Solitons of the (2+2)-dimensional Toda lattice

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Abstract
We use the generalized Cauchy matrix approach to derive the N-soliton solutions for the (2+2)-dimensional Toda lattice.

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1. Introduction
In this note, we discuss the soliton solutions for the (2+2)-dimensional Toda lattice.

The Toda model, one of the best-studied integrable systems which has applications in various fields of mathematics and physics, was introduced in [1] as a nonlinear chain, i.e. a (1+1)-dimensional system with one continuous (time) and one discrete (index) variables. One of the questions that has attracted much interest during its fifty-year history is to find generalizations of this model, first of all, to describe problems in high dimensions (dynamics of two- and three-dimensional lattices, coupled fields in two- or three-dimensional spaces, etc). There are several known three-dimensional extensions of the Toda model: the (2+1)-dimensional (2 continuous + 1 discrete variable) chains [2–7], the (1+2)-dimensional (1 continuous + 2 discrete variables) lattices [3, 8–10], completely discrete 3-dimensional lattice [7]. However, the attempts to proceed to higher dimensions demonstrate that almost all methods of generalization which preserve the integrability (with, probably, the only exception [11]) lead either to the models with extra fields [12, 13] (the ‘matter type fields’, in the terminology of [12]) or to the equations that lose some typical features of the integrable systems, like trilinear equations studied in [14–16].

The equations we study in this paper can be written as

$$\frac{\partial^2 u}{\partial x \partial y} = e^{\Delta_1 u} - e^{\Delta_2 u}$$ (1.1)

where $u$ is a function of two continuous ($x$ and $y$) and two discrete ($n_1$ and $n_2$) variables,
and $\Delta_1$ and $\Delta_2$ are the one-dimensional second-order difference (discrete Laplace) operators,

$$
\begin{align*}
\Delta_1 u(n_1, n_2) &= u(n_1 + 1, n_2) - 2u(n_1, n_2) + u(n_1 - 1, n_2), \\
\Delta_2 u(n_1, n_2) &= u(n_1, n_2 + 1) - 2u(n_1, n_2) + u(n_1, n_2 - 1).
\end{align*}
$$

(1.3)

Here, we do not discuss the questions related to the integrability of (1.1), such as the zero-curvature (or Lax) representation, the conservation laws, the Hamiltonian structure etc. We restrict ourselves, similar to the cited works above [7, 14, 16], to the problem of finding some class of particular solutions. We formulate an ansatz and use it to derive the N-soliton solutions for (1.1).

2. Soliton matrices

In what follows, we use the soliton matrices that were studied in [17]. We define the soliton tau-functions as the determinants

$$
\tau = \det \left| 1 + A\bar{A} \right|
$$

(2.1)

of the matrices defined by

$$
\begin{align*}
MA - A\bar{M} &= \langle \alpha | a | , \\
M\bar{A} - \bar{A}M &= \langle \bar{\alpha} | \bar{a} |
\end{align*}
$$

(2.2)

where $M$ and $\bar{M}$ are constant $N \times N$ diagonal matrices, $| \alpha \rangle$ and $| \bar{\alpha} \rangle$ are constant $N$-column vectors while $\langle a |$ and $\langle \bar{a} |$ are $N$-component rows that depend on the coordinates describing the model. It should be noted that the overbar does not indicate the complex conjugation: we consider all matrices, rows and columns to be real.

We define shifts $T_\zeta$ as

$$
\begin{align*}
T_\zeta \langle a | &= \langle a | H_\zeta , \\
T_\zeta \langle \bar{a} | &= \langle \bar{a} | \bar{H}_\zeta
\end{align*}
$$

(2.3)

with

$$
\begin{align*}
H_\zeta &= (M - \zeta) (M + \zeta)^{-1}, \\
\bar{H}_\zeta &= (\bar{M} + \zeta) (\bar{M} - \zeta)^{-1}
\end{align*}
$$

(2.4)

(we do not indicate the unit matrix explicitly and write $M - \zeta$ instead of $M - \zeta 1$, etc) which determines the action of the shifts on the matrices $A$ and $\bar{A}$,

$$
\begin{align*}
T_\zeta A &= A H_\zeta , \\
T_\zeta \bar{A} &= \bar{A} \bar{H}_\zeta
\end{align*}
$$

(2.5)

and, hence, the tau-functions $\tau$.

Note that the shifts used in this work are different from the ones used in [17]. As a result, the tau-functions (2.1) satisfy a different set of equations. What is important is that in the case of (2.5) we do not have the simple three-point equations that were, in some sense, the base of most of the results presented in [17]. Thus, the calculations of this work are somewhat more cumbersome, with more emphasis upon the algebraic properties of the matrices (2.2).
In what follows, we study the results of the combined action of several shifts, which we denote as
\[ T_{\xi \eta} = T_{\xi} T_{\eta}, \quad T_{\xi \eta \zeta} = T_{\xi} T_{\eta} T_{\zeta}, \quad \ldots \] (2.6)
or, by means of the set notation, as
\[ X = \{ \xi_1, \ldots, \xi_N \}, \quad \mathcal{T}_X = \prod_{n=1}^{N} T_{\xi_n}. \] (2.7)

After some simple algebra, one can present the shifted tau-function as
\[ T_{\xi} \tau = E_{\xi \xi} \tau \] (2.8)
where quantities \( E_{\xi \eta} \) are defined by
\[ E_{\xi \eta} = 1 + (\xi + \eta) \langle b_\xi | GA | \beta_\eta \rangle \] (2.9)
with
\[ G = (1 + AA)^{-1} \] (2.10)
and
\[ \langle b_\xi | = \langle \bar{a} | (\bar{M} - \xi)^{-1}, \quad | \beta_\eta \rangle = \langle \bar{M} + \eta)^{-1} | \bar{a} \rangle \] (2.11)
(see appendix A).

Equation (2.8) can be generalized to
\[ \frac{\mathcal{T}_X \tau}{\tau} = \frac{D_X}{C_X} \] (2.12)
where \( D_X \) and \( C_X \) are the determinants given by
\[ D_X = \det \left| \frac{E_{\xi \eta}}{\xi + \eta} \right|_{\xi, \eta \in X}, \quad C_X = \det \left| \frac{1}{\xi + \eta} \right|_{\xi, \eta \in X} \] (2.13)
(see appendix A).

Equation (2.12) can be used to derive various identities for \( \mathcal{T}_X \tau \) with different \( X \). The ones that we need in this paper can be formulated as the following two propositions.

**Proposition 1.** The tau-functions (2.1) satisfy the Miwa (discrete BKP) equation
\[ \tau T_{\alpha \beta \gamma} \tau = \Gamma_{\alpha \beta \gamma} (T_{\alpha} \tau)(T_{\beta} \gamma \tau) + \Gamma_{\beta \alpha \gamma} (T_{\beta} \tau)(T_{\alpha} \gamma \tau) + \Gamma_{\gamma \alpha \beta} (T_{\gamma} \tau)(T_{\alpha} \beta \tau) \] (2.14)
with constants \( \Gamma \) defined by
\[ \Gamma_{\xi \gamma} = \prod_{\eta \in \gamma} \frac{\xi + \eta}{\xi - \eta}. \] (2.15)

We outline a proof of this statement in appendix A.

**Proposition 2.** Each solution of equation (2.14) with (2.15) delivers a solution for the second equation of the discrete BKP hierarchy
\[ \tau T_{\alpha \beta \gamma} \tau = \Gamma_{\alpha \beta \gamma} (T_{\alpha} \tau)(T_{\beta} \gamma \tau) + \Gamma_{\beta \alpha \gamma} (T_{\beta} \tau)(T_{\alpha} \gamma \tau) + \Gamma_{\gamma \alpha \beta} (T_{\gamma} \tau)(T_{\alpha} \beta \tau) \] (2.16)
with constants $\Gamma$ defined by

$$
\Gamma_{X,Y} = \prod_{\xi \in X} \prod_{\eta \in Y} \frac{\xi + \eta}{\xi - \eta}.
$$

(2.17)

We prove this statement in appendix B.

Finally, introducing the operators

$$
\mathbb{D}_\zeta = \lim_{\alpha \to \zeta} \frac{1}{\varepsilon_{\alpha, \zeta}} \left( \tau_{\alpha} \tau_{\zeta}^{-1} - 1 \right)
$$

(2.18)

with

$$
\varepsilon_{\xi, \eta} = \frac{\xi - \eta}{\xi + \eta}.
$$

(2.19)

(note that $\varepsilon_{\alpha, \zeta} \to 0$ as $\alpha \to \zeta$) one can derive from equation (2.16) and definition (2.17) the following result:

**Proposition 3.** The tau-functions (2.1) satisfy the bilinear equation

$$
\tau (\mathbb{D}_\lambda \mathbb{D}_\mu \tau) - (\mathbb{D}_\lambda \tau) (\mathbb{D}_\mu \tau) = \Gamma_{\lambda, \mu}^2 \left( \mathbb{T}_{\lambda \mu} \tau \mathbb{T}_{\lambda \mu}^{-1} \tau - \mathbb{T}_{\lambda \mu} \mathbb{T}_{\lambda \mu}^{-1} \tau \mathbb{T}_{\lambda \mu} \tau \right) - \left( \Gamma_{\lambda, \mu}^2 - \Gamma_{\lambda, \mu}^{-2} \right) \tau^2
$$

(2.20)

with $\mathbb{T}_\zeta = \mathbb{T}_\zeta^{-1}$.

This result gives us a possibility to obtain a family of particular solutions for (1.1) by some simple algebraic calculations.

### 3. N-soliton solutions

Let us compare equation (2.20) with fixed $\lambda$ and $\mu$,

$$
\lambda, \mu = \text{constant}
$$

(3.1)

(which we consider as two parameters of our solution), and the bilinear form of (1.1), which can be obtained by the substitution $u = \ln \omega$,

$$
\omega (n_1, n_2) \frac{\partial^2 \omega (n_1, n_2)}{\partial x \partial y} = \frac{\partial \omega (n_1, n_2)}{\partial x} \frac{\partial \omega (n_1, n_2)}{\partial y} - \omega (n_1, n_2 - 1) \omega (n_1 + 1, n_2) - \omega (n_1, n_2 - 1) \omega (n_1, n_2 + 1),
$$

(3.2)

where the dependence of $\omega$ on $x$ and $y$ is not indicated explicitly, and see how one can modify $\tau$, defined in section 2, to transform it into a solution for (3.2).

First, it is easy to note that the first two terms in the right-hand side of (2.20) coincide, up to the constants, with the right-hand side of (3.2) provided we introduce the dependence on $n_1$ and $n_2$ in such a way that the translations $n_1 \to n_1 + 1$ and $n_2 \to n_2 + 1$ lead to the same result as application of the shifts $\mathbb{T}_{\lambda} \mathbb{T}_{\mu}^{-1}$ and $\mathbb{T}_{\lambda} \mathbb{T}_{\mu}$. This is easy to achieve by introducing the $(n_1, n_2)$-dependence of the rows $\langle a |$ and $\langle \bar{a} |$ (and hence of the matrices $A$ and $\bar{A}$) as follows:

$$
\langle a | = \langle a (n_1, n_2) | = \langle c | H_{\lambda}^{n_1 + m_2} H_{\mu}^{n_2 - m_1},
$$

$$
\langle \bar{a} | = \langle \bar{a} (n_1, n_2) | = \langle \bar{c} | H_{\lambda}^{n_1 + m_2} H_{\mu}^{n_2 - m_1}.
$$

(3.3)
with constant \( \langle c | \) and \( \langle \bar{c} | \).

Next, it should be noted that the operators \( \mathbb{D}_\xi \) defined in (2.18) are, in fact, differential operators. Thus, it is possible to introduce the \( x \)- and \( y \)-dependence of the rows \( \langle a | \) and \( \langle \bar{a} | \) (and hence of the matrices \( A \) and \( \bar{A} \)) so that the action of \( \mathbb{D}_\lambda \) and \( \mathbb{D}_\mu \) defined in terms of the \( T \)-shifts lead to the same results as the differentiating with respect to \( x \) and \( y \). Applying \( \mathbb{D}_\xi \) to \( \langle a | \) and \( \langle \bar{a} | \),

\[
\mathbb{D}_\xi \langle a | = \langle a | L_\xi, \quad \mathbb{D}_\xi \langle \bar{a} | = \langle \bar{a} | L_\bar{\xi}
\]

(3.4)

with

\[
L_\xi = \lim_{\alpha \to \xi} \frac{\alpha + \xi}{\alpha - \xi} \left( H_\alpha H_\xi^{-1} - 1 \right) = H_\xi - H_\xi^{-1},
\]

\[
\bar{L}_\xi = \lim_{\alpha \to \bar{\xi}} \frac{\alpha + \xi}{\alpha - \xi} \left( H_\alpha H_\xi^{-1} - 1 \right) = \bar{H}_\xi - \bar{H}_\xi^{-1}
\]

(3.5)

and taking

\[
\langle a | = \langle a(x,y,n_1,n_2) | = \langle a(n_1,n_2) | \exp (xL_\lambda + yL_\mu),
\]

\[
\langle \bar{a} | = \langle \bar{a}(x,y,n_1,n_2) | = \langle \bar{a}(n_1,n_2) | \exp (x\bar{L}_\lambda + y\bar{L}_\mu)
\]

(3.6)

we ensure \( \mathbb{D}_\lambda \langle a | = \frac{\partial}{\partial x} \langle a | \) and \( \mathbb{D}_\mu \langle a | = \frac{\partial}{\partial y} \langle a | \) with similar result for \( \langle \bar{a} | \).

Finally, one has to take into account the factors \( \Gamma_{\lambda,\mu}^{\pm 2} \) in the first two terms and to ‘eliminate’ the last two terms in the right-hand side of (2.20). This can be done by introducing a function \( \varphi \),

\[
\omega = e^{\varphi} \tau,
\]

(3.7)

which satisfies

\[
\Delta_1 \varphi = 2 \ln |\Gamma_{\lambda,\mu}|,
\]

\[
\Delta_2 \varphi = -2 \ln |\Gamma_{\lambda,\mu}|,
\]

\[
\frac{\partial^2 \varphi}{\partial x \partial y} = \Gamma_{\lambda,\mu}^{2} - \Gamma_{\bar{\lambda},\bar{\mu}}^{2}.
\]

(3.8)

Now we can formulate the main result of this paper.

**Theorem 1.** The \( N \)-soliton solutions for the \((2+2)\)-dimensional Toda lattice (1.1) can be written as

\[
u(x,y,n_1,n_2) = \varphi(x,y,n_1,n_2) + \ln \det [1 + A(x,y,n_1,n_2) \bar{A}(x,y,n_1,n_2)]
\]

(3.9)

where

\[
\varphi(x,y,n_1,n_2) = 8xy \frac{\lambda \mu (\lambda^2 + \mu^2)}{(\lambda^2 - \mu^2)^2} + (n_1^2 - n_2^2) \ln \frac{\lambda + \mu}{\lambda - \mu},
\]

(3.10)

the matrices \( A(x,y,n_1,n_2) \) and \( \bar{A}(x,y,n_1,n_2) \) are given by

\[
A(x,y,n_1,n_2) = A_0 \exp (xL_\lambda + yL_\mu) H_{\lambda}^{n_1+n_2} H_{\mu}^{n_1-n_2},
\]

\[
\bar{A}(x,y,n_1,n_2) = \bar{A}_0 \exp (x\bar{L}_\lambda + y\bar{L}_\mu) \bar{H}_{\lambda}^{n_1+n_2} \bar{H}_{\mu}^{n_1-n_2}
\]

(3.11)

with the matrices \( H_\lambda, H_\mu, L_\lambda \) and \( L_\mu \) defined in (2.4) and (3.5), and the elements of \( A_0 \) and \( \bar{A}_0 \).
given by
\[
(A_0)_{j,k} = \frac{c_k}{M_j - M_k}, \quad j, k = 1, ..., N, \tag{3.12}
\]
\[
(\bar{A}_0)_{j,k} = \frac{\bar{c}_k}{M_j - M_k}, \quad j, k = 1, ..., N. \tag{3.13}
\]

Here, \(\lambda, \mu, M_j, M_j, c_j, \bar{c}_j (j = 1, ..., N)\) are arbitrary constants that play the role of parameters in (3.9).

Note that we have put all components of the columns \(|\alpha\rangle\) and \(|\bar{\alpha}\rangle\) equal to the unity, which can be done without loss of generality by redefining the constants \(c_k\) and \(\bar{c}_k\) (the components of the rows \(\langle c |\) and \(\langle \bar{c} |\)). Also, we have written \(\varphi\) as the simplest solution of (3.8). One can, in principle, ‘generalize’ (3.10) by adding two functions of one variable, \(\varphi_1(x)\) and \(\varphi_2(y)\), as well as terms proportional to \(n_1, n_2\) and \(n_1n_2\), which is a manifestation of the trivial symmetries of (1.1).

Of 4\(N\) + 2 constants mentioned in theorem 1, the parameters \(\lambda\) and \(\mu\) play a special role. They completely determine the background solution \(\varphi\) which completely determines the asymptotic behaviour of \(u\). Even if we ‘forget’ about the background and consider \(\bar{u} = u - \varphi\), then, by changing the values of \(\lambda\) and \(\mu\) with respect to the values of \(M_j\) and \(M_j\), one can control the signs of the elements of the \(L\)- and \(\bar{L}\)-matrices and the moduli (compared with the unity) of the elements of the \(H\)- and \(\bar{H}\)-matrices that, in its turn, determines which elements of the \(A\)- and \(\bar{A}\)-matrices grow or vanish in which sector of the asymptotic region, i.e. the asymptotic properties of \(\bar{u}\).

### 3.1. One-soliton case

Here, we discuss the simplest soliton solutions. First, let us consider the one-soliton case. After assuming \(c_1\bar{c}_1 < 0\) (which ensures absence of singularities) and imposing a technical restriction \((M_1^2 - \lambda^2) (M_1^2 - \lambda^2) (M_1^2 - \mu^2) (M_1^2 - \mu^2) > 0\), (which excludes sign alternating) the function \(u\) can be presented as
\[
u(x, y, n_1, n_2) = \varphi(x, y, n_1, n_2) + \ln \det \left| 1 + e^{2f(x, y, n_1, n_2)} \right| \tag{3.14}
\]
where \(f\) is a linear, with respect to all its arguments, function,
\[
f(x, y, n_1, n_2) = f_0 + k_1x + k_1y + \gamma_1n_1 + \gamma_2n_2. \tag{3.15}
\]
\(f_0\) is an arbitrary constant while the constants \(k_1\), \(k_2\), \(\gamma_1\) and \(\gamma_2\) (which are combinations of \(M_1\), \(M_1\), \(c_1\), \(\bar{c}_1\), \(\lambda\) and \(\mu\) that are not presented here explicitly) satisfy the ‘dispersion relation’
\[
k_1k_2 = \Gamma_{\lambda, \mu}^2 \sinh^2 \gamma_1 - \Gamma_{\lambda, \mu}^{-2} \sinh^2 \gamma_2. \tag{3.16}
\]
It is easy to see that, even if we forget about the background \(\varphi\), the behaviour of the soliton part of the solution, \(u - \varphi\), is different from what is expected of a soliton:
\[
u - \varphi \sim \begin{cases} e^{-2|f|} & \rightarrow 0 \quad \text{as } f \rightarrow -\infty, \\ 2f & \rightarrow \infty \quad \text{as } f \rightarrow +\infty. \end{cases} \tag{3.17}
\]
However, after calculating the second derivatives one arrives at the famed sech-expression:

\[
\frac{\partial^2}{\partial x \partial y} (u - \varphi) = \frac{k_x k_y \cosh^2 f}{\cosh^2 f}.
\]  

(3.18)

Similarly, one can derive the ‘standard’ soliton formulae for the second-order differences:

\[
e^{\Delta_x u} = \Gamma_{\lambda, \mu}^2 \left[ 1 + \frac{\sinh^2 \gamma_1}{\cosh^2 f} \right].
\]

\[
e^{\Delta_y u} = \Gamma_{\lambda, \mu}^{-2} \left[ 1 + \frac{\sinh^2 \gamma_2}{\cosh^2 f} \right].
\]

(3.19)

3.2. Two-soliton case

To demonstrate the structure of the two-soliton solutions we calculate (3.9) for some fixed set of soliton parameters: \( M_1 = 7.25, \bar{M}_1 = 9.25, \hat{c}_1 = 1.0, \hat{c}_1 = -2.0, \ M_2 = 1.75, \ M_2 = 2.25, \)
\( c_2 = 1.0, \bar{c}_2 = -0.5, \lambda = 1 \) and \( \mu = 10 \). To make the plots more clear we present there not
the function \( u \) itself but the second derivative of its soliton part (without the background \( \varphi \)),
\[
\frac{\partial^2}{\partial x \partial y} (u - \varphi),
\]
(3.20)
which in the one-soliton case is given by (3.18).

It is easy to see from figure 1 that with continuous coordinates being fixed we have typical
two-discrete soliton configuration and, correspondingly, with discrete coordinates being fixed
we have typical two-continuous soliton configuration. In both cases, there is superposition of
two solitons in the asymptotical regions with typical soliton shifts in the zones of crossings.

In figure 2 we illustrate the interplay between the discrete and continuous coordinates. The
most obvious effect is the shift of the distributions as a whole. Of course, this effect is not the
only one: the different dependence of different elements of the matrices \( A \) and \( \bar{A} \) can notice-
ably change the value of the determinant in (3.9) and hence the form of the solution.

4. Discussion

As one can see from the above presentation, we have derived the N-soliton solutions using the
well-known construction based on matrices (2.2) which are the so-called ‘almost-intertwin-
ing’ matrices, or matrices that satisfy the ‘rank one condition’ which is a particular case of the
Sylvester equation (see [18–25]).

The procedure we have used can be viewed as a version of the direct linearization method based
on the Cauchy matrices (see [26, 27], the book [28] and references therein). From the viewpoint
of the discrete BKP equation (see proposition 1), our generalization consists in using the product
of the matrices \( A \) and \( \bar{A} \) instead of only one of them, as in [29], where the authors obtained, in the
framework of the direct linearizing transform, the so-called pfaffian solutions [30].

The appearance of the pair \( A \) and \( \bar{A} \) is a characteristic feature of the complex models like
the Ablowitz–Ladik (discrete nonlinear Schrödinger) equations [31] (see section 6 of [17]).
However, the relationship between the complex Ablowitz–Ladik equations and the real Toda
model is not new. In some sense, the result of this paper can be viewed as an extension of the
one of [32] where the correspondence between the \((2+1)\)-dimensional Toda lattice and the
Ablowitz–Ladik hierarchy has been discussed.

Finally, we would like to make a comment on the use of the word ‘soliton’ throughout this
paper. We have employed the typical soliton ansatz: if we expand the determinant in (2.1),
we arrive at the sum of \( \exp \)-functions which is a version of the Hirota ansatz. Thus, from this
viewpoint, we have typical soliton tau-functions and, hence, the solutions presented in this
paper are solitons. On the other hand, \( u(x, y, n_1, n_2) \) given by (3.9) grows in almost all direc-
tions, while the solitons are usually expected to vanish at the infinity or, at least, be bounded
(as dark solitons). Nevertheless, in our opinion, such solutions (constructed of the soliton tau-
functions but unbounded) may still be termed as ‘solitons’ without leading to much confusion.

Appendix A. Proof of proposition 1

Application of (2.2) together with (2.5) and (2.4) leads to
\[
(T_{\zeta} - 1) A \bar{A} = 2 \zeta | \gamma_{\zeta} \rangle \langle b_{\zeta} | \tag{A.1}
\]
which implies
\[ T_\zeta \tau = \det |1 + \bar{A}A + 2\zeta |\bar{\gamma}_\zeta \rangle \langle \beta_\zeta | \] (A.2) 

\[ = \tau \det |1 + 2\zeta G|\bar{\gamma}_\zeta \rangle \langle \beta_\zeta | \] (A.3)

where \( G \) is defined in (2.10) and
\[ |\bar{\gamma}_\mu \rangle = A |\bar{\beta}_\mu \rangle. \] (A.4)

Calculating the determinant of the ‘almost rank-one’ matrix in (A.3) one arrives at (2.8).

Using (2.5) and (A.1) we can calculate the ‘evolution’ of \( G, \langle b_\lambda | \) and \( |\bar{\gamma}_\mu \rangle \):
\[ T_\zeta G = G - 2\zeta E^{-1}_{\zeta \xi} G |\bar{\gamma}_\zeta \rangle \langle b_\xi | G, \] (A.5)
\[ (\zeta - \lambda) T_\zeta \langle b_\lambda | = 2\zeta |b_\zeta | - (\zeta + \lambda) |b_\lambda |, \] (A.6)
\[ (\zeta - \mu) T_\zeta |\bar{\gamma}_\mu \rangle = 2\zeta |\bar{\gamma}_\zeta \rangle - (\zeta + \mu) |\bar{\gamma}_\mu \rangle \] (A.7)

which leads to
\[ (\zeta - \lambda)(\zeta - \mu) E_{\zeta \xi} T_\zeta E_{\lambda \mu} = (\zeta + \lambda)(\zeta + \mu) E_{\zeta \xi} E_{\lambda \mu} - 2\zeta(\lambda + \mu) E_{\zeta \xi} E_{\xi \mu} \] (A.8)

and then (after replacing \( \lambda, \mu \rightarrow \xi \) and \( \zeta \rightarrow \eta \)) to the two-shift determinant formula
\[ \frac{T_{\xi \eta} \tau}{\tau} = -4\xi \eta \Gamma_{\xi,\eta} \begin{vmatrix} D_{\xi \xi} & D_{\xi \eta} \\ D_{\eta \xi} & D_{\eta \eta} \end{vmatrix}, \quad D_{\xi \eta} = \frac{E_{\xi \eta}}{\xi + \eta}. \] (A.9)

This relation can be recursively generalized to the \( N \)-determinants which yields the \( N \)-shift formulae (2.12). The last fact that we need to prove the proposition 1 is not obvious, though rather easy to demonstrate: the functions of two variables \( E_{\lambda \mu} \) can be factorized into the scalar products of two-vectors, each of which depends on only one of them. Indeed, by simple algebra one can verify the identity
\[ E_{\lambda \mu} = \left( \vec{\psi}_\lambda, \vec{\phi}_\mu \right) \] (A.10)

with
\[ \vec{\psi}_\lambda = \left( 1 - \langle a | G A | \bar{\beta}_\lambda \rangle \right), \quad \vec{\phi}_\mu = \left( 1 - \langle \bar{a} | G A | \beta_\mu \rangle \right) \] (A.11)

where
\[ \bar{G} = (1 + \bar{A}A)^{-1}, \quad |\bar{\beta}_\lambda \rangle = (\bar{M} - \lambda)^{-1} |\alpha \rangle, \] (A.12)

while \( G \) and \( |\bar{\beta}_\mu \rangle \) are defined in (2.10) and (2.11). Since any three two-vectors are linearly dependent, it is easy to conclude that
\[ \det \left| E_{\xi,\eta} \right|_{j,k=1,\ldots,N} = 0 \quad N \geq 3. \] (A.13)

Now, to prove the fact that \( \tau \) satisfies (2.14) one has just to write (2.12) with \( X = \{ \xi, \eta, \zeta \} \), to substitute \( E_{\xi \xi} E_{\eta \eta} E_{\zeta \zeta} + E_{\xi \zeta} E_{\zeta \eta} E_{\xi \eta} \) from (A.13) with \( \{ \xi_1, \xi_2, \xi_3 \} = \{ \xi, \eta, \zeta \} \) and to rewrite the remaining combinations \( E_{\xi \eta} E_{\eta \xi} \) and \( E_{\xi \zeta} \) using (A.9) and (2.8).
Appendix B. Proof of proposition 2

The proof of proposition 2 is based on simple algebraic calculations. It is straightforward to show that the bilinear combination of the tau-functions that appears in (2.16),

\[ h_{\alpha\beta\gamma\delta} = \Gamma_{\alpha\delta,\beta\gamma} (T_{\alpha\delta} \tau) (T_{\beta\gamma} \tau) + \Gamma_{\gamma\delta,\alpha\beta} (T_{\gamma\delta} \tau) (T_{\alpha\beta} \tau) - \tau T_{\alpha\beta\gamma\delta} \tau \]  

(B.1)

is a linear combination,

\[ (T_{\delta} \tau) h_{\alpha\beta\gamma\delta} = +\Gamma_{\alpha\beta\gamma} (T_{\alpha\beta} \tau) f_{\beta\gamma\delta} + \Gamma_{\beta\alpha\gamma} (T_{\beta\alpha} \tau) f_{\alpha\gamma\delta} + \Gamma_{\gamma\alpha\beta} (T_{\gamma\alpha} \tau) f_{\alpha\beta\delta} + \tau T_{\delta} f_{\alpha\beta\gamma\delta}, \]  

(B.2)

of the bilinear expressions \( f_{\xi\eta\zeta} \),

\[ f_{\xi\eta\zeta} = \Gamma_{\xi\zeta,\eta} (T_{\xi} \tau) (T_{\eta} \tau) + \Gamma_{\eta\xi,\zeta} (T_{\eta} \tau) (T_{\xi} \tau) + \Gamma_{\zeta\xi,\eta} (T_{\zeta} \tau) (T_{\xi} \tau) - \tau T_{\xi\eta\zeta} \tau, \]  

(B.3)

that constitute the Miwa equation (see (2.14)). Thus,

\[ f_{\xi\eta\zeta} = 0 \quad (\forall \xi, \eta, \zeta) \quad \Rightarrow \quad h_{\alpha\beta\gamma\delta} = 0 \]  

(B.4)

i.e. if \( \tau \) is a solution of the (2.14) \( (f_{\xi\eta\zeta} = 0) \), then it is a solution of the (2.16) \( (h_{\alpha\beta\gamma\delta} = 0) \).

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