ON STRONG SOLUTIONS OF TIME INHOMOGENEOUS ITÔ’S EQUATIONS WITH “SUPERCRITICAL” DIFFUSION AND DRIFT

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Abstract. We prove strong existence and uniqueness of solutions of Itô’s stochastic time dependent equations with irregular diffusion and drift terms of Morrey class type. In a sense we are treating a “supercritical” case.

1. Introduction

Let $\mathbb{R}^d$ be a $d-$dimensional Euclidean space of points $x = (x^1, ..., x^d)$ with $d \geq 3$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, carrying a $d_1$-dimensional Wiener process $w$, where $d_1 \geq d$.

Assume that on $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$ we are given $\mathbb{R}^d$-valued Borel functions $b = (b^i), \sigma^k = (\sigma^{ik}), k = 1, ..., d_1$. Set $\sigma = (\sigma^k) = (\sigma^{ik})$. We are going to investigate the equation

$$x_s = x + \int_0^s \sigma^k(t + u, x_u) \, dw^k_u + \int_0^s b(t + u, x_u) \, du, \quad (1.1)$$

where and everywhere below the summation over repeated indices is understood.

We are interested in the so-called strong solutions, that is solutions such that, for each $s \geq 0$, $x_s$ is $\mathcal{F}^{w}_s$-measurable, where $\mathcal{F}^{w}_s$ is the completion of $\sigma(w_u : u \leq s)$. We present sufficient conditions for the equation to have a strong solution completing those from [6], where only $b$ was allowed to satisfy the most general conditions of summability. Here we extend these conditions on the first derivatives $\sigma_x$ of $\sigma$ in $x$. We also deal with the issue of strong uniqueness. Acquaintance with the results of [6] is necessary for understanding the arguments in the present article.

A very popular condition in the literature is that $\sigma_x, b \in L_{p,q}$ with, perhaps, different $p, q$ for $\sigma_x$ and $b$ such that $p, q \in [2, \infty)$

$$\frac{d}{p} + \frac{2}{q} = 1. \quad (1.2)$$

We refer to [1], [6], [7] for a discussion of what really fascinated development was achieved under this condition. It turned out that there is a weaker condition than the above one under which one still can prove the strong

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solvability in the time inhomogeneous case. This condition is presented in [6], where it was used only in what concerns \( b \) and even in the case of constant \( \sigma \) produced new results.

Here is an example in which we prove existence of strong solutions. For simplicity we give an example with \( \sigma, b \) independent of time. Take \( d = 3, d_1 = 12 \), and for some numbers \( \alpha, \beta, \gamma \geq 0 \) let \( \sigma^k \) be the \( k \)th column of the matrix given by \((0/0 := 3^{-1/2})\)

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha \\
\end{pmatrix}
+ \frac{\beta}{|x|} \begin{pmatrix}
x^1 \\
x^2 \\
x^3 \\
0 \\
x^1 \\
x^2 \\
x^3 \\
0 \\
x^1 \\
x^2 \\
x^3 \\
0 \\
\end{pmatrix},
\]

\[b(x) = -\gamma \left( \frac{x}{|x|} \right) _{I_{0<|x|\leq 1}}.
\]

Our result shows that if \( \alpha = 1 \) and \( \beta \) and \( \gamma \) are sufficiently small, then (1.1) has a strong solution. By the way, if \( \alpha = \gamma = 0 \) and \( \beta = 1 \), there exist strong solutions of (1.1) only if the starting point \( x \neq 0 \) (see [3]). In case \( \alpha = 1 \) and \( \beta = 0 \) strong solutions exist only if \( \gamma \) is sufficiently small. Observe that for \( \beta \neq 0 \) and \( \gamma \neq 0 \) we have \( D\sigma^k, b \in L_{d-\varepsilon,\text{loc}}(\mathbb{R}^d) \) for any \( \varepsilon \in (0, 1) \) but not for \( \varepsilon = 0 \). Therefore, \( D\sigma^k, b \notin L_{p,q,\text{loc}} \) with \( p, q \) satisfying (1.2), and we say that such coefficients have “supercritical” singularities.

Recall that the case of time independent \( \sigma, b \) with \( D\sigma^k, b \in L_{d,\text{loc}} \) is investigated in [3] the main idea of which is somewhat different from what we use here.

We conclude the introduction by some notation. We set

\[D_{x^i} u = D_i u = u_{x^i} = \frac{\partial}{\partial x^i} u, \quad Du = u_x = (D_i u),\]

\[D_{ij} u = u_{x^i x^j} = D_i D_j u, \quad u_{xx} = (D_{ij} u), \quad \partial_t u = \frac{\partial}{\partial t} u.\]

If \( \sigma = (\sigma^{ij}) \) by \( |\sigma|^2 \) we mean the sum of squares of all entries.

For \( p \in [1, \infty) \), and domain \( \Gamma \subset \mathbb{R}^d \) by \( L_p(\Gamma) \) we mean the space of Borel (real-, vector- or matrix-valued) functions on \( \Gamma \) with finite norm given by

\[\|f\|_{L_p(\Gamma)}^p = \int_\Gamma |f(x)|^p \, dx.\]

Set \( L_p = L_p(\mathbb{R}^d) \).

For \( p, q \in [1, \infty) \) and domain \( Q \subset \mathbb{R}^{d+1} \) by \( L_{p,q}(Q) \) we mean the space of Borel (real-, vector- or matrix-valued) functions on \( Q \) with finite norm given by

\[\|f\|_{L_{p,q}(Q)}^q = \|f I_Q\|_{L_{p,q}}^q = \int_\mathbb{R} \left( \int_{\mathbb{R}^d} |f I_Q(t, x)|^p \, dx \right)^{q/p} \, dt.\]

Set \( L_{p,q} = L_{p,q}(\mathbb{R}^{d+1}) \).

By \( W^2_p \) we mean the space of Borel functions \( u \) on \( \mathbb{R}^d \) whose Sobolev derivatives \( u_x \) and \( u_{xx} \) exist and \( u, u_x, u_{xx} \in L_p \). The norm in \( W^2_p \) is given
by
\[ \|u\|_{W^2_p} = \|u_{xx}\|_{L^p} + \|u\|_{L^p}. \]

Similarly \( W^1_p \) is defined. As usual, we write \( f \in L^p_{\text{loc}} \) if \( f \zeta \in L^p \) for any \( \zeta \in C_0^\infty (= C_0^\infty (\mathbb{R}^d)) \).

By \( W^{1,2}_p(Q) \) we mean the collection of \( u \) such that \( \partial_t u, u_{xx}, u_x, u \in L^p_{\text{loc}}(Q) \). The norm in \( W^{1,2}_p(Q) \) is introduced in an obvious way.

Introduce
\[ B_R(x) = \{ y \in \mathbb{R}^d : |x - y| < R \}, \quad B_R = B_R(0) \]
and let \( B_R \) be the collection of balls of radius \( R \). Also let
\[ C_{\tau,\rho}(t, x) = [t, t + \tau] \times B_\rho(x), \quad C_\rho = C_\rho(0,0), \]
and let \( C_\rho \) be the collection of \( C_\rho(t, x) \).

In the proofs of our results we use various (finite) constants called \( N \) which may change from one occurrence to another and depend on the data only in the same way as it is indicated in the statements of the results.

2. Main result

Fix \( \delta, \rho_b, \rho_\sigma \in (0,1] \). The value of \( \theta > 0 \) in Assumption 2.1 below will be specified later.

**Assumption 2.1.** For any \( \rho \leq \rho_\sigma \) and any \( C \in C_\rho \)
\[ \int_C |\sigma(t, x) - \sigma(t)| \, dx \, dt \leq \theta, \quad (2.1) \]
where
\[ \sigma(t) = \int_C \sigma(t, x) \, dx \, ds \quad (\text{note } t \text{ and } ds). \]

Observe that \( \sigma(t) \) is the average of \( \sigma(t, x) \) over a ball of radius \( \rho \) and, if \( \sigma \) is independent of \( x \), the left-hand side of (2.1) is zero.

This assumption is absent and is not needed in [6]. In this regard our situation seems to be more restrictive than in [6]. However, we allow \( \sigma \) to be more general than in [6], where \( D\sigma_B \equiv 0 \).

**Assumption 2.2.** (i) The matrix-valued \( a = (a^{ij}) = (\sigma^{ik}\sigma^{jk}) \) takes its values in \( S_\delta \), that is the set of \( d \times d \) symmetric matrices with eigenvalues between \( \delta \) and \( \delta^{-1} \);

(ii) For any \( t, \sigma(t, \cdot) \in W^1_{1,\text{loc}} \) and the tensor-valued \( D\sigma \) admits a representation \( D\sigma = D\sigma_M + D\sigma_B \) with Borel summands ("Morrey part" of \( D\sigma \)}
plus the “bounded part”) such that there exist a finite constant $\bar{D}\sigma_M$ and $p_{D\sigma} \in (2, d]$ for which
\[
\left( \int_B |D\sigma_M(t, x)|^{p_{D\sigma}} \, dx \right)^{1/p_{D\sigma}} \leq \rho^{-1} \bar{D}\sigma_M,
\]
whenever $t \in \mathbb{R}$, $B \in \mathbb{B}_\rho$, and $\rho \leq \rho_\sigma$, and there exists a constant $\bar{D}\sigma_B < \infty$ such that
\[
\int_{\mathbb{R}} |D\sigma_B(t)|^2 \, dt \leq (\bar{D}\sigma_B)^2, \quad \bar{D}\sigma_B(t) := \text{ess sup}_{x \in \mathbb{R}^d} |D\sigma_B(t, x)|;
\]
(iii) The vector-valued $b = (b^i)$ admits a representation $b = b_M + b_B$ with Borel summands such that there exist $p_b \in (d/2 + 1, d]$ and a constant $\hat{b}_M < \infty$ for which
\[
\left( \int_B |b_M(t, x)|^{p_b} \, dx \right)^{1/p_b} \leq \hat{b}_M \rho^{-1}, \quad (2.2)
\]
whenever $t \in \mathbb{R}$, $B \in \mathbb{B}_\rho$, and $\rho \leq \rho_b$, and there exists a constant $\bar{b}_B \in (0, \infty)$ such that
\[
\int_{\mathbb{R}} \bar{b}_B^2(t) \, dt \leq \bar{b}_B^2, \quad \bar{b}_B(t) := \sup_{x \in \mathbb{R}^d} |b_B(t, x)|.
\]
Introduce
\[
\beta_b(t) = \sup_s \int_s^{s+t} \bar{b}_B^2(u) \, du.
\]
To state our main result fix some $p_0 \in (1, p_b), q_0 \in (1, \infty)$ satisfying
\[
\frac{d}{p_0} + \frac{2}{q_0} < 2,
\]
(recall that $p_b > d/2$ even $> d/2 + 1$).

**Theorem 2.3.** There exist constants $\alpha_\sigma, \alpha_b > 0$, depending only on $d, d_1, \delta, p_{D\sigma}, p_b$, with $\alpha_0$ also depending on $p_0, q_0, \rho_\sigma$, and there exists a constant $\theta(d, d_1, \delta, p_b, q_0) > 0$ such that, if Assumption 2.1 is satisfied with this $\theta$ and
\[
\bar{D}\sigma \leq \alpha_\sigma, \quad \hat{b}_M \leq \alpha_b, \quad (2.3)
\]
then for any $(t, x) \in \mathbb{R}^{d+1}$ on $[0, \infty)$ there exists a strong solution of (1.1) such that for any $m = 1, 2, \ldots,$ and Borel nonnegative $f$ on $\mathbb{R}^{d+1}$ we have
\[
E_{t, x} \left( \int_0^T f(t + s, x_s) \, ds \right)^m \leq m! N^m \|f\|_{L_{p_0, q_0}}^m \exp \int_t^{t+T} mN(\delta) \bar{b}_B^2(s) \, ds,
\]
where $N$ depends only on $d, d_1, \delta, p_b, q_0, \rho_\sigma, \rho_b, T,$ and $E_{t, x}$ is the expectation sign with subscripts $t, x$ indicating that we are dealing with the solution of (1.1). Furthermore, on any given probability space carrying a $d_1$-dimensional Wiener process any solution of (1.1) with given starting point $(t, x)$, admitting estimate (2.4) with $m = 1$ for any Borel $f \geq 0$ and $T \in (0, \infty)$ with a constant $N$ independent of $f$, is unique and is strong.
This theorem is proved in Section 4 after we develop necessary tools in Section 3 treating equations with coefficients smooth in $x$.

**Remark 2.4.** We fixed the probability space. Observe that one knows that, once the assertion of Theorem 2.3 about existence holds on at least one probability space carrying a Wiener process, it also holds on any other such space.

**Remark 2.5.** Our conditions are global in time and, of course, considering equation (1.1), say with $(t, x) = (0, 0)$ we only need these conditions to be satisfied on $\mathbb{R}^d$. This is because one can always set $b = 0$ and $\sigma = \text{const}$ outside $[0, \infty)$.

**Remark 2.6.** It is shown in Remark 2.5 of [6] that the condition $b \in L_{p,q}$ with $p, q$ satisfying (1.2) is stronger than condition (iii) in Assumption 2.2.

**Remark 2.7.** By using Poincaré’s inequality, it is not hard to see that condition (2.1) follows from Assumption 2.2 if there we take $N(d, d_1, \delta) \hat{D}_\sigma \leq \theta$ for an appropriate $N(d, d_1, \delta)$ and make $\rho_\sigma$ in condition (2.1) sufficiently small depending on the modulus of continuity at zero of $\beta_B(t) = \sup_s \int_s^{s+t} |D\sigma_B(u)|^2 \, du$, $t \geq 0$.

3. **Specifying $\theta$ and $\alpha$ in the case of smooth bounded $\sigma$ and $b$**

In this section we suppose that $\sigma^k$ and $b$ are smooth in $x$ and bounded in $(t, x)$ along with any derivative of any order with respect to $x$.

By classical results, since our coefficients are regular enough, given a $d_1$-dimensional Wiener process $w_t$, for any $(t, x) \in \mathbb{R}^{d+1}$ one can uniquely solve equation (1.1) and the solutions $(t + s, x_s)$ form a strong Markov process. Sometimes we write $x_s = x_s(t, x)$ to emphasize that $x_s$ is a solution of (1.1) with given $(t, x)$.

Observe that for $\rho \leq \rho_\sigma$, $C = C_\rho(\tau, \xi) \in \mathbb{C}_\rho$, and

$$a(t) = \int_C a(t, x) \, dxds$$

we have

$$\int_C |a(t, x) - a(t)| \, dxdt$$

$$\leq \int_\tau^{\tau+\rho^2} \int_{B_\rho(\xi)} \int_{B_\rho(\xi)} |\sigma(t, x_1) - \sigma(t, x_2)| \, dx_1 dx_2 dt$$

$$\leq N_1(d, d_1, \delta) \int_\tau^{\tau+\rho^2} \int_{B_\rho(\xi)} \int_{B_\rho(\xi)} |\sigma(t, x_1) - \sigma(t, x_2)| \, dx_1 dx_2 dt$$

$$\leq 2N_1(d, d_1, \delta) \int_C |\sigma(t, x) - \sigma(t)| \, dxdt \leq 2N_1(d, d_1, \delta)\theta.$$
In short,
\[ \int_C |a(t, x) - a(t)| \, dx \, dt \leq 2N_1(d, d_1, \delta) \theta. \] (3.1)

In Theorem 3.3 of [6] there is the assumption that (3.1) holds with \( \theta \) in place of \( 2N_1(d, d_1, \delta) \theta \), and this theorem allows us to introduce \( \theta_0 = \theta_0(d, d_1, \delta, p, q) > 0 \) for any \( p, q \in (1, \infty) \) in the following way. Set
\[ L_0 u(t, x) = \partial_t u(t, x) + (1/2)a^{ij}(t, x)D_{ij} u(t, x). \]

For \( T \in (0, \infty) \) set \( \mathbb{R}^d_T = (0, T) \times \mathbb{R}^d \) and fix some \( p, q \in (1, \infty) \). Here is an adaptation of Theorem 3.3 of [6].

**Theorem 3.1.** There exists \( \theta_0 = \theta_0(d, d_1, \delta, p, q) > 0 \) such that if Assumption 2.1 is satisfied with \( \theta = \theta_0 \), then for any \( f \in L_{p,q}(\mathbb{R}^d_T) \) and \( \lambda \geq 0 \) there exists a unique \( u \in W^{1,2}_p(\mathbb{R}^d_T) \) such that \( u(T, x) = 0 \) and \( L_0 u - \lambda u = f \) in \( \mathbb{R}^d_T \) (a.e.). Moreover, there exists constants \( (\lambda_0, N_0) = (\lambda_0, N_0)(d, d_1, \delta, p, q, \rho_0) \) such that for \( \lambda \geq \lambda_0 \)
\[ \|\lambda u, \sqrt{\lambda}Du, D^2 u\|_{L_{p,q}(\mathbb{R}^d_T)} \leq N_0 \|f\|_{L_{p,q}(\mathbb{R}^d_T)}. \] (3.2)

Then follows the first restriction on \( \bar{b}_M \).

**Theorem 3.2** (Theorem 3.5 of [6]). Assume that \( p \in (1, p_b), q \in (1, \infty) \) satisfy
\[ \frac{d}{p} + \frac{2}{q} < 2, \] (3.3)
and Assumption 2.1 is satisfied with \( \theta = \theta_0(d, d_1, \delta, p, q) \). Also suppose that
\[ N_0 N(d, p, p_b) \bar{b}_M \leq 1/2, \] (3.4)
where \( N_0 = N_0(d, \delta, p, q, \rho_0) \) is introduced in Theorem 3.3 of [6] and \( N(d, p, p_b) \) is introduced in Theorem 3.4 of [6]. Then for any \( (t, x) \in \mathbb{R}^{d+1}, m = 1, 2, \ldots, \) and Borel nonnegative \( f \) on \( \mathbb{R}^{d+1} \) we have
\[ E_{t,x} \left( \int_0^T f(t+s, x_s) \, ds \right)^m \leq m!N^m \|f\|_{L_{p,q}}^m \exp \int_t^{t+T} mN(\delta)\bar{b}_M^2(s) \, ds, \] (3.5)
where \( N \) depends only on \( d, d_1, \delta, p, p_b, q, \rho_2, \rho_0, T \).

Now we use \( \theta_0(d, d_1, \delta, p, q) \) to construct \( \theta(d, d_1, \delta, p_b, p_0, q_0) \) in Theorem 2.3 in the following way. We fix \((p_1, q_1)\) such that
\[ p_b > p_1 = q_1 > d/2 + 1. \]
We may consider \((p_1, q_1)\) as a function of \( d, p_b \) and then set
\[ \theta_1(d, d_1, \delta, p_b, p_0, q_0) = \min \left[ \theta_0(d, d_1, \delta, p_i, q_i) : i = 0, 1 \right]. \] (3.6)

Below in the article we suppose that Assumption 2.1 is satisfied with \( \theta = \theta_1 \) defined in (3.6) until the point when it will be explicitly reduced further down.

Since \( \theta_1 \leq \theta_0(d, d_1, \delta, p_1, q_1) \), the following result is derived in the same way as Theorem 3.2 of [6] is derived from Theorem 3.3 of [6].
Theorem 3.3. Let $b_B = 0$. Then there exists $\theta = \theta(d, \delta, p_b) > 0$ and, for any $\varepsilon > 0$, there exist $\hat{b}_M, \varepsilon > 0$, depending only on $d, \delta, p_b, \rho, \varepsilon$, such that, if Assumption 2.1 is satisfied with the above $\theta$ and $\hat{b}_M \leq \hat{b}_M, \varepsilon$, then for any $\rho \leq \rho_b$, $C \in \mathbb{C}_\rho$, and $(t, x) \in \mathbb{R}^{d+1}$

$$E_{t,x} \int_0^{\tau_C} |b(t+s, x_s)| \, ds \leq \varepsilon \rho,$$

where $\tau_C$ is the first exit time of $(t+s, x_s)$ from $C$.

We also need the following.

Theorem 3.4 (Theorem 3.7 of [6]). There exists $\varepsilon_0 = \varepsilon_0(d, \delta) > 0$ such that, if $\hat{b}_M \leq \hat{b}_M, \varepsilon_0(d, \delta)$, then for any $n > 0$, $(t, x) \in \mathbb{R}^{d+1}$, and $0 \leq r \leq s$ we have

$$E_{t,x} \sup_{u \in [r,s]} |x_u - x_r|^n \leq N e^{N(d) \beta_b(s-r)}(|s-r|^{n/2} + |s-r|^n),$$

where $N = N(n, \rho_b, d, \delta)$.

According to this theorem we add to (3.4) another restriction on $\hat{b}_M$:

$$\hat{b}_M \leq \hat{b}_M, \varepsilon_0(d, \delta),$$

which guarantees that (3.8) holds.

Next, for $n = 1, 2, \ldots$ and $r \in (0, \infty)$ introduce

$$\Gamma^n_r = \{(t_1, \ldots, t_n) : r > t_1 > \ldots > t_n > 0\},$$

and introduce $W^n_r$ as the closed linear subspace of $L_2(\Omega, \mathcal{F}, P)$ generated by constants if $m = 0$, by the set of constants and

$$\int_0^r f(t) \, dw_t^k$$

if $m = 1$ or, if $m \geq 2$, by the set of constants and

$$\int_{\Gamma^n_r} f(t_n, \ldots, t_1) \, dw_{t_n}^{k_n} \ldots dw_{t_1}^{k_1}$$

$$:= \int_0^r \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{n-1}} f(t_n, \ldots, t_1) \, dw_{t_n}^{k_n} \ldots dw_{t_1}^{k_1},$$

where $k, (k_1, \ldots, k_n), n \leq m$ are arbitrary and $f(t)$ and $f(t_n, \ldots, t_1)$ are arbitrary Borel bounded functions of their arguments. The projection operator in $L_2(\Omega, \mathcal{F}, P)$ on $W^n_r$ we denote by $\Pi^n_r$.

The final piece of information we need is the following result in which we suppose that

$$\widehat{D\sigma_M, b_M} \leq \beta,$$

where $\beta = \beta(d, d_1, \delta, p_{D\sigma}, p_b) > 0$ is taken from Theorem 4.5 of [6].
Theorem 3.5 (Theorem 4.9 of [6]). Take \( f \in C_0^\infty \) and \( r > 0 \) and set \( x_r = x_r(0), \xi = f(x_r) \). Then

\[
\sum_{n=1}^{\infty} E|\xi - \Pi_n^\delta \xi|^2 \leq N(d, \delta, pD\sigma, p_b, \bar{D}\sigma_B, \tilde{b}_B, r) \left( \int_{\mathbb{R}^d} |Df|^2p_b \, dx \right)^{1/p_b}. \tag{3.11}
\]

4. Proof of Theorem 2.3

In (2.1) we take \( \theta = \theta(d, d_1, \delta, p_b, p_b, q_0) = \theta_1(d, d_1, \delta/4, p_b, p_0, q_0) \), where \( \theta_1 \) is defined in (3.6) and take the \( \alpha \)'s, depending on the data in the way stated in the theorem, such that (3.4), (3.9), and (3.10) with \( \delta/4 \) in place of \( \delta \) hold as consequences of (2.3). Then Theorem 3.5 becomes available even if \( \sigma \) satisfies Assumption 2.2 (i) with \( \delta/4 \) in place of \( \delta \), but we do not change the original \( \sigma \).

Next we proceed to a final restriction of \( \bar{D}\sigma_M \) when \( \delta/4 \) appears. Take nonnegative \( \xi \in C_0^\infty(\mathbb{R}) \), \( \eta \in C_0^\infty(\mathbb{R}^d) \) with unit integrals and supports in unit balls, set \( \zeta(t, x) = \xi(t)*\eta(x) \), \( \zeta_n(t, x) = n^{d+1}\zeta(nt, nx) \), \( \eta_n(x) = n^d\eta(nx) \) and define

\[
\sigma^{(n)} = \sigma \ast \eta_n, \quad D\sigma_M^{(n)} = (D\sigma_M) \ast \eta_n, \quad D\sigma_B^{(n)} = (D\sigma_B) \ast \eta_n,
\]

where the convolution is performed with respect to \( x \), and

\[
b^{(n)} = b \ast \zeta_n, \quad b_M^{(n)} = b_M \ast \zeta_n, \quad b_B^{(n)} = b_B \ast \zeta_n,
\]

where the convolution is performed with respect to \( (t, x) \). As is easy to see, for each \( n \), \( \sigma^{(n)} \) is bounded along with its any derivative of any order with respect to \( x \), \( b^{(n)} \) is a smooth bounded function. Furthermore, Minkowski’s inequality easily shows that \( D\sigma^{(n)}, b^{(n)} \) (with the same \( p_{D\sigma}, p_b \)) satisfy

\[
\hat{D}\sigma_M^{(n)} \leq \hat{D}\sigma_M, \quad \hat{D}\sigma_B^{(n)} \leq \hat{D}\sigma_B, \quad \hat{b}_M^{(n)} \leq \hat{b}_M, \quad \hat{\beta}_b^{(n)} \leq \hat{\beta}_b, \quad \hat{b}_B^{(n)} \leq \hat{b}_B.
\]

Finally, introduce

\[
\Gamma_m = \{ t : \hat{D}\sigma_B(t) \leq m \}, \quad \sigma_m^{(n)}(t, x) = \sigma^{(n)}(t, x)I_{\Gamma_m}(t) + \kappa I_{\Gamma_m^c}(t),
\]

where \( \kappa \) is any fixed \( d \times d_1 \)-matrix such that \( \kappa \kappa^* = (\delta^{ij}) \).

Lemma 4.1. Set \( a_m^{(n)} = \sigma_m^{(n)}(\sigma_m^{(n)})^* \). Then there is a sequence \( m(n) \to \infty \) as \( n \to \infty \) such that for sufficiently large \( n \) the eigenvalues of \( a_m^{(n)} \) are between \( \delta/4 \) and \( 4/\delta \) if

\[
N(d)\hat{D}\sigma_M \leq \delta^{1/2}/4, \tag{4.1}
\]

where \( N(d) \) is specified in the proof.

Proof. Obviously, we only need to deal with \( t \in \Gamma_m^{(n)} \). Observe that for such \( t \)

\[
|\sigma_m^{(n)*}\lambda| = |\sigma^{(n)*}\lambda| \leq \eta_n \ast |\sigma^*\lambda| \leq \delta^{-1/2}|\lambda|.
\]
Therefore, we need only prove that for sufficiently large $n$ and $t \in \Gamma_{m(n)}$
\[|\sigma^{(n)*}(t, x)\lambda| \geq |\lambda|\delta^{1/2}/2. \tag{4.2}\]

For any $y$ we have
\[|\sigma^{(n)*}(t, x, y)| \geq |\sigma^{*}(t, x, y)\lambda| - |(\sigma^{(n)*}(t, x) - \sigma^{*}(t, x - y))\lambda|
\[\geq |\lambda|\left(\delta^{1/2} - |\sigma^{(n)*}(t, x) - \sigma^{*}(t, x - y)|\right).\]

Furthermore, by using Poincaré’s inequality and recalling that $t \in \Gamma_{m(n)}$, we obtain
\[I := \int_{\mathbb{R}^d} |\sigma^{(n)*}(t, x) - \sigma^{*}(t, x, y)|\eta_n(y) \, dy \, ds\]
\[\leq \int_{B_1} \int_{B_1} |\sigma(t, x - z/n) - \sigma(x - y/n)|\eta(y)\eta(z) \, dy \, dz\]
\[\leq N(d) \int_{B_{1/n}(x)} \int_{B_{1/n}(x)} |\sigma(t, y) - \sigma(t, z)| \, dy \, dz\]
\[\leq N(d)n^{-1} \int_{B_{1/n}(x)} |D\sigma(t, y)| \, dy =: N(d)I_1 + N(d)I_2,\]
where
\[I_1 = n^{-1} \int_{B_{1/n}(x)} |D\sigma_M(t, y)| \, dy\]
\[\leq n^{-1}\left(\int_{B_{1/n}(x)} |D\sigma(t, y)|^{p_D}\, dy\right)^{1/p_D} \leq \widehat{D}\sigma_M\]
if $1/n \leq \rho_\sigma$, and
\[I_2 = n^{-1} \int_{B_{1/n}(x)} |D\sigma_B(t, y)| \, dy \leq n^{-1}m(n)\]

Hence,
\[I \leq N(d)\widehat{D}\sigma_M + N(d)n^{-1}m(n)\]
and we get the desired result by defining $m(n)$ so that $N(d)n^{-1}m(n) \leq \delta^{1/2}/4$. The lemma is proved.

After we imposed the last restriction we proceed further with the proof in which, for notation simplicity, we suppose that $t = 0$ and $x = 0$. We define $x^n$ as solutions of
\[x^n_s = \int_0^s \sigma^{(n)k}(u, x^n_u) \, dw^k_u + \int_0^s b^{(n)}(u, x^n_u) \, du,\]
and repeating literally the corresponding arguments in [6] based on Skorokhod’s embedding method prove the existence part in Theorem 2.3 and estimate (2.4).

To prove the assertion about uniqueness we need some more preparation. Introduce
\[Lv(t, x) = \partial_t v(t, x) + (1/2)\sigma^{ij}(t, x)Dv(t, x) + b^i(t, x)D_i v(t, x).\]
Theorem 4.2 (Itô’s formula). Let $T \in (0, \infty)$, $v \in W^{1,2}_{p_0,q_0}(\mathbb{R}^d_T)$, and let $x_s$ be a solution of (1.1) with $t = 0, x = 0$ for which (2.4) holds with $m = 1$. Then (a.s.) for all $s \in [0, T]$

\[ v(s, x_s) = v(0, 0) + \int_0^s L v(u, x_u) \, du + \int_0^s \sigma^{ki} D_i v(u, x_u) \, dw^k_u \quad (4.3) \]

and the last term is a square-integrable martingale.

Proof. Set

\[ \gamma = \frac{d}{p_0} + \frac{2}{q_0}, \quad \kappa = \frac{\gamma}{\gamma - 1}. \]

Observe that $2 > \gamma > 1$ ($p_0 < p_b \leq d$) and $\kappa > 2$. It follows from Corollary 5.3 of [4] that

\[ \|Dv\|_{L_{p_0,q_0}}(\mathbb{R}^d_T) \leq N\|v\|_{W^{1,2}_{p_0,q_0}}(\mathbb{R}^d_T), \quad (4.4) \]

where (and below) $N$ is independent of $v$. In turn, (4.4) and (2.4) imply that

\[ E \int_0^T |Dv(s, x_s)|^\kappa \, ds \leq N\|v\|_{W^{1,2}_{p_0,q_0}}^{\kappa}(\mathbb{R}^d_T) \quad (4.5) \]

and this proves the last statement of the theorem.

We prove (4.3) as usual by taking smooth $v_n$ converging to $v$ in $W^{1,2}_{p,q}(\mathbb{R}^d_T)$. Since $\gamma < 2$, $v_n$ converge uniformly in $\mathbb{R}^d_T$ in light of the embedding theorems. In what concerns the integral terms in (4.3), estimate (2.4) shows that the only term of interest is

\[ I_n := E \int_0^T |b(s, x_s)| |D(v_n - v)(s, x_s)| \, ds = I'_n + I''_n, \]

where, owing to Hölder’s inequality and (4.5),

\[ I'_n := E \int_0^T |b_B(s, x_s)| |D(v_n - v)(s, x_s)| \, ds \]

\[ \leq \left( \int_0^T b^*_B(s) \, ds \right)^{1/\gamma} \left( E \int_0^T |D(v_n - v)(s, x_s)|^\kappa \, ds \right)^{1/\kappa} \rightarrow 0 \]

as $n \rightarrow \infty$, and

\[ I''_n := E \int_0^T |b_M(s, x_s)| |D(v_n - v)(s, x_s)| \, ds \]

\[ \leq N \left( \int_0^T \|b_M(s, \cdot)| |D(v_n - v)(s, \cdot)|\|_{L_{p_0,q_0}}(\mathbb{R}^d) \, ds \right)^{1/q_0}. \]

By Lemma 3.5 of [5] the integrand in the above term is dominated by a constant times

\[ \hat{b}^{q_0}_M \|(v_n - v)(s, \cdot)|\|_{W^{1,2}_{p_0}}^{q_0}(\mathbb{R}^d) \]

and this shows that $I''_n \rightarrow 0$ as $n \rightarrow \infty$ as well. The theorem is proved.

For $n \geq 0$ introduce

\[ L^n v(t, x) = \partial_t v(t, x) + (1/2)\alpha^{ij}(t, x)D_{ij} v(t, x) + b^i_M(t, x)D_i v(t, x) \]
solves the original equation on \([0, T]\) the new \(L_v(\cdot)\) condition. Then by uniqueness Remark there.

Remark 4.4. In Theorem 4.3 the terminal condition is zero. However, it also holds if it is any \(n > n_0\) in which the assumption that \(a \) and \(b \) are regular is irrelevant and was not used. Adding bounded lower order terms does not affect the proof there.

Proof of uniqueness in Theorem 2.3. Again we will treat only solutions of (1.1) with \(t = 0, x = 0\) and first we deal with any such solution for which (2.4) holds with \(m = 1\). Take a Borel bounded \(c\) on \(\mathbb{R}^{d+1}\) with compact support, take a smooth function \(f\) on \(\mathbb{R}^d\) with compact support, take \(n > 0\), and let \(v_n\) be the solution of the equation \(L^n v + cv_n = 0\) in \(\mathbb{R}^d_T\) with terminal condition \(v_n(T, \cdot) = f\). Also set

\[
\gamma_{n,s} = I_{(n, \infty)}(b_B(s, x_s))(\sigma^* a^{-1} b_B)(s, x_s),
\]

observe that

\[
\sigma^k(s, x_s) \gamma_{n,s} = I_{(n, \infty)}(b_B(s, x_s)) b_B(s, x_s)
\]

and introduce

\[
\phi_{n,s} = - \int_0^s \gamma_{n,u} d\omega_u^k - (1/2) \int_0^s |\gamma_{n,u}|^2 du + \int_0^s c(u, x_u) du.
\]

By Itô’s formula, applied to \(v(s, x_s) \exp \phi_{n,s}\), owing to (4.6), we obtain

\[
f(x_T) e^{\phi_{n,T}} = v_n(0,0) + \int_0^T e^{\phi_{n,u}} (\sigma^k D_i v(u, x_u) + v(u, x_u) \gamma_{n,u}^k) du_k.
\]

Notice that \(|\gamma_{n,u}| \leq N b_B(u)\) and \(b_B\) is a deterministic function of class \(L_2(\mathbb{R})\). It follows that all moments of \(\exp \phi_{n,u}\) are bounded on \([0, T]\) and since, as we have seen in the proof of Theorem 4.2, estimate (2.4) leads to (4.5) with \(\kappa > 2\), the expectation of the last term in (4.7) vanishes. Hence,

\[
Ef(x_T) e^{\phi_{n,T}} = v_n(0,0).
\]

By sending \(n \to \infty\) and using the above mentioned properties of \(\exp \phi_{n,u}\) we find

\[
Ef(x_T) \exp \left( \int_0^T c(u, x_u) du \right) ds = \lim_{n \to \infty} v_n(0,0).
\]
Since the last limit is independent of what solution \( x_s \) we take and we have the arbitrariness in \( c \) and \( f \), we conclude that all solutions (not necessarily strong) of (1.1) possessing property (2.4) with \( m = 1 \) have the same finite-dimensional distributions. This is a conditional ((2.4) assumed) weak uniqueness. At this moment one is tempted to refer to the result of A. Cherny [2] saying that weak uniqueness and strong existence imply the uniqueness of strong solutions. However, in this result one needs unconditional weak uniqueness which we do not know how to prove. Therefore, we proceed differently still using the idea from [2].

Define \( \tau = \sigma^* \sigma \). This is a symmetric nonnegative definite matrix and the following is well defined

\[
\Sigma = \lim_{\varepsilon \downarrow 0} (\tau + \varepsilon I)^{-1},
\]

where \( I \) is the \( d_1 \times d_1 \) identity matrix. As is easy to see by using the diagonal forms, \( \Sigma^2 = \Sigma, \Sigma \tau = \tau, \) and (tr \( AB = \) tr \( BA \))

\[
\text{tr} (\Sigma \sigma^* - \sigma^*) (\sigma \Sigma - \sigma) = \text{tr} (\Sigma \sigma^* \sigma \Sigma - \sigma^* \sigma \Sigma - \Sigma \sigma^* \sigma + \sigma^* \sigma) = \text{tr} (-\Sigma \tau + \tau) = 0,
\]

so that \( \sigma \Sigma = \sigma \). Next, by extending our probability space, if necessary, we suppose that we are also given a \( d_1 \)-dimensional Wiener process \( \tilde{w}_t \) independent of \( w_t \). Define

\[
\xi_s = \int_0^s \Sigma(u, x_u) \, dw_u + \int_0^s (I - \Sigma(u, x_u)) \, dw_u.
\]

An easy application of the Lévy theorem shows that \( \xi_s \) is a \( d_1 \)-dimensional Wiener process.

The crucial step is to prove that the processes \( x \) and \( \xi \) are independent on \([0, T]\) because (dropping arguments \((s, x_s)\))

\[
dx_s^i d\xi_s^k = \sigma^{ir} \sigma^{rk} (\delta^{kn} - \Sigma^{kn}) \, dw_s^r = \sigma^{ir} (\delta^{kr} - \Sigma^{kr}) \, ds = (\sigma^{ik} - \sigma^{ir} \sigma^{rk}) \, ds = 0.
\]

To do that, take two bounded Borel functions \( c' \) and \( c'' \) with compact support on \( \mathbb{R}^{d+1} \) and \( \mathbb{R}^{d_1+1} \), respectively, and take a smooth compactly supported function \( f \) on \( \mathbb{R}^d \). Then for \( n > 0 \) define \( v'_n \) as a solution of

\[
L^n v'_n + c' v'_n = 0 \quad \text{on} \quad [0, T] \times \mathbb{R}^d, \quad \text{and} \quad v''_n \text{ as a solution of} \quad (\partial_t + (1/2) \Delta + c'') v''_n = 0 \quad \text{on} \quad [0, T] \times \mathbb{R}^{d_1}.
\]

By Itô’s formula applied to

\[
v'_n(s, x_s) v''_n(s, \xi_s) e^{\phi_{n, s}},
\]

where

\[
\phi_{n, s} = -\int_0^s \gamma_{n, u}^k \, dw_u^k - (1/2) \int_0^s |\gamma_{n, u}|^2 \, du + \int_0^s [c'(u, x_u) + c''(u, \xi_u)] \, du
\]

we get

\[
f(x_T) e^{\phi_{n, T}} = v'_n(0, 0) v''(0, 0) + \int_0^T e^{\phi_{n, u}} [v''(u, \xi_u) \sigma^{ik} D_k v'_n(u, x_u) \, dw_u^k + v'_n(u, x_u) D_k v''(u, \xi_u) \, d\tilde{w}_u^k].
\]
By taking expectations and arguing as above we see that
\[ Ef(x_T) \exp \left( \int_0^T c'(s, x_s) ds \right) \exp \left( \int_0^T c''(s, \xi_s) ds \right) = \lim_{n \to \infty} v'_n(0,0) v''(0,0). \]  
(4.9)

The limit here we find from (4.8) and conclude that
\[ Ef(x_T) \exp \left( \int_0^T c'(s, x_s) ds \right) \exp \left( \int_0^T c''(s, \xi_s) ds \right) = Ef(x_T) \exp \left( \int_0^T c'(s, x_s) ds \right) v''(0,0). \]

After that the arbitrariness of \( f \) shows that
\[ E \exp \left( \int_0^T c'(s, x_s) ds \right) \exp \left( \int_0^T c''(s, \xi_s) ds \right) = E \exp \left( \int_0^T c'(s, x_s) ds \right) v''(0,0). \]

By taking \( c' = 0 \) we identify \( v''(0,0) \) and then the arbitrariness of \( c' \) and \( c'' \) proves that \( x \) and \( \xi \) are independent indeed.

Then we observe that
\[ w_s = I_s + \int_0^s \left( I - \Sigma(u, x_u) \right) d\xi_u, \]
where
\[ I_s = \int_0^s \Sigma(u, x_u) d\xi_u = \lim_{\varepsilon \downarrow 0} \int_0^s \left( \tau(u, x_u) + \varepsilon I \right)^{-1} \sigma^*(u, x_u) dm_u, \]
\[ m_s = \int_0^s \sigma(u, x_u) d\xi_u = x_s - \int_0^s b(u, x_u) du. \]

We see that \( I_s \) is a functional of \( x_\cdot \), so that the distribution of \( I \) are defined uniquely. Since the Wiener process \( \xi \cdot \) is independent of \( x_\cdot \), the conditional distribution of \( w_\cdot \) given \( x_\cdot \) and the joint distribution of \( (w_\cdot, x_\cdot) \) are unique (provided (2.4)).

The remaining part of the proof of the uniqueness of strong solutions can now follow exactly as in [2]. The theorem is proved.

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