Study of $\gamma\pi \to \pi\pi$ *

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Abstract

The problem of $\gamma\pi \to \pi\pi$ is studied using the axial anomaly, elastic unitarity, analyticity and crossing symmetry. Single variable dispersion relation is assumed. Using elastic unitarity relation, an integral equation equation for the lowest partial wave amplitude is obtained. The solution for this integral equation is obtained by an iteration procedure corresponding to that obtained from the vector meson dominance model but with multiple scattering effects taken into account.

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1 Introduction

One of the fundamental calculation in particle theory is the $\pi^0 \to \gamma\gamma$ decay rate $[1]$. It is a combination of Partial Conserved Axial Current (PCAC) and the short distance behavior of Quantum Chromodynamics (QCD):

$$A(\pi^0 \to \gamma\gamma) = iF_{\gamma\gamma}\epsilon^\mu\sigma^\tau\epsilon'^\mu_k\epsilon'^\sigma_{k'}$$  \hspace{1cm} (1)

with

$$F_{\gamma\gamma} = \frac{e^2 N_c}{12\pi^2 f_\pi} = 0.025 GeV^{-1} \hspace{1cm} (2)$$

where $e$ is the electric charge, $f_\pi = 93 MeV$ and $N_c = 3$ is the number of color in QCD. This calculation is valid in a world where the $\pi^0$ is massless. Some corrections have to be made in order to take into account of the finite value of the pion mass. It turns out that the massless pion anomaly formula is in very good agreement with the pion life time data, implying that the correction to the physical pion mass is very small.

Another Axial Anomaly result is the process $\gamma\pi \to \pi\pi$ or its analytical continuation $\gamma \to 3\pi$ $[2]$. This last process requires more corrections because, for practical consideration, measurements will be done at energy far from the chiral limit where the anomaly formula is applicable; furthermore the analytical continuation from one process to the other is a delicate procedure due to the presence of the complex singularity which is absent in the former reaction. The calculation of the process $\gamma\pi \to \pi\pi$ is in itself interesting because of future experiments being proposed at various accelerator facilities and also of its important role in the calculation of $\pi^0 \to \gamma\gamma^*$ $[citeTruong1]$.

The $\gamma\pi \to \pi\pi$ amplitude is given as:

$$A(\gamma(k)\pi^0(p_0) \to \pi^+(p_1)\pi^-(p_2)) = iF_{3\pi}(s, t, u)\epsilon^\mu\sigma^\tau\epsilon_p0\epsilon_p1\pi\epsilon_p2\pi \hspace{1cm} (3)$$

The kinematics are as follows: $s = (k + p_0)^2, t = (k - p_1)^2, u = (k - p_2)^2$ with $k$ refers to the photon 4-momentum and $p_0$ refers to the neutral pion 4-momentum and $p_1$ and $p_2$ are those of the charge pions. In terms of $s$ and the c.o.m. scattering angle $\theta$, we have $t = a(s) - b(s)\cos\theta, u = a(s) + b(s)\cos\theta$ with $a(s) = (3m^2_\pi - s)/2$ and $b(s) = 1/2(s - m^2_\pi)\sqrt{(1 - 4m^2_\pi/s)}$.

In the chiral limit (the zero limit of the pion 4-momenta), the matrix element is given by the anomaly equation:

$$F_{3\pi}(0) = \lambda = \frac{e}{4\pi^2 f_\pi^3} = 9.7 GeV^{-3} \hspace{1cm} (4)$$

where the zero in the argument of $F_{3\pi}$ refers to the chiral limit of the massless pions; the number of colors $N_c$ is equal to 3.

Experimentally, $\lambda$ is measured at an average photon pion energy of 0.4GeV and assuming that there is no momentum dependence $F_{3\pi}$, it is equal to $[4]$:

$$\lambda^{expt} = 12.9 \pm 0.9 \pm 0.5 GeV^{-3} \hspace{1cm} (5)$$
The agreement between experiment and theory is not very good. Taking into account the momentum dependence of $F_{3\pi}$, the following value of $\lambda$ is obtained [5]:

$$\lambda_{\text{expt}} = 11.9 \pm 0.9 \pm 0.5 \text{GeV}^{-3}$$ \hspace{1cm} (6)

There is still disagreement between theory and experiment.

The calculations of this process are usually done within the Vector Meson Dominance models (VMD) [7, 8, 9]; recently it is discussed within the framework of Chiral Perturbation Theory (ChPT) [10] and also a combination of ChPT and VMD [6].

### 2 Integral Equation Approach using Elastic Unitarity Relation

In this talk, the process $\gamma\pi \rightarrow \pi\pi$ is studied using dispersion relation and elastic unitarity. An integral equation, similar to the pion form factor Muskhelishvili-Omnes integral equation [12] is obtained. The difference is that the integral equation to be treated here is much more complicated due to crossing symmetry; no exact solution has been found. We shall get the solution of this integral equation by an iterative procedure, but with the crucial property that the iterative solution satisfies the phase theorem at every steps as required by unitarity.

We make here the same assumption as in the pion form factor calculation. Because the pion form factor can be calculated to within 15% (in the amplitude) [11], we expect to have the same degree of reliability for this new integral equation.

If a better accuracy is demanded, similar to the calculation of the pion form factor, a more precise low energy measurement for $\gamma\pi \rightarrow \pi\pi$ is needed.

The strong P-wave $\pi\pi$ scattering phase shifts are supposed to be known and are given by the experimental data up to 1 GeV or higher which show the existence of the $\rho$ resonance at 0.77 GeV with a width of 0.151 GeV. We shall not directly make an assumption on the Vector Meson Dominance, but try to find a solution consistent with the constraints of the elastic unitarity, crossing symmetry and also of the low energy theorem, Eq. (4).

The process $\gamma\pi \rightarrow \pi\pi$ is a completely symmetric reaction in the three variables $s,t,u$ i.e. the same amplitude describes not only the reaction $\gamma\pi^0 \rightarrow \pi^+\pi^-$, but also the two other amplitudes involving the permutations of the pions. It is assumed that the scattering amplitude $\gamma\pi^0 \rightarrow \pi^+\pi^-$ can be represented by a single spectral function dispersion relation:

$$F(s, t, u) = \overline{\lambda} + \left[s - m^2_{\pi} + \frac{1}{\pi} \int_{4m^2_{\pi}}^{\infty} \frac{\sigma(z)dz}{(z - m^2_{\pi})(z - s - i\epsilon)}\right] + [s \leftrightarrow t] + [s \leftrightarrow u] \hspace{1cm} (7)$$

where for simplicity we drop the subscript $3\pi$ in $F(s, t, u)$ and set $\overline{\lambda} = F(m^2_{\pi}, m^2_{\pi}, m^2_{\pi})$, the value of the scattering amplitude at the symmetry point; the relation between $\lambda$ and $\overline{\lambda}$ will be discussed below. Eq. (7) shows explicitly the symmetry in $s, t, u$ variables.
Because the $\rho$ resonance occurs at a much higher energy than $m^2_{\pi}$, we expect that $\lambda$ differs little from $\lambda$. We can first estimate the value of $\lambda$ by examining the large $N_c$ limit of QCD. In this limit, the pion loop corrections are neglected and the vector meson propagator appears as a pole with zero width. Introducing a contact term to satisfy the anomaly equation, the Vector Meson model yields $\lambda = \lambda(1 + 3m^2_{\pi}/2s_{\rho}) \simeq 1.05\lambda$. We shall use this value in the following.

### 2.1 Partial Wave Amplitude and Integral Equation

Taking into account of the lowest partial wave projection of $F(s, t, u)$, $F(s)$, together with the strong P-wave $\pi\pi \rightarrow \pi\pi$ amplitude, the elastic unitarity relation gives:

$$\sigma(s) = F(s)e^{-i\delta(s)}\sin\delta(s)$$  

where $\delta$ is the P-wave $\pi\pi$ phase shift obtained from the available experimental data which show that they pass through $90^\circ$ at the $\rho$ mass with a width of 151 MeV and that there is no measurable inelastic effect below 1 GeV. We have

$$F(s) = \lambda + \frac{(s - m^2_{\pi})}{\pi} \int_{4m^2_{\pi}}^{\infty} \frac{F(z)e^{-i\delta(z)}\sin\delta(z)}{(z - m^2_{\pi})(z - s - i\epsilon)}dz$$

$$+ \frac{1}{\pi} \int_{4m^2_{\pi}}^{\infty} F(z)e^{-i\delta(z)}\sin\delta(z)\left\{\frac{1}{b(s)}\ln \frac{z - a(s) + b(s)}{z - a(s) - b(s)} - \frac{2}{z - m^2_{\pi}}\right\}dz$$

Eq. (9) is a complicated integral equation. It is similar to, but more complicated than the Muskelishvili-Omnes (MO) type [12], because the $t$ and $u$ channel contributions are also expressed in terms of the unknown function $F(s)$. It should be noticed that the first term has a cut from $4m^2_{\pi}$ to $\infty$ and the second one has a cut from 0 to $-\infty$. This remark enables one to solve the integral equation by the following iteration scheme which converges very fast.

The iterative and final solutions can be expressed in terms of the function $D(s, 0)$, normalized to unity at $s = 0$ and defined in terms of the phase shift $\delta$ as given by:

$$\frac{1}{D(s)} = \exp\frac{s}{\pi \int_{4m^2_{\pi}}^{\infty} \frac{\delta(z)dz}{z(z - s - i\epsilon)}}$$  

(10)

Other functions $D$ normalised to unity at $s = s_0$ can be expressed in terms of the function $D(s, 0)$ by the simple relation $D(s, s_0) = D(s, 0)/D(s_0, 0)$.

### 2.2 Iterative Solutions

As it was remarked above, the Integral Equation (9) has both right and left cuts. Because of this analytic structure, we can define an iteration procedure which consists in splitting Eq. (9) into two separate equations:
\[ F^{(i)}(s) = \frac{\bar{\lambda}}{3} + T_B^{(i-1)}(s) + \frac{s - m_{\pi}^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{F^{(i)}(z)e^{-i\delta(z)}\sin\delta(z)}{(z - m_{\pi}^2)(z - s - i\epsilon)}dz \]  

(11)

and

\[ T_B^{(i-1)}(s) = \frac{2\bar{\lambda}}{3} + \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} F^{(i-1)}(z)e^{-i\delta(z)}\sin\delta(z)\left\{ \frac{1}{b(s)} \ln \left| \frac{z - a(s) + b(s)}{z - a(s) - b(s)} \right| - \frac{2}{(z - m_{\pi}^2)} \right\}dz \]  

(12)

where \( i \geq 1 \) and \( F^i \) is the value of the function \( F(s) \) calculated at the \( i \)th step in the iteration procedure; the Born term \( T_B^{i-1}(s) \) is calculated at the \( i \)th - 1 step. The Born term is real for \( s \geq 0 \) and has a left cut in \( s \) for \( s < 0 \). In writing Eqs. (11,12), care was taken to preserve the symmetry in the \( s, t, u \) variables for the function \( F(s, t, u) \) which requires us to split symmetrically the subtraction constant \( \bar{\lambda} \) in Eq. (3) into three equal pieces, one contributes to Eq. (13) the other two to Eq. (14).

The solution of the integral equation Eq. (11) is of the MO type [12]:

\[ F^{(i)}(s) = \frac{\bar{\lambda}}{3D(s, m_{\pi}^2)} + T_B^{(i-1)}(s) + \frac{s - m_{\pi}^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{D(z, m_{\pi}^2)e^{i\delta(z)}\sin\delta(z)T_B^{(i-1)}(z)dz}{(z - m_{\pi}^2)(z - s - i\epsilon)} \]  

(13)

where it is assumed that the well-known polynomial ambiguity inherited in the MO integral equation is absent [12] except the constant term which is required by the low energy theorem. It can be shown that the phase theorem for \( F^{(i)}(s) \) is satisfied. The first term on the R.H.S. of Eq. (13) represents the \( \rho \) vector meson contribution to \( F^{(i)}(s) \), the second term is roughly the same as the \( \rho \pi\gamma \) vertex correction due to the exchange of a resonant pair of P-wave pions (\( \rho \) vector meson). (The higher polynomial ambiguity could represent some uncalculable inelastic effect occuring above the inelastic threshold).

One arbitrarily defines the convergence of the iteration scheme at the \( i \)th iteration step when \( |F^{(i)}| / |F^{(i-1)}| \) differs from 1 by less than 1% or so in the energy range from the two pion threshold to 1 GeV. (Alternatively one can also require that the ratio \( |T_B^{(i)}| / |T_B^{(i-1)}| \) to be unity within an accuracy of 1% or so).

Once the solution for the partial wave is obtained we should return to the calculation of the full amplitude. This can be done by combining the \( T_B^{(i-1)} \) Born term in Eq. (13) with higher uncorrected partial waves (for rescattering) from the \( t \) and \( u \) channels to get the final solution:

\[ F^{(i)}(s, t, u) = \frac{\bar{\lambda}}{3}\left\{ \frac{1}{D(s, m_{\pi}^2)}(1 + 3I^{(i-1)}(s)) \right\} + \{(s \leftrightarrow t)\} + \{(s \leftrightarrow u)\} \]  

(14)

where the function \( I^{(i-1)} \) denotes the multiple rescattering correction:

\[ I^{(i-1)}(s) = \frac{s - m_{\pi}^2}{\pi} \int_{4m_{\pi}^2}^{\infty} \frac{D(z, m_{\pi}^2)e^{i\delta(z)}\sin\delta(z)T_B^{i-1}(z)dz}{(z - m_{\pi}^2)(z - s - i\epsilon)} \]  

(15)

Projecting out the \( l = 0 \) partial wave from Eq. (14), we arrive at Eq. (13) with \( T_B^{i-1}(s) \) replaced by \( T_B^{i}(s) \). Because of the assumed criteria for the convergence of the iteration
scheme, $T^{-1}_B(s) \simeq T_B(s)$ it is easily seen that $F^{(i)}(s)$ has the phase $\delta$, using the result of Eq. (13). The remaining (even) higher partial waves $l > 0$ are all real because we have assumed that the strong final state interaction of the higher partial waves are negligible. The final solution Eq. (14) is completely symmetric in the $s, t, u$ variables.

3 Numerical Solutions

In order to carry out the iteration procedure to find the solution of the integral equation one has to parametrize the function $D(s, 0)$, normalized to be unity at $s = 0$, in terms of the experimental P-wave phase shift. In the following we shall make three different parametrisations for the $D$ function. They are all expressed in terms of the P-wave $\pi\pi$ phase shifts $\delta$ as given by Eq. (10).

The function $D^{-1}(s)$ can be parametrised as follows [14, 11]:

$$D^{-1}(s) = \frac{1}{1 - \frac{s}{s_R} - \frac{1}{9\pi^2 f^2} \left\{ (s - 4m^2_{\pi})H_{\pi\pi}(s) + 2s/3 \right\}}$$  \hspace{1cm} (16)

The $\rho$ mass is defined as the vanishing of the real part of the denominator and is equal to $s_R$ in the narrow width approximation. Using $\sqrt{s_\rho} = 0.770 GeV$ in Eq. (16), we have $\Gamma_\rho = 155.6 MeV$ for $f_\pi = 0.093 GeV$, which is very near to the experimental value of $150.7 \pm 1.2 MeV$. If we want to fit the $D_1(s)$ to the experimental width, we can phenomenologically change $f_\pi = 0.0945 GeV$ in Eq. (16).

In the above subsection we have discussed the comparison between our result and the low energy experimental data. Our result is about 1 standard deviation too low if the the value of $\lambda = 12.9 \pm 0.9 \pm 0.5 GeV^{-3}$ is used. Taking into account of the momentum dependence of $F_{3\pi}$ then the experimental value of $\lambda = 11.9 \pm 0.9 \pm 0.5 GeV^{-3}$ and is in excellent agreement with our calculation. At low energy, the effect of the multiple scattering is important and the good agreement with the experimental data is obtained thanks to this effect which is fully taken into account in solving the integral equation.

We want to discuss now the result of our integral equation in the $\rho$ resonance region where there are some experimental measurements. The experimental data are usually analyzed in term of the $\rho \rightarrow \pi\gamma$ width using the Breit-Wigner formulae:

$$\sigma(\gamma\pi \rightarrow \rho) = \frac{24\pi s_\rho}{s_\rho - m^2_{\pi}} \frac{s_\rho \Gamma(\rho \rightarrow 2\pi) \Gamma(\rho \rightarrow \pi\gamma)}{(s_\rho - s)^2 + s_\rho \Gamma^2_{\rho}}$$  \hspace{1cm} (17)

where $\Gamma_\rho$ is the total width of the vector meson $\rho$. In terms of the matrix element for $\gamma\pi \rightarrow \pi\pi$, $F_{3\pi}(s, t, u)$, we have:

$$\frac{d\sigma}{d\cos\theta}(\gamma\pi \rightarrow \pi\pi) = \frac{|F_{3\pi}|^2}{1024\pi} \frac{(s - m^2_{\pi}) (s - 4m^2_{\pi})^{3/2}}{\sqrt{s} \sin^2\theta}$$  \hspace{1cm} (18)

In arriving at Eq. (17) we must assume that there is no background term in the cross section which might interfere with the Breit-Wigner term which may not negligible. By
background term we mean amplitudes which do not have the Breit Wigner form which might be present. In our formulation the background term from the $t$ and $u$ channels are automatically taken into account as can be seen from Eq. (18). It turns out because the $\rho$ width is fairly small, the Breit-Wigner approximation, Eq. (17), is accurate to few percents, and hence we can use it with confidence. Our method is therefore to equate Eq. (15) and Eq. (17) to calculate the $\rho \to \pi \gamma$ width which is a convenient way to express our solution in terms of physically measurable quantity. We shall use this method when we compare our result with those given by other models.

Using the D-function as given by Eq. (10) which satisfies the KSRF relation, we obtain:

$$\Gamma(\rho \to \pi \gamma) = 63 \text{KeV}$$

(19)

This value is to be compared with the world average for the charge $\rho \to \pi \gamma$ width, $\Gamma(\rho^+ \to \pi^+ \gamma) = 68 \pm 7 \text{KeV}$. The corresponding for the the neutral $\rho$ is very different, $\Gamma(\rho^0 \to \pi^0 \gamma) = 120 \pm 30 \text{KeV}$. As explained above, there should be no difference between the values of the charge and neutral radiative widths.

Using $\Gamma(\omega \to \pi \gamma) = 716 \pm 42 \text{KeV}$ and the SU$(3)$ relation, we get $\Gamma(\rho \to \pi \gamma) = 80 \pm 5 \text{KeV}$ which is about 20% larger than the present data.

How accurate is our calculation with the assumption of the elastic unitarity relation? From our experience of using the same assumption for the pion form factor calculation, the maximum value of the square of the absolute value of the pion form factor at the $\rho$ peak is 32 which is 20-25% too low compared with the data. We expect to commit the same magnitude of error using the elastic unitarity for our problem. If the experimental data on $\rho \to \pi \gamma$ width was changed to the SU$(3)$ value of 80KeV, our value would still be consistent with the data.

To put it more quantitatively, the pion form factor $V(s)$ can be multiplied by a real polynomial, say $(1 + 0.13s/s_\rho)$, which phenomenologically represents the inelastic effect occurring at a higher energy. This modification gives an excellent fit to the experimental data on the time-like pion form factor up to $1 \text{GeV}^2$.

We can likewise multiply our solution by the same factor which might represent some correction to our assumption of the elastic unitarity relation for our problem. This discussion is purely phenomenology and is not needed at present because our calculation is in good agreement with data. More accurate measurement of $F_{3\pi}$ at low energy will be necessary in order to limit the size of this term.

In conclusion, the process $\gamma \pi \to \pi \pi$ is calculated using the low energy theorem, analyticity, elastic unitarity and crossing symmetry which are fundamental conditions for a theory involving strong interaction. The final result shows that one effectively takes into account of the (unstable) $\rho$ model in the $s$, $t$ and $u$ channels and their rescattering effect treated in a self-consistent way. It is important to put these results to experimental tests.
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