CRITICAL FACTORISATION IN SQUARE-FREE WORDS

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Abstract. A position $p$ in a word $w$ is critical if the minimal local period at $p$ is equal to the global period of $w$. According to the Critical Factorisation Theorem all words of length at least two have a critical point. We study the number $\eta(w)$ of critical points of square-free ternary words $w$, i.e., words over a three letter alphabet. We show that the sufficiently long square-free words $w$ satisfy $\eta(w) \leq |w| - 5$ where $|w|$ denotes the length of $w$. Moreover, the bound $|w| - 5$ is reached by infinitely many words. On the other hand, every square-free word $w$ has at least $|w|/4$ critical points, and there is a sequence of these words closing to this bound.

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1. Introduction

The Critical Factorisation Theorem [2, 4] is one of the gems in combinatorics on words. It states that each word $w$ with $|w| \geq 2$ has a critical point, i.e., a position where the local period $\partial(w, p)$ is equal to the global period $\partial(w)$ of the word. For a word $w$ with a factorisation $w = xy$, $\partial(w, |x|)$ denotes the length of the shortest word $u$ such that of $u$ and $x$ one is a suffix of the other, and of $u$ and $y$ one is a prefix of the other.

In the binary case, say $w \in \{0, 1\}^*$, it was shown in [5] that there are words having only one critical point; e.g., the Fibonacci words of length at least five are such. Also, it was shown there that each binary word $w$ of length $|w| \geq 5$ and period $\partial(w) > |w|/2$ has less than $|w|/2$ critical points.

We shall now study the number of critical points in ternary square-free words. We show that, each sufficiently long square-free word $w$ can have at most $|w| - 5$ critical points, and the bound $|w| - 5$ is obtained by infinitely many square-free $w$. Also, we prove that a square-free word $w$ has at least $|w|/4$ critical points, and that there is a sequence of square-free words closing to this bound.

2. Preliminaries

For a more extensive introduction to combinatorics on words, including square-freeness and critical factorisation, we refer to Lothaire [6].

For a finite alphabet $\Sigma$, let $\Sigma^*$ denote the monoid of all finite words over $\Sigma$ under concatenation. The empty word is denoted by $\varepsilon$. Let $w \in \Sigma^*$. The length $|w|$ of $w$ is the number of the occurrences of its letters. If $w = w_1uw_2$ then $u$ is a factor of $w$. It is a prefix if $w_1 = \varepsilon$, and a suffix if $w_2 = \varepsilon$. The word $w$ is said to be bordered if there exists a nonempty word $v$, with $v \neq w$, that is both a prefix and a suffix of $w$.

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A word \( w \in \Sigma^* \) is square-free if it has no factors of the form \( vv \) for nonempty words \( v \). Axel Thue [8] showed in 1912 that there are square-free words over a ternary alphabet \( \Sigma_3 = \{0, 1, 2\} \). One such word is obtained by iterating the following morphism \( \tau: \Sigma_3^* \to \Sigma_3^* \) on the initial letter 0:

\[
\tau(0) = 012, \quad \tau(1) = 02, \quad \tau(2) = 1.
\]

The iteration ultimately gives an infinite square-free word

\[
m = 012021012102012 \cdots
\]

By the form of the morphism \( \tau \), the word \( m \) does not contain the short words 010, 212 and 01201 as its factors. The infinite word \( m \) is sometimes called a variation of Thue-Morse word; see [1].

**Lemma 2.1.** Let \( x \) be a nonempty factor of a square-free word \( w \). Then “\( x \) does not overlap with itself in \( w \)”, meaning that if \( w = ux_1x_2x_3v \) where \( x = x_1x_2 = x_2x_3 \) and \( x_2 \neq \varepsilon \) then \( x_1 = \varepsilon = x_3 \).

**Proof.** Overlapping means, see e.g. [6], that \( x_1 \) and \( x_3 \) are conjugates: \( x_1 = rs \), \( x_3 = sr \) and \( x_2 = (rs)^k r \) for some \( r, s \) and \( k \geq 0 \). But \( x_1x_2x_3 = (rs)^k+2 r \) does contain a square even if \( k = 0 \). \( \square \)

### 3. Critical factorisations

We follow the main notations of [5].

An integer \( p \), with \( 1 \leq p \leq |w| \), is a period of \( w \) if for the prefix \( u \) of \( w \) of length \( p \), \( w \) is a prefix of \( u^n \) for some \( n \). The minimal period of \( w \) is denoted by \( \partial(w) \). We have that \( w \) is unbordered if and only if \( \partial(w) = |w| \).

An integer \( p \) with \( 1 \leq p < |w| \) is called a position or a point in \( w \). It denotes the place after the prefix \( x \) of length \( p \): \( w = x \cdot y \), \( |x| = p \). Thus there are \( |w| - 1 \) positions in \( w \). A nonempty word \( u \) is a repetition word of \( w \) at \( p \) if there are words \( x' \) and \( y' \) (possibly empty) such that \( u = x'x \) or \( x = x'u \), and \( u = yy' \) or \( y = uy' \). If here \( |u| > |x| \) (resp., \( |u| > |y| \)) then \( u \) is said to have left overflow (resp., right overflow) at \( p \); see Figure 1.

The length of a repetition word \( w \) at \( p \) is called a local period at \( p \). The minimal local period of \( w \) at \( p \) is denoted by

\[
\partial(w, p) = \min \{ q \mid q \text{ a local period of } w \text{ at } p \}.
\]

Clearly, the (global) period \( \partial(w) \) is a local period at every point, and hence \( \partial(w, p) \leq \partial(w) \) for all \( p \). A position \( p \) of \( w \) is said to be critical if \( \partial(w, p) = \partial(w) \).

The following result follows from the minimality assumption on \( \partial(w, p) \).

**Lemma 3.1.** A repetition word \( u \) of \( w \) at \( p \) of length \( \partial(w, p) \) is unique and it is unbordered.

For a word \( w \), we let

\[
\eta(w) = \text{the number of critical points of } w.
\]
The number

\[ \frac{\eta(w)}{|w| - 1} \]

is called the density of the critical points in \( w \).

**Example 3.2.** Let \( w = 01202012021021021 \) be an unbordered word of length 19, i.e., \( \partial(w) = |w| \). It is not square-free. The minimal local periods of \( w \) are in order of the 18 positions

\[ 3, 5, 5, 2, 5, 5, 2, 5, 19, 19, 3, 3, 3, 3, 3, 3, 3. \]

In this example, \( \eta(w) = 4 \), and the density of critical points is \( \frac{4}{18} = 0.222 \ldots \)

The Critical Factorisation Theorem is due to Césari and Vincent [2]. The present form of the theorem was developed by Duval [4]; for the proofs, see also [3], [5] and Chapter 8 in [7].

**Theorem 3.3 (Critical Factorisation Theorem).** Every word \( w \) of length \( |w| \geq 2 \) has a critical point. Moreover, there is a critical point \( p \) satisfying \( p \leq \partial(w) \).

**Lemma 3.4.** Let \( u \) be a repetition word of \( w \) at \( p \) with \( |w| \geq 2 \) of length \( \partial(w, p) \). If \( u \) has both left and right overflows at \( p \) then \( p \) is a critical point.

**Proof.** Let \( w = xy \) where \( u = x'x = yy' \) for nonempty words \( x', y' \); see Figure 2. By symmetry, we may assume that \( |x'| \leq |y| \) (otherwise \( |y'| \leq |x| \)). Therefore \( y = x'z \) and \( x = zy' \) for some \( z \). Now, \( w = xy = zy'x'z \), and hence \( |zy'x'| \) is a period of \( w \), i.e., \( \partial(w) \leq |zy'x'| \). But \( |zy'x'| = |x'zy'| = |u| \) which shows that \( \partial(w, p) = |u| = \partial(w) \) implying that \( p \) is a critical point.

## 4. Maximum number of critical points

We notice first that if \( w \) is a square-free word with \( |w| \geq 2 \), then \( \partial(w, p) \geq 2 \) for all positions \( p \), since \( \partial(w, p) = 1 \) would imply a factor of the form \( aa \) in \( w \).

The next lemma follows from the observation that if a point \( p \) of \( w \) has neither left nor right overflow, the minimal repetition word \( u \) at \( p \) supplies a square \( uu \) in \( w \).

**Lemma 4.1.** A word \( w \) with \( |w| \geq 2 \) is square-free if and only if each repetition word at each position \( p \) has left or right overflow, or both.

**Example 4.2.** The square-free word \( w = 01202012021021021 \) of length 17 is unbordered, i.e., \( \partial(w) = 17 \). It has 9 critical points at the consecutive positions \( p = 5, 6, \ldots, 13 \). This gives the density number \( 9/16 \approx 0.56 \). For instance, the position \( p = 4 \) has the minimal repetition word \( u = 012021021021021 \), since \( u \) is the shortest factor after the prefix 0102 that ends with 0102. Thus \( \partial(w, 4) = 12 \).

For a word \( w \), let

\[ M(w) = \left\lfloor \frac{|w| + 1}{2} \right\rfloor \]

denote the midpoint of \( w \). For odd length \( |w| \), it is just a choice of the two points nearest to the centre of \( w \).
Lemma 4.3. For a square-free word \( w \in \Sigma_3^* \), the position \( M(w) \) is critical.

Proof. For even \(|w|\), the claim is clear from Lemma 4.1.

Suppose then that \(|w| = 2k + 1\), and let \( u \) be the minimal local repetition word of \( w \) at \( M(w) = k + 1 \). Suppose \( u \) has right but not left overflow. Then \(|u| = k + 1\), and hence \( w = vav \) where \( u = va \) for a prefix \( v \) and an overflow letter \( a \). But then \( \partial(w, k + 1) = |u| = \partial(w) \), and the claim follows. \( \square \)

Theorem 4.4. The minimal local periods form a unimodular sequence for square-free ternary words \( w \in \Sigma_3^* \) with \(|w| \geq 2\), i.e.,

\[
\partial(w, p - 1) \leq \partial(w, p) \quad \text{for} \quad 2 \leq p \leq M(w),
\]

\[
\partial(w, p) \leq \partial(w, p - 1) \quad \text{for} \quad p - 1 \geq M(w).
\]

In particular, the critical points \( p \) of \( w \) form an interval \( q_0 \leq p \leq q_1 \) for some \( q_1 \leq M(w) \) and \( q_2 \geq M(w) \).

Proof. Let \( 2 \leq p \leq M(w) \). The cases for \( p \geq M(w) \) follow by considering the reverse of the word \( w \) which is also square-free. Let the minimal repetition word of \( w \) at \( p \) be \( u \), i.e., \(|u| = \partial(w, p)\). Since \( w \) is square-free and \(|u| \geq 2\), \( u \) has left overflow. If it also has right overflow then \( p \) is critical by Lemma 3.4. Let \( a \) be the letter such that \( u = va \). Then \(|av|\) is a local period at \( p - 1 \) since the position \( p - 1 \) has a repetition word \( av \). (It need not be minimal.) Hence \( \partial(u, p - 1) \leq \partial(w, p) \).

For the second claim, by Lemma 4.3, \( w \) has a critical point \( p \) with \( p \leq M(n) \) and a critical point \( q \geq M(w) \).

This proves the claim. \( \square \)

Example 4.5. Consider the prefix \( w = \tau^5(0) \) of the square-free word \( m \), i.e.,

\[
w = 01202101210201202102120121.
\]

It is unbordered with \(|w| = 24\). The sequence of the 23 minimal local periods is

\[
3, 6, 6, 12, 12, 12, 12, 24, \ldots, 24, 14, 14, 6, 2.
\]

Thus \( \eta(w) = 12 \), i.e., just over one half of the positions are critical.

Theorem 4.6. For each square-free ternary word \( w \) of length \(|w| \geq 26 \), we have \( \eta(w) \leq |w| - 5 \).

Proof. Let \( w \in \Sigma_3^* \) be a square-free ternary word of length \( n \geq 26 \). We show that \( w \) has at least four non-critical points among the \( n - 1 \) positions. The points 1 and \( n - 1 \) are always non-critical, since every letter of \( \Sigma_3 \) occurs in every factor of length four. Let us then assume that \( w \) has exactly three non-critical points. Therefore at least one of the positions 2 or \( n - 2 \) is critical. Without restriction, we can say that \( p = 2 \) is critical.

Without restriction we may assume that \( 01 \) is a prefix of \( w \). It can be checked that there are no such square-free words of length 15 where \( 01 \) occurs only as a prefix. After inspection, we find that the only such word of length 14 is \( v = 01201202102020 \).

Thus since \(|w| \geq 15\), we have \( w = 01x01y \) for some words \( x \) and \( y \) such that \( 01 \) does not occur in \( x \). Now, the word \( x01 \) is a repetition of \( w \) at position 2 and hence, by the criticality of \( p = 2 \), we have \(|x01| = \partial(w) \). Because \( w \) is square-free, \( y \) must be a proper prefix of \( x \). The word \( x \) does not have any occurrences of \( 01 \) and it cannot end in the letter 0. Thus \(|01x| \leq 13 \). But now \( n \leq 25 \); a contradiction. \( \square \)

Example 4.7. In contrast to Theorem 4.6, the word \( w = 01210212021021201202012020120201 \) of length 23 with \( \partial(w) = 13 \) has only three non-critical points, \( p = 1, 2, 22 \).

The upper bound on the critical points is optimal:

Theorem 4.8. There are arbitrarily long square-free words \( w \in \Sigma_3^* \) with \( \eta(w) = |w| - 5 \).
Table 1. Local periods of non-critical points.

| p | ∂(w, p) | Rep. word |
|---|---------|-----------|
| 1 | 2       | 10        |
| 2 | 4       | 0201      |
| | | 1202      |
| | | 21        |

Proof. We rely on the infinite square-free word $m$ that is a fixed point of the morphism $\tau$. Consider the factors of $m$ of the form $\beta = 10201\alpha 12021$. For our purpose, it suffices to choose the words $\beta$ that start after the position 9 of $m$, i.e., just after the prefix 01201012. There are infinitely many words $\beta$ since the suffix 12021 is a factor of $\tau^2(0)$.

For fixed middle word $\alpha$, consider $w = 0\beta 2 = 010201\alpha 12021\alpha$ that begins and ends in the ‘forbidden’ words 010 and 212 that do not occur in $m$. It is, clearly, square-free and unbordered. Each point $p$ with $2 < p < |w| - 2$ is critical, since the minimal repetition word at $p$ must have both left and right overflow in order to leap over a factor 010 or 212; see Lemma 3.4. Table 1 lists the local periods and the minimal repetition words for the remaining four (non-critical) points.

5. Minimum number of critical points

We now turn to the minimality problem of critical points in square-free words.

Theorem 5.1. For each square-free word $w \in \Sigma_3^*$ with $|w| \geq 2$, we have $\eta(w) \geq |w|/4$.

Proof. Let $w \in \Sigma_3^*$ be a square-free word of length $|w| = n$. We remind that, by Lemma 4.3, the middle point $M(w)$ is always critical in $w$. We show that the distance between two non-critical points on the opposite sides of the middle point is at least $n/4$. The claim then follows from Theorem 4.4.

Assume, contrary to the claim, that $p$ and $q$ are non-critical points such that

$$p < n/2 < q \quad \text{and} \quad q - p < n/4.$$  \hspace{1cm} (5.1)

Let $u$ and $v$ be the minimal repetition words at $p$ and $q$, respectively. Consequently, the word $u$ has left overflow, and $v$ has right overflow. Observe that $p > n/4$ and $q < 3n/4$. From $p > n/4$ it follows that $|u| > n/4$. Similarly $|v| > n/4$ and $q - |v| < n/2$. Since $q - p < n/4$, we have $p + |u| \geq q$, i.e., the second occurrence of $u$ reaches over the position $q$. Similarly the first occurrence of $v$ starts before the position $p$; see Figure 3, where $|z| = q - p$.

We now rely on the notations of the factors in Figure 3.

The words $u_3$ and $v_1$ are both prefixes of $v$ and suffixes of $u$. If $|v_1| > |u_3|$ then, as prefixes of $v$, we have $v_1 = u_3x$ for some nonempty $x$. But then $x$ is a border of $u$ since $u_3$ cannot overlap with itself at the end of the first occurrence of $u$; a contradiction. Similarly, if $|u_3| > |v_1|$ then, as suffixes of $u$, we have $u_3 = xv_1$ for some
nonempty $x$ yielding that $x$ is a border of $v$; a contradiction. Therefore $v_1 = u_3$. In this case $z = u_1u_2 = v_2v_3$, and

$$w = u_2v_1zv_1v_2 = u_2v_1u_1u_2v_1v_2.$$  

Since $u_1u_2 = v_2v_3$, one of $u_1$ or $v_2$ is a prefix of the other. To avoid $(u_2v_1u_1)^2$ in $w$, the word $v_2$ must be a proper prefix of $u_1$. But now $\partial(w) \leq |u_2v_1u_1| = |u| = \partial(w, p)$ contradicting the assumption that $p$ was not critical. This proves the claim.  

For the existence part of the next theorem, we take a quick technical analysis of the prefixes of the word $m$. An induction argument gives $|\tau^n(0)| = 3 \cdot 2^{n-1}$, $|\tau^n(1)| = 2^n$ and $|\tau^n(2)| = 2^{n-1}$. For instance,

$$|\tau^{n+1}(0)| = |\tau^n(012)| = 3 \cdot 2^{n-1} + 2^n + 2^{n-1} = 3 \cdot 2^n.$$  

Define the words $m_n$, for $n \geq 1$, as follows

$$m_n = \tau^{2n-1}(0)\tau^{2n-3}(0) \cdots \tau^3(0)\tau(0).$$  

We show that $m_n0$ is a prefix of $m$ of length $4^n$. First $m_10 = 0120 = \tau(0)0$ is a prefix of $m$. Inductively, we have

$$\tau^2(m_n0) = \tau^{2n+1}(0)\tau^{2n-1}(0) \cdots \tau^3(0)\tau(0) = m_{n+1}021.$$  

and hence also $m_{n+1}0$ is a prefix of $m$.

For the length of $m_n$, we obtain

$$|m_n| = \sum_{i=1}^{n} 3 \cdot 2^{2n-i} = 3 \sum_{i=1}^{n} 4^n - 1.$$  

As a prefix of $m$, the word $m_n$ is square-free.

**Theorem 5.2.** For all real numbers $\delta > 0$, there exists a square-free ternary word $w = w(\delta)$ the density of which satisfies

$$0.25 < \frac{\eta(w)}{|w|} < 0.25 + \delta.$$  

**Proof.** For any square-free word $x \in \Sigma^*_3$, let

$$w_x = 0x02x10x02x0.$$  

Suppose first that $w_x$ is square-free, and thus that $x$ does not overlap with itself in $w_x$. The suffix $2x0$ of $w_x$ does not occur elsewhere in $w_x$, and hence the point $3|x| + 6$ is critical, since it must have both overflows. It is the rightmost critical point. Indeed, $\partial(w_x, 3|x| + 7) = |x| + 2$. For the point $2|x| + 3$, the minimal repetition word is $10x02x$ of length $2|x| + 4 < \partial(w)$ since $\partial(w) > 3|x| + 7$. By Lemma 4.3, the middle point is critical, and hence the position $2|x| + 4$ is the leftmost critical point. It follows that $w_x$ has $(3|x| + 7) - (2|x| + 4) = |x| + 3$ critical points. Thus

$$\frac{\eta(w)}{|w_x|} = \frac{|x| + 3}{4|x| + 8} = 0.25 + \frac{1}{|w_x|},$$
which has the limit 0.25 as \(|x| \to \infty\).

It remains to show that there are arbitrarily long square-free words \(x\) for which \(w_x\) is square-free. Again, we lean on the word \(m\). We consider the words \(w_xn\) where

\[ x_n = 120102m_n. \]

We have

\[ w_x = 0\cdot120102m_n \cdot 02\cdot120102m_n \cdot 10\cdot120102m_n \cdot 02\cdot120102m_n \cdot 0. \]

Since 010 and 212 do not occur in \(m\), both 010 and 212 would have to be aligned in any square \(uv\) of \(w_xn\), which is not possible by the ‘markers’ 02, 1 and 0 dividing the word. Also, since \(m_n\) has a border \(\tau(0)\), one easily checks that there are no short squares \(uv\) in \(w_xn\) for \(|u| \leq 4\). Hence a possible square must be inside one of the words (a) \(102m_n021\), (b) \(102m_n101201\), or (c) \(102m_n0\). We consider these cases separately. Recall that \(m_1 = 012 = \tau(0)\). Also, since \(m\) is a fixed point of the morphism \(\tau\), whenever \(v\) is a factor of \(m\), so is \(\tau(v)\).

(a) Let \(\alpha_n = 102m_n021\). The word \(\alpha_1 = 102012021\) occurs in \(m\) after position 9. We prove by induction that each \(\alpha_n\) is a factor of \(m\), and thus they are square-free. Suppose, using (5.2), that

\[ \alpha_i = 102m_i021 = 102\tau^{2i-1}(0) \cdots \tau(0)021 \]

is a factor of \(m\). Then

\[ \tau(\alpha_i) = 0201\cdot21\tau^{2i}(0) \cdots \tau^2(0)012\cdot021 \]

where the indicated factor will be denoted by \(z = 21\tau^{2i}(0) \cdots 012\). By mapping with \(\tau\), we obtain

\[ \tau(z) = 102\tau^{2i+1}(0) \cdots \tau^3(0)\tau(0)021 = 102m_{i+1}021 = \alpha_{i+1}. \]

Hence \(\alpha_n\) is a factor of \(m\) for all \(n\).

(b) We employ in this case the same techniques as in (a) except that we need to eliminate the last letter 1 of the word. In order for \(102m_{n+1}101201\) to have a square \(uv\), the former occurrence of \(u\) in the square must be a factor of \(102m\). However, \(m\) does not have a factor 01201 since it would have to be part of the square 012012. Therefore we can, and must, choose \(\beta_n = 102m_n101202\).

The first occurrence of \(\beta_1 = 102m_1101202 = 102\tau(0)101202\) in \(m\) starts after position 17. We proceed inductively as in case (a). Suppose that

\[ \beta_i = 102m_i101202 = 102\tau^{2i-1}(0) \cdots \tau(0)101202 \]

is a factor of \(m\). Mapping by \(\tau\) gives

\[ \tau(\beta_i) = 0201\cdot21\tau^{2i}(0) \cdots \tau^2(0)0201\cdot2021021, \]

where the indicated portion \(z = 21\tau^{2i}(0) \cdots \tau^2(0)0201\) gives

\[ \tau(z) = 102\tau^{2i+1}(0) \cdots \tau^3(0)\tau(0)101202 = 102m_{i+1}101202 = \beta_{i+1}. \]

Hence \(\beta_n\) is a factor of \(m\), and thus square-free, for all \(n\).

(c) The word \(102m_n0\) is a factor of \(\alpha_n\) and thus square-free. This proves the claim.
The chosen words $x_n = 120102m_n$ are not the only ones that give a square-free word $w_x$.

Problem 5.3. Does there exist, for all sufficiently large $n$, a word $x$ of length $n$ such that $w_x$ is square-free?

Problem 5.4. Does there exist a word $w$ such that $\eta(w) = |w|/4$?

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