PEAK SECTIONS AND BERGMAN KERNELS ON KÄHLER MANIFOLDS WITH COMPLEX HYPERBOLIC CUSPS

SHENGXUAN ZHOU

Abstract. By revisiting Tian’s peak section method, we obtain a localization principle of the Bergman kernels on Kähler manifolds with complex hyperbolic cusps, which is a generalization of Auvray-Ma-Marinescu’s localization result Bergman kernels on punctured Riemann surfaces [4]. Then we give some further estimates when the metric on the complex hyperbolic cusp is a Kähler-Einstein metric or when the manifold is a quotient of the complex ball. By applying our method directly to Poincaré type cusps, we also get a partial localization result.

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1. Introduction

Let $(M, \omega)$ be an $n$-dimensional Kähler manifold, and $L$ be a positive line bundle on $M$ equipped with a hermitian metric $h$. Then we can define the Bergman space for each $m \in \mathbb{N}$ as the following Hilbert space

$$H^0_{L^2} (M, \omega, L^m, h^m) = \left\{ S \in H^0 (M, L^m) : \int_M \| S \|^2_{h^m} d\text{Vol}_\omega < \infty \right\},$$

with the $L^2$ inner product $\langle S_1, S_2 \rangle_{L^2, M, \omega, h^m} = \int_M \langle S_1, S_2 \rangle_{h^m} d\text{Vol}_\omega$. Clearly, this space will also have a standard $L^2$ norm $\| S \|_{L^2, M, h^m, \omega} = \left( \langle S, S \rangle_{L^2, M, h^m, \omega} \right)^{\frac{1}{2}}$. Sometimes we denote the inner
product and norm briefly by \( \langle s_1, s_2 \rangle_{L^2} \) and \( \|s\|_{L^2} \), respectively. Hence the Bergman kernel can be defined as
\[
B_{M,\omega,\mathcal{L},h,m}(x,y) = \sum_{i \in I} S_{m,i}(x) \otimes (S_{m,i}(y))^*, \quad \forall x, y \in M,
\]
and the Bergman kernel function can be defined as
\[
\rho_{M,\mathcal{L},\omega,h,m}(x) = B_{M,\mathcal{L},\omega,h,m}(x,x) = \sup_{S \in \mathcal{H}^0_{L^2}(M,\mathcal{L},\omega)} \frac{\|S\|^2_{h,m}}{\int_M \|S\|^2_{h,m} \, dVol_{\omega}} = \sum_{i \in I} \|S_{m,i}(x)\|^2_{h,m},
\]
for any \( x \in M \), where \( \{S_{m,i}\}_{i \in I} \) is an \( L^2 \) orthonormal basis associated with \( h^m \) and \( dVol_{\omega} = \frac{\omega^n}{n!} \) in the Hilbert space \( \mathcal{H}^0_{L^2}(M,\omega,\mathcal{L},h^m) \). For abbreviation, we denote the Bergman space, Bergman kernel and Bergman kernel function briefly by \( \mathcal{H}^0_{L^2}(M,\mathcal{L},\omega) \), \( B_{\omega,m} \) and \( \rho_{\omega,m} \), respectively. Note that \( B_{\omega,m} \) is a \( (\pi_1^*\mathcal{L}^m \otimes \pi_2^*\mathcal{L}^{-m}) \)-valued function on \( M \times M \), and \( \rho_{\omega,m} \) are functions on \( M \), where \( \pi_i \) is the projection of \( M \times M \) onto the \( i \)-th component.

The Bergman kernel plays an important role in Kähler geometry. In the pioneering work [34], Tian used his peak section method to prove that Bergman metrics converge to the original polarized metric in the \( C^2 \)-topology. By a similar method, Ruan [30] proved that this convergence is \( C^\infty \). Later, Zelditch [36], also Catlin [6] independently, used the Szegö kernel to obtain an alternative proof of the \( C^\infty \)-convergence of Bergman metrics, moreover, they gave an asymptotic expansion of Bergman kernel function, which is the Kähler potential of the Bergman metric, on polarized Kähler manifolds. Such an asymptotic expansion is called the Tian–Yau–Zelditch expansion. This expansion can be also obtained by Tian’s peak section method, see Liu-Lu [21]. By using the heat kernel, Dai-Liu-Ma [7] gave another proof of the Tian–Yau–Zelditch expansion and they further considered the asymptotic behavior of Bergman kernels on symplectic manifolds and Kähler orbifolds (see also Ma-Marinescu’s book [25]). There are many important applications of the Tian–Yau–Zelditch expansion, for example, [10] and [35].

More recently, Auvray-Ma-Marinescu [4, 5] and Sun-Sun [32] independently considered the asymptotic behavior of the Bergman kernels on punctured Riemann surfaces. Sun-Sun also discusses the logarithmic K-semistability of punctured Riemann surfaces by considering the asymptotic behavior of the Bergman kernels in [32]. Through revisiting Tian’s peak section method, we generalize the results of Auvray-Ma-Marinescu to Kähler manifolds with complex hyperbolic cusps. The results and methods presented in this paper can be applied to more general cases and may be helpful in understanding log-K stability or the geometry of Shimura varieties.

In this paper, an \( n \)-dimensional \( (n \geq 1) \) complex hyperbolic cusp is defined in terms of a collection of data \( (V/\Gamma_{V}, \omega_V, \mathcal{L}_V/\Gamma_{V}, h_V) \) with the following property. Let \( (D, \omega_D) \) be a flat Abelian variety of complex dimension \( n-1 \), and \( (\mathcal{L}_D, h_D) \) be a negative Hermitian line bundle on \( D \) whose curvature form is \( -\omega_D \). If \( n = 1 \), the space \( D \) becomes a point, and \( \mathcal{L}_D \cong D \times \mathbb{C} \). Then \( \partial\bar{\partial}-\text{lemma} \) implies that \( h_D \) is unique up to scaling. It is clear that \( v \mapsto h_D(v, v) \) gives a smooth function on the total space \( \mathcal{L}_D \). By a straightforward computation, we see that \( \omega_V = -\sqrt{-1} \partial\bar{\partial} \log (\omega_D) \) is a Kähler form on \( V = \{0 < h_D < 1\} \) with \( \text{Ric}(\omega_V) = -(n+1)\omega_V \), where \( \pi_D : \mathcal{L}_D \to D \) is the projection of line bundle. For \( r > 0 \), we will denote by \( V_r \) the open subset \( \{0 < h_D < r\} \) of the total space of \( \mathcal{L}_D \). Let \( \mathcal{L}_V \) be a trivial line bundle on \( V \) with a Hermitian metric \( h_V \), such that there exists a section \( e_{\mathcal{L}_V} \in H^0(V, \mathcal{L}_V) \) satisfying that \( \|e_{\mathcal{L}_V}\|^2_{h_V} = |\log h_D| \). Let \( \Gamma_V \) be a finite subgroup of \( \text{Aut}(V, \omega_V) \). Since \( h_D \) is invariant under the action of \( \text{Aut}(V, \omega_V) \) on \( V \), the action of \( \Gamma_V \) on \( V \) can be extended to the total space of \( \mathcal{L}_D \) (see also Lemma 3.3). Now we further extend the action of \( \Gamma_V \) on \( V \) to the total space of the line bundle \( \mathcal{L}_V \) such that the biholomorphic maps corresponding to the action are automorphisms of the line bundle \( \mathcal{L}_V \), and \( h_V \) is invariant under this action. Then there exists
a homomorphism $\Theta_{V} : \Gamma_{V} \to U(1) \subset \mathbb{C}^*$ such that $g(\varepsilon_{L_{V}}) = \Theta_{V}(g)\varepsilon_{L_{V}}$, $\forall g \in \Gamma_{V}$. If $\Gamma_{V} = 0$, we can omit it in the notations. More details about complex hyperbolic cusps can be found in [27].

Now we define the asymptotic complex hyperbolic cusp, which is our main local model in this paper.

**Definition 1.1** (Asymptotic complex hyperbolic cusp). Let $(M, \omega)$ be a Kähler manifold, let $(\mathcal{L}, h)$ be a Hermitian line bundle on $M$ and let $\alpha > 0$ be a constant. We say that an open subset $U$ of $M$ is an asymptotic complex hyperbolic cusp of order $\alpha$ if it satisfies that:

1) $\text{Ric}(h) = \omega$ on $U$.

2) There exist a complex hyperbolic cusp $(V/\Gamma_{V}, \omega_{V}, \mathcal{L}_{V}/\Gamma_{V}, h_{V})$, a constant $r \in (0, e^{-1})$, a biholomorphic map $\varphi_{U} : V_{r}/\Gamma_{V} \to U$ and a section $\varepsilon_{L} \in H^{0}(U, \mathcal{L}_{V}|_{U})$ such that $\|\varepsilon_{L}|_{h} \circ \varphi_{U} = e^{u}|\log h_{D}|$, where $u = O\left(|\log h_{D}|^{-\alpha}\right)$ as $h_{D} \to 0^{+}$ to all orders with respect to $\omega_{V}$.

Moreover, we say that $U$ is an asymptotic complex hyperbolic cusp of order $\infty$ if for any $p \in \mathbb{N}$, $u = O\left(|\log h_{D}|^{-p}\right)$ as $h_{D} \to 0^{+}$ to all orders with respect to $\omega_{V}$.

By abuse of notation, it is sometimes convenient to write $V_{r}/\Gamma_{V}$ instead of $U$.

Since $M$ is a manifold, we see that the action of the group $\Gamma_{V}$ on $V$ is free in Definition 1.1, and $\mathcal{L}_{V}/\Gamma_{V}$ is a line bundle. In general, the quotient space $V/\Gamma_{V}$ is only an orbifold, and $\mathcal{L}_{V}/\Gamma_{V}$ is only a line orbibundle. Before we state our first main result, we need the following assumption.

**Definition 1.2** (Admissible pair). Let $m \in \mathbb{N}$ and $\xi, \kappa, r > 0$. Then a pair $(\gamma, \sigma) \in \mathbb{R}_{+}^{2}$ is said to be $(m, \xi, \kappa, r)$-admissible if $r \geq \sigma$, $\gamma \geq e^{r-\xi-k}\kappa^{m}$ and $\log \sigma \geq e^{-\xi}\kappa^{m} \log \gamma$.

Now we can state our first main result, which is the following localization principle.

**Theorem 1.1.** Let $(M, \omega)$ be a Kähler manifold, let $(\mathcal{L}, h)$ be a Hermitian line bundle on $M$ and $\alpha > 0$ be a constant such that $U \cong V_{r}/\Gamma_{V}$ is an asymptotic complex hyperbolic cusp of order $\alpha$, and $\text{Ric}(h) = \omega$ and $\text{Ric}(\omega) = -\Lambda \omega$ for some constants $\epsilon, \Lambda > 0$. Suppose that if $(M, \omega)$ is not complete, then $M$ is pseudoequivalent and $\bar{U}$ is complete.

Let $B_{V_{r}/\Gamma_{V}, \omega, m}(x, y)$ denote the Bergman kernel on $(V_{r}/\Gamma_{V}, \omega, \mathcal{L}_{V}, h)$. Given constants $\xi, \kappa, \beta > 0$. Then there exists a large constant $C > 1$ satisfying the following property.

Let $m \in \mathbb{N}$, and $(\gamma_{m}, \sigma_{m})$ be an $(m, \xi, \kappa, r)$-admissible. Then for any $m \in \mathbb{N}$, the Bergman kernel satisfying that

1) $\|B_{M, \omega, m}(x, y)\|_{C^{k,h_{m},\omega}} \leq Cm^{-l}\|\log h_{D}(x)\|^{-\beta}\|\log h_{D}(y)\|^{-\beta}$,

for any $(x, y) \in (\pi_{V}(V_{\gamma_{m}}) \times (U \setminus \pi_{V}(V_{\sigma_{m}}))) \cup ((U \setminus \pi_{V}(V_{\gamma_{m}})) \times \pi_{V}(V_{\gamma_{m}}))$, and

2) $\|B_{V_{\gamma_{m}}/\Gamma_{V}, \omega, m}(x, y) - B_{M, \omega, m}(x, y)\|_{C^{k,h_{m},\omega}} \leq Cm^{-l}\|\log h_{D}(x)\|^{-\beta}\|\log h_{D}(y)\|^{-\beta}$,

for any $(x, y) \in \pi_{V}(V_{\gamma_{m}}) \times \pi_{V}(V_{\gamma_{m}})$, where $\pi_{V}$ is the quotient map $V \to V/\Gamma_{V} \cong U$, $\|\cdot\|_{C^{k,h_{m},\omega}}$ is the $C^{k}$-norm associate with the Hermitian metric $\pi_{V}^{*}h^{m} \otimes \pi_{V}^{*}h^{-m}$ and the Kähler metric $\omega$ on $U \times U$.

Moreover, when $m \geq C$, the Bergman kernel functions satisfying that

3) $\sup_{V_{\gamma_{m}}/\Gamma_{V}}\left|\frac{\rho_{M, \omega, m}(x)}{\rho_{V_{\gamma_{m}}/\Gamma_{V}, \omega, m}(x)} - 1\right| \leq Cm^{-l}$. 
Remark. Since there exists a constant $\delta > 0$ independent of $m$ such that $\text{inj} (U \setminus V_m) \geq \frac{\delta \log m}{m}$ (see Lemma 4.3), we see that the asymptotic behavior of $B_{M,\omega,m}(x,y)$ on $(U \setminus V_m) \times (U \setminus V_m)$ is same to the standard case. See $[15, 25]$ for details in the standard case.

Our proof of Theorem 1.1 is based on Tian’s Peak section method. The key step, Proposition 4.1, is to show that the decay speed of the peak sections is sufficiently fast at a distance. We can use the Agmon type estimate (see Corollary 4.1) to obtain this estimate by observing the Bergman kernels can be represented simply by the peak sections.

Now we will give some refinements of Theorem 1.1 in two special cases. The first case is about the complex hyperbolic cusps with Kähler-Einstein metrics. We now recall some properties of it.

Fix a polarized $(n-1)$-dimensional flat Abelian variety $(D, \omega_D)$ with polarization $(\mathcal{L}_D^{1/2}, h_D^{-1/2})$. Now we can construct an $n$-dimensional complex hyperbolic cusp $(V, \omega_V, \mathcal{L}_V, h_V)$ by the statement before Theorem 1.1, where $V = \{ 0 < h_D < 1 \}$ is an open subset of the total space of the line bundle $\mathcal{L}_D$, $\omega_V = -\sqrt{-1} \partial \bar{\partial} \log (-\log h_D)$, and $(\mathcal{L}_V, h_V) \cong (\mathcal{C}, -\log h_D)$ is a Hermitian line bundle on $V$. Let $r \in (0,1)$ be a constant.

For any given smooth real-valued function $f \in C^\infty (\partial V_r)$, Datar-Fu-Song $[8]$ proved that there exists a unique Kähler-Einstein metric $\omega_{r,f} = \omega_V + \sqrt{-1} \partial \bar{\partial} u_{r,f}$ on $V_r$ by considering the Dirichlet problem of the Monge-Ampère equation, where $V_r = \{ 0 < h_D < r \} \subset V$, and $u_{r,f} \in C (\bar{V}_r)$ is a smooth real-valued function on $V_r$ such that $u_{r,f}|_{\partial V_r} = f$, and $|u_{r,f}| = o(1)$ as $h_D \to 0^+$. When $n \geq 2$, Fu-Hein-Jiang $[11]$ further studied the asymptotic behavior of $\omega_{r,f}$ and $u_{r,f}$ by establishing sharper estimates. For any $k \in \mathbb{Z}_{\geq 0}$, they proved that there exists a constant $c_{r,f} \in \mathbb{R}$ such that

$$
\left\| \nabla^k \left( u_{r,f} + \log \left( 1 + c_{r,f} \log h_D \right) \right) \right\|_{\omega_V} = O \left( \log h_D \right)^{\frac{k-1}{2}} e^{-2\sqrt{\lambda_1 D \log h_D}}
$$

as $h_D \to 0^+$, where $\lambda_1 D > 0$ denotes the first eigenvalue of $(D, \omega)$. See $[11, \text{Main Theorem}]$ for more details. By considering the change of the Hermitian line bundle under the scalar multiplication $v \mapsto e^{-\frac{\lambda_1 D}{2}} v$ on the total space of $\mathcal{L}_D$, (4) implies that the data $(V_r, \omega_{r,f}, \mathcal{L}_V, e^{-u_{r,f}}/h_V)$ is an asymptotic hyperbolic cusp of order $\infty$. See also Lemma 5.1. Since the Kähler metrics on asymptotic hyperbolic cusps can always be expressed as $\omega_V + \sqrt{-1} \partial \bar{\partial} u$ for some function $u$, Fu-Hein-Jiang’s estimate holds for all asymptotic hyperbolic cusps with Kähler-Einstein metrics.

Now we can state our result for the Bergman kernels on asymptotic complex hyperbolic cusps with Kähler-Einstein metric. For abbreviation, we denote $e^{-\frac{\lambda_1 D}{2}} \log h_D$ briefly by $r_m$ for any $m \geq 2$.

**Theorem 1.2.** Let $(M, \omega)$ be a Kähler manifold and let $(\mathcal{L}, h)$ be a Hermitian line bundle on $M$ such that $\text{Ric}(h) \geq \epsilon \omega$ and $\text{Ric}(\omega) \geq -\Lambda \omega$ for some constants $\epsilon, \Lambda > 0$. Suppose that $M$ is pseudoconvex if $(M, \omega)$ is not complete.

Assume that $U \cong V_r/\Gamma_V$ is an asymptotic complex hyperbolic cusp of order $\infty$ on $(M, \omega, \mathcal{L}, h)$ such that $U$ is complete. Then for any given constants $k, l \in \mathbb{N}$ and $\beta > 0$, there are constants $m_0, C > 0$ such that for any integer $m \geq m_0$, the Bergman kernels satisfying that

$$
\|B_{M,\omega,m}(x,y)\|_{C^{k,h_V^{1/2}},\omega_V} \leq C m^{-l} |\log h_D(x)|^{-\beta} |\log h_D(y)|^{-\beta},
$$

for any $(x,y) \in (\pi_V (V_{r_m}) \times (U \setminus \pi_V (V_{r_m})) \cup (U \setminus \pi_V (V_{r_m})) \times \pi_V (V_{r_m}))$, and

$$
\|B_{M,\omega,m}(x,y) - B_{V/\Gamma_V,\omega_V,m}(x,y)\|_{C^{k,h_V^{1/2}},\omega_V} \leq C m^{-l} |\log h_D(x)|^{-\beta} |\log h_D(y)|^{-\beta},
$$
for any \((x, y) \in \pi_V (V_{r,n}) \times \pi_V (V_{m,n})\). Moreover, the Bergman kernel functions satisfying that
\[
\sup_{V_{r,m}} \left| \frac{\rho_{M, \omega, m}(x)}{\rho_{V/\Gamma_V, \omega_{V,m}}(x)} - 1 \right| \leq Cm^{-l},
\]
where \(\| \cdot \|_{C^k, h_{r,m}}\) is the \(C^k\)-norm associate with the Hermitian metric \(\pi_1^*h^m \otimes \pi_2^*h^{-m}\) and the Kähler metric \(\pi_1^*\omega + \pi_2^*\omega\) on \(V \times V\).

The next case is about the quotients of the complex hyperbolic ball. Let \((\mathbb{B}^n, \omega_B)\) be the \(n\)-dimensional complex hyperbolic ball with constant bisectional curvature \(-1\), and let \(\Gamma\) be a lattice in \(\text{Aut}(\mathbb{B}^n, \omega_B) \cong PU(n, 1)\). Here \(\Gamma\) is a lattice means that \(\Gamma\) is a discrete subgroup of \(PU(n, 1)\) such that \((\mathbb{B}^n/\Gamma, \omega_B)\) has finite volume. When all parabolic elements of \(\Gamma\) are unipotent, there exist a finite collection of disjoint open subsets \(U_i \subset \mathbb{B}^n/\Gamma, i = 1, \cdots, N\), and complex hyperbolic cusps \((V_i, \omega_{V_i}, L_{V_i}, h_{V_i})\) such that \((U_i, \omega_{\mathbb{B}^n}) \cong (V_i, \omega_{V_i})\), and \((\mathbb{B}^n/\Gamma) \setminus (\cup_{i=1}^N U_i)\) is compact. Now we suppose that \(\mathbb{B}^n/\Gamma\) is a manifold, that is to say the lattice \(\Gamma\) is torsion free. At this time, \(\mathbb{B}^n/\Gamma\) satisfies the following assumptions at infinity.

**Theorem 1.3.** Let \((M, \omega)\) be a Kähler manifold and let \((\mathcal{L}, h)\) be a Hermitian line bundle on \(M\) such that \(\text{Ric}(h) \geq \epsilon \omega\) and \(\text{Ric}(\omega) \geq -\Lambda \omega\) for some constants \(\epsilon, \Lambda > 0\). Suppose that \(M\) is pseudoconvex if \((M, \omega)\) is not complete.

We follow the notations used in Definition 1.1. Assume that \(U \cong V\) is an asymptotic complex hyperbolic cusp on \((M, \omega, \mathcal{L}, h)\) that satisfies \(u = 0\) and \(\bar{U}\) is complete. Then for any given constants \(k, l \in \mathbb{N}\) and \(\beta > 0\), there are constants \(m_0, C > 0\) such that for any integer \(m \geq m_0\), the Bergman kernels satisfying that
\[
\sup_{V_{r,2} \times V_{r,2}} \left| \log h_D(x) \right|^\beta \left| \log h_D(y) \right|^\beta \| B_{M, \omega, m}(x, y) - B_{V/\Gamma_V, \omega_{V,m}}(x, y) \|_{C^k, h_{r,2}} \leq Cm^{-l},
\]
and the Bergman kernel functions satisfying that
\[
\sup_{V_{r,2}} \left| \frac{\rho_{M, \omega, m}(x)}{\rho_{V/\Gamma_V, \omega_{V,m}}(x)} - 1 \right| \leq Cm^{-l},
\]
where \(\| \cdot \|_{C^k, h_{r,2}}\) is the \(C^k\)-norm associate with the Hermitian metric \(\pi_1^*h^m \otimes \pi_2^*h^{-m}\) and the Kähler metric \(\pi_1^*\omega + \pi_2^*\omega\) on \((V/\Gamma_V) \times (V/\Gamma_V)\).

**Remark.** In the case \(n = 1\), we give a new proof of Auvray-Ma-Marinescu’s estimate for Bergman kernels on punctured Riemann surfaces [4, Theorem 1.1] that relies on Tian’s peak section method.

**Remark.** When \(\Gamma_V = 0\) and \(\mathcal{L}_D^{-1}\) is globally generated, the \(C^0\)-estimates for the quotients between Bergman kernel functions, (7) and (9), can be improved to the \(C^k\) estimates. We will prove them in Proposition 5.2 and Proposition 5.4, respectively.

Let \(X = \mathbb{B}^n/\Gamma\), and let \(\mathcal{K}_X\) be the canonical bundle on \(X\). Then the Kähler metric \(\omega_{\mathbb{B}^n}\) gives a Hermitian metric \(h_X\) on \(\mathcal{K}_X\). Since \(\text{Ric}(\omega_{\mathbb{B}^n}) = -(n + 1)\omega_{\mathbb{B}^n}\), we have \(\text{Ric}(h_X) = (n + 1)\omega_{\mathbb{B}^n}\), and
hence \((X, (n+1)\omega_B, K_X, h_X)\) is a polarized complete Kähler orbifold. Now we can describe the asymptotic behavior of the supremum of the Bergman kernel functions of \((X, \omega_B, K_X, h_X)\). Note that Theorem 1.1 also holds for Kähler orbifolds with asymptotic complex hyperbolic cusps, so we don’t need to assume that \(\Gamma\) is torsion free here.

**Theorem 1.4.** Under the conditions stated above, there exist an integer \(N = N(n, \Gamma) \in \mathbb{N}\), a constant \(C = C(n, \Gamma) > 0\), and a sequence of positive constants \(\{c_i\}_{i=1}^{\infty}\), such that \(c_i = c_{i+N}\), \(\forall i \in \mathbb{N}\), and satisfy the following properties.

If \(\Gamma\) is cocompact, then

\[
\sup_X \rho_{(X, \omega_B, K_X, h_X), m} = c_m m^n + O(m^{n-1}), \quad \text{as } m \to \infty.
\]

If \(\Gamma\) is not cocompact, then

\[
\sup_X \rho_{(X, \omega_B, K_X, h_X), m} = c_m m^{n+1} + O(m^n), \quad \text{as } m \to \infty.
\]

Moreover, if \(\Gamma\) is neat, then \(N = 1\).

Estimates for the upper bound of the Bergman kernel functions of the Shimura varieties are relevant in arithmetic geometry [1, 4, 19, 20]. It is possible to generalize our argument for Theorem 1.4 to the more general Shimura varieties, and possibly useful for arithmetic geometry.

Now we apply the arguments in the proof of Theorem 1.1 to Poincaré type cusps. The basic model of the Poincaré type cusp consists of a Kähler manifold \((X, \omega_B, K_X, h_X)\) and a Kahler metric \(-\sqrt{-1} \sum_{i=1}^{k} \partial \bar{\partial} \log \left(\log |z_i|^2\right) + \pi_{k,n}^* \omega_{k,n}\) on \((\mathbb{D}_r^*)^k \times B_{r^{2n-2k}}(0)\), where \(\omega_{k,n}\) is the projection of \((\mathbb{D}_r^*)^k \times B_{r^{2n-2k}}(0)\) onto \(B_{r^{2n-2k}}(0)\), \(\omega_{k,n}\) is a Kahler metric on \(B_{r^{2n-2k}}(0)\), and \(\mathbb{D}_r^* = \{z \in \mathbb{C} | 0 < |z| < r\}\) for any \(r > 0\). By abuse of notation, we will write \(\bar{\partial} \omega_{k,n}\) above simply \(\omega_{\text{mod}, k,n}\) when \(\omega_{k,n}\) is an Euclidean metric.

Fix a simple normal crossing divisor \(D\) in an \(n\)-dimensional compact Kähler manifold \(M\). By definition, we see that for each \(x \in D\), there exists a neighborhood \(U_x\) of \(x\) and a biholomorphic map

\[
\varphi_x : U_x \setminus D \to \prod_{i=1}^{k} \mathbb{D}_r^* \times B_{r^{2n-2k}}(0) \subset (\mathbb{D}_r^*)^k \times \mathbb{C}^{n-k},
\]

where \(r_x, j \in (0, 1]\) and \(r_x, k, n > 0\) are constants. Let \(\omega\) be a Kahler metric on \(\tilde{M} \setminus D\). Then we say that \(\omega\) is a metric with Poincaré type cusp along \(D\) if for any \(x \in U\), the data \((U_x, \varphi_x)\) constructed above satisfying that

\[
(\varphi_x^{-1})^* \omega = -\sqrt{-1} \sum_{i=1}^{k} \partial \bar{\partial} \log \left(\log |z_i|^2\right) + \pi_{k,n}^* \omega_{k,n} + \sqrt{-1} \partial \bar{\partial} u_x,
\]

where \(u_x\) is a smooth function on \(\prod_{i=1}^{k} \mathbb{D}_r^* \times B_{r^{2n-2k}}(0)\), and there exists a constant \(\alpha > 0\) such that \(u_x \in \cap_{i=1}^{k} O \left(\log |z_i|^{-\alpha}\right)\) as \(\inf_{1 \leq i \leq k} |z_i| \to 0^+\) to all orders with respect to \((\varphi_x^{-1})^* \omega\).

Let \(M\) denote the manifold \(\tilde{M} \setminus D\). Suppose that \((M, \omega)\) is a complete Kähler manifold with Poincaré type cusp along \(D\) and \((\mathcal{L}, h)\) is a Hermitian line bundle on \(M\) such that \(\text{Ric}(h) \geq \epsilon \omega\) and \(\text{Ric}(\omega) \geq -\Lambda \omega\) for some constants \(\epsilon, \Lambda > 0\). We further assume that for any \(x \in D\), there exists a local frame \(e_x \in H^0(U_x \setminus D, \mathcal{L})\) such that \(\|e_x\|^2 = e^{-u_x} - \tau_{k,n} \left(\prod_{i=1}^{k} \log |z_i|^2\right)\), where \(\tau_{k,n}\) is
the projection of \((\mathbb{D}_1^+)^k \times \mathbb{C}^{n-k}\) onto \(\mathbb{C}^{n-k}\), and \(\tau_{k,n}\) is a Kähler potential of \(\omega_{k,n}\). It follows that \(\text{Ric}(h) = \omega\) on a neighborhood of \(D\).

By imitating the argument in the proof of Theorem 1.1, one can give an analogy of Theorem 1.1 on Kähler manifolds with Poincaré type cusps.

**Proposition 1.1.** Let \(M\) be a compact Kähler manifold, \(D\) be a simple normal crossing divisor, and \((M, \omega) = (M\setminus D, \omega)\) be a complete Kähler manifold with Poincaré type cusp along \(D\). Assume that \((L, h)\) be a positive line bundle on \(M\) that satisfies the above conditions. For any open subset \(U \subset M\), we will denote by \(B_{U,\omega,m}(x, y)\) the Bergman kernel on \((U, \omega, L, h)\). Fix a Riemannian metric \(\bar{g}\) on \(M\). Write \(\bar{d}(x) = \text{dist}_{\bar{g}}(x, D)\), \(\forall x \in M\). Given constants \(\xi, \kappa, \beta > 0\). Then there exists a constant \(C > 0\) satisfying the following property.

Let \(m \geq C\) be an integer, and \((U_m, U'_m)\) be a pair of open subsets of \(M\) such that the distance \(\text{dist}_{\omega}(U'_m, M\setminus U_m) \geq \xi \frac{\log m}{\sqrt{m}}\), and \(\left\{ x \in M \mid \bar{d}(x) \leq e^{-\kappa \frac{\log m}{\sqrt{m}}} \right\} \subset U'_m\). Then the Bergman kernel satisfying that

\[
\sup_{U_m \times (M\setminus U_m) \times U'_m} \left| \log \bar{d}(x) \right|^{\beta} \left| \log \bar{d}(y) \right| \| B_{\omega,m}(x, y) \|^C_k \leq C^{-1}m^{-l},
\]

\[
\sup_{U_m \times U'_m} \left| \log \bar{d}(x) \right|^{\beta} \left| \log \bar{d}(y) \right| \| B_{U_m,\omega,m}(x, y) - B_{\omega,m}(x, y) \|^C_k \leq C^{-1}m^{-l}.
\]

This paper is organized as follows. In Section 2, we collect some preliminary results that will be used many times. Then we will calculate the Bergman kernels on the complex hyperbolic cusp in Section 3. The proofs of Theorem 1.1 and Proposition 1.1 are presented in Section 4. Finally, the proofs of Theorems 2.1, 2.3, and Proposition 1.4 are contained in Section 5.

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## 2. Preliminaries

Now we state the Hörmander’s \(L^2\) estimate as follows without proof.

**Proposition 2.1.** Let \((M, \omega)\) be an \(n\)-dimensional complete Kähler manifold. Let \((L, h)\) be a hermitian holomorphic line bundle, and let \(\psi\) be a function on \(M\), which can be approximated by a decreasing sequence of smooth functions \(\{\psi_i\}_{i=1}^\infty\). Suppose that

\[
\sqrt{-1} \partial \bar{\partial} \psi_i + \text{Ric}(\omega) + \text{Ric}(h) \geq \gamma \omega
\]

for some positive continuous function \(\gamma\) on \(M\), \(\forall i \in \mathbb{N}\). Then for any \(L\)-valued \((0,1)\)-form \(\zeta \in L^2\) on \(M\) with \(\bar{\partial} \zeta = 0\) and \(\int_M |\zeta|^2 e^{-\psi} \omega^n\) finite, there exists a \(L\)-valued function \(u \in L^2\) such that \(\bar{\partial} u = \zeta\) and

\[
\int_M |u|^2 e^{-\psi} \omega^n \leq \int_M \gamma^{-1} |\zeta|^2 e^{-\psi} \omega^n,
\]

where \(\| \cdot \|\) denotes the norms associated with \(h\) and \(\omega\).

The proof of Hörmander’s \(L^2\) estimate can be found in [9, 16]. Note that the \(L^2\) estimate also holds for complete Kähler orbifolds [24, Proposition 2] and weakly pseudoconvex Kähler manifolds [9, Corollary 5.3].

We introduce the Cheeger-Gromov \(C^{m, a}\)-norm for Kähler manifolds now.
Definition 2.1 (Holomorphic Norms). Let $(M, \omega, x)$ be a pointed Kähler manifold. We say that the holomorphic $C^{m,\alpha}$-norm on the scale of $r$ at $x$: 

$$
\| (M, \omega, x) \|_{C^{m,\alpha}, r}^{\text{holo}} \leq Q,
$$

provided there exists a biholomorphic chart $\phi : B_r(0) \to (U, x) \subset M$ such that $\phi(0) = x$, $|D\phi| \leq e^Q$ on $B_r(0)$ and $|D\phi^{-1}| \leq e^Q$ on $U$, and for all multi-indices $I$ with $0 \leq |I| \leq m$, $r^{(|I|+\alpha)} \| D^I \omega \|_\alpha \leq Q$. Globally we define 

$$
\| (M, \omega) \|_{C^{m,\alpha}, r}^{\text{holo}} = \sup_{x \in M} \| (M, \omega, x) \|_{C^{m,\alpha}, r}^{\text{holo}}.
$$

By using Hörmander’s $L^2$ estimate appropriately, Tian initiate his peak section method in [34]. One of the key steps of Tian’s peak section method is to construct the following global section.

Lemma 2.1 ([34], Lemma 1.2, Lemma 2.3). For an $n$-tuple of integers $P = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^+_n$, an integer $p' > p = \sum_{j=1}^n p_j$, and constants $\Lambda, \epsilon, Q > 0$, there are constants $m_0, C$ which depending on $\Lambda, n, p, p', \epsilon, Q$ with the following property.

Let $(M, \omega, L, h)$ be a polarized manifold such that $\text{Ric}(\omega) \geq -\Lambda \omega$, $\text{Ric}(h) \geq \epsilon \omega$, and $x \in M$. Suppose that $M$ is pseudoconvex if $(M, \omega)$ isn’t complete. Assume that there exists a local coordinate $(z_1, \cdots, z_n) : B_{\log Q}^{\frac{1}{p'}}(x) \to U \subset \mathbb{C}^n$, such that $x = (0, \cdots, 0)$, $e^{-Q \omega_{\text{Euc}}} \leq \omega \leq e^{Q \omega_{\text{Euc}}}$, and the hermitian matrix $(g_{ij})$ satisfies that $g_{ij}(0) = \delta_{ij}$, $dg_{ij}(0) = 0$, and $\|g_{ij}\|_{L^1, \omega} \leq Q$, where $\|f\|_{k, \omega}$ is the interior norms on a domain in $\mathbb{R}^{2n}$. We further assume that there exists a holomorphic frame $e_L$ on this coordinate such that the local representation function of $e_L$ is 

$$
\frac{2}{m} \left( 1 + \frac{a_m}{m^{2p}} \right) (z_1^{p_1} \cdots z_n^{p_n} + \varphi_m) e_L,
$$

and locally at $x$,

$$(0) = 1,\quad \frac{2}{m^{2p}},$$

and $\lambda_m = \lambda_m \left( 1 + \frac{a_m}{m^{2p}} \right)$, 

$$
\| \varphi_m \|_{h_m} \leq C, \quad |a_m| \leq C, \quad \varphi_m \text{ is holomorphic on } \left\{ |z| \leq \frac{\log(m)}{Q^{\frac{1}{2n}}} \right\},
$$

and $\|a_m\|_{h_m} \leq b_m \|z\|_{2p'}$, moreover 

$$
\lambda_m^{-2} = \int_{\left\{ |z| \leq \frac{\log(m)}{Q^{\frac{1}{2n}}} \right\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 a m d\nu,
$$

and hence we can assume that $|m^{n+p} \lambda_m^{-2} - n^p P| \leq C m^{-\frac{1}{2}}$, where $d\nu = \left( \sqrt{-1} \right)^n \det(g_{ij}) dz_1 \wedge \ldots \wedge dz_n$ is the volume form.
Remark. By the theory of Cheeger-Gromov convergence, we can find such coordinates and holomorphic frames on polarized manifolds with a bound of Ricci curvature and a lower bound of injective radius. See [2, 28] for details.

Remark. By the theory of Cheeger-Gromov convergence, we can find such coordinates and holomorphic frames on polarized manifolds with a bound of Ricci curvature and a lower bound of injective radius. See [2, 28] for details.

As a corollary, Tian shows that the global holomorphic sections constructed above have the following almost-orthogonal property.

**Corollary 2.1** ([34], Lemma 3.1). Let $S_m$ be a holomorphic global section constructed in Lemma 2.1, and $T$ be another holomorphic global section of $L^m$ with $\int_M \|T\|^2_{h_m} \, d\text{Vol}_\omega = 1$, which contains no term $z^p$ in its Taylor expansion at $x_0$. Then

$$\int_M \langle S_m, T \rangle_{h_m} \, d\text{Vol}_\omega \leq Cm^{-\frac{1}{2}}, \quad \forall m \geq m_0,$$

where $m_0, C$ are positive constants depending on $\epsilon, \Lambda, Q$.

Then we recall the Tian-Yau-Zelditch expansion theorem of Bergman kernels on a given manifold. The proof can be found in [6, 7, 21, 36].

**Theorem 2.1.** For constants $\Lambda, \epsilon, \xi, \delta k, Q > 0$, there are constants $m_0, m_1, C$ which depending on $\Lambda, \epsilon, \xi, \delta, k, Q$ with the following property.

Let $(M, \omega)$ be a Kähler manifold such that $\text{Ric}(\omega) \geq -\Lambda \omega$, let $(L, h)$ be a Hermitian line bundle on $(M, \omega)$ such that $\text{Ric}(h) \geq \omega$, and let $U$ be an open subset of $M$. If $(M, \omega)$ isn’t complete, we suppose that $\bar{U}_{m_0}$ is complete, $\text{Ric}(h) = \omega$ on $U_m$, $\inf_{x \in U_m} \text{infj}(x) \geq \frac{\delta \log m}{\sqrt{m}}$, and $\sum_{j=0}^{2k} \sup_{U_m} \|\nabla^j \text{Ric}(\omega)\| \leq Q$.

Then there exist coefficients $a_j \in C^\infty(M), \forall j \in \mathbb{N}$, such that

$$\left\| \rho_{\omega, m} - \sum_{j=0}^{k} a_j m^{n-j} \right\|_{C^k(U)} \leq Cm^{n-k-1}, \quad \forall m > m_1.$$

**Remark.** In Lemma 2.1, Corollary 2.1 and Theorem 2.1, we assume the complete and pseudoconvex condition just so that Hörmander’s $L^2$ estimate can be used. So these results still hold when $M$ is replaced by an open subset of $M$.

3. **Bergman kernels on complex hyperbolic cusp**

In this section, we calculate the Bergman kernels on the complex hyperbolic cusp by find a special $L^2$ orthonormal basis, and then we consider the asymptotic behavior of Bergman kernels. We only consider the case $n \geq 2$ in this section, because the expression of Bergman kernels on the punctured unit disc is known. For more details, see Subsection 3.1 in [4].

3.1. **Expression of the Bergman kernels on complex hyperbolic cusps.**

First we consider the expansion of holomorphic functions on the total space of line bundle. We will use the letter $\mathcal{L}$ to denote a holomorphic line bundle on a complex manifold $X$. Let $V$ be a connected open neighborhood of $X$ in the total space $\mathcal{L}$. Since $\mathcal{L}^{-1} \otimes \mathcal{L} \cong \mathcal{C}$, we see that for each section $S \in H^0(X, \mathcal{L}^{-1})$, the map $v \mapsto S(v)$ gives a holomorphic function on the total space of $\mathcal{L}$. Then we show that any holomorphic function on $V \setminus X$ has a similar structure.
Lemma 3.1. Let $f$ be a holomorphic function on $V \setminus X$. Then there exists a sequence of holomorphic sections $S_k \in H^0(M, \mathcal{L}^{-k})$, $\forall k \in \mathbb{Z}$, such that $f$ has the following power series expansion on $V \setminus X$:

$$f(v) = \sum_{k \in \mathbb{Z}} f_{S_k}(v) = \sum_{k \in \mathbb{Z}} S_k(v^k), \quad \forall v \in V \setminus X \subset \mathcal{L},$$

where $v^k = v^{\otimes k}$ is a point in the total space of $\mathcal{L}^k$.

Moreover, if $X$ is compact and $\mathcal{L}$ is a negative line bundle, then we have $S_k = 0$, $\forall k < 0$.

Proof. Let $\{U_j\}_{j \in J}$ be an open neighborhood of $X$ such that for each $j$, there exists a trivialization $\psi_j : \mathcal{L}|_{U_j} \cong U_j \times \mathbb{C}$ satisfies that $U_j \times \mathbb{D}_1 \subset \psi_j(V \cap \mathcal{L}|_{U_j})$, where $\mathbb{D}_1$ is the unit disc in $\mathbb{C}$. By the Laurent series expansion on $\mathbb{D}_1 = \mathbb{D}_1 \setminus \{0\}$, we can find a sequence of holomorphic functions $\{\varphi_{j,k}\}_{k \in \mathbb{Z}} \subset \mathcal{O}(U_j)$ for each $j$, such that

$$f \circ \psi_j^{-1}(z, w) = \sum_{k \in \mathbb{Z}} \varphi_{j,k}(z) w^k, \quad \forall z \in U_j, \ w_j \in \mathbb{D}_1^*,$$

and the coefficients satisfy

$$\varphi_{j,k}(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f \circ \psi_j^{-1}(z, w)}{w^{k+1}} dw.$$

Assume that $U_{j_1} \cap U_{j_2} \neq \emptyset$ for some $j_1, j_2 \in J$. Then the transition function of $\mathcal{L}$ on $U_{j_1} \cap U_{j_2}$ is a holomorphic function $\psi_{j_2,j_1} \in \mathcal{O}(U_{j_1} \cap U_{j_2})$ defined by $\psi_{j_2,j_1}^{-1}(z, \psi_{j_2,j_1}(z)) = \psi_{j_1}^{-1}(z, 1), \forall z \in U_{j_1} \cap U_{j_2}$. The Cauchy integral formula now shows that

$$\varphi_{j_1,k}(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f \circ \psi_{j_1}^{-1}(z, w_{j_1})}{w_{j_1}^{k+1}} dw_{j_1}.$$

It follows that $\{\varphi_{j,k}\}_{j \in J}$ gives a holomorphic section $S_k \in H^0(X, \mathcal{L}^{-k})$. Then we can find an open neighborhood $V_1$ of $X$ in $V$, such that $f$ has the above expansion in $V_1 \setminus X$. Then the connectedness of $V$ implies that $f$ has this expansion in the whole $V \setminus X$.

Now we assume that $X$ is compact and $\mathcal{L}$ is negative. By the Kodaira embedding theorem, we can find a constant $k_0 > 0$ such that $H^0(X, \mathcal{L}^{-k}) \neq 0$ for each $k > k_0$. If $H^0(X, \mathcal{L}^k) \neq 0$ for some $k > 0$, then there exists an integer $k_1 > k_0$ such that $H^0(X, \mathcal{L}^{k_1}) \neq 0$. It follows that we can find a holomorphic function $\phi = S \otimes S' \in H^0(X, \mathcal{L}^{-k_1} \otimes \mathcal{L}^{k_1}) \cong H^0(X, \mathbb{C})$ such that $\phi \neq 0$, but $\phi(x) = 0$ for some $x$. But the compactness of $X$ implies that $H^0(X, \mathbb{C}) \cong \mathbb{C}$, contradiction. Hence we have $H^0(X, \mathcal{L}^k) \neq 0, \forall k > 0$. This gives $S_k = 0, \forall k < 0$. This completes the proof.

Let $(\mathcal{L}_D, h_D)$ be a negative line bundle on $D$, and $(V, \omega_V, \mathcal{L}_V, h_V)$ be an $n$-dimensional ($n \geq 2$) complex hyperbolic cusp defined in Section 1 now.
For each $x \in D$, we can find an open neighborhood $U$ of $x$ in $D$, a biholomorphic chart $z = (z_1, \cdots, z_{n-1}): U \cong B_1^{n-1} \subset \mathbb{C}^{n-1}$, and a non-vanishing section $e_U \in H^0(U, \mathcal{L}_D)$. Let $w$ be a holomorphic map defined by $v = w(v)e_U, \forall v \in \mathcal{L}_D|_U$. Recall that $V$ is a subset of the total space of $\mathcal{L}_D$, we see that the holomorphic map $(z, w)$ gives a holomorphic chart:

$$(z, w): \mathcal{L}_D|_U \cap V \cong \{(z, w) \in \mathbb{B}_1^{n-1} \times \mathbb{C} \mid 0 < |w| \cdot \|e_U\|_{h_D} < 1\}.$$ 

Set $\varphi = \log \left(\frac{\|e_U\|^2}{h_D}\right)$. Then the volume form on $\pi^{-1}_D(U) \cap V, d\text{Vol}_\omega_v$, can be calculated as following:

$$d\text{Vol}_{\omega_v} = \frac{\left(-\sqrt{-1}\partial \bar{\partial} \log (-\log h_D)\right)^n}{n!}$$

$$= \frac{1}{n!} \left(\sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log h_D} + \sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log h_D^2}\right)^n$$

$$= \frac{1}{n!} \left(\sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log h_D} + \sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log h_D^2}\right)^n$$

$$= \frac{1}{n!} \frac{(\partial \bar{\partial} \varphi)^{n-1} \wedge \partial \left(\log |w|^2 + \varphi\right) \wedge \bar{\partial} \left(\log |w|^2 + \varphi\right)}{|\log h_D|^{n+1}}$$

$$= \frac{\sqrt{-1} \pi_D^* (d\text{Vol}_{\omega_D}) \wedge dw \wedge d\bar{w}}{|\log h_D|^n |\log h_D|^{n+1}} = \frac{\pi_D^* (d\text{Vol}_{\omega_D}) \wedge \omega_V}{n |\log h_D|^n |\log h_D|^{n-1}}.$$ 

Now we can find an orthonormal basis of $H^0_L(2, \mathcal{L}_V^m)$.

**Lemma 3.2.** For $m \geq n+1$, the set

$$\left\{\left(\frac{n}{2\pi (m-n-1)!} \int_{S_{q,j}} e_{\mathcal{L}_V}^m \right)^{\frac{1}{2}} f_{S_{q,j}} e_{\mathcal{L}_V}^m \mid 1 \leq j \leq N_q, q \geq 1\right\}$$

forms an orthonormal basis of $H^0_L(D, \mathcal{L}_V^m)$, where $\{S_{q,j}\}_{j=1}^{N_q}$ is an $L^2$ orthonormal basis in the Hilbert space $H^0_L(D, \mathcal{L}_D^{-q})$, and $f_{S_{q,j}}$ denote the holomorphic function corresponding to $S_{q,j}$ defined in Lemma 3.1.

**Proof.** Let $S_1$ and $S_2$ be the holomorphic functions corresponding to the holomorphic sections $S_1 \in H^0(D, \mathcal{L}_D^{-q})$ and $S_2 \in H^0(D, \mathcal{L}_D^{-q})$, respectively. Then we have

$$\int_V \langle f_{S_1} e_{\mathcal{L}_V}^m, f_{S_2} e_{\mathcal{L}_V}^m \rangle_{h_D^m} d\text{Vol}_{\omega_v} = \int_V f_{S_1} f_{S_2} |\log h_D|^m d\text{Vol}_{\omega_v}$$

$$= \frac{1}{n} \int_V f_{S_1} f_{S_2} |\log h_D|^{m+1-n} d\text{Vol}_{\omega_D} \wedge \omega_V$$

$$= \int_D \left(\frac{1}{n} \int_{V_z} f_{S_1} f_{S_2} |\log h_D|^{m+1-n} \omega_V \right) d\text{Vol}_{\omega_D},$$
where \( V_\delta = \pi_D^{-1}(x) \cap V \), \( z \in D \), and \( \pi_D : \mathcal{L}_D \to D \) is the projection of line bundle. For each \( z \in D \), we can find \( e \in \pi_D^{-1}(x) \) such that \( h_D(e) = 1 \). Then we have

\[
\frac{1}{n} \int_{V_\delta} f_{S_1, S_2} | \log h_D |^{m + 1 - n} \omega_V = \sqrt{-1} S_1 (e^n) \overline{S_2 (e^n)} \int_{D^*_1} \frac{w'^{q_1} w'^{q_2} \log |w|^2 |^{m + 1 - n}}{n |w|^2} dw \wedge dw
\]

\[
= \sqrt{-1} \delta_{q_1, q_2} h^{-q_1} (S_1, S_2) \int_{D^*_1} |w|^{2q_2 - 2} \log |w|^2 |^{m + 1 - n} \frac{dw \wedge dw}{n},
\]

where \( \delta_{q_1, q_2} \) is the Kronecker symbol. It follows that

\[
\int_V \langle f_{S_1, S_2} e_{L, V}^{m}, f_{S_1, S_2} e_{L, V}^{m} \rangle_{h_V^n} d\text{Vol}_{\omega_V} = 0,
\]

if \( q_1 \neq q_2 \). Now we assume that \( q_1 = q_2 = q \). Since \( m \geq n + 1 \), we have

\[
\int_{D^*_1} \sqrt{-1} |w|^{-2} \log |w|^2 |^{m + 1 - n} dw \wedge dw = \int_0^{2\pi} 2^{m-n-1} d\theta \int_0^1 r^{-1} \log r |^{m+1-n} dr = \infty,
\]

and hence \( e_{L, V}^{m} \notin H_{L, V}^0 (V, \mathcal{L}_V^n) \). When \( q \geq 1 \), by a direct computation, we obtain

\[
\int_{D^*_1} |w|^{2q-2} \log |w|^2 |^{m + 1 - n} \frac{dw \wedge dw}{\sqrt{-1}} = \int_0^{2\pi} 2^{m-n} d\theta \int_0^1 r^{2q-1} \log r |^{m+1-n} dr
\]

\[
= 2^{m-n+1} \pi \int_0^\infty e^{-2qt} r^{m+1-n} dt
\]

\[
= 2^{m-n+1} \pi \cdot (2q)^{n-m} \cdot (m-n-1)! = \frac{2\pi (m-n-1)!}{q^{m-n}}.
\]

Then we have

\[
\int_V \langle f_{S_1, S_2} e_{L, V}^{m}, f_{S_1, S_2} e_{L, V}^{m} \rangle_{h_V^n} d\text{Vol}_{\omega_V} = \int_D \left( \frac{1}{n} \int_{V_\delta} f_{S_1, S_2} | \log h_D |^{m+1-n} \omega_V \right) d\text{Vol}_{\omega_D}
\]

\[
= \int_D h^{-q} (S_1, S_2) d\text{Vol}_{\omega_D} \int_{D^*_1} \sqrt{-1} |w|^{2q-2} \log |w|^2 |^{m + 1 - n} \frac{dw \wedge dw}{n}
\]

\[
= \frac{2\pi (m-n-1)!}{n \cdot q^{m-n}} \int_D h^{-q} (S_1, S_2) d\text{Vol}_{\omega_D}.
\]

Combining Lemma 3.1 and the above calculation, we can conclude that the set (15) forms an orthonormal basis of \( H_{L, V}^0 (D, \mathcal{L}_V^n) \).

**Remark.** In the case \( n = 1 \), analysis similar to that in the proof of Lemma 3.2 shows that the set

\[
\left\{ \left( \frac{n \cdot q^{m-n}}{2\pi (m-n)!} \right) \frac{1}{q} e_{L, V}^{m} \mid q \geq 1 \right\}
\]

forms an orthonormal basis of \( H_{L, V}^0 (V, \mathcal{L}_V^n) \) when \( m \geq 2 \).

As another application of the calculation of the volume form on \( (V, \omega_V) \), we can estimate the injectivity radius on \( (V, \omega_V) \).

**Lemma 3.3.** There exists a constant \( \delta = \delta (D) > 0 \), such that the injectivity radius \( \text{inj}_V (x) \) becomes a strictly increasing function of \( h_D (x) \) on \( V_\delta \), and

\[
\frac{\delta}{\log (h_D (x))} \leq \text{inj}_V (x) \leq \frac{\pi}{\log (h_D (x))}, \quad \forall v \in V_\delta
\]

where \( \text{inj}_V (x) \) is the injectivity radius at \( x \in V \).
Proof. Let \( \pi_Q : \mathbb{C}^n \to D \) be the universal covering map of \( D \). Then the morphism \( \pi_Q^* \mathcal{L}_D \to \mathcal{L}_D \) induced by \( \pi_Q \) is also a universal covering map. For abbreviation, we use the same letter \( \pi_Q \) to designate it. Since \( \text{Pic} (\mathbb{C}^n) = 0 \), we have \( (\pi_Q^* \mathcal{L}_D, \pi^* h_D) \cong (\mathbb{C}, e^{|z|^2}) \), and the isomorphism gives a holomorphic chart \((z, w) \in \mathbb{C}^n \times \mathbb{C} \) of \( \pi_Q^{-1}(V) \). It is easy to show that \( \text{inj}_{\pi_Q^{-1}(V)}(z, w) \) is a function of \( h_D \circ \pi_Q(z, w) \) on \( \pi_Q^{-1}(V) \). For each smooth curve \( \gamma : [0, 1] \to \pi_Q^{-1}(V) \), we can construct a family of curves by \( \gamma_t(s) = (z(\gamma(s)), tw(\gamma(s))) \), \( \forall t \in (0, 1) \). By a straightforward computation, we obtain \( |\gamma'(s)| \leq |\gamma'(s)| \), and equality holds only when \( |\gamma'(s)| = 0 \). It follows that \( \text{inj}(z, w) \) is a strictly increasing function of \( h_D \circ \pi_Q(z, w) \) on \( \pi_Q^{-1}(V) \).

Let \( \gamma(s) = (0, e^{\sqrt{-1}t}w), s \in [0, 2\pi] \). Clearly, \( [\gamma] \neq 0 \) in the fundamental group \( \pi_1 \left( \pi_Q^{-1}(V) \right) \).

Then \( L(\gamma) = \frac{-2\pi}{\log(|w|)} \) implies that \( \text{inj}_{\pi_Q^{-1}(V)}(z, w) \leq \frac{-\pi}{\log(h_D(x))} \). Choosing \( x_1, x_2 \in \pi_Q^{-1}(x) \) and a smooth curve \( \gamma : [0, 1] \to \pi_Q^{-1}(V) \) such that \( x_1 \neq x_2 \), \( \gamma(0) = x_1 \) and \( \gamma(1) = x_2 \). Assume that \( L(\gamma) \leq 1 \). Then we have \( \text{inj}_{\pi_Q^{-1}(V)}(z, w) \leq \frac{-\pi}{\log(h_D(x))} \), and there exists a constant \( \epsilon = \epsilon(n) > 0 \) such that \( \text{dist}_{\pi_Q^{-1}(V)}(x_1, x_2) \geq \frac{\epsilon}{\sqrt{-e \log(h_D(x))}} \). Then we can find a constant \( \delta_1 = \delta_1(D) > 0 \) such that \( \text{inj}_{\pi_Q^{-1}(V)}(z, w) \geq \text{inj}_Q(z, w) \).

Since the holomorphic sectional curvature of \( \omega_V \) is a constant, we can conclude that the sectional curvature of \( \omega_V \) is bounded. Recall that the volume form on \( V \) can be expressed as \( dV_{\omega_V} = \frac{\pi^* \omega_D(h_D \circ \pi_Q(x), x)}{n \log(h_D(x))^{n-1}} \).

Thus we can find a constant \( \delta_2 > 0 \) depending only on \((V, \omega_V)\), such that \( \text{Vol}_{\omega_V} \left( B_{\pi_Q^{-1}(V)}(x) \right) \geq \delta_2 \left( \frac{-1}{\log(h_D(x))} \right)^{2n}, \forall x \in V_{\delta_2} \).

Then the Cheeger’s lemma implies this lemma.

Let \( \Gamma_V \) be a finite subgroup of \( \text{Aut}(V, \omega_V) \). Then we see that the injectivity radius function is invariant under the action of \( \Gamma_V \). Hence \( h_D \) is invariant under the action of \( \text{Aut}(V, \omega_V) \) on \( V \), and the action of \( \Gamma_V \) on \( V \) can be extended to the total space of \( \mathcal{L}_D \). Now we further extend the action of \( \Gamma_V \) on \( V \) to the line bundle \( \mathcal{L}_V \) such that \( h_V \) is invariant under this action. Notice that by the invariance of \( h_D \) under the action of \( \Gamma_V \), we have a homomorphism \( \Theta_{\Gamma_V} : \Gamma_V \to U(1) \subset \mathbb{C}^* \) such that \( g(e_{\mathcal{L}_V}) = \Theta_{\Gamma_V}(g)e_{\mathcal{L}_V}, \forall g \in \Gamma_V \).

Fix \( m \geq n + 1 \). Now we consider the group action \( \Gamma_V \times H^0(V, \mathcal{L}_V^m) \to H^0(V, \mathcal{L}_V^m) \) given by

\[
\Delta_{\Gamma_V, m} : \left\{ \begin{array}{ccc}
\Gamma_V & \times & H^0(V, \mathcal{L}_V^m) \\
ge & \to & H^0(V, \mathcal{L}_V^m)
\end{array} \right. \]

(17)

for any \( g \in \Gamma_V \subset \text{Aut}(V, \omega_V) \) and \( f \in \mathcal{O}(V) \).

Let \( H^0_{L^2}(V, \mathcal{L}_V^m)_q \) be the subspace of \( H^0_{L^2}(V, \mathcal{L}_V^m) \) generated by \( \{ f_{S_q, j} e_{\mathcal{L}_V^m} \}_{j=1}^{N_q} \), and \( H^0_{L^2}(V, \mathcal{L}_V^m)_{\Gamma_V} \) be the subspace of \( H^0_{L^2}(V, \mathcal{L}_V^m) \) containing all holomorphic sections that are invariant under the action \( \Delta_{\Gamma_V, m} \). Since \( H^0_{L^2}(V, \mathcal{L}_V^m)_q \) is invariant under the action \( \Delta_{\Gamma_V, m} \), we see that the Hilbert space \( H^0_{L^2}(V, \mathcal{L}_V^m)_{\Gamma_V} \) can be generated by \( \oplus_{q=1}^{N_q} \left( H^0_{L^2}(V, \mathcal{L}_V^m)_{\Gamma_V} \cap H^0_{L^2}(V, \mathcal{L}_V^m)_q \right) \).

Then there are non-negative integers \( N_{m,q} \leq N_q \) and \( L^2 \) orthonormal sets \( \{ S_{m,q,j} \}_{j=1}^{N_{m,q}} \subset H^0_{L^2}(D, \mathcal{L}_D^m)_{\Gamma_V} \) such
that the set
\[
\left\{ \frac{n \cdot q^{m-n}}{2\pi(m-n-1)!} f_{S_{m,q,j},^mL_V} \middle| 1 \leq j \leq N'_{m,q}, q \geq 1 \right\}
\]
forms an orthonormal basis of $H^0_{L_V}(V, L_V^m)_{\Gamma_V}$. Now we can define the $\Delta_{\Gamma_V,m}$-invariant Bergman kernel functions by $\rho_{D,\omega_D,\Delta_{\Gamma_V,m,q}} = \sum_{j=1}^{N'_{m,q}} \|S_{m,q,j}\|_{h_D^{-q}}^2$.

Then we can obtain the expression of Bergman kernel functions on $V/\Gamma_V$ for $m \geq n + 1$. **Proposition 3.1.** For $m \geq n + 1$, the Bergman kernel functions on $(V/\Gamma_V, \omega_V, L_V/\Gamma_V, h_V)$ can be expressed as
\[
\rho_{V/\Gamma_V,\omega_V,m}(\pi_V(x)) = \frac{n |\Gamma_V| |\log h_D(x)|^m}{2\pi(m-n-1)!} \sum_{q=1}^{\infty} q^{m-n} \rho_{D,\omega_D,\Delta_{\Gamma_V,m,q}}(\pi_D(x)) h_D(x)^q,
\]
where $x \in V$, $\pi_V : V \to V/\Gamma_V$ is the quotient map, and $\pi_D : V \subset L_D \to D$ is the projection.

**Proof.** By Lemma 3.2, we have
\[
\rho_{V/\Gamma_V,\omega_V,m}(\pi_V(x)) = |\Gamma_V| \sum_{q=1}^{\infty} \frac{n q^{m-n}}{2\pi(m-n-1)!} \left( \sum_{j=1}^{N'_{m,q}} |\tilde{S}_{m,j,q}(x)|^2 \cdot \|e_{\pi_V}^m\|_{h_D^m}^2 \right)
= |\Gamma_V| \sum_{q=1}^{\infty} \frac{n q^{m-n} \log h_D(x)^q}{2\pi(m-n-1)!} \left( \sum_{j=1}^{N'_{m,q}} \|S_{m,j,q}(\pi_D(x))\|_{h_D^{-q}}^2 \right)
= \frac{n |\Gamma_V| |\log h_D(x)|^m}{2\pi(m-n-1)!} \sum_{q=1}^{\infty} q^{m-n} \rho_{D,\omega_D,\Delta_{\Gamma_V,m,q}}(\pi_D(x)) h_D(x)^q,
\]
which proves the proposition. $\square$

**Remark.** As in the proof of Proposition 3.1, we see that for any integer $k \geq 0$ and $m \geq n + 1$, the Bergman kernel functions on $(V/\Gamma_V, \omega_V, L_V/\Gamma_V, |\log h_D|^k h_V)$ can be expressed as
\[
(19) \quad \rho^{(k)}_{V/\Gamma_V,\omega_V,m}(\pi_V(x)) = \frac{n |\Gamma_V| |\log h_D(x)|^{m(k+1)}}{2\pi(m(k+1)-n-1)!} \sum_{q=1}^{\infty} q^{m(k+1)-n} \rho_{D,\omega_D,\Delta_{\Gamma_V,m,q}}(\pi_D(x)) h_D(x)^q,
\]
where $x \in V$, $\pi_V : V \to V/\Gamma_V$ is the quotient map, and $\pi_D : V \subset L_D \to D$ is the projection.

### 3.2. Asymptotic behavior of Bergman kernel functions on complex hyperbolic cusps.

In this subsection, we will consider the asymptotic behavior of Bergman kernel functions on complex hyperbolic cusp.

Let $(V/\Gamma_V, \omega_V, L_V/\Gamma_V, h_V)$ be a complex hyperbolic cusp defined on an polarized flat Abelian variety $(D, \omega_D)$ with a polarization $(L_D^{-1}, h_D^{-1})$. Now we consider the Bergman kernel function near cusp. We can use an argument similar to that in [4].

For each integer $m \geq n + 1$, set
\[
b_m(a, t) = \frac{n |\Gamma_V| t^m}{2\pi(m-n-1)!} \sum_{q=1}^{\infty} q^{m-n} \rho_{D,\omega_D,\Delta_{\Gamma_V,m,q}}(a) e^{-qt}, \quad \forall a \in D, \forall t \in (0, \infty).
\]
It is clear that $\rho_{V/\Gamma_V,\omega_V,m}(x) = b_m(\pi_D(x), -\log h_D(x))$, for any $x \in V$. Then we consider the Taylor expansion of each term in this series.
Let \( f(t) = (1 + t)e^{-t} \). Then we see that
\[
b_m(a, mt) = \frac{n |\Gamma_V| m^me^{-m} \sum_{q=1}^{\infty} q^{-n} \rho_{D, \omega_D, \Delta_{G, m, q}}(a) (f(qt - 1))^m }{2\pi (m - n - 1)!} \sum_{q=1}^{\infty} q^{-n} \rho_{D, \omega_D, \Delta_{G, m, q}}(a) (f(qt - 1))^m , \forall x \in D, \forall t \in (0, \infty). \]

Now we consider the asymptotic behavior of \( f_m \) as \( m \to \infty \). We can assume that
\[
\sum_{j=1}^{\infty} \frac{1}{j!} \left[ m \sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{k} (m^t)^j \right] = \sum_{l=3}^{\infty} \lambda_{m,l} m^t \in \mathbb{R}[t],
\]
where \( \lambda_{m,l} \in \mathbb{R} \), and \( \mathbb{R}[t] \) is the formal power series ring over \( \mathbb{R} \). Clearly, we have \( \lambda_{m,3} = \frac{m}{3} \).

**Lemma 3.4.** Let \( N \geq 3 \) be an integer. Then we can find a constant \( C = C(N) > 0 \) such that
\[
(f(t))^m - \left( 1 + \sum_{l=3}^{N} \lambda_{m,l} m^t \right) e^{-\frac{mt^2}{2}} \leq \frac{C}{m^{\frac{N+1}{3}} |1 + |t| + 2|N|^\frac{1}{2}}
\]
for any integer \( m \geq n + 1 \) and constant \( t \in (-1, \infty) \).

**Proof.** It is easy to see that for each given \( m \in \mathbb{N} \), there exists a constant \( C = C(m, N) > 0 \) such that (20) holds. So we only need to prove it for sufficiently large \( m \geq e^{10N^2} \). We will divide the proof into three parts according to the value of \( t \).

When \( |t| \leq \frac{\log(m)}{\sqrt{m}} \), the Taylor expansion gives a constant \( C_1 = C_1(N) > 0 \) such that
\[
(f(t))^m - \left( 1 + \sum_{l=3}^{N} \lambda_{m,l} m^t \right) e^{-\frac{mt^2}{2}} \leq C_1 m^{\left\lfloor \frac{N+1}{3} \right\rfloor - \frac{N+1}{2} (mt^2)^{N+1} e^{-\frac{mt^2}{2}} \leq C_1 m^{\left\lfloor \frac{N+1}{3} \right\rfloor - \frac{N+1}{2} } ,
\]
where \( \lfloor . \rfloor \) is the greatest integer function.

By a straightforward computation, we see that \( t^l e^{-\frac{mt^2}{2}} \leq C_2 e^{-\frac{mt^2}{4}} \) for \( |t| \geq \frac{\log(m)}{\sqrt{m}} \) and \( l = 3, \cdots, N \), where \( C_2 = C_2(N) > 0 \) is a constant. It is sufficient to prove that there are constants \( m_0, C > 0 \) depending only on \( N \), such that \( f(t))^m \leq C m^{-N} (1 + t)^{-N} \) for \( m \geq m_0 \), and \( t \geq \frac{\log(m)}{\sqrt{m}} \).

When \( t \leq -\frac{\log(m)}{\sqrt{m}} \), we see that \( f(t) \) is increasing, and hence the Taylor expansion implies that
\[
(f(t))^m \leq \left( f \left( -\frac{\log(m)}{\sqrt{m}} \right) \right)^m \leq e^{m \log \left( 1 - \frac{\log(m)}{\sqrt{m}} \right)} \leq e^{-\frac{m \log(m)}{\sqrt{m}}} ,
\]
where \( C_3 = C_3(N) > 0 \) is a constant. Clearly, we can find a constant \( m_0 = m_0(N) > 0 \) such that \( (f(t))^m \leq m^{-N} \) for \( m \geq m_0 \) and \( t \leq -\frac{\log(m)}{\sqrt{m}} \).

Now we assume that \( t \geq \frac{\log(m)}{\sqrt{m}} \). By a direct calculation, we can conclude that \( f_{m,N}(t) \) is descending for \( t \geq \frac{N}{m} \), and hence we can assume that \( (1 + t)^N f_{m,N} \) is descending for \( t \geq \frac{\log(m)}{\sqrt{m}} \). Apply the Taylor expansion again, we have
\[
f_{m,N}(t) \leq f_{m,N} \left( -\frac{\log(m)}{\sqrt{m}} \right) e^{(m+N) \log \left( 1 + \frac{\log(m)}{\sqrt{m}} \right)} \leq e^{-\frac{m \log(m)}{\sqrt{m}}} ,
\]
where \( C_4 = C_4(N) > 0 \) is a constant. Clearly, we can find a constant \( m_1 = m_1(N) > 0 \) such that \( f_m(t) \leq m^{-N} (1 + t)^{-N} \) for \( m \geq m_0 \) and \( t \geq \frac{\log(m)}{\sqrt{m}} \). This completes the proof. \( \square \)
Lemma 3.5. There exists a constant \( C = C(N, V, |\Gamma_V|) > 0 \), such that

\[
\sup_{a \in D} \left| \frac{2\pi (m - n - 1)!}{n|\Gamma_V|^m} b_m(a, mt) - t \sum_{q=1}^\infty q^{-n} \rho_{D, \omega_D, \Delta_{\Gamma_V, m, q}}(a) e^{-qt} G_{m-1,N}(qt - 1) \right| \leq \frac{C}{m^{N+1 - \frac{N}{4}}} t,
\]

for any integer \( m \geq n + 1 \) and \( t \in (0, \infty) \).

Proof. By Lemma 3.4, we see that

\[
\left| \frac{2\pi (m - n - 1)!}{n|\Gamma_V|^m} b_m(a, mt) - t \sum_{q=1}^\infty q^{-n} \rho_{D, \omega_D, \Delta_{\Gamma_V, m, q}}(a) e^{-qt} G_{m-1,N}(qt - 1) \right| \leq C m^{-N+1} t \int_0^\infty e^{-t} \left( 1 + |q|^2N \right)^{-1} d\xi \leq C m^{-N+1} t e^{-t},
\]

where \( C = C(N, V, |\Gamma_V|) > 0 \) is a constant, and then we get (21). \( \square \)

As a corollary, we can estimate the sup of Bergman kernel functions.

Corollary 3.1. There exists a constant \( C = C(N, V, |\Gamma_V|) > 0 \), such that

\[
\left| \left( \frac{2\pi}{4} \right)^{\frac{3}{2} - \frac{N}{2}} m^{-\frac{N}{2} - \frac{1}{2}} \sup_{x \in \Gamma_V} \rho_{V/\Gamma_V, \omega_V, m}(x) - |\Gamma_V| \alpha_{D,m} \right| \leq \frac{C}{\sqrt{m}},
\]

where \( \alpha_{D,m} = \sup_{a \in D} \sup_{q \in \mathbb{N}} q^{-n} \rho_{D, \omega_D, \Delta_{\Gamma_V, m, q}}(a) \). Moreover, \( \alpha_{D,m} = \alpha_{D,m+N,|\Gamma_V|}, \forall m \in \mathbb{N} \), and hence there are constants \( c_1, \cdots, c_{|\Gamma_V|} > 0 \) depending only on \( (D, E_D), \Gamma_V \) and \( \Theta_{\Gamma_V} \), such that

\[
\left| c_j \left( k |\Gamma_V| + j \right)^{-N+1} \sup_{V} \rho_{V/\Gamma_V, \omega_V, k|\Gamma_V|+j - 1} \right| \leq \frac{C}{\sqrt{k}}, \forall k \in \mathbb{N},
\]

for \( j = 1, \cdots, |\Gamma_V| \).

Remark. By the standard Tian-Yau-Zelditch expansion, we see that

\[
\lim_{q \to \infty} q^{-n} \left( \sup_{a \in D} \rho_{D, \omega_D, \Delta_{\Gamma_V, m, q}}(a) \right) \leq |\Gamma_V| \lim_{q \to \infty} q^{-n} \left( \sup_{a \in D} \rho_{D, \omega_D, m, q}(a) \right) = \lim_{l \to \infty} (2\pi)^{n-1/2} l^{-1} = 0.
\]

Hence we can find an integer \( q_m \in \mathbb{N} \) and a point \( a \in D \), such that \( \alpha_{D,m} = q_m^{-n} \rho_{D, \omega_D, \Gamma_V, q_m}(a) \).
Proof. Let $t = q_m^{-1}$ such that $\alpha_{D,m} = q_m^{-n} \rho_{D,\omega_D,\Delta_{V,m},q_m}(a)$. Then Stirling’s formula implies that

$$
(2\pi)^{\frac{n}{2}} n^{-n-\frac{1}{2}} \sup_{x \in V/\Gamma_V} \rho_{V/\Gamma_V,\omega_V,m}(x) \geq (2\pi)^{\frac{n}{2}} n^{-n-\frac{1}{2}} b_m(a, mq_m^{-1})
$$

(24)

$$
\geq (1 - Cm^{-1}) |\Gamma_V| \sum_{j=1}^{\infty} j^{-n} \rho_{D,\omega_D,\Delta_{V,m},j}(a) (f(jq_m^{-1} - 1))^m
$$

$$
\geq (1 - Cm^{-1}) |\Gamma_V| \alpha_{D,m}(a),
$$

where $C > 0$ is a constant independent of $m$. For abbreviation, we use the same letter $C$ for constants independent of $m$. By Lemma 3.5, to prove (22), we only need to prove that

$$
\sup_{t > 0} \left( t \sum_{q=1}^{\infty} q^{1-n} \rho_{D,\omega_D,\Delta_{V,m},q}(a) e^{1-qt} G_{m-1,3}(qt - 1) \right) \leq \alpha_{D,m}(a) + \frac{C}{\sqrt{m}}.
$$

Note that the standard Tian-Yau-Zelditch expansion implies that $\rho_{D,\omega_D,\Delta_{V,m},q}(a) \leq C' q^{n-1}$ for some constant $C' > 0$ independent of $m$ and $q$. An easy computation now shows that

$$
\sup_{t > 0} \left( t \sum_{|q-t| \leq 1} q^{1-n} \rho_{D,\omega_D,\Delta_{V,m},q}(a) e^{1-qt} G_{m-1,3}(qt - 1) \right)
$$

$$
\leq \alpha_{D,m} + \sup_{t > 0} \left( C e^{-\frac{(m-1)^2}{m}} + C m t^3 + C m t^3 e^{-\frac{(m-1)^2}{m}} \right) \leq \alpha_{D,m} + \frac{C}{\sqrt{m}}.
$$

Then we estimate the remaining terms as follows.

$$
\sup_{t > 0} \left( t \sum_{|q-t| \geq 2} q^{1-n} \rho_{D,\omega_D,\Delta_{V,m},q}(a) e^{1-qt} G_{m-1,3}(qt - 1) \right)
$$

$$
\leq \sup_{t > 0} \left( C \sum_{q=1}^{\infty} (1 + m|q|)^3 e^{-\frac{(m-1)|q|^2}{2}} \right) \leq C \int_{0}^{\infty} t (1 + m|t\xi|^3) e^{-\frac{(m-1)|t\xi|^2}{2}} d\xi
$$

$$
= C \int_{0}^{\infty} (1 + my^3) e^{-\frac{(m-1)y^2}{2}} dy + C \sup_{\zeta > 0} \left( m\zeta^3 e^{-\frac{(m-1)^2}{2}} \right) \leq \frac{C}{\sqrt{m}}
$$

and (22) is proved.

Since $\Theta_{\Gamma_V}(g)|_{\Gamma_V} = 1$, $\forall g \in \Gamma_V$, we see that $\rho_{D,\omega_D,\Delta_{V,m},q} = \rho_{D,\omega_D,\Delta_{V,m},+|\Gamma_V|,q}$, $\forall m \in \mathbb{N}$. It follows that $\alpha_{D,m} = \alpha_{D,m+|\Gamma_V|}$, $\forall m \in \mathbb{N}$, which completes the proof. \qed

Remark. For any integer $k \geq 0$ and $m \in \mathbb{N}$, let $\rho^{(k)}_{V/\Gamma_V,\omega_V,m}(x)$ be the $m$-th Bergman kernel function on $\left(V/\Gamma_V, \omega_V, \mathcal{L}_V/\Gamma_V, |\log h_D|^{k} h_V\right)$.

As in the proof of Corollary 23, the series expansion (19) shows that there exists a constant $C = C(k, N, V, |\Gamma_V|) > 0$, such that

$$
(2\pi)^{\frac{n}{2}} n^{-n-\frac{1}{2}} \sup_{x \in V/\Gamma_V} \rho^{(k)}_{V/\Gamma_V,\omega_V,m}(x) - |\Gamma_V| \alpha_{D,m} \leq \frac{C}{\sqrt{m}},
$$

where $\alpha_{D,m} = \sup_{a \in D} \sup_{q \in \mathbb{N}} q^{-n} \rho_{D,\omega_D,\Delta_{V,m},q}(a)$.\]
4. Localization of Bergman kernels

In this section, we establish some basic estimates for peak sections, and then use it to estimate the distance of global Bergman kernel and the Bergman kernel on asymptotic complex hyperbolic cusp. We will also show that this localization approach is also valid for the Poincaré type metrics.

4.1. Localization of peak sections.

Let $(M, \omega)$ be a Kähler manifold, $\mathcal{L}$ be a holomorphic line bundle on $M$ equipped with a hermitian metric $h$ whose curvature form is $\omega$. For any closed subspace $X \subset H^0_{L^2}(M, \mathcal{L})$ and complex bounded linear functional $0 \neq T \in X^*$, we say that a section $S \in X$ is the peak section of $T$, if $\int_M \|S\|^2 \mathrm{dVol}_\omega = 1$, and $T(S) = \|T\|$. By the Riesz representation theorem, $S$ is the peak section of $T$ if and only if $T(S) \in \mathbb{R}$, and $S \perp \ker T$ in $X \subset H^0_{L^2}(M, \mathcal{L})$. Note that $H^0_{L^2}(M, \mathcal{L})$ is a Hilbert space, and so is $X$. It follows that there exists a unique peak section of $T$ in $X$.

Let $U$ be an open subset of $M$, $X \subset H^0_{L^2}(U, \mathcal{L})$ be a closed subspace, and $0 \neq T \in X^*$. Then the restriction map gives a bounded linear embedding $H^0_{L^2}(M, \mathcal{L}) \to H^0_{L^2}(U, \mathcal{L})$. When $X \cap H^0_{L^2}(M, \mathcal{L}) \not\subset \ker T$, we see that $T \neq 0$ on $X \cap H^0_{L^2}(M, \mathcal{L})$, and hence we can find a unique peak section of $T|_{X \cap H^0_{L^2}(M, \mathcal{L})}$ in $X \cap H^0_{L^2}(M, \mathcal{L})$. The peak section of $T|_{X \cap H^0_{L^2}(M, \mathcal{L})}$ is called the peak section on $M$. The following lemma gives an estimate of the $L^2$ distance between the peak sections of $T \in X^*$ on $U$ and on $M$.

**Lemma 4.1.** Let $S_U \in X$ and $S_M \in X \cap H^0_{L^2}(M, \mathcal{L})$ be the peak sections on $U$ and $M$ respectively. Assume that $\left\|T\|_{X \cap H^0_{L^2}(M, \mathcal{L})}\right\| \geq (1 - \epsilon) \|T\| > 0$ for some $\epsilon > 0$. Then

$$\int_{M \setminus U} \|S_M\|^2 \mathrm{dVol}_\omega + \int_U \|S_U - S_M\|^2 \mathrm{dVol}_\omega \leq 2\epsilon.$$

**Proof.** By definition, we have $T(S_M) = \left\|T\|_{X \cap H^0_{L^2}(M, \mathcal{L})}\right\| \leq \|T\| = T(S_U)$. Hence there exists a constant $\delta \in (0, \epsilon]$ such that $T(s_M) = (1 - \delta) T(s_U)$. It follows that $s_M - (1 - \delta) s_U \in \ker T$. Then we have

$$\int_U \langle S_U, S_M - (1 - \delta) S_U \rangle \mathrm{dVol}_\omega = 0.$$

Combining these, we can conclude that

$$\int_U \|S_U - S_M\|^2 \mathrm{dVol}_\omega = \int_U \|S_U\|^2 + \|S_M\|^2 - 2\Re \langle S_U, S_M \rangle \mathrm{dVol}_\omega$$

$$= \int_U \|S_U\|^2 + \|S_M\|^2 - 2(1 - \delta) \|S_U\|^2 \mathrm{dVol}_\omega$$

$$= 2\delta - \int_{M \setminus U} \|S_M\|^2 \mathrm{dVol}_\omega \leq 2\epsilon - \int_{M \setminus U} \|S_M\|^2 \mathrm{dVol}_\omega,$$

and the lemma follows. \qed

Now we show how to use the peak section to give some estimates about Bergman kernels. Taking a vector $e_p \in \mathcal{L}^m$ at $p \in U$ such that $\|e_p\|_{h^m} = 1$. Then $T_p(S) = e_p^* (S)$ gives a bounded linear functional of $H^0_{L^2}(U, \mathcal{L}^m)$. If $T_p \neq 0$, then the peak section $S_U \in H^0_{L^2}(U, \mathcal{L}^m)$ satisfies that $B_U(p, x) = S_U(p) \otimes S_U(x)^*$, $\forall x \in U$, where $B_U$ is the Bergman kernel on $U$. So the Bergman kernel can be represented by a peak section. By combining Lemma 2.1, Theorem 2.1 and Lemma 4.1, we can give a proof of the following Agmon type estimate.
Corollary 4.1. Given constants \( n, \delta, \Lambda, \epsilon, Q > 0 \) and \( k \geq 0 \), there exists a constant \( C \) which depending only on \( n, \delta, \Lambda, \epsilon, k, Q \) with the following property.

Let \((M, \omega)\) be a Kähler manifold, \( \mathcal{L} \) be a holomorphic line bundle on \( M \) equipped with a hermitian metric \( h \) such that \( \text{Ric} (\omega) \geq - \Lambda \omega, \text{Ric}(h) \geq \epsilon \omega \), and \( x \in M \). If \((M, \omega)\) isn’t complete, we suppose that \( M \) is pseudoconvex.

Fix \( m \geq C \). Write \( U_m = B_{\varepsilon_m^m}(x_0) \). Assume that \( U_m \) is compact, \( \text{Ric}(h) = \omega \) on \( U_m \),
inf_{y \in U_m} \text{inj}(y) \geq \frac{\delta \log m}{\sqrt{m}} \quad \text{and} \quad \sum_{j=0}^{2k} \sup_{U_m} \left\| \nabla^j \text{Ric}(\omega) \right\| \leq Q. \text{ Then we have}

\[
\| B_{M,\omega,m}(x,y) \|_{C^l; h_m, \omega} \leq C \varepsilon_m^{-k+n+l},
\]

for any \( x, y \in B_{\varepsilon_m^m}(x_0) \) satisfying that \( \text{dist}_\omega(x,y) \geq \frac{\delta \log m}{\sqrt{m}} \), where \( B_{M,\omega,m}(x,y) \) is the \( m \)-th Bergman kernel on \((M, \omega, \mathcal{L}, h)\).

Proof. Without loss of generality, we can assume that \( \delta = 1 \). For each \( m \geq 2 \), we denote \( \varepsilon_m \) briefly by \( \varepsilon_m \). By the theory of Cheeger-Gromov convergence, we can assume that for any point \( y \in U_m \), there exists a biholomorphic map \((z_1, \cdots, z_n) : B_{\varepsilon_m^m}(y) \to W_y \subset \mathbb{C}^n \) around \( y \) such that \( z_1(y) = \cdots = z_n(y) = 0 \), \( e^{-Q'} \omega_{Euc} \leq \omega = e^{Q'} \omega_{Euc} \), and the hermitian matrix \((g_{ij})\) satisfies that \( g_{ij}(0) = \delta_{ij} \), \( dg_{ij}(0) = 0 \), and \( \| g_{ij} \|_{2^{k+1} + \frac{1}{2}; W_y} \leq Q' \), where \( \mu \in (0, 1) \), \( Q' \geq 10 \) are constants depending only on \( n, k, Q \), and \( \| f \|_{k, \alpha} \) is the interior norms on a domain in \( \mathbb{R}^{2n} \). We further assume that there exists a holomorphic frame \( e_y \) of \( \mathcal{L} \) on this coordinate such that the local representation function of \( h, a = \log h(e_y, e_y) \), satisfying that \( a(0) = 0 \), \( \frac{\partial a}{\partial z^j}(0) = 0 \) for each multi-index \( I \) with \( |I| \leq 3 \), and \( \| a \|_{2^{k+4} + \frac{1}{2}; U_y} \leq Q' \varepsilon_m^2 \).

For notational convenience the same letter \( C \geq 10 \) will be used to denote large constants depending only on \( n, \Lambda, k, \epsilon, Q \).

Let \( \{ S_j \}_{j \in J} \) be an orthonormal basis in \( H^0_{L^2}(M, \mathcal{L}^m) \), and \( x, y \) be points in \( B_{2\varepsilon_m^m}(x_0) \) satisfying that \( \text{dist}_\omega(x,y) \geq \varepsilon_m \). Assume that \( S_j = f_j x e^m_x \) on \( B_{\varepsilon_m^m}(x) \), and \( S_j = f_j y e^m_y \) on \( B_{\varepsilon_m^m}(y) \). Then we have \( B_{M,\omega,m} = a^m_y \left( \sum_j f_j x f_j y \right) \| e^m_x \otimes e^m_y \|_{B_{\varepsilon_m^m}(x) \times B_{\varepsilon_m^m}(y)} \).

Since \( \| g_{ij} \|_{2^{k+1} + \frac{1}{2}; W_y} \leq 2Q' \| a \|_{2^{k+4} + \frac{1}{2}; W_y} \leq 2Q' \varepsilon_m^2 \), it is sufficient to show that

\[
\sum_j \frac{\partial^{|P_1|} f_j x}{\partial x^1} \frac{\partial^{|P_2|} f_j y}{\partial y^2} \leq C \varepsilon_m^{-k+n+|P_1|+|P_2|}, \quad \forall |P_1| + |P_2| \leq k,
\]

where \( P_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i}) \in \mathbb{Z}_+^n \) are \( n \)-tuples of integers.

For each \( S \in H^0(W_y, \mathcal{L}_m) \), \( f = e^m_y(S) \) is a holomorphic function on \( W_y \), and hence the map \( T_{P,y}(S) = \frac{\partial^{|P_1|}}{\partial z^1} \frac{\partial^{|P_2|}}{\partial \bar{z}^2} \) gives a linear functional \( T_{P,y} \) on \( H^0(W_y, \mathcal{L}^m) \), where \( P = (p_1, p_2, \ldots, p_n) \) is an \( n \)-tuple of integers. Then Lemma 2.1 shows that \( T_{P,y} \| H^0_{L^2}(M, \mathcal{L}^m) \) \( \neq 0 \) for sufficiently large \( m \). For any \( y \in M \) and open neighborhood \( U \) of \( y \), let \( S_{P,y,U} \) denotes the peak section of \( T_{P,y} \| H^0_{L^2} (U, \mathcal{L}^m) \) \( \neq 0 \). Then we only need to show that

\[
| T_{P_1,x,M}(S_{P_1,x,M}) \cdot T_{P_2,y,M}(S_{P_2,x,M}) | \leq C \varepsilon_m^{-k+n+|P_1|+|P_2|}, \quad \forall |P_1| + |P_2| \leq k.
\]
Combining Theorem 2.1 with Lemma 4.1, we can conclude that
\[
\int_{M \setminus B_{\epsilon m} (y_1)} \|S_{0,y_1,M} \|^2 \, dVol_\omega \leq Cm^{-k}, \quad \forall m \geq C, \quad \forall y_1 \in B_{\epsilon m} (x_0).
\]
Hence we see that
\[
(27) \quad |T_{P_1,y_1} (S_{0,y_2,M})| \leq \left( \int_{M \setminus B_{\epsilon m} (y_2)} \|S_{0,y_2,M} \|^2 \, dVol_\omega \right)^{\frac{1}{2}} |T_{P_1,y_1} (S_{P_1,y_1,M} \setminus B_{\epsilon m} (y_2))| \leq Cm^{-\frac{k}{2}} |T_{P_1,y_1} (S_{P_1,y_1,B_{\epsilon m} (y_1)})|,
\]
for any \( m \geq C \), and \( y_1, y_2 \in B_{\epsilon m} (x_0) \) satisfying that \( \text{dist}_\omega (y_1, y_2) \geq \frac{\epsilon m}{100} \).

By Lemma 2.1, we can find a holomorphic section
\[
S'_{P_1,y_1,B_{\epsilon m} (y_1)} \in H^0 \left( B_{\epsilon m} (y_1), \omega, \lambda_m, h^m \right),
\]
such that \( \left\| S'_{P_1,y_1,B_{\epsilon m} (y_1)} \right\|_{L^2, \lambda_m, \omega} = 1 \), \( 0 < T_{P_1,y_1} (S'_{P_1,y_1,B_{\epsilon m} (y_1)}) \leq Cm^{-\frac{k}{2}} \), and Corollary 2.1 now implies that
\[
\left| \int_{B_{\epsilon m} (y_1)} \left\langle S'_{P_1,y_1,B_{\epsilon m} (y_1)}, S'' \right\rangle \, dVol_\omega \right| \leq Cm^{-\frac{k}{2}} \|S''\|_{L^2, \lambda_m, \omega},
\]
for any \( m \geq C \), and \( S'' \in \ker T_{P_1,y_1} \cap H^0 \left( B_{\epsilon m} (y_1), \omega, \lambda_m, h^m \right) \). Let
\[
\partial_{P_1,y_1} = \int_{B_{\epsilon m} (y_1)} \left\langle S'_{P_1,y_1,B_{\epsilon m} (y_1)}, S_{P_1,y_1,B_{\epsilon m} (y_1)} \right\rangle \, dVol_\omega.
\]
Then we have \( S'_{P_1,y_1,B_{\epsilon m} (y_1)} - \partial_{P_1,y_1} S_{P_1,y_1,B_{\epsilon m} (y_1)} \in \ker T_{P_1,y_1} \), and we have
\[
(28) \quad 1 - \partial_{P_1,y_1} \leq \left\| S'_{P_1,y_1,B_{\epsilon m} (y_1)} - \partial_{P_1,y_1} S_{P_1,y_1,B_{\epsilon m} (y_1)} \right\|_{L^2, \lambda_m, \omega} \leq \left\langle S'_{P_1,y_1,B_{\epsilon m} (y_1)}, S'_{P_1,y_1,B_{\epsilon m} (y_1)} - \partial_{P_1,y_1} S_{P_1,y_1,B_{\epsilon m} (y_1)} \right\rangle_{L^2, \lambda_m, \omega} \leq Cm^{-\frac{k}{2}}, \quad \forall m \geq C.
\]
We thus get \( \partial_{P_1,y_1} \geq 1 - Cm^{-\frac{k}{2}}, \forall m \geq C \). Now (27) becomes
\[
(29) \quad |T_{P_1,y_1} (S_{0,y_2,M})| \leq Cm^{-\frac{k}{2}} \left| T_{P_1,y_1} (S_{P_1,y_1,B_{\epsilon m} (y_1)}) \right| \leq Cm^{-\frac{k}{2}} \partial_{P_1,y_1} \leq C2^{m-\frac{k}{2}} + \frac{\epsilon m}{40},
\]
for any \( m \geq C \), and \( y_1, y_2 \in B_{\epsilon m} (x_0) \) satisfying that \( \text{dist}_\omega (y_1, y_2) \geq \frac{\epsilon m}{40} \).

It is easy to check that
\[
T_{P_1,y_1} (S_{0,y_2,M}) T_{0,y_2} (S_{0,y_2,M}) = \sum_j T_{P_1,y_1} (S_j) T_{0,y_2} (S_j) = T_{P_1,y_1} (S_{P_1,y_1,M}) T_{0,y_2} (S_{P_1,y_1,M}).
\]
Let $y_1 = x$. Then we have
\[
\|SP_{1,x,M}(y_2)\|_{S_m} = |T_{0,y_2}(SP_{1,x,M})|
\]
\[
= |TP_{1,x}(S_{0,y_2,M})| \cdot |T_{0,y_2}(S_{0,y_2,M})| \cdot |TP_{1,x}(SP_{1,x,M})|^{-1}
\]
\[
\leq Cm^{-\frac{k-n-|P_1|}{2}} \cdot m^{\frac{n}{2} - k} \cdot m^{-\frac{n+|P_1|}{2}}
\]
\[
= Cm^{-\frac{k+n}{2}}, \quad \forall m \geq C, \quad \forall y_2 \in B_{S_m}(x_0) \setminus B_{S_m}(x).
\]

It follows that
\[
|TP_{2,y}(SP_{1,x,M})|\]
\[
\leq \left( \int_{B_{2s_m}(x) \setminus B_{S_m}(x)} \|SP_{1,x,M}\|^2 dVol_{\omega} \right)^{\frac{1}{2}} \left|TP_{2,y}(SP_{1,x,M})\right|
\]
\[
\leq Cr_n m^{-\frac{1}{2}} \left|TP_{2,y}(SP_{2,y,M})\right|
\]
\[
\leq C^2 m^{-\frac{k+n}{2}}(\log m)^n,
\]
and (26) is proved. \qed

Remark. It is also shown in the above argument that for each given constant $\mu \in (0,1)$, the peak section, $S_{P_1,x,M}$, satisfying that
\[
\int_{B_{2s_m}(x) \setminus B_{S_m}(x)} \|SP_{1,x,M}\|^2 dVol_{\omega} \leq Cm^{-\frac{1}{2}}(\log m)^n, \quad \forall m \geq C,
\]
where $C$ is a constant depending only on $n, \Lambda, \epsilon, \delta, \mu, k, Q$.

### 4.2. Approximate the peak section by approximating the metric.

Let $M$ be a Kähler manifold, $(\mathcal{L}, h)$ be a Hermitian line bundle on $M$, let $\omega$ and $\omega'$ be Kähler metrics on $M$, and let $u$ be a smooth real-valued function on $M$. Suppose that there are constants $\delta, \epsilon > 0$ such that $e^{-\epsilon} \omega'' \leq \omega' \leq e^{\epsilon} \omega''$ and $|u| \leq \delta$. Write $h' = e^{-u}h$. Clearly, we see that $H^0_{L^2}(M, \omega, \mathcal{L}, h) = H^0_{L^2}(M, \omega', \mathcal{L}, h')$ as linear subspaces of $H^0(M, \mathcal{L})$, and the $L^2$-norms are equivalent. So we can use the same notation $H^0_{L^2}(M, \mathcal{L})$ for them if we don’t emphasize the norms. Let $X$ be a closed subspace of $H^0_{L^2}(M, \mathcal{L})$. Fix a bounded linear functional $0 \neq T : H^0_{L^2}(M, \mathcal{L}) \rightarrow \mathbb{C}$.

Let $S_T$ be the peak section of $T$ in $X \subset H^0_{L^2}(M, \omega, \mathcal{L}, h)$, and $S'_T$ be the peak section of $T$ in $X \subset H^0_{L^2}(M, \omega', \mathcal{L}, h')$. Then we can estimate the $L^2$-norm of $S_T - S'_T$ as follows.

**Lemma 4.2.** Under the above assumptions, we have
\[
\|S_T - S'_T\|_{L^2,h',\omega'}^2 \leq e^{\delta + \epsilon} - e^{-\delta - \epsilon},
\]
and
\[
e^{-\frac{\delta + \epsilon}{2}}T(S'_T) \leq T(S_T) \leq e^{\frac{\delta + \epsilon}{2}}T(S'_T).
\]

**Proof.** By definition, we can obtain
\[
\|S_T\|_{L^2,h',\omega'}^2 = \frac{1}{n!} \int_M \|S_T\|_{h'}^2 \omega'' \leq \frac{e^{\delta + \epsilon}}{n!} \int_M \|S_T\|_{h'}^2 \omega'' = e^{\delta + \epsilon}.
\]
Hence we have
\[
T(S'_T) \geq T\left(\|S_T\|_{L^2,h',\omega'}^{-1} S_T\right) = \|S_T\|_{L^2,h',\omega'}^{-1} T(S_T) \geq e^{-\frac{\delta + \epsilon}{2}}T(S_T).
\]
Similarly, we can conclude that
\[
\|S_T\|_{L^2; h, \omega}^2 = \frac{1}{n!} \int_M \|S_T\|_{h}^2 \omega^n = \frac{e^{\delta + \epsilon}}{n!} \int_M \|S_T\|_{h}^2 \omega^n = e^{\delta + \epsilon},
\]
and then we see that
\[
T(S_T) \geq T\left(\|S_T\|_{L^2; h, \omega}^{-1} S_T\right) = \|S_T\|_{L^2; h, \omega}^{-1} T(S_T) \geq e^{-\frac{\delta}{2}} T(S_T).
\]
Combining (36) with (38), we get
\[
e^{-\frac{\delta}{2} + \epsilon} T(S_T) \leq T(S_T) \leq e^{\frac{\epsilon}{2}} T(S_T).
\]
Let \( \lambda \in \left[e^{-\frac{\delta}{2} + \epsilon}, e^{\frac{\epsilon}{2}}\right] \) be the unique constant such that \( T(S_T') = \lambda T(S_T) \). Then we have \( S_T' - \lambda S_T \in \ker T \cap X \). It follows that
\[
\int_M \langle S_{T}', S_T \rangle_h d\Vol_\omega - \lambda = \int_M \langle S_T - \lambda S_T, S_T \rangle_h d\Vol_\omega = 0,
\]
and hence
\[
\|S_T - S_T'\|_{L^2; h, \omega}^2 = \int_M \langle S_T - \lambda S_T, S_T - \lambda S_T \rangle_h d\Vol_\omega
\]
\[
= \|S_T\|_{L^2; h, \omega}^2 - 2\lambda \int_M \langle S_T', S_T \rangle_h d\Vol_\omega + \lambda^2
\]
\[
= \|S_T\|_{L^2; h, \omega}^2 - \lambda^2 \leq e^{\delta + \epsilon} - e^{-\delta - \epsilon},
\]
and the proof is complete. \( \square \)

4.3. Peak sections on asymptotic complex hyperbolic cusps.

We will prove Theorem 1.1 in this part. In this subsection, we let \((M, \omega)\) be a Kähler manifold with \(\operatorname{Ric}(\omega) \geq -\Lambda \omega\) and let \((\mathcal{L}, h)\) be a Hermitian line bundle with \(\operatorname{Ric}(h) \geq \epsilon \omega\), where \(\epsilon, \Lambda > 0\) are constants. Suppose that \(M\) is pseudoconvex if \((M, \omega)\) is not complete. Assume that there exists an asymptotic complex hyperbolic cusp \((U, \omega, \mathcal{L}, h) \equiv \left(V_r/\Gamma_V, \omega_V - \sqrt{-1} \partial \bar{\partial} u, \mathcal{L}_V/\Gamma_V, e^u h_V\right) \) on \((M, \omega)\) such that \(U\) is complete, where \(r \in (0, 1)\) is a positive constant, and \(u = o(1)\) as \(h_D \to 0^+\) to all orders with respect to \(\omega_V\). Note that \((\mathcal{L}_V, h_V) \cong (\mathbb{C}, |log h_D|)\) on \(V\).

It will cause no confusion if we use the same letter to designate a subset of \(V_r/\Gamma_V\) and the open subset in \(U\) corresponding to it. Assume that \(U\) is complete in \((M, \omega)\). Since \(u = o(1)\) as \(h_D \to 0^+\) to all orders with respect to \(\omega_V\), we can estimate the lower bounds of injectivity radius on \(U\). For any subset \(W \subset M\), we write \(\operatorname{inj}(W) = \inf_{x \in W} \operatorname{inj}_M(x)\).

**Lemma 4.3.** Let \(r\) be the constant defined in above, and \(k \in \mathbb{N}\) be a constant. Then there exists a constant \(C = C(k, M, \omega) > 0\) such that the curvature \(\sum_{j=0}^{5k+10n} \|\nabla^j \operatorname{Ric}(\omega)\|_{\omega} \leq C \) on \(U\), and for each \(t \in (0, r)\), we have \(\operatorname{inj}(U \setminus \pi_V(V_t)) \geq C^{-1} |\log t|^{-1}\), where \(\pi_V : V \to V/\Gamma_V\) is the quotient map.

**Proof.** Since the holomorphic sectional curvature of \(\omega_V\) is a constant, we can conclude that the sectional curvature of \(\omega_V\) is bounded. Then \(u = o(1)\) implies that the sectional curvature of \(\omega\) is bounded on \(U\). Similarly, for each \(j \in \mathbb{Z}_{\geq 0}\), \(\|\nabla^j \operatorname{Ric}(\omega)\|_{\omega} \) is bounded on \(U\). As in the proof of Lemma 3.3, Cheeger’s lemma now shows that there exists a constant \(\zeta > 0\) depending only on \((M, \omega, L, h)\), such that \(\operatorname{inj}(x) \geq \zeta |\log t|^{-1}\) on \(U \setminus \pi_V(V_t)\), for any \(t \in (0, r)\). Note that \(U\) is complete. \( \square \)
Now we can construct suitable quasi-plurisubharmonic functions for using $L^2$ estimate.

**Lemma 4.4.** Let $r$ be the constant defined in above, and $\beta > 0$, $\delta \in (0, e^{-1})$ be constants. Suppose that $\delta < \min\{e^{-1}, r\}$. Then for any point $y \in V_{\beta}^{1+n}$, there exists a quasi-plurisubharmonic function $\psi \in L^1(M, \omega)$ such that $\psi \leq 0$, $\lim_{x \to y} (\psi(x) - \log (\text{dist}_\omega(x, y))) = -\infty$, $\sqrt{-1} \partial \bar{\partial} \psi \geq -C |\log \delta| \omega$, and $\text{supp} \psi \subset \tilde{V}_\delta/\Gamma_\nu$, where $C > 0$ is a constant depending only on $(M, \omega)$, and $\beta$.

**Proof.** We assume that $\Gamma_\nu = 0$ at first. By Theorem 2.1, for sufficiently large integer $k$, we have $(2\pi)^{-n} \leq k^{-n} \rho_{D, \omega D, k} \leq (2\pi)^n$. Choosing an $L^2$ orthonormal basis $\{S_j\}_{j=1}^{N_k}$ of $H^0_{L^2}(D, \mathcal{L}_D^{-k})$ satisfying that $S_j(\pi_D(y)) = 0$, $\forall j \geq 2$. Note that we can construct holomorphic functions $\tilde{S}_j$ from $S_j$ as in Lemma 3.1. Let

$$f(x) = |\tilde{S}_1(x) - \tilde{S}_1(y)|^2 + \sum_{j \geq 2} |\tilde{S}_j(x)|^2, \quad \forall x \in V.$$ 

Then we have $f(x) = \rho_{D, k} (\pi_{D}(x)) h_D(x)^k + |\tilde{S}_1(y)|^2 - 2 \text{Re} \left(\bar{\tilde{S}}_1(x)\tilde{S}_1(y)\right)$. Clearly, we see that $|\tilde{S}_1(y)|^2 \leq \delta^k (1+2\beta) (2\pi)^n$. Since $u = o(1)$, we can conclude that there exists a constant $k_0 > 0$ depending only on $(M, \omega)$ and $\beta$, such that $\|\nabla \log f(x)\|_{\omega} \leq C_1 |\log \delta|$, for any $x \in V_\delta \setminus V_{\beta}^{1+n}$ and $k \geq k_0$, where $C_1$ is a constant depending only on $(M, \omega)$, $k$ and $\beta$. Fix a large integer $k$.

By a straightforward calculation, we see that $\text{dist}_\omega(V_{\beta}^{1+n}, M \setminus V_\delta) \geq C_2^{-1} \log (1 + \beta)$ for some constant $C_2$ depending only on $(M, \omega)$. Then we can choose a cut-off function $\eta \in C_\infty (M)$ such that $\eta = 0$ on $V_\delta \setminus V_{\beta}^{1+n}$, $0 \leq \eta \leq 1$, and $\sum_{i=0}^{2} \|\nabla^i \eta\|_{\omega} \leq C_3$, where $C_3$ is a constant depending only on $(M, \omega)$.

Let $C_f = \sup_{V_\delta} f$, and $\psi_0(x) = \eta(x) (\log f(x) - \log C_f)$. It is easy to see that

$$\sup_{V_\delta \setminus V_{\beta}^{1+n}} \sum_{i=0}^{1} \|\nabla^i (\log f(x) - \log C_f)\|_{\omega} \leq C_4 |\log \delta|,$$

where $C_4$ is a constant depending only on $(M, \omega)$ and $\beta$. Hence we have

$$\sqrt{-1} \partial \bar{\partial} \psi_0(x) = \sqrt{-1} (\log f(x) - \log C_f) \partial \bar{\partial} \eta(x) + \sqrt{-1} \partial \log f(x) \wedge \bar{\partial} \eta + \sqrt{-1} \bar{\partial} \eta(x) \wedge \partial \log f(x) + \sqrt{-1} \bar{\partial} \eta \partial \delta \log f(x) \geq -3C_3 C_4 |\log \delta| \omega, \quad \forall x \in V_\delta \setminus V_{\beta}^{1+n}.$$ 

Here we used $\sqrt{-1} \partial \bar{\partial} \log f(x) \geq 0$ on $V$.

Now we consider the case $\Gamma_\nu \neq 0$. Let $\psi_0$ be the quasi-plurisubharmonic function constructed on $(V_\nu, \pi_\nu^* \omega)$. Since $\psi = \sum_{g \in \Gamma_\nu} \psi_0 \circ g$ is invariant under the action of $\Gamma_\nu$, we see that $\psi$ gives the function we need. $\square$

**Remark.** If we choose $f(x) = \sum_{j=1}^{N_k} |S_j(x)|^2$ in the proof of Lemma 4.4, then we can find a quasi-plurisubharmonic function $\psi \in L^1(M, \omega)$ such that $\psi \leq 0$, $\lim_{h_D(x) \to 0} (\psi(x) - \log (h_D(x))) = -\infty$, $\sqrt{-1} \partial \bar{\partial} \psi \geq -C |\log \delta| \omega$, $\text{supp} \psi \subset \tilde{V}_\delta$, and $\psi$ is invariant under the action of $\Gamma_\nu$ on $V_\nu \cong U$, where $C > 0$ is a constant depending only on $(M, \omega)$, and $\beta$.

Since $(U, \omega, \mathcal{L}, h) \cong (V_\nu/\Gamma_\nu, \omega V - \sqrt{-1} \partial \bar{\partial} u, \mathcal{L}_V/\Gamma_\nu, e^u \mathcal{L}_V)$ and $(\mathcal{L}_V, h_V) \cong (C, |\log h_D|)$ on $V$, we can conclude that any $L^m$-valued holomorphic section on $U$ can be represented by a $\Delta_{t \nu, m}$-invariant holomorphic function on $V_\nu$, $\forall m \in \mathbb{N}$. It follows that any finite linear combination of $\delta_x$ and its derivatives gives a bounded linear functional on $H^0_{H^2} (U, \mathcal{L}^m)$, where $\delta_x$ is the Dirac function.
at $x$. By applying Hörmander’s $L^2$ estimate in a similar way as in [34], we can give the following estimates for such peak sections. For abbreviation, we denote $e^{-\frac{r_m}{\sqrt{m}}}$ and $\log m$ briefly by $r_m$ and $\varepsilon_m$, respectively, for any $m \geq 2$.

**Proposition 4.1.** Let $\kappa$ and $\delta$ be given positive constants. Then there are constants $C, m_0$ that depend only on $(M, \omega, L, h)$, $k, \kappa, \delta$ and $l$, and satisfies the following property.

Let $m \geq n + 1$ be an integer satisfies that $r_m^k < e^{-1}r$. Choosing a constant $\gamma_m \in [r_m^k, e^{-1}r]$, a point $x_m \in \pi_V(V_{\gamma_m})$, and a linear functional $T_m$ defined by finite linear combination of $\delta_{x_m}$ and its derivatives, where $\pi_V : V \to V/\Gamma_V$ is the quotient map. Assume that the order of derivatives contained in $T_m$ is at most $k$. Let $U_m$ be an open neighborhood of $x_m \in M$. Suppose that $U_m \subset U$ and $\text{dist}_{\omega}(\pi_V(V_{\gamma_m}), M \setminus U_m) \geq \varepsilon_m$. Then

$$\left\| T_m \right\|_{L^2(M, \mathcal{L}^m)}^2 \geq \left( 1 - \frac{C}{m^l} \right) \left\| T_m \right\|_{L^2(U_m, \mathcal{L}^m)}^2 \cdot$$

**Proof.** Without loss of generality, we can assume that $\kappa = \delta = 1$. When $m \geq m_0$, this proposition is obvious. So we can assume that $T_m \mid_{L^2(U_m, \mathcal{L}^m)} \neq 0$ from now.

The proof falls naturally into two parts by whether $x_m$ belongs to the set $\pi_V(V_{r_m})$.

**Part 1.** We start by prove the estimate (42) when $x_m \in \pi_V(V_{r_m}) \setminus \pi_V(V_{\gamma_m})$.

By Lemma 4.3, we see that $\| (M, \omega, x_m) \|_C^{k, \alpha}$-norm defined in Definition 2.1, and $Q' > 0$ is a constant depending only on $(M, \omega, L, h)$, $k$ and $r$. Then we can find a holomorphic coordinate $(z_1, \cdots, z_n) : W_{\xi, x_m} \to B_{\varepsilon_{x_m}}(0) \subset \mathbb{C}^n$ around $x_m$, such that the hermitian matrices $(g_{ij}(\xi, x_m))_{C^{k+10, 2, x_m}}$ and there are holomorphic frames $e_{1, m}$ of $C\omega$ on $W_{\xi, x_m}$ such that

$$a = \log h(e_{1, m}, e_{1, m}) \text{ satisfying that } a(0) = 0, \frac{\partial a}{\partial z_{i}}(0) = 0 \text{ for each multi-index } I \text{ with } |I| \leq 3, \text{ and } \|a\|^{k+10, 2, \omega}_{W_{\xi, x_m}} \leq Q'_{\varepsilon_{x_m}} \text{, where } \xi, Q > 0 \text{ are constants depending only on } (M, \omega, L, h), k \text{ and } r.$$

For each $\lambda \in (0, \xi)$, the inverse image of $B_{\varepsilon_{x_m}}(0)$ about the map $\phi_{x_m}$ will be denoted by $W_{\lambda, x_m}$.

Let $\eta(t) \in C^\infty(\mathbb{R})$ be a cut-off function such that $\eta = 1$ on $(-\infty, 1]$, $\eta = 0$ on $[2, \infty)$, $-2 \leq \eta' \leq 0$, $|\eta'| \leq 8$. Set $\eta_{\lambda, m}(z) = \eta \left( \frac{|z|^2}{\lambda^2 r_{x_m}} \right)$, and $\psi(z) = 4(\varepsilon_{x_m} + n) \log \left( \frac{|z|^2}{\lambda^2 r_{x_m}} \right)$, and $\eta_{\lambda, m}(z)$. Then we have $\sqrt{3}d\bar{\omega} \geq -C_1 \lambda^2 \varepsilon_{x_m} \omega_{\mathcal{E}uc}$, where $C_1 > 0$ is a constant. It will cause no confusion if we use the same letter to designate a function on $B_{\lambda, x_m}(0)$ and the composite of it with the map $\phi_{x_m}$.

Since $T_m \mid_{L^2(U_m, \mathcal{L}^m)} \neq 0$, there exists a peak section $S_{T_m, W_{\lambda, x_m}}$ of $T_m$ on $W_{\lambda, x_m}$. By the Hörmander’s $L^2$ estimate, we can find a smooth $\mathcal{L}^m$-valued section $u_1 \in L^2$ on $M$ such that $\partial u_1 = \bar{\partial} (\eta_{\lambda, m}S_{T_m, W_{\lambda, x_m}})$ that satisfies $\text{Vol}_{\mathcal{L}^m}(W_{\lambda, x_m}) = \text{Vol}_{\mathcal{L}^m}(T_m, W_{\lambda, x_m})$.

$$\int_M \left\| u_1 \right\|_{\mathcal{L}^m}^2 e^{-\psi}d\text{Vol}_{\omega} \leq \frac{C_2}{\varepsilon_{x_m} - C_1 e^{Q\lambda^2 \varepsilon_{x_m}}} \int_M \left\| \bar{\partial} u_1 \right\|_{\mathcal{L}^m}^2 e^{-\psi}d\text{Vol}_{\omega}$$

$$\leq \frac{C_2}{\varepsilon_{x_m} - C_1 e^{Q\lambda^2 \varepsilon_{x_m}}} \int_M \left\| S_{T_m, W_{\lambda, x_m}} \right\|_{\mathcal{L}^m}^2 e^{-\psi}d\text{Vol}_{\omega}$$

$$\leq \frac{C_2}{\varepsilon_{x_m} - C_1 e^{Q\lambda^2 \varepsilon_{x_m}}} \int_M \left\| S_{T_m, W_{\lambda, x_m}} \right\|_{\mathcal{L}^m}^2 d\text{Vol}_{\omega} = C_2 \left( \log m \right)^{-2},$$

for any $m \geq 1 + e^{10C_1 e^{Q\lambda^2 \varepsilon_{x_m}}}$, where $C_2 > 1$ is a constant depending only on $\varepsilon, C_1, \lambda, \kappa, \delta$ and $l$. We assume that $m \geq 1 + e^{10C_1 e^{Q\lambda^2 \varepsilon_{x_m}}}$ from now. Then the integrability of $u_1$ implies that
$T_m(u_1) = 0$, and hence we have

$$
\left\| T_m \right\|_{L^2_L(M,\mathcal{L}_m)} \geq \left\| \eta_{\lambda,m} S_{T_m,W,\lambda,x,m} - u_1 \right\|_{L^2_L(h^{-\lambda},\omega)}^{-1} \left\| T_m \left( \eta_{\lambda,m} S_{T_m,W,\lambda,x,m} - u_1 \right) \right\| \\
= \left( \left\| \eta_{\lambda,m} S_{T_m,W,\lambda,x,m} \right\|_{L^2_L(h^{-\lambda},\omega)} + \left\| u_1 \right\|_{L^2_L(h^{-\lambda},\omega)} \right)^{-1} \left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) \right\| \\
\geq \left( 1 + C_2^3 (\log m)^{-1} \right)^{-1} \left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) \right\|.
$$

(44)

Analysis similar to that in the proof of Corollary 4.1 shows that for any $\lambda \in \left( 0, \frac{\epsilon}{10} \right)$,

$$
\left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) S_{T_m,W,\lambda,x,m} (y) \left\| = \left\| T_m \left( S_{y,W,\lambda,x,m} \right) S_{y,W,\lambda,x,m} (y) \right\| \\
\leq \left( \int_{W,\lambda,x,m} \left\| S_{y,W,\lambda,x,m} \right\|^2 dVol_\omega \right)^{\frac{1}{2}} \left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) \right\| \left( \rho_{m,\lambda,W,\lambda,x,m} (y) \right)^{\frac{1}{2}} \\
\leq C_3 m^{-2k-4n-1} \left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) \right\|, \quad \forall y \in W_{\lambda,x,m} \setminus W_{3\lambda,x,m},
$$

where $C_3 = C_3 (\lambda, C_1, C_2, \epsilon, \Lambda, k, r) > 0$ is a constant, and $S_{y,W}$ is the peak section of $\delta_y$ on $W$. It follows that $\left\| S_{T_m,W,\lambda,x,m} \right\| \leq C_3 m^{-2k-4n-1}$ on $W_{\lambda,x,m} \setminus W_{3\lambda,x,m}$. By applying the Hörmander’s $L^2$ estimate on $W_{\lambda,x,m} \setminus W_{3\lambda,x,m}$, we can conclude that

$$
\left\| T_m \right\|_{L^2_L(M,\mathcal{L}_m)} \geq \left( 1 - C_4 m^{-2k-2n-1} \right) \left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) \right\|, \quad \forall y \in W_{\lambda,x,m} \setminus W_{3\lambda,x,m},
$$

(46)

where $C_4 = C_4 (\lambda, C_1, C_2, \epsilon, \Lambda, k, r) > 0$ is a constant. Fix $\lambda = \frac{\epsilon}{10}$. It is easy to see that

$$
\left\| T_m \left( S_{T_m,W,\lambda,x,m} \right) \right\| = \left\| T_m \right\|_{L^2_L(M,\mathcal{L}_m)} \geq \left\| T_m \right\|_{L^2_L(U_m,\mathcal{L}_m,\mathcal{L}_m)}.
$$

(47)

Combining (46) with (47), we can conclude that the estimate (42) holds for $x_m \in V_{\gamma,m} \setminus V_{\epsilon,m}$ and $C = e^{(2k+2n+\delta)}(C_1+10C_2\lambda^{-2}\epsilon^{-1})$.

Part 2. To complete the proof, we need to prove the estimate (42) when $x_m \in \pi_V \left( V_{\epsilon,m} \right)$.

By Lemma 4.4, we can find a quasi-plurisubharmonic function $\psi_m$ on $M$ by choosing $\delta_m = r_m$ and $\beta_m = 0$. Then we have $\sqrt{-1} \partial \bar{\partial} \psi_m \geq -C_5 z^{-1} \omega$, where $C_5 > 0$ is a constant depends only on $(M,\omega)$. Let $\eta_m \in \mathcal{C}^\infty (M)$ be a cut-off function such that $\eta_m = 1$ on $\pi_V \left( V_{\delta_m} \right)$, $\eta_m = 0$ on $M \setminus \pi_V \left( V_{\delta_m} \right)$, and $\sum_{i=1}^{\infty} \left\| \nabla^i \eta_m \right\|^2 \leq C_6$, where $C_6 > 0$ is a constant depends only on $(M,\omega)$.

Choosing a large integer $m_0$ such that $m_0 > 2C_5 \sqrt{m_0} + 1$. By the Hörmander’s $L^2$ estimate, we can find a smooth $\mathcal{L}_m$-valued section $u_2$ on $M$ such that $\tilde{\partial} u_2 = \tilde{\partial} \left( \eta_m S_{T_m,V_{\epsilon,m}^{1,5}/\pi_V} \right)$, and

$$
\int_M \left\| u_2 \right\|^2_{L^m} e^{-5(n+k)} \psi_m dVol_\omega \leq \frac{1}{m - C_3 \epsilon m} \int_M \left\| \tilde{\partial} u_2 \right\|^2_{L^m} e^{-5(n+k)} \psi_m dVol_\omega \\
\leq \frac{C_4}{m - C_3 \sqrt{m}} \int_{\pi_V \left( V_{\epsilon,m}^{1,5} \right) \setminus \pi_V \left( V_{\delta_m} \right)} \left\| S_{T_m,V_{\epsilon,m}^{1,5}/\pi_V} \right\|^2_{L^m} dVol_\omega \leq \frac{C_7}{m - C_3 \sqrt{m}},
$$

(48)

where $S_{T_m,V_{\epsilon,m}^{1,5}/\pi_V}$ is the peak section of $T_m$ on $V_{\epsilon,m}^{1,5}/\pi_V$, and $C_7 > 0$ is a constant depending only on $(M,\omega,\mathcal{L},h)$ and $k$. Then there exists a constant $C_8 > 0$ depending only on $(M,\omega,\mathcal{L},h)$.
and \( k \), such that \( \int_M \| u_2 \|^2_{h_m} \, d\text{Vol}_\omega \leq C_k m^{-1}, \forall m \geq m_0 \). By the integrability of \( u_2 \), we see that \( T_m(u_2) = 0 \), and hence we have

\[
\left\| T_m \right\|_{L^2(M,\mathcal{E}^n)} \geq \left( \int_M \left[ \eta_m S_{T_m,V_{1,5}/\Gamma_V} - u_2 \right]^2_{h_m} \, d\text{Vol}_\omega \right)^{\frac{1}{2}} \left| T_m \left( \eta_m S_{T_m,V_{1,5}/\Gamma_V} - u_2 \right) \right| \geq (1 + C_k m^{-1})^{-\frac{1}{2}} \left| T_m \left( S_{T_m,V_{1,5}/\Gamma_V} \right) \right|.
\]

(49)

Analysis similar to that in the Part 1 now shows that for any integer \( l \geq 0 \),

\[
\| T_m \left( S_{T_m,U_m} \right) S_{T_m,U_m}(y) \| = \| T_m \left( S_{y,U_m} \right) S_{y,U_m}(y) \|
\]

\[
\leq \left( \int_{\Gamma_V} \| S_{y,U_m} \|_{h_N}^2 \, d\text{Vol}_\omega \right) \left| T_m \left( S_{T_m,V_{1,5}/\Gamma_V} \right) \right| \left( \rho_{m,\omega,U_m}(y) \right)^{\frac{1}{2}}
\]

\[
\leq C_9 m^{-2k-4n-l} \left| T_m \left( S_{T_m,U_m} \right) \right|, \; \forall y \in \pi_V \left( V_{r_m} \right),
\]

where \( C_9 > 0 \) is a constant depending only on \( (M,\omega,L) \), \( k \) and \( l \), and \( S_{y,U} \) is the peak section of \( \delta_\mu \) on \( W \). Since \( T_m \left| H^0_{L^2}(V_{r_m}/\Gamma_V,\mathcal{E}^n) \right| \neq 0 \), we can see that \( \| S_{T_m,U_m}(y) \| \leq C_7 m^{-2k-4n-l} \), for any \( y \in \pi_V \left( V_{r_m} \right) \).

Let \( \chi_m \in C^\infty(M) \) be a cut-off function such that \( \chi_m = 1 \) on \( \pi_V \left( V_{r_m} \right), \chi_m = 0 \) on \( M \setminus \pi_V \left( V_{r_m} \right), \) and \( \sum_{i=1}^2 \| \nabla^i \chi_m \| \leq C_{10} \), where \( C_{10} > 0 \) is a constant depending only on \( (M,\omega,L,h) \), \( k \) and \( l \). By using the Hörmander’s \( L^2 \) estimate to the weight function \( e^{-5(n+k)\psi_m} \) and the smooth \( \mathcal{E}^n \)-valued \( (0,1) \)-form \( \bar{\partial} (\chi_m S_{T_m,U_m}) \), we can conclude that

\[
\left\| T_m \right\|_{L^2(M,\mathcal{E}^n)} \geq \left( 1 - C_{11} m^{-2k-2n-l} \right) \left\| T_m \right\|_{H^0_{L^2}(U_m,\mathcal{E}^n)} \right|,
\]

(51)

where \( C_{11} > 0 \) is a constant depending only on \( (M,\omega,L,h) \), \( k \) and \( l \). This completes the proof.

We assume that \( u = O \left( \| \log h_D \|^{-\alpha} \right) \) as \( h_D \to 0^+ \) to all orders with respect to \( \omega_V \) for some constant \( \alpha > 0 \) from now. Then we can give an estimate for the derivatives of holomorphic sections.

**Lemma 4.5.** We follow the above assumption about \( u \). Given constants \( n, k \in \mathbb{N} \) and \( Q > 0 \). There exists a constant \( C > 0 \) with the following property.

Let \( (M,\omega) \) be an \( n \)-dimensional Kähler manifold, \( \mathcal{L} \) be a holomorphic line bundle on \( M \) equipped with a hermitian metric \( h \), and \( x \in M \). Assume that \( \text{inf} (B_r(x)) \geq r, \text{Ric}(h) = \omega \) on \( B_r(x) \), and \( \sum_{i=0}^k \sup_{B_r(x)} \left\| \nabla^i \text{Ric}(\omega) \right\| \omega \leq Q \), where \( r > 0 \) is a constant. Then for any \( k, m \in \mathbb{N} \) and \( S \in H^0_{L^2}(M,\mathcal{E}^m) \), we have \( \| \nabla^i S(x) \|^2 + \| \nabla^i S^*(x) \|^2 \leq C m^{n+k} \left( \min \left\{ \sqrt{m}r, 1 \right\} \right)^{-2n-2k} \| S \|_{L^2(B_r(x))}^2 \).

**Proof.** Without loss of generality, we can assume that \( \| S \|_{L^2;B_r(x)} \neq 0 \). By replacing \((M,\omega) \) with \((M,m\omega) \), we can reduce this lemma to the case of \( m = 1 \). By the theory of Cheeger-Gromov \( C^{k,\alpha} \)-convergence, we can find a coordinate \((z_1,\cdots,z_n) : B_r(x) \to U_x \subset \mathbb{C}^n \) such that the image of \( x \) is 0, \( e^{-Q^*} \omega_{Euc} \leq \omega \leq e^{Q^*} \omega_{Euc} \), and the hermitian matrix \( \left( g_{ij} \right) \) satisfies \( g_{ij}(0) = \delta_{ij}, dg_{ij}(0) = 0 \), and \( \| g_{ij} \|_{k+1,\frac{1}{2};U_x} \leq Q' \), where \( \| f \|_{k,\alpha} \) is the interior norms on a domain in \( \mathbb{R}^{2n} \), and \( Q' = Q'(n,k,Q) > 0 \) is a constant. We further assume that there exists a holomorphic frame \( e_x \) of \( \mathcal{L} \) on this coordinate such that the local representation function of \( h = h(e_x,e_x) \), satisfying that \( a(0) = 1 \) and \( \| a - 1 \|_{k+1,\frac{1}{2};U_x} \leq Q' \rho^2 \).
Write \( S = f \varepsilon_z \), where \( f \in \mathcal{O}(U_x) \) is a holomorphic function. Now we are reduced to establishing the estimate \( \sum_{l=0}^{k} \| \nabla^l f(0) \|_{\omega_{\mathbb{R}^n}}^2 \leq C (\min \{ r, 1 \})^{-2n-2k} \int_{U_x} |f|^2 d\text{Vol}_{\mathbb{C}^n} \) on the Euclidean domain \( U_x \subset \mathbb{C}^n \) containing \( B_{c^{-\alpha}}(0) \). This is an immediate consequence of the standard interior estimates for derivatives of harmonic functions. This is our claim. \( \square \)

As a corollary of Lemma 4.5, we can give the following estimate of the derivatives of the holomorphic sections around the complex hyperbolic cusps. Here we denote \( e^{-\frac{\varepsilon_m}{\sqrt{m}}} \) and \( \log \frac{m}{\sqrt{m}} \) briefly by \( r_m \) and \( \varepsilon_m \), respectively, for any \( m \geq 2 \).

**Corollary 4.2.** We follow the above assumption about \( u \). Given \( \delta, \kappa, \beta, k > 0 \), there exists a constant \( C > 1 \) depending only on \( (M, \omega, \mathcal{L}, h) \), \( \delta, \kappa, \beta \) and \( k \) satisfying the following property.

Let \( m \geq C \) be an integer such that \( \pi_V(V_{r_m}) \subset U \), \( U_m \) be a sequence of open subsets of \( U \) containing \( \pi_V(V_{r_m}) \), and \( \bar{U}_m \) be open subsets of \( U_m \) such that \( \text{dist}_\omega(\bar{U}_m, M \setminus U_m) \geq \delta \varepsilon_m \). Then for any \( S \in H^0_L(U_m, \mathcal{L}^m) \), we have \( \| \nabla^k S(x) \|_{h^m, \omega}^2 + \| \nabla^k S^*(x) \|_{h^m, \omega}^2 \leq C m \| \log h_D(x) \|^{-\beta} \| S \|_{L^2,h^m,\omega}^2 \), \( \forall x \in \bar{U}_m \), where \( S^* \) is the smooth section of \( \mathcal{L}^{-m} \) given by \( S^*(e) = h^m(e, S) \), and \( \nabla^* \) is the Chern connection of the Hermitian line bundle \( (\mathcal{L}^{-m}, h^{-m}) \).

**Proof.** Without loss of generality, we can assume that \( \| S \|_{L^2,h^m,\omega} = 1 \) and \( \kappa = \delta = 1 \).

Note that that \( u = O \left( \| \log h_D \|^{-\alpha} \right) \) as \( h_D \to 0^+ \) to all orders with respect to \( \omega_V \) for some constant \( \alpha > 0 \). Let \( \tau_m = e^{-\frac{1}{2}+2a_1^m} \). By Lemma 4.3, we see that there exists a constant \( C_1 > 1 \) depending only on \( (M, \omega) \) and \( k \), such that for any \( m \in \mathbb{N} \), we have \( \text{inj}(U \setminus \pi_V(V_{r_m})) \geq C_1^{-1} m^{-2^2+2a_1^{-1}} \), and \( \sum_{j=0}^{2k} \| \nabla^j \text{Ric}(\omega) \|_{\omega} \leq C_1 \) on \( U \). Then we can use Lemma 4.5 to find a constant \( C_2 > 0 \) depending only on \( (M, \omega) \) and \( k \), such that

\[
(52) \quad \sup_{\bar{U}_m \setminus \pi_V(V_{r_m})} \left( \| \nabla^k S \|_{h^m, \omega}^2 + \| \nabla^k S^* \|_{h^m, \omega}^2 \right) \leq C_2 m^{C_2}.
\]

Since \( \| \log h_D \| \leq m^{2^2+2a_1^{-1}} \) on \( \bar{U}_m \setminus \pi_V(V_{r_m}) \), we can conclude that

\[
(53) \quad \| \nabla^k S(x) \|_{h^m, \omega}^2 + \| \nabla^k S^*(x) \|_{h^m, \omega}^2 \leq C_2 m^{C_2+2^2+2a_1^{-1} \beta} \| \log h_D(x) \|^{-\beta}, \quad \forall x \in \bar{U}_m \setminus \pi_V(V_{r_m}).
\]

By the definition, we have \( \| S \|_{h}^2 \leq \rho_{U_m, \omega, \mathcal{L}, h} \) on \( U_m \). Since \( h = \| \log h_D \| e^u, \omega = \omega_V - \sqrt{-1} \partial \bar{\partial} u \) and \( u = O \left( \| \log h_D \|^{-\alpha} \right) \), we see that

\[
(54) \quad \sup_{V_m/\Gamma_V} \left( \log \left( \frac{h^m}{\| \log h_D \|^m} \right) + \left( \frac{\omega^m}{\omega^m_V} \right) \right) \leq C_3 m \| \log \tau_m \|^{-\alpha} \leq C_3 m^{-1},
\]

where \( C_3 > 1 \) is a constant depending only on \( (M, \omega) \). Lemma 4.2 now shows that

\[
(55) \quad \rho_{V_m/\Gamma_V, m}(x) \leq e^{C_5} \rho_{U_m, \omega, \mathcal{L}, h}(x), \quad \forall x \in V_m/\Gamma_V,
\]

where \( \rho_{V_m/\Gamma_V, m} \) is the \( m \)-th Bergman kernel on \( (V_m/\Gamma_V, \omega, \mathcal{L}, h) \).

Analysis similar to that in the proof of Proposition 3.1 shows that for any \( m \geq n + 1 \),

\[
(56) \quad \rho_{V_m/\Gamma_V, \omega, \mathcal{L}, h}(x) \leq C_4 \| \log h_D(x) \|^m \sum_{q=1}^{\infty} q^{n+1} \tau_m^{-q} h_D(x)^q, \quad \forall x \in V_m/\Gamma_V,
\]
where $C_4 > 1$ is a constant depending only on $(M, \omega)$. Then we can conclude that

$$\rho_{\nu_m^{\tau}}/\Gamma \nu \nu_n, \omega(x) \leq C_4 \log h_D(x)^{\frac{1}{2}} \leq \sum_{q=1}^m q^n \left( \tau_m \log \tau_m \right)^{\frac{m}{2}} \leq C_4 \log h_D(x)^{\frac{1}{2}},$$

(57)

for any $m \geq n + 1$ and $x \in \nu_m^{\tau}/\Gamma \nu$, where $C_5 > 1$ is a constant depending only on $(M, \omega)$.

It follows that there exists a constant $C_6 > 1$ depending only on $(M, \omega)$, such that for any $x \in \nu_m^{\tau}/\Gamma \nu$, we have $\|S\|^2 \leq C_6 \log h_D(x)^{\frac{1}{2}} \leq C_6 \log h_D(x)^{\frac{1}{2}}$. Applying Lemma 4.5 again, we can find a constant $C_7$ depending only on $(M, \omega)$, $k$ and $\beta$ such that $\|\nabla k S(x)\| + \|\nabla k S(x)\|^2 \leq C_7 \log h_D(x)^{\frac{1}{2}}$ on $\nu_m^{\tau}$. This completes the proof.

Before proving Theorem 1.1, we now prove that $\rho_{\nu_m^{\tau}}/\Gamma \nu \nu_n, \omega, m > 0$ everywhere. Hence the fraction in inequality (3) is well defined.

**Lemma 4.6.** Let $x \in \nu^{V}/\Gamma \nu$, $m_0 \geq n + 1$ be an integer, and $T \in H^0(\nu^{V}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^{m_0})^*$ be a linear functional defined by finite linear combination of $\delta_\nu$. Assume that $T \neq 0$ as a distribution, and the quotient space $\nu^{\nu}/\Gamma \nu$ is a manifold. Then $\|T\|^2_{H^0(\nu^{V}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^{m_0})} \neq 0$.

**Proof.** To shorten notation, we write $N$ instead of $|\Gamma \nu|$.

Since $\Theta_{\nu} (g^N) = 1, \forall g \in \Gamma \nu$, we see that $(V^{\nu}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^N) \cong (C, \log h_D[N])$. Then there exists a holomorphic section $e \in H^0 (V^{\nu}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^N)$ such that $\|e\|^2 = \log h_D[N]$. For each $p \in \mathbb{Z}_{\geq 0}$, we can define $T_p \in H^0 (V^{\nu}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^{m_0})$ by

$$T_p (S) = T (e^{-p}(S)) e^p, \forall S \in H^0 \left( V^{\nu}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^{m_0+p}\right).$$

Since $(V^{\nu}/\Gamma \nu, \omega \nu, \mathcal{L}_V/\Gamma _V, h_V)$ is a polarized complete Kähler manifold with constant Ric curvature $\text{Ric}(\omega \nu) = -(n + 1)\omega \nu$, we see that Lemma 2.1 implies that there exist a large integer $p_{1}$ and a section $S_{p_1} \in H^0_2 \left( V^{\nu}/\Gamma \nu, (\mathcal{L}_V/\Gamma _V)^{m_0+p_{1}}\right)$ such that $T_{p_1} (S_{p_1}) \neq 0$. It follows that $T (e^{-p_1}(S)) \neq 0$.

Let $S^{'} = f e_{L}^{m_0} \in H^0_2 (V, \mathcal{L}_V^{m_0})_{\Gamma \nu}$ be the pullback of $e^{-p_1}(S)$. By the statement above Proposition 3.1, we can find constants $a_{q,j}$ such that $\sum q = 1 \sum_{j=1}^{N_{m_0,q}} a_{q,j} S_{m_0,q,j}$ converge to $f$ uniformly on a neighborhood of $x$, where $\left\{S_{m_0,q,j}^{'}\right\}_{j=1}^{N_{m_0,q}}$ is an $L^2$ orthonormal set in the Hilbert space $H^0_2 (D, \mathcal{L}_D^q)$, and $\left\{S_{m_0,q,j}\right\}$ denote the holomorphic function corresponding to $S_{m_0,q,j}^{'}$ defined in Lemma 3.1. Now we see that there exists a holomorphic function $S_{m_0,q,j}$ such that $T \left( S_{m_0,q,j}^{'} e_{L}^{m_0}\right) \neq 0$, and the lemma follows.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1:** Take a fixed $l \in \mathbb{N}$ at first. We only need to show this theorem in the case where $\xi = \kappa = 1$, the more general case is then a fairly immediate consequence.

The proof will be divided into three parts, proving estimates (1), (2) and (3) respectively.

**Part 1:** First, we need to prove (1).

Set $(x_m, y_m) \in \pi_V (\nu^{\nu}_m) \times (U \pi_V (\nu^{\nu}_m))$. Choosing vectors $e_{x_m} \in \mathcal{L}_m | x_m$ and $e_{y_m} \in \mathcal{L}_m | y_m$, such that $\|e_{x_m}\|_{l_m} = \|e_{y_m}\|_{l_m} = 1$. Let $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 \leq k$. Consider the unit tangent
vectors \(v_{1,m}, \ldots, v_{k,m} \in T_{x_m} M \oplus 0 \subset \mathcal{T}_{(x_m,y_m)}(M \times M)\) and \(v_{k+1,m}, \ldots, v_{k+2,m} \in 0 \oplus T_{y_m} M \subset T_{(x_m,y_m)}(M \times M)\). Now we can define linear functionals on the space of holomorphic sections on small neighborhoods of \(x_m\) and \(y_m\) as following:

\[
\nabla^k S(v_{1,m}, \ldots, v_{k,m}) = T_{1,m}(S)e_{x,m},
\]

\[
\nabla^{k_2}S^*(v_{k+1,m}, \ldots, v_{k+2,m}) = T_{2,m}(S)e_{y,m}^*,
\]

for any holomorphic section \(S\) of \(\mathcal{L}^m\) around \(x_m\) and \(y_m\), respectively. Then we have

\[
\sum_j \left\| \nabla^k S_j\right\|_{h^{m,\omega}} |x_m| (v_{1,m}, \ldots, v_{k,m}) \otimes \nabla^{k_2}S_j^*|y_m| (v_{k+1,m}, \ldots, v_{k+2,m})
\]

where \(\left\| \cdot \right\|_{h^{m,\omega}}\) means the norms corresponding to the Hermitian metric \(\pi^1 h^m \otimes \pi^2 h^{-m}\) and Kähler metric \(\pi^1 \omega + \pi^2 \omega\) on \(M \times M\), the family \(\{S_j\}_{j \in J}\) is an \(L^2\)-orthonormal basis of \(H_{L^2}^0(M, \mathcal{L}^m)\), and \(S_{T_{1,m}} \in H_{L^2}^0(M, \mathcal{L}^m)\) is the peak section of the functional \(T_{1,m}\) on \(M\). Since \(u = o(1)\) as \(h_D \to 0^+\), we can find a constant \(\epsilon' > 0\) depending only on \((M, \omega, L, h)\) such that

\[
\min_{\pi V} \left\{ \text{dist}_{\omega} (\pi V(V_{y_m}), M \setminus \pi V(V_{\sqrt{\log h_D(x_m)/\epsilon}})) \right\}
\]

and Lemma 4.3 now shows that there exists a constant \(\delta \in (0, \epsilon')\) depending only on \((M, \omega)\), such that \(\text{inj}(U \setminus \pi V(V_{y_m})) \geq \delta \epsilon_m\). By Corollary 4.2, we see that there exists a constant \(C_0 > 0\) depending only on \((M, \omega, L, h)\), \(k\) and \(\beta\), such that

\[
|\log h_D(x_m)|^{2\beta} \left\| T_{1,m}|_H^{L^2,\omega} (M, \mathcal{L}^m) \right\|
\]

\[
+ |\log h_D(y_m)|^{2\beta} \left\| T_{2,m}|_H^{L^2,\omega} (B_{\delta\epsilon_m}(y_m), \mathcal{L}^m) \right\|
\]

\[
\leq C_0 m^{C_0}.
\]

By using Lemma 4.1 and Proposition 4.1 to \(V_{y_m}\) and \(V_{\sqrt{\log h_D(x_m)}}\), we can conclude that for each integer \(q \in \mathbb{N}\), there exists a constant \(C_1 > 0\) depending only on \((M, \omega, L, h)\), \(k\) and \(q\), such that

\[
\int_{M \setminus \pi V(V_{y_m})} \left\| S_{T_{1,m}} \right\|_{h^{m,\omega}}^2 \text{dVol}_\omega \leq C_1 m^{-q}.
\]

Choosing \(q = 2l + 4 + 4\lfloor C_0 \rfloor\), where \(\lfloor \cdot \rfloor\) is the greatest integer function. It follows that

\[
\left\| T_{1,m}\right\|_{H^{L^2,\omega} (M, \mathcal{L}^m)} \leq \left\| T_{1,m}\right\| \left\| T_{2,m}\right\|_{H^{L^2,\omega} (B_{\delta\epsilon_m}(y_m), \mathcal{L}^m)} \left\| S_{T_{1,m}} \right\|_{L^2, B_{\delta\epsilon_m}(y_m), h^{m,\omega}}
\]

\[
\leq C_2 C_1 m^{2C_0 - 1 - 2 - 2\lfloor C_0 \rfloor} |\log h_D(x_m)|^{-\beta} |\log h_D(y_m)|^{-\beta}
\]

\[
\leq C_2 C_1 m^{-1} |\log h_D(x_m)|^{-\beta} |\log h_D(y_m)|^{-\beta}.
\]

Hence we can conclude that

\[
\sup_{\pi V(V_{y_m}) \times (U \setminus \pi V(V_{y_m}))} \left\{ |\log h_D(x)|^{2\beta} |\log h_D(y)|^{2\beta} \right\} \left\| B_{\omega,m} (x, y) \right\|_{C^{1,\epsilon_m,\omega}} \leq C_2 m^{-1}.
\]
where a constant $C_2 > 0$ depending only on $(M, \omega, \mathcal{L}, h), k, l$ and $\beta$. By exchanging $x_m$ and $y_m$, one can see that

$$\sup_{(U \setminus \pi_V(V_{x_m})) \times \pi_V(V_{y_m})} \| \log h_D(x) \|^\beta \| \log h_D(y) \|^\beta \| B_{x_m, y} \|_{\mathcal{C}^{k, h_m, \omega}} \leq C_2 m^{-l}. $$

Part 2. Our task now is to prove (2).

Let $(x_m, y_m) \in \pi_V(V_{\gamma_m}) \times \pi_V(V_{\gamma_m})$. Let $e_{x_m} \in \mathcal{L}_{x_m}$ and $e_{y_m} \in \mathcal{L}_{y_m}$ such that $\| e_{x_m} \|_{h_m} = \| e_{y_m} \|_{h_m} = 1$. Choosing $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 \leq k$, unit tangent vectors $v_{1, m}, \ldots, v_{k_1, m} \in T_{x_m} M \oplus 0 \subset T_{x_m}(M \times M)$, and unit tangent vectors $v_{k_1+1, m}, \ldots, v_{k_1+k_2, m} \in \mathbb{T}(y_m) M \subset T_{(x_m, y_m)}(M \times M)$. Now we can define a linear functional on $H^0_{L^2}(\pi_V(V_{\gamma_m}), \mathcal{L}^m)$ as following:

$$\nabla^{k_1} S (v_{1, m}, \ldots, v_{k_1, m}) = T_{1, m}(S) e_{x_m},$$

$$\nabla^{k_2} S^* (v_{k_1+1, m}, \ldots, v_{k_1+k_2, m}) = T_{2, m}(S) e_{y_m},$$

for any $S \in H^0_{L^2}(\pi_V(V_{\gamma_m}), \mathcal{L}^m)$. Let $S_{T_{1, m}} \in H^0_{L^2}(M, \mathcal{L}^m)$ and $S_{T_{m}} \in H^0_{L^2}(M, \mathcal{L}^m)$ denote the peak sections of the functional $T_{1, m}$ on $\pi_V(V_{\gamma_m})$ and $M$, respectively. As in the argument in Part 1, a straightforward computation shows that

$$\left\| \nabla_x \nabla_y^{k_2} (B_{x_m, y_m} (x_m, y_m) - B_{x_m, y_m}) \right\|_{h_m, \omega} = \left\| T_{1, m}(S_{T_{1, m}} - T_{1, m}(S_{T_{1, m}})) \right\|_{h_m, \omega} \leq T_{1, m}(S_{T_{1, m}} - T_{1, m}(S_{T_{1, m}})) \| T_{2, m}(S_{T_{1, m}}) \| \left\| T_{2, m}(S_{T_{1, m}} - T_{1, m}(S_{T_{1, m}})) \right\|.$$

By Corollary 4.2, we see that there exists a constant $C_3 > 0$ depending only on $(M, \omega, \mathcal{L}, h)$ and $k$, such that

$$\| \log h_D(x_m) \|^{2\beta} \left\| T_{1, m}(S_{T_{1, m}} - T_{1, m}(S_{T_{1, m}})) \right\| \leq C_3 m^{C_3}.$$

Combining Lemma 4.1 and Proposition 4.1, we see that for each given integer $q \in \mathbb{N}$, there exists a constant $C_4 > 0$ depending only on $(M, \omega, \mathcal{L}, h), k$ and $q$, such that

$$\int_{\pi_V(V_{\gamma_m})} \left\| S_{T_{1, m}} - S_{T_{1, m}} \right\|^2 \mathrm{dVol}_{\omega} \leq C_3 m^{-q}.$$

Choosing $q = 2l + 4 + 4[2C_2]$, where $[\cdot]$ is the greatest integer function. Then we have

$$\left\| T_{1, m}(S_{T_{1, m}} - T_{1, m}(S_{T_{1, m}})) \right\| \leq 2 \left\| T_{1, m}(S_{T_{1, m}} - T_{1, m}(S_{T_{1, m}})) \right\| \leq 2C_3 m^{-l-2+2C_2} | \log h_D(x_m) |^{-\beta} | \log h_D(y_m) |^{-\beta}.$$
Then we have
\begin{equation}
\frac{\rho_{M,\omega,m}(x)}{\rho_{V_{\omega,m}(x)}} = \frac{\|T_{0,m}\|_{H^{0}_{L^{2}}(M,\omega,\mathcal{L}^{m},h^{m})}}{\|T_{0,m}\|_{H^{0}_{L^{2}}(V_{\omega,m},\mathcal{L}^{m},h^{m})}},
\end{equation}
and (3) follows from Proposition 4.1.

4.4. Peak sections on Poincaré type cusps.

Now we consider the peak sections on Poincaré type cusps. This subsection is an analog of the previous subsection on Poincaré type cusp. Let \( M \) be a compact Kähler manifold, \( D \) be a simple normal crossing divisor, and \((M,\omega) = (M\backslash D,\omega)\) be a complete Kähler manifold with Poincaré type cusp along \( D \). Fix a Riemannian metric \( \bar{g} \) on \( M \). Write \( d(x) = \text{dist}_{\bar{g}}(x,D) \), \( \forall x \in M \). Assume that \((M,\omega)\) is a complete Kähler manifold with \( \text{Ric}(\omega) \geq -\Lambda \omega \) and \( (\mathcal{L},h) \) is a Hermitian line bundle with \( \text{Ric}(h) \geq \epsilon \omega \), where \( \epsilon,\Lambda > 0 \) are constants.

The proof of the following estimate of the injectivity radius on \( M \) is similar to that of Lemma 4.3, so we omit it.

**Lemma 4.7.** Given an integer \( k \in \mathbb{N} \). Then there exists a constant \( C = C(k,M,\omega) > 0 \) such that the curvature \( \sum_{j=k}^{3k+10n} \|\nabla^{j}\text{Ric}(\omega)\|_{\omega} \leq C \) on \( M \), and for any \( x \in M \), the injectivity radius \( \text{inj}_{M}(x) \geq C^{-1} \left( |\log d(x)| + 1 \right)^{-1} \).

Choosing \( x \in D \). By definition, we can find a neighborhood \( U_{x} \) of \( x \) and a biholomorphic map \( \varphi_{x} : U_{x} \backslash D \rightarrow \prod_{i=1}^{k} \mathbb{D}_{r_{x,i}}^{*} \times B_{r_{x,k_{x},n}}^{2n-2k_{x}}(0) \subset (\mathbb{D}_{r_{x,i}}^{*})^{k_{x}} \times \mathbb{C}^{n-k_{x}} \), such that \((\varphi_{x}^{-1})^{*}\omega = -\sqrt{-1} \sum_{i=1}^{k_{x}} \partial \bar{\partial} \log \left( |z_{i}|^{2} \right) + \pi_{r_{x,i},n}^{*} \omega_{r_{x,k_{x},n}} + \sqrt{-1} \partial \bar{\partial} u_{x} \), where \( r_{x,i}, r_{x,k_{x},n} \) are positive constants, \( \omega_{r_{x,k_{x},n}} \) is a Kähler metric on \( B_{r_{x,k_{x},n}}(0) \), \( u_{x} \) is a smooth function on \( U_{x} \backslash D \), and there exists a constant \( \alpha > 0 \) such that \( u_{x} \circ \varphi_{x}^{-1} \in C^{0}(\mathbb{D}_{r_{x,i}}^{*})^{k_{x}} \times B_{r_{x,k_{x},n}}^{2n-2k_{x}}(0) \) for any constants \( t,s \in (0,1) \). It will cause no confusion if we use the same letter to designate a subset of \( U_{x} \backslash D \) and its image under \( \varphi_{x} \). Now we can construct quasi-plurisubharmonic functions on \( U_{x} \backslash D \). This is an analog of Lemma 4.4 on Poincaré type cusp.

**Lemma 4.8.** Let \( \beta,\delta,\epsilon > 0 \) be constants. Assume that \( \delta < \epsilon^{-2} n^{-2} r_{x,j} \), \( j = 1, \ldots, k_{x} \), and \( \epsilon \leq \epsilon^{-2} n^{-2} r_{x,n} \). Then for any \( p \in U_{x,\delta^{1+\delta},\epsilon} \), there exists a quasi-plurisubharmonic function \( \psi \in L^{1}(M,\omega) \cap C^{\infty}(M \backslash \{p\}) \) such that \( \psi \leq 0 \), \( \text{supp} \psi \subset U_{x,\delta,4\epsilon} \), \( \sqrt{-1} \partial \bar{\partial} \psi \geq -C (|\log \delta| + \epsilon^{-2}) \omega \) and \( \lim_{y \to p}(\psi(y) - \log(\text{dist}(\omega,p,y))) = -\infty \), where \( C > 0 \) is a constant dependent only on \( (M,\omega) \) and \( \beta \).

**Proof.** Fix a point \( p = (p_{1}, \ldots, p_{n}) \in (\mathbb{D}_{1/2}^{*})^{k_{x}} \times B_{1}^{2n-2k_{x}}(0) = U_{x,\beta^{1+\beta},\epsilon} \). Let
\[
 f(y) = \sum_{i=1}^{k_{x}} \delta^{2} |y_{i} - p_{i}|^{2} + \sum_{i=k_{x}+1}^{n} \epsilon^{-2} |y_{i} - p_{i}|^{2}, \quad \forall y = (y_{1}, \ldots, y_{n}) \in U_{x}.
\]

It is easy to see that \( f(y) \geq \min \{ \delta^{2} \beta, \epsilon^{-2} \} \), \( \forall y \in U_{x} \backslash U_{x,\beta^{1+\beta},2\epsilon} \), and \( f(y) \leq \epsilon^{-2} n \), \( \forall y \in U_{x,\beta,4\epsilon} \).
Since $u = o(1)$, we can conclude that $|\nabla \log f(x)| \leq C_1 \left( |\log \delta| + \epsilon^{-1} \right)$, $\forall x \in U_{x,\delta} \setminus U_{x,\delta + \beta}$, where $C_1$ is a constant depending only on $(M, \omega)$ and $\beta$.

Now we consider the model metric $\omega_{\text{mod}} = -\sum_{i=1}^{k_x} \partial \bar{\partial} \log |z_i|^2$ on $(\mathbb{D}_\beta)^{k_x}$. By a straightforward calculation, we see that $\text{dist}_{\omega_{\text{mod}}} \left( (\mathbb{D}_\beta)^{k_x}, (\mathbb{D}_1)^{k_x} \setminus (\mathbb{D}_\beta)^{k_x} \right) \geq C_2^{-1} \log (1 + \beta)$ for some constant $C_2$ depending only on $(M, \omega)$. Then we can choose a cut-off function $\eta_1 \in C^\infty \left( (D_1)^{k_x} \right)$ such that $\eta_1 = 1$ on $(D_{\beta}^{k_x})$, $\eta_1 = 0$ on $(D_{1}^{k_x}) \setminus (D_{\beta}^{k_x})$, $0 \leq \eta_1 \leq 1$, and $\sum_{i=0}^{2} \|\nabla \eta_1\|_{\omega_{\text{mod}}} \leq C_3$, where $C_3 > 0$ is a constant depending only on $n$ and $\beta$. Choosing another cut-off function $\eta_2 \in C^\infty \left( B^{2n-2k_x}_4(0) \right)$ such that $\eta_2 = 1$ on $B^{2n-2k_x}_e(0)$, $0 \leq \eta_2 \leq 1$, and $\sum_{i=0}^{2} \|\nabla \eta_2\|_{\omega_{\text{mod}}} \leq C_4$, where $C_4$ is a constant depending only on $n$. By the biholomorphic map $\varphi_x$, we see that $\eta_1$ and $\eta_2$ can be viewed as functions on $U_x$.

Let $M = \sup_{U_x,\delta} f$, and $\psi(x) = \eta_1(x)\eta_2(x) (\log f(x) - \log M)$. Analysis similar to that in the proof of Lemma 4.4 shows that $\varphi_x$ satisfies the properties we need. 

We will denote by $W_r$ the domain $\{x \in M | \bar{d}(x, D) < r\} \subset M \setminus D$. Now we consider the peak sections around Poincaré type cusps. To apply Lemma 4.8 to points around $D$, we need the following decomposition result for the manifold $(M, \omega)$. For abbreviation, we denote $e^{-\frac{m \log m}{\sqrt{m}}}$ briefly by $r_m$ and $\frac{\log m}{\sqrt{m}}$ by $m$, respectively, for any $m \geq 2$.

**Lemma 4.9.** There are constants $m_0, b_1, b_2, b_3, C_1, C_2 > 0$, and a decreasing sequence $\{\xi_m\}_{m=1}^{\infty}$ that converges to 0 as $m \to \infty$ satisfying the following property.

Let $m > m_0$ be a constant. Then there are points $y_{i,m} \in D$, $i = 1, \ldots, N_m$, positive constants $C_{1,i,m} \leq C_1$, $C_{2,i,m} \leq C_2$, open neighborhoods $U_{i,m}$ of $y_{i,m}$ in $M$ and biholomorphic maps $\varphi_{i,m} : U_{i,m} \setminus D \to \left(D_{r_m}^{k_{i,m}} \setminus B_{b_{2}\xi_m}^{2n-2k_x}(0) \right)$ such that $U_{i,m} \setminus D \subset W_{r_m}$, $W_{r_m} \subset \bigcup_{i=1}^{N_m} \varphi_{i,m}^{-1} \left(D_{r_m}^{k_{i,m}} \setminus B_{b_{2}\xi_m}^{2n-2k_x}(0) \right)$, and the Kähler metrics $\left(\varphi_{i,m}^{-1}\right)^* \omega = \omega_{\text{mod},k_{i,m},n} + \sqrt{-1} \partial \bar{\partial} u_{i,m}$ on $U_{i,m} \setminus D$ for some real valued function $u_{i,m}$ satisfying that $\|u_{i,m}\|_{C^2_{\omega}} \leq \xi_m$ on $U_{i,m} \setminus D$, $i = 1, \ldots, N_m$.

**Proof.** By the compactness of the divisor $D \subset M$, we can find points $\tilde{y}_j \in D$, $j = 1, \ldots, N$, open neighborhoods $\tilde{U}_j$ of $\tilde{y}_j$ in $M$, and biholomorphic maps $\varphi_j : (z_{j,1}, \ldots, z_{j,n}) \in \tilde{U}_j \to \left(D_{\epsilon_j}^{k_j} \setminus B_{\epsilon_j}^{2n-2k_j}(0) \right)$ such that $\varphi_j \left(\tilde{y}_j\right) = 0$, $\varphi_j \left(\tilde{U}_j \setminus D\right) = \left(D_{\epsilon_j}^{k_j} \setminus B_{\epsilon_j}^{2n-2k_j}(0) \right)$, and $\left(\varphi_{i,m}^{-1}\right)^* \omega = -\sqrt{-1} \sum_{i=1}^{k_{i,m}} \partial \bar{\partial} \log \left|z_{j,i}\right|^2 + \pi_{k_{i,m},n} + \sqrt{-1} \partial \bar{\partial} u_j$, where $\pi_i$ is the projection of $C^n$ onto the $i$-th component, $\epsilon_j > 0$ are constants, $\omega_j$ is a Kähler metric on $B_{\epsilon_j}^{2n-2k_j}(0)$, and $u_j = 0(1)$ as $\inf_{z \in \mathbb{D}} |z_{j,i}| \to 0^+$ to all orders with respect to $\omega$.

Fix $j \in \{1, \ldots, N\}$. It is sufficient to construct the open subsets and biholomorphic maps on $\tilde{U}_j$. Without loss of generality, we can assume that $\omega = \varphi_{j*} \omega_{\text{mod}}$. For each nonnegative integer $q \leq k_j$,
we consider the set

\[ Z_q = \bigcup_{\{i_1, \ldots, i_q\} \subset\{1, \ldots, k_j\}} \left( \bigcup_{l=1}^q \{ z_{j,i_l} = 0 \} \right) \bigcup_{l \in\{1, \ldots, k_j\} \setminus \{i_1, \ldots, i_q\} } \{ z_{j,l} = 0 \} \subset U_j. \]

Then we have \( \bigcup_{q=0}^{k_j} Z_q = \bar{U}_j \cap D \), and \( Z_q \cap Z_{q'} = \emptyset \) for each \( q \neq q' \).

By Lemma 4.7, we can find a constant \( \delta > 0 \) such that for any point \( z \in D^*_1 \), there exist an open neighborhood \( U_z \) of \( z \) in \( D^*_1 \), an increasing function \( \rho(t) \) on \((0, 1)\), and a biholomorphic map \( \phi_z : U_z \to D_{\delta|\log|z|}|^{-1} \) such that \( \phi_z(z) = 0 \), \( \lim_{t \to 0^+} \rho(t) = 0 \), and \( \omega_{mod,1,1} - \phi_z^* \omega_{Euc} = \sqrt{-1} \partial \bar{\partial} u_{U_z} \) for some function \( u_{U_z} \) satisfying that \( \|u_{U_z}\|_{C^2_{\omega_1,1}} \leq \rho(|z|) \) on \( U_z \), where \( \omega_{Euc} \) is the Euclidean Kähler form on \( \mathbb{C}^n \).

Choosing \( y \in Z_q \cap \varphi_j^{-1} \left( (\mathbb{D}_{C_j})^{k_j} \times B^{2n-2k_j}_2(0) \right) \). Without loss of generality, we can assume that \( y \in \cap_{i=1}^q \{ z_{j,i} = 0 \} \). Let \( t \leq n^{-2} e^{-2e^2} \) be a positive constant. When \( y \notin B_{i}^{Euc,n} \left( \bigcup_{k=p+1}^{n} Z_k \right) \), then we can find an increasing function \( \tilde{\rho}(t) \) on \((0, 1)\), an open neighborhood \( \tilde{U}_{y,t} \) of \( y \) in \( U_j \) and a biholomorphic map \( \tilde{\phi}_{y,t} : \tilde{U}_{y,t} \to (\mathbb{D}_{\delta\log t})^p \times B^{2n-2p\delta}_{\delta|\log t|} -1(0) \) such that \( \lim_{t \to 0^+} \rho(t) = 0 \), \( \lim_{t \to 0^+} \tilde{\phi}_{y,t} = 0 \), \( \lim_{t \to 0^+} \omega_{mod,1,1} - \omega = \sqrt{-1} \partial \bar{\partial} u_{\tilde{U}_{y,t}} \) for some function \( u_{\tilde{U}_{y,t}} \) satisfying that \( \|u_{\tilde{U}_{y,t}}\|_{\omega_{Euc,1}} \leq \rho(t) \) on \( U_{y,t} \), where \( B_{t}^{Euc,n}(x) \) is the Euclidean ball in \( \mathbb{C}^n \) with radius \( t \) and center \( x \in \mathbb{C}^n \), and \( B_{t}^{Euc,n}(A) = \cup_{x \in A} B_{t}^{Euc,n}(x) \).

Now we see that for each constant \( \mu > 0 \), there are constants \( C_j > 0 \) and \( \epsilon_{j,l} > 0 \), \( l = 1, 2, \ldots, k_j \), such that

\[ \varphi_j^{-1} \left( (\mathbb{D}_{C_j})^{k_j} \times B^{2n-2k_j}_2(0) \right) \subset \bigcup_{l=1}^{k_j} \bigcup_{y \in Z_l \setminus B_{i_{l+1}}^{Euc,n} \left( \bigcup_{q=p+1}^{n} Z_q \right)} \tilde{U}_{y,t_{l+1},r_m} \]

for sufficiently large \( m \).

Clearly, we can find a small constant \( \mu > 0 \) such that \( \varphi_j^{-1} \left( (\mathbb{D}_{\mu r_m})^{k_j} \times B^{2n-2k_j}_2(0) \right) \subset W_{r_m} \), for sufficiently large \( m \). Then the open \( \tilde{U}_{y,t_{l+1},r_m} \) subsets in \( (74) \) are the open subsets we need.

Then we can prove the lemma by choosing \( j = 1, \ldots, N \) in the above argument. \( \square \)

This is the Poincaré type cusp version of Proposition 4.1. The proof of this proposition is similar to the proof of Proposition 4.1, except that Lemma 4.4 is replaced by Lemma 4.8.

**Proposition 4.2.** Given constants \( k, \delta, \kappa, l > 0 \), there exists a constant \( C > 0 \) depending only on \( (M, \omega, \mathcal{L}, h) \), \( k, \delta, \kappa, l \), satisfying the following property.

For any integer \( m \in \mathbb{N} \), choose a constant \( \gamma_m \geq \gamma_m^* \), a point \( p_m \in W_{\gamma_m} \), and a linear functional \( T_m \) defined by finite linear combination of \( \delta_{p_m} \) and its derivatives. Assume that the order of derivatives contained in \( T_m \) is at most \( k \), \( \forall m \in \mathbb{N} \). Suppose that there exists a sequence of open subsets \( \{ U_m \}_{m=1}^{\infty} \) of \( M \) such that \( \text{dist}_\omega(W_{\gamma_m}, M \setminus U_m) \geq \delta \gamma_m, \forall m \in \mathbb{N} \). Then we have

\[ \left\| T_m \big|_{H^p_{L^2}(M, \mathcal{L}^m)} \right\| \geq \left( 1 - \frac{C}{m^t} \right) \left\| T_m \big|_{H^p_{L^2}(U_m, \mathcal{L}^m \setminus U_m)} \right\|. \]
Remark. Note that for any point \( p \) on the manifold \( M \), we can choose a trivialization of \( \mathcal{L} \) around \( p \), so \( \delta_p \) can still represent a linear functional on \( H^0_{L^2}(M, \mathcal{L}) \).

Proof. Without loss of generality, we can assume that \( \kappa = \delta = 1 \). When \( T_m \mid_{H^0_{L^2}(U, \mathcal{L}^m)} = 0 \), this proposition is obvious. So we can assume that \( T_m \mid_{H^0_{L^2}(U, \mathcal{L}^m)} \neq 0 \) from now.

Let \( \beta > 0 \) be a large constant. When \( p_m \notin W_{\gamma^m, \delta} \), analysis similar to that in the first part of the proof of Proposition 4.1 shows that (75) holds for some constant \( C \) depending only on \((M, \omega, \mathcal{L}, h), k, \delta, \beta, \kappa \) and \( l \).

Now we assume that \( p_m \in W_{\gamma^m, \delta} \). In this case, by replacing Lemma 4.4 with Lemma 4.8, we can apply the argument in the second part in the proof of Proposition 4.1 to give a proof of this proposition, and the proof is complete. \( \square \)

The next result is an analogy of Corollary 4.2 on Kähler manifolds with Poincaré type cusps, and it is also a corollary of Lemma 4.5.

**Corollary 4.3.** Given \( \epsilon, \kappa, \beta, k > 0 \), there exists a constant \( C > 0 \) depending only on \((M, \omega), \epsilon, \kappa, \beta \) and \( k \) satisfying the following property.

Let \( U_m \) be an open subset of \( M \) containing \( W_{r_m} \) for some integer \( m \geq C \), and \( \tilde{U}_m \) be an open subset of \( U_m \) such that \( \text{dist}_{\omega} (\tilde{U}_m, M \setminus U_m) < \epsilon \). Then for any \( S \in H^0_{L^2}(U, \mathcal{L}^m) \), we have

\[
\| \nabla^k S(x) \|_2^2 + \| \nabla^k S^*(x) \|_2^2 \leq C m^C \| \log \rho(x) \|_2^2 \| S \|^2_{L^2(U, \mathcal{L}^m)}, \quad \forall x \in \tilde{U}_m,
\]

where \( S^* \) is the smooth section of \( \mathcal{L}^{-m} \) given by \( S^*(e) = h^m(e, S) \), and \( \nabla^* \) is the Chern connection of the Hermitian line bundle \( (\mathcal{L}^{-m}, h^{-m}) \).

Proof. By the compactness of the divisor \( D \subset \tilde{M} \), we can find points \( \tilde{y}_j \in D, j = 1, \ldots, N \), open neighborhoods \( \tilde{U}_j \) of \( \tilde{y}_j \) in \( \tilde{M} \), and biholomorphic maps

\[
\varphi_j = (z_{j,1}, \ldots, z_{j,n}) : \tilde{U}_j \to (\mathbb{D}_{e_j})^{k_j} \times B_{e_j}^{2n-2k_j} \quad (0)
\]

such that \( \varphi_j (\tilde{y}_j) = 0, \varphi_j (\tilde{U}_j \setminus D) = (\mathbb{D}_{e_j})^{k_j} \times B_{e_j}^{2n-2k_j} \), \( D \subset \cup_{j=1}^N \varphi_j^{-1} \left( (\mathbb{D}_{e_j})^{k_j} \times B_{e_j}^{2n-2k_j} \right) \), and \( (\varphi_j^{-1})^* \omega = -\sqrt{-1} \sum_{i=1}^{k_j} \partial \bar{\partial} \log (\log |z_i|^2) + \pi_{e_j,n}^* \omega_j + \sqrt{-1} \partial \bar{\partial} u_j \), where \( \epsilon_j > 0 \) are constants, \( \omega_j \) is a Kähler metric on \( B_{e_j}^{2n-2k_j} \), and \( u_j = \epsilon_{k_j} O \left( \log |z_{j,i}| \right)^{-\alpha} \) as \( \inf_{1 \leq i \leq k_j} |z_{j,i}| \to 0^+ \) to all orders with respect to \( \omega_{\text{mod}} \).

Fix \( j \in \{1, \ldots, N\} \). Let \( \gamma_m = e^{-m^{2+2n-1}} \). Now we consider the Bergman kernel functions on \( \tilde{U}_{\tilde{y}_j, \gamma_m, e_j, l} = \varphi_j^{-1} \left( (\mathbb{D}_{e_j})^{l-1} \times \mathbb{D}_{\gamma_m}^{k_j-l} \times B_{e_j}^{2n-2k_j} \right) \subset \tilde{U}_j \setminus D, l = 1, \ldots, k_j \).

As in the proof of Corollary 4.2, we only need to find a constant \( C_{j,l, \beta} > 0 \) independent of \( m \), such that for sufficiently large integer \( m \),

\[
\rho_{\tilde{U}_{\tilde{y}_j, \gamma_m, e_j, l}; \omega_{\text{mod}, m}(x)} \leq C_{j,l, \beta} m^{C_{j,l, \beta}} |\log |z_{j,l}(x)||^{-\beta-2n}, \quad \forall x \in \tilde{U}_{\tilde{y}_j, \gamma_m, e_j, l}.
\]
By a straightforward calculation, one can obtain

\[
\rho_{\tilde{z}_j, \gamma_m, \iota, \iota \tilde{\iota}, \iota \tilde{\iota}} \omega_{\text{mod}, m}(x) = \left( \prod_{1 \leq i \leq k_j} \rho_{D^*_{x,i}, \omega_{\text{mod}, m}}(z_j,i(x)) \right) \cdot \rho_{D^*_{y_m}, \omega_{\text{mod}, m}}(z_j,l(x)) \cdot \rho_{B^2_{z_j, 2k_j}(0), \omega_{B_{2r}, m}}(\tilde{z}_j, k_j(x), \ldots, \tilde{z}_j, n(x)) .
\]

Combining Corollary 3.1 with Tian-Yau-Zelditch expansion, we see that there exists a constant \( C_1 = C_1(n) > 0 \) such that \( \rho_{\tilde{z}_j, \gamma_m, \iota, \iota \tilde{\iota}, \iota \tilde{\iota}} \omega_{\text{mod}, m}(x) \leq C_1 m^{\frac{2}{n}} \rho_{D^*_{y_m}, \omega_{\text{mod}, m}}(z_j,l(x)) \) for sufficiently large \( m \). It is sufficient to show that \( \rho_{D^*_{y_m}, \omega_{\text{mod}, m}}(z) \leq C_2 m^{\frac{2}{n}} |\log |z||^{-2n}, \forall z \in \mathbb{D}^*_{y_m}, \) which is shown in the proof of Corollary 4.2. This completes the proof. \( \square \)

We conclude this section by proving Proposition 1.1.

**Proof of Theorem 1.1:** Fix \( l \in \mathbb{N} \) at first. We only need to show this theorem in the case where \( \xi = \kappa = 1 \), the more general case is then a fairly immediate consequence.

Similar to the proof of Proposition 1.1, we also split the proof of this theorem into two parts, proving estimates (12) and (13) respectively.

**Part 1:** First, we need to prove (12).

Set \((x_m, y_m) \in V_m \times (M \setminus U_m)\). Choosing vectors \( e_{x_m} \in \mathcal{L}^m_{|x_m} \) and \( e_{y_m} \in \mathcal{L}^m_{|y_m} \) such that \( \|e_{x_m}\| = \|e_{y_m}\| = 1 \). Let \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \) such that \( k_1 + k_2 \leq k \). Consider the unit tangent vectors \( v_{1,m}, \ldots, v_{k_1,m} \in T_{x_m} M \oplus 0 \subset T_{(x_m,y_m)} (M \times M) \), and \( v_{k_1+1,m}, \ldots, v_{k_1+k_2,m} \in 0 \oplus T_{y_m} M \subset T_{(x_m,y_m)} (M \times M) \). Now we can define bounded linear functionals on small neighborhoods of \( x_m \) and \( y_m \) as following:

\[
\nabla^{k_1} S (v_{1,m}, \ldots, v_{k_1,m}) = T_{1,m}(S) e_{x_m},
\]

\[
\nabla^{k_2} S^* (v_{k_1+1,m}, \ldots, v_{k_1+k_2,m}) = T_{2,m}(S^*) e_{y_m},
\]

for any holomorphic section \( S \) of \( \mathcal{L}^m \) around \( x_m \) and \( y_m \), respectively. Then we have

\[
\left\| \nabla^{k_1}_x \nabla^{k_2}_y B_{\omega,m} (x_m, y_m) (v_{1,m}, \ldots, v_{k_1+k_2,m}) \right\| = \left| T_{1,m}(S_{T_{1,m}}) T_{2,m}(S_{T_{1,m}}) \right|,
\]

where \( \{S_j\}_{j \in J} \) is an \( L^2 \)-orthonormal basis of \( H^0_{L^2} (M, \mathcal{L}^m) \), and \( S_{T_{1,m}} \in H^0_{L^2} (M, \mathcal{L}^m) \) is the peak section of the functional \( T_{1,m} \) on \( M \). Since \( u = o(1) \) as \( h_D \rightarrow 0^+ \), we can find a sequence of open subsets \( \{U_m\}_{m=2}^{\infty} \) and a constant \( \epsilon > 0 \) independent of \( m \) such that

\[
\text{inf} \left( \text{dist}_\omega \left( V_m, M \setminus U_m \right), \text{dist}_\omega \left( U_m, M \setminus U_m \right) \right) \geq \epsilon \frac{\log m}{\sqrt{m}}, \forall m \in \mathbb{N},
\]

and Lemma 4.7 now shows that there exists a constant \( \delta \in (0, \epsilon) \) independent of \( m \) such that \( \text{inj} (M \setminus V_m) \geq \delta \frac{\log m}{\sqrt{m}} \). By Corollary 4.3, we see that there exists a constant \( C_0 > 0 \) depending only on \((M, \omega, \mathcal{L}, h)\) and \( k \), such that

\[
|\log \tilde{d} (x_m)|^{2\beta} \left\| T_{1,m} \right\|_{H^0_{L^2} (M, \mathcal{L}^m)} + |\log \tilde{d} (y_m)|^{2\beta} \left\| T_{2,m} \right\|_{H^0_{L^2} \left( B_{\delta \log m, \sqrt{m}}, \mathcal{L}^m \right)} \leq C_0 m^{C_0}, \forall m \in \mathbb{N},
\]
where \( \epsilon > 0 \) is a constant independent of \( m \). By using Lemma 4.1 and Proposition 4.2 to \( V_m \) and \( \bar{U}_m \), we can conclude that for each integer \( q \in \mathbb{N} \), there exists a constant \( C_1 > 0 \) depending only on \((M, \omega, \mathcal{L}, \mathcal{H})\), \( k \) and \( q \), such that
\[
\int_{M \setminus \bar{U}_m} \|S_{T, m}\|^2 d\operatorname{Vol}_\omega \leq C_1 m^{-q}, \quad \forall m \in \mathbb{N}.
\]
Choosing \( q = 2l + 4 + 4[C_0] \), where \([ \cdot ]\) is the greatest integer function. It follows that
\[
\left| T_{1, m} \left( S_{T, m} \right) T_{2, m} \left( S_{T, m} \right) \right| \leq \|T_{1, m}\| \cdot \left\| T_{2, m} \right\|_{H^0_{L_2} \left( B_{\frac{m}{2} \log m} \right)} \cdot \|S_{T, m}\|_{L^2_{L_2 \log m} \left( \frac{m}{2} \right)} \leq C_0^2 C_1 m^{2C_0 m^{1-2-2(C_0)}} \left| \log d(x_m) \right|^{-\beta} \left| \log d(y_m) \right|^{-\beta} \leq C_0^2 C_1 m^{-1} \left| \log \tilde{d}(x_m) \right|^{-\beta} \left| \log \tilde{d}(y_m) \right|^{-\beta}.
\]
Hence we can conclude that
\[
\sup_{V_m \times (M \setminus \bar{U}_m)} \left| \log \tilde{d}(x) \right|^\beta \left| \log \tilde{d}(y) \right|^\beta \left\| B_{\omega, m}(x, y) \right\|_{C^\beta} \leq C m^{-l}.
\]
By exchanging \( x_m \) and \( y_m \), one can see that the proof of
\[
\sup_{\left( M \setminus \bar{U}_m \right) \times V_m} \left| \log \tilde{d}(x) \right|^\beta \left| \log \tilde{d}(y) \right|^\beta \left\| B_{\omega, m}(x, y) \right\|_{C^\beta} \leq C m^{-l}
\]
is similar to the argument above.

**Part 2.** It remains to prove (13). Set \((x_m, y_m) \in V_m \times V_m\). Let \( e_{x_m} \in \mathcal{L}^m \mid x_m \) and \( e_{y_m} \in \mathcal{L}^m \mid y_m \) such that \( \|e_{x_m}\| = \|e_{y_m}\| = 1 \).
Choosing \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \) such that \( k_1 + k_2 \leq k \), unit tangent vectors \( v_{1, m}, \ldots, v_{k_1, m} \in T_{x_m} M + 0 \subset T_{(x_m, y_m)} (M \times M) \), and \( v_{k_1+1, m}, \ldots, v_{k_1+k_2, m} \in 0 + T_{y_m} M \subset T_{(x_m, y_m)} (M \times M) \). Now we can define bounded linear functionals on \( V_m \) as following:
\[
\nabla^{k_1} S (v_{1, m}, \ldots, v_{k_1, m}) = T_{1, m}(S) e_{x_m},
\]
\[
\nabla^{k_2} S^* (v_{k_1+1, m}, \ldots, v_{k_1+k_2, m}) = T_{2, m}(S^*) e_{y_m},
\]
for any \( S \in H^0_{L_2} (V_m, \mathcal{L}^m) \). Let \( S_{T_{1, m}, \mathcal{L}^m} \in H^0_{L_2} (U_m, \mathcal{L}^m) \) and \( S_{T_{2, m}, \mathcal{L}^m} \in H^0_{L_2} (M, \mathcal{L}^m) \) denote the peak sections of the functional \( T_{1, m} \) on \( U_m \) and \( M \), respectively. As in the argument in Part 1, a straightforward computation shows that
\[
\left\| \nabla^{k_1} \nabla^{k_2} (B_{\omega, m}(x_m, y_m) - B_{\omega, \omega, m}(x_m, y_m))(v_{1, m}, \ldots, v_{k_1+k_2, m}) \right\| = \left\| \left( T_{1, m}(S_{T_{1, m}}) T_{2, m}(S_{T_{2, m}}) - T_{1, m}(S_{T_{1, m}, \mathcal{L}^m}) T_{2, m}(S_{T_{2, m}, \mathcal{L}^m}) \right) e_{x_m} e_{y_m} \right\| \leq \left| T_{1, m}(S_{T_{1, m}} - S_{T_{1, m}, \mathcal{L}^m}) \right| \cdot \left| T_{2, m}(S_{T_{2, m}, \mathcal{L}^m}) \right| \cdot \left| T_{2, m}(S_{T_{1, m} - S_{T_{1, m}, \mathcal{L}^m}}) \right| \cdot \left| T_{2, m}(S_{T_{2, m} - S_{T_{2, m}, \mathcal{L}^m}}) \right|.
\]
By Corollary 4.3, we see that there exists a constant \( C_2 > 0 \) depending only on \((M, \omega, \mathcal{L}, \mathcal{H})\) and \( k \), such that
\[
\left| \log \tilde{d}(x_m) \right|^{2\beta} \left| T_{1, m} \right|_{H^0_{L_2} (U_m, \mathcal{L}^m)}^{2\beta} + \left| \log \tilde{d}(y_m) \right|^{2\beta} \left| T_{2, m} \right|_{H^0_{L_2} (U_m, \mathcal{L}^m)}^{2\beta} \leq C_2 m^{C_2}, \quad \forall m \in \mathbb{N},
\]
where \( \epsilon > 0 \) is a constant independent of \( m \). Combining Lemma 4.1 and Proposition 4.2, we see that for each given integer \( q \in \mathbb{N} \), there exists a constant \( C_3 > 0 \) depending only on \((M, \omega, \mathcal{L}, \mathcal{H})\), \( k \)
and $q$, such that
\[
\int_{V^m} \|S_{T_1,m} - S_{T_{V^m,1,m}}\|^2 d\text{Vol}_\omega \leq C_3 m^{-q}, \quad \forall m \in \mathbb{N}.
\]
Choosing $q = 2l + 4 |C_2|$, where $|\cdot|$ is the greatest integer function. Then we have
\[
\begin{align*}
|T_{1,m} (S_{T_1,m} - S_{T_{V^m,1,m}})| &+ |T_{2,m} (S_{T_{T_{V^m,1,m}}}| &\leq 2 \left| T_{1,m} |h^0_{L^2(U,\mathcal{L})}\right| \left| T_{2,m} |h^0_{L^2(U,\mathcal{L})}\right| \|S_{T_1,m} - S_{T_{V^m,1,m}}\|_{L^2(U,\mathcal{L})} \\
&\leq 2C^2 C_3 m^{2|C_2| - 2|C_2|} \left| \log d(x_m) \right| - \beta \left| \log d(y_m) \right| - \beta \\
&\leq 2C^2 C_3 m^{-l} \left| \log d(x_m) \right| - \beta \left| \log d(y_m) \right| - \beta.
\end{align*}
\]
Then we can obtain the estimate (13), and the theorem follows. 

\[ \square \]

5. **Bergman kernels in two special cases**

In this section, we consider the Bergman kernels on complex hyperbolic cusps with Kähler-Einstein metrics and quotients of complex ball.

5.1. **Bergman kernels on complex hyperbolic cusps with Kähler-Einstein metrics.**

Let $(M, \omega)$ be a Kähler manifold and let $(\mathcal{L}, h)$ be a Hermitian line bundle on $M$ such that \( \text{Ric}(h) \geq \epsilon \omega \) and \( \text{Ric}(\omega) \geq -\Lambda \omega \) for some constants \( \epsilon, \Lambda > 0 \). Suppose that $M$ is pseudoconvex if $(M, \omega)$ is not complete. Assume that $U \cong V / \Gamma_V$ is an asymptotic complex hyperbolic cusp on $(M, \omega, \mathcal{L}, h)$ such that $(U, \omega)$ is a Kähler-Einstein manifold with \( \text{Ric}(\omega) = -(n + 1)\omega \), and $U$ is complete.

First, we show that $U$ is an asymptotic complex hyperbolic cusp of order $\infty$. Since $\pi_V^* \omega$ is a Kähler-Einstein metric on $V$, we only need to consider the case $\Gamma_V = 0$.

For each constant $t > 0$, we denote the restriction of the scalar multiplication $v \to tv$ on the domain $V_{t^{-1}} \subset \mathcal{L}_D$ as $\Phi_t$. By definition, $\Phi_t : V_{t^{-1}} \to V_1 = V$ is a biholomorphic map. The pullbacks $\Phi_t^* \mathcal{L}_V$ and $\Phi_t^* \mathcal{L}_V^{-1}$ are holomorphic line bundles on $V_{t^{-1}}$, and the Bergman kernels satisfying that $\Phi_t^* B_{V,\omega_V,m} = B_{V_{t^{-1}}, \Phi_t^* \omega_V,m}$ on $V_{t^{-1}} \times V_{t^{-1}}$. Note that $\Phi_t^* \omega_V$ is a non-vanishing holomorphic section on $V_{t^{-1}}$ and $\|\Phi_t^* \omega_V\| = |\log h_V \circ \Phi_t|$, so we see that $\Phi_t^* \mathcal{L}_V$ and $\Phi_t^* \mathcal{L}_V^{-1}$ are trivial line bundles on $V_{t^{-1}}$. Now we regard $(\Phi_t^* \mathcal{L}_V, \Phi_t^* \mathcal{L}_V^{-1})$ as the restriction of holomorphic line bundles $\mathcal{L}_V$ and $\mathcal{L}_V^{-1}$ on $V \cap V_{t^{-1}}$ respectively, just replace the Hermitian metrics $h_V$ and $h_V^{-1}$ with the Hermitian metrics $\|\Phi_t^* \omega_V\| \cdot \|e_{\mathcal{L}_V}\|^{-1} h_V$ and $\|\Phi_t^* \omega_V\| \cdot \|e_{\mathcal{L}_V}\|^{-1} h_V^{-1}$, respectively. Therefore, the pull back of the Bergman kernels, $\Phi_t^* B_{V,\omega_V,m}$, can be regarded as the $\pi_V^* \mathcal{L}_V \otimes \pi_V^* \mathcal{L}_V^{-1}$-valued functions on $(V \cap V_{t^{-1}}) \times (V \cap V_{t^{-1}})$. By considering the change of the Hermitian line bundle under the mapping $\Phi_t$, we can prove the following lemma.

**Lemma 5.1.** Let $(V, \omega_V, \mathcal{L}_V, h_V)$ be an $n$-dimensional complex hyperbolic cusp, $t > 0$ be a constant, and $\Phi_t : V_{t^{-1}} \to V$ be the restriction of the scalar multiplication $v \to tv$ on $V_{t^{-1}} \subset \mathcal{L}_D$.

Then we have $\Phi_t^* \omega_V = \omega_V - \sqrt{-1} \partial \bar{\partial} \log \left( 1 - 2 t \left| \log h_V \right|^{-1} \right)$ on $V \cap V_{t^{-1}}$. 

\[ \square \]
Proof. By definition, we have $\omega_V = -\sqrt{-1} \partial \bar{\partial} \log |\log h_D|$, and hence

\[
\Phi^*_i \omega_V = -\sqrt{-1} \Phi^*_i \left( \partial \bar{\partial} \log |\log h_D| \right) = -\sqrt{-1} \partial \bar{\partial} \log |\log h_D \circ \Phi_i|
\]

\[
= -\sqrt{-1} \partial \bar{\partial} \log |\log t^2 h_D| = -\sqrt{-1} \partial \bar{\partial} \log (|\log h_D| - 2 \log t)
\]

\[
= -\sqrt{-1} \partial \bar{\partial} \log |\log h_D| - \sqrt{-1} \partial \bar{\partial} \log \left( 1 - 2 \log t |\log h_D|^{-1} \right)
\]

\[
= \omega_V - \sqrt{-1} \partial \bar{\partial} \log \left( 1 - 2 \log t |\log h_D|^{-1} \right), \quad \text{on } V \cap V_{i-1}.
\]

This establishes the formula. \hfill \Box

Combining Lemma 5.1 with Fu-Hein-Jiang’s estimate (4), we see that $U$ is an asymptotic complex hyperbolic cusp of order $\infty$ by choosing $t = e^{-\frac{c_r f}{x}}$, where $c_{r,f}$ is the constant in (4).

We now come to the proof of Theorem 1.2. Here $r_m$ denotes $e^{-\frac{c_m}{x}}$ for $m \geq 2$.

**Proof of Theorem 1.2:** Without loss of generality, we can assume that $\Gamma_V = 0$. By definition, there exists a smooth function $u$ on $U$ such that $(U, \omega, L, h) \cong (V_\gamma, \omega_V - \sqrt{-1} \partial \bar{\partial} u, V_\gamma, e^{h V_\gamma})$ and for any $\alpha > 0$, $u = O \left( |\log h_D|^{-\alpha} \right)$ as $h_D \to 0^+$ to all orders with respect to $\omega_V$.

We divide our proof into three parts.

*Part 1.* At first, we prove estimate (5).

By definition, we see that the estimate (5) is a consequence of the localization principle (1).

*Part 2.* We now start to consider estimate (6).

By the localization principle (2), one can see that the estimate (6) can be localized, and hence we only need to show that there are constants $m_0, C > 0$ independent of $m$, such that

\[
\sup_{V_{r,m} \times V_{r,m}} |\log h_D(x)|^\beta |\log h_D(y)|^\beta \left\| B_{V_{r,m} \omega,m}(x, y) - B_{V_{r,m} \omega_V,m}(x, y) \right\|_{C^{k,H_{r,m}^\omega} \omega_V} \leq C m^{-\frac{1}{4}},
\]

for any $m \geq m_0$. Choosing a large integer $m_1$ such that $r_m \leq r^4$, $\forall m \geq m_1$. Fix an integer $m \geq m_1$ and a point $(x_m, y_m) \in V_{r,m} \times V_{r,m}$. Let $e_{x,m} \in L_{V,x_m}^m$ and $e_{y,m} \in L_{V,y_m}^m$ such that $\|e_{x,m}\|_{h_{V,x_m}^\omega} = \|e_{y,m}\|_{h_{V,y_m}^\omega} = 1$. Choosing nonnegative integers $k_1, k_2$ such that $k_1 + k_2 \leq k$, unit tangent vectors $v_{1,m}, \ldots, v_{k_1,m} \in T_{x_m} V \oplus 0 \subset T_{(x_m, y_m)}(V \times V)$, and unit tangent vectors $v_{k_1+1,m}, \ldots, v_{k_1+k_2,m} \in 0 \oplus T_{y_m} V \subset T_{(x_m, y_m)}(V \times V)$. Here the metric we consider is $\omega_V$. It is easy to see that $H_{r,m}^L(V_{r,m}, \omega, L_{V}^m, h^m) = H_{r,m}^L(V_{r,m}, \omega_V, L_{V}^m, h_{V}^m)$ as vector subspaces of $H^0(V_{r,m}, L^m)$, and the $L^2$-norms are equivalent. Now we can define bounded linear functionals on $V_{r,m}$ as following:

\[
\nabla^{k_1} S(v_{1,m}, \ldots, v_{k_1,m}) = T_{1,m}(S)e_{x,m},
\]

\[
\nabla^{k_2} S^*(v_{k_1+1,m}, \ldots, v_{k_1+k_2,m}) = T_{2,m}(S^*)e_{y,m}^*,
\]

for any $S \in H_{r,m}^0(V_{r,m}, L_{V}^m)$. Let $S_{T_{1,m}} \in H_{r,m}^L(V_{r,m}, L_{V}^m)$ and $S'_{T_{1,m}} \in H_{r,m}^L(V_{r,m}, L_{V}^m)$ denote the peak sections of the functional $T_{1,m}$ on $V_{r,m}$ associated with $(h, \omega)$ and $(h_V, \omega_V)$, respectively. By
a straightforward computation, we can conclude that
\[
\left\| \nabla_x^{k_1} \nabla_y^{k_2} \left( B_{V, \omega, m}(x, y) - B_{V, \omega, k_2, m}(x, y) \right) (x_m, y_m) (v_1, \cdots, v_{k_1+k_2, m}) \right\|_{h^V_{x, m}}
\]
\[
= \left\| \left( T_{1, m} \left( S_{T_{1, m}} \right) - T_{1, m} \left( S'_{T_{1, m}} \right) \right) - T_{2, m} \left( S'_{T_{1, m}} \right) \right\|_{h^V_{x, m}}
\]
\[
\leq \left| T_{1, m} \left( S_{T_{1, m}} - S'_{T_{1, m}} \right) \right| \cdot \left| T_{2, m} \left( S'_{T_{1, m}} \right) \right| + \left| T_{1, m} \left( S_{T_{1, m}} \right) \right| \cdot \left| T_{2, m} \left( S_{T_{1, m}} - S'_{T_{1, m}} \right) \right|
\].

By Corollary 4.2, we see that there exists a constant \( C_1 > 0 \) independent of \( m \), such that
\[
|\log h_D (x_m)|^{2\beta} \left| T_{1, m} \left|_{H^0_L(V_{x, m}, \mathcal{L}^0)} \right| + |\log h_D (y_m)|^{2\beta} \left| T_{2, m} \left|_{H^0_L(V_{x, m}, \mathcal{L}^0)} \right| \right| \leq C_1 m^{C_1}, \ \forall m \in \mathbb{N},
\]
where \( \epsilon > 0 \) is a constant independent of \( m \). The norm we used here is the norm associated with the metric \( \omega_V \) and the Hermitian metric \( h^0_{x, m} \). Combining Fu-Hein-Jiang’s estimate (4) with Lemma 4.2, we see that for each given integer \( q \in \mathbb{N} \), there exists a constant \( C_2 > 0 \) depending only on \( V, r \) and \( q \), such that
\[
\int_{V_{x, m}} \left| S_{T_{1, m}} - S'_{T_{1, m}} \right|^2_{h^V_{x, m}} \ dVol_{\omega_V} \leq C_2 m^{-q}, \ \forall m \in \mathbb{N}.
\]
Choosing \( q = 2l + 4 + 4 \lfloor C_2 \rfloor \), where \( \lfloor . \rfloor \) is the greatest integer function. Then we have
\[
\left| T_{1, m} \left( S_{T_{1, m}} - S'_{T_{1, m}} \right) \right| \cdot \left| T_{2, m} \left( S'_{T_{1, m}} \right) \right| + \left| T_{1, m} \left( S_{T_{1, m}} \right) \right| \cdot \left| T_{2, m} \left( S_{T_{1, m}} - S'_{T_{1, m}} \right) \right|
\]
\[
\leq 2 \left| T_{1, m} \left|_{H^0_L(V_{x, m}, \mathcal{L}^0)} \right| \right| \cdot \left| T_{2, m} \left|_{H^0_L(V_{x, m}, \mathcal{L}^0)} \right| \right| \cdot \left| S_{T_{1, m}} - S'_{T_{1, m}} \right|_{L^2; V_{x, m}, h^0_{x, m}, \omega_V}
\]
\[
\leq 2 C_1^2 C_2 m^{2C_0 m^{-l-2-2\lfloor C_2 \rfloor}} |\log h_D (x_m)|^{-\beta} |\log h_D (y_m)|^{-\beta}
\]
\[
\leq 2 C_1^2 C_2 m^{-l} |\log h_D (x_m)|^{-\beta} |\log h_D (y_m)|^{-\beta}.
\]
Then we can obtain the estimate (6).

Part 3. In this part, we prove the inequality (7).

For any point \( x \in V_r \), we choose a vector \( e_x \in \mathcal{L}_V^r \), such that \( \|e_x\|_{h^V} = 1 \). Hence we can define bounded linear functionals \( T_{x, m} \in H^0_L(V_{x, m}, \mathcal{L}_V^m)^* \) by \( S |_x = T_{x, m}(S)e_x, \ \forall S \in H^0_L(V_{x, m}, \mathcal{L}_V^m) \), where \( m \) is an integer such that \( r^3 \leq r^4 \), and \( x \in V_{r^4} \). For any open subset \( U' \) of \( V \) containing \( x \), we denote the peak sections of \( T_{x, m} \) on \( H^0_L(U', \omega_V, \mathcal{L}_V^m, h^m) \) as \( S_{U', x, m} \). Similarly, for any open subset \( U' \) of \( M \) containing \( x \), we will denote by \( S_{U', x, m} \) the peak sections of \( T_{x, m} \) on \( H^0_L(U', \omega, \mathcal{L}^m, h^m) \). It is easy to see that
\[
\frac{\rho_{M, \omega, m}(x)}{\rho_{V, \omega_V, m}(x)} = \left| \frac{T_{x, m}(S_{M, x, m})}{T_{x, m}(S_{V, x, m})} \right|^2 e^{m \mu}.
\]

By definition, there are constants \( m_2, C_3 > 0 \) depending only on \( V, r \) and \( l \), such that
\[
\sup_{V_{r^4}} |e^{m \mu} - 1| \leq C_3 m^{-\frac{l}{2}}, \ \forall m \geq m_2.
\]
Applying Proposition 4.1 to $T_{x,m} (S_{M,x,m})$ and $T_{x,m} (S'_{V,x,m})$, one can see that there are constants $m_3, C_4 > 0$ depending only on $V, r, f$ and $l$, such that
\[
\sup_{x \in V, m} \left| \frac{T_{x,m} (S_{M,x,m})}{T_{x,m} (S'_{V,x,m})} - 1 \right| \leq C_4 m^{-l}, \ \forall m \geq m_3.
\]

By definition, $\omega = \omega_V - \sqrt{-1} \partial \bar{\partial} u$, and $h = e^{u} h_V$. Lemma 4.2 now implies that there are constants $m_4, C_5 > 0$ depending only on $V, r$ and $l$, such that
\[
\sup_{x \in V, m} \left| \frac{T_{x,m} (S'_{V,x,m})}{T_{x,m} (S'_{V,x,m})} - 1 \right| \leq C_5 m^{-l}, \ \forall m \geq m_4.
\]

It follows that
\[
\sup_{x \in V, m} \left| \frac{\rho_{M,x,m}(x)}{\rho_{V,x,m}(x)} - 1 \right| = \sup_{x \in V, m} \left| \frac{T_{x,m} (S_{M,x,m})}{T_{x,m} (S_{V,x,m})} \right|^2 - 1
\]
\[
= \sup_{x \in V, m} \left| \frac{T_{x,m} (S_{M,x,m})}{T_{x,m} (S_{V,x,m})} \right|^2 \cdot \left| \frac{T_{x,m} (S'_{V,x,m})}{T_{x,m} (S'_{V,x,m})} \right|^2 \cdot \left| \frac{T_{x,m} (S'_{V,x,m})}{T_{x,m} (S'_{V,x,m})} \right|^2 - 1
\]
\[
\leq (1 + 2C_4 m^{-l})^4 (1 + 2C_5 m^{-l})^2 (1 + 2C_3 m^{-l}) - 1 \leq (1 + 2C_4)^4 (1 + 2C_3)^2 (1 + 2C_3) m^{-l},
\]
for any integer $m \geq \sum_{j=2}^4 (m_j + 10C_{j+1})$. \hfill \Box

Before considering the $C^k$-estimate of the quotients of Bergman kernel functions, we need the following lemma.

**Lemma 5.2.** Let $(V, \omega_V, L_V, h_V)$ be a complex hyperbolic cusp defined in Section 1, and $S \in H^2_{\omega_V} (V, L_V)$ such that $\|S\|_{L^2 \omega_V} = 1$. Then for each $k \in \mathbb{N}$, there exists a constant $C > 0$ depending only on $(V, \omega_V, L_V, h_V)$ and $k$, such that
\[
\|S(x)\|_{C^k \omega_V} \leq C m^C \|\log h(x)\| C \rho_{D^2, m} \left( \sqrt{h_D(x)} \right),
\]
for any $x \in V_{\epsilon^{-2}}$ and integer $m \geq C$, where $\rho_{D^2, m}$ is $m$-th Bergman kernel on the polarized punctured unit disc $(\mathbb{D}^2, \omega_{D^2}, C, \log |z|^2)$.

**Proof.** Combining Lemma 4.3 with Lemma 4.5, we can find a constant $C_1 > 0$ depending only on $(V, \omega_V, L_V, h_V)$ and $k$, such that
\[
\|\nabla^k S(x)\|_{C^k \omega_V} \leq C_1 m^{C_1} \|\log h_D(x)\|_{C^1} \int_{B_{C_1^{-1} \log h_D(x)^{-10}}(x)} \rho_{\omega_V, m}(y) d\text{Vol}_{\omega_V},
\]
for any $x \in V_{\epsilon^{-2}}$ and integer $m \geq C_1$. By a direct computation, we can conclude that there exists a constant $C_2 > 0$ depending only on $(V, \omega_V, L_V, h_V)$, such that
\[
B_{C_1^{-1} \log h_D(x)^{-10}}(x) \subset \left\{ y \in V \left| \frac{h_D(x)}{h_D(y)} \right|^{C_2 \log h_D(x)^{-10}} \leq h_D(y) \leq h_D(x)^{1-C_2 \log h_D(x)^{-10}} \right\},
\]
for any $x \in V_{e^{-2}}$. Combining Theorem thmregularpart with Lemma 4.3, we see that there exists a constant $\epsilon \in (0, e^{-1})$ depending only on $(V, \omega_V, L_V, h_V)$, such that

$$\frac{(m + n)!}{(2\pi)^{n+1} m!} \leq \rho_{\omega_V,m}(x) \leq \frac{(m + n)!}{(2\pi)^{n-1} m!},$$

for any $x \in V_{e^{-2}} \setminus V_{e^{-\sqrt{m}}}$ and integer $m \geq \epsilon^{-1}$. Hence there exists a constant $C_3 > 0$ depending only on $(V, \omega_V, L_V, h_V)$, such that

$$\|\nabla^k f(x)\|_{C^0; h_\omega^\mathcal{U}(a)} \leq C_3 m C_3 \|\log h_D(x)\|_{C^3} \leq C_3 m^2 C_3 \left|\log h_D(x)\right|^{2C_3} \rho_{\omega_V,m} \left(\sqrt{h_D(x)}\right),$$

for any $x \in V_{e^{-2}} \setminus V_{e^{-\sqrt{m}}}$ and integer $m \geq C_3$.

Now we will give an estimate of $\rho_{\omega_V,m}(y)$ for $y \in B_{C_1^{-1} \left|\log h_D(x)\right|^{-10}}(x)$ and $x \in V_{e^{-\sqrt{m}}}$. By Proposition 3.1, we can conclude that

$$\rho_{\omega_V,m}(y) = \frac{n \left|\log h_D(y)\right|^{m}}{2\pi (m - n - 1)!} \sum_{l=1}^{\infty} \left|l^{m-n} \rho_{\omega_D,l}(\pi_D(y)) h_D(y)^l\right|^{m} \leq \frac{C_4 n \left|\log h_D(x)\right|^{m}}{2\pi (m - n - 1)!} \sum_{l=1}^{\infty} \left|l^{m-1} h_D(x)^l (1 - C_2 \left|\log h_D(x)\right|^{-10})\right|^{m} \leq \frac{C_4^2 n \left|\log h_D(x)\right|^{m}}{2\pi (m - n - 1)!} \sum_{l=1}^{\infty} \left|l^{m-1} h_D(x)^l (1 - C_2 \left|\log h_D(x)\right|^{-10})\right|^{m} \leq C_4^2 m^m \rho_{\omega_V,m} \left(\sqrt{h_D(x)}\right),$$

for any $m \geq C_4$, where $C_4 > 0$ is a constant depending only on $(V, \omega_V, L_V, h_V)$. Combining these inequalities, we can prove this lemma. 

Now we need to estimate the $C^k$ norm on $(V, \omega_V + \pi_D^* \omega_D)$. Let $a \in D$ and $z = (z_1, \cdots, z_{n-1}) : U_a \to \mathbb{C}^{n-1}$ be a holomorphic coordinate around $a \in D$ such that $z(a) = 0$ and $\omega_D = \omega_E$. Fix a holomorphic section $e_{\mathcal{L}_{D}} \in H^0(U_a, \mathcal{L}_{D})$ satisfying that $\|e_{\mathcal{L}_{D}}\|_{h_D} = e^{\left|z\right|^2}$ on $U_a$. Now we can construct a holomorphic map $\varphi : U_a \times \mathbb{R}_+ \times \mathbb{R} \to \pi_D^{-1}(U_a)$ by $\varphi(z,r,\theta) = r e^{\sqrt{-1}\theta} e_{\mathcal{L}_{D}}(z)$, $\forall (z,r,\theta) \in U_a \times \mathbb{R}_+ \times \mathbb{R}$. It is clear that $rank_{(z,r,\theta)}(d\varphi) = 2n$, $\forall (z,r,\theta) \in U_a \times \mathbb{R}_+ \times \mathbb{R}$, and $\varphi^{-1} \left(\pi_D^{-1}(U_a) \cap V\right) = \{(z,r,\theta) \in U_a \times \mathbb{R}_+ \times \mathbb{R} | r^2 < e^{-\left|z\right|^2}\}$.

Write $x_{2i-1} = \text{Re} z_i$ and $x_{2i} = \text{Im} z_i$, $i = 1, \cdots, n-1$. Let

$$\mathcal{X}^j = \left|\log h_D\right|^{\frac{1}{2}} \mathcal{Y}^j = \mathcal{Z}^j = \frac{\partial}{\partial x^j}, \quad j = 1, \cdots, 2n - 2,$$

(87)

$$\mathcal{X}^{2n-1} = \mathcal{Y}^{2n-1} = r \left|\frac{\partial}{\partial x^j} Z_{2n-1} = r \left|\frac{\partial}{\partial y} Z_{2n} = \left|\frac{\partial}{\partial \theta} Z_{2n} = \left|\frac{\partial}{\partial \theta} Z_{2n} = \left|\frac{\partial}{\partial \theta} Z_{2n} = \left|\frac{\partial}{\partial \theta},$$

Then we can obtain the following estimate.

**Lemma 5.3.** Under the above assumptions, for any given $k \in \mathbb{N}$, there exists a constant $C > 0$ depending only on $(V, \omega_V)$ and $U_a$, such that for any point $x \in V_{e^{-2}} \cap \pi_D^{-1}(U_a)$ and function
Proposition 5.1. Let \( M, \omega, \mathcal{L}, h \) and \( U \cong V_r \) be as in Theorem 1.2. Assume that \( \Gamma_V = 0 \) and \( \mathcal{L}^{-1}_D \) is globally generated. Then there exists a constant \( b > 0 \), such that for any given integer \( l \geq 0 \),

\[
\sum_{p=0}^{k} \| \nabla^p f(x) \|_{\omega_V + \pi_D \omega_D} \leq C \sum_{p=0}^{k} \| \nabla^p f(x) \|_{\omega_V} \leq C^2 \sum_{p=0}^{k} \| \nabla^p f(x) \|_{\omega_V}.
\]

Moreover, if \( f = f_D \circ \pi_D \) for some function \( f_D \in C^k(D, \mathbb{R}) \), then

\[
\| \nabla^k f(x) \|_{\omega_V + \pi_D \omega_D} \leq C \sum_{l=0}^{k} \| \nabla^l f_D(\pi_D(x)) \|_{\omega_D}.
\]

The proof of this lemma is not difficult but is too long to give it.

Then we can give an estimate for the \( C^k \)-norm of the quotient of Bergman kernel functions on \( V_r \) with respect to the Kähler metric \( \omega_V \).

**Proposition 5.1.** Let \((M, \omega, \mathcal{L}, h)\) and \( U \cong V_r \) be as in Theorem 1.2. Assume that \( \Gamma_V = 0 \) and \( \mathcal{L}^{-1}_D \) is globally generated. Then there exists a constant \( b > 0 \), such that for any given integer \( l \geq 0 \),

\[
\left\| \frac{\rho_{M, \omega, m}(x)}{\rho_{V, \omega, m}(x)} - 1 \right\|_{C^k(\omega_V)} \leq C \| \log h_D \|^b m^{-l}, \quad \forall m \geq C, \quad \forall x \in V_r,
\]

where \( C > 0 \) is a constant depending only on \((M, \omega, \mathcal{L}, h), V, k \) and \( l \).

**Proof.** By choosing open subsets \( U_a \) of \( D \) as stated before Lemma 5.3, we can find a finite covering \( \{ \pi_D^{-1}(U_a) \}_{a \in A} \) of \( V \). Fix \( x \in V_r \). Then there are vector fields \( \mathcal{X}_i \) and \( \mathcal{Y}_i \) around \( x \) satisfying the estimates in Lemma 5.3. Let \( \{ \mathcal{S}_{M,m,j} \}_{j \in I_m} \) and \( \{ \mathcal{S}'_{M,m,j} \}_{j \in I_m} \) be \( L^2 \) orthonormal bases in the spaces \( H^0_L(M, \omega, \mathcal{L}^m, h^m) \) and \( H^0_L(V, \omega_V, \mathcal{L}^m, h_V^m) \), respectively. Then we have

\[
\frac{\rho_{M, \omega, m}(x)}{\rho_{V, \omega, m}(x)} = \frac{\sum_{j \in I_m} \| \mathcal{S}_{M,m,j}(x) \|_{h^m}^2}{\sum_{j \in I_m} \| \mathcal{S}'_{M,m,j}(x) \|_{h_V^m}^2} = e^{\mu u} \frac{\sum_{j \in I_m} \| \mathcal{S}_{M,m,j}(x) \|_{h_V^m}^2}{\sum_{j \in I_m} \| \mathcal{S}'_{M,m,j}(x) \|_{h_V^m}^2}, \forall x \in U \cong V_r.
\]
By Lemma 5.3, we can conclude that for any constant $l > 0$, there exists a constant $C_1 > 0$ depending only on $(M, \omega, L, h)$, $V$, $l$ and $k$, such that

$$\sum_{p=0}^{k} \sup_{V_m} \| \nabla^p (e^{m^a} - 1) \|_{\omega V} \leq C_1 m^{-l}, \forall m \geq C_1.$$ 

For abbreviation, we write $\mathcal{Y}_{i_1, \ldots, i_p}$ instead of $\mathcal{Y}_{i_1} \cdots (\mathcal{Y}_{i_p} \left( \sum_{j \in I_m} \| S_{M, m,j} \|_{h V}^2 \right) ) \cdots)$ ($x$), and we write $\mathcal{Y}'_{i_1, \ldots, i_p}$ instead of $\mathcal{Y}_{i_1} \cdots (\mathcal{Y}_{i_p} \left( \sum_{j \in I'_m} \| S'_{M, m,j} \|_{h V}^2 \right) ) \cdots)$ ($x$). By induction on $k$, for any integer $k \geq 1$, one can conclude that

$$\sum_{1 \leq i_1, \ldots, i_k \leq 2n} \mathcal{Y}_{i_1} \left( \cdots \mathcal{Y}_{i_k} \left( \sum_{j \in I_m} \| S_{M, m,j} \|_{h V}^2 \right) \right) \cdots \right)(x)$$

$$= \sum_{p=0}^{k} \sum_{1 \leq i_1, \ldots, i_p \leq 2n} \left( \sum_{j \in I_m} \| S'_{M, m,j} \|_{h V}^2 \right) - k \mathcal{Y}_{i_1, \ldots, i_p} \mathcal{Y}_k \mathcal{Y}_i \cdots \mathcal{Y}_{i_p}$$

where $p \leq k$ is a nonnegative integer, $P_{k,p}$ are polynomials in variables $\mathcal{Y}_{i_1, \ldots, i_p}, \mathcal{Y}'_{i_1, \ldots, i_p}$, and of degree $k - 1$, and $1 \leq i_1, \ldots, i_p \leq 2n$ are integers. Since $L^{-1}$ is globally generated, we can find a constant $C_2 > 0$ depending only on $V$, such that $\rho_{p \geq m} \left( \sqrt{h_D(x)} \right) \leq C_2 \rho_{V, \omega, V, m}(x)$, $\forall x \in V$, $\forall m \geq n + 1$. By Lemma 5.2, we see that for any integer $k \in \mathbb{N}$ and point $x \in V_{e-2}$,

$$\left| \nabla^k \left( \sum_{j \in I_m} \| S_{M, m,j} \|_{h V}^2 - 1 \right) \right|_{\omega V} \leq C_3 m^{C_3} \log h_D \sum_{p=0}^{k} \sum_{i_1, \ldots, i_p \leq 2n} \mathcal{Y}_{i_1, \ldots, i_p} - \mathcal{Y}'_{i_1, \ldots, i_p}$$

where $C_3 \geq 0$, $C_3 > 0$ is a constant depending only on $(M, \omega, L, h)$, $V$ and $k$. Now we estimate the function $|\mathcal{Y}_{i_1, \ldots, i_p} - \mathcal{Y}'_{i_1, \ldots, i_p}|$. Since Chern connection is compatible with the Hermitian metric $h V$, we have

$$\sum_{p=0}^{k} \sum_{1 \leq i_1, \ldots, i_p \leq 2n} \mathcal{Y}_{i_1, \ldots, i_p} - \mathcal{Y}'_{i_1, \ldots, i_p}$$

For any point $x \in U \cong V$, and finite sequence of integers $1 \leq i_1, \cdots, i_p \leq 2n$, we can define a linear functional $T_{x,m,i_1, \ldots, i_p}$ on $H^0_{L} (V, L V)$ by

$$\nabla \mathcal{Y}_{i_1} \cdots \nabla \mathcal{Y}_{i_p} S(x) = T_{x,m,i_1, \ldots, i_p}(S) e^m, \forall S \in H^0_{L} (V, L V),$$
where \( e_x \in \mathcal{L}_V \) is a vector such that \( \|e_x\|_{h_V} = 1 \). Let \( S_{x,m,i_1,...,i_p} \) and \( S'_{x,m,i_1,...,i_p} \) be the peak sections of \( T_{x,m,i_1,...,i_p} \) in \( H^0_{L^2}(M, \omega, \mathcal{L}^m, h^m) \) and \( H^0_{L^2}(V, \omega_V, \mathcal{L}^m_{h^V}) \), respectively. Then we have

\[
\sum_{j \in I_m} \left\langle \nabla y_{i_1} \cdots \nabla y_{i_p}, S_{M,m,j}, \nabla y_{i_p+1} \cdots \nabla y_{i_p} S_{M,m,j} \right\rangle_{h^V_V} \\
- \sum_{j \in I_m} \left\langle \nabla y_{i_1} \cdots \nabla y_{i_p}, S'_{M,m,j}, \nabla y_{i_p+1} \cdots \nabla y_{i_p} S'_{M,m,j} \right\rangle_{h^V_V} = \left\langle T_{x,m,i_1,...,i_p} \left( S_{x,m,i_1,...,i_p} - S'_{x,m,i_1,...,i_p} \right), T_{x,m,i_1,...,i_p} \right\rangle_{h^V_V} \\
\quad \quad \quad \quad - \left\langle T_{x,m,i_1,...,i_p} \left( S'_{x,m,i_1,...,i_p} - S_{x,m,i_1,...,i_p} \right), T_{x,m,i_1,...,i_p} \right\rangle_{h^V_V} \leq 2 \left\| T_{x,m,i_1,...,i_p} \right\| \left\| T_{x,m,i_1,...,i_p} \right\| \left\| S_{x,m,i_1,...,i_p} - S'_{x,m,i_1,...,i_p} \right\|,
\]

where \( \| \cdot \| \) are norms corresponding to the Hilbert space \( H^0_{L^2}(V_{\mathcal{L}^m_{h^V}}, \omega_V, \mathcal{L}^m_{h^V}) \). Combining Lemma 4.1, Proposition 4.1 and Lemma 4.2, we can assert that for any integer \( l > 0 \), there exists a constant \( C_4 > 0 \) depending only on \((M, \omega, \mathcal{L}, h)\), \( V \), \( k \) and \( l \), such that

\[
\int_{V_{\mathcal{L}^m_{h^V}}} \left\| S_{x,m,i_1,...,i_p} - S'_{x,m,i_1,...,i_p} \right\|_{h^V_V}^2 d\nu \leq C_4 m^{-1}, \quad \forall m \geq C_4, \forall x \in V_{r_m}.
\]

Hence for any integers \( k, l \in \mathbb{N} \),

\[
\left\| \nabla^k \left( \sum_{j \in I_m} \left\| S_{M,m,j} \right\|_{h^V_V}^2 \right) - 1 \right\|_{\omega_V} \leq C_6 m^{-l} |\log h_D| C_5, \quad \forall m \geq C_6, \forall x \in V_{r_m},
\]

where \( C_6 > 0 \) is a constant depending only on \((M, \omega, \mathcal{L}, h)\), \( V \) and \( k \), and \( C_6 > 0 \) is a constant depending only on \((M, \omega, \mathcal{L}, h)\), \( V \) and \( k \). This is our claim. \( \Box \)

Now we are ready to prove the \( C^k \)-estimate of the quotients of Bergman kernel functions.

**Proposition 5.2.** Let \((M, \omega, \mathcal{L}, h)\) and \( U \cong V_r \) be as in Theorem 1.2. Assume that \( \Gamma_V = 0 \) and \( \mathcal{L}^{-1}_D \) is globally generated. Then for any given integer \( l \geq 0 \), there exists a constant \( C > 0 \) depending only on \((M, \omega, \mathcal{L}, h)\), \( V \) and \( l \), such that

\[
\sup_{V_{r_m}} \left| \frac{\rho_{M,\omega,m}(x)}{\rho_{V,\omega_V,m}(x)} - 1 \right| \leq C m^{-l}, \quad \forall m \geq C.
\]

**Proof.** By Proposition 5.1, we only need to prove it when \( x \in V_{r_{m+10}} \). Without loss of generality, we can assume that \( m \) is sufficiently large such that \( r_m < r^2 \).

Let \( X_m = H^0_{L^2}(V_{r_m}^4, \omega_V, \mathcal{L}^m_{h^V}) \) be the subspace of \( H^0_{L^2}(V_{r_m}^4, \omega_V, \mathcal{L}^m_{h^V}) \) generated by the orthogonal set \( \left\{ f_{S_{q,j}}^m, 1 \leq j \leq N_q, q \geq 2 \right\} \), where \( \left\{ S_{q,j} \right\}_{j=1}^{N_q} \) is an orthonormal basis of the Hilbert space \( H^0_{L^2}(D, \omega_D, \mathcal{L}^{-q}_D, h^q_D) \), and \( f_{S_{q,j}} \) denote the holomorphic function corresponding to \( S_{q,j} \) defined in Lemma 3.1. Then we define \( Y_m = X_m \cap H^0_{L^2}(M, \omega, \mathcal{L}^m, h^m) \), and \( Z_m = X_m \cap H^0_{L^2}(V, \omega_V, \mathcal{L}^m_{h^V}) \). Note that \( (V, \mathcal{L}_V) \cong (U, \mathcal{L}) \), and hence \( H^0_{L^2}(V, \omega_V, \mathcal{L}^m_{h^V}) = \)}}
$H^0_{L^2}(U, \omega, L^m, h^m)$ as subspaces of $H^0(V_m, L^m, \omega)$, $\forall m \in \mathbb{N}$. Let $X^\perp_m$, $Y^\perp_m$ and $Z^\perp_m$ be the orthogonal complements of $X_m$, $Y_m$ and $Z_m$ in $H^0_{L^2}(V_m, \omega, L^m, h^m)$, $H^0_{L^2}(M, \omega, L^m, h^m)$ and $H^0_{L^2}(V, \omega, L^m, h^m)$, respectively. By applying Lemma 3.2 to $H^0_{L^2}(V, \omega)$, $\forall m \geq n + 1$, we see that \(\{f_{S_j,m} e^{n \omega}_V \mid 1 \leq j \leq N_q\}\) is an orthogonal basis of $X^\perp_m$ and $Z^\perp_m$ at the same time. Let $a_m = \left(\frac{n}{2\pi(m-n)}\right)^{-\frac{1}{2}}$. It is easy to see that for any integer $l > 0$, we have

$$
\int_{V^\perp_m} \left\| a_m f_{S_1,j} e^{n \omega}_V \right\|^2_{H^0_{\omega, V}} dVol_{\omega} - 1 + \sup_{V^\perp_m} \left\| a_m f_{S_1,j} e^{n \omega}_V \right\|^2_{H^0_{\omega, V}} \leq C_1 m^{-l},
$$

where $C_1 > 0$ is a constant depending only on $V$ and $l$.

Analysis similar to that in the proof of Lemma 4.4 shows that there are quasi-plurisubharmonic functions $\psi_m \in L^1(V, \omega V)$ such that $\psi_m \leq 0$, $supp_\psi_m \subseteq V_{r_2}$, $-\Delta \psi_m \geq -C_2 \sqrt{m} \omega V$, and $\lim_{h \to 0} (\psi_m(x) - \log h (\omega V)) = -\infty$, where $C_2 > 0$ is a constant depending only on $V$. See also the remark below Lemma 4.4. Let $\eta_m \in C^\infty(V)$ such that $\eta_m = 1$ on $V_{r_2}$, $\eta_m = 0$ on $V \setminus V_{r_2}$, and $|\nabla \eta_m|_{\omega V} \leq C_3$, where $C_3 > 0$ is a constant depending only on $V$. By Hörmander’s $L^2$ estimate, we can find a smooth $\mathbb{C}^m$-valued function $\zeta_{j,m}$ on $M$ such that $\bar{\partial} (a_m \eta_m f_{S_1,j} e^{n \omega}_V - \zeta_{j,m}) = 0$, and for each integer $l > 0$,

$$
\int_M \|\zeta_{j,m}\|_{h^m} e^{-10\omega_m} dVol \leq C_4 \int_M \left\| \bar{\partial} (a_m \eta_m f_{S_1,j} e^{n \omega}_V) \right\|^2_{h^m,\omega} e^{-10\omega_m} dVol \leq C_4 m^{-l},
$$

for any integer $m \geq C_4$, where $C_4 > 0$ is a constant depending only on $(M, \omega, L, h)$ and $l$. It follows that $\zeta_{j,m} |_{V^\perp_m} \in X_m$, and $|\zeta_{j,m}|_{L^2,h^m,\omega} \leq C_5 m^{-l}$. Similarly, there are holomorphic sections $\zeta'_{j,m} \in Y_m$ such that $a_m \eta_m f_{S_1,j} e^{n \omega}_V - \zeta_{j,m} - \zeta'_{j,m} \in Y^\perp_m$, and for any given integer $l$, we have $|\zeta_{j,m}|_{L^2,h^m,\omega} \leq C_5 m^{-l}$, $\forall m \geq C_5$, where $C_5 > 0$ is a constant depending only on $(M, \omega, L, h)$ and $l$. By Schmidt orthonormalization process, we can find constants $b_{i,j,m}$ such that the set $\{\sum_{j=1}^{N_1} b_{i,j,m} (a_m \eta_m f_{S_1,j} e^{n \omega}_V - \zeta_{j,m} - \zeta'_{j,m})\}_{i=1}^{N_1}$ becomes an orthonormal basis of $Y^\perp_m$, and for each given integer $l$, there exists a constant $C_6 > 0$ depending only on $(M, \omega, L, h)$ and $l$, such that $|b_{i,j,m} - \delta_{i,j}| \leq C_6 m^{-l}$, $\forall m \geq C_6$.

An argument similar to the one used in Lemma 5.2 shows that for each integer $k > 0$, there exists a constant $C_7 > 0$ depending only on $V$ and $k$, such that

$$
\|S(x)\|_{C^k h^{-1/2}_m, \omega^V} \leq C_7 m C_7 \|\log h (\omega V)\|_{C^k} \rho_{h_m} \left(\sqrt{h (\omega V)}\right)^{2k} \|S\|_{L^2 h^{-1/2}_m, \omega^V},
$$

for any $m \geq C_7$, $S \in X_m$ and $x \in V_{r_10}$. As in Proposition 5.1, this estimate shows that for any integer $k \geq 0$, we have

$$
\left\| \nabla^k \left( \sum_{j=1}^{N_1} \left( a_m f_{S_1,j} e^{n \omega}_V \right)^2_{h^m,\omega} - 1 \right) \right\|_{\omega^V} + \left\| \nabla^k \left( \sum_{j=1}^{N_1} b \left( a_m f_{S_1,j} e^{n \omega}_V \right)^2_{h^m,\omega} - 1 \right) \right\|_{\omega^V} \leq C_8 m^{-l} \|\log h (\omega V)\|_{C^k} \rho_{h_m} \left(\sqrt{h (\omega V)}\right)^{2k}, \forall m \geq C_8, \forall x \in V_{r_10},
$$

where $C_8 > 0$ is a constant depending only on $(M, \omega, L)$ and $k$. Since $|b_{i,j,m} - \delta_{i,j}| \leq C_6 m^{-l}$, we can apply Lemma 5.3 to show that for any integers $k, l > 0$, there exists a constant $C_9 > 0$ and
depending only on \((M, \omega, \mathcal{L})\), \(k\) and \(l\), such that
\[
\left\| \nabla^k \left( \sum_{j=1}^{N_1} \left\| \sum_{j=1}^{N_1} b_{i,j,m} f_{S_{i,j}, m} e_L e_{V_i} \right\|_{h_0}^2 - 1 \right) \right\|_{\omega + \pi_D} \leq C_9 m^{-l}, \ \forall m \geq C_9, \ \forall x \in V_{e^m}. 
\]
Combining these inequalities, one can conclude that there exists a constant \(C_10 > 0\) depending only on \((M, \omega, \mathcal{L}, h)\), \(k\) and \(l\), such that
\[
\left\| \frac{\rho_{M, \omega, m}(x)}{\rho_{V, \omega, V, m}(x)} - 1 \right\|_{C^k \omega_{D}} \leq C_{10} m^{-l}, \ \forall m \geq C_10, \ \forall x \in V_{e^m},
\]
which completes the proof.

5.2. Bergman kernels on quotients of complex ball.

Let \((\mathbb{B}^n, \omega_B)\) be the \(n\)-dimensional complex hyperbolic ball with constant bisectional curvature \(-1\), and let \(\Gamma\) be a torsion free lattice in \(\text{Aut}(\mathbb{B}^n, \omega_B) \cong PU(n,1)\). Then the quotient space \(\mathbb{B}^n/\Gamma\) is a manifold, and there exist a finite collection of disjoint open subsets \(U_i \subset \mathbb{B}^n/\Gamma\), \(i = 1, \cdots, N\), and complex hyperbolic cusps \((V_i/\Gamma_{V_i}, \omega_{V_i}, L_{V_i}/\Gamma_{V_i}, h_{V_i})\) such that \((U_i, \omega_{B^n}) \cong (V_i/\Gamma_{V_i}, \omega_{V_i})\), and \((\mathbb{B}^n/\Gamma) \setminus \bigcup_{i=1}^N U_i\) is compact.

Now we consider the Bergman kernels on \(U_i\). Since \(U_i\) and \(\mathbb{B}^n/\Gamma\) satisfy the conditions of Theorem 1.3, we see that Theorem 1.3 describes the asymptotic behavior of the Bergman kernels on \(U_i\).

**Proof of Theorem 1.3:** Let \(\gamma_m = r^2\) and \(\sigma_m = r\) in Theorem 1.1. Note that \(U \cong V_i/\Gamma_{V_i}\). By applying Theorem 1.1 to \(U \subset M\) and \(V_i/\Gamma_{V_i} \subset V/\Gamma_{V_i}\), respectively, we can prove the theorem.

Our next goal is to prove the \(C^k\)-estimate of the quotients of Bergman kernel functions.

By combining Lemma 5.3 with Proposition 5.1, one can get the following proposition.

**Proposition 5.3.** Let \((M, \omega, \mathcal{L}, h)\) and \(U \cong V_i\) be as in Theorem 1.3. Assume that \(\Gamma_V = 0\) and \(L_D^{-1}\) is globally generated. Then there exists a constant \(b > 0\) depending only on \((M, \omega, \mathcal{L}, h)\), such that for any given integer \(l \geq 0\),
\[
\left\| \frac{\rho_{M, \omega, m}(x)}{\rho_{V, \omega, V, m}(x)} - 1 \right\|_{C^k \omega_{D}} \leq C |h_D(x)|^{-b} m^{-l}, \ \forall x \in V_{e^m}, \ \forall m \geq C,
\]
where \(\omega_0 = \pi_D \omega_{D} + \sqrt{-1} \partial \overline{\partial} h_D\) is a Kähler metric on the total space of \(\mathcal{L}_D\), and \(C > 0\) is a constant depending only on \((M, \omega, \mathcal{L}, h)\), \(V\), \(k\) and \(l\).

When \(h_D\) is small, we cannot derive the estimate we need from Proposition 5.3. In this case, we need to make some finer estimates by approximating the \(L^2\) orthonormal basis of \(H^0_{L^2}(M, \omega, \mathcal{L}^m, h^m)\) using the \(L^2\) orthonormal basis of \(H^0_{L^2}(V/\Gamma_{V_i}, \omega_{V_i}, (L_{V_i}/\Gamma_{V_i})^m, h_{V_i})\). Suppose that \(\Gamma_V = 0\). The following lemma gives an estimate about the norm of holomorphic sections on \(V\).

**Lemma 5.4.** Let \((V, \omega_V, L_V^n, h_V^m)\) be a complex hyperbolic cusp, and \(\kappa \in \left(0, e^{-1}\right)\) be a constant. Then for any given constant \(\mathcal{R} > 0\), there exists a constant \(C > 0\) depending only on \(\kappa\) and \((V, \omega_V, L_V, h_V)\), such that
\[
\left\| a_{m,q} e_{L_V} \right\|_{L^2_{\mathcal{R}, L^2, V \setminus V_d, h_{V_d}. \omega_V}} \leq C e^{-4Rm} \left\| S_q \right\|_{L^2_{\mathcal{R}, L^2, D, h_{D}^{-1}. \omega_{D}}},
\]
for any positive integers \(m \geq n + 1\), \(q \leq e^{-4(|\kappa|+\log|\kappa|)+n+4})m\) and holomorphic sections \(S_p \in H^0(D,\mathcal{L}_D^p)\), where \(a_{m,q} = \sqrt{\frac{nq^m - m}{2\pi(m-n-1)!}}\), and \(f_{S_p}\) is the holomorphic function on \(V\) defined in Lemma 3.1.

**Remark.** Since \(q \in \mathbb{N}\), we see that Lemma 5.4 is trivial when \(m \leq e^{-4(|\kappa|+\log|\kappa|)+n+4})m\).

**Proof.** Without loss of generality, we can assume that \(|S_q|_{L^2,\mathcal{E}_D^{-\rho},\omega_D} = 1\). Fix \(x \in V \setminus V_\kappa\). Let \(t = h_D(x) \in (\kappa, 1)\). By Theorem 2.1, there exists a constant \(C_1 > 0\) depending only on \((D,\omega,\mathcal{L}_D, h_D)\), such that \(\rho_{D,\omega_D,\mathcal{E}_D^{-\rho},\omega_D} \leq C_1 t^q n^{-1}\), \(\forall q \in \mathbb{N}\). Hence

\[
\|f_{S_p}(x)\|^2 = h_D(x)^q \|S_q(\pi D(x))\|^2_{h_D} \leq t^q \rho_{D,\omega_D,\mathcal{E}_D^{-1},\omega_D}(\pi D(x)) \leq C_1 t^q n^{-1}.
\]

By Stirling’s formula, we can conclude that

\[
\left\| a_{m,q} f_{S_p}(x)e^{m\mathcal{L}_V}(x) \right\|^2_{h_D} \leq C_1 \frac{nq^m - m}{2\pi(m-n-1)!} t^q |\log t|^m \leq C_1 \frac{m^m q^m - m^m}{2\pi |\log t|^m},
\]

for any positive integers \(m \geq n + 1\) and \(q \leq e^{-4(|\kappa|+\log|\kappa|)+n+4})m\). Now (14) implies that

\[
\left\| a_{m,q} f_{S_p}(x)e^{m\mathcal{L}_V} \right\|^2_{L^2(V \setminus V_\kappa, h_D, \omega_V)} \leq C_1 e^{-4\delta m} \left( Vol_{\omega_V} (V_{c^{-1}} \setminus V_\kappa) + \int_{V \setminus V_{c^{-1}}} |\log h_D|^m dVol_{\omega_V} \right) \leq C_2 e^{-4\delta m} \left( 1 + \int_{D \setminus V_{c^{-1}}} \sqrt{-1} |\omega|^2 |\log \omega|^m d\xi \right) \leq C_3 e^{-4\delta m},
\]

where \(C_2, C_3\) are positive constants depending only on \(r\) and \((V, \omega_V, \mathcal{L}_V, h_V)\). This is our assertion \(\square\)

The following basic fact of linear algebra will also be used in the proof of the \(C^k\)-estimate of the quotients of Bergman kernel functions.

**Lemma 5.5.** Let \(N \in \mathbb{N}\), and let \(A = (a_{ij})_{i,j=1}^N\) be an \(N \times N\) Hermitian matrix. Suppose that there exists a positive constant \(\delta \leq \frac{1}{\sqrt{\det(\Lambda)}}\) such that \(\max_{1 \leq i,j \leq N} |a_{ij} - \delta_{ij}| \leq \delta\). Then there exists an \(N \times N\) Hermitian matrix \(B = (b_{ij})_{i,j=1}^N\) such that \(A = BB^T\), and \(\max_{1 \leq i,j \leq N} |b_{ij} - \delta_{ij}| \leq 2N\delta\).

**Proof.** Recall that any Hermitian matrix can be diagonalized by a unitary matrix, so we can find \(Q \in U(N)\) such that \(A = QA^TQ^T\), where \(\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N)\) is a diagonal matrix. Since \(\max_{1 \leq i,j \leq N} |a_{ij} - \delta_{ij}| \leq \delta\), the characteristic polynomial of \(A - I\), \(p_A(\lambda) = \lambda^n + c_n \cdots + c_1 \lambda^2 + c_2 \lambda + c_3\), satisfying that \(c_i \leq N^i \delta^i, i = 1, \cdots, n\). Note that \(p_A(\lambda_1) = \cdots = p_A(\lambda_N) = 0\). Then we can conclude that \(\max_{1 \leq i,j \leq N} |\lambda_i - 1| \leq 2N\delta \leq \frac{1}{\sqrt{\det(\Lambda)}}\). Clearly, \(\Lambda\) is real matrix. Let \(B = Q\sqrt{\Lambda}Q^T\), where \(\sqrt{\Lambda} = \text{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_N})\) is a diagonal matrix. Then we have \(A = BB^T\), and \(\sqrt{\Lambda} - 1 \leq |\lambda_i - 1|\) implies that \(\max_{1 \leq i,j \leq N} |b_{ij} - \delta_{ij}| \leq 2N\delta\) \(\square\)

Now we are ready to prove the \(C^k\)-estimate of the quotients of Bergman kernel functions.
Proposition 5.4. Let \((M, \omega, \mathcal{L}, h)\) and \(U \cong V_r\) be as in Theorem 1.3. Assume that \(\Gamma_V = 0\) and \(\mathcal{L}_D^{-1}\) is globally generated. Then

\[
\sup_{V_{r2}} \left\| \frac{\rho_{M, \omega, m}(x)}{\rho_{V, \omega, m}(x)} - 1 \right\|_{C^k, \omega_0} \leq Cm^{-1}, \quad \forall m \geq C,
\]

where \(\omega_0 = \pi_D^* \omega_D + \sqrt{-1} \partial \overline{\partial} h_D\) is a Kähler metric on the total space of \(\mathcal{L}_D\), and \(C > 0\) is a constant depending only on \((M, \omega, \mathcal{L}, h), \omega_0, V, k\) and \(l\).

Proof. For each integer \(q \geq 1\), we use \(X_{m,q}\) to denote the linear subspace of \(H^0_{L^2}(V_r, \omega_V, \mathcal{L}_V^{m}, h_V^{r})\) generated by \(\{f_{S_{q,j}} e_{m_V}^{m} \mid q \geq 1, 1 \leq j \leq N_q\}\), where \(\{S_{q,j}\}_{j=1}^{N_q}\) is an \(L^2\) orthonormal basis in the Hilbert space \(H^0_{L^2}(D, \mathcal{L}_D^{m})\), and \(f_{S_{q,j}}\) denote the holomorphic function corresponding to \(S_{q,j}\) defined in Lemma 3.1. Write \(Y_{m,q} = X_{m,q} \cap H^0_{L^2}(V, \omega_V, \mathcal{L}_V^{m}, h_V^{r})\), and \(Z_{m,q} = X_{m,q} \cap H^0_{L^2}(M, \omega, \mathcal{L}^{m}, h^{m})\). Let \(\delta \geq 10\) be a constant that will be determined later. Set \(q_{m,R} = \left\lfloor e^{-4(|R| + |\log |r|| + n + 6)} m \right\rfloor\), for any integer \(m \geq e^{4(|R| + |\log |r|| + n + 6)}\). Then Lemma 5.4 shows that there exists a constant \(C_1 \geq 1\) depending only on \(r \) and \((V, \omega_V, \mathcal{L}_V, h_V)\), such that

\[
\left\| a_{m,q} f_{S_{q,j}} e_{m_V}^{m} \right\|^2_{L^2(V \setminus V_{r2}; h_V^{r}, \omega_V)} \leq C_1 e^{-4R m}, \quad \forall q \leq q_{m,R}, \; 1 \leq j \leq N_q,
\]

where \(a_{m,q} = \sqrt{\frac{m!}{2\pi (m-n-j)!}}\). Let \(\eta \in C^\infty(V)\) such that \(0 \leq \eta \leq 1\), \(|\nabla \eta|_{\omega_V} \leq 10, \eta = 1\) on \(V_{r2}\) and \(\eta = 0\) on \(V \setminus V_r\). Then we can apply Hörmander’s \(L^2\) estimate to show that there are \(\mathcal{L}^m\)-valued smooth functions \(u_{m,q,j}\) on \(M\), such that \(\tilde{\partial} u_{m,q,j} = \overline{\partial} (a_{m,q} \eta f_{S_{q,j}} e_{m_V}^{m})\), and

\[
\int_M \left\| u_{m,q,j} \right\|^2_{m} dV_{\omega} \leq \int_M \left\| \overline{\partial} (a_{m,q} \eta f_{S_{q,j}} e_{m_V}^{m}) \right\|^2_{m, \omega} dV_{\omega} \leq 100C_1 e^{-4Rm},
\]

for any integers \(q \leq q_{m,R}\) and \(1 \leq j \leq N_q\). Write \(\tilde{S}_{m,q,j} = a_{m,q} \eta f_{S_{q,j}} e_{m_V}^{m} - u_{m,q,j}\) for any integers \(q \leq q_{m,R}\) and \(1 \leq j \leq N_q\). Since \(a_{m,q} f_{S_{q,j}} e_{m_V}^{m} \perp X_{m,q,m,n+1}\), \(\forall q \leq q_{m,R}\), we see that the image of \(\tilde{S}_{m,q,j}\) under the orthogonal projection of \(H^0_{L^2}(M, \mathcal{L}^m)\) onto \(Z_{m,q,m,n+1}\), \(\tilde{S}_{m,q,j} Z\), satisfying that

\[
\left\| \tilde{S}_{m,q,j} Z \right\|^2_{L^2(M; h^m, \omega)} \leq \left\| u_{m,q,j} \right\|^2_{L^2(M; h^m, \omega)} + \left\| a_{m,q} \eta f_{S_{q,j}} e_{m_V}^{m} \right\|^2_{L^2(V \setminus V_{r2}; h^m, \omega)} \leq 200C_1 e^{-2Rm},
\]

Set \(\tilde{S}_{m,q,j} = \tilde{S}_{m,q,j} Z\), and \(N_{s, m, R, \lambda} = \sum_{q=1}^{q_{m,R}} N_q\). Then \(v_{m,q,j,q',j'} \equiv \left\{ \tilde{S}_{m,q,j} Z, \tilde{S}_{m,q,j} Z \right\}\) gives an \(N_{s, m, R, \lambda} \times N_{s, m, R, \lambda}\) Hermitian matrix \(Y_m\) such that \(\| v_{m,q,j,q',j} - \delta_{q,q'} \delta_{j,j'} \| \leq 400C_1 e^{-R m}\). Note that \(N_{s, m, R, \lambda} \leq C_2 q^n\) for some constant \(C_2 > 0\) depending only on \((D, \omega_D, \mathcal{L}_D, h_D)\). So we can assume that \(e^{Rm} \geq e^{400(C_1 + C_2) q^n}\) by increasing \(m\) if necessary. Lemma 5.5 now shows that there exists a constant \(C_3 \geq 1\) depending only on \(r\) and \((V, \omega_V, \mathcal{L}_V, h_V)\), such that we can find constants \(b_{m,q,j,q',j'}\) satisfying that \(|b_{m,q,j,q',j'}| \leq C_3 q^n e^{-Rm}\), and

\[
\tilde{S}_{m,q,j} = \tilde{S}_{m,q,j} + \sum_{q'=1}^{q_{m,R}} \sum_{j'=1}^{N_{q'}} b_{m,q,j,q',j'} \tilde{S}_{m,q,j} Z \quad 1 \leq q \leq q_{m,R}, 1 \leq j \leq N_q
\]

forms an orthonormal basis of \(X_{m,q,n+1}\). It follows that

\[
\tilde{S}_{m,q,j} = a_{m,q} f_{S_{q,j}} e_{m_V}^{m} + \sum_{q'=1}^{q_{m,R}} \sum_{j'=1}^{N_{q'}} c_{m,q,j,q',j'} a_{m,q} f_{S_{q',j'}} e_{m_V}^{m} + \tilde{S}_{m,q,j} Z
\]
where \( c_{m,q,j,q',j'} \) are constants satisfying that \( |c_{m,q,j,q',j'}| \leq 200C_3q^{3n}e^{-\rho m} \), and \( \bar{S}^Z_{m,q,j} \in \mathcal{Z}_{m,q,m,R}+1 \) satisfying that \( \left\| \bar{S}^Z_{m,q,j} \right\|_{L^2(M;h^{m,\omega})} \leq 200C_3q^{3n}e^{-\rho m} \).

Fix \( k \in \mathbb{N} \) and \( x \in V_{r,2} \). Then there are vector fields \( \mathcal{Z}_i \) around \( x \) satisfying the estimates in Lemma 5.3. For any finite sequence of integers \( 1 \leq i_1, \ldots, i_p \leq 2n \), we can define a linear functional \( T_{x,m,i_1,\ldots,i_p} \) on \( H^0_{L^2}(V_r,L^m_V) \) by

\[
T_{x,m,i_1,\ldots,i_p}(S) = \mathcal{Z}_{i_1} \cdot \left( \cdots \left( \mathcal{Z}_{i_p} \left( h_D^{-2} e^{-m}_L(S) \right) \right) \cdots \right), \forall S \in H^0_{L^2}(V_r,L^m_V).
\]

By a straightforward calculation, one can obtain

\[
\left\| T_{x,m,i_1,\ldots,i_p} \right\|_{L^2(Y_{m,q,m,R}+1)}^2 = \sum_{q>q_m,R} \sum_{j=1}^{N_q} \left| T_{x,m,i_1,\ldots,i_p} \left( a_{m,q} \mathcal{Z}_{i_j} e^{-m}_L(S) \right) \right|^2 \leq \sum_{q>q_m,R} C_4N_q^2 a_{m,q}^{p+n+1} h_D(x)^{q-p-1} \leq \sum_{q>q_m,R} \frac{nC_2C_4}{2\pi(m-n-1)!} q^{m+p+1} h_D(x)^{q-p-1}.
\]

When \( \log h_D(x) \leq -10 - \frac{m+p+n}{q_m,R} \), (105) implies that

\[
\left\| T_{x,m,i_1,\ldots,i_p} \right\|_{L^2(Y_{m,q,m,R}+1)}^2 \leq \frac{nC_2C_4}{2\pi(m-n-1)!} q^{m+p+n} h_D(x)^{q_m,R} h_D(x)^{q_m,R}.
\]

Note that \( \rho_{\omega,1} > 0 \) on \( D \). Then we can conclude that

\[
\left\| T_{x,m,i_1,\ldots,i_p} \left( a_{m,q} \mathcal{Z}_{i_j} e^{-m}_L(S) \right) \right|^2 \leq C_4e^{km} q^{m,R} \sum_{q=1}^{q_m,R} \sum_{j=1}^{N_q} \left| a_{m,q} \mathcal{Z}_{i_j} e^{-m}_L(S) \right|^2,
\]

for any integers \( q \leq q_m,R \) and \( 1 \leq j \leq N_q \).

By the construction in much the same way as in the proof of Lemma 4.4, we see that there exist a quasi-plurisubharmonic function \( \psi \in L^1(V,\omega_V) \) and a constant \( C_5 \) depending only on \( r \) and \( (V,\omega_V,L^m_V,h_D) \), such that \( \psi \leq 0 \), \( \sqrt{-1} \partial \bar{\partial} \psi \geq -C_5\omega_V \), suppo\( \psi \subset V_{r,2} \), and

\[
\lim_{h_D(y) \to 0} (\psi(y) - \log (\text{dist}_{\omega_V}(x,y))) = \lim_{y \to x} (\psi(y) - \log (h_D(y))) = -\infty.
\]

Assume that \( m-n-2 \geq C_5 (q_m,R+4n+4p) \). Then \( \sqrt{-1} (2n+2k+q_m,R) \partial \bar{\partial} \psi + m\omega + \text{Ric}(\omega) \geq \omega \).

Let \( S_{x,m,Y,i_1,\ldots,i_p} \) be the peak section of \( T_{x,m,i_1,\ldots,i_p} \). Hence we can use Hörmander’s \( L^2 \) estimate to show that there exists a smooth \( \mathcal{L}^m \)-values \( u_{x,m,Y,i_1,\ldots,i_p} \) on \( M \) such that \( \partial u_{x,m,Y,i_1,\ldots,i_p} = \partial (\eta S_{x,m,Y,i_1,\ldots,i_p}) \), and

\[
\int_M \left\| u_{x,m,Y,i_1,\ldots,i_p} \right\|_{h^m}^2 d\text{Vol}_\omega \leq \int_M \left\| \partial (\eta S_{x,m,Y,i_1,\ldots,i_p}) \right\|_{h^{m,\omega}}^2 d\text{Vol}_\omega = \int_M \left\| \partial \eta \otimes S_{x,m,Y,i_1,\ldots,i_p} \right\|_{h^{m,\omega}}^2 d\text{Vol}_\omega \leq 100.
\]
Similarly, we have

\[ T_{x,m,i_1,\ldots,i_p} = S_{x,m,Y,i_1,\ldots,i_p} - u_{x,m,Y,i_1,\ldots,i_p} \]

(109)

It follows that

\[ \left\| T_{x,m,i_1,\ldots,i_p} \right\|_{V,\omega,m}^2 \leq 200 \left\| T_{x,m,i_1,\ldots,i_p} \right\|_{Z_{m,qm},\tilde{\omega}+1}^2. \]

(110)

Similarly, we have

\[ \left\| T_{x,m,i_1,\ldots,i_p} \right\|_{Z_{m,qm},\tilde{\omega}+1}^2 \leq 200 \left\| T_{x,m,i_1,\ldots,i_p} \right\|_{Y_{m,qm},\tilde{\omega}+1}^2. \]

(111)

Let \( \{ \tilde{S}_i \}_{i \in I_{m,\tilde{\omega}}} \) be an \( L^2 \) orthonormal basis of \( Z_{m,\tilde{\omega}} \). Then we have \( \frac{\rho_{M,\omega,m}}{\rho_{V,\omega,V,m}} \)

\( \left( \sum_{q=1}^{\infty} \sum_{j=1}^{N_q} \left\| \tilde{S}_{m,q,j} \right\|_{h_m}^2 \sum_{i \in I_{m,\tilde{\omega}}} \left\| \tilde{S}_i \right\|_{h_m}^2 \right)^{-1} \)

(112)

\[ \left( \sum_{q=1}^{\infty} \sum_{j=1}^{N_q} \left( \rho_{m,q} \tilde{S}_{m,q,j} \right) \right) \left( h_D^{-\frac{1}{2}} \right)^2 \left( \sum_{q=1}^{\infty} \sum_{j=1}^{N_q} \left| a_{m,q} f_{S,q,j} \right| \right) \left( h_D^{-\frac{1}{2}} \right)^2 \]

\[ \left( \sum_{i \in I_{m,\tilde{\omega}}} \left| \tilde{S}_i \right| \right) \left( h_D^{-\frac{1}{2}} \right)^2 \left( \sum_{q=1}^{\infty} \sum_{j=1}^{N_q} \left| a_{m,q} f_{S,q,j} \right| \right) \left( h_D^{-\frac{1}{2}} \right)^2 \]

By Lemma 3.1, there exists a constant \( C_6 \) depending only on \( k, r \) and \( (V, \omega, L_V, h_V) \), such that

\[ \sum_{p=0}^{k} \left\| \nabla^p f(x) \right\|_{\omega_0} \leq C_6 \sum_{p=0}^{k} \sum_{1 \leq i_1, \ldots, i_p \leq 2n} \left| Z_{i_1} \cdots (Z_{i_p}(f)) \cdots \right| (x), \]

(113)
for any smooth function around $x$. As in the proof of Proposition 5.1, we see that the estimates (111), (106) and (107) imply that

\[
\left\| \left( \sum_{q=q_m,\kappa+1}^{N_q} \sum_{j=1}^{N_q} \left| a_{m,q,j} f_{S_{q,j}} h_D^{-\frac{1}{2}} \right|^2 \right) \cdot \left( \sum_{q=1}^{N_q} \sum_{j=1}^{N_q} \left| a_{m,q,j} f_{S_{q,j}} h_D^{-\frac{1}{2}} \right|^2 \right)^{-1} \right\|_{C^k,\omega_0} (x)
\leq C_6 \sum_{p=0}^{k} \sum_{1 \leq i_1, \ldots, i_p \leq 2n} \left| Z_{i_1} \right| \cdots \left| Z_{i_p} \right| \left( \sum_{q=q_m,\kappa+1}^{N_q} \sum_{j=1}^{N_q} \left| a_{m,q,j} f_{S_{q,j}} h_D^{-\frac{1}{2}} \right|^2 \right)^{-1} \right( x \right)
\leq C_7 e^{k(k-1)m} q_{m,\kappa}^{2k(k+n)} \left( \sum_{p=0}^{k} \sum_{1 \leq i_1, \ldots, i_p \leq 2n} \left| T_{x,m,i_1, \ldots, i_p} \right|_{V_{m,q_{m,\kappa}+1}}^2 \right)^{2} \cdot \left( \sum_{q=1}^{N_q} \sum_{j=1}^{N_q} \left| a_{m,q,j} f_{S_{q,j}} f_{L_{V,\kappa}} (x) h_D (x) \right|^2 \right)^{-1} \right)
\leq C_7^2 \left( q_{m,\kappa}^{m+k+n} h_D (x)^{q_{m,\kappa}-k-1} \right)^k,
\]

where $C_7 > 0$ is a constant depending only on $k$, $r$ and $(V, \omega_V, L_V, h_V)$. Similarly, we have

\[
\left\| \left( \sum_{i \in I_{m,\kappa}} e^{-m} \left( S_i \right) h_D^{-\frac{1}{2}} \right)^{-1} \right\|_{C^k,\omega_0} (x)
\leq e^{200kC_7^2} \left( q_{m,\kappa}^{m+k+n} h_D (x)^{q_{m,\kappa}-k-1} \right)^k.
\]

Then we can use (102) to find a constant $C_8 > 0$ depending only on $k$, $r$ and $(V, \omega_V, L_V, h_V)$, such that

\[
\left\| \left( \sum_{q=1}^{N_q} \sum_{j=1}^{N_q} e^{-m} \left( \tilde{S}_{m,q,j} \right) h_D^{-\frac{1}{2}} \right)^{-1} \right\|_{C^k,\omega_0} (x)
\leq C_8 e^{2k^2 m - k m} + C_8 \left( q_{m,\kappa}^{m+k+n} h_D (x)^{q_{m,\kappa}-k-1} \right)^k.
\]

Let $\mathcal{R} > 100 (n + k + C_5)^2$ be a large constant, let $m_1 \geq e^4 (\mathcal{R} + \log |\log r| + |\log r| + 2n + 2k + 6)$ be an integer, and let $\mathcal{C} > m_1^2$ be a constant. Then for any $m \geq m_1$ and $x \in V_{m-c}$, we have

\[
q_{m,\kappa}^{m+k+n} h_D (x)^{q_{m,\kappa}-k-1} \leq e^{-m}.
\]

It follows that

\[
\sup_{V_{m-c}} \left\| \rho_{M,\omega,m} (x) - 1 \right\|_{C^k,\omega_0} \leq C_9 e^{-m}, \quad \forall m \geq m_1,
\]
where $C_9 > 0$ is a constant depending only on $k$, $r$ and $(V, \omega, L_V, h_V)$.

When $x \in V \setminus V_{m-c}$, the estimate (97) is a consequence of Proposition 5.3. This completes the proof. □

Now we consider Theorem 1.4. In order to consider the case where $\Gamma$ is not torsion free, we need to recall some basic notations about orbifolds and orbifold line bundles.

An $n$-dimensional complex orbifold is a composite concept formed of a second countable Hausdorff paracompact topological space $X$ and a collection of data $\{\tilde{U}_x, G_x, U_x, p_x\}_{x \in X}$ with the following property. For each point $x \in X$, there exist an open neighborhood $U_x$ of $x$ in $X$, a connected open subset $\tilde{U}_x \subset \mathbb{C}^n$ and a continuous map $p_x : \tilde{U}_x \to U_x$ together with a finite group $G$ of biholomorphic automorphisms of $\tilde{U}_x$ such that $p_x$ induces an homeomorphism between $U_x$ and $\tilde{U}_x/G$. If $x \in U_y$ for some $y \in X$, then there exist a neighborhood $V_x$ of $x$ and a holomorphic map $\phi : p_x^{-1}(V_x) \to \tilde{U}_y$ such that $p_y = p_x \circ \phi$.

Let $G_x = \{g \in G : g(x) = x\}$. Then $G_x$ is unique up to isomorphism, and the complex manifold $X_{\text{reg}} = \{x \in X : |G_x| = 1\}$ is an open dense subset of $X$. A Kähler metric $\omega$ (resp. line orbibundle $\mathcal{L}$) on $X$ is a Kähler metric $\tilde{\omega}$ (resp. line bundle $\tilde{\mathcal{L}}$) on $X_{\text{reg}}$ such that for each $x \in X$, $p_x^*\omega$ is a restriction of a Kähler metric $\tilde{\omega}$ (resp. line bundle $\tilde{\mathcal{L}}$) on $\tilde{U}_x$ for $p_x^{-1}(U_x \cap X_{\text{reg}})$. Similarly, a hermitian metric $h$ on $\mathcal{L}$ is a hermitian metric $\tilde{h}$ on $\tilde{\mathcal{L}}|_{\tilde{U}_x}$ such that for each $x \in X$, $p_x^*h$ is a restriction of a hermitian metric $\tilde{h}|_{\tilde{U}_x}$ on $\tilde{\mathcal{L}}|_{\tilde{U}_x}$.

A Kähler orbifold is defined in terms of a complex orbifold $X$ and a Kähler metric $\omega$ on $X$. We say that a line orbibundle $\mathcal{L}$ on a complex orbifold $X$ is positive, if there exists a hermitian $h$ on $\mathcal{L}$ such that $\frac{1}{2\pi}\text{Ric}(h)$ is a Kähler metric on $X$, where $\text{Ric}(h) = -\sqrt{-1}\partial\bar{\partial}(\log(h))$. This $\omega = \frac{1}{2\pi}\text{Ric}(h)$ is a polarized Kähler metric on $X$. A global section $S$ of $\mathcal{L}$ on $X$ is a section $S \in H^0(X_{\text{reg}}, \mathcal{L}|_{X_{\text{reg}}})$ that can be extended on each $\tilde{U}_x$. Let $H^0(X, \mathcal{L})$ denote the linear space of all these global sections of $\mathcal{L}$. Then we can define the Bergman kernels on orbifolds just as in the beginning of Section 1.

Then we recall some basic properties of Bergman kernels on orbifolds. Let $(X, \omega)$ be a complete Kähler orbifold, and $(\mathcal{L}, h)$ be a Hermitian line orbibundle on $X$. Fix $x \in X$. Then there exists an open neighborhood $U_x$ of $x$ such that $U_x \cong \tilde{U}_x/G_x$ for some open neighborhood $\tilde{U}_x$ of $0 \in \mathbb{C}^n$. Let $p_x$ denote the quotient map $\tilde{U}_x \to U_x/G_x \cong U_x$. Assume that $p_x(0) = x$, $\tilde{\omega}$ is an extension of $p_x^*\omega |_{U_x \cap X_{\text{reg}}}$ on $U_x$, and $(\tilde{\mathcal{L}}, \tilde{h})$ is an extension of $(p_x^*\mathcal{L} |_{U_x \cap X_{\text{reg}}}, p_x^*h |_{U_x \cap X_{\text{reg}}})$ on $U_x$. Then we see that the action of $G_x$ on $U_x$ can be extended to an action on $\tilde{\mathcal{L}}$ that preserves $\tilde{h}$, and $\tilde{\mathcal{L}}/G_x \cong \mathcal{L}$. Let $H^0_{\tilde{\mathcal{L}}}((\tilde{U}_x, \tilde{\omega}, \tilde{\mathcal{L}}^m, \tilde{h}^m))_{G_x}$ be the subspace of $H^0_{\tilde{\mathcal{L}}}((\tilde{U}_x, \tilde{\omega}, \tilde{\mathcal{L}}^m, \tilde{h}^m))$ containing all holomorphic sections that are invariant under the action of $G_x$. Then there exist an orthonormal basis $\{S_{m,i}\}_{i \in I_m}$ of $H^0_{\tilde{\mathcal{L}}}((\tilde{U}_x, \tilde{\omega}, \tilde{\mathcal{L}}^m, \tilde{h}^m))$ and a subset $I_{m,1} \subset I_m$, such that $\{S_{m,i}\}_{i \in I_{m,1}}$ is an orthonormal basis of $H^0_{\mathcal{L}}((\tilde{U}_x, \tilde{\omega}, \tilde{\mathcal{L}}^m, \tilde{h}^m))_{G_x}$. Then

\begin{equation}
(119) \quad p_{x,\omega,\mathcal{L}^m,h^m}(p_x(z)) = |G_x| \sum_{i \in I_{m,1}} \|S_{m,i}(z)\|_{h^m}^2 = \sum_{g \in G_x} \sum_{i \in I_m} h^m(g(S_{m,i})(z), S_{m,i}(z)),
\end{equation}

and similarly,

\begin{equation}
(120) \quad p_{x}^*B_{U_x,\omega,\mathcal{L}^m,h^m}(p_x(z), p_x(w)) = \sum_{g \in G_x} \sum_{i \in I_m} g(S_{m,i})(z) \odot S_{m,i}(w)^*.
\end{equation}
Assume that there exists a section $e_\mathcal{Z} \in H^0(\overline{U}_x, \overline{\mathcal{L}})$ such that $e_\mathcal{Z} \neq 0$ on $\overline{U}_x$, and $\|e_\mathcal{Z}\|_x$ is invariant under the action of $G_x$. Then there exists a homomorphism $\Theta_G^{orb} : G_x \to U(1) \subset \mathbb{C}^*$ such that $g(e_\mathcal{Z}) = \Theta_G^{orb}(g)e_\mathcal{Z}$, $\forall g \in G_x$. Clearly, $\Theta_G^{orb}$ is independent of the choice of $e_\mathcal{Z}$.

Now we can give an estimate of the Bergman kernel function on $\mathbb{B}^n / \Gamma$ far away from the cusp. The following lemma has actually been proved by Dai-Liu-Ma [7] using the heat kernel method, but here we give a proof that only relies on the peak section method.

**Lemma 5.6.** We follow the notations and assumptions above. For any constants $n, \Lambda, k, \epsilon > 0$, there a constant $\delta, C, m_0 > 0$ with the following property. Let $(X, \omega)$ be a complete Kähler orbifold with $\text{Ric}(\omega) \geq -\Lambda \omega$, $(\mathcal{L}, h)$ be a Hermitian line orbibundle on $X$ with $\text{Ric}(h) \geq \epsilon \omega$, $x \in X$ and $m \geq m_0$. Assume that $G_x \leq U(n)$, and there exists a constant $r \geq \frac{\epsilon \log(m)}{\log m}$ such that the open set $U_x$ containing $B_r(x)$. Suppose that the bisectional curvature $\mathcal{B}K = -1$ on $U_x$, and $\text{Ric}(h) = (n + 1)\omega$ on $U_x$. Then

\[ \rho_{\omega, m} \circ p_x(z) = \frac{(2\pi)^{-n}(mn - 1)!}{(mn - n - 1)!} \sum_{g \in G_x} \Theta_G^{orb}(g)^m (1 - |z|^2)^mn \leq Ce^{-\delta mr^2}, \]

where $\rho_{\omega, m}$ is the Bergman kernel function on $(X, \omega, \mathcal{L}, h)$, and $(z, w) = \sum_{i=1}^n z_iw_i$, $\forall z, w \in \mathbb{C}^n$.

**Proof.** By assumptions, we can consider $\overline{U}_x = p_x^{-1}(U_x)$ as an open neighborhood of $0 \in \mathbb{B}^n$ such that $\overline{U}_x$ is invariant under the action of $G_x \leq U(n)$. Then $(\overline{U}_x, \overline{\omega}, \overline{\mathcal{L}}, \overline{h}) \cong (T_x, \omega_\mathcal{B}^n, K_\mathcal{B}^n, h_\mathcal{B}^n)$, where $(K_\mathcal{B}^n, h_\mathcal{B}^n)$ is the Hermitian line bundle in Theorem 1.4.

Let $e_{K_\mathcal{B}^n} \in H^0(\mathbb{B}^n, K_\mathcal{B}^n)$ such that $\|e_{K_\mathcal{B}^n}\|_{K_\mathcal{B}^n}^2 = (1 - |z|^2)^n$. Then we can describe the action $g : H^0(\mathbb{B}^n, K_\mathcal{B}^n) \to H^0(\mathbb{B}^n, K_\mathcal{B}^n)$ by $g(f_{K_\mathcal{B}^n}) = (\Theta_G^{orb}(g) \cdot f \circ g^{-1})_{K_\mathcal{B}^n}$, for any holomorphic section $f_{K_\mathcal{B}^n} \in H^0(\mathbb{B}^n, K_\mathcal{B}^n)$. Write

\[ S_{m, P} = \sqrt{\frac{(mn + |P| - 1)!}{2^{m+1}m!}} \sum_{P \in \mathbb{Z}_{\geq 0}^m} h^m (g(S_{m, P})(z), S_{m, P}(z)) \]

for each multi-index $P \in \mathbb{Z}_{\geq 0}^m$ and integer $m \geq 2$. Since $\{S_{m, P}\}_{P \in \mathbb{Z}_{\geq 0}^m}$ is an orthonormal basis of $H^0_{\mathcal{L}_x}(\mathbb{B}^n, \omega_\mathcal{B}^n, K_\mathcal{B}^n, h_\mathcal{B}^n)$, we can apply (119) and (120) to show that for any integer $m \geq 2$,

\[
\rho_{\mathbb{B}^n/G_x, \omega_\mathcal{B}^n, K_\mathcal{B}^n} \circ (p_x(z), p_x(w)) \leq \sum_{g \in G_x} \Theta_G^{orb}(g)^m \left( \sum_{P \in \mathbb{Z}_{\geq 0}^m} h^m (f_{m, P}(g^{-1}(z)), f_{m, P}(z)) \right) \left(1 - |z|^2\right)^{mn}
\]

and

\[
P_{\mathbb{B}^n/G_x, \omega_\mathcal{B}^n, K_\mathcal{B}^n} \circ (p_x(z), p_x(w)) \leq \sum_{g \in G_x} \Theta_G^{orb}(g)^m \left(1 - |z|^2\right)^{mn} e_{K_\mathcal{B}^n}^m(z) \otimes e_{K_\mathcal{B}^n}^m(w),
\]
Now we only need to show that
\[ (124) \quad \left\| \rho_{x,\omega,m} \circ p_x(z) - \rho_{\mathbb{B}_n^*/G,\omega_{\mathbb{B}_n}^*}(K_{\mathbb{B}_n}/G_x)^m \circ p_x(z) \right\|_{C^k(B_{\mathbb{B}_n}(0))} \leq C e^{-\delta m r^2}. \]

We can see that the proof of Theorem 1.1 actually leads to the estimate (124), noting that (122) and (123) implies that the integral of peak sections defined by a distribution that supports is a point in $B_{\mathbb{B}_n}(x)$ on $B_{\mathbb{B}_n}(x) \setminus B_{\mathbb{B}_n}(x)$ decays exponentially as $mr^2 \to \infty$.

Now we are in place to prove Theorem 1.4.

**Proof of Theorem 1.4:** Since $\Gamma$ is a lattice, the quotient space $X = \mathbb{B}_n^*/\Gamma$ is a complete Kähler orbifold, and there exist a finite collection of disjoint open subsets $U_i \subset X$, $i = 1, \cdots, N$, and complex hyperbolic cusps $(V_i/\Gamma V_i, \omega V_i, \mathcal{L}_{V_i}/\Gamma V_i, h_{V_i})$ such that $(U_i, \omega_{\mathbb{B}_n}) \cong (V_i/\Gamma V_i, \omega_{V_i})$, and $X \setminus \bigcup_{i=1}^N U_i$ is compact. When $\Gamma$ is cocompact, the orbifold $X$ is compact, and hence $N = 0$.

Note that the localization principle, Theorem 1.1, holds for complete Kähler orbifolds with asymptotic complex hyperbolic cusps. Then we can apply Lemma 5.6 to conclude that
\[ \sup_{X \setminus \bigcup_{i=1}^N U_i} \rho_{X,\omega_{\mathbb{B}_n},K_{X,h_{X,m}}} = c_m m^n + O \left( m^{n-1} \right), \text{ as } m \to \infty. \]

where $(c_i)_{i=1}^\infty$ is a sequence of positive constants depending only on $n$ and $\Gamma$, such that $c_i = c_{i+N_1}$ for some integer $N_1 = N_2(n, \Gamma) > 0$, $\forall i \in \mathbb{N}$.

Similarly, Corollary 23 now implies that there exist an integer $N_2 = N_2(n, \Gamma) > 0$ and a sequence of positive constants $(c_i')_{i=1}^\infty$ depending only on $n$ and $\Gamma$, such that $c_i' = c_{i+N_2}'$, $\forall i \in \mathbb{N}$, and
\[ (126) \quad \sup_{\bigcup_{i=1}^N U_i} \rho_{X,\omega_{\mathbb{B}_n},K_{X,h_{X,m}}} = c'_m m^{n+\frac{1}{2}} + O \left( m^n \right), \text{ as } m \to \infty. \]

Combining (125) with (126), we can obtain (10) and (11). When $\Gamma$ is neat, $X$ becomes a manifold, and the all group $\Gamma_i$ are trivial. It follows that $N = 1$ in this case, which proves the theorem.

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BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, Peking University, Beijing, 100871, China
Email address: zhoushx19@pku.edu.cn