A Weil–Petersson Type Metric on the Space of Fano Kähler–Ricci Solitons

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Received: 1 March 2022 / Accepted: 4 August 2022 / Published online: 22 September 2022
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Abstract
In this paper, we define a Weil–Petersson type metric on the space of (shrinking) Kähler–Ricci solitons and prove a necessary and sufficient condition on when it is independent of the choices of Kähler–Ricci soliton metrics. We also show that the Weil–Petersson metric is Kähler when it defines a metric on the Kuranishi space of small deformations of Kähler–Ricci solitons. Finally, we establish the first and second order deformation of Fano Kähler–Ricci solitons and show that, essentially, the first effective term in deforming Kähler–Ricci solitons leads to the Weil–Petersson metric.

Keywords Kähler–Ricci solitons · Weil–Petersson metric · Deformation of complex structures · Twisted Kuranishi-divergence gauge

Mathematical subject classification: 32G07 · 32Q15 · 53C55

1 Introduction
The Weil–Petersson metric is a powerful tool in understanding the geometry of moduli spaces of various geometric objects equipped with certain canonical metrics. It was
first introduced by Weil \cite{28} in the late 1950s by using the Petersson pairing on modular forms and it is a natural $L^2$ metric on the moduli space $\mathcal{M}_g$ of hyperbolic Riemann surfaces of genus $g \geq 2$. More precisely, for any point $p$ in $\mathcal{M}_g$ represented by a Riemann surface $\Sigma$, it is well known that the holomorphic tangent space of $\mathcal{M}_g$ at $p$ can be identified with $H^{0,1}(\Sigma, T^1\Sigma)$, and that the dual space $\Omega^{1,0}_p\mathcal{M}_g$ can be identified with $H^0(K^2_\Sigma)$, the space of holomorphic quadratic differentials on $\Sigma$. The Weil–Petersson co-metric, as defined by Weil, is given by

$$\|\eta\|^2_{WP} = \int_{\Sigma} |\eta|^2 \lambda dV,$$

where $\lambda$ is a hyperbolic metric on $\Sigma$ and $\eta \in H^0(K^2_\Sigma)$. It follows that the Weil–Petersson metric on $H^{0,1}(\Sigma, T^1\Sigma)$ is defined by

$$\|\varphi\|^2_{WP} = \int_{\Sigma} |\varphi|^2 \lambda dV$$

for any harmonic Beltrami differential $\varphi$ with respect to $\lambda$.

The Weil–Petersson metric on $\mathcal{M}_g$ is well understood. In the 1960s, Ahlfors \cite{1, 2} showed that the Weil–Petersson metric on $\mathcal{M}_g$ is Kähler and has negative curvature; see also Wolpert’s work \cite{30}. Chu \cite{7} and Masur \cite{18} computed the asymptotics of the Weil–Petersson metric and showed that the Weil–Petersson metric is incomplete and has finite volume.

Subsequently, the Weil–Petersson metric has been generalized to study the geometry of moduli spaces of higher-dimensional varieties and vector bundles which admit certain canonical metrics. Koiso \cite{15} first studied the Weil–Petersson metric on the moduli space of Kähler–Einstein manifolds and proved that it is Kähler. Royden \cite{20} and Siu \cite{22} developed the canonical lifting method to compute the curvature of the Weil–Petersson metric on the moduli spaces of Kähler–Einstein manifolds of general type. Their method was adopted by Nannicini \cite{19} to study the Weil–Petersson metric on the moduli spaces of polarized Calabi–Yau manifolds. Candelas et al. \cite{3} showed that the Weil–Petersson metric on the moduli spaces of polarized Calabi–Yau manifolds is the curvature of the $L^2$ metric on the first Hodge bundle and its curvature can be derived from the work of Todorov \cite{24}. In these cases, the Weil–Petersson metric on the parameter space of a family of Kähler–Einstein manifolds is defined as the $L^2$ metric on the space of harmonic Beltrami differentials on each fiber of the family with respect to fiberwise Kähler–Einstein metrics. The lifting method was later refined by Schumacher \cite{21} and, together with Fujiki, they introduced the generalized Weil–Petersson metric on the (coarse) moduli space of csck metrics \cite{10}.

The Weil–Petersson metric is well defined as long as the Kähler–Einstein metric on each $X_t$ is unique. This is the case when the fibers are of general type, Calabi–Yau type, or Fano type with discrete automorphism groups. However, when the fibers are Fano Kähler–Einstein manifolds with non-discrete automorphism groups, there is a family of Kähler–Einstein metrics on each fiber. In this situation, the $L^2$ inner product and harmonic representatives of the Kodaira–Spencer classes may vary according to the choices of Kähler–Einstein metrics. Thus, it is not a priori clear that the Weil–Petersson metric is well defined. In fact, a necessary and sufficient condition was provided in \cite{6} to guarantee the well-definedness of the Weil–Petersson metric in the Fano Kähler–Einstein case (see also Theorem 3.2). There is also a canonical Weil–
Petersson metric on the (complexified) Kähler moduli spaces which was defined in a similar fashion [26, 29]. Such Weil–Petersson metrics play an important role in mirror symmetry.

In this paper, we study the Fano Kähler–Ricci soliton case and extend the work in [6] to define and investigate a Weil–Petersson type metric on the Kuranishi space of small deformations of Fano Kähler–Ricci solitons. This metric is the natural $L^2$ metric on the space of $f$-twisted harmonic Beltrami differentials, where $f$ is the Ricci potential of a Kähler–Ricci soliton metric. We also provide a necessary and sufficient condition for the Weil–Petersson metric to be well defined in the Fano Kähler–Ricci soliton case; see Theorem 3.3. It turns out, as shown in Theorem 3.6, that this Weil–Petersson metric is closely tied to the variation of Kähler–Ricci solitons as the underlying complex structures vary.

We remark that Donaldson, in [9], defined a new Kähler metric on the space $J$ of complex structures that are compatible with a given symplectic structure $\omega$ on the background smooth manifold of a Fano manifold. In particular, the moment map of the action of the symplectomorphism group on $J$, with respect to this new Kähler structure, is given by the volume form of $\omega$ twisted by the Ricci potential. It follows that the zeros of the moment map correspond to Kähler–Einstein metrics, while Fano Kähler–Ricci solitons are just the critical points of the $H$-functional defined in [14], which can be viewed as the “norm square” of the moment map. In this situation, the Weil–Petersson metric we defined can be viewed as the restriction of Donaldson’s metric on the Kuranishi space of Fano Kähler–Ricci solitons when the twisted Kuranishi-divergence gauge is imposed.

The paper is organized as follows. In Sect. 2, we review the twisted Kuranishi-divergence gauge for general Fano manifolds that was established by the third author in her 2014 thesis [31]. In Sect. 3, we first recall small deformation of Fano Kähler–Einstein manifolds and the Weil–Petersson metric in the Fano Kähler–Einstein setting, and then explain Donaldson’s construction of his new metric on $J$ and the associated moment map. We then define the Weil–Petersson metric on the space of Fano Kähler–Ricci solitons. To get a useful metric on the moduli space of Fano Kähler–Ricci solitons, the Weil–Petersson metric we introduced has to be independent of the choices of Kähler–Ricci soliton metrics. We will prove a sufficient and necessary condition on such independence and show that the Weil–Petersson metric is Kähler when it defines a metric on the Kuranishi space. Finally, we establish the first and second order deformation of Fano Kähler–Ricci solitons and show that essentially the first effective term in deforming Kähler–Ricci solitons leads to the Weil–Petersson metric.

2 The Twisted Kuranishi-Divergence Gauge

Let $(M^n, \omega_g)$ be a Fano manifold of dimension $n$ such that $[\omega_g] = 2\pi c_1 (M)$ with Ricci potential $f \in C^\infty (M, \mathbb{R})$:

$$\text{Ric} (\omega_g) = \omega_g + \sqrt{-1} \partial \bar{\partial} f.$$
We normalize $f$ so that \( \int_M e^f \, dV_g = \int_M (2\pi c_1 (M))^n \), where 

\[
(\omega_g^n) := \frac{\omega_g^n}{n!}
\]

is the volume form of $\omega_g$. Note that, by definition, $\omega_g$ is a (shrinking) Kähler–Ricci soliton if the vector field $X = \nabla^1_0 f \in A^0 (M, T^{1,0} M)$ is holomorphic, namely if $X \in H^0 (M, T^{1,0} M)$. We refer the reader to [4, 16, 27] for examples of Fano Kähler–Ricci solitons, and to [5] for a basic overview of Ricci solitons.

The twisted Laplacian on smooth functions is defined by 

\[
\Delta_f u = \Delta u + \overline{X} (u) = g^{i\overline{j}} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} + g^{i\overline{j}} \frac{\partial f}{\partial z_i} \frac{\partial u}{\partial \overline{z}_j}, \quad u \in C^\infty (M, \mathbb{C})
\]

where \((z_1, \ldots, z_n)\) is any system of local holomorphic coordinates on $M$. We also denote by $\overline{\partial}^*_f$ the twisted adjoint operator of $\overline{\partial}$ with respect to the weighted volume form $e^f \, dV_g$. It is natural in the current situation since 

\[
\text{Ric} (e^f \, dV_g) = \omega_g.
\]

The twisted Laplacian $\Delta_f$ is not a real operator and we define its conjugate operator $\overline{\Delta}_f$ by 

\[
\overline{\Delta}_f u = \overline{\Delta_f u} = \Delta u + \overline{X} (u).
\]

Similarly, we can define the twisted divergence operator $\text{div}_f$ on vector fields by 

\[
\text{div}_f (V) = \text{div} (V) + V (f).
\]

Although $\Delta_f$ is not a real operator, it is self-adjoint with respect to the Hermitian inner product 

\[
(u, v) = \int_M u \overline{v} \, e^f \, dV_g
\]

and thus its eigenvalues are real. Furthermore, the eigenvalues are all positive (except the zero eigenvalue when acting on constant functions). Let $\lambda_1$ be the smallest positive eigenvalue of $\Delta_f$ and denote by $\Lambda^1_f$, in case $\lambda = 1$ is an eigenvalue, the eigenspace of eigenvalue 1:

\[
\Lambda^1_f = \{ u \in C^\infty (M, \mathbb{C}) \mid (1 + \Delta_f) u = 0 \}.
\]

Extending the Lichnerowicz theorem, Futaki [12] showed that
Theorem 2.1 Let $(M, \omega_g)$ be a Fano manifold with Ricci potential $f$. Then $\lambda_1 \equiv \lambda_1(\Delta f) \geq 1$. Moreover, if $H^0(M, T^{1,0}M) \neq 0$ then $\lambda_1 = 1$ and $H^0(M, T^{1,0}M) \cong \Lambda^1_f$. In particular,

$$\text{div}_f : H^0(M, T^{1,0}M) \rightarrow \Lambda^1_f$$

and $\nabla^{1,0} : \Lambda^1_f \rightarrow H^0(M, T^{1,0}M)$ are linear isomorphisms with $\nabla^{1,0} \circ \text{div}_f = -\text{Id}$.

Next we consider the Kodaira-Spencer-Kuranishi deformation theory of Fano manifolds. On the space of Beltrami differentials $A^{0,1}(M, T^{1,0}M)$, the twisted adjoint of the $\partial$ operator is given by

$$\partial^* f \phi = \partial^* \phi - \mathbf{X} \lrcorner \phi$$

and the twisted divergence operator is

$$\text{div}_f \phi = \text{div} \phi + \phi \lrcorner \partial f$$

for any $\phi \in A^{0,1}(M, T^{1,0}M)$. In the following, we will use $\mathbb{H}_f$ to denote the spaces of $f$-harmonic forms or the $L^2$ projection, with respect to the metric $g$ and the weighted volume form $e^f dV_g$, to such harmonic spaces. For example,

$$\mathbb{H}^{0,1}_f(M, T^{1,0}M) = \left\{ \phi \in A^{0,1}(M, T^{1,0}M) \mid \overline{\partial} \phi = 0, \overline{\partial}_f^* \phi = 0 \right\}.$$

Note that, by the Kodaira vanishing theorem and the Serre duality, we have

$$H^{0,k}(M, T^{1,0}M) = 0 \quad \text{for all} \quad k \geq 2,$$

thus the deformation of the complex structures on $M$ is unobstructed.

In 2014, the third author in her thesis [31] obtained the following result on the deformation of complex structures with respect to the twisted Kuranishi gauge.

Theorem 2.2 Let $(M_0, \omega_0)$ be a Fano manifold, with $\omega_0$ in the class $2 \pi c_1(M_0)$, and let $f_0$ be the normalized Ricci potential. Let $m = h^{0,1}(M_0, T^{1,0}M_0)$ and let $0 \in B \subset \mathbb{C}^m$ be the open ball of radius $\varepsilon$ with coordinates $t = (t_1, \ldots, t_m)$. Let $\{\phi_1, \ldots, \phi_m\} \subset \mathbb{H}^{0,1}_f(M_0, T^{1,0}M_0)$ be a basis. Then, there exists a unique power series

$$\phi(t) = \sum_{i=1}^m t_i \phi_i + \sum_{|I| \geq 2} t^I \phi_I \subset A^{0,1}(M_0, T^{1,0}M_0)$$

such that $\phi(t)$ is convergent, when $\varepsilon$ is small, and satisfies the equations

$$\begin{align*}
\overline{\partial}_0 \phi(t) &= \frac{1}{2} [\phi(t), \phi(t)]; \\
\overline{\partial}^*_f \phi(t) &= 0; \\
\mathbb{H}_f^0(\phi(t)) &= \sum_{i=1}^m t_i \phi_i.
\end{align*}$$

(2.1)
Furthermore, let $\mathcal{X} = M_0 \times B$ be the smooth manifold with the projection map $\pi : \mathcal{X} \to B$, and

$$\Omega^{1,0}_{p,t,\mathcal{X}} = \pi^* T^{1,0}_t B \oplus (I + \varphi(t)) \Omega^{1,0}_{p,M_0}$$

for any point $(p, t) \in \mathcal{X}$. Then, $\mathcal{X}$ is a complex manifold; $(\mathcal{X}, B, \pi)$ is a Kuranishi family of $M_0$; and $t$ is a flat coordinate system, unique up to the choice of the metric $\omega_0$ in $2\pi c_1(M_0)$ and affine transformations.

In [23], the divergence gauge was introduced by the second author to study complex deformations of Kähler–Einstein manifolds of general type and pluricanonical forms. In particular, it was shown that, with respect to Kähler–Einstein metrics, the divergence gauge is equivalent to the Kuranishi gauge. Such an equivalence of the twisted Kuranishi and the divergence gauges with respect to general Kähler metrics on Fano manifolds was proved by the third author [31] in 2014; see also Remark 5 in [6].

**Theorem 2.3** Let $(M_0, \omega_0)$ and $f_0$ be as above. Suppose $\varphi \in A^{0,1}(M_0, T^{1,0}M_0)$ satisfies $\overline{\partial}_0 \varphi = \frac{1}{2} [\varphi, \varphi]$. Then,

$$\overline{\partial}^*_{f_0} \varphi = 0 \quad \text{if and only if} \quad \text{div}_{f_0} \varphi = 0.$$

Furthermore, $\varphi \cdot \omega_0 = 0$ when either one of these equivalent conditions is imposed.

In addition, it follows from the proof of the above theorem that

**Corollary 2.1** Let $(M_0, \omega_0)$ and $f_0$ be as above, and let $\varphi \in A^{0,1}(M_0, T^{1,0}M_0)$ such that $\overline{\partial}_0 \varphi = 0$ and $\varphi \cdot \omega_0 = 0$. Then, $\overline{\partial}_0 (\text{div}_{f_0} \varphi) = 0$ and

$$\overline{\partial}^*_{f_0} \varphi = -g^{ij} \left( \text{div}_{f_0} \varphi \right)_j \frac{\partial}{\partial z_i}.$$

### 3 The Weil–Petersson Metric

In this section, we first recall the Weil–Petersson metric defined on the space of Fano Kähler–Einstein manifolds and then explain Donaldson’s construction of his new metric on $\mathcal{J}$ and the associated moment map. We then define the Weil–Petersson metric on the space of Fano Kähler–Ricci solitons and prove a necessary and sufficient condition on when such a Weil–Petersson metric is independent of the choices of Kähler–Ricci soliton metrics. We also show that the Weil–Petersson metric is Kähler when it defines a metric on the Kuranishi space. Finally, we establish the first and second order deformation of Fano Kähler–Ricci solitons and show that, essentially, the first effective term in deforming Kähler–Ricci solitons leads to the Weil–Petersson metric.
3.1 The Weil–Petersson Metric on the Space of Fano Kähler–Einstein Manifolds

First of all, let us recall the Weil–Petersson metric on the Kuranishi space of Fano Kähler–Einstein manifolds and some key results proved in [6]. Let \((M_0, \omega_0)\) be a Fano Kähler–Einstein manifold and let \((\mathcal{X}, B, \pi)\) be the Kuranishi family of \((M_0, \omega_0)\) as constructed in Theorem 2.2 (with \(f_0\) being a constant potential function). As shown in [6], there is a deep relationship between the existence of Kähler–Einstein metrics on each small deformation \(M_t\) of \(M_0\) and the stability of the action of \(\text{Aut}_0(M_0)\) on the Kuranishi space \(B\).

**Theorem 3.1** (Cao et al. [6]) Let \((M_0, \omega_0)\) be a Fano Kähler–Einstein manifold and let \((\mathcal{X}, B, \pi)\) be the Kuranishi family with respect to \(\omega_0\). By shrinking \(B\) if necessary, the following statements are equivalent:

1. \(M_t\) admits a Kähler–Einstein metric for each \(t \in B\);
2. The dimension \(h^0(M_t, T^{1,0}M_t)\) of the space of holomorphic vector fields on \(M_t\) is independent of \(t\) for all \(t \in B\);
3. The automorphism group \(\text{Aut}_0(M_t)\) is isomorphic to \(\text{Aut}_0(M_0)\) for all \(t \in B\).

In particular, the above conditions are also equivalent to the condition that the action of \(\text{Aut}_0(M_0)\) on \(B\) is trivial.

The Weil–Petersson metric on the Kuranishi space \(B\) is defined as

\[
\left< \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right>_{WP} (t) = \int_{M_t} \left< \mathbb{H}_t \left( K_{S_t} \left( \frac{\partial}{\partial t_i} \right) \right), \mathbb{H}_t \left( K_{S_t} \left( \frac{\partial}{\partial t_j} \right) \right) \right> \omega_t^n,
\]

where \(\omega_t\) is a Kähler–Einstein metric on \(M_t\), \(K_{S_t}\) is the Kodaira–Spencer map and \(\mathbb{H}_t\) is the harmonic projection with respect to \(\omega_t\). Since Kähler–Einstein metrics on \(M_t\) are non-unique when \(\text{Aut}_0(M_t)\) is nontrivial, a natural question is whether the above Weil–Petersson metric depends on the choice of Kähler–Einstein metrics or not. In [6], we showed the following result.

**Theorem 3.2** (Cao et al. [6]) The Weil–Petersson metric on \(B\) is independent of the choice of Kähler–Einstein metrics if and only if one of the equivalent conditions in Theorem 3.1 holds.

We remark that the above results fit in with the moment map picture of Donaldson–Fujiki [8, 11]. Another description of the Weil–Petersson metric in the Kähler–Einstein case is Schumacher’s work on the Deligne pairing [21]. Let \(\pi : \mathcal{X} \to B\) be a family of Fano manifolds with a smooth family of Kähler–Einstein metrics. The volume forms of these Kähler–Einstein metrics define a metric on the relative anti-canonical bundle \(K_{\mathcal{X}/B}^{-1}\). In this case, the Weil–Petersson metric is just the canonical metric on the Deligne pairing \(\langle K_{\mathcal{X}/B}^{-1}, \ldots, K_{\mathcal{X}/B}^{-1} \rangle\), up to a multiplicative constant. We note that the smoothness assumptions in Schumacher’s work are in fact related to the actions of the automorphism groups of fibers on the base space \(B\) as discussed in [6].
3.2 Weil–Petersson Type Metric on the Space of Fano Kähler–Ricci Solitons

Donaldson [9] constructed a new Kähler structure on the space of complex structures on the underlying smooth manifold of a Fano manifold which is better adapted to the Kähler–Einstein geometry. This work of Donaldson sheds some light on the appropriate definition of Weil–Petersson type metrics on the space of Kähler–Ricci solitons. We first describe this Kähler structure.

Let $(M, L)$ be the background smooth pair of a Fano manifold $M_0$ and its anticanonical bundle $K_{M_0}^{-1}$. Let $\omega$ be a symplectic form on $M$ representing the class $2\pi c_1(M_0)$ and let $h$ be a metric on $L$ with a compatible connection $\nabla$ such that $\text{curv}(\nabla) = -\sqrt{-1}\omega$. Now let $J = \{\text{almost complex structures on } M \text{ compatible with } \omega\}$ and $J_{\text{int}} \subset J$ be the space of integrable ones. Each element $J \in J$ determines an $L$-valued $n$-form $\alpha \in \Omega^n(M, L)$, unique up to scaling, and $J$ is integrable if and only if $d\nabla\alpha = 0$. Explicitly, if $J \in J_{\text{int}}$ then, up to a multiplicative constant, $\alpha$ coincides with $d\bar{z}_1 \wedge \cdots d\bar{z}_n \otimes \left( \frac{\partial}{\partial z_1} \wedge \cdots \frac{\partial}{\partial z_n} \right)$ for any local holomorphic coordinates $(z_1, \ldots, z_n)$ on the complex manifold $M_J = (M, J)$. Let $\hat{J}_{\text{int}} \subset \Omega^n(M, L)$ be the space of all such $\alpha$ and denote by $T_\alpha$ the tangent space of $\hat{J}_{\text{int}}$ at $\alpha$. Naturally $\hat{J}_{\text{int}}$ is a $\mathbb{C}^n$-bundle over $J_{\text{int}}$. The metric $h$ on $L$ leads to a natural $L^2$ metric on $\Omega^n(M, L)$. For $\alpha, \beta \in \Omega^n(M, L)$, set

$$\langle \langle \alpha, \beta \rangle \rangle = -c_n \int_M (\alpha \wedge \bar{\beta})_h$$

(3.1)

where $c_n = \sqrt{-1}$ if $n$ is odd, and $c_n = 1$ if $n$ is even. Donaldson showed that, for $\alpha \in \Omega^n(M, L)$ with $d\nabla\alpha = 0$, $\langle \langle \cdot, \cdot \rangle \rangle$ is positive definite on the orthogonal complement of $\alpha$ in $T_\alpha$ and descends to a Kähler metric on $\hat{J}_{\text{int}}$.

Now we can rewrite Donaldson’s metric in terms of Beltrami differentials (see also [13]). For each $J \in \hat{J}_{\text{int}}$, we have the identification

$$T_{J,0}^{1,0}\hat{J}_{\text{int}} \cong \left\{ \varphi \in A^{0,1}(M_J, T_{J,0}^{1,0}M_J) \mid \bar{\partial}_J \varphi = 0, \varphi \omega = 0 \right\}.$$ 

Let $\Omega_J$ be the unique volume form on $M$ with

$$\text{Ric}_J(\Omega_J) = \omega \quad \text{and} \quad \int_M \Omega_J = \int_M \omega^{[n]}.$$ 

Namely, if we let $f_J$ be the normalized Ricci potential of the Kähler manifold $(M_J, \omega)$ then $\Omega_J = e^{f_J} \omega^{[n]}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Diagram of the Kähler–Ricci soliton structure.}
\end{figure}
By Corollary 2.1, we know that $\overline{\partial}_J \text{div}_J \varphi = 0$ for each $\varphi \in T^{1,0}_J \mathcal{J}^{\text{int}}$. Since $h^{0,1}(M_J) = 0$, there exists a unique function $\xi_\varphi \in C^\infty(M, \mathbb{C})$ such that
\[
\text{div}_J \varphi = \overline{\partial}_J \xi_\varphi \quad \text{and} \quad \int_{M_J} \xi_\varphi \Omega_J = 0.
\]
Then, for any $\varphi, \psi \in T^{1,0}_J \mathcal{J}^{\text{int}}$, the Donaldson metric is given by
\[
\langle \langle \varphi, \psi \rangle \rangle = \int_{M_J} \left( \text{Tr}(\varphi \overline{\psi}) - \xi_\varphi \xi_\psi \right) \Omega_J. \tag{3.2}
\]
We note that if $g_J$ is the Kähler metric on $M_J$ with Kähler form $\omega$ then
\[
\text{Tr}(\varphi \overline{\psi}) = \langle \langle \varphi, \psi \rangle \rangle_{g_J}
\]
because $\varphi \omega = \psi \omega = 0$. Now we let $G = \text{Symp}_0(M, \omega)$. Since $M_0$ is a Fano manifold, we know that $H^1(M) = 0$. It follows that $G$ consists of Hamiltonian diffeomorphisms. We identify the Lie algebra of $G$ by
\[
g \cong \left\{ u \in C^\infty(M, \mathbb{R}) \mid \int_M e^u \omega^n = \int_M \omega^n \right\}.
\]
Then, the dual of $g$ can be identified with
\[
g^* \cong \left\{ \Omega \in \Omega^{2n}(M, \mathbb{R}) \mid \int_M \Omega = \int_M \omega^n \right\}.
\]
In this case, the moment map $\mu$ of the action of $G$ on $(\mathcal{J}^{\text{int}}, \langle \langle \cdot, \cdot \rangle \rangle)$ is given by $\mu(J) = \Omega_J$.

In view of Donaldson’s construction above and the twisted Kuranishii-divergence gauge discussed in Sect. 2, it is natural to define the Weil–Petersson metric on the deformation space of Fano Kähler–Ricci solitons to be the restriction of the Donaldson metric to a Kuranishi space with respect to a twisted Kuranishii-divergence gauge determined by Kähler–Ricci solitons.

Precisely, let $p : \mathcal{Y} \to C$ be a holomorphic family of Fano manifolds such that each fiber $Y_s = p^{-1}(s)$ admits a Kähler–Ricci soliton metric $\omega_s$ and the family $\{\omega_s\}$ depends smoothly on $s$. Then, the Weil–Petersson metric is the Hermitian metric on $C$ defined in the following way.

**Definition 3.1** For any $s \in C$ and any tangent vectors $u, v \in T^{1,0}_s C$, consider
\[
\text{KS}_s(u), \text{KS}_s(v) \in H^{0,1}(Y_s, T^{1,0}_s Y_s)
\]
where KS is the Kodaira–Spencer map. Let $f_s$ be the (normalized) Ricci potential of $\omega_s$ and let $\varphi, \psi \in H^{0,1}_{f_s}(Y_s, T^{1,0}_s Y_s)$ be the $f_s$-harmonic representatives of $\text{KS}_s(u)$.
and KS, (v), respectively. Then, the \textit{Weil–Petersson inner product} of \( u \) and \( v \) is

\[
\langle u, v \rangle_{WP} := \int_{Y_s} \text{Tr} \left( \varphi \overline{\psi} \right) e^{f_s} \omega^n_s. \tag{3.3}
\]

**Remark 3.1** In the above definition, if the family \( \mathcal{Y} \) is a Kuranishi family of \( Y_s \) then the Weil–Petersson metric at \( s \in C \) is essentially a Hermitian metric on the cohomology \( H^{0,1}(Y_s, T^{1,0} Y_s) \). Note that in general, it may depend on the choice of the Kähler–Ricci soliton metric on \( Y_s \).

Now we investigate the relation between the Donaldson metric and the Weil–Petersson metric for general Fano manifolds.

**Proposition 3.1** Let \((M, \omega)\) be a Fano manifold such that \( \text{Ric} (\omega) = \omega + \sqrt{-1} \partial \overline{\partial} f \). For any (cohomology classes) \( A, B \in H^{0,1}(M, T^{1,0} M) \) and \( \varphi, \psi \in A^{0,1}(M, T^{1,0} M) \), with \( \overline{\partial} \varphi = \overline{\partial} \psi = 0, \varphi \wedge \omega = \psi \wedge \omega = 0 \) and \([\varphi] = A, [\psi] = B\), we have

\[
\langle A, B \rangle_{WP} = \langle \langle \varphi, \psi \rangle \rangle + \int_M \left( (1 + \Delta f)^{-1} \xi \right) \overline{\xi} e^f \omega^n,
\]

where \( \langle \langle \varphi, \psi \rangle \rangle \) is the Donaldson metric as defined in Eqs. (3.1) and (3.2).

**Proof** Since \( \overline{\partial} \varphi = 0 \), by Hodge theory, we know that there exists a unique smooth vector field \( W_\varphi \in A^0(M, T^{1,0} M) \) such that

\[
\begin{cases}
\varphi - \overline{\partial} W_\varphi = \mathbb{H}_f (\varphi) \\
W_\varphi \perp_{L^2} H^0(M, T^{1,0} M).
\end{cases}
\]

By Theorems 2.2 and 2.3, we know that \( \text{div}_f \mathbb{H}_f (\varphi) = 0 \). It follows that

\[
\text{div}_f \varphi - \text{div}_f \overline{\partial} W_\varphi = \text{div}_f \mathbb{H}_f (\varphi) = 0
\]

and

\[
\text{div}_f \overline{\partial} W_\varphi = \overline{\partial} \text{div}_f W_\varphi = \sqrt{-1} W_\varphi \wedge \omega.
\]

By Corollary 2.1, we know that \( \overline{\partial} (\text{div}_f \varphi) = 0 \) and thus \( \overline{\partial} (W_\varphi \wedge \omega) = 0 \), which implies that there exists a unique smooth function \( u_\varphi \) such that

\[
\begin{cases}
W_\varphi \wedge \omega = \sqrt{-1} \overline{\partial} u_\varphi \\
\int_M u_\varphi e^f \omega^n = 0.
\end{cases}
\]

It then follows that \( W_\varphi = \nabla^{1,0} u_\varphi \), \( \text{div}_f W_\varphi = \Delta f u_\varphi \), and

\[
\overline{\partial} \xi = \text{div}_f \varphi + \overline{\partial} u_\varphi = \overline{\partial} ((1 + \Delta f) u_\varphi).
\]
By the normalization of $\xi_\psi$ and $u_\psi$, we conclude that $\xi_\psi = (1 + \Delta_f) u_\psi$.

Similarly, we have $\xi_\psi = (1 + \Delta_f) u_\psi$. Thus,

$$
\langle A, B \rangle_{WP} = \langle \langle \mathbb{H}_f(\varphi), \mathbb{H}_f(\psi) \rangle \rangle = \int_M \text{Tr} (\varphi \psi) e^f \omega^{[n]} - \int_M \langle \partial W_\varphi, \partial W_\psi \rangle_\omega e^f \omega^{[n]}
= \int_M \text{Tr} (\varphi \psi) e^f \omega^{[n]} - \int_M \Delta_f u_\varphi (1 + \Delta_f) u_\psi e^f \omega^{[n]}
= \langle \langle \varphi, \psi \rangle \rangle + \int_M u_\varphi \xi_\psi e^f \omega^{[n]}
= \langle \langle \varphi, \psi \rangle \rangle + \int_M \left( (1 + \Delta_f)^{-1} \xi_\psi \right) \xi_\psi e^f \omega^{[n]}.
$$

\[ \square \]

For a Fano manifold $M$ admitting a Kähler–Ricci soliton metric, we know that Kähler–Ricci soliton metrics on $M$ are not unique. However, for any two such soliton metrics $\omega_1$ and $\omega_2$ on $M$, by a result of Tian and Zhu [25], there exists a biholomorphism $\sigma \in \text{Aut}_0(M)$ such that $\omega_2 = \sigma^* \omega_1$. In order to obtain a metric on the moduli space of Kähler–Ricci solitons over $M$, it is natural to require that the Weil–Petersson metric defined by (3.3) is independent of the choices of Kähler–Ricci solitons in the spirit of Theorem 3.2. If so, we shall say that the Weil–Petersson metric is well defined.

Our following result provides a necessary and sufficient condition on the well-definedness of the Weil–Petersson metric.

**Theorem 3.3** Let $M$ be a Fano manifold that admits a Kähler–Ricci soliton metric. Then, the Weil–Petersson metric is well defined if and only if the action of $\text{Aut}_0(M)$ on the Kuranishi space of $M$ is trivial.

**Proof** Let $\omega_0$ be any Kähler–Ricci soliton metric on $M$ with normalized Ricci potential $f_0$, and let $\Omega_0 = e^{f_0} \omega_0^{[n]}$ be the weighted volume form. For any smooth family $\{\sigma_s\}_{-\varepsilon < s < \varepsilon} \subset \text{Aut}_0(M)$ with $\sigma_0 = I_d$, let $\omega_s = \sigma_s^* \omega_0$. Then, $\omega_s$ is a Kähler–Ricci soliton metric with normalized Ricci potential $f_s$. We denote $\Omega_s = e^{f_s} \omega_s^{[n]}$.

For any cohomology classes $A, B \in H^{0,1}(M, T^{1,0}M)$, we let $\varphi_s, \psi_s \in H^{0,1}_f(M, T^{1,0}M)$ with $[\varphi_s] = A$ and $[\psi_s] = B$. Since $\omega_0$ is an arbitrary Kähler–Ricci soliton metric, the Weil–Petersson metric is well defined if and only if

$$
\frac{d}{ds} \bigg|_{s=0} \int_M \text{Tr} (\varphi_s \psi_s) \Omega_s = 0.
$$

Let $w = \frac{d}{ds} \bigg|_{s=0} \sigma_s$. Then, $w$ is a real holomorphic vector field on $M$. Since $[\varphi_s] = [\varphi_0]$ and $[\psi_s] = [\psi_0]$, there exist smooth families of vector fields $X(s), Y(s) \subset A^0 (M, T^{1,0}M)$, respectively, with $X(0) = Y(0) = 0$, such that

$$
\varphi_s = \varphi_0 + \partial X(s) \quad \text{and} \quad \psi_s = \psi_0 + \partial Y(s).
$$
It follows from a direct computation that
\[
\frac{d}{ds}\Big|_{s=0} \int_M \text{Tr} \left( \varphi_s \overline{\psi}_s \right) \Omega_s = \int_M \left\langle \overline{\varphi} \left( \frac{d}{ds}\Big|_{s=0} X(s) \right), \psi_0 \right\rangle_{\omega_0} \Omega_0 \\
+ \int_M \left\langle \varphi_0, \overline{\varphi} \left( \frac{d}{ds}\Big|_{s=0} Y(s) \right) \right\rangle_{\omega_0} \Omega_0 \\
+ \int_M \text{Tr} \left( \varphi_0 \overline{\psi}_0 \right) \left( \frac{d}{ds}\Big|_{s=0} \right) \\
= \int_M \text{Tr} \left( \varphi_0 \overline{\psi}_0 \right) \left( \text{div}_{f_0} w \right) \Omega_0.
\]

Replacing \(w\) by \(Jw\) and combining, we see that the Weil–Petersson metric is well defined provided, for any Kähler–Ricci soliton metric \(\omega_0\), any basis \(\{\varphi_1, \ldots, \varphi_m\}\) of \(H_{f_0}^{0,1} (M, T^{1,0}M)\) and any holomorphic vector field \(v \in H^0 (M, T^{1,0}M)\), we have
\[
\int_M \left( \text{div}_{f_0} v \right) \text{Tr} \left( \varphi_i \overline{\varphi}_j \right) \Omega_0 = 0, \quad 1 \leq i, j \leq m.
\]

Namely,
\[
\{ \text{Tr} \left( \varphi_i \overline{\varphi}_j \right) \mid 1 \leq i, j \leq m \} \perp_{L^2} \Lambda^1_{f_0},
\]

where we use the volume form \(\Omega_0\) to define the \(L^2\) inner product. By integration by parts, we see that Eq. (3.4) is equivalent to
\[
\int_M \left[ [v, \varphi_i], \varphi_j \right]_{\omega_0} \Omega_0 = 0.
\]

Since \([v, \varphi_i] = L_v \varphi_i, \overline{\varphi} (L_v \varphi_i) = 0\) and \(\varphi_j\) is \(f_0\)-harmonic, we see that Eq. (3.6) is in turn equivalent to the vanishing of the cohomology classes
\[
[L_v \varphi_i] = 0, \quad \text{for} \ 1 \leq i \leq m.
\]

Now we let \(\pi : X \to B\) be a Kuranishi family of \(M\) such that \(\pi^{-1}(0) \cong M\). Then, \(\text{Auto}_0 (M)\) acts on \(B\) and fixes 0. Hence, \(\text{Auto}_0 (M)\) acts on \(T_{0,1}^0 B \cong H^{0,1} (M, T^{1,0}M)\) linearly, and the corresponding action of its Lie algebra, \(\text{Lie} (\text{Auto}_0 (M)) \cong H^0 (M, T^{1,0}M)\), on \(H^{0,1} (M, T^{1,0}M)\) is given by \(v ([\varphi]) = [L_v \varphi]\). Thus, Eq. (3.7) is equivalent to the condition that the action of \(\text{Auto}_0 (M)\) on \(B\) is trivial. \(\square\)

A direct corollary, as in [6], is the following sufficient condition on the well-definedness of the Weil–Petersson metric.

**Corollary 3.1** Let \((X, B, \pi)\) be a Kuranishi family of \(M \cong M_0 = \pi^{-1}(0)\) such that \(M\) admits a Kähler–Ricci soliton. If \(h^0 (M_t, T^{1,0}M_t)\) remains constant as \(t\) varies in \(B\) then the Weil–Petersson metric is well defined.
Remark 3.2 As stated in Theorem 3.1, the conditions of the trivial action of $\text{Aut}_0(M)$ on the Kuranishi space of $M$ and the constancy of $h^0(M_t, T^{1,0}M_t)$ in $t$ are equivalent in the case of Fano Kähler–Einstein metrics, and they are also equivalent to the existence of Kähler–Einstein metrics on each fiber $M_t$. Such equivalence relations in the case of Fano Kähler–Ricci solitons will be discussed in a forthcoming paper. We also point out that the automorphism group of a Fano Kähler–Ricci soliton is not reductive in general. However, it is not hard to show that if the action of the group generated by the soliton vector field on the Kuranishi space is trivial, then the action of the unipotent radical is also trivial. This is clear on the level of Lie algebra and follows from Mabuchi’s work [17] on the Calabi–Matsushima type theorem and the Jacobi identity.

Now we assume that there exists a smooth family of Kähler–Ricci solitons on a Kuranishi family $(X, B, \pi)$ of Fano manifolds and the Weil–Petersson metric is well defined at each point of $B$. In this case, the Weil–Petersson metric is a Hermitian metric on the Kuranishi space $B$.

In general, a basic question is whether Weil–Petersson metrics are Kähler. In the case of Riemann surfaces, the Kählerian property of the Weil–Petersson metric was stated by Weil [28] and proved by Ahlfors [1]. Koiso [15] showed that the Weil–Petersson metrics on the spaces of Kähler–Einstein manifolds with nonzero first Chern class are Kähler. Later, Siu [22] provided a simpler proof by using the canonical lift. Nannicini [19] showed that the Weil–Petersson metrics on the moduli spaces of polarized Calabi–Yau manifolds are Kähler. Schumacher [21] gave a unified proof by using the harmonic lift which depends on the variation of Kähler–Einstein metrics.

In the case of Fano Kähler–Ricci solitons, to compute the derivatives of the Weil–Petersson metric and check its Kählerian property, we need to study the deformation of Kähler–Ricci solitons. When the Weil–Petersson metric is well defined, we have freedom to impose any gauge and to modify the given family of Kähler–Ricci solitons. Now let $(X, B, \pi)$ be a Kuranishi family of $M_0 = \pi^{-1}(0)$ as constructed in Theorem 2.2 with $\varphi(t)$ the unique solution to (2.1) and let $\{\omega_t\}_{t \in B}$ be a smooth family of Kähler metrics such that $\omega_t$ is a Kähler–Ricci soliton metric on $M_t$ with normalized Ricci potential $f_t$. We impose the $f_0$-twisted Kuranishi-divergence gauge on $X$. As before, for each $t \in B$, we let $\Omega_t = e^{f_t} \omega_t^{[n]}$ and also $X_t = \nabla_1^{1,0} f_t \in H^0(M_t, T^{1,0}M_t)$ be the soliton vector field on $M_t$. It follows that $\text{Im} (X_t)$ is a Killing field on $(M_t, \omega_t)$. We denote by $T$ the subgroup of $\text{Isom}_0(M_0, \omega_0)$ generated by $\text{Im} (X_0)$.

The twisted Kuranishi-divergence gauge allows us to identify each $M_t$ with $M_0$ as smooth manifolds and we can view $\{\Omega_t\}_{t \in B}$ as a family of volume forms on $M_0$ and thus there exists a smooth function $\rho = \rho (t, z) \in C^\infty (B \times M_0, \mathbb{R})$ such that

$$\Omega_t = e^\rho \det \left( I - \varphi(t) \varphi(t) \right) \Omega_0. \quad (3.8)$$

Here, we keep the factor $\det \left( I - \varphi(t) \varphi(t) \right)$ in (3.8) simply for the convenience of computations. It follows that $\rho(0, z) = 0$ and we have

$$\rho = \sum_{i=1}^m t_i \mu_i + \sum_{j=1}^m \bar{t}_j \bar{\mu}_j + O \left( |t|^2 \right), \quad (3.9)$$
where \( m = h^{0,1}(M_0, T^{1,0}M_0) \) and \( \mu_i \in C^\infty(M_0, \mathbb{C}) \), \( 1 \leq i \leq m \), are smooth functions. In this situation, we have

**Theorem 3.4** The Kuranishi family \((X, B, \pi)\) is \(T\)-equivariant and the potential functions \( f_t \) have the expansion

\[
    f_t = f_0 + \sum_{i=1}^m t_i (1 + \Delta_0) \mu_i + \sum_{j=1}^m t_j (1 + \Delta_0) \mu_j + O(|t|^2). \tag{3.10}
\]

Furthermore, each \( \mu_i \) satisfies the equations

\[
\begin{align*}
    \left( 1 + \Delta_{f_0} \right) \left( 1 + \overline{\Delta_{f_0}} \right) \mu_i &= 0, \\
    \left( 1 + \Delta_{f_0} \right) \left( 1 + \overline{\Delta_{f_0}} \right) \mu_j &= 0.
\end{align*} \tag{3.11}
\]

**Proof** By Theorems 2.2 and 2.3, we know that \( \text{div}_{f_0} \varphi(t) = 0 \) and \( \varphi(t) \omega_0 = 0 \) for each \( t \). By using these results and the facts that \( \omega_t = -\sqrt{-1} \partial_t \overline{\partial}_t \log \Omega_t \) and \( f_t = \log \left( \Omega_t / \omega_t^{[n]} \right) \), formula (3.10) follows from a direct computation.

Now, since \( \omega_t \) is a Kähler–Ricci soliton metric and thus \( \overline{\partial}_t \nabla_t^{1,0} f_t = 0 \) for each \( t \), we have

\[
\frac{\partial}{\partial t_i} \bigg|_{t=0} \overline{\partial}_t \nabla_t^{1,0} f_t = 0 \quad \text{and} \quad \frac{\partial}{\partial t_j} \bigg|_{t=0} \overline{\partial}_t \nabla_t^{1,0} f_t = 0, \quad \text{for each } i, j.
\]

Then, the first equation leads to

\[
L_{\text{Im}(X_0)} \varphi_i + \overline{\partial}_0 \nabla_0^{1,0} \left( 1 + \overline{\Delta_{f_0}} \right) \mu_i = 0 \tag{3.12}
\]

for each \( i \), and the second equation leads to

\[
\overline{\partial}_0 \nabla_0^{1,0} \left( 1 + \overline{\Delta_{f_0}} \right) \mu_j = 0 \tag{3.13}
\]

for each \( j \).

By Theorem 2.3, we know that \( \varphi_i \omega_0 = 0 \) and \( \text{div}_{f_0} \varphi_i = 0 \). A direct computation then shows that

\[
L_{\text{Im}(X_0)} \varphi_i \in \mathbb{H}^{1,0}_{f_0} \left( M_0, T^{1,0}M_0 \right)
\]

is \( f_0 \)-harmonic. Thus, by Eq. (3.12), we know that

\[
L_{\text{Im}(X_0)} \varphi_i = 0 \tag{3.14}
\]

for each \( i \). This implies that the action of \( T \) on \( \mathbb{H}^{1,0}_{f_0} \left( M_0, T^{1,0}M_0 \right) \) is trivial. Since \( T \subset \text{Isom}_0(M_0, \omega_0) \), by the uniqueness part of Theorem 2.2, we know that \( T \) preserves \( \varphi(t) \) for each \( t \in B \) and thus \( T \subset \text{Aut}_0(M_0) \).

Finally, formula (3.11) follows from Eqs. (3.12)–(3.14) and Theorem 2.1. \qed
Remark 3.3  The $T$-equivariance of the Kuranishi family allows us to modify the family $\{\omega_t\}$ of Kähler–Ricci solitons. In fact, by averaging the family $\{\omega_t\}$ over $T$ with respect to the Haar measure, we get a $T$-invariant family $\{\tilde{\omega}_t\}$ of Kähler–Ricci soliton metrics on $\mathcal{X}$ which restrict to $\omega_0$ on $M_0$ since $\omega_0$ is $T$-invariant.

From now on, we will assume the family $\{\omega_t\}$ of Kähler–Ricci soliton metrics is $T$-invariant. In this case, the functions $\mu_i$’s in the expansion (3.9) are very special, hence $\rho$ takes a much simpler form.

Corollary 3.2  Let $\{\omega_t\}$ be a $T$-invariant family of Kähler–Ricci soliton metrics on the Kuranishi family $(\mathcal{X}, B, \pi)$. Then, each $\mu_i$ in the expansion (3.9) has the property that

$$\mu_i, \bar{\mu}_i \in \Lambda^1_{f_0}. $$

Proof  Since each $\omega_t$ is $T$-invariant, we know that $\rho$ is $T$-invariant and thus $\text{Im} (X_0)(\rho) = 0$. This implies $\text{Im} (X_0)(\mu_i) = 0$ and $\text{Im} (X_0)(\bar{\mu}_j) = 0$. It then follows that

$$\left(1 + \Delta_{f_0}\right) \mu_i = \left(1 + \Delta_0\right) \mu_i + X_0(\mu_i) $$
$$= \left(1 + \Delta_0\right) \mu_i + \bar{X}_0(\mu_i) $$
$$= \left(1 + \Delta_{f_0}\right) \mu_i, $$

hence

$$0 = \int_{M_0} \left(\left(1 + \Delta_{f_0}\right) \left(1 + \Delta_{f_0}\right) \mu_i\right) \bar{\mu}_j \Omega_0 $$
$$= \int_{M_0} \left(1 + \Delta_{f_0}\right) \mu_i \left(1 + \Delta_{f_0}\right) \bar{\mu}_j \Omega_0. $$

Thus, $\left(1 + \Delta_{f_0}\right) \mu_i = 0$, and the same argument works for $\bar{\mu}_i$. \qed

Remark 3.4  By the above corollary, we know that $\text{Re} (\mu_i) \in \Lambda^1_{f_0}$ and $\text{Im} (\mu_i) \in \Lambda^1_{f_0}$. It follows that, for each $i$, the holomorphic vector field $\nabla_0^{1,0} \mu_i$ lies in the reductive part of $H^0 (M_0, T^{1,0}M_0)$ that is determined by $\omega_0$. In particular, $\nabla_0^{1,0} \mu_i$ commutes with $X_0$ and $\text{Im} (X_0)$ since $X_0$ lies in the center of the reductive part.

By using Theorem 3.4, we can now prove the Kählerian property of the Weil–Petersson metric.

Theorem 3.5  Assume that each small deformation of a Fano Kähler–Ricci soliton admits a Kähler–Ricci soliton metric and the Weil–Petersson metric is well defined. Then, the Weil–Petersson metric on the Kuranishi space is Kähler. Furthermore, the flat coordinate system described in Theorem 2.2 is a normal coordinate system of the Weil–Petersson metric at 0.
Proof We fix a Kähler–Ricci soliton \((M_0, \omega_0)\) with normalized Ricci potential \(f_0\) and let \((\mathfrak{X}, B, \pi)\) be the Kuranishi family of \(M_0\) constructed in Theorem 2.2. By the assumption in Theorem 3.5 and Remark 3.3, we can extend \(\omega_0\) to a \(T\)-invariant family of soliton metrics \(\{\omega_t\}\) on \(\mathfrak{X}\) such that \(\omega_t\) is a Kähler–Ricci soliton on \(M_t\) for each \(t \in B\). Let

\[
h_{ij}(t) = \left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right)_{WP}(t) = \int_{M_t} \text{Tr} \left( \mathbb{H}_{f_t} \left( KS_t \left( \frac{\partial}{\partial t_i} \right) \right) \right) \mathbb{H}_{f_t} \left( KS_t \left( \frac{\partial}{\partial t_j} \right) \right) \Omega_t.
\]

To prove the theorem, it is enough to show that \(\frac{\partial}{\partial t_k} h_{ij}(0) = 0\) for all \(i, j, k\).

We first consider the Kodaira–Spencer class \(KS_t \left( \frac{\partial}{\partial t_i} \right)\). Let

\[
\psi_i(t) = \left(\frac{\partial}{\partial t_i} \varphi(t)\right) \left( I - \varphi(t)\overline{\varphi(t)} \right)^{-1}.
\]

Then, a direct computation shows that \(\overline{\partial}_t \psi_i(t) = 0\) and \(KS_t \left( \frac{\partial}{\partial t_i} \right) = [\psi_i(t)]\). To find the harmonic representative \(\mathbb{H}_{f_t}(\psi_i(t))\), we recall the linear isomorphism

\[
\tau_t : A^0 \left( M_0, T^{1,0} M_0 \right) \to A^0 \left( M_t, T^{1,0} M_t \right)
\]

that was defined in [6],

\[
\tau_t(v) = \left( I - \varphi(t)\overline{\varphi(t)} \right)^{-1} (v) - \varphi(t) \left( I - \varphi(t)\overline{\varphi(t)} \right)^{-1} (v).
\]

By using this isomorphism, there exists a smooth family \(\{V_i(t)\}_{t \in B} \subset A^0 \left( M_0, T^{1,0} M_0 \right)\) such that \(V_i(0) = 0\) and

\[
\psi_i(t) + \overline{\partial}_t \left( \tau_t (V_i(t)) \right) = \mathbb{H}_{f_t}(\psi_i(t)).
\]

By direct computations, we have

\[
\frac{\partial}{\partial t_k} \bigg|_{t=0} \psi_i(t) = \frac{\partial^2}{\partial t_i \partial t_k} \bigg|_{t=0} \varphi(t),
\]

\[
\frac{\partial}{\partial t_l} \bigg|_{t=0} \psi_i(t) = 0,
\]

\[
\frac{\partial}{\partial t_k} \bigg|_{t=0} \overline{\partial}_t \left( \tau_t (V_i(t)) \right) = \overline{\partial}_0 \left( \frac{\partial}{\partial t_k} \bigg|_{t=0} V_i(t) \right),
\]

\[
\frac{\partial}{\partial t_l} \bigg|_{t=0} \overline{\partial}_t \left( \tau_t (V_i(t)) \right) = \overline{\partial}_0 \left( \frac{\partial}{\partial t_l} \bigg|_{t=0} V_i(t) \right).
\]
We note that if \( v_1, v_2 \in A^{0,1}(M_t, T^{1,0}M_t) \) are two Beltrami differentials such that either \( v_1 \cdot \omega_t = 0 \) or \( v_2 \cdot \omega_t = 0 \), then \( \text{Tr} \ (v_1 \bar{v_2}) = (v_1, v_2)_{\omega_t} \). It follows that

\[
\frac{\partial}{\partial t_k} h_{ij}(0) = \int_{M_0} \text{Tr} \left( \frac{\partial}{\partial t_k} \left|_{t=0} \right. \left( \psi_i(t) + \bar{\psi}_i(t) \right) \right) \Omega_0 \\
+ \int_{M_0} \text{Tr} \left( \frac{\partial}{\partial t_0} \left|_{t=0} \right. \left( \psi_j(t) + \bar{\psi}_j(t) \right) \right) \Omega_0 \\
+ \int_{M_0} \text{Tr} \left( \psi_i \bar{\psi}_j \right) \left( \frac{\partial}{\partial t_k} \Omega_t \right)
\]

\[
= \int_{M_0} \left( \frac{\partial^2}{\partial t_i \partial t_k} \right) \left|_{t=0} \right. \varphi(t), \varphi_j \right) \Omega_0 \\
+ \int_{M_0} \left( \frac{\partial}{\partial t_i} \bar{\varphi}_j \right) \left( \frac{\partial}{\partial t_0} \right) \Omega_0 \\
+ \int_{M_0} \text{Tr} \left( \psi_i \bar{\psi}_j \right) \mu_k \Omega_0
\]

By the construction of the power series \( \varphi(t) \) in Theorem 2.2, we know that \( \frac{\partial^2}{\partial t_i \partial t_k} \varphi(t) \) is in the image of \( \bar{\varphi}_{f_0} \), and thus, the first term of the right side of the above formula vanishes. To handle the second term, by Corollary 3.2, we know that \( \mu_k \in \Lambda^1_{f_0} \). Since the Weil–Petersson metric is well defined, Eq. (3.5) implies that the second term vanishes.

Thus, the Weil–Petersson metric is Kähler and the flat coordinates \((t_1, \ldots, t_m)\) is a normal coordinate system of the Weil–Petersson metric at 0. \(\Box\)

Finally, we show that the Weil–Petersson metric is closely related to the second order deformation of Kähler–Ricci solitons. To see this clearly and to simplify computations, we need to modify the \( T \)-invariant family \( \{\omega_t\} \) of Kähler–Ricci soliton metrics on the Kuranishi family \( \mathcal{X} \) once more.

We consider the holomorphic vector field \( \nabla^{1,0}_0 \mu_k \) for \( 1 \leq k \leq m \). By Remark 3.4, we know that it is \( T \)-invariant. Now we extend \( \nabla^{1,0}_0 \mu_k \) to a holomorphic section \( Z_k \in H^0 \left( T^{1,0}_{\mathcal{X}/B} \right) \). By the work of Kodaira, this can be done provided the dimension of the reductive part of \( \text{Aut}_0(M_t) \) remains constant as \( t \) varies in \( B \), and this is closely related to the existence of Kähler–Ricci soliton metrics on each \( M_t \) in the spirit of Theorem 3.1. Again, by the averaging trick which leaves \( \nabla^{1,0}_0 \mu_k \) intact, we can assume that each \( Z_k \) is \( T \)-invariant. Now we let \( \sigma \) be the time 1 flow of the vector field \( Z = \sum_{k=1}^m t_k Z_k \) and we replace the family \( \{\omega_t\} \) of Kähler–Ricci soliton metrics by the family \( \{\sigma^* \omega_t\} \). A direct computation shows that all \( \mu_k \)'s as in the expansion (3.9) with respect to the new family \( \{\sigma^* \omega_t\} \) vanish. We call such a family of Kähler–Ricci solitons a normalized family. In this case, the function \( \rho \) defined by Eq. (3.8) is special that it satisfies \( \rho = O \left( |t|^2 \right) \).
Theorem 3.6 Let \((\mathcal{X}, B, \pi)\) be the Kuranishi family of a Kähler–Ricci soliton \((M_0, \omega_0)\) given by Theorem 2.2, and let \(\{\omega_t\}_{t \in B}\) be a \(T\)-invariant normalized family of Kähler–Ricci solitons on \(\mathcal{X}\). Let \(\rho\) be the deformation function defined by Eq. (3.8) and set

\[
\eta_{i\bar{j}} = \left. \frac{\partial^2}{\partial t_i \partial \bar{t}_j} \right|_{t=0} \rho.
\]

Then, we have

(a) \[
\left. \frac{\partial^2}{\partial t_i \partial \bar{t}_j} \right|_{t=0} f_t = (1 + \Delta_0) \eta_{i\bar{j}} - \text{Tr} \left( \varphi_i \bar{\varphi}_j \right),
\]

where \(f_t\) is the normalized Ricci potential of \(\omega_t\).

(b) \[
(1 + \Delta f_0) \eta_{i\bar{j}} - \text{Tr} \left( \varphi_i \bar{\varphi}_j \right) \in \Lambda^1_{f_0},
\]

In particular,

\[
\int_{M_0} \eta_{i\bar{j}} \Omega_0 = \left\langle \frac{\partial}{\partial t_i} , \frac{\partial}{\partial \bar{t}_j} \right\rangle_{WP} (0).
\]

Proof The proof of formula (3.15) is similar to the first order expansion of \(f_t\) in the proof of Theorem 3.4. To prove (3.16), by the Kähler–Ricci soliton equation, we note that

\[
\left. \frac{\partial^2}{\partial t_i \partial \bar{t}_j} \right|_{t=0} \bar{\nabla}^{1,0}_i f_t = 0.
\]

A simple computation shows that

\[
\bar{\partial}_0 \bar{\nabla}^{1,0}_0 \left( (1 + \Delta_{f_0}) \eta_{i\bar{j}} - \text{Tr} \left( \varphi_i \bar{\varphi}_j \right) \right) = 0.
\]

Since \(\{\omega_t\}\) is \(T\)-invariant, we know that \(\rho\), and hence also \(\eta_{i\bar{j}}\), are \(T\)-invariant. This implies that \(\Delta_{f_0} \eta_{i\bar{j}} = \Delta f_0 \eta_{i\bar{j}}\) and then formula (3.16) follows from Theorem 2.1.

Finally, by formula (3.16), we know that

\[
\int_{M_0} \left( (1 + \Delta_{f_0}) \eta_{i\bar{j}} - \text{Tr} \left( \varphi_i \bar{\varphi}_j \right) \right) \Omega_0 = 0
\]

and thus

\[
\int_{M_0} \eta_{i\bar{j}} \Omega_0 = \int_{M_0} (1 + \Delta_{f_0}) \eta_{i\bar{j}} \Omega_0 = \int_{M_0} \text{Tr} \left( \varphi_i \bar{\varphi}_j \right) \Omega_0 = \left\langle \frac{\partial}{\partial t_i} , \frac{\partial}{\partial \bar{t}_j} \right\rangle_{WP} (0).
\]
**Remark 3.5** Theorem 3.6 indicates the close relation between the variation of Kähler–Ricci solitons and the Weil–Petersson metric we have defined. By finding higher order expansions of the function $\rho$ defined by (3.8), we can derive the curvature formula of the Weil–Petersson metric. However, note that there is no apparent bound of the curvature of the Weil–Petersson metric even in the case of Fano Kähler–Einstein manifolds.

**Declarations**

**Conflict of interest** There is no conflict of interest to disclose.

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