Research Article

On Laplacian Equienergetic Signed Graphs

Qingyun Tao¹ and Lixin Tao²

¹College of Mathematics and Physics, Hunan University of Arts and Science, Changde, Hunan 415000, China
²School of Computer Science, Hunan University of Technology, Zhuzhou, Hunan 412000, China

Correspondence should be addressed to Lixin Tao; tlxing@sina.com

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The Laplacian energy of a signed graph is defined as the sum of the distance of its Laplacian eigenvalues from its average degree. Two signed graphs of the same order are said to be Laplacian equienergetic if their Laplacian energies are equal. In this paper, we present several infinite families of Laplacian equienergetic signed graphs.

1. Introduction

A signed graph is an ordered pair \( \Gamma = (G, \sigma) \), where \( G = (V, E) \) is a simple unsigned graph called the underlying graph of \( \Gamma \), and \( \sigma: E \rightarrow \{+,-\} \), called a signing (or a signature), is a sign function from the edge set \( E \) to the set \( \{+,-\} \) of signs.

The sign of a signed graph is defined as the product of signs of its edges. A signed graph is said to be positive (respectively, negative) if its sign is positive (respectively, negative), i.e., it contains an even (respectively, an odd) number of negative edges. A signed graph is said to be all-positive (respectively, all-negative) if all its edges are positive (respectively, negative), denoted by \((G,+)(G,-)\). A signed graph is said to be balanced if each of its cycles is positive, otherwise unbalanced.

Let \( A(G) = (a_{ij})_{n \times n} \) be adjacency matrix of unsigned graph \( G \), where \( a_{ij} = 1 \) whenever vertices \( i \) and \( j \) are adjacent and \( a_{ij} = 0 \) otherwise. And, let \( D(G) \) be its diagonal matrix of vertices degrees; then, \( L(G) = D(G) - A(G) \) and \( Q(G) = D(G) + A(G) \) are called Laplacian and signless Laplacian matrix of \( G \), respectively.

Similarly, for a signed graph \( \Gamma = (G, \sigma) \), its adjacency matrix \( A(\Gamma) = (a_{ij})_{n \times n} \) is defined as

\[
 a_{ij} = \begin{cases} 
 \sigma_{ij}, & \text{if vertices } i \text{ and } j \text{ are adjacent,} \\
 0, & \text{otherwise,} 
\end{cases}
\]

and \( L(\Gamma) = D(\Gamma) - A(\Gamma) \) is called its Laplacian matrix, where \( D(\Gamma) \) is the same as \( D(G) \). Note that \( L(G,+)=L(G) \) and \( L(G,-)=Q(G) \).

Suppose \( \Gamma = (G, \sigma) \) is a signed graph and \( \theta: V \rightarrow \{+1,-1\} \) is any sign function. Switching \( \Gamma \) by \( \theta \) means constructing a new signed graph \( \Gamma^\theta = (G, \sigma^\theta) \) whose underlying graph is also \( G \), while sign function specified on an edge \( e = uv \) is \( \sigma^\theta(e) = \theta(u)\sigma(u)e\theta(v) \).

Let \( \Gamma_1 = (G, \sigma_1) \) and \( \Gamma_2 = (G, \sigma_2) \) be two signed graphs on the same underlying graph \( G \). We call \( \Gamma_1 \) and \( \Gamma_2 \) are switching equivalent; write \( \Gamma_1 \sim \Gamma_2 \), if there exists a switching function \( \theta \) such that \( \Gamma_2 = \Gamma_1^\theta \). Actually, switching equivalent signed graphs can be considered as switching isomorphic, and their signatures are thought to be equivalent. Switching leaves many signed-graphic invariant, such as the set of positive cycles.

Two matrices \( M_1 \) and \( M_2 \) of order \( n \) are called signature similar if there exists a signature matrix, i.e., a diagonal matrix \( S = \text{diag}\{s_1, \ldots, s_n\} \) with diagonal entries \( s_i = \pm 1 \) such that \( M_2 = SM_1S \). Notice that two signature similar matrices have the same eigenvalues. From the definitions of switching equivalent and signature similar, we have the following.

Lemma 1. Suppose \( \Gamma_1 = (G, \sigma_1) \) and \( \Gamma_2 = (G, \sigma_2) \) be two signed graphs on the same underlying graph \( G \); then, \( \Gamma_1 \sim \Gamma_2 \) if and only if \( L(\Gamma_1) \) and \( L(\Gamma_2) \) are signature similar.
Signed graphs were first introduced by HARARY, in [1], in connection with the study of the theory of social balance in social psychology, see [2]. After that, there are many works on signed graph, see [3–5].

The energy of unsigned graph $G$ is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. The Laplacian energy of a graph $G$ is defined as $LE(G) = \sum_{i=1}^{\lambda_i(G)} |\lambda_i(G) - \overline{\lambda_i(G)}|$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$ are the eigenvalues of Laplacian matrix of $G$, $\overline{\lambda_i(G)} = (2m(G)/n)$ is the average degree of $G$. For its basic properties, see [10–15]. The signless Laplacian energy of a graph $G$ is defined as $QE(G) = \sum_{i=1}^{\lambda_i(G)} |\lambda_i(G) - \overline{\lambda_i(G)}|$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ are the eigenvalues of signless Laplacian matrix of $G$.

The Laplacian energy of a signed graph $G$ is defined similarly as $LE(G) = \sum_{i=1}^{\lambda_i(G)} |\lambda_i(G) - \overline{\lambda_i(G)}|$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ are the eigenvalues of Laplacian matrix of $G$ and $\overline{\lambda_i(G)}$ is the average degree of $G$, which is the same as that of $G$, so we will denote it by $\overline{\lambda}$ simply.

Two (signed) graphs of the same order are said to be Laplacian equienergetic if they have the same Laplacian energy. Two (signed) graphs are said to be Laplacian cospectral if they have the same Laplacian eigenvalues. From Lemma 1, we know that switching equivalent signed graphs must be Laplacian cospectral and Laplacian equienergetic.

Finding Laplacian equienergetic graphs is interesting, and there are so many research works on Laplacian equienergetic unsigned graphs already, see [17–20], for example, but it is very few on signed ones. In this paper, we present several infinite families of Laplacian equienergetic signed graphs which are not Laplacian cospectral.

### 2. Main Results

It seems more difficult to find Laplacian equienergetic signed graphs than unsigned ones, since we know too little about Laplacian spectrum of signed graphs. It is known that an unsigned graph could be considered as an all-positive signed graph; therefore, two Laplacian equienergetic unsigned graphs is also a pair of Laplacian equienergetic signed ones. And, we will not deal with this case. In the paper, we try to find Laplacian equienergetic signed graph pairs with common underlying graph. That is, we are interested in graphs $G$ which satisfy $LE(G, \sigma_1) = LE(G, \sigma_2)$. At first, we consider graphs which satisfy $LE(G, +) = LE(G, -)$; in other words, we consider graphs which satisfy $LE(G) = QE(G)$. In fact, there are such graphs trivially. It is well known that, for a regular graph, $LE(G) = QE(G)$ holds. Besides, for bipartite graphs, the Laplacian spectrum coincides with the signless Laplacian spectrum; obviously, in this case, we have $LE(G) = QE(G)$ also. Now, we present two infinite families of connected nonregular and nonbipartite such graphs.

Let $G_1 \vee G_2$ denote the join of graphs $G_1$ and $G_2$, obtained from the union of $G_1$ and $G_2$ by joining every vertex of $G_1$ with every vertex of $G_2$. The following lemma is from [21].

**Lemma 2.** Let $G_1$ and $G_2$ be graphs on $n_1$ and $n_2$ vertices, respectively. Let $L_1$ and $L_2$ be the Laplacian matrices for $G_1$ and $G_2$, respectively, and let $L$ be the Laplacian matrix for $G_1 \vee G_2$. If $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n_1} = 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n_2} = 0$ are the eigenvalues of $L_1$ and $L_2$, respectively, then the eigenvalues of $L$ are $\{n_1 + n_2, n_1 + \beta_1, n_1 + \beta_2, \ldots, n_1 + \beta_{n_2 - 1}, n_2 + \alpha_1, n_2 + \alpha_2, \ldots, n_2 + \alpha_{n_1 - 1}, 0\}$.

The first family of such graphs is from [22].

**Theorem 3.** For each integer $\rho \geq 1$, unsigned graph $G \equiv K_1 \vee (K_2 \cup pK_3)$ satisfying $LE(G) = QE(G)$.

The second family of such graphs is as follows.

**Theorem 4.** For each integer $\rho \geq 3$, unsigned graph $G \equiv K_2 \vee (pK_3)$ satisfying $LE(G) = QE(G)$.

**Proof.** On the one hand, by Lemma 2, the Laplacian spectrum of $K_2 \vee (pK_3)$ is $\{2p + 2^{(2)}, 4^{(p)}, 2^{(p - 1)}, 0\}$. Noting that its average degree is $\overline{\lambda} = ((5p + 1)/((p + 1)))$, it can be easily verified that $LE(K_2 \vee (pK_3)) = 4((2p^2 + p + 1) / (p + 1))$.

On the other hand, the signless Laplacian matrix of $K_2 \vee (pK_3)$ with suitable labeling has the form (with zeros omitted)

\[
Q(K_2 \vee (pK_3)) = \begin{pmatrix}
2p + 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 2p + 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 3 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 3 & 1 \\
1 & 1 & 1 & 1 & 3 \\
\end{pmatrix}
\]

Thus, its signless Laplacian characteristic polynomial is $|Q(K_2 \vee (pK_3)) - \lambda I| = \cdots$
Lemma 5. Suppose \( \Gamma \) has exactly \( k \) Laplacian eigenvalues greater than average degree \( \bar{d} \), that is, \( \lambda_1(\Gamma) \geq \cdots \geq \lambda_k(\Gamma) > \bar{d} \geq \lambda_{k+1}(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) \); then, \( \text{LE}(\Gamma) = 2 \sum_{i=1}^{k} \lambda_i(\Gamma) - \bar{d} \). Besides, if two signed graphs \( \Gamma_1 = (G, \sigma_1) \) and \( \Gamma_2 = (G, \sigma_2) \) both have \( k \) Laplacian eigenvalues greater than \( \bar{d} \) and the same sum of first \( k \)-greatest Laplacian eigenvalues, then \( \Gamma_1 \) and \( \Gamma_2 \) are Laplacian equienergetic.

Proof. Notice that \( \sum_{i=1}^{n} \lambda_i(\Gamma) = 2m(\Gamma) \) and \( \bar{d} = (2m(\Gamma))/n \); by the definition of Laplacian energy,

\[
\begin{bmatrix}
2p + 1 - x & 1 & 1 & \cdots & \cdots & 1 \\
1 & 2p + 1 - x & 1 & \cdots & \cdots & 1 \\
1 & 1 & 3 - x & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 3 - x & 1 \\
1 & 3 - x & \cdots & \cdots & 1 \\
\end{bmatrix}
\]

Performing \( C_1 - (1/(4 - x))C_3 \cdots + C_n \) and \( C_2 - (1/(4 - x))C_3 \cdots + C_n \) where \( C_i \) is the \( i \)-th column of the above determinant, then according to the Laplace theorem, expand the resulting determinant along the first two columns we obtain \(|Q(K_G \cup pK_2) - xI| = (2p + 4 - x)(2p - x)(4 - x)^{p-1}/(2 - x)^{p+1}\). Therefore, if \( k \) satisfies \( \text{LE}(\sigma G, \Gamma) \geq \text{average degree} \), then \( \text{LE}(\sigma G, \Gamma) \geq \text{average degree} \), and \( \text{LE}(\sigma G, \Gamma) \geq \text{average degree} \). Besides, if two signed graphs \( \Gamma_1 = (G, \sigma_1) \) and \( \Gamma_2 = (G, \sigma_2) \) both have \( k \) Laplacian eigenvalues greater than \( \bar{d} \) and the same sum of first \( k \)-greatest Laplacian eigenvalues, then \( \Gamma_1 \) and \( \Gamma_2 \) are Laplacian equienergetic.

Theorem 6. For each odd integer \( n = 2k + 1 \), \( k \geq 2 \), let \( G \equiv K_n \cup (k - 1)K_1 \), where \( H_k \) is an arbitrary graph on \( k \) vertices (connected or disconnected), the sign function \( \sigma \) specifies exactly \( k \) edges to be negative: the only edge of \( K_2 \); \( k - 1 \) edges connect one vertex of \( K_2 \) to \( k - 1 \) distinct isolated vertices \( K_1 \). And, all remaining edges are positive; then, \( G \) satisfies \( \text{LE}(G, \sigma) = \text{LE}(G, +) \).

Proof. With suitable labeling of vertices of \( G, \sigma \), its Laplacian matrix has the form (with zeros omitted)
Thus, its Laplacian characteristic polynomial is

\[ |L(G, \sigma) - xI| = \]

\[
\begin{vmatrix}
 n-1-x & 1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\
 1 & n-1-x & -1 & \cdots & -1 & -1 & \cdots & -1 \\
 1 & -1 & 2-x & & & & & \\
 \vdots & \vdots & \ddots & & & & & \\
 1 & -1 & 2-x & & & & & \\
 -1 & -1 & & & & & & \\
 \vdots & \vdots & & & & & & \\
 -1 & -1 & & & & & & \\
\end{vmatrix} \tag{6}
\]

Noting that the row sum of \((2-x)I_k + L(H_k)\) is \(2-x\), performing \(C_1 + C_2\) and \(C_1 + (2/(2-x))(C_{k+2} + \cdots + C_n)\), the resulting determinant is

\[
\begin{vmatrix}
 n-x - \frac{n-1}{2-x} & \frac{1-x}{2-x} & 1 & \cdots & 1 & -1 & \cdots & -1 \\
 n-x - \frac{n-1}{2-x} & n-1-x - \frac{n-2}{2-x} & -1 & \cdots & -1 & -1 & \cdots & -1 \\
 0 & 0 & 2-x & & & & & \\
 \vdots & \vdots & \ddots & & & & & \\
 2-x & & & & & & & \\
 \vdots & \vdots & & & & & & \\
 0 & 0 & & & & & & \\
\end{vmatrix} \tag{7}
\]

Now, according to the Laplace theorem, expanding the above determinant along the first two columns, we obtain

\[
\begin{vmatrix}
 n-x - \frac{n-1}{2-x} & \frac{1-x}{2-x} & 1 & \cdots & 1 & -1 & \cdots & -1 \\
 n-x - \frac{n-1}{2-x} & n-1-x - \frac{n-2}{2-x} & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\end{vmatrix} \times (-1)^{1+2+1+2} \times (2-x)^{k-1} \times |(2-x)I_k + L(H_k)|. \tag{8}
\]
Suppose the Laplacian spectrum of $H_k$ is $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k = 0$; then, we have $\|2 - x\|H_k + L(H_k)\| = \prod_{i=1}^{k} (2 + \alpha_i - x)$; thus, we obtain $|L(G, \sigma) - xi| = (n + 1 - x)(n - 1 - x)(1 - x)^2 (2 - x)^{(n-7)/2} \prod_{i=1}^{k} (2 + \alpha_i - x)$. Therefore, the Laplacian spectrum of $(G, \sigma)$ is $\{n + 1, n - 1, 2 + \alpha_1, \ldots, 2 + \alpha_{k-1}, 2(n-5)/2, 1^{(2)}\}$.

On the contrary, by Lemma 2, we can get the Laplacian spectrum of $(G, +)$ is $\{n^{(2)}, 2 + \alpha_1, \ldots, 2 + \alpha_{k-1}, 2^{(k-1)}, 0\}$. Notice that $(G, +)$ and $(G, \sigma)$ have the same average degree, which is $\overline{d} \geq (4m - 6)/n > 2$, and both have $i$: for some $2 \leq i \leq k + 1$, Laplacian eigenvalues are greater than $\overline{d}$ and the same sum of first $i$-greatest Laplacian eigenvalues; by Lemma 5, we can conclude that $LE(G, \sigma) = LE(G, +)$. This completes the proof.

**Theorem 7**

(1) For each even integer $n = 4k + 2$, $k \geq 1$, let $G \equiv K_3 \vee (H_{k-1} \cup 3K_3)$, where $H_{k-1}$ is an arbitrary graph on $k - 1$ vertices (connected or disconnected); the sign function $\sigma$ specifies exactly $3k$ edges to be negative, where $k$ edges connecting the $i$th vertex of $K_3$ to $k$ distinct isolated vertices $K_3, i = 1, 2, 3$, and all remaining edges are positive; then, $G$ satisfies $LE(G, \sigma) = LE(G, +)$.

Proof. With suitable labeling of vertices of $(G, \sigma)$, its Laplacian characteristic polynomial is $|L(G, \sigma) - xi| = |

| $n - 1 - x$ | $-1$ | $-1$ | $k - 1$ | $-1 \cdots - 1$ | $k$ | $-1 \cdots - 1$ | $k - 1$ | $-1 \cdots - 1$ | $k$ |
|---|---|---|---|---|---|---|---|---|---|
| $n - 1 - x$ | $-1$ | $k - 1$ | $-1 \cdots - 1$ | $1$ |
| $-1$ | $-1$ | $n - 1 - x$ | $-1 \cdots - 1$ | $3 - x$ |
| $-1$ | $-1$ | $1$ | $-1 \cdots - 1$ | $1$ |
| $-1$ | $-1$ | $1$ | $-1 \cdots - 1$ | $1$ |
| $-1$ | $-1$ | $3 - x$ |

Performing, in turn, $C_1 + (1/(3 - x))(C_4 + \cdots + C_{3k+2} - C_{3k+3} - \cdots - C_n)$, $C_2 + (1/(3 - x))(C_4 + \cdots + C_{2k+2} - C_{2k+3} - \cdots - C_{3k+2} + C_{3k+3} + \cdots + C_n)$, and $C_3 + (1/(3 - x))(C_4 + \cdots + C_{k+2} - C_{k+3} - \cdots - C_{2k+2} + C_{2k+3} + \cdots + C_n)$, where $C_i$ is the $i$th column of the determinant, up to now, except for the first three rows, all elements of the first three columns are
simplified to zero. Then, according to the Laplace theorem, expanding the resulting determinant along the first three columns, we obtain

\[
\begin{vmatrix}
\frac{n-2-x}{3-x} & 0 & 0 \\
0 & \frac{x-2}{3-x} & 0 \\
0 & 0 & \frac{n-1-x}{3-x} \\
\end{vmatrix}
\]

\[
\times \left[(3-x)I_{k-1} + L(H_{k-1})\right] \times (3-x)^{3k}. \tag{10}
\]

Noting that \( n = 4k + 2 \), the above expression can be simplified as \((n + 1 - x)^2 (n - 2 - x) (2 - x)^3 (3 - x) I_{k-1} + L(H_{k-1})\) \((3-x)^{3k}\). Suppose the Laplacian spectrum of \( H_{k-1} \) is \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{k-1} = 0 \); then, we have \(|(3-x)I_{k-1} + L(H_{k-1})| = \prod_{i=1}^{k-1} (3 + \alpha_i - x)\); consequently, the Laplacian spectrum of \((G, \sigma)\) is \([n + 1^{(2)}, n - 2, 3, \alpha_1, \ldots, 3 + \alpha_{k-1}, 3^{(3k-3)}, 2^{(3)}]\).

By Lemma 2, we can easily get the spectrum of \( L(G, +) \) is \([n^{(2)}, 3 + \alpha_1, \ldots, 3 + \alpha_{k-1}, 3^{(3k-3)}, 0]\). Notice that \((G, +)\) and \((G, \sigma)\) have the same average degree, which is
\[ d \geq (\lfloor (6n-12)/n \rfloor) \geq 4, \text{ and both have } i; \text{ for some } 3 \leq i \leq k+2, \]
Laplacian eigenvalues are greater than \( d \) and the same sum of first \( i \)-greatest Laplacian eigenvalues; by Lemma 5, we can conclude that \( LE(G, \sigma) = LE(G, +) \). Therefore, case (1) holds.

As for the proof of case (2), it is similar to that of case (1). We just present the Laplacian spectrum of \( (G, \sigma) \), which is \( \{n+2, n-1, n, 3+\alpha_1, \ldots, 3+\alpha_s, 3^{(k-2)}, 3^{(k-3)}, 2^{(2)} \} \), and the spectrum of \( L(G, +) \) is \( \{n, 3+\alpha_1, \ldots, 3+\alpha_s, 3^{(k-4)}, 0\} \), where \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s = 0 \) is the Laplacian spectrum of \( H_k \).
And, it is not difficult to verify that \( LE(G, \sigma) = LE(G, +) \) holds.

**Theorem 8.** For each even integer \( n = 2k + 2, k \geq 2 \), let \( G \equiv K_4 \cup (H_{k-2} \cup K_1) \), where \( H_{k-2} \) is an arbitrary graph on \( k-2 \) vertices (connected or disconnected); the sign function \( \sigma \) specifies exactly \( 2k \) edges to be negative, where \( 2k \) edges connect two vertices of \( K_4 \) to \( k \) distinct isolated vertices \( K_1 \) and all remaining edges are positive; then, \( G \) satisfies \( LE(G, \sigma) = LE(G, +) \).

**Proof.** The proof of Theorem 8 is similar to that of Theorem 6. We just present the Laplacian spectrum of \( (G, \sigma) \), which is \( \{n+2, n-2, n-1, 4+\alpha_1, \ldots, 4+\alpha_{k-2}, 4^{(k-2)}, 2^{(2)} \} \), where \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{k-2} = 0 \), is the Laplacian spectrum of \( H_k \). While the Laplacian spectrum of \( (G, +) \) is \( \{n, 4+\alpha_1, \ldots, 4+\alpha_{k-2}, 4^{(k-1)}, 0\} \), and it is not difficult to verify that \( LE(G, \sigma) = LE(G, +) \) holds.

**Example 1.** We give four instances with property \( LE(G, \sigma) = LE(G, +) \) corresponding to Theorems 6, 7 (1) and (2), and 8, respectively, see Figure 1, where bold lines denote positive edges and dotted lines denote negative edges. \( G_1 \equiv K_4 \cup (S_1 \cup 2K_1) \), \( G_2 \equiv K_4 \cup (K_2 \cup 9K_1) \), \( G_3 \equiv K_4 \cup (K_3 \cup 6K_1) \), and \( G_4 \equiv K_4 \cup (S_1 \cup 5K_1) \). Their Laplacian spectra are shown in Table 1.

**Data Availability**
No data were used to support the findings of the study.

**Conflicts of Interest**
The authors declare that they have no conflicts of interest.

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