A collocation method for numerical solution of Telegraph equation

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Abstract

In this paper, B-spline collocation method is developed for the solution of one-dimensional hyperbolic telegraph equation. The convergence of the method is proved. Also the method is applied on some test examples, and the numerical results have been compared with the analytical solutions. The $L_\infty, L_2$ and Root-Mean-Square errors (RMS) in the solutions show the efficiency of the method computationally.

Keywords: B-spline; Telegraph equation; Collocation method; Convergence.

1 Introduction

Hyperbolic partial differential equations are commonly used in signal analysis for transmission and propagation of electrical signals [1] and also has applications in other fields [2, 3]. In the present paper, a collocation approach based on quintic B-spline functions is utilized for the numerical solution of the one-dimensional hyperbolic telegraph equation. In recent years, many different methods have been used to estimate the solution of the one-dimensional hyperbolic telegraph equation; see, for example, [4-9]. Consider the second-order linear hyperbolic partial differential equation in one-space dimension:

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x, t), \ a \leq x \leq b, \ t \geq 0,$$

with the initial conditions

$$u(x, 0) = f_0(x),$$
$$\frac{\partial u}{\partial t}(x, 0) = f_1(x),$$

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and boundary conditions
\begin{align}
 u(a, t) &= g_0(t), \quad u(b, t) = g_1(t), \quad (1.4) \\
 \frac{\partial u}{\partial x}(a, t) &= g_2(t), \quad \frac{\partial u}{\partial x}(b, t) = g_3(t), \quad (1.5)
\end{align}

where \( \alpha \) and \( \beta \) are constants.

The balance of this paper is organized as follows. In Section 2, the quintic B-spline collocation method for the numerical solution of the one-dimensional hyperbolic telegraph equation is described. In Section 3 we derive convergence of the B-spline collocation method. In Section 4, the results of numerical experiments are presented. A summary is given at the end of the paper in Section 5.

\section{Quintic B-spline collocation method}

The interval \([a, b]\) is partitioned into a mesh of uniform length \( h := \frac{b-a}{N} \) by the knots \( x_i, i = 0, 1, \ldots, N \) such that \( x_i := x_0 + ih \) and \( a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b \). To solve the equation (1.1) by collocation method with quintic B-splines as basis functions, we define the approximation \( U^n(x) \) as following
\begin{equation}
 U^n(x) = \sum_{i=-2}^{N+2} c_i^n B_i(x). \quad (2.1)
\end{equation}

where \( U^n(x) \) is a shape function that approximates \( u(x, t_n) \) for the time level \( t_n = nk \) where \( k \) is a time step size. For each time level \( t_n \), the set \( \{c^n_{-2}, c^n_{-1}, \ldots, c^n_{N+1}, c^n_{N+2}\} \) are unknown real coefficients, which are to be found, and the \( B_i(x) \) are the quintic B-spline functions defined by [10, 11]
\begin{equation}
 B_i(x) = \frac{1}{h^5} \left\{ \begin{array}{ll}
 (x-x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}], \\
 (x-x_{i-3})^5 - 6(x-x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}], \\
 (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5, & x \in [x_{i-1}, x_i], \\
 (x_{i+3} - x_i)^5 - 6(x_{i+2} - x_i)^5 + 15(x_{i+1} - x_i)^5, & x \in [x_i, x_{i+1}], \\
 (x_{i+3} - x_i)^5 - 6(x_{i+2} - x_i)^5, & x \in [x_{i+1}, x_{i+2}], \\
 (x_{i+3} - x_i)^5, & x \in [x_{i+2}, x_{i+3}], 
\end{array} \right. \quad (2.2)
\end{equation}

where \( B_{-2}, B_{-1}, B_0, B_1, \ldots, B_{N+1}, B_{N+2} \) form a basis over the region \( a \leq x \leq b \). The values of \( B_i(x) \) and its derivatives may be tabulated as in Table 1. Using approximate function (2.1) and Table 1, we have
\begin{equation}
 u(x_i, t_n) \approx U^n_i = c^n_{i-2} + 26c^n_{i-1} + 66c^n_i + 26c^n_{i+1} + c^n_{i+2}, \quad (2.3)
\end{equation}
\begin{equation}
 \frac{\partial u}{\partial x}(x_i, t_n) \approx (U')_i^n = \frac{1}{h^5}(-5c^n_{i-2} - 50c^n_{i-1} + 50c^n_{i+1} + 5c^n_{i+2}), \quad (2.4)
\end{equation}
\begin{equation}
 \frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx (U'')_i^n = \frac{1}{h^5}(20c^n_{i-2} + 40c^n_{i-1} - 120c^n_i + 40c^n_{i+1} + 20c^n_{i+2}). \quad (2.5)
\end{equation}
Table 1: $B_i, B'_i$ and $B''_i$ at the node points.

| $x$     | $x_{i-3}$ | $x_{i-2}$ | $x_{i-1}$ | $x_i$ | $x_{i+1}$ | $x_{i+2}$ | $x_{i+3}$ |
|---------|----------|----------|----------|------|----------|----------|----------|
| $B_i(x)$ | 0        | 1        | 26       | 66   | 26       | 1        | 0        |
| $hB'_i(x)$ | 0      | 5        | 50       | 0    | -50      | -5       | 0        |
| $h^2B''_i(x)$ | 0     | 20       | 40      | -120 | 40       | 20       | 0        |

To apply the proposed method, discretizing the time derivative in the usual finite difference way, with using following finite difference formulae \[12\], we can write:

\[
\left( \frac{\partial^2 u}{\partial t^2} \right)_i^n \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^{n-1}}{\Gamma(k)^2}, \tag{2.6}
\]

\[
\left( \frac{\partial u}{\partial t} \right)_i^n \approx \frac{u_{i+1}^{n+1} - u_i^{n-1}}{2\Gamma(k)}, \tag{2.7}
\]

\[
\left( \frac{\partial^2 u}{\partial x^2} \right)_i^n \approx \frac{(\frac{\partial^2 u}{\partial x^2})_{i}^{n-1} + (\frac{\partial^2 u}{\partial x^2})_{i}^{n+1}}{2}, \tag{2.8}
\]

where $k$ is a time step size, $(\frac{\partial^2 u}{\partial x^2})_{i}^n := \frac{\partial^2 u}{\partial x^2}(x_i, t_n)$, $u_i^n := u(x_i, t_n)$ and $\Gamma(k)$ is a selected function of $k$ satisfying the following equation

\[
\Gamma(k)^2 = (k)^2(1 + O(k)^j), \quad j = 0, 1, \ldots \tag{2.9}
\]

In the numerical computations, we applied the following two possible choices for $\Gamma(k)$ to improve the accuracy: $k$ and $2\sin(\frac{k}{2})$. Hence (1.1) can be written as:

\[
\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^{n-1}}{\Gamma(k)^2} + 2\alpha \frac{u_{i+1}^{n+1} - u_i^{n-1}}{2\Gamma(k)} + \beta^2 u_i^n = \frac{(\frac{\partial^2 u}{\partial x^2})_{i}^{n-1} + (\frac{\partial^2 u}{\partial x^2})_{i}^{n+1}}{2} + f(x_i, t_n). \tag{2.10}
\]

Rearranging the terms and simplifying we get

\[
u u_i^{n+1} + w(\frac{\partial^2 u}{\partial x^2})_{i}^{n+1} = \Phi^n(x_i), \tag{2.11}
\]

where

\[
\Phi^n(x_i) := 2 - (\beta \Gamma(k))^2 u_i^n + (\alpha \Gamma(k) - 1) u_i^{n-1} + \frac{\Gamma(k)^2}{2} (\frac{\partial^2 u}{\partial x^2})_{i}^{n-1} + \Gamma(k)^2 f(x_i, t_n), \tag{2.12}
\]

\[
v := 1 + \alpha \Gamma(k), \tag{2.13}
\]

\[
w := -\frac{\Gamma(k)^2}{2}. \tag{2.14}
\]
Substituting the approximate solution $U$ for $u$ and putting the values (2.3) and (2.5) in (2.11) yields the following difference equation with the variables $c_i, i = -2, \ldots, N + 2$.

\[
u U_{i}^{n+1} + w (U'^{n})_{i}^{n+1} = (v + 20 \frac{w}{h^2}) c_{i+2}^{n+1} + (2v + 40 \frac{w}{h^2}) c_{i+1}^{n+1} + (6v - 120 \frac{w}{h^2}) c_{i}^{n+1} + (26v + 40 \frac{w}{h^2}) c_{i-1}^{n+1} + (v + 20 \frac{w}{h^2}) c_{i-2}^{n+1} = \Psi_{i}^{n}, \quad i = 0, \ldots, N,
\]

(2.15)

where

\[
\Psi_{i}^{n} := \left(2 - (\beta \Gamma(k))^2\right) U_{i}^{n} + (\alpha \Gamma(k) - 1) U_{i}^{n-1} + \frac{\Gamma(k)^2}{2} (U'^{n})_{i}^{n-1} + \Gamma(k)^2 f(x_i, t_n).
\]

(2.16)

The system (2.15) consists of $(N + 1)$ linear equations in $(N + 5)$ unknowns $\tilde{C} := \{c_{-2}, c_{-1}, \ldots, c_{N+1}, c_{N+2}\}$. To obtain a unique solution for $\tilde{C}$ we must use the boundary conditions. From the boundary conditions we can write

\[
c_{-2}^{n+1} + 26c_{-1}^{n+1} + 66c_{0}^{n+1} + 26c_{1}^{n+1} + 5c_{2}^{n+1} = g_0(t_{n+1}),
\]

(2.17)

\[
c_{N-2}^{n+1} + 26c_{N-1}^{n+1} + 66c_{N}^{n+1} + 26c_{N+1}^{n+1} + 5c_{N+2}^{n+1} = g_1(t_{n+1}),
\]

(2.18)

and

\[
\frac{1}{h} \left(-5c_{-2}^{n+1} - 50c_{-1}^{n+1} + 50c_{1}^{n+1} + 5c_{2}^{n+1}\right) = g_2(t_{n+1}),
\]

(2.19)

\[
\frac{1}{h} \left(-5c_{N-2}^{n+1} - 50c_{N-1}^{n+1} + 50c_{N}^{n+1} + 5c_{N+2}^{n+1}\right) = g_3(t_{n+1}).
\]

(2.20)

By using (2.17)-(2.20), we obtain

\[
c_{-1}^{n+1} = \frac{33}{8} c_{0}^{n+1} - \frac{9}{4} c_{1}^{n+1} - \frac{1}{8} c_{2}^{n+1} + \frac{h g_2(t_{n+1})}{80} + \frac{g_0(t_{n+1})}{16},
\]

(2.21)

\[
c_{-2}^{n+1} = \frac{165}{4} c_{0}^{n+1} + \frac{65}{2} c_{1}^{n+1} + \frac{9}{4} c_{2}^{n+1} - \frac{13 h g_2(t_{n+1})}{40} - \frac{5 g_0(t_{n+1})}{8},
\]

(2.22)

\[
c_{N+1}^{n+1} = \frac{33}{8} c_{N}^{n+1} - \frac{9}{4} c_{N-1}^{n+1} - \frac{1}{8} c_{N-2}^{n+1} - \frac{h g_3(t_{n+1})}{80} + \frac{g_1(t_{n+1})}{16},
\]

(2.23)

\[
c_{N+2}^{n+1} = \frac{165}{4} c_{N}^{n+1} + \frac{65}{2} c_{N-1}^{n+1} + \frac{9}{4} c_{N-2}^{n+1} + \frac{13 h g_3(t_{n+1})}{40} - \frac{5 g_1(t_{n+1})}{8}.
\]

(2.24)

Hence we have the following system consists of $(N + 1)$ linear equations in $(N + 1)$ unknowns $\{c_0, c_1, \ldots, c_{N-1}, c_N\}$. The B-spline method in matrix form can be written as follows:

\[
AC = Q,
\]

(2.25)
Theorem 3.1. Suppose that \( u(x,t) \) be the exact solution of (1.1) and \( u(x,t) \in \mathcal{C}^5[a,b] \) also \( \left| \frac{\partial^5 u(x,t)}{\partial x^5} \right| \leq L \) and \( U(x,t) \) be the numerical approximation by our methods, then we can write
\[
\| u(x,t) - U(x,t) \|_{\infty} = \mathcal{O}(h^2 + k^3).
\]
Before we prove, we recall following theorem and lemma.

**Theorem 3.2.** Suppose that \( f(x) \in C^5[a, b] \). Then for the unique quintic spline \( S(x) \) associated with \( f \), we have

\[
\| f^{(j)} - S^{(j)} \|_\infty \leq K_j \omega_5(h) h^{4-j}, \quad j = 0, 1, 2, 3.
\]  

(3.2)

where \( \omega_5(h) \) denotes the modulus of continuity of \( f^{(5)} \) and the coefficients \( \lambda_j \) are independent of \( f \) and \( h \).

**Proof.** For the proof see [13].

**Remark 3.3.** By using Theorem 3.2 and definition of the modulus of continuity, we can say that if \( |f^{(5)}(x)| \leq L \), we can write (3.2) as

\[
\| f^{(j)} - S^{(j)} \|_\infty \leq \lambda_j L h^{4-j}, \quad j = 0, 1, 2, 3.
\]

(3.3)

**Lemma 3.4.** For the B-splines \( \{B_{-2}, \cdots, B_{N+2}\} \) we have the following inequality:

\[
\left| \sum_{i=-2}^{N+2} B_i(x) \right| \leq 186, \quad (a \leq x \leq b).
\]

(3.4)

**Proof.** From the real analysis we have \( |\sum_{i=-2}^{N+2} B_i(x)| \leq \sum_{i=-2}^{N+2} |B_i(x)| \). If \( x = x_i, \ i = 1, \ldots, N \), then, we have

\[
\left| \sum_{i=-2}^{N+2} B_i(x) \right| = 120 \leq 186,
\]

(3.5)

and if \( x_{i-1} \leq x \leq x_i \), then, we can write

\[
\left| \sum_{i=-2}^{N+2} B_i(x) \right| \leq |B_{i-3}(x)| + |B_{i-2}(x)| + |B_{i-1}(x)| + |B_i(x)|
\]

\[
+ |B_{i+1}(x)| + |B_{i+2}(x)| \leq 1 + 26 + 66 + 66 + 26 + 1 \leq 186
\]

Now we prove theorem 3.1.

**Proof.** Suppose that \( \varepsilon_i = u(t_i) - U^i \) be the local truncation error for (2.10) at the \( i \)th. By using the truncation error, we can write

\[
|\varepsilon_i| \leq q_1 k^2, \quad i \geq 2.
\]

(3.6)

In addition we have \( |\varepsilon_1| \leq q_1 k^3 \). To continue we assume that \( e_{n+1} \) be the global error in time discretizing process and \( q = \max\{q_1, \ldots, q_n\} \). We can write the following global error estimate at \( n + 1 \) level

\[
e_{n+1} = \sum_{i=1}^{n} \varepsilon_i, \quad (k \leq T/n),
\]

(3.7)
with the help of (3.6)-(3.7), we can write
\[
|e_{n+1}| = \left| \sum_{i=1}^{n} \varepsilon_i \right| \leq n \rho k^2 \leq n \frac{T}{n} k = \rho k,
\]
where \( \rho = \varrho T \).

Now at the \((n+1)\)th time step we assume that \( u(x) \) be the exact solution of (2.11) and \( U(x) = \sum_{i=-2}^{N+2} c_i B_i(x) \) be the B-spline approximation to \( u(x) \). Also we assume that \( S^*(x) = \sum_{i=-2}^{N+2} c_i^* B_i(x) \) be the unique spline interpolate to the exact solution. In order to derive a bound for \( \| u(x) - S^*(x) \|_{\infty} \) and \( \| S^*(x) - U(x) \|_{\infty} \), now we substituting \( S^* \) in (2.11) the we get the following result
\[
AC^* = Q^*,
\]
(3.9)

With considering (2.25) and (3.9), we get
\[
A(C^* - C) = (Q^* - Q).
\]
(3.10)

From (2.15), we can write
\[
|\Psi_i^* - \Psi_i| \leq v \left| S^*(x_i) - U(x_i) \right| + w \left| S''^*(x_i) - U''(x_i) \right|.
\]
(3.11)

By using (3.11) and Theorem 3.2, we can write
\[
\| Q^* - Q \|_{\infty} \leq M_1 h^2,
\]
(3.12)

where \( M_1 = v \lambda_0 L h^2 + w \lambda_2 L \). In this step from (3.10), we can write
\[
(C^* - C) = A^{-1}(Q^* - Q).
\]
(3.13)

By taking the infinity norm from (3.13) and applying (3.12), we get
\[
\| C^* - C \|_{\infty} \leq \| A^{-1} \|_{\infty} \| Q^* - Q \|_{\infty} \leq M_1 h^2 \| A^{-1} \|_{\infty}.
\]
(3.14)

By using the theory of matrices, we can write
\[
\sum_{i=1}^{N+1} a_{ki}^{-1} \eta_i = 1,
\]
(3.15)

where \( a_{ki}^{-1} \) are the elements of \( A^{-1} \) and \( \eta_i \) (1 \( \leq i \leq N + 1 \)) is the summation of the \( i \)th row of the matrix \( A \). As a result we can write
\[
\| A^{-1} \|_{\infty} = \sum_{i=1}^{N+1} |a_{ki}^{-1}| \leq \frac{1}{\min_{1 \leq i \leq N} \eta_i} \leq \frac{1}{\Lambda},
\]
(3.16)

where \( \Lambda \) is is constant. Following result is obtained by substituting (3.16) into (3.14), we get
\[
\| C^* - C \|_{\infty} \leq \frac{M_1 h^2}{\Lambda} \leq M_2 h^2,
\]
(3.17)
where $M_2 = \frac{M}{N}$ is constant. Considering the B-spline collocation approximation and the computed spline approximation, we can write:

$$S^*(x) - U(x) = \sum_{i=-2}^{N+2} (c_i^* - c_i) B_i(x),$$  \hspace{1cm} (3.18)

taking norm from (3.18) and by using (3.17) and lemma 3.4, we obtain

$$\| S^*(x) - U(x) \|_\infty \leq \| \sum_{i=-2}^{N+2} (c_i^* - c_i) B_i(x) \|_\infty \| C^* - C \|_\infty \leq 186M_2h^2.$$ \hspace{1cm} (3.19)

Also from Theorem 3.2 we can write

$$\| u - S^*(x) \|_\infty \leq \lambda_0 L h^4,$$ \hspace{1cm} (3.20)

and therefore with helping (3.19) and (3.20) we get

$$\| u - U(x) \|_\infty \leq \omega h^2,$$ \hspace{1cm} (3.21)

where $\omega = \lambda_0 L h^2 + 186M_2$.

\section{Numerical examples}

In order to illustrate the performance of the quintic B-spline collocation method in solving the One-dimensional hyperbolic telegraph equation and justify the accuracy and efficiency of the present method, we consider the following examples. To show the efficiency of the present method for our problem in comparison with the exact solution, we report the RMS error, $L_\infty$ and $L_2$ using formulae

\begin{equation*}
RMS = \frac{\left(\sum_{i=1}^{N} |u(x_i, t) - U_n(x_i)|^2\right)^{\frac{1}{2}}}{N^{\frac{1}{2}}},
\end{equation*}

\begin{equation*}
L_\infty = \max_i |U_n(x_i) - u(x_i, t)|, \quad L_2 = h \sum_{i=1}^{N} (u(x_i, t) - U_n(x_i))^2,
\end{equation*}

where $U(x, t)$ denotes numerical solution and $u(x, t)$ denotes analytical solution.

\textbf{Example 1.} Consider the hyperbolic telegraph equation (1.1) with $\alpha = \pi, \beta = \pi$, in the interval $[0, 1]$. In this case we have $f(x, t) = \pi^2 \sin(\pi x)(\sin(\pi t) + 2 \cos(\pi t))$. The analytical solution given by $u(x, t) = \sin(\pi t) \sin(\pi x)$. The boundary conditions and the initial conditions are taken from the exact solution. Table 2 shows the absolute error between the analytical solution and the numerical solution at different points for $t = 0.5$. Table 3 shows the $L_2$ errors at different partitions. The graph of the solution is given in Figure 1. Also, Figure 2 shows that the solution obtained by our method is close to the exact solution.
Table 2: A comparison of absolute errors of Example 1 at different points with $h = 1/100, k = 1/200$.

| Method          | present method | method in [14] |
|-----------------|----------------|----------------|
| $x$             | $\Gamma : 2\sin\left(\frac{x}{2}\right)$ | $\Gamma : k$ | $\eta = \frac{1}{10}, \gamma = \frac{1}{4}$ grid |
| 0.2             | $3.23677\times10^{-3}$ | $3.26277\times10^{-5}$ | $5.858898718658607\times10^{-1}$ |
| 0.4             | $5.32737\times10^{-5}$ | $5.32737\times10^{-5}$ | $9.479897263432836\times10^{-1}$ |
| 0.6             | $5.32737\times10^{-5}$ | $5.32737\times10^{-5}$ | $9.479897263432840\times10^{-1}$ |
| 0.8             | $5.32737\times10^{-5}$ | $3.26277\times10^{-5}$ | $5.858898718658610\times10^{-1}$ |

Table 3: A comparison of $L_2$ errors of Example 1 at different partitions.

| Partitions | N=100, k=0.01 | N=400, k=0.001 |
|------------|---------------|----------------|
| $Time$     | $\Gamma : 2\sin\left(\frac{x}{2}\right)$ | $\Gamma : k$ | $\Gamma : 2\sin\left(\frac{x}{2}\right)$ | $\Gamma : k$ |
| 0.5        | $1.5799\times10^{-5}$ | $1.58168\times10^{-5}$ | $1.57942\times10^{-7}$ | $7.91938\times10^{-8}$ |
| 1          | $6.94858\times10^{-6}$ | $6.61076\times10^{-6}$ | $6.94312\times10^{-8}$ | $3.33013\times10^{-8}$ |
| 1.5        | $1.50368\times10^{-5}$ | $1.51490\times10^{-5}$ | $1.50334\times10^{-7}$ | $7.57557\times10^{-8}$ |
| 2          | $7.25141\times10^{-6}$ | $6.89868\times10^{-6}$ | $7.24547\times10^{-8}$ | $3.4743\times10^{-8}$ |

Example 2. We consider the hyperbolic telegraph equation (1.1) with $f(x, t) = (3-4\alpha+\beta^2)\exp(-2t)\sinh(x)$ and the analytical solution $u(x, t) = \exp(-2t)\sinh(x)$, in the interval $[0, 1]$. The boundary conditions and the initial conditions are taken from the exact solution. Tables 4 and 5 give a comparison between the $L_\infty$ errors found by our method and the method in [7]. Also Table 5 shows $RMS$ and $L_2$ errors. Figure 3, shows absolute error for different values of time with $N = 200, k = 0.001$.

Example 3. In this example we consider the hyperbolic telegraph equation (1.1) with $\alpha = 10, \beta = 5$ in $x \in [0, 1]$ and $f(x, t) = -2\alpha\sin(t)\sin(x)$ +
Figure 1: The graph of the solution for Example 1 with $\Gamma(k) = k$, $N = 200$ and $k=0.01$.

Figure 2: Comparisons between numerical and analytical solutions with $\Gamma(k) = k$ (left) and $\Gamma(k) = 2\sin(k/2)$ (right) for Example 1 at different times with $N = 200, k = 0.01$. 
Table 4: A comparison of $L_\infty$ errors of Example 2 for $\alpha = 20$, $\beta = 10$ at different time and $N = 21$, $k = 0.01$.

| Method          | present method | method in [7] |
|-----------------|----------------|---------------|
| $Time$          | $\Gamma : 2\sin(\frac{\pi}{4})$ | $\Gamma : k$ |
| $\Gamma : 2\sin(\frac{\pi}{4})$ | $1.9517 \times 10^{-9}$ | $1.92002 \times 10^{-9}$ | $2.3874 \times 10^{-6}$ | $2.21693 \times 10^{-6}$ |
| $\Gamma : k$   | $1.2478 \times 10^{-9}$ | $1.2675 \times 10^{-9}$ | $1.72404 \times 10^{-5}$ | $1.22688 \times 10^{-6}$ |

Table 5: A comparison of $L_\infty$ errors of Example 2 for $\alpha = 20$, $\beta = 10$ at different time and $N = 21$, $k = 0.0001$.

| Method          | present method | method in [7] |
|-----------------|----------------|---------------|
| $Time$          | $\Gamma : 2\sin(\frac{\pi}{4})$ | $\Gamma : k$ |
| $\Gamma : 2\sin(\frac{\pi}{4})$ | $1.9517 \times 10^{-9}$ | $1.92002 \times 10^{-9}$ | $2.3874 \times 10^{-6}$ | $2.21693 \times 10^{-6}$ |
| $\Gamma : k$   | $1.2478 \times 10^{-9}$ | $1.2675 \times 10^{-9}$ | $1.72404 \times 10^{-5}$ | $1.22688 \times 10^{-6}$ |

Table 6: $L_2$ and RMS errors of Example 2 for $\alpha = 20$, $\beta = 10$ at different time and $N = 21$, $k = 0.01$.

| $\Gamma$ | $2\sin(\frac{\pi}{4})$ | $k$ | $L_2$ | RMS | $L_2$ | RMS |
|----------|----------------|-----|-------|-----|-------|-----|
| $Time$   | $\Gamma$ | $2\sin(\frac{\pi}{4})$ | $k$ | $L_2$ | RMS | $L_2$ | RMS |
| 0.2      | $2.3651 \times 10^{-6}$ | $1.08383 \times 10^{-9}$ | $2.2565 \times 10^{-6}$ | $1.03406 \times 10^{-9}$ |
| 0.4      | $2.82003 \times 10^{-6}$ | $1.2923 \times 10^{-5}$ | $2.67987 \times 10^{-6}$ | $1.22807 \times 10^{-5}$ |
| 0.8      | $2.0589 \times 10^{-6}$ | $9.4359 \times 10^{-6}$ | $1.95292 \times 10^{-6}$ | $8.94939 \times 10^{-6}$ |
| 1        | $1.5714 \times 10^{-6}$ | $7.20105 \times 10^{-6}$ | $1.48999 \times 10^{-6}$ | $6.82797 \times 10^{-6}$ |

$\beta^2 \cos(t) \cos(x)$. The exact solution for this case is $u(x, t) = \cos(t) \sin(x)$. The boundary conditions and the initial conditions are taken from the exact solution. In order to compare the solutions with [7], we have taken $k = 0.001$ and $N = 21$. Table 7 gives a comparison between the $L_\infty$ error found by our method and by method in [7]. Table 8 shows $L_2$ in different partitions. Table 9 shows RMS and $L_2$ errors. Figure 4, shows absolute error for different values of time with $N = 50$, $k = 0.001$. 

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Figure 3: Absolute errors with $\Gamma(k) = k$ (left) and $\Gamma(k) = 2\sin(k/2)$ (right) for Example 2 with $k = 0.001$, $N = 200$ and $\alpha = 20$, $\beta = 10$.

Figure 4: Absolute errors with $\Gamma(k) = k$ (left) and $\Gamma(k) = 2\sin(k/2)$ (right) for Example 3 with $k = 0.001$, $N = 50$. 
Table 7: A comparison of $L_{\infty}$ errors for Example 3 at different time and $N = 21, k = 0.001$.

| Method                     | present method | method in [7] |
|----------------------------|----------------|---------------|
| $Time$ $\Gamma : 2\sin(\frac{k}{2})$ $\Gamma : k$ Uniform Grid Nonuniform grid |
| 0.5 $7.59175 \times 10^{-9}$ $5.35357 \times 10^{-9}$ $1.67216 \times 10^{-8}$ $1.28551 \times 10^{-5}$ | |
| 1 $3.5984 \times 10^{-9}$ $2.39252 \times 10^{-8}$ $4.71302 \times 10^{-4}$ $3.20793 \times 10^{-5}$ | |
| 1.5 $1.61922 \times 10^{-8}$ $4.2016 \times 10^{-8}$ $8.62796 \times 10^{-4}$ $5.71454 \times 10^{-5}$ | |
| 2 $2.49884 \times 10^{-8}$ $5.20784 \times 10^{-8}$ $8.62796 \times 10^{-4}$ $5.71454 \times 10^{-5}$ | |

Table 8: A comparison of $L_{2}$ errors of Example 3 at different partitions.

| Partitions | N=200,k=0.01 | N=400,k=0.001 |
|------------|--------------|---------------|
| $Time$ $\Gamma : 2\sin(\frac{k}{2})$ $\Gamma : k$ $\Gamma : 2\sin(\frac{k}{2})$ $\Gamma : k$ |
| 0.5 $3.69769 \times 10^{-8}$ $2.46374 \times 10^{-8}$ $2.65293 \times 10^{-10}$ $1.73258 \times 10^{-10}$ | |
| 1 $1.40829 \times 10^{-8}$ $1.16929 \times 10^{-7}$ $9.95706 \times 10^{-11}$ $8.27160 \times 10^{-10}$ | |
| 1.5 $8.1959 \times 10^{-8}$ $2.09527 \times 10^{-7}$ $5.80334 \times 10^{-10}$ $1.48296 \times 10^{-10}$ | |
| 2 $1.28423 \times 10^{-7}$ $2.61864 \times 10^{-7}$ $9.09458 \times 10^{-10}$ $1.85373 \times 10^{-9}$ | |

Table 9: $L_{2}$ and RMS errors of Example 3 at different time and $N = 80, k = 0.05$.

| $\Gamma$ $Time$ $L_{2}$ $\Gamma : 2\sin(\frac{k}{2})$ RMS $\Gamma : 2\sin(\frac{k}{2})$ RMS |
|-----------------|--------------|-------------|----------------|-------------|
| 0.5 $1.37362 \times 10^{-6}$ $1.2286 \times 10^{-6}$ | $9.4623 \times 10^{-7}$ | $8.89618 \times 10^{-6}$ |
| 1 $5.5496 \times 10^{-7}$ | $4.96372 \times 10^{-6}$ | $4.61873 \times 10^{-6}$ | $4.13112 \times 10^{-5}$ |
| 1.5 $3.22627 \times 10^{-6}$ | $2.88566 \times 10^{-5}$ | $8.26127 \times 10^{-6}$ | $7.38911 \times 10^{-5}$ |
| 2 $5.05609 \times 10^{-6}$ | $4.52231 \times 10^{-5}$ | $1.03189 \times 10^{-5}$ | $9.22947 \times 10^{-5}$ |

5 Conclusion

The quintic B-spline collocation method is used to solve the one-dimensional hyperbolic telegraph equation with initial and boundary conditions. The convergence analysis of the method is shown. The numerical solutions are compared with the exact solution by finding the RMS, $L_{2}$ and $L_{\infty}$ errors. The numerical results given in the previous section demonstrate the good accuracy of the scheme proposed in this research.

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