Scattering of Noncommutative \((n, 1)\) Solitons

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Abstract

We study scattering of noncommutative solitons in 2 + 1 dimensional scalar field theory. In particular, we investigate a system of two solitons with level \(n\) and \(n'\) (the \((n, n')\)-system) in the large noncommutativity limit. We show that the scattering of a general \((n, n')\)-system occurs at right angles in the case of zero impact parameter. We also derive an exact Kähler potential and the metric of the moduli space of the \((n, 1)\)-system. We examine numerically the \((n, 1)\) scattering and find that the closest distance for a fixed scattering angle is well approximated by a function \(a + b\sqrt{n}\) where \(a\) and \(b\) are some numerical constants.
It has been recognized that noncommutative field theory is interesting because it has various non-trivial solitons [1, 2, 3, 4, 5, 6] which include those do not appear in commutative theories. The noncommutative soliton in 2 + 1 dimensional scalar field theory [3, 7, 8, 9] is one of important examples. Noncommutative field theory arises also in open string theory with a B-field background [10, 11]; the effective field theory on D-branes becomes noncommutative. Noncommutative solitons of such field theory can be identified with various D-brane configurations [12, 13, 14, 15, 6]. For example, one may use tachyonic noncommutative solitons to construct lower dimensional D-branes from unstable higher dimensional D-branes in the effective field theoretic description of open string field theory [13]. In order to study dynamical aspects of D-branes, scattering of noncommutative solitons [16] would play an important role. Recently, a systematic construction of a multi-soliton solution was proposed [17, 18]. In these works, the level one multi-soliton solutions and finite $\theta$ correction have been explored. Here, a level one soliton is a radially symmetric Gaussian lump solution.

In this letter, we will study scattering of noncommutative solitons with higher levels in the limit of the large noncommutativity ($\theta \to \infty$). In particular, we will consider a system which contains two solitons with level $n$ and $n'$, respectively. A “level $n$” soliton means $n$ coincident level one solitons. We call this system the $(n, n')$-system. We investigate the Kähler potential and metric of the moduli space of the $(n, n')$-system. In this work, using the expansion around the origin of the moduli space, we will show that the scattering of the $(n, n')$-system occurs at right angles in the case of zero impact parameter. We next derive an exact Kähler potential and the metric of the moduli space of the $(n, 1)$-system. Using this metric we investigate some global aspects of the moduli space. We calculate numerically the scattering angle for the $(n, 1)$-system as a function of the closest distance and find that the closest distance is well approximated by a function $a + b\sqrt{n}$ ($a$ and $b$ are some constants) for a fixed scattering angle.

We begin with real scalar field theory on 2 + 1 dimensional noncommutative spacetime with coordinates $(t, x, y)$. It has spatial noncommutativity such as

$$[\hat{x}, \hat{y}] = i\theta. \quad (1)$$
The action is
\[ S = - \int dt d^2 x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right\}, \] (2)

where fields multiplication are defined by using the star product
\[ (\phi_1 \ast \phi_2)(x, y) = \exp \left( i \frac{\theta}{2} (\partial_{\xi_1} \partial_{\eta_2} - \partial_{\eta_1} \partial_{\xi_2}) \right) \phi_1(\xi_1, \eta_1) \phi_2(\xi_2, \eta_2) \bigg|_{\xi_1=\xi_2=x, \eta_1=\eta_2=y}. \] (3)

For simplicity, we assume that the potential function \( V \) has only one local minimum at \( \phi = \lambda \) other than \( \phi = 0 \), and \( V(0) = 0 \). In the following, we will only consider the case of the large noncommutativity limit, \( \theta \to \infty \). Rescaling \( x, y \to \sqrt{\theta} x, \sqrt{\theta} y \), the action is dominated by the potential term so that the static field equation becomes
\[ \frac{\partial V}{\partial \phi}(\phi) = 0. \] (4)

One class of solutions to this equation can be constructed by using a function which satisfies the condition
\[ (\phi_0 \ast \phi_0)(x, y) = \phi_0(x, y). \] (5)

A solution to eq.(4) is given by \( \lambda \phi_0(x, y) \). A function \( \phi(x, y) \) on the noncommutative plane can be mapped to an operator \( \Phi(\hat{x}, \hat{y}) \) acting on the Hilbert space \( \mathcal{H} \) of one particle on the line. The relation between \( \phi(x, y) \) and \( \Phi(\hat{x}, \hat{y}) \) is given by using the Weyl-Moyal correspondence:
\[ \Phi(\hat{x}, \hat{y}) = \int \frac{d^2 k}{(2\pi)} \tilde{\phi}(k) e^{-i(k_x \hat{x} + k_y \hat{y})}, \] (6)
\[ \tilde{\phi}(k) = \int d^2 x \phi(x, y) e^{i(k_x x + k_y y)}. \] (7)

With this correspondence, the energy of a solution \( \phi \) can be written as
\[ E[\phi] = \theta \int d^2 x V(\phi) = 2\pi \theta \text{Tr}_{\mathcal{H}}(V(\Phi)). \] (8)

This formula tells us that if we find a solution \( \Phi \), then another solution which has the same energy can be obtained by acting a unitary operator \( U \) on \( \Phi \) as \( U \Phi U^\dagger \).

An operator which satisfies the condition (3) is a projection operator. Thus the most general solution of this class can be written using a set of orthogonal projection operators \( \{P_i\} \):
\[ \lambda (P_1 + P_2 + \cdots). \] (9)
Taking $a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$ and $a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$, the Hilbert space can be regarded as the Fock space of a harmonic oscillator. Any projection operator can be written by the Fock basis $\{ |n\rangle \}$. A rank $k$ projection operator can always be written in the form

$$U(\sum_{i=0}^{k-1} |i\rangle\langle i|)U^\dagger.$$  \hspace{1cm} (10)

A diagonal projection operator $|n\rangle\langle n|$ corresponds to a radially symmetric configuration, because it has the same number of creation and annihilation operators and $a^\dagger a \approx r^2/2$, where $r^2 = x^2 + y^2$. The most basic solution is $\lambda|0\rangle\langle 0|$, which corresponds to the Gaussian lump configuration centered at the origin. More generally, a field configuration $\phi_n(x, y)$ which corresponds to $|n\rangle\langle n|$ is

$$\phi_n(x, y) = 2(-1)^ne^{-r^2/2}L_n(2r^2),$$  \hspace{1cm} (11)

where $L_n$ is the $n$-th Laguerre polynomial. If $U = 1$, the projection operator corresponds to a radially symmetric configuration centered at the origin whose width is $\sim \sqrt{n}$. If $U$ is a translation operator; $U(z) \equiv e^{za^\dagger - \bar{z}a}$ where $z = \frac{1}{\sqrt{2}}(x + iy)$, the corresponding field configuration has the same profile but centered at $(x, y)$ \[4\]. A rank $k$ projection operator is called a level $k$ soliton in \[4\], but in this letter we define a level $k$ soliton at $z$ as a projection operator onto a subspace spanned by $\{ U(z)|0\rangle, U(z)|1\rangle, \ldots, U(z)|k-1\rangle \}$. This can be shown to be equivalent to $k$ coincident level 1 solitons by the coordinate transformation in \[17\].

The solution for a system of $k$ level 1 solitons \[17, 18\] each centered at $z_i$ ($i = 1, \ldots, k$) on the complex $z$-plane can be constructed using coherent states

$$|z_i\rangle \equiv U(z_i)|0\rangle = e^{-\frac{1}{2}|z_i|^2}e^{z_i a^\dagger}|0\rangle, \quad U(z_i) = e^{z_i a^\dagger - \bar{z}_i a},$$  \hspace{1cm} (12)

and is given as

$$\Phi = \lambda |z_i\rangle G^{ij}|z_j\rangle,$$  \hspace{1cm} (13)

or equivalently,

$$\phi(z) = \lambda \cdot 2G^{ij}G_{ji}e^{-2(\bar{z} - \bar{z}_j)(z - z_i)}.$$  \hspace{1cm} (14)

Here $G_{ij}$ is the $n \times n$ hermitian matrix

$$G_{ij} = \langle z_i|z_j\rangle = e^{-\frac{1}{2}|z_i|^2 - \frac{1}{2}|z_j|^2 + \bar{z}_i z_j},$$  \hspace{1cm} (15)
and $G^{ij}$ is its inverse such that $G^{il}G_{lj} = \delta^i_j$. The moduli space of this solution is parametrized by $z_i$. Its metric can be obtained by

$$g_{ij} = \frac{1}{2\pi\lambda^2} \int d^2x \partial_i \phi \partial_j \phi = \frac{1}{\lambda^2} \text{Tr}[\partial_i \Phi \partial_j \Phi]$$ (16)

which comes from the time derivative term in the action when we regard $z_i$ depending (slowly) on time $t$ [13, 14]. The metric (16) may also be expressed as

$$g_{ij} = G^{ij} \langle z_j | a\phi_\perp a^\dagger | z_i \rangle,$$ (17)

where $\phi_\perp \equiv 1 - |z_l \rangle G_{lm} \langle z_m |$ and there is no summation over $i$ and $j$. This moduli space has a Kähler structure and the Kähler potential is given by the formula

$$K = \ln \left( \exp \left( \sum_{l=1}^{k} |z_l|^2 \right) \cdot \det G \right) = \ln \det(e^{\bar{z}_i z_j}).$$ (18)

This Kähler potential has coordinate singularities when two or more $z_i$’s coincide. These singularities of Kähler potential would appear as conical singularities of the metric. These conical singularities have been explicitly examined in the case of $k = 2$ and $k = 3$ at the origin of the respective moduli space. For example, in the case of $k = 2$ ($z_1 \neq z_2$), the metric is given as [16]

$$d^2s = \frac{1}{2} f(r)(dr^2 + r^2 d\theta^2),$$

$$f(r) = \coth(r^2/4) - \frac{r^2/4}{\sinh^2(r^2/4)},$$ (19)

where we have taken $z_1 = -z_2 = re^{i\theta}/2\sqrt{2}$ so that the relative distance between two level 1 solitons is $r$. $f(r)$ behaves $\sim r^2$ as $r \to 0$, so if we take new coordinates $\rho = r^2$ and $\tilde{\theta} = 2\theta$, the metric becomes a flat one near the origin: $ds^2 \propto d\rho^2 + \rho^2 d\tilde{\theta}^2$. Thus a soliton coming from $\tilde{\theta} = 0$ will pass through the origin (and the other soliton) smoothly and go in the direction of $\tilde{\theta} = \pi$, i.e. $\theta = \pi/2$. That is, the scattering of two level 1 solitons occurs at right angles. It is difficult to see the conical singularities explicitly in the Kähler metric (or potential) and to determine the deficit angle for higher $k$ or for a solution which contains higher level solitons, because the Kähler potential and metric is so complicated. The only exception is the case of scattering of two level $n$ solitons and it was conjectured in [16] that the scattering occurs also at right angles.
Now we will explore the locus of coincidence in the moduli space of a \((n,n')\)-system by expanding a Kähler metric around the origin of the moduli space of a multi-soliton solution which consists of \(k\) level 1 solitons.

A Kähler potential for the moduli space of the \(k\) soliton solution is given by (18).

Using the translational invariance of the theory, we can set the center of mass position \(c = \frac{1}{k} \sum_{i=1}^{k} z_i\) simply at the origin. Let \(y_i\) be the relative coordinates \(z_i - c\), then we have

\[
K = \ln \det(e^{y_i y_j}).
\]

(20)

Expanding the exponentials in the determinant, we obtain

\[
\det(e^{y_i y_j}) = \frac{1}{k!} \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{1}{m_1! \ldots m_k!} |F_m(y)|^2,
\]

(21)

where

\[
F_m(y) \equiv \begin{vmatrix} y_1^{m_1} & \cdots & y_k^{m_1} \\ \vdots & \ddots & \vdots \\ y_1^{m_k} & \cdots & y_k^{m_k} \end{vmatrix}, \quad m = (m_1, \ldots, m_k).
\]

(22)

For \(\delta = (0,1,\ldots,k-1)\), \(F_\delta(y)\) becomes the Vandermonde determinant

\[
F_\delta(y) = \prod_{i>j}(y_i - y_j).
\]

(23)

\(F_m(y)\) is non-zero for \(0 \leq m_1 < m_2 < \ldots < m_k\).

Let \(\mu = m - \delta = (\mu_1, \ldots, \mu_k)\) \((0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k)\). Then \(F_m(y)\) can be expressed as

\[
F_m(y) = S_\mu(y) F_\delta(y),
\]

(24)

where \(S_\mu(y)\) is a symmetric polynomial in \(y_j\) of degree \(\sum_{i=1}^{k} \mu_i\) and known as the Schur function \([20]\). It is defined for a partition of \(\sum_{i=1}^{k} \mu_i\). So we may specify a Schur function with a partition instead of \(\mu\): for example, \(S_{(0,0,\ldots,0,1)} = S_1, S_{(0,\ldots,0,1,1,2)} = S_{2,1,1}, \) etc. Their explicit examples are given as follows:

\[
S_1 = e_1,
\]

\[
S_2 = e_1^2 - e_2, \quad S_{1,1} = e_2,
\]

\[
S_3 = e_1^3 - 2e_1 e_2 + e_3, \quad S_{2,1} = e_1 e_2 - e_3, \quad S_{1,1,1} = e_3.
\]

\[
S_4 = e_1^4 - 3e_2 e_1^2 + 2e_3 e_1 + e_2^2 - e_4, \quad S_{3,1} = e_2 e_1^2 - e_2^2 - e_3 e_1 + e_4,
\]

\[
S_{2,2} = e_2^2 - e_3 e_1, \quad S_{2,1,1} = e_3 e_1 - e_4, \quad S_{1,1,1,1} = e_4.
\]
where $e_m$ denotes the $m$-th elementary symmetric polynomial:

$$e_m = \sum_{i_1 < \cdots < i_m} y_{i_1} \cdots y_{i_m}, \quad (25)$$

Suppose that all of $y_i$ are small. The Kähler potential $K = \ln \det(e^{\tilde{g}_i y_j})$ can be rewritten as

$$K = \ln \left( \sum_{0 \leq \mu_1 \leq \cdots \leq \mu_k} \frac{1}{\mu_1!(\mu_2 + 1)! \cdots (\mu_k + k - 1)!} |S_\mu(y)|^2 |F_\delta(y)|^2 \right)$$

$$= \ln \left( \left( 1 + \frac{1}{k} |S_1|^2 + \frac{1}{k(k + 1)} |S_2|^2 + \frac{1}{(k - 1)k} |S_{1,1}|^2 + \cdots \right) |F_\delta(y)|^2 \right) - \ln(1! \ldots (k - 1)!)$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} G^m + \ln \prod_{i<j} |y_i - y_j|^2 - \ln(1! \ldots (k - 1)!), \quad (26)$$

where

$$G = \frac{1}{k} |S_1|^2 + \frac{1}{k(k + 1)} |S_2|^2 + \frac{1}{(k - 1)k} |S_{1,1}|^2$$

$$+ \frac{1}{k(k + 1)(k + 2)} |S_3|^2 + \frac{1}{(k - 1)k(k + 1)} |S_{2,1}|^2 + \frac{1}{(k - 2)(k - 1)k} |S_{1,1,1}|^2$$

$$+ \frac{1}{(k - 1)k(k + 1)(k + 3)} |S_4|^2 + \frac{1}{(k - 2)(k - 1)k(k + 2)} |S_{3,1}|^2$$

$$+ \frac{1}{(k - 1)k(k + 1)} |S_{2,2}|^2 + \frac{1}{(k - 2)(k - 1)k(k + 1)} |S_{2,1,1}|^2$$

$$+ \frac{1}{(k - 3)(k - 2)k(k - 1)} |S_{1,1,1,1}|^2 + \cdots. \quad (27)$$

Because we are working in the center of mass system, we have $e_1 = \sum_i y_i = 0$, $S_2 = -e_2$, $S_{1,1} = e_2$ and so on. Thus $G$ can be simplified. Furthermore, note that $e_m$ can be rewritten by $p_m = \sum_{i=1}^k y_i^n$, e.g. $e_1 = p_1$, $e_2 = -\frac{1}{2}p_2 + \frac{1}{2}p_1^2$. Finally, we get the following expression for $K$:

$$K = \sum_{i<j} \ln |y_i - y_j|^2 - \ln(1! \ldots (k - 1)!) + \frac{1}{2(k^2 - 1)} |p_2|^2 + \frac{k}{3(k^2 - 4)(k^2 - 1)} |p_3|^2$$

$$+ \frac{2}{(k - 1)k(k + 2)(k + 3)} \left| \frac{1}{8}p_2 - \frac{1}{4}p_4 \right|^2 - \frac{k^2 + 1}{16(k^2 - 1)k^2} |p_2|^4$$

$$+ \frac{2}{(k - 3)(k - 2)k(k + 1)} \left| \frac{1}{8}p_2^2 - \frac{1}{4}p_4 \right|^2 + \cdots. \quad (28)$$
The first term in the above expression will diverge when some of \( y_i \) are coincident, but this divergence can be removed by the Jacobian of the coordinate transformation \([17]\). Let us briefly recall how this can be achieved. For instance, when \( n \) of \( z_i \) coincide at \( z = c + w \), we should redefine the basis from \( \{ e^{|z_i}\alpha|0\} \) to
\[
\left\{ \sum_{a=1}^{n} (V^{-1})_{ab} e^{z_a\alpha}|0\right\}_{b=1,...,n}, \quad \{ e^{z_j\alpha}|0\} \}_{j=n+1,...,k},
\]
where \( V_{ab} \equiv (z_a - z)^{b-1} = (y_a - w)^{b-1} \). The new basis vectors for \( b = 1,\ldots, n \) become
\[
\frac{(a_1)^{b-1}}{(b-1)!} e^{z_a\alpha}|0\rangle, \quad (b = 1,\ldots, n)
\]
as \( z_a - z = y_a - w \to 0 \). It corresponds to the merging process of \( n \) level 1 solitons into a single level \( n \) soliton. This coordinate transformation changes the Kähler potential from \( K \) to
\[
K' = K + \ln(\det(V^{-1})^\ast \det(V^{-1})) = K - \sum_{a < b} \ln |y_a - y_b|^2,
\]
in this limit. Here we have used eq.(23). \( K \) and \( K' \) are equivalent and give the same metric. Thus the singular terms in \( K \) could be removed by this coordinate transformation.

This procedure is further applicable to the remaining \( z_j \) \( (j = n + 1,\ldots, k) \) repeatedly. Therefore, we can get an expansion of a Kähler potential corresponding to an arbitrary level-\( (n_1, n_2, \ldots) \) soliton solution \( (n_1 + n_2 + \cdots = k) \) in the locus of coincidence from eq.(28).

Now we can explore how the scattering of one level \( n \) soliton and one level \( n' \) soliton is \( (n + n' = k) \). We call this system the \( (n, n') \)-system. We will take \( y_1 = \cdots = y_n = n'y/k, \ y_{n+1} = \cdots = y_k = -ny/k \), such that the relative distance is always \( r = \sqrt{2}|y| \).

Then eq.(31) becomes
\[
K' = \frac{n^2 n'^2}{2(k^2 - 1) k^2} |y|^4 + \frac{n^2 n'^2 (n - n')^2}{3(k^2 - 4)(k^2 - 1) k^3} |y|^6
+ \frac{n^2 n'^2 c_8(k, n)}{4(k^2 - 9)(k^2 - 4)(k^2 - 1)^2 k^4} |y|^8 + \cdots,
\]
where,
\[
c_8(k, n) = k^6 - 10k^5 n + 34k^4 n^2 + 10k^3 n - k^2 - 46k^2 n^2 + 25k^2 m^4 + 72kn^3 - 36k^4.
\]
Then we get the Kähler metric by \( g_{y\bar{y}} = \partial_y \partial_{\bar{y}} K' \);

\[
ds^2 = \left( \frac{n^2 n'^2}{2(k^2 - 1) k^2} r^2 + \cdots \right) (dr^2 + r^2 d\theta^2), \quad y = \frac{1}{\sqrt{2}} r e^{i\theta} \quad (34)
\]

This behavior is the same as the case of the (1,1)-system as we have seen in eq.(19). So we conclude that the scattering of the \((n, n')\)-system occurs at right angles in the center of mass system when the impact parameter is zero.

This behavior of the \((n, n')\) scattering can be explicitly examined by the Kähler potential using new basis. If we take \( k \) linearly independent (not necessarily orthogonal) basis vectors \( \{|\psi_i\rangle\} \) which span a subspace of \( \mathcal{H} \), the rank \( k \) projection operator onto this subspace is given by [17, 18]

\[
P = |\psi_i\rangle h^{ij} \langle \psi_j|. \quad (35)
\]

Here, \( h^{ij} \) is the inverse matrix of the \( k \times k \) hermitian matrix

\[
h_{ij} \equiv \langle \psi_i | \psi_j \rangle, \quad (36)
\]

so that \( h^{ik} h_{kj} = \delta^i_j \). If the \( k \) basis vectors \( \{|\psi_i\rangle\} \) holomorphically depend on complex parameters \( z_a \), a natural metric on the moduli space is

\[
g_{\alpha\bar{\beta}} = \text{Tr}[\partial_\alpha P \partial_{\bar{\beta}} P], \quad (37)
\]

and a Kähler potential which gives this metric is given by [17]

\[
K = \ln \det(h_{ij}). \quad (38)
\]

The above metric is the same one as the physical metric

\[
g_{\alpha\bar{\beta}} = \frac{1}{2\pi \lambda^2} \int d^2 \phi \partial_\alpha \phi \partial_{\bar{\beta}} \phi. \quad (39)
\]

up to a overall constant.

Now take the \( k = n + n' \) basis vectors as

\[
\{|\psi_i\rangle\} \equiv \left\{ e^{z_{i1} a^1} |0\rangle, a^1 e^{z_{i1} a^1} |0\rangle, \ldots, a^{n-1} e^{w_{i1} a^1} |0\rangle, e^{w_{i2} a^1} |0\rangle, a^1 e^{w_{i2} a^1} |0\rangle, \ldots, a^{n'-1} e^{w_{i2} a^1} |0\rangle \right\}
\]

\[
= \left\{ a^{i_n} e^{z_i a^1} |0\rangle \right\}_{i=1, \ldots, k}. \quad (40)
\]

\(^1\)For the \( k \) level 1 solitons, if we define \(|\psi_i\rangle \equiv e^{z_{i1} a^1} |0\rangle \) \((i = 1, \ldots, k)\) as in ref.[17] which is differ from \(|z_i\rangle \) in eq.(12) by the normalization factor, then eq.(38) gives the same result as eq.(18).
For this basis, the inner product matrix $h_{ij}$ is expressed as follows:

$$h_{ij} = \langle \psi_i | \psi_j \rangle = \begin{cases} n_i! z_i^{n_j-n_i} L_{n_i}^{(n_j-n_i)}(-z_i z_j) \cdot e^{z_i z_j}, & n_i \leq n_j, \\ n_j! z_j^{n_i-n_j} L_{n_j}^{(n_i-n_j)}(-z_i z_j) \cdot e^{z_i z_j}, & n_j \leq n_i. \end{cases}$$

(42)

Here, $L_n^{(\alpha)}$ is the Laguerre polynomial:

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^r \left( \frac{n + \alpha}{n - r} \right) \frac{x^r}{r!}.$$  

(43)

From the Weyl-Moyal correspondence, we obtain the solution of the $(n, n')$-system as

$$\phi(z) = \lambda h^{ij} \phi_{ij}(z),$$

(44)

where $\phi_{ij}(z)$ is a field configuration corresponding to $|\psi_i\rangle\langle \psi_j|$. This is given by the Weyl-Moyal correspondence:

$$\phi_{ij}(z) = \begin{cases} (-)^{n_j} n_i! (2z - z_i)^{n_j-n_i} L_{n_i}^{(n_j-n_i)}((2z - z_j)(2z - z_i)) \cdot \phi_{0ij}, & n_i \leq n_j, \\ (-)^{n_i} n_j! (2z - z_j)^{n_i-n_j} L_{n_j}^{(n_i-n_j)}((2z - z_i)(2z - z_j)) \cdot \phi_{0ij}, & n_j \leq n_i, \end{cases}$$

(45)

Here, $\phi_{0ij}(z) \equiv 2e^{z_i z_j} e^{-2(z_i z_j)(z-z_i)}$. Fig.1 is an example of a field configuration for a $(n, n')$-system plotted by using the above formulae.

In the case of the $(n, 1)$-system, the Kähler potential can be calculated as follows. Let us change the basis from $|\psi_i\rangle$ to

$$|\tilde{\psi}_i\rangle \equiv U(z_i)|n_i\rangle \ (i = 1, \ldots, k), \quad U(z_i) = e^{z_i a^1 - \bar{z}_i a},$$

(46)

where $|n_i\rangle$ are the Fock basis. $z_i$ and $n_i$ are the same as (11). The matrix $h_{ij}$ can be re-expressed with the $k \times k$ matrices $B_{ij} \equiv \langle \tilde{\psi}_i | \psi_j \rangle$ and $\tilde{h}_{ij} \equiv \langle \tilde{\psi}_i | \tilde{\psi}_j \rangle$ which have structures simpler than $h$;

$$h_{ij} = \langle \psi_i | \psi_j \rangle = \langle \tilde{\psi}_i | \tilde{\psi}_j \rangle \tilde{h}^{lm} \langle \tilde{\psi}_m | \psi_j \rangle = B_{il}^{m} \tilde{h}^{lm} B_{mj} = (B^* \tilde{h}^{-1} B)_{ij}.$$  

(47)
exact solution for a $(n,n')$-system in case of $r = 6$.

Here, we have introduced the inverse matrix $\tilde{h}^{ij}$ of $\tilde{h}_{ij}$ such that $\tilde{h}^{ji}\tilde{h}_{ij} = \delta^i_j$. The second equality in eq. (47) holds because $\{|\psi_i\rangle\}$ and $\{|\tilde{\psi}_i\rangle\}$ span the same subspace of $\mathcal{H}$ and $|\tilde{\psi}_i\rangle\tilde{h}^{lm}|\tilde{\psi}_m\rangle$ is the identity operator in this subspace. With the relation, we have

$$\det h = \frac{\det B^* \det B}{\det h}. \quad (48)$$

From (48) and (49), $B_{ij}$ is given by

$$B_{ij} = \begin{cases} \sqrt{n_i!}e^{-\frac{1}{2}|z_i|^2}z_i^{n_j-n_i}L_{n_i}^{(n_j-n_i)}(-z_i(z_j-z_i)), & n_i \leq n_j; \\ \frac{n_i!}{\sqrt{n_i!}}e^{-\frac{1}{2}|z_i|^2}z_i^{n_j-n_i}L_{n_j-n_i}^{(n_i)}(-z_i(z_j-z_i)), & n_j \leq n_i. \end{cases} \quad (49)$$

For the $(n,1)$-system, note that $z_i = z_j = w_1$ ($i,j \leq n$) and $L^{(\alpha)}_n(0) = \binom{n+\alpha}{n}$. Then we obtain

$$B = \begin{pmatrix} C_1 \bar{w}_1C_1 & \bar{w}_1^2C_1 & \cdots & \bar{w}_1^{n-1}C_1 & C_2 \\ \binom{1}{1}C_1 & \binom{2}{1}w_1C_1 & \cdots & \binom{n-1}{1}w_1^{n-2}C_1 & (w_2-w_1)C_2 \\ \sqrt{2!}\binom{2}{2}C_1 & \cdots & \sqrt{2!}\binom{n-1}{2}w_1^{n-3}C_1 & \frac{(w_2-w_1)^2}{\sqrt{2!}}C_2 \\ 0 & \cdots & \cdots & \cdots & \cdots \\ C_3 \bar{w}_2C_3 & \bar{w}_2^2C_3 & \cdots & \bar{w}_2^{n-1}C_3 \\ \end{pmatrix}, \quad (50)$$

where $C_1 = e^{\frac{1}{2}|w_1|^2}$, $C_2 = e^{-\frac{1}{2}(1+w_1^2+w_1w_2)}$, $C_3 = e^{-\frac{1}{2}(w_1^2+w_2w_1)}$, $C_4 = e^{\frac{1}{2}|w_2|^2}$. Noting

Figure 1: An example of the $(n,n')$-systems. This is a $(3,2)$-system plotted by using the exact solution for a $(n,n')$-system in case of $r = 6$. 
where $G_{12} = \langle w_1|w_2 \rangle$, $G_{21} = \langle w_2|w_1 \rangle$. From these matrix forms, we can easily calculate their determinants:

$$
\det B = \left( \prod_{m=0}^{n-1} \sqrt{m!} \right) e^{\frac{n}{2}|w_1|^2} e^{\frac{n}{2}|w_2|^2} \left( 1 - e^{-|w_1-w_2|^2} \sum_{m=0}^{n-1} \frac{|w_1-w_2|^{2m}}{m!} \right),
$$

$$
\det \tilde{h} = 1 - e^{-|w_1-w_2|^2} \sum_{m=0}^{n-1} \frac{|w_1-w_2|^{2m}}{m!}.
$$

From eqs. (51), (52) and (53), the determinant of $h$ or the Kähler potential is determined as follows:

$$
K = n|w_1|^2 + |w_2|^2 + \ln \left( 1 - e^{-|w_1-w_2|^2} \sum_{m=0}^{n-1} \frac{|w_1-w_2|^{2m}}{m!} \right) + \ln \left( \prod_{m=0}^{n-1} m! \right)
$$

$$
= \frac{n}{n+1}|y|^2 + \ln \left( 1 - e^{-|y|^2} \sum_{m=0}^{n-1} \frac{|y|^{2m}}{m!} \right) + \ln \left( \prod_{m=0}^{n-1} m! \right).
$$

In the second line, we have set $w_1 = y/(n+1)$, $w_2 = -ny/(n+1)$. The Kähler metric can be calculated from this as

$$
g_{yy} = \partial_y \partial_y K = \frac{n}{n+1} + \frac{|y|^{2(n-1)}}{(n-1)! Q(|y|^2)} \left( n - |y|^2 - \frac{|y|^{2n}}{(n-1)! Q(|y|^2)} \right),
$$

where

$$
Q(x) \equiv 1 - e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!}.
$$

We introduce coordinates $(r, \theta)$ by $y = re^{i\theta}/\sqrt{2}$ as before. The metric becomes $ds^2 = f(r)(dr^2 + r^2 d\theta^2)$, where

$$
f(r) \equiv \frac{1}{2} \left\{ \frac{n}{n+1} + \frac{(r^2/2)^{n-1}}{(n-1)! Q(r^2/2)} \left( n - \frac{r^2}{2} - \frac{(r^2)^n}{(n-1)! Q(r^2/2)} \right) \right\}.
$$
Expanding this around $r \sim 0$, we get

$$ds^2 \approx \frac{n}{2(n+1)^2(n+2)} r^2 (dr^2 + r^2 d\theta^2).$$

(58)

This indeed agrees with the result (34). When the relative distance $r$ between the level $n$ soliton and the level 1 soliton is very large, $Q(r^2/2)$ goes to 1. Immediate consequence of this fact is that the Kähler potential (54) behave as $K \approx n |y|^2/(n + 1)$ and thus the metric goes to a flat one in this limit;

$$ds^2 \approx \frac{n}{2(n+1)} (dr^2 + r^2 d\theta^2).$$

(59)

That is, two solitons do not affect each other when they are remote. This is the same as the case of the (1,1)-system.

Let us consider the $(n,1)$ scattering more. Using the above $f(r)$, we can calculate numerically the scattering angle in case that the impact parameter is nonzero. The scattering angle $\theta_{\text{ext}}$ is given by the formula [16]

$$\theta_{\text{ext}} = -2 \int_{r_0}^{\infty} \frac{ds}{s \sqrt{(s^2 f(s)/r_0^2 f(r_0)) - 1}},$$

(60)

where $r_0$ is the closest distance and related to the impact parameter $b$ by $b = r_0 \sqrt{f(r_0)}$. For $n \leq 6$, the scattering angles are plotted in fig. [4]. When the closest distance goes to zero i.e. the zero impact parameter limit, a scattering angle is $\pi/2$. As the closest distance or an impact parameter becomes large, a scattering angle closes to $\pi$ (no scattering limit). This
Figure 3: The closest distance \((n \leq 30)\) for \(\theta_{\text{ext}} = 2.0, 2.5, 3.0\) (from bottom to top in the graph). The curves are the fitting curves in the text.

The figure shows that for a fixed \(\theta_{\text{ext}}\) the closest distance \(r_0\) seems to be roughly proportional to \(\sqrt{n}\). This is expected from the fact that the (radially symmetric) level \(n\) soliton has size \(\sqrt{n}\). In fact, when \(\theta_{\text{ext}}\) is fixed to be 2.0, we observe that \(r_0(\theta_{\text{ext}} = 2.0) \approx -0.05 + 1.52\sqrt{n}\). Similarly, when \(\theta_{\text{ext}}\) is fixed to be 2.5 and 3.0, \(r_0(\theta_{\text{ext}} = 2.5) \approx 0.81 + 1.48\sqrt{n}\) and \(r_0(\theta_{\text{ext}} = 3.0) \approx 1.81 + 1.44\sqrt{n}\) respectively. These fittings are drawn in fig.3. These observations are natural reflection of the width of the soliton, though it is difficult to see this from the formula (60).

In this letter, we have seen some properties of a general \((n,n')\)-system. We conclude that the \((n,n')\) scattering occurs at right angles in the case of the zero impact parameter. We have confirmed that the right angle scattering is a universal property of two-body scattering of noncommutative solitons. Especially for the \((n,1)\)-system, we have shown directly the metric of the relative moduli space goes to a flat one far from the origin. This could be generalized to arbitrary \((n,n')\)-systems. Finally we have numerically calculated the scattering angle for the \((n,1)\)-system and found the closest distance for a fixed scattering angle is well approximated by a function \(a + b\sqrt{n}\) \((a\) and \(b\) are some numerical constants). It may exist the similar relation for a \((n,n')\) scattering.

Note that as \(n\) becomes large, the Kähler potential (54) for the \((n,1)\)-system diverges, since \(Q(x) \to 0\). But even in this case, expanding the potential in terms of \(|y|^2/n\), and neglecting a coordinate singularity \(\ln|y|^2\), we can get the same formula for the metric as eq.(58). The metric tends to be zero everywhere when \(n\) goes to infinity. If we rescale
$y \to \sqrt{n}y$, then we see that in this new coordinate the scattering behavior of this system is qualitatively the same as the case of finite $n$.

It is interesting to consider corrections to the metric when the noncommutativity parameter is large but finite. It is also interesting to explore the multi soliton solutions and their properties when gauge degrees of freedom are exist, where the solutions we have considered in this letter can be a part of exact solutions even at finite $\theta$ \cite{6}. These subjects are left for future studies.

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