Geometric Engineering of $\mathcal{N}=2$ CFT$_4$s based on Indefinite Singularities: Hyperbolic Case

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Abstract

Using Katz, Klemm and Vafa geometric engineering method of $\mathcal{N}=2$ supersymmetric QFT$_4$s and results on the classification of generalized Cartan matrices of Kac-Moody (KM) algebras, we study the un-explored class of $\mathcal{N}=2$ CFT$_4$s based on indefinite singularities. We show that the vanishing condition for the general expression of holomorphic beta function of $\mathcal{N}=2$ quiver gauge QFT$_4$s coincides exactly with the fundamental classification theorem of KM algebras. Explicit solutions are derived for mirror geometries of CY threefolds with hyperbolic singularities.

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1 Introduction

During the last few years supersymmetric $d$ dimension conformal field theories (CFT$_d$) have been subject to an intensive interest in connection with superstring compactifications on Calabi-Yau manifolds (CY) and AdS/CFT correspondence. An important class of these CFTs correspond to those embedded in type II string compactifications on elliptic fibered CY threefolds with $ADE$ singularities preserving eight supersymmetries. These field models, which give exact solutions for the moduli space of the Coulomb branch and which admit a very nice geometric engineering in terms of quiver diagrams, were shown to be classified into two categories according to the type of singularities: (i) $\mathcal{N}=2$ CFT$_4$ based on finite $ADE$ singularities; with gauge group $G = \prod \text{SU}(n_i)$ and matters in both fundamental $n_i$ and bi-fundamental $(n_i, n_j)$ representations of $G$. (ii) $\mathcal{N}=2$ CFT$_4$ with gauge group $G = \prod \text{SU}(s_i n)$ and bi-fundamental matters only. This second category of scale invariant field models are classified by affine $ADE$ Lie algebras. The positive integers $s_i$ appearing in $G$ are the usual Dynkin weights; they form a special positive definite integer vector $s = (s_i)$ satisfying $K_{ij} s_j = 0$

$$K_{ij} n_j = 0,$$

where $n_j = n s_j = 0$ and where $K$ is the Cartan matrix. The appearance of this remarkable eq in the geometric engineering of $\mathcal{N}=2$ CFT$_4$s is very exciting; first because $4d$ conformal invariance, requiring the vanishing of the holomorphic beta function, is now translated into a condition on allowed Kac-Moody-Lie algebras eq(1). Second, even for $\mathcal{N}=2$ CFT$_4$ based on finite $ADE$ with $m_i n_i$ fundamental matters, the condition for scale invariance may be also formulated in terms of $K$ as,

$$K_{ij} n_j = m_i.$$

The identities can be rigorously derived by starting from mirror geometry of type IIA string on Calabi-Yau threefolds and taking the field theory limit in the weak gauge couplings $g_r$ regime associated with large volume base ($V_r = 1/\varepsilon$ with $\varepsilon \to 0$). In this limit, one shows that complex deformations $a_r l$ of the mirror geometry scale as $\varepsilon^{l-k_r-1}$ and the universal coupling parameters $Z^{(a_r)}$ (given by special ratios of $a_{r,l}$ and $a_{r, \pm 1,l}$) behave as $\varepsilon^{-b_r}$ where $b_r$ is the beta function coefficient of the $r$-th $U(n_r)$ gauge sub-group factor of $G$. Scale invariance of the CFT$_4$s requires $b_r = 0 \forall r$ and turns out to coincide exactly with eqs(1,2) depending on the type of $ADE$ singularities one is considering; i.e affine or finite. The third exciting feature we want to give here is that both eqs(1,2) can be viewed as just the two leading relations of the triplet

$$K^{(q)}_{ij} n_j = q m_i; \quad q = 0, +1, -1.$$

The extra relation namely $K^{(-)}_{ij} n_j = -m_i$ describes the so called indefinite subset in the classification of Kac-Moody-Lie algebras in terms of generalized Cartan matrices. If one forgets for a while the algebraic geometry aspect of singular surfaces and focus on the classification of $\mathcal{N}=2$
CFTs listed above, one learns from the striking resemblance between the three sectors of eqs(3) that it is legitimate to ask why not a third class of $\mathcal{N} = 2$ CFTs associated with the third sector of eq(3). A positive answer to this question will not only complete the picture on $\mathcal{N} = 2$ CFTs embedded in type II strings on CY threefolds given in [10]; but will also open an issue to approach singularities based on indefinite Lie algebras.

The triplet (3) is not the unique motivation for our interest in this third kind of $\mathcal{N} = 2$ supersymmetric scale invariant field theory. There is also another strong support coming from geometric engineering of $\mathcal{N} = 2$ QFTs [10, 11, 12]. There, the geometric engineering of fundamental matters requires the introduction of the so called trivalent vertex. This vertex has a Mori vector $q_\tau = (1, -2, 1; 1)$ with five entries (sometimes called also CY charges). The first three ones are common as they are involved in the geometric engineering of gauge fields and bi-fundamental matters. They lead to eq(1). The fourth entry is used in the engineering of fundamental matters of CFTs based on finite ADE and lead to eq(2). But until now, the fifth entry has been treated as a spectator only needed to ensure the CY condition $\sum_{\tau=1}^{5} q_\tau = 0$. Handling this vertex on equal footing as the four previous others gives surprisingly the missing third sector of eqs(3) we are after.

In this paper, we study the remarkable class of the yet un-explored $\mathcal{N} = 2$ CFTs associated with the third sector of eq(3). To achieve this goal, we have to prove that this kind of $\mathcal{N} = 2$ CFTs really exist by computing the general expression of the holomorphic beta function and then show that the moduli space of their solutions is non trivial. To do so, we have to develop the following: (i) study indefinite singularities of CY threefolds in relation with the third sector of eqs(3). (ii) develop the geometric engineering of $\mathcal{N} = 2$ QFTs embedded in type II strings on these class of singular CY 3-folds and then solve $\mathcal{N} = 2$ QFT scale invariance constraint eqs. Moreover, as the algebraic geometry of CY manifolds with indefinite singularities, like the classification of indefinite Lie algebras, are still open questions, we will approach our problem throughout explicit examples. Fortunately, we dispose actually of partial results dealing the classification of a subset of indefinite Lie algebras. This concerns the so called hyperbolic subset having Dynkin diagrams very closed to the usual finite and affine Lie algebras. Since we know much about finite and affine ADE geometries, one suspects to get here also exact results for hyperbolic mirror geometries. In addition to the explicit results one expects, the study of hyperbolic singularities is particularly interesting because it will give us more insight on the way Katz, Mayr and Vafa basic result regarding ADE bundles on elliptic curve could be generalized to include indefinite singularities.

The presentation of this paper is as follows: In section 2, we review the computation of the general expression of beta function of $\mathcal{N} = 2$ QFTs based on trivalent geometry. Here we show that the general solutions for $\mathcal{N} = 2$ CFT scale invariance condition coincides exactly with the Lie algebraic classification eq(3). The missing sector of $\mathcal{N} = 2$ CFT turns out to be intimately related to the extra fifth entry of $q_\tau$. In section 3, we study indefinite Lie algebras and
too particularly their hyperbolic subset as well links with $\mathcal{N} = 2$ supersymmetric field theories in 4 dimensions. In section 4, we expose our explicit solutions for $\mathcal{N} = 2$ CFTs based on hyperbolic singularities and in section 5, we give the conclusion and perspectives.

2 Trivalent geometry and Beta function

In [10], trivalent geometry has been introduced to engineer $\mathcal{N} = 2$ supersymmetric QFTs (with fundamental matter) embedded in type IIA string theory on CY threefolds with ADE singularities. This geometry extends the standard ADE Dynkin diagrams and involves higher dimension vertices [10, 12]. In this section, we give the necessary tools one needs for the computation of the holomorphic beta function of $\mathcal{N} = 2$ QFTs. Details and techniques regarding this geometry can be found in [10, 11] and subsequent works on this subject [12, 13].

2.1 Trivalent geometry

To illustrate the ideas, we start by considering the case of a unique trivalent vertex; then we give the results for chains of trivalent vertices.

a) Case of one trivalent vertex

Since trivalent vertices depend on the kind of the Dynkin diagrams one is using, we will fix our attention on those appearing in linear chains involving $A_k$ type singularities. A quite similar analysis is also valid for $D_k$ graphs. In the case of $A_k$ type diagrams, trivalent geometry is described by the typical three dimensional vertices $V_i$,

$$V_0 = (0,0,0); \quad V_1 = (1,0,0); \quad V_2 = (0,1,0); \quad V_3 = (0,0,1); \quad V_4 = (1,1,1) \quad (4)$$

satisfying the following toric geometry relation

$$\sum_{i=0}^4 q_i V_i = -2V_0 + V_1 + V_2 + V_3 - V_4 = 0 \quad (5)$$

The vector charge $(q_i) = (-2,1,1,1,-1)$ is known as the Mori vector and the sum of its $q_i$ components is zero as required by the CY condition; $\sum_i q_i = 0$. In type IIB strings on mirror CY3, the $(V_0, V_1, V_2, V_3, V_4)$ vertices are represented by complex variables $(u_0, u_1, u_2, u_3, u_4)$ constrained as $\prod_i u_i^{q_i} = 1$ and solved by $(1, x, y, z, xyz)$; see figure 1. In terms of these variables, the algebraic eq describing mirror geometry, associated to $\text{eq}(5)$, is given by the following complex surface,

$$P(X^*) = e + ax + by + (c - dxy)z, \quad (6)$$

where $a, b, c, d$ and $e$ are complex moduli. Note that upon eliminating the $z$ variable, the above (trivalent) algebraic geometry eqs reduces exactly to the standard bivalent vertex of $A_1$ geometry,

$$P(X^*) = ax + e + \frac{bc}{d} \frac{1}{x} \quad (7)$$
The monomials $y_0 = x$, $y_1 = 1$ and $y_2 = \frac{4}{x}$ satisfy the well known $su(2)$ relation namely $y_0 y_2 = y_1^2$.

To get algebraic geometry eq of the CY3, one promotes the non zero coefficients $a, b, c, d$ and $e$ to holomorphic polynomials in $CP^1$ as follows:

$$e = \sum_{i=0}^{n_r} c_i w^i; \quad a = \sum_{i=0}^{n_{r-1}} a_i w^i; \quad b = \sum_{i=0}^{n_{r+1}} b_i w^i,$$

$$c = \sum_{i=0}^{m_r} c_i w^i; \quad d = \sum_{i=0}^{m'_r} d_i w^i; \quad e_0, a_0, b_0, c_0, d_0 \neq 0.$$

(8)

These analytic polynomials encode the fibrations of $SU(1 + n_r - 1) \times SU(1 + n_r) \times SU(1 + n_{r+1})$ gauge and $SU(1 + m_r) \times SU(1 + m'_r)$ flavor symmetries; of the underlying $N = 2$ QFT4; engineered over the nodes of the trivalent vertex. For instance $SU(1 + n_{r-1})$ gauge symmetry is fibered over $V_0$ and $SU(1 + m_r)$ and $SU(1 + m'_r)$ flavor invariances are fibered over the nodes $V_3$ and $V_4$ respectively, see figure 2.

Note that the functions $a, b, c, d$ and $e$ are not all of them independent as one can usually fix one of them. In [10], the coefficient $d$ of eq(7) was set to one and $c$ kept arbitrary in order to geometric engineer the needed fundamental matters for finite ADE CFT4s. Here we will keep all of these moduli arbitrary and fix one of them only at appropriate time. The reason is that by fixing one of these moduli from the beginning; one rules out a full sector in the moduli space of CFT4s as it has been the case for the CFT4 sector associated with indefinite Lie algebras we want to study here.

$N = 2$ QFT4 limit

To get the various $N = 2$ CFT4s embedded in type II strings on CY3-folds, we have to study the field theory limit one gets from mirror geometry of type IIA string on CY3 and look for the scaling properties of the gauge coupling constants moduli. We will do this for the case of one trivalent vertex eq(6) and then give the general result for a chain of several trivalent vertices. To that purpose, we proceed in three steps: First determine the behaviour of the complex moduli $f_i$
This graph describes a typical vertex one has in trivalent geometric engineering of $\mathcal{N} = 2$ supersymmetric QFT$_4$. SU $(1 + l)$ gauge and flavour symmetries are fibered over the five black nodes. Flavor symmetries require large base volume.

appearing in the expansion eq(8) under a shift of $w$ by $1/\varepsilon$ with $\varepsilon \to 0$. Doing this and requiring that eqs(8) should be preserved, that is still staying in the singularity described by eqs(8), we get the following,

$$
e_l \sim \varepsilon^{l-n_r}; \quad a_l \sim \varepsilon^{l-n_r-1}; \quad b_l \sim \varepsilon^{l-n_r+1};$$

$$c_l \sim \varepsilon^{l-m_r}; \quad d_l \sim \varepsilon^{l-m'_r}. \quad (9)$$

Second compute the scaling of the gauge coupling constant moduli $Z^{(g)}$ under the shift $w' = w + 1/\varepsilon$. Putting eqs(9) back into the explicit expression of $Z^{(g)}$ namely

$$Z^{(g)} = \frac{a_0 b_0 c_0}{d_0^2}, \quad (10)$$

we get the following behaviour $Z^{(g)} \sim \varepsilon^{-b_r}$ with $b_r$ given by,

$$b_r = 2n_r - n_{r-1} - n_{r+1} - (m_r - m'_r). \quad (11)$$

In the limit $\varepsilon \to 0$, finiteness of $Z^{(g)}$ requires then that the field theory limit should be asymptotically free; that is $b_r \leq 0$. Satured bound $b_r = 0$ corresponds to scale invariance we are interested in here. With these relations at hand, let us see how they extend to the case of several trivalent vertices.

b) Chains of trivalent vertices
Thinking about the vertices of eq(5) as a generic trivalent vertex of a linear chain of $N$ trivalent vertices, that is

$$
\begin{align*}
V_0 & \to V_0^0; & V_3 & \to V_0^+; & V_4 & \to V_0^-; \\
V_1 & \to V_{\alpha-1}^0; & V_2 & \to V_{\alpha+1}^0,
\end{align*}
$$

(12)

and varying $\alpha$ on the set $\{1, \ldots, N\}$ together with intersections between $V_0^\alpha$ and $V_{\alpha\pm1}^\alpha$ specified by Mori vector $q^\alpha_i$, one can build more general toric geometries. In the generic case, the analogue of eq(5) extends as

$$
\sum_{\alpha \geq 0} (q^\alpha_i V_0^\alpha + V_0^i - V_i^-) = 0.
$$

(13)

Note that the $\pm$ upper indices carried by the $V_i^\pm$ vertices refer to the fourth +1 and five −1 entries of the Mori vector $q^\alpha_i = (q^\alpha_i^+; +1, -1)$ of trivalent vertex. In practice, the Mori vectors $q^\alpha_i$ s form a $N \times (N + s)$ rectangular matrix whose $N \times N$ square sub-matrix $q^\alpha_i$ is minus the generalized Cartan matrix $K_{ij}$. For the example of affine $A_{N-1}$, the Mori charges read as $q^\alpha_i = 2\delta^\alpha_i - \delta^i_0 - \delta^i_0$ with the usual periodicity of affine $SU(N)$. The remaining $N \times s$ part of $q^\alpha_i$ is fixed by the Calabi-Yau condition $\sum_{\alpha} q^\alpha_i = 0$ and the corresponding vertices are interpreted as dealing with non compact divisors defining the singular space on which live singularities. In mirror geometry where $x_{\alpha-1}$, $x_{\alpha}$, $x_{\alpha+1}$, $y_{\alpha}$, and $\frac{x_{\alpha-1}x_{\alpha+1}y_{\alpha}}{y_{\alpha}^2}$ are the variables associated with the vertices $\{12\}$, eq(6) extends as

$$
\begin{align*}
&\alpha - 1 \times x_{\alpha-1} + a_0 x_{\alpha} + a_{\alpha+1} x_{\alpha+1} + c_0 y_{\alpha} + d_0 \frac{x_{\alpha-1}x_{\alpha+1}y_{\alpha}}{y_{\alpha}^2} = 0 \\
&\text{where } a_0, c_0, \text{ and } d_0 \text{ are complex moduli. Summing over the vertices of the chain and setting } y_{\alpha} = x_{\alpha} z_{\alpha}, \text{ one gets}
\end{align*}
$$

$$
P(X^*) = a_0 x_0 + \sum_{\alpha \geq 1} \left( a_0 x_\alpha + c_0 x_\alpha z_\alpha + d_0 \frac{x_{\alpha-1}x_{\alpha+1}z_\alpha}{x_\alpha} \right). 
$$

(14)

Eliminating the variable $z_\alpha$ as we have done for eq(7), we obtain

$$
P(X^*) = \sum_{\alpha \geq 0} x^\alpha a_\alpha (w) \prod_{\beta \geq 1} \left( c_\beta (w) \right)^{\alpha-\beta} d_\beta (w).
$$

(15)

We will use this expression as well as eqs(8) when building mirror geometries of CY threefolds with hyperbolic singularities.

2.2 Classification of $\mathcal{N} = 2$ CFT$_3$s

Using eq(11), and focusing on the interesting situation where

$$
n_\alpha = m_\alpha = m'_\alpha = 0 \quad \text{for } \alpha > N,
$$

(16)

and all remaining others are non zero, the condition for conformal invariance of the underlying $\mathcal{N} = 2$ supersymmetric gauge theory reads, in terms of generalized Cartan matrices $K$ of Lie algebras, as

$$
K_{ij} n_j - (m_i - m'_i) = 0; \quad 1 \leq i \leq N.
$$

(17)
This is a very remarkable relation; first because it can be put into the form eq(3) and second its solutions, which depend on the sign of \((m_i - m'_i)\), are exactly given by the fundamental theorem on the classification of Lie algebras. Let us first recall this theorem and then give our solutions.

a) Theorem I: Classification of Lie algebras

A generalized indecomposable Cartan matrix \(K\) obey one and only one of the following three statements:

i) \textit{Finite type} (\(\det K > 0\)): There exist a real positive definite vector \(u\) (\(u_i > 0; i = 1, 2, \ldots\)) such that

\[
K_{ij}u_j = v_j > 0.
\]  

(18)

ii) \textit{Affine type}, \(\text{corank}(K) = 1, \det K = 0\): There exist a unique, up to a multiplicative factor, positive integer definite vector \(n\) (\(n_i > 0; i = 1, 2, \ldots\)) such that

\[
K_{ij}n_j = 0.
\]  

(19)

iii) \textit{Indefinite type} (\(\det K \leq 0\), \(\text{corank}(K) \neq 1\)): There exist a real positive definite vector \(u\) (\(u_i > 0; i = 1, 2, \ldots\)) such that

\[
K_{ij}u_j = -v_i < 0.
\]  

(20)

Eqs (18-20) combine together to give eq(3). These eqs tell us, amongst others, that whenever there exist a real (integer) positive definite vector \(u\) such that \(K_{ij}u_j < 0\), then the Cartan matrix is of indefinite type.

b) Solutions of eq(17)

Setting \(u_i = n_i\) and \(v_i = |m_i - m'_i|\) in the constraint eq(17) required by the vanishing of the beta function, one gets the general solutions for \(N = 2\) supersymmetric conformal invariance in four dimensions. The previous theorem teaches us that there should exist three kinds of \(N = 2\) CFTs in one to one correspondence with finite, affine and indefinite Lie algebras. These CFTs are as follows:

(i) \(N = 2\) CFTs based on \textit{finite} Lie algebras; this subset is associated with the case \(m_i > m'_i \geq 0\).

(ii) \(N = 2\) CFTs based on \textit{affine} Lie algebras; they correspond to the case \(m_i = m'_i\) and too particularly \(m_i = m'_i = 0\) considered in QFT literature.

(iii) \(N = 2\) CFTs based on \textit{indefinite} Lie algebras; they correspond to the case \(0 \leq m_i < m'_i\).

This result generalizes the known classification concerning \(N = 2\) CFTs using \textit{finite} and \textit{affine} Lie algebras which are recovered here by setting \(m'_i = 0\). For the other remarkable case where \(m_i = 0\); but \(m'_i > 0\), we get the missing third class of \(N = 2\) CFTs involving indefinite Lie algebras. Before giving the geometric engineering of these \(N = 2\) CFTs, note that the beta function eqs (11) is invariant under the change \(m_i \rightarrow m_i - m^*_i\) and \(m'_i \rightarrow m'_i - m^*_i\). By appropriate choices of the positive numbers \(m^*_i\); for instance by taking \(m^*_i = m_i\) or setting \(m^*_i = m'_i\) one can usually rewrite eq(17) as in eq(6) namely \(K^{(q)}_{ij}n_j = qm_j\) with \(q = 0, \pm 1\). This symmetry reflects just the freedom to fix one of the complex moduli of eq(14).
3 More on Indefinite Lie algebras

As the subject of indefinite Lie algebras is still a mathematical open problem since the full classification of their generalized Cartan matrices has not yet been achieved, we will focus our attention here on the Wanglai Li special subset [14, 15], known also as hyperbolic Lie algebras. This is a subset of indefinite Kac-Moody-Lie algebras which is intimately related to finite and affine ones and on which we know much about their classification. The results we will derive here concerns these hyperbolic algebras; but they apply as well to other kinds of indefinite Lie algebras that are not of Wanglai Li type. For other applications of hyperbolic Lie algebras in string theory; see [17, 18] - [19, 20].

To start, note that the derivation of hyperbolic Lie algebras is based on the same philosophy one uses in building affine Lie algebras \( \hat{g} \) from finite ones \( g \) by adding a node to the Dynkin diagram of \( g \). Using this method, Wanglai Li constructed and classified the 238 possible Dynkin diagrams of the hyperbolic Lie algebras from which one derives their generalized Cartan matrices. The corresponding diagrams, which were denoted in [14] as \( \mathcal{H}_i^n; i = 1, ..., \) contain as a sub-diagram of co-order 1, the usual Dynkin graphs associated with \( \hat{g} \) and \( g \) Lie algebras. In other words, by cutting a node of an order \( n \) hyperbolic Dynkin diagram, the resulting \((n - 1)\)-th sub-diagrams one gets is one of the two following: (i) either it coincides with one of the Dynkin graphs of \( g \); or (ii) it coincides with an affine \( \hat{g} \) one. The general structure of hyperbolic Dynkin diagrams are then of the form figure 3;

![Figure 3](image)

Figure 3: Figures 3a and 3b represent the two kinds of Dynkin diagrams of Hyperbolic Lie algebras. These diagrams are built by adding one node to given finite and affine Dynkin graphs. The full set of hyperbolic Dynkin diagrams may be found in [18].

In what follows we give some features of \( \mathcal{H}_i^n \) hyperbolic algebras that are close to finite and affine ADE.

3.1 Hyperbolic Lie algebras

Following [14, 15, 16], an indecomposable generalized Cartan matrix \( K \) is said to be hyperbolic (resp strictly hyperbolic) if: (1) it is of indefinite type and (2) any connected proper
sub-diagram of the Dynkin diagram, associated with the matrix $K$, is of finite or affine (rep finite) type.

**Classification**

Motivated by physical applications, we prefer to rearrange the $H^n_i$ Wanglai Li hyperbolic algebras into two types as follows:

(1) **Type I hyperbolic Lie algebras, ($Type I H$).** They concern the $H^n_i$'s that contain order $(n - 1)$ affine Lie algebras. These $H^n_i$ algebras are given by the following list:

- **Simply laced $Type I H$:** A careful inspection of the Wanglai Li classification shows that simply laced diagrams are:
  
  $H^4_1, H^4_2, H^4_3, H^5_1, H^5_2, H^5_3, H^6_4, H^6_5, H^7_1$. 

  These hyperbolic algebras contain the well known affine $ADE$ as maximal Lie subalgebras; some of them have internal discrete automorphisms. Note in passing that $H^n_i$ Dynkin graphs with $D$ and $E$ sub-diagrams are denoted, in trivalent geometry as $T(p,q,r)$ or again as $DE_s$.

- **Non simply laced $Type I H$:** There are 63 non simply laced diagrams of $Type I H$; those having an order greater than 6 are as follows
  
  $H^8_6, H^8_7, H^9_8, H^9_9, H^9_{10}, H^{10}_5$. 

  the others can be found in [15].

(2) **Type II hyperbolic Lie algebras, ($Type II H$).** They concern the $H^n_i$'s that are not in $Type I H$ list; their Dynkin diagrams do contain no order $(n - 1)$ affine sub-diagram once cutting a node.

**Hyperbolic Symmetries**

In studying Wanglai Li hyperbolic symmetries, one should distinguish two cases: hyperbolic algebras of order two and those with orders greater than two. For the first ones there are infinitely many, whereas the number of the second type is finite. The two following theorems classify these algebras.

- **Theorem II:** Let $K$ be an indecomposable generalized Cartan matrix of order 2 with $K_{11} = K_{22} = 2$ and $K_{12} = -a$ and $K_{21} = -b$ where $a$ and $b$ are positive integers; then, we have the following classification

  (i) $K$ is of **finite** type if and only if $det K > 0$; i.e $ab < 4$

  (ii) $K$ is of **affine** type if and only if $det K = 0$; i.e $ab = 4$

  (iii) $K$ is of **indefinite** type if and only if $det K < 0$; i.e $ab > 4$

  From this result, one sees that while $ab \leq 4$ has a finite number of solutions for positive integers
There are however infinitely many for \( ab > 4 \); but no one of the corresponding algebra is simply laced. For orders greater than two, there is a similar classification; but the following theorem shows that, for \( n \geq 3 \), there exist however a finite set of hyperbolic algebras.

**b) Theorem III**

The full list of the Dynkin diagrams of hyperbolic Cartan matrices for \( 3 \leq n \leq 10 \) is given in [14, 15]; see also [16]. There are altogether 238 hyperbolic diagrams, 35 diagrams are strictly hyperbolic ones and 142 diagrams are symmetric or symmetrisable.

(i) The orders \( n \) of strictly hyperbolic Cartan matrices are bounded as \( 2 \leq n \leq 5 \)

(ii) The orders \( n \) of a hyperbolic Cartan matrices are bounded as \( 2 \leq n \leq 10 \)

### 3.2 From Affine to Hyperbolic

In this subsection, we study simply laced typeI\( H \) hyperbolic Lie algebras [21]; see also figure 2. This subset is more a less simple to handle when building mirror geometries of \( \mathcal{N} = 2 \) CFT\(_4\)s embedded in type IIA strings on CY Threefolds. Simply laced hyperbolic geometries has no branch cuts; but can be also extended to the case of general geometries associated with non simply laced hyperbolic symmetries; especially those obtained from simply laced typeI\( H \) algebras by following the idea of folding used in [12, 21]-[22]. Moreover, as simply laced typeI\( H \) algebras contain affine ADE symmetries as maximal subalgebras, we will use results on affine geometries to derive the mirror geometries associated with typeI\( H \). To do so, we start from the known results on affine ADE geometries and look for generalizations that solve the constraint eqs required by typeI\( H \) invariance. For pedagogical reasons, we will illustrate our method on the four examples of figure 4.

Generalization to other diagrams is straightforward. Let us first give some particular features on these special hyperbolic Lie algebras, then consider the building of the corresponding mirror geometries and hyperbolic \( \mathcal{N} = 2 \) CFT\(_4\)s.

- \( \mathcal{H}_3^4 \) hyperbolic Lie algebra

From the Dynkin diagram of figure 4a, one derives the following generalized Cartan matrix with diagonal entries 2 associated with the \((1 + 3)\) nodes \( N_i \) and off diagonal negative integers describing intersections

\[
\mathbf{K} \left( \mathcal{H}_3^4 \right) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & -1 \\
0 & -1 & -1 & 2
\end{pmatrix}.
\]

This matrix \( \mathbf{K}_{ij}, 0 \leq i, j \leq 3 \) has a negative determinant \( \det \mathbf{K} = -3 \) and exhibits remarkable properties in agreement with the classification of [16]. \( \mathbf{K} \left( \mathcal{H}_3^4 \right) \) contains the Cartan matrix of affine \( A_2 \) and that of finite \( A_3 \) as \( 3 \times 3 \) sub-matrices. The first one is obtained by subtracting
Figure 4: Figures 4a, 4b, 4c and 4d represent four examples of simply laced Dynkin diagrams of order three and four Hyperbolic Lie algebras. The simple root system as well as Cartan matrices associated with these graphs may be easily read from the \( \widehat{A}_2 \), \( \widehat{A}_3 \) and \( \widehat{D}_5 \) affine ones.

the first row and column of (23) or equivalently by cutting the node \( N_0 \) of figure 4a. The second is recovered by subtracting the last row and column which correspond to cutting \( N_3 \).

- \( \mathcal{H}^5_1 \) hyperbolic Lie algebra

Figure 4b leads to the following \( \mathcal{H}^5_1 \) generalized Cartan matrix ,

\[
K (\mathcal{H}^5_1) = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & -1 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & -1 & 2
\end{pmatrix}.
\]  

The determinant of this matrix is equal to \(-4\) and its Dynkin diagram contains those of affine \( A_3 \), finite \( A_4 \) and finite \( D_4 \) as three sub-graphs of order 4. Like for TypeI\( \mathcal{H}^4_0 \), the Dynkin graph of \( \mathcal{H}^5_1 \) has a \( Z_2 \) internal automorphism fixing three nodes and interchanging two.

- \( \mathcal{H}^7_1 \) and \( \mathcal{H}^8_1 \) hyperbolic Lie algebras
Denoted also as $DE_7$ in [16], the generalized Cartan matrix for $H_7$ reads as,

\[
K(H_7) = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

This is a hyperbolic Lie algebra of order 7 and has three order six subalgebras namely affine $D_5$, finite $D_6$ and finite $E_6$. Similarly, one can check that the order seven subalgebras of $H_8$ ($\equiv DE_8$) are affine $D_6$, finite $D_7$ and finite $E_7$.

4 Hyperbolic CFTs

Following [10], mirror geometries of type IIA string on CY threefolds with ADE singularities are described by an algebraic geometry eq of the form

\[
P(X^*) = \sum \alpha a_\alpha y_\alpha,
\]

where $a_\alpha = a_\alpha (w)$ are complex moduli with expansion of type eq[13] and where the $y_\alpha$ complex variables are constrained as

\[
\prod_{j=1}^{n} y_\alpha^q_j = \prod_{\alpha=n+1}^{n+4} y_\alpha^{-q_\alpha}.
\]

In this eq $q_j$ is minus $K_{ij}$ and $y_\alpha$, with $n < \alpha < n + 5$, are some extra complex variables that define the elliptic curve on which lives the singularity.

4.1 Flat ADE bundles on elliptic curve

In the case of affine ADE singularities, the $n$ first variables $y_i$ are solved in terms of the the four extra ones, $y_{n+1}, y_{n+2}, y_{n+3}$ and $y_{n+4}$, which themselves are realized as

\[
y_{n+1} = y^2, \quad y_{n+2} = x^3, \quad y_{n+3} = z^6, \quad y_{n+4} = xyz,
\]

where $(y, x, z)$ are the homogeneous coordinates of the weighted projective space $\mathbb{W}P^2 (3, 2, 1)$. These monomials generate an elliptic curve,

\[
y^2 + x^3 + z^6 + xyz = 0,
\]

embedded in $\mathbb{W}P^2 (3, 2, 1) \subset \mathbb{W}P^3 (3, 2, 1, 6 - N)$. Using $y, x, z$ and $v$ coordinates of $\mathbb{W}P^3 (3, 2, 1, 6 - N)$ and considering the generalized Cartan matrix $K_{ij}$ of an affine Lie algebra, one also gets by help
of eqs (27), the other y,s. Putting altogether in eqs (26), we obtain the following result for the cases of affine \( \hat{A}_2 \) and affine \( \hat{A}_3 \) geometries respectively

\[
\hat{A}_2 : y^2 + x^3 + z^6 + xyz + v (b_2 z^3 + c xz + d y) = 0, \tag{30}
\]

\[
\hat{A}_3 : y^2 + x^3 + z^6 + xyz + v (b_2 z^4 + c xz^2 + d yz + e x^2) = 0. \tag{31}
\]

For the case of affine \( \hat{D}_5 \), we have

\[
\hat{D}_5 : 0 = y^2 + x^3 + z^6 + xyz + v (b_2 z^4 + c_1 y x z^3 + c_2 x^2 z^4 + a_1 z^{10} + a_2 x^5). \tag{32}
\]

The generic relations for eq (26), with generic finite and affine \( \hat{ADE} \) singularities, including eqs (30), eqs (31), eqs (32), may be found in [10, 11, 12]. Before going ahead note that for affine singularities, the powers \( t \) of the complex variable \( v \) appearing in the mirror geometries are given by the Dynkin weights of the corresponding affine algebra. These integers have a remarkable interpretation in toric geometry representation of CY threefolds. The vertices \( V_\alpha = (l_\alpha, r_\alpha, s_\alpha) \) are arranged into subsets belonging to parallel square lattices of \( \mathbb{Z}^3 \). For the case of eq (32), the vertices of the elliptic curve have \( l_\alpha = 0 \); those associated with the nodes with Dynkin weights 1 have \( l_\alpha = 1 \) and finally those Dynkin nodes with weights 2 have \( l_\alpha = 2 \). Note moreover, that homogeneity of the polynomials \( P (X^*) \) describing mirror geometries with affine singularities requires \( v \) to have, in general, a homogeneous dimensions \( 6 - N \). This dimension vanishes for \( N = 6 \), a remarkable feature which give the freedom to have arbitrary powers of \( v \) without affecting the homogeneity of \( P (X^*) \). The solutions we will derive here below for the case of hyperbolic singularities use, amongst others, this special property. To avoid confusion, from now on, we denote the variable \( v \) of the case \( N = 6 \) by \( t \) and thinking about it as parameterizing the complex plane.

### 4.2 Flat Hyperbolic bundles on elliptic K3

In the case of hyperbolic singularities we are after, the previous elliptic curve is replaced by the following special elliptic fibered K3 surface,

\[
y^2 + x^3 + t^{-1} z^6 + xyz = 0, \tag{33}
\]

where the fourth variable \( t \) parameterizes the base of K3. This complex surface describes the base manifold of the mirror CY threefold with hyperbolic singularities we are interested in here. As these kind of unfamiliar manifolds look a little bit unusual, let us give also the solutions for the vertices these CY threefolds. For the case of eq (33), the corresponding four vertices read as

\[
\begin{align*}
y x z & \leftrightarrow (0, 0, 0) \\
y^2 & \leftrightarrow (0, 0, -1) \\
x^3 & \leftrightarrow (0, -1, 0) \\
t^{-1} z^6 & \leftrightarrow (-1, 2, 3)
\end{align*} \tag{34}
\]
To get the algebraic geometry eq

$$P(X^*_{hyp}) = y^2 + x^3 + t^{-1}z^6 + xyz + \sum_{i=1}^n a_i y_i,$$

(35)

describing mirror geometries of CY threefolds with hyperbolic singularities we solve eqs (27) for hyperbolic generalized Cartan matrices of order $n$. The $y_i$s one gets are given by monomials of type $y_i = y^{\alpha_i} x^{\beta_i} z^{\gamma_i} t^{l_i}$ and so

$$P(X^*_{hyp}) = y^2 + x^3 + t^{-1}z^6 + xyz + \sum_{i=1}^n a_i (w) y^{\alpha_i} x^{\beta_i} z^{\gamma_i} t^{l_i}.$$ 

(36)

The vertices $V_i$ corresponding to the monomials $y^{\alpha_i} x^{\beta_i} z^{\gamma_i} t^{l_i}$ have first entries equal to the $l_i$ powers of the variable $t$; i.e,

$$y^{\alpha_i} x^{\beta_i} z^{\gamma_i} t^{l_i} \leftrightarrow (l_i, r_{\alpha_i}, s_{\alpha_i}).$$

(37)

The remaining others are determined by solving eq $\sum_{i=0}^{n+4} q_{\alpha_i} V_{\alpha_i} = 0$. To see how the machinery works in practice, let us illustrate our solutions on the four hyperbolic Lie algebra examples of figure 4.

### 4.3 Explicit solutions

Here we consider the mirror geometry of CY threefolds with $H^4_3$, $H^5_1$, $H^7_1$ and $H^8_1$ hyperbolic singularities of figures 4. First we describe the surfaces with $H^4_3$ and $H^5_1$ singularities; then consider the case of Calabi-Yau threefolds based on $H^3_4$ and $H^3_1$ geometries. After, we give the mirror geometries with gauge groups and fundamental matters that lead, in the weak gauge coupling regime, to $\mathcal{N} = 2$ supersymmetric CFT4$\delta$s with $H^4_3$ and $H^5_1$ singularities. As the analysis for $H^7_1$ and $H^8_1$ is quite similar to $H^4_3$ and $H^5_1$ respectively, we give just the results for these algebras.

- **Mirror geometry of surfaces with $H^4_3$ singularity**

  To get the mirror geometry of surface with $H^4_3$ singularity, we have to solve eqs (27). To that purpose, we have to specify the $q_{\alpha_i}$ integers associated with $H^4_3$. They are given by the following $4 \times (4+4)$ rectangular matrix

$$q_{\alpha} = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 0 & -1 \\
0 & 1 & -2 & 1 & 3 & 1 & 0 & -4 \\
0 & 1 & 1 & -2 & 0 & 1 & 0 & -1
\end{pmatrix}.$$  

(38)

The first $4 \times 4$ block is just the Cartan matrix for $H^4_3$ and the other $3 \times 4$ block is associated with the $y_5, y_6, y_7$ and $y_8$ realized as,

$$y_5 = y^2, \quad y_6 = x^3, \quad y_7 = t^{-1} z^6, \quad y_8 = xyz.$$  

(39)
Putting eqs (39) and (38) back into eqs (27), we get, after some straightforward computations, the following result:

\[ P (X^*_H) = y^2 + x^3 + z^6t^{-1} + xyz + [az^6 + btz^6 + ctxz^4 + dyz^3t], \] (40)

where a, b, c and d are complex structures. This is a compact homogeneous complex two-dimensional surface embedded in \( WP^3 (3, 2, 1, 0) \); it shares features with affine \( A_2 \) mirror geometry (30). The vertices associated with the monomials of the second line of eq (28) are,

\[ z^6 \leftrightarrow (0, 2, 3) \]
\[ tz^6 \leftrightarrow (1, 2, 3) \]
\[ txz^4 \leftrightarrow (1, 1, 2) \]
\[ tyz^3 \leftrightarrow (1, 1, 1). \] (41)

In the case of \( H_4^3 \), the Mori vectors are given by

\[ q^i_a = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \cr 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & -1 \cr 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \cr 0 & 0 & 1 & 0 & -2 & 1 & 2 & 0 & 0 & -2 \cr 0 & 0 & 1 & 1 & -2 & 1 & 2 & 0 & -3 \end{pmatrix}, \] (42)

and eq (40) extends as,

\[ P (X^*_H) = y^2 + x^3 + z^6t^{-1} + xyz + [az^6 + btz^6 + ctxz^4 + dyz^3t + etx^2z^2], \] (43)

Here also one can write down the analogue of the vertices (41).

- **Calabi-Yau threefolds with \( H_4^3 \) and \( H_1^5 \) singularities.**

To get the mirror of Calabi-Yau threefolds with \( H_4^3 \) and \( H_1^5 \) singularities, we have to vary the complex structures \( f \) ( \( f \) stands for a, b, c, d and e ) on the \( CP^1 \) base parameterized by \( w \)

\[ P (X^*_H) = y^2 + x^3 + z^6t^{-1} + xyz + [az^6 + btz^6 + ctxz^4 + dyz^3t + etx^2z^2], \] (44)

where \( f(w) \)s are as in (30). A similar eq is also valid for \( H_1^5 \) hyperbolic singularity. In the \( N = 2 \) supersymmetric QFT, the degree \( n_f \) of these polynomials defines the rank of the gauge group \( SU (n_f + 1) \) fibered on the corresponding node.

- **Hyperbolic CFT\( _4 \)s**
The $\mathcal{N}=2$ supersymmetric QFT$_4$ limit may be made scale invariant by introducing fundamental matters. Since, the $K_{ij} n_j$ vector should be equal to $-m_i$; with $m_i$ positive integers, we have

\begin{align}
2n_a - n_b &= m_a \\
-n_a + 2n_b - n_c - n_d &= m_b \\
-n_b + 2n_c - n_d &= m_c \\
-n_b - n_c + 2n_d &= m_d.
\end{align}

(45)

In geometric engineering method, this is equivalent to engineer $m_i n_i$ fundamental matters of SU($n_i$) on the trivalent nodes of the hyperbolic diagram with negative Mori charges. The simplest configuration one may write down corresponds to take $n_a = 0$, $n_b = n_c = n_d = n$; i.e ( $m_a = n$, $m_b = m_c = m_d = 0$ ) and engineer a SU($n$) flavor symmetry on the hyperbolic node by using trivalent geometry techniques of [10]. The resulting mirror geometry reads as

\begin{equation}
\begin{aligned}
P \left( X_{\mathcal{H}_4^{\alpha}}^* \right) &= y^2 + x^3 + t z^6 + xyz \\
&+ \left[ \frac{1}{\alpha(w)} z^6 + t z^6 b(w) + t x z^4 c(w) + y z t d(w) \right]
\end{aligned}
\end{equation}

(46)

where $\alpha(w)$, $b(w)$, $c(w)$ and $d(w)$ are all of them polynomials of degree $n$. A quite similar result is valid for $\mathcal{N}=2$ CFT$_4$ based on the $\mathcal{H}_1^5$ singularity.

**CY threefolds with $\mathcal{H}_1^5$ and $\mathcal{H}_1^8$ singularities**

These solutions share features with affine $D_5$ and affine $D_6$ mirror geometries of [10]. We will derive here the mirror geometry associated with $\mathcal{H}_1^5$ singularity and the underlying $\mathcal{N}=2$ supersymmetric CFT$_4$. Then we give the results for $\mathcal{H}_1^8$.

- **Surfaces with $\mathcal{H}_1^5$ and $\mathcal{H}_1^8$ singularities**

Complex surface with $\mathcal{H}_1^5$ singularity is described by

\begin{equation}
P \left( X_{\text{hyp}}^* \right) = y^2 + x^3 + t z^6 + xyz + \sum_{i=1}^{7} a_i y_i = 0,
\end{equation}

(47)

where the the $y_i$s are constrained as $\prod_{j=1}^{7} y_j^{q_{ij}} = \prod_{\alpha=8}^{11} y_\alpha^{-q_{\alpha i}}$ and where $q_{ij}$ is given by,

$$
q_{ij} = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -2 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0
\end{pmatrix}.
$$

(48)
Using this explicit expression (48), one can solve the expression of the \( y_i \)s in terms of \( x, y, z \) and \( t \). Putting these expressions back into (47), we get the following result:

\[
P \left( X_{\mathcal{H}_1^7}^* \right) = y^2 + x^3 + z^6 t^{-1} + x y z
\]

\[
+ h z^6 + b_2 t z^6 + b_2 y z^3 t + c_1 t x y z + c_2 t x z^2
\]

\[
+ a_1 t^2 z^6 + a_2 t^2 x z^4
\]

where \( a_i, b_i, c_i \) and \( h \) are complex moduli. For the case of \( \mathcal{H}_1^8 \) singularity, the result we get is

\[
P \left( X_{\mathcal{H}_1^8}^* \right) = y^2 + x^3 + z^6 t^{-1} + x y z
\]

\[
+ h z^6 + b_2 t z^6 + b_2 y z^3 t + c_1 t x y z + c_2 t x^3
\]

\[
+ a_1 t^2 z^6 + a_2 t^2 x z^4 + a_3 t^2 x^2 z^2.
\]

- **Calabi-Yau threefolds with \( \mathcal{H}_1^7 \) and \( \mathcal{H}_1^8 \) singularities**

Varying the complex moduli on \( \mathbb{CP}^1 \) base, we get the mirror of CY threefolds with \( \mathcal{H}_1^7 \) and \( \mathcal{H}_1^8 \) singularities. We obtain

\[
P \left( X_{\mathcal{H}_1^7}^* \right) = y^2 + x^3 + z^6 t^{-1} + x y z
\]

\[
+ h (w) z^6 + b_2 (w) t z^6 + b_2 (w) y z^3 t + c_1 (w) t x y z + c_2 (w) t x^3
\]

\[
+ a_1 (w) t^2 z^6 + a_2 (w) t^2 x z^4 + a_3 (w) t^2 x^2 z^2,
\]

and a quite similar eq for \( \mathcal{H}_1^8 \).

- **Hyperbolic \( \text{CFT}_4^s \)**

The \( \mathcal{N} = 2 \) supersymmetric QFT\(_4\) limit may be made scale invariant by introducing fundamental matters. As before the \( K_{ij} n_j \) vector,

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}
\]

should be a negative integer vector; i.e \( \mathbf{v} = -\mathbf{m} \). Its entries are interpreted as the numbers of fundamental matters one should engineer on the trivalent nodes of \( \mathcal{H}_1^7 \) with negative Mori vector weight. There are various kinds of solutions one can write down; but the simplest one corresponds to take

\[
\begin{pmatrix}
n_h \\
n_{b_2} \\
n_{a_1} \\
n_{a_1} \\
n_{a_2} \\
n_{c_1} \\
n_{c_2}
\end{pmatrix}
= \begin{pmatrix}
2n_h - n_{b_2} \\
-n_{b_2} + 2n_{a_1} - n_{a_2} \\
-n_{a_1} + 2n_{b_1} \\
-n_{a_1} + 2n_{a_2} - n_{c_2} \\
-n_{a_2} + 2n_{c_1} \\
-n_{a_2} + 2n_{c_2}
\end{pmatrix},
\]

(52)
\[ n_h = 0, \]
\[ n_{b_1} = n_{b_1} = n_{c_1} = n_{c_2} = n, \]
\[ n_{a_1} = n_{a_2} = 2n, \]

which correspond to have \( m_h = n \) and \( m_{b_1} = m_{b_1} = m_{c_1} = m_{c_2} = m_{a_1} = m_{a_2} = 0 \). Therefore, the full symmetry of this \( \mathcal{N} = 2 \) hyperbolic CFT\(_4\) has \( SU^4 (n) \times SU^2 (2n) \) as a gauge group and \( SU (n) \) flavor symmetry carried by the negative node of the trivalent vertex of the \( \mathcal{H}_1^7 \) hyperbolic chain. The corresponding mirror geometry is described by the following algebraic eq

\[
P\left( X_{\mathcal{H}_1^7}^* \right) = y^2 + x^3 + z^6 t^{-1} + xyz \\
+ \frac{1}{\alpha (w)} z^6 + b_2 (w) t z^6 + b_1 (w) y z^3 t + c_1 (w) t x y z + c_2 (w) t x^3 \\
+ a_1 (w) t^2 z^6 + a_2 (w) t^2 x z^4 + a_3 (w) t^2 x^2 z^2
\]

(54)

A similar result may be also written down for \( \mathcal{N} = 2 \) CFT\(_4\) based on the \( \mathcal{H}_1^8 \) singularity.

5 Conclusion and Outlook

In this paper, we have derived a new class of \( \mathcal{N} = 2 \) CFT\(_4\)s embedded in type IIA superstring on CY threefolds with indefinite singularities. We have also given the geometric engineering of \( \mathcal{N} = 2 \) QFT\(_4\) based on a special set of hyperbolic singularities as well as their infrared limit. This development has consequences on the understanding \( \mathcal{N} = 2 \) QFT\(_4\)s embedded in type IIA string theories; but also on the classification of indefinite singularities of complex surfaces. For the first point and besides the fact that most of the results, obtained in the context of ADE geometries during the last few years, extends naturally to the indefinite hyperbolic sector, there is also a remarkable and new fact which appear in this kind of field theories. Indeed viewed from type II superstring compactifications, our result tells that one may also have non perturbative gauge symmetries engineered by Lie algebras of indefinite type. It would be an interesting task to deeper this issue. From algebraic geometry point of view, our present study gives the first explicit steps towards the understanding of Calabi-Yau manifolds with indefinite singularities. Though type IIA defining algebraic geometry eqs of such singularities are still lacking, one may usually get precious informations on them by using the mirror geometry eqs

\[
\prod_{j=1}^{n} y_j^{K_{ij}} = \prod_{\alpha \geq 1} \xi_\alpha^{-q_\alpha}. \tag{55}
\]

where \( K_{ij} \) is the generalized Cartan matrix. From these eqs, one may solve the \( y_j \)s in term of \( \xi_\alpha \); but this program requires what \( \xi_\alpha \)s are? In the case of finite ADE singularities, these \( \xi_\alpha \)s parameterize a real two sphere \( S^2 \sim \mathbb{CP}^1 \) and for affine ADE; they parameterize an elliptic
curve embedded in \( \mathbb{W} \mathbb{P}^2 \). For the indefinite (hyperbolic) singularities, we have learnt from our present work that \( \xi \alpha \) parameterize the (elliptic) \( K3 \) surface \( y^2 + x^3 + z^6 t^{-1} + x y z \) with the remarkable pole at \( t = 0 \). We suspect that this is a signature for complex surfaces with indefinite singularities. We also suspect that, instead of the simple poles encountered the examples we have considered in this paper, one may have in general higher orders poles as well. More details regarding this special issue are considered in \([23]\).

We end this conclusion by discussing a link between the Lie algebraic solutions we have developed in the present study and representation theory of the gauge quiver diagrams. A naive way to see how representations of quiver graphs can be implemented in our analysis is to think about the three relations,

\[
\sum_{j=1}^{r} K_{ij}^{(q)} n_j = q m_i; \quad q = 0, +1, -1,
\]

as the Lie algebraic set up behind some representation theory identities carrying same informations. Results from KM algebra representations show that the identities in question are given by the duality property between positive simple roots \( a_\nu \) and fundamental weights \( \Lambda_\mu \) which, we prefer to write it, for generalised symmetrisable Cartan matices \( K_{\mu \nu} \) with corank \( (K_{\mu \nu}) = 0 \) and order \( (r + l + 1) \), as follows,

\[
\sum_{\nu=-l}^{r} K_{\mu \nu} \Lambda_\mu = a_\nu; \quad \mu = -l, \ldots, -1, 0, 1, 2, \ldots, r.
\]

Notice that the duality identity between \( \Lambda_\nu \) and \( a_\nu \) reads in general \( \langle \Lambda_\mu, a_\nu \rangle = \delta_{\mu \nu} \) with \( \langle,\rangle \) is the non degenerate symmetric bilinear form on KM algebra. For invertible \( K_{\mu \nu} \)s, this duality eq can be usually put in the above form. To get the constraint relation describing the passage between eqs (56) and eqs (57), we will restrict ourselves here to give the main lines of the method which we illustrate below on the rank \( (r + 2) \) hyperbolic Lie algebras subset we have been studying.

To that purpose, recall first that the commutation relations of hyperbolic algebras \( \mathcal{H}^{(r+2)} \), with Chevalley generators \( e_\mu, f_\mu, h_\mu; -1 \leq \mu \leq r \), are

\[
[h_\mu, e_\nu] = K_{\mu \nu} e_\nu; \quad [h_\mu, f_\nu] = -K_{\mu \nu} f_\nu; \quad [e_\mu, f_\nu] = \delta_{\mu \nu} h_\mu;
\]

\[
(ad e_\mu)^{1-K_{\mu \nu}} e_\nu = (ad f_\mu)^{1-K_{\mu \nu}} f_\nu = 0; \quad ; \mu \neq \nu, \quad -1 \leq \mu, \nu \leq r
\]

where \( h_i; 1 \leq i \leq r \) are the usual commuting Cartan generators and the two extra \( h_0 \) and \( h_{-1} \) are as: \( h_0 = k - \sum_{i=1}^{r} \tilde{s}_i h_i \) and \( h_{-1} = -k - d \) with \( k \) and \( d \) respectively the central element and the derivation of the affine subalgebra \( \tilde{g}_r \) of \( \mathcal{H}^{(r+2)} \) \([24], [27]\). The \( \tilde{s}_i \)s are roughly speaking the Dynkin weights and \( k \) and \( d \) generators have scalar products given by \( \langle k, k \rangle = < d, d > = 0 \) and \( < k, d > = 1 \). Second, note that like in ordinary and affine Lie algebras, the \( (r + 2) \) fundamental weights \( \Lambda_\mu \) and positive simple roots \( a_\mu \) of hyperbolic algebras are immediately deduced by extending the \( (r + 1) \) affine ones. In addition to the ordinary positive simple roots \( a_i = \alpha_i \);
future occasion.

as extensions and applications in supersymmetric quiver gauge theories will be considered in a

opposite of its projections on simple roots. More details on this representation analysis as well

lattice having projections along simple roots given by negative integers; i.e.

it is manifested in the present approach by the remarkable property

fundamental matter in affine case has a nice geometric interpretation in hyperbolic root lattice;

i = 1, ..., r, there are two extra ones namely \( a_{-1} \) and \( a_0 \) which read as \( a_{-1} = -k - \delta \), \( a_0 = \delta - \psi \)

with \( \psi = \sum_{i=1}^{r} s_i \alpha_i \) being the usual highest root. Similarly, we have for the fundamental weights

the realisation \( \Lambda_{-1} = -\delta \), \( \Lambda_0 = k - \delta \) and \( \Lambda_i = \lambda_i + s_i k - s_i \delta \). Both of these \( a_\mu \) and \( \Lambda_\mu \) satisfy

duality property and may be rewritten in a more convenient form as extended objects like,

\[
\begin{align*}
a_{-1} &= (0, -1, -1); \quad a_0 = (-\psi, 0, 1); \quad a_i = (\alpha_i, 0, 0), \quad i = 1, ..., r;
\Lambda_{-1} &= (0, 0, -1); \quad \Lambda_0 = (0, 1, -1); \quad \Lambda_i = (\lambda_i, s_i, -s_i), \quad i = 1, ..., r;
\end{align*}
\]

(59)

where the two extra directions refer to the \( k \) and \( \delta \) generators. To bring eq (57) into eq (58), we use a nice property of the Lorentzian structure of the root lattice \( \mathcal{L}_{r+2} \) for hyperbolic \( \mathcal{H}(r+2) \)

and do it in three steps by performing projections of eq (57) along root lattice vectors \( r_q \) of the

three regions of \( \mathcal{L}_{r+2} \) namely space like vectors \( (r_+^2 > 0) \), light like vectors \( (r_0^2 = 0) \) and time

like vectors \( (r_0^2 < 0) \). These three sectors turns out to correspond exactly to the three relations

of eqs (50) we are after. We first treat the case of finite ADE algebras, then affine ADE and

finally their hyperbolic over extension. (a) For finite ADE, the representation identity involving ordinary \( \alpha_i \)'s and \( \lambda_i \)'s and contained\(^1\) in eq (57), reads as,

\[
\sum_{j=1}^{r} K_{ij}^{(+)\lambda_j} = \alpha_i; \quad i, j = 1, 2, ..., r;
\]

(60)

where \( K_{ij}^{(+)\lambda_j} \) is as before. To put this relation into the form \( K_{ij}^{(+)\lambda_j} n_j = m_i \), we consider a positive integer space like vector \( r_+ = \sum_{i=1}^{r} n_i \alpha_i \) of the ADE root lattice such that \( < r_+, \alpha_i >= m_i \) is a positive integer but \( < r_+, \Lambda_\mu >= < r_+, \alpha_\mu >= 0 \) for \( \mu = -1, 0 \). Projecting eq (60) along this vector \( r_+ \), we get the desired eq (11); thanks to the duality relation \( < \alpha_i, \lambda_\mu >= \delta_{ij} \). Therefore, we learn that the ranks \( n_i \) of the quiver gauge group \( \prod_{i=1}^{r} U(n_i) \) are just the projections of \( r_+ \) on the \( \lambda_i \)

fundamental weights and the numbers \( m_i \) of fundamental matters are the projections of \( r_+ \) on the \( \alpha_i \)

simple roots. (b) For affine case, we get a similar interpretation in term of representation theory by taking the projection of \( K_{\mu\nu}\Lambda_\nu = a_\nu \) along the imaginary root lattice vector \( r_0 = n \delta \). Using the identities \( < r_0, \alpha_\mu >= -n \delta_{-1, \mu} \) and \( < r_0, \Lambda_{-1} >= 0, < r_0, \Lambda_0 >= n \) and \( < r_0, \Lambda_i >= ns_i \) as one may check from eqs (59), one obtains two eqs, one of which is trivial and the second is just \( K_{ij}^{(0)} n_j = 0 \) eq (12) with the right integer values \( n_j = ns_j \). Note that the absence of fundamental matter in affine case has a nice geometric interpretation in hyperbolic root lattice;

it is manifested in the present approach by the remarkable property \( < r_0, \alpha_i >= < r_0, \delta >= 0 \).

(c) Finally considering a positive integer time like vector \( r_- = \sum_{i=1}^{r} n_i \alpha_i \) of the hyperbolic root lattice having projections along simple roots given by negative integers; i.e \( < r_-, \alpha_i >= -m_i \), and projecting eq (57) along \( r_- \), we get the desired result once more. Here also the \( n_i \)'s are the projections of \( r_- \) on the fundamental weights and the number of fundamental matter is the opposite of its projections on simple roots. More details on this representation analysis as well as extensions and applications in supersymmetric quiver gauge theories will be considered in a future occasion.

\(^1\) Eqs (59) for ADE subalgebras of \( \mathcal{H}(r+2) \) can be rederived under projection of eqs (57) along \( r_+ \).
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