Semileptonic decays $B/B_s \to (D^{(*)}, D_{s}^{(*)}) l \nu_l$ in the PQCD approach with the lattice QCD input

Xue-Qing Hu\textsuperscript{1,*}, Su-Ping Jin\textsuperscript{1,†} and Zhen-Jun Xiao\textsuperscript{1,2‡}

1. Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing, Jiangsu 210023, People’s Republic of China, and

2. Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University, Nanjing 210023, People’s Republic of China

(Dated: December 10, 2019)

Abstract

In this paper, we studied the semileptonic $B/B_s \to (D^{(*)}, D_{s}^{(*)}) l \nu_l$ decays in the framework of the standard model (SM), by employing the perturbative QCD (PQCD) factorization formalism combining with the lattice QCD inputs of the relevant transition form factors. We calculated the branching ratios $B(B_s \to D_{s}^{(*)} l \nu_l)$ with $l = (\text{e}, \mu, \tau)$, the ratios of the branching ratios $R(D^{(*)})$ and $R(D_{s}^{(*)})$ and other physical observables $P_r(D_{s}^{(*)})$, $F_L(D_{s}^{(*)})$ and $A_{FB}(\tau)$. The “PQCD+Lattice” predictions for $B(B \to D^{(*)} l \nu_l)$ and $R(D^{(*)})$ do agree well with those currently available experimental measurements within the errors. For the ratios $R(D_{s})$ and $R(D_{s}^{(*)})$, the "PQCD+Lattice" predictions agree well with other known predictions. For both $P_r(D^{(*)})$ and $F_L(D^{(*)})$, our theoretical predictions agree well with the measured values within errors. Our theoretical predictions about the considered semileptonic $B/B_s$ decays could be tested in the near future LHCb and the Belle II experiments.

PACS numbers: 13.20.He, 12.38.Bx, 14.40.Nd

* hu-xueqing@qq.com
† 2223919088@qq.com
‡ xiaozhenjun@njnu.edu.cn
I. INTRODUCTION

The studies for the semileptonic decays of $B/B_s$ meson do play an essential role in testing the standard model (SM) as well as in searching for new physics (NP) beyond the SM, since the lepton flavor universality (LFU) can be examined through this kind of decay modes. The LFU means that all electroweak gauge bosons ($Z^0$, $\gamma$ and $W^{\pm}$) have equivalent couplings to three generation leptons, and the only difference arises due to the mass differences: $m_e < m_\mu \ll m_\tau$. Therefore, if the experiments could find some signals for the lepton flavor violation (LFV), it would be a true challenge to the SM.

The first observation of the $R(D)$ and $R(D^*)$ anomalies for the semileptonic decays $B \to D^{(*)}l\nu_l$ with $l^- = (e^-, \mu^-, \tau^-)$ in 2012 by BaBar Collaboration [1, 2] has invoked intensive studies for $B \to D^{(*)}l\nu_l$ decays in the framework of the SM [3–10] and various new NP models[11–18]. When more measurements are reported by both Belle and LHCb Collaboration [19–25], however, the deviations between the measured $R(D)$ and $R(D^*)$ and the SM predictions become a little narrow now [26–29]:

1. The latest Belle measurements [24] alone exhibit an excellent consistency with the averaged SM predictions [3–5, 29]:

$$R(D) = \begin{cases} 0.307 \pm 0.037(\text{stat.}) \pm 0.016(\text{syst.}), \quad \text{Belle} \quad [24] \\ 0.299 \pm 0.003, \quad \text{SM} \quad [29] \end{cases}$$

$$R(D^*) = \begin{cases} 0.283 \pm 0.018(\text{stat.}) \pm 0.014(\text{syst.}), \quad \text{Belle} \quad [24] \\ 0.258 \pm 0.005, \quad \text{SM} \quad [29] \end{cases}$$

they are compatible within one standard deviation [24, 29].

2. The combined analysis of currently available measured $R(D)$ and $R(D^*)$ with the inclusion of the new Belle results [24] gives the following world averaged values:

$$R(D)^{\text{Exp}} = 0.340 \pm 0.027 \pm 0.013, \quad R(D^*)^{\text{Exp}} = 0.295 \pm 0.011 \pm 0.008,$$

the discrepancy decreases from the previous 3.8$\sigma$ to 3.1$\sigma$ with respect to the SM expectations [26, 27, 29].

3. Besides the ratios $R(D^{(*)})$, some relevant physical observables, such as the longitudinal polarization of the tau lepton $P_\tau(D^*)$, the fraction of $D^*$ longitudinal polarization $F_L(D^*)$, have been measured very recently by Belle Collaboration [30–32]:

$$P_\tau(D^*) = -0.38 \pm 0.51(\text{stat.})^{0.21}_{0.16}(\text{syst.}), [30, 31],$$

$$F_L(D^*) = 0.60 \pm 0.08(\text{stat.}) \pm 0.04(\text{syst.}), [32].$$

They are compatible with the SM predictions $P_\tau(D^*) = -0.497 \pm 0.013$ [33], $F_L(D^*) = 0.441(6)$ [34] and 0.457(10) [35] for $B \to D^*\tau\bar{\nu}_\tau$ decays.

It is evident to see from above points that although the anomalies about $R(D^{(*)})$ become less serious recently, it is still a sizeable discrepancy with the SM expectations, and it must be investigated with complementary and more precise measurements in order to make a conclusion about it. In addition to the $B \to D^{(*)}l\nu_l$ decays, the $\bar{B}_s \to D^{(*)}_s l\nu_l$ decay mode is one of the best choices for
crossing examination. The systematic theoretical studies and experimental measurements for the semileptonic decays of \( B_s \) meson are therefore very important and worth of doing immediately.

As illustrated by the Feynman diagrams in Fig. 1, the \( B_s \rightarrow D_s^{(*)} l^+ \nu \) decays are closely related with those \( B \rightarrow D^{(*)} l^+ \nu \) decays through the SU(3) flavor symmetry. They are all controlled by the same \( b \rightarrow c l \nu \) transitions at the quark level, but with a different spectator quark: \((u, d)\) or \( s \) quark. In the limit of SU(3) flavor symmetry, consequently, these two sets of semileptonic decays must have very similar properties. If current anomalies in \( B \rightarrow D^{(*)} l\nu \) decays are indeed induced by the new physics contributions, it must show up in the \( B_s \rightarrow D_s^{(*)} l\nu \) decays. It is therefore necessary and very interesting to study \( B_s \rightarrow D_s^{(*)} l\nu \) decays and to measure the corresponding ratios \( R(D_s^{(*)}) \) and other relevant physical observables such as \( P_L(D_s^{(*)}), F_L(D_s^0) \) and \( A_{FB}(\tau) \), in order to check if there exist any similar deviations.

In the framework of the SM, as is well known, the central issue for the calculations of such semileptonic decays is the estimation for the values and shapes of the relevant form factors \( F_{+,0}(q^2) \), \( V(q^2) \) and \( A_{0,1,2}(q^2) \) for the \( B(s) \rightarrow D_s^{(*)} \) transitions. However, the calculations for these form factors are not an easy task and can not be estimated reliably in the whole range of the momentum \( q^2 \) carried by the lepton pair by using one method, the extrapolation is indispensable. There are many traditional methods or approaches to estimate the relevant form factors and then provide their own predictions for the ratios \( R(D^{(*)}) \), for examples, the heavy quark effective theory (HQET)[4, 5], the light cone sum rules (LCSR)[36–38], the lattice QCD (LQCD)[39–43] and the perturbative QCD factorization approach (PQCD) [9, 10].

In recent years, the PQCD factorization approach has been used to study the various kinds of the semileptonic decays of \( B/B_s/B_c \) mesons for example as being done in Refs. [9, 10, 44–50]. However, just like many other theoretical approaches, the form factors evaluated by using the PQCD approach are only reliable at the low \( q^2 \) region. Therefore, the extrapolations must be done in order to cover the whole range of \( q^2 \) of the form factors \( f_i(q^2) \). In order to improve the reliability of the size and shape of all the six relevant form factors obtained by employing the PQCD approach, we here will include the lattice QCD inputs at the end point \( q^2_{\text{max}} \) so that the extrapolation from the PQCD predictions at the low \( q^2 \) region to the high \( q^2 \) region become reliable too.

In Refs. [9, 10], the \( B \rightarrow D^{(*)} l\nu \) decays have been studied by employing the PQCD approach only [9] or with the inclusion of the lattice QCD input [10]. In Ref. [44], the \( \bar{B}^0_s \rightarrow D_s^{(*)} l^- \bar{\nu} \) decays have been studied by employing the PQCD approach without the lattice QCD input. In this paper, we will study the semileptonic decays of \( B \) and \( B_s \) mesons simultaneously and focus on the following three tasks:

1. For both \( B_s \rightarrow D_s^{(*)} \) and \( B \rightarrow D^{(*)} \) and transitions, we evaluate the relevant form factors in the low \( 0 \leq q^2 \leq m_t^2 \) region by using the PQCD factorization approach, and then include the lattice QCD inputs at the end point \( q^2_{\text{max}} \) in the fitting process and extrapolate those form factors to the entire momentum region by employing the Bourrely-Caprini-Lellouch (BCL) parametrization [51, 52] instead of the pole models used previously in Refs. [9, 10, 44].

2. In addition to the calculation for the branching ratios and the ratios \( R(D_s^{(*)}) \) and \( R(D^{(*)}) \), we will also calculate other three kinds of physical observables (they are not considered in previous works [9, 10, 44]) for the decays of \( B \) and \( B_s \) mesons: the longitudinal polarization of the tau lepton \( P_L(D_s^{(*)}) \), the fraction of \( D_s^{(*)} \) longitudinal polarization \( F_L(D_s^0) \) and the forward-backward asymmetry of the tau lepton \( A_{FB}(\tau) \) in both the ordinary PQCD approach and the “PQCD +Lattice” approach (i.e. the PQCD approach with the inclusion
of the lattice QCD input for form factors).

(3) We will present our theoretical predictions, compare them with those currently available experimental measurements or the theoretical predictions obtained by using other different theories or models.

This paper is organized as follows: In section II, we briefly review the kinematics of the $B^0_s \rightarrow D^{(*)}_s l^+\nu_l$. The calculations of the form factors for the $B^0_s \rightarrow D^{(*)}_s$ transitions are given then. We use the Lattice QCD results at $q^2_{\text{max}}$ given by the HPQCD group[43] as the inputs in our extrapolation. The explicit expressions of the differential decay rates and additional physical observables are also given in Sec. II. In section III, we present the theoretical predictions for all considered physical observables obtained by using the PQCD approach, the “PQCD+Lattice” approach and some typical different models. A short summary will be given in the final section.

II. THEORETICAL FRAMEWORK

A. Kinematics and the wave functions

\begin{equation}
\begin{aligned}
p_1 &= \frac{m_B}{\sqrt{2}}(1, 1, 0_\perp), \\
p_2 &= \frac{r \cdot m_B}{\sqrt{2}}(\eta^+, \eta^-, 0_\perp), \\
\epsilon_L &= \frac{1}{\sqrt{2}}(\eta^+, -\eta^-, 0_\perp), \\
\epsilon_T &= (0, 0, 1).
\end{aligned}
\end{equation}

while $\epsilon_L$ and $\epsilon_T$ denotes the longitudinal and transverse polarization of the $(D^*, D^*_s)$ mesons, respectively. The parameter $\eta^\pm$ and $r$ are defined as:

\begin{equation}
\begin{aligned}
\eta^\pm &= \eta \pm \sqrt{\eta^2 - 1}, \\
\eta &= \frac{1}{2r}(1 + r^2 - \frac{q^2}{m_B^2}), \\
r &= \frac{m_D}{m_B}.
\end{aligned}
\end{equation}

$^1$ Throughout this paper the symbol $B_{(s)}$ describes both $B = (B_u, B_d)$ and $B_s$ mesons.
where \( q = p_1 - p_2 \) is the momentum of the lepton pair. The momenta of the spectator quarks in \( B_{(s)} \) and \( D_{(s)}^{(*)} \) mesons are parameterized as

\[
k_1 = \frac{m_{B_{(s)}}}{\sqrt{2}}(0, x_1, k_{1\perp}), \quad k_2 = \frac{r \cdot m_{B_{(s)}}}{\sqrt{2}}(x_2\eta^+, x_2\eta^-, k_{2\perp}), \tag{8}\]

where \( x_1 \) and \( x_2 \) are the fraction of the momentum carried by the light spectator quark in the initial \( B/B_s \) meson and the final state meson \( D^{(*)}/D_s^{(*)} \), respectively.

For the \( B/B_s \) meson wave function, we use the same one as being used in Refs. [53, 54].

\[
\Phi_{B_{(s)}}(x, b) = \frac{i}{\sqrt{2N_c}}(\phi_{B_{(s)}} + m_{B_{(s)}})\gamma_5\phi_{B_{(s)}}(x, b), \tag{9}\]

\[
\phi_{B_{(s)}}(x, b) = N_{B_{(s)}} \cdot x^2(1-x)^2 \cdot \exp \left[ -\frac{x^2 \cdot M_{B_{(s)}}^2}{2\omega_{B_{(s)}}^2} - \frac{1}{2}(\omega_{B_{(s)}} \cdot b)^2 \right]. \tag{10}\]

The normalization factor \( N_{B_{(s)}} \) depends on the values of the parameter \( \omega_{B_{(s)}} \) and decay constant \( f_{B_{(s)}} \) through the normalization relation: \( \int_0^1 dx \phi_{B_{(s)}}(x, b = 0) = f_{B_{(s)}}/(2\sqrt{6}) \). In order to estimate the uncertainties of theoretical predictions, we set the shape parameter \( \omega_{B} = 0.40 \pm 0.04 \) GeV and \( \omega_{B_s} = 0.50 \pm 0.05 \) GeV.

For the \( D_{(s)} \) and \( D_{s}^{(*)} \) meson, we use the same wave functions as being used in Ref. [55]

\[
\Phi_{D_{(s)}}(p, x) = \frac{i}{\sqrt{6}}\gamma_5(\phi_{D_{(s)}} + m_{D_{(s)}})\phi_{D_{(s)}}(x), \tag{11}\]

\[
\Phi_{D_{s}^{(*)}}(p, x) = \frac{-i}{\sqrt{6}} \left[ f_L(\phi_{D_{s}^{(*)}}^L + m_{D_{s}^{(*)}}^L)\phi_{D_{s}^{(*)}}^L(x) + f_T(\phi_{D_{s}^{(*)}}^T + m_{D_{s}^{(*)}}^T)\phi_{D_{s}^{(*)}}^T(x) \right]. \tag{12}\]

with the distribution amplitudes

\[
\phi_{D_{(s)}}^{D_{(s)}}(x) = \frac{f_{D_{(s)}}^{D_{(s)}}}{2\sqrt{6}} \cdot 6x(1-x) \left[ 1 + C_{D_{(s)}}(1-2x) \right] \cdot \exp \left[ -\frac{\omega_{D_{(s)}}^2 b^2}{2} \right]. \tag{13}\]

where we set the parameters \( C_D = C_{D_s} = C_{D_s} = C_{D_s} = 0.5 \) and \( \omega = 0.1 \). From the heavy quark limit, we here assume that

\[
f_{D_{s}^{(*)}}^L = f_{D_{s}^{(*)}}^T = f_{D_{s}^{(*)}}, \quad f_{D_{s}^{(*)}}^L = f_{D_{s}^{(*)}} = f_{D_{s}^{(*)}}, \tag{14}\]

\[
\phi_{D_{s}^{(*)}}^{D_{(s)}^{(*)}} = \phi_{D_{s}^{(*)}}^{D_{(s)}^{(*)}}, \quad \phi_{D_{s}^{(*)}}^{D_{(s)}^{(*)}} = \phi_{D_{s}^{(*)}}^{D_{(s)}^{(*)}}. \tag{15}\]

**B. Form Factors in PQCD approach**

The form factors of the \( B_{(s)} \rightarrow D_{(s)} \) transition are defined in the same form as in Ref. [56]

\[
\langle D_{(s)}(p_2)|\bar{c}(0)\gamma_\mu b(0)|B_{(s)}(p_1)\rangle = \left[ (p_1 + p_2)_\mu - \frac{m_{B_{(s)}}^2 - m_{D_{(s)}}^2}{q^2}q_\mu \right] F_+(q^2) + \left[ \frac{m_{B_{(s)}}^2 - m_{D_{(s)}}^2}{q^2}q_\mu \right] F_0(q^2) \tag{16}\]

\[5\]
In order to cancel the poles at $q^2 = 0$, $F_+ (0)$ should be equal to $F_0 (0)$. For convenience, we also define the auxiliary form factors $f_1 (q^2)$ and $f_2 (q^2)$,

$$\langle D_{(s)} (p_2) | \bar{c}(0) \gamma_\mu b(0) | B_{(s)} (p_1) \rangle = f_1 (q^2) p_{1 \mu} + f_2 (q^2) p_{2 \mu}, \tag{17}$$

where the auxiliary form factors $f_1 (q^2)$ and $f_2 (q^2)$ are related to $F_+ (q^2)$ and $F_0 (q^2)$ through the following relations,

$$F_+ (q^2) = \frac{1}{2} \left[ f_1 (q^2) + f_2 (q^2) \right], \tag{18}$$

$$F_0 (q^2) = \frac{1}{2} f_1 (q^2) \left[ 1 + \frac{q^2}{m_{B(s)}^2 - m_{D(s)}^2} \right] + \frac{1}{2} f_2 (q^2) \left[ 1 - \frac{q^2}{m_{B(s)}^2 - m_{D(s)}^2} \right]. \tag{19}$$

As for the vector meson $D_{(s)}^*$ at the final state, the form factors involved in $B_{(s)} \to D_{(s)}^*$ transitions are $V (q^2)$ and $A_{0,1,2} (q^2)$. And they are defined in the following forms [56]:

$$\langle D_{(s)}^* (p_2) | \bar{c}(0) \gamma_\mu b(0) | B_{(s)} (p_1) \rangle = \frac{2i V (q^2)}{m_{B(s)} + m_{D_{(s)}^*}} \epsilon_{\mu \nu \alpha \beta} e^{* \nu} p_1^\alpha p_2^\beta, \tag{20}$$

$$\langle D_{(s)}^* (p_2) | \bar{c}(0) \gamma_\mu \gamma_5 b(0) | B_{(s)} (p_1) \rangle = 2 m_{D_{(s)}^*} A_0 (q^2) \frac{e^* \cdot q}{q^2} q_\mu$$

$$+ (m_{B(s)} + m_{D_{(s)}^*}) A_1 (q^2) \left( \epsilon_\mu^* - \frac{e^* \cdot q}{q^2} q_\mu \right)$$

$$- A_2 (q^2) \frac{e^* \cdot q}{m_{B(s)} + m_{D_{(s)}^*}} \left[ (p_1 + p_2)_\mu - \frac{m_{B(s)}^2 - m_{D_{(s)}^*}^2}{q^2} q_\mu \right]. \tag{21}$$

We calculated the relevant form factors mentioned above in PQCD approach, and the analytical expressions are like follows:

$$f_1 (q^2) = 8 \pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_2 b_1 b_2 \phi_{B_{(s)}} (x_1, b_1) \phi_{D_{(s)}} (x_2, b_2)$$

$$\times \left\{ [2r (1 - rx_2)] \cdot H_1 (t_1)$$

$$+ 2r (2r_c - r) + x_1 r \left( -2 + 2 \eta + \sqrt{\eta^2 - 1} - \frac{2 \eta}{\sqrt{\eta^2 - 1}} + \frac{\eta^2}{\sqrt{\eta^2 - 1}} \right) \right\} \cdot H_2 (t_2) \right\} \tag{22}$$

$$f_2 (q^2) = 8 \pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_2 b_1 b_2 \phi_{B_{(s)}} (x_1, b_1) \phi_{D_{(s)}} (x_2, b_2)$$

$$\times \left\{ [2 - 4 x_2 r (1 - \eta)] \cdot H_1 (t_1) + \left[ 4 r - 2 r_c - x_1 + \frac{x_1}{\sqrt{\eta^2 - 1}} (2 - \eta) \right] \cdot H_2 (t_2) \right\}, \tag{23}$$

$$V (q^2) = 8 \pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_2 b_2 b_2 \phi_{B_{(s)}} (x_1, b_1) \phi_{D_{(s)}^*} (x_2, b_2) \cdot (1 + r)$$

$$\times \left\{ [1 - rx_2] \cdot H_1 (t_1) + \left[ r + \frac{x_1}{2 \sqrt{\eta^2 - 1}} \right] \cdot H_2 (t_2) \right\}, \tag{24}$$
\[ A_0(q^2) = 8\pi m_{B(s)}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B(s)}(x_1, b_1) \phi_{D_s}^*(x_2, b_2) \]
\[ \times \left\{ [1 + r - rx_2(2 + r - 2\eta)] \cdot H_1(t_1) + r^2 + r_c + \frac{x_1}{2} + \frac{\eta x_1}{2\sqrt{\eta^2 - 1}} + \frac{rx_1}{2 \sqrt{\eta^2 - 1}}(1 - 2\eta(\eta + \sqrt{\eta^2 - 1})) \right\} \cdot H_2(t_2), \] (25)

\[ A_1(q^2) = 8\pi m_{B(s)}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B(s)}(x_1, b_1) \cdot \phi_{D_s}^*(x_2, b_2) \cdot \frac{r}{1 + r} \]
\[ \times \left\{ 2[1 + \eta - 2rx_2 + r\eta x_2] \cdot H_1(t_1) + [2r_c + 2\eta r - x_1] \cdot H_2(t_2) \right\}, \] (26)

\[ A_2(q^2) = \frac{(1 + r)^2(\eta - r)}{2r(\eta^2 - 1)} \cdot A_1(q^2) \]
\[ - 8\pi m_{B(s)}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B(s)}(x_1, b_1) \cdot \phi_{D_s}^*(x_2, b_2) \cdot \frac{1 + r}{\eta^2 - 1} \]
\[ \times \left\{ [(1 + \eta)(1 - r) - rx_2(1 - 2r + \eta(2 + r - 2\eta))] \cdot H_1(t_1) + r + r_c(\eta - r) - \eta r + rx_1\eta - \frac{x_1}{2}(\eta + r) + x_1(\eta r - \frac{1}{2})\sqrt{\eta^2 - 1} \right\} \cdot H_2(t_2), \] (27)

where the color factor \( C_F = 4/3 \), the mass ratio \( r_c = m_c/m_{B(s)} \), and the functions \( H_i(t_i) \) are in the following form

\[ H_i(t_i) = h_i(x_1, x_2, b_1, b_2) \cdot \alpha_s(t_i) \exp [-S_{ab}(t_i)], \quad \text{for} \quad i = 1, 2. \] (28)

The explicit expressions of the hard functions \( h_i(x_1, x_2, b_1, b_2) \), the hard scales \( t_i \) and the Sudakov factors \([-S_{ab}(t_i)]\) will be given in the Appendix.

\section*{C. Lattice inputs at \( q_{max}^2 \)}

The lattice QCD has its own advantages to calculate the relevant form factors at large \( q^2 \) region. We generally believe that the lattice QCD predictions for those relevant form factors close or at the \( q_{max}^2 \) are reliable. In this work, we make use of this reliability and take the lattice QCD predictions for all relevant form factors at the endpoint \( q_{max}^2 \) as the additional inputs in the fitting process so that the extrapolation of these form factors from the low \( q^2 \) region to \( q_{max}^2 \) could become more reliable.

The form factors used in the lattice QCD are parameterized as follows [57, 58]:

\[ \langle D(s) | \bar{c}V^{\mu b} | B(s) \rangle = \sqrt{m_{B(s)} m_{D(s)}} [h_+(w)(v + v')^\mu + h_-(w)(v - v')^\mu] \]
\[ \langle D(s) | \bar{c}A^{\mu b} | B(s) \rangle = \sqrt{m_{B(s)} m_{D(s)}} \epsilon_{\mu \rho \sigma}^b \epsilon_{v v'}^{a \sigma} h_V(w), \]
\[ \langle D(s) | \bar{c}A^{\mu b} | B(s) \rangle = \sqrt{m_{B(s)} m_{D(s)}} \epsilon_{\mu \rho \sigma}^b [g^{\mu \nu}(1 + w)h_A(w) - v^\nu(v^\mu h_{A2}(w) + v^\mu h_{A3}(w))]. \] (29)

where \( v = p_{B(s)}/m_{B(s)} \), \( v' = p_{D(s)}/m_{D(s)} \), the velocity transfer \( w = v \cdot v' \), and \( \epsilon \) is the polarization vector of the \( D^*_s \) meson. Through a simple transformation, we can relate them to the form factors
used in our work:

\[
F_\pm(q^2) = \frac{1}{2\sqrt{r}} \left[(1 + r)h_+(w) - (1 - r)h_-(w)\right],
\]

\[
F_0(q^2) = \sqrt{r} \left[\frac{1 + w}{1 + r}h_+(w) - \frac{w - 1}{1 - r}h_-(w)\right],
\]

where \(w = (m_{B(s)}^2 + m_{D(s)}^2 - q^2)/(2m_{B(s)}m_{D(s)})\), and

\[
V(q^2) = \frac{1 + r}{2\sqrt{r}}h_V(w),
\]

\[
A_0(q^2) = \frac{1}{2\sqrt{r}} \left[(1 + w)h_{A_1}(w) - (1 - wr)h_{A_2}(w) + (r - w)h_{A_3}(w)\right],
\]

\[
A_1(q^2) = \sqrt{r} \frac{1}{1 + r}(1 + w)h_{A_1}(w),
\]

\[
A_2(q^2) = \frac{1 + r}{2\sqrt{r}}(r h_{A_2}(w) + h_{A_3}(w)),
\]

where \(w = (m_{B(s)}^2 + m_{D(s)}^2 - q^2)/(2m_{B(s)}m_{D(s)})\).

At the end point \(q^2 = q_{\text{max}}^2\), we have

\[
h_V(1) = h_{A_1}(1) = h_{A_3}(1), \quad h_{A_2}(1) = 0, \quad h_{A_1}(1) = \mathcal{F}(1),
\]

\[
w = 1, \quad \left[h_+(1) - \frac{(1 - r)}{(1 + r)}h_-(1)\right] = \mathcal{G}(1).
\]

Therefore, those relevant form factors at the endpoint \(q_{\text{max}}^2\) will become:

\[
F_+(q_{\text{max}}^2) = \frac{1 + r}{2\sqrt{r}}\mathcal{G}(1),
\]

\[
V(q_{\text{max}}^2) = A_0(q_{\text{max}}^2) = A_2(q_{\text{max}}^2) = \frac{1}{A_1(q_{\text{max}}^2)} = \frac{1 + r}{2\sqrt{r}}\mathcal{F}(1),
\]

By using the formulae above and including the lattice inputs [43, 59]

\[
\mathcal{G}^{B \to D}(1) = 1.033 \pm 0.095, \quad \mathcal{F}^{B \to D^*}(1) = 0.895 \pm 0.010 \pm 0.024,
\]

\[
\mathcal{G}^{B_s \to D_s}(1) = 1.052 \pm 0.046, \quad \mathcal{F}^{B_s \to D_s^*}(1) = 0.883 \pm 0.012 \pm 0.028,
\]

with the relation between \(F_0\) and \(F_+\) near the endpoint \(q_{\text{max}}^2\) as evaluated in Ref. [59]:

\[
F_0^{B \to D}/F_+^{B \to D} = 0.73 \pm 0.04, \quad F_0^{B_s \to D_s}/F_+^{B_s \to D_s} = 0.77 \pm 0.02,
\]

We find the following values of the relevant form factors at the endpoint \(q_{\text{max}}^2\):

\[
F_0^{B \to D}(q_{\text{max}}^2) = 0.86 \pm 0.08, \quad F_0^{B_s \to D_s}(q_{\text{max}}^2) = 0.91 \pm 0.05,
\]

\[
F_+^{B \to D}(q_{\text{max}}^2) = 1.17 \pm 0.10, \quad F_+^{B_s \to D_s}(q_{\text{max}}^2) = 1.19 \pm 0.05,
\]

\[
V^{B \to D^*}(q_{\text{max}}^2) = 1.01 \pm 0.05, \quad V^{B_s \to D_s^*}(q_{\text{max}}^2) = 0.98 \pm 0.05,
\]

\[
A_0^{B \to D^*}(q_{\text{max}}^2) = 1.01 \pm 0.05, \quad A_0^{B_s \to D_s^*}(q_{\text{max}}^2) = 0.98 \pm 0.05,
\]

\[
A_1^{B \to D^*}(q_{\text{max}}^2) = 0.80 \pm 0.04, \quad A_1^{B_s \to D_s^*}(q_{\text{max}}^2) = 0.79 \pm 0.04,
\]

\[
A_2^{B \to D^*}(q_{\text{max}}^2) = 1.01 \pm 0.05, \quad A_2^{B_s \to D_s^*}(q_{\text{max}}^2) = 0.98 \pm 0.05.
\]

The uncertainty of the form factors comes from the errors of \(\mathcal{G}(1)\) as given in Eq. (34) is around 5 \(~\) 10\%, while the errors of \(\mathcal{F}(1)\) is around 2 percent only. We here set conservatively the common error of 5\% for the form factors \(V, A_{0,1,2}\) in Eq. (36) by taking into account approximately the small variations of the central values of \(\mathcal{G}(1)\) and \(\mathcal{F}(1)\) in recent years [43, 57–59].
D. Extrapolations and Differential decay rates

We know that the form factors calculated by the PQCD approach are only reliable at the low $q^2$ region. In order to cover the whole momentum region $m_l^2 \leq q^2 \leq q_{\text{max}}^2$ we have to do the extrapolations, and we also use the values of the relevant form factors at the endpoint $q_{\text{max}}^2$ evaluated based on the lattice QCD for the purpose of improving the extrapolation.

In the previous works [9, 44], we used the pole model parametrization [60] to do the fitting. In this work, we will use the Bourely-Caprini-Lellouch (BCL) parametrization [51] instead of the pole model one, in order to match the lattice inputs and improve the fitting process. We use our PQCD predictions for all relevant form factors $f_i(q^2)$ at the sixteen points of $0 \leq q^2 \leq m_l^2$ as inputs and the lattice QCD results as additional inputs at the endpoints $q_{\text{max}}^2$, then make the extrapolation from the low $q^2$ region to the endpoint $q_{\text{max}}^2$ by using the BCL parametrization [51]. Analogous to Ref. [52], we here also consider only the first two terms of the series in the parameter $z$:

$$ f_i(t) = \sum_{k=0}^{1} \alpha_k^i z^k(t, t_0) = \frac{1}{1 - t/m_R^2} \left( \alpha_0^i + \alpha_1^i \frac{\sqrt{t+ - t - \sqrt{t+ - t_0}}}{\sqrt{t+ - t + \sqrt{t+ - t_0}}} \right), \quad (37) $$

with

$$ 0 \leq t_0 = t_+ \left(1 - \frac{1}{\sqrt{1 + \frac{m_R^2}{t_+}}}\right) \leq t_-, \quad (38) $$

where $t = q^2$, $t_\pm = (m_{B(s)} \pm m_{\bar{D}(s)})^2$, and $m_R$ are the masses of the low-laying resonance. The optimized value of $t_0$ and the values of $m_R$ are chosen to be the same ones as in Ref. [52]. With the extrapolations above, now we have the access to the full momentum dependence of these relevant form factors, and the branching ratios of the semileptonic decays $B(s) \rightarrow D(s)_l\bar{\nu}$ can be calculated. The quark level transition of these semileptonic decays is the $b \rightarrow cl\bar{\nu}$ transition with the effective Hamiltonian [61]

$$ \mathcal{H}_{\text{eff}}(b \rightarrow cl\bar{\nu}) = \frac{G_F}{\sqrt{2}} V_{cb} \bar{c} \gamma_\mu (1 - \gamma_5) b \cdot \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu_l. \quad (39) $$

where $G_F = 1.16637 \times 10^{-5} GeV^{-2}$ is the Fermi-coupling constant. And the differential decay rates of the decay mode $B(s) \rightarrow D(s)_l\bar{\nu}$ can be expressed as [62]:

$$ \frac{d\Gamma(B(s) \rightarrow D(s)_l\bar{\nu})}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B(s)}^3} \left(1 - \frac{m_l^2}{q^2}\right)^2 \frac{\lambda^{1/2}(q^2)}{2q^2} \cdot \left\{ 3m_l^2 \left( m_{B(s)}^2 - m_{D(s)}^2 \right)^2 |F_0(q^2)|^2 + (m_l^2 + 2q^2) \lambda(q^2) |F_+(q^2)|^2 \right\}, \quad (40) $$

where $m_l$ is the mass of the leptons $e, \mu$ or $\tau$, and $\lambda(q^2) = (m_{B(s)}^2 + m_{D(s)}^2 - q^2)^2 - 4m_{B(s)}^2m_{D(s)}^2$ is the phase space factor.

For the decay mode $B(s) \rightarrow D^*_l\bar{\nu}$, the corresponding differential decay widths can be written as [60]:

$$ \frac{d\Gamma_{L}(B(s) \rightarrow D^*_l\bar{\nu})}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B(s)}^3} \left(1 - \frac{m_l^2}{q^2}\right)^2 \frac{\lambda^{1/2}(q^2)}{2q^2} \cdot \left\{ 3m_l^2 \lambda(q^2) A_0^2(q^2) + \frac{m_l^2 + 2q^2}{4m_l^2} \left( m_{B(s)}^2 - m_{D^*_l}^2 - q^2 \right) \left( m_{B(s)} + m_{D^*_l} \right) A_1(q^2) - \frac{\lambda(q^2)A_2(q^2)}{m_{B(s)} + m_{D^*_l}} \right\}^2, \quad (41) $$
\[
\frac{d\Gamma_T(B_s \rightarrow D^*_s)(\nu)}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m^3_{B(s)}} \left(1 - \frac{m^2_{\tau}}{q^2}\right)^2 \lambda^{3/2}(q^2) \left(m^2_\tau + 2q^2\right) \left(\frac{m^2_{B(s)} + m^2_{D^*_s}}{\lambda(q^2)}\right)^2 + \left(\frac{m^2_{B(s)} + m^2_{D^*_s}}{\lambda(q^2)}\right)^2 A^2(q^2),
\]

with the phase space factor \(\lambda(q^2) = (m^2_{B(s)} + m^2_{D^*_s} - q^2)^2 - 4m^2_{B(s)} m^2_{D^*_s}\). The total differential decay widths are defined as:

\[
\frac{d\Gamma}{dq^2} = \frac{d\Gamma_L}{dq^2} + \frac{d\Gamma_T}{dq^2}.
\]

E. Additional physical observables: \(P_\tau, F_L(D^*_s)\) and \(A_{FB}(\tau)\)

Besides the decay rates and the ratios \(R(X)\), there are other three additional physical observables for the considered \(B/B_s\) semileptonic decays: the longitudinal polarization of the tau lepton \(P_\tau\), the fraction of \(D^*_s\) longitudinal polarization \(F_L(D^*_s)\) and the forward-backward asymmetry of the tau lepton \(A_{FB}(\tau)\). These additional physical observables are sensitive to some kinds of new physics beyond the SM [63–65].

As for the definitions of these additional physical observables, we follow Refs. [33, 66, 67]. For the \(\tau\) longitudinal polarization, for example, the authors of Refs. [33, 66] have studied them in the \(Q\) rest frame where the spacial components of the momentum transfer \(Q = p_B - p_M\) vanish. They used a coordinate system in the \(Q\) rest frame so that the direction of the \(B\) momenta is along the \(z\) axis, and the \(\tau\) momentum lies in the \(x-z\) plane. Here, we use the same definition:

\[
P_\tau(D^*_s) = \frac{\Gamma_(D^*_s) - \Gamma_(D^*_s)}{\Gamma_(D^*_s) + \Gamma_(D^*_s)}.
\]

(44)

where \(\Gamma_\pm(D^*_s)\) denotes the decay rate of the decay \(B(s) \rightarrow D^*_s \tau \bar{\nu}_\tau\) with the \(\tau\) lepton helicity of \(\pm 1/2\). The explicit expressions of \(\Gamma_\pm\) in the SM can be found for example in the Appendix of Ref. [67]. For \(B(s) \rightarrow D^*_s \tau \bar{\nu}_\tau\) decays, we find

\[
\frac{d\Gamma_+}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m^3_{B(s)}} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m^2_{\tau}}{q^2}\right)^2 \frac{m^2_\tau}{2q^2} \left(H^s_{V,0} + 3H^s_{V,t}\right),
\]

(45)

\[
\frac{d\Gamma_-}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m^3_{B(s)}} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m^2_{\tau}}{q^2}\right)^2 \left(H^s_{V,0}\right).
\]

(46)

For \(B(s) \rightarrow D^*_s \tau \bar{\nu}_\tau\) decays, we have

\[
\frac{d\Gamma_+}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m^3_{B(s)}} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m^2_{\tau}}{q^2}\right)^2 \frac{m^2_\tau}{2q^2} \left(H^s_{V,+} + H^s_{V,-} + H^s_{V,0} + 3H^s_{V,t}\right),
\]

(47)

\[
\frac{d\Gamma_-}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m^3_{B(s)}} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m^2_{\tau}}{q^2}\right)^2 \left(H^s_{V,+} + H^s_{V,-} + H^s_{V,0}\right).
\]

(48)
where the functions \((H_{V,0}^s, H_{V,t}^s, H_{V,\pm}, H_{V,0}, H_{V,t})\) can be written as the combinations of the six form factors as defined in Eqs. \((18,19,22-27)\):

\[
H_{V,0}^s(q^2) = \sqrt{\frac{\lambda(q^2)}{q^2}} F_+(q^2),
\]
\[
H_{V,t}^s(q^2) = \frac{m_B^2 - m_D^2}{\sqrt{q^2}} F_0(q^2),
\]
\[
H_{V,\pm}(q^2) = (m_B^2 + m_{D*}^2) \frac{\lambda(q^2)}{m_B^2 + m_{D*}^2} V(q^2),
\]
\[
H_{V,0}(q^2) = \frac{m_B^2}{2m_{D*}} \sqrt{\frac{\lambda(q^2)}{q^2}} \left[ - (m_B^2 - m_{D*}^2 - q^2) A_1(q^2) + \frac{\lambda(q^2)}{m_B^2 + m_{D*}^2} A_2(q^2) \right],
\]
\[
H_{V,t}(q^2) = - \frac{\lambda(q^2)}{q^2} A_0(q^2).
\]

The phase space factor \(\lambda\) is the same ones as defined in the Eqs. \((40-42)\).

For the fraction of \(D^*\) longitudinal polarization \(F_L(D^*_s)\), it is defined through the secondary decay chain \(D^*_s \rightarrow D_s^0 \pi^0\) of the considered semileptonic \(B_s^0\) decays. And the \(F_L(D^*_s)\) can be expressed as the same form in Ref. [67]:

\[
F_L(D^*_s) = \frac{\Gamma^0}{\Gamma^0 + \Gamma^+ + \Gamma^-},
\]

and the corresponding differential decay rates are of the following form:

\[
\frac{d\Gamma^\pm}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_{B^+}^3} q^2 \sqrt{\lambda(q^2)} \left( 1 - \frac{m_{\tau}^2}{q^2} \right)^2 \left( 1 + \frac{m_{\tau}^2}{2q^2} \right) (H_{V,\pm}^s),
\]
\[
\frac{d\Gamma^0}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_{B^0}^3} q^2 \sqrt{\lambda(q^2)} \left( 1 - \frac{m_{\tau}^2}{q^2} \right)^2 \left( 1 + \frac{m_{\tau}^2}{2q^2} \right) \left( H_{V,0}^s + \frac{3m_{\tau}^2}{2q^2} H_{V,t}^s \right).
\]

As for the \(\tau\) lepton forward-backward asymmetry \(A_{FB}(\tau)\), it’s a little complicated since it is defined in the \(\tau\bar{\nu}\) rest frame. The explicit expression is of the following form [67]:

\[
A_{FB} = \frac{\int_0^1 \frac{d\tau}{d\cos\theta} d\cos\theta - \int_{-1}^0 \frac{d\tau}{d\cos\theta} d\cos\theta}{\int_{-1}^1 \frac{d\tau}{d\cos\theta} d\cos\theta} = \frac{\int b_\theta(q^2) dq^2}{\Gamma_{B^0}},
\]

where the angle \(\theta\) is the angle between the 3-momenta of the lepton \(\tau\) and the initial \(B\) or \(B_s\) in the \(\tau\bar{\nu}\) rest frame. The function \(b_\theta(q^2)\) is the angular coefficient which can be written as [67]:

\[
b_\theta(D^*_s)(q^2) = \frac{G_F^2 |V_{cb}|^2}{128 \pi^3 m_{B^0}^3} q^2 \sqrt{\lambda(q^2)} \left( 1 - \frac{m_{\tau}^2}{q^2} \right)^2 \frac{m_{\tau}^2}{q^2} (H_{V,0}^s H_{V,t}^s),
\]
\[
b_\theta(D^*)(q^2) = \frac{G_F^2 |V_{cb}|^2}{128 \pi^3 m_{B^0}^3} q^2 \sqrt{\lambda(q^2)} \left( 1 - \frac{m_{\tau}^2}{q^2} \right)^2 \left[ \frac{1}{2} (H_{V,0}^s - H_{V,\pm}^s) + \frac{m_{\tau}^2}{q^2} (H_{V,0}^s H_{V,t}^s) \right].
\]

With the above definitions and formulae, we are now ready to give our numerical results and phenomenological analysis.
III. NUMERICAL RESULTS AND DISCUSSIONS

In the numerical calculations, we use the following input parameters (here masses and decay constants in units of GeV) [28, 29, 68, 69]:

$$m_B = 5.28, \quad m_{B_s} = 5.367, \quad m_{D_0} = 1.865, \quad m_{D_+} = 1.870, \quad m_{D_s^0} = 2.007,$$

$$m_{D_s^+} = 2.010, \quad m_{D_{s0}} = 1.968, \quad m_{D_{s2}} = 2.112, \quad m_{\tau} = 1.777, \quad m_c = 1.275 \pm 0.025,$$

$$f_D = 0.212, \quad f_{D_s} = 0.249, \quad f_{B_{D+}} = 0.187, \quad f_{B_0} = 0.191, \quad f_{B_s} = 0.227,$$

$$|V_{cb}| = (42.2 \pm 0.8) \times 10^{-3}, \quad \tau_{B_+} = 1.638 \text{ps}, \quad \tau_{B_0} = 1.520 \text{ps}, \quad \tau_{B_s} = 1.509 \text{ps},$$

$$f_{D^*} = (1.078 \pm 0.036) \cdot f_D, \quad f_{D_s^*} = (1.087 \pm 0.020) \cdot f_{D_s}, \quad \Lambda_{MS}^{(f=4)} = 0.287. \quad (60)$$

A. Form Factors

For the considered semileptonic $B/B_s$ meson decays, it is obvious that the theoretical predictions for the differential decay rates and other physical observables are strongly dependent on the form factors $F_{0,0}(q^2)$, $V(q^2)$ and $A_{0,1,2}(q^2)$. To be specific, $F_{0,0}(q^2)$ controls the process $B(s) \to D(s)l\nu_l$ while $V(q^2)$ and $A_{0,1,2}(q^2)$ play the key roles in the process $B(s) \to D^*(s)l\nu_l$. The value of these form factors at $q^2 = 0$ and their $q^2$ dependence in the whole range of $0 \leq q^2 \leq q_{max}^2$ contain lots of information of the relevant physical process.

In Refs. [9, 53, 55], the authors examined the applicability of the PQCD approach to $(B \to D^{(*)})$ transitions, and have shown that the PQCD approach with the inclusion of the Sudakov effects is applicable to the semileptonic decays $B \to D^{(*)}l\nu_l$ at the low $q^2$ region. Therefore, we present our PQCD predictions for the relevant form factors of $B(s) \to D^{(*)}(s)$ transitions at the point $q^2 = 0$ in Table I.

| $B^+ \to D^0$ | $B^0 \to D^-$ | $B^0_s \to D_{s0}^-$ | $B^+ \to D^{*0}$ | $B^0 \to D^{*-}$ | $B^0_s \to D_{s*}^-$ |
|-------|-------|-------|-------|-------|-------|
| $F_{0,0}(0)$ | $0.53 \pm 0.10$ | $0.54 \pm 0.10$ | $0.52 \pm 0.10$ | $0.64 \pm 0.11$ | $0.65 \pm 0.11$ |
| $V(0)$ | $0.49 \pm 0.08$ | $0.50 \pm 0.09$ | $0.48 \pm 0.09$ | $0.51 \pm 0.09$ | $0.50 \pm 0.09$ |
| $A_{0}(0)$ | $0.51 \pm 0.09$ | $0.52 \pm 0.08$ | $0.50 \pm 0.09$ | $0.54 \pm 0.09$ | $0.55 \pm 0.10$ |
| $A_{1}(0)$ | $- $ | $- $ | $- $ | $- $ | $- $ |
| $A_{2}(0)$ | $- $ | $- $ | $- $ | $- $ | $- $ |

From the numerical values in Table I, one can see easily that the same form factors corresponding to different $B \to D^{(*)}$ or $B_s \to D_{s}^{(*)}$ transitions are very similar in magnitude at $q^2 = 0$, which implies that the effect of the SU(3) flavor symmetry breaking is really small, less than 10%. In order to cover whole $q^2$ region, one has to make an extrapolation for all relevant form factors from the $q^2 = 0$ region to $q^2 = q_{max}^2$ region. In this work we will make the extrapolation by using two different methods.

(1) The first method is analogous to the one used in Refs. [9, 44], we first calculate explicitly the values of the relevant form factors at several points in the lower region $0 \leq q^2 \leq m_c^2$.
FIG. 2. (Color online) The theoretical predictions for the $q^2$-dependence of the six form factors for $B^+\to (D^0, D^{*0})$ transitions in the PQCD approach (the blue solid curves), and the “PQCD+Lattice” method (the red dashed curves).

by using the expressions as given in Eqs. (22-27) and the definitions in Eqs. (18,19). In the fitting process, we will use the BCL parametrization formula as given in Eqs. (37,38) instead of the pole model parametrization being used in Refs. [9, 44].

2) The second one is the so-called “PQCD+Lattice” method, similar with what we did in Ref. [10]. Since the lattice QCD results for the form factors are reliable and accurate at $q^2 \approx q_{\text{max}}^2$ region, we take the lattice QCD predictions for all relevant form factors at the endpoint $q_{\text{max}}^2$ as the additional inputs in the fitting process. In order to match the lattice inputs, we use the BCL parametrization [51, 52] to make the extrapolation.

In Figs. 2 and 3, we show the theoretical predictions for the $q^2$-dependence of the six relevant form factors for $B^+\to (D^0, D^{*0})$ and $B_s\to (D_{s}^{-}, D_{s}^{*0})$ transitions, obtained by employing the PQCD approach and the “PQCD + Lattice” method. In these two figures, the blue solid curves (red dashed curves) show the theoretical predictions for the $q^2$-dependence of the form factors ($F_{0,+}(q^2)$, $V(q^2)$, $A_{0,1,2}(q^2)$) for the $B/B_s\to D^{(*)}/D_{s}^{(*)}$ transitions by using the traditional PQCD approach (the “PQCD+Lattice” approach). The blue and red band show the theoretical uncertainties. One can see from the theoretical predictions that (a) the values of these six form factors are very similar in value in the low $q^2$ region due to the SU(3) flavor symmetry, but become a little different at large $q^2$ region; (b) the difference between the theoretical predictions between the traditional PQCD approach and the “PQCD+Lattice” method is small in the low $q^2$ region and remain not large even in the large $q^2$ region.

B. Branching ratios and the ratio of Brs

By inserting the form factors we have obtained above into the differential decay rates equations as given in Eqs. (40-43), it is straightforward to make the integration over the range of $m_t^2 \leq q^2 \leq$
the uncertainties of the decay constants of the initial \( B_s \) where the major theoretical errors come from the uncertainties of the input parameters small and have been neglected. In Table II, we list our PQCD and “PQCD+Lattice” predictions for the branching ratios of the considered semileptonic decays of \( B_s^0 \) meson, where the total errors are obtained by adding the individual errors in quadrature. As comparison, we also show some theoretical predictions from other theoretical approaches in the framework of the SM. Unfortunately, there still be no experimental results available at present. In Table III, we show the theoretical predictions for the ratios of the branching ratios \( R(D_s) \) and \( R(D_s^*) \) defined in the same way as

\[
(m_{B(s)} - m_{D(s)})^2. \quad \text{For the four semileptonic decays } B_s^0 \to D_s^{(s)} \tau^+ \nu_\tau \text{ and } B_s \to D_s^{(s)} l^+ \nu_l \text{ with } l^+ = (e^+, \mu^+) \text{, for example, the theoretical predictions for their branching ratios are the following:}
\]

\[
B(B_s^0 \to D_s^- \tau^+ \nu_\tau) = \left\{ \begin{array}{l}
0.72^{+0.32}_{-0.23}(\omega_{B_s}) \pm 0.03(V_{cb}) \pm 0.02(m_c), \text{ PQCD,} \\
0.63^{+0.17}_{-0.13}(\omega_{B_s}) \pm 0.03(V_{cb}) \pm 0.02(m_c), \text{ PQCD + Lattice,}
\end{array} \right. \quad (61)
\]

\[
B(B_s^0 \to D_s^- l^+ \nu_l) = \left\{ \begin{array}{l}
1.97^{+0.88}_{-0.65}(\omega_{B_s}) \pm 0.08(V_{cb}) \pm 0.03(m_c), \text{ PQCD,} \\
1.84^{+0.77}_{-0.50}(\omega_{B_s}) \pm 0.08(V_{cb}) \pm 0.03(m_c), \text{ PQCD + Lattice,}
\end{array} \right. \quad (62)
\]

\[
B(B_s^0 \to D_s^{*-} \tau^+ \nu_\tau) = \left\{ \begin{array}{l}
1.45^{+0.45}_{-0.39}(\omega_{B_s}) \pm 0.06(V_{cb}) \pm 0.06(m_c), \text{ PQCD,} \\
1.20^{+0.25}_{-0.22}(\omega_{B_s}) \pm 0.05(V_{cb}) \pm 0.02(m_c), \text{ PQCD + Lattice,}
\end{array} \right. \quad (63)
\]

\[
B(B_s^0 \to D_s^{*-} l^+ \nu_l) = \left\{ \begin{array}{l}
5.04^{+1.60}_{-1.41}(\omega_{B_s}) \pm 0.20(V_{cb}) \pm 0.16(m_c), \text{ PQCD,} \\
4.42^{+1.26}_{-0.98}(\omega_{B_s}) \pm 0.17(V_{cb}) \pm 0.06(m_c), \text{ PQCD + Lattice,}
\end{array} \right. \quad (64)
\]

where the major theoretical errors come from the uncertainties of the input parameters \( \omega_{B_s} = 0.50 \pm 0.05 \text{ GeV, } |V_{cb}| = (42.2 \pm 0.8) \times 10^{-3} \text{ and } m_c = 1.275^{+0.025}_{-0.035} \text{ GeV. The possible errors from the uncertainties of the decay constants of the initial } B_s \text{ meson and the final and } D_s^{(s)} \text{ mesons are small and have been neglected.} \]
\( R(D^{(*)}) \):

\[
R(D_s) = \frac{\mathcal{B}(B_s^0 \to D_s^- \tau^+ \nu_\tau)}{\mathcal{B}(B_s^0 \to D_s^- l^+ \nu_l)} = \begin{cases} 
0.365^{+0.009}_{-0.012}, & \text{PQCD}, \\
0.341^{+0.024}_{-0.025}, & \text{PQCD + Lattice}, 
\end{cases} \quad (65)
\]

\[
R(D_s^*) = \frac{\mathcal{B}(B_s^0 \to D_s^{*-} \tau^+ \nu_\tau)}{\mathcal{B}(B_s^0 \to D_s^{*-} l^+ \nu_l)} = \begin{cases} 
0.287^{+0.008}_{-0.011}, & \text{PQCD}, \\
0.271^{+0.015}_{-0.016}, & \text{PQCD + Lattice}, 
\end{cases} \quad (66)
\]

where \( l \) denotes an electron or a muon.

TABLE II. The theoretical predictions (in unit of \( 10^{-2} \)) for the branching ratios of the considered semileptonic decays of \( B_s^0 \) meson with \( l = (e, \mu) \) obtained by using various theoretical approaches [44, 70–76].

| Approach      | \( \mathcal{B}(B_s^0 \to D_s^- l^+ \nu_l) \) | \( \mathcal{B}(B_s^0 \to D_s^- \tau^+ \nu_\tau) \) | \( \mathcal{B}(B_s^0 \to D_s^{*-} l^+ \nu_l) \) | \( \mathcal{B}(B_s^0 \to D_s^{*-} \tau^+ \nu_\tau) \) |
|---------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| PQCD          | 1.94^{+0.30}_{-0.23}           | 0.73^{+0.32}_{-0.23}           | 5.04^{+1.62}_{-1.43}          | 1.45^{+0.46}_{-0.40}          |
| PQCD+Lattice  | 1.84^{+0.51}_{-0.37}           | 0.63^{+0.17}_{-0.13}           | 4.42^{+1.27}_{-1.00}         | 1.20^{+0.26}_{-0.23}         |
| PQCD[44]      | 2.13^{+0.77}_{-0.76}           | 0.84^{+0.38}_{-0.28}           | 4.76^{+1.87}_{-1.49}         | 1.44^{+0.31}_{-0.42}         |
| IAMF[70]      | 1.4 – 1.7                      | 0.47 – 0.55                    | 5.1 – 5.8                    | 1.2 – 1.3                     |
| RQM[71]       | 2.1 ± 0.2                      | 0.62 ± 0.05                    | 5.3 ± 0.5                    | 1.3 ± 0.1                     |
| LCSR[72]      | 1.0 ± 0.4 – 0.3                | 0.33^{+0.14}_{-0.11}           | –                             | –                             |
| LFQM[73]      | –                              | –                              | 5.2 ± 0.6                    | 1.3^{+0.2}_{-0.1}             |
| CQM[74]       | 2.73 – 3.00                    | –                              | 7.49 – 7.66                  | –                             |
| QCDSR[75]     | 2.8 – 3.8                      | –                              | 1.89 – 6.61                  | –                             |
| Lattice[76]   | 2.013 – 2.469                  | 0.619 – 0.724                  | –                             | –                             |

TABLE III. The theoretical predictions for the ratios \( R(D_s) \) and \( R(D_s^*) \) obtained by using the various theoretical approaches [44, 71–73, 76].

| Ratios       | PQCD       | PQCD+Lattice | PQCD[44] | RQM[71] | LCSR[72] | LFQM[73] | CQM[74] | QCDSR[75] | Lattice[76] |
|--------------|------------|--------------|----------|---------|----------|----------|---------|-----------|-------------|
| \( R(D_s) \) | 0.365^{+0.009}_{-0.012} | 0.341^{+0.024}_{-0.025} | 0.392(22) | 0.295   | 0.33     | –        | 0.295   | 0.027     |
| \( R(D_s^*) \)| 0.287^{+0.008}_{-0.011} | 0.271^{+0.015}_{-0.016} | 0.302(11) | 0.245   | –        | 0.25     | –       | –         |

Following the same procedure as we did for \( B_s \) decays, we can calculate the branching ratios and the ratios \( R(D^{(*)}) \) for \( B \to (D, D^*) (l^+ \nu_l, \tau^+ \nu_\tau) \) decays by employing the PQCD and the “PQCD+Lattice” approaches, respectively. The numerical results are listed in Table IV for branching ratios and in Table V for ratios \( R(D) \) and \( R(D^*) \). In the estimation for errors, the parameter \( \omega_B = 0.40 \pm 0.04 \text{ GeV} \) is used.

From the numerical results as listed in Tables II – V, one can see the following points:

1. For all considered semileptonic decays \( B \to D^{(*)} l^+ \nu_l \) with \( l = (e, \mu, \tau) \), the “PQCD+Lattice” predictions for their branching ratios and the ratios \( R(D^{(*)}) \) agree well with those currently available experimental measurements within the errors, which can be treated as one evidence for the reliability of this “PQCD+Lattice” method to deal with the semileptonic decays of \( B \) meson.
TABLE IV. The PQCD and “PQCD+Lattice” predictions for the branching ratios (in unit of 10^{-2}) of the eight semileptonic decays B \to D^{(*)}\tau^+\nu_{\tau} and B \to D^{(*)}\ell^+\nu_{\ell} with \ell = (e, \mu). As a comparison, we also list the previous “PQCD+Lattice” predictions[10], the SM predictions based on HQET [11], and the world average of the measured values as given in PDG 2018 [68].

| Channels          | PQCD              | PQCD+Lattice       | PQCD[10]          | HQET[11]        | PDG[68] |
|-------------------|-------------------|--------------------|-------------------|-----------------|---------|
| B^+ \to D^0\tau^+\nu_{\tau} | 0.86^{+0.34}_{-0.25} 0.69^{+0.21}_{-0.17} | 0.95^{+0.37}_{-0.31} 0.66 \pm 0.05 | 0.77 \pm 0.25 |
| B^+ \to D^0\ell^+\nu_{\ell}  | 2.29^{+0.91}_{-0.72} 2.10^{+0.85}_{-0.62} | 2.19^{+0.76}_{-0.69} |                  |       |
| B^+ \to D^{*0}\tau^+\nu_{\tau} | 1.60^{+0.39}_{-0.37} 1.34^{+0.26}_{-0.23} | 1.47^{+0.43}_{-0.40} 1.43 \pm 0.05 | 1.88 \pm 0.20 |
| B^+ \to D^{*0}\ell^+\nu_{\ell}  | 5.53^{+1.45}_{-1.25} 4.89^{+1.21}_{-1.00} | 4.87^{+1.64}_{-1.40} |                  | 4.88 \pm 0.10 |
| B^0 \to D^-\tau^+\nu_{\tau}  | 0.82^{+0.33}_{-0.24} 0.62^{+0.19}_{-0.14} | 0.87^{+0.34}_{-0.28} 0.64 \pm 0.05 | 1.03 \pm 0.22 |
| B^0 \to D^-\ell^+\nu_{\ell}  | 2.19^{+0.91}_{-0.68} 1.95^{+0.77}_{-0.66} | 2.03^{+0.92}_{-0.70} |                  | 2.20 \pm 0.10 |
| B^0 \to D^{*-}\tau^+\nu_{\tau} | 1.53^{+0.37}_{-0.35} 1.25^{+0.25}_{-0.21} | 1.36^{+0.38}_{-0.37} 1.29 \pm 0.06 | 1.67 \pm 0.13 |
| B^0 \to D^{*-}\ell^+\nu_{\ell}  | 5.32^{+1.37}_{-1.20} 4.63^{+1.15}_{-0.95} | 4.52^{+1.44}_{-1.31} |                  | 4.88 \pm 0.10 |

TABLE V. The PQCD and “PQCD+Lattice” predictions for the ratios \(R(D)\) and \(R(D^*)\). As a comparison, we also show the previous “PQCD+Lattice” predictions as given in Ref. [10], the average of the SM predictions as given in Ref. [29], several measured values as reported by BaBar, Belle and LHCb Collaborations [1, 24, 25], as well as the world averaged results from HFLAV group [29].

| Ratios | PQCD          | PQCD+Lattice | PQCD[10] | SM[29] | BaBar[1] | Belle[24] | LHCb[25] | HFLAV[29] |
|--------|---------------|--------------|----------|--------|----------|-----------|----------|-----------|
| R(D)   | 0.376^{+0.011}_{-0.012} 0.324^{+0.020}_{-0.022} | 0.337(38) 0.299(3) | 0.440(72) 0.307(40) | – | 0.340(30) |
| R(D^*) | 0.288^{+0.008}_{-0.010} 0.272^{+0.013}_{-0.014} | 0.269(21) 0.258(5) | 0.332(30) 0.283(23) | 0.291(35) | 0.295(14) |

(2) For the four \(B_s\) decay modes and the eight \(B\) decay modes, the “PQCD+Lattice” predictions for the branching ratios are generally smaller than the conventional PQCD predictions, but the differences between them are relatively small: less than 20% in magnitude. These predictions also agree well with previous PQCD predictions as given in Ref. [9, 10, 44] within the still large theoretical uncertainties.

(3) For the ratios \(R(D_s)\) and \(R(D_s^*)\), the theoretical errors in branching ratios are largely cancelled, the PQCD and “PQCD+Lattice” predictions for both ratios \(R(D_s)\) and \(R(D_s^*)\) have a very small error only: around 5% in magnitude. These predictions could be tested in the near future LHCb and Belle-II experiments.

(4) For the branching ratios, the theoretical predictions from different theoretical approaches can be a little different for the same decay mode, but they still agree within errors due to the large theoretical uncertainties. For the ratios \(R(D)\) and \(R(D^*)\), our “PQCD+Lattice” predictions also agree well with the average of the SM predictions obtained by employing the HQET plus the available Lattice QCD input for the relevant form factors [3–6, 29].

(5) By comparing the values of the two sets of the ratios \((R(D), R(D_s))\) and \((R(D^*), R(D_s^*))\), as listed in Table III and V, one can see that the SU(3) flavor symmetry keeps very well in
both the PQCD and the “PQCD+Lattice” approaches. This point could also be tested by experiments.

C. $P_\tau(D_{(s)}^{(*)})$, $F_L(D_{(s)}^*)$ and $A_{FB}(\tau)$

Besides the branching ratios and the ratios $R(X)$ of the branching ratios, there are other three additional physical observables: such as the longitudinal polarization of the tau lepton $P_\tau(D_{(s)}^{(*)})$, the fraction of $D_{(s)}^*$ longitudinal polarization $F_L(D_{(s)}^*)$ and the forward-backward asymmetry of the tau lepton $A_{FB}(\tau)$. These physical quantities can be measured in the LHCb and Belle experiments, and may be sensitive to some kinds of new physics [33, 63–66]. The calculations and investigations for these additional physical observables may provide new clues to understand the $R(D^{(*)})$ puzzle, and it is therefore necessary and interesting.

In Refs. [33–35], the authors calculated these three physical observables in the framework of the SM and examined possible new physics effects on them. In this paper, we calculate these observables by using the PQCD and the “PQCD+Lattice” approach, respectively. Based on the formulae as given explicitly in Eqs. (44–59), we make the calculations and show the numerical predictions for $P_\tau(D_{(s)}^{(*)})$, $F_L(D_{(s)}^*)$ and $A_{FB}(\tau)$ in Table VI. The measured values of $P_\tau(D^*)$ and $F_L(D^*)$ [30–32] as given in Eqs. (4,5) are also listed in Table VI. As a comparison, we also show other SM predictions for these physical observables as given in Refs. [33, 34] in Table VI.

| Observable | Approach | $B^0 \to D^- \pi^+ \nu_\tau$ | $B_s^0 \to D_s^- \pi^+ \nu_\tau$ | $B^0 \to D^{*-} \pi^+ \nu_\tau$ | $B_s^0 \to D_s^{*-} \pi^+ \nu_\tau$ |
|------------|----------|----------------|----------------|----------------|----------------|
| $P_\tau(D_{(s)}^{(*)})$ | PQCD | 0.32(1) | 0.31(1) | $-0.54(1)$ | $-0.54(1)$ |
| | PQCD+Lat. | 0.30(1) | 0.30(1) | $-0.53(1)$ | $-0.53(1)$ |
| | SM[33] | – | – | $-0.497(13)$ | – |
| | SM[34] | 0.325(3) | – | $-0.508(4)$ | – |
| | Belle[30] | – | – | $-0.38 \pm 0.51_{+0.21}^{-0.16}$ | – |
| $F_L(D_{(s)}^*)$ | PQCD | – | – | 0.42(1) | 0.42(1) |
| | PQCD+Lat. | – | – | 0.43(1) | 0.43(1) |
| | SM[35] | – | – | 0.457(10) | – |
| | SM[34] | – | – | 0.441(6) | – |
| | Belle[32] | – | – | $0.60 \pm 0.08 \pm 0.04$ | – |
| $A_{FB}(\tau)$ | PQCD | 0.35(1) | 0.36(1) | $-0.085(2)$ | $-0.083(2)$ |
| | PQCD+Lat. | 0.36(1) | 0.36(1) | $-0.054(2)$ | $-0.050(2)$ |
| | SM[34] | 0.361(1) | – | $-0.084(13)$ | – |

From the numerical results as listed in Eqs. (4,5) and in Table VI, one can find the following points:

1. The uncertainties of all theoretical predictions for the physics observables $P_\tau(D_{(s)}^{(*)})$, $F_L(D_{(s)}^*)$ and $A_{FB}(\tau)$ are very small when compared with the ones for the branching ratios, since the
IV. SUMMARY

In this paper, we studied the semileptonic decays $B_s \rightarrow D_s l^+ \nu_l$ in the frame work of the SM by employing both the conventional PQCD factorization approach and the “PQCD+Lattice” approach. In the second approach, we take into account the Lattice QCD results of the relevant form factors as an input in the extrapolation from the low $q^2$ approach. In the second approach, we take into account the Lattice QCD results of the relevant SM by employing both the conventional PQCD factorization approach and the “PQCD+Lattice” form factors.

(1) For the twelve considered $B/B_s$ semileptonic decay modes, the “PQCD+Lattice” predictions for the branching ratios are generally smaller than the conventional PQCD predictions, but the differences between them are relatively small: less than 20% in magnitude. The “PQCD+Lattice” predictions for their branching ratios and the ratios $R(D^{(*)})$ do agree well with those currently available experimental measurements within the errors.

(2) For the ratios $R(D_s)$ and $R(D_s^*)$, the PQCD and “PQCD+Lattice” predictions are the following:

$$R(D_s) = \begin{cases} 0.365^{+0.009}_{-0.012}, & \text{PQCD}, \\ 0.341^{+0.024}_{-0.025}, & \text{PQCD + Lattice}, \end{cases} \quad (67)$$

$$R(D_s^*) = \begin{cases} 0.287^{+0.008}_{-0.011}, & \text{PQCD}, \\ 0.271^{+0.015}_{-0.016}, & \text{PQCD + Lattice}, \end{cases} \quad (68)$$

Theoretical uncertainties are largely cancelled in ratios.

(2) The PQCD and “PQCD+Lattice” predictions for the physics observables $P_\tau(D_s^{(*)})$, $F_L(D_s^{(*)})$ and $A_{FB}(\tau)$ for $B_s \rightarrow D_s l^+ \nu_l$ decays are very similar with each other: the difference for a fixed decay mode is less than 5%. For the $A_{FB}(\tau)$ for the $B_s \rightarrow D_s^{(*)} l^+ \nu_l$ decays, however, the difference is about 40%. The reason lies in the definition of the angular coefficient function $b_0^{(D_s^{(*)})}(q^2)$, where the term $(H_{V,+,+}^2 - H_{V,+,+}^2)$ in Eq. (59) can be affected moderately by the different high $q^2$ behaviour of the form factors in the PQCD and “PQCD+Lattice” approach.

(3) For both $P_\tau(D^{(*)})$ and $F_L(D^{(*)})$, our theoretical predictions agree well with the measured ones within errors, partially due to the still large experimental errors. For all considered decay modes, the PQCD and “PQCD+Lattice” predictions for the three kinds of physics observables are consistent with the results from other approaches in the framework of the SM as well. We expect that these physical observables could be measured in high precision at the future LHCb and Belle-II experiments and it can help us to test the theoretical models or approaches.

In addition to the branching ratios and the ratios $R(D^{(*)})$, the fraction of $D_s^{(*)}$ longitudinal polarization $F_L(D_s^{(*)})$ and the forward-backward asymmetry of the tau lepton $A_{FB}(\tau)$. From the numerical calculations and phenomenological analysis we found the following points:

(1) For both $P_\tau(D^{(*)})$ and $F_L(D^{(*)})$, our theoretical predictions agree well with the measured ones within errors, partially due to the still large experimental errors. For all considered decay modes, the PQCD and “PQCD+Lattice” predictions for the three kinds of physics observables are consistent with the results from other approaches in the framework of the SM as well. We expect that these physical observables could be measured in high precision at the future LHCb and Belle-II experiments and it can help us to test the theoretical models or approaches.

IV. SUMMARY

In this paper, we studied the semileptonic decays $B_s \rightarrow D_s l^+ \nu_l$ in the frame work of the SM by employing both the conventional PQCD factorization approach and the “PQCD+Lattice” approach. In the second approach, we take into account the Lattice QCD results of the relevant form factors as an input in the extrapolation from the low $q^2$ to the endpoint $q^{2}_{\text{max}}$. We calculated the form factors $F_{0,+,+}(q^2)$, $V(q^2)$ and $A_{0,1,2}(q^2)$ of the $B_s \rightarrow D_s^{(*)}$ transitions, provided the theoretical predictions for the branching ratios of the considered $B/B_s$ semileptonic decays and the ratios $R(D^{(*)})$ and $R(D_s^{(*)})$. In addition to the branching ratios and the ratios $R(X)$, we also gave our theoretical predictions for the additional physical observables: the longitudinal polarization of the tau lepton $P_\tau(D_s^{(*)})$, the fraction of $D_s^{(*)}$ longitudinal polarization $F_L(D_s^{(*)})$ and the forward-backward asymmetry of the tau lepton $A_{FB}(\tau)$.

From the numerical calculations and phenomenological analysis we found the following points:

(1) For the twelve considered $B/B_s$ semileptonic decay modes, the “PQCD+Lattice” predictions for the branching ratios are generally smaller than the conventional PQCD predictions, but the differences between them are relatively small: less than 20% in magnitude. The “PQCD+Lattice” predictions for their branching ratios and the ratios $R(D^{(*)})$ do agree well with those currently available experimental measurements within the errors.

(2) For the ratios $R(D_s)$ and $R(D_s^*)$, the PQCD and “PQCD+Lattice” predictions are the following:

$$R(D_s) = \begin{cases} 0.365^{+0.009}_{-0.012}, & \text{PQCD}, \\ 0.341^{+0.024}_{-0.025}, & \text{PQCD + Lattice}, \end{cases} \quad (67)$$

$$R(D_s^*) = \begin{cases} 0.287^{+0.008}_{-0.011}, & \text{PQCD}, \\ 0.271^{+0.015}_{-0.016}, & \text{PQCD + Lattice}, \end{cases} \quad (68)$$
They also agree well with other SM predictions based on different approaches. These predictions could be tested by the forthcoming LHCb and Belle-II experiments.

(3) For most observables $P_\tau(D_{(s)}^{(*)})$, $F_L(D_{(s)}^{(*)})$, and $A_{FB}(\tau)$, the PQCD and “PQCD+Lattice” predictions agree very well with each other: the difference is less than $5\%$. The relatively large difference for the $A_{FB}(\tau)$ of the $B_{(s)} \rightarrow D_{(s)}^{(*)}\tau^+\nu_{\tau}$ decays can be understood reasonably.

(4) For both $P_\tau(D^*)$ and $F_L(D^*)$, our theoretical predictions agree well with the measured ones within errors. For all considered decay modes, the PQCD and “PQCD+Lattice” predictions for the three kinds of physics observables are consistent with the results from other approaches in the framework of the SM as well. The future LHCb and Belle-II experiments can help us to test the above theoretical predictions.

**ACKNOWLEDGMENTS**

We wish to thank Wen-Fei Wang and Ying-Ying Fan for valuable discussions. This work was supported by the National Natural Science Foundation of China under Grant No. 11775117 and 11235005.

**Appendix A: Relevant functions**

In this appendix, we present the explicit expressions for some functions which have already appeared in the previous sections. The hard functions $h_1$ and $h_2$ come form the Fourier transform, and they can be written as:

$$h_1(x_1, x_2, b_1, b_2) = K_0(\beta_1 b_1)[\theta(b_1 - b_2)I_0(\alpha_1 b_2)K_0(\alpha_1 b_1) + \theta(b_2 - b_1)I_0(\alpha_1 b_1)K_0(\alpha_1 b_2)]S_t(x_2),$$

(A1)

$$h_2(x_1, x_2, b_1, b_2) = K_0(\beta_2 b_1)[\theta(b_1 - b_2)I_0(\alpha_2 b_2)K_0(\alpha_2 b_1) + \theta(b_2 - b_1)I_0(\alpha_2 b_1)K_0(\alpha_2 b_2)]S_t(x_2),$$

(A2)

where $K_0$ and $I_0$ are modified Bessel functions, and

$$\alpha_1 = m_{B_{(s)}} \sqrt{x_2 r \xi}, \quad \alpha_2 = m_{B_{(s)}} \sqrt{x_1 r \xi - r^2 + r_c^2}, \quad \beta_1 = \beta_2 = m_{B_{(s)}} \sqrt{x_1 x_2 r \xi}. \quad (A3)$$

The threshold resummation factor $S_t(x)$ is adopted from Ref. [77]:

$$S_t = \frac{2^{1+2c}\Gamma(3/2 + c)}{\sqrt{\pi}\Gamma(1 + c)}[x(1 - x)]^c,$$

(A4)

and we here set the parameter $c = 0.3$.

The factor $\exp[-S_{ab}(t)]$ contains the Sudakov logarithmic corrections and the renormalization group evolution effects of both the wave functions and the hard scattering amplitude with $S_{ab}(t) = S_B(t) + S_M(t)$ [77, 78],

$$S_B(t) = s \left( x_1 \frac{m_{B_{(s)}}}{\sqrt{2}}, b_1 \right) + \frac{5}{3} \int_{1/b_1}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \quad (A5)$$

$$S_M(t) = s \left( x_2 \frac{m_{B_{(s)}}}{\sqrt{2}} r \xi, b_2 \right) + s \left( 1 - x_2 \right) \frac{m_{B_{(s)}}}{\sqrt{2}} r \xi, b_2 \right) + 2 \int_{1/b_2}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \quad (A6)$$

19
where $\eta^+$ is defined in Eq. (7). The hard scale $t$ and the quark anomalous dimension $\gamma_q = -\alpha_s/\pi$ governs the aforementioned renormalization group evolution. The explicit expressions of the functions $s(Q, b)$ can be found for example in Appendix A of Ref. [78]. The hard scales $t_i$ in Eqs.(22-27) are chosen as the largest scale of the virtuality of the internal particles in the hard $b$-quark decay diagram,

$$t_1 = \max\{m_{B(q)} \sqrt{x_2 r \eta^+}, 1/b_1, 1/b_2\},$$

$$t_2 = \max\{m_{B(q)} \sqrt{x_1 r \eta^+ - r^2 + r_c^2}, 1/b_1, 1/b_2\}. \quad (A7)$$

[1] J. P. Lees et al. (BABAR Collaboration), Evidence for an Excess of $\bar{B} \to D^{(*)+}\tau^+\bar{\nu}_\tau$ Decays, Phys. Rev. Lett. 109, 101802 (2012).
[2] J.P. Lees et al. (BABAR Collaboration), Measurement of an excess of $\bar{B} \to D^{(*)+}\tau^-\bar{\nu}_\tau$ decays and implications for charged Higgs bosons, Phys. Rev. D 88, 072012 (2013).
[3] D.Bigi, P. Gambino, Revisiting $B \to D\ell\nu$, Phys. Rev. D 94, 094008 (2016).
[4] F.U. Bernlochner, Z. Ligeti, M. Papucci, D.J. Robinson, Combined analysis of semileptonic $B$ decays to $D$ and $D^*$: $R(D^{(*)})$, $|V_{cb}|$, and new physics, Phys. Rev. D 95, 115008 (2017).
[5] S. Jaiswal, S. Nandi, S.K. Patra, Extraction of $|V_{cb}|$ from $B \to D^{(*)}\ell\nu$ and the Standard Model predictions of $R(D^{(*)})$, JHEP 1712, 060 (2017).
[6] D. Bigi, P. Gambino, S. Schacht, $R(D^{(*)})$, $|V_{cb}|$, and the Heavy Quark Symmetry relations between form factors, JHEP 1711, 061 (2017).
[7] J.A. Bailey et al. (Fermilab Lattice and MILC Collaborations), $B \to D\ell\nu$ form factors at nonzero recoil and $|V_{cb}|$ from $2+1$-flavor lattice QCD, Phys. Rev. D 92, 034506 (2015).
[8] H. Na et al. (HPQCD Collaboration), $B \to D\ell\nu$ form factors at nonzero recoil and extraction of $|V_{cb}|$, Phys. Rev. D 92, 054510 (2015).
[9] Y.Y. Fan, W.F. Wang, S. Cheng and Z.J. Xiao, Semileptonic decays $B \to D^{(*)}\ell\nu$ in the perturbative QCD factorization approach, Chin. Sci. Bull. 59, 125 (2014).
[10] Y.Y. Fan, Z.J. Xiao, R.M. Wang, B.Z. Li, The $B \to D^{(*)}\ell\nu$ decays in the pQCD approach with the Lattice QCD input, Sci. Bull. 60, 2009 (2015).
[11] S. Fajfer, J. F. Kamenik, I. Nisandzic, On the $B \to D^{*}\tau\bar{\nu}_\tau$ Sensitivity to New Physics, Phys. Rev. D 85, 094025 (2012).
[12] A. Datta, M. Duraisamy, D. Ghosh, Diagnosing New Physics $b \to c\tau\ell\nu$ in decays in the light of the recent BaBar result, Phys. Rev. D 86, 034027 (2012).
[13] Y. Sakaki, M. Tanaka, A. Tayduganov, R. Watanabe, Probing New Physics with $q^2$ distributions in $B \to D^{(*)}\tau\bar{\nu}$, Phys. Rev. D 91, 114028 (2015).
[14] M. Freytsis, Z. Ligeti, J. T. Ruderman, Flavor models for $\bar{B} \to D^{(*)}\tau\bar{\nu}$, Phys. Rev. D 92, 054018 (2015).
[15] S. Bhattacharya, S. Nandi, S. K. Patra, Optimal-observable analysis of possible new physics in $B \to D^{(*)}\tau\bar{\nu}$, Phys. Rev. D 93, 034011 (2016).
[16] X.-Q. Li, Y.-D. Jang, X. Zhang, Revisiting the one leptoquark solution to the $R(D^{(*)})$ anomalies and its phenomenological implications, JHEP 1608,054 (2016).
[17] D. Bardhan, P. Byakti, D. Ghosh, A closer look at the $R_{D}$ and $R_{D^*}$ anomalies, JHEP 1701, 125 (2017).
[18] A. Celis, M. Jung, X.-Q. Li, A. Pich, Scalar contributions to $b \to c(u)\tau\ell\nu$ transitions, Phys. Lett. B 771, 168 (2017).
[19] M. Huschle et al. (Belle Collaboration), Measurement of the branching ratio of $\bar{B} \to D^{(*)}\tau^{-}\bar{\nu}_{\tau}$ relative to $\bar{B} \to D^{(*)}l^{-}\bar{\nu}_{l}$ decays with hadronic tagging at Belle, Phys. Rev. D 92, 072014 (2015).

[20] S. Hirose et al. (Belle Collaboration), Measurement of the $\tau$ lepton polarization and $R(D^*)$ in the decay $\bar{B} \to D^*\tau^{-}\bar{\nu}_{\tau}$, Phys. Rev. Lett. 118, 211801 (2017).

[21] S. Hirose et al. (Belle Collaboration), Measurement of the $\tau$ lepton polarization and $R(D^*)$ in the decay $\bar{B} \to D^*\tau^{-}\bar{\nu}_{\tau}$ with one-prong hadronic decays at Belle, Phys. Rev. D 97, 012004 (2018).

[22] R. Aaij et al. (LHCb Collaboration), Measurement of the ratio of the $B^0 \to D^{*-}\tau^+\nu_{\tau}$ and $B^0 \to D^{*-}\mu^+\nu_{\mu}$ branching fractions using three-prong $\tau$-lepton decays, Phys. Rev. Lett. 120, 171802 (2018).

[23] A. Abdesselam et al. (Belle Collaboration), Measurement of $R(D)$ and $R(D^*)$ with a semileptonic tagging method, arXiv:1904.08794[hep-ex];

[24] G. Caria et al. (Belle Collaboration), Measurement of $R(D)$ and $R(D^*)$ with a semileptonic tagging method, arXiv:1910.05864[hep-ex].

[25] R. Aaij et al. (LHCb Collaboration), Test of Lepton Flavor Universality by the measurement of the $B^0 \to D^{*-}\tau^+\nu_{\tau}$ branching fraction using three-prong $\tau$ decays, Phys. Rev. D 97, 072013 (2018).

[26] G. Caria et al. (Belle Collaboration), Measurement of $R(D)$ and $R(D^*)$ with a semileptonic tag at Belle, talk given at 54th Rencontres de Moriond, EW, Mar. 22, 2019.

[27] A. Hicheur (LHCb Collaboration), Flavour anomalies in $B$ decay at LHCb, talk given at NuFact2019, Aug. 26 -31, 2019, Daegu, Korea; arXiv:1910.13121[hep-ex].

[28] Y. Amhis et al. (HFLAV Collaboration), Averages of $b$-hadron, $c$-hadron, and $\tau$ lepton properties as of summer 2016, Eur. Phys. J. C 77, 895 (2017); and references therein.

[29] Y. Amhis et al. (HFLAV Collaboration), Averages of $b$-hadron, $c$-hadron, and $\tau$ lepton properties as of summer 2018, arXiv:1909.12524[hep-ex]; and references therein.

[30] S. Hirose et al., (Belle Collaboration), Measurement of the $\tau$ lepton polarization in the decay $\bar{B} \to D^*\tau^{-}\bar{\nu}_{\tau}$, Phys. Rev. Lett. 118, 211801 (2017).

[31] S. Hirose et al., (Belle Collaboration), Measurement of the $\tau$ lepton polarization and $R(D^*)$ in the decay $\bar{B} \to D^*\tau^{-}\bar{\nu}_{\tau}$ with one-prong hadronic $\tau$ decays at Belle, Phys. Rev. D 97, 012004 (2018).

[32] A. Abdesselam et al., (Belle Collaboration), Measurement of the $D^*$ polarization in the decay $B^0 \to D^{*-}\tau^+\nu_{\tau}$, talk given at CKM 2018, Bell-Conf-1805, arXiv: 1903.03102[hep-ex].

[33] M. Tanaka, R. Watanabe, New physics in the weak interaction of $\bar{B} \to D^{(*)}\tau\nu$, Phys. Rev. D 87, 034028 (2013).

[34] Z.R. Huang, Y. Li, C.D. Lü, M. Ali Paracha and C. Wang, Footprints of new physics in $b \to c\tau\nu$ transitions, Phys. Rev. D 98, 095018 (2018).

[35] S. Bhattacharya, S. Nandi and S. K. Patra, $b \to c\tau\nu_{\tau}$ decays: a catalogue to compare, constrain, and correlate new physics effects, Eur. Phys. J. C 79, 268 (2019).

[36] S. Faller, A. Khodjamirian, Ch. Klein, Th. Mannel, $B \to D^{(*)}$ Form Factors from QCD Light-Cone Sum Rules, Eur. Phys. J. C 60, 603 (2009).

[37] Y.M. Wang, Y.B. Wei, Y.L. Shen, C.D. Lü, Perturbative corrections to $B \to D$ form factors in QCD, JHEP 1706, 062 (2017).

[38] Y. Zhang, T. Zhong, X.G. Wu, K. Li, H.B. Fu, T. Huang, Uncertainties of the $B \to D$ transition form factor from the $D$-meson leading-twist distribution amplitude, Eur. Phys. J. C 78, 76 (2018).

[39] T. Bhattacharya et al. (LANL/SWME Collaboration), Update on $B \to D^{(*)}\nu\bar{\nu}$ form factor at zero-recoil using the Oktay-Kronfeld action, arXiv:1812.07675.

[40] Jon A. Bailey et al. (LANL-SWME Collaboration), Calculation of $\bar{B} \to D^{(*)}\nu\bar{\nu}$ form factor at zero recoil using the Oktay-Kronfeld action, EPJ Web Conf. 175, 13012 (2018).

[41] T. Kaneko et al. (JLQCD Collaboration), $B \to D^{(*)}\nu\bar{\nu}$ form factors from $N_f=2+1$ QCD with Mobius
domain-wall quarks, arXiv:1811.00794.

[42] C.J. Monahan, C.M. Bouchard, G.P. Lepage, H. Na, J. Shigemitsu, (HPQCD Collaboration), Form factor ratios for $B_s \rightarrow K\ell\nu$ and $B_s \rightarrow D_s\ell\nu$ semileptonic decays and $|V_{ub}|/|V_{cb}|$, Phys. Rev. D 98, 114509 (2018).

[43] Judd Harrison et al. (HPQCD Collaboration), Lattice QCD calculation of the $B_{(s)} \rightarrow D_{(s)}^*\ell\nu$ form factors at zero recoil and implications for $|V_{cb}|$, Phys. Rev. D 97, 054502 (2018).

[44] Y.Y. Fan, W.F. Wang and Z.J. Xiao, Study of decays $B_s^0 \rightarrow (D^+_s, D^{*+}_s)\ell^-\bar{\nu}_\ell$ in the pQCD factorization approach, Phys. Rev. D 89, 014030 (2014).

[45] W.F. Wang, Z.J. Xiao, The semileptonic decays $B/B_s \rightarrow (\pi, K)(?, ?^?, ?, \nu, \nu\bar{\nu})$ in the perturbative QCD approach beyond the leading-order, Phys. Rev. D 86, 114025 (2012).

[46] W.M. Wang, M. Liu and Z.J. Xiao, Semileptonic decays $B_{(s)} \rightarrow (\eta, \eta^{(l)}_G)(ll, \ell\nu, \nu\bar{\nu})$ in the perturbative QCD approach beyond the leading-order, Phys. Rev. D 87, 097501 (2013).

[47] Z.J. Xiao, Y.Y. Fan, W.F. Wang and S. Cheng, The semileptonic decays $B/B_s$ meson in the perturbative QCD approach: A short review, Chin. Sci. Bull. 59, 3787 (2014).

[48] W.F. Wang, Y.Y. Fan and Z.J. Xiao, Semileptonic decays $B_c \rightarrow (\eta_c, J/\psi)\ell\nu$ in the perturbative QCD approach, Chin. Phys. C 37, 093102 (2013).

[49] W.F. Wang, X. Yu, C.D. Lü and Z.J. Xiao, The semileptonic decays $B^+_c \rightarrow D_{(s)}^{(*)}(l+\nu, l^+l^-, \nu\bar{\nu})$ in the perturbative QCD approach, Phys. Rev. D 90, 094018 (2014).

[50] X.Q. Hu, S.P. Jin and Z.J. Xiao, Semi-leptonic decays $B_c \rightarrow (\eta_c, J/\psi)\ell\nu$ in the ”PQCD + Lattice” approach, Chin. Phys. C 44 (2020) in press; arXiv:1904.07530 [hep-ph].

[51] C. Bourrely, I. Caprini and L. Lellouch, Model-independent description of $B \rightarrow \pi\ell\nu$ decays and a determination of $|V_{ub}|$, Phys. Rev. D 79, 013008 (2009). [Erratum ibid. 82, 099902 (2010)].

[52] D. Leljak, B. Melic and M. Patra, On lepton flavour universality in semileptonic $B_c \rightarrow \eta_c, J/\psi$ decays, JHEP 05, 094 (2019).

[53] T. Kurimoto, H.N. Li, A.I. Sanda, $B \rightarrow D^{(*)}$ form factors in perturbative QCD, Phys. Rev. D 67, 054028 (2003).

[54] Z.J. Xiao, W.F. Wang, and Y.Y. Fan, Revisiting the pure annihilation decays $B_s \rightarrow \pi^+\pi^-$ and $B^0 \rightarrow K^+K^-$: The data and the perturbative QCD predictions, Phys. Rev. D 85, 094003 (2012).

[55] R. H. Li, C. D. Lü, H. Zou, $B/B_s \rightarrow D_{(s)}P, D_{s}V, D_{(s)}^* P, D_{s}^{*} V$ decays in the pQCD approach, Phys. Rev. D 78, 014018 (2008).

[56] M. Beneke, Th. Feldmann, Symmetry breaking corrections to heavy to light $B$ meson form-factors at large recoil, Nucl. Phys. B 592, 3 (2001).

[57] Jon A. Bailey et al. (Fermilab Lattice and MILC Collaborations), Update of $|V_{ub}|$ from the $B \rightarrow D^{(*)}\ell\nu$ form factor at zero recoil with three-flavor lattice QCD, Phys. Rev. D 89, 114504 (2014).

[58] Jon A. Bailey et al. (Fermilab Lattice and MILC Collaborations), $B_s \rightarrow D_s/B \rightarrow D$ semileptonic form-factor ratios and their application to $BR(B_s^0 \rightarrow \mu^+\mu^-)$, Phys. Rev. D 85, 114502 (2012).

[59] M. Atoui, V. Morenas, D. Becirevic and F. Sanfilippo, $B_s \rightarrow D_s\ell\bar{\nu}_\ell$ near zero recoil in and beyond the Standard Model, Eur. Phys. J. C 74, 2861 (2014).

[60] W. Wang, Y.L. Shen and C.D. Lü, Covariant light-front approach for $B_c$ transition form factors, Phys. Rev. D 79, 054012 (2009).

[61] G. Buchalla, A.J. Buras, and M.E. Lautenbacher, Weak decays beyond leading logarithms, Rev. Mod. Phys. 68, 1125-1144 (1996).

[62] R.H. Li, C.D. Lü, W. Wang, and X.X. Wang, $B \rightarrow S$ transition form factors in the perturbative QCD approach, Phys. Rev. D 79, 014013 (2009).

[63] J. P. Lees et al. [The BABAR Collaboration], Measurement of an Excess of $\bar{B} \rightarrow D^{(*)}\tau^-\bar{\nu}_\tau$ Decays and Implications for Charged Higgs Bosons, Phys. Rev. D 88, 072012 (2013).
[64] M. A. Ivanov, Jurgen G. Korner, C. T. Tran, Exclusive decays $B \to l^- \bar{\nu}$ and $B \to D^{(s)}l^- \bar{\nu}$ in the covariant quark model, *Phys. Rev. D* **92**, 114022 (2015).

[65] M. A. Ivanov, Jurgen G. Korner, C. T. Tran, Probing new physics in $B \to D^{(s)}\tau^- \bar{\nu}$ using the longitudinal, transverse, and normal polarization components of the tau lepton, *Phys. Rev. D* **95**, 036021 (2017).

[66] M. Tanaka, R. Watanabe, $\tau$ longitudinal polarization in $B \to D\tau\nu$ and its role in the search for charged Higgs boson, *Phys. Rev. D* **82**, 034027 (2010).

[67] Y. Sakaki, R. Watanabe, M. Tanaka, A. Tayduganov, Testing leptoquark models in $B \to D^{(*)}\tau\bar{\nu}$, *Phys. Rev. D* **88**, 094012 (2013).

[68] M. Tanabashi et al. [Particle Data Group], Review of Particle Physics, *Phys. Rev. D* **98**, 030001 (2018).

[69] V. Lubicz et al. [ETM Collaboration], Masses and decay constants of $D^{(*)}_s$ and $B^{(*)}_s$ mesons with $N_f = 2 + 1 + 1$ twisted mass fermions, *Phys. Rev. D* **96**, 034524 (2017).

[70] R.H. Li, C.D. Lu, and Y.M. Wang, Exclusive $B_s$ decays to the charmed mesons $D^+_s(1968, 2317)$ in the standard model, *Phys. Rev. D* **80**, 014005 (2009).

[71] G. Li, F.L. Shao, and W. Wang, $B_s \to D_s(3040)$ form factors and $B_s$ decays into $D_s(3040)$. *Phys. Rev. D* **82**, 094031 (2010).

[72] S.M. Zhao, X. Liu, and S.J. Li, Study of $B_s \to D_{sJ}(2317, 2460)\ell\nu$ semileptonic decays in the CQM model, *Eur. Phys. J. C* **51**, 601 (2007).

[73] K. Azizi, M. Bayar, Semileptonic $B_q \to D_q^*\nu(q = s, d, u)$ decays in QCD sum rules, *Phys. Rev. D* **78**, 054011 (2008); K. Azizi, QCD sum rules study of the semileptonic $B_s(B^{\pm}) (B^0) \to D_s(1968)(D^0)(D^\pm)\ell\nu$ Decays, *Nucl. Phys. B* **801**, 70 (2008).

[74] R. Dutta, N. Rajeev, Signature of lepton flavor universality violation in $B_s \to D_s\tau\nu$ semileptonic decays, *Phys. Rev. D* **97**, 095045 (2018).

[75] T. Kurimoto, H. N. Li, and A. I. Sanda, Leading power contribution to $B \to \pi\rho$ transition form factors, *Phys. Rev. D* **65**, 014007 (2001).

[76] C.D. Lü, K. Ukai, and M.Z. Yang, Branching ratio and CP violation of $B \to \pi\pi$ decays in the perturbative QCD approach, *Phys. Rev. D* **63**, 074009 (2001).