On finite groups whose power graph is a cograph

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Abstract

A $P_4$-free graph is called a cograph. In this paper we partially characterize finite groups whose power graph is a cograph. As we will see, this problem is a generalization of the determination of groups in which every element has prime power order, first raised by Graham Higman in 1957 and fully solved very recently.

First we determine all groups $G$ and $H$ for which the power graph of $G \times H$ is a cograph. We show that groups whose power graph is a cograph can be characterised by a condition only involving elements whose orders are prime or the product of two (possibly equal) primes. Some important graph classes are also taken under consideration.
finite simple groups we show that in most of the cases their power graphs are not cographs: the only ones for which the power graphs are cographs are certain groups \(\text{PSL}(2, q)\) and \(\text{Sz}(q)\) and the group \(\text{PSL}(3, 4)\). However, a complete determination of these groups involves some hard number-theoretic problems.

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1 Introduction

There are various graphs we can define for a group using different group properties [4]. These graphs include the commuting graph, the generating graph, the power graph, the enhanced power graph, deep commuting graph, etc. The power graphs were first seen in early 2000’s as the undirected power graphs of semigroups [18]. For a semigroup \(S\), the directed power graph of \(S\), denoted by \(\vec{P}(S)\), is a directed graph with vertex set \(V(\vec{P}(S)) = S\); and two distinct vertices \(x\) and \(y\) are having an arc \(x \rightarrow y\) if \(y\) is a power of \(x\).

The corresponding undirected graph is called the undirected power graph of \(S\), denoted by \(P(S)\). The undirected power graph of a semigroup was introduced by Chakrabarty et al. [10] in 2009. So the undirected power graph of \(S\) is the graph with vertex set \(V(P(S)) = S\), with an edge between two vertices \(u\) and \(v\) if \(u \neq v\) and either \(v\) is a power of \(u\) or \(u\) is a power of \(v\). These concepts are defined for groups as a special case of semigroups. In the sequel, we only consider groups; “power graph” will mean “undirected power graph”, and all the groups in this paper are finite.

The power graphs were well studied in literature [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 16, 21, 23]. We find several research papers in which researchers give complete or partial characterization of different graph parameters for the power graphs. We mention few notable works in this context:

- \(P(G)\) is a complete graph if and only if either \(G\) is trivial or a cyclic group of prime power order. (Chakrabarty et al. [10])

- \(P(G)\) is always connected and we can compute the number of edges in \(P(G)\) by the formula
\[
|E(P(G))| = \frac{1}{2} \left[ \sum_{a \in G} (2\sigma(a) - \phi(\sigma(a)) - 1) \right].
\]
• The power graph of a finite group is Eulerian if and only if $G$ has odd number of elements.

• Curtin et al. [15] introduced the concept of proper power graphs. They determine the diameter of the proper power graph of $S_n$.

• Chattopadhyay et al. [11] have provided suitable bounds for the vertex connectivity $P(G)$ where $G$ is a cyclic group.

• Cameron [5] proved that, for any two finite groups $G_1$ and $G_2$, if power graphs of $G_1$ and $G_2$ are isomorphic then $\bar{P}(G_1)$ and $\bar{P}(G_2)$ are also isomorphic.

In our previous paper [23], we partially characterized finite groups whose power graphs forbid certain induced subgraphs. These subgraphs include $P_4$ (the path on 4 vertices); $C_4$ (the cycle on 4 vertices); $2K_2$ (the complement of $C_4$); etc. A graph forbidding $P_4$ is called a cograph. In other words, a graph $\Gamma$ is a cograph if it does not contain the 4-vertex path as an induced subgraph. Cographs have various important properties. For example, they form the smallest class of graphs containing the 1-vertex graph and closed under complementation and disjoint union. (The complement of a connected cograph is disconnected.) See [3, 4] for more about these concepts.

We will use the term power-cograph group, sometimes abbreviated to PCG-group, for a finite group whose power graph is a cograph.

In [23], we completely characterized finite nilpotent power-cograph groups. We proved the following theorem:

**Theorem 1.1** ([23], Theorem 3.2). Let $G$ be a finite nilpotent group. Then $P(G)$ is a cograph if and only if either $|G|$ is a prime power, or $G$ is cyclic of order $pq$ for distinct primes $p$ and $q$.

For a given group $G$, the power graph of any subgroup of $G$ is an induced subgraph of $P(G)$. Thus if power graph of a group is a cograph then the power graph of any of its subgroups is also a cograph. In other words, the class of finite power-cograph groups is subgroup-closed.

For that reason, we have a necessary condition for a group to be a power-cograph group: any nilpotent subgroup of such a group is either a $p$-group or isomorphic to $C_{pq}$, where $p$ and $q$ are distinct primes. So if $G$ has a nilpotent subgroup which is neither a $p$-group nor isomorphic to $C_{pq}$, then $P(G)$ is not a cograph. In our previous paper, we have asked the following question:
Classify the finite groups $G$ for which $P(G)$ is a cograph. In this paper we provide further results towards the answer to this question.

We now give several equivalent conditions on a finite group which are known to imply that the power graph is a cograph. First we require a few definitions.

- For a finite group $G$, let $\pi(G)$ denote the set of all prime divisors of $|G|$. The prime graph or Gruenberg–Kegel graph of $G$ is a graph with $V = \pi(G)$ and two distinct elements $p$ and $q$ of $\pi(G)$ are connected if and only if $G$ contains an element of order $pq$.

- The enhanced power graph of $G$ is the graph with vertex set $G$, in which vertices $x$ and $y$ are joined if there exists $z \in G$ such that both $x$ and $y$ are powers of $z$. Clearly the power graph is a spanning subgraph of the enhanced power graph.

- The group $G$ is an EPPO group if every element of $G$ has prime power order.

**Theorem 1.2.** For a finite group $G$, the following conditions are equivalent:

1. $G$ is an EPPO group;
2. the Gruenberg–Kegel graph of $G$ has no edges;
3. the power graph of $G$ is equal to the enhanced power graph.

If these conditions hold, then the power graph of $G$ is a cograph.

For the equivalence of (a)–(c), see Aalipour et al. [1]. If these hold, then any edge of the power graph not containing the identity joins elements whose orders are powers of the same prime; so the reduced power graph of $G$ (obtained by removing the identity) is a disjoint union of reduced power graphs of groups of prime power order, which are cographs by Theorem 1.1. The class of EPPO groups was first investigated (though not under that name) by Graham Higman in 1957 [17], but the complete determination of these groups only appeared in a paper not yet published [9].

We note that the condition that $G$ is a power-cograph group does not imply (a)–(c). Moreover two groups may have the same prime graph, yet one and not the other is a power-cograph group. For example, consider $G_1 = C_{12}$ and $G_2 = D_6$. 
In this paper we explore various graph classes and try to identify whether their power graphs are cographs or not. First we discuss direct product two groups. We are able to identify certain solvable groups whose power graph is a cograph. Finally we consider finite simple groups. Our result is as follows:

**Theorem 1.3.** Let $G$ be a non-abelian finite simple group. Then $G$ is a power-cograph group if and only if one of the following holds:

(a) $G = \text{PSL}(2, q)$, where $q$ is an odd prime power with $q \geq 5$, and each of $(q - 1)/2$ and $(q + 1)/2$ is either a prime power or the product of two distinct primes;

(b) $G = \text{PSL}(2, q)$, where $q$ is a power of 2 with $q \geq 4$, and each of $q - 1$ and $q + 1$ is either a prime power or the product of two distinct primes;

(c) $G = \text{Sz}(q)$, where $q = 2^{2e+1}$ for $e \geq 2$, and each of $q - 1$, $q + \sqrt{2q} + 1$, and $q - \sqrt{2q} + 1$ is either a prime power or the product of two distinct primes;

(d) $G = \text{PSL}(3, 4)$.

We end the introduction with some remarks about this theorem. In the first three cases, determining precisely which groups occur is a purely number-theoretic problem which is likely to be quite difficult. For example, the values of $d$ (at least 2) for which the power graph of $\text{PSL}(2, 2^d)$ is a cograph for $d \leq 200$ are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, and 199, and the values of $e$ (at least 1) for which the power graph of $\text{Sz}(2^{2e+1})$ is a cograph for $e \leq 100$ are 1, 2, 3, 4, 5, 6, 8, 44.

**Problem 1.4.** Are there infinitely many non-abelian finite simple groups $G$ which are power-cograph groups?

Secondly, there is a big gap between finding the simple groups satisfying the condition and finding all groups. This can be seen in the somewhat similar property of being an EPPO group, where the list of simple EPPO groups follows from the work of Suzuki [24, 25] but the complete determination of these groups is much more recent.
2 Direct products

Recall that a finite nilpotent group can be written as a direct product of its Sylow subgroups. Thus if $P(G \times H)$ is a cograph then, by using Theorem 1.1 we have only a few choices for the orders of $G$ and $H$. The following theorem gives a complete characterization for all direct products $G \times H$ such that $P(G \times H)$ is a cograph.

**Theorem 2.1.** Let $G$ and $H$ be non-trivial groups. Then $G \times H$ is a power-cograph group if and only if one of the following holds:

(a) the orders of $G$ and $H$ are powers of the same prime;

(b) $G$ and $H$ are cyclic groups of distinct prime orders;

(c) there are primes $p$ and $q$ and an integer $m \geq 1$ such that $q^m \mid (p - 1)$; one of $G$ and $H$ is a cyclic group of order $q$, and the other is the non-abelian group

$$\langle a, b : a^p = 1, b^{q^m} = 1, b^{-1}ab = a^k \rangle,$$

where $k$ is an integer with multiplicative order $q^m \pmod{p}$.

**Proof.** Let $P(G \times H)$ be a cograph. If the orders of $G$ and $H$ are each divisible by exactly one prime then $G \times H$ is nilpotent; by Theorem 1.1 we have one of the first two cases. So we can suppose that at least one of $G$ and $H$ has order divisible by two primes.

Suppose that $|G|$ is divisible by primes $p$ and $q$. Then no prime except possibly $p$ or $q$ can divide $H$. For suppose that $r \mid |H|$; let $a$ and $b$ be elements of orders $p$ and $q$ in $G$, and $c$ an element of order $r$ in $H$. Then $(b, bc, c, ac)$ is an induced $P_4$ in $P(G \times H)$.

It follows that at most two primes divide each of $|G|$ and $|H|$.

Suppose first that both $p$ and $q$ divide each of $|G|$ and $|H|$. The direct product of a Sylow $p$-subgroup of $G$ and a Sylow $q$-subgroup of $H$ is nilpotent. So Theorem 1.1 implies that $|G| = |H| = pq$. Each is non-abelian; so, without loss of generality, $q \mid p - 1$. Let $a$ be an element of order $p$ in $G$, and $b$, $c$ elements of order $q$ in $H$ which are not joined in the power graph. Then $(b, ab, a, ac)$ is an induced $P_4$.

In the remaining case, one of $G$ and $H$ (say $G$, without loss of generality) is a group of prime power order $q^m$. By assumption, $H$ is not a $q$-group, and so contains a subgroup $P$ of prime order $p \neq q$. Then $G \times P$ is a nilpotent
subgroup of $G \times H$ with forbidden structure, unless $|G| = q$. Now $p$ must divide $|H|$ to the first power only, and $|H| = pq^m$ for some $m \geq 1$.

We claim next that $H$ has a normal Sylow $p$-subgroup $P$. For suppose not, and let $b$ and $c$ be elements of order $p$ not adjacent in the power graph, and $a$ a non-identity element of $G$. Then $(b, ab, a, ac)$ is an induced path in the power graph of $G \times H$, a contradiction. As before, we conclude that $C_H(P) = P$, and so the Sylow $q$-subgroup of $H$ (which is a complement to $P$) is cyclic of order dividing $p - 1$. This yields the claimed structure for $G$ and $H$.

For the converse, we begin with some preliminary remarks. If $x, y$ are elements of a group $G$, and $x \to y$ in the directed power graph of $G$, then $y$ is a power of $x$, and so $o(y) | o(x)$. If also $y \not\leftrightarrow x$, then the divisibility is proper. Moreover, since $\to$ is a transitive relation, if $(a, b, c, d)$ is an induced power graph in $P(G)$, then either $a \to b \leftarrow c \to d$ or the reverse.

So let $H = \langle a, b : a^p = b^{pq} = 1, b^{-1}ab = a^k \rangle$, and $G = C_q = \langle c \rangle$. Then all non-identity elements in $G \times H$ have orders a power of $q$, $p$, or $pq$. Also, if $(g, h)$ has order a power of $q$, then $(g, h)^q = (1, h^q)$.

Suppose if possible that $(x_1, x_2, x_3, x_4)$ is an induced path in $P(G \times H)$, where $x_i = (g_i, h_i)$, and suppose that $x_1 \to x_2 \leftarrow x_3 \to x_4$ in $\tilde{P}(G)$. Then none of $x_1, \ldots, x_4$ is the identity, so $x_1$ and $x_3$ each have order a power of $q$ or $pq$.

If $x_3$ has order a power of $q$, then $g_2 = g_4 = 1$ and $h_2$ and $h_4$ belong to the same (cyclic) Sylow $q$-subgroup of $H$, so $x_2$ and $x_4$ are adjacent in $P(G \times H)$, a contradiction. So $x_3$ has order $pq$, and $x_2$ and $x_4$ have orders $p$ or $q$. If $x_2$ has order $p$, then $x_1$ has order $pq$. On the other hand, if $x_2$ has order $q$, then $g_2 \neq 1$, so $x_2$ cannot be a $q$th power, so again $x_1$ has order $pq$. But this implies that $x_1$ and $x_3$ are joined.

Remark 2.2. The converse of the above is not true in general. Consider $G = C_4$ and $H = C_6$. Then by Theorem [1.1], both $P(G)$ and $P(H)$ are cographs, whereas $P(G \times H)$ is not a cograph.
3 Minimal non-power-cograph groups

Let $\mathcal{C}$ be the class of finite groups $G$ for which $P(G)$ is a cograph. As noted earlier, $\mathcal{C}$ is subgroup-closed; so it can be characterised by finding all minimal non-$\mathcal{C}$ groups.

**Theorem 3.1.** Let $G$ be a finite group. Then $P(G)$ is not a cograph if and only if $G$ contains elements $g$ and $h$ with orders $pr$ and $pq$ respectively, where $p, q, r$ are prime numbers and $p \neq q$, such that

(a) $g^r = h^q$;

(b) if $q = r$, then $g^p \notin \langle h^p \rangle$.

**Proof.** Let $G$ be a minimal non-$\mathcal{C}$ group. Suppose first that $G$ is abelian. By Theorem 1.1, it has order the product of three primes which are not all equal. We distinguish three cases.

- Suppose that $|G| = pqr$ where $p, q, r$ are all distinct. Then $G$ is cyclic; say $G = \langle x \rangle$. Now if we put $g = x^q$ and $h = x^r$, we see that the conditions of the theorem are satisfied.

- Suppose that $|G| = p^2q$, and that the Sylow $p$-subgroup of $G$ is cyclic, generated by $g$. Let $z$ be an element of order $q$, and $h = g^pz$. Take $r = p$ in the conditions of the theorem.

- Finally, suppose that $|G| = p^2q$ and the Sylow $p$-subgroup is elementary abelian, generated by $x$ and $y$. Let $z$ be an element of order $q$. Now take $g = xz$ and $h = yz$. Then $g$ and $h$ have order $pq$; $g^p = z^p = h^p$, but $g^q = x^q \notin \langle x^p \rangle$. So these elements satisfy the conditions of the theorem, if we take $r = q$ and reverse the roles of $p$ and $q$.

So we can suppose that $G$ is nonabelian.

Since $P(G)$ is not a cograph, there is an induced path $(a, b, c, d)$ in $P(G)$. As we saw in the proof of Theorem 2.1 we may assume that $a \rightarrow b \leftarrow c \rightarrow d$ in $\bar{P}(G)$.

Now $\langle c \rangle$ is a cyclic group and contains $b$ and $c$. Since $G$ is nonabelian, it is a proper subgroup, and hence its power graph is a cograph. So the order of $c$ is either a prime power or of the form $pq$ where $p$ and $q$ are distinct primes. The former case is impossible. For the power graph of a cyclic group of prime power order is complete, but $b$ is not joined to $d$. So the order of
c is $pq$, with $p \neq q$. We may suppose without loss that $b = c^q$ has order $p$ while $d = c^p$ has order $q$.

Now consider the element $a$. We know that the order of $a$ is divisible by $p$ (the order of $b$). By replacing $a$ by a power of itself, we can assume that the order of $a$ is $pr$, where $r$ is a prime which may or may not be equal to $p$. (This power is still joined to $b$, but it cannot be joined to $d$. For if $a$ and $d$ are joined, then $d \in \langle a \rangle \cap \langle c \rangle = \langle b \rangle$, contradicting the fact that $d$ has order $q$ whereas $b$ has order $p$. Also $a$ cannot be joined to $c$, for this would imply that $a \rightarrow c$ and hence $a \rightarrow d$.)

We have now verified all the conditions of the theorem.

Conversely, if these conditions hold, then $(g, g^r = h^q, h, h^p)$ is an induced path of length 3, so $P(G)$ is not a cograph.

Remark 3.2. A minimal non-PCG group has nontrivial centre. For such a group is generated by elements $g$ and $h$ as in the theorem, and $g^r = h^q$ is in the centre.

Corollary 3.3. Let $G$ be a finite group. Let $P_2(G)$ be the set of non-identity elements of $G$ whose orders are either prime or the product of two (not necessarily distinct) prime numbers. Then $P(G)$ is a cograph if and only if the induced subgraph on $P_2(G)$ is a cograph.

Here is an application, which we will require later. Suppose that $G$ is a finite group containing elements $a$ of order 4 and $b$ of order 6 such that $a^2$ and $b^3$ are conjugate. Replacing $b$ by a conjugate, we may assume that $a^2 = b^3$. Now the theorem above implies that $G$ is not a power-cograph group. These conditions can be verified for the simple groups $M_{11}$ and $PSU(3, 8)$ using the ATLAS of finite groups [13]. We will use this argument several times, so we refer to it as the 4-6 test.

4 Examples

Below we let $P^*(G)$ be the reduced power graph of $G$, the induced subgraph on the set $G^# = G \setminus \{1\}$. Note that $P(G)$ is a cograph if and only if $P^*(G)$ is a cograph.

We also make the following observation.

Theorem 4.1. Let $G$ be a finite group in which any two distinct maximal cyclic subgroups intersect in the identity. Then $P(G)$ is a cograph if and
only if the orders of the maximal cyclic subgroups are either prime powers or products of two distinct primes.

Proof. Every edge of $P(G)$ is contained in a maximal cyclic subgroup of $G$. The hypothesis implies that $P^*(G)$ is the union of $P^*(C)$ as $C$ runs over the maximal cyclic subgroups of $G$.

Theorem 4.2. The symmetric group $S_n$ on $n$ symbols is a power-cograph group if and only if $n \leq 5$.

Proof. for $n \geq 6$, $P(S_n)$ contain a path $(pq)(xy) \sim (xy) \sim (qrz)(xy) \sim (qzr)$ and thus $P(S_6)$ is not a cograph.

For $n \leq 5$, the maximal cyclic subgroups intersect in the identity, and their orders are in the sets $\{2\}$ (for $n = 2$), $\{2, 3\}$ (for $n = 3$), $\{2, 3, 4\}$ (for $n = 4$), or $\{4, 5, 6\}$ (for $n = 5$), so these symmetric groups are all power-cograph groups, by Theorem 4.1.

Theorem 4.3. Let $p$ and $q$ ($< p$) be primes and $G$ be the semidirect product of $C_p$ by $C_{q^m}$ acting faithfully on $C_p$. Then $P(G)$ is a cograph.

Proof. By assumption, there are no elements of order $pq$, so the orders of the maximal cyclic subgroups are $p$ and $q^m$.

Theorem 4.4. If $G$ is a dihedral group of order $2m$, then $G$ is a power-cograph group if and only if $m$ is either a prime power or the product of two distinct primes.

Proof. The orders of maximal cyclic subgroups are 2 and $m$, and intersection of any two cyclic subgroup is the identity.

4.1 Remarks on solvable groups

Let $G$ be a solvable group and $G \in \mathcal{C}$. Let $F(G)$ be the Fitting subgroup of $G$. Then by Theorem 4.1, $F(G)$ is either of prime power order or a cyclic group of order $pq$, where $p$ and $q$ are distinct primes.

First, let $F(G) = C_{pq}$ for distinct primes $p$ and $q$. Then $F(G)$ contains its centraliser, and so is equal to it; so $G/F(G)$ acts as a group of automorphisms of $F(G)$. Thus $G$ is contained in the group $(C_p : C_{p-1}) \times (C_q : C_{q-1})$. If $G$ contains a direct product larger than $C_p \times C_q$, then this product is described by Theorem 2.1; it has the form $(C_p : C_{qm}) \times C_q$. If $G$ is strictly larger than

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this, then it contains an element of prime order \( r \) with \( r \mid p - 1 \) and \( r \mid q - 1 \), acting non-trivially on both \( C_p \) and \( C_q \). But then \( G \) contains a subgroup \( C_q \times (C_q : C_r) \), contrary to Theorem 2.1.

Otherwise the structure of \( G \) is \((C_p \times C_q).C_r\) where \( r \) divides both \( p - 1 \) and \( q - 1 \), and \( r \) is either a prime power or the product of two primes. Such a group is a PCG group, since its maximal cyclic subgroups have orders \( pq \) or \( r \).

Next suppose that \( F(G) \) be a \( p \)-group. We divide this case into two subcases.

If all the elements of \( G \) are of prime power order then the prime graph of \( G \) is a null graph, and hence \( G \in \mathcal{C} \). Higman [17] gave a nice characterization of such groups. And in that case \(|G|\) has at most two prime divisors and \( G/F(G) \) is one of the following:

(a) a cyclic group whose order is a power of a prime other than \( p \).

(b) a generalized quaternion group, \( p \) being odd; or

(c) a group of order \( p^aq^b \) with cyclic Sylow subgroups, \( q \) being a prime of the form \( kp^a + 1 \).

But difficulties arise when \( F(G) \) is a \( p \)-group and \( G \) contains elements whose order is not a prime power. By Theorem 1.1, the order of any element in a group in \( \mathcal{C} \) is either a prime power or the product of two primes. This case can occur; here are two examples:

**Example 4.5.** The Frobenius group \( F_7 \) of order 42 has \( P(F_7) \) a cograph. Here \(|F_7|\) is divisible by 3 primes, \( F_7 \) contains an element of order 6, and it’s Fitting subgroup \( C_7 \).

**Example 4.6.** Let \( G \) be the semidirect product of the Heisenberg group \( H_3 \) of order 27 by \( C_2 \). Then \( G \) is solvable and \( G \in \mathcal{C} \). In this case, the Fitting subgroup \( F(G) = H_3 \), and \( G \) contains elements of order 6.

**Problem 4.7.** Classify all solvable \( \mathcal{C} \)-groups whose Fitting subgroup is a \( p \)-group.
5 Finite simple groups

In this section we discuss simple groups whose power graphs are cographs. For each prime $p$, the simple group $C_p$ has complete power graph, therefore it is a power-cograph group. In the next theorem we classify alternating groups which are power-cograph groups.

**Theorem 5.1.** The alternating group $A_n$ is a power-cograph group if and only if $n \leq 6$.

**Proof.** For $n \geq 7$, the 4-6 test is applicable, with $a = (1, 2, 3, 4)$ and $b = (1, 3)(2, 4)(5, 6, 7)$.

Now we consider $n \leq 6$.

If $n = 3$ then $A_3$ is nothing but the cyclic group $C_3$ and hence its power graph is the complete graph $K_3$ and hence a cograph.

For $n = 4, 5, 6$ then prime graph of $A_n$ is a null graph and by Theorem 1.2 the power graph is a cograph. \qed

In the next few sections we discuss simple groups of Lie type of low rank or over small fields and sporadic simple groups. Information about specific groups is found in the ATLAS [13], and further information about the simple groups and their subgroups is in Rob Wilson’s book [26].

We also use the fact that $C$ is subgroup-closed; so, if a group $G$ contains a subgroup not in $C$, then $G \notin C$.

5.1 Simple groups of Lie type of rank 1

The simple groups of Lie type of rank 1 are $A_1(q) = \text{PSL}(2, q)$, $^2A_2(q) = \text{PSU}(3, q)$, $^2B_2(q) = \text{Sz}(q)$ where $q = 2^{2e+1}$, and $^2G_2(q) = R_1(q)$ where $q = 3^{2e+1}$.

In [5], Cameron proved that, if $q$ is an odd prime power, then the power graph of $\text{PSL}(2, q)$ is a cograph if a only if $(q - 1)/2$ and $(q + 1)/2$ are either prime powers or product of two primes. And if $q \geq 4$ is a power of 2 then the power graph of $\text{PSL}(2, q)$ is a cograph if and only if $q - 1$ and $q + 1$ are either prime powers or products of two distinct primes.

Next we show that power graph of $\text{PSU}(3, q)$ is not a cograph for $q \neq 2$. Since $\text{PSU}(3, 2)$ is not simple, there are no simple power-cograph groups of this type.
Theorem 5.2. Let $q$ be a power of an odd prime $p$. Then power graph of PSU$(3, q)$ is not a cograph.

Proof. We use the fact that PSU$(3, q)$, $q$ odd, has a cyclic subgroup of order $(q^2 - 1)/\gcd(q + 1, 3) = (q - 1) \cdot (q + 1)/\gcd(q + 1, 3)$. So, if the power graph is a cograph, then both $(q - 1)$ and $(q + 1)/\gcd(q + 1, 3)$ are primes, or else both are powers of the same prime. But both these numbers are even; so they must both be powers of 2. Since one of $q - 1$ and $q + 1$ is not divisible by 4, we must have $(q - 1, q + 1) = (2, 4)$ or $(4, 6)$, so $q = 3$ or 5.

Now for $q = 3$ the group PSU$(3, 3)$ contains elements of order 12, so the power graph is not a cograph. On the other hand PSU$(3, 5)$ contains $A_7$. Therefore the power graph of PSU$(3, q)$ is not a cograph. \\

Theorem 5.3. Let $q \geq 4$ be a power of 2. Then the power graph of PSU$(3, q)$ is not a cograph.

Proof. Let $\beta$ be a generator of the multiplicative group of GF$(q^2)$. Then $\beta^{q-1}$ has order $q + 1$. Let $p$ be a prime factor of $q + 1$ greater than 3, and let $d = (q + 1)/p$. Then $\alpha = \beta^{d(q-1)}$ has order $p$. Then $\alpha \beta = \beta^{d(q^2-1)}$, so $\alpha \beta = \beta^{d(q^2-1)} = 1$ in GF$(q^2)$. Consider the elements

$$
g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
h = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}
$$

$$
k = \begin{pmatrix} 0 & \alpha & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}
$$

Then $g$ is a element of order 2 and it commutes with $h$. So $o(gh) = o(hk) = 2p$. On the other hand $k^2 = h$.

Therefore the elements the elements $g, gh, h, k$ induce a path of length 3 in SU$(2, q)$.

Now observe that $g, h, k \in SU(3, q) \setminus Z$. Take $\gamma = gZ$ and $\eta = hZ$, $\kappa = kZ$. Then the elements $\gamma, \gamma \eta, \eta, \kappa$ induce a path of length 3 in the power graph of PSU$(3, q)$.

The argument fails for $q = 8$. But we saw earlier that PSU$(3, 8)$ is not a power-cograph group, using the 4-6 test. \qed
Theorem 5.4. Let $G = ^2B_2(q) = Sz(q)$, $q = 2^{2e+1}$. Then $G \in \mathcal{C}$ if and only if each of $q - 1$, $q + \sqrt{2q} + 1$ and $q - \sqrt{2q} + 1$ is either a prime power or the product of two distinct primes.

Proof. Any edge of the power graph is contained in a maximal cyclic subgroup. The maximal cyclic subgroups of Sz($q$) have orders 4, $q - 1$, $q + \sqrt{2q} + 1$ and $q - \sqrt{2q} + 1$. These four numbers are pairwise coprime. (The last three are odd. The difference between the third and fourth is a power of 2, but 2 does not divide either. Suppose that $p$ is a prime dividing both $q - 1 = 2^{2e+1} - 1$ and $q + \sqrt{2q} + 1 = 2^{2e+1} + 2^{e+1} + 1$. Then $p$ divides their difference, $2^{e+1} + 2$; since it is odd, it divides $2^{e+1} + 1$, and hence it divides $2^{2e} - 1$, and also $2^{2e+1} - 2$. This $p$ divides 1. The argument for $q - 1$ and $q - \sqrt{2q} + 1$ is similar.) Thus no element can lie in maximal cyclic subgroups of different orders. So, if the power graph contains $P_4$, then this $P_4$ must be contained in a maximal cyclic subgroup, so this subgroup must have three prime divisors, not all equal. The converse is clear. \hfill \Box

Now let $G = ^2G_2(q) = R_1(q)$, $q = 3^{2e+1}$. The centraliser of an involution in $G$ is $C_2 \times PSL(2, q)$, which contains subgroups $C_2 \times C(q \pm 1)/2$. So, if $G \in \mathcal{C}$, then $(q \pm 1)/2$ is either prime or a power of 2. If it is a power of 2, then we have a solution to Catalan’s equation, contradicting the result of Mihăilescu’s Theorem: see [12, Section 6.11]. The numbers $(q \pm 1)/2$ have opposite parity, so cannot both be prime. So $G$ is not a power-cograph group.

5.2 Simple groups of Lie type of rank 2

The rank 2 simple groups of Lie type are $A_2(q) = PSL(3, q)$, $C_2(q) = PSp(4, q)$, $^2A_3(q) = PSU(4, q)$, $^2A_4(q) = PSU(5, q)$, $G_2(q)$, $^2F_4(q)$ and $^3D_4(q)$. We examine each of the above cases. In the case of $A_2(q)$, we prove a slightly stronger result, for later use.

Theorem 5.5. Let $G$ be a quotient of $SL(3, q)$ by a subgroup of the group of scalars. If $G$ is a power-cograph group, then $q = 2$ or $q = 4$.

Proof. We work in $SL(3, q)$. Suppose that $q$ is odd. Consider the elements

$$
g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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It is easily checked that
\[ g^2 = h^3 = \begin{pmatrix} -I_2 & O \\ O & 1 \end{pmatrix} \]

So \((h^2, h, h^3 = g^2, g)\) is an induced path of length 3 in the power graph.

Now observe that neither \(g\) nor \(h\) contains any non-identity scalar matrix. So these elements project onto elements with the same property in the quotient when a group of scalars is factored out.

Now we consider \(q\) to be a power of 2, with \(q > 4\). If \(q\) is an odd power of 2, then \(q - 1\) is not divisible by 3, while if \(q\) is an even power of 2, then \(q - 1\) cannot be a power of 3 (according to the solution of Catalan’s equation) and so must have a larger prime divisor.

Let \(\alpha\) be an element of the multiplicative group of \(\text{GF}(q)\) of prime order \(p\) greater than 3. Consider the elements
\[
g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad k = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-2} & 0 \\ 0 & 0 & \alpha \end{pmatrix}.
\]

It is routine to check that
\[ g^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

an element of order 2; and that \(g^2\) commutes with \(k\), so that \(g^2k\) has order \(2p\), and \((g^2k)p = g^2\).

Putting \(h = g^2k\), we have \(h^p = g^2\), so the elements \(g, g^2 = h^p, h, h^2\) induce a path of length 3.

No power of any of these elements except the identity is a scalar. (For this we need \(p > 3\), since if \(p = 3\) then \(\alpha^{-2} = \alpha\).) So factoring out a group of scalars we get elements with the same properties.

Finally we note that \(\text{PSL}(3, 2)\) and \(\text{PSL}(3, 4)\) are power-cographs (as their Gruenberg–Kegel graphs are null). However, \(\text{PSL}(3, 2) \cong \text{PSL}(2, 7)\), so this group does not need to be included in the statement of the theorem.

\begin{center}\textbf{Theorem 5.6.} Let \(G = \text{PSp}(4, q)\). Then \(P(G)\) is not a cograph.\end{center}
Proof. A 4-dimensional symplectic space is the direct sum of two 2-dimensional symplectic spaces; and the 2-dimensional symplectic group is the special linear group. So $G = \text{PSp}(4,q)$ contains a subgroup which is the direct product of two copies of PSL$(2,q)$ if $q$ is even, or the central product of two copies of SL$(2,q)$ if $q$ is odd.

Thus $G$ contains the direct product of cyclic groups of orders $q \pm 1$ if $q$ is even, and a quotient of this by a subgroup of order 2 if $q$ is odd.

For $q$ even, $q - 1$ and $q + 1$ are coprime, so $P(G)$ is a cograph only if both are primes; since one is divisible by 3, this requires $q = 2$ or $q = 4$.

For $q$ odd, one of $(q - 1)/2$ and $(q + 1)/2$ is even, so the order of the cyclic subgroup is divisible by 4 and (if $q > 3$) by at least one further prime. So $P(G)$ is a cograph only if $q = 3$.

Now PSp$(4,2) \cong S_6$ is not simple; PSp$(4,3)$ contains elements of order 12; and PSp$(4,4)$ is ruled out by the 4-6 test.

**Theorem 5.7.** The power graph of $G_2(q)$ is not a cograph.

Proof. The group $G_2(q)$ contains both SL$(3,q)$ and SU$(3,q)$ [14, 20]. Now SL$(3,q) = \text{PSL}(3,q)$ if $q \not\equiv 1 \pmod{3}$, while SU$(3,q) = \text{PSL}(3,q)$ if $q \not\equiv -1 \pmod{3}$. So, for any $q$, $G_2(q)$ contains either PSL$(3,q)$ or PSU$(3,q)$. Now the former is in $\mathcal{C}$ only for $q = 2$ or $q = 4$, and the latter is never in $\mathcal{C}$ except for $q = 2$ (this group is not simple). So the only case needing further consideration is $q = 2$; but $G_2(2)$ is not simple, and is not in $\mathcal{C}$ (it contains PSU$(3,3)$ as a subgroup of index 2).

Below we give arguments for the rest of the simple groups of Lie type of rank 2. We find that in each of the following cases the power graph is not a cograph.

- Let $G = ^2A_3(q) = \text{PSU}(4,q)$. This group contains PSp$(4,q)$, so we only need consider $q = 2$. But PSp$(4,2) \cong \text{PSp}(4,3)$.
- The group $G = ^2A_4(q) = \text{PSU}(5,q)$ contains PSU$(4,q)$. So $G \notin \mathcal{C}$.
- The group $^2F_4(2^d)$ contains $^2F_4(2)$ for all odd $d$ (Malle [22]), and $^2F_4(2)$ is ruled out by the 4-6 test.
- The group $G = ^3D_4(q)$ contains $G_2(q)$ (see Kleidman [19]).
5.3 Higher rank

Let \( G \) be a simple group of Lie type of higher rank. We show that \( P(G) \) is not a cograph.

Since the Dynkin diagram of \( G \) contains a single bond in all cases, \( G \) has a subgroup of a Levi factor which is a quotient of \( \text{SL}(3, q) \) by a group of scalars. The results of the preceding section give the desired conclusion if \( q \notin \{2, 4\} \).

It remains to deal with groups over the fields of 2 or 4 elements.

Now \( \text{PSL}(4, 2) \cong A_8 \), so its power graph is not a cograph, while \( \text{PSp}(6, 2) \) is excluded by the 4-6 test. Moreover, \( \text{PSL}(4, 4) \) contains \( \text{PSL}(4, 2) \), and \( \text{PSp}(6, 4) \) contains \( \text{PSp}(6, 2) \) (by restricting scalars). The orthogonal and unitary groups of Lie rank 3 all contain \( \text{PSp}(4, q) \) for \( q = 2 \) or \( q = 4 \). So \( P(G) \) is not a cograph.

5.4 Sporadic simple groups

Now we prove that there exist no sporadic simple group whose power graph is cograph. Recall that there are 26 sporadic simple groups [13], namely, the five Mathieu groups \((M_{11}, M_{12}, M_{22}, M_{23} \text{ and } M_{24})\), four Janko groups \((J_1, J_2, J_3 \text{ and } J_4)\), three Conway groups \((Co_1, Co_2 \text{ and } Co_3)\), three Fischer groups \((Fi_{22}, Fi_{23} \text{ and } Fi_{24})\), Higman–Sims group \((HS)\), the McLaughlin group \((M^c L)\), the Held group \(He\), the Rudvalis group \(Ru\), the Suzuki group \((Suz)\), the O’Nan group \((O’N)\), the Harada–Norton group \(HN\), the Lyons group \((Ly)\), the Thompson group \((Th)\) the Baby Monster group \((B)\) and the Monster group \((M)\). Amongst these 26 groups the the Mathieu group \(M_{11}\) is of smallest order \(|M_{11}| = 7920 = 2^4 \cdot 3^3 \cdot 5 \cdot 11\).

Observation 5.8. We observe, using information in the ATLAS of Finite Groups [13], that \(M_{11}\) is not a power-cograph group, by the 4-6 test; it contains elements \(a, b\) of orders 4 and 6 respectively with \(a^2 = b^3\).

Theorem 5.9. Let \( G \) be a sporadic simple group. Then \( P(G) \) is not a cograph.

Proof. Observation 5.8 shows that the power graph of the Mathieu group \(M_{11}\) is not a cograph. Now the Mathieu group \(M_{11}\) is a subgroup of all the other sporadic simple groups except \(J_1, M_{22}, J_2, J_3, He, Ru\) and \(Th\). So the power graphs of these groups are also not cographs.
For the other seven groups we look for subgroups which are not power-cograph groups. We observe that $J_1$ contains $D_3 \times D_5$, $M_{22}$ contains $A_7$, $J_2$ contains $A_4 \times A_5$, $J_3$ contains $C_3 \times A_6$, $He$ contains $S_7$, $Ru$ contains $A_8$ and $Th$ contains $PSL(2, 19) : C_2$. By Theorems 2.1, 4.2 and 5.1, the power graphs of these subgroups are not cographs. Hence the power graphs of the original groups are not cographs.

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