Demonstrating anyonic fractional statistics with a six-qubit quantum simulator

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Anyons are exotic quasiparticles living in two dimensions that do not fit into the usual categories of fermions and bosons, but obey a new form of fractional statistics. Following a recent proposal [Phys. Rev. Lett. 98, 150404 (2007)], we present an experimental demonstration of the fractional statistics of anyons in the Kitaev spin lattice model using a photonic quantum simulator. We dynamically create the ground state and excited states (which are six-qubit graph states) of the Kitaev model Hamiltonian, and implement the anyonic braiding and fusion operations by single-qubit rotations. A phase shift of $\pi$ related to the anyon braiding is observed, confirming the prediction of the fractional statistics of Abelian 1/2-anyons.

Quantum statistics classifies fundamental particles in three dimensions as bosons and fermions. Interestingly, in two dimensions the laws of physics permit the existence of exotic quasiparticles—anyons—which obey a new statistical behavior, called fractional or braiding statistics \cite{1}. That is, upon exchange of two such systems, the system wave function will acquire a statistical phase which can take any value—hence this name. Anyons have been predicted to live as excitations in fractional quantum Hall (FQH) systems \cite{2,3,4}. Alternatively, quantum states with anyonic excitations can be artificially designed in spin model systems that possess highly nontrivial ground states with topological order. A prominent example is the Kitaev spin lattice model \cite{5,6}, which opened the avenue of fault-tolerant topological quantum computing \cite{7,8}.

It is an important goal to manipulate the anyons and demonstrate their exotic statistics. Towards this goal, a number of theoretical schemes have been proposed, both in the FQH regime \cite{9} and in the Kitaev models \cite{10,11,12,13,14}. However, it has proved extremely difficult to experimentally detect the fractional statistics associated with anyon braiding. While recent experiments in the FQH systems have indeed revealed some signatures of anyonic statistics \cite{15,16}, resolving individual anyons remains elusive \cite{7}. Here we take a different approach to this challenge — exploiting the spin models to study the anyonic statistics. Following a recent proposal \cite{14}, we demonstrate the fractional statistics of anyons by simulation of the Kitaev model on a six-photon graph state. The method is to dynamically create the ground state and excited state of the anyonic model Hamiltonian, and implement the braiding and fusion operations by single-qubit rotations.

How can the statistical nature of elementary particles be experimental observable? First let us recall the concept of quantum statistics. The wave function of a two-particle system $\psi_{12}(r_1,r_2)$ will acquire a statistical phase $\theta$ upon an adiabatic exchange of two particles, that is, $\psi_{12}(r_2,r_1) = e^{i\theta} \psi_{12}(r_1,r_2)$, where $\theta = 0$ for bosons, $\theta = \pi$ for fermions, and $\theta$ can be any value ($0 \leq \theta \leq \pi$) for anyons. It can be seen that a full circulation of a particle around the other one is equivalent to two successive particle exchanges $\pi$. After such a circulation, both bosons and fermions show a trivial phase ($\phi = 2\theta = 0, 2\pi$), but anyons will get an observable non-trivial phase $\phi$. To realize this idea, we need a system where anyons can be created and braiding operations can be carried out experimentally. The Kitaev model is well suited for this.

The first Kitaev model was designed on a spin lattice with qubits living on the edges (see Fig. 1a). For each vertex $v$ and face $f$, we consider operators of the form

$$A_v = \prod_{j \in \text{star}(v)} X_j, \quad B_f = \prod_{j \in \text{boundary}(f)} Z_j, \quad (1)$$

where $X (Z)$ denotes the standard Pauli matrix $\sigma_x (\sigma_z)$. These operators $A_v, B_f$ are put together to make up the model Hamiltonian

$$H = -\sum_v A_v - \sum_f B_f, \quad (2)$$

The ground state $|\psi_g\rangle$ of this Hamiltonian (2) is given by $A_v|\psi_g\rangle = |\psi_g\rangle$ and $B_f|\psi_g\rangle = |\psi_g\rangle$ for all vertices and faces. Violations of these conditions cost energy and generate excited states $|\psi_e\rangle$. A quasiparticle is created on the vertex $v_i$ (face $f_i$) if $A_{v_i} (B_{f_i})$ acting on the excited state $|\psi_e\rangle$ yields an eigenvalue $-1$ instead of $+1$ for the ground state. In Ref. \cite{13,14}, the quasiparticles on vertices are called “electric charges” (e-particles for short) and those on faces are called “magnetic vortices” (m-particles). It is shown that these particles have unusual mutual statistical properties, as we can get a phase flip $-1$ if we move one particle around the other, which are thus called abelian $1/2$-anyons \cite{8} (see Fig. 1b).
Recently, Han, Raussendorf and Duan [14] exploited the fact that the statistical properties of anyons are manifested by the underlying ground and excited states [17]. So, instead of direct engineering the interactions of the Hamiltonian $H$ (2) and ground-state cooling which are extremely demanding experimentally, an easier way is to dynamically create the ground state and the excitations of this model Hamiltonian, encoding the underlying anyonic model in a multiparticle entangled state which can used to simulate the dynamics of the anyonic system. The quasiparticles are then defined by the negative eigenstate of a stabilizer element $A_i$ or $B_f$. Specifically, as illustrated in Fig. 1b, with the ground state $|\psi_g\rangle$ prepared, one can first create a pair of e-particles by applying a single-qubit $Z$ rotation. The system wave function will be in the excited state $|\psi_e\rangle$. To make fractional phase experimentally detectable in a later stage, we apply a $\sqrt{Z}$ rotation and get a superposition state $(1/\sqrt{2})(|\psi_g\rangle + |\psi_e\rangle)$. Then we create another pair of m-particles and move one of them around an e-particle along a closed loop, and finally annihilate the m-particles. After doing so, it is predicted that a fractional phase $\pi$ will be added to $|\psi_e\rangle$, thus the superposition state will become $(1/\sqrt{2})(|\psi_g\rangle - |\psi_e\rangle)$.

As the anyons are perfectly localized quasiparticles in this model Hamiltonian, a small spin lattice containing six qubits shown in Fig. 2a allows for a proof-of-principle experimental demonstration [2, 14]. The Hamiltonian of this model is $H_6 = -A_1 - A_2 - B_1 - B_2 - B_3 - B_4$, where $A_1 = X_1X_2X_3$, $A_2 = X_5X_4X_6$, $B_1 = Z_1Z_3Z_1$, $B_2 = Z_2Z_3Z_5$, $B_3 = Z_4Z_6$, $B_4 = Z_5Z_6$ (the subscripts of the Pauli matrices label the qubits). The ground state $|\psi_6\rangle$ of this Hamiltonian $H_6$ is locally equivalent to a six-qubit graph state [15, 14], which can be represented by the graph as depicted in Fig. 2b. This equivalence follows from the fact that the operators $A_1, \ldots, B_4$ in the Kitaev model can be uniquely derived from the stabilizer operators $g_i$ (see Fig. 2b) of the graph state.

Now we proceed with the experiment in three steps: (1) create and analyze of the ground state $|\psi_6\rangle$, (2) verify the anyonic excitations, (3) implement the braiding
operations and detect the anyonic phase. We use single photons as a real physical system to simulate the creation and control the anyons. The quantum states are encoded in the polarization of the photons which are robust to decoherence. The experimental set-up is illustrated in Fig. 3c. We start from three pairs of entangled photons produced by spontaneous parametric down-conversion (SPDC) [20]. The photons in spatial modes $a-b$ and $c-f$ are prepared in the states $|\phi^+\rangle_{ij} = (1/\sqrt{2})(|H\rangle_i|H\rangle_j + |V\rangle_i|V\rangle_j)$, while those in mode $c-d$ are disentangled using polarizers and then prepared in the states $|\phi^\pm\rangle_i = (1/\sqrt{2})(|H\rangle_i + |V\rangle_i)$, where $H(V)$ denotes horizontal (vertical) polarization, and $i$ and $j$ label the spatial modes. We then pass the photons through a linear optics network (see Fig. 3c). A coincidence detection of all six outputs corresponds exactly to the ground state

$$|\psi\rangle_6 = \frac{1}{2}(|H\rangle_1|H\rangle_2|H\rangle_3|H\rangle_4|H\rangle_5|H\rangle_6 + |V\rangle_1|V\rangle_2|V\rangle_3|H\rangle_4|H\rangle_5|H\rangle_6 + |H\rangle_1|H\rangle_2|V\rangle_3|V\rangle_4|V\rangle_5|V\rangle_6 + |V\rangle_1|V\rangle_2|H\rangle_3|V\rangle_4|V\rangle_5|V\rangle_6). \quad (3)$$

To verify that the ground state $|\psi\rangle_6$ has been obtained, first we experimentally measure the expectation values of its stabilizer operators $A_1, \cdots, B_4$. These stabilizer operators describe the intrinsic correlations in the state $|\psi\rangle_6$ and uniquely define it, thus their expectation values could serve as a good experimental signature. For an ideal state $|\psi\rangle_6$, all expectation values should give +1. In our experiment however, the ground state was created imperfectly. Figure 3a shows the measurement results, with all expectation values being positive in a range between 0.51±0.04 and 0.74±0.03, in qualitative agreement with the theoretical prediction. For a more complete and quantitative analysis, we aim to estimate the fidelity of the produced state, that is, its overlap with the desired one. This quantity is given by $F_{\psi_6} = \langle \psi | \rho_{\text{exp}} | \psi \rangle$, which is equal to one for an ideal state, and 1/64 for a completely mixed six-qubit state. To do so, we consider a special observable, which allows for lower bounds on the fidelity, while being easily measurable with few correlation measurements [22, 23]. By making these measurements, we estimate the fidelity of the created ground state to be $F_{\psi_6} \geq 0.532 \pm 0.041$ [24]. The imperfection of this state is mainly caused by the high-order photon emissions during the SPDC and the partial distinguishability of independent photons [23].

We now move to the step (2). With the ground state $|\psi\rangle_6$ created, we apply a Z (X) rotation on qubit 3 (4), creating an excited state $|\psi_{\text{em}}\rangle_6$ on which a pair of e-particles live on the vertices $v_1$ and $v_2$, and another pair of m-particles on faces $f_1$ and $f_3$ (see Fig. 2a). The Z and X rotations are experimentally realized using HWPs oriented at 0° and 45°, respectively. As discussed before, the anyonic excitations are signaled by violations of stabilizer conditions, that is, $A_3 |\psi_{\text{em}}\rangle_6 = -|\psi_{\text{em}}\rangle_6$, $B_3 |\psi_{\text{em}}\rangle_6 = -|\psi_{\text{em}}\rangle_6$. Thus in our case, theoretically, the expectation values of $A_1$ and $A_2$ should become −1 because of the e-particles, and the same for $B_1$ and $B_3$ due to the m-particles. To verify this, we measure the expectation values of the operators $A_1, \cdots, B_4$. The results are shown in Fig. 3b, where the values of $A_1$, $A_2$, $B_1$ and $B_3$ flip compared to those shown in Fig. 3a which supports the presence of anyonic excitations [14].

Now we proceed to the step (3). On the ground state $|\psi\rangle_6$, first we apply a $\sqrt{Z}$ operation using a QWP oriented at 0° on the qubit 3 of the ground state $|\psi\rangle_6$, yielding a superposition state $|\psi\rangle_6 = (1/\sqrt{2})(|\psi\rangle_6 + |\psi_e\rangle_6)$, where $|\psi_e\rangle_6$ is the excited state with a pair of e-particles on $v_1$ and $v_2$. With an X rotation on the qubit 4 we further create a pair of m-particles on $f_1$ and $f_3$. Then we perform four $X$ operations on the qubits $6-5-3-4$ to implement the braiding operation, that is, the m-particle on $f_3$ is moved around the e-particle on $v_2$ along an anticlockwise closed loop. We note that the crossing at the qubit 3, which is unavoidable in two dimensions, is relevant for the unusual statistics. Finally, the pair of m-particles is annihilated with an X operation on qubit 4 (fusion).

After these, if there is a fractional phase $\phi$ acquired, the state $|\psi\rangle_6$ will become $|\psi_f\rangle_6 = (1/\sqrt{2})(|\psi\rangle_6 + e^{i\phi}|\psi_e\rangle_6)$. To determine this $\phi$, we look at the correlation measurement outcomes of the six photons where the photons 1 and 2 are fixed at $|\pm\rangle$ polarization, 4, 5 and 6 at $|H\rangle$ and the photon 3 is measured in the basis $(|\pm\rangle \pm e^{i\alpha}|\mp\rangle)/\sqrt{2}$ with $\alpha$ varying in $\pi/4$ steps. In this setting, the six-fold coincidence counts should follow the relation $C(\phi, \alpha) \propto 1 + \sin(\phi - \alpha)$ for the state $|\psi_f\rangle_6$, thus

![FIG. 3: The measured expectation values of the operators $A_1, \cdots, B_4$ of the ground state $|\psi\rangle_6$ (a) and the excited state $|\psi_{\text{em}}\rangle_6$ (b). The excited state $|\psi_{\text{em}}\rangle_6$ has a pair of e-particles on $v_1, v_2$ and a pair of m-particles on $f_1, f_3$, thus the values for $A_1, A_2, B_1, B_3$ become negative. Each expectation value is derived from a complete set of 64 six-fold coincidence events in 15h in measurement basis $Z^\otimes 6$ or $X^\otimes 6$. The error bars represent one standard deviation, deduced from propagated Poissonian statistics of the raw detection events.]
an unknown phase $\phi$, if occurs, can be revealed. Figure 4a shows the measurement results for both the state $|\psi_f\rangle_6$ and $|\psi_f\rangle_5$, before and after the process of m-particle creation, braiding and fusion. These two curves clearly exhibit a phase difference of $\pi$, confirming the prediction of the fractional statistics.

For a more complete proof, we implement a $\sqrt{Z}$ transformation on the qubit 3 of the remaining state $|\psi_f\rangle_6$. The state $|\psi_f\rangle_6$ will be converted to $|\psi_f\rangle_6$ if there is a fractional phase $\pi$, otherwise it will go to $|\psi_f\rangle_6$. To test this, again we measure the expectation values of the operators $A_1, \cdots, B_4$ of the state after the $\sqrt{Z_3}$ transformation. The experimental results are shown in Fig. 4b, which are in agreement with that the final state is $|\psi_f\rangle_6$ and thus prove the fractional phase change of $\phi = \pi$. Here we note that the facts that the present setup use free-flying non-interacting photons and that the timescale of the braiding operations is extremely small ($\sim$ picoseconds for photons passing through the HWPs and QWPs) implies that this acquired phase cannot be a dynamical phase. Moreover, the creating, braiding and annihilating of the m-particles which corresponds to the operation $X_4X_5X_3$ do not give rise to a phase either, as the ground state $|\psi_f\rangle_6$ is an eigenstate of this observable $X_4X_5X_3$. Similar arguments also apply to the e-particles’ case. Consequently, this excludes possible geometrical phases and proves the observed phase is purely statistical.

In summary, we have demonstrated the creation and manipulation of anyons in the Kitaev spin lattice model, and observed the fractional statistics of the abelian 1/2-anyons. This has been done without generating the four-body interactions in the model Hamiltonian but alternatively, in an easier way—by encoding the underlying anyonic system on a six-photon graph state, or equivalently, realizing the six-qubit circuit shown in Fig. 2c of ref. [14]. It should be noted that the absence of the Hamiltonian does not prevent us from studying the topological and statistical properties of the anyons here; however, topological quantum computing in a fault-tolerant manner would eventually require such a Hamiltonian. From a quantum-information prospective, our experiment may be seen as using quantum computers which have already well understood physics as a tool to simulate other difficult quantum systems. Such quantum simulators can in principle provide exponential speedup in the simulation of quantum physics, and may offer a more controllable and clean access to study strongly correlated behaviors than natural complex solid-state systems. Possible future work along this line may include investigation of the robustness of anyonic braiding and other possible phases using hyper-entangled graph states, and demonstration of some basic elements of cluster-state topological quantum computing.

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[24] Though some connection between the ground-state fidelity of the Kitaev code and its topological order has been explored in literatures (see e.g. A. Hamma, R. Ioniciou, P. Zanardi, Phys. Lett. A 337, 22 (2005); M. Aguado, G. Vidal, Phys. Rev. Lett. 100, 070404 (2008); L. Amico et al., Rev. Mod. Phys. 80, 517 (2008)), a strict quantitative criteria remains unclear concerning the fidelity requirements to prove the fractional statistics. Nevertheless, our results shown in Fig. 3,4 display reasonably high visibility for experimental confirmation.

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[30] Note added: After the submission of our results to arXiv (0710.0278), we became aware of two related work: J.K. Pachos et al. arXiv:0710.0895 J. Du et al. arXiv:0712.2694