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1. Introduction

Phenomenological studies of lipid bilayer membranes have become imperative, for a large amount of experiments reveal that the matrix of biomembrane plays an important role in some non-linear phenomena such as shape fluctuations of cells and oscillations of the membrane potential in an external electric field (1-3). In approaching the problem, however, one is immediately confronted with complexities because of the concurrence of the elastic-viscous motion and the change in shape of the material. The lack of a theoretical framework tailored for non-Newtonian fluid membranes impedes progress in this research area, though the general geometrical method in continuum mechanics has been well established. The purpose of this paper is to present such a framework.

The membranes constituted by two layers of rod-shaped molecules are identified with hydrotropic liquid crystals in the laminar phase. From this point of view, one assumes in the present paper that the dynamic theory of liquid crystals developed during the last 30 years is applicable to the membranes. In fact, the experimental observations mentioned above would not be surprising if one thinks of the anisotropy feature of the material, which, as is well known, leads to instabilities in the liquid crystal bulk.

Due to the pioneering research of Oseen [4], the fundamental contribution of Frank [5] and the important supplementary work of Nehring and Saupe [6], a curvature elasticity theory of liquid crystals has been established. It is essentially a phenomenological theory. A dimensionless unit vector named “director” is defined to indicate the preferred orientation of the molecules. The energy stored in the material during elastic deformation is thus expressed in terms of this vector. Extending the theory to thin films in particular yields the well-known curvature elasticity equation of membranes [7] which has been used in fruitful studies on the hydrostatic form of lipid vesicles [8].

The dynamic theory of liquid crystals [9-13], mainly uniaxial crystals, stems from such a mechanical base. It assumes rotation of the molecules, indicated by the director, superposed on the mean flow of the molecules themselves. The theory was used to investigate instabilities of liquid crystals and has interpreted successfully the Williams’ domain structure and the homoetropic textures as well in nematics [14-22].

One adapts the theory to the particular context by neglecting non-uniformities across the membrane thickness and replacing the three-dimensional film by a two-dimensional surface. It is exactly the method used in the theory of thin shells [23]. The simplification is acceptable in most cases of biocells, for the membrane thickness (40-80 Å) is sufficiently small with regard to the cell diameter (1-10 μm). Moreover, the tilt angle of the long molecular axes with respect to the vector normal to the surface is supposed to be small and hence the elasticity theory is restricted within the extent of linear approximation. With these considerations, one obtains
a particular version of the dynamic equations, in which geometric invariants such as the fundamental magnitudes and the curvatures are used to describe the shape of the membrane, while the displacement of the configuration of the membrane is treated separately from the internal movement of the particles inside the membrane. The attempt to construct such a group of equations, so far as one knows, was made 10 years ago by Peterson [24]. He established, however, only the dynamic equations of a solid film and the hydrodynamic equations of a Newtonian fluid membrane, separately. This left a gap between these two limiting cases. One will see that by introducing the theory of liquid crystals the gap is naturally bridged.

Recent tether experiments support the opinion that slip between the two molecular layers causes interlayer viscous dissipation [25] which might not be negligible in dynamical studies of bilayer membranes [26]. One recognizes that this kind of internal dissipation is excluded from the present theoretical model.

This paper is the first part of a series devoted to present the general electrohydrodynamic equations of polar viscoelastic fluid membranes isolated from their surroundings. The hydrodynamic theory of liquid crystals will be recapitulated in the next section, then it will be adapted to the context of bilayer membranes in the third section. The connection of such a membrane to the liquid surroundings will be considered in subsequent papers.

2. Outline of the Ericksen-Leslie theory

The hydrodynamics of liquid crystals was primarily established by Ericksen [9]. He assumed an intrinsic motion of the non-spherical molecules and used a time-dependent director to indicate it. Because of the material symmetry, he constructed the constitutive relationships of the dynamic variables involved and hence interpreted explicitly the dynamic conservation laws for uniaxial liquid crystals. Leslie [10] was able to take into account the antisymmetric stress of the anisotropic continuum [27,28], which had been overlooked in the expressions of Ericksen, and thus provided us a refined version of the theory.
2.1. Mass balance

The principle of mass conservation holds in the convected volume \( R \)

\[
\frac{d}{dt} \iiint_R \rho \, dR = 0 \quad (1)
\]

where \( \rho \) is the mass density and \( t \) is the time.

2.2. Conservation of momentum

Let \( \mathbf{v} \) denote the barycenter velocity of the molecules. Conservation of linear momentum in the motion of the continuum is assumed, which asserts

\[
\frac{d}{dt} \iiint_R \rho \mathbf{v} \, dR = \iiint_R \mathbf{F} \, dR + \iint_s ds \cdot \mathbf{T} \quad (2)
\]

where \( \mathbf{F} \) is the body force, \( \mathbf{T} \) the stress tensor and \( s \) is the surface enclosing the considered volume.

The body force consists of the gravitational attraction and electromagnetic forces \([9]\)

\[
\mathbf{F} = \mathbf{F}^g + \mathbf{F}^{\text{e.m}} \quad (3)
\]

The gravitational force is naturally written as

\[
\mathbf{F}^g = \rho \mathbf{\hat{g}} \quad (4)
\]

where \( \mathbf{\hat{g}} \) is the gravitational acceleration.

The force generated in an external magnetic field is given as \([9]\)

\[
\mathbf{F}^m = (\mathbf{M} \cdot \nabla) \mathbf{H} \quad (5)
\]

where \( \mathbf{H} \) is the external magnetic field, \( \mathbf{M} \) is the magnetization and \( \nabla \) is the vector derivative operator.

The most general relationship of \( \mathbf{M} \) linear with the magnetic field is \([9]\)

\[
\mathbf{M} = \chi_\perp \mathbf{H} + (\chi_\parallel - \chi_\perp) (\mathbf{d} \cdot \mathbf{H}) \mathbf{d} = \chi_\parallel \mathbf{H} + \chi_a (\mathbf{d} \cdot \mathbf{H}) \mathbf{d} \quad (6)
\]

where \( \mathbf{d} \) is the director, \( \chi_\parallel \) and \( \chi_\perp \) are the constant magnetic susceptibilities in the directions parallel and perpendicular to the director, respectively, and \( \chi_a \) is the anisotropy.

The force generated by the electric field is not expressed explicitly in the Ericksen-Leslie theory. In electrohydrodynamic studies of liquid crystals, however, the Lorentz force is commonly considered as a cause of instabilities \([17-22]\). We do not go far towards calculating completely the force in the electric field, but, inaccurately, equate it to the Lorentz force, which amounts partly to the ponderomotive force in liquid dielectrics \([29]\). That is

\[
\mathbf{F}^e = \rho_e \mathbf{E} \quad (7)
\]
where \( \rho_e \) is the density of injected electric charges and \( \mathbf{E} \) is the electric field.

The stress tensor consists of three parts [9]

\[
\mathbf{T} = \mathbf{T}^i + \mathbf{T}^e + \mathbf{T}^v
\]  

(8)

where \( \mathbf{T}^i \) is the isotropic part of the stress, \( \mathbf{T}^e \) is the stress arising from elastic deformation and \( \mathbf{T}^v \) is that from viscous motion.

The isotropic part is expressed as

\[
\mathbf{T}^i = -\sigma \mathbf{I}
\]  

(9)

where \( \sigma \) is a constant and \( \mathbf{I} \) is the unit tensor.

The elastic stress may be deduced from the density of reversible deformation work \( W^d \) [9], i.e.

\[
\mathbf{T}^e = - (\nabla \mathbf{d}) \cdot \frac{\partial W^d}{\partial (\mathbf{d} \nabla)}
\]  

(10)

It is suggested in the Oseen-Frank-Nehring-Saupe theory that

\[
W^d = k_1 (\nabla \cdot \mathbf{d}) - k_2 (\mathbf{d} \cdot \nabla \times \mathbf{d}) + \frac{1}{2} (k_{11} - 2k_{13}) (\nabla \cdot \mathbf{d})^2 + \\
+ \frac{1}{2} k_{22} (\mathbf{d} \cdot \nabla \times \mathbf{d})^2 + \frac{1}{2} (k_{33} + 2k_{13}) (\mathbf{d} \cdot \nabla \mathbf{d})^2 - \\
- (k_{12} - k_{23}) (\mathbf{d} \cdot \nabla \times \mathbf{d}) (\nabla \cdot \mathbf{d}) - k_{13} \nabla \cdot (\mathbf{d} \nabla \cdot \mathbf{d}) - \\
- \frac{k_{23}}{2} \nabla \cdot [(\mathbf{d} \cdot \nabla \times \mathbf{d}) \mathbf{d}]
\]  

(11)

where \( k_i \)'s are the elastic moduli (Frank’s notation [5] is used in this paper), among which the relationship \( k_{24} = \frac{1}{2} (k_{11} - k_{22}) \) holds.

In relation (11), the first term originally represented the splay structure resulting from the change of symmetry of the material, although liquid crystals having ferroelectric features are rarely found in nature. The elastic moduli \( k_2, k_{12} \) and \( k_{23} \) are relevant to the chirality of the molecules. The three squared terms stand for, in sequence, the splay, twist and bending modes of the elastic deformation. The last three terms relate to boundary distortions.

The viscous stress is supposed to be a linear function of the rate of strain tensor \( \mathbf{S} \) and the rotational velocity of the director \( \mathbf{N} \) [9,10]

\[
\mathbf{T}^v = \alpha_1 (\mathbf{d} \cdot \mathbf{S} \cdot \mathbf{d}) \mathbf{d} \mathbf{d} + \alpha_2 \mathbf{d} \mathbf{N} + \alpha_3 \mathbf{N} \mathbf{d} + \alpha_4 \mathbf{S} + \alpha_5 (\mathbf{S} \cdot \mathbf{d}) + \alpha_6 (\mathbf{S} \cdot \mathbf{d}) \mathbf{d}
\]  

(12)

in which the independent scalar variables such as the mass density and the temperature have been absorbed in the viscosity coefficients. The viscosity coefficients satisfy the relationship [11]

\[
\alpha_2 + \alpha_3 = \alpha_5 - \alpha_6
\]  

(13)
The two considered variables are given by

\[ S = \frac{1}{2} (v \nabla + \nabla v) \]  

(14)

and

\[ N = \dot{d} - \frac{1}{2} \nabla \times (v - \dot{r}) \times d \]  

(15)

where \( r \) is the position vector, the overdot indicates the convected derivative.

The movement of the director is governed by the equation [9,10]

\[ \frac{d}{dt} \iiint_R \rho \dot{d} dR = \iiint_R (G + g) dR + \iint_s d \cdot \Pi \]  

(16)

where \( \rho \) is the moment of inertia per unit volume (ML\(^{-1}\)), \( G \) and \( g \) are respectively the external director body force and the intrinsic director body force, having the dimensions of torque per unit volume (ML\(^{-1}\)T\(^{-2}\)), and \( \Pi \) is the intrinsic stress (with the dimensions of ML\(^{-2}\)T\(^{-2}\)).

The magnetic field and electric field forces may, respectively, be obtained from the magnetization energy [9]

\[ W^m = -\frac{1}{2} \mathbf{M} \cdot \mathbf{H} \]  

(17)

and the polarization energy [29]

\[ W^p = -\frac{1}{2} \mathbf{P} \cdot \mathbf{E} \]  

(18)

where \( \mathbf{P} \) is the electric dipole moment.

The constitutive relationship of \( \mathbf{P} \) linear with the electric field is given as [9]

\[ \mathbf{P} = \frac{\epsilon_\parallel - \frac{1}{4\pi} \mathbf{E}}{\epsilon_\parallel - \frac{1}{4\pi}} (\mathbf{d} \cdot \mathbf{E}) \mathbf{d} = \frac{\epsilon_\perp - \frac{1}{4\pi} \mathbf{E}}{\epsilon_\perp} + \frac{\epsilon_a}{4\pi} (\mathbf{d} \cdot \mathbf{E}) \mathbf{d} \]  

(19)

where \( \epsilon_\parallel \) and \( \epsilon_\perp \) are the constant dielectric permeabilities with the interpretations indicated, and \( \epsilon_a \) is the dielectric anisotropy.

With the help of relations (6) and (17)-(19), the field-generated forces defined by [9]

\[ \mathbf{G}^{m,e} = \frac{\partial W^{m,p}}{\partial \mathbf{d}} \]  

(20)

are interpreted as

\[ \mathbf{G}^m = -\chi_a (\mathbf{d} \cdot \mathbf{H}) \mathbf{H} \]  

(21)

\[ \mathbf{G}^e = -\frac{\epsilon_a}{4\pi} (\mathbf{d} \cdot \mathbf{E}) \mathbf{E} \]  

(22)
The intrinsic body force $g$ consists also of three parts

$$
g = \lambda d + g^e + g^v \tag{23}$$

where $\lambda$ is a constant.

The conservative and non-conservative body forces exerted on the director are [9] respectively

$$
g^e = -\frac{\partial W^d}{\partial d} \tag{24}$$

and

$$
g^v = \gamma_1 N + \gamma_2 S \cdot d \tag{25}$$

where $\gamma_1$ and $\gamma_2$ are the viscosity coefficients relevant to the intrinsic motion.

Relationships between the $\gamma$ coefficients in Eq.(25) and the $\alpha$ coefficients in Eq.(12) are [9]

$$
\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5 \tag{26}
$$

The stress $\Pi$ has only a conservative part [9]

$$
\Pi = \frac{\partial W^d}{\partial (d\nabla)} \tag{27}
$$

The extra-body force $g^e + g^v$ and the extra-stress $\Pi$ generate a torque on the director.

2.3. Energy balance

The general form of the energy conservation law in the continuum is [9]

$$
\frac{d}{dt} \int \int \int_R \rho \frac{v \cdot v}{2} + \frac{\rho}{2} \dot{d} \cdot \dot{d} + \rho U \, dR = \int \int \int_R (F \cdot v + G \cdot \dot{d} + Q^h) \, dR + \int \int ds \cdot \left( T \cdot v + \Pi \cdot \dot{d} - J^h \right) \tag{28}
$$

where $U$ is the internal energy per unit mass, $Q^h$ is the heat supply per unit volume per unit time, and $J^h$ is the heat current flowing out of the volume per unit area per unit time.

For uniaxial liquid crystals, the heat flux is supposed to have a linear relationship with the thermal field [9]

$$
J^h = \beta_0 \nabla \Theta + \beta_1 (d \cdot \nabla \Theta) d \tag{29}
$$

where $\Theta$ is the temperature and $\beta_0$ and $\beta_1$ are the heat conductivities.
3. Adaptation to membranes

Now we will adapt the Ericksen-Leslie dynamic theory to a bilayer membrane. Unless otherwise stated, the Latin suffix runs over 1, 2 and 3, whilst that the Greek suffixes run over 1 and 2. Italic symbols indicate the components of a vector or tensor in the moving local frame (see below) and block symbols represent those in fixed global coordinates (inertial reference system). Summation convention is assumed throughout.

As suggested in the theory of thin shells, one defines a geometric surface situated at the middle position in the membrane and names it the “middle surface” (see Fig 1). The word “surface” in the following means, uniquely, the middle surface.

Let \( \mathbf{r} \) be the vector from an origin in Euclidean space to a point in the middle surface. Any spatial position in the membrane is then determined by

\[
x^i = r^i (\theta^\alpha, t) + \theta^3 n^i (\theta^\alpha, t)
\]

where \( x^i \) denotes the spatial coordinates, \( \theta^i \) are arbitrary parameters and \( n^i \) is the normal vector to the middle surface.

If \( \theta^1 \) and \( \theta^2 \) are identified with the curvilinear coordinates on the middle surface, then the local base vectors are given by

\[
\mathbf{e}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}, \quad \mathbf{e}_3 = \mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = \frac{1}{\sqrt{a}} \frac{\partial \mathbf{r}}{\partial \theta^1} \times \frac{\partial \mathbf{r}}{\partial \theta^2}
\]

where \( a \) is the determinant of the metric tensor

\[
a_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta
\]

The second- and the third-order magnitude of the surface are

\[
b_{\alpha\beta} = -\frac{\partial \mathbf{e}_3}{\partial \theta^\alpha} \cdot \mathbf{e}_\beta = \frac{\partial \mathbf{e}_\alpha}{\partial \theta^\beta} \cdot \mathbf{e}_3
\]

\[
c_{\alpha\beta} = \frac{\partial \mathbf{e}_3}{\partial \theta^\alpha} \cdot \frac{\partial \mathbf{e}_3}{\partial \theta^\beta}
\]

For uniquely determined surface, the fundamental magnitudes satisfy the identity

\[
c_{\alpha\beta} - 2H b_{\alpha\beta} + Ka_{\alpha\beta} = 0
\]

where

\[
2H = a^{\beta\alpha} b_{\alpha\beta}
\]

is the first curvature (mean curvature) and

\[
K = \frac{|b_{\alpha\beta}|}{|a_{\alpha\beta}|} = \frac{b}{a}
\]

the second curvature (gaussian curvature).
The director vector $\mathbf{d}$ (a unit vector parallel to the long molecular axes) is considered to radiate from the middle surface on the side to which the normal vector is positive (Fig. 1).

The molecules displace in the Euclidean space at a velocity $\mathbf{v}$. With respect to the local bases $\mathbf{e}_i$ which move with the surface, the velocity has the components

$$v^\alpha = \dot{\theta}^\alpha, \quad v^3 = 0$$

because the particle flow is supposed to be parallel to the surface. With respect to the spatial coordinates $x^i$ the velocity is given by

$$v^i = \dot{x}^i = t^i_\alpha v^\alpha + \dot{r}^i$$

where

$$t^i_\alpha = \frac{\partial r^i}{\partial \theta^\alpha}$$

is a time-dependent hybrid tensor. The tensor $t^i_\alpha$ is used to associate the flexible and movable surface with the inertial reference system. For example, if the director is known by its components $d^k$ with respect to the local bases, then its components in the global coordinates must be $d^k = t^k_\alpha d^\alpha + n^k d^3$. Conversely, while one has the expansion of a field vector at a certain position of the middle surface with respect to the fixed coordinates, say, $E_k$, one knows naturally, with respect to the moving frame, its tangential parts $E_\alpha = t^K_\alpha E_K$ and its normal part $E_{(n)} = E_3 = E_K n^K$. The subscript $(n)$ denotes the normal part of the vectors.

One defines the varying configuration of the surface in a way that any point of the surface at the considered instant $(\theta^1, \theta^2, t)$ is the endpoint of a translation of the point which occupied the same curvilinear coordinates on the surface at the preceding instant $(\theta^1, \theta^2, t')$ along the normal vector $\mathbf{n}$. The displacement speed of the surface is thus determined by

$$\dot{r}^i = \left( \frac{\partial r^i}{\partial t} \right)_{\theta^1, \theta^2} = wn^i$$

where $w$ is the rate of displacement.

Across the thickness of the membrane the non-uniformities, such as that of the mass, the temperature and the internal energy, as well as that of the velocity and the stress, are supposed negligible. (The assumption $v^\alpha, v^3 = 0$ implies that the two molecular layers do not mutually slip.) It follows that the differentiation of any tensor property of the material is a covariant differentiation along the surface. The surface derivative operator is defined by [30]

$$\nabla_s = e^3 \frac{\partial}{\partial \theta^3}$$

Replacing the three-dimensional operator $\nabla$ in Eq.(11) by the two-dimensional operator $\nabla_s$ and multiplying both sides of the expression by the membrane thickness
\( h \) yields the surface density of elastic energy of the membrane. To the first-order terms (to small tilting angle of the director to the normal), it is

\[
W^d_{(1)} = -k_1' \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\sigma \partial_\sigma a \right) + k_2' \varepsilon^{\alpha\beta} \partial_\beta d_\alpha +
\]

\[
+ k_{11}' H \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\sigma \partial_\sigma a \right) -
\]

\[
- (k_{22}' + k_{24}') \left[ K - 2H \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) + d_{\alpha,\beta(\tau)} b^{\beta\alpha} \right] -
\]

\[
- 2k_{12}' H \varepsilon^{\alpha\beta} \partial_\beta d_\alpha - 2k_{13}' d^\beta \partial_\beta H
\]

(43)

where the symbol \( \partial_\alpha \) denotes the partial derivative with respect to the parameter \( \theta^\alpha \), \( \varepsilon^{\alpha\beta} \) is the two-parametric permutation tensor, the prime indicates the surface covariant derivative and the subscript \((\tau)\) the tangential part (see Appendix A). Here the elastic moduli in Eq.(11) have been replaced by

\[
k_i' = h k_i, \quad k_{ij}' = h k_{ij}
\]

(44)

Relationship (43) includes the energy contributed by the non-uniform orientation of the molecules, which was emphasized to be important but excluded from consideration in the Helfrich’s theory [7].

Letting \( d = n \), one recovers Helfrich’s formula [7]

\[
W^d_{(0)} = k_c H + k_c' H^2 + \bar{k}_c K
\]

(45)

where

\[
k_c = -2k_1', \quad k_c' = 2k_{11}', \quad \bar{k}_c = -(k_{22}' + k_{24}')
\]

(46)

Hence the framework encompasses Helfrich’s theory as a special case where the long molecular axes are parallel to the normal vector. Recently, Ou-Yang and Liu deduced Eq.(45) from formula (11) [31].

The first term in Eq.(43) and in Eq.(45) now stands for the non-symmetry of the two molecular leaves [7]. The last three terms in Eq.(43) and the last term in Eq.(45) are actually meaningful only when the membrane has a free edge such as a helical strips (e.g. [32,33]). Otherwise, the orientation of the molecules at the periphery is determined by the veins of the solid boundary on which the membrane is braced rather than the internal molecular field. For a closed vesicle, these terms are vacuous of physical content.

The following kinematic theorem will frequently be used later, which is, in fact, an analogue of the Reynolds’ transport theorem in two-dimensional fluids [28].

If \( \phi \) is any function of position on a surface and of time, which can be any scalar or tensor component, and \( s \) is a material part of the surface, then

\[
\frac{d}{dt} \iint_s \phi \, ds = \iint_s \left[ \dot{\phi} + \phi \left( v'^\alpha, \alpha + \frac{\dot{a}}{2a} \right) \right] \, ds
\]

(47)
where $v_{\alpha, \alpha} + \dot{a}/(2a)$ is the dilation of area.

The change in area of the membranes arises from, besides extension and compression, tilting of the molecules also [34-37]. For the surface configuration defined above, the rate of change of area is related to the displacement speed by [38]

$$\frac{\dot{a}}{2a} = -2Hw$$

(48)

If there is neither a source nor a sink in the membrane, then

$$v_{\alpha, \alpha} = 0$$

(49)

On account of (48) and (49), one rewrites (47) as

$$\frac{d}{dt} \iint s \phi ds = \iint s \left( \dot{\phi} - 2Hw\phi \right) ds$$

(50)

3.1. Mass balance

Let $\gamma$ be the surface density of mass. The mass conservation principle requires

$$\frac{d}{dt} \iint s \gamma ds = 0$$

(51)

Recalling theorem (50) and taking off the integration one obtains

$$\partial_t \gamma + v^\beta \partial_\beta \gamma - 2Hw\gamma = 0$$

(52)

where the convected derivative $d_t = \partial_t + v^\beta \partial_\beta$ has been taken into consideration.

If $\xi(i)$ is the mass percentage of the $i$th species inlaid in the membrane, then the balance of this species gives

$$\partial_t \xi(i) + v^\alpha \partial_\alpha \xi(i) - 2Hw\xi(i) = Q(i) - J^{\alpha}_{(i), \alpha}$$

(53)

where $Q(i)$ is the surface chemical source and $J^{\alpha}_{(i)}$ the surface diffusion flux of the species.

3.2. Conservation of momentum

The net force on an arbitrary area with periphery $l$ resolved in the direction $m_\beta$ is given by

$$\frac{d}{dt} \iint s \gamma w ds = \iint s F^3 ds + \oint_l m_\beta T^{33} dl$$

(54)

$$\frac{d}{dt} \iint s \gamma v^\alpha ds = \iint s F^\alpha ds + \oint_l m_\beta T^{\beta\alpha} dl$$

(55)
Making use of theorem (50) and Eq.(52), one obtains Cauchy’s equation for the displacement movement of the surface

\[ \gamma \dot{w} = F^3 + T_{,\beta}^{\beta} \]  

(56)

and that for the internal flow of the particles

\[ \gamma i^{\alpha} = F^{\alpha} + T_{,\beta}^{\beta \alpha} \]  

(57)

In view of the perturbation analysis, one expands Eqs.(56) and (57) below on the assumption that the tilt angle of the director with respect to the normal vector is small. All the expressions in the previous section are valid once the spatial operator \( \nabla \) is replaced everywhere by the surface operator \( \nabla_s \) and the energy densities Eqs.(11), (17) and (18) are multiplied by the membrane thickness. The details of the derivation are given in Appendix C.

As is shown in Eq.(3), the body force \( F \) consists of the gravitational attraction and electromagnetic field-generated forces. The expansion of these vectors expressed in Eqs.(4)-(7) with respect to the local bases gives

\[ F^g_{3} = \gamma g_{(n)} \]  

(58)

\[ F^g_{\alpha} = \gamma g_{i} t_{i}^{\alpha} \]  

(59)

\[ F^{m}_{3} = H_{j,i} n_{j} \left[ \chi_{\perp} (H_{i} - H_{(n)} n_{i}) + \chi_{a} H_{(n)} t_{j}^{i} d_{j}^{\beta} \right] \]  

(60)

\[ F^{m}_{\alpha} = H_{j,i} t_{j}^{i} \left[ \chi_{\perp} (H_{i} - H_{(n)} n_{i}) + \chi_{a} H_{(n)} t_{j}^{i} d_{j}^{\beta} \right] \]  

(61)

\[ F^{e}_{3} = \gamma e E_{(n)} \]  

(62)

\[ F^{e}_{\alpha} = \gamma e i_{i} t_{i}^{\alpha} \]  

(63)

where \( \gamma_e \) is the surface density of electric charges.

The stress tensor \( T \) consists of (as shown in (8)) an isotropic part, a conservative part and a non-conservative part. The application of the surface divergence operator to the isotropic stress yields

\[ T_{,\beta}^{\beta \alpha} = -a^{\beta \alpha} \partial_{\beta} \sigma \]  

(64)

where \( \sigma \) is identified with the surface tensor. The elastic restoring force of the membrane arises from the tilt of the molecules and the curvature of the sheet. When the long molecular axes have a small angle to the normal vector while the shape of the membrane is flat, then the surface force reads

\[ T_{,\beta}^{\beta \alpha} = 0 \]  

(65)

\[ T_{,\beta}^{\beta \alpha} = -k'_{1} a^{\beta \sigma} d_{\sigma, \beta(\tau)}^{\alpha} + k'_{2} e^{\alpha \sigma} a^{\beta \gamma} d_{\sigma, \gamma \beta(\tau)} \]  

(66)

Conversely, when the membrane is flexible while the long molecular axes remain parallel to the normal vector, then the surface force is given by

\[ T_{,\beta}^{\beta \alpha} = (k'_{1} - 2H k'_{11}) (4H^2 - 2K) + (k'_{22} + k'_{24}) 2HK \]  

(67)

\[ T_{,\beta}^{\beta \alpha} = 2k'_{1} a^{\alpha \beta} \partial_{\beta} H - 2k'_{2} e^{\alpha \beta} \partial_{\beta} H - 2k'_{11} (b^{\alpha \beta} + 2H a^{\alpha \beta}) \partial_{\beta} H + \]  

\[ + (k'_{22} + k'_{24}) a^{\alpha \beta} \partial_{\beta} K + 2k'_{12} e^{\alpha \beta} (b^{\sigma}_{\beta} \partial_{\sigma} H + \partial_{\beta} H^2) \]  

(68)
One considers next the viscous stress. It is easy to verify that for the two-dimensional fluid the rate of strain given by Eq.(14) is
\[ S = \frac{1}{2} (v_{\alpha, \beta} + v_{\beta, \alpha} - 2w_{\alpha\beta}) e^\alpha e^\beta \]  
(69)
and the rotational velocity of the director given in Eq.(15)
\[ N = \left[ \frac{1}{2} v^\beta b^\alpha + \dot{d}^\alpha - \frac{1}{2} \varepsilon^{\alpha\beta} d^\beta (\varepsilon^{\lambda\mu} \partial_\lambda v_\mu) \right] e^\alpha + \frac{1}{2} v^\beta b^\alpha d^\beta e^3 \]  
(70)
where \( \dot{d}^\alpha = \partial_t d^\alpha + \dot{v}^\sigma d^\alpha, \sigma(\tau) \). In writing Eq.(69) one took into account the formula [38]
\[ \dot{a}_\lambda = -2w b_{\lambda\mu} \]  
(71)
(see Appendix C.) Substituting expressions (69) and (70) into (12) gives the viscous resistance in a two-dimensional fluid. In the limiting case \( 2H = K = 0 \) it is given by
\[ T^{\nu\beta}_{\alpha\beta} = \alpha_3 \left\{ \partial_t \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) + v^\alpha_{\beta, \alpha(\tau)} + v^{\alpha\beta}_{\alpha\beta(\tau)} - \frac{1}{2} \varepsilon^{\beta\sigma} (d_\sigma \partial_\beta + \partial_\beta d_\sigma) (\varepsilon^{\lambda\mu} \partial_\lambda v_\mu) \right\} + \frac{\alpha''}{2} E^{\alpha\beta\lambda\mu} \left[ (v_{\mu, \lambda\beta} + v_{\lambda, \mu\beta}) d_\alpha (v_{\mu, \lambda} + v_{\lambda, \mu}) d_\alpha, \beta(\tau) \right] \]  
(72)
\[ T^{\nu\beta}_{\alpha\beta} = \frac{\eta}{2} E^{\alpha\beta\lambda\mu} (v_{\lambda, \mu\beta} + v_{\mu, \lambda\beta}) \]  
(73)
and in the other limiting case \( \mathbf{d} \parallel \mathbf{n} \) it is given by
\[ T^{\nu\beta}_{\alpha\beta} = - (\mu + \eta) 4H^2 w + \eta \left[ 4K w + (b^\lambda \mu - H a^\lambda \mu) (v_{\lambda, \mu} + v_{\mu, \lambda}) \right] + \frac{\alpha_3}{2} \left[ v^{\alpha\beta}_{\alpha\beta} + v^\beta \partial_\beta (2H) \right] \]  
(74)
\[ T^{\nu\alpha}_{\beta\beta} = - \mu a^{\alpha\beta} \partial_\beta (2H w) + \frac{\eta}{2} E^{\alpha\beta\lambda\mu} (v_{\lambda, \mu\beta} + v_{\mu, \lambda\beta}) - 2\eta \left[ \partial_\beta w \left( b^{\alpha\beta} - H a^{\alpha\beta} \right) + w a^{\alpha\beta} \partial_\beta H \right] + \frac{\alpha_2}{2} H v^\beta b^\alpha_{\beta} + \frac{\alpha_3}{2} (2H v^\beta b^\alpha_{\beta} - K v^\alpha) \]  
(75)
where \( \mu \) is the dilatation viscosity, \( \eta \) is the shear viscosity and \( E^{\alpha\beta\lambda\mu} \) is the fourth-order tensor \( E^{\alpha\beta\lambda\mu} = a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} - a^{\alpha\beta} a^{\lambda\mu} \). In deducing (72)-(75), formula (48) was used.

The momentum balance for the intrinsic motion over the area \( s \) Eq.(16) is written as
\[ 0 = \int_s \int (G^3 + g^3) \, ds + \oint m_\beta \Pi^{\beta\beta} \, dl \]  
(76)
\[ \frac{d}{dt} \int_s \tilde{\gamma} d^\alpha = \int_s (G^\alpha + g^\alpha) \, ds + \oint m_\beta \Pi^{\beta\alpha} \, dl \]  
(77)
where \( \tilde{\gamma} = \gamma h \). In the usual way, Eqs.(76) and (77) lead directly to the point expression of the law

\[
0 = G^3 + g^3 + \Pi_{\alpha,\beta}^3
\]  
(78)

\[
\tilde{\gamma} \ddot{d}^\alpha = G^\alpha + g^\alpha + \Pi_{\alpha,\beta}^\alpha
\]  
(79)

When a polar fluid is exposed to the external magnetic field, the director is reoriented by the force \( \mathbf{G}^m \), as given by Eq.(21)

\[
G_{m3}^3 = -\chi a H_{(n)} \left( H_{i_i^j} d^\sigma + H_{(n)} \right)
\]  
(80)

\[
G_{m\alpha}^\alpha = -\chi a H_{i_i^j} \left( H_{i_i^j} d^\sigma + H_{(n)} \right)
\]  
(81)

In an external electric field, it is subjected to the force \( \mathbf{G}^e \), as given by Eq.(22)

\[
G_{e3}^3 = -\frac{\epsilon_a}{4\pi} E_{(n)} \left( E_{i_i^j} d^\sigma + E_{(n)} \right)
\]  
(82)

\[
G_{e\alpha}^\alpha = -\frac{\epsilon_a}{4\pi} E_{i_i^j} \left( E_{i_i^j} d^\sigma + E_{(n)} \right)
\]  
(83)

For a two-dimensional fluid, the extra-body forces given by Eqs.(24) and (25), in the limiting case \( 2H = K = 0 \), are given by

\[
g_{3}^3 = -k_2^\epsilon \alpha^\beta \partial_{\beta} d_\alpha
\]  
(84)

\[
g_{\alpha}^\alpha = -k_{13}^\epsilon \alpha^\beta \partial_{\beta} \left( \partial_{\sigma} d^\sigma + \frac{1}{2a} d^\sigma \partial_{\sigma} a \right) + k_{23}^\epsilon \alpha^\beta \partial_{\beta} \left( \varepsilon^{\lambda\mu} \partial_\lambda d_\mu \right)
\]  
(85)

\[
g_{3}^3 = 0
\]  
(86)

\[
g_{\alpha}^\alpha = \gamma_1 \left( d^\alpha + \frac{1}{2} \varepsilon^{\lambda\beta} \partial_\lambda v_\mu d_\beta \varepsilon^{\beta\alpha} \right) + \frac{\gamma''_2}{2} E^{\alpha\beta\lambda\mu} \left( v_{\lambda, \mu} + v_{\mu, \lambda} \right) d_\beta
\]  
(87)

and, in the other limiting case \( \mathbf{d} \parallel \mathbf{n} \),

\[
g_{3}^3 = 0
\]  
(88)

\[
g_{\alpha}^\alpha = k_{13}^\epsilon \alpha^\beta \partial_{\beta} \left( 2H \right)
\]  
(89)

\[
g_{3}^3 = 0
\]  
(90)

\[
g_{\alpha}^\alpha = \frac{\gamma_1}{2} v_\beta b_\beta^\alpha
\]  
(91)

where \( \gamma''_2 \) is referred to the shear motion.

The surface force to which the director is subjected during the elastic deformation, given by Eq.(27), takes the form

\[
\Pi_{\alpha,\beta}^3 = k_2^\epsilon \beta^\alpha \partial_{\beta} d_\alpha
\]  
(92)

\[
\Pi_{\alpha,\beta}^\alpha = \frac{1}{2} \left( k_{11}^\epsilon - k_{22}^\epsilon \right) a^\alpha^\beta \partial_{\beta} \left( \partial_{\sigma} d^\sigma + \frac{1}{2a} d^\sigma \partial_{\sigma} a \right) + k_{22}^\epsilon \beta^\alpha \partial_{\beta} \left( \varepsilon^{\lambda\mu} \partial_\lambda d_\mu \right) + k_{24}^\epsilon \gamma \partial_{\gamma} d_\alpha^{\alpha\beta} - k_{12}^\epsilon \left[ \varepsilon^{\alpha^\beta} \partial_{\beta} \left( \partial_{\sigma} d^\sigma + \frac{1}{2a} d^\sigma \partial_{\sigma} a \right) + a^\alpha^\beta \partial_{\beta} \left( \varepsilon^{\lambda\mu} \partial_\lambda d_\mu \right) \right]
\]  
(93)
for $2H = K = 0$ and

$$\Pi^{\beta\gamma}_{\alpha,\beta} = (k'_{11} - 2HK_1) 2H + (k'_{22} + k'_{24}) 2K$$  \hspace{1cm} (94)

$$\Pi^{\beta\alpha}_{\alpha,\beta} = -k'_{11}a^{\alpha\beta}\partial_\beta (2H) + k_{12}^{\alpha\beta}\partial_\beta (2H)$$  \hspace{1cm} (95)

for $d \parallel n$.

### 3.3. Energy balance

The energy balance Eq.(28) for any material area of the membrane reads

$$\frac{d}{dt} \int_s \int_s \frac{\gamma}{2} (v^\alpha v_\alpha + w^2) + \frac{\gamma}{2} \hat{\alpha} \hat{d}_\alpha + \gamma U \, ds = \int_s \int_s \left( F^\alpha v_\alpha + F^3 w + G^\alpha \dot{d}_\alpha + Q^h \right) \, ds +$$

$$+ \oint l m_\beta \left( T^{\beta\alpha} v_\alpha + T^{\beta3} w + \Pi^{\beta\alpha} \dot{d}_\alpha - J^{h\beta} \right) \, dl$$ \hspace{1cm} (96)

where $Q^h$ is the rate of local production of heat per unit area and $J^h$ is the surface diffusion current across a unit linear element of the periphery of the considered area.

Taking account of Eqs.(56)-(57), and (78)-(79), one obtains from Eq.(96) a differential expression of the law

$$\gamma \dot{U} = Q^h - J^{h\alpha}, \alpha + T^{\beta\alpha} v_\alpha, \beta + T^{\beta3} \partial_\beta w + \Pi^{\beta\alpha} \dot{d}_\alpha, \beta - g^\alpha \dot{d}_\alpha$$ \hspace{1cm} (97)

For practical uses, one replaces the internal energy by a testable physical quantity and obtains

$$\gamma C_l \dot{\Theta} - T^{e\beta\alpha} v_\alpha, \beta - T^{e\beta3} \partial_\beta w - \Pi^{\beta\alpha} \dot{d}_\alpha, \beta + g^{e\alpha} \dot{d}_\alpha = Q^h - J^{h\alpha}, \alpha + (-\sigma a^{\beta\alpha} + T^{e\beta\alpha}) v_\alpha, \beta + T^{e\beta3} \partial_\beta w - g^{e\alpha} \dot{d}_\alpha$$ \hspace{1cm} (98)

where $C_l$ is the heat capacity per unit mass in the static state [39], the convected derivative of the temperature is considered as

$$\dot{\Theta} = \partial_t \Theta + v^\alpha \delta^j_\alpha \partial_j \Theta + \omega n^j \partial_j \Theta$$ \hspace{1cm} (99)

(see Appendix D.)

The non-vanishing terms of Eq.(98) containing $T^{e,v}$, $g^{e,v}$ and $\Pi$ are

$$T^{e\beta\alpha} v_\alpha, \beta = -\left( k'_{11} a^{\beta\sigma} \sigma, \sigma, \sigma - k'_{22} a^{\beta\lambda} \varepsilon^{\alpha\mu} d_{\mu, \lambda} \right) v_\alpha, \beta$$ \hspace{1cm} (100)

$$\Pi^{\beta\alpha} \dot{d}_\alpha, \beta = \frac{\eta}{2} E^{\beta\gamma} \lambda_{\alpha \mu} (v_{\lambda, \mu} + v_{\mu, \lambda}) v_\alpha, \beta$$ \hspace{1cm} (101)

$$T^{e\beta3} \partial_\beta w = \left\{ \frac{\alpha_3}{2} \left[ \dot{d}_\beta + \frac{1}{2} (\varepsilon_{\lambda \mu} \partial_\lambda v_\mu) \varepsilon^{\beta\alpha} d_\alpha \right] + \right.$$

$$\left. + \frac{\alpha''}{2} E^{\beta\gamma} \lambda_{\alpha \mu} (v_{\lambda, \mu} + v_{\mu, \lambda}) d_\alpha \right\} \partial_\beta w$$ \hspace{1cm} (102)
for $2H = K = 0$, and

$$T^{e\beta\alpha}v_{\alpha,\beta} = \left[(k_1' - 2Hk_{11})b^{\beta\alpha} + (k_2' - k_{12}')b^\beta_\gamma \varepsilon^{\gamma\alpha} + 
+ (k_{22}' + k_{24}')Ka^{\beta\alpha}\right]v_{\alpha,\beta} \tag{104}$$

$$g^{e\alpha}\dot{d}_\alpha = k_{13}'d_\alpha a^{\alpha\beta} \partial_\beta (2H) \tag{2H}$$

$$\Pi^{e\alpha}\dot{d}_\alpha,\beta = \left[(k_1' - 2Hk_{11})a^{\beta\alpha} + (k_2' - 2Hk_{12}')\varepsilon^{\beta\alpha} + 
+ (k_{22}' + k_{24}') (2Ha^{\beta\alpha} - b^{\beta\alpha})\right]d_{\alpha,\beta} \tag{106}$$

$$T^{w\beta\alpha}v_{\alpha,\beta} = \left\{ (\eta - \mu)2Hwa^{\beta\alpha} + 
+ \eta \left[ \frac{1}{2}E^{\alpha\beta\lambda\mu} (v_{\lambda,\mu} + v_{\mu,\lambda}) - 2wb^{\beta\alpha}\right]\right\}v_{\alpha,\beta} \tag{107}$$

$$T^{w\beta\alpha}d_\beta w = - \frac{\alpha_3}{2} v^\sigma b_\sigma^\beta \partial_\beta w \tag{108}$$

$$g^{e\alpha}\dot{d}_\alpha = \frac{\gamma_1}{2} v_\beta b^{\beta\alpha} \dot{d}_\alpha \tag{109}$$

for $d \parallel n$. Here the covariant derivative of $\dot{d}_\alpha$ is given by

$$\dot{d}_{\alpha,\beta} = \partial_\tau (d_{\alpha,\beta}(\tau)) + v^\beta d_{\alpha,\gamma}(\tau) + v^\gamma d_{\alpha,\gamma}(\tau) \tag{110}$$

(see Appendix C.)

For the local production of heat it is commonly written as [39]

$$Q^h = \sum_r \sum_k H_k \nu_{kr} b_r \tag{111}$$

where $H_k$ is the enthalpy of the $k$th species per unit mass, $\nu_{kr}$ is the stoichiometric number of species $k$ in chemical reaction $r$ and $b_r$ is the reaction rate of chemical reaction $r$ referred to a unit area.

The divergence of the heat flux Eq.(29) is given for the two-dimensional fluid, by

$$J^{h\alpha}_{\tau,\alpha} = - \kappa_\perp (\partial_\tau \Theta)_{,k} (g^{ik} - n^i n^k) + \kappa_a 2H (n^i \partial_\tau \Theta) + 
+ \kappa_a \left[ (t^i_{\beta} \partial_\Theta) (2Hd^\beta + b_\alpha^\beta d^\alpha) - n^i (\partial_\tau \Theta)_{,k} t^k_{\alpha} d^\alpha - 
- (n^i \partial_\Theta) \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) \right] \tag{112}$$

where $\kappa_\perp$ is the heat conductivity in the direction perpendicular to the director vector and $\kappa_a \equiv \kappa_\parallel - \kappa_\perp$ the thermal anisotropy.
In the above derivations, one takes an arbitrary three-dimensional coordinate frame as the inertial reference system and two arbitrary lines on the middle surface as the curvilinear coordinates. For most purposes, however, one may choose a simple coordinate system. For instance, if one defines the two principal curvature lines as the curvilinear coordinates, which leads to an orthogonal conjugate system, then the expressions will be much simplified.

4. Concluding remarks

By adapting the electrohydrodynamic theory of uniaxial liquid crystals to thin films, one obtains a group of dynamic equations, including basic Eqs.(52)-(53), (56)-(57), (78)-(79) and (98), as well as relevant expressions (58)-(70), (72)-(75), (80)-(95), (100)-(109) and (111)-(112), which may conveniently be used in the study of hydrodynamic and electrodynamic phenomena of biomembrane matrix. The surface density of elastic energy Eq.(43) may be used to the hydrostatic studies of membranes with particular regard to molecule tilting.

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Appendix A. Preliminaries

The Gauss formula is given by

$$\frac{\partial e_\alpha}{\partial \theta^\beta} = \left\{ \mu \atop \alpha \beta \right\} e_\mu + b_{\alpha \beta} e_3$$

$$\frac{\partial e^\alpha}{\partial \theta^\beta} = - \left\{ \alpha \atop \mu \beta \right\} e^\mu + b^\alpha_{\beta} e^3$$

and the derivative of the normal vector

$$\frac{\partial e_3}{\partial \theta^\beta} = - b^\mu_{\beta} e_\mu = - b_{\beta \mu} e^\mu$$

where \( \left\{ \alpha \atop \sigma \beta \right\} \) is the second kind of Christoffel symbol.

Some surface differential invariants of the director vector are given as

$$\nabla_s \mathbf{d} = \left( d^\alpha_{\beta(\tau)} - d^3 b^\alpha_{\beta} \right) e^\alpha e_\alpha + \left( \partial_\beta d^3 + d^\sigma b_{\sigma\beta} \right) e^3 e_3$$

$$= d^\alpha_{\beta} e^\beta e_\alpha + d^3_{\beta} e^\beta e_3$$

$$\nabla_s \cdot \mathbf{d} = d^\alpha_{\alpha(\tau)} - 2 H d^3 = d^\alpha_{\alpha}$$

$$\nabla_s \times \mathbf{d} = \varepsilon^{\alpha \beta} \left[ (\partial_\beta d_3 + d^\sigma b_{\sigma\beta}) e_\alpha - \partial_\beta d_\alpha e_3 \right]$$

$$= \varepsilon^{\alpha \beta} \left( d_3,_{\beta} e_\alpha - \partial_\beta d_\alpha e_3 \right)$$

16
where $\varepsilon^{\alpha\beta}$ is a two parametric permutation tensor, having the components $\varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a}$. The subscript $(\tau)$ distinguishes the tangential part

$$d^{\alpha,\beta(\tau)} = \partial_{\beta}d^{\alpha} + d^{\sigma} \left\{ \begin{array}{c} \alpha \\ \sigma \beta \end{array} \right\}$$

from the whole covariant derivative

$$d^{\alpha,\beta} = \partial_{\beta}d^{\alpha} + d^{\sigma} \left\{ \begin{array}{c} \alpha \\ \sigma \beta \end{array} \right\} - d^{\beta}b^{\alpha}_{\beta} = d^{\alpha,\beta(\tau)} - d^{\beta}b^{\alpha}_{\beta}$$

$$d^{3,\beta} = \partial_{\beta}d^{3} + d^{\sigma}b_{\sigma\beta}$$

Supposing $A$ is a field vector. The surface derivative of the vector is associated with the spatial derivative of the vector by

$$\nabla_{s}A = A_{k,j}t^{i}_{\alpha}e^{\alpha} \left( t^{k}_{\beta}e^{\beta} + n^{k}e^{3} \right)$$

Let $B$ be a second-order tensor ($B = T$ or $\Pi$). The contraction of the third-order tensor $\nabla_{s}B$ gives

$$\nabla_{s} \cdot B = \left( B^{3\alpha}_{,\beta(\tau)} - B^{3\beta}b^{\alpha}_{\beta} - 2HB^{3\alpha} \right) e_{\alpha} +$$

$$+ \left( B^{3\beta}_{,\beta(\tau)} + B^{3\alpha}b_{\alpha\beta} - 2HB^{33} \right) e_{3}$$

with

$$B^{3\alpha}_{,\beta(\tau)} = \partial_{\beta}B^{3\alpha} + B^{3\beta} \frac{1}{2a} \partial_{\beta}a + B^{3\gamma} \left\{ \begin{array}{c} \alpha \\ \gamma \beta \end{array} \right\}$$

$$B^{33}_{,\beta(\tau)} = \partial_{\beta}B^{33} + B^{33} \frac{1}{2a} \partial_{\beta}a$$

The normal vector to the middle surface has the spatial components

$$n_{i} = \frac{1}{2} \varepsilon_{ijk}t^{j}_{\alpha}t^{k}_{\beta}e^{\alpha\beta}$$

where $\varepsilon_{ijk}$ is the three-parametric permutation tensor, which is equal to $\sqrt{g}$ when $ijk$ is an even permutation of 1, 2, 3, equal to $-\sqrt{g}$ when it is an odd permutation and 0 in other cases, $g$ being the determinant of the metric tensor of the spatial coordinates $g_{ij}$.

**Appendix B. Surface density of elasticity energy**

Volume density of elasticity energy Eq.(11) can otherwise be expressed as

$$W^{d} = k_{1} \left( \nabla \cdot d \right) - k_{2} \left( d \cdot \nabla \times d \right) + \frac{1}{2} k_{11} \left( \nabla \cdot d \right)^{2} +$$

$$+ \frac{1}{2} k_{22} \left( d \cdot \nabla \times d \right)^{2} + \frac{1}{2} \left( k_{33} + 2k_{13} \right) \left( d \cdot \nabla d \right)^{2} -$$

$$- \frac{1}{2} \left( k_{22} + k_{24} \right) \left[ \left( \nabla \cdot d \right)^{2} - d \nabla \cdot \nabla d \right] -$$

$$- k_{12} \left( \nabla \cdot d \right) \left( d \cdot \nabla \times d \right) +$$

$$+ k_{13} d \cdot \nabla \left( \nabla \cdot d \right) - k_{23} d \cdot \nabla \left( d \cdot \nabla \times d \right)$$

(A12)
Replacing $\nabla$ by the surface operator $\nabla_s$ defined in Eq.(42), and multiplying the moduli by the membrane thickness $h$, one obtains from Eq.(A12) the surface density of the elasticity energy

$$W^d = -k'_1 \left( 2H - d^\alpha,_{\alpha(\tau)} \right) + k'_2 \varepsilon^{\alpha\beta} \partial_\beta d_\alpha + k'_{11} H \left( 2H - d^\alpha,_{\alpha(\tau)} \right) -$$

$$- \left( k'_{22} + k'_{24} \right) \left( K - 2Hd^\alpha,_{\alpha(\tau)} + d_{\alpha,\beta(\tau)} b^{\beta\alpha} \right) -$$

$$- k'_{12} 2H \varepsilon^{\alpha\beta} \partial_\beta d_\alpha - k'_{13} d^\beta \partial_\beta \left( 2H \right) + O \left( d^{\alpha 2} \right)$$

(A13)

where $k'_i = hk_i$, $k'_{ij} = hk_{ij}$.

Appendix C. Forces sustained in membranes

C1. Forces being exerted on the director

C1.1. External forces

For a two-dimensional fluid, the force to which the director is subjected to in an external magnetic field is given by

$$G^m = -\chi_a \left( d \cdot H \right) H = -\chi_a \left( H_t^k d^\sigma + H_{(n)} \right) \left( H_t^j e^\alpha + H_{(n)} e^3 \right)$$

(A14)

and that in an external electric field is given by

$$G^e = -\frac{\varepsilon_a}{4\pi} \left( d \cdot E \right) E = -\frac{\varepsilon_a}{4\pi} \left( E_k t^k d^\sigma + E_{(n)} \right) \left( E_j t^j e^\alpha + E_{(n)} e^3 \right)$$

(A15)

C1.2. Extra body forces

C1.2.1 Elastic restoring force. Inserting Eq.(A12) in relation (24), one obtains

$$g^e = -\frac{\partial W^d}{\partial d}$$

$$= k_2 \nabla \times d - k_{22} \left( d \cdot \nabla \times d \right) \left( \nabla \times d \right) -$$

$$- \left( k_{33} + 2k_{13} \right) d \cdot \left( \nabla d \right) \cdot \left( d \nabla \right) + k_{12} \left( \nabla \cdot d \right) \left( \nabla \times d \right) -$$

$$- k_{13} \nabla \left( \nabla \cdot d \right) + k_{23} \left[ \nabla \left( d \cdot \nabla \times d \right) + d \cdot \nabla \left( \nabla \times d \right) \right]$$

(A16)

With respect to the local reference frame, it is expressed as

$$g^e = k'_2 \varepsilon^{\alpha\beta} \left( d^\sigma b_{\sigma\beta} e_\alpha - \partial_\beta d_\alpha e_3 \right) -$$

$$- \left( k'_{33} + 2k'_{13} \right) \left( 2H b^{\alpha\beta} d_\beta - K d^\alpha \right) e_\alpha -$$

$$- k'_{12} 2H \varepsilon^{\alpha\beta} \left( d^\sigma b_{\sigma\beta} e_\alpha - \partial_\beta d_\alpha e_3 \right) +$$

$$+ k'_{13} a^{\alpha\beta} \partial_\beta \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\tau \partial_\sigma a \right) e_\alpha +$$

$$+ k'_{23} a^{\alpha\beta} \partial_\beta \left( \varepsilon^{\lambda\mu} \partial_\lambda d_\mu \right) e_\alpha + O \left( d^{\alpha 2} \right)$$

(A17)
C1.2.2. Viscous resistance  The rate of strain $\mathbf{S}$ given in Eq.(14) is written for a two-dimensional fluid

$$\mathbf{S} = \frac{1}{2} (\mathbf{v} \nabla_s + \nabla_s \mathbf{v})$$

$$= \frac{1}{2} \left( \partial_\beta \mathbf{v} \epsilon^\beta + \epsilon^\alpha \partial_\alpha \mathbf{v} \right)$$

$$= \frac{1}{2} g_{ij} \left( t^i_\alpha v^j_\beta + t^j_\beta v^i_\alpha \right) \epsilon^\alpha \epsilon^\beta$$

$$= \frac{1}{2} g_{ij} \left[ t^i_\alpha \left( t^j_\mu v^\mu + t^j_\nu v^\nu \right)^\beta + t^j_\beta \left( t^i_\nu v^\nu + t^i_\mu v^\mu \right)^\alpha \right] \epsilon^\alpha \epsilon^\beta$$

$$= \frac{1}{2} (v_{\alpha, \beta} + v_{\beta, \alpha} - 2w b_{\alpha \beta}) \epsilon^\alpha \epsilon^\beta$$

(A18)

for [28]

$$g_{ij} t^i_\lambda t^j_\mu = a_{\lambda \mu}, \quad g_{ij} \left( t^i_\lambda \dot{r}^\mu + t^j_\mu \dot{r}^i_\lambda \right) = \dot{a}_{\lambda \mu}$$

and [38]

$$\dot{a}_{\lambda \mu} = -2w b_{\lambda \mu}$$

(71)

The rotational velocity of the director $\mathbf{N}$ defined in Eq.(15) is written as

$$\mathbf{N} = \dot{\mathbf{d}} - \frac{1}{2} \nabla_s \times (\mathbf{v} - \dot{\mathbf{r}}) \times \mathbf{d}$$

$$= \left[ \dot{d}^\alpha - \frac{1}{2} v^\beta b^\alpha_\beta + \frac{1}{2} \varepsilon^{\lambda \mu} \partial_\lambda v_\mu d_\beta \varepsilon^{\beta \alpha} \right] \epsilon^\alpha +$$

$$+ \frac{1}{2} v^\beta b_{\beta \alpha} d^\alpha \epsilon^3 + O \left( d^{\alpha 2} \right)$$

(A19)

where

$$\dot{d}^\alpha = \partial_t d^\alpha + v^\sigma d^\alpha_{, \sigma (r)}.$$

Inserting Eqs.(A18) and (A19) into relation (25), one obtains

$$g^v = \gamma_1 \left( \dot{d}^\alpha - \frac{1}{2} v_\beta b^\alpha_\beta - \frac{1}{2} \varepsilon^{\lambda \mu} \partial_\lambda v_\mu d_\beta \varepsilon^{\beta \alpha} \right) \epsilon^\alpha - \gamma_2' 2H w d^\alpha \epsilon^\alpha +$$

$$+ \gamma_2'' \left[ \frac{1}{2} E^{\alpha \beta \lambda \mu} (v_{\lambda, \beta} + v_{\mu, \lambda}) - 2w (b^{\beta \alpha} - H a^{\beta \alpha}) \right] d_\beta \epsilon^\alpha +$$

$$+ \frac{1}{2} \gamma_1 v^\alpha b_\alpha d^\beta \epsilon_3 + O \left( d^{\alpha 2} \right)$$

(A20)

where, $\gamma_2'$ is referred to the area dilation, $\gamma_2''$ to the shear motion, the fourth-order isotropic tensor $E^{\alpha \beta \lambda \mu} = a^{\alpha \lambda} a^{\beta \mu} + a^{\alpha \mu} a^{\beta \lambda} - a^{\lambda \mu} a^{\alpha \beta}$ is used for the tensor operation compatible with the transverse isotropy of the material [28].
C1.3. Extra surface force

The substitution of Eq.(A12) into relation (27) gives

$$\Pi = \frac{\partial W}{\partial (d \nabla)}$$

\[= k_1 I + k_2 d \cdot \varepsilon + k_{11} (\nabla \cdot d) I - k_{22} (d \cdot \nabla \times d) d \cdot \varepsilon + \]
\[+ (k_{33} + 2k_{13}) d \cdot (\nabla d) d - 2 (k_{22} + k_{24}) [(\nabla \cdot d) I - \nabla d] + \]
\[+ k_{12} [(\nabla \cdot d) d \cdot \varepsilon - (d \cdot \nabla \times d) I] - \]
\[- k_{23} [(\nabla \times d) d - d \cdot (\nabla d) \cdot \varepsilon] \quad (A21)\]

where \(\varepsilon\) is the third-order permutation tensor. With respect to the local bases it is given by

$$\Pi = k_1' a^\beta \alpha e_\beta e_\alpha + k_2' \varepsilon^\beta \alpha (e_\beta e_\alpha - d_\alpha e_\beta e_3) -$$

\[- k_1' \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\sigma \partial_\sigma a \right) a^\beta \alpha e_\beta e_\alpha - \]
\[- k_2' \varepsilon^\lambda \mu \partial_\lambda d_\mu e^\beta \alpha e_\beta - (k'_{33} + 2k'_{13}) d^\sigma b^\beta \alpha e_\beta e_3 + \]
\[+ 2 (k'_{22} + k'_{24}) \left[ 2H a^\beta \alpha - b^\beta \alpha - \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) a^\beta \alpha + \right] \]
\[+ a^\beta \gamma d^\gamma \partial_\gamma (\varepsilon^\tau) \right] e_\beta e_\alpha + b^\beta \alpha d_\alpha e_\beta e_3 \right\} - \]
\[- k'_{12} \left[ 2H \varepsilon^\beta \alpha - \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) \varepsilon^\beta \alpha + \right] \]
\[+ \varepsilon^\lambda \mu \partial_\lambda d_\mu e^\beta \alpha \right] e_\beta e_\alpha + 2H \varepsilon^\alpha \beta d_\alpha e_\beta e_3 \right\} - \]
\[- k'_{23} \varepsilon^\beta \alpha \partial_\beta d_\alpha e_3 e_3 + O \left( d^\alpha 2 \right) \quad (A22) \]

The application of the surface divergence operator to Eq.(A22) yields

$$\Pi^\beta \alpha = k'_{22} \varepsilon^\beta \sigma d_\sigma b^\alpha \beta - k_{11} a^\beta \alpha \partial_\beta \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\sigma \partial_\sigma a \right) -$$

\[- k'_{22} \varepsilon^\beta \alpha \partial_\beta \left( e^\lambda \mu \partial_\lambda d_\mu \right) + (k'_{33} + 2k'_{13}) \left( 2H b^\beta \alpha - K d^\alpha \right) + \]
\[+ 2 \left( k'_{22} + k'_{24} \right) \left[ K d^\alpha - 2H b^\beta \alpha d_\beta - a^\beta \alpha \partial_\beta \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) + \right] \]
\[+ a^\beta \gamma d^\gamma \partial_\gamma (\varepsilon^\beta) \right] - \]
\[- k'_{12} \left[ \varepsilon^\beta \alpha \partial_\beta \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\sigma \partial_\sigma a \right) + \right] \]
\[+ a^\beta \alpha \partial_\beta \left( e^\lambda \mu \partial_\lambda d_\mu \right) + 2H \varepsilon^\beta \alpha b^\alpha \beta d_\sigma \right] + O \left( d^\alpha 2 \right) \quad (A23) \]
\[ \Pi_{\beta, \beta}^{33} = k_1' 2H - k_2' \varepsilon^{\beta \alpha} \partial_{\beta} d_{\alpha} - k_{11}' \left[ 4H^2 - 2H \left( \partial_{\sigma} d^\sigma + \frac{1}{2a} d^\sigma \partial_{\sigma} a \right) \right] - \\
- (k_{33}' + 2k_{13}') \left[ b^\beta \varepsilon_{\beta \tau} + d^\beta \partial_{\beta} (2H) \right] + \\
+ 4 (k_{22}' + k_{24}') \left[ K - H \left( \partial_{\sigma} d^\sigma + \frac{1}{2a} d^\sigma \partial_{\sigma} a \right) \right] + \\
+ d^\beta \partial_{\beta} H + b^{\beta \sigma} d_{\sigma, \beta} \right] - \\
+ k_{12}' \varepsilon^{\beta \alpha} d_{\alpha} \partial_{\beta} (2H) + k_{23}' 2H \varepsilon^{\beta \alpha} \partial_{\beta} d_{\alpha} + O \left( d^\alpha \right)^2 \] (A24)

C2. Forces exerting on the continuum

C2.1. Gravitational attraction

The gravity given in Eq.(4) is written for the two-dimensional fluid

\[ \mathbf{F}^g = \gamma \mathbf{g} = \gamma \mathbf{g}_k \left( t_{\alpha}^k \mathbf{e}^\alpha + n^k \mathbf{e}^3 \right) \] (A25)

C2.2. Ponderomotive forces

The force the external magnetic field exerts on the body of the polar fluid, given by Eq.(5), takes the form

\[ \mathbf{F}^m = H_{k, i} \left[ \chi_{\perp} \left( H^j - n^i H_{(n)} \right) + \chi_{(n)} t_{\beta}^j d^\beta \right] \left( t_{\alpha}^k \mathbf{e}^\alpha + n^k \mathbf{e}^3 \right) \] (A26)

In deducing (A26), one used relation (6) and the formula [28]

\[ t_{\alpha}^i t_{\beta}^j \delta^{\alpha \beta} = g^{ij} - n^i n^j \] (A27)

The Lorentz force in an electric field (7) is given by

\[ \mathbf{F}^e = \gamma_\epsilon \mathbf{E} = \gamma_\epsilon \left( E_k t_{\alpha}^k \mathbf{e}^\alpha + E_{(n)} \mathbf{e}^3 \right) \] (A28)

C2.3. Isotropic force

The application of the surface divergence operator to the isotropic stress expressed in Eq.(9) gives

\[ \nabla_s \cdot \mathbf{T} = - a^{\alpha \beta} \partial_{\beta} \sigma_{\alpha} \] (A29)

C2.4. Elastic restoring force

Inserting Eq.(A12) into relation (10) yields

\[ \mathbf{T}^e = - \left( \nabla d \right) \frac{\partial W^d}{\partial (d \nabla)} \]

\[ = - k_1 \nabla d + k_2 \left( \nabla d \right) \times d - k_{11} \left( \nabla \cdot d \right) \left( \nabla d \right) - \\
- k_{22} \left( d \cdot \nabla \times d \right) \left( \nabla d \right) \times d - \\
- (k_{33} + 2k_{13}) \left( \nabla d \right) \cdot (d \nabla) \cdot (d d) + \\
+ 2 (k_{22} + k_{24}) \left[ \left( \nabla \cdot d \right) \nabla d - \left( \nabla d \right) \cdot (\nabla d) \right] + \\
+ k_{12} \left[ \left( \nabla \cdot d \right) \left( \nabla d \right) \times d + (d \cdot \nabla \times d) \left( \nabla d \right) + \\
+ k_{23} \left[ \left( \nabla d \right) \cdot (d \nabla) \cdot d + (\nabla d) \cdot (\nabla \times d) d \right] \right) \] (A30)
With respect to the local bases it is given by

\[ T^e = k'_1 \left[ \left( b^{\beta \alpha} - a^{\beta \gamma} d^{\alpha}_{, \gamma(\tau)} \right) e_\beta e_\alpha - d_\alpha b^{\sigma \beta} e_\beta e_3 + \right. \]

\[ \left. + k'_2 \varepsilon^{\sigma \alpha} \left[ \left( b^{\beta \gamma} - a^{\beta \gamma} d^{\sigma}_{, \gamma(\tau)} \right) e_\beta e_\alpha + d_\sigma b^{\alpha}_\sigma e_\beta e_3 \right] - \right. \]

\[ - k'_1 \left\{ \left[ 2H b^{\beta \alpha} - \frac{1}{\sqrt{a}} \partial_\sigma \left( d^\sigma \sqrt{a} \right) b^{\beta \alpha} - 2H a^{\beta \sigma} d^{\alpha}_{, \sigma(\tau)} \right] e_\beta e_\alpha - \right. \]

\[ \left. \quad - 2H b^{\beta \sigma} d_\sigma e_\beta e_3 \right\} + k'_2 \varepsilon^{\lambda \mu} \partial_\lambda d_\mu \varepsilon^{\alpha \beta} b^{\beta}_\alpha e_\beta e_3 - \right. \]

\[ - (k'_{33} + 2k'_{13}) \left( 2H b^{\beta \alpha} d_\alpha - K d^\beta \right) e_\beta e_3 + \]

\[ + 2 \left( k'_{22} + k'_{24} \right) \left\{ K a^{\beta \alpha} + a^{\beta \gamma} d_{, \gamma(\tau)} b^{\sigma \alpha} - \frac{1}{\sqrt{a}} \partial_\sigma \left( d^\sigma \sqrt{a} \right) b^{\beta \alpha} + \right. \]

\[ \left. + \left( b^{\beta \sigma} - 2H a^{\beta \sigma} \right) d^{\alpha}_{, \sigma(\tau)} \right\} e_\beta e_\alpha - K d^\beta e_\beta e_3 \right\} - \]

\[ - k'_1 \left\{ \left( 2H b^\beta_{, \beta} - \frac{1}{\sqrt{a}} \partial_\sigma \left( d^\sigma \sqrt{a} \right) b^\beta - 2H a^{\beta \sigma} d_\mu_{, \sigma(\tau)} \right) \varepsilon^{\mu \alpha} + \right. \]

\[ + \varepsilon^{\lambda \mu} \partial_\lambda d_\mu b^{\alpha \beta} \right\} e_\beta e_\alpha - 2H \varepsilon^{\lambda \mu} b^\beta_{, \lambda} d_\mu e_\beta e_3 \right\} - \]

\[ - k'_2 K \varepsilon^{\beta \alpha} d_\alpha e_\beta e_3 + O \left( d^\alpha a^2 \right) \]  

\[ \text{(A31)} \]

The dot product of the surface derivative operator and the stress \( T^e \) gives

\[ T^{e \beta \alpha}_{, \beta} = k'_1 \left[ a^{\beta \alpha} \partial_\beta \left( 2H \right) - d^{\alpha}_{, \beta(\tau)} a^{\beta \sigma} + 2H b^{\beta \alpha} d_\beta - K d^\alpha \right] - \]

\[ - k'_2 \left\{ \varepsilon^{\alpha \beta} \left[ \partial_\beta \left( 2H \right) + K d_\beta - a^{\sigma \gamma} d_{, \gamma(\tau)} b^{\beta \sigma} d_\mu \right] \right\} - \]

\[ - k'_1 \left\{ \left( b^{\beta \alpha} + 2H a^{\beta \alpha} \right) \partial_\beta \left( 2H \right) - a^{\beta \sigma} d^{\alpha}_{, \sigma(\tau)} \partial_\beta \left( 2H \right) - \right. \]

\[ \left. - \left( b^{\beta \alpha} \partial_\beta + 2a^{\beta \alpha} b^{\beta}_{, \beta} H \right) \left[ \frac{1}{\sqrt{a}} \partial_\sigma \left( d^\sigma \sqrt{a} \right) \right] + \right. \]

\[ + 2H \left( 2H b^{\beta \alpha} d_\beta - a^{\beta \sigma} d^{\alpha}_{, \beta(\tau)} - K d^\alpha \right) \right\} + \]

\[ + k'_2 \left\{ \varepsilon^{\alpha \sigma} b^{\beta}_{, \sigma} \partial_\beta \left( \varepsilon^{\lambda \mu} \partial_\lambda d_\mu \right) + \varepsilon^{\alpha \beta} \partial_\beta \left( 2H \right) \left( \varepsilon^{\lambda \mu} \partial_\lambda d_\mu \right) \right\} + \]

\[ + \left( k'_{33} + 2k'_{13} \right) \left( 4H^2 b^{\beta \alpha} d_\beta - 2HK d^\alpha - K b^{\alpha \beta} d_\beta \right) + \]

\[ + 2 \left( k'_{22} + k'_{24} \right) \left\{ a^{\beta \alpha} \partial_\beta K + K b^{\beta \alpha} d_\beta + a^{\beta \sigma} \left( b^{\alpha \gamma} d_{, \gamma(\tau)} \right) \right\} \]

\[ + \left( b^{\beta \sigma} - 2H a^{\beta \sigma} \right) d^{\alpha}_{, \beta(\tau)} - \]

\[ - \left( b^{\alpha \beta} \partial_\beta + 2a^{\alpha \beta} \partial_\beta H \right) \left[ \frac{1}{\sqrt{a}} \partial_\sigma \left( \sqrt{a} d^\sigma \right) \right] \right\} - \]

\[ - k'_2 \left\{ 2\varepsilon^{\beta \alpha} \left( b^{\beta}_{, \beta} H + \partial_\beta H^2 \right) + 4H^2 \varepsilon^{\beta \sigma} b^{\alpha}_{, \sigma} d_\beta - \right. \]
\[
\begin{align*}
-\epsilon^{\beta \alpha} & \left[ (2 \partial_\beta H + b^\beta_\sigma \partial_\sigma) \left( \frac{1}{\sqrt{a}} \partial_\gamma (\sqrt{a} d^\gamma) \right) \right] + \\
+2a^{\gamma \sigma} & \partial_\gamma H d_\beta, \sigma(\tau) + 2Ha^{\gamma \sigma} d_\beta, \sigma(\tau) - 2HKd_\beta \right] - \\
- \left( b^{\beta \alpha} \partial_\beta + 2a^{\beta \alpha} \partial_\beta H \right) \left( \epsilon^{\lambda \mu} \partial_\lambda d_\mu \right) \right] + \\
+ k''_{23} K b^\beta_\gamma \epsilon^{\beta \gamma} d_\gamma + O \left( d^{\alpha 2} \right)
\end{align*}
\] (A32)

\[
T^{\beta \alpha} = \left. \kappa \right| \left. T^{\epsilon^{\beta \alpha}} \right| = \left. \kappa \right| \left. \epsilon^{\beta \alpha} \right| \left( 4H^2 - 2K - 2b^\beta_\sigma d^\sigma, (2H) \right) + k''_{2} \epsilon^{\beta \alpha} d_\beta \partial_\alpha (2H) - \\
- k''_{11} \left[ (4H^2 - 2K) \left( 2H - \partial_\sigma d^\sigma - \frac{1}{2a} d^\sigma \partial_\sigma a \right) \right] - \\
- \left( d^\sigma b^\sigma_\beta + 2Hd^\beta \right) \partial_\beta (2H) - 4Hb^\beta_\alpha d^\alpha, (2H) \right] - \\
- \left( k''_{33} + 2k''_{13} \right) \left[ 2Hb^\beta_\sigma d^\sigma, (2H) + \left( b^\beta_\sigma d^\sigma + 2Hd^\beta \right) \partial_\beta (2H) - \\
- d^\beta \partial_\beta K - K \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) \right] + \\
+ 2 \left( k''_{22} + k''_{24} \right) \left[ 2HK - d^\beta \partial_\beta K + 2Hb^\beta_\alpha d^\alpha, (2H) \right] - \\
- \left( 4H^2 + K \right) \left( \partial_\sigma d^\sigma + \frac{1}{2a} d^\sigma \partial_\sigma a \right) \right] + \\
+ 2K''_{12} \left[ \epsilon^{\alpha \beta} \left[ \left( b^\sigma_\alpha \partial_\sigma H + \partial_\alpha H^2 \right) d_\beta + Hb^\beta_\alpha d^\alpha, (2H) \right] - \\
- \left( 2H^2 - K \right) \epsilon^{\lambda \mu} \partial_\lambda d_\mu \right] + \\
+ k''_{23} \epsilon^{\alpha \beta} \left( d_\alpha \partial_\beta K + Kd_\alpha, (2H) \right) + O \left( d^{\alpha 2} \right)
\] (A33)

C2.5. Viscous resistance

With the help of Eqs.(A18)-(A19), the viscous stress Eq.(12) is written as

\[
T'' = - \left\{ 2\mu H w a^{\alpha \beta} + 2\eta w \left( b^{\alpha \beta} - Ha^{\alpha \beta} \right) - \frac{\eta}{2} E^{\alpha \beta \lambda \mu} (v_\lambda, _\mu + v_\mu, _\lambda) + \\
+ \frac{\alpha_2}{2} v^\sigma b^\beta_\sigma d^\alpha + \frac{\alpha_3}{2} v^\sigma b^\alpha_\sigma d^\beta \right\} e_\alpha e_\beta + \\
+ \left\{ \alpha_3 \left[ d^\alpha - \frac{1}{2} v^\beta b^\alpha_\beta - \frac{1}{2} (\epsilon^{\mu \lambda} v_\lambda, _\mu) \epsilon^{\beta \alpha} d_\beta \right] - \alpha_6' 2H w d^\alpha + \\
+ \alpha_6'' \left[ \frac{1}{2} E^{\alpha \beta \lambda \mu} (v_\lambda, _\mu + v_\mu, _\lambda) d_\beta - 2w \left( b^{\alpha \beta} d_\beta - H d^\alpha \right) \right] \right\} e_\alpha e_3 + \\
\right.
\]
\[
+ \left\{ \alpha_2 \left[ d^\beta - \frac{1}{2} v^{\alpha} b^{\beta}_\alpha - \frac{1}{2} (\varepsilon^{\mu\lambda} v_{\lambda,\mu}) \varepsilon^{\alpha\beta} d_\alpha \right] - \alpha'_5 2H w d^\beta + \\
+ \alpha''_5 \left[ \frac{1}{2} E^{\alpha\beta\lambda\mu} (v_{\lambda,\mu} + v_{\mu,\lambda}) d_\alpha - 2w \left( d_\alpha b^{\alpha\beta} - H d^\beta \right) \right] \right\} e_3 e_\beta + \\
+ \frac{1}{2} (\alpha_2 + \alpha_3) v_\alpha b^{\alpha\beta} d_\beta e_3 e_3 + O \left( d^{\alpha 2} \right)
\]

where the viscosity coefficient \( \alpha_4 \) in Eq.(12) has been split into the dilatation viscosity \( \mu \) and the shear viscosity \( \eta \); the \( \alpha' \) coefficients refer to the dilatation of area and the \( \alpha'' \) coefficients to the shear motion.

The dot product of \( \nabla_s \) and \( T^v \) gives

\[
T^{\nu\beta,\alpha}_{\beta,\beta} = - \mu a^{\alpha\beta} d_\beta (2H w) + \frac{\eta}{2} E^{\alpha\beta\lambda\mu} (v_{\lambda,\mu} + v_{\mu,\lambda}) - \\
- 2\eta [\partial_\beta w (b^{\beta\alpha} - Ha^{\beta\alpha}) + w a^{\alpha\beta} \partial_\beta H] + \\
+ \alpha_2 H v_\beta b^{\sigma\alpha} + \frac{\alpha_3}{2} (2H b^{\alpha\sigma} v_\sigma - K v^\alpha) - \\
- \alpha_2 \left[ 2H d^\alpha + \frac{1}{2} (v_\sigma b^{\sigma\beta} d^\beta),_{\beta,(\tau)} - H (\varepsilon^{\mu\lambda} v_{\lambda,\mu}) \varepsilon^{\beta\alpha} b^\beta \right] - \\
- \alpha_3 \left[ d^\beta b^\alpha + \frac{1}{2} b^{\beta\sigma} (v_\sigma d^\alpha),_{\beta,(\tau)} + d^\alpha v^\sigma \partial_\sigma H - \\
- \frac{1}{2} (\varepsilon^{\mu\lambda} v_{\lambda,\mu}) \varepsilon^{\sigma\beta} d_\sigma b^\beta + \right] + \right.
\]

\[
T^{\nu,\beta,3}_{\beta,(\tau)} = - \mu 4H^2 w + \eta \left( b^{\lambda\mu} - Ha^{\lambda\mu} \right) (v_{\lambda,\mu} + v_{\mu,\lambda}) - \\
- 4\eta w (H^2 - K) - \frac{\alpha_3}{2} (b^\beta \varepsilon^\sigma (v_\sigma + 2v^\beta \partial_\beta H) + \\
+ \alpha_3 \left\{ \partial_t \left( \partial_\beta d^\beta + \frac{1}{2} d^\beta \partial_\beta a \right) + v^\sigma d^\beta,_{\sigma,(\tau)} + \right.
\]

\[
\left. + v^\sigma d^\beta,_{\sigma,\beta,(\tau)} - \frac{1}{2} \varepsilon^{\beta\sigma} (d_\sigma \partial_\beta + \partial_\beta d_\sigma) (\varepsilon^{\lambda\mu} \partial_\lambda v_\mu) \right\} - \\
- \alpha'_6 \left[ d^\beta \partial_\beta (2H w) + 2H w \left( \partial_\beta d^\beta + \frac{1}{2} d^\beta \partial_\beta a \right) \right] + \\
+ \alpha''_6 \left[ \frac{1}{2} E^{\alpha\beta\lambda\mu} (v_{\mu,\lambda} + v_{\lambda,\mu}) d_\alpha,_{\beta,(\tau)} - \right.
\]

\[
24
\]
-2\partial_\beta w (d_\alpha b^{\beta\alpha} - Hd^\beta) - 2w \left[ d_{\alpha, \beta(r)} b^{\beta\alpha} + d^3 \partial_\beta H - H \left( \partial_\beta d^\beta + \frac{1}{2\alpha} d^3 \partial_\beta a \right) \right] -

- (\alpha_2 + \alpha_3) Hv^\alpha b_\alpha b^{\beta} d^\beta + O \left( d^\alpha \right) \tag{A36}

Appendix D. Energy balance

In the following, one looks for a testable parameter to alter the internal energy \( U \) in the energy balance equation.

With the help of Eqs. (8), (9) and (23), Eq. (97) is written as

\[
\gamma \dot{U} - \left( T^{\epsilon, \alpha} v_{\alpha, \beta} + T^{\epsilon, \beta} \partial_\beta w + \Pi^{\beta\alpha} \dot{d}_{\alpha, \beta} - g^{\epsilon, \alpha} \dot{d}_{\alpha} \right) = Q^h - J^{h, \alpha} + \left[ (-\sigma^{\alpha, \beta} + T^{v, \beta} v_{\alpha, \beta} + T^{v, \beta} \partial_\beta w - g^{v, \alpha} \dot{d}_{\alpha} \right] \tag{A37}
\]

From the first law of thermodynamics, one knows that

\[
dU - \sum_i L_i dl_i = dQ + dW_{diss} \tag{A38}
\]

where \( W_{diss} \) is the dissipated work, \( l_i \) the \( i \)th work coordinate, \( L_i \) the conjugate work coefficient, and \( U \) and \( Q \) are defined as usual.

As \( U \) is the characteristic function of the temperature \( \Theta \) and of the work coefficient \( l_i \), one has

\[
dU = \left( \frac{\partial U}{\partial \Theta} \right)_{l_i} d\Theta + \sum_i \left( \frac{\partial U}{\partial l_i} \right)_{\Theta, j} dl_i.
\]

Equality (A38) may thus be written as

\[
\left( \frac{\partial U}{\partial \Theta} \right)_{l_i} d\Theta + \sum_i \left[ \left( \frac{\partial U}{\partial l_i} \right)_{\Theta, j} - L_i \right] dl_i = dQ + dW_{diss} \tag{A39}
\]

From the fundamental equations

\[
dU = \Theta dS + \sum_i L_i dl_i
\]

\[
dF = -S d\Theta + \sum_i L_i dl_i,
\]

one has

\[
\left( \frac{\partial U}{\partial l_i} \right)_{l_j} = \Theta \left( \frac{\partial S}{\partial l_i} \right)_{l_j} + L_i
\]

\[
\left( \frac{\partial S}{\partial l_i} \right)_{\Theta, l_j} = - \left( \frac{\partial L_i}{\partial \Theta} \right)_{l_i},
\]

25
where $F$ is the Helmholtz function and $S$ is the entropy. Substitution of the two expressions above into Eq.(A39) leads to

$$C_l d\Theta - \Theta \sum_i \left( \frac{\partial L_i}{\partial \Theta} \right) l_i \, dl = dQ + dW_{diss} \quad (A40)$$

where

$$C_l \equiv \left( \frac{\partial U}{\partial \Theta} \right)_{l_i} = \Theta \left( \frac{\partial S}{\partial \Theta} \right)_{l_i} \quad (A41)$$

is defined as the heat capacity at constant work coordinates [39]. Dividing Eq.(A40) by the infinitesimal time interval, one obtains the instantaneous energy balance

$$C_l \frac{d\Theta}{dt} - \Theta \sum_i \left( \frac{\partial L_i}{\partial \Theta} \right)_{l_i} \frac{dl_i}{dt} = \frac{dQ}{dt} + \frac{dW_{diss}}{dt} \quad (A42)$$

We turn back now to Eq.(A37). The terms in the parentheses on the left-side of Eq.(A37), i.e. the rate of change of the work of the elastic deformation, is identical to the second term on the left-side of Eq.(A42), whereas the right-sides of Eqs.(A37) and (A42) are identical. It follows thus that

$$\gamma C_l \dot{\Theta} - T e^{\beta \alpha} v_{\alpha, \beta} - T e^{\beta 3} \partial \beta w - \Pi^{\beta \alpha} \dot{d}_{\alpha, \beta} + g^{\alpha} \dot{d}_\alpha = Q^h - J^{h, \alpha} + \left( -\sigma a^{\beta \alpha} + T v^{\beta \alpha} \right) v_{\alpha, \beta} + T v^{\beta 3} \partial \beta w - g v^{\alpha} \dot{d}_\alpha \quad (A43)$$

Here the convected derivative of the temperature is given by

$$\dot{\Theta} = \partial_t \Theta + (v^\alpha t^i_\alpha + w n^j) \partial_j \Theta.$$

The internal energy $U$ in Eq.(A37) has now been replaced by the product $C_l \dot{\Theta}$.

The heat flux Eq.(29) is expanded with respect to the local bases

$$J^h = - \partial_j \Theta \left\{ \left( \kappa^i t^j_\alpha + \kappa a n^j d_\alpha \right) e^\alpha + \left( \kappa^i t^j_\alpha d^\alpha + \kappa || n^j \right) e^3 \right\} + O \left( d^{\alpha 2} \right) \quad (A44)$$

where the coefficient $\beta_0$ has been replaced by the heat conductivity perpendicular to the director vector $-\kappa_\perp$ and the coefficient $\beta_1$ by the thermal anisotropy $-\kappa_a \equiv \kappa_\perp - \kappa ||$.

The divergence of the heat flux is given by

$$J^{h, \alpha}_{, \alpha} = J^{h, \alpha}_{, \alpha(\tau)} - 2H J^{h3}$$

$$= - \kappa_\perp \left( \partial_j \Theta \right)_{, k} \left( g^{kj} - n^k n^j \right) + \kappa_a 2H \left( n^j \partial_j \Theta \right) +$$

$$+ \kappa_a \left[ \left( t^i_\beta \partial_j \Theta \right) \left( 2H d^\beta + b^\beta_{\alpha} d^\alpha \right) -$$

$$- \left( n^j \partial_j \Theta \right) \left( \partial_\alpha + \frac{1}{2a} \partial_\alpha a \right) d^\alpha -$$

$$- n^i \left( \partial_j \Theta \right)_{, k} t^k_\alpha d^\alpha \right] + O \left( d^{\alpha 2} \right) \quad (A45)$$
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Figure captions

Fig.1 Illustration of the middle surface and the local coordinate axes.