Existence and uniqueness theorems for massless fields on a class of spacetimes with closed timelike curves.

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Abstract

We study the massless scalar field on asymptotically flat spacetimes with closed timelike curves (CTC's), in which all future-directed CTC's traverse one end of a handle (wormhole) and emerge from the other end at an earlier time. For a class of static geometries of this type, and for smooth initial data with all derivatives in $L^2$ on $I^-$, we prove existence of smooth solutions
which are regular at null and spatial infinity (have finite energy and finite $L_2$-norm) and have the given initial data on $I^-$. A restricted uniqueness theorem is obtained, applying to solutions that fall off in time at any fixed spatial position. For a complementary class of spacetimes in which CTC’s are confined to a compact region, we show that when solutions exist they are unique in regions exterior to the CTC’s. (We believe that more stringent uniqueness theorems hold, and that the present limitations are our own.) An extension of these results to Maxwell fields and massless spinor fields is sketched. Finally, we discuss a conjecture that the Cauchy problem for free fields is well defined in the presence of CTC’s whenever the problem is well-posed in the geometric-optics limit. We provide some evidence in support of this conjecture, and we present counterexamples that show that neither existence nor uniqueness is guaranteed under weaker conditions. In particular, both existence and uniqueness can fail in smooth, asymptotically flat spacetimes with a compact nonchronal region.
I. INTRODUCTION

Although spacetimes with closed timelike curves occur as solutions to the vacuum Einstein equations, they have been regarded as unphysical, in part because the most familiar examples have no well-defined initial value problem. Morris and Thorne [1], however, introduced a class of wormhole geometries in which, although there are many closed timelike curves, the set of closed timelike and null geodesics has measure zero. On these spacetimes, Morris et. al. [2] noted that the evolution of free fields is well defined in the limit of geometrical optics; and this in turn makes it seem likely that a multiple scattering series converges to a solution for arbitrary initial data. [2–4] The simplest of the spacetimes they considered are static, with CTC’s present at all times, and we reported an existence and a restricted uniqueness theorem for the massless scalar field on such static time-tunnel spacetimes. [4]

The present paper provides details of these latter results and outlines their extension to spinor and vector fields. A final section discusses the Cauchy problem on spacetimes in which CTCs are confined to a compact region. We prove a uniqueness theorem and present a conjecture on the existence of free fields whose data are specified on a spacelike hypersurface (partial Cauchy surface) to the past of any CTCs.

Until the final section, the geometry $\mathcal{N}$, $g$ we consider is static in the sense that there is a timelike Killing vector $t^\alpha$ that is everywhere locally hypersurface orthogonal. The manifold has topology $\mathcal{N} = M \times \mathbb{R}$, where $M$ is a plane with a handle (wormhole) attached: $M = \mathbb{R}^3 \#(S^2 \times S^1)$. The metric $g_{\alpha\beta}$ on $\mathcal{N}$ is smooth ($C^\infty$), and, for simplicity in treating

1Spacetime indices will be lower case Greek, spatial indices lower case Latin, spinor indices upper case Latin. For those familiar with the abstract index notation, letters near a (or $\alpha$) in the alphabet will be abstract while those after i ($i$) will be concrete, so that $\xi^\mu$ is the $\mu^{th}$ component of the vector $\xi^\alpha$. Our signature is $- + ++$, and Riemann tensor conventions follow. We will use “manifold” as a shorthand for smooth manifold, with or without boundary.
the asymptotic behavior of the fields, we will assume that outside a compact region $\mathcal{R}$ the geometry is flat, with metric $\eta_{\alpha\beta}$.

**II. PRELIMINARIES**

**A. Foliation of a related spacetime with boundary.**

One can construct the 3-manifold $M$ from $\mathbb{R}^3$ by removing two balls and identifying their spherical boundaries, $\Sigma_I$ and $\mathcal{T}(\Sigma_I)$, by a map $\mathcal{T}$, as shown in Fig. 1. In Fig. 1, the map $\mathcal{T}$ involves an improper rotation of $\Sigma_I$, and it yields an orientable handle. Maps $\mathcal{T}$ which involve a proper rotation of $\Sigma_I$ yield non-orientable handles. For the rest of this paper, we will in fact assume that $\mathcal{T}$ is improper and that the handle is consequently orientable. This matters, however, only to our treatment of a two-component Weyl spinor. Were we to allow non-orientable handles as well, everything would be the same, except that we should have to consider 4-component Dirac spinors. The sphere obtained by the identification of $\Sigma_I$ and $\mathcal{T}(\Sigma_I)$ will be called the “throat” of the handle. Its location is arbitrary: after removing any sphere $\Sigma$, from the handle of $M$ one is left with a manifold homeomorphic to $\mathbb{R}^3 \setminus (B^3 \# B^3)$, whose boundary is the disjoint union of two spheres. Let $C \cong S^2 \times \mathbb{R}$ be the history of the throat in the spacetime $\mathcal{N}$, its orbit under the group of time-translations generated by $t^\alpha$. Then, after removing the cylinder $C$ from the spacetime $\mathcal{N}$, one is similarly left with a manifold homeomorphic to $\mathbb{R}^4 \setminus (B^3 \# B^3) \times \mathbb{R}$, whose boundary, $\partial(\mathcal{N} \setminus C)$, is the disjoint union, $C_I \cup C_{II}$, of two timelike cylinders. We will again use the symbol $\mathcal{T}$ to denote the map from $C_I$ to $C_{II}$ that relates identified points; its restriction to a single sphere $\Sigma_I \subset C_I$ is the map denoted above by $\mathcal{T}$.

For the spacetimes we will consider, identified points of $C_I$ and $C_{II} = \mathcal{T}(C_I)$ will be timelike separated, and copies of $M$ in the spacetime cannot be everywhere spacelike. Thus, although $t^\alpha$ is locally hypersurface-orthogonal, no complete hypersurface of $\mathcal{N}$ with a single asymptotically flat region is orthogonal to every trajectory of $t^\alpha$. One can, however, foliate
$\mathcal{N}\backslash\mathcal{C}$ by spacelike hypersurfaces $\mathcal{M}_t$ orthogonal to $t^\alpha$. Each $\mathcal{M}_t$ can be chosen to agree asymptotically with a $t =$ constant surface of the flat metric $\eta_{\alpha\beta}$ and to intersect $C_I$ and $C_{II}$ in isometric spheres $\Sigma_I$ and $\Sigma_{II}$. The sphere $\mathcal{T}(\Sigma_I)$ identified with $\Sigma_I$ is a time-translation of $\Sigma_{II}$, obtained by moving $\Sigma_{II}$ a parameter distance $\tau$ along the trajectories of $t^\alpha$ (see Fig. 2).

The manifold $\mathcal{M}_t$ near $\Sigma_I$ is a smooth extension of $\mathcal{M}_{t+\tau}$ near $\mathcal{T}(\Sigma_I)$. That is, if $U_I \subset \mathcal{M}_t$ and $U_{t+\tau} \subset \mathcal{M}_{t+\tau}$ are spacelike neighborhoods of $\Sigma_I$ and $\mathcal{T}(\Sigma_I)$, their union $U_I \cup U_{t+\tau}$ is a smooth, spacelike submanifold of $\mathcal{N}$. Because the time-translation of $U_{t+\tau}$ to $\mathcal{M}_t$ is a neighborhood $U_{II} \subset \mathcal{M}_t$ of $\Sigma_{II}$ isometric to $U_{t+\tau}$, by artificially identifying the spheres $\Sigma_I$ and $\Sigma_{II}$ of $\mathcal{M}$, one obtains a copy of $\mathcal{M}$ (a plane with a handle) with a metric that is everywhere smooth and spacelike. Of course this spacelike copy of $\mathcal{M}$ is not a submanifold of the spacetime $\mathcal{N}$.

A static metric on $\mathcal{N}$ is given by

$$g_{\alpha\beta} = -e^{-2\nu}t_\alpha t_\beta + h_{\alpha\beta}, \quad (2.1)$$

where $h_{\alpha\beta}t^\beta = 0$. If the Minkowski coordinate $t$ is extended to $\mathcal{N}\backslash\mathcal{C}$ by making $\mathcal{M}_t$ a $t=$constant surface, then $t^\alpha \nabla_\alpha t = 1$, $\nabla^\alpha t = -e^{-2\nu}t^\alpha$, and the metric (2.1) can be written on $\mathcal{N}\backslash\mathcal{C}$ in the form

$$g_{\alpha\beta} = -e^{2\nu}\partial_\alpha t\partial_\beta t + h_{\alpha\beta}. \quad (2.2)$$

It will be convenient to single out a representative hypersurface,

$$\mathcal{M} := \mathcal{M}_0. \quad (2.3)$$

We will denote by $h_{ab}$ the corresponding spatial metric on $\mathcal{M}$; that is, $h_{ab}$ is the pullback of $h_{\alpha\beta}$ (or $g_{\alpha\beta}$) to $\mathcal{M}$.

**B. An initial value problem**

Although $\mathcal{N}$ has no spacelike hypersurface which could play the role of a Cauchy surface, one can pose initial data to the massless scalar wave equation
\[ \square \Phi = 0 \] \hspace{1cm} (2.4)

at past null infinity, \( \mathcal{I}^- \). We will show that for all data in \( H_\infty(\mathcal{I}^-) \), there is a solution \( \Phi \) to Eq. (2.4).

Because the geometry is flat outside a region of fixed spatial size, \( \mathcal{I} \) is a copy of the Minkowski space \( I \). In the null chart \( (v = t + r, \hat{r}) \), \( \mathcal{I}^- \) has coordinates \( (v, \hat{r}) \). In Minkowski space, a smooth solution \( \Phi_0 \) with finite energy to the wave equation (2.4) has as initial data on \( \mathcal{I}^- \) the single function \( \[8\] 

\[ f_0(v, \hat{r}) = \lim_{r \to \infty} r \Phi_0(v, r \hat{r}). \] \hspace{1cm} (2.5)

It is helpful to write the solution \( \Phi_0 \) in terms of its positive frequency part, \( \Psi_0 \):

\[ \Phi_0 = 2 \text{Re} \Psi_0 = (2\pi)^{-3/2} \int d^3k \left[ a(k)e^{i(k \cdot x - \omega t)} + a^*(k)e^{-i(k \cdot x - \omega t)} \right]. \] \hspace{1cm} (2.6)

Then the function,

\[ \Psi_0 = (2\pi)^{-3/2} \int d^3k \; a(k)e^{i(k \cdot x - \omega t)}, \] \hspace{1cm} (2.7)

has initial data on \( \mathcal{I}^- \) given by \[8\]

\[ \lim_{r \to \infty} r \Psi_0(v, r \hat{r}) = \frac{i}{(2\pi)^{3/2}} \int_0^\infty d\omega \; \omega a(-\omega \hat{r})e^{-i\omega v}. \] \hspace{1cm} (2.8)

The corresponding initial data \( f_0(v, \hat{r}) \) for \( \Phi_0 \), has Fourier transform,

\[ \tilde{f}_0(\omega, \hat{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dv f_0(v, \hat{r})e^{i\omega v}, \] \hspace{1cm} (2.9)

related to \( a(k) \) by

\[ \tilde{f}_0(\omega, \hat{r}) = i\omega a(-\omega \hat{r}), \quad \omega \geq 0, \] \hspace{1cm} (2.10)

\[ \tilde{f}_0(-\omega, \hat{r}) = \tilde{f}_0^*(\omega, \hat{r}). \] \hspace{1cm} (2.11)

If we define \( L^2(\mathcal{I}^-) \) by the norm,
\[
||f||^2_{L^2(I^-)} = \int dv d\Omega |f|^2, \quad (2.12)
\]
then the \(L_2\) norm of \(\Psi_0\) on a spacelike hyperplane is equal to the \(L_2\) norm of its initial data on \(I^-\):
\[
\lim_{r \to \infty} \int dv d\Omega |r \Psi_0|^2 = \int dk |a|^2 = \int dV |\Psi_0|^2. \quad (2.13)
\]
The \(L_2\) norm of \(\Phi_0\) depends on hypersurface, but it is bounded by the constant \(L_2\) norm of \(\Psi_0\).

The flux of energy at \(I^-\) is given in terms of the stress tensor,
\[
T_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \Phi \nabla^\gamma \Phi, \quad (2.14)
\]
by
\[
\int_{I^-} d\Sigma_\alpha T^\alpha_\beta t^\beta = \lim_{r \to \infty} \int dv d\Omega \left[ \partial_\nu (r \Phi_0) \right]^2 \quad (2.15)
\]
\[
= \int d\omega d\Omega \omega^2 |\tilde{f}|^2 \quad (2.16)
\]
\[
= \int dk \omega^2 |a(k)|^2. \quad (2.17)
\]

In the spacetime \(\mathcal{N}\) that we are considering, we seek a solution \(\Phi\) to the scalar wave equation,
\[
\nabla_\alpha \nabla^\alpha \Phi = 0, \quad (2.18)
\]
with initial data \(f\) for which \(f\) and its derivatives are in \(L_2(I^-)\). It will again be convenient to relate \(f\) to a function \(a(k)\) as in Eqs. (2.9) - (2.11), in this case defining \(a(k)\) by Eqs. (2.10) and (2.11).

Initial data for vector and spinor fields have a similar character. An electromagnetic field \(F_{\alpha\beta}\) can be written in terms of a vector potential \(A_\alpha\) satisfying the Lorentz gauge condition,
\[
F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha, \quad \nabla_\alpha A^\alpha = 0. \quad (2.19)
\]
The equation \(\nabla_\beta F^\beta_\alpha = 0\), governing a free Maxwell field, is then equivalent to
\[ \nabla_\beta \nabla^\beta A^\alpha - R^\alpha_\beta A^\beta = 0. \]  

(2.20)

For a field with finite energy on Minkowski space, initial data on \( \mathcal{I}^- \) has the form,

\[ \lim_{r \to \infty} r A_\alpha(v, r\hat{r}) = 2\text{Re}\frac{i}{(2\pi)^{\frac{1}{2}}} \int d\omega \omega a_\alpha(\omega \hat{r}) e^{-i\omega v}, \]  

(2.21)

with \( a_\alpha(k) k^\alpha = 0 = a_\alpha t^\alpha \).

We will adopt the 2-component spinor notation given in Penrose and Rindler [6], with \( \nabla_{AA'} = \sigma^\alpha_{AA'} \nabla_\alpha \), where \( \sigma^\alpha_{AA'} \) has components equal to entries of (the usual Pauli spin matrices)/\( \sqrt{2} \). The free-field equation for a massless spinor \( \Phi^A \) is given by

\[ \nabla_{AA'} \Phi^A = 0, \]  

(2.22)

with initial data on \( \mathcal{I}^- \)

\[ \lim_{r \to \infty} r \Phi^A(v, r\hat{r}) = 2\text{Re}\frac{i}{(2\pi)^{\frac{1}{2}}} \int d\omega \omega a^A(\omega \hat{r}) e^{-i\omega v}, \]  

(2.23)

where \( a^A(k) k_{AA'} = 0 \). Because the spacetime \( \mathcal{N} \) is not simply connected, one must specify a choice of spinor structure in order make sense of Eq. (2.22). On the simply connected spacetime \( \mathcal{N}\setminus\mathcal{C} \), the two spinor structures on \( \mathcal{N} \) correspond to a choice of sign in identifying a spinor at a point \( P_I \in \mathcal{C}_I \) with a spinor at the corresponding point \( P_{II} \in \mathcal{C}_{II} \). The choice of spinor structure thus becomes a choice of boundary condition (see Eq. (2.33), below). The two choices give two inequivalent spinor fields on \( \mathcal{N} \).

As in (2.13), the \( L^2 \) norm of the initial data for vector and spinor fields is equal in flat space to the \( L^2 \) norm on a spacelike hyperplane:

\[ \lim_{r \to \infty} \int dv d\Omega |r A_0|^2 = \int dk |a|^2 = \int dV |A|^2, \]  

(2.24)

\[ \lim_{r \to \infty} \int dv d\Omega |r \Phi_0|^2 = \int dk |a|^2 = \int dV |\Phi|^2. \]  

(2.25)

From the form of the energy-momentum tensor for vector and spinor fields,

\[ T_{\alpha\beta} = \frac{1}{4\pi} (F_{\alpha\gamma} F^\gamma_\beta - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}), \]  

(2.26)
\[ T_{\alpha\beta} = i\sigma_\alpha^{A'A'}\sigma_\beta^{BB'}(\Phi(A\nabla_B)_{A'}\Phi_{B'} - \Phi_{(A'\nabla_B')}_{A}\Phi_B), \quad (2.27) \]

the energy flux of the fields \( F_{\alpha\beta} \) and \( \Phi^A \) at \( \mathcal{I}^- \) is

\[ \frac{1}{4\pi} \int_{\mathcal{I}^-} dS_\alpha \, F^{\alpha\gamma} F_{\beta\gamma} \, t^\beta = \frac{1}{4\pi} \int dk \, \omega^2 a_j(k)^* a^j(k), \quad (2.28) \]

and

\[ \int_{\mathcal{I}^-} dS_\alpha \, \sigma^{A'A'} \sigma_\beta^{BB'}(\Phi(A\nabla_B)_{A'}\Phi_{B'} - \Phi_{(A'\nabla_B')}_{A}\Phi_B) \, t^\beta = \int dk \, \omega^2 |a(k)|^2. \quad (2.29) \]

C. Boundary conditions

A scalar field on \( \mathcal{N} \) satisfies at the cylindrical boundaries of \( \mathcal{N} \setminus C \) conditions expressing the continuity of \( \Phi \) and its normal derivative along a path that traverses the wormhole. Let \( P_I \) and \( P_{II} = T(P_I) \) be identified points on the cylinders \( C_I \) and \( C_{II} \). The tangent vector to a path that traverses the wormhole, entering at \( P_I \) and leaving at \( P_{II} \), points inward at \( P_I \) and outward at \( P_{II} \). When the cylinders \( C_I \) and \( C_{II} \) are identified, a unit inward normal to \( C_I \) is thus identified with a unit outward normal to \( C_{II} \). If we denote by \( \hat{n}_I \) and \( \hat{n}_{II} \) the unit outward normals to \( C_I \) and \( C_{II} \), the boundary conditions can be written

\[ \Phi(P_{II}) = \Phi(P_I) \quad (2.30a) \]

\[ \hat{n}_{II} \cdot \nabla \Phi(P_{II}) = -\hat{n}_I \cdot \nabla \Phi(P_I). \quad (2.30b) \]

The analogous conditions for vector and spinor fields can be stated in terms of the differential map \( \mathcal{T} \) induced by \( T \). If \( \{\hat{e}_\mu(P_I)\} = \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{n}_I\} \) is a right-handed orthonormal frame at \( P_I \), then the corresponding right-handed frame at \( P_{II} \) (if \( T \) were a proper rotation, then the corresponding frame would be left-handed) is

\[ \{\hat{e}_\mu(P_{II})\} = \{\mathcal{T}_* \hat{e}_0, \mathcal{T}_* \hat{e}_1, \mathcal{T}_* \hat{e}_2, -\hat{n}_{II}\}. \quad (2.31) \]

The boundary conditions for a vector field can be expressed in terms of its components along the frame \( \{\hat{e}_\mu\} \):
\[ A_\mu(P_{II}) = A_\mu(P_I) \]  

(2.32a)

\[ \hat{n}_{II} \cdot \nabla A_\mu(P_{II}) = -\hat{n}_I \cdot \nabla A_\mu(P_I). \]  

(2.32b)

A spinor field has components along a spinor frame, an element of the double covering \((\simeq SL(2,C))\) of the space of right-handed orthonormal frames \((\simeq SO(3,1))\) at a point. Two spinor frames correspond to the same orthonormal frame, and the choice of spinor structure is the choice of which spinor frame at \(P_{II}\) to identify with a spinor frame at \(P_I\). Let \((o^A, i^A)\) be a field of spinor frames covering a field of frames \(\hat{e}_\mu\) that satisfies Eq. (2.31) on \(\mathcal{N}\setminus C\). We can choose as boundary conditions for the corresponding components \(\Phi^J\) of a spinor field

\[ \Phi^J(P_{II}) = \Phi^J(P_I) \]  

(2.33a)

\[ \hat{n}_{II} \cdot \nabla \Phi^J(P_{II}) = -\hat{n}_I \cdot \nabla \Phi^J(P_I). \]  

(2.33b)

The opposite spinor structure would be selected by changing the sign of the right hand side of these equations or, equivalently, by keeping the same sign but choosing a homotopically different frame field \(\{\hat{e}_\mu\}\).

**D. Eigenfunction expansions**

Because the geometry is static, we can express solutions as a superposition of functions with harmonic time dependence.

\[ \Phi(t, x) = \int d\omega \phi(\omega, x)e^{-i\omega t}, \]  

(2.34a)

\[ A_\alpha(t, x) = \int d\omega A_\alpha(\omega, x)e^{-i\omega t}, \]  

(2.34b)

\[ \Phi^A(t, x) = \int d\omega \phi^A(\omega, x)e^{-i\omega t}. \]  

(2.34c)

Here \(x\) is naturally a point of the manifold of trajectories of \(t^\alpha\), but we can identify it with a point of the simply connected spacelike hypersurface \(\mathcal{M} = \mathcal{M}_0\), with spherical boundaries.
Σ_I and Σ_{II}. Let \((t, x_I)\) and \((t + \tau, x_{II})\) be points of \(\mathcal{N}\setminus C\) that are identified in \(\mathcal{N}\). The harmonic components of fields on \(\mathcal{N}\) can be regarded as fields on \(\mathcal{M}\) satisfying the boundary conditions,

\[
\phi(\omega, x_{II}) = e^{i\eta} \phi(\omega, x_I),
\]

\[
\hat{n}_{II} \cdot \nabla \phi(\omega, x_{II}) = -e^{i\eta} \hat{n}_I \cdot \nabla \phi(\omega, x_I),
\]

\[
A_\mu(\omega, x_{II}) = e^{i\eta} A_\mu(\omega, x_I)
\]

\[
\hat{n}_{II} \cdot \nabla A_\mu(\omega, x_{II}) = -e^{i\eta} \hat{n}_I \cdot \nabla A_\mu(\omega, x_I),
\]

\[
\Phi^J(\omega, x_{II}) = e^{i\eta} \Phi^I(\omega, x_I),
\]

\[
\hat{n}_{II} \cdot \nabla \phi^J(\omega, x_{II}) = -e^{i\eta} \hat{n}_I \cdot \nabla \phi^J(\omega, x_I),
\]

with phase \(\eta = \omega \tau\).

The harmonic components, \(\phi\), of the scalar field satisfy on \(\mathcal{M}\) elliptic equations of the form

\[
(\omega^2 + L)\phi = 0,
\]

with boundary conditions \((2.35)\), where \(L\) can be defined by the action of \(\nabla_\alpha \nabla^\alpha\) on time independent fields \(f\) on \(\mathcal{N}\); that is

\[
L f := e^{2\nu} \nabla_\alpha \nabla^\alpha f|_M,
\]

for fields satisfying \(\mathcal{L}_t f = 0\), where \(\mathcal{L}_t\) is the Lie derivative along \(t^\alpha\). Then

\[
\mathcal{L} = e^\nu D^a e^\nu D_a,
\]
where $D_a$ is the covariant derivative of the 3-metric $h_{ab}$ on $\mathcal{M}$. We will denote by $\mathcal{L}_\eta$ the operator $\mathcal{L}$ with boundary conditions (2.35). The analogous operators for spinor and scalar fields are discussed in Sect. [IIIC]. We show in Lemma 4 below that one can construct solutions $F(\eta, k, x)$, to the wave equation (2.18), having, on the flat geometry outside $\mathcal{R}$, the form of a plane wave plus a purely outgoing wave, corresponding to the scattering of the plane wave by the interior geometry. The existence of solutions for initial data on $\mathcal{I}^-$ then follows if one can show that a spectral decomposition of the form

$$
\Phi(x, t) = \int E(k, x)[e^{-i\omega t} a(k) + e^{i\omega t} a^*(k)]d^3k,
$$

(2.41)

converges, where $E(k, x) = F(\eta = \omega \tau, k, x)$. The major difficulty lies in the fact that, because the boundary conditions (2.30) involve a time-translation, the corresponding boundary conditions (2.35) depend on the frequency $\omega$. If $\eta$ were independent of frequency, the result would follow from the usual spectral theorem for self-adjoint operators. Here, however, $E(k, x)$ and $E(k', x)$ are, for $|k| \neq |k'|$, eigenfunctions of different operators; they are not orthogonal, and their completeness is not guaranteed by the spectral theorem. The main job of Section III is to show that the solution to the scalar wave equation for arbitrary initial data on $\mathcal{I}^-$ can nevertheless be constructed as a spectral integral of the form (2.41).

We adopt throughout the common usage in which the term "eigenfunction" refers not to an element of the domain of $\mathcal{L}_\eta$, but to a function in a weighted $L_2$ space, whose $L_2$ norm diverges (an example is the function $e^{ikx}$ in $\mathbb{R}^3$).

**E. Sobolev spaces**

We shall need some standard properties of Sobolev spaces, including the Sobolev embedding and trace theorems. These can be found in Reed and Simon [7]. Denote by $H_s(N)$ the Sobolev space on a manifold $N$ with volume form $\epsilon$, so that for $s$ a positive integer, $H_s(N)$ is the space of functions on $N$ for which the function and its first $s$ derivatives are square integrable. More generally, for any real $s$ and any chart $y : U \to \mathbb{R}^n$ of $N$, we have:

**Definition.** The Hilbert space $H_s(U)$ is the completion of $C_c^\infty(U)$ in the norm
\[ \|f\|_s = \int_{\mathbb{R}^n} d\xi \left| \hat{f}(\xi) \right|^2 (1 + |\xi|^2)^s, \]  

(2.42)

where \( \hat{f} \) is the Fourier transform on \( \mathbb{R}^n \) of \( f \circ y^{-1} \), regarded as a function on \( \mathbb{R}^n \) with support on \( y(U) \).

We use several spaces of functions based on the Sobolev spaces \( H_s(N) \).

**Definition.** \( H^{\text{loc}}_s(N) \) is the space of functions in \( H_s(U) \) for all compact \( U \subset N \).

For the spacelike 3-manifold \( M \equiv M_0 \) a single chart \( x: M \to \mathbb{R}^3 \) will map the flat metric outside some region \( \mathcal{R} \) to the flat metric of \( \mathbb{R}^3 \), allowing a simple definition of Sobolev spaces and weighted \( L^2 \) spaces on \( M \):

**Definition.** \( H_s(M) \) is the completion of \( C_0^\infty(M) \) in the norm (2.42), with \( \hat{f} \) the Fourier transform of \( f \circ x^{-1} \). \( L^2_{2,r}(M) \) is the completion of \( C_0^\infty(M) \) in the norm

\[ \|f\|_{2,r} = \int_M dV e^{-\nu} |f|^2 (1 + |x|^2)^r. \]  

(2.43)

The lapse function, \( e^{-\nu} \), appearing in the measure is required to make the operator \( \mathcal{L} \) symmetric.

The spaces \( H_s(\mathbb{R}^3) \) and \( L^2_{2,r}(\mathbb{R}^3) \) are defined in the same way, with \( M \) replaced by \( \mathbb{R}^3 \).

A tensor field will be said to be in any of the spaces defined above if its components with respect to the charts \( y \) or \( x \) are in the space. A spinor field will be said to be in any of these spaces if its components in a spinor frame are in the spaces, for a spinor frame associated with a smooth orthonormal frame that has bounded covariant derivatives of all orders.

Finally, it is helpful to define spaces that incorporate the boundary conditions on \( \partial M = \Sigma_I \sqcup \Sigma_{II} \).

**Definition.** \( \mathbb{H}_1 \) is the intersection of \( H_1(M) \) with functions satisfying the boundary condition (2.30a). \( \mathbb{H}_2 \) is the intersection of \( H_2(M) \) with functions satisfying the boundary conditions (2.30a-2.30b).

The Sobolev trace theorem implies that elements of \( H_1(M) \) have well-defined values on \( \partial M \), so \( \mathbb{H}_1 \) and \( \mathbb{H}_2 \) are completions in the \( H_1 \) and \( H_2 \) norms of \( C^\infty \) functions satisfying the boundary conditions specified in the definition.
III. EXISTENCE AND UNIQUENESS THEOREMS

A. Existence theorem for a massless scalar field

Let $R$ be large enough that for $r > R$ the geometry is flat. Define an exterior region $\mathcal{E}$ by $\mathcal{E} = \{ p \in \mathcal{N} | r(p) > R \}$ and an interior region $\mathcal{R} = \mathcal{N} \setminus \mathcal{E}$. It will be helpful in what follows to introduce a smoothed step function, $\chi$, that vanishes on $\mathcal{R}$:

$$\chi(p) = \begin{cases} 0, & \text{if } p \in \mathcal{R} \\ 1, & \text{if } r(p) \geq R + \epsilon, \end{cases}$$

for some $\epsilon > 0$.

**Definition.** A scalar field $\Phi$ on $\mathcal{N}$ is asymptotically regular at spatial infinity if $\Phi \circ \chi \in H_1(\mathcal{M})$; it is asymptotically regular at null infinity if the limits,

$$f(v, \hat{r}) = \lim_{r \to \infty} r\Phi(v, r\hat{r}),$$
$$g(u, \hat{r}) = \lim_{r \to \infty} r\Phi(u, r\hat{r}),$$

exist, with $f \in H_1(\mathcal{I}^-)$, $g \in H_1(\mathcal{I}^+)$, where $u$ is the null coordinate $t - r$. That is, $\Phi$ is asymptotically regular if it is an $L_2$-function with finite energy on $\mathcal{N}$ and has well-defined data with finite energy on $\mathcal{I}^-$ and $\mathcal{I}^+$.

Our result on existence of solutions to the scalar wave equation has the following form. **Proposition 1.** For almost all spacetimes $\mathcal{N}, g$ of the kind described in Sec. I A (for almost all parameters $\tau$), the following existence theorem holds. Let $f$ be initial data on $\mathcal{I}^-$ for which $f$ and all its derivatives are in $L_2(\mathcal{I}^-)$. Then there exists a solution $\Phi$ to the scalar wave equation which is smooth and asymptotically regular at null and spatial infinity and which has $f$ as initial data.

The proof is given as a sequence of lemmas. For boundary conditions (2.35) specified by a fixed phase, $\eta$, we show that the operators $\mathcal{L}_\eta$ are self-adjoint on a dense subspace of $L_2$. We follow a method given by Wilcox [8] to obtain explicit eigenfunctions $F(\eta, k, x)$ in a weighted $L_2$ space. For fixed boundary phase $\eta$ the eigenfunctions are complete and
orthonormal, but, as mentioned earlier, a solution $\Phi$ of Eq. (2.18) is a superposition of eigenfunctions,

$$E(k, x) = F(\eta = \omega \tau, k, x),$$

that are not orthonormal. For each $\omega$, the function $E(k, x)$ is an eigenfunction of a different operator $L_\eta$ because the boundary condition depends on $\omega$ through the relation $\eta = \omega \tau$.

Lemma 1. The operator $L_\eta$ with boundary conditions (2.35) is self-adjoint on the space $L_2(\mathcal{M})$ with domain $\mathcal{H}_2$.

Proof. Recall that for an operator $A$ with domain $D(A) \subset L_2(\mathcal{M})$, a function $f \in L_2(\mathcal{M})$ is in $D(A^\dagger)$ if and only if $\exists h \in L_2(\mathcal{M})$ such that $\langle f | Ag \rangle = \langle h | g \rangle \forall g \in D(A)$. To prove that $\mathcal{H}_2 \subset D(L_\eta^\dagger)$ is easy: for $f \in \mathcal{H}_2$, $L_\eta f \in L_2(\mathcal{M})$. Take $h = L_\eta f$. Then, writing $\Sigma := \Sigma_I \sqcup \Sigma_{II}$, we have

$$\langle f | L_\eta g \rangle = \langle L_\eta f | g \rangle + \int_{\Sigma_I \sqcup \Sigma_{II}} dS_a e^{-\nu} (\overline{f e^{i\eta}}) \hat{D} \cdot g e^{i\eta}. \quad (3.4)$$

Because $f$ and $g$ satisfy the boundary conditions (2.35), we have

$$\int_{\Sigma_{II}} dS_a e^{-\nu} \hat{D} \cdot g = -\int_{\Sigma_I} dS_a e^{-\nu} (\overline{f e^{i\eta}}) \hat{D} \cdot g e^{i\eta} = -\int_{\Sigma_I} dS_a e^{-\nu} \hat{f} \hat{D} \cdot g. \quad (3.5)$$

Thus $L_\eta$ is symmetric,

$$\langle f | L_\eta g \rangle = \langle L_\eta f | g \rangle = \langle h | g \rangle,$$

and $\mathcal{H}_2 \subset D(L_\eta^\dagger)$.

To prove that $D(L_\eta^\dagger) \subset \mathcal{H}_2$, we proceed as follows. For $f \in D(L_\eta^\dagger)$, there is an $h \in L_2(\mathcal{M})$ with

$$\langle f | L_\eta g \rangle = \langle h | g \rangle \forall g \in D(L_\eta). \quad (3.7)$$

That is, the equation $L_\eta^\dagger f = h$ is satisfied weakly on $\mathcal{M}$. Then, by elliptic regularity, we have $f \in H_2^{\text{loc}}(\mathcal{M})$ and $L_\eta f = h$ is satisfied strongly on $\mathcal{M}$.

Asymptotic behavior of $f$ follows from the fact that the exterior region $\mathcal{E}$ can be regarded as a subspace of Euclidean 3-space, $E^3$. Let $\chi$ be the smooth step function of Eq. (3.1), and
let $\tilde{f} = f\chi$. Then $\tilde{f}$ is a function on $E^3$ satisfying $\nabla^2 \tilde{f} = \tilde{h}$, with $h \in L_2(E^3)$. Because $\nabla^2$ is self-adjoint on $E^3$ with domain $H_2(E^3)$, we have $\tilde{f} \in H_2(E^3)$, implying $f_{E} \in H_2(E)$. 

To check that $f$ satisfies the boundary conditions, we use the fact, noted in Sect. I (before Eq. (2.1)), that by identifying $\Sigma_I$ and $\Sigma_{II}$, one obtains a spacelike copy $\tilde{M}$ of $M$ with a smooth spatial metric, $\tilde{g}_{ab}$. ($\tilde{M}$ is an artificially constructed spacelike surface with a handle; it is not a submanifold of our spacetime, $\mathcal{N}$.) The operator $\tilde{L}$ constructed from $\tilde{g}_{ab}$ is smooth and elliptic. Because the projection, $p : \mathcal{M}_t \to M$, is an isometry there, it agrees with $L$ on the interior of $\mathcal{M}_t$. Let $U$ be a neighborhood of $\Sigma$ for which $p^{-1}(U) = U_I \cup U_{II}$, with $U_I$ and $U_{II}$ disjoint neighborhoods of $\Sigma_I$ and $\Sigma_{II}$. If, as expected, the function $f \circ p^{-1}$ has a phase that changes by $e^{i\eta}$ across $\Sigma$, then we should get a smooth function $\hat{f}$ on $U$ by requiring

$$
\hat{f}(P) = \begin{cases} 
  f \circ p^{-1}, & P \in p(U_I) \\
  e^{-i\eta}f \circ p^{-1}, & P \in p(U_{II}) 
\end{cases}
$$

(3.8)

To see that this is true, first smoothly truncate $f$ so that it vanishes outside of $U_I$ and $U_{II}$, and then define $\hat{f}$ as above. Let $g$ of (3.4) similarly have support on $U_I \cup U_{II}$, and let $h$ be the function in $L_2(M)$ given by (3.8). Define $\hat{g}$ and $\hat{h}$ on $U_I \cup U_{II}$ according to (3.8), with $f$ replaced by $g$ and $h$, respectively. Then $g \in H_2(M) \Rightarrow \hat{g} \in H_2(\hat{M})$, and we have

$$
< \hat{f} | L_\eta \hat{g} > = < \hat{h} | \hat{g} >,
$$

(3.9)

$\forall \hat{g} \in H_2(\hat{M})$ with support on $U_I \cup U_{II}$, implying (again by elliptic regularity) that $\hat{f}$ is smooth across $\Sigma$. Thus any $f \in D(L_\eta^\dagger)$ is a function in $H_2$ satisfying the boundary conditions (2.35), meaning $D(L_\eta^\dagger) \subset \mathcal{H}_2$. $\Box$

We now find a set of eigenfunctions that are complete for data on $\mathcal{I}^-$. We consider solutions $F(\eta, k, x)$ to Eq. (2.38) which, for $r > R$, have the form

$$
F = (2\pi)^{-3/2} e^{ik \cdot x} + \text{outgoing waves}.
$$

(3.10)

To prove existence of the eigenfunctions $F$, one first rewrites Eq. (2.38) in the inhomogeneous form
\[ (\omega^2 + \mathcal{L}_\eta)\varphi = \rho \quad (3.11) \]

where \( \varphi \) is purely outgoing and \( \rho \) has compact support. One can do this by using the steplike function \( \chi(r) \) of Eq. (3.1), writing

\[ F = \chi(2\pi)^{-3/2}e^{ik\cdot x} + F_{\text{out}}. \quad (3.12) \]

Then

\[ (\omega^2 + \mathcal{L}_\eta)F = (\omega^2 + \mathcal{L}_\eta)F_{\text{out}} - \rho, \quad (3.13) \]

with

\[ \rho = -(2\pi)^{-3/2}e^{ik\cdot x}((\nabla^\beta \nabla_\beta - 2ik^\beta \nabla_\beta)\chi. \quad (3.14) \]

The homogeneous equation, \((\omega^2 + \mathcal{L}_\eta)F = 0\), is equivalent to the inhomogeneous equation (3.11) with \( \varphi = F_{\text{out}} \).

**Lemma 2.** Let \( \lambda_k \) be a sequence of complex numbers with positive imaginary part, such that \( \lambda_k \to \omega^2 \). Consider families \( \{\varphi_k, \rho_k\} \) of smooth fields on \( \mathcal{M} \), where for each \( k \), the field \( \rho_k \) has support on the region \( r \leq R + \epsilon \), and \( \varphi_k \) is the unique asymptotically regular solution to Eq. (3.11) with \( \omega^2 \) replaced by \( \lambda_k \). If \( \rho_k \to \rho \) in \( L^2(\mathcal{M}) \), then a subsequence \( \{\varphi_m\} \) converges in an \( H_1 \) norm to a smooth outgoing solution \( \varphi \) to Eq. (3.11).

**Proof.** Let \( D \) be a real number greater than \( R + \epsilon \), \( \mathcal{M}_D = \{x \in \mathcal{M}, \ r \leq D\} \). Denote by \( || \ |_{s,D} \) the norm of \( H^s(\mathcal{M}_D) \).

Suppose first that \( ||\varphi_k||_{2,D} \) has a bound \( C \) independent of \( k \). Since \( H^2(\mathcal{M}_D) \hookrightarrow H^1(\mathcal{M}_D) \) is a compact embedding (by the Sobolev embedding theorem) the set \( \{\varphi_k\} \) belongs to a compact set of \( H^1(\mathcal{M}_D) \). Thus there is a subsequence \( \{\varphi_m\} \) that converges in \( H^1(\mathcal{M}_D) \):

\[ \varphi_m \to \varphi. \quad (3.15) \]

We must show that \( \varphi \) satisfies (i) \((\omega^2 + \mathcal{L})\varphi = \rho \) on \( \mathcal{M}_D \), (ii) the boundary conditions (2.35), and (iii) we must extend \( \varphi \) outside \( \mathcal{M}_D \).

(i): To see that \( \varphi \) weakly satisfies (3.11), define \( \forall \psi \in C^\infty_0(\mathcal{M}_D) \)
\[ Q \equiv -\langle D_a(e^\nu \psi)|D^a(e^\nu \varphi) \rangle + \omega^2 \langle \psi|\varphi \rangle - \langle \psi|\rho \rangle. \]  

(3.16)

Add to this \( Q \) to the quantity \( \langle D_a(e^\nu \psi)|D^a(e^\nu \varphi_m) \rangle - \lambda_m \langle \psi|\varphi_m \rangle + \langle \psi|\rho_m \rangle = 0. \) Then

\[
|Q| = | - \langle D_a(e^\nu \psi)|D^a(e^\nu \varphi) \rangle + \omega^2 \langle \psi|\varphi_m \rangle - \langle \psi|\rho \rangle \\
+ \langle D_a(e^\nu \psi)|D^a(e^\nu \varphi_m) \rangle - \lambda_m \langle \psi|\varphi_m \rangle + \langle \psi|\rho_m \rangle | \\
\leq |\langle D_a(e^\nu \psi)|D^a[e^\nu (\varphi - \varphi_m)] \rangle| + \omega^2 |\langle \psi|\varphi - \varphi_m \rangle| + \\
|\langle \omega^2 - \lambda_m \rangle \langle \psi|\varphi_m \rangle| + |\langle \psi|\rho - \rho_m \rangle|. 
\]  

(3.17)

The limit of the right hand side of (3.17) is zero as \( m \to \infty \):

\[
\lim_{m\to\infty} \| \varphi - \varphi_m \|_{1, R} = 0, 
\]

(3.18a)

\[
\lim_{m\to\infty} \| \rho - \rho_m \|_0 = 0, 
\]

(3.18b)

\[
\lim_{m\to\infty} |\omega^2 - \lambda_m | = 0. 
\]

(3.18c)

Then \( Q \) independent of \( m \) implies \( Q = 0 \). Hence \( \varphi \) is a weak solution to \( (\omega^2 + \mathcal{L}) \varphi = \rho \) in \( \mathcal{M}_D \), and elliptic regularity implies that it is a strong solution.

(ii): Next define a neighbourhood \( U \) on \( \hat{M} \), as in the proof of Lemma 1. And define \( \hat{\varphi}_k \) on \( U \) from \( \varphi_k \) on \( \mathcal{M}_D \) in exactly the same way that \( \hat{f} \) was defined in Lemma 1 from \( f \). Then, because \( \varphi_k \) is smooth and satisfies the boundary conditions (2.35) on \( \mathcal{M}_D \), we have that \( \hat{\varphi}_k \) is smooth on \( U \). Also, because \( \rho_k \) has support only outside of \( U \) (away from the boundaries), \( \hat{\varphi}_k \) satisfies \( (\lambda_k + \hat{\mathcal{L}}) \hat{\varphi}_k = 0 \) on \( U \). Now, a subsequence \{\( \hat{\varphi}_m \)\} converges in \( H_1(U) \) to a solution \( \hat{\varphi} \), which is continuous. But, by elliptic regularity \( \hat{\varphi} \) is smooth in \( U \) and therefore \( \varphi \) satisfies both boundary conditions (2.35) \( \forall \rho \in C^\infty(\mathcal{M}_D) \).

(iii): For \( r > R + \epsilon \), we have \( \rho_m = 0 \). Because the space is flat outside \( r = R \), and \( \varphi_m \) is asymptotically regular, we can use the explicit Green function for \( \nabla^2_{\text{flat}} + \omega^2 \) to write

\[
\varphi_m(x) = \int_{|y| = R} dS_y \varphi_m(y) \leftrightarrow e^{i\sqrt{\lambda_m}|x-y|} \frac{\partial_y}{|x-y|}, 
\]

(3.19)

where \( \text{Im} \sqrt{\lambda_m} > 0 \). Then for \( R + \epsilon < r < D \),
\[
\varphi(x) = \lim_{m \to \infty} \varphi_m(x) = \int_{|y|=R} dS_y \varphi(y) \stackrel{y \to x}{\to} e^{i\omega|x-y|}.
\] (3.20)

an outgoing wave. Defining \( \varphi \) by Eq. (3.20) for \( r > D \) we obtain an outgoing \( C^\infty \) solution to (3.11) as claimed.

The construction has so far relied on the assumption that \( \| \varphi_k \|_{2,D} \), was bounded. If not, the sequence \( \tilde{\varphi}_k = \varphi_k / \| \varphi_k \|_{2,D} \) has unit norm and a source \( \tilde{\rho}_k = \rho_k / \| \varphi_k \|_{2,D} \) whose norm converges to zero:

\[
\lim_{k \to \infty} \| \tilde{\rho}_k \| = 0.
\] (3.21)

This leads to a contradiction. From the previous paragraph, there is a subsequence \( \tilde{\phi}_m \) converging to an outgoing solution \( \tilde{\varphi} \) of \( L \eta \tilde{\varphi} = 0 \). But Rellich’s uniqueness theorem (Lemma 3 below) implies \( \tilde{\varphi} = 0 \), whence \( \lim_{m \to \infty} \| \varphi_m \|_{0,D} = 0 \). From \( L \tilde{\varphi}_m = \tilde{\rho}_m \), we have \( \| \tilde{\varphi}_m \|_{2,D} \leq C \| \tilde{\varphi}_m \|_{0,D} + \| \tilde{\rho}_m \|_0 \). Then \( \lim_{m \to \infty} \| \tilde{\varphi}_m \|_{2,D} = 0 \) contradicting \( \| \tilde{\varphi}_m \|_{2,D} = 1 \).

**Lemma 3.** (Rellich’s uniqueness theorem.) Let \( \varphi \in H^2 \) be outgoing at spatial infinity and satisfy \((L \eta + \omega^2) \varphi = 0\). Then \( \varphi = 0 \).

**Proof.** In the exterior region \( E \), \( \varphi \) is a smooth solution to the flat space equation \((\omega^2 + \nabla^2) \varphi = 0\) and can therefore be written as a sum

\[
\sum \alpha_{lm} h_l^{(1)}(\omega r \Omega) + \beta_{lm} h_l^{(2)}(\omega r \Omega),
\] (3.22)

converging in \( L^2_{loc} \). Here \( h_l^{(1)} \) and \( h_l^{(2)} = h_l^{(1)*} \) are spherical Hankel functions, satisfying the Wronskian relation,

\[
h_l^{(1)}(y) \partial_y h_l^{(2)}(y) - h_l^{(2)}(y) \partial_y h_l^{(1)}(y) = \frac{2}{iy^2}
\] (3.23)

Using this relation and orthonormality of the spherical harmonics \( Y_{lm} \), we have,

\[
0 = \int_{\mathcal{R}} dV \varphi^*(\omega^2 + L) \varphi - \int_{\mathcal{R}} dV (\omega^2 + L) \varphi^* \varphi = \int_{r=R} dS \varphi^* \frac{\partial}{\partial r} \varphi
\]

\[
= \sum_{lm} \left[ |\alpha_{lm}|^2 h_l^{(2)} \frac{\partial}{\partial r} h_l^{(1)} + |\beta_{lm}|^2 h_l^{(1)} \frac{\partial}{\partial r} h_l^{(2)} \right] R^2
\]

\[
= \frac{2i}{\omega} \sum_{lm} |\alpha_{lm}|^2 - |\beta_{lm}|^2
\] (3.24)

\[
\implies \sum |\alpha_{lm}|^2 = \sum |\beta_{lm}|^2.
\] (3.25)
In other words, ingoing and outgoing fluxes are equal. Then \( \varphi \) outgoing implies \( \alpha_{lm} = 0 \implies \beta_{lm} = 0 \implies \varphi = 0 \) outside \( \mathcal{R} \). Aronszjan’s elliptic continuation theorem [9] then implies \( \varphi = 0 \) everywhere on \( \mathcal{M} \). \( \square \)

**Lemma 4.** There is a unique solution, \( F(\eta, k, x) \), to the equation \( (\mathcal{L}_\eta + \omega^2)F = 0 \), for which

\[
F = (2\pi)^{-3/2}e^{ik \cdot x} + \text{outgoing waves.} \tag{3.26}
\]

The map

\[
L_2(\mathcal{M}) \rightarrow L_2(\mathbb{R}^3), \tag{3.27}
\]

given by

\[
f(x) \mapsto \hat{f}(k) = \int dk F(\eta, k, x)f(x), \tag{3.28}
\]

is unitary.

**Proof.** An immediate consequence of Lemmas 2 and 3 is that there exists a unique outgoing smooth solution \( \varphi \) to (3.11). Then Eq. (3.12), relating \( F \) to \( F_{\text{out}} \), gives us existence and uniqueness of a smooth solution \( F \) of the claimed form. Unitarity is implied by the self-adjointness of \( \mathcal{L}_\eta \) for fixed \( \eta \) and the fact that \( \int dk \) is a spectral measure. A detailed proof of unitarity, applicable with essentially trivial changes to our case is given in Chap. 6 of Wilcox [8]. (For example, the “generalized Neumann condition,” \( \int d^3x [f \nabla^2 g + \nabla f \cdot \nabla g] = 0 \), is replaced by \( \int dV [f e^\nu D_a (e^\nu D^a g) + D_a (e^\nu f) D^a (e^\nu g)] = 0 \).) \( \square \)

**Lemma 5.** For almost all \( \tau \) the following holds: Let \( a(k) \in L_{2,n}(\mathbb{R}^3) \), and let \( \psi(x) = \int dk F(\eta = \omega \tau, k, x)a(k) \).

Then \( \psi(x) \in H_{n-3/2-\epsilon}(\mathcal{M}_D) \), all \( \epsilon > 0 \).

**Proof.**

Fourier transform \( F(\eta, k, x) \), truncating smoothly at \( x = R \): With \( \chi \) the smoothed step function of Eq. (3.1), let \( g_y \in L_2(\mathcal{M}) \) be given by
\( g_y(x) := (2\pi)^{3/2} e^{ixy} [1 - \chi(x)] \), \( (3.29) \)

with norm

\[
\|g_y\|_{L^2(M)} = CR^3, \tag{3.30}
\]

some \( C \) independent of \( y \). From the fact that \( F \), regarded as a map from \( L^2(M) \) to \( L^2(\mathbb{R}^3) \) is norm-preserving, the function

\[
\hat{F}(\eta, k, y) := \int dVe^{-\nu} F(\eta, k, x) g_y(x) \tag{3.31}
\]

has norm in \( k \)-space

\[
\|\hat{F}(\eta, \cdot, y)\|_{L^2(\mathbb{R}^3)} = CR^{3/2}, \quad \forall \eta, y. \tag{3.32}
\]

In order to bound the integral of the Lemma, we will bound a norm of \( \hat{F}(\omega\tau, k, y) \) in \( \tau - k \) space, for any finite interval \( I = [\tau_0, \tau_1] \) of \( \tau \). It will be convenient to take \( \tau_1 = m\tau_0 \), for some integer \( m \). Writing

\[
Q(\eta, k, y) = \frac{|\omega^n \hat{F}(\eta, k, y)|^2}{(1 + \omega^2)^n}, \tag{3.33}
\]

we have

\[
\int_0^\infty d\omega \int_I d\tau Q(\omega\tau, k, y) = \int_0^\infty d\omega \int_{\omega\tau_0}^{\omega\tau_1} d\eta \omega^{-1} Q(\eta, k, y) \\
\leq \sum_{j=0}^\infty \int_0^{2\pi} d\eta \int_{2\pi j/\tau_0}^{2\pi (j+1)/\tau_0} d\omega \ (j + 1) \frac{\tau_1}{\tau_0} \omega^{-1} Q \\
\leq \int_0^{2\pi} d\eta \int_0^\infty d\omega \ (\frac{\tau_1}{\tau_0} + \frac{\omega \tau_1}{2\pi}) \omega^{-1} Q. \tag{3.34}
\]

In the first inequality, we have used the fact that \( Q \) is periodic in \( \eta \).

From this relation, we obtain

\[
\int dk \int_I d\tau Q(\omega\tau, k, y) = \left[ \int_{\omega < 2\pi/\tau_0} dk + \int_{\omega > 2\pi/\tau_0} dk \right] \int_I d\tau Q \\
\leq C' + \int_{\omega > 2\pi/\tau_0} dk \int_0^{2\pi} d\eta \left( \frac{\tau_1}{\tau_0} + \frac{\omega \tau_1}{2\pi} \right) \omega^{-1} Q(\eta, k; y) \\
\leq C' + \frac{\tau_1}{\pi} \int_{\omega > 2\pi/\tau_0} dk \int_0^{2\pi} d\eta Q(\eta, k; y) \\
\leq C' + C'' R^3, \tag{3.35}
\]
where the last inequality follows from Eq. (3.32).

Integrating over $y$, we have,

$$\int d\tau dk dy \frac{\omega^2 |\hat{F}(\omega \tau, k; y)|^2}{(1 + \omega^2)^n (1 + y^2)^{3/2+\epsilon}} < \infty$$  \hspace{1cm} (3.36)

$$\implies \omega^n \hat{F}(\omega \tau, k; y) \in L_2(I) \otimes L_{2,-n}(\mathbb{R}^3) \otimes L_{-3/2-\epsilon}(\mathbb{R}^3)$$

$$\implies \nabla^n F(\omega \tau, k, x) \in L_2(I) \otimes L_{2,-n}(\mathbb{R}^3) \otimes H_{-3/2-\epsilon}(\mathcal{M}_D)$$

$$\implies F(\omega \tau, k, x) \in L_2(I) \otimes L_{2,-n}(\mathbb{R}^3) \otimes H_{n-3/2-\epsilon}(\mathcal{M}_D).$$  \hspace{1cm} (3.37)

Thus, for almost all $\tau$,

$$F(\omega \tau, k, x) \in L_{2,-n}(\mathbb{R}^3) \otimes H_{n-3/2-\epsilon}(\mathcal{M}_D)$$

$$\implies \int dk a(k) F(\omega \tau, k, x) \in H_{n-3/2-\epsilon}(\mathcal{M}_D)$$  \hspace{1cm} (3.38)

for $a(k) \in L_{2,n}(\mathbb{R}^3)$. \hfill \Box

Lemma 6. Let $\Psi_{\text{out}}$ be the outgoing field in Minkowski space of a smooth, spatially bounded source, $\rho$. Then data for $\Psi_{\text{out}}$ on $\mathcal{I}^+$ is well-defined and smooth.

Proof. The lemma follows from the explicit form of the flat space retarded solution,

$$\Psi_{\text{out}}(t = u + r, \vec{x}) = \int d^3 y \frac{\rho(t = u + r - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|},$$  \hspace{1cm} (3.39)

where $r = |\vec{x}|$. Writing $|\vec{x} - \vec{y}| = r - \hat{x} \cdot \vec{y} + O(|\vec{y}/\vec{x}|)$, we have

$$|\rho(t = u + r - |\vec{x} - \vec{y}|, \vec{y}) - \rho(t = u + \hat{x} \cdot \vec{y}, \vec{y})| < K|\vec{y}/\vec{x}| \max |\dot{\rho}|.$$  \hspace{1cm} (3.40)

This bound holds for all values of $r$ along the null ray at constant $u, \theta, \phi$, with $\max |\dot{\rho}|$ the maximum value of $|\dot{\rho}|$ on the (compact) intersection of the support of $\rho$ with the past light cones from points of the null ray. Then data on $\mathcal{I}^+$ takes the explicit form,

$$\lim_{r \to \infty} r\Psi_{\text{out}}(t = u + r, \vec{x}) = \int d^3 y \rho(t = u + \hat{x} \cdot \vec{y}, \vec{y}).$$  \hspace{1cm} (3.41)
Lemma 7. Let $\tau$ be such that the conclusion to Lemma 5 holds. Let $a(k) \in L_{2,n}(\mathbb{R}^3)$, all $n \in \mathbb{Z}$, and let

$$
\Psi(t,x) = \int dk a(k) F(\omega \tau, k, x) e^{-i\omega t}.
$$

(3.42)

Then $\Psi$ is smooth on $\mathcal{N}$, and has data on $\mathcal{I}^-$ given by Eq. (2.8),

$$
\lim_{r \to \infty} r \Psi(v, r \hat{r}) = \frac{i}{(2\pi)^{1/2}} \int_0^\infty d\omega \omega a(-\omega \hat{r}) e^{i\omega v}.
$$

(3.43)

That is, if $\Psi$ has for each harmonic the same ingoing part as does a solution $\Psi_0$ in Minkowski space, then $\Psi$ has the same data on $\mathcal{I}^-$ that $\Psi_0$ has.

Proof. The smoothness of $a(k)$ implies, by Lemma 5, that the integral of Eq. (3.42) defines $\Psi(0, \cdot) \in H_n(\mathcal{M}_D) \ \forall n$. Since $a(k) e^{-i\omega t} \in L_2(\mathbb{R}^3)$, we have $\Psi(t, \cdot) \in H_n(\mathcal{M}_{t,D}) \ (\forall n, t)$, and $\Psi(t, \cdot)$ is smooth on each surface $\mathcal{M}_t$, because it is smooth on $\mathcal{M}_{t,D}$ for all $D$. To see that $\Psi$ is smooth on $\mathcal{N}$, note first that it is smooth on the throat $\Sigma$ that joints $\mathcal{M}_t$ to its extension $\mathcal{M}_t + \tau$. This follows from the fact that the location of the cylinder $C$ removed from $\mathcal{N}$ is arbitrary: if, instead of removing $C$ from $\mathcal{N}$, one removes a different, cylinder $C'$, disjoint from $C$, one obtains the same $\Psi$, because the eigenfunctions on $M$ from which $\Psi$ is constructed are unique by Lemma 3. Thus $\Psi(t, \cdot)$ is smooth on each $\mathcal{M}'_t$ and, in particular, on the throat $\Sigma$.

Because $\omega a(k)$ is similarly in $L_{2,n}(\mathbb{R}^3) \ \forall n$, $\partial_t \Psi$ is smooth on each $\mathcal{M}_t$ and on the throat. Finally, $\Psi$ is smooth on $\mathcal{N}$, because $\mathcal{N}$ is covered by globally hyperbolic subspacetimes $(U, g|_U)$ which have as Cauchy surfaces $U \cap (\mathcal{M}_t \cup \Sigma \cup \mathcal{M}_{t+\tau})$, for some $t$; and on $(U, g|_U)$, $\Psi$ satisfies the smooth hyperbolic equation $\nabla^a \nabla_a \Psi = 0$ with smooth initial data.

We will first relate data on $\mathcal{I}^+$ to $a(k)$ and then reverse the argument, deducing the data on $\mathcal{I}^-$ from that on $\mathcal{I}^+$.

As in Eq. (3.12), we can write

$$
F(\omega \tau, k, x) = \chi(x) F_0(k, x) + F_{out}(\omega \tau, k, x),
$$

(3.44)
where \( F_0(k, x) = (2\pi)^{-3/2} e^{ik \cdot x} \). Then

\[
\Psi = \chi(x) \Psi_0^- + \Psi_{\text{out}},
\]

(3.45)

where

\[
\Psi_{\text{out}} = \int dk \, a(k) F_{\text{out}}(\omega \tau, k, x) e^{-i\omega t}, \quad \Psi_0^- = \int dk \, a(k) F_0(k, x) e^{-i\omega t}
\]

(3.46)

The integral defining \( \Psi_{\text{out}} \) converges in \( L_2(\mathcal{M}_D) \) to a smooth function, because the integrals defining \( \Psi \) and the Minkowski-space solution \( \Psi_0 \) so converge. \( \Psi \) and \( \Psi_{\text{out}} \) are smooth on \( \mathcal{M} \) because they are smooth on \( \mathcal{M}_D \) for all \( D \).

We can use the spherical harmonic basis \( \{ Y_{lm} \} \) for \( L_2(S^2) \) to write, for the exterior region \( \mathcal{E} \),

\[
F_0(k, x) = \left( \frac{2}{\pi} \right)^{1/2} \sum_{lm} i^l j_l(\omega r) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{x})
\]

(3.47)

\[
F_{\text{out}}(\eta, k, x) = \sum_{lm} \gamma_{lm}(\eta, k) h_l^{(1)}(\omega r) Y_{lm}(\hat{x}).
\]

(3.48)

with the sums converging in \( L_2(S^2) \). Then

\[
\Psi_{\text{out}} = \frac{1}{\sqrt{2\pi}} \int d\omega \omega^2 \sum_{lm} c_{lm}(\omega) h_l^{(1)}(\omega r) Y_{lm}(\hat{x}) e^{-i\omega t},
\]

(3.49)

where

\[
c_{lm}(\omega) = \sqrt{2\pi} \int d\Omega_k a(k) \gamma_{lm}(\omega \tau, k).
\]

(3.50)

We can similarly rewrite the convergent integrals for \( \Psi_0 \) and \( \Psi \) in \( \mathcal{E} \):

\[
\Psi_0^- = \frac{1}{\sqrt{2\pi}} \int d\omega \omega^2 \sum_{lm} 2a_{lm}(\omega) j_l(\omega r) Y_{lm}(\hat{x}) e^{-i\omega t},
\]

(3.51)

\[
\Psi = \frac{1}{\sqrt{2\pi}} \int d\omega \omega^2 \sum_{lm} [b_{lm}(\omega) h_l^{(1)}(\omega r) + a_{lm}(\omega) h_l^{(2)}(\omega r)] Y_{lm}(\hat{x}) e^{-i\omega t},
\]

(3.52)

where

\[
a_{lm}(\omega) = i^l \int d\Omega_k a(k) Y_{lm}^*(\hat{k})
\]

(3.53)
\begin{equation}
    b_{lm} = a_{lm} + c_{lm}.
    \tag{3.54}
\end{equation}

The construction of \( \Psi_{\text{out}} \) from outgoing waves \( F_{\text{out}} \) of Eq. (3.12), satisfying the inhomogeneous equation (3.11) expresses \( \Psi_{\text{out}} \) on \( \mathcal{E} \) as the retarded solution on Minkowski space, (3.41), to \( \Box \Psi_{\text{out}} = \rho \), with \( \rho := \Box \Psi_0 \chi \). If one writes \( \rho \) of Eq. (3.41) as a sum of spherical harmonics and uses Eq. (3.39) to relate \( c_{lm} \) to \( \rho_{lm} \), data (3.41) on \( \mathcal{I}^+ \) for \( \Psi_{\text{out}} \) becomes

\begin{equation}
    \lim_{r \to \infty} r \Psi_{\text{out}}(t = u + r, \hat{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega \omega \sum_{lm} i^{-(l+1)} c_{lm}(\omega) Y_{lm}(\hat{x}) e^{-i\omega u}.
    \tag{3.55}
\end{equation}

A finite-energy solution on Minkowski space of the form (3.51) has data on \( \mathcal{I}^+ \) given by

\begin{equation}
    \lim_{r \to \infty} r \Psi^+_0(u + r, \hat{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega \omega \sum_{lm} i^{-(l+1)} a_{lm}(\omega) Y_{lm}(\hat{x}) e^{-i\omega u}.
    \tag{3.56}
\end{equation}

The first part of the proof of Lemma 3 implies for the solution to \( L_\eta \Psi = 0 \) on \( \mathcal{E} \), eigenfunctions of \( L_\eta \) satisfy, for each \( \omega \),

\begin{equation}
    \sum_{lm} |a_{lm}(\omega)|^2 = \sum_{lm} |b_{lm}(\omega)|^2.
    \tag{3.57}
\end{equation}

Thus the data induced by \( \Psi \) on \( \mathcal{I}^+ \) satisfies the constraints imposed on \( a(k) \): \( b(k) \in L_{2,n}(\mathbb{R}^3) \) all \( n \in \mathbb{Z} \). As a result, the argument just given, with \( \mathcal{I}^- \) and \( \mathcal{I}^+ \) reversed, implies

\begin{equation}
    \Psi = \Psi^+_0 + \Psi^\text{in},
    \tag{3.58}
\end{equation}

with \( \Psi^+_0 \) and \( \Psi^\text{in} \) smooth, and given on \( \mathcal{E} \) by expressions

\begin{align}
    \Psi^+_0 &= \frac{1}{\sqrt{2\pi}} \int d\omega \omega^2 \sum_{lm} 2b_{lm}(\omega) j_l(\omega r) Y_{lm}(\hat{x}) e^{-i\omega t}, \tag{3.59} \\
    \Psi^\text{in} &= \frac{-1}{\sqrt{2\pi}} \int d\omega \omega^2 c_{lm}(\omega) h_l^{(2)}(\omega r) Y_{lm}(\hat{x}) e^{-i\omega t}, \tag{3.60}
\end{align}

converging in \( L_2(\mathcal{M}_D) \) for each \( D \); and data on \( \mathcal{I}^- \) is well-defined, with

\begin{equation}
    \lim_{r \to \infty} r \Psi(r - v, r \hat{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega \omega \sum_{lm} i^{(l+1)} (b_{lm} + c_{lm}) Y_{lm}(\hat{x}) e^{-i\omega v},
    \tag{3.62}
\end{equation}

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whence, using Eq. (3.54), we recover (3.43).

**Lemma 8.** Let \( \psi \) be a smooth retarded solution on Minkowski space to the massless scalar wave equation whose source \( \rho \) has support within a compact spatial region \( r < R \), and suppose that \( \psi \) has zero data on \( I^- \) and data \( f \) on \( I^+ \) with finite energy norm, \( ||f||_{H_1(I^+)} \). Then \( \psi \) has finite energy norm \( ||\psi||_{H_1(H)} \) on a spacelike hyperplane \( H \) of Minkowski space.

**Proof.** The proof will follow from conservation of energy, but, because the energy density of a massless scalar field does not include a term proportional to \( \psi^2 \), we will need to consider both the energy of \( \psi \) and of a time integral of \( \psi \) to bound \( ||\psi||_1 \) on a spacelike hyperplane.

Let \( H \) be the surface \( t = 0 \). We need only consider the retarded field of a source \( \rho \) that vanishes for \( t > 0 \), because the field on \( t = 0 \) depends only on values of \( \rho \) for \( t < 0 \). Let \( t^\alpha \) be the timelike Killing vector \( \partial_t \), and let \( J^\alpha = -T^\alpha_\beta t^\beta \), with \( T^\alpha_\beta \) the energy-momentum tensor of \( \psi \). Let \( E(u_0) \) be the energy of \( \psi \) on the part \( H(u_0) \) of \( H \) with \( r < |u_0| \):

\[
E(u_0) = \int_{H(u_0)} dS_\alpha J^\alpha, \tag{3.63}
\]

with \( dS_\alpha \) along the normal \( \nabla_\alpha t \). Let \( I_{u_0,v_0} \) be the part of the past null cone \( v = v_0 \) lying to the future of \( t = (u_0 + v_0)/2 \); and let \( J_{u_0,v_0} \) be the part of the future null cone \( u = u_0 \) between \( t = 0 \) and \( t = (u_0 + v_0)/2 \). Then \( H(u_0), I_{u_0,v_0}, \) and \( J_{u_0,v_0} \) form the boundary of a source-free region of Minkowski space. Denote by \( F_I(u_0, v_0) \) and \( F_J(u_0, v_0) \) the flux through \( I_{u_0,v_0} \) and \( J_{u_0,v_0} \), respectively,

\[
F_I = \int_{I_{u_0,v_0}} dS_\alpha J^\alpha, \quad F_J = \int_{J_{u_0,v_0}} dS_\alpha J^\alpha, \tag{3.64}
\]

choosing \( dS_\alpha \) along the normals \( \nabla_\alpha u \) and \( \nabla_\alpha v \). Then \( \nabla_\alpha J^\alpha = 0 \) implies

\[
E(u_0) + F_J(u_0, v_0) = F_I(u_0, v_0). \tag{3.65}
\]

A massless scalar field, \( \psi \) satisfies the dominant energy condition: the vector \( J^\alpha \) is future-directed non-spacelike. Consequently \( E(u_0), F_I(u_0, v_0), \) and \( F_J(u_0, v_0) \) are all positive, and we have
The retarded field of a smooth, spatially compact source has asymptotic behavior

\[
\psi(u, x) = f(u, \hat{x})/r + O(r^{-2}),
\]
\[
\nabla^\alpha v \nabla_\alpha \psi = \partial_u f(u, \hat{x})/r + O(r^{-2}),
\]
\[
h^{\alpha\beta} \partial_\beta \psi = O(r^{-2}),
\]
(3.67)

where \( h^{\alpha\beta} = \eta^{\alpha\beta} + t^\alpha t^\beta \) is the projection orthogonal to \( t^\alpha \). Consequently,

\[
\lim_{v_0 \to \infty} F_I(u_0, v_0) = F_I^+(u_0),
\]
(3.68)

where \( F_I^+(u_0) \) is the flux through the part of \( I^+(u_0) \) lying to the future of \( u = u_0 \):

\[
F_I^+(u) = \int_{-u_0}^{0} du \int d\Omega |\partial_u f(u, \hat{x})|^2.
\]
(3.69)

Since \( E(u_0) \) is independent of \( v_0 \), we have for all \( u_0 \), \( E(u_0) < F_I^+(u_0) \). Hence the scalar field \( \psi \) has finite energy on \( \mathcal{H} \):

\[
E = \lim_{u_0 \to -\infty} E(u_0) \leq \lim_{u_0 \to -\infty} F_I^+(u_0) < ||f||_{H_1(I^+)}.
\]
(3.70)

In order to bound \( \int_{\mathcal{H}} dV \psi^2 \), we essentially repeat the argument for

\[
\tilde{\psi} = \int_0^u du' \psi(u', x).
\]
(3.71)

Eq. (3.67) implies that \( \tilde{\psi} \) has data \( \tilde{f} \) on \( I^+ \), where

\[
\tilde{f} = \int_0^u du' f(u, \hat{x}).
\]
(3.72)

Bounding the energy \( \tilde{E} \) of \( \psi \) on \( \mathcal{H} \) bounds the \( L_2 \) norm of \( \psi \), because

\[
\tilde{E} = \int_{\mathcal{H}} dS_\alpha \tilde{J}_\alpha = \int_{\mathcal{H}} dV \left[ \frac{1}{2} (\partial_u \tilde{\psi})^2 + (\partial_u \tilde{\psi} - \partial_r \tilde{\psi})^2 + r^{-2} (\partial_\theta \tilde{\psi})^2 + (r \sin \theta)^{-2} (\partial_\varphi \tilde{\psi})^2 \right]
\geq \int_{\mathcal{H}} dV \frac{1}{2} \psi^2.
\]
(3.73)

Because \( \psi \) vanishes for \( u \geq 0 \), \( \tilde{\psi} \) satisfies the scalar wave equation with source \( \tilde{\rho} \): Using
\[ \square = -2\partial_u \partial_r + \partial_r^2 + \frac{1}{r^2} (\partial_{\theta}^2 + \frac{1}{\sin^2 \theta} \partial_{\varphi}^2), \]  
(3.74)

and

\[ \partial_n \tilde{\psi}(u, x) = \psi(u, x) = \int_0^u du' \partial_{u'} \psi(u', x), \]  
(3.75)

we have

\[ \square \tilde{\psi} = \int_0^u du' \square \psi(u', x) \]

\[ = \tilde{\rho}(u, x). \]  
(3.76)

Then Equations (3.64, 3.65, 3.66) hold for the energy fluxes \( \tilde{F}_I(u_0, v_0) \), and \( \tilde{F}_J(u_0, v_0) \), and, in particular,

\[ \tilde{E}(u_0) \leq \tilde{F}_I(u_0, v_0). \]  
(3.77)

However, \( \tilde{\psi} \) is not the \textit{retarded} solution to the scalar wave equation with source \( \tilde{\rho} \), so we must check that the asymptotic conditions (3.67) on \( \psi \) hold for \( \tilde{\psi} \). This is easy: A function \( O(r^{-n}) \) has, for \( r \) greater than some \( r_0 \), the form \( g(u, x)/r^n \) where, for fixed \( u, \hat{r} \), \( g \) is a bounded function of \( r \). In our case, \( \psi \) is smooth, and the corresponding functions \( g \) are smooth and bounded in a compact domain \([0, u]\) for \( u \) (and hence for \( u, \hat{r} \)). Thus \( \int_0^u g(u') du' = O(r^{-n}) \), and it follows that \( \tilde{\psi} \) satisfies (3.67). The analogues of Eqs. (3.69) and (3.70),

\[ \tilde{F}_{I^+(u_0)} = \int_{u_0}^0 du \int d\Omega |\partial_u \tilde{f}(u, \hat{x})|^2 = \int_{u_0}^0 du \int d\Omega |f(u, \hat{x})|^2 < \infty, \]  
(3.78)

and

\[ \tilde{E} = \lim_{u_0 \to -\infty} \tilde{E}(u_0) \leq \lim_{u_0 \to -\infty} \tilde{F}_{I^+(u_0)} < ||f||_{L^2(I^+)}, \]  
(3.79)

together with Eq. (3.73), imply \( ||\psi||_{L^2(H)} < \infty \). Finally, Eq. (3.70) and the bound on \( ||\psi||_{L^2(H)} \) implies \( ||\psi||_{H^1(H)} < \infty \).  
\( \square \)
Corollary Under the assumptions of Lemma 7, $\Psi_{\text{out}}$ of Eq. (3.46) is asymptotically regular at spatial infinity.

Proof. As in Lemma 7, $\Psi_{\text{out}}$ is on $\mathcal{E}$ the retarded solution of Eq. (3.39) with smooth spatially bounded source $\rho$. Then Lemma 8 implies $\Psi_{\text{out}}$ is regular at spatial infinity.

Proof of Proposition 1. The proof is essentially immediate from the Lemmas. By Lemmas 6 and 7, with $\Phi = 2\text{Re}\Psi$, for almost all $\tau \Phi$ is a smooth solution on $\mathcal{N}$ to the scalar wave equation, with data $f$ on $\mathcal{I}^-$. Finally, on $\mathcal{E}$, $\Psi = \Psi_0 + \Psi_{\text{out}}$ where $\Psi_0$ is asymptotically regular at spatial infinity because it is a solution to the flat-space wave equation with data having finite energy-norm on $\mathcal{I}^-$ and $\Psi_{\text{out}}$ is regular by the Corollary to Lemma 8.

B. Restricted uniqueness theorem for a massless scalar field

Because the system is linear, uniqueness means that the only solution to Eq. (2.18) with zero data at $\mathcal{I}^-$ is $\Phi = 0$. This is not true in the geometrical optics limit, because closed null geodesics $c(\lambda)$ can loop through the wormhole and never reach $\mathcal{I}$; and one might worry that there are analogous smooth solutions that vanish in past and future and have zero data on $\mathcal{I}^-$. The following restricted uniqueness theorem rules them out: smoothed versions of a looping zero-rest mass particle spread and reach $\mathcal{I}$. Denote by $\mathcal{E}_{K_t}$ an energy norm of the field on a compact submanifold $K_t \subset \mathcal{M}_t$, where manifolds in the family $\{K_t\}$ are related by time-translation along the Killing trajectories:

$$\mathcal{E}_{K_t} = \int_{K_t} dV e^{-\nu} \left[ |\nabla \Phi|^2 + |\Phi|^2 \right] = ||\Phi||_{H_1(K_t)}.$$  (3.80)

Proposition 2. If $\Phi(t,x)$ is a smooth solution to $\nabla_\alpha \nabla^\alpha \Phi = 0$, having finite energy and zero initial data on $\mathcal{I}^-$, and if $\lim_{t \to \pm \infty} \mathcal{E}_{K_t} = 0$ for any family of compact (time-translation
related) $K_t$, then $\Phi = 0$.

**Proof.** The result is a corollary of Rellich’s uniqueness theorem for each mode. The requirement that the energy norm on any compact $K$ vanishes as $t \to \infty$ allows one to transform the solution as follows. Denote by $H_T$ a smoothed step function,

$$H_T(t) = \begin{cases} 1, & |t| \leq T \\ 0, & |t| \geq T + \epsilon, \end{cases}$$

and write the Fourier transform of $\Phi$ in the form

$$\hat{\Phi}(\omega, x) = \lim_{T \to \infty} (2\pi)^{-1/2} \int dt H_T(t) e^{-i\omega t} \Phi(t, x).$$

We have

$$0 = \int dt H_T(t)e^{-i\omega t} \nabla_{\alpha} \nabla^\alpha \Phi(t, x) = \int e^{-\nu} dt \ e^{-i\omega t} \left[ H_T(\omega^2 + \mathcal{L}) + 2i\omega \partial_t H_T - \partial_t^2 H_T \right] \Phi(t),$$

whence

$$\left| \int dt e^{-i\omega t} \left[ H_T(\omega^2 + \mathcal{L}) \Phi(t) \right] \right|^2 \leq \left[ \int dt \ \left( |2\omega \partial_t H_T \Phi(t)| + |\partial_t^2 H_T \Phi(t)| \right) \right]^2.$$  

Because $\partial_t H_T$ and $\partial_t^2 H_T$ are bounded functions of $t$ with compact support, we can write

$$\int_K dV \left| \int dt H_T e^{-i\omega t} (\omega^2 + \mathcal{L}) \Phi(t, x) \right|^2 \leq C \max_{|t| \in [T, T+\epsilon]} \int_K dV |\Phi(t, x)|^2,$$

or

$$\left\| \int dt H_T e^{-i\omega t} (\omega^2 + \mathcal{L}) \Phi(t, x) \right\|_{L^2(K_t)}^2 \leq C \max_{t \in [T, T+\epsilon]} \|\Phi\|_{L^2(K_t)}^2,$$

where $C$ is a constant independent of $T$. From our assumption that $\|\Phi\|_{L^2(K_t)}^2 \to 0$ as $t \to \infty$, we have

$$\lim_{T \to \infty} \max_{t \in [T, T+\epsilon]} \|\Phi\|_{L^2(K_t)}^2 = 0$$

implying
\[
\lim_{T \to \infty} \left\| \int dt H_T e^{-i\omega t} (\omega^2 + \mathcal{L}) \Phi(t, x) \right\|_{L^2(K_t)}^2 = 0. \quad (3.88)
\]

It follows that for all \( \omega \), \( \int dt e^{-i\omega t} (\omega^2 + \mathcal{L}) \Phi(t, x) = (\omega^2 + \mathcal{L}) \hat{\Phi}(\omega, x) \) vanishes for almost all \( x \). Finally, since finite energy and zero initial data imply purely outgoing at spatial infinity, by Rellich’s theorem \( \Phi \) must vanish as well. \( \square \)

C. Other massless fields

Extending these results to Weyl and Maxwell fields appears straightforward, at least for hyperstatic spacetimes, with \( t^\alpha t_\alpha = -1 \). The statement of the existence theorem is identical, with the scalar field \( \phi \) replaced by a Weyl spinor \( \phi^A \) and (say) a vector potential \( \phi^\alpha \) for a free Maxwell field \( F^{\alpha \beta} \) in a Lorentz gauge with \( A_\alpha t^\alpha = 0 \). The statement of uniqueness for a Weyl field is identical to that for a scalar field; for a Maxwell field it must be modified to exclude the time independent solutions that have nonzero flux threading the handle:

If \( F^{\alpha \beta} \) is a smooth solution to

\[
\nabla_\alpha F^{\alpha \beta} = 0, \quad \nabla_{[\alpha} F^{\beta \gamma]} = 0 \quad (3.89)
\]

with

\[
\int_S F_{\alpha \beta} dS^{\alpha \beta} = 0 = \int_S *F_{\alpha \beta} dS^{\alpha \beta},
\]

having finite energy and zero initial data on \( \mathcal{I}^- \), and if \( \lim_{t \to \pm \infty} \mathcal{E}_{K_t} = 0 \) for any family of compact (time-translation related) \( K_t \), then \( F^{\alpha \beta} = 0 \).

For a hyperstatic spacetime, the proofs appear to require only minor changes, because we can again decompose the wave operator for vector and spinor fields in the manner

\[
\nabla_\beta \nabla^\beta = -\mathcal{L}_t^2 + \mathcal{L}, \quad (3.90)
\]

with \( \mathcal{L} = D^a D_a \), as in Eq. 2.40. Then the harmonic components \( \phi^\alpha, \phi^A \) satisfy

\[
(\omega^2 + \mathcal{L}) \phi^\alpha = 0, \quad (\omega^2 + \mathcal{L}) \phi^A = 0. \quad (3.91)
\]
The operator $\mathcal{L}$ is symmetric on $L^2(M)$, defined for vector and spinor fields $\phi$ on $M$ by the norm $<\phi|\phi> = \int_M dV e^{-\nu} |\phi|^2$, with $|\phi|^2 \equiv \bar{\phi}^a \phi_a$ and $|\phi|^2 \equiv t_{AA'} \bar{\phi}^{A'} \phi_A$ for vectors and spinors, respectively. The definition of the Sobolev spaces of Sect. II E are similarly extended automatically to vectors and spinors by contracting vector indices with $h_{ab}$ and spinor indices with $t_{AA'}$ (In a spinor frame associated with an orthonormal frame for which $t^a$ is the timelike frame vector, $t_{AA'}$ has components $t_{II'} = \delta_{II'}$. Sobolev spaces of Sect. II E are similarly extended to vector and spinor fields. Finally, the space $H_2$, is defined for vectors and spinors as the set of fields in $H^2(M)$ satisfying the boundary conditions (2.32,2.33).

Here is a sketch of the proofs. The existence theorem involves the same set of lemmas. Lemma 1, self-adjointness of $\mathcal{L}_\eta$ on the spaces $H_2$ of fields in $H^2(M_t)$ again follows from the symmetry of $\mathcal{L}_\eta$, reflecting the fact that boundary conditions imply smoothness of the fields $\hat{\phi}^A$ and $\hat{\phi}^\alpha$ on $\hat{M}$.

Lemma 2 and its proof can be repeated as written with $\phi$ and $\rho$ regarded as vectors or spinors.

In the proof of Lemma 3, each of the components of $\phi^\alpha (\phi^A)$ with respect to covariantly constant frames (spinor frames) on the exterior region $E$ are scalars satisfying Eqs. (3.22), (3.24), and (3.25). They therefore vanish on $E$ and, by elliptic continuation, vanish on all of $M$.

In Lemma 4, eigenfunctions for vector and spinor fields are defined by

$$F^A = \chi (2\pi)^{3/2} \epsilon^A e^{ik \cdot x} + \varphi^A \quad (3.92a)$$

$$F^\alpha = \chi (2\pi)^{3/2} \epsilon^\alpha e^{ik \cdot x} + \varphi^\alpha \quad (3.92b)$$

where $\epsilon^\alpha$ and $\epsilon^A$ are covariantly constant on $E$ and satisfy $k_\alpha \epsilon^\alpha = 0 = k_{AA'} \epsilon^A$. The first part of Lemma 4, existence and uniqueness of eigenfunctions $F^\alpha, F^A$, is again immediate from Lemmas 2 and 3. The second part, unitarity, again appears to be a straightforward extension of Chap. 6 in Wilcox [8], but here there are details we have not checked.

The proof of Lemma 5 goes through as written if $a$ in Lemma 5 is interpreted as a vector (spinor) with a covariant index and their product is read as a dot product: e.g., $f(x) =$
\[ \int dk \, F^{\alpha}(\eta = \omega \tau, k, x)a_\alpha(k) \]

In Lemma 6, the equations are correct as written for the components of \( \phi^\alpha \) and \( \phi^A \), but one must use, in addition, the fact that the fields satisfy the peeling theorem for flat space \([10]\) to complete the characterization of their behavior. In Lemma 7, the proof of smoothness can be read as written. The proof of regularity at \( I^- \) and the recovery of initial data relies on a spherical harmonic decomposition that can be modified in a standard way for spinors and vectors. Finally, Lemma 8 and the proof of Propositions 1 and 2, can be read as written, with the change in the statement of Proposition 2 given above.

IV. THE CAUCHY PROBLEM FOR MORE GENERAL SPACETIMES

The work reported above shows the existence of an unexpected class of spacetimes for which an existence theorem and at least a partial uniqueness theorem can be proved. How broad is the class of spacetimes for which a generalized Cauchy problem is well-defined? Examples of spacetimes with CTCs for which one can prove existence and uniqueness for linear wave equations are not difficult to find, if one allows singularities and does not require that solutions have finite energy \([13]\), and we will display some examples below. Finding examples of nonsingular geometries with CTCs and a well-defined Cauchy problem is more difficult, but the earlier work by Morris et. al. \([2]\) is persuasive: Their time-tunnel examples are asymptotically flat spacetimes in which CTCs are confined to a compact region and for which there appears to be a well-defined initial value problem for data on a spacelike hypersurface to the past of the nonchronal region, the set of points through which CTCs pass. In the present section we present a uniqueness result complementary to that of Sect. \([III\, B]\) a conjecture on existence and uniqueness, and examples of spacetimes that do or do not have a well-posed initial value problem.
A. A result on uniqueness for spacetimes with a compact nonchronal region

In Sect. III B we ruled out, for the static spacetimes considered, a lack of uniqueness corresponding to a field forever trapped inside the nonchronal region, a smooth analog of a closed null geodesic. Here we show, for spacetimes with a compact nonchronal region, $\mathcal{A}$, that when solutions to the Cauchy problem exist, they are unique outside $\mathcal{A}$. Because data is now given on a spacelike surface (instead of $I^-$), we need no longer restrict consideration to massless wave equations. Initial data on a spacelike hypersurface $\mathcal{M}$ for a solution $\Phi$ to the scalar wave equation,

$$(-\nabla_\alpha \nabla^\alpha + m^2)\Phi = 0,$$

will mean the pair of functions

$$\phi = \Phi|_{\mathcal{M}}, \quad \pi = n^\alpha \nabla_\alpha \Phi|_{\mathcal{M}},$$

where $n^\alpha$ is a unit normal to $\mathcal{M}$.

**Proposition 3.** Let $\mathcal{N}, g_{\alpha\beta}$ be a smooth, asymptotically flat spacetime with regions $\mathcal{F}$ and $\mathcal{P}$ to the future and past of a compact 4-dimensional submanifold $\mathcal{A}$ defined by $\mathcal{F} = \mathcal{N}\setminus J^-(\mathcal{A})$ and $\mathcal{P} = \mathcal{N}\setminus J^+(\mathcal{A})$, where both $\mathcal{F}$ and $\mathcal{P}$ are globally hyperbolic and foliated by complete spacelike 3-manifolds. Suppose for arbitrary smooth data with finite energy and a Cauchy surface $\mathcal{M}_F$ of $\mathcal{P}$ that the scalar wave equation has a solution on $\mathcal{N}$ with finite energy on $\mathcal{F}$, and suppose that for arbitrary smooth data with finite energy and a Cauchy surface $\mathcal{M}_F$ of $\mathcal{F}$ that the scalar wave equation has a solution on $\mathcal{N}$ with finite energy on $\mathcal{P}$. Then, solutions on $\mathcal{N}$ with finite energy in $\mathcal{N}\setminus \mathcal{A}$ are unique in $\mathcal{N}\setminus \mathcal{A}$.

**Proof.** The proof relies on the nondegeneracy of the symplectic form,

$$\omega_{\mathcal{M}}(f, g) := \int_{\mathcal{M}} dS_\alpha (f \nabla^\alpha g - g \nabla^\alpha f),$$

and the fact that $\omega$ is independent of hypersurface. That is, let $\mathcal{B} \subset \mathcal{N}$ be a slab, a 4-dimensional submanifold of $\mathcal{N}$ bounded by two submanifolds $\mathcal{M}$ and $\mathcal{M}'$ in the foliations of
\( \mathcal{P} \) and \( \mathcal{F} \) that coincide outside of a compact region. Then, for any two solutions with finite energy,

\[
0 = \int_B d^4V f(\nabla_\alpha \nabla^\alpha - m^2)g \\
= \int_{\partial B} dS_\alpha (f \nabla^\alpha g - g \nabla^\alpha f) \\
= \omega_M(f, g) - \omega_{M'}(f, g). \tag{4.4}
\]

Suppose the theorem is false. Then there is a solution \( \Phi \) to Eq. (4.1) with zero initial data on \( \mathcal{M}_P \) (say) and with \( \Phi \) nonzero and with finite energy somewhere on \( \mathcal{N} \backslash \mathcal{A} \). Thus a hypersurface \( \mathcal{M} \) in the foliation \( \mathcal{F} \) has nonzero data for \( \Phi \) and we can deform \( \mathcal{M} \) outside the support of \( \Phi \) (in the intersection of \( \mathcal{F} \) and \( \mathcal{P} \)) to coincide with \( \mathcal{M}_P \). Because \( \omega \) is non-degenerate, there is data \((\Psi, \dot{\Psi})\) on \( \mathcal{M} \) such that

\[
\omega_M(\Phi, \Psi) \neq 0. \tag{4.5}
\]

But, by hypothesis, a solution \( \Psi \) exists on \( \mathcal{N} \), corresponding to the initial data on \( \mathcal{M} \); and the fact that \( \omega \) is independent of hypersurface implies \( \omega_{M'}(\Phi, \Psi) \neq 0 \), contradicting the assumption that \( \Phi \) has vanishing initial data on \( \mathcal{M}_P \). \( \Box \)

**Corollary.** Proposition 3 holds for the Maxwell, Dirac, and Weyl fields.

**Proof.** Each of the three fields has a conserved symplectic product \( \omega \). The proof goes through as stated, with the symplectic product and initial data of a scalar field replaced by that of the Maxwell and Dirac fields and with the energy norm for a scalar field replaced by vector and spinor energy norms. \( \Box \)

Because the billiard-ball examples considered by Echeverria et al. [3][4][11] have a multiplicity of solutions for the same initial data, uniqueness in spacetimes with CTCs is likely to hold only for free or weakly interacting fields. Because solutions seem always to exist for the billiard ball examples in the spacetimes they considered, it may be that classical interacting fields have solutions on spacetimes for which solutions to the free field equations exist.
B. A conjecture

The solution to the problem posed at the beginning of this section – to delineate the class of spacetimes for which a generalized Cauchy problem is well-defined – is, of course, not known. We present a conjecture here, motivated by examples of geometries which appear to have a well-defined Cauchy problem and by examples of others for which either existence or uniqueness fails. We will motivate the conjecture with a brief reminder of some examples that are by now well known; a more detailed discussion of additional spacetimes will be given in Sect. [IV.C].

A helpful 2-dimensional example of a spacetime where the Cauchy problem is not well defined is Misner space. This is the quotient of the half of Minkowski space on one side of a null line $L$ by the subgroup $\{1, B^{\pm 1}, B^{\pm 2}, \ldots\}$ generated by a boost $B$ about a point of the null line. Equivalently, if $\ell$ is a null line parallel to $L$, and $B\ell$ is its boosted image, then Misner space is the strip between $\ell$ and $B\ell$, where boundary points related by $B$ are identified (see Fig. 3). Misner space has a single closed null geodesic, $C = CC'$ in the Figure, and the past $P$ of $C$ is globally hyperbolic. The future of $C$ is nonchronal, so $C$ is a chronology horizon, a Cauchy horizon that bounds the nonchronal region. Initial data for the scalar-wave equation can be posed on a Cauchy surface $M$ of $P$, but solutions have divergent energy on the chronology horizon.

The reason solutions diverge is clear in the geometrical optics limit. A light ray $\gamma$, starting from $M$, loops about the space and is boosted each time it loops. Because $\gamma$ loops an infinite number of times before reaching $C$, its frequency and energy diverge as it approaches the horizon. The ray $\gamma$ is an incomplete geodesic: it reaches the horizon in finite affine parameter length, because each boost decreases the affine parameter by the blueshift factor $\alpha \equiv [(1 + V)/(1 - V)]^{1/2}$, with the velocity of the boost $V$. This behavior is not

\[^2\text{That is, trajectories of a (locally-defined) timelike Killing vector cross the null geodesic at a sequence of points. The Killing vector can be used to compare the affine parameter at successive}\]
unique to Misner space: A theorem due to Tipler [12] shows that geodesic incompleteness is generic in spacetimes like Misner space in which CTCs are “created” – spacetimes with a nonchronal region to the future of a spacelike hypersurface.

In more than two dimensions, however, the existence of incomplete null geodesics like $\gamma$ does not always imply that the chronology horizon is unstable. This is because there may be only a set of measure zero of such geodesics so that the energy may remain finite on the chronology horizon. For the time-tunnel spacetimes considered in refs [2,3], a congruence of null rays initially parallel to $\gamma$ spreads as the rays approach the chronology horizon. When the spreading of the rays overcomes the successive boosts (when the fractional decrease in flux is greater than the fractional increase in squared frequency), the horizon is stable in the geometrical optics approximation. Because the instability of the chronology horizon (or of the spacetime to its future) appears to be the obstacle to a well defined Cauchy problem, we are led to the following conjecture.

**Conjecture.** Consider a spacetime $\mathcal{N}, g_{\alpha\beta}$ that is

(i) a smooth, asymptotically flat, and for which past and future regions $\mathcal{P} = \mathcal{N} \setminus J^+(\mathcal{A})$ and $\mathcal{F} = \mathcal{N} \setminus J^-(\mathcal{A})$ of a compact 4-dimensional submanifold $\mathcal{A}$ are globally hyperbolic. Suppose that

(ii) the Cauchy problem for massless fields is well-defined in the geometric optics limit. Then the Cauchy problem for massless wave equations (for scalar, Maxwell, and Weyl fields) is well-defined.

Because the instability of massive fields also corresponds in the geometric optics limit to an instability of particles moving along trajectories that become null as one approaches the chronology horizon, it may be that massive wave equations also have a well-defined Cauchy crossing points by time-translating a segment of the geodesic to successively later segments. Compared in this way, the affine parameter of a given segment will will be less than that of the next segment by the blueshift factor $[(1 + V)/(1 - V)]^{1/2}$. 

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problem for the same spacetimes.

C. Examples and counterexamples

For no asymptotically flat spacetime in 4-dimensions, in which CTCs are confined to a compact region, are we aware of a rigorous demonstration that finite-energy solutions to the scalar wave equation do exist for arbitrary initial data, or that solutions are unique. There may also be no published counterexamples, 4-dimensional asymptotically flat spacetimes with a compact nonchronal region (more precisely, spacetimes satisfying condition (i) of the conjecture) for which the nonexistence or nonuniqueness of solutions to free-field equations is proven; but counterexamples like this are not difficult to find. We present below two examples of asymptotically flat spacetimes which are globally hyperbolic to the past and future of a compact region; one can prove for one of them that no solution exists for generic data and for the other that solutions are not unique. The examples show that, without some requirement akin to the well-defined geometric optics limit of our conjecture, there can be too many closed geodesics to allow a well-defined initial value problem. It might also be straightforward to show, for the time-tunnel spacetimes of Refs. [1–3], that whenever the chronology horizon is unstable in the geometric optics limit, it is genuinely unstable for fields with smooth initial data.

The first example is a time-tunnel spacetime like those of Morris et al. but with a metric that is everywhere smooth and is chosen to induce a flat 2-metric on the part of the identified spheres that face each other. The spacetime is also flat between the flat pieces of the spheres, so the geometry includes a region isometric to a piece of (Misner space) × $E^2$, where $E^2$ is flat Euclidean 2-space. For definiteness, we shall identify it with the following piece of Minkowski space. Regard Misner space as the strip of 2-dimensional Minkowski space between the parallel null lines, $v = -A$ and $v = -A\alpha(V)$, with boundary identified by a boost with velocity $V$ as above; and take a finite section $S$ of that strip bounded by the bottom half of the hyperbola $uv = A^2$ and by the left half of the hyperbola $uv = -A^2$, as in
Fig. 4 (Here $u = t - z$, $v = t + z$ are the usual null coordinates). Finally, take as the piece of 4-dimensional Minkowski space $S \times D$, where $D$ is the disk $x^2 + y^2 < B^2$, and $B > A$. Spacelike sections of $S \times D$ prior to the horizon, $C \times D$, are cylinders with circular cross section in $E^2$. In the full spacetime, $\mathcal{N}$, the surface $C \times D$ is again a part of the chronology horizon.

The key to showing that the chronology horizon of $\mathcal{N}$ is unstable is to note that the past of points on $C \times D$ with $x = y = 0$ lies entirely in $C \times D$. This is most easily seen using the universal covering space of $S \times D$. Misner space has as its universal covering space the half of 2-dimensional Minkowski space to the left of $v = 0$. The corresponding cover $\tilde{S} \times D$ of $S \times D$ is the part of 4-dimensional Minkowski space bounded by the 3-surfaces $v = -A$; $v = -A\alpha$; $uw = A$, lower branch; $uw = -A$, left branch; and $x^2 + y^2 < B^2$. The past light cone in $\tilde{S} \times D$ of a point on $\tilde{C} \times (0, 0)$ has maximum value of $x^2 + y^2$ where it meets the boundary $uw = A$, and calculation shows that the intersection is a surface with $x^2 + y^2 < A^2$. Since, by construction, $A^2 < B^2$, the past light cone of a point of $\tilde{C} \times \{(0, 0)\}$ never intersects the boundary $x^2 + y^2 = B^2$. Thus every point $\tilde{P} \in \tilde{S} \times D$ to the past of $\tilde{C} \times \{(0, 0)\}$ is in the domain of dependence of the spacelike boundary $uw = A$ of $\tilde{S} \times D$; and every point $P \in S \times D$ to the past of $C \times (0, 0)$ is then in the domain of dependence of the boundary $uw = A$ of $S \times D$. Data on $uw = A$ that is independent of $x$ and $y$ yields a solution that diverges in $S \times D$ because the solution is identical in the domain of dependence of $uw = A$ to the divergent solution in Misner space. Finally, by picking a spacelike hypersurface of the full spacetime that agrees with the $uw = A$ surface in $S \times D$, we obtain data on a spacelike hypersurface to the past of the chronology horizon for which no finite-energy solution exists to the scalar wave equation for generic smooth initial data with finite energy.

The example of a geometry with compact nonchronal region for which uniqueness fails depends on a construction suggested by Geroch [17]. Although 2-dimensional geometries obtained by removing slits and identifying sides are singular [18], it is possible in 4-dimensions to build smooth geometries in a similar way. The construction relies on the following observation.
Lemma 9 Any smooth compact 4-dimensional spacetime with boundary can be embedded (i) in a smooth compact spacetime without boundary, and (ii) in a smooth spacetime that is isometric to Minkowski space outside a compact region.

Proof. The technique is borrowed from references [16,19] (see also [20]). (i): Any manifold \( M \) with boundary \( \Sigma \) can be embedded in a compact manifold \( \tilde{M} \) by attaching a second copy \( M' \) of \( M \) to the outward side of \( \Sigma \). Let \( U \) be a collar of \( M \), a neighborhood \( U \cong \Sigma \times I \) with boundary \( \Sigma' \sqcup \Sigma \). A lorentzian metric \( g \) on \( M \) can be extended to a lorentzian metric \( \tilde{g} \) on \( \tilde{M} \) precisely when a direction field \( \hat{t} \) of timelike directions on \( U \) can be extended to a timelike direction field on \( \tilde{M} \). Now \( \hat{t} \) can always be extended to a Morse direction field on \( \tilde{M} \), a direction field that has isolated zeroes, at each of which the line-element field is tangent to a vector field with index \( \pm 1 \). By cutting out a ball \( B^4 \) containing each zero and gluing in a copy of \( \mathbb{R}P^4 \setminus B^4 \) for each zero of index 1 and a copy of \( \mathbb{C}P^2 \setminus B^4 \) for each zero of index \(-1\), we can extend the line element to a nonvanishing field on the interior. (ii): The proof here is nearly identical. First embed \( M \) in \( \tilde{M} \) as in (i). If one removes a a ball \( B^4 \) from \( M' \) then one can put a flat metric on a neighborhood \( V \) of the spherical boundary \( \partial B^4 \) that makes \( V \) into a copy of Minkowski space outside a ball. One is again asking to extend a direction field on a new boundary, \( U \sqcup V \) to the 4-manifold that it bounds, and the construction proceeds as in (i). Finally any Lorentzian metric \( g \) on the compact manifold-with-boundary, \( M' (M' \setminus B^4) \) that is smooth on \( U (U \sqcup V) \) can be deformed to a smooth metric that agrees with \( g \) on a neighborhood of the boundary of \( U \) and \( V \). □

Using this construction, we exhibit a smooth, asymptotically flat spacetime with compact nonchronal region, for which solutions with finite energy do not exist for generic initial data. Begin with a 4-torus \( T^4 \) with a flat metric \( \eta \) chosen to make two of the generators null and the other two spacelike. Explicitly, identify by translation opposite faces of the rectangular 4-cell in Minkowski space, \( 0 \leq u \leq A, 0 \leq v \leq A, 0 \leq x \leq A, 0 \leq y \leq A \). The geometry \( T^4, \eta \) is chosen because it has solutions to the wave equation whose support is not all of...
$T^4$; examples are smooth plane waves, functions $\Phi = \Phi(u)$, with $\Phi(u) = 0, u < A/2$. Cut a ball out of $T^4$ and embed it in a spacetime that is isometric to Minkowski space outside a compact region $U$, using Lemma 9. The resulting spacetime satisfies condition (i) of the Conjecture, but solutions to the wave equation for data on a Cauchy surface $\mathcal{M}$ for the past of the Cauchy horizon are not unique: Zero data on $\mathcal{M}$ is consistent with arbitrary solutions $\Phi(u)$ whose support on $T^4$ is disjoint from the removed ball.

Although our primary interest is in smooth geometries with CTCs confined to a compact region, it is worth pointing out that if one allows singularities, there are simple examples of spacetimes with CTCs for which one can easily prove the existence of solutions to free-field equations for arbitrary initial data. For these geometries, however, solutions for smooth data with finite energy are not smooth and do not in general have finite energy; generic solutions are in $L^2_{\text{loc}}$. Consider, for example, two-dimensional Minkowski space with two parallel timelike or spacelike segments of equal length removed, as in Fig. 5, and each side of each segment identified with a side of the other segment after translation by a timelike vector $V$, which will be taken to point up and to the right.

3 An example of suitable manifold is $T^4 \# R^1 \# CP^2 \# CP^2$.

4 More precisely, to construct the first spacetime, let $\tilde{\mathcal{N}}$ be the manifold obtained from Minkowski space by removing the two timelike segments $\bar{L}_1$ and $\bar{L}_2 = \mathcal{T}(\bar{L}_1)$ where $\mathcal{T}$ is translation by a timelike vector $V$. Call the segments with their endpoints removed $L_1$ and $L_2$. Formally reattach $L_1$ and $L_2$ by writing $\mathcal{N} = \tilde{\mathcal{N}} \sqcup L_1 \sqcup L_2$. The topology of the first spacetime, $\mathcal{N}$, is generated by the open sets of $\tilde{\mathcal{N}}$ together with neighborhoods of points of $L_1$ and $L_2$ defined as follows: Let $\mathcal{O}$ be any open set in an atlas for Minkowski space intersecting the line through $\bar{L}_1$ in an open interval $\ell \subset L_1$. Let $\mathcal{O}_R$ be the part of $\mathcal{O}$ lying to the left (right) of the line through $L_1$. Let $\mathcal{O}'_R$ be the part of the translated open set $\mathcal{O}' = \mathcal{T}(\mathcal{O})$ that lies to the right of the line through $L_2$; and let $\mathcal{O}'_L$ be the part of the translated open set $\mathcal{O}' = \mathcal{T}(\mathcal{O})$ that lies to the left of the line through $L_2$. 

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The resulting geometries are flat with two conical singularities, corresponding to the
removed endpoints of the segments. Similar spacetimes have been discussed by Geroch and
Horowitz \[15\] and Politzer \[13\]. Some identified points are timelike related, and timelike
curves in Minkowski space joining points \( P_I \in L_I \) to \( T(P_I) \in L_{II} \) become CTCs in \( \mathcal{N} \). For
each spacetime \( \mathcal{N} \) there is a spacelike hypersurface \( \mathcal{M} \) to the past of the nonchronal region
on which initial data can be set, and we will see it is easy to find a solution for arbitrary
initial data on \( \mathcal{M} \).

**Proposition 4** For any initial data \( \phi, \dot{\phi} \) in \( L_2(\mathcal{M}) \otimes L_2(\mathcal{M}) \) there is a unique solution in
\( L_2^\text{loc} \) to the massless scalar wave equation on spacetimes of the form described above. In a
neighborhood of spatial infinity the solution agrees with the Minkowski space solution for
the same initial data.

**Proof.** We can divide initial data on \( \mathcal{M} \) into a sum of data for right-moving and left-moving
waves, \( f(x - t) \) and \( g(x + t) \), by writing

\[
  f(x) = \frac{1}{2}[\phi(x) - \int_{-\infty}^{x} dx' \dot{\phi}(x')], \quad g(x) = \frac{1}{2}[\phi(x) + \int_{-\infty}^{x} dx' \dot{\phi}(x')].
\]  

(4.6)

We separately construct solutions for right-moving and left moving data. On Minkowski
space, right-moving data, \( (f, \dot{f} = -f') \) gives the solution \( f(x - t) \); equivalently, \( f(P) = f(p) \),
where \( p \in \mathcal{M} \) is the past endpoint of the right moving null ray from \( \mathcal{M} \) to \( P \). Note that data
in \( L_2 \) that is discontinuous across a finite set of points \( p, q, \cdots r, s \) of \( \mathcal{M} \) yields a solution
that is discontinuous across the boundaries of the strip between the two right-moving null
lines through endpoints \( p, q \). Each point of \( \mathcal{N} \) similarly lies on a unique right-moving null
geodesic, and all but four of these rays, followed back to the past, intersect \( \mathcal{M} \). Define a
solution in \( L_2^\infty(\mathcal{N}) \) by \( f(P) = f(p) \), where \( p \in \mathcal{M} \) is the past endpoint of the right moving
null ray from \( \mathcal{M} \) to \( P \). The four rays that fail to meet \( \mathcal{M} \) are the future parts of null lines

...and to the right of the line through \( L_1 \). Then \( \mathcal{O}_L \cup \ell \cup \mathcal{O}_R \) and \( \mathcal{O}_R' \cup \ell' \cup \mathcal{O}_L' \) are open sets of \( \hat{\mathcal{N}} \).

With the obvious maps to subsets of \( \mathbb{R}^4 \), the open sets just enumerated form an atlas. Because of
the deleted endpoints, \( \mathcal{N} \) is not complete.
that emerge from the (removed) endpoints of the identified segments. These lines are the future and past parts of right-moving null rays that are geodesically incomplete, leaving the manifold at the removed endpoints. They divide $\mathcal{N}$ into five strips which intersect $\mathcal{M}$ in five segments
\[ (-\infty, p_2'), (p_2', p_2), (p_2, p_1'), (p_1', p_1), (p_1, \infty), \]
where $p_i$ and $p_i'$ are the points where the right-moving null rays from the bottom and top endpoints of $L_i$, respectively, meet $\mathcal{M}$. Since each strip is isometric to a strip bounded by null rays in Minkowski space, the function $f$ satisfies the scalar wave equation with the given initial data everywhere except at the null boundaries of the strips.

The proof of existence for left-moving solutions is identical, where five new strips intersect $\mathcal{M}$ in five new segments
\[ (-\infty, q_1), (q_1, q_1'), (q_1', q_2), (q_2, q_2'), (q_2', \infty), \]
where $q_i$ and $q_i'$ are the points where left-moving null rays from the bottom and top endpoints of $L_i$, respectively, meet $\mathcal{M}$. Outside the chronology horizon, the left- and right-moving solutions have their Minkowski space values because past directed null rays from points outside the horizon never intersect $L_1 \cup L_2$. Thus the solution outside a spatially compact region of any asymptotically spacelike hypersurface agrees with the Minkowski space solution for the same initial data.

Proving uniqueness of these solutions appears to be straightforward: Assuming that any solution in $L^2_{\text{loc}}$ is locally a sum of right-moving and left-moving solutions, one can trace it back to nonzero data on $\mathcal{M}$. Similar spacetimes can be constructed in 4-dimensions by removing two parallel planar 3-disks from Minkowski space and identifying their boundaries as in the two dimensional example. Again it seems clear that solutions in $L^2_{\text{loc}}$ exist and that they do not in general have finite energy.

Curiously, in 4-dimensions these singular spacetimes can be made into smooth spacetimes by using a construction essentially equivalent to that of Lemma 9. In addition to removing two 3-disks, one removes a small solid torus ($D^3 \times S^1$) at the edge of each disk. Then when
the sides of the disk are glued back in, one is left with a spacetime with boundary $S^3 \times S^1$, which one can glue to a compact spacetime.

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FIGURES

FIG. 1. An orientable 3-manifold $M$ is constructed by identifying points of $\Sigma_I$ and points of $\Sigma_{II}$ that are labeled by the same letter, with subscripts I and II: $P_{II} = T(P_I)$.

FIG. 2. The spacelike hypersurfaces $\mathcal{M}_t$ foliate the spacetime whose boundary is the union of two cylinders, $C_I \cup C_{II}$. In the spacetime $\mathcal{N}$, the cylindrical boundaries are identified, and $\mathcal{M}_{t+\tau}$ is a smooth continuation of $\mathcal{M}_t$ across the identified spheres $\Sigma_I$ and $T(\Sigma_I)$.

FIG. 3. Misner space is the region between the two null rays $\ell$ and $\mathcal{B}\ell$, with points of the null boundaries identified by the boost $\mathcal{B}$. The curve $\mathcal{C} = CC'$ is a chronology horizon, a closed null geodesic that separates the nonchronal region above it from the globally hyperbolic spacetime to its past. Equivalently, Misner space is the quotient of the half of Minkowski space lying to the left of the null line $L$ by the group of boosts generated by $\mathcal{B}$.

FIG. 4. A piece of Misner space is used in the construction of a 4-dimensional spacetime $\mathcal{N}$ with a compact nonchronal region and an unstable chronology horizon. The lines $pq$ and $p'q'$ lie in the covering space, the half of Minkowski space lying to the left of $v = 0$, and they are identified by the boost that defines Misner space. In $\mathcal{N}$, these lines can be regarded as lying along the trajectory of the wormhole mouth.

FIG. 5. Two simple spacetimes with CTCs and a well defined Cauchy problem are shown these two figures. Two parallel slits are removed from Minkowski space and points labelled by the same letter are identified.

FIG. 6. Each shaded region is a strip isometric to a piece of Minkowski space. A right-moving solution to the massless wave equation is smooth and well-defined on each strip.

FIG. 7. An example of an asymptotically flat spacetime with compact nonchronal region is depicted here with two dimensions suppressed. Balls are removed from $\mathcal{N}$ and the torus, and their boundary 3-spheres $\Sigma_I$ and $\Sigma_{II}$ are identified. Arrows at $P$ point along null generators of the torus. The shaded region is the support of a solution to the massless scalar wave equation.