Minimal Surfaces in $S^3$ with Constant Contact Angle

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Abstract

We provide a characterization of the Clifford Torus in $S^3$ via moving frames and contact structure equations. More precisely, we prove that minimal surfaces in $S^3$ with constant contact angle must be the Clifford Torus. Some applications of this result are then given, and some examples are discussed.

Keywords: minimal surfaces, Clifford Torus, three sphere, contact angle

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1 Introduction

The study of minimal surfaces played a formative role in the development of mathematics over the last two centuries. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have

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also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds. We mention for example the two papers, [2] and [6], where the authors classify Legendrian minimal surfaces in $S^5$ with constant Gaussian curvature. Besides, interesting characterizations of the Clifford torus in spheres are given in [3], [8], and [9].

The scope of this note is to use a geometric invariant in order to study immersed surfaces in three dimensional sphere. This invariant (the contact angle ($\beta$)) is the complementary angle between the contact distribution and the tangent space of the surface.

We show that the Gaussian curvature $K$ of a minimal surface in $S^3$ with contact angle $\beta$ is given by:

$$K = 1 - |\nabla \beta + e_1|^2$$

Moreover, the contact angle satisfies the following Laplacian equation

$$\Delta(\beta) = -\tan(\beta)|\nabla \beta + 2e_1|^2$$

where $e_1$ is the characteristic field defined in section 2 and introduced by Bennequin, in [1] pages 190 - 206.

Using the equations of Gauss and Codazzi, we have proved the following two theorems:

**Theorem 1.** The Clifford Torus is the only minimal surface in $S^3$ with constant contact angle.

**Theorem 2.** The Clifford Torus is the only compact minimal surface in $S^3$ with contact angle $0 \leq \beta < \frac{\pi}{2}$ (or $-\frac{\pi}{2} < \beta \leq 0$)

More in general, we have the following congruence result:

**Theorem 3.** Consider $S$ a Riemannian surface, $e$ a vector field on $S$, and $\beta : S \to ]0, \frac{\pi}{2}[ \cup ]-\frac{\pi}{2}, 0[ \cup ]\pi, 2\pi[$ a function over $S$ that verifies the following equation:

$$\Delta \beta = -\tan(\beta)(|\nabla \beta|^2 + 4(e(\beta) + 1))$$

then there exist one and only one minimal immersion of $S$ into $S^3$ such that $e$ is the characteristic vector field, and $\beta$ is the contact angle of this immersion.

Finally, in section 5, we give two examples of minimal surfaces in $S^3$. The first one, we determine that the contact angle ($\beta$) of the Clifford Torus is ($\beta = 0$) and the second one we determine that the contact angle of the totally geodesic sphere is ($\beta = \arccos(x_2)$), and therefore, non constant.
2 Contact Angle for Immersed Surfaces in $S^3$

Consider in $\mathbb{C}^2$ the following objects:

- the Hermitian product: $(z, w) = z^1\bar{w}^1 + z^2\bar{w}^2$;
- the inner product: $\langle z, w \rangle = Re(z, w)$;
- the unit sphere: $S^3 = \{ z \in \mathbb{C}^2 | (z, z) = 1 \}$;
- the Reeb vector field in $S^3$, given by: $\xi(z) = iz$;
- the contact distribution in $S^3$, which is orthogonal to $\xi$:

$$\delta_z = \{ v \in T_zS^3 | \langle \xi, v \rangle = 0 \}.$$  

Note that $\delta$ is invariant by the complex structure of $\mathbb{C}^2$.

Let now $S$ be an immersed orientable surface in $S^3$.

**Definition 1.** The contact angle $\beta$ is the complementary angle between the contact distribution $\delta$ and the tangent space $TS$ of the surface.

Let $(e_1, e_2)$ be a local frame of $TS$, where $e_1 \in TS \cap \delta$. Then $\cos \beta = \langle \xi, e_2 \rangle$.

Let $(f_1 = z^\perp, f_2 = iz^\perp$ and $f_3 = iz)$ be an orthonormal frame of $S^3$, where $z^\perp = (-\bar{z}_2, \bar{z}_1)$.

The covariant derivative is given by:

\[
\begin{align*}
Df_1 & = w_1^2 f_2 + w_2 f_3 \\
Df_2 & = w_1^3 f_1 - w_1^2 f_3 \\
Df_3 & = -w_2 f_1 + w_1 f_2
\end{align*}
\]  

(2)

where $(w_1, w_2, w_3)$ is the coframe associated to $(f_1, f_2, f_3)$.

Let $e_1$ be an unitary vector field in $TS \cap \delta$, where $\delta$ is the contact distribution. Thus follows that:

\[
\begin{align*}
e_1 & = f_1 \\
e_2 & = \sin(\beta) f_2 + \cos(\beta) f_3 \\
e_3 & = -\cos(\beta) f_2 + \sin(\beta) f_3
\end{align*}
\]  

(3)

where $\beta$ is the angle between $f_3$ and $e_2$, $(e_1, e_2)$ are tangent to $S$ and $e_3$ is normal to $S$. 


3 Equations for the Gaussian Curvature and for the Laplacian of a Minimal Surface in $S^3$

In this section, we will give formulas for the Laplacian and for the Gaussian curvature of a minimal surface immersed in $S^3$.

The reader can see [7], and [4] for further details.

Let $(\theta^1, \theta^2, \theta^3)$ be the coframe associated to $(e_1, e_2, e_3)$.

Thus, from equations (3), it follows that:

\[
\begin{align*}
\theta^1 &= w_1 \\
\theta^2 &= \sin(\beta) w_2 + \cos(\beta) w_3 \\
\theta^3 &= -\cos(\beta) w_2 + \sin(\beta) w_3
\end{align*}
\] (4)

We know that $\theta^3 = 0$ on $S$, then we obtain the following equation:

\[
\sin(\beta) w^3 = \cos(\beta) w^2
\] (5)

we have also

\[
\begin{align*}
w^2 &= \sin(\beta) \theta^2 \\
w^3 &= \cos(\beta) \theta^2
\end{align*}
\]

It follows from (4) that:

\[
\begin{align*}
d\theta^1 + \sin(\beta)(w^1_2 - \cos(\beta) \theta^2) \land \theta^2 &= 0 \\
d\theta^2 + \sin(\beta)(w^1_1 + \cos(\beta) \theta^2) \land \theta^1 &= 0 \\
d\theta^3 &= d\beta \land \theta^2 - \cos(\beta) w^1_2 \land w^1 + (1 + \sin^2(\beta)) \theta^1 \land \theta^2
\end{align*}
\]

Therefore the connection form of $S$ is given by

\[
\theta^1_2 = \sin(\beta)(w^1_2 - \cos(\beta) \theta^2)
\] (6)

Differentiating $e_3$ at the basis $(e_1, e_2)$, we have fundamental second forms coefficients

\[
De_3 = \theta^1_3 e_1 + \theta^2_3 e_2
\]

where

\[
\begin{align*}
\theta^1_3 &= -\cos(\beta) w^1_2 - \sin(\beta) \theta^2 \\
\theta^2_3 &= d\beta + \theta^1
\end{align*}
\]

It follows from $d\theta^3 = 0$, that

\[
w^1_2(e_2) = -\frac{\beta_1}{\cos \beta} - \frac{(1 + \sin^2(\beta))}{\cos \beta}
\] (7)
onde $d\beta(e_1) = \beta_1$.

The condition of minimality is equivalent to the following equation

$$\theta_3^1 \wedge \theta^2 - \theta_2^3 \wedge \theta^1 = 0$$

we have

$$w_2^1(e_1) = \frac{\beta_2}{\cos(\beta)}$$

where $d\beta(e_2) = \beta_2$.

It follows from (6), (7) and (8),

$$\theta_1^2 = \tan(\beta)(\beta_2 \theta^1 - (\beta_1 + 2)\theta^2)$$
$$\theta_3^1 = -\beta_2 \theta^1 + (\beta_1 + 1)\theta^2$$
$$\theta_3^2 = (\beta_1 + 1)\theta^1 + \beta_2 \theta^2$$

If $J$ is the complex structure of $S$ we have $Je_1 = e_2$ e $Je_2 = -e_1$.

Using $J$, the forms above reduce to:

$$\theta_2^1 = \tan(\beta)(d\beta \circ J - 2\theta^2)$$
$$\theta_3^1 = -d\beta \circ J + \theta^2$$
$$\theta_3^2 = d\beta + \theta^1$$

Gauss equation is

$$d\theta_2^1 = \theta^2 \wedge \theta^1 + \theta_1^3 \wedge \theta_2^3$$

which implies

$$d\theta_2^1 = (|\nabla \beta|^2 + 2\beta_1) (\theta^2 \wedge \theta^1)$$

and therefore

$$K = 1 - |\nabla \beta + e_1|^2$$

Differentiating $\theta_2^1$, we have

$$d\theta_2^1 = \sec^2(\beta)(|\nabla \beta|^2 + 2\beta_1) (\theta^2 \wedge \theta^1)$$
$$+ (\tan(\beta) \Delta(\beta) + 2 \tan^2(\beta)(\beta_1 + 2))(\theta^2 \wedge \theta^1)$$

Using (10) and (11), we obtain the following formula for the Laplacian of $S$

$$\Delta(\beta) = -\tan(\beta)((\beta_1 + 2)^2 + \beta_2^2)$$

Or

$$\Delta(\beta) = -\tan(\beta)|\nabla \beta + 2e_1|^2$$

Codazzi equations are

$$d\theta_3^3 + \theta_3^1 \wedge \theta_1^1 = 0$$
$$d\theta_3^3 + \theta_3^1 \wedge \theta_2^1 = 0$$

A straightforward computation in the first equation gives (12) and the second equation is always verified.
4 Main Results

4.1 Proof of the Theorem 1

Suppose that $\beta$ is constant, it follows from \( (10) \) that $d\theta_1^2 = 0$ and, therefore, $K = 0$, i.e., Gaussian curvature is identically null, hence $S \subset S^3$ is the Clifford Torus, which prove the Theorem 1.

4.2 Proof of the Theorem 2

For $0 \leq \beta < \frac{\pi}{2}$, we have $\tan \beta \geq 0$, hence $\Delta(\beta) \leq 0$ and using that $S$ is a compact surface, we conclude by Hopf’s Lemma that $\beta$ is constant, and therefore, $K = 0$ and $S$ is the Clifford Torus, which prove the Theorem 2.

4.3 Proof of the Theorem 3

Let $S$ be an orientable surface in $S^3$, and let $e$ be an unit vector field on $S$. We choose an orthonormal positive basis $(e_1, e_2)$ with $e_1 = e$, and let $(\theta^1, \theta^2)$ be a coframe on $S$. For each function $\beta : S \rightarrow [0, \frac{\pi}{2}]$ that satisfies the following Laplacian equation:

$$\Delta(\beta) = -\tan(\beta) |\nabla \beta + 2e_1|^2$$

We define the following fundamental second form:

$$\begin{align*}
\theta_1^3 &= (d\beta + \theta^1) \circ J \\
\theta_2^3 &= -(d\beta + \theta^1)
\end{align*}$$

(13)

Now, the proof follows from Gauss-Codazzi equations.

5 Examples

Examples of minimal surfaces in $S^3$ was discovered by Lawson, in \[5\]. Here we will use the notion of the contact angle to give a characterization of these known examples.
5.1 Contact Angle of Clifford Torus in $S^3$

Let us consider the following torus in $S^3$:

$$T^2 = \{(z_1, z_2) \in C^2/ z_1 \bar{z}_1 = \frac{1}{2}, z_2 \bar{z}_2 = \frac{1}{2}\}$$

Let $f$ be the following immersion:

$$f(u_1, u_2) = \frac{\sqrt{2}}{2}(e^{iu_1}, e^{iu_2})$$

Tangent space $T(T^2)$ is given by $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$, thus we have:

$$a \frac{\partial}{\partial u_1} + b \frac{\partial}{\partial u_2} = \lambda z^\perp$$

using the condition above and the fact that $|\lambda| = 1$, we obtain:

$$\lambda = ie^{i(u_1+u_2)}$$

Unit vector fields are:

$$\begin{cases} e_1 = ie^{i(u_1+u_2)} z^\perp \\ e_2 = iz \\ e_3 = iz^\perp \end{cases}$$

The contact angle is the angle between $e_2$ and $f_3$,

$$\cos(\beta) = \langle e_2, f_3 \rangle$$

$$\beta = 0$$

Therefore, the contact angle is:

Fundamental second form is given by the following:

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

5.2 Minimal surface in $S^3$ with non constant contact angle

Let us consider the following surface in $S^3$:

$$\begin{cases} z_2 - \bar{z}_2 \\ (x_1)^2 + (y_1)^2 + (x_2)^2 + (y_2)^2 \end{cases} = 0$$

$$\begin{cases} z_2 - \bar{z}_2 \\ (x_1)^2 + (y_1)^2 + (x_2)^2 + (y_2)^2 \end{cases} = 1$$
We see that the vector fields are:

\[
\begin{align*}
e_1 &= \frac{1}{\sqrt{1-x^2}} (-x_1 x_2, -y_1 x_2, 1 - x_2^2, 0) \\
e_2 &= \frac{1}{\sqrt{x_1^2 + y_1^2}} (y_1, -x_1, 0, 0) \\
e_3 &= (0, 0, 0, 1)
\end{align*}
\]

The contact angle is the angle between \(e_2\) and \(f_3\), that is,

\[
\cos(\beta) = \langle e_2, f_3 \rangle = \frac{x_2}{2}
\]

Therefore, the contact angle is non constant:

\[
\beta = \arccos(x_2)
\]

**Remark 1.** For higher dimensions, when we have a compact minimal surface immersed in \(S^5\), we proved, in [6], that the case \(\beta = \frac{\pi}{2}\) gives an alternative proof of the classification of a Theorem from Blair in [2], for Legendrian minimal surfaces in \(S^5\) with constant Gaussian curvature.

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