Summation formulas of hyperharmonic numbers with their generalizations II

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MR Subject Classifications: Primary 05A19, 11B65; Secondary 11B73

Abstract

In 1990, Spieß gave some identities of harmonic numbers including the types of \( \sum_{\ell=1}^{n} \ell^k H_\ell \), \( \sum_{\ell=1}^{n} \ell^k H_{n-\ell} \) and \( \sum_{\ell=1}^{n} \ell^k H_\ell H_{n-\ell} \). In this paper, we derive several formulas of hyperharmonic numbers including \( \sum_{\ell=0}^{n} \ell^p h_\ell (r) h_{n-\ell} (s) \) and \( \sum_{\ell=0}^{n} \ell^p (h_\ell (r))^2 \). Some more formulas of generalized hyperharmonic numbers are also shown.

Keywords: hyperharmonic numbers, Stirling numbers, summation formulae

1 Introduction and preliminaries

Many researchers have been considering varieties of identities involving harmonic numbers \( H_n := \sum_{j=1}^{n} 1/j \). One of the most famous types is the so-called Euler sum like

\[
\sum_{k=1}^{\infty} \frac{(H_k)^m}{(k+1)^n} \quad (m, n \geq 1),
\]
that Euler considered in response to a letter from Goldbach in 1742 (see, e.g., [4, p.253]). It is interesting that the Riemann zeta functions and their generalizations often appear in such expressions. Borwein-Borwein-Girgensohn theorem [5] is useful to compute some kinds of Euler sums.

There are many generalizations of harmonic numbers. One of the most useful ones is the generalized harmonic numbers, defined by

\[ H_n^{(r)} = 1 + \frac{1}{2^r} + \cdots + \frac{1}{n^r}. \]

Zave [19] showed the coefficients of some power series expansions in terms of the generalized harmonic numbers. One of the most interesting generalized Euler sums is of the form

\[ S_{\pi,q} = \sum_{k=1}^{\infty} \frac{H_k^{(\pi_1)}H_k^{(\pi_2)}\cdots H_k^{(\pi_r)}}{k^q} \]

and the quantity \( \pi_1 + \pi_2 + \cdots + \pi_r + q \) is called the weight of the sum. Note that one part of weights \( k^q \) exists in the denominator.

On the contrary, Spieß [18] gave some identities including the types of \( \sum_{\ell=1}^{n-1} \ell^k H_\ell \), \( \sum_{\ell=1}^{n-1} \ell^k H_{n-\ell} \) and \( \sum_{\ell=1}^{n-1} \ell^k H_\ell H_{n-\ell} \). Here, weights exist in the numerator, that is, they are of the form in the multiplication.

Another kind of generalization of harmonic numbers is the hyperharmonic numbers [2, 3, 7, 9, 10, 11, 14, 15, 16], defined by

\[ h_n^{(r)} = \sum_{\ell=1}^{n} h_\ell^{(r-1)} \quad \text{with} \quad h_1^{(1)} = H_n. \]  

In particular, we have

\[ h_n^{(r)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1}) \quad [9] \]

\[ = \sum_{j=1}^{n} \binom{n+r-j-1}{r-1} \frac{1}{j} \quad [3], \]

Let \( x \) be an indeterminate. Then following Riordan [17], we write

\[ (x)_n = x(x-1)\cdots(x-n+1) \quad (n \in \mathbb{N}), \]

for the falling factorial, with \( (x)_0 = 1 \), and \( \mathbb{N} := \{1, 2, 3, \cdots\} \). With this notation, the Stirling numbers of the first kind, denoted by \( s(n,k) \), are defined as

\[ (x)_n = \sum_{k=0}^{n} s(n,k)x^k \quad (n \in \mathbb{N}). \]
The Stirling numbers of the second kind, denoted by \( S(n, k) \), are defined as

\[
x^n = \sum_{k=0}^{n} S(n, k)(x)_k, \quad (n \in \mathbb{N}).
\]

These numbers may also be defined recursively by

\[
s(n + 1, k) = s(n, k - 1) - n \cdot s(n, k),
\]

\[
S(n + 1, k) = S(n, k - 1) + k \cdot S(n, k)
\]

with boundary values

\[
s(n, 0) = s(0, n) = S(n, 0) = S(0, n) = \delta_{n0} \quad (n \geq 0),
\]

where \( \delta_{nm} \) is the Kronecker delta, that is, \( \delta_{nn} = 1 \), \( \delta_{nm} = 0 \) for \( n \neq m \). Tables of the Stirling numbers can be found in Abramowitz and Stegun [1].

The unsigned Stirling numbers of the first kind, denoted by \( s_u(n, k) \), are defined as

\[
(x)_{(n)} = \sum_{k=0}^{n} s_u(n, k)x^k \quad (n \in \mathbb{N}).
\]

where \( (x)_{(n)} = x(x + 1) \cdots (x + n - 1) \ (n \geq 1) \) denotes the rising factorial with \( (x)_{(0)} = 1 \). It is well-known that \( s_u(n, k) = (-1)^{n+k}s(n, k) \).

The generalized hyperharmonic numbers are defined by (see [10, 13, 16])

\[
H^{(p, r)}_n := \sum_{j=1}^{n} H^{(p, r-1)}_j \quad (n, p, r \in \mathbb{N}),
\]

with

\[
H^{(p, 1)}_n = H^{(p)}_n.
\]

Observing \( H^{(1, r)}_n = h^{(r)}_n \), we see that the generalized hyperharmonic numbers are unified extensions of both generalized harmonic numbers and hyperharmonic numbers.

According to the spirit of Spieß, the authors [12] derived several formulas of hyperharmonic numbers of type \( \sum_{\ell=1}^{n} \ell^p H^{(r)}_{\ell} \) and \( \sum_{\ell=0}^{n} \ell^p H^{(r)}_{n-\ell} \). Several formulas of \( q \)-hyperharmonic numbers are also derived as \( q \)-generalizations. The purpose of this paper is to show several identities

\[
\sum_{\ell=0}^{n} \ell^p h^{(r)}_{\ell} h^{(s)}_{n-\ell}, \quad \sum_{\ell=0}^{n} (\ell)^p h^{(r)}_{\ell} h^{(s)}_{n-\ell}, \quad \sum_{\ell=0}^{n} \ell^p (h^{(r)}_{\ell})^2, \quad \sum_{\ell=0}^{n} (\ell)^p (h^{(r)}_{\ell})^2
\]
and
\[
\sum_{\ell=0}^{n} \ell^p H_\ell^{(q,r)}, \quad \sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(q,r)}, \quad \sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(q,r)}.
\]

This paper is also motivated from the summation \(\sum_{\ell=1}^{n} \ell^k\), which is related to Bernoulli numbers (e.g., [6, 8]), because we can think of harmonic numbers or hyperharmonic numbers as the weight of the summation \(\sum_{\ell=1}^{n} \ell^k\).

2 Some fundamental results and lemmata

2.1 Some fundamental results

Spieß [18] gives some identities including the types of \(\sum_{\ell=0}^{n} \ell^k H_\ell\) and \(\sum_{\ell=0}^{n} \ell^k H_{n-\ell}\) in terms of \(H_{n+1}\). More precisely, it is shown [18] that for \(n, k, r \in \mathbb{N}\),

\[
\sum_{\ell=0}^{n} \ell^k = A(k, n),
\]

\[
\sum_{\ell=0}^{n} \ell^k H_\ell = A(k, n)H_{n+1} - B(k, n),
\]

\[
\sum_{\ell=0}^{n} \ell^k H_{n-\ell} = A(k, n)H_{n+1} - C(k, n),
\]

\[
\sum_{\ell=0}^{n} \ell^k H_{\ell} H_{n-\ell} = A(k, n)S_{n+1} - (B(k, n) + C(k, n))H_{n+1} + D(k, n),
\]

\[
\sum_{\ell=0}^{n} \ell^k H_\ell^{(2)} = A(k, n)(H_n)^2 - (-1)^pB^+_p H_n + E(k, n),
\]

\[
\sum_{\ell=0}^{n} \ell^k (H_\ell)^2 = A(k, n)(H_n)^2 - F(k, n)H_n + G(k, n),
\]

\[
\sum_{\ell=0}^{n} \ell^k H_\ell^{(r)} = A(k, n)H_n^{(r)} - \sum_{t=1}^{k+1} H_n^{(r-t)} H(k, t),
\]

where \(A(k, n), B(k, n), C(k, n), D(k, n), E(k, n), F(k, n)\) and \(G(k, n)\) are polynomials in \(n\), \(S_n = (H_n)^2 - H_n^{(2)}\), \(B^+_n\) is the well-known Bernoulli numbers. The Bernoulli numbers \(B^+_n\) are determined by the recurrence formula

\[
\sum_{j=0}^{k} \binom{k+1}{j} B^+_j = k + 1 \quad (k \geq 0)
\]
or by the generating function
\[ \frac{t}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \]

The \( A(k, n) \) are well-known (see, e.g. Riordan \[17\]). Spieß \[18\] gives explicit forms for \( k = 0, 1, 2, 3 \). More generally, it is shown that for \( n, k \in \mathbb{N} \),

\[
A(k, n) = \sum_{\ell=0}^{k} S(k, \ell) n^\ell \frac{1}{\ell + 1} H_{\ell + 1} \left( \frac{n + 1}{\ell + 1} \right),
\]

\[
B(k, n) = \sum_{\ell=0}^{k} \frac{1}{\ell + 1} S(k, \ell) n^\ell \frac{1}{\ell + 1} H_{\ell + 1} \left( \frac{n + 1}{\ell + 1} \right),
\]

\[
C(k, n) = \sum_{\ell=0}^{k} S(k, \ell) n^\ell \frac{1}{\ell + 1} H_{\ell + 1} \left( \frac{n + 1}{\ell + 1} \right) H_{\ell + 1},
\]

\[
D(k, n) = \sum_{\ell=0}^{k} S(k, \ell) n^\ell \frac{1}{\ell + 1} H_{\ell + 1} \left( \frac{n + 1}{\ell + 1} \right) \left( \frac{1}{\ell + 1} H_{\ell + 1} + H_{\ell + 1}^2 \right),
\]

\[
E(k, n) = \sum_{\ell=0}^{k} S(k, \ell) n^\ell A_\ell(n),
\]

\[
F(k, n) = \sum_{\ell=0}^{k} S(k, \ell) n^\ell \frac{1}{\ell + 1} \left( 2 \left( \frac{n}{\ell + 1} \right) + (-1)^\ell \right),
\]

\[
G(k, n) = \sum_{\ell=0}^{k} S(k, \ell) n^\ell \left( \frac{2}{(\ell + 1)^2} \left( \frac{n}{\ell + 1} \right) - \frac{1}{(\ell + 1)!} \sum_{k=2}^{\ell + 1} s(\ell + 1, k) H_n^{(2-\ell)} \right),
\]

\[
H(k, t) = \sum_{\ell=t}^{k} \frac{1}{\ell + 1} S(k, \ell) s(\ell + 1, t),
\]

where \( A_\ell(n) \) is a polynomial in \( n \) of degree \( \ell \) (see \[18\]).

For a positive integer \( n \), we have

\[
\sum_{\ell=1}^{n} \ell H_\ell H_{n-\ell} = \frac{n(n + 1)}{2} S_n - n^2 H_n + n^2, \tag{4}
\]

where

\[
S_n := (H_n)^2 - H_n^{(2)} = \frac{2}{n!} s_u(n + 1, 3) = \sum_{k=1}^{n} \frac{H_{n-k}}{k}.
\]

5
Note that
\[ \sum_{k=1}^{n} H_k \left( \frac{1}{k} \right) = \frac{(H_n)^2 + H_n^{(2)}}{2}. \]  

We see that
\[ \sum_{\ell=1}^{n} \ell^2 H_\ell H_{n-\ell} = \frac{n(n+1)(2n+1)}{6} S_n 
- \frac{n(13n^2 + 6n - 1)}{18} H_n + \frac{n(71n^2 + 30n + 7)}{108}, \]  

\[ \sum_{\ell=1}^{n} \ell^3 H_\ell H_{n-\ell} = \left( \frac{n(n+1)}{2} \right)^2 S_n 
- \frac{n^2(n+1)(7n-1)}{12} H_n + \frac{n^2(35n^2 + 30n + 7)}{72}. \]  

Combining the identities (4), (5), (6), we have
\[ \sum_{\ell=1}^{n} \ell H_\ell^{(2)} H_{n-\ell}^{(2)} = \frac{n(n+1)(n+2)(n+3)}{12} S_n 
- \frac{n^2(11n^2 + 48n + 49)}{36} H_n + \frac{n^2(85n^2 + 312n + 251)}{216}. \]

### 2.2 Some lemmata

Spieß [18] gives the following lemma, which will be frequently used in later sections.

**Lemma 1** ([18]). Given summation formulas \( \sum_{\ell=0}^{n} \binom{\ell}{p} c_\ell = F(n, p), n, p \in \mathbb{N}, \) one has
\[ \sum_{\ell=0}^{n} \ell^p c_\ell = \sum_{\ell=0}^{p} S(p, \ell)! \cdot \ell! \cdot F(n, \ell). \]

**Lemma 2** ([11, Lemma 2.4]). The following identities hold:
\[ \binom{n+r-1}{r-1} = \frac{1}{(r-1)!} \sum_{k=1}^{r} s_u(r, k)n^{k-1}. \]  

\[ \binom{n+r-1}{r-1} \left( \frac{1}{n+1} + \cdots + \frac{1}{n+r-1} \right) = \frac{1}{(r-1)!} \sum_{k=1}^{r-1} s_u(r, k+1)k!n^{k-1}. \]
We now prove some lemmas which will be needed in later sections.

**Lemma 3.** For $r, s, n \in \mathbb{N}$ with $0 \leq \ell \leq n$, we have

$$\binom{\ell + r - 1}{r - 1}\binom{n - \ell + s - 1}{s - 1} = \sum_{t=0}^{r+s-2} a(r, s, n, t) \ell^t,$$

where

$$a(r, s, n, t) = \frac{1}{(r-1)! (s-1)!} \times \sum_{k_1=1}^{r-1} s_u(r, t_1 + 1) \sum_{k_2=0}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2 n^{k_2-t_2}}.$$

**Proof.** By using [7] we have

$$\binom{\ell + r - 1}{r - 1}\binom{n - \ell + s - 1}{s - 1} = \frac{1}{(r-1)! (s-1)!} \sum_{k_1=1}^{r-1} s_u(r, t_1 + 1) \sum_{k_2=0}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2 n^{k_2-t_2}}$$

$$= \frac{1}{(r-1)! (s-1)!} \times \sum_{t_1=0}^{r-1} s_u(r, t_1 + 1) \sum_{k_2=0}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2 n^{k_2-t_2}}$$

$$= \frac{1}{(r-1)! (s-1)!} \times \sum_{t_1=0}^{r-1} s_u(r, t_1 + 1) \ell^{t_1} \sum_{k_2=0}^{s-1} s_u(s, k_2 + 1) \sum_{t_2=0}^{s-1} \binom{k_2}{t_2} (-1)^{t_2 n^{k_2-t_2}}$$

$$= \frac{1}{(r-1)! (s-1)!} \times \sum_{t_1=0}^{r+s-2} \ell^{t_1} \sum_{t_2=0}^{s-1} s_u(r, t_1 + 1) \sum_{k_2=t_2}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2 n^{k_2-t_2}}.$$
Lemma 4. For $r, s, n \in \mathbb{N}$ with $0 \leq \ell \leq n$, we have

$$
\binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right)
= \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \ell^t,
$$

where

$$
a_1(r, s, n, t) = \frac{1}{(r-1)!(s-1)!} \times \sum_{t_1 + t_2 = t \atop 0 \leq t_1 \leq r-1 \atop 0 \leq t_2 \leq s-2} s_u(r, t_1 + 1) \sum_{k_2 = t_2}^{s-2} s_u(s, k_2 + 2)(k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} n^{k_2 - t_2}.
$$

Proof. By using (7) and (8), we have

$$
\binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right)
= \frac{1}{(r-1)!(s-1)!} \sum_{k_1 = 1}^{r} s_u(r, k_1) \ell^{k_1-1} \frac{1}{(s-1)!} \sum_{k_2 = 1}^{s-1} s_u(s, k_2 + 1) k_2 (n - \ell)^{k_2-1}
= \frac{1}{(r-1)!(s-1)!} \sum_{t_1 = 0}^{r-1} s_u(r, t_1 + 1) \ell^{t_1}
\times \sum_{k_2 = 0}^{s-2} s_u(s, k_2 + 2)(k_2 + 1) \sum_{t_2 = 0}^{k_2} \binom{k_2}{t_2} \ell^{t_2} (-1)^{t_2} n^{k_2 - t_2}
= \frac{1}{(r-1)!(s-1)!} \sum_{t_1 = 0}^{r-1} s_u(r, t_1 + 1) \ell^{t_1}
\times \sum_{t_2 = 0}^{s-2} \ell^{t_2} \sum_{k_2 = t_2}^{s-2} s_u(s, k_2 + 2)(k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} n^{k_2 - t_2}
= \frac{1}{(r-1)!(s-1)!} \sum_{t = 0}^{r+s-3} \ell^t \sum_{t_1 + t_2 = t \atop 0 \leq t_1 \leq r-1 \atop 0 \leq t_2 \leq s-2} s_u(r, t_1 + 1)
\times \sum_{k_2 = t_2}^{s-2} s_u(s, k_2 + 1)(k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} n^{k_2 - t_2}.
$$
Lemma 5. For $r, s, n \in \mathbb{N}$ with $0 \leq \ell \leq n$, we have

$$(\ell + r - 1) \left( \frac{n - \ell + s - 1}{s - 1} \right) \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right) = \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \ell^t,$$

where

$$a_2(r, s, n, t) = \frac{1}{(r-1)!(s-1)!} \times \sum_{t_1+t_2=t}^{t_1 \leq r-2 \atop 0 \leq t_2 \leq s-1} s_u(r, t_1 + 2)(t_1+1) \sum_{k_2=t_2}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} n^{k_2-t_2}.$$ 

Proof. By using (7) and (8), we have

$$(\ell + r - 1) \left( \frac{n - \ell + s - 1}{s - 1} \right) \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right) = \frac{1}{(r-1)!(s-1)!} \sum_{k_1=1}^{r-1} s_u(r, k_1 + 1) k_1 \ell^{k_1-1} \frac{1}{(s-1)!} \sum_{k_2=1}^{s} s_u(s, k_2)(n-\ell)^{k_2-1}$$

$$= \frac{1}{(r-1)!(s-1)!} \sum_{t_1=0}^{r-2} s_u(r, t_1 + 2)(t_1+1) \ell^{t_1} \times \sum_{k_2=0}^{s-1} s_u(s, k_2 + 1) \sum_{t_2=0}^{k_2} \binom{k_2}{t_2} \ell^{t_2} (-1)^{t_2} n^{k_2-t_2}$$

$$= \frac{1}{(r-1)!(s-1)!} \sum_{t_1=0}^{r-2} s_u(r, t_1 + 2)(t_1+1) \ell^{t_1} \times \sum_{t_2=0}^{s-1} \ell^{t_2} \sum_{k_2=t_2}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} n^{k_2-t_2}$$

$$= \frac{1}{(r-1)!(s-1)!} \sum_{t=0}^{r+s-3} \ell^t \sum_{t_1+t_2=t}^{t_1 \leq r-2 \atop 0 \leq t_2 \leq s-1} s_u(r, t_1 + 2)(t_1+1) \times \sum_{k_2=t_2}^{s-1} s_u(s, k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} n^{k_2-t_2}.$$ 

$\square$
Lemma 6. For $r, s, n \in \mathbb{N}$ with $0 \leq \ell \leq n$, we have

$$
\left( \frac{\ell + r - 1}{r - 1} \right) \left( \frac{n - \ell + s - 1}{s - 1} \right) \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right)
\times \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right) = \sum_{t=0}^{r+s-4} a_3(r, s, n, t) \ell^t,
$$

where

$$a_3(r, s, n, t) = \frac{1}{(r-1)!(s-1)!} \sum_{\substack{t_1 + t_2 = t \\ 0 \leq t_1 \leq r-2 \\ 0 \leq t_2 \leq s-2}} s_u(r, t_1 + 2)(t_1 + 1)
\times \sum_{k_2-t_2}^{s-1} s_u(s, k_2 + 2)(k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} \ell^{k_2-t_2}.
$$

Proof. By using (7) and (8), we have

$$
\left( \frac{\ell + r - 1}{r - 1} \right) \left( \frac{n - \ell + s - 1}{s - 1} \right) \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right)
\times \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right) = \frac{1}{(r-1)!(s-1)!}
\sum_{k_1=1}^{r-2} s_u(r, k_1 + 1) k_1 \ell^{k_1 - 1} \frac{1}{(s-1)!} \sum_{k_2=1}^{s-1} s_u(s, k_2 + 1) k_2 (n - \ell) k_2 - 1
$$

$$= \frac{1}{(r-1)!(s-1)!} \sum_{t_1=0}^{r-2} s_u(r, t_1 + 2)(t_1 + 1) \ell^{t_1}
\times \sum_{k_2=0}^{s-2} s_u(s, k_2 + 2)(k_2 + 1) \sum_{t_2=0}^{k_2} \binom{k_2}{t_2} \ell^{t_2} (-1)^{t_2} \ell^{k_2-t_2}
$$

$$= \frac{1}{(r-1)!(s-1)!} \sum_{t_1=0}^{r-2} s_u(r, t_1 + 2)(t_1 + 1) \ell^{t_1}
\times \sum_{t_2=0}^{s-2} \sum_{k_2=t_2}^{s-2} s_u(s, k_2 + 2)(k_2 + 1) \binom{k_2}{t_2} (-1)^{t_2} \ell^{k_2-t_2}
$$

$$= \frac{1}{(r-1)!(s-1)!} \sum_{t_1=0}^{r-2} \ell^{t_1} \sum_{\substack{t_1 + t_2 = t \\ 0 \leq t_1 \leq r-2 \\ 0 \leq t_2 \leq s-2}} s_u(r, t_1 + 2)(t_1 + 1)
$$

$$= \sum_{t=0}^{r+s-4} a_3(r, s, n, t) \ell^t.
$$
\[ \times \sum_{k_2=t_2}^{s-2} s_u(s, k_2 + 2)(k_2 + 1)\binom{k_2}{t_2}(-1)^{t_2} n^{k_2-t_2}. \]

3 Main results

In this section, we study the summations

\[ \sum_{\ell=0}^{n} \ell p h_\ell^{(r)} h_{n-\ell}^{(s)}, \sum_{\ell=0}^{n} (\ell)^p h_\ell^{(r)} h_{n-\ell}^{(s)}, \sum_{\ell=0}^{n} \ell^p (h_\ell^{(r)})^2, \sum_{\ell=0}^{n} (\ell)^p (h_\ell^{(r)})^2. \]

Theorem 1. For \( n, r, p \geq 1 \), we have

\[ \sum_{\ell=0}^{n} \ell p h_\ell^{(r)} h_{n-\ell}^{(s)} = A(p, r, s, n) S_n + B(p, r, s, n) H_n + C(p, r, s, n), \quad (9) \]

where

\[ A(p, r, s, n) = \sum_{t=0}^{r+s-2} a(r, s, n, t) A(p + t, n), \]

\[ B(p, r, s, n) = \sum_{t=0}^{r+s-3} (a_1(r, s, n, t) + a_2(r, s, n, t)) A(p + t, n) \]

\[ + \sum_{t=0}^{r+s-2} a(r, s, n, t) \left( A(p + t, n) \frac{2}{n+1} - H_{s-1} - H_{r-1} - B(p + t, n) - C(p + t, n) \right), \]

\[ C(p, r, s, n) = \sum_{t=0}^{r+s-3} a(r, s, n, t) \left( D(p + t, n) - \frac{1}{n+1} (B(p + t, n) - C(p + t, n)) \right) \]

\[ + \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \left( A(p + t, n) \frac{1}{n+1} - B(p + t, n) \right) \]

\[ - H_{s-1} \sum_{t=0}^{r+s-2} a_1(r, s, n, t) \left( A(p + t, n) \frac{1}{n+1} - B(p + t, n) \right) \]

\[ + \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \left( A(p + t, n) \frac{1}{n+1} - C(p + t, n) \right) \]
\[ \begin{align*}
& \sum_{t=0}^{r+s-4} a_3(r,s,n,t) \sum_{\ell=0}^{n} \ell^{p+t} \\
& - H_{s-1} \sum_{t=0}^{r+s-3} a_2(r,s,n,t) \sum_{\ell=0}^{n} \ell^{p+t} \\
& - H_{r-1} \sum_{t=0}^{r+s-2} a(r,s,n,t) \left( A(p+t,n) \frac{1}{n+1} - C(p+t,n) \right) \\
& - H_{r-1} \sum_{t=0}^{r+s-3} a_1(r,s,n,t) \sum_{\ell=0}^{n} \ell^{p+t} + H_{r-1} H_{s-1} \sum_{t=0}^{r+s-2} a(r,s,n,t) \sum_{\ell=0}^{n} \ell^{p+t}. 
\end{align*} \]

**Proof.** With the help of (2), we have

\[ \sum_{\ell=0}^{n} \ell^{p} h^{(r)}_{\ell} h^{(s)}_{n-\ell} \]

\[ \begin{align*}
& = \sum_{\ell=0}^{n} \ell^{p} \left( \binom{\ell + r - 1}{r - 1} (H_{\ell+r-1} - H_{r-1}) \binom{n - \ell + s - 1}{s - 1} (H_{n-\ell+s-1} - H_{s-1}) \right) \\
& = \sum_{\ell=0}^{n} \ell^{p} \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} \\
& \times \left( H_{\ell} H_{n-\ell} + H_{\ell} \left( \frac{1}{n-\ell+1} + \cdots + \frac{1}{n-\ell+s-1} \right) \right) \\
& - H_{\ell} H_{s-1} + \left( \frac{1}{\ell+1} + \cdots + \frac{1}{\ell+r-1} \right) H_{n-\ell} \\
& + \left( \frac{1}{\ell+1} + \cdots + \frac{1}{\ell+r-1} \right) \left( \frac{1}{n-\ell+1} + \cdots + \frac{1}{n-\ell+s-1} \right) \\
& - \left( \frac{1}{\ell+1} + \cdots + \frac{1}{\ell+r-1} \right) H_{s-1} - H_{r-1} H_{n-\ell} \\
& - H_{r-1} \left( \frac{1}{n-\ell+1} + \cdots + \frac{1}{n-\ell+s-1} \right) + H_{r-1} H_{s-1} \right). 
\end{align*} \]

The above sum can be divided into 9 parts. With the help of Lemmas 3, 4, 5 and 6 we have the following 9 identities:

**Part I**

\[ \begin{align*}
& = \sum_{\ell=0}^{n} \ell^{p} \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} H_{\ell} H_{n-\ell} 
\end{align*} \]
\[
\sum_{\ell=0}^{r+s-2} \ell^p H_{r+s-\ell} \sum_{t=0}^{n} a(r, s, n, t) \ell^t
= \sum_{t=0}^{r+s-2} a(r, s, n, t) \sum_{\ell=0}^{n} \ell^{p+t} H_{r+s-\ell}
= \sum_{t=0}^{r+s-2} a(r, s, n, t)
\times \left( A(p + t, n)S_{n+1} - (B(p + t, n) + C(p + t, n))H_{n+1} + D(p + t, n) \right)
\]

\[
S_n \sum_{t=0}^{r+s-2} a(r, s, n, t) A(p + t, n)
+ H_n \sum_{t=0}^{r+s-2} a(r, s, n, t) \left( 2A(p + t, n) \frac{1}{n+1} - B(p + t, n) - C(p + t, n) \right)
+ \sum_{t=0}^{r+s-2} a(r, s, n, t) \left( D(p + t, n) - \frac{1}{n+1} (B(p + t, n) - C(p + t, n)) \right).
\]

**Part II**

\[
\sum_{\ell=0}^{n} \ell^p \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} H_{\ell} \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right)
= \sum_{\ell=0}^{n} \ell^p H_{\ell} \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \ell^t
= \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \sum_{\ell=0}^{n} \ell^{p+t} H_{\ell}
= \sum_{t=0}^{r+s-3} a_1(r, s, n, t) (A(p + t, n)H_{n+1} - B(p + t, n))
\]
\[ H_n \sum_{t=0}^{r+s-3} a_1(r, s, n, t) A(p + t, n) \]
\[ + \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \left( A(p + t, n) \frac{1}{n + 1} - B(p + t, n) \right). \]

**Part III**
\[ = - \sum_{\ell=0}^{n} \ell^p \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} H_\ell \]
\[ = - \sum_{\ell=0}^{n} \ell^p H_\ell H_{s-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) \ell^t \]
\[ = - H_{s-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) \sum_{\ell=0}^{n} \ell^{p+t} H_\ell \]
\[ = - H_n H_{s-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) A(p + t, n) \]
\[ - H_{s-1} \sum_{t=0}^{r+s-2} a_1(r, s, n, t) \left( A(p + t, n) \frac{1}{n + 1} - B(p + t, n) \right). \]

**Part IV**
\[ = \sum_{\ell=0}^{n} \ell^p \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right) H_{n-\ell} \]
\[ = \sum_{\ell=0}^{n} \ell^p H_{n-\ell} \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \ell^t \]
\[ = \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \sum_{\ell=0}^{n} \ell^{p+t} H_{n-\ell} \]
\[ = \sum_{t=0}^{r+s-3} a_2(r, s, n, t) (A(p + t, n) H_{n+1} - C(p + t, n)) \]
\[ = H_n \sum_{t=0}^{r+s-3} a_2(r, s, n, t) A(p + t, n) \]
\[ + \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \left( A(p + t, n) \frac{1}{n + 1} - C(p + t, n) \right). \]
Part V

\[ \sum_{\ell=0}^{n} \ell^p \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} \times \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right) \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right) \]

\[ = \sum_{\ell=0}^{n} \ell^p \sum_{t=0}^{r+s-4} a_3(r, s, n, t) \ell^t \]

\[ = \sum_{t=0}^{r+s-4} a_3(r, s, n, t) \sum_{\ell=0}^{n} \ell^{p+t}. \]

Part VI

\[ = - \sum_{\ell=0}^{n} \ell^p \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right) H_{s-1} \]

\[ = -H_{s-1} \sum_{\ell=0}^{n} \ell^p \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \ell^t \]

\[ = -H_{s-1} \sum_{t=0}^{r+s-3} a_2(r, s, n, t) \sum_{\ell=0}^{n} \ell^{p+t}. \]

Part VII

\[ = - \sum_{\ell=0}^{n} \ell^p \binom{\ell + r - 1}{r - 1} \binom{n - \ell + s - 1}{s - 1} H_{r-1} H_{n-\ell} \]

\[ = -H_{r-1} \sum_{\ell=0}^{n} \ell^p H_{n-\ell} \sum_{t=0}^{r+s-2} a(r, s, n, t) \ell^t \]

\[ = -H_{r-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) (A(p + t, n) H_{n+1} - C(p + t, n)) \]

\[ = -H_{r-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) A(p + t, n) \]

\[ - H_{r-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) \left( A(p + t, n) \frac{1}{n + 1} - C(p + t, n) \right). \]
Part VIII

\[\begin{align*}
&= - \sum_{\ell=0}^{n} \ell^p \left( \frac{\ell + r - 1}{r - 1} \right) \left( \frac{n - \ell + s - 1}{s - 1} \right) H_{r-1} \left( \frac{1}{n - \ell + 1} + \cdots + \frac{1}{n - \ell + s - 1} \right) \\
&= -H_{r-1} \sum_{\ell=0}^{n} \ell^p \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \ell^t \\
&= -H_{r-1} \sum_{t=0}^{r+s-3} a_1(r, s, n, t) \sum_{\ell=0}^{n} \ell^p + t.
\end{align*}\]

Part IX

\[\begin{align*}
&= \sum_{\ell=0}^{n} \ell^p \left( \frac{\ell + r - 1}{r - 1} \right) \left( \frac{n - \ell + s - 1}{s - 1} \right) H_{r-1} H_{s-1} \\
&= H_{r-1} H_{s-1} \sum_{\ell=0}^{n} \ell^p \sum_{t=0}^{r+s-2} a(r, s, n, t) \ell^t \\
&= H_{r-1} H_{s-1} \sum_{t=0}^{r+s-2} a(r, s, n, t) \sum_{\ell=0}^{n} \ell^p + t.
\end{align*}\]

Combining all these 9 parts together, we get the desired result. \(\square\)

**Theorem 2.** For positive integers \(n, p\) and \(r\), we have

\[\sum_{\ell=0}^{n} (\ell^p h_{\ell}^{(r)} h_{n-\ell}^{(s)}) = A_1(p, r, s, n) S_n + B_1(p, r, s, n) H_n + C_1(p, r, s, n),\]

where

\[A_1(p, r, s, n) = \sum_{m=0}^{p} s_u(p, m) A(m, r, s, n),\]

\[B_1(p, r, s, n) = \sum_{m=0}^{p} s_u(p, m) B(m, r, s, n),\]

\[C_1(p, r, s, n) = \sum_{m=0}^{p} s_u(p, m) C(m, r, s, n).\]
Proof.

\[
\sum_{\ell=0}^{n} (\ell)! h_{\ell}^{(r)} h_{n-\ell}^{(s)}
= \sum_{\ell=0}^{n} \sum_{m=0}^{p} s_u(p, m) \ell^m h_{\ell}^{(r)} h_{n-\ell}^{(s)}
= \sum_{m=0}^{p} s_u(p, m) \sum_{\ell=0}^{n} \ell^m h_{\ell}^{(r)} h_{n-\ell}^{(s)}
= \sum_{m=0}^{p} s_u(p, m) \left( A(m, r, s, n) S_n + B(m, r, s, n) H_n + C(m, r, s, n) \right)
= \left( \sum_{m=0}^{p} s_u(p, m) A(m, r, s, n) \right) S_n + \left( \sum_{m=0}^{p} s_u(p, m) B(m, r, s, n) \right) H_n
+ \left( \sum_{m=0}^{p} s_u(p, m) C(m, r, s, n) \right).
\]

Before going further, we introduce some more notations.

\[
a_4(r, r, t) := \frac{1}{(r!)^2} \sum_{t_1 + t_2 = t} s_u(r, t_1 + 1) s_u(s, t_2 + 1),
\]

\[
a_5(r, r, t) := \frac{1}{(r-1)!^2} \sum_{t_1 + t_2 = t} s_u(r, t_1 + 1) (t_2 + 1) s_u(s, t_2 + 2).
\]

**Theorem 3.** For positive integers \(n, p\) and \(r\), we have

\[
\sum_{\ell=0}^{n} \ell^p (h_{\ell}^{(r)})^2 = A(2, p, r, n)(H_n)^2 + B(2, p, r, n) H_n + C(2, p, r, n),
\]

where

\[
A(2, p, r, n) = \sum_{t=0}^{2r-2} a_4(r, r, t) A(p + t, n),
\]

\[
B(2, p, r, n) = -\sum_{t=0}^{2r-2} a_4(r, r, t) F(p + t, n) - 2H_{r-1} \sum_{t=0}^{2r-2} a_4(r, r, t) A(p + t, n)
\]

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\[ + 2 \sum_{t=0}^{2r-3} a_5(r, r, t) A(p + t, n), \]

\[ C(2, p, r, n) = \sum_{t=0}^{2r-2} a_4(r, r, t) G(p + t, n) \]

\[ + \sum_{\ell=0}^{n} \ell^p \left( \frac{\ell + r - 1}{r - 1} \right)^2 \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} - H_{r-1} \right)^2 \]

\[ - 2H_{r-1} \sum_{t=0}^{2r-2} a_4(r, r, t) \left( A(p + t, n) \frac{1}{n + 1} - B(p + t, n) \right) \]

\[ + 2 \sum_{t=0}^{2r-3} a_5(r, r, t) \left( A(p + t, n) \frac{1}{n + 1} - B(p + t, n) \right). \]

**Proof.** With the help of (2), we have

\[ \sum_{\ell=0}^{n} \ell^p \left( h_{\ell}^{(r)} \right)^2 \]

\[ = \sum_{\ell=0}^{n} \ell^p \left( \frac{\ell + r - 1}{r - 1} \right)^2 \left( H_{\ell+r-1} - H_{r-1} \right)^2 \]

\[ = \sum_{\ell=0}^{n} \ell^p \left( \frac{\ell + r - 1}{r - 1} \right)^2 \left( H_{\ell}^2 + H_{r-1}^2 - 2H_{r-1}H_{\ell} + \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right)^2 \right) \]

\[ + 2H_{\ell} \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right) - 2H_{r-1} \left( \frac{1}{\ell + 1} + \cdots + \frac{1}{\ell + r - 1} \right). \]

The above sum can be divided into 4 parts.

**Part I**

\[ = \sum_{\ell=0}^{n} \ell^p \left( \frac{\ell + r - 1}{r - 1} \right)^2 H_{\ell}^2 \]

\[ = \sum_{\ell=0}^{n} \ell^p H_{\ell}^2 \left( \frac{1}{(r - 1)!} \sum_{k_1=1}^{r} s_u(r, k_1) \ell^{k_1-1} \right)^2 \]

\[ = \sum_{\ell=0}^{n} \ell^p H_{\ell}^2 \left( \frac{1}{(r - 1)!} \sum_{t=0}^{2r-2} \sum_{t_1+t_2=t} s_u(r, t_1 + 1) s_u(r, t_2 + 1) \right) \]
\[
\begin{align*}
&= \sum_{t=0}^{2r-2} a_4(r, r, t) \frac{n}{\ell^p} H^2_{\ell} \\
&= \sum_{t=0}^{2r-2} a_4(r, r, t) \left( A(p + t, n)(H_n)^2 - F(p + t, n) H + G(p + t, n) \right) \\
&= (H_n)^2 \sum_{t=0}^{2r-2} a_4(r, r, t)A(p + t, n) - H_n \sum_{t=0}^{2r-2} a_4(r, r, t)F(p + t, n) \\
&\quad + \sum_{t=0}^{2r-2} a_4(r, r, t)G(p + t, n).
\end{align*}
\]

Part II
\[
\left( r - 1 \right)^2 \left( \frac{1}{r + 1} + \cdots + \frac{1}{\ell + r - 1} - H_r \right)^2.
\]

Part III
\[
\begin{align*}
&= -2 \sum_{t=0}^{2r-2} a_4(r, r, t) \frac{n}{\ell^p} H^2_{\ell} \\
&= -2H_{r-1} \sum_{t=0}^{2r-2} a_4(r, r, t) \sum_{t=0}^{n} \frac{n}{\ell^p} H^2_{\ell} \\
&= -2H_{r-1} \sum_{t=0}^{2r-2} a_4(r, r, t) \sum_{t=0}^{n} \frac{n}{\ell^p} H_{\ell} \\
&= -2H_n H_{r-1} \sum_{t=0}^{2r-2} a_4(r, r, t) A(p + t, n) \\
&\quad - 2H_{r-1} \sum_{t=0}^{2r-2} a_4(r, r, t) \left( A(p + t, n) \frac{1}{n + 1} - B(p + t, n) \right).
\end{align*}
\]

Part IV
\[
\begin{align*}
&= 2 \sum_{t=0}^{2r-2} a_4(r, r, t) \frac{n}{\ell^p} H^2_{\ell} \\
&= 2H_{r-1} \sum_{t=0}^{2r-3} a_5(r, r, t) \frac{n}{\ell} H_{\ell} \\
&= 2 \sum_{t=0}^{2r-3} a_5(r, r, t) \frac{n}{\ell} H_{\ell}.
\end{align*}
\]

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\[
2 \sum_{t=0}^{2r-3} a_5(r, r, t) \sum_{\ell=0}^{n} \ell^{p+t} H_{n-\ell}
\]
\[
= 2 \sum_{t=0}^{2r-3} a_5(r, r, t)(A(p + t, n)H_{n+1} - B(p + t, n))
\]
\[
= 2H_n \sum_{t=0}^{2r-3} a_5(r, r, t)A(p + t, n)
\]
\[
+ 2 \sum_{t=0}^{2r-3} a_5(r, r, t) \left( A(p + t, n) \frac{1}{n + 1} - B(p + t, n) \right).
\]

Combining all these 4 parts together, we get the desired result. □

**Theorem 4.** For positive integers \( n, p \) and \( r \), we have

\[
\sum_{\ell=0}^{n} (\ell \, p \, h_\ell^{(r)})^2 = A_2(p, r, n)(H_n)^2 + B_2(p, r, n)H_n + C_2(p, r, n),
\]

where

\[
A_2(p, r, n) = \sum_{m=0}^{p} s_u(p, m)A(2, m, r, n),
\]

\[
B_2(p, r, n) = \sum_{m=0}^{p} s_u(p, m)B(2, m, r, n),
\]

\[
C_2(p, r, n) = \sum_{m=0}^{p} s_u(p, m)C(2, m, r, n).
\]

**Proof.**

\[
\sum_{\ell=0}^{n} (\ell \, p \, h_\ell^{(r)})^2
\]
\[
= \sum_{\ell=0}^{n} \sum_{m=0}^{p} s_u(p, m)\ell^{m} (h_\ell^{(r)})^2
\]
\[
= \sum_{m=0}^{p} s_u(p, m) \sum_{\ell=0}^{n} \ell^{m} (h_\ell^{(r)})^2
\]
\[
\begin{align*}
&\sum_{m=0}^{p} s_u(p, m) \left( A(2, m, r, n)(H_n)^2 + B(2, m, r, n)H_n + C(2, m, r, n) \right) \\
&= \left( \sum_{m=0}^{p} s_u(p, m)A(2, m, r, n) \right) S_n + \left( \sum_{m=0}^{p} s_u(p, m)B(2, m, r, n) \right) H_n \\
&\quad + \left( \sum_{m=0}^{p} s_u(p, m)C(2, m, r, n) \right).
\end{align*}
\]

\[\square\]

4 Summations involving generalized hyperharmonic numbers

In this section, we consider some summations involving generalized hyperharmonic numbers. We give expressions of the summations

\[
\sum_{\ell=0}^{n} \ell^p H_{n}^{(q, r)} , \quad \sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(r)} , \quad \sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(q, r)} .
\]

Lemma 7 ([13]). For \(r, n, p \in \mathbb{N}\), we have

\[
H_{n}^{(p, r)} = \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j)n^j H_n^{(p-m)}.
\]

The coefficients \(\hat{a}(r, m, j)\) satisfy the following recurrence relations:

\[
\begin{align*}
\hat{a}(r + 1, r, 0) &= -\sum_{m=0}^{r-1} \hat{a}(r, m, r - m - 1) \frac{1}{r - m}, \\
\hat{a}(r + 1, m, \ell) &= \sum_{j=\ell-1}^{r-1-m} \hat{a}(r, m, j) \left( \frac{j + 1}{j - \ell + 1} \right) B_{j-\ell+1} \quad (0 \leq m \leq r - 1, 1 \leq \ell \leq r - m), \\
\hat{a}(r + 1, m, 0) &= -\sum_{y=0}^{r-1-y} \sum_{j=\max\{0, m-y-1\}}^{r-1-y} \hat{a}(r, y, j)D(r, m, j, y) \quad (0 \leq m \leq r - 1),
\end{align*}
\]

where

\[
D(r, m, j, y) = \sum_{\ell=\max\{0, m-y-1\}}^{j} \frac{1}{j+1} \left( \frac{j + 1}{j - \ell} \right) B_{j-\ell} \left( \frac{\ell + 1}{m - y} \right)(-1)^{1+\ell-m+y}.
\]
The initial value is given by $\hat{a}(1, 0, 0) = 1$.

**Theorem 5.** For positive integers $n, p, q$ and $r$ with $q \geq r$, we have

$$\sum_{\ell=0}^{n} \ell^p H_{\ell}^{(q,r)} = \sum_{y=0}^{p+r} b(p, r, n, y) H_{n}^{(q-y)},$$

where

$$b(p, r, n, 0) = \sum_{j=0}^{r-1} \hat{a}(r, 0, j) A(p + j, n),$$

$$b(p, r, n, y) = \sum_{j=0}^{r-1-y} \hat{a}(r, y, j) A(p + j, n) - \sum_{m=0}^{r-1-m} \hat{a}(r, m, j) H(p + j, t) \quad (1 \leq y \leq r - 1),$$

$$b(p, r, n, y) = - \sum_{m=0}^{r-1-m} \hat{a}(r, m, j) H(p + j, t) \quad (r \leq y \leq p + r).$$

**Proof.** With the help of Lemma 7, we obtain

$$\sum_{\ell=0}^{n} \ell^p H_{\ell}^{(q,r)}$$

$$= \sum_{\ell=0}^{n} \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j) \ell^p H_{\ell}^{(q-m)}$$

$$= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j) \sum_{\ell=0}^{n} \ell^p H_{\ell}^{(q-m)}$$

$$= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j) \left( A(p + j, n) H_{n}^{(q-m)} - \sum_{t=1}^{p+j+1} H_{n}^{(q-m-t)} H(p + j, t) \right)$$

$$= \sum_{m=0}^{r-1} H_{n}^{(q-m)} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j) A(p + j, n)$$

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\[-\sum_{m=0}^{r-1} \sum_{t=1}^{p+r-m} H_n^{(q-m-t)} \sum_{j=\max(0,t-p-1)}^{r-1-m} \hat{a}(r, m, j)H(p+j, t)\]

\[= \sum_{m=0}^{r-1} H_n^{(q-m)} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j)A(p+j, n)\]

\[-\sum_{x=1}^{p+r} H_n^{(q-x)} \sum_{m+t=x}^{\max(0,t-p-1)} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j)H(p+j, t).\]

\[\square\]

**Lemma 8.** For \(r, p, n \in \mathbb{N}\), we have

\[\sum_{k=1}^{n} \binom{k}{p} H_{n-k}^{(r)} = \sum_{t=0}^{p+1} a_6(p, n, t)H_{n-1}^{(r-t)},\]

where

\[a_6(p, n, t) = \frac{1}{(p+1)!} \sum_{i=t}^{p+1} s(p+1, i) \binom{i}{t} (-1)^i (n+1)^{i-t}.\]

**Proof.**

\[\sum_{k=1}^{n} \binom{k}{p} H_{n-k}^{(r)} = \sum_{k=1}^{n} \binom{k}{p} \sum_{j=1}^{n-k} \frac{1}{j^t}\]

\[= \sum_{j=1}^{n-1} \frac{1}{j^t} \sum_{k=1}^{n-j} \binom{k}{p}\]

\[= \sum_{j=1}^{n-1} \frac{1}{j^t} \binom{n-j+1}{p+1}\]

\[= \sum_{j=1}^{n-1} \frac{1}{j^t} \frac{1}{(p+1)!} \sum_{i=0}^{p+1} s(p+1, i) (n-j+1)^i\]

\[= \sum_{j=1}^{n-1} \frac{1}{j^t} \frac{1}{(p+1)!} \sum_{i=0}^{p+1} s(p+1, i) \sum_{t=0}^{i} \binom{i}{t} j^t (-1)^t (n+1)^{i-t}\]

\[= \sum_{j=1}^{n-1} \frac{1}{j^t} \frac{1}{(p+1)!} \sum_{t=0}^{p+1} j^t \sum_{i=t}^{p+1} s(p+1, i) \binom{i}{t} (-1)^t (n+1)^{i-t}\]

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\[
\sum_{t=0}^{p+1} \sum_{j=1}^{n-1} \frac{1}{j^{t-j}} \frac{1}{(p+1)!} \sum_{i=t}^{i+t} s(p+1, i) \left( \begin{array}{c} i \\ t \end{array} \right) (-1)^t (n+1)^{i-t}.
\]

\[\text{Theorem 6.} \quad \text{For positive integers } n, p \text{ and } r, \text{ we have}
\]
\[
\sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(r)} = \sum_{\ell=0}^{p+1} \sum_{\ell=1}^{p} S(p, \ell) \ell! a_6(\ell, n, t).
\]

\[\text{Proof.} \quad \sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(r)} = \sum_{\ell=0}^{p} \sum_{k=0}^{n} \left( \begin{array}{c} k \\ \ell \end{array} \right) H_{n-k}^{(r)}.
\]

\[
= \sum_{\ell=1}^{p} \sum_{k=1}^{n} \left( \begin{array}{c} k \\ \ell \end{array} \right) H_{n-k}^{(r)}
\]

\[
= \sum_{\ell=1}^{p} \sum_{t=0}^{p+1} S(p, \ell) \ell! a_6(\ell, n, t) H_{n-1}^{(r-t)}
\]

\[
= \sum_{\ell=0}^{p+1} H_{n-1}^{(r-t)} \sum_{\ell=1}^{p} S(p, \ell) \ell! a_6(\ell, n, t).
\]

\[\text{Before going further, we introduce some more notations.}
\]

\[a_7(p, n, t) := \sum_{\ell=1}^{p} S(p, \ell) \cdot \ell! \cdot a_6(\ell, n, t),
\]

\[a_8(r, m, k, n) := \sum_{j=k}^{r-1-m} \hat{a}(r, m, j) \left( \begin{array}{c} j \\ k \end{array} \right) (-1)^k n^{j-k}.
\]

\[\text{Theorem 7.} \quad \text{For positive integers } n, p, q \text{ and } r, \text{ we have}
\]
\[
\sum_{\ell=0}^{n} \ell^p H_{n-\ell}^{(q,r)} = \sum_{y=0}^{p+r} c(p, r, n, y) H_{n-1}^{(q-y)}.
\]

\[\text{where}
\]
\[
c(p, r, n, y) = \sum_{\substack{m+t=y \\ 0 \leq m \leq r-1 \\ 0 \leq t \leq p+r-m}} \sum_{k=\max\{0, t-p-1\}}^{r-1-m} a_7(p + k, n, t) a_8(r, m, k, n).
\]
Proof. With the help of Lemma 7 and Theorem 6, we obtain

\[
\sum_{\ell=0}^{n} \beta^p H_{n-\ell}^{(q,r)} \\
= \sum_{\ell=0}^{n} \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j)(n - \ell)^j H_{n-\ell}^{(q-m)} \\
= \sum_{\ell=0}^{n} \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \hat{a}(r, m, j) \sum_{k=0}^{j} \binom{j}{k} \ell^k (-1)^k n^{j-k} H_{n-\ell}^{(q-m)} \\
= \sum_{\ell=0}^{n} \sum_{m=0}^{r-1} H_{n-\ell}^{(q-m)} \sum_{k=0}^{r-1-m} \ell^k \sum_{j=k}^{r-1-m} \hat{a}(r, m, j) \binom{j}{k} (-1)^k n^{j-k} \\
= \sum_{m=0}^{r-1} \sum_{k=0}^{r-1-m} \sum_{\ell=0}^{n} \ell^k H_{n-\ell}^{(q-m)} a_8(r, m, k, n) \\
= \sum_{m=0}^{r-1} \sum_{k=0}^{r-1-m} H_{n-1}^{(q-m-t)} a_7(p + k, n, t) a_8(r, m, k, n) \\
= \sum_{m=0}^{r-1} \sum_{t=0}^{r-1-m} \sum_{k=\max\{0,t-p-1\}}^{r-1-m} a_7(p + k, n, t) a_8(r, m, k, n) \\
= \sum_{y=0}^{p+r} H_{n-1}^{(q-y)} \sum_{m+t=y}^{m+t=y} \sum_{k=\max\{0,t-p-1\}}^{r-1-m} a_7(p + k, n, t) a_8(r, m, k, n). 
\]

\[\square\]

Remark 1. Summation formulas of type \(\sum_{\ell=0}^{n} (\ell)^p H_{n-\ell}^{(q,r)}\), \(\sum_{\ell=0}^{n} (\ell)^p H_{n-\ell}^{(r)}\) and \(\sum_{\ell=0}^{n} (\ell)^p H_{n-\ell}^{(q,r)}\) can be obtained in the same way with Theorem 2.

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