Non-Perturbative Models For
The Quantum Gravitational Back-Reaction On Inflation

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ABSTRACT

We consider a universe in which inflation commences because of a positive cosmological constant, the effect of which is progressively screened by the interaction between virtual gravitons that become trapped in the expansion of spacetime. Perturbative calculations have shown that screening becomes non-perturbatively large at late times. In this paper we consider effective field equations which can be evolved numerically to provide a non-perturbative description of the process. The induced stress tensor is that of an effective scalar field which is a non-local functional of the metric. We use the known perturbative result, constrained by general principles and guided by a physical description of the screening mechanism, to formulate a class of ansätze for this functional. A scheme is given for numerically evolving the field equations which result from a simple ansatz, from the beginning of inflation past the time when it ends. We find that inflation comes to a sudden end, producing a system whose equation of state rapidly approaches that of radiation. Explicit numerical results are presented.

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1 Introduction

Perturbation theory is an immensely gratifying tool. It almost always provides quantitative answers for how a known system changes with the inclusion of a small, new effect. The great frustration in using the technique is that its answers become unreliable precisely when they are most interesting: when the new effect causes major changes. Our recent study of the quantum gravitational back-reaction on inflation [1] illustrates both the utility of perturbation theory, and the frustration of not being able to push it further.

The unperturbed system in our case is classical general relativity, the Lagrangian for which is:

\[ \mathcal{L} = \frac{1}{16\pi G} (R - 2\Lambda) \sqrt{-g}. \]  

(1)

Here \( G \) is Newton’s constant and \( \Lambda \) is the cosmological constant, assumed positive. On a spatially flat manifold the invariant element for a homogeneous and isotropic universe can be written in co-moving coordinates:

\[ \hat{g}_{\mu\nu}(t, \vec{x}) dx^\mu dx^\nu = -dt^2 + e^{2b(t)} d\vec{x} \cdot d\vec{x}. \]  

(2)

And the classical solution is:

\[ b_{\text{class}}(t) = Ht, \]  

(3)

where \( H \equiv (\Lambda/3)^{1/2} \) is the Hubble constant. If we specialize to the manifold \( T^3 \times \mathbb{R} \), where each of the coordinate radii is \( H^{-1} \), then the 3-volume:

\[ V(t) = H^{-3} e^{3Ht}, \]  

(4)

is finite but grows exponentially.

The perturbation we seek to study is the gravitational interaction between virtual infrared gravitons that become trapped in the expansion of spacetime and get pulled apart. Although we have computed this exactly at the lowest non-trivial order in perturbation theory [1, 2], an intuitive understanding of the effect is necessary if we are to abstract it beyond the perturbative regime. The physical picture is that virtual gravitons of sufficiently long physical wave length are torn apart by inflation. This is the phenomenon of superadiabatic amplification, first studied by Grishchuk in 1974 [3]. Although infrared gravitons are continually produced in this way, the volume
of space expands so rapidly that the energy density of these gravitons remains a constant — and rather small — fraction of $\Lambda/(8\pi G)$.\textsuperscript{1} However, as each graviton pair recedes, the intervening space is filled by their long range gravitational potentials. These potentials persist even after the gravitons that engendered them have reached cosmological separations. As new pairs are ripped apart, their potentials add to those already present. This is a secular effect and it obviously continues as long as inflation does. Because gravity is attractive the effect tends to counteract inflation — and hence to screen the cosmological constant.

One might expect similar results from quanta other than gravitons but this is not so. To experience superadiabatic amplification a particle must be effectively massless with respect to $H$, and it must not possess classical conformal invariance.\textsuperscript{2} One or the other of these two conditions excludes every other known particle and most of the conjectured ones. The only contender, besides the graviton, is a massless, minimally coupled scalar. These do experience superadiabatic amplification, but global conformal invariance prevents them from inducing a gravitational interaction comparable to that of gravitons.\textsuperscript{3} One might get a strong effect if such a scalar had non-derivative self-interactions, but it is difficult to understand why these would not also induce a substantial mass.

Screening affords a simple and satisfying reformulation of inflationary cosmology and a beautiful resolution to the associated problems of fine tuning. Inflation starts, in this scheme, because the cosmological constant is positive and not unreasonably small. Inflation eventually ends due to the self-gravitation of virtual gravitons which have become trapped in the superluminal expansion of spacetime. Inflation lasts for a very long time because gravitational interactions are weak, even at the GUT scale. One can be indifferent about adding matter because gravitons are the unique phenomenologically viable quanta which induce screening. The only thing to avoid, in this scheme, is introducing an inflaton field and fine tuning its potential!\textsuperscript{4} Best of all, the infrared character of the screening mechanism means that it can

\textsuperscript{1}It is easy to show that there is on average one infrared graviton per Hubble volume. Even for inflation on the GUT scale this is only about $10^{-13}$ of the energy density of the cosmological constant.

\textsuperscript{2}Massive particles are short range, so their virtual quanta seldom get far enough apart to become trapped in the expansion of spacetime. Conformally invariant particles are incapable, locally, of distinguishing between the conformally flat classical background and flat space.
be studied reliably using quantum general relativity, in spite of the theory’s lack of perturbative renormalizability and without regard to what happens at the Planck scale.

But there is a problem: the quantum gravitational back-reaction can only be studied perturbatively so long as it is weak. This regime is not without interest. For example, perturbative analysis shows that inflation lasts a long time and that conventional matter is incapable of competing with the quantum gravitational back-reaction. If one assumes a sudden end to inflation — which is certainly supported by the perturbative results — then it should be possible to predict the spectrum of density fluctuations in the perturbative regime. However, the most interesting questions lie frustratingly beyond the point where perturbation theory is valid. Hence the need for a non-perturbative model.

Our strategy for creating such a model is to infer the induced stress tensor:

\[ T_{\mu\nu}[g] = \frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda \right), \]  

(5)

as a non-local functional of the metric which correctly reproduces the known perturbative effect and which captures the physical origin of screening generally. Of course there is some ambiguity in this, but surprisingly little of any significance. Given an ansatz we can numerically integrate the field equations as far into the future as is desired. What we find, for a simple ansatz, is that inflation ends suddenly over a period of about five e-foldings, following which the equation of state rapidly approaches that of pure radiation.

This paper consists of eight sections, of which the first is drawing to a close. In Section 2 we show that, for the purposes of cosmology, the induced stress tensor can be parameterized as that of an effective scalar field which is a non-local functional of the metric. We also derive what this functional must be during the perturbative regime. In Section 3 we enumerate six principles which constrain the scalar functional generally. Section 4 gives a semi-quantitative model of screening which is of course the ultimate physical motivation for the choice of scalar. We discuss various ansätze for the scalar

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3Infrared phenomena can always be studied using the low energy effective theory. This is why Bloch and Nordsieck were able to resolve the infrared problem in QED before the theory’s renormalizability was suspected. It is also why Weinberg was able to give a similar resolution for the infrared problem of quantum general relativity with zero cosmological constant. And it is why Feinberg and Sucher were able to compute the long range force induced by neutrino exchange using Fermi Theory.
in Section 5. Section 6 describes our scheme for numerically integrating the dynamical system resulting from a simple ansatz. We also report explicit results. In Section 7 we reconstruct the potential of the effective scalar, analytically for large or small values of the scalar and numerically for any value. Section 8 is a discussion of our results.

2 Effective scalar stress tensor

The point of this section is to show that, for the purposes of cosmology, we can model the induced stress tensor \( T_{\mu\nu}[g] \) as that of a scalar field which is itself a non-local functional of the metric:

\[
T_{\mu\nu}[g] = \partial_\mu \phi[g] \partial_\nu \phi[g] - g_{\mu\nu} \left( \frac{1}{2} g^{\rho\sigma} \partial_\rho \phi[g] \partial_\sigma \phi[g] + V(\phi[g]) \right). \tag{6}
\]

We will also show that, since the potential can always be chosen to enforce conservation, one really needs only the functional \( \phi[g] \). And we will use the known perturbative results \([9]\) to derive what \( \phi[g] \) must be when specialized to the classical background \([3]\).

There are three senses in which one might discuss the induced stress tensor, or any other functional of the metric. The first sense is generally for an arbitrary metric; the second is as a functional of \( b(t) \) for a spatially flat, homogeneous and isotropic metric; and the third is as an explicit function of time for the case of perturbative corrections. When the basic symbol appears unadorned we mean the general quantity; the presence of a hat implies specialization to the spatially flat, homogeneous and isotropic form; and the perturbative version of the same quantity is denoted by a tilde. For example, a general invariant element is represented thus:

\[
ds^2 = g_{\mu\nu}(t, \vec{x}) dx^\mu dx^\nu. \tag{7}\]

Specializing to flat, homogeneous and isotropic spacetimes gives:

\[
d\hat{s}^2 = -dt^2 + \exp[2b(t)] d\vec{x} \cdot d\vec{x}. \tag{8}\]

And we express the perturbative result as follows \([9]\):

\[
d\tilde{s}^2 = -dt^2 + \sqrt{1 + A(t)} \exp[2Ht \{1 + D(t)\}], \tag{9}\]
where perturbative expansions for the functions $D(t)$ and $A(t)$ are:

$$D(t) = \frac{19}{2}(\epsilon Ht)^2 + O\left((\epsilon Ht)^3\right), \quad (10)$$

$$A(t) = -\frac{172}{9}\epsilon^2(Ht)^3 + O\left(\epsilon^3(Ht)^4\right), \quad (11)$$

The small parameter in these expansions is $\epsilon \equiv G\Lambda/3\pi$. We will assume that it can be as big as $10^{-12}$ or as small as $10^{-68}$.

Although deep intuition about the induced stress tensor derives from its dependence upon a general metric, we cannot hope to say much about the micro-structure of quantum gravity. Nor is this necessary. Screening is a phenomenon of cosmological scales, so we need only a model that is accurate for spatially flat, homogeneous and isotropic metrics (8). The isometries of these metrics imply that only $\hat{T}_{00}[b]$ and $\hat{T}_{ij}[b]$ can be non-zero, and that both are functions of time alone. We shall parameterize them in the usual way as an induced energy density $\rho(t)$ and an induced pressure $p(t)$:

$$\hat{T}_{00}[b](t) \equiv \rho(t) \quad , \quad \hat{T}_{ij}[b](t) \equiv g_{ij}p(t). \quad (12)$$

We first show that $\rho(t)$ and $p(t)$ can be chosen so as to support any evolution for $b(t)$. Then we show that it is always possible to choose the scalar field $\phi[g]$ and its potential $V(\phi)$ to give the desired energy density and pressure.

The non-trivial components of the effective field equations are:

$$3b^2 = 3H^2 + 8\pi G\rho, \quad (13)$$

$$-2\ddot{b} - 3\dot{b}^2 = -3H^2 + 8\pi Gp. \quad (14)$$

Although one usually regards these as equations for $b(t)$ in terms of $\rho(t)$ and $p(t)$, we can take the reverse view:

$$\rho(t) = \frac{1}{8\pi G} \left(3\dot{b}^2(t) - 3H^2\right), \quad (15)$$

$$p(t) = \frac{1}{8\pi G} \left(-2\ddot{b}(t) - 3\dot{b}^2(t) + 3H^2\right). \quad (16)$$

The physical import of these equations is that one can find $\rho(t)$ and $p(t)$ so as to support any evolution for $b(t)$.

In a homogeneous and isotropic background, the energy density and pressure of a scalar $\phi[b]$ are:

$$\rho = \frac{1}{2} \left(\frac{d\phi}{dt}\right)^2 + V(\phi), \quad (17)$$
Combining this with (15-16) we see find that an arbitrary evolution $b(t)$ can be supported by making the following choices for the scalar and its potential:

\[
\left( \frac{d\hat{\phi}}{dt} \right)^2 = \frac{1}{8\pi G} \left( -2\ddot{b} \right), \\
V = \frac{1}{8\pi G} \left( \dot{b} + 3\dot{b}^2 - 3H^2 \right). 
\]

Given an explicit function $b(t)$ one constructs $V(\phi)$ by solving the differential equation (19) for the scalar as an explicit function of time, call it $\hat{\phi}[b](t) = f(t)$. We then invert this relation to express time as a function of $\hat{\phi}$, $t = f^{-1}(\hat{\phi})$. The potential $V(\phi)$ is found by evaluating relation (20) for $t = f^{-1}(\hat{\phi})$.

The construction is completed by giving a functional of the metric which agrees with $\hat{\phi}(t)$ for the particular choice of $b(t)$. There are many solutions. Perhaps the simplest is obtained from $P[g](t, \vec{x})$, the invariant volume of the past light cone of the point $(t, \vec{x})$. For a spatially flat, homogeneous and isotropic metric, this is a monotonically increasing function of the co-moving time $t$:

\[
\hat{P}[b](t) = \frac{4}{3}\pi \int_0^t dt' e^{3b(t')} \left( \int_t^{t'} dt'' e^{-b(t'')} \right)^3, 
\]

so we can invert the relation. Suppose that for the specific function $b(t)$ we get $\hat{P}[b](t) = \pi(t)$. Then time is $t = \pi^{-1}(\hat{P})$, and the scalar for a general metric could be taken as:

\[
\phi[g] = f \left( \pi^{-1}(P[g]) \right). 
\]

Since all this can be done for any function $b(t)$, we lose nothing by assuming that the induced stress tensor has the scalar form (6).

In the preceding discussion we inverted the proper order of things to show that the scalar stress tensor (6) can describe any homogeneous and isotropic geometry. That point having been made, we can return to the usual dynamical problem of inferring $g_{\mu\nu}$ from $T_{\mu\nu}[g]$. For a homogeneous and isotropic universe the non-trivial equations are:

\[
3\dot{\phi}^2 = 3H^2 + 8\pi G \left\{ \frac{1}{2} \left( \frac{d\hat{\phi}[b]}{dt} \right)^2 + V(\hat{\phi}) \right\}, 
\]

\[\text{(23)}\]
\[-2\ddot{b} - 3\dot{b}^2 = -3H^2 + 8\pi G \left \{ \frac{1}{2} \left ( \frac{d\hat{\phi}[b]}{dt} \right )^2 - V(\hat{\phi}) \right \}. \quad (24)\]

In dynamical terms, equation (23) is actually a constraint. If it is true initially then time evolution and energy conservation conspire to keep it true. The dynamical equation for \(b(t)\) could be taken to be (24), but it is more convenient to add (23):

\[\dddot{b} = -4\pi G \left ( \frac{d\hat{\phi}[b]}{dt} \right )^2. \quad (25)\]

Note that the scalar potential has dropped out. It therefore follows that a model is specified by giving the scalar \(\phi[g]\) as a functional of the metric.

If the potential is desired it can be determined by the round-about process of first solving (23) for \(b(t)\) and substituting to find \(\hat{\phi}[b](t)\). One then inverts to express \(t\) as a function of \(\hat{\phi}\), and finally substitutes into the constraint (23):

\[V(\hat{\phi}) = -\frac{1}{2} \left ( \frac{d\hat{\phi}}{dt} \right )^2 + \frac{3}{8\pi G} \left ( \dot{b}^2 - H^2 \right ). \quad (26)\]

This turns out to be much easier than it might seem. In Section 7 we will obtain analytic expressions for \(V(\phi)\) in the perturbative regime and in the regime of asymptotically late times. We will also carry out the process numerically over the full range of evolution.

It remains to work out \(\hat{\phi}(t)\) during the perturbative regime. Using relation (9) we can express the second time derivative of \(b(t)\) in terms of the functions \(D(t)\) and \(A(t)\):

\[\ddot{b}(t) = H \left [ 2\dot{D}(t) + t\ddot{D} \right ] + \frac{1}{2} \left ( \frac{\dot{A}(t)}{1 + A(t)} \right ) - \frac{1}{2} \left ( \frac{\dot{A}(t)}{1 + A(t)} \right )^2. \quad (27)\]

It is easy to see from the perturbative expansions (10) and (11) that only the third term matters at any time during the perturbative regime. Comparing with expression (25) we infer:

\[\left ( \frac{d\hat{\phi}}{dt} \right )^2 \approx \frac{1}{8\pi G} \left ( \frac{\dot{A}}{1 + A} \right )^2. \quad (28)\]
Making an arbitrary choice of sign and using the fact that only the first term in the expansion of $A(t)$ matters, we obtain the following formula for the scalar during the perturbative regime:

$$\tilde{\phi}(t) \approx -\frac{1}{\sqrt{8\pi G}} \ln [1 + A(t)], \quad (29)$$

$$\approx -\frac{1}{\sqrt{8\pi G}} \ln \left[1 - \frac{172}{9} \epsilon^2 (Ht)^3 \right], \quad (30)$$

where the small parameter is $\epsilon \equiv G\Lambda/(3\pi)$.

### 3 General principles

In the previous section we saw that, on cosmological scales, the induced stress tensor can be taken as that of a scalar field, $\tilde{\phi}[g]$ which is itself a non-local functional of the metric. We saw further that it is really only necessary to give this functional, since the associated potential is determined by conservation. In fact we actually require only the restriction $\tilde{\phi}[h]$ of this functional to a flat, homogeneous and isotropic geometry. However, powerful constraints exist on how the scalar can depend upon a general metric. The purpose of this section is to state these constraints.

1. **Causality**

   We are actually going to **guess** the induced stress tensor but, were we to **compute** it, $T_{\mu\nu}[g](x)$ would come from Schwinger’s effective action for expectation values $[10, 11]$, not from the more common, “in”–“out” effective action. One consequence is that $T_{\mu\nu}[g](x)$ — and hence also $\tilde{\phi}[g](x)$ — can only depend upon the metric at points $y^\mu$ which lie within the past light cone of $x^\mu$. It is worth noting that the effective field equations for “in”–“out” matrix elements must be symmetric: if they depend upon a field at $x^\mu - \Delta x^\mu$ then they must also depend upon fields at $x^\mu + \Delta x^\mu$. This is avoided in Schwinger’s method because his effective action really depends upon **two** fields, one the background during forward time evolution and the other the background during evolution back to the initial state. The effective field equations are obtained by varying with respect to (either) one of these fields and then setting them equal. This is what breaks the forward-backwards symmetry of the “in”–“out” field equations. Causality arises because the forward and backward evolutions interfere destructively outside the past light cone.
2. General Coordinate Invariance

Because the dynamics of quantum general relativity are general coordinate invariant, non-invariance can only enter the effective field equations from the gauge in which the initial state was specified. In other words, $\phi[g]$ must be invariant up to surface terms. This means that it must consist of a covariant local part plus non-local operators, such as the retarded propagator, acting on local functions of the Riemann tensor and its covariant derivatives. Although one can form many covariants from the Riemann tensor and its derivatives, only a few are distinct in a homogeneous and isotropic universe.

To see this, note first that the spacetime is conformally flat. This means the Weyl tensor vanishes and one can express the Riemann tensor in terms of the Ricci tensor:

$$\hat{R}_{\rho\sigma\mu\nu} = \frac{1}{2} \left( \hat{g}_{\rho\mu} \hat{R}_{\sigma\nu} - \hat{g}_{\rho\sigma} \hat{R}_{\nu\mu} - \hat{g}_{\nu\rho} \hat{R}_{\mu\sigma} + \hat{g}_{\nu\mu} \hat{R}_{\rho\sigma} \right) - \frac{1}{6} \left( \hat{g}_{\rho\mu} \hat{g}_{\sigma\nu} - \hat{g}_{\rho\sigma} \hat{g}_{\nu\mu} \right) \hat{R}.$$  

(31)

Although we will express $\phi[g]$ as an invariant functional of a general metric, we need not worry about the distinction between ansätze which agree for a spatially flat, homogeneous and isotropic universe.

3. Stability of the initial value problem

Since the quantum field theoretic problem was well posed given only the initial state wavefunctional, it must be that the associated effective field equations can be evolved forward from $t = 0$ knowing only the metric and its first time derivative. This limits the local terms in $T_{\mu\nu}[g]$ to those which contain at most second time derivatives of the metric. Since the induced stress tensor contains derivatives of the effective scalar, the local part of $\phi[g]$ can have at most first derivatives. Stability also imposes requirements on non-local terms which are differentiated or which can give local terms by partial integration.

4. Non-locality

A universe which is initially inflating will continue to inflate unless $T_{\mu\nu}[g]$ is a non-local functional of the metric. To see this, assume the converse. Using the previous principle we can then constrain $T_{\mu\nu}[g]$ to consist of second rank functions of the Riemann tensor. Note that in a locally de Sitter geometry the Riemann tensor can be written in terms of the Ricci scalar. One way of expressing this is by saying that the following tensor vanishes:

$$V_{\rho\sigma\mu\nu} \equiv R_{\rho\sigma\mu\nu} - \frac{1}{12} \left( g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \right) R.$$  

(32)
This fact can be used to write any second rank tensor function of the Riemann tensor as a term which vanishes in a locally de Sitter geometry plus a function of the Ricci scalar times the metric. For example, consider the partially contracted product of two Riemann tensors:

$$R_{\mu}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} = V_{\mu}^{\alpha\beta\gamma} V_{\nu\alpha\beta\gamma} + \frac{1}{3} R V_{\mu\rho \nu} + \frac{1}{16} R^2 g_{\mu\nu} .$$

(33)

But the initial condition of our problem is a locally de Sitter geometry. Therefore all terms of the first type vanish, and terms of the second type simply renormalize the cosmological constant. We have already defined our cosmological constant to absorb any terms of the second type, so local effective field equations would have $R_{\mu\nu} = \Lambda g_{\mu\nu}$ as a solution for all time. Since we can actually see inflation begin to slow in perturbation theory it follows that non-local terms must be responsible. The same argument works later on, even after the effective Hubble constant has been substantially reduced: the effect would stop without non-local terms. So the important part of the induced stress tensor must be non-local.

5. Dimensional Analysis

The induced stress tensor has the dimensions of length$^{-4}$, so $\phi[g]$ goes like length$^{-1}$. This seemingly trivial fact conceals a surprisingly powerful constraint. The most important quantities from which $\phi[g]$ can be constructed have the following dimensionalities:

$$G \sim \text{length}^2 , \quad R_{\rho\sigma\mu\nu} \sim \text{length}^{-2} ,$$

$$\Lambda \sim \text{length}^{-2} , \quad \frac{1}{\Box} \sim \text{length}^2 ,$$

(34)

where $\Box$ is the scalar d'Alembertian:

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_{\mu} \left( g^{\mu\nu} \sqrt{-g} \partial_{\nu} \right) .$$

(35)

Note that the dimensionless quantity $G\Lambda$ is less than about $10^{-11}$, even for GUT scale inflation. Further, the curvature is guaranteed to be of order $\Lambda$ during the perturbative regime, and it had better be considerably smaller at late times. This means that terms with too many powers of $G$ are likely to be negligible unless they accompany non-local growth factors such as $1/\Box$.

6. The Flat Space Limit

When $\Lambda = 0$ we know that the vacuum is stable, so the “in”-“out” matrix
element of an operator agrees with its expectation value. We also know that “in”-“out” amplitudes with \( \ell \) loop amplitudes contain at most \( \ell \) infrared logarithms [7]. This means that the most infrared singular term which can remain in the \( \Lambda = 0 \) “in”-“out” effective action has the form:

\[
\Gamma_{\text{flat}}[g] \sim \int d^4x \sqrt{-g} R \left[ \ln(\Box) \right]^\ell R.
\] (36)

Terms whose field dependence is more singular must possess positive powers of \( \Lambda \), for example:

\[
\Lambda \int d^4x \sqrt{-g} R \frac{1}{\Box} R.
\] (37)

And we must of course avoid inverse powers of \( \Lambda \).

4 The physics of screening

The most important constraint on the functional \( \phi[g] \) is that it should correctly reflect the physics of screening. Of course choosing the scalar so that its perturbative restriction agrees with (30) automatically enforces this during the perturbative regime, so any additional information must come from understanding the mechanism of screening for an arbitrary homogeneous and isotropic background. That is the purpose of this section. Our procedure is to work first in the classical background, where results can be checked against perturbation theory, and then generalize. We begin by giving a simple derivation of the phenomenon of superadiabatic amplification [3, 4], whereby the 0-point energy of infrared graviton modes is vastly enhanced over the familiar \( \frac{1}{2} \hbar \omega \) of flat space. The next step is to work out the Newtonian approximation for the gravitational self-interaction of this 0-point energy. Comparison with the known perturbative result indicates how to include relativistic effects. Then the analysis is generalized for an arbitrary homogeneous and isotropic background.

Let us recall some facts about the classical background:

\[
d s_{\text{class}}^2 = -dt^2 + e^{2Ht} d\vec{x} \cdot d\vec{x}.
\] (38)

Because it is not physically sensible to assume that coherent inflation commences over a region of more than about one Hubble volume, we work on the manifold \( T^3 \times \mathbb{R} \), where each of the coordinate radii is \( H^{-1} \). The 3-volume of this manifold is finite but expands exponentially:

\[
V(t) = H^{-3} e^{3Ht}.
\] (39)
By setting $ds^2_{\text{class}} = 0$ we find the world line of a light ray which passes through $\vec{x} = 0$, directed along the unit vector $\hat{r}$ at $t = t_0$:

$$\vec{r}(t) = \frac{\hat{r}}{H} \left( e^{-Ht_0} - e^{-Ht} \right).$$  \hspace{1cm} (40)

Multiplying by $\exp(HT)$ and taking the norm gives the physical distance from the origin along the surface of simultaneity:

$$e^{HT} \| \vec{r}(t) \| = \frac{1}{H} \left( 1 - e^{-H(t_0-t)} \right).$$  \hspace{1cm} (41)

From this we infer the existence of a causal horizon of physical distance $H^{-1}$, beyond which even a signal traveling at the speed of light can never reach. We can also compute the invariant 4-volume of the past light cone from the point $(t, \vec{x}) = (t_0, 0)$ to the surface of simultaneity at $t = 0$:

$$P_{\text{class}}(t_0) = \int_0^{t_0} dt \ e^{3Ht} \times \frac{4}{3} \pi \| \vec{r}(t) \|^3,$$

$$= \frac{4}{3} \pi H^{-4} \left( Ht_0 - \frac{11}{6} + 3e^{-Ht_0} - \frac{3}{2} e^{-2Ht_0} + \frac{1}{3} e^{-3Ht_0} \right).$$  \hspace{1cm} (43)

Finally, note that the coordinate transformation $\eta = -H^{-1} \exp(-HT)$ makes the classical background conformal to flat space:

$$ds^2_{\text{class}} = \Omega^2 \left( -d\eta^2 + d\vec{x} \cdot d\vec{x} \right),$$  \hspace{1cm} (44)

where the conformal factor is $\Omega \equiv -1/(H\eta)$. Note that the surface of simultaneity at $t = 0$ corresponds to $\eta = -1/H$, and that the infinite future corresponds to $\eta \to 0^-$.  

The next step is the kinematics of free gravitons. Graviton modes are described by a polarization and by a co-moving wave number of the form $\vec{k} = 2\pi H \vec{n}$, where $\vec{n}$ is a 3-tuple of integers. The integral approximation to a mode sum is:

$$2 \sum_{\vec{n}} f(2\pi H \vec{n}) \approx 2 \int d^3n \ f(2\pi H \vec{n}) = 2 \int \frac{d^3k}{(2\pi H)^3} \ f(\vec{k}),$$  \hspace{1cm} (45)

where the infrared cutoff is at $k \equiv \|\vec{k}\| = H$. Since physical distances expand by $\exp(HT)$, physical wave numbers redshift by $\exp(-HT)$. We are most
interested in infrared modes, defined as those whose physical wave lengths have expanded beyond the causal horizon:

\[ \text{Infrared} \iff H \lesssim k \lesssim H e^{Ht}. \tag{46} \]

We shall refer to the higher modes as “ultraviolet.”

Now consider the dynamics of free gravitons. For any homogeneous and isotropic geometry, these are the same as those of a massless, minimally coupled scalar. Suppose we call such a field \( \psi(\eta, \vec{x}) \). In the classical background its Lagrange density is:

\[ \mathcal{L} = \frac{1}{2} \Omega^2 \left( \psi'^2 - \nabla^2 \psi \right), \tag{47} \]

where a prime denotes differentiation with respect to the conformal time \( \eta \). The mode coordinates are obtained by taking the spatial Fourier transform and multiplying by a factor of \( H \):

\[ q_{\vec{k}}(\eta) \equiv H \int d^3 x \ e^{i\vec{k} \cdot \vec{x}} \psi(\eta, \vec{x}). \tag{48} \]

These variables allow us to recognize the Lagrangian as a sum of independent harmonic oscillators:

\[ L \equiv \int d^3 \mathcal{L} = \frac{1}{2} H \Omega^2 \sum_{\vec{k}} \left( q_{\vec{k}}^* q_{\vec{k}}' - k^2 q_{\vec{k}}^2 \right). \tag{49} \]

Since \( q_{-\vec{k}} = q_{\vec{k}}^* \) we can treat this system as if there were a single real mode for each wave number \( \vec{k} \).

It is straightforward to express the mode coordinate and its conjugate momentum in terms of creation and annihilation operators. The Heisenberg equation of motion is:

\[ q_{\vec{k}}'' - \frac{2}{\eta} q_{\vec{k}}' + k^2 q_{\vec{k}} = 0. \tag{50} \]

It follows that the negative frequency mode solution is:

\[ u(\eta, k) = \frac{\Omega^{-1}}{\sqrt{2k}} \left( 1 - i \frac{\eta}{k} \right) e^{-ik\eta}, \tag{51} \]

and we define its time derivative as:

\[ u'(\eta, k) \equiv -ik \Omega^{-2} v(\eta, k) = -ik \frac{\Omega^{-1}}{\sqrt{2k}} e^{-ik\eta}. \tag{52} \]
The Wronskian formed from $u(\eta, k)$ and $v(\eta, k)$ is constant in consequence of the equation of motion, and with our normalization its value is:

$$u(\eta, k)v^*(\eta, k) + u^*(\eta, k)v(\eta, k) = \frac{1}{k}.$$  \hspace{1cm} (53)

We can express $q_{\vec{k}}(\eta)$ as a linear combination of $u(\eta, k)$ and $u^*(\eta, k)$:

$$q_{\vec{k}}(\eta) = u(\eta, k)a_{\vec{k}} + u^*(\eta, k)a_{\vec{k}}^\dagger.$$  \hspace{1cm} (54)

The conjugate momentum is:

$$p_{\vec{k}}(\eta) = H\Omega^2 q'_{\vec{k}}(\eta),$$

$$= -ikHv(\eta, k)a_{\vec{k}} + ikHv(\eta, k)a_{\vec{k}}^\dagger.$$  \hspace{1cm} (55)

Requiring that $q_{\vec{k}}(\eta)$ commute canonically with $p_{\vec{k}}(\eta)$ and making use of the Wronskian determines the commutator of $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$ to be:

$$[a_{\vec{k}}, a_{\vec{k}}^\dagger] = \frac{1}{H}.$$  \hspace{1cm} (57)

The “Hamiltonian” which generates the conformal time evolution of mode $\vec{k}$ is:

$$H^\eta_{\vec{k}} = \frac{p^2_{\vec{k}}}{2H\Omega^2} + \frac{1}{2}Hk^2\Omega^2 q^2_{\vec{k}}.$$  \hspace{1cm} (58)

This is just a harmonic oscillator with frequency $k$ and mass $H\Omega^2$. Because the mass is time dependent there are no stationary states but one can of course compute the expectation value of $H^\eta_{\vec{k}}$ in the presence of some state. In the far ultraviolet curvature is obviously a small effect, so we may assume the flat space vacuum:

$$a_{\vec{k}}|0\rangle = 0.$$  \hspace{1cm} (59)

Note that Heisenberg states do not evolve, and that the operator $a_{\vec{k}}$ was constructed to be time independent. Hence condition (59) persists, even after the originally ultraviolet mode has red shifted to the infrared.

The expectation value of $H_\eta$ is simple to take in the presence of this state:

$$\langle 0|H^\eta_{\vec{k}}(\eta)|0\rangle = \frac{1}{2H\Omega^2} \times H^2k^2v(\eta, k)v^*(\eta, k) \times \frac{1}{H}$$

$$+ \frac{1}{2}Hk^2\Omega^2 \times u(\eta, k)u^*(\eta, k) \times \frac{1}{H},$$  \hspace{1cm} (60)
\begin{align}
\frac{1}{4} k + \frac{1}{4k} \left(1 + \frac{1}{k^2 \eta^2}\right), \\
= \frac{1}{2} k + \frac{1}{4k \eta^2}.
\end{align}

Now exploit the relation between co-moving time and conformal time to relate the co-moving Hamiltonian to the conformal one:

\[ H^t = i \frac{\partial}{\partial t} = i \Omega^{-1} \frac{\partial}{\partial \eta} = \Omega^{-1} H^\eta \]  

(63)

It follows that the physical energy in mode \( k \) at co-moving time \( t \) is:

\[ E_k = \frac{1}{2} k e^{-Ht} + \frac{H^2}{4k} e^{Ht} \]  

(64)

The first term is just the familiar 0-point energy, appropriately red shifted. One way to understand the second term is that virtual gravitons whose physical wave lengths exceed the Hubble radius cannot recombine; they are pulled apart by the expansion of spacetime. The energy of any one such graviton redshifts, but there are so many produced that the total energy contributed by each infrared mode actually increases.

Since there are an infinite number of ultraviolet modes, the total 0-point energy diverges. This is not consistent with the assumption that the background is initially undergoing inflation with Hubble constant \( H \). To make the assumption consistent we must subtract the original 0-point energy by normal ordering. However, this has only a minuscule effect on the superadiabatically amplified 0-point energy of the infrared modes. We can use simple Newtonian ideas to obtain a crude estimate of the energy density induced by their self-gravitation.

To obtain the energy density of mode \( k \) we divide by the 3-volume:

\[ \rho_k = \left( E_k - \frac{1}{2} k e^{-Ht} \right) \div V(t) = \frac{H^5}{4k} e^{-2Ht}. \]  

(65)

Although this red shifts to zero, it does so more slowly than pure radiation. One consequence is that the associated Newtonian potential remains constant:

\[ -e^{-2Ht} k^2 \phi_k = 4\pi G \rho_k \quad \Rightarrow \quad \phi_k = -\frac{\pi G H^5}{k^3}. \]  

(66)
The total Newtonian potential from all infrared modes is accordingly:

\[ \varphi_{IR} = \frac{1}{\pi^2 H^3} \int_H^{H \exp[Ht]} dk k^2 \varphi_k = -\frac{G H^2}{\pi} H t. \]  (67)

This combines with the total 0-point energy density of infrared modes:

\[ \rho_{IR} = \frac{1}{\pi^2 H^3} \int_H^{H \exp[Ht]} dk k^2 \rho_k = \frac{H^4}{8\pi^2}, \]  (68)

to produce an \textit{increasing} gravitational interaction energy density:

\[ \rho_{Newt}(t) = \varphi_{IR} \cdot \rho_{IR} = -\frac{G H^6}{8\pi^3} H t. \]  (69)

This is down from \( \rho_{IR} \) by a factor of the small number \( G \Lambda / (3\pi) \lesssim 10^{-12} \), but its time dependence eventually makes it the more important effect.

The Newtonian estimate we have just obtained compares fairly well with the exact result of the lowest non-trivial order in perturbation theory [1]:

\[ \rho(t) = -\frac{G H^6}{8\pi^3} \left\{ (Ht)^2 + O(Ht) \right\} + O(G^2). \]  (70)

The extra factor of \( Ht \) derives from the inclusion of four relativistic effects which were omitted in the Newtonian approximation:

1. There is a 0-point pressure in addition to the 0-point energy density.
2. The Newtonian potential is not the only gravitational field.
3. The various gravitational potentials can carry momentum.
4. The gravitational interaction is not linear.

Although one must really do the quantum field theoretic calculation to get the right answer, it is simple enough to indicate how each of these effects modifies the Newtonian estimate.

If we assume that the 0-point stress-energy of each mode is separately conserved then the 0-point pressure works out to be:

\[ p_k = -\frac{1}{3} \rho_k = -\frac{H^5}{12k} e^{-2Ht}. \]  (71)
Like the 0-point energy density, the total 0-point pressure is only a constant — and very small — fraction of the pressure in the cosmological constant. However, unlike the 0-point energy, the 0-point pressure serves as the source for a gravitational potential whose homogeneous equation of motion is that of a massless, minimally coupled scalar \[12\]. It is straightforward to compute the retarded Green’s function for this potential:

\[
G(\eta, \vec{x}; \eta', \vec{x}') = -\frac{\theta(\Delta \eta)}{4\pi} \left\{ \frac{\Omega^{-1} \Omega'^{-1}}{\Delta x} \delta (\Delta \eta - \Delta x) + H^2 \theta (\Delta \eta - \Delta x) \right\}, \tag{73}
\]

where \(\Delta \eta \equiv \eta - \eta'\) and \(\Delta x \equiv \|\vec{x} - \vec{x}'\|\). The first term is just like its flat space cognate: the only contribution comes from sources on the actual light cone, so there is no growth for a constant source. However, the second term superposes over the entire past light cone, whose invariant volume \((43)\) grows like \(Ht\) in the classical background.

The third relativistic effect means that we should not view the precipitating event as the creation of two gravitons with opposite 3-momenta, whose stress energy then induces a gravitational potential containing zero 3-momentum. What really happens is the creation of a graviton with 3-momentum \(\vec{k}_1\) and another with 3-momentum \(\vec{k}_2\), which together induce a potential with 3-momentum \(- (\vec{k}_1 + \vec{k}_2)\). Although this still leaves two mode sums, they are not cleanly split between a 0-point stress energy and a potential term, as was the case for our Newtonian estimate.

The final relativistic effect means that we must include a bewildering variety of interactions where the potential scatters off one of the gravitons, or where it interacts with itself. This is one of the things that makes the quantum field theoretic calculation so difficult. We can nonetheless say that the effect is still due to the gravitational interaction between virtual gravitons which are pulled apart by inflation. Superadiabatic amplification can still be used to estimate the rate at which these gravitons are created and the stress energy which they carry. And it is generally the case that one factor of \(Ht\) derives from one of the two mode sums while the other factor of \(Ht\) comes from an integration over interaction times.

Correcting the Newtonian estimate for the induced energy density is not quite the end. Just as a pressure is associated with \(\rho_k\), so there is a pressure associated with \(\rho(t)\). We can find it by using energy conservation, which...
reads as follows in the classical background (3):

\[
\dot{\rho} = -3H(\rho + p) .
\] (74)

Combined with (70) this implies that most rapidly growing part of the induced pressure is exactly opposite that of the energy density:

\[
p(t) = \left( \frac{G}{8\pi^3} \right) \left\{ (Ht)^2 + O(Ht) \right\} + O(G^2) .
\] (75)

It follows that the interaction between infrared gravitons acts to screen inflation by an amount that becomes non-perturbatively large at late times.

At this point it is useful to recall some standard facts about inflation [13, 14] in order to form a proper impression both of the effect’s magnitude and of the time scale over which it acts. What is usually termed, the “scale of inflation,” is the mass \( M \) defined so that \( M^4 \) equals the energy density of the cosmological constant, \( \Lambda/(8\pi G) \). Since the Planck mass is \( M_p = G^{-1/2} \) we have:

\[
G\Lambda = 8\pi \left( \frac{M}{M_p} \right)^4 .
\] (76)

It is traditional to assume that \( M \) is GUT scale, which makes \( G\Lambda \sim 10^{-11} \). One sometimes encounters models with scales as low as that of electroweak symmetry breaking. (Past that point there is not enough CP violation to explain the observed baryon asymmetry.) Inflation on the electroweak scale would give \( G\Lambda \sim 10^{-67} \). These numbers mean that the gravitational interaction energy density is a very small fraction of \( M^4 \) unless \( Ht \) is enormous.

Although there are higher order corrections to the induced energy density (70) and pressure (75), it turns out that the lowest order effect becomes non-perturbatively strong when these higher terms are still insignificant [3]. The way this works is that the induced stress tensor serves as a source for corrections to the classical background (3). The pressure again engenders an extra factor of \( Ht \) from the invariant volume of the past light cone [12], and this causes the lowest order effect to throttle inflation before higher orders can become significant. When the background is expressed in co-moving coordinates (3), the function \( b(t) \) has the form:

\[
b(t) = Ht[1 + D(t)] + \frac{1}{2}\ln[1 + A(t)] ,
\] (77)
where we recall from (10-11) the perturbative expansions of $D(t)$ and $A(t)$ for small $\epsilon \equiv G\Lambda/(3\pi)$ and large $Ht$ [9]:

\begin{align}
D(t) &= + \frac{19}{2} (\epsilon Ht)^2 + O \left((\epsilon Ht)^3\right), \\
A(t) &= - \frac{172}{9} \epsilon^2 (Ht)^3 + O \left(\epsilon^3 (Ht)^4\right),
\end{align}

Perturbation theory breaks down when $A(t)$ approaches $-1$, at which time only the first term in the expansion of $A(t)$ is relevant and no term in $D(t)$ is significant [9].

The function $A(t)$ is of great importance because we saw, at the close of Section 2, that it gives the scalar during the perturbative regime:

$$\tilde{\phi}(t) = - \frac{1}{\sqrt{8\pi G}} \ln [1 + A(t)].$$

Summarizing and abstracting the preceding analysis, we may conclude that $A(t)$ derives, during the perturbative regime, from acting the retarded scalar Green’s function, $\Box^{-1}$, on a source that grows like $(Ht)^2$:

$$A = -8\pi G \int_0^t dt' e^{-3Ht'} \int_0^{t'} dt'' e^{3Ht''} \text{Source}(t'') = 8\pi G \frac{1}{\Box} (\text{Source}) ,$$

$$\text{Source} = 172 \frac{G H^6}{8\pi^3} (Ht)^2 + \ldots .$$

This source is the stress energy induced by the gravitational interaction between gravitons produced by superadiabatic amplification. It consists of a variety of stress energy-potential and potential-potential terms, each of which contains two mode sums over the 3-momenta of the gravitons. One factor of $Ht$ is attributable to one of these mode sums, the other comes from the superposition over interaction times in a potential. Part of the task of generalization is straightforward since the potentials are obtained from the retarded Green’s function, which can be defined for any homogeneous and isotropic background:

$$\frac{1}{\Box} f \equiv - \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} f(t'').$$

What remains is to understand superadiabatic amplification on a general homogeneous and isotropic background.
We begin by comparing the co-moving element with the conformal one:

\[ ds^2 = -dt^2 + e^{2b(t)} d\bar{x} \cdot d\bar{x} = \Omega^2 \left( -d\eta^2 + d\bar{x} \cdot d\bar{x} \right) , \tag{84} \]

to infer general relations for the conformal factor and the coordinate transformation:

\[ \Omega(\eta) = \exp[b(t)] \quad dt = \Omega d\eta . \tag{85} \]

Let us denote by \( \eta = \eta_i \) the image of \( t = 0 \). Although we cannot be explicit for a general \( b(t) \), any superluminally expanding spacetime will have \( \eta_i < 0 \). We can also assume that the approach to the infinite future is \( \eta \to 0^- \), as before. The manifold is still \( T^3 \times \mathbb{R} \), with each of the coordinate radii equal to \( H^{-1} \). We therefore conclude that the 3-volume and the graviton mode sum are:

\[ V(t) = H^{-3} e^{3b(t)} , \tag{86} \]
\[ 2 \sum_{\bar{n}} f \left( 2\pi H \| \bar{n} \| \right) \approx \frac{1}{\pi^2 H^3} \int dk k^2 f(k) . \tag{87} \]

It turns out that there is always a factor of \( V^{-1} \) associated with each mode sum so that the factors of \( H^3 \) cancel and any physical dependence upon the range of co-moving coordinates must come from the limits of integration.

Dynamical graviton modes still obey the equation of motion of a massless, minimally coupled scalar [3]. The mode coordinates for such a scalar are still obtained by spatial Fourier transforming according to (48), and the formula for the Lagrangian is unchanged from (49), provided the general conformal factor is understood. What changes is the Heisenberg equations of motion:

\[ q''_{\bar{k}} + 2 \frac{\Omega'}{\Omega} q'_{\bar{k}} + k^2 q_{\bar{k}} = 0 . \tag{88} \]

Redefining the field variable as \( Q(\eta) \equiv \Omega q_{\bar{k}}(\eta) \) gives the following suggestive equation:

\[ Q'' + \left( k^2 - \frac{\Omega''}{\Omega} \right) Q = 0 . \tag{89} \]

There are two regimes in which good approximate solutions can be found: the far ultraviolet, where \( k^2 \gg \Omega''/\Omega \), and the far infrared where \( k^2 \ll \Omega''/\Omega \). Before going on to study these regimes we should note the important fact
that they can be given an invariant specification. In view of the relation (85) between conformal and co-moving coordinates we can write:

$$\frac{\Omega''}{\Omega} = \frac{d}{dt} e^b \frac{d}{dt} e^b = e^{2b} \left( \ddot{b} + 2\dot{b}^2 \right) = \frac{1}{6} e^{2b} \hat{R}.$$  (90)

The two regimes are therefore characterized by how the physical (i.e., red shifted) momentum compares with the square root of the Ricci scalar. In view of our definition (46) for the classical background, it is reasonable to make the following general definition for “infrared” gravitons:

$$\text{Infrared} \iff e^{-2b(t)} H^2 \lesssim e^{-2b(t)} k^2 \lesssim \frac{1}{12} \hat{R}(t).$$  (91)

As before, we refer to the higher modes as “ultraviolet.”

In the ultraviolet regime the Ricci scalar term is a perturbation:

$$Q'' + k^2 Q = \frac{\Omega''}{\Omega} Q.$$  (92)

We can obtain the mode functions by iterating those of flat space:

$$Q_{uv}(\eta, k) = \frac{1}{\sqrt{2k}} e^{-ik\eta} + \int_{-\infty}^{\eta} d\eta' \frac{1}{k} \sin [k(\eta - \eta')] \frac{\Omega''(\eta')}{\Omega(\eta')} Q_{uv}(\eta', k).$$  (93)

This obviously results in a series in powers of $1/k$, the successive terms of which are less and less significant for large, negative $\eta$.

The mode functions associated with $Q_{uv}(\eta, k)$ are:

$$u(\eta, k) = \Omega^{-1}(\eta) Q_{uv}(\eta, k),$$  (94)

$$v(\eta, k) = \frac{i}{k} [\Omega(\eta) Q'_{uv}(\eta, k) - \Omega'(\eta) Q_{uv}(\eta, k)].$$  (95)

The associated Wronskian:

$$u(\eta, k) v^*(\eta, k) + u^*(\eta, k) v(\eta, k) =$$

$$\frac{i}{k} [Q'_{uv}(\eta, k) Q^*_{uv}(\eta, k) - Q_{uv}(\eta, k) Q'^*_{uv}(\eta, k)].$$  (96)

is constant in consequence of the equation of motion (89). Since $Q_{uv}(\eta, k)$ approaches the flat space mode functions at early times, we see that the
constant is $1/k$, the same as before. The subsequent operator expansions and commutation relations:

$$q_{\vec{k}}(\eta) = u(\eta, k)a_{\vec{k}} + u^*(\eta, k)a_{\vec{k}}^\dagger,$$

$$p_{\vec{k}}(\eta) = -ikHv(\eta, k)a_{\vec{k}} + ikHv(\eta, k)a_{\vec{k}}^\dagger,$$

$$[a_{\vec{k}}, a_{\vec{k}}^\dagger] = \frac{1}{H},$$

are the same as for the classical background. And the expectation value of the Hamiltonian in the presence of the state annihilated by $a_{\vec{k}}$ is:

$$\langle 0 | H_{\vec{k}}^0 | 0 \rangle = \frac{1}{2} \left[ Q_{uv}' - \frac{\Omega'_2}{\Omega} Q_{uv} \right] \ast \left[ Q_{uv}' - \frac{\Omega'_2}{\Omega} Q_{uv} \right] + \frac{1}{2} k^2 Q_{uv}' Q_{uv}. \tag{100}$$

Although this formula is correct for all times, obtaining the leading behavior for small $\eta$ requires that we develop an infrared expansion for the mode functions.

In the infrared regime the momentum term is a perturbation:

$$Q'' - \frac{\Omega''}{\Omega} Q = -k^2 Q,$$ \tag{101}

At zeroth order the two independent solutions are:

$$Q_{10}(\eta) = \Omega(\eta) \quad Q_{20}(\eta) = -\Omega(\eta) \int_\eta^0 \frac{d\eta'}{\Omega^2(\eta')}.$$ \tag{102}

For superluminal expansion the limit $\eta \to 0^-$ carries $Q_{10}(\eta)$ to infinity, while $Q_{20}(\eta)$ goes to zero as $\sim \eta \Omega^{-1}$. The limit $\eta \to -\infty$ has the opposite effect: $Q_{10}(\eta)$ approaches zero and $Q_{20}(\eta)$ goes to infinity as $\sim \eta \Omega^{-1}$.

In view of the limiting forms for $Q_{10}(\eta)$ and $Q_{20}(\eta)$ it is the advanced Green’s function:

$$G_{\text{adv}}(\eta, \vec{\eta}) = \theta(\eta - \vec{\eta}) \left[ Q_{10}(\eta)Q_{20}(\vec{\eta}) - Q_{20}(\eta)Q_{10}(\vec{\eta}) \right], \tag{103}$$

which gives a reasonable equation to iterate for the full solution based on $Q_{20}$:

$$Q_2(\eta, k) = Q_{20}(\eta) - k^2 \int_{-\infty}^0 d\vec{\eta} G_{\text{adv}}(\eta, \vec{\eta}) Q_{20}(\vec{\eta}, k). \tag{104}$$
The resulting series is:
\[ Q_2 = \sum_{\ell=0}^{\infty} (-k^2)^\ell G^\ell_{\text{adv}} \cdot Q_{20}, \tag{105} \]
where the \( \ell \)-th “power” of the Green’s function denotes the \( \ell \)-fold integration:
\[ G^\ell_{\text{adv}} \cdot Q_{20} \equiv \int_{-\infty}^{0} d\eta_1 G_{\text{adv}}(\eta, \eta_1) \cdots \int_{-\infty}^{0} d\eta_\ell G_{\text{adv}}(\eta_{\ell-1}, \eta) Q_{20}(\eta) \tag{106} \]
For asymptotically small \( \eta \) this \( \ell \)-th power goes to zero like \( \sim \eta^{2\ell}\Omega^{-1} \). The expansion (105) is therefore in terms which are less and less significant at late times.

The square of \( Q_{10}(\eta) \) is not integrable at \( \eta = 0 \). To obtain the full solution based on \( Q_{10} \) we must therefore begin iterating with a Green’s function which is intermediate between advanced and retarded:
\[ G^\ell_{\text{int}}(\eta, \eta) = \theta(\eta - \eta) Q_{20}(\eta) Q_{10}(\eta) + \theta(\eta - \eta) Q_{10}(\eta) Q_{20}(\eta) + \theta(\eta - \eta) Q_{10}(\eta) Q_{20}(\eta). \tag{107} \]
The result is a series whose terms have the asymptotic behavior \( \sim \eta^{2\ell}\Omega \). For high enough \( \ell \) the integral of such a term times \( Q_{10} \) no longer converges at large \( \eta \). At this point we must continue the iteration using the advanced Green’s function. The full series is:
\[ Q_1 = \sum_{\ell=0}^{N} (-k^2)^\ell G_{\text{int}}^\ell \cdot Q_{10} + \sum_{\ell=N+1}^{\infty} (-k^2)^\ell G_{\text{adv}}^{\ell-N} \cdot G_{\text{int}}^N \cdot Q_{10}, \tag{108} \]
where \( N \) is the order of perturbation theory at which we must change Green’s functions. As with the expansion for \( Q_2 \), the small \( \eta \) behavior of each successive term is weaker than that of its predecessor by a factor of \( \eta^2 \).

Since \( Q_1(\eta, k) \) and \( Q_2(\eta, k) \) are independent solutions of the same linear, second order differential equation as \( Q_{uv}(\eta, k) \), we can find complex numbers \( \alpha \) and \( \beta \) to enforce the condition:
\[ Q_{\text{ir}}(\eta, k) \equiv \alpha Q_1(\eta, k) + \beta Q_2(\eta, k), \tag{109} \]
\[ = Q_{uv}(\eta, k). \tag{110} \]
Although \( Q_{\text{ir}}(\eta, k) = Q_{uv}(\eta, k) \) over the full range of \( \eta \), we can estimate \( \alpha \) by matching the zeroth order solutions at the boundary between ultraviolet and infrared. This gives the relation:
\[ \alpha \sim \frac{\Omega_{1}^{-1}}{\sqrt{2k}} e^{-ik\eta}, \tag{111} \]
where \( \eta_1 \) is the conformal time at which \( 2k^2 = \Omega''_1/\Omega_1 \).

Our estimate (111) can be checked whenever explicit solutions exist. For example, suppose the scale factor is a power law which obeys the same initial conditions \( (b(0) = 0 \text{ and } \dot{b}(0) = H) \) as the classical background:

\[
e^b(t) = \left(1 + \frac{Ht}{p}\right)^p.
\]  

(We assume \( p > 1 \) to make the expansion superluminal.) The conformal factor is:

\[
\Omega = \left(\frac{\eta}{\eta_0}\right)^{p-1},
\]  

(113)

where \( \eta_0 = -p/(p - 1) \ H^{-1} \) and the properly normalized solution is:

\[
Q(\eta, k) = \frac{1}{2\sqrt{\pi \eta}}e^{-i\frac{\pi}{2}(\nu + \frac{1}{2})}H^{(2)}_{\nu}(k\eta), \quad \nu = \frac{1}{2}\left(\frac{3p - 1}{p - 1}\right).
\]  

(114)

The infrared solutions \( Q_1(\eta, k) \) and \( Q_2(\eta, k) \) give the two Bessel functions from which the Hankel function is constructed:

\[
\alpha Q_1(\eta, k) = \frac{i}{2}\csc(\nu\pi)e^{-i\frac{\pi}{2}(\nu + \frac{1}{2})}\sqrt{\pi \eta}J_{-\nu}(k\eta),
\]  

(115)

\[
\beta Q_2(\eta, k) = -\frac{1}{2}\csc(\nu\pi)e^{i\frac{\pi}{2}(\nu + \frac{1}{2})}\sqrt{\pi \eta}J_{\nu}(k\eta).
\]  

(116)

By making the first terms of the respective series expansions agree with \( Q_{10}(\eta) \) and \( Q_{20}(\eta) \) we infer the following expressions for \( \alpha \) and \( \beta \):

\[
\alpha = \frac{\Gamma(\nu)}{\sqrt{2\pi k}} \left(\frac{2i}{k\eta_0}\right)^{\nu - \frac{1}{2}} = \frac{\Gamma(\nu)}{\sqrt{\pi}} \left(\frac{1}{32} - \frac{1}{8\nu^2}\right)^{\frac{1}{2} - \frac{1}{4}\nu} \frac{\Omega_1^{-1}}{\sqrt{2k}},
\]  

(117)

\[
\beta = -i\nu \sqrt{\frac{2\pi}{k}} \csc(\nu\pi) \left(\frac{ik\eta_0}{2}\right)^{\nu - \frac{1}{2}}
\]

\[
= -2\nu i \sqrt{\pi} \csc(\nu\pi) \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1)} \left(\frac{1}{32} - \frac{1}{8\nu^2}\right)^{\frac{1}{2} - \frac{1}{4}\nu} \frac{\Omega_1^{-1}}{\sqrt{2k}}.
\]  

(118)

Except for powers very close to the superluminal bound \( (p = 1) \) the index \( \nu \) is of order one and we see that \( \alpha \) agrees with estimate (111). Unless \( \nu \)
happens to be very close to an integer or half integer we also see that $\alpha$ and $\beta$ are comparable.

Our series expansions imply the following relations for late times:

$$ Q_{ir}(\eta, k) \to \alpha \Omega(\eta) ,$$

$$ Q'_{ir}(\eta, k) - \frac{\Omega'(\eta)}{\Omega(\eta)} Q_{ir}(\eta, k) \to \alpha \eta \Omega(\eta) .$$

Substitution in (100) gives the late time behavior of the (conformal) energy in mode $\vec{k}$:

$$ \langle 0| H_{\vec{k}}^0(\eta)|0 \rangle \to \frac{1}{2} \alpha^* \Omega^2(\eta) \sim \frac{R_1}{12k} \Omega^2(\eta) ,$$

where $R_1$ is the value of the Ricci scalar at the time mode $\vec{k}$ crosses from the ultraviolet regime to the infrared. The form of $R_1(k)$ depends upon $b(t)$. For general power law inflation (112) the Ricci scalar is:

$$ \hat{R}(t) \equiv 6 \left( \ddot{b}(t) + 2 \dot{b}^2(t) \right) = \left( \frac{H}{1 + H t / p} \right)^2 \left( -\frac{6}{p} + 12 \right) ,$$

and we find:

$$ R_1(k) = 12 \left( \frac{2p - 1}{2p} \right)^{\frac{1}{p+1}} \left( \frac{H}{k} \right)^{\frac{2}{p+1}} H^2 .$$

Taking $p$ to infinity recovers the de Sitter result, $R_1(k) = 12H^2$. We therefore expect that $R_1(k)$ is generally a slowly decreasing or constant function of $k$.

It is now straightforward to generalize what was done in classical background. The co-moving energy is down by a factor of $\Omega$ so it approaches:

$$ E_{\vec{k}} \to \frac{1}{2} \alpha^* k^2 e^{b(t)} \sim \frac{R_1}{48k} e^{b(t)} .$$

Dividing by the 3-volume gives the energy density:

$$ \rho_{\vec{k}} \to \frac{1}{2} \alpha^* k^3 e^{-2b(t)} \sim \frac{H^3 R_1}{48k} e^{-2b(t)} .$$

If the stress energy in mode $\vec{k}$ is separately conserved it is straightforward to infer the pressure:

$$ \dot{\rho}_{\vec{k}} = -3 \dot{b}(\rho_{\vec{k}} + p_{\vec{k}}) \implies p_{\vec{k}} = -\frac{1}{3} \rho_{\vec{k}} .$$
We can even parallel our previous estimate for the Newtonian potential of mode \( \bar{k} \):

\[
-e^{-2b(t)}k^2\varphi_{\bar{k}} = 4\pi G \rho_{\bar{k}} \quad \Rightarrow \quad \varphi_{\bar{k}} = -2\pi G \alpha \alpha^* H^3 \sim -\frac{\pi G R_1 H^3}{12k^3} .
\]  

(127)

Of course all these results reduce to those obtained for the classical background when we set \( R_1 = 12H^2 \). The difference is that we can now recognize the Ricci scalar as the source of the stress energy and the gravitational potentials whose various interactions lead to screening.

It remains to see what becomes of the two mode sums. Although we have already explained that the Newtonian estimate misses some essential features of the fully relativistic result, it can still be used to get a rough idea. In the Newtonian approximation the total infrared energy density and potential are the following mode sums:

\[
\rho_{IR} = \frac{1}{\pi^2 H^3} \int_{H}^{K(t)} dk k^2 \rho_{\bar{k}} \sim \frac{1}{48\pi^2} e^{-2b(t)} \int_{H}^{K(t)} dk k R_1(k) ,
\]

(128)

\[
\varphi_{IR} = \frac{1}{\pi^2 H^3} \int_{H}^{K(t)} dk k^2 \varphi_{\bar{k}} \sim -\frac{G}{12\pi} \int_{H}^{K(t)} \frac{dk}{k} R_1(k) ,
\]

(129)

where \( K^2(t) \equiv \hat{R}(t) \exp[2b(t)]/12 \). The point which generalizes about this is that, for the dominant terms, one of the mode sums goes like \( \exp[2b]dk \), whereas the other has the form \( dk/k \).

The relation:

\[
k^2 = \frac{1}{12} R_1 e^{2b_1} ,
\]

(130)

can be used to convert these mode sums into integrations over time. For example, the general power law inflation (112) gives:

\[
kdk = \left( \frac{p - 1}{p} \right) \frac{1}{12} R_1 e^{2b} \tilde{b}_1 dt_1 .
\]

(131)

Except for very slow inflation \( (p \sim 1) \) the time dependence of the Ricci scalar is negligible compared with that of the scale factor. We can therefore write:

\[
kdk \approx \frac{1}{12} R_1 e^{2b} \tilde{b}_1 dt_1 .
\]

(132)

In the same approximation the total infrared energy density becomes:

\[
\rho_{IR} \approx \frac{1}{576\pi^2} e^{-2b(t)} \int_0^t dt' \tilde{b}(t') e^{2b(t')} \hat{R}^2(t') \approx \frac{\hat{R}^2(t)}{1152\pi^2} .
\]

(133)
Hence we conclude that the $\exp[2b]kd\kappa$ mode sum engenders an extra factor of the Ricci scalar.

The $dk/k$ mode sum is more subtle. Of course we can convert to an integration over time as before:

$$\varphi_{IR} \approx -\frac{G}{12\pi} \int_0^t dt' \dot{b}(t') \tilde{R}(t').$$

(134)

However, further progress requires that we neglect $\ddot{b}$ relative to $\dot{b}^2$. For the general power law (112) the ratio $\ddot{b}/\dot{b}^2$ is $-1/p$, which suggests that the approximation is also valid for rapid inflation. We can use this to introduce a second integration, and write the result in terms of $\Box^{-1}$:

$$\varphi_{IR} \approx -\frac{G}{\pi} \int_0^t dt' \dot{b}^3(t'),$$

(135)

$$\approx -\frac{3G}{\pi} \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} \dot{b}^4(t''),$$

(136)

$$\approx \frac{G}{48\pi} \Box \left( \tilde{R}^2 \right).$$

(137)

Our conclusion is accordingly that the $dk/k$ mode sum goes to $\Box^{-1} R$.

It would be a mistake to take the Newtonian approximation too seriously. For example, although the source term of equation (81) contains two mode sums, they are not associated one with a 0-point stress energy and the other with a potential. Nor must the source have the Newtonian form of a stress energy times a potential; one can also have products of derivatives of potentials, and there are terms in which the two virtual gravitons interact with the potential at different times. What can reasonably be concluded is that the source term of equation (81) involves two factors of $\Box^{-1}$ acting in some order on five Ricci scalars.

5 Ansätze for the scalar

The purpose of this section is to discuss possibilities for the scalar $\phi[g]$ that are consistent with the results of the last three sections. We begin by reviewing these results, then we tabulate the 33 candidate invariants which can be constructed from the retarded scalar Green’s function $\Box^{-1}$ and the Ricci scalar $R$. The section closes with a discussion of the possibilities for
involving other Green’s functions, and for replacing some of the five Ricci scalars by perturbatively equivalent factors of $4\Lambda$.

Relation (30) of Section 2 suggests that we take the scalar to be the logarithm of a dimensionless invariant:

$$
\phi[g] = -\frac{1}{\sqrt{8\pi G}} \ln(1 - F[g]) \tag{138}
$$

In the classical background (3) this invariant must reduce to:

$$
\tilde{F}(t) = \frac{172}{9} \left( \frac{G\Lambda}{3\pi} \right)^2 (Ht)^3 + \text{subdominant} \tag{139}
$$

From Section 3 we learned that $F[g](x)$ must be a dimensionless invariant which depends causally upon the fields within the past lightcone of $x^\mu$. The requirement of a stable initial value problem implies that any factors of the Riemann tensor must be protected by at least two inverse derivatives, with derivatives of the curvature protected by correspondingly more inverse derivatives. This is consistent with the fact that the important part of $F[g](x)$ must be non-local. However, correspondence with the known results for $\Lambda = 0$ puts severe restrictions on how many inverse derivatives can appear. To be precise, the induced stress tensor (6) must either vanish with $\Lambda$ or else it must not be more infrared singular than $\partial^4 \ln(\partial^2)$ when subjected to the derivative expansion for weak fields in a flat space background.

Section 4 used the physics of our mechanism to show that $F[g]$ has the form of the scalar retarded Green’s function $\square^{-1}$ acting on a source constructed from two more factors of $\square^{-1}$ and five factors of the Ricci scalar $R$. This reduces the problem to combinatorics. There are 21 ways of placing the factors of $R$ when the two retarded Green’s functions act in series, and there are 12 distinct placements when they act in parallel. Table 1 lists the 33 possibilities. If one requires that at least one factor of $R$ must reside at each of the three locations then only ten terms remain.

All of the terms in Table 1 are manifestly invariant, causal and non-local. Note also that various the factors of $R$ are protected by enough factors of $\square^{-1}$ so as not to jeopardize the stability of the initial value problem. There is no problem with the flat space limit since the source terms behave generically like $\partial^6$ in the weak field derivative expansion. This means that the scalar —

---

4In fact $\tilde{F}(t)$ is just minus the perturbative coefficient function $A(t)$ of expression (11).
| #  | Candidate Source                                      | #  | Candidate Source                                      |
|----|------------------------------------------------------|----|------------------------------------------------------|
| 1a | $2 \, R^3(\Box^{-1}1(\Box^{-1}1))$                 | 1b | $R^3(\Box^{-1}1) (\Box^{-1}1)$                       |
| 2a | $2 \, R^4(\Box^{-1}R(\Box^{-1}1))$                 | 2b | $R^4(\Box^{-1}R) (\Box^{-1}1)$                       |
| 3a | $2 \, R^4(\Box^{-1}1(\Box^{-1}R))$                 | 3b | $R^4(\Box^{-1}R^2) (\Box^{-1}1)$                     |
| 4a | $2 \, R^3(\Box^{-1}R^2(\Box^{-1}1))$               | 4b | $R^3(\Box^{-1}1) (\Box^{-1}1)$                       |
| 5a | $2 \, R^3(\Box^{-1}R(\Box^{-1}1))$                 | 5b | $R^3(\Box^{-1}R) (\Box^{-1}1)$                       |
| 6a | $2 \, R^3(\Box^{-1}1(\Box^{-1}R^2))$               | 6b | $R^3(\Box^{-1}R^2) (\Box^{-1}1)$                     |
| 7a | $2 \, R^2(\Box^{-1}R^3(\Box^{-1}1))$               | 7b | $R(\Box^{-1}R^4) (\Box^{-1}1)$                       |
| 8a | $2 \, R^2(\Box^{-1}R(\Box^{-1}1))$                 | 8b | $R(\Box^{-1}R^3) (\Box^{-1}1)$                       |
| 9a | $2 \, R^2(\Box^{-1}R(\Box^{-1}R^2))$               | 9b | $R(\Box^{-1}R^2) (\Box^{-1}R^2)$                     |
| 10a| $2 \, R^2(\Box^{-1}1(\Box^{-1}R^3))$               | 10b| $(\Box^{-1}R^5) (\Box^{-1}1)$                        |
| 11a| $2 \, R(\Box^{-1}R^4(\Box^{-1}1))$                 | 11b| $(\Box^{-1}R^4) (\Box^{-1}1)$                        |
| 12a| $2 \, R(\Box^{-1}R^3(\Box^{-1}1))$                 | 12b| $(\Box^{-1}R^3) (\Box^{-1}R^2)$                     |
| 13a| $2 \, R(\Box^{-1}R^2(\Box^{-1}R^2))$               | 14a| $2 \, R(\Box^{-1}1(\Box^{-1}R^3))$                 |
| 15a| $2 \, R(\Box^{-1}1(\Box^{-1}R^4))$                 | 15b| $2 \, R(\Box^{-1}R^5(\Box^{-1}1))$                 |
| 16a| $2 \, R(\Box^{-1}R^4(\Box^{-1}R^2))$               | 17a| $2 \, R(\Box^{-1}R^4(\Box^{-1}R))$                 |
| 18a| $2 \, R(\Box^{-1}R^3(\Box^{-1}R^2))$               | 19a| $2 \, R(\Box^{-1}R^2(\Box^{-1}R^3))$               |
| 20a| $2 \, R(\Box^{-1}R(\Box^{-1}R^4))$                 | 21a| $2 \, R(\Box^{-1}1(\Box^{-1}R^5))$                 |

Table 1: Candidate sources for $F[g]$. To obtain the functional $F[g]$ act $-\frac{43}{48} \left(\frac{G}{12\pi}\right)^2 \Box^{-1}$ on the source.
and hence the induced stress tensor — goes like $\partial^4$, which is perfectly allowed. Though it would not have been legimate to impose this as a requirement, it is reassuring to note that all of the terms approach a constant for subluminal expansion.

It is important to note that the actual source need not consist of a single term from Table 1; it could equally well be a linear combination. Though we will not attempt it here, an explicit derivation of the source seems possible in two different ways. First, we might re-compute the leading two loop diagrams for a general homogeneous and isotropic background, using the methods of Section 4 to isolate the dominant terms and to express them in invariant form. This would be difficult, but no more so than the original computation. The second technique exploits the fact that the terms of Table 1 survive in the limit $\Lambda \to 0$. They must therefore appear in the usual effective action, which can be invoked to further constrain the allowed combinations. One might even be able to pick off the coefficients by subjecting the two loop effective action to an expansion in powers of curvature along the lines developed by Barvinsky and Vilkovisky.

It remains to discuss two issues, the first of which is why the non-locality is confined to inverses of the scalar d’Alembertian:

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right)$$

$$\longrightarrow \Box = -e^{-3b} \frac{d}{dt} \left( e^{3b} \frac{d}{dt} \right) \longrightarrow \Box = -e^{-3Ht} \frac{d}{dt} \left( e^{3Ht} \frac{d}{dt} \right). \quad (140)$$

Of course we derived its presence from the physics of the mechanism, but why do we not need to allow for the possibility of inverses of other differential operators? It has been pointed out that the kinematics of free gravitons on a homogeneous and isotropic background is governed by $\Box$ but there are also non-dynamical, constrained modes which possess a different kinetic operator $D_B$:

$$\hat{D}_B \equiv -e^{-b} \frac{d}{dt} \left( e^{-b} \frac{d}{dt} e^{2b} \right). \quad (141)$$

We will additionally consider the kinetic operator of a massless, conformally

---

5This sounds paradoxical — in view of the fact that the “in” vacuum does not evolve into the “out” vacuum — but it isn’t really. The distinction between the two vacua simply means that the imaginary part of the effective action is non-zero (in fact, infrared divergent) when evaluated at the classical background.
coupled scalar:

\[
\hat{\mathcal{D}}_{\text{conf}} \equiv \widehat{\square} - \frac{1}{6} \hat{\mathcal{R}} = -e^{-2b} \frac{d}{dt} \left( e^b \frac{d}{dt} e^b \right),
\]  

although there is no dynamical reason to suspect its presence.

The reason we ignore \( \hat{\mathcal{D}}^{-1}_B \) and \( \hat{\mathcal{D}}^{-1}_{\text{conf}} \) is that they act to produce constants during the perturbative regime, and they are even less relevant for slower expansion. To see this, consider how each of the three inverse differential operators acts on a function of time \( f(t) \) for an arbitrary homogeneous and isotropic background:

\[
\begin{align*}
\hat{\square} f &= -\int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} f(t'') , \quad (143) \\
\frac{1}{\hat{D}}_B f &= -e^{-2b(t)} \int_0^t dt' e^{b(t')} \int_0^{t'} dt'' e^{b(t'')} f(t'') , \quad (144) \\
\frac{1}{\hat{D}}_{\text{conf}} f &= -e^{-b(t)} \int_0^t dt' e^{-b(t')} \int_0^{t'} dt'' e^{2b(t'')} f(t'') . \quad (145)
\end{align*}
\]

In the perturbative regime \( b(t) = Ht \) and \( \hat{\mathcal{R}}(t) = 12H^2 \). Therefore acting each of the three operators on the Ricci scalar gives:

\[
\begin{align*}
\hat{\square} \hat{\mathcal{R}} &= -4Ht + \frac{4}{3} - \frac{4}{3} e^{-3Ht} , \quad (146) \\
\frac{1}{\hat{D}}_B \hat{\mathcal{R}} &= -6 \left( 1 - 2e^{-Ht} + e^{-2Ht} \right) , \quad (147) \\
\frac{1}{\hat{D}}_{\text{conf}} \hat{\mathcal{R}} &= -6 \left( 1 - 2e^{-Ht} + e^{-2Ht} \right) . \quad (148)
\end{align*}
\]

Except for the beginning of inflation, the operators \( \hat{\mathcal{D}}_B \) and \( \hat{\mathcal{D}}_{\text{conf}} \) are indistinguishable from constants during the perturbative regime! On the other hand, \( \hat{\square}^{-1} \) grows like \( Ht \). When we recall that this factor must be at least \( 10^7 \) before anything interesting happens, the other operators are negligible in comparison.

The other operators perform even worse during the slower expansion that should follow the end of inflation. Suppose we call the transition time \( t_z \). It is useful to break the double integrations of the operators into periods which are before and after \( t_z \):

\[
\int_0^t dt' \int_0^{t'} dt'' = \int_0^{t_z} dt' \int_0^{t'} dt'' + \int_{t_z}^t dt' \int_0^{t'} dt'' + \int_{t_z}^t dt' \int_{t_z}^{t'} dt'' . \quad (149)
\]
Since there are so many more e-foldings before $t_z$ than afterwards the largest contribution for each of the three operators comes from the first of the three integrals to the right of (149). Since this integral is restricted to the perturbative regime we can estimate the post inflationary behavior of the three operators:

$$\frac{1}{\Box} \hat{R} \approx -4Ht_z, \quad (150)$$

$$\frac{1}{D_B} \hat{R} \approx -6e^{-2b(t)+2b(t_z)}, \quad (151)$$

$$\frac{1}{D_{\text{conf}}} \hat{R} \approx -6e^{-b(t)+b(t_z)}. \quad (152)$$

Whereas $\Box^{-1}$ gives an enormous constant, the other operators are only of order unity at $t = t_z$, and they fall off thereafter. It follows that these operators are irrelevant.

So much for other operators; the final issue is whether some of the five factors of $R$ should be replaced by $4\Lambda$. Although these terms are distinct for a general homogeneous and isotropic background, it is not possible to distinguish them during the perturbative regime:

$$R \rightarrow \hat{R} = 6(\ddot{b} + 2\dot{b}^2) \rightarrow \bar{R} = 12H^2. \quad (153)$$

For the factors of $R$ which occur furthest back in time it probably makes little difference whether or not they are replaced by $4\Lambda$. The integrals are dominated by the inflationary period, during which the two terms agree. On the other hand, the factors of $R$ which appear latest might play an important role in the post inflationary regime where they are insignificant compared with $4\Lambda$.

Of course one can consider the physics of screening on a general background, as we did in Section 4. This favors $R$ over $4\Lambda$. The preference is very strong for the two factors of $R$ from the 0-point energy of the virtual gravitons (123) and for the factor from the $dk$ mode sum (122). It is somewhat weaker for the factor of $R$ from the $dk/k$ mode sum (137), and it is very thin for the factor that was introduced to compensate the dimensions of the other $\Box^{-1}$ in the source.

We can therefore find some justification to consider sources involving four or only three factors of the Ricci scalar. There are 15 ways of placing four $R$'s when the two factors of $\Box^{-1}$ act sequentially, and there are 9 distinct
Table 2: Candidate sources for $F[g]$ with only three Ricci scalars. To obtain the functional $F[g]$ act $-\frac{43}{48} \left( \frac{G \Lambda}{3 \pi} \right)^2 \Box^{-1}$ on the source.

placements for the parallel case. We shall not bother listing them all. Table 2 gives the 10 series and 6 parallel candidates that contain only three factors of $R$. It is worth remarking that the presence of explicit factors of $\Lambda$ would preclude deriving the induced stress tensor from the usual effective action, however, one could still derive the actual source by re-computing the dominant two loop contribution in a general homogeneous and isotropic background.

6 Numerical evolution

Despite the enormous restriction we have obtained on the possible forms for the scalar $\phi[g]$, it is still not quite unique. The purpose of this section is to develop and to implement a scheme for numerically evolving whatever model is defined when one finally settles upon a choice for $\phi[g]$. Our procedure will be to select one of the ansätze of the previous section and then evolve it far enough past the end of inflation to infer the asymptotic behavior at late times. We shall also obtain explicit results for the number of e-foldings which occur before the end of inflation, the rapidity of the transition, and for the
Recall from Section 2 that the evolution equation is:
\[ \ddot{b} = -4\pi G \left( \frac{d\dot{\phi}[b]}{dt} \right)^2, \]  
(154)

subject to the initial condition \( b(0) = 0 \) and \( \dot{b}(0) = H \). This gives \( b(t) \). From the field effective field equation (13-14) we see that the total energy density and pressure, including the contribution from the cosmological constant, are:
\[ \rho_{\text{tot}} = \frac{1}{8\pi G} 3\dot{b}^2, \]  
(155)
\[ p_{\text{tot}} = \frac{1}{8\pi G} \left( -2\ddot{b} - 3\dot{b}^2 \right). \]  
(156)

This means that the instantaneous equation of state can be reconstructed as follows:
\[ \frac{p_{\text{tot}}(t)}{\rho_{\text{tot}}(t)} = -\frac{2\ddot{b}(t)}{\dot{b}^2(t)} - 1. \]  
(157)

Recall as well that the scalar can be written in terms of a dimensionless invariant \( F[g] \):
\[ \phi[g] = -\frac{1}{\sqrt{8\pi G}} \ln \left( 1 - F[g] \right). \]  
(158)

Substitution gives the following general evolution equation:
\[ \ddot{b} = -\frac{1}{2} \left( \frac{d\tilde{F}/dt}{1 - F[b]} \right)^2. \]  
(159)

In Section 5 we argued that \( F[g] \) has the form of a retarded scalar Green’s function acting on a source composed of two more retarded scalar Green’s functions which act in some order on three to five factors of the Ricci scalar.

Before specializing to a particular ansatz, we should comment on the qualitative behavior of all models. During inflation the time derivative of the dynamical variable \( b(t) \) is a large positive constant. Inflation is brought to an end when the inherently negative second derivative becomes large enough to force \( \dot{b}(t) \) down to nearly zero. It is clear from (159) that this must occur.
when $\hat{F}[b](t)$ approaches unity. All viable models show the following growth for $\hat{F}(t)$ during the perturbative regime:

$$\hat{F}(t) = \frac{172}{3} \epsilon^2 (Ht)^3 + \text{subdominant},$$

where we define the dimensionless parameter $\epsilon \equiv GA/(3\pi)$. Our perturbative estimate for the number of inflationary e-foldings is accordingly [9]:

$$N_{\text{pert}} = \left( \frac{9}{172} \right)^{\frac{1}{3}} \left( \frac{3\pi}{GA} \right)^{-\frac{2}{3}} \left( \frac{81}{11008} \right)^{\frac{1}{3}} \left( \frac{M_P}{M} \right)^{\frac{5}{3}},$$

where $M_P$ is the Planck mass and $M$ is the scale of inflation. It is intriguing to plug in the numbers. For inflation on the GUT scale one has $N_{\text{pert}} \sim 10^7$ e-foldings before screening becomes effective. For electroweak inflation we predict $N_{\text{pert}} \sim 10^{15}$ e-foldings.

For technical and historical reasons we chose to develop the scheme for source 4d of Table 2:

$$F[g] = -\frac{43}{48} \epsilon^2 \left( R \left( \frac{1}{\Box} R \right) \right)^2.$$

The dimensionless parameter $\epsilon$ is only about $10^{-12}$, even for GUT scale inflation. Specializing to a homogeneous and isotropic background we have:

$$\hat{F}[b](t) = \frac{172}{3} \epsilon^2 \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} \left( \frac{3}{2} b(t'') + 3b^2(t'') \right) B^2[b](t''),$$

where we define the functional $B[b](t)$ as follows:

$$B[b](t) \equiv \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} \left( \frac{3}{2} b(t'') + 3b^2(t'') \right).$$

The time derivative of $\hat{F}[b](t)$ is simple to compute:

$$\frac{d\hat{F}}{dt} = \frac{172}{3} \epsilon^2 \int_0^t dt' e^{-3b(t')} \left( \frac{3}{2} b(t') + 3b^2(t') \right) B^2[b](t').$$

---

6 A minor point is that the parameter $\epsilon$ characterizes all models, not just those with three factors of the Ricci scalar. Even when the explicit factors of $\Lambda$ are replaced by $R$’s one still gets $\epsilon^2$ when the co-moving time is rescaled to the dimensionless variable $\tau \equiv Ht$. 35
Although the evolution equation we have obtained is both non-local and non-linear, it is simple to solve numerically by naive discretization. The independent variable $t$ is characterized by an integer $i$ and a dimensionless step size $\Delta \tau$:

$$ t \rightarrow iH^{-1}\Delta \tau. \quad (166) $$

Functions of $t$ become discrete in the usual way:

$$ f(t) \rightarrow f_i. \quad (167) $$

Derivatives are discretized using the difference operator:

$$ \dot{f}(t) \rightarrow H \left( \frac{f_{i+1} - f_i}{\Delta \tau} \right) \equiv H \frac{\Delta f_i}{\Delta \tau}. \quad (168) $$

In order to preserve the fundamental theorem of integral calculus we must discretize integrals by summing one step backwards:

$$ \int_0^t dt' f(t') \rightarrow \sum_{j=0}^{i-1} H^{-1}\Delta \tau f_j. \quad (169) $$

It turns out that all the factors of $H^{-1}\Delta \tau$ cancel for our ansatz (this is its nice technical feature) and we obtain the following discretized evolution equation:

$$ \Delta^2 b_i = -\frac{1}{2} \left( \frac{\Delta F_i}{1 - F_i} \right)^2, \quad (170) $$

where the discrete versions of $\hat{F}[b](t)$ and $B[b](t)$ are:

$$ F_i \equiv \frac{172}{3} e^2 \sum_{j=0}^{i-1} e^{-3b_j} \sum_{k=0}^{j-1} e^{3b_k} \left( \frac{3}{2} \Delta^2 b_k + 3(\Delta b_k)^2 \right) (B_k)^2, \quad (171) $$

$$ B_i \equiv \sum_{j=0}^{i-1} e^{-3b_j} \sum_{k=0}^{j-1} e^{3b_k} \left( \frac{3}{2} \Delta^2 b_k + 3(\Delta b_k)^2 \right). \quad (172) $$

We have simplified the notation by dropping the hat on the discretized version of $\hat{F}[b](t)$ because $F[g](t, \vec{x})$ and $\hat{F}(t)$ are never discretized.

Although the discretization we have achieved is plausible, it would be tedious to iterate on account of the need to compute summations at each
step. This can be avoided by simply keeping $\Delta F_i$ and $\Delta B_i$ as auxiliary variables. The resulting recursion scheme is:

$$B_i = B_{i-1} + \Delta B_{i-1}, \quad (173)$$
$$\Delta B_i = e^{-3\Delta b_i} \left( \Delta B_{i-1} + \frac{3}{2} \Delta^2 b_{i-1} + 3(\Delta b_{i-1})^2 \right), \quad (174)$$
$$F_i = F_{i-1} + \Delta F_{i-1}, \quad (175)$$
$$\Delta F_i = e^{-3\Delta b_i} \left( \Delta F_{i-1} + \frac{172}{3} \epsilon^2 \left( \frac{3}{2} \Delta^2 b_{i-1} + 3(\Delta b_{i-1})^2 \right)(B_{i-1})^2 \right), \quad (176)$$
$$b_i = b_{i-1} + \Delta b_{i-1}, \quad (177)$$
$$\Delta b_i = \Delta b_{i-1} + \Delta^2 b_{i-1}, \quad (178)$$
$$\Delta^2 b_i = -\frac{1}{2} \left( \frac{\Delta F_i}{1 - F_i} \right)^2. \quad (179)$$

Note that only differences of $b$ enter the scheme; the exponentials of $\pm 3b$ in the continuum theory are all absorbed into such terms. Note also that all the variables are initialized to zero except $\Delta b_0 = \Delta \tau$. It is only through this initial value that the scheme depends on the step size $\Delta \tau$.

Although $\epsilon$ should be about $10^{-12}$ or less, the evolution is very slow for such small values. In this first study we accordingly ran the scheme for somewhat larger, although still sub-Planckian values: $\epsilon = 10^{-3}, 10^{-4}, 10^{-5}$ and $10^{-6}$. The evolution was carried out with a step size of $\Delta \tau = 10^{-3}$ using the package Mathematica [10]. The results for the effective Hubble constant $H_{\text{eff}}(t) = \dot{b}(t)$ are displayed in Figure 1. In each case the end of inflation is rapid and requires about five e-foldings, in good agreement with the perturbative prediction [9].

Figure 2 shows the instantaneous equation of state $p_{\text{tot}}/\rho_{\text{tot}}$ for the four values of $\epsilon$. The asymptotic equation of state is clearly that of pure radiation: $p_{\text{tot}} = \frac{1}{3}\rho_{\text{tot}}$, corresponding to a scale factor which grows as the square root of the co-moving time. The reason for this is that the functional $\tilde{F}$ continues to approach one as long as $\tilde{R} = 6(\ddot{b} + 2\dot{b}^2)$ is positive. As $\tilde{F}$ approaches one, the second derivative of $b(t)$ become ever more negative, which drives $\tilde{R}$ to zero. But this implies square root expansion:

$$\tilde{R}(t) = 0 \quad \implies \quad \dot{b}(t) = \frac{1}{2(t - t_z)}, \quad (180)$$

where the shift $t_z$ provides a reasonable definition for the time at which inflation ends.
Figure 1: $H_{\text{eff}}(t) \div H$ versus $Ht$ for $\epsilon = 10^{-3}, 10^{-4}, 10^{-5}$ and $10^{-6}$.
Figure 2: $p_{tot}/\rho_{tot}$ versus $Ht$ for $\epsilon = 10^{-3}$, $10^{-4}$, $10^{-5}$ and $10^{-6}$. 
Table 3: Parameters from the various runs.

| $\epsilon$ | $M$ (GeV) | $N_{\text{pert}}$ | $Ht_z$ | $\alpha$ | $Ht_F$ |
|------------|-----------|-----------------|--------|----------|--------|
| $10^{-3}$  | $1.7 \times 10^{18}$ | 37.4            | 38.1   | 0.2085   | 0.04959|
| $10^{-4}$  | $9.6 \times 10^{17}$  | 173.6           | 174.9  | 0.2347   | 0.01208|
| $10^{-5}$  | $5.3 \times 10^{17}$  | 805.8           | 808.7  | 0.2425   | 0.002680|
| $10^{-6}$  | $3.0 \times 10^{17}$  | 3740.0          | 3748.0 | 0.2447   | 0.0005837|

Of course the limiting form is approached asymptotically. The first corrections in the series are:

$$\dot{b}(t) = \frac{1}{2(t-t_z)} - \frac{\alpha \ln[H(t-t_z)]}{H(t-t_z)^2} + \ldots$$  \hspace{1cm} (181)

$$\hat{F}(t) = 1 - \frac{t_F}{t-t_z} + \ldots$$  \hspace{1cm} (182)

where the higher corrections are down by inverse powers of $H(t-t_z)$, possibly offset by logarithms of same. The parameters $Ht_z$, $\alpha$ and $Ht_F$ are determined by fitting the curves for each of the four runs. Their numerical values are reported in Table 3. Points to note are the close agreement between $N_{\text{pert}}$ and $Ht_z$, and the near constancy of $\alpha$ over three decades of variation in $\epsilon$. The first fact means that the number of e-foldings for inflation to end is well predicted by perturbation theory; the second fact means that the transition to radiation domination is almost independent of the scale of inflation. That the transition is also quite rapid is illustrated by Figure 3. Note that it is extremely rapid with respect to the evolving time constant provided by $H_{\text{eff}}(t)$ because the corrections fall off with the time constant $H \gg H_{\text{eff}}(t)$.

Once square root expansion is accepted one can actually derive the asymptotic series from the evolution equation. To see this, substitute the asymptotic forms:

$$\dot{b}(t) = \frac{1}{2(t-t_z)} + \delta \ddot{b}(t)$$  \hspace{1cm} (183)

$$\hat{F}[b](t) = 1 - \delta F(t)$$  \hspace{1cm} (184)

into the evolution equation (159). Taking the square root gives a differential equation for $\delta F(t)$:

$$\frac{d}{dt} \ln(\delta F) = -\frac{1}{t-t_z} + (t-t_z) \delta \ddot{b}(t) + \ldots$$  \hspace{1cm} (185)

Figure 3: Asymptotic approach to radiation domination for $\epsilon = 10^{-6}$. Left-hand graph shows pure radiation (solid) versus the numerical result (dashed). The right-hand graph shows the asymptotic formula (solid) versus the numerical result (dashed).

Integration gives the constant $t_F$ in (182). To get (181) we differentiate expression (165):

$$\frac{d^2 \tilde{F}}{dt^2} = -3\dot{b}d\tilde{F}dt + 86\epsilon^2 (\ddot{b} + 2\dot{b}^2) B^2[b](t).$$

(186)

Now substitute (182) and the leading term in (181) to obtain the following leading order result:

$$\frac{d^2 \tilde{F}}{dt^2} + 3\dot{b} \frac{d\tilde{F}}{dt} = -\frac{t_F}{2(t - t_F)^3} + \ldots.$$  

(187)

This must be equal to $86\epsilon^2 (\ddot{b} + 2\dot{b}^2) B^2[b]$. The time dependence must come from the term $\dot{b}(t) + 2\dot{b}^2(t)$ since $B[b](t)$ is dominated by what went on during inflation:

$$B[b](t) = B[b](t_z) + \ldots,$$

$$= H t_z + \ldots.$$  

(188)  

(189)

The asymptotic form (183) gives:

$$\ddot{b}(t) + 2\dot{b}^2(t) = \delta \frac{b(t)}{t - t_z} + \ldots.$$  

(190)
Substituting everything in (186) gives a differential equation for $\delta b(t)$:

$$\frac{t_F}{2(t-t_z)^3} + \ldots = 86\epsilon^2(Ht_z)^2\left(\ddot{\delta b}(t) + \frac{2\dot{\delta b}(t)}{t-t_z}\right) + \ldots . \quad (191)$$

The solution is the second term in (181) with the relation:

$$Ht_F \approx 172\epsilon^2(Ht_z)^2\alpha . \quad (192)$$

The independently fitted parameters of Table 3 obey (192) to a few percent, with agreement better for smaller values of $\epsilon$. The slight discrepancy is probably due to the approximation $B[b](t) \approx B[b_{\text{class}}](t_z) = Ht_z$, which is an overestimate. If we assume that $\alpha$ is almost constant then $Ht_z \approx N_{\text{pert}} \sim \epsilon^{-2/3}$ implies that $Ht_F$ varies as the two thirds power of $\epsilon$. A consequence is that the post-inflation value of $\dot{F}[g](t)$ must be very close to one for low scale inflation.

It is perhaps significant that the effective Hubble constant $\dot{b}(t)$ approaches the expansion rate for a radiation dominated universe from below. Although matter is negligible during inflation, it cannot be ignored afterwards. We must therefore expect that matter radiation is produced during the transition. The asymptotic expansion rate means that this matter radiation is progressively enhanced with respect to the purely quantum gravitational stress energy we have been discussing.

7 The scalar potential

The purpose of this section is to reconstruct the scalar potential for the model we have just evolved. We begin by reviewing the general technique, then we obtain analytic expressions during the perturbative and late time regimes. The section closes with an explicit numerical reconstruction over the full period of evolution.

Recall from Section 2 that the scalar potential $V(\phi)$ does not enter the evolution equation and is therefore not required to determine $b(t)$. We instead reconstruct the potential from $b(t)$ by imposing stress energy conservation. The procedure is first to get the potential as a function of time:

$$\dot{V}(t) = \frac{1}{8\pi G} \left( \ddot{b}(t) + 3b^2(t) - 3H^2 \right) . \quad (193)$$
One then inverts the relation:

\[ \hat{\phi}(t) = -\frac{1}{\sqrt{8\pi G}} \ln \left( 1 - \hat{F}[b](t) \right), \tag{194} \]

to find time as function of \( \hat{\phi} \). Substituting \( t(\phi) \) in (193) gives \( V(\phi) \).

This procedure can be carried out analytically during the perturbative regime where we can find explicit expressions for (193) and (194):

\[ \tilde{V}(t) = -\frac{\Lambda}{8\pi G N_{\text{pert}}} \frac{3}{1 - (Ht/N_{\text{pert}})^2} \left( 1 - \frac{1}{4N_{\text{pert}}} \frac{(Ht/N_{\text{pert}})^2}{1 - (Ht/N_{\text{pert}})^2} \right), \tag{195} \]

\[ \tilde{\phi}(t) = -\frac{1}{\sqrt{8\pi G}} \ln \left( 1 - (Ht/N_{\text{pert}})^2 \right). \tag{196} \]

(Recall that \( N_{\text{pert}} \) is our perturbative estimate (161) for the number of e-foldings of inflation.) Solving for the time as a function of the scalar:

\[ \frac{Ht}{N_{\text{pert}}} = \left( 1 - e^{-\sqrt{8\pi G} \tilde{\phi}} \right)^{\frac{1}{4}}, \tag{197} \]

gives the following relation for the potential during the perturbative regime:

\[ V_{\text{pert}}(\tilde{\phi}) = -\frac{\Lambda}{8\pi G} \frac{3e^{\sqrt{8\pi G} \tilde{\phi}}}{N_{\text{pert}}} \left( 1 - e^{-\sqrt{8\pi G} \tilde{\phi}} \right)^{\frac{3}{2}} \left[ 1 - \frac{e^{\sqrt{8\pi G} \tilde{\phi}}}{4N_{\text{pert}}} \left( 1 - e^{-\sqrt{8\pi G} \tilde{\phi}} \right)^{\frac{3}{2}} \right]. \tag{198} \]

This expression should be valid for \( 0 \leq \tilde{\phi} \ll \ln(N_{\text{pert}})/\sqrt{8\pi G} \). Note that it is the same for all models since it follows from the known perturbative results.

The other regime where explicit expressions can be obtained is that of late times. For the model of Section 6 we have:

\[ \hat{V}(t) = -\frac{\Lambda}{8\pi G} \left( 1 - \frac{1}{12H^2(t+t_z)^2} + \ldots \right), \tag{199} \]

\[ \hat{\phi}(t) = -\frac{1}{\sqrt{8\pi G}} \ln \left( \frac{t_F}{t-t_z} + \ldots \right). \tag{200} \]

During this period the time can be expressed in terms of the scalar as:

\[ t = t_z + t_F e^{\sqrt{8\pi G} \tilde{\phi}} + \ldots, \tag{201} \]
The Effective Scalar Potential for \( \epsilon = 0.001, 0.0001, 0.00001, \) and \( 0.000001 \)

Figure 4: \( V(\phi) \) versus \( \phi \) for \( \epsilon = 10^{-3} \) (leftmost), \( 10^{-4}, 10^{-5} \) and \( 10^{-6} \) (rightmost).

which gives the following result for the late time potential:

\[
V_{\text{late}}(\phi) = -\frac{\Lambda}{8\pi G} \left[ 1 - \frac{1}{12} \left( \frac{e^{-\sqrt{8\pi G}\phi}}{Ht_F} \right)^2 + \ldots \right]. \tag{202}
\]

This expression should be valid for \( \phi \gtrsim -\ln(Ht_F)/\sqrt{8\pi G} \). Using (192) we see that this is about \( \phi \gtrsim \ln(N_{\text{pert}})/\sqrt{8\pi G} \), so most of the range is covered by the two asymptotic expressions.

Full coverage can be obtained by computing the values of \( \phi \) and \( V(\phi) \) at each step in the numerical evolution, and then plotting the resulting curve. The discretized formulae are:

\[
\frac{\phi_i}{M_P} = -\frac{1}{\sqrt{8\pi}} \ln(1 - F_i), \tag{203}
\]
\[
\frac{V_i}{M^4} = 1 - \left( \frac{\Delta b_i}{\Delta \tau} \right)^2 - \frac{1}{3} \frac{\Delta^2 b_i}{\Delta \tau^2}.
\]  
(204)

The curves for each of the four runs are plotted in Figure 4. Note that they confirm the results of our asymptotic expansions, including the fact that the transition from inflation occurs at \( \phi \sim \ln(N_{\text{pert}})/\sqrt{8\pi G} \).

8 Discussion

We have developed a compelling physical picture for a particular second order back-reaction that quantum gravity has on inflation. The first order effect is that long wavelength virtual gravitons can be ripped apart by the superluminal expansion of spacetime. This is the phenomenon of superadiabatic amplification, first studied by Grishchuk [3]. This first order effect is not secular. As more and more graviton pairs are injected into the inflating universe, the growth of their energy is cancelled by the expansion of the 3-volume to produce a small, constant energy density (and pressure \( p = -\rho \)) of magnitude about \( H^4 \). The secular effect comes at the next order from the gravitational interaction between the receding virtual gravitons. As graviton pairs are pulled apart their long range gravitational potentials fill the intervening space, and these potentials remain to add with those of new pairs even after the old pairs have been redshifted into insignificance. The second order effect is suppressed by the small dimensionless coupling constant \( G\Lambda < \sim 10^{-11} \), but it is cumulative. And it slows inflation because gravity is attractive.

An explicit two loop computation has already confirmed that the quantum gravitational back-reaction slows inflation by an amount which eventually becomes non-perturbatively large [2]. The question is, what happens next? We have argued in this paper that the question can be answered by inferring and then numerically evolving the most cosmologically significant terms in the effective field equations. Section 2 proved that the important part of the quantum gravitationally induced stress tensor is that of an effective scalar field \( \phi[g] \) which is itself a non-local functional of the metric. We also showed that a model is completely specified by \( \phi[g] \), since the potential \( V(\phi) \) does not enter the evolution equation, and we obtained an explicit expression (30) for the scalar during the perturbative regime.

In Sections 3 and 4 we showed that general considerations and the physics
of screening very largely constrain the scalar. It must have the form:

$$\phi[g] = -\frac{1}{\sqrt{8\pi G}} \ln (1 - F[g]) ,$$  \(205\)

where the functional \(F[g]\) consists of a retarded scalar Green’s function \(-1\) acting on a source composed of two more factors of \(-1\) acting in some order on from three to five Ricci scalars. We did not determine whether the two inner Green’s functions act in series or in parallel, nor did we fix how the various Ricci scalars are located with respect to them. The resulting combinatorics yields 73 possibilities, many of which were tabulated in Section 5. We also identified two procedures for actually deriving the scalar from the result of perturbative computations of the same complexity as the one already done.

In Section 6 we selected one of the candidate models:

$$F[g] = -\frac{43}{48} \left( \frac{G\Lambda}{3\pi} \right)^2 \frac{1}{\Box} \left( R \left( \frac{1}{\Box} R \right)^2 \right) ,$$  \(206\)

and numerically evolved it through the end of inflation into the regime of asymptotically late times. We found that inflation ends over a period of about five e-foldings, following which the universe asymptotically approaches the square root expansion of a radiation dominated universe:

$$\dot{b}(t) \longrightarrow \frac{1}{2(t - t_z)} .$$  \(207\)

Almost identical results were found for the series analog model:

$$F[g] = -\frac{43}{24} \left( \frac{G\Lambda}{3\pi} \right)^2 \frac{1}{\Box} \left( R \left( \frac{1}{\Box} R \frac{1}{\Box} R \right) \right) ,$$  \(208\)

although we did not report them. We were able to confirm the asymptotic forms by deriving analytic expressions from the non-linear and non-local evolution equation. An interesting consequence of this analysis is that the approach is from below, implying that any matter radiation produced in the transition would be relatively enhanced by the slower redshift.

The asymptotic analysis can be used to categorize models by the number of factors of \(R\) which lie immediately to the right of the outer \(-1\). Suppose there are \(f\) outer factors of \(R\). We can expose them by taking derivatives:

$$-\Box \hat{F} = \frac{d^2 \hat{F}}{dt^2} + 3b \frac{d \hat{F}}{dt} .$$  \(209\)
The two internal retarded Green’s functions will be effectively constant, dominated by what went on during inflation, so the time dependence must come from the $f$ outer factors of $\hat{R}$. Suppose that the asymptotic expansion rate is a general power law $p > 0$:

$$\dot{b}(t) \longrightarrow \frac{p}{t - t_z}. \quad (210)$$

The Ricci scalar goes to:

$$\hat{R}(t) = 6\dot{b}^2(t) + 12\ddot{b}^2(t) \longrightarrow \frac{2p^2 - p}{(t - t_z)^2}, \quad (211)$$

so the lefthand side of (209) goes like $(t - t_z)^{-2f}$, unless $p = 1/2$. To find the righthand side, consider the evolution equation:

$$\ddot{b}(t) = -\frac{1}{2} \left( \frac{d\tilde{F}/dt}{1 - \tilde{F}} \right)^2, \quad (212)$$

It follows that $\tilde{F}[b](t)$ must have the form:

$$\tilde{F}[b](t) \longrightarrow 1 - \left( \frac{t_F}{t - t_z} \right)^{\sqrt{2p}}. \quad (213)$$

Hence the righthand side of (209) goes like $(t - t_z)^{-2 - \sqrt{2p}}$, and we must have $2 + \sqrt{2p} = 2f$, unless $p = 1/2$.

It follows that the case of no outer Ricci scalars ($f = 0$) is not consistent with stable evolution. What happens for these models is that $\tilde{F}[b](t)$ actually reaches one and $\dot{b}(t)$ goes to minus infinity. For $f = 1$ we get $p = 0$, which is also not consistent. However, taking account of next order terms in the asymptotic expansion of $\dot{b}$ gives $p = 1/2$. The other cases are all consistent: for $f = 2$ we get $p = 2$; $f = 3$ gives $p = 8$; $f = 4$ results in $p = 18$; and $f = 5$ produces $p = 32$. Since the actual model is likely to be a linear combination of the tabulated candidates, and since the term with the lowest value of $f$ dominates the asymptotic behavior, we conclude that the approach to a radiation dominated universe is generic for models which include at least one $f = 1$ term while avoiding any $f = 0$ terms.

\footnote{The $f = 0$ terms on Table 1 are 16a-21a and 10b-12b; the $f = 1$ terms are 11a-15a and 7b-9b. On Table 2 the $f = 0$ terms are 7c-10c and 5d-6d; the $f = 1$ terms are 4c-6c and 3d-4d.}
Of course fixing the asymptotic time dependence does not provide a physical interpretation for what happens after the end of inflation. For example, the stress energy of our model is not likely to consist of gravitons, in spite of the fact that it approaches the equation of state of radiation. There is simply no way to have created them. The energy density of gravitons produced by superadiabatic amplification is a small constant for as long as it can be reliably tracked using perturbation theory — which is almost to the end of inflation. What terminates inflation is the buildup of gravitational interaction stress energy and it must be this which adds with the stress tensor of the cosmological constant to produce a residual obeying the equation of state of radiation.

The effective scalar \( \phi[g] \) poses a similar interpretational dilemma. It is certainly not a fundamental particle but one might plausibly interpret screening in terms of the formation of a scalar bound state on cosmological scales. The “scalars” would be the virtual graviton pairs which are ripped apart by inflation. Although they are separated to cosmological distances it is their binding energy which eventually arrests inflation. Since the pairs tend to annihilate on sub-horizon scales there are no new massless quanta to embarrass phenomenology. One very attractive feature of this interpretation is that we can compute the spectrum of density perturbations using the standard formalism, just as if the scalar was fundamental.

Finally, there is the question of how to couple matter and what effect doing so will have, both on the geometry and on reheating. Gravitons dominate screening through their unique combination of masslessness without conformal invariance, however, ordinary matter becomes important at the end of inflation. The simplest assumption would that the matter and the gravitational stress tensors are separately conserved except for a brief period at the end of inflation when some of the gravitational stress energy excites matter degrees of freedom. It remains to see if there can be sufficient reheating without the phase of coherent oscillations that characterizes scalar based inflation \([13, 14]\).

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