How Many Machines Can We Use in Parallel Computing for Kernel Ridge Regression?

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Abstract

This paper attempts to solve a basic problem in distributed statistical inference: how many machines can we use in parallel computing? In kernel ridge regression, we address this question in two important settings: nonparametric estimation and hypothesis testing. Specifically, we find a range for the number of machines under which optimal estimation/testing is achievable. The employed empirical processes method provides a unified framework, that allows us to handle various regression problems (such as thin-plate splines and nonparametric additive regression) under different settings (such as univariate, multivariate and diverging-dimensional designs). It is worth noting that the upper bounds of the number of machines are proven to be un-improvable (upto a logarithmic factor) in two important cases: smoothing spline regression and Gaussian RKHS regression. Our theoretical findings are backed by thorough numerical studies. This work extends [13] to random and multivariate design.

Key Words: Computational limit, divide and conquer, kernel ridge regression, minimax optimality, nonparametric testing.

1 Introduction

In the parallel computing environment, a common practice is to distribute a massive dataset to multiple processors, and then aggregate local results obtained from separate machines into global counterparts. This Divide-and-Conquer (D&C) strategy often requires a growing number of machines to deal with an increasingly large dataset. An important question to statisticians

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is "how many machines can we use in parallel computing to guarantee statistical optimality?"
The present work aims to explore this basic yet fundamentally important question in a classical
nonparametric regression setup, i.e., kernel ridge regression (KRR). This can be done by carefully
analyzing statistical versus computational trade-off in the D&C framework, where the number of
deployed machines is treated as a simple proxy for computing cost.

Recently, researchers have made impressive progress about KRR in the modern D&C framework
with different conquer strategies; examples include median-of-means estimator proposed by [8],
Bayesian aggregation considered by [12, 19, 15, 17], and simple averaging considered by [25] and
[13]. Upper bounds for the number of machines $s$ have been studied in such strategies to guarantee
good property. For instance, [25] showed that, when $s$ processors are employed with $s$ in a suitable
range, D&C method still preserves minimax optimal estimation. In smoothing spline regression
(a special case of KRR), [13] derived critical, i.e., un-improvable, upper bounds for $s$ to achieve
either optimal estimation or optimal testing, but their results are only valid in univariate fixed
design. The critical bound for estimation obtained by [13] significantly improves the one given in
[25]. Nonetheless, it remains unknown if results obtained in [13] continues to hold in a more general
KRR framework where the design is either multivariate or random. On the other hand, there is a
lack of literature dealing with nonparametric testing in general KRR. To the best of our knowledge,
[13] is the only reference but in the special smoothing spline regression with univariate fixed designs.

In this paper, we consider KRR in the D&C regime in a general setup: design is random and
multivariate. We characterize the upper bounds for $s$ for achieving optimal estimation and testing
based on empirical processes (EP) method. Our EP method can handle various function spaces
including Sobolev space, Gaussian RKHS, or spaces of special structures such as additive functions,
in a unified manner. It is worthy noting that our upper bound for $s$ is always larger than or equal
to the one obtained by [25] using the same averaging local results approach. In the particular
smoothing spline regression, our upper bound is almost identical as [13] (upto a logarithmic factor)
for optimal estimation, which is proven to be un-improvable. In Gaussian RKHS, our upper bound
for $s$ is almost identical to [25].

The second main contribution of this paper is to propose a Wald type test statistic for nonpara-
metric testing in D&C regime. Asymptotic null distribution and power behaviors of the proposed
test statistic are carefully analyzed. One important finding is that the upper bounds of $s$ for optimal
testing are dramatically different from estimation, indicating the essentially different natures of the
two problems. Our testing results are derived in a general framework that cover the aforementioned
important function spaces. As an important byproduct, we derive a minimax rate of testing for
nonparametric additive models with diverging number of components which is new in literature.
Such rate is crucial in deriving the upper bound for $s$ for optimal testing, and is of independent
2 Distributed kernel ridge regression

Suppose that data \( \{(Y_i, X_i) : i = 1, \ldots, N\} \) are iid generated from the following regression model

\[
Y_i = f(X_i) + \epsilon_i, \ i = 1, \ldots, N, \tag{2.1}
\]

where \( \epsilon_i \) are random errors with \( E\{\epsilon_i\} = 0, \ E\{\epsilon_i^2|X_i\} = \sigma^2(X_i) > 0 \), the covariates \( X_i \in \mathcal{X} \subseteq \mathbb{R}^d \) follows a distribution \( \pi(x) \), and \( Y_i \in \mathbb{R} \) is a real-valued response. Here, \( d \geq 1 \) is either fixed or diverging with \( N \), and \( f \) is unknown. Throughout we assume \( f \in \mathcal{H} \), where \( \mathcal{H} \subset L^2_\pi(\mathcal{X}) \) is a reproducing kernel Hilbert space (RKHS) associated with an inner product \( \langle \cdot, \cdot \rangle_\mathcal{H} \) and corresponding norm \( \| \cdot \|_\mathcal{H} \).

For estimating \( f \), we consider the kernel ridge regression (KRR) in a divide-and-conquer (D&C) regime. First, randomly divide the \( N \) samples into \( s \) subsamples. Let \( I_j \) denote the set of indices of the observations from subsample \( j \) for \( j = 1, \ldots, s \). For simplicity, suppose \( |I_j| = n \), i.e., all subsamples are of equal sizes. Hence, the total sample size is \( N = ns \). Then, we estimate \( f \) based on subsample \( j \) through the following KRR method:

\[
\hat{f}_j = \arg\min_{f \in \mathcal{H}} \ell_{j, \lambda}(f) \equiv \arg\min_{f \in \mathcal{H}} \frac{1}{2n} \sum_{i \in I_j} (Y_i - f(X_i))^2 + \frac{\lambda}{2} \|f\|_\mathcal{H}^2, \ j = 1, \ldots, s,
\]

where \( \lambda > 0 \) is a smoothing parameter. The D&C estimator of \( f \) is defined as the average of \( \hat{f}_j \)'s, that is, \( \bar{f} = \sum_{j=1}^s \hat{f}_j / s \).

Based on \( f \), we further propose a Wald-type statistic \( T_{N, \lambda} := \|\bar{f}\|^2 \) for testing the hypothesis

\[
H_0 : f = 0, \text{ vs. } H_1 : f \in \mathcal{H}\backslash\{0\}. \tag{2.2}
\]

In general, testing \( f = f_0 \) (for a known \( f_0 \)) is equivalent to testing \( f_* \equiv f - f_0 = 0 \). So, (2.2) has no loss of generality.

3 Main results

In this section, we derive some general results relating to \( \bar{f} \) and \( T_{N, \lambda} \). Let us first introduce some regularity assumptions.

3.1 Assumptions

The following Assumptions A1 and A2 require that the design density is bounded and the error \( \epsilon \) has finite fourth moment.
Assumption A1. There exists a constant $c_\pi > 0$ such that for all $x \in \mathcal{X}$, $0 \leq \pi(x), \sigma^2(x) \leq c_\pi$.

Assumption A2. There exists a positive constant $\tau$ such that $E\{\epsilon^4|x\} < \tau$ almost surely.

For $f, g \in \mathcal{H}$, define
\[
\langle f, g \rangle = V(f, g) + \lambda(f, g)_\mathcal{H},
\]
where $V(f, g) = E\{f(X)g(X)\}$ and $\lambda > 0$ is the penalization parameter. Clearly, $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathcal{H}$. Let $\| \cdot \|$ be the corresponding norm. Define $\|f\|_{sup} = \sup_{x \in \mathcal{X}} |f(x)|$ as the supremum norm of $f$. We assume the following simultaneous diagonalization condition on $V$ and $\langle \cdot, \cdot \rangle_\mathcal{H}$, which is commonly used in KRR literature, e.g., [11, 26].

Assumption A3. There exist $\varphi_\nu \in \mathcal{H}$ with $c_\varphi \equiv \sup_{\nu \geq 1} \|\varphi_\nu\|_{sup} < \infty$ and positive nondecreasing sequence $\gamma_\nu, \nu \geq 1$, such that $V(\varphi_\nu, \varphi_\mu) = \delta_{\nu\mu}$ and $\langle \varphi_\nu, \varphi_\mu \rangle_\mathcal{H} = \gamma_\nu \delta_{\nu\mu}$, for all $\nu, \mu \geq 1$, where $\delta_{\nu\mu}$ is Kronecker’s $\delta$. Furthermore, any $f \in \mathcal{H}$ admits generalized Fourier expansion $f = \sum_{\nu \geq 1} V(f, \varphi_\nu)\varphi_\nu$ with convergence held in $\| \cdot \|$-norm.

Under Assumption A3, it is easy to prove that $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is also a RKHS with reproducing kernel function given by $K(x, y) = \sum_{\nu \geq 1} \frac{\varphi_\nu(x)\varphi_\nu(y)}{1 + \gamma_\nu}$, for all $x, y \in \mathcal{X}$; see [11] for more details. Define $K_x(\cdot) = K(x, \cdot)$ for $x \in \mathcal{X}$, which is clearly an element in $\mathcal{H}$, and satisfies the following so-called reproducing property:
\[
\langle f, K_x(\cdot) \rangle = f(x), \quad \text{for all } f \in \mathcal{H}.
\]

Define $h^{-1} := \left( \sum_{\nu \geq 1} \frac{1}{1+\gamma_\nu} \right)^{-1}$, which is known as effective dimension; see [24]. There is an explicit relationship between $h$ and $\lambda$ as illustrated in various concrete examples in Section 3.4. Define $Pf = E_X\{f(X)\}$, $P_j f = n^{-1} \sum_{i \in I_j} f(X_i)$ and
\[
\xi_j = \sup_{f, g \in \mathcal{H}} \| f, g \| = 1 \| P_j f - Pf g \|, \quad 1 \leq j \leq s.
\]

Here, $\xi_j$ is the supremum of the empirical processes based on subsample $j$. The quantity $\max_{1 \leq j \leq s} \xi_j$ plays a vital role in determining the critical upper bound of $s$ to guarantee statistical optimality. Assumption A4 below says that $\xi_j$ are uniformly bounded by $\sqrt{\log b N \over n h^a}$, $a, b$ are constants that can be specified in various kernels. Verification of this condition is deferred to Section 3.4 in concrete settings based on an empirical processes (EP) method, where the values of $a, b$ will be explicitly specified.

Assumption A4. There exist nonnegative constants $a, b$ such that
\[
\max_{1 \leq j \leq s} \xi_j = O_P \left( \sqrt{\log b N \over n h^a} \right).
\]
Besides, we require that the series \( \sum_{\nu \geq 1} (1 + \lambda \gamma_\nu)^{-2} \) has the same order as \( h^{-1} \). This condition holds in various examples as discussed in Section 3.4.

**Assumption A5.** It holds that \( \sum_{\nu \geq 1} (1 + \lambda \gamma_\nu)^{-2} \asymp h^{-1} \).

### 3.2 Minimax optimal estimation

In this section, we derive a general error bound for \( \hat{f} \). Let \( X_j = \{X_i : i \in I_j\} \) and \( X = \{X_1, \ldots, X_s\} \). Suppose that (2.1) holds under \( f = f_0 \). For convenience, let \( P_\lambda \) be a self-adjoint operator from \( H \) to itself such that \( \langle P_\lambda f, g \rangle = \lambda \langle f, g \rangle_H \) for all \( f, g \in H \). The existence of \( P_\lambda \) follows by [11, Proposition 2.1]. We first obtain a uniform error bound for \( \hat{f}_j \)'s in the following Lemma 3.1.

**Lemma 3.1.** Suppose Assumptions A1, A3, A4 are satisfied and \( \log b N = o(nh^a) \) with \( a, b \) given in Assumption A4. Then with probability approaching one, for any \( 1 \leq j \leq s \),

\[
E\{\| \hat{f}_j - E\{\hat{f}_j|X_j\} \|_2^2 | X_j \} \leq \frac{4c_\varphi c^2 \xi_j^2}{n h}, \tag{3.2}
\]

\[
\| E\{\hat{f}_j|X_j\} - f_0 + P_\lambda f_0 \| \leq 2 \xi_j \lambda^{1/2} \| f_0 \|_H, \tag{3.3}
\]

where \( c_\varphi \equiv \sup_{\nu \geq 1} \| \varphi_\nu \|_{\text{sup}} \) and \( C \geq 0 \) is any (nonrandom) constant.

(3.2) quantifies the deviation from \( \hat{f}_j \) to its conditional mean through a higher order remainder term, and (3.3) quantifies the bias of \( \hat{f}_j \). Both error bounds are non-asymptotic. Lemma 3.1 immediately leads to the following result on \( \bar{f} \). Specifically, (3.2) and (3.3) lead to (3.4), which, together with the rates of \( \sum_{i=1}^N \epsilon_i K_{X_i} \) and \( P_\lambda f_0 \) in Lemma 6.1, leads to (3.5).

**Theorem 3.2.** If the conditions in Lemma 3.1 hold, then with probability approaching one,

\[
E\{\| \bar{f} - \frac{1}{N} \sum_{i=1}^N \epsilon_i K_{X_i} - f_0 + P_\lambda f_0 \|_2^2 | X \} \leq 4 \left( \frac{c_\varphi c^2}{Nh} + \lambda \| f_0 \|_H^2 \right) \max_{1 \leq j \leq s} \xi_j^2, \tag{3.4}
\]

\[
E\{\| \bar{f} - f_0 \|_2^2 | X \} \leq \frac{4c_\varphi c^2}{Nh} + 8 \lambda \| f_0 \|_H^2. \tag{3.5}
\]

Theorem 3.2 is a general result that holds for many commonly used kernels. Note that \( n = N/s \), the condition \( \log b N = o(nh^a) \) directly implies that as long as \( s \) is dominated by \( Nh^a / \log b N \), the conditional mean squared errors can be upper bounded by the variance term \( (Nh)^{-1} \) and the squared bias term \( \lambda \| f_0 \|_H^2 \). Then the minimax optimal estimation can be obtained through such bias-variance tradeoff in (3.5). Section 3.4 further illustrates concrete and interpretable guarantees on the conditional mean squared errors to particular kernels.
It is worth noting that the upper bound provided in (3.5) is always smaller than or equal to the one obtained by [25]. The range of \( s \) is completely determined through the condition 
\[
\log^b N = o(Nh^a/s),
\]
where \( a, b \) are determined by the uniform bound of \( \xi_j \) in Assumption A4. In other words, a broader range of \( s \) can be achieved by a sharper uniform bound of \( \xi_j \), which is guaranteed by the empirical process methods in this work. For instance, in smoothing spline regression, we achieve an upper bound for \( s \) almost identical to the critical one obtained by [13] (upto a logarithmic factor).

### 3.3 Minimax optimal testing

In this section, we derive the asymptotic distribution of \( T_{N,\lambda} := \|\bar{f}\|^2 \) and further investigate its power behavior. For simplicity, assume that \( \sigma^2(x) \equiv \sigma^2 \) is known. Otherwise, we can replace \( \sigma^2 \) by its consistent estimator to fulfill our procedure. We will show that the distributed test statistic \( T_{N,\lambda} \) can achieve minimax rate of testing (MRT), provided that the number of divisions \( s \) belongs to a suitable range. Here, MRT is defined as the minimal distance between the null and the alternative hypotheses such that valid testing is possible. The range of \( s \) is determined based on the criteria that the proposed test statistic can asymptotically achieve correct size and high power.

We first obtain testing consistency of \( T_{N,\lambda} \). The following theorem shows that \( T_{N,\lambda} \) is asymptotically normal under \( H_0 \). The key condition to obtain such a result is 
\[
\log^b N = o(nh^a+1).
\]
This condition in turn leads to upper bounds for \( s \) to achieve MRT; see Section 3.4 for detailed illustrations.

**Theorem 3.3.** Suppose Assumptions A1 to A5 are all satisfied, and as \( N \to \infty \), \( Nh^2 \to \infty \), and \( \log^b N = o(nh^a+1) \). Then, as \( N \to \infty \),
\[
\frac{N^2}{\sigma(N)} \left( T_{N,\lambda} - \frac{\sigma^2}{Nh} \right) \xrightarrow{d} N(0,1),
\]
where \( \sigma^2(N) = 2\sigma^4(N-1) \sum_{\nu \geq 1} \frac{1}{(1+\lambda \nu)^2} \asymp N^2 h^{-1} \)

By Theorem 3.3 we can define an asymptotic testing rule with \( (1 - \alpha) \) significance level as follows:
\[
\psi_{N,\lambda} = I \left( |T_{N,\lambda} - \sigma^2/(Nh)| \geq z_{1-\alpha/2} \sigma(N)/N^2 \right),
\]
where \( z_{1-\alpha/2} \) is the \( (1 - \alpha/2) \times 100 \) percentile of standard normal distribution.

For any \( f \in \mathcal{H} \), define
\[
\begin{align*}
    b_{N,\lambda} &= (\lambda^{1/2} \|f\|_{\mathcal{H}} + (Nh)^{-1/2}) \sqrt{\frac{\log^b N}{nh^a}}, \quad \text{and} \\
    d_{N,\lambda} &= \lambda^{1/2} \|f\|_{\mathcal{H}} + (Nh^{1/2})^{-1/2} + N^{-1/2} + b_{N,\lambda}^1(Nh)^{-1/4} + b_{N,\lambda}.
\end{align*}
\]
$d_{N,\lambda}$ is used to measure the distance between the null and the alternative hypotheses. The following Theorem 3.4 shows that, if the alternative signal $f$ is separated from zero by an order $d_{N,\lambda}$, then the proposed test statistic asymptotically achieves high power. To achieve optimal testing, it is sufficient to minimize $d_{N,\lambda}$. As long as $s$ is dominated by $\left(Nh^{a+1}/\log^b N\right)$, $d_{N,\lambda}$ can be simplified as

$$d_{N,\lambda} \approx \lambda^{1/2}\|f\|_H + \left(\frac{Nh^{1/2}}{\log N}\right)^{-1/2}$$

(3.6)

Then, MRT can be achieved by selecting $\lambda$ to balance the tradeoff between the bias of $\bar{f}$ and the standard derivation of $T_{N,\lambda}$. It is worth noting that, such a tradeoff in (3.6) for testing is different from the bias-variance tradeoff in (3.2) for estimation, thus leading to different optimal testing rate.

**Theorem 3.4.** If the conditions in Theorem 3.3 hold, then for any $\varepsilon > 0$, there exist $C_\varepsilon$ and $N_\varepsilon$ s.t.

$$\inf_{\|f\| \geq C_\varepsilon d_{N,\lambda}} P_f(\psi_{N,\lambda} = 1) \geq 1 - \varepsilon, \text{ for any } N \geq N_\varepsilon.$$  

Section 3.4 will develop upper bounds for $s$ in various concrete examples based on the above general theorems. Our results will indicate that the ranges for $s$ to achieve MRT are dramatically different from ones to achieve optimal estimation.

### 3.4 Examples

In this section, we derive upper bounds for $s$ in four featured examples to achieve optimal estimation/testing, based on the general results obtained in Sections 3.2 and 3.3. Our examples cover the settings of univariate, multivariate and diverging-dimensional designs.

#### 3.4.1 Example 1: Smoothing spline regression

Suppose $\mathcal{H} = \{f \in S^m(\mathbb{I}) : \|f\|_H \leq C\}$ for a constant $C > 0$, where $S^m(\mathbb{I})$ is the $m$th order Sobolev space on $\mathbb{I} \equiv [0, 1]$, i.e.,

$$S^m(\mathbb{I}) = \{f \in L^2(\mathbb{I}) : f^{(j)} \text{ are abs. cont. for } j = 0, \ldots, m - 1, \text{ and } \int_\mathbb{I} |f^{(m)}(x)|^2 dx < \infty\},$$

and $\|f\|_H = \int_\mathbb{I} |f^{(m)}(x)|^2 dx$. Then model (2.1) becomes the usual smoothing spline regression. In addition to Assumption A1, we assume that

$$c^{-1}_\pi \leq \pi(x) \leq c_\pi, \text{ for any } x \in \mathbb{I}. \quad (3.7)$$
We call the design satisfying (3.7) as quasi-uniform, a common assumption on many statistical problems; see [3]. Quasi-uniform assumption excludes cases where design density is (nearly) zero at certain data points, which may cause estimation inaccuracy at those points.

It is known that when \( m > 1/2 \), \( S^m(\mathbb{I}) \) is a RKHS under the inner product \( \langle \cdot, \cdot \rangle \); see [11], [4]. Meanwhile, Assumption A3 holds with kernel eigenvalues \( \gamma_\nu \propto \nu^{2m}, \nu \geq 1 \). Hence, Assumption A5 holds with \( h \propto \lambda^{1/(2m)} \). It follows from [3, Corollary 5.41] that the following holds.

**Proposition 3.5.** Under (3.7), there exist universal positive constants \( c_1, c_2, c_3 \) such that for any \( 1 \leq j \leq s \),

\[
P(\xi_j \geq t) \leq 2n \exp \left( - \frac{nt^2}{c_1 + c_2 t} \right), \text{ for all } t \geq c_3(nh)^{-1}.
\]

An immediate consequence of Proposition 3.5 is that Assumption A4 holds with \( a = b = 1 \).

Then based on Theorem 3.2 and Theorem 3.4, we have the following results.

**Corollary 3.6.** Suppose that \( \mathcal{H} = S^m(\mathbb{I}) \), (3.7), Assumptions A1 and A2 hold.

\( a \)

- If \( m > 1/2 \), \( s = o(N^{2m/(2m+1)} / \log N) \) and \( \lambda \propto N^{-2m/(2m+1)} \), then \( \| \hat{f} - f_0 \| = O_P(N^{-m/(2m+1)}) \).

\( b \)

- If \( m > 3/4 \), \( s = o(N^{(4m-3)/(4m+1)} / \log N) \) and \( \lambda \propto N^{-4m/(4m+1)} \), then the Wald-type test achieves minimax rate of testing \( N^{-2m/(4m+1)} \).

It is known that the estimation rate \( N^{-m/(2m+1)} \) is minimax-optimal; see [16]. Furthermore, the testing rate \( N^{-2m/(4m+1)} \) is also minimax optimal, in the sense of [6]. It is worth noting that the upper bound for \( s = o(N^{2m/(2m+1)} / \log N) \) matches (up to a logarithmic factor) the critical one by [13] in evenly spaced design, which is substantially larger than the one obtained by [25], i.e., \( s = o(N^{(2m-1)/(2m+1)} / \log N) \) for bounded eigenfunctions.

### 3.4.2 Example 2: Nonparametric additive regression

Consider the function space

\[
\mathcal{H} = \{ f(x_1, \ldots, x_d) = \sum_{k=1}^{d} f_k(x_k) : f_k \in S^m(\mathbb{I}), \| f_k \|_{\mathcal{H}} \leq C \text{ for } k = 1, \ldots, d \},
\]

where \( C > 0 \) is a constant. That is, any \( f \in \mathcal{H} \) has an additive decomposition of \( f_k \)'s. Here, \( d \) is either fixed or slowly diverging. Such additive model has been well studied in many literatures; see [16], [7], [10], [23] among others. For \( x = (x_1, \ldots, x_d) \in \mathcal{X} \), suppose \( x_i, x_j \) are independent for \( i \neq j \in \{1, \ldots, d\} \), and each \( x_i \) satisfies (3.7). For identifiability, assume \( E\{ f_k(x_k) \} = 0 \) for all
1 ≤ k ≤ d. For f = \sum_{k=1}^{d} f_k and g = \sum_{k=1}^{d} g_k, define

\langle f, g \rangle_H = \sum_{k=1}^{d} \langle f_k, g_k \rangle_H = \sum_{k=1}^{d} \int f_k^{(m)}(x)g_k^{(m)}(x)dx, \quad \text{and}

V(f, g) = \sum_{k=1}^{d} V_k(f_k, g_k) \equiv \sum_{k=1}^{d} E\{f_k(X_k)g_k(X_k)\}.

It is easy to verify that H is an RKHS under \langle \cdot, \cdot \rangle defined in (3.1). Lemma 3.7 below summarizes the properties for the H with d additive components.

Lemma 3.7. (a) There exist eigenfunctions \( \varphi_\nu \) and eigenvalues \( \gamma_\nu \) that satisfying Assumption A3.

(b) It holds that \( \sum_{\nu \geq 1}(1 + \lambda \gamma_\nu)^{-1} = h^{-1} \asymp d\lambda^{-1/(2m)} \), and Assumption A5 holds with \( \sum_{\nu \geq 1}(1 + \lambda \gamma_\nu)^{-2} \asymp h^{-1} \) accordingly.

(c) For \( f \in H \), \( \|P_\lambda f\|^2 \leq cd\lambda \), where \( c \) is a bounded constant.

(d) Assumption A4 holds with \( a = b = 1 \).

Combining Lemma 3.7, Theorems 3.2, 3.3 and 3.4, we have the following result.

Corollary 3.8. Suppose Assumption A1, A2 holds.

(a) If \( m > 1/2 \), \( d = o(N^{2m+1}/\log N) \), \( s = o(d^{-1}N^{2m+1}/\log N) \), \( \lambda \asymp N^{-2m+1} \), then \( \|\bar{f} - f_0\| = O_P(d^{1/2}N^{-m/2+1}) \).

(b) If \( m > 3/4 \), \( d = \min\{4m-3, 4m+1\}/N^{2m+1} \), \( s = o(d^{-1}N^{2m+1}/\log N) \), \( \lambda \asymp d^{-1}N^{-2m+1} \), then the Wald-type test achieves minimax rate of testing with \( d^{2m+1}N^{-4m+1} \).

Remark 3.1. It was shown by [10] that \( d^{1/2}N^{-m/2+1} \) is the minimax estimation rate in nonparametric additive model. Part (a) of Corollary 3.8 provides an upper bound for \( s \) such that \( \bar{f} \) achieves this rate. Meanwhile, Part (b) of Corollary 3.8 provides a different upper bound for \( s \) such that our Wald-type test achieves minimax rate of testing \( d^{2m+1}N^{-4m+1} \). It should be emphasized that such minimax rate of testing is a new result in literature which is of independent interest. The proof is based on a local geometry approach recently developed by [20]. When \( d = 1 \), all results in this section reduce to Example 1 on univariate smoothing splines.
3.4.3 Example 3: Gaussian RKHS regression

Suppose that $H$ is an RKHS generated by the Gaussian kernel $K(x, x') = \exp(-c\|x - x'\|^2), x, x' \in \mathbb{R}^d$, where $c, d > 0$ are constants. Then Assumption A3 holds with $\gamma_\nu \asymp [(\sqrt{5} - 1)/2]^{2\nu + 1}, \nu \geq 1$; see [14]. It can be shown that $h \asymp (-\log \lambda)^{-1/2}$ holds which verifies Assumption A5. To verify Assumption A4, we need the following lemma.

**Lemma 3.9.** For Gaussian RKHS, Assumption A4 holds with $a = 2, b = d + 2$.

Following Theorem 3.2, Theorems 3.3 and 3.4, we get the following consequence.

**Corollary 3.10.** Suppose that $H$ is a Gaussian RKHS and Assumptions A1 and A2 hold.

(a) If $s = o(N^{(2m-3)/2} \log N)$ and $\lambda \asymp N^{-2m/(2m+d)}$, then $\|\bar{f} - f_0\| = O_P(N^{m/(2m+d)})$.

(b) If $s = o(N^{2m-7d/(4m+d)} \log N)$ and $\lambda \asymp N^{-4m/(4m+d)}$, then the Wald-type test achieves minimax rate of testing $N^{-2m/(4m+d)}$.

Corollary 3.10 shows that one can choose $s$ to be of order $N$ (upto a logarithmic factor) to obtain both optimal estimation and testing. This is consistent with the upper bound obtained by [25] for optimal estimation, which is of a different logarithmic factor. Interestingly, Corollary 3.10 shows that one can also choose $s$ to be almost identical to $N$ to obtain optimal testing.

3.4.4 Example 4: Thin-Plate spline regression

Consider the $m$th order Sobolev space on $\mathbb{I}^d$, i.e., $H = S^m(\mathbb{I}^d)$, with $d < 2m$ being fixed. The condition $d < 2m$ is necessary for $S^m(\mathbb{I}^d)$ to be an RKHS; see [2, 18] for more details. It is known that Assumption A3 holds with $\gamma_\nu \asymp \nu^{2m/d}$; see [5]. Hence, Assumption A5 holds with $h \asymp \lambda^{d/(2m)}$.

The following lemma verifies Assumption A4.

**Lemma 3.11.** For thin-plate splines, Assumption A4 holds with $a = 3 - d/(2m), b = 1$.

Following Theorem 3.2, Theorem 3.3 and Theorem 3.4, we have the following result.

**Corollary 3.12.** Suppose $f \in S^m(\mathbb{I}^d)$ with $2m - d > 0$, and Assumption A1 and Assumption A2 holds.

(a) If $s = o(N^{(2m-3d)/2} \log N)$ and $\lambda \asymp N^{-2m/(2m+d)},$ then $\|\bar{f} - f_0\| = O_P(N^{-m/(2m+d)}).

(b) If $s = o(N^{(4m-7d+4d^2)/(4m+4d)} \log N)$ and $\lambda \asymp N^{-4m/(4m+d)},$ then the Wald-type test achieves minimax rate of testing $N^{-2m/(4m+d)}$.

Corollary 3.12 demonstrates upper bounds on $s$. Compared with Corollary 3.6 in the univariate case, these upper bounds are smaller due to the curse of dimensionality.
4 Simulation

In this section, we examined the performance of our proposed estimation and testing procedures versus various choices of number of machines in three examples based on simulated datasets.

4.1 Smoothing splne regression

The data were generated from the following regression model

\[ Y_i = c \ast (0.6 \sin(1.5\pi X_i)) + \epsilon_i, \quad i = 1, \cdots, N, \]  

(4.1)

where \( X_i \sim \text{Unif}[0, 1] \), \( \epsilon_i \sim N(0, 1) \) and \( c \) is a constant. Cubic spline (i.e., \( m = 2 \) in Section 3.4.1) was employed for estimating the regression function. To display the impact of the number of divisions \( s \) on statistical performance, we set sample sizes \( N = 2^l \) for \( 9 \leq l \leq 13 \) and chose \( s = N^\rho \) for \( 0.1 \leq \rho \leq 0.8 \). To examine the estimation procedure, we generated data from model (4.1) with \( c = 1 \). Mean squared errors (MSE) were reported based on 100 independent replicated experiments. The left panel of Figure 4.1 summarizes the results. Specifically, it displays that the MSE increases as \( s \) does so; while the MSE increases suddenly when \( \rho \approx 0.7 \), where \( \rho \equiv \log(s)/\log(N) \). Recall that the theoretical upper bound for \( s \), i.e., \( N^{0.8} \); see Corollary 3.6. Hence, estimation performance becomes worse near this theoretical boundary.

We next consider the hypothesis testing problem \( H_0 : f = 0 \). To examine the proposed Wald test, we generated data from model (4.1) at both \( c = 0, 1 \); \( c = 0 \) used for examining the size of the test, and \( c = 1 \) used for examining the power of the test. Significance level was chosen as 0.05. Both size and power were calculated as the proportions of rejections based on 500 independent replications. The middle and right panels of Figure 4.1 summarize the results. Specifically, the right panel shows that the size approaches the nominal level 0.05 under various choices of \((s, N)\), showing the validity of the Wald test. The middle panel displays that the power increases when \( \rho \) decreases; the power maintains at 100% when \( \rho \leq 0.5 \) and \( N \geq 4096 \). Whereas the power quickly drops to zero when \( \rho \geq 0.6 \). This is consistent with our theoretical finding. Recall that the theoretical upper bound for \( s \) is \( N^{0.56} \); see Corollary 3.6. The numerical results also reveal that the upper bound of \( s \) to achieve optimal testing is indeed smaller than the one required for optimal estimation.

4.1.1 Nonparametric additive regression

We generated data from the following nonparametric model of two additive components

\[ Y_i = c \ast f(X_{i1}, X_{i2}) + \epsilon_i, \quad i = 1, \cdots, N, \]  

(4.2)

where \( f(x_1, x_2) = 0.4 \sin(1.5\pi x_1) + 0.1(0.5 - x_2)^3 \), and \( X_{i1}, X_{i2} \sim \text{Unif}[0, 1] \), \( \epsilon_i \sim N(0, 1) \), and \( c \) is a constant. To examine the estimation procedure, we generated data from (4.2) with \( c = 1 \). To
examine the testing procedure, we generated data at $c = 0, 1$. $N, s$ were chosen to be the same as the smoothing spline example in Section 4. Results are summarized in Figure 4.1.1. The interpretations are again similar to Figure 4.1, only with a slightly different asymptotic trend. Specifically, the MSE suddenly increases at $\rho \approx 0.6$, and the power quickly approaches one at $\rho \approx 0.5$. The sizes are around the nominal level 0.05 for all cases.

4.1.2 Gaussian RKHS regression

We consider Gaussian RKHS and generated data from the following model

$$Y_i = c * f(X_{i1}, X_{i2}, X_{i3}) + \epsilon_i, \quad i = 1, \ldots, N,$$

where $f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_2x_3$, $X_{i1}, X_{i2}, X_{i3} \overset{iid}{\sim} \text{Unif}[0, 1]$, $\epsilon_i \overset{iid}{\sim} N(0, 1)$, and $c$ is a constant. For estimation, we generated data from model (4.3) at $c = 1$; for testing $H_0 : f = 0$, we generated data at $c = 0, 1$. $N, s$ were chosen similar to the smoothing spline example in Section
4. Results are summarized in Figure 4.1.2. Overall the interpretations are similar as Section 4. The only difference is that the estimation or testing performance changes at different values of $\rho$. Specifically, the left panel shows that the MSE suddenly increases at $\rho \approx 0.7$, greater than $\rho \approx 0.6$ observed in Section 4. The middle panel shows that the power approaches 100% when $\rho \leq 0.6$, greater than $\rho \approx 0.5$ observed in Section 4. Such findings are consistent with Corollary 3.10 which indicates that upper bound of $s$ in Gaussian RKHS is greater. Besides, the sizes are close to 0.05 which shows the validity of the test.

Figure 3: Gaussian RKHS. (a) MSE of $\hat{f}$ versus $\rho \equiv \log(s) / \log(N)$. (b) Power of the Wald test versus $\rho$. (c) Size of the Wald test versus $\rho$.

5 Conclusion

Our work offers theoretical insights on how to allocate data in parallel computing for KRR in both estimation and testing procedures. In comparison with [25] and [13], our work provides a general and unified treatment of such problems in modern high-dimensional or big data settings. Furthermore, using a rather different empirical processes (EP) technique, we have improved the upper bound of the number of machines in smoothing spline regression by [25] from $N^{(2m-1)/(2m+1)}/\log N$ to $N^{2m/(2m+1)}/\log N$ for optimal estimation, which is proven un-improvable in [13] (upto a logarithmic factor). In the end, we would like to point out that our theory is useful in designing a distributed version of generalized cross validation method that is developed to choose tuning parameter $\lambda$ and the number of machines $s$; see [21].

6 Proofs of main results

6.1 Some preliminary results

Lemma 6.1. (a) For any $x, y \in \mathcal{X}$, $K(x, y) \leq c_0^2 h^{-1}$.
(b) For any $f \in \mathcal{H}$, $\|P_\lambda f\| \leq \lambda^{1/2}\|f\|_{\mathcal{H}}$.

Proof. (a)

$$K(x, y) = \sum_{\nu \geq 1} \varphi_{\nu}(x)\varphi_{\nu}(y) / (1 + \lambda \gamma_{\nu}) \leq c^2 \varphi_{h^{-1}},$$

where the last inequality is by Assumption A3 and the definition of $h^{-1}$.

(b)

$$\|P_\lambda f\| = \sup_{g \in \mathcal{H}, \|g\| \leq 1} \langle P_\lambda f, g \rangle = \lambda\langle f, g \rangle_{\mathcal{H}} \leq \lambda^{1/2}\|f\|_{\mathcal{H}}\lambda^{1/2}\|g\|_{\mathcal{H}} \leq \lambda^{1/2}\|f\|_{\mathcal{H}}$$

\[\Box\]

### 6.2 Proofs in Section 3.2

Our theoretical analysis relies on a set of Fréchet derivatives to be specified below: for $j = 1, 2, \ldots, s$, the Fréchet derivative of $\ell_{j, \lambda}$ can be identified as: for any $f, f_1, f_2 \in \mathcal{H}$,

$$D\ell_{j, \lambda}(f)f_1 = -\frac{1}{n} \sum_{i \in I_j} (Y_i - f(X_i))\langle K_{X_i}, f_1 \rangle + \langle P_\lambda f, f_1 \rangle := \langle S_{j, \lambda}(f), f_1 \rangle,$$

$$DS_{j, \lambda}(f)f_1f_2 = \frac{1}{n} \sum_{i \in I_j} f_2(X_i)\langle K_{X_i}, f_1 \rangle + \langle P_\lambda f_2, f_1 \rangle = \langle DS_{j, \lambda}(f)f_2, f_1 \rangle,$$

$$D^2S_{j, \lambda}(f) = 0.$$

More specifically,

$$S_{j, \lambda}(f) = -\frac{1}{n} \sum_{i \in I_j} (Y_i - f(X_i))K_{X_i} + P_\lambda f,$$

$$DS_{j, \lambda}(f)g = \frac{1}{n} \sum_{i \in I_j} g(X_i)K_{X_i} + P_\lambda g.$$

Define $S_{\lambda}(f) = E\{S_{j, \lambda}(g)\}$, hence, $DS_{\lambda}(f) = E\{DS_{j, \lambda}(f)\}$. It follows from [11] that $\langle DS_{\lambda}(f)f_1, f_2 \rangle = \langle f_1, f_2 \rangle$ for any $f, f_1, f_2 \in \mathcal{H}$ which leads to $DS_{\lambda}(f) = id$.

**Proof of Lemma 3.1.** Throughout the proof, let $\tilde{f}_j = E\{\tilde{f}_j|X_j\}$. It is easy to see that

$$0 = S_{j, \lambda}(\tilde{f}_j) = -\frac{1}{n} \sum_{i \in I_j} (Y_i - \tilde{f}_j(X_i))K_{X_i} + P_\lambda \tilde{f}_j,$$

$$0 = \frac{1}{n} \sum_{i \in I_j} (\tilde{f}_j(X_i) - f_0(X_i))K_{X_i} + P_\lambda \tilde{f}_j.$$

Subtracting the two equations one gets that

$$\frac{1}{n} \sum_{i \in I_j} (\tilde{f}_j - \tilde{f}_j)(X_i)K_{X_i} + P_\lambda (\tilde{f}_j - \tilde{f}_j) = \frac{1}{n} \sum_{i \in I_j} \epsilon_i K_{X_i}.$$
Equation (6.1) shows that
\[
\hat{f}_j - \tilde{f}_j = \arg\min_{f \in \mathcal{H}} \ell_j^*(f) \equiv \arg\min_{f \in \mathcal{H}} \frac{1}{2n} \sum_{i \in I_j} (\epsilon_i - f(X_i))^2 + \frac{\lambda}{2} \|f\|_2^2.
\]

Let \( e_j = \frac{1}{n} \sum_{i \in I_j} \epsilon_i K_{X_i} \) and \( \epsilon_j = \hat{f}_j - \tilde{f}_j \). Then consider Taylor’s expansion
\[
\ell_j^*(e_j) - \ell_j^*(\epsilon_j) = \frac{1}{2} D^2 \ell_j^*(\epsilon_j)(e_j - \epsilon_j)(e_j - \epsilon_j)
\]
\[
= \frac{1}{2} \langle P \epsilon_j - \epsilon_j \rangle^2 + \frac{1}{2} \langle P \lambda(\epsilon_j - \epsilon_j), e_j - \epsilon_j \rangle,
\]
\[
\ell_j^*(\epsilon_j) - \ell_j^*(\epsilon_j) = D \ell_j^*(\epsilon_j)(e_j - \epsilon_j) + \frac{1}{2} D^2 \ell_j^*(\epsilon_j)(e_j - \epsilon_j)(e_j - \epsilon_j)
\]
\[
= (P_j - P)(\epsilon_j - \epsilon_j) + \frac{1}{2} P_j(\epsilon_j - \epsilon_j)^2 + \frac{1}{2} \langle P \lambda(\epsilon_j - \epsilon_j), e_j - \epsilon_j \rangle.
\]

Adding the two equations one obtains that
\[
P_j(\epsilon_j - \epsilon_j)^2 + \langle P \lambda(\epsilon_j - \epsilon_j), e_j - \epsilon_j \rangle + (P_j - P)(\epsilon_j - \epsilon_j) = 0.
\]

Uniformly for \( j \), it holds that
\[
|(P_j - P)(\epsilon_j - \epsilon_j))| \leq \xi_j \|e_j\| \cdot \|\epsilon_j - e_j\|,
\]
\[
P_j(\epsilon_j - \epsilon_j)^2 + \langle P \lambda(\epsilon_j - \epsilon_j), (e_j - e_j) \rangle \geq (1 - \xi_j) \|\epsilon_j - \epsilon_j\|^2.
\]

Combining the two inequalities one gets that
\[
(1 - \xi_j) \|\epsilon_j - \epsilon_j\|^2 \leq \xi_j \|e_j\| \cdot \|\epsilon_j - e_j\|.
\]

Taking expectations conditional on \( X_j \) on both sides and noting that \( \xi_j \) is \( \sigma(X_j) \)-measurable, one gets that
\[
(1 - \xi_j) E\{\|\epsilon_j - e_j\|^2 | X_j \} \leq \xi_j E\{\|e_j\| \cdot \|\epsilon_j - e_j\| | X_j \} \leq \xi_j E\{\|e_j\|^2 | X_j \}^{1/2} E\{\|\epsilon_j - e_j\|^2 | X_j \}^{1/2}.
\]

By assumption \( \log^b N = o(nh^a) \) and Assumption A4, \( \max_{1 \leq j \leq s} \xi_j = o_P(1) \), i.e., with probability approaching one \( \max_{1 \leq j \leq s} \xi_j \leq 1/2 \), hence,
\[
E\{\|\epsilon_j - e_j\|^2 | X_j \} \leq 4 \xi_j^2 E\{\|e_j\|^2 | X_j \}
\]
\[
= \frac{4 \xi_j^2}{n^2} \sum_{i, i' \in I_j} E\{\epsilon_i \epsilon_i' K(X_i, X_{i'}) | X_j \}
\]
\[
= \frac{4 \xi_j^2}{n^2} \sum_{i \in I_j} \sigma^2(X_i) K(X_i, X_i)
\]
\[
\leq \frac{4c_\pi e_\sigma^2 \xi_j}{nh},
\]
(6.2)
where the last inequality follows from Assumption A1 and Lemma 6.1 that $K(x,x) \leq c_\psi^2 h^{-1}$. This proves (3.2).

By (6.2) it is easy to derive

$$E\{\|\tilde{f}_j - f_j\|^2 | X_j\} \leq \frac{4c_\psi c_\phi^2}{nh}. \quad (6.3)$$

Now we look at $\|\tilde{f}_j - f^*_0\|$. It is easy to see that $\tilde{f}_j$ is the minimizer of the following problem.

$$\tilde{f}_j = \arg\min_{\tilde{f}_j \in \mathcal{H}} \tilde{\ell}_{j,\lambda}(f) = \arg\min_{f \in \mathcal{H}} \frac{1}{2n} \sum_{i \in I_j} (f_0(X_i) - f(X_i))^2 + \frac{\lambda}{2} \|f\|^2_{\mathcal{H}}.$$  

We use a similar strategy for handling part (3.2). Note that

$$\tilde{\ell}_{j,\lambda}(f^*_0) - \tilde{\ell}_{j,\lambda}(\tilde{f}_j) = \frac{1}{2} D^2 j,\lambda(\tilde{f}_j)(f^*_0 - \tilde{f}_j)(f^*_0 - \tilde{f}_j)$$

$$= \frac{1}{2} P_j(f^*_0 - \tilde{f}_j)^2 + \frac{1}{2} \langle P_\lambda(f^*_0 - \tilde{f}_j), f^*_0 - \tilde{f}_j \rangle,$$

$$\tilde{\ell}_{j,\lambda}(\tilde{f}_j) - \tilde{\ell}_{j,\lambda}(f^*_0) = P_j(f^*_0 - f_0)(\tilde{f}_j - f^*_0) + \langle P_\lambda f^*_0, \tilde{f}_j - f^*_0 \rangle$$

$$+ \frac{1}{2} P_j(\tilde{f}_j - f^*_0)^2 + \frac{1}{2} \langle P_\lambda(\tilde{f}_j - f^*_0), \tilde{f}_j - f^*_0 \rangle.$$  

Adding the two equations, one gets that

$$P_j(\tilde{f}_j - f^*_0)^2 + \langle P_\lambda(\tilde{f}_j - f^*_0), \tilde{f}_j - f^*_0 \rangle$$

$$= P_j(f_0 - f^*_0)(\tilde{f}_j - f^*_0) - \langle P_\lambda f^*_0, \tilde{f}_j - f^*_0 \rangle$$

$$= (P_j - P)(f_0 - f^*_0)(\tilde{f}_j - f^*_0) + P(f_0 - f^*_0)(\tilde{f}_j - f^*_0) - \langle P_\lambda f^*_0, \tilde{f}_j - f^*_0 \rangle$$

$$= (P_j - P)(f_0 - f^*_0)(\tilde{f}_j - f^*_0) + \langle f_0 - f^*_0, \tilde{f}_j - f^*_0 \rangle - \langle P_\lambda(f_0 - f^*_0), \tilde{f}_j - f^*_0 \rangle$$

$$= (P_j - P)(f_0 - f^*_0)(\tilde{f}_j - f^*_0) + \langle f_0 - f^*_0 - P_\lambda(f_0 - f^*_0) - P_\lambda f^*_0, \tilde{f}_j - f^*_0 \rangle$$

$$= (P_j - P)(f_0 - f^*_0)(\tilde{f}_j - f^*_0).$$

Therefore,

$$(1 - \xi_j)\|\tilde{f}_j - f^*_0\|^2 \leq \xi_j\|f_0 - f^*_0\| \times \|\tilde{f}_j - f^*_0\| = \xi_j\|P_\lambda f_0\| \times \|\tilde{f}_j - f^*_0\| \leq C\xi_j \lambda^{1/2} \|\tilde{f}_j - f^*_0\|,$$

implying that, with probability approaching one, for any $1 \leq j \leq s$, $\|\tilde{f}_j - f^*_0\| \leq 2C\xi_j \lambda^{1/2}$. This proves (3.3).

Proof of Theorem 3.2. Recall $f^*_0 = (id - P_\lambda)f_0$ and $\tilde{f}_j = E\{\tilde{f}_j|X_j\}$. Also notice that $\frac{1}{N} \sum_{i=1}^{N} \epsilon_i K_{X_i} =
\[ \frac{1}{s} \sum_{j=1}^{s} e_j. \] By direct calculations and Lemma 3.1, we have with probability approaching one,
\[ E\{\|\bar{f} - f_0^* - \frac{1}{N} \sum_{i=1}^{N} \epsilon_i K_{X_i} \|^2 | \mathbf{X} \} \]
\[ = \frac{1}{s^2} \sum_{j=1}^{s} E\{\|\hat{f}_j - \bar{f}_j - e_j \|^2 | \mathbf{X}_j \} + \frac{1}{s^2} \| \sum_{j=1}^{s} (\bar{f}_j - f_0^*) \|^2 \]
\[ \leq 4 \left( \frac{c_\pi c_\phi^2}{Nh} + \lambda \|f_0\|_H^2 \right) \max_{1 \leq j \leq s} \xi_j^2. \]
This proves (3.4). The result (3.5) immediately follows by the assumption \( \max_{1 \leq j \leq s} \xi_j^2 = o_p(1). \)

6.3 Proofs in Section 3.3

Before proving consistency of the test statistics \( T_{N, \lambda} \), i.e., Theorem 3.3, let us state a technical lemma. Define \( W(N) = \sum_{1 \leq i < k \leq N} W_{ik} \), with \( W_{ik} = 2 \epsilon_i \epsilon_k K(X_i, X_k) \), and let \( \sigma^2(N) = \text{Var}(W(N)) \). Define the empirical kernel matrix \( \mathbf{K} = [K(X_i, X_j)]_{i,j=1}^{N} \) and \( \mathbf{e} = (\epsilon_1, \ldots, \epsilon_N)^T \).

Lemma 6.2. Suppose Assumptions A1, A2, A3, A5 are all satisfied, and \( N \to \infty, h = o(1), Nh^2 \to \infty \). Then it holds that
\[ \mathbf{e}^T \mathbf{K} \mathbf{e} = \sigma^2 N h^{-1} + W(N) + O_P(\sqrt{Nh^{-2}}). \] (6.4)
Furthermore, as \( N \to \infty, \sigma^2(N) = 2\sigma^4 N(N - 1) \sum_{\nu \geq 1} \frac{1}{(1 + \lambda_{\nu})^2} \propto N^2 h^{-1}, \frac{W(N)}{\sigma(N)} \xrightarrow{d} N(0, 1) \).

Proof of Lemma 6.2. It is easy to see that
\[ \mathbf{e}^T \mathbf{K} \mathbf{e} = \sum_{i=1}^{N} \epsilon_i^2 K(X_i, X_i) + W(N). \]
Since
\[ \text{Var} \left( \sum_{i=1}^{N} \epsilon_i^2 K(X_i, X_i) \right) \leq N E\{\epsilon_i^4 K(X_i, X_i)^2 \} \leq \tau c_\phi^4 Nh^{-2}, \]
where the last “\( \leq \)” follows by Assumption A2 and Lemma 6.1 that \( K(x, x) \leq c_\phi^2 h^{-1} \), we get that
\[ \sum_{i=1}^{N} \epsilon_i^2 K(X_i, X_i) = E\{\sum_{i=1}^{N} \epsilon_i^2 K(X_i, X_i) \} + O_P \left( \sqrt{c_\phi^4 Nh^{-2}} \right) \]
\[ = \sigma^2 Nh^{-1} + O_P(\sqrt{c_\phi^4 Nh^{-2}}). \]
Next we prove asymptotic normality of $W(N)$. Note $\sigma^2(N) = E\{W(N)^2\}$. Let $G_I, G_{II}, G_{IV}$ be defined as

$$
G_I = \sum_{1 \leq i < t \leq n} E\{W_{it}^4\},
$$

$$
G_{II} = \sum_{1 \leq i < t < k \leq n} (E\{W_{it}^2 W_{tk}^2\} + E\{W_{it}^2 W_{ik}^2\} + E\{W_{ik}^2 W_{tk}^2\})
$$

$$
G_{IV} = \sum_{1 \leq i < t < k < l \leq n} (E\{W_{it} W_{ik} W_{lt} W_{lk}\} + E\{W_{it} W_{il} W_{kt} W_{kl}\} + E\{W_{ik} W_{il} W_{tk} W_{tl}\}).
$$

Since $K(x, x) \leq c^2 \phi^{-1}$, we have $G_I = O(N^2 h^{-4})$ and $G_{II} = O(N^3 h^{-4})$. It can also be shown that for pairwise distinct $i, k, t, l$,

$$
E\{W_{ik} W_{it} W_{lk} W_{lt}\}\]
= 2^4 \sigma^8 \sum_{\nu=1}^{\infty} \frac{1}{(1 + \lambda_{\nu})^8} = O(h^{-1}),
$$

which implies that $G_{IV} = O(N^4 h^{-1})$. In the mean time, a straight algebra leads to that

$$
\sigma^2(N) = 4\sigma^4 \left( \frac{N}{2} \right) \sum_{\nu=1}^{\infty} \frac{1}{(1 + \lambda_{\nu})^2}
= 2\sigma^4 N(N - 1) \sum_{\nu \geq 1} \frac{1}{(1 + \lambda_{\nu})^2} \asymp N^2 h^{-1},
$$

where the last conclusion follows by Assumption A5. Thanks to the conditions $h \to 0$, $Nh^2 \to \infty$, $G_I, G_{II}$ and $G_{IV}$ are all of order $o(\sigma^4(N))$. Then it follows by [1] that as $N \to \infty$,

$$
\frac{W(N)}{\sigma(N)} \xrightarrow{d} N(0, 1).
$$

The above limit leads to that $W(N) = O_P(Nh^{-1/2})$.

**Proof of Theorem 3.3.** The proof is based on Lemma 6.2. Under $f_0 = 0$, it follows from Corollary 3.2 and Assumption A4 that

$$
E\{\|\bar{f} - \frac{1}{N} \sum_{i=1}^{N} \epsilon_{i} K_{X_i}\|^2 | \mathbf{X} \} = O_P \left( \frac{c^2 \log b N}{N n h^{1+a}} \right),
$$

leading to

$$
\|\bar{f} - \frac{1}{N} \sum_{i=1}^{N} \epsilon_{i} K_{X_i}\|^2 = O_P \left( \frac{c^2 \log b N}{N n h^{1+a}} \right).
$$
Following the proof of Lemma 3.1 and the trivial fact $\hat{f}_j = 0$ when $f_0 = 0$, we have for any $1 \leq j \leq s$,

$$E\{\|\hat{f}_j - e_j\|^2|X_j\} \leq \frac{4c_\pi \epsilon^2 \varphi^2}{nh}, \quad E\{\|e_j\|^2|X_j\} \leq \frac{c_\pi \epsilon^2}{nh}, \text{ a.s.}$$

Therefore, by Cauchy-Schwartz inequality,

$$E\{|\langle \hat{f}_j - e_j, e_j \rangle|X_j\} \leq \sqrt{E\{\|\hat{f}_j - e_j\|^2|X_j\}E\{\|e_j\|^2|X_j\}} \leq \frac{2c_\pi \epsilon^2}{nh} \xi_j,$$

and hence,

$$E\left\{\sum_{j=1}^{s} |\langle \hat{f}_j - e_j, e_j \rangle|X\right\} \leq \frac{2c_\pi \epsilon^2}{nh} \max_{1 \leq j \leq s} \xi_j.$$

By Assumption A4, the above leads to that

$$\sum_{j=1}^{s} \langle \hat{f}_j - e_j, e_j \rangle = O_P \left(\frac{sc_\pi \epsilon^2}{nh} \sqrt{\log b N nh^a} \right).$$

Meanwhile, it holds that

$$\sum_{j \neq l} \langle \hat{f}_j - e_j, e_l \rangle = \sum_{j < l} \langle \hat{f}_j - e_j, e_l \rangle + \sum_{j > l} \langle \hat{f}_j - e_j, e_l \rangle \equiv R_1 + R_2,$$

with

$$R_1 = O_P \left(\frac{sc_\pi \epsilon^2}{nh} \sqrt{\log b N nh^a} \right), \quad R_2 = O_P \left(\frac{sc_\pi \epsilon^2}{nh} \sqrt{\log b N nh^a} \right).$$

To see this, note that

$$E\{R_1^2|X\} = \sum_{j < l} E\{|\langle \hat{f}_j - e_j, e_l \rangle|^2|X\} \leq \sum_{j < l} E\{\|\hat{f}_j - e_j\|^2\|e_l\|^2|X\} = \sum_{j < l} E\{\|\hat{f}_j - e_j\|^2|X_j\}E\{\|e_l\|^2|X_l\} \leq \left(\frac{s}{2}\right) \frac{4c_\pi \epsilon^4 \varphi^4}{n^2 h^2} \max_{1 \leq j \leq s} \xi_j^2.$$

19
Similar result holds for $R_2$. Hence, by Lemma 6.2 and direct algebra, we get that

$$T_{N,\lambda} = N^{-2} \epsilon' \mathbf{K} \epsilon + \frac{2}{s^2} \sum_{j,l=1}^{s} \langle \hat{f}_j - e_j, e_l \rangle + \| \bar{f} - \frac{1}{N} \sum_{i=1}^{N} \epsilon_i K X_i \| ^2$$

$$= N^{-2} \epsilon' \mathbf{K} \epsilon + \frac{2}{s^2} \sum_{j=1}^{s} \langle \hat{f}_j - e_j, e_j \rangle + \frac{2}{s^2} (R_1 + R_2) + \| \bar{f} - \frac{1}{N} \sum_{i=1}^{N} \epsilon_i K X_i \| ^2$$

$$= \frac{\sigma^2}{N h} + \frac{W(N)}{N^2} + O_P \left( \frac{2 c^2}{N h \sqrt{\log b N}} \right) + O_P \left( \frac{2 c^2 \log b N}{N h^a} \right)$$

The last equality follows from the condition $\log b N = o(\sqrt{nh a + 1})$. Therefore, by $c^2/(Nh) = o(1)$, $Nh \to \infty$ (from $Nh^2 \to \infty$ and $h \to 0$), condition $\log b N = o(\sqrt{nh a + 1})$ and $\sigma^2(N) \preceq N^2 h^{-1}$ (Lemma 6.2), as $N \to \infty$,

$$\frac{N^2}{\sigma(N)} \left( T_{N,\lambda} - \frac{\sigma^2}{N h} \right) = \frac{W(N)}{\sigma(N)} + O_P \left( \frac{c^2}{\sqrt{N h}} + \frac{c^2}{\sqrt{\log b N}} \right)$$

Proof is completed. \(\square\)

Proof of Theorem 3.4. For any $f \in \mathcal{H}$, define $R_f = \bar{f} - N^{-1} \sum_{i=1}^{N} \epsilon_i K X_i + f + \mathcal{P}_\lambda f$. By direct examinations, it holds that

$$\| \bar{f} \|^2 - \sigma^2/(Nh)$$

$$= \| R_f + \frac{1}{N} \sum_{i=1}^{N} \epsilon_i K X_i - f + \mathcal{P}_\lambda f \|^2 - \sigma^2/(Nh)$$

$$\geq \{ \epsilon' \mathbf{K} \epsilon/N^2 - \sigma^2/(Nh) \} + \| f - \mathcal{P}_\lambda f \|^2 - \frac{2}{N} \sum_{i=1}^{N} \epsilon_i (f - \mathcal{P}_\lambda f)(X_i)$$

$$+ \frac{2}{N} \sum_{i=1}^{N} \epsilon_i R_f(X_i) - 2(f - \mathcal{P}_\lambda f, R_f)$$

$$\equiv T_1 + T_2 + T_3 + T_4 + T_5.$$
It follows by (6.4), Theorem 3.2, Assumption A4 that, uniformly for \( f \in \mathcal{H} \),

\[
T_1 = W(N)/N^2 + O_P((N^{3/2}h)^{-1}), \quad \text{(by (6.4))}
\]

\[
P_f \left( |T_3| \geq \sigma \| f - \mathcal{P}_\lambda f \| / (\varepsilon \sqrt{N}) \right) \leq \varepsilon^2, \quad \text{for arbitrary } \varepsilon > 0
\]

\[
T_4 = O_P(b_{N,\lambda}/\sqrt{Nh}), \quad \text{(by Theorem 3.2, Assumption A4 and (6.4))}
\]

\[
T_5 = \| f - \mathcal{P}_\lambda f \| \times O_P(b_{N,\lambda}), \quad \text{(by Theorem 3.2 and Assumption A4)}
\]

Note that \( \| \mathcal{P}_\lambda f \| \leq \lambda^{1/2} \| f \|_{\mathcal{H}} \) for any \( f \in \mathcal{H} \). Therefore, to achieve high power, i.e., power is at least \( 1 - \varepsilon \), one needs to choose a large \( N_\varepsilon \) and \( C_\varepsilon \) s.t. \( N \geq N_\varepsilon \) and

\[
\| f \| \geq C_\varepsilon / \sqrt{Nh^{1/2}}, \quad \| f \| \geq C_\varepsilon / \sqrt{N}, \quad \| f \| \geq C_\varepsilon \sqrt{b_{N,\lambda}/\sqrt{Nh}},
\]

\[
\| f \| \geq C_\varepsilon b_{N,\lambda}, \quad \| f \| \geq C_\varepsilon \lambda^{1/2} \| f \|_{\mathcal{H}}.
\]

Proof is completed. \( \square \)

### 6.4 Proof of Section 3.4.2

**Proof of Lemma 3.7 (a).** For each \( \nu \geq 1 \), there exist \( p \in \mathbb{N} \) and \( 1 \leq k \leq d \), such that \( \nu = pd + k \). Suppose \( x = (x_1, \ldots, x_d) \), then for each \( x_k \), there exists \((\varphi_p^{(k)}, \gamma_p^{(k)})\) and \((\varphi_{p'}^{(k)}, \gamma_{p'}^{(k)})\) satisfying \( V_k(\varphi_p^{(k)}, \varphi_{p'}^{(k)}) = \delta_{pp'} \) and \( \int_1 \varphi_p^{(k)}(x)\varphi_{p'}^{(k)}(x)dx = \gamma_p^{(k)} \delta_{pp'} \). In fact, the eigenfunctions \( \varphi_\nu \) and eigenvalues \( \gamma_\nu \) can be constructed by an ordered sequence of \( \varphi_p^{(k)}, \gamma_p^{(k)} \) as \( \varphi_\nu(x) = \varphi_p^{(k)}(x_k) \) and \( \gamma_\nu = \gamma_p^{(k)} \).

Next, we verify such construction of eigenfunctions \( \varphi_\nu \) and eigenvalues \( \gamma_\nu \) satisfy Assumption A3. When \( \nu \neq \mu \), then there exist \( p_1, q_1, p_2, q_2 \), such that \( \nu = p_1d + q_1, \mu = p_2d + q_2 \), then

\[
V(\varphi_{p_1d+q_1}, \varphi_{p_2d+q_2}) = V(\varphi_{p_1}^{q_1}(x_{q_1}), \varphi_{p_2}^{q_2}(x_{q_2}))
\]

\[
= \begin{cases} 0 & p_1 \neq p_2, q_1 = q_2 \\ V_{q_1}(\varphi_{p_1}^{q_1}(x_{q_1}), 0) + V_{q_2}(0, \varphi_{p_2}^{q_2}(x_{q_2})) = 0 & q_1 \neq q_2 \end{cases}
\]

On the other hand,

\[
\langle \varphi_\nu, \varphi_\mu \rangle_{\mathcal{H}} = \langle \varphi_{p_1}^{q_1}, \varphi_{p_2}^{q_2} \rangle_{\mathcal{H}} = \begin{cases} \gamma_{p_1}^{q_1} = \gamma_\nu & p_1 = p_2, q_1 = q_2 \\ 0 & \nu \neq \mu \end{cases}
\]

For any \( f \in \mathcal{H} \),

\[
f(\mathbf{x}) = f_1(x_1) + \cdots + f_d(x_d) = \sum_{k=1}^{d} \sum_{\nu=1}^{\infty} V_k(f_k, \varphi_\nu^{(k)}) \varphi_\nu^{(k)}(x_k)
\]

\[
= \sum_{k=1}^{d} \sum_{\nu=1}^{\infty} V(f, \varphi_\nu^{(k)}) \varphi_\nu^{(k)}(x_k) = \sum_{\nu=1}^{\infty} V(f, \varphi_\nu) \varphi_\nu(x)
\]

\( \square \)

21
Proof of Lemma 3.7 (b). It is easy to see that
\[ \sum_{\nu \geq 1} (1 + \lambda \gamma_{\nu})^{-1} = \sum_{q=1}^{d} \sum_{p \geq 1} (1 + \lambda \gamma_{p}^{(k)})^{-1} \asymp d \lambda^{-1/(2m)} =: h^{-1}. \]

Proof of Lemma 3.7 (c). Notice that \( \|f\|_{H}^{2} \leq \sum_{i=1}^{d} \|f_{k}\|_{H}^{2} \leq Cd \), then by Lemma 6.1 (b), \( \|P_{\lambda}f\|_{H}^{2} \leq \lambda\|f\|_{H}^{2} \leq Cd\lambda \).

Next, we prove Lemma 3.7 (d). To prove Lemma 3.7 (d), it is sufficient to prove the following Lemma 6.3.

**Lemma 6.3.** Under (3.7), there exist universal positive constants \( c_{1}, c_{2}, c_{3} \) such that for any \( 1 \leq j \leq s \),
\[ P(\xi_{j} \geq t) \leq 2n \exp\left( - \frac{nht^{2}}{c_{1} + c_{2}t} \right), \text{ for all } t \geq c_{3}(nh)^{-1}, \]
where \( h^{-1} \asymp d\lambda^{-1/(2m)}. \)

To prove Lemma 6.3, based on Lemma 3.5 and Lemma 3.4 in Chapter 21 in [3], we only need to bound
\[ \| \frac{1}{n} \sum_{i=1}^{n} K_{h}(X_{i}, \cdot) - E[K_{h}(X_{i}), \cdot] \|_{\infty}, \]
and
\[ \| \frac{1}{n} \sum_{i=1}^{n} hK'_{h}(X_{i}, \cdot) - h E[K'_{h}(X_{i}), \cdot] \|_{\infty}, \]
where \( K_{h}(\cdot, \cdot) \) is the kernel function equipped with \( \| \cdot \|_{H} \). For \( x, z \in \mathbb{R}^{d} \), define the exponential family \( g_{h}(x - z) = \sum_{j=1}^{d} g_{h_{0,j}}(x_{j} - z_{j}) \), where \( g_{h_{0,j}}(x_{j} - z_{j}) \) satisfies equation (5.12) in [3] with \( h_{0} = dh \).
Let \( K_{h}(x, z) = \sum_{j=1}^{d} K_{h_{0,j}}(x_{j}, z_{j}) \), where \( K_{h_{0,1}}(\cdot, \cdot) = \cdots = K_{h_{0,d}}(\cdot, \cdot) \) is the kernel function of the RKHS \( S^{m}(\mathbb{I}) \). Then \( K_{h_{0,j}}(\cdot, \cdot) \) and \( g_{h_{0,k}}(\cdot) \) satisfies equation (5.14) in [3]. Next, we show that \( K_{h}(\cdot, \cdot) \) can be approximated by \( g_{h}(\cdot) \) in the following Lemma 6.4.

**Lemma 6.4.** Assume that the family \( K_{h} = \sum_{j=1}^{d} K_{h_{0,j}} \) with \( K_{h_{0,j}} \), \( 0 < h_{0} \leq 1 \) is convolution-like. Then there exists a constant \( c \), such that for all \( h, 0 < h \leq 1 \), and for every strictly positive design \( X_{1}, X_{2}, \cdots, X_{n} \in (0, 1)^{d} \),
\[ \| \frac{1}{n} \sum_{i=1}^{n} K_{h}(X_{i}, \cdot) \|_{\infty} \leq c \| \frac{1}{n} \sum_{i=1}^{n} g_{h}(X_{i} - \cdot) \|_{\infty}. \]
Proof. For \( t = (t_1, \cdots, t_d) \in [0,1]^d \) and \( x = (x_1, \cdots, x_d) \in [0,1]^d \), let \( S_{nh}(t) = \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - t) \), and \( s_{nh}(t) = \frac{1}{n} \sum_{i=1}^{n} g_h(X_i - t) \). For \( j = 1, \cdots, d \), \( K_{h,j} \) satisfies

\[
K_{h_0,j}(t_j, x_j) = h_0 g_{h,j}(x) K_{h_0,j}(t_j) + \int_0^1 g_{h,j}(x_j - z_j) \{ h_0 K'_{h_0,j}(t_j, z_j) + K_{h_0,j}(t_j, z_j) \} dz_j,
\]

where \( h_0 = dh \). Note that \( K_{h_0,j}, h_0 K'_{h_0,j} \) are all convolutional-like, then \( |h_0 K'_{h_0,j}(t_j, z_j)| \leq c h_0^{-1} \) and \( |K_{h_0,j}(t_j, z_j)| \leq c h_0^{-1} \). Therefore,

\[
\int_0^1 g_{h_0,j}(x_j - z_j) \{ h_0 K'_{h_0,j}(t_j, z_j) + K_{h_0,j}(t_j, z_j) \} dz_j \leq 2c \cdot h_0^{-1} \int_0^1 g_{h_0,j}(x_j - z_j) dz_j
\]

\[= 2c \cdot h_0^{-2} \int_0^1 e^{-h_0^{-1}(x_j - z_j)} dz_j \leq 2c \cdot (g_{h_0,j}(x_j) - g_{h_0,j}(x_j - 1)) \leq 2c \cdot g_{h_0,j}(x_j).
\]

Then, we have \( K_{h_0,j}(t_j, x_j) \leq h_0 \cdot g_{h_0,j}(x) K_{h_0,j}(t_j) + c \cdot g_{h_0,j}(x_j) \).

\[
K_h(x, t) = \sum_{j=1}^{d} K_{h_0,j}(t_j, x_j) \leq h_0 \sum_{j=1}^{d} g_{h_0,j}(x) K_{h_0,j}(t_j) + c \sum_{j=1}^{d} g_{h_0,j}(x_j)
\]

\[\leq c_1 \sum_{j=1}^{d} g_{h_0,j}(x_j) + c \sum_{j=1}^{d} g_{h_0,j}(x_j) \leq c' \sum_{j=1}^{d} g_{h_0,j}(x_j) = c' g_h(x),
\]

where \( c_1 = \max \{ h_0 K_{h_0,1}(t_1), \cdots, h_0 K_{h_0,d}(t_d, 0) \} \) is a bounded constant by the convolution-like assumption. Let \( X_i = x \) and substitute the formula above into the expression for \( S_{nh}(t) \) and \( s_{nh}(t) \), this gives \( S_{nh}(t) \leq c' s_{nh}(0) \). Therefore, \( \| S_{nh} \|_{\infty} \leq c' \| s_{nh}(0) \| \leq \| s_{nh} \|_{\infty} \). The last inequality is due to the fact that all \( X_i \) are strictly positive, then \( s_{nh}(t) \) is continuous at \( t = 0 \), and so \( s_{nh}(0) \leq \| s_{nh} \|_{\infty} \).

Let \( P_n \) be the empirical distribution function of the design \( X_1, X_2, \cdots, X_n \), and let \( P_0 \) be the design distribution function. Here \( P_0 = \pi(x) \). Define

\[
[gh \circ (dP_n - dP_0)](t) = \int_{[0,1]^d} g_h(x - t) (dP_n(x) - dP_0(x)),
\]

then based on Lemma 6.4, we only need to show the following result to prove Lemma 6.3.

Lemma 6.5. For all \( x = (x_1, \cdots, x_d) \in [0,1]^d, t > 0, \)

\[
P\left([gh \circ (dP_n - dP_0)](x) > t\right) \leq 2 \exp\left\{-\frac{nht^2}{w_2 + 2/3t}\right\}, \quad (6.5)
\]

where \( w_2 \) is an upper bound on the density \( P_0(x) \).
Proof. Consider for fixed $x$, $\frac{1}{n} \sum_{i=1}^{n} g_h(X_i - x) = \sum_{k=1}^{d} \sum_{i=1}^{n} \theta_{ik}$, with $\theta_{ik} = \frac{1}{n} g_{h_{0,k}}(x_{i,k} - x_k)$. Then $\theta_{ik}$ ($i = 1, \ldots, n; k = 1, \ldots, d$) are i.i.d. and $|\theta_{ik}| \leq (nh_0)^{-1}$, where $h_0 = d^{-1}h$. For the variance $\text{Var}(\theta_{ik})$, $$\text{Var}(\theta_{ik}) = \frac{1}{n^2} \left\{ \left[ g_{h_{0,k}} \otimes dP_0 \right](x_k) - \left\{ g_{h_{0,k}} \otimes dP_0 \right\}(x) \right\}^2 \right\}$$ $$\leq \frac{1}{n^2} \left[ g_{h_{0,k}} \otimes dP_0 \right](x_k)$$ $$= n^{-2} \int_0^1 \int h_0^{-2} e^{-h_0^{-1}(x_{ik} - x_k)} dP_0(x_k)$$ $$\leq \frac{1}{2} w_2 n^{-2} h_0^{-1}.$$ Therefore, $V := \sum_{i=1}^{n} \sum_{k=1}^{d} \text{Var}(\theta_{ik}) \leq \frac{1}{2} w_2 n^{-1} h^{-1}$. Then by Bernstein’s inequality, (6.5) has been proved.

Lemma 6.6. For all $j = 1, \ldots, n$,
$$\mathbb{P}\{ |g_h \otimes (dP_n - dP_0)|(X_j) > t \} \leq 2 \exp\left\{ -\frac{1/4nh^2}{w_2 + 2/3t} \right\},$$
provided $t \geq 2(1 + w_2)(nh)^{-1}$, where $w_2$ is an upper bound on the density.

Proof. Consider $j = n$. Note that
$$[g_h \otimes dP_n](X_n) = \frac{1}{n} g_h(0) + \frac{1}{n} \sum_{i=1}^{n-1} g_h(X_i - X_n)$$ $$= \frac{1}{n} \sum_{k=1}^{d} g_{h_{0,k}}(0) + \frac{1}{n} \sum_{i=1}^{n-1} g_h(X_i - X_n)$$ $$= d(nh_0)^{-1} + \frac{n-1}{n} [g_h \otimes dP_{n-1}](X_n),$$
so that its expectation, conditional on $X_n$, equals
$$\mathbb{E}[g_h \otimes dP_n](X_n) = (nh)^{-1} + \frac{n-1}{n} [g_h \otimes dP_0](X_n).$$

Then $\mathbb{P}\{ |g_h \otimes (dP_{n-1} - dP_0)|(X_n)| > t|X_n \} \leq 2 \exp\left\{ -\frac{(n-1)ht^2}{w_2 + 2/3t} \right\}$. Note that this upper bound does not involve $X_n$, it follows that
$$\mathbb{P}\{ |g_h \otimes (dP_{n-1} - dP_0)|(X_n)| > t \} = \mathbb{E}\left[ \mathbb{P}\{ |g_h \otimes (dP_{n-1} - dP_0)|(X_n)| > t|X_n \} \right]$$
has the same bound. Finally, note that
$$[g_h \otimes (dP_n - dP_0)](X_n) = \varepsilon_{nh} + \frac{n-1}{n} [g_h \otimes (dP_{n-1} - dP_0)](X_n),$$

24
where \( |\epsilon_{nh}| = |(nh)^{-1} - \frac{1}{n}[g_n \otimes dP_0](X_n)| \leq (nh)^{-1} + (nh)^{-1}w_2 \leq c_2(nh)^{-1} \). Therefore,

\[
P \left\{ \left| (nh)^{-1} \right| \right\} \leq \frac{n}{n-1} \left( t - c_2(nh)^{-1} \right)^2 \}
\]

\[
\leq 2 \exp \left\{ - \frac{nh(t - c_2(nh)^{-1})^2}{w_2 + 2/3(t - c_2(nh)^{-1})} \right\}.
\]

6.4.1 Proof of Corollary 3.8

Note that for any \( x, y \in [0, 1]^d \), by Lemma 6.1, we have \( K(x, y) \leq c_0^2 h^{-1} \), where \( h^{-1} \asymp d\lambda^{1/(2m)} \), and \( \|P_0f\|_2 \leq \lambda \|f\|_{H}^2 \leq C d\lambda \), then Corollary 3.8 can be easily achieved by applying Theorem 3.2 and Theorem 3.4.

Next, we show that \( d_{N,\lambda, d}^* = d^{\frac{2m+1}{2}} N^{-\frac{2m}{4m+1}} \) is the minimax testing rate. Consider the model

\[
\tilde{y} = \theta + w,
\]

where \( \theta \in \mathbb{R}^n \) satisfies the ellipse constraint \( \sum_{j=1}^n \frac{\theta_j^2}{\mu_j} \leq d \), where \( \mu_1 \geq \mu_2 \geq \cdots \geq 0 \), and the noise vector \( w \) is zero-mean with variance \( \frac{\sigma^2}{n} \). Note that model (2.1) is equivalent to model (6.6) (see Example 3 in [20] for details), thus we only need to prove the minimax testing rate under model (6.6) for the testing problem \( \theta = 0 \) with \( \mu_j \asymp \left[ \frac{j}{2} \right]^{-2m} \). Let \( m_u(\delta; \varepsilon) := \arg\max_{1 \leq k \leq d} \{d\mu_k \geq \frac{1}{2} \delta^2\} \), and \( m_l(\delta; \varepsilon) := \arg\max_{1 \leq k \leq d} \{d\mu_{k+1} \geq \frac{9}{16} \delta^2\} \). Then by Corollary 1 in [20], we have

\[
\sup\{\delta | \delta \leq \frac{1}{4} \sigma^2 \sqrt{m_u(\delta; \varepsilon) \delta} \} \leq d_{N,\lambda, d}^* \leq \inf\{\delta | \delta \geq c\sigma^2 \sqrt{m_u(\delta; \varepsilon) \delta} \}.
\]

Let \( \delta^* \) satisfies \( \delta^* \asymp \sqrt{m_l(\delta; \varepsilon)} \asymp \sqrt{m_u(\delta; \varepsilon)} \), we have \( \delta^* = d_{N,\lambda, d}^* \asymp d^{\frac{2m+1}{2}} N^{-\frac{2m}{4m+1}} \).

6.5 Proofs in Section 3.4

Proof of Lemma 3.9. For \( p, \delta > 0 \), define \( \mathcal{G}(p) = \{ f \in \mathcal{H} : \|f\|_{\sup} \leq 1, \|f\|_{\mathcal{H}} \leq p \} \) and the corresponding entropy integral

\[
J(p, \delta) = \int_0^\delta \psi_1^{-1}(D(\varepsilon, \mathcal{G}(p), \|\cdot\|_{\sup})) d\varepsilon + \delta \psi_2^{-1}(D(\delta, \mathcal{G}(p), \|\cdot\|_{\sup}))^2,
\]

where \( \psi(s) = \exp(s^2) - 1 \) and \( D(\varepsilon, \mathcal{G}(p), \|\cdot\|_{\sup}) \) is the \( \varepsilon \)-packing number of \( \mathcal{G}(p) \) in terms of \( \|\cdot\|_{\sup} \)-metric. In what follows, we particularly choose \( p = c_K^{-1}(h/\lambda)^{1/2} \), where \( c_K \equiv \sup_{g \in \mathcal{H}} h^{1/2}\|g\|_{\sup}/\|g\| \) is finite, according to [22].
Define \( \psi_i(g) = c^{-1}_k h^{1/2} g(X_i) \) and \( Z_j(g) = n^{-1/2} \sum_{i \in I_j} [\psi_i(g) K X_i - E\{\psi_i(g) K X_i\}] \). Following [22, Lemma 6.1], for any \( 1 \leq j \leq s \), for any \( t \geq 0 \),

\[
P \left( \sup_{g \in \mathcal{G}(p)} \| Z_j(g) \| \geq t \right) \leq 2 \exp \left( -\frac{t^2}{C^2 J(p, 1)^2} \right),
\]

(6.8)

for an absolute constant \( C > 0 \). Since \( \| f \| = 1 \) implies that \( c^{-1}_k h^{1/2} f \in \mathcal{G}(p) \). Then it can be shown that

\[
\sqrt{n} \xi_j \leq c^{-2}_k h^{-1} \sup_{g \in \mathcal{G}(p)} \| Z_j(g) \|, \quad j = 1, \ldots, s.
\]

Following (6.8) we have

\[
P \left( \sqrt{n} \max_{1 \leq j \leq s} \xi_j \geq t \right) \leq 2s \exp \left( -\frac{c^{-4}_k h^2 t^2}{C^2 J(p, 1)^2} \right),
\]

which implies that

\[
\sqrt{n} \max_{1 \leq j \leq s} \xi_j = O_P \left( \sqrt{\frac{\log N}{h^2}} J(p, 1) \right).
\]

(6.9)

It follows by [27, Proposition 1] that \( J(p, 1) = O \left( \frac{\log (h/\lambda)}{\sqrt{d+1/2}} \right) = O \left( \frac{\log N}{\sqrt{d+1/2}} \right) \). Then

\[
\max_{1 \leq j \leq s} \xi_j = O_P \left( \sqrt{\frac{\log^{d+2} N}{nh^{2/3}}} \right).
\]

That is, Assumption A4 holds with \( a = 2 \) and \( b = d + 2 \). Proof completed. \( \square \)

Proof of Lemma 3.11.

\[
J(p, 1) \leq \int_0^1 \sqrt{\log D(\varepsilon, \mathcal{G}, \| \cdot \|_{\sup})} \, d\varepsilon + \sqrt{\log D(1, \mathcal{G}, \| \cdot \|_{\sup})}
\]

\[
\leq \int_0^1 \sqrt{\left( \frac{p}{\varepsilon} \right)^{d/2m} + 1} \, d\varepsilon + \sqrt{2p^{d/2m}}
\]

\[
\leq c_d p^{d/(2m)}
\]

where the penultimate step is based on [9]. Therefore, \( J(p, 1) = O(p^{d/(2m)}) \), where \( p = (h/\lambda)^{1/2} \). From e.q. (6.9), we have

\[
\max_{1 \leq j \leq s} \xi_j = O_P \left( \sqrt{\frac{\log N}{nh^{3-d/(2m)}}} \right)
\]

\( \square \)
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