Characterizing scalable measures of quantum resources

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The question of how quantum resources, like entanglement and coherence, depend on the number of copies is addressed. We define scalable quantities \( \mathcal{E}(\rho^\otimes N) \) as physically consistent measures that can be described as functions of the variables \( \{ \mathcal{E}(\rho^\otimes i_1), \mathcal{E}(\rho^\otimes i_2), \ldots, \mathcal{E}(\rho^\otimes i_j); N \} \) for \( N > i_j \). If analyticity around vanishing resources is assumed, recursive relations can be derived for the Maclaurin series of \( \mathcal{E}(\rho^\otimes N) \), which enable us to determine the possible functional forms. In particular, we find that if \( \mathcal{E}(\rho^\otimes 2^n) \) depends only on \( \mathcal{E}(\rho), \mathcal{E}(\rho^\otimes 2) \), and \( n \), then it is completely determined by Fibonacci polynomials, to leading order. We show that the one-shot distillable (OSD) entanglement is well described as a scalable measure for several families of states. For a particular two-qutrit state \( \rho \), we determine the OSD entanglement for \( \rho^\otimes 96 \) from smaller tensorings, to leading order, with an accuracy of 97\% and no extra computational effort.

I. INTRODUCTION

If, in the future, quantum resources are to be distributed to a number of users, industrial setups would be required, that is, large scale production of copies of standard states. Thus, a physically and economically relevant question concerns how the amount of resources embodied by several copies of a state relates to those of a single copy. The exponential growth of the Hilbert space dimension as the number of copies increases (linearly) is a prohibitive obstacle in this program, which justifies the interest in asymptotic results. However, in actual situations the number of copies is always finite and, importantly, the asymptotic regime may become dominant only for an impracticable number of copies [1, 3].

In the particular case of non separability, whether or not the entanglement of \( N \) copies of a certain state coincides with \( N \) times the entanglement of a single copy, constitutes the additivity problem [4]. Several quantifiers have been studied under this perspective. The squashed entanglement [5] and the logarithmic negativity [6], for instance, are additive measures, \( \mathcal{E}(\rho^\otimes N) = N\mathcal{E}(\rho) \). For some time, this question remained open regarding the entanglement of formation [7], which has been ultimately shown to be non-additive [8] (see [9] for a broader discussion). This is the case of most of the measures, e. g., the distillable entanglement [10, 11], the Schmidt number [9], the relative entropy of entanglement [12], and the geometric distance [13, 14], to cite a few.

A dramatic manifestation of non-additivity is the phenomenon of superactivation, for which, given a particular quantifier \( \mathcal{E} \), we may find states \( \rho \), such that \( \mathcal{E}(\rho) = 0 \) and \( \mathcal{E}(\rho^\otimes \rho) > 0 \), as is the case of the distillable entanglement [15, 16]. This makes it clear that the question in the first paragraph is too limited, since \( \mathcal{E}(\rho^\otimes N) \) cannot possibly depend only on \( \mathcal{E}(\rho) \), in general.

More generally, we may ask what are the variables that determine \( \mathcal{E}(\rho^\otimes N) \) and how it depends on them. These variables for additive measures are, of course, \( \{ e_1, N \} \), \( \mathcal{E}(\rho^\otimes N) = Ne_1 \), with \( e_1 = \mathcal{E}(\rho) \). It seems natural to investigate whether other measures are related to more complex sets of variables, say, \( \{ e_1, e_2, N \} \), \( e_2 = \mathcal{E}(\rho^\otimes 2) \), and nonlinear functional dependencies. In this manuscript we investigate quantum figures of merit such that \( \mathcal{E}(\rho^\otimes N) \) can be expressed as a function of \( \{ \mathcal{E}(\rho^\otimes 4), \mathcal{E}(\rho^\otimes 12), \ldots, \mathcal{E}(\rho^\otimes k) \} \subset \{ \mathcal{E}(\rho^\otimes 2) \}^{N/N_2} \) and \( N \).

In these equalities, we assumed that the number of copies is a power of 2, \( N = 2^n \), so that the numbers \( N/2, N/4, \ldots, N/2^{n-1}, N/2^n \) are always integers.

In general, given an arbitrary positive integer \( a \), one can take \( N = a^n \) and \( K = a^k \), with \( n, k \in \{ 0, 1, 2, \ldots \} = \mathbb{N} \), \( k \leq n \), such that

\[
\rho^\otimes N = a^{\otimes (N/K)}, \quad \text{with } a \equiv \rho^{\otimes K}.
\]

is well defined. In what follows we will denote the set of all integer powers of \( a \) by \( \mathbb{P}_a = \{ 1, a, a^2, \ldots \} \) and use the notation \( e_1 = \mathcal{E}(\rho), e_2 = \mathcal{E}(\rho^\otimes 2), \ldots, e_i = \mathcal{E}(\rho^\otimes i) \). We may have \( \mathcal{E}(\rho^\otimes N) = 0 \), even when the state \( \rho^\otimes N \) does contain some finite amount of the considered resource. We reserve the term “zero-resource state” for those states that indeed contain no resource, e. g., separable states.
for entanglement, incoherent states for coherence, and so on. We will denote measures which are functions of $e_1$ and $N$ only by

$$\mathcal{E}(\rho^{\otimes N}) = E^{(N)}(\mathcal{E}(\rho)) = E^{(N)}(e_1). \quad (2)$$

By definition, the condition $E^{(1)}(e_1) = e_1$ must be satisfied. For any quantifier we will assume that $\mathcal{E}(\rho^{\otimes N}) = 0$ for zero-resource states $\rho$ (but not the other way around). Note that, while $\mathcal{E}$ maps states $\rho$ in the Hilbert-Schmidt space $\mathcal{B}(\mathcal{H})$ into non-negative real numbers, $E^{(N)}$ takes subsets of $\mathbb{R}_+$ into subsets of $\mathbb{R}_+$. As we remarked, several measures cannot be described via (2), e.g., those that allow for superactivation. This motivates the following definition:

**Definition 1:** Let $\rho \in \mathcal{B}(\mathcal{H})$ be an arbitrary state, $N \in \mathbb{N}$ and $\mathcal{E} : \mathcal{B}(\mathcal{H})^{\otimes N} \to \mathbb{R}_+$ a quantity that depends exclusively on $\rho^{\otimes N}$. If there exists an ordered set of $q$ positive integers $\{i_1, i_2, \ldots, i_q\}$ for $i_q < N$, such that one can express $\mathcal{E}(\rho^{\otimes N})$ as a function of $(e_{i_1}, e_{i_2}, \ldots, e_{i_q}) \equiv e$, and $N$:

$$\mathcal{E}(\rho^{\otimes N}) = E^{(N)}(e_{i_1}, e_{i_2}, \ldots, e_{i_q}) = E^{(N)}(e), \quad (3)$$

we say that $\mathcal{E}$ is $q$-extensible ($q$-$E$) with respect to $\rho$ and $N$. We assume that $q$ is the smallest integer that makes (3) valid.

There is a simple, but non-trivial constraint that follows from the previous considerations and from Eq. (3). For the sake of clarity, we initially refer to 1-$E$ measures with $e_{i_1} = e_1$.

**Proposition 1:** Let $N, K \in \mathbb{N}_+$ with $K < N$. Let $\rho$ be an arbitrary quantum state in $\mathcal{B}(\mathcal{H})$ and $\mathcal{E}$ an 1-$E$ function respecting relation (3), then

$$E^{(N)}(e_1) = E^{(N/K)} \left( E^{(K)}(e_1) \right). \quad (4)$$

**Proof:** Given the equality (3), it follows immediately that $\mathcal{E}(\rho^{\otimes N}) = E(\sigma^{\otimes N/K})$, which, in the notation introduced in [2], corresponds to $E^{(N)}(e_1) = E^{(N/K)}(\mathcal{E}(\sigma))$. One can use relation (2) again to write $\mathcal{E}(\sigma) = E(\rho^{\otimes K}) = E^{(K)}(e_1) \Box$

The proof is given in appendix A. This statement generalizes Eq. (3).

**Definition 2:** We say that a $q$-extensible function (with respect to $\rho$ and for all $N \in \mathbb{N}_+$) that satisfy condition (3), is a $q$-scalable ($q$-$S$) measure with respect to $\rho$ and $\mathbb{N}_+$.

It is clear that physically consistent $q$-extensible

Additive measures trivially satisfy relation (4), which, however, allows for more general dependencies. An example of a nonlinear 1-$E$ function, satisfying (4), is $E^{(N)}(e_1) = \lambda^{1-N}(e_1)^N$, $\lambda \in \mathbb{R}_+$. Indeed for $\lambda = 1$ this describes a multiplicative measure $E^{(N)}(e_1) = (e_1)^N$ as in the case of the pure-state entanglement measure defined in [17, 18] for $N$ even. We will have more to say about the solutions of (4) later.

A consequence of proposition 1 is that the function $E^{(a)}(e_1)$ completely determines $E^{(N)}(e_1)$, with $N = a^n$. To see this, set $K = a$, $k = 1$, in (4), $E^{(N)}(e_1) = E^{(N/a)}(E^{(a)}(e_1))$, which leads to $E^{(a)}(e_1) = E^{(a)}(E^{(a)}(e_1))$, and $E^{(a^n)}(e_1) = E^{(a^n)}(E^{(a)}(e_1))$, and so on. It is immediate, via induction, that

$$E^{(N)}(e_1) = E^{(a^n)}(E^{(a)}(e_1)) \quad (5)$$

where $\circ$ denotes composition and $n = \log_a N$.

This means that by picking $a = 2$, e.g., the behaviors of $E^{(\rho^{\otimes 4})}$, $E^{(\rho^{\otimes 8})}$, $E^{(\rho^{\otimes 16})}$, etc, are completely specified by the properties of $E^{(\rho^{\otimes 2})}$. In other words, for nonlinear 1-$E$ functions, the way $E^{(2)}(e_1)$ deviates from linearity completely determines the functions $E^{(2^n)}(e_1)$.

Now we provide the generalizations of the previous results to arbitrary $q$-extensible functions.

**Theorem 1:** Let $a \in \mathbb{N}_+$, and $N, K \in \mathbb{P}_a$ with $K < N$. Let $\rho \in \mathcal{B}(\mathcal{H})$ and $\mathcal{E}$ a $q$-$E$ function compatible with identity (4), then

$$E^{(N)}(e) = E^{(N/K)} \left( E^{(i_1 K)}(e), E^{(i_2 K)}(e), \ldots, E^{(i_q K)}(e) \right), \quad (6)$$

for $N \geq i_q K$.

**Proof:** The demonstration is analogous to that of proposition 1. Relation (4) implies $E(\rho^{\otimes N}) = E(\sigma^{\otimes N/K})$ and, since $\mathcal{E}$ is extensible, $E^{(N)}(e) = E^{(N/K)}(E(\sigma^{\otimes i_1}), E(\sigma^{\otimes i_2}), \ldots, E(\sigma^{\otimes i_q}))$. But $E(\sigma^{\otimes i}) = E(\rho^{\otimes i}) = E^{(i)}(e) \Box$

**Corollary 1:** The function $E^{(N)}(e)$ is completely determined by $E^{(i_1 \sigma)}(e)$, $E^{(i_2 \sigma)}(e)$, ..., $E^{(i_q \sigma)}(e)$:

$$E^{(N)}(e) = E^{(i_1 \sigma)}(e) \left( E^{(i_2 \sigma)}(e), \ldots, E^{(i_q \sigma)}(e) \right) \frac{(n-2) \text{ times}}{\ldots \left( E^{(i_{n-1} \sigma)}(e), \ldots, E^{(i_q \sigma)}(e) \right) \ldots \left( E^{(i_{n-1} \sigma)}(e), \ldots, E^{(i_q \sigma)}(e) \right)} \frac{(n-2) \text{ times}}{\ldots \left( E^{(i_{n-1} \sigma)}(e), \ldots, E^{(i_q \sigma)}(e) \right) \ldots \left( E^{(i_{n-1} \sigma)}(e), \ldots, E^{(i_q \sigma)}(e) \right)}.$$

quantum-resource functions must be $q$-scalable. Otherwise, we might find $E(\rho^{\otimes N}) \neq E(\sigma^{\otimes N/K})$, with $\sigma = \rho^{\otimes K}$. As an illustration, we show that the one-shot distillable (OSD) entanglement [2] is an example of a $2$-$S$ function, in the regime of a large number of copies. The OSD entanglement of a bipartite state $\rho$ is related to the maximal dimension $\kappa$ of the maximally entangled
state $|\Psi_e\rangle = \sum |k\rangle / \sqrt{K}$ (within some error tolerance $\epsilon$) that can be obtained from $\rho^{\otimes N}$ via non-entangling operations, for $N$ finite (for details see [3]). We denote this quantity by $E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes N})$, which has been analytically determined for the family of Bell diagonal states $\rho_{\text{Bell}} = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|\Psi^-\rangle\langle\Psi^-|$, where $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. The result for an arbitrary number $N$ of copies reads:

$$E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes N}) = N(1 - h(p)) + \sqrt{Np(1-p)} \log \left(\frac{1-p}{p}\right) \Phi^{-1}(\epsilon) + O(\log N),$$

where $h(p) = -p \log p \pm (1-p) \log(1-p)$ and $\Phi^{-1}$ is the inverse of the cumulative normal distribution $\Phi$.

Let us consider integers $N > M > L$ sufficiently large, so that the terms $O(\log N)$ can be neglected. By setting $e_L = E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes L}) \equiv e$ and $e_M = E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes M}) \equiv f$, it is a simple exercise to show that

$$E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes N}) = (M - \sqrt{MN})e + (\sqrt{LN} - L)f \sqrt{M(\sqrt{M/N} - \sqrt{L/N})},$$

for any fixed $M$ and $L$ and arbitrary $N$. Note that for $N = L$ and $N = M$ we get the results $e$ and $f$, respectively, as it should be. Therefore, for a sufficiently large number of copies, the OSD entanglement can be determined as a function of the variables $e$, $f$, and $N$. Note that, since $N > M$, the term proportional to $e = E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes L})$ gives a negative contribution to $E_{\text{OSD}}^{\epsilon}(\rho_{\text{Bell}}^{\otimes N})$. Since the later must be non-negative, we have $(M - \sqrt{MN})e + (\sqrt{LN} - L)f \geq 0$.

Equation (7) only demonstrates that this function is 2-$E$, however, it is easy to show that it is also a 2-$S$ measure with respect to $\rho_{\text{Bell}}$ and $P_a$, for any $a$. In fact, the details of the factors multiplying the terms in $N$ and $N^{1/2}$ are irrelevant to the proof (completely analogous results hold for arbitrary pure states [3]). Therefore we can state the following result.

**Proposition 2:** Any quantum-resource measure that can be expressed as $E(\rho^{\otimes N}) = 3N + \Theta(\sqrt{N}) + O(\log N)$, where $\Theta$ and $\Phi$ depend on the state $\rho$ and on fixed parameters, in the limit of a large, but finite number of copies $N$, is a 2-$S$ function up to logarithmic order.

The proof is given in Appendix B.

### III. ANALYTICITY

We proceed by considering functions $E^{(N)}(\epsilon)$ which are analytic in the vicinity of $\epsilon = 0$. We will address 1-$S$ and 2-$S$ quantifiers, while the treatment of general $q$-$S$ analytic functions will be deferred to a future publication.

#### A. 1-$S$ case

Since it is clearly the most relevant case for 1-$S$ functions, we take $i_1 = 1, e_1 \equiv e$ and consider that $E^{(N)}(\epsilon)$ is analytic at $e = 0$. More precisely, we will assume that the function $E^{(N)}(\epsilon)$ has a Maclaurin series that converges in the non-vanishing interval $[0, e_N], e_N > 0$.

$$E^{(N)}(\epsilon) = d_1(N)e + d_2(N)e^2 \cdots = \sum_{j=1}^{\infty} d_j(N)e^j, \quad (8)$$

for $e \in [0, \inf\{e_N, e_{N/\epsilon}, e_{K}\}]$. We defined $d_j(N) = \frac{d_j}{de}E^{(N)}(0)\big|_{e=0}$, with $d_0(N) = E^{(N)}(0) = 0$ and $d_j(1) = \delta_{1,j}$. From the right-hand side of (8) we must have $E^{(N)}(\epsilon) = E^{(N/K)}(d_1(K)e + d_2(K)e^2 \cdots)$, for $e \in [0, e_K)$. By expanding $E^{(N/K)}(\epsilon)$ itself we get

$$E^{(N)}(\epsilon) = d_1(N/K)(d_1(K)e + d_2(K)e^2 \cdots)^2 + \cdots$$

$$d_\ell(N/K)(d_1(K)e + d_2(K)e^2 \cdots)^\ell + \cdots \quad (9)$$

for $E^{(N/K)}(\epsilon) \in [0, e_{N/K})$. Comparing the terms of same order in $e$ in the last equation and in equation (8), we get a recursive way to determine the coefficients of the series. Before presenting the general result we recall the definition of compositions. A composition of an integer $j$ in $\ell$ parts is an ordered sum $j = \mu_1 + \mu_2 + \cdots + \mu_\ell$, of strictly positive integers. For instance, there are 5 compositions of the integer $j = 6$ in $\ell = 2$ parts: 1+5, 5+1, 2+4, 4+2, and 3+3. A well-known result in enumerative combinatorics is that there are $(\ell - 1)$ such compositions [10].

**Theorem 2:** The general recursive relation satisfied by the Maclaurin coefficients $d_j(N)$ of 1-$S$ analytic (at $e = 0$) measures is given by

$$d_j(N) = \sum_{\ell=1}^{j} d_{\ell}(N/K) \sum_{i=1}^{(j-1)} \pi_i(j, \ell; K), \quad (10)$$

where $\pi_i(j, \ell; K) = d_{\mu_1}(K)d_{\mu_2}(K)\cdots d_{\mu_i}(K)$ is a product with $(\mu_1, \mu_2, \cdots, \mu_i)$ being the $i$-th composition of $j$ into $\ell$ parts.

**Proof:** Let us consider the terms of order $e^j$ in (8) and (9). It is immediate that the first contributing term from (10) is $d_1(N/K)d_1(K)$. Note that this term is the product of $d_{e=1}(N/K)$ and $\pi_1(1, 1; K) = d_1(K)$, since there is one way to compose $j$ as the sum of a single integer ($\mu_1 = j$). The last contributing term is $d_j(N/K)(d_1(K))^j$, which is the product of $d_{e=1}(N/K)$ and $\pi_1(j, j; K) = d_1(K)d_1(K)\cdots d_1(K) = (d_1(K))^j$, since there is only one way to compose $j$ as the sum of $j$ positive integers ($\mu_1 = 1, \mu_2 = 1, \ldots, \mu_j = 1$). It is clear from (9) that the other contributing terms must contain products of $d$’s such that the sum of the subindexes equals $j$ (since these are the only terms proportional to $e^j$). The number of terms in the products is $\ell$. All possible compositions will be present, and thus, there are $(\ell - 1)$ products of $d$’s for each $\ell$ [10].
Specifically, for the first order coefficient we have \( d_1(N) = d_1(N/K) d_1(K) \) or, \( g_1(N) = g_1(N/K) + g_1(K) \), with \( g_1(N) = \log_\alpha d_1(N) \). By setting \( K = a \) for increasing values of \( N \), it is easy to get the general solution in terms of \( g_1(a) \), \( g_1(N) = \log_\alpha N g_1(a) \). That is \( d_1(N) = (d_1(a))^\nu = N^\nu, \ nu \equiv \log_\alpha d_1(a) \).

Next, we compare the second order coefficients, to get the recursive relation
\[
d_2(N) = \left( \frac{N}{K} \right)^\nu d_2(K) + K^{2\nu} d_2(N/K),
\]
which can be easily iterated, leading to
\[
E^{(N)}(e) = N^\nu e + \left( \frac{N^\nu - 1}{a^\nu - 1} \right) \left( \frac{N}{a} \right)^\nu d_2(a) e^2 + O(e^3),
\]
for \( \nu = \log_\alpha d_1(a) \). Therefore, we can state that any \( 1 \)-\( S \) measure which is analytic around \( e = 0 \) must obey the above expansion. In principle, the iteration can be continued up to arbitrary order. As we remarked before, once we find the series \( E^{(N)}(e) = d_1(a) e + d_2(a) e^2, \ldots \), the coefficients of the expansion of \( E^{(N)}(e) \) are determined. Note that the \( 1 \)-\( S \) analytical quantifiers whose regularized counterparts \( E_{\text{reg}} = \lim_{N \to \infty} E^{(\rho^{\otimes N})}/N \) are finite and non-zero are necessarily additive, and, thus \( E_{\text{reg}} = e \), for \( e \) sufficiently small (this is not necessarily true for \( 2 \)-\( S \) measures).

### B. 2-\( S \) case

We proceed by considering 2-\( S \) measures \( E^{(\rho^{\otimes N})} = E^{(N)}(e, f), e_1 = e \) and \( e_2 = f \). To alleviate the notation we set \( a = 2 \), so that the relations that follow refer to number of copies \( N = 4, 8, 16, \) etc. The results derived below can be easily extended to the more general case of \( e_b = e \) and \( e_b n = f \) (we will soon consider a case in which \( b = 6 \) and \( a = 2 \)). The measure is analytic at \((e, f) = (0, 0)\) if there is a ball (restricted to \( e \geq 0 \) and \( f \geq 0 \)) with finite radius around the origin such that
\[
E^{(N)}(e, f) = \sum_{i,j} d_{ij}(N) e^i f^j = d_{10}(N) e + d_{01}(N) f + O(2)
\]
If \( \rho \) is a zero-resource state, we must have \( E^{(N)}(e, f) = 0 \), therefore, \( E^{(N)}(0,0) = d_{00} = 0 \) in general. In addition, \( E^{(1)}(e, f) = e \) and \( E^{(2)}(e, f) = f \), implying \( d_{11}(1) = \delta_{11} d_{01} \) and \( d_{21}(2) = \delta_{02} d_{21} \). In the present case, the recurrence relation (3) reads \( E^{(N)}(e, f) = E^{(N/K)}(E^{(K)}(e, f), E^{(2K)}(e, f)) \). The following statement is valid.

**Theorem 3:** If \( E \) is a 2-\( S \) resource measure such that \( E^{(\rho^{\otimes 2})} \) depends on \( e = E(\rho), \ f = E(\rho^{\otimes 2}) \) and on \( n \), where \( E(\rho^{\otimes 2}) = xe + yf + O(2) \), with \( x \) and \( y \) known, then, for \( e \) and \( f \) sufficiently small, we have
\[
E(\rho^{\otimes 2}) = \sqrt{x^{n-1}} [\sqrt{x} F_{n-1}(\xi) + F_n(\xi)] + O(2), \quad (12)
\]
for arbitrary \( n \), where \( \xi = y/\sqrt{x} \), and \( F_n(\xi) \) are the Fibonacci polynomials [20, 21]. We note that, after inserting the explicit form of the polynomials in (12), only integer powers of \( x \) appear in the final expressions, thus, negative values of \( x \) are possible (see appendix C).

**Proof:** Using the series expansion in Eq. (6) and collecting terms of first order only, we get the coupled relations
\[
\begin{align*}
\lambda(N) &= \lambda(N/K) \lambda(K) + y(N/K) \lambda(2K), \\
\lambda(N) &= y(N/K) \lambda(2K) + \lambda(N/K) \lambda(K),
\end{align*}
\]
where we set \( \lambda(N) \equiv d_{10}(N) \) and \( \lambda(N) \equiv d_{01}(N) \), so that \( E^{(N)}(e, f, \lambda) = \lambda(N) e + \lambda(N) f + O(2) \), \( \lambda(1) = 1, \lambda(0) = 0, \lambda(2) = 1 \). Now, since \( e \) and \( f \) are free variables, the first non-trivial expansion is for \( N = 4 \). Therefore, once we know \( \lambda(\rho^{\otimes 4}) = E^{(4)}(e, f) = \sum d_{ij}(4) e^i f^j \), then \( \lambda(\rho^{\otimes N}) \), \( N > 4 \) is determined. For this reason we set \( x = \lambda(4) \) and \( y = \lambda(4) \).

To derive expressions for all \( N = 2^n \) it suffices to set \( k = 1, K = a = 2 \). With this, the recurrence relations simplify to \( \lambda(N) = x \lambda(N/2) \) and \( \lambda(N) = y \lambda(N/2) + \chi(N/2) \). Decoupling the \( \gamma \) relation we get the linear homogeneous recurrence relation with constant coefficients \( \gamma_n = y \gamma_{n-1} + x \gamma_{n-2} \), with \( \gamma_j = \lambda(4^j) \).

By setting \( \gamma_n = \sqrt{x^{n-1}} F_n \), we get the one-parameter recurrence relation
\[
F_n(\xi) = \xi F_{n-1}(\xi) + F_{n-2}(\xi)
\]
which defines the Fibonacci polynomials [20, 21], with \( \xi = y/\sqrt{x} \) [the famous Fibonacci numbers are given by \( F_n(1) \)]. Therefore, by setting \( E(\rho) = e, E(\rho^{\otimes 2}) = f \), knowing \( x \) and \( y \) in \( E(\rho^{\otimes 4}) = E(4)(e, f) = xe + yf \), and recalling that \( \lambda_n = x \gamma_n \), we get (12) □

The explicit form of the polynomials is
\[
F_n(\xi) = \sum_{j=0}^{(n-1)/2} \binom{n-j-1}{j} \xi^{n-2j-1},
\]
where \( \ldots \) denotes the floor function.

It is not easy to find sufficient data in the literature to put these formulas to test. Recently, however, the authors of [3] managed to numerically calculate, via linear programming, the OSD entanglement of up to \( N = 100 \) copies of the symmetric state \( \rho_F = F \rho_b + (1-F)(I - \rho d) / (d^2 - 1) \), where \( \rho d \) is the maximally entangled states in \( d \) dimensions. The parameters used in the calculations are \( d = 3, F = 0.9 \), an error tolerance of \( e = 0.001 \), and the first non-vanishing result was obtained for six copies. Since this same figure of merit has displayed 2-scalability for Bell-diagonal and pure states, we will assume the same property here.

It turns out that the validity of theorem 1 can be easily extended to \( N \in \{ b, ba, ba^2, \ldots \} = \mathbb{P}_b \) with \( K \in \mathbb{P}_d \) [see relation (11)]. By setting \( N = 6 \times 2^n \in \mathbb{P}_b \), using \( E^{(N)}(e) = E^{(N/K)}(E^{(6K)}(e), E^{(12K)}(e)) \), with \( e = (e_6, e_{12}) \equiv (e, f) \), and \( K = 2^b \in \mathbb{P}_d \), we get exactly the result of Eq. (12), with the left-hand side replaced...
by $\mathcal{E}(\rho^\otimes 6 \times 2^n)$. We, therefore, have $e = \mathcal{E}_{\text{OSD}}(\rho_F^6)$, $f = \mathcal{E}_{\text{OSD}}(\rho_F^{12})$, $x e + y f = \mathcal{E}_{\text{OSD}}(\rho_F^{24})$, and $xy e + (y^2 + x) f = \mathcal{E}_{\text{OSD}}(\rho_F^{48})$. We used the numeric values for $N = 6, 12, 24, 48$ in [2] to determine $e$, $f$, $x$, and $y$ from these equations and predict the value of (see appendix D):

$$\mathcal{E}_{\text{OSD}}(\rho_F^{96}) = x(y^2 + x)e + (y^3 + 2xy)f = (0.7 \pm 0.1) \times 96.$$  

The value obtained directly from [3] is $0.683 \pm 0.001$, which corresponds to an agreement of 97% with the central value predicted by formula (12). This is a quite precise results since $F = 0.9$ may not be exactly in the regime of weak resources, and no free parameter has been adjusted. It would be very interesting to investigate data with slightly smaller values of $F$ and also to develop second order recurrences for $2$-S monotones. Finally, we note that superactivation of non additivity may happen in $q$-S functions. In the present case we may have $\mathcal{E}(\rho^\otimes 2) = 2\mathcal{E}(\rho)$, that is, $f = 2e$, however, with $\mathcal{E}(\rho^\otimes 4) = (x + 2y)e \neq 4\mathcal{E}(\rho)$ whenever $x \neq 4 - 2y$.

IV. CLOSING REMARKS

Any strategy that helps to circumvent the direct evaluation of functions whose domains are high-dimensional Hilbert spaces is potentially useful in all fields of quantum information. In this manuscript we have introduced the concept of scalability for any physical figure of merit which is solely determined by the $N$-fold tensor product of a quantum state $\rho$. Although we referred to entanglement, the presented results are equally valid for coherence measures [22, 28], for example, with minor adaptations (e.g., replacing “vanishing on separable states” with “vanishing on incoherent states”). This approach enabled us to employ elementary tools from analysis and combinatorics to the study of a broad class of quantum resources. It is worth mentioning that the same ideas may also be extended to resources that depend on information not contained in $\rho^\otimes N$. Several measures of Bell nonlocality [29, 31] and steering [32], e.g., refer to the state plus the Bell scenario, which can fit into the presented formalism provided that the number of observables per party remains fixed as the number of copies grow. For direct quantifications of quantum behaviors [32] the extension is not emmendate.

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Appendix A: Proof of corollary 1

By setting $K = a$ and for $N = i_q a^n$, we get $E^{(i_q a^n)} = E^{(i_q a)}(E^{(i_q a)}(e), E^{(i_q a)}(e), \ldots, E^{(i_q a)}(e))$ and

$$E^{(i_q a^n)} = E^{(i_q a^n)} \left( E^{(i_q a)}(e), \ldots, E^{(i_q a)}(e) \right)$$

Using theorem 1 again we get

$$E^{(i_q a^n)} = E^{(i_q a^n)} \left( E^{(i_q a)}(e), \ldots, E^{(i_q a)}(e) \right),$$

and so on. It is immediate that by assuming the validity of eq. (6) of the main text for $N = i_q a^n$ it will be valid for $N = i_q a^{n+1}$. Since corollary 1 is trivially valid for $N = i_q a$, due to the principle of finite induction, we get:

$$E^{(N)}(e) = E^{(i_q a)}(E^{(i_q a)}(E^{(i_q a)}(e), \ldots, E^{(i_q a)}(e)), \ldots, E^{(i_q a)}(E^{(i_q a)}(e), \ldots, E^{(i_q a)}(e))).$$
Appendix B: Proof of proposition 2

From the definitions in the main text, we have

$$\mathfrak{F} = \frac{\sqrt{L}f - \sqrt{Me}}{M\sqrt{L} - L\sqrt{M}} \quad \text{and} \quad \mathcal{G} = \frac{Lf - Me}{L\sqrt{M} - M\sqrt{L}}.$$  

With these relations one can explicitly write Eq. (7) in the main text as

$$E^{(N)}(e,f) = \frac{\sqrt{N}}{\sqrt{M} - \sqrt{L}} \left[ \left( \frac{\sqrt{M} - \sqrt{N}}{\sqrt{L}} \right) e + \left( \frac{\sqrt{N} - \sqrt{L}}{\sqrt{L}} \right) f \right]. \quad (B1)$$

It is a simple exercise to show that the above relation is valid up to logarithmic order for any resource function which can be written as \(\mathcal{E} (\rho^{\otimes N}) = \mathfrak{F} N + \mathcal{G} \sqrt{N} + O(\log N)\). The proof of proposition 2 consists in showing that the above function satisfy theorem 1, Eq. (6) in the main text, that is

$$E^{(N/K)}(e,f) = E^{(N)}(e,f) = E^{(KL)}(e,f), E^{(KM)}(e,f)$$

$$= \frac{\sqrt{N/K}}{\sqrt{M} - \sqrt{L}} \left[ \left( \frac{\sqrt{M} - \sqrt{N/K}}{\sqrt{L}} \right) E^{(KL)}(e,f) + \left( \frac{\sqrt{N/K} - \sqrt{L}}{\sqrt{L}} \right) E^{(KM)}(e,f) \right]$$

$$= \frac{1}{(\sqrt{M} - \sqrt{L})^2} \left\{ \mathcal{N} \left( \frac{\sqrt{M} - \sqrt{N/K}}{\sqrt{L}} \right) \left[ \left( \frac{\sqrt{M} - \sqrt{KL}}{\sqrt{L}} \right) e + \left( \frac{\sqrt{KL} - \sqrt{L}}{\sqrt{L}} \right) f \right] \right. ~+ \left. \sqrt{KM} \left( \frac{\sqrt{N/K} - \sqrt{LM}}{\sqrt{M}} \right) \right\}.$$  

With some further algebraic manipulations one shows that all terms containing \(K\) cancel out and, in addition, that the remaining terms exactly coincide with Eq. (B1) above, which finishes the proof.

Appendix C: Some explicit formulas for equation (12)

Let us consider a 2-S measure, for which one can express \(\mathcal{E}(\rho^{\otimes b}) = e, \mathcal{E}(\rho^{\otimes 2b}) = f, \) and \(\mathcal{E}(\rho^{\otimes 4b}) = x e + y f + O(2),\) for a constant integer \(b\). This is a slightly more general case for which theorem 1 can be extended trivially. Application of theorem 3, Eq. (12) in the main text, leads to

$$\mathcal{E}(\rho^{\otimes 8b}) = x ye + (y^2 + x)f + O(2),$$

$$\mathcal{E}(\rho^{\otimes 16b}) = x(y^2 + x)e + (y^3 + 2xy)f + O(2),$$

$$\mathcal{E}(\rho^{\otimes 32b}) = x(y^3 + 2xy)e + (y^4 + 3xy^2 + x^2)f + O(2),$$

$$\mathcal{E}(\rho^{\otimes 64b}) = x(y^4 + 3xy^2 + x^2)e + (y^5 + 4xy^3 + 3x^2y)f + O(2),$$ etc,

where \(O(2)\) denotes terms proportional to \(e^2, ef,\) and \(f^2\).

Appendix D: Numeric parameters used to estimate \(\mathcal{E}_{OSD}(\rho_F^{\otimes 6})\)

We used a software to carefully extract the data from a zoomed copy of figure 2 of reference [3] of the main text, for \(N = 6, 7, \ldots, 99, 100\). In particular the OSD entanglement per copy for the four points mentioned in the main text is

$$\frac{\mathcal{E}_{OSD}^{(6)}(\rho_F^{\otimes 6})}{6} = 0.167 \pm 0.001, \quad \frac{\mathcal{E}_{OSD}^{(12)}(\rho_F^{\otimes 12})}{12} = 0.308 \pm 0.001, \quad \frac{\mathcal{E}_{OSD}^{(24)}(\rho_F^{\otimes 24})}{24} = 0.439 \pm 0.001, \quad \frac{\mathcal{E}_{OSD}^{(48)}(\rho_F^{\otimes 48})}{48} = 0.573 \pm 0.001.$$
This leads to $x = -3.1 \pm 0.3$, $y = 3.71 \pm 0.08$. For $N = 96$ we obtain

\[
\frac{E_{OSD}(\rho_9^\otimes 96)}{96} = \frac{x(y^2 + x) e + (y^3 + 2xy) f}{96} = 0.7 \pm 0.1 \quad \text{(from theorem 3)}
\]

The numeric value obtained from reference [3] is

\[
\frac{E_{OSD}(\rho_9^\otimes 96)}{96} \approx 0.683 \pm 0.001.
\]

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