SUPPORT THEOREM FOR THE LIGHT-RAY TRANSFORM OF VECTOR FIELDS ON MINKOWSKI SPACES

SIAMAK RABIENIAHARATBAR

Abstract. We study the Light-Ray transform of integrating vector fields on the Minkowski time-space $\mathbb{R}^{1+n}$, $n \geq 2$, with the Minkowski metric. We prove a support theorem for vector fields vanishing on an open set of light-like lines. We provide examples to illustrate the application of our results to the inverse problem for the hyperbolic Dirichlet-to-Neumann map.

1. Introduction

Let $(M,g)$ be a Lorentzian manifold of dimension $1+n$, $n \geq 2$, with a Lorentzian metric $g$ of signature $(-,+,\ldots,+)$. Given a weight $\kappa \in C^\infty(M \times S^{n-1})$, in general, the weighted Light-Ray transform of a vector field $f$ is defined by:

$$L_\kappa f(\gamma) = \int \kappa(\gamma(s), \gamma'(s)) f_i(\gamma(s)) \gamma'^i(s) \, ds,$$

where $\gamma = \gamma_x, \theta$ is the family of future pointing light-like geodesics (null-geodesics) on $M$ in the direction of $(1, \theta)$, $\theta \in S^{n-1}$. We choose a certain parametrization for the family of light-like geodesics $\gamma$ and require the weight function $\kappa$ to be positively homogeneous of degree zero in its second variable. The homogeneity of $\kappa$ makes the parameterization of light-like geodesics independent.

Light-ray transform has been attracting a growing interest recently, due to its wide range of applications. One major application of this transform is in the study of hyperbolic equations with time-dependent coefficients to recover the lower order terms from boundary or scattering information, see, e.g., [40, 32, 31, 48, 49, 33, 4, 15] and also recovering the lower order terms of time-independent hyperbolic equations [3, 26].

In the case where $f$ is a function and supported in the cylinder $\mathbb{R} \times B(0,R)$ with tempered growth in the Minkowski space, Stefanov in [39] has shown that $Lf$ determines $f$ uniquely. The fact that $Lf$ recovers the Fourier transform $\hat{f}$ of $f$ (w.r.t. all variables) in the space-like cone $|\tau| < |\xi|$ in a stable way is used to show that the potential in the wave equation is uniquely determined by the scattering data. Moreover, since $\hat{f}(\tau, \xi)$ is analytic in the $\xi$ variable (with values distributions in the $\tau$ variable), then one can fill in the missing cone by analytic continuation in the $\xi$ variable. In a recent work by Stefanov [39], analytic microlocal methods is applied to show support theorems and injectivity of $L_\kappa$ for analytic metrics and weights (on an analytic manifolds $M$) for functions. In particular, the results in [39] are generalized to a local data and independent of the tempered growth for large $t$.

Analytic microlocal method were already used in many works. Boman and Quinto in [6, 7] proved support theorems for Radon transforms with flat geometry and analytic weights (see also [59]). For related results using same techniques, we refer to [22, 23], where the support theorem is proved on simple analytic manifolds (see also [11]). Uhlmann and Vasy in [46] used the scattering calculus to prove a support theorem in the Riemannian case near a strictly convex point of a hypersurface in dimensions $n \geq 3$ without the analyticity condition (see also [42, 43, 44]).

Analytic and non-analytic microlocal analysis have been used to prove the injectivity and stability estimates for tensor fields of order two and higher. For the tensor fields of order two, Pestov and Uhlmann in [28] proved the unique recovery of Riemannian metric on a compact and simple Riemannian surface with...
boundary. See also Sharafutdinov \cite{27} for related results. More general result on injectivity up to potential fields of the geodesic ray transform for tensor fields of any order on Riemannian manifold can be found in \cite{27}. In \cite{15, 11}, a generic s-injectivity up to potential fields and a stability estimate are established on a compact Riemannian manifold $M$ with non-necessarily convex boundary and with possible conjugate points. Microlocal method for tomographic problems is used to detect singularities of the Lorentzian metric of the Universe using measurements of the Cosmic Microwave Background (CMB) radiation in \cite{21}. It is described in \cite{21} which singularities are visible and which cannot be detected. In \cite{23}, a Helgasons type of support theorem is proved for the geodesic ray transform of symmetric 2-tensor fields on a Riemannian manifolds (with boundary) with a real-analytic metric $g$. It is shown that the tensor field can be recovered up to a potential fields. In \cite{21}, authors studied the problem of recovery of a covector fields on simple Riemannian manifold with weight. Under some condition on weight, the recovery up to potential field and uniqueness are shown. See also Sharafutdinov \cite{21}, for the inversion of three dimensional X-ray transform of symmetric tensor fields of any order with sources on a curve.

Our main goal in this paper is to study the local and analytic microlocal invertibility of the operator $L$ acting on vector fields on the Minkowski time-space $\mathbb{R}^{1+n}$ when the weight $\kappa$ is simply one. We study the following operator $L$:

$$
L f(x, \theta) = \int f(s, x + s\theta) \cdot (1, \theta) \, ds,
$$

where $\gamma = \gamma_{x, \theta} = (s, x + s\theta)$ is the family of future pointing light-like lines (light-rays) on the Minkowski time-space $\mathbb{R}^{1+n}$ in the direction of $(1, \theta)$, with $|\theta| = 1$. Note that above operator is a special case of the operator defined in \cite{11}.

The main novelty of our work is that $Lf$ is known only over an open set of light-like lines, $\Gamma$, on the Minkowski time-space $\mathbb{R}^{1+n}$ (the Incomplete data case). Our results can be considered as a Helgason’s type support theorems. The global invertibility (injectivity) of the operator $L$ (the Complete data case) up to potential fields is already established, see for example \cite{10, 12}.

We generalize the method used in \cite{10} to study the stable recovery of the analytic wave front set of vector field $f$ instead of functions, and prove the support theorems in the Minkowski time-space $\mathbb{R}^{1+n}$. For our purpose, we apply the analytic stationary phase approach by Sjöstrand \cite{28} already used by the Stefanov \cite{29} and Uhlmann in \cite{11} (see also \cite{11, 22, 23}), to prove our results.

This paper is organized as follow: Section one is an introduction. In section two we state some definitions and our main result. Fourier analysis of the light-ray transform $L$ in the Minkowski time-space $\mathbb{R}^{1+n}$ is studied in section three. In section four, we first introduce the notion of Analytic Wave Front Set of a vector-valued distribution $f$. We then consider the partial data and apply analytic microlocal analysis argument and stationary phase method to recover the analytic wave front of vector field $f$. In section five, we state an analytic continuation result and we prove our main theorem. Our examples in the last section illustrate how our results imply the inverse recovery of a smooth potential field for the hyperbolic Dirichlet-to-Neumann map.

2. MAIN RESULT

We first state some definitions and a proposition which are necessary for our main result.

**Definition 2.1.** We call a vector $u = (u_0, u')$ space-like if $|u_0| < |u'|$. Vectors with $|u_0| > |u'|$ are called time-like. Light-like vectors are those for which we have $|u_0| = |u'|$.

**Definition 2.2.** We say the set $K$ is light-like convex if for any two points in $K$, the light-like geodesic connecting them lies in $K$.

**Definition 2.3.** Let $K$ be a subset of Minkowski time-space $\mathbb{R}^{1+n}$. We say $K$ expands with a speed less than one if

$$
K \subset \{(t, x) : |x| \leq C|t| + R\}, \quad \text{for some} \quad 0 < C < 1, \quad R > 0.
$$
Remark 2.1. This definition allows us to integrate over a compact interval. In fact, if the supp \( \text{supp} f \) in such a set expands with speed less than one, then the operator defined by (2.2) is integrating over a compact set including \( (x_0, \theta_0) \in \mathbb{R}^n \times S^{n-1} \). In other words, the integral of \( f \) and \( \chi f \) have the same light-ray transform near \( (x_0, \theta_0) \), where the function \( \chi \) is a smooth cut-off with property \( \chi = 1 \) in a neighborhood of \( (x_0, \theta_0) \).

From now on, we study the operator defined by (2.2). We know that, any three-dimensional vector field \( f = (f_0, f_1, f_2) \), has a three-dimensional curl \( f \). In other words, one may work with the curl \( f \) to do the analytic recovery of the analytic wave front set. This, however, is not the case for any vector field \( f \) with dimension \( n > 3 \) as the generalized curl \( f \), \( df \), does not have the same dimension as the vector field \( f \) does. This motivates us to introduce an appropriate operator where it forms an \( n \)-dimensional parametrized vector field with all the necessary components of \( df \) for the analytic recovery process. We now state our first proposition.

Proposition 2.1. Let \( f = (f_0, f_1, \ldots, f_n) \in \mathcal{C}^1(\mathbb{R}^{1+n}, \mathbb{C}^{1+n}) \) be such that \( |f| \) and \( |\partial f_i/\partial x^j| \) are bounded by \( C(1 + |x|)^{-1-\epsilon} \) with some \( \epsilon > 0 \) and constant \( C > 0 \). Then for any \( (x, \theta) \in (\mathbb{R}^n \times S^{n-1}) \) and \( v \in \mathbb{R}^n \)

\[
(v, \nabla_x) Lf(x, \theta) = \int_{\gamma_{x, \theta}} \tilde{f}_v(x) \cdot (1, \theta) \ ds,
\]

where \( \tilde{f}_v(x) = (\tilde{f}_{v_0}, \tilde{f}_{v_1}, \ldots, \tilde{f}_{v_n})(x) \in S(\mathbb{R}^{1+n}) \) a parametrized vector field with

\[
\tilde{f}_{v_i}(x) = \sum_{0 \leq j \leq n} f_{ij}(x)(0, v)^j_{(i)} = \sum_{0 \leq j \leq n} (\partial_j f_i - \partial_i f_j)(x)(0, v)^j_{(i)}, \quad i = 1, 2, \ldots, n.
\]

Here by \( (0, v)_{(i)} \) we mean the \( i \)-th component of \( (0, v) \) is excluded.

Proof. We show this result for \( n = 3 \). The proof for higher dimension is analogous. Let \( f \in \mathcal{C}^1(\mathbb{R}^{1+3}, \mathbb{C}^{1+3}) \) and fix \( (1, \theta) \in \mathbb{R} \times S^{n-1} \). For \( v \in \mathbb{R}^3 \), we take the directional derivative of the operator \( Lf \). Therefore we have

\[
(v, \nabla_x) Lf(x, \theta) = \int_R (v, \nabla_x) f(s, x + s\theta). (1, \theta) ds.
\]

On the other hand, by the Fundamental Theorem of Calculus one has

\[
\int_R \frac{d}{ds} [f(s, x + s\theta). (0, v)] \ ds = 0.
\]

Subtracting above identities, we get

\[
(v, \nabla_x) Lf(x, \theta) = \int_R ((0, v). \nabla_x) f(s, x + s\theta). (1, \theta) - \frac{d}{ds} [f(s, x + s\theta). (0, v)] \ ds.
\]

Note here that we used \( z = (t, x) \in \mathbb{R}^{1+n} \) to balance the dimension of the two terms on the right hand side of above equation. Expanding the right hand side and rearranging all terms with respect to components of \( (1, \theta) \), i.e., \( 1, \theta^1, \theta^2, \theta^3 \), we get

\[
\int_{\gamma_{x, \theta}} [v^1 (\partial_1 f_0 - \partial_0 f_1) + v^2 (\partial_2 f_0 - \partial_0 f_2) + v^3 (\partial_3 f_0 - \partial_0 f_3)] + [v^2 (\partial_2 f_1 - \partial_1 f_2) + v^3 (\partial_3 f_1 - \partial_1 f_3)] \theta^1 + [v^1 (\partial_1 f_2 - \partial_2 f_1) + v^3 (\partial_3 f_2 - \partial_2 f_3)] \theta^2 + [v^1 (\partial_1 f_3 - \partial_3 f_1) + v^2 (\partial_2 f_3 - \partial_3 f_2)] \theta^3 \ ds.
\]

Therefore,

\[
(v, \nabla_x) Lf(x, \theta) = \int_{\gamma_{x, \theta}} \tilde{f}_v(x) \cdot (1, \theta) \ ds.
\]

Here \( \gamma_{x, \theta} \) is the light-like lines parameterized by their points of intersection with \( t = 0 \) and direction \( (1, \theta) \) \( \square \)

Remark 2.2. i) For \( n = 2 \), setting \( v \) to be \( (1, 0) \) and \( (0, 1) \) yields to the following identities.

\[
\partial_1 Lf(x, \theta) = \int_{\gamma_{x, \theta}} (\partial_1 f_0 - \partial_0 f_1) + \theta^2 (\partial_2 f_1 - \partial_2 f_1) \ ds = \int_{\gamma_{x, \theta}} (-c_2 + \theta^2 c_0) \ ds.
\]
\[
\partial _{2}L f(x, \theta) = \int _{\gamma _{x, \theta}} (\partial _{2}f_{0} - \partial _{2}f_{2}) + \theta ^{1}(\partial _{2}f_{1} - \partial _{2}f_{2}) \, ds = \int _{\gamma _{x, \theta}} (c_{1} - \theta ^{1}c_{0}) \, ds,
\]
where \((c_{0}, c_{1}, c_{2}) =: \text{curl } f\). Similar results can be seen in [11, Proposition 2.8].

ii) The vector field \(\tilde{f}_{v}\) has the following property: for any \(v \in \mathbb{R}^{n}\),

\[
(0, v) \cdot \tilde{f}_{v}(x) = v^{1} \tilde{f}_{1,v}(x) + v^{2} \tilde{f}_{2,v}(x) + \cdots + v^{n} \tilde{f}_{n,v}(x) = 0.
\]

This is analogous to solenoidal condition for vector fields in the Fourier domain.

iii) Each component of \(\tilde{f}_{v}\) is a superposition of components of curl \(f\) (for \(n = 2\)) and of the generalized curl, \(df\) (for \(n \geq 3\)). This is a very important property since it forms an overdetermined system of equations which helps us to recover the curl and generalized curl, \(df\).

iv) Clearly

\[
Lf = 0 \implies (v \cdot \nabla_{x})Lf = 0, \quad \forall v \in \mathbb{R}^{n}.
\]

For the case where the vector field \(f\) is compactly supported, we have

\[
Lf = 0 \iff (v \cdot \nabla_{x})Lf = 0, \quad \forall v \in \mathbb{R}^{n}.
\]

In fact, the directional derivative of \(Lf\) with respect to \(x\) is zero for all \(v \in \mathbb{R}^{n}\), which implies that \(Lf\) is constant. Now \(f\) is compactly supported, therefore \(Lf = 0\).

Above properties motivate us to work with \(\tilde{f}_{v}\) and \((v \cdot \nabla_{x})L\) instead of \(f\) and \(Lf\) in the following sections. Our main result is a support theorem in the spirit of Theorem 2.1 as follows:

**Theorem 2.1.** Let \(n \geq 2\) and \(f \in \mathcal{E}'(\mathbb{R}^{1+n})\) be so that \(\text{supp } f\) expands with a speed less than one. Let \(G\) be an open and connected neighborhood of \((x_{0}, \theta _{0}) \in \mathbb{R}^{n} \times \mathbb{S}^{n-1}\) and \(\gamma _{x_{0}, \theta _{0}}\) be a light-like line with direction \(\theta _{0}\) passing through the point \(x_{0}\).

i) For \(n = 2\), if \(Lf(x, \theta) = 0\) in \(G_{\pm}\) and if \(\gamma _{x_{0}, \theta _{0}}\) does not intersect \(\text{supp } \text{curl } f\), then none of the light-like lines \(\gamma _{x, \theta} : (x, \theta) \in G_{\pm}\), does. Here \(G_{\pm}\) is an open and connected neighborhood of \((x_{0}, \pm \theta _{0}) \in \mathbb{R}^{2} \times \mathbb{S}^{1}\).

ii) For \(n \geq 3\), if \(Lf(x, \theta) = 0\) in \(G\) and if \(\gamma _{x_{0}, \theta _{0}}\) does not intersect \(\text{supp } df\), then none of the light-like lines \(\gamma _{x, \theta} : (x, \theta) \in G\), does.

**Remark 2.3.** For \(n = 2\) we require the operator \(L\) to be known in two different directions \((1, \theta)\) and \((1, -\theta)\). For discussion, we refer the reader to Fourier analysis in the following section. It is shown that, when the operator \(L\) is known only for one direction \((1, \theta)\), the ellipticity is lost for the case.

3. **Fourier Analysis in the Minkowski case**

In this section, we consider the case where light-ray transform is known over all light-like lines (complete data). This allows us to do the Fourier analysis by fixing the initial point and stay in a small neighborhood of the direction \((1, \theta)\). As we mentioned above, this case has been already studied. We do this analysis to have some insight for the microlocal analysis part of our study.

The following proposition is some preliminary results which states that in the space-like cone \(\{(\tau, \xi) : |\tau| < |\xi|\}\), the operator \((v \cdot \nabla_{x})L\) recovers the Fourier transform of the curl \(f\) and the generalized curl, \(df\), for \(n = 2\) and \(n \geq 3\), respectively.

**Proposition 3.1.** Let \(f \in \mathcal{S}(\mathbb{R}^{1+n})\).

i) For \(n = 2\), if \(Lf(x, \theta) = 0\) for all \(x\) and for \(\theta\) near \(\pm \theta _{0}\), then \(\mathcal{F}(\text{curl } f) = 0\) for \(\zeta\) close \(\zeta _{0}\), where \(\zeta _{0}\) is the unique space-like vector up to re-scaling with the property \((1, \pm \theta _{0}) \cdot \zeta _{0} = 0\).

ii) For \(n \geq 3\), if \(Lf(x, \theta) = 0\) for all \(x\) and for \(\theta\) near \(\theta _{0}\), then \(\mathcal{F}(df) = 0\) for all \(\zeta\) near the set \(\{\zeta : (1, \theta) \cdot \zeta = 0\}\).

**Proof.** i) Let \(\zeta _{0}^{0} = (\tau _{0}^{0}, (\xi _{1}^{0})^{0}, (\xi _{2}^{0})^{0})\) be a fixed space-like vector and without loss of generality assume that \(\theta _{0} = \pm \tau _{2} \in \mathbb{R}^{2}\) such that \((1, \pm \theta _{0}) \cdot \zeta _{0}^{0} = 0\). One has

\[
\tau _{0}^{0} \pm (\xi _{2}^{0})^{0} = 0 \quad \text{which implies that } \tau _{0}^{0} = (\xi _{2}^{0})^{0} = 0.
\]
This implies that ζ⁰ has to be in the form of \((0, (ζ¹)^{0}, 0)\), which means, up to re-scaling it is a unique ζ⁰ with property \((1, \pm θ₀) \cdot ζ⁰ = 0\). Therefore, one may choose ζ⁰ = e¹ ∈ ℝ¹⁺². Note that this choice of ζ⁰ can be done since we may apply Lorentzian transformation to any fixed space-like vector and transform it to e¹. We first state the **Vectorial Fourier Slice Theorem** for a general set of lines:

\[
\hat{f}(ζ) \cdot ω = \int_{ω ⊥} e^{-iz · ζ} Lf(z, ω) \, dS, \quad ∀ω ⊥ ζ, \quad ∀f ∈ L¹(ℝⁿ).
\]

To prove this, note that the integral on the RHS equals

\[
\int_{ω ⊥} \int_{ℝ} e^{-iz · ζ} f₁(z + sω)ψ^j dSḍ = \int_{ω ⊥} \int_{ℝ} e^{-iz · ζ} f₁(z + sω) ds dSḍ.
\]

Set \( x = z + sω \). Then, it is easy to see that when \( ω ⊥ ζ \), we have \( x \cdot ζ = z \cdot ζ \) and therefore above integral equals \( \hat{f}(ζ) \). In this paper, we apply the Vectorial Fourier Slice theorem when the set of lines is restricted to a set of light-like lines.

By assumption \((v, ∇ζ)Lf = 0\) for all \( v ∈ ℝ² \). Let \( v \) be an arbitrary fixed vector in \( ℝ² \). For \( ζ = e¹ \), by Vectorial Fourier Slice theorem we have

\[
0 = \hat{f}_v(ζ) \cdot (1, θ) = \hat{f}_{v₀}(ζ) + \hat{f}_{v₁}(ζ)θ¹ + \hat{f}_{v₂}(ζ)θ², \quad ∀(1, θ) ⊥ ζ.
\]

For a fixed \( v ∈ ℝ² \), since \((1, ±θ₀) \cdot ζ = 0\), above equation implies

\[
\begin{cases}
\hat{f}_{v₀}(ζ) = v¹(ζ¹f₀ − τζ₂) + v²(ζ²f₀ − τζ₁) = 0, \\
\hat{f}_{v₁}(ζ) = v¹(ζ¹f₁ − θ¹ζ₂) + v²(ζ²f₁ − θ²ζ₁) = 0.
\end{cases}
\]

The vector \( v \) is arbitrary in \( ℝ² \), therefore one may choose two linearly independent vectors, say \( v₁ = θ⁺ ⊥ \) and \( v₂ = θ \), to conclude

\[
ζ¹f₁ − τζ₂ = ζ²f₂ − ξ²f₁ = 0 \implies F(\text{curl } f)(ζ) = 0.
\]

Now let \( ζ = (τ, ξ) ∈ ℝ¹⁺² \) be any non-zero space-like vector. We solve the equation \((1, θ) \cdot ζ = 0\), for \( θ \).

Set \( θ = aξ + bξ⁺ \). Therefore,

\[
−τ = θ \cdot ξ = (aξ + bξ⁺) \cdot ξ = a|ξ|^² \implies a = −τ/|ξ|^².
\]

On the other hand,

\[
1 = |θ|^² = (a² + b²)|ξ|^² \implies b = ± 1/|ξ|^² √−τ² + |ξ|^².
\]

For \( ξ ∈ ℝ² \), we set

\[
\theta = θ_±(ζ) = \frac{1}{|ξ|^²}(−τξ¹ ± √−τ² + |ξ|^²ξ², −τξ² ± √−τ² + |ξ|^²ξ¹).
\]

Clearly \((1, θ_±(ζ)) \cdot ζ = 0\) and \( θ_± \) are the only two choices with the property \(|θ_±(ζ)| = 1\). In order to have ζ close to \( ζ₀ = e¹ \), we require \( θ² = 0 \), so we have

\[
−τξ¹ ± √−τ² + |ξ|^²ξ² = 0.
\]

Since ζ is a space-like vector, \( √−τ² + |ξ|^² \) is non-zero which implies that \( τξ¹ = ξ² = 0 \). Note that \( ξ¹ \) is not zero, otherwise \( θ_±(ζ) \) would be undefined. This forces \( τ \) to be zero, and therefore ζ ≈ ζ₀ = e¹. In particular, this implies that \( θ = θ_±(ζ) \) is analytic near \( ζ₀ = e¹ \). With \( θ_±(ζ₀) = ±e₂ \) ∈ \( ℝ² \). Therefore \( θ_±(ζ) \) is within a neighborhood of \( ±θ₀, θ ≈ ±θ₀ \) if ζ is within a neighborhood of ζ₀, ζ ≈ ζ₀. Considering our choices of direction \( θ_±(ζ) \), the equation (3.1) can be written as

\[
\hat{f}_v(ζ) = \frac{1}{|ξ|^²}(−τξ¹ ± √−τ² + |ξ|^²ξ²)\hat{f}_{v₁}(ζ) + \frac{1}{|ξ|^²}(−τξ² ± √−τ² + |ξ|^²ξ¹)\hat{f}_{v₂}(ζ) = 0.
\]
and

\[ f_{0v}(\zeta) + \frac{1}{|v|^2} (-\tau \xi^1 - \sqrt{-\tau^2 + |\xi|^2}) \hat{f}_{1v}(\zeta) + \frac{1}{|v|^2} (-\tau \xi^2 + \sqrt{-\tau^2 + |\xi|^2}) \hat{f}_{2v}(\zeta) = 0. \]

Subtract (3.4) from (3.3) to get

\[ f_{0v}(\zeta) + \frac{1}{|v|^2} (-\xi^1 - \sqrt{-\tau^2 + |\xi|^2}) \hat{f}_{1v}(\zeta) + \frac{1}{|v|^2} (-\xi^2 + \sqrt{-\tau^2 + |\xi|^2}) \hat{f}_{2v}(\zeta) = 0. \]

Multiply (3.3) by \( \xi^1 \) and using (3.5), we get

\[ f_{0v}(\zeta) + \frac{1}{|v|^2} (-\xi^1 \xi^2) \hat{f}_{1v}(\zeta) + \frac{1}{|v|^2} (\xi^2 \xi^1) \hat{f}_{2v}(\zeta) = 0. \]

Note that for \( i = 0,1,2 \), we expanded \( \hat{f}_{1v}(\zeta) \) in (3.5) and (3.6), and rearranged both equations in terms of \( \hat{f}_{1v}(\zeta) \), to get the RHS of above equations. Now set \( v = \xi \), therefore \( \xi \cdot v = |\xi|^2 \neq 0 \) and

\[ \xi^2 \hat{f}_{1v}(\zeta) - \xi^1 \hat{f}_{2v}(\zeta) = 0. \]

Clearly \( \xi^2 \hat{f}_{1v}(\zeta) - \xi^1 \hat{f}_{2v}(\zeta) = 0 \) implies that \( \hat{f}_{1v}(\zeta) = \hat{f}_{2v}(\zeta) \) for \( v \in \mathbb{R}^2 \). Plugging \( \hat{f}_{1v}(\zeta) = 0 \) into the LHS of (3.6), we conclude \( \xi^1 \hat{f}_{0v}(\zeta) = 0 \). The vector \( \zeta \) is space-like with property \( \xi^1 \neq 0 \), therefore

\[ \hat{f}_{0v}(\zeta) = v^1(\xi^1 \hat{f}_0 - \tau \hat{f}_1) + v^2(\xi^2 \hat{f}_0 - \tau \hat{f}_2) = 0. \]

Since \( v \) is arbitrary in \( \mathbb{R}^2 \), any two linearly independent vectors \( v_1 \) and \( v_2 \) implies that \( \xi^2 \hat{f}_0 - \tau \hat{f}_2 = 0 \). Notice that one may use the equation on the RHS of (3.7) and the fact that \( v \) is an arbitrary vector to have the same conclusion. This shows that all three components of curl \( f \) in Fourier domain are zeros, and thus \( \mathcal{F}(\text{curl } f)(\zeta) = 0 \).

ii) Let first \( n = 3 \) and \( \zeta = \zeta^0 \) be a fixed non-zero space-like vector. Applying the Lorentzian transformation, we may assume that \( \zeta^0 = e^2 := (0,0,1,0) \in \mathbb{R}^{1+3} \). Set

\[ \theta(a) = \sin(a)e_1 + \cos(a)e_3 = (\sin(a),0,\cos(a)). \]

Clearly \( |\theta(a)| = 1 \), \( \theta(0) = e_3 \), and \( (1,\theta(a)) \cdot \zeta = 0 \).

By assumption \( (v \cdot \nabla_x)Lf = 0 \) for all \( v \in \mathbb{R}^3 \). Let \( v \) be an arbitrary fixed vector in \( \mathbb{R}^3 \). For \( \zeta = e^2 \), by Vectorial Fourier Slice theorem

\[ 0 = \hat{f}_v(\zeta) \cdot (1,\theta) = \hat{f}_{0v}(\zeta) + \hat{f}_{1v}(\zeta)\theta^1 + \hat{f}_{2v}(\zeta)\theta^2 + \hat{f}_{3v}(\zeta)\theta^3, \quad \forall (1,\theta) \perp \zeta \]

Plugging \( \theta = \theta(a) \) into above equation we get

\[ 0 = \hat{f}_v(\zeta) \cdot (1,\theta) = \hat{f}_{0v}(\zeta) + \hat{f}_{1v}(\zeta)\sin(a) + \hat{f}_{3v}(\zeta)\cos(a). \]

Differentiating above equation with respect to parameter \( a \) once and twice, we get

\[
\begin{align*}
\hat{f}_{1v}(\zeta)\cos(a) - \hat{f}_{3v}(\zeta)\sin(a) &= 0, \\
-\hat{f}_{1v}(\zeta)\sin(a) - \hat{f}_{3v}(\zeta)\cos(a) &= 0.
\end{align*}
\]

It is easy to see that the last two equations imply that \( \hat{f}_{1v}(\zeta) = \hat{f}_{3v}(\zeta) = 0 \) for \( v \in \mathbb{R}^2 \). Now by equation (3.8) we conclude that \( \hat{f}_{0v}(\zeta) = 0 \) for \( v \in \mathbb{R}^2 \).

Our goal is to show the Fourier transform of the generalized curl of \( f, \mathcal{F}(df) \), is zero. Let \( v = (0,0,\cos(a)) \in \theta^\perp \) and plug it into \( \hat{f}_{0v}(\zeta) = 0 \) for \( i = 0,1,3 \). We have

\[
\begin{align*}
-\xi^1 \hat{f}_0 - \tau \hat{f}_1 &= \xi^2 \hat{f}_0 - \xi^3 \hat{f}_3 = \xi^4 \hat{f}_0 - \xi^0 \hat{f}_3 = 0.
\end{align*}
\]

One may repeat above differentiation argument for the first equation to conclude that

\[
\xi^1 \hat{f}_0 - \tau \hat{f}_1 = \xi^2 \hat{f}_0 - \xi^3 \hat{f}_3 = \xi^4 \hat{f}_0 - \xi^0 \hat{f}_3 = 0.
\]
Using the same argument simultaneously for the second and third equations implies that

\[ \xi^2 \hat{f}_1 - \xi^1 \hat{f}_2 = \xi^2 \hat{f}_3 - \xi^3 \hat{f}_2 = \xi^3 \hat{f}_1 - \xi^1 \hat{f}_3 = 0. \]

Therefore, \( \mathcal{F}(df)(\zeta) = 0 \) for a fixed \( \zeta = e^2 \). Note that, one may choose three linearly independent vectors \( v_1, v_2, v_3 \in \mathbb{R}^3 \), and conclude the same result.

To have an arbitrary \( \zeta \), we use the fact that the Lorentzian transformation is transitive and rotates every space-like vector to a space-like vector. Let \( \mathcal{L}_0 \) be a Lorentzian transformation with the property \( \mathcal{L}_0 \zeta = \zeta^0 = e^2 \) and let \( \mathcal{L} \) with \( \mathcal{L}y = x \) be a Lorentzian transformation whose representation in Fourier domain is given by \( \mathcal{L}_0 \). By the definition of Fourier transform one has

\[ \mathcal{F}[(df)(\mathcal{L}(.))](\zeta) = \int (df)(y)e^{iy\cdot \zeta}dy = \int (df)(x)e^{iL^{-1}x\cdot \zeta}|detL^{-1}|dx = \int (df)(x)e^{ix\cdot \zeta}|detL^{-1}|dx. \]

Therefore

\[ \mathcal{F}[(df)(\mathcal{L}(.))](\zeta) = |det \mathcal{L}^{-1}|\mathcal{F}(df)(\mathcal{L}^{-T}\zeta). \]

But \( \mathcal{L}^T = \mathcal{L} \) and \( \mathcal{L}^{-1}(\zeta) = \zeta^0 \), hence

\[ \mathcal{F}[(df)(\mathcal{L}(.))](\zeta) = |det \mathcal{L}^{-1}|\mathcal{F}(df)(\zeta^0) = 0, \]

since \( \mathcal{F}(df)(\zeta^0) = 0 \). This proves for any space-like vector \( \zeta \) near \( \zeta^0 \), the Fourier transform of generalized curl of \( f \) vanishes as desired. For the general case \( n > 3 \), one needs to choose \( n \) linearly independent vectors \( v_1, v_2, \ldots, v_n \in \mathbb{R}^n \) to show \( \mathcal{F}(df)(\zeta) = 0 \).

**Corollary 3.1.** Let \( f \in C_0^\infty(\mathbb{R}^{1+n}) \).

i) For \( n = 2 \), if \( Lf(x, \theta) = 0 \) for \( \theta \) near \( \pm \theta_0 \), then \( f \) is a smooth potential field with compact support, that is, \( f = d\phi \) with some \( \phi(x) \rightarrow 0 \), as \( |x| \rightarrow \infty \).

ii) For \( n \geq 3 \), if \( Lf(x, \theta) = 0 \) for \( \theta \) near \( \theta_0 \), then \( f \) is a smooth potential field with compact support, that is, \( f = d\phi \) with some \( \phi(x) \rightarrow 0 \), as \( |x| \rightarrow \infty \).

**Proof.** i) The first part of Proposition 3.1 implies that \( \mathcal{F}(\text{curl } f) = 0 \). Since \( f \in C_0^\infty(\mathbb{R}^{1+2}) \), we extend \( f \) as zero outside of the supp \( f \). Now by analyticity of Fourier transform, \( \mathcal{F}(\text{curl } f) \) is zero everywhere. Applying the inverse Fourier transform implies that \( \text{curl } f = 0 \) everywhere. Since the time-space \( \mathbb{R}^{1+2} \) is a simply connected domain, there exists a finite smooth function \( \phi \) such that \( f = d\phi \); in fact let \( x \in \text{supp } f \) and \( x_0 \) be a point outside of \( \text{supp } f \). Let \( c(t) \) be a path connecting \( x_0 \) to \( x \). We define \( \phi \) as follow:

\[ \phi(x) = \int_{x_0}^{x} f(c(t)) \cdot c'(t)dt + \phi(x_0), \]

which is smooth and satisfies \( f = d\phi \).

ii) By the second part of Proposition 3.1 we know that \( \mathcal{F}(df) = 0 \). Similar argument as part (i) shows \( df = 0 \) and therefore, \( f \) is a smooth potential field with compact support. \( \square \)

**Remark 3.1.** i) For \( n = 2 \) there are two discrete choices of directions, \( \pm \theta_0 \), and this is necessary to have the result. Following example shows that one cannot decrease the number of directions from two to one. Let \( \zeta \) be a space-like vector and \( \phi \in \mathcal{S} \) be supported in the interior of open cone \( \{|\tau| < |\xi|\} \). Consider \( \theta_{\pm}(\zeta) \) defined by (3.5) and set

\[ \eta = \hat{f}(\zeta) = (1, \frac{\xi^2}{\sqrt{-\xi^2 + |\xi|^2}}, -\frac{\xi^1}{\sqrt{-\xi^2 + |\xi|^2}}) \hat{\phi}(\zeta). \]

Clearly \( \eta \) is a non-zero vector field in the Schwartz space \( \mathcal{S} \) and is in the kernel of light-ray transform as it solves \( (1, \theta_{+}(\zeta)) \cdot \hat{f}(\zeta) = 0 \). Notice that, \( (1, \theta_{-}(\zeta)) \cdot \hat{f}(\zeta) \neq 0 \). This example does not provide a compactly supported vector field, however, it shows this is an obstruction to consider only one light-ray and stay close to it for the reconstruction. (the intersection of light-like cone and time variable is just two points)

For the Minkowski spaces of signature \( 1 + 3 \) or higher, however, this is not an obstruction. For instance when \( n = 3 \), one may consider a two-parameter family of direction, \( \theta(a, b) \), near a fixed \( \theta_0 \) and do the reconstruction process by perturbation.

ii) In above proposition for \( n \geq 3 \), to show the uniqueness results we performed differentiation which is not
problematic. However, for the stable inversion results, a differentiation may not preserve the stability. In other words, one may choose several discrete values of non-zero parameters near zero to create an invertible linear system to get stability estimate results. For discussion, we refer the reader to proof of theorem 4.1 for \( n \geq 3 \).

In the next section we state a theorem on the recovery of analytic space-like singularities in the Minkowski case which is a tool to prove our main result.

4. Microlocal Recovery of Analytic Wave Front Set

In this section, we mainly follow the analytic microlocal analysis argument to show that we can recover all space-like analytic singularities of \( f \) conormal to the light-like lines along with integration of operator \((v, \nabla_x)L\). (see also [38, Lemma 3.1])

We first introduce the Analytic Wave Front Set (or analytic singular spectrum) of a vector-valued distribution. For the case of a scalar-valued distribution, the definition can be found in [9, Hörmander [39], Sjöstrand [40], and Sato [41]. Bony [3] and Sjöstrand have shown the equivalence of all these definitions. For a vector-valued distribution \( f = (f_0, f_1, f_2, \ldots, f_n) \in \mathcal{D}'(X, \mathbb{C}^{1+n}) \), we define the analytic wave front set of \( f \), \( \text{WF}_A(f) \), as the union of \( \text{WF}_A(f_i) \). Note that, for the vector-valued distribution \( f \), the analytic wave front set \( \text{WF}_A(f) \) does not specify in which component \( f \) is singular. In our work, we follow the Sjöstrand’s exposition.

**Theorem 4.1.** Let \( f \in \mathcal{E}'(\mathbb{R}^{1+n}) \) and let \( \gamma_{x_0, \theta_0} \) be a fixed light-like line so that \( \gamma_{x, \theta}(s) \not\equiv \sup f \) for \( |s| \geq 1/C \) with some \( C \) for all \((x, \theta) \) near \((x_0, \theta_0)\).

i) For \( n = 2 \), if \( Lf(x, \theta) = 0 \) for all \((x, \theta) \) near \((x_0, \pm \theta_0)\), then \( \text{WF}_A(\text{curl } f) \) contains no space-like vectors conormal to \( \gamma_{x_0, \pm \theta_0} \).

ii) For \( n \geq 3 \), if \( Lf(x, \theta) = 0 \) for all \((x, \theta) \) near \((x_0, \theta_0)\), then \( \text{WF}_A(df) \) contains no space-like vectors conormal to \( \gamma_{x_0, \theta_0} \).

**Proof.** i) Let first \( f \in \mathcal{C}^1(\mathbb{R}^{1+2}) \). By assumption \( Lf = 0 \) only near \( \gamma_{x_0, \theta_0} \), therefore the localization is needed. We choose a local chart for the lines close to \( \gamma_{x_0, \theta_0} \), and without loss of generality we may assume that \( x_0 = 0 \) and \( \theta_0 = \pm e_2 \). So we have \( \gamma_0 = \gamma_{0, e_2} = (s, 0, 0) \).

Since \( Lf = 0 \), we have \( (v, \nabla_x)Lf = 0 \) for \( v \in \mathbb{R}^n \). Let \( v \) be an arbitrary fixed vector in \( \mathbb{R}^n \) and \( \zeta^0 \neq 0 \) be a space-like vector, conormal to \( \gamma_0 \) at \( x_0 = 0 \) with property \((1, \theta_0) \perp \zeta^0 \). Applying the Lorentz transformation, we may assume that \( \zeta^0 = e^1 := (0, 1, 0) \in \mathbb{R}^{1+2} \). Our goal is to show that \((0, \zeta^0) \not\in \text{WF}_A(\text{curl } f) \).

Let \( \chi_N \in \mathcal{C}^\infty_c(\mathbb{R}^2) \) be supported in \( B(0, \varepsilon), \varepsilon > 0 \), with \( \chi_N = 1 \) near \( x_0 = 0 \) so that

\[
|\partial_\alpha^\varepsilon \chi_N| \leq (CN)^{|\alpha|}, \quad \text{for } |\alpha| \leq N.
\]

Then for \( 0 < \varepsilon \ll 1, \lambda > 0 \), and \( \theta \) near \( \theta_0 \),

\[
0 = \int e^{i\lambda x \cdot \xi}(\chi_N(v, \nabla_x)Lf)(x, \theta) \, dx = \iint e^{i\lambda x \cdot \xi} \chi_N(x) \tilde{f}_\varepsilon(x)(\gamma_{x, \theta}(s)) \cdot (1, \theta) \, ds \, dx.
\]

If \((1, \theta) \cdot \zeta = 0 \) with \( \zeta = (\tau, \xi) \), then \( \gamma_{x, \theta} \cdot \zeta = (s, x + s\theta) \cdot \zeta = x \cdot \xi \). Making a change of variable \( z = \gamma_{x, \theta} \) in above integral yields to

\[
0 = \int e^{i\lambda z \cdot \xi}(\chi_N(v, \nabla_x)Lf)(x, \theta) \, dx = \int e^{i\lambda z \cdot \theta}(\zeta_\theta \tilde{a}_N(z, \theta)\tilde{f}_\varepsilon(z)(1, \theta)\, dz = \int e^{i\lambda z \cdot \zeta} \tilde{a}_N(z, \theta)\tilde{f}_\varepsilon(z)(1, \theta)\, dz,
\]

when \((1, \theta) \cdot \zeta = 0 \). Notice that \( a_N(0, \theta) = 1 \).

Now let \( \zeta \) be a space-like vector near \( \zeta^0 \) and set \( \theta = \theta_{\pm}(\zeta) \) (see (5.2)). Plugging \( \theta_{\pm}(\zeta) \) into (4.2), we get

\[
\int e^{i\lambda z \cdot \zeta} \tilde{a}_N(z, \theta_{\pm}(\zeta))\tilde{f}_\varepsilon(z)(1, \theta_{\pm}(\zeta))\, dz = 0, \quad \text{near } \zeta = e^1.
\]

Here \( \tilde{a}_N(z, \zeta) = a_N(z, \theta) \) where \( \tilde{a}_N(0, \zeta) = 1 \). Note also that for \( \zeta \approx \zeta^0 \), we have \( \theta(\zeta) \approx \theta_0 \).

In the next step we will apply the complex stationary phase method [Sjöstrand [38], similar to the case where it is applied to the Calderón problem with partial data in [42] and to the integral geometry problem in [43, 44].] We need to analyze the phase function and its critical points.
Fix $0 < \delta \ll 1$ and let $\chi_\delta$ be the characteristic function of the unit ball $B(0, \delta)$ in $\mathbb{R}^{1+2}$. With some $w, \eta \in \mathbb{R}^{1+2}$ close to $w = 0$, $\eta = e^1$, multiply the LHS of (4.3) by
\[ \chi_\delta(\zeta - \eta) e^{i\lambda((\zeta - \eta)^2/2 - w \cdot \zeta)} \]
and integrate w.r.t. $\zeta$ to get
\[ \int \int e^{i\lambda \Phi(z,w,\zeta,\eta)} b_N(z,\zeta,\eta) \tilde{f}_w(z)(1,\theta_\pm(\zeta))^i \, dz \, d\zeta = 0, \]
where $b_N = \chi_\delta(\zeta - \eta) \tilde{a}_N$ is a new amplitude and
\[ \Phi = (z-w) \cdot \zeta + i(\zeta - \eta)^2/2. \]
Consider the phase function $\zeta \to \Phi$. If $w = z$, there is a unique real critical point $\zeta_\eta = \eta$, and it satisfies $\Im \Phi_{\zeta\zeta} > 0$ at $\zeta = \zeta_\eta$. For $w \neq z$, the phase $\Phi$, as function of $\zeta$, has a unique critical point $\zeta_\eta = \eta + i(z-w)$. We now split the $z$-integral in (4.4) into two parts: over the set $\Sigma = \{ z; |z-w| \leq \delta/C^0 \}$, for some $C^0 > 1$, and then over the complement of $\Sigma$. Since $|\Phi_\zeta|$ has a $(\delta$-dependent) positive lower bound for $z \in \Sigma$ (for $\zeta$ real) and there is no real critical point for the function $\zeta \to \Phi$ in this set, we can estimate that part of integral. Using the estimate (4.1), integration by parts $N$-times w.r.t. $\zeta$, and the fact that on the boundary $|\zeta - \eta| = \delta$, the factor $e^{i\lambda \Phi}$ is exponentially small with $\lambda$, we get
\[ \left| \int \int_{\Sigma} e^{i\lambda \Phi(z,w,\zeta,\eta)} b_N(z,\zeta,\eta) \tilde{f}_w(z)(1,\theta_\pm(\zeta))^i \, dz \, d\zeta \right| \leq C(CN/\lambda)^N + CN e^{-\lambda/C}. \]
Note also that we used the fact that
\[ e^{i\lambda \Phi} = \frac{(\Phi_\zeta)_i \cdot \partial \zeta}{i\lambda |\Phi_\zeta|^2} e^{i\lambda \Phi}. \]
Now on the set $\{ z; |z-w| \leq \delta/C^\eta \}$, $C^\eta \gg 1$, we apply the complex stationary phase method for the rest of the $\zeta$-integral in (4.4). To estimate (4.5) for $z \in \Sigma$ we set: $\psi(z,w,\eta) = \Phi_{\zeta=\zeta_\eta}$. Therefore,
\[ \psi = \eta \cdot (z-w) + i|z-w|^2 - \frac{i}{2} |z-w|^2 = \eta \cdot (z-w) + \frac{i}{2} |z-w|^2. \]
Clearly the new phase function $\psi(z,w,\eta)$ satisfies
\[ \psi_\zeta(z,\zeta_\eta) = \zeta, \quad \psi_w(z,\zeta_\eta) = -\zeta, \quad \psi(z,\zeta_\eta) = 0. \]
For $(z,\zeta)$ close to $(0, e^1)$ we use this phase function and apply (3.8), Theorem 2.8] and the remark after it to the $\zeta$-integral above to get
\[ \int \int_{\Sigma} e^{i\lambda \psi(z,w,\zeta,\eta)} b_N(z,\zeta,\eta) \tilde{f}_w(z)(1,\theta_\pm(\zeta))^i \, dz \, d\zeta = \int \int_{\Sigma} e^{i\lambda \Phi(z,w,\zeta,\eta)} b_N(z,\zeta,\eta) \tilde{f}_w(z)(1,\theta_\pm(\zeta))^i \, dz \]
\[ = \int_{\Sigma} e^{i\lambda \psi(z,w,\zeta,\eta)} b_\lambda(z,w,\eta) \tilde{f}_w(z)(1,\theta_\pm(z,\zeta_\eta))^i \, dz \]
\[ = \int_{\Sigma} e^{i\lambda \psi(z,w,\zeta,\eta)} b_\lambda(z,w,\eta) \tilde{f}_w(z)(1,\theta_\pm(z,\eta))^i \, dz = \mathcal{O}(\lambda^{n/2}(CN/\lambda)^N + CN e^{-\lambda/C}), \]
where $\beta = (w,\eta)$, and $\tilde{B}_{\lambda \pm}$ is a classical elliptic analytic symbol of order 0 with principal part equal
\[ \sigma_{\beta}(\tilde{B}_{\lambda \pm}(z,\zeta,\beta)) = (1,\theta_\pm(\zeta)), \quad \text{up to an elliptic factor near } (z,\beta) = (0,0,e^1), \]
where $\theta_\pm(\zeta) = (\theta^1_+ (\zeta), \theta^2_+(\zeta))$ is defined in (3.2). In particular, for $(z,w,\zeta) = (0,0,e^1)$ we have
\[ \sigma_\beta(\tilde{B}_{\lambda \pm}(0,0,e^1)) = (1,\theta_\pm(e^1)) = (1,0,\pm 1) = (1,\pm e_2), \quad \text{up to an elliptic factor}. \]
For $z \in \Sigma$ with $\delta \ll 1$ and $|w| \ll 1$, $\eta$ close to $e^1$, the variable $(z,\beta)$ in (4.5) is near $(0,0,e^1)$ and therefore $\tilde{B}_{\lambda \pm}$ is independent of $N$ as $\chi_N = 1$ near the origin. Choose now $N$ so that $N \leq \lambda/(Ce) \leq N + 1$ to get the following exponential error on the right,
\[ \int_{\Sigma} e^{i\lambda \psi(z,w,\zeta,\eta)} [\tilde{f}_{w}(z) \tilde{B}_{\lambda \pm}^0(z,\beta) + \tilde{f}_{w}(z) \tilde{B}_{\lambda \pm}^1(z,\beta) + \tilde{f}_{w}(z) \tilde{B}_{\lambda \pm}^2(z,\beta)] \, dz = \mathcal{O}(\lambda^{n/2}(CN/\lambda)^N + CN e^{-\lambda/C}). \]
(4.6) \[ e^{i\lambda \psi(z,\beta)} \tilde{f}_{\alpha}(z) \tilde{B}_{\lambda \beta}^*(z,\beta) \, dz = \mathcal{O}(e^{-\lambda/C}). \]

**Microlocal Ellipticity.** Now we show that for \((1, \theta_{\pm}(\zeta))\), equations in (4.6) form an elliptic system of equations at \((0, 0, \zeta^0)\). Let \((z, \zeta)\) near \((0, 0, \zeta^0)\) and \(v\) be fixed, and consider the principal symbols \(\sigma_p(\tilde{B}_{\lambda \beta}(z, \zeta)) \equiv (1, \theta_{\pm}(\zeta))\). Microlocal version of ellipticity is equivalent to show that for a constant vector field \(f_v = (\tilde{f}_{0v}, \tilde{f}_{1v}, \tilde{f}_{2v})\),

\[(1, \theta_{\pm}(\zeta))^T \tilde{f}_{\alpha} = 0,\]

forms an elliptic system. Above equations can be written as

\[
\begin{align*}
\tilde{f}_{\alpha} + \frac{1}{|\xi|^2}(-\tau \xi^1 + \sqrt{-\tau^2 + |\xi|^2} \xi^2) \tilde{f}_{\beta} &= 0, \\
\tilde{f}_{\alpha} + \frac{1}{|\xi|^2}(-\tau \xi^1 - \sqrt{-\tau^2 + |\xi|^2} \xi^2) \tilde{f}_{\beta} &= 0.
\end{align*}
\]

By similar arguments as it is shown in Proposition 3.1 for \(n = 2\), one may conclude that

\[
\begin{align*}
0 &= \tau \tilde{f}_{\alpha} - \xi \tilde{f}_{\beta}
\end{align*}
\]

Clearly \(\partial_2 f_1 - \partial_1 f_2 = 0\) implies that \(\tilde{f}_{1\alpha} = \tilde{f}_{2\alpha} = 0\) (for definition of \(\tilde{f}_{\alpha}\) see Proposition 2.1) and therefore

\[
\tilde{f}_{0\alpha} = v^1(\partial_1 f_0 - \partial_0 f_1) + v^2(\partial_2 f_0 - \partial_0 f_2) = 0.
\]

Since \(v\) is arbitrary in \(\mathbb{R}^2\), any two linearly independent vectors \(v_1\) and \(v_2\) implies that \(\partial_2 f_0 - \partial_0 f_2 = 0\).

Therefore, the equation (4.6) leads to the following system of equations

\[
\begin{align*}
\int_{\mathbb{R}^n} e^{i\lambda \psi(z, \beta)} &|v^1(\partial_1 f_0 - \partial_0 f_1)(x) + v^2(\partial_2 f_0 - \partial_0 f_2)(x)| B^0_{\lambda \beta}(z, \beta) \, dz = \mathcal{O}(e^{-\lambda/C}) \\
\int_{\mathbb{R}^n} e^{i\lambda \psi(z, \beta)} &|v^1(\partial_2 f_1 - \partial_1 f_2)| B^1_{\lambda \beta}(z, \beta) \, dz = \mathcal{O}(e^{-\lambda/C}) \\
\int_{\mathbb{R}^n} e^{i\lambda \psi(z, \beta)} &|v^1(\partial_1 f_2 - \partial_2 f_1)| B^2_{\lambda \beta}(z, \beta) \, dz = \mathcal{O}(e^{-\lambda/C}),
\end{align*}
\]

where up to an elliptic factor,

\[
\sigma_p(B^0_{\lambda \beta}(z, \zeta)) \equiv \begin{cases} 
1 & i = 0 \\
\theta^1_{\pm}(\zeta) & i = 1 \\
\theta^2_{\pm}(\zeta) & i = 2.
\end{cases}
\]

Therefore, at \((z, \zeta) = (0, 0, \zeta^0)\), above elliptic systems of equations implies that \((0, \zeta^0) \not\in WF_A(c)\) as desired. Notice that above system is an overdetermined system of equations since the term \(\partial_2 f_0 - \partial_0 f_2\) is repeated in the second and third equations. This is due to the property of \((v, \nabla_x) L f \) and \(f_v\) as we pointed out on Remark 2.2.

Now if \(f \in \mathcal{E}'(\mathbb{R}^{1+2})\) is a distribution, as stated in the theorem, the result still holds in the sense of distributions. In fact, one may take a sequence of \(f^\alpha\)-smooth functions which converges to the distribution \(f\). The equation (4.6) holds for each smooth function. Now the \(z\)-integral in (4.6) can be thought in the sense of distributions as the integrand can be considered as the action of a distribution on a smooth function.

ii) Let first \(f \in \mathcal{C}^1(\mathbb{R}^{1+n})\). By assumption, \(L f = 0\) only near \(\gamma_{x_0, \theta_0}\), so we choose a local chart for the lines close to \(\gamma_{x_0, \theta_0}\). Since \(L f = 0\), we have \((v, \nabla_x) L f = 0\) for \(v \in \mathbb{R}^n\). Let \(v\) be an arbitrary fixed vector in \(\mathbb{R}^n\) and let \(x_0 = 0\) and \(\theta_0 = \pm e_1\). Our goal is to show \((0, \zeta^0) \not\in WF_A(h)\) for \(\zeta^0\) a non-zero space-like vector and conormal to \(\gamma_0\) at \(x_0 = 0\). Applying the Lorentz transformation, we may assume that \(\zeta^0 = e^{a-1} := (0, \ldots, 0, 1, 0) \in \mathbb{R}^{1+n}\). Let \(\chi_N \in C_0^\infty(\mathbb{R}^n)\) be supported in \(B(0, \varepsilon), \varepsilon > 0\), with \(\chi_N = 1\) near \(x_0 = 0\) so that

\[
(4.7) \quad |\partial_x \chi_N| \leq (CN)^{|\alpha|}, \quad |\alpha| \leq N.
\]

Then for \(0 < \varepsilon \ll 1, \lambda > 0\), and \(\theta\) near \(\theta_0\),

\[
0 = \int e^{i\lambda x \xi} \chi_N(v, \nabla_x) L f(x, \theta) \, dx = \int \int e^{i\lambda x \xi} \chi_N(x) \tilde{f}_{\alpha}(x, \theta, s) \cdot (1, \theta) \, ds \, dx.
\]
Similar to the first part of theorem, we make a change of variable $z = \gamma_{x,0}$ to get
\[ (4.8) \quad 0 = \int e^{iz x \xi} (\chi_N(x, \nabla x) Lf)(x, \theta) dx = \int e^{iz(x, \theta) \xi} a_N(z, \theta) \tilde{f}_{i_n}(z)(1, \theta)^i dz = \int e^{iz x \xi} a_N(z, \theta) \tilde{f}_{i_n}(z)(1, \theta)^i dz, \]
when $(1, \theta) \not\perp \zeta$. Notice that $a_N(0, \theta) = 1$.

Let $a_1, a_2, \ldots, a_{n-1}$, be $n - 1$ non-zero parameters near zero. We set $\theta(a_1, a_2, \ldots, a_{n-1})$ to be the $n$-dimensional spherical coordinates where
\[
\begin{align*}
\theta^1 &= \sin(a_{n-1}) \sin(a_{n-2}) \sin(a_{n-3}) \ldots \sin(a_4) \sin(a_3) \sin(a_2) \\
\theta^2 &= \sin(a_{n-1}) \sin(a_{n-2}) \sin(a_{n-3}) \ldots \sin(a_4) \sin(a_3) \cos(a_2) \\
\theta^3 &= \sin(a_{n-1}) \sin(a_{n-2}) \sin(a_{n-3}) \ldots \sin(a_4) \cos(a_3) \\
\vdots & \\
\theta^{n-2} &= \sin(a_{n-1}) \sin(a_{n-2}) \cos(a_{n-3}) \\
\theta^{n-1} &= \sin(a_{n-1}) \cos(a_{n-2}) \\
\theta^n &= \cos(a_{n-1})
\end{align*}
\]
Clearly $|\theta(a_1, a_2, \ldots, a_{n-1})| = 1$, $\theta(a_1, a_2, \ldots, 0) = e_n$. Considering the $n$-dimensional spherical coordinate, one may solve the equation $(1, \theta) \cdot \zeta = 0$ for $\zeta = (\tau, \xi)$ to get
\[
\zeta((a_1, a_2, \ldots, a_{n-1}), \xi) = (-\theta(a_1, a_2, \ldots, a_{n-1}) \cdot \xi, \xi).
\]
To simplify our analysis, we show the rest of proof for $n = 3$. For $n > 3$, one may repeat the following arguments to conclude the result. Let
\[
\theta(a, b) = \sin(a) \sin(b) e_1 + \sin(a) \cos(b) e_2 + \cos(a) e_3
\]
be the 3-dimensional spherical coordinates. Plugging $\theta(a, b)$ into (4.8) we get
\[
\int e^{i\lambda \phi(z, \zeta)} a_N(z, \theta(a, b)) \tilde{f}_{i_n}(z)(1, \theta(a, b))^i dz = 0, \quad \text{near } a = 0,
\]
where $\phi(z, \xi) = z \cdot \zeta((a, b), \xi)$ and $a_N(z, \theta(a, b)) = \chi_N(x - s\theta(a, b))$ with $a_N(0, (a, b)) = 1$. Note that
\[
\phi_z(0, \xi) = \zeta, \quad \phi_\zeta = \text{Id}.
\]
It is more convenient to work with $\zeta$ variable instead of $(a, b, \xi)$. Let $b$ be a non-zero fixed parameter near zero. We show that the map $((a, b), \xi) \to \zeta \in \mathbb{R}^{1+3}$ is a local analytic diffeomorphism near $(0, b, e^2)$. More precisely, the determinant of Jacobian associated with the map $((a, b), \xi) \to \zeta \in \mathbb{R}^{1+3}$ is
\[
-\cos(a) \sin(b) \xi^1 - \cos(a) \cos(b) \xi^2 + \sin(a) \xi^3,
\]
which is equal to $-\sin(b) \xi^1 - \cos(b) \xi^2$ near $a = 0$. Now the fixed parameter $b$ near zero and our choice of $\zeta^0$ imply that the determinant is $-\cos(b)$ which is non-zero. Hence, one may apply the Implicit function theorem near $a = 0$ to locally invert the map to $\zeta \to ((a, b), \xi) \in \mathbb{R}^{1+3}$. One may compute $a$ explicitly to get
\[
a = a(\zeta) = -\tan^{-1}(\frac{\xi^3}{\sin(b) \xi^1 + \cos(b) \xi^2}) + \sin^{-1}(-\frac{\tau}{\sqrt{(\sin(b) \xi^1 + \cos(b) \xi^2)^2 + \xi^3)^2}},
\]
which maps $a = 0$ to $\zeta^0$ diffeomorphically. Notice that for the fixed parameter $b$ near zero and $\zeta \approx \zeta^0$, $a(\zeta)$ is the unique solution of the equation
\[
-\tau = -\theta(a, b) \cdot \xi = \sin(a) \sin(b) \xi^1 + \sin(a) \cos(b) \xi^2 + \cos(a) \xi^3,
\]
near $a = 0$. Therefore, we may work in the $\zeta$ variables instead of the $((a, b), \xi)$ to get
\[
\int e^{i\lambda \phi(z, \zeta)} \tilde{a}_N(z, \zeta) \tilde{f}_{i_n}(z)(1, \theta(\zeta))^i dz = 0, \quad \text{near } \zeta = e^2,
\]
where $\tilde{a}_N(z, \zeta) = a_N(z, \theta(a, b))$ and $\tilde{a}_N(0, \zeta) = 1$. 
In the next step, we will analyze the phase function and its critical points. (A similar argument as in the first part of theorem by applying the complex stationary phase method of Sjöstrand)

Fix 0 < \( \delta \ll 1 \) and let \( \chi_\delta \) be the characteristic function of the unit ball \( B(0, \delta) \) in \( \mathbb{R}^{1+3} \). With some \( w, \eta \in \mathbb{R}^{1+3} \) close to \( w = 0, \eta = e^2 \), multiply the l.h.s. of above integral equation by

\[
\chi_\delta(\zeta - \eta)e^{i\lambda((\zeta - \eta)^2/2 - w \cdot \zeta)}
\]

and integrate w.r.t. \( \zeta \) to get

\[
\int \int e^{i\lambda \Phi(z,w,\zeta,\eta)}b_N(z,\zeta,\eta)\tilde{f}_{iv}(z)(1,\theta(\zeta))^t d\zeta = 0, \quad \text{near } \zeta = e^2,
\]

where \( b_N = \chi_\delta(\zeta - \eta)a_N \) is a new amplitude and \( \Phi = (z - w) \cdot \zeta + i(\zeta - \eta)^2/2 \).

Now consider the phase function \( \zeta \to \Phi \). If \( w = z \), there is a unique real critical point \( \zeta_c = \eta \), and it satisfies \( \Im\Phi \zeta > 0 \) at \( \zeta = \zeta_c \). For \( w \neq z \), the phase \( \Phi \), as function of \( \zeta \), has a unique critical point \( \zeta_c = \eta + i(z - w) \).

Now we split the \( \zeta \)-integral (4.9) into two parts: over \( \Sigma = \{z; |z - w| \leq \delta/C^0\} \), for some \( C^0 > 1 \), and then over the complement of \( \Sigma \). Since \( |\Phi_\zeta| \) has a \( (\delta\text{-dependent}) \) positive lower bound for \( |z - w| > \delta/C^0 \) (for \( \zeta \) real) and there is no real critical point for the function \( \zeta \to \Phi \) in this set, we can estimate that part of integral. Using the estimate (4.10), integration by parts \( N \times \) w.r.t. \( \zeta \), and the fact that on the boundary \( |\zeta - \eta| = \delta \), the factor \( e^{i\lambda \Phi} \) is exponentially small with \( \lambda \), we get

\[
\left| \int \int_{\Sigma} e^{i\lambda \Phi(z,w,\zeta,\eta)}b_N(z,\zeta,\eta)\tilde{f}_{iv}(z)(1,\theta(\zeta))^t d\zeta \right| \leq C(CN/\lambda)^N + CN e^{-\lambda/C}.
\]

Again for above inequality, we used the fact that

\[
e^{i\lambda \Phi} = \frac{(\Phi)_\zeta \cdot \zeta}{i\lambda|\Phi_\zeta|^2} e^{i\lambda \Phi}.
\]

Now on the set \( \{z; |z - w| \leq \delta/C\} \), \( C \gg 1 \), we apply stationary phase method for the rest of \( \zeta \)-integral in (4.10). To estimate (4.10) for \( z \in \Sigma \) we set: \( \psi(z,w,\eta) = \Phi|_{\zeta = \zeta_c} \). Therefore,

\[
\psi = \eta \cdot (z - w) + i|z - w|^2 - \frac{1}{2}|z - w|^2 = \eta \cdot (z - w) + \frac{i}{2}|z - w|^2.
\]

Notice that \( \psi(z,w,\eta) \) satisfies

\[
\psi_z(z,w,\eta) = \eta = \phi_z(0,\eta), \quad \psi_w(z,w,\eta) = -\eta = -\phi_w(0,\eta), \quad \psi(z,w,\eta) = 0.
\]

For \( (z,\zeta) \) close to \( (0,e^2) \), we use this phase function and apply [38, Theorem 2.8] and the remark after it to the \( \zeta \)-integral above to get

\[
\int \int_{\Sigma} e^{i\lambda \Phi(z,w,\zeta,\eta)}b_N(z,\zeta,\eta)\tilde{f}_{iv}(z)(1,\theta(\zeta))^t d\zeta = \int \int_{\Sigma} e^{i\lambda \Phi(z,w,\zeta,\eta)}b_N(z,\zeta,\eta)\tilde{f}_{iv}(z)(1,\theta(\zeta))^t d\zeta
\]

\[
= \int \int_{\Sigma} e^{i\lambda \psi(z,w,\eta)}b_N(z,w,\eta)\tilde{f}_{iv}(z)(1,\theta(z,w))^t d\zeta = \int \int_{\Sigma} e^{i\lambda \psi(z,\beta)}b_N(z,\beta)\tilde{f}_{iv}(z)(1,\theta(z,\beta))^t d\zeta
\]

\[
\int \int_{\Sigma} e^{i\lambda \psi(z,\beta)}\tilde{f}_{iv}(z)\tilde{B}_\lambda(z,\beta) d\zeta = \mathcal{O}(\lambda^{n/2}(CN/\lambda)^N + CN e^{-\lambda/C}).
\]

Here \( \beta = (w,\eta) \) and \( \tilde{B}_\lambda \) is a classical elliptic analytic symbol of order 0. For \( z \in \Sigma \) with \( \delta \ll 1 \) and \( |w| \ll 1, \eta \) near \( e^2 \), the variable \( (z,\beta) \) in (4.11) is near \( (0,0,e^2) \) and then \( \tilde{B}_\lambda \) is independent of \( N \) because \( \chi_N = 1 \) near the origin. We choose \( N \) so that \( N \leq \lambda/(Ce) \leq N + 1 \). Therefore we get the following exponential error on the right,

\[
\int \int_{\Sigma} e^{i\lambda \psi(z,\beta)}\tilde{f}_{iv}(z)\tilde{B}_\lambda(z,\beta) d\zeta = \mathcal{O}(e^{-\lambda/C}).
\]
Since the phase function satisfies the properties in (4.10), on a small neighborhood of $\zeta^0$, we perform the following change of variable in above integral equation:

$$(z, w, \eta) \rightarrow (z, w, \zeta) = (z, w, \phi_z(w, \eta)),$$

which yields to

$$\int_{\Sigma} e^{i\lambda \phi(z, w, \zeta)} [\hat{f}_{0_{\alpha}}(z) \hat{B}^0_{\lambda}(z, w, \zeta) + \hat{f}_{1_{\alpha}}(z) \hat{B}^1_{\lambda}(z, w, \zeta) + \hat{f}_{2_{\alpha}}(z) \hat{B}^2_{\lambda}(z, w, \zeta) + \hat{f}_{3_{\alpha}}(z) \hat{B}^3_{\lambda}(z, w, \zeta)] dz \approx 0,$$

(4.12)

Here $\hat{B}^i_{\lambda}$ is a new classical elliptic symbol of order zero with the principal part of $\sigma_p(\hat{B}_{\lambda}(z, z, \zeta)) \equiv (1, \theta(\zeta))$, up to an elliptic factor. In particular, for $(z, w, \zeta) = (0, 0, 0)$, we have

$$\sigma_p(\hat{B}_{\lambda}(0, 0, 0, \zeta)) \equiv (1, \theta(\zeta)) = (1, 0, 0, 1) = (1, e_3).$$

As it is shown above, the map $(a, \xi) \rightarrow \zeta$ is a local diffeomorphism near $a = 0$ (equivalently near $\zeta^0 = e^3$). Therefore, we work with the principal symbol in terms of $(a, \xi)$ instead, which means up to an elliptic factor

$$\sigma_p(\hat{B}_{\lambda}(z, z, (a, \xi))) \equiv (1, \theta(a, b)).$$

To show $(0, \zeta^0) \notin \WF_A(df)$, we need to form an elliptic system of equations using $\hat{f}^3_{\alpha}$. Let $(z, z, (a, \xi))$ near $(0, 0, (0, \zeta^0))$ and $v$ be fixed. For our goal, we slightly perturb $\theta(a, b)$ near $a \approx 0$ and $b$. Let

$$\{\Theta_k\}_{k=0}^{3} = \{(1, \theta(a, b)), (1, \theta(-a, b)), (1, \theta(a, -b)), (1, \theta(0, b))\}$$

be the set of perturbations of $\theta(a, b)$, with property $\sigma_p(\hat{B}_{\lambda}(z, z, (a, \xi))) = \Theta_k$, for $k = 0, 1, 2, 3$. Microlocal version of ellipticity is equivalent to show that for a constant vector field $\hat{f}_v = \hat{f}_{0_v} + \hat{f}_{1_v} + \hat{f}_{2_v} + \hat{f}_{3_v}$,

$$\Theta \hat{f}_v = 0,$$

forms an elliptic system of equations. Here the matrix $[\Theta]_{4 \times 4}$ is the associated matrix with above principal symbols, $\Theta_k$. The matrix $[\Theta]$ is invertible since

$$\det(\Theta) = \det \begin{pmatrix} 1 & \sin(a) \sin(b) & \sin(a) \cos(b) & \cos(a) \\ 1 & -\sin(a) \sin(b) & -\sin(a) \cos(b) & \cos(a) \\ 1 & -\sin(a) \sin(b) & \sin(a) \cos(b) & \cos(a) \\ 0 & 0 & 0 & 1 \end{pmatrix} = 4 \sin^2(a) \sin(b) \cos(b) (1 - \cos(a)) \neq 0,$$

for $a$ and our fixed parameter $b$ near zero. Therefore $\Theta \hat{f}_v = 0$ implies that $\hat{f}_v = 0$. Therefore, the equation $[\Theta]_{4 \times 4}$ with $\{\Theta_k\}_{k=0}^{3}$ leads to the following system of equations related to $\theta_0 = e_3$:

$$\int_{\Sigma} e^{i\lambda \phi(z, w, \zeta)} [v^1(\partial_1 f_0 - \partial_2 f_1) + v^2(\partial_2 f_0 - \partial_3 f_2) + v^3(\partial_3 f_0 - \partial_1 f_3)] B^0_{\lambda}(z, \beta) dz = O(e^{-\lambda/C})$$

$$\int_{\Sigma} e^{i\lambda \phi(z, w, \zeta)} [v^1(\partial_1 f_1 - \partial_2 f_2) + v^2(\partial_2 f_1 - \partial_3 f_3) + v^3(\partial_3 f_2 - \partial_1 f_3)] B^1_{\lambda}(z, \beta) dz = O(e^{-\lambda/C})$$

$$\int_{\Sigma} e^{i\lambda \phi(z, w, \zeta)} [v^1(\partial_1 f_2 - \partial_2 f_3) + v^2(\partial_2 f_2 - \partial_3 f_3)] B^2_{\lambda}(z, \beta) dz = O(e^{-\lambda/C})$$

with respect to an elliptic factor

$$\sigma_p(B^2_{\lambda}(z, z, \zeta)) = \begin{cases} 1 & i = 0 \\ \sin(a) \sin(b) & i = 1 \\ \sin(a) \cos(b) & i = 2 \\ \cos(a) & i = 3. \end{cases}$$

Here the phase function $\psi$ satisfies the conditions $\lambda \phi \geq 2$ and $\Re \psi > C_0 |z - w|^2$ as $\psi_{2\xi} = \Id$. Note that, some components of $df$ are repeated in above equations. This forms an overdetermined system of equations for our fixed vector $v \in \mathbb{R}^3$. Since $v$ is arbitrary, for any choice of three linearly independent vectors $\{v_i\}_{i=1}^3 \subseteq \mathbb{R}^3$, for $(z, z, (z, \xi)) = (0, 0, (0, \zeta^0))$ one may conclude that $(0, \zeta^0) \notin \WF_A(df)$, which proves the second part of the theorem for $n = 3$. For the general case $n > 3$, one needs to slightly perturb $\theta(a_1, a_2, \ldots, a_n)$ with respect to parameters $a_1, a_2, \ldots, a_{n-1}$. This forms an elliptic system $\Theta \hat{f}_v = 0$ for the microlocal ellipticity.
discussion and therefore concluding the result. Now for any vector-valued distribution \( f \in \mathcal{E}'(\mathbb{R}^{1+n}) \), as we pointed out in the proof of part (i), the result remains true in the sense of distributions. \( \square \)

**Remark 4.1.** By Fundamental Theorem of Calculus, the potential field is in the kernel of operator \( L \), so one could only hope to recover \( \text{curl} f \) for \( n = 2 \). For the Riemannian case with dimension \( n \geq 3 \), foliation (slicing method) can be used to achieve the uniqueness results. One may restrict \( x \) to a two-dimensional plane, say \( \Pi = \{ (t, x^3, \cdots, x^n) = \text{const} \} \), and apply the results in Theorem 4.1 when \( n = 2 \). This only recovers some components of the generalized curl of the vector field \( f \) even if different permutations are chosen to fix different components of \( x \). In order to recover all other components, one needs to perturb above two-dimensional planes. Therefore, such a slicing technique can be done as the transform is overdetermined, however, additional assumption which is the information of light-ray for two discrete directions \((1, \pm \theta)\) is required. Even though the foliation method is a simpler approach for the recovery of the vector field \( f \), we do not perform foliation to achieve stronger results.

## 5. Proof of Main Results

For our main result we need the following lemma which is a unique analytic continuation result across a timelike hypersurface in the Minkowski time-space.

**Lemma 5.1.** Let \( f \in C^\infty(\mathbb{R}^{1+n}) \) and let \( \gamma_{x_0, \theta_0} \) be a fixed light-like lines in the Minkowski time-space so that \( \gamma_{x, \theta} \) does not intersect \( \text{supp} f \) for \( |s| \geq 1/C \) with some \( C \) for all \((x, \theta)\) near \((x_0, \theta_0)\). Fix \( z_0 = (s_0, x_0 + s_0 \theta_0) \in \gamma_{x_0, \theta_0} \), let \( S \) be an analytic timelike hypersurface near \( z_0 \) and assume that \( \gamma_{x_0, \theta_0} \) is tangent to \( S \) at \( z_0 \).

i) For \( n = 2 \), if \( Lf(x, \theta) = 0 \) near \((x_0, \pm \theta_0)\) and if \( \text{curl} f = 0 \) on one side of \( S \) near \( z_0 \), then \( \text{curl} f = 0 \) near \( z_0 \).

ii) For \( n \geq 3 \), if \( Lf(x, \theta) = 0 \) near \((x_0, \theta_0)\) and if \( df = 0 \) on one side of \( S \) near \( z_0 \), then \( df = 0 \) near \( z_0 \).

**Proof.** Let \( n \geq 3 \) and assume that \( z_0 = \text{supp} df \). By assumption \((v, \nabla_x) Lf(x, \theta) = 0 \) near \((z_0, \theta_0)\). Since \( df \) is non-zero only in half space \( S \), then there exists \( \zeta_0 \) such that \((z_0, \zeta_0) \in \text{WF}_A(df)\), as \( df \) cannot be analytic at \( z_0 \). By the definition of analytic wave front set for vector-valued distributions, there exist a component of \( df \), say \( f_{ij} = \partial_j f_i - \partial_i f_j \), where \((z_0, \zeta_0) \in \text{WF}_A(f_{ij})\). In other words, if the half space \( S \) intersects the \( \text{supp} df \), it must intersect at least one of the components of \( df = f_{ij} \), as \( df \) cannot be analytic at the intersection point. Now by Sato-Kawai-Kashiwara Theorem (see [30], [31]), \((z_0, \zeta_0 + sv(z_0)) \in \text{WF}_A(f_{ij})\), where \( v(z_0) \) is one of the two unit conormals to \( S \) at \( z_0 \). This in turn implies that \((z_0, \frac{\zeta_0}{c} + v(z_0)) \in \text{WF}_A(f_{ij})\) as the wave front set is a conic set. Now passing to limit we have \((z_0, v(z_0)) \in \text{WF}_A(f_{ij})\) since the analytic wave front set is closed. By assumption on \( S \), that vector is space-like and is conormal to \( \gamma_{x_0, \theta_0}(s_0) \). This contradicts Theorem 4.1 part (ii), which implies that \( df = 0 \) and completes the proof. For \( n = 2 \), one may repeat above arguments and use the first part of Theorem 4.1 to conclude the result. \( \square \)

We now ready to state the proof of the main result.

**Proof of Theorem 2.1.** Let \( n \geq 3 \). By assumption \((v, \nabla_x) Lf(x, \theta) = 0 \). The proof follows from [Theorem 2.1, 30, 31] replacing \( Lf(x, \theta) \) by \((v, \nabla_x) Lf(x, \theta)\) and applying the second part of Lemma 5.1.

To conclude the result for \( n = 2 \), one may use Lemma 5.1 part (i) and repeat the proof of [Theorem 2.1, 30, 31]. \( \square \)

## 6. Examples

In the following examples we illustrate how our result imply the recovery of vector field up to a smooth potential field.

**Example 6.1.** Let \( f \) be a vector field (distribution) supported in cone \( \{(t, x) \in \mathbb{R}^{1+n} \mid |x| < c|t|\} \) and let \( \Gamma_{\rho_0} \) be the following surface:

\[
\Gamma_{\rho_0} = \{(t, x) \in \mathbb{R}^{1+n} \mid \psi(t, x) = |x - x_0|^2 - c^2|t - t_0|^2 - \rho_0^2 = 0\},
\]
for some \( \rho_0 \geq 0 \) and \( 0 < c < 1 \). Assume now that \( f \) integrates to zero over all light-like lines \( \gamma \) in the exterior of \( \Gamma_{\rho_0}, \operatorname{ext}(\Gamma_{\rho_0}) \).

**Figure 1.** \( \Gamma_{\rho_0} \) with \( 0 < c < 1 \).

We show the vector field \( f \) can be recovered up to a potential field in the \( \operatorname{ext}(\Gamma_{\rho_0}) \). By definition \( (z, \zeta) \) is conormal to \( \Gamma_{\rho_0} \) if and only if

\[
(z, \zeta) \in N^*\Gamma_{\rho_0} = \{(t, x, \tau, \xi) \in T^*(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}) \mid (t, x) \in \Gamma_{\rho_0}, \quad (\tau, \xi) = 0 \text{ on } T_{(t,x)}\Gamma_{\rho_0}\}.
\]

Clearly the gradient of \( \psi, \nabla \psi \), is normal to surface \( \Gamma_{\rho_0} \) at \( (t, x) \). So by definition \( \sigma \nabla \psi \cdot (dt, dx) \) is the conormal vector to surface \( \Gamma_{\rho_0} \) at \( (t, x) \). In fact to find the conormal we compute the total differential of \( \psi \),

\[
d\psi(t, x) = -2c^2(t - t_0)dt + 2(x - x_0)dx,
\]

and therefore the covector:

\[
(z, \zeta) = (t, x, \tau, \xi) = (t, x, -2\sigma c^2(t - t_0), 2\sigma(x - x_0)) \in N^*\Gamma_{\rho_0}, \quad \text{for } \sigma \in \mathbb{R},
\]

is conormal to \( \Gamma_{\rho_0} \). Clearly the \( \zeta = (\tau, \xi) \) is space-like in the \( \operatorname{ext}(\Gamma_{\rho_0}) \) as it is easy to show \( |\tau| = 2\sigma c^2|t - t_0| \leq 2\sigma|x - x_0| = |\xi| \) in the \( \operatorname{ext}(\Gamma_{\rho_0}) \). Therefore, for any decreasing family of \( \rho \) with the property \( \rho \to \rho_0 \), the surfaces \( \Gamma_{\rho} \) will be a family of analytic time-like hypersurfaces in the \( \operatorname{ext}(\Gamma_{\rho_0}) \).

Let \( \sigma = \frac{1}{2}, \quad (t_0, x_0) = (0, 0) \), and fixed \( \rho > 0 \) be the smallest one with the property that \( \operatorname{supp} f \cap \Gamma_{\rho} \neq \emptyset \). This is true since \( \operatorname{supp} f, \Gamma_{\rho} \) is a compact set. Now assume that \( \gamma_0 \) is tangent to \( \Gamma_{\rho} \) at \( z_0 \) (i.e. \( (z_0, \zeta_0) \) is conormal to \( \gamma_0 \)). By compactness of \( \operatorname{supp} f \), we have \( \gamma \not\in \operatorname{supp} f \) for any \( \gamma \) approaching \( \gamma_0 \) in the \( \operatorname{ext}(\Gamma_{\rho_0}) \). Theorem 4.1 implies \( \operatorname{WF}_\Lambda(df) \) contains no space-like vector conormal to \( \gamma_0 \) since by assumption \( Lf = 0 \) over all light-like lines \( \gamma \) near \( \gamma_0 \) on one side of analytic time-like hypersurface \( \Gamma_{\rho} \) (see Figure 1.) Now using the analytic continuation result, Lemma 5.1, we can recover the vector field \( f \) up to a potential field.

**Example 6.2.** Let \( f \in \mathcal{E}'(\mathbb{R}^{1+n}) \) be so that \( \operatorname{supp} f \) expands with a speed less than one and let \( \Gamma_0 \) be the following surface:

\[
\Gamma_0 = \{(t, x) \in \mathbb{R}^{1+n} \mid \psi(t, x) = (|x - x_0| - R)^2 - c^2|t - t_0|^2 = 0\}, \quad \text{for some } 0 < c < 1.
\]

Assume now that \( f \) integrates to zero over all light-like lines \( \gamma \) intersecting \( \operatorname{supp} f \) outside of the surface \( \Gamma_0 \). We show the vector field \( f \) can be recovered up to a potential field in the \( \operatorname{ext}(\Gamma_{\rho}) \), where the surface \( \Gamma_{\rho} \) with \( \rho > 0 \) is defined as follow:

\[
\Gamma_{\rho} = \{(t, x) \in \mathbb{R}^{1+n} \mid \psi(t, x) = (|x - x_0| - R)^2 - c^2|t - t_0|^2 - \rho^2 = 0\}.
\]
By definition the covector:

\[
(z, \zeta) = (t, x, \tau, \xi) = \left( \frac{x - x_0}{|x - x_0|} \right) (|x - x_0| - R) \in N^* \Gamma_\rho, \quad \text{for } \sigma \in \mathbb{R},
\]

is conormal to \( \Gamma_\rho \). Note that \(|\tau| = \frac{2\sigma c^2}{|x - x_0|} \) and \(|\xi| = \frac{2\sigma}{|x - x_0|} \). So in the \( \text{ext}(\Gamma_\rho) \), the covector \( \zeta = (\tau, \xi) \) is space-like (i.e. \(|\tau| \leq |\xi|\)). Thus, for \( \rho > 0 \), the surfaces \( \Gamma_\rho \) will be a family of analytic timelike hypersurfaces.

Example 6.3. Let \( f \in \mathcal{E}'(\mathbb{R}^{1+n}) \) be so that \( \text{supp} f \) expands with a speed less than one, and consider the cylinder \([0, T] \times \Omega\) for some \( T > 0 \) and \( \Omega \subset \mathbb{R}^n \). In Example 6.2 we showed that the vector field \( f \) can be recovered up to a smooth potential field in the exterior of \( \Gamma_\rho \). Now consider the surface \( \Gamma \) as union of all those exteriors of timelike hypersurfaces for \( t \in [0, T] \). This surface includes all light-like lines \( \gamma_{x, \theta} = (s, x + s\theta), \quad (z, \theta) \in \mathbb{R}^n \times S^{n-1}, \) not intersecting the top and the bottom of the cylinder \([0, T] \times \Omega\) (see Figure 3.) By Theorem 2.1, we can recover \( f \) up to \( d\phi, \phi = 0 \) on \([0, T] \times \Omega\) in the set covered by those lines for \( n \geq 3 \). As we pointed out on Remark 3.1, for \( n = 2 \) there are two discrete choices of directions which means that for recovery of \( f \) up to a potential \( d\phi \) one needs to know \( Lf \) along light-like \( \gamma_{x, \theta} \) as well as knowing \( Lf \) along \( \gamma_{x, -\theta} \). Note that our uniqueness result do not require the vector field to be compactly supported in time. Moreover, we are not considering any Cauchy data on circles on top and bottom of the cylinder, which means there is no internal measurement. This is the optimal way one can wish to recover the vector field \( f \) in this set. Uniqueness result in this paper and result in \( \cite{46} \) (recovery of the unknown potential \( q \) in \( \Gamma \)) generalize the uniqueness results in \( \cite{40, 32, 31, 49, 33, 4, 3} \).
REFERENCES

[1] A. Begmatov. A certain inversion problem for the ray transform with incomplete data. Siberian Math. Journal, 42(3):428-434, 2001.

[2] M. Bellassoued and I. Ben Aicha. Stable determination outside a cloaking region of two time-dependent coefficients in an hyperbolic equation from Dirichlet to Neumann map. arXiv:1605.03466, 2016.

[3] M. Bellassoued and D. Dos Santos Ferreira. Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map. Inverse Problem. Imaging, 5(4):745-773, 2011.

[4] I. Ben Aicha. Stability estimate for hyperbolic inverse problem with time dependent coefficient. arXiv:1506.01935, 2015.

[5] J. Boman. Helgasons support theorem for Radon transforms a new proof and a generalization. In Mathematical methods in tomography (Oberwolfach, 1990), volume 1497 of Lecture Notes in Math., pages 15. Springer, Berlin, 1991.

[6] J. Boman and E. T. Quinto. Support theorems for real-analytic Radon transforms. Duke Math. J., 55(4):943-948, 1987.

[7] J. Boman and E. T. Quinto. Support theorems for Radon transforms on real analytic line complexes in three-space. Trans. Amer. Math. Soc., 335(2):877-890, 1993

[8] J. M. Bony. Equivalence des diverses notions de spectre singulier analytique. Séminaire Goulaouic-Schwartz, 1976-77, no.3.

[9] J. Bros, D. Iagolnitzer. Support essentiel et structure analytique des distributions. Séminaire Goulaouic-Lions-Schwartz, 1975-76, no. 18.

[10] A. Denisiuk. Inversion of the x-ray transform for 3D symmetric tensor fields with sources on a curve. Inverse Problems, 22 (2006) 399411.
[11] B. Frigyik, P. Stefanov, and G. Uhlmann. The X-ray transform for a generic family of curves and weights. J. Geom. Anal., 18(1):89108, 2008.

[12] A. Greenleaf and G. Uhlmann. Nonlocal inversion formulas for the X-ray transform. Duke Math. J., 58(1):205240, 1989.

[13] A. Greenleaf and G. Uhlmann. Estimates for singular Radon transforms and pseudodifferential operators with singular symbols. J. Funct. Anal., 89(1):202232, 1990.

[14] A. Greenleaf and G. Uhlmann. Microlocal techniques in integral geometry. In Integral geometry and tomography (Arcata, CA, 1989), volume 113 of Contemp. Math., pages 121135. Amer. Math. Soc., Providence, RI, 1990.

[15] V. Guillemin. On some results of Gelfand in integral geometry. In Pseudodifferential operators and applications (Notre Dame, Ind., 1984), volume 43 of Proc. Sympos. Pure Math., pages 149155. Amer. Math. Soc., Providence, RI, 1985.

[16] V. Guillemin and S. Sternberg. Geometric asymptotics. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[17] S. Helgason. The Radon transform, volume 5 of Progress in Mathematics. Birkh"auser Boston Inc., Boston, MA, second edition, 1999.

[18] S. Holman and P. Stefanov. The Weighted Doppler Transform. Inverse Problem. Imaging, 4(1):111-130, 2010.

[19] L. Hörmander. Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients. Comm. Pure Appl. Math. 24 (1971), 671-704.

[20] C. E. Kenig, J. Sjöstrand., and G. Uhlmann. The Calderón problem with partial data. Ann. of Math. (2), 165(2):567591, 2007.

[21] Y. Kian. Recovery of time-dependent damping coefficients and potentials appearing in wave equation from partial data. arXiv:1603.09600, 2016.

[22] V. P. Krishnan. A support theorem for the geodesic ray transform on functions J. Fourier Anal. Appl., 15(4):515520, 2009.

[23] V. P. Krishnan and P. Stefanov. A support theorem for the geodesic ray transform of symmetric tensor fields. Inverse Probl. Imaging, 3(3):453464, 2009.

[24] M. Lassas, L. Oksanen, P. Stefanov, and G. Uhlmann. On the Inverse Problem of Finding Cosmic Strings and Other Topological Defects. arXiv:1505.03123.

[25] R. B. Melrose. Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. Inverse Probl. In Spectral and scattering theory (Sanda, 1992), volume 161 of Lecture Notes in Pure and Appl. Math., pages 85130. Dekker, New York, 1994.

[26] C. Montalto. Stable determination of a simple metric, a covector field and a potential from the hyperbolic Dirichlet-to-Neumann map. Partial Differential Equations, 39(1):120145, 2014.

[27] G.P. Paternain, M. Salo and G. Uhlmann. Tensor tomography on simple surfaces. arXiv:1109.0505 to appear in Invent. Math.

[28] L. Pestov and G. Uhlmann. Two dimensional simple compact manifolds with boundary are boundary rigid. Ann.Math.161(2)(2005),10891106.

[29] A. Z. Petrov. Einstein Spaces. Translated from the Russian by R. F. Kelleher. Translation edited by J. Woodrow. Pergamon Press, Oxford-Edinburgh-New York, 1969.

[30] E.T. Quinto. Real analytic Radon transforms on Rank one symmetric spaces. Proc. Math. Soc. 117(1993), 179-186.

[31] A. G. Ramm and Rakesh. Property C and an inverse problem for a hyperbolic equation. J. Math. Anal. Appl., 156(1):209219, 1991.
[32] A. G. Ramm and J. Sjöstrand. *An inverse problem of the wave equation*. Math. Z., 206(1):119130, 1991.

[33] R. Salazar. *Determination of time-dependent coefficients for a hyperbolic inverse problem*. Inverse Problems, 29(9):095015, 17, 2013.

[34] R. Salazar. *Stability estimate for the relativistic Schrödinger equation with time-dependent vector potentials*. Inverse Problems, 30(10):105005, 18, 2014.

[35] M. Sato. *Hyperfunctions and partial differential equations*. Proc. Int. Conf. Funct. Anal. Tokyo 1969, 9194.

[36] M. Sato, T. Kawai, and M. Kashiwara. *Microfunctions and pseudo-differential equations*. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, pages 265529. Lecture Notes in Math., Vol. 287. Springer, Berlin, 1973.

[37] V. A. Sharafutdinov. *Integral Geometry of Tensor Fields, Inverse and Ill-Posed Problems Series*. VSP, Utrecht, 1994

[38] J. Sjöstrand. *Singularités analytiques microlocales. In Astérisque, 95, volume 95 of Astérisque, pages 1166*. Soc. Math. France, Paris, 1982.

[39] P. Stefanov. *Support Theorems for the Light Ray Transform on Analytic Lorentzian Manifolds*. Proc. Amer. Math. Soc. 145(3), 1259-1274, 2017.

[40] P. Stefanov. *Uniqueness of the multi-dimensional inverse scattering problem for time dependent potentials*. Math. Z., 201(4):541559, 1989.

[41] P. Stefanov and G. Uhlmann. *Microlocal Analysis and Integral Geometry*. Book in Progress

[42] P. Stefanov and G. Uhlmann. *Stability estimates for the X-ray transform of tensor fields and boundary rigidity*. Duke Math. J., 123(3):445467, 2004.

[43] P. Stefanov and G. Uhlmann. *Boundary rigidity and stability for generic simple metrics*. J. Amer. Math. Soc., 18(4):9751003, 2005.

[44] P. Stefanov and G. Uhlmann. *Integral geometry of tensor fields on a class of non-simple Riemannian manifolds*. Amer. J. Math., 130(1):239268, 2008.

[45] P. Stefanov and G. Uhlmann. *The geodesic X-ray transform with fold caustics*. Anal. PDE, 5-2:219260, 2012.

[46] P. Stefanov and Y. Yang. *The Inverse Problem for The Dirichlet-to-Neumann Map on Lorentzian Manifolds*. 2016.

[47] F. Trèves. *Introduction to Pseudodifferential and Fourier Integral Operators, Vol. 1. Pseudodifferential Operators*. The University Series in Mathematics. Plenum Press, New York, 1980.

[48] G. Uhlmann and A. Vasy. *The inverse problem for the local geodesic ray transform*. Inventiones mathematicae, pages 138, 2015.

[49] A. Waters. *Stable determination of X-Ray transforms of time dependent potentials from partial boundary data*. Comm. Partial Differential Equations, 39(12):21692197, 2014.

[50] A. Waters and R. Salazar. *Global stability for time dependent X-ray transforms on simple manifolds and applications*. [arXiv:1311.1591] 2013.

Department of Mathematics, Purdue University, West Lafayette, IN 47907