K\), we obtain
\[
-\frac{i}{2w\pi} d(b^{-1} db) = \delta(b) * 1
\]
(26)
a \(\delta\)-like singularity in the curvature\(^3\). Together with the basis \(\partial\) of the tangential space with \(dx^k(\partial_j) = \delta_j^k\) we obtain for (24):
\[
(\text{Ric})_{ik} - \frac{1}{2} g_{ik} R = \delta_i^j \left\{ [(d(b^{-1} db))(\partial_j, \partial_k)]_i^j \right\}
\]
\[
+ \frac{1}{2} g_{ik} (g^{lm} [(d(b^{-1} db))(\partial_j, \partial_l)]_m^j)
\]
(27)
\(^3\)Such distribution-valued differential forms are known as currents in mathematics.
If we use the abbreviation \((\delta(b) \ast 1)(\partial_i, \partial_k) = \delta(b)_{ik}\) then we get the final result

\[
(Ric)_{ik} - \frac{1}{2} g_{ik} R = 2\pi w \delta^i_j (\delta(b^j_i)_{jk} + \frac{1}{2} g_{ik} (g^{jm} \delta(b^j_m)_{jl})
\]

which means that

\[
Ric(X,Y) - \frac{1}{2} g(X,Y) R \neq 0 \quad \text{in} \quad M'.
\]

But the right hand side of this equation represents the source of the gravitational field given by the energy-momentum tensor. In our case this term represents the embedding of a surface in a 4-dimensional manifold, because the support of the \(\delta\) function is a two-dimensional manifold. One can interprets such a term as a embedded (compact) surface in a 4-manifold with the energy given by the product of the volume of the surface and the winding number of the map. If one fix one coordinate to be the time coordinate then one can interpret such a term as a string moving through the 4-dimensional manifold. So we have the main result:

The change of the differential structure leads to a singularity in the curvature considered in the standard differential structure. The change corresponds to a source of a gravitational field given by a embedded surface

That means if we live in the coordinate system with standard differential structure and look at a system with an exotic one, we observe this system with an additional gravitational field corresponding to the change. In four dimensions this source can be interpreted as a string.

**Acknowledgments**

I wish to thank H. Rosé for fruitful discussions which clarify the physical interpretation of differential structures. Special thanks to G. Heß for many discussions about the work of Harvey and Lawson. Furthermore I thank Prof. Nietsch and T. Mautsch for many talks about the mathematical details. Last but not least I want to thank Prof. Brans for many discussions, his helpful remarks and corrections of mistakes.

**References**

[1] Milnor J 1956 On manifolds homeomorphic to the 7-sphere. *Ann. Math.* 64 399 – 405

[2] Milnor J 1965 *Lectures on the h-cobordism theorem* (Princeton: Princeton Univ. Press)

[3] Kervaire M A and Milnor J 1963 Groups of homotopy spheres: I. *Ann. Math.* 77 504 – 537
[4] Kirby R and Siebenmann L C 1977 *Foundational essays on topological manifolds, smoothings, and triangulations* (Princeton: Ann. Math. Studies. Princeton University Press)

[5] Freedman M H 1982 The topology of four-dimensional manifolds. *J. Diff. Geom.* 17 357–454

[6] Donaldson S K 1983 An application of gauge theory to the topology of 4-manifolds. *J. Diff. Geom.* 18 269–316

[7] Gompf R 1985 An infinite set of exotic $\mathbb{R}^4$'s. *J. Diff. Geom.* 21 283–300

[8] Taubes C H 1987 Gauge theory on asymptotically periodic 4-manifolds. *J. Diff. Geom.* 25 363–430

[9] Brans C H and Randall D 1993 Exotic differentiable structures and general relativity. *Gen. Rel. Grav.* 25 205

[10] Brans C H 1994 Localized exotic smoothness. *Class. Quant. Grav.* 11 1785–1792

[11] Brans C H 1994 Exotic smoothness and physics. *J. Math. Phys.* 35 5494–5506

[12] Dubois-Violette M 1994 A bigraded version of the Weil algebra and of the Weil homomorphism for Donaldson invariants. [hep-th/9402063](http://arxiv.org/abs/hep-th/9402063)

[13] Harwey F R and Lawson H B 1993 *A theory of characteristic currents associated to a singular connection* Société Mathématique De France, astérisque 213 edition

[14] Donaldson S K 1990 Polynomial invariants for smooth four manifolds. *Topology* 29 257–315

[15] Donaldson S K and Kronheimer P B 1990 *The Geometry of Four-Manifolds* (Oxford: Oxford Univ. Press)

[16] Seiberg N and Witten E 1994 Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory. *Nucl. Phys.* B 426 19–52

[17] Witten E 1994 Monopoles and four-manifolds. *Math. Research Letters* 1 769–796

[18] Kronheimer P B and Mrowka T S 1994 Recurrence relations and asymptotics for four-manifold invariant. *Bull. AMS* 30 215–221

[19] Milnor J 1964 Microbundle I. *Topology* 3(Suppl. 1) 53 – 80

[20] Spanier E H 1966 *Algebraic topology* (McGraw-Hill)
[21] Griffiths P A and Morgan J W 1981 *Rational homotopy theory and differential forms* (Boston: Birkhäuser)

[22] Salamon D 1995 *Spin Geometry and Seiberg-Witten Invariants* University of Warwick, manuscript, (October 1995)

[23] Hirzebruch F and Hopf H 1958 Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. *Math. Annalen* **136** 156

[24] Fintushel R and Stern R 1995 Rational blowdowns of smooth 4-manifolds. *AMS-preprint, alg-geom 9505018*

[25] Friedman R and Morgan J W 1994 *Smooth Four-Manifolds and Complex Surfaces* (Berlin-Heidelberg-New York: Ergebnisse der Mathematik und ihrer Grenzgebiete.3.Folge/A Series of Modern Surveys in Mathematics vol. 27. Springer Verlag)

[26] Fintushel R and Stern R 1995 Donaldson invariants of 4-manifolds with simple type. *J. Diff. Geom.* **42** 577–633

[27] Kronheimer P B and Mrowka T S 1993 Gauge theory for embedded surfaces, I. *Topology* **32** 773–826

[28] Kronheimer P B and Mrowka T S 1995 Gauge theory for embedded surfaces, II. *Topology* **34** 37–97