Dunkl–Klein–Gordon Equation in Three-Dimensions: The Klein–Gordon Oscillator and Coulomb Potential

Abstract Recent studies show that deformations in quantum mechanics are inevitable. In this contribution, we consider a relativistic quantum mechanical differential equation in the presence of Dunkl operator-based deformation and we investigate solutions for two important problems in three-dimensional spatial space. To this end, after introducing the Dunkl quantum mechanics, we examine the Dunkl–Klein–Gordon oscillator solutions with the Cartesian and spherical coordinates. In both coordinate systems, we find that the differential equations are separable and their eigenfunctions can be given in terms of the associate Laguerre and Jacobi polynomials. We observe how the Dunkl formalism is affecting the eigenvalues as well as the eigenfunctions. As a second problem, we examine the Dunkl–Klein–Gordon equation with the Coulomb potential. We obtain the eigenvalue, their corresponding eigenfunctions, and the Dunkl-fine structure terms.

1 Introduction

Nearly a century after the establishment of quantum mechanics, we observe an increasing number of studies on the deformations of the classical formalism of Heisenberg algebra [1–5]. All considered deformations rely on different physical motivations, for example, a minimal observable length, maximally observable momentum and length, etc... [6–9]. In the last decade, a new deformation is under investigation in quantum mechanics by substituting the Dunkl quantum mechanics, we examine the Dunkl–Klein–Gordon oscillator solutions with the Cartesian and spherical coordinates. In both coordinate systems, we find that the differential equations are separable and their eigenfunctions can be given in terms of the associate Laguerre and Jacobi polynomials. We observe how the Dunkl formalism is affecting the eigenvalues as well as the eigenfunctions. As a second problem, we examine the Dunkl–Klein–Gordon equation with the Coulomb potential. We obtain the eigenvalue, their corresponding eigenfunctions, and the Dunkl-fine structure terms.
Coulomb potential problem on the plane [22], Dirac [23], Klein–Gordon [24], pseudo-harmonic [25], and Duffin–Kemmer–Petiau [26] oscillator problems are examined comprehensively. Besides, Chung et al. discussed one dimensional Dunkl–quantum mechanics with the simplest example of a confined particle in one-dimensional box in [27,28]. Kim et al. extended electrostatics to Dunkl formalism in [29], while Ghazouani et al. solved Dunkl–Coulomb problem in three dimensions [30]. Mota et al. derived the Landau energy levels of the relativistic Dunkl oscillator in [31]. Very recently, statistical mechanics are also revisited with Dunkl deformation [32,33].

In this contribution we work in three spatial dimensions, at first, we examine the Dunkl–Klein–Gordon oscillator within Cartesian coordinates, then in spherical coordinates. Then, we consider the Coulomb potential and derive a solution out of the Dunkl–Klein–Gordon equation. To our best knowledge, Mota et al. studied a similar problem in two dimensions with the polar coordinates [24], and nobody handled this problem in three-dimensional geometry. Since we live in three spatial dimensions, we think our findings will be very important.

We organize the manuscript as follows: In the next section, we introduce the three Dunkl algebra and quantum mechanics. Then, in Sect. 3, we solve Dunkl–Klein–Gordon oscillator in three dimension in Cartesian and spherical coordinates, respectively. Next, in Sect. 3.2, we study Coulomb potential in Klein–Gordon equation within Dunkl formalism. Finally, we conclude the manuscript in Sect. 5.

2 Dunkl-Quantum Mechanics

In the Dunkl quantum mechanics the ordinary partial derivatives are substituted with the Dunkl derivatives

\[ D_j = \frac{\partial}{\partial x_j} + \frac{\mu_j}{x_j} (1 - R_j). \]

(1)

Here, Wigner constants, \( \mu_j \) are positive real numbers. \( R_j \) are the reflection operators which satisfy the following action [17,18]:

\[ R_j f (x_j) = f (-x_j); \quad R_i R_j = R_j R_i; \quad R_j x_i = -\delta_{ij} x_i R_j; \]

\[ R_j \frac{\partial}{\partial x_j} = -\delta_{ij} \frac{\partial}{\partial x_i} R_j. \quad \text{(no summation)} \]

(2)

Therefore, the Dunkl operators obey the following algebra:

\[ R_j D_j = -D_j R_j; \quad [D_j, D_j] = 0; \quad [x_i, D_j] = \delta_{ij} \left( 1 + 2\mu_{ij} R_{ij} \right). \quad \text{(no summation)} \]

(3)

Before proceeding to the next section, it would be useful to highlight some other features of the Dunkl operator.

- It is a linear operator:

\[ D_j \left[ af (x_j) + bg (x_j) \right] = a D_j f (x_j) + b D_j g (x_j), \]

(4)

- For any \( f (x_j), g (x_j) \) the Dunkl derivative satisfies the general Leibniz rule [34]:

\[ D_j \left( f (x_j) g (x_j) \right) = \left( D_j f (x_j) \right) g (x_j) + f (x_j) \left( D_j g (x_j) \right) - \frac{\mu_j}{x_j} \left[ (1 - R_j) f (x_j) \right] \left[ (1 - R_j) g (x_j) \right]. \]

(5)

If one of these functions is even, \( f (x_j) \) or \( g (x_j) \), then we obtain the usual Leibniz formula.

\[ D_j \left( f (x_j) g (x_j) \right) = \left( D_j f (x_j) \right) g (x_j) + f (x_j) \left( D_j g (x_j) \right). \]

(6)

- In the presence of Dunkl derivatives, the ordinary definition of the scalar product must be modified with [27]:

\[ \langle f | g \rangle = \int dx g^* (x) f (x) |x|^{2\mu} dx, \]

(7)
The expectation value of an operator $O$ with respect to the state, $\psi$, can be defined by
\[
\langle \psi | O | \psi \rangle = \int dx \psi^* (x) O \psi (x) |x|^{2\mu} dx.
\] (8)

The square of the Dunkl derivative can be expressed in the form of
\[
D_i^2 = \frac{\partial^2}{\partial x_i^2} - \frac{\mu_i}{x_i^2} (1 - R_i) + \frac{2\mu_i}{x_i} \frac{\partial}{\partial x_i}.
\] (9)

3 Dunkl–Klein–Gordon Oscillator in Three Dimensions

In this section, we solve the Dunkl–Klein–Gordon oscillator in three dimensions. To this end, we employ the Dunkl momentum, $\frac{1}{i} D_j$, instead of the ordinary momentum operator, $p_j$, in natural units. In that case, the stationary Klein–Gordon oscillator equation reads
\[
\left\{ E^2 - \left( \frac{1}{i} D_j + i m \omega x_j \right) \left( \frac{1}{i} D_j - i m \omega x_j \right) - m^2 \right\} \psi = 0; \quad \text{with } j = 1, 2, 3,
\] (10)

where $m$ and $\omega$ indicate the rest mass and oscillator frequency, respectively. When we follow the Dunkl algebra which is summarized in the previous section, we get the Dunkl–Klein–Gordon oscillator equation in Cartesian coordinates.
\[
\left\{ -D_1^2 - D_2^2 - D_3^2 - 2m \omega \left( \mu_1 R_1 + \mu_2 R_2 + \mu_3 R_3 + \frac{3}{2} \right) + m^2 \omega^2 (x_1^2 + x_2^2 + x_3^2) \right\} \psi = (E^2 - m^2) \psi.
\] (11)

This equation is manifestly separable in Cartesian and spherical coordinates even in the presence of the reflection operators.

3.1 Solution in Cartesian Coordinates

By using the Cartesian coordinates, we can obtain three one-dimensional Dunkl–Klein–Gordon oscillators out of Eq. (11). To this end, we define
\[
E^2 - m^2 = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,
\] (12)
\[
\psi = \psi (x_1) \psi (x_2) \psi (x_3),
\] (13)
\[
\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3,
\] (14)

where
\[
\mathcal{H}_j = -D_j^2 - m \omega \left( 1 + 2 \mu_j R_j \right) + m^2 \omega^2 x_j^2, \quad j = 1, 2, 3.
\] (15)

Then, we could be able to express the Dunkl–Klein–Gordon oscillator as it is one dimension with the given energy eigenvalue constraint.
\[
\left\{ \frac{\partial^2}{\partial x_j^2} + \frac{2\mu_j}{x_j} \frac{\partial}{\partial x_j} - \frac{\mu_j}{x_j^2} (1 - R_j) - m^2 \omega^2 x_j^2 + m \omega \left( 1 + 2 \mu_j R_j \right) - \mathcal{E}_j \right\} \psi (x_j) = 0.
\] (16)

Since $[\mathcal{H}_j, R_j] = 0$, the eigenfunctions could be selected as they have a definite parity, $R_j \psi (x_j) = s_j \psi (x_j)$, with $s_j = \pm 1$. Then, we set
\[
\xi_j = m \omega x_j^2; \quad \psi (\xi_j) = \xi_j^{1-s_j} e^{-\xi_j^2/2} Y (\xi_j),
\] (17)
thus, Eq. (16) becomes
\[
\left\{ \xi_j \frac{\partial^2}{\partial \xi_j^2} + \left( 1 + \mu_j - \frac{s_j}{2} - \xi_j \right) \frac{\partial}{\partial \xi_j} + n_j \right\} \Upsilon(\xi_j) = 0,
\]
(18)
where \( n_j \) is non-negative integer quantum numbers. In this case, the energy eigenvalue function, which depends on parity, is quantized as follows:
\[
E_{n_j}^{sj} = 2m\omega \left[ 2n_j + \left( \frac{1}{2} + \mu_j \right) (1 - s_j) \right].
\]
(19)
Then, we can express the general solution of Eq. (18) in terms of the associated Laguerre polynomials as
\[
\Upsilon(\xi_j) = C_{s_j} L_{n_j}^{\mu_j - \frac{s_j}{2}}(m\omega x_j^2),
\]
(20)
where \( C_{s_j} \) is a normalization constant that can be determined from Eq. (7) as follows [35]:
\[
\int x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) = \delta_{nm} (n + \alpha)! \frac{(n + \alpha)!}{n!}.
\]
(21)
After a straightforward calculation, we obtain the following expression for the normalization constant
\[
C_{s_j} = \sqrt{\frac{2m\omega n_j!}{(n_j + \mu_j - \frac{s_j}{2})!}},
\]
(22)
To evaluate the total expression of the energy eigenvalues, we replace \( E_{n_j}^{sj} \) by their expressions defined in Eq. (12). We find
\[
E_n^{(s_1,s_2,s_3)} = \pm \sqrt{2m\omega \left[ 2n + \left( \mu_1 + \frac{1}{2} \right) (1 - s_1) + \left( \mu_2 + \frac{1}{2} \right) (1 - s_2) + \left( \mu_3 + \frac{1}{2} \right) (1 - s_3) \right] + m^2},
\]
(23)
where \( n = n_1 + n_2 + n_3 \).

Our findings show that the total energy eigenvalue function of the Dunkl–Klein–Gordon oscillator depends not only on the quantum numbers but on the Wigner parameters and parities. The maximal contribution from the additional terms is obtained for \( s_1 = s_2 = s_3 = -1 \), while the minimal contribution is achieved for \( s_1 = s_2 = s_3 = +1 \).

Before we end this section, we present the plots of probability densities of the ground and first two excited states in Fig. 1. Here, we take \( m = 0.5, \omega = 1 \) and \( \mu_1 = 0.5 \). We demonstrate \( s = 1 \) and \( s = -1 \) on the Fig. 1a, b, respectively.

We observe that for the positive parity case, the highest probability of detecting particle detection is around the center. Contrary, in the negative parity case, the highest probability of detection particle on the center is zero.

### 3.2 Solution in Spherical Coordinates

In spherical coordinates, we adopt
\[
x_1 = r \sin \theta \cos \varphi; \quad x_2 = r \sin \theta \sin \varphi; \quad x_3 = r \cos \theta.
\]
(24)
Therefore, Eq. (11) takes the form
\[
\left\{ \frac{\partial^2}{\partial r^2} + \frac{2(1 + \mu_1 + \mu_2 + \mu_3)}{r} \frac{\partial}{\partial r} - m^2 \omega^2 r^2 + 2m\omega \left( \frac{3}{2} + \mu_1 R_1 + \mu_2 R_2 + \mu_3 R_3 \right) + \frac{J_\varphi}{r^2 \sin^2 \theta} + \frac{\mathcal{J}_\theta}{r^2} + E^2 - m^2 \right\} \psi = 0,
\]
(25)
which appeared in the study of the three dimensional Dunkl-harmonic oscillator problem [21]. Therefore, we give their solutions in terms of the Jacobi polynomials that are characterized with parity quantum numbers, $J$ where

\[
\text{Fig. 1 Reduced probability densities versus coordinate}
\]

\[\frac{\partial^2 J}{\partial \varphi^2} + 2 \left[ \mu_2 \cot \varphi - \mu_1 \tan \varphi \right] \frac{\partial}{\partial \varphi} - \frac{\mu_1}{\cos^2 \varphi} \left( 1 - R_1 \right) - \frac{\mu_2}{\sin^2 \varphi} \left( 1 - R_2 \right), \quad (26)\]

\[\frac{\partial^2 J}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + 2 \left[ (\mu_1 + \mu_2) \cot \theta - \mu_3 \tan \theta \right] \frac{\partial}{\partial \theta} - \frac{\mu_3}{\cos^2 \theta} \left( 1 - R_3 \right), \quad (27)\]

At this point we must note that in spherical coordinates the reflection operators act on the wave function as follows:

\[R_1 \psi (r, \theta, \varphi) = \psi (r, \theta, \pi - \varphi); \quad R_2 \psi (r, \theta, \varphi) = \psi (r, \theta, -\varphi); \quad R_3 \psi (r, \theta, \varphi) = \psi (r, \pi - \theta, \varphi). \quad (28)\]

Then, we assume $\Psi = F (r) \Theta (\theta) \Phi (\varphi)$ and substitute it in Eq. (25) to separate it to three ordinary differential equations. We find

\[\left\{ \frac{\partial^2}{\partial \varphi^2} + 2 \left( \mu_1 + \mu_2 \right) \cot \varphi - \mu_3 \tan \varphi - \frac{\mu_1}{\cos^2 \varphi} \left( 1 - R_1 \right) - \frac{\mu_2}{\sin^2 \varphi} \left( 1 - R_2 \right) + \Omega^2 \right\} \Phi (\varphi) = 0, \quad (29)\]

\[\left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + 2 \left[ (\mu_1 + \mu_2) \cot \theta - \mu_3 \tan \theta \right] \frac{\partial}{\partial \theta} - \frac{\mu_3}{\cos^2 \theta} \left( 1 - R_3 \right) - \frac{\Omega^2}{\sin^2 \theta} + \sigma^2 \right\} \Theta (\theta) = 0, \quad (30)\]

\[\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \left( 1 + \mu_1 + \mu_2 + \mu_3 \right) \frac{\partial}{\partial r} - m^2 \omega^2 r^2 + 2m\omega \left( \frac{3}{2} + \mu_1 R_1 + \mu_2 R_2 + \mu_3 R_3 \right) \right\} F (r) = 0, \quad (31)\]

where $\Omega^2$, and $\sigma^2$ are two separation constants. We observe that Eqs. (29) and (30) are similar to the ones which appeared in the study of the three dimensional Dunkl-harmonic oscillator problem [21]. Therefore, we give their solutions in terms of the Jacobi polynomials that are characterized with parity quantum numbers, $s_1, s_2, s_3$, and their reflection operator correspondences $k = \frac{1-s_1}{2}$, $p = \frac{1-s_2}{2}$, and $\sigma = \frac{1-s_3}{2}$.

\[\Phi_{s_1,s_2} (\varphi) = C_\varphi \cos^k (\varphi) \sin^p (\varphi) P_{s_1-\frac{1}{2}, s_2-\frac{1}{2}}^{(\mu_2+p-\frac{1}{2}, \mu_1+k-\frac{1}{2})} (\cos (2\varphi)), \quad (32)\]

\[\Theta_{s_3} (\theta) = C_\theta \cos^\sigma (\theta) \sin^{2\nu} (\theta) P_{s_3-\frac{1}{2}}^{(2\nu+\mu_1+\mu_2, \mu_3+\sigma-\frac{1}{2})} (\cos (2\theta)). \quad (33)\]

Here $C_\varphi$ and $C_\theta$ are the normalization constants. When $s_1, s_2 = 1$, $\nu$ is a positive integer whereas it is a positive half-integer when $s_1, s_2 = -1$. If $s_1 = s_2 = 1$, we are in the special case where $\nu = 0$. Similarly if $s_3 = 1$, then
the quantum number \( \ell \) takes only positive integer values whereas they have non-negative half integer values when \( s_3 = -1 \). For the validity of the given solutions, the separation constants have to satisfy the following conditions:

\[
\begin{align*}
\Omega^2 &= 4v (v + \mu_1 + \mu_2), \\
\sigma^2 &= 4 \left( \ell + v + \mu_1 + \mu_2 + \mu_3 + \frac{1}{2} \right).
\end{align*}
\] (34)  (35)

Next, we look for an exact solution to the radial equation. To this end, we first substitute Eqs. (34) and (35) into Eq. (31). We arrive at

\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} + \frac{2 (1 + \mu_1 + \mu_2 + \mu_3)}{r} \frac{d}{dr} - m^2 \omega^2 r^2 + 2m \omega \left( \frac{3}{2} + \mu_1 s_1 + \mu_2 s_2 + \mu_3 s_3 \right) \\
- 4 \left( \ell + v \right) \left( \ell + v + 1 \right) \left( \frac{1}{2} + \frac{1}{2} \right) + (E^2 - m^2) \right\} F(r) = 0.
\end{align*}
\] (36)

With the aid of the new variable definition

\[
\rho = m \omega r^2; \quad F = \rho^{v+\ell} e^{-\frac{\rho}{2}} \Xi(\rho),
\] (37)

Equation (36) takes the form

\[
\begin{align*}
\left\{ \rho \frac{d^2}{d\rho^2} + \left( \frac{3}{2} + 2 (v + \ell) + \mu_1 + \mu_2 + \mu_3 - \rho \right) \frac{d}{d\rho} + \left( \mu_1 k + \mu_2 p + \mu_3 \sigma + (v + \ell) \right) + \frac{E^2 - m^2}{4m \omega} \right\} \Xi(\rho) = 0.
\end{align*}
\] (38)

The solution of the above equation can be given in terms of the Laguerre polynomials as follows

\[
\Xi(\rho) = C_L^N L_N^{2(v+\ell)+\mu_1+\mu_2+\mu_3+\frac{1}{2}}(\rho),
\] (39)

where the quantum number, \( N \), has the form

\[
N = -\frac{1}{2} \left[ \mu_1 (1 - s_1) + \mu_2 (1 - s_2) + \mu_3 (1 - s_3) + 2 (v + \ell) \right] + \frac{E^2 - m^2}{4m \omega}.
\] (40)

Thus, the radial eigenfunctions of the three dimensional Dunkl–Klein–Gordon oscillator becomes

\[
F(\rho) = C_F \rho^{v+\ell} e^{-\frac{\rho}{2}} L_N^{2(v+\ell)+\mu_1+\mu_2+\mu_3+\frac{1}{2}}(\rho),
\] (41)

where the energy spectrum reads

\[
E_{N,v,\ell}^{s_1,s_2,s_3} = \pm \sqrt{2m \omega [2 (N + v + \ell) + \mu_1 (1 - s_1) + \mu_2 (1 - s_2) + \mu_3 (1 - s_3)] + m^2}.
\] (42)

We observe that the energy spectrum explicitly depends not only on the quantum numbers \( (N, v, \ell) \) but the other parameters, \( (\mu_j, s_j) \), which characterize the Dunkl derivative. Therefore, we conclude that the energy spectrum is dependent on a term originating from the conventional Klein–Gordon oscillator and an additional term originating from the Dunkl derivative. It is worth noting that, as in the previous section, the maximal contribution of the Dunkl term is obtained for \( s_1 = s_2 = s_3 = -1 \), while the minimal contribution is achieved from \( s_1 = s_2 = s_3 = +1 \). In addition such a correction term, which depends explicitly on \( s_j \), lifts the degeneracy of energy levels.

Before we conclude this section, we briefly would like to introduce the orthogonality relation of the angular and radial parts of the wavefunction. Using the following orthogonality relation (43)

\[
\int \psi_{N,v,\ell}^{s_1,s_2,s_3} \psi_{N',v',\ell'}^{s_1',s_2',s_3'} | r \sin \theta \cos \phi |^{2\mu_1} | r \sin \theta \sin \phi |^{2\mu_2} | r \cos \theta |^{2\mu_3} r^2 dr d\theta d\phi = \delta_{N,N'} \delta_{v,v'} \delta_{\ell,\ell'} \delta_{s_1,s_1'} \delta_{s_2,s_2'} \delta_{s_3,s_3'}.
\] (43)
and the integral property given in (21) with the following property of the Jacobi polynomials [35]

\[
\int (1 - x)^\alpha (1 + x)^\beta P^\alpha_\beta (x) P^\alpha_\beta (x) \, dx = \frac{2^{1+\alpha+\beta} \Gamma (1 + \alpha + N) \Gamma (1 + N + \beta)}{N! (1 + \alpha + \beta + 2N) \Gamma (1 + N + \alpha + \beta)},
\]

we find the normalization constants as follows:

\[
C_\psi = \sqrt{\frac{4 (v - \frac{k + p}{2})! (1 + 2v + \mu_2 + \mu_1 - 1) \Gamma (1 + v + \frac{k + p}{2} + \mu_1 + \mu_2 - 1)}{\Gamma (1 + v + \mu_2 + \frac{p - k}{2} - \frac{1}{2}) \Gamma (1 + \frac{k + p}{2} + \mu_1 - \frac{1}{2})},}
\]

\[
C_\theta = \sqrt{\frac{2 (\ell - \frac{\nu}{2})! (1 + 2\ell + 2\nu + \mu_1 + \mu_2 + \mu_3 - \frac{1}{2}) \Gamma (1 + \ell + 2\nu + \mu_1 + \mu_2 + \mu_3 + \frac{\sigma - 1}{2})}{\Gamma (1 + \ell + 2\nu + \mu_1 + \mu_2 - \frac{\sigma}{2}) \Gamma (1 + \ell + \mu_3 + \frac{\sigma - 1}{2})}},
\]

\[
C_r = \sqrt{\frac{2^m n!}{(n + 2\nu + 2\ell + \mu_1 + \mu_2 + \mu_3 + \frac{1}{2})!}}.
\]

Before we proceed to next section, we would like to emphasize that the obtained Dunkl-energy spectrum functions with Cartesian and spherical coordinates, given in Eqs. (23) and (42), are consistent with the existing results of literature [36], when the wigner parameters vanish.

4 Dunkl–Coulomb Potential in Three Dimensions

In this section, we consider the Coulomb potential \( V (r) = -\frac{Ze^2}{r} \) and intend to derive a solution in three-dimensional Dunkl–Klein–Gordon equation. We start by expressing the stationary Dunkl–Klein–Gordon equation according to [37]

\[
\left\{ \left( E + \frac{Ze^2}{r} \right)^2 \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial q^2} + \frac{2 (1 + \mu_1 + \mu_2 + \mu_3)}{r} \frac{\partial}{\partial r} + \frac{J_\psi}{r^2 \sin^2 \theta} + \frac{J_\theta}{r^2} - m^2 \right\} \psi = 0
\]

Then, we define \( \varphi = 2\sqrt{m^2 - E^2} r \), and use the angular operator’s \((J_\psi, J_\theta)\) eigenvalues, \((\Omega^2, \sigma^2)\), we get the Dunkl–Klein–Gordon radial equation in the following form

\[
\left\{ \frac{d^2}{dq^2} + \frac{2 (1 + \mu_1 + \mu_2 + \mu_3)}{r} \frac{d}{dq} - \frac{Ze^2}{q} + \frac{Z^2 e^4}{q^2} - 4 (\ell + v) (\ell + v + \mu_1 + \mu_2 + \mu_3 + \frac{1}{2}) - \frac{1}{4} \right\} \psi (r) = 0.
\]

For the solution we make an ansatz

\[
\psi (q) = e^{-\frac{q}{2}} q^\eta \chi (q),
\]

where the constant \( \eta \) has to be determined. To this end, we substitute Eq. (50) into Eq. (49). We find

\[
\left\{ \frac{d^2}{dq^2} + \left( \frac{2 (1 + \eta + \mu_1 + \mu_2 + \mu_3) - 1}{q} \right) \frac{d}{dq} - \frac{(1 + \eta + \mu_1 + \mu_2 + \mu_3) + \frac{Ze^2}{q}}{q} + \frac{\eta (\eta - 1) + 2\eta (1 + \mu_1 + \mu_2 + \mu_3) + Z^2 e^4 - 4 (\ell + v) (\ell + v + \mu_1 + \mu_2 + \mu_3 + \frac{1}{2})}{q^2} \right\} \chi (r) = 0.
\]

Then, we cancel the term proportional to \( \frac{1}{q^2} \). This gives a second order equation for \( \eta \).

\[
\eta (\eta - 1) + 2\eta (1 + \mu) + Z^2 e^4 - 4 (\ell + v) (\ell + v + \mu + 1/2) = 0.
\]
Therefore, the radial equation reduced to the confluent type hypergeometric equation

\[
\eta = -\mu_1 - \mu_2 - \mu_3 - \frac{1}{2} + \sqrt{\left(\mu_1 + \mu_2 + \mu_3 + 2\nu + 2\ell + \frac{1}{2}\right)^2 - Z^2e^4}.
\] (53)

Therefore, the radial equation reduced to the confluent type hypergeometric equation

\[
\left\{\frac{d^2}{dq^2} + \left[1 + \left(1 + 2(\eta + \mu_1 + \mu_2 + \mu_3)\right) - \eta \right] \frac{d}{dq} - \frac{EZe^2}{\varsigma} - \left(1 + \eta + \mu_1 + \mu_2 + \mu_3\right)\right\} \chi = 0.
\] (54)

which has solution in terms of the generalized Laguerre polynomials, \(L_n^{1+2(\eta+\mu_1+\mu_2+\mu_3)}(\varrho)\), where \(n\) is a positive-integer number.

\[
n = -\frac{EZe^2}{\varsigma} - (1 + \eta + \mu_1 + \mu_2 + \mu_3).
\] (55)

Accordingly we express the energy spectrum of the Dunkl–Klein–Gordon equation with Coulomb potential energy as follows:

\[
E_{n,\ell,\nu}(Z) = m \left\{1 + \frac{Z^2e^4}{n + \frac{1}{2} + \sqrt{\left(\mu_1 + \mu_2 + \mu_3 + 2\nu + 2\ell + \frac{1}{2}\right)^2 - Z^2e^4}}\right\}^{-1/2}.
\] (56)

We find that the energy spectrum is quantized in terms on the quantum numbers \(n, \ell, \nu\) and the Wigner parameter \(\mu_j\) which characterizes the Dunkl algebra. Moreover we derive a constraint,

\[
\left(\mu_1 + \mu_2 + \mu_3 + 2\nu + 2\ell + \frac{1}{2}\right) > Z\epsilon^2,
\] (57)

which is necessary for the existence of physical energy eigenvalues.

Finally, we explore the Dunkl-fine structure energy. To this end, we Taylor expand Eq. (56) in powers of \(Z^2e^4\). We find

\[
E_{n,\ell,\nu}(Z) \approx m \left\{1 - \frac{1}{2} \frac{Z^2e^4}{(n + \mu_1 + \mu_2 + \mu_3 + 2\nu + 2\ell + 1)^2} \right. \\
\left. - \frac{(Z^2e^4)^2}{(n + \mu_1 + \mu_2 + \mu_3 + 2\nu + 2\ell + 1)^4} \left(\frac{1}{2(n + \mu_1 + \mu_2 + \mu_3 + 2\nu + 2\ell + 1) - \frac{3}{8}}\right)\right\}.
\] (58)

Here, the first term corresponds to the rest energy in natural units. Dunkl formalism’s contribution to the nonrelativistic energy spectrum arises by the second term. We obtain the Dunkl-fine structure contribution as the third term in Eq. (58). It is worth noting that for particular parameter values

\[
\mu_1 + \mu_2 + \mu_3 + 2(\nu + \ell) = 5/6,
\] (59)

the Dunkl-fine structure contribution vanishes. We note that this theoretical result can be converged to the experimental results by choosing the appropriate Wigner parameter values.

We also observe that in the absence of Wigner parameters Eq. (58) reduces to

\[
E_{n,\ell,\nu}(Z) \approx m \left\{1 - \frac{1}{2} \frac{(Ze^2)^2}{(n + 2\ell + 1)^2} - \frac{(Ze^2)^4}{2(n + 2\ell + 1)^4} \left(\frac{1}{\ell + \frac{1}{2}} - \frac{3}{4}\right)\right\}.
\] (60)

which is the same with the given results in [38] up to a re-definition of a principle quantum number.
5 Conclusions

In this manuscript, we examine the Dunkl–Klein–Gordon oscillator in three-dimensional spatial space. To this end, we obtain the solutions by employing at first the Cartesian, and then, spherical coordinates. In both cases, we observe that the Dunkl–Klein–Gordon oscillator can easily be separated to three one-dimensional equations. After straightforward algebra, we obtain the energy eigenvalue functions in both coordinate systems. We find that the Dunkl operator modifies the energy eigenvalues. We also derive the normalized eigenfunctions in terms of associate(generalized) Laguerre and Jacobi polynomials. Finally, we study the Coulomb potential in the Dunkl–Klein–Gordon equation. We show how the energy eigenvalue function and Dunkl-fine structure values are changed under the effect of Dunkl deformation.

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