Quantum Mechanics on Periodic and Non-Periodic Lattices and Almost Unitary Schwinger Operators

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Abstract

In this work we uncover the mathematical structure of the Schwinger algebra and introduce an almost unitary Schwinger operators which are derived by considering translation operators on a finite lattice. We calculate mathematical relations between these algebras and show that the almost unitary Schwinger operators are equivalent to the Schwinger algebra. We introduce new representations for $M_N(C)$ in terms of these algebras.

1 Introduction

Quantum mechanics on a finite periodic lattice is a well known subject which has been studied repeatedly since Schwinger’s famous 1960 paper \cite{Schwinger60}. He developed the generators of a complete unitary operator basis. Applications of Schwinger approach have been used in quantum optics, quantum communications, quantum probability and Galois quantum systems \cite{Vourdas13}. In addition one can find the review of the literature on quantum systems with finite Hilbert space and the link between this theory and the other research fields in Vourdas \cite{Vourdas13}.

Schwinger considered a periodic lattice on which the translation operator $U$ is unitary due to the periodicity of the lattice. On such a lattice the position can be again expressed by a unitary operator $V$ such that

\begin{align}
VU &= qUV \\
V^d &= U^d = 1 \quad \text{and} \quad VV^\dagger &= UU^\dagger = 1
\end{align}

where $d$ is the number of points on the periodic lattice. Schwinger chose the integer $d$ to be prime and in this case the relation $q = \exp \frac{2\pi i}{d}$ can be omitted since it is already implied by the other equations.

In \cite{Vourdas13} it has been shown that a finite lattice has an almost unitary quasi-translation operator $a$ which satisfies

\begin{align}
aa^\dagger &= 1 - a^{d-1}a^{d-1}, \quad \text{and} \quad a^{d} = 0 \\
a^\dagger a &= 1 - a^{d-1}a^{d-1}, \quad \text{and} \quad a^{d} = 0.
\end{align}

The operators $a^\dagger$ and $a$ in the above relations can be respectively regarded as the right quasi-translation operator and the left quasi-translation operator since an end point can be translated only in one direction. A point which lies at the right end of the finite lattice can only be translated left end vice versa. Equation (2) gives the minimal set of relations that define the algebra generated by $a$ and $a^\dagger$. The second set of the relations written in equation (3) can be derived using (2). Although the algebra defined by equation (1) and by equation (2) look very different, physically they accomplish basically the same concept. Therefore the exact mathematical relation between them should be unveiled.

In this paper we construct the projection operators in terms of the almost unitary translation operators and in terms of the unitary Schwinger operators. Since projection operators play the key role in relations between these two algebras we investigate their properties. Then we are able to write each algebra in terms of the other one. We also find two new representations where the standard basis of $M_N(C)$ is constructed in terms of the projection operators in each algebra. Finally, we formulate an algebra which is related to representing a multi-dimensional lattice in terms of one-dimensional lattices in each direction.

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2 Mathematical structure of the almost unitary translation operators

In our previous work [14] projection operators \( P_n \) were defined as,
\[
P_n = a^\dagger_n a^n, \quad \text{where} \quad P_0 = 1
\]
and it was shown that
\[
a^n a^\dagger_n = 1 - P_{d-n}.
\]
Thus we can also define another projection operator such as,
\[
R_n = a^n a^\dagger_n, \quad \text{where} \quad R_0 = 1.
\]
Therefore one can easily see the following relations between the projection operators \( P_n \) and \( R_n \)
\[
P_n = 1 - R_{d-n} \quad \text{and} \quad R_n = 1 - P_{d-n}.
\]
Their properties which are calculated in the appendix are summarized as,
\[
P_n = a^\dagger_n a^n, \quad P_0 = 1, \quad P_m a^\dagger = a^\dagger P_{m-1}, \quad aP_m = P_{m-1} a,
\]
\[
P_n P_m = P_j \quad \text{where} \quad j = \max(n, m),
\]
\[
P_m a^n = a^\dagger_n P_m = 0 \quad \text{for} \quad n + m \geq d.
\]
\[
R_n = a^n a^\dagger n, \quad R_0 = 1, \quad R_m a^\dagger = a^\dagger R_{m+1}, \quad aR_m = R_{m+1} a,
\]
\[
R_n R_m = R_j \quad \text{where} \quad j = \max(n, m),
\]
\[
a^n R_m = R_m a^\dagger n = 0 \quad \text{for} \quad n + m \geq d.
\]

In [14] we have already considered a position space of \( d \) points where a particle located at position \( X = \beta n \) is described by the ket vector \( |n\rangle \), \( n = 0, 1, \ldots, d - 1 \) where
\[
X |n\rangle = \beta n |n\rangle
\]
with \( \beta \) as the grid spacing. We can define the position operator as
\[
X = \beta \sum_{m=1}^{d-1} P_m,
\]
\[
X = \beta \{ a^\dagger a + \cdots + a^n a^n + \cdots + a^{d-1} a^{d-1}\}
\]
Applying this to \( |n\rangle \) one gets the desired result.

3 The Schwinger algebra in terms of the almost unitary translation operators

We define unitary operators \( U \) and \( V \) that cyclically permutes the vectors of a given system in terms of almost unitary operators \( a \) and \( a^\dagger \)
\[
U = a^\dagger + a^{d-1}, \quad \text{(12)}
\]
\[
V = \sum_{n=0}^{d-1} q^n (P_n - P_{n+1}). \quad \text{(13)}
\]
Then we will show that these definitions satisfy the Schwinger algebra given by equation (1). The first relation we will prove is

\[ VV^k = \sum_{n=0}^{d-1} q^{(k+1)n} (P_n - P_{n+1}) \]  

(14)

which has the inclusion relation

\[ V^d = VV^{d-1} = \sum_{n=0}^{d-1} q^{dn} (P_n - P_{n+1}) \]

(15)

\[ = \sum_{n=0}^{d-1} (P_n - P_{n+1}) \]

\[ = (1 + a^1 a + a^2 a^2 + \cdots + a^{d-1} a^{d-1}) - (a^1 a + a^2 a^2 + \cdots + a^{d-1} a^{d-1} + a^d a^d) \]

\[ = 1 \]

where we used the algebra relation \( a^{id} = 0 \).

We will prove the equation (14) using the method of proof by induction. For \( k = 1 \) we have

\[ VV = \sum_{n,m=0}^{d-1} q^{n+m} (P_n - P_{n+1})(P_m - P_{m+1}) \]

(16)

\[ = \sum_{n,m=0}^{d-1} q^{n+m} (P_n P_m - P_{n+1} P_{m+1}) \]

\[ = \sum_{n,m=0}^{d-1} q^{n+m} (P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]

\[ = \sum_{n,m=0}^{d-1} q^{n+m} (P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]

\[ + \sum_{n,m=0}^{d-1} q^{n+m} (P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]

\[ + \sum_{n,m=0}^{d-1} q^{n+m} (P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]

\[ + \sum_{n,m=0}^{d-1} q^{n+m} (P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]

Using the property \( P_n P_m = P_j \) where \( j = \max(n, m) \), we obtain

\[ VV = 0 + \sum_{n,m=0}^{d-1} q^{n+m} (P_n - P_{n+1}) + 0 \]

(17)

\[ = \sum_{n=0}^{d-1} q^{2n} (P_n - P_{n+1}). \]

We assume that for \( k = l \)

\[ VV^l = \sum_{n=0}^{d-1} q^{l(n+1)} (P_n - P_{n+1}). \]  

(18)
For $k = l + 1,$

\[ VV^{l+1} = (VV^l)V \]  
\[ = \sum_{n=0}^{d-1} q^{(l+1)n} (P_n - P_{n+1}) \sum_{m=0}^{d-1} q^m (P_m - P_{m+1}) \]
\[ = \sum_{n,m=0}^{d-1} q^{(l+1)n+m} (P_n P_m - P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]
\[ = \sum_{n,m=0}^{d-1} q^{(l+1)n+m} (P_n P_m - P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]
\[ + \sum_{n,m=0}^{d-1} q^{(l+1)n+m} (P_n P_m - P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]
\[ + \sum_{n,m=0}^{d-1} q^{(l+1)n+m} (P_n P_m - P_n P_{m+1} - P_{n+1} P_m + P_{n+1} P_{m+1}) \]

Using the same property $P_n P_m = P_j$ where $j = \max(n, m),$ we obtain

\[ VV^{l+1} = 0 + \sum_{n,m=0}^{d-1} q^{(l+1)n+m} (P_n - P_{n+1}) + 0 \]
\[ = \sum_{n=0}^{d-1} q^{(l+2)n} (P_n - P_{n+1}). \square \]

Then we will show that $VV^\dagger = 1$ by using definition of $V$

\[ V = \sum_{n=0}^{d-1} q^n (P_n - P_{n+1}). \]  
\[ (21) \]
Since $P_n = P_n^\dagger$ and $q^n = q^{d-n}$ we have

\[
V V^\dagger = \sum_{m=0}^{d-1} \sum_{n=0}^{d-1} q^m q^{d-n} (P_m - P_{m+1})(P_n - P_{n+1})
\]

(22)

\[
= \sum_{m,n=0}^{d-1} q^m q^{d-n} (P_m P_n - P_m P_{n+1} - P_{m+1} P_n + P_{m+1} P_{n+1})
\]

\[
= \sum_{m,n=0}^{d-1} q^{m+d-n} (P_m P_n - P_m P_{n+1} - P_{m+1} P_n + P_{m+1} P_{n+1})
\]

\[
+ \sum_{m,n=0}^{d-1} q^{m+d-n} (P_m P_n - P_m P_{n+1} - P_{m+1} P_n + P_{m+1} P_{n+1})
\]

\[
+ \sum_{m,n=0}^{d-1} q^{m+d-n} (P_m P_n - P_m P_{n+1} - P_{m+1} P_n + P_{m+1} P_{n+1})
\]

\[
= 0 + \sum_{m=0}^{d-1} q^d (P_m - P_{m+1}) + 0
\]

\[
= 0 + \sum_{m=0}^{d-1} q^d (P_m - P_{m+1}) + 0
\]

\[
= (1 + a^d + a^{d-1} + \cdots + a^{d-n}) - (a^d + a^{d-1} + \cdots + a^{d-n})
\]

\[
= 1.
\]

□

Next, we will show that

\[
U U^n = a^{i+n+1} + a^{d-(n+1)} \\
\text{with } n = 0, 1, \ldots, d - 1
\]

(23)

which implies

\[
U^d = U U^{d-1} = a^{i+d} + a^{d-d} = a^{i+d}
\]

(24)

\[
= 1.
\]

Our method is proof by induction. For $n = 1$ we have

\[
U U = (a^d + a^{d-1})(a^d + a^{d-1})
\]

(25)

\[
= a^d a^d + a^d a^{d-1} + a^{d-1} a^d + a^{d-1} a^{d-1}
\]

\[
= a^{d+2} + (a^d a) a^{d-2} + a^{d-2} (a a^d)
\]

\[
= a^{d+2} + a^{d+2} + a^{d-2} R_1
\]

\[
= a^{d+2} + a^{d+2} P_{d-2} + a^{d-2} (1 - P_{d-1})
\]

\[
= a^{d+2} + a^{d+2} P_{d-1} + a^{d-2} - a^{d-2} P_{d-1}
\]

\[
= a^{d+2} + a^{d-2}.
\]

For $n = l$ we assume that

\[
U U^l = a^{l+l+1} + a^{d-(l+1)}.
\]

(26)
For \( n = l + 1 \) we obtain

\[
UU^{l+1} = (UU^l)U
\]

(27)

\[
= (a^{l+1} + a^{d-(l+1)}) (a^\dagger + a^{d-1})
\]

\[
= a^{l+2} + a^{l+1} a^{d-1} + a^{d-(l+1)} a^\dagger + a^{2d-(l+2)}
\]

\[
= a^{l+2} + (a^{l+1} a^{l+1}) a^{d-1-(l+1)} + a^{d-(l+2)} (aa^\dagger)
\]

\[
= a^{l+2} + P_{l+1} a^{d-1-(l+1)} + a^{d-(l+2)} R_1 \text{ use (8)}
\]

\[
= a^{l+2} + a^{d-(l+2)} P_{l+1} + d-l-2 + a^{d-(l+2)} (1 - P_{d-1})
\]

\[
= a^{l+2} + a^{d-(l+2)} P_{d-1} + a^{d-(l+2)} - a^{d-(l+2)} P_{d-1}
\]

\[
= a^{l+2} + a^{d-(l+2)}.
\]

The last term in the third line is zero because at most \( l = d - 2 \) according to equation (23).

We will obtain \( UU^\dagger = 1 \) just by substitution

\[
UU^\dagger = (a^\dagger + a^{d-1}) (a + a^{d-1})
\]

(28)

\[
= a^\dagger a + a^{l} + a^{d} + a^{d-1} a^{d-1}
\]

\[
= P_1 + R_{d-1}
\]

\[
= 1. \square
\]

where we have used (2), (3) and (7).

We have the formula for \( U \) and \( V \), so we will show that \( VU = qUV \) just by substitution. Thus left hand side of the formula is equal to

\[
VU = \sum_{n=0}^{d-1} q^n (P_n - P_{n+1}) (a^\dagger + a^{d-1})
\]

(29)

\[
= \sum_{n=0}^{d-1} q^n (P_n a^\dagger + P_n a^{d-1} - P_{n+1} a^\dagger - P_{n+1} a^{d-1})
\]

Since \( P_n = a^\dagger a \) and \( P_0 = 1 \), \( P_n a^{d-1} = 0 \) except for \( n = 0 \) and \( P_{n+1} a^{d-1} = 0 \) for all \( n \). Therefore we obtain

\[
VU = \sum_{n=0}^{d-1} q^n (P_n a^\dagger - P_{n+1} a^\dagger) + q^0 a^{d-1}
\]

(30)

\[
= P_0 a^\dagger - P_1 a^\dagger + \sum_{n=1}^{d-1} q^n (a^\dagger P_{n-1} - a^\dagger P_n) + a^{d-1}
\]

\[
= a^\dagger - a^\dagger P_0 + \sum_{n=1}^{d-1} q^n (a^\dagger P_{n-1} - a^\dagger P_n) + a^{d-1}
\]

\[
= \sum_{n=1}^{d-1} q^n a^\dagger (P_{n-1} - P_n) + a^{d-1}
\]
where we have used (8). To obtain right hand side of the formula \( VU = qUV \) we calculate,

\[
UV = (a^\dagger + a^{d-1}) \sum_{n=0}^{d-1} q^n (P_n - P_{n+1})
\]

\[
= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + a^{d-1} (P_n - P_{n+1})
\]

\[
= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + \sum_{n=0}^{d-1} q^n a^{d-1} (P_n - P_{n+1})
\]

\[
= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + \sum_{n=0}^{d-1} q^n a^{d-1} (1 - R_{d-n} - (1 - R_{d-n-1}))
\]

\[
= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + \sum_{n=0}^{d-1} q^n a^{d-1} (R_{d-n-1} - R_{d-n}).
\]

Since \( R_m = a^m a^\dagger m \), \( a^{d-1} R_m = 0 \) except \( m = 0 \). The element \( R_{d-n-1} = R_0 = 1 \) for \( n = d - 1 \) so we have,

\[
UV = \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + q^{d-1} a^{d-1}
\]

\[
qUV = \sum_{n=0}^{d-1} q^{n+1} a^\dagger (P_n - P_{n+1}) + q^d a^{d-1}
\]

\[
= \sum_{n=1}^{d} q^n a^\dagger (P_{n-1} - P_n) + a^{d-1}
\]

\[
= \sum_{n=1}^{d-1} q^n a^\dagger (P_{n-1} - P_n) + a^{d-1}
\]

where we have used the facts that \( q^d = 1 \), \( P_d = 0 \) and \( a^\dagger P_{d-1} = 0 \). which is the same result given by equation (30).

4 Almost unitary operators in terms of the Schwinger algebra

It is also possible to write the almost unitary operators \( a \) and \( a^\dagger \) in terms of \( U \) and \( V \)

\[
a^\dagger = U - \left( \frac{1 + V + V^2 + \cdots + V^{d-1}}{d} \right) U
\]

The expression in the parentheses is called \( \mathcal{P}_0 \). It is shown that \( \mathcal{P}_0 \) is a projection operator (see appendix). We have found that more general projection operators \( \mathcal{P}_n \) is written as

\[
\mathcal{P}_n = \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}
\]

From the definition it is easily seen that

\[
\mathcal{P}_{d-l} = \mathcal{P}_{-l} \quad \text{and} \quad \mathcal{P}_{d+l} = \mathcal{P}_l \quad \text{and} \quad \mathcal{P}_d = \mathcal{P}_0.
\]

The relationships between unitary operators and projection operators are found as

\[
\mathcal{P}_m U^m = U^m \mathcal{P}_{n+m},
\]

\[
U^\dagger m \mathcal{P}_n = \mathcal{P}_{n+m} U^\dagger m
\]
in the appendix. In addition, multiplication of $\mathcal{P}_n$ and $\mathcal{P}_m$ is found in the appendix as
\begin{align}
\mathcal{P}_n \mathcal{P}_m &= 0 \quad \text{for } m \neq n \\
\mathcal{P}_n \mathcal{P}_n &= \mathcal{P}_n
\end{align}
(37)

In terms of $U$ we may define $a^\dagger$ as
\begin{equation}
a^\dagger \equiv U - \mathcal{P}_0 U.
\end{equation}
(38)

We will show that
\begin{equation}
a^\dagger a^{|l|} = U^{l+1}(1 - \mathcal{P}_1 - \mathcal{P}_2 - \cdots - \mathcal{P}_{l+1})
\end{equation}
(39)
which implies the following relation
\begin{align}
a^{|d|} &= a^\dagger a^{|d-1|} \\
&= U^d[1 - (\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_d)] \\
&= U^d[1 - (\mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_{d-1})] \\
&= \mathbb{1}(1 - 1) \\
&= 0
\end{align}
(40)

where we have used (35) and (A-23). The equation (40) is the second equation of the unitary algebra defined in (2).

We will prove the equation (39) by the method of proof by induction. For $n = 1$ we have
\begin{align}
a^\dagger a^\dagger &= (U - \mathcal{P}_0 U)(U - \mathcal{P}_0 U) \\
&= U(1 - \mathcal{P}_1)U(1 - \mathcal{P}_1) \\
&= U^2(1 - \mathcal{P}_2)(1 - \mathcal{P}_1) \\
&= U^2(1 - \mathcal{P}_1 - \mathcal{P}_2)
\end{align}
(41)

where we have used (36) and (37). For $n = l$ we assume that
\begin{equation}
a^\dagger a^{|l|} = U^{l+1}(1 - \mathcal{P}_1 - \mathcal{P}_2 - \cdots - \mathcal{P}_{l+1}).
\end{equation}
(42)

Therefore for $n = l + 1$ we obtain
\begin{align}
a^\dagger a^{|l+1|} &= (a^\dagger a^{|l|}) a^\dagger \\
&= U^{l+1}(1 - \mathcal{P}_1 - \mathcal{P}_2 - \cdots - \mathcal{P}_{l+1})U(1 - \mathcal{P}_1) \\
&= U^{l+2}(1 - \mathcal{P}_2 - \mathcal{P}_3 - \cdots - \mathcal{P}_{l+2})(1 - \mathcal{P}_1) \\
&= U^{l+2}(1 - \mathcal{P}_1 - \mathcal{P}_2 + \mathcal{P}_2 \mathcal{P}_1 - \cdots - \mathcal{P}_{l+2} + \mathcal{P}_{l+2} \mathcal{P}_1) \\
&= U^{l+2}(1 - \mathcal{P}_1 - \mathcal{P}_2 - \cdots - \mathcal{P}_{l+2}). \square
\end{align}
(43)

Now we will show that $aa^\dagger = 1 - a^{|d-1|}a^{|d-1|}$ in terms of $U$ and $V$, at the left hand side we have,
\begin{align}
aa^\dagger &= U^\dagger(1 - \mathcal{P}_0)(1 - \mathcal{P}_0)U \\
&= U^\dagger(1 - \mathcal{P}_0 - \mathcal{P}_0 + \mathcal{P}_0 \mathcal{P}_0)U \\
&= U^\dagger(1 - \mathcal{P}_0)U \\
&= U^\dagger U(1 - \mathcal{P}_1) \\
&= 1 - \mathcal{P}_1
\end{align}
(44)
where we have used (36) and (1). By using (39) we easily obtain
\begin{align}
a^{d-1} &= U^{d-1}(1 - \mathcal{P}_1 - \cdots - \mathcal{P}_{d-1}) \\
&= U^{d-1}(1 - (\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_{d-1})) \\
&= U^{d-1}(1 - (1 - \mathcal{P}_0)) \\
&= U^{d-1}\mathcal{P}_0
\end{align}

where we have used (A-23). Taking hermitian conjugate of this equation and using hermicity property of the projection operators which is shown by (A-21) we get
\begin{align}
a^{d-1} &= \mathcal{P}_0 U^{d-1}.
\end{align}

Then for the right hand side of \( a a^\dagger = 1 - a^{d-1} a^{d-1} \) we have
\begin{align}
1 - a^{d-1} a^{d-1} &= 1 - U^{d-1} \mathcal{P}_0 ^\dagger \mathcal{P}_0 U^{d-1} \\
&= 1 - U^{d-1} \mathcal{P}_0 ^\dagger U^{d-1} \\
&= 1 - \mathcal{P}_1 \\
&= a a^\dagger
\end{align}

at the second line we used the relations given by (37) and at the last line we have used (44). This is the first equation defining the almost unitary algebra given by (2).

5 New Representations for Basis of \( M_N(C) \)

The \( e_{ij} \) satisfying (46) form the standard basis of \( M_N(C) \)
\begin{align}
e_{ij} e_{kl} &= \delta_{jk} e_{il} \\
e_{ij}^\dagger &= e_{ji}.
\end{align}

In this section we give two new representations of the basis matrices. One of them is written in terms of the almost unitary algebra as
\begin{align}
e_{mn} &= a^{m} R_{d-1} a^{n} \quad \text{where} \quad R_{n} = a^{n} a^{n\dagger} \quad \text{and} \quad m, n = 0, 1, \cdots, d - 1.
\end{align}

The other one is written in terms of Schwinger \( U \) and \( V \) operators
\begin{align}
e_{mn} &= \begin{cases} 
U^{m-n} \mathcal{P}_{d-n} & \text{for } m > n \\
\mathcal{P}_{d-n} & \text{for } m = n \\
U^{n-m} \mathcal{P}_{d-n} & \text{for } m < n
\end{cases} \\
&\text{with } \mathcal{P}_n = \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}
\end{align}

We prove these representations satisfy (48) in the last part of the appendix.

6 Multi-dimensional lattice in terms of lower dimensional lattices

Denoting a linear lattice with \( d \) elements by \( \mathcal{L}_d \), we can show the cartesian product \( \mathcal{L}_{d_1} \times \mathcal{L}_{d_2} \) by the dots in the following figure.

\begin{align}
4 \times 3 \text{ Lattice}
\end{align}
Corresponding to this cartesian product of the lattices we have the tensor product of the algebra \( \mathcal{A}_{d_2} \otimes \mathcal{A}_{d_1} \). On the cartesian product shown in the figure the right translation operator corresponds to \( a_1^{\dagger} \otimes 1 \) and the up translation operator corresponds to \( 1 \otimes a_1^{\dagger} \). We denote the (right) translation operator on \( \mathcal{A}_{d_1}, \mathcal{A}_{d_2}, \mathcal{A}_{d_1d_2} \) respectively by \( a_1^{\dagger}, a_2^{\dagger}, a_{d_1d_2}^{\dagger} \) and consider \( \mathcal{L}_{d_1}, \mathcal{L}_{d_2} \) as a one one-dimensional lattice as shown by the arrows in the figure. This satisfies an isomorphism

\[
\mathcal{A}_{d_2d_1} \xrightarrow{\Delta} \mathcal{A}_{d_2} \otimes \mathcal{A}_{d_1}
\]  

(51)

and one can write

\[
\Delta(a_2^{\dagger}_{d_2d_1}) = 1_{d_2} \otimes a_1^{\dagger} + a_2^{\dagger} \otimes a_1^{d_{1}^{\dagger} - 1}.
\]  

(52)

One immediately can check that the action of \( a_{d_2d_1}^{\dagger} \) is given by the arrows in the figure and satisfies the correct algebraic relations.

Similarly, we can express the translation operator for a \( d_1 \times d_2 \) dimensional periodic lattice by

\[
\Delta(U_{d_2d_1}) = 1_{d_2} \otimes a_1^{\dagger} + a_2^{\dagger} \otimes a_{d_1}^{d_{1}^{\dagger} - 1} + a_{d_2}^{d_{2}^{\dagger} - 1} \otimes a_{d_1}^{d_{1}^{\dagger} - 1}.
\]  

(53)

7 Conclusion

We have shown that the Schwinger algebra can also be given by almost unitary operators which are physically related to the shift operators on a finite lattice. We have named these operators as almost unitary operators because relations

\[
UU^{\dagger} = 1, \quad VV^{\dagger} = 1 \quad \text{and} \quad VU = qUV \quad \text{where} \quad q = e^{\frac{2\pi}{d}}
\]  

(54)

are replaced by

\[
aa^{\dagger} = 1 - a^{d_{1}^{\dagger} - 1} a_{d_{1}^{\dagger} - 1} \quad \text{and} \quad a^{\dagger}a = 1 - a^{d_{1}^{\dagger} - 1} a_{d_{1}^{\dagger} - 1}
\]  

(55)

and the terms \( a^{d_{1}^{\dagger} - 1} a_{d_{1}^{\dagger} - 1} \), \( a^{d_{1}^{\dagger} - 1} a_{d_{1}^{\dagger} - 1} \) reflect violation of unitarity for \( a \) and \( a^{\dagger} \). For \( U \) we have the relation \( U^{\dagger} = U^{-1} \) due to the periodic nature of the lattice. Similarly \( a \) and \( a^{\dagger} \) can be considered as inverse of each other except at the end points. Note that \( a \) and \( a^{\dagger} \) play the role of \( U \) and \( U^{\dagger} \) where as \( V \) can be defined in terms of \( a \) and \( a^{\dagger} \). It takes quite an effort to construct \( V \) which is given by (13) in terms of \( a \) and \( a^{\dagger} \).

In usual quantum mechanics where the position and the angular momentum operators have continuous eigenvalues the following equations are equivalent to each other

\[
V(p_0)U(x_0) = e^{-\frac{ip_0}{\hbar}x_0}U(x_0)V(p_0)
\]  

(56)

\[
[X, P] = i\hbar
\]  

(57)

\[
[X, U(x_0)] = x_0 U(x_0)
\]  

(58)

here \( U(x_0) = \exp(-\frac{ip_0}{\hbar}x_0) \), \( V(p_0) = \exp(\frac{ip_0}{\hbar}X) \)

(59)

For the discrete periodic case \( X \) is defined only modulo \( 2\pi r \). However \( U \) and \( V \) are well defined. Thus we have only one corresponding equation

\[
VU = qUV \quad \text{where} \quad q = \exp\frac{2\pi i}{d}.
\]  

(60)

(61)

On the other hand for the discrete non-periodic case the position operator \( X \), the right quasi-translation operator \( a^{\dagger} \) and the left quasi-translation operator \( a \) are well defined. Therefore we have only one equation corresponding to (58)

\[
[X, a^{\dagger}] = \beta a^{\dagger}
\]  

(62)
where \( \beta \) is the grid spacing.

We have shown how to construct almost unitary translation operators \( a, a^\dagger \) in terms of \( U, V \) and vice versa. In addition we have found the relation between basis matrices of \( M_N(C) \) and the almost unitary operators and the relation between basis matrices of \( M_N(C) \) and the Schwinger algebra. Furthermore we established an isomorphism between a multi-dimensional and periodic or non periodic linear lattices.

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**Appendix**

**Projection operators in terms of \( a \) and \( a^\dagger \)**

We have found two projection operators \( P_n \) and \( R_n \);

\[
P_n = a^\dagger n a^n \quad \text{and} \quad R_n = a^n a^\dagger n
\]

The properties

\[
P_m a^n = a^\dagger n P_m = 0 \quad \text{for} \quad n + m \geq d
\]

\[
a^n R_m = R_m a^\dagger n = 0 \quad \text{for} \quad n + m \geq d
\]

are immediate results of definition of the projection operators and the algebra property \( a^d = a^d = 0 \).

The following properties have already been proved in [14].

\[
P_n P_m = P_m \quad \text{where} \quad m \geq n,
\]

\[
P_m a^\dagger = a^\dagger P_{m-1} \quad \text{and} \quad P_m a = a P_{m+1}.
\]

Now, we will prove \( R_n R_m = R_m \) where \( m \geq n \). Our method is proof by induction.

For \( n = 1 \)

\[
R_1 R_m = (aa^\dagger)(a^m a^\dagger m)
\]

\[
= a(a^\dagger a)a^{n-1}a^\dagger m
\]

\[
= a(1 - a^d a^d_{-1})a^{m-1}a^\dagger m
\]

\[
= aa^{m-1}a^\dagger m
\]

\[
= a^m a^\dagger m
\]

\[
R_1 R_m = R_m
\]

where we have used \( a^d = 0 \). For \( n = l \), we assume

\[
R_l R_m = R_m \quad \text{for} \quad m \geq l.
\]
For $n = l + 1$, we have

$$R_{l+1}R_m = (a^{l+1}a^{l+1})(a^m a^{m})$$

$$(A-7)$$

$$= a(a^l a^l) a(a^m a^{m-1}) a^\dagger$$

$$= aR_l(a a) R_{m-1} a^\dagger$$

$$= aR_l (1 - a^{d-1} a^{d-1}) R_{m-1} a^\dagger$$

$$= aR_l R_{m-1} a^\dagger - aR_l R_{m-1} R_{m-1} a^\dagger$$

$$= aR_l R_{m-1} a^\dagger - aR_{m-1} R_{m-1} a^\dagger$$

where we have used (2) and (A-6). By using $aR_{d-1} = a^d a^{d-1} = 0$ and the assumption for $n = l$ with the fact that $l + 1 \leq m$ implies $l \leq m - 1$, so we have $R_l R_{m-1} = R_{m-1}$ and $aR_{m-1} a^\dagger = R_m$. Therefore

$$R_{l+1}R_m = R_m \Box$$

(A-8)

Then we will show that $R_m R_n = R_m$ for $m \geq n$ by using the relation $R_n = 1 - P_{d-n}$ which is equation (7).

$$R_m R_n = (1 - P_{d-m})(1 - P_{d-n})$$

$$(A-10)$$

$$= 1 - P_{d-n} - P_{d-m} + P_{d-m} P_{d-n}$$

using $P_m P_m = P_m$ where $m \geq n$ with the fact that $m \geq n$ implies $d - n \geq d - m$

$$R_m R_n = 1 - P_{d-n} - P_{d-m} + P_{d-n}$$

$$(A-11)$$

$$= 1 - P_{d-m}$$

$$= R_m \Box$$

By using the last two proofs we conclude that

$$R_n R_m = R_j \quad \text{where} \quad j = \max(n, m).$$

(A-12)

Similarly we will show that $P_m P_n = P_m$ where $m \geq n$ by using $P_n = 1 - R_{d-n}$ which is equation (7).

$$P_m P_n = (1 - R_{d-m})(1 - R_{d-n})$$

$$(A-13)$$

$$= 1 - R_{d-m} - R_{d-n} + R_{d-m} R_{d-n}$$

using $R_n R_m = R_m$ where $m \geq n$ with the fact that $m \geq n$ implies $d - n \geq d - m$

$$P_m P_n = 1 - R_{d-m} - R_{d-n} + R_{d-n}$$

$$(A-14)$$

$$= 1 - R_{d-m}$$

$$= P_m \Box$$

This result and equation given by (A-3) can be expressed in one equation as

$$P_n P_m = P_j \quad \text{where} \quad j = \max(n, m).$$

(A-15)
Now, we will calculate relations between the projection operator $R_m$ and shift operators $a$ and $a^\dagger$.

\[ R_m a = a^m a^l m a \]
\[ = a^m a^l m - 1 (a^l a) \]
\[ = a (a^m - 1) a^l m - 1 P_1 \]
\[ = a R_{m-1} (1 - R_{d-1}) \]
\[ = a R_{m-1} - a R_{m-1} R_{d-1} \]
\[ = a R_{m-1} - a R_{d-1} \]
\[ = a R_{m-1} \]

where we have used (7), (A-11) and the fact that $a R_{d-1} = a^d a^l d - 1 = 0$

\[ a^l R_m = a^l a^m a^l m \]
\[ = (a^l a) (a^m - 1 a^l m - 1) a^l \]
\[ = P_1 R_{m-1} a^l \]
\[ = (1 - R_{d-1}) R_{m-1} a^l \]
\[ = R_{m-1} a^l - R_{d-1} R_{m-1} a^l \]
\[ = R_{m-1} a^l - R_{d-1} a^l \]
\[ = R_{m-1} a^l \]

where we have used $R_{d-1} a^l = a^l d - 1 a^d = 0$. Let’s summarize what we have derived about the projection operators $P_n$ and $R_n$ up to now:

\[ P_n = a^l n a^n, \quad P_0 = 1, \quad P_m a^l = a^l P_{m-1}, \quad a P_m = P_{m-1} a, \quad P_n P_m = P_j \quad \text{where} \quad j = \max(n, m), \]
\[ a^n a^l n a^m = a^l n P_m = 0 \quad \text{for} \quad n + m \geq d, \]

\[ R_n = a^n a^l n, \quad R_0 = 1, \quad R_m a^l = a^l R_{m+1}, \quad a R_m = R_{m+1} a, \quad R_n R_m = R_j \quad \text{where} \quad j = \max(n, m), \]
\[ a^n R_m = R_m a^l n = 0 \quad \text{for} \quad n + m \geq d, \]

\[ P_n = 1 - R_{d-n} \quad \text{and} \quad R_n = 1 - P_{d-n}. \]

**Projection operators in terms of $U$ and $V$**

We have found two projection operators $\mathcal{P}_n$ and $\mathcal{R}_n$:

\[ \mathcal{P}_n = \frac{1 + q^n V + q^{2n} V^2 + \ldots + q^{(d-1)n} V^{(d-1)}}{d} \quad \text{(A-18)} \]
\[ \mathcal{R}_n = 1 - \mathcal{P}_n \quad \text{(A-19)} \]
In the last equation we used the fact that the sum of roots of unity gives zero,

\[ (\mathcal{P}_n)^2 = \left( 1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)} \right) \frac{d}{d} \]

\[ + \frac{(q^n V + q^{2n} V^2 + q^{3n} V^3 \cdots + q^{dn} V^d)}{d} \]

\[ + \cdots \]

\[ + q^{(d-1)n} V^{(d-1)} + q^{dn} V^d + q^{(d+1)n} V^{(d+1)} \cdots + 2^{2(d-n)} V^{2(d-1)} \]

\[ = \frac{d}{d} \left( 1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)} \right) \]

\[ = \mathcal{P}_n \]

where we have used \( V^d = 1 \) and \( q^d = 1 \).

By using definition of \( \mathcal{P}_n \) and Schwinger equation \( VU = qUV \), one can easily obtain

\[ \mathcal{P}_n U^m = U^m \mathcal{P}_{n+m}. \]  

(A-21)

Hermitian conjugate of \( \mathcal{P}_n \) is easily calculated as,

\[ (\mathcal{P}_n)^\dagger = \left( 1 + (q^n)^\dagger V^\dagger + (q^{2n})^\dagger (V^2)^\dagger + \cdots + (q^{(d-1)n})^\dagger (V^{(d-1)})^\dagger \right) \frac{d}{d} \]

\[ = \frac{1 + q^{d-n} V^{d-1} + q^{(d-2)n} V^{d-2} + \cdots + q^n V}{d} \]

\[ = \mathcal{P}_n \]

(A-22)

where we have used the unitary property of \( V \) and properties of complex number \( q = exp(\frac{2\pi i}{d}) \). By using hermicity property of \( \mathcal{P}_n \) we will calculate,

\[ (\mathcal{P}_n U^m = U^m \mathcal{P}_{n+m})^\dagger \]

\[ U^\dagger m \mathcal{P}_n = \mathcal{P}_{n+m} U^\dagger m. \]  

(A-23)

The other property of \( \mathcal{P}_n \) is found by the following steps,

\[ \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_{(d-1)} = \left( 1 + V + V^2 + \cdots + V^{(d-1)} \right) \frac{d}{d} \]

\[ + \frac{1 + qV + q^2 V^2 + \cdots + q^{(d-1)} V^{(d-1)}}{d} \]

\[ + \frac{1 + q^2 V + q^4 V^2 + \cdots + q^{2(d-1)} V^{(d-1)}}{d} \]

\[ + \cdots \]

\[ + \frac{1 + q^{d-1} V + q^{2(d-1)} V^2 + \cdots + q^{(d-1)(d-1)} V^{(d-1)}}{d} \]

\[ = \frac{d/d + (1 + q + q^2 + \cdots + q^{(d-1)}) V/d}{d} \]

\[ + \frac{(1 + q^2 + q^4 + \cdots + q^{2(d-1)}) V^2/d}{d} \]

\[ + \cdots \]

\[ + \frac{(1 + q^{(d-1)} + q^{2(d-1)} + \cdots + q^{(d-1)(d-1)}) V^{(d-1)}/d}{d} \]

\[ = 1 \]

In the last equation we used the fact that the sum of roots of unity gives zero,

\[ (1 + q + q^2) + \cdots + q^{d-1} = 0 \quad \text{with} \quad q = exp(\frac{2\pi i}{d}). \]  

(A-25)

Furthermore, when we take unity as \( 1 = exp(2\pi in) \) where \( n \) is a integer, its roots will be \( 1, q^n, q^{2n}, \ldots, q^{n(d-1)} \). Thus
the sum of the terms appear in parentheses in the last lines are equal to 0.

Then sum of $\mathcal{P}_n$ given by equation (A-17) is found easily,

$$
\mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_{d-1} = (1 - \mathcal{P}_0) + (1 - \mathcal{P}_1) + \cdots + (1 - \mathcal{P}_{d-1}) = d - 1.
$$

(A-26)

Now we will show that $\mathcal{P}_n \mathcal{P}_m = 0$ unless $n \neq m$ by direct substitution of definition of projection operators

$$
\mathcal{P}_n \mathcal{P}_m = (1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{d-1})(1 + q^m V + q^{2m} V^2 + \cdots + q^{(d-1)m} V^{d-1})/d^2
$$

(A-27)

$$
= \left\{ (1 + q^m V + q^{2m} V^2 + \cdots + q^{(d-1)m} V^{d-1}) \\
+ (q^n V + q^{n+m} V^2 + q^{n+2m} V^3 + \cdots + q^{(d-1)m+n} V^d) \\
+ (q^{2n} V^2 + q^{2n+m} V^3 + q^{2n+2m} V^4 + \cdots + q^{(d-1)m+2n} V^{d+1}) \\
+ \cdots \\
+ (q^{(d-1)n} V^{d-1} + q^{(d-1)n+m} V^d + q^{(d-1)n+2m} V^{(d+1)} + \cdots + q^{(d-1)(m+n)} V^{2d-2}) \right\}/d^2
$$

$$
= \left\{ 1 + (q^{(d-1)m+n} + q^{(d-2)m+2n} + \cdots + q^{(d-1)n+m}) V^d \right. \\
+ (q^m + q^n + q^{(d-1)m+2n} + \cdots + q^{(d-1)(n+2m)}) V \\
+ (q^{2m} + q^{n+m} + q^{2n} + \cdots + q^{(d-1)n+3m}) V^2 \\
+ \cdots \\
+ (q^{(d-1)m} + q^{(d-2)m+n} + \cdots + q^{(d-1)n}) V^{d-1} \right\}/d^2
$$

with the help of $V^{d+n} = V^n$. Now replace $n - m$ by $k$ and use $q^d = 1$

$$
\mathcal{P}_n \mathcal{P}_m = \left\{ (1 + q^k + q^{2k} + \cdots + q^{(d-1)k}) \\
+ q^m(1 + q^k + q^{2k} + \cdots + q^{(d-1)k}) V \\
+ q^{2m}(1 + q^k + q^{2k} + \cdots + q^{(d-1)k}) V^2 \\
+ \cdots \\
+ q^{(d-1)m}(1 + q^k + q^{2k} + \cdots + q^{(d-1)k}) V^{d-1} \right\}/d^2
$$

(A-28)

Furthermore taking unity as 1, $1 + q^k + \cdots + q^{(d-1)k}$ will be equal to equation (A-24) in different order. Hence $(1 + q^k + \cdots + q^{(d-1)k}) = 0$. As a result each parentheses in the equation (A-27) are equal to zero except $k = 0$ (corresponds $n = m$). At the exception each parentheses sum to $d$. We can summarize our results as

$$
\mathcal{P}_n \mathcal{P}_m = 0 \quad \text{for} \quad m \neq n
$$

$$
\mathcal{P}_n \mathcal{P}_n = d(1 + q^n V + \cdots + q^{(d-1)n} V^{d-1})/d^2.
$$

(A-29)

$$
M_N(C) \text{ in terms of } a \text{ and } a^\dagger
$$

The standard basis of $M_N(C)$ is given by the operator $e_{ij}$ which satisfies

$$
e_{ij} e_{kl} = \delta_{jk} e_{il} \quad e_{ij}^\dagger = e_{ji}
$$

(A-30)

$e_{mn}$ can be expressed in terms of $a$ and $a^\dagger$ as,

$$
e_{mn} = a^{\dagger m} R_{d-1} a^n \quad m, n = 0, 1, \cdots, d - 1
$$

(A-31)
Then,

\[
e_{ij}e_{kl} = (a_{i}^\dagger R_{d-1}a_{j})(a_{k}^\dagger R_{d-1}a_{l}) \\
= (a_{i}^\dagger a_{l}R_{d-1-j})(R_{d-1-k}a_{k}^\dagger a_{l}) \\
= a_{i}^\dagger a_{l}R_{d-1-j}R_{d-1-k}a_{k}^\dagger a_{l} \tag{A-32}
\]

if \( j = k \)

\[
= a_{i}^\dagger a_{l}R_{d-1-j}a_{j}^\dagger a_{l} \\
= a_{i}^\dagger R_{d-1}a_{l} \\
= e_{il}
\]

if \( j > k \)

\[
= a_{i}^\dagger a_{j}R_{d-1-1}a_{k}^\dagger a_{l} \\
= a_{i}^\dagger a_{j}a_{k}^\dagger R_{d-1}a_{l} \\
= a_{i}^\dagger a_{j}R_{d-1}a_{k}R_{d-1}a_{l} \\
= a_{i}^\dagger a_{j}^{-k}R_{d-1}a_{l} \\
= 0
\]

if \( j < k \)

\[
= a_{i}^\dagger a_{j}R_{d-1-j}a_{k}^\dagger a_{l} \\
= a_{i}^\dagger R_{d-1}a_{k}^\dagger a_{l} \\
= a_{i}^\dagger R_{d-1}R_{j}a_{k}^{-j}a_{l} \\
= a_{i}^\dagger R_{d-1}a_{k}^{-j}a_{l} \\
= a_{i}^\dagger R_{d-1}R_{j}a_{k}^{-j}a_{l} \\
= 0
\]

where we have used (9). Thus \( e_{ij}e_{kl} = \delta_{jk}e_{il} \). It is the time to check second part of equation (A-29)

\[
(e_{ij} = a_{i}^\dagger R_{d-1}a_{j})^* \\
e_{ij} = a_{j}^\dagger R_{d-1}a_{i} \\
e_{ij}^* = a_{j}^\dagger R_{d-1}a_{i} \\
e_{ij} = e_{ji}
\]

\[M_{N}(C) \text{ in terms of } U \text{ and } V\]

\( e_{mn} \) can be written in terms of operators \( U \) and \( V \) as

\[
e_{mn} = \begin{cases} 
U^{m-n}P_{d-n} & \text{for } m > n \\
P_{d-n} & \text{for } m = n \\
U^{n-m}P_{d-n} & \text{for } m < n 
\end{cases}
\]

with \( P_{n} = \frac{(1 + q^{n}V + q^{2n}V^{2} + \cdots + q^{(d-1)n}V^{(d-1)})}{d} \). \( \tag{A-34} \)

For \( m > n \)

\[
e_{ij}e_{kl} = U^{i-j}P_{d-j}U^{k-l}P_{d-l} \\
= U^{i-j}U^{k-l}P_{d-j+k-l}P_{d-l} \\
\]

if \( j = k \)

\[
= U^{i-j}U^{k-j}P_{d-j+k-l}P_{d-l} \\
= U^{i-j}P_{d-j+k-l}P_{d-l} \\
= e_{il}
\]

if \( j \neq k \)

\[
= 0
\]

Thus \( e_{ij}e_{kl} = \delta_{jk}e_{il} \).
Now let us check second part of the equation (A-29) for $m > n$

\[
\begin{align*}
(e_{mn} &= U^{m-n} P_{d-n})^\dagger \\
 e_{mn}^\dagger &= P_{d-n}^\dagger U^{m-n} \\
 e_{mn}^\dagger &= P_{d-n} U^{m-n} \\
 e_{mn}^\dagger &= U^{m-n} P_{d-n-(m-n)} \\
 e_{mn}^\dagger &= U^{m-n} P_{d-m} \\
 e_{mn}^\dagger &= U^{n-m} P_{d-m} \\
 e_{mn}^\dagger &= e_{nm}
\end{align*}
\] (A-36)

where we have used (A-21), (A-22) and (1). Next for $m = n$ we have

\[
\begin{align*}
e_{ii}e_{kk} &= P_{d-i} P_{d-k} \\
\text{if } i = k &= 1 \\
\text{if } i \neq k &= 0
\end{align*}
\] (A-37)

where we have used (A-28). Thus $e_{ij}e_{kl} = e_{il}$. Now, hermicity of property of the projection operators given by equation (A-21) imply the second part of the matrix algebra given by the equation (A-29).

Finally for $m < n$ we have

\[
\begin{align*}
e_{ij}e_{kl} &= U^{l-j-i} P_{d-j} U^{l-k} P_{d-l} \\
&= U^{l-j-i} U^{l-k} P_{d-j-l+k} P_{d-l} \\
\text{if } j = k &= U^{l-j-i} U^{l-k} P_{d-l} P_{d-l} \\
&= U^{l-j-i} P_{d-l} \\
&= e_{il} \\
\text{if } j \neq k &= U^{l-j-i} U^{l-k} P_{d-j-l+k} P_{d-l} \\
&= 0
\end{align*}
\] (A-38)

Thus we can conclude that $e_{ij}e_{kl} = \delta_{jk}e_{il}$

Now let us check second part of the equation (A-29) for $m < n$ we have

\[
\begin{align*}
(e_{mn} &= U^{n-m} P_{d-n})^\dagger \\
 &= P_{d-n}^\dagger U^{n-m} \\
 &= U^{n-m} P_{d-j+j-i} \\
 &= U^{n-m} P_{d-m} \\
 &= e_{nm} \Box
\end{align*}
\] (A-39)

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