On Noncentral Tanny-Dowling Polynomials and Generalizations of Some Formulas for Geometric Polynomials

Mahid M. Mangontarum$^1$ and Norlailah M. Madid$^2$
Department of Mathematics
Mindanao State University–Main Campus
Marawi City 9700
Philippines

$^1$mmangontarum@yahoo.com
$^1$mangontarum.mahid@msumain.edu.ph
$^2$norlailahmadid07@gmail.com

Abstract
In this paper, we establish some formulas for the noncentral Tanny-Dowling polynomials including sums of products and explicit formulas which are shown to be generalizations of known identities. Other important results and consequences are also discussed and presented.

1 Introduction
The geometric polynomials $[16]$, denoted by $w_n(x)$, are defined by

$$w_n(x) = \sum_{k=0}^{n} k! \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k,$$

(1)

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ are the well-celebrated Stirling numbers of the second kind $[6, 15]$. These polynomials are known to satisfy the exponential generating function

$$\sum_{n=0}^{\infty} w_n(x) \frac{z^n}{n!} = \frac{1}{1 - x(e^z - 1)}$$

(2)

and the recurrence relation

$$w_{n+1}(x) = x \frac{d}{dx} [w_n(x) + xw_n(x)].$$

(3)

The case when $x = 1$ yields

$$w_n := w_n(1) = \sum_{k=0}^{n} k! \left\{ \begin{array}{c} n \\ k \end{array} \right\},$$

(4)
the geometric numbers (or ordered Bell numbers). Recall that the numbers \( \binom{n}{k} \) count the number of partitions of a set \( X \) with \( n \) elements into \( k \) non-empty subsets. These numbers can also be interpreted as the number of ways to distribute \( n \) distinct objects into \( k \) identical boxes such that no box is empty. On the other hand, the numbers \( k! \binom{n}{k} \) can be combinatorially interpreted as the number of distinct ordered partitions of \( X \) with \( k \) blocks, or the numbers of ways to distribute \( n \) distinct objects into \( k \) distinct boxes. It follows immediately that the geometric numbers count the number of distinct ordered partitions of the \( n \)-set \( X \).

The study of geometric polynomials and numbers has a long history. Aside from the paper of Tanny [16], one may also see the works of Boyadzhiev [4], Dil and Kurt [8], and the references therein for further readings. Benoumhani [3] studied two equivalent generalizations of \( w_n(x) \) given by

\[
F_{m,1}(n; x) = \sum_{k=0}^{n} m^k k! W_m(n, k)x^k
\]

and

\[
F_{m,2}(n; x) = \sum_{k=0}^{n} k! W_m(n, k)x^k,
\]

where \( W_m(n, k) \) denote the Whitney numbers of the second kind of Dowling lattices [2]. These are called Tanny-Dowling polynomials and are known to satisfy the following exponential generating functions:

\[
\sum_{n=0}^{\infty} F_{m,1}(x) z^n \frac{x^n}{n!} = \frac{e^z}{1 - x(e^{mx} - 1)},
\]

\[
\sum_{n=0}^{\infty} F_{m,2}(x) z^n \frac{x^n}{n!} = \frac{e^z}{1 - \frac{x}{m}(e^{mx} - 1)}.
\]

More properties can be seen in [2, 3]. In a recent paper, Kargın [9] established a number of explicit formulas and formulas involving products of geometric polynomials, viz.

\[
(x + 1) \sum_{k=0}^{n} \binom{n}{k} w_k(x) w_{n-k}(x) = w_{n+1}(x) + w_n(x),
\]

\[
\sum_{k=0}^{n} \binom{n}{k} w_k(x_1) w_{n-k}(x_2) = \frac{x_2 w_n(x_2) - x_1 w_n(x_1)}{x_2 - x_1},
\]

\[
w_n(x) = x \sum_{k=1}^{n} \binom{n}{k} (-1)^{n+k}k!(x+1)^{k-1},
\]

and

\[
w_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k \frac{2^{n+1}(x+1)x^k + (-1)^{k+1}}{(2x+1)^{k+1}}.
\]
This was done with the aid of the two-variable geometric polynomials \( w_k(r; x) \) defined by

\[
\sum_{n=0}^{\infty} w_n(r; x) \frac{z^n}{n!} = \frac{e^{rx}}{1 - x(e^z - 1)}, \tag{13}
\]

A natural generalization of \( F_{m,1}(x) \) and \( F_{m,2}(x) \) are the noncentral Tanny-Dowling polynomials introduced by Mangontarum et al. \[12\] defined as

\[
\tilde{f}_{m,a}(n; x) = \sum_{k=0}^{n} k! \tilde{W}_{m,a}(n, k) x^k, \tag{14}
\]

where \( \tilde{W}_{m,a}(n, k) \) are the noncentral Whitney numbers of the second kind with an exponential generating function given by

\[
\sum_{n=k}^{\infty} \tilde{f}_{m,a}(n; x) \frac{z^n}{n!} = \frac{me^{-az}}{m - x(e^{mz} - 1)}. \tag{15}
\]

Looking at (15), it is readily observed that

\[
\tilde{f}_{m,0}(n; x) = w_n \left( \frac{x}{m} \right),
\]

\[
\tilde{f}_{m,-1}(n; x) = F_{m,2}(n; x)
\]

and

\[
\tilde{f}_{1,-r}(n; x) = w_n(r, x).
\]

The numbers \( \tilde{W}_{m,a}(n, k) \) admit a variety of combinatorial properties which can be seen in [12]. These numbers appear to be a common generalization of \( \{n\} \) and \( W_m(n, k) \), as well as other notable numbers reported by the respective authors in [1, 5, 10, 11, 13]. It is important to note that the noncentral Whitney numbers of the second kind is equivalent to the \((r, \beta)\)-Stirling numbers by Corcino \[7\] and the \(r\)-Whitney numbers of the second kind by Mező \[14\].

In the present paper, we establish some formulas for the noncentral Tanny-Dowling polynomials including sums of products and explicit formulas. These formulas are shown to generalize the above-mentioned identities obtained by Kargın \[9\] for the geometric polynomials when the parameters are assigned with specific values. We also discuss some identities resulting from the said formulas.

## 2 Formulas for Sum of Products

Now, the exponential generating function in (15) can be rewritten as

\[
\sum_{n=0}^{\infty} \tilde{f}_{m,a}(n; x) \frac{z^n}{n!} = \frac{1}{1 - \frac{x}{m}(e^{mz} - 1)} \cdot e^{-az}.
\]
Hence, by applying (2) and using Cauchy’s product for two series, we obtain
\[
\sum_{n=0}^{\infty} \tilde{F}_{m,a}(n; x) \frac{z^n}{n!} = \sum_{n=0}^{\infty} m^n w_n \left( \frac{x}{m} \right) \frac{z^n}{n!} \sum_{n=0}^{\infty} (-a)^n \frac{z^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} w_k \left( \frac{x}{m} \right) m^k (-a)^{n-k} \right) \frac{z^n}{n!}.
\]

Comparing the coefficients of \(\frac{z^n}{n!}\) yields the result in the next theorem.

**Theorem 1.** The noncentral Tanny-Dowling polynomials \(\tilde{F}_{m,a}(n; x)\) satisfy the following identity:
\[
\tilde{F}_{m,a}(n; x) = \sum_{k=0}^{n} \binom{n}{k} m^k w_k \left( \frac{x}{m} \right) (-a)^{n-k}.
\]  

(16)

**Alternative proof of Theorem 1.** From [12, Theorem 10], the noncentral Whitney numbers of the second kind satisfy the following formula in terms of the Stirling numbers of the second kind:
\[
\tilde{W}_{m,a}(n, k) = \sum_{j=0}^{n} \binom{n}{j} (-a)^{n-j} m^j - k \left\{\binom{j}{k}\right\}.
\]

Multiplying both sides by \(k! x^k\) and summing over \(k\) gives the desired result. 

Before proceeding, we see that when \(m = 1\) and \(a = -r\), (16) becomes
\[
\tilde{F}_{1,-r}(n; x) = \sum_{k=0}^{n} \binom{n}{k} w_k(x) r^{n-k} = w_n(r; x),
\]
which is precisely an identity obtained by Kargın [9, Equation (13)].

By applying the exponential generating function in (15),
\[
\sum_{n=0}^{\infty} \left[ \tilde{F}_{m,a-m}(n; x) - \tilde{F}_{m,a}(n; x) \right] \frac{z^n}{n!} = \frac{me^{-(a-m)z}}{m-x(e^{mz} - 1)} - \frac{me^{-az}}{m-x(e^{mz} - 1)}
\]
\[
= \frac{m}{x} \left[ \frac{me^{-az}}{m-x(e^{mz} - 1)} - e^{-az} \right]
\]
\[
= \frac{m}{x} \sum_{n=0}^{\infty} \tilde{F}_{m,a}(n; x) \frac{z^n}{n!} - \sum_{n=0}^{\infty} (-a)^n \frac{z^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \frac{m}{x} \left( \tilde{F}_{m,a}(n; x) - (-a)^n \right) \frac{z^n}{n!}.
\]

Comparing the coefficients of \(\frac{z^n}{n!}\) gives
\[
\tilde{F}_{m,a-m}(n; x) - \tilde{F}_{m,a}(n; x) = \frac{m}{x} \left[ \tilde{F}_{m,a}(n; x) - (-a)^n \right].
\]

The result in the next theorem follows by solving for \(x \tilde{F}_{m,a-m}(n; x)\).
Theorem 2. The noncentral Tanny-Dowling polynomials $\tilde{F}_{m,a}(n; x)$ satisfy the following recurrence relation:

$$x\tilde{F}_{m,a-m}(n; x) = (m + x)\tilde{F}_{m,a}(n; x) - (-a)^n m. \quad (17)$$

Setting $m = 1$ and $a = -r$ in (17) gives

$$x\tilde{F}_{1,-r-1}(n; x) = (1 + x)\tilde{F}_{1,-r}(n; x) - r^n$$

which is exactly the following identity [9, Equation (14)]:

$$xw_n(r+1; x) = (1 + x)w_n(r; x) - r^n.$$  

On the other hand, when $a = 0$ and $a = m$ in (17), we get

$$x\tilde{F}_{m,-m}(n; x) = (m + x)w_n\left(\frac{x}{m}\right) \quad (18)$$

and

$$(m + x)\tilde{F}_{m,m}(n; x) = xw_n\left(\frac{x}{m}\right) - (-m)^{n+1}, \quad (19)$$

respectively. Applying (16) yields

$$xm^n \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} w_k\left(\frac{x}{m}\right) = (m + x)w_n\left(\frac{x}{m}\right) \quad (20)$$

and

$$(m + x)m^n \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} w_k\left(\frac{x}{m}\right) (-1)^{n-k} = xw_n\left(\frac{x}{m}\right) - (-m)^{n+1}. \quad (21)$$

These are generalizations of the results obtained by Dil and Kurt [8] using the Euler-Seidel matrix method. That is, setting $x = 1$ and $m = 1$ gives

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} w_k = 2w_n$$

and

$$2 \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k w_k = (-1)^n w_n + 1.$$

The next theorem contains a formula for the sum of product of noncentral Tanny-Dowling polynomials for different values of $a$.

Theorem 3. The noncentral Tanny-Dowling polynomials satisfy the following relation:

$$x \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \tilde{F}_{m,a_1}(k; x)\tilde{F}_{m,a_2}(n - k; x) = \tilde{F}_{m,\bar{A}}(n + 1; x) + \bar{A}\tilde{F}_{m,\bar{A}}(n; x), \quad (22)$$

where $\bar{A} = a_1 + a_2 + m$ for real numbers $a_1$ and $a_2$.  

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Proof. We start by taking the derivative of (15) with respect to $z$. That is,
\[
\frac{\partial}{\partial z} \left( \frac{me^{-az}}{m - x(e^{mz} - 1)} \right) = \frac{me^{-az}}{m - x(e^{mz} - 1)} \cdot \frac{x me^{mz}}{m - x(e^{mz} - 1)} - \frac{ame^{-az}}{m - x(e^{mz} - 1)}.
\]
Replacing $a$ with $\bar{A} = a_1 + a_2 + m$ yields
\[
\frac{\partial}{\partial z} \left( \frac{me^{-\bar{A}z}}{m - x(e^{mz} - 1)} \right) = \sum_{n=k}^{\infty} \bar{F}_{m,\bar{A}}(n + 1; x) \frac{z^n}{n!}
\]
in the left-hand side while we get
\[
\frac{me^{-\bar{A}z}}{m - x(e^{mz} - 1)} \cdot \frac{x me^{mz}}{m - x(e^{mz} - 1)} = \frac{me^{-a_1z}}{m - x(e^{mz} - 1)} \cdot \frac{me^{-a_2z}}{m - x(e^{mz} - 1)}
\]
\[
= x \sum_{n=k}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \bar{F}_{m,a_1}(k; x) \bar{F}_{m,a_2}(n - k; x) \frac{z^n}{n!}
\]
and
\[
\frac{\bar{A}me^{-\bar{A}z}}{m - x(e^{mz} - 1)} = \bar{A} \cdot \sum_{n=k}^{\infty} \bar{F}_{m,\bar{A}}(n; x) \frac{z^n}{n!}
\]
in the right-hand side. Combining the above equations and comparing the coefficients of $\frac{z^n}{n!}$ gives the desired result.

When $a_1 = a_2 = 0$ in (22),
\[
x \sum_{k=0}^{n} \binom{n}{k} w_k \left( \frac{x}{m} \right) w_{n-k} \left( \frac{x}{m} \right) = \bar{F}_{m,m}(n + 1; x) + m \bar{F}_{m,m}(n; x).
\]
Applying (17) to the right-hand side of this equation gives
\[
x \sum_{k=0}^{n} \binom{n}{k} w_k \left( \frac{x}{m} \right) w_{n-k} \left( \frac{x}{m} \right) = \frac{x w_{n+1} \left( \frac{x}{m} \right) - (-m)^{n+2}}{m + x} + m \frac{x w_n \left( \frac{x}{m} \right) - (-m)^{n+1}}{m + x}
\]
which can be simplified into the following identity:
\[
(m + x) \sum_{k=0}^{n} \binom{n}{k} w_k \left( \frac{x}{m} \right) w_{n-k} \left( \frac{x}{m} \right) = w_{n+1} \left( \frac{x}{m} \right) + m w_n \left( \frac{x}{m} \right).
\]
Obviously, this identity boils down to the result obtained by Kargin [9] in (9) when $m = 1$.

**Theorem 4.** For $x_1 \neq x_2$, the following formula holds:
\[
\sum_{k=0}^{n} \binom{n}{k} \bar{F}_{m,a_1}(k; x_1) \bar{F}_{m,a_2}(n - k; x_2) = \frac{x_2 \bar{F}_{m,a_1+a_2}(n; x_2) - x_1 \bar{F}_{m,a_1+a_2}(n; x_1)}{x_2 - x_1}.
\]
Proof. Note that we can write
\[
\frac{me^{-a_1 z}}{m - x_1(e^{m z} - 1)} \cdot \frac{me^{-a_2 z}}{m - x_2(e^{m z} - 1)} = \frac{1}{x_2 - x_1} \left[ \frac{x_2 me^{-(a_1+a_2) z}}{m - x_2(e^{m z} - 1)} - \frac{x_1 me^{-(a_1+a_1) z}}{m - x_1(e^{m z} - 1)} \right].
\]
Following the same method used in the previous theorem leads us to the desired result. \(\square\)

This theorem contains a formula for the sums of products of noncentral Tanny-Dowling polynomials for different values of \(x\). When \(a_1 = a_2 = 0\), (24) reduces to
\[
\sum_{k=0}^{n} \binom{n}{k} w_k \left( \frac{x_1}{m} \right) w_{n-k} \left( \frac{x_2}{m} \right) = \frac{x_2 w_n \left( \frac{x_2}{m} \right) - x_1 w_n \left( \frac{x_1}{m} \right)}{x_2 - x_1}. \tag{25}
\]
It is clear to see that when \(m = 1\), we recover the sum of products of geometric polynomials in (10).

3 Explicit formulas

In Theorem 1, we obtained an explicit formula that expresses the noncentral Tanny-Dowling polynomials in terms of the geometric polynomials. Now, with \(g_n = \frac{1}{a^n} \tilde{F}_{m,a}(n; x)\) and \(f_j = \left( \frac{m}{a} \right) ^ j w_j \left( \frac{x}{m} \right)\), the binomial inversion formula
\[
f_n = \sum_{j=0}^{n} \binom{n}{j} g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f_j. \tag{26}
\]
allows us to express the geometric polynomials \(w_n \left( \frac{x}{m} \right)\) in terms of the noncentral Tanny-Dowling polynomials as follows.
\[
w_n \left( \frac{x}{m} \right) = \frac{1}{m^n} \sum_{j=0}^{n} \binom{n}{j} a^{n-j} \tilde{F}_{m,a}(j; x). \tag{27}
\]
In this section, we will derive more explicit formulas for both polynomials.

Using \(x - m\) in place of \(x\) in (15) gives
\[
\sum_{n=k}^{\infty} \tilde{F}_{m,a}(n; x-m) \frac{z^n}{n!} = \frac{me^{-(-a-m)(-z)}}{m + x(e^{-m z} - 1)} = \sum_{n=k}^{\infty} \tilde{F}_{m,-a-m}(n; -x) \frac{(-z)^n}{n!}.
\]
By comparing the coefficients of \(\frac{z^n}{n!}\), we get
\[
\tilde{F}_{m,a}(n; x-m) = (-1)^n \tilde{F}_{m,-a-m}(n; -x). \tag{28}
\]
Applying (17) to the right-hand side gives
\[ \tilde{F}_{m,a}(n; x - m) = (-1)^n \left[ \frac{(m - x)\tilde{F}_{m,-a}(n; -x) - a^n m}{-x} \right]. \]
Replacing \(-x\) and \(-a\) with \(x\) and \(a\), respectively, and solving for \(\tilde{F}_{m,a}(n; x)\) yields
\[ \tilde{F}_{m,a}(n; x) = \frac{(-1)^n x\tilde{F}_{m,-a}(n; -x - m) + (-a)^n m}{m + x}. \]

By (14), we get the next theorem.

**Theorem 5.** The noncentral Tanny-Dowling polynomials satisfy the following explicit formula:
\[ \tilde{F}_{m,a}(n; x) = x \sum_{k=0}^{n} (-1)^{n+k} k! \tilde{W}_{m,-a}(n, k)(m + x)^{k-1} + \frac{(-a)^n m}{m + x}. \] (29)

Setting \(a = 0\) in \(\tilde{W}_{m,a}(n, k)\) allows us to express the noncentral Whitney numbers of the second kind in terms of \(\binom{n}{k}\). More precisely, when \(a = 0\) in [12, Proposition 7], we can see that
\[ \tilde{W}_{m,0}(n, k) = m^{n-k} \binom{n}{k}. \]

Thus, (29) becomes
\[ w_n \left( \frac{x}{m} \right) = x \sum_{k=0}^{n} (-1)^{n+k} k! m^{n-k} \binom{n}{k} (m + x)^{k-1} \] (30)
when \(a = 0\). Moreover, when \(m = 1\), we recover the explicit formula in [11]. The expression \(m^{n-k} \binom{n}{k}\) is actually called translated Whitney numbers of the second kind and is denoted by \(\binom{n}{k}^{(m)}\). These numbers satisfy the recurrence relation given by [11, Theorem 8]
\[ \binom{n}{k}^{(m)} = \binom{n-1}{k-1}^{(m)} + mk \cdot \binom{n-1}{k}^{(m)} \]
and the explicit formula [13, Proposition 2]
\[ \binom{n}{k}^{(m)} = \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (m^j)^n. \]

More properties of these numbers can be seen in [11]. With these, we may also write
\[ w_n \left( \frac{x}{m} \right) = x \sum_{k=0}^{n} (-1)^{n+k} k! \binom{n}{k}^{(m)} (m + x)^{k-1}, \] (31)
an explicit formula for the geometric polynomials \(w_n \left( \frac{x}{m} \right)\) in terms of the translated Whitney numbers of the second kind.
Now, it can be shown that
\[
\frac{y^2 - 1}{2y} \left( e^{-a(2z)} + \frac{e^{-a(2z)}}{y + e^{mz}} \right) = \frac{e^{-a(2z)}}{1 - \left( \frac{1}{y^2 - 1} \right) (e^{m(2z)} - 1)}.
\]

Notice that the right-hand side is
\[
\frac{e^{-a(2z)}}{1 - \left( \frac{1}{y^2 - 1} \right) (e^{m(2z)} - 1)} = \frac{me^{-a(2z)}}{m - \left( \frac{m}{y^2 - 1} \right) (e^{m(2z)} - 1)} = \sum_{n=0}^{\infty} 2^n \tilde{F}_{m,a} \left( n; \frac{m}{y^2 - 1} \right) \frac{z^n}{n!}.
\]

Also, in the left-hand side, we have
\[
\frac{e^{-a(2z)}}{y - e^{mz}} = \frac{1}{y - 1} \sum_{n=0}^{\infty} \tilde{F}_{m,2a} \left( n; \frac{m}{y - 1} \right) \frac{z^n}{n!}
\]
and
\[
\frac{e^{-a(2z)}}{y + e^{mz}} = \frac{1}{y + 1} \sum_{n=0}^{\infty} \tilde{F}_{m,2a} \left( n; \frac{-m}{y + 1} \right) \frac{z^n}{n!}.
\]

Combining these equations and comparing the coefficients of \( \frac{z^n}{n!} \) results to
\[
2^{n+1} \tilde{F}_{m,a} \left( n; \frac{m}{y^2 - 1} \right) = \frac{y + 1}{y} \tilde{F}_{m,2a} \left( n; \frac{m}{y - 1} \right) + \frac{y - 1}{y} \tilde{F}_{m,2a} \left( n; \frac{-m}{y + 1} \right).
\]

Note that if we set \( x = \frac{m}{y^2 - 1} \), then \( y = \frac{m + x}{x} \). Hence, skipping the tedious computations allow us to write
\[
(m + 2x)\tilde{F}_{m,2a} \left( n; \frac{x}{m + 2x} \right) = 2^{n+1} (m + x)\tilde{F}_{m,a} \left( n; \frac{x^2}{m + 2x} \right) - m\tilde{F}_{m,2a} \left( n; \frac{-mx}{m + 2x} \right).
\]

The next theorem is obtained by applying (14).

**Theorem 6.** The noncentral Tanny-Dowling polynomials satisfy the following explicit formula:
\[
\tilde{F}_{m,2a} \left( n; \frac{x}{m} \right) = \sum_{k=0}^{n} k!x^k \left[ \frac{2^{n+1}(m+x)x^k\tilde{W}_{m,a}(n,k) + (-m)^{k+1}\tilde{W}_{m,2a}(n,k)}{(m+2x)^{k+1}} \right].
\]  

(32)

Since it is already known that \( \tilde{W}_{m,0}(n,k) = \binom{n}{k}^{(m)} \), then when \( a = 0 \), the right-hand side can be expressed in terms of the translated Whitney numbers of the second kind. That is,
\[
w_n \left( \frac{x}{m} \right) = \sum_{k=0}^{n} k!x^k \binom{n}{k}^{(m)} \left[ \frac{2^{n+1}(m+x)x^k + (-m)^{k+1}}{(m+2x)^{k+1}} \right].
\]  

(33)
Lastly, when $m = 1$, we recover the explicit formula in \cite{12}.

Finally, we will end by mentioning an explicit formula for $\tilde{F}_{m,a}(n; x)$ established in \cite{12, Theorem 19} that is given by

$$\tilde{F}_{m,a}(n; x) = \frac{m}{m + x} \sum_{k=0}^{\infty} \left( \frac{x}{m + x} \right)^k (mk - a)^n. \quad (34)$$

This explicit formula entails interesting particular cases. For instance, when $a = 0$,

$$w_n \left( \frac{x}{m} \right) = \frac{m^{n+1}}{m + x} \sum_{k=0}^{\infty} \left( \frac{x}{m + x} \right)^k k^n. \quad (35)$$

When $m = 1$ and then $x = 1$, we get formulas for the ordinary geometric polynomials and numbers. That is,

$$w_n(x) = \frac{1}{x + 1} \sum_{k=0}^{\infty} \left( \frac{x}{x + 1} \right)^k k^n \quad (36)$$

and

$$w_n = \sum_{k=0}^{\infty} \frac{k^n}{2k+1}. \quad (37)$$

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