Tensor perturbations of generalized Eddington-inspired Born-Infeld branes

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The Palatini $f(|\hat{\Omega}|)$ gravity is a generalized theory of the Eddington-inspired Born-Infeld gravity, where $|\hat{\Omega}|$ is constructed with the spacetime metric and independent connection. In this paper, we study $f(|\hat{\Omega}|)$ theory in the thick brane scenario. The analytic solution of the thick brane generated by a single scalar field is obtained for $f(|\hat{\Omega}|) = |\hat{\Omega}|^{1/2+n}$. For the case of $n > 1/2$, the warp factor is divergent at the boundary of the extra dimension, while the brane system is asymptotically anti-de Sitter. Besides, the asymptotically de Sitter bulk spacetime can be obtained at the region of $0 < n < 1/2$ and of $-1/2 < n < 0$, with the warp factor convergent at the boundary of the extra dimension. In particular, because of the constraints we use, the cases $n = 0$ and $n = 1/2$ are excluded from our analytic solution. Since the integral of the energy density of the brane should not diverge, the case $n < -1/2$ is excluded, too. The investigation around the negative energy density we obtained implies the localization of ordinary matter on the brane. It is shown that the tensor perturbation of the brane is stable and the massless graviton is localized on the thick brane. Therefore, the effective Einstein-Hilbert action on the brane can be rebuilt in low-energy approximation.

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I. INTRODUCTION

As the most successful gravitational theory, General Relativity (GR) is able to explain the gravitational phenomena in the scale ranging from sub-millimeter to Solar System scales [1]. For examples, the deflection of light [2], the precession of Mercury’s perihelion, and so on. In particular, the recently detected gravitational waves provides the final piece of the puzzle for general relativity [3–6]. However, as time goes by, despite its observational successes, there are a lot of problems that beyond the ability of GR, such as the present accelerated expansion of the Universe [7–15] and the unavoidable existence of spacetime singularities [16–18]. Because of these drawbacks, there arise many alternative gravity theories. These so-called modified theories of gravity provide us many new approaches to solve the long-standing problems in GR.

In the early years before the full development of quantum electrodynamics, physicists found a serious problem existed in classical Maxwell’s theory, that is, the divergence of the self-energy of a point-like charged particle. This problem violates the principle of finiteness which states that the physical quantities should not become infinite in a satisfactory theory. One way to solve this problem is to use a non-linear electromagnetic theory called Born-Infeld electromagnetism, which was introduced by Born and Infeld in 1933 [19–21]. In addition, with the symmetry arguments, this theory requires a square root structure and hence can recover Maxwell’s theory at lower energy. The singularity problem also occurs in GR. A similar way to deal with this problem is to construct a pure metric Born-Infeld theory of gravity [22], which shares the same idea of Born and Infeld. However, the story is a little bit complicated. The main problem is that, when the Einstein-Hilbert action is changed, the ghost problem will occur [23, 24]. Nevertheless, based on the previous work [19–22, 25, 26], a remarkable generalization of Born-Infeld theory of gravity called Eddington-inspired Born-Infeld (EiBI) gravity was introduced by Banados and Ferreira [27]. By including a Palatini formalism in the action, the connection is independent of the spacetime metric, and hence

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this theory can avoid the ghosts as well as the singular solutions, such as the big bang singularity [27] and the charged black hole singularity [28, 29], successfully. Moreover, the leading order correction of the action was found to be able to reproduce the Einstein-Hilbert action when the curvature was much smaller than the parameter $b$ in the theory [30]. This theory has received plenty of attentions and has been discussed in many aspects. In a homogeneous and isotropic spacetime, the tensor perturbations is unstable in the Eddington regime, while it is hard to distinguish the evolution of tensor perturbations in the Einstein regime [32]. Besides, with a proper choice of the parameter $b$, the vector modes and scalar modes will not diverge in the Eddington regime [33]. The power spectra of the tensor and scalar perturbations were investigated in Refs. [34, 35], and it was found that they are well consistent with the standard chaotic inflation model. Li and Wei found that, with the existence of the homogeneous and inhomogeneous scalar perturbations, the Einstein static universe is unstable in EiBI gravity [36]. The singularity-avoiding behaviour of the universe was found in both cases of the domination by a scalar field or a perfect fluid [37]. However, when the homogeneous and isotropic universe is filled with phantom energy and the dark and baryonic matter, the Big Rip singularity is unavoidable [38]. In particular, in the early time of the universe, the growth rate of clustering in EiBI gravity deviates from that in the CDM model [39]. It should be pointed out that, at the surface of a polytropic star, where the discontinuity exists in the matter density or even in its first derivative, the curvature singularity and the unacceptable Newtonian limit occur [40]. Besides, with an analysis on the phase transitions of the compact star and the spatial discontinuity in the sound speed, the Ricci scalar was also found to contain singularity on the surface of the star [41]. Nevertheless, by considering the gravitational back-reaction on the particles building up the polytropic star, this problem can be cured [42]. For the case of the neutron star, the structure of the magnetic field inside and the mass-radius relation were discussed in Refs. [43, 44]. To give a physical interpretation of the fundamental quantities, the pressureless cold dark matter fluid was investigated in Ref. [45]. It was found that this fluid has a non-zero effective sound speed and a large change in the dynamics of the early universe is expected in EiBI theory [45]. The recent study on the realistic dark matter shows that there is a contradictory result in EiBI gravity. Thus, one wonders whether EiBI gravity is available to describe gravitational phenomena on all astrophysical and cosmological scales [46]. Fortunately, it was found to be able to predict all the astrophysical quantities involved in the investigation of dark matter halos, and therefore can give a direct comparison with the corresponding observational parameters [46].

Following this research, Potapov et al. obtained a quantified upper limit on the central density of dark matter [47]. The value of this upper limit, which is necessary to guarantee the stability of the circular orbit, was proved to be consistent with the samples of 111 spiral galaxies in a conformal gravity and with the Navarro-Frenk-White and Burkert density profiles [47-49]. It is remarkable that, for a charged EiBI black hole, the radii of the horizon and photon sphere decrease with the parameter $b$ [29]. Therefore, the detection of this feature may shine a light on EiBI gravity in the near future. The recent research on the gravitational waves in EiBI gravity can be seen in Refs. [50, 51].

On the other hand, to unify Maxwell’s electromagnetism and Einstein’s gravity, the idea of extra dimensions was proposed. The first realization was given by Kaluza and Klein (KK) in the well-known KK theory [52-54], where they assumed that there exists a compact spacelike fifth dimension. Inspired by the conception of brane in string theory, Arkani-Hamed, Dimopoulos, and Dvali (ADD) proposed the braneworld model of large extra dimensions with an assumption that the Standard Model particles are localized on the four-dimensional manifold while gravitons can propagate freely in the extra dimensions [55]. In this model, the number of compact spatial extra dimensions is more than one. Unfortunately, it was found that this model could not solve the gauge hierarchy problem completely, since a new hierarchy appear between the fundamental length and the size of the extra dimensions. To deal with this problem, Randall and Sundrum (RS) introduced a nonfactorizable warped extra dimension in RS-I thin braneworld model by considering the reaction of the branes [56]. With interests in higher-dimensional theories and EiBI theory, the authors in Refs. [57-62] discussed many new features of EiBI theory in the braneworld scenario. The perturbations of background spacetime and the localization of gravity were discussed in Refs. [57-59] while the localization of matter fields was considered in Refs. [60-62].

Recently, based on EiBI theory, Odintsov, Olmo, and Rubiera-Garcia introduced a generalized theory called $f(|\Omega|)$ theory [63]. With the functional extension of the square root structure, it was found to be able to present nonsingular solutions [63-65], which are the same as the ones in EiBI theory. Hence, it is reasonable to consider $f(|\Omega|)$ theory in the higher-dimensional spacetime scenario. Besides, one can ask whether there exist any braneworld solutions and whether its four-dimensional effective theory could recover GR in low-energy approximation. Although the stability problem of tensor perturbation in EiBI brane model was discussed in Ref. [57], the conclusion depends on the specific brane solution. The stability problem was further solved in Ref. [58] without using any background solution. However, in the viewpoint of $f(|\Omega|)$ theory, EiBI theory is a particular case with $f(|\Omega|) = |\Omega|^{1/2}$, and therefore, it is interesting and important to verify the stability of the tensor perturbation in the family of functional extensions of EiBI theory.

In this paper, we will investigate the brane model in $f(|\Omega|)$ theory. We will give thick brane solutions in the five-dimensional $f(|\Omega|)$ theory. The four-dimensional flat thick brane is in fact a domain wall generated by the background scalar field and its configuration is determined by $f(|\Omega|)$ gravity, the scalar field, and the way of coupling between
scalar field and gravity. We will investigate the stability of the tensor perturbation of the brane system and derive the low-energy effective theory of \( f(|\hat{\Omega}|) \) gravity on the brane by considering the contribution of the massless graviton.

This paper is organized as follows: In Sec. II, we give a brief review of EiBI theory. In Sec. III, we introduce the generalized theory of EiBI gravity with a short discussion. In Sec. IV, we construct the thick braneworld model and give the analytic brane solution. In Sec. V, we first investigate the stability problem of tensor perturbation. We also study the localization of the graviton zero mode on the brane and then derive the four-dimensional effective theory of \( f(|\hat{\Omega}|) \) theory in lower energy. After that, we investigate the correction to the usual Newtonian gravitational potential from the massive gravitons. Finally, in Sec. VI, we give a short conclusion.

II. THE \( \hat{\Omega} \) REPRESENTATION OF EIBI THEORY

We start with the action of the \( D \)-dimensional EiBI theory [27]

\[
S = \frac{1}{k^2 b} \int d^Dx \left[ \sqrt{-g_{MN} + bR_{MN}(\Gamma)} - \lambda \sqrt{-g_{MN}} \right] + S_m(g_{MN}, \phi),
\]

where the constant \( \kappa \) satisfies \( \kappa^2 = 8\pi G \) with \( G \) the \( D \)-dimensional Newtonian gravitational constant, \( b \) is a free parameter in this theory, \( \lambda \) is a parameter related to the cosmological constant, \( g_{MN} \) is the spacetime metric, \( R_{MN}(\Gamma) \) is the symmetric part of the Ricci tensor formed from the independent connection \( \Gamma_{MN}^P \), and \( S_m(g_{MN}, \phi) \) is the action of the matter fields coupled only to the metric \( g_{MN} \). Note that, here and after, for the sake of simplicity, we use the notation \( \Gamma \) to represent \( \Gamma_{MN}^P \). Being a Palatini formalism, the connection is dynamically independent of the spacetime metric, which leads to a variety of interesting geometric phenomena. In EiBI theory, the gravitational field equations are

\[
q_{MN} = g_{MN} + bR_{MN}(\Gamma),
\]

\[
\sqrt{-g} q^{MN} = \lambda g^{MN} - bk^2 T^{MN},
\]

where \( q_{MN} \) is the auxiliary metric compatible with the independent connection \( \Gamma_{MN}^P \). Note that \( q^{MK}q_{ML} = \delta^K_L \) with \( q^{MN} \) the inverse of \( q_{MN} \). In EiBI theory, there is not any high-order term of spacetime metric in the field equations, and hence it can avoid ghosts instability [30]. The parameter \( \lambda \) is related to the effective cosmological constant, \( \Lambda_{\text{eff}} = \frac{\lambda}{2b^2} \). By setting \( \lambda = 1 \), one can obtain an asymptotically Minkowski spacetime. What’s more, the theory will recover the Eddington’s in the high curvature regime (called as Eddington regime, i.e., \( R_{MN} \gg 1/b \)) while recover GR in the low curvature regime. For homogeneous and isotropic spacetimes, there is a maximum density and minimum length at early times [27], which is just the Eddington regime. The dynamics of homogeneous and isotropic universe shows that, in the case of \( b > 0 \), one could avoid the singular behavior of the universe no matter it is dominated by a scalar field nor a perfect fluid, and, in the case of \( b < 0 \), one is able to obtain an oscillating universe [37]. Therefore, it can be concluded that EiBI theory supports a non-singular description of the universe. However, it should be pointed out that, in the case of some polytropic matter, the curvature scalar of spacetime metric is divergent at the surface [40]. Indeed, even though one neglects the complicated inner structure of the stellar such as a white dwarf, this property is still satisfied. It is important to stress that EiBI theory is unable to describe neither a degenerate gas of nonrelativistic electrons nor a monatomic isentropic gas described within Newtonian theory, which indicates the incompleteness of EiBI theory [40]. Nevertheless, with an analysis on the gravitational backreaction, the stable surface of a polytropic star was found to be formed without the polytropic matter, implying that the singularity problem on the surface will never occur [42]. In addition, the stability of circular material orbits around our Milky Way galaxy gives a constraint on the central density of perfect fluid dark matter, which is assumed to full fill in the galaxy halo [49]. It was shown that this limit is consistent with the data given by Navarro-Frenk-White and Burkert profiles for dark matter, implying a testable prediction of EiBI theory. Recently, some groups have investigated the phenomenological aspects and dynamics of EiBI theory in braneworld scenario. Assuming an empty bulk spacetime with the ordinary matter living on the brane, the mass and radius of neutron stars, compatible with observational data within the cosmologically and astrophysically accepted ranges, can be restored [66]. On the other hand, by using an assumption, \( \phi'(y) = Ka^2(y) \), an analytical background solution could be obtained and the corresponding thick brane was constructed [57]. It was shown that the massless graviton is localized near the thick brane and the stability of the tensor perturbation can be ensured. However, it should be pointed out that the conclusion for the existence of the stable tensor perturbation is related to the specific assumption \( \phi'(y) = Ka^2(y) \), and hence one should investigate the stability problem of the tensor perturbation in a more general way. Because of this issue, two more general assumptions, \( \phi'(y) = Ka^{2n}(y) \) and \( \phi'(y) = K_1a^n(y)(1 - K_2a^2(y)) \), were
considered in Ref. [58]. The analytical and numerical thick brane solutions were obtained. The localization of the massless graviton on the thick brane was also investigated in both cases. It is remarkable that the tensor perturbation was finally proved to be stable without using any assumption. In this paper, we will discuss the stability problem in a more general gravity, i.e., $f(|\hat{\Omega}|)$ theory, which is a generalization of EiBI gravity.

The action (1) can also be expressed in the $\hat{\Omega}$ representation as [63]

$$S = \frac{1}{\kappa^2 b} \int d^D x \sqrt{-g} [\hat{\Omega}^{1/2} - \lambda] + S_m(g_{MN}, \phi),$$  \hspace{1cm} (4)

where $\Omega^K_N \equiv \delta^K_N + bg^{KL}R_{LN}$. The gravitational field equations are given by

$$q_{MN} = g_{MN} + bR_{MN}(\Gamma),$$  \hspace{1cm} (5)

$$|\hat{\Omega}|^{2} q^{MN} = \lambda g^{MN} - b\kappa^2 T^{MN},$$  \hspace{1cm} (6)

where $|\hat{\Omega}|^{1/2} = \sqrt{-q}/\sqrt{-\hat{g}}$. It is obvious that the above field equations (5) and (6) are the same as the origin ones (2) and (3). However, the $\hat{\Omega}$ representation is easier to be generalized.

III. BACKGROUND EQUATIONS OF $f(|\hat{\Omega}|)$ THEORY

In Ref. [63], the authors generalized EiBI theory to $f(|\hat{\Omega}|)$ theory with the action given by

$$S = \frac{1}{\kappa^2 b} \int d^D x \sqrt{-g} [f(|\hat{\Omega}|) - \lambda] + S_m(g_{MN}, \phi).$$  \hspace{1cm} (7)

It is clear that, with a specific choice of $f(|\hat{\Omega}|) = |\hat{\Omega}|^{1/2}$, this theory will recover EiBI theory at the action level. It should be noted that $f(|\hat{\Omega}|)$ theory can recover GR at lower energy ($|bR_{MN}| \ll 1$), while it will deviate from GR when the curvature is much larger than $1/b$. It is interesting to assume that

$$f(|\hat{\Omega}|) = |\hat{\Omega}|^{1/2 + n},$$  \hspace{1cm} (8)

where the parameter $n$ vanishes for the case of EiBI theory. This assumption was first introduced in Ref. [63] with a slight modification on the index. It was found that, for the case of $f(|\hat{\Omega}|) = |\hat{\Omega}|^n$, the action will recover the Einstein-Hilbert action in low-energy approximation. The application of $f(|\hat{\Omega}|)$ theory in the early-time cosmology was also discussed [63]. It was shown that this theory could always avoid big bang singularities without any fine-tuning. Besides, two types of nonsingular solutions found in EiBI theory were also presented in $f(|\hat{\Omega}|)$ theory [64]. In the black hole, by replacing the point-like singularity with a wormhole structure, one can construct a modified inner structure of the black hole compared to the one in EiBI theory, and hence obtain a nonsingular spacetime [63]. In this paper, we will consider the following matter action of a scalar field

$$S_m = \int d^D x \sqrt{-g} \left[ -\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \right].$$  \hspace{1cm} (9)

In order to compare the auxiliary metric in EiBI theory with the one in $f(|\hat{\Omega}|)$ theory, we define an auxiliary tensor

$$\rho_{MN} = g_{MN} + bR_{MN}(\Gamma).$$  \hspace{1cm} (10)

To obtain the field equations, we vary the full action:

$$\delta S = \frac{1}{\kappa^2 b} \int d^D x \left[ \frac{1}{2} \sqrt{-g} g^{MN} (f - \lambda) \delta g_{MN} + \sqrt{-\hat{g}} f_{|\hat{\Omega}|} |\hat{\Omega}|^{-1} \rho \delta \Omega^K_P \right] + \delta S_m,$$  \hspace{1cm} (11)

where the variation of the matter action can be written as

$$\delta S_m = \int d^D x \left\{ \sqrt{-g} \left[ \frac{1}{2} \partial_M \phi \partial^N \phi - \frac{1}{4} g^{MN} \partial^K \phi \partial_K \phi - \frac{1}{2} g^{MN} V \right] \delta g_{MN} ight. \right.$$  
$$+ \left[ \partial_K \left( \sqrt{-g} \partial^K \phi \right) - \sqrt{-g} V \phi \right] \delta \phi \right\},$$  \hspace{1cm} (12)
with $V_\phi \equiv dV(\phi)/d\phi$. With the following relations
\[
\begin{align*}
p_{MN} &= g_{MK}\Omega^K_N, \\
\delta \Omega^K_P &= \delta(p_{PL}g^{LK}) = g^{LK}\delta g_{PL} + bg^{LK}\delta R_{PL} - \Omega^M_P g^{KN}\delta g_{MN}, \\
\delta R_{MN}(\Gamma) &= \nabla^{(T)}_K(\delta \Omega^K_M) - \nabla^{(T)}_N(\delta \Omega^K_M),
\end{align*}
\]
the variation can be expressed as
\[
\delta S = \frac{1}{k^2} \int d^Dx \left\{ \sqrt{-g} \left[ \frac{1}{2} g^{MN}(f - \lambda) + |\hat{\Omega}| f_{|\hat{\Omega}|} p^{MN} - |\hat{\Omega}| f_{|\hat{\Omega}|} g^{MN} \right] \delta g_{MN} \\
- b \nabla^{(T)}_K \left( \sqrt{-g} |\hat{\Omega}| f_{|\hat{\Omega}|} p^{MN} \right) \delta \Omega^K_M + b \nabla^{(T)}_N \left( \sqrt{-g} |\hat{\Omega}| f_{|\hat{\Omega}|} p^{MN} \right) \delta \Omega^K_M \right\} + \delta S_m,
\]
where the covariant derivative, $\nabla^{(T)}$, is compatible with the independent connection. By varying the full action with respect to the metric field, connection field, and scalar field, respectively, the equations of motion can be obtained as follows
\[
\begin{align*}
\nabla^{(T)}_K \left( \sqrt{-g} |\hat{\Omega}| f_{|\hat{\Omega}|} p^{MN} \right) &= 0, \\
(f - \lambda) g^{MN} + 2|\hat{\Omega}| f_{|\hat{\Omega}|} p^{MN} - 2|\hat{\Omega}| f_{|\hat{\Omega}|} g^{MN} &= -bk^2 T^{MN}, \\
\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} \partial^M \phi \right) &= V_\phi,
\end{align*}
\]
where the energy-momentum tensor is defined as
\[
T^{MN} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{MN}} = \partial^M \phi \partial^N \phi - g^{MN} \left( \frac{1}{2} \partial_K \phi \partial^K \phi + V \right).
\]
Equation (16) is equivalent to
\[
(f - \lambda) g^{MN} + 2|\hat{\Omega}| f_{|\hat{\Omega}|} g_{MK} g_{NL} p^{KL} - 2|\hat{\Omega}| f_{|\hat{\Omega}|} g_{MN} = -bk^2 T^{MN}.
\]
Note that $p_{MN} \neq g_{MK} g_{NL} p^{KL}$. By defining another auxiliary tensor
\[
q_{MN} \equiv \left( 2|\hat{\Omega}|^{\frac{1}{2}} f_{|\hat{\Omega}|} \right)^{\frac{2}{D-2}} p_{MN},
\]
which leads to
\[
\begin{align*}
p^{MN} &= \left( 2|\hat{\Omega}|^{\frac{1}{2}} f_{|\hat{\Omega}|} \right)^{\frac{2}{D-2}} q^{MN}, \\
|p_{MN}| &= \left( 2|\hat{\Omega}|^{\frac{1}{2}} f_{|\hat{\Omega}|} \right)^{\frac{2}{D-2}} |q_{MN}|,
\end{align*}
\]
one finds that Eq. (15) can be transformed as
\[
\nabla^{(T)}_K \left( \sqrt{-g} q^{MN} \right) = 0,
\]
for equivalently,
\[
\nabla^{(T)}_K q^{MN} = 0.
\]
It implies that the auxiliary tensor $q_{MN}$ is compatible with the independent connection $\Gamma$ referred before. Therefore, we will call the tensor $q_{MN}$ the auxiliary metric, and regard the definition $p_{MN} \equiv g_{MN} + bR_{MN}(\Gamma)$ as an equation of motion. For the case of EiBI theory, $f(|\hat{\Omega}|) = |\hat{\Omega}|^{1/2}$, Eqs. (20) and (23) will reduce to $q_{MN} \equiv p_{MN}$ and $\nabla^{(T)}_K \left( \sqrt{-g} p^{MN} \right) = 0$, respectively. Therefore, the two tensors are the same in EiBI theory. It is worth noting that the auxiliary metric in $f(|\hat{\Omega}|)$ theory is usually different from that in EiBI theory, while they are related by Eq. (20). What’s more, there exist some relations between the spacetime metric and the auxiliary one. The definition of the auxiliary metric
\[
q_{MN} \equiv \left( 2|\hat{\Omega}|^{\frac{1}{2}} f_{|\hat{\Omega}|} \right)^{\frac{2}{D-2}} (g_{MN} + bR_{MN}(\Gamma)),
\]
denotes the geometric relationship between them while Eq. (16) implies that the two metrics are also connected by the matter field.

In this paper, we mainly focus on the five-dimensional \( f(|\hat{\Omega}|) \) theory \( (D = 5) \). To preserve the four-dimensional Poincaré invariance on the brane, we assume the metrics given as follows

\[
\begin{align*}
 ds^2 &= g_{MN} dx^M dx^N = a^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \\
 ds^2 &= q_{MN} dx^M dx^N = u(y) \eta_{\mu\nu} dx^\mu dx^\nu + v(y) dy^2.
\end{align*}
\]  

(26)  

(27)

Although Eqs. \( (26) \) and \( (27) \) seem to imply that there are two kinds of line elements, we should note that, in Palatini formalism, only the one measured by the spacetime metric, \( g_{MN} \), is physical. In other words, one should consider the physical metric, i.e., the spacetime metric, is measurable and the geodesic of a free falling particle is corresponding to it. That is to say, the physical connection on the manifold is the Levi-Civita connection of physical metric, other than the independent connection. Hence, only the physical curvature tensor, \( R^{P}_{MN}(g) \), has the usual relation to the parallel transport, which is defined by the Levi-Civita connection of the physical metric, in the manifold. Note that, as discussed in Refs. [67, 68], the auxiliary metric does not have any direct physical content, and the independent connection is therefore only explained as an auxiliary field. What’s more, the Ricci tensor, \( R_{MN}(\Gamma) \), is related to the parallel transport corresponding to the auxiliary metric.

Nevertheless, one may insist that the independent connection does define the parallel transport while the spacetime metric is still measurable. It’s obvious that this point of view is reasonable, because once we construct an independent connection, which is considered to be a dynamical quantity, there could be a covariant derivative and some curvature quantities compatible to it. However, as we know, in Palatini formalism, the matter action should be independent of the independent connection and matter is covariantly coupled to the spacetime metric which leads to the conservation of the energy-momentum tensor [40]. So this viewpoint will restrict us to some specific fields, such as the scalar field and the electromagnetic field [67]. What’s more, with this standpoint, one can find that the norm of vectors and the sense of orthogonality are not preserved as is shown in the following short calculation

\[
\frac{D}{d\lambda} (g_{\mu\nu} V^\mu W^\nu) = \left( \frac{D}{d\lambda} g_{\mu\nu} \right) V^\mu W^\nu + g_{\mu\nu} \left( \frac{D}{d\lambda} V^\mu \right) W^\nu + g_{\mu\nu} V^\mu \left( \frac{D}{d\lambda} W^\nu \right) \\
= \left( \frac{D}{d\lambda} g_{\mu\nu} \right) V^\mu W^\nu \neq 0,
\]

(28)  

(29)

where the directional covariant derivative is defined as

\[
\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla^\mu (\Gamma),
\]

(30)

and \( V^\mu \) and \( W^\mu \) are parallel-transported vectors along a curve \( x^\mu(\lambda) \) (i.e., \( \frac{D}{d\lambda} V^\mu = 0 = \frac{D}{d\lambda} W^\mu \)). And this is the reason we consider the parallel transport defined by the Levi-Civita connection of the physical metric.

Now, for the sake of simplicity, we recall the equations of motion as follows

\[
(2|\hat{\Omega}|)^{\frac{3}{2}} f_{|\hat{\Omega}|} \hat{\Omega}^{\frac{3}{2}} q_{MN} = g_{MN} + bR_{MN}(\Gamma),
\]

(31)

\[
(f - \lambda) g^{MN} + 2 \hat{\Omega} |\hat{\Omega}|^{\frac{3}{2}} f_{|\hat{\Omega}|} q^{MN} - 2 |\hat{\Omega}| f_{|\hat{\Omega}|} g^{MN} = -b\kappa^2 T^{MN},
\]

(32)

\[
\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} \partial^M \phi) = V_\phi.
\]

(33)

IV. THICK BRANE SOLUTIONS

In this section, we mainly focus on the construction of thick branes in \( f(|\hat{\Omega}|) \) theory. We will first give the background solutions with a short discussion and further analyse whether they are thick brane solutions. With the
metric ansatz (26) and (27), the explicit form of Eqs. (31), (32), and (33) are shown as follows

\[ a^2 + b \frac{uv'u' - 2v(u'^2 + uu'')} {4uv} = \frac{au^2} {2\frac{1}{2} f|_{\Omega} v^2}, \]
\[ 1 + b \frac{u'^2} {u^2} + b \frac{u'v' - 2u''v} {uv} = \frac{av^2} {2\frac{1}{2} f|_{\Omega} u^2}, \]
\[ 2uv^{\frac{1}{2}} + a^2 [2(f - \lambda) - b\kappa^2 (2V + \phi'^2)] = \frac{2\frac{1}{2} u^2 v^{\frac{1}{2}}} {a f|_{\Omega}}, \]
\[ 2 \frac{u^2} {a^2 v^{\frac{1}{2}}} + a^2 [2(f - \lambda) - b\kappa^2 (2V + \phi'^2)] = \frac{2\frac{1}{2} u^2 v^{\frac{1}{2}}} {a f|_{\Omega}}, \]
\[ \frac{4a'\phi'} {a} + \phi'' = V_\phi, \]

where the primes denote the derivative with respect to the extra-dimensional coordinate \( y \). It can be seen that there are six functions (a, u, v, f, \phi, and V) but with five equations. However, the law of local energy-momentum conservation breaks the independence among the last three equations. So one can usually use two of them to express the other one and finally has four independent equations with six functions. Now, we have two superfluous variables, and therefore can introduce some relations among these six functions, or even some assumptions of the forms of them.

In this paper, we regard the expression \( f(|\Omega|) = |\Omega|^\frac{2}{2} + n \) as an assumption. Further, to coincident with the thick brane model, we assume that the warp factor has the typical form of \( a(y) = \text{sech}^m(ky) \). After a tedious calculation, the background solution is finally obtained:

\[ a(y) = \text{sech}^n \frac{1}{2} \frac{1}{2} (ky), \]
\[ u(y) = K \text{sech}^{\frac{1}{2} + \frac{1}{2}} (ky), \]
\[ v(y) = \frac{2nK} {3 - 2n} \text{sech}^{2 + \frac{4}{2}} (ky), \]
\[ \phi(y) = \frac{(2n + 3)^\frac{1}{2} \kappa} {2^{n + \frac{1}{2}}} \int \left[ 1 + \frac{2n}{3} \tanh^2(ky) \text{sech}^{2 + \frac{1}{2}} (ky)dy, \right] \]
\[ V(y) = -\frac{\lambda} {b\kappa^2} + \frac{2\sqrt{2K\kappa^2} \text{sech}^{2n - 1}(ky)} {4bn^2\sqrt{n}\sqrt{2n + 3}(2n + 1)} \left[ 3 + 4n(n + 2) + 2n(3 - 2n)\text{sech}^2(ky) \right], \]

where

\[ k = \frac{2} {2n + 3} \sqrt{\frac{3} {b}}, \]
\[ K = \frac{(2n - 1)\frac{n + 1}{2} + 1(2n + 1)\frac{1}{2}} {2^{n + 1} + 1(2n + 3)\frac{1}{2}}, \]

and \( b \) is fixed as a negative to guarantee that the parameter \( k \) is real. One can find that, for the given background solution, there are some constrains on \( n \), i.e., \( n \neq 0, 1/2, -3/2 \). It seems that, the constraints on \( n \) prevent the \( f(|\Omega|) \) theory considered here from recovering EiBI theory. However, the constraints are actually arisen from the particular assumptions we have chosen to eliminate the above two superfluous variables, and they may vanish if one uses other ways to solve the equations of motion. So, if \( n = 0 \), or equivalently \( f = |\Omega|^\frac{2}{2} \), using the assumptions given by Liu et al. [57, 58], there should exist a family of thick brane solutions. Although we finally obtain an unshown analytical solution of the scalar field by integrating Eq. (42), we find that there exists the Appell series \( F_1 \) in the solution of (42), from which we can conclude that the solution is finite only for the case of \( n > 1/2 \). While, because of the existence of the hypergeometric function, it is hard to obtain analytic solution of the scalar potential \( V(\phi) \). One can choose some values, such as 5/2, 9/2, and 13/2, of the parameter \( n \) to obtain the simple analytical expressions of the scalar field,
which are given as follows

\[
\phi(y) = A \sqrt{\frac{5 \tanh^2(ky) + 3}{4 \cosh(2ky)-1}} \left[ 5 \sqrt{4 \cosh(2ky)-1} \tanh(ky) + 3 \sqrt{5 \sinh^2(ky)} \cosh(ky) \right], \quad (n = \frac{5}{2})
\]

\[
\phi(y) = B \cosh(ky) \sqrt{\frac{30 \tanh^2(ky) + 10}{2 \cosh(2ky)-1}} \left[ 13 \arctanh \left( \frac{3 \sinh^2(ky)}{2 \cosh(2ky)-1} \right) \right] + \sqrt{6 \cosh(2ky) - 3(6 \sinh^2(ky) + 5) \tanh(ky) \tanh(ky)} \], \quad (n = \frac{9}{2})
\]

\[
\phi(y) = C_1 \text{sech}^3(ky) \sqrt{\frac{13 \tanh^2(ky) + 3}{8 \cosh(2ky)-5}} \left[ C_2 \arctanh \left( \frac{13 \sinh^2(ky)}{8 \cosh(2ky)-5} \right) \text{sech}^3(ky) \right] + 13 \sqrt{8 \cosh(2ky) - 5} \sum_{i=1}^{3} d_i \sinh((2i - 1)ky) \], \quad (n = \frac{13}{2})
\]

where

\[
A = \frac{2048}{78125 \kappa}, \quad B = \frac{536870912}{94143178827 \kappa}, \quad C_1 = \frac{470184984576 \sqrt{47}}{865041591381337933 \kappa}, \quad C_2 = 218448 \sqrt{13}, \quad d_1 = 28598, \quad d_2 = 7901, \quad d_3 = 935.
\]

Note that, on the boundary of the extra dimension, the scalar field approaches to a constant, i.e., \( \phi|_{y \to \pm \infty} = \pm \nu_0 = h(n)(2n + 3)^{3/4} K^{3/4}/(2^{5/4} \kappa n^{1/4}) \), where \( h(n) \) is a function of the parameter \( n \) only. For the case of \( n = 5/2, n = 9/2, \) and \( n = 13/2, h(n) \) equals to 1.23205, 0.828521, and 0.664938, respectively.

Next, we discuss the behavior of the scalar potential. By using Eq. (33), we have the following relations

\[
V_\phi = \frac{4a^2 \phi'}{a^2 \phi'} + \phi'''
\]

\[
V_{\phi\phi} = \frac{4a^4 \phi'' + 4aa'' \phi'^2 - 4a^2 \phi'^2}{a^2 \phi'} + \phi'''
\]

where \( V_{\phi\phi} \) represents the second derivative of the scalar potential with respect to the scalar field. It is obvious that the concrete forms of \( V_\phi \) and \( V_{\phi\phi} \) with respect to \( y \) can be obtained by using Eqs. (39) and (42). Since the scalar field \( \phi \) is an odd function and monotonously increases with the extra dimension \( y \), one can conclude that when \( \phi = 0, V_\phi|_{\phi=0} = 0 \) and \( V_{\phi\phi}|_{\phi=0} = k^2 \left( 3 \times 2^{3/4} k K^{3/4}(2n - 1)(2n + 3)^{3/4} + 2\kappa(3 - 2n)n^{1/4} \right)/(12n^{1/4} \kappa) \). Hence, with a proper value of \( k \), we can have \( V_{\phi\phi}|_{\phi=0} \), which denotes a minimum of the scalar potential at \( \phi = 0 \). When \( y \) approaches to infinity, or equivalently \( \phi = \pm \nu_0 \), one can also find that \( V_{\phi=\pm \nu_0} = 0 \). It is hard to decide whether \( \phi = \pm \nu_0 \) are the extreme points because of the cut-off on the scalar field. Nevertheless, as shown in Fig. 1, by comparing the values of the scalar potential \( V(\phi = 0) \) and \( V(\phi = \pm \nu_0) \), we find that there exists a vacuum at the origin of the extra dimension corresponding to the lowest energy state of the scalar field.

To check whether the background solution corresponds to a thick braneworld model, we should discuss the energy density of the brane. As we know, in most of braneworld models [69, 70], the vacuum of the scalar field is usually located at the boundary of the extra dimension, and the energy density of the brane is contributed by the scalar field only. So, one should subtract the contribution from the cosmological constant, which may exist in the scalar potential and leads to a nonzero vacuum energy density, to fix the energy density to be zero at the boundary. However, the story is usually not this case in modified gravities. Because once we try to modify the Einstein-Hilbert action, we are actually including some unknown effects on the geometry. Hence, it is acceptable to change the scalar potential while
keeping some of the others unchanged [71]. In this paper, the vacuum is no longer located at the boundary but at the origin of the extra dimension. Therefore, instead of subtracting the non-zero vacuum energy density, we find that it is important to drop a constant term, $-\frac{\lambda}{bc^2}$, in the energy density to fix it to be zero far from the brane. Further, for a static observer with four-velocity $\omega^M$, the brane’s energy density is defined as $\rho = T_{MN}\omega^M\omega^N + \frac{\lambda}{bc^2}$ and has the following explicit expression

$$
\rho(y) = -T_0^0 + \frac{\lambda}{bc^2} = -\frac{(2n - 1)^{5n+\frac{3}{2}(2n + 3)}2^{-n}k^2}{3 \times 4^{2n+1}(n+1)k^2}\text{sech}^{2n+1}(k y). \tag{52}
$$

It is important to note that the integral of the energy density over the extra dimension does not diverge, if and only if $n > -1/2$. In addition, in the region of $-1/2 < n < 1/2$, one should be careful to choose the value of $n$ in order to guarantee that the energy density is real. Now, the thick brane is located near the origin of the extra dimension with its thickness determined by $k$. What’s more, the brane can hide its thickness from low-energy tests on the brane by enlarging $k$.

After the analysis of the localization of the thick brane, we should further investigate the effects of the thick brane on the geometry of five-dimensional spacetime. Using the above thick brane solution and the definition of the physical curvature scalar, $R \equiv g^{MN}R_{MN}(g)$, one obtains

$$
R = -\frac{1}{4}k^2 (2n - 1) \left[ (10n - 13) \tanh^2(k y) + 8 \right]. \tag{53}
$$

It is obvious that, when $n > 1/2$, the physical curvature scalar of the bulk spacetime approaches to a negative constant, i.e., $2k^2(1 - 2n)$, far from the brane, which implies an asymptotically anti-de Sitter (AdS) spacetime. However, for the case of $n < 1/2$ the bulk spacetime changes to an asymptotically de Sitter (dS) spacetime. Note that, because of the constraint $n \neq 1/2$, one cannot obtain the asymptotically Minkowski spacetime. The extremum of the physical curvature scalar implies that, being consistent with the configuration of the thick brane, the matter is mainly distributed on the brane. The shapes of the thick brane solution, energy density, and physical curvature scalar are shown in Fig. 1.

V. METRIC PERTURBATIONS

A. Localization of the massless graviton

In this subsection, we investigate the metric perturbations and the localization of the massless graviton. With great interest in the tensor perturbation, we use the transverse-traceless (TT) gauge to decouple the vector perturbations and the scalar perturbations from the tensor perturbation in physical metric, as done in Ref. [57]. Thus, the perturbed metrics should be

$$
ds^2 = g_{MN}dx^Mdx^N = (\bar{g}_{MN} + \triangle g_{MN})dx^Mdx^N = a^2(y)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2, \tag{54}
$$

$$
ds^2 = q_{MN}dx^Mdx^N = (\bar{q}_{MN} + \triangle q_{MN})dx^Mdx^N = \eta_{\mu\nu} + \gamma_{\mu\nu}dx^\mu dx^\nu + 2\gamma_{\mu5}dx^\mu dy + v(y)(1 + \gamma_{55})dy^2, \tag{55}
$$

with

$$
\bar{g}_{MN} = \begin{pmatrix} a^2\eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \quad \triangle g_{MN} = \begin{pmatrix} a^2h_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}, \tag{56}
$$

and

$$
\bar{q}_{MN} = \begin{pmatrix} u\eta_{\mu\nu} & 0 \\ 0 & v \end{pmatrix}, \quad \triangle q_{MN} = \begin{pmatrix} u\gamma_{\mu\nu} & \gamma_{\mu5} \\ \gamma_{5\mu} & v\gamma_{55} \end{pmatrix}, \tag{57}
$$

where the notations $\gamma$ and $h$ respectively represent the perturbations of the auxiliary and physical metrics, $\bar{g}_{MN}$ and $\bar{q}_{MN}$ are the background metrics. From now on, we will mainly focus on the TT part of $h$, which satisfies the TT gauge condition

$$
\partial_\rho(h^{TT})^\rho_\mu = 0 = (h^{TT})^\rho_\rho = h^{TT}. \tag{58}
$$
FIG. 1: The shapes of the thick brane solution, energy density, and physical curvature scalar. The parameters are set to $\kappa = 1$, $k = 1$, $\lambda = 1$, and $n = 5/2$ (blue solid line), $n = 9/2$ (red dashed line), and $n = 13/2$ (black dotted line).

To obtain the inverse of the metrics, one should use the relations $g_{MN}g_{MK} = \delta^K_N$ and $q_{MN}q_{MK} = \delta^K_N$. Moreover, aiming to obtain the linear perturbation equations, we drop the high-order perturbation terms in fields for convenience. Finally, one has

$$\triangle^{(1)} g_{PK} \equiv \delta g_{PK} = -\bar{g}^{NP}\bar{g}^{MK}\triangle g_{MN},$$

$$\triangle^{(1)} q_{PK} \equiv \delta q_{PK} = -\bar{q}^{NP}\bar{q}^{MK}\triangle q_{MN}. \tag{59}$$

Note that, here and after, we use the notation $\delta$ together with a tensor to denote its first-order perturbation. The inverse of metrics can be expressed as

$$g^{MN} = \begin{pmatrix} a^{-2}[\eta^{\mu\nu} - (h^{TT})^{\mu\nu}] & 0 \\ 0 & 1 \end{pmatrix}, \quad q^{MN} = \begin{pmatrix} u^{-1}(\eta^{\mu\nu} - \gamma^{\mu\nu}) & -u^{-1}v^{-1}\gamma_5^{\mu} \\ -u^{-1}v^{-1}\gamma_5^{\nu} & v^{-1}(1 - \gamma_5^{55}) \end{pmatrix}. \tag{60}$$

For the sake of simplicity, we give the expressions of the perturbed auxiliary tensor $\Omega^N_M$ and its determinant as follows

$$\Omega^N_M \approx \tilde{\Omega}^N_M + \delta \Omega^N_M, \tag{62}$$

$$|\Omega| \approx |\tilde{\Omega}| + \delta |\Omega|. \tag{63}$$
\[ \delta \Omega^N_M + b \delta R_{MN} = 2 - \frac{4}{3} \frac{\tilde{\Omega}}{\Omega} f_{\Omega,\Omega} \delta q_{MN} - \frac{1}{3} \frac{2}{3} \frac{\tilde{\Omega}}{\Omega} f_{\Omega,\tilde{\Omega}} \tilde{q}_{MN} \delta \tilde{\Omega} \]
\[ - \frac{1}{3} - \frac{1}{3} f_{\Omega,\Omega} \delta q_{MN} \delta \tilde{\Omega}, (64) \]

\[ - b^n \delta T^{MN} = f_{\Omega,\Omega} \tilde{q}^M \delta \tilde{\Omega} + (f - \lambda) \delta g^{MN} + \frac{1}{3} \frac{2}{3} \frac{\tilde{\Omega}}{\Omega} f_{\Omega,\tilde{\Omega}} \tilde{q}^M \delta \tilde{\Omega} \]
\[ + \frac{5}{3} \frac{2}{3} \frac{\tilde{\Omega}}{\Omega} f_{\Omega,\tilde{\Omega}} \tilde{q}^M \delta \tilde{\Omega} + 2 \frac{2}{3} \frac{\tilde{\Omega}}{\Omega} f_{\Omega,\tilde{\Omega}} \tilde{q}^M \delta \tilde{\Omega} \]
\[ - 2 f_{\Omega,\Omega} \tilde{q}^M \delta \tilde{\Omega} - 2 \tilde{\Omega} f_{\Omega,\Omega} \tilde{q}^M \delta \tilde{\Omega} - 2 \tilde{\Omega} f_{\Omega,\Omega} \delta g^{MN}. (65) \]

It is important to note that, as discussed in Ref. [57], the TT components of the perturbed metrics are decoupled with the scalar field perturbation. So, in this paper, we only consider \( h \) in the perturbed energy-momentum tensor:

\[ \delta T^{MN} = - b^n \delta g^{MN} \delta \tilde{\Omega} + g^{MN} \delta \tilde{\Omega} \]
\[ - \frac{1}{2} \delta g^{MN} \delta \tilde{\Omega} \]
\[ - \frac{1}{2} \delta g^{MN} \delta \tilde{\Omega} - \delta g^{MN} \delta \tilde{\Omega}. (66) \]

By calculating Eqs. (65) and (66), one can obtain the following relations:

\[ \gamma_{\mu\nu} = h_{\mu\nu}^{TT}, \quad \gamma_{\mu5} = 0, \quad \gamma_{55} = 0. (67) \]

It is interesting that the above relations (67) are the same as the ones obtained in EiBI theory [57]. Actually, before the calculation of the above relations, as shown in Eq. (64), we have expected that perturbations on the metrics should be related by the function \( f \). But it seems that, once we use the TT gauge on the physical metric perturbations, no matter what kinds of \( f \) we employ, the auxiliary metric perturbations are always decoupled from \( f \) and equal to the TT components of the physical metric perturbations. It means that, once we use the TT gauge condition on the physical metric perturbations, we are actually using the same gauge condition on the auxiliary metric perturbations. Note that the above relations will lead to \( \delta \tilde{\Omega} = 0 \), and one can further simplify Eq. (64) as

\[ \Box^{(4)} h_{\mu\nu}^{TT} + \frac{u}{v} (h_{\mu\nu}^{TT})'' + \left( \frac{2u'}{v} - \frac{uv'}{v^2} \right) (h_{\mu\nu}^{TT})' = 0, (68) \]

which can be reduced to the result given in EiBI theory [58] if one sets \( f(\tilde{\Omega}) = \tilde{\Omega}^{1/2} \). Further, one can use the coordinate transformation \( dy = \sqrt{u(y)/v(y)} dz \) and the decomposition of the \( n \)th mode of the tensor perturbation

\[ h_{\mu\nu}^{TT(n')}(x^{\rho}, z) = \epsilon_{\mu\nu}^{(n')}(x^{\rho}) u(z)^{-3/4} \psi_{n'}(z) (69) \]

to obtain the Schrödinger-like equation of the \( n \)th graviton KK mode. The fifth component of Eq. (68) is

\[ - \partial_z^2 \psi_{n'}(z) + U(z) \psi_{n'}(z) = m_{n'}^2 \psi_{n'}(z) (70) \]

with the effective potential \( U(z) \) given by

\[ U(z) = - \frac{3}{16} \left( \frac{\partial_z u}{u} \right)^2 - 4u \frac{\partial_z^2 u}{u^2}. (71) \]

The four-dimensional mass of the \( n \)th graviton KK mode, \( m_{n'} \), is defined by

\[ \Box^{(4)} \epsilon_{\mu\nu}^{(n')}(x^{\mu}) = m_{n'}^2 \epsilon_{\mu\nu}^{(n')}(x^{\mu}). (72) \]

The Schrödinger-like equation (70) implies that, for the case of infinite extra dimension, the unbound graviton KK modes can propagate in the extra dimension while the bound ones are trapped in the effective potential. Moreover, the effective potential, which is resulted from the warped extra dimension and usually acts like a potential well, affects the behavior of the KK gravitons. In general, the graviton zero mode is usually bounded in the potential well while the massive graviton is unbounded. Note that the factorization of Eq. (70):

\[ \left( \frac{d}{dz} + 3 \frac{\partial_z u}{u} \right) \left( \frac{d}{dz} + 3 \frac{\partial_z u}{u} \right) \psi_{n'} = m_{n'}^2 \psi_{n'}. (73) \]
ensures that the mass square of every graviton KK mode can not be negative, which highlights the absence of tachyon states and the stability of the tensor perturbations. Here, we are interested in the behavior of the massless graviton whose localization on the thick brane is an important indicator of reconstruction of the four-dimensional Newtonian potential. The corresponding equation of the massless graviton is given as follows

$$\left(\frac{d}{dz} + \frac{3}{4} \partial_z u\right)\left(\frac{d}{dz} - \frac{3}{4} \partial_z u\right)\psi_0 = 0. \quad (74)$$

Then, one can easily obtain the function of the zero mode KK graviton:

$$\psi_0 = Au^{3/4}. \quad (75)$$

Using the normalization condition \(\int_{z_{\min}}^{z_{\max}} \psi_0^2 dz = 1\), one can further determine the normalization coefficient as \(A^{-2} = \int_{-\infty}^{+\infty} u^{1/2} dy\).

In order to recover the four-dimensional Newtonian potential, we investigate the localization of the massless graviton. The corresponding analytical function of the massless graviton is given as

$$\psi_0(y) = AK^{3/4} \text{sech}^{3/8} (\frac{3}{8} (ky)), \quad (76)$$

where the parameter \(K\) is given in Eq. (45) and the normalization coefficient reads \(A^{-2} = \frac{K^{3/2} \sqrt{2\pi} \Gamma(\frac{3}{4} + \frac{n}{2})}{\kappa \sqrt{4 + 2n} \Gamma(\frac{5}{4} + \frac{n}{2})}\).

The shapes of the effective potential, the massless graviton, and the function \(z(y)\) are shown in Fig. 2. We find that the effective potential is infinite at the boundary of the extra dimension, and hence, the ground state of gravitons should vanish at the boundary. Therefore, the massless graviton is localized near the brane. On the other hand, we find that, with the solution \((39)-(41)\), the coordinate transformation leads to a finite extra-dimensional coordinate \(z\). Unlike the infinite one obtained in Refs. [57, 58], this finite extra-dimensional coordinate makes all the gravitons bounded in the extra dimension, and the shapes of gravitons are determined by the effective potential.

FIG. 2: The shapes of the effective potential, the massless graviton, and the function \(z(y)\). The parameter \(n\) is set to \(n = 5/2\) (blue solid line), \(9/2\) (red dashed line), and \(13/2\) (black dotted line), and the parameters \(\kappa, k,\) and \(\lambda\) are set to unity for convenience.
B. Four-dimensional low-energy effective theory

In this subsection, we investigate the low-energy effective theory of a specific family of theories, i.e., \( f(|\hat{\Omega}|) = |\hat{\Omega}|^{1/2 + n} \), on the brane. With a small value of \( bR_{MN} \), by expanding the action (7) to second order in \( b \), the action reads

\[
S \approx \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \hat{R} - 2\Lambda_{\text{eff}} + \frac{(2n + 1)}{4} b \hat{R}^2 - \frac{b}{2} R_{MN} R^{MN} + O(b^2) \right] + S_m(g_{MN}, \phi),
\]

where \( \Lambda_{\text{eff}} = \frac{\lambda_{MN}}{M_{2n+1}} \) is the effective cosmological constant, \( \kappa^2 = \frac{2(n+1)}{\lambda^2} \), and \( \hat{R} \) is the auxiliary metric. Recalling Eq. (20) and expanding the determinant of the auxiliary tensor \( \Omega^N_M \), the relation between the auxiliary metric and the physical one is given as follows

\[
q_{MN} \approx (1 + 2n) \frac{\pi^2}{\lambda^2} g_{MN} + b(1 + 2n) \frac{\pi^2}{\lambda^2} \left( R_{MN} + \frac{2n}{D - 2} g_{MN} \hat{R} \right) + O(b^2).
\]

It means that, with a lowest-order approximation, the independent connection is also compatible with the physical metric. In other words, the action (77) will reproduce the Einstein-Hilbert action together with the effective cosmological constant in the low-energy approximation. What’s more, with a tiny parameter \( n \), the auxiliary metric approximates to the physical one in lower energy. After considering the TT gauge condition on the metric perturbations and using thick brane solutions, one can obtain a four-dimensional effective theory in lower energy. Besides, neglecting the contributions of the effective cosmological constant and matter for convenience, the four-dimensional effective action in low-energy approximation is finally given as follows

\[
S \supset \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-|g_{\mu\nu}^{(4)}| R^{(4)}},
\]

where \( g_{\mu\nu}^{(4)}(x^\rho) \) is the four-dimensional metric on the brane. The four-dimensional curvature scalar is now defined as \( R^{(4)}(x^\rho) = g^{(4)\mu\nu}(x^\rho) R_{\mu\nu}^{(4)}(\hat{g}_{\lambda\rho}) \). Moreover, the integral of the action over the extra dimension is absorbed in the constant \( \kappa_4 \) with

\[
\kappa^2 = \kappa_4^2 \frac{2n + 1}{k} D_1(y) |^{+\infty}_{-\infty} = \kappa_4^2 \sqrt{\pi} \frac{2n + 1}{k} \frac{\Gamma \left( n - \frac{1}{2} \right)}{\Gamma(n)},
\]

where

\[
D_1(y) = \sinh(ky) \, _2F_1 \left( \frac{1}{2}, n; \frac{3}{2}; -\sinh^2(ky) \right).
\]

Further, the ratio between the \( D \)-dimensional and the four-dimensional Newtonian gravitational constants is

\[
\frac{G}{G_4} = \frac{\lambda_1(n)}{k},
\]

\[
\lambda_1(n) = \sqrt{\pi} \frac{(2n + 1) \Gamma \left( n - \frac{1}{2} \right)}{\Gamma(n)}.
\]

Hence, the effective four-dimensional (reduced) Planck scale of this low-energy effective theory should be \( M_{Pl} = \kappa_4^{-1} \sim 10^{19} \text{GeV} \), while the fundamental five-dimensional mass scale is \( M_* = \kappa^{-2/3} \).

C. Correction to Newtonian gravitational potential

To give a prediction on the correction to the effective Newtonian gravitational potential in this \( f(|\hat{\Omega}|) \) theory, we should investigate the contribution from the massive gravitons. As we have mentioned above, a finite extra-dimensional coordinate \( z \) is finally obtained. Thus, in a qualitative analysis, the KK spectrum of gravitons is discrete and the function of the \( n \)th graviton KK mode should be given as

\[
\psi_n(z) \sim A_n \sin \frac{\pi n'}{L} z + B_n \cos \frac{\pi n'}{L} z,
\]
where $A_n = \sqrt{2/L}$ and $B_n = \sqrt{2/L}$ are the normalization coefficients for the odd and even modes, respectively, and the size $L$ of the extra dimension in the conformal coordinate $z$ can be calculated from the coordinate transformation $dz = \sqrt{y(u)} dy$ as:

$$L = \frac{\lambda_3(n)}{k},$$  \hspace{1cm} (85)

where

$$\lambda_2(n) = \sqrt{\frac{2n}{2n + 3}} D_2(y)|^{+\infty}_{-\infty} = \sqrt{\frac{2n\pi}{2n + 3}} \frac{\Gamma\left(\frac{n}{2} + \frac{3}{8}\right)}{\Gamma\left(\frac{n}{2} + \frac{5}{8}\right)},$$  \hspace{1cm} (86)

$$D_2(y) = \sinh(ky) {}_2F_1\left(\frac{1}{2}, \frac{2n + 7}{8} ; \frac{3}{2} ; -\sinh^2(ky)\right).$$  \hspace{1cm} (87)

Besides, the contribution from the massive gravitons to the four-dimensional effective Newtonian potential on our brane at $z = 0$ will only come from the even modes. Then, we conclude that the gravitational potential between two masses on the brane is $[56, 72, 73]$

$$V_{\text{eff}}(r) \approx G_4 \frac{m_1^2 m_2^2}{r} + \sum_{n'=1}^{\infty} G \frac{m_1^2 m_2^2 e^{-m_{n'} r}}{r} \left| \psi_{n'}(0) \right|^2$$

$$= G_4 \frac{m_1^2 m_2^2}{r} \left[ 1 + \frac{G}{G_4 L e^{\pi r/L} - 1} \right],$$  \hspace{1cm} (88)

where $m_1$ and $m_2$ are the mass of the particles, $G_4$ is the four-dimensional Newtonian gravitational constant, and $m_{n'} = n'\pi/L$ is the mass of the $n'$th graviton KK mode. It is now clear that the first term above is contributed from the massless graviton and that massive gravitons will give some corrections to the effective Newtonian potential. Moreover, the correction term expressed by the exponential function will vanish in a large scale, and then recovers the usual Newtonian potential between the particles. On the other hand, if the distance between particles are short enough ($r \ll L$), the total correction contributed from the massive gravitons cannot be neglected. For this case, the effective Newtonian potential will be

$$V_{\text{eff}}(r) \approx G_4 \frac{m_1^2 m_2^2}{r} \left[ 1 + \frac{2\lambda_1}{\pi r_0} \right],$$  \hspace{1cm} (89)

where we have used Eq. (82). The recent test of the gravitational inverse-square law showed that, at a 95% confidence level, the usual Newtonian potential will hold down to a length scale at $59 \mu m$ [74]. It means that, the contribution from the second term of the effective Newtonian potential (89) should be negligible at $r \geq 59 \mu m$. Based on this fact, we give a strong constraint on the parameter $k$ in $f(\Omega)$ theory:

$$k > \frac{2\lambda_1}{\pi r_0},$$  \hspace{1cm} (90)

where we assume $r_0 = 59 \mu m$ being the critical distance of breaking the gravitational inverse-square law between the masses. As is shown in Fig. 3, the parameter $\lambda_1$ has a minimum value $\lambda_{\text{min}} = 7.84871$ at the point $n = 1.90554$. Therefore, $k$ should have a mass scale of at least $10^{-2} eV$ and will increase with the deviation of $n$ from that point. Then, the constraint on the fundamental mass scale can be obtained by using Eq. (82):

$$M_*^4 > \frac{2}{\pi r_c} M_{Pl}^4,$$  \hspace{1cm} (91)

It means that the fundamental mass scale should be at least $10^3 TeV$. Recall Eqs. (85) and (90), one further gives the constraint on the size of the extra dimension, $L$, in the conformal coordinate $z$ as

$$L < \frac{\pi r_c}{2} \lambda_3(n),$$  \hspace{1cm} (92)

where

$$\lambda_3(n) = \frac{\lambda_2}{\lambda_1}$$  \hspace{1cm} (93)
From Fig. 3, one finds that when \( n \) approaches to 1/2, the parameter \( \lambda_3 \) approaches to zero. Besides, \( \lambda_3 \) gets its maximal value, \( \lambda_{3 \text{max}} = 0.212763 \), at the point \( n = 1.64715 \), which implies that \( L \) has a length scale of at most 0.1 mm. The mass of the \( n' \)th graviton KK mode and Eq. (92) give another constraint:

\[
m_{n'} > \frac{2}{r_c} \lambda_4(n),
\]

with

\[
\lambda_4(n) = \frac{n}{\lambda_3}.
\]

It is obvious that the parameter \( \lambda_4 \) has a minimum value, \( \lambda_{4 \text{min}} = 5.45515 \), when \( n = 0.912792 \), and one concludes that the mass spectrum of the massive gravitons is at least \( 10^{-2}\text{eV} \). Note that the value of the mass gap is depend on the value of \( \lambda_4 \), and, as shown in Fig. 3, it will increase to infinity when \( n \) approaches to infinity and 1/2. In addition, we only give the lower limit of the fundamental mass scale, which means that the coupling strength of each massive graviton to matter is about \( 1/M_s \sim 10^{-5}\text{TeV}^{-1} \) at most. However, one could always assume that the fundamental mass scale is much larger than that limit, for example, \( M_s \sim 10^{16}\text{TeV} \). Therefore, the effect of each massive graviton on particle physics can be negligible, and then one should count the combined effect of all the massive gravitons to give an observable correction to both the usual Newtonian potential and particle physics.

![Graphs showing the shapes of parameters](image)

**FIG. 3:** The shapes of the parameters \( \lambda_1 \), \( \lambda_2 \), \( \lambda_3 \), and \( \lambda_4 \). The blue dashed lines denote the value of the parameters with \( n \) approaching to 1/2, while the orange dashed lines localize the extreme points of the parameters.

**VI. CONCLUSION**

Summarizing, in this paper, we investigated the thick \( f(|\hat{\Omega}|) \) brane model and stability of the tensor perturbations of the brane system. We first reviewed EiBI theory and its generalized theory, i.e., \( f(|\hat{\Omega}|) \) theory, in Palatini formalism. It is worth noting that the auxiliary tensor \( p_{MN} = g_{MN} + bR_{MN}(\Gamma) \) is the auxiliary metric compatible with the independent connection \( \Gamma \) if and only if \( f(|\hat{\Omega}|) = |\hat{\Omega}|^{1/2} + \text{const.} \), where the constant term can be absorbed into the cosmological constant. In fact, this special choice corresponds to EiBI theory. In addition, we investigated the five-dimensional brane model in \( f(|\hat{\Omega}|) \) theory and gave an analytic thick brane solution for \( f(|\hat{\Omega}|) = |\hat{\Omega}|^{\pm n} \). Although the scalar field generating the thick brane still has a kink configuration like in other gravity theories, the vacuum is located...
at $\phi = 0$ but not at $\phi = \pm v_0$. It should be noticed that our solution did require some constraints on the parameter $n$, i.e., $n \neq 0$, $1/2$, $-3/2$, to make itself meaningful. Moreover, the statement that the brane’s energy density should not diverge gives a lowest limit on the parameter $n$, i.e., $n > -1/2$. It is interesting that the energy density in this model goes to negative near the origin of the extra dimension, which is a little bit stranger than other gravity theories. Nevertheless, one can still judge the brane configuration from the localized energy density. We analysed the physical curvature scalar of the bulk spacetime and concluded that, for the cases of $n > 1/2$ and $n < 1/2$, the spacetime is asymptotically AdS and dS, respectively. In particular, because of the constraint on the parameter $n$, we could never fix it to $n = 1/2$, and therefore the asymptotically Minkowski spacetime can not be obtained.

We also investigated the tensor perturbations of the two metrics, and showed that the choice of TT gauge on the physical metric perturbations would lead to the TT gauge condition on the auxiliary metric perturbations. It is important to note that, unlike what we have assumed before the calculation of the linear perturbation equations, the tensor perturbations of the two metrics were found to be connected with each other and to be independent of $f(|\hat{\Omega}|)$. Besides, from the obtained Schrödinger-like equation of the extra-dimensional part of the graviton KK mode, we proved that there is no tachyon state existed and the tensor mode was stable. The massless graviton was found to be localized on the thick brane, so the four-dimensional Newtonian potential could be recovered. It is remarkable that a finite conformal extra-dimensional coordinate $z$ was found and all the KK gravitons were restricted to be bounded states. At last, we obtained the low-energy effective theory and the correction to the usual Newtonian gravitational potential on the brane and gave a strong constraint on the parameters in this generalized EiBI theory.

The localization of the fermion fields and the gravitons is a conspicuous problem in the braneworld model. Many efforts have been paid in this area, and it is now well known that the localization problem could be addressed well in some thick braneworld models [60–62, 75–84]. On the other hand, the investigation around the fermion resonances makes it possible to find the massive modes with finite lifetimes [83–90]. As we have concluded, in our model, the gravity can be localized on the constructed thick brane. Therefore, it is interesting to give a further study on the localization mechanism and the resonance problem of the fermion fields in this model.

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