On the cover Turán number of Berge hypergraphs

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Abstract

For a fixed set of positive integers \( R \), we say \( \mathcal{H} \) is an \( R \)-uniform hypergraph, or \( R \)-graph, if the cardinality of each edge belongs to \( R \). For a graph \( G = (V, E) \), a hypergraph \( \mathcal{H} \) is called a Berge-\( G \), denoted by \( \text{BG} \), if there exists a bijection \( f : E(G) \to E(\mathcal{H}) \) such that for every \( e \in E(G) \), \( e \subseteq f(e) \). In this paper, we define a variant of Turán number in hypergraphs, namely the cover Turán number, denoted as \( \hat{\text{ex}}_R(n, G) \), as the maximum number of edges in the shadow graph of a Berge-\( G \) free \( R \)-graph on \( n \) vertices. We show a general upper bound on the cover Turán number of graphs and determine the cover Turán density of all graphs when the uniformity of the host hypergraph equals to 3.

1 Introduction

A hypergraph is a pair \( \mathcal{H} = (V, E) \) where \( V \) is a vertex set and \( E \subseteq 2^V \) is an edge set. For a fixed set of positive integers \( R \), we say \( \mathcal{H} \) is an \( R \)-uniform hypergraph, or \( R \)-graph for short, if the cardinality of each edge belongs to \( R \). If \( R = \{k\} \), then an \( R \)-graph is simply a \( k \)-uniform hypergraph or a \( k \)-graph. Given an \( R \)-graph \( \mathcal{H} = (V, E) \) and a set \( S \in \binom{V}{s} \), let \( d(S) \) denote the number of edges containing \( S \) and \( \delta_s(\mathcal{H}) \) be the minimum \( s \)-degree of \( \mathcal{H} \), i.e., the minimum of \( d(S) \) over all \( s \)-element sets \( S \in \binom{V}{s} \). When \( s = 2 \), \( \delta_2(\mathcal{H}) \) is also called the minimum co-degree of \( \mathcal{H} \). Given a hypergraph \( \mathcal{H} \), the 2-shadow (or shadow) of \( \mathcal{H} \), denoted by \( \partial(\mathcal{H}) \), is a simple 2-uniform graph \( G = (V, E) \) such that \( V(G) = V(\mathcal{H}) \) and \( uv \in E(G) \) if and only if \( \{u, v\} \subseteq h \) for some \( h \in E(\mathcal{H}) \). Note that \( \delta_2(\mathcal{H}) \geq 1 \) if and only if \( \partial(\mathcal{H}) \) is a complete graph. In this case, we say \( \mathcal{H} \) is covering.

There are several notions of a path or a cycle in hypergraphs. A Berge path of length \( t \) is a collection of \( t \) hyperedges \( h_1, h_2, \ldots, h_t \in E \) and \( t + 1 \) vertices \( v_1, \ldots, v_{t+1} \) such that \( \{v_i, v_{i+1}\} \subseteq h_i \) for each \( i \in [t] \). Similarly, a \( k \)-graph \( \mathcal{H} = (V, E) \) is called a Berge cycle of length \( t \) if \( E \) consists of \( t \) distinct edges \( h_1, h_2, \ldots, h_t \) and \( V \) contains \( t \) distinct vertices \( v_1, v_2, \ldots, v_t \) such that \( \{v_i, v_{i+1}\} \subseteq h_i \) for every \( i \in [t] \) where \( v_{t+1} = v_1 \). Note that there may be other vertices than \( v_1, \ldots, v_t \) in the edges of a Berge cycle or path. Gerbner and Palmer [14] extended the definition of Berge paths and Berge cycles to general graphs. In particular, given a simple graph \( G \), a hypergraph \( \mathcal{H} \) is called Berge-\( G \) if there is a bijection \( f : E(G) \to E(\mathcal{H}) \) such that for all \( e \in E(G) \), we have \( e \subseteq f(e) \).

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We say an $R$-graph $\mathcal{H}$ on $n$ vertices contains a Hamiltonian Berge cycle (path) if it contains a Berge cycle (path) of length $n$ (or $n - 1$). We say $\mathcal{H}$ is Berge-Hamiltonian if it contains a Hamiltonian Berge cycle. Bermond, Germa, Heydemann, and Sotteau [3] showed a Dirac-type theorem for Berge cycles. We showed in [25] that for every finite set $R$ of positive integers, there is an integer $n_0 = n_0(R)$ such that every covering $R$-uniform hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices contains a Berge cycle $C_s$ for any $3 \leq s \leq n$. In particular, every covering $R$-graph on sufficiently large $n$ vertices is Berge-Hamiltonian.

Extremal problems related to Berge hypergraphs have been receiving increasing attention lately. For Turán-type results, let $ex_k(n, G)$ denote the maximum number of hyperedges in $k$-uniform Berge-$G$-free hypergraph. Győri, Katona and Lemons [16] showed that for a $k$-graph $\mathcal{H}$ containing no Berge path of length $t$, if $t \geq k + 2 + 5$, then $e(\mathcal{H}) \leq \frac{t}{5}\binom{n}{t}$; if $3 \leq t \leq k$, then $e(\mathcal{H}) \leq \frac{n(t-1)}{k+1}$. Both bounds are sharp. The remaining case of $t = k + 1$ was settled by Davoodi, Győri, Methuku and Tompkins [6]. For cycles of a given length, Győri and Lemons [17, 18] showed that $ex_k(n, C_{2t}) = \Theta(n^{1+1/t})$. The same asymptotic upper bound holds for odd cycles of length $2t + 1$ as well. The problem of avoiding all Berge cycles of length at least $k$ has been investigated in a series of papers [22, 10, 11, 9, 19]. For general results on the maximum size of a Berge-$G$-free hypergraph for an arbitrary graph $G$, see for example [13, 15, 27].

For Ramsey-type results, define $R^k_c(BG_1, \ldots, BG_c)$ as the smallest integer $n$ such that for any $c$-edge-coloring of a complete $k$-uniform hypergraph on $n$ vertices, there exists a Berge-$G_i$ subhypergraph with color $i$ for some $i$. Axenovich and Gyárfás [2] who focus on the Ramsey number of small fixed graphs where the number of colors may go to infinity.

Very recently, we [26] defined a new type of Ramsey number, namely the cover Ramsey number, denoted as $\tilde{R}^k_c(BG_1, BG_2)$, as the smallest integer $n_0$ such that for every covering $R$-uniform hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices and every $2$-edge-coloring (blue and red) of $\mathcal{H}$, there is either a blue Berge-$G_1$ or a red Berge-$G_2$ subhypergraph. We show that for every $k \geq 2$, $R(G_1, G_2) \leq \tilde{R}^k_c(BG_1, BG_2) \leq c_k \cdot R(G_1, G_2)$ for some $c_k$. Moreover, $\tilde{R}^k_c(K_t) > (1 + o(1))\frac{\sqrt{2}}{e}t^{2t^{1/2}}$ for sufficiently large $t$ and $\tilde{R}^k_c(BG, BG) \leq c(d, k)n$ if $\Delta(G) \leq d$. It occurs to us that the cover Ramsey number for Berge graphs behaves more like the classical Ramsey number than the Ramsey number of Berge hypergraphs defined in [2, 30, 12]. This inspires us to extend the investigation to the analogous cover Turán number for Berge hypergraphs. In particular, given a fixed graph $G$ and a finite set of positive integers $R \subseteq [k]$, we define the $R$-cover Turán number of $G$, denoted as $\hat{ex}_R(n, G)$, as the maximum number of edges in the shadow graph of a Berge-$G$-free $R$-graph on $n$ vertices. The $R$-cover Turán density, denoted as $\hat{\pi}_R(G)$, is defined as

$$\hat{\pi}_R(G) = \limsup_{n \to \infty} \frac{\hat{ex}_R(n, G)}{\binom{n}{2}}.$$ 

When $R$ is clear from the context, we ignore $R$ and use cover Turán number and cover Turán density for short. A graph is called $R$-degenerate if $\hat{\pi}_R(G) = 0$. For the ease of reference,
when \( R = \{k\} \), we simply denote \( \pi_k(G) \) as \( \pi_k(G) \) and call \( G \) \( k \)-degenerate if \( \pi_k(G) = 0 \).

We remark that the Turán number of graphs only differ by a constant factor when the host hypergraph is uniform compared to non-uniform. In particular, we show the following proposition.

**Proposition 1.** If \( R \) is a finite set of positive integers such that \( \min(R) = m \geq 2 \) and \( \max(R) = M \). Then given a fixed graph \( G \),

\[
\max_{r \in R} \hat{\chi}_r(n, G) \leq \hat{\chi}_R(n, G) \leq \frac{M}{m} \hat{\chi}_m(n, G).
\]

Indeed, the first inequality is clear from definition. For the second inequality, suppose we have an \( R \)-graph \( \mathcal{H} \) with more than \( \frac{M}{m} \hat{\chi}_m(n, G) \) edges in its shadow. For each hyperedge \( h \) in \( \mathcal{H} \), shrink it to a hyperedge of size \( m \) by uniformly and randomly picking \( m \) vertices in \( h \). Call the resulting hypergraph \( \mathcal{H}' \). It is easy to see that for any edge in \( e \in E(\partial(\mathcal{H})) \), \( Pr(e \in E(\partial(\mathcal{H}'))) \geq \frac{m}{M} \). Hence by linearity of expectation, the expected number of edges in \( \partial(\mathcal{H}') \) is more than \( \hat{\chi}_m(n, G) \). It follows that there exists a way to shrink \( \mathcal{H} \) to a \( m \)-graph with at least \( \hat{\chi}_m(n, G) + 1 \) edges in its shadow. Thus, by definition of the cover Turán number, \( \mathcal{H}' \) contains a Berge copy of \( G \), which corresponds to a Berge-\( G \) in \( \mathcal{H} \).

**Remark 1.** Note that Proposition 1 implies that if a graph \( G \) is \( k \)-degenerate (where \( k \geq 2 \), then it is \( R \)-degenerate for any \( R \) satisfying \( \min(R) \geq k \). In particular, a bipartite graph is \( k \)-degenerate for all \( k \geq 2 \).

In this paper, we determine the cover Turán density of all graphs when the uniformity of the host graph equals to 3. We first establish a general upper bound for the cover Turán density of graphs.

**Theorem 1.** For any fixed graph \( G \) and any fixed \( \epsilon > 0 \), there exists \( n_0 \) such that for any \( n \geq n_0 \),

\[
\hat{\chi}_k(n, G) \leq \left( 1 - \frac{1}{\chi(G) - 1} + \epsilon \right) \binom{n}{2}.
\]

We remark that Theorem 1 holds when the host hypergraph is non-uniform as well, i.e. we can replace \( k \) with any fixed finite set of positive integers \( R \). If \( \chi(G) > k \), there is a construction giving the matching lower bound. Partition the vertex set into \( t := \chi(G) - 1 \) equitable parts \( V = V_1 \cup V_2 \cup \cdots \cup V_t \). Let \( \mathcal{H} \) be the \( k \)-uniform hypergraph on the vertex set \( V \) consisting of all \( k \)-tuples intersecting each \( V_i \) on at most one vertex. The shadow graph is simply the Turán graph with \( (1 - \frac{1}{\chi(G) - 1} + o(1)) \binom{n}{2} \) edges. The shadow graph is \( K_{t+1} \)-free, thus contains no subgraph \( G \). It follows that \( \mathcal{H} \) is Berge-\( G \)-free. Therefore, we have the following theorem:

**Theorem 2.** For any \( k \geq 2 \), and any fixed graph \( G \) with \( \chi(G) \geq k + 1 \), we have

\[
\hat{\pi}_k(G) = 1 - \frac{1}{\chi(G) - 1}.
\]

Given a simple graph \( G \) on \( n \) vertices \( v_1, \ldots, v_n \) and a sequence of \( n \) positive integers \( s_1, \ldots, s_n \), we denote \( B = G(s_1, \ldots, s_n) \) the \( (s_1, \ldots, s_n) \)-blowup of \( G \) obtained by replacing
every vertex $v_i \in G$ with an independent set $I_i$ of $s_i$ vertices, and by replacing every edge $(v_i, v_j)$ of $G$ with a complete bipartite graph connecting the independent sets $I_i$ and $I_j$. If $s = s_1 = s_2 = \cdots = s_n$, we simply write $G(s_1, \ldots, s_n)$ as $G(s)$ where $s$ is called the blowup factor. We also define a generalized blowup of $G$, denoted by $G(s_1, \ldots, s_n; M)$ where $M \subseteq E(G) \subseteq \left(\binom{n}{2}\right)$, as the graph obtained by replacing every vertex $v_i \in G$ with an independent set $I_i$ of $s_i$ vertices, and by replacing every edge $(v_i, v_j) \in E(G) \setminus M$ with a complete bipartite graph connecting $I_i$ and $I_j$. When $M = \emptyset$, we simply write $G(s_1, \ldots, s_n; M)$ as the standard blowup $G(s_1, \ldots, s_n)
$.

We first want to characterize the class of degenerate graphs when the host hypergraph is 3-uniform. Observe that $\hat{\text{ex}}_k(n, G) \leq \left(\frac{k}{3}\right) e_{x_k}(n, G)$. This implies that any graph $G$ satisfying $e_{x_k}(n, G) = o(n^2)$ is $k$-degenerate. In particular, by results of [17, 18, 14, 27], any cycles of fixed length at least 4 and $K_{2,t}$ are 3-degenerate. For triangles, Grósz, Methuku and Tompkins [15] showed that the uniformity threshold of a triangle is 5, which implies that $C_3$ is 5-degenerate. Moreover, there are constructions which show that $C_3$ is not 3-degenerate or 4-degenerate. For $K_{s,t}$ where $s, t \geq 3$, it is shown [27, 15, 1] that $\text{ex}_r(n, K_{s,t}) = \Theta(n^{r-\frac{r(r-1)}{2s}})$. Thus in this case, the corresponding results on Berge Turán number do not imply the degeneracy of $K_{s,t}$ in the cover Turán density.

In this paper, we classify all degenerate graphs when the host hypergraph is 3-uniform.

**Theorem 3.** Given a simple graph $G$, $\hat{\pi}_3(G) = 0$ if and only if $G$ satisfies the following conditions:

1. $G$ is triangle-free, and there exists an induced bipartite subgraph $B \subseteq G$ such that $V(G) - V(B)$ is a single vertex.
2. There exists a bipartite subgraph $B \subseteq G$ such that $E(G) - E(B)$ is a matching (possibly empty) in one of the partition of $B$.

**Corollary 1.** Given a simple graph $G$, $\hat{\pi}_3(G) = 0$ if and only if $G$ is contained in both $C_5(1, s, s, s, s)$ and $C_5(s, s, s, s, s; \{\{1, 2\}\})$ for some positive integers $s$.

![Figure 1: $C_5(1, s, s, s, s)$ and $C_5(s, s, s, s, s; \{\{1, 2\}\})$](image)

**Corollary 2.** Given a simple graph $G$, $\hat{\pi}_3(G) = 0$ if and only if $G$ is contained in one of the three graphs $C_5(1, s, s, s, s, 1)$, $C_5(1, s, s, s, s; \{\{2, 3\}\})$ or $C_5(1, s, s, s, s; \{\{3, 4\}\})$.

With Theorem 1 and Theorem 3, we can then determine the cover Turán density of all graphs when $k = 3$. The results are summarized in the following theorem.
Theorem 4. Given a simple graph $G$,

$$\hat{\pi}_3(G) = \begin{cases} 
1 - \frac{1}{\chi(G)-1} & \text{if } \chi(G) \geq 4, \\
0 & \text{if } G \text{ satisfies the condition in Theorem 3,} \\
\frac{1}{2} & \text{otherwise.}
\end{cases}$$

For 3-cover Turán number, we also show the following proposition:

Proposition 2. Let $G$ be a bipartite graph such that every edge is contained in a $C_4$. Then

$$e\hat{x}_3(n, G) = \Theta(ex(n, G)).$$

Proof. The fact that $e\hat{x}_3(n, G) = O(ex(n, G))$ is a consequence of Proposition 1. For the lower bound, consider an extremal $G$-free graph $H$ with $ex(n, G)$ edges. It follows that there is a bipartite subgraph $H' = A \cup B$ of $H$ which is $G$-free and contains at least $\frac{1}{2}ex(n, G)$ edges. We then construct a 3-graph $\mathcal{H}$ as follows. For each $a \in A$, replace $a$ with two new vertices $a_1, a_2$. The vertex set $B$ remains the same. For each $e = \{a, b\} \in E(H)$ with $a \in A$, $b \in B$, we have a hyperedge $\{a_1, a_2, b\}$ in $\mathcal{H}$. We claim that $\mathcal{H}$ contains no Berge-$G$. Indeed, it is easy to see that if there is any Berge-$G$ in $\mathcal{H}$, one of the edges of $G$ must be $a_1a_2$ for some $a \in A$. However, note that there is no $C_4$ containing $a_1a_2$ in $\partial(\mathcal{H})$ while every edge of $G$ is contained in a $C_4$. Therefore, $\mathcal{H}$ contains no Berge copy of $G$. $\square$

Using Proposition 2, we have the following corollary on the asymptotics of the cover Turán number of $K_{s,t}$.

Corollary 3. For positive integers $t \geq s \geq 2$, we have

$$e\hat{x}_3(n, K_{s,t}) = \Theta(ex(n, K_{s,t})).$$

The following questions would be interesting for further investigations:

Question 1. Characterize all $k$-degenerate graphs or determine the $\{k\}$-cover Turán density of all graphs for $k \geq 4$.

Question 2. Determine the asymptotics of the Berge Turán number of the 3-degenerate graphs in Theorem 3.
2 Proof of Theorem 1

Let \( k \geq 2 \) and \( G \) be a fixed graph with \( \chi(G) \geq 2 \). Let \( \epsilon > 0 \). Suppose \( \mathcal{H} \) is an edge-minimal \( k \)-uniform hypergraph on sufficiently large \( n \) vertices such that
\[
|E(\partial(\mathcal{H}))| \geq \left( 1 - \frac{1}{\chi(G) - 1} + \epsilon \right) \left( \binom{n}{2} \right).
\]

Our goal is to show that \( \mathcal{H} \) contains a Berge copy of \( G \). For ease of reference, set \( H = \partial(\mathcal{H}) \).

Let \( M = k^2/\epsilon \). Let \( H' \) be the subgraph of \( H \) obtained by deleting all the edges \( uv \) from \( H \) with co-degree \( d(\{u, v\}) \geq M \) in \( \mathcal{H} \).

**Claim 1.** \( |E(H')| \geq \left( 1 - \frac{1}{\chi(G) - 1} + \epsilon/2 \right) \left( \binom{n}{2} \right) \).

**Proof.** Let \( L = E(H) \setminus E(H') \). By double counting, the number of hyperedges containing some edge in \( L \) is at least \( LM/(\binom{k}{2}) \). Since \( \mathcal{H} \) is assumed to be edge-minimal, it follows that every hyperedge \( h \) contains a vertex pair that is only contained in \( h \). Hence \( |E(\mathcal{H})| \leq \left( \binom{n}{2} \right) \).

It follows that
\[
LM/\left( \binom{k}{2} \right) \leq |E(\mathcal{H})| \leq \left( \binom{n}{2} \right),
\]
which implies that
\[
L \leq \frac{k^2}{2M} \left( \binom{n}{2} \right) \leq \frac{\epsilon}{2} \left( \binom{n}{2} \right).
\]

This completes the proof of the claim. \( \square \)

Let \( G' \) be the blowup of \( G \) by a factor of \( b = Mv(G)^2k \), i.e., \( G' = G(b) \). Suppose \( V(G) = \{v_1, \ldots, v_s\} \) and \( V_i \) is the blowed-up independent set in \( G' \) that corresponds to \( v_i \).

Recall the celebrated Erdős-Stone-Simonovits theorem [7, 8], which states that for a fixed simple graph \( F \), \( ex(n, F) = \left( 1 - \frac{1}{\chi(F) - 1} + o(1) \right) \left( \binom{n}{2} \right) \). Since \( \chi(G') = \chi(G) \), it follows by the Erdős-Stone-Simonovits theorem that for sufficiently large \( n \), \( H' \) contains \( G' \) as a subgraph.

Our goal is to give an embedding \( f \) of \( G \) into \( G' \) so that \( f(v_i) \in V_i \) for all \( 1 \leq i \leq s \) and every edge of \( G \) is embedded in a distinct hyperedge in \( \mathcal{H} \). For ease of reference, set \( L_j = \{v_1, \ldots, v_j\} \). For \( 1 \leq t \leq s \) and \( v \in V(G) \), set \( N_i(v) = N_G(v) \cap L_t \). For \( i = 1 \), just embed \( v_1 \) to an arbitrary vertex in \( V_1 \). Suppose that \( v_1, \ldots, v_t \) are already embedded and edges in \( G[L_t] \) are already embedded in distinct hyperedges. We now want to embed \( v_{t+1} \) into an appropriate vertex in \( V_{t+1} \), i.e., we want to find a vertex \( u \in V_{t+1} \) such that there are distinct unused hyperedges embedding the edges from \( u \) to \( f(N_t(v_{t+1})) \). Note that each vertex \( u \) in \( V_{t+1} \) is adjacent to all vertices in \( f(N_t(v_{t+1})) \) in \( G' \). Let \( S_t(u) = \{u\} \times f(N_t(v_{t+1})) \), i.e., \( S_t(u) \) is the set of vertex pairs which contain \( u \) and another vertex in \( f(N_t(v_{t+1})) \).

Recall that \( |V_{t+1}| = Mv(G)^2k \). At most \( e(G)(k - 2) \) vertices in \( V_{t+1} \) are contained in hyperedges that are already used. For any of the remaining vertices \( u \in V_{t+1} \), if there are no distinct hyperedges embedding all vertex pairs in \( S_t(u) \), that means some hyperedge contains at least two vertex pairs \( uw_1, uw_2 \) in \( S_t(u) \). Note that \( d_{H'}(\{w_1, w_2\}) \leq M \) by the definition of \( H' \). Thus the number of vertices \( u \in V_{t+1} \) such that there exists some hyperedge containing at least two vertex pairs in \( S_t(u) \) is at most
\[
\binom{t}{2} M(k - 2) \leq \frac{Mv(G)^2k}{2}.
\]
Since $|V_{t+1}| = Mv(G)^2k$, it follows that there exists some $u \in V_{t+1}$ such that $u$ is not contained in any hyperedge already used and there is no hyperedge containing at least two vertex pairs in $S_i(u)$. It follows that there are distinct unused hyperedges containing all vertex pairs in $S_i(u)$. Set $f(v_{t+1})$ to be this $u$.

By induction, we can then conclude that $\mathcal{H}$ contains a Berge copy of $G$. This completes the proof of Theorem 1.

3 Proof of Theorem 3

3.1 Regularity Lemma

The proof of Theorem 3 uses the Szemerédi Regularity Lemma. Given a graph $G$, and two disjoint vertex sets $X, Y \subseteq V(G)$, let $e(X, Y)$ denote the number of edges intersecting both $X$ and $Y$. Define $d(X, Y) = e(X, Y)/|X||Y|$ as the edge density between $X$ and $Y$. $(X, Y)$ is called $\epsilon$-regular if for all $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have $|d(X, Y) - d(X', Y')| \leq \epsilon$. We say a vertex partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ equipartite (with the exceptional set $V_0$) if $|V_i| = |V_j|$ for all $i, j \in [k]$. The vertex partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ is said to be $\epsilon$-regular if all but at most $\epsilon k^2$ pairs $(V_i, V_j)$ with $1 \leq i < j \leq k$ are $\epsilon$-regular and $|V_0| \leq \epsilon n$. The extremely powerful Szemerédi’s regularity lemma states the following:

**Theorem 5.** [32] For every $\epsilon$ and $m$, there exists $N_0$ and $M$ such that every graph $G$ on $n \geq N_0$ admits an $\epsilon$-regular partition $V_0 \cup V_1 \cup \cdots \cup V_k$ satisfying that $m \leq k \leq M$.

A $\epsilon$-regular pair satisfies the following simple lemma.

**Lemma 1.** Suppose $(X, Y)$ is an $\epsilon$-regular pair of density $d$. Then for every $Y' \subseteq Y$ of size $|Y'| \geq \epsilon|Y|$, there exists less than $\epsilon|X|$ vertices in $X$ that have less than $(d - \epsilon)|Y'|$ neighbors in $Y'$.

**Proof.** Let $Y' \subseteq Y$ with $|Y'| \geq \epsilon|Y|$. Let $X'$ be the set of vertices of $X$ that have less than $(d - \epsilon)|Y'|$ neighbors in $Y'$. Note that $d(X', Y') < (d - \epsilon)$, which can only happen if $|X'| < \epsilon|X|$.

Using Lemma 1, we will show the following lemma using the standard embedding technique.

**Lemma 2.** Fix a positive integer $s$. Suppose $(X, Y)$ is an $\epsilon$-regular pair of density $d$ such that $\epsilon \leq 1/4s$, $(d - \epsilon)^s \geq 4\epsilon$ and $|X|, |Y| \geq 4s/(d - \epsilon)^s$. Then there exist disjoint subsets $A, C \subseteq X$ and $B, D \subseteq Y$ such that $|A| = |B| = s$, $|C| \geq \epsilon|X|$, $|D| \geq \epsilon|Y|$, and there is a complete bipartite graph connecting $A$ and $D$, $B$ and $C$ as well as $A$ and $B$.

**Proof.** Denote $A = \{a_1, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_s\}$. For each $i \in [s]$, we will first embed $a_i$ to $X$ one vertex at a time. After embedding the $t$-th vertex, we will show that the following condition is satisfied:

$$\left|Y \cap \bigcap_{i=1}^{t} N(a_i)\right| \geq (d - \epsilon)^t|Y|.$$ 

The condition is trivially satisfied when $t = 0$. Suppose that we already embedded the vertices $a_1, \ldots, a_t$ for some $t > 0$. Let $Y'_t = Y \cap \bigcap_{i=1}^{t} N(a_i)$. By induction, $|Y'_t| \geq (d - \epsilon)^t|Y| > \epsilon|Y|$.
Hence by Lemma 1, at least \((1 - \epsilon)|X| - s\) vertices in \(X\) have at least \((d - \epsilon)|Y'_t|\) neighbors in \(Y'_t\). Embed \(a_{t+1}\) to one of these \((1 - \epsilon)|X| - s\) vertices and it’s easy to see that

\[
\left|Y \cap \bigcap_{i=1}^{t+1} N(a_i)\right| \geq (d - \epsilon)|Y'_t| \geq (d - \epsilon)^{t+1}|Y|.
\]

Now we want to embed \(b_t\) to \(Y'_s\) one vertex at a time. The process is entirely the same as long as

\[
(d - \epsilon)^s(|X| - s) \geq \epsilon|X|,
\]

\[
(d - \epsilon)^s|Y| - \epsilon|Y| - s \geq 1,
\]

and

\[
(d - \epsilon)^s|Y| - s \geq \epsilon|Y|,
\]

which are satisfied by our assumption on \(d\), \(|X|\) and \(|Y|\).

\[\square\]

### 3.2 Constructions for Theorem 3

Before we prove Theorem 3, we first give two constructions and show that if \(G\) does not satisfy the conditions (1) and (2) in Theorem 3, then at least one of the constructions do not contain a Berge copy of \(G\). In particular, suppose \(A, B\) are two disjoint set of vertices enumerated as \(A = \{a_1, \ldots, a_{n/2}\}\) and \(B = \{b_1, \ldots, b_{n/2}\}\). Let \(H_1\) be a 3-uniform hypergraph such that \(V(H_1) = A \cup B\) and \(E(H_1) = \{\{a_i, b_j, b_{j+1}\} : j \text{ is odd}\}\). Let \(H_2\) be a 3-uniform hypergraph such that \(V(H_2) = A \cup B\) and \(E(H_2) = \{\{b_1, a_i, b_j\} : a_i \in A, b_j \in B\setminus\{b_1\}\}\). Observe that

\[
\lim_{n \to \infty} \frac{|E(\partial(H_1))|}{\binom{n}{2}} = \lim_{n \to \infty} \frac{|E(\partial(H_2))|}{\binom{n}{2}} = \frac{1}{2}.
\]

**Claim 2.** If \(\hat{\pi}_3(G) = 0\), then condition (1) and (2) of Theorem 3 must hold.

**Proof.** Suppose that \(\hat{\pi}_3(G) = 0\). We claim that (1) and (2) must hold. First observe that \(H_1\) contains no Berge triangle. Hence \(G\) must be triangle-free otherwise \(H_1\) is Berge-\(G\)-free. Now note that given a hypergraph \(\mathcal{H}\), if \(\partial(\mathcal{H})\) is \(G\)-free, then \(\mathcal{H}\) must be Berge-\(G\)-free. Observe that \(\partial(H_1)\) contains a bipartite subgraph \(B \subseteq \partial(H_1)\) such that \(E(\partial(H_1)) - E(B)\) is a matching (possibly empty) in one of the partition of \(B\). Hence if there is no such bipartite subgraph in \(G\), then \(\partial(H_1)\) is \(G\)-free, implying that \(H_1\) is Berge-\(G\)-free. Since \(\hat{\pi}_3(G) = 0\), it follows that \(G\) must satisfy condition (1). Similarly, observe that \(\partial(H_2)\) satisfies condition (2). Hence if \(G\) doesn’t satisfy condition (2), then \(H_2\) is Berge-\(G\)-free, which contradicts that \(\hat{\pi}_3(G) = 0\).

Therefore we can conclude that (1) and (2) must hold for \(G\).

\[\square\]

### 3.3 Proof of Theorem 3

The forward direction is proved in Claim 2. It remains to show that if \(G\) satisfies the conditions (1) and (2) in Theorem 3, then \(\hat{\pi}_3(G) = 0\). Suppose not, i.e., \(\hat{\pi}_3(G) \geq d\) for some \(d > 0\). Our goal is to show that for every 3-graph \(\mathcal{H}\) on (sufficiently large) \(n\) vertices and at least \(d\binom{n}{2}\) edges in \(\partial(\mathcal{H})\), \(\mathcal{H}\) contains a Berge copy of \(G\).
Assume first that $\mathcal{H}$ is edge-minimal while maintaining the same shadow. It follows that in every hyperedge $h$ of $\mathcal{H}$, there exists some $e \in \binom{h}{2}$ such that $e$ is contained only in $h$. Moreover, note that since each hyperedge covers at most 3 edges in $\partial(\mathcal{H})$, we have that
\[
|E(\mathcal{H})| \geq \frac{1}{3} |E(\partial(\mathcal{H}))| \geq \frac{d(n)}{3}.
\]
Call an edge $e \in \partial(\mathcal{H})$ uniquely embedded if there exists a unique hyperedge $h \in E(\mathcal{H})$ containing $e$. Now randomly partition $V(\mathcal{H})$ into three sets $X, Y, Z$ of the same size. Let $e(X, Y, Z)$ denote the number of hyperedges of $\mathcal{H}$ intersecting each of the sets $X, Y, Z$ on at most one vertex. It’s easy to see that $E[e(X, Y, Z)] = \frac{2}{9}|E(\mathcal{H})|$. Hence there exists a 3-partite subhypergraph $\mathcal{H}_1 = X \cup Y \cup Z$ of $\mathcal{H}$ such that $|E(\mathcal{H}_1)| \geq \frac{2}{9}|E(\mathcal{H})|$. Note that each hyperedge $h$ of $\mathcal{H}_1$ contains some $e \in \binom{h}{2}$ that is uniquely embedded. Hence there are at least $\frac{2}{9}|E(\mathcal{H})|$ uniquely embedded edges in $\partial(\mathcal{H}_1)$. Without loss of generality, assume that there are at least $\frac{2}{27}|E(\mathcal{H})|$ uniquely embedded edges between the vertex sets $X$ and $Y$ in $\partial(\mathcal{H}_1)$. Let $\mathcal{H}'$ be the subhypergraph of $\mathcal{H}_1$ with only hyperedges containing a uniquely embedded edge between $X$ and $Y$.

For ease of reference, let $H' = \partial(\mathcal{H}')$ and let $H'[X \cup Y]$ be the subgraph of $\partial(\mathcal{H}')$ induced by $X \cup Y$. Note that $H'[X \cup Y]$ is bipartite with at least $\frac{2}{27}|E(\mathcal{H})| \geq \frac{2d^2(n)}{27} = d^2(n)$ edges.

Let $\epsilon = \epsilon(s, d'/2)$ be small enough so that $\epsilon$ satisfies the assumptions in Lemma 2. Applying the regularity lemma on $H'[X \cup Y]$, we can find an $\epsilon$-regular partition in which there exist two parts $X' \subseteq X, Y' \subseteq Y$ such that $(X', Y')$ is an $\epsilon$-regular pair with edge density at least $d'/2$. Moreover, $|X'|, |Y'| \geq n/M$ for some constant $M > 0$. By Lemma 2, we can find disjoint subsets $A, C \subseteq X'$ and $B, D \subseteq Y'$ such that $|A| = |B| = s$, $|C| \geq \epsilon |X'|$, $|D| \geq \epsilon |Y'|$, and there is a complete bipartite graph connecting $A$ and $D$, $B$ and $C$ as well as $A$ and $B$.

Now consider the subhypergraph $\hat{\mathcal{H}} = \mathcal{H}'[C \cup D \cup Z]$ of $\mathcal{H}'$ induced by the vertex set $C \cup D \cup Z$, i.e., all hyperedges in $\hat{\mathcal{H}}$ contain vertices only in $C \cup D \cup Z$. Given a vertex set $S \subseteq V(\hat{\mathcal{H}})$, define $\hat{d}_S(v)$ as the number of neighbors of $v$ in $S$ in $\partial(\hat{\mathcal{H}})$.

**Claim 3.** If there exists some $z \in Z$ such that $\hat{d}_C(v) \geq 2s$ and $\hat{d}_D(v) \geq 2s$, then $\mathcal{H}'$ contains a Berge-$C_5(1, s, s, s, s)$ as subhypergraph.

**Proof.** Denote the $C_5(1, s, s, s, s)$ that we wish to embed as $\{v_1\} \cup V_2 \cup V_3 \cup V_4 \cup V_5$. Let $v_1 = z$ and $V_4 = A, V_3 = B$. Let $C_2, D_2$ be the set of neighbors of $z$ in $C$ and $D$ respectively in $\partial(\hat{\mathcal{H}})$. We wish to embed $V_2$ in $C_2$ and $V_5$ in $D_2$. Note that $|C_2|, |D_2| \geq 2s$ by our assumption. Pick arbitrary $s$ of them to be $V_2$. For each vertex pair $\{z, w\}$ where $w \in V_2$, there exists a hyperedge $h \subseteq C \cup D$ containing $\{z, w\}$. Use $h$ to embed $\{z, w\}$. Observe that at most $s$ vertices in $D_2$ are contained in hyperedges embedding the edges from $z$ to $V_2$. Since $|D_2| \geq 2s$, we can set $V_5$ to be arbitrary $s$ vertices among vertices in $D_2$ that are not contained in any hyperedge embedding the edges from $z$ to $V_2$. We then have distinct hyperedges (in $\hat{\mathcal{H}}$ only) embedding the edges from $z$ to $V_2$ and $V_5$ respectively. Moreover, recall that by our choice of $X'$ and $Y'$, vertex pairs between $V_2$ and $V_3$, $V_3$ and $V_4$, $V_4$ and $V_5$ are uniquely embedded (with the third vertex in $Z$), i.e., there exist distinct hyperedges embedding them. Hence, we obtain a Berge-$C_5(1, s, s, s, s)$ in $\mathcal{H}'$. 

Now observe that $|C| \geq \epsilon |X'|$, $|D| \geq \epsilon |Y'|$. Hence by the $\epsilon$-regularity of $(X', Y')$, the
number of edges $e(C, D)$ in $\partial(\mathcal{H})$ satisfies that

$$e(C, D) \geq (d' \frac{d}{2} - \epsilon)|C||D| \geq (d' \frac{d}{2} - \epsilon)e^2|X'||Y'| \geq (d' \frac{d}{2} - \epsilon)e^2 \frac{n^2}{M^2} = cn^2$$

where $c$ is a constant depending on $\epsilon$ and $d'$.

**Claim 4.** If $\mathcal{H}'$ contains no Berge-$C_5(1, s, s, s, s)$ as subhypergraph, it must contain a Berge-$C_5(\pi, \pi, s, s, s, s; \{1, 2\})$ as subhypergraph.

**Proof.** By claim 3, since $\mathcal{H}'$ contains no Berge-$C_5(1, s, s, s, s)$ as subhypergraph, it follows that given any $v \in Z$, one of $d_C(v)$, $d_D(v)$ must be smaller than $2s$. Let $Z_1$ be the set of vertices $z \in Z$ with $d_C(v) < 2s$, and $Z_2$ be the set of vertices $z \in Z$ with $d_D(v) < 2s$. Let $e(Z_1, D)$ and $e(Z_2, C)$ denote the number of edges between $Z_1$ and $D$, $Z_2$ and $C$ respectively in $\partial(\mathcal{H})$. Since $e(C, D) \geq cn^2$ and all hyperedges in $\mathcal{H}$ contains a vertex in $Z$, it follows that at least one of the $e(Z_1, D)$ and $e(Z_2, C)$ must be at least $\Omega(n^2)$. WLOG, suppose $e(Z_1, D) \geq c'n^2$ for some $c' > 0$. Recall the classical result of Kővári, Sós and Turán [24], who showed that $ex(n, K_{r,t}) = O(n^{2-1/r})$ where $r \leq t$. By the Turán number of complete bipartite graphs, we have that for sufficiently large $n$, $\partial(\mathcal{H})[D \cup Z_1]$ contains a complete bipartite graph $K_{s',s'}$.

It remains to find a Berge copy of a matching between $Z_1$ and $C$ in $\mathcal{H}'$. Let $C_1$ be the collection of vertices $v \in C$ such that there is some hyperedge containing $v$ and one of the edges in the $K_{s',s'}$ above. Observe that for each $v \in C_1$, $d_{Z_1}(v) \leq s$, otherwise we obtain a Berge-$C_5(1, s, s, s, s)$ in $\mathcal{H}'$.

It then follows by our choice of the order of the complete bipartite graph that there exists a Berge copy of a matching of size $s$ with vertex set $Z' \cup C'$ where $Z' \subseteq Z_1$ and $C' \subseteq C_1$. Moreover, at most $s$ vertices of the $K_{s',s'}$ in $D$ are contained in the hyperedges that embed the matching. It follows that we could also find a $D' \subseteq D$ of size $s$ such that there is Berge complete bipartite graph on $Z' \cup D'$. Denote the $C_5(s, s, s, s, s; \{1, 2\})$ we wish to embed as $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$. Set $V_1 = Z'$, $V_2 = C'$, $V_3 = B$, $V_4 = A$, and $V_5 = D'$, we then obtain a Berge-$C_5(s, s, s, s, s; \{1, 2\})$ in $\mathcal{H}'$.

In summary, if $\mathcal{H}$ is 3-graph with at least $d(n^2)$ edges in $\partial(\mathcal{H})$ for some $d > 0$ and $n$ sufficiently large, then $\mathcal{H}$ contains either a Berge-$C_5(1, s, s, s, s)$ or a Berge-$C_5(s, s, s, s, s; \{1, 2\})$. Observe that if $G$ satisfies the conditions (1) and (2) in Theorem 3, then $G$ is a subgraph of both $C_5(1, s, s, s, s)$ and $C_5(s, s, s, s, s; \{1, 2\})$. Hence it follows that $\hat{\pi}_3(G) = 0$. This completes the proof of the theorem.

\[\square\]

4 Proof of Theorem 4

If $\chi(G) \geq 4$, we are done by Theorem 2. If $\chi(G) \leq 3$ and $G$ is not degenerate, the two hypergraphs we constructed in Section 3.2 provide the lower bound 1/2, which is also an upper bound by Theorem 1. Theorem 3 resolves the case when $G$ is degenerate.

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