THE CHROMATIC NUMBER OF DENSE RANDOM BLOCK GRAPHS

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Abstract. The chromatic number \(\chi(G)\) of a graph \(G\), that is, the smallest number of colors required to color the vertices of \(G\) so that no two adjacent vertices are assigned the same color, is a classic and extensively studied topic in computer science and mathematics. Here we consider the case where \(G\) is a random block graph, also known as the stochastic block model. The vertex set is partitioned into \(k \in \mathbb{N}\) parts \(V_1, \ldots, V_k\), and for each \(1 \leq i \leq j \leq k\), two vertices \(u \in V_i, v \in V_j\) are connected by an edge with some probability \(p_{i,j} \in (0,1)\) independently. Our main result pins down the typical asymptotic value of \(\chi(G)\) and establishes the distribution of the sizes of the color classes in optimal colorings. In contrast to the case of a binomial random graph \(G(n,p)\), that corresponds to \(k = 1\) in our model, where the size of an “average” color class in an (almost) optimal coloring essentially coincides with the independence number, the block model reveals a far more diverse and rich picture: there, in general, the “average” class in an optimal coloring is a convex combination of several “extreme” independent sets that vary in total size as well as in the size of their intersection with each \(V_i\), \(1 \leq i \leq k\).

1. Introduction & Main Result

Chromatic Number of Random Graphs. Given a graph \(G\), its chromatic number \(\chi(G)\) is defined as the smallest number of colors required for coloring the vertices such that no two adjacent vertices of \(G\) are assigned the same color. The chromatic number is a central parameter in graph theory with applications in various areas. Moreover, the chromatic number of a binomial random graph \(G(n,p)\), in which each possible edge between any two vertices is included independently with probability \(p\), is among the most studied properties in random graph theory. Since the seminal papers of Erdős and Rényi [14, 15], understanding properties of the distribution of \(\chi(G(n,p))\) has been a fundamental problem. In a breakthrough paper from 1978, Bollobás [5] obtained the first asymptotically tight result: he established that for \(p \in (0,1)\), with high probability (w.h.p. for short)\(^1\) as \(n \to \infty\)

\[
\chi(G(n,p)) = (1 + o(1)) \frac{n}{c(p) \log n}, \quad \text{where} \quad c(p) = \frac{2}{\log(1-p)}. \tag{1}
\]

Logarithms are always natural logarithms in this paper. Actually, and since this will become important later, much more than (1) is true. In particular, it has long been known that w.h.p. the independence number \(\alpha(G(n,p))\), that is, the size of the largest independent set, in \(G(n,p)\) equals 

\[(1 + o(1))c(p) \log n \quad (\text{see [26]}).\]

The main contribution of [5] was to establish that w.h.p. essentially all vertices of \(G(n,p)\) can be covered by independent sets that are essentially as large as \(\alpha(G(n,p))\). In other words, w.h.p. any (almost) optimal coloring of \(G(n,p)\) consists of color classes with asymptotic size \(c(p) \log n\) that cover \(n - o(n/\log n)\) vertices.

The aforementioned paper of Bollobás initiated a long line of research that is concerned with studying various properties of the distribution of the chromatic number. The currently most accurate result about the asymptotic value of \(\chi(G(n,p))\) for \(p \in (0,1)\) is due to Heckel [18], who

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\(^{1}\)We say that a series of events \(E_1, E_2, \ldots\) holds with high probability, if \(\lim_{n\to\infty} \Pr[E_n] = 1\).
improved previous results by several authors, e.g. [16, 27, 28, 31], and where she showed upper and lower bounds for $\chi(G(n,p))$ that are within $o(n/\log^2 n)$; note that (1) gives bounds within $o(n/\log n)$ only.

Apart from the probable asymptotic value of $\chi(G(n,p))$, other parameters of it have been of considerable interest and difficulty. Most notably, the question about the concentration of $\chi(G(n,p))$, that is, the smallest size of an interval in which $\chi(G(n,p))$ is located w.h.p. has been in the focus since the seminal papers of Erdős and Rényi (see also [6]). In a recent remarkable breakthrough, Heckel [19] showed polynomial non-concentration bounds for $\chi(G(n,p))$, thus answering a long-standing open question.

Moreover, the regime of sparse random graphs, that is, when $p = p(n) \to 0$ is a function of $n$, has been studied intensively. Luczak [24] showed tight asymptotic bounds, proving that $\chi(G(n,p)) = (1 + o(1))np/(2\log(np))$ for $p = \omega(1/n)$. Alon and Krivelevich [3], improving on previous results of Shamir and Spencer [33] and Luczak [25], proved that the chromatic number should be concentrated on at most two consecutive integers for $p < n^{-1/2-\varepsilon}$; Coja-Oghlan, Panagiotou, and Steger [10] determined an explicit interval of length three that contains $\chi(G(n,p))$ w.h.p. for $p \leq n^{-3/4-\varepsilon}$. Achlioptas and Naor [2] brought possible values for the chromatic number down to two explicit values for $p = d/n$, where $d$ is fixed. Coja-Oghlan and Vilenchik [11] determined the threshold for colorability with a given number of colors for almost all values of $d$, but determining the exact threshold for all values of $d$ still remains open.

Stochastic Block Model. In this paper we study the chromatic number of random graphs in the so-called stochastic block model (also known as the planted partition model). The model is a generalization of $G(n,p)$ and is defined as follows. Given $k \in \mathbb{N}$, let $P = (p_{ij})_{i,j \in [k]}$ be a symmetric matrix with all entries $p_{ij} \in (0,1)$. Moreover, let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a vector of $(1)$-norm $|\alpha| = 1$ and with all $\alpha_i \in (0,1]$. For an integer $n \in \mathbb{N}$ we let $G(n,\alpha,P)$ be a random graph $(V,E)$ obtained as follows. The vertex set $V = V_1 \cup \cdots \cup V_k$ consists of $k$ disjoint parts such that $|V_i| = \alpha_i n$ for every $i \in [k]$ and $\sum_{i \in [k]} \alpha_i n = n$. Furthermore, for $i,j \in [k]$, two distinct vertices $u \in V_i$ and $v \in V_j$ form an edge $uv \in E$ with probability $p_{ij}$ independently. Throughout the paper we think of $k \geq 1$ as a fixed integer, i.e. the number of parts $V_i$ is fixed and independent of $n$, and $P$ as a fixed matrix, that is, we only consider (dense) graphs with w.h.p. $\Omega(n^2)$ edges. We call $G \sim G(n,\alpha,P)$ a random block graph. Clearly, such a graph model is a direct generalization of the Erdős-Rényi binomial random graph $G(n,p)$, which is obtained by choosing $k = 1$.

The stochastic block model is rather flexible and it enables us to describe a variety of situations; it is very much interconnected with the clustering problem, where we want to partition the vertices of a given graph into “strongly connected” parts with “weak” inter-class interactions. It is thus no surprise that it appears as a natural model in several contexts, for example in statistics, machine learning, physics, and computer science. Its applications range from social networks to image processing and to genetics, see e.g. [30, 32, 34] for some influential papers in this context, and various properties of the model have been studied in physics [12, 20] and mathematics and computer science [7, 8, 9, 21, 29]. For further history, reference, and discussion, we refer to the amazing survey [1].

Recently there have been several papers that establish generalizations of well-known results about properties of the binomial random graph in the more general stochastic block model. For instance, Hamiltonicity [4] and the size of the largest independent set/clique [13] (in a richer model that we will also discuss). Our focus here is to determine the asymptotic value of the chromatic number of random block graphs. As we will see shortly, a direct consequence of the methods we employ are precise bounds on the types (sizes) of independent sets these random graphs have.
1.1. Main results. As we previously mentioned, an (almost) optimal coloring of the binomial random graph $G(n, p)$ for $p \in (0, 1)$ typically has a rather simple structure and can be constructed greedily: almost all $n$ vertices are covered by independent sets that are of nearly maximum size, that is, of size roughly $c(p) \log n$, where $c(p) = -2/\log(1 - p)$ is defined in (1). As we shall see, the structure of optimal colorings is more intricate and diverse when we consider the broader model of random block graphs.

In order to formulate our results we first introduce some notation. Let $k \in \mathbb{N}$ and $G = G(n, \alpha, P)$ where $\alpha \in \mathbb{R}^k$ and $P \in \mathbb{R}^{k \times k}$. Before we consider the chromatic number of $G$ we look at the distribution of independent sets in $G$, as the two parameters are inherently dependent on each other. For a vector $c \in \mathbb{R}^k$ and $I \subseteq [k]$ define the map

$$g(c, I) := \sum_{i \in I} c_i + \frac{1}{2} \sum_{i,j \in I} c_i c_j \log(1 - p_{ij}).$$

The quantity has a natural interpretation. Let $X_{c,I}$ be the number of independent sets in $G$ that intersect each $V_i$, $i \in I$, at roughly $c_i \log n$ vertices. Then, as it turns out (see Section 2)

$$\frac{1}{\log^2 n} \log \mathbb{E}[X_{c,I}] = g(c, I) + o(1).$$

From Markov’s inequality we readily obtain that w.h.p. $X_{c,I} = 0$ if $g(c, I) < 0$. Thus, let $A \subseteq \mathbb{R}^k$ be defined as

$$A := \{ c \in \mathbb{R}^k_{\geq 0} : g(c, I) \geq 0 \text{ for all } \emptyset \neq I \subseteq [k] \},$$

where $[k] = \{1, \ldots, k\}$. As we will show, the vectors in $\log n \cdot A$ essentially contain exactly all admissible types of independent sets that we encounter in $G$ w.h.p., where “type” refers to a vector $t = (t_1, \ldots, t_k)$, and an independent set is of type $t$ if it intersects each $V_i$, $i \in [k]$, in $t_i$ vertices. Actually, we will show in Section 2 even more, namely that w.h.p. every sufficiently large subset of $G$ contains an independent set of any type in $(1 - o(1)) \log n \cdot A$; this paves the way for greedily coloring $G$ with great flexibility. Our main result reads as follows.

**Theorem 1.1.** Let $k \geq 1$, $\alpha \in (0, 1]^k$ with $|\alpha| = 1$, and let $P = (p_{ij})_{i,j \in [k]}$ be a symmetric matrix with all $p_{ij} \in (0, 1)$. Consider a random block graph $G \sim G(n, \alpha, P)$. Then w.h.p.

$$\chi(G) = (1 + o(1)) \frac{n}{c^* \log n},$$

where $c^*$ is given as

$$c^* = c^*(\alpha, P) = \max \{ |c| : c \in \text{conv}(A) \cap \{ t \cdot \alpha : t \in \mathbb{R}_{\geq 0}^k \} \}. \quad (4)$$

Let us for a moment dwell on the definition of $c^*$. Consider an optimal coloring of a typical instance of $G$, which is nothing else than a partition of the vertices of $G$ into independent sets $S_1, \ldots, S_{\chi(G)}$. We assume that $\chi(G) = n/(c \log n)$ and want to determine $c$. As already mentioned, the $S_i$’s are in $\log n \cdot A$. Moreover, each $|V_i|$ can be recovered as $\sum_{1 \leq j \leq \chi(G)} |V_i \cap S_j|$. Thus, the average intersection of the color classes with the part $V_i$ is $\overline{1} := \alpha_i n / \chi(G) = c \alpha_i \log n$. In conclusion, in any (in particular, in an optimal) coloring, the average intersection of the color classes with each $V_i$ is proportional to $\log n \cdot \alpha$ and is furthermore a convex combination of some types in $\log n \cdot A$. Hence, it comes as no surprise that in order to determine $\chi(G)$, $c$ should be chosen to be maximal under these side constraints, and this is exactly (4).

In the proof, which is conducted in Section 3, we greedily construct an explicit coloring of $G$ with the claimed number of colors, that is, we carefully pick different types of independent sets from $\log n \cdot A$ and cover with them all but at most $o(n / \log n)$ vertices. In particular, depending very much on the shape of $A$, this may result in different types of colorings: we may end up coloring all parts $V_1, \ldots, V_k$ independently with different colors, or at the other end of the spectrum, we may choose just a single type $t$ such that $t_i/t_j \sim \alpha_i/\alpha_j$ for all $i, j \in [k]$ and thus
cover (almost) all vertices just with sets of type $t$. These two cases are not exhaustive and as it turns out, in general we may cover (almost) all vertices with independent sets of $k + 1$ different types.

As a remark, by taking $k = 1$, $\alpha_1 = 1$, and $p_{11} = 1/2$, we recover the classic result of Bollobás [5]. In Section 4 we present various applications of the result. Among others, we study the case $k = 2$ in detail and characterize explicitly in all cases the structure of the optimal colorings. Moreover, for general $k \in \mathbb{N}$ we characterize the cases in which

$$\chi(G) \sim \sum_{1 \leq i \leq k} \chi(G[V_i]),$$

that is, an optimal coloring of $G$ is essentially obtained by coloring each of the $k$ subgraphs individually; as we show, this happens if and only if $p_{ij} \geq 1 - \sqrt{(1 - p_i)(1 - p_j)}$. Our last application concerns one more relevant case, namely when there is some homogeneity with respect to the edge probabilities. In particular, we assume that all inter-class probabilities $p_{ij}$, for $i \neq j$, are equal, and all intra-class probabilities are also equal. In that case, we determine explicitly the asymptotic value of the chromatic number.

Note that we determine $\chi(G)$ in the case when $p_{ij}$’s and $\alpha_i$’s are fixed and independent of $n$. Extending this for $p_{ij} = p_{ij}(n)$ and $\alpha_i = \alpha_i(n)$ remains an open problem for further research.

1.2. Graph limits. In an even more general setting we may look at limits of dense graph sequences. For a detailed introduction to the topic we refer the reader to the wonderful book of Lovász [22]. A graphon is a symmetric measurable function $W : \Omega \times \Omega \rightarrow [0, 1]$, where $\Omega$ is a probability space. In order to show that every graphon $W$ is attained as a limit of a sequence of finite graphs, Lovász and Szegedy [23] introduced a random graph model $G(n, W)$ defined as follows. The vertex set of a graph $G \sim G(n, W)$ is $[n]$. In order to sample a graph $G \sim G(n, W)$, one first generates a sequence of $n$ points $x_1, \ldots, x_n \in \Omega$ and subsequently makes $x_i x_j$ an edge with probability $W(x_i, x_j)$ independently of everything else. Clearly, the class of stochastic block models $G(n, \alpha, P)$ is (essentially) a special case of $G(n, W)$.

As in the block model, we define a continuous version of the function $g$ in (2), where we replace sums by integrals and values by densities. More specifically (and compare also with [13]), define for $I \subseteq \Omega$ and a non-negative $L^1$-function $c$,

$$\tilde{g}(c, I) = \int_{x \in I} c(x) d\nu + \frac{1}{2} \int_{(x,y) \in I \times I} c(x)c(y) \log(1 - W(x,y)) d\nu^2.$$ 

Moreover, the set of “admissible types” is defined analogously by

$$\tilde{A} = \{ c : c \text{ is a non-negative } L^1 \text{-function on } \Omega \text{ such that } \tilde{g}(c, I) \geq 0 \text{ for all } I \subseteq \Omega \}. $$

One would guess that the chromatic number of $G \sim G(n, W)$ is then obtained in a way similar to the one from Theorem 1.1.

**Conjecture 1.2.** Let $W : \Omega \times \Omega \rightarrow [0, 1]$ be a graphon with essential infimum in $(0, 1)$ and consider a graph $G \sim G(n, W)$. Then w.h.p.

$$\chi(G) = (1 + o(1)) \frac{n}{c^* \log n},$$

where $c^*$ is given as

$$c^* = c^*(W) = \sup \{ \|c\|_1 : c \in \text{conv}(\tilde{A}) \cap \{ t \cdot \nu : t \in \mathbb{R}_{\geq 0} \} \}. $$

However, this seemed to be out of reach for our techniques and we leave it as a question for further research.
2. Independent Sets

In this section we study the distribution of independent sets in the random block graph \( G(n, \alpha, P) \). This serves as a main ingredient towards deriving the desired bounds on the chromatic number later on. Throughout, for \( t \in \mathbb{N}_0^k \) we say that a set \( S \subseteq V(G) \) is a \( t \)-set or of type \( t \) in \( G(n, \alpha, P) \) if \( S \cap V_i = t_i \) for every \( i \in [k] \). Vectors are denoted by lower-case bold letters. Given vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{N}_0^k \), we write \( \mathbf{u} \leq \mathbf{v} \) if \( u_i \leq v_i \) for all \( i \in [k] \).

Our starting point and main technical tool in this section is a simple consequence of Janson’s inequality, see [17, Section 21.6], which we restate in a variant convenient for our application.

**Theorem 2.1** (Janson’s inequality). Let \( k \in \mathbb{N}, \alpha \in (0,1)^k \) with \( |\alpha| = 1 \), and let \( P = (p_{ij})_{i,j \in [k]} \) be a symmetric matrix with all \( p_{ij} \in (0,1) \). Consider a family \( \{ S_i \}_{i \in \mathbb{I}} \) of subsets of the vertex set \( [n] \) and let \( G \sim G(n, \alpha, P) \). For each \( i \in \mathbb{I} \), let \( X_i \) denote the indicator random variable for the event \( \{ S_i \text{ is an independent set in } G \} \) and, for each ordered pair \( (i,j) \in \mathbb{I} \times \mathbb{I} \), write \( X_i \sim X_j \) if \( E(S_i) \cap E(S_j) \neq \emptyset \). Let

\[
X := \sum_{i \in \mathbb{I}} X_i, \quad \mu := \mathbb{E}[X], \quad \text{and} \quad \Delta := \sum_{(i,j) \in \mathbb{I} \times \mathbb{I}} \mathbb{E}[X_i X_j].
\]

Then

\[
\Pr[X = 0] \leq e^{-\mu^2/(2\Delta)}.
\]

The next lemma is the central result of this section. In simple terms, it states that w.h.p. whenever we take a sufficiently large subset of vertices of \( G \sim G(n, \alpha, P) \), there is an independent \( t \)-set, for any \( t \) that “falls” within the set \( \mathcal{A} \), that is, \( t \in (1 - o(1)) \log n \cdot \mathcal{A} \). This lemma alone allows us to greedily take out independent sets (color classes) from \( G(n, \alpha, P) \) as long as there is some “large” set of vertices remaining in each \( V_i \).

**Lemma 2.2.** Let \( G \sim G(n, \alpha, P) \). Let \( s \in \mathbb{N}_0^k \) be such that \( s_i \geq \alpha_i n / \log^2 n \) for all \( i \in [k] \) and \( S \subseteq V(G) \) be an \( s \)-set. Then, for every \( t \in (\log(n) - 7 \log \log n) \cdot \mathcal{A} \cap \mathbb{N}_0^k \) and \( X_t \) being the random variable denoting the number of independent \( t \)-sets in \( G[S] \)

\[
\Pr[X_t = 0] = e^{-\Omega(n^2 / \log^8 n)}.
\]

**Proof.** Let us write \( X_t = \sum_{I \in \mathcal{S}} X_I \), where \( \mathcal{S} \) is the family of all subsets of \( S \) that intersect each \( V_i \), \( i \in [k] \), in exactly \( t_i \) vertices, and \( X_I \) is an indicator random variable for the event that \( I \) is an independent set in \( G[S] \). Set now \( \mu := \mathbb{E}[X_I] \) and \( \Delta := \sum_{(I,J) \in \mathcal{S} \times \mathcal{S}} \mathbb{E}[X_I X_J] \). This puts us directly into the setup of Janson’s inequality (Theorem 2.1) with the goal to show

\[
\Pr[X_t = 0] \leq e^{-\mu^2/(2\Delta)} \leq e^{-\Omega(n^2 / \log^8 n)}.
\]

The whole proof boils down to showing that the \( \Delta \) term can be bounded by

\[
\Delta = O\left( \mu^2 \cdot \frac{\log^8 n}{n^2} \right),
\]

which is what we accomplish in the remainder.

Firstly, it is convenient to determine \( \mu = \mathbb{E}[X_I] \) as it helps simplify some calculations. For each \( i \in [k] \), there are \( \binom{s_i}{t_i} \) choices for the intersection of a \( t \)-set with \( S \cap V_i \). Additionally, in order for such a \( t \)-set to be an independent set in \( G[S] \), none of the \( \binom{s_j}{2} \) pairs can form an edge, which happens with probability \( (1 - p_{ij})^{\binom{s_j}{2}} \). Lastly, no two vertices \( u, v \) in the \( t \)-set with \( u \in V_i \) and \( v \in V_j \) can form an edge in \( G[S] \), which happens with probability \( (1 - p_{ij})^{s_is_j} \). Putting all of this together, we directly get

\[
\mu = \prod_{1 \leq i \leq k} \frac{s_i}{t_i} \cdot \prod_{1 \leq i \leq k} (1 - p_{ij})^{\binom{s_i}{2}} \cdot \prod_{1 \leq i < j \leq k} (1 - p_{ij})^{s_is_j}. \tag{6}
\]
We now turn our attention in bounding the $\overline{\sum}$ term as promised. Note that the $\overline{\sum}$ term depends only on those sets which “overlap” in at least two vertices. We denote this “overlap” vector by $\mathbf{o}$ and note that $\mathbf{o} \leq \mathbf{t}$ and $|\mathbf{o}| \geq 2$. Each $o_i$, for $i \in [k]$, measures the “overlap” of the sets inside of the part $V_i$. For a fixed overlap vector $\mathbf{o}$ and a fixed $i \in [k]$, there are at most

$$\left(\binom{s_i}{t_i}\right)\left(\binom{t_i}{o_i}\right)\left(\binom{s_i-t_i}{t_i-o_i}\right),$$

choices for two $t$-sets which intersect in exactly $o_i$ vertices within $V_i$. Similarly as above when deriving the expectation, the probability of both such $t$-sets being independent is given by a term for intra-class edges and inter-class edges and is exactly

$$(1 - p_i)^{2\binom{t_i}{2} - \binom{o_i}{2}} \cdot \prod_{j \neq i} (1 - p_{ij})^{2t_j - o_i o_j}.$$

Thus, the contribution to the $\overline{\sum}$ term of a fixed overlap vector $\mathbf{o}$ is given by

$$\prod_{1 \leq i \leq k} \left(\binom{s_i}{t_i}\right)\left(\binom{t_i}{o_i}\right)\left(\binom{s_i-t_i}{t_i-o_i}\right) \cdot \prod_{1 \leq i \leq k} (1 - p_i)^{2\binom{t_i}{2} - \binom{o_i}{2}} \cdot \prod_{1 \leq i < j \leq k} (1 - p_{ij})^{2t_j - o_i o_j}.$$ 

Then, by summing up over all choices of $\mathbf{o}$, we get

$$\overline{\sum} = \sum_{\mathbf{o} \leq \mathbf{t}, |\mathbf{o}| \geq 2} \left(\prod_{1 \leq i \leq k} \left(\binom{s_i}{t_i}\right)\left(\binom{t_i}{o_i}\right)\left(\binom{s_i-t_i}{t_i-o_i}\right) \cdot \prod_{1 \leq i \leq k} (1 - p_i)^{2\binom{t_i}{2} - \binom{o_i}{2}} \cdot \prod_{1 \leq i < j \leq k} (1 - p_{ij})^{2t_j - o_i o_j}\right)$$

$$= \mu^2 \cdot \sum_{\mathbf{o} \leq \mathbf{t}, |\mathbf{t}| \geq 2} \left(\prod_{1 \leq i \leq k} \left(\binom{t_i}{o_i}\right)\left(\binom{o_i}{t_i-o_i}\right) \cdot \prod_{1 \leq i \leq k} (1 - p_i)^{\binom{o_i}{2}} \cdot \prod_{1 \leq i < j \leq k} (1 - p_{ij})^{-o_i o_j}\right)$$

$$= \mu^2 \cdot \sum_{\mathbf{o} \leq \mathbf{t}, |\mathbf{t}| \geq 2} f(\mathbf{o}).$$

To complete the proof we aim to give a bound on $\sum_{\mathbf{o} \leq \mathbf{t}, |\mathbf{t}| \geq 2} f(\mathbf{o})$ of the order $\log^8 n/n^2$. We first show this the whole sum is essentially dominated by those $f(\mathbf{o})$ with $|\mathbf{t}| = 2$.

Consider an arbitrary $\mathbf{o}$ and let $\mathbf{e}_z \in \{0, 1\}^k$ be a unit vector with $(\mathbf{e}_z)_z = 1$ for some $z \in [k]$. We derive

$$\frac{f(\mathbf{o})}{f(\mathbf{e}_z + \mathbf{o})} \geq \frac{(t_z)^{s_z-t_z}(t_z)^{s_z-t_z}}{(t_z+1)^{s_z-t_z}(t_z+1)(t_z)} \cdot (1 - p_z)^{\binom{o_z+1}{2} - \binom{o_z}{2}} \cdot \prod_{j \neq z} (1 - p_{jz})^{o_j - o_z o_j},$$

Using the fact that $t_z \leq c(p_z \log n)$ (by definition (3) of $A$) and $s_z \geq o_z n/\log^2 n$, this can be simplified (by standard manipulations of binomial coefficients) to

$$\frac{f(\mathbf{o})}{f(\mathbf{e}_z + \mathbf{o})} \geq \frac{\delta n}{\log^4 n} \cdot (1 - p_z)^{o_z} \cdot \prod_{j \neq z} (1 - p_{jz})^{o_j},$$

for some constant $\delta > 0$ which depends only on $\alpha_i$’s and $p_i$’s. Let $\mathbf{t} \leq \mathbf{t}$ be such that $|\mathbf{t}| = 2$ and $\mathbf{t} = \mathbf{e}_x + \mathbf{e}_y$, for some (not necessarily distinct) $x, y \in [k]$. Then from (7), for every $\mathbf{t} \leq \mathbf{o} \leq \mathbf{t}$ with $|\mathbf{t}| \geq 3$, we obtain

$$\frac{f(\mathbf{t})}{f(\mathbf{t})} \geq \left(\frac{\delta n}{\log^4 n}\right)^{|\mathbf{t}| - 2} \cdot \prod_{1 \leq i \leq k} (1 - p_i)^{\binom{o_i}{2}} \cdot \prod_{1 \leq i < j \leq k} (1 - p_{ij})^{n o_i o_j}.$$
Therefore, which lie “outside” of $\mathcal{O}$. On the other hand, since $d \leq o \leq t$ and $t \in (1 - \varepsilon) \log n \cdot \mathcal{A}$, using the definition of $\mathcal{A}$, we get

$$
\sum_{1 \leq i \leq k} \frac{d_i}{(1 - \varepsilon)^2 \log n} + \frac{1}{2} \sum_{1 \leq i < j \leq k} \frac{d_i d_j}{(1 - \varepsilon)^2 \log^2 n} \log(1 - p_{ij}) \geq 0.
$$

Multiplying the whole inequality by $(1 - \varepsilon)^2 \log^2 n$ gives

$$
h(d) := \sum_{1 \leq i \leq k} d_i \log n + \frac{1}{2} \sum_{1 \leq i \leq k} d_i^2 \log(1 - p_i) + \sum_{1 \leq i < j \leq k} d_i d_j \log(1 - p_{ij}) \geq \varepsilon |d| \log n. \quad (9)
$$

On the other hand, since $d = o - \bar{o}$ and $\bar{o} = e_x + e_y$, we have

$$
h(o) - 2 \log n \geq h(d) - 2 \log n + \frac{1}{2} \left(2o_x \log(1 - p_x) + 2o_y \log(1 - p_y)\right) + \sum_{j \neq x} o_x o_j \log(1 - p_{xj}) + \sum_{j \neq y} o_x o_j \log(1 - p_{yj}).
$$

Therefore, $h(o) - 2 \log n \geq h(d) - O(|o|)$. By plugging in (9) into (8), and as $|o| \geq 3$, we get

$$
\frac{f(o)}{f(o)} \geq \exp \left( (|o| - 2)(\log 4 - 4 \log \log n) + 7(|o| - 2) \log \log n - O(|o|) \right) \geq \log^3(|o| - 2) n + o(\log^3|o| - 2).\n$$

Clearly, by the fact that $t \in (1 - \varepsilon) \log n \cdot \mathcal{A}$ and as $\mathcal{A}$ is bounded, the norm of $o$ is at most $C \log n$ for some (large) constant $C > 0$ depending only on $\alpha_i$’s and $p_i$’s. (We may, e.g., choose $C$ as $C = \sum_{1 \leq i \leq k} c(p_{ii})$. This finally implies

$$
\sum_{0 < t < |o| \geq 2} f(o) = O \left( \sum_{1 \leq i \leq k} \frac{n}{\log n} \sum_{|\bar{o}| = 2} f(\bar{o}) \right) = O \left( \sum_{|\bar{o}| = 2} f(\bar{o}) \right).
$$

Hence, it remains to show that $f(\bar{o}) = O(\log^8 n/n^2)$. Note that each such $\bar{o}$ with $|\bar{o}| = 2$ can be represented as $\bar{o} = e_x + e_y$ for some $x, y \in [k]$, where $e_i$ stands for a unit vector $e_i \in \{0, 1\}^k$ with $(e_i)_i = 1$. In case $x \neq y$ with $\bar{o}_x = 1$ and $\bar{o}_y = 1$, we have

$$
f(\bar{o}) \leq t_x \left( \frac{s_x - t_x}{t_x - 1} \right) \cdot t_y \left( \frac{s_y - t_y}{t_y - 1} \right) \cdot (1 - p_{xy})^{-1}.
$$

Simple manipulations with binomial coefficients give

$$
f(\bar{o}) \leq \frac{(t_x t_y)^2}{s_x s_y (1 - p_{xy})}.
$$

On the other hand, if $x = y$ and $\bar{o}_x = 2$, then

$$
f(\bar{o}) = \left( \frac{t_x}{2} \right) \left( \frac{s_x - t_x}{t_x - 2} \right) \cdot (1 - p_x)^{-1} \leq \frac{t_x^4}{s_x^2 (1 - p_x)}.
$$

Recalling that $s_i \geq \alpha_i n / \log^2 n$ and $t_i = O(\log n)$ shows the desired bound and completes the proof of the lemma.}

The next lemma establishes the fact that $G(n, \alpha, P)$ w.h.p. contains no independent $t$-sets which lie “outside” of $\mathcal{A}$. We start by making a useful observation about $\mathcal{A}$.
\textbf{Claim 2.3.} There exists a constant $C = C(P) > 0$ such that any vector $c \in \mathbb{R}^k_{\geq 0}$ with $|c| \leq C$ is contained in $\mathcal{A}$.

\textbf{Proof.} For any $I \subseteq [k]$, we have
\[ g(c, I) \geq \sum_{i \in I} c_i + \frac{1}{2} \sum_{i \neq j} c_i c_j \log(1 - \max_{i,j} p_{ij}) \geq |c| + \frac{1}{2} |c|^2 \log(1 - \max_{i,j} p_{ij}). \]
It follows that $g(c, I) \geq 0$ for all $I \subseteq [k]$ if $|c| \leq C := -2/\log(1 - \max_{i,j} p_{ij})$. \hfill \Box

\textbf{Lemma 2.4.} Let $G \sim G(n, \alpha, P)$. Let $X$ be the random variable denoting the number of independent $t$-sets in $G$ with $t \in \mathbb{N}_0^k \setminus (\log n \cdot \mathcal{A})$. Then
\[ \Pr[X > 0] \leq O(n^{-1}). \]
\textbf{Proof.} Fix any $t \not\in \mathbb{N}_0^k \setminus (\log n \cdot \mathcal{A})$ and let $X_t$ count the number of independent $t$-sets for that fixed $t$. Observe that, by the definition (3) of $\mathcal{A}$, there is a $I \subseteq [k]$ such that $g((t_i/\log n)_{i \in I}, I) < 0$. Let $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_k)$ be defined as
\[ \tilde{t}_i = \begin{cases} t_i, & \text{if } i \in I, \\ 0, & \text{otherwise}, \end{cases} \]
and let $X_{\tilde{t}}$ be the random variable denoting the number of independent $\tilde{t}$-sets in $G$. Then
\[ \mathbb{E}[X_{\tilde{t}}] = \prod_{i \in I} \left( \alpha_i n \right)^{\tilde{t}_i} \cdot \prod_{i \in I} (1 - p_i)^{\tilde{t}_i} \cdot \prod_{i,j \in I, i \neq j} (1 - p_{ij})^{\tilde{t}_i \tilde{t}_j}. \]
Since $\alpha_i < 1$, using that $\binom{n}{t} \leq \left(\frac{en}{t}\right)^t$, this can further be bounded by
\[ \mathbb{E}[X_{\tilde{t}}] \leq \exp \left( \sum_{i \in I} \tilde{t}_i \log n - \sum_{i \in I} t_i \log t_i + \frac{1}{2} \sum_{i \neq j} \tilde{t}_i \tilde{t}_j \log(1 - p_i) + \sum_{i,j \in I, i \neq j} t_i t_j \log(1 - p_{ij}) \right). \]
Recall, $g((t_i/\log n)_{i \in I}, I) < 0$, and thus
\[ \sum_{i \in I} \frac{t_i}{\log n} - \frac{1}{2} \sum_{i \neq j} \frac{t_i t_j}{\log^2 n} \log(1 - p_i) + \sum_{i,j \in I, i \neq j} \frac{t_i t_j}{\log^2 n} \log(1 - p_{ij}) < 0, \]
which yields
\[ \frac{1}{2} \sum_{i \in I} \tilde{t}_i \log(1 - p_i) + \sum_{i,j \in I, i \neq j} t_i t_j \log(1 - p_{ij}) < -\sum_{i \in I} \tilde{t}_i \log n. \]
By Jensen’s inequality and the fact that $|\tilde{t}| = \Omega(\log n)$ by Claim 2.3, it further follows that
\[ \mathbb{E}[X_{\tilde{t}}] \leq \exp \left( -\frac{|\tilde{t}|}{k} \log(\tilde{t}) \right) \leq O(n^{-1}). \]
Hence, by Markov’s inequality $\Pr[X_{\tilde{t}} > 0] \leq \mathbb{E}[X_{\tilde{t}}] \leq O(n^{-1})$.

Let $\mathcal{M}$ be the set of minimal $t$’s and $m$’s for some $m \in \mathcal{M}$. Similarly as in the proof of Lemma 2.2 (as $\mathcal{A}$ is bounded), we know $|m| \leq C \log n$ for some $C > 0$ depending only on $\alpha_i$’s and $p_i$’s. Therefore, $|\mathcal{M}| \leq (C \log n)^k$ and by a union bound over all vectors in $\mathcal{M}$, we get
\[ \Pr[X > 0] \leq \sum_{m \in \mathcal{M}} \Pr[X_m > 0] \leq O(n^{-1}). \]
In particular, w.h.p. there is also no independent $t$-set for any $t \in \mathbb{N}_0^k \setminus (\log n \cdot \mathcal{A})$, which completes the proof. \hfill \Box
3. Chromatic Number

In this section we provide the proof of our main theorem. Recall, the goal is to give a precise (up to lower order terms) bound on the chromatic number of a random block graph $G(n, \alpha, P)$. For the convenience of the reader we restate our main result.

**Theorem 1.1.** Let $k \geq 1$, $\alpha \in (0, 1]^k$ with $|\alpha| = 1$, and let $P = (p_{ij})_{i,j \in [k]}$ be a symmetric matrix with all $p_{ij} \in (0, 1)$. Consider a random block graph $G \sim G(n, \alpha, P)$. Then w.h.p.

$$\chi(G) = (1 + o(1)) \frac{n}{c^* \log n},$$

where $c^*$ is given as

$$c^* = c^*(\alpha, P) = \max \{|c| : c \in \text{conv}(A) \cap \{t \cdot \alpha : t \in \mathbb{R}_{\geq 0}\}\}. \tag{4}$$

**3.1. Upper bound.** Set $\varepsilon = 7 \log \log n / \log n$ and let $c^* = \arg \max \{|c| : c \in \text{conv}(A) \cap \{t \cdot \alpha : t \in \mathbb{R}_{\geq 0}\}\}$. While constructing a coloring to give an upper bound on $\chi(G)$ we need to distinguish two cases: $c^* \in A$ and $c^* \notin A$.

Assume the former, that is, $c^* \in A$ and let $t = (1 - \varepsilon) \log n \cdot c^*$. The probability that there is a set $S \subseteq V(G)$ with $s_i := |S \cap V_i| = \alpha_i n / \log^2 n$ for all $i \in [k]$ and such that $G[S]$ does not contain an independent $t$-set is, by Lemma 2.2 and a union bound over all choices for $S$, at most

$$\prod_{1 \leq i \leq k} \left( \frac{\alpha_i n}{s_i} \right) e^{-\Omega(n^2 / \log^8 n)} \leq e^{\sum_{1 \leq i \leq k} s_i \log n} \cdot e^{-\Omega(n^2 / \log^8 n)} = o(1).$$

As a direct consequence, w.h.p. as long as there are at least $\alpha n$ vertices remaining in every $V_i$, there is an independent $t$-set where $t = (1 - \varepsilon) \log n \cdot c^*$. We construct the coloring in the usual way: repeatedly take out an independent $t$-set and assign all the vertices in it a new color. By the argument above, this is possible until there are $n / \log^2 n$ vertices left in total, at which point we assign to each of those uncolored vertices a color different from all the previously used ones. Therefore, the total number of colors used is at most

$$\frac{n}{|t|} + \frac{n}{\log^2 n} = \frac{n}{|c^*|(\log n - 7 \log \log n)} + \frac{n}{\log^2 n} = (1 + o(1)) \frac{n}{c^* \log n},$$

as claimed.

On the other hand, if $c^* \notin A$, w.h.p. no independent $(c^* \log n)$-set exists in $G$ by Lemma 2.4. In order to circumvent this, we represent $c^*$ as a convex combination

$$c^* = \sum_{1 \leq i \leq k+1} \lambda_i t_i, \tag{10}$$

where $t_i \in A$, $\lambda_i \in [0, 1]$ for all $i \in [k+1]$, and $\sum_{1 \leq i \leq k+1} \lambda_i = 1$.

We then construct a coloring of $G$ as follows. Greedily and sequentially select $\lambda_i n / (c^* \log n)$ independent $(t_i(1 - \varepsilon) \log n)$-sets and assign all the vertices in it a new color. We can indeed do this as even after taking all such sets, the number of vertices remaining in every $V_i$ is

$$\alpha_i n - \sum_{1 \leq j \leq k+1} \frac{\lambda_j n}{c^* \log n} \cdot (t_j)_i \cdot (1 - \varepsilon) \log n = \alpha_i n - \frac{(1 - \varepsilon)n}{c^*} \cdot \sum_{1 \leq j \leq k+1} \lambda_j (t_j)_i = \alpha_i n - \frac{(1 - \varepsilon)n}{c^*} \cdot (c^*)_i = \frac{7\alpha_i n \log n}{\log n},$$

where the last equality follows from the fact that $c^* = c^* \cdot \alpha$ and our choice of $\varepsilon$.

Let $Q_i$ be the set of uncolored vertices in every $V_i$. Since $G[V_i]$ is distributed as $G(\alpha_i n, p_{ij})$ and $Q_i$ is a subset of $V_i$ of size $|Q_i| \geq \varepsilon \alpha_i n$ w.h.p. by Lemma 2.2 it follows that as long as there are at least $\alpha_i n / \log^2 n$ vertices remaining, generously rounding for $\alpha$, we find an independent set of size at least $c(p_i) \log n / 2$ in $G[Q_i]$. We greedily take such sets one by one and assign all
the vertices in each a new color. Lastly, assign every uncolored vertex a new color which was previously unused. Therefore,
\[ \chi(G[Q_i]) \leq \frac{14\alpha_i n \log \log n}{c(p_i) \log^2 n} + \frac{n}{\log^2 n} = o\left(\frac{n}{\log n}\right). \]
Consequently, the number of different colors used for the whole graph \( G \) is at most
\[ \sum_{1 \leq i \leq k+1} \frac{\lambda_i n}{c^* \log n} + \sum_{1 \leq i \leq k} \chi(G[Q_i]) = \frac{n}{c^* \log n} + o\left(\frac{n}{\log n}\right), \]
as \( \sum_{1 \leq i \leq k+1} \lambda_i = 1 \). This confirms the claimed upper bound.

3.2. Lower bound. Set \( N = \chi(G) \) and consider any coloring with color classes \( S_1, \ldots, S_N \). Trivially, for every \( j \in [k] \) we have \( \sum_{1 \leq i \leq N} |S_i \cap V_j| = \alpha_j n \). So by Lemma 2.4 we may assume every color class is an independent \( t \)-set for some \( t \in \log n \cdot \mathcal{A} \). Let \( \overline{t} = \frac{1}{N} \sum_{i \in [N]} t_i \) and note that \( \overline{t} \in \{ t \cdot \alpha \} \) for some \( t \in \mathbb{R}_{\geq 0} \) and \( \overline{t} \in \text{conv}(\mathcal{A}) \). Therefore,
\[ N = \frac{n}{|\overline{t}| \log n} \geq \frac{n}{c^* \log n}, \]
by maximality of \( c^* \) (see (4)). \( \square \)

4. Special Cases

4.1. Two-block case. Throughout this subsection we assume that \( k = 2 \) and try to in detail describe the possible structure of the set \( \mathcal{A} \) and independent sets of \( G(n, \alpha, P) \). Recall, the set \( \mathcal{A} \) is defined in order to describe “feasible” sizes of independent sets the graph \( G(n, \alpha, P) \) can have and in the case \( k = 2 \) is given as follows:
\[ \mathcal{A} = \{ 0 \leq c_1 \leq c(p_1), 0 \leq c_2 \leq c(p_2), \]
\[ c_1 + c_2 + \frac{c_1^2}{2} \log(1 - p_1) + \frac{c_2^2}{2} \log(1 - p_2) + c_1 c_2 \log(1 - p_{12}) \geq 0 \} \]
The first two equations determine the size of the largest independent set inside each of the parts \( V_1 \) and \( V_2 \) on their own by treating them as \( G(\alpha_1 n, p_1) \) and \( G(\alpha_2 n, p_2) \), respectively. The third inequality is what determines the shape of \( \mathcal{A} \).

In particular, having \( p_1 \) and \( p_2 \) fixed, the shape of \( \mathcal{A} \) varies significantly depending on \( p_{12} \). Note that, by using (1), the inequality
\[ c_1 + c_2 + \frac{c_1^2}{2} \log(1 - p_1) + \frac{c_2^2}{2} \log(1 - p_2) + c_1 c_2 \log(1 - p_{12}) \geq 0 \]
can be rewritten as
\[ (c_1 + c_2)\left(1 - \frac{c_1}{c(p_1)} - \frac{c_2}{c(p_2)}\right) + c_1 c_2 \left(\log(1 - p_{12}) + \frac{1}{c(p_1)} + \frac{1}{c(p_2)}\right) \geq 0. \]
Therefore, \( \mathcal{A} \) is convex if and only if \( \log(1 - p_{12}) + 1/c(p_1) + 1/c(p_2) \geq 0 \), or, equivalently, \( p_{12} \leq 1 - \sqrt{(1 - p_1)(1 - p_2)} \). In this case, the constant \( c^* \) defined as
\[ c^* = \max \left\{|c| : c \in \text{conv}(\mathcal{A}) \cap \{\alpha t : t \in \mathbb{R}_{\geq 0}\}\right\} \]
is given by a vector \( c^* \) which actually belongs to the set \( \mathcal{A} \) itself. In other words, independent \( t \)-sets with \( t_1 = c^*_1(1 - o(1)) \log n \) and \( t_2 = c^*_2(1 - o(1)) \log n \) w.h.p. exist in \( G(n, \alpha, P) \), and a coloring can be found by greedily picking these sets as long as possible and then coloring all remaining vertices with a new color.
On the other hand, if \( p_{12} > 1 - \sqrt{(1 - p_1)(1 - p_2)} \), the situation is quite different. In this case, the set \( \mathcal{A} \) is \textit{concave} and the vector \( \mathbf{c}^* \) which determines the constant \( c^* \) does not belong to the set \( \mathcal{A} \), but lies on its convex hull. In particular, it is given by

\[
\mathbf{c}^* = \left( \frac{\alpha_1}{c(p_1)} + \frac{\alpha_2}{c(p_2)} \right) \left( \frac{\alpha_1}{c(p_1)} + \frac{\alpha_2}{c(p_2)} \right)^{-1}.
\]

The optimal coloring is then achieved by looking at the “extremal points” \( \mathbf{c}_1 = (c(p_1), 0) \) and \( \mathbf{c}_2 = (0, c(p_2)) \) and using independent \((c_1(1 - \varepsilon) \log n)\)-sets and \((c_2(1 - \varepsilon) \log n)\)-sets as color classes. Perhaps unsurprisingly, it is shown in Proposition 4.1 below that w.h.p. the chromatic number of \( G(n, \alpha, P) \) is then and only then the sum of the chromatic number of the two parts \( G[V_1] \) and \( G[V_2] \), that is

\[
\chi(G(n, \alpha, P)) = (1 + o(1)) \left( \chi(G(n, \alpha_1, p_1)) + \chi(G(n, \alpha_2, p_2)) \right) = (1 + o(1)) \frac{\alpha_1 c(p_2) + \alpha_2 c(p_1)}{c(p_1) c(p_2)} \frac{n}{\log n}.
\]

For \( p_{12} = 1 - \sqrt{(1 - p_1)(1 - p_2)} \) we have that \( \mathcal{A} \) is both convex and concave since it is limited by a line—so any convex combination of \( \mathbf{c}\)-sets along this line yields a correct chromatic number asymptotically.

The shape of the set \( \mathcal{A} \) for fixed \( 0 < p_1 \leq p_2 < 1 \) and depending on \( p_{12} \in (0, 1) \) is depicted on Figure 1 below.

![Figure 1. Possibilities for \( \mathcal{A} \) in case \( k = 2 \), assuming \( p_1 \leq p_2 \) and then varying \( p_{12} \). The dashed lines correspond to \( c(p_1) \) and \( c(p_2) \), that is, \( c_1 = \frac{2}{\log(1 - p_1)} \) and \( c_2 = \frac{2}{\log(1 - p_2)} \).](image)

Clearly, the constant \( c^* \) and the vector \( \mathbf{c}^* \) that defines it do not only depend on the set \( \mathcal{A} \) but also on the vector \( \alpha \). In Figure 2 we show how the vector \( \mathbf{c}^* \) is defined.

Worth noting is that if \( p_{12} < 1 - \sqrt{(1 - p_1)} \) or \( p_{12} < 1 - \sqrt{(1 - p_2)} \) the inequalities \( c_i < c(p_i) \) become relevant for the shape of \( \mathcal{A} \). What this means for the chromatic number is that if \( p_{12} \leq 1 - \sqrt{(1 - p_1)} \) and \( \alpha_1 \leq c(p_1) \left( \log(1 - p_{12}) - \frac{1}{2} \log(1 - p_2) \right) \),

\[
\chi(G(n, \alpha, P)) = (1 + o(1)) \left( \chi(G(n, c_1, p_1)) + \chi(G(n, c_2, p_2)) \right) = (1 + o(1)) \frac{\alpha_1 c(p_2) + \alpha_2 c(p_1)}{c(p_1) c(p_2)} \frac{n}{\log n}.
\]
then the vertex set $V_1$ has become so sparse and small, that we can color it for free. So the chromatic number of $G(n, \alpha, P)$ is asymptotically bounded by the chromatic number of $G[V_2]$ and is therefore

$$\chi(G(n, \alpha, P)) = (1 + o(1)) \frac{\alpha_2 n}{c(p_2) \log n}.$$  

The equivalent holds if $p_{12} \leq 1 - \sqrt{(1 - p_2)}$ and $\alpha_2 \leq c(p_2) \left( \log(1 - p_{12}) - (1/2) \log(1 - p_1) \right)$ for the vertex set $V_2$.

4.2. Concave set and the union of random graphs. In this subsection we further explore how the random block graph $G(n, \alpha, P)$ behaves in the general case $k \geq 2$ and when

$$p_{ij} > 1 - \sqrt{(1 - p_i)(1 - p_j)} \quad \text{for all } 1 \leq i < j \leq k.$$  

In other words, when all of the bipartite graphs between the parts $G[V_i, V_j]$ are significantly denser than the densest graph $G[V_i]$. In case $k = 2$, this is depicted on the rightmost parts of Figure 1 and Figure 2.

**Proposition 4.1.** With high probability

$$\chi(G(n, \alpha, P)) = (1 + o(1)) \left( \sum_{1 \leq i \leq k} \chi(G(\alpha_i n, p_i)) \right)$$  

if and only if $p_{ij} \geq 1 - \sqrt{(1 - p_i)(1 - p_j)}$ for all $1 \leq i < j \leq k$.

**Proof.** We first show that $p_{ij} \geq 1 - \sqrt{(1 - p_i)(1 - p_j)}$ for all $1 \leq i < j \leq k$ implies the desired bound on the chromatic number. Recall, for a vector $c \in \mathbb{R}^k$ and $I \subseteq [k]$, the function $g(c, I)$ is defined as

$$g(c, I) = \sum_{i \in I} c_i + \frac{1}{2} \sum_{i,j \in I} c_i c_j \log(1 - p_{ij}).$$  

Note that we can reformulate this as

$$g(c, I) = \left( \sum_{i \in I} c_i \right) \left( 1 - \sum_{i \in I} \frac{c_i}{c(p_i)} \right) + \sum_{i \neq j \in I} c_i c_j \left( \log(1 - p_{ij}) + \frac{1}{c(p_i)} + \frac{1}{c(p_j)} \right).$$  

By assumption of $p_{ij} \geq 1 - \sqrt{(1 - p_i)(1 - p_j)}$ we have that the term $\log(1 - p_{ij}) + 1/c(p_i) + 1/c(p_j)$ is negative or zero for all $i, j \in [k]$. Consequently, $A \subseteq B := \{ c \in \mathbb{R}_{\geq 0}^k : 1 - \sum_{i \in [k]} \frac{c_i}{c(p_i)} \geq 0 \}$ and $B$ has as boundary a hyperplane and therefore is a convex set.

For every $i \in [k]$, let

$$t_i = c(p_i) \cdot e_i, \quad h = \sum_{1 \leq j \leq k} \frac{\alpha_j}{|t_j|}, \quad \text{and} \quad \lambda_i = \frac{\alpha_i}{h |t_i|} \quad (12)$$  

$\text{Figure 2.}$ The red line represents the vector $\alpha$. The red point represents its intersection with $\text{conv}(A)$, i.e. the vector $\alpha^*$. 


Note that each $\lambda_i \in [0, 1]$ and $\sum_{1 \leq i \leq k} \lambda_i = 1$. We claim that we can represent $c^*$ as a convex combination of $t_i$'s like

$$c^* = \sum_{1 \leq i \leq k} \lambda_i t_i.$$  

Observe that, by definition

$$c^* = |c^*| = \sum_{1 \leq i \leq k} \lambda_i |t_i| = \sum_{1 \leq i \leq k} \frac{\alpha_i}{h} = \frac{1}{h}, \quad (13)$$

Each $t_i \in A \subseteq B$ is the intersection point of the hyperplane of $B$ with the corresponding axis. So, in fact, $A \subseteq \text{conv}(A) = B$ and since $c^*$ is a linear combination of the $t_i$'s it must lie on the boundary of $B$. That gives the upper bound on $c^*$ and $c^* \in B$ gives the lower bound. The rest now follows from the same strategy as in Theorem 1.1 and the fact that the number of different colors used is at most

$$\sum_{1 \leq i \leq k} \frac{\alpha_i n}{(1 - \varepsilon) |t_i| \log n} + o\left(\frac{n}{\log n}\right) \quad (12)$$

and

$$h \cdot \frac{n}{(1 - \varepsilon) \log n} + o\left(\frac{n}{\log n}\right) \quad (13) = c^* \log n + o\left(\frac{n}{\log n}\right).$$

As for the other direction, whenever there is a $p_{ij} < 1 - \sqrt{(1 - p_i)(1 - p_j)}$ for a fixed $i \neq j \in [k]$ we can color $G(n, \alpha, P)$ in the following way. For every $h \in [k] \setminus \{i, j\}$ color each $V_h$ separately with $\chi(G(\alpha_i n h, p_{hj}))$ colors. Then look at the graph induced by $V_i \cup V_j$. Clearly, $G[V_i \cup V_j]$ is distributed as the block graph $G(\alpha_i n + \alpha_j n j, \alpha^*, P')$, where $\alpha^* = (\alpha_i, \alpha_j)$ and $P' = \left(\frac{p_{ij}}{p_{ij}}, \frac{p_{ji}}{p_{ij}}\right)$.

Our observations from analysing the two-block case in Section 4.1 tell us that we can w.h.p. color this graph with asymptotically less colors than the sum of the chromatic numbers of the parts thus proving the proposition. \hfill \Box

4.3. Convex set with homogeneous balanced partition. The case where $A$ is convex can quickly turn out to be quite complicated. Perhaps one of the cases worth mentioning is when the probability matrix $P$ contains only two different values, one for the diagonal and one for the off-diagonal, and additionally $|V_1| = \cdots = |V_k| = n/k$. So, for $P$ we have

$$p_{ii} = p \quad \forall i \in [k] \quad \text{and} \quad p_{ij} = q \quad \forall i \neq j \in [k],$$

with $p \geq q$. Then $A$ takes a convex shape since all the equations form convex sets and the vector $c^*$ must on the boundary of $A$. In other words, $c^* = (\frac{c}{k}, \ldots, \frac{c}{k})$ where $c^* = |c^*|$ and, in particular, it must hold that

$$\sum_{i \in [k]} \frac{c^*}{k} + \frac{1}{2} \sum_{i \neq j \in [k]} \left(\frac{c^*}{k}\right)^2 \log(1 - p) + \sum_{1 \leq i \leq j \leq k} \left(\frac{c^*}{k}\right)^2 \log(1 - q) \leq 0.$$  

By rearranging we get that $c^* \leq -2\left(\frac{1}{k} \log(1 - p) + \frac{k-1}{k} \log(1 - q)\right)$. Therefore, in case the previous is satisfied with an equality, since $p \geq q$ and due to $\alpha_i = 1/k$, the equations $g(\cdot, I)$ are automatically satisfied for all subsets of the indices $I \subseteq [k]$, and hence $c^*$ is maximal and in $A$.

Applying Theorem 1.1, we get

$$\chi(G(n, \alpha, P)) = \left(1 + o(1)\right) \frac{n}{2 \log n} \left(-\frac{1}{k} \log(1 - p) - \frac{k-1}{k} \log(1 - q)\right).$$

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