Turán- and Ramsey-type results for unavoidable subgraphs

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Abstract
We study Turán- and Ramsey-type problems on edge-colored graphs. A two-edge-colored graph is called \( \varepsilon \)-balanced if each color class contains at least an \( \varepsilon \)-proportion of its edges. Given a family \( \mathcal{F} \) of two-edge-colored graphs, the Ramsey-type function \( R(\varepsilon, \mathcal{F}) \) is the smallest \( n \) for which any \( \varepsilon \)-balanced \( K_n \) must contain a copy of an \( F \in \mathcal{F} \), and the Turán-type function \( \text{ex}(\varepsilon, n, \mathcal{F}) \) is the maximum number of edges in an \( n \)-vertex \( \varepsilon \)-balanced graph which avoids all of \( \mathcal{F} \). In this paper, we show that for any \( \varepsilon' < \varepsilon \leq 1/2 \), \( \text{ex}(\varepsilon, n, \mathcal{F})/(\frac{n}{2}) \leq 1 - \Omega(1/R(\varepsilon', \mathcal{F})) \), as long as \( R(\varepsilon', \mathcal{F}) \) is finite. We use this result to show that the Ramsey-type function is linear for bounded degree graphs. Finally, we consider the Turán-type function for several classes of edge-colored graphs. In particular, we show that for any \( k \), sufficiently large \((1/2)\)-balanced graphs with edge-density above 1/2 must contain a \( k \)-cycle with a non-monochromatic coloring of its edge set. The same result when \( k = 3 \) was proved by DeVos, McDonald, and Montejano.

Keywords
Ramsey numbers, Turán-type problem, unavoidable graphs

1 INTRODUCTION

The central parameter in (graph) Ramsey theory is the Ramsey number \( R(k) \), quantifying the smallest integer \( n \) for which every 2-edge-coloring of the complete graph \( K_n \) contains a monochromatic clique on \( k \) vertices. It is known that \( 2^{k/2+o(k)} \leq R(k) \leq 2^{2k+o(k)} \), but the
constants that appear in the exponents have resisted improvements for decades. For an overview, we refer the reader to the survey by Conlon, Fox, and Sudakov [9]. We will refer to a graph endowed with a red–blue edge-coloring as a \textit{bicolored graph}. Throughout the paper, a coloring of a graph refers to a two-edge-coloring of the graph.

Recently, there has been interest in finding non-monochromatic patterns in two-edge-colorings of \( K_n \). Of course, to do so, one needs to assume that both color classes are sufficiently well-represented. Bollobás thus asked (see [10]) which non-monochromatic subgraphs can be guaranteed in a two-edge-coloring of \( K_n \) where both color classes have at least \( \varepsilon \cdot e(K_n) \) edges. A bicolored graph \( G \) where both color classes have at least \( \varepsilon \cdot e(G) \) edges will be henceforth called \( \varepsilon \)-\textit{balanced}. Call a bicolored graph \( K_n \) an \textit{unavoidable} \( t \)-graph if one of the color classes is a clique of size \( t \) (Type 1), or two disjoint cliques of size \( t \) (Type 2). As the edge-coloring of \( K_n \) can itself be unavoidable, subgraphs of unavoidable graphs are the only non-monochromatic subgraphs we can hope to find. Bollobás conjectured [10] that any large enough \( \varepsilon \)-balanced complete graph would contain an unavoidable \( t \)-graph.

Cutler and Montágh [10] confirmed Bollobás’s conjecture. For \( \varepsilon \in \left( 0, \frac{1}{2} \right] \) and \( t \in \mathbb{N} \), let \( R(\varepsilon, t) \in \mathbb{N} \) be the smallest integer such that, whenever \( n \geq R(\varepsilon, t) \), every \( \varepsilon \)-balanced \( K_n \) contains an unavoidable \( t \)-graph. Sharp bounds for \( R(\varepsilon, t) \) were later obtained by Fox and Sudakov [15], who showed that \( R(\varepsilon, t) = (1/\varepsilon)^{\Theta(t)} \). Caro, Hansberg, and Montejano [7] proved Bollobás’s conjecture with even weaker hypotheses, letting \( \varepsilon \to 0 \). They additionally studied the minimum density of each color class in a bicolored \( K_n \) required for finding forbidden bicolored graphs with a fixed number of red edges, giving the problem a Turán-type flavor. Subsequently, similar problems were studied in [3,11,16] (see also [5]). Multicolor and infinite variants of the problem were studied in [4]. In [7], the connection between this problem and zero-sum labelings of graphs was explored (see also [6]).

In this paper, we continue the trend from [7] and consider an extremal version of Bollobás’s problem. The celebrated theorem of Turán states that any graph \( G \) with more than \( \left( 1 - \frac{1}{k-1} \right)^{v(G)/2} \) edges necessarily contains a \( k \)-clique. By analogy with Bollobás’s problem and with Turán’s theorem, we ask which non-monochromatic subgraphs can be guaranteed in a dense enough \( \varepsilon \)-balanced graph. It is clear as before that subgraphs of unavoidable graphs are the only patterns that we can hope to find. Our first theorem says that in fact unavoidable graphs can be found in dense enough \( \varepsilon \)-balanced graphs, and the required density is closely related to the corresponding Ramsey-type function.

**Theorem 1.1.** Let \( \varepsilon \in (0,1/2] \) and \( t \in \mathbb{N} \). For any \( 0 < \varepsilon' < \varepsilon \) there exists a constant \( K = K(\varepsilon, \varepsilon') \) such that any \( \varepsilon \)-balanced bicolored graph \( G \) with \( e(G) \geq \left( 1 - \frac{1}{K \cdot R(\varepsilon', t)} \right)^{v(G)/2} \) edges (and sufficiently many vertices) contains an unavoidable \( t \)-graph.

On the other hand, one can construct a bicolored graph without an unavoidable \( t \)-graph with density \( \left( 1 - \frac{1}{R(\varepsilon, t)} \right) \) as follows: Start with an \( \varepsilon \)-balanced complete graph on \( R(\varepsilon, t) - 1 \) vertices that contain no unavoidable \( t \)-graphs, and replace each vertex by equally sized independent sets, and each edge by a complete bipartite graph (of the same color). This shows that the asymptotics of this extremal problem is closely related to the Ramsey-type function \( R(\varepsilon, t) \).

Next we consider forbidding graphs other than the complete graph, and we see that our Theorem 1.1 has an interesting application in the aforementioned Ramsey-type setting. We call
a bicolored graph *inevitable* if for large enough \( t \) it can be embedded in both a Type 1 \( t \)-graph and Type 2 \( t \)-graph. For example, graphs where one color class form a star which spans all the vertices, and cycles whose coloring alternates between two red edges followed by two blue edges are inevitable. (We characterize inevitable graphs in Proposition 3.2.)

We stress here that inevitable and unavoidable are not synonymous in this context. In fact, an unavoidable \( t \)-graph of either Type 1 or Type 2 is actually not inevitable, as inevitable graphs must be embeddable in both types of unavoidable graphs (for large enough \( t \)). Furthermore, unavoidable graphs are only a specific family of two-edge-colored complete graphs, whereas inevitable graphs need not be complete. The word “unavoidable” is perhaps not the clearest choice, but since it has been used in all previous papers on the subject we will continue its use.

If \( H \) is an inevitable graph, \( \varepsilon(H) \) denotes the least integer such that every \( \varepsilon \)-balanced clique on at least \( \varepsilon(H) \) vertices contains \( H \) as a subgraph. The function \( \varepsilon(H) \) was studied in [4] for dense \( H \); it was shown that in this case \( \varepsilon(H) \) grows exponentially with \( v(H) \). For sparse \( H \), one would expect \( \varepsilon(H) \) to grow much slower. In the context of monochromatic graphs, Chvátal–Rödl–Szemerédi–Trotter showed [8] that bounded degree graphs have linear Ramsey functions. Using Theorem 1.1, we obtain the bicolored analogue of their result:

**Theorem 1.2.** For any \( \varepsilon > 0 \) and \( \Delta \) there exists a constant \( C = C(\varepsilon, \Delta) \) such that for any inevitable \( H \) with maximum degree \( \Delta \), \( \varepsilon(H) \leq C \cdot v(H) \).

Lastly, we consider an extremal analogue of the \( \varepsilon(H) \) function. For \( \mathcal{F} \) a family of bicolored graphs, we call \( \mathcal{F} \) *color-consistent* if it is unchanged by a permutation of the colors.

**Definition 1.3.** Let \( \mathcal{F} \) be a color-consistent family of bicolored graphs. We define \( \varepsilon(\mathcal{F}) \) as the maximum integer \( M \) such there exists an \( n \)-vertex \( \varepsilon \)-balanced graph \( G \) with \( e(G) = M \) such that \( G \) avoids all \( F \in \mathcal{F} \). If \( H \) is a single bicolored graph, \( \varepsilon(H) \) denotes \( \varepsilon(\mathcal{F}) \), where \( \mathcal{F} \) denotes \( H \) with the colors inverted. Finally, we let \( \varepsilon(\mathcal{F}) := \lim_{n \to \infty} \varepsilon(\mathcal{F}) \).

We note here that Definition 1.3 can also be extended to families of graphs that are not color-consistent, but that we will only consider color-consistent families in this paper. We prove the basic properties of this function in Section 3, such as the existence of the limit \( \varepsilon(\mathcal{F}) \). We generalize the celebrated Erdős–Stone Theorem to our setting, which allows us to extend the following result of DeVos, McDonald, and Montejano (rephrased in our language).

**Theorem 1.4 (DeVos–McDonald–Montejano [12]).** Let \( T_1 \) and \( T_2 \) be the bicolored triangles that are not monochromatic. Then, \( \varepsilon(T_1) = 2/3 \).

Thus, a density of 1/3 in both color classes is required to ensure the existence of a non-monochromatic triangle. We extend this result to cycles of any fixed size:

**Theorem 1.5.** Let \( C_k \) denote the family of all non-monochromatic bicolored \( k \)-cycles. Then, if \( k \geq 4 \), \( \varepsilon(C_k) = 1/2 \).

For non-monochromatic cliques, we show that the Turán density which guarantees an uncolored clique also guarantees a bicolored clique of size differing by at most an additive constant.
Theorem 1.6. Let \( k \geq 3 \), and let \( \mathcal{K}_k \) denote the family of all non-monochromatic bicolored \( k \)-cliques. Then, \( \text{ex}\left(\frac{1}{2}, \mathcal{K}_k\right) = 1 - \frac{1}{k + O(1)} \).

We believe Theorem 1.6 can be improved as follows.

Conjecture 1. Let \( \mathcal{K}_k \) be the family of all non-monochromatic bicolored \( k \)-cliques. For \( k \) sufficiently large, \( \text{ex}\left(\frac{1}{2}, \mathcal{K}_k\right) = 1 - \frac{1}{k - 1} \).

1.1 Organization

In Section 2, we first prove our main extremal result, Theorem 1.1, and then we show its application in the Ramsey-type setting, Theorem 1.2, in Section 2.1. Afterwards, in Section 3 we start studying the function \( \text{ex}(\epsilon, \mathcal{F}) \). After proving some basic properties, we characterize when \( \text{ex}(\epsilon, \mathcal{F}) < 1 \), and prove a bicolored version of the Erdős–Stone Theorem. In Section 4, we use this theory to prove Theorems 1.5 and 1.6.

For \( f, g : \mathbb{N} \to \mathbb{N} \), we use \( f \ll g \) to mean that \( f = o(g) \) and \( f \leq g \) to mean \( f \leq (1 + o(1))g \). For a bicolored graph \( G \), we will use subscripts \( R \) and \( B \) to denote graph invariants restricted to the red and blue edges, respectively, for example, \( E_B(G) \) will denote the set of blue edges in \( G \) and \( d_R(v) \) will denote the red degree of a vertex \( v \). Similarly, \( d_B(X, Y) \) denotes the total number of red edges between vertex sets \( X \) and \( Y \).

2 BALANCED CLIQUES IN BALANCED GRAPHS

The proof of Theorem 1.1 is almost immediate after the following lemma.

Lemma 2.1. For any \( 0 < \epsilon' < \epsilon \leq 1/2 \), there exists a \( C, K_0 \) such that for all \( k \geq K_0 \) and \( n \) sufficiently large, we have that any \( \epsilon \)-balanced \( n \)-vertex graph with \( \left( 1 - \frac{1}{\epsilon} \right) \left( \frac{n}{2} \right) \) edges contains an \( \epsilon' \)-balanced clique on at least \( \frac{k}{C} \) vertices.

We do not know if in the above statement \( \frac{k}{C} \) can be replaced by a quantity of the form \( (1 - o(1))k \). We consider this problem in Section 5 further. We will first show how to derive Theorem 1.1 using the above lemma, and we will prove Lemma 2.1 right afterwards. We note that \( \epsilon' \) must be strictly less than \( \epsilon \) in Lemma 2.1 for technical reasons that can be seen during its proof.

Proof of Theorem 1.1, assuming Lemma 2.1. Given \( \epsilon \) and \( \epsilon' \), let \( C \) and \( K_0 \) be the constants obtained by applying Lemma 2.1 with \( \epsilon \) and \( \epsilon' \). Choose \( K := \max\left\{ \frac{K_0}{R(\epsilon', t)}, C \right\} \) and consider a graph \( G \) with \( e(G) \geq \left( 1 - \frac{1}{K \cdot R(\epsilon', t)} \right) \left( \frac{v(G)}{2} \right) \) and \( v(G) \) sufficiently large. Since by our choice, \( K \cdot R(\epsilon', t) \geq K_0 \), we have that \( G \) contains an \( \epsilon' \)-balanced clique on \( \frac{K \cdot R(\epsilon', t)}{C} \geq R(\epsilon', t) \) vertices. Such a subgraph by definition contains an unavoidable \( t \)-graph, as desired. \( \square \)
Before the proof of Lemma 2.1, we record the version of Chebyshev's inequality and the Chernoff bound that we will use (see [2]).

**Lemma 2.2** (Chebyshev’s inequality). Let $A$ be a random variable with finite mean and variance. Then the following inequality holds for any real $\delta > 0$:

$$\Pr(|A - \mathbb{E}[A]| \geq \delta \cdot \mathbb{E}[A]) \leq \frac{\text{Var}[A]}{\delta^2 \mathbb{E}(A)^2}.$$ 

**Lemma 2.3** (Chernoff bound). Let $X := \sum_{i=1}^{m}X_i$, where $(X_i)_{i \in [m]}$ is a sequence of independent indicator random variables with $\Pr(X_i = 1) = p_i$. Let $\mathbb{E}[X] = \mu$. Then, for any $0 < \gamma < 1$, we have that $\Pr(|X - \mu| \geq \gamma \mu) \leq 2e^{-\gamma^2 \mu^2 / 3}$.

**Proof of Lemma 2.1.** Let $\varepsilon$ and $\varepsilon'$ be fixed with $0 < \varepsilon' < \varepsilon \leq 1/2$ and let $G$ be a graph on $n$ vertices that is $\varepsilon$-balanced and has at least $(1 - \frac{1}{k})(\binom{n}{2})$ edges. We will sample a subset of vertices randomly, alter it to be a clique, and show that with positive probability that it is $\varepsilon'$-balanced. We will choose $C$ to be a large constant and $K_0$ will be chosen after $C$ is fixed.

First, we choose a subset $S$ of vertices of $G$ by picking each vertex independently with probability $p := \frac{4k}{Cn}$. Here $n$ is assumed to be large enough that $p < 1$.

For every pair of nonadjacent vertices in $S$, we remove both of them. We call the resulting set $T$. We will show that there is a positive probability that $T$ satisfies the conclusion of the statement, that is, $T$ is an $\varepsilon'$-balanced clique on at least $\frac{k}{C}$ vertices.

The bad events which we hope to simultaneously avoid are as follows:

(a) $|T| < \frac{k}{C}$,

(b) $e_B(T) \leq \varepsilon'\binom{|T|}{2}$,

(c) $e_R(T) \leq \varepsilon'\binom{|T|}{2}$.

Since $T$ is a clique by construction, if we avoid conditions (a)–(c) then we are done.

Since $G$ has a total edge density $1 - \frac{1}{k}$ the expected number of pairs of vertices in $S$ which are not adjacent is

$$p^2 \frac{1}{k} \binom{n}{2} \leq \frac{8k}{C^2}.$$ 

Since we will remove at most two vertices for each pair of nonadjacent vertices in $S$, by Markov’s inequality the probability that we remove more than $\frac{160k}{C^2}$ vertices from $S$ is at most $\frac{1}{10}$. We also have that $\mathbb{E}(|S|) = \frac{4k}{C}$ and so by the Chernoff bound we have that

$$\Pr \left( \frac{2k}{C} < |S| < \left( \frac{1}{1 - \frac{1}{C}} \right) \frac{4k}{C} \right) > \frac{9}{10},$$

(1)
as long as $k$ is a large enough constant (note again that we choose $K_0$ after $C$). For ease of notation, let $M = \left(\frac{1}{1 - \frac{1}{C}}\right)^{\frac{4k}{C}}$, and we note that the slightly unnatural-looking inequalities in (1) are chosen with foresight. Choosing $C$ large enough so that $\frac{8k}{C^2} < \frac{k}{C}$ gives that (a) occurs with probability less than $\frac{1}{5}$.

Next we give an upper bound on the probability that (b) occurs. Let $U$ be the event that the upper bound in (1) occurs, that is, $|\mathcal{S}| < M$. Recall that $K_0$ will be chosen so that $k$ is large enough to ensure $\mathbb{P}(U) > \frac{9}{10}$. To estimate the probability that (b) occurs, we see that

$$
\mathbb{P}((b) \text{ occurs}) = \mathbb{P}((b) \text{ occurs} \land U) + \mathbb{P}((b) \text{ occurs} \land U^c)
$$

$$
\leq \mathbb{P}\left(e_B(T) \leq \varepsilon\left(\frac{M}{2}\right)\right) + \mathbb{P}(U^c),
$$

where the last inequality follows because if $U$ occurs then $|\mathcal{S}| \leq M$. Therefore, we have that

$$
\mathbb{P}((b) \text{ occurs}) \leq \mathbb{P}\left(e_B(T) \leq \varepsilon\left(\frac{M}{2}\right)\right) + \frac{1}{10},
$$

and so it suffices to give a good enough upper bound on

$$
\mathbb{P}\left(e_B(T) \leq \varepsilon\left(\frac{M}{2}\right)\right).
$$

To do this, we first wish to give a lower bound on $\mathbb{E}[e_B(T)]$. Let $\overline{N}(u)$ denote the nonneighborhood of a vertex and $d_u = |\overline{N}(u)|$. Suppose $uv$ is a blue edge and note:

$$
P(uv \in T) = P(uv \in S) \cdot P(uv \in T|uv \in S) = p^2 \cdot \prod_{w \in \overline{N}(u) \cup N(v)} (1 - P(w \in S))
$$

$$
\geq p^2 (1 - p)^{\overline{d}_u + \overline{d}_v}.
$$

Therefore, we have that

$$
\mathbb{E}[e_B(T)] \geq \sum_{uv \in E_b(G)} p^2 (1 - p)^{\overline{d}_u + \overline{d}_v} \geq p^2 \sum_{uv \in E_b(G)} (1 - (\overline{d}_u + \overline{d}_v)p)
$$

$$
= p^2 e_b(G) - p^3 \sum_{v \in V(G)} \overline{d}_v \cdot d_{v, \text{blue}}
$$

$$
\geq p^2 e_b(G) - \frac{2p^3 n}{k} \left(\frac{n}{2}\right)
$$

$$
\geq p^2 e_b(G) \left(1 - \frac{16}{\varepsilon C}\right) =: P,
$$

where the inequalities are by linearity of expectation, Bernoulli’s inequality, $d_{v, \text{blue}} \leq n$, and simplification, respectively. It follows that
Thus, assuming $K_0 \geq C$, we may choose $C$ large enough with respect to $\varepsilon$ and $\varepsilon'$ so that

$$\frac{\mathbb{E}[e_B(T)]}{\binom{M}{2}} \geq \varepsilon \left(1 - \frac{1}{k}\right) \left(1 - \frac{16}{\varepsilon C}\right) \left(1 - \frac{1}{C}\right)^2.$$ 

where $\delta := 1/C^{l/2}$ is an extra factor that we will use soon when applying Chebyshev’s inequality.

Now, we wish to estimate $\text{Var}[e_B(T)]$ to apply Chebyshev’s inequality. Let $I_{uv}$ denote the indicator random variable denoting whether the blue edge $uv$ is inside $S$. Then $e_B(S) = \sum_{uv \in E_B(G)} I_{uv}$. Note that for disjoint edges $uv$ and $xy$, $I_{uv}$ and $I_{xy}$ are independent random variables. Recall that $\mathbb{E}[e_B(T)] \geq P$.

$$\text{Var}[e_B(T)] = \mathbb{E}[e_B(T)^2] - \mathbb{E}[e_B(T)]^2 \leq \mathbb{E}[e_B(S)^2] - \mathbb{E}[e_B(T)]^2 \leq (e_B(G)^2 p^4 + n^3 p^3 + n^2 p^2) - \mathbb{E}[e_B(T)]^2 \leq (e_B(G)^2 p^4 - P^2) + n^3 p^3 + n^2 p^2 \leq P^2/5 + 64k^3/C^3 + 16k^2/C^2 \leq P^2/2,$$

where the second to last inequality is true if we choose $K_0$ to be a large enough constant, as $P^2 = \Theta(k^4/C^4)$, hence much larger than the other two terms for $k$ large.

With all the inequalities we have collected, we can now bound the probability that bad event (b) happens via Chebyshev:

$$\mathbb{P}(\text{(b) occurs}) \leq \mathbb{P}\left(\frac{e_B(T)}{\binom{M}{2}} \leq \varepsilon' \left(1 - \frac{1}{C}\right)^2\right) + \frac{1}{10} \leq \mathbb{P}\left(\frac{|e_B(T) - \mathbb{E}[e_B(T)]|}{\mathbb{E}[e_B(T)]} \geq \delta \cdot \mathbb{E}[e_B(T)]\right) + \frac{1}{10} \leq \frac{\delta^2 \text{Var}[e_B(T)]}{\mathbb{E}[e_B(T)]^2} + \frac{1}{10} \leq \frac{1}{C} + \frac{1}{10} < \frac{1}{5}.$$

By symmetry, the probability that (c) occurs is also less than $\frac{1}{5}$. Using the union bound, there is a positive probability that none of the events occur. 

2.1 $\varepsilon$-Balanced Ramsey numbers of sparse graphs

In this section we show how to prove Theorem 1.2 using Theorem 1.1. We begin by formally stating the analogous result in the ordinary Ramsey setting. Recall that when $H$ is a graph
Theorem 2.4 (Chvátal–Rödl–Szemerédi–Trotter). For every $\Delta$ there exists a $c = c(\Delta) > 0$ such that $R(H) \leq c \cdot v(H)$ for all graphs $H$ with maximum degree at most $\Delta$.

The proof of Theorem 1.2 will follow the original regularity-based proof of Theorem 2.4 quite closely. In fact, except for the differences we outline in this section, the proof remains mostly identical. Hence, we provide the details of the proof only in the appendix. Here, we will record the extra tools we will need in the proof, most of which will be reused in Section 3.

We first state the Regularity lemma, starting with the necessary terminology. Let $G$ be a bipartite graph with partite sets $A$ and $B$, and $|A| = |B| = n$. For $X \subseteq A$ and $Y \subseteq B$ define $d(X, Y) = \frac{e(X, Y)}{|X||Y|}$. We call $G$ $\varepsilon$-regular if for all subsets $X \subseteq A$ and $Y \subseteq B$ with $|X|, |Y| \geq \varepsilon n$ we have $|d(X, Y) - d(A, B)| \leq \varepsilon$.

Lemma 2.5 (Szemerédi [19]). For any $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ such that any graph $G$ can be (vertex) partitioned into $k$ (where $\frac{1}{2} \leq k \leq M$) equal-sized parts $(V_i)_{i \in [k]}$ and a junk set $J$ with $|J| < \varepsilon n$ such that all but $\varepsilon$-fraction of the pairs $(V_i, V_j)$ are $\varepsilon$-regular.

We sometimes call the pairs of the form $(V_i, V_j)$ super-edges, the $V_i$ super-vertices. Given $\varepsilon$ and $d$, the graph with vertex set the super-vertices and $V_i$ joined to $V_j$ when the pair $(V_i, V_j)$ is $\varepsilon$-regular with density at least $d$ is called the cluster graph.

Now, we will point out the key differences of our proof of Theorem 1.2 with the original proof of Theorem 2.4. First, the main goal in the proof of Theorem 2.4, after an application of the Regularity lemma, is finding a $(\Delta + 1)$-clique in the cluster graph where each super-edge of the clique has a density at least $1/2$ in the same color class. Our goal will change to finding an unavoidable $(\Delta + 1)$-graph in the cluster graph. By this, we mean a $(\Delta + 1)$-clique in the cluster graph where super-edges are colored with one of the unavoidable patterns, and the color of a super-edge signals a high density for that color for the corresponding pair. To achieve this goal, we will apply our Theorem 1.1.

Second, a technicality that arises is that we need to keep track of how balanced our graph is as we associate to super-edges with the correct color class. In particular, coloring by the majority color class, as in the proof of Theorem 2.4, does not work in our setting. To achieve this we state a lemma that allows us to extract a balanced set of red and blue super-edges from the cluster graph of a balanced graph. We will use this result again in Section 3. We will use the version of the Chernoff bound from Lemma 2.3 in our proof.

Before stating the lemma we need to define density in each color class. Given a bicolored graph $G$ and two subsets $X$ and $Y$ of vertices, define $e_R(X, Y)$ (resp., $e_B(X, Y)$) to be the number of red (resp., blue) edges with one endpoint in $X$ and one endpoint in $Y$. Let $d_R(X, Y) = \frac{e_R(X, Y)}{|X||Y|}$ and $d_B(X, Y) = \frac{e_B(X, Y)}{|X||Y|}$. Finally, let $d_R(G)$ and $d_B(G)$ denote $d_R(V(G), V(G))$ and $d_B(V(G), V(G))$.

Lemma 2.6. Let $\varepsilon, \varepsilon_0 > 0$, $k \geq 1/\varepsilon_0$, and $G$ be a $k$-partite bicolored graph with parts $\mathcal{R} := \{P_1, ..., P_k\}$ such that $d_R(P_i, P_j) \geq \varepsilon$ or $d_B(P_i, P_j) \geq \varepsilon$ or $d(P_i, P_j) = 0$ for any two parts $P_i$
and $P_i$ with $i \neq j$. Then, there exists an edge-coloring of a graph on vertex set $\mathcal{R}$ such that a red (resp., blue) edge $\{P_i, P_j\}$ implies $d_R(P_i, P_j) \geq \varepsilon$ (resp., $d_B(\cdot)$), and furthermore, $|d_R(\mathcal{R}) - d_R(G)| \leq \gamma_R$, where $\gamma_R = 4\varepsilon_0/\sqrt{d_R(G)}$, and the same holds for blue.

**Proof.** We produce the coloring randomly, and bound the failure probability via the Chernoff bound. For any pair in $\mathcal{R}$ with positive density of edges, we add a red edge $\{P_i, P_j\}$ with probability $d_R(P_i, P_j)/d(P_i, P_j)$ (for $i \neq j$), and we add a blue edge otherwise.

Let $X_{i,j}$ be the indicator random variable for whether $\{P_i, P_j\}$ is red or not in $\mathcal{R}$. Then,

$$\mathbb{E}[X] = \sum_{\{i,j\} \in \binom{[k]}{2}} \mathbb{P}(X_{i,j} = 1) = d_R(G)\left(\frac{k}{2}\right).$$

By the Chernoff bound,

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq \gamma \mathbb{E}[X]\right) \leq 2e^{-\mathbb{E}[X]/3} \leq 2e^{-d_R(G)\frac{16k^2}{6} - \varepsilon_0 d_B(G)^{-1}} \leq 2e^{-2} \leq \frac{2}{7}$$

Where in the second to last inequality we used $k \geq \frac{1}{\varepsilon_0}$. Repeating the same calculation with the blue edges, we conclude that with at least $3/7$ probability, there exists a colored edge assignment in which neither the red nor the blue density deviate more than $\gamma_R$ or $\gamma_B$ from their prior densities in $G$.

Finally, we will need to use a bicolored version of the Regularity lemma (Lemma 2.5). By simply iterating the Regularity lemma to the red edges, and then the blue edges, we can obtain the following corollary.

**Corollary 2.7.** For any bicolored graph $G$, we can obtain a single regularity partition where the red and the blue graphs both satisfy the conclusions of the Regularity lemma (Lemma 2.5), with the same exact hypotheses.

For a detailed proof of Theorem 1.2, see the appendix.

### 3. THE BALANCED EXTREMAL FUNCTION

In this section we systematically study the function $\text{ex}(\varepsilon, n, \mathcal{F})$ defined in Section 1. First, recall that a bicolored graph $G$ is called $\varepsilon$-balanced if both color classes have at least $\varepsilon \cdot e(G)$ edges, and that a family $\mathcal{F}$ of bicolored graphs is color-consistent if it is unchanged by a permutation of the colors. As defined in Section 1, given a color-consistent family of bicolored graphs $\mathcal{F}$, we let $\text{ex}(\varepsilon, n, \mathcal{F})$ be the maximum integer $M$ such that there exists an $n$-vertex $\varepsilon$-balanced bicolored graph with $M$ edges that avoids all $F \in \mathcal{F}$. We also defined $\text{ex}(\varepsilon, \mathcal{F}) = \lim_{n \to \infty} \text{ex}(\varepsilon, n, \mathcal{F})/\binom{n}{2}$. 


In this paper, we will be concerned with the case when \( \mathcal{F} \) is composed of connected, non-monochromatic graphs. Thus \( \text{ex}(\varepsilon, n, \mathcal{F}) \) will always be at least quadratic in \( n \), as two disjoint cliques of the same size but different colors will not contain any forbidden subgraphs. Indeed, we have \( \text{ex}(\varepsilon, n, \mathcal{F}) \geq \frac{1}{4} \binom{n}{2} \). Hence we are simply interested in the limit value, \( \text{ex}(\varepsilon, \mathcal{F}) \).

Of course, it is not a priori clear that this limit exists. We deal with this technicality in the next lemma.

**Proposition 3.1.** Let \( \mathcal{F} \) be a color-consistent family of graphs and \( \varepsilon > 0 \) be arbitrary. Then \( \lim_{n \to \infty} \frac{\text{ex}(\varepsilon, n, \mathcal{F})}{\binom{n}{2}} \) exists and we define this limit to be \( \text{ex}(\varepsilon, \mathcal{F}) \).

**Proof.** Assume to the contrary. Since \( \text{ex}(\varepsilon, n, \mathcal{F}) \) is bounded between 0 and 1, we may fix two subsequences which converge to different limits. Say the two limits are \( \delta \) apart, for some \( \delta > 0 \). We can then fix \( n_1 \) from the sequence with the larger limit and \( n_2 \) from the other sequence so that \( n_1 > n_2 \) and

\[
\frac{\text{ex}(\varepsilon, n_1, \mathcal{F})}{\binom{n_1}{2}} > \frac{\text{ex}(\varepsilon, n_2, \mathcal{F})}{\binom{n_2}{2}} + \delta,
\]

We note that we may choose \( n_2 \) to be as large as we need. Let \( G_1 \) be an \( \mathcal{F} \)-avoiding graph on \( n_1 \) vertices and \( \varepsilon \)-balanced \( \mathcal{F} \)-avoiding graph with density strictly larger than \( \frac{\text{ex}(\varepsilon, n_2, \mathcal{F})}{\binom{n_2}{2}} \), yielding a contradiction.

We sample a subset of \( n_2 \) vertices from \( G_1 \) (without repetition) uniformly at random, and call it \( G_2 \). Let \( p(n) := \prod_{0 \leq i < n} \frac{n - i}{n_1} \). Note that \( \mathbb{E}[e_B(G_2)] = e_B(G_1)p(2) \).

We now estimate \( \text{Var}[e_B(G_2)] \). Let \( 1_{uv} \) denote the random variable indicating whether the blue edge \( uv \) is contained in \( G_2 \) so that \( e_B(G_2) = \sum_{uv \in E_b(G_1)} 1_{uv} \).

\[
\text{Var}[e_B(G_2)] = \mathbb{E}[e_B(G_2)^2] - \mathbb{E}[e_B(G_2)]^2
\]

\[
= \sum_{uv, x \in E_b(G_1)} \mathbb{E}[1_{uv}1_{xy}] + \sum_{uv, x \in E_b(G_1)} \mathbb{E}[1_{uv}1_{xy}] + \sum_{uv, x \in E_b(G_1)} \mathbb{E}[1_{uv}1_{xy}] - \mathbb{E}[e_B(G_2)]^2
\]

\[
\leq \left( e_B(G_1)^2p(4) + n_1^3p(3) + n_1^2p(2) \right) - \mathbb{E}[e_B(G_2)]^2 = o(\mathbb{E}[e_B(G_2)]^2).
\]

Thus, since the variance is low, for any \( \gamma > 0 \), if \( n_2 \) is sufficiently large, we have by Chebyshev’s inequality (Lemma 2.2) that

\[
\mathbb{P}(|e_B(G_2) - \mathbb{E}[e_B(G_2)]| \geq \gamma \cdot \mathbb{E}[e_B(G_2)]) < 1/2.
\]

It follows that we may choose \( n_2 \) large enough that with probability more than 1/2, \( |d_B(G_2) - d_B(G_1)| \leq \delta/5 \).

We can similarly show that with probability more than 1/2, the red density of \( G_2 \) is at most \( \delta/5 \) away from that of \( G_1 \), and so with positive probability, both events happen
simultaneously. For such a $G_2$, by deleting at most $\frac{2\delta}{5}n^2$ edges, we can create an $\varepsilon$-balanced $G_2'$. Overall, we can thus ensure that $G_2'$ has density strictly larger than $\frac{\text{ex}(\varepsilon, n, F)}{n^2}$, is $\varepsilon$-balanced, and is $F$-avoiding (as a subgraph of $G_1$), which gives us the desired contradiction.

We finally remark that it is clear from the definition that $\text{ex}(\varepsilon, n, F)$ is monotone increasing for decreasing $\varepsilon$.

### 3.1 Inevitable graphs and characterization

Now that we know $\text{ex}(\varepsilon, F)$ exists for every color-consistent family $F$, the first natural question is to determine for which graphs $\text{ex}(\varepsilon, F) < 1$. The answer turns out to be closely related to families which contain an inevitable graph. Here we give a structural characterization of inevitable graphs.

**Proposition 3.2.** Let $H$ be a bicolored graph. Then, $H$ is inevitable if and only if there exists a vertex partition $L \sqcup R = V(H)$ with the edges contained in $L$ and $R$ entirely red, and edges that go across entirely blue, such that either:

1. $H$ does not contain a blue–red–blue walk on three edges.
2. $H$ does not contain a red–blue–red path,

or the same except permuting red and blue.

Note that the first case can be replaced equivalently with: There exist subsets $X \subseteq L$, $Y \subseteq R$ such that $X$ and $Y$ are independent and all the edges that are between $L$ and $R$ are contained between $X$ and $Y$.

**Proof.** Any $H$ that falls in one of the two cases can be embedded into both types of unavoidable graphs and thus is inevitable. Indeed, embedding such $H$ into a Type 2 graph is trivial because of the first part of the proposition. To embed a Case 1 $H$ into a Type 1 graph we simply embed the independent sets inside the isolated color class. To embed a Case 2 $H$ into a Type 1 graph we can greedily embed all the (e.g.) red edges inside the isolated (red) clique, and as there cannot be any blue edge between two red edges, we will not have a problem embedding the blue edges as well. We now focus on the other direction of the proof.

Assume that $H$ is inevitable. The first part of the proposition follows immediately from the fact that there exists an embedding of $H$ into a Type 2 unavoidable graph, where (without loss of generality) the edges of the bipartite graph are blue. We also now $H$ can be embedded in a Type 1 unavoidable graph. We case on the color of the bipartite graph in the Type 1 graph $H$ can be embedded to.

Case 1. $H$ has an embedding to a Type 1 unavoidable graph where the edges that go across are red. Let us call the set of vertices embedded in the blue part $B$, and the red part
R. Now, observe that if any two vertices from $L$ are embedded in $B$, they cannot be adjacent, as edges contained in $L$ are red, but edges contained in $B$ are blue. The same goes for any two vertices from $R$. Therefore, any blue edge between vertices from $L$ and $R$ is contained in independent sets.

**Case 2.** $H$ has an embedding to a Type 1 unavoidable graph where the edges that go across are blue. Now it is clear that any path that starts red–blue will end on a vertex adjacent only to blue edges. □

We finally state the following theorem, the proof of which now follows from the definitions.

**Theorem 3.3.** Let $\mathcal{F}$ be a color-consistent family of bicolored graphs. The following are equivalent.

1. $\text{ex}(\varepsilon, \mathcal{F}) < 1$,
2. $R(\varepsilon, \mathcal{F}) < \infty$,
3. There exists $F, G \in \mathcal{F}$ such that $F$ and $G$ are contained in a Type 1 graph, and a Type 2 graph, respectively.

**Proof.** Let $\mathcal{F}$ be a color-consistent family of bicolored graphs. First we show that (2) implies (3) and that (1) implies (3), both by contrapositive. Assume that there is and $X \in \{1, 2\}$ such that no $F \in \mathcal{F}$ is contained in a Type $X$ graph. Since for any arbitrarily large $n$, a coloring of $K_n$ can itself be Type $X$ and furthermore can be $\varepsilon$-balanced for any $\varepsilon \leq 1/2$, we have that $R(\varepsilon, \mathcal{F}) = \infty$ and $\text{ex}(\varepsilon, \mathcal{F}) = 1$.

Now we show (3) implies (2). Assume (3). Then there exists a $t$ such that an unavoidable $t$-graph contains either $F$ (if the graph is Type 1) or $G$ (if the graph is Type 2). By the finiteness of the function $R(\varepsilon, t)$, there is an $N$ such that any $\varepsilon$-balanced coloring of $K_N$ contains an unavoidable $t$-graph, and hence contains either $F$ or $G$. Therefore, $R(\varepsilon, \mathcal{F}) \leq N$.

We complete the proof by showing that (3) implies (1). Let $t$ be large enough that $F, G \in \mathcal{F}$ are contained in an unavoidable $t$-graph of Types 1 and 2, respectively. Then by Theorem 1.1, there exists a $K$ such that for sufficiently large $n$ we have that any $\varepsilon$-balanced $n$ vertex graph with at least $\left(1 - \frac{1}{K \cdot R(\varepsilon / 2, t)}\right)^n\binom{n}{2}$ edges contain an unavoidable $t$-graph and hence contain either $F$ or $G$. Therefore, $\text{ex}(\varepsilon, \mathcal{F}) \leq \frac{1}{K \cdot R(\varepsilon / 2, t)}$. □

We note that it may be that $FG = \in$ (3), in which case $\mathcal{F}$ contains an inevitable graph.

### 3.2 Erdős–Stone type theorem

In this subsection, we state a two-color Erdős–Stone Theorem using the parameter $\text{ex}(\varepsilon, \cdot)$. This theorem will be useful in Section 4 as we study the parameter $\text{ex}(\varepsilon, \mathcal{F})$ for certain families of bicolored graphs $\mathcal{F}$. As the proof does not require any new ideas compared to the standard regularity based proof of Erdős–Stone Theorem (other than an application of Lemma 2.6), we provide the details only in the appendix.
We now state our two-color generalization of the celebrated Erdős–Stone Theorem. Note that by a $t$-blow-up of an edge-colored graph $F$, we mean the edge-colored graph where each vertex of $F$ is replaced by an independent set of size $t$ and if $\{v, w\}$ is an edge of color $c$ in $F$, we add a complete bipartite graph of color $c$ between the independent sets corresponding to $v$ and $w$.

**Theorem 3.4.** Let $\epsilon, \delta > 0$ and $t \in \mathbb{N}$ be arbitrary. Let $\mathcal{F}$ be some finite family of bicolored graphs. Let $c = \text{ex}(\epsilon, \mathcal{F})$. Then, for all sufficiently large $n$ (with respect to $\mathcal{F}, \epsilon, t,$ and $\delta$) we have that any $\epsilon$-balanced $n$-vertex graph with $(c + \delta)^2/n$ edges contains a $t$-blow-up of some $F \in \mathcal{F}$.

Although it is standard, we give a proof of Theorem 3.4 in the appendix.

### 4 | Non-Monochromatic Cliques and Cycles

In this section, we will study the densities of $\frac{1}{2}$-balanced graphs that ensure the existence of either non-monochromatic cliques or non-monochromatic cycles, proving Theorems 1.5 and 1.6.

#### 4.1 | Cliques

Let $K_k$ denote the family of non-monochromatic cliques on $k$ vertices. In this subsection we will show that

$$\text{ex}\left(\frac{1}{2}, K_k\right) \leq 1 - \frac{1}{k + O(1)},$$

proving Theorem 1.6.

**Proof of Theorem 1.6.** We modify a proof of Turán’s theorem attributed to Aigner [1].

Let $G$ be a $\frac{1}{2}$-balanced bicolored graph with density $1 - \frac{1}{k + C} := \delta$, and suppose $C$ is large enough so that $\delta > \frac{9}{10}$. Furthermore, assume that $G$ contains no non-monochromatic copy of $K_k$. We will show that $C$ must be bounded by an absolute constant (independent of $k$).

Consider the following algorithm to produce a non-monochromatic clique. First, sample the vertices with repetition until all vertices are seen, creating a sequence of vertices. Observe that this also induces a uniformly sampled permutation of the vertex set (simply by deleting subsequent appearances of a vertex in the sequence), and we will consider both the sequence with repetition and the corresponding permutation. Define $S$ to be the set of all vertices which appear before all of their nonneighbors in the permutation, and define $w$ to be the number of vertices in the sequence (not the permutation) before a non-monochromatic triangle appears (note that $w$ will always be well defined, as $\delta > \frac{2}{3}$ implies that there is at least one non-monochromatic triangle). Let $T$ be a non-monochromatic triangle that appears first, and let $W$ be the set of vertices that appear in the first $w$ terms of the sequence.

By definition of $S$, the set $S$ induces a clique in the graph. Therefore, the set $T \cup (S \setminus W)$ forms a non-monochromatic clique in $G$. Since $G$ is $K_k$-free, this implies that
\[ T \cup (S \setminus W') < k. \]

By linearity of expectation, we have that
\[ E[|T \cup (S \setminus W')|] < k, \]
and in particular \( E[|S|] - E[w] \leq E[|S|] - E[w] < k. \)

Now, we have that
\[ E[|S|] = \sum_{v \in V(G)} \frac{1}{n - d(v)}. \]

By convexity of the function \( f(x) = 1/x \), we have that
\[ E[|S|] \geq n \left( \frac{1}{n - \frac{1}{n} \sum d(v)} \right) = n \left( \frac{1}{n - \delta(n - 1)} \right) \sim \frac{1}{1 - \delta} = k + C. \]

Combining inequalities gives that \( C < E[w] \) and hence it suffices to show that \( E[w] \) is bounded by a constant.

To do this, we claim that there are \( \Omega(n^3) \) non-monochromatic triangles in \( G \). To see this, in a graph on \( n \) vertices with \( e \) edges, there are at least \( \frac{e(4e - n)}{3n} \) triangles (see [18, p. 275]). Since \( \delta > \frac{9}{10} \), this implies that there are at least \( \frac{12}{100} n^3 \) triangles in \( G \). On the other hand, in a graph with \( \binom{n}{2} \) edges, there are at most \( \left( \frac{n}{2} \right) \) triangles (see [17, Chap. 13, Exercise 31b]). Since the red edges and the blue edges each individually have density less than \( 1/2 \), this implies that there are at most \( \frac{1}{6\sqrt{2}} n^3 \) monochromatic triangles in \( G \). Noting that \( \frac{1}{6\sqrt{2}} < \frac{1}{12} \) proves the claim.

Now, to bound \( E[w] \) by a constant, we observe that in the sequence of vertices selected with repetition, every consecutive and disjoint triple is an independently and uniformly selected sequence of three vertices. Since \( G \) has \( \Omega(n^3) \) non-monochromatic triangles, each of these sequences of three vertices has a positive probability (independent of \( k \)) of inducing a non-monochromatic triangle, and hence the expected waiting time to see a non-monochromatic triangle is \( O(1) \).

\[ \square \]

### 4.2 Cycles

Here, our goal is to prove Theorem 1.5, establishing that with the exception of triangles, the extremal threshold for finding a non-monochromatic cycle of any length in \((1/2)\)-balanced graphs is \( 1/4 \). We will end up proving a result significantly stronger, by finding the extremal threshold of all cycles where one of the color classes is a disjoint union of even-length paths. By greedily walking around the cycle while placing maximal length red paths on the same side of the partition while alternating sides of the partition during each maximal blue path, one can see that these cycles satisfy the hypotheses of Proposition 3.2. We did not see an easy way to
characterize all cycles satisfying the hypotheses of Proposition 3.2, and there are cycles with inevitable colorings which are not of this form (the smallest example is on 8 vertices).

**Theorem 4.1.** Let \( C \) be an inevitable cycle such that all maximal (without loss of generality) blue paths are of even edge length. Then we have the extremal value dichotomy:

1. If the red (or blue) color class in \( C \) contains any isolated edges, then \( \text{ex}(\frac{1}{2}, n, C) = \frac{2}{3} \).
2. Otherwise, all maximal red (or blue) paths are of length at least 2, and \( \text{ex}(\frac{1}{2}, n, C) = \frac{1}{2} \).

Theorem 1.5 follows from the second case of the dichotomy, as there exist non-monochromatic inevitable colorings of cycles (of length greater than 3) with the blue color class only containing even-length maximal paths and no isolated edges in red.

The theorem largely depends on the next lemma. As before, let \( T_1 \) and \( T_2 \) be the two types of non-monochromatic triangles, and let \( H_1 \) and \( H_2 \) be a red triangle incident with a blue edge and a blue triangle incident with a red edge, respectively, and call these bicolored graphs handles.

**Lemma 4.2.** \( \text{ex}(\frac{1}{2}, \{T_1, T_2, H_1, H_2\}) = \frac{1}{2} \). We delay the proof of the lemma to Section 4.3, and we first show how it implies the main theorem of this section. Our technique is to use our Erdős–Stone type theorem (Theorem 3.4) to find blow-ups of either non-monochromatic triangles or handles and finding appropriate embeddings inside the blow-ups.

**Proof of Theorem 4.1.** There are two cases, depending on whether the cycle \( C \) has isolated edges.

Case 1. If the cycle \( C \) contains isolated edges, we will use the same construction from DeVos, McDonald, and Montejano [12] which is as follows. Split the vertex set into three equal-sized parts, add red edges between a pair of parts and add blue edges for another pair of parts. Let the part incident on both red and blue edges be independent. Let the other pair of the red bipartite graph be a red clique, and the other pair of the blue bipartite graph a blue clique. Add no edges between the red and the blue clique. It is easy to see that this construction contains no cycles with an isolated edge in either color class. And the density of this construction is 2/3 whenever \( n \) is a multiple of 3. This establishes the lower bound.

Now, we consider a large enough (1/2)-balanced graph with density \( 2/3 + \delta \) for some fixed positive \( \delta \). In [12] it is shown that \( \text{ex}(\frac{1}{2}, K_3) = 2/3 \). This means that in any 1/2-balanced bicolored graph with more than \( (2/3)\binom{n}{2} \) edges necessarily contains a non-monochromatic triangle. If the graph is large enough, by Theorem 3.4 we can find a \( v(C) \)-blow-up of a non-monochromatic triangle. To conclude this case, we just have to find an embedding of \( C \) (whose blue edges are a disjoint union of even-length paths) into this blow-up. Suppose the blow-up is on parts \( A, B, C \), and edges between \( A \) and \( B \) are red and all other edges are blue.
The cycle is a collection of consecutive paths \( P \) of the form \( 2k \) blue edges followed by \( j \) red edges. We will show that any such \( P \) can be embedded on \((A, B, C)\) with the start and the end vertex in the same partition. Note embeddings of this form can be combined to embed any bicolored cycle of the specified form.

Suppose \( j \) is odd. Then, starting at \( A \), we follow an odd path between \( A \) and \( B \), ending at \( B \). Then, we follow a path of length 2 from \( B \) to \( A \) through \( C \). Then, we follow an even-length blue path between \( A \) and \( B \) as long as necessary.

Suppose now that \( j \) is even. Then, we follow an even-length red path starting at \( A \), alternating \( A \) and \( B \), ending at \( A \). Then, we do the same for a blue path alternating between \( A \) and \( C \).

Case 2. In this case, the blue paths are still of even length, and red paths are of length at least 2. As 1/2 is the trivial lower bound by the disjoint red and blue clique construction, we only need to show the upper bound. So assume we have a large enough (1/2)-balanced graph whose density exceeds 1/2.

By our Lemma 4.2 and Theorem 3.4 we can find a large enough blow-up of either a non-monochromatic triangle (Case 1A) or a handle (Case 1B).

To complete the embedding, just like in the previous case, it suffices to show how to embed a bicolored path \( P \) whose edge color sequence is an even number of blue edges followed by at least two red edges such that the endpoints of the path are embedded to the same part of the blow-up. In Case 1A, the embedding we give in the previous case applies here as well. In Case 1B, label the parts of the blow-up of the handle \((A, B, C, D)\) where \((A, B, C)\) induces a blow-up of a red triangle. We embed the even-length blue path on the tail of the handle, that is starting at \( C \) and alternating between \( C \) and \( D \), ending at \( C \). If the red path is of even length, we can just embed it alternatingly using \( C \) and \( B \) (or \( A \)). Otherwise, the red path is of odd length, and of length at least 3 by assumption. Then, we embed the first three edges of the red path by traversing a path from \( C \) to \( A \) to \( B \) back to \( C \). Then, we follow an even-length red path between \( C \) and \( B \) as long as necessary, ending again at \( C \). This concludes our embedding.

4.3 Handles and nonmonochromatic triangles

Our goal is now reduced to proving Lemma 4.2, that is, finding either handles or non-monochromatic triangles in all 1/2-balanced bicolored graphs with edge density 1/2. Note the lower bound in Lemma 4.2 simply follows from coloring the edges of a \( K_{n/2,n/2} \) half red, half blue, in an arbitrary fashion.

Throughout this section, we fix \( G \) to be a sufficiently large 1/2-balanced graph containing neither a handle nor a nonmonochromatic triangle. We are done if we can show that \( e(G) \lesssim n^2/4 \). We fix a partition of the vertices of \( G \):

\[
V(G) = R \uplus B \uplus M.
\]

Here, \( R \) (resp., \( B \)) denotes the vertices of \( G \) that are only incident on red (resp., blue) edges. \( M \) denotes the vertices that are incident on at least one red and one blue edge. Any vertex not in one of these sets would be an isolated vertex, the deletion of which would create a denser
graph. Hence, we may assume \( R, B, \) and \( M \) partition \( V(G) \). Let \( r, b, \) and \( m \) denote the sizes \( R, B, \) and \( M \), respectively.

**Lemma 4.3.** For every triangle \( T \) in \( G, T \cap M = \emptyset \). In particular, \( e(G[M]) \leq m^2/4 \), and for any edge \( \{u, v\} \) contained in \( R \) or \( B \), \( N(u) \cap N(v) \cap M = \emptyset \).

**Proof.** Assume for the sake of contradiction there exists a triangle \( T \) which has a vertex from \( M \), call it \( x \). We may assume the triangle is monochromatic, as \( G \) is handle and nonmonochromatic triangle free. As \( x \) is incident on both a red and a blue edge, regardless of the color of \( T \), we may extend it to a handle through \( x \). The bound on \( e(G[M]) \leq m^2/4 \) immediately follows by Mantel’s theorem. Adjacent vertices in \( R \) or \( B \) cannot have a common neighborhood in \( M \), as this gives a triangle with a vertex inside \( M \). \( \square \)

**Lemma 4.4.** If a triangle free graph \( H \) on \( m \) vertices has an independent set of size \( d \) where \( d \geq m/2 \), then \( H \) has at most \( d(m - d) \) edges.

**Proof.** If \( d \geq m/2 \) then the quantity \( d(m - d) \) is decreasing in \( d \), so without loss of generality we may assume that the independence number of \( H \) is \( d \). Let \( I \) be an independent set of size \( d \). Since \( H \) is triangle-free, the neighborhood of any vertex is an independent set, and so \( d(v) \leq d \) for all \( v \). Since \( I \) is an independent set, any edge must have at least one vertex not in \( I \). Therefore

\[
e(H) \leq \sum_{v \notin I} d(v) \leq \sum_{v \notin I} d = (m - d) \cdot d.
\]

Our approach is to modify \( G \) into \( G' \), decreasing neither the red nor the blue edges, until \( G' \) has an easy-to-understand structure. Using the bound on \( e(G[M]) \) from Lemmas 4.3 and 4.4, we will be able to bound the density of \( G' \) from above.

For \( v \in V \), let \( G \cup \text{clone}(v) \) denote the graph obtained by adding to \( G \) a new vertex \( v' \) with \( N(v') = N(v) \), and the color of an edge \( \{v', x\} \) is the same as that of \( \{v, x\} \). In particular, \( v \) and \( v' \) are not adjacent in \( G \cup \text{clone}(v) \).

**Lemma 4.5.** For any bicolored \( H \) avoiding handles and nonmonochromatic triangles, and \( v \in V(H), H \cup \text{clone}(v) \) still avoids handles and nonmonochromatic triangles.

**Proof.** It is obvious that any new forbidden substructure in \( H \cup \text{clone}(v) \) has to use both \( v \) and the clone, \( v' \). Since \( v \) and \( v' \) are nonadjacent, this structure can only be a handle. Yet, for any monochromatic triangles \( v \) takes part in, \( v' \) is adjacent to all three vertices of this triangle in the same color. Hence there cannot be handles in the new graph either. \( \square \)

Fix a permutation of \( V(G) \) arbitrarily and set \( G_0 := G \). While there is still a nonedge \( \{u, v\} \) contained in either \( R \) or \( B \) such that \( N(u) \neq N(v) \) in \( G_i \) we perform the following operation to obtain \( G_{i+1} \): we take \( x \in \{u, v\} \) with larger degree (if the degree of \( u \) equals the degree of \( v \), choose the vertex that is ordered first by the permutation), delete the lower degree vertex to
acquire $G'_i$, and set $G_{i+1} := G'_i \cup \text{clone}(x)$. When the procedure terminates, we set the final graph to be $G'$.

By Lemma 4.5, $G'$ does not contain any forbidden substructure, and since we perform the cloning within $R$ and $B$, neither the red nor the blue edge count decreases at any point during the procedure.

**Observation 1.** In $G'$, nonadjacency is an equivalence relation in $R$ and $B$. Thus, both $G'[R]$ and $G'[B]$ are complete multipartite graphs.

Indeed, if $\{x,y\}$ and $\{y,z\}$ are nonedges in $R$ (or $B$), by virtue of the procedure, $N(x) = N(y) = N(z)$, meaning that there cannot be an edge between $x$ and $z$. Since vertices in each maximal independent set of $R$ (and $B$) are clones of each other, their neighborhoods in $M$ are also identical, that is, each equivalence class of nonedges in $R$ (and $B$) induces a complete bipartite graph between $R$ (or $B$) and $M$. Call $R_1, ..., R_k$ be the parts of $R$, and $B_1, ..., B_l$ the parts of $B$. The next observation follows from Lemma 4.3.

**Observation 2.** For any $i \neq j$, the sets $N(B_i) \cap N(B_j) \cap M$ and $N(R_i) \cap N(R_j) \cap M$ are empty.

This observation implies that every vertex in $M$ sends edges to at most one $R_i$ and at most one $B_j$. Using this, we may refine the structure even more. Without loss of generality, assume that $|R_1| \geq \cdots \geq |R_k|$ and $|B_1| \geq \cdots \geq |B_l|$. Define

$$d := \max_{ij} |N(R_i) \cap M|, |N(B_j) \cap M|.$$  

By Lemma 4.3, $M$ contains an independent set of size $d$. Since $R$ and $B$ induce complete multipartite graphs, we have

$$e(R) = \binom{r}{2} - \sum_{i=1}^{k} \left( \frac{|R_i|}{2} \right) \leq \binom{r}{2} - \binom{|R_1|}{2} - \binom{|R_2|}{2}, \quad (2)$$

$$e(B) = \binom{b}{2} - \sum_{i=1}^{l} \left( \frac{|B_i|}{2} \right) \leq \binom{b}{2} - \binom{|B_1|}{2} - \binom{|B_2|}{2}. \quad (3)$$

Because each vertex in $M$ sends edges to at most one independent set in $R$ and at most one independent set in $B$, and because $|R_1| \geq \cdots \geq |R_k|$ and $|B_1| \geq \cdots \geq |B_l|$, we have

$$e(R, M) + e(B, M) \leq d(|R_1| + |B_1|) + (m - d)(|R_2| + |B_2|). \quad (4)$$

Consider the following graph $G''$ (see Figure 1). Let $V(G'') = R' \bigcup M' \bigcup B'$, where $|R'| = r$, $|B'| = b$, and $|M'| = m$. Let $M' = M_1 \bigcup M_2$ where $M_1 = \max\{m/2, d\}$. We will define the edges of $G''$ in $R'$ and $B'$, those with one endpoint in $M$ and then those with both endpoints in $M'$. 
Let $R'$ induce a complete multipartite graph with parts of size $R_1$, $R_2$, and all the rest of size 1. Let $B'$ induce a complete multipartite graph with partite sets of sizes $B_1$, $B_2$, and all the rest of size 1.

Place a complete bipartite graph between the independent set of size $R_1$ and $M_1$, between the independent set of size $R_2$ and $M_2$, between the independent set of size $B_1$ and $M_1$, and between the independent set of size $B_2$ and $M_2$.

Let $M_1$ and $M_2$ induce a complete bipartite graph in $M'$.

By Lemma 4.3, we have that $e_{M}(M') \leq n^2/4$. If $d \geq m/2$, by Lemma 4.4 we have that $e(M) \leq d(m - d)$. Therefore, by Equations (2)–(4), we have $e(G'') \geq e(G')$, and so it suffices to show that $e(G'') \leq n^2/4$.

We claim that $|R'|$ and $|B'|$ must each be bounded above by $n/2$. Otherwise, say $|R'| > n/2$, we would have $M \cup B < n/2$ and so that number of blue edges in $G$ would be less than $\left(\frac{n}{2}\right)$. We will show that $e(G'') \leq n^2/4$ for any possible sizes of the parts, subject to the constraint that $|R'|, |B'| \leq n/2$. Assume that $G''$ has the maximum number of edges possible. Since we are only interested in an asymptotic result, we will work with densities to ease the calculations. Let $r' = |R'|/n = r/n, b' = |B'|/n = b/n, \eta = |R_1|/n, b_1 = |B_1|/n$, and $m_i = |M_i|/n$. We will suppress negligible error terms, so for example if $|R'| \sim n/4$ we will write that $r' = \frac{1}{4}$ instead of $r' = \frac{1}{4} + o(1)$. Note that as long as $|R'| \neq |R_1|$ and $|B'| \neq |B_1|$, the sets $R_1$ and $B_1$ will be nonempty, since they are the sizes of the largest independent sets in $R$ and $B$ in $G$.

**Case 1.** $b' = 0$ or $r' = 0$: Without loss of generality, assume $b' = 0$. If we have $r' = r_1 - r_2 = 0$ and $b' = 0$, then $G''$ is $o(n^2)$ edges from being bipartite, and hence we have $e(G'') \leq n^2/4$. Otherwise $R' \setminus (R_1 \cup R_2)$ is nonempty. Moving a vertex from $R \setminus (R_1 \cup R_2)$ to $R_i$ changes the number of edges in $G''$ by $(m_i - \eta)n$. Since $e(G'')$ is maximized, we must have

![Figure 1](https://example.com/f1.png) Graph $G''$. Solid indicates red, dotted indicates blue color. There are no restrictions on color of edges between $M_1$ and $M_2$, they can be red or blue. All labeled parts are independent sets, and the leftmost and rightmost sets are red and blue cliques, respectively. The leftmost three sets union to $R'$, and the rightmost three sets union to $B'$. 


\(m_1 = r_1\) and \(m_2 = r_2\). But since \(b' = 0\) and \(r' \leq 1/2\), we have \(m_1 + m_2 \geq 1/2\) and so \(n_1 + r_2\) must equal 1/2. But now again \(G''\) is \(o(n^2)\) edges from being bipartite and we are done.

**Case 2.** \(r' > 0, b' > 0\): If \(r' - r_1 - r_2 = 0\), then relabeling vertices in \(R_1\) to be in \(M_2\) and vertices in \(R_2\) to be in \(M_1\) changes the graph by \(o(n^2)\) edges, and reduces to Case 1. Similarly, if \(b' - b_1 - b_2 = 0\), we may reduce it to Case 1. So we may assume that both \(r' - r_1 - r_2\) and \(b' - b_1 - b_2\) are positive. Without loss of generality, assume that \(r' \geq b'\).

Now, moving a vertex from \(R_1\) to \(R_2\) (recall that both are nonempty) changes the number of edges by \((n - r_1 + m_2 - m_1)n\) and moving a vertex from \(B_1\) to \(B_2\) changes the number of edges by \((b_1 - b_2 + m_2 - m_1)n\). Hence we have

\[
n_1 - r_2 = m_1 - m_2 = b_1 - b_2.
\]

If \(b' = 1/2\), then \(r' = 1/2\) (since we assumed \(r' \geq b'\)) and \(m = 0\), and \(e(G'') \leq 2\left(\frac{n}{2}\right)^2\). So assume that \(b' < 1/2\), and in particular \(B'\) could gain vertices without violating the constraint. Then moving a vertex from \(M_2\) to \(B_1\) changes \(e(G'')\) by \((b' - b_1 - b_2 - r_2)n\) and moving a vertex from \(M_1\) to \(B_2\) changes \(e(G'')\) by \((b' - b_1 - b_2 - n_1)n\). Thus we have

\[
n_1 = r_2 = b' - b_1 - b_2,
\]

which implies that \(m_1 = m_2\) and \(b_1 = b_2\). Now, moving a vertex from \(B'\setminus (B_1 \cup B_2)\) to \(B_1\) changes \(e(G'')\) by \((m_1 - b_1)n\) and moving from \(R'\setminus (R_1 \cup R_2)\) to \(R_i\) changes \(e(G'')\) be \((m_i - n_1)n\). Therefore

\[
r_1 = r_2 = b_1 = b_2 = m_1 = m_2 = b' - b_1 - b_2.
\]

Finally, moving a vertex from \(B'\setminus (B_1 \cup B_2)\) to \(R'\setminus (R_1 \cup R_2)\) (if it is possible without violating the constraint), changes the number of edges by \((r - b)n\). We therefore must have that either \(r' = b'\) or \(r = 1/2\).

When \(r' = b' < 1/2\), moving a vertex from \(M_2\) to \(R_1\) or \(M_1\) to \(R_2\) shows that \(b_1 = b_2 = r' - n_1 - r_2\). In this case we have that all eight parts have the same size, and \(e(G'') \sim \frac{3}{16}n^2\).

Otherwise, \(r' = 1/2\) and so \(m_1 + m_2 + b' = 1/2\). It follows that the seven parts which are the same size all have size \(\frac{n}{10}\), and in this case \(e(G'') \sim \frac{4}{25}n^2\).

In all cases, we have that \(e(G) \leq e(G'') \leq n^2/4\), completing the proof of Lemma 4.2.

### 5 | DISCUSSION

Our Theorem 1.1 gives an upper bound on the \(\varepsilon\)-balanced Turán-type number of a family as a function of the \(\varepsilon\)-balanced Ramsey-type number of a complete graph, and blow-ups of \(\varepsilon\)-balanced Ramsey graphs can also give lower bounds on the \(\varepsilon\)-balanced Turán-type number of a family. In general, we do not have any methods for determining the exact values of either the \(\varepsilon\)-balanced Ramsey-type function or the \(\varepsilon\)-balanced Turán-type function, and it would be very interesting to understand when the extremal graphs for the \(\varepsilon\)-balanced Turán-type function are given by blow-ups of \(\varepsilon\)-balanced Ramsey graphs. In particular, we believe this to be true
asymptotically when avoiding a non-monochromatic $K_k$ with $k$ sufficiently large (Conjecture 1).

**Problem 1.** Prove or disprove Conjecture 1.

One potential difficulty in proving Conjecture 1 is that the equality does not hold for $k \in \{3, 4\}$. The theorem of DeVos, McDonald, and Montejano [12], that $\text{ex}(n, K_3) = 2/3$, shows that a $1/2$-balanced complete bipartite graph is not extremal for avoiding a non-monochromatic triangle. For $k = 4$, consider the following construction due to Urschel [20]. Let $A$ be a clique and let $B, C, D$ be the partite sets of a complete 3-partite graph. Join $A$ to the graph induced by $C \cup D$. Any $K_4$ in this graph must use at least one vertex from $A$. Therefore, to avoid a nonmonochromatic $K_4$, color blue all edges within $A$, between $A$ and $C \cup D$, and between $C$ and $D$. Color the edges between $B$ and $C \cup D$ so that the whole graph is $1/2$-balanced (note, this puts constraints on the sizes of $A, B, C, D$). Optimizing over the sizes of the parts gives that $\text{ex}(K_4) > 0.67508 > \frac{2}{3}$.

**Problem 2.** Determine the exact value of $\text{ex}(K_4)$.

In a more general setting, removing the constant factor in our Theorem 1.1 would be quite interesting, as already stated in Section 1. Denoting the family of $t$-unavoidable graphs by $\mathcal{I}_t$, Theorem 1.1 says $\text{ex}(\epsilon, \mathcal{I}_t) \leq 1 - \frac{1}{O(t)}$, where the suppressed constant term depends on how close $\epsilon'$ is to $\epsilon$. On the other hand, taking a blow-up of $\epsilon$-balanced Ramsey graphs shows that $\text{ex}(\epsilon, \mathcal{I}_t) \geq 1 - \frac{1}{R(\epsilon, t) - 1}$.

Removing the constant term that appears in the upper bound for $\text{ex}(\epsilon, \mathcal{I}_t)$ would show that blow-up of Ramsey graphs is asymptotically the densest construction avoiding unavoidable graphs. One way of achieving this would be by removing the constant factor $C$ in Lemma 2.1.

**Problem 3.** Determine if for any $\epsilon' < \epsilon$, any sufficiently large graph with density $1 - \frac{1}{k}$ contains an $\epsilon'$-balanced graph on $(1 - o(1))k$ vertices.

We mention a conjecture of a similar flavor by Diwan and Mubayi [14]. They conjecture that given any red–blue edge-coloring of a $K_k$, then any union of a red and blue graph each with at least $\text{ex}(n, K_k)$ edges (the union will be a multigraph) will contain this colored $K_k$. They prove their conjecture when the coloring contains a monochromatic clique on $k - 1$ vertices. They also ask whether an Erdős–Simonovits–Stone Theorem is possible in this context. Our Theorem 3.4 can give some information in this setting, but we were not able to define a chromatic number-like parameter that would satisfactorily answer their question.

In the present paper, we investigated the extremal problem for bicolored cycles, restricting our attention to 1/2-balanced host graphs. It would be interesting to study this problem in a wider range. In particular, the following problem is of interest.

**Problem 4.** Determine $\text{ex}(\epsilon, C_k)$ when $\epsilon < \frac{1}{2}$.

Finally, in this paper we studied what happens when one forbids a color-consistent family of graphs. It would be interesting to understand what happens when forbidding, for example, a nontrivial subset of the family of non-monochromatic complete graphs or cycles.
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APPENDIX A

A.1 | Proof of Theorem 1.2

Here we provide the details for the proof of Theorem 1.2 which we omitted in the paper, as it closely resembles the original proof of Theorem 2.4.

Proof of Theorem 1.2. Let $H$ be an inevitable graph with maximum degree $\Delta$. Given $\varepsilon$, and $H$, we will fix a regularity parameter $\varepsilon_0$, whose value is chosen later. We also fix an $\varepsilon$-balanced graph $G$ with $\nu(G) \geq C \cdot \nu(H)$, where $C$ is a constant that depends on $\varepsilon_0$ and $\Delta$, the exact value to be specified later. Our goal is to find a copy of $H$ in $G$.

We apply Corollary 2.7 to $G$ with parameter $\varepsilon_0$. We delete all edges incident to junk set (at most $\varepsilon_0 n^2$) and between pairs which are not $\varepsilon_0$-regular (at most $\varepsilon_0^2 n^2/20$), and all red edges between parts with red density less than $\varepsilon_10$ (at most $\varepsilon_0^2 n^2/2$), and similarly for blue. In total, we have lost at most (being generous) $\varepsilon_0 (5 + 4\varepsilon_0) n^2$ edges. In particular, choosing $\varepsilon_0$ small will ensure that the graph is still at least $\varepsilon_0^2$-balanced. Call this resulting graph $G'$. Note that $G'$ has colored edges between every $\varepsilon_0$-regular pair, and for every such pair, at least one of the color classes will have density greater than $\varepsilon_10$.

We use Lemma 2.6 with the auxiliary graph $\mathcal{R}$ whose vertices $P_i (i \in [k])$ correspond to the $k$ parts in the cluster graph to find a coloring of $\mathcal{R}$ where the color of a pair indicates at least $\varepsilon_0/10$ density of that color in the pair. And further, the density of the smaller color class changed at most by $\varepsilon_0^2$. Since we choose $\varepsilon_0$ after $\varepsilon$ is fixed, we can choose $\varepsilon_0$ small enough to bound $\gamma < \varepsilon/100$. Thus, since $G'$ was $\varepsilon/2$ balanced, the coloring of $\mathcal{R}$ will surely be at least $\varepsilon/4$-balanced. And furthermore, we know that the density of this bicolored $\mathcal{R}$ is at least $1 - \varepsilon_0$.

Thus, we can apply our Theorem 1.1 to say there exists a constant $K$ such that for any $t$ which satisfies

$$\varepsilon_0 \leq \frac{1}{K \cdot R(\varepsilon/4, t)}$$

we can find an unavoidable $t$-graph in the color cluster graph. Using bounds on $R(\varepsilon/4, t)$, we obtain that for a constant $K'$, we can set $t \geq K' \log(\varepsilon_0^{-1})$. Note this allows us to make $t$ as large as we want by making $\varepsilon_0$ small. Let us choose $\varepsilon_0$ so that $K'\varepsilon_0^{-1} \geq \Delta + 1$.

We are at the last section of the proof where we just have to embed $H$ into the clusters. Let us say that the unavoidable graph in the cluster graph we found was of Type 1 (the proof is similar if it is of Type 2). As $H$ is inevitable, $H$ has an embedding to a large enough Type 1 graph. In both the left-hand side and right-hand side of this embedding, the maximum degree is at most $\Delta$, hence there exists a partition of each side into $\Delta + 1$ independent sets. Now, we will go through the independent sets and embed each vertex in a greedy fashion to the corresponding cluster, using only $\varepsilon_0$-regularity, and that the density between each pair is at least $\varepsilon/10$.

Our goal is to find an embedding $f : V(H) \rightarrow V(G)$ which respects incidence. We will find the embedding iteratively and greedily. For each vertex $u$ we will maintain a set of “potential” vertices in $G$ such that if $u$ were mapped to any of these vertices, the incidence would be respected. We do this as follows.
For any vertex \( u \in V(H) \), denoted by \( C_{u,i} \), the set of potential vertices in the corresponding cluster to which \( u \) can be embedded after \( i \) vertices have already been embedded. More precisely, \( C_{u,i} \) is composed of vertices \( v \in C_{u,i-1} \) such that for all \( x \in V(H) \) whose embedding to the graph \( f(x) \in V(G) \) is defined by the \( i \)th step \( u \sim x \Rightarrow v \sim f(x) \). This condition ensures that the embedding preserves incidence. We will prove by induction that \( C_{u,i} \) is always larger than \( v(H) \), and thus we may always choose an \( f(u) \) such that \( f \) is both an injection and respects incidence. We set \( C_{u,0} \) to be just the entirety of the cluster we plan to embed \( u \) inside. Say we are in the \( (i+1) \)th stage of the algorithm. Let \( g(u, i) := \# \text{ of neighbors of } u \text{ already embedded before step } i \). Assume inductively that

\[
|C_{u,i}| \geq |C_{u,0}| \cdot \left( \frac{\epsilon}{10} - \epsilon_0 \right)^{g(u,i)}
\]

for all unembedded vertices \( u \). So for some fixed unembedded \( u \), we have a set \( C \) with \( |C| \geq |C_{u,0}| \left( \frac{\epsilon}{10} - \epsilon_0 \right)^{g(u,i)} - v(H) \) that we can choose for \( f(u) \), accounting for vertices from the same cluster that might already have been used. To select \( f(u) \) in a way that will preserve the induction hypothesis, we invoke \( \epsilon_0 \)-regularity between \( C \) and all the \( C_{u',i} \) for \( u' \sim u \) and \( u' \) is not embedded yet. We may do so, as long as \( |C_{u,0}| \epsilon_0 < |C_{u,0}| \left( \frac{\epsilon}{10} - \epsilon_0 \right)^{g(u,i)} - v(H) \). Since \( |C_{u,0}| = v(G)/k \geq C \cdot v(H)/k \) where \( k \) is the number of clusters, we may choose \( \epsilon_0 \) small enough, and then \( C \) large enough (the number of clusters is bounded above depending only on \( \epsilon_0 \)) so that the inequality holds. Thus, we can apply \( \epsilon_0 \) regularity which tells us that all but at most \( \Delta \epsilon_0 \) fraction of \( C \) has at least \( (\frac{\epsilon}{10} - \epsilon_0)|C_{u',i}| \) neighbors in each \( C_{u',i} \) for each unembedded \( u' \). As \( (1 - \Delta \epsilon_0)|C| \geq 1 \), we can choose an \( f(u) \) that preserves our inductive invariant. Hence we will be able to embed all of \( H \) in \( G \).

**A.2 | Proof of Theorem 3.4**

The proof uses the Regularity lemma (Lemma 2.5 and Corollary 2.7) and additionally the following embedding lemma.

**Lemma A.1.** Let \( H \) be some graph with \( \chi(H) \leq n \). For any \( d > 0 \) there exists an \( \epsilon > 0 \) and \( N_0 \) with the following property. Let \( R := \bigsqcup_{i \in [n]} R_i \) be an \( n \)-partite graph where each part \( R_i \) has cardinality at least \( N_0 \) and for all edges \( \{v_i, v_j\} \) of \( H \), the pair \( (R_i, R_j) \) is \( \epsilon \)-regular with density at least \( d \). Then, \( H \) is embeddable in \( R \).

The proof and explicit bounds follow from a simple inductive argument. For example, see Lemma 7.3.2 in Diestel [13]. We use the embedding lemma with bicolored graphs as follows.

**Lemma A.2.** Let \( H \) be a bicolored graph whose uncolored version has \( \chi(H) \leq n \). For any \( d > 0 \) there exists an \( \epsilon > 0 \) and \( N_0 \) with the following property. Let \( R := \bigsqcup_{i \in [n]} R_i \) be a bicolored \( n \)-partite graph where each part \( R_i \) has cardinality at least \( N_0 \) and if \( \{v_i, v_j\} \) is an edge of color \( c \) in \( H \), then the pair \( (R_i, R_j) \) is \( \epsilon \)-regular with \( c \)-density at least \( d \). Then, \( H \) is embeddable (as a colored graph) in \( R \).

We may now start the proof of Theorem 3.4.
Proof. We give a brief outline of the proof first. We take a graph satisfying our assumptions, and apply the colored Regularity lemma (Corollary 2.6) to it. Then we create an auxiliary cluster graph with vertices corresponding to the parts of the regularity partition. We show there is a choice of red and blue edges in this cluster graph such that whenever there is a colored edge in the cluster graph, the pair of vertex sets in the regularity partition corresponding to that edge will have a large density in that color. We also show that with this choice, the resulting cluster graph is $\varepsilon$-balanced and has density greater than $c$. We may then apply Lemma A.2 to find our $t$-blow-up. We now go through the details.

Let $G$ be an $n$ vertex $\varepsilon$-balanced graph with \((c + \delta)\binom{n}{2}\) edges where $\varepsilon$ is chosen as large as possible (so that the smaller color class has exactly $\varepsilon(c + \delta)\binom{n}{2}$ edges). $\varepsilon_0$ will be a positive constant depending on $\varepsilon, \delta, F$ that we can choose small enough for all of the subsequent calculations, and let $d := \frac{\varepsilon_0\delta^2}{4}$. Apply Corollary 2.7 to $G$ with parameter $\varepsilon_0$ to acquire a regularity partition of $G$ with parts $\{R_i\}_{i=1}^k$. We apply the standard clean-up process:

- Delete all edges incident on the junk set. There are at most $\varepsilon_0n^2$ such edges.
- Delete all edges between pairs which are not $\varepsilon_0$-regular. There are at most $\varepsilon_0\left(\begin{array}{c}k \\ \frac{n}{k}\end{array}\right)^2 \leq \frac{\varepsilon_0n^2}{2}$ such edges.
- Delete all red edges between pairs with red density less than $d$. There are at most $\left(\begin{array}{c}k \\ \frac{n}{k}\end{array}\right)d\left(\frac{n}{k}\right)^2 \leq \frac{dn^2}{2}$ such edges.
- Repeat the previous step for blue edges. Again, there are at most $\frac{dn^2}{2}$ such edges.

Call the resulting graph $G'$. Note we deleted at most \(\left(d + \frac{3\varepsilon_0}{2}\right)n^2 \leq \frac{\varepsilon_0\delta^2}{2}\binom{n}{2}\) edges, where the inequality holds for a small enough choice of $\varepsilon_0$. Without loss of generality, call the smaller color class in the graph $G$ blue. We thus know that in $G'$ blue still has density at least $\varepsilon\left(c + \frac{\delta}{2}\right)$ and red still has density at least $(1 - \varepsilon)\left(c + \frac{\delta}{2}\right)$. Denote these quantities $d_{BG}(G')$ and $d_{RG}(G')$.

Consider an auxiliary graph $\mathcal{R}$ whose vertices $P_i$ ($i \in [k]$) correspond to the $k$ parts in the regularity partition. We wish to add colored edges in $\mathcal{R}$ so that a red edge indicates high red density between the $\varepsilon_0$-regular pair in $G'$, and similarly for blue. We wish to ensure this yields an $\varepsilon$-balanced graph with density greater than $c$. To do so, we use Lemma 2.6. We can thus fix a coloring of the edges of $\mathcal{R}$ where the color of a pair indicates at least $d$ density of that color in the pair. Furthermore, we know that the density of red and blue in the coloring of $\mathcal{R}$ is at most $\gamma$ away from their respective densities in $G'$ where $\gamma := \max(\gamma_R, \gamma_B)$ (these parameters are defined in the statement of Lemma 2.6).

If necessary, delete red edges until the red density in $\mathcal{R}$ is at most $d_{R(G')}$. Further, if necessary, delete more edges from the appropriate color class until the resulting graph on $\mathcal{R}$ is $\varepsilon$-balanced. As $d_{B(G')} = \frac{\varepsilon}{1 - \varepsilon}d_{R(G')}$, we delete at most $\frac{(1 - \varepsilon)}{\varepsilon}\gamma\left(\begin{array}{c}k \\ 2\end{array}\right)$ edges in this step. Call the resulting graph $\mathcal{R}'$. 
Now, note that we may choose $\varepsilon_0$ small enough to control $\gamma$ so that $\mathcal{R}'$ has more than $(c + \delta/4)\binom{k}{2}$ edges. Since $\mathcal{R}'$ is $\varepsilon$-balanced, we have that for $\varepsilon_0$ small enough (so that $\mathcal{R}'$ has enough vertices), by definition of $c$, $\mathcal{R}'$ must contain a copy of some $F \in \mathcal{F}$. Since $\varepsilon_0$ may also be chosen small enough with respect to $d$, we can apply Lemma A.2 with the copy of $F$ in $\mathcal{R}'$ and conclude that $G'$ contains a $t$-blow-up of $F$, completing the proof. □