HOMEOMORPHIC MODEL FOR THE POLYHEDRAL SMASH PRODUCT OF
DISKS AND SPHERES

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Abstract. In this paper we present unpublished work by David Stone on polyhedral smash products. He proved that the polyhedral smash product of the CW-pair \((D^2, S^1)\) over a simplicial complex \(K\) is homeomorphic to an iterated suspension of the geometric realization of \(K\). Here we generalize his technique to the CW-pair \((D^{k+1}, S^k)\), for an arbitrary \(k\). We generalize the result further to a set of disks and spheres of different dimensions.

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1. Introduction

In all the following, \(m \in \mathbb{N}\) is any natural number and \([m] = \{1, \ldots, m\}\). Also, we set \(K\) to be an abstract simplicial complex whose vertex set is contained in \([m]\), that is \(K\) is a family of subsets \(\sigma \subseteq [m]\), called simplices, such that whenever \(\sigma \in K\) and \(\tau \subseteq \sigma\), then \(\tau \in K\).

**Definition 1.1** (Polyhedral smash product). Let \((X, A) = \{(X_i, A_i)\}_{i \in [m]}\) be a family of pointed CW-pairs, that is, the \(X_i\) are CW-complexes and \(A_i \hookrightarrow X_i\) are subcomplexes, for all \(i \in [m]\). The **polyhedral smash product** of \((X, A)\) over \(K\), denoted \(\hat{Z}(K; (X, A)) \subseteq \bigwedge_{i=1}^{m} X_i\), is the space given by

\[
\hat{Z}(K; (X, A)) = \bigcup_{\sigma \in K} \hat{D}(\sigma),
\]

where \(\hat{D}(\sigma) = \bigwedge_{i=1}^{m} Y_i\) with \(Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{otherwise} \end{cases}, \quad \forall \sigma \in K.
\]

Using categorical language, consider \(\text{CAT}(K)\) to be the face category of \(K\), that is, objects are simplices and morphisms are inclusions. Define the \(\text{CAT}(K)\)-diagram given by

\[
\hat{D} : \text{CAT}(K) \to \text{Top}
\]

\[
\sigma \mapsto \hat{D}(\sigma),
\]

where \(\hat{D}(\sigma)\) is given by (1) and the functor \(\hat{D}\) maps the morphism \(\rho \subseteq \sigma\) to the inclusion \(\hat{D}(\rho) \subseteq \hat{D}(\sigma)\). Then

\[
\hat{Z}(K; (X, A)) = \text{colim}_{\sigma \in K} \hat{D}(\sigma).
\]

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Below we recall some well-known operations on spaces.

**Definition 1.2.** [4, §0] For \( n \in \mathbb{N} \), let \((X, x_0)\) and \((Y, y_0)\) be two pointed topological spaces.

- The **join** \( X \ast Y \) of \( X \) and \( Y \) is the quotient space defined by \( X \ast Y = X \times Y / \sim \), where \( I = [0, 1] \) and \( \sim \) is the equivalence relation generated by
  \[
  (x, y, 0) \sim (x, y', 0), \ \forall x \in X \text{ and } \forall y, y' \in Y,
  \]
  \[
  (x, y, 1) \sim (x', y, 1), \ \forall x, x' \in X \text{ and } \forall y \in Y.
  \]
- The **wedge sum** \( X \vee Y \) of \( X \) and \( Y \) is the quotient space defined by \( X \vee Y = X \amalg Y / (x_0 \sim y_0) \).
- The **smash product** \( X \wedge Y \) of \( X \) and \( Y \) is the quotient space defined by \( X \wedge Y = X \times Y / X \vee Y \).
- The (unreduced) **suspension** \( \Sigma X \) of \( X \) is the space defined by \( \Sigma X = S^0 \ast X \), where \( S^0 \) denotes the 0-sphere.
- The (unreduced) **cone** \( CX \) of \( X \) is the space defined by \( CX = c \ast X \), where \( c \) is a single point.

David stone made the following conjecture.

**Conjecture 1.3.** If \( F \) is a compact subspace of \( \mathbb{R}^n \), then there is a homeomorphism

\[
\widehat{Z}(K; (c \ast \Sigma F, \Sigma F)) \cong \Sigma \left( \ast^m F \right) \ast |K|,
\]

where \( \ast^m F \) is defined as the \( m \)-fold join of \( m \) copies of \( F \).

As it is mentioned in [1, Remark 2.20], David Stone used a kind of geometrical argument to prove a particular case of his conjecture by taking \( F = S^0 \). Hence he proved the following.

**Theorem 1.4.** [5] There is a homeomorphism

\[
\widehat{Z}(K; (D^2, S^1)) \cong \Sigma^{m+1} |K|.
\]

In this paper we apply the same technique to a more general case. For \( k \in \mathbb{N} \), we consider \( F = S^{k-1} \), which is compact (as a closed and bounded subspace of \( \mathbb{R}^k \)) and we have \( \Sigma^i F \cong S^k \), \( c \ast \Sigma F \cong D^{k+1} \) and \( \ast^m F \cong S^{km} \) (since \( S^i \ast S^j \cong S^{i+j+1} \)). Hence

\[
\left( \ast^m F \right) \ast |K| \cong S^{km-1} \ast |K| \\
\cong \left( \ast^m S^0 \right) \ast |K| \\
\cong \Sigma^{km} |K|.
\]

So we can state a generalization of Stone’s result.

**Theorem 1.5.** For any \( k \in \mathbb{N} \cup \{0\} \), there is a homeomorphism

\[
\widehat{Z}(K; (D^{k+1}, S^k)) \cong \Sigma^{km+1} |K|.
\]

The goal of this paper is first to generalize David Stone’s technique for the proof of **Theorem 1.5** and secondly to provide a further generalization (see **Theorem 6.6** of the latter result for a set of disks and spheres of different dimensions.

**Theorem 1.6.** For any \( m \)-tuple \( J = (j_1, \cdots, j_m) \) in \( (\mathbb{N} \cup \{0\})^m \), there is a homeomorphism

\[
\widehat{Z}(K; (D^{J+1}, S^J)) \cong \Sigma^{j_1 + \cdots + j_m + 1} |K|,
\]

where \( (D^{J+1}, S^J) = \{(D^{j_1+1}, S^{j_1}), \cdots, (D^{j_m+1}, S^{j_m})\} \).

In order to prove **Theorem 1.5** we need to put together some topological and combinatorial tools, hence the rest of the paper is organized as follows. In **Sections 2** and **3**, we describe respectively the necessary topological and combinatorial tools. **Section 4** is devoted to the proof of **Theorem 1.5** for \( k \geq 1 \). The case \( k = 0 \), namely **Theorem 5.1** is treated in **Section 5** using a
more categorical argument. Finally, in Section 6 we prove the main result, Theorem 6.6, using an inductive argument based on the case \( k = 0 \).

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2. Topological tools

Among the tools we use in the proof of Theorem 1.5, the homeomorphisms \( \Psi : C\Delta^{n-1} \to C^n \) and \( \overline{\Psi} : \Sigma\Delta^{n-1} \to \widetilde{D}^n \), described below, are both playing an important role. They were defined by David Stone in [5] and we recycle them here to prove this more general case. Before we introduce them, let us first recall the usual homeomorphism \( \widetilde{\Theta} : CX/X \to \Sigma X \).

Given a space \( X \), we identify \( X \) with the base \( c \times X \times \{1\} \) of \( CX = c \ast X \). Let \( S^0 = \{s_1, s_2\} \) to be the 0-sphere and consider the map

\[
\Theta : CX \to \Sigma X
\]

\[
[c, x, \lambda] \mapsto \Theta[c, x, \lambda] = \begin{cases} 
(s_1, x, 2\lambda), & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\
(s_2, x, 2 - 2\lambda), & \text{if } \frac{1}{2} \leq \lambda \leq 1.
\end{cases}
\]

Then \( \Theta \) factors through a map \( \widetilde{\Theta} : CX/X \to \Sigma X \); see Figure 1.

**Figure 1.** Factorization of \( \Theta \) through \( \widetilde{\Theta} \).

**Lemma 2.1.** The map \( \widetilde{\Theta} \) is a homeomorphism.

**Notation 2.2.** In \( \mathbb{R}^n \),

- let \( e_i \) be the \( i \)-th standard basis vector. Let \( c \) denote the origin and let \( t_1, \ldots, t_n \) denote the coordinates of a point \( x \in \mathbb{R}^n \). We identify \( x \sim cx \) and so \( x = \sum_{i=1}^n t_i e_i \).
• Set $C = [0, 2]$, with based point 2 and consider
  \[ C^n = [0, 2]^n = \{ \sum_{i=1}^{n} t_i e_i \in \mathbb{R}^n : 0 \leq t_i \leq 2 \} \text{, the } n\text{-cube of side 2.} \]
  \[ \partial_+ C^n = \{ \sum_{i=1}^{n} t_i e_i \in C^n : \max t_i = 2 \} \text{, the outer boundary of } C^n. \]
  \[ \partial_- C^n = \{ \sum_{i=1}^{n} t_i e_i \in C^n : \min t_i = 0 \} \text{, the inner boundary of } C^n. \]
  \[ \partial C^n = \partial_+ C^n \cup \partial_- C^n \text{, the boundary of } C^n. \]
  \[ \tilde{D}^n = C^n / \partial_+ C^n \text{, with the quotient map } \omega : C^n \to \tilde{D}^n. \]

**Notation 2.4.** Let us consider the following setup.

- For any $X = \{ x_1, \cdots, x_p \} \subseteq \mathbb{R}^n$, let $cx(X)$ denote the convex hull of $X$, that is
  \[ cx(X) = \left\{ \sum_{i=1}^{p} t_i x_i \in \mathbb{R}^n : t_i \geq 0, \sum_{i=1}^{p} t_i = 1 \right\}. \]
- Set $\Delta^{n-1} = cx(\{ e_1, \cdots, e_n \})$ to be the standard $(n-1)$-simplex.
- For any $J \subseteq [n]$, set $\Delta(J) = cx(\{ e_i : i \in J \}) \cong \Delta^{|J|-1}$, where $|J|$ denotes the cardinality of $J$.

**Remark 2.5.** The abstract cone $C \Delta^{n-1}$ can be realized as a subspace of $\mathbb{R}^n$, a subspace which is homeomorphic to the $n$-cube $C^n$ by reparametrization as we can observe in Figure 2. This motivates the existence of a bijection $\Psi : C \Delta^{n-1} \to C^n$, defined by equation (4).

\[ \Psi : C \Delta^{n-1} \to C^n \]

where $C^n = [0, 2]^n$ is the $n$-cube of side 2 set in **Notation 2.2**

**Remark 2.6.** As mentioned in **Notation 2.2**, the basepoint of $C = [0, 2]$ is 2. The above defined map $\Psi$ does not send the cone point to the basepoint $(2, \cdots, 2)$ of the $n$-cube $C^n$, as one might expect, but to the origin $c$ of $\mathbb{R}^n$ for convenience.

By Lemma 2.1 we have $C \Delta^{n-1} / \Delta^{n-1} \cong \Sigma \Delta^{n-1} \cong \Delta^n$. Also $\Psi(c \times \Delta^{n-1} \times \{1\}) = \partial_+ C^n$ and hence, $\Psi$ factors through the map

\[ \overline{\Psi} : \Sigma \Delta^{n-1} \to \tilde{D}^n, \]

where $\tilde{D}^n = C^n / \partial_+ C^n$ is the topological disk introduced in **Notation 2.2**
Lemma 2.7. The maps $\Psi$ and $\overline{\Psi}$ are both homeomorphisms.

Proof. As a continuous bijection from the compact space $C\Delta^{n-1}$ to the Hausdorff space $C^n$, $\Psi$ is a homeomorphism. Hence, $\Psi$ gives us the homeomorphism of the pairs $(C\Delta^{n-1}, \Delta^{n-1}) \cong (C^n, \partial_+ C^n)$, so that the induced map $\overline{\Psi} : \Sigma\Delta^{n-1} \to \overline{D}^n$ is a homeomorphism. \hfill $\square$

Remark 2.8. If we consider $([0,2], 2)$ to be a pointed space, then collapsing $\partial_+ C^{2(k+1)}$ in $C^{2(k+1)} \cong C^{k+1} \times C^{k+1}$, we get $\overline{D}^{k+1} \cup \overline{D}^{k+1} \cong D^{k+1} \cup D^{k+1} \cong D^{2(k+1)}$. This can be generalized to the case of $C^{p(k+1)} \cong C^{k+1} \times \cdots \times C^{k+1}$ and so collapsing $\partial_+ C^{p(k+1)}$ corresponds to $\bigwedge^p \overline{D}^{k+1} \cong \bigwedge^p D^{k+1} \cong D^{p(k+1)}$. Hence

$$\omega \left( \prod_{i=1}^{p} C^{k+1} \right) \cong \bigwedge^p \overline{D}^{k+1} \cong \bigwedge^p D^{k+1} \text{ and}$$

$$\omega \left( \prod_{i=1}^{p} \partial_+ C^{k+1} \right) \cong \bigwedge^p \partial_+ D^{k} \cong \bigwedge^p S^{k}, \text{ where } \partial_+ D^{k} \text{ denotes the boundary of } \overline{D}^{k+1}.$$
Lemma 2.9. For any compact and Hausdorff spaces $X$ and $Y$, there is a homeomorphism
\[ \varphi : C(X * Y) \to CX \times CY. \]

Proof. We follow the proof of [2, Lemma 8.1]. We can represent a point in $C(X * Y)$ by $[c, [x, y, t], \lambda]$. We define the homeomorphism $\varphi$ by
\[ \varphi([c, [x, y, t], \lambda]) = ([c, x, 2\lambda \cdot \min\{t, 1/2\}], [c, y, 2\lambda \cdot \min\{1 - t, 1/2\}]) \in CX \times CY, \]
where the cone point is at $\lambda = 0$. At $\lambda = 1$, $\varphi$ reduces to the usual homeomorphism
\[ X * Y \cong (CX \times Y) \cup (X \times CY). \]
The map $\varphi$ is a homeomorphism as a continuous bijection from the compact space $C(X * Y)$ to the Hausdorff space $CX \times CY$. \hfill \Box

3. Combinatorial tools

One of the main goals of this section is to embed the simplicial complex $K$ in a bigger simplex with vertex set $[(k + 1)m]$. We start by introducing a linearized version of the join of spaces.

**Definition 3.1** (Geometrically joinable).

- Two compact subspaces $X$ and $Y$ of $\mathbb{R}^n$ are said to be **geometrically joinable** if whenever $x, x' \in X$, $y, y' \in Y$ and $\lambda, \lambda' \in I$ are such that $\lambda x + (1 - \lambda)y = \lambda' x' + (1 - \lambda')y'$, then we have one of the three following possibilities
  - $\lambda = \lambda' = 0$, and so $y = y'$;
  - $\lambda = \lambda' = 1$, and so $x = x'$;
  - $0 \neq \lambda = \lambda' \neq 1$, $x = x'$ and $y = y'$.
- More generally, $p$ compact subspaces $X_1, \cdots, X_p \subseteq \mathbb{R}^n$ are **geometrically joinable** if whenever we have an equality between two convex combinations of points of $X_1, \cdots, X_p$, that is, whenever
  \[ \sum_{i=1}^{p} \lambda_i x_i = \sum_{i=1}^{p} \lambda'_i x'_i, \]
  for some $x_i, x'_i \in X_i$ and $\lambda_i, \lambda'_i \in I$ with $\sum_{i=1}^{p} \lambda_i = 1 = \sum_{i=1}^{p} \lambda'_i$; then for all $i = 1, \cdots, p$ such that $\lambda_i \neq 0$ or $\lambda'_i \neq 0$, we have $\lambda_i = \lambda'_i$ and $x_i = x'_i$.
- If $p$ compact subspaces $X_1, \cdots, X_p \subseteq \mathbb{R}^n$ are geometrically joinable, then we define their **geometric join** $\bigwedge_{i=1}^{p} X_i$ to be the set of all convex combinations of elements of $X_i$, that is,
  \[ \bigwedge_{i=1}^{p} X_i = \left\{ \sum_{i=1}^{p} \lambda_i x_i \in \mathbb{R}^n : x_i \in X_i \text{ and } \lambda_i \in I \text{ such that } \sum_{i=1}^{p} \lambda_i = 1 \right\}. \]

The notion of geometrically joinable introduced here is also called “in general position”. In the following, when we use the notation $X \bigwedge Y$, it is to be understood that $X$ and $Y$ are indeed geometrically joinable.

**Example 3.2.** Single points $X_1 = \{x_1\}, \cdots, X_p = \{x_p\}$ in $\mathbb{R}^n$ are geometrically joinable if and only if they are affinely independent. In that case, their geometric join $\bigwedge_{i=1}^{p} X_i$ is their convex hull, which is a $(p - 1)$-simplex, that is,
\[ \bigwedge_{i=1}^{p} X_i = \text{cx}\{x_1, \cdots, x_p\} \cong \Delta^{p-1}. \]
Lemma 3.3.

(1) Let $X$ and $Y$ be geometrically joinable subspaces of $\mathbb{R}^n$. The map
\[
\Phi : X \ast Y \to X \bar{\times} Y
\]
\[
[x, y, \lambda] \mapsto \lambda x + (1 - \lambda)y
\]
is a homeomorphism.

(2) More generally, for geometrically joinable subspaces $X_1, \ldots, X_p$ of $\mathbb{R}^n$, the map
\[
\Phi_p : \bigstar_{i=1}^p X_i \to \bigstar_{i=1}^p X_i
\]
\[
[x_i, \lambda_{i1}, \ldots, \lambda_{ip}] \mapsto \sum_{i=1}^p \lambda_i x_i
\]
is a homeomorphism.

Remark 3.4. Observe that if $U$ and $V$ are respective subspaces of geometrically joinable spaces $X$ and $Y$, then $U$ is geometrically joinable to $V$.

Lemma 3.5. Let $X, Y_1$ and $Y_2$ be three compact subspaces of $\mathbb{R}^n$. If $X$ is geometrically joinable to each $Y_i$ and
\[
X \bar{\times} Y_1 \cap X \bar{\times} Y_2 = X \bar{\times} (Y_1 \cap Y_2),
\]
then $X$ is geometrically joinable to $Y_1 \cup Y_2$.

Proof. Let $x, x' \in X$, $w, w' \in Y_1 \cup Y_2$ and $\lambda, \lambda' \in I$ be such that
\[
\lambda x + (1 - \lambda)w = \lambda' x' + (1 - \lambda')w'.
\]
If we have either $w, w' \in Y_1$ or $w, w' \in Y_2$, then there is nothing to show since $X$ is geometrically joinable to both $Y_1$ and $Y_2$. Without lost of generality suppose $w \in Y_1$ and $w' \in Y_2$, and so (6) gives us
\[
X \bar{\times} Y_1 \ni \lambda x + (1 - \lambda)w = \lambda' x' + (1 - \lambda')w' \in X \bar{\times} Y_2.
\]
Then by (5), there are $x'' \in X$, $w'' \in Y_1 \cap Y_2$ and $\lambda'' \in I$ such that
\[
\lambda x + (1 - \lambda)w = \lambda'' x'' + (1 - \lambda'')w'',
\]
\[
\lambda' x' + (1 - \lambda')w' = \lambda'' x'' + (1 - \lambda'')w''.
\]

If $\lambda'' = 0$ (respectively $\lambda'' = 1$) then
\begin{itemize}
\item $\lambda = 0$ (respectively $\lambda = 1$) and $w = w''$ (respectively $x = x''$) by (7), since $X$ and $Y_1$ are geometrically joinable.
\item $\lambda' = 0$ (respectively $\lambda' = 1$) and $w' = w''$ (respectively $x' = x''$) by (8), since $X$ and $Y_2$ are geometrically joinable.
\end{itemize}
Thus $\lambda = 0 = \lambda'$ and $w = w'$ (respectively $\lambda = 1 = \lambda'$ and $x = x'$).
If $0 \neq \lambda' \neq 1$ then
\begin{itemize}
\item $\lambda = \lambda''$ and, $x = x''$ and $w = w''$ by (7), since $X$ and $Y_1$ are geometrically joinable.
\item $\lambda' = \lambda''$ and, $x' = x''$ and $w' = w''$ by (8), since $X$ and $Y_2$ are geometrically joinable.
\end{itemize}
Thus $\lambda = \lambda'$ and, $x = x'$ and $w = w'$. Therefore $X$ is geometrically joinable to $Y_1 \cup Y_2$. $\square$

Notation 3.6. Now let $n = (k + 1)m$ and for $i \in [m]$, consider the following notations:
\begin{itemize}
\item $v_i^\ell = e_{(k+1)(\ell-1)+\ell}$, for any $\ell = 1, \ldots, k + 1$,
\item $a_i = \frac{1}{k + 1} \sum_{\ell=1}^{k+1} v_{\ell}$ \textup{bar}\{v_{\ell}\}_{\ell=1}^{k+1}$, that is, $a_i$ is the barycenter of $\{v_{\ell}\}_{\ell=1}^{k+1}$,
\item $\Delta_i = \text{ex} \{v_{\ell}\}_{\ell=1}^{k+1}$,
\item $S_i = \partial \Delta_i$.
\end{itemize}
Lemma 3.7. For any subset $\sigma \subseteq [m]$, the collections $\{\Delta_i\}_{i \in \sigma}$, $\{S_i\}_{i \in \sigma}$ and $\{a_i\}_{i \in \sigma}$ are respectively families of geometrically joinable compact subspaces of $\mathbb{R}^n$.

Proof. Consider the following identity of convex combinations

$$\sum_{i=1}^{\vert \sigma \vert} \lambda_i x_i = \sum_{i=1}^{\vert \sigma \vert} \lambda_i' x_i', \quad (9)$$

for some $x_i, x_i' \in \Delta_i$ and $\lambda_i, \lambda_i' \in I$ with $\sum_{i=1}^{k} \lambda_i = 1 = \sum_{i=1}^{k} \lambda_i'$. The equation (9) is equivalent to

$$\sum_{i=1}^{\vert \sigma \vert} \sum_{\ell=1}^{k+1} \lambda_i t_i^\ell v_i^\ell = \sum_{i=1}^{\vert \sigma \vert} \sum_{\ell=1}^{k+1} \lambda_i' s_i^\ell v_i^\ell,$$

where $x_i = \sum_{\ell=1}^{k+1} t_i^\ell v_i^\ell$ and $x_i' = \sum_{\ell=1}^{k+1} s_i^\ell v_i^\ell$ are convex combinations. Since $\{v_i^\ell : i \in \sigma, \ell = 1, \cdots, k+1\}$ is affinely independent, then $\lambda_i t_i^\ell = \lambda_i' s_i^\ell$, for all $i \in \sigma, \ell = 1, \cdots, k+1$. Without loss of generality if $\lambda_i \neq 0$ (the proof is similar if we assume $\lambda_i' \neq 0$), then $t_i^\ell = \frac{\lambda_i'}{\lambda_i} s_i^\ell$. Hence we have

$$\sum_{\ell=1}^{k+1} t_i^\ell = 1 \implies \sum_{\ell=1}^{k+1} \frac{\lambda_i'}{\lambda_i} s_i^\ell = 1 \implies \frac{\lambda_i'}{\lambda_i} \sum_{\ell=1}^{k+1} s_i^\ell = 1 \implies \lambda_i' = \lambda_i \implies t_i^\ell = s_i^\ell, \text{ for all } i \in \sigma, \ell = 1, \cdots, k+1 \implies x_i = \sum_{\ell=1}^{k+1} t_i^\ell v_i^\ell = \sum_{\ell=1}^{k+1} s_i^\ell v_i^\ell = x_i' \implies x_i = x_i'.$$

Therefore the collection $\{\Delta_i\}_{i \in \sigma}$ is geometrically joinable. As the collection of boundaries of disjoint $k$-simplices $\Delta_i$ in $\mathbb{R}^n$ respectively, $\{S_i\}_{i \in \sigma}$ is a family of geometrically joinable compact subspaces of $\mathbb{R}^n$. Likewise, the collection of barycenters $\{a_i\}_{i \in \sigma}$ of the $k$-simplices $\Delta_i$ is geometrically joinable. \qed

Notation 3.8. For any $\sigma \subseteq [m]$ and by Lemma 3.7 consider the setting

- $J(\sigma) = \cup_{i \in \sigma} \{(k + 1)(i - 1) + \ell\}_{\ell=1}^{k+1} \subseteq [n]$,
- $\Delta_\sigma = \Delta(J(\sigma)) = \mathfrak{F}_{i \in \sigma} \Delta_i$,
- $S_\sigma = \mathfrak{F}_{i \in \sigma} S_i$,
- $S^*_{\sigma} = \mathfrak{F}_{j \in \sigma} S_j$,
- $a_\sigma = \text{cx}\{a_i : i \in \sigma\}$.

An example of this setup is illustrated in Figure 3.

Lemma 3.9. For $\sigma \subseteq [m]$, the compact spaces $a_\sigma$ and $S_\sigma$ are geometrically joinable, and we have $\Delta_\sigma = a_\sigma \mathfrak{F} S_\sigma$.

Proof. Let us prove $a_\sigma$ and $S_\sigma$ are geometrically joinable for $k = 1$ and for $\sigma = \{1, 2\}$; the general case can be proved similarly. In this setup, $S_\sigma$ can be split as follows

$$S_\sigma = \underbrace{[e_1, e_2]}_{F_1} \cup \underbrace{[e_1, e_3]}_{F_2} \cup \underbrace{[e_2, e_3]}_{F_3} \cup \underbrace{[e_2, e_4]}_{F_4} \cup \underbrace{[e_3, e_4]}_{F_4} \text{ and}$$
The complexes $a_\sigma$ and $F_i$, for each $i = 1, 2, 3, 4$, are both 1-simplices and all their four vertices are not coplanar. So $a_\sigma$ and $F_i$, for each $i = 1, 2, 3, 4$, are geometrically joinable and their join $a_\sigma \bar{x} F_i$ is a 3-simplex. Also we have

$$a_\sigma \bar{x} (F_1 \cap F_2) = a_\sigma \bar{x} \{e_1\} = (a_\sigma \bar{x} F_1) \cap (a_\sigma \bar{x} F_2).$$

Hence by Lemma 3.5 $a_\sigma$ is geometrically joinable to $F_1 \cup F_2$. Similarly, we have

$$a_\sigma \bar{x} (F_1 \cup F_2) = a_\sigma \bar{x} \{e_3\} = (a_\sigma \bar{x} F_1 \cup F_2) \cap (a_\sigma \bar{x} F_3).$$

Hence $a_\sigma$ is geometrically joinable to $F_1 \cup F_2 \cup F_3$. Similarly, we have

$$a_\sigma \bar{x} (F_1 \cup F_2 \cup F_3) = a_\sigma \bar{x} \{e_2, e_4\} = (a_\sigma \bar{x} F_1 \cup F_2 \cup F_3) \cap (a_\sigma \bar{x} F_4).$$

Hence $a_\sigma$ is geometrically joinable to $S_\sigma = F_1 \cup F_2 \cup F_3 \cup F_4$.

For any $i \in [m]$, $S_i = \partial \Delta_i$ and $a_i = \overline{\text{bar}} \{v_{i,l}^{k+1}\}_{l=1}^4$. Then $a_i$ and $S_i$ are geometrically joinable, and we have $\Delta_i = a_i \bar{x} S_i$. We deduce

$$\Delta_\sigma = \bigcup_{i \in \sigma} \Delta_i = \bigcup_{i \in \sigma} (a_i \bar{x} S_i) = (a_\sigma \bar{x} S_i) = a_\sigma \bar{x} S_\sigma.$$

□

4. PROOF OF THEOREM 1.5

Now we have all the tools to write down the proof of Theorem 1.5 for $k \geq 1$. The case of $k = 0$ will be treated in the next section. In the following, we consider the notations introduced in Sections 2 and 3.

Proof. Setting $W_\sigma = \Delta_\sigma \bar{x} S_\sigma^*$ for each $\sigma \in K$, we have

$$(10) \quad \bigcup_{\sigma \in K} W_\sigma = \bigcup_{\sigma \in K} \left( \bigcup_{i \in \sigma} \Delta_i \right) \bar{x} \left( \bigcup_{j \notin \sigma} S_j \right).$$
Consider the collection of simplices \( A(K) = \{a_\sigma\}_{\sigma \in K} \), which is a geometric realization of the simplicial complex \( K \) and

\[
|A(K)| = \bigcup_{\sigma \in K} a_\sigma
\]

to be the underlying subspace of \( \mathbb{R}^n = \mathbb{R}^{(k+1)m} \). The complexes \( a_{[m]} \) and \( S_{[m]} \) are geometrically joinable by Lemma 3.9 and also \( A(K) \) is a subcomplex of \( a_{[m]} \). Therefore \( |A(K)| \) and \( S_{[m]} \) are geometrically joinable by Remark 3.4 and we have \( S_{[m]} \| A(K) \| \cong S^{km-1} \times |K| \cong \Sigma^{km} |K| \), since \( S_{[m]} \cong S^{km-1} \). Then

\[
W_\sigma = \Delta_\sigma \| S_{\sigma}^*
\]

\[= a_\sigma \| (S_{\sigma} \| S_{\sigma}^*) \] by Lemma 3.9

\[= a_\sigma \| S_{[m]} \].

This is illustrated by Figure 4, where \( W_\sigma \) is the union of the two blue triangular surfaces. In this example, we have \( S_{[m]} \cong \partial W_\sigma \).

![Figure 4](image_url)

**Figure 4.** Examples of \( W_\sigma \), for \( m = 2 \), \( \sigma = \{1\} \) and \( k = 1 \), that is \( n = 4 \).

Hence from equation (11) we also have

\[
\bigcup_{\sigma \in K} W_\sigma \cong \Sigma^{km} |K|.
\]

Therefore we have

\[
\Psi(C(\bigcup_{\sigma \in K} W_\sigma)) \cong \bigcup_{\sigma \in K} \Psi(CW_\sigma)
\]

\[
= \bigcup_{\sigma \in K} \Psi \left( C \left( \bigcup_{i \in \sigma} \Delta_i \bigcup_{j \not\in \sigma} S_j \right) \right) \] by (10)

\[
\cong \bigcup_{\sigma \in K} \Psi \left( \prod_{i \in \sigma} (C\Delta_i) \times \prod_{j \not\in \sigma} (CS_j) \right) \] by Lemma 2.9

\[
\cong \bigcup_{\sigma \in K} \Psi \left( \prod_{i \in \sigma} C^{k+1}_i \times \prod_{j \not\in \sigma} \partial C^{k+1}_j \right) \] by Lemma 2.7

\[
(13)
\]

where \( C^{k+1}_i \) and \( C^{k+1}_j \) are copies of the \((k+1)\)-cube.
Then we get
\[ \Psi(\bigcup_{\sigma \in K} W_{\sigma}) \cong \bigcup_{\sigma \in K} \Psi(\Sigma W_{\sigma}) \]
\[ \cong \bigcup_{\sigma \in K} \omega \left( \prod_{i \in \sigma} C_i^{k+1} \times \prod_{j \not\in \sigma} \partial C_j^{k+1} \right) \] by (13) and Lemma 2.7
\[ \cong \bigcup_{\sigma \in K} \left( \bigwedge_{i \in \sigma} \tilde{D}_i^{k+1} \wedge \bigwedge_{j \not\in \sigma} \partial \tilde{D}_j^{k+1} \right) \] by Remark 2.8 where \( \tilde{D}_i^{k+1} \) are copies of the nonstandard \((k + 1)\)-disk \( \tilde{D}_i^{k+1} \) defined in Lemma 2.3.

\[ \Psi(\bigcup_{\sigma \in K} W_{\sigma}) \cong \bigcup_{\sigma \in K} \left( \bigwedge_{i \in \sigma} D_i^{k+1} \wedge \bigwedge_{j \not\in \sigma} S_j^k \right) \] by Lemma 2.3 where \( D_i^{k+1} \) and \( S_j^k \) are copies of the \((k + 1)\)-disk \( D_i^{k+1} \) and the \( k \)-sphere \( S_j^k \) respectively.

\[ \hat{Z}(K; (D^{k+1}, S^k)) \cong \sum_{\sigma \in K} \left( \sum_{i \in \sigma} D_i^{k+1} \wedge \bigwedge_{j \not\in \sigma} S_j^k \right) \] by Lemma 2.3.

On the other hand, we obtain
\[ \Psi(\bigcup_{\sigma \in K} W_{\sigma}) \cong \Psi(\Sigma km|K|) \] by (12)
\[ \cong \Psi \left( \sum_{\sigma \in K} D^{k+1}_i \wedge \bigwedge_{j \not\in \sigma} \partial \tilde{D}_j^{k+1} \right) \] by Remark 2.8 where \( \tilde{D}_i^{k+1} \) are copies of the nonstandard \((k + 1)\)-disk \( \tilde{D}_i^{k+1} \) defined in Lemma 2.3.

Therefore by (14) and (15), we have
\[ \hat{Z}(K; (D^{k+1}, S^k)) \cong \sum_{\sigma \in K} \left( \sum_{i \in \sigma} D_i^{k+1} \wedge \bigwedge_{j \not\in \sigma} S_j^k \right) \] by Lemma 2.7.

Hence we have the result.

5. THE CASE \( k = 0 \)

In the previous section, we have proved Theorem 1.5 for \( k \geq 1 \). Here we prove the remaining case, namely \( k = 0 \), given by the following.

**Theorem 5.1.** There is a homeomorphism
\[ \hat{Z}(K; (D^1, S^0)) \cong \Sigma|K|. \]

For the purpose of the proof, we will consider the categorical definition of the polyhedral smash product given by (6). As in the third bullet of Notation 2.4, consider the functor
\[ \Delta : \text{CAT}(K) \to \text{Top} \]
\[ \sigma \mapsto \Delta(\sigma) \cong \Delta^{\sigma - 1}. \]

The geometric realization of \( K \) is given by
\[ |K| = \text{colim}_{\sigma \in K} \Delta(\sigma). \]

**Proof.** Consider the composite functor \( \Sigma \Delta \) given by
\[ \Sigma \Delta : \text{CAT}(K) \to \text{Top} \]
\[ \sigma \mapsto \Sigma \Delta(\sigma) \cong \Sigma \Delta^{\sigma - 1}. \]

Let \( \tau \subseteq \sigma \) be a face inclusion in \( K \), with \( |\tau| = p \leq \ell = |\sigma| \). We will look at the case \( \tau = \{1, \cdots, p\} \subseteq \{1, \cdots, \ell\} = \sigma \) for simplicity; the same argument works for the general case. Consider the two following maps
\[ \phi_1 : C\Delta^{p-1} \to C\Delta^{\ell-1} \]
\[ [c, (t_1, \ldots, t_p), \lambda] \mapsto [c, (t_1, \ldots, t_p, 0, \ldots, 0), \lambda] \]
and
\[ \phi_2 : C^p \to C^{\ell} \]
\[ (t_1, \ldots, t_p) \mapsto (t_1, \ldots, t_p, 0, \ldots, 0). \]

The transformation \( \Psi : \Sigma\Delta \Rightarrow \hat{D} \), where the functor \( \hat{D} \) is defined by (2) for the pair \((D^1, S^0)\), is a natural isomorphism if the left hand side diagram commutes since it induces the commutative diagram on the right hand side below:

\[
\begin{array}{ccc}
C\Delta^{p-1} & \xrightarrow{\Psi_p} & C^p \\
\phi_1 & \cong & \phi_2 \\
C\Delta^{\ell-1} & \xrightarrow{\Psi_\ell} & C^{\ell}
\end{array}
\]

where \( \overline{\phi}_1([s_i, x, \lambda]) = [\phi_1([c, x, \lambda])] \), for all \([s_i, x, \lambda] \in \Sigma\Delta^{p-1}\), \( \tilde{D}^p = C^1 \wedge \cdots \wedge C^1 \) and so \( \overline{\phi}_2([y]) = [\phi_2(y)] \), for all \( y \in C^p \). For \([c, (t_1, \ldots, t_p), \lambda] \in C\Delta^{p-1}\) we have

\[
\Psi_\ell \phi_1([c, (t_1, \ldots, t_p), \lambda]) = \Psi_\ell([c, (t_1, \ldots, t_p, 0, \ldots, 0), \lambda])
\]
\[
= \begin{cases}
2\lambda(t_1, \ldots, t_p, 0, \ldots, 0), & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\
\left( (2 - 2\lambda) + (2\lambda - 1) \frac{2}{\max_{1 \leq i \leq p} t_i} \right) (t_1, \ldots, t_p, 0, \ldots, 0), & \text{if } \frac{1}{2} \leq \lambda \leq 1
\end{cases}
\]

by (4)

\[
\phi_2 \Psi_p([c, (t_1, \ldots, t_p), \lambda]) = \begin{cases}
\phi_2(2\lambda(t_1, \ldots, t_p)), & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\
\phi_2 \left( \left( (2 - 2\lambda) + (2\lambda - 1) \frac{2}{\max_{1 \leq i \leq p} t_i} \right) (t_1, \ldots, t_p) \right), & \text{if } \frac{1}{2} \leq \lambda \leq 1
\end{cases}
\]
\[
= \begin{cases}
2\lambda(t_1, \ldots, t_p, 0, \ldots, 0), & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\
\left( (2 - 2\lambda) + (2\lambda - 1) \frac{2}{\max_{1 \leq i \leq p} t_i} \right) (t_1, \ldots, t_p, 0, \ldots, 0), & \text{if } \frac{1}{2} \leq \lambda \leq 1
\end{cases}
\]
\[
\text{by (4)}
\]

Then the two diagrams commute and therefore \( \Psi \) is a natural isomorphism. Passing to the colimit, we have

(17)\[ \operatorname{colim} \Psi \colon \colim_{\sigma \in K} \Sigma\Delta(\sigma) \xrightarrow{\cong} \colim_{\sigma \in K} \hat{D}(\sigma). \]

But we have

\[ \colim_{\sigma \in K} \Sigma\Delta(\sigma) \cong \Sigma \colim_{\sigma \in K} \Delta(\sigma) = \Sigma|K| \text{ by (16)} \]
and also considering identity (3), the homeomorphism (17) yields a homeomorphism
\[ \Sigma|K| \cong \hat{Z}(K; (X, A)). \]

6. Generalization of Theorem 1.5

In this section, we generalize Theorem 1.5 further, using an argument kindly provided by
the referee. Instead of doubling all the vertices of \( K \) simultaneously as in David Stone’s original
construction, we double one vertex at a time and argue inductively, starting from the case \( k = 0 \) in
Theorem 5.1. Let us start by setting some notation and stating intermediate results.

Notation 6.1.

- For \( J = (j_1, \cdots, j_m) \) an \( m \)-tuple from \((\mathbb{N} \cup \{0\})^m\), denote the family of CW-pairs
  \[ (D_{J}^{j_1+1}, S_{J}^{j_1}) = \{(D^{j_1+1}, S^{j_1}), \cdots, (D^{j_m+1}, S^{j_m})\}. \]
- Set \( J_i = (0, \cdots, 1, \cdots, 0) \) to be the \( m \)-tuple having 1 only at the \( i \)-th position and 0 elsewhere.
  For a simplicial complex \( K \) over \([m]\) and \( i \in [m] \), consider the new simplicial complex \( K(J_i) \) with \( m + 1 \) vertices labeled \( \{1, \cdots, i - 1, i_a, i_b, i + 1, \cdots, m\} \) and defined by
  \[ K(J_i) := \{(\sigma \setminus \{i\}) \cup \{i_a, i_b\} \mid \sigma \in K \text{ and } i \in \sigma \} \cup \{\sigma \cup \{i_b\} \mid \sigma \in K \text{ and } i \notin \sigma\} \]
  \[ \cup \{\sigma \cup \{i_b\} \mid \sigma \in K \text{ and } i \notin \sigma\} \cup \{\text{all their subsets}\}. \]

The meaning behind the introduction of \( K(J_i) \) is illustrated in the following example.

Example 6.2. For \( m = 2 \), consider \( K = \{\emptyset, \{1\}, \{2\}\} \) and \( J_1 = (1, 0) \). We have
\[
\hat{Z}(K; (D_{J_1}^{j_1+1}, S_{J_1}^{j_1})) = \hat{Z}(K; ((D^2, S^1), (D^1, S^0)))
\]
\[ = \hat{D}({1}) \cup \hat{D}({2}) \quad \text{as a subspace of } D^2 \land D^1 \]
\[ = D^2 \land S^0 \cup S^1 \land D^1 \]
\[ \cong (D^1 \land D^1) \land S^0 \cup (D^1 \land S^0 \land S^0 \land D^1) \land D^1 \]
\[ = (D^1 \land D^1 \land S^0) \cup (D^1 \land S^0 \land D^1) \land (S^0 \land D^1 \land D^1) \]
\[ = \hat{D}({1_a, 1_b}) \cup \hat{D}({2} \cup \{1_a\}) \cup \hat{D}({2} \cup \{1_b\}) \]
\[ = \hat{Z}(K(J_1); ((D^1, S^0), (D^1, S^0), (D^1, S^0))) \]
\[ = \hat{Z}(K(J_1); (D^1, S^0)), \text{ see Figure 5} \]

\[ \begin{array}{c}
1 \\
\bullet \\
1_a \\
\bullet \\
2 \\
\end{array}
\]

\[ \begin{array}{c}
K(J_1) = \{\{1_a, 1_b\}, \{1_a, 2\}, \{1_b, 2\}, \text{their subsets}\}. \]

The next lemma suggests that the polyhedral smash product \( \hat{Z}(K; (D_{J_i}^{j_1+1}, S_{J_i}^{j_1})) \) can be computed iteratively with steps involving \( K(J_i) \) for some \( i \in [m] \).

Lemma 6.3. Let \( J = (j_1, \cdots, j_m) \) to be an \( m \)-tuple and \( i \in [m] \) such that \( j_i \neq 0 \). There is a homeomorphism
\[ \hat{Z}(K; (D_{J_i}^{j_1+1}, S_{J_i}^{j_1})) \cong \hat{Z}(K(J_i); (D_{J_i}^{j_i+1}, S_{J_i}^{j_i})), \]
where \( J' \) is the \((m + 1)\)-tuple \( J' = (j_1, \cdots, j_i - 1, 0, \cdots, j_m) \).
Proof. The polyhedral smash product $\hat{Z}(K; (D^{j+1}, S^{j}))$ is defined as follows

\[
\hat{Z}(K; (D^{j+1}, S^{j})) = \hat{Z}(K; \{(D_{j_1}^{j+1}, S_{j_1}), \ldots, (D_{j_m}^{j+1}, S_{j_m})\})
\]

\[
= \bigcup_{\sigma \in K} \left( \hat{D}(\sigma) \right) \quad \text{as a subspace of } \bigwedge_{l=1}^{m} D^{j_l+1} \cong \tilde{D}^{j_1+\cdots+j_m+m}
\]

\[
= \bigcup_{\sigma \in K} \left( \bigwedge_{l=1}^{m} Y_\ell \right), \text{ where } Y_\ell \text{ depends on } \sigma \text{ and is defined in (1)}
\]

\[
= \bigcup_{\sigma \in K} \left( \bigwedge_{l=1}^{m} Y_\ell \right) \bigcup \bigcup_{\sigma \in K, i \neq \sigma} \left( \bigwedge_{l=1}^{m} Y_\ell \right)
\]

\[
= \bigcup_{\sigma \in K, i \neq \sigma} \left( Y_1 \wedge \cdots \wedge D^{j_1+1} \wedge \cdots \wedge Y_m \right) \bigcup \bigcup_{\sigma \in K} \left( Y_1 \wedge \cdots \wedge S^{j_1} \wedge \cdots \wedge Y_m \right)
\]

\[
\cong \bigcup_{\sigma \in K, i \neq \sigma} \left( Y_1 \wedge \cdots \wedge (D^{j_1} \wedge D^1) \wedge \cdots Y_m \right)
\]

\[
= \bigcup_{\sigma \in K, i \neq \sigma} \left( Y_1 \wedge \cdots \wedge D^{j_1} \wedge D^1 \wedge \cdots Y_m \right) \bigcup \bigcup_{\sigma \in K, i \neq \sigma} \left( Y_1 \wedge \cdots \wedge D^{j_1} \wedge S^0 \wedge \cdots Y_m \right)
\]

\[
= \bigcup_{\sigma \in K, i \neq \sigma} \left( Y_1 \wedge \cdots \wedge S^{j_1-1} \wedge D^1 \wedge \cdots Y_m \right)
\]

where $J'$ is the $(m+1)$-tuple $J' = (j_1, \ldots, j_i - 1, 0, \ldots, j_m)$. \hfill \qed

Example 6.4.

(1) For $m = 3$, consider $K = \{1, 2\}, \{3\}$, their subsets and $J = (1, 1, 0)$. We have

\[
\hat{Z}(K; (D^{j+1}, S^{j})) = \hat{Z}(K; \{(D^2, S^1), (D^2, S^1), (D^1, S^0)\})
\]

\[
= \hat{D}(\{1, 2\}) \cup \hat{D}(\{3\}) \quad \text{as a subspace of } D^2 \wedge D^2 \wedge D^1
\]

\[
= D^2 \wedge D^2 \wedge S^0 \cup S^1 \wedge S^1 \wedge D^1
\]

\[
\cong D^2 \wedge (D^1 \wedge D^1) \wedge S^0 \cup S^1 \wedge (D^1 \wedge S^0 \cup S^0 \wedge D^1) \wedge D^1
\]

\[
= (D^2 \wedge D^1 \wedge S^0) \cup (S^1 \wedge D^1 \wedge S^0 \wedge D^1) \cup (S^1 \wedge S^0 \wedge D^1 \wedge D^1)
\]

\[
= \hat{D}(\{\{1, 2a, 2b\}\} \cup \hat{D}(\{2a, 3\}) \cup \hat{D}(\{2b, 3\})
\]

\[
= \hat{Z}(K(J_2); \{(D^2, S^1), (D^1, S^0), (D^1, S^0), (D^1, S^0)\})
\]

\[
= \hat{Z}(K(J_2); (\underbrace{D^{j+1}, S^{j}})), \text{ with } J_1 = (1, 0, 0, 0)
\]

\[
= \hat{Z}(K(J_2)(J_1); (D^1, S^0))
\]

\[
= \hat{Z}(K(J); (D^1, S^0)), \text{ where } K(J) = K(J_2)(J_1).
\]

See Figure 6.
Lemma 6.5. For any $i \in [m]$, we have

$$|K(J_i)| \cong \Sigma|K|.$$  

Proof. Set $S^0 = \{s_1, s_2\}$ to be the 0-sphere. We have

$$\Sigma|K| = S^0 * |K|$$

$$= \{s_1, s_2\} * \left( \bigcup_{\sigma \in K} |\sigma| \right)$$

$$= \bigcup_{\sigma \in K} (\{s_1, s_2\} * |\sigma|)$$

$$= \left( \bigcup_{\sigma \in K, i \in \sigma} (\{s_1, s_2\} * |\sigma|) \right) \cup \left( \bigcup_{\sigma \in K, i \notin \sigma} (\{s_1, s_2\} * |\sigma|) \right)$$

$$\cong \left( \bigcup_{\sigma \in K, i \in \sigma} (|\sigma\setminus\{i\} \cup \{s_1, s_2\}) \right) \cup \left( \bigcup_{\sigma \in K, i \notin \sigma} (|\sigma \cup \{s_1\}| \cup |\sigma \cup \{s_2\}|) \right)$$

$$\cong \left( \bigcup_{\sigma \in K, i \in \sigma} (|\sigma\setminus\{i\} \cup \{s_1, s_2\}) \right) \cup \left( \bigcup_{\sigma \in K, i \notin \sigma} (|\sigma \cup \{s_1\}|) \right) \cup \left( \bigcup_{\sigma \in K, i \notin \sigma} (|\sigma \cup \{s_2\}|) \right)$$

$$\cong \left( \bigcup_{\sigma \in K, i \in \sigma} (|\sigma\setminus\{i\} \cup \{i_a, i_b\}) \right) \cup \left( \bigcup_{\sigma \in K, i \notin \sigma} (|\sigma \cup \{i_a\}|) \right) \cup \left( \bigcup_{\sigma \in K, i \notin \sigma} (|\sigma \cup \{i_b\}|) \right)$$

$$= \bigcup_{\tau \in K(J_i)} |\tau|$$

$$= |K(J_i)|.$$ 

The lemma is illustrated in Figure 7 for the simplicial complex $K$ from Example 6.4. Now we can state and prove the main result.
Theorem 6.6. For any $m$-tuple $J = (j_1, \cdots, j_m)$ in $(\mathbb{N} \cup \{0\})^m$, there is a homeomorphism

\[ Z(K; (D^{j_1+1}, S^0)) \cong \Sigma |K|. \]

Proof. Applying Lemma 6.3 $\sum_{i=1}^m j_i$ times, we get

\[ Z(K, (D^{j_1+1}, S^0)) \cong Z(K(J); (D^1, S^0)), \]

where $K(J)$ is a simplicial complex obtained by applying the basic move (doubling a single vertex) $\sum_{i=1}^m j_i$ times. By the base case $k = 0$ in Theorem 5.1, we have

\[ Z(K(J); (D^1, S^0)) \cong \Sigma |K(J)|. \]

Finally, by applying Lemma 6.5 $\sum_{i=1}^m j_i$ times, we have

\[ \Sigma |K(J)| \cong \Sigma \Sigma |K(J)| \]

\[ = \Sigma |K(j_1 + \cdots + j_m)|. \]

Therefore by putting equations (18), (19) and (20) together, we obtain

\[ Z(K; (D^{j_1+1}, S^0)) \cong \Sigma |K(j_1 + \cdots + j_m)|. \]

\[ \square \]

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