Abstract

We compute the $p$--adic Abel-Jacobi map of the product of a Hilbert modular surface and a modular curve at a null-homologous (modified) embedding of the modular curve in this product, evaluated on differentials associated to a Hilbert cuspidal form $f$ of weight $(2, 2)$ and a cuspidal form of weight 2. We generalize this computation to suitable null-homological cycles in the fibre products of the universal families on the surface and the curve, evaluated at differentials associated to $f$ and $g$ of higher weights.

We express the values of the $p$--adic Abel-Jacobi map at these weights in terms of a $p$--adic $L$--function associated to a Hida family of Hilbert modular forms and a Hida family of cuspidal forms. Our function is a Hilbert modular analogue of the $p$--adic $L$--function introduced in [17].

1 Introduction

The present work starts a series devoted to studying null-homologous cycles on the product of a modular curve and a Hilbert modular surface, and higher dimensional generalizations of this, with a view to producing distinguished cohomology classes in the Bloch-Kato Selmer group of the representation associated to an elliptic curve $E/Q$ twisted by an Artin representation associated to a Hilbert modular form, and to relate these classes with the dimension of the Bloch-Kato Selmer group. Following the philosophy of [17], of which our program is a prolongation to the Hilbert modular setting, our aim is to obtain a $p$--adic Gross-Zagier type formula relating the image under the $p$--adic Abel-Jacobi map of certain null-homologous cycles with the special values of a $p$--adic $L$--function arising out of a $p$--adic family of Hilbert cuspidal forms passing by a given one $f$ of weight $(2, 2)$ and a $p$--adic family of cuspidal forms passing...
by a rational one \( g \) of weight 2, for a prime \( p \), generic in the sense we will specify.

To state the main result, we need some preliminary notions and notations. Let \( R \) be a real quadratic field and fix an inclusion \( \text{inc} : R \hookrightarrow \mathbb{R} \), and assume, just for convenience, that \( R \) has a unit of norm \(-1\), so that \( \mathfrak{u}^+/(\mathfrak{u}^+)^2 \) is trivial (otherwise, the content of this article holds after replacing our cohomologies by their invariant part under the natural action of this two-elements group on the moduli schemes we consider). Let \( \mathfrak{o} \) be the ring of integers of \( R \) and choose a narrow class of it and \( \mathfrak{a} \subseteq \mathfrak{o} \) a primitive integral ideal representing it. Assume that \( N_2 \geq 4 \), the discriminant \( d_R = Nm(\mathfrak{o}_R) \), and \( N_1 = Nm(\mathfrak{a}) \) are mutually coprime, and write \( N = d_R N_1 N_2 \). Consider a ring \( R \supseteq \mathfrak{o} \) in which \( N \) is invertible. Denote by \( Y_R(N_2) \) the Hilbert modular surface of level \( N_2 \), chosen toroidal desingularization of the Satake minimal compactification \( \mathcal{M}_{R}(\mathfrak{a}, N_2) \) of the geometrically normal and irreducible surface \( \mathcal{M}_R(\mathfrak{a}, N_2) \) over \( R \) which represents the functor assigning to schemes \( S \) over \( R \) the set of abelian surfaces over \( R \) with \( \mathfrak{o} \)–real multiplication, ordered \( \mathfrak{a} \)–polarization and \( N_2 \)–level structure (see the definitions in section 2). The Rapoport rank one \( \mathfrak{o} \otimes \mathbb{Z} \mathcal{O}_{\mathcal{M}_R(N_2)} \) –sheaf \( \mathcal{R}_{\mathcal{M}_R(N_2)} := pr_{\mathcal{M}_R(N_2)*}\mathcal{O}_{\mathcal{M}_R(N_2)}^{A_{\mathcal{M}_R(N_2)}}/\mathcal{M}_R(N_2) \) of relative differentials of the universal object \( A_{\mathcal{M}_R(N_2)}^{\mathfrak{a}} \) is locally free, because \( d_R \) is invertible in \( R \). The inclusion and the conjugate inclusion of \( \mathfrak{o} \) in \( R \) associate to \( \mathcal{R}_{\mathcal{M}_R(N_2)} \) two line bundles whose \( k, k' \) powers are the modular line bundles \( L_{\mathcal{M}_R(N_2)}^{(k,k')} \), whose sections are the \( \mathfrak{a} \)-Hilbert modular forms of weight \( (k, k') \) over \( R \). These extend to line bundles \( L_{Y_R(N_2)}^{(k,k')} \) on the whole \( Y_R(N_2) \) because the Rapoport sheaf \( \mathcal{R}_{\mathcal{M}_R(N_2)} \) extends to a rank one locally free \( \mathfrak{o} \otimes \mathbb{Z} \mathcal{O}_{Y_R(N_2)} \) –sheaf \( \mathcal{R}_{Y_R(N_2)} \). Hilbert modular forms of parallel weight \( (k, k') \) can be defined equivalently as sections of these line bundles on \( \mathcal{M}_R(\mathfrak{a}, N_2) \) or on the whole \( Y_R(N_2) \) (Koecher principle), and those vanishing on the transform divisor \( D^c \) of the cusps \( \mathcal{M}_R(N_2) = \mathcal{M}_R(N_2) \) are called cuspidal. The complex manifold \( \mathcal{M}_C(\mathfrak{a}, N_2)(\mathbb{C}) \) is the quotient of the Poincaré square half-plane \( \mathfrak{H}^2 \) by the Hilbert modular group \( \Gamma_1(\mathfrak{a}, N_2) \) consisting of matrices of \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) whose determinant belongs to \( \mathfrak{u}^+ \). Since we assume \( \mathfrak{u}^+/(\mathfrak{u}^+)^2 \) trivial, it is also the quotient by the action of the subgroup \( \Gamma_1^1(\mathfrak{a}, N_2) \subseteq \Gamma_1(\mathfrak{a}, N_2) \) of unimodular matrices.

The bundle of logarithmic differentials \( \Omega_{Y_R(N_2)}(\log D^c) \approx L_{Y_R(N_2)}^{(2,0)} \oplus L_{Y_R(N_2)}^{(0,2)} \) has determinant \( \omega_{Y_R(N_2)}(D^c) \approx L_{Y_R(N_2)}^{(2,2)} \) so that a cuspidal \( \mathfrak{a} \)-Hilbert modular form \( f \) of weight \( (2, 2) \) corresponds bijectively to a 2-differential \( \omega_f \) on \( Y_R(N_2) \) (also denoted \( \omega(f) \) if the expression of \( f \) is long), this generalizing to an \( L_{Y_R(N_2)}^{(k,k')} \)-valued \( \log D^c \)-differential \( \omega_f \) if \( f \)
has weight \((k,k') \geq (2,2)\).

There is a generically injective embedding \(j_{X_R(N)} : X_R(N) \rightarrow Y_R(N_2)\) of the modular curve \(X_R(N)\) into \(Y_R(N_2)\) for the congruence group \(\Gamma(N) = \Gamma_0(d_RN_1) \cap \Gamma_1(N_2)\) after compactification, desingularization and base change to \(Spec R\), and the restriction \(j_{X_R(N)}^* L_{Y_R(N_2)}^{(k,k')}\) is \(L_{0,R}^{k+k'}\), for the modular line bundle \(L_0\) on \(X(N)\) (denoted \(\omega\) in the literature) such that \(L_0^2 \approx \omega_{X(N)}(\text{cusps})\).

Let \(K \supseteq \mathfrak{R}\) be a number field having ring of integers \(R \supseteq \mathfrak{o}\). Choose \(p\) splitting in \(K\) so that it splits in \(\mathfrak{R}\) as \(p = \pi \pi'\). Then \(p\) does not divide the discriminant \(d_R\), and we choose it prime with \(N = d_RN_1N_2\). The embedding \(\mathfrak{R} \hookrightarrow \mathfrak{R}_\pi \approx \mathbb{Q}_p\) extends to an embedding \(K \hookrightarrow \mathbb{Q}_p\) which restricts to \(R_m \approx \mathbb{Z}_p\) for a maximal ideal \(m\) of \(R\) lying on \(\pi\), so that \(\mathbb{Z} \subseteq \mathfrak{o} \subseteq R\) induces isomorphisms \(\mathbb{Z}_p \approx \mathfrak{o}_\pi \approx R_m\) and \(\mathbb{Q}_p \approx \mathfrak{R}_\pi \approx K_m\); and we analogously obtain \(\mathbb{Z}_p \approx \mathfrak{o}_{\pi'} \approx R_{m'}\) and \(\mathbb{Q}_p \approx \mathfrak{R}_{\pi'} \approx K_{m'}\). The locus of points in \(\mathcal{M}_{\mathbb{Q}_p}(N_2)\) corresponding to Abelian surfaces with ordinary \(\mathbb{F}_p\)-specialization is \(\mathcal{A} = \mathcal{M}_\mathbb{Q}_p(N_2)\) for the mod. \(p\)-nonvanishing locus \(\mathcal{A}\) of the Hasse section of the Hasse modular line bundle on \(Y_{\mathbb{Q}_p}(N_2)\).

This is an open set of the associated \(\mathbb{Q}_p\)-rigid space and has as base of strict neighborhoods the mod. \(p^n\)-nonvanishing loci \(\mathcal{W}_{p^n}\) of the Hasse section. The complement of the Hasse divisor \(D^h\) where this section vanishes is \(Y_{\mathbb{Q}_p}(N_2) = \bigcup_{\epsilon > 0} \mathcal{W}_\epsilon\), a Zariski open set which generalizes to the Hilbert-modular context the open set \(X'_{\mathbb{Q}_p}(N) = j_{X_R(N)}^{-1}(Y_{\mathbb{Q}_p}(N_2))\).

The \(p\)-adic Hilbert modular forms defined as rigid sections \(f\) of the restricted modular line bundles \(L_\mathcal{A}^{(k,k')}\) are also \(p\)-adic Hilbert modular forms in the sense of Katz in [31] so they have a \(q\)-expansion at the nonramified cusps, which at the one called "standard cusp" is denoted \(f(q) := \sum_{\nu \in \mathbb{Z}_p} a_{\nu} q^\nu\), with \(a_{\nu} \in \mathbb{Z}_p\) and \(a_0 = 0\) null when \(f\) is cuspidal, being \(f \mapsto f(q)\) an injective map; those extending to some \(\mathcal{W}_\epsilon\) are called cooverconvergent. The Frobenius morphism \(\phi\) on the ordinary locus \(Y_{\mathbb{F}_p}(N_2) \subseteq Y_{\mathbb{Q}_p}(N_2)\) has a natural cover \(\phi : \phi^* L_{Y_{\mathbb{Q}_p}}^{(k,k')} \rightarrow L_{Y_{\mathbb{F}_p}}^{(k,k')}\), and a lifting \(\phi : \mathcal{A} \rightarrow \mathcal{A}\) with natural cover \(\phi : \phi^* L_{\mathcal{A}}^{(k,k')} \rightarrow L_{\mathcal{A}}^{(k,k')}\) so it acts on \(p\)-adic \(a\)-Hilbert modular forms \(f\), being \(\sum_{\nu \in \mathbb{Z}_p} a_{\nu} q^{\nu}\) the \(q\)-expansion of the transform; by a recent result of Conrad [13], this lifting extends to a morphism between open sets \(\mathcal{W}_\epsilon\), so that \(\phi\) preserves overconvergence.

Let \(\mathfrak{o}\) be a rational point of \(X_{\mathbb{Q}}(N)\) and let \(X_{\mathbb{Q}}(N)_{\mathfrak{o}}\) be the modified
operators as defined for instance in \cite{40} of their complex points. Let \( \omega \) weight \( 2 \), modular group \( \Gamma(2) \) and assume that the prime \( p \) chosen ordinary at \( \pi \) ordinary at \( \pi' \) and \( \omega \) positive "slope" \( \sigma = \text{ord}_m(\omega) \) and \( \sigma' = \text{ord}_{m'}(\omega') \).

Assume we are given a rational cuspidal form \( g \) of weight 2 for the modular group \( \Gamma(N) \), normalized eigenform for the Hecke operators \( T_n \) and assume that the prime \( p \) is chosen ordinary for \( g \) and that \( f \) is non-

ordinary at \( \pi \) and \( \pi' \), i.e. it has Hecke eigenvalue \( a_\pi \) being non-unit of \( R_m = \mathbb{Z}_p \) and \( a_{\pi'} \) being non-unit of \( R_{m'} = \mathbb{Z}_p \), i.e. having positive "slope" \( \sigma = \text{ord}_m(a_\pi) \) and \( \sigma' = \text{ord}_{m'}(a_{\pi'}) \).

Take a model of the above curve, surface and null-homologous cycle over \( \mathbb{Q}_p \), as smooth and projective over \( \mathbb{Z}_p \) (and fix an inclusion \( \mathbb{Q}_p \to \mathbb{C} \) once and for all). We can use then the Besser theory developed in \cite{7}, to compute in section 3 the \( p \)-adic Abel-Jacobi map of \( X_{\mathbb{Q}_p}(N) \times Y_{\mathbb{Q}_p}(N_2) \) at the null-homologous cycle \( X_{\mathbb{Q}_p}(N) \) evaluated on the de Rham class of \( \omega_\phi \otimes \eta_g^{-r} \) for the differential \( \omega_\phi \) on \( Y_{\mathbb{Q}_p}(N_2) \) attached to \( f \) and the unique "unit root" differential \( \eta_g^{-r} \) on \( X_{\mathbb{Q}_p}(N) \) having cup product \( \langle \omega_\phi, \eta_g^{-r} \rangle = 1 \). Denote, just as in \cite{17} \( e_g \) the projection to the "ordinary subspace" of the de Rham cohomology space \( H^1_{dR}(X_{\mathbb{Q}_p}(N)/\mathbb{Q}_p) \) where \( \phi \) acts with eigenvalue \( p \), and \( e_g \) the projection to the 2-dimensional isotypic component of \( \omega_g \), this lying in the subspace \( H^1_{dR}(X_{\mathbb{Q}_p}(N)/\mathbb{Q}_p) \).

The main result in section 3 is

\[ AJ_p(X_{\mathbb{Q}_p}(N)_{\omega})(\omega_\phi \otimes \eta_g^{-r}) = e_g e_{\text{ord}} Q_\phi(\phi)^{-1} f_X_{\mathbb{Q}_p}(N) g_\phi^\sharp, \eta_g^{-r} \quad (1) \]

where \( Q_\phi(x) \) is the characteristic polynomial of \( \phi \) acting on the isotypic component \( H^2(Y_{\mathbb{Q}_p}(N_2)/\mathbb{Q}_p)(f) \) of \( \omega_\phi \) in \( H^2(Y_{\mathbb{Q}_p}(N_2)/\mathbb{Q}_p) \), and \( g_\phi^\sharp \) is a primitive of the overconvergent 2-differential \( Q_\phi(\phi)\omega_\phi \) whose existence we prove in section 3. Under the above non-ordinary conditions, there is in fact (as recalled in sec. 4) by \cite{36}, a unique primitive \( g_\phi^\sharp \) of type \((1,0)\), i.e. overconvergent section of the first direct summand of the decomposition \( \Omega_{\log(D)}(\mathbb{Q}_p) \approx L^{(2,0)}(\mathbb{Q}_p) \oplus L^{(0,2)}(\mathbb{Q}_p) \).

We give the proof of (1), more in general, for a 1-differential \( \eta \) not necessarily ‘of the form \( \eta_g^{-r} \), but then \( e_g e_{\text{ord}} \) does not appear in the first factor of the the product (1), so we force this factor to stay in the kernel.
the Verschiebung operator $U$ with coefficients still lying in $\mathcal{O}$ as sum $f$ of $n$ factors. Computation of the residue map of $H^1_{dR}(X_{QP}(N)/\mathbb{Q}_p)$ of the residue map of $H^1_{dR}(X_{QP}^*(N)/\mathbb{Q}_p)$ by replacing $Q_f(\phi)$ by $P_f(\phi) = (1 - p^{-1}\phi)Q_f(\phi)$, as the first factor kills the target of the residue map and the product $\prod$ remains the same while replacing $Q_f(\phi)$ by any multiple polynomial.

In section 4, we generalize $\prod$ for the evaluation of a $p$–adic Abel-Jacobi at Hilbert modular forms $f$ and $g$ of higher weights $(k, k) = (2 + n, 2 + n)$ and $k_0 = 2 + n_0$ for $0 \leq n_0 = 2n - 2t$, with $t > 0$ or $t = n = n_0 = 0$ at certain null-homologous cycles $\Delta_{n,n_0}$ of higher dimension. Choose a rational prime $p$ just as above, i.e. not dividing $N$ and splitting in $K$ and in $\frak{o}$ as $p = \pi \pi'$ being ordinary for $g$ and nonordinary, i.e. of positive slope $\text{ord}_{\frak{m}}(a_{\pi})$ and $\text{ord}_{\frak{m}'}(a_{\pi'})$, for $f$. Consider the $L^{(n,n)}$-valued 2–differential $\omega_f$ of $Y(N_2)$ and the $L_0$-valued 1–differential $\omega_g$ of $X(N)$ (the definition is over $\mathbb{Q}_p$, when omitted). The product analogous to $\prod$ would need a covariant derivative for differentials valued in modular line bundles, but we have at least the Gauss-Manin covariant log $D^c$-derivative $\nabla^{GM}$ on a bundle $\mathcal{L}$ in which $\mathcal{R}$ is included with a splitting projection $sp$ on just $\mathcal{A}$. This $\mathcal{L}_n$ is the $n$–symmetric power of the four-bundle $\mathcal{L}$ of relative first de Rham cohomology which generalizes to the Hilbert modular context the direct factor $\mathcal{L}_0$ of its restriction $j^*_X(N)\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_0$ , a decomposition which follows from the fact that the universal semiabelian surface splits on $X(N)$ as fibre product of two copies of the universal generalized elliptic curve, because of our assumptions. Under our non-ordinary hypothesis, by [36], for the overconvergent ”$\pi'$–depletion” $f^{[\pi']}$ of $f$ of $q$–expansion $f^{[\pi']} (q) = \sum_{\nu \in (a + 1)^+} a_\nu q^\nu$, the associated $L^{(n,n)}$-valued 2–differential $\omega_{f^{[\pi']}}$ has a unique overconvergent $L^{2n}$–valued differential $\partial_{\mathcal{A}[\pi']}\nabla^{GM}$ of type $(1, 0)$ such that $\nabla^{GM} \partial_{\mathcal{A}[\pi']} = \omega_{f^{[\pi']}}$. The split projection in $\mathcal{A}$ of $\partial_{\mathcal{A}[\pi']}\nabla^{GM}$ provides a unique nearly overconvergent form $F^{[\pi']}_{[2+n,n]}$ of weight $(2 + n, n)$ whose associated $L^{(n,n)}$–differential $\partial_{F^{[\pi']}_{[2+n,n]}}\nabla^{GM}$ has $\omega_{f^{[\pi']}}$ as its ”split derivative” $\nabla^{GM}_{sp} := sp \circ \nabla^{GM}$. In terms of the $\theta, \theta'$ operators on $p$–adic forms acting as $\sum \nu a_\nu q^\nu$ and $\sum \nu' a_\nu q^\nu$ on $q$–expansions, this amounts to $F^{[\pi']}_{[2+n,n]} = \theta^{-1} f^{[\pi']}$, i.e. it has $q$–expansion $\sum_{\nu \in (a + 1)^+} \nu \nu^{-1} a_\nu q^\nu$ with coefficients still lying in $\mathcal{R}$ as the $\nu'$ inverted in them are units of $\frak{o}_s \approx \mathcal{R}$; and the analogous holds for $\pi$–depletion.

We can express the explicit value of the product $\prod$ by a purely formal computation which we perform in section 5. For this, we decompose $f$ as sum $f_{\nu_0} + f_{\nu_1} + f_{\nu_2} + f_{\nu_3}$ of its four stabilizations or eigenvectors of the Verschiebung operator $U_p$ on $p$–adic $a$–Hilbert modular forms ex-
pressed on \(q\)-expansions by \((U_p f) (q) = \sum_{\nu \in (a^{-1})^+} a_{\nu} q^{\nu}\) and preserving overconvergence. Each of the four root factors of the four dimensional polynomial \(Q_f (\phi)\) acts on one of these four summands as a \(p\)-depletion operator, and the other three factors become in the cup-product (\(\Pi\)) (up to summands with vanishing contribution to the final product) the multiplication by a scalar factor, which is the cause of the Euler factors \(\mathcal{E}(f, g)\) and \(\mathcal{E}_1 (g)\) appearing in the following theorem computing suitable \(p\)-adic Abel-Jacobi maps. Denoting \(\mathcal{E}^{n_0}\) the fibred power of the universal generalized elliptic curve \(\mathcal{E}\) over \(X(N)\), and \(\hat{A}^{U, n}\) the fibred power of the smooth compactification \(\hat{A}^U\) of the universal semiabelian scheme over \(Y(N_2)\), we construct in section 4 null-homologous cycles \(\Delta_{n, n_0}\) in \(\hat{A}^{U, n} \times \mathcal{E}^{n_0}\) for integers \(n, n_0\) as above. These generalize, with the help of the Scholl operators killing the nonintermediate homology of a product of generalized elliptic curves, the null-homologous cycle \(\Delta_{0, 0}\) which is the modified diagonal \(X(N)_{0}\) viewed as a cycle in \(Y(N_2) \times X(N)\).

**Theorem 1** With \(\mathfrak{A}, \mathfrak{O}, R, \mathfrak{O}, \mathfrak{a}\), and integers \(N = d_{\mathfrak{A}} N_1 N_2\) as previously introduced, let \(f\) be an \(\mathfrak{a}\)-Hilbert cuspidal form of level \(N_2\) and weight \((k, k) = (n + 2, n + 2) \geq (2, 2)\) over \(R \supseteq \mathfrak{O}\), and let \(f(q) = \sum_{\nu \in (a^{-1})^+} a_{\nu} q^{\nu}\) be its \(q\)-expansion at the standard cusp. Assume we are given a \(\mathfrak{a}\)-Hilbert cuspidal Hecke eigenform \(f\) of level \(N_2 \geq 4\) and weight \((k, k) = (n + 2, n + 2) \geq (2, 2)\) over the ring of integers \(R \supseteq \mathfrak{O}\) of a number field \(K\) extending \(\mathfrak{A}\), and let \(f(q) = \sum_{\nu \in (a^{-1})^+} a_{\nu} q^{\nu}\) be its \(q\)-expansion at the standard cusp; suppose also that we are given a rational cuspidal Hecke eigenform \(g\) for the modular group \(\Gamma (N) = \Gamma_0 (d_{\mathfrak{A}} N_1) \cap \Gamma_1 (N_2)\) of weight \(k_0 = n_0 + 2 \geq 2\) with \(n_0 = 2n - 2t\) being \(t > 0\) or \(t = n_0 = n = 0\), and let \(g(q) = \sum_{n \geq 1} b_{n} q^{n}\) be its \(q\)-expansion at the infinity cusp. Denote \(\omega_f\) the attached \(L^{(n,n)}\)-differential of order 2 on a chosen toroidal desingularization \(Y_K (N_2)\) of the Satake compactification of the fine moduli \(\mathcal{M}_K (\mathfrak{a}, N_2)\). Denote \(\omega_g\) the attached \(L^{n_0}\)-differential on the modular curve \(X^{\mathfrak{O}}(N) = X^{\mathfrak{O}}(\Gamma (N))\) and \(\eta^{\nu - r}_{g}\), the unique class of \(\mathcal{L}_{\nu}^r\)-valued de Rham cohomology such that \(< \omega_g, \eta^{\nu - r}_g > = 1\).

Let \(p\) be a prime not dividing \(N\) and splitting in \(K\), and in \(\mathfrak{A}\) as \(p = \pi \pi'\), such that \(p\) is ordinary for \(g\) and nonordinary for \(f\), i.e. of positive slope \(\sigma, \sigma'\), at \(\pi\) and \(\pi'\). Writting \(j_{X^{\mathfrak{O}}(N)} : X^{\mathfrak{O}}_Q (N) \hookrightarrow X^{\mathfrak{O}}_p (N) \rightarrow Y_{N_2} (N_2)\) the complement of the locus of the points of the moduli with non-ordinary specialization to \(X_{\mathfrak{F}_p} (N)\), the value of the \(p\)-adic Abel-
Jacobi map at the null-homologous cycle $\Delta_{n,n_0}$ is

$$AJ_p(\Delta_{n,n_0})(\omega_f \otimes \eta_g^{u-r}) =$$

$$(-1)^{t!} \frac{\mathcal{E}_1(g)}{\mathcal{E}(f,g)} < e_g e_{ord} \mathcal{J}_{X^*(N)} (\frac{\theta^{1-t}\beta_{\pi'}^f [p^e]}{2} - \frac{\theta^{1-t}\beta_{\pi'}^g [p^e]}{2}), \eta_g^{u-r} > ,$$

with the non-zero factors

$$\mathcal{E}_1(g) = 1 - \beta_1^2 p^{-k_0} \quad \text{and} \quad \mathcal{E}(f,g) = \prod_{i,i'} (1 - \alpha_i \alpha_{i'} \beta_1 p^{2-k_0})$$

where $\alpha_0, \alpha_1$ and $\alpha_{\pi'}, \alpha_{\pi'}$ and $\beta_1$ are the roots of the polynomials $1 - a_{\pi} x + p^{k_1-1}x^2$ and $1 - a_{\pi'} x + p^{k_1-1}x^2$; and $\beta_1$ is the root of the polynomial $1 - b_p x + p^{k_0-1}x^2$ having $p$-valuation $k_0 - 1$.

In section 6, in theorem\[2\], we prove a Gross-Zagier type formula. For given $f$ and $g$ of weights $(2, 2)$ and $2$, and choice of splitting prime $p = \pi \pi'$ of positive slope $\sigma, \sigma'$, at $\pi$ and $\pi'$, there is a Hida family $g$ of ordinary cuspidal forms passing by $g$, and, assuming an essentially nonrestrictive condition \[126\] on $\sigma, \sigma'$, it passes by $f$ a $p$-adic family (version of Yamagami \[6\]) $f$ of Hilbert cuspidal forms nonordinary at $\pi, \pi'$ of the same slope $\sigma, \sigma'$. We construct a $p$-adic $L$-function $L_p(f,g)$ of the tensor product $\bigwedge_f \otimes \bigwedge_g$ of the two algebras defining these familie, such that for classical weights $s$ and $r$ lying on $(k, k) \geq (2, 2)$ with even $0 \leq k_0 < 2k-2$ or $k_0 = k = 2$, the value $L_p(f,g)(r, s)$ agrees with the $p$-adic Abel-Jacobi map computed in section 5, up to nonzero Euler factors; while for $k_0 \geq 2k$, the value of $L_p(f,g)(r, s)$ interpolates, up to nonzero Euler factors, a ratio of periods of modular forms, which, by the second case of \[27\], theorem 1.1, is a special value of the complex $L$-function of the representation associated to $g$, tensored by the representation associated to the overconvergent $\theta^{1-t} f_s - \theta^{1-t} f_s$ (for $t \leq -1$) restricted to $X(N)$.

**Theorem 2** Consider an $a$-Hilbert cuspidal form $f$ and a cuspidal form $g$ of weight $(2, 2)$ and $2$ and choose $p = \pi \pi'$ as in theorem\[1\], ordinary for $g$ and nonordinary of positive slope $\sigma, \sigma'$ at $\pi, \pi'$. Assuming condition \[126\], we associate to $f$ and $g$ families $f$ and $g$ as above so that $f_s = f$ and $g_r = g$ for some classical $(r, s) \in \Omega_{\mathfrak{r}, \mathfrak{g}}$. There is a Garrett-Hida $p$-adic $L$-function $L_p(f,g) : \Omega_{\mathfrak{r}, \mathfrak{g}} \rightarrow \mathbb{C}_p$, defined in \[132\], with the following property: For any pair of classical characters $(r, s) \in \Omega_{\mathfrak{r}, \mathfrak{g}}$ lying over integers $(k_0, k)$ such that $k_0$ is even with $2k - 2 > k_0 > 2$ or $2k - 2 \geq k_0 = 2$, the ordinary cuspidal form $g$, and the $a$-Hilbert cuspidal form
nonordinary of slope \( \sigma, \sigma' \) at \( \pi, \pi' \), have value under the \( p \)-adic Abel Jacobi map

\[
AJ_p(\Delta_{k-2,k_0-2})(\omega_{f_s} \otimes \eta_{g_r}^{u-r}) = (-1)^t t! \frac{\mathcal{E}_0(g_r)\mathcal{E}_1(g_r)}{\mathcal{E}(f_s, g_r)} L_p(f, g)(r, s),
\]

where \( t = k - 1 - (k_0/2) \), and \( \mathcal{E}_0(g_r) \) is the non-null factor \( 1 - \beta_1^2 p^{1-k_0} \), as well as \( \mathcal{E}_1(g_r), \mathcal{E}(f_s, g_r) \) are the factors already defined and proved non-null in [2].

We have been aware of the work of Y. Liu [23], essentially complementary to ours. We will discuss the relation of this article with Liu in a future work. It is a pleasure to thank Henri Darmon for having proposed this problem to us, which opens, jointly with his recent work with Massimo Bertolini and Victor Rotger, a promising line of forthcoming research. We thank Adel Betina by critical reading and suggestions which have improved the presentation of the article, and thank David Loeffler and Sarah Zerbes by pointing us a mistake in a previous version of it, and how to fix it. Thanks also to Donu Arapura, Mladen Dimitrov, Eyal Goren, Tomás Gómez, Haruzo Hida, Ignasi Mundet and Fran Presas by their answers to technical questions. The authors are partially supported by the MICINN project MTM2010-17389 (Spain). The first author is also partially supported by the Science Foundation Ireland project 13/IA/1914.

To end this introduction, we provide, as preparatory material, a basic observation on rational homology. Consider an embedding of an algebraic curve into a surface \( j_C : C \hookrightarrow S \), both smooth and projective over \( \mathbb{C} \), and assume \( S \) has Betti numbers \( b_1 = b_3 = 0 \). For a point \( o \in C \), consider in \( C \times C \) the cycle

\[
C_o = C - o \times C - C \times o
\]

\( o \)-modification of the diagonal cycle \( C \) of \( C \times C \). The cycle, equally denoted \( C_o \), induced in \( S \times C \) by

\[
j_C \times C : C \times C \longrightarrow S \times C
\]
is then null-homologous, since

\[
C_o = \partial W_3 + \sum E_i \times F_i = \partial(W_3 + \sum A_i \times F_i),
\]

where the 1-cycles \( E_i \) represent the homology classes of a basis of \( H_1(C) \), the 1-cycles \( F_i \) represent the homology classes of their intersection-dual

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basis, the $A_i$ are 2-chains of $S$ such that $\partial A_i = E_i$, and $W_3$ denotes a 3-chain in $C \times C$. In fact, for $C_o$ to be null-homologous, it is not necessary to assume that $j_C$ is an embedding, but only generically injective, and it is not necessary to assume that $S$ has null 1-homology, but just that the generically injective morphism $j_C : C \to S$ kills the 1-homology in the sense that the image of any 1-cycle of $C$ is null-homologous in $S$.

## 2 Hilbert modular forms and differentials

### 2.1 Hilbert modular surfaces

Let $K$ be a real quadratic field with different ideal $\mathfrak{d}$, and fix an inclusion $\text{inc} : K \hookrightarrow \mathbb{R}$. Its ring of integers $\mathfrak{o}$ is invariant under the conjugation $\sigma : \nu \mapsto \nu'$ of $K$ over $\mathbb{Q}$. Denote by $\mathfrak{o}^+$ the positive cone of $\mathfrak{o}$ consisting of the totally positive elements $\nu \in \mathfrak{o}$, i.e. those $\nu$ such that $\nu, \nu' \in \mathbb{R}^+$, and $\mathfrak{u}^+ \subseteq \mathfrak{o}^+$ the totally positive units. Assume, just for convenience, that there is a unity of norm $-1$, so that $\mathfrak{u}^+/(\mathfrak{u}^+)^2$ is trivial (which can be assumed since the inclusion of the modular curve in the Hilbert modular surface attached to some $K$ is just a tool, and under this assumption we will avoid restricting to the $\mathfrak{u}^+/(\mathfrak{u}^+)^2$–invariant part of the cohomologies we consider). Choose once and for all a narrow class of ideals and a primitive ideal $\mathfrak{a} \subseteq \mathfrak{o}$ representing it, whose norm $N_1 := Nm(\mathfrak{a})$ is prime with the discriminant $d_\mathfrak{a} = Nm(\mathfrak{d}_\mathfrak{a})$, and denote $\mathfrak{a}^+ = \mathfrak{a} \cap \mathfrak{o}^+$. Choose an integer $N_2 \geq 4$ prime to $d_\mathfrak{a} N_1$, and denote $N := d_\mathfrak{a} N_1 N_2$.

All schemes in this article are separated and of finite type over its base. For a ring $R$, let $\mathcal{M}_R(\mathfrak{a}, N_2)$, or just $\mathcal{M}_R(N_2)$ understanding $\mathfrak{a}$, the irreducible scheme of dimension 2 (i.e. surface) over $R$ which is fine moduli for the Abelian surfaces $A$ over $R$ with a ”real multiplication” or ring monomorphism $\alpha : \mathfrak{o} \hookrightarrow \text{End}(A)$; an ordered $\mathfrak{a}$–polarization or $\mathfrak{o}$–linear isomorphism $\delta : \text{Sym}_\mathfrak{o}(A, A^\vee) \approx \mathfrak{a}$ from symmetric $\mathfrak{o}$–homomorphisms from $A$ to the dual Abelian surface $A^\vee$ applying the cone of positive polarizations $\text{Sym}_\mathfrak{o}(A, A^\vee)^+ \to \mathfrak{a}^+$; and a $N_2$–level structure or $\mathfrak{o}$–linear monomorphism of group schemes

$$\varepsilon : \mu_{N_2/R} \otimes_{\mathbb{Z}} \mathfrak{o}^{-1}_R \hookrightarrow A.$$ (5)

Here $\mu_{N_2/R} = \text{Spec} R[t]/(t^{N_2} - 1)$ is the group-scheme over $R$ of $N_2$–roots of unity$^2$, i.e. the kernel of the $N_2$–power endomorphism of the multiplicative group scheme $\mathbb{G}_m/R$ (cf. 3.2, 3.3 and 5.4 of [1] or lemma 1.1 of

$^2$To be more precise, for any scheme $S$ over $R$, it is $(\mu_{N_2} \otimes_{\mathbb{Z}} \mathfrak{o}^{-1}))(S) = \mu_{N_2}(S) \otimes_{\mathbb{Z}} \mathfrak{o}^{-1}$. Because of [2] under the assumption $\mathfrak{o} \subseteq R$, an $N_2$–level structure is a subgroup scheme over $R$ of $A$ isomorphic to $\mu_{N_2} \times \mu_{N_2}$ preserved by the action of $\mathfrak{o}$.
For $R \rightarrow R'$, it is
\[ M_{R'}(a, N_2) = M_R(a, N_2) \times_{\text{Spec } R} \text{Spec } R'. \]

For level $N_2 \leq 3$, there is a coarse moduli for these objects, still denoted $M_R(N_2)$, although no longer fine (see theorem 4.9 of [30], for instance). In the particular case of ”no level” $M_R(1)$, the fact that $M_R(N_2)$, for $N_2 \geq 4$, has a universal family, provides us thus with a finite morphism
\[ pr_{M_R} : M_R(N_2) \rightarrow M_R(1) \]
because of the weak universal property of coarse moduli spaces.

For $N_2 \geq 4$, denote
\[ pr_{M_R(N_2)} : A^U_{M_R(N_2)} \rightarrow M_R(N_2) \]
the representing object, i.e. the universal Abelian surface with real $\mathfrak{o}$–multiplication $\alpha^U_R$, ordered $a$–polarization $\delta^U_R$ and $N_2$–level structure $\varepsilon^U_R$.

Because of the assumption that $d_K$ is invertible in $R$, the ”Rapoport” rank one $\mathfrak{o} \otimes \mathbb{Z} O_{M_R(N_2)}$–sheaf
\[ \mathcal{R}_{M_R(N_2)} := pr_{M_R(N_2)} \ast \Omega^1_{M_R(N_2)/M_R(N_2)} \]
is locally free, i.e. the ”Rapoport” open subscheme where it is locally free is the whole moduli, so that $\mathcal{R}_{M_R(N_2)}$ is locally free of rank two as a $O_{M_R(N_2)}$ –sheaf. The moduli $M_R(N_2)$ is smooth over $R$, as it is so, in general, for the Rapoport locus. Assume, from now on, that $\mathfrak{o} \subseteq R$. Then the monomorphism provided by the two natural morphisms $\mathfrak{o} \otimes \mathbb{Z} R \rightarrow R$, inclusion and conjugation $\sigma$, is an isomorphism:
\[ \mathfrak{o} \otimes \mathbb{Z} R \approx R \oplus R \]
and in fact $M \otimes \mathbb{Z} j \approx M \oplus M$ for any $R$-module $M$ and fractional ideal $j$ (cf. 2.0.3 and 2.1.1 of [31]. The two natural homomorphisms
\[ \mathfrak{o} \otimes \mathbb{Z} O_{M_R(N_2)} \rightarrow O_{M_R(N_2)} \]
induce two homomorphisms of sheaves of multiplicative groups
\[ (\mathfrak{o} \otimes \mathbb{Z} O_{M_R(N_2)})^* \rightarrow O_{M_R(N_2)}^* \]
\[ \text{[3] Since we will refer frequently to the article of Katz [31], we warn the reader that our smooth moduli } M_R(N_2) \text{ is denoted } \mathcal{M}(a, \mathfrak{b}_{00}(N_2))_R \text{ in that reference.} \]
which associate to any invertible \( \mathcal{O} \otimes \mathcal{O}_{\mathcal{M}_R(N_2)} \)-sheaf, two invertible \( \mathcal{O}_{\mathcal{M}_R(N_2)} \)-sheaves of which it is thus the direct sum. Applying this to the Rapoport invertible \( \mathcal{O} \otimes \mathcal{O}_{\mathcal{M}_R(N_2)} \)-sheaf, we obtain a decomposition into line bundles (cf. 2.0.9 of [31])

\[
\mathcal{R}_{\mathcal{M}_R(N_2)} \cong L^{(1,0)}_{\mathcal{M}_R(N_2)} \oplus L^{(0,1)}_{\mathcal{M}_R(N_2)}.
\]

This allows us to define the modular line bundles

\[
L^{(k,k')}_{\mathcal{M}_R(N_2)} := k L^{(1,0)}_{\mathcal{M}_R(N_2)} \otimes k' L^{(0,1)}_{\mathcal{M}_R(N_2)}
\]

of weight \((k, k') \in \mathbb{Z} \times \mathbb{Z}\). The \(n\)-th tensor power \( \mathcal{R}^{\otimes n}_{\mathcal{M}_R(N_2)}\) is an invertible \( \mathcal{O}_{\mathcal{M}_R(N_2)} \otimes \mathcal{O} \)-sheaf which thus decomposes (cf. 2.0.10 of [31]) as a direct sum of line bundles

\[
\mathcal{R}^{\otimes n}_{\mathcal{M}_R(N_2)} \cong L^{(n,0)}_{\mathcal{M}_R(N_2)} \oplus L^{(0,n)}_{\mathcal{M}_R(N_2)}.
\]

This must not be confused with the \( \mathcal{O}_{\mathcal{M}_R(N_2)} \)-symmetric \(n\)-th power of the rank two \( \mathcal{O}_{\mathcal{M}_R(N_2)}\)-bundle \( \mathcal{R}_{\mathcal{M}_R(N_2)}\), which is a \( \mathcal{O}_{\mathcal{M}_R(N_2)}\)-bundle of rank \(n + 1\) denoted

\[
\mathcal{R}^n_{\mathcal{M}_R(N_2)} \cong \bigoplus_{k,k' \geq 0, k+k'=n} L^{(k,k')}_{\mathcal{M}_R(N_2)}
\]

According to the material recalled, for instance, at the beginning of proof 11.9 of [1], there is a surface \( \overline{\mathcal{M}}_R(N_2)\) normal and projective over \( R \), called ”Satake compactification”, which minimally compactifies the moduli \( \mathcal{M}_R(N_2)\). The singularities of \( \overline{\mathcal{M}}_R(N_2)\) are the points \( \mathcal{M}_R(N_2) - \mathcal{M}_R(N_2)\) outside the moduli. We choose, once and for all, a toroidal desingularization, over \( R \),

\[
Y_R(a, N_2) \to \overline{\mathcal{M}}_R(a, N_2),
\]

We denote by \( D^c \) the transform divisor of the cusps (this should be denoted \( D^c_{Y_R(N_2)} \) to recall to which surface it belongs, but we spare, in general, subindexes of surface divisors, as always clear). The morphism \( \Phi \) extends obviously to

\[
pr_{Y_R} : Y_R(N_2) \to Y_R(1) = Y_R
\]

(we will denote \( Y_R(a, N_2) \) by just \( Y_R(N_2) \), when \( a \) is clear by the context). According to an idea of Mumford (recalled in (4.4) of [22] and in the
these are the liftings of the line bundles
ment preceding theorem 4.2 in [22]). In the case of parallel weight,
\( L \) coincides with invariance needs not to be imposed, cf. [20], 1.6 and 5.2 iii) of [21]). This
the action extending the translation in the Abelian fibres (in which the
extends to a universal family of dimension 2 with \( \mathfrak{O}_{\mathcal{M}_R(N_2)} \)-sheaf
of relative differentials which are \( A_{Y_R(N_2)}^{\mathcal{U}} \)-invariant, i.e. invariant by the action extending the translation in the Abelian fibres (in which the invariance needs not to be imposed, cf. [20], 1.6 and 5.2 iii) of [21]). This
the universal family of level \( N \),
\( \mathfrak{O}_{\mathcal{M}_R(N_2)} \)-sheaf, this having rank
as rank two locally free \( \mathfrak{O}_{\mathcal{M}_R(N_2)} \)-sheaf
The rank-one locally free \( \mathcal{O}_{Y_R(N_2)} \)-sheaves \( \mathcal{O}_{Y_R(N_2)}^{L^{(k,k')}}_{Y_R(N_2)} \) attached to \( \mathcal{R}_{Y_R(N_2)} \) as above, extend the modular line bundles \( L^{(k,k')}_{\mathcal{M}_R(N_2)} \) (cf. (4.4) and comments preceding theorem 4.2 in [22]). In the case of parallel weight, these are the liftings of the line bundles \( L^{(k,k)}_{\mathcal{M}_R(N_2)} \) on the minimal compactification (cf. remark 5.6 of [1]).
Extending what has been said for the non-compactified moduli, the rank two locally free \( \mathcal{O}_{Y_R(N_2)} \)-sheaf \( \mathcal{R}_{Y_R(N_2)} \) splits as
\( \mathcal{R}_{Y_R(N_2)} \cong \mathcal{L}_{Y_R(N_2)}^{(1,0)} \oplus \mathcal{L}_{Y_R(N_2)}^{(0,1)} \) (11)
Again, we will consider the rank one \( \mathfrak{O}_{\mathcal{M}_R(N_2)} \)-sheaf \( n \)-tensor power \( \mathcal{R}_{Y_R(N_2)}^{\otimes n} \) of \( \mathcal{R}_{Y_R(N_2)} \) which, as rank two locally free \( \mathcal{O}_{Y_R(N_2)} \)-sheaf, is
\( \mathcal{R}_{Y_R(N_2)}^{\otimes n} \cong \mathcal{L}_{Y_R(N_2)}^{(n,0)} \oplus \mathcal{L}_{Y_R(N_2)}^{(0,n)} \)
and the \( n \)-symmetric power
\( \mathcal{R}_{Y_R(N_2)}^{n} \cong \bigoplus_{k+k' \geq 0 \atop k+k'=n} L_{Y_R(N_2)}^{(k,k')} \) for \( n \geq 0 \) (12)
of \( \mathcal{R}_{Y_R(N_2)} \) as rank two locally free \( \mathcal{O}_{Y_R} \)-sheaf, this having rank \( n+1 \).
The \( R \)-module \( M_{k,k'}(R, a, N_2) \) of \( a \)-polarized Hilbert modular forms of level \( N_2 \) over \( R \), and weight \( (k, k') \), or just \( a \)-Hilbert modular forms if the rest is understood, can be defined as the \( R \)-module of sections
\( M_{k,k'}(a, N_2, R) := \Gamma L_{\mathcal{M}_R(N_2)}^{(k,k')} = \Gamma L_{Y_R(N_2)}^{(k,k')} \)
The equality is obvious by normality of $M_{R(N_2)}$ in the parallel case $k = k'$, and is called Koecher principle in general. As this principle is proved in [38] 4.9 by checking it on open affine subsets of $Y_{k,k'}(N_2)$, it holds also for sections of $L_{Y_{k,k'}}(N_2)$ on any open set $U \subseteq Y_{k,k'}(N_2)$, i.e. they coincide with the sections on $U \cap M_{R(N_2)}$. For general weight, we can provisionally define the $R-$submodule of the Hilbert cuspidal forms, as

$$S_{k,k'}(a,N_2, R) = \Gamma L_{Y_{k,k'}}(-D_c)$$  \hspace{1cm} (13)$$

(which will correspond to the familiar definition in terms of $q-$expansions).

Lemma 6.3 of [40] asserts that $M_{k,k'}(a,N_2, C)$ is null whenever $k$ or $k'$ are nonpositive.

The smooth variety $M_{R(N_2)}(C)$ of complex points of $M_{R(N_2)}$, which we sloppy denote $M_{R(N_2)}$, can be seen as the quotient $\Gamma_1(a,N_2)/\mathcal{H}_\mathbb{R}$ (cf. 2.1 of [21], for instance) of

$$\mathcal{H}_\mathbb{R} := \{(z_1, z_2) \in \mathbb{R} \otimes \mathbb{Z} \mathbb{C} \mid im(z_i) \in (\mathbb{R} \otimes \mathbb{Z} \mathbb{R})_+\} \approx \mathcal{H}^2$$

by the left action of the subgroup $\Gamma_1(a,N_2)$ of the group $\Gamma_0(a,N_2)$ of unimodular matrices of the form

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} o & a^* \\ a \circ \mathbb{R} N_2 \circ \end{pmatrix}_{\det=1}$$

given by condition $d \equiv 1 \mod. N_2$, a quotient which is uniformized by an algebraic variety defined over $\mathbb{Z}[\frac{1}{N_2}]$, thus defined over $R$. The reason is that $N$ is divisible by all primes whose localization do not make the subgroup the full special linear group, cf. X. 4 [40]. We have kept the notations $\Gamma_1(a,N_2) \subseteq \Gamma_0(a,N_2)$ just for the sake of generality, although these groups are equal, in our case $u^+/(u^+)^2 = 1$, to the groups $\Gamma_1(a,N_2) \subseteq \Gamma_0(a,N_2)$ which in general are bigger, having $u^+/(u^+)^2$ as a two-elements quotient which acts on $\Gamma_1(a,N_2)/\mathcal{H}_\mathbb{R}$ having $\Gamma_1(a,N_2)/\mathcal{H}_\mathbb{R}$ as a quotient (This last surface is coarse moduli of the analogous data but replacing "polarizations" by "classes of polarizations")

In terms of this modular group, $M_{k,k'}(a,N_2, \mathbb{C})$ consists of the holomorphic functions $f : \mathcal{H}_\mathbb{R} \longrightarrow \mathbb{C}$ such that

$$f(\begin{pmatrix} a z_1 + b \\ c z_1 + d \end{pmatrix} = (cz_1 + d)^k(c'z_2 + d')^{k'}f(z_1, z_2)$$  \hspace{1cm} (14)$$

for all $\gamma \in \Gamma_1(a,N_2)$. For $R \subseteq \mathbb{C}$, the $R-$submodule $M_{k,k'}(a,N_2, R)$ consists of those $f$ whose Fourier coefficients, as periodic function, lie in $R$. 

13
For general $R$ as in our hypothesis, [1], a set of cusps of $\mathcal{M}_R(\mathcal{N}_2)$ called "non-ramified cusps", is considered in bijection with 4-tuples $(b_1, b_2, \varepsilon, j)$ where $b_1, b_2$ are fractional ideals such that $b_1b_2^{-1} = \mathfrak{a}$; $\iota : N_2^{-1}\mathfrak{a}/\mathfrak{a} \approx N_2^{-1}b_1^{-1}/b_1^{-1}$ is a $\mathfrak{a}$-linear isomorphism; and $j : b_1 \otimes R \approx \mathfrak{o} \otimes R$ is an $\mathfrak{o} \otimes R$-isomorphism (cf. [1] (6.4) for instance). Among them, we will always refer to the standard cusp given by $b_1 = \mathfrak{o}$ and $b_2 = \mathfrak{a}^{-1}$, and we will consider as isomorphism $\iota$ the identity in $N_2^{-1}\mathfrak{o}/\mathfrak{o}$ and as isomorphism $j \iota$ the identity in $\mathfrak{o} \otimes R$. We associate to an $\mathfrak{a}$-Hilbert modular form $f$ of weight $(k, k')$ defined over $R$, as expansion near this cusp, a formal $q$-expansion (cf. [1] 6.8 and [31] (1.2.12), for instance)

$$f(q) := \sum_{\nu \in (\mathfrak{a}^{-1})^+ \cup \{0\}} a_\nu q^\nu, \text{ with } a_\nu \in R. \quad (15)$$

where the set of subindexes $(\mathfrak{a}^{-1})^+ \cup \{0\}$ contains $($ contains $(\mathfrak{o}^{-1})^+ \cup \{0\}$, thus $\mathfrak{o}^+ \cup \{0\}$, because $\mathfrak{a} \subseteq \mathfrak{o}$ (Here $a_\nu = a_{\varepsilon^2 \nu}$, for all positive units $\varepsilon$, so , under our assumption that $u^+ = (u^+)^2$, the coefficient $a_\nu$ only depends on the principal ideal $(\nu)$). This $q$-expansion can be seen as attached to a nowhere vanishing relative differential $\omega_{can} = \varepsilon_*(\frac{dt}{t})$ given by the level $\varepsilon$ in (5) (called $\omega_\mathfrak{a}(j)$ in comment to (1.2.10) of [31]) in the punctured complete neighborhood of the standard cusp of $\mathcal{M}_R(\mathcal{N}_2)$, i.e. a section trivializing $\mathcal{R}_{Y_R(\mathcal{N}_2)}$ as rank one $\mathfrak{o} \otimes \mathcal{O}_{Y_R(\mathcal{N}_2)}$-sheaf, which induces a canonical trivialization of each locally free sheaf $L^{(k,k')}_{Y_R(\mathcal{N}_2)}$ on it (so that a section becomes a function, and the $q$-expansion $f(q)$ is the evaluation of the extended function $f$ at the Tate point. In fact, $f(q)$ belongs to the complete local ring of the standard cusp in $\mathcal{M}_R(\mathcal{N}_2)$ and is independent of the toroidal embedding chosen to desingularize the standard cusp, cf. 6.5 to 6.8 of [1]). Now the submodule $S_{k,k'}(\mathfrak{a}, \mathcal{N}_2, R) \subseteq M_{k,k'}(\mathfrak{a}, \mathcal{N}_2, R)$ of cuspidal forms can be recovered by the condition $a_0 = 0$ in the expansion of $f(q)$ at the standard cusp, and the analogous vanishing at all the other cusps. In fact all Hilbert modular forms of unparallel weight are cuspidal, as immediate consequence of 11.1 iii) in [1](cf. comment before theorem 4.2 of [22]). Being $R \supseteq \mathfrak{o}$ of zero characteristic, the homomorphism from $M_{k,k'}(\mathfrak{a}, \mathcal{N}_2, R)$ to the $R$-module of formal series indexed by $(\mathfrak{a}^{-1})^+ \cup \{0\}$ is injective.

2.2 Derivatives

The Kodaira-Spencer isomorphism (obtained, for instance, from the Gauss-Manin connection, which we will recall in section 4) takes, under our assumption that $d_\mathfrak{a}$ is invertible in $R$, the form of an isomorphism of invertible $\mathfrak{o} \otimes \mathcal{O}_{\mathcal{M}(\mathcal{N}_2)}$-sheaves

$$KS : \Omega_{\mathcal{M}_R(\mathcal{N}_2)/R} \approx \mathcal{R}_{\mathcal{M}_R(\mathcal{N}_2)} \otimes_{\mathfrak{o} \otimes \mathcal{O}_{\mathcal{M}_R(\mathcal{N}_2)}} (\mathcal{R}_{\mathcal{M}_R(\mathcal{N}_2)} \otimes_\mathfrak{o} \mathfrak{d}_\mathfrak{a} \mathfrak{a}^{-1}).$$
In case \( d_a^{-1} \) is prime to any integer prime \( p \) which is a non-unit in \( R \), this is isomorphic as \( O_{\mathcal{M}_R(N_2)} \)-sheaf to \( R^{\otimes 2}_{\mathcal{M}_R(N_2)} \), so we have the equally denoted isomorphism of locally free \( O_{\mathcal{M}(N_2)} \)-sheaves

\[
KS : \Omega_{\mathcal{M}_R(N_2)/R} \simeq L_{\mathcal{M}_R(N_2)}^{(2,0)} \oplus L_{\mathcal{M}_R(N_2)}^{(0,2)}
\]

(cf. 2.1.1 to 2.1.3 in [31] and [21] 5.2 iii). We will make from now on this assumption which will be empty in case \( R \) is a field, or equivalent to \( p \) being coprime with \( a \) in case \( R \) is isomorphic to \( \mathbb{Z}_p \), the two cases considered in our applications. The Kodaira-Spencer isomorphism induces

\[
\omega_{\mathcal{M}(N_2)/R} \approx L_{\mathcal{M}(N_2)}^{(2,2)}.
\]

This extends to the whole of \( Y_R(N_2) \) as the sheaf

\[
\Omega_{Y_R(N_2)/R}(\log D^c) \simeq R^{\otimes 2}_{Y_R(N_2)} \approx L_{Y_R(N_2)}^{(2,0)} \oplus L_{Y_R(N_2)}^{(0,2)},
\]

of log \( D^c \)-differentials on \( Y_R(N_2) \) over \( R \), i.e. those having, as well as their exterior derivatives, possible simple poles at \( D^c \) (cf. for instance [10] p. 62 and [21] 5.2 iii). As a consequence of (16) and of the fact that \( M_{2,0}(a,N_2,\mathbb{C}) \) and \( M_{2,0}(a,N_2,\mathbb{C}) \) are null (10) ch. I), the irregularity \( q = h^{1,0} = h^{0,1} \) of the complex surface \( Y\mathbb{C}(N_2)(\mathbb{C}) \) is null, so that this surface has null Betti numbers \( b_1 = b_3 = 2q = 0 \) the key fact of our program.

The ”residue” epimorphism which assigns, to each log \( D^c \)-differential \( 1 \)-form, its eventual simple pole, provides a short exact sequence

\[
0 \longrightarrow \Omega_{Y_R(N_2)/R} \longrightarrow \Omega_{Y_R(N_2)/R}(\log D^c) \longrightarrow \mathcal{O}_{D^c} \longrightarrow 0.
\]

From this we obtain, by comparing first Chern classes,

\[
\omega_{Y_R(N_2)/R} \approx L_{Y_R(N_2)}^{(2,2)}(-D^c).
\]

This associates bijectively a 2-differential of the scheme \( Y_R(N_2) \) over \( R \) to each \( a \)-Hilbert cuspidal form of weight \( (2, 2) \) and level \( N_2 \) over \( R \).

Recall that the exterior derivative

\[
d = (\partial, \partial') = O_{\mathcal{M}(N_2)} \longrightarrow \Omega_{\mathcal{M}(N_2)} \approx L_{\mathcal{M}(N_2)}^{(2,0)} \oplus L_{\mathcal{M}(N_2)}^{(0,2)}
\]

\[\text{17}\]
is the unique homomorphism of sheaves of $R$-modules satisfying, on local sections, the Leibniz law (this way of writing derivatives is a slight abuse of notation in that we identify vector bundles with the associated locally free sheaves of $R$-modules). It acts on the $q$-expansion of local sections of these bundles near (in the punctured complete neighborhood of) the standard cusp, as

$$f(q) = \sum_{\nu \in (a^{-1})^+ \cup \{0\}} a_\nu q^\nu \mapsto (\theta f(q), \theta' f(q)),$$

where

$$\theta f(q) = \sum_{\nu \in (a^{-1})^+ \cup \{0\}} \nu a_\nu q^\nu \text{ and } \theta' f(q) = \sum_{\nu \in (a^{-1})^+ \cup \{0\}} \nu' a_\nu q^\nu. \quad (18)$$

This derivative (17) has null composition with the exterior derivative

$$d = \partial' - \partial : \Omega_{\mathcal{M}(N_2)} \approx L_{\mathcal{M}(N_2)}^{(2,0)} \oplus L_{\mathcal{M}(N_2)}^{(0,2)} \longrightarrow \omega_{\mathcal{M}(N_2)} \approx L_{\mathcal{M}(N_2)}^{(2,2)}, \quad (19)$$

sum of

$$\partial' : L_{\mathcal{M}(N_2)}^{(2,0)} \longrightarrow L_{\mathcal{M}(N_2)}^{(2,2)} \text{ and } - \partial : L_{\mathcal{M}(N_2)}^{(0,2)} \longrightarrow L_{\mathcal{M}(N_2)}^{(2,2)},$$

It acts on $q$-expansions as

$$\left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} b_\nu q^\nu, \sum_{\nu \in (a^{-1})^+ \cup \{0\}} c_\nu q^\nu \right) \mapsto \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} (\nu' b_\nu - \nu c_\nu) q^\nu \right) =$$

$$\theta' \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} b_\nu q^\nu \right) - \theta \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} c_\nu q^\nu \right) \quad (20)$$

thus providing the complex $\Omega_{\mathcal{M}(N_2)}^\bullet$ of sheaves of $R$-modules on $\mathcal{M}(N_2)$

$$\mathcal{O}_{\mathcal{M}(N_2)} \overset{d=(\partial, \partial')}{\longrightarrow} \Omega_{\mathcal{M}(N_2)} = L_{\mathcal{M}(N_2)}^{(2,0)} \oplus L_{\mathcal{M}(N_2)}^{(0,2)} \overset{d=\partial'-\partial}{\longrightarrow} \omega_{\mathcal{M}(N_2)} = L_{\mathcal{M}(N_2)}^{(2,2)}.$$ 

This extends to the complex $\otimes_{Y_R(N_2)}^\bullet(\log D^c)$ of log $D^c$-differentials on $Y_R(N_2)$

$$\mathcal{O}_{Y_R(N_2)} \overset{d=(\partial, \partial')}{\longrightarrow} \otimes_{Y_R(N_2)}(\log D^c) =$$

$$L_{Y_R(N_2)}^{(2,0)} \oplus L_{Y_R(N_2)}^{(0,2)} \overset{d=\partial'-\partial}{\longrightarrow} \omega_{Y_R(N_2)}(D^c) = L_{Y_R(N_2)}^{(2,2)}.$$
2.3 Morphisms from modular curves

Recall we denote rather sloppy by $\mathcal{M}_C(N_2)$ the smooth variety of complex points of this scheme, and that it is the quotient of $\mathfrak{H}_K$ by the left action of $\Gamma^1(a, N_2)$. This action factors through the projectivized group $\mathbb{P}\Gamma^1(a, N_2) = \Gamma^1(a, N_2)/\{\pm 1\}$.

It is proved in prop. 1.1 of chap V in [40] the existence of a morphism

$$\mathcal{M}_C(\Gamma_0(N_1)) \rightarrow \mathcal{M}_C(a, N_1)$$

inducing in smooth compactifications

$$j_{X_C(\Gamma_0(N_1))} : X_C(\Gamma_0(N_1)) \rightarrow Y_C$$

(21)

This morphism is in fact generically injective. With a similar proof, we can see that there is a morphism

$$\mathcal{M}_C(\Gamma_0(N_1)) \rightarrow \mathcal{M}_C(a, N_2)$$

inducing

$$j_{X_C(\Gamma(N))} : X_C(\Gamma(N)) \rightarrow Y_C(N_2)$$

(22)

for the congruence group $\Gamma_1(N) \subseteq \Gamma(N) := \Gamma_0(d_K N_1) \cap \Gamma_1(N_2) \subseteq \Gamma_0(N)$ because this is the subgroup of $\Gamma^1(a, N_2)$ stabilizing the diagonal in its action on $\mathfrak{H}_K$. Indeed, a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma^1(a, N_2)$ stabilizes the diagonal $\mathfrak{H}$ if and only if $\gamma = \gamma'$ or $\gamma = -\gamma'$. It cannot be $\gamma = -\gamma$ because then $d$ belongs to $\mathbb{Z}[\sqrt{d_{K}}]$ so it cannot be $1$ mod. $N_2$. Therefore $\gamma$ is selfconjugate, i.e. a matrix in

$$\Gamma^1(a, N_2) \cap \text{SL}(2, \mathbb{Q}) = \Gamma(N)$$

Both quotients $\Gamma^1(a, N_2) \backslash \mathfrak{H}_K$ and $\Gamma_0(N) \backslash \mathfrak{H}_K$ by the left actions of $\Gamma^1(a, N_2)$ and of the stabilizer $\Gamma_0(N)$ of the diagonal are defined over $\mathbb{Z}[\frac{1}{N}]$, thus over $R$.

The morphism described between the smooth varieties of complex points, of both curve and surface, is in fact induced by a morphism of schemes. Indeed, in [35] it is justified that Hirzebruch-Zagier cycles in Hilbert modular surfaces are in fact given by morphisms of schemes over $\mathbb{Z}[\frac{1}{N}]$, thus over our ring $R \supseteq \mathfrak{o}$. The construction is made more generally for Hilbert-Zagier cycles in "twisted" Hilbert modular surfaces, which in the particular case of a Hilbert modular surface are just the
usual Hirzebruch-Zagier cycles. These are the images of suitable morphisms from Shimura curves associated to quaternion algebras, and their transforms by the Hecke correspondences. In our particular case, the morphism $X_R(N) \rightarrow Y_R(N_2)$ is from a Shimura curve associated to a quaternion algebra $$(\frac{a,b}{Q}) \subseteq M_2(\mathbb{Q}(\sqrt{a}))$$ which is in fact the matrix algebra $M_2(\mathbb{Q})$ (so that $a = 1$), because it is the quaternion algebra $Q_B$ associated by V. 1. [40] to the skew-hermitian matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ whose $B$—diagonal $\mathcal{H}_B \subseteq \mathcal{H}^2$ of equation (1.3) [40] is the true diagonal $z_1 = z_2$ of $\mathcal{H}^2$.

If $E^U_{X_R(N)}$ is the universal family on $X_R(N)$ as modular curve, then the pullback of the universal family $A^U_{X_R(N_2)}$ by $j_{X_R(N)}$ is

$$A^U_{X_R(N_2)} := E^U_{X_R(N)} \otimes \mathfrak{o}_R^{-1} \approx E^U_{X_R(N)} \times_{X_R(N)} E^U_{X_R(N)}$$

(23)

where the isomorphism is the relative version of the isomorphism $E \times_{\mathcal{Z}} \mathfrak{o}_R^{-1} \approx E \times E$ for any elliptic curve $E$ over $R$, as $M \otimes_{\mathcal{Z}} j \approx M \otimes_{\mathcal{Z}} \mathfrak{o} \approx M \oplus M$ for any fractional ideal $j$ as we assume $R \supseteq \mathfrak{o}$ (cf. (5.4) of and [31] 2.1.1 ). The diagonal $E^U_{X_R(N)} \rightarrow A^U_{X_R(N)}$ induces the ”restriction” morphism between bundles of relative differentials $pr_\ast \Omega_{A^U_{X_R(N_2)}/X_R(N)} \rightarrow pr_\ast \Omega_{E^U_{X_R(N_2)}/X_R(N)}$, i.e. $j_{X_R(N)}^\ast \mathcal{R}_0 \rightarrow L_{0,R}$ restricting isomorphisms

$$j_{X_R(N)}^\ast L_{Y_R(N_2)}^{1(0)} \approx L_{0,R} \quad \text{and} \quad j_{X_R(N)}^\ast L_{Y_R(N_2)}^{0(1)} \approx L_{0,R} \ ,$$

so that

$$j_{X_R(N)}^\ast L_{Y_R(N_2)}^{(k,k')} = L_{0,R}^{k+k'} \ .$$

[5] Just to put this in relation with more general contexts, we observe that the diagonal can be recovered in terms of the $\mathfrak{o}_B$—multiplication in $A^U_{X_R(N)}$ as kernel of $$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{o}_B.$$ The quotient line bundle $L_{0,R}$ on this particular Shimura curve still has sense on a Shimura curve on a non-split quaternionic algebra $Q_B$, the global sections of its powers being the Hilbert modular forms, according to the Atkin-Lehner correspondence. The notation $L_{0,R}$ for this line bundle on the modular curve $X_R(N)$ usually denoted $\omega_{X_R(N)}$ recalls that $\Gamma_0$ is the essential part of the level $\Gamma(N) = \Gamma_0(d_\mathfrak{m}N_1) \cap \Gamma_1(N_2)$, as the additional $N_2$—level is just to have universal families; our notation avoids confusions, as $\omega_{X_R(N)}(−\text{cusps})$ has a Kodaira- Spencer isomorphism with the cotangent line bundle $\omega_{X_R(N)}$ of the modular curve; and avoids an excessive use of the omega symbol in this article.
2.4 $p$–adic Hilbert modular forms

Let $K \supseteq \mathfrak{K}$ be a number field, having ring of integers $R \supseteq \mathfrak{o}$. Choose $p$ splitting in $K$, so that it splits in $\mathfrak{K}$ as $p = \pi \pi'$. Then $p$ does not divide the discriminant $d_{\mathfrak{K}}$, and we choose it prime with $N = d_{\mathfrak{K}}N_1N_2$. The embedding $\mathfrak{K} \hookrightarrow \mathfrak{K}_\pi \cong \mathbb{Q}_p$ extends to an embedding $K \hookrightarrow \mathbb{Q}_p$ which restricts to $R_m \cong \mathbb{Z}_p$ for a maximal ideal $m$ of $R$ lying on $\pi$, so that $\mathbb{Z} \subseteq \mathfrak{o} \subseteq R$ induces isomorphisms

$$\mathbb{Z}_p \cong \mathfrak{o}_\pi \cong R_m \text{ and } \mathbb{Q}_p \cong \mathfrak{K}_\pi \cong K_m$$

We get, analogously for $\pi'$,

$$\mathbb{Z}_p \cong \mathfrak{o}_{\pi'} \cong R_{m'} \text{ and } \mathbb{Q}_p \cong \mathfrak{K}_{\pi'} \cong K_{m'}$$

Denote $\kappa$ the residual field $\mathbb{F}_p$. Recall from section 2 of [23], for instance, that the Hasse invariant $H_{\mathfrak{K},p}$ is a $\mathfrak{a}$–Hilbert modular form over $\kappa$ of level $N_2$ of parallel weight $(p - 1, p - 1)$, i.e. a section of $L_{(p-1,p-1)}^{\mathfrak{M}_\kappa(N_2)}$, whose $q$–expansion is 1. The open subscheme $\mathfrak{M}_\kappa(N_2)^{ord}$ of the ordinary points is the complement of the support of the ample divisor $D^h$ of $\mathfrak{M}_\kappa(N_2)$ where $H_{\mathfrak{K},p}$ vanished. We still denote by $D^h$ the non-ample divisor which is strict transform of it in $Y_\kappa(N_2)$

**Proposition.** There is an integer $n_0 \geq 0$, which we take minimal, such that an $\mathfrak{a}$–Hilbert modular form $H_{\mathfrak{K},p}$ section of $L_{\mathfrak{M}_\kappa(N_2)}^{(n_0(p-1), n_0(p-1))}$ reduces modulo $p$ to the section $H_{\mathfrak{K},p}^{n_0}$ of $L_{\mathfrak{M}_\kappa(N_2)}^{(n_0(p-1), n_0(p-1))}$.

**Proof:** From the ampleness of $L_{\mathfrak{M}_\mathfrak{K},p}^{(1,1)}$, it follows the exacteness of the sequence, for $n > 0$,

$$0 \longrightarrow L_{\mathfrak{M}_\mathfrak{K},p}(N_2)^{(n,n)} \xrightarrow{p} L_{\mathfrak{M}_\mathfrak{K},p}(N_2)^{(n,n)} \longrightarrow L_{\mathfrak{M}_\mathfrak{K},p}(N_2)^{(n,n)} \longrightarrow 0$$

and, from this, the exactness of the sequence

$$0 \longrightarrow H^0(L_{\mathfrak{M}_\mathfrak{K},p}(N_2)^{(n,n)}) \xrightarrow{p} H^0(L_{\mathfrak{M}_\mathfrak{K},p}(N_2)^{(n,n)}) \longrightarrow H^1(L_{\mathfrak{M}_\mathfrak{K},p}(N_2)^{(n,n)}) = 0$$

\[ \square \]

\[ \text{\footnotesize{6 Points in the support of } } D^h \text{, i.e. nonordinary for the reduction } \mathfrak{M}_R(N_2) \times \mathbb{Z} \mathfrak{M}_{\mathfrak{K}/p}(N_2) \text{ are union of points in the support of two divisors } D_{\pi} \text{ and } D_{\pi'} \text{ nonordinary for the reductions } \mathfrak{M}_R(N_2) \times \mathfrak{o} \mathfrak{M}_{\mathfrak{K}/\pi}(N_2) \text{ and } \mathfrak{M}_R(N_2) \times \mathfrak{o} \mathfrak{M}_{\mathfrak{K}/\pi'}(N_2) \text{, the two larger strata of the stratification of } D^h \text{ [25] considered in def. 2.1 of} \]
We still denote by $D^h$ the ample divisor of $\overline{M}_{Z_p}(N_2)$ where $H_{Z_p}$ vanishes, which is a normal crossing divisor, and denote the same its strict transform in $Y_{Z_p}(N_2)$, although it is not longer ample as it is disjoint with the strict transforms of the cusps (The subscripts of the divisors will not be needed, as is always clear where they belong) The schemes

\[ Y_Q'(N_2) = Y_{Q_p}(N_2) - D^h \quad \text{and} \quad Y_\kappa'(N_2) = Y_\kappa(N_2) - D^h, \]

smooth and projective over $Q_p$ and $\kappa$, are not affine but lie thus on affine varieties

\[ \overline{M}_{Q_p}(N_2)' = \overline{M}_{Q_p}(N_2) - D^h \quad \text{and} \quad \overline{M}_\kappa(N_2) = \overline{M}_\kappa(N_2) - D^h \]

being $\overline{M}_\kappa^{\text{ord}}(N_2) = \overline{M}_\kappa(N_2)'$ the intersection of this last with $\overline{M}_\kappa(N_2)$. The scheme $Y_Q'(N_2)$ has $Y_{Z_p}(N_2)$ as a flat, smooth and projective model over $Z_p$, with special fibre $Y_\kappa(N_2)$ (this uses that $N_2$ is invertible in $Z_p$). We will denote $Y_{Z_p}(N_2)$ the formal completion of $Y_{Z_p}(N_2)$ along this special fibre, i.e. its $p$–adic completion. Because of properness, the rigid points of the formal scheme $Y_{Z_p}(N_2)$ correspond bijectively with the points of the rigidification $Y_{Q_p}(N_2)^{\text{rig}}$ of the $Q_p$–scheme $Y_{Q_p}(N_2)$ (cf. 2.4 prop. 3, and 2.7 def.1, prop. 7 and 1.13 prop.4 of [9]). When rigid is clear by the context -for instance while taking differentials on a rigid analytic open subset- we just write $Y_{Q_p}(N_2)$; and we will also denote the same the rigid analytic coherent sheaf $F^{\text{rig}}$ on $Y_{Q_p}(N_2)^{\text{rig}}$ corresponding in a unique way to a coherent sheaf $F$ on $Y_{Q_p}(N_2)$ by the rigid version of the GAGA principle (1.16 theory. 11-13 of Loc.cit), if clear by the context, for instance while taking sections of it over a rigid open subset.

The specialization map

\[ sp : Y_{Q_p}(N_2)^{\text{rig}} \longrightarrow Y_\kappa(N_2)(\kappa) \]

induces

\[ red : Y_{Q_p}(N_2)(Q_p) = Y_{Z_p}(N_2)(Z_p) \longrightarrow Y_\kappa(N_2)(\kappa) \]

extending to the equally denoted reduction

\[ red : Y_{Q_p}(N_2)(C_p) = Y_{Z_p}(N_2)(O_{C_p}) \longrightarrow Y_\kappa(N_2)(\kappa). \]

For a locally closed subset $S$ of $Y_\kappa(N_2)(\kappa)$, the tube $]S[ \subseteq Y_{Q_p}(N_2)^{\text{rig}}$ is the rigid $Q_p$–space (denoted equally $]S[$ the corresponding rigid $C_p$–space) which is counterimage of $S$ by the specialization map, so that $]Y_\kappa(N_2)[= Y_{Q_p}(N_2)^{\text{rig}}$. 

20
Since $D^h$ is defined over $\mathbb{Z}_p$, we obtain a flat and smooth scheme $Y'_{\mathbb{Z}_p}(N_2) = Y_{\mathbb{Z}_p}(N_2) - D^h$ over $\mathbb{Z}_p$, having special fibre $Y'^{\ast}_\wp(N_2) = Y^{\ast}_\wp(N_2) - D^h$. Let $\mathcal{A} \subseteq Y'^{\ast}_{\mathbb{Z}_p}(N_2)_{rig} \subseteq Y_{\mathbb{Z}_p}(N_2)_{rig}$ be the rigid $\mathbb{Q}_p$-space $Y'^{\ast}_\wp(N_2)$, and denote equally by $\mathcal{A}$ the corresponding rigid $\mathbb{C}_p$-space.

The Zariski open subset $Y'_{\mathbb{Z}_p}(N_2)(\mathbb{C}_p) \subseteq Y_{\mathbb{Z}_p}(N_2)(\mathbb{C}_p)$ can be seen, near the divisor $D^h$, as the set of $\mathbb{C}_p$-points where a $\mathbb{C}_p$-valued function, still denoted $H_{\mathbb{C}_p}$, does not vanish. This alternative way of looking at the $\alpha$-Hilbert modular forms is explained for instance in Katz [31] 1.2.1 to 1.9.3, recalled in 11 of [1].

$p$-adic filtering of $\mathbb{Z}_p$ , the equally denoted function is defined over $\mathbb{C}_p$. The analogous holds for $Y'_{\mathbb{Z}_p}(N_2)(\mathbb{C}_p)$. The rigid open set $\mathcal{A} \subseteq Y_{\mathbb{Z}_p}(N_2)_{rig}$ appears as the set of $\mathbb{C}_p$-points whose value by $H_{\mathbb{C}_p}$ has $p$-adic valuation 0. It can be enlarged

$$\mathcal{A} \subseteq \mathcal{W}_\varepsilon \subseteq Y'_{\mathbb{Z}_p}(N_2)(\mathbb{C}_p)$$

by the wide open sets $\mathcal{W}_\varepsilon$ in the rigid topology of $Y'_{\mathbb{Z}_p}(N_2)(\mathbb{C}_p)$ where the $\mathbb{C}_p$-value of $H_{\mathbb{C}_p}$ has $p$-adic order strictly smaller than $\varepsilon > 0$. Clearly,

$$\mathcal{W}_\varepsilon \subseteq \mathcal{W}_{\varepsilon'} \quad \text{for } \varepsilon < \varepsilon', \quad \text{and } \bigcup_{\varepsilon > 0} \mathcal{W}_\varepsilon = Y'_{\mathbb{Z}_p}(N_2)(\mathbb{C}_p).$$

Denote $\mathcal{M}_{\mathbb{Q}_p}(N_2)^{ord}$ the projection of $\mathcal{A}$ to $\mathcal{M}_{\mathbb{Q}_p}(N_2)$, and denote $\mathcal{M}_{\mathbb{Q}_p}(N_2)^{ord}$ its intersection with $\mathcal{M}_{\mathbb{Q}_p}(N_2)$. Analogously, denote by $\mathcal{M}_{\mathbb{Q}_p}(N_2)(\varepsilon)$ the projection of $\mathcal{W}_\varepsilon$ to $\mathcal{M}_{\mathbb{Q}_p}(N_2)$, and $\mathcal{M}_{\mathbb{Q}_p}(N_2)(\varepsilon)$, or just $\mathcal{M}_\varepsilon$, its intersection with $\mathcal{M}_{\mathbb{Q}_p}(N_2)$. The $\mathbb{Z}_p$-module of $p$-adic $\alpha$-Hilbert modular forms of weight $(k, k') \in \mathbb{Z} \times \mathbb{Z}$

$$M^{p,rig}_{k, k'}(N_2, \mathbb{Z}_p) := \Gamma_{rig}(L^{(k, k')}_A/\mathbb{Z}_p) \supseteq \Gamma L^{(k, k')}_{\mathcal{M}(N_2)} = M^{p,rig}_{k, k'}(N_2, \mathbb{Z}_p)$$

is the one of rigid analytic sections on $\mathcal{A}$, defined over $\mathbb{Z}_p$, of $L^{(k, k')}_{Y_{\mathbb{Z}_p}(N_2)}$, as a rigid analytic line bundle. These are $p$-adic in the sense of Katz in [31] 1.9.1 to 1.9.3, recalled in 11 of [1], and thus have a $q$-expansion

$$\sum_{\nu \in (a^{-1})^+ \cup \{0\}} a_\nu q^\nu,$$

This is because abelian schemes over $R_m/m^n$ have ordinary $p$-reduction if and only if they admit a $\mu_{p^n}$-level structure $\varepsilon_{p^n}$, thus a $\mu_{p^{n\alpha}}$-level structure $\varepsilon_{p^{n\alpha}}$, for some $\alpha$ (cf. 1.11 of [31]). Since the tangent map of $\varepsilon_{p^{n\alpha}} : \mu_{p^{n\alpha}} \otimes \mathbb{Z} \rightarrow A_{R_m/m^n}$ is an isomorphism we can push to $A_{R_m/m^n}$ the canonical differential $\omega_{can} = \frac{d\varepsilon}{\varepsilon}$ of $\mu_{p^{n\alpha}}$, so to obtain a nowhere vanishing section $\omega_{can}$ trivializing the Rapoport
with all \( a_\nu \in \mathbb{Z}_p \), which agrees with the \( q \)-expansion formerly defined, in case it is classical.

Denote \( S_{p,k,k'}^{p-\text{adic}}(N_2, \mathbb{Z}_p) \) the \( \mathbb{Z}_p \)-module of the \( p \)-adic Frechet subspace of the cuspidal ones, i.e. those having \( a_\nu = 0 \) for this expansion at the standard cusp and at all cusps. The \( \mathbb{Z}_p \)-module \( M_{p,k,k'}^{p-\text{adic}}(N_2, \mathbb{Z}_p) \) contains the submodule of overconvergent \( \mathfrak{a} \)-Hilbert modular forms, i.e. rigid analytic sections on some wide open set. This is the filtered union

\[
M_{k,k'}^{\text{oc}}(N_2, \mathbb{Z}_p) := \bigcup_{\varepsilon > 0} \Gamma_\text{rig}(L_{\mathbb{W}_\varepsilon}(N_2, \mathbb{Z}_p)) \supseteq S_{k,k'}^{\text{oc}}(N_2, \mathbb{Z}_p) \tag{27}
\]

made by the sections of the sheaf \( j^! L_{\mathcal{M}_p^\varepsilon(N_2)} \) where we denote

\[
j^! \mathcal{F} := \lim_{\varepsilon \to 0} j_*(\mathcal{F} |_{\mathcal{M}_\varepsilon}) \tag{28}
\]

for any coherent sheaf \( \mathcal{F} \) on \( \mathcal{M}_{\mathbb{Q}_p}(N_2) \) (and the notation holds for a coherent sheaf on \( Y_{\mathbb{Q}_p}(N_2) \)). The inclusion (27) is an equality in the unparallel case. We define in an analogous way \( p \)-adic and overconvergent \( \mathfrak{a} \)-Hilbert modular forms, and cuspidal forms, with coefficients in any complete extension of \( \mathbb{Z}_p \), as for instance \( \mathbb{Q}_p \) so to obtain \( p \)-adic Frechet spaces \( M_{k,k'}^{p-\text{adic}}(N_2, \mathbb{Q}_p) \), \( M_{k,k'}^{\text{oc}}(N_2, \mathbb{Q}_p) \), etc.

An \( \mathfrak{a} \)-Hilbert modular form \( f \) of \( q \)-expansion (15), classical, \( p \)-adic, or overconvergent, of weight \( (k, k') \) defined over \( \mathbb{Z}_p \)-“restricts” to a classic, \( p \)-adic, or overconvergent modular form \( f \downharpoonright \) of weight \( k + k' \), defined over \( \mathbb{Z}_p \), of \( q \)-expansion

\[
f \downharpoonright (q) = \sum_{\nu \in \left(\mathfrak{a}^{-1}\right)^+ \cup \{0\}} a_\nu q^{Tr(\nu)}. \tag{29}
\]

This is because \( \mathcal{A}_0 := \overline{\mathcal{A}}_{X(N)\mathbb{Q}_p} \) and \( \mathcal{W}_\varepsilon,0 := \overline{\mathcal{W}}_{X(N)\mathbb{Q}_p} \) are the affinoid and the wide open sets of the modular curve \( X(N)\mathbb{Q}_p \) considered in the usual definition of \( p \)-adic and overconvergent modular forms.

The \( k \)-linear Frobenius \( \phi : \mathcal{M}_\kappa(N_2) \to \mathcal{M}_\kappa(N_2) \) is realized on geometric points \( x \) of \( \mathcal{M}_\kappa(N_2)^{\text{ord}} \) in the following way: For an abelian surface \( A_x \) with \( \mathfrak{a} \)-multiplication, ordered \( \mathfrak{a} \)-polarization and \( N_2 \)-level, corresponding to a point \( x \), i.e. admitting a level \( \varepsilon_{p^\alpha} : \mu_{p^\alpha} \otimes_{\mathbb{Z}} \mathfrak{a}^{-1} \)

bundle \( \mathcal{R}_{\mathcal{M}_{\mathbb{Q}_p}/\mathcal{M}_{\mathbb{Q}_p}(N_2p\mathbb{Q}_p)} \) and thus the associated modular line bundles. For those of parallel weight this extends to a trivialization at the cusps, so that their sections become functions, this giving, as projective limit over \( \alpha \), a section of the structure sheaf of the formal scheme \( \mathcal{M}_{\mathbb{Q}_p}(N_2p\mathbb{Q}_p) \), i.e. a \( p \)-adic Hilbert modular form in the sense of Katz, and the association is in fact biunivocal. In the unparallel case, just add the fact that all forms are then cuspidal.
\[
\rightarrow A_x , \text{ the kernel of the } \kappa-\text{linear Frobenius (cf. lemma 4.23 and cor. 4.22 of } [8]) \phi : A_x \rightarrow A_x \text{ is the torsion } p \text{ group subscheme } H_x \cong \mu_p \times \mu_p, \text{ which is } \sigma-\text{invariant because the product by } \sigma \text{ in } A_x \text{ preserves the } p-\text{torsion. The ordered } \sigma-\text{polarization on } A_x \text{ is a symmetric bilinear form which descends to } A_x / H_x \text{ as, being the target torsion free, it kills any torsion subgroup. Furthermore, } \mu_{N_2} \otimes \mathbb{Z} \xrightarrow{p} A_x \rightarrow A_x / H_x \text{ is still a monomorphism as } N_2 \text{ is prime to } p. \text{ We obtain in this way } \phi(x) \text{ (cf. [31], 1. 11). Among the zero characteristic liftings of } \phi, \text{ all of them inducing homotopic maps on the complex of differentials (cf. prop. 1 of [6]), we will choose and denote equally a fixed morphism }
\]
\[
\phi : M_{Q_p}(N_2)^{ord} \rightarrow M_{Q_p}(N_2)^{ord}
\]
extending to
\[
\phi : M_{Q_p}(N_2)(\varepsilon) \rightarrow M_{Q_p}(N_2)(\varepsilon')
\]
for some \( \varepsilon' > \varepsilon > 0 \), which is provided by lemma 3.1.1 and 3.1.7 of [31], or alternatively by theorem. 4.3.1 ii) of [13], where a finite etale subgroup \( G_1 \subseteq A_{M_{Q_p}(N_2)(\varepsilon)}[p] \) of rank \( p^d \) is found, with \( d \) the relative dimension, in our case \( d = 2 \). The morphism \( \phi \) is then covered by a homomorphism \( \Phi \) of abelian surfaces on rigid spaces
\[
\begin{array}{c}
A_\varepsilon := A_{M_{Q_p}(N_2)(\varepsilon)} \xrightarrow{\Phi} A_{M_{Q_p}(N_2)(\varepsilon')} := A_{\varepsilon'} \\
\downarrow \\
M_\varepsilon := M_{Q_p}(N_2)^{ord}(\varepsilon) \xrightarrow{\phi} M_{Q_p}(N_2)(\varepsilon') := M_{\varepsilon'}
\end{array}
\] (30)
factoring by the quotient \( A_{M_{Q_p}(N_2)(\varepsilon)} / G_1 \) isomorphic to the pullback \( \phi^*(A_{M_{Q_p}(N_2)(\varepsilon')}) \) of \( A_{M_{Q_p}(N_2)(\varepsilon')} \) by \( \phi \), followed by the natural projection from this pullback, just as quoted in (2.1) of Coleman’s [12] from sect. 3.10 of Katz’ [32]. The construction of the lifting \( \phi \) uses the additional fact, mentioned in the comment preceding lemma 3.1.1 of [34], that this quotient inherits a natural \( \sigma \)-multiplication, \( \sigma \)-polarization and \( \mu_{N_2} \)-level, a fact which can be proved as above.

The diagram (30) analogous to (2.1) in [12], induces as in Loc. cit. morphisms
\[
\Phi^* \Omega_{A_{\varepsilon'}/M_{\varepsilon'}} \rightarrow \Omega_{A_{\varepsilon}/M_{\varepsilon}} \text{ and } \Phi^* \Omega_{A_{\varepsilon'}/M_{\varepsilon'}} \rightarrow \Omega_{A_{\varepsilon}/M_{\varepsilon}}
\] (31)
inducing
\[
\phi^* \mathcal{R}_{M_{\varepsilon'}} \rightarrow \mathcal{R}_{M_{\varepsilon}} \text{ and } \phi^* \mathcal{R}_{M_{\varepsilon'}} \rightarrow \mathcal{R}_{M_{\varepsilon}}
\] (32)

thus
\[
\phi^* L^{(k,k')}_{M_{\varepsilon'}} \rightarrow L^{(k,k')}_{M_{\varepsilon}} \text{ and } \phi^* L^{(k,k')}_{M_{\varepsilon}} \rightarrow L^{(k,k')}_{M_{\varepsilon'}}
\]
and so induce on sections
\[ \Gamma L^{(k,k')}_{M_L} \rightarrow \Gamma \phi^* L^{(k,k')}_{M_L} \rightarrow \Gamma L^{(k,k')}_{M_L} \rightarrow \Gamma L^{(k,k')}_{M_L}, \]
or, by the Koecher' principle also valid for rigid analytic sections (cf. lemma 4.14 of [34]),
\[ \phi : \Gamma L^{(k,k')}_{W_L} \rightarrow \Gamma L^{(k,k')}_{W_L} \quad \text{and} \quad \text{Ver} : \Gamma L^{(k,k')}_{W_L} \rightarrow \Gamma L^{(k,k')}_{W_L}. \] (33)

These induce equally denoted endomorphisms of the sheaves \( j^{\dagger}L^{(k,k')}_{M_{Lp}(N_2)} \) and \( j^{\dagger}L^{(k,k')}_{Y_{Lp}(N_2)} \), defined as in [28], so they induce equally denoted operators \( \phi \) and \( \text{Ver} \) on the \( \mathbb{Z}_p \)-module \( M^{oc}_{k,k'}(N_2, \mathbb{Z}_p) \) of its sections, and on the submodule \( S^{oc}_{k,k'}(N_2, \mathbb{Z}_p) \) of the cuspidal (called Verschubung operator because it is induced by a morphism which on fibres of the universal family and mod. \( p \) is the dual of the Frobenius isogeny). Just as in theorem. 5.1 of [12], where the arguments using diagram (2.2)[12], analogous to our diagram (30), do not really depend on the dimension \( d \), their composition is
\[ \text{Ver} \circ \phi = p^{k+k'+d}. \]

and, based in 1.11.21 and lemma 1.11.22 of [31], \( \phi \) acts on \( q \)-expansions as \( p^{k+k'+2}V_p \) where
\[ V_p : \sum_{\nu \in (a^{-1})^+\cup\{0\}} a_\nu q^{\nu} \mapsto \sum_{\nu \in (a^{-1})^+\cup\{0\}} a_\nu q^{p\nu} = \sum_{\nu \in (a^{-1})^+\cup\{0\}} a_{\nu/p} q^{\nu} \]
(restriction of the equally denoted ”Katz operator” of [1], 13.10, which shows that this Katz operator preserves overconvergent forms). Analogously, the Verschubung operator \( \text{Ver} \) acts as
\[ U_p : \sum_{\nu \in (a^{-1})^+\cup\{0\}} a_\nu q^{\nu} \mapsto \sum_{\nu \in (a^{-1})^+\cup\{0\}} a_\nu q^{\nu/p} = \sum_{\nu \in (a^{-1})^+\cup\{0\}} a_{p\nu} q^{\nu} \]
being clearly \( U_p V_p = 1. \)

Since \( \text{Ver} \) and \( \phi \) act on \( j^{\dagger}L^{(k,k')}_{Y_{Lp}(N_2)} \) preserving cuspidality, so in fact on \( j^{\dagger}\Omega^i_{Y_{Lp}(N_2)} \) for \( i = 0, 1, 2 \), and this is compatible with the exterior derivative
\[ j^{\dagger}\Omega^i_{Y_{Lp}} \rightarrow j^{\dagger}\Omega^1_{Y_{Lp}} \rightarrow j^{\dagger}\omega|_{Y_{Lp}} \] (34)
they also act on the hypercohomology of this complex, which is the rigid cohomology \( H^i_{rig}(Y_{\kappa_m}^i(N_2), \mathbb{Q}_p) \). As this \( \phi \)-action reduces to the \( k \)-linear
Frobenius of $Y'_m(N_2)$, it computes the natural Frobenius action on this rigid cohomology, because it can be computed equivalently by any zero characteristic lifting of the $k-$linear Frobenius. The natural restriction gives a homomorphism of complexes

$$
\begin{align*}
&\mathcal{O}_{Y_m} \rightarrow \Omega^1_{Y_m} \rightarrow \omega_{Y_m} \\
&j^! \mathcal{O}_{Y_m} \rightarrow j^! \Omega^1_{Y_m} \rightarrow j^! \omega_{Y_m}
\end{align*}
$$

inducing a $\mathbb{Q}_p-$linear map

$$
H^i_{dR}(Y_m(N_2), \mathbb{Q}_p) \approx H^i_{rig}(Y_m(N_2), \mathbb{Q}_p) \rightarrow H^i_{rig}(Y'_m(N_2), \mathbb{Q}_p)
$$

compatible with the $\phi-$action.

Associated to the splitting $p = \pi \pi'$ in $\mathfrak{o}$, the $\mathfrak{o}-$morphisms (31) and (32) give morphisms

$$
\phi^*L_{M_{\mathfrak{p}'}}^{(1,0)} \oplus \phi^*L_{M_{\mathfrak{p}'}}^{(0,1)} \rightarrow L_{M_{\mathfrak{p}'}}^{(1,0)} \oplus L_{M_{\mathfrak{p}'}}^{(0,1)} \text{ and } \phi^*L_{M_{\mathfrak{p}'}}^{(1,0)} \oplus \phi^*L_{M_{\mathfrak{p}'}}^{(0,1)} \rightarrow L_{M_{\mathfrak{p}'}}^{(1,0)} \oplus L_{M_{\mathfrak{p}'}}^{(0,1)}
$$

each of them direct sum of the $\mathbb{Z}_p$ -morphisms obtained by extension of (32) via the two embeddings $\mathfrak{o} \rightarrow R$ and $\mathfrak{o} \approx \mathfrak{F} \rightarrow R$ which induce isomorphisms $\mathfrak{o}_\pi \approx R_m$ and $\mathfrak{o}_\pi' \approx R_m'$, all of them isomorphic to $\mathbb{Z}_p$. The direct factors $\phi^*L_{M_{\mathfrak{p}'}}^{(1,0)} \rightarrow L_{M_{\mathfrak{p}'}}^{(1,0)}$ and $\phi^*L_{M_{\mathfrak{p}'}}^{(1,0)} \rightarrow L_{M_{\mathfrak{p}'}}^{(1,0)}$ induce $\phi_{\pi}$ and $Ver_{\pi'}$ on $j^! L_{M_{\mathfrak{p}'}}^{(k,0)}$ analogous to (33). The endomorphism $\phi_{\pi}$ acts on $q-$expansions as $p^{k}V_{\pi}$ for

$$
V_{\pi} : \sum_{\nu \in (a^{-1})+\cup \{0\}} a_\nu q^{\nu} \mapsto \sum_{\nu \in (a^{-1})+\cup \{0\}} a_\nu q^{\pi \nu} = \sum_{\nu \in (a^{-1})+\cup \{0\}} a_{\nu / \pi} q^{\nu} , \quad (35)
$$

and $Ver_{\pi}$ acts as

$$
U_{\pi} : \sum_{\nu \in (a^{-1})+\cup \{0\}} a_\nu q^{\nu} \mapsto \sum_{\nu \in (a^{-1})+\cup \{0\}} a_\nu q^{\nu / \pi} = \sum_{\nu \in (a^{-1})+\cup \{0\}} a_{\nu / \pi} q^{\nu} \quad (36)
$$

Endomorphisms $\phi_{\pi'}$ and $Ver_{\pi'}$ of $j^! L_{M_{\mathfrak{p}'}}^{(0,k')}_{M_{\mathfrak{p}'}}$ are analogously induced by the second direct factors, and they act on $q-$expansions via operators $p^{k'}V_{\pi'}$ and $U_{\pi'}$. We can still denote $\phi_{\pi}$ the endomorphism of

$$
\begin{align*}
&j^! L_{M_{\mathfrak{p}'}}^{(k,k')}_{M_{\mathfrak{p}'}} = j^! L_{M_{\mathfrak{p}'}}^{(k,0)}_{M_{\mathfrak{p}'}} \otimes j^! L_{M_{\mathfrak{p}'}}^{(0,k')}_{M_{\mathfrak{p}'}}
\end{align*}
$$
obtained as limit of the behavior on sections of
\[
(\phi^* L_{M_{k'}}^{(k,0)} \otimes L_{M_{k'}}^{(0,k')}) \text{ rest.} \otimes \text{id.} \rightarrow (\phi^* L_{M_{k'}}^{(k,0)} \otimes L_{M_{k'}}^{(0,k')}) \phi_{\pi} \otimes \text{id.} \rightarrow L_{M_{k'}}^{(k,0)} \otimes L_{M_{k'}}^{(0,k')}.
\]
It induces an operator on \( M_{oc}^{k,k'}(N_2, Z_p) \) and \( S_{oc}^{k,k'}(N_2, Z_p) \), whose behavior on \( q \)-expansions is again \( p^k V_\pi \) as in (35); and analogously with \( \phi_{\pi'} \) acting on \( q \)-expansions as \( p^k V_{\pi'} \). Ditto for the Verschiebung operators \( \text{Ver}_\pi \) and \( \text{Ver}_{\pi'} \) which act on \( q \)-expansions as \( U_\pi \) and \( U_{\pi'} \).

Clearly,
\[
U_\pi U_{\pi'} = U_{\pi}, U_{\pi'} = U_p,
\]
\[
V_\pi V_{\pi'} = V_{\pi}, V_{\pi'} = V_p \text{ so that } \phi_{\pi} \phi_{\pi'} = \phi_{\pi'} \phi_{\pi} = \phi_p,
\]
\[
U_{\pi} V_{\pi} = U_{\pi}, V_{\pi'} = U_p V_p = 1.
\]

3 The \( p \)-adic Abel-Jacobi map

3.1 Besser theory

We recall first the essentials of Besser theory. There is a \( p \)-adic version of the Abel-Jacobi map in terms of the syntomic cohomology, but since it is not possible to mimic integration in this cohomology, this framework does not allow us to compute this map in terms of \( p \)-adic integration. Besser has solved this problem in [7] by embedding the syntomic cohomology into the finite-polynomial cohomology, where the integration is possible. As well known, \( p \)-adic integration was already available for 1-differentials in Coleman’s theory by using Frobenius as monodromy path to integrate, and Besser has generalized (in cohomological guise) this use of Frobenius to integrate differentials of higher order. Let \( X_{Q_p} \) be a smooth, irreducible scheme of dimension \( d \) over \( Q_p \), having a smooth, flat model \( X_{Z_p} \) over \( Z_p \). Denote by \( X_\kappa \) its reduction to \( \kappa = F_p \). In [7], Amnon Besser embeds the syntomic cohomology of \( X_{Z_p} \) into a \( Q_p \)-space which he calls “finite polynomial cohomology”

\[
H^i_{syn}(X_{Z_p}, n) \subseteq H^i_{fp}(X_{Z_p}, n)
\]
in a way which depends functorially on \( X_{Z_p} \), but is not a true cohomology theory, in the sense of providing long exact sequences as expected. To construct this wider space, Amnon Besser considers in (2.1 to 2.4 of [7]), for a positive integer \( u \) and \( P(x) \) in the multiplicative set \( \mathcal{P}_\kappa \) of

\[\text{Since, according a former footnote, } D^b \text{ is union of the two larger strata } D_\pi \text{ and } D_{\pi'}, \text{ it can be naturally defined overconvergence respect } D_\pi, \text{ a property preserved by } U_\pi \text{ and } V_\pi, \text{ and overconvergence respect } D_{\pi'}, \text{ but we will deal here with just overconvergence in the whole sense.}\]
rational polynomials \( P(x) = \prod (1 - \alpha_j x) \) with first coefficient 1 and roots \( \alpha_j \) of complex norm \( p^{\frac{n}{2}} \), and for any integer \( n \geq 0 \), the syntomic \( P \)-complex \( \mathbb{R} \Gamma_P(X_{\mathbb{Z}_p}, n) \) defined as the mapping fibre complex\(^9\) of the homomorphism of complexes

\[
\text{Fil}^n \mathbb{R} \Gamma_{dR}(X_{\mathbb{Q}_p}/\mathbb{Q}_p) \xrightarrow{P(\phi)} \mathbb{R} \Gamma_{rig}(X_{\kappa}/\mathbb{Q}_p)
\]

composition of

\[
\text{Fil}^n \mathbb{R} \Gamma_{dR}(X_{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{R} \Gamma_{dR}(X_{\mathbb{Q}_p}, \mathbb{Q}_p) \to \mathbb{R} \Gamma_{rig}(X_{\kappa}/\mathbb{Q}_p)
\]

(the last being a map only in the derived category, as it is composition of a morphism and the inverse of a quasi-isomorphism) with the evaluation of the polynomial \( P(x) \) in the endomorphism

\[
\mathbb{R} \Gamma_{rig}(X_{\kappa}/\mathbb{Q}_p) \xrightarrow{\phi} \mathbb{R} \Gamma_{rig}(X_{\kappa}/\mathbb{Q}_p)
\]

of the complex of sheaves of \( \mathbb{Q}_p \)-spaces (def. 4.13 of [8]) induced, according cor. 4.22 and lemma 4.23 of Loc.cit., by the \( \kappa \)-linear Frobenius \( \phi : X_{\kappa} \to X_{\kappa} \). This we equally denote \( \phi \), as well as the \( \mathbb{Q}_p \)-linear map

\[
\phi : H^i_{rig}(X_{\kappa}/\mathbb{Q}_p) \to H^i_{rig}(X_{\kappa}/\mathbb{Q}_p)
\]

which it induces on the finite dimensional \( \mathbb{Q}_p \)-space of rigid cohomology of \( X_{\kappa} \). If \( X_{\mathbb{Z}_p} \) is proper over \( \mathbb{Z}_p \) so that \( X_{\mathbb{Q}_p} \) and \( X_{\kappa} \) are proper over \( \mathbb{Q}_p \) and \( \kappa \) resp., the \( \mathbb{Q}_p \)-linear map in cohomologies

\[
H^i_{rig}(X_{\kappa}/\mathbb{Q}_p) \to H^i_{dR}(X_{\mathbb{Q}_p}/\mathbb{Q}_p)
\]

induced by the map in the derived category is an isomorphism.

The finite polynomial complex \( \mathbb{R} \Gamma_{f_{p,m}}(X_{\mathbb{Z}_p}, n) \) of weight \( n \) is defined by Besser (def. 2.4 of [7]) as the direct limit of \( \mathbb{R} \Gamma_P(X_{\mathbb{Q}_p}, n) \) for all polynomials \( P \in \mathfrak{P}_{u} \) ordered by the divisorial relation, and the ”finite polynomial” cohomology \( H^i_{f_{p,m}}(X_{\mathbb{Z}_p}, n) \) of weight \( u \) is the cohomology of this complex. Since direct limit is an exact functor, \( H^i_{f_{p,m}}(X_{\mathbb{Z}_p}, n) \) is the direct limit of the cohomologies \( H^i_p(X_{\mathbb{Z}_p}, n) \) of the complexes \( \mathbb{R} \Gamma_P(X_{\mathbb{Q}_p}, n) \). Besser writes \( H^i_{f_{p}}(X_{\mathbb{Z}_p}, n) \) for \( H^i_{f_{p,m}}(X_{\mathbb{Z}_p}, n) \). As a piece of the long exact sequence

---

\(^9\)We recall that the mapping fibre of a homomorphism of complexes \( \psi : A^\bullet \to B^\bullet \) is the complex given in degree \( i \) by \( A^i \oplus B^{i-1} \), with coboundary map

\[
d(a^i, b^{i-1}) = (da^i, \psi(a^i) - db^{i-1})
\]

so its cocycles are given by a cocycle \( a^i \) in \( A^i \) and a preimage \( b^{i-1} \) in \( B^{i-1} \) of \( \psi(a) \).
\[ ... \to \Fil^{n} H_{dR}^{i-1}(X_{Q_p}/\mathbb{Q}_p) \xrightarrow{P(\phi)} H_{rig}^{i-1}(X_{\kappa}/\mathbb{Q}_p) \]
\[ \to H_{fp}^{i}(X_{\mathbb{Z}_p}, n) \to \Fil^{n} H_{dR}^{i-1}(X/\mathbb{Q}_p) \xrightarrow{P(\phi)} H_{rig}^{i}(X_{\kappa}/\mathbb{Q}_p) \to \ldots \]

we take the short exact sequence

\[ 0 \to H_{rig}^{i-1}(X_{\kappa}/\mathbb{Q}_p)/P(\phi) \Fil^{n} H_{dR}^{i-1}(X_{Q_p}/\mathbb{Q}_p) \to H_{fp}^{i}(X_{\mathbb{Z}_p}, n) \]
\[ \to \Fil^{n} H_{dR}^{i}(X_{Q_p}/\mathbb{Q}_p) P(\phi) = 0 \to 0 \quad (40) \]

as in (12) of [7]. Assume \( X_{\mathbb{Z}_p} \) is proper over \( \mathbb{Z}_p \). The polynomials \( P \in \mathfrak{P}_i \subseteq \mathbb{Q}[x] \) which are a multiple in \( \mathbb{Q}_p[x] \) of a given polynomial \( Q(x) = \prod (1 - \alpha_j x) \in \mathbb{Q}_p[x] \) with first coefficient 1 and roots \( \alpha_j^{-1} \) of complex norm \( p^{\frac{1}{2}} \)-make a multiplicative subset \( \mathfrak{P}_i^Q \) of \( \mathfrak{P}_i \), which is cofinal in the sense that any polynomial in \( \mathfrak{P}_i \), divides one of them, so that \( H_{fp}^{i} \) can be seen as \( \lim_{P \in \mathfrak{P}_i^Q} H_{fp}^{i} \). The sequence (40) holds in particular for any \( P \in \mathfrak{P}_i^Q \), where \( Q_\phi \) is the characteristic polynomial of \( \phi \) acting on \( H_{dR}^{i}(X_{Q_p}/\mathbb{Q}_p) \), so we obtain an exact sequence

\[ 0 \to H_{dR}^{i-1}(X_{Q_p}/\mathbb{Q}_p)/\Fil^{n}H_{dR}^{i-1}(X/\mathbb{Q}_p) \xrightarrow{i} H_{fp}^{i}(X_{\mathbb{Z}_p}, n) \xrightarrow{P} \Fil^{n}H_{dR}^{i}(X_{Q_p}/\mathbb{Q}_p) \to 0 \quad (41) \]

because this sequence holds for any \( P \in \mathfrak{P}_i^Q \) with \( i = P(\phi) \). Indeed, properness implies \( H_{rig}^{i-1} \approx H_{dR}^{i-1} \) and \( P(\phi) \) acts invertible on it because \( P \in \mathfrak{P}_i \) has eigenvalues of complex norm \( p^{\frac{1}{2}} \); while \( \phi \) acts on this space with eigenvalues of complex norm \( p^{\frac{1}{2d+1}} \), thus \( P(\phi) \Fil^{n} H_{dR}^{i-1}(X_{Q_p}/\mathbb{Q}_p) \approx \Fil^{n} H_{dR}^{i-1}(X_{Q_p}/\mathbb{Q}_p) \). As a consequence, \( H_{fp}^{i}(X_{\mathbb{Z}_p}, n) = 0 \) if \( i > 2d + 1 \), and there is an isomorphism

\[ H_{dR}^{2d}(X_{Q_p}/\mathbb{Q}_p) \xrightarrow{i} H_{fp}^{2d+1}(X_{\mathbb{Z}_p}, d + 1) \quad (42) \]

since both \( \Fil^{d+1} H_{dR}^{2d+1}(X_{Q_p}/\mathbb{Q}_p) \) and \( \Fil^{d+1} H_{dR}^{2d}(X_{Q_p}/\mathbb{Q}_p) \) are null. Composing its inverse with the trace map in de Rham cohomology it is obtained in prop. 2.5. 4 of [7] the ”trace map” in finite polynomial cohomology

\[ tr_{X_{\mathbb{Z}_p}} : H_{fp}^{2d+1}(X_{\mathbb{Z}_p}, d + 1) \xrightarrow{i^{-1}} H_{dR}^{2d}(X_{Q_p}/\mathbb{Q}_p) \xrightarrow{tr_{X_{\mathbb{Z}_p}}} \mathbb{Q}_p. \]

In various respects, finite polynomial cohomology behaves as de Rham cohomology: by prop. 2.5. 3 of [7] there is a bilinear cup-product

\[ H_{fp}^{i}(X_{\mathbb{Z}_p}, n) \otimes H_{fp}^{j}(X_{\mathbb{Z}_p}, m) \xrightarrow{\cup} H_{fp}^{i+j}(X_{\mathbb{Z}_p}, n + m) \]
which, composed with the trace map, gives a "Poincaré type" perfect pairing

\[ <\_,\_>: H^i_{fp}(X_{Z_p}, n) \otimes H^{2d-i+1}_{fp}(X_{Z_p}, d - n + 1) \rightarrow \text{tr}_{X_{Z_p}} \rightarrow \Q_p \quad (43) \]

The commutativity of the diagram

\[
\begin{array}{ccc}
H^i_{fp}(X_{Z_p}, n) \otimes H^{2d-i}_{dR}(X_{Q_p})/F i\iota^{d-i-n+1} & \overset{1 \otimes i}{\longrightarrow} & H^i_{fp}(X_{Z_p}, n) \otimes H^{2d-i+1}_{fp}(X_{Z_p}, d - n + 1) \\
\downarrow & & \downarrow \quad <\_,\_>: \rightarrow \text{tr}_{X_{Z_p}} \rightarrow \Q_p\\
F i\iota^n H^i_{dR}(X_{Q_p}) \otimes H^{2d-i}_{dR}(X_{Q_p})/F i\iota^{d-i-n+1} & <\_,\_>: \rightarrow \text{tr}_{X_{Z_p}} \rightarrow \Q_p
\end{array}
\]

in [7] (14) provides an equivalent definition of the perfect pairing (43) in terms of the analogous pairing \(<\_,\_>: \rightarrow \text{tr}_{X_{Z_p}} \rightarrow \Q_p\) in de Rham cohomology, since \(1 \times i\) is an isomorphism, so the finite polynomial pairing inherits the usual properties of it, as the fact that of being equivalently defined by reduction to the diagonal. This means that \(<\_,\_>: \rightarrow \text{tr}_{X_{Z_p}} \rightarrow \Q_p\) coincides with the composition

\[
H^i_{fp}(X_{Z_p}, n) \otimes H^{2d-i+1}_{fp}(X_{Z_p}, d - n + 1) \rightarrow H^{2d+1}_{fp}(X_{Z_p} \times X_{Z_p}, d + 1) \\
\rightarrow H^{2d+1}_{fp}(X_{Z_p}, d + 1) \rightarrow \text{tr}_{X_{Z_p}} \rightarrow \Q_p
\]

i.e. for \(\tilde{\alpha} \in H^i_{fp}(X_{Z_p}, n)\) and \(\tilde{\beta} \in H^{2d-i+1}_{fp}(X_{Z_p}, d + 1 - n)\), it is

\[
<\tilde{\alpha}, \tilde{\beta}>_{fp} = \text{tr}_{X_{Z_p}}(j^*_{X_{Z_p} \times X_{Z_p}}(\tilde{\alpha} \otimes \tilde{\beta}))
\]

(44)

where \(j_{X_{Z_p} \times X_{Z_p}} : X_{Z_p} \hookrightarrow X_{Z_p} \times X_{Z_p}\) is the diagonal map.

**Lemma.** For \(j_{X_{Z_p} \times Y_{Z_p}} : X_{Z_p} \hookrightarrow Y_{Z_p}\), and finite-polynomial classes \(\tilde{\alpha} \in H^i_{fp}(X_{Z_p}, n)\) and \(\beta \in H^{2d-i+1}(Y_{Z_p}, d + 1 - n)\) on \(X_{Z_p}\) and \(Y_{Z_p}\), it is

\[
<\tilde{\alpha}, j^*_{X_{Z_p} \times Y_{Z_p}}(\beta)>_{fp} = \text{tr}_{X_{Z_p}}(j^*_{X_{Z_p} \times Y_{Z_p}}(\tilde{\alpha} \otimes \beta))
\]

(45)

**Proof:** both are equal to \(\text{tr}_{X_{Z_p}} j^*_{X_{Z_p} \times X_{Z_p}}(\tilde{\alpha} \otimes j^*_{X_{Z_p} \times Y_{Z_p}}(\beta))\), the left hand side because of (44), and the other because \(j^*_{X_{Z_p} \times X_{Z_p}}(\tilde{\alpha} \otimes \beta) = j^*_{X_{Z_p} \times Y_{Z_p}}(\tilde{\alpha} \otimes j^*_{X_{Z_p} \times Y_{Z_p}}(\beta)). \Box

**Lemma.** For \(X_{Z_p}\) of relative dimension 1, finite-polynomial classes \(\tilde{\alpha}, \tilde{\beta} \in H^2_{fp}(X_{Z_p}, 2)\), and a point \(o\) of \(X\) defined over \(Z_p\), it is

\[
j^*_{(X \times o)_{Z_p}}(\tilde{\alpha} \otimes \tilde{\beta}) = j^*_{(X \times o)_{Z_p}}(\tilde{\alpha} \otimes \tilde{\beta}) = 0
\]
(in the notation of a restriction morphism \( j \) we drop the bigger scheme, when clear).

**Proof:** It is, for instance, 
\[
\begin{align*}
\hat{j}_{(X \times o) Z_p} (\tilde{\alpha} \otimes \tilde{\beta}) &= \hat{j}_{X Z_p}^* \tilde{\alpha} \otimes \hat{j}_{Z_p}^* \tilde{\beta}
\end{align*}
\]

belonging to \( H^2_{fp}(o Z_p, 2) = 0 \). □

Besser theorem gives a "class map" from Chow groups to finite polynomial cohomology \( cl_{fp} \) compatible with the familiar class map \( cl_{dR} \) to de Rham cohomology:

\[
\begin{align*}
CH^i(X Z_p) \xrightarrow{cl_{fp}} H^{2i}_{fp}(X Z_p, i) \\
\downarrow \\
CH^i(X \mathbb{Q}_p) \xrightarrow{cl_{dR}} Fil^{2i} H^{2i}_{dR}(X \mathbb{Q}_p)
\end{align*}
\]

Let us denote by \( CH^i(X Z_p)_0 \rightarrow CH^i(X \mathbb{Q}_p)_0 \) the subgroups of classes applying to zero, say of "null-homologous" classes. Let \( Z_{Z_p} = \sum r_l Z_{l, Z_p} \) represent a class in \( CH^i(X Z_p)_0 \), with the coefficients \( r_l \) being rational and the schemes \( Z_{l, Z_p} \) being proper and smooth of dimension \( d - i \) over \( \mathbb{Z}_p \). Let \( Z = \sum r_l Z_l \) be the corresponding class in \( CH^i(X \mathbb{Q}_p)_0 \), obtained by taking generic points. The \( p \)-adic or syntomic Abel Jacobi map evaluated at \( Z \) is computed in [7] theory. 1.2 (and taken here as a definition) as the functional \( Fil^{2d-i+1} H^{2d-2i+1}_{dR}(X \mathbb{Q}_p/\mathbb{Q}_p) \rightarrow \mathbb{Q}_p \) applying the class of \( \omega \) into

\[
AJ_p(Z)(\omega) = \int_{Z_{Z_p}} \omega := \sum r_l tr_{Z_{l, Z_p}}(\hat{j}_l^* \tilde{\omega}) \in \mathbb{Q}_p \quad (46)
\]

where \( \tilde{\omega} \in H^{2d-2i+1}_{fp}(X Z_p, d - i + 1) \) is any lifting of \( \omega \) by the Besser epimorphism (33), and \( j_l \) is the morphism \( j_{Z_{l, Z_p}, X Z_p} : Z_{l, Z_p} \rightarrow X Z_p \) which Besser assumes injective, but it is enough generically injective.

Finally, let us recall for the complement \( X_{Z_p}' \) of a subscheme \( Z_{Z_p} \) also proper over \( \mathbb{Z}_p \), the Mayer-Vietoris exact sequence

\[
\begin{align*}
\ldots \rightarrow H^{i}_{rig, Z_{\kappa}}(X_{\kappa}/\mathbb{Q}_p) &\rightarrow H^{i}_{rig}(X_{\kappa}/\mathbb{Q}_p) \rightarrow H^{i}_{rig}(X'_{\kappa}/\mathbb{Q}_p) \\
\rightarrow H^{i+1}_{rig, Z_{\kappa}}(X_{\kappa}/\mathbb{Q}_p) &\rightarrow \ldots
\end{align*}
\]

of \( \mathbb{Q}_p \)-spaces acted by the \( \mathbb{Q}_p \)-linear maps \( \phi \) induced from the \( \kappa \) -linear Frobenius \( \phi : X_{\kappa} \rightarrow X_{\kappa} \) restricting \( \phi : Z_{\kappa} \rightarrow Z_{\kappa} \). Berthelot has proved in [4] (cor. 5.7) a purity theorem for \( Z_{Z_p} \) with \( n \) irreducible components of pure codimension \( d \)

\[
H^{2d-1}_{rig, Z_{\kappa}}(X_{\kappa}/\mathbb{Q}_p) = 0 \quad \text{and} \quad H^{2d}_{rig, Z_{\kappa}}(X_{\kappa}/\mathbb{Q}_p) = \mathbb{Q}_p(-d)^n, \quad (48)
\]
where the Tate’s twist notation means that the natural action of \( \phi \) on this copy of \( \mathbb{Q}_p \) is given by multiplication by \( p^d \) (cf. lemma 7.2 in \[7\] and preceding comment). As a consequence, we have, in case \( d = 1 \), the following inclusions \( i^1, i^2 \) (followed of what we call residue maps)

\[
0 = H^1_{\text{rig}, Z_n}(X_\kappa/\mathbb{Q}_p) \rightarrow H^1_{\text{rig}}(X_\kappa/\mathbb{Q}_p) \xrightarrow{i^1} H^1_{\text{rig}}(X'_\kappa/\mathbb{Q}_p)
\]

\[
\text{res}_1^* H^2_{\text{rig}, Z_n}(X_\kappa/\mathbb{Q}_p) \approx \mathbb{Q}_p(-1)^n \rightarrow H^2_{\text{rig}}(X_\kappa/\mathbb{Q}_p)
\]

\[
\text{res}_2^* H^2_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \rightarrow H^3_{\text{rig}, Z_n}(X_\kappa/\mathbb{Q}_p).
\]

This allows to present a class in \( H^i_{\text{rig}}(X_\kappa/\mathbb{Q}_p) \) as a class in \( H^i_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \) having null residue, i.e. null image in \( H^{i+1}_{\text{rig}, Z_n}(X_\kappa/\mathbb{Q}_p) \), a presentation which is unique in case \( i = 1 \). The advantage of this presentation is that the cohomology \( H^i_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \) has a particularly handable description in terms of a particular inclusion \( X'_\kappa \hookrightarrow X_\kappa \) in the special fibre of a proper smooth scheme \( X_{\mathbb{Z}_p} \) over \( \mathbb{Z}_p \), as follows. First recall that wide open sets, and thus overconvergence, can be mimicked in this general setting by the notion of strict neighborhood: Write \( X^\text{rig}_{\mathbb{Q}_p} \) for the rigid analytic space associated to the formal completion of the generic fibre of \( X_{\mathbb{Z}_p} \) at the special fibre \( X_\kappa \) (or \( p \)-completion), and consider the specialization map \( sp : X^\text{rig}_{\mathbb{Q}_p} \rightarrow X_\kappa(\kappa) \) from this rigid analytic space to the set of \( \kappa \)-points of the special fibre; for any locally closed subset \( S \) of this special fibre, associate a rigid analytic \( \mathbb{Q}_p \)-subspace \( \lfloor S \rfloor \subseteq X^\text{rig}_{\mathbb{Q}_p} \) whose underlying set \( sp^{-1}(S) \) consists of all points specializing to some point in \( S \) (cf.\[8\], 4). A neighborhood \( U \) of \( \lfloor X_\kappa \rfloor \) in the rigid analytic topology of \( \lfloor X_\kappa \rfloor \) is said strict if \( \{ U, Z_n \} \) is a covering of \( \lfloor X_\kappa \rfloor \). Generalizing a notation we have already used in our particular case, any rigid coherent sheaf \( \mathcal{F} \) on the rigid analytic space \( \lfloor X_\kappa \rfloor \) has associated the sheaf \( j^! \mathcal{F} \) on \( \lfloor X_\kappa \rfloor \) of overconvergent sections of \( \mathcal{F} \), i.e. sections over some strict neighborhood \( U \) of \( \lfloor X_\kappa \rfloor \), i.e. the direct limit \( j^! \mathcal{F} = \lim_{\longrightarrow} j_{U,*}(\mathcal{F}|_U) \) taken over all strict neighborhoods \( U \). In these notations, the cohomology of the rigid analytic complex \( \mathbb{R} \Gamma(\lfloor X_\kappa \rfloor, j^! \Omega^\bullet_{\lfloor X_\kappa \rfloor}) \) is \( H^i_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \) (we take advantage of the fact that the inclusion of \( X'_\kappa \) in the special fibre \( X_\kappa \) of \( X_{\mathbb{Z}_p} \), provides us, taking formal completion along the special fibre, with a particular "rigid datum" among those considered in sec. 4 of \[8\], all the analogous "rigid data" providing complexes which are mutually quasi-isomorphic, and quasi-isomorphic to \( \mathbb{R} \Gamma_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \), i.e. have \( H^i_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \) as cohomology). If, furthermore, we are giving a morphism \( \lfloor X'_\kappa \rfloor \rightarrow \lfloor X_\kappa \rfloor \) reducing to \( \phi : X'_\kappa \rightarrow X'_\kappa \) and extending to a morphism \( U \rightarrow V \) between strict neighborhoods \( U \subseteq V \) (as will happen in the case we are interested) then the natural action of this lifting on \( H^i_{\text{rig}}(X'_\kappa/\mathbb{Q}_p) \) is just the \( \phi \)-action, so the lifting provides an operative way to handle this action.
3.2 Application of Besser theory

We keep $N_2, N_1$ and $N = d_1 N_1 N_2$ as in our hypothesis. Recall that the modular curve $X(N)$ has a model smooth, flat and projective over $\mathbb{Z}[\frac{1}{N}]$, thus over $\mathbb{Q}$ and over $\mathbb{Z}_p$. Choose a point $o$ of $X(N)$ with this definition. The fact that $Y(N_2)$ has null odd homology ([40], IV, 6.1) implies that the modified diagonal $X(N)_o$ of

$$X(N) \times X(N) \rightarrow Y(N_2) \times X(N)$$

is null-homologous null in $Y(N_2) \times X(N)$ (we have adopted the convention that omission of the subindex means $X \mathbb{Q}_p(N)$ and $Y \mathbb{Q}_p(N_2)$). This is a consequence of the topological observation made at the end of the introduction, applied to the complex curve $X(N)$ at the null-homologous null cycle $X(N)_o$, which we write sometimes $Y \times X$ and $X_o$ for short.

The Hecke operators $T_n$ act on $a$–Hilbert modular forms for all ideals $n \subseteq \mathfrak{o}$ because of the hypothesis we have made that $u^+ = (u^+)^2$, since Hecke operators act on the $a$–Hilbert modular forms of group $\Gamma_1(a, N_2)$ invariant by $u^+/u^+(u^+)^2$, i.e. on those which are modular for the bigger group $\Gamma_1(a, N_2)$, and in this case both groups are the same. In fact it acts null on $a$–Hilbert modular forms unless $n$ is in the narrow class of $a$, i.e. $n = \nu a$ for some $\nu \in \mathfrak{o}^+$ defined up to totally positive units, and we denote $T_\nu$ this Hecke operator acting on $a$–Hilbert modular forms $f$. If $f$ is in fact an eigenform, the coefficient $a_\nu$ of its $q$–expansion is precisely the eigenvalue of $T_\nu$. We are going to make an explicit use in this article of two of such operators, namely

$$T_\pi := U_\pi + p^{k-1}V_\pi \quad \text{and} \quad T_\pi' := U_\pi + p^{k-1}V_\pi'$$

(as follows from the computation of the Fourier coefficients of the transforms for instance in VI. 1 of [40]), so we can take here such expressions 49 as definition of these particular operators, as well as of the Hecke operator $T_p = U_p + p^{k-1}V_p$ acting on modular forms of weight $k$, i.e. on sections of the line bundle $L_0^k$ on the modular curve $X\mathbb{Q}_p(N)$.

**Definition 3** In the notations above, for nonnegative rationals $\sigma, \sigma'$, we denote $S_{k,k'}(a, N_2, R)^{\sigma, \sigma'} \subseteq S_{k,k'}(a, N_2, R)$ the $R$–submodule of forms of slope $(\sigma, \sigma')$, i.e. eigenforms of $T_\pi$ and $T_\pi'$ of eigenvalues $a_\pi$ and $a_\pi'$ in $R$ such that $\text{ord}(a_\pi) = \sigma$ in $R_m \approx \mathfrak{o}_\pi \approx \mathbb{Z}_p$ and $\text{ord}(a_\pi') = \sigma'$ in $R_m' \approx \mathfrak{o}_\pi' \approx \mathbb{Z}_p$. Denote
and analogously \( S_{k,k'}(a,N_2,R)^{\sigma,\ast} \). In the case of \( \sigma = 0 \) we call such forms ordinary for \( T_\pi \), otherwise nonordinary, and analogously for \( T_{\pi'} \).

Let \( f \) be an \( a \)-Hilbert cuspidal eigenform of weight \((2,2)\) and level \( N_2 \geq 4 \) defined over \( K \), thus over \( K_m \approx \mathbb{Q}_p \), to which corresponds a section

\[
\omega_f \in H^0(L_{Y_{q_p}(N_2)}^{(2,2)}(-D^c)) = H^0(\omega_Y) \\
\subseteq H^2_{dR}(Y_{\mathbb{Q}_p}(N_2)/\mathbb{Q}_p) \approx H^2_{rig}(Y_\kappa/\mathbb{Q}_p) \xrightarrow{\phi} H^2_{rig}(Y'_\kappa/\mathbb{Q}_p)
\]

in the smallest filter \( \text{Fil}^2 H^2_{dR}(Y_{q_p}(N_2)/\mathbb{Q}_p) \) of de Rham cohomology, since a section of the line bundle \( \mathcal{L}^{(2,2)}_{\mathcal{M}_{q_p}(N_2)} \) extends, by normality, to a section of the extended line bundle \( \mathcal{L}^{(2,2)}_{\mathcal{M}_{q_p}(N_2)} \). Let

\[
H^2_{dR}(Y_{\mathbb{C}_p}(N_2)/\mathbb{C}_p)(f) \subseteq H^2_{dR}(Y_{\mathbb{C}_p}(N_2)/\mathbb{C}_p)
\]

be the isotypic component of \( f \) in cohomology, i.e. the subspace where \( \omega_f \) lies, irreducible invariant for the natural action of \( \phi \) on de Rham cohomology. Consider the algebraic values \( \gamma_j \) inverse of the roots of the characteristic polynomial

\[
Q_f(x) = \prod (1 - \gamma_j x) = \det(1 - \phi^{-1} x \mid H^2_{dR}(Y_{q_p}(N_2)/\mathbb{Q}_p)(f)) \in \mathbb{Q}_p[x]
\]

with \( Q_f(\phi) = 0 \), all such roots \( \gamma_j^{-1} \) of complex norm \( p \). We will use Besser theory with the cofinal multiplicative system \( \mathfrak{p}_2^f \subseteq \mathfrak{p}_2 \) for

\[
P_f(x) := (1 - p^{-1} \phi)Q_f(x) \in \mathbb{Q}_p[x]
\]

Consider the embedding \( j_{Y_{\kappa}(N_2)} : Y'_\kappa(N_2) \hookrightarrow Y_\kappa(N_2) \), with \( Y_\kappa(N_2) - Y'_\kappa(N_2) = D^{h_{\kappa}} \), in which the rigid analytic \( \mathbb{Q}_p \)-space \( ]Y'_\kappa[ \) is \( \mathcal{A} \). For the morphism solving singularities

\[
pr : Y' \rightarrow \mathcal{M}_{q_p}(N_2) - D^h,
\]

it is \( pr_* \mathcal{O}_{Y'} = \mathcal{O}_{\mathcal{M}_{q_p}(N_2) - D^h} \), so that, in the spectral sequence abutting to the rigid cohomology of \( H^\bullet_{rig}(Y'_\kappa/\mathbb{Q}_p) \),

\[
E_1^{1,0} = H^1_{rig}(Y', \mathcal{O}_{Y'}) = H^1_{rig}(\mathcal{M}_{q_p}(N_2) - D^h, \mathcal{O}_{\mathcal{M}_{q_p}(N_2) - D^h}) = 0,
\]

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because the higher cohomology of a rigid analytic coherent sheaf on the rigid analytification of an affine scheme is null (Kiehl’s rigid analytic version of Cartan’s theorem’
). Therefore \( E_2^{1,0} = 0 \), and thus the elements of

\[
\text{Fil}^2 H^2_{\text{rig}}(Y'/\mathbb{Q}_p) = \frac{E_2^{2,0}}{\text{Im}[E_1^{1,0} \xrightarrow{d_2} E_2^{0,2}]} = E_2^{2,0} = (54)
\]

are just classes of overconvergent 2-differentials module exterior derivatives of overconvergent differentials.

Let \( P(x) \in \mathfrak{P}_2 \). We prefer to see, for what follows, the de Rham class of \( \omega_f \) as a class in \( \text{Fil}^2 H^2_{\text{rig}}(Y'/\mathbb{Q}_p) \) thanks to the vertical monomorphisms

\[
\begin{align*}
\text{Fil}^2 H^2_{dR}(Y_{\mathbb{Q}_p}/\mathbb{Q}_p) & \cong \text{Fil}^2 H^2_{\text{rig}}(Y'/\mathbb{Q}_p) \\
\downarrow & \downarrow \\
\text{Fil}^2 H^2_{dR}(Y'/\mathbb{Q}_p) & \longrightarrow \text{Fil}^2 H^2_{\text{rig}}(Y'/\mathbb{Q}_p)
\end{align*}
\]

The class of \( P(\phi)(\omega_f) \in \Gamma_{\text{rig}}([Y_{\kappa}[\cdot, j_{Y'_{\kappa}}^\dagger \omega_{\bullet}]_{Y_{\kappa}[\cdot]}]) \) in this space \((54)\) vanishes, so by \((54)\) there is a one-form \( \varrho \) on some wide open set \( \mathcal{W}_\varepsilon \) such that

\[
d\varrho = P(\phi)\omega_f = (56)
\]

(we always equally denote a differential and its restriction to an open set). Then

\[
\tilde{\omega}_f = (\omega_f, \varrho) \in H^2_P(Y'_{\mathbb{Q}_p}, 2)
\]

(57)
is a lift of \( \omega_f \) by the epimorphism

\[
H^2_P(Y'_{\mathbb{Q}_p}, 2) \longrightarrow \text{Fil}^2 H^2_{dR}(Y'/\mathbb{Q}_p) \overset{P(\phi)=0}{\longrightarrow} 0,
\]

(58)

particular case of \((40)\). Applying to the lift \( \tilde{\omega}_f \) the ”restriction” homomorphism

\[
j^*_{X'(N)}: H^2_P(Y'_{\mathbb{Q}_p}, 2) \longrightarrow H^2_P(X'_{\mathbb{Q}_p}, 2),
\]

we obtain an element in the \( H^2_P \) cohomology, thus in the \( H^2_{\text{fp}} \) cohomology, of

\[
X'_{\mathbb{Q}_p} := j^{-1}_{X_{\mathbb{Q}_p}(N)}(Y'_{\mathbb{Q}_p})
\]

which provides in turn an element \( P(\phi)^{-1}j^*_{X'_{\mathbb{Q}_p}(N)}\varrho \) of \( H^1_{\text{rig}}(X'_{\kappa}/\mathbb{Q}_p) \) via the isomorphism

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0 \rightarrow H^1_{\text{rig}}(X'_\kappa/Q_p) \xrightarrow{P(\phi)} H^2_f(X'_Zp, 2) \rightarrow Fil^2H^2_{dR}(X'_Q_p/Q_p) = 0 , \quad (59)

Since \( P(\phi) \) is a multiple of \( (1-p^{-1}\phi) \), this element is in fact in the kernel of the Berthelot residue map, because \( (1-p^{-1}\phi) \) vanishes on the target \( Q_p(-1) \) of this map; i.e. it is an element in the first space of the sequence (cf. 41)

\[
0 \rightarrow H^1_{dR}(X/Q_p) \xrightarrow{P(\phi)} H^2_f(X_Zp, 2) \rightarrow Fil^2H^2_{dR}(X/Q_p) = 0 . \quad (60)
\]

**Lemma 4** In these notations, and for any 1–form

\[
\eta \in H^1_{dR}(X_{Q_p}(N)/Q_p)
\]

and polynomial \( P(x) \in \mathbb{Q}_p^{P_f} \), the value on \( \omega_f \otimes \eta \) of the p-adic Abel-Jacobi map of \( Y_{Q_p}(N_2) \times X_{Q_p}(N) \) at the null-homologous cycle \( X(N)_o \) is given by the product in \( H^1_{dR}(X_{Q_p}(N)/Q_p) \)

\[
AJ_p(X(N)_o)(\omega_f \otimes \eta) = < P(\phi)^{-1}j^*_{X(N)}(\varrho - \sigma), \eta > . \quad (61)
\]

**Proof:** First, observe that the expression on the right hand of (61) is the same as for any other de Rham 1-cochain \( \sigma \) in \( Y_{Q_p}'(N_2) \) applying to \( P(\phi)\omega_f \). Indeed, the difference \( \varrho - \sigma \) would be a de Rham cocycle, thus defining a de Rham 1-class in \( Y_{Q_p}'(N_2) \), in fact image of a de Rham class in \( Y_{Q_p}(N_2) \) because \( P(\phi) \) vanishes on the target \( Q_p(-1) \) of the residue map. By a base change argument, the fact that \( b_1(Y_{Q_p}(N_2)(\mathbb{C})) = 0 \) entails that this is in fact the null class, so that \( < P(\phi)^{-1}j^*_{X(N)}(\varrho - \sigma), \eta > = 0 \).

Denote by \( \tilde{\eta} \in H^2_f(X(N)_{Z_p}, 2) \) the class in finite-polynomial cohomology corresponding to \( \eta \) by the isomorphism (60), given by multiplication by \( P(\phi) \). Let \( \tilde{\omega}_f = (\omega_f, \sigma) \) be a lifting of \( \omega_f \) by the Besser epimorphism \( H^2_f(Y(N_2)_{Z_p}, 2) \rightarrow H^2_{dR}(X_{Q_p}/Q_p) \).

By (46), and lemma (3.1),
\( AJ_p(X(N)_o)(\omega_f \otimes \eta) = \int_{X(N)_o} \omega_f \otimes \eta = (62) \)

\[
\begin{align*}
\text{tr}_{X(N)_{Z_p}} j^*_X(N)_{Z_p} \cdot Y(N_2)_{Z_p} \times X(N)_{Z_p} (\tilde{\omega}_f \otimes \tilde{\eta}) \\
- \text{tr}(X(N)_{\times o})_{Z_p} j^*_X(N)_{Z_p} \cdot Y(N_2)_{Z_p} \times X(N)_{Z_p} (\tilde{\omega}_f \otimes \tilde{\eta}) \\
- \text{tr}(o \times X(N))_{Z_p} j^*_X(N)_{Z_p} \cdot Y(N_2)_{Z_p} \times X(N)_{Z_p} (\tilde{\omega}_f \otimes \tilde{\eta}) \\
= \text{tr}_{X(N)_{Z_p}} j^*_X(N)_{Z_p} \cdot Y(N_2)_{Z_p} \times X(N)_{Z_p} (\tilde{\omega}_f \otimes \tilde{\eta}) \quad (63)
\end{align*}
\]

The vanishing of the first substracting terms is because

\[ j^*_X(N)_{\times o} \cdot Z_p \times X(N)_{Z_p} (\tilde{\omega}_f \otimes \tilde{\eta}) \\
= j^*_X(N)_{\times o} \cdot X(N)_{Z_p} \times X(N)_{Z_p} (j^*_X(N)_{Z_p} \cdot Y(N_2)_{Z_p} \times X(N)_{Z_p} (\tilde{\omega}_f \otimes \tilde{\eta})) \]

so that we can apply lemma \((3.1)\) and analogously with the vanishing of other substracting terms.

By \((45)\), this can be rewritten as the pairing in finite polynomial cohomology

\[ AJ_p(X(N)_o)(\omega_f \otimes \eta) = < j_X(N)^{*} \tilde{\omega}_f, \tilde{\eta} >_{fp} \]

which by \((7)\) \((14)\) equals the paring of their isomorphically corresponding classes in first de Rham cohomology (cf. \((60)\)).

\[ AJ_p(X(N)_o)(\omega_f \otimes \eta) = < P(\phi)^{-1} j_X^*(N) \sigma, \eta > = < P(\phi)^{-1} j_X^*(N) \vartheta, \eta >, \]

the last equality following from the remark made at the beginning of the proof \((10)\) and from the observation that \( < P(\phi)^{-1} j_X^*(N) \vartheta, \eta > \) does not depend on the choice of \( P(x) \) in \( \mathfrak{P}_2^{P_f} \).

Indeed, for any other \( Q(x) \in \mathfrak{P}_2^{P_f} \), there are \( P', Q' \in \mathbb{Q}_p[x] \) so that \( P'P' = Q'Q \in \mathfrak{P}_2^{F_f} \), and the equality \( d(P'(\phi)\vartheta) = P' P(\phi) \omega_f \) provides a lifting of the class of \( \omega_f \) in the Besser epimorphism \((58)\) relative to the polynomial \( P'P \), ”restricting” the de Rham class \( P'(\phi) j_X^*(N) \vartheta \) so that

\[ < (P'P(\phi))^{-1} (P' \phi) j_X^*(N) \vartheta, \eta > = < P(\phi)^{-1} j_X^*(N) \vartheta, \eta >. \]

and analogously for \( Q \) and \( Q' \)

\(^{10}\)The idea here is that, being the product independent of choice of cochain, the advantage of \( \sigma \) is that it is a cochain on the whole \( Y_{K_m}(N_2) \), so it serves to compute the \( p \)-adic Abel-Jacobi map, and the cochain \( \vartheta \) has the advantage of being a differential.
The evaluation in $\phi$ and $\omega_f$

$$f^\sharp := Q_f(p^2V_p)(f) \quad \text{and} \quad f^\flat := P_f(p^2V_p)(f)$$

of the two polynomials \(51\) provides overconvergent 2-differentials

$$\omega_f^\sharp := Q_f(\phi)\omega_f \quad \text{and} \quad \omega_f^\flat = P_f(\phi)\omega_f = (1 - p^{-1}\phi)\omega_{f^\flat},$$

Choose, once and for all, overconvergent 1-differentials $\varrho^\flat$ and $\varrho^\sharp = (1 - p^{-1}\phi)\varrho^\flat$, corresponding to pairs of a-Hilbert modular forms $F^\sharp = (F^\sharp_{(2,0)}, F^\sharp_{(0,2)})$ and $F^\flat = (F^\flat_{(2,0)}, F^\flat_{(0,2)})$, such that

$$d\varrho^\flat = \omega_f^\flat, \quad \text{i.e.} \quad \theta'F^\flat_{(2,0)} - \theta F^\flat_{(0,2)} = f^\flat$$

or equivalently

$$d\varrho^\sharp = \omega_f^\sharp, \quad \text{i.e.} \quad \theta'F^\sharp_{(2,0)} - \theta F^\sharp_{(0,2)} = f^\sharp$$

We will deal, in the framework of the next section, with a primitive $F^\sharp = (F^\sharp_{(2,0)}, 0)$ which we will denote $F^\sharp_{[2,0]}$ and analogous $F^\sharp_{[0,2]}$, $F^\flat_{[2,0]}$, $F^\flat_{[0,2]}$. Lemma \(4\) allows us to write

$$AJ(X(N)_o)(\omega_f \otimes \eta) = \langle P_f(\phi)^{-1}j^*_{X^\flat_{[0,2]}(N)}\varrho^\flat, \eta \rangle, \quad (64)$$

a product in $H^1_{dR}(X_{Q_p}(N)/\mathbb{Q}_p)$.

4 Differentiation of $p$–adic Hilbert Modular Forms

4.1 Gauss-Manin covariant derivative

Consider, in the variety $Y(N_2)$, smooth and projective over $\mathbb{Q}_p$, the rank two locally free $\mathfrak{o} \otimes \mathcal{O}_{Y(N_2)}$–sheaf

$$\mathcal{L}_{Y(N_2)} = \mathcal{H}^1_{dR}(\hat{A}^U) := \mathbb{R}^1pr_{Y(N_2)*}\Omega^\bullet_{\hat{A}^U/Y(N_2)}(\log D^c_{\hat{A}^U})$$

(cf. \(10\) and \(31\)) of first log-de Rham cohomology of a smooth compactification $\hat{A}^U$ of the universal family $A^U$

$$pr_{Y(N_2)}: \hat{A}^U \longrightarrow Y(N_2) \quad \text{.}$$

where $D^c_{\hat{A}^U}$ denotes the divisor which is counterimage of the cuspidal divisor $D^c$ by this projection. The Frobenius action preserves its natural filtration (cf. \(21\) 5.2 iii) by rank one $\mathfrak{o} \otimes \mathcal{O}_{Y(N_2)}$–sheaves

$$0 \longrightarrow \mathcal{R}_{Y(N_2)} \longrightarrow \mathcal{L}_{Y(N_2)} \longrightarrow \mathcal{S}_{Y(N_2)} \longrightarrow 0. \quad (65)$$
where \( S_{Y(N_2)} := R_{Y(N_2)}^\vee \otimes \mathfrak{a}^{-1} \) is isomorphic as \( O_{Y(N_2)} \)-sheaf to
\[
R_{Y(N_2)}^\vee \approx L_{Y(N_2)}^{(-1,0)} \oplus L_{Y(N_2)}^{(0,-1)}
\]
(cf. [21] 1.9 and [31] 1.0.3 and 1.0.13). For brevity, we adopt from now on the convention of omitting the subindex \( Y(N_2) \) in the notation of vector bundles over \( Y(N_2) \). Let
\[
\mathcal{L} = \mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(0,1)}
\]
be the decomposition of the rank 2 locally free \( \mathfrak{o} \otimes O_{Y(N_2)} \)-sheaf \( \mathcal{L} \) as direct sum of two rank 2 locally free \( O_{Y(N_2)} \)-sheaves. Using [31] 2.0.10, we can split (65), viewed as a short exact sequence of locally free \( O_{Y(N_2)} \)-sheaves, as a direct sum of the two exact sequences
\[
\begin{align*}
0 & \longrightarrow L^{(1,0)} \longrightarrow \mathcal{L}^{(1,0)} \longrightarrow L^{(-1,0)} \longrightarrow 0 \\
0 & \longrightarrow L^{(0,1)} \longrightarrow \mathcal{L}^{(0,1)} \longrightarrow L^{(0,-1)} \longrightarrow 0
\end{align*}
\]
Each of these two short exact sequences ”restricts” via \( j_{X(N)} : X(N) \longrightarrow Y(N_2) \) to the well known sequence
\[
0 \longrightarrow L_0 \longrightarrow \mathcal{L}_0 \longrightarrow L_0^{-1} \longrightarrow 0 \tag{66}
\]
on the compact modular curve \( X(N) \), where, as in section 2, we denote \( L_0 \) the (weight 1 ) modular line bundle on \( X(N) \) and \( \mathcal{L}_0 \) the rank two bundle of (relative, log-cusps) first de Rham cohomology of the universal generalized elliptic curve for the group \( \Gamma(N) = \Gamma_0(dR N_1) \cap \Gamma_1(N_2) \) . Clearly \( j_{X(N)}^* \mathcal{L} \cong \mathcal{L}_0 \oplus \mathcal{L}_0 \), since, compactifying [23], the pull back of \( \widehat{A}_{Y(N_2)}^U \) by \( j_{X(N)} \) is
\[
\widehat{A}_{Y(N_2)}^U |_{X(N)} \cong \widehat{E}_{X(N)}^U \times_{X(N)} \widehat{E}_{X(N)}^U \tag{67}
\]
so this isomorphism is the relative version of the Küneth decomposition \( H^1_{dR}(A) \cong H^1_{dR}(E) \oplus H^1_{dR}(E) \) for \( A = E \times E \), with \( E \) a generalized elliptic curve .

We now take \( n \)-th symmetric powers
\[
\mathcal{R}^n \subseteq \mathcal{R} \otimes \mathfrak{a}^n \quad \text{and} \quad \mathcal{L}^n \subseteq \mathcal{L} \otimes \mathfrak{a}^n
\]
in the short sequence of locally free \( O_Y \)-sheaves
\[
0 \longrightarrow \mathcal{R} \approx L^{(1,0)} \oplus L^{(0,1)} \longrightarrow \mathcal{L} \longrightarrow S \approx L^{(-1,0)} \oplus L^{(0,-1)} \longrightarrow 0 ,
\]
namely ( [31], 2.1.5)
\[
0 \longrightarrow \mathcal{R}^n \approx \bigoplus_{k+k' = n, k,k' \geq 0} L^{(k,k')} \longrightarrow \mathcal{L}^n \longrightarrow S^{(n)} \longrightarrow 0 , \tag{68}
\]
where the notation $S^{(n)}$ used for the quotient is just to avoid confusion with the symmetric power $S^n$ of $S$. This short sequence (68) on $Y(N_2)$ is the analogous of the filtration

$$0 \rightarrow L_0^n \rightarrow \mathcal{L}_0^n \rightarrow S_0^{(n)} \rightarrow 0$$

of the rank 2 bundle $\mathcal{L}_0^n$ on the smooth compactified modular curve $X(N)$, by the rank one bundle $L_0^n$ and a rank $n$ bundle denoted $S_0^{(n)}$ as quotient (and we warn the reader that $L_0^n \subseteq L_0^n \otimes S_0^n$ is denoted $L_n \subseteq L_n$ in [17]).

By Ex. II. 5.1.c of [26], there is a filtration of locally free $\mathcal{O}_Y$–sheaves

$$\mathcal{L}_n = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq ... \supseteq \mathcal{F}^n \supseteq \mathcal{F}^{n+1} = 0$$

by bundles $\mathcal{F}^m$ image of

$$\mathcal{R}^m \otimes \mathcal{L}^{n-m} \rightarrow \mathcal{L}^m \otimes \mathcal{L}^{n-m} \rightarrow \mathcal{L}^n$$

with quotients

$$\mathcal{F}^m/\mathcal{F}^{m+1} \cong \mathcal{R}^m \otimes S^{n-m} \approx \bigoplus_{k+k' \geq 0} L^{(k,k')} \otimes \bigoplus_{l+l' = n-m} L^{(-l,-l')} \approx \bigoplus_{l,l' \geq 0} L^{(k-l,k'-l')}$$

(70)

so that $\mathcal{R}^n = \mathcal{F}^n$.

**Proposition 5** The bundle $\mathcal{L}_n$ satisfies the Koecher principle

$$\Gamma(U, \mathcal{L}_n) = \Gamma(U \cap \mathcal{M}(N_2), \mathcal{L}_n)$$

for all open sets $U \subseteq Y(N_2)$, i.e. $\mathcal{L}_n^{Y(N_2)} = \text{inc}_* \mathcal{L}_n^{\mathcal{M}(N_2)}$ for the inclusion $\text{inc} : \mathcal{M}(N_2) \rightarrow Y(N_2)$.

**Proof:** We prove that all $\mathcal{F}^m$ satisfy this principle, in particular $\mathcal{F}^0 = \mathcal{L}_n$, proceeding by induction. If $\mathcal{F}^{m+1}$ does, then as $\mathcal{F}^m/\mathcal{F}^{m+1}$ clearly does because of (70), and as a consequence $\mathcal{F}^m$ satisfies also this principle. Indeed, in the diagram

$$\begin{array}{c}
\Gamma(U, \mathcal{F}^{m+1}) \hookrightarrow \Gamma(U, \mathcal{F}^m) \rightarrow \Gamma(U, \mathcal{F}^m/\mathcal{F}^{m+1}) \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
\Gamma(U \cap \mathcal{M}(N_2), \mathcal{F}^{m+1}) \hookrightarrow \Gamma(U \cap \mathcal{M}(N_2), \mathcal{F}^m) \rightarrow \Gamma(U \cap \mathcal{M}(N_2), \mathcal{F}^m/\mathcal{F}^{m+1})
\end{array}$$

the left and the right vertical arrows are isomorphisms, and the middle vertical arrow is a monomorphism, so it is also an isomorphism.□

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The rank four bundle $L$ bears a connection
\[
\nabla^{GM} = (\nabla, \nabla') : L \longrightarrow L \otimes \Omega_{Y(N_2)} \otimes \Omega_{Y(N_2)}(\log D^c) \approx (L \otimes L^{(2,0)}_{Y(N_2)}) \oplus (L \otimes L^{(0,2)}_{Y(N_2)}),
\]
with $\log D^c$-poles (the Gauss-Manin connection). The composition of the second covariant derivative
\[
\nabla^{GM} : (L \otimes L^{(2,0)}_{Y(N_2)}) \oplus (L \otimes L^{(0,2)}_{Y(N_2)} - \nabla : L \otimes L^{(0,2)}_{Y(N_2)} \longrightarrow L \otimes L^{(2,2)}_{Y(N_2)}
\]
so we denote $\nabla^{GM} = \nabla' - \nabla$. The Gauss-Manin connection is integrable in the sense that
\[
L \nabla^{GM} = (\nabla, \nabla') \longrightarrow L \otimes \Omega_{Y(N_2)} \otimes \Omega_{Y(N_2)}(\log D^c)
\]
is a complex of sheaves of $\mathbb{Q}_p$-spaces on $Y(N_2)$.

The hypercohomology of this complex is the $L^n$-valued log $D^c$-de Rham cohomology $H_{\log -dR}(Y(N_2), L^n/\mathbb{Q}_p)$ of $Y(N_2)$. There is again a $\mathbb{Q}_p$-linear map
\[
H^i_{\log -dR}(Y(N_2), L^n/\mathbb{Q}_p) \approx H^i_{\log -rig}(Y_{\kappa}(N_2), L^n/\mathbb{Q}_p)
\]
to the analogously defined $L^n$-valued log $D^c$-rigid cohomology of $Y_{\kappa}(N_2)$ over $\mathbb{Q}_p$, induced by the restriction homomorphism
\[
\nabla_{\kappa}^n \longrightarrow \Omega^1_{\kappa} \otimes \Omega^n \otimes \omega_{\kappa} \otimes \Omega^n
\]
between the $\nabla^{GM}$-complexes defining both log $D^c$-rigid cohomologies.

Taking the first direct image $R^1 pr_*$ in (31), we obtain
\[
\phi^* L_{M_{\epsilon}} \longrightarrow L_{M_{\epsilon}} \text{ and } \phi_* L_{M_{\epsilon}} \longrightarrow L_{M_{\epsilon}}
\]
arbitrary to (32), and from this
\[
\phi^* (\Omega^i_{\mathbb{Q}_p} \otimes L^n_{M_{\epsilon}}) \longrightarrow \Omega^i_{\mathbb{Q}_p} \otimes L^n_{M_{\epsilon}}
\]
so that $Ver$ and $\phi$ act on $j^!\mathcal{L}^n_{|\mathcal{M}_n|}$, thus also on $j^!\mathcal{L}^n_{|\mathcal{Y}_n|}$, because of the Koecher principle for $\mathcal{L}^n$, and, being these actions compatible with the covariant derivative, it induces actions of $Ver$ and $\phi$ on $H^i_{log-rig}(Y'_\kappa(N_2), \mathcal{L}^n/\mathbb{Q}_p)$.

These cohomologies are too big for our purposes, as in case $n = 0$, they not the de Rham and the rigid cohomology but their log $D^c$-versions, so we switch to the parabolic-de Rham and parabolic-rigid cohomology, which for $n = 0$ gives the usual de Rham and the usual rigid cohomologies. That is to say, in the de Rham case, the cohomology

$H^i_{par-dR}(Y(N_2), \mathcal{L}^n/\mathbb{Q}_p)$ of the subcomplex of sheaves of $\mathbb{Q}_p$-spaces

$(\Omega^i_{Y(N_2)} \otimes \mathcal{L}^n)_{par} := (\Omega^i_{Y(N_2)} \otimes \mathcal{L}^n) + \nabla^GM(\Omega^{i-1}_{Y(N_2)} \otimes \mathcal{L}^n) \subseteq \Omega^i_{Y(N_2)}(\log D^c) \otimes \mathcal{L}^n$

i.e. the cohomology of the de Rham ”parabolic” complex

$\mathcal{L}^n_{Y(N_2)} \longrightarrow (\Omega_{Y(N_2)}(\log D^c) \otimes \mathcal{L}^n) / (\Omega_{Y(N_2)}(\log D^c) \otimes \mathcal{O}_{D^c})$

Note that $(\Omega^i_{Y(N_2)} \otimes \mathcal{L}^n)_{par}$ is a locally free sheaf as kernel of the projection

$\Omega^i_{Y(N_2)}(\log D^c) \otimes \mathcal{L}^n \longrightarrow (\Omega^i_{Y(N_2)}(\log D^c) \otimes \mathcal{L}^n) / (\Omega^i_{Y(N_2)}(\log D^c) \otimes \mathcal{O}_{D^c})$

Note also that $\nabla^GM(\mathcal{L}^n \otimes \Omega_{Y(N_2)})_{par} = \nabla^GM(\mathcal{L}^n \otimes \Omega_{Y(N_2)})$ because $\nabla^GM$ is integrable.

The parabolic rigid cohomology $H^i_{par-rig}(Y'_\kappa(N_2), \mathcal{L}^n/\mathbb{Q}_p)$ is analogously defined as hypercohomology of the $j^!(\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|})_{par}$ complex, i.e.

$$j^!\mathcal{L}^n_{|\mathcal{Y}_n|} \longrightarrow j^!(\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|}) + j^!\nabla^GM \mathcal{L}^n_{|\mathcal{Y}_n|}$$

$$\longrightarrow j^!(\omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|}) + j^!\nabla^GM (\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|})$$

and it is acted by $\phi$ and $Ver$ because the parabolic complex inherits such actions.

There is a $\mathbb{Q}_p$-linear map, compatible with the $\phi$ action,

$$H^i_{par-dR}(Y(N_2), \mathcal{L}^n/\mathbb{Q}_p) \approx H^i_{par-rig}(Y'_\kappa(N_2), \mathcal{L}^n/\mathbb{Q}_p)$$

induced by the restriction homomorphism

$$\mathcal{L}^n_{|\mathcal{Y}_n|} \cong (\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|}) + \nabla^GM \mathcal{L}^n_{|\mathcal{Y}_n|} \cong (\omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|}) + \nabla^GM (\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|})$$

$$j^!\mathcal{L}^n_{|\mathcal{Y}_n|} \longrightarrow j^!(\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|}) + j^!\nabla^GM \mathcal{L}^n_{|\mathcal{Y}_n|}$$

and

$$j^!(\omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|}) + j^!\nabla^GM (\Omega^i_{|\mathcal{Y}_n|} \otimes \mathcal{L}^n_{|\mathcal{Y}_n|})$$

(75)
between the complexes defining these cohomologies.

Being just a formal homological tool, there is a Frobenius-invariant Mayor-Vietoris sequence relating these parabolic cohomologies (de Rham and rigid) for \( Y(N_2) \) and for \( Y'(N_2) \) with their versions as parabolic cohomologies for \( Y(N_2) \) supported in \( D^h \) (i.e. parabolic version of the rigid supported cohomology defined in [7] 7.8)

The natural inclusion \( R \subseteq L \) has over the region \( A \) in (24), a splitting

\[
sp : L_A \rightarrow R_A
\]
equivalent to a decomposition in Frobenius-invariant components

\[
L_A = R_A \oplus S_A = (L_A^{(1,0)} \oplus L_A^{(0,1)}) \oplus (L_A^{(-1,0)} \oplus L_A^{(0,-1)})
\]

defined in the noncuspidal points of \( A \) by the slope of the Frobenius action, and extended by normality to the noncuspidal points. The sub-bundle \( S_A \subseteq L_A \) is horizontal, i.e. \( \nabla^{GM} (S_A) \subseteq S_A \) (cf. [31] 1.11.26 and 27). This provides in turn a split of the symmetric power of the rank four bundle \( L_A \)

\[
sp : L_A^n \rightarrow R_A^n = \bigoplus_{k+k' = n, k,k' \geq 0} L_A^{(k,k')},
\]
i.e. the exact sequence (68) splits in Frobenius-stable direct factors

\[
L_A^n = R_A^n \oplus S_A^{(n)}
\]

**Lemma.** The complement \( S_A^{(n)} \) is \( \nabla^{GM} \)-horizontal, i.e. \( \nabla^{GM} (S_A^{(n)}) \subseteq S_A^{(n)} \)

**Proof:** The filtration \( \mathcal{F}^* \) splits on \( A \), i.e.

\[
L_A^n = \bigoplus_{m=0}^n (R_A^m \otimes S_A^{n-m}) = R_A^n \oplus S_A^{(n)} \quad \text{with} \quad S_A^{(n)} = \bigoplus_{m=0}^{n-1} (R_A^m \otimes S_A^{n-m})
\]

For all \( m \leq n - 1 \),

\[
\nabla^{GM} (R_A^m \otimes S_A^{n-m}) \subseteq (\nabla^{GM} (R_A^m) \otimes S_A^{n-m}) \oplus (R_A^m \otimes \nabla^{GM} (S_A^{n-m}))
\]

The second direct factor is contained in \( R_A^m \otimes S_A^{n-m} \subseteq S_A^{(n)} \) and the first direct factor is contained in

\[
L_A^m \otimes S_A^{n-m} = \bigoplus_{j=0}^m (R_A^j \otimes S_A^{m-j}) \otimes S_A^{n-m} = \bigoplus_{j=0}^m (R_A^j \otimes S_A^{n-j}) \subseteq S_A^{(n)}
\]

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Remark. Each tangent field \( t \) on \( M = M(N_2) \) induces a derivation \( \nabla^G_M : \mathcal{L}_M \rightarrow \mathcal{L}_M \), this providing, as explained in [31] 1.0.17 to 1.0.21, the isomorphism, as rank two \( \mathcal{O}_M \)-sheaves, of the invertible \( \mathfrak{o} \otimes \mathcal{O}_M \)-sheaves \( \mathcal{T}_M \) and \( \mathcal{R}_M^{\otimes (-2)} \) so that the first decomposes as

\[
\mathcal{T}_M \approx L_M^{(-2,0)} \oplus L_M^{(0,-2)} = \text{Hom}_{\mathcal{O}_M}(L_M^{(1,0)}, L_M^{(-1,0)}) \oplus \text{Hom}_{\mathcal{O}_M}(L_M^{(0,1)}, L_M^{(0,-1)})
\]

For a tangent field \( \tau \) in the first direct factor \( L_M^{(-2,0)} \) of \( \mathcal{T}_M \), and a local section \( s \) of \( \mathcal{L} \), this says the following: if \( s \) lies in \( L_M^{(1,0)} \), then then the projection to \( \mathcal{S}_M \) of \( \nabla^G_M(s) \) lies in \( L_M^{(-1,0)} \); if \( s \) lies in \( L_M^{(0,1)} \) then \( \nabla^G_M(s) = 0 \), and the analogous holds for \( \tau' \) in \( L_M^{(0,-2)} \).

Extending the remark to the whole of \( Y(N_2) \) (understood as subindex in the notations), the dual \( \mathcal{O}_{Y(N_2)} \)-isomorphism is \( \Omega_{Y(N_2)}(\log D^c) \approx \mathcal{R}_c^{\otimes 2} \), and all three compositions

\[
L^{(0,1)} \hookrightarrow \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes L^{(2,0)} \twoheadrightarrow L^{(-1,0)} \otimes L^{(2,0)}
\]

\[
L^{(0,1)} \hookrightarrow \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes L^{(2,0)} \twoheadrightarrow L^{(0,-1)} \otimes L^{(2,0)}
\]

\[
L^{(1,0)} \hookrightarrow \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes L^{(2,0)} \twoheadrightarrow L^{(0,-1)} \otimes L^{(2,0)}
\]

are null, and

\[
L^{(1,0)} \hookrightarrow \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes L^{(2,0)} \twoheadrightarrow L^{(-1,0)} \otimes L^{(2,0)}
\]

is an isomorphism; and analogously \( \nabla' \) provides the isomorphism \( L^{(0,1)} \approx L^{(0,-1)} \otimes L^{(0,2)} \). In particular, both \( \nabla \) and \( \nabla' \) preserve the decomposition \( \mathcal{L} \approx \mathcal{L}^{(1,0)} \oplus \mathcal{L}^{(0,1)} \), a comment in [31] to 2.3.18.

To understand the behavior of the covariant derivative on the symmetric powers \( \mathcal{L}^n \) of the rank four bundle \( \mathcal{L} \) and its filtrations, let us consider the canonical trivialization of the rank two bundle \( \mathcal{R}_M \approx L_M^{(1,0)} \oplus L_M^{(0,1)} \) in the punctured formal neighborhood of the standard cusp by the canonical section \( \omega_{\text{can}} \) (cf. comment of [13]), now just written \( \omega \) for simplicity, and its conjugate \( \omega' \). This gives a trivialization of the tangent bundle \( \mathcal{T}_M \approx L_M^{(-2,0)} \oplus L_M^{(0,-2)} \) by the square dual bases \( \tau = \omega^{-2} \) and \( \tau' = \omega'^{-2} \). The rank four bundle \( \mathcal{L} \) is trivialized, by adding to \( \omega \) and \( \omega' \) the local sections

\[
\eta := \nabla^G_M(\omega) \quad \text{and} \quad \eta' := \nabla'^G_M(\omega')
\]

and there are also the identities \( \nabla^G_M(\omega) = 0 \) and \( \nabla^G_M(\omega') = 0 \). Therefore,
\[ \nabla^{GM} \eta = 0 \quad \text{and} \quad \nabla^{GM} \eta' = 0 \]

The vector bundle \( \mathcal{L}^n \) is consequently trivialized by sections

\[ (\omega)^k (\omega')^{k'} (\eta)^l (\eta')^{l'} \quad \text{with} \quad k + k' + l + l' = n \]

in terms of which

\[ \nabla^{GM}_f (\omega)^k (\omega')^{k'} (\eta)^l (\eta')^{l'} = 0 \quad \text{and analogously for} \quad \nabla^{GM}_f. \]

We express the outcome of this remark in terms of the above filtration \( \mathcal{F}^\bullet \) and of a finer filtration \( \mathcal{F}^{\bullet \bullet} \) defined by taking \( \mathcal{F}^{(k,0)} \subseteq \mathcal{L}^n \) the image

\[ L^{(k,0)} \otimes \mathcal{L}^{n-k} \hookrightarrow \mathcal{R}^k \otimes \mathcal{L}^{n-k} \hookrightarrow \mathcal{L}^k \otimes \mathcal{L}^{n-k} \rightarrow \mathcal{L}^n \]

and analogous \( \mathcal{F}^{(0,k')} \subseteq \mathcal{L}^n \), and by being \( \mathcal{F}^{(k,k')} := \mathcal{F}^{(k,0)} \cap \mathcal{F}^{(0,k')} \subseteq \mathcal{L}^n \). The outcome is that these filtrations are compatible with the Gauss-Manin-connection in the sense that

\[ \nabla^{GM}(\mathcal{F}^m) \subseteq \mathcal{F}^{m-1} \otimes \Omega_{Y(N_2)}(\log D_c) \]
\[ \nabla(\mathcal{F}^{(k,k')}) \subseteq \mathcal{F}^{(k-1,k')} \otimes \mathcal{L}^{(2,0)} \]
\[ \nabla'(\mathcal{F}^{(k,k')}) \subseteq \mathcal{F}^{(k,k'-1)} \otimes \mathcal{L}^{(0,2)} \]

so to induce isomorphisms

\[ \nabla : \mathcal{F}^{(k,k')}/\mathcal{F}^{(k+k'+1)} \xrightarrow{\cong} \mathcal{F}^{(k-1,k')}/\mathcal{F}^{(k+k')} \otimes \mathcal{L}^{(2,0)} \]
\[ \nabla' : \mathcal{F}^{(k,k')}/\mathcal{F}^{(k+k'+1)} \xrightarrow{\cong} \mathcal{F}^{(k,k'-1)}/\mathcal{F}^{(k+k')} \otimes \mathcal{L}^{(0,2)} \]

\[ (\mathcal{F}^m_{W_c}/\mathcal{F}^{m+1}_{W_c}) \otimes \omega_{W_c}(\log D_c) \approx \bigoplus_{i+j=m} \mathcal{F}^{(i,j)}_{W_c}/\mathcal{F}^{(i,j)}_{W_c} \cap \mathcal{F}^{m+1}_{W_c} \otimes \mathcal{L}^{(2,2)}_{W_c} \]

**Definition.** We say that a rigid \( L^{(k,k')}_{A} \)-differential is *nearly overconvergent* \( \varrho \) when it is in the image of

\[ \Gamma_{rig}(\Omega^i_{A} \otimes \mathcal{L}^n_{A})_{par} \longrightarrow \Gamma_{rig}(\Omega^i_{A} \otimes \mathcal{L}^n_{A})_{par} \xrightarrow{\text{sp}^n} \Gamma_{rig}(\Omega^i_{A} \otimes \mathcal{R}^n_{A}) \]
been an analogue. We recall at this point that the Hida projector $\Gamma_{rig}(\Omega_{\mathcal{A}}^i \otimes \mathcal{L}_{\mathcal{A}}^n) \rightarrow \Gamma_{rig}(\Omega_{\mathcal{A}}^i \otimes \mathcal{R}_{\mathcal{A}}^n)$, (The image of $\Gamma_{rig}(\Omega_{\mathcal{A}}^i \otimes \mathcal{L}_{\mathcal{A}}^n)_{par}$ lies in $\Gamma_{rig}(\Omega_{\mathcal{A}}^i \otimes \mathcal{R}_{\mathcal{A}}^n)$ because of the description of the $\nabla^{GM}$-derivative of local sections, in the former remark, in terms of our choice of a local base of $\mathcal{L}$).

We say that two nearly overconvergent $1$-differentials are equivalent $\varrho \approx \sigma$ when $e_{ord}j_{X(N)}^*(\varrho - \sigma) = 0$ (the reason is that they will have same contribution to the cup-product computing the $p$–adic Abel Jacobi map); and say two overconvergent $L^{(k,k')}$-2-differentials are equivalent when its difference has a $\nabla^{GM}$-primitive equivalent to $0$.

Considering now the generically injective morphism $j_{X(N)} : X(N) \rightarrow Y(N_2)$, the splitting $sp$ induces, via $j_{X(N)}^*$, the equally denoted splitting $sp : \mathcal{L}_{0,A_0} \rightarrow L_{0,A_0}$ of the restriction to $A_0 := j_{X(N)}^{-1}(A)$ of each of the two direct factors of $j_{X(N)}^* \mathcal{L} \approx \mathcal{L}_0 \oplus \mathcal{L}_0$, hence an splitting

$$sp : \mathcal{L}_{0,A_0}^n \rightarrow L_{0,A_0}^n$$

of $[69]$. Therefore, a nearly overconvergent $L^{(k,k')}$-differential in our sense ”restricts”, i.e. has image by $j_{X(N)}^*$, to a nearly overconvergent $L_{0+k'}^{(k,k')}$-differential, in the sense of $[17]$ def. 4.3, of which our definition has been an analogue. We recall at this point that the Hida projector

$$e_{ord} = \lim_n U^n_p$$

from the space of overconvergent modular forms to the subspace of the ordinary ones, extends to the space of nearly overconvergent forms, and has the same image space, cf. lemma 2.7 in $[17]$.

**Proposition 6 :**

a) For a wide open set $\mathcal{W}_\varepsilon$, any $\varrho \in \Gamma_{rig}(\mathcal{L}_{\mathcal{W}_\varepsilon}^n \otimes \Omega_{\mathcal{W}_\varepsilon})_{par} \subseteq \Gamma_{rig}\mathcal{L}_{\mathcal{W}_\varepsilon}^n \otimes \Omega_{\mathcal{W}_\varepsilon}(\log D^c)$ is sum $\varrho = \varrho_n + \nabla^{GM} \alpha$ for some $\varrho_n \in \Gamma_{rig}\mathcal{R}_{\mathcal{W}_\varepsilon}^n \otimes \Omega_{\mathcal{W}_\varepsilon}$ and $\alpha \in \Gamma_{rig}\mathcal{L}_{\mathcal{W}_\varepsilon}^n$.
b) $sp(\varrho) \approx \varrho_n$

**Proof:**

Proof: a) This is particular case of the more general fact that any $\varrho_m \in \Gamma_{rig}\mathcal{F}_{\mathcal{W}_\varepsilon}^m \otimes \Omega_{\mathcal{W}_\varepsilon}(\log D^c)$ is sum $\varrho_m = \varrho_{m+1} + \nabla \alpha$ for some $\varrho_{m+1} \in \Gamma_{rig}\mathcal{F}_{\mathcal{W}_\varepsilon}^{m+1} \otimes \Omega_{\mathcal{W}_\varepsilon}$ and $\alpha \in \Gamma_{rig}\mathcal{L}_{\mathcal{W}_\varepsilon}$, which we prove by induction. For each summand $\overline{u}_{i,j}$ of the class $\overline{\varrho}_m = \sum_{i+j = m} \overline{u}_{i,j}$ of $\varrho_m$ in the decomposition

$$(\mathcal{F}_{\mathcal{W}_\varepsilon}^m/\mathcal{F}_{\mathcal{W}_\varepsilon}^{m+1}) \otimes \Omega_{\mathcal{W}_\varepsilon}(\log D^c) \approx \bigoplus_{i+j = m} \mathcal{F}_{\mathcal{W}_\varepsilon}^{(i,j)}/\mathcal{F}_{\mathcal{W}_\varepsilon}^{(i,j)} \cap \mathcal{F}_{\mathcal{W}_\varepsilon}^{m+1} \otimes L_{\mathcal{W}_\varepsilon}^{(2,2)}$$

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one of the two indexes \( i, j \) is positive, for instance \( i > 0 \), i.e. \( j < m \), so it is the image \( \nabla'(\overline{\zeta}_{i,j}) = \nabla^{GM}(\overline{\zeta}_{i,j+1}) \) for some

\[
\overline{\zeta}_{i,j+1} \in \Gamma_{rig}(\mathcal{F}^{(i,j+1)}_{W_e} / \mathcal{F}^{(i,j+1)}_{W_e} \cap \mathcal{F}^{m+2}_{W_e}) \otimes L^{(2,0)}_{W_e}
\]

because of the above isomorphism \( \mathcal{L} \) (tensored by \( L^{(2,0)}_{W_e} \))

\[
\partial': (\mathcal{F}^{(i,j+1)}_{W_e} / \mathcal{F}^{(i,j+1)}_{W_e} \cap \mathcal{F}^{m+2}_{W_e}) \otimes L^{(2,0)}_{W_e} \approx (\mathcal{F}^{(i,j)}_{W_e} / \mathcal{F}^{(i,j)}_{W_e} \cap \mathcal{F}^{m+1}_{W_e}) \otimes L^{(2,2)}_{W_e}
\]

b) From part a) we obtain

\[
\partial'_{X'(N)} = \partial'_{X'(N)} + \nabla(\partial'_{X'(N)} s),
\]

so that, by lemma 2.7 of \( \mathcal{L} \),

\[
e_{ord}(\partial'_{X'(N)} n) = e_{ord}(sp(\partial'_{X'(N)})) = e_{ord}(\partial'_{X'(N)} sp(n))
\]

\( \square \)

Composing the covariant derivative of a rigid analytic section \( \alpha \) of \( \mathcal{R}_A^n \subseteq \mathcal{L}_A^n \) with the split projection \( sp \) in \( \mathcal{L} \), we obtain what may be called its ”split derivative”

\[
\nabla^{GM}_{sp} \alpha := sp(\nabla^{GM} \alpha) = (\nabla_{sp} \alpha, \nabla'_{sp} \alpha),
\]

which is a rigid analytic section on \( A \), defined over \( \mathbb{Q}_p \), of

\[
\mathcal{R}_A^n \otimes \Omega_A(\log D^c) \approx (\mathcal{R}_A^n \otimes L^{(2,0)}_A) \oplus (\mathcal{R}_A^n \otimes L^{(0,2)}_A)
\]

(cf. \( \mathcal{L} \) 2.5.7, \( p \)-adic analogue of 2.3.2 and 2.3.12). A key fact which follows of the behavior \( \mathcal{L} \) of the covariant derivative (cf. also \( \mathcal{L} \) 2.5.12) is that, if \( \alpha \) is in fact a section of a modular line subbundle \( L^{(k,k')}_{A} \subseteq \mathcal{R}_A^n \), i.e. corresponds to a \( p \)-adic form of weight \((k, k')\), then

\[
\nabla^{GM}_{sp} \alpha = (\nabla_{sp} \alpha, \nabla'_{sp} \alpha) \text{ lies in }
\]

\[
L^{(k,k')}_{A} \otimes \Omega_A(\log D^c) \approx L^{(k+2,k')}_{A} \oplus L^{(k,k'+2)}_{A},
\]

i.e. it corresponds to a pair of \( p \)-adic \( \mathfrak{a} \)-Hilbert modular forms of weights \((k + 2, k')\) and \((k, k' + 2)\). We can thus write this split derivative, by abusing notation, as

\[
\nabla_{sp}: L^{(k,k')}_{A} \rightarrow L^{(k+2,k')}_{A} \quad \text{and} \quad \nabla'_{sp}: L^{(k,k')}_{A} \rightarrow L^{(k,k'+2)}_{A}.
\]

Generalizing \( \mathcal{L} \), if \( \alpha \) has \( q \)-expansion \( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} a_{\nu} q^\nu \), then the split derivatives \( \nabla_{sp} \alpha \) and \( \nabla'_{sp} \alpha \) have \( q \)-expansions in the given trivializations.
\[ \theta \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} a_\nu q^\nu \right) = \sum_{\nu \in (a^{-1})^+ \cup \{0\}} \nu a_\nu q^\nu \quad (81) \]

and
\[ \theta' \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} a_\nu q^\nu \right) = \sum_{\nu \in (a^{-1})^+ \cup \{0\}} \nu' a_\nu q^\nu . \quad (82) \]

This follows from the remark of Andreatta-Goren in [1] 12.26 that the Gauss-Manin covariant derivative studied by Katz in 2.6 [31] acts on \(p\)-adic modular forms as the theta operators studied by Andreatta-Goren in sec. 12 of [1], i.e. those acting on \(q\)-expansions as \(81\) (the two basic cases among those considered in 12.23.2 of [1]).

Obviously, if \(\alpha\) is an overconvergent \(L^a\)-differential, then \(\nabla^{GM}_\alpha\) is overconvergent, i.e. both \(\nabla \alpha\) and \(\nabla' \alpha\) are overconvergent, so that \(\nabla_{sp}^{GM} \alpha\) is nearly overconvergent. We can say more:

**Lemma 7** If \(\alpha\) is a nearly overconvergent \(L^{(k,k')}\)-differential, then \(\nabla^{GM}_\alpha\) is nearly overconvergent, i.e. both \(\nabla_{sp} \alpha\) and \(\nabla'_{sp} \alpha\) are nearly overconvergent.

**Proof:** Write \(n = k + k'\). It is enough to prove it for a rigid section \(\alpha\) of \(\mathcal{R}_A^n \otimes \Omega^i_A\) which is restriction of some rigid section \(\sigma\) of \(\mathcal{L}_{W^i}^n \otimes \Omega^i_{W^i}\), i.e. \(\sigma_A = \alpha + \beta\) with \(\alpha \in \mathcal{R}_A^n\) and \(\beta \in S_{A_i}^{(n)}\). The rigid section \(\nabla^{GM}_\alpha\) of \(\mathcal{R}_A^n \otimes \Omega^{i+1}_{A_i}\) is the split projection of the rigid section \(\nabla^{GM}_\sigma\) of \((\mathcal{L}_{W^i}^n \otimes \Omega^{i+1}_{W^i})_{\text{par}}\) because \(\nabla^{GM}(S_{W^i}^{(n)} \otimes \Omega^i_{W^i}) \subseteq (S_{W^i}^{(n)} \otimes \Omega^{i+1}_{W^i})_{\text{par}}\).

Observe that the ”key fact” works also for the second log \(D^c\)-covariant derivative \(\nabla^{GM}\) as it is induced by the first: any overconvergent section \((\beta, \gamma)\) of \(L^{(k,k')} \otimes \Omega(\log D^c) = L^{(k+2,k')} \oplus L^{(k,k'+2)}\) defined over \(\mathbb{Q}_p\), corresponding to a pair of \(p\)-adic \(a\)-Hilbert modular forms over \(\mathbb{Q}_p\) of weights \((k + 2, k')\) and \((k, k' + 2)\) with \(q\)-expansions \(\sum_{\nu \in (a^{-1})^+ \cup \{0\}} b_\nu q^n\)

and \(\sum_{\nu \in (a^{-1})^+ \cup \{0\}} c_\nu q^n\), has as split derivative \(\nabla^{GM}_{sp}(\beta, \gamma) = \nabla'_{sp} \beta - \nabla_{sp} \gamma\)

a rigid analytic section of \(\mathcal{L}_{A_i}^{k+k'} \otimes \omega_A(D^c)\), defined over \(\mathbb{Q}_p\), lying in fact inside \(L_{A_i}^{(k+2,k')} \otimes \omega_A(D^c) \approx L_{A_i}^{(k,k'+2)}\), thus corresponding to a nearly overconvergent \(a\)-Hilbert modular form over \(\mathbb{Q}_p\) of weight \((k + 2, k' + 2)\) and \(q\)-expansion
\[ \theta' \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} b_\nu q^\nu \right) - \theta \left( \sum_{\nu \in (a^{-1})^+ \cup \{0\}} c_\nu q^\nu \right) = \sum_{\nu \in (a^{-1})^+ \cup \{0\}} (\nu' b_\nu - \nu c_\nu) q^n . \quad (83) \]
This generalizes (20).

Finally, it will be useful also to recall that the second exterior power

\[ \bigwedge^2 \mathcal{L}_{Y(N_2)} = H_{dR}^2(\hat{A}^U) := \mathbb{R}^2 \text{pr}_{Y(N_2)*} \Omega^*_\mathcal{A}_{\hat{A}^U/Y(N_2)}(\log D_{c\hat{A}^U}) \]

inherits an equally denoted Gauss-Manin connection, and that the smallest filter \( \text{Fil}^2 \) of its Hodge filtration is the "Norm" modular line bundle \( L_{Y(N_2)}^{(1,1)} \) in the sequence

\[ 0 \rightarrow L_{Y(N_2)}^{(1,1)} \rightarrow \bigwedge^2 \mathcal{L}_{Y(N_2)} \rightarrow \bigwedge^2 \mathcal{L}_{Y(N_2)} \rightarrow S_2 \rightarrow 0 \]

analogous to (65), from which we have that \( L_{Y(N_2)}^{(n,n)} \) is the smallest filter \( \text{Fil}^{2n} \) of the Hodge filtration of \( \bigwedge^2 \mathcal{L}_{Y(N_2)} \).

### 4.2 Null-homologous cycles of higher dimension.

On the \( n \)-th product of a generalized elliptic curve \( E \) consider \( \epsilon_{E^n} = \epsilon_{E^n}^{\text{sym}} \circ \epsilon_{E^n}^{\text{inv}} \) with

\[ \epsilon_{E^n}^{\text{sym}} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma) \sigma \in \mathbb{Q}[\text{Aut}E^n] \quad (84) \]

\[ \epsilon_{E^n}^{\text{inv}} = \left( \frac{1 - u_1}{2} \right) \cdots \left( \frac{1 - u_n}{2} \right) \in \mathbb{Q}[\text{Aut}E^n] \quad (85) \]

where all \( u_i \) are equal to the involution \( u \) of \( E \), and denote equally the idempotent operator it induces on rational cycles of \( E^n \); this is alternatively presented in [3] (1.4.4) in terms of the map \( j : (\mu_2)^n \ltimes \Sigma_n \rightarrow \pm 1 \) product of the identity and the sign character:

\[ \epsilon_{E^n} = \frac{1}{2^n n!} \sum_{\tau \in (\mu_2)^n \ltimes \Sigma_n} j(\tau) \tau \in \mathbb{Q}[\text{Aut}E^n] \quad (86) \]

It is observed in lemma 1.8 of [3] that \( \epsilon_E \) vanishes at all the Künneth pieces of the de Rham homology \( H_{dR}^\bullet(E^n) \) of \( E^n \), except the one \( \bigotimes^n H^1(E) \) of the intermediate dimension \( n \), where it acts as the projection to its symmetric part

\[ \epsilon_{E^n} H_{dR}^n(E^n) = \text{Sym}^n H_{dR}^1(E) \]

or, by duality, in terms of the singular homology \( H_\bullet(E^n) \), it is null at all pieces \( H_i(E^n) \) of order \( i \neq n \), and
\[ \epsilon_{E^n} H_n(E^n) = Sym^n H_1(E) \]

We will denote by \( \mathcal{E} \) the smooth compactification \( \hat{E}_{X(N)}^U \) (universal generalized elliptic curve) of the universal abelian scheme \( E_{X(N)}^U \) over \( X(N) \), and denote \( \mathcal{E}^n \) its \( n \)-th product fibred over \( X(N) \) (but will keep the notation \( \hat{A}_{Y(N_2)}^U \to Y(N_2) \) for the proper, smooth variety of dimension \( 2n + 2 \) smooth \( n \)-th fibred product of the compactified universal family \( \hat{A}_{Y(N_2)}^U \) over \( Y(N_2) \)). The operator \( \epsilon_{E^n} \) is defined fibrewise, and as a consequence of the above killing of nonintermediate cohomology, it is

\[ \epsilon_{E^n} H_{dR}^{n+1}(\mathcal{E}^n) = H_{dR}^1(L^0_0) \]

and \( \epsilon_{E^n} H_{dR}^i(\mathcal{E}^n) = 0 \) for \( i \neq n + 1 \) \hspace{1cm} (87)

(lemma 2.2 of [3]), i.e. \( \epsilon_{E^n} H_i(\mathcal{E}^n) = 0 \) for all \( i \neq n + 1 \).

For an integer \( n \geq 0 \) and even integer \( 0 \leq n_0 < 2n \), denote \( m = n + \frac{n_0}{2} \). We now construct null-homologous \((m+1)\)-algebraic cycles \( \Delta_{n,n_0} \) of \( \hat{A}_{Y(N_2)}^U \times \mathcal{E}^{n_0} \), on which to compute the \( p \)-adic Abel-Jacobi map, using the generically injective embedding

\[ \mathcal{E}^{2n} = \mathcal{E}^n \times_{X(N)} \mathcal{E}^n = (\mathcal{E} \times_{X(N)} \mathcal{E})^n \to \hat{A}_{Y(N_2)}^U \]

given by [67]. We will use the idempotent operator \( \epsilon = \epsilon_{E^n} \times_{X(N)} \mathcal{E}^n \) in cohomology of \( \mathcal{E}^n \times_{X(N)} \mathcal{E}^n \), so restricting \( \epsilon_{E^n} \) to the diagonal \( \mathcal{E}^n \to \mathcal{E}^n \times_{X(N)} \mathcal{E}^n \). It is restricted from the equally denoted operator on the whole of \( \hat{A}_{Y(N_2)}^U \) with a definition on its fibres analogous to [54] but which does not have the property of killing all nonintermediate cohomology, as \( \epsilon = \epsilon_{E^n} \times_{X(N)} \mathcal{E}^n \) does: this last can be proved just as for lemma 2.2 of [3] i.e. \( \epsilon H^i(\mathcal{E}^n \times_{X(N)} \mathcal{E}^n) = 0 \) for \( i \neq 2n + 1 \) and \( \epsilon H^{2n+1}(\mathcal{E}^n \times_{X(N)} \mathcal{E}^n) = H^1(L^0_0 \otimes L^0_0) \) so \( \epsilon \) fixes the cohomology classes lying inside this subspace.

We consider first the case \( n_0 = 0 \), so that \( m = n \). In case \( n = 0 \), the cycle \( \Delta_{n,0} \) of \( Y(N_2) \times X(N) \) is the one already studied, so we can assume \( n \geq 1 \). Consider \( \mathcal{G} r_n \approx \mathcal{E}^n \) the graph of the projection \( pr_n : \mathcal{E}^n \to X(N) \) of the diagonal cycle

\[ \mathcal{E}^n \to \mathcal{E}^n \times_{X(N)} \mathcal{E}^n \to \hat{A}_{Y(N_2)}^U \]

and define the cycle of \( \hat{A}_{Y(N_2)}^U \times X(N) \)

\[ \Delta_{n,0} := (\epsilon \times id_{X(N)})(\mathcal{G} r_n) = (\epsilon_{E^n} \times id_{X(N)})(\mathcal{G} r_n) \]

Since \( \epsilon_{E^n} \) kills the homology of \( \mathcal{E}^n \) of dimension different of \( n + 1 \), the operator induced by \( \epsilon_{E^n} \times id_{X(N)} \) in the homology of \( \mathcal{E}^n \times X(N) \) kills its
homology different of \( n + 3, n + 2, n + 1 \). Since \( \mathcal{G}_{rn} \subseteq \mathcal{E}^n \times X(N) \) has dimension \( 2(n + 1) \), its homology class is killed by the operator, if \( n \geq 2 \).

In case \( n = 1, n_0 = 0 \), we choose a rational point \( o \in X(N) \) and write \( E_o \) the generalized elliptic curve \( pr^{-1}(o) \), so to have a decomposition

\[
\mathcal{G}_{r1} = \mathcal{E} \times \{o\} + \sum i_j E_i \times F_j + (E_o \times X(N))
\]

analogous to (4) for cycles \( F_j \) making a base of the 1-homology of \( X(N) \), and cycles \( E_i \) making a base of the 3-homology of \( \mathcal{E} \). Since this last homology is killed by the action of \( \epsilon_{\mathcal{E}} \), we obtain that the homology of \( \Delta_{1,0} := (\epsilon \times id_{X(N)})(\mathcal{G}_{r} - \mathcal{E} \times \{o\} - E_o \times X(N)) \) is null.

Finally, consider the case \( 0 < n_0 < 2n \). Write \( \{1, \ldots, m\} \) as union \( A \cup B \) of two subsets of cardinality \( n \), so with an intersection \( T = A \cap B \) of cardinality \( t = n - \frac{n_0}{2} = m - n_0 > 0 \). The projections \( pr_A : \mathcal{E}^m \to \mathcal{E}^n \) and \( pr_B : \mathcal{E}^m \to \mathcal{E}^n \) to the arguments in \( A \) and \( B \), induce a generically injective embedding

\[
\varphi_{m;2n} : \mathcal{E}^m \hookrightarrow \mathcal{E}^n \times_{X(N)} \mathcal{E}^n \cong \mathcal{E}^{2n} \to \hat{\mathcal{A}}_{Y(N_2)}^{U,n}
\]

the first being essentially the diagonal embedding

\[
\mathcal{E}^m \cong \mathcal{E}^t \times_{X(N)} \mathcal{E}^{n_0} \overset{(\text{diag})^t}{\hookrightarrow} (\mathcal{E} \times_{X(N)} \mathcal{E})^t \times_{X(N)} \mathcal{E}^{n_0}
\]

in the \( t \) arguments of \( T \). Observe that \( \epsilon \) restricts \( \epsilon_{\mathcal{E}^t} \times_{X(N)} \epsilon_{\mathcal{E}^{n_0}} \) to \( \mathcal{E}^m \), so it kills all the nonintermediate cohomology of \( \mathcal{E}^m \), i.e. \( \epsilon H^i(\mathcal{E}^m) = 0 \) for \( i \neq m + 1 \), and \( \epsilon H^{m+1}(\mathcal{E}^m) = H^1(L_0 \otimes L_0^{n_0}) \) so that \( \epsilon \) fixes the cohomology classes lying in this subspace. We define the cycle of \( \hat{\mathcal{A}}_{Y(N_2)}^{U,n} \times \mathcal{E}^{n_0} \) transform

\[
\Delta_{n,n_0} := (\epsilon \times \epsilon_{\mathcal{E}^{n_0}})(\mathcal{G}_{r_{m,n_0}})
\]

of the graph \( \mathcal{G}_{r_{m,n_0}} \subseteq \mathcal{E}^m \times \mathcal{E}^{n_0} \) of the projection

\[
\varphi_{m;n_0} : \mathcal{E}^m \to \mathcal{E}^{n_0}
\]

to the \( n_0 \) arguments in \( (A - T) \sqcup (B - T) \). It vanishes on the homology of \( \mathcal{E}^m \times \mathcal{E}^{n_0} \) of order different of its intermediate dimension \( (m + 1) + (n_0 + 1) \). In particular, it vanishes on the homology class of \( \mathcal{G}_{r_{m,n_0}} \cong \mathcal{E}^m \) as it has topological dimension \( 2(m + 1) \), thus differing \( t > 0 \) from the intermediate dimension.
The other ingredient for the computation of the \( p \)-adic Abel-Jacobi map will be a cuspidal form \( g \) of weight \( k_0 = n_0 + 2 \geq 2 \) and \( \Gamma_0 \)-level \( N \), thus modular for the group \( \Gamma(N) \), and a \( \mathfrak{a} \)-Hilbert cuspidal form \( f \) of level \( N_2 \) and weight \( (k, k) = (n + 2, n + 2) \geq (2, 2) \) to which we associate the \( \mathcal{L}^{2n} \)-differential,

\[
\omega_f \in \text{Fil}^{2+2n} H^{2+2n}_{dR}(\tilde{A}_{Y(N_2)}) = H^0(\omega_{\tilde{A}_{Y(N_2)}}) = H^0(\text{pr}^* \omega_{Y(N_2)} \otimes \omega_{\tilde{A}_{Y(N_2)}/Y(N_2)}) = H^0(\omega_{Y(N_2)} \otimes \text{pr}^* (\omega_{\tilde{A}_{Y(N_2)}/Y(N_2)})^n) = H^0(\omega_{Y(N_2)} \otimes \mathcal{L}^{(n,n)}). \tag{90}
\]

**Remark.** It will be used in the proof of the proposition below that the making the product \( \mathcal{L}_0 \otimes \mathcal{L}_0 \rightarrow \mathcal{O}_{X(N)}(-1) \) in each of the \( t \) fibred diagonal factors of \( \mathcal{E}^m \rightarrow \mathcal{E}^{2n} \), an \( \mathcal{L}_0^{2n} \)-differential in \( X(N) \), thus differential of order \( 2n + 1 \) in \( \mathcal{E}^{2n} \), transforms, after restriction to \( \mathcal{E}^m \rightarrow \mathcal{E}^{2n} \) an equally denoted \( \mathcal{L}_0^{n_0} \)-differential in \( X(N) \), so differential of order \( n_0 + 1 \) in \( \mathcal{E}^{n_0} \), via

\[
\mathcal{L}_0 \otimes \mathcal{L}_0 \rightarrow \mathcal{L}_0^{n_0/2} \otimes (\mathcal{L}_0 \otimes \mathcal{L}_0) \otimes t \otimes \mathcal{L}_0^{n_0/2} \rightarrow \mathcal{L}_0^{n_0/2} \otimes \mathcal{L}_0^{n_0/2} \rightarrow \mathcal{L}_0^{n_0} \rightarrow \mathcal{L}_0^{n_0}(-t) \rightarrow \mathcal{L}_0^{n_0} \quad \tag{91}
\]

(as in proof of prop. 2.9 in [17]).

**Proposition 8**

a) The \( p \)-adic Abel-Jacobi map is computed in terms of the product

\[
\langle \ldots \rangle: H^1_{dR-par}(X(N), \mathcal{L}_0^{n_0}(-t)) \times H^1_{dR-par}(X(N), \mathcal{L}_0^{n_0}) \rightarrow \mathbb{C}_p(-1-m)
\]

by

\[
AJ_p(\Delta_{n,n_0}) (\omega_f \otimes \eta_g^{u-r}) = \langle Q_f(\phi)^{-1} e_g e_{ord} j^*_{X'(N)} \partial_{\mathcal{F}_2}, \eta_g^{u-r} > \tag{92}
\]

for the \( \mathcal{L}^{2n} \)-differential \( \partial_{\mathcal{F}_2} \) with \( \nabla^{GM}(\partial_{\mathcal{F}_2}) = \omega_{\mathcal{F}_2} \).

b) It can also be computed as

\[
AJ_p(\Delta_{n,n_0}) (\omega_f \otimes \eta_g^{u-r}) = \langle Q_f(\phi)^{-1} e_g e_{ord} j^*_{X'(N)} \partial_{\mathcal{F}_2}, \eta_g^{u-r} > \tag{92}
\]

**Proof:** a) Besser theory is applied similarly as in section 3, generalizing the polynomial \( P_f(x) \) used in that computation by

\[
P_f(x) := (1 - p^{1-k} \phi) Q_f(x) \in \mathbb{Q}_p[x]
\]

By the same reason as in section 3, this polynomial kills the target of the residue map, so that \( \langle \langle 92 \rangle \rangle \) equals \( \langle P_f(\phi)^{-1} e_g e_{ord} j^*_{X'(N)} \partial_{\mathcal{F}_2}, \eta_g^{u-r} > \)

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for the overconvergent $\mathcal{L}^{2n}$-differential $\varrho_{\mathcal{F}^3}$ with $\nabla^{GM} (\varrho_{\mathcal{F}^3}) = \omega_{f^2}$, which is equivalent to $d\varrho_{\mathcal{F}^3} = \omega_{f^2}$ as $\varrho_{\mathcal{F}^3}$ and $\omega_{f^2}$ can be seen as overconvergent differentials in $\widehat{A}^{U_{\infty}}_{Y(N_2)}$ of order $2n + 1$ and $2n + 2$.

To prove \( \text{a)}\), we can assume $t > 0$ because this has been already proved for $n = n_0 = 2$. As analogous generalization of this case, and just as in the proof of the analogous in theor. 3.8 of [17], the $p$–adic Abel Jacobi map at the null-homologous cycle $\Delta_{n,n_0} = (\epsilon \times \epsilon_{n_0})\mathcal{E}^m$ of components isomorphic to $\mathcal{E}^m$ and defined over $\mathbb{Z}_p$ (by [86]), is given by the integral [46]

$$AJ_p(\Delta_{n,n_0})(\omega_f \otimes \eta^{u-r}_g) = \langle cl_{fp}(\Delta_{n,n_0}), \tilde{\omega}_f \otimes \tilde{\eta}^{u-r}_g \rangle_{fp}$$

in terms of liftings of the given differentials by the Besser epimorphism. A lifting $\tilde{\omega}_f$ is given by the primitive $\varrho_{\mathcal{F}^3}$ of $\omega_{f^2} = P(\phi)\omega_f$, and the product equals

$$tr\epsilon^{m}_{fp}(\varphi^{*}_{m;2n,n_0}(\tilde{\omega}_f \otimes \tilde{\eta}^{u-r}_g)) = \langle \varphi^{*}_{m;2n}(\tilde{\omega}_f), \varphi^{*}_{m;n_0}(\tilde{\eta}^{u-r}_g) \rangle_{fp}$$

by the same argument as decomposition (86) [17] in components of $\Delta_{n,n_0}$ (here $\varphi^{*}_{m;2n,n_0} : \mathcal{E}^m \rightarrow \mathcal{E}^{2n} \times \mathcal{E}^{n_0}$ is the product of $\varphi_{m;2n}$ and $\varphi_{m;n_0}$, defined over $\mathbb{Z}_p$). Just as in the argument of section 3, a preimage of the finite polynomial class $\varphi^{*}_{m;2n}(\tilde{\omega}_f)$ for the map $\mathcal{i}$ of [11] is the de Rham class of $P_f(\phi)^{-1}j^*_{X'(N)}\varrho_{\mathcal{F}^3}$, so that

$$AJ_p(\Delta_{n,n_0})(\omega_f \otimes \eta^{u-r}_g) = \langle P_f(\phi)^{-1}e_{\text{ord}}j^*_{X'(N)}\varrho_{\mathcal{F}^3}, \eta^{u-r}_g \rangle_{dR}$$

$$= \langle Q_f(\phi)^{-1}e_{\text{ord}}j^*_{X'(N)}\varrho_{\mathcal{F}^5}, \eta^{u-r}_g \rangle_{dR}$$

proving a) in this case. To make sense of this product, observe that restriction to $\mathcal{E}^m$ of the primitive $\varrho_{\mathcal{F}^3}$ of $\omega_{f^2} = P(\phi)\omega_f$ appearing in the its factor corresponds to an overconvergent section of $\mathcal{L}^m_0 \otimes \mathcal{L}^m_0$ (as class acted by $\epsilon$ thus in its image $\mathcal{L}^m_0 \otimes \mathcal{L}^m_0$), and it is then pulled-back by $pr_{m;2n} : \mathcal{E}^m \rightarrow \mathcal{E}^{2n}$ to a differential of $\mathcal{E}^m$ or section of $\mathcal{L}^m_0$; and observe that the second factor is an overconvergent section of $\mathcal{L}^m_0$ or overconvergent differential of $\mathcal{E}^{n_0}$, pulled-back by $pr_{m;n_0} : \mathcal{E}^m \rightarrow \mathcal{E}^{n_0}$ to a section of $\mathcal{L}^m$ or differential of $\mathcal{E}^m$. The product of a de Rham class in $\mathcal{E}^m$ and the de Rham class $\eta^{u-r}_g$ in $\mathcal{E}^{n_0}$ makes sense because, by projection formula, it can be equivalently understood with pull-back by $pr_{m;n_0}$ in the second argument or with push-forward in the first argument.

As for the remaining case $n_0 = 0$, $n = 1$, two analogous terms to those \([62]\) in the proof of \([4]\) must be subtracted, and only one term if $n_0 = 0, n > 1$, but in both cases, and by analogous reason, they do not contribute to the computation of product \([92]\).
b) We decompose $\varrho_{\pi^2} = \varrho_n + \nabla s$ with $\varrho_n \approx \text{sp}(\varrho)$ by (6), and remark that $\varrho_n$ is also a primitive of $\varrho$ so that it can be replaced en (92), so also $\text{sp}(\varrho)$, instead of $\varrho_{\pi^2}$.

Remark. Observe that we have not used as polynomial $P_f(x)$ the analogous to the one considered by [17] (to fulfill condition ii) of subsection 3.2) as this would be $(1 - p^{-n_{1/2}})Q_f(x) = (1 - p^{n+1})Q_f(x)$. As some of its roots have complex norm $p^{\pm n_{1/2}} = p^{\pm k_{1/2}}$ and some have $p^{\pm n_{1/2}} = p^{\pm k_{1/2}}$, so not all of them of the same complex norm, it would not be allowed for our use in Besser theory. The first factor is used in order to vanish, while evaluated in $\phi$, in the target of the residue map for the first argument of the cup product (92). Since this argument lies in $H^1_{dR - \text{par}}(X(N), L^{\omega_0}_n(\mathbb{G}))$ the target of its residue map is twisted by $Q_p(-n_{1/2} - t) = Q_p(-n - 1)$, so the factor $(1 - p^{-2n_{1/2}}x) = (1 - p^{1-k}x)$ we consider in our definition of $P_f(x)$ does the job.

### 4.3 Primitives for depleted Hilbert modular forms

Keeping our notations, if an overconvergent $\mathfrak{a}$-Hilbert cuspidal form $f$ of weight $(k, k')$, level $N_2$, defined over $\mathbb{Q}_p$, has $q$-expansion (15)

$$f(q) = \sum_{\nu \in (a^{-1})^{+}} a_{\nu} q^{\nu},$$

its $p$-depletion and $\pi$-depletion are the overconvergent $\mathfrak{a}$-Hilbert modular forms

$$f[p] = (1 - V_p U_p) f \quad \text{and} \quad f[\pi] = (1 - V_\pi U_\pi) f$$

of same weight and level, and $q$-expansion

$$f[p](q) = \sum_{\nu \in (a^{-1})^{+}, \ p | \nu} a_{\nu} q^{\nu} \quad \text{and} \quad f[\pi](q) = \sum_{\nu \in (a^{-1})^{+}, \ \pi | \nu} a_{\nu} q^{\nu}.$$ (94)

An overconvergent $\mathfrak{a}$-Hilbert modular form $f$ is said $p$-depleted if $f = f[p]$ and $\pi$-depleted if $f = f[\pi]$. Obviously these are idempotent operators, and $\pi$-depleted implies $p$-depleted. We assume from now on that $f$ is nonordinary at both $\pi$ and $\pi'$.

Proposition. Under this nonordinarity condition for cuspidal $f$ of weight $(k, k') = (n + 2, n' + 2)$,

a) there is a unique $\mathcal{L}^{n+n'}$-valued $(1, 0)$-differential $\varrho_{\pi[\pi']} \in \Omega^{(1,0)}_{W_c} \otimes \mathcal{L}^{n+n'}_{W_0} \subseteq L^{(2,0)}_{W_c} \otimes \mathcal{L}^{n+n'}_{W_c}$ such that

$$\nabla^{GM}(\varrho_{\pi[\pi']}) = \nabla' \varrho_{\pi[\pi']} = \omega_{f[\pi']}$$
and is rational sum of \(n'\) nearly overconvergent forms \((\theta')^{-j-1}f^{[\pi']}\) in the sense that, near the standard cusp, it trivializes (in the above base trivializing \(L^{n+n'}\)) as sum of \(q\)-expansions

\[
\mathcal{F}^{[\pi']} (q) = \sum_{j=0}^{n'} (-1)^j j! \binom{n'}{j} (\theta')^{-j-1} f^{[\pi']} (q) \omega^n (\omega')^{n'-j} (\eta')^j .
\]  

(95)

b) There is a unique nearly convergent \(L^{(n,n')} - (1,0)\)-differential \(\varrho(F^{[\pi']}_{[k,k'-2]})\) associated to overconvergent cuspidal form \(F^{[\pi']}_{[k,k'-2]}\) of weight \((k,k' - 2) = (n + 2, n')\) such that

\[
\nabla^G \varrho(F^{[\pi']}_{[k,k'-2]}) = \nabla' \varrho(F^{[\pi']}_{[k,k'-2]}) = \omega f^{[\pi']}
\]

thus \(F^{[\pi']}_{[k,k'-2]}\) having \(q\)-expansion

\[
F^{[\pi']}_{[k,k'-2]} (q) = \sum_{\nu \in (a-1)^+, \pi' | \nu} (\nu')^{-1} a_{\nu} q^{\nu'}
\]  

(96)

i.e. \(\theta'(F^{[\pi']}_{[k,k'-2]}) = f^{[\pi']}\), so that we write (because of the uniqueness)

\(F^{[\pi']}_{[k,k'-2]} = (\theta')^{-1} f^{[\pi']}\)

**Proof:**

a) The purely formal fact that (95) is the \(q\)-expansion of the \(\nabla^G\) primitive if it exists, thus the unicity, is an obvious computation which goes exactly as for depleted modular forms, since we only derive one of the two variables (this is made in detail in comment preceding lemma 9.2 of [12], in the particular case where \(f_i = 0\) in notations of Loc. cit.) The essential fact of its existence is proved, under the nonordinarity hypothesis of \(f\) at both \(\pi\) and \(\pi'\), in [36] (this nonordinarity condition was not imposed in a previous version of our article, and is in fact unnecessary according a conjecture in Loc. cit.). In particular, it is proved in Loc. cit. that \((\theta')^{-n'-1} f^{[\pi']}\) is overconvergent, so it follows that its iterated \(\theta'\) derivatives \((\theta')^{-j-1} f^{[\pi']}\), for all \(j = 0, ..., n' - 1\), are all nearly-overconvergent, by lemma 7.

b) It follows from a) by just taking \(\varrho(F^{[\pi']}_{[k,k'-2]}) = sp(\varrho(F^{[\pi']}))\), i.e. the term \(j = 0\) in the sum (95) of \(q\)-expansions.

Observe that the coefficients of the expansion (96) stay all in \(\mathbb{Z}_p\) as \(\pi \nmid \nu'\) in all of them, so that \(\nu'\) is invertible in \(\mathfrak{p}_\pi \approx \mathbb{Z}_p\).

**Remark.** Applying (91) to the \(L^{2n}\)-differential \(j_{X'}^{*} T(N) \varrho (\mathcal{F}^{[\pi']}) \) in (95), we obtain an equally denoted overconvergent differential in \(L^{2n}_0 (-t)\) whose split projection is the nearly overconvergent \((-1)^t t! j_{X'}^{*} T(N) \varrho (\theta^t - t f^{[\pi']})\), thus with an overconvergent \(e_{ord}\) projection which we will prove to compute the \(p\)-adic Abel-Jacobi map for higher dimensional null-homologous cycles.
5 Computation of the $p$–adic Abel–Jacobi map

Let us recall first how the isotypic component and the stabilizations are made for cuspidal forms, so to imitate it for $a$-Hilbert cuspidal forms. Assume that $g$ is a cuspidal form over $\mathbb{Q}$ of weight $k_0 = n_0 + 2$ with $n_0 \geq 0$ for the modular group $\Gamma(N)$, and consider it as defined over $\mathbb{Q}_p$, for $p \nmid N$. Let

$$
\sum_{n \geq 1} b_n q^n
$$

be its $q$--expansion. Assume also that $g$ is an eigenform of eigenvalue $b_n$ for all the Hecke operators $T_n$, and that it is normalized, so that all $b_n \in \mathbb{Z}$. In particular $b_p g = T_p g = U_p g + p^{k_0 - 1} V_p g$. As a consequence, and using $U_p V_p = 1$, the operator

$$
1 - V_p b_p + p^{k_0 - 1} V_p^2 = (1 - \beta_0 V_p)(1 - \beta_1 V_p)
$$

with $\beta_0 + \beta_1 = b_p$ and $\beta_0 \beta_1 = p^{k_0 - 1}$

acts on the cuspidal form $g$ as the $p$–depletion operator $1 - V_p U_p$, where $\beta_0, \beta_1 \in \mathbb{Q}$ are the reciprocal roots of this polynomial in $V_p$, i.e. $\beta_0^{-1} = p^{1 - k_0} \beta_1$ and $\beta_1^{-1} = p^{1 - k_0} \beta_0$ are its roots. Again we will not work in $X(N)$ with all the $L_0$–valued log $D^e$–de Rham cohomology, but with its parabolic piece

$$
H^1_{dR-par}(X(N), L_0^{n_0} / \mathbb{Q}_p) \subseteq H^1_{dR}(X(N), L_0^{n_0} / \mathbb{Q}_p)
$$

which inherits from de Rham an equally denoted product $< -, - >$, and an action of $V_p$ as $p^{1 - k_0} \phi$, having $U_p$ as inverse (as $U_p V_p = 1$ in this finite-dimensional space). The polynomial in $V_p$ vanishes on the class of the $L_0^{n_0}$–valued differential $\omega_g$, so $V_p$ leaves invariant a 2-dimensional subspace or isotypic component of $g$

$$
H^1_{dR-par}(X(N), L_0^{n_0} / \mathbb{Q}_p)(g) \subseteq H^1_{dR-par}(X(N), L_0^{n_0} / \mathbb{Q}_p)
$$

the inclusion following from the fact that the parabolic cohomology is $V_p$–invariant.

On the $\mathbb{C}_p$-extended isotypic component $H^2_{dR-par}(X_{C_p}(N), L_0^{n_0} / \mathbb{C}_p)(g)$,

$$
V_p = p^{1 - k_0} \phi \text{ acts as } \begin{pmatrix} p^{1 - k_0} \beta_1 \\ p^{1 - k_0} \beta_0 \end{pmatrix}
$$

$$
U_p = V_p^{-1} \text{ acts as } \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}
$$

(98)
We define the two $p$–stabilizations $g_i$ of $g$, $i \in \{0,1\}$, as

$$g_i := (1 - \beta_j V_p) g \in S_{k_0}^c(\Gamma(N), \mathbb{C}_p) \text{ for } j \in \{0,1\} \text{ with } j \neq i \quad (99)$$

and decompose $g$ as linear combination of these two stabilizations

$$g = \frac{\beta_0}{\beta_0 - \beta_1} g_0 - \frac{\beta_1}{\beta_0 - \beta_1} g_1. \quad (100)$$

Observe that $g_i$ has still eigenvalue $b_n$ for all operators $T_n$ with $p \nmid n$. Indeed,

$$T_n g_i = T_n g - \beta_j V_p T_n g = b_n (g - \beta_j V_p g) = b_n g_i,$$

since $V_p$ commutes with $U_n$, thus also with $T_n$. Observe also that $g_i$ is an eigenform for $U_p$ of eigenvalue $\beta_i$. Indeed, since $T_p g = \beta_p g = \beta_i g + \beta_j g$, it is

$$U_p g_i = U_p g - \beta_j g = T_p g - p^{k_0-1} V_p g - \beta_j g = \beta_i g - p^{k_0-1} V_p g = \beta_i g_i,$$

the last equality following from $\beta_i \beta_j = p^{k_0-1}$. The decomposition $(100)$ tells us that $g$ belongs to the 2-dimensional span

$$S_{k_0}^c(\Gamma(N), \mathbb{C}_p)(g) \subseteq S_{k_0}^c(\Gamma(N), \mathbb{C}_p)$$

of $g_0$ and $g_1$, a subspace acted by $U_p$ as

$$U_p = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad (101)$$

(in the strong sense, not only in the sense of cohomology classes, what we already had in $(98)$). Clearly, the natural homomorphism

$$S_{k_0}^c(\Gamma(N), \mathbb{Q}_p) \longrightarrow H^{1}_{\text{rig-par}}(X'_{\text{(N)}}, \mathcal{L}_{0}^{n_0}/\mathbb{Q}_p)$$

restricts to an isomorphism

$$S_{k_0}^c(\Gamma(N), \mathbb{Q}_p)(g) \approx H^{1}_{\text{rig-par}}(X_{\text{(N)}}, \mathcal{L}_{0}^{n_0}/\mathbb{Q}_p)(g) \subseteq H^{1}_{\text{rig-par}}(X'_{\text{(N)}}, \mathcal{L}_{0}^{n_0}/\mathbb{Q}_p), \quad (102)$$

the inclusion (up to isomorphism) being analogous to $(55)$. Assume now that $g$ is ordinary for $p$, i.e. $U_p$ acts on it via a $p$–adic unit $\beta_0$. This is equivalent to saying that $U_p$ acts via this unit on the cohomology class of its associated $L^{n_0}_0$–valued differential $\omega_g$. Therefore, its inverse operator $V_p$ in cohomology acts also via a $p$–adic unit $\beta_0^{-1} = p^{1-k_0} \beta_1$ on this class, i.e. $\phi = p^{k_0-1} V_p$ acts with algebraic eigenvalue $\beta_1$ of $p$–adic value $k_0 - 1$, and this is said ”acting with slope $k_0 - 1$”. In other words, $g$ lies in the
first eigenspace of the diagonalization\(^{[104]}\) with \(\beta_0\) being a \(p\)-adic unit. Since both algebraic eigenvalues \(\beta_0\) and \(\beta_1\) can be distinguished, in the ordinary case, by having \(p\)-adic value 0 or \(k_0 - 1\) (which explains our notation, taken from the case \(k_0 = 2\)), we could properly write \(\beta_0(g)\) and \(\beta_1(g)\), but spare this for convenience except when we want to emphasize it. We then denote

\[
H^1_{dR-par}(X_{\mathbb{C}^p}(N), \mathcal{L}_0^{\nu_0})_{\beta_0,1}(g) = H^1_{dR-par}(X_{\mathbb{C}^p}(N), \mathcal{L}_0^{\nu_0})_{\beta_0,1}(g) = H^0_{ord}(g) \oplus H^{u-r}(g)
\]

the decomposition of the \(\mathbb{C}_p\)-isotypic component of \(g\) in the two monodimensional ("ordinary" and "unit root) subspaces diagonalizing \(^{[108]}\), as already recalled for the special case \(k_0 = 2\) in \(^{[103]}\). The class of the corresponding \(L_0^{\nu_0}\)-valued differential \(\omega_g\) lies in the first direct factor. Since the the \(\mathbb{C}_p\)-valued product establishes a perfect pairing between the two factors, there is a unique class \(\eta_g^{u-r}\) in the second direct factor such that

\[
< \omega_g, \eta_g^{u-r} >= 1 ,
\]

(This class is presented in \(^{[17]}\) as lift of the class in \(H^1(X(N), L_0^{-\nu_0})\) represented by the \(L_0^{-\nu_0}\)-valued antiholomorphic differential which is hermitian-dual to \(\omega_g \in H^0(X(N), \omega_X(N) \otimes L_0^{\nu_0})\)).

Observe that this works also if the ordinary cuspidal form \(g\) of level \(N\) is not classical but only overconvergent, thus classical of level \(Np\) (Coleman classicality theorem), except that its isotypic component in \(H^2_{dR-par}(X'(N), \mathcal{L}_0^{\nu_0}/\mathbb{Q}_p)\) does not lie then inside \(H^2_{dR-par}(X(N), \mathcal{L}_0^{\nu_0}/\mathbb{Q}_p)\) as in \(^{[102]}\).

Denote still by \(e_{ord}\) the action of the Hida projector induced on cohomology. The obvious equation \(e_{ord}(g_0) = e_{ord}g - \beta_1 e_{ord} V_pg\) induces the equation in cohomology

\[
e_{ord}(g_0) = e_{ord}g - \beta_1 V e_{ord} g = \mathcal{E}_0(g) e_{ord} g ,
\]

for \(\mathcal{E}_0(g) = (1 - \beta_1 \beta_0^{-1}) = (1 - \beta_1^2 p^{1-k_0})\),

since \(V_p\) commutes in cohomology with \(U_p\), so also with \(e_{ord}\).

Since we are assuming that \(g\) is ordinary at \(p\), it makes sense to call \(g_0\) the ordinary \(p\)-stabilization \(g^{(p)}\) of \(g\), so the equality \(^{[105]}\) becomes \(e_{ord}g^{(p)} = \mathcal{E}_0(g) g\).

Next, we describe next similar notions for an \(\alpha\)-Hilbert modular form \(f\) defined over \(R\), thus over \(\mathbb{Q}_p\), of parallel weight \((k, k) \geq (2, 2)\) and level \(N \geq 4\) which is eigenform for all Hecke operators \(T_\nu\), in particular eigenform for \(T_\pi = U_\pi + p^{k-1} V_\pi\), i.e.

\[
T_\pi f = U_\pi f + p^{k-1} V_\pi f = a_\pi f
\]

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This decomposition -kind of Eichler relation since $\phi_\pi$ acts as $p^{k-1}V_\pi$ on the corresponding $L^{(k-2,k-2)}$-valued differential $\omega_f$ - follows from the computation of the coefficients of the $q$-expansion of the $T_\pi$ transform (found for instance in the discussion before (3) in VI. 1 of [40]). Since $U_\pi V_\pi = 1$, the operator

\[
1 - a_\pi V_\pi + p^{k-1}V_\pi^2 = (1 - \alpha_0 V_\pi)(1 - \alpha_1 V_\pi) \text{ with } \alpha_0 + \alpha_1 = a_\pi \text{ and } \alpha_0 \alpha_1 = p^{k-1} \tag{106}
\]

acts on $f$ as $\pi-$depletion operator $1 - V_\pi U_\pi$, where the algebraic values $\alpha_0, \alpha_1$ are the reciprocal roots of this polynomial in $V_\pi$, i.e. $\alpha_0^{-1} = p^{1-k} \alpha_1$ and $\alpha_1^{-1} = p^{1-k} \alpha_0$ are the roots; and $\alpha_{0'}, \alpha_{1'}$ are defined analogously for the prime $\pi'$. Recall we are under the nonordinarity hypothesis that all $\alpha_0, \alpha_1, \alpha_{0'}, \alpha_{1'}$ are $p-$adic nonunits.

The two ”$\pi$-stabilizations” $f_i \in S_{k,k}^\infty(N_2, \mathbb{C}_p)$ of $f$ are

\[
f_i := (1 - \alpha_j V_\pi) f \text{ for } j \neq i, \tag{107}
\]

i.e. $f_i = \sum_{\nu \in (a^{-1})^\times} c_{\nu} q^\nu$ with $c_{\nu} = a_\nu - \alpha_j a_{\nu/\pi}$, if $\pi \mid \nu$, and $a_{\nu}$ otherwise. Observe that $f_i$ has still eigenvalue $a_\nu$ for the Hecke operators $T_\nu$ with $\pi \nmid \nu$ (thus same eigenvalue $a_\nu$ for all the Hecke operators $T_\nu$ with $\pi \nmid \nu$). Indeed,

\[
T_\nu f_i = T_\nu g - \alpha_j V_\pi T_\nu f = a_\nu (g - \alpha_j V_\pi f)
\]

since $V_\pi$ commutes with $U_\nu$, thus also with $T_\nu$. Observe furthermore that $f_i$ is an eigenform for $U_\pi$ of eigenvalue $\alpha_i$. Indeed, since $T_\pi f = a_\pi f = \alpha_i f + \alpha_j f$,

\[
U_\pi f_i = U_\pi f - \alpha_j f = T_\pi f - p^{k-1} V_\pi f - \alpha_j f = \alpha_i f - p^{k-1} V_\pi f = \alpha_i f_i,
\]

the last equality because $\alpha_i \alpha_j = p^{k-1}$. We have then the decomposition

\[
f = \frac{\alpha_0}{\alpha_0 - \alpha_1} f_0 - \frac{\alpha_1}{\alpha_0 - \alpha_1} f_1, \tag{108}
\]

and the analogous

\[
f = \frac{\alpha_{0'}}{\alpha_{0'} - \alpha_{1'}} f_{0'} - \frac{\alpha_{1'}}{\alpha_{0'} - \alpha_{1'}} f_{1'},
\]

in terms of the two $\pi'$-stabilizations $f_{i'}$ having eigenvalue $\alpha_{i'}$ for $U_\pi'$ and eigenvalue $a_{\nu}$ for all $T_\nu, \pi' \nmid \nu$.

We can define also the four ”$\pi, \pi'$-stabilizations” $f_{ii'} \in S_{k,k}^\infty(N_2, \mathbb{C}_p)$ of $f$ as

\[
f_{ii'} := (1 - \alpha_{i'} V_{\pi'})(1 - \alpha_i V_\pi) f = (1 - \alpha_i V_\pi)(1 - \alpha_{i'} V_{\pi'}) f. \tag{109}
\]
This is an eigenform for $U_\pi$ and $U_{\pi'}$ of eigenvalue $\alpha_i$ and $\alpha_i'$, thus for $U_p = U_\pi U_{\pi'}$ of eigenvalue $\alpha_i \alpha_i'$; and it is an eigenform of $T_\nu$ of eigenvalue $a_\nu$ whenever $\pi, \pi' \nmid \nu$. In particular, the $\nu$—coefficients of the $q$—expansions of both $f_{i\nu}$ and $f$ are the same, for $\pi, \pi' \nmid \nu$.

From the two decompositions above, we obtain the four terms decomposition
\[
f = \sum_{i,i'} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_{0'} - \alpha_{1'})} f_{i\nu'}\]
in eigenforms of the $U_p$ operator, all having same eigenvalue $a_\nu$ for the operators $T_\nu$ with $\pi, \pi' \nmid \nu$.

Putting this together, $f$ belongs to the 2-dimensional span
\[
S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)(U_\pi, f) \subseteq S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)
\]
of $f_0$ and $f_1$ which is acted by $U_\pi$ as
\[
\left( \begin{array}{c} \alpha_0 \\ \alpha_1 \end{array} \right),
\]
and to the span of $f_0'$, $f_1'$ acted by $U_{\pi'}$ as
\[
\left( \begin{array}{c} \alpha_0' \\ \alpha_1' \end{array} \right);
\]
And on the other hand, $f$ belongs to the 4-dimensional span
\[
S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)(f) \subseteq S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)
\]
of $f_{00'}$, $f_{10'}$, $f_{10'}$, $f_{11'}$, which is acted by $U_p = U_\pi U_{\pi'}$ as
\[
\left( \begin{array}{cc} \alpha_0 \alpha_0' & \alpha_0 \alpha_1' \\ \alpha_1 \alpha_0' & \alpha_1 \alpha_1' \end{array} \right)
\]
so that
\[
S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)(f) \approx S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)(U_\pi, f) \otimes S^\text{oc}_{k,k}(N_2, \mathbb{C}_p)(U_{\pi'}, f)
\]
as $U_p = U_\pi U_{\pi'}$—acted space.
As a consequence, in $H^2_{dR-par}(Y(N_2), \mathcal{L}^n/\mathbb{C}_p)$, for $n = 2k - 2$, where $V_\pi$ acts as $p^{1-k}\phi_\pi$ and has inverse $U_\pi$ (because $U_\pi V_\pi = 1$ in this finite-dimensional space), the polynomial \([106]\) in $V_\pi$ vanishes on the class of the $L^{(k-2,k-2)}$-valued differential $\omega_f$, so $V_\pi$ leaves invariant a 2-dimensional subspace

\[ H^2_{dR-par}(Y(N_2), \mathcal{L}^n/\mathbb{C}_p)(\phi_\pi, f) \subseteq H^2_{dR-par}(Y(N_2), \mathcal{L}^n/\mathbb{C}_p) , \]

in which it acts with minimal polynomial \([106]\). On the $\mathbb{C}_p$-extension $H^2_{dR-par}(Y_{\mathbb{C}_p}(N_2), \mathcal{L}^n/\mathbb{C}_p)(\phi_\pi, f)$ of this subspace,

\[ V_\pi = p^{1-k}\phi_\pi = U_p^{-1} \text{ acts as } \begin{pmatrix} p^{1-k}\alpha_1 \\ p^{1-k}\alpha_0 \end{pmatrix} \]

and on the isotypic component $H^2_{dR-par}(Y(N_2), \mathcal{L}^{k-2}/\mathbb{C}_p)(f)$ of $f$ in $H^2_{dR-par}(Y(N_2), \mathcal{L}^n/\mathbb{C}_p)$, i.e. the minimal subspace invariant by $\phi = \phi_\pi\phi_{\pi'}$ in which the class of $\omega_g$ lies,

\[ V_\pi = V_\pi V_{\pi'} = U_p^{-1} = p^{2-2k}\phi \text{ acts as } \begin{pmatrix} p^{2-2k}\alpha_1\alpha_1' \\ p^{2-2k}\alpha_1\alpha_0' \\ p^{2-2k}\alpha_0\alpha_1' \\ p^{2-2k}\alpha_0\alpha_0' \end{pmatrix} \]

so that

\[ H^2_{dR-par}(Y(N_2), \mathcal{L}^{k-2}/\mathbb{C}_p)(f) \cong H^2_{dR-par}(Y(N_2), \mathcal{L}^n/\mathbb{C}_p)(\phi_\pi, f) \]

as $\phi_\pi = \phi_\pi\phi_{\pi'}$-acted space. Clearly, \([114]\) applies isomorphically to \([113]\) by the homomorphism

\[ S_{k,k}^{oc}(N_2, \mathbb{C}_p) \longrightarrow H^2_{par-rig}(Y_{\mathbb{C}_p}(N_2), \mathcal{L}^n/\mathbb{C}_p)(f) . \]

This works also if $f$ is only overconvergent, with $H^2_{par-dR}(Y_{\mathbb{C}_p}(N_2), \mathcal{L}^n/\mathbb{C}_p)(f)$ instead of $H^2_{dR-par}(Y_{\mathbb{C}_p}(N_2), \mathcal{L}^n/\mathbb{C}_p)(f)$.

Consider the degree four characteristic polynomial of $\phi$ on the isotypic component of $\omega_f \in H^0(L_{Y(N_2)}^{(k,k)}(-D^e))$, with coefficients in $R$, thus in $\mathbb{Q}_p$,

\[ Q_f(x) = \prod (1 - \alpha_i^{-1}\alpha_{i'}^{-1}x) = \prod (1 - \alpha_i\alpha_{i'}p^{2-2k}x) \in \mathbb{Q}_p[x] \]

generalizing the polynomial given in \([50]\) (Its coefficients must be symmetric in $\alpha_0$ and $\alpha_1$, and also symmetric in $\alpha_0'$ and $\alpha_1'$, thus must be rational expressions of $\alpha_0 + \alpha_1 = \alpha$ and $\alpha_0\alpha_1 = p^{k-1}$ and $\alpha_0' + \alpha_1' = \alpha_{\pi'}$ and
\[ \alpha_0 \alpha_V = p^{k-1}, \text{all values in } R. \] As \( Q_f(\phi) = 0 \) it divides the characteristic polynomial of \( \phi \) acting on \( H^{2+2n}_{dR}(\mathcal{A}_{Y(N_2)}^U) \), so its roots \( \alpha_i \alpha_{i'} \) have complex norm \( p^{\frac{u}{2}} = p^{k-1} \), for \( u = 2 + 2n \). As operative tool, we will denote
\[
Q_{ii'}(x) := \frac{Q_f(x)}{(1 - \alpha_i \alpha_{i'} p^{2-2k} x)}.
\]

Denote \( \omega_f \) the \( L^{(k-2,k-2)} \)-differential \( \omega_f \) associated to the overconvergent \( a \)-Hilbert cuspidal form
\[
f^2 := Q_f(\phi)(f) = \sum (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_{0'} - \alpha_{1'})} Q_f(\phi) f_{ii'}
\]
as follows from (110) (where, abusing notation, we write \( p \) for the operator \( p^{2k-2} V_p \) acting in \( S_{k,k}^c(N_2, \mathbb{C}_p) \)). In the following proof of the main theorem 2 we find, explicitly up to the equivalence \( \approx \) defined in (111), an \( L^{2n} \)-differential \( \varrho_f^o \) with \( \nabla^{\mathbb{G}M}(\varrho_f^o) = \omega_f \).

**Proof of theorem 2**

To compute
\[
Q_f(\phi) f = \sum_{i=0}^{4} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_{0'} - \alpha_{1'})} Q_f(\phi) f_{ii'}
\]
decompose
\[
(1 - \alpha_i \alpha_{i'} V_{\pi} V_{\pi'}) f_{ii'} = \frac{1}{2}(1 - \alpha_i V_{\pi})(1 + \alpha_{i'} V_{\pi'}) f_{ii'} + \frac{1}{2}(1 - \alpha_{i'} V_{\pi})(1 + \alpha_i V_{\pi}) f_{ii'}.
\]
(116)
The first summand equals
\[
\frac{1}{2}(1 - \alpha_i V_{\pi}) f_{ii'} + \frac{1}{2} \alpha_i V_{\pi'}(1 - \alpha_{i'} V_{\pi}) f_{ii'} = \frac{1}{2}(1 - V_{\pi} U_{\pi}) f_{ii'} + \frac{1}{2} \alpha_i V_{\pi'}(1 - V_{\pi} U_{\pi}) f_{ii'}.
\]
(117)
Observe that
\[
(1 - V_{\pi} U_{\pi}) f_{ii'} = (1 - V_{\pi} U_{\pi}) f_i - \alpha_{i'}(1 - V_{\pi} U_{\pi}) V_{\pi'} f_i
\]
\[
= (1 - V_{\pi} U_{\pi}) f_i - \alpha_{i'}(1 - V_{\pi} U_{\pi}) V_{\pi'} f_i,
\]
because \( (1 - V_{\pi} U_{\pi}) V_{\pi} = 0 \), as \( U_{\pi} V_{\pi} = 1 \). Therefore, (117) equals
\[
\frac{1}{2}(1 - V_{\pi} U_{\pi}) f - \frac{\alpha_{i'}}{2}(1 - V_{\pi} U_{\pi}) V_{\pi'} f + \frac{\alpha_i}{2}(1 - V_{\pi} U_{\pi}) V_{\pi'} f_{ii'}
\]
(118)
Analogously, the second summand in (116) equals
\[
\frac{1}{2}(1 - V_{\pi'} U_{\pi'}) f - \frac{\alpha_j}{2}(1 - V_{\pi'} U_{\pi'}) V_{\pi} f + \frac{\alpha_j}{2}(1 - V_{\pi'} U_{\pi'}) V_{\pi} f_{ii'} \quad (119)
\]

\textit{Lemma.} For any overconvergent Hilbert modular form \( f \) of weight \((k, k)\), denote by \( f_{\pi'[^{\pi}]} \) the \( \pi \)-depletion of \( f_{\pi'} := V_{\pi} f \).

a) \( \theta^{-1} f_{\pi'[^{\pi}]} \approx 0 \) and \( \theta'^{-1} f_{\pi'[^{\pi}]} \approx 0 \)

b) \( \mathcal{F}_{\pi'[^{\pi}]} \approx 0 \), where \( \nabla_{\scriptscriptstyle GM} \mathcal{F}_{\pi'[^{\pi}]} = \nabla' \mathcal{F}_{\pi'[^{\pi}]} = f_{\pi[^{\pi}]} \), i.e. \( f_{\pi[^{\pi}]} \approx 0 \); analogously \( \mathcal{F}_{\pi'[\pi]} \approx 0 \), i.e. \( f_{\pi'[^{\pi}]} \approx 0 \)

c) \( f_{ii'}^{[p]} \approx f^{[p]} \approx \frac{1}{2} f^{[^{\pi}]} + \frac{1}{2} f^{[\pi']} \)

\textit{Proof of the lemma:}

a) If \( f(q) = \sum_{\nu \in (a^{-1})^+} b_{\nu} q^{\nu} \), then

\[
f_{\pi[^{\pi}]}(q) = \sum_{\nu \in (a^{-1})^+, \pi | \nu} b_{\nu} q^{\nu}
\]

so that

\[
\theta^{-1} f_{\pi[^{\pi}]}(q) = \sum_{\nu \in (a^{-1})^+, \pi | \nu} (\pi' \nu)^{-1} b_{\nu} q^{\nu}
\]

and

\[
\theta^{-1} f_{\pi'[^{\pi}]}(q) = \sum_{\nu \in (a^{-1})^+, \pi | \nu} (\pi' \nu)^{-1} b_{\nu} q^{\nu} \text{Tr}(\pi' \nu) = \sum_{n \geq 1} c_n q^n
\]

with

\[
c_n = \sum_{\nu \in (a^{-1})^+, \pi | \nu} (\pi' \nu)^{-1-t} b_{\nu}
\]

This is in the kernel of \( U_p \) because \( p \nmid n \) whenever \( c_n \neq 0 \). Indeed, if \( p \) divides \( n = \pi' \nu + \pi' x' \), then also \( \pi \) does, so it divides \( \nu \), a contradiction. Therefore, it is also in the kernel of \( e_{\text{ord}} = \lim_n U_{p^n} \).

b) It is a consequence of a) and of the expression (93) of the \( \nabla' - \text{primitive} \).

c) It is a consequence of b), and of (110), (118) and (119) \( \square \)

We use this lemma in combination with the fact that, for the overconvergent primitive \((1,0)\)-differential \( \omega_{\mathcal{F}[^{\pi'}]} \) of \( \omega_{f^{[^{\pi}]}'} \) (i.e. with \( \nabla_{\scriptscriptstyle GM} \omega_{\mathcal{F}[^{\pi'}]} = \nabla' \omega_{\mathcal{F}[^{\pi'}]} = \omega_{f^{[^{\pi}]}'} \)), the first argument of the product proved in \( \S \) b) to compute the \( p \)-adic Abel Jacobi map is

\[
e_{\text{ord}} J_{X'(N)}^* \mathcal{P}(\omega_{\mathcal{F}[^{\pi'}]}) = (-1)^t ! e_{\text{ord}} J_{X'(N)}^* \omega(\theta'^{-1-t} f^{[^{\pi}]}')
\]
(cf. last remark of the former section). Since \((1 - \alpha_i \alpha_{i'} V_p) f_{i'i'} = f_{i'i'}^{[p]}\),

\[
 f^\sharp = Q_f(\phi) f = \sum_{i,i'} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_0' - \alpha_1')} Q_{i'i'}(\phi) (1 - \alpha_i \alpha_{i'} V_p) f_{i'i'}
\]

\[
 \approx \sum_{i,i'} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_0' - \alpha_1')} Q_{i'i'}(\phi) \left( \frac{f^{[\pi]}}{2} + \frac{f^{[\pi']}}{2} \right)
\]

so that \(\omega_{f^\sharp}\) has a \(\nabla^G\) primitive \(\vartheta_{f^\sharp}\) with the equivalence

\[
 \vartheta(F^\sharp) \approx (-1)^t! \sum_{i,i'} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_0' - \alpha_1')} Q_{i'i'}(\phi) \vartheta\left( \frac{\theta^{t-1-t} f^{[\pi]}}{2} - \frac{\theta^{t-1-t} f^{[\pi']}}{2} \right)
\]

Therefore, the expression computing the \(p\)-adic Abel-Jacobi by \([92]\) is

\[
 (-1)^t! \sum_{i,i'} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{(\alpha_0 - \alpha_1)(\alpha_0' - \alpha_1')}
\]

\[
 < (1 - \alpha_3 \alpha_{i'} p^{2k} \phi)^{-1} e_{g \text{ord}} j_{X'(N)}^* \vartheta\left( \frac{\theta^{t-1-t} f^{[\pi]}}{2} - \frac{\theta^{t-1-t} f^{[\pi']}}{2} \right), \eta_g^{u-r} >
\]

\[
 = (-1)^t! \frac{C_t(f,g)}{(\alpha_0 - \alpha_1)(\alpha_0' - \alpha_1')} \sum_{i,i'} (-1)^{i+i'} \frac{\alpha_i \alpha_{i'}}{1 - \alpha_i \alpha_{i'} \beta_1 p^{-t} p^{2k-2k}}
\]

where

\[
 C_t(f,g) := < e_{g \text{ord}} j_{X'(N)}^* \vartheta\left( \frac{\theta^{t-1-t} f^{[\pi]}}{2} - \frac{\theta^{t-1-t} f^{[\pi']}}{2} \right), \eta_g^{u-r} >
\]

and \(\beta_1 = \beta_1(g)\). Taking

\[
 \mathcal{E}(f,g) := \prod_{i,i'} (1 - \alpha_i \alpha_{i'} \beta_1 p^{2k-2k-t})
\]

as common denominator in the summation, two numerators add up

\[
 \alpha_0 (\alpha_0' - \alpha_1)(1 - \alpha_1 \alpha_0' \beta p^{2k+t})(1 - \alpha_1 \alpha_1 \beta p^{2-2k+t})
\]

and the other two add up

\[
 -\alpha_1 (\alpha_0' - \alpha_1)(1 - \alpha_0 \alpha_0' \beta p^{2-2k+t})(1 - \alpha_0 \alpha_1' \beta p^{2-2k+t}),
\]

so all four numerators add up
\[(\alpha_0 - \alpha_1)(\alpha_0' - \alpha_1')(1 - \alpha_0\alpha_0'\alpha_1'\beta^2 p^{4-4k+2t}) \]
\[= (\alpha_0 - \alpha_1')(\alpha_0' - \alpha_1)(1 - \beta^2 p^{2-2k+2t}) \]
since \(\alpha_0\alpha_1\alpha_0'\alpha_1' = p^{2k-2}\) (and the cancellation of the missing terms has also used \(\alpha_0\alpha_1 = \alpha_0'\alpha_1' = p^{k-1}\)). Therefore the factor multiplying to \(C_p(f, g)\) in (120) is \(E_1(g) / E(f, g)\) for
\[E_1(g) := (1 - \beta^2 p^{2-2k+2t}) = (1 - \beta^2 p^{-k_0}) \]

We finally check that \(E_1(g)\) does not vanish. Observe that it vanishes if and only if \(\beta_1 = \pm p^{k-1}\). Recall that \(\beta_0\) is a \(p\)-adic unit, so \(\beta_1\) has valuation \(p^{k-1}\) as \(\beta_0\beta_1 = p^{k-1}\). In case the Euler factor vanished, it would be \(b_p = \beta_0 + \beta_1 = (1 + p^{k-1})\), contradicting the fact that \(|b_p| \leq 2p^{k-1}\) according to the Ramanujan-Petersson conjecture for Hecke newforms of arbitrary weight, proved by Deligne in [114].

As pointed out in the introduction, the right hand side of the equality of theorem 3.12 of [17] has the opposite sign to ours. This is because each of the four summands \(g_{\alpha} \otimes h_{\beta}\) in which the authors split the product \(gh\), are transformed, after Frobenius and restriction, to the sum of \(-g_{\alpha}^{[p]} h_{\beta}\) plus the derivative of a \(p\)-depleted power series, which lies in the kernel of \(e_{ord}\).

### 6 Hida families and a \(p\)-adic Gross-Zagier formula

Decompose the \(p\)-profinite group
\[Z(N_2) := \lim Cl_R(N_2p^n) \approx W \times Z(N_2)_{tor} \]
in torsion free part \(W\) and torsion part \(Z(N_2)_{tor}\). The action of \(Z(N_2) := \lim Cl_R(N_2p^n)\) on the \(p^n\)-roots of unity yields a homomorphism \(Z(N_2) \rightarrow Gal(\mathfrak{K}(\mu_{p^\infty}), \mathfrak{K})\), and taking limit we obtain the cyclotomic character
\[\chi : Z(N_2) \rightarrow Gal(\mathfrak{K}(\mu_{p^\infty}), \mathfrak{K}) \rightarrow Gal(\mathbb{Q}(\mu_{p^\infty}), \mathbb{Q}) \approx Aut(\mu_{p^\infty}) = \mathbb{Z}_p^* \]
This restricts to isomorphisms between their torsion free parts
\[\chi| : W \rightarrow Gal(\mathfrak{K}(\mu_{p^\infty}), \mathfrak{K}(\mu_p)) \approx \Gamma_p := 1 + p\mathbb{Z}_p, \quad (124) \]
the one in the middle being justified because
1 \rightarrow Gal(\mathcal{K}(\mu_p), \mathcal{K}) \rightarrow Gal(\mathcal{K}(\mu_{p^\infty}), \mathcal{K}) \rightarrow Gal(\mathcal{K}(\mu_{p^\infty}), \mathcal{K}) \rightarrow 1

(125)
is the projection to the torsion part. The fact that \chi_| is an isomorphism is a particular case of the Leopoldt conjecture, which holds for totally real fields, in particular for \mathcal{K}. The classical characters \mathbb{Z}(N_2) \rightarrow \mathbb{Z}_p^* \subseteq \mathbb{C}_p^* are the integer powers \chi^l of this cyclotomic character.

The Iwasawa \mathbb{Z}_p-algebra \Lambda is \mathbb{Z}_p[[W]] . Inside the group of characters

\mathcal{X}(\mathcal{X}(\ast)) = Hom_{\mathbb{Z}_p}(\Lambda, \mathcal{X}(\ast)) \approx Hom_{cts}(\Gamma_p, \mathcal{X}(\ast)) ,
denote \mathcal{X}_{\text{alg}}(\ast) \subseteq \mathcal{X}(\ast) the dense subset of characters of the form \chi^l\varepsilon for a torsion character \varepsilon : \Gamma_p \rightarrow \mathbb{C}_p^* , so the classical characters are those in

\mathcal{X}_{\text{clas}}(\ast) = \mathcal{X}_{\text{alg}}(\ast) \cap \mathcal{X}(\ast)_{tf}
i.e. algebraic with torsion factor \varepsilon = 1. This is dense in the open torsion free part \mathcal{X}(\ast)_{tf} \subseteq \mathcal{X}(\ast).

We recall next the following definitions from \cite{17} and \cite{5}.

**Definition.** A Hida family of tame level \( N > 1 \), i.e. not divisible by \( p \), is \( g = (\Lambda_g, \Omega_g, \Omega_g^{cl}, g(q)) \), where

- \( \Lambda_g \) is a finite flat extension of \( \Lambda \), thus inducing \( \zeta_g : Hom(\Lambda_g, \mathbb{C}_p) \rightarrow \mathcal{X}(\ast) \)
- \( \Omega_g \) is a non-empty open subset of \( Hom(\Lambda_g, \mathbb{C}_p) \).
- \( \Omega_g^{cl} \) is a \( p \)-adically dense subset of \( \Omega_g \) whose image under \( \zeta_g \) lies in \( \mathcal{X}_{\text{clas}}(\ast) \).
- \( g(q) = \sum_{n \geq 1} a_n q^n \in \Lambda_g[[q]] \) is such that, for all \( r \in \Omega_g^{cl} \), the power series \( g_r^{(p)}(q) = \sum_{n \geq 1} a_n(r)q^n \) is the \( q \)-expansion of the ordinary \( p \)-stabilization \( g_r^{(p)} \) of a normalised newform \( g_r \) of weight \( \zeta(r) \) for \( \Gamma_0(N) \).

**Definition.** A \( \Lambda \)-adic modular form of tame level \( N \) is a quadruple \( (\Lambda_\phi, \Omega_\phi, \Omega_\phi^{cl}, \phi(q)) \) where

- \( \Lambda_\phi \) is a complete finitely generated (but not necessarily finite), flat extension of \( \Lambda \), thus inducing \( \zeta_\phi : Hom(\Lambda_\phi, \mathbb{C}_p) \rightarrow \mathcal{X}(\ast) \)
- \( \Omega_\phi \) is a non-empty open subset of \( Hom(\Lambda_\phi, \mathbb{C}_p) \).
- \( \Omega_\phi^{cl} \) is a \( p \)-adically dense subset of \( \Omega_\phi \) whose image under \( \zeta_\phi \) lies in \( \mathcal{X}_{\text{clas}}(\ast) \).
- \( \phi(q) = \sum_{n \geq 1} a_n q^n \in \Lambda_\phi[[q]] \) is such that, for all \( r \in \Omega_\phi^{cl} \), the power
series \( \phi_r^{(p)}(q) := \sum_{n \geq 1} a_n(r) q^n \) is the \( q \)-expansion of a classical ordinary cusp form in \( S_{k_0(r)}(\Gamma_1(N) \cap \Gamma_0(p), \mathbb{Q}) \otimes \mathbb{C}_p \).

Next, we present the following material on \( p \)-adic families of non-ordinary \( a \)-Hilbert cuspidal forms, as in [41] sections 3.2 and 4. First, consider the increasing function \( \vartheta : \mathbb{Q}_\geq 0 \rightarrow \mathbb{Q}_\geq 0 \)
\[
\vartheta(x) := \left[ 2xh^+ \left( \frac{18(p-1)}{2(p-2)} x + 2 \right) \right]
\]
where \( h^+ \) is the narrow class number of \( \mathfrak{R} \) and the bracket is the integer part. For a positive rational \( \sigma \) denote
\[
\Lambda_\sigma = R_m \left< \frac{X}{p^{\vartheta(\sigma)}} \right> := \left\{ \sum_{n \geq 0} a_n \left( \frac{X}{p^{\vartheta(\sigma)}} \right)^n \mid \lim |a_n|_p = 0 \right\} \approx \mathbb{Z}_p[[X]]_{p^{\vartheta(\sigma)}}
\]
and for \( z \in \mathcal{O}_{\mathbb{C}_p} \), consider the \( p \)-adic ball
\[
B_{z,\vartheta(\sigma)} = \{ w \in \mathcal{O}_{\mathbb{C}_p} \mid |w - z|_p \leq p^{-\vartheta(\sigma)} \}.
\]
Any element of \( \Lambda_\sigma \) defines a rigid analytic function on \( B_{z,\vartheta(\sigma)} \) defined over \( K_m \).

**Definition** (Yamagami) A \( p \)-adic family of \( a \)-Hilbert cuspidal forms of tame level \( N_2 \geq 4 \), i.e. not divisible by \( p \), and positive rationals \( \sigma \geq \sigma' \) as slope, is \( f = (\Lambda_f, \Omega_f, \Omega_{f}^d, f(q)) \) with

- \( \Lambda_f = \Lambda_\sigma \)
- \( \Omega_f = B_{n_0, \vartheta(\sigma)} \) for some \( n_0 \in \mathbb{N} \)
- \( \Omega_f^d = \{ n \in B_{n_0, \vartheta(\sigma)} \cap \mathbb{N} \mid n \equiv n_0 \pmod{2} \} \)
- \( f(q) = \sum_{\nu \in \mathcal{A}_\tau^{-1}} a_\nu q^\nu \in \Lambda_f[[q^\nu]] \) specializing to each \( n \in \Omega_f^d \) with \( n + 1 > \sigma \) to the \( q \)-expansion \( f_n(q) = \sum_{\nu \in \mathcal{A}_\tau^{-1}} a_\nu(n) q^\nu \) of an eigenform \( f_n \)

\( \in S_{n+2,n+2}(a,N,\mathbb{Q})^{\sigma,\sigma'} \) of the full Hecke algebra.

The assumption in the following theorem is not essentially restrictive, as it dissapears while working adelicaly and imposing on \( f \) the condition of being \( \pi \) and \( \pi' \)-new, which is a mild condition playing in the finite slope setting the role of the primitivity condition in the ordinary setting (for details cf. Theorem 4.5 in [41]).

**Theorem.** Suppose \( \sigma \) and \( \sigma' \) are chosen so that
\[
\dim_{\mathbb{C}_p} S_{k,k}(a,N_2,\mathbb{C}_p)^{\sigma,\star} = \dim_{\mathbb{C}_p} S_{k,k}(a,N,\overline{\mathbb{Q}})^{*,\sigma'} = 1 \quad (126)
\]
and let \( f \in S_{k,k}(\mathfrak{a},N_2,\overline{\mathbb{Q}})^{\sigma,\sigma'} \) be a Hecke-eigenform (i.e. for the full Hecke algebra of operators). There exists a \( p \)-adic family \( f = (\Lambda_f, \Omega_f, \Omega^{cl}_f, f(q)) \) of \( \mathfrak{a} \)-Hilbert cuspidal forms of level \( N_2 \) and slope \((\sigma,\sigma')\) such that \( f_{k-2} = f^{11} \).

Given a rational cuspidal Hecke eigenform \( g \), normalized and ordinary at \( p \), new of weight \( 2 \) and of \( \Gamma_0 \)-level \( N = d_K N_1 N_2 \) the one-dimensional Hida theorem (quoted in [17] 2.15, cf. also [16](12)) associates a Hida family \( g = (\Lambda_g, \Omega_g, \Omega^{cl}_g, g(q)) \) of \( \Gamma_0 \)-level \( N \) such that, for some classical weight \( r \) with \( \zeta_g(r) = 2 \), it is \( g_r = g \).

Consider now the \( \Lambda \)-adic family \( h = (\Lambda_h, \Omega_h, \Omega^{cl}_h, h(q)) \) of cuspidal forms of \( \Gamma_0 \)-level \( N \), given in the following way:

- \( \Lambda_h \) is the flat \( \Lambda \)-algebra
  \[
  \Lambda \longrightarrow \Lambda \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow \Lambda \otimes \Lambda_{\sigma} = \Lambda_h
  \]
  the left homomorphism being given by \([z] \mapsto [z^2] \otimes [z^2]\)

- \( \Omega_h \subseteq Hom_{\mathbb{Z}_p}(\Lambda_h, \overline{\mathbb{Q}}_p) \) is the \( p \)-adic open
  \[
  \text{Hom}_{cts}(\Gamma_p, \overline{\mathbb{Q}}_p) \times D_{0,\nu(\sigma)} \subseteq \text{Hom}_{cts}(\Gamma_p, \overline{\mathbb{Q}}_p) \times B_{0,\vartheta(\sigma)}
  \subseteq \text{Hom}_{\mathbb{Z}_p}(\Lambda \otimes_{\mathbb{Z}_p} \Lambda_{\sigma}, \overline{\mathbb{Q}}_p),
  \]
  where \( D_{0,\vartheta(\sigma)} = \{ z \in \overline{\mathbb{Q}}_p \mid |z|_p \leq p^{-\vartheta(\sigma)} \} \), and the last inclusion is given by multiplication in \( \overline{\mathbb{Q}}_p \).

- To define \( \Omega^{cl}_h \), recall that any \( z \in \mathbb{Z}_p^* \) is decomposable as
  \[
  z = \mu(z) < z >,
  \]

\[11\] In [11] theor. 4.4, \( p \)-adic families of Hilbert modular forms are presented as collections of rigid analytic functions \( \{a_T\} \) on \( p \)-adic balls of the form \( B_{n_0,\psi(\sigma)} \), indexed by all the Hecke operators, such that their evaluation at classical weights are Hecke eigenvalues of (adelic) Hilbert modular forms.

We restrict ourselves to \( \mathfrak{a} \)-Hilbert modular forms, embedded into adelic ones as \( f \mapsto (0,...,0,f,0,...,0) \) (in the \( \mathfrak{a} \)-argument). Yamagami’s \( p \)-adic families applied to our particular case are equivalent to ours. Indeed, it is enough to keep only the subfamily of rigid analytic functions \( \{a_\nu\} \) with \( \nu \in (\mathfrak{a}^{-1})^+ \), (since the other operators act trivially on our \( \mathfrak{a} \)-Hilbert modular forms embedded into the direct sum, as we explain section 3) and to consider the \( \mathfrak{a} \)-component of Yamagami’s specialisations. The functions \( \{a_\nu\} \) belong to the Tate algebra which we have taken as first input in the quadruple defining \( f \) (def. 6).

Finally, we pack together these functions into a formal power series to be in accordance with the presentation in [17] of Hida families and \( \Lambda \)-adic modular forms (recalled in def. 6 and 4). With this we produce, at the end of our construction, a \( \Lambda \)-adic modular form which is taken as an input together with the Hida family of modular cusp forms passing through \( g \), to apply [17] prop. 2.19 and obtain our \( p \)-adic \( L \)-function.
with $< z > \in \Gamma_p$ and $\mu(z)$ the only (after Hensel lemma) $p - 1$ root of 1 with $\mu(z)$ congruent to $z$ mod. $p$. Our dense subset $\Omega^l_\mathcal{H} \subseteq \Omega_\mathcal{H}$ will be the subset of $(-1) \times \Omega^l_\mathcal{T}$ consisting of pairs $(j, s)$, with $\zeta_f(s)$ classical lying over $k$, such that $k + j \geq 1$ (Here we identify an integer $k$ with the continuous homomorphism $\Gamma_p \rightarrow \mathbb{Q}_p^*$ rising to the $k$-th power, and $-1$ denotes the class of $-1$ mod. $p - 1$). The image by the epimorphism $\zeta_\mathcal{H}$ lies indeed in the set $\mathbb{Z}_p^\geq 2$ of classical characters $\Gamma_p \rightarrow \mathbb{Q}_p^*$. It is a consequence of the fact that, for $(j, s) \in \Omega^l_\mathcal{H}$ and $\zeta_f(s) = \chi^k$, it is

$$
\zeta_\mathcal{H}(j, s)(z) = j(z^2)s(z^2) = z^{2j+2k}
$$

• First, recall that

$$
f^\pi_s(\pi, \pi')[\pi] + f^\pi_s(\pi, \pi')[\pi] \approx f^\pi_s[\pi] + f^\pi_s[\pi']
$$

by the lemma inside the proof of [1] so that

$$
e_{ord}(\frac{\theta^{t-1} - f^\pi_s[\pi']}{\theta^{t-1} - f^\pi_s[\pi]}) \equiv (q)
$$

$$
= \frac{1}{2} e_{ord}(\sum_{\nu \in (a^{-1})^+, \nu \mid \nu'} \nu^{t-1} a_\nu(s)q^{Tr(\nu)} - \sum_{\nu \in (a^{-1})^+, \nu \mid \nu'} \nu^{t-1} a_\nu(s)q^{Tr(\nu)})
$$

(127)

With this in mind, we define $h(q)$ as

$$
\frac{1}{2} e_{ord}(\sum_{\nu \in (a^{-1})^+, \nu \mid \nu'} \mu(\nu')^{-1}[< \nu' >] \otimes a_\nu q^{Tr(\nu)})
$$

$$
- \frac{1}{2} e_{ord}(\sum_{\nu \in (a^{-1})^+, \nu \mid \nu'} \mu(\nu)^{-1}[< \nu >] \otimes a_\nu q^{Tr(\nu)})
$$

(129)

It has, indeed, all coefficients in $\Lambda_\mathcal{H}$, since $\pi \mid \nu, \nu'$ implies $\pi \mid \mu(\nu), \mu(\nu')$ so that $\mu(\nu)^j, \mu(\nu')^j \in R_m \approx \mathbb{Z}_\nu$ also for negative values of $j$ (as $m$ lies over $\pi$) The key point is that, for $(j, s) \in \Omega^l_\mathcal{H}$ with $j = -1 + (p - 1)m \geq -1$ (so that $j$ is $-1$ in $(\mathbb{Z}/p)^*$) and $\zeta_f(s) = \chi^k$, it is

$$
(\mu(\nu)^{-1}[< \nu >] \otimes a_\nu)(j, s) = \mu(\nu)^{-1}[< \nu >] a_\nu(s) = \mu(\nu)^j a_\nu(s) = \nu^j a_\nu(s)
$$

and analogously $(\mu(\nu')^{-1}[< \nu' >] \otimes a_\nu') = \nu'^j a_\nu'$, so that

$$
h^{(p)}_{(j, s)}(q) = -\frac{1}{2} e_{ord}(\sum_{\nu \in (a^{-1})^+, \nu \mid \nu'} \nu^j a_\nu(s)q^{Tr(\nu)}) + \frac{1}{2} e_{ord}(\sum_{\nu \in (a^{-1})^+, \nu \mid \nu'} \nu'^j a_\nu(s)q^{Tr(\nu)})
$$

68
The overconvergent cuspidal form \( h_{(j,s)}^{(p)} \) has \( \zeta_h(j, s) = 2n + 2j \geq 2 \) and, being overconvergent of level \( N \) and ordinary, it is classical of level \( Np \): 

\[ h_{(j,s)}^{(p)} \in S_{2j+2n}(\Gamma_0(Np), \overline{Q}) \subseteq S_{2j+2n}(\Gamma_1(N) \cap \Gamma_0(p), \overline{Q}) . \]

Denote \( \Omega_{f,g} \) and \( \Omega^{cl}_{f,g} \) the two fibre products

\[
\begin{align*}
\Omega_{f,g} \longrightarrow & \Omega_g & \Omega^{cl}_{f,g} \longrightarrow & \Omega^{cl}_g \\
\downarrow & \zeta_g & \downarrow & \zeta_g \\
\Omega_h & \zeta_h \rightarrow \mathcal{X}^{(*)} & \Omega^{cl}_h & \zeta_h \rightarrow \mathcal{X}^{(*)}_{\text{cl}}.
\end{align*}
\]

(130)

For any pair of such families \( f \) and \( g \), and classical weights \( r \in \Omega^{cl}_g \) lying over \( k_0 \) and \( s \in \Omega^{cl}_f \) lying over \( k \), with

\[ k_0 = \zeta_g(r) = \zeta_h(j, s) = 2k + 2j \geq 2 \]

i.e. with \((r, s) \in \Omega^{cl}_{f,g}\) in [130], the ordinary cuspidal form \( e_{g,h_{r,s}}^{(p)} \) of level \( Np \) is, by the comment preceding lemma 2.19 in [17], the ordinary stabilization of an ordinary cuspidal form \( h_{r,s} \) of level \( N \), i.e.

\[ e_{g,h_{r,s}}^{(p)} = h_{r,s} - \beta_1(g_r) V_p h_{r,s} \]  

(131)

Lemma 2.19 of [17] tells us that there is a partial "Garrett-Hida \( p \)-adic \( L \)-function", i.e.a function \( L_p(f, g) : \Omega_{f,g} \longrightarrow \mathbb{C}_p \) having poles at finitely many nonclassical weights and satisfying at classical weights \((r, s) \in \Omega^{cl}_{f,g}\):

\[ L_p(f, g)(r, s) = \left[ e_{g,h_{r,s}}^{(p)}, g_r^{(p)} \right]_{Np} \]

(132)

\[ \left[ h_{r,s}, g_r \right]_N = < \omega_{h_{r,s}}, \eta^{u-r}_{g_r} > \in \mathbb{C}_p . \]  

(133)

The brackets here indicate the Petersson inner product of cuspidal forms of weight \( k_0 \) and indicated level, and \(< \omega_{h_{r,s}}, \eta^{u-r}_{g_r} >\) denotes, as always, the Poincaré pairing in \( H^1_{dR}(X(N), L^{(k_0-2)}_{dR}) \) of the classes of the overconvergent \( L^{(k_0-2)}_{dR} \)-differentials \( \omega_{h_{r,s}} \) and \( \eta^{u-r}_{g_r} \). This last defined in [104] by \(< \omega_{g_r}, \eta^{u-r}_{g_r} > = 1 \) (This class is presented in [17] as lift of the class in \( H^1_{dR}(X(N), L^{-1_{(k_0-2)})} \) represented by the \( L^{-1_{(k_0-2)}} \)-antiholomorphic form hermitian-dual to the \( L_{(k_0-2)} \)-holomorphic form \( \omega_{g_r} \), and last equality is just [17] (111), formula [132]).

We are now in position to prove the second theorem stated at the introduction.
Proof of theorem 2. The proof that \( E_0(g_r) \) is non-null is analogous to the proof of the analogous statement for \( E_1(g_r) \). The last assertion belongs to the theorem 1 already proved.

Denote \( \alpha_1(g_r) \) and \( \beta_1(g_r) \) as \( \alpha_1 \) and \( \beta_1 \). Applying the Hida projector \( e_{ord} \) to \( 131 \), we obtain

\[
e_{g_r,h_r}^{(p)} = e_{ord} h_r - e_{ord} \beta_1 V_p h_r.
\]

This represents the same cohomology class as

\[
e_{ord} h_r - \beta_1 V_p e_{ord} h_r = (1 - \beta_1 \beta_0^{-1}) e_{ord} h_r,
\]

since \( V_p \) commutes with \( U_p \), thus with \( e_{ord} \), in cohomology. Now, by \( \ref{132} \),

\[
L_p(f, g)(r, s) = \langle \omega e_{ord} h_r, \eta g_r^{u-r} \rangle = E_0(g_r)^{-1} \langle \omega e_{gr,h_r}, \eta g_r^{u-r} \rangle \in \mathbb{C}_p.
\]

By theorem 1,

\[
AJ_p(\Delta_{-2,k_0-2})(\omega_f \otimes \eta g_r^{u-r}) = (-1)^{t!} \frac{E_1(g_r)}{E(f_s, g_r)} \langle e_{g,h_r}^{(p)}, \eta g_r^{u-r} \rangle,
\]

so that \( \ref{134} \) proves \( \ref{3} \). □

We end by justifying our denomination of \( L_p(f, g) \) as a \( p \)-adic \( L \)-function. First, we recall that at classical weights \( r \) and \( s \) above \( (k, k) \) and \( k_0 \) with \( k - (k_0/2) \geq 1 \), the value \( L_p(f, g)(r, s) \) computes the \( p \)-adic Abel-Jacobi map at the cycle \( \Delta_{n,n_0} \) evaluated at suitable differential forms. The case \( k - (k_0/2) = 1 \) is excluded unless \( k = k_0 = 2 \), in which case \( p \)-adic Abel-Jacobi map at the cycle \( \Delta_{0,0} \) is computed. This range of classical weights plays here the role of the set of balanced weights in \( \ref{17} \).

Second, we observe that, in general, at any pair of classical weights \( (r, s) \), the value \( L_p(f, g)(r, s) = \frac{[g_r,h_r]}{[g_r,g_r]} \) belongs to \( \mathbb{Q} \), which we view in \( \mathbb{C}_p \), via the \( p \)-adic embedding fixed beforehand.
Third, for classical weights $r, s$ above $(k, k)$ and $k_0$ with $k - (k_0/2) \leq 0$, it is

$$L_p(f, g) = \frac{[g_r^{(p)}, h_s^{(p)}]_N}{[g_r^{(p)}, g_r^{(p)}]_N} = \frac{1}{2} \left[ g_r^{(p)}, e_g, e_{ord} j_{X(N)}^{+} (\theta^{1-t} f_s^{[\pi]} - \theta^{-1-t} f_s^{[\pi]}) \right]_{Np}$$

which is proved, as in the balanced case, to be equal to

$$\frac{1}{2} \frac{(-1)^t}{t!} \mathcal{E}_0(f_s, g_r) \left\{ g_r^{(p)}, e_g, j_{X(N)}^{*} (\theta^{1-t} f_s^{[\pi]} - \theta^{-1-t} f_s^{[\pi]}) \right\}_{Np}$$

$$= \frac{1}{2} \frac{(-1)^t}{t!} \mathcal{E}_0(f_s, g_r) \left\{ g_r^{(p)}, j_{X(N)}^{*} (\theta^{1-t} f_s^{[\pi]} - \theta^{-1-t} f_s^{[\pi]}) \right\}_{Np}$$

By theorem 1.1. of [27], this is a special value of the complex $L-$function of the representation associated to $g_r$ tensored by the representation associated to $\theta^{1-t} f_s - \theta^{-1-t} f_s$ restricted to $X(N)$. This corresponds with the second of the three cases dealt with in the quoted theorem. Our ongoing research tackles the construction of an Euler system out of $L_p(f, g)$, in the same spirit as in [18].

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