Noisy soccer balls

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In her Comment \cite{Hossenfelder:2014vya} Hossenfelder proposes a generalization of the results we reported in Ref.\cite{Amelino-Camelia:2013bja} and argues that thermal fluctuations introduce incurable pathologies for the description of macroscopic bodies in the relative-locality framework. We here show that Hossenfelder’s analysis, while raising a very interesting point, is incomplete and leads to incorrect conclusions. Her estimate for the fluctuations did not take into account some contributions from the geometry of momentum space which must be included at the relevant order of approximation. Using the full expression here derived one finds that thermal fluctuations are not in general large for macroscopic bodies in the relative-locality framework. We find that such corrections can be unexpectedly large only for some choices of momentum-space geometry, and we comment on the possibility of developing a phenomenology suitable for possibly ruling out such geometries of momentum space.
I. INTRODUCTION

In a recent paper [2] we considered the description of macroscopic bodies within the relative-locality framework based on curved-momentum-space. Our main result was to show that in an idealized case, in which the constituents of a macroscopic body are modelled as being in rigid motion, an apparently troublesome aspect of the description of macroscopic bodies, the so-called “soccer-ball problem”, is actually absent. This is essentially because the deformation scale reflecting curvature of momentum space, which is assumed to be of order $M_P$ for a single constituent, scales like $NM_P$ for a body with $N$ constituents in rigid motion. This was reassuring, but for full assurance we should consider the case in which the momenta of the constituents fluctuate around their mean. This can be called the quasi-rigid motion. In the closing remarks of Ref. [2] we offered a simple argument encouraging the hypothesis that also for bodies in quasi-rigid motion our solution of the soccer-ball problem should stand. However, in a Comment [1], Hossenfelder criticized our argument. Using a random-walk model of thermal fluctuations in a macroscopic body, Hossenfelder claimed that non-linear terms arising from the curvature of momentum space can accumulate leading to apparently paradoxical result that the fluctuations overwhelm the contributions of averages. If this was true the case of constituents in quasi-rigid motion would be unmanageably pathological.

Here we address this significant challenge to the cogency of the relative locality hypothesis by giving a more precise analysis of the relevant thermal fluctuations.

We find, first of all, that the question posed by Hossenfelder is interesting and leads to a new type of investigation in which thermal fluctuations probe the geometry of momentum space away from small momenta. This may potentially open up new opportunities to constrain the geometry of momentum space from the properties of macroscopic bodies.

Our analysis leads to the identification of three classes of momentum space geometries. There are cases in which Hossenfelder’s worry is realized and the fluctuations of the momenta of the constituents of a hot macroscopic body have effects which scale like positive powers of $N$ in ways that seem to contradict our observations. On the other hand there are also many cases in which the dangerous terms cancel and the fluctuations of the deformed theory behave like those of standard statistical physics. Most interesting is, however, the third, in between, case, in which there may be new physical effects by which the thermodynamics of large bodies may be able to serve as a probe of the geometry of momentum space for macroscopic values of momentum and energy.

Our analysis is more careful than both our brief remarks in [2] and Hossenfelder’s approach [1], which did not keep all terms necessary to do a consistent perturbative analysis of the effects of the fluctuations. Taking all terms into account at each order means that we can constrain the geometry of the addition rule to higher order that was possible before, and still leave us with a open set of non trivial possibilities.

II. TOTAL-MOMENTUM FLUCTUATIONS WITHIN A CONSISTENT PERTURBATIVE APPROACH

We take as starting point $P^{(N)} = \oplus_{a=1}^N (\bar{p} + \delta p_a)$ to be the total momentum of our soccer ball in the relative-locality framework. Here, $\bar{p}$ is the average value of the momenta for each constituent, and $\delta p_a$ is the fluctuation of this value for the constituent $a$. The precise definition is by recurrence $P^{(N+1)} = P^{(N)} \oplus (\bar{p} + \delta p^{N+1})$, with $P^{(0)} = 0$. The deformed addition rule $\oplus$ is a manifestation of the geometry of momentum space, which, as done in [1], we describe in terms of a perturbative series in inverse powers of $M_P$:

\[(p \oplus q)_a = p_a + q_a + M_P^{-1} \Gamma^{\mu\nu}_a p_\mu q_\nu + M_P^{-2} \Delta^{\mu\nu\rho}_a p_\mu q_\nu p_\rho + M_P^{-2} \tilde{\Delta}^{\mu\nu\rho\sigma}_a q_\mu q_\nu q_\rho q_\sigma + \cdots \] (1)

Notice that we included terms up to quadratic order in $M_P$, whereas Hossenfelder only considered the linear order. We shall soon see that a correct estimation of the size of fluctuations of the total momentum of a body reveals that the leading contribution is quadratic in $M_P$, so that truncations of the composition law to linear order, such as Hossenfelder’s, cannot give a consistent description.

Also note that we did not include in (1) terms of the form $\Gamma pp$ or $\Delta ppp$ or $\Delta qqq$ because, as already done in [2], [1], we work in connection-normal coordinates. This are the coordinates in which the composition laws is linear for collinear momenta, $(a p \oplus b p)_\mu = (a + b)p_\mu$ (for any real number $a, b$). It also implies that the following conditions must be satisfied:

\[\Gamma^{(\mu\nu)}_a = 0, \quad \Delta^{(\mu\nu\rho)}_a + \tilde{\Delta}^{(\mu\nu\rho)}_a = 0, \] (2)

where the bracket means symmetrization of indices.

For the analysis of how small fluctuations of the momenta of the constituents affect the total momentum of the body we consider contributions to the total momentum which are linear in the fluctuations:

\[P^{(N)}_\alpha = N\bar{p}_\alpha + \sum_a (W^{(N)}_a(\bar{p}))^\beta_\alpha \delta p^\beta_\alpha + \cdots , \] (3)
where the dots stand for higher-order terms in $\delta p$. The first term takes the simple form $N\bar{p}_a$, thanks to the adoption of connection-normal coordinates. $W_a^{(N)}(\bar{p})$ in the second term is the coefficient of $\delta p^a$ in the expansion of $P_a^{(N)}$ and can be explicitly calculated from the knowledge of $\oplus$. In particular on the basis of (1) our task will be to compute $W_a^{(N)}$ to quadratic order in $1/M_P$.

For genuine fluctuations we demand that the second term in (3) is much smaller than the first one and average to zero in the large $N$ limit. Hossenfelder’s observes that it is not enough to demand that the fluctuations are small. For consistency we need to evaluate the size of fluctuations around the mean value and show that it is negligible as well.

The size of fluctuations depends crucially on

$$\langle \beta \rangle \sim \frac{\sigma^2}{m T \delta_{ij}}$$

(4)

where $\langle \cdots \rangle$ denotes the average that will be described in the next section.

We definitely agree with Hossenfelder that a fully satisfactory analysis of macroscopic bodies must consider the size of fluctuations around the mean value. However, we here show that the type of contributions to $\sigma^2$ on which Hossenfelder’s analysis focused, which have quadratic dependence on the Planck scale, require an analysis of the composition law to the $M_P^{-2}$ order. As we shall show, the fact that Ref. [1] only included order-$M_P^{-1}$ corrections to the composition law led Hossenfelder to incorrect results.

III. SIZE OF FLUCTUATIONS IN THE NON-RELATIVISTIC REGIME

Having characterized the needed properties of $W_a(\bar{p})$ we are now ready for estimating the size of the fluctuations. For this we shall assume [3], as done in Hossenfelder’s Comment [1], that the momenta $\bar{p}$ fluctuate randomly, $\langle \delta p_i \rangle = 0$, $\langle \delta p_i \delta p_j \rangle \propto \delta^{ab} \delta_{ij}$. The point to be stressed is that the fluctuations must be ‘physical”, and when binding forces are neglected (as here and in [1]) the on-shellness of each constituent particle must be preserved. This condition was not imposed, in [1] so in general the fluctuations described in [1] would be unphysical virtual fluctuations. We shall here improve the analysis also in this respect.

Since the bulk of our experimental knowledge of the properties of macroscopic bodies is in the non-relativistic regime, we shall focus on the case where $\bar{p}_i \ll \bar{p}_0 = m$ (and assume again for simplicity the all constituents have the same mass $m$). Here we assume for simplicity that in the non-relativistic limit the on-shellness relation for constituent particles is undeformed to second order. This means that we can take $e \equiv \bar{p}_0 = m + \bar{p}^2/(2m)$ and $e\delta e - \bar{p}_0 \delta \bar{p} = 0$, i.e. $\delta e = \bar{p}_i \delta p_i/e$. In the non-relativistic limit we can rely on the Boltzmannian distribution, with the one particle probability density given by $f(p) \sim \exp(-\frac{p^2 e}{2m e})$, and therefore

$$\langle \delta p_i \delta p_j \rangle = m T \delta^{ab} \delta_{ij}.$$  

(5)

The fluctuation in energy follows from the mass-shell relation $\delta e = u_i \delta p_i$ where $u_i = p_i/m$ is the spatial velocity. Even if we are in the non relativistic limit we can use a relativistic notation to write down the fluctuation of both energy and momenta, we introduce a 4-velocity $u_\alpha = (1, u_i)$ and we can write the non relativistic fluctuation as:

$$\langle \delta p^\alpha \rangle = 0, \quad \langle \delta p^\alpha \delta p^\beta \rangle = m T \delta^{ab} (\eta_{a\beta} + 2u_a u_\beta).$$  

(6)

We can now use this fluctuation model to evaluate the size of fluctuation to be given by the average $N^2 \langle \sigma^2 \rangle_{\alpha\beta} = \langle \sum_{a,b} W_a^{(N)}(\bar{p})^2 W_b^{(N)}(\bar{p}) \delta^a_b \delta p^a \delta p^b \rangle$. With the Boltzmannian model this gives

$$\langle \sigma^2 \rangle_{\alpha\alpha'} = \frac{m T^2}{N^2} \left( \sum_a \langle W_a^{(N)}(\bar{p})^2 \rangle \eta_j W_a^{(N)}(\bar{p})_{\alpha\alpha'} + \frac{2}{m^2} \sum_a \langle W_a^{(N)}(\bar{p}) \bar{p} \rangle_{\alpha} \langle W_a^{(N)}(\bar{p}) \bar{p} \rangle_{\alpha'} \right).$$  

(7)

where $^T$ denotes the transposition. The first key point is that in our connection-normal coordinates the second term in (7) vanishes identically:

$$\langle W_a^{(N)}(\bar{p}) \rangle_{\alpha} = 0.$$  

(8)

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1 We agree with Ref. [3] that the idealization of completely random fluctuations might be a good starting point for the investigation of the problem at hand. Still it should be appreciated that for some macroscopic bodies of definite interest this idealization is unrealistic. In particular in a solid there are long-range binding forces and a better approximation is to assume gaussian fluctuations where $\langle \delta p_i \rangle = 0$, but now $\langle \delta p_i \delta p_j \rangle = \delta^{ab} \Delta_{ij}$, where $\Delta_{ij}$ is the inverse interaction kernel. This kernel depends on where the atoms are in the solid.
The size of fluctuations is therefore simply given by

$$(a^2)_{\alpha \alpha'} = \frac{mT}{N^2} \left( \sum_a [(W^{(N)}_a(\bar{p}))^T \eta W^{(N)}_a(\bar{p})]_{\alpha \alpha'} \right).$$  \hfill (9)

The goal is now to evaluate this object up to quadratic order.

IV. PERTUBATIVE EXPANSION

Let us first presents the results we obtain for the expansion of $W_a(\bar{p})$ and the fluctuations to second order. Some relevant results are collected in the Appendix. From the expansion \((15)\), the definition of $W^{(N)}_a(\bar{p})$, and restricting our focus to a rest frame analysis, so that $\bar{p}_\alpha = m\theta_\alpha$, we get

$$(W^{(N)}_a)^\beta_\alpha = \delta^\beta_\alpha - \frac{m}{M_p} (N + 1 - 2a) \Gamma^{0\beta}_\alpha + \frac{m^2}{M_p} \left( A^N_a \Gamma^0 \eta^0 + B^N_a S^{00} + C^N_a \tilde{\Delta}^{00} + D^N_a \Delta^{00} \right)^\beta_\alpha .$$

where $A^N_a, B^N_a, C^N_a, D^N_a$ are $N$ dependent coefficients, explicitly derived in the Appendix. Here we use the matrix notation $(\Gamma^0)^\beta_\alpha \equiv \Gamma^{0\beta}_\alpha$, similarly $(S^{00})^\beta_\alpha \equiv S^{00\beta}_\alpha$ etc... and we have defined $S^{00 \beta \gamma \delta}_\alpha \equiv 3 \Delta^{0 \beta \gamma \delta}_\alpha$. Since $N$ can be very large for a macroscopic body it is particularly important to keep track of the order $N$ of each term. We find that the dominant contributions come from

$$\sum_{a=1}^N (N + 1 - 2a)^2 \approx N^3, \quad \sum_{a=1}^N B^N_a \approx \frac{N^3}{3}, \quad \sum_{a=1}^N D^N_a \approx \frac{3}{4} N^2.$$  \hfill (10)

We see that the $D$ term is only of order $N^2$ instead of $N^3$. The other sums involved in the evaluation of the fluctuation are all of order less or equal to $N^2$.

Equipped with this, we can now evaluate the the first term in \((11)\) finding that

$$\frac{(\sigma^2)_{\alpha \alpha'}}{m^2} = \frac{T}{Nm} \delta_{\alpha \alpha'} - \frac{T}{M_p} \left[ (\Gamma^0 \eta)^T + \eta \Gamma^0 \right]_{\alpha \alpha'} + \frac{mTN}{3M_p^2} \left[ (\Gamma^0 \eta \Gamma^0) + [(S^{00} \eta)^T + \eta S^{00}] + O(1/N) \right]_{\alpha \alpha'}.$$  \hfill (11)

\(T\) denotes the transposition.

Let us pause briefly for comparing our constructive result to the estimate proposed by Hossenfelder in \[(11)\]. Within our notation that estimate would take the form

$$\frac{\sigma^2_{\text{Hoss}}}{m^2} \sim (\Gamma^0)^2 \frac{T^2 N}{M_p^2} .$$

It should be noticed first of all that Hossenfelder’s heuristic estimate does not reproduce exactly any of the terms in the result \[(11)\] we have derived. The single term in Hossenfelder’s estimate does agree roughly with our term $(\Gamma^0 \eta \Gamma^0)$ in \[(11)\], however the size of fluctuations is controlled by a $mT$ dependence in our case while she uses $T^2$. More importantly the single estimate is not a consistent approximation of our result since it misses the contribution going like $(S^{00} \eta)^T + \eta S^{00}$, which is of the same order. This is of course connected to our earlier remarks about the consistency of a perturbative approach to this sort of derivation: our $(\Gamma^0 \eta \Gamma^0)$ term originates from the $O(M_p^{-1})$ contribution to the composition law while our $(S^{00} \eta)^T + \eta S^{00}$ term originates from the $O(M_p^{-2})$ contribution to the composition law, but they both appear at the same order of approximation of $\sigma^2$.

The difference is not only quantitative but also very importantly qualitative: whereas Hossenfelder was using her evaluation as motivation for excluding all composition laws at leading order, we find that, even if one did insist on excluding the corresponding type of corrections, there is only a constraint on the relationship between leading-order form of the composition law, affecting $\Gamma^0 \eta \Gamma^0$, and next-to-leading-order form of the composition law, affecting $(S^{00} \eta)^T + \eta S^{00}$.

Having derived the full $(1/M_p^2$-order expression we can also analyze the structure of \[(11)\] from a wider perspective. The first term is the usual fluctuation term which goes away in the large $N$ limit. The second term is proportional to the linear evaluation of the non-metricity tensor $N_{ij} \equiv \Gamma_{0ij}^0 + \Gamma_{ij}^0$. It corresponds to a correction to the usual fluctuations which is suppress by a very small factor $T/M_p$. If we demand the non-metricity to vanish this implies that the connection coefficient are entirely determined by the torsion.
The only problematic term in this expansion is the last term proportional to \((\Gamma^0)^T \eta \Gamma^0 + [S^{00}]^T \eta S^{00}\), it is of order \(NmT/M_P^2\). The demand that the fluctuations do not scale with \(N\), results in a condition for one of the second order coefficients given by \(((\Gamma^0)^T \eta \Gamma^0) + [S^{00}]^T \eta S^{00} = 0\), in components this simply reads
\[
\Gamma_{\alpha}^0 \partial_{\alpha}^k \eta + 2S_{(ij)}^0 = 0,
\]
which tells us that the symmetric part of \(S^{00}\) is fixed, while the rest of the components of \(\Delta^{00}, \tilde{\Delta}^{00}\) are free. When this condition is satisfied we do still have additional corrections proportional to \(nmT/M_P^2\Delta^{00}\), but evidently none of these contributions is problematic.

V. OUTLOOK

We had shown in [2] that, contrary to earlier naive arguments, there is no interpretational challenges nor any phenomenological paradoxes in the idealized case of body composed of constituents in rigid motion.

We here showed that even for bodies whose constituents are in quasi-rigid motion the relative locality framework does not in general encounter any pathologies, extending the analysis up to quadratic order in the (inverse of) the Planck scale for effects linear in the fluctuations of the momenta of the constituents. Within our analysis the description of macroscopic bodies is completely unproblematic for the large class of non-trivial momentum-space geometries that satisfy the condition (12).

It is interesting to contemplate the possibility that geometries that do not satisfy condition (12) might be ruled out experimentally exploiting the \(N\) dependence of \(\sigma^2\). But here, we could also advocate a more phenomenological approach. We start by noticing that corrections of order \(NmT/M_P^2\) are still extremely small (smaller than one part in \(10^{22}\)) for an actual soccerball at soccer-playing temperatures of about 300 Kelvin, so we can confirm that the original ‘soccer-ball problem’ [2] is not going to be a matter of contention with any choice of momentum-space geometry. As suggested by Hossenfelder there might still be a problem for macroscopic bodies in some extreme cases, such as ultra-hot and ultra-massive bodies studied in astrophysics [1]. But instead of jumping to the conclusion that mere existence of such astrophysical bodies suffices for excluding some momentum-space geometries, we feel a more prudent scientific approach would be to seek the opportunity to analyze in future works an actual measurement result on such bodies which is affected by large corrections, for suitable choices of momentum-space geometry. A first step toward this goal would be to generalize the analysis we here reported so that it could apply to case with \(T \gtrsim m\) (within the nonrelativistic limit, on which we here focused, only cases with \(m \ll T\) can be consistently studied). Moreover, it will be necessary to gain some control also about the interactions among macroscopic bodies in the relative-locality framework: in principle for some momentum-space geometries large corrections might be present for the abstract notion of total momentum of a body and yet not be present in our measurements of properties of that body, which inevitably require a role for interactions with the body. This is clearly a line of research worth pursuing since the opportunity of excluding at least some geometries of momentum space would be extremely valuable for the relative-locality research program.

Another interesting challenge we leave for future studies concerns the geometric interpretation of results such as our condition (12). It is very tempting to assume that conditions such as (12), when analyzed beyond the level of dry relationships among tensors, might be characterizing in meaningful geometric way desirable (or needed) properties of momentum-space geometry. In closing we sketch out an argument suggesting that this could be the case and which opens new investigations.

We know from general theory of relative locality that \(W_a^{0(N)}\) is a transport operator from 0 to \(N\bar{p}\). we also know that this transport operator is determined to first order by a connection evaluated at 0. Let us assume, for the sake of our argument, that this is true beyond first order. That is lets suppose that \(W_a^{0(N)}\) can be written as the parallel transport of a connection \(\Gamma\) along a path, labelled by \(a\), from 0 to \(N\bar{p}\). That is we assume that \(W_a^{0}(\bar{p}) = P_a \exp J_0^{N\bar{p}} \Gamma\), where \(P_a\) denotes the path ordering along a path \(a\). Even if that might not be generally true, assuming it gives us a key insight.

Let us also suppose that the momentum space geometry is such that the connection is compatible with the metric, that is that is \(\nabla \eta = 0\), in other words the connection is the Levi-Civita connection plus torsion possibly. Integrating out this equation can be straightforwardly done and leads to the following metric compatibility condition on the transport operator:
\[
[(W_a^{0(N)}(\bar{p}))^T \eta W_a^{0(N)}(\bar{p})]_{\alpha \alpha'} = \eta_{\alpha \alpha'}(N\bar{p})
\]
where \(\eta(p)\) is the metric at \(p\) while \(\eta = \eta(0)\). This suggest that, under the conditions we specified, the fluctuations would be exploring the geometry of momentum space away from the origin. This would lead to the exciting opportunity
of exploring the geometry of momentum space away from the origin by harnessing the power of the large number $N$. It also opens a new set of mathematical investigation on when our hypothesis can be realize.

Appendix A: Properties of $W_a(p)$ to order $M_{p}^{-2}$

Here we establish the dependence of $W_a(p)$ on the form of the composition law to second order in (inverse of) the Planck scale. It is useful to start by observing that from (1) we get:

\[
(a_p + \delta p) \oplus (b_p + \delta q)_ \alpha = (a + b) \tilde{p}_\alpha + (\delta p + \delta q)_\alpha + M_p^{-1} a \Gamma_\alpha^{\mu} \delta q_\mu + M_p^{-1} b \Gamma_\alpha^{\mu} \delta p_\mu
\]

\[
+ M_p^{-2} (ab \Delta_\alpha^{\mu\nu} + ab \Delta_\alpha^{\mu\nu}) \delta p_\mu + M_p^{-2} (ba \Delta_\alpha^{\mu\nu} + ba \Delta_\alpha^{\mu\nu}) \delta q_\mu + \ldots
\]

where we used the notation $T^\mu_{\nu\rho} \equiv T^{\mu\nu\rho} \tilde{p}_\alpha$. We can conveniently rewrite this expression by defining

\[
S_\alpha^{\beta\mu
u} = 3 \Delta_\alpha^{(\beta\mu\nu)}.
\]

Using the fact that we are working with connection-normal coordinates, finding

\[
[(a_p + \delta p) \oplus (b_p + \delta q)_\alpha = (a + b) \tilde{p}_\alpha + (\delta p + \delta q)_\alpha
\]

\[
+ \Gamma_\alpha^{\mu}(b \delta p + a \delta q)_\mu + ab S_\alpha^{\mu\nu}(\delta p - \delta q)_\mu + (b \Delta - a \Delta)_\alpha^{\mu\nu}(b \delta p - a \delta q)_\mu.
\]  

(A2)

From this we can now get the recursion equation determining the perturbation to order $M_p^{-2}$:

\[
[W_a^{(N+1)}]^{\mu}_{\alpha} = \left( \delta_\beta^{\mu} - \Gamma_\alpha^{\beta\mu} + NS_\alpha^{\beta\mu\nu} + (\tilde{\Delta}_\alpha^{\beta\mu\nu} - N \Delta_\alpha^{\beta\mu\nu}) \right) [W_a^{(N)}]^{\mu}_{\beta},
\]  

(A3)

\[
[W_N^{(N+1)}]^{\mu}_{\alpha} = \delta_\alpha^{\mu} + NT_\alpha^{\mu\nu} - NS_\alpha^{\mu\nu} - N(\tilde{\Delta}_\alpha^{\mu\nu} - N \Delta_\alpha^{\mu\nu}).
\]  

(A4)

In the first line $a = (1, \ldots, N)$, while the second line gives the initial condition for the recurrence. This can be solved as

\[
W_a^{(N)} = 1 - (N + 1 - 2a) \Gamma_\alpha^{\beta} + A_a^N \Gamma_\alpha^{\beta} + B_a^N S_\alpha^{\beta} + C_a^N \tilde{\Delta}_\alpha^{\beta} + D_a^N \Delta_\alpha^{\beta} + \ldots
\]  

(A5)

where $\Gamma_\alpha^{\beta}$ denotes the matrix $(\Gamma_\alpha^{\beta})_\alpha^\beta = \Gamma_\alpha^{\beta\alpha}$, and similarly for the other symbols, while the coefficients are

\[
A_a^N = (-1)^{N-a}(a-1),
\]  

(A6)

\[
B_a^N = \frac{N(N-1)}{2} - \frac{a(a+1)}{2} + 1,
\]  

(A7)

\[
C_a^N = N + 1 - 2a,
\]  

(A8)

\[
D_a^N = - \frac{N(N-1)}{2} + \frac{3a-2(a-1)}{2}.
\]  

(A9)

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