The Operator Manifold Formalism. I

G.T.Ter-Kazarian
Byurakan Astrophysical Observatory, Armenia 378433
E-mail:gago@bao.sci.am
December 20, 1998

Abstract
The suggested operator manifold formalism enables to develop an approach to the unification of the geometry and the field theory. The secondary quantization and differential geometric aspects are studied. The former is equivalent to a configuration space wave mechanics incorporated with geometric properties leading to the quantization of geometry, different in principle from earlier suggested schemes. We show that the matrix elements of operator tensors produce the Cartan’s exterior forms, define one parameter group of operator diffeomorphisms, consider the operator differential forms and their integration, also operator exterior differentiation. We elaborate the formalism of operator multimanifold yielding the multiworld geometry involving the spacetime continuum and internal worlds, where the subquarks are defined implying the Confinement and Gauge principles. This formalism in Part II is used to develop further the microscopic approach to the field theory.

1 Introduction
A number of alternative approaches have been proposed towards the unified gauge field theory, e.g. [1-7]. Each of them has its own advantages and difficulties. The key problem is to find out the mathematical structures enabling an insight to the concepts of particle physics. An alternative approach is developed in our recent work on the operator manifold formalism [8], elaborated in analogy of secondary quantization incorporated with the geometric properties. In the present paper we continue to study its background including the rigorous definition of operator manifold and infer the matrix elements of field operators used for calculation of matrix elements of geometric objects. We generalize this formalism via the concept of operator multimanifold- a multiworld geometry decomposed into the spacetime continuum and internal worlds.

The operator manifold formalism has the following features: 1. It provides a natural unification of the geometry yielding Special and General Relativity principles and fermions serving as the basis for the subquarks (Part II). 2. They emerge in the geometry only in certain permissible combinations, which utilizes the idea of Subcolour (Subquark) Confinement principle, and undergo the transformations yielding the internal symmetries and Gauge principle (subsec.3.4,3.5).

This approach still should be considered as a preliminary one and numerous issues still remain to be solved. The only argument forcing us to consider it seriously is the fact that some important properties of particle physics can be derived naturally within this approach.

2 Preliminaries
This article is the continuation of [8], so we adopt its all ideas and notations, except the change in the order of vector and covector indices to fit conventional notations used in
differential geometry (subsec.2.4). It is convenient to describe our approach in terms of manifold $G = G_\eta \oplus G_u$, (subsec.2.1) $\text{Dim} G = 12$, $\text{Dim} G_i = 6$ ($i = \eta, u$). But, one may readily return to conventional terms of Minkowski spacetime continuum (subsec.2.1). To be brief we often suppress the indices without notice.

In Part II and further we deal with multiworld geometry, except for the change of the concept of quark used in [8,9] as well as in Part I, to subquark defined in the given internal world.

### 2.1 Operator Vector and Covector Fields

Consider a curve $\lambda(t) : \mathbb{R}^1 \to G$ passing through a point $p = \lambda(0) \in G$ with tangent vector $A|_{\lambda(t)}$, where the $G$ is 12 dimensional smooth differentiable manifold. The set $\{\zeta\}$ are local coordinates in open neighbourhood of $p \in \mathcal{U}$. The 12 dimensional smooth vector field $A_p = A(\zeta)$ belongs to the section of tangent bundle $\mathcal{T}_p$ at the point $p(\zeta)$. The one parameter group of diffeomorphisms $A^t$ is given for the curve $\zeta(t)$ passing through point $p$ and $\zeta(0) = \zeta_p$, $\dot{\zeta}(0) = A_p : \frac{d}{dt} A^t_p(A) = \frac{d}{dt} \bigg|_{t=0} A^t(\zeta(t)) = A_p(\zeta)$. Hence, $dA^t : \mathcal{T}(G) \to \mathcal{T}(G) = \bigcup_{p(\zeta)} \mathcal{T}_p$. The $\{e_{(\lambda, \mu, \alpha)} = O_{\lambda, \mu} \otimes \sigma_\alpha \} \subset G (\lambda, \mu = 1, 2; \alpha = 1, 2, 3)$ is a set of linear independent 12 unit vectors at the point $p$, provided with the linear unit bispseudovectors $O_{\lambda, \mu}$ and the ordinary unit vectors $\sigma_\alpha$ implying

$$<O_{\lambda, \mu}, O_{r, \nu} > = *\delta_{\lambda, r} *\delta_{\mu, \nu} \quad <\sigma_\alpha, \sigma_\beta > = \delta_{\alpha \beta}, \quad *\delta = 1 - \delta,$$

where $\delta$ is Kronecker symbol, the $\{O_{\lambda, \mu} = O_\lambda \otimes O_\mu\}$ is the basis for tangent vectors of $2 \times 2$ dimensional linear pseudospace $^*\mathbb{R}^4 = ^*\mathbb{R}^2 \otimes ^*\mathbb{R}^2$, the $\sigma_\alpha$ refers to three dimensional ordinary space $\mathbb{R}^3$. Henceforth we always let the first two subscripts in the parentheses to denote the pseudovector components, while the third refers to the ordinary vector components. The metric on $G$ is $\hat{g} : \mathcal{T}_p \otimes \mathcal{T}_p \to C^\infty(G)$ a section of conjugate vector bundle $S^2\mathbb{T}$. Any vector $A_p \in \mathcal{T}_p$ reads $A = eA$, provided with components $A$ in the basis $\{e\}$. In holonomic coordinate basis $(\partial/\partial \zeta)_p$ one gets $A = \frac{d\zeta}{dt} \bigg|_p$ and $\hat{g} = gd\zeta \otimes d\zeta$.

The manifold $G$ is decomposed as follows:

$$G = ^*\mathbb{R}^2 \otimes ^*\mathbb{R}^2 \otimes \mathbb{R}^3 = G_\eta \oplus G_u = \sum_{\lambda, \mu = 1}^2 \oplus R_\lambda^3 = R_x^3 \oplus R_x^3 \oplus R_u^3 \oplus R_u^3$$

with corresponding basis vectors $e_{i(\lambda \alpha)} = O_{i \lambda} \otimes \sigma_\alpha \subset G (\lambda = \pm, \ i = \eta, u)$ of tangent sections, where

$$O_{i+} = \frac{1}{\sqrt{2}} (O_{1,1} + \varepsilon_i O_{2,1}), \quad O_{i-} = \frac{1}{\sqrt{2}} (O_{1,2} + \varepsilon_i O_{2,2}), \quad \varepsilon_\eta = 1, \quad \varepsilon_u = -1.$$ 

Then $<O_{i \lambda}, O_{i \tau}> = \varepsilon_i \delta_{ij} *\delta_{\lambda \tau}$. The $G$ is decomposed into three dimensional ordinary and time flat spaces $G_\eta = R_x^3 \oplus R_u^3$ with signatures $\text{sgn}(R_x^3) = (+ + +)$ and $\text{sgn}(R_u^3) = (---)$ (the same holds for $G_u$). The positive metric forms are defined on manifolds $G_i : \quad \eta^2 \in$
\( G, \ u^2 \in G \). The passage to Minkowski space is a further step as follows: Since all
directions in \( \mathbf{R}^3 \) are equivalent, then by notion *time* one implies the projection of time-
coordinate on fixed arbitrary universal direction in \( \mathbf{R}^3 \). By the reduction \( \mathbf{R}^3 \to \mathbf{R}^1 \) the
passage \( G \to M^1 = \mathbf{R} \oplus \mathbf{R}^1 \) may be performed whenever it will be needed. For more
discussion of properties of \( G \) we refer to [10,11].

Unifying the geometry and particles into one framework the operator manifold formalism
is analogous to the method of secondary quantization with appropriate expansion over
the geometric objects. For the secondary quantization of geometry, first we substitute
the basis elements by the creation and annihilation operators acting in the configuration
space of occupation numbers. Instead of pseudo vectors \( O_\lambda \) we introduce the operators
supplied by additional index (\( r \)) referring to the quantum numbers of corresponding state

\[
\hat{O}_1^r = O^r_1 \alpha_1, \quad \hat{O}_2^r = O_2^r \alpha_2, \quad \hat{O}^r_\lambda = \star \delta^{\lambda \mu} \hat{O}^r_\mu = (\hat{O}^r_\lambda)^+, \\
\{\hat{O}^r_\lambda, \hat{O}^r_\eta \} = \delta_{rr'} \delta_{\lambda, \lambda'} I_2, \quad < O^r_\lambda, O^r_\eta \rangle = \delta_{rr'} \delta_{\lambda, \lambda'}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(2.1.1)

The matrices \( \alpha_\lambda \) satisfy the condition \( \{ \alpha_\lambda, \alpha_\tau \} = \star \delta_{\lambda, \tau} I_2 \), where \( \alpha^\lambda = \star \delta^{\lambda \mu} \alpha_\mu = (\alpha_\lambda)^+ \). For
example \( \alpha_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Creation operator \( \hat{O}_1^r \) generates one occupied
state \( | 1 \rangle_{(0)} \equiv | 0, \ldots, 1, \ldots \rangle > \) and the basis vector \( O_1^r \) with the quantum number \( r \)
through acting on nonoccupied vacuum state \( | 0 \rangle \equiv | 0, 0, \ldots \rangle > \). 
Accordingly, the action of annihilation operator \( \hat{O}_2^r \) on one occupied state yields the
vacuum state and the basis vector \( O_2^r \) \( \hat{O}_2^r | 1 \rangle = O_2^r | 0 \rangle > \). So \( \hat{O}_1^r | 1 \rangle = 0 \),
\( \hat{O}_2^r | 0 \rangle = 0 \). For instance, a matrix realization of the states is \( | 0 \rangle \equiv \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \),

\( | 1 \rangle \equiv \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). The vacuum state reads \( \chi_0 \equiv | 0 \rangle \equiv \prod_{r=1}^N (\chi_1)_r \). The one occupied
state is \( \chi_{r'} \equiv | 1 \rangle = (\chi_2)_{r'} \prod_{r \neq r'} (\chi_1)_r \). Instead of ordinary basis vectors we introduce the
operators \( \hat{\sigma}_\alpha^r \equiv \delta_{\alpha, \beta \gamma} \sigma_\beta^r \sigma_\gamma \), where \( \sigma_\gamma \) are Pauli’s matrices, and

\[
< \sigma_\alpha^r, \sigma_\beta^r > = \delta_{rr'} \delta_{\alpha, \beta}, \quad \hat{\sigma}_r^\alpha = \delta_{\alpha, \beta} \hat{\sigma}_r^\beta = (\hat{\sigma}_r^\alpha)^+ = \hat{\sigma}_r^\alpha, \quad \{ \hat{\sigma}_r^\alpha, \hat{\sigma}_r^\beta \} = 2 \delta_{rr'} \delta_{\alpha, \beta} I_2.
\]

(2.1.2)

A matrix realization of the vacuum state \( | 0 \rangle \equiv \varphi_{1(a)} \) and one occupied state \( | 1_{(a)} \rangle \equiv \varphi_{2(a)} \) is as follows: \( \varphi_{1(a)} \equiv \chi_1, \ \varphi_{2(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \varphi_{2(2)} = \begin{pmatrix} -i \\ 0 \end{pmatrix}, \ \varphi_{2(3)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \). Then

\[
\hat{\sigma}_r^\alpha \varphi_{1(a)} = \sigma_\alpha^r \varphi_{2(1)} = (\sigma_\alpha^r \sigma_\alpha^r) \varphi_{1(a)}, \quad \hat{\sigma}_r^\alpha \varphi_{2(a)} = \sigma_\alpha^r \varphi_{1(a)} = (\sigma_\alpha^r \sigma_\alpha^r) \varphi_{2(a)}.
\]

Hence, the single eigenvalue \( (\sigma_\alpha^r \sigma_\alpha^r) \) associates with different \( \varphi_{\lambda(a)} \), namely it is degenerated with
degeneracy degree equal 2. So, among quantum numbers \( r \) there is also
the quantum number of the half integer spin \( \bar{\sigma} \) \( (\sigma_3 = \frac{1}{2} s, \ s = \pm 1) \). As it will be seen,
this consequently gives rise to the spins of particles. One occupied state reads
\( \varphi_{r''(a)} = (\varphi_{2(2)})_{r'} \prod_{r \neq r'} (\chi_1)_r \). Next we introduce the operator

\[
\hat{\gamma}_{(\lambda, \mu, a)}^r \equiv \hat{O}^r_\lambda \otimes \hat{O}^r_\mu \otimes \hat{\sigma}_\alpha^r
\]
and the state vector

\[ \chi_{\lambda, \mu, \tau(\alpha)} \equiv \chi_{\lambda, \mu, \tau} \equiv \chi_{\lambda} \otimes \chi_{\mu} \otimes \varphi_{\tau(\alpha)}, \]

where \( \lambda, \mu, \tau, \nu = 1, 2 \); \( \alpha, \beta = 1, 2, 3 \) and \( r \equiv (r_1, r_2, r_3) \). Omitting two valuedness of state vector we apply \( |\lambda, \tau, \delta(\beta) > \equiv |\lambda, \tau, \tau > \), and remember that always the summation must be extended over the double degeneracy of the spin states \( (s = \pm 1) \). One infers the explicit form of corresponding matrix elements

\[ < \lambda, \mu | \hat{\gamma}_r^{(\tau, \nu, \alpha)} | \tau, \nu > = * \delta_{\lambda \tau} * \delta_{\mu \nu} \epsilon_r^{(\tau, \nu, \alpha)}, \quad < \tau, \nu | \hat{\gamma}_r^{(\tau, \nu, \alpha)} | \lambda, \mu > = * \delta_{\lambda \tau} * \delta_{\mu \nu} \epsilon_r^{(\tau, \nu, \alpha)}, \]

for given \( \lambda, \mu \). The operators of occupation numbers are

\[ \hat{N}_{1, \alpha} = \hat{\gamma}^{(1,1,\alpha)} \hat{n}^{(2,2,\alpha)}; \quad \hat{N}_{2, \alpha} = \hat{\gamma}^{(2,1,\alpha)} \hat{n}^{(1,2,\alpha)}, \]

with the expectation values implying Pauli’s exclusion principle

\[ < 2, 2 | \hat{N}_{1, \alpha} | 2, 2 > = \delta_{r^2} \delta_{\alpha, \beta}, \quad < 1, 2 | \hat{N}_{2, \alpha} | 1, 2 > = \delta_{r^2} \delta_{\alpha, \beta}, \]

\[ < 1, 1 | \hat{N}_{1, \alpha} | 1, 1 > = 0, \quad < 2, 1 | \hat{N}_{2, \alpha} | 2, 1 > = 0. \]

The set of operators \( \{ \hat{\gamma}_r \} \) is the basis for tangent operator vectors \( \hat{\Phi}(\zeta) = \hat{\gamma}_r \Phi_r(\zeta) \) of the 12 dimensional flat operator manifold \( \hat{G} \), where we introduce the vector function belonging to the ordinary class of functions of \( C^\infty \) smoothness defined on the manifold \( G: \Phi_r^{(\lambda, \mu, \alpha)}(\zeta) = \zeta^{(\lambda, \mu, \alpha)} \Phi_r^{(\lambda, \mu)}(\zeta), \quad \zeta \in G \). But a set of operators \( \{ \hat{\gamma}_r \} \) is a dual basis for operator covectors \( \hat{\Phi}(\zeta) = \hat{\gamma}_{r'} \Phi_{r'}(\zeta) \), where \( \Phi_{r'} = \Phi_r \) (charge conjugated). One gets

\[ < \lambda, \mu | \hat{\Phi}(\zeta) \hat{\Phi}(\zeta) | \lambda, \mu > = * \delta_{\lambda \tau} * \delta_{\mu \nu} \Phi_r^{(\tau, \nu, \alpha)}(\zeta) \Phi_{r'}^{(\tau, \nu, \alpha)}(\zeta), \]

for given \( \lambda, \mu \). Considering the state vectors

\[ \chi^0(\nu_1, \nu_2, \nu_3, \nu_4) = | 1, 1 >^\nu_i \cdot | 1, 2 >^\nu_2 \cdot | 2, 1 >^\nu_3 \cdot | 2, 2 >^\nu_4, \]

\[ \nu_i = \begin{cases} 1 & \text{if } \nu = \nu_i \text{ for some } i, \\ 0 & \text{otherwise}, \end{cases} \]

\[ | \chi_{-}(1) > = \chi^0(1, 0, 0, 0), \quad | \chi_{+}(1) > = \chi^0(0, 0, 0, 1), \quad \chi_{\lambda}(\zeta) | \chi_{\mu}(\zeta) >= \delta_{\lambda \mu}, \]

\[ | \chi_{-}(2) > = \chi^0(0, 0, 1, 0), \quad | \chi_{+}(2) > = \chi^0(0, 1, 0, 0), \quad \chi_{\lambda}(\zeta) | \chi_{\mu}(\zeta) >= \delta_{\lambda \mu}, \]

provided \( < \chi_{\pm} | A | \chi_{\pm} > \equiv \sum \lambda < \chi_{\lambda}(\zeta) | A | \chi_{\pm}(\zeta) > \), we get the matrix elements

\[ < \chi_{+} | \hat{\Phi}(\zeta) \hat{\Phi}(\zeta) | \chi_{+} > \equiv \Phi^+_{\lambda}(\zeta) = \Phi_{\lambda}(\lambda, 1, \alpha)(\zeta) \Phi_{\lambda}(\lambda, 1, \alpha)(\zeta), \]

\[ < \chi_{-} | \hat{\Phi}(\zeta) \hat{\Phi}(\zeta) | \chi_{-} > \equiv \Phi^-_{\lambda}(\zeta) = \Phi_{\lambda}(\lambda, 2, \alpha)(\zeta) \Phi_{\lambda}(\lambda, 2, \alpha)(\zeta). \]

The basis \( \{ \hat{\gamma}_r \} \) is decomposed into \( \{ \hat{\gamma}_i \} \) \( (\lambda = \pm; \quad \alpha = 1, 2, 3; \quad i = \eta, u) \):

\[ \hat{\gamma}_i^{(\lambda, \alpha)} = \frac{1}{\sqrt{2}}(\gamma_r^{(i, 1, \alpha)} + \epsilon_i \gamma_r^{(2, 2, \alpha)}), \quad \hat{\gamma}_i^{(\lambda, \alpha)} = \frac{1}{\sqrt{2}}(\gamma_r^{(i, 1, \alpha)} - \epsilon_i \gamma_r^{(2, 2, \alpha)}). \]
\( \hat{\Psi} \in \hat{G} \) and operator covectors \( \tilde{\Psi} \) are written \( \hat{\Psi} = \hat{\gamma}^r \Psi \), \( \tilde{\Psi} = \hat{\gamma} \tilde{\Psi}^r \), where the vector functions of \( C^\infty \) smoothness are defined on the manifolds \( \hat{G} \):

\[
\Psi_{\eta r}^{(\pm \alpha)}(\eta, p_\eta) = \eta^{(\pm \alpha)} \Psi_{\eta r}^{(\pm \eta, p_\eta)}, \quad \Psi_{u r}^{(\pm \alpha)}(u, p_u) = u^{(\pm \alpha)} \Psi_{u r}^{(\pm u, p_u)}. \tag{2.1.9}
\]

It is assumed that the probability of finding the vector function in the state \( r \) with given sixvector of coordinate \( \eta \) or \( u \) and momentum \( p_\eta \) or \( p_u \) is determined by the square of its state wave function \( \Psi_{\eta r}^{\pm}(\eta, p_\eta) \), or \( \Psi_{u r}^{\pm}(u, p_u) \). Due to the spin states, the \( \Psi_{\eta r}^{\pm} \) is the Fermi field of the positive and negative frequencies \( \Psi_{\eta r}^{\pm} = \Psi_{i r}^{\pm} \).

### 2.2 Realization of the Flat Manifold \( G \)

The bispinor \( \Psi(\zeta) \) defined on manifold \( G = G \oplus G \) is written \( \Psi(\zeta) = \Psi(\eta) \Psi(u) \), where the \( \Psi \) is a bispinor defined on the manifold \( G \). The free state of \( i \)-type fermion with definite values of momentum \( p_i \) and spin projection \( s \) is described by means of plane waves,

\[
(\hbar = 1, c = 1)[8]: \quad \Psi_{\eta p_\eta}^{i}(\eta) = \left( \frac{m}{E_\eta} \right)^{1/2} U(\eta, s) e^{-ip_\eta \eta}, \quad \text{etc where } E_i \equiv p_{i0} = |\vec{p}_{i0}|, \quad p = \sqrt{\frac{1}{2}(p_{i(+)\alpha} + p_{i(-)\alpha})}, \quad \vec{p}_i = \frac{1}{\sqrt{2}}(\vec{p}_{i+} - \vec{p}_{i-}), \quad p_\eta^2 = E_\eta^2 - p_{\eta i}^2 = p_u^2 = E_u^2 - \vec{p}_u^2 = m_i^2.
\]

We consider also the solutions of negative frequencies. For the spinors the useful relations of orthogonality and completeness hold. We make use of localized wave packets constructed by means of superposition of plane wave solutions furnished by creation and annihilation operators in agreement with Pauli’s principle

\[
\hat{\Psi}_{\eta r} = \sum_{\pm s} \int \frac{d^3 p_i}{(2\pi)^{3/2}} \left( \hat{\gamma}_{\eta r}^{i(+)\alpha} \Psi_{i r}^{(\alpha)} + \hat{\gamma}_{\eta r}^{i(-)\alpha} \Psi_{i r}^{(-\alpha)} \right),
\]

and where the summation is extended over all dummy indices. The matrix element of the anticommutator of expansion coefficients reads

\[
\langle \chi_- | \{ \hat{\gamma}_{\eta r}^{i(+)\alpha} (p_i, s), \hat{\gamma}_{j r}^{i(-)\beta} (p'_j, s') \} | \chi_- \rangle = \epsilon_{i j} \delta_{s s'} \delta_{\alpha \beta} \delta_{(3)} (\vec{p}_i - \vec{p}_j). \tag{2.2.1}
\]

We also consider wave packets of operator vector fields \( \Phi(\zeta) \). In this manner we get the important relation

\[
\sum_{\lambda = \pm} < \chi_\lambda \mid \Phi(\zeta) \Phi(\zeta) \mid \chi_\lambda > = \sum_{\lambda = \pm} < \chi_\lambda \mid \tilde{\Phi}(\zeta) \tilde{\Phi}(\zeta) \mid \chi_\lambda > = -i \lim_{\zeta' \to \zeta} (\zeta') G(\zeta - \zeta') = -i \left[ \lim_{\eta' \to \eta} G(\eta - \eta') - \lim_{u' \to u} G(u - u') \right], \tag{2.2.2}
\]

where the Green’s function \( G(i - i') = -(i \hat{\partial} + m) \Delta(i - i') \) is used, provided with the invariant singular functions \( \Delta(i - i') \) \( (i = \eta, u) \). Realization of the manifold \( G \) is due to the constraint imposed upon the matrix element eq.(2.2.2) which is, as a geometric object, required to be finite

\[
\sum_{\lambda = \pm} < \chi_\lambda \mid \Phi(\zeta) \Phi(\zeta) \mid \chi_\lambda > = \zeta^2 G(0, \zeta_F) < \infty. \tag{2.2.3}
\]
Thereto [8] (see eq.(3.1.3))
\[
G(0) = G(0) = G(0) = \lim_{u \to u'} \left[ -i \sum_{p_u} |u|_{p_u} \Psi (u) \Psi (u') \theta (u_0 - u'_0) + i \sum_{p_u} \bar{\Psi} (u') \Psi (u) \theta (u_0 - u_0) \right].
\]

The G, G, and G are causal Green’s functions characterized by the boundary condition that only positive frequency occur for \(\eta_0 > 0\) \((u_0 > 0)\), only negative for \(\eta_0 < 0\) \((u_0 < 0)\). Here \(\eta_0 = |\bar{\eta}_0|, \eta_0 = \frac{1}{\sqrt{2}} (\eta_+ - \eta_-)\) and the same holds for \(u_0\). Then, satisfying the condition eq.(2.2.3) a length of each vector \(\zeta = e \zeta \in G\) (see eq.(2.2.2)) compulsory should be equal zero
\[
\zeta^2 = \eta^2 - u^2 = 0. \tag{2.2.5}
\]

Thus the requirement eq.(2.2.3) provided by eq.(2.2.4) yields the realization of the flat manifold \(G\), which subsequently leads to Minkowski flat space \(M^4\) (subsec.2.1), where according to eq.(2.2.5) the Relativity principle holds
\[
d\eta^2 \big|_{6-4} \equiv d^2 s = d u^2 = inv.
\]

### 2.3 Mathematical Background: Field Aspect

The quantum field theory of operator manifold is equivalent to configuration space wave mechanics employing the antisymmetric state functions incorporated with geometric properties of corresponding objects [12]. In this subsection we reach to rigorous definition of concept of operator manifold \(\hat{G}\), construct the explicit forms of wave state functions and calculate the matrix elements of field operators.

Suppose that the \(i\)-th fermion is found in the state \(r_i\) with the vector function \(\Phi^{(\lambda_i, \mu_i, \alpha_i)} = \zeta^{(\lambda_i, \mu_i, \alpha_i)} \Phi^\lambda_{\mu_i} (\zeta_{r_i})\) (eq.(2.1.6)) and
\[
\zeta_{r_i} = \sum_{\lambda_i, \mu_i} e^{r_i}_{(\lambda_i, \mu_i, \alpha_i)} \zeta^{(\lambda_i, \mu_i, \alpha_i)} \in \mathcal{U}_{r_i},
\]

the \(\mathcal{U}_{r_i}\) is the open neighbourhood of the point \(\zeta_{r_i}\); the \(r_i\) implies a set \((r_i^{11}, r_i^{12}, r_i^{21}, r_i^{22})\). Let the \(\mathcal{H}^{(1)}\) is a Hilbert space used for quantum mechanical description of one particle, namely \(\mathcal{H}^{(1)}\) is a finite or infinite dimensional complex space provided with scalar product \((\Phi, \Psi)\), which is linear with respect to \(\Psi\) and antilinear to \(\Phi\). The \(\mathcal{H}^{(1)}\) is complete in norm \(|\Phi| = (\Phi, \Phi)^{1/2}\), i.e. each fundamental sequence \(\{ \Phi_n \}\) of vectors of \(\mathcal{H}^{(1)}\) is converged by norm on \(\mathcal{H}^{(1)}\). One particle state function is written \(\Phi^{(1)} = \prod_{\lambda_i, \mu_i = 1} \Phi^{(1)}_{r_i, \lambda_i, \mu_i} \in \mathcal{H}^{(1)}_{r_i}\), where
\[
\mathcal{H}^{(1)}_{r_i} = \prod_{\lambda_i, \mu_i = 1} \otimes \mathcal{H}^{(1)}_{r_i, \lambda_i, \mu_i}.\]  Define
\[
\bar{\Phi}^{(1)} = \zeta \Phi^{(1)} = \bar{\mathcal{G}}^{(1)}_{r_i} = \bar{\mathcal{U}}^{(1)}_{r_i} \otimes \mathcal{H}^{(1)}_{r_i}.
\]

For description of \(n\) particle system we introduce Hilbert space
\[
\mathcal{H}^{(n)}_{(r_1, \ldots, r_n)} = \mathcal{H}^{(1)}_{r_1} \otimes \cdots \otimes \mathcal{H}^{(1)}_{r_n}, \tag{2.3.2}
\]
by considering all sequences
\[ \Phi_{(r_1,\ldots,r_n)}^{(n)} = \{ \Phi^{(1)}_{r_1}, \ldots, \Phi^{(1)}_{r_n} \} = \Phi^{(1)}_{r_1} \otimes \cdots \otimes \Phi^{(1)}_{r_n}, \]  
where \( \Phi^{(1)}_{r_i} \in \mathcal{H}^{(1)}_{r_i} \) provided, as usual, with the scalar product
\[ (\Phi_{(r_1,\ldots,r_n)}^{(n)}, \Psi_{(r_1,\ldots,r_n)}^{(n)}) = \prod_{i=1}^{n} (\Phi^{(1)}_{r_i} \cdot \Psi^{(1)}_{r_i}). \]

We consider the space \( \mathcal{H}^{(n)}_{(r_1,\ldots,r_n)} \) of all limited linear combinations of eq.\((2.3.2)\) and continue by linearity the scalar product eq.\((2.3.4)\) on \( \mathcal{H}^{(n)}_{(r_1,\ldots,r_n)} \). The wave function \( \Phi^{(n)}_{(r_1,\ldots,r_n)} \in \mathcal{H}^{(n)}_{(r_1,\ldots,r_n)} \) must be antisymmetrized over its arguments. We distinguish the antisymmetric part \( A\mathcal{H}^{(n)} \) of Hilbert space \( \mathcal{H}^{(n)} \) by considering the functions
\[ A\Phi^{(n)}_{(r_1,\ldots,r_n)} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S(n)} sgn(\sigma)\Phi^{(n)}_{\sigma(r_1,\ldots,r_n)}. \]

The summation is extended over all permutations of indices \((r_1^{\lambda_1}, \ldots, r_n^{\lambda_n})\) of the integers \(1, 2, \ldots, n\), where the antisymmetrical eigenfunctions are sums of the same terms with alternating signs in dependence of a parity \( sgn(\sigma) \) of transposition. One continues the reflection \( \Phi^{(n)} \rightarrow A\Phi^{(n)} \) by linearity on \( \mathcal{H}^{(n)} \), which is limited and enables the expansion by linearity on \( A\mathcal{H}^{(n)} \). The region of values of this reflection is a \( A\mathcal{H}^{(n)} \), namely an antisymmetrized tensor product of \( n \) identical samples of \( \mathcal{H}^{(1)} \). We introduce
\[ A\Phi_{(r_1,\ldots,r_n)}^{(n)} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S(n)} sgn(\sigma)\Phi_{\sigma(r_1,\ldots,r_n)}^{(n)} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S(n)} sgn(\sigma)\Phi_{r_1}^{(1)} \otimes \cdots \otimes \Phi_{r_n}^{(1)} \in A\mathcal{G}_{(r_1,\ldots,r_n)}^{(n)} = A\mathcal{G}_{(r_1,\ldots,r_n)}^{(n)} \otimes A\mathcal{H}_{(r_1,\ldots,r_n)}^{(n)}. \]

and consider a set \( A\mathcal{F} \) of all sequences \( A\Phi = \{ A\Phi^{(0)}, A\Phi^{(1)}, \ldots, A\Phi^{(n)} \} \), with a finite number of nonzero elements. Therewith, the set \( A\mathcal{F} A\Phi = \{ A\Phi^{(0)}, A\Phi^{(1)}, \ldots, A\Phi^{(n)} \} \) is provided with the structure of Hilbert subspace employing the composition rules
\[ A(\lambda \Phi + \mu \Psi)^{(n)} = \lambda A\Phi^{(n)} + \mu A\Psi^{(n)}, \quad \forall \lambda, \mu \in C, \]
\[ (A\Phi, A\Psi) = \sum_{n=0}^{\infty} (A\Phi^{(n)}, A\Psi^{(n)}). \]

The wave manifold \( \mathcal{G} \) stems from the \( A\mathcal{F} \) is due to the expansion by metric induced as a scalar product on \( A\mathcal{F} \)
\[ \mathcal{G} = \sum_{n=0}^{\infty} \mathcal{G}^{(n)} = \sum_{n=0}^{\infty} \left( \mathcal{U}^{(n)} \otimes A\mathcal{H}^{(n)} \right). \]

The creation \( \hat{\gamma}^r \) and annihilation \( \hat{\gamma}^r \) operators for each \( \mathcal{H}^{(1)} \) can be defined as follows: one must modify the basis operators in order to provide an anticommutation in both the same and different states
\[ \hat{\gamma}^r_{(\lambda,\mu,\alpha)} \rightarrow \hat{\gamma}^r_{(\lambda,\mu,\alpha)} \eta^\lambda_r, \quad (\eta^\lambda_r)^+ = \eta^\lambda_r, \]
for given \( \lambda, \mu, \alpha \), where \( \eta_r \) is a diagonal operator in the space of occupation numbers, while, at \( r_i < r_j \) one gets \( \hat{\gamma}^r_{r_i} \eta_{r_j} = -\eta_{r_j} \hat{\gamma}^r_{r_i}, \quad \hat{\gamma}^r_{r_i} \eta_{r_j} = \eta_{r_i} \hat{\gamma}^r_{r_j}. \) The operators of corresponding
occupation numbers (for given \(\lambda, \mu, \alpha\)) are \(\hat{N}_r = \hat{\gamma}^r \hat{\gamma}_r\). Since the diagonal operators \((1 - 2\hat{N}_r)\) anticommute with the \(\hat{\gamma}^r\), then \(\eta_{r_i} = \prod_{r=1}^{r_i-1} (1 - 2\hat{N}_r)\), where

\[
\eta_{r_i}^{11} \Phi(n_1, \ldots, n_N; 0; 0) = \prod_{r=1}^{r_i-1} (-1)^{n_r} \Phi(n_1, \ldots, n_N; 0; 0),
\]

(2.3.10)

etc. Here the occupation numbers \(n_r(m_r, q_r, t_r)\) are introduced, which refer to the \(r\)-th states corresponding to operators \(\hat{\gamma}_{(1,1,\alpha)}\), etc either empty \((n_r, \ldots, t_r = 0)\) or occupied \((n_r, \ldots, t_r = 1)\). To save writing we abbreviate the modified operators by the same symbols. For example, acting on free state \(|0\rangle_{r_i}\) the creation operator \(\hat{\gamma}_{r_i}\) yields the one occupied state \(|1\rangle_{r_i}\) with the phase + or − depending of parity of the number of quanta in the states \(r < r_i\). Modified operators satisfy the same anticommutation relations of the basis operators (subsec.2.1). It is convenient to make use of notation \(\hat{\gamma}_{(r,\mu,\alpha)} \equiv e_{r}(\lambda, \mu, \alpha) \hat{\gamma}^\lambda_{(\alpha)}\), and abbreviate the pair of indices \((\alpha)\) by the single symbol \(r\).

Then for each \(\Phi \in A^\mathcal{H}^{(n)}\) and any vector \(f \in \mathcal{H}^{(1)}\) the operators \(\hat{b}(f)\) and \(\hat{b}^*(f)\) imply

\[
\hat{b}(f) \Phi = \frac{1}{\sqrt{(n-1)!}} \sum_{\sigma \in S(n)} \text{sgn}(\sigma) \left( f \Phi_{(\sigma(1))}^{(1)} \otimes \cdots \otimes \Phi_{(\sigma(n))}^{(1)} \right),
\]

\[
\hat{b}^*(f) \Phi = \frac{1}{\sqrt{(n+1)!}} \sum_{\sigma \in S(n+1)} \text{sgn}(\sigma) \Phi_{(\sigma(0))}^{(1)} \otimes \Phi_{(\sigma(1))}^{(1)} \otimes \cdots \otimes \Phi_{(\sigma(n))}^{(1)},
\]

(2.3.11)

where \(\Phi_{(0)}^{(1)} \equiv f\). One continues the \(\hat{b}(f)\) and \(\hat{b}^*(f)\) by linearity to linear reflections, which are denoted by the same symbols acting respectively from \(A^\mathcal{H}^{(n)}\) into \(A^\mathcal{H}^{(n-1)}\) or \(A^\mathcal{H}^{(n+1)}\). They are limited over the values \(\sqrt{n}|f|\) and \(\sqrt{(n+1)|f|}\) and can be expanded by continuation up to the reflections acting from \(A^\mathcal{H}^{(n)}\) into \(A^\mathcal{H}^{(n-1)}\) or \(A^\mathcal{H}^{(n+1)}\). Finally, they must be continued by linearity up to the linear operators acting from \(A^\mathcal{F}\) into \(A^\mathcal{F}\) defined on the same closed region in \(A^\mathcal{H}^{(n)}\), namely in \(A^\mathcal{F}\), which is invariant with respect to reflections \(\hat{b}(f)\) and \(\hat{b}^*(f)\). Hence, at \(f_i, g_i \in \mathcal{H}^{(1)}\) \((i = 1, \ldots, n; j = 1, \ldots, m)\) all polynomials over \(\{\hat{b}^*(f_i)\}\) and \(\{\hat{b}(g_j)\}\) are completely defined on \(A^\mathcal{F}\). While, for given \(\lambda, \mu, \) one has

\[
< \lambda, \mu | \{ \hat{b}_{\alpha}^{\lambda \mu}(f), \hat{b}_{\beta}^{\lambda \mu}(g) \} | \lambda, \mu > = \delta_{\alpha \beta} \epsilon_{\epsilon}. \tag{2.3.12}
\]

The mean values \(< \phi; \hat{b}_{\alpha}^{\lambda \mu}(f) \hat{b}_{\beta}^{\lambda \mu}(g) >\) calculated at fixed \(\lambda, \mu\) for any element \(\Phi \in A^\mathcal{F}\) equal to mean values of the symmetric operator of occupation number in terms of \(\hat{N}_r = \hat{\gamma}_r\) \(\hat{\gamma}_r\), with a wave function \(f\) in the state described by \(\Phi\). Here, as usual, it is denoted \(< \phi; A \Phi > = Tr \rho \rho A = (\Phi, A \Phi)\) for each vector \(\Phi \in \mathcal{H}\) with \(|\Phi| = 1\), while the \(P_{\phi}\) is projecting operator onto one dimensional space \(\{\lambda \Phi | \lambda \in C\}\) generated by \(\Phi\). Therewith, the probability of transition \(\phi \rightarrow \psi\) is given \(P \rho_{\phi}^{\psi} = |< \psi, \phi >|^2\). The linear operator \(A\) is defined on the elements of linear manifold \(\mathcal{D}(\mathcal{A})\) of \(\mathcal{H}\) taking the values in \(\mathcal{H}\). The \(\mathcal{D}(\mathcal{A})\) is an everywhere closed region of definition of \(\mathcal{A}\), namely the closure of \(\mathcal{D}(\mathcal{A})\) by the norm given in \(\mathcal{H}\) coincides with \(\mathcal{H}\). Meanwhile, the \(\mathcal{D}(\mathcal{A})\) is included in \(\mathcal{D}(\mathcal{A}^*)\) and \(\mathcal{A}\) coincides with the reduction of \(\mathcal{A}^*\) on \(\mathcal{D}(\mathcal{A})\), because the \(\mathcal{D}(\mathcal{A})\) is a symmetric operator and the linear operator \(\mathcal{A}^*\) is maximal conjugated to \(\mathcal{A}\). That is, any operator \(\mathcal{A}'\) conjugated to \(\mathcal{A}\) - \((\Psi, A^\dagger \Psi) = (A^\dagger \Psi, A)\) at all \(\Phi \in \mathcal{D}(\mathcal{A})\) and \(\Psi \in \mathcal{D}(\mathcal{A}')\) coincides with the reduction of \(\mathcal{A}^*\) on some linear manifold \(\mathcal{D}(\mathcal{A}')\) included in \(\mathcal{D}(\mathcal{A}^*)\). Thus, the operator \(\mathcal{A}^{**}\) is closed symmetric expansion of operator \(\mathcal{A}\), namely it is a closure of \(\mathcal{A}\). Self conjugated operator \(\mathcal{A}\) (the closure of which is self conjugated too) allows only one self conjugated expansion \(\mathcal{A}^{**}\).
Thus, self conjugated closure $\hat{N}$ of operator $\sum_{i=1}^{\infty} \hat{b}^*(f_i)\hat{b}(f_i)$, where $\{f_i \mid i = 1, \ldots, n\}$ is an arbitrary orthogonal basis on $H^{(1)}$, is regarded as the operator of occupation number. For the vector $\chi^0 \in A F$ and $\chi_i^{0(n)} = \delta_{0n}$ one gets $< \chi_i^{0(n)}, \hat{N}(f) > \equiv 0$ for all $f \in H^{(1)}$. So, the $\chi^0$ is a vector of vacuum state: $\hat{b}(f)\chi^0 = 0$ for all $f \in H^{(1)}$. If $f = \{f_i \mid i = 1, 2, \ldots\}$ is an arbitrary orthogonal basis on $H^{(1)}$, then due to irreducibility of operators $\hat{b}^*(f_i) \mid f_i \in f$, the $A H$ includes the 0 and whole space $A H$ as invariant subspaces with respect to all $\hat{b}^*(f)$. To define the 12 dimensional operator manifold $\hat{G}$ we consider a set $\hat{F}$ of all sequences $\hat{\Phi} = \{\hat{\Phi}(0), \hat{\Phi}(1), \ldots, \hat{\Phi}(n), \ldots\}$ with a finite number of nonzero elements provided

$$
\hat{\Phi}^{(n)}{(r_1, \ldots, r_n)} = \hat{\Phi}_{r_1}^{(1)} \otimes \cdots \otimes \hat{\Phi}_{r_n}^{(1)} \in \hat{G}^{(n)}, \quad \hat{\Phi}_{r_i}^{(1)} = \hat{\zeta}_{r_i} \Phi_i^{(1)} \in \hat{G}^{(1)} = \hat{U}_{r_i}^{(1)} \otimes H_i^{(1)},
$$

$$
\hat{\zeta}_{r_i} = \sum_{\alpha_i = 1}^{3} \hat{\gamma}_{r_i}^{(\lambda_i, \mu_i, \alpha_i)} \Phi_i^{(1)} \in \hat{U}_{r_i}^{(1)}, \quad \hat{G}^{(n)} = \hat{U}^{(n)} \otimes H, \quad \hat{U}^{(n)} = \hat{U}_{r_1}^{(1)} \otimes \cdots \otimes \hat{U}_{r_n}^{(1)}.
$$

(2.3.13)

Then on the analogy of eq.(2.3.8) the operator manifold $\hat{G}$ ensues

$$
\hat{G} = \sum_{n=0}^{\infty} \hat{G}^{(n)} = \sum_{n=0}^{\infty} \left(\hat{U}^{(n)} \otimes H^{(n)}\right).
$$

(2.3.14)

To define the secondary quantized form of one particle observable $A$ on $H$, following [13] let consider a set of identical samples $\hat{H}_i$ of one particle space $H^{(1)}$ and operators $A_i$ acting in them. To each closed linear operator $A^{(1)}$ in $H^{(1)}$ with everywhere closed region of definition $D(A^{(1)})$ following operators are corresponded:

$$
A^{(1)}_1 = A^{(1)} \otimes I \otimes \cdots \otimes I,
$$

$$
A^{(1)}_n = I \otimes I \otimes \cdots \otimes A^{(1)},
$$

(2.3.15)

Their sum $\sum_{j=1}^{n} A^{(1)}_j$ is given on the intersection of regions of definition of operator terms including a linear manifold $D(A^{(1)}) \otimes \cdots \otimes D(A^{(n)})$ being closed in $\hat{H}^{(n)}$. While, the $A^{(n)}$ is a minimal closed expansion of this sum with $D(A^{(n)})$. One considers a linear manifold $D(\Omega(A))$ in $H = \sum_{n=0}^{\infty} \hat{H}^{(n)}$ defined as a set of all vectors $\Psi \in H$ such as $\Psi^{(n)} \in D(A^{(n)})$ and

$$
\sum_{n=0}^{\infty} \left| A^{(n)} \Psi^{(n)} \right|^2 < \infty. \quad \text{The manifold } D(\Omega(A)) \text{ is closed in } H. \quad \text{On this manifold one defines a closed linear operator } \Omega(A) \text{ acting as } \Omega(A)^{(n)} = A^{(n)} \Psi^{(n)}, \text{ namely } \Omega(A)\Phi = \sum_{n=0}^{\infty} A^{(n)} \Psi^{(n)},
$$

while the $\Omega(A)$ is self conjugated operator with everywhere closed region of definition. We suppose that the vector $\Phi^{(n)} \in H^{(n)}$ is in the form eq.(2.3.3), where $\Phi_i \in D(A)$. Then

$$
< \varphi^{(n)}; A^{(n)} > = \sum_{i=1}^{n} < \varphi_i; A >, \quad \text{which enables the expansion by continuing onto } D(A).
$$

Thus $A^{(n)}$ is the $n$ particle observable corresponding to one particle observable $A$. So

$$
< \varphi; \Omega(A) > = \sum_{n=0}^{\infty} < \varphi^{(n)}; A^{(n)} > \quad \text{for any } \Phi_i \in D(\Omega(A)). \quad \text{While, the } \Omega(A) \text{ reflects } A^D = D(\Omega(A)) \hookrightarrow A^H \text{ into } A^H. \quad \text{The reduction of } \Omega(A) \text{ on } A^H \text{ is self conjugated in the region } A^D, \text{ because of the fact that } A^H \text{ is a closed subspace of } H. \quad \text{Hence, the } \Omega(A) \text{ is a}$$
secondary quantized form of one particle observable $A$ on $\mathcal{H}$.

The vacuum state reads eq. (2.1.7) with the normalization requirement

$$
< \chi^0 (\nu'_1, \nu'_2, \nu'_3, \nu'_4) \mid \chi^0 (\nu_1, \nu_2, \nu_3, \nu_4) >= \prod_{i=1}^{4} \delta_{\nu_i \nu'_i}.
$$

(2.3.16)

The state vectors

$$
\chi (\{n_r\}_1^N; \{m_r\}_1^M; \{q_r\}_1^Q; \{t_r\}_1^T; \nu_r) \equiv (\hat{b}_{11}^1)^{n_1} \cdots (\hat{b}_{11}^M)^{n_M} \cdot (\hat{b}_{12}^1)^{m_1} \cdots (\hat{b}_{12}^M)^{m_M} \cdot (\hat{b}_{22}^1)^{q_1} \cdots (\hat{b}_{22}^Q)^{q_Q} \cdots (\hat{b}_{TT}^1)^{t_1} \cdots (\hat{b}_{TT}^T)^{t_T} \chi^0 (\nu_1, \nu_2, \nu_3, \nu_4),
$$

(2.3.17)

where $\{n_r\}_1^N = n_1, \ldots, n_N$, etc are the eigenfunctions of modified operators. They form a whole set of orthogonal vectors

$$
< \chi; \chi' >= \prod_{r=1}^{N} \delta_{n_r n'_r} \cdot \prod_{r=1}^{M} \delta_{m_r m'_r} \cdot \prod_{r=1}^{Q} \delta_{q_r q'_r} \cdot \prod_{r=1}^{T} \delta_{t_r t'_r} \cdot \prod_{r=1}^{4} \delta_{\nu_r \nu'_r}.
$$

(2.3.18)

Considering an arbitrary superposition

$$
\chi = \sum_{a=\{n_r\}_1^N, \{m_r\}_1^M, \{q_r\}_1^Q, \{t_r\}_1^T=0}^{1} c'(a) \chi (a),
$$

(2.3.19)

the coefficients $c'$ of the expansion are the corresponding amplitudes of probabilities. Taking into account eq. (2.3.12), the nonvanishing matrix elements of operators $\hat{b}_{rk}^{11}$ and $\hat{b}_{11}^{11}$ read

$$
< \chi (\{n'_r\}_1^N; 0; 0; 1, 0, 0, 0) \mid \hat{b}_{rk}^{11} \chi (\{n_r\}_1^N; 0; 0; 1, 0, 0, 0) >=
$$

$$
= 1, 1 | \hat{b}_{rk}^{11} \cdots \hat{b}_{11}^{11} \cdot \hat{b}_{rk}^{11} \cdots \hat{b}_{11}^{11} | 1, 1 >=
$$

$$
= \left\{ \begin{array}{l}
(-1)^{n'-k'} \text{ if } n_r = n'_r \text{ for } r \neq r_k \text{ and } n_{r_k} = 0; n'_{r_k} = 1, \\
0 \text{ otherwise,}
\end{array} \right.
$$

(2.3.20)

$$
< \chi (\{n'_r\}_1^N; 0; 0; 1, 0, 0, 0) \mid \hat{b}_{11}^{11} \chi (\{n_r\}_1^N; 0; 0; 1, 0, 0, 0) >=
$$

$$
= 1, 1 | \hat{b}_{11}^{11} \cdots \hat{b}_{11}^{11} \cdot \hat{b}_{rk}^{11} \cdots \hat{b}_{11}^{11} | 1, 1 >=
$$

$$
= \left\{ \begin{array}{l}
(-1)^{n-k} \text{ if } n_r = n'_r \text{ for } r \neq r_k \text{ and } n'_{r_k} = 0; n_{r_k} = 1, \\
0 \text{ otherwise,}
\end{array} \right.
$$

(2.3.21)

where one denotes $n = \sum_{r=1}^{N} n_r$, $n' = \sum_{r=1}^{N} n'_r$, the $r_k$ and $r'_k$ are $k$-th and $k'$-th terms of regulated sets of $\{r_1, \ldots, r_n\}$ ($r_1 < r_2 < \cdots < r_n$) and $\{r'_1, \ldots, r'_n\}$ ($r'_1 < r'_2 < \cdots < r'_n$), respectively. Continuing along this line we get a whole set of explicit forms of matrix elements of the rest of operators $\hat{b}_{rk}$ and $\hat{b}_{k}^{*}$. Thus

$$
\sum_{\{n_r\}=0}^{1} < \chi^0 \mid \hat{\Phi} (\zeta) \mid \chi >= \sum_{r=1}^{N} c^r \cdot e^{n_r} (1, 1, \alpha) \Phi (1, 1, \alpha) + \cdots,
$$

(2.3.21)
provided
\[ c'_{n_r} \equiv \delta_{1n_r} c'(0, \ldots, n_r, \ldots, 0; 0; 0), \ldots \] (2.3.22)

Hereinafter we change the notation
\[ \tilde{c}(r^{11}) = c'_{n_r}, \quad \tilde{c}(r^{21}) = c'_{q_r}, \quad N_{11} = N, \quad N_{21} = Q, \]
\[ \tilde{c}(r^{12}) = c'_{m_r}, \quad \tilde{c}(r^{22}) = c'_{t_r}, \quad N_{12} = M, \quad N_{22} = T, \] (2.3.23)

and make use of
\[ F_{\lambda\mu} = \sum_\alpha c^{(\lambda\mu)}_\alpha \Phi^{(\lambda\mu,\alpha)}; \quad \sum_{\{r_1\}}^1 < \chi^0 | \hat{A} | \chi > \equiv < \chi^0 | \hat{A} \parallel \chi >, \] (2.3.24)

The matrix elements of operator vector and covector fields take the forms
\[ < \chi^0 \parallel \hat{\Phi}(\zeta) \parallel \chi > = \sum_{\lambda\mu=1}^{N_{\lambda\mu}} \sum_{r^{\lambda\mu}_{i1}}^{N_{\lambda\mu}} \tilde{c}^{(r^{\lambda\mu}_{i1})} F^{(r^{\lambda\mu}_{i1})}(\zeta), \] (2.3.25)
\[ < \chi \parallel \overline{\Phi}(\zeta) \parallel \chi^0 > = \sum_{\lambda\mu=1}^{N_{\lambda\mu}} \sum_{r^{\lambda\mu}_{i1}}^{N_{\lambda\mu}} \tilde{c}^{*(r^{\lambda\mu}_{i1})} F^{*(r^{\lambda\mu}_{i1})}(\zeta). \]

In the following we shall use: \[ \left\{ \sum_{\lambda\mu=1}^{N_{\lambda\mu}} \sum_{r^{\lambda\mu}_{i1}}^{N_{\lambda\mu}} \right\}^n \equiv \sum_{\lambda\mu=1}^{N_{\lambda\mu}} \sum_{r^{\lambda\mu}_{i1}}^{N_{\lambda\mu}}, \quad \tilde{c}^{(r^{\lambda\mu}_{i1})} \equiv \tilde{c}^{(r^{\lambda\mu}_{i1})} \quad \text{and} \quad \tilde{c}^{(r^{\lambda\mu}_{11})} = c'(n_1, \ldots, n_N; 0; 0; 0), \] etc. The anticommutation relations ensue
\[ < \chi_- | \{ \tilde{b}^+_{i_r}, \tilde{b}^+_{i_+} \} | \chi_- > = < \chi_+ | \{ \tilde{b}^-_{i_r}, \tilde{b}^-_{i_+} \} | \chi_+ > = \delta_{\tilde{r}'} \] (2.3.26)

provided \[ \tilde{\gamma}^{(\lambda\alpha)}_{i_r} = \tilde{c}^{(\lambda\alpha)}_{i_r} \tilde{b}^\lambda_{i_r} \] \[ (r\alpha) \rightarrow r. \] The state functions
\[ \chi = (\tilde{b}^+_{\eta N})^{n_N} \cdots (\tilde{b}^+_{\eta 1})^{n_1} \cdots (\tilde{b}^-_{\eta M})^{m_M} \cdots (\tilde{b}^-_{\eta 1})^{m_1} \cdots (\tilde{b}^+_{u Q})^{q_Q} \cdots (\tilde{b}^-_{u 1})^{q_1} \cdots (\tilde{b}^+_{u T})^{t_T} \cdots (\tilde{b}^-_{u 1})^{t_1} \cdot \chi_-(\lambda) \chi_+(\mu), \] (2.3.27)

form a whole set of orthogonal eigenfunctions of corresponding operators of occupation numbers \[ \tilde{N}^\lambda_{r_\mu} = \tilde{b}^\lambda_{i_r + \lambda} \] with the expectation values 0,1.

### 2.4 Differential Geometric Aspect

The set of operators \( \{ \gamma^r \} \) is the basis for all operator vectors of tangent section \( \hat{T}_{\Phi_p} \) of principle bundle with the base \( \hat{G} \) at the point \( \Phi_p = \Phi(\zeta(t)) \). The smooth field of tangent operator vector \( \hat{A}(\Phi(\zeta)) \) is a class of equivalency of the curves \( f(\Phi(\zeta)) = \Phi_p \). While, the operator differential \( \hat{d} A_p^t \) of the flux \( A_p^t : \hat{G} \rightarrow \hat{G} \) at the point \( \Phi_p \) with the velocity fields \( \hat{A}(\Phi(\zeta)) \) is defined by one parameter group of operator diffeomorphisms given for the curve \( \Phi(\zeta(t)) : R^1 \rightarrow \hat{G} \), provided \( \Phi(\zeta(0)) = \Phi_p \) and \( \hat{\Phi}(\zeta(0)) = \hat{A}_p \)
\[ \hat{d} A_p^t(\mathbf{A}) = \frac{d}{dt} \bigg|_{t=0} A_p^t(\Phi(\zeta(t))) = \hat{A}(\Phi(\zeta)) = \gamma^r A_p, \] (2.4.1)
where the \( \{A_p\} \) are the components of \( \hat{A} \) in the basis \( \{ \hat{r}^r \} \). According to eq.(2.4.1), in holonomic coordinate basis \( \hat{r}^r \rightarrow (\partial / \partial r^r(\zeta(t)))_p \) one gets \( A_p = \left. \frac{\partial \Phi \rho}{\partial \zeta^r} \right|_p \).

The operator tensor \( \hat{T} \) of \((n,0)\)-type at the point \( \hat{\Phi}_p \) is a linear function of the space \( \hat{T}_n^p = \hat{T}_{\Phi p} \otimes \cdots \otimes \hat{T}_{\Phi p} \), where \( \otimes \) stands for tensor product. It enables a correspondence between the element \( (\hat{A}_1, \ldots, \hat{A}_n) \) of \( \hat{T}_n^p \) and the number \( T(\hat{A}_1, \ldots, \hat{A}_n) \), provided by linearity. An explicit form of matrix element of operator tensor reads

\[
\frac{1}{\sqrt{n!}} < \chi^0 \parallel \hat{\Phi}(\zeta_1) \otimes \cdots \otimes \hat{\Phi}(\zeta_n) \parallel \chi > = \left\{ \begin{array}{c}
\sum_{\lambda\mu=1}^{N_{\lambda\mu}} \bar{c}(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}) F_{\lambda\mu}(\zeta_1) \wedge \cdots \wedge F_{\lambda\mu}(\zeta_n),
\end{array} \right.
\]

(2.4.2)

where \( \wedge \) stands for exterior product. So, constructing matrix elements of operator tensors of \( \hat{G} \) one produces the Cartan’s exterior forms on wave manifold \( \mathcal{G} \). Whence, the matrix elements of symmetric operator tensors equal zero.

The linear operator form of 1 degree \( \hat{\omega}^1 \) is a linear operator valued function on \( \hat{T}_{\Phi p} \), namely \( \hat{\omega}^1(\hat{A}_p) : \hat{T}_{\Phi p} \rightarrow \hat{R} \), where \( \hat{A}_p \in \hat{T}_{\Phi p} \), and the operator \( \hat{\omega}^1(\hat{A}) = < \hat{\omega}^1, \hat{A} > \in \hat{R} \) is corresponded to \( \hat{A}_p \) at the point \( \hat{\Phi}_p \), provided, according to eq.(2.3.26), with

\[
< \chi \parallel \hat{\omega}^1 \parallel \chi^0 > = \sum_{\lambda\mu=1}^{N_{\lambda\mu}} \sum_{r^{\lambda\mu}=1}^{r^{\lambda\mu}} \bar{c}(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}) \omega^1_{\lambda\mu},
\]

(2.4.3)

where \( \omega^1_{\lambda\mu} = \epsilon_{\lambda\mu} \omega^1(\zeta) \), the \( < \omega^1_{\lambda\mu}, \hat{A} > = \omega^1_{\lambda\mu}(\hat{A}) \) is a linear form on \( \hat{T}_p \), and

\[
\hat{\omega}^1(\lambda_1 \hat{A}_1 + \lambda_2 \hat{A}_2) = \lambda_1 \hat{\omega}^1(\hat{A}_1) + \lambda_2 \hat{\omega}^1(\hat{A}_2),
\]

\[
\forall \lambda_1, \lambda_2 \in \hat{R}, \quad \hat{A}_1, \hat{A}_2 \in \hat{T}_{\Phi p}.
\]

(2.4.4)

The set of all linear operator forms defined at the point \( \hat{\Phi}_p \) fill the operator vector space \( \hat{T}_{\Phi p}^* \) dual to \( \hat{T}_{\Phi p} \). While, the \( \{ \hat{r}^r \} \) serves as a basis for them. The operator \( n \) form is defined as the exterior product of operator 1 forms

\[
\hat{\omega}^n(\hat{A}_1, \ldots, \hat{A}_n) = (\hat{\omega}^1 \wedge \cdots \wedge \hat{\omega}^n)(\hat{A}_1, \ldots, \hat{A}_n) = \begin{array}{c}
\hat{\omega}^1(\hat{A}_1) \cdots \hat{\omega}^1(\hat{A}_n), \\
v \hat{\omega}^1(\hat{A}_n) \cdots \hat{\omega}^1(\hat{A}_n)
\end{array}
\]

(2.4.5)

Here as well as for the rest of this section we abbreviate the set of indices \( (\lambda_i, \mu_i, \alpha_i) \) by the single symbol \( i \). If \( \{ \hat{r}^i \} \) and \( \{ \hat{r}^i \} \) are dual bases respectively in \( \hat{T}_{\Phi p} \) and \( \hat{T}_{\Phi p}^* \), then the \( \{ \hat{\omega}^1 \otimes \cdots \otimes \hat{\omega}^i \otimes \cdots \otimes \hat{\omega}^1 \} \) will be the basis in operator space \( \hat{T}_q = \hat{T}_{\Phi p} \otimes \cdots \otimes \hat{T}_{\Phi p} \otimes \cdots \otimes \hat{T}_{\Phi p} \). Any operator tensor \( \hat{T} \in \hat{T}_q(\Phi) \) can be written

\[
\hat{T} = T_{r_1 \cdots r_p}(r_1, \ldots, r_p, s_1, \ldots, s_q) \hat{\omega}^1 \otimes \cdots \otimes \hat{\omega}^i \otimes \cdots \otimes \hat{\omega}^1 \otimes \cdots \otimes \hat{\omega}^1,
\]

where \( T_{r_1 \cdots r_p}(r_1, \ldots, r_p, s_1, \ldots, s_q) = T \left( \hat{\omega}^1 \otimes \cdots \otimes \hat{\omega}^i \otimes \cdots \otimes \hat{\omega}^1 \otimes \cdots \otimes \hat{\omega}^1 \right) \) are the components of \( \hat{T} \) in \( \{ \hat{\omega}^1 \} \) and \( \{ \hat{\omega}^i \} \). For any function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) of the ordinary class
of functions of $C^\infty$ smoothness on $\hat{G}$, according to eq.(2.4.1), one defines an operator differential

$$<\hat{d} f, \hat{A}> = (Af)^\wedge,$$

by means of smooth reflection

$$\hat{d} f : \mathcal{T}(\hat{G}) \to \hat{R} \quad \left(\mathcal{T}(\hat{G}) = \bigcup_{\Phi_p} \mathcal{T}_{\Phi_p}\right),$$

where $\langle \chi\|B\|\chi^0 \rangle = \sum_{\lambda,\mu=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu})B(r^{\lambda\mu})$ (see eq.(2.4.1)). Then

$$\langle \chi\|\hat{d} f, \hat{A}\|\chi^0 \rangle = \sum_{\lambda,\mu=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu}) < f, A >_{r^{\lambda\mu}} = \sum_{\lambda,\mu=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu})(Af)_{r^{\lambda\mu}}. \quad (2.4.6)$$

In coordinate basis $\langle d\hat{\Phi}^i, \hat{\partial} \rangle / \partial\Phi^j = \hat{\delta}^j_i$, provided $d\hat{\Phi}^i \equiv \hat{\Phi}^i$ and $\langle \chi\|\hat{\delta}^j_i\|\chi^0 \rangle = \sum_{\lambda,\mu=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu})\hat{\delta}^j_i$. We define the differential operator $n$ form $\hat{\omega}^n|_{\Phi_p}$ at the point $\Phi_p \in \hat{G}$ as the exterior operator $n$ form on tangent operator space $\mathcal{T}_{\Phi_p}$ of tangent operator vectors $\hat{A}_1, \ldots, \hat{A}_n$. That is, if the $\wedge \mathcal{T}_{\Phi_p}(\hat{G})$ means the exterior algebra on $\mathcal{T}_{\Phi_p}(\hat{G})$, then operator $n$ form $\hat{\omega}^n|_{\Phi_p}$ is an element of $n$-th degree out of $\wedge \mathcal{T}_{\Phi_p}$ depending upon the point $\Phi_p \in \hat{G}$. Hence $\hat{\omega}^n = \bigcup_{\Phi_p} \hat{\omega}^n|_{\Phi_p}$. Any differential operator $n$ form of dual operator space $\hat{\mathcal{T}}_{\Phi_p} \times \cdots \times \hat{\mathcal{T}}_{\Phi_p}$ may be written

$$\hat{\omega}^n = \sum_{i_1 \cdots i_n} \alpha_{i_1 \cdots i_n}(\Phi)d\hat{\Phi}^i_1 \wedge \cdots \wedge d\hat{\Phi}^i_n, \quad (2.4.7)$$

provided by the smooth differentiable functions $\alpha_{i_1 \cdots i_n}(\Phi) \in C^\infty$ and basis $d\hat{\Phi}^i \wedge \cdots \wedge d\hat{\Phi}^i_n = \sum_{\sigma \in S_n} sgn(\sigma)\gamma^\sigma_i \otimes \cdots \otimes \gamma^\sigma_n$. Any antisymmetric operator tensor of $(0,n)$ type reads

$$\hat{\mathcal{T}}^* = T_{i_1 \cdots i_n} \gamma^i_1 \otimes \cdots \otimes \gamma^i_n = \sum_{i_1 < \cdots < i_n} T_{i_1 \cdots i_n} d\hat{\Phi}^i_1 \wedge \cdots \wedge d\hat{\Phi}^i_n. \quad (2.4.8)$$

Let the $\hat{D}_1$ and $\hat{D}_2$ are two compact convex parallelepipeds in oriented $n$ dimensional operator space $\hat{R}^n$ and the $f : \hat{D}_1 \to \hat{D}_2$ is differentiable reflection of interior of $\hat{D}_1$ into $\hat{D}_2$ retaining an orientation, namely for any function $\varphi \in C^\infty$ defined on $\hat{D}_2$ it holds $\varphi \circ f \in C^\infty$ and $f^*\varphi(\hat{A}_p) = \varphi(f(\hat{A}_p))$, where $f^*$ is an image of function $\varphi (f(\hat{A}_p))$ on $\hat{D}_1$ at the point $\Phi_p$. Hence, the function $f$ induces a linear reflection $\hat{d} f : \hat{\mathcal{T}}(\hat{D}_1) \to \hat{\mathcal{T}}(\hat{D}_2)$ as an operator differential of $f$ implying $\hat{d} f (\hat{A}_p) = \hat{A}_p(\varphi \circ f)$ for any operator vector $\hat{A}_p \in \hat{\mathcal{T}}_{\Phi_p}$ and for any function $\varphi \in C^\infty$ defined in the neighbourhood of $\Phi_p = f(\Phi_p)$. If the function $f$ is given in the form $\Phi^i = \varphi^i(\hat{A}_p)$ and $\hat{A}_p = (A^i \hat{\partial} / \partial\Phi^i)_p$, then in terms
of local coordinates one gets $\left(\hat{d}f\right)\hat{A}_p = A^i \left(\frac{\partial \Phi^j}{\partial \Phi^i}\right)_p \left(\frac{\hat{\partial}}{\partial \Phi^j}\right)_p$. So, if $f_1 : \hat{D}_1 \to \hat{D}_2$ and $f_2 : \hat{D}_2 \to \hat{D}_3$ then $\hat{d} \left(f_2 \circ f_1\right) = \hat{d} f_2 \circ \hat{d} f_1$. The differentiable reflection $f : \hat{D}_1 \to \hat{D}_2$ induces the reflection $\hat{f}^* : \hat{T}^* \left(\hat{D}_2\right) \to \hat{T}^* \left(\hat{D}_1\right)$ conjugated to $\hat{f}_*$. The latter is the operator differential of $f$, while

\[
< \hat{f}^* \hat{\omega}^1, \hat{A} >_{\Phi_p} = < \hat{\omega}^1, \hat{f}_* \hat{A} >_{\left(f(\Phi_p)\right)} ,
\]

where $\hat{A} \big|_{f(\Phi_p)} = (\hat{d} f) \hat{A}_p$ and $\hat{\omega}^1 \in \hat{T}^* \big|_{f(\Phi_p)}$. Hence $\hat{f}^* \left(\hat{d} \varphi\right) = \hat{d} \left(\hat{f}^* \varphi\right)$ and

\[
\hat{f}^* \left(T \left(\hat{A}_1, \ldots, \hat{A}_n\right)\right)_{\Phi_p} = T \left(\hat{f}_* \hat{A}_1, \ldots, \hat{f}_* \hat{A}_n\right)_{f(\Phi_p)},
\]

\[
T \left(\hat{f}_* \hat{\omega}_1, \ldots, \hat{f}_* \hat{\omega}_n\right)_{\Phi_p} = \hat{f}_* T \left(\hat{\omega}_1, \ldots, \hat{\omega}_n\right)_{f(\Phi_p)}.
\]

For any differential operator $n$ form $\hat{\omega}^n$ on $\hat{D}_2$ the reflection $f$ induces the operator $n$ form $\hat{f}^* \hat{\omega}^n$ on $\hat{D}_1$,

\[
\left(\hat{f}^* \hat{\omega}^n\right) \left(\hat{A}_1, \ldots, \hat{A}_n\right)_{\Phi_p} = \hat{f}_* \hat{\omega}^n \left(\hat{f}_* \hat{A}_1, \ldots, \hat{f}_* \hat{A}_n\right)_{f(\Phi_p)}. \tag{2.4.11}
\]

If $\hat{\omega}^1 = \alpha_i d\Phi^i$ then $\hat{f}^* \left(\alpha_i^j d\Phi^i\right) = \alpha_i^j \frac{\partial \Phi'^i}{\partial \Phi^j} d\Phi'^i$. This can be extended up to $\hat{\omega}^n \to \hat{\omega}^n$

\[
\hat{f}^* \left(\sum_{i_1 < \cdots < i_n} T'_{i_1 \cdots i_n} d\Phi'^{i_1} \wedge \cdots \wedge d\Phi'^{i_n}\right) = \sum_{i_1 < \cdots < i_n} T'_{i_1 \cdots i_n} \frac{\partial \Phi'^{i_1}}{\partial \Phi^{i_1}} \cdots \frac{\partial \Phi'^{i_n}}{\partial \Phi^{i_n}} d\Phi'^{i_1} \wedge \cdots \wedge d\Phi'^{i_n},
\]

namely $\hat{f}^* \hat{\omega}^n = J_{\Phi} \hat{\omega}^n = (det df) \hat{\omega}^n$, where $J_{\Phi}$ is the Jacobian of reflection $J_{\Phi} = \left\| \frac{\partial \Phi'^i}{\partial \Phi^j} \right\|$.

While

\[
\left(\hat{f}_1 \circ \hat{f}_2\right)^* = \hat{f}_1^* \circ \hat{f}_2^* , \quad \hat{f}^* \left(\hat{\omega}_1 \wedge \hat{\omega}_2\right) = \hat{f}^* \left(\hat{\omega}_1\right) \wedge \hat{f}^* \left(\hat{\omega}_2\right).
\]

We may consider the integration of operator $n$ form implying $\int_{\hat{D}_1} \hat{f}^* \hat{\omega}^n = \int_{\hat{D}_2} \hat{\omega}^n$. In general, let the $\hat{D}_1$ is a limited convex $n$ dimensional parallelepiped in $n$ dimensional operator space $\hat{R}^n$. One defines the $n$ dimensional $\hat{i}$-th piece of integration path $\tilde{\hat{\sigma}}^i$ in $\hat{G}$ as $\hat{\sigma}^i = \left(\hat{D}_i, f_i, Or_i\right)$, where $\hat{D}_i \in \hat{R}^n$, $f_i : \hat{D}_i \to \hat{G}$ and the $Or_i$ is an orientation of $\hat{R}^n$. Then, the integral over the operator $n$ form $\hat{\omega}^n$ along the operator $n$ dimensional chain $\hat{c}_n = \sum m_i \hat{\sigma}^i$ may be written

\[
\int_{\hat{c}_n} \hat{\omega}^n = \sum m_i \int_{\hat{\sigma}^i} \hat{\omega}^n = \sum m_i \int_{\hat{D}_i} \hat{f}^* \hat{\omega}^n ,
\]

where the $m_i$ is a multiple number. Taking into account the eq.(2.4.3), the matrix element yields

\[
< \chi \| \int_{\hat{c}_n} \hat{\omega}^n \| \chi^0 > \to \left\{ \sum_{\lambda_\mu=1}^2 \sum_{\gamma_\mu=1}^{N_{\lambda_\mu}} \left( \prod_{\lambda_\mu=1}^2 \sum_{r_{\lambda_\mu}^1}^{r_{\lambda_\mu}^\mu} \right) \right. \int_{\hat{D}_i} \hat{f}^* \hat{\omega}^n \left( r_{\lambda_\mu}^1, \ldots, r_{\lambda_\mu}^\mu \right) \left. \right) < \chi \| \chi^0 > \right., \tag{2.4.13}
\]
Next we employ the analog of exterior differentiation. We define the operator \((n + 1)\) form \(\hat{d}\hat{\omega}^n\) on \((n + 1)\) operator vectors \(\hat{A}_1, \ldots, \hat{A}_{n+1} \in \hat{\mathbf{T}}_{\Phi_p}\) by considering diffeomorphic reflection \(f\) of the neighbourhood of the point 0 in \(\hat{\mathbb{R}}^n\) into neighbourhood of the point \(\Phi_p\) in \(\hat{G}\). The prototypes of operator vectors \(\hat{A}_1, \ldots, \hat{A}_{n+1} \in \hat{\mathbf{T}}_{\Phi_p}(\hat{G})\) at the operator differential of \(f\) belong to tangent operator space \(\hat{\mathbb{R}}^n\) in 0. Then, the prototypes are the operator vectors \(\hat{\xi}_1, \ldots, \hat{\xi}_{n+1} \in \hat{\mathbb{R}}^n\). Let \(f\) reflects the parallelepiped \(\Pi^n\), stretched over the \(\hat{\xi}_1, \ldots, \hat{\xi}_{n+1}\), into \((n + 1)\) dimensional piece \(\hat{\Pi}\) on \(\hat{G}\). While the border of \(n\) dimensional chain \(\partial\Pi\) in \(\mathbb{R}^{n+1}\) defined as follows: the pieces \(\hat{\sigma}^i\) of the chain \(\partial\Pi\) are \(n\) dimensional facets \(\partial\Pi_i\) of parallelepiped \(\partial\Pi\) with the reflections embedding the facets into \(\mathbb{R}^{n+1}\): \(f_i : \hat{\Pi}_i \to \hat{\mathbb{R}}^{n+1}\), and the orientations \(\text{Ori}_i\) defined \(\partial\Pi = \sum \hat{\sigma}^i\), \(\hat{\sigma}^i = (\hat{\Pi}_i, f_i, \text{Ori}_i)\).

Considering the curvilinear parallelepiped

\[
F(A_1, \ldots, A_n) = \int_{\partial\Pi} \hat{\omega}^n,
\]

one may state that the unique operator of \((n + 1)\)-form \(\hat{\Omega}\) exists on \(\hat{\mathbf{T}}_{\Phi_p}\), which is the principle \((n + 1)\) linear part in 0 of integral over the border of \(F(A_1, \ldots, A_n)\), namely

\[
F(\varepsilon A_1, \ldots, \varepsilon A_n) = \varepsilon^{n+1} \hat{\Omega}(A_1, \ldots, A_{n+1}) + O(\varepsilon^{n+1}),
\]

where the \(\hat{\Omega}\) is independent of the choice of coordinates used in definition of \(F\). The prove of it is similar to those of corresponding theorem of differential geometry [14]. If in local coordinates \(\hat{\omega}^n = \sum_{i_1 \prec \cdots \prec i_n} T_{i_1 \cdots i_n} d\Phi_{i_1} \wedge \cdots \wedge d\Phi_{i_n}\), then

\[
\hat{\Omega} = \hat{d}\hat{\omega}^n = \sum_{i_1 \prec \cdots \prec i_n} \hat{d}T_{i_1 \cdots i_n} d\Phi_{i_1} \wedge \cdots \wedge d\Phi_{i_n}.
\]

The operator of exterior differential \(\hat{d}\) commutes with the reflection \(f : \hat{G} \to \hat{G}\)

\[
\hat{d} (\hat{f}^* \hat{\omega}^n) = \hat{f}^* (\hat{d}\hat{\omega}^n).
\]

So define the exterior differential by operator \((n+1)\) form

\[
\hat{d}\hat{\omega}^n = \sum_{i_0} \frac{\partial T_{i_1 \cdots i_n}}{\partial \Phi_{i_0}} d\Phi_{i_0} \wedge d\Phi_{i_1} \wedge \cdots \wedge d\Phi_{i_n} = \sum_{i_1 \prec \cdots \prec i_n} (\hat{d}T_{i_1 \cdots i_n}) \wedge d\Phi_{i_1} \wedge \cdots \wedge d\Phi_{i_n},
\]

then

\[
< \chi \parallel \hat{d}\hat{\omega}^n \parallel \chi^0 \longrightarrow \sum_{i_1 \prec \cdots \prec i_n} \sum_{\lambda, \mu=1}^{2} \sum_{n_{\lambda, \mu}^N} \tilde{c}(r_1^\lambda, \ldots, r_n^\lambda)(dT(r_1^\lambda, \ldots, r_n^\lambda))_{i_1 \cdots i_n} \wedge d\Phi_{i_1}^\lambda \wedge \cdots \wedge d\Phi_{i_n}^\lambda.
\]

We may draw a conclusion that the matrix element of any geometric object of operator manifold \(\hat{G}\) yields corresponding geometric object of wave manifold \(\mathcal{G}\).
3 Primordial Structures and Link Establishing Processes

To facilitate the physical picture and provide sufficient background it seems worth to bring few formal matters in concise form which one will have to know in order to understand the general structure of our approach without undue hardship. Here we only outline briefly the relevant steps. In the mean time we refer to [8] for more detailed justification of some of the procedures and complete exposition.

Before proceeding further, it is profitable to define the pulsating gauge functions and fields denoted by wiggles as follows:

1. An invariant with respect to the coordinate transformations function \( \tilde{W}(x) \) defined on the space \( M \) (\( x \in M \)) is called the pulsating gauge function if it undergoes local gauge transformations

\[
\tilde{W}'(x) = U(x)\tilde{W}(x).
\] (3.0.1)

Here \( U(x) \) is the element of some simple Lie group \( G \) the generators of which imply the algebra \([F^a, F^b] = iC^{abc}F^c\), where \( C^{abc} \) are wholly antisymmetric structure constants.

2. A smooth function \( \tilde{\Phi}(\tilde{W}(x)) \) belonged to some representation of the group \( G \), where the generators are presented by the matrices \( T^a \), is called the pulsating field if under the transformation eq.(3.0.1) it transforms

\[
\tilde{\Phi}' \equiv \tilde{\Phi}(\tilde{W}'(x)) = U(x)\tilde{\Phi}(\tilde{W}(x)).
\] (3.0.2)

Let \( L_0(\Phi, \partial \Phi) \) is the invariant Lagrangian of free field \( \Phi \) defined on \( M \). Then, a simple gauge invariant Lagrangian of the pulsating field \( \tilde{\Phi} \) can be written

\[
L = \tilde{\Phi}^+ \tilde{\Phi} L_0(\Phi, \partial \Phi),
\] (3.0.3)

which reduces to

\[
L \equiv L \left( \tilde{\Phi}, \tilde{\Phi}^\dagger \right) = L_0 \left( \tilde{\Phi}(\tilde{W}(x)), D\tilde{\Phi}(\tilde{W}(x)) \right).
\] (3.0.4)

Here we have noticed that due to eq.(3.0.2) and eq.(3.0.1) \( \tilde{\Phi}(\tilde{W}) = \tilde{W}\Phi \), and introduced the covariant derivative \( \tilde{D}\Phi \equiv D\tilde{\Phi}(W) = \tilde{W}\partial\Phi \). Whence

\[
D = \partial - igT^aW^a, \quad T^aW^a = -i\frac{1}{g}\partial \ln \tilde{W}, \quad D\tilde{W} = (D\tilde{W})^+ = 0,
\] (3.0.5)

where \( W^a \) is the gauge field, \( g \) is the coupling constant. Thus, the conventional matter fields interacting by gauge fields are the pulsating fields.

3.1 The Regular Primordial Structures

In [8] we have chosen a simple setting and considered the primordial structures designed to possess certain physical properties satisfying the stated general rules. These structures are the substance out of which the geometry and particles are made. We distinguish \( \eta \)- and \( u \)-types primordial structures involved in the linkage establishing processes occurring between the structures of different types. Let us recall that the \( \eta \)-type structure may accept the linkage only from \( u \)-type structure, which is described by the link function \( \Psi(s) \) belonging to the ordinary class of functions of \( C^\infty \) smoothness, where \( s \equiv \eta = \)
Thus, under local gauge transformations

$$s' = e^{-i\alpha} s, \quad \partial \alpha \neq 0,$$

the link function $\Psi(s)$ transforms

$$\Psi(s') = e^{-i\alpha}\Psi(s),$$

and the Lagrangian eq.(3.0.3) is invariant under gauge transformations. It includes the covariant derivative $D(s) = \partial + igb(s)$ and gauge field $b(s) = \frac{i}{g}\partial \ln s$ undergone gauge transformations $b(s') = b(s) + \frac{1}{g}\partial \alpha$. Then $\Psi(s) = s\Psi_u = e^{i\alpha}\Psi_{i(\lambda\alpha)} (i = \eta, u)$, where the eq.(2.1.9) holds

$$\Psi_{i(\pm\alpha)}(\eta, p) = u_{\eta} (\pm\alpha)\Psi^\pm_{\eta}(\eta, p_{\eta}), \quad \Psi_{i(\pm\alpha)}(u, p) = u_{i(\pm\alpha)}\Psi^\pm_{i}(u, p_{u}), \quad (3.1.1)$$

a bispinor $\Psi^\pm_i$ is the invariant state wave function of positive or negative frequencies, $p_i$ is the corresponding link momentum. Thus, a primordial structure can be considered as a fermion found in external gauge field $b(s)$.

The simplest system made of two structures of different types becomes stable only due to the stable linkage, namely

$$\begin{vmatrix} p_{\eta} \end{vmatrix} = (p_{\eta(\lambda\alpha)}, p_{\eta(\lambda\alpha)})^{1/2} = \begin{vmatrix} p_{u} \end{vmatrix} = (p_{u(\lambda\alpha)}, p_{u(\lambda\alpha)})^{1/2}. \quad (3.1.2)$$

Otherwise they are unstable. There is not any restriction on the number of primordial structures of both types getting into the link establishing processes simultaneously. In the stable system the link stability condition must be held for each linkage separately.

The persistent processes of creation and annihilation of the primordial structures occur in different states $s, s', s'', ...$. The ”creation” of structure in the given state $(s)$ is due to its transition to this state from other states $(s', s'', ...)$, while the ”annihilation” means a vice versa. Satisfying eq.(3.1.2) the primordial structures from the given state as well as different states can establish a stable linkage. Among the states $(s, s', s'', ...)$ there is a lowest one $(s_0)$, in which all structures are regular. That is, they are in free (pure) state and described by the plane wave functions $\Psi_{i(\pm\alpha)}(\eta, p_{\eta})$ or $\Psi^\pm_{i}(u, p_{u})$ defined respectively on flat manifolds $G$ and $G$. The index $(f)$ specifies the points of corresponding flat manifolds $\eta_f \in G, u_f \in G$. For example, in accordance with subsec.2.2, the equation of regular structure $\Psi(s_+)$ $(s = s_+ + s_-)$ reads

$$[i\gamma_f(\partial + igb(s_+)) - m]\Psi(s_+) = 0,$$

the matrices $\gamma_f$ are given in eq.(3.3.3). Whence the equation of plane wave function $\Psi^+_p$ of positive frequencies stems

$$(i\gamma_f\partial - m)\Psi^+_p = 0.$$
3.2 The Distorted Primordial Structures

In all higher states the primordial structures are distorted (interaction states) and described by distorted link functions defined on distorted manifolds \( \tilde{G} \) and \( \tilde{\Sigma} \). The distortion \( G \to \tilde{G} \) with hidden Abelian local group \( G = U^{loc}(1) = SO^{loc}(2) \) and one dimensional trivial algebra \( \tilde{g} = R^1 \) is considered in \([8,15]\). It involves a drastic revision of a role of local internal symmetries in the concept of curved geometry. Under the reflection of fields and their dynamics from Minkowski space to Riemannian a standard gauge principle of local internal symmetries is generalized. The gravitation gauge group is proposed, which is generated by hidden local internal symmetry. This suggests an opportunity for the unification of all interactions on an equal footing.

Our scheme is implemented as follows: Considering the principle bundle \( p : E \to G \) the basis \( e^f \) is transformed \( e = De^f \), under massless gauge distortion field \( \alpha_f \) associated with \( U^{loc}(1) \). The matrix \( D \) is in the form \( D = C \otimes R \), where the distortion transformations \( O_{(\lambda \alpha)} = C_{(\lambda \alpha)} O_\tau \) and \( \sigma_{(\lambda \alpha)} = R^\alpha_{(\lambda \alpha)} \sigma_\beta \) are defined. Here \( C_{(\lambda \alpha)} = \delta^\tau_\alpha + \kappa a_{(\lambda \alpha)^*} \delta^\tau_\lambda \), but \( R \) is a matrix of the group \( SO(3) \) of ordinary rotations of the planes involving two arbitrary basis vectors of the spaces \( R^3_{\pm} \) around the orthogonal third axes. The rotation angles are determined from the constraint imposed upon distortion transformations that a sum of distorted parts of corresponding basis vectors \( O_\lambda \) and \( \sigma_\beta \) should be zero at given \( \lambda \)

\[
< O_{(\lambda \alpha)}, O_\tau >_{\tau \neq \lambda} + \frac{1}{2} \varepsilon_{\alpha \beta \gamma} < \frac{\sigma_{(\lambda \beta)}}{\sigma_{(\lambda \gamma)}} \frac{\sigma_\gamma}{\sigma_\beta} > = 0, \quad (3.2.1)
\]

where \( \varepsilon_{\alpha \beta \gamma} \) is an antisymmetric unit tensor. Thereupon \( \tan \theta_{(\lambda \alpha)} = -\kappa a_{(\lambda \alpha)} \), where \( \theta_{(\lambda \alpha)} \) is the particular rotation around the axis \( \sigma_\alpha \). Since the \( R \) is independent of the sequence of rotation axes, then it implies the mean value \( R = \frac{1}{6} \sum_{i \neq j \neq k} R^{(ijk)} \), where \( R^{(ijk)} \) the matrix of rotations occurring in the given sequence \( (ijk) \) \( (i,j,k = 1,2,3) \). The field \( \alpha_f \) is due to the distortion of basis pseudovector \( O_\lambda \), while the distortion of \( \sigma_\alpha \) follows from eq.(3.2.2).

Next we construct the diffeomorphism \( G \to \tilde{G} \) and introduce the invariant action of the fields. The passage from six dimensional curved manifold \( \tilde{G} \) to four dimensional Riemannian geometry \( R^4 \) is straightforward by making use of reduction of three time components \( e_{0\alpha} = \frac{1}{\sqrt{2}}(e_{(+\alpha)} + e_{(-\alpha)}) \) of basis sixvector \( e_{(\lambda \alpha)} \) to the single one \( e_0 \) in the given universal direction, which merely fixed a time coordinate. Actually, since Lagrangian of the fields defined on \( \tilde{G} \) is a function of scalars, namely, \( A_{(\lambda \alpha)}B_{(\lambda \alpha)} = A_{0\alpha}B^{0\alpha} + A_\alpha B^\alpha \), so taking into account that \( A_{0\alpha}B^{0\alpha} = A_{0\alpha} < e^{0\alpha}, e^{0\beta} > B_{0\beta} = A_0 < e^0, e^0 > B_0 = A_0 B^0 \), one readily may perform the required passage. In this case, instead of eq.(2.2.5), one has

\[
d\zeta^2 = d\eta^2 - du^2 = 0, \quad d\eta^2 \bigg|_{6 \to 4} \equiv d s^2 = g_{\mu\nu} dx^\mu dx^\nu = d u^2 = inv. \quad (3.2.2)
\]

3.3 Reflection of the Fermi Fields

Within this approach we may consider the reflection of the Fermi fields and their dynamics from the flat manifold \( G \) to distorted manifold \( \tilde{G} \), and vice versa. Thereat we construct a diffeomorphism \( u(\alpha_f) : G \to \tilde{G} \), where the holonomic functions \( u(\alpha_f) \) satisfy defining relation

\[
e\psi = e^\lambda + \chi^\lambda (B_f). \quad (3.3.1)
\]
Here $e^f$ and $e$ are the basis vectors on $G$ and $\tilde{G}$. The $\psi$ is taken to denote $\psi \equiv \frac{\partial u}{\partial u_f}$. The covector
\[
\chi_{(\tau\beta)}(B_f) = e_{(\lambda\alpha)}\chi^{(\lambda\alpha)}_{(\tau\beta)} - \frac{1}{2}e_{(\lambda\alpha)} \int_0^{u_f} (\partial^f_{\alpha}\bar{D}^{(\lambda\alpha)}_{\tau\beta} - \partial^f_{\alpha}\bar{D}^{(\lambda\alpha)}_{\tau\beta}) du_f^{(\beta)}
\]
(3.3.2)
realizes the coordinates $u$ by providing a criteria of integration and undegeneration [16,17].

A Lagrangian $L(x)$ of fields $\Psi(u)$ may be obtained under the reflection from a Lagrangian $L_f(u_f)$ of corresponding shadow fields $\Psi_f(u_f)$ and vice versa. The $\Psi_f(u_f)$ is defined as the section of vector bundle associated with the primary gauge group $G$ by reflection $\Psi_f : G \to E$ that $p\Psi_f(u_f) = u_f$, where $u_f \in G$ is a point of flat manifold $G$ (specified by index $(f)$). The $\Psi_f$ takes value in standard fiber $F_{u_f}$ upon $u_f : p^{-1}(U(f)) = U(f) \times F_{u_f}$, where $U(f)$ is a region of base of principle bundle upon which an expansion into direct product $p^{-1}(U(f)) = U(f) \times G$ is defined. The fiber is Hilbert vector space on which a linear representation $U_f$ of the group $G$ is given. Respectively $\Psi(u) \subset F_u$, where $F_u$ is the fiber upon $u : p^{-1}(U) = U \times F_u$, $U$ is the region of base $\tilde{G}$. Thus, the reflection of bispinor fields may be written down
\[
\Psi(u) = R(B_f)(\Psi_f(u_f)), \quad \bar{\Psi}(u) = \bar{\Psi}_f(u_f)\bar{R}^+(B_f),
\]
\[
g(u)\nabla\Psi(u) = S(B_f)R(B_f)\gamma_f D\Psi_f(u_f),
\]
\[
(\nabla\bar{\Psi}(u)) g(u) = S(B_f) (D\bar{\Psi}_f(u_f)) \gamma_f \bar{R}^+(B_f).
\]

Reviewing the notation $B(u_f) = T^aB^a(u_f)$ is the gauge field of distortion with the values in Lie algebra of group $G$, $R(B_f)$ is the reflection matrix (see eq.(3.3.6)), $\bar{R} = \gamma^0R\gamma^0$, $D = \partial^f - igB$, $g^{(\lambda\alpha)}(\theta) = V^{(\lambda\alpha)}_{(i,l)}(\theta)\gamma^i_f$, $V^{(\lambda\alpha)}_{(i,l)}(\theta)$ are congruence parameters of curves (Latin indices refer to tetrad components). The matrices $\gamma^f_{(\pm\alpha)} = \frac{1}{\sqrt{2}}(\gamma^0\sigma^\alpha \pm \gamma^\alpha)$, $\gamma^0, \gamma^\alpha$ are Dirac matrices. $\nabla$ is covariant derivative defined on $\tilde{G}$: $\nabla = \partial + \Gamma$, where the connection $\Gamma(\theta)$ in terms of Ricci rotation coefficients reads $\Gamma_{(\lambda\alpha)}(\theta) = \frac{1}{4}\Delta_{(\lambda\alpha)(i,l)}(m,p)\gamma^i_f\gamma^f_{(m,p)}\gamma_{(i,l)}$, $\Gamma_{(\lambda\alpha)}(\theta) = \frac{1}{4}\Delta_{(\lambda\alpha)(i,l)(m,p)}\gamma^i_f\gamma^f_{(m,p)}\gamma_{(i,l)}$.

According to the general gauge principle [8,15], the physical system of the fields $\Psi(u)$ is required to be invariant under the finite local gauge transformations
\[
\Psi'(u) = U_R\Psi(u),
\]
\[
(g(u)\nabla\Psi(u))' = U_R(g(u)\nabla\Psi(u)), \quad U_R = R(B'_f)U_fR^{-1}(B_f),
\]
(3.3.4)
of the Lie group of gravitation $G_R(\varnothing U_R)$ generated by $G$, where the gauge field $B_f(u_f)$ is transformed under $G$ in standard form. The physical meaning of the general principle is as follows: one has conventional $G$-gauge theory on flat manifold in terms of curvilinear coordinates if curvature tensor is zero, to which the zero vector eq.(3.3.2) is corresponded. Otherwise it yields the gravitation interaction.

\[1\] I wish to thank S.P.Novikov for valuable discussion of this point
Out of a set of arbitrary curvilinear coordinates in $\tilde{G}$ the real curvilinear coordinates may be distinguished, which satisfy eq.(3.3.1) under all possible Lorentz and gauge transformations. There is a single-valued conformity between corresponding tensors with various suffixes on $\tilde{G}$ and $G$. While, each index transformation is incorporated with function $\psi$.

The transformation of real curvilinear coordinates $u \rightarrow u'$ is due to some Lorentz ($\Lambda$) and gauge ($B_f \rightarrow B'_f$) transformations

$$\frac{\partial u'}{\partial u} = \psi(B'_f)\psi(B_f)\Lambda.$$  \hspace{1cm} (3.3.5)

There would then exist preferred systems and group of transformations of real curvilinear coordinates in $\tilde{G}$. The wider group of transformations of arbitrary curvilinear coordinates in $\tilde{G}$ would then be of no consequence for the field dynamics. A straightforward calculation gives the reflection matrix

$$R(u, u_f) = R_f(u_f)R_g(u) = \exp \left[ -i\Theta_f(u_f) - \Theta_g(u) \right],$$

where

$$\Theta_f(u_f) = g \int_0^{u_f} B(u_f) du_f, \quad \Theta_g(u) = \frac{1}{2} \int_0^u R_f^+ \{g\Gamma R_f, g\} du.$$  \hspace{1cm} (3.3.6)

Then

$$S(B_f) = \frac{1}{8K} \psi \left\{ \tilde{R}^+ g R, \gamma^0 \right\} = inv,$$

where

$$K = \tilde{R}^+ R = \tilde{R}^+_g R_g = 1.$$  \hspace{1cm} (3.3.7)

The Lagrangian of shadow Fermi field may be written

$$L_f(u_f) = J_\psi L(u) =$$

$$J_\psi \left\{S(B_f) \frac{i}{2} \left[ \Psi_f(u_f)\gamma_0 D\Psi_f(u_f) - (D\bar{\Psi}_f(u_f))\gamma_0 \Psi_f(u_f) \right] - m\bar{\Psi}_f(u_f)\Psi_f(u_f) \right\}.$$  \hspace{1cm} (3.3.11)

provided by $J_\psi = \|\psi\| \sqrt{-g} \equiv \left( 1 + 2\| < e^f, \chi^f > + \| < \chi^f, \chi^f > \right)^{1/2}$. The Lagrangian $L(u)$ of the field $\Psi(u)$ reads

$$\sqrt{-g} L(u) = \frac{\sqrt{-g}}{2} \left\{ -i\bar{\Psi}(u)g(\partial_u - \Gamma)\Psi(u) + i\bar{\Psi}(u)(\partial_u - \bar{\Gamma})g\Psi(u) + 2m\bar{\Psi}(u)\Psi(u) \right\},$$

yielding the field equations

$$\left[ ig(\partial_u - \Gamma) - m \right] \Psi(u) = 0, \quad \bar{\Psi}(u) \left[ i(\partial_u - \bar{\Gamma})g - m \right] = 0.$$  \hspace{1cm} (3.3.12)

The solution enables to write down the relation between the wave functions of distorted and regular structures [8]

$$\Psi^\lambda_{u\lambda}(\theta_+) = f_{(+)}(\theta_+)\Psi^\lambda_u, \quad \Psi^\lambda_{u\lambda}(\theta_-) = \Psi^\lambda_{u\lambda} f_{(-)}(\theta_-).$$  \hspace{1cm} (3.3.14)
The $\Psi_{u,\lambda}^\lambda(\Psi_u)$ is the plane wave function of regular ordinary structure (antistructure) and

$$f_+(\theta_{+k}) = e^{\chi_R(\theta_{+k}) - i\chi_J(\theta_{+k})}, \quad f_-(\theta_{-k}) = f_+(\theta_{+k})|_{\theta_{+k} = \theta_{-k}}, \quad (3.3.15)$$

where the $\chi_R$ and $\chi_J$ are given in Appendix.

Next we shall admit that the $\eta$-type (fundamental) regular structure can not directly form a stable system with the regular $u$-type (ordinary) structures. Instead of it the $\eta$-type regular structure forms a stable system with the infinite number of distorted ordinary structures, where the link stability condition held for each linkage separately. Such structures take part in realization of flat manifold $G$ (subsec.2.2). The laws regarding to this apply in use of functions of distorted ordinary structures

$$\Psi_{u,\lambda}^{(\lambda\alpha)}(\theta_+) = u^{(\lambda\alpha)}_{(\lambda)} \Psi_u^\lambda(\theta_+), \quad \Psi_{u,\lambda}^{(\lambda\alpha)}(\theta_-) = u^{(\lambda\alpha)}_{(\lambda)} \Psi_u^\lambda(\theta_-), \quad (3.3.16)$$

where $u \in \bar{G}_u$. We employ the wave packets constructed by superposition of these functions furnished by generalized operators of creation and annihilation as the expansion coefficients

$$\hat{\Psi}_u(\theta_+) = \sum_{\pm} \int \frac{d^3p_0}{(2\pi)^{3/2}} \left( \hat{\gamma}^k_{u(\mp\alpha)} \Psi_{u,\lambda}^{(+\alpha)}(\theta_{+k}) + \gamma^k_{u(-\alpha)} \Psi_{u,\lambda}^{(-\alpha)}(\theta_{+k}) \right),$$

$$\bar{\Psi}_u(\theta_-) = \sum_{\pm} \int \frac{d^3p_0}{(2\pi)^{3/2}} \left( \hat{\gamma}^k_{u(+\alpha)} \Psi_{u,\lambda}^{(-\alpha)}(\theta_{-k}) + \gamma^k_{u(-\alpha)} \Psi_{u,\lambda}^{(+\alpha)}(\theta_{-k}) \right), \quad (3.3.17)$$

where as usual the summation is extended over all dummy indices. The matrix element of anticommutator of generalized expansion coefficients reads

$$\langle \chi_- | \{ \hat{\gamma}_{u,k}^{(+\alpha)}(p, s), \hat{\gamma}_{u,k'}^{(\pm\beta)}(p', s') \} | \chi_- \rangle = -\delta_{ss'}\delta_{\alpha\beta}\delta_{kk'}\delta^3(\vec{p} - \vec{p}'). \quad (3.3.18)$$

The wave packets eq. (3.3.17) yield the causal Green’s function $G^\theta_{u,F}(\theta_+ - \theta_-)$ of distorted ordinary structure. Geometry realization requirement (eq.(2.2.4)) now must be satisfied for each ordinary structure in terms of

$$G^\theta_{u,F}(0) = \lim_{\theta_+ \to \theta_-} G^\theta_{u,F}(\theta_+ - \theta_-) = G^\theta_{\eta,F}(0) = \lim_{\eta_f' \to \eta_f} G^\theta_{\eta_f,F}(\eta_f' - \eta_f). \quad (3.3.19)$$

They are valid if following relations hold for each distorted ordinary structure:

$$\sum_k \Psi_{u}(\theta_{+k}) \bar{\Psi}_{u}(\theta_{-k}) = \sum_k \Psi_{u}(\theta_{+k}) \bar{\Psi}_{u}(\theta_{-k}) = \cdots = inv. \quad (3.3.20)$$

Namely, the distorted ordinary structures only in permissible combinations realize the geometry in a stable system. Below, in schematic manner we exploit the background of the Colour Confinement and Gauge principles.

### 3.4 Quarks and Colour Confinement

We may think of the function $\Psi_{u,\lambda}^\lambda(\theta_{+k})$ at fixed $(k)$ as being $u$-component of bispinor field of quark $q_k$, and of $\bar{\Psi}_{u,\lambda}(\theta_{-k})$ - an $u$-component of conjugated bispinor field of antiquark.
Hence, the following transformations may be implemented upon distorted ordinary structures in the case of local and global rotations we respectively distinguish two types of quarks: local $q_k$ and global $q'_k$, which will be in use in the next Part II as the local and global subquark fields. Thus, the quark is a fermion with the integer spin and certain colour degree of freedom. There are exactly three colours. The rotation through the angle $\theta_{+k}$ yields a total quark field defined on the flat manifold $G = G_u \oplus G_\eta$

$$q_k(\theta) = \Psi(\theta_{+k}) = \Psi^0(\eta_u)\Psi(\theta_{+k})$$

(3.4.1)

where $\Psi^0$ is a plane wave defined on $G_\eta$. According to eq.(3.3.14), one gets

$$q_k(\theta) = \Psi^0 q(\theta) = q(\theta)\Psi^0, \quad q(\theta) \equiv f_{(+)}(\theta_{+k})\Psi^0,$$

(3.4.2)

where $\Psi^0$ is a plane wave, $q(\theta)$ and $q(\theta)$ may be considered as the quark fields with the same quantum numbers defined respectively on flat manifolds $G_u$ and $G_\eta$. By making use of the rules stated in subsec.2.1 one may readily return to Minkowski space $G_\eta \rightarrow M^4$.

In the sequel, a conventional quark field defined on $M^4$ will be ensued $q(\theta) \rightarrow q_k(x), \quad x \in M^4$. Due to eq.(3.3.20) and eq.(3.4.1) they imply

$$\sum_k q_kq_k = \sum_k q'_kq'_k = \cdots = \text{inv},$$

(3.4.3)

and

$$\sum_k f_{(+)}(\theta_{+k})f_{(-)}(\theta_{-k}) = \sum_k f'_{(+)}(\theta'_{+k})f'_{(-)}(\theta'_{-k}) = \cdots = \text{inv}.$$  

(3.4.4)

The eq.(3.4.3) utilizes the idea of Colour (Quark) Confinement principle: the quarks emerge in the geometry only in special combinations of colour singlets. Only two colour singlets are available (see below)

$$(qq) = \frac{1}{\sqrt{3}}\delta_{kk'}\hat{q}_k\hat{q}_{k'} = \text{inv}, \quad (qqq) = \frac{1}{\sqrt{6}}\varepsilon_{kml}\hat{q}_k\hat{q}_l\hat{q}_m = \text{inv}. \quad (3.4.5)$$

### 3.5 Gauge Principle; Internal Symmetries

Following [8], the principle of identity holds for ordinary regular structures, namely each regular structure in the lowest state can be regarded as a result of transition from an arbitrary state, in which they assumed to be distorted. This is stated below

$$\Psi_{\eta}\Psi_{\eta} = f_{(+)}^{-1}(\theta_{+k})\Psi_{\eta}(\theta_{+k}) = f_{(+)}^{-1}(\theta'_{+l})\Psi_{\eta}(\theta'_{+l}) = \cdots.$$  

(3.5.1)

Hence, the following transformations may be implemented upon distorted ordinary structures

$$\Psi_{\eta}(\theta'_{+l}) = \Psi_{\eta}(\theta_{-l}, \theta_{+k}) = f(\theta'_{+l}, \theta_{+k})\Psi_{\eta}(\theta_{+k}),$$

$$\Psi_{\eta}(\theta'_{-l}) = \Psi_{\eta}(\theta_{-l}, \theta_{-k}) = f(\theta'_{-l}, \theta_{-k})\Psi_{\eta}(\theta_{-k}) = f^*_{(-)}(\theta'_{+l}, \theta_{+k});$$

(3.5.2)

$$\theta'_{+l} = \theta_{-l}, \quad \theta'_{+k} = \theta_{+k}.$$
transformation operators

where \( l, k, c, d, m, n \) form transformations implemented upon the quark field, which in matrix notation take the form local and global rotations. Making use of eq.(3.4.1), eq.(3.4.2) and eq.(3.5.2), one gets the \( \theta_\parallel \) reduce to identity \( f_\theta \) gives rise to nontrivial conditions (3.5.5) reduce to identity \( f_\theta \). Next we consider a particular case of two dimensional local transformations through electric charge. The invariance under the local group \( U(1) \) leads to electromagnetic field, the massless quanta of which - photons are electrically neutral, just because of the condition eq.(3.4.4):

\( f^{(+)} = \exp\{\chi^R_{lk} - i\chi^J_{lk}\} \),
\( f^{(-)} = (f^{(+)})^* \) \( \theta_{-l} = \theta_\perp^{+l} \),
\( \theta_{-k} = \theta_\perp^{+k} \),
\[ \chi^R_{lk} = \chi_\perp^{(+l)} - \chi_\perp^{(+k)} \), \( \chi^J_{lk} = \chi_\perp^{(+l)} - \chi_\perp^{(+k)} \).

The transformation functions are the operators in the space of internal degrees of freedom labeled by \((\pm k)\) corresponding to distortion rotations around the axes \((\pm k)\) by the angles \( \theta_{\pm k} \). We make proposition that the distortion rotations are incompatible, namely the transformation operators \( f^{(+)} \) obey the incompatibility relations

\[ \begin{align*}
    f^{(+)}_l f^{(+)}_d - f^{(+)}_d f^{(+)}_l &= \|f^{(+)}\|_{\text{cm}} \epsilon_{kdn} f^{(-)}_m, \\
    f^{(-)}_l f^{(-)}_d - f^{(-)}_d f^{(-)}_l &= \|f^{(-)}\|_{\text{cm}} \epsilon_{kdn} f^{(+)}_m,
\end{align*} \]

where \( l, k, c, d, m, n = 1, 2, 3 \). The relations eq.(3.5.5) hold in general for both local and global rotations. Making use of eq.(3.4.1), eq.(3.4.2) and eq.(3.5.2), one gets the transformations implemented upon the quark field, which in matrix notation take the form \( q'(\zeta) = U(\theta(\zeta)) q(\zeta) \), \( \bar{q}'(\zeta) = \bar{q}(\zeta) U^+(\theta(\zeta)) \), where \( q = \{q_k\} \), \( U(\theta) = \{f^{(+)}_l\} \).

Due to the incompatibility commutation relations (3.5.5), the transformation matrices \( \{U\} \) generate the unitary group of internal symmetries \( U(1), SU(2), SU(3) \). As far as distorted ordinary structures have taken participation in the realization of geometry \( G \) instead of regular ones, stated somewhat differently the principle of identity of regular structures directly leads to the equivalent one: an action integral of any physical system must be invariant under arbitrary transformations eq.(3.5.2) (the Gauge principle). Below we discuss different possible models.

1. In the simple case of one dimensional local transformations, through the local angles \( \theta_{+1}(\zeta) \) and \( \theta_{-1}(\zeta) \) \( f^{(+)} = \begin{pmatrix} f^{(+)}_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \) \( f^{(-)} = (f^{(+)})^* \). The incompatibility relations (3.5.5) reduce to identity \( f^{(+)}_{11} = \|f^{(+)}\| \). If \( \chi_\perp^{(+1)} = \chi_\perp^{(+1)} \), and transformations

\[ f^{(+)}_{11} = U(\theta) = f(\theta_{+1}(\zeta), \theta_{-1}(\zeta)) = \exp\{-i\chi^{(+)}_{Jl}(\theta_{+1}) + i\chi^{(-)}_{Jl}(\theta_{-1})\}. \]

generate a commutative Abelian unitary local group of electromagnetic interactions realized as the Lie group \( U^{\text{loc}}(1) = SO^{\text{loc}}(2) \) with one dimensional trivial algebra \( \hat{q}_l = R_l \): \( U(\theta) = e^{-i\theta} \), where \( \theta \equiv \chi^{(+)}_{Jl}(\theta_{+1}) - \chi^{(-)}_{Jl}(\theta_{-1}) \). The strength of interaction is specified by the coupling \( Q \) of electric charge. The invariance under the local group \( U^{\text{loc}}(1) \) leads to electromagnetic field, the massless quanta of which - photons are electrically neutral, just because of the condition eq.(3.4.4):

\[ f(\theta_{+1}, \theta_{-1}) = f(\theta'_{+1}, \theta'_{-1}) = \cdots \text{inv}. \]

2. Next we consider a particular case of two dimensional local transformations through the angles \( \theta_{\pm m}(\zeta) \) around two axes \((m = 1, 2)\). The matrix function of transformation is written down \( f^{(+)} = \begin{pmatrix} f^{(+)}_{11} & f^{(+)}_{12} & 0 \\ f^{(+)}_{21} & f^{(+)}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \) \( f^{(-)} = (f^{(+)})^* \). The incompatibility relations (3.5.5) give rise to nontrivial conditions

\[ \begin{align*}
    f^{(+)}_{11} &= \|f^{(+)}\| (f^{(+)}_{22})^*, & f^{(+)}_{21} &= -\|f^{(+)}\| (f^{(+)}_{12})^*, \\
    f^{(+)}_{12} &= -\|f^{(+)}\| (f^{(+)}_{21})^*, & f^{(+)}_{22} &= \|f^{(+)}\| (f^{(+)}_{11})^*,
\end{align*} \]
Hence $\|f(+)\| = 1$. One readily infers the matrix $U(\theta)$ of gauge transformations of collection of fundamental fields $U = e^{-i\vec{\theta}\vec{\theta}} = \begin{pmatrix} f_{11}^{(+)} & f_{12}^{(+)} \\ f_{21}^{(+)} & f_{22}^{(+)} \end{pmatrix}$, where $T_i$ ($i = 1, 2, 3$) are the generators of the group $SU(2)$. The fields will come in multiplets forming a basis for representations of the isospin group $SU(2)$. Meanwhile

$$
\begin{align*}
\theta_1 &= \frac{e^{\chi_{12}^R} \sin \chi_{12}^J}{\sqrt{1 - e^{2\chi_{11}^R} \cos^2 \chi_{11}^J}}, \\
\theta_2 &= -\frac{e^{\chi_{12}^R} \cos \chi_{12}^J}{\sqrt{1 - e^{2\chi_{11}^R} \cos^2 \chi_{11}^J}}, \\
\theta_3 &= \frac{e^{\chi_{11}^R} \sin \chi_{11}^J}{\sqrt{1 - e^{2\chi_{11}^R} \cos^2 \chi_{11}^J}},
\end{align*}
\quad \theta_1 = |\vec{\theta}| = 2 \arccos\left(e^{\chi_{11}^R} \cos \chi_{11}^J\right),
$$

provided

$$
\chi_{11}^R = \chi_{22}^R, \quad \chi_{12}^R = \chi_{21}^R, \quad \chi_{11}^J + \chi_{22}^J = 0, \quad \chi_{21}^J + \chi_{12}^J = \pi, \quad \chi_{12}^R = \frac{1}{2} \ln \left(1 - e^{2\chi_{11}^R}\right). \quad (3.5.9)
$$

That is, three functions $\chi_{11}^R, \chi_{11}^J$ and $\chi_{12}^R$ or the angles $\theta_{+1}, \theta_{+1}$ and $\theta_{+2}$ are parameters of the group $SU^{loc}(2)$

$$
\chi_{11}^R = \chi_R(\theta_{+1}' - \chi_R(\theta_{+1}), \quad \chi_{11}^J = \chi_J(\theta_{+1}) - \chi_J(\theta_{+1}), \quad \chi_{12}^J = \chi_J(\theta_{+1}' - \chi_J(\theta_{+2}). \quad (3.5.10)
$$

3. In a case of gauge transformations occurred around all three axes ($l, k = 1, 2, 3$):

$$
\begin{pmatrix} f_{11}^{(+)} & f_{12}^{(+)} & f_{13}^{(+)} \\ f_{21}^{(+)} & f_{22}^{(+)} & f_{23}^{(+)} \\ f_{31}^{(+)} & f_{32}^{(+)} & f_{33}^{(+)} \end{pmatrix}, \quad f^{(-)} = (f^{(+)})^+, \quad \text{the incompatibility relations} \ (3.5.5) \ \text{yield}
$$

the unitary condition $U^{-1} = U^{+}$, $f^{(+) \equiv U}$, and also $\|U\| = 1$. Then $U(\theta) = e^{-\frac{1}{2}\lambda\bar{\theta}}$, where

$$
\left(\frac{\lambda_i}{2}\right) (i = 1, \ldots, 8) \ \text{are the matrix representation of generators of the group} \ SU(3). \ \text{Right through differentiation one infers} \ \lambda d\bar{\theta} = 2i U dU, \ \text{or} \ \bar{\theta} = -\int Im \left(\lambda \left(\tilde{f}^{(-) df^{(+)}}\right)\right), \ \text{provided} \ Re \left(\lambda \left(\tilde{f}^{(-) df^{(+)}}\right)\right) \equiv 0. \ \text{At the infinitesimal transformations} \ \theta_i \ll 1, \ \text{we get}
$$

$$
\begin{align*}
\theta_1 &\approx 2e^{\chi_{12}^R} \sin \chi_{12}^J, \quad \theta_3 \approx \sin \chi_{33}^J + 2 \sin \chi_{11}^J, \quad \theta_5 \approx 2(1 - e^{\chi_{13}^R} \cos \chi_{13}^J), \\
\theta_2 &\approx 2(1 - e^{\chi_{12}^R} \cos \chi_{12}^J), \quad \theta_4 \approx 2e^{\chi_{13}^R} \sin \chi_{13}^J, \quad \theta_6 \approx 2e^{\chi_{21}^R} \sin \chi_{23}^J, \\
\theta_7 &\approx 2(1 - e^{\chi_{23}^R} \cos \chi_{23}^J), \quad \theta_8 \approx -\sqrt{3} \sin \chi_{33}^J,
\end{align*}
\quad (3.5.12)
$$

provided

$$
\begin{align*}
\chi_{il}^R &\approx 0, \quad \chi_{lk}^R \approx \chi_{il}^R, \quad \chi_{lk}^J \approx \chi_{kl}^J, \quad (l \neq k) \ \sin \chi_{11}^J + \sin \chi_{22}^J + \sin \chi_{33}^J \approx 0. \quad (3.5.13)
\end{align*}
$$

The internal symmetry group $SU^{loc}(3)$ enables to introduce a gauge theory in colour space, with the colour charges as exactly conserved quantities. The local colour transformations are implemented on the coloured quarks right through a $SU^{loc}(3)$ rotation matrix $U$ in the fundamental representation.

4 Operator Multimanifold $\hat{G}_N$
4.1 Operator Vector and Covector Fields

The formalism of operator manifold \( \hat{G} = \hat{G} \oplus \hat{G} \) is built up by assuming an existence only of ordinary primordial structures of one sort (one u-channel). Being confronted by our major goal to develop the microscopic approach to field theory based on multiverse geometry, henceforth instead of one sort of ordinary structures we are going to deal with different species of ordinary structures. That is, before we enlarge the previous model we must make an additional assumption concerning an existence of infinite number of \( i \)\textsuperscript{u}-type ordinary structures of different species \( i = 1, 2, \ldots, N \) (multi-u channel). These structures will be specified by the superscript to the left. This hypothesis, as it will be seen in the Part II, leads to the progress of understanding of the properties of particles. At the very outset we consider the processes of creation and annihilation of regular structures of one sort (one channel). Being confronted by the outset we consider the processes of creation and annihilation of regular structures of one sort (one channel). Being confronted by the processes of creation and annihilation of regular structures of one sort (one channel).

\[
\hat{O}_{\lambda, \mu}^{r_1 r_2} = \hat{O}_{\lambda}^{r_1} \otimes \hat{O}_{\mu}^{r_2} \equiv \hat{O}_{\lambda, \mu} = \hat{O}_{\lambda, \mu}(\alpha_\lambda \otimes \alpha_\mu),
\]

provided \( r = (r_1, r_2) \) and

\[
\begin{align*}
\hat{O}_{1, 1}^{r_1} &= \frac{1}{\sqrt{2}}(\hat{O}_{\eta_+}^{r_1} + \hat{O}_{\eta_+}^{r_2}), \\
\hat{O}_{2, 1}^{r_1} &= \frac{1}{\sqrt{2}}(\hat{O}_{\eta+}^{r_1} - \hat{O}_{\eta+}^{r_2}), \\
\hat{O}_{1, 2}^{r_1} &= \frac{1}{\sqrt{2}}(\hat{O}_{\eta-}^{r_1} + \hat{O}_{\eta-}^{r_2}), \\
\hat{O}_{2, 2}^{r_1} &= \frac{1}{\sqrt{2}}(\hat{O}_{\eta-}^{r_1} - \hat{O}_{\eta-}^{r_2}),
\end{align*}
\]

where

\[
< \nu_i, \nu_j > = \delta_{ij}, \quad < \hat{O}_{\eta_+}^{r_1}, \hat{O}_{\eta_+}^{r_2} > = -\delta_{ij} \delta_{rr}, \delta_{\lambda\tau}, \quad < \hat{O}_{\eta_+}^{r_1}, \hat{O}_{\eta_+}^{r_2} > = 0.
\]

In analogy with subsec.2.1 we consider the operator \( i\hat{\gamma}^r_{(\lambda, \mu, \alpha)} = i\hat{O}_{\lambda, \mu}^{r_1 r_2} \otimes \hat{\sigma}_\alpha^r \), and calculate nonzero matrix element

\[
< \lambda, \mu | i\hat{\gamma}^r_{(\tau, \nu, \alpha)} | \tau, \nu > = \delta_{\lambda\tau} \delta_{\mu\nu} i\hat{\sigma}_\alpha^r,
\]

where \( i\hat{\sigma}_\alpha^r = i\hat{O}_{\lambda, \mu}^{r_1} \otimes \sigma_\alpha^r \). The set of operators \( \{ \hat{\gamma}^r \} \) is the basis for all operator vectors \( \Phi(\zeta) = i\hat{\gamma}^r i\Phi(\zeta) \) of tangent section of principle bundle with the base of operator multimanifold \( \hat{G}_N = \left( \sum_i \oplus^* \hat{R}_i^4 \right) \otimes \hat{R}^3 \). Here \( \hat{R}_i^4 \) is the 2 \times 2 dimensional linear pseudo operator space, with the set of the linear unit operator pseudo vectors eq.(4.1.3) as the basis of tangent vector section, and \( \hat{R}^3 \) is the three dimensional real linear operator space with the basis consisted of the ordinary unit operator vectors \( \{ \hat{\sigma}_\alpha^r \} \). The \( \hat{G}_N \) is decomposed as follows:

\[
\hat{G}_N = \hat{G}_\eta \oplus \hat{G}_u \oplus \cdots \oplus \hat{G}_u_N,
\]

where \( \hat{G} \) is the six dimensional operator manifold of the species \( (i) \) with the basis

\[
\left\{ \hat{\gamma}^r_{(\lambda, \alpha)} = i\hat{O}_{\lambda}^{r_1} \otimes \hat{\sigma}_\alpha^r \right\}.
\]

The expansions of operator vectors and covectors are written

\[
\hat{\Psi}_\eta = \hat{\gamma}^r \hat{\Psi}, \quad \hat{\Psi}_u = i\hat{\gamma}^r i\hat{\Psi}, \quad \bar{\hat{\Psi}}_\eta = \hat{\gamma}^r \bar{\hat{\Psi}}, \quad \bar{\hat{\Psi}}_u = i\hat{\gamma}^r i\bar{\hat{\Psi}},
\]

where the components \( \hat{\Psi} \) (\( \eta \)) and \( i\hat{\Psi} \) (\( u \)) are respectively the link functions of \( \eta \)-type and \( i\)\textsuperscript{u}-type structures.
4.2 Field Aspect

The quantum field and differential geometric aspects of $\hat{G}_N$ may be discussed on the analogy of $G_{N=1}$. Here we turn to some points of the field aspect. We consider the special system of regular structures, which is made of fundamental structure of $\eta$-type and infinite number of $^i u$-type ordinary structures of different species ($i = 1, \ldots, N$). To become stable the primordial structures in this system establish a stable linkage

$$p^2 = p^2_\eta - \sum_{i=1}^N p^2_{u_i} = 0. \quad (4.2.1)$$

The free field defined on multimanifold $G_N = \underbrace{G \oplus G \oplus \cdots \oplus G}_{u_N}$ is written

$$\Psi = \Psi(\eta) \Psi(u), \quad \Psi(u) = \Psi(u_1) \cdots \Psi(u_N),$$

where $\Psi$ is the bispinor defined on the internal manifold $u_i$. A Lagrangian of free field reads

$$\bar{L}_0(D) = \frac{i}{2} \{ \bar{\Psi}_e(\zeta)^i \gamma(\lambda,\mu,\alpha) \partial \Psi_e(\zeta) - \partial \bar{\Psi}_e(\zeta)^i \gamma(\lambda,\mu,\alpha) \Psi_e(\zeta) \}. \quad (4.2.2)$$

We adopt the following conventions:

$$\Psi_e(\zeta) = e \otimes \Psi(\zeta) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \Psi(\zeta), \quad \bar{\Psi}_e(\zeta) = e \otimes \bar{\Psi}(\zeta), \quad \bar{\Psi}(\zeta) = \Psi^+(\zeta) \gamma^0,$$

$$i \gamma(\lambda,\mu,\alpha) = i \bar{\partial} \lambda \mu \otimes \bar{\sigma}^\alpha, \quad i \bar{\partial} \lambda \mu = \frac{1}{\sqrt{2}} (\nu_\lambda \xi_0 \otimes \bar{\partial}^\mu + \varepsilon_\lambda \xi \otimes i \bar{\partial}^\mu),$$

$$\varepsilon_\lambda = \begin{cases} 1 & \lambda = 1 \\ -1 & \lambda = 2 \end{cases}, \quad \nu_\lambda, \nu_\mu \Rightarrow \delta_\lambda^\mu, \quad \{ i \bar{\partial}^\lambda, j \bar{\partial}^\mu \} = \delta_\lambda^\mu \delta_\lambda^\mu,$$

$$\bar{\partial} = \partial / i \gamma(\lambda,\mu,\alpha), \quad \xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\xi_0^2 = -\xi^2 = -\xi_0^2 = \xi_i^2 = 1, \quad \{ \xi_0, \xi \} = \{ \xi_0, \xi_0i \} = \{ \xi_0i, \xi \} = \{ \xi, \xi_0 \} = \{ \xi, \xi_0i \} = \{ \xi_0i, \xi \} = \{ \xi, \xi_0i \} = \{ \xi_0i, \xi \} = \{ \xi, \xi_0i \} = \{ \xi, \xi_0i \} = \{ \xi, \xi_0i \} = \{ \xi, \xi_0i \},$$

$$\partial = \partial / i \gamma(\lambda,\mu,\alpha), \quad \xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

Field equations are written

$$\bar{\Psi}(\eta)(\hat{p} - m) = 0, \quad \bar{\Psi}(\eta)(\hat{p} - m) = 0, \quad \Psi(\eta)(\hat{p} - m) = 0, \quad \Psi(\eta)(\hat{p} - m) = 0, \quad (4.2.4)$$

where

$$\hat{p} = i \hat{\eta}, \quad \hat{p} = i \hat{u}, \quad \hat{\partial} = i \gamma(\lambda,\mu,\alpha) \frac{\partial}{\partial \eta^\lambda}, \quad \partial = \partial / \partial \eta^\lambda, \quad \partial = \partial / \partial u_i^\lambda,$$

$$i \gamma(\lambda,\mu,\alpha) = i \bar{\partial} \lambda \mu \otimes \bar{\sigma}^\alpha = \nu_\lambda \xi_0 \otimes \gamma(\lambda,\mu), \quad \nu_\lambda \xi_0 \otimes \bar{\partial} \lambda \mu \otimes \bar{\sigma}^\alpha,$$

$$i \gamma(\lambda,\mu,\alpha) = i \bar{\partial} \lambda \mu \otimes \bar{\sigma}^\alpha = \xi \otimes i \gamma(\lambda,\mu,\alpha) = \xi \otimes i \bar{\partial} \lambda \mu \otimes \bar{\sigma}^\alpha,$$

$$\left( \gamma(\lambda,\mu,\alpha) \right)^+ = \delta^\lambda_\tau \delta^\alpha_\beta \gamma(\tau,\beta) = \gamma(\lambda,\mu,\alpha), \quad \left( i \gamma(\lambda,\mu,\alpha) \right)^+ = -i \gamma(\lambda,\mu,\alpha). \quad (4.2.5)$$

The state of free ordinary structure of $^i u$-type with the given values of link momentum $p$ and spin projection $s_i$ is described by means of plane wave. It is also necessary to consider the solution of negative frequencies with the normalized bispinor amplitude.
4.3 Realization of Multimanifold $G_N$

We consider a special stable system eq.(4.2.1). In analogy with subsec.2.1 we make use of localized wave packets by means of superposition of plane wave solutions furnished by creation and annihilation operators in agreement with Pauli’s principle. Straightforward calculation gives the relation

$$\sum_{\lambda=\pm} <\chi_\lambda | \hat{\Phi}(\zeta) \bar{\Phi}(\zeta) | \chi_\lambda> = \sum_{\lambda=\pm} <\chi_\lambda | \bar{\Phi}(\zeta) \hat{\Phi}(\zeta) | \chi_\lambda> =$$

$$= -i \lim_{\zeta \to \zeta'} (\zeta') G(\zeta - \zeta') = -i \lim_{\eta \to \eta'} (\eta \eta') G(\eta - \eta') - i \lim_{u_i \to u_i'} \sum_{i=1}^{N} (u_i^2 u_i') G(u_i - u_i'),$$

(4.3.1)

provided by the Green’s function $G(u_i - u_i') = -(i \hat{\partial} + m) \Delta(u_i - u_i')$, where the $\Delta(u_i - u_i')$ is the invariant singular function. Thus

$$\zeta^2 G(0) = \eta^2 G(0) - \sum_{i=1}^{N} u_i^2 G(0),$$

(4.3.2)

where $G_{\eta F}$, $G_{\zeta F}$, and $G_{u F}$ are causal Green’s functions of the $\eta-$, $u-$ and $\zeta$-type structures. The realization of the multimanifold stems from the condition imposed upon the matrix element eq.(4.3.1), that as the bilinear form on operator vectors it is required to be finite

$$\sum_{\lambda=\pm} <\chi_\lambda | \hat{\Phi}(\zeta) \bar{\Phi}(\zeta) | \chi_\lambda> = \zeta^2 G(0) < \infty.$$ 

(4.3.3)

Let denote $u_i^2 G(0) = \lim_{u_i \to u_i'} \sum_{i=1}^{N} (u_i^2 u_i') G(u_i - u_i')$ and consider a stable system eq.(4.2.1).

Hence

$$G_{\eta F}(0) = G_{\zeta F}(0) = G_{u F}(0),$$

(4.3.4)

provided $m \equiv |p_u| = \left( \sum_{i=1}^{N} p_{u_i}^2 \right)^{1/2} = |p_{\eta}|$. According to eq.(4.3.5) and eq.(4.3.4), the length of each vector $\zeta = i e^{-i \zeta} \in G_N$ should be equal zero $\zeta^2 = \eta^2 - u^2 = \eta^2 - \sum_{i=1}^{N} (u_i^2)^2 = 0$, where use is made of

$$u_i^G \equiv u_i \left[ \lim_{u_i \to u_i'} G_{\eta F}(u_i - u_i') / \lim_{\eta \to \eta'} G_{\eta F}(\eta - \eta') \right]^{1/2},$$

and $u_i^G = i e^{-i \zeta} u_i^{G(\lambda, \alpha)}$. Thus, the multimanifold $G_N$ comes into being, which is decomposed as follows:

$$G_N = G_{\eta} \oplus G_{u_i^G} \oplus \cdots \oplus G_{u_i^G}.$$  

(4.3.5)

It brings us to the conclusion: the major requirement eq.(4.3.3) provided by stability condition eq.(4.3.4) or eq.(4.2.1) yields the flat multimanifold $G_N$. Meanwhile the Minkowski flat space $M^4$ stems from the flat submanifold $G_{\eta}$ (subsec. 2.1), in which the line element turned out to be invariant. That is, the principle of Relativity comes into being with $M^4$ ensued from the multiworld geometry $G_N$. In the subsequent paper (Part II) we shall use a notion of $i$-th internal world for the submanifold $G_{u_i}$.
5 Concluding Remarks

Our aim is to develop the operator manifold formalism, which is the mathematical basis for the presented approach to describe the microscopic structures of particles (Part II). It is a generalization of secondary quantization method with appropriate expansion over the geometric objects leading to the quantization of geometry, different from all existing schemes. Based on configuration space mechanics with antisymmetric state functions, we discuss in detail the quantum field and differential geometric aspects of the method of operator manifold. We develop the formalism of operator multi manifold yielding the multiworld geometry. The value of the present version of hypothesis of existence of multiworld structures resides in solving some key problems of particle physics (Part II).

Acknowledgements

I am pleased to mention the most valuable discussions with S.P.Novikov and the late V.Ambartsumian on the various issues treated in this paper. I express my gratitude to G.Jona-Lasinio for fruitful comments and suggestions. I’m indebted to V.Gurzadyan, A.M.Vardanian and K.L.Yerknapetian for support.

Appendix

The Solution of Wave Equation of Distorted Structure

To solve the equation (3.3.13)

\[ ig(\partial u - \Gamma) - m \Psi(u) = 0, \]  
(A.1)

we transform it into

\[ \{-\partial^2 - m^2 - (g\Gamma)^2 + 2(\Gamma g) + (g\partial)(g\Gamma)\} \Psi = 0, \]  
(A.2)

where we abbreviate the indices \((\lambda\alpha)\) by the single symbol \(\mu\), and Latin indices \((im)\) \((i = \pm, m = 1, 2, 3)\) by \(i\), also denote \(\hat{p}_u \equiv \hat{p}\) and

\[
\begin{align*}
\frac{1}{2} & \sigma^{\mu\nu} F_{\mu\nu} = (g\partial)(g\Gamma) - (\partial\Gamma), \\
\frac{1}{2} & \sigma^{\mu\nu}[\Gamma_{\mu}, \Gamma_{\nu}] = (g\Gamma)^2 - \Gamma^2, \\
2 & \sigma^{\mu\nu} = \{g^\mu, g^\nu\}, \\
2 & \sigma^{\mu\nu} = [g^\mu, g^\nu], \\
F_{\mu\nu} & = \partial_\mu \Gamma_{\nu} - \partial_\nu \Gamma_{\mu}.
\end{align*}
\]

(A.3)

We are looking for a solution given in the form \(\Psi = e^{-ipuF(\varphi)}\), where \(p_\mu\) is a constant sixvector \(pu = p_\mu u_\mu\), and admit that the field of distortion is switched on at \(u_0 = -\infty\) smoothly. Then the function \(\Psi\) must match onto the wave function of ordinary regular structure. Smoothness requires that the numbers \(p_\mu\) become the components of link momentum of regular structure and satisfy the boundary condition \(p_\mu p_\mu = m^2 = p_0^2\) eq.(3.1.3). Due to it we cancel unwanted solutions and clear up the normalization of wave functions

\[
\int \Psi^*_p \Psi_p d^3u = \int \bar{\Psi}_{p'} \gamma^0 \Psi_{p'} d^3u = (2\pi)^3 \delta(p' - p).
\]

(A.4)

We suppose that at \(\sqrt{-g} \neq 1\) the gradient of the function \(\varphi\) reads

\[
\partial_\mu \varphi = V^{i}_\mu k_i, \quad \partial^\mu \varphi = V^i_\mu k^i,
\]
where \(k_i\) are arbitrary constant numbers satisfying the condition \(k_i k_i = 0\). Thus \(\partial^\mu \varphi \partial_\mu \varphi = (V^\mu_i V^\mu_j) k^i k^j = 0\). Then the eq.(14.4) gives rise to \(F' = A(\theta) F\), where \((\cdots)'\) stands for the derivative with respect to \(\varphi\), and

\[
A(\theta) = \frac{2i(p\Gamma) + m^2 - p^2 + (g\Gamma)^2 - (g\partial)(g\Gamma)}{2i(kVp) - (kDV)};
\]

(A.5)

where

\[
(kVp) = k^i V^\mu_i p_\mu, \quad (kDV) = k^i D_\mu V^\mu_i, \quad D_\mu = \partial_\mu - 2\Gamma_\mu,
\]

\[
p^2 = p^\mu p_\mu = g^{\mu\nu}(\theta) p_\mu p_\nu, \quad (kVdu) = k_i V^\mu_i du^\mu.
\]

(A.6)

We are interested in the right-handed eigenvectors \(F_r\) \((r = 1, 2, 3, 4)\) corresponding to eigenvalues \(\mu_r\) of matrix \(A\): \(AF_r = \mu_r F_r\), which are the roots of polynomial characteristic equation

\[
c(\mu) = ||(\mu I - A)|| = 0.
\]

Thus, one gets \(F'_r = \mu_r F_r\) and \(F = \prod_{r=1}^4 F_r\). Hence \((\ln F)' = \sum_{r=1}^4 \mu_r = tr A\) and \((\ln F)' = X_R(\theta) - iX_J(\theta)\), provided

\[
X_R(\theta) = tr A_R(\theta) = tr \left\{ \frac{-(kDV) [m^2 - p^2 + (g\Gamma)^2 - (g\partial)(g\Gamma)] + (kVp)(p\Gamma)}{(kDV)^2 + 4(kVp)^2} \right\},
\]

\[
X_J(\theta) = tr A_J(\theta) = 2tr \left\{ \frac{(kVp) [m^2 - p^2 + (g\Gamma)^2 - (g\partial)(g\Gamma)] + (kDV)(p\Gamma)}{(kDV)^2 + 4(kVp)^2} \right\}.
\]

(A.7)

The solution of eq.(A.1) reads

\[
F(\theta) = C \left( \frac{m}{E_u} \right)^{1/2} U \exp\{\chi_R(\theta) - i\chi_J(\theta)\},
\]

(A.8)

where \(C = 1\) is the normalization constant, \(U\) is the constant bispinor, and

\[
\chi_R(\theta) = \int_0^{u^\mu} (kVdu) X_R(\theta), \quad \chi_J(\theta) = \int_0^{u^\mu} (kVdu) X_J(\theta).
\]

(A.9)

References

[1] A.Ashtekar, J.Lewandowski, Class.Quant.Grav., 14 A55 (1997); [gr-qc/9711031].

[2] A.Ashtekar, Int.J.Mod.Physics., D5 629 (1996).

[3] E.Witten, Int.J.Mod.Phys., A10 1247 (1995); Nucl.Physics., B471 135 (1996); B471 195 (1996); B474 343 (1996); B500 3 (1997).

[4] S.Weinberg, Phys.Rev., D56 2303 (1997).

[5] R.Penrose, “Fundamental issues of curved-space quantization”, Talk given at VII Marcel Grossmann Meeting, Jerusalem, 1997

[6] M.Shifman, Prog.Part.Nucl.Phys., 39 1 (1997).
[7] B.S.De Witt, R.D.Graham (Eds.) *The Many-Worlds Interpretation of Quantum Mechanics*, Princeton Univ.Press, 1973.

[8] G.T.Ter-Kazarian, *Astrophys. and Space Sci.*, 241 161 (1996).

[9] G.T.Ter-Kazarian, IC/94/290, ICTP (preprint), Trieste, Italy, 1994.

[10] G.T.Ter-Kazarian, dg-ga/9710010.

[11] G.T.Ter-Kazarian, *Nuovo Cimento*, 112 825 (1997).

[12] G.T.Ter-Kazarian, *Comm. Byurakan Obs.* 62 1 (1989).

[13] G.T.Ter-Kazarian, *Astrophys. and Space Sci.* 194 1 (1992).

[14] J.M.Cook, *Trans. Amer. Math. Soc.*, 74 222 (1953).

[15] V.I.Arnold, *Mathematical Methods of Classical Mechanics*, Nauka, Moscow, 1989.

[16] B.A.Dubrovin, S.P.Novikov and A.T.Fomenko, *The Contemporary Geometry; The Methods and Applications*, Nauka, Moscow, 1986.

[17] L.S.Pontryagin, *The Continuous Groups*, Nauka, Moscow, 1984.