AN ITERATIVE METHOD FOR ESTIMATION THE ROOTS OF REAL-VALUED FUNCTIONS

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Abstract. In this paper we study the recursive sequence $x_{n+1} = \frac{x_n + f(x_n)}{2}$ for each continuous real-valued function $f$ on an interval $[a, b]$, where $x_0$ is an arbitrary point in $[a, b]$. First, we present some results for real-valued continuous function $f$ on $[a, b]$ which have a unique fixed point $c \in (a, b)$ and show that the sequence $\{x_n\}$ converges to $c$ provided that $f$ satisfies some conditions. By assuming that $c$ is a root of $f$ instead of being its fixed point, we extend these results. We define two other sequences by $x_0^+ = x_0^- = x_0 \in [a, b]$ and $x_{n+1}^+ = x_n^+ + \frac{f(x_n^+)}{2}$ and $x_{n+1}^- = x_n^- - \frac{f(x_n^-)}{2}$ for each $n \geq 0$. We show that for each real-valued continuous function $f$ on $[a, b]$ with $f(a) > 0 > f(b)$ which has a unique root $c \in (a, b)$, the sequence $\{x_n^+\}$ converges to $c$ provided that $f' \geq -2$ on $(a, b)$. Accordingly we show that for each real-valued continuous function $f$ on $[a, b]$ with $f(a) < 0 < f(b)$ which has a unique root $c \in (a, b)$, the sequence $\{x_n^-\}$ converges to $c$ provided that $f' \leq 2$ on $(a, b)$. By an example we also show that there exists some continuous real-valued function $f : [a, b] \to [a, b]$ such that the sequence $\{x_n\}$ does not converge for some $x_0 \in [a, b]$.

1. Introduction and preliminaries

Fixed point theory is one of the most applied branch of mathematics which is a combination of geometry, topology and analysis [2]. Most of the time in applied mathematics and engineering a problem is converted into an equation and finding the roots of these equations solves the related problem. One typical method to solve these equations (which means finding the root of these equations) is to convert them into fixed point problems. For any real-valued function $f : A \subset \mathbb{R} \to \mathbb{R}$, $x \in A$ is called a fixed point of $f$ if $f(x) = x$ [1]. An example of finding the roots of a function through solving a fixed point problem is as follows: For the given function $f$ define $F(x) = \frac{x + f(x)}{2}$ for each $x \in A$. It is easy to see that each fixed point of $F$ is a root of $f$. In this case for finding a fixed point of $F$ the recurrence sequence $\{x_n\}$ is defined...
by \( x_0 = a \in A \) and
\[
x_{n+1} = \frac{x_n + f(x_n)}{2} \quad \text{for each } n \geq 0.
\] (1.1)

The sequence \( \{x_n\} \) is called the Picard sequence of \( F \) based at \( x_0 = a \) [1]. For a continuous function \( f \), \( F \) is also continuous and if \( x_n \) converges to \( c \), then \( F(c) = c \) and as we mentioned before this follows that \( f(c) = 0 \). In this paper we study the convergence of above sequence for a given function \( f \) on \([a, b]\) and show that if \( f \) satisfies some differentiability conditions on \([a, b]\), then this sequence is convergent.

Let \( f : [0, 1] \rightarrow [0, 1] \) be a continuous function. A question naturally rises is that: is the Sequence 1.1 defined above is convergent for each \( x_0 = a \in [0, 1] \)? At the beginning we hoped that to answer this question positively. But, we found a counter example to confirm that the answer is not yes all the time (Example 4.1). Then, we begin our study to find some suitable conditions to make sure the convergence of this sequence.

This paper has organized as follows. First, we study real-valued continuous function \( f \) on an interval \([a, c]\) or \([c, b]\), where \( c \) is the unique fixed point of \( f \) which also satisfies a slight differentiability condition at \( c \). We show that the sequence \( \{x_n\} \) converges to \( c \) provided that the initial point \( x_0 \) be close enough to \( c \) (Lemma 2.1 and Lemma 2.2). Then, assuming extra differentiability condition on \( f \) we establish some results which make sure the convergence of the Sequence 1.1 on the whole interval \([a, c]\) or \([c, b]\) (Lemma 2.3 and Lemma 2.5). Then our adventure of hunting for roots begins. By defining the following sequence we convert the above results to results about roots (Corollaries 2.7).

\[
x_{n+1} = x_n + \frac{f(x_n)}{2} \quad \text{for each } n \geq 0, \text{ } x_0 \text{ is an arbitrary initial point.} \tag{1.2}
\]

Until this point, some kinds of roots are missed by our approach. Indeed, the Sequence 1.2 never converges to any such kinds of roots (Remark 2.8). In dealing with these kinds of roots the following sequence emerges. Then we establish some other results to cover them too.

\[
x_{n+1} = x_n - \frac{f(x_n)}{2} \quad \text{for each } n \geq 0, \text{ } x_0 \text{ is an arbitrary initial point.} \tag{1.3}
\]

Now, we provide a result to show that each unique root of a given function with some additional conditions is the limit of one of these two sequences (Theorem 2.11). Then, we make a comparison between our methods and the Newton-Raphson method. We discuss functions that satisfies our conditions and have a root \( c \in (a, b) \). Finally, we illustrate our results by giving some examples. We also give a practical example to show that our method is working where Newton-Raphson method does
Newton-Raphson Method is depended firmly on how the initial point is chosen. But in our approach we don’t rely so much on how the initial points is chosen or how far they are from the related roots. Also, the conditions under which the Newton-Raphson sequence is convergent is somehow hard (Theorem 1.1). We give a very simple conditions for our sequences to converge. We give a brief introduction of Newton-Raphson Method as we need it in the sequel. Newton introduced a method to find the root of a function $f$ through an iterative sequence as follows.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for all } n \geq 0. \tag{1.4}$$

First, an initial guess is made which is denoted by $x_0$. Then we intersect the tangent line at $(x_0, f(x_0))$ with $X$ axis. This intersection point will be $x_1$. Continuing this process, the Sequence 1.4 is defined. There are lots of examples which shows that this sequence does not always converges to the guessed root. (Example 3.1). In fact, Newton-Raphson method depends a lot on the initial guess to hunt the roots down. But, under some conditions this sequence always converges to the related roots. The following result shows that when Newton-Raphson sequence converges.

**Theorem 1.1.** [Convergence of Newton-Raphson method] (Endre Süli and David F. Mayers [4]) Let $c \in \mathbb{R}$. Suppose that $f$ is a continuous function with second continuous derivative on the closed interval $[c - \delta, c + \delta]$ where $\delta > 0$. Also, suppose that $f''(c) \neq 0$ and $f(c) = 0$ and there exists a constant number $K > 0$ such that:

$$| \frac{f''(x)}{f'(x)} | \leq K \text{ for each } x \in [c - \delta, c + \delta]. \tag{1.5}$$

If $| x_0 - c | \leq \min \{ \delta, \frac{1}{K} \}$, then the Newton-Raphson sequence converges to $c$.

In this manuscript we show the set of real numbers by $\mathbb{R}$. We denote the set $\{0, 1, 2, \ldots \}$ of none negative integers by $\mathbb{N}$. For each $a, b \in \mathbb{R}$ with $a < b$, $[a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$ and $(a, b) = \{ x \in \mathbb{R} : a < x < b \}$ will denote closed and open interval from $a$ to $b$. Let $f$ be a real-valued function on $\mathbb{R}$ we show the left and right derivative of $f$ at $a$ by $f'(a-), f'(a+)$ respectively.
2. Main results

To begin our study, we present the following two results which briefly indicate that if \( x_0 \) is close enough to a fixed point of \( f \), then the sequence \( x_{n+1} = \frac{x_n + f(x_n)}{2} \) converges to \( c \).

**Lemma 2.1.** Let \( \delta > 0 \), \( c \in R \) and \( f \) be a continuous real-valued function on \([c-\delta, c]\) which is left differentiable at \( c \) with \( f'(c-) \geq -1 \) such that \( f(x) > x \) for each \( x \in (c-\delta, c) \) and \( c \) is the unique fixed point of \( f \) in \([c-\delta, c]\). Let \( c \neq x_0 \in (c-\delta, c) \) and for each \( n \geq 0 \) define:

\[
x_{n+1} = \frac{x_n + f(x_n)}{2}.
\]

If \( \delta \) is chosen small enough, then the sequence \( \{x_n\} \) converges to \( c \).

**Proof.** Define \( g(x) = \frac{f(x)-x}{x-c} \) for each \( x \in (c-\delta, c) \). By our assumption \( g'(c) \geq -2 \), so there exists some \( \delta > \sigma > 0 \) such that \( \frac{f(x)-x}{x-c} \geq -2 \) for each \( x \in (c-\sigma, c) \). Therefore, for each \( x \in (c-\sigma, c) \) we have:

\[
\frac{f(x)-x}{2} \leq c-x.
\]  

If \( x_0 \in (c-\sigma, c) \), then \( x_1 = \frac{f(x_0)+x_0}{2} > x_0 \) and \( x_1 - x_0 = \frac{f(x_0)-x_0}{2} \leq c-x_0 \) which follows that \( x_1 \leq c \). Arguing by induction we see that \( x_n \leq c \) and \( x_{n+1} \geq x_n \) for each \( n \geq 0 \). Therefore, \( \{x_n\} \) converges to some \( a \in [c-\sigma, c] \). It is easy to see that \( f(a) = a \), so by uniqueness assumption \( a = c \) and the proof is complete. \( \square \)

**Lemma 2.2.** Let \( \delta > 0 \), \( c \in R \) and \( f \) be a continuous real-valued function on \([c, c+\delta]\) which is right differentiable at \( c \) with \( f'(c+) \geq -1 \) such that \( f(x) < x \) for each \( x \in (c, c+\delta) \) and \( c \) is the unique fixed point of \( f \) in \([c, c+\delta]\). Let \( c \neq x_0 \in (c, c+\delta) \) and for each \( n \geq 0 \) define:

\[
x_{n+1} = \frac{x_n + f(x_n)}{2}.
\]

If \( \delta \) is chosen small enough, then the sequence \( \{x_n\} \) converges to \( c \).

**Proof.** Define \( g(x) = \frac{f(x)-x}{x-c} \) for each \( x \in (c, c+\delta) \). By our assumption \( g'(c+) \geq -2 \), so there exists some \( \delta > \sigma > 0 \) such that \( \frac{f(x)-x}{x-c} \geq -2 \) for each \( x \in (c, c+\sigma) \). Therefore, for each \( x \in (c, c+\sigma) \) we have:

\[
\frac{f(x)-x}{2} \geq c-x.
\]  

If \( x_0 \in (c, c+\sigma) \), then \( x_1 = \frac{f(x_0)+x_0}{2} < x_0 \) and \( x_1 - x_0 = \frac{f(x_0)-x_0}{2} \geq c-x_0 \) which follows that \( x_1 \geq c \). Arguing by induction we deduce that \( x_n \geq c \) and \( x_{n+1} \leq x_n \) for
each \( n \geq 0 \). Therefore, \( \{x_n\} \) converges to some \( a \in [c, c+\sigma] \). It is easy to see that \( f(a) = a \), so by uniqueness assumption \( a = c \) and the proof is complete. \( \square \)

In above two lemmas to make sure the convergence of the sequence \( x_{n+1} = \frac{x_n + f(x_n)}{2} \) in a neighborhood of \( c \) we just needed the one-sided derivative of \( f \) at \( c \). But, practically, when we start looking for a fixed point \( c \) of a function \( f \) on some interval, we have no idea if the condition \( f'(c-) \geq -1 \) or \( f'(c+) \geq -1 \) holds or not. Apart from that we don’t know where to choose our initial point \( x_0 \). Therefore, we change the Lemma 2.1 as follows to make it practically more useful. But it will still need some other modifications as we will see in the sequel.

**Lemma 2.3.** Let \( f \) be a continuous real-valued function on \([a, c]\) which is differentiable on \((a, c)\) with \( f'(x) \geq -1 \) on \((a, c)\) and \( f(x) > x \) for each \( x \in [a, c] \) and \( c \) is the unique fixed point of \( f \) in \([a, c]\). Let \( x_0 \in [a, c] \) and for each \( n \geq 0 \) define:

\[
x_{n+1} = \frac{x_n + f(x_n)}{2},
\]

then the sequence \( \{x_n\} \) converges to \( c \).

**Proof.** Define \( g(x) = \frac{f(x) - x}{x - c} \) for each \( x \in [a, c] \). By our assumption \( g'(x) \geq -2 \) for each \( x \in (a, c) \). For each \( x \in [a, c] \) we have \( \frac{f(x) - x}{x - c} \geq -2 \). Since otherwise we define \( h(x) = f(x) - x \) for each \( x \in [a, c] \) and applying Lagrange mean value theorem for \( h \) [3], there exists \( \sigma \in (c, x) \) such that:

\[
-2 \leq h'(
\sigma\) = \frac{h(x) - h(c)}{x - c} = \frac{f(x) - x}{x - c} < -2,
\]

which is a contradiction. Therefore, for each \( x \in [a, c] \) we have \( \frac{f(x) - x}{x - c} \geq -2 \). Equivalently, we have:

\[
\frac{f(x) + x}{2} \leq c \text{ for each } x \in [a, c].
\]

Now, define \( \{x_n\} \) as in equation 2.5. For each \( n \geq 0 \), \( x_n \leq c \) and \( x_{n+1} \geq x_n \). Therefore, \( \{x_n\} \) converges to some point \( b \in [a, c] \). So \( f(b) = b \) and since \( c \) is the unique fixed point of \( f \) in \([a, c]\) we deduce that \( b = c \) and the proof is complete. \( \square \)

**Remark 2.4.** Example 4.1 shows that the differentiability condition, \( f'(x) \geq -1 \) on \((a, c)\), in Lemma 2.3 is necessary.

**Lemma 2.5.** Let \( f \) be a continuous real-valued function on \([c, b]\) which is differentiable on \((c, b)\) with \( f'(x) \geq -1 \) on \((c, b)\) and \( f(x) < x \) for each \( x \in (c, b) \) and \( c \) is the unique fixed point of \( f \) in \([c, b]\). Let \( x_0 \in (c, b] \) and for each \( n \geq 0 \) define:

\[
x_{n+1} = \frac{x_n + f(x_n)}{2},
\]
then the sequence $\{x_n\}$ converges to $c$.

Proof. Similar to the proof of Lemma 2.3, for each $x \in [a, c]$ we have $\frac{f(x) - x}{x - c} \geq -2$. Since $f(x) < x$ and $x > c$ for each $x \in (c, b]$, this follows that $\frac{x - f(x)}{2} \leq x - c$. So we have:

$$\frac{f(x) + x}{2} \geq c \text{ for each } x \in [c, b). \quad (2.9)$$

Now, define $\{x_n\}$ as in equation 2.8. For each $n \geq 0$, $x_n \geq c$ and $x_{n+1} \leq x_n$. Therefore, $\{x_n\}$ converges to some point $d \in [c, b)$. So $f(d) = d$. Since $c$ is the unique fixed point of $f$ in $[c, b]$, we deduce that $d = c$ and the proof is complete. □

Remark 2.6. By Example 4.3, in Lemma 2.3 the assumption $f(x) > x$ for each $x \in [a, c]$ can not be replaced with $f(x) < x$ for each $x \in [a, c]$. Also, in Lemma 2.5 the assumption $f(x) < x$ for each $x \in (c, b]$ can not be replaced with $f(x) > x$ for each $x \in (c, b]$.

Now it is easy to convert the above results to look for the roots of a given function instead of its fixed points.

Corollary 2.7. Let $f$ be a continuous real-valued function on $[a, b]$ and $c \in (a, b)$ be the unique root of $f$ in $[a, b]$. Suppose that $f$ is differentiable on $A = (a, c) \cup (c, b)$ with $f'(x) \geq -2$ for each $x \in A$. Also, suppose that $f(x) > 0$ for each $x \in (a, c)$ and $f(x) < 0$ for each $x \in (c, b)$. For each $x_0 \in [a, b]$ and for each $n \geq 0$ define:

$$x_{n+1} = x_n + \frac{f(x_n)}{2}, \quad (2.10)$$

then the sequence $\{x_n\}$ converges to $c$.

Proof. Define $F(x) = f(x) + x$ for each $x \in [a, b]$. $F(x) > x$ for each $x \in [a, c)$ and $F$ satisfies all conditions of Lemma 2.3 on $[a, c]$. Let $x_0 \in [a, c]$, define $\{x_n\}$ by:

$$x_{n+1} = \frac{F(x_n) + x_n}{2} = x_n + \frac{f(x_n)}{2} \text{ for all } n \geq 0, \quad (2.11)$$

$\{x_n\}$ converges to $c$ by Lemma 2.3. If $x_0 \in [c, b]$, then $F$ satisfies all conditions of Lemma 2.5 on $[c, b]$ and again the above sequence converges to $c$. So the proof is complete. □

Remark 2.8. In Corollary 2.7 if we had $f(x) < 0$ for each $x \in [a, c)$ and $f(x) > 0$ for each $x \in (c, b]$. Then, the sequence 2.10 does not converge to $c$ for any $x_0 \in A = [a, c) \cup (c, b]$. To see this, assume that $x_0 \in [a, c)$. $f(x_0) < 0$, therefore, $x_1 = x_0 + \frac{f(x_0)}{2} < x_0$. By induction we see that $x_{n+1} < x_n < \cdots < x_0 < c$. Therefore, $x_n \not\rightarrow c$ as $n \rightarrow \infty$. The same is true when $x_0 \in (c, b]$. We define the sequence 2.12 in the following Corollary to converges to these kind of roots.
Corollary 2.9. Let $f$ be a continuous real-valued function on $[a, b]$ and $c \in (a, b)$ be the unique root of $f$ in $[a, b]$. Suppose that $f$ is differentiable on $A = (a, c) \cup (c, b)$ with $f'(x) \leq 2$ for each $x \in A$. Also, suppose that $f(x) < 0$ for each $x \in [a, c)$ and $f(x) > 0$ for each $x \in (c, b]$. For each $x_0 \in [a, b]$ and for each $n \geq 0$ define:

$$x_{n+1} = x_n - \frac{f(x_n)}{2},$$

then the sequence $\{x_n\}$ converges to $c$.

Proof. Define $F(x) = -f(x) + x$ for each $x \in [a, b]$. $F(x) > x$ for each $x \in [a, c)$ and $F$ satisfies all conditions of Lemma 2.3 on $[a, c]$. Let $x_0 \in [a, c]$ and define $\{x_n\}$ by:

$$x_{n+1} = \frac{F(x_n) + x_n}{2} = x_n - \frac{f(x_n)}{2} \text{ for all } n \geq 0,$$

$\{x_n\}$ converges to $c$ by Lemma 2.3. If $x_0 \in [c, b]$, then $F$ satisfies all conditions of Lemma 2.5 on $[c, b]$ and again the above sequence converges to $c$. So the proof is complete. □

Remark 2.10. To be able to refer to them easily, we denote the sequence $\{x_n\}$ in 2.10 and 2.12 by $\{x^+_n\}$ and $\{x^-_n\}$ respectively.

Now we are ready to provide a practical result to look for the roots of a given function on a given interval which satisfies some conditions as follows.

Theorem 2.11. Let $f$ be a continuous real-valued function on $[a, b]$ which is differentiable on $(a, b)$. Then,

(i) if $f'(x) \geq -2$ for each $x \in (a, b)$, $f(a) > 0$, $f(b) < 0$ and $f$ has a unique root $c$ in $(a, b)$. Then, the sequence 2.14 converges to $c$ for each $x_0 \in [a, b]$.

$$x_{n+1} = x_n + \frac{f(x_n)}{2} \text{ for each } n \geq 0,$$

(ii) if $f'(x) \leq 2$ for each $x \in (a, b)$, $f(a) < 0$, $f(b) > 0$ and $f$ has a unique root $c$ in $(a, b)$. Then, the sequence 2.15 converges to $c$ for each $x_0 \in [a, b]$.

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \text{ for each } n \geq 0.$$

Proof. First, suppose that $f$ satisfies in (i). Let $x < c$. If $f(x) < 0$, since $f(x)f(a) < 0$, $f$ has a root in $(a, x)$ which contradicts with the uniqueness of $c$. Therefore, $f(x) > 0$ for each $x \in [a, c)$. Similarly, $f(x) < 0$ for each $x \in (c, b]$. Now, $f$ satisfies all conditions of Corollary 2.7 on the interval $[a, b]$, therefore the sequence 2.14 converges to $c$ for each $x_0 \in [a, b]$. Now, suppose that $f$ satisfies in (ii). We see that $f(x) < 0$ for each $x \in (a, c)$ and
For each $x \in (c, b]$. Consider that $f$ satisfies all conditions of Corollary 2.9. So, the sequence 2.15 converges to $c$. If $x_0 = c$ the proof is evident.

\[\square\]

**Remark 2.12.** It is worth to mention that in Theorem 2.11 we are not allowed to replace the Equation 2.15 with Equation 2.14. See Example 4.4 for a counter example in this regard.

**Corollary 2.13.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $-2 \leq f'(x) \leq 2$ for each $x \in (a, b)$ which has at least some roots in $[a, b]$. Then for each point $x_0 \in [a, b]$ one of the following sequences converges to a root of $f$. If $[a, b]$ be an infinite interval the result again holds.

\[(a) \quad x_{n+1} = x_n + \frac{f(x_n)}{2},\]
\[(b) \quad x_{n+1} = x_n - \frac{f(x_n)}{2}.\]

**Proof.** Let $x_0 \in (a, b)$ and $f(x_0) > 0$. By assumption there exists a root on the right or left side of $x_0$. First, suppose that $c$ be the nearest root of $f$ which is on the right side of $x_0$. Since there is no root $x \in [x_0, c)$, therefore $f(x) > 0$ for each $x \in [x_0, c)$. Now, $f$ satisfies all conditions of Corollary 2.7. Therefore, the sequence $\{x_n\}$ converges to $c$ which is defined by (a).

Now, suppose that $c$ is the nearest fixed point of $f$ which is on the left side of $x_0$. Since there should be no other root $x \in (c, x_0]$, $f(x) > 0$ for each $x \in (c, x_0]$. Therefore, by Corollary 2.9, the sequence $\{x_n\}$, defined by (b), converges to $c$. If $f(x_0) < 0$ a similar arguments hold, so the proof is complete. \[\square\]

3. Numerical results

In this part we present some numerical examples to show how our method can be applied to estimate roots. We present different situations where our introduced recursive sequences converges to the related roots. Through these numerical example we reveal some other aspects of our methods by the time we are dealing with them.

In the following example we make a comparison between Newton-Raphson and our method. Sometime Newton-Raphson sequence does not converge to the root although the initial point is very close to it. But, we see that our method still works even when the initial point $x_0$ is very faraway from the guessed root.

In what follows we denote the Newton-Raphson sequence by $\{x_n^N\}$ and we denote our sequences by $\{x_n^-\}, \{x_n^+\}$ as we declared in Remark 2.10.
Example 3.1. Suppose that \( f(x) = x^3 - 2x + 2 \) for each \( x \in \mathbb{R} \). \( f \) has a root \( c \in (-2, 0) \). For \( x_0 = 0 \) we make a comparison between Newton method and our method. For Newton method we have: \( x_0^N = 0, x_1^N = 1, x_2^N = 0, \ldots \). Therefore, we see that the Newton sequence \( \{x_n^N\} \) oscillates between 0 and 1. The sequence \( \{x_n\} \) also does not converges if we choose \( x_0 = 0 \). The reason is that by Corollary 2.9 we should have \( f'(x) = 3x^2 - 2 \leq 2 \) for each \( x \in (c, 0] \). To overcome this problem, define the function \( g \) by \( g = \frac{f}{5} \). For each \( x \in (c, 0) \) we have:

\[
g'(x) = \frac{3}{5} : x \in (-2, 0) \leq 2. \tag{3.1}
\]

Now, \( g \) satisfies all conditions of Corollary 2.9 on \( (c, 0) \). Therefore, the sequence \( \{x_n^+\} \) for \( g \) with initial point \( x_0 = 0 \) should converges to \( c \). Here it is the first 30 terms of the sequence \( \{x_n^-\} \) for \( g \):

0, -0.2,
-0.43920000000000003, -0.7185679795712001, -1.025179040648767, -1.32246934505463805,
-1.5556732218319653, -1.690316407071083, -1.7454276306539045, -1.7627655547359082,
-1.7675670505642251, -1.7688406751853836, -1.7691744113097234, -1.76926157774243,
-1.7692843247213965, -1.7692902594529567, -1.7692918077457465, -1.769292116686607,
-1.7692923170447714, -1.7692923445354456, -1.769292351707251, -1.7692923535782414,
-1.7692923540663479, -1.7692923541936858, -1.769292354226906, -1.7692923542355725,
-1.7692923542378334, -1.7692923542384231, -1.769292354238577, -1.7692923542386172.

Remark 3.2. In above calculation of the root, dividing the given function \( f \) by a constant changed the situation. So to make the inequality \( f'(x) \leq 2 \) hold for all \( x \in (c, b] \) we can divide \( f \) by a suitable constant \( M > 0 \) and find the root of \( \frac{f}{M} \) instead of \( f \) which is the same. Notice that dividing the function by a constant \( M \) does not make any change to the Newton-Raphson sequence. Also notice that dividing by very large constant \( M > 0 \) is not advised because \( x_{n+1} - x_n = \frac{f(x_n^-)}{2} \). Therefore, Dividing \( f \) by a large constant might make \( x_{n+1} - x_n \) smaller which means that this might slow down the convergence pace of the sequence. Also notice that in Example 3.1, by Remark 2.8, we see that the sequence \( \{x_n^+\} \) is divergent for any \( x_0 \neq c \). That is why we didn’t use it to look for the root.

As we explained it in above remark we choose a suitable constant \( M > 0 \) and calculate the sequence \( \{x_n^-\} \) for some other initial points as follows to estimate the root in above example.

Our method also converges on the left side of the root. Here it is the first 30 terms of the sequence \( \{x_n^-\} \) for \( f/4 \) with \( x_0 = -2 \).
As we mentioned our method is not depended so much on choosing initial point $x_0$ for the root $c$ provided that we devide $f$ by a constant $M > 0$ which make sure that $f'(x)$ remains smaller than 2 on $[-2, x_0]$. The next 100 terms of $\{x^{-}_n\}$ for $f/13$ with $x^{-}_0 = 3$ is as follows. The convergence speed is not very good but it works:

$$
3,
2.1153846153846154, 1.83710520909282, 1.663030685559212, 1.537132059563976, 1.4387624802728445, 1.3579637383218315, 1.28918502204226, 1.2290214485296653, 1.1752373606891795, 1.1262857079761661, 1.0810494017195553, 1.0386921389409374, 0.9985674732071567, 0.9601607952584468, 0.9230508317707333, \ldots
$$

Now we choose $x_0 = 10$ which is very distant from the root. First, we divide $f$ by 150. We have $f'(x) = \frac{3x^2-2}{150} \leq 2$ on $[-2, 10]$. Here it is the first 2000 terms of $x^{-}_n$ which shows that the result is desirable:

$$
10,
6.7266666666666667, 5.7502827367901235, 5.1481598872541925, 4.7209992393798945, 4.395069745964866, 4.1347103056594925, 3.919987368905806, 3.73868265632258, 3.582735187579203, 3.446658762614731, 3.326488376782681, 3.2193004991871463, 3.1228808538029766, 3.0355149399772943, 2.955850706332395, \ldots
$$

\text{...}
we should use the sequence on \([0,1]\). Consequently our method can not be applied.

Example 3.3. Let \(f(x) = x^{1/3}\). We know that \(x = 0\) is the unique root of \(f\) in \((-1,1)\). \(f(x) < 0\) for each \(x \in (-1,0)\) and \(f(x) > 0\) for each \(x \in (0,1)\). By Corollary 2.9 we should use the sequence \(\{x_n\}\) to find the root. Examining the sequence \(\{x_n^-\}\) for some \(x_0 \in (-1,1)\) we see that the sequence is not convergent. The reason is that \(\lim_{x \to 0} f'(x) = +\infty\) and whatever \(\delta > 0\) we choose \(f'\) remains unbounded on \((-\delta,\delta)\). Therefore, the differentiability condition does not hold and consequently our method can not be applied.

4. Examples

In this section we illustrate our results by some examples.

The following example shows that for a continuous function \(f : [a, b] \to [a, b]\) the sequence \(x_{n+1} = \frac{x_n + f(x_n)}{2}\) does not converge to a fixed point of \(f\) for some \(x_0 \in [a, b]\).

Example 4.1. Define \(f : [0,1] \to [0,1]\) as follows:

\[
\begin{align*}
1 & \quad \text{if } 0 \leq x \leq \frac{3}{8} \\
-2x + \frac{7}{4} & \quad \text{if } \frac{3}{8} \leq x \leq \frac{7}{12} \\
-6x + \frac{49}{12} & \quad \text{if } \frac{7}{12} \leq x \leq \frac{49}{72} \\
0 & \quad \text{if } \frac{49}{72} \leq x \leq 1
\end{align*}
\]

\(f\) is continuous and \(x = \frac{7}{12}\) is the unique fixed point of \(f\). Let \(x_0 = \frac{23}{63}\) and for each \(n \geq 1\) define:

\[
x_{n+1} = \frac{x_n + f(x_n)}{2}.
\] (4.1)

By calculation we see that \(x_1 = \frac{86}{126}, x_2 = \frac{43}{126}, x_3 = \frac{169}{252}, x_4 = \frac{23}{63} = x_0\). Therefore, \(\{x_n\}\) is not convergent. It is worth mentioning that \(f\) also satisfies all conditions of Lemma 2.3 on \([\frac{3}{8}, \frac{7}{12}]\) except for the differentiability condition of \(f\) on \([-1,1]\). More clearly, \(c = \frac{7}{12}\) is the unique fixed point of \(f\) in \([0,c]\), and \(f(x) > x\) for each \(x \in [0,c]\). But, \(f\) is not differentiable at \(x = \frac{3}{8}\) and \(f' \not\leq -1\) on \((\frac{3}{8},c)\).

Example 4.2. For each \(x \in R\) define \(f(x) = x^3 + 3x + 1\) and \(g(x) = f(x) - x\). \(g(-1)g(0) < 0\). By mean value theorem \(g\) has a root in \((-1,0)\). Equivalently \(f\) has a fixed point in \((-1,0)\). By Rolle’s theorem this fixed point is unique. Denote this fixed point by \(c\). We see that \(g(x) < x\) for each \(x < c\) and \(g(x) > x\) for each \(x > c\).

Now, let \(\{x_n\}\) be the sequence which is defined by Equation 2.5. If \(x_0 < c\), then by induction we see that \(x_{n+1} = \frac{x_n + f(x_n)}{2} < x_n < x_0\). So in this case this sequence
does not converges to $c$. Similarly, this is the case when $x_0 > c$. Therefore, for each $x_0 \neq c$ the sequence $\{x_n\}$ does not converges to $c$.

The following example shows that, in Lemma 2.3 it is necessary to have $f(x) > x$ on $[a, c)$.

**Example 4.3.** For each $x \in [2/3, 1]$ define:

$$f(x) = 2x - 1.$$ 

$f$ is a continuous function on $[2/3, 1]$ and $x = 1$ is the unique fixed point of $f$ in $[2/3, 1]$. $f(x) < x$ for each $x < 1$. For each $x_0$ in $[2/3, 1)$ we have $f(x_0) < x_0$. Therefore, $x_1 = \frac{f(x_0) + x_0}{2} < x_0$. Through induction we see that $x_{n+1} < x_n$ for all $n \geq 0$. Therefore, $\{x_n\}$ does not converges to 1. So in Lemma 2.3 the assumption $f(x) > x$ for each $x \in [a, c)$ can not be replaced with $f(x) < x$ for each $x \in [a, c)$.

Also suppose that $f(x) = 2x - 1$ for each $x \in [1, 2]$. $c = 1$ is the unique fixed point of $f$ in $[1, 2]$ and $f(x) > x$ for each $x \in (1, 2]$. Let $x_0 \in (1, 2]$, similarly we see that $x_{n+1} > x_n > \cdots > x_0 > 1$ for each $n \geq 0$. Therefore, $\{x_n\}$ does not converges to 1. So in Lemma 2.5, the condition $f(x) < x$ on $(c, b]$ can not be replaced with $f(x) > x$ on $(c, b]$.

**Example 4.4.** Let $f : [-1, 1] \to [-1, 1]$ be defined by:

$$f(x) = \begin{cases} 
\frac{2}{3}(x - \frac{1}{2}) & \text{if } -1 \leq x \leq \frac{1}{2} \\
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1 
\end{cases}$$

Let $g : [-1, 1] \to [-1, 1]$ be defined by:

$$g(x) = \begin{cases} 
-\frac{2}{3}(x - \frac{1}{2}) & \text{if } -1 \leq x \leq \frac{1}{2} \\
-2x + 1 & \text{if } \frac{1}{2} \leq x \leq 1 
\end{cases}$$

$x = \frac{1}{2}$ is the unique root of $f$ in $[-1, 1]$. Let $\{x_n\}$ be the sequence defined in the Equation 2.14 in Theorem 2.11 and $x_0 = -1$. Then, $x_1 = x_0 + \frac{f(x_0)}{2} = \frac{-3}{2} \notin [-1, 1]$. The same is true if $x_0 = 1$. So Equation 2.15 can not be replaced with Equation 2.14 in Theorem 2.11. Similarly using $g$ shows that the Equation 2.14 can not be replaced with Equation 2.15 in Theorem 2.11.

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