A derivation of the beam equation

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Abstract
The Euler–Bernoulli equation describing the deflection of a beam is a vital tool in structural and mechanical engineering. However, its derivation usually entails a number of intermediate steps that may confuse engineering or science students at the beginning of their undergraduate studies. We explain how this equation may be deduced, beginning with an approximate expression for the energy, from which the forces and finally the equation itself may be obtained. The description is begun at the level of small ‘particles’, and the continuum level is taken later on. However, when a computational solution is sought, the description turns back to the discrete level again. We first consider the easier case of a string under tension, and then focus on the beam. Numerical solutions for several loads are obtained.

Online supplementary data available from stacks.iop.org/

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(Some figures may appear in colour only in the online journal)

1. Motivation

The Euler–Bernoulli beam equation is of paramount importance in civil engineering, being a simplified theory that yields relevant results for the dynamics and statics of beams [1–3]. From the mathematical point of view, it is perhaps the simplest differential equation of a high order (fourth) that has a clear practical relevance. However, a simple derivation is not easy to find, since most often a number of intermediate concepts, such as bending moments and shear forces, are introduced. We have designed a seminar in which the beam equation is obtained from an expression for the energy. As a preliminary step, the equation for the string under tension is derived, in a manner slightly different from what is usual. Computational methods are employed for both systems, in order to find numerical solutions.
This article explains the method used. Its contents would fit in about two hours, ideally followed by a practical computing session of about two hours.

2. Introduction

Depending on the level of the class, some introduction to elementary Newtonian particle mechanics may be needed. In particular, Newton’s second law, and the case when forces are conservative and can therefore be obtained as derivatives of some potential energy.

However, Newton’s laws deal with point particles and forces between them. They are very accurate at scales such as the solar system, at which the planets and the Sun are almost point-like. Another regime in which Newtonian physics works very well is at very small scales, where molecules or atoms are again almost point-like (even if the interactions themselves have a quantum origin).

Everyday objects, on the other hand, are usually extended (continuum). Historically, ‘particles’ have been introduced in order to apply point dynamics to these systems. Particles are portions of material that are ‘very’ small, yet still ‘large enough’ to have macroscopic features, such as density, volume, temperature, etc. In practical terms, they can be as small as small cells, about 1 μm in size. If they are smaller, thermal effects can cause such effects as Brownian motion. Of course, the students should not relate these particles, with real particles, such as electrons.

The idea behind this approach is to apply Newtonian physics to each of these particles. Then, one takes the continuum limit, in which the number of these particles, \( N \), is taken to infinity, but some of their features are taken zero (others may stay tend to infinity or reach a finite value.) As a simple example, the mass of each particle, \( m \), could tend to zero in such a way that the total mass, given by

\[
M = Nm
\]

remains constant. The same applies for lengths and other quantities, even if in some cases, as we will see, the correct limit is not obvious to anticipate.

This way, one may derive laws expressed as partial differential equations. For example, we will derive here the wave equation:

\[
\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}
\]

and the beam equation:

\[
\frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4} + q.
\]

Traditionally, many special mathematical methods have been devised to solve these equations, leading to huge advances in mathematics, physics, and engineering [4]. The advent of computing has changed the situation somewhat, often providing a more direct and easier way to find solutions (or more precisely, a numerical approximation to the solutions). On the other hand computers are discrete by nature. A continuum equation can therefore not be computed as is: it must be discretized. The conclusion, as we will discuss, is that we return to where we started, two centuries ago [5].
3. The string

This is a system covered in many textbooks, but we treat it here for two reasons. The first is that the procedure will be mirrored when discussing the beam, but the expressions are simpler in this case. The second is that some of the quantities that are introduced in elementary textbooks (chiefly, the tension) arise naturally in this procedure.

We model a string as a line of point particles (we will call ‘beads’) joined by massless springs. Of course, most strings are not built this way, but the point is that the continuum limit will be the same for a wide number of systems, and we choose a simple one to work with.

3.1. Energy

The expression for the energy of a spring comes from Hooke’s law:

\[ U = \frac{1}{2} \kappa (\ell - \ell_0)^2 , \]

where \( \ell_0 \) is the spring’s natural length and \( \ell \), its actual length. The elastic parameter \( \kappa \) is Hooke’s spring constant. This is the best known expression, but here we will prefer to use an expression that features the relative deformation, \((\ell - \ell_0)/\ell_0\), called the ‘strain’:

\[ U = \frac{1}{2} \frac{B}{\ell_0} (\ell - \ell_0)^2 , \]

where \( B = \kappa \ell_0 \) is a parameter with units of force, whose meaning will be discussed later.

Let us first consider the string under tension, but unperturbed otherwise. Its shape will be a straight line as in the upper part of figure 1, and all the spring lengths will be equal to \( d \), with \( d > \ell_0 \). The energy will therefore be

\[ U_0 = N \times \frac{1}{2} \frac{B}{\ell_0} (d - \ell_0)^2 = \frac{1}{2} \frac{B}{\ell_0} N (dN - \ell_0N)^2 = \frac{1}{2} \frac{B}{L_0} (L - L_0)^2 . \]

In order to reach the later equality we have multiplied and divided by \( N \), and we have written \( Nd = L \) for the length of the string and \( N\ell_0 = L_0 \) for its natural length. This is actually our first application of the continuum limit, since we begin with an expression that depends on the particles’ magnitudes \( d \) and \( \ell_0 \) and we end up with another one that depend on the macroscopic magnitudes \( L \) and \( L_0 \). Also, the parameter \( B \) seems to require no change with \( N \) in order the energy \( U \) be finite. Since \( B = \kappa \ell_0 \), the string constant \( \kappa \) should then tend to infinity (not to zero!) in this limit.

If, on the other hand, the string is distorted, as in the lower part of figure 1, the energy will now be:

\[ U = \frac{1}{2} \frac{B}{\ell_0} (\ell_{i-1,i} - \ell_0)^2 + \frac{1}{2} \frac{B}{\ell_0} (\ell_{i,i+1} - \ell_0)^2 + \cdots , \]

where, out of the \( N \) terms, we just write the two of them that are related to the \( i \)th bead at the centre of figure 1. We will use \( i \) subindices to refer to beads and \( i - 1 \), \( i \) to refer to quantities between bead \( i - 1 \) and bead \( i \), such as the length of the spring between them (similarly for \( i, i + 1 \)).
3.2. Forces and tension

Let us find first the horizontal component of the force on bead $i$:

$$f_i^x = -\frac{\partial U}{\partial x_i} = -\frac{B}{\ell_0} (\ell_{i-1,i} - \ell_0) \frac{\partial \ell_{i-1,i}}{\partial x_i} - \frac{B}{\ell_0} (\ell_{i,i+1} - \ell_0) \frac{\partial \ell_{i,i+1}}{\partial x_i},$$

where he have applied the chain rule of differentiation to $\ell_{i-1,i}$ and $\ell_{i,i+1}$. The Pythagorean theorem tells us:

$$\ell_{i-1,i}^2 = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \quad \ell_{i,i+1}^2 = (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2.$$

From this, we can obtain these interesting identities:

$$\frac{\partial \ell_{i-1,i}}{\partial x_i} = \cos \theta_{i-1,i} \quad \frac{\partial \ell_{i,i+1}}{\partial x_i} = -\cos \theta_{i,i+1}.$$

Therefore:

$$f_i^x = -\frac{B}{\ell_0} \left[ (\ell_{i-1,i} - \ell_0) \cos \theta_{i-1,i} - (\ell_{i,i+1} - \ell_0) \cos \theta_{i,i+1} \right].$$

We can write this as

$$f_i^x = -T_{i-1,i} \cos \theta_{i-1,i} + T_{i,i+1} \cos \theta_{i,i+1},$$

where the tension between bead $i - 1$ and bead $i$ being $T_{i-1,i} = \frac{B}{\ell_0} (\ell_{i-1,i} - \ell_0)$, similarly for $T_{i,i+1}$.

Now, if deflections are small: $\ell_{i-1,i}, \ell_{i,i+1} \approx d$, therefore $T_{i-1,i} = T_{i,i+1} = T$, where

$$T = \frac{B}{\ell_0} d - \ell_0 = \frac{B L - L_0}{L_0}.$$

Moreover, $\cos \theta_{i,i+1} \approx 1$ for all beads, therefore $f_i \approx 0$.

Notice this tension $T$ is the external force to be applied to the ends of the string to keep it tout. Indeed, the left end particle ($i = 1$) has no neighbour at its left to pull from it, and the tension $T$ must be applied from the outside. The same applies to the other end. Moreover, recalling the total energy is $U = \frac{1}{2} \ell_0 \left( L - L_0 \right)^2$, we see $T' = -\frac{dU}{dL} = -T$. What this means is that the string is trying to shrink with a force $T'$ that we must overcome with another one $T$ which is equal but pointing outwards.

\[ \text{Figure 1. The string under tension, unperturbed (upper graph), and perturbed (lower graph).} \]
The $B$ parameter is then seen to be
\[ B = \frac{T}{L - L_0} = \frac{T}{(L - L_0)/L_0}. \]

One of the important elastic properties of a material is its Young’s modulus (also known as tensile modulus, or elastic modulus), a magnitude with units of pressure that is defined as the stress/strain ratio:
\[ E = \frac{T/A_0}{(L - L_0)/L_0}. \]

The strain is, as in Hooke’s law, $(L - L_0)/L_0$, and the stress is the tension divided by the cross section of the string under no tension, $A_0$. Therefore, our $B$ parameter is related to Young’s modulus:
\[ B = EA_0. \]

As a simple experiment, students can try to measure experimentally values of Young’s modulus from these equations, see appendix B.

The vertical component of the force follows from the identity
\[ f_i^y = -T_i \sin \theta_i + T_{i+1} \sin \theta_{i+1}, \]

with the end result:
\[ f_i^y = -T_i \sin \theta_i + T_{i+1} \sin \theta_{i+1}. \]

Many physics books, such as [6–8], basically start with this equation for the vertical force, from considerations of the net vertical on each bead. Our derivation has the advantage of providing more insight on the meaning of the tension $T$. It is also less likely to result in errors in signs. On the other hand, in [9] we find a derivation similar to ours (although it is given in terms of rods coupled by torsion.)

In the limit of small vertical deflections, the force may be written as
\[ f_i^y \approx -\frac{T}{d} \sin \theta_i + T \sin \theta_{i+1}. \]

In this limit, the sines are also similar to the tangents, so:
\[ f_i^y \approx -\frac{T}{d} (y_i - y_{i-1}) + \frac{T}{d} (y_{i+1} - y_i) = \frac{T}{d} (y_{i+1} - 2y_i + y_{i+1}). \]

### 3.3. The wave equation

Let us continue with the equations of motion for our bead. Newton’s second law gives (only the $y$ direction is changing, so we will drop the $y$ superindices):
\[ ma_i = \frac{T}{d} (y_{i-1} - 2y_i + y_{i+1}) \]

that can be written as
\[ a_i = \frac{T}{m/d} \frac{y_{i-1} - 2y_i + y_{i+1}}{d^2} = \frac{T}{\mu} \frac{y_{i-1} - 2y_i + y_{i+1}}{d^2}, \]

with $\mu = m/d$ the mass per unit length.
The last ratio is a discrete, finite differences, version of the second spatial derivative \[10\]. Therefore, in the limit \(d \to 0\) we may write the wave equation

\[
a = \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}.
\]

It can be shown that the phase velocity of traveling waves is given by \(v^2 = T/\mu\).

### 3.4. The loaded string

It is often interesting to find the equilibrium solution to equations, setting the time derivatives equal to zero. In this case, it is rather dull: the solution to \(\frac{\partial^2 y}{\partial x^2} = 0\) is just a straight line. For homogeneous Dirichlet boundary conditions \(y(0) = y(L) = 0\), the unique solution is simply \(y(x) = 0\).

To make things more interesting, we may add a vertical force \(F(x)\) to each bead. This force is constant in the vertical direction, but may vary along the string. The energy equation would be modified as:

\[
U = \sum - F(x)y.
\]

The wave equation is now:

\[
\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} + F/m
\]

The static solution is given by the equation:

\[
\frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} = -F/m
\]

This is a Poisson equation. For example, for the case of gravity one would have

\[
F = -mg \quad \rightarrow \quad \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} = g.
\]

Setting \(y(0) = y(L) = 0\), the unique solution is a parabola:

\[
y = -\frac{mg}{2T}x(L - x).
\]

Some students may know the solution to this sort of problems involving hanging strings is often more involved, with shapes such as the catenary resulting. That is the case, but in this limit of small deformations the solution is simply an upward parabola, which is the usual limit of any curve close to a minimum.

### 3.5. Computing the loaded string

A computer may be used in order to find the equilibrium shape of a string under general loads. However, in order to apply computational methods we need to go back to the discretized equations, which are the ones that are readily implemented on a computer. This actually takes us back to the historic derivation of these equations, as we have discussed.

For example, we would have the equation of motion:

\[
a_i = \frac{T}{\mu} \frac{y_{i-1} - 2y_i + y_{i+1}}{d^2} + F/m.
\]
For the static case:
\[
\frac{y_{i-1} - 2y_i + y_{i+1}}{d^2} = -\frac{F_i}{Td} = q_i,
\]
where \( q_i = -\frac{F_i}{(Td)} \) is the string load.

Notice this is a linear equation that cannot readily be solved only for \( y_i \), since it involves \( y_{i-1} \) and \( y_{i+1} \), which are also unknown. In fact, one has a system of \( N \) linear equations, which be written in matrix form:
\[
\begin{pmatrix}
1 & \cdots & 0 & \cdots & 0 \\
\cdots & 1 & -2 & 1 & \cdots \\
0 & \cdots & 0 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & -1 \\
\end{pmatrix}
\begin{pmatrix}
y_{i-1} \\
y_i \\
y_{i+1} \\
\vdots \\
\vdots \\
\end{pmatrix}
= 
\begin{pmatrix}
q_{i-1} \\
q_i \\
q_{i+1} \\
\vdots \\
\vdots \\
\end{pmatrix}.
\]

It can also be summarized as the symbols under the under-braces suggest:
\[
\nabla^2 \vec{y} = \vec{q},
\]
where \( \nabla^2 \) is a matrix for second derivatives, \( \vec{y} \) is a vector containing the vertical positions, and \( \vec{q} \) is a vector containing the loads.

This linear algebra problem is implemented in all major computational environments. We choose to carry out the calculations using Python, it being a powerful emerging ‘scientific ecosystem’ with many advantages \[11\]. One of them is that it is free and open source, and is included in all major Linux distributions. Another choice with the same advantages is Octave, which is designed to be a clone of Matlab, itself a viable choice but not free. Other options such as Maple or Mathematica are also possible.

We recommend IPython in notebook form, to open an interactive session within a web browser in the lecture room, and run the programs ‘live’. Relevant files can be obtained in the online supplementary materials of this article.

In figure 2 (above) we plot the solution for the shape of the equilibrium string under uniform load. The agreement with the exact parabolic solution of (1) is seen to be excellent. Indeed, this approximation to the second derivative is known to be exact (up to machine precision) for quadratic functions.

Some students may notice that, for certain choice of parameters, the numerical \( y \) values may not be ‘small’ and can be actually comparable to \( L \). However, they should be reminded that variables are usually cast into non-dimensional form in computations. For example, writing (1) in the form:
\[
y = -\frac{\mu g L^2}{2T} \frac{x}{L} \left( 1 - \frac{x}{L} \right),
\]
we see it is not \( x \) but rather \( x/L \) which is relevant, and that the scale of \( y \) is fixed by \( \mu g L^2 / T \). It is the later length scale, and how it compares with \( L \), which tells us whether our string has little deformation or not.

It would seem not so useful to obtain a solution that is already known, but this numerical method still works for loads that are not so simple. As an example, we show results for a Gaussian load \( q(x) = -\exp(-((x - 0.7)/0.1)^2) \) (in reduced units), which which may model, e.g. some deformation due to a blunt object. In figure 2 (below) we show that the solution has a shape that could be expected, with linear parts on the zones where little load is applied.
Figure 2. Numerical result for the uniformly loaded string (above) and for a complex loading (below). Dots: numerical results, line: theoretical solution. Horizontal length scaled by $L$, and vertical displacement by $A \equiv \frac{\mu g L^2}{T}$.

Figure 3. An accordion, a mental image of our model for a beam [12].
4. The beam

In the beam we are concerned with bending, not compression or expansion. We may picture a physical beam as a succession of slabs that are subject to bending. It would be like a accordion, an instrument featuring a bellows that resembles a succession of slabs, see figure 3. It would be a silent one, since the instrument emits sounds when compressing or expanding the bellows, as air enters or leaves it, but not when it is bent.

4.1. Energy and forces

We begin by writing the energy as:

\[ U = \frac{1}{2} C \left[ (\Delta \theta_{i-1})^2 + (\Delta \theta_i)^2 + (\Delta \theta_{i+1})^2 + \cdots \right], \]

where \( C \) is a stiffness parameter with units of force \( \times \) area. Each \( \Delta \theta_i \) is the difference of angles limiting each slab, see figure 6. What we are supposing here is that each slab has its energy increased when its two limiting angles differ, and that this dependence is quadratic to lowest order (it cannot be linear since a reversal in sign of the angles should lead to the same energy increase).

As we can see in figure 5

\[ \Delta \theta_i = \theta_{i+1} - \theta_{i-1} \approx \frac{y_{i+1} - y_i}{d} = \frac{y_i - y_{i-1}}{d} = \frac{y_{i+1} - 2y_i + y_{i-1}}{d}. \]

We therefore find again the discrete version of the second derivative (but for a \( 1/d \) factor).
The energy can now be written as

\[ U = \frac{1}{2} \frac{C}{d^3} \left[ (2y_{i-1} - y_i - y_{i+1})^2 + \left(2y_i - y_{i+1} - y_{i-1}\right)^2 + \left(2y_{i+1} - y_{i+2} - y_i\right)^2 + \cdots \right]. \]

It is not too difficult to obtain the \( y \) component of the force on slab \( i \):

\[ f_i^y = -\frac{\partial U}{\partial y_i} = \frac{C}{d^3} \left[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}\right]. \] (2)

**Figure 6.** Numerical result for the uniformly loaded string (above), centrally loaded beam (middle) and for a complex loading (below). Dots: numerical results, lines: theoretical solutions. Horizontal length scaled by \( L \), and vertical displacement by \( A = \frac{q_0L^4}{EI} \).
This is an approximation to the fourth derivative (but for a \(-1/d\) factor) \([10]\). Therefore:

\[
f(x) \approx -Cd \frac{\partial^4 y}{\partial x^4}
\]

### 4.2. The beam equation

It is again easy to add vertical forces to each slab:

\[
U = -F(x)y.
\]

The final dynamical equation is

\[
m \frac{\partial^2 y}{\partial t^2} = -Cd \frac{\partial^4 y}{\partial x^4} + F,
\]

and its static solution is given by:

\[
C \frac{\partial^4 y}{\partial x^4} = \frac{F}{d} = q,
\]

where \(q = F/d\) is the beam load. As shown in the appendix A, the \(C\) parameter is \(C = EI\), where \(E\) is Young’s modulus (again) and \(I\) a quantity known as the second moment of inertia.

We therefore obtain the dynamic beam equation:

\[
\mu \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4} + q
\]

and its static version, which is probably better known:

\[
EI \frac{\partial^4 y}{\partial x^4} = q.
\]

### 4.3. Computing the loaded beam

Again, our discrete problem involving the slabs can be cast as a linear algebra problem:

\[
\nabla^4 y = \bar{q},
\]

with a fourth derivative matrix \(\nabla^4\) having: a diagonal with \(-6\) values, subdiagonals above and below it with values of \(4\), and finally subdiagonals above and below the two former ones with values of \(-1\), with a common factor of \(1/d^4\), as seen in equation (2).

In supplementary materials the code to solve the static loaded beam can be found. We consider here a double clamped beam, in which the ends are fixed to be 0 and both ends, and so are the first derivatives. The result for uniform loading, figure 6 (above), is not so accurate when compared with the exact solution [3] as the string was, since this is a higher order derivative for which the approximation is not so good. Nevertheless, the numerical solution can be seen to converge to the exact one as \(N\) is increased.

We may also consider the case of central loading, where the whole load is placed in the middle of the beam. The resulting beam shape is plotted in figure 6 (middle), and compared with the exact solution. Finally, we reuse our Gaussian load function for the string and apply it to the beam. Since a beam is different from a string, the resulting shape, figure 6 (below) is not so obvious to guess, with the maximum deformation away from \(x = 0.7L\), the point at which the load is greater. The students are encouraged to perform simple experiments, as described in appendix B.
5. Conclusions

We have shown in this article how the concept of ‘particle’ may be used in order to obtain physical laws written as differential equations. A traditional point of view is to take these laws as the ultimate expressions, for which solutions should be obtained in different situations. Mathematically this entails that a given differential equation has different solutions corresponding to different boundary conditions and initial conditions. However, in later years the emergence of computers makes it easy to obtain numerical solutions to the equations. Since computers are discrete, the equations must be brought into discrete form, which actually brings us back to particles.

This situation may seem paradoxical, but most experienced researchers will agree that computational techniques do not replace, but rather compliment, traditional mathematical analysis. However, the direct simulation of a particle description can, in our opinion, be a powerful teaching resource for first year college courses.

There are many ways in which this article may be extended. Additional simple experiments may be proposed in addition to the ones given at the appendix B.

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Appendix A. The stiffness parameter

A slab is compressed in different amount at different values of y. Indeed the compression (or extension) at height $y$ is:

$$c = y \Delta \theta \quad y \in (-h/2, h/2).$$

In the later range for $y$ we are already assuming deviations are small.

The energy cost of a compression for a strip with area $dA$ (see figure A1) is:

$$dU = \frac{1}{2} (EdA) \frac{c^2}{d},$$

where $E$ is Young’s modulus.

The total energy cost for this slab is therefore

$$U_0 = \frac{1}{2} \frac{E}{d} \int_{-h/2}^{h/2} c^2 dA = \frac{1}{2} \frac{E}{d} (\Delta \theta)^2 \left( \int_{-h/2}^{h/2} y^2 dA \right)$$

The last integral is purely geometric, and is called $I$ ‘the second moment of area’ (aka moment of inertia of plane area, area moment of inertia, or second area moment), with units of area squared. Therefore

$$U_0 = \frac{1}{2} \frac{EI}{d} (\Delta \theta)^2.$$
Recall our original expression for each of the slabs:

\[ U_0 = \frac{1}{2} \frac{C}{d} (\Delta \theta)^2. \]

Clearly, \( C \) is given by \( C = EI \).

Appendix B. Experiments

As a simple experiment, students can try to measure experimentally values of Young’s modulus from our equations \( B = EA_0 \) and \( T = B(L - L_0)/L_0 \). This means we can obtain \( B \) and \( E \) by measuring the length of the string under no tension, its length when tuned, and its tension. Ideally, all of these quantities should be measured, but some of them can be taken from the manufacturer. For example, a standard 0.036 in guitar string means a diameter of 0.9144 mm that may be not easy to measure. This experiment can also be combined with elementary wave theory. If the string is tuned to a known fundamental frequency \( f \), then

\[ f = \frac{1}{2L} \sqrt{\frac{T}{\mu}}, \]

a fact that may let us measure \( T \) given \( L \) and \( \mu \).

A simple experiment for the beam is to clamp the end of a flexible object, such as a ruler, and measure its deflection at the hanging end. This would be a uniformly loaded cantilever beam if the load results simply from the weight of the object. Other loadings can of course be explored. The experiments are quite easy to carry out, but the correct mathematical and numerical description of the hanging end needs to be carefully addressed. If the object vibrates, there is also a relationship for the frequency, which is more complicated than for the string, but whose solution can be found in standard books [3].

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