A COROLLARY OF RIEMANN HYPOTHESIS

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Abstract. This paper use the results of the value distribution theory , got a significant conclusion by Riemann hypothesis

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First, we give some signs, definition and theorem in the value distribution theory, its contents see the references [1] and [2] .

Definition.

\[
\log^+ x = \begin{cases} 
\log x & 1 \leq x \\
0 & 0 \leq x < 1 
\end{cases}
\]

It is easy to see that \( \log x \leq \log^+ x \).

Set \( f(z) \) is a meromorphic function in the region \( |z| < R, 0 < R \leq \infty \), and not identical to zero.

\( n(r, f) \) represents the poles number of \( f(z) \) on the circle \( |z| \leq r (0 < r < R) \), multiple poles being repeated. \( n(0, f) \) represents the order of pole of \( f(z) \) in the origin. For arbitrary complex number \( a \neq \infty \), \( n(r, \frac{1}{f-a}) \) represents the zeros number of \( f(z) - a \) in the circle \( |z| \leq r (0 < r < R) \), multiple zeros being repeated. \( n(0, \frac{1}{f-a}) \) represents the order of zero of \( f(z) - a \) in the origin.

Definition.

\[
m(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\varphi})| \, d\varphi
\]
\[ N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r \]

Definition. \( T(r, f) = m(r, f) + N(r, f) \).

\( T(r, f) \) is called the characteristic function of \( f(z) \).

**Lemma 1.** If \( f(z) \) is an analytical function in the region \( |z| < R \ (0 < R \leq \infty) \), then

\[ T(r, f) \leq \log^+ M(r, f) \leq \frac{\rho + r}{\rho - r} T(\rho, f)(0 < r < \rho < R) \]

where \( M(r, f) = \max_{|z|=r} |f(z)| \)

The proof of the lemma see the page 57 of the references [1].

**Lemma 2.** Set \( f(z) \) is a meromorphic function in the region \( |z| < R \ (0 < R \leq \infty) \), not identical to zero. Set \( |z| < \rho \ (0 < \rho < R) \) is a circle, \( a_\lambda \ (\lambda = 1, 2, ..., h) \) and \( b_\mu \ (\mu = 1, 2, ..., k) \) respectively is the zeros and the poles of \( f(z) \) in the circle, appeared number of every zero or every pole and its order the same, and that \( z = 0 \) is not the zero or the pole of function \( f(z) \), then in the circle \( |z| < \rho \), we have the following formula

\[ \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\varphi})| \, d\varphi - \sum_{\lambda=1}^h \log \frac{\rho}{|a_\lambda|} + \sum_{\mu=1}^k \log \frac{\rho}{|b_\mu|} \]

this formula is called Jensen formula.

The proof of the lemma see the page 48 of the references [1].

**Lemma 3.** Set function \( f(z) \) is the meromorphic function in \( |z| \leq R \), and

\[ f(0) \neq 0, \infty, 1, \quad f'(0) \neq 0 \]

then when \( 0 < r < R \), have

\[ T(r, f) < 2 \left\{ N(R, \frac{1}{f}) + N(R, f) + N(R, \frac{1}{f-1}) \right\} \]
This is a form of Nevanlinna second basic theorems.

The proof of the lemma see the theorem 3.1 of the page 75 of the references [1].

The need for behind, We will make some preparations.

**LEMMA 4.** If when \( x \geq a \), \( f(x) \) is a nonnegative degressive function, then below limits exist

\[
\lim_{{N \to \infty}} \left( \sum_{{n=a}}^{{N}} f(n) - \int_{a}^{N} f(x) \, dx \right) = \alpha
\]

where \( 0 \leq \alpha \leq f(a) \). in addition, if when \( x \to \infty \), have \( f(x) \to 0 \), then

\[
\left| \sum_{{a \leq n \leq \xi}} f(n) - \int_{a}^{\xi} f(\nu) \, d\nu - \alpha \right| \leq f(\xi - 1), \quad (\xi \geq a + 1)
\]

The proof of the lemma see the theorem 2 of page 91 of the references [3].

Set \( s = \sigma + it \) is the complex number, when \( \sigma > 1 \), the definition of Riemann Zeta function is

\[
\zeta(s) = \sum_{{n=1}}^{\infty} \frac{1}{n^s}
\]

When \( \sigma > 1 \), from the page 90 of the references [4], have

\[
\log \zeta(s) = \sum_{{n=2}}^{\infty} \frac{\Lambda(n)}{n^s \log n}
\]

where \( \Lambda(n) \) is Mangoldt function.
LEMMA 5. For any real number $t$, have

(1) $0.0426 \leq | \log \zeta(4 + it) | \leq 0.0824$

(2) $| \zeta(4 + it) - 1 | \geq 0.0426$

(3) $0.917 \leq | \zeta(4 + it) | \leq 1.0824$

(4) $| \zeta'(4 + it) | \geq 0.012$

PROOF.

(1)

$$| \log \zeta(4 + it) | \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} \leq \sum_{n=2}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} - 1 \leq 0.0824$$

$$| \log \zeta(4 + it) | \geq \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4} = 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \geq 0.0426$$

(2)

$$| \zeta(4 + it) - 1 | = \left[ \sum_{n=2}^{\infty} \frac{1}{n^4 + it} \right] \geq \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4}$$

$$= 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \geq 0.0426$$

(3)

$$| \zeta(4 + it) | = \left| \sum_{n=1}^{\infty} \frac{1}{n^4 + it} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \leq 1.0824$$

$$| \zeta(4 + it) | \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^4} = 2 - \sum_{n=1}^{\infty} \frac{1}{n^4} = 2 - \frac{\pi^4}{90} \geq 0.917$$
from lemma 4, have

\[
\sum_{n=3}^{\infty} \frac{\log n}{n^4} = \int_{3}^{\infty} \frac{\log x}{x^4} dx + \alpha
\]

where \(0 \leq \alpha \leq \frac{\log 3}{3^4}\)

\[
\int_{3}^{\infty} \frac{\log x}{x^4} dx = -\frac{1}{3} \int_{3}^{\infty} \log x \, dx \, x^{-3} = \frac{\log 3}{3^4} + \frac{1}{3} \int_{3}^{\infty} x^{-4} \, dx
\]

\[
= \frac{\log 3}{3^4} - \frac{1}{3^2} \int_{3}^{\infty} dx \, x^{-3} = \frac{\log 3}{3^4} + \frac{1}{3^5}
\]

therefore

\[
\sum_{n=3}^{\infty} \frac{\log n}{n^4} \leq \frac{\log 3}{3^4} + \frac{1}{3^5} + \frac{\log 3}{3^4}
\]

therefore

\[
|\zeta'(4 + it)| \geq \frac{\log 2}{2^4} - \frac{2 \log 3}{3^4} - \frac{1}{3^5} \geq 0.012
\]

The proof is complete.

Set \(0 < \delta \leq \frac{1}{100} \), \(c_1, c_2, \ldots\), represents positive constant with only \(\delta\) relevant in the article below.

**LEMMA 6.** When \(\sigma \geq \frac{1}{2}, \, |t| \geq 2\), have

\[
|\zeta(\sigma + it)| \leq c_1 |t|^{\frac{1}{2}}
\]

The proof of the lemma see the theorem 2 of page 140 and the theorem 4 of page 142, of the references [4].
**Lemma 7.** Set \( f(z) \) is the analytic function in the circle \(|z - z_0| \leq R\), then for any \( 0 < r < R \), in the circle \(|z - z_0| \leq r\), have

\[
|f(z) - f(z_0)| \leq \frac{2r}{R - r} \left( A(R) - Ref(z_0) \right)
\]

where \( A(R) = \max_{|z - z_0| \leq R} Ref(z) \)

The proof of the lemma see the theorem 2 of page 61 of the references [4].

Now assume Riemann hypothesis is correct, abbreviation for RH. In other words, when \( \sigma > \frac{1}{2} \), the function \( \zeta(\sigma + it) \) has no zeros. Set the union set of the region \( \sigma > \frac{1}{2}, |t| > 1 \) and the region \( \sigma > 2, |t| \leq 1 \) is the region \( D \).

Therefore, the function \( \zeta(\sigma + it) \) have neither zero nor poles in the region \( D \), so, function \( \log \zeta(\sigma + it) \) is a defined multi-valued analytic function in the region \( D \). Every single value analytic branch differ \( 2\pi i \) integer times.

Assuming there are the points \( s_0 \) in the region \( D \), satisfy \( \zeta(s_0) = 1 \) (If there is not such point \( s_0 \), then the result of lemma 9 turns into \( N(\rho, \frac{1}{\zeta-1}) = 0 \), the results of the theorem of this article can be obtained directly). For different single value analytic branch, the value of \( \log \zeta(s_0) = \log 1 \) are different, it can value \( 0, 2\pi ki, (k = \pm 1, \pm 2, \ldots) \). We select the single valued analytic branch of \( \log \zeta(s_0) = \log 1 = 0 \).

Because the region \( D \) is simple connected region, so the according to the single value theorem of analytic continuation (the theorem see the theorem 2 of page 276 of the references [5] and theorem 1 of page 155 of the references [6]), \( \log \zeta(\sigma + it) \) is the single valued analytic function in the region \( D \). in addition, when and only when \( \zeta(\sigma + it) = 1 \), have \( \log \zeta(\sigma + it) = 0 \). In other words, 1 value point of \( \zeta(\sigma + it) \) is the zero of \( \log \zeta(\sigma + it) \), the opposite is true.

Below, \( \log \zeta(\sigma + it) \) always express a single valued analytic branch for we selected.

**Lemma 8.** If RH is correct, then when \( 0 < \delta \leq \frac{1}{100}, \sigma \geq \frac{1}{2} + 2\delta, |t| \geq 16 \), we have

\[
|\log \zeta(\sigma + it)| \leq c_2 \log |t| + c_3
\]

**Proof.** In the lemma 7, we choose \( z_0 = 0 \), \( f(z) = \log \zeta(z + 4 + it) \), \( |t| \geq 16 \), \( R = \frac{7}{2} - \delta \), \( r = \frac{7}{2} - 2\delta \). Because \( \log \zeta(z + 4 + it) \) is the analytic function
in the circle $|z - z_0| \leq R$, so, from the lemma 7, in the circle $|z - z_0| \leq r$, we have

$$| \log \zeta(z + 4 + it) \log \zeta(4 + it) | \leq \frac{7}{\delta} \left( A(R) - R \log \zeta(4 + it) \right)$$

hence

$$| \log \zeta(z + 4 + it) | \leq \frac{7}{\delta} \left( A(R) + | \log \zeta(4 + it) | \right) + | \log \zeta(4 + it) |$$

from the lemma 6, have

$$A(R) = \max_{|z - z_0| \leq R} \log | \zeta(z + 4 + it) | \leq \frac{1}{2} \log |t| + \log c_1$$

from the lemma 5, have

$$| \log \zeta(z + 4 + it) | \leq c_2 \log |t| + c_3$$

because $|t| \geq 16$ is real number arbitrarily, so when $\sigma \geq \frac{1}{2} + 2\delta$, we have

$$| \log \zeta(\sigma + it) | \leq c_2 \log |t| + c_3$$

The proof is complete.

**Lemma 9.** If RH is correct, then when $0 < \delta \leq \frac{1}{100}$, $|t| \geq 16$, $\rho = \frac{7}{2} - 2\delta$, in the circle $|z| \leq \rho$, we have

$$N\left( \rho, \frac{1}{\zeta(z + 4 + it) - 1} \right) \leq \log \log |t| + c_4$$

**Proof.** In the lemma 2, we choose $f(z) = \log \zeta(z + 4 + it)$, $R = \frac{7}{2} - \delta$, $\rho = \frac{7}{2} - 2\delta$, $a_\lambda (\lambda = 1, 2, ..., h)$ is the zeros of function $\log \zeta(z + 4 + it)$ in the circle $|z| < \rho$, multiple zeros being repeated. The function $\log \zeta(z + 4 + it)$ has no poles in the the circle $|z| < \rho$, and $\log \zeta(4 + it)$ not equal to zero, therefore we have

$$\log | \log \zeta(4 + it) | = \frac{1}{2\pi} \int_0^{2\pi} \log | \log \zeta(4 + it + e^{i\varphi}) | d\varphi - \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|}$$

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from the lemma 5 and the lemma 8, have
\[ \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|} \leq \log \log |t| + c_4 \]

because \( z = 0 \) is neither the zero, nor pole of the function \( \log \zeta(z + 4 + it) \), so if \( r_0 \) is a sufficiently small positive number, then

\[ \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|} = \int_{r_0}^{\rho} \left( \log \frac{\rho}{t} \right) \frac{d n(t, \frac{1}{f})}{t} = \left[ \left( \log \frac{\rho}{t} \right) n(t, \frac{1}{f}) \right]_{r_0}^{\rho} 
+ \int_{r_0}^{\rho} \frac{n(t, \frac{1}{f})}{t} \, dt = \int_{0}^{\rho} \frac{n(t, \frac{1}{f})}{t} \, dt = N\left(\rho, \frac{1}{f}\right) \]

\[ = N\left(\rho, \frac{1}{\log \zeta(z + 4 + it)}\right) = N\left(\rho, \frac{1}{\zeta(z + 4 + it) - 1}\right) \]

The proof is complete.

**THEOREM.** If RH is correct, then when \( \sigma \geq \frac{1}{2} + 4\delta \), \( 0 < \delta \leq \frac{1}{100} \), \( |t| \geq 16 \), we have

\[ |\zeta(\sigma + it)| \leq c_8 \left(\log |t|\right)^{c_8} \]

**proof.** In the lemma 3, we choose \( f(z) = \zeta(z + 4 + it) \), \( |t| \geq 16 \), from the lemma 5, have \( f(0) = \zeta(4 + it) \neq 0, \infty, 1, \quad f'(0) = \zeta'(4 + it) \neq 0 \), and \( f''(0) = \zeta''(4 + it) \geq 0.012, \quad |f(0)| = |\zeta(4 + it)| \leq 1.0824 \). We choose \( R = \frac{7}{2} - 2\delta \), \( r = \frac{7}{2} - 3\delta \). Because \( \zeta(z + 4 + it) \) is the analytic function, and have neither zero nor the poles in the circle \( |z| \leq R \), therefore

\[ N\left(R, \frac{1}{f}\right) = 0, \quad N\left(R, f\right) = 0 \]

from the lemma 9, have

\[ T\left(r, \zeta(z + 4 + it)\right) \leq 2 \log \log |t| + c_5 \]
In the lemma 1, we choose $R = \frac{7}{2} - 2\delta$, $\rho = \frac{7}{2} - 3\delta$, $r = \frac{7}{2} - 4\delta$, from the maximal principle, in the the circle $|z| \leq r$, we have

$$\log^+ |\zeta(z + 4 + it)| \leq c_6 \log \log |t| + c_7$$

Since $|t| \geq 16$ is arbitrary real number, so when $\sigma \geq \frac{1}{2} + 4\delta$, have

$$\log^+ |\zeta(\sigma + it)| \leq c_6 \log \log |t| + c_7$$

therefore

$$\log |\zeta(\sigma + it)| \leq c_6 \log \log |t| + c_7$$

therefore

$$|\zeta(\sigma + it)| \leq c_8 (\log |t|)^{c_\alpha}$$

The proof is complete.

The result of this theorem is better than known results.

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