VARIATIONAL SPLINES AND PALEY-WIENER SPACES ON COMBINATORIAL GRAPHS

ISAAC PESENSON

ABSTRACT. Notions of interpolating variational splines and Paley-Wiener spaces are introduced on a combinatorial graph $G$. Both of these definitions explore existence of a combinatorial Laplace operator on $G$. The existence and uniqueness of interpolating variational spline on a graph is shown. As an application of variational splines the paper presents a reconstruction algorithm of Paley-Wiener functions on graphs from their uniqueness sets.

1. Introduction and Main Results

The paper introduces variational splines and Paley-Wiener spaces on combinatorial graphs. Variational splines are defined as minimizers of Sobolev norms which are introduced in terms of a combinatorial Laplace operator. It is shown that variational splines not only interpolate functions but also provide optimal approximations to them. Paley-Wiener spaces on combinatorial graphs are defined by using spectral resolution of a combinatorial Laplace operator. The main result of the paper is a reconstruction algorithm of Paley-Wiener functions from their uniqueness sets using variational splines.

The following is a summary of main notions and results. We consider finite or infinite and in this case countable connected graphs $G = (V(G), E(G))$, where $V(G)$ is its set of vertices and $E(G)$ is its set of edges. We consider only simple (no loops, no multiple edges) undirected unweighed graphs. A number of vertices adjacent to a vertex $v$ is called the degree of $v$ and denoted by $d(v)$. We assume that degrees of all vertices are bounded from above and we use notation $d(G) = \max_{v \in V(G)} d(v)$.

The space $L_2(G)$ is the Hilbert space of all complex-valued functions $f : V(G) \to \mathbb{C}$ with the following inner product

$$\langle f, g \rangle = \sum_{v \in V(G)} f(v) \overline{g(v)}$$

and the following norm

$$\|f\| = \|f\|_0 = \left( \sum_{v \in V(G)} |f(v)|^2 \right)^{1/2}.$$

1991 Mathematics Subject Classification. 42C99, 05C99, 94A20, 41A15; Secondary 94A12.

Key words and phrases. Combinatorial graph, combinatorial Laplace operator, variational splines, Paley-Wiener spaces, interpolation, approximation, reconstruction.
The discrete Laplace operator $\mathcal{L}$ is defined by the formula \[ \mathcal{L}f(v) = \frac{1}{\sqrt{d(v)}} \sum_{u \sim v} \left( \frac{f(v)}{\sqrt{d(v)}} - \frac{f(u)}{\sqrt{d(u)}} \right), f \in L_2(G), \]
where $v \sim u$ means that $v, u \in V(G)$ are connected by an edge. It is known that the Laplace operator $\mathcal{L}$ is a bounded operator in $L_2(G)$ which is self-adjoint and positive definite. Let $\sigma(\mathcal{L})$ be the spectrum of a self-adjoint positive definite operator $\mathcal{L}$ in $L_2(G)$, then $\sigma(\mathcal{L}) \subset [0, 2]$. In what follows we will use the notations \[
\omega_{\min} = \inf_{\omega \in \sigma(\mathcal{L})} \omega, \omega_{\max} = \sup_{\omega \in \sigma(\mathcal{L})} \omega.
\]

For a fixed $\varepsilon \geq 0$ the Sobolev norm is introduced by the following formula
\[
\|f\|_{t, \varepsilon} = \left\| (\varepsilon I + \mathcal{L})^{t/2} f \right\|, t \in \mathbb{R}.
\]

The Sobolev space $H_{t, \varepsilon}(G)$ is understood as the space of functions with the norm \[(1.1)\]

Variational splines in spaces $L_2(\mathbb{R}^d)$ are introduced as functions which minimize certain Sobolev norms \[37, 5\]. Sobolev spaces in $L_2(\mathbb{R}^d)$ can be defined as domains of powers of the Laplace operator $\Delta$ in $L_2(\mathbb{R}^d)\] \[42\]. To construct variational splines on a graph $G$ we are going to use the same idea by replacing the classical Laplace operator $\Delta$ by the combinatorial Laplacian $\mathcal{L}$ in $L_2(G)$.

For a given set of indices $I$ (finite or infinite) the notation $l_2$ will be used for the Hilbert space of all sequences of complex numbers $y = \{y_i\}, i \in I$, for which $\sum_{i \in I} |y_i|^2 < \infty$.

**Variational Problem**

Given a subset of vertices $W = \{w\} \subset V(G)$, a sequence of complex numbers $y = \{y_w\} \in l_2, w \in W$, a positive $t > 0$, and an non-negative $\varepsilon \geq 0$ we consider the following variational problem:

*Find a function $Y$ from the space $L_2(G)$ which has the following properties:*
*1) $Y(w) = y_w, w \in W$,
*2) $Y$ minimizes functional $Y \rightarrow \|(\varepsilon I + \mathcal{L})^{t/2} Y\|$.\*

**Remark 1.** It is convenient to have such a functional in the Variational Problem which is equivalent to a norm. Thus, if the operator $\mathcal{L}$ has a bounded inverse in $L_2(G)$ (it is a situation on homogeneous trees of order $q + 1, q \geq 2$) then we will assume that $\varepsilon$ is zero. Otherwise we assume that $\varepsilon$ is a ”small” positive number. In what follows we will write $\varepsilon \geq 0$ with understanding that $\varepsilon = 0$, if the operator $\mathcal{L}$ is invertible in $L_2(G)$ and that $\varepsilon > 0$, if $\mathcal{L}$ is not invertible in $L_2(G)$.

We show that the above variational problem has a unique solution $Y_{t, \varepsilon}^{W, y}$. We say that $Y_{t, \varepsilon}^{W, y}$ is a variational spline of order $t$. It is also shown that every spline is a linear combination of fundamental solutions of the operator $(\varepsilon I + \mathcal{L})^t$ and in this sense it is a polyharmonic function with singularities. Namely it is shown that every spline satisfies the following equation
\[
(\varepsilon I + \mathcal{L})^t Y_{t, \varepsilon}^{W, y} = \sum_{w \in W} \alpha_w \delta_w,
\]
where \( \{ \alpha_w \}_{w \in W} = \{ \alpha_w(Y_{t,\varepsilon}^W,y) \}_{w \in W} \) is a sequence from \( l_2 \) and \( \delta_w \) is the Dirac measure at a vertex \( w \in W \). The set of all such splines for a fixed \( W \subset V(G) \) and fixed \( t > 0, \varepsilon \geq 0 \), will be denoted as \( \mathcal{Y}(W,t,\varepsilon) \).

A fundamental solution \( E_{2t,\varepsilon}^w, \ n \in V(G) \), of the operator \((\varepsilon I + L)^t\) is the solution of the equation

\[
(\varepsilon I + L)^t E_{2t,\varepsilon}^w = \delta_w,
\]

where \( \delta_w \) is the Dirac measure at \( w \in V(G) \).

It is shown in the paper that for every set of vertices \( W = \{ w \} \), every \( t > 0, \varepsilon \geq 0 \), and for any given sequence \( y = \{ y_w \} \in l_2 \), the solution \( Y_{t,\varepsilon}^{W,y} \) of the Variational Problem has a representation

\[
Y_{t,\varepsilon}^{W,y} = \sum_{w \in W} y_w L_{t,\varepsilon}^W w,
\]

where \( L_{t,\varepsilon}^W \) is the so called Lagrangian spline, i.e. it is a solution of the same Variational Problem with constraints \( L_{t,\varepsilon}^W(v) = \delta_{w,v}, w \in W \), where \( \delta_{w,v} \) is the Kronecker delta. Another representation is

\[
Y_{t,\varepsilon}^{W,y} = \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^{W,y}) E_{2t,\varepsilon}^w,
\]

where \( \{ \alpha_w(Y_{t,\varepsilon}^{W,y}) \}_{w \in W} \) is a sequence in \( l_2 \).

Given a function \( f \in L_2(G) \) we will say that the spline \( Y_{t,\varepsilon}^{W,f} \) interpolates \( f \) on \( W \) if \( Y_{t,\varepsilon}^{W,f}(w) = f(w) \) for all \( w \in W \). It is shown in the Theorem 2.4 that for a given function \( f \in L_2(G) \) its interpolating spline \( Y_{t,\varepsilon}^{W,f} \) is always an optimal approximation (modulo given information).

**Remark 2.** It is important to realize that for a fixed set \( W \subset V(G) \) and fixed \( t,\varepsilon \geq 0 \), the correspondence

\[
\{ y_w \} \to \{ \alpha_w(Y_{t,\varepsilon}^{W,y}) \}, y = \{ y_w \} \in l_2,
\]

where \( Y_{t,\varepsilon}^{W,y} \) is a spline, depends just on the geometry of \( G \) and \( W \). In other words the map (1.3) is responsible for the connection between "analysis" on \( G \) and its geometry.

Our main goal is to develop spline interpolation and approximation in the so-called Paley-Wiener spaces.

Paley-Wiener spaces on \( \mathbb{R}^d \) are denoted \( PW_\omega(\mathbb{R}), \omega > 0 \), and contain functions \( f \in L_2(\mathbb{R}) \) whose \( L_2 \)-Fourier transform

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx
\]

has support in \([ -\omega, \omega ] \). The classical sampling theorem says that if \( f \in PW_\omega(\mathbb{R}) \) then \( f \) is completely determined by its values at points \( k\pi/\omega, k \in \mathbb{Z} \), and can be reconstructed in a stable way from the samples \( f(k\pi/\omega) \) by using the so-called cardinal series

\[
f(x) = \sum_{k \in \mathbb{Z}} f \left( \frac{k\pi}{\omega} \right) \frac{\sin(\omega x - k\pi)}{\omega x - k\pi}.
\]
where convergence is understood in the $L_2$-sense. In papers [35], [36], [15], [18], [24], [28], splines were used as a tool for reconstruction of Paley-Wiener functions from their uniqueness sets.

The Paley-Wiener spaces and in particular sampling problems in these spaces attracted attention of many mathematicians [1], [2], [6], [17], [22], [21], [19]. C. Shannon [38], [41], suggested to use the sampling theory in Paley-Wiener spaces as a theoretical foundation for practical problems in signal analysis and information theory. Since then the sampling theory found many other applications in particular in image reconstruction and learning theory [39], [40]. Some of the ideas and methods of the sampling theory of Paley-Wiener functions were recently extended to the cases of Riemannian manifolds, groups, and quantum graphs [8], [9], [10], [11], [13], [14], [23]-[34]. Splines on manifolds and quantum graphs were developed in [32], [33].

To define Paley-Wiener spaces on combinatorial graphs we use the fact that the Laplace operator $\mathcal{L}$ is a self-adjoint positive definite operator in the Hilbert space $L^2(G)$. According to the spectral theory [3] there exist a direct integral of Hilbert spaces $X = \int X(\lambda)dm(\lambda)$ and a unitary operator $F$ from $L^2(G)$ onto $X$, which transforms domain of $\mathcal{L}^s$, $s \geq 0$, onto $X_s = \{x \in X | \lambda^s x \in X\}$ with norm

$$\|x(\tau)\|_{X_s} = \left( \int_{\sigma(\mathcal{L})} \lambda^{2s}\|x(\lambda)\|_X^2 dm(\lambda) \right)^{1/2}$$

and $F(\mathcal{L}^s f) = \lambda^s (Ff)$. We introduce the following notion of discrete Paley-Wiener spaces.

**Definition 1.** Given an $\omega \geq 0$ we will say that a function $f$ from $L^2(G)$ belongs to the Paley-Wiener space $PW_\omega(G)$ if its "Fourier transform" $Ff$ has support in $[0, \omega]$.

To be more consistent with the definition of the classical Paley-Wiener spaces we should consider the interval $[0, \omega^2]$ instead of $[0, \omega]$. We prefer our choice because it makes formulas and notations simpler.

Since the operator $\mathcal{L}$ is bounded every function from $L^2(G)$ belongs to a certain Paley-Wiener space $PW_\omega(G)$ for some $\omega \in \sigma(\mathcal{L})$ and we have the following stratification

$$L^2(G) = PW_{\omega_{\text{max}}}(G) = \bigcup_{\omega \in \sigma(\mathcal{L})} PW_\omega(G), PW_{\omega_1}(G) \subseteq PW_{\omega_2}(G), \omega_1 < \omega_2.$$ 

Different properties of the spaces $PW_\omega(G)$ and in particular a generalization of the Paley-Wiener Theorem are collected in the Theorem 3.1.

For a subset $S \subset V(G)$ (finite or infinite) the notation $L^2(S)$ will denote the space of all functions from $L^2(G)$ with support in $S$:

$$L^2(S) = \{\varphi \in L^2(G), \varphi(v) = 0, v \in V(G) \setminus S\}.$$ 

**Definition 2.** We say that a set of vertices $U \subset V(G)$ is a uniqueness set for a space $PW_\omega(G), \omega > 0$, if for any two functions from $PW_\omega(G)$ the fact that they coincide on $U$ implies that they coincide on $V(G)$.

**Definition 3.** We say that a set of vertices $S \subset V(G)$ is a $\Lambda$-set if for any $\varphi \in L^2(S)$ it admits a Poincare inequality with a constant $\Lambda > 0$

$$\|\varphi\| \leq \Lambda \|\mathcal{L}\varphi\|, \varphi \in L^2(S), \Lambda > 0.$$
The infimum of all $\Lambda > 0$ for which $S$ is a $\Lambda$-set will be called the Poincare constant of the set $S$ and denoted by $\Lambda(S)$.

It is shown in the Theorem 3.4 that if a set $S \subseteq V(G)$ is a $\Lambda$-set, then its complement $U = V(G) \setminus S$ is a uniqueness set for any space $PW_{\omega}(G)$ with $\omega < 1/\Lambda$. Since $L_2(G) = PW_{\omega_{\max}}(G)$ every function in $L_2(G)$ belongs to a certain Paley-Wiener space and one cannot expect that non-trivial uniqueness sets there exist for every Paley-Wiener subspace. But it is reasonable to expect that uniqueness sets exist for Paley-Wiener spaces $PW_{\omega}(G)$ with relatively small $\omega > 0$.

It will be shown (see Section 3) that for every graph $G$ there exists a constant $\Omega_G \geq 1$ such that for $0 < \omega < \Omega_G$ functions from $PW_{\omega}(G)$ can be determined by using their values only on certain subsets of vertices. Namely, it is shown that for any graph $G$ spaces $PW_{\omega}(G)$ with $0 < \omega < \sqrt{1 + \frac{1}{d(G)}} = \Omega_G > 1$ have non-trivial uniqueness sets. A more detailed description of uniqueness sets will be given in a separate paper.

The main result of the present article is obtained in Section 4 and can be stated in the following form.

**Theorem 1.1.** 1) Assume that $\mathcal{L}$ is invertible in $L_2(G)$. If $S$ is a $\Lambda$-set then any $f \in PW_{\omega}(G)$ with $\omega < 1/\Lambda$ can be reconstructed from its values on $U = V(G) \setminus S$ as the following limit

$$f = \lim_{k \to \infty} Y_{k}^{U,f}, k = 2^l, l \in \mathbb{N},$$

where $Y_{k}^{U,f}$ is a spline interpolating $f$ on the set $U = V(G) \setminus S$ and the error estimate is

$$\|f - Y_{k}^{U,f}\| \leq 2 \gamma_k \|f\|, \gamma = \Lambda \omega < 1, k = 2^l, l \in \mathbb{N}.$$

2) If the operator $\mathcal{L}$ in $L_2(G)$ is not invertible, then for any $\Lambda$-set $S$ and any $0 < \varepsilon < 1/\Lambda$, every function $f \in PW_{\omega}(G)$, where

$$0 < \omega < \frac{1}{\Lambda} - \varepsilon,$$

can be reconstructed from its values on $U = V(G) \setminus S$ as the following limit

$$f = \lim_{k \to \infty} Y_{k,\varepsilon}^{U,f}, k = 2^l, l \in \mathbb{N},$$

where $Y_{k,\varepsilon}^{U,f}$ is a spline interpolating $f$ on the set $U = V(G) \setminus S$ and the error estimate is given by

$$\|f - Y_{k,\varepsilon}^{U,f}\| \leq 2 \gamma_k \|f\|, \gamma = \Lambda(\omega + \varepsilon) < 1, k = 2^l, l \in \mathbb{N}.$$

We know two papers [12], [16], in which authors consider sampling on $\mathbb{Z}_N$ and $\mathbb{Z}_N$ and one paper [20] were sampling on $\mathbb{Z}_N$ was used to prove some deep results in discrete harmonic analysis. But our approach to the problem and our results are very different from the methods and results of these papers.
2. Variational splines on combinatorial graphs

We are going to use the same notations and the same Variational Problem which were defined in the Introduction.

**Theorem 2.1.** For every set of vertices $W = \{w\}$, all $k > 0$, $\varepsilon \geq 0$, and for any given sequence $y = \{y_w\} \in l_2$, the Variational Problem has a unique solution.

**Proof.** Consider the set $M_0(W) \subset L_2(G)$, of all functions from $L_2(G)$ whose restriction to $W = \{w\}$ is zero. This is a closed subspace of $L_2(W)$.

Given a sequence of complex numbers $y = \{y_w\} \in l_2$, the linear manifold $\mathcal{M}(W, y)$ of all functions $f$ from $L_2(G)$ such that $f(w) = y_w$ is a shift of the closed subspace $M_0(W)$, i.e.

\[
M(W, y) = M_0(W) + g,
\]

where $g$ is any function from $L_2(G)$ such that for all $w \in W$ one has $g(w) = y_w$.

Consider the orthogonal projection $h_{t, \varepsilon}$ of the function $g \in L_2(G)$ from (2.1) onto the space $M_0(W)$ with respect to the inner product in $H_{t, \varepsilon}(G), t > 0$:

\[
\langle f_1, f_2 \rangle_{H_{t, \varepsilon}(G)} = \left\langle (\varepsilon I + \mathcal{L})^{t/2} f_1, (\varepsilon I + \mathcal{L})^{t/2} f_2 \right\rangle_{L_2(G)}.
\]

The function $Y_{t, \varepsilon} \in \mathcal{M}(W, y)$ is the solution to the above variational problem. Indeed, it is clear that $Y_{t, \varepsilon} \in \mathcal{M}(W, y)$. To show that $Y_{t, \varepsilon}$ is the minimizer, one has

\[
\mathcal{Y} \rightarrow \| (\varepsilon I + \mathcal{L})^{t/2} \mathcal{Y} \|
\]

on the set $\mathcal{M}(W, y)$ we note that any function from $\mathcal{M}(W, y)$ can be written in the form $Y_{t, \varepsilon} + \psi$, where $\psi \in M_0(W)$. Since $Y_{t, \varepsilon} = g - h_{t, \varepsilon}$ is orthogonal to $M_0(W)$ in $H_{t, \varepsilon}(G)$ we obtain for any $\sigma \in \mathbb{C}

\| (\varepsilon I + \mathcal{L})^{t/2} (Y_{t, \varepsilon} + \sigma \psi) \|^2 = \| (\varepsilon I + \mathcal{L})^{t/2} Y_{t, \varepsilon} \|^2 + |\sigma|^2 \| (\varepsilon I + \mathcal{L})^{t/2} \psi \|^2, \psi \in M_0(W),

that means that the function $Y_{t, \varepsilon}$ is the minimizer.

The fact that the minimizer is unique follows from the well-known properties of Hilbert spaces. The proof is complete. 

The following result shows that every solution of the Variational Problem 1)-2) should be a ”polyharmonic function” with ”singularities” on the set $W$.

**Theorem 2.2.** For every set of vertices $W = \{w\}, w \in V(G)$, every $t > 0$, $\varepsilon \geq 0$, and for any given sequence $y = \{y_w\} \in l_2$, the solution $Y_{t, \varepsilon} \in \mathcal{M}(W, y)$ of the Variational Problem satisfies the following equation

\[
(\varepsilon I + \mathcal{L})^t Y_{t, \varepsilon} = \sum_{w \in W} \alpha_w \delta_w,
\]

where $\{\alpha_w\}_{w \in W} = \{\alpha_w(Y_{t, \varepsilon})\}_{w \in W}$ is a sequence from $l_2$. Conversely, if a function satisfies equation (2.2) then it is a spline.

**Proof.** If $\delta_w$ is a Dirac function concentrated at a point $w \in W$ then for any $\phi \in L_2(G)$ the function

\[
\psi = \phi - \sum_{w \in W} \phi(w) \delta_w
\]
belongs to $\mathcal{M}_0(W)$ and because every solution of the above Variational Problem 1)-2) is orthogonal to $\mathcal{M}_0(W)$ in the Hilbert space $H_{2t,\varepsilon}(G)$ we obtain

$$0 = \sum_{v \in V(G)} (\varepsilon I + L)^{t/2} Y_{t,\varepsilon}^{W,y}(v)(\varepsilon I + L)^{t/2} \psi(v).$$

It implies that

$$\sum_{v \in V(G)} (\varepsilon I + L)^t Y_{t,\varepsilon}^{W,y}(v)\phi(v) = \sum_{w \in W} \left< Y_{t,\varepsilon}^{W,y}, \delta_w \right>_{H_{2t,\varepsilon}(G)} \phi(w), \phi \in L_2(G).$$

In other words $(\varepsilon I + L)^t Y_{t,\varepsilon}^{W,y}$ is a function of the form

$$(\varepsilon I + L)^k Y_{t,\varepsilon}^{W,y} = \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^{W,y})\delta_w,$$

where $\alpha_w(Y_{t,\varepsilon}^{W,y}) = \left< Y_{t,\varepsilon}^{W,y}, \delta_w \right>_{H_{2t,\varepsilon}(G)} \in l_2$. Thus we proved that every solution of the Variational Problem is a solution of (2.2). The converse is obvious. The Theorem is proved. \(\square\)

A fundamental solution $E_{2t,\varepsilon}^v, v \in V(G)$, of the operator $(\varepsilon I + L)^t$, is a solution of the equation

$$(\varepsilon I + L)^t E_{2t,\varepsilon}^v = \delta_v,$$

where $\delta_v$ is the Dirac measure at $v \in V(G)$. The following Theorem explains the structure of splines and it follows from (2.2) and linearity of the set of splines which is a consequence of the last Theorem.

**Theorem 2.3.** For every set of vertices $W = \{w\}$, every $t > 0, \varepsilon \geq 0$, and for any given sequence $y = \{y_w\} \in l_2$, the solution $Y_{t,\varepsilon}^{W,y}$ of the Variational Problem has a representation

$$Y_{t,\varepsilon}^{W,y} = \sum_{w \in W} y_w L_{t,\varepsilon}^{W,w},$$

where $L_{t,\varepsilon}^{W,w}$ is the so called Lagrangian spline, i.e. it is a solution of the same Variational Problem with constrains $L_{t,\varepsilon}^{W,w}(v) = \delta_{w,v}, w \in W$, where $\delta_{w,v}$ is the Kronecker delta. Another representation is

$$(2.4) Y_{t,\varepsilon}^{W,y} = \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^{W,y}) E_{2t,\varepsilon}^w,$$

where $\{\alpha_w(Y_{t,\varepsilon}^{W,y})\}_{w \in W}$ is a sequence in $l_2$.

Now we are going to show that variational interpolating splines provide an optimal approximation.

**Definition 4.** For the given $W \subset V(G), f \in L_2(G), t > 0, \varepsilon \geq 0, K > 0$, the notation $Q(W, f, t, \varepsilon, K)$ will be used for a set of all functions $g$ in $L_2(G)$ such that

1) $g(w) = f(w), w \in W$,

and

2) $\|((\varepsilon I + L)^{t/2}g\| \leq K$.

It is easy to verify that every set $Q(W, f, k, \varepsilon, K)$ is convex, bounded, and closed. The next Theorem shows that for a given function $f \in L_2(G)$ its interpolating spline $Y_{t,\varepsilon}^{W,f}$ is always an optimal approximation (modulo given information).
Theorem 2.4. The following statements hold true:

1) If $K < \| (\varepsilon I + \mathcal{L})^{1/2} Y_{t,\varepsilon}^W, f \|_{\mathcal{H}_{t,\varepsilon}(\mathcal{G})}$ then the set $Q(W, f, t, \varepsilon, K)$ is empty.

2) Every variational spline $Y_{t,\varepsilon}^W$ is the center of the convex set $Q(W, f, t, \varepsilon, K)$. As a result the following inequalities holds true for any $g \in Q(W, f, t, \varepsilon, K)$

$$\| Y_{t,\varepsilon}^W - g \|_{\mathcal{H}_{t,\varepsilon}(\mathcal{G})} \leq \frac{1}{2} \text{diam}Q(W, f, t, \varepsilon, K),$$

and

$$\| Y_{t,\varepsilon}^W - g \|_{L_2(\mathcal{G})} \leq \frac{1}{2} \| (\varepsilon I + \mathcal{L})^{-1/2} \| \text{diam}Q(W, f, t, \varepsilon, K),$$

where diam is taken with respect to the norm of the Sobolev space $\mathcal{H}_{t,\varepsilon}(\mathcal{G})$.

Proof. Given a function $f \in L_2(\mathcal{G})$ the linear manifold $\mathcal{I}(W, f)$ is the set of all functions $g$ from $L_2(\mathcal{G})$ such that $f(w) = g(w), w \in W$. Let us note that the distance from zero to the subspace $\mathcal{I}(W, f)$, in the metric of the space $\mathcal{H}_{t,\varepsilon}(\mathcal{G})$ is exactly the Sobolev norm of the unique spline $Y_{k,\varepsilon}^W \in \mathcal{I}(W, f)$. This norm can be expressed in terms of the sequence $(Y_{t,\varepsilon}^W)(w) = f(w), w \in W$, and the sequence $\{\alpha_w(Y_{t,\varepsilon}^W)\}, w \in W$, from the representation

$$Y_{t,\varepsilon}^W = \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^W) E^w_{t,\varepsilon},$$

Indeed,

$$\| Y_{t,\varepsilon}^W \|_{\mathcal{H}_{t,\varepsilon}(\mathcal{G})} = \left\langle (\varepsilon I + \mathcal{L})^{1/2} Y_{t,\varepsilon}^W, (\varepsilon I + \mathcal{L})^{1/2} Y_{t,\varepsilon}^W \right\rangle^{1/2} =$$

$$\left\langle (\varepsilon I + \mathcal{L})^{1/2} Y_{t,\varepsilon}^W, Y_{t,\varepsilon}^W \right\rangle^{1/2} = \left\langle \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^W) \delta_w, Y_{t,\varepsilon}^W \right\rangle^{1/2} =$$

$$\left( \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^W) f(w) \right)^{1/2}.$$ 

It shows that the intersection

$$Q(W, f, t, \varepsilon, K) = \mathcal{I}(W, f) \cap B_{t,\varepsilon}(0, K),$$

where $B_{t,\varepsilon}(0, K)$ is the ball in $\mathcal{H}_{t,\varepsilon}(\mathcal{G})$ whose center is zero and the radius is $K$, is not empty if and only if

$$K \geq \| Y_{t,\varepsilon}^W \|_{\mathcal{H}_{t,\varepsilon}(\mathcal{G})} = \left( \sum_{w \in W} \alpha_w(Y_{t,\varepsilon}^W) f(w) \right)^{1/2}.$$ 

The first part of the Theorem is proved.

Now we are going to show that for a given function $f$ the interpolating spline $Y_{t,\varepsilon}^W$ is the center of the convex, closed and bounded set $Q(W, f, t, \varepsilon, K)$ for any $K \geq \| Y_{t,\varepsilon}^W \|_{\mathcal{H}_{t,\varepsilon}(\mathcal{G})}$. In other words it is sufficient to show that if

$$Y_{t,\varepsilon}^W + h \in Q(W, f, t, \varepsilon, K)$$

for some function $h$ from the Sobolev space $\mathcal{H}_{t,\varepsilon}(\mathcal{G})$ then the function $Y_{t,\varepsilon}^W - h$ also belongs to the same intersection. Indeed, since $h$ is zero on the set $W$ one has
\[
\left\langle (\varepsilon I + \mathcal{L})^{t/2} Y^{W,f}_t, (\varepsilon I + \mathcal{L})^{t/2} h \right\rangle = \left\langle (\varepsilon I + \mathcal{L})^{t} Y^{W,f}_t, h \right\rangle = 0.
\]

But then
\[
\left\| (\varepsilon I + \mathcal{L})^{t/2} (Y^{W,f}_t + h) \right\|_{L^2(G)} = \left\| (\varepsilon I + \mathcal{L})^{t/2} (Y^{W,f}_t - h) \right\|_{L^2(G)}.
\]

In other words,
\[
\left\| (\varepsilon I + \mathcal{L})^{t/2} (Y^{W,f}_t - h) \right\|_{L^2(G)} \leq K
\]

and because \( Y^{W,f}_t + h \) and \( Y^{W,f}_t - h \) take the same values on \( W \) the function \( Y^{W,f}_t - h \) belongs to \( Q(W, f, t, \varepsilon, K) \). It is clear that the following inequality holds true
\[
\| Y^{W,f}_t - g \|_{H^t(G)} \leq \frac{1}{2} \text{diam} Q(W, f, t, \varepsilon, K)
\]

for any \( g \in Q(W, f, t, \varepsilon, K) \). Using this inequality one obtains
\[
\left\| Y^{W,f}_t - g \right\|_{L^2(G)} = \left\| (\varepsilon I + \mathcal{L})^{-t/2} (\varepsilon I + \mathcal{L})^{t/2} (Y^{W,f}_t - g) \right\|_{L^2(G)} \leq \frac{1}{2} \| (\varepsilon I + \mathcal{L})^{-t/2} \| \text{diam} Q(W, f, t, \varepsilon, K).
\]

The Theorem is proven. \( \square \)

3. Paley-Wiener spaces on combinatorial graphs

The Paley-Wiener spaces \( PW_\omega(G), \omega > 0 \), were introduced in the Definition 1 of the Introduction. Since the operator \( \mathcal{L} \) is bounded it is clear that every function from \( L^2(G) \) belongs to a certain Paley-Wiener space. Note that if
\[
\omega_{\text{min}} = \inf_{\omega \in \sigma(\mathcal{L})} \omega
\]

then the space \( PW_\omega(G) \) is not trivial if and only if \( \omega \geq \omega_{\text{min}} \).

Using the spectral resolution of identity \( P_\lambda \) we define the unitary group of operators by the formula
\[
e^{it\mathcal{L}} f = \int_{\sigma(\mathcal{L})} e^{it\tau} dP_\tau f, f \in L^2(G), t \in \mathbb{R}.
\]

The next theorem can be considered as a form of the Paley-Wiener theorem and it essentially follows from a more general result in [25].

**Theorem 3.1.** The following statements hold true:

1) \( f \in PW_\omega(G) \) if and only if for all \( s \in \mathbb{R}_+ \) the following Bernstein inequality takes place
\[
\| \mathcal{L}^s f \| \leq \omega^s \| f \|; \tag{3.1}
\]

2) the norm of the operator \( \mathcal{L} \) in the space \( PW_\omega(G) \) is exactly \( \omega \);

3) \( f \in PW_\omega(G) \) if and only if the following holds true
\[
\lim_{s \to \infty} \| \mathcal{L}^s f \|^{1/s} = \omega, s \in \mathbb{R}_+.
\]
4) $f \in PW_\omega(G)$ if and only if for every $g \in L^2(G)$ the scalar-valued function of the real variable $t \in \mathbb{R}^1$

$$(e^{it\mathcal{L}}f, g) = \sum_{v \in V} e^{it\mathcal{L}}f(v)\overline{g(v)}$$

is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type $\omega$;

5) $f \in PW_\omega(G)$ if and only if the abstract-valued function $e^{it\mathcal{L}}f$ is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type $\omega$;

6) $f \in PW_\omega(G)$ if and only if the solution $u(t,v), t \in \mathbb{R}, v \in V(G)$, of the Cauchy problem for the corresponding Schrodinger equation

$$i \frac{\partial u(t,v)}{\partial t} = \mathcal{L}u(t,v), u(0,v) = f(v), i = \sqrt{-1},$$

has analytic extension $u(z,v)$ to the complex plane $\mathbb{C}$ as an entire function and satisfies the estimate

$$\|u(z,\cdot)\|_{L^2(G)} \leq e^{\omega|z|}\|f\|_{L^2(G)}.$$  

We prove here only the first part of the Theorem.

**Lemma 3.2.** A function $f \in L^2(G)$ belongs to $PW_\omega(G)$ if and only if the following Bernstein inequality holds true for all $s \in \mathbb{R}^+$

$$(3.2) \quad \|\mathcal{L}^s f\| \leq \omega^s \|f\|.$$

**Proof.** We use the spectral theorem for the operator $\mathcal{L}$ in the space $L^2(G)$ in the form it was presented in the Introduction.

Let $f$ belongs to the space $PW_\omega(G)$ and $\mathcal{F}_\mathcal{L} f = x \in X$. Then

$$\left(\int_0^\infty \lambda^{2s}\|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda)\right)^{1/2} = \left(\int_0^\infty \lambda^{2s}\|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda)\right)^{1/2} \leq \omega^s \|x\|_X, s \in \mathbb{R}^+, \omega$$

which gives Bernstein inequality for $f$.

Conversely, if $f$ satisfies Bernstein inequality then $x = \mathcal{F}_\mathcal{L} f$ satisfies $\|x\|_X \leq \omega^s \|x\|_X$. Suppose that there exists a set $\sigma \subset [0, \infty) \setminus [0, \omega]$ whose $m$-measure is not zero and $x|_\sigma \neq 0$. We can assume that $\sigma \subset [\omega + \epsilon, \infty)$ for some $\epsilon > 0$. Then for any $s \in \mathbb{R}^+$ we have

$$\int_\sigma \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \leq \int_{\omega + \epsilon}^\infty \lambda^{-2s}\|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \leq \|x\|_X^2 (\omega/\omega + \epsilon)^{2s},$$

which shows that or $x(\lambda)$ is zero on $\sigma$ or $\sigma$ has measure zero.

The Theorem 3.1 shows that the notion of Paley-Wiener functions of type $\omega$ on a combinatorial graph can be completely understood in terms of familiar entire functions of exponential type $\omega$ bounded on the real line.

The notion of $\Lambda$-sets was introduced in the Definition 3 in the Introduction. The role of $\Lambda$-sets is explained in the following Theorem.

**Theorem 3.3.** If a set $S \subset V(G)$ is a $\Lambda$-set, then the set $U = V(G) \setminus S$ is a uniqueness set for any space $PW_\omega(G)$ with $\omega < 1/\Lambda$. 

Proof. If \( f, g \in PW_\omega(G) \) then \( f - g \in PW_\omega(G) \) and according to the Theorem 3.1 the following Bernstein inequality holds true

\[
\|L(f - g)\| \leq \omega \|f - g\|.
\]

If \( f \) and \( g \) coincide on \( U = V(G) \setminus S \) then \( f - g \) belongs to \( L_2(S) \) and since \( S \) is a \( \Lambda \)-set we have

\[
\|f - g\| \leq \Lambda \|L(f - g)\|.
\]

Assume that \( \omega < 1/\Lambda \) and that \( f \) is not identical to \( g \). We have the following inequalities

\[
\|f - g\| \leq \Lambda \|L(f - g)\| \leq \Lambda \omega \|f - g\| < \|f - g\|, \omega < 1/\Lambda,
\]

which provide the desired contradiction if \( f - g \) is not identical zero. It proves the Theorem. \( \square \)

As it was mentioned in the Introduction one cannot expect that non-trivial uniqueness sets there exist for functions from every Paley-Wiener subspace. But it is reasonable to expect that uniqueness sets exist for Paley-Wiener spaces \( PW_\omega(G) \) with relatively small \( \omega > 0 \). Indeed, a direct calculation shows that for any graph \( G \) spaces \( PW_\omega(G) \) with

\[
0 < \omega < \sqrt{1 + \frac{1}{d(G)}} = \Omega_G > 1, d(G) = \max_{v \in V(G)} d(v),
\]

have non-trivial uniqueness sets.

Here are two examples of Paley-Wiener spaces on graphs and their uniqueness sets.

1. **Finite graphs.** If a set of vertices \( V(G) \) of a graph \( G \) is finite then the spectrum of the Laplace operator is discrete and the space \( PW_\omega(G) \) is a span of eigenfunctions whose eigenvalues \( \leq \omega \). In this case if \( U \) is a uniqueness sets for a space \( PW_\omega(G) \) then \( |U| \) is at least a number of eigenvalues (with multiplicities) of \( L \) on the interval \([0, \omega]\).

2. **Lattice \( \mathbb{Z}^n \).** We consider a one-dimensional lattice \( \mathbb{Z} \). In this case there is a version of the Fourier transform \( \mathcal{F} \) on the space \( L_2(\mathbb{Z}) \) which is defined by the formula

\[
\mathcal{F}(f)(\xi) = \sum_{k \in \mathbb{Z}} f(k) e^{ik\xi}, f \in L_2(\mathbb{Z}), \xi \in [-\pi, \pi).
\]

It gives a unitary operator from \( L_2(G) \) on the space \( L_2(\mathbb{T}) = L_2(\mathbb{T}, d\xi/2\pi) \), where \( \mathbb{T} \) is the one-dimensional torus and \( d\xi/2\pi \) is the normalized measure. One can verify the following formula

\[
\mathcal{F}(L_\mathbb{Z}f)(\xi) = 2 \sin^2 \frac{\xi}{2} \mathcal{F}(f)(\xi),
\]

where \( L_\mathbb{Z} \) is the Laplace operator on the graph \( \mathbb{Z} \). The next result is obvious.

**Theorem 3.4.** The spectrum of the Laplace operator \( L_\mathbb{Z} \) on the one-dimensional lattice \( \mathbb{Z} \) is the set \([0, 2]\). A function \( f \) belongs to the space \( PW_\omega(\mathbb{Z}) \), \( 0 < \omega < 2 \), if and only if the support of \( \mathcal{F}f \) is a subset \( \Omega_\omega \) of \([\pi, \pi)\) on which \( 2 \sin^2 \frac{\xi}{2} \leq \omega \).
Our nearest goal is to show that for a one-dimensional line graph $Z$ the estimates in Poincare inequalities of finite successive sets of vertices can be computed explicitly.

Consider a set of successive vertices $S = \{v_1, v_2, ..., v_N\}$, and the corresponding space $L_2(S)$. If $bS = \{v_0, v_{N+1}\}$ is the boundary of $S$, then for any $\varphi \in L_2(S)$ the function $\mathcal{L}_Z\varphi$ has support on $S \cup bS$ and

$$\mathcal{L}_Z\varphi(v_0) = -\varphi(v_1), \mathcal{L}_Z\varphi(v_1) = 2\varphi(v_1) - \varphi(v_2),$$

$$\mathcal{L}_Z\varphi(v_N) = 2\varphi(v_N) - \varphi(v_{N-1}), \mathcal{L}_Z\varphi(v_{N+1}) = -\varphi(v_N),$$

and for any other $v_j$ with $2 \leq j \leq N - 1$,

$$\mathcal{L}_Z\varphi(v_j) = -\varphi(v_{N-1}) + 2\varphi(v_j) - \varphi(v_{N+1}).$$

Let $C_{2N+2} = \Gamma(S)$ be a cycle graph

$$C_{2N+2} = \{u_{-N-1}, u_{-N}, ..., u_{-1}, u_0, u_1, u_2, ..., u_N, u_{N+1}\}$$

with the following identification

$$u_{-N-1} = u_{N+1}.$$ 

Thus the total number of vertices in $C_{2N+2}$ is $2N + 2$. We introduce an embedding of $S \cup bS$ into $C_{2N+2}$ by the following identification

$$v_0 = u_0, v_1 = u_1, ..., v_N = u_N, v_{N+1} = u_{N+1}.$$ 

This embedding gives a rise to an embedding of $L_2(S)$ into $L_2(C_{2N+2})$, namely every $\varphi \in L_2(S)$ is identified with a function $F_\varphi \in L_2(C_{2N+2})$ for which

$$F_\varphi(u_0) = 0, F_\varphi(u_1) = \varphi(v_1), ..., F_\varphi(u_N) = \varphi(v_N), F_\varphi(u_{N+1}) = 0,$$

and also

$$F_\varphi(u_{-1}) = -\varphi(v_1), ..., F_\varphi(u_{-N}) = -\varphi(v_N).$$

It is important to note that

$$\sum_{u \in C_{2N+2}} F_\varphi(u) = 0.$$

If $\mathcal{L}_C$ is the Laplace operator on the cycle $C_{2N+2}$ then a direct computation shows that for the vector $F_\varphi$ defined above the following is true

$$2\|\varphi\| = \|F_\varphi\|, 2\|\mathcal{L}_Z\varphi\| = \|\mathcal{L}_C F_\varphi\|, \varphi \in L_2(S), F_\varphi \in L_2(C_{2N+2}).$$

The operator $\mathcal{L}_C$ in $L_2(C_{2N+2})$ has a complete system of orthonormal eigenfunctions

$$\psi_n(k) = \exp 2\pi i \frac{n}{2N + 2} k, 0 \leq n \leq 2N + 1, 1 \leq k \leq 2N + 2,$$

with eigenvalues

$$\lambda_n = 1 - \cos \frac{2\pi n}{2N + 2}, 0 \leq n \leq 2N + 1.$$ 

The definition of the function $F_\varphi \in L_2(C_{2N+2})$ implies that it is orthogonal to all constants and its Fourier series does not contain a term which corresponds to the index $n = 0$. It allows to obtain the following estimate

$$\|\mathcal{L}_C F_\varphi\|^2 = \sum_{n=1}^{2N+1} \lambda_n^2 |\langle F_\varphi, \psi_n \rangle|^2 \geq 4 \sin^4 \frac{\pi}{2N + 2} \|\mathcal{L}_C F_\varphi\|^2.$$
It gives the following estimate for functions \( \varphi \) from \( L_2(S) \)

\[
\| \varphi \| \leq \frac{1}{2} \sin^{-2} \frac{\pi}{2N + 2} \| L_Z \varphi \|.
\]

Thus we proved the following Lemma.

**Lemma 3.5.** If \( S = \{v_1, v_2, \ldots, v_N\} \) consists of \( |S| = N \) successive vertices of a line graph \( Z \) then it is a \( \Lambda \)-set for

\[
\Lambda = \frac{1}{2} \sin^{-2} \frac{\pi}{2|S| + 2}.
\]

In other words, for any \( \varphi \in L_2(S) \) the following inequality holds true

\[
\| \varphi \| \leq \Lambda \| L_Z \varphi \|.
\]

**Remark 3.** The last inequality which can be written as

\[
\| L_Z \varphi \| \geq 2 \sin^2 \frac{\pi}{2|S| + 2} \| \varphi \|, \varphi \in L_2(S),
\]

is similar to one of inequalities in [7].

Note that in the case \( |S| = 1 \) the last Lemma gives the inequality

\[
\| \delta_v \| \leq \| L_Z \delta_v \|, S = \{v\},
\]

but direct calculations give a better value for \( \lambda \):

\[
\| \delta_v \| = \sqrt{\frac{2}{3}} \| L_Z \delta_v \|, v \in V.
\]

Let us note that if \( \{S_j\} \) is a finite or infinite sequence of disjoint subsets of vertices \( S_j \subset V \) such that the sets \( S_j \cup bS_j \) are pairwise disjoint and every \( S_j \) has type \( \Lambda_j \), then their union \( S = \bigcup_j S_j \) is a set of type \( \Lambda = \sup_j \Lambda_j \). Indeed, since the sets \( S_j \) are disjoint every function \( \varphi \in L_2(S), S = \bigcup_j S_j \), is a sum of functions \( \varphi_j \in L_2(S_j) \) which are pairwise orthogonal. Moreover because the sets \( S_j \cup bS_j \) are disjoint the functions \( L_Z \varphi_j \) are also orthogonal. Thus we have

\[
\| \varphi \|^2 = \sum_j \| \varphi_j \|^2 \leq \sum_j \Lambda_j^2 \| L_Z \varphi_j \|^2 \leq \Lambda^2 \| L_Z \varphi \|^2,
\]

where \( \Lambda = \sup_j \Lambda_j \).

A combination of this observation along with the last Lemma 3.6 gives the following result for any \( 0 < \omega < \sqrt{3/2} \).

**Theorem 3.6.** If \( S \) is a finite or infinite union of disjoint sets \( \{S_j\} \) of successive vertices such that

1) the sets \( S_j \cup bS_j \) are disjoint

and

2) for every \( j \) the following inequality holds

\[
|S_j| < \frac{\pi}{2 \arcsin \sqrt{\frac{3}{2}}} - 1,
\]

then every function \( f \in PW_\omega(Z) \) is uniquely determined by its values on the set \( U = V(Z) \setminus S \).
A similar result holds true for a lattice $\mathbb{Z}^n$ of any dimension. Consider for example the case $n = 2$. In this situation the Fourier transform $\mathcal{F}$ on the space $L_2(\mathbb{Z}^2)$ is the unitary operator $\mathcal{F}$ which is defined by the formula
\[
\mathcal{F}(f)(\xi_1, \xi_2) = \sum_{(k_1, k_2) \in \mathbb{Z}^2} f(k_1, k_2) e^{ik_1\xi_1 + ik_2\xi_2}, f \in L_2(\mathbb{Z} \times \mathbb{Z}),
\]
where $(\xi_1, \xi_2) \in [-\pi, \pi) \times [-\pi, \pi)$. The operator $\mathcal{F}$ is isomorphism of the space $L_2(G)$ on the space $L_2(T \times T) = L_2(T \times T, d\xi_1 d\xi_2/4\pi^2)$, where $T$ is the one-dimensional torus. the following formula holds true
\[
\mathcal{F}(\mathcal{L}_{\mathbb{Z}^2} f)(\xi) = \left( \sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} \right) \mathcal{F}(f)(\xi),
\]
where $\mathcal{L}_{\mathbb{Z}^2}$ is the Laplace operator on the graph $\mathbb{Z}^2$. We have the following result.

**Theorem 3.7.** The spectrum of the Laplace operator on the lattice $\mathbb{Z}^2$ is the set $[0, 2]$. A function $f$ belongs to the space $PW_\omega(\mathbb{Z} \times \mathbb{Z}), 0 < \omega < 2$, if and only if the support of $\mathcal{F} f$ is a subset $\Omega_\omega$ of $[-\pi, \pi) \times [-\pi, \pi)$ on which
\[
\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} \leq \omega.
\]

Given a set $S = \{v_{n,m}\}, 1 \leq n \leq N, 1 \leq m \leq M$, we consider embedding of $S$ into two-dimensional discrete torus of the size $T = (2N + 2) \times (2M + 2) = \{u_{n,m}\}$. Every $f \in L_2(S)$ is identified with a function $g \in L_2(T)$ in the following way
\[
g(u_{n,m}) = f(v_{n,m}), 1 \leq n \leq N, 1 \leq m \leq M,
\]
and
\[
g(u_{n,m}) = 0, N < n \leq N + 2, M < m \leq M + 2.
\]

We have
\[
\|\mathcal{L}_{\mathbb{Z}^2} f\| = \|\mathcal{L}_T g\|
\]
where $\mathcal{L}_T$ is the combinatorial Laplacian on the discrete torus $T$. Since eigenfunctions of $\mathcal{L}_T$ are products of the corresponding functions (3.1) a direct calculation gives the following inequality
\[
\|\varphi\| \leq \frac{1}{4} \min \left( \sin \frac{\pi}{2N_1 + 2}, \sin \frac{\pi}{2N_2 + 2}, ..., \sin \frac{\pi}{2N_n + 2} \right) \|\mathcal{L}_{\mathbb{Z}^2} \varphi\|, \varphi \in L_2(S).
\]

In a similar way one can obtain corresponding results for a lattice $\mathbb{Z}^n$ of any dimension. Note that the spectrum of the Laplace operator on $\mathbb{Z}^n$ is $[0, 2]$ and $\Omega_{2^n} = \sqrt{\frac{2^n + n}{2n}}/2$.

Let $N_j = \{N_{1,j}, ..., N_{n,j}\}, j \in \mathbb{N}$, be a sequence $n$-tuples of natural numbers. For every $j$ the notation $S(N_j)$ will be used for a "rectangular solid" of "dimensions" $N_{1,j} \times N_{2,j} \times ... \times N_{n,j}$.

Using these notations we formulate the following sampling Theorem.

**Theorem 3.8.** If $S$ is a finite or infinite union of rectangular solids $\{S(N_j)\}$ of vertices of dimensions $N_{1,j} \times N_{2,j} \times ... \times N_{n,j}$ such that
1) the sets $S_j = S(N_j) \cup bS(N_j)$ are disjoint,
and
2) the following inequality holds true for all $j$
\[
\omega < 4 \min \left( \sin \frac{\pi}{2N_1,j + 2}, \sin \frac{\pi}{2N_2,j + 2}, ..., \sin \frac{\pi}{2N_n,j + 2} \right),
\]
then every \( f \in PW_\omega(\mathbb{Z}^n) \) is uniquely determined by its values on \( U = V(\mathbb{Z}^n) \setminus S \).

4. Reconstruction of Paley-Wiener spaces using splines

Now we are going to use variational splines \( Y_{k,\varepsilon}^{U,f} \) as a reconstruction tool of Paley-Wiener functions \( f \in PW_\omega(G) \) from their values on uniqueness sets of the form \( U = V(G) \setminus S \), where \( S \) is a \( \Lambda \)-set and \( \Lambda < 1/\omega \). We will need the following Lemma.

Lemma 4.1. If \( A \) is a bounded self-adjoint positive definite operator in a Hilbert space \( H \) and a positive \( a > 0 \), the following inequality holds true:

\[
\|\varphi\| \leq a\|A\varphi\|,
\]

then for the same \( \varphi \in H \), and all \( k = 2^l, l = 0, 1, 2, \ldots \) the following inequality holds:

\[
\|\varphi\| \leq a^k\|A^k\varphi\|.
\]

Proof. By the spectral theory [3] there exist a direct integral of Hilbert spaces

\[
X = \int_0^{\|A\|} X(\tau) d\mu(\tau)
\]

and a unitary operator \( F \) from \( H \) onto \( X \), which transforms domain of \( A^t, t \geq 0 \), onto \( X_t = \{ x \in X \mid \tau^tx \in X \} \) with norm

\[
\|A^t\|_H = \left( \int_0^{\|A\|} \tau^{2t}\|Ff(\tau)\|^2_{X(\tau)} d\mu(\tau) \right)^{1/2}
\]

and \( F(A^t f) = \tau^t(Ff) \). According to our assumption we have for a particular \( \varphi \in H \)

\[
\int_0^{\|A\|} |F\varphi(\tau)|^2 d\mu(\tau) \leq a^2 \int_0^{\|A\|} \tau^2|F\varphi(\tau)|^2 d\mu(\tau)
\]

and then for the interval \( B = B(0, a^{-1}) \) we have

\[
\int_B |F\varphi(\tau)|^2 d\mu(\tau) + \int_{[0,\|A\|] \setminus B} |F\varphi|^2 d\mu(\tau) \leq a^2 \left( \int_B \tau^2|F\varphi|^2 d\mu(\tau) + \int_{[0,\|A\|] \setminus B} \tau^2|F\varphi|^2 d\mu(\tau) \right).
\]

Since \( a^2\tau^2 < 1 \) on \( B(0, a^{-1}) \)

\[
0 \leq \int_B (|F\varphi|^2 - a^2\tau^2|F\varphi|^2) d\mu(\tau) \leq \int_{[0,\|A\|] \setminus B} \left( a^2\tau^2|F\varphi|^2 - |F\varphi|^2 \right) d\mu(\tau).
\]

This inequality implies the inequality

\[
0 \leq \int_B (a^2\tau^2|F\varphi|^2 - a^4\tau^4|F\varphi|^2) d\mu(\tau) \leq \int_{[0,\|A\|] \setminus B} \left( a^4\tau^4|F\varphi|^2 - a^2\tau^2|F\varphi|^2 \right) d\mu(\tau)
\]

or

\[
a^2 \int_{[0,\|A\|]} \tau^2|F\varphi|^2 d\mu(\tau) \leq a^4 \int_{B_+} \tau^4|F\varphi|^2 d\mu(\tau),
\]

which means

\[
\|\varphi\| \leq a\|A\varphi\| \leq a^2\|A^2\varphi\|.
\]

Now, by using induction one can finish the proof of the Lemma. The Lemma is proved. \( \square \)
Proof of the first part of the Theorem 1.1.

We assume that the operator $L$ has bounded inverse and $\varepsilon = 0$. If $f \in PW_\omega(G)$ and $Y_{k,f}(Y_{k,f} = Y_{k,f})$ is a variational spline which interpolates $f$ on a set $U = V(G) \setminus S$, where $S$ is a $\Lambda$-set $(0 < \omega < 1/\Lambda)$, then $f - Y_{k,f} \in L_2(S)$ and we have

\begin{equation}
\|f - Y_{k,f}\| \leq \Lambda \|L(f - Y_{k,f})\|.
\end{equation}

At this point we can apply the last Lemma with $A = L$, $a = \Lambda$ and $\varphi = f - Y_{k,f}$. It gives the inequality

\begin{equation}
\|f - Y_{k,f}\| \leq \Lambda^k \|L^k(f - Y_{k,f})\|
\end{equation}

for all $k = 2^l, l = 0, 1, 2, \ldots$. Since the interpolant $Y_{k,f}$ minimizes the norm $\|L^k \cdot \|$ it gives

\[\|f - Y_{k,f}\| \leq 2\Lambda^k \|L^k f\|, k = 2^l, l \in \mathbb{N}.
\]

Because for functions $f \in PW_\omega(G)$ the Bernstein inequality holds

\[\|L^m f\| \leq \omega^m \|f\|, m \in \mathbb{N},\]

it implies the first part of the Theorem 1.1:

\[\|f - Y_{k,f}\| \leq 2\gamma^k \|f\|, \gamma = \Lambda \omega < 1, k = 2^l, l \in \mathbb{N}.
\]

Proof of the second part of the Theorem 1.1.

Now we assume that the operator $L$ is not invertible (it is a typical situation on any finite graph). We fix an

\[0 < \varepsilon < \frac{1}{\Lambda}\]

and assume that

\[0 < \omega < \frac{1}{\Lambda} - \varepsilon.
\]

If $f \in PW_\omega(G)$ and $Y_{k,\varepsilon}$ is a variational spline which interpolates $f$ on a set $U = V(G) \setminus S$ where $S$ is a $\Lambda$-set then $f - Y_{k,\varepsilon} \in L_2(S)$ and we have

\begin{equation}
\|f - Y_{k,\varepsilon}\| \leq \Lambda \|L(f - Y_{k,\varepsilon})\|.
\end{equation}

For any $g \in L_2(G)$ the following inequality holds true

\begin{equation}
\|Lg\| \leq \|(\varepsilon I + L)g\|.
\end{equation}

Thus the inequalities (4.3) and (4.4) imply the inequality

\[\|f - Y_{k,\varepsilon}\| \leq \Lambda \|(\varepsilon I + L)(f - Y_{k,\varepsilon})\|.
\]

We apply the Lemma 4.1 with $A = \varepsilon I + L$, $a = \Lambda$ and $\varphi = f - Y_{k,\varepsilon}$. It gives the inequality

\[\|f - Y_{k,\varepsilon}\| \leq \Lambda^k \|(\varepsilon I + L)^k(f - Y_{k,\varepsilon})\|
\]

for all $k = 2^l, l = 0, 1, 2, \ldots$. Using the minimization property of $Y_{k,\varepsilon}$ we obtain

\[\|f - Y_{k,\varepsilon}\| \leq 2\Lambda^k \|(\varepsilon I + L)^k f\|, k = 2^l, l \in \mathbb{N}.
\]

If $f \in PW_\omega(G)$, then the Bernstein inequality

\[\|L^m f\| \leq \omega^m \|f\|, m \in \mathbb{N},
\]

implies the inequality

\[\|(\varepsilon I + L)^m f\| \leq (\omega + \varepsilon)^m \|f\|, m \in \mathbb{N}.
\]
After all we have the following inequality
\[ \| f - Y_{k,l}^{U,f} \| \leq 2\gamma^k \| f \|, \gamma = \Lambda(\omega + \varepsilon) < 1, k = 2^l, l \in \mathbb{N}. \]

The proof of the Theorem 1.1 is complete.

References
[1] A. Beurling, Local Harmonic analysis with some applications to differential operators, Some Recent Advances in the Basic Sciences, vol. 1, Belfer Grad. School Sci. Annu. Sci. Conf. Proc., A. Gelbart, ed., 1963-1964, 109-125.
[2] A. Beurling and P. Malliavin, On the closure of characters and the zeros of entire functions, Acta Math., 118, (1967), 79-95.
[3] M. Birman and M. Solomyak, Spectral theory of selfadjoint operators in Hilbert space, D.Reidel Publishing Co., Dordrecht, 1987.
[4] F. R. K. Chung, Spectral Graph Theory, CBMS 92, AMS, 1994.
[5] J. Duchon, Splines minimizing rotation-invariant seminorms in Sobolev spaces, in ”Constructive Theory of Functions of Several Variables” (W.Schempp and K.Zeller, eds.), pp. 85-100, Springer-Verlag, New York/Berlin, 1977.
[6] R. Duffin, A. Schaeffer, A class of nonharmonic Fourier series, Trans. AMS, 72, (1952), 341-366.
[7] K. Fan, O. Taussky, J. Todd, Discrete analogs of inequalities of Wirtinger, Monatsh. fur Mathematik, 59, (1955), 73-90.
[8] M. Ebata, M. Eguchi, S. Koizumi, K. Kumahara, On sampling formulas on symmetric spaces, J. Fourier Anal. Appl. 12 (2006), no. 1, 1–15.
[9] M. Ebata, M. Eguchi, S. Koizumi, K. Kumahara, Analogues of sampling theorems for some homogeneous spaces, Hiroshima Math. J. 36 (2006), no. 1, 125–140.
[10] H. Feichtinger and I. Pesenson, Iterative recovery of band limited functions on manifolds, in Wavelets, Frames and Operator Theory, Contemp. Math., 345, AMS, (2004), 137-153.
[11] H. Feichtinger and I. Pesenson, A reconstruction method for band-limited signals on the hyperbolic plane, Samp. Theory Signal Image Process. 4 (2005), no. 2, 107–119.
[12] M.W. Frazier, R. Torres, The sampling theorem, \( \varphi \)-transform, and Shannon wavelets for \( \mathbb{R}, \mathbb{Z}, \mathbb{T}, \) and \( \mathbb{Z}^d \). Wavelets: Mathematics and Applications, J.J. Benedetto and M.W. Frazier, ed., 221-246, Stud. Adv. Math., CRC, Boca Raton, FL, 1994.
[13] H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Lecture Notes in Mathematics, 1863, Springer, 2005.
[14] H. Führ and K. Gröchenig, Sampling theorems on locally compact groups from oscillation estimates, Math. Z., 255 (2007), no. 1, 177-194.
[15] M. Golitschek, On the convergence of interpolating periodic spline functions of high degree, Numer. Math., 19 (1972), 146-154.
[16] K. Gröchenig, A Discrete Theory of Irregular Sampling, Linear Algebra and its Applications, 193(1993), 129-150.
[17] H. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, Acta. Math., 117, (1967), 37-52.
[18] Y. Lyubarskii, W. R. Madych, The Recovery of Irregularly Sampled Band Limited Functions via Tempered Splines, J. of Functional Analysis 125 (1994), 201–222.
[19] Y. Lyubarskii, K. Seip, Weighted Paley-Wiener spaces, J. Amer. Math. Soc. 15(2002), no. 4, 979-1006.
[20] A. Magyar, E. M. Stein, S. Wainger, Discrete analogues in harmonic analysis: spherical averages. Ann. of Math. (2) 155 (2002), no. 1, 189–208.
[21] J. Ortega-Cerda, K. Seip, Fourier frames, Annals of Math., 155 (2002), 789-806.
[22] R.E.A.C. Paley and N. Wiener, Fourier Transforms in the Complex Domain, Coll. Publ., 19, Providence: Amer. Math. Soc., (1934).
[23] I. Pesenson, Sampling of Paley-Wiener functions on stratified groups, J. of Fourier Analysis and Applications 4 (1998), 269–280.
[24] I. Pesenson, Reconstruction of band-limited functions in \( L^2(\mathbb{R}^d) \), Proceed. of AMS, Vol.127(12), (1999), 3593- 3600.
[25] I. Pesenson, *A sampling theorem on homogeneous manifolds*, Trans. of AMS, Vol. 352(9), (2000), 4257-4270.

[26] I. Pesenson, *Poincare-type inequalities and reconstruction of Paley-Wiener functions on manifolds*, J. of Geometric Analysis 4(1), (2004), 101-121.

[27] I. Pesenson, *Deconvolution of band limited functions on symmetric spaces*, Houston J. of Math., 32, No. 1, (2006), 183-204.

[28] I. Pesenson, *Sampling sequences of distributions in $L_2(\mathbb{R})$*, Int. J. Wavelets Multiresolut. Inf. Process. 3(2005), no. 3, 417-434.

[29] I. Pesenson, *Band limited functions on quantum graphs*, Proc. Am. Math. Soc. 133, No.12, 3647-3655 (2005).

[30] I. Pesenson, *Analysis of band-limited functions on quantum graphs*, Appl. Comput. Harmon. Anal. 21 (2006), no. 2, 230-244.

[31] I. Pesenson, *Frames for spaces of Paley-Wiener functions on Riemannian manifolds*, in Integral Geometry and Tomography, Contemp. Math., 405, AMS, (2006), 137-153.

[32] I. Pesenson, *Variational splines on Riemannian manifolds with applications to integral geometry*, Adv. in Appl. Math. 33 (2004), no. 3, 548–572.

[33] I. Pesenson, *Polynomial splines and eigenvalue approximations on quantum graphs*, J. Approx. Theory 135 (2005), no. 2, 203–220.

[34] I. Pesenson, *Sampling in Paley-Wiener spaces on combinatorial graphs*, will appear in Trans. of AMS.

[35] I.J. Schoenberg, *Notes on spline functions I. The limits of the interpolating periodic spline functions as their degree tends to infinity*, Indag. Math., 34 (1972), 412-422.

[36] I.J. Schoenberg, *Notes on spline functions III. On the convergence of the interpolating cardinal splines as their degree tends to infinity*, Israel J. Math. 16 (1973) 87–93.

[37] I. Schoenberg, *Cardinal Spline Interpolation*, CBMS, 12 SIAM, Philadelphia, 1973.

[38] C. Shannon, W. Weaver, *The Mathematical Theory of Communication*, Univ. of Illinois Press, 1963.

[39] S. Smale, D. X. Zhou, *Shannon sampling and function reconstruction from point values*, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 3, 279–305.

[40] S. Smale, D. X. Zhou, *Shannon sampling. II. Connections to learning theory*, Appl. Comput. Harmon. Anal. 19 (2005), no. 3, 285–302.

[41] G. Strang, *Signal processing for everyone*, Computational mathematics driven by industrial problems (Martina Franca, 1999), 365–412, Lecture Notes in Math., 1739, Springer, Berlin, 2000.

[42] H. Triebel, *Theory of function spaces II*, Monographs in Mathematics, 84. Birkhuser Verlag, Basel, 1992.

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122

E-mail address: pesenson@math.temple.edu