Numerical Study on Entrance Length in Thermal Counterflow of Superfluid $^4$He

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Received: 28 June 2018 / Accepted: 21 February 2019 / Published online: 28 February 2019
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Abstract
Three-dimensional numerical simulations in a square duct were conducted to investigate entrance lengths of normal fluid and superfluid flows in a thermal counterflow of superfluid $^4$He. The two fluids were coarse-grained by using the Hall–Vinen–Bekharevich–Khalatnikov (HVBK) model and were coupled through mutual friction. We solved the HVBK equations by parameterizing the coefficient of the mutual friction to consider the vortex line density. A uniform mutual friction parameter was assumed in the streamwise direction. Our simulation showed that the entrance length of the normal fluid from a hot end becomes shorter than that of a single normal fluid due to the mutual friction with the parabolically developed superfluid flow near the hot end. As the mutual friction increases, the entrance length decreases. Same as that, the entrance length of the superfluid from a cold end is affected by the strength of the mutual friction due to the parabolically developed normal fluid flow near the cold end. Aside from the entrance effect, the realized condition of a tail-flattened flow is discussed by parameterizing the superfluid turbulent eddy viscosity and the mutual friction.

Keywords Entrance length · Superfluid · Thermal counterflow · HVBK model · Two-fluid model · Mutual friction

1 Introduction
An inviscid fluid flow of $^4$He realizes below 2.17 K, and the superfluid $^4$He consists of the inviscid superfluid and viscous normal fluid components. The superfluidity is caused by Bose–Einstein condensation, and thus, the superfluid vortex is quantized.
In a duct flow filled with the superfluid $^4$He connected to a large helium bath, when the duct end opposed to the helium bath is heated, the normal fluid flow goes to the helium bath and the superfluid flow moves to the heated end to satisfy the total mass conservation. This experimental setup is termed as a thermal counterflow. As increasing a heat flux at the heated end, a relative velocity of the two fluids increases and mutual friction emerges in the thermal counterflow [1]. The mutual friction is due to the tangle of the quantized vortex in the superfluid flow. The tangle structure was numerically demonstrated by using the vortex filament method (VFM) in a computational box under the periodic boundary condition [2,3].

A solid boundary effect was considered in flows between parallel walls [4] and in square duct flows [5]. In the simulations, although velocity profiles of the normal fluid flows were prescribed as parabolic, tail-flattened and turbulent flows, vortex line density distributions become bimodal in the wall-normal direction. Such a non-uniform distribution of the vortex line density was observed near the hot end in the experiments of the thermal counterflows [6]. Recently, the VFM for the superfluid and Navier–Stokes equations for the normal fluid were fully coupled by the mutual friction and it was found that the vortex line density strength and its non-uniform distribution quite affect the velocity profiles of the two fluids [7].

Entrance lengths of the normal fluid and the superfluid at the opposite pipe ends of the thermal counterflows were studied by the Hall–Vinen–Bekharevich–Khalatnikov (HVBK) model [8], and it was found that the entrance length of the normal fluid increases linearly with the Reynolds number whereas that of the superfluid is enhanced by up to one order of magnitude when compared to the normal fluid [9]. At the cooled pipe end connected to the helium bath, a large recirculation region of the superfluid flow appears and the superfluid in the central region reenters into the helium bath. This flow configuration induces a tail-flattened flow of the normal fluid recently discovered in the duct flow experiments [10].

In this study, we investigate the influences of the mutual friction and its non-uniform distribution on the entrance lengths of the two fluids by means of the HVBK model. Aside from the entrance effect, we will discuss the condition realizing the tail-flattened flow by parameterizing the superfluid turbulent eddy viscosity and the mutual friction.

2 Governing Equations and Numerical Methods

We adopted the two-fluid model in which the superfluid $^4$He is composed of an inviscid superfluid component and a viscous normal fluid component. The density ratio of each component depends on temperature.

We used the HVBK model that is the coarse-grained two-fluid model. The momentum equations are described as follows:

\[
\rho_n \left[ \frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n \right] = -\nabla p_n + \mu_n \nabla^2 \mathbf{v}_n + \mathbf{F}_{ns},
\]

\[
\rho_s \left[ \frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s \right] = -\nabla p_s - \mathbf{F}_{ns},
\]
where the subscripts of $n$ and $s$ denote the normal fluid and superfluid components, $\rho$ is the density, $v$ is the velocity, $p$ is the pressure, $\mu$ is the molecular viscosity, and $F_{ns}$ denotes the mutual friction. The mutual friction is formulated as

$$F_{ns} = -a_f (v_n - v_s), \quad a_f = g \rho_s \kappa \alpha L,$$

(3)

where $a_f$ is defined as a coefficient of the mutual friction, $g$ is the anisotropic parameter, $\kappa$ is the quantum circulation of the superfluid vortex, $\alpha$ is the coefficient as a function of temperature [11], and $L$ is the vortex line density [1,7]. In the present study, we use $a_f$ as an external parameter changing the mutual friction, although in real flow $a_f$ is uniquely determined by the relative velocity $v_n - v_s$ and temperature that also determine the vortex line density $L$. The mutual friction depends on the vortex line density as shown in the two-fluid coupled simulation [7]. Here, the superfluid $^4$He is treated as an incompressible flow, and continuity equations are adopted as $\nabla \cdot v_n = \nabla \cdot v_s = 0$.

We adopted commonly used numerical methods for the incompressible flow. The second-order finite difference method and the second-order Adams–Bashforth method are utilized for the spatial discretization and temporal integration, respectively. The velocity and pressure are coupled by using the MAC (marker and cell) scheme [12], and the Poisson equation of pressure is solved by the BiCGSTAB (biconjugate gradient stabilized) method [13].

Figure 1 shows the configurations (a) and (b) of the experimental setup of the duct for thermal counterflow and (c) computational domain. In the configuration (a), the duct end is connected just beneath the helium bath colored on blue. In this configuration, the normal fluid flow abruptly expands into the helium bath and it causes the strong recirculation of the superfluid flow. It is reported that the recirculation affects the entrance length near the cold end and the superfluid boundary layer [9]; hereafter, we call the reference [9] as the Bertolaccini, Lévêque and Roche (BLR) study. By contrast, we assume the configuration (b) as shown in the recent experiment [10]. The duct end is inserted into the inside of the helium bath. In the present study, we assume that the superfluid enters at the cold end without recirculation.

We conducted three-dimensional two-fluid simulations of the thermal counterflow in the duct filled with the superfluid $^4$He. The computational domain is the square duct of $L \times D \times D$, $L$ is the duct length and $D$ is the duct width and height in the cross section. We assume the hot end at $x = 0$ and the cold end at $x = L$. We chose the duct size of $D = 1$ cm and $L = 30$ cm, same as the experiment [10]. The slip condition for superfluid and the no-slip condition for normal fluid are used on the
walls and uniform temperature at 2 K is adopted in the overall computational domain. Uniform distributions are applied to the superfluid and normal fluid flows for the inlet condition. It should be noted that the hot end at $x = 0$ is not the wall but a location slightly away from the heated wall. The convective outflow condition [14] is used for the outlet condition of the two fluids.

We carried out two conditions of mean normal fluid velocities of $\langle v_n \rangle = 0.35 \text{ mm/s (Re = 210)}$ for the parabolic flows and $\langle v_n \rangle = 3.7 \text{ mm/s (Re = 2227)}$ for the tail-flattened flow, where the Reynolds number of the normal fluid flow $Re$ is based on the kinematic viscosity $v_n = \mu_n/\rho_n$, the duct width $D$ and the mean normal fluid velocity. The grid points are $(N_x, N_y, N_z) = (481, 15, 15)$ for the parabolic flows with $a_f = 0.5$, $5$ and $(N_x, N_y, N_z) = (31, 15, 15)$ for the tail-flattened flow with $a_f = 57$. The mutual friction coefficient of $a_f = 0.5$ corresponds to the condition of $\langle v_n \rangle = 0.7 \text{ mm/s [7]}$ and that of $a_f = 57$ is equivalent to the condition of a tail-flattened flow in the experiment [10]. We yielded not only a uniform $a_f$ distribution but also a non-uniform $a_f$ distribution due to the vortex line density mimicked as Fig. 2. $L' = -12.8y(y - 0.5)^2(y - 1) - 12.8z(z - 0.5)^2(z - 1) + 0.8$ is a non-uniform distribution normalized by the constant $a_f$ and the mean value of $L'$ is 1.01. Such a non-uniform distribution has been observed in the VFM numerical simulation [4,5] and the experiments [6] of the thermal counterflows.

### 3 Results and Discussion

Velocity profiles of the normal fluid and superfluid flows in the $y$ direction are shown in Fig. 3. A bunch of positive velocities and a bunch of negative velocities correspond to the normal fluid velocities and the superfluid velocities, respectively. We considered the effect of the non-uniform vortex line density through the non-uniform distribution of the mutual friction coefficient. The effect appears in the centerline velocity of the superfluid and normal fluid flows. At the hot end ($x = 0\text{ cm}$) of Fig. 3, the normal fluid velocity profiles are uniform by the inlet condition whereas the outlet superfluid velocity profiles become parabolic for uniform mutual friction distribution and
Fig. 3 Velocity profiles of normal fluid (positive velocity) and superfluid (negative velocity) flows at a $x = 0$ cm (hot end), b $x = 15$ cm (center of the duct) and c $x = 30$ cm (cold end) in the $y$ direction. Red color and green color show the mutual friction coefficient of $a_f = 0.5$ and 5. Solid line and dashed line denote the uniform and non-uniform (multiplied by $L'$ in Fig. 2) distributions of the mutual friction coefficient (Color figure online)

![Velocity Profiles](image)

Fig. 4 Centerline velocity profiles of a normal fluid and b superfluid flows in the duct of $L = 30$ cm. See the figure caption in Fig. 3 for the symbols (Color figure online)

bimodal for the non-uniform mutual friction distribution. These profiles are different from the BLR study [9]. Our study examines the flow regime above $T_1$ transition in which the superfluid component transits to turbulence but the normal fluid is still laminar, whereas the BLR study mostly focuses on the flow regime below $T_1$ transition in which both fluid components are in the laminar flow regime. Therefore, we should consider $\omega_s$ directly or $L$ obtained from integration of $\omega_s$ in the mutual friction as mentioned in [15].

Let us look at the centerline velocity profiles of the superfluid and normal fluid flows as shown in Fig. 4. In the present study, the entrance length is measured as the length from the entrance where the centerline velocity in the streamwise direction corresponds to 99% of maximum centerline velocity. The strong mutual friction shortens the entrance lengths of the normal fluid and superfluid flows at the same Reynolds number, e.g., $0.65 X_n \rightarrow 0.59 X_n$ for the normal fluid and $0.99 X_n \rightarrow 0.31 X_n$ for the superfluid ($X_n$ denotes the entrance length of the single normal fluid), although in fact the mutual friction is uniquely determined by the relative velocity $v_{ns}$. The influence of non-uniform mutual friction is slight except for the cold end in the normal fluid flow while the influence is considerable in the overall region of the superfluid flow. This is due to zero viscosity of the superfluid flow. It is found that the entrance length of the superfluid flow is longer than that of the normal fluid flow for weak
mutual friction. As shown in Fig. 4b, the centerline velocity profiles of the superfluid flow for $a_f = 0.5$ from the cold end overlap with the profiles from the hot end, and thus, there is a possibility that the entrance lengths for $a_f = 0.5$ are estimated to be short. The examined results in a long duct of $L = 60$ cm are displayed in Fig. 5. The overlap of the entrance lengths is resolved in the long duct. It is confirmed that the entrance length of the superfluid flow in the short duct is underestimated due to the overlap, i.e., $0.65X_n \rightarrow 0.74X_n$ for the normal fluid and $0.99X_n \rightarrow 1.19X_n$ for the superfluid. It is also confirmed that in the superfluid flow, the entrance length for the non-uniform mutual friction becomes longer than that for the uniform mutual friction, i.e., $1.49X_n \leftarrow 1.19X_n$. In the present study, the mutual friction parameter is assumed to remain constant in the streamwise direction. This assumption means that the vortex line density is fully developed. Therefore, it should be noted that the assumption may yield the underestimated entrance lengths due to preventing a feedback mechanism to develop the vortex line density.

A tail-flattened profile is interpreted as an entry effect from superfluid entering the duct from the helium bath in the BLR study [9]. However, by using the present
simulations, aside from the entrance effect, we examined the condition to realize a tail-flattened flow as shown in Fig. 6. The non-uniform mutual friction was given to all the results because the uniform mutual friction showed a center-flattened flow and never provided the tail-flattened flow. We introduce the turbulent eddy viscosity of superfluid $\nu_s$ that is the coefficient of the turbulent viscous term $\nu_s \nabla^2 v_s$ originated from the Reynolds averaged stress tensor of the nonlinear term $v_s \cdot \nabla v_s$. As increasing the turbulent eddy viscosity, the superfluid velocity profile becomes flat. The weak mutual friction near the walls and at the center yields the tail-flattened profile, but the large turbulent viscosity and the strong mutual friction are needed.

4 Conclusion

The entrance lengths of normal fluid and superfluid flows in a thermal counterflow of superfluid $^4$He at 2 K are studied by using the coarse-grained HVBK two-fluid model for three-dimensional numerical simulations in a square duct. A uniform mutual friction parameter was assumed in the streamwise direction. It is found that the entrance length of the normal fluid flow from a hot end shortens when compared to that of a single normal fluid flow. This is due to the parabolically developed superfluid flow near the hot end by the mutual friction. The entrance length of the superfluid is longer than that of the normal fluid flow for weak mutual friction. As the mutual friction increases, the entrance lengths of the two fluids decrease at the same Reynolds number. A tail-flattened profile is realized by large turbulent viscosity and strong mutual friction.

Acknowledgements This work was supported by JSPS KAKENHI Grant Number JP18K03935.

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