A generalization of Gauss’ divergence theorem.

Vieri Benci*     Lorenzo Luperi Baglini†

June 18, 2014

Abstract

This paper is devoted to the proof Gauss’ divergence theorem in the framework of “ultrafunctions”. They are a new kind of generalized functions, which have been introduced recently [2] and developed in [4], [5] and [6]. Their peculiarity is that they are based on a Non-Archimedean field namely on a field which contains infinite and infinitesimal numbers. Ultrafunctions have been introduced to provide generalized solutions to equations which do not have any solutions not even among the distributions.

Contents

1 Introduction 2

1.1 Notations 2

2 Λ-theory 4

2.1 Non Archimedean Fields 4

2.2 The Λ-limit 4

2.3 Natural extension of sets and functions 6

3 Ultrafunctions 8

3.1 Definition of Ultrafunctions 8

3.2 The canonical ultrafunctions 10

3.3 Canonical extension of functions and measures 14

4 Generalization of some basic notions of calculus 15

4.1 Derivative 15

4.2 Gauss’ divergence theorem 17

4.3 A simple application 19

---

*Dipartimento di Matematica, Università degli Studi di Pisa, Via F. Buonarroti 1/c, 56127 Pisa, ITALY; e-mail: benci@dma.unipi.it

†University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Vienna, AUSTRIA, e-mail: lorenzo.luperi.baglini@univie.ac.at, supported by grant P25311-N25 of the Austrian Science Fund FWF.
1 Introduction

In many problems of mathematical physics, the notion of function is not sufficient and it is necessary to extend it. Among people working in partial differential equations, the theory of distribution of Schwartz and the notion of weak solution are the main tools to be used when equations do not have classical solutions.

Usually, these equations do not have classical solutions since they develop singularities. The notion of weak solutions allows to obtain existence results, but uniqueness may be lost; also, these solutions might violate the conservation laws. As an example let us consider the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \text{div} F(t, x, u) = 0,$$

(1)

where $F : \mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^u \to \mathbb{R}^N_x$ satisfies the following assumption: $F(t, x, 0) = 0$.

A classical solution $u(t, x)$ is unique and, if it has compact support, it preserves the quantity $Q = \int u \, dx$. However, at some time a singularity may appear and the phenomenon cannot be longer described by a classic solution. The notion of weak solution becomes necessary, but the problem of uniqueness becomes a central issue. Moreover, in general, $Q$ is not preserved. From a technical point of view, the classical proof of conservation of $Q$ fails since we cannot apply Gauss’ divergence theorem to weak solutions.

In this paper we suggest a method to overcomes these problems. This method consists in using a different kind of generalized solutions, namely functions which belong to the space of ”ultrafunctions”. The ultrafunctions have been introduced recently in [2] and developed in [1], [5], [6], [7] and [8]. The peculiarity of ultrafunctions is that they are based on a non Archimedean field, namely a field which contains infinite and infinitesimal numbers. The ultrafunctions have been introduced to provide generalized solutions to equations which do not have any solutions, not even among the distributions. However they provide also uniqueness in problems which have more than one weak solution. Moreover, we will state a generalization of Gauss’ divergence theorem which can be applied to the study of partial differential equations (see e.g. [9]). Here we give a simple application to equation (1) using an elementary notion of generalized solution (see section 4.3). In a paper in preparation, we will give a more appropriate notion of generalized solution of an evolution problem and we will study in details the properties of the generalized solutions of Burgers’ equation.

1.1 Notations

If $X$ is a set then

- $\mathcal{P}(X)$ denotes the power set of $X$ and $\mathcal{P}_{fin}(X)$ denotes the family of finite subsets of $X$;
\( \mathcal{F}(X,Y) \) denotes the set of all functions from \( X \) to \( Y \) and \( \mathcal{F}(\mathbb{R}^N) = \mathcal{F}(\mathbb{R},\mathbb{R}) \).

Let \( \Omega \) be a subset of \( \mathbb{R}^N \): then

- \( \mathcal{C}(\Omega) \) denotes the set of continuous functions defined on \( \Omega \subset \mathbb{R}^N \);
- \( \mathcal{C}_c(\Omega) \) denotes the set of continuous functions in \( \mathcal{C}(\Omega) \) having compact support in \( \Omega \);
- \( \mathcal{C}^k(\Omega) \) denotes the set of functions defined on \( \Omega \subset \mathbb{R}^N \) which have continuous derivatives up to the order \( k \);
- \( \mathcal{C}^k_c(\Omega) \) denotes the set of functions in \( \mathcal{C}^k(\Omega) \) having compact support;
- \( \mathcal{D}(\Omega) \) denotes the set of the infinitely differentiable functions with compact support defined on \( \Omega \subset \mathbb{R}^N \);
- \( \mathcal{D}'(\Omega) \) denotes the topological dual of \( \mathcal{D}(\Omega) \), namely the set of distributions on \( \Omega \);
- if \( A \subset \mathbb{R}^N \) is a set, then \( \chi_A \) denotes the characteristic function of \( A \);
- for any \( \xi \in (\mathbb{R}^N)^* \), \( \rho \in \mathbb{R}^* \), we set \( B_\rho(\xi) = \{ x \in (\mathbb{R}^N)^* : |x - \xi| < \rho \} \);
- \( \text{supp}(f) = \{ x \in (\mathbb{R}^N)^* : f(x) \neq 0 \} \);
- \( \text{mon}(x) = \{ y \in (\mathbb{R}^N)^* : x \sim y \} \);
- \( \text{gal}(x) = \{ y \in (\mathbb{R}^N)^* : x \sim f(y) \} \);
- \( \forall^{a.e.} x \in X \) means "for almost every \( x \in X \);
- if \( a, b \in \mathbb{R}^* \), then
  - \( [a,b]_{\mathbb{R}^*} = \{ x \in \mathbb{R}^* : a \leq x \leq b \} \);
  - \( (a,b)_{\mathbb{R}^*} = \{ x \in \mathbb{R}^* : a < x < b \} \);
  - \( ]a,b[ = [a,b]_{\mathbb{R}^*} \setminus (\text{mon}(a) \cup \text{mon}(b)) \);
- if \( W \) is a generic function space, its topological dual will be denoted by \( W' \) and the pairing by \( W' \langle \cdot, \cdot \rangle_{W} \) or simply by \( \langle \cdot, \cdot \rangle \);
- if \( \mathfrak{M} = [\mathcal{C}_c(\Omega)]' \) is the space of Radon measures, \( \mu \in \mathfrak{M} \) and \( f \) is a Borel function, we will use the notation
  \[ \langle f, \mu \rangle_{\mathfrak{M}} \quad \text{or} \quad (f, \mu) \]
  rather than \( \int f(x) \, d\mu \). For example, the \( \delta \) is the Dirac measure, we will be will use the notation
  \[ \langle f, \delta \rangle = f(0) \].
2 \( \Lambda \)-theory

In this section we present the basic notions of non Archimedean mathematics and of nonstandard analysis following a method inspired by [3] (see also [1], [2] and [3]).

2.1 Non Archimedean Fields

Here, we recall the basic definitions and facts regarding non Archimedean fields. In the following, \( K \) will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 1. Let \( K \) be an ordered field. Let \( \xi \in K \). We say that:
- \( \xi \) is infinitesimal if, for all positive \( n \in \mathbb{N} \), \( |\xi| < \frac{1}{n} \);
- \( \xi \) is finite if there exists \( n \in \mathbb{N} \) such that \( |\xi| < n \);
- \( \xi \) is infinite if, for all \( n \in \mathbb{N} \), \( |\xi| > n \) (equivalently, if \( \xi \) is not finite).

Definition 2. An ordered field \( K \) is called non Archimedean if it contains an infinitesimal \( \xi \neq 0 \).

It is easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number and that the inverse of a nonzero infinitesimal number is infinite.

Definition 3. A superreal field is an ordered field \( K \) that properly extends \( \mathbb{R} \).

It is easy to show, due to the completeness of \( \mathbb{R} \), that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of ”closeness”:

Definition 4. We say that two numbers \( \xi, \zeta \in K \) are infinitely close if \( \xi - \zeta \) is infinitesimal. In this case we write \( \xi \sim \zeta \).

Clearly, the relation ”\( \sim \)” of infinite closeness is an equivalence relation.

Theorem 5. If \( K \) is a superreal field, every finite number \( \xi \in K \) is infinitely close to a unique real number \( r \sim \xi \).

2.2 The \( \Lambda \)-limit

In this section we will introduce a particular superreal field \( K \) and we will analyze its main properties by mean of the \( \Lambda \)-theory, in particular by mean of the notion of \( \Lambda \)-limit (for complete proofs and for further properties of the \( \Lambda \)-limit, the reader is referred to [2], [4], [5], [6], [7]).

We set

\[ \mathcal{L} = \mathcal{P}_\omega(\mathbb{R}), \]
where $\mathcal{P}_\omega(\mathbb{R}^N)$ denotes the family of finite subsets of $\mathbb{R}$. We will refer to $\mathcal{L}$ as the "parameter space". Clearly $(\mathcal{L}, \subset)$ is a directed set. A function $\varphi : D \to E$ defined on a directed set will be called net (with values in $E$). A net $\varphi$ is the generalization of the notion of sequence and it has been constructed in such a way that the Weierstrass definition of limit makes sense: if $\varphi_\lambda$ is a real net, we have that

$$
\lim_{\lambda \to \infty} \varphi_\lambda = L
$$

if and only if

$$
\forall \varepsilon > 0 \exists \lambda_0 > 0 \text{ such that, } \forall \lambda > \lambda_0, \ |\varphi_\lambda - L| < \varepsilon. \tag{2}
$$

The key notion of the $\Lambda$-theory is the $\Lambda$-limit. Also the $\Lambda$-limit is defined for real nets but it differs from the Weierstrass limit defined by (2) mainly for the fact that there exists a non Archimedean field in which every real net admits a limit.

Now, we will present the notion of $\Lambda$-limit axiomatically:

**Axioms of the $\Lambda$-limit**

- **(A-1) Existence Axiom.** There is a superreal field $\mathbb{K} \supset \mathbb{R}$ such that every net $\varphi : \mathcal{L} \to \mathbb{R}$ has a unique limit $L \in \mathbb{K}$ (called the "$\Lambda$-limit" of $\varphi$.) The $\Lambda$-limit of $\varphi$ will be denoted as

$$
L = \lim_{\Lambda \uparrow} \varphi(\lambda).
$$

Moreover we assume that every $\xi \in \mathbb{K}$ is the $\Lambda$-limit of some real function $\varphi : \mathcal{L} \to \mathbb{R}$.

- **(A-2) Real numbers axiom.** If $\varphi(\lambda)$ is eventually constant, namely $\exists \lambda_0 \in \mathcal{L}, r \in \mathbb{R}$ such that $\forall \lambda \supset \lambda_0, \ \varphi(\lambda) = r$, then

$$
\lim_{\Lambda \uparrow} \varphi(\lambda) = r.
$$

- **(A-3) Sum and product Axiom.** For all $\varphi, \psi : \mathcal{L} \to \mathbb{R}$:

$$
\begin{align*}
\lim_{\Lambda \uparrow} \varphi(\lambda) + \lim_{\Lambda \uparrow} \psi(\lambda) &= \lim_{\Lambda \uparrow} (\varphi(\lambda) + \psi(\lambda)); \\
\lim_{\Lambda \uparrow} \varphi(\lambda) \cdot \lim_{\Lambda \uparrow} \psi(\lambda) &= \lim_{\Lambda \uparrow} (\varphi(\lambda) \cdot \psi(\lambda)).
\end{align*}
$$

The proof that this set of axioms $\{(A-1),(A-2),(A-3)\}$ is consistent can be found e.g. in [2] or in [5].

---

1We recall that a directed set is a partially ordered set $(D, \prec)$ such that, $\forall a, b \in D, \exists c \in D$ such that $a \prec c$ and $b \prec c$. 

5
2.3 Natural extension of sets and functions

The notion of Λ-limit can be extended to sets and functions in the following way:

**Definition 6.** Let $E_\lambda$, $\lambda \in \mathcal{L}$, be a family of sets in $\mathbb{R}^N$. We pose

$$\lim_{\lambda \uparrow \Lambda} E_\lambda := \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi(\lambda) \in E_\lambda \right\}.$$

A set which is a Λ-limit is called internal. In particular if, $\forall \lambda \in \mathcal{L}$, $E_\lambda = E$, we set $\lim_{\lambda \uparrow \Lambda} E_\lambda = E^*$, namely

$$E^* := \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi(\lambda) \in E \right\}.$$

$E^*$ is called the natural extension of $E$.

Notice that, while the Λ-limit of a constant sequence of numbers gives this number itself, a constant sequence of sets gives a larger set, namely $E^*$. In general, the inclusion $E \subseteq E^*$ is proper.

This definition, combined with axiom (Λ-1), entails that $\mathcal{K} = \mathbb{R}^*$.

Given any set $E$, we can associate to it two sets: its natural extension $E^*$ and the set $E^\sigma$, where

$$E^\sigma := \{ x^* \mid x \in E \}. \quad (3)$$

Clearly $E^\sigma$ is a copy of $E$; however it might be different as set since, in general, $x^* \neq x$. Moreover $E^\sigma \subset E^*$ since every element of $E^\sigma$ can be regarded as the Λ-limit of a constant sequence.

Let us note that the ”limit” terminology has to be intended, in this paper, just as suggestive\(^2\), since the Λ−limit is not intended (here) as a topological limit; nevertheless, it is possible to show that the Λ−limit can also be interpreted as a topological limit with respect to a particular topology defined on the space $\mathcal{L} \cup \{ \Lambda \}$, where Λ here has to be intended as a ”point at infinity”; $\mathcal{L} \cup \{ \Lambda \}$ is built in the same spirit of the Alexandroff one point compactification except that $\mathcal{L} \cup \{ \Lambda \}$ will be equipped with a different topology. We plan to show this topological characterization in a forthcoming paper.

**Definition 7.** Let

$$f_\lambda : E_\lambda \to \mathbb{R}, \quad \lambda \in \mathcal{L},$$

be a family of functions. We define a function

$$f : \left( \lim_{\lambda \uparrow \Lambda} E_\lambda \right) \to \mathbb{R}^*$$

\(^{2}\)In fact, it is easy to prove that $\mathcal{K}$ is isomorphic to $\mathbb{R}^X / \sim$, where we set $\varphi_1 \sim \varphi_2 \iff \lim_{\lambda \uparrow \Lambda} \varphi_1(\lambda) = \lim_{\lambda \uparrow \Lambda} \varphi_2(\lambda)$. This identification can be used to prove that, in what follows, all our definitions do not depend on the choice of representatives.
as follows: for every $\xi \in (\lim_{\Lambda} E_\lambda)$ we pose

$$f(\xi) := \lim_{\lambda \uparrow \Lambda} f_\lambda(\psi(\lambda)),$$

where $\psi(\lambda)$ is a net of numbers such that

$$\psi(\lambda) \in E_\lambda \quad \text{and} \quad \lim_{\Lambda} \psi(\lambda) = \xi.$$

A function which is a $\Lambda$-limit is called **internal**. In particular if, $\forall \lambda \in \mathcal{L}$,

$$f_\lambda = f, \quad f : E \to \mathbb{R},$$

we set

$$f^* = \lim_{\lambda \uparrow \Lambda} f_\lambda.$$

$f^* : E^* \to \mathbb{R}^*$ is called the **natural extension** of $f$.

More in general, the $\Lambda$-limit can be extended to a larger family of nets; to this aim, we recall that the superstructure on $\mathbb{R}$ is defined as follows:

$$U = \bigcup_{n=0}^{\infty} U_n$$

where $U_n$ is defined by induction as follows:

$$U_0 = \mathbb{R}; \quad U_{n+1} = U_n \cup \mathcal{P}(U_n).$$

Here $\mathcal{P}(E)$ denotes the power set of $E$. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $U$ contains almost every usual mathematical object.

We can extend the definition of the $\Lambda$-limit to any bounded net of mathematical objects in $U$. To this aim, let us consider a net

$$\varphi : \mathcal{X} \to U_n.$$ (4)

We will define $\lim_{\Lambda} \varphi(\lambda)$ by induction on $n$. For $n = 0$, $\lim_{\Lambda} \varphi(\lambda)$ is defined by the axioms (A-1),(A-2),(A-3); so by induction we may assume that the limit is defined for $n - 1$ and we define it for the net \( \varphi \) as follows:

$$\lim_{\Lambda} \varphi(\lambda) = \left\{ \lim_{\Lambda} \psi(\lambda) \mid \psi : \mathcal{X} \to U_{n-1} \quad \text{and} \quad \forall \lambda \in \mathcal{X}, \, \psi(\lambda) \in \varphi(\lambda) \right\}. \quad (5)$$

**Definition 8.** A mathematical entity (number, set, function or relation) which is the $\Lambda$-limit of a net is called **internal**.

---

\[3\] We recall that a net $\varphi : \mathcal{X} \to U$ is bounded if there exists $n$ such that $\forall \lambda \in \mathcal{X}, \varphi(\lambda) \in U_n$. 
Let us note that, if \((f_\lambda), (E_\lambda)\) are, respectively, a net of functions and a net of sets, the \(\Lambda\)-limit of these nets defined by (5) coincides with the \(\Lambda\)-limit given by Definitions 6 and 7. The following theorem is a fundamental tool in using the \(\Lambda\)-limit:

**Theorem 9. (Leibniz Principle)** Let \(R\) be a relation in \(\mathbb{U}_n\) for some \(n \geq 0\) and let \(\varphi, \psi : X \to \mathbb{U}_n\). If

\[
\forall \lambda \in X, \varphi(\lambda) R \psi(\lambda)
\]

then

\[
\left( \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \right) R^* \left( \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \right).
\]

When \(R\) is \(\in\) or \(=\) we will not use the symbol \(\ast\) to denote their extensions, since their meaning is unaltered in universes constructed over \(\mathbb{R}^*\). To give an example of how Leibniz Principle can be used to prove facts about internal entities, let us prove that if \(K \subseteq \mathbb{R}\) is a compact set and \((f_\lambda)\) is a net of continuous functions then \(f = \lim_{\lambda \uparrow \Lambda} f_\lambda\) has a maximum on \(K^*\). For every \(\lambda\) let \(\xi_\lambda\) be the maximum value attained by \(f_\lambda\) on \(K\), and let \(x_\lambda \in K\) be such that \(f_\lambda(x_\lambda) = \xi_\lambda\). For every \(\lambda\), for every \(y_\lambda \in K\) we have that \(f_\lambda(y_\lambda) \leq f_\lambda(x_\lambda)\). By Leibniz Principle, if we pose

\[
x = \lim_{\lambda \uparrow \Lambda} x_\lambda
\]

we have that

\[
\forall y \in K, f(y) \leq f(x),
\]

so \(\xi = \lim_{\lambda \uparrow \Lambda} \xi_\lambda\) is the maximum of \(f\) on \(K\) and it is attained on \(x\).

3 Ultrafunctions

3.1 Definition of Ultrafunctions

Let \(W \subset \mathfrak{F}(\mathbb{R}^N, \mathbb{R})\) be a function vector space such that \(D \subseteq W \subseteq L^2\).

**Definition 10.** We say that \((W_\lambda)_{\lambda \in \Sigma}\) is an approximating net for \(W\) if

1. \(W_\lambda\) is a finite dimensional vector subspace of \(W\) for every \(\lambda \in \Sigma\);
2. \(\lambda_1 \subseteq \lambda_2 \Rightarrow W_{\lambda_1} \subseteq W_{\lambda_2}\);
3. if \(Z \subset W\) is a finite dimensional vector space then \(\exists \lambda\) such that \(Z \subseteq W_\lambda\) (hence \(W = \bigcup_{\lambda \in \Sigma} W_\lambda\)).

**Example 11.** Let

\[
\{e_a\}_{a \in \mathbb{R}}
\]
be a Hamel basis of $W$. For every $\lambda \in \mathcal{L}$ let

$$W_\lambda = \text{Span} \{e_a \mid a \in \lambda\}.$$  

Then $(W_\lambda)$ is an approximating net for $W$.

**Definition 12.** Let $(W_\lambda)$ be an approximating net for $W$. We call space of ultrafunctions generated by $(W,(W_\lambda))$ the $\Lambda$-limit

$$W_\Lambda := \left\{\lim_{\lambda \uparrow \Lambda} f_\lambda \mid f_\lambda \in W_\lambda\right\}.$$

In this case we will also say that the space $W_\Lambda$ is based on the space $W$.

So a space of ultrafunctions based on $W$ depends on the choice of an approximating net for $W$. Nevertheless, different spaces of ultrafunctions based on $W$ have a lot of properties in common. In what follows, $W_\Lambda$ is any space of ultrafunctions based on $W$.

Since $W_\Lambda \subset [L^2]^*$, we can equip $W_\Lambda$ with the following inner product:

$$(u,v) = \int_{\Omega}^* u(x)v(x) \, dx,$$

where $f^*$ is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1 \to \mathbb{R}.$$  

The norm of an ultrafunction will be given by

$$\|u\| = \left(\int_{\Omega}^* |u(x)|^2 \, dx\right)^{\frac{1}{2}}.$$  

So, given any vector space of functions $W$, we have the following properties:

1. the ultrafunctions in $W_\Lambda$ are $\Lambda$-limits of nets $(f_\lambda)$ of functions, with $f_\lambda \in W_\lambda$ for every $\lambda$;

2. the space of ultrafunctions $W_\Lambda$ is a vector space of hyperfinite dimension, since it is a $\Lambda$-limit of a net of finite dimensional vector spaces;

3. if we identify every function $f \in W$ with the ultrafunction $f^* = \lim_{\lambda \uparrow \Lambda} f$, then $W \subset W_\Lambda$;

4. $W_\Lambda$ has a $\mathbb{R}^*$-valued scalar product.

---

4We recall that $\{e_a\}_{a\in \mathbb{R}}$ is a Hamel basis for $W$ if $\{e_a\}_{a\in \mathbb{R}}$ is a set of linearly independent elements of $W$ and every element of $W$ can be (uniquely) written as a finite sum (with coefficients in $\mathbb{R}$) of elements of $\{e_a\}_{a\in \mathbb{R}}$. Since a Hamel basis of $W$ has the continuum cardinality we can use the points of $\mathbb{R}$ as indices for this basis.
Hence the ultrafunctions are particular internal functions
\[ u : (\mathbb{R}^N)^* \to \mathbb{R}^*. \]

**Remark 13.** For every \( f \in (\mathbb{R}^N, \mathbb{R}) \) and for every space of ultrafunctions \( W_\Lambda \) based on \( W \) we have that \( f^* \in W_\Lambda \) if and only if \( f \in W \).

**Proof.** Let \( f \in W \). Then, eventually, \( f \in W_\Lambda \) and hence
\[ f^* = \lim_{\Lambda \uparrow \Lambda} f \in \lim_{\Lambda \uparrow \Lambda} W_\Lambda = W_\Lambda. \]
Conversely, if \( f \notin W \) then by the Theorem \( \square \) it follows that \( f^* \notin W^* \) and, since \( W_\Lambda \subset W^* \), this entails the thesis. \( \square \)

### 3.2 The canonical ultrafunctions

In this section we will introduce a space \( V \) and an approximating net \((V_\lambda)\) of \( V \) such that the space of ultrafunction \( V_\Lambda \) generated by \((V, V_\lambda)\) is adequate for many applications, particularly to PDE. The space \( V \) will be called the canonical space.

Let us recall the following standard terminology: for every function \( f \in L^1_{\text{loc}} \) we say that a point \( x \in \mathbb{R}^N \) is a Lebesgue point for \( f \) if
\[
 f(x) = \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)dy = 0,
\]
where \( m(B_r(x)) \) is the Lebesgue measure of the ball \( B_r(x) \); we recall the very important Lebesgue differentiation theorem (see e.g. [11]), that we will need in the following:

**Theorem 14.** If \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \) then a.e. \( x \in \mathbb{R}^N \) is a Lebesgue point for \( f \).

We fix once for ever an infinitesimal number \( \eta \neq 0 \). Given a function \( f \in L^1_{\text{loc}} \), we set
\[
 \overline{f}(x) = \text{st} \left( \frac{1}{m(B_{\eta}(x))} \int_{B_{\eta}(x)} f(y)dy \right),
\]
where \( m(B_{\eta}(x)) \) is the Lebesgue measure of the ball \( B_{\eta}(x) \). We will refer to the operator \( f \mapsto \overline{f} \) as the Lebesgue operator.

**Lemma 15.** The Lebesgue operator \( f \mapsto \overline{f} \) satisfies the following properties:

1. if \( x \) is a Lebesgue point for \( f \) then \( \overline{f}(x) = f(x) \);
2. \( f(x) = \overline{f}(x) \) a.e.;
3. if \( f(x) = g(x) \) a.e. then \( \overline{f}(x) = \overline{g}(x) \);
4. \( \overline{\overline{f}}(x) = \overline{f}(x) \).
Proof: If \( x \) is a Lebesgue point for \( f \) then

\[
\frac{1}{m(B_\eta(x))} \int_{B_\eta(x)} f(y)dy \sim f(x),
\]
so \( \overline{f}(x) = f(x) \).

This follows immediately by Theorem 14 and 1.

Let \( x \in \mathbb{R}^N \). Since \( f(x) = g(x) \) a.e., we obtain that \( \int_{B_\eta(x)} f(y)dy = \int_{B_\eta(x)} g(y)dy \), so

\[
\overline{f}(x) = \text{st} \left( \frac{1}{m(B_\eta(x))} \int_{B_\eta(x)} f(y)dy \right)
= \text{st} \left( \frac{1}{m(B_\eta(x))} \int_{B_\eta(x)} g(y)dy \right) = \overline{g}(x).
\]

This follows easily by 2 and 3. □

Example 16. If \( E = \Omega \) is an open set with smooth boundary, we have that

\[
\overline{\chi_\Omega}(x) = \begin{cases} 
1 & \text{if } x \in \Omega; \\
0 & \text{if } x \notin \Omega; \\
\frac{1}{2} & \text{if } x = \partial \Omega.
\end{cases}
\]  

(7)

We recall the following definition:

Definition 17. Let \( f \in L^1 \). \( f \) is a bounded variation function (BV for short) if there exists a finite vector Radon measure \( \text{grad} f \) such that, for every \( g \in C_1^c(\mathbb{R}^N, \mathbb{R}^N) \), we have

\[
\int f(x) \div g(x)dx = - \langle \text{grad} f, g \rangle.
\]

Let us note that \( \text{grad} f \) is the gradient of \( f(x) \) in the sense of distribution. Thus, the above definition can be rephrased as follows:

\( f \) is a bounded variation function if \( \text{grad} f \in \mathcal{M} \).

We now set

\[
V = \{ u \in BV_c \cap L^\infty \mid \overline{\pi}(x) = u(x) \},
\]
where \( BV_c \) denotes the set of function of bounded variation with compact support. So, by Lemma 15, we have that if \( u \in BV_c \cap L^\infty \) then \( \overline{\pi} \in V \). Let us observe that the condition \( \overline{\pi}(x) = u(x) \) entails that the essential supremum of any \( u \in V \) coincides with the supremum of \( u \), namely \( \|u\|_{L^\infty} = \sup |u(x)| \).

We list some properties of \( V \) that will be useful in the following:

Theorem 18. The following properties hold:

1. \( V \) is a vector space and \( C_1^c \subset V \subset L^p \) for every \( p \in [1, +\infty] \);
2. if \( u \in V \) then the weak partial derivative \( \partial_j u = \frac{\partial u}{\partial x_j} \) is a Radon finite signed measure;

3. if \( u,v \in V \) then \( u = v \text{ a.e. if and only if } u = v \);

4. the \( L^2 \) norm is a norm for \( V \) (and not a pseudonorm).

Proof: 
1 This follows easily by the definitions and the fact that \( BV_c, L^\infty \) are vector spaces.
2 This holds since \( V \subset BV_c \).
3 Let \( u,v \in V \). If \( u = v \) then clearly \( u = v \text{ a.e.} \); conversely, let us suppose that \( u = v \text{ a.e.} \); by Theorem 15 we deduce that \( \mathfrak{m}(x) = \mathfrak{n}(x) \). But \( u,v \in V \), so \( u(x) = \mathfrak{m}(x) = \mathfrak{n}(x) = v(x) \).
4 Let \( u \in V \) be such that \( \| u \|_{L^2} = 0 \). Then \( u = 0 \text{ a.e.} \) and, since \( 0 \in V \), by 3 we deduce that \( u = 0 \). □

Remark 19. By Theorem 18 it follows that, for every \( f \in V \), \( \partial_j f \in V' \) where \( V' \) denotes the (algebraic) dual of \( V \). This relation is very important to define the ultrafunction derivative (see section 4.1). In fact if \( f,g \in V \), then

\[ \langle f, \partial_j g \rangle \]

is well defined, since \( \partial_j g \) is a finite Radon measure and \( f \) is a bounded Borel-measurable function and hence \( f \in M' \).

Definition 20. A bounded Caccioppoli set \( E \) is a Borel set such that \( \chi_E \in BV_c \), namely such that \( \text{grad}(\chi_E) \) (the distributional gradient of the characteristic function of \( E \)) is a finite radon measure. The number

\[ \langle 1, \text{grad}(\chi_E) \rangle \]

is called Caccioppoli perimeter of \( E \).

We set

\[ \mathcal{B} = \{ \Omega \text{ is a bounded, open, Caccioppoli set} \} \]

Let us note \( \mathcal{B} \) is closed under unions and intersections and that, by definition, if \( \Omega \in \mathcal{B} \) then \( \chi_{\Omega} \in V \).

Lemma 21. If \( f,g \in V \) and \( \Omega, \Theta \in \mathcal{B} \), then,

\[ \int_{\Omega} f \chi_{\Theta} \ y \chi_{\Theta} \in V \]

moreover, we have that

\[ \int_{\Omega} f \chi_{\Theta} \ y \chi_{\Theta} \ dx = \int_{\Omega \cap \Theta} f(x)g(x)dx. \]

12
Proof. $BV_c \cap L^\infty$ is an algebra, so $f \chi_{\Omega} g \chi_{\Theta} \in BV_c \cap L^\infty$; by Lemma 15.3 $\int f \chi_{\Omega} g \chi_{\Theta} \in BV_c \cap L^\infty$ and by Lemma 15.4 $\int f \chi_{\Omega} g \chi_{\Theta} \in V$. Using again Lemma 15.3, we have that

$$\int f \chi_{\Omega} g \chi_{\Theta} = \int f \chi_{\Omega} g \chi_{\Theta} = \int_{\Omega \cap \Theta} f g.$$ 

\[ \square \]

For some applications we would need an analogous of Lemma 21 to hold for our space of ultrafunctions. This is possible if we take, as space of ultrafunctions, the $\Lambda$-limit of a particular net of finite dimensional subspaces of $V$ constructed as follows: let $\{e_a | a \in \mathbb{R}\}$ be an Hamel basis for $V$. Let

$$I_\mathcal{B} = \{ \chi_\Omega \mid \Omega \in \mathcal{B} \}.$$

Since $\mathcal{B}$ is closed under intersection then $I_\mathcal{B}$ is closed under multiplication. Let

$$S_\mathcal{B} = \text{Span}(I_\mathcal{B})$$

and let

$$\{ \chi_\Omega \}_{a \in \mathbb{R}} \subseteq I_\mathcal{B}$$

be a Hamel basis for $S_\mathcal{B}$. Let $\mathcal{B}_\lambda$ be the closure under intersections of the set $\{ \Omega_a | a \in \lambda \}$. Let us note that $\mathcal{B}_\lambda$ is, by construction, finite. Let

$$V_\lambda = \text{Span}\{ e_a \cdot \chi_\Omega \mid a \in \lambda, \Omega \in \mathcal{B}_\lambda \}.$$

Lemma 22. If $w \in V_\lambda$ and $\Omega \in \mathcal{B}_\lambda$ then $w \cdot \chi_\Omega \in V_\lambda$.

Proof. This follows because $\mathcal{B}_\lambda$ is closed under intersections: if $w = e_a \cdot \chi_\Omega$, then $w \cdot \chi_\Omega = e_a \cdot \chi_\Omega \cdot \chi_\Omega = e_a \cdot \chi_{\Omega \cap \Omega} \in W_\lambda$ since $\Omega \cap \Omega \in \mathcal{B}_\lambda$. \[ \square \]

Now we let $V_\lambda = \lim_{\lambda \uparrow \Lambda} V_\lambda$ and $\mathcal{B}_\Lambda = \lim_{\lambda \uparrow \Lambda} \mathcal{B}_\lambda$.

Theorem 23. If $u \in V_\lambda$ and $\Omega \in \mathcal{B}_\lambda$ then $u \cdot \chi_\Omega \in V_\lambda$.

Proof. Let $u = \lim_{\lambda \uparrow \Lambda} u_\lambda$ and $\Omega = \lim_{\lambda \uparrow \Lambda} \Omega_\lambda$. By Lemma 22 for every $\lambda$ we have that $u_\lambda \cdot \chi_{\Omega_\lambda} \in V_\lambda$, so $\lim_{\lambda \uparrow \Lambda} u_\lambda \cdot \chi_{\Omega_\lambda} = u \cdot \chi_\Omega \in V_\lambda$. \[ \square \]

By our construction, $(V_\lambda)$ is an approximating net for $V$.

Definition 24. The space of ultrafunctions $V_\Lambda$ generated by $(V, (V_\lambda))$ is called the canonical space of ultrafunctions, and its elements are called canonical ultrafunctions.

The canonical space of ultrafunctions has three important properties for applications:

1. $V_\lambda \subseteq (L^2)^\ast$;
2. since $V_\Lambda \subseteq V^*$ we have that

$$u \in V_\Lambda \Rightarrow \partial_j u \in V'_\Lambda,$$  \hspace{1cm} (8)

where $V'_\Lambda$ denotes the dual of $V_\Lambda$;

3. if $\Omega$ is a bounded open set with smooth boundary, then $\chi_{\Omega^c} \in V_\Lambda$.

Property 1 is in common with (almost) all the space of ultrafunctions that we considered in our previous works (see [2], [4], [6], [7]); it is important since it gives a duality which corresponds to the scalar product in $L^2$. This fact allows to relate the generalized solutions in the sense of ultrafunctions with the weak solutions in the sense of distributions.

Property 2 follows by the construction of $V_\Lambda$, since $V_\Lambda \subseteq V^*$. This relation is used to define the ultrafunction derivative (see section 4.1). There are other spaces such as $C^1_c$, $H^1$ or the fractional Sobolev space $H^{1/2}$ which satisfy (8); the fractional Sobolev space $H^{1/2}$ is the optimal Sobolev space with respect to this request (in the sense that it is the biggest space). However our choice of the space is due to the request 3. This request seems necessary to get a definite integral which satisfies the properties which allows to prove Gauss’ divergence theorem (see section 4.2) and hence to prove some conservation laws. Also this property implies that the extensions of local operators\footnote{By local operator we mean any operator $F : V(\Omega) \to V(\Omega)$ such that $\text{supp}(F(f)) \subseteq \text{supp}(f) \forall f \in V(\Omega)$.} are local.

Let us note that there are other spaces which satisfy 1, 2, 3, e.g the space generated by functions of the form $u(x) = f(x)\chi_\Omega(x)$ with $f \in C^2$. Clearly this space is included in $V$ and so it seems more convenient to take $V$. In any case, we think that $V$ is a good framework for our work.

### 3.3 Canonical extension of functions and measures

We denote by $\mathcal{M}$ the vector space of (signed) Radon measure on $\mathbb{R}^N$.

We start by defining a map

$$P_\Lambda : \mathcal{M} \to V_\Lambda$$

which will be very useful in the extension of functions. As usual we will suppose that $L^1_{loc} \subseteq \mathcal{M}$ identifying every locally integrable function $f$ with the measure $f(x)dx$.

**Definition 25.** If $\mu \in \mathcal{M}$, $\bar{\mu} = P_\Lambda \mu$ denotes the unique ultrafunction such that

$$\forall v \in V_\Lambda, \int \bar{\mu}(x)v(x)dx = \langle v, \mu \rangle .$$
In particular, if $u \in [L^1_{\text{loc}}]^*$, $\widetilde{u} = P_{\Lambda} u$ denotes the unique ultrafunction such that
\[
\forall v \in V_{\Lambda}, \quad \int \widetilde{u}(x)v(x)dx = \int u(x)v(x)dx.
\]

Let us note that this definition is well posed since every ultrafunction $v \in V_{\Lambda}$ is $\mu$-integrable for every $\mu \in M^*$ and hence $v \in (M^*)'$. 

**Remark 26.** Notice that, if $u \in [L^2(\mathbb{R})]^*$, then $P_{\Lambda}$ is the orthogonal projection.

In particular, if $f \in L^1_{\text{loc}}$, the function $\widetilde{f}$ is well defined. From now on we will simplify the notation just writing $\widetilde{f}$.

**Example 27.** Take $\frac{1}{|x|}$, $x \in \mathbb{R}^N$; if $N \geq 2$, then $\frac{1}{|x|} \in L^1_{\text{loc}}$, and it is easy to check that the value of $\frac{1}{|x|}$ for $x = 0$ is an infinite number. Notice that the ultrafunction $\frac{1}{|x|}$ is different from $\left(\frac{1}{|x|}\right)^*$ since the latter is not defined for $x = 0$. Moreover they differ "near infinity" since $\frac{1}{|x|}$ has its support is contained in an interval (of infinite length).

**Example 28.** If $E$ is a bounded borel set, then
\[
\widetilde{\chi_E} = (\chi_E)^*.
\]

### 4 Generalization of some basic notions of calculus

#### 4.1 Derivative

As we already mentioned, the crucial property that we will use to define the ultrafunctions derivative is that the weak derivative of a $BV$ function is a Radon measure. This allows to introduce the following definition:

**Definition 29.** Given an ultrafunction $u \in V_{\Lambda}$, we define the **ultrafunction derivative** as follows:
\[
D_j u = P_{\Lambda} \partial_j u = \partial_j \widetilde{u},
\]
where $P_{\Lambda}$ is defined by Definition 25.

The above definition makes sense since $\partial_j u \in M^*$. More explicitly if $u \in V_{\Lambda}$ then, $\forall v \in V_{\Lambda}$,
\[
\int D_j uv \, dx = \langle v, \partial_j u \rangle.
\]

The right hand side makes sense since $|v|$ is bounded and $\partial_j u$ is a finite measure.

**Theorem 30.** The derivative is antisymmetric; namely, for every ultrafunctions $u, v \in V_{\Lambda}$ we have that
\[
\int D_j u(x)v(x)dx = - \int u(x)D_j v(x)dx. \quad (9)
\]
Proof: Let us observe that $BV_c \cap L^\infty$ is an algebra, so $u \cdot v \in BV_c \cap L^\infty$. Let $\Omega \in \mathcal{B}$ contain the support of $u \cdot v$. Then

$$0 = \langle uv, \partial_j \chi_{\Omega} \rangle = \langle \chi_{\Omega} \cdot \partial_j (uv) \rangle = \langle 1, \partial_j (uv) \rangle$$

Hence

$$\int D_j u(x)v(x)dx + \int u(x)D_j v(x)dx = \langle u, \partial_j v \rangle + \langle v, \partial_j u \rangle$$

Since $u,v \in V^*$, then $u\partial_j v$ and $v\partial_j u$ are Radon measures, and hence

$$\langle u, \partial_j v \rangle + \langle v, \partial_j u \rangle = \langle 1, u\partial_j v \rangle + \langle 1, v\partial_j u \rangle$$

Then

$$\int D_j u(x)v(x)dx + \int u(x)D_j v(x)dx = \langle 1, u\partial_j v \rangle + \langle 1, v\partial_j u \rangle = \langle 1, \partial_j (uv) \rangle = 0$$

so we obtain the thesis. \(\square\)

Actually, the ultrafunctions derivative coincides with the classical one for a large class of functions:

Theorem 31. Let $\Omega \in \mathcal{B}$ and let $u \in L^1_{loc}$. If $\partial_j u \in BV(\Omega)$ then

$$\forall x \in \Omega^*, \ D_j \tilde{u}(x) = (\partial_j u)^*(x).$$

Proof: Since $\partial_j (u) \in BV(\Omega)$ we have that $\partial_j u \cdot \chi_{\Omega} \in V$. We claim that

$$D_j \tilde{u}(x) \cdot (\chi_{\Omega}^*)^*(x) = (\partial_j (u(x)) \chi_{\Omega}(x))^*.$$ 

In fact, for every $v \in V_{\Lambda}$ we have

$$\int (D_j \tilde{u}(x) \cdot \chi_{\Omega}^* (x))v(x)dx = \int D_j \tilde{u}(x) \cdot (\chi_{\Omega}^* (x))v(x)dx$$

$$= \langle \chi_{\Omega}^* v, \partial_j^* u \rangle = \langle v, \chi_{\Omega} \partial_j^* u \rangle$$

$$= \langle v, (\chi_{\Omega} \partial_j u)^* \rangle = \int vP_{\Lambda} (\chi_{\Omega} \partial_j u)^* dx;$$

hence $D_j \tilde{u} \cdot \chi_{\Omega}^* = P_{\Lambda} (\chi_{\Omega} \cdot \partial_j u)^*$. Since $\chi_{\Omega} \cdot \partial_j u \in V_{\Lambda}$, then

$$D_j \tilde{u} \cdot \chi_{\Omega}^* = (\chi_{\Omega} \cdot \partial_j u)^* = (\partial_j u)^* \cdot \chi_{\Omega}^*.$$ 

By the previous equality, our claim follows. \(\square\)
4.2 Gauss’ divergence theorem

Definition 32. If \( u \in (L^1_{\text{loc}})^* \) and \( \Omega \in \mathcal{B}_\Lambda \) then we set

\[
\int_\Omega u \, dx := \int u \, \chi_\Omega \, dx
\]

The above definition makes sense for any internal open set (the lambda-limit of a net of open sets) and more in general for any internal Borel set. However, the integral extended to a set in \( \mathcal{B}_\Lambda \) has nicer properties, as it will be shown below. For example if \( u \) and \( \Omega \) are standard, the above integral coincides with the usual one.

Next we want to deal with some classical theorem in field theory such as Gauss’ divergence theorem. To do this we need some new notations. The gradient and the divergence of a standard function, (distribution) or of an internal function (distribution) will be denoted by \( \text{grad} \), \( \text{div} \) respectively; their generalization to ultrafunctions will be denoted by

\[
\nabla, \nabla \cdot
\]

Namely, if \( u \in V_\Lambda \), we have that

\[
\text{grad} \, u = (\partial_1 u, ..., \partial_N u); \quad \nabla \, u = (D_1 u, ..., D_N u).
\]

Similarly, if \( \phi = (\phi_1, ..., \phi_N) \in (V_\Lambda)^N \), we have that

\[
\text{div} \, \phi = \partial_1 \phi_1 + ... + \partial_N \phi_N; \quad \nabla \cdot \phi = D_1 \phi_1 + ... + D_N \phi_N.
\]

If \( \Omega \in \mathcal{B}_\Lambda \), then \( \text{grad} \, \chi_\Omega = (\partial_1 \chi_\Omega, ..., \partial_N \chi_\Omega) \) is a vector-valued Radon measure such that, \( \forall \phi \in (C^1)^N \),

\[
\langle \text{grad} \, \chi_\Omega, \phi \rangle = - \int_\Omega \text{div} \, \phi \, dx \tag{10}
\]

As usual, we will denote by \( |\text{grad} \, \chi_\Omega| \) the total variation of \( \chi_\Omega \), namely a Radon measure defined as follows: for any Borel set \( A \),

\[
|\text{grad} \, \chi_\Omega| \, (A) = \sup \left\{ \langle \text{grad} \, \chi_\Omega, \phi \rangle \mid \forall \phi \in (C^1)^N, \ |\phi(x)| \leq 1 \right\}.
\]

\( |\text{grad} \, \chi_\Omega| \) is a measure concentrated on \( \partial \Omega \) and the quantity

\[
\langle \text{grad} \, \chi_\Omega, \mathbf{1} \rangle
\]

is called Caccioppoli perimeter of \( \Omega \) (see e.g. [10]). If \( \partial \Omega \) is smooth, then \( |\text{grad} \, \chi_\Omega| \) agrees with the usual surface measure and hence if \( f \) is a Borel function, \( \langle f, |\text{grad} \, \chi_\Omega| \rangle \) is a generalization of the surface integral \( \int_{\partial \Omega} f(x) \, d\sigma \). This generalization suggests a further generalization in the framework of ultrafunctions:
Definition 33. If $u \in V^*$ and $\Omega \in \mathcal{B}_\Lambda$ then we set
\[
\int_{\partial \Omega} u \, d\sigma := \int u \, |\nabla \chi_\Omega| \, dx,
\]
where $|\nabla \chi_\Omega|$ is the ultrafunction defined by the following formula: $\forall v \in V_\Lambda$
\[
\int |\nabla \chi_\Omega| \, v(x) \, dx = \langle v, |\text{grad} \chi_\Omega| \rangle.
\]

Lemma 34. If $\phi \in (V_\Lambda)^N$ and $\Omega \in \mathcal{B}^*$ then
\[
\int_{\Omega} \nabla \cdot \phi \, dx = -\int \phi \cdot \nabla \chi_\Omega \, dx.
\]

Proof: We have that
\[
\int_{\Omega} \nabla \cdot \phi \, dx = \sum_j \int_{\Omega} D_j \phi_j \, dx = \sum_j \int D_j \phi_j \chi_\Omega \, dx = -\sum_j \int \phi_j \, D_j \chi_\Omega \, dx = -\int \phi \cdot \nabla \chi_\Omega \, dx.
\]

We can give to the Gauss theorem a more meaningful form: let $\nu_\Omega(x) = \begin{cases} -\frac{\nabla \chi_\Omega(x)}{|\nabla \chi_\Omega(x)|} & \text{if } |\nabla \chi_\Omega(x)| \neq 0; \\ 0 & \text{if } |\nabla \chi_\Omega(x)| = 0. \end{cases}$

Let us note that, by construction, $\nu_\Omega(x)$ is an internal function whose support is infinitely close to $\partial \Omega$.

Theorem 35. (Gauss’ divergence theorem for ultrafunctions) If $\phi \in (V_\Lambda)^N$ and $\Omega \in \mathcal{B}_\Lambda$ then
\[
\int_{\Omega} \nabla \cdot \phi \, dx = \int_{\partial \Omega} \phi \cdot \nu_\Omega(x) \, d\sigma.
\]

Proof: We have that $\nabla \chi_\Omega = -\nu_\Omega |\nabla \chi_\Omega|$ and, by using Theorem 34 and Definition 33 we get:
\[
\int_{\Omega} \nabla \cdot \phi \, dx = -\int \phi \cdot \nabla \chi_\Omega \, dx = \int \phi \cdot \nu_\Omega |\nabla \chi_\Omega| \, dx = \int_{\partial \Omega} \phi \cdot \nu_\Omega \, d\sigma.
\]
4.3 A simple application

Let us consider the following Cauchy problem:

\[ \frac{\partial u}{\partial t} + \text{div} F(t, x, u) = 0; \]
\[ u(0, x) = u_0(x), \]

where \( x \in \mathbb{R}^N \). It is well known that this problem has no classical solutions since it develops singularities.

One way to formulate this problem in the framework of ultrafunctions is the following:

\[ \text{find } u \in C^1(\mathbb{R}^N) \text{ such that:} \]
\[ \forall v \in V, \quad \int [\partial_t u + \nabla \cdot F(t, x, u)] v(x) dx = 0; \]
\[ u(0, x) = u_0^*(x) \]

where \( \partial_t = \left( \frac{\partial}{\partial t} \right)^* \) and \( u_0 \in C^1_c \).

We assume that

\[ F \in C^1 \]

and that

\[ |F(t, x, u)| \leq c_1 + c_2|u|. \]

**Theorem 36.** Problem (14) has a unique solution and it satisfies the following conservation law:

\[ \partial_t \int_{\Omega} u(t, x) \, dx = - \int_{\partial\Omega} F(t, x, u(t, x)) \cdot \nu_{\Omega}(x) \, d\sigma \]

for every \( \Omega \in \mathcal{B}_\Lambda \). In particular if \( F(t, x, 0) = 0 \) for every \( (t, x) \in [0, T] \times \mathbb{R}^N \), then

\[ \partial_t \int_{\Omega} u(t, x) \, dx = 0. \]

**Proof:** First let us prove the existence. For every \( \mu \in \mathcal{L} \), let us consider the problem

\[ \text{find } u \in C^1([0, T], V_\Lambda) \text{ such that:} \]
\[ \partial_t u + P_\Lambda \text{div} F(t, x, u) = 0; \]
\[ u(0, x) = u_0^*(x), \]

where \( P_\Lambda : \mathfrak{M} \to V_\Lambda \) is the "orthogonal projection", namely, for every \( \mu \in \mathfrak{M} \), \( P_\Lambda \mu \) is the only element in \( V_\Lambda \) such that,

\[ \forall v \in V_\Lambda, \quad \int P_\Lambda \mu v \, dx = (v, \mu). \]

In the above equation we have assumed that \( \Lambda \) is so large that \( u_0^*(x) \in V_\Lambda \). Equation (19) reduces to an ordinary differential equation in a finite dimensional
space and hence, by (15) and (16), it has a unique global solution \( u_\lambda \). Equation (19) can be rewritten in the following equivalent form:

\[
\forall v \in V_\lambda, \int [\partial_t u + \text{div} F(t, x, u)] v(x) dx = 0.
\]

Taking the A-limit, we get a unique solution of (14).

Equation (17) follows, as usual, from Gauss’ theorem:

\[
\partial_t \int_\Omega u(t, x) dx = \int_\Omega \partial_t u(t, x) dx = (\text{by eq. } (14) \text{ with } v = \bar{1})
- \int_\Omega \nabla \cdot F(t, x, u(t, x)) dx = \int_{\partial \Omega} F(t, x, u(t, x)) \cdot \nu_\Omega(x) d\sigma.
\]

In particular, if \( F(t, x, 0) = 0 \), since \( u \) has compacr support, we have that \( F(t, x, u(t, x)) = 0 \) if \( |x| \geq R \) with \( R \) is sufficiently large. Then, taking \( \Omega = B_R \), (18) follows. □

References

[1] Benci V., *An algebraic approach to nonstandard analysis*, in: Calculus of Variations and Partial differential equations, (G. Buttazzo, et al., eds.), Springer, Berlin (1999), 285–326.

[2] Benci V., *Ultrafunctions and generalized solutions*, in: Adv. Nonlinear Stud. 13, (2013), 461–486, arXiv:1206.2257.

[3] Benci V., Di Nasso M., *Alpha-theory: an elementary axiomatic for non-standard analysis*, Expo. Math. 21, (2003), 355–386.

[4] Benci V., Luperi Baglini L., *A model problem for ultrafunctions*, in: Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems. Electron. J. Diff. Eqns., Conference 21 (2014), pp. 11-21.

[5] Benci V., Luperi Baglini L., *Basic Properties of ultrafunctions*, to appear in the WNDE2012 Conference Proceedings, arXiv:1302.7156.

[6] Benci V., Luperi Baglini L., *Ultrafunctions and applications*, to appear on DCDS-S (Vol. 7, No. 4) August 2014, arXiv:1405.4152.

[7] Benci V., Luperi Baglini L., *A non archimedean algebra and the Schwartz impossibility theorem*, Monatsh. Math. (2014), DOI 10.1007/s00605-014-0647-x.

[8] Benci V., Luperi Baglini L., *Generalized functions beyond distributions*, to appear on AJOM (2014), arXiv:1401.5270.
[9] Benci V., Luperi Baglini L., Generalized solutions of the Burgers’ equations, in preparation.

[10] Caccioppoli R., Sulla quadratura delle superfici piane e curve, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (6), (1927), 142–146.

[11] Lebesgue H., (1910). Sur l’intégration des fonctions discontinues,, Ann. Scientifiques Éc. Norm. Sup. (27), (1910), 361–450.

[12] Keisler H.J., Foundations of Infinitesimal Calculus, Prindle, Weber & Schmidt, Boston, (1976).

[13] Robinson A., Non-standard Analysis, Proceedings of the Royal Academy of Sciences, Amsterdam (Series A) 64, (1961), 432–440.