A FAMILY OF FINITE $p$-GROUPS SATISFYING CARLSON’S DEPTH CONJECTURE

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Abstract. Let $p > 3$ be a prime number and let $r$ be an integer with $1 < r < p - 1$. For each $r$, let moreover $G_r$ denote the unique quotient of the maximal class pro-$p$ group of size $p^{r+1}$. We show that the mod-$p$ cohomology ring of $G_r$ has depth one and that, in turn, it satisfies the equalities in Carlson’s depth conjecture [3].

1. Introduction

Let $p$ be a prime number, let $G$ be a finite $p$-group and let $\mathbb{F}_p$ denote the finite field of $p$ elements with trivial $G$-action. Then, the mod-$p$ cohomology ring $H^*(G; \mathbb{F}_p)$ is a finitely generated, graded commutative $\mathbb{F}_p$-algebra (see [7, Corollary 7.4.6]), and so many ring-theoretic notions can be defined; Krull dimension, associated primes and depth, among others. Some of the aforementioned concepts have a group-theoretic interpretation; for instance, the Krull dimension $\dim H^*(G; \mathbb{F}_p)$ of $H^*(G; \mathbb{F}_p)$ equals the $p$-rank $\text{rk}_p G$ of $G$, i.e., the largest integer $s \geq 1$ such that $G$ contains an elementary abelian subgroup of rank $s$. However, the depth of $H^*(G; \mathbb{F}_p)$, written as $\text{depth} H^*(G; \mathbb{F}_p)$, is the length of the longest regular sequence in $H^*(G; \mathbb{F}_p)$, and it seems to be far more difficult to compute. There are, however, lower and upper bounds for this number. For instance, in [6], Duflot proved that the depth of $H^*(G; \mathbb{F}_p)$ is at least as big as the $p$-rank of the centre $Z(G)$ of $G$, i.e., $\text{depth} H^*(G; \mathbb{F}_p) \geq \text{rk}_p Z(G)$ and, in [19], Notbohm proved that for every elementary abelian subgroup $E$ of $G$ with centralizer $C_G(E)$ in $G$, the inequality $\text{depth} H^*(G; \mathbb{F}_p) \leq \text{depth} H^*(C_G(E); \mathbb{F}_p)$ holds. In [3], J. Carlson provided further upper bounds for the depth (see Theorem 2.4) and stated a conjecture that still remains open (see Conjecture 2.5).

The aim of the present work is to compute the depth of the mod-$p$ cohomology rings of certain quotients of the maximal class pro-$p$ group that moreover satisfy the equalities in the aforementioned conjecture. Let $p$ be an odd prime number, let $\mathbb{Z}_p$ denote the ring of $p$-adic integers and let $\zeta$ be a...
primitive $p$-th root of unity. Consider the cyclotomic extension $\mathbb{Z}_p[\zeta]$ of degree $p - 1$ and note that its additive group is isomorphic to $\mathbb{Z}_p^{p-1}$. The cyclic group $C_p = \langle \sigma \rangle$ acts on $\mathbb{Z}_p[\zeta]$ via multiplication by $\zeta$, i.e., for any $x \in \mathbb{Z}_p$, the action is given as $x^\sigma = \zeta x$. Using the ordered basis $1, \zeta, \ldots, \zeta^{p-2}$ in $\mathbb{Z}_p[\zeta] \cong \mathbb{Z}_p^{p-1}$, this action is given by the matrix
\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-1 & -1 & -1 & \ldots & -1
\end{pmatrix}
\]
We form the semidirect product $S = C_p \rtimes \mathbb{Z}_p^{p-1}$, which is the unique pro-$p$ group of maximal nilpotency class. Note that this is the analogue of the infinite dihedral pro-$2$ group for the $p$ odd case. Moreover, $S$ is a uniserial $p$-adic space group with cyclic point group $C_p$ (compare [15, Section 7.4]).

We write $[x, k \sigma] = [x, \sigma, k, \ldots, \sigma]$ for the iterated group commutator. Set $T_0 = \mathbb{Z}_p[\zeta]$ and define, for each integer $i \geq 1$, $T_i = (\zeta - 1)^i \mathbb{Z}_p[\zeta] = [T_0, i \sigma] = \gamma_{i+1}(S)$.

These subgroups are all the $C_p$-invariant subgroups of $T_0$, and the successive quotients satisfy
\[T_i / T_{i+1} \cong \mathbb{Z}_p[\zeta] / (\zeta - 1)\mathbb{Z}_p[\zeta] \cong C_p.\]

Hence, $|T_0 : T_i| = p^i$ for every $i \geq 0$. For each integer $r$ with $1 < r < p - 1$, consider the finite quotient $T_0 / T_r = \mathbb{Z}_p[\zeta] / (\zeta - 1)^r \mathbb{Z}_p[\zeta]$ and choose a generating set for $T_0 / T_r$ as follows,
\[a_1 = 1 + T_r, \quad a_2 = (\zeta - 1) + T_r, \quad \ldots, \quad a_r = (\zeta - 1)^{r-1} + T_r.\]

Using the multiplicative notation, we obtain that
\[T_0 / T_r = \langle a_1, \ldots, a_r \rangle \cong C_p \times \cdots \times C_p,\]
and since for all $i \geq 0$, the subgroups $T_i$ are $C_p$-invariant, we can form the semidirect product
\[G_r = C_p \rtimes T_0 / T_r \cong C_p \rtimes (C_p \times \cdots \times C_p).\]

The finite $p$-groups $G_r$ have size $p^{r+1}$ and exponent $p$. Note that in particular, $G_2$ is the extraspecial group of size $p^3$ and exponent $p$. We state the main result.

**Theorem A.** Let $p > 3$ be a prime number, let $r$ be an integer with $1 < r < p - 1$ and let $G_r$ be given as in (1). Then, $H^*(G_r; \mathbb{F}_p)$ has depth one.
Notation. Throughout let $p$ be an odd prime number and let $G$ denote a finite group. Let $R$ be a commutative ring with unity. A $G$-module $A$ will be a right $RG$-module. For such $G$-modules, we shall use additive notation in Sections 2 and 3, and multiplicative notation in Section 4, for our convenience. Moreover, if $a \in A$ and $g \in G$, we write $a^g$ to denote the action of $g$ on $a$.

Let $A$ be a $G$-module and let $P_n \rightarrow R$ be a projective resolution of the trivial $G$-module $R$, then for every $n \geq 0$, the $n$-th cohomology group $H^n(G; A)$ is defined as $\text{Ext}^n(R, A) = H^n(\text{Hom}_G(P_n, A))$. Let $K \leq G$ be a subgroup of $G$ and let $\iota: K \rightarrow G$ denote an inclusion map. This map induces the restriction map in cohomology, which will be denoted by $\text{res}_K^G: H^*(G; A) \rightarrow H^*(K; A)$.

Group commutators are given as $[g, h] = g^{-1} h^{-1} gh = g^{-1} h^g$ and for every $k \geq 1$, iterated commutators are written as $[x, y, \ldots, y] = [x, y^k y]$, where we use left normed group commutators, i.e., $[x, y, z] = [[x, y], z]$. Also, the $k$-th term of the lower central series of $G$ is denoted by $\gamma_k(G) = [G, \ldots, G]$.

2. Preliminaries

2.1. Depth. In this section we give background on the depth of mod-$p$ cohomology rings of finite $p$-groups and we also state one of the key results for the proof of Theorem A.

Let $n \geq 1$ be an integer number. We say that a sequence of elements $x_1, \ldots, x_n \in H^*(G; \mathbb{F}_p)$ is regular if, for every $i = 1, \ldots, n$, the element $x_i$ is not a zero divisor in the quotient $H^*(G; \mathbb{F}_p)/(x_1, \ldots, x_{i-1})$, where $(x_1, \ldots, x_{i-1})$ denotes the ideal generated by the elements $x_1, \ldots, x_{i-1}$ in $H^*(G; \mathbb{F}_p)$.

Definition 2.1. The depth of $H^*(G; \mathbb{F}_p)$, denoted by depth $H^*(G; \mathbb{F}_p)$, is the maximal length of a regular sequence in $H^*(G; \mathbb{F}_p)$.

Recall that a prime ideal $p \subseteq H^*(G; \mathbb{F}_p)$ is an associated prime of $H^*(G; \mathbb{F}_p)$ if, for some $\varphi \in H^*(G; \mathbb{F}_p)$, it is of the form

$$p = \{\psi \in H^*(G; \mathbb{F}_p) \mid \varphi \cup \psi = 0\}.$$  

The set of all associated primes of $H^*(G; \mathbb{F}_p)$ is denoted by $\text{Ass} H^*(G; \mathbb{F}_p)$. It is known that for every $p \in \text{Ass} H^*(G; \mathbb{F}_p)$, the following inequality holds

$$\text{depth } H^*(G; \mathbb{F}_p) \leq \dim H^*(G; \mathbb{F}_p)/p.$$

In particular, $\text{depth } H^*(G; \mathbb{F}_p) \leq \dim H^*(G; \mathbb{F}_p)$ ([4, Proposition 12.2.5]) and, when the two values coincide, the mod-$p$ cohomology ring is said to be Cohen-Macaulay. In the following proposition, we recall the lower and upper bounds for the depth of $H^*(G; \mathbb{F}_p)$ by Duflot [6] and Notbohm [19], respectively.

Proposition 2.2. Let $G$ be a finite $p$-group. The following inequalities hold

$$1 \leq \text{rk}_p Z(G) \leq \text{depth } H^*(G; \mathbb{F}_p) \leq \text{depth } H^*(C_G(E); \mathbb{F}_p).$$
Before stating the crucial result for our construction, we introduce the concept of detection in cohomology.

**Definition 2.3.** Let \( G \) be a finite \( p \)-group and let \( \mathcal{H} \) be a collection of subgroups of \( G \). We say that \( \mathcal{H}^\ast(G; \mathbb{F}_p) \) is detected by \( \mathcal{H} \) if
\[
\bigcap_{H \in \mathcal{H}} \text{Ker res}_H^G = 0.
\]

Given a finite \( p \)-group \( G \) and a subgroup \( E \leq G \), let \( C_G(E) \) denote the centralizer of \( E \) in \( G \). For \( s \geq 1 \), define
\[
\mathcal{H}_s(G) = \{ C_G(E) \mid E \text{ is an elementary abelian subgroup of } G, \text{rk}_p E = s \},
\]
\[
\omega_a(G) = \min \{ \dim \text{H}^\ast(G; \mathbb{F}_p)/p \mid p \in \text{Ass H}^\ast(G; \mathbb{F}_p) \},
\]
\[
\omega_d(G) = \max \{ s \geq 1 \mid \text{H}^\ast(G; \mathbb{F}_p) \text{ is detected by } \mathcal{H}_s(G) \}.
\]

**Theorem 2.4** ([3]). Let \( G \) be a finite \( p \)-group. Then, the following inequalities hold
\[
\text{depth H}^\ast(G; \mathbb{F}_p) \leq \omega_a(G) \leq \omega_d(G).
\]

In fact, in the same article, J. F. Carlson conjectured that the previous inequalities are actual equalities.

**Conjecture 2.5** (Carlson). Let \( G \) be a finite \( p \)-group. Then,
\[
\text{depth H}^\ast(G; \mathbb{F}_p) = \omega_a(G) = \omega_d(G).
\]

A particular case of the above conjecture was proven by D. Green in [9].

### 2.2. Yoneda extensions

Let \( G \) be a finite group and let \( R \) be a commutative ring with unity. We describe the mod-\( p \) cohomology ring \( \text{H}^\ast(G; R) \) in terms of Yoneda extensions and the Yoneda product. For a more detailed account on this topic, we refer to [16, Chapter IV] and [18].

**Definition 2.6.** Let \( A \) and \( B \) be \( G \)-modules. For every integer \( n \geq 1 \), a *Yoneda \( n \)-fold extension* \( \varphi \) of \( B \) by \( A \) is an exact sequence of \( G \)-modules of the form
\[
\varphi : 0 \longrightarrow A \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow B \longrightarrow 0.
\]

Given two Yoneda \( n \)-fold extensions
\[
\varphi : 0 \longrightarrow A \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow B \longrightarrow 0
\]
and
\[
\varphi' : 0 \longrightarrow A' \longrightarrow M'_n \longrightarrow \cdots \longrightarrow M'_1 \longrightarrow B' \longrightarrow 0,
\]
we say that there is a *morphism between the Yoneda \( n \)-fold extensions* \( \varphi \) and \( \varphi' \), if there exist \( G \)-module homomorphisms \( f_0 : B \longrightarrow B' \), \( f_{n+1} : A \longrightarrow A' \)
and, for every \( i = 1, \ldots, n \), \( f_i : M_i \to M_i' \), making the following diagram commute

\[
\begin{array}{c}
\varphi : 0 \rightarrow A \rightarrow M_n \rightarrow \cdots \rightarrow M_1 \rightarrow B \rightarrow 0 \\
\downarrow f_{n+1} \downarrow f_n \downarrow f_i \downarrow f_0 \\
\varphi' : 0 \rightarrow A' \rightarrow M_n' \rightarrow \cdots \rightarrow M_1' \rightarrow B' \rightarrow 0.
\end{array}
\]

In particular, if \( \varphi, \varphi' \) are both Yoneda \( n \)-fold extensions of \( A \) by \( B \) and, there is a morphism from \( \varphi \) to \( \varphi' \) with identity maps \( f_0 = \text{id}_B \) and \( f_{n+1} = \text{id}_A \), we write \( \varphi \Rightarrow \varphi' \).

**Definition 2.7.** Let \( n \geq 1 \) be an integer and let \( \varphi, \varphi' \) as above. We say that \( \varphi \) is equivalent to \( \varphi' \), denoted by \( \varphi \equiv \varphi' \), if there are Yoneda \( n \)-fold extensions \( \varphi_1, \ldots, \varphi_r \) of \( B \) by \( A \) such that

\[
\varphi \Rightarrow \varphi_1 \Leftarrow \varphi_2 \Rightarrow \cdots \Leftarrow \varphi_{r-1} \Rightarrow \varphi_r \Leftarrow \varphi'.
\]

Moreover, we denote by \( \text{YExt}^n(B, A) \) the set of all Yoneda \( n \)-fold extensions of \( B \) by \( A \) up to equivalence.

We recall the uniqueness of pushouts and pullbacks of Yoneda extensions [10, Section II.6] and endow the set \( \text{YExt}^n(B, A) \) with the Baer sum so that \( \text{YExt}^n(B, A) \) becomes an abelian group. The proof of the following result can be found in [10, Section IV.9].

**Proposition 2.8.** Let \( \varphi \in \text{YExt}^n(B, A) \) be represented by a Yoneda extension

\[
0 \rightarrow A \rightarrow M_n \rightarrow \cdots \rightarrow M_1 \rightarrow B \rightarrow 0.
\]

(a) Given a \( G \)-module homomorphism \( \alpha : A \rightarrow A' \), there is a unique equivalence class \( \alpha \ast \varphi \in \text{YExt}^n(B, A') \) represented by a Yoneda extension

\[
0 \rightarrow A' \rightarrow M_n' \rightarrow \cdots \rightarrow M_1' \rightarrow B \rightarrow 0,
\]

admitting a morphism of Yoneda extensions of the following form:

\[
\begin{array}{c}
0 \rightarrow A \rightarrow M_n \rightarrow \cdots \rightarrow M_1 \rightarrow B \rightarrow 0 \\
\downarrow \alpha \downarrow \downarrow \downarrow \downarrow \\
0 \rightarrow A' \rightarrow M_n' \rightarrow \cdots \rightarrow M_1' \rightarrow B \rightarrow 0.
\end{array}
\]

We say that the Yoneda extension \( \alpha \ast \varphi \) is the pushout of \( \varphi \) via \( \alpha \).

(b) Given a \( G \)-module homomorphism \( \beta : B' \rightarrow B \) there is a unique equivalence class \( \beta \ast \varphi \in \text{YExt}^n(B', A) \) represented by a Yoneda extension

\[
0 \rightarrow A \rightarrow M_n'' \rightarrow \cdots \rightarrow M_1'' \rightarrow B' \rightarrow 0,
\]
admitting a morphism of Yoneda extensions of the following form:

\[
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & M''_n & \rightarrow & \cdots & \rightarrow & M''_1 & \rightarrow & B' & \rightarrow & 0 \\
\parallel & & \downarrow & & \downarrow & & & & \downarrow & & \beta & & \\
0 & \rightarrow & A & \rightarrow & M_n & \rightarrow & \cdots & \rightarrow & M_1 & \rightarrow & B & \rightarrow & 0.
\end{array}
\]

We say that the Yoneda extension $\beta^*\varphi$ is the pullback of $\varphi$ via $\beta$.

Let now $A$ and $B$ be $G$-modules and let

\[\nabla_A: A \times A \rightarrow A\] and \[\Delta_B: B \rightarrow B \times B\]

denote the codiagonal and the diagonal homomorphism, respectively.

**Definition 2.9.** Let $n \geq 1$ be an integer and let $\varphi, \varphi' \in \text{YExt}^n(B, A)$ be two Yoneda extension classes. We define the *Baer sum* of $\varphi$ and $\varphi'$ as

\[\varphi + \varphi' = (\nabla_A)^*((\Delta_B)^* (\varphi \times \varphi')) \in \text{YExt}^n(B, A).\]

Then, for every integer $n \geq 1$, the set $\text{YExt}^n(B, A)$ endowed with the Baer sum is an abelian group. Indeed, the zero element of $\text{YExt}^1(B, A)$ is the split extension

\[0 \rightarrow A \rightarrow A \times B \rightarrow B \rightarrow 0,\]

and for $n > 1$, the zero element of $\text{YExt}^n(B, A)$ is the Yoneda extension

\[0 \rightarrow A \rightarrow A \rightarrow 0 \rightarrow^{n-2} 0 \rightarrow B \rightarrow B \rightarrow 0.\]

**Theorem 2.10 ([16, Theorem 6.4]).** For every $G$-module $A$ and integer $n \geq 1$, there is a group isomorphism $H^n(G; A) \cong \text{YExt}^n(R, A)$ that is natural in $A$.

Let $A$, $B$ and $C$ be $G$-modules and let $n, m \geq 1$ be integers. Given $\varphi \in \text{YExt}^n(B, A)$ represented by the Yoneda extension

\[0 \rightarrow A \rightarrow N_n \rightarrow \cdots \rightarrow N_1 \rightarrow B \rightarrow 0,\]

and $\varphi' \in \text{YExt}^m(C, B)$ represented by the Yoneda extension

\[0 \rightarrow B \rightarrow M_m \rightarrow \cdots \rightarrow M_1 \rightarrow C \rightarrow 0,\]

we define their *Yoneda product* $\varphi \cup \varphi' \in \text{YExt}^{n+m}(G, B)$ as the Yoneda extension

\[0 \rightarrow A \rightarrow N_n \rightarrow \cdots \rightarrow N_1 \rightarrow M_m \rightarrow \cdots \rightarrow M_1 \rightarrow C \rightarrow 0.\]

This product defines a bilinear pairing. In particular, the Yoneda product in $H^*(G; R) \cong \text{YExt}^*(R, R)$ coincides with the usual cup product (see [1, Proposition 3.2.1]).
2.3. Crossed extensions. Let $G$ be a finite group. In this section we describe, for every integer $n \geq 2$, the cohomology group $H^n(G; R)$ using crossed extensions. For more information about this subject, see [11], [12] and [18].

Definition 2.11. Let $M_1$ and $M_2$ be groups with $M_1$ acting on $M_2$. A crossed module is a group homomorphism $\rho: M_2 \to M_1$ satisfying the following properties:

(i) $y_2^{\rho(y'_2)} = y_2^{y'_2}$ for all $y_2, y'_2 \in M_2$, and
(ii) $\rho(y_2^{y_1}) = \rho(y_2)^{y_1}$ for all $y_1 \in M_1$ and $y_2 \in M_2$.

Definition 2.12. Let $n \geq 1$ be an integer and let $A$ be a $G$-module. A crossed $n$-fold extension $\psi$ of $G$ by $A$ is an exact sequence of groups of the form

$$\psi: 0 \to A \xrightarrow{\rho_0} M_n \xrightarrow{\cdots} M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} G \to 1,$$

satisfying the following conditions:

(i) $\rho_1: M_2 \to M_1$ is a crossed module,
(ii) $M_i$ is a $G$-module for every $i = 3, \ldots, n$, and
(iii) $\rho_i$ is a $G$-module homomorphism for every $i = 2, \ldots, n$.

Definition 2.13. A morphism of crossed $n$-fold extensions $\psi$ and $\psi'$ is a morphism of exact sequences of groups

$$\psi: 0 \to A \xrightarrow{\rho_0} M_n \xrightarrow{\cdots} M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} G \to 1$$

$$\psi': 0 \to A' \xrightarrow{\rho_0'} M'_n \xrightarrow{\cdots} M'_2 \xrightarrow{\rho_1'} M'_1 \xrightarrow{\rho_0'} G' \to 1,$$

where for each $i = 3, \ldots, n + 1$, the morphism $f_i$ is a $G$-module homomorphism and, $f_1$ and $f_2$ are compatible with the actions of $M_1$ on $M_2$ and of $M'_1$ on $M'_2$, respectively.

In particular, if $\psi$ and $\psi'$ are both crossed $n$-fold extensions of $G$ by $A$ and, there is a morphism from $\psi$ to $\psi'$ with identity maps $f_0 = \text{id}_G$ and $f_{n+1} = \text{id}_A$, we write $\psi \equiv \psi'$.

Moreover, we can define an equivalence relation on crossed $n$-fold extensions of $G$ by $A$ as for Yoneda extensions in Definition 2.7, denoted by $\psi \equiv \psi'$. We will also denote by $\text{XExt}^n(G, A)$ the set of all crossed $n$-fold extensions of $G$ by $A$ up to equivalence.

For the $n = 2$ case, we can use the following characterization of equivalent crossed extensions.

Proposition 2.14 ([11]). Let $G$ be a finite group and let $A$ be a $G$-module. Then, two crossed 2-fold extensions of $G$ by $A$

$$\psi: 0 \to A \xrightarrow{\rho_2} M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} G \to 1 \text{ and } \psi': 0 \to A \xrightarrow{\tau_2} N_2 \xrightarrow{\tau_1} N_1 \xrightarrow{\tau_0} G \to 1,$$
are equivalent if and only if there exist a group $X$ and a commutative diagram

![Diagram](image)

satisfying the following properties:

(a) $-\tau_2: A \to N_2$ is given by $(-\tau_2)(a) = \tau_2(-a)$ for $a \in A$,
(b) the diagonals are short exact sequences,
(c) $\mu_1 \circ \rho_2(A) = \mu_1(M_2) \cap \mu_2(N_2)$, and
(d) conjugation in $X$ coincides with the actions of both $M_1$ on $M_2$ and $N_1$ on $N_2$.

Analogous to Yoneda extensions, for an integer $n \geq 1$, given an $n$-crossed extension $\varphi \in \text{XExt}^n(G, A)$ and a $G$-module homomorphism $\alpha: A \to A'$, we can find a unique pushout $\alpha^* \varphi \in \text{XExt}^n(G, A')$ of $\varphi$ via $\alpha$, and given a group homomorphism $\beta: G' \to G$ we can find a unique pullback $\beta^* \varphi \in \text{XExt}^n(G', A)$ of $\varphi$ via $\beta$ (see [11, Proposition 4.1]).

We can also endow $\text{XExt}^n(G, A)$ with an abelian group structure. Given two crossed $n$-fold extension classes $\varphi, \varphi' \in \text{XExt}^n(G, A)$, we define their Baer sum as

$$\varphi + \varphi' = (\nabla_A)_*(\Delta_G)^*(\varphi \times \varphi').$$

The zero element of $\text{XExt}^1(G, A)$ is represented by the split extension

$$0 \to A \to G \rtimes A \to G \to 1,$$

and for $n > 1$, the zero element of $\text{XExt}^n(G, A)$ is represented by the Yoneda extension

$$0 \to A \to A \to 0 \to \cdots \to 0 \to G \to 0 \to 1.$$

**Theorem 2.15** ([11, Theorem 4.5]). Let $G$ be a finite group. For every $G$-module $A$ and every integer $n \geq 1$, there is a group isomorphism $H^{n+1}(G; A) \cong \text{XExt}^n(G, A)$ that is natural in both $G$ and $A$. 


3. Product between extensions

3.1. Product of Yoneda extensions and crossed extensions. We now describe the Yoneda product between two cohomology classes, one of them represented by a Yoneda extension and the other one by a crossed extension.

Definition 3.1. Let $G$ be a finite group, let $A$ and $B$ be $G$-modules and let $n, m \geq 1$ be integer numbers. Given a Yoneda $n$-fold extension class $\varphi \in \text{YExt}^n(A, B)$ represented by

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow A \longrightarrow 0,$$

and a crossed $m$-fold extension class $\psi \in \text{XExt}^m(G, A)$ represented by

$$0 \longrightarrow A \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1,$$

we define their Yoneda product $\varphi \cup \psi$ as the extension

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

Remark 3.2. It can be readily checked that

$$\text{YExt}^n(A, B) \times \text{XExt}^m(G, A) \longrightarrow \text{XExt}^{n+m}(G, B)$$

given by $(\varphi, \psi) \mapsto \varphi \cup \psi$ is well defined.

The following result shows that this product respects the pushouts and pullbacks.

Lemma 3.3. Let $G$ and $G'$ be finite groups, let $A, A', B$ and $B'$ be $G$-modules and let $n, m \geq 1$ be integer numbers. Let, moreover, $\varphi \in \text{YExt}^n(A, B)$, $\varphi' \in \text{YExt}^n(A', B)$ and $\psi \in \text{XExt}^m(G, A)$. Then, the following relations are satisfied.

1. Given a $G$-module homomorphism $\alpha: A \longrightarrow A'$, we have that
   $$(\alpha^* \varphi') \cup \psi \equiv \varphi' \cup (\alpha_* \psi) \in \text{XExt}^{n+m}(G, B).$$

2. Given a $G$-module homomorphism $\beta: B \longrightarrow B'$, we have that
   $$(\beta_* \varphi) \cup \psi \equiv \beta_* (\varphi \cup \psi) \in \text{XExt}^{n+m}(G, B').$$

3. Given a group homomorphism $\tau: G' \longrightarrow G$, we have that
   $$\varphi \cup (\tau^* \psi) \equiv \tau^* (\varphi \cup \psi) \in \text{XExt}^{n+m}(G', B).$$

Proof. The proofs of 2 and 3 are straightforward. For 1, follow the proof of the analogous result for Yoneda extensions mutatis mutandis (compare [16, Proposition III.5.2]).

Proposition 3.4. Let $G$ be a finite group, let $A$ and $B$ be $G$-modules and let $n, m \geq 1$ be integers. Then, the Yoneda product induces a well-defined bilinear pairing

$$\text{YExt}^n(A, B) \otimes \text{XExt}^m(G, A) \longrightarrow \text{XExt}^{n+m}(G, B).$$
Proof. Let \( \varphi, \varphi' \in \text{YExt}^n(A, B) \) and \( \psi, \psi' \in \text{XExt}^n(G, A) \). On the one hand, using that \((\Delta_A)_* \psi \equiv (\Delta_G)^* (\psi \times \psi)\), we have that
\[
(\varphi + \varphi') \cup \psi \equiv [(\nabla_B)_*(\Delta_A)^* (\varphi \times \varphi')] \cup \psi \\
\equiv (\nabla_B)_*(\Delta_G)^* [(\varphi \times \varphi')] \cup (\psi \times \psi)] \\
\equiv (\nabla_B)_*(\Delta_G)^* [(\varphi \cup \psi) \times (\varphi' \cup \psi)] \\
\equiv \varphi \cup \psi + \varphi' \cup \psi.
\]
On the other hand, we have that
\[
\varphi \cup (\psi + \psi') \equiv \varphi \cup [(\nabla_A)_*(\Delta_G)^* (\psi \times \psi')] \\
\equiv (\nabla_B)_*(\Delta_G)^* [(\varphi \times \varphi) \cup (\psi \times \psi')] \\
\equiv (\nabla_B)_*(\Delta_G)^* [(\varphi \cup \psi) \times (\varphi \cup \psi')] \\
\equiv \varphi \cup \psi + \varphi \cup \psi'.
\]
\[\square\]

3.2. Yoneda and cup products coincide. In order to show that the Yoneda product of Yoneda extensions with crossed extensions coincides with the usual cup product, we will follow a construction by B. Conrad [5], giving an explicit correspondence between crossed extensions and Yoneda extensions.

Let \( G \) be a finite group and let \( A \) be a \( G \)-module. Let \( \psi \in \text{XExt}^n(G, A) \) be a class represented by a crossed \( n \)-fold extension
\[
0 \longrightarrow A \overset{\rho_n}{\longrightarrow} M_n \longrightarrow \cdots \longrightarrow M_2 \overset{\rho_1}{\longrightarrow} M_1 \overset{\rho_0}{\longrightarrow} G \longrightarrow 1,
\]
with \( M_2 \) abelian (such a representative always exists, see [11, Proposition 2.7]). Consider the \( G \)-module \( \text{Im} \rho_1 \leq M_1 \). Then, we have an extension \( \psi_0 \in \text{XExt}^1(G, \text{Im} \rho_1) \) of the form
\[
\psi_0 : 0 \longrightarrow \text{Im} \rho_1 \longrightarrow M_1 \longrightarrow G \longrightarrow 1.
\]
Now, we can embed \( \text{Im} \rho_1 \) into an injective \( G \)-module \( I \). As \( I \) is injective, we have that \( \text{XExt}^1(G, I) \cong H^2(G, I) = 0 \), and so the pushout of \( \psi_0 \) via the embedding of \( \text{Im} \rho_1 \) into \( I \) splits, i.e., there is a group homomorphism \( \Phi : M_1 \longrightarrow G \rtimes I \) such that the following diagram commutes:
\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Im} \rho_1 \\
\downarrow & & \downarrow \Phi \\
0 & \longrightarrow & I \\
\end{array}
\begin{array}{ccc}
& & G \\
\longrightarrow & G \rtimes I & \longrightarrow G \\
& & 1 \end{array}
\]
We can find a group homomorphism \( \nu : M_1 \longrightarrow G \) and a map \( \chi : M_1 \longrightarrow I \) that for every \( x, y \in M_1 \) satisfies
\[
\chi(xy) = \chi(x)^{\nu(y)} \chi(y),
\]
(3)
such that for every $x, y \in M_1$ we can write
$$\Phi(x) = (\nu(x), \chi(x)).$$

Moreover, if we denote by $\pi: I \rightarrow I/\text{Im}\rho_1$ the canonical projection, there is a unique map $\tau: G \rightarrow I/\text{Im}\rho_1$ such that $\tau \circ \nu = \pi \circ \chi$. Furthermore, because $\chi$ satisfies (3) and $\nu = \rho_0$ is surjective, we have that for every $g, h \in G$,
$$\tau(gh) = \tau(g)^h + \tau(h),$$
and so $\tau$ is a 1-cocycle. Hence, $\tau$ can be represented as a cohomology class in $H^1(G, I/\text{Im}\rho_1) \cong \text{YExt}^1(R, I/\text{Im}\rho_1)$ by a Yoneda extension of the form
$$0 \rightarrow I/\text{Im}\rho_1 \rightarrow E_\tau \rightarrow R \rightarrow 0.$$

**Remark 3.5.** The choices of the $G$-module $I$ and the cocycle $\tau$, and consequently $E_\tau$, only depend on $\text{Im}\rho_1 \leq M_1$.

Finally, we can construct the element $\Upsilon(\psi) \in \text{YExt}^{n+1}(R, A)$ given by the Yoneda extension
$$0 \rightarrow A \rightarrow M_n \rightarrow \cdots \rightarrow M_2 \rightarrow I \rightarrow E_\tau \rightarrow R \rightarrow 0.$$ This construction gives rise to a group isomorphism
$$\Upsilon: \text{XExt}^n(G, A) \rightarrow \text{YExt}^{n+1}(R, A).$$

**Proposition 3.6.** Let $G$ be a finite group and let $n, m \geq 1$ be integer numbers. Then, the Yoneda product
$$\text{YExt}^n(A, B) \otimes \text{XExt}^m(G, A) \rightarrow \text{XExt}^{n+m}(G, B)$$
coincides with the Yoneda product
$$\text{YExt}^n(A, B) \otimes \text{YExt}^{n+1}(R, A) \rightarrow \text{XExt}^{n+m+1}(R, B).$$
In particular, if $A = B = R$, the above product coincides with the cup product
$$\cup: H^n(G; R) \otimes H^{n+1}(G; R) \rightarrow H^{n+m+1}(G; R).$$

**Proof.** Let $\varphi \in \text{YExt}^n(A, B)$ be a class represented by an extension
$$0 \rightarrow B \rightarrow N_n \rightarrow \cdots \rightarrow N_1 \rightarrow A \rightarrow 0,$$
and let $\psi \in \text{XExt}^m(G, A)$ be a class represented by an extension
$$0 \rightarrow A \rightarrow M_m \rightarrow \cdots \rightarrow M_2 \rightarrow M_1 \rightarrow G \rightarrow 1,$$
with $M_2$ abelian. We need to prove that $\Upsilon(\varphi \cup \psi) = \varphi \cup \Upsilon(\psi)$.

By (4), for $m > 1$, the extension $\Upsilon(\psi) \in \text{YExt}^{m+1}(R, A)$ is of the form
$$0 \rightarrow A \rightarrow M_m \rightarrow \cdots \rightarrow M_2 \rightarrow I \rightarrow E_\tau \rightarrow R \rightarrow 0,$$
and \( \varphi \cup \psi \in \text{XExt}^{n+m}(G, A) \) is represented by the crossed \((n + m)\)-fold extension

\[
0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G 
\]

By Remark 3.5, we can use the same \( I \) and \( \tau \) in the construction of \( \Upsilon(\psi) \). Therefore, \( \Upsilon(\psi) \in \text{YExt}^{n+m+1}(G, A) \) is represented by

\[
0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_2 \longrightarrow I \longrightarrow E_r \longrightarrow R 
\]

which coincides with \( \varphi \cup \Upsilon(\psi) \).

For \( m = 1 \), we have that \( \psi \in \text{XExt}^1(G, A) \) is represented by a crossed 1-fold extension of the form

\[
0 \longrightarrow A \longrightarrow \gamma_1 \longrightarrow M_1 \longrightarrow G \longrightarrow 1.
\]

Then, \( \varphi \cup \psi \) is given by the crossed \((n + 1)\)-fold extension

\[
0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow \gamma_1 \longrightarrow M_1 \longrightarrow G \longrightarrow 1,
\]

where \( \gamma_1 = \rho_1 \circ \mu_0 \). Now, we have that \( \text{Im} \gamma_1 = \text{Im} \rho_1 \), and so we can once again use the same \( I \) and \( \tau \) in the construction of both \( \Upsilon(\psi) \) and \( \Upsilon(\varphi \cup \psi) \). Therefore, both \( \varphi \cup \Upsilon(\psi) \) and \( \Upsilon(\varphi \cup \psi) \) are given by the same extension

\[
0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow I \longrightarrow E_r \longrightarrow R \longrightarrow 0.
\]

Finally, if \( A = B = R \) then the Yoneda product of Yoneda extensions coincides with the cup product of cohomology classes.

\[ \square \]

4. Finite \( p \)-groups of depth one mod-\( p \) cohomology

Let \( p \) be an odd prime. For each integer \( r \) with \( 1 < r < p - 1 \), the finite \( p \)-group \( G_r \), described in (1), is generated by the elements \( \sigma, a_1, \ldots, a_r \) satisfying the following relations:

- \( \sigma^p = a_i^p = [a_i, a_j] = [a_r, \sigma] = 1 \), for \( i = 1, \ldots, r \) and \( j = 1, \ldots, r - 1 \),
- \( [a_j, \sigma] = a_{j+1} \) for \( j = 1, \ldots, r - 1 \).

The aim of this section is to prove Theorem A. Consider the elementary abelian \( p \)-group \( E = \langle \sigma, a_r \rangle \) with centralizer \( C_{G_r}(E) \) in \( G_r \) equal to \( E \). By Proposition 2.2, we have that

\[
1 = \text{rk}_p(\text{Z}(G_r)) \leq \text{depth} \text{H}^*(G_r; \mathbb{F}_p) \leq \text{depth} \text{H}^*(C_{G_r}(E); \mathbb{F}_p) = 2.
\]

To show the result, we construct a non-trivial mod-\( p \) cohomology class of \( G_r \) that restricts trivially to the mod-\( p \) cohomology of the centralizers of all rank 2 elementary abelian subgroups of \( G_r \). Then, \( \omega_d(G_r) = 1 \) and Theorem 2.4 yields that \( \text{depth} \text{H}^*(G_r; \mathbb{F}_p) = 1. \)
4.1. Construction. We follow the assumptions in Notation and, additionally, suppose that $p > 3$. In this section, we construct for each integer $r$ with $1 < r < p - 1$, a cohomology class $\theta_r \in H^3(G_r; \mathbb{F}_p)$ that is a cup product of a Yoneda 1-fold extension and a crossed 2-fold extension.

We start by defining a cohomology class $\sigma^* \in H^1(G_r; \mathbb{F}_p) = \text{Hom}(G_r, \mathbb{F}_p)$. To that aim, for each $r$, consider the homomorphism $\sigma^*: G_r \rightarrow \mathbb{F}_p$ satisfying $\sigma^*(\sigma) = 1$, $\sigma^*(a_1) = \cdots = \sigma^*(a_r) = 0$.

The class $\sigma^*$ can be represented by the Yoneda extension

$$1 \longrightarrow C_p = \langle a_{r+2} \rangle \longrightarrow C_p \times C_p \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow 1,$$

where the action of $G_r$ on $C_p \times C_p = \langle a_{r+1}, a_{r+2} \rangle$ is described by

$$g \in G_r \mapsto a_{r+1}^g = a_{r+1}a_r^{\sigma^*(g)}, \quad a_{r+2}^g = a_{r+2}.$$

We continue by defining a crossed 2-fold extension $\eta_r \in H^2(G_r; \mathbb{F}_p)$ as follows. For $r > 1$, let

$$\lambda_r: T_0/T_{r+1} \times T_0/T_{r+1} \longrightarrow T_0/T_{r+1}$$

be the alternating bilinear map satisfying $\lambda_r(a_{r-1}, a_r) = a_{r+1}$. Now, define $(T_0/T_{r+1}) \lambda_r$ to be the group with underlying set $T_0/T_{r+1}$ and with group operation given by

$$x, y \in T_0/T_{r+1} \mapsto x \cdot \lambda_r y = x \lambda_r(x, y)^{1/2}.$$

Finally, define the $p$-group $\widehat{G}_r = C_p \rtimes (T_0/T_{r+1}) \lambda_r$ of size $|\widehat{G}_r| = p^{r+2}$ and exponent $p$. Let $\eta_r \in H^2(G_r, \mathbb{F}_p)$ be the cohomology class represented by the crossed 2-fold extension

(6) $$1 \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow \widehat{G}_r \longrightarrow G_r \longrightarrow 1.$$

Then, we define the cohomology class $\theta_r = \sigma^* \cup \eta_r \in H^3(G_r; \mathbb{F}_p)$, which is represented by the crossed 3-fold extension

(7) $$1 \longrightarrow C_p \longrightarrow C_p \times C_p \longrightarrow \widehat{G}_r \longrightarrow G_r \longrightarrow 1.$$

4.2. Non-triviality. In the present section we will show that the cohomology class $\theta_r$ described in (7) is non-trivial.

Proposition 4.1. Let $p > 3$ be a prime number. For each integer $r$ with $1 < r < p - 1$, let $\theta_r \in H^3(G_r; \mathbb{F}_p)$ be the cohomology class constructed in (7). Then, $\theta_r \neq 0$. 
Proof. Assume by contradiction that $\theta_r = 0$. Then, by Proposition 2.14 there exists a group $X$ such that the following diagram commutes:

\[
\begin{array}{c}
\text{1} \\
\downarrow \mu \\
C_p \times C_p \\
\downarrow \mu \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \nu \\
\tilde{G}_r \\
\downarrow \nu \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \phi \\
G_r \\
\downarrow \phi \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \psi \\
\tilde{G}_r \\
\downarrow \psi \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \phi \\
G_r \\
\downarrow \phi \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \phi \\
\tilde{G}_r \\
\downarrow \phi \\
\text{1} \\
\end{array}
\]

We have that $X = \langle \bar{\sigma}, \bar{a}_1, \ldots, \bar{a}_{r+2} \rangle$ with elements $\bar{\sigma}, \bar{a}_1, \ldots, \bar{a}_{r+1}, \bar{a}_{r+2} \in X$ that satisfy

$$\bar{a}_{r+2} = \mu(a_{r+2}), \nu(\bar{\sigma}) = \sigma \quad \text{and} \quad \nu(\bar{a}_i) = a_i \text{ for all } i = 1, \ldots, r+1,$$

and we have that $Z(X) = \langle \bar{a}_{r+2} \rangle$ and $\gamma_r(X) = \langle \bar{a}_r, \bar{a}_{r+1}, \bar{a}_{r+2} \rangle$. Consider the normal subgroup

$$Y = \langle \bar{a}_{r-1}, \bar{a}_{r}, \bar{a}_{r+1}, \bar{a}_{r+2} \rangle \triangleleft X,$$

which fits into the following commutative diagram:

\[
\begin{array}{c}
\text{1} \\
\downarrow \\
\langle a_{r+1}, a_{r+2} \rangle \\
\downarrow \\
\langle a_{r-1}, a_r, a_{r+1} \rangle \\
\downarrow \\
\langle a_{r+2} \rangle \\
\downarrow \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \\
\langle a_{r+1}, a_{r+2} \rangle \\
\downarrow \\
\langle a_{r-1}, a_r, a_{r+1} \rangle \\
\downarrow \\
\langle a_{r+2} \rangle \\
\downarrow \\
\text{1} \\
\end{array} \quad \begin{array}{c}
\text{1} \\
\downarrow \\
\langle a_{r+1}, a_{r+2} \rangle \\
\downarrow \\
\langle a_{r-1}, a_r, a_{r+1} \rangle \\
\downarrow \\
\langle a_{r+2} \rangle \\
\downarrow \\
\text{1} \\
\end{array}
\]

Then, we have that $Z(Y) = \langle \bar{a}_{r+1}, \bar{a}_{r+2} \rangle$, and moreover,

$$[\bar{\sigma}, Y, \gamma_r(X)] = [\gamma_r(X), \gamma_r(X)] = 1 \quad \text{and} \quad [\gamma_r(X), \bar{\sigma}, Y] = [Z(Y), Y] = 1.$$

Therefore, the three subgroup lemma (see [20, 5.1.10]) leads us to the conclusion that $[Y, \gamma_r(X), \bar{\sigma}] = 1$. Nevertheless, a direct computation shows
that
\[ [Y, \gamma_r(X), \tilde{\sigma}] = [Z(Y), \sigma] = Z(X) \neq 1, \]
which gives a contradiction. Hence, \( \theta_r \neq 0 \).

4.3. Trivial restriction. In this section we show that for every elementary abelian subgroup \( E \) of \( G_r \) of \( p \)-rank \( \text{rk}_p E = 2 \), the image of \( \theta_r \) via the restriction map,
\[ \text{res}_{C_{Gr}}^{E} : H^3(G_r; \mathbb{F}_p) \rightarrow H^3(C_{Gr}(E); \mathbb{F}_p), \]
is trivial, i.e., \( \text{res}_{C_{Gr}}^{E} \theta_r = 0 \). This will imply that the cohomology class \( \theta_r \) is not detected by \( H_2(G_r) \), and so \( \omega_d(G_r) = 1 \).

**Proposition 4.2.** Let \( p > 3 \) be a prime number and let \( r \) be an integer such that \( 1 < r < p - 1 \). Let \( E \leq G_r \) be an elementary abelian subgroup with \( \text{rk}_p E = 2 \). Then, \( \text{res}_{C_{Gr}}^{E} \theta_r = 0 \).

**Proof.** There are two types of elementary abelian subgroups \( E \leq G_r \), either \( E \leq \langle a_1, \ldots, a_r \rangle \) or \( E \nleq \langle a_1, \ldots, a_r \rangle \). Assume first that \( E \leq \langle a_1, \ldots, a_r \rangle \). Then, \( C_{Gr}(E) = \langle a_1, \ldots, a_r \rangle \) and we have that \( \text{res}_{C_{Gr}}^{E} \sigma^* = 0 \). Therefore,
\[ \text{res}_{C_{Gr}}^{E} \theta_r = (\text{res}_{C_{Gr}}^{E} \sigma^*) \cup (\text{res}_{C_{Gr}}^{E} \eta_r) = 0. \]

Assume now that \( E \nleq \langle a_1, \ldots, a_r \rangle \). Then, \( E = \langle b, a_r \rangle \) with \( b = \sigma x \) for some \( x \in \langle a_1, \ldots, a_{r-1} \rangle \), and \( C_{Gr}(E) = E \). Moreover, \( \text{res}_{C_{Gr}}^{E} \eta_r \) is represented by the extension that is obtained by taking the pullback of \( \eta_r \) via the inclusion \( E \hookrightarrow G_r \), as illustrated in the following diagram
\[
\begin{array}{c}
1 & \longrightarrow & \langle a_{r+1} \rangle & \longrightarrow & \hat{E} = \langle b, a_r, a_{r+1} \rangle & \longrightarrow & E & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \langle a_{r+1} \rangle & \longrightarrow & \hat{G}_r & \longrightarrow & G_r & \longrightarrow & 1.
\end{array}
\]
Observe that \( \hat{E} \cong C_p \ltimes (C_p \times C_p) \) is the extraspecial group of order \( p^3 \) and exponent \( p \). Hence, \( \text{res}_{C_{Gr}}^{E} \eta_r \) is represented by the extension
\[ (8) \quad 1 \rightarrow C_p = \langle a_{r+1} \rangle \rightarrow \hat{E} = C_p \ltimes (C_p \times C_p) \rightarrow C_p \times C_p = \langle b, a_r \rangle \rightarrow 1. \]
It can be readily checked (following the construction in [2, Section IV.3]) that the extension class of (8) coincides with the cup-product \( b^* \cup a_r^* \), and so \( \text{res}_{C_{Gr}}^{E} \eta_r = b^* \cup a_r^* \). Consequently,
\[ \text{res}_{C_{Gr}}^{E} \theta_r = (\text{res}_{C_{Gr}}^{E} \sigma^*) \cup b^* \cup a_r^* = 0, \]
as the product of any three elements of degree one is trivial in \( H^3(E; \mathbb{F}_p) \). \( \square \)

**Proof of Theorem A.** In (5), we obtained that \( 1 \leq \text{depth} H^*(G_r; \mathbb{F}_p) \leq 2 \). In Proposition 4.1, we constructed a cohomology class \( \theta_r \in H^*(G_r; \mathbb{F}_p) \) that is non-trivial and that, for every elementary abelian subgroup \( E \leq G_r \) of
rank 2, satisfies that $\text{res}^G_{C_{G_r}(E)}(\theta_r) = 0$ (see Proposition 4.2). This implies that $\omega_d(G) = \text{rk}_p Z(G) = 1$. Then, by Theorem 2.4, we conclude that depth $H^*(G_r; \mathbb{F}_p) = \text{rk}_p Z(G) = 1$. □

5. Remarks and further work

Let $p$ be an odd prime number and let $r$ be an integer with $r \geq p - 1$. Consider the finite $p$-groups $G_r = C_p \rtimes T_0/T_r$ defined in (1). For each prime $p$, if $r = p - 1$, then $G_r$ has size $p^{r+1}$, has exponent $p$ and is of maximal nilpotency class; while if $r > p - 1$, then $G$ has size $p^{r+1}$ and exponent bigger than $p$. In particular, for the $p = 3$ and $r = 2$ case, $G_2$ is the extraspecial 3-group of order 27 and exponent 3, and it is known that the depth of its mod-3 cohomology ring is 2 (compare [13] and [17]). We believe that this phenomena will occur with more generality; namely, for every prime number $p \geq 3$ and $r \geq p - 1$, the following equality will hold depth $H^*(G_r; \mathbb{F}_p) = 2$. For these groups, if we mimic the construction of the mod-$p$ cohomology class $\theta_r$ in Section 4.1, it is no longer true that its restriction in the mod-$p$ cohomology of the centralizer of all elementary abelian subgroups of rank 2 vanishes. We propose the following conjecture.

Conjecture 5.1. Let $p$ be an odd prime, let $r \geq p - 1$ be an integer, and let $G_r = C_p \rtimes T_0/T_r$ be as in (1). Then $H^*(G_r; \mathbb{F}_p)$ has depth 2.

The above conjecture is known to be true for the particular cases where $p = 3$ and $r = 2$ or $r = 3$. In these two cases the mod-$p$ cohomology rings have been calculated using computational sources (see [14]). Another argument supporting the conjecture is that for a fixed prime $p$ and $r \geq p - 1$, the groups $G_r$ have isomorphic mod-$p$ cohomology groups; not as rings, but as $\mathbb{F}_p$-modules (see [8]). This last isomorphism comes from a universal object described in the category of cochain complexes together with a quasi-isomorphism that induces an isomorphism at the level of spectral sequences.

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