Statistical fluctuations under resetting: rigorous results

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Abstract
In this paper we investigate the normal and the large fluctuations of additive functionals associated with a stochastic process under a general non-Poissonian resetting mechanism. Cumulative functionals of regenerative processes are very close to renewal-reward processes and inherit most of the properties of the latter. Here we review and use the classical law of large numbers and central limit theorem for renewal-reward processes to obtain same theorems for additive functionals of a stochastic process under resetting. Then, we establish large deviation principles for these functionals by illustrating and applying a large deviation theory for renewal-reward processes that has been recently developed by the author. We discuss applications of the general results to the positive occupation time, the area, and the absolute area of the reset Brownian motion. While introducing advanced tools from renewal theory, we demonstrate that a rich phenomenology accounting for dynamical phase transitions emerges when one goes beyond Poissonian resetting.

Keywords: renewal-reward processes, stochastic processes with resetting, additive functionals, law of large numbers, central limit theorem, large deviation principles, dynamical phase transitions

1. Introduction
Recent years have witnessed a growing interest of physicists in stochastic processes under resetting, whereby a process starts anew from its initial position at random times. Resetting mechanisms have been proven to represent a simple way of generating non-equilibrium stationary states \[1, 2\] and anomalous diffusion while keeping stationarity \[3\]. They have been also verified to speed up some searching tasks by defining intermittent search strategies, which interspace periods of random exploration with random returns to a starting point \[2\].

Physicists have focused much of their work on the Brownian motion and related processes under Poissonian resetting, studying deviations from the typical behavior and addressing
first-passage time problems in connection with intermittent search strategies [2]. In particular, the ability of stochastic resetting to shape fluctuations has been tested for additive functionals, such as the positive occupation time of the Brownian motion [4], its area and absolute area [4], the area of the fractional Brownian motion [5], the area of the Ornstein–Uhlenbeck process [6], and the local time of the Brownian motion [7,8]. For the Brownian motion subjected to Poissonian resetting, also positive and negative excursions have been investigated [9]. As resetting has the effect of confining the process around the initial position, leading to the emergence of a stationary state for the reset Brownian motion [2], one natural question is whether or not the resetting mechanism can bring about a large deviation principle for an additive functional when it does not satisfy a large deviation principle without resetting. Poissonian resetting has been discovered to be frequent enough to bring about a large deviation principle in some cases, such as the positive occupation time of the Brownian motion [4], but not in other cases, such as its area [4]. These findings have finally prompted physicists to inspect moderate deviations for the area of the Brownian motion and the fractional Brownian motion under Poissonian resetting [5].

With this paper we drop the Poissonian hypothesis and answer the above question when the resetting protocol is allowed to be any, and in particular when it is allowed to be adapted to the additive functional under consideration. To date, there are no contributions in this direction as far as we know. Most likely, one of the reasons that has made Poissonian resetting very attractive is the fact that it is the only resetting mechanism for which the moment generating function of additive functionals can be obtained explicitly at any time [6], so that their asymptotic properties can be investigated by means of elementary mathematical tools. Outside the Poissonian framework, advanced mathematical arguments are needed to get around the impossibility to work out every single step. We resort to the mathematical theory of renewal-reward processes to go beyond Poissonian resetting.

From the mathematical point of view, the mechanism of resetting is nothing but an aspect of renewal theory, which has been developed since the 40s by Doob, Feller, Cox, and many others [10]. In fact, the resetting process is a renewal process, which is meant to describe some event that is renewed randomly over time. In renewal theory the occurrences of an event often involve some broad-sense reward, and in this case one speaks of renewal-reward systems. For example, renewals can represent the random arrivals of customers at a store and rewards can be the random amount of money that each customer spends in the store. Renewal-reward systems find classical application in queuing theory [11], insurance [12], finance [13], and statistical physics of polymers [14,15] among others. A renewal-reward system is characterized by the renewal times \( T_1, T_2, \ldots \) at which the event occurs and by the corresponding rewards \( X_1, X_2, \ldots \). If \( S_1, S_2, \ldots \) denote the waiting times for a new occurrence of the event, then the renewal time \( T_n \) can be expressed for each \( n \geq 1 \) in terms of the waiting times as \( T_n = S_1 + \cdots + S_n \). Standard assumptions, which we also make, are that the waiting time and reward pairs \( (S_1, X_1), (S_2, X_2), \ldots \) form an independent and identically distributed (i.i.d.) sequence of random vectors on a probability space \((\Omega, \mathcal{F}, P)\), the waiting times taking non-negative real values and the rewards taking values in \( \mathbb{R}^d \) with some dimension \( d \geq 1 \). Any dependence between \( S_n \) and \( X_n \) is allowed. To avoid trivialities we suppose that the waiting times are not concentrated at zero, that is \( P[S_1 = 0] < 1 \). This implies in particular that almost all the trajectories account for only finitely many renewals in finite time\(^1\). The number of renewals up to the time \( t \geq 0 \) is \( N_t := \sup \{ n \geq 0 : T_n \leq t \} \) with \( T_0 := 0 \).

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\(^1\) Since \( P[S_1 > 0] > 0 \), there exists \( \delta > 0 \) with the property that \( P[S_1 \geq \delta] > 0 \). The second Borel–Cantelli lemma states that \( S_n \geq \delta \) for infinitely many \( n \) almost surely, so that \( \lim_{n \to \infty} T_n = +\infty \) \( P \)-a.s.
A quantity of great interest in renewal-reward systems is the cumulative reward by the time \( t \geq 0 \), which is the random variable

\[
W_t := \sum_{n=1}^{N_t} X_n
\]

with \( W_t := 0 \) if \( N_t = 0 \). The stochastic process \( \{W_t\}_{t \geq 0} \) is the so-called renewal-reward process, or compound renewal process. Insisting on the above example, \( W_t \) would be the amount of money earned by the store up to the time \( t \). The typical value \( \mu \) of the cumulative reward in the large \( t \) limit is identified by the following strong law of large numbers (SLLN) [16]. Hereafter, \( E \) denotes expectation with respect to the law \( P \) and \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^d \):

\[
\| x \| := \sqrt{x \cdot x} \text{ for every } x \in \mathbb{R}^d.
\]

**Theorem 1.1.** If \( \mathbb{E}[S_1] < +\infty \) and \( \mathbb{E}[\|X_1\|] < +\infty \), then

\[
\lim_{t \to +\infty} \frac{W_t}{t} = \frac{\mathbb{E}[X_1]}{\mathbb{E}[S_1]} =: \mu \quad \text{P.-a.s.}
\]

The normal fluctuations around \( \mu \) are described by the following central limit theorem (CLT) [16], which introduces the asymptotic covariance matrix \( \Sigma \) of \( W_t \).

**Theorem 1.2.** If \( \mathbb{E}[S_1] < +\infty \) and \( \mathbb{E}[\|X_1\|^2] < +\infty \), then for each Borel set \( A \subseteq \mathbb{R}^d \)

\[
\lim_{t \to +\infty} \mathbb{P} \left[ \frac{W_t - \mu t}{\sqrt{t}} \in A \right] = \int_A \frac{\frac{1}{\sqrt{2\pi}^d} \exp\left( -\frac{1}{2} x \cdot \Sigma^{-1} x \right) \, dx}{\sqrt{\det \Sigma}}
\]

provided that the matrix \( \Sigma \) is positive definite, with

\[
\Sigma := \frac{\text{cov}(X_1 - \mu S_1)}{\mathbb{E}[S_1]}.
\]

Theorems 1.1 and 1.2 are well-established properties of renewal-reward processes that descend from basic tools of probability theory, the former being a simple manipulation of the Kolmogorov’s law of large numbers and the latter being a combination of the classical central limit theorem with the Anscombe’s theorem for random sequences with random index. The typical fluctuations can be also described when the variances of \( S_1 \) and \( \|X_1\| \) are not finite by introducing proper scaling and possibly Lévy stable laws [16].

While the study of the typical behaviors of the cumulative reward \( W_t \) can be considered concluded, the study of its large fluctuations is still in progress and some definitive results have been achieved only very recently. In fact, many mathematicians have investigated the large fluctuations of \( W_t \) with increasing level of generality [17–23]. At last, the author has provided large deviation principles with optimal hypotheses for lattice waiting-time distributions first [24, 25] and any waiting-time distribution later [26]. We shall review the findings of [26] in the next section.

In this paper we make use of the theory of renewal-reward processes to investigate normal and large fluctuations of additive functionals associated with a stochastic process under a resetting mechanism. The reference stochastic process and the resetting mechanism are allowed to be any. While providing general results that characterize both typical and rare behaviors, we show that resetting at a higher frequency than Poissonian resetting brings about a large deviation principle where the Poissonian protocol fails.

We apply our general results to the positive occupation time, the area, and the absolute area of the reset Brownian motion, continuing the study initiated in [4]. A large deviation theory beyond Poissonian resetting allows to appreciate the emergence of singularities in the graph of their rate functions, which are interpreted as dynamical phase transitions. Dynamical
phase transitions have been already documented for time-averaged quantities such as the heat exchanged after a quench [27, 28], the heat exchanged between two thermal walls [29, 30], the active work in active matter [31], the total displacement of random walks [32–35], occupation times of a Brownian particle [36, 37], and the entropy production [38–42]. Physicists often invoke the Gärtner–Ellis theorem to justify their calculations that lead to dynamical phase transitions, although it is necessary for a dynamical phase transition to occur that the hypotheses of that theorem are violated. We do not suffer from this contradiction because we are able to develop a theory for the large fluctuations of additive functionals of stochastic processes under resetting that does not require the smoothness hypotheses of the Gärtner–Ellis theorem.

The paper is organized as follows. In section 2 we review the large deviation principles for renewal-reward processes established by the author in [26]. In this section we also propose a formula for the rate function that facilitates applications. In section 3 we introduce general stochastic processes under a resetting mechanism and the associated additive functionals. For general additive functionals we prove the law of large numbers, the central limit theorem, and a large deviation principle. In section 4 we apply these results to the positive occupation time, the area, and the absolute area of the reset Brownian motion, supplying a complete characterization of their rate functions. In this section we also discuss how rate functions behave in the limit of zero resetting through some examples. Conclusions and prospects for future research are finally reported in section 5. This paper is essentially a math work. The most technical proofs are postponed in the appendices and are not necessary to understand the main line of discussion.

2. Large deviations for renewal-reward processes

The cumulative reward \( W_t \) is said to satisfy a weak large deviation principle (weak LDP) with the rate function \( I \) if there exists a lower semicontinuous function \( I \) on \( \mathbb{R}^d \) such that

(a) \( \liminf_{t \to +\infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_t}{t} \in G \right] \geq -\inf_{w \in G} \{ I(w) \} \) for each open set \( G \subseteq \mathbb{R}^d \),

(b) \( \limsup_{t \to +\infty} \frac{1}{t} \ln \mathbb{P}\left[ \frac{W_t}{t} \in K \right] \leq -\inf_{w \in K} \{ I(w) \} \) for each compact set \( K \subseteq \mathbb{R}^d \).

Bounds (a) and (b) are called lower large deviation bound and upper large deviation bound, respectively. The cumulative reward \( W_t \) is said to satisfy a full large deviation principle (full LDP) if the upper large deviation bound holds for all closed sets \( K \), and not only for compact sets. The rate function \( I \) is denoted good if it has compact level sets, namely if the set \( \{ w \in \mathbb{R}^d : I(w) \leq a \} \) is compact for each \( a \geq 0 \). We refer to [43, 44] for the language of large deviation theory. In this section we review the LDPs for renewal-reward processes established by the author in [26]. For simplicity, and for the applications we have in mind, here we assume that the limit

\[
\lim_{t \to +\infty} \frac{1}{t} \ln \mathbb{P}[S_1 > s] =: \ell
\]

exists as an extended real number. Notice that \( \ell \in [0, +\infty] \). This assumption is not made in [26], where, besides, possible infinite-dimensional rewards are considered.

The results of [26] rely on Cramér’s theory for waiting times and rewards. The joint Cramér’s rate function of waiting times and rewards is the function \( J \) that maps each \( (s, w) \in \mathbb{R} \times \mathbb{R}^d \) in the extended real number

\[
J(s, w) := \sup_{(\zeta, k) \in \mathbb{R} \times \mathbb{R}^d} \left\{ s \zeta + w \cdot k - \ln \mathbb{E} [e^{\zeta S_1 + k \cdot K_1}] \right\}.
\]
The lower-semicontinuous regularization $\Upsilon$ of $\inf_{\gamma > 0}\{\gamma J(\cdot/\gamma, \cdot/\gamma)\}$ is the function that associates every $(\beta, w) \in \mathbb{R} \times \mathbb{R}^d$ with

$$\Upsilon(\beta, w) := \lim_{\delta \to 0} \inf_{\beta, \delta} \inf_{v} \inf_{\gamma > 0} \left\{ \gamma J\left(\frac{v - \beta}{\gamma} \right) \right\},$$

$\Delta_{w, \delta} := \{v \in \mathbb{R}^d : ||v - w|| < \delta\}$ being the open ball of center $w$ and radius $\delta$. We make use of $\Upsilon$ to construct a 'rate function' $I$ according to the formula

$$I(w) := \begin{cases} \inf_{\beta \in [0, 1]} \{\Upsilon(\beta, w) + (1 - \beta)\ell\} & \text{if } \ell < +\infty, \\ \Upsilon(1, w) & \text{if } \ell = +\infty \end{cases}$$

for all $w \in \mathbb{R}^d$. The following theorem states a weak and a full LDP for the cumulative reward $W_t$ and is (part of) theorem 1.1 of [26]. The function $I$ turns out to really be the rate function.

**Theorem 2.1.** The following conclusions hold:

(a) the function $I$ is lower semicontinuous and convex;
(b) $W_t$ satisfies a weak LDP with the rate function $I$. Furthermore, the upper large deviation bound holds for each Borel convex set $K$ whenever $\ell < +\infty$ or $I(0) < +\infty$;
(c) if $\mathbb{E}[e^{\|X_t\|^2}] < +\infty$ for some number $\rho > 0$, then $I$ has compact level sets and $W_t$ satisfies a full LDP with the good rate function $I$.

We stress that no assumption on the law of waiting times and rewards is needed for the validity of a weak LDP. Moreover, the upper large deviation bound extends to convex sets if $\ell < +\infty$ or, whereas it fails in general when $\ell = +\infty$ and $I(0) = +\infty$ as shown in [26] by means of some counterexamples. Regarding a full LDP with a good rate function, we can observe that it does not require an hypothesis on waiting times but only an exponential moment condition on rewards: $\mathbb{E}[e^{\|X_t\|^2}] < +\infty$ for some $\rho > 0$.

We conclude the section by providing a formula for the rate function $I$ that facilitates the use of theorem 2.1 in applications. Let $\varphi$ be the function that maps any $k \in \mathbb{R}^d$ in the extended real number

$$\varphi(k) := \sup \left\{ \varsigma \in \mathbb{R} : \mathbb{E}\left[e^{\varsigma S_t + kX_t}\right] \leq 1 \right\},$$

where the supremum over the empty set is customarily interpreted as $-\infty$. The following proposition shows that $I$ is the convex conjugate, i.e. the Legendre–Fenchel transform, of $-(\varphi \wedge \ell)$. As usual, given two extended real numbers $a$ and $b$, we denote min$\{a, b\}$ by $a \wedge b$ and max$\{a, b\}$ by $a \vee b$ for brevity. The proof of the proposition is presented in appendix A.

**Proposition 2.1.** The following conclusions hold:

(a) the function $\varphi$ is upper semicontinuous and concave;
(b) $I(w) = \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k + \varphi(k) \wedge \ell \right\}$ for all $w \in \mathbb{R}^d$.

3. Stochastic processes under resetting and fluctuations

A stochastic process under resetting can be constructed from independent copies of the reference process as follows. Let $\{B_{1,t}\}_{t \geq 0}, \{B_{2,t}\}_{t \geq 0}, \ldots$ be i.i.d. $\mathbb{R}$-valued stochastic processes on a probability space $(\Omega, \mathcal{F}, P)$ where a renewal process $\{T_n\}_{n \geq 0}$ is already given. We think of
these processes as fresh realizations of the same stochastic dynamics, say \( \{B_{1,t}\}_{t \geq 0} \), and we defined the dynamics under resetting \( \{Z_t\}_{t \geq 0} \) by pasting them together at each renewal time:

\[
Z_t := B_{N_1 + 1, t - T_{N_1}}
\]

for all \( t \geq 0 \). We remind that \( N_t := \sup \{ n \geq 0 : T_n \leq t \} \) is the number of renewals by the time \( t \), so that \( Z_t = B_{n+1, t - T_n} \) for \( t \in [T_{n-1}, T_n) \) and \( n \geq 1 \). The stochastic process under resetting \( \{Z_t\}_{t \geq 0} \) is the reset Brownian motion or the reset fractional Brownian motion when \( \{B_{1,t}\}_{t \geq 0}, \{B_{2,t}\}_{t \geq 0}, \ldots \) are Brownian motions or fractional Brownian motions, respectively. It is the reset Ornstein–Uhlenbeck process if \( \{B_{1,t}\}_{t \geq 0}, \{B_{2,t}\}_{t \geq 0}, \ldots \) are Ornstein–Uhlenbeck processes. According to the existing literature [2, 4–9], we assume that the processes \( \{B_{1,t}\}_{t \geq 0}, \{B_{2,t}\}_{t \geq 0}, \ldots \) are independent of the waiting times. We also assume that their sample paths are measurable in order to enable integration over time.

An additive functional of the process \( \{Z_t\}_{t \geq 0} \) is the random variable

\[
F_t := \int_0^t f(Z_r) \, dr
\]

where \( f \) is some real measurable function over \( \mathbb{R} \). For instance, the positive occupation time and the absolute area studied in [4] for the reset Brownian motion with Poissonian resetting are the additive functionals corresponding to \( f(z) := 1_{\{z > 0\}} \) and \( f(z) := |z| \), respectively. The area investigated in [4] for the reset Brownian motion with Poissonian resetting, in [5] for the reset fractional Brownian motion with Poissonian resetting, and in [6] for the reset Ornstein–Uhlenbeck process with Poissonian resetting, is found with \( f(z) := z \). We recall that Poissonian resetting amounts to a pure exponential waiting time distribution: \( \mathbb{P}[S_1 > s] = e^{-rs} \) for all \( s \geq 0 \) with some resetting rate \( r > 0 \).

In this section we provide general results for the fluctuations of the additive functional \( F_t \) by resorting to the theory of renewal-reward processes. The random variable \( F_t \) is closely related to a cumulative reward. In fact, if we associate the ‘reward’

\[
X_n := \int_0^{S_n} f(B_{n, r}) \, dr
\]

with the \( n \)th renewal at the time \( T_n \), then \( F_t \) differs from the cumulative reward \( W_t := \sum_{n=1}^{N_t} X_n \) by the contribution \( \int_0^{T_n} f(B_{N_1+1, r}) \, dr \) of the backward recurrence time. The backward recurrence time, or current lifetime, is the time \( t - T_n \) elapsed from the last renewal. Explicitly, for all \( t \geq 0 \) we can write

\[
F_t = \sum_{n=1}^{N_t} \int_{T_{n-1}}^{T_n} f(B_{n, r}) \, dr + \int_{T_{N_t}}^{t} f(B_{N_1+1, r}) \, dr
\]

\[
= \sum_{n=1}^{N_t} X_n + \int_0^{T_N} f(B_{N_1+1, r}) \, dr = W_t + \int_0^{t - T_N} f(B_{N_1+1, r}) \, dr.
\]

Such relationship suggests that some limit theorems for \( F_t \) can be obtained from limit theorems for \( W_t \). This is quite immediate for the SLLN and the CLT, where the ‘incomplete reward’ \( \int_0^{T_N} f(B_{N_1+1, r}) \, dr \) disappears upon rescaling under mild assumptions on the resetting mechanism. In other words, the backward recurrence time cannot contribute to the typical behavior of \( F_t \). The issue of large deviations is more involved since the backward recurrence time comes into play. The typical behavior of the additive functional \( F_t \) is investigated in section 3.2, whereas the large fluctuations are addressed in section 3.2.
3.1. Typical behaviors of additive functionals

Set for brevity
\[ M_n := \int_0^{S_n} |f(B_{n,\tau})| \, d\tau \geq |X_n|. \]

Under the hypotheses \( \mathbb{E}[S_1] < +\infty \) and \( \mathbb{E}[M_1^\rho] < +\infty \) with some \( \rho > 0 \), the difference between the additive functional \( F_t \) and the cumulative reward \( W_t := \sum_{n=1}^{\infty} X_n \) satisfies
\[
\lim_{t \uparrow +\infty} t^{-\frac{1}{\rho}} |F_t - W_t| = 0 \quad \mathbb{P}\text{-a.s.} \tag{3.2}
\]

In fact, for every \( \epsilon > 0 \) we have
\[
\sum_{n=1}^{\infty} \mathbb{P}\{n^{-\frac{1}{\rho}} M_n > \epsilon\} = \sum_{n=1}^{\infty} \mathbb{P}\{M_n^\rho / \epsilon^\rho > n\}
\leq \int_0^{+\infty} \mathbb{P}\{M_t^\rho / \epsilon^\rho > \lambda\} \, d\lambda = \mathbb{E}[M_t^\rho / \epsilon^\rho] < +\infty.
\]
Thus, the Borel–Cantelli lemma yields \( \lim_{t \uparrow +\infty} n^{-\frac{1}{\rho}} M_n = 0 \) \( \mathbb{P}\text{-a.s.} \). At the same time, theorem 1.1 with unit rewards states that \( \lim_{t \uparrow +\infty} N_t / t = 1 / \mathbb{E}[S_1] > 0 \) \( \mathbb{P}\text{-a.s.} \). By combining these two limits we realize that \( \lim_{t \uparrow +\infty} t^{-\frac{1}{\rho}} M_{N_t+1} = 0 \) \( \mathbb{P}\text{-a.s.} \). At this point, (3.2) is evident since (3.1) and the inequality \( t - T_{N_t} < T_{N_t+1} - T_{N_t} = S_{N_t+1} \) give for all \( t > 0 \)
\[
|F_t - W_t| = \left| \int_0^{t-T_{N_t}} f(B_{N_t+1,\tau}) \, d\tau \right| \leq \int_0^{S_{N_t+1}} |f(B_{N_t+1,\tau})| \, d\tau = M_{N_t+1}.
\]

Limit (3.2) has consequences that are easy to see.

Theorem 1.1 tells us that \( \lim_{t \uparrow +\infty} W_t / t = \mu \) \( \mathbb{P}\text{-a.s.} \) if \( \mathbb{E}[S_1] < +\infty \) and \( \mathbb{E}[|X_1|] < +\infty \). Under the slightly more restrictive condition \( \mathbb{E}[S_1] < +\infty \) and \( \mathbb{E}[M_1] < +\infty \) we have \( \lim_{t \uparrow +\infty} t^{-\frac{1}{\rho}} |F_t - W_t| = 0 \) by (3.2). Thus, the following SLLN for \( F_t \) holds true.

**Theorem 3.1.** If \( \mathbb{E}[S_1] < +\infty \) and \( \mathbb{E}[M_1] < +\infty \), then
\[
\lim_{t \uparrow +\infty} \frac{F_t}{\mathbb{E}[S_1]} =: \mu \quad \mathbb{P}\text{-a.s.}
\]

If \( \mathbb{E}[S_1^2] < +\infty \) and \( \mathbb{E}[M_1^2] < +\infty \), then \( \mathbb{E}[X_1^2] < +\infty \) and, provided that \( \mathbb{E}[(X_1 - \mu S_1)^2] \neq 0 \), theorem 1.2 ensures us that \( (W_t - \mu t) / \sqrt{\nu t} \) converges in distribution as \( t \) is sent to infinity to a Gaussian variable with mean 0 and variance 1. The number \( \nu \) is the asymptotic variance of \( W_t \):
\[
\nu := \frac{\mathbb{E}[(X_1 - \mu S_1)^2]}{\mathbb{E}[S_1^2]} . \tag{3.3}
\]

At the same time, (3.2) shows that \( \lim_{t \uparrow +\infty} t^{-\frac{1}{\rho}} |F_t - W_t| = 0 \) \( \mathbb{P}\text{-a.s.} \). Thus, Slutsky’s theorem tells us that \( (F_t - \mu t) / \sqrt{\nu t} \) converges in distribution to the same limit as \( (W_t - \mu t) / \sqrt{\nu t} \). These arguments prove the following CLT for \( F_t \).

**Theorem 3.2.** If \( \mathbb{E}[S_1^2] < +\infty \) and \( \mathbb{E}[M_1^2] < +\infty \), then for every \( z \in \mathbb{R} \)
\[
\lim_{t \uparrow +\infty} \mathbb{P}\left[ \frac{F_t - \mu t}{\sqrt{\nu t}} \leq z \right] = \int_{-\infty}^z e^{-\frac{1}{2}x^2} \, dx \]

provided that \( \nu \) given by (3.3) is not zero.
When the function $f$ that defines the additive functional is positive or negative, such as for the positive occupation time and the absolute area, the condition $\mathbb{E}[X_t^2] < +\infty$ that makes the asymptotic variance $\nu$ finite also is the condition $\mathbb{E}[M_t^2] < +\infty$ that neutralizes the contribution of the backward recurrence time. In section 4 we shall see that the conditions $\mathbb{E}[X_t^2] < +\infty$ and $\mathbb{E}[M_t^2] < +\infty$ are tantamount even in cases where $f$ changes sign, such as the case of the area of the reset Brownian motion.

3.2. Large fluctuations of additive functionals

We discuss the large fluctuations of $F_t$ by applying same definitions of section 2. The additive functional $F_t$ satisfies a full LDP with the rate function $I$ if there exists a lower semicontinuous function $I$ on $\mathbb{R}$ such that

(a) $\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{t} \in G \right] \geq - \inf_{w \in G} \{ I(w) \}$ for each open set $G \subseteq \mathbb{R}$,
(b) $\limsup_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{t} \in K \right] \leq - \inf_{w \in K} \{ I(w) \}$ for each closed set $K \subseteq \mathbb{R}$.

The rate function $I$ is good if it has compact level sets. A remark is in order about the question raised in the introduction regarding the ability of the resetting mechanism to bring about an LDP for $F_t$ when it does not satisfy an LDP without resetting. In most cases both $F_t$ and the additive functional without resetting, namely $\int_0^t f(B_1, \tau) \, d\tau$, are expected to satisfy an LDP according to the above definition. Then, the question actually is whether the rate function of $F_t$ turns out to be positive, so that the large fluctuations of $F_t$ really occur with probability that is exponentially small in the time $t$, whereby the rate function of $\int_0^t f(B_1, \tau) \, d\tau$ is zero. The property of goodness of $I$ excludes that the region of its zeros extends to infinity.

As in section 2, we require that the following limit exists:

$$\ell := \lim_{s \uparrow +\infty} -\frac{1}{s} \ln \mathbb{P} \left[ S_t > s \right].$$

Then, by theorem 2.1 and proposition 2.1 the cumulative reward $W_t := \sum_{n=1}^{N_t} X_n$ satisfies a weak LDP and its rate function is the convex conjugate of $-(\varphi \wedge \ell)$, $\varphi$ being the function that maps $k \in \mathbb{R}$ in

$$\varphi(k) := \sup \left\{ \zeta \in \mathbb{R} : \mathbb{E} \left[ e^{\zeta S_t + kX_t} \right] \leq 1 \right\}.$$ 

When adding the contribution $\int_{T_{N_t}}^{T_{N_t+1}} f(B_{N_t+1, \tau}) \, d\tau$ of the backward recurrence time to $W_t$ in order to form $F_t$, a putative rate function is the lower semicontinuous convex function $I$ that associates $w \in \mathbb{R}$ in

$$I(w) := \sup_{k \in \mathbb{R}} \{ w\ell + \varphi(k) \wedge \varphi(k) \},$$

where

$$\varphi(k) := \liminf_{t \uparrow +\infty} -\frac{1}{t} \ln \left\{ \mathbb{E}(k) \mathbb{P} \left[ S_t > t \right] \right\}$$

and $\mathbb{E}(k) := \mathbb{E}[e^{k \int_0^t f(B_1, \tau) \, d\tau}]$ is the value at $k$ of the moment generating function of the additive functional without resetting. If we drop $\mathbb{E}_t(k)$, then $\varphi(k) = \ell$ and formula (3.4) returns the original rate function of $W_t$. Thus, we can get a cue for introducing $\mathbb{E}_t(k)$ by observing that the occurrence of the event $S_t > t$, i.e. $N_t = 0$, entails $F_t = \int_0^t f(B_1, \tau) \, d\tau$ whereas $W_t = 0$. The following proposition demonstrates an estimate of the asymptotic logarithmic moment generating function of $F_t$ and the upper large deviation bound with the rate function $I$. The proof is
presented in appendix B. We recall that the asymptotic logarithmic moment generating function of \( F \) is the function \( g \) that maps \( k \in \mathbb{R} \) in
\[
g(k) := \lim_{t \uparrow + \infty} \frac{1}{t} \ln \mathbb{E} \left[ e^{tF} \right].
\]

The asymptotic logarithmic moment generating function \( g \) does not exist if this limit does not exist at some point \( k \). Investigation of \( g \) will allow to compare our theory with the Gärtner–Ellis theory.

**Proposition 3.1.** The following conclusions hold with \( I \) defined by (3.4):

\[
\begin{align*}
(a) \quad & \limsup_{t \uparrow + \infty} \frac{1}{t} \ln \mathbb{E} \left[ e^{tF} \right] \leq -\varphi(k) \land \varpi(k) \text{ for all } k \in \mathbb{R}; \\
(b) \quad & \limsup_{t \uparrow + \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F}{t} \in K \right] \leq -\inf_{w \in k} \{ I(w) \} \text{ for each closed set } K \subseteq \mathbb{R}.
\end{align*}
\]

There is no possibility to promote the upper large deviation bound stated by part (b) of proposition 3.1 to an LDP without some hypothesis on the contribution of the backward recurrence time. In all cases concerning the reset Brownian motion that we have considered we have found \( \varpi(k) \geq \varphi(k) \) for all \( k \in \mathbb{R} \), meaning that the fluctuations of the incomplete reward \( \int_0^{t-T_0} f(B_{N_0+1,t}) \, d\tau \) do not overcome the fluctuations of complete rewards. Actually, we are not able to exhibit an example that violates \( \varpi(k) \geq \varphi(k) \) for some \( k \). Thus, we are going to assume that the condition \( \varpi(k) \geq \varphi(k) \) is fulfilled for all \( k \in \mathbb{R} \). This condition gives
\[
I(w) := \sup_{k \in \mathbb{R}} \{ wk + \varphi(k) \}.
\] (3.6)

Importantly, this condition suffices to obtain an LDP for the additive functional \( F \), as stated by the following theorem. The proof of the lower large deviation bound is based on theorem 2.1 and proposition 2.1 and is reported in appendix C.

**Theorem 3.3.** The following conclusions hold with \( I \) defined by (3.6):

\[
\begin{align*}
(a) \quad & \text{if there exists } \rho > 0 \text{ such that } \mathbb{E}[e^{\rho|X_1|}] < +\infty, \text{ then } I \text{ is a good rate function. If moreover } \\
& \ell > 0, \text{ then } I(w) = 0 \text{ if and only if } w = \mu := \mathbb{E}[X_1]/\mathbb{E}[S_1]; \\
(b) \quad & \text{if } \varpi(k) \geq \varphi(k) \text{ for all } k \in \mathbb{R}, \text{ then } F_i \text{ satisfies a full LDP with the rate function } I; \\
(c) \quad & \text{if } \varpi(k) \geq \varphi(k) \text{ for all } k \in \mathbb{R}, \text{ then the asymptotic logarithmic moment generating function } \\
& g \text{ of } F_i \text{ exists and equals } -\varphi.
\end{align*}
\]

We stress that the condition \( \varpi(k) \geq \varphi(k) \) for all \( k \in \mathbb{R} \) does not mean that the contribution of the backward recurrence time is negligible. It means that the fluctuations of the incomplete reward \( \int_0^{t-T_0} f(B_{N_0+1,t}) \, d\tau \) are not dominant, but they could be comparable to those of complete rewards. In confirmation of this, formula (3.6) is not the rate function without the contribution of the backward recurrence time, which is the rate function of \( W_i \), i.e. the convex conjugate of \(- (\varphi \land \ell) \). The rate function (3.6) of \( F_i \) coincides with the rate function of \( W_i \) only if \( \varphi(k) \leq \ell \) for all \( k \), which holds trivially if \( \ell = +\infty \). If instead \( \ell < +\infty \) and \( \varphi(k) > \ell \) for some \( k \), then the large fluctuations of the incomplete reward contribute to the large fluctuations of \( F_i \).

Whatever the reference stochastic process is, we can always find a renewal process that satisfies the condition \( \varpi(k) \geq \varphi(k) \) for all \( k \in \mathbb{R} \), thus bringing about an LDP for the additive functional \( F_i \). Indeed, this condition is automatically met if the waiting times are designed to fulfill the exponential moment condition \( \mathbb{E}[e^{\lambda M_1}] < +\infty \) for all \( \rho > 0 \) with \( M_1 := \int_0^{\bar{S}_1} |f(B_{1,t})| \, d\tau \). As \( |X_1| \leq M_1 \), this condition also implies that the rate function is good.
Therefore, a full LDP with a good rate function can always be achieved by means of a resetting protocol adapted to the additive functional \( F_t \). This fact is finally verified in appendix D, where the following corollary of theorem 3.3 is demonstrated.

**Corollary 3.1.** If \( \mathbb{E}[e^{\rho M_t}] < +\infty \) for all \( \rho > 0 \), then \( F_t \) satisfies a full LDP with the good rate function \( I \) given by (3.6). Moreover, its asymptotic logarithmic moment generating function \( g \) exists and equals \( -\varphi \).

There are additive functionals that fulfill \( \varpi(k) \geq \varphi(k) \) for all \( k \in \mathbb{R} \) for any renewal process. They have the property that \( \frac{1}{t} \ln \mathcal{E}_t(k) \) has a limit as \( t \) is sent to infinity for each \( k \), meaning that the value \( \mathcal{E}_t(k) \) of the moment generating function of \( \int_0^t f(B_{1,t}) \, d\tau \) does not oscillate too much at large \( t \). These additive functionals satisfy a full LDP under any resetting mechanism, although the rate function may not be good. We anticipate that the positive occupation time, the area, and the absolute area of the Brownian motion belong to this class of additive functionals. To get insights into the issue we observe that \( \mathbb{E}[e^{\varphi(k)\mathcal{E}_t(k) + k\mathcal{X}_t}] \leq 1 \) if \( \varphi(k) > -\infty \). In fact, by definition of \( \varphi(k) \) there exists a sequence \( \{\xi_i\}_{i \geq 1} \) with the property that \( \lim_{t \to \infty} \mathcal{E}_t(\xi_i) = \varphi(k) \) and \( \mathbb{E}[e^{\varphi(k)\mathcal{E}_t(k) + k\mathcal{X}_t}] \leq 1 \) for all \( i \), so that the Fatou’s lemma implies \( \mathbb{E}[e^{\varphi(k)\mathcal{E}_t(k) + k\mathcal{X}_t}] \leq \liminf_{t \to \infty} \mathbb{E}[e^{\varphi(k)\mathcal{E}_t(k) + k\mathcal{X}_t}] \leq 1 \). The independence between \( S_t \) and \( \mathcal{B}_{1,t} \) allows to recast the relationship \( \mathbb{E}[e^{\varphi(k)\mathcal{E}_t(k) + k\mathcal{X}_t}] \leq 1 \) as

\[
\int_{[0,+\infty)} e^{\varphi(k)s} \mathcal{E}_t(k) P(ds) = \mathbb{E}[e^{\varphi(k)S_t + kX_t}] \leq 1,
\]

where \( P := \mathbb{P}[S_t \in \cdot] \) is the probability measure induced on \([0, +\infty)\) by \( S_t \). Thus, while the inequality \( \varpi(k) \geq \varphi(k) \) is trivial if \( \varphi(k) = -\infty \), when \( \varphi(k) > -\infty \) it reads \( \limsup_{t \to +\infty} \frac{1}{t} \ln \mathcal{E}_t(k) \mathbb{P}[S_t > t] \leq 0 \) conditional on (3.7). This push the idea that \( \varpi(k) \geq \varphi(k) \) is likely to be satisfied provided that \( \mathcal{E}_t(k) \) is not affected by too large oscillations as \( t \) goes on. The following corollary of theorem 3.3 makes these arguments rigorous. The proof is provided in appendix E.

**Corollary 3.2.** Assume that for all \( k \in \mathbb{R} \) the limit \( \lim_{t \to +\infty} \frac{1}{t} \ln \mathcal{E}_t(k) \) exists and that \( \frac{1}{t} \ln \mathcal{E}_t(k) \) is eventually non-decreasing with respect to \( t \) if such limit is infinite. Then, \( F_t \) satisfies a full LDP with the rate function \( I \) given by (3.6). Moreover, its asymptotic logarithmic moment generating function \( g \) exists and equals \( -\varphi \).

We conclude the section by drawing a comparison with the Gärtner–Ellis theory. The Gärtner–Ellis theory aims to obtain an LDP based on the sole knowledge of the asymptotic logarithmic moment generating function \( g \). Supposing that \( \varpi(k) \geq \varphi(k) \) for all \( k \in \mathbb{R} \), part (c) of theorem 3.3 yields \( g = -\varphi \), so that \( g \) is convex and lower semicontinuous. Set \( k_- := \inf\{k \in \mathbb{R} : g(k) < +\infty\} \) and \( k_+ := \sup\{k \in \mathbb{R} : g(k) < +\infty\} \) and notice that \( g(k) < +\infty \) for all \( k \in (k_-, k_+) \) by convexity. The Gärtner–Ellis theorem gives a full LDP for \( F_t \) with the convex conjugate of \( g \) as the rate function if the following additional conditions are met [43, 44]:

(a) \( k_- < 0 < k_+ \);
(b) \( g \) is differentiable throughout the open interval \( (k_-, k_+) \);
(c) \( g \) is steep, i.e. \( \lim_{k \downarrow k_-} g'(k) = -\infty \) if \( k_- > -\infty \) and \( \lim_{k \uparrow k_+} g'(k) = +\infty \) if \( k_+ < +\infty \).

These conditions qualify the lower semicontinuous convex function \( g \) as essentially smooth. Essentially smoothness of the asymptotic logarithmic moment generating function is crucial in the Gärtner–Ellis theory to deduce the lower large deviation bound via an exponential change of measure. Thus, the use of the Gärtner–Ellis theorem without essentially smoothness of \( g \) is definitely incorrect, although its conclusions may be true. In fact, theorem 3.3...
shows that essentially smoothness of \( g = -\varphi \) is not necessary at all for the conclusions of the Gärtner–Ellis theorem to hold. Theorem 3.3 is ultimately based on Cramér’s theory, whose modern construction relies on a sub-additive argument that has nothing to do with exponential changes of measure \([43, 44]\).

A general property of convex conjugation states that if \( g = -\varphi \) is essentially smooth, then the rate function \( I \) given by (3.6) is strictly convex on the effective domain where it is finite (see [45], theorem 26.3). Therefore, all models that involve a non-strictly convex rate function cannot be faced with the Gärtner–Ellis theory. These models are exactly the models that exhibit a dynamical phase transition. A dynamical phase transition is a point in the effective domain of \( I \) that marks the beginning of an affine stretch in its graph, thus breaking strictly convexity of \( I \). Summing up, no model that exhibits a dynamical phase transition falls within the scope of the Gärtner–Ellis theorem. We shall find several dynamical phase transitions in the context of the reset Brownian motion, which are perfectly tackled by theorem 3.3.

4. Applications to the Brownian motion

In this section we discuss applications of theorems 3.1, 3.2, and 3.3 to the positive occupation time, the area, and the absolute area of the reset Brownian motion. From now on, \( \{B_{1,t}\}_{t \geq 0}, \{B_{2,t}\}_{t \geq 0}, \ldots \) are assumed to be standard Brownian motions. No restriction is imposed on the renewal process defining the reset mechanism. The fundamental property of the Brownian motion that allows explicit calculations is the fact that, for each number \( a \geq 0 \), the process \( \{B_{1,a}\}_{t \geq 0} \) is distributed as \( \{\sqrt{a}B_{1,t}\}_{t \geq 0} \). We shall tacitly make use of this property of self-similarity many times.

Before going into specific issues, we provide an overall idea of what assumptions on waitings times are required by theorems 3.1 and 3.3 and by corollary 3.1 to apply to an additive functional of the reset Brownian motion with function \( f \) of at most polynomial growth. In fact, the following lemma states conditions on the waiting times that imply \( \mathbb{E}[M_1^\alpha] < +\infty \) for some \( \rho > 0 \) or \( \mathbb{E}[e^{\rho M_1}] < +\infty \) for all \( \rho > 0 \). The proof is presented in appendix F.

**Lemma 4.1.** Assume that \( |f(z)| \leq A(1 + |z|^\alpha) \) for all \( z \in \mathbb{R} \) with constants \( A > 0 \) and \( \alpha \geq 0 \). The following conclusions hold:

(a) if \( \mathbb{E}[S_{1,\rho}^{\alpha/(1+\alpha/2)}] < +\infty \) for some \( \rho > 0 \), then \( \mathbb{E}[M_1^\alpha] < +\infty \);
(b) if \( \alpha \in [0, 2) \) and \( \limsup_{s \to +\infty} s^{-\alpha/2} \ln \mathbb{P}[S_1 > s] = -\infty \), then \( \mathbb{E}[e^{\rho M_1}] < +\infty \) for all \( \rho > 0 \).

Suppose that \( |f(z)| \leq A(1 + |z|^\alpha) \) for all \( z \in \mathbb{R} \). Theorems 3.1 and 3.2 in combination with lemma 4.1 show that the SLLN and the CLT hold for \( F_t := \int_0^t f(Z_r) \, dr \) provided that \( \mathbb{E}[S_{1,\rho}^{\alpha/(1+\alpha/2)}] < +\infty \) and \( \mathbb{E}[S_{1,\rho}^{\alpha/2}] < +\infty \), respectively. If moreover \( \alpha \in [0, 2) \) and \( \limsup_{s \to +\infty} s^{-\alpha/2} \ln \mathbb{P}[S_1 > s] = -\infty \), then corollary 3.1 of theorem 3.3 and lemma 4.1 imply that \( F_t \) satisfies a full LDP with the good rate function \( I \) given by (3.6). The restriction \( \alpha < 2 \) is due to the Gaussian tails of the Brownian motion. We have \( \alpha = 0 \) for the positive occupation time and \( \alpha = 1 \) for the area and the absolute area.

4.1. Positive occupation time of the reset Brownian motion

The positive occupation time of the reset Brownian motion is the additive functional

\[
F_t := \int_0^t \mathbb{1}_{\{Z_r > 0\}} \, dt.
\]
The associated reward \( X_t := \int_0^{S_t} \mathbb{1}_{(B_t, > 0)} \, d\tau = S_t \int_0^1 \mathbb{1}_{(B_t, > 0)} \, d\tau \) has the property that \( X_t / S_t \) is statistically independent of \( S_t \) and distributed as \( \int_0^1 \mathbb{1}_{(B_t, > 0)} \, d\tau \in [0, 1] \). The Lévy’s arcsine law for the Wiener process \([46]\) gives the distribution of \( \int_0^1 \mathbb{1}_{(B_t, > 0)} \, d\tau \), and hence of \( X_t / S_t \); for each \( x \in [0, 1] \)

\[
P\left[ \frac{X_t}{S_t} \leq x \right] = \frac{2}{\pi} \arcsin \sqrt{x}.
\]

We find \( E[X_t] = E[S_t] / 2 \), \( E[S_t X_t] = E[S_t^2] / 2 \), and \( E[X_t^2] = 3E[S_t^2] / 8 \). As far as the typical behavior of \( F_t \) is concerned, theorem 3.1 tells us that the positive occupation time \( F_t \) satisfies

\[
\lim_{t \to +\infty} \frac{F_t}{t} = \frac{1}{2} \text{ P-a.s.}
\]

provided that \( E[S_t] < +\infty \). If \( E[S_t^2] < +\infty \), then theorem 3.2 gives for every \( z \in \mathbb{R} \)

\[
\lim_{t \to +\infty} \mathbb{P} \left[ \frac{F_t - t/2}{\sqrt{t}} \leq z \right] = \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx
\]

with

\[
v = \frac{1}{8} \frac{E[S_t^2]}{E[S_t]}.
\]

Regarding the large fluctuations, we can invoke the Lévy’s arcsine law again to get for every \( t > 0 \) and \( k \in \mathbb{R} \)

\[
\mathcal{E}_t(k) := \mathbb{E} \left[ e^{\frac{k}{2} \int_0^t \mathbb{1}_{(B_s, > 0)} \, ds} \right] = \mathbb{E} \left[ e^{\frac{kt}{2} \int_0^t \mathbb{1}_{(B_s, > 0)} \, ds} \right] = \int_0^1 e^{\frac{kx}{2} \mathbb{1}_{(1-x, > 0)}} \, dx.
\]

Starting from this formula, it is a simple exercise to prove that \( \lim_{t \to +\infty} \frac{1}{2} \ln \mathcal{E}_t(k) = 0 \lor k \) for all \( k \), which states in particular that \( \frac{1}{2} \ln \mathcal{E}_t(k) \) has a finite limit when \( t \) is sent to infinity. Thus, corollary 3.2 of theorem 3.3 yields the following LDP.

**Proposition 4.1.** The positive occupation time \( F_t \) of the reset Brownian motion satisfies a full LDP with the rate function \( I \) given by (3.6) for any resetting protocol. Moreover, its asymptotic logarithmic moment generating function exists and equals \( -\varphi \).

The rest of the section is devoted to characterize \( I \). All the information about the large fluctuations of the positive occupation time \( F_t \) is contained in the function \( \Phi \) that associates the pair \((\zeta, k) \in \mathbb{R}^2\) with

\[
\Phi(\zeta, k) := \mathbb{E} \left[ e^{i\zeta X_t + ik X_t} \right] = \int_0^1 \frac{1}{\pi \sqrt{x(1-x)}} \mathbb{E} \left[ e^{i\zeta \xi_s + ik s} \right] \, dx.
\]

This function determines \( \varphi \) through the formula \( \varphi(k) = \sup \{ \zeta \in \mathbb{R} : \Phi(\zeta, k) \leq 1 \} \) and \( I \) by convex conjugation of \( -\varphi \). We recall that \( \varphi \) is concave and upper semicontinuous according to proposition 2.1. We notice that if \( \ell := \lim_{x \to +\infty} -\frac{1}{2} \ln \mathbb{P}[S_t > x] < +\infty \), then \( \Phi(\zeta, k) = +\infty \) for \( \zeta > \ell \), so that \( \varphi(k) \leq \ell \) for all \( k \in \mathbb{R} \). As a consequence, \( F_t \) and the associate cumulative reward \( W \) share the same rate function, meaning that the backward recurrence time cannot affect the large fluctuations of the positive occupation time on an exponential scale. The function \( \varphi \) is finite and satisfies the symmetry \( \varphi(-k) = \varphi(k) + k \) for all \( k \in \mathbb{R} \). In fact, on the one hand \( \Phi(0, k) \leq 1 \) for \( k < 0 \) and \( \Phi(-k, k) \leq 1 \) for \( k \geq 0 \), so that \( \varphi(k) \geq 0 \lor k \) for any \( k \) by definition, and on the other hand \( \Phi(\zeta, -k) = \Phi(\zeta - k, k) \). The symmetry of \( \varphi \) endows \( I \) with the symmetry \( I(1/2 - w) = I(1/2 + w) \) for every \( w \in \mathbb{R} \). We have \( I(w) := \sup_{k \in \mathbb{R}} \{ kw + \varphi(k) \} \geq
\( \sup_{k \in \mathbb{R}} \{ wk - 0 \vee k \} = +\infty \) for \( w < 0 \) or \( w > 1 \), which is really expected as \( F_t/t \in [0, 1] \). The values of \( I \) over the interval \([0, 1]\) containing its effective domain can be computed explicitly for Poissonian resetting.

**Example 4.1.** Under Poissonian resetting, i.e. \( \mathbb{P}[S_1 > s] = e^{-rs} \) for all \( s > 0 \) with some resetting rate \( r > 0 \), for each pair \((\zeta, k)\) we find

\[
\Phi(\zeta, k) = \int_0^1 \frac{\mathbb{E}[e^{(\zeta + k\lambda)S_1}]}{\pi \sqrt{x(1-x)}} \, dx = \begin{cases} 
\int_0^1 \frac{r}{r - \zeta - kx} \frac{dx}{\pi \sqrt{x(1-x)}} & \text{if } r - \zeta > 0 \vee k, \\
+\infty & \text{otherwise}
\end{cases}
\]

Thus, for all \( k \in \mathbb{R} \)

\[
\varphi(k) = r - \frac{k + \sqrt{k^2 + 4r^2}}{2}
\]

and for every \( w \in [0, 1] \)

\[
I(w) = r \left[ 1 - 2\sqrt{w(1-w)} \right].
\]

The rate function \( I \) depends linearly on the resetting rate. This rate function has been previously determined in [4] by direct inspection at finite times \( t \) of the distribution of the positive occupation time. In fact, an explicit formula exists for this distribution in the special case of Poissonian resetting.

In order to describe the rate function \( I \) beyond Poissonian resetting we need to examine in detail the properties of the function \( \varphi \). When \( 0 < \ell < +\infty \) these properties are characterized by the two extended real numbers

\[
\Lambda := \mathbb{E} \left[ \frac{e^{\ell S_1}}{\sqrt{1 + S_1}} \right]
\]

and

\[
\Xi := \mathbb{E} \left[ \sqrt{S_1 e^{\ell S_1}} \right].
\]

We let the following lemma to present the relevant features of \( \varphi \). The proof is reported in appendix G. The lemma shows in particular that the asymptotic logarithmic moment generating function \( g = -\varphi \) is not differentiable in the interesting case \( 0 < \ell < +\infty, \, \Lambda < +\infty, \) and \( \Xi < +\infty \). In this case, the large fluctuations of the positive occupation time of the reset Brownian motion cannot be faced with the Gärtner–Ellis theory.

**Lemma 4.2.** The following conclusions hold:

(a) if \( \ell = +\infty \) or \( 0 < \ell < +\infty \) and \( \Lambda = +\infty \), then \( \varphi(k) \) solves the equation \( \mathbb{E}[e^{(k)S_1 + kX_1}] = 1 \) for every \( k \in \mathbb{R} \). The function \( \varphi \) is analytic throughout \( \mathbb{R} \) with

\[
-\varphi'(k) = \frac{\mathbb{E}[X_1 e^{(k)S_1 + kX_1}]}{\mathbb{E}[S_1 e^{(k)S_1 + kX_1}]}.
\]

and the limits \( \lim_{k \downarrow -\infty} -\varphi'(k) = 0 \) and \( \lim_{k \uparrow +\infty} -\varphi'(k) = 1 \) hold true.
(b) if $0 < \ell < +\infty$ and $\Lambda < +\infty$, then there exists $\lambda > 0$ such that $\mathbb{E}[e^{S_1-\lambda X_1}] = 1$ and $\varphi(k)$ solves the equation $\mathbb{E}[e^{\varphi(k) S_1 + \Lambda X_1}] = 1$ for $|k| < \lambda$, whereas $\varphi(k) = \ell \wedge (\ell - k)$ for $|k| \geq \lambda$.

The function $\varphi$ is analytic on the open interval $(-\lambda, \lambda)$ with

$$-\varphi'(k) = \frac{\mathbb{E}[X_1 e^{\varphi(k) S_1 + \Lambda X_1}]}{\mathbb{E}[S_1 e^{\varphi(k) S_1 + \Lambda X_1}]}.$$ 

If $\Xi = +\infty$, then $\varphi$ is differentiable throughout $\mathbb{R}$. If $\Xi < +\infty$, then $\varphi$ is not differentiable at $-\lambda$ and $\lambda$ with finite left derivatives

$$-\varphi_-(k) = \begin{cases} 0 & \text{for } k = -\lambda, \\ \lim_{h \uparrow \lambda} -\varphi'(k) = \frac{\mathbb{E}[X_1 e^{\varphi(k) S_1 + \Lambda X_1}]}{\mathbb{E}[S_1 e^{\varphi(k) S_1 + \Lambda X_1}]} < 1 & \text{for } k = \lambda, \end{cases}$$

and finite right derivatives

$$-\varphi_+(k) = \begin{cases} \lim_{h \downarrow \lambda} -\varphi'(k) = \frac{\mathbb{E}[X_1 e^{\varphi(k) S_1 + \Lambda X_1}]}{\mathbb{E}[S_1 e^{\varphi(k) S_1 + \Lambda X_1}]} > 0 & \text{for } k = -\lambda, \\ 1 & \text{for } k = \lambda. \end{cases}$$

(c) if $\ell = 0$, then $-\varphi(k) = 0$ or $k$.

We are now in the position to discuss the graph of $I$. We remind that if $w$ is a subgradient of the convex function $-\varphi$ at the point $k$, then $wh + \varphi(h) \leq wk + \varphi(k)$ for all $h \in \mathbb{R}$ and $I(w) := \sup_{h \in \mathbb{R}} \{wh + \varphi(h)\} \leq wk + \varphi(k)$, which implies $I(w) = wk + \varphi(k)$. We refer to [45] for general properties of convex functions that sometimes we use. If $\ell = +\infty$ or $0 < \ell < +\infty$ and $\Lambda = +\infty$, then for each given $w \in (0, 1)$ there exists $k \in \mathbb{R}$ such that $w = -\varphi'(k)$ as part (a) of lemma 4.2 states that $-\varphi'$ increases continuously from 0 at $k = -\infty$ to 1 at $k = +\infty$. Thus, $w$ is a subgradient of $-\varphi$ at $k$ and we have $I(w) = wk + \varphi(k)$. Then, in the case $\ell = +\infty$ or $0 < \ell < +\infty$ and $\Lambda = +\infty$, $I$ turns out to be analytic on the open interval $(0, 1)$ by the analytic implicit function theorem. The values of $I$ at the points $w = 0$ and $w = 1$ are determined by the limits $I(0) = \lim_{w \downarrow 0} I(w)$ and $I(1) = \lim_{w \uparrow 1} I(w)$, which hold because $I$ is convex and lower semicontinuous (see [45], corollary 7.5.1). The same conclusions are valid when $0 < \ell < +\infty$, $\Lambda < +\infty$, and $\Xi = +\infty$. In fact, in this case $-\varphi'$ increases continuously from 0 at $k = -\lambda$ to 1 at $k = \lambda$ by part (b) of lemma 4.2. The rate function is again analytic on $(0, 1)$ and, moreover, $I(0) = \varphi(-\lambda)$ as $0$ is a subgradient of $-\varphi$ at $k = -\lambda$ and $I(1) = \lambda + \varphi(\lambda)$ as $1$ is a subgradient of $-\varphi$ at $k = \lambda$. We stress that Poissonian resetting falls in the class $0 < \ell = r < +\infty$ and $\Lambda = +\infty$.

The graph of the rate function $I$ is characterized by two singularities and two affine stretches when $0 < \ell < +\infty$, $\Lambda < +\infty$, and $\Xi < +\infty$. For instance, this occurs with the waiting times described by the laws

$$P[S_1 > s] = \frac{e^{-s}}{1 + s^\alpha}$$

and

$$P[S_1 > s] = e^{-\beta s}$$

for all $s \geq 0$ with $\alpha > 3/2$ and $\beta \in (0, 1)$. In fact, according to part (b) of lemma 4.2, in this case all points in $[0, w_-]$ with $w_- := -\varphi_-'(-\lambda)$ are subgradient of $-\varphi$ at $-\lambda$ and all points in $[w_+, 1]$ with $w_+ := -\varphi_+'(\lambda)$ are subgradient of $-\varphi$ at $\lambda$ (see [45], theorem 23.2). Thus,
\[ I(w) = -w\lambda + \varphi(-\lambda) \] for all \( w \in [0, w_-] \) and \( I(w) = w\lambda + \varphi(\lambda) \) for all \( w \in [w_+, 1] \), so that two affine stretches emerge. Moreover, as \(-\varphi'\) increases continuously from \( w_- \) at \( k = -\lambda \) to \( w_+ \) at \( k = \lambda \), for each \( w \in (w_-, w_+) \) there exists \( k \in (-\lambda, \lambda) \) such that \( w = -\varphi'(k) \), and \( I(w) = wk + \varphi(k) \) follows. The rate function \( I \) is analytic over the open interval \((w_-, w_+)\) by the analytic implicit function theorem. It is clear that \( w_- < \mu = 1/2 \) and \( w_+ > \mu \) as \( I(\mu) = 0 \) by theorem 3.3. In conclusion, the points \( w_- \) and \( w_+ \) are singular points for \( I \) that mark some form of dynamical phase transition.

Dynamical phase transitions under resetting have been previously documented for the total displacement of certain random walks [33]. The same type of singular behavior has been found for the macroscopic observables of renewal models of statistical mechanics, such as the pinning model of polymers or the Poland-Scheraga model, under equilibrium conditions [24, 47]. All these models share a common regenerative structures, whether they are kinetic models with few degrees of freedom or equilibrium models with many degrees of freedom. Although the characterization of these phase transitions is beyond the scope of the present work, we can try to give an explanation by drawing an analogy with sums of i.i.d. random variables and the phenomenon of condensation of their fluctuations [48–51]. After all, \( F_t \) is close to the cumulative reward \( W_t \), which is a random sum of i.i.d. random variables. If \( F_t \) were exactly a sum of i.i.d. random variables, then we would conclude that a fluctuation \( w \) of \( F_t / t \) above \( w_+ > \mu \) is realized by the combination of two different mechanisms: many small deviations of the waiting times all in the same direction for a partial excursion up to \( w_+ \), and a big jump of only one of the waiting times for the remaining part \( w - w_+ \) of the fluctuation. The same would occur for a fluctuation of \( F_t / t \) below \( w_- < \mu \). We believe that this picture also describes the dynamical phase transitions of the positive occupation time of the reset Brownian motion.

Finally, the case \( \ell = 0 \) corresponds to a heavy-tailed waiting time and gives \( I(w) = 0 \) for all \( w \in [0, 1] \). This rate function simply tells us that the decay with time of the probability of a large fluctuation of the positive occupation time is slower than exponential for heavy-tailed waiting times. In this case, the description of the large fluctuations requires more precise asymptotic results for specific classes of waiting times. Still regarding renewal-reward processes as primary tools, we mention that precise large deviation principles for renewal-reward processes have been established when rewards are independent of waiting times for several types of heavy-tailed waiting times and rewards [52–57]. Some forms of dependence between rewards and waiting times have been considered in [59, 60].

4.2. Area of the reset Brownian motion

The area of the reset Brownian motion is the additive functional

\[ F_t := \int_0^t Z_{-}\,d\tau. \]

The corresponding reward \( X_1 := \int_0^t B_{1,\tau} \,d\tau \) enjoys the property that \( \sqrt{3/S_1}X_1 \) is statistically independent of \( S_1 \) and distributed as \( \sqrt{3/S_1} \int_0^t B_{1,\tau} \,d\tau \), which in turn is distributed as a Gaussian random variable with mean 0 and variance 1 [61]. In order to characterize the typical behavior of the area \( F_t \) we observe that \( \mathbb{E}[X_1] = 0 \) if \( \mathbb{E}[S_1^{3/2}] < +\infty \) and \( \mathbb{E}[X_1] = \mathbb{E}[S_1^{3/2}]/3 \) if \( \mathbb{E}[S_1] < +\infty \). Furthermore, since \( \mathbb{E}\left[ \int_0^t |B_{1,\tau}| \,d\tau \right] = \sqrt{8/9\pi} \) and \( \mathbb{E}\left[ \left( \int_0^t |B_{1,\tau}| \,d\tau \right)^2 \right] = 3/8 \) [61], we realize that \( \mathbb{E}[M_1^1] = \mathbb{E}\left[ \int_0^t |B_{1,\tau}| \,d\tau \right] = \mathbb{E}[S_1^{3/2}] \int_0^t |B_{1,\tau}| \,d\tau = \sqrt{8/9\pi} \mathbb{E}[S_1^{3/2}] \) and, similarly, that \( \mathbb{E}[M_2^1] = (3/8) \mathbb{E}[S_1^{3/2}] \). Thus, theorem 3.1 states that \( F_t \) satisfies
\[
\lim_{t \to +\infty} \frac{F_t}{t} = 0 \quad \mathbb{P}\text{-a.s.}
\]
whenever \(E[S_{1}^{3/2}] < +\infty\). Under the hypothesis \(E[S_{1}^{2}] < +\infty\), theorem 3.2 shows that for every \(z \in \mathbb{R}\)
\[
\lim_{t \to +\infty} \mathbb{P} \left[ \frac{F_t}{\sqrt{V_t}} \leq z \right] = \int_{-\infty}^{z} e^{-t^{2}/2} \, \text{d}x
\]
with
\[
\nu = \frac{1}{3} \frac{E[S_{1}^{3}]}{E[S_{1}^{2}]},
\]
As mentioned at the end of section 3.1, the hypothesis \(E[M_{1}^{2}] < +\infty\) is the same as \(E[X_{1}^{2}] < +\infty\) although the function \(f(z) := z\) that defines the area takes both positive and negative values.

Moving to the large fluctuations of \(F_t\), we observe that for all \(t \geq 0\) and \(k \in \mathbb{R}\)
\[
E_{t}(k) := \mathbb{E} \left[ e^{k \int_{0}^{t} B_{1,-} \, \text{d}\tau} \right] = \mathbb{E} \left[ e^{k^{1/2} \int_{0}^{t} B_{1,-} \, \text{d}\tau} \right] = e^{k^{2} t}.
\]
Thus, \(\frac{1}{t} \ln E_{t}(k) = (\frac{k^{2}}{6}) t\) is non-decreasing with respect to \(t\) and corollary 3.2 of theorem 3.3 gives the following result.

**Proposition 4.2.** The area \(F_t\) of the reset Brownian motion satisfies a full LDP with the rate function \(I\) given by (3.6) for any resetting protocol. Moreover, its asymptotic logarithmic moment generating function exists and equals \(-\varphi\).

The function \(\varphi\) that defines \(I\) by convex conjugation now reads
\[
\varphi(k) := \sup \left\{ \zeta \in \mathbb{R} : \mathbb{E} \left[ e^{\zeta S_{1} + k X_{1}} \right] \leq 1 \right\} = \sup \left\{ \zeta \in \mathbb{R} : \mathbb{E} \left[ e^{\zeta S_{1} + k^{2} S_{1}^{2}} \right] \leq 1 \right\}
\]
and satisfies the symmetry \(\varphi(-k) = \varphi(k)\). This symmetry gives \(I(-w) = I(w)\) for all \(w \in \mathbb{R}\).
We point out that if \(\ell < +\infty\), then \(\varphi(k) = 0\) for \(k = 0\) and \(\varphi(k) = -\infty\) for \(k \neq 0\). Thus, as in the case of the positive occupation time, we have \(\varphi(k) \leq \ell\) for all \(k \in \mathbb{R}\), showing that \(F_t\) and \(W_t\) satisfy a full LDP with the same rate function. However, if \(\ell < +\infty\), then \(I(w) = 0\) for every \(w\) and the probability of a large fluctuation decays slower than exponential in time.

We want to characterize the rate function \(I\), and a preliminary study of \(\varphi\) is needed to this aim. The relevant rate is not \(\ell\), as for the positive occupation time, but rather \(r\) defined by
\[
r := \liminf_{t \to +\infty} \frac{1}{s^{3}} \ln \mathbb{P}[S_{1} > s].
\]
When \(r\) is finite, we set for brevity \(\xi := \varphi(\sqrt{6r})\). When \(r\) and \(\xi\) are finite, we introduce the real number
\[
\Lambda := \mathbb{E} \left[ e^{\xi S_{1} + r X_{1}} \right] = \mathbb{E} \left[ e^{\varphi(\sqrt{6r}) S_{1} + \sqrt{6r} X_{1}} \right] \leq 1
\]
and the extended real number
\[
\Xi := \mathbb{E} [S_{1}^{2} e^{-\xi S_{1} + r X_{1}}].
\]
The following lemma, which is proved in appendix H, collects the main features of the function \(\varphi\) and puts \(\xi\), \(\Lambda\), and \(\Xi\) into context. It demonstrates in particular that the asymptotic logarithmic moment generating function \(g = -\varphi\) is not steep in the interesting case \(0 < r < +\infty, \xi > -\infty, \Lambda = 1, \text{and} \Xi < +\infty\). The Gärtner–Ellis theory does not apply in this case.
Lemma 4.3. The following conclusions hold:

(a) if \( r = +\infty \), then \( \varphi(k) \) solves the equation \( \mathbb{E}[\varphi(k)S_k + \frac{1}{2}k^2 \xi^2] = 1 \) for every \( k \in \mathbb{R} \). The function \( \varphi \) is analytic throughout \( \mathbb{R} \) with

\[
-\varphi'(k) = \frac{k}{3} \mathbb{E}[\varphi(k)S_k + \frac{1}{2}k^2 \xi^2],
\]

and the limits \( \lim_{k \to -\infty} -\varphi'(k) = -\infty \) and \( \lim_{k \to +\infty} -\varphi'(k) = +\infty \) hold true;

(b) if \( 0 < r < +\infty \), then \( \varphi(k) \) solves the equation \( \mathbb{E}[\varphi(k)S_k + \frac{1}{2}k^2 \xi^2] = 1 \) for \( |k| < \sqrt{6r} \), whereas \( \varphi(k) = -\infty \) for \( |k| > \sqrt{6r} \). The function \( \varphi \) is analytic on the open interval \((-\sqrt{6r}, \sqrt{6r})\) with

\[
-\varphi'(k) = \frac{k}{3} \mathbb{E}[\varphi(k)S_k + \frac{1}{2}k^2 \xi^2],
\]

The derivative \( \varphi' \) has the limits

\[
\lim_{k \to -\sqrt{6r}} -\varphi'(k) = \begin{cases} -\sqrt{\frac{2r}{3} \mathbb{E}[\varphi(k)S_k + \frac{1}{2}k^2 \xi^2]} & \text{if } \xi = -\infty, \Lambda = 1, \text{ and } \Xi < +\infty, \\ -\infty & \text{otherwise} \end{cases}
\]

and

\[
\lim_{k \to +\sqrt{6r}} -\varphi'(k) = \begin{cases} \sqrt{\frac{2r}{3} \mathbb{E}[\varphi(k)S_k + \frac{1}{2}k^2 \xi^2]} & \text{if } \xi = -\infty, \Lambda = 1, \text{ and } \Xi < +\infty, \\ +\infty & \text{otherwise} \end{cases}
\]

(c) if \( r = 0 \), then \( \varphi(k) = 0 \) for \( k = 0 \) and \( \varphi(k) = -\infty \) for \( k \neq 0 \).

We can now discuss the graph of \( I \). If \( r = +\infty \), or \( 0 < r < +\infty \) and \( \xi = -\infty \), or \( 0 < r < +\infty \) and \( \xi > -\infty \) and \( \Lambda < 1 \), or \( 0 < r < +\infty \) and \( \xi > -\infty \) and \( \Lambda = 1 \) and \( \Xi = +\infty \), then parts (a) and (b) of Lemma 4.3 state that the convex function \( -\varphi \) is differentiable with continuously increasing derivative from \( -\infty \) at \( k = -\sqrt{6r} \) to \( +\infty \) at \( k = \sqrt{6r} \). We agree that \( -\sqrt{6r} = -\infty \) and \( \sqrt{6r} = +\infty \) when \( r = +\infty \). As a consequence, for each \( w \in \mathbb{R} \) there exists \( k \in (-\sqrt{6r}, \sqrt{6r}) \) such that \( w = -\varphi'(k) \), and \( I(w) = wk + \varphi(k) \) follows. The rate function \( I \) turns out to be analytic throughout \( \mathbb{R} \) and good in all these cases.

Singularity emerges when \( 0 < r < +\infty \), \( \xi > -\infty \), \( \Lambda = 1 \), and \( \Xi < +\infty \), such as for the waiting times of example 4.2 below. By setting

\[
w_{\pm} := \pm \sqrt{\frac{2r}{3} \mathbb{E}[\varphi^2 S_k + \frac{1}{2}k^2 \xi^2]},
\]

and by invoking part (b) of Lemma 4.3, we have that for each \( w \in (w_-, w_+) \) there exists \( k \in (-\sqrt{6r}, \sqrt{6r}) \) such that \( w = -\varphi'(k) \), in such a way that \( I(w) = wk + \varphi(k) \). The rate function is analytic on the open interval \((w_-, w_+)\). For \( w \geq w_+ \), the function that maps \( k \in [-\sqrt{6r}, \sqrt{6r}] \) to \( wk + \varphi(k) \) is non-decreasing, whereas \( \varphi(k) = -\infty \) for \( k \notin [-\sqrt{6r}, \sqrt{6r}] \). Thus, \( I(w) = \sup_{k \in \mathbb{R}} (wk + \varphi(k)) \) is \( w_{\sqrt{6r}} + \varphi(\sqrt{6r}) = w_{\sqrt{6r}} + \xi \) for all \( w \geq w_+ \). Similarly, \( I(w) = -w_{\sqrt{6r}} + \xi \) for all \( w \leq w_- \). The graph of \( I \) is now characterized by two affine stretches.
with extremes $w_-$ and $w_+$, which are associated with some dynamical phase transitions. As for the positive occupation time, we believe that the mechanism underlying these dynamical phase transitions is the phenomenon of condensation of fluctuations. We leave a quantitative description as an open problem to be addressed in future research.

To conclude, we observe that $I(w) = 0$ for all $w \in \mathbb{R}$ if $r = 0$. This case, which includes Poissonian resetting as already observed in [4], needs more precise asymptotic results to be devised for specific classes of waiting times.

**Example 4.2.** The simplest waiting time distribution that brings about a full LDP with good rate function for the area is the super-exponential distribution

$$P[S_1 > s] = e^{-rs}$$

for all $s \geq 0$ with some ‘resetting rate’ $r > 0$. We discuss this example in detail. While $\varphi(k) = -\infty$ for $|k| > \sqrt{6r}$, for $k \in [-\sqrt{6r}, \sqrt{6r}]$ it turns out that $\varphi(k)$ solves the equation

$$E[e^{\varphi(k)S_1 + \frac{1}{6}k^3S_1^3}] = \int_0^{+\infty} 3rs^2e^{\varphi(k)s + \frac{1}{3}k^3s^3} ds = 1.$$ (4.2)

We find $\xi := \varphi(\sqrt{6r}) = -(6r)^{\frac{1}{2}} > -\infty$, $\Lambda = 1$, and $\Xi = 10r^{-2} < +\infty$. Thus, two singularities and two affine stretches occur. In fact, for every $w \in \mathbb{R}$ the rate function reads

$$I(w) = \begin{cases} -w\sqrt{6r} + \xi & \text{if } w \leq w_-, \\ wk + \varphi(k) & \text{if } w \in (w_-, w_+), \\ w\sqrt{6r} + \xi & \text{if } w \geq w_+ \end{cases}$$

with $w_+ = \pm \frac{20}{3}(6r)^{-\frac{1}{2}}$ and $k \in (-\sqrt{6r}, \sqrt{6r})$ such that $w + \varphi'(k) = 0$.

This example offers a possibility to understand how the rate function behaves in the limit of zero resetting. Let us write $\varphi_-$ and $I_\varphi$ in place of $\varphi$ and $I$ to stress the dependence on $r$. We already know that $I_0 = 0$. The change of variable $s \mapsto r^{-4}s$ in (4.2) shows that $\varphi_-(k) = r^4\varphi_1(r^{-2}k)$ for all $k$. It follows that for every $w \in \mathbb{R}$

$$I_r(w) := \sup_{k \in \mathbb{R}} \{wk + \varphi_-(k)\} = r^4 \sup_{k \in \mathbb{R}} \{r^4wk + \varphi_1(k)\} = r^4I_r(r^4w).$$

The analysis of $I_1$ in a neighborhood of the origin is a simple exercise, which allows one to conclude that there exists a positive constant $C$ such that

$$|I_1(w) - \frac{\Gamma(1/3)}{2}w^2| \leq Cw^4$$

for any $w$, $\Gamma$ being the Euler gamma function. Thus, for all $r > 0$ and $w \in \mathbb{R}$ we find

$$|I_r(w) - \frac{\Gamma(1/3)}{2}r^4w^2| \leq Crw^4,$$

which states that $I_r(w)$ approaches the value 0 with speed $r^4$ when $r$ goes to 0.

### 4.3. Absolute area of the reset Brownian motion

The absolute area of the reset Brownian motion is the additive functional

$$F_r := \int_0^t |Z_r| \, dr$$

with extremes $w_-$ and $w_+$, which are associated with some dynamical phase transitions. As for the positive occupation time, we believe that the mechanism underlying these dynamical phase transitions is the phenomenon of condensation of fluctuations. We leave a quantitative description as an open problem to be addressed in future research.

To conclude, we observe that $I(w) = 0$ for all $w \in \mathbb{R}$ if $r = 0$. This case, which includes Poissonian resetting as already observed in [4], needs more precise asymptotic results to be devised for specific classes of waiting times.
with reward \( X_1 := \int_0^S |B_{1,\tau}| \, d\tau \). Repeating some of the arguments of the beginning of section 4.2, we find \( \mathbb{E}[X_1] = \sqrt{8/9\pi} \mathbb{E}[S_{1/2}^2] \), \( \mathbb{E}[X_1^2] = (3/8) \mathbb{E}[S_1^1] \), and \( \mathbb{E}[S_1 X_1] = \sqrt{8/9\pi} \mathbb{E}[S_{1/2}^2] \). Then, regarding the typical behavior of the absolute area \( F_t \), theorem 3.1 states that

\[
\lim_{t \to +\infty} \frac{F_t}{t} = \mu = \sqrt{\frac{8}{9\pi} \frac{\mathbb{E}[S_{1/2}^2]}{\mathbb{E}[S_1]}} \quad \text{P.-a.s.}
\]

if \( \mathbb{E}[S_{1/2}^2] < +\infty \). When \( \mathbb{E}[S_1^1] < +\infty \), theorem 3.2 tells us that for all \( z \in \mathbb{R} \)

\[
\lim_{t \to +\infty} \mathbb{P} \left[ \frac{F_t}{\sqrt{\mathbb{E}[S_1]}} \geq z \right] = \int_{-\infty}^z e^{-\frac{1}{2} x^2} \, dx
\]

with

\[
v = 3 \frac{\mathbb{E}[S_1^3]}{\mathbb{E}[S_1]} - \mu \sqrt{\frac{32}{9\pi}} \frac{\mathbb{E}[S_{1/2}^2]}{\mathbb{E}[S_1]} + \mu^2 \frac{\mathbb{E}[S_{1/2}^2]}{\mathbb{E}[S_1]}
\]

The study of the large fluctuations of \( F_t \) requires to introduce some further mathematical details. The Airy function \( \text{Ai} \) is the analytic function that maps \( x \in \mathbb{R} \) in

\[
\text{Ai}(x) := \frac{1}{\pi} \int_0^{+\infty} \cos \left( xu + \frac{u^3}{3} \right) \, du.
\]

The derivative \( \text{Ai}' \) of \( \text{Ai} \) has countably many simple zeros \( 0 > z_1 > z_2 > \cdots \) on the negative real axis [61, 62]. For each \( i \geq 1 \) we set \( \nu_i := 2^{-1/3} |z_i| \) and

\[
c_i := \frac{1 + 3 \int_0^i \text{Ai}(x) \, dx}{3 \text{Ai}''(z_i)} = \frac{1 + 3 \int_0^i \text{Ai}(x) \, dx}{3 |z_i| \text{Ai}(z_i)}.
\]

We have \( \nu_1 = 0.80861 \ldots \) and \( c_1 = 1.48257 \ldots [62] \). The asymptotics of \( \nu_i \) and \( c_i \) is characterized by the limits \( \lim_{i \to \infty} (2\pi i)^{-2/3} \nu_i = 2 \) and \( \lim_{i \to \infty} (-1)^i \sqrt{3i/2} c_i = -1 [62] \). Our interest in the Airy function stems from the fact that the moment generating function of \( \int_0^1 |B_{1,\tau}| \, d\tau \) takes the value [61, 62]

\[
\mathbb{E} \left[ e^{k |B_{1,\tau}|} \right] = \sum_{i=1}^\infty c_i e^{-\nu_i |k|^{2/3}}
\]

for \( k < 0 \). The following lemma is proved in appendix I and completes the picture for \( k \geq 0 \).

**Lemma 4.4.** There exists a positive constant \( L \) such that for all \( k \geq 0 \)

\[
e^{k^2} \leq \mathbb{E} \left[ e^{k |B_{1,\tau}|} \right] \leq LE^{k^2}.
\]

**Formula 4.3** shows that for all \( t > 0 \) and \( k < 0 \)

\[
\mathbb{E}_t(k) := \mathbb{E} \left[ e^{k^{3/2} |B_{1,\tau}|} \right] = \mathbb{E} \left[ e^{k^{1/2} |B_{1,\tau}|} \right] = \sum_{i=1}^\infty c_i e^{-\nu_i |k|^{2/3}},
\]

so that \( \lim_{t \to +\infty} \frac{1}{t} \ln \mathbb{E}_t(k) = -\nu_i |k|^{2/3} \) exists and is finite for \( k \leq 0 \). Lemma 4.4 entails \( \lim_{t \to +\infty} \frac{1}{t} \ln \mathbb{E}_t(k) = +\infty \) for every \( k > 0 \). We claim that \( \frac{1}{t} \ln \mathbb{E}_t(k) \) is non-decreasing with respect to \( t \) for \( k > 0 \). To show this, fix \( k > 0 \) and \( q > 1 \). Hölder’s inequality gives for all \( t > 0 \)
\[
\{ \mathcal{E}_t(k) \}^q = \left\{ \mathbb{E} \left[ e^{k \int_0^t \mathbb{1}_{[0,t)}(dr)} \right] \right\}^q \leq \mathbb{E} \left[ e^{k \int_0^t \mathbb{1}_{[0,t)}(dr)} \right] = \mathbb{E} \left[ e^{k t / 2} \right] - \mathcal{E}_t(k).
\]

Choosing \( q = (s/t)^{3/2} \) with \( s > t \) and observing that \( \ln \mathcal{E}_t(k) \geq 0 \) we find

\[
\frac{1}{t} \ln \mathcal{E}_t(k) \leq \sqrt{\frac{1}{s}} \ln \mathcal{E}_t(k) \leq \frac{1}{s} \ln \mathcal{E}_t(k).
\]

We have thus verified the hypothesis of corollary 3.2 of theorem 3.3, deducing the following LDP.

**Proposition 4.3.** The absolute area \( F_t \) of the reset Brownian motion satisfies a full LDP with the rate function \( I \) given by (3.6) for any resetting protocol. Moreover, its asymptotic logarithmic moment generating function exists and equals \(-\varphi\).

In contrast to the cases of the positive occupation time and the area, the rate functions of \( F_t \) and \( W_t \) can differ for the absolute area, meaning that the backward recurrence time can affect the large fluctuations of \( F_t \). For instance, this occur for Poissonian resetting, where \( \varphi(k) > \ell = r \) for all \( k \leq 0 \) sufficiently large in magnitude. Indeed, in the case of Poissonian resetting, formula (4.3) shows that for \( k < 0 \) and \( \zeta < r + \nu_1 |k|^{2/3} \)

\[
\mathbb{E} [e^{\zeta S_1 + kX_1}] = \int_0^{+\infty} \mathbb{E} \left[ e^{\zeta t / 2} \right] e^{r \zeta - s \zeta} ds = \int_0^{+\infty} \sum_{i=1}^{+\infty} \frac{rc_i}{r + \nu_1 |k|^{2/3} - \zeta}.
\]

The last equality is justified by the fact that \( |c_i| / \nu_1 \) goes to zero as \( i^{-7/6} \) when \( i \) is sent to infinity. Thus, \( \mathbb{E} [e^{\zeta S_1 + kX_1}] < +\infty \) for \( k < 0 \) and \( \zeta < r + \nu_1 |k|^{2/3} \) with the limit \( \lim_{i \to +\infty} \mathbb{E} [e^{\zeta S_1 + kX_1}] = +\infty \), and \( \mathbb{E} [e^{\zeta S_1 + kX_1}] = |k|^{-2/3} \sum_{i=1}^{+\infty} (rc_i / \nu_1) < 1 \) for \( k < 0 \) sufficiently large in magnitude. It follows that \( \varphi(k) \) is for each \( k < 0 \) the unique real number \( \zeta \) that solves the equation \( \mathbb{E} [e^{\zeta S_1 + kX_1}] = 1 \), and that \( \varphi(k) > r \) for all \( k < 0 \) sufficiently large in magnitude.

The rate function \( I \) of the absolute area satisfies \( I(w) = +\infty \) for \( w < 0 \) because the absolute area takes non-negative values. For a formal proof it is enough to observe that \( \varphi(k) \geq 0 \) for all \( k \leq 0 \) by definition, so that \( I(w) := \sup_{k \in \mathbb{R}} \{ wk + \varphi(k) \} \geq \sup_{k \leq 0} \{ wk \} = +\infty \) for \( w < 0 \). On the positive semiaxis the function \( \varphi \) shares the same properties of the corresponding function associated with the area. Consequently, above the mean the rate function \( I \) has the same features of the rate function of the area. To describe them we set

\[
r := \lim \inf_{t \to +\infty} - \frac{1}{s} \ln \mathbb{P}[S_1 > s].
\]

When \( r \) is finite, we write \( \zeta := \varphi(\sqrt{6r}) \) for brevity. When \( r \) and \( \zeta \) are finite, we introduce the real number

\[
\Lambda := \mathbb{E} [e^{\zeta S_1 + \sqrt{6r}X_1}] \leq 1
\]

and the extended real number

\[
\Xi := \mathbb{E} [S_1^3 e^{\zeta S_1 + \sqrt{6r}X_1}].
\]

The following lemma characterizes the function \( \varphi \) through the values of \( \zeta \), \( \Lambda \), and \( \Xi \). The proof is presented in appendix J. Similarly to the area, the asymptotic logarithmic moment
The following conclusions hold:

(a) if \( r = +\infty \), then \( \varphi(k) \) solves the equation \( \mathbb{E}[e^{\varphi(k)S_1+\ell kX}] = 1 \) for every \( k \in \mathbb{R} \). The function \( \varphi \) is analytic throughout \( \mathbb{R} \) with

\[
-\varphi'(k) = \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1+\ell kX}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1+\ell kX}]}.
\]

and the limits \( \lim_{k \to -\infty} -\varphi'(k) = 0 \) and \( \lim_{k \to +\infty} -\varphi'(k) = +\infty \) hold true;

(b) if \( 0 < r < +\infty \), then \( \varphi(k) \) solves the equation \( \mathbb{E}[e^{\varphi(k)S_1+\ell kX}] = 1 \) for \( k < \sqrt{6r} \), whereas \( \varphi(k) = -\infty \) for \( k > \sqrt{6r} \). The function \( \varphi \) is analytic on \( (-\infty, \sqrt{6r}) \) with

\[
-\varphi'(k) = \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1+\ell kX}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1+\ell kX}]}.
\]

The derivative \( \varphi' \) has the limits \( \lim_{k \to -\infty} -\varphi'(k) = 0 \) and

\[
\lim_{k \to \sqrt{6r}} -\varphi'(k) = \begin{cases} \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1+\ell kX}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1+\ell kX}]} & \text{if } \xi > -\infty, \Lambda = 1, \text{ and } \Xi < +\infty, \\ +\infty & \text{otherwise}; \end{cases}
\]

(c) if \( r = 0 \) and \( \ell = +\infty \), or \( \ell < +\infty \) and \( \mathbb{E}[e^{\ell S_1}] = +\infty \), or \( \ell < +\infty \) and \( \mathbb{E}[e^{\ell + \nu|k|^{3/2}}S_1+\ell kX_1] > 1 \) for all \( k < 0 \), then \( \varphi(k) \) solves the equation \( \mathbb{E}[e^{\varphi(k)S_1+\ell kX}] = 1 \) for \( k \leq 0 \), whereas \( \varphi(k) = -\infty \) for \( k > 0 \). The function \( \varphi \) is analytic on \( (0, +\infty) \) with

\[
-\varphi'(k) = \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1+\ell kX}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1+\ell kX}]}.
\]

The derivative \( \varphi' \) has the limits \( \lim_{k \to -\infty} -\varphi'(k) = 0 \) and

\[
\lim_{k \downarrow 0} -\varphi'(k) = \begin{cases} \mu = \sqrt{\frac{\sqrt{8}}{\pi} \mathbb{E}[S_1^{3/2}]} & \text{if } \mathbb{E}[S_1^{3/2}] < +\infty, \\ +\infty & \text{if } \mathbb{E}[S_1^{3/2}] = +\infty. \end{cases}
\]

We point out that part (c) of lemma 4.5 does not consider the case \( \ell < +\infty \), \( \mathbb{E}[e^{\ell S_1}] < +\infty \), and \( \mathbb{E}[e^{(\ell + \nu|k|^{3/2})S_1+\ell kX_1}] \leq 1 \) for some \( k < 0 \). Actually, it seems that this circumstance cannot occur. In fact, numerical simulations suggest that the function that maps \( s > 0 \) in

\[
\mathbb{E}\left[e^{\nu_1 s^{3/2} - s/3}\int_{0}^{[1]}[1_{B_1}]d\tau\right] = \sum_{i=1}^{\infty} c_i e^{-(\nu_i - \nu_1)s^{3/3}}
\]

has limit 1 when \( s \) approaches 0 and is strictly increasing. It follows that

\[
\mathbb{E}\left[e^{(\ell + \nu_1|k|^{3/2})s} + k\int_{0}^{[1]}[1_{B_1}]d\tau\right] \geq \mathbb{E}\left[e^{\nu_1 s^{3/3} - s/3}\int_{0}^{[1]}[1_{B_1}]d\tau\right] > 1
\]

for all \( k < 0 \) and \( s > 0 \), which implies \( \mathbb{E}[e^{(\ell + \nu_1|k|^{3/2})S_1+\ell kX_1}] > 1 \) for every \( k < 0 \). Unfortunately, we are not able to prove rigorously that \( \sum_{i=1}^{\infty} c_i e^{-(\nu_i - \nu_1)s^{3/3}} \) is strictly increasing with respect to \( s > 0 \).
Let us discuss briefly the graph of the rate function $I$ on the region $[0, +\infty)$ containing its effective domain. If $r = +\infty$, or $0 < r < +\infty$ and $\xi = -\infty$, or $0 < r < +\infty$ and $\xi > -\infty$ and $\Lambda < 1$, or $0 < r < +\infty$ and $\xi > -\infty$ and $\Lambda = 1$ and $\Xi = +\infty$, then parts (a) and (b) of lemma 4.5 show that the convex function $-\varphi$ is differentiable with continuously increasing derivative from 0 at $k = -\infty$ to $+\infty$ at $k = 2\sqrt{lr}$. We agree that $\sqrt{lr} = +\infty$ when $r = +\infty$.

As a consequence, for each $w \in (0, +\infty)$ there exists $k \in (\xi, 2\sqrt{lr})$ such that $w = -\varphi'(k)$ and $I(w) = wk + \varphi(k)$. The rate function $I$ turns out to be analytic on $(0, +\infty)$ and good in all these cases. We have $I(0) = \lim_{w \to 0} I(w)$ as $I$ is convex and lower semicontinuous (see [45], corollary 7.5.1).

A singularity emerges when $0 < r < +\infty$, $\xi > -\infty$, $\Lambda = 1$, and $\Xi < +\infty$, as for the area. By setting

$$w_+ := \frac{E[X_1 e^{\xi S_1 + 2\sqrt{lr}X_1}]}{E[S_1 e^{\xi S_1 + 2\sqrt{lr}X_1}]}$$

(4.4)

and by invoking part (b) of lemma 4.3, we have that for each $w \in (0, w_+]$ there exists $k \in (\xi, 2\sqrt{lr})$ such that $w = -\varphi'(k)$, in such a way that $I(w) = wk + \varphi(k)$. The rate function is analytic on the open interval $(0, w_+]$. As before, $I(0) = \lim_{w \to 0} I(w)$. For $w \geq w_+$, the function that maps $k \in (\xi, 2\sqrt{lr})$ in $wk + \varphi(k)$ is non-decreasing, whereas $\varphi(k) = -\infty$ for $k > 2\sqrt{lr}$. Thus, $I(w) = \sup_{k \in (\xi, 2\sqrt{lr})} (wk + \varphi(k)) = w\sqrt{lr} + \varphi(2\sqrt{lr}) = w\sqrt{lr} + \xi$ for all $w \geq w_+$.

The graph of $I$ is now characterized by an affine stretch and a singularity at the point $w_+$. Example 4.3 below discusses a situation where $0 < r < +\infty$, $\xi > -\infty$, $\Lambda = 1$, and $\Xi < +\infty$.

Finally, we consider the case $r = 0$, which differs from the corresponding case of the area because the fluctuations of the absolute area below the mean are characterized by a positive rate function. This fact has been already found in [4] while studying Poissonian resetting. Assuming that $E[e^{(1+\nu)k^{1/2}}X_1 + kX_1] > 1$ for all $k < 0$ when $\ell < +\infty$ and $E[e^{\xi S_1}] < +\infty$ and setting $\mu := +\infty$ if $E[S_1^{1/2}] = +\infty$ for brevity, part (c) of lemma 4.5 states that the convex function $-\varphi$ is differentiable on $(-\infty, 0)$ with continuously increasing derivative from 0 at $k = -\infty$ to $\mu$ at $k = 0$, whereas $\varphi(k) = -\infty$ for $k > 0$. Thus, for each $w \in (0, \mu)$ there exists $k \in (-\infty, 0)$ such that $w = -\varphi'(k)$ and $I(w) = wk + \varphi(k)$. The rate function turns out to be analytic on $(0, \mu)$ and $I(0) = \lim_{w \to 0} I(w)$. We stress that these conclusions are valid even for heavy-tailed waiting times, for which $\ell = 0$. If $\mu < +\infty$, then $I(w) = 0$ for $w \geq \mu$ since the function that maps $k \in (-\infty, 0]$ in $wk + \varphi(k)$ is non-decreasing in this case.

Example 4.3. The rate function $I$ of the absolute area of the reset Brownian motion with Poissonian resetting was studied in [4]. Here we discuss in detail the case

$$P[S_1 > s] = e^{-rs}$$

for all $s \geq 0$ with some ‘resetting rate’ $r > 0$. We have $\varphi(k) = -\infty$ for $k > \sqrt{lr}$ and

$$E[e^{\varphi(k)S_1 + kX_1}] = \int_0^{+\infty} 3rs^2 e^{\varphi(k)\mu - rs} \mathbb{E}[e^{\varphi_0 L^2 \sqrt{\xi} |B_1 + \sqrt{\xi}r}] \, ds = 1$$

(4.5)

for $k < \sqrt{lr}$. Unlike the area, the case $k = \sqrt{lr}$ is not immediate. Lemma 4.4 tells us that for every $< 0$

$$\frac{3r}{\sqrt{\xi}} = 3r \int_0^{+\infty} s^2 e^{\xi s} \, ds \leq E[e^{\xi S_1 + \sqrt{lr}X_1}] \leq 3Lr \int_0^{+\infty} s^2 e^{\xi s} \, ds \leq \frac{3Lr}{\sqrt{\xi}}$$

with some positive constant $L$. Thus, the function that maps $\xi \in \mathbb{R}$ in $E[e^{\xi S_1 + \sqrt{lr}X_1}]$ is finite and continuous for $< 0$ and goes to infinity when $\xi$ goes to 0 from below. This gives
\( \xi := \varphi(\sqrt{6k}) > -\infty \) and \( \Lambda := \mathbb{E}[e^{\xi + \sqrt{6}\xi}] = 1. \) Since \( \xi < 0, \) we find \( \Xi := \mathbb{E}[\xi] = 72r[\xi]^{-3} < +\infty. \) According to the above discuss, we finally realize that the rate function takes the value

\[
I(w) = \begin{cases} 
+\infty & \text{if } w \leq 0, \\
wk + \varphi(k) & \text{if } w \in (0, w_+), \\
w\sqrt{6r + \xi} & \text{if } w \geq w_+ 
\end{cases}
\]

with \( k < \sqrt{6r} \) such that \( w + \varphi'(k) = 0 \) and \( w_+ \) as in (4.4). This rate function exhibits one singularity and one affine stretch.

Once again, we can make use of this example to discuss the limit of zero resetting. As for the area, writing \( \varphi, \) and \( I, \) in place of \( \varphi \) and \( I \) to stress the dependence on \( r, \) the change of variable \( s \mapsto r^{-1}s \) in (4.5) shows that \( \varphi_r(k) = r^4 \varphi_1(r^{-1}k) \) for all \( k \) and \( I_r(w) = r^4 I_1(r^4w) \) for all \( w. \) The analysis of \( I_1 \) in a neighborhood of the origin allows to conclude that there exists a positive constant \( C \) such that

\[
|I_1(w) - \frac{4\nu_1^3}{27w^2}| \leq C(1 + w)
\]

for any \( w > 0. \) Thus, for all \( r > 0 \) and \( w > 0 \) we obtain

\[
|I_r(w) - \frac{4\nu_1^3}{27w^2}| \leq C(r^4 + r^4w).
\]

As \( r \) goes to zero, \( I_r(w) \) approaches a non-trivial limit with speed \( r^4. \) The result is consistent with the known asymptotics of the absolute area without resetting [61]:

\[
\lim_{r \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{1}{t} \int_0^t |B_{1,r}| \, d\tau \leq w \right] = \lim_{r \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \int_0^t |B_{1,r}| \, d\tau \leq \frac{w}{\sqrt{t}} \right] = -\frac{4\nu_1^3}{27w^2}.
\]

5. Conclusions

The mathematical theory of renewal-reward processes boasts a long tradition. In this paper we have taken advantage of this theory to investigate the fluctuations, both normal and large, of additive functionals associated with a stochastic process under a general non-Poissonian resetting mechanism. While providing the law of large numbers, the central limit theorem, and a large deviation principle, we have demonstrated that a suitable resetting protocol can always bring about a large deviation principle with good rate function. In other words, it is always possible to find a resetting mechanism that makes the probability of a large fluctuation of these functionals exponentially small in time. This confirms the ability of resetting to confine the process around the initial position.

We have used our general results to characterize the fluctuations of the positive occupation time, the area, and the absolute area of the reset Brownian motion under any resetting mechanism. Our general results constitute a solid background for researchers that aim to analyze the fluctuations of other additive functionals and other stochastic processes than the Brownian motion. We stress that there are no restrictions to the resetting protocol that can be implemented.

Our study of the positive occupation time, the area, and the absolute area of the reset Brownian motion has shown that a rich phenomenology accounting for dynamical phase transitions emerges beyond Poissonian resetting. This paper opens to investigation of the mechanisms that originate such dynamical phase transitions. Although we leave to future research
the task of providing precise results, we suggest that these phase transitions are due to a phenomenon of condensation of fluctuations similarly to sums of independent and identically distributed random variables [48–51]. After all, an additive functional $F_t$ of the reset Brownian motion is close to the associated cumulative reward $W_t$, which is a random sum of independent and identically distributed rewards. According to this interpretation, a fluctuation of $F_t/t$ that crosses a singular point of the rate function would be realized by the combination of two different mechanisms: many small deviations of the rewards all in the same direction for a partial excursion up to the singular point, and a big jump of only one of the rewards for the remaining part of the fluctuation. We point out that even if in some cases the fluctuations of rewards are solely determined by the waiting times, such as for the positive occupation time of the reset Brownian motion, in general they are the result of the interplay between the fluctuations of the waiting times and the fluctuations of the reference process, such as for the area and the absolute area of the reset Brownian motion.

Data availability statement

No new data were created or analyzed in this study.

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Appendix A. Proof of proposition 2.1

The proof of the proposition requires a little of convex analysis and we refer to [45] for the needed results. It $\psi$ is a proper convex function on $\mathbb{R}^d$ taking extended real values, we denote by $\text{dom}\psi := \{x \in \mathbb{R}^d : \psi(x) < +\infty\}$ the effective domain of $\psi$. The convex conjugate of $\psi$ is the function $\psi^*$ that maps $x^* \in \mathbb{R}^d$ in $\psi^*(x^*) := \sup_{x \in \mathbb{R}^d} \{x^* \cdot x - \psi(x)\}$. The function $\psi^*$ is convex and lower semicontinuous and $\psi^{**}$ turns out to be the lower semicontinuous hull of $\psi$ (see [45], theorem 12.2), i.e. the largest lower semicontinuous function not greater than $\psi$. Consequently, $\psi^{**} = \psi$ when $\psi$ is lower semicontinuous.

If $x^*$ is a subgradient of $\psi$ at $x$, then $\psi^*(x^*) = x^* \cdot x - \psi(x)$ (see [45], theorem 23.5). If $\psi$ is lower semicontinuous, then for each $x^*$ in the relative interior of $\text{dom} \psi^*$ there exists $x \in \text{dom} \psi$ such that $x^* \in \partial \psi(x)$ (see [45], theorems 23.4 and 23.5). The relative interior of $\text{dom} \psi^*$ is the interior that results when $\text{dom} \psi$ is regarded as a subset of its affine hull.

A.1. Proof of part (a)

If $k \in \mathbb{R}^d$ is such that $\varphi(k) > -\infty$, then $\mathbb{E}[e^{\varphi(k)S_{1}+kX_{1}}] \leq 1$. Indeed, by definition of supremum, there exists a sequence $\{\zeta_i\}_{i \geq 1}$ of real numbers that satisfies $\mathbb{E}[e^{\zeta_i S_{1}+kX_{1}}] \leq 1$ for all $i$ and $\lim_{i \to \infty} \zeta_i = \varphi(k)$. Fatou’s lemma gives $\mathbb{E}[e^{\varphi(k)S_{1}+kX_{1}}] \leq \liminf_{i \to \infty} \mathbb{E}[e^{\zeta_i S_{1}+kX_{1}}] \leq 1$. We shall use this property of $\varphi$ many times along the paper.

Let us prove concavity of $\varphi$ by verifying that $\varphi(ah + bk) \geq a\varphi(h) + b\varphi(k)$ for any given $h \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $a \in (0, 1)$, and $b \in (0, 1)$ such that $\varphi(h) > -\infty$, $\varphi(k) > -\infty$, and $a + b = 1$. Since $\mathbb{E}[e^{\varphi(h)S_{1}+hX_{1}}] \leq 1$ and $\mathbb{E}[e^{\varphi(k)S_{1}+kX_{1}}] \leq 1$, Hölder’s inequality gives
We deduce $\varphi(ah + bk) \geq a\varphi(h) + b\varphi(k)$ from here by the definition of $\varphi(ah + bk)$. Thus, $-\varphi$ is convex with effective domain $\text{dom}(-\varphi) = \{k \in \mathbb{R}^d : \varphi(k) > -\infty\}$.

The function $\varphi$ is upper semicontinuous if the level set $L_a := \{k \in \mathbb{R}^d : \varphi(k) \geq a\}$ is closed for any $a \in \mathbb{R}$. Given $a \in \mathbb{R}$, let us show that if $\{k_i\}_{i \geq 1} \subseteq L_a$ is a sequence converging to $k$, then $k \in L_a$. This demonstrates that $L_a$ is closed. As $\varphi(k_i) \geq a > -\infty$, we have $\mathbb{E}[e^{\psi_{S_i} + k} X_i] \leq \mathbb{E}[e^{\psi(k_i) S_i + k} X_i] \leq 1$ for all $i$. Then, Fatou’s lemma yields $\mathbb{E}[e^{\psi_{S_i} + k} X_i] \leq \liminf_{i \to \infty} \mathbb{E}[e^{\psi_{S_i} + k} X_i] \leq 1$. This bound shows that $\varphi(k) \geq a$.

A.2. Proof of part (b)

Basically, the proof of part (b) amounts to verify that for all $\beta \geq 0$ and $w \in \mathbb{R}^d$

$$\Upsilon(\beta, w) = \sup_{k \in \text{dom}(\varphi)} \{w \cdot k + \beta \varphi(k)\}. \quad (A.1)$$

In fact, if (A.1) is valid and $\ell = +\infty$, then part (b) follows by setting $\beta = 1$ in (A.1). If (A.1) is valid and $\ell < +\infty$, then part (b) follows by applying Sion’s minimax theorem to the function that, for a given $w \in \mathbb{R}^d$, maps $(\beta, k) \in [0, 1] \times \text{dom}(\varphi)$ in $w \cdot k + \beta \varphi(k) + (1 - \beta)\ell$. This function is convex and continuous with respect to $\beta$ for each fixed $k \in \text{dom}(\varphi)$ and concave and upper semicontinuous with respect to $k$ for each fixed $\beta \in [0, 1]$. Thus, Sion’s minimax theorem allows us to exchange an infimum with respect to $\beta$ and a supremum with respect to $k$ by stating that

$$I(w) := \inf_{\beta \in [0, 1]} \left\{ \Upsilon(\beta, w) + (1 - \beta)\ell \right\} = \inf_{\beta \in [0, 1]} \sup_{k \in \text{dom}(\varphi)} \{w \cdot k + \beta \varphi(k) + (1 - \beta)\ell\}$$

$$= \sup_{k \in \text{dom}(\varphi)} \inf_{\beta \in [0, 1]} \{w \cdot k + \beta \varphi(k) + (1 - \beta)\ell\}$$

$$= \sup_{k \in \text{dom}(\varphi)} \left\{w \cdot k + \varphi(k) \wedge \ell\right\}.$$  

We stress that Sion’s minimax theorem applies because $\beta$ lies in a compact set.

Let us move to the proof of identity (A.1). The lower bound

$$\Upsilon(\beta, w) \geq \sup_{k \in \text{dom}(\varphi)} \{w \cdot k + \beta \varphi(k)\}. \quad (A.2)$$

is immediate for all $\beta \geq 0$ and $w \in \mathbb{R}^d$. In fact, if $(\zeta, k) \in \mathbb{R} \times \mathbb{R}^d$ is such that $\mathbb{E}[e^{\psi_{S_i} + k} X_i] \leq 1$, then $J(s, w) \geq s + w \cdot k$ by definition for every $(s, w) \in \mathbb{R} \times \mathbb{R}^d$. In particular, we have $J(s, w) \geq w \cdot k + s \varphi(k)$ if $k \in \text{dom}(\varphi)$. It follows that $\Upsilon(\beta, w) \geq w \cdot k + \beta \varphi(k)$ for all $\beta \geq 0$, $w \in \mathbb{R}^d$, and $k \in \text{dom}(\varphi)$.

The upper bound

$$\Upsilon(\beta, w) \leq \sup_{k \in \text{dom}(\varphi)} \{w \cdot k + \beta \varphi(k)\}. \quad (A.3)$$

is much more involved, and we need to split the proof in several steps. We solve the case $\beta = 0$ first, then we address the case $\beta > 0$ by means of a truncation argument. The function that maps $(\zeta, k) \in \mathbb{R} \times \mathbb{R}^d$ in $\ln\mathbb{E}[e^{\psi_{S_i} + k} X_i]$ is proper convex, so that there exist $(\beta_o, w_o) \in \mathbb{R} \times \mathbb{R}^d$ and $c \in \mathbb{R}$ such that $\ln\mathbb{E}[e^{\psi_{S_i} + k} X_i] \geq \beta_o \zeta + w_o \cdot k - c$ for all $(\zeta, k) \in \mathbb{R} \times \mathbb{R}^d$ (see [45], corollary 12.1.2). Let $D := \{(\zeta, k) \in \mathbb{R} \times \mathbb{R}^d : \mathbb{E}[e^{\psi_{S_i} + k} X_i] < +\infty\}$ be its effective domain. For every $w \in \mathbb{R}^d$, $\delta > 0$, and $\gamma_o \in (0, \delta)$ such that $\gamma_o |\beta_o| < \delta$ and $\gamma_o ||w_o|| < \delta$ we have
latter inequality entails that lemma, which shows that \( \phi \) is non-increasing for any \( k \in \mathbb{R}^d \) with the unique solution \( \varphi(k) \) of the equation
\[
\mathbb{E} \left[ e^{\varphi(k)} S_1 \right] \mathbb{I}_{\{S_1 \leq 1\}} = 1.
\]
The function \( \varphi_2 \) is finite throughout \( \mathbb{R}^d \) and analytic by the analytic implicit function theorem. The gradient \( \nabla \varphi_i(k) \) and the Hessian matrix \( H_i(k) \) of \( \varphi_i \) at \( k \) satisfies for every \( k \in \mathbb{R}^d \) and \( u \in \mathbb{R}^d \) the relationships
\[
\nabla \varphi_i(k) = - \frac{\mathbb{E}[X_1 e^{\varphi_i(k)} S_1 + u \cdot X_1]}{\mathbb{E}[S_1 e^{\varphi_i(k)} S_1 + u \cdot X_1]} \mathbb{I}_{\{S_1 \leq 1\}} \] (A.4)
and
\[
u \cdot H_i(k) u = - \frac{\mathbb{E}[u \cdot \nabla \varphi_i(k) S_1 + u \cdot X_1]^2 e^{\varphi_i(k)} S_1 + u \cdot X_1]}{\mathbb{E}[S_1 e^{\varphi_i(k)} S_1 + u \cdot X_1]} \mathbb{I}_{\{S_1 \leq 1\}} \leq 0.
\]
The latter shows that \( \varphi_i \) is concave.

Clearly, the sequence \( \{ \varphi_i(k) \} \) is non-increasing for any \( k \in \mathbb{R}^d \). Let \( \varphi_\infty \) be the function that maps \( k \) to \( \varphi_\infty(k) \) as desired. In fact, the bound \( \varphi_\infty(k) \geq \varphi(k) \) is non-trivial only for \( k \in \text{dom}(\varphi) \), if \( k \in \text{dom}(\varphi) \), then it follows from the fact that \( \mathbb{E}[e^{\varphi(k)} S_1 + u \cdot X_1] \mathbb{I}_{\{S_1 \leq 1\}} \leq 1 \) for all \( i \) by construction of \( \varphi \). The bound \( \varphi_\infty(k) \leq \varphi(k) \) is non-trivial only when \( \varphi_\infty(k) > -\infty \) and is due to Fatou’s lemma, which shows that \( \mathbb{E}[e^{\varphi(k)} S_1 + u \cdot X_1] \mathbb{I}_{\{S_1 \leq 1\}} = 1 \). The latter inequality entails that \( \varphi_\infty(k) \leq \varphi(k) \) by definition of \( \varphi(k) \).

We shall verify that for all \( \beta > 0 \) and \( w \in \mathbb{R}^d \)
\[
\Upsilon(\beta, w) \leq \inf_{i \geq 1 \beta^s} \{ w \cdot k + \beta \varphi_i(k) \} = \inf_{i \geq 1} (-\beta \varphi_i)^s(w).
\] (A.5)
This bound gives the upper bound (A.3) as follows. Fix \( \beta > 0 \) and consider the function \( \vartheta \) that maps \( w \in \mathbb{R}^d \) in \( \vartheta(w) := \inf_{i \geq 1} (-\beta \varphi_i)^s(w) \). The limit \( \vartheta(w) = \lim_{\gamma \to \infty} (-\beta \varphi_i)^s(w) \) holds true because \( \{ (-\beta \varphi_i)^s(w) \}_i \) is a non-increasing sequence, and the function \( \vartheta \) turns out to be convex as it is the pointwise limit of a sequence of convex functions. Since \( \vartheta^* \) is the largest lower semicontinuous function not greater than \( \vartheta \) and since \( \Upsilon(\beta, \cdot) \) is lower semicontinuous by construction and \( \Upsilon(\beta_0, w) \leq \vartheta(w) \) for all \( w \in \mathbb{R}^d \) by (A.5), we have \( \Upsilon(\beta_0, w) \leq \vartheta^*(w) \).
for every \( w \in \mathbb{R}^d \). Let us show that \( \vartheta^*(w) \leq (\beta_o \varphi)^*(w) \) for all \( w \in \mathbb{R}^d \). The fact that \( \vartheta(w) \leq (\beta_o \varphi)^*(w) \) for all \( w \in \mathbb{R}^d \) and \( i \geq 1 \) yields \( \vartheta^*(k) \geq (\beta_o \varphi_i)^*(k) = -\beta_o \varphi_i(k) \) for all \( k \in \mathbb{R}^d \). We have exploited the identity \((\beta_o \varphi_i)^* = -\beta_o \varphi_i\), which is valid because \(-\beta_o \varphi_i\) is convex and continuous. By sending \( i \) to infinity we find \( \vartheta^*(k) \geq -\beta_o \varphi_i(k) \) for all \( k \in \mathbb{R}^d \). Thus, \( \vartheta^*(w) \leq (\beta_o \varphi)^*(w) \) for each \( w \in \mathbb{R}^d \).

Let us move to demonstrate (A.5). To this aim, for each integer \( i \geq 1 \) we introduce ‘truncated’ functions \( J_i \) and \( T_i \), which are defined for every \((s, w) \in \mathbb{R} \times \mathbb{R}^d \) and \((\beta, w) \in \mathbb{R} \times \mathbb{R}^d \) by

\[
J_i(s, w) := \sup_{(\zeta, k) \in \mathbb{R} \times \mathbb{R}^d} \left\{ s \zeta + w \cdot k - \ln \mathbb{E}\left[e^{\zeta S_i + k X_i} 1_{\{S_i \lor |X_i| \leq i\}}\right]\right\}
\]

and

\[
T_i(\beta, w) := \lim_{\varepsilon \downarrow 0} \inf_{(\beta, w) \in (\beta - \varepsilon, \beta + \varepsilon)} \inf_{v \in \Delta_{w, \gamma}} \inf_{\gamma > 0} \left\{ \gamma J_i \left( \frac{s}{\gamma}, \frac{v}{\gamma} \right) \right\}.
\]

We have \( J_i(s, w) \leq J_i(s, w) \) and \( T_i(\beta, w) \leq T_i(\beta, w) \) as a consequence. Bound (A.5) is due to the identity

\[
T_i(\beta, w) = \sup_{k \in \mathbb{R}^d} \left\{ w \cdot k + \beta \varphi_i(k) \right\}
\]

(A.6)

valid for all \( \beta > 0, w \in \mathbb{R}^d \), and \( i \geq 1 \), which we are going to verify by exploiting the differentiability of \( \varphi_i \).

By repeating the arguments we used to deduce the lower bound (A.2), we realize that \( T_i(\beta, w) \geq \sup_{k \in \mathbb{R}^d} \{ w \cdot k + \beta \varphi_i(k) \} \) for all \( \beta > 0, w \in \mathbb{R}^d \), and \( i \geq 1 \). The main task to achieve (A.6) is then to prove that \( T_i(\beta, w) \leq \sup_{k \in \mathbb{R}^d} \{ w \cdot k + \beta \varphi_i(k) \} \). The differentiability of \( \varphi_i \) comes into play at this point. Fix \( \beta_o > 0 \) and \( i \geq 1 \). The problem is to demonstrate that \( T_i(\beta_o, w) \leq (\beta_o \varphi_i)^*(w) \) for all \( w \in \text{dom} (\beta_o \varphi_i) \). Let \( D_o \) be the relative interior of \( \text{dom} (\beta_o \varphi_i) \). We prove at first that \( T_i(\beta_o, w) \leq (\beta_o \varphi_i)^*(w) \) for all \( w \in D_o \), and then we extend this bound from \( D_o \) to \( \text{dom} (\beta_o \varphi_i) \). We know from the general properties of convex functions mentioned at the beginning that if \( w \in D_o \), then there exists \( k_o \in \mathbb{R}^d \) such that \( w = -\nabla \varphi_i(k_o) \) and \( (\beta_o \varphi_i)^*(w) = w \cdot k_o + \beta_o \varphi_i(k_o) \). Set \( \zeta_o := \varphi_i(k_o) \) and

\[
\gamma_o := \frac{\beta_o}{\mathbb{E}[c^{\zeta_o} S_i + k_o X_i 1_{\{S_i \lor |X_i| \leq i\}}]}.
\]

By definition we have

\[
T_i(\beta_o, w) \leq \inf_{\gamma > 0} \left\{ \gamma J_i \left( \frac{\beta_o}{\gamma}, \frac{w}{\gamma} \right) \right\} \leq \gamma_o J_i \left( \frac{\beta_o}{\gamma_o}, \frac{w}{\gamma_o} \right).
\]

On the other hand, \( J_i \) is the convex conjugate of the function that associates the pair \((\zeta, k) \in \mathbb{R} \times \mathbb{R}^d \) with \( \ln \mathbb{E}[e^{\zeta S_i + k X_i} 1_{\{S_i \lor |X_i| \leq i\}}] \), and the gradient of this function at \((\zeta_o, k_o) \) is exactly \((\beta_o \gamma_o, w/\gamma_o) \) as one can easily verify by appealing to (A.4). Then, bearing in mind that \( \mathbb{E}[e^{\zeta_o S_i + k_o X_i} 1_{\{S_i \lor |X_i| \leq i\}}] = 1 \), we get

\[
T_i(\beta_o, w) \leq \gamma_o J_i \left( \frac{\beta_o}{\gamma_o}, \frac{w}{\gamma_o} \right) = \beta_o \zeta_o + w \cdot k_o - \gamma_o \ln \mathbb{E}\left[e^{\zeta_o S_i + k_o X_i} 1_{\{S_i \lor |X_i| \leq i\}}\right] = w \cdot k_o + \beta_o \varphi_i(k_o) = (\beta_o \varphi_i)^*(w).
\]
Thus, \( \Upsilon_i(\beta_0, w) \leq (-\beta_0 \varphi_1)^*(w) \) for any \( w \in D_\phi \). In order to extend this bound from \( D_\phi \) to \( \text{dom}(-\beta_0 \varphi_1)^* \), pick \( w_0 \in D_\phi \) and any \( w \in \text{dom}(-\beta_0 \varphi_1)^* \). The half-open line segment \( \{ \lambda w + (1 - \lambda)w_0 : \lambda \in [0, 1) \} \) lies in \( D_\phi \) (see [45], theorem 6.1), so that we can state

\[
\Upsilon_i(\beta_0, \lambda w + (1 - \lambda)w_0) \leq (-\beta_0 \varphi_1)^*(\lambda w + (1 - \lambda)w_0)
\]

for all \( \lambda \in [0, 1) \). We have \( \lim_{\lambda \uparrow 1} (-\beta_0 \varphi_1)^*(\lambda w + (1 - \lambda)w_0) = (-\beta_0 \varphi_1)^*(w) \) as \( (-\beta_0 \varphi_1)^* \) is convex and lower semicontinuous (see [45], corollary 7.5.1). At the same time, we have \( \liminf_{\lambda \uparrow 1} \Upsilon_i(\beta_0, \lambda w + (1 - \lambda)w_0) \geq \Upsilon_i(\beta_0, w) \) since \( \Upsilon_i(\beta_0, \cdot) \) is lower semicontinuous by construction. Thus, we find \( \Upsilon_i(\beta_0, w) \leq (-\beta_0 \varphi_1)^*(w) \) by sending \( \lambda \) to one from below in (A.7). Resuming, we have shown that \( \Upsilon_i(\beta_0, w) \leq (-\beta_0 \varphi_1)^*(w) \) for all \( w \in \text{dom}(-\beta_0 \varphi_1)^* \), and hence for all \( w \in \mathbb{R}^d \). Since \( \beta_0 > 0 \) and \( i \geq 1 \) are arbitrary, the proof of (A.6) is concluded.

**Appendix B. Proof of proposition 3.1**

Let \( \varpi \) be the function that maps \( k \in \mathbb{R} \) in \( \varpi(k) \) and set \( \vartheta := \varphi \wedge \varpi \) for brevity.

**B.1. Proof of part (a)**

Fix \( k \in \mathbb{R} \). Let us demonstrate the bound

\[
\lim_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{E}[e^{\vartheta F}] \leq -\vartheta(k). \tag{B.1}
\]

Assume that \( \varphi(k) > -\infty \) and \( \varpi(k) > -\infty \), otherwise \( \vartheta(k) = \varphi(k) \wedge \varpi(k) = -\infty \) and the bound is trivial. Pick numbers \( \epsilon > 0 \) and \( \vartheta < \varpi(k) \) and denote by \( P := \mathbb{P}[S_i \in \cdot] \) the probability measure induced by \( S_i \) over \([0, +\infty)\). By definition of \( \varpi(k) \), there exists a positive constant \( C \) such that for all \( t \geq 0 \)

\[
\mathcal{E}_i(k) P[S_i > \tau] = \int_{(t, +\infty)} \mathcal{E}_i(k) P(dx) \leq C e^{\vartheta - \vartheta t}
\]

with \( \mathcal{E}_i(k) := \mathbb{E}[e^{k \int_0^t \varphi(B_s, r) ds}] \). Moreover, we have

\[
a := \mathbb{E}[e^{\varphi(k) S_i^1 - S_i^2}] = e^{e^\varphi(k) S_i^1 - e^\varphi(k) S_i^2} < 1
\]

by the independence between \( S_i \) and \( \{B_{1,i}\}_{i \geq 0} \) and the definition of \( \varphi(k) \). Then, invoking again the independence between the waiting times and the processes \( \{B_{1,i}\}_{i \geq 0}, \{B_{2,i}\}_{i \geq 0}, \ldots \), we can write for every \( t \)

\[
\mathbb{E}[e^{\vartheta F}] = e^{\sum_{n=0}^\infty \mathbb{E}\left[ \prod_{i=1}^n \mathcal{E}_i(k) \mathbb{1}_{\{T_i \leq \tau\}} \mathcal{E}_{T_i} (k) \mathbb{1}_{\{S_{i+1} > t - T_i\}} \right]}
\]

\[
= \sum_{n=0}^\infty \mathbb{E}\left[ \prod_{i=1}^n \mathcal{E}_i(k) \mathbb{1}_{\{T_i \leq \tau\}} \mathcal{E}_{T_i} (k) \mathbb{1}_{\{S_{i+1} > t - T_i\}} \right]
\]

\[
= \sum_{n=0}^\infty \mathbb{E}\left[ \prod_{i=1}^n \mathcal{E}_i(k) \mathbb{1}_{\{T_i \leq \tau\}} \int_{(t - T_i, +\infty)} \mathcal{E}_{T_i}(k) P(ds) \right]
\]

\[
\leq C \sum_{n=0}^\infty \mathbb{E}\left[ \prod_{i=1}^n \mathcal{E}_i(k) \mathbb{1}_{\{T_i \leq \tau\}} e^{(t - T_i) - \vartheta(t - T_i)} \right]. \tag{B.2}
\]
Hereafter we make the usual convention that an empty sum is 0 and an empty product is 1. If $\varphi(k) \leq \varrho$, then bound (B.2) gives
\[
\mathbb{E}[e^{kF}] \leq C \sum_{a=0}^{\infty} \mathbb{E} \left[ \prod_{i=1}^{n} \mathcal{E}_{k_i}(k) \mathbb{E}_{\{T_i \leq t\}} e^{c(t-T_i) - \varphi(k)(t-T_i)} \right] \\
\leq C e^{ct - \varphi(k)t} \sum_{a=0}^{\infty} \mathbb{E} \left[ \prod_{i=1}^{n} e^{\varphi(k)S_i - \varphi(k)C} \mathcal{E}_{k_i}(k) \right] \\
= C e^{ct - \varphi(k)t} \sum_{a=0}^{\infty} a^n = Ce^{ct - \varphi(k)t} \frac{1}{1-a}.
\]

If $\varphi(k) > \varrho$, then bound (B.2) gives
\[
\mathbb{E}[e^{kF}] \leq C e^{ct - \varphi(k)t} \sum_{a=0}^{\infty} \mathbb{E} \left[ \prod_{i=1}^{n} e^{\varphi(k)S_i - \varphi(k)C} \mathcal{E}_{k_i}(k) \right] = Ce^{ct - \varphi(k)t} \frac{1}{1-a}.
\]

In conclusion, we find
\[
\limsup_{t \to +\infty} \frac{1}{t} \ln \mathbb{E}[e^{kF}] \leq -\varphi(k) \wedge \varrho + \epsilon,
\]
which implies (B.1) thanks to the arbitrariness of $\epsilon > 0$ and $\varrho < \varpi(k)$. 

**B.2. Proof of part (b)**

We are going to use part (a) of the proposition to demonstrate the upper large deviation bound for closed sets with rate function $I$ equal to the convex conjugate of $-\vartheta$. To begin with, we observe that the function $\varpi$ is concave, so that $\vartheta$ is concave. Similarly to part (a) of proposition 2.1, this fact follows from the Hölder’s inequality and we do not repeat the proof. Then, we notice that inf $w \in \mathbb{R} \{I(w)\} = 0$. Indeed, since $\varphi(0) = 0$ and $\varpi(0) \geq 0$ by definition, we have $I(w) \geq \vartheta(0) = \varphi(0) \wedge \varpi(0) = 0$ for all $w \in \mathbb{R}$. At the same time, denoting by $(-\varphi)^*$ the convex conjugate of $-\varphi$ and by $(-\varphi)^{**}$ the convex conjugate of $(-\varphi)^*$ we find
\[
\inf_{w \in \mathbb{R}} \{I(w)\} = \inf_{w \in \mathbb{R}} \sup_{k \in \mathbb{R}} \{wk + \vartheta(k)\} \leq \inf_{w \in \mathbb{R}} \sup_{k \in \mathbb{R}} \{wk + \varphi(k)\} \\
= \inf_{w \in \mathbb{R}} \{(-\varphi)^*(w)\} = -(-\varphi)^*(0).
\]

This bound is inf $w \in \mathbb{R} \{I(w)\} \leq 0$ since $(-\varphi)^*(0) = \varphi(0) = 0$ as $-\varphi$ is convex and lower semicontinuous.

We are ready to prove the upper large deviation bound for the additive functional $F_t$. Fix a closed set $K$ such that $\inf_{w \in K} \{I(w)\} > 0$, otherwise the upper large deviation bound trivially holds. If $\vartheta(k) = -\infty$ for $k \neq 0$, then $I(w) = 0$ for all $w \in \mathbb{R}$ and $\inf_{w \in K} \{I(w)\} = 0$. Thus, we are actually considering a model where $\vartheta(k)$ is finite for some $k \neq 0$. We need to distinguish three different situations: $\vartheta(k) = -\infty$ for all $k < 0$, $\vartheta(k) = -\infty$ for all $k > 0$, and $\vartheta(k) > -\infty$ for $k$ in an open neighborhood of the origin.

If $\vartheta(k) = -\infty$ for all $k < 0$, then $I(w) = \sup_{k \geq 0} \{wk + \vartheta(k)\}$ is non-decreasing with respect to $w$ and $\lim_{w \to -\infty} I(w) = 0$ since $\inf_{w \in \mathbb{R}} \{I(w)\} = 0$. Thus, the assumption $\inf_{w \in K} \{I(w)\} > 0$ implies $\nu := \inf_{K} \{I\} > -\infty$. As $K$ is a closed set, $\nu \in K$ and consequently $I(\nu) \geq \inf_{w \in K} \{I(w)\}$. Since $K \subseteq [\nu, +\infty)$, the Chernoff bound and limit (B.1) imply for every $k \geq 0$
Theorem B.3 We have $I(w) = 0$ if and only if $w \leq \vartheta$ (see [45], theorem 23.5). Given any point $w_0 \in \vartheta$, we can state that $I(w) = \sup_{k \geq 0} \{wk + \vartheta(k)\}$ for $w \leq w_0$ and $I(w) = \sup_{k \geq 0} \{wk + \vartheta(k)\}$ for $w > w_0$. In fact, $w_0k + \vartheta(k) \leq I(w_0) = 0$ for all $k \in \mathbb{R}$, so that $wk + \vartheta(k) = (w - w_0)k + w_0 + \vartheta(k) \leq 0$ for all $k > 0$ if $w \leq w_0$ and $wk + \vartheta(k) = (w - w_0)k + w_0 + \vartheta(k) \leq 0$ for all $k < 0$ if $w > w_0$.

Since $\inf_{w \in K} \{I(w)\} > 0$, $\vartheta$ belongs to the complement $K'$ of $K$. Let $(v_-, v_+)$ be the union of all the open intervals in $K^c$ that contain $\vartheta$. Notice that either $v_-$ or $v_+$ must be finite since $K$ is non-empty. Clearly, $v_- \leq w_0$ and $v_+ \geq w_0$, so that $I(v_-) = \sup_{k \geq 0} \{v_-k + \vartheta(k)\}$ and $I(v_+) = \sup_{k \geq 0} \{v_+k + \vartheta(k)\}$. If $v_+ < +\infty$, then $v_+ \in K$ as $K$ is closed and consequently $I(v_+) \geq \inf_{w \in K} \{I(w)\}$. As before, the Chernoff bound and limit (B.1) give

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{I} \geq v_+ \right] \leq -\sup_{k \geq 0} \left\{ v_+k + \vartheta(k) \right\} = -I(v_+) \leq -\inf_{w \in K} \{I(w)\}. \quad (B.3)$$

Similarly, if $v_- > -\infty$, then $I(v_-) \geq \inf_{w \in K} \{I(w)\}$ and the Chernoff bound combined with limit (B.1) yields

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{I} \leq v_- \right] \leq -I(v_-) \leq -\inf_{w \in K} \{I(w)\}. \quad (B.4)$$

Bounds (B.3) and (B.4) prove the upper large deviation bound for the closed set $K$ since $K \subseteq (-\infty, w_-] \cup [w_+, +\infty)$.

**Appendix C. Proof of theorem 3.3**

**C.1. Proof of part (a)**

The rate function $I = (-\varphi)^*$ has compact level sets if some minorant, such as the function that maps $w$ in $\sup_{k \geq 0} \{wk + \varphi(k) \wedge \ell\} \leq I(w)$, has compact level sets. By theorem 2.1 and proposition 2.1, the latter has compact level sets if $\mathbb{E}[e^{\ell|X|}] < +\infty$ for some $\rho > 0$.

As $I$ is the convex conjugate of the convex and lower semicontinuous function $-\varphi$ that takes value zero at the origin, $I(w) = 0$ if and only if $w$ is a subgradient of $-\varphi$ at the origin (see [45], theorem 23.5). Assume that $\ell > 0$ and that there exists $\rho > 0$ such that $\mathbb{E}[e^{\ell|X|}] < +\infty$. We claim that $\varphi$ is differentiable at the origin with $-\varphi'(0) = \mu$. Thus, $\mu$ is the only subgradient of $-\varphi$ at the origin. To begin with, we pick a finite number $\ell_0 > 0$ such that $\mathbb{E}[e^{\ell_0|X|}] > 1$. The dominated convergence theorem shows that there exists $\rho_0 > 0$ such that $1 < \mathbb{E}[e^{\ell_0|X| + k|X|}] < +\infty$ for $|k| < \rho_0$. This way, the function $\Phi$ that maps the pair $(\zeta, k) \in (-\infty, \ell_0) \times (-\rho_0, \rho_0)$ to $\Phi(\zeta, k) := \mathbb{E}[e^{\ell_0|X| + k|X|}]$ is analytic and $\Phi(\ell_0, k) > 1$ for $|k| < \rho_0$. As $\varphi(k) := \sup\{\zeta \in \mathbb{R} : \Phi(\zeta, k) \leq 1\}$, for each $k \in (-\rho_0, \rho_0)$ the number $\varphi(k)$ turns out to be the
unique solution \( \zeta \) of the equation \( \Phi(\zeta, k) = 1 \), and \( \varphi \) turns out to be analytic on \((-\rho_o, \rho_o)\) by the analytic implicit function theorem. The identity \( \Phi(\varphi(k), k) = 1 \) for \( |k| < \rho_o \) and the fact that \( \varphi(0) = 0 \) give

\[
-\varphi'(0) = \frac{\partial \varphi}{\partial k}(0, 0) = \frac{\mathbb{E}[X_1]}{\mathbb{E}[\xi]} =: \mu.
\]

C.2. Proof of part (b)

Assume that \( \varpi(k) \geq \varphi(k) \) for all \( k \in \mathbb{R} \). According to part (b) of proposition 3.1, \( F_t \) satisfies the upper large deviation bound with the rate function \( I = (-\varphi)^* \). Let us show that a lower large deviation bound holds with the same rate function. We prove such bound by a truncation argument. For each integer \( i \geq 1 \), let \( \varphi_i \) be the function that maps \( k \in \mathbb{R} \) in

\[
\varphi_i(k) := \sup \bigg\{ \zeta \in \mathbb{R} : \mathbb{E}[e^{\zeta S_i + kX_i} \mathbb{1}_{\{S_i > j_i\}}] \leq 1 \bigg\}.
\]

By repeating some arguments of the proof of proposition 2.1, one can easily verify that the function \( \varphi_i \) is concave and upper semicontinuous. One can also verify that, for any \( k \), the sequence \( \{\varphi_i(k)\}_{i\geq1} \) is non-increasing and converges to \( \varphi(k) \). It follows in particular that

\[
\lim_{i \to \infty} (-\varphi_i)^*(w) =: J(w)
\]

exists and \( J(w) \leq (-\varphi_i)^*(w) \) for all \( i \). We shall show that for each open set \( G \subseteq \mathbb{R} \) and point \( w \in G \)

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{t} \in G \right] \geq -J(w).
\]

This bound implies the lower large deviation bound as follows. The function \( J \) that associates each \( w \) with \( J(w) \) is convex since it is the pointwise limit of a sequence of convex functions. Convexity entails that the convex conjugate \( J^*(w) := \sup_{k \in \mathbb{R}} \{ kw - J^*(k) \} \) of the convex conjugate \( J^*(k) := \sup_{w \in \mathbb{R}} \{ kw - J(w) \} \) of \( J \) equals the lower-semicontinuous regularization of \( J \) (see [45], theorem 12.2), which is the function that maps \( w \) in \( \lim_{\delta \downarrow 0} \inf_{v \in \Delta_{w, \delta}} \{J(v)\} \). Fix an open set \( G \) and a point \( w \in G \). Since there exists \( \delta_o > 0 \) such that \( \Delta_{w, \delta_o} \subseteq G \), we have from (C.2)

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{t} \in G \right] \geq - \inf_{v \in \Delta_{w, \delta_o}} \{J(v)\} \geq - \inf_{v \in \Delta_{w, \delta_o}} \{J^*(v)\}
\]

for all \( \delta \in (0, \delta_o) \), in such a way that

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{t} \in G \right] \geq - \lim_{\delta \downarrow 0} \inf_{v \in \Delta_{w, \delta}} \{J(v)\} = -J^*(w).
\]

At the same time, \( J(w) \leq (-\varphi_i)^*(w) \) for all \( w \) and \( i \) gives \( J^*(k) \geq (-\varphi_i)^*(k) = -\varphi(k) \) for all \( k \) and \( i \) since \(-\varphi_i \) is convex and lower semicontinuous. By sending \( i \) to infinity we realize that \( J^*(k) \geq -\varphi(k) \) and \( J^*(w) \leq (-\varphi)^*(w) = I(w) \). Thus, (C.3) implies

\[
\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F_t}{t} \in G \right] \geq - I(w),
\]

which yields the lower large deviation bound after optimizing over \( w \).

Let us now verify (C.2) by introducing a truncated probability distribution for waiting times and rewards. Fix once and for all an open set \( G \subseteq \mathbb{R} \) and a point \( w \in G \). Assume that \( J(w) < +\infty \), otherwise there is nothing to prove. Let \( s > 0 \) be a number such that \( m_1 := \frac{1}{2} \mathbb{E}[S_1 \wedge s] > 0 \)
and \( m_2 := 2E[(S_1 \wedge s)^2] > 0 \). By the monotone convergence theorem and (C.1) there exists \( i_o \geq 1 \) such that for all \( i \geq i_o \),

\[
\begin{align*}
& \mathbb{P}[S_1 \lor M_1 \leq i] > 0, \\
& \mathbb{E}[(S_1 \wedge s)(S_1 \lor M_1 \leq i)] > m_1, \\
& \mathbb{E}[(S_1 \wedge s)^2(S_1 \lor M_1 \leq i)] < m_2, \\
& \sup_{i \geq i_o} \{(-\varphi_i)^*(w)\} < +\infty.
\end{align*}
\]

(C.5)

For each \( i \geq i_o \), we construct the truncated probability distribution \( Q_i \) for \((S_1, X_1)\) by conditioning on the event \( S_1 \lor M_1 \leq i \):

\[
Q_i(A) := \mathbb{P}[(S_1, X_1) \in A | S_1 \lor M_1 \leq i] = \frac{\mathbb{P}[(S_1, X_1) \in A, S_1 \lor M_1 \leq i]}{\mathbb{P}[S_1 \lor M_1 \leq i]}
\]

for each Borel set \( A \subseteq \mathbb{R}^2 \). The Kolmogorov extension theorem tells us that there exists a probability space \((\Omega, F, \mathbb{P})\) that hosts infinitely many i.i.d. waiting time and reward pairs \((S_1, X_1), (S_2, X_2), \ldots\) distributed according to the law \( Q_i \). Pay attention to not confuse \( X_n \) on \((\Omega, F, \mathbb{P})\) with \( X_n := \int_0^\infty f(B_{n, t}) \, dt \) on \((\Omega, F, \mathbb{P})\). Denoting expectation with respect to \( \mathbb{P} \) by \( \mathbb{E} \), set for \( k \in \mathbb{R} \)

\[
\psi_i(k) := \sup \left\{ \zeta \in \mathbb{R} : \mathbb{E}_i[e^{\zeta S_1 + kX_1}] \leq 1 \right\}.
\]

Since \( \lim_{s \to +\infty} \frac{1}{s} \ln \mathbb{P}[(S_1 > s)] = +\infty \), theorem 2.1 and proposition 2.1 state that \( W_t \) satisfies a weak LDP with the rate function \( J_t = (-\psi_i)^* \) with respect to the model \((\Omega, F, \mathbb{P})\).

To proceed we observe that, by the last of (C.5), there exists a large number \( \eta \geq 1/m_1 \) such that for all \( i \geq i_o \),

\[
\frac{(m_1\eta - 1)^2}{2m_2\eta} \geq (-\varphi_i)^*(w) + 1.
\]

(C.6)

Fix \( i \geq i_o \) and let \( q_i \) be the integer part of \( 1 + \eta t \). As \( G \) is open, there exists \( \delta > 0 \) such that \((w - 2\delta, w + 2\delta) \subset G \). Bearing in mind that \( |F_t - W_t| \leq \int_0^{S_{t+1}} |f(B_{n+1, t})| \, dt = M_{N_{t+1}} \), for \( i \geq i_o \) and \( t > i/\delta \) we find \( F_t / \in G \) if \( W_t / \in (w - \delta, w + \delta) \) and \( M_{N_{t+1}} < i \). Thus, for \( i \geq i_o \) and \( t > i / \delta \) we have

\[
\mathbb{P}\left[ \frac{F_t}{t} \in G \right] \geq \mathbb{P}\left[ \frac{F_t}{t} \in G, N_t < q_i \right] \\
\geq \mathbb{P}\left[ \frac{F_t}{t} \in G, N_t < q_i, \max\{S_1 \lor M_1, \ldots, S_{q_i} \lor M_{q_i}\} \leq i \right] \\
\geq \mathbb{P}\left[ \frac{W_t}{t} \in (w - \delta, w + \delta), N_t < q_i, \max\{S_1 \lor M_1, \ldots, S_{q_i} \lor M_{q_i}\} \leq i \right] \\
= \mathbb{P}_t \left[ \frac{W_t}{t} \in (w - \delta, w + \delta), N_t < q_i \right] \left\{ \mathbb{P}[S_1 \lor M_1 \leq i] \right\}^{q_i} \\
\geq \left\{ \mathbb{P}[S_1 \lor M_1 \leq i] \right\}^{q_i} \left\{ \mathbb{P}_t \left[ \frac{W_t}{t} \in (w - \delta, w + \delta) \right] - \mathbb{P}_t[N_t \geq q_i] \right\}. \tag{C.7}
\]

The lower large deviation bound for \( W_t \) with respect to the model \((\Omega, F, \mathbb{P})\) gives

\[
\liminf_{t \to +\infty} \frac{1}{t} \mathbb{E}_i \left[ \frac{W_t}{t} \in (w - \delta, w + \delta) \right] \geq -J_t(w) = -(-\varphi_i)^*(w). \tag{C.8}
\]

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We now point out that for each $k$

$$\psi(k) := \sup \left\{ \xi \in \mathbb{R} : \mathbb{E}_x \left[ e^{\xi S_1 + kX_k} \right] \leq 1 \right\}$$

$$= \sup \left\{ \xi \in \mathbb{R} : \mathbb{E}_x \left[ e^{\xi S_1 + kX_k} \mathbb{1}_{\{S_1 \vee M_1 \leq i\}} \right] \leq \mathbb{P}[S_1 \vee M_1 \leq i] \right\}$$

$$\leq \sup \left\{ \xi \in \mathbb{R} : \mathbb{E}_x \left[ e^{\xi S_1 + kX_k} \mathbb{1}_{\{S_1 \vee M_1 \leq i\}} \right] \leq 1 \right\} \Rightarrow: \varphi_i(k),$$

so that (C.8) entails

$$\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}_i \left[ \frac{W_t}{t} \in (w - \delta, w + \delta) \right] \geq -(-\varphi_i)^*(w) \geq -(-\varphi)^*(w). \quad (C.9)$$

In order to estimate $\mathbb{P}_i [N_t \geq q_t]$ we make use of the Chernoff bound, which yields for each $\lambda \geq 0$

$$\mathbb{P}_i [N_t \geq q_t] = \mathbb{P}_i [T_{q_t} \leq i] \leq e^{\lambda \mathbb{E}_x \left[ e^{-\lambda T_{q_t}} \right]} = e^{\lambda \mathbb{E}_x \left[ e^{-\lambda S_1} \right]} \leq e^{\lambda \mathbb{E}_x \left[ e^{-\lambda (S_1 \wedge t)} \right]},$$

so that

$$\limsup_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}_i [N_t \geq q_t] \leq \lambda + \eta \ln \mathbb{E}_x \left[ e^{-\lambda (S_1 \wedge t)} \right].$$

On the other hand, by combining the inequalities $e^{-x} \leq 1 + x + \frac{1}{2}x^2$ valid for $x \geq 0$ and $1 + x \leq e^x$ valid for any $x$ with the second and third of (C.5), we realize that

$$\mathbb{E}_x \left[ e^{-\lambda (S_1 \wedge t)} \right] = \mathbb{E}_x \left[ e^{-\lambda (S_1 \wedge t)} \mathbb{1}_{\{S_1 \vee M_1 \leq i\}} \right]$$

$$\leq 1 - \lambda \mathbb{E}_x \left[ (S_1 \wedge t) \mathbb{1}_{\{S_1 \vee M_1 \leq i\}} \right] + \frac{\lambda^2}{2} \mathbb{E}_x \left[ (S_1 \wedge t)^2 \mathbb{1}_{\{S_1 \vee M_1 \leq i\}} \right]$$

$$\leq 1 - \lambda m_1 + \frac{\lambda^2}{2} m_2 \leq e^{-\lambda m_1 + \frac{\lambda^2}{2} m_2}.$$

Thus, we find

$$\limsup_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}_i [N_t \geq q_t] \leq (1 - m_1 \eta) \lambda + m_2 \eta \frac{\lambda^2}{2}.$$ 

By optimizing over $\lambda$, i.e. by taking $\lambda := (m_1 \eta - 1)/(m_2 \eta) \geq 0$, and by recalling (C.6) we finally obtain

$$\limsup_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P}_i [N_t \geq q_t] \leq -\frac{(m_1 \eta - 1)^2}{2m_2 \eta} \leq -(-\varphi)^*(w) - 1. \quad (C.10)$$

Bounds (C.9) and (C.10) show that $\mathbb{P}_i [N_t \geq q_t]$ decays at a faster rate than $\mathbb{P}_i [W_i / t \in (w - \delta, w + \delta)]$. Then, (C.7) gives

$$\liminf_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{P} \left[ \frac{F}{t} \in G \right] \geq \eta \ln \mathbb{P}[S_1 \vee M_1 \leq i] - (-\varphi)^*(w).$$

We get (C.2) from here by sending $i$ to infinity as $\lim_{t \uparrow +\infty} \mathbb{P}[S_1 \vee M_1 \leq i] = 1$.

### C.3. Proof of part (c)

Suppose that $\varpi(k) \geq \varphi(k)$ for all $k \in \mathbb{R}$. Part (a) of proposition 3.1 shows that for every $k$

$$\limsup_{t \uparrow +\infty} \frac{1}{t} \ln \mathbb{E}_x \left[ e^{xF} \right] \leq -\varphi(k).$$
Let us verify that for every $k$ we also have

$$\liminf_{t\to+\infty} \frac{1}{t} \ln \mathbb{E}[e^{\rho F_i}] \geq -\varphi(k). \quad (C.11)$$

Pick $k \in \mathbb{R}$ and fix numbers $w \in \mathbb{R}$ and $\epsilon > 0$. As $kF_i \geq twk - t\epsilon |k|$ when $F_i/t \in (w - \epsilon, w + \epsilon)$, for each $t$ we find the bound

$$\mathbb{E}[e^{\rho F_i}] \geq \mathbb{E} \left[ e^{\rho F_i} \mathbb{1}_{\{F_i/t \in (w - \epsilon, w + \epsilon)\}} \right] \geq e^{\rho wk - tw|k|} \mathbb{P} \left[ F_i/t \in (w - \epsilon, w + \epsilon) \right].$$

Thus, the lower large deviation bound gives

$$\liminf_{t\to+\infty} \frac{1}{t} \ln \mathbb{E}[e^{\rho F_i}] \geq wk - I(w) - \epsilon |k|,$$

and the arbitrariness of $w$ and $\epsilon$ implies

$$\liminf_{t\to+\infty} \frac{1}{t} \ln \mathbb{E}[e^{\rho F_i}] \geq \sup_{w \in \mathbb{R}} \{ wk - I(w) \}.$$ 

This result is (C.11) because the convex conjugate of $I$ is $-\varphi$ since $\varphi$ is upper semicontinuous.

## Appendix D. Proof of corollary 3.1

Assume that $\mathbb{E}[e^{\rho M}] < +\infty$ for every $\rho > 0$. In order to prove this corollary of theorem 3.3 we show that the condition $\varpi(k) \geq \varphi(k)$ is satisfied for all $k \in \mathbb{R}$, with $\varpi(k)$ defined by (3.5). Fix $k \in \mathbb{R}$ such that $\varphi(k) > -\infty$ and pick $\epsilon \in (0, 1)$. Hölder’s inequality gives for all $t > 0$

$$\mathcal{E}_t(k) \mathbb{P}[S_1 > \ell] = \mathbb{E} \left[ e^{\int_0^t f \{ |B_{1,r}^\ell| \} \mathbb{1}_{\{S_1 > \ell\}} \right] \leq \mathbb{E} \left[ e^{\int_0^t f \{ |B_{1,r}^\ell| \} \mathbb{1}_{\{S_1 > \ell\}} \right] \leq \mathbb{E} \left[ e^{\int_0^t f \{ |B_{1,r}^\ell| \} \mathbb{1}_{\{S_1 > \ell\}} \right]^{1-\epsilon},$$

so that

$$\varpi(k) := \liminf_{t\to+\infty} \frac{1}{t} \ln \mathbb{E}[\mathcal{E}_t(k)\mathbb{P}[S_1 > \ell]] \geq \ell \left( 1 - \epsilon \right).$$

By sending $\epsilon$ to zero we get $\varpi(k) \geq \ell$. Thus, $\varpi(k) \geq \varphi(k)$ is verified if we demonstrate that $\varphi(k) \leq \ell$, which trivially holds when $\ell = +\infty$. Although it is hard to exhibit a function $f$ for which $\mathbb{E}[e^{\rho M}] < +\infty$ for all $\rho > 0$ when $\ell < +\infty$, we are not able to exclude that such a function exists.

Suppose that $\ell < +\infty$. Set $f_{\pm} := (\pm f) \vee 0$ and $g_{\pm}(t) := \mathbb{E} \int_0^t f_{\pm}(B_{1,r}^\ell) \mathbb{d}r \geq 0$ for all $t \geq 0$. We show at first that $\limsup_{t\to+\infty} g_{\pm}(t)/t \leq 0$. Pick $\rho > 0$ and denote by $P := \mathbb{P}[S_1 \in \cdot]$ the probability measure induced on $[0, +\infty)$ by $S_1$. Jensen’s inequality gives $e^{\rho g_{\pm}(t)} \leq \mathbb{E} \left[ e^{\rho f_{\pm}(B_{1,r}^\ell) \mathbb{d}r} \right] \leq \mathbb{E} \left[ e^{\rho f_{\pm}(|B_{1,r}^\ell|) \mathbb{d}r} \right]$ for all $s \geq 0$. We use this bound and the fact that $g_{\pm}$ is non-decreasing to state that for every $t > 0$

$$e^{\rho g_{\pm}(t)} \mathbb{P}[S_1 > \ell] = \int_{(0, +\infty)} e^{\rho g_{\pm}(t)} \mathbb{P}(ds) \leq \int_{[0, +\infty)} e^{\rho g_{\pm}(t)} \mathbb{P}(ds) \leq \int_{[0, +\infty)} \mathbb{E} \left[ e^{\rho f_{\pm}(|B_{1,r}^\ell|) \mathbb{d}r} \right] \mathbb{P}(ds) \leq \mathbb{E} \left[ e^{\rho f_{\pm}(\ell) \mathbb{d}r} \right] = \mathbb{E} \left[ e^{\rho M} \right] < +\infty.$$
It follows that
\[
\rho \limsup_{\ell \to +\infty} \rho_\pm(t) - \ell = \limsup_{\ell \to +\infty} \frac{1}{\ell} \ln \left\{ e^{\rho_\pm(t)P[S_1 > \ell]} \right\} \leq 0,
\]
and the arbitrariness of \( \rho \) implies \( \limsup_{\ell \to +\infty} \rho_\pm(t)/\ell \leq 0 \).

We are now ready to prove that \( \varphi(k) \leq \ell \). Fix any \( \epsilon > 0 \). Since \( \limsup_{\ell \to +\infty} \rho_\pm(t)/\ell \leq 0 \), there exists \( \ell > 0 \) with the property that \( 0 \leq \rho_\pm(s) \leq \epsilon \) for all \( s > t \), in such a way that \( \mathbb{E}[k \, (f(B, \tau) - \hat{f}) \, d\tau] = -k g_- (s) + k g_+ (s) \geq -\epsilon k |s| \) for \( s > t \). Once again, we appeal to Jensen’s inequality to get
\[
\mathbb{E} \left[ e^{(\varphi(k) - \epsilon)|k|} \, S_1 \, \mathbb{I}_{\{S_1 > \ell\}} \right] \leq \mathbb{E} \left[ e^{\varphi(k)|S_1 - \hat{f}|} + (S_1) \right] \leq \mathbb{E} \left[ e^{\varphi(k)|S_1 + \hat{f}|} + (S_1) \right] = \mathbb{E} \left[ e^{\varphi(k)|S_1 + \hat{f}|} + (S_1) \right] \leq 1.
\]
This bound demonstrates that \( \varphi(k) - \epsilon |k| \leq \ell \) since \( \mathbb{E}[e^{\zeta} \, \mathbb{I}_{\{S_1 > \ell\}}] = +\infty \) for \( \zeta > \ell \). The arbitrariness of \( \epsilon \) gives \( \varphi(k) \leq \ell \).

**Appendix E. Proof of corollary 3.2**

According to theorem 3.3, it suffices to verify that \( \varpi(k) \geq \varphi(k) \) for any given \( k \) such that \( \varphi(k) > -\infty \). Recall that \( \int_{[0, +\infty)} e^{\varphi(k) \, S_1} P(ds) \leq 1 \) for such a \( k \), where \( P := P[S_1 \in \cdot] \) is the probability measure induced on \([0, +\infty)\) by \( S_1 \).

Set \( \lambda := \lim_{t \to +\infty} \frac{1}{t} \ln \mathbb{E}_t(k) \), which exists by assumption. If \( \lambda = -\infty \) or \( \ell = +\infty \), then \( \varpi(k) = +\infty \) and the inequality \( \varpi(k) \geq \varphi(k) \) trivially holds. Suppose that \( \lambda > -\infty \) and that \( \ell < +\infty \). We address first the case \( \lambda = +\infty \). In this case we have \( \varpi(k) = \ell - \lambda \) and \( \mathbb{E}_t(k) \geq e^{(\lambda - \epsilon)|k|} \) for an arbitrary number \( \epsilon > 0 \) and all sufficiently large \( t \). Thus, for all sufficiently large \( t \) we find
\[
\mathbb{E} \left[ e^{\varphi(k)|S_1 + (\lambda - \epsilon)|k|} \, \mathbb{I}_{\{S_1 > \ell\}} \right] = \int_{(t, +\infty)} e^{\varphi(k)|S_1 + (\lambda - \epsilon)|k|} P(ds) \leq \int_{(t, +\infty)} e^{\varphi(k)|S_1 + (\lambda - \epsilon)|k|} P(ds) \leq 1.
\]
This bound implies \( \varphi(k) + (\lambda - \epsilon) \leq \ell \) since \( \mathbb{E}[e^{\zeta} \, \mathbb{I}_{\{S_1 > \ell\}}] = +\infty \) for \( \zeta > \ell \). The arbitrariness of \( \epsilon \) gives \( \ell - \lambda \geq \varphi(k) \), i.e. \( \varpi(k) \geq \varphi(k) \).

Let us consider now the case \( \lambda = +\infty \). In this case \( \frac{1}{t} \ln \mathbb{E}_t(k) \) is eventually non-decreasing with respect to \( t \) by assumption. We show that \( e^{\varphi(k) \, S_1} P[S_1 > \ell] \leq 1 \) for all sufficiently large \( t \), which gives \( \varpi(k) \geq \varphi(k) \). In fact, let \( t \) be a sufficiently large time such that \( \frac{1}{t} \ln \mathbb{E}_t(k) \leq \frac{1}{t} \ln \mathbb{E}_t(k) \) for every \( s > t \). If \( \varphi(k) + \frac{1}{t} \ln \mathbb{E}_t(k) < 0 \), then the bound \( e^{\varphi(k) \, S_1} P[S_1 > \ell] \leq 1 \) is trivial. If \( \varphi(k) + \frac{1}{t} \ln \mathbb{E}_t(k) > 0 \), then \( \varphi(k)|t + \ln \mathbb{E}_t(k) \leq \varphi(k)|s + \frac{1}{t} \ln \mathbb{E}_t(k) \) for each \( s > t \), so that
\[
e^{\varphi(k)|S_1 + (\lambda - \epsilon)|k|} P(ds) \leq \int_{(t, +\infty)} e^{\varphi(k)|S_1 + (\lambda - \epsilon)|k|} P(ds) \leq 1.
\]

**Appendix F. Proof of lemma 4.1**

The lemma relies on the fact that the uniform norm \( \lVert B \rVert := \sup_{t \in [0, 1]} \lVert B(t, \tau) \rVert \) of the Brownian motion has a Gaussian tail:
\[
\lim_{x \to +\infty} \frac{1}{x^2} \ln \mathbb{P} \left[ \lVert B \rVert > x \right] = -\lambda
\]
with some \( \lambda > 0 \) (see [63], corollary 3.2).
F.1. Proof of part (a)

Part (a) is proved as follows. Pick $\rho > 0$ and denote by $P := \mathbb{P}[S_1 = 1]$ the probability measure induced on $[0, +\infty)$ by $S_1$. Since $|f(z)| \leq A(1 + |z|^\alpha)$ for all $z \in \mathbb{R}$ with $A > 0$ and $\alpha > 0$, we have $M_1 := \int_{Q_1} (\beta^M |B_{1,\tau}|)^{\alpha} d\tau \leq A\mathbb{E}_1 + A\int_{Q_1} |B_{1,\tau}|^{\alpha} d\tau$. Since $\int_{Q_1} |B_{1,\tau}|^{\alpha} d\tau = 1/1 + s \int_{Q_1} |B_{1,\tau}|^{\alpha} d\tau$ for each $s > 0$, we can state that

\[
\mathbb{E}[M_1^\rho] \leq A^\rho \int_{(0, +\infty)} \mathbb{E} \left[ \left( s + \int_{Q_1} |B_{1,\tau}|^{\alpha} d\tau \right)^\rho \right] P(ds) = A^\rho \int_{(0, +\infty)} \mathbb{E} \left[ \left( s + \int_{Q_1} |B_{1,\tau}|^{\alpha} d\tau \right)^\rho \right] P(ds)
\]

Thus, $\mathbb{E}[S_1^{(1 + \rho/2)}] < +\infty$ implies $\mathbb{E}[M_1^\rho] < +\infty$ since $\mathbb{E}[(1 + |B_1|^{\alpha})^\rho] < +\infty$ for each $\alpha > 0$ and $\rho > 0$.

F.2. Proof of part (b)

Regarding part (b), we suppose that $\alpha \in (0, 2)$, we fix $\rho > 0$, and we write

\[
\mathbb{E}[e^{\rho M_1}] \leq A^\rho \int_{(0, +\infty)} \mathbb{E} \left[ e^{\rho s + \rho |B_1|^{\alpha/2}} \int_{Q_1} |B_{1,\tau}|^{\alpha} d\tau \right] P(ds)
\]

The function that maps $s \geq 0$ in $s^{\alpha/2}$ is concave as $\alpha \in (0, 2)$. Then, for every $\beta > 0$ we have

\[
\|B_1\|_{\infty}^\alpha = (\|B_1\|_{\infty}^2)^{\alpha/2} \leq \beta^{\alpha/2} + \alpha \beta^{\alpha/2 - 1}(\|B_1\|_{\infty}^2 - \beta) \leq \beta^{\alpha/2} + \frac{\alpha}{2} \beta^{-\frac{2}{\alpha}} \|B_1\|_{\infty}^2.
\]  

(F.2)

By taking $\beta := s^{\frac{\alpha}{\alpha - 2}} (\alpha \rho A / \lambda)^{\frac{2}{\alpha - 2}}$ in (F.2) with some $s \geq 0$ we realize that

\[
\rho A s + \rho A s^{\frac{2}{\alpha - 2}} \|B_1\|_{\infty}^\alpha \leq \rho A s + (\alpha / \lambda)^{\frac{2}{\alpha - 2}} (\rho A)^{\frac{2}{\alpha - 2}} s^{\frac{2}{\alpha - 2}} + \frac{\alpha}{2} \|B_1\|_{\infty}^2
\]

\[
\leq \eta \left( 1 + s^{\frac{2}{\alpha - 2}} \right) + \frac{\alpha}{2} \|B_1\|_{\infty}^2,
\]

where we have set $\eta := \rho A + (\alpha / \lambda)^{\frac{2}{\alpha - 2}} (\rho A)^{\frac{2}{\alpha - 2}}$ for brevity. Thus

\[
\mathbb{E}[e^{\rho M_1}] \leq e^\eta \mathbb{E} \left[ e^{s \frac{2}{\alpha - 2}} \right] \mathbb{E} \left[ e^{\frac{\alpha}{2} \|B_1\|_{\infty}^\alpha} \right].
\]

Fubini’s theorem and (F.1) give

\[
\mathbb{E} \left[ e^{\frac{\alpha}{2} \|B_1\|_{\infty}^\alpha} \right] = \mathbb{E} \left[ 1 + \frac{\lambda}{2} \int_0^{+\infty} e^{\frac{s}{x}} \mathbb{P} \left[ \|B_1\|_{\infty} > \sqrt{x} \right] dx \right] = 1 + \frac{\lambda}{2} \int_0^{+\infty} e^{\frac{s}{x}} \mathbb{P} \left[ \|B_1\|_{\infty} > \sqrt{x} \right] dx < +\infty.
\]
Fubini’s theorem and the hypothesis \( \limsup_{x \to +\infty} s^{-1/z} \ln \mathbb{P}[S_1 > s] = -\infty \) give
\[
E \left[ e^{\eta S_1} \right] = E \left[ 1 + \eta \int_0^{+\infty} e^{\eta s} \mathbb{P}_{\{S_1 > s \}} \, ds \right] = 1 + \eta \int_0^{+\infty} e^{\eta s} \mathbb{P}_{\{S_1 > s \}} \, ds < +\infty.
\]

Appendix G. Proof of lemma 4.2

The function \( \Phi \) that maps \( (\zeta, k) \in \mathbb{R}^2 \) in
\[
\Phi(\zeta, k) := E \left[ e^{\zeta S_1 + k S_2} \right] = \int_0^1 \mathbb{E}_x \left[ e^{(\zeta + k)s} \right] \, dx
\]
is finite and analytic throughout \( \mathbb{R}^2 \) if \( \ell := \lim_{t \to +\infty} -1/t \ln \mathbb{P}[S_1 > s] = +\infty \), since \( E[e^{\zeta S_1}] < +\infty \) for all \( \zeta \in \mathbb{R} \) in such case. If instead \( \ell < +\infty \), then \( E[e^{\zeta S_1}] < +\infty \) for \( \zeta < \ell \) and \( E[e^{\zeta S_1}] = +\infty \) for \( \zeta > \ell \). In this second case \( \Phi \) is finite and analytic in the region of pairs \( (\zeta, k) \) with \( \zeta < \ell \wedge (\ell - k) \), whereas \( \Phi(\zeta, k) = +\infty \) for \( \zeta > \ell \wedge (\ell - k) \). It follows that \( \varphi(k) = \sup \{ \zeta \in \mathbb{R} : \Phi(\zeta, k) \leq 1 \} \leq \ell \wedge (\ell - k) \) for all \( k \). Thus, \( \varphi(k) \leq -0 \vee k \) when \( \ell = 0 \), which gives part (c) of the lemma as we already know that \( \varphi(k) \geq -0 \vee k \).

We are going to identify \( \varphi \), and then to prove parts (a) and (b) of the lemma. If \( \ell = +\infty \), then for each \( k \in \mathbb{R} \) the value \( \varphi(k) \) of \( \varphi \) turns out to be the unique real number \( \zeta \) that solves the equation \( \Phi(\zeta, k) = 1 \), and the function \( \varphi \) is analytic throughout \( \mathbb{R}^2 \) by the analytic implicit function theorem. If \( 0 < \ell < +\infty \) and \( \Phi(\ell \wedge (\ell - k), k) > 1 \), then \( \varphi(k) \) is the unique real number \( \zeta \) such that \( \Phi(\zeta, k) = 1 \), whereas if \( 0 < \ell < +\infty \) and \( \Phi(\ell \wedge (\ell - k), k) \leq 1 \), then \( \varphi(k) = \ell \wedge (\ell - k) \). These arguments show that the case \( 0 < \ell < +\infty \) requires to investigate the behavior of \( \Phi(\zeta, k) \) along the curve \( \zeta = \ell \wedge (\ell - k) \) in order to assess \( \varphi(k) \). The identity
\[
\Phi(\ell \wedge (\ell - k), k) = \int_0^1 \mathbb{E}_x \left[ e^{(\ell - |k|)s} \right] \, dx
\]
states that \( \Phi(\ell \wedge (\ell - k), k) \) is a convex function of \( |k| \). Let us demonstrate that there exist two positive constants \( C_- \) and \( C_+ \) such that for all \( k \neq 0 \)
\[
C_- = \frac{e^{\pi}}{\sqrt{|k|}} \leq \Phi(\ell \wedge (\ell - k), k) \leq C_+ \mathbb{E}_x \left[ e^{\ell S_1} \right] \leq C_+ \frac{A}{\sqrt{|k| \wedge 1}} \tag{G.1}
\]
with
\[
A := \mathbb{E}_x \left[ e^{\ell S_1} \right] / (\sqrt{1 + S_1}).
\]
The limit
\[
\lim_{s \to +\infty} \sqrt{s} \int_0^1 e^{-sx} \frac{e^{-ax}}{\pi \sqrt{x(1-x)}} \, dx = \int_0^{+\infty} e^{-x} \frac{e^{-ax}}{\pi \sqrt{x}} \, dx
\]
shows that there exist two constants \( C_- > 0 \) and \( C_+ > 0 \) such that
\[
C_- \leq \frac{1}{\sqrt{s}} \int_0^1 e^{-sx} \frac{e^{-ax}}{\pi \sqrt{x(1-x)}} \, dx \leq \frac{C_+}{\sqrt{1 + \alpha}} \tag{G.2}
\]
the lower bound being valid for all \( s > 1 \) and the upper bound being valid for all \( s \geq 0 \). Then, Fubini’s theorem yields for \( k \neq 0 \)

\[
\Phi(\ell \wedge (\ell - k), k) = \int_0^1 \frac{\mathbb{E}[e^{i(\ell - kS_1)}]}{\pi \sqrt{x(1 - x)}} \, dx \geq \mathbb{E}\left[ \frac{C - e^{\ell S_1}}{\sqrt{kS_1}} \right] \geq C - \frac{e^{\pi \ell}}{\sqrt{|k|}}
\]

and

\[
\Phi(\ell \wedge (\ell - k), k) = \int_0^1 \frac{\mathbb{E}[e^{i(\ell - kS_1)}]}{\pi \sqrt{x(1 - x)}} \, dx \leq \mathbb{E}\left[ \frac{C + e^{\ell S_1}}{\sqrt{1 + |kS_1|}} \right].
\]

Bounds (G.1) together with the identity \( \Phi(\ell \wedge (\ell - k), k) = \mathbb{E}[e^{\ell S_1}] \) for \( k = 0 \) show that \( \Phi(\ell \wedge (\ell - k), k) = +\infty \) for all \( k \in \mathbb{R} \) if \( \Lambda = +\infty \). Then, as in the case \( \ell = +\infty \), also in the case \( 0 < \ell < +\infty \) and \( \Lambda = +\infty \) it turns out that \( \varphi(k) \) for every \( k \in \mathbb{R} \) is the unique real number \( \zeta \) that solves the equation \( \Phi(\zeta, k) = 1 \), and the function \( \varphi \) is analytic throughout \( \mathbb{R} \) by the analytic implicit function theorem. If instead \( 0 < \ell < +\infty \) and \( \Lambda < +\infty \), then bounds (G.1) tell us that \( \Phi(\ell \wedge (\ell - k), k) \) is a finite convex function of \( |k| \neq 0 \), which goes to zero as \( |k| \) is sent to infinity by the dominated converge theorem. Convexity and finiteness imply continuity and the monotone converge theorem states that \( \lim_{|k| \to 0} \Phi(\ell \wedge (\ell - k), k) = \mathbb{E}[e^{\ell S_1}] > 1 \). Thus, if \( 0 < \ell < +\infty \) and \( \Lambda < +\infty \), then there exists \( \lambda > 0 \) with the property that \( \Phi(\ell \wedge (\ell - k), k) > 1 \) for \( |k| < \lambda \), \( \Phi(\ell \wedge (\ell - k), k) = 1 \) for \( |k| = \lambda \), and \( \Phi(\ell \wedge (\ell - k), k) < 1 \) for \( |k| > \lambda \). The number \( \lambda \) solves the equation

\[
1 = \Phi(\ell, -\lambda) = \mathbb{E}\left[e^{\ell S_1 - \lambda X_1}\right].
\]

In conclusion, if \( 0 < \ell < +\infty \) and \( \Lambda < +\infty \), then \( \varphi(k) \) is the unique real number \( \zeta \) that solves the equation \( \Phi(\zeta, k) = 1 \) for \( |k| < \lambda \), whereas \( \varphi(k) = \ell \wedge (\ell - k) \) for \( |k| \geq \lambda \). The analytic implicit function theorem gives that \( \varphi \) is analytic on the open interval \( (-\lambda, \lambda) \), but \( \varphi \) may be not differentiable at the boundary points \( -\lambda \) and \( \lambda \).

G.1. Proof of part (a)

If \( \ell = +\infty \) or \( 0 < \ell < +\infty \) and \( \Lambda = +\infty \), then \( \varphi \) is an analytic function throughout \( \mathbb{R} \) that satisfies \( \mathbb{E}[e^{\varphi(k)S_1 + kX_1}] = \Phi(\varphi(k), k) = 1 \) and

\[
\varphi'(k) = -\frac{\partial}{\partial k} \left( \frac{\varphi(k)}{\varphi(k)}, k \right) = -\frac{\mathbb{E}[X_1 e^{\varphi(k)S_1 + kX_1}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1 + kX_1}]}\]

for all \( k \in \mathbb{R} \). Let us verify that \( \lim_{k \to -\infty} \varphi'(k) = 0 \) and \( \lim_{k \to +\infty} \varphi'(k) = -1 \). The symmetry \( \varphi(-k) = \varphi(k) + k \) makes it sufficient to address \( \lim_{k \to +\infty} \varphi'(k) = -1 \) only. We have \( \lim_{k \to +\infty} \varphi'(k) \geq -1 \) since \( X_1 \leq S_1 \). We show that \( \lim_{k \to +\infty} \varphi'(k) \leq -1 \) by contradiction. Assume that \( \lim_{k \to +\infty} \varphi'(k) > -1 \), so that there exists \( \eta > 0 \) such that \( \varphi'(k) > -1 + 2\eta \) for all sufficiently large \( k \). Let \( \delta > 0 \) be such that \( P\{S_1 > \delta\} > 0 \). Since \( \varphi(0) = 0 \), concavity gives \( \varphi(k) \geq \varphi'(k)k \geq (-1 + 2\eta)k \) for all sufficiently large \( k \). Then, under the conditions \( S_1 > \delta \) and \( X_1 > (1 - \eta)S_1 \), for all sufficiently large \( k \) we find \( \varphi(k)S_1 + kX_1 \geq (-1 + 2\eta)kS_1 + (1 - \eta)kS_1 = \eta kS_1 \geq \eta k \), so that

\[
1 = \mathbb{E}\left[e^{\varphi(k)S_1 + kX_1}\right] \geq e^{\eta k \delta} \mathbb{P}\{S_1 > \delta, X_1 > (1 - \eta)S_1\} = e^{\eta k \delta} \mathbb{P}\{S_1 > \delta\} \left( 1 - \frac{2}{\pi} \arcsin \sqrt{1 - \eta} \right).
\]

This bound is a contradiction since the r.h.s. goes to infinity as \( k \) is sent to infinity.
G.2. Proof of part (b)

In the case $0 < \ell < +\infty$ and $\Lambda < +\infty$ the function $\Phi$ is analytic on the open interval $(-\Lambda, \Lambda)$, and for each $k \in (-\Lambda, \Lambda)$ fulfills $\mathbb{E}[e^{\varphi(k)S_t + kX_t}] = \Phi'(k), S_t = 1$ and

$$\varphi'(k) = -\frac{\mathbb{E}[X_t e^{\varphi(k)S_t + kX_t}]}{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]}.$$  \hfill (G.3)

We have $\varphi(k) = \ell \wedge (\ell - k)$ for $k \notin (-\Lambda, \Lambda)$. Let us compute the left derivative $\varphi'_-(k)$ and the right derivative $\varphi'_+(k)$ of $\varphi$ at the points $k = -\Lambda$ and $k = \Lambda$, which exist because $\varphi$ is concave and finite on the whole $\mathbb{R}$. Once again, the symmetry $\varphi(-k) = \varphi(k) + k$ tells us that it suffices to address only the instance $k = \Lambda$. As $\varphi(k) = \ell - k$ for $k \geq \Lambda$ we find $\varphi'_-(\lambda) = -1$ and, by concavity, $\varphi'_-(\lambda) \geq \varphi'_+(\lambda) = -1$ (see [45], theorem 24.1). In order to calculate the left derivative we observe at first that bounds (G.2) can be used as before to obtain for $k > 0$

$$C_\Xi \Xi - k^{-1} e^k \Xi \leq \mathbb{E}[S_t e^{(\ell-k)S_t + kX_t}] = \int_0^1 \mathbb{E}[S_t e^{(\ell-k)S_t + kX_t}] \frac{\mathbb{E}[X_t e^{\varphi(k)S_t + kX_t}]}{\pi \sqrt{x(1-x)}} dx \leq C_\Xi \Xi + \frac{\Xi}{\sqrt{k}}$$

with 

$$\Xi := \mathbb{E}[\sqrt{S_t e^{\varphi(S_t)}}].$$

Thus, $\mathbb{E}[S_t e^{(\ell-k)S_t + kX_t}] = +\infty$ or $\mathbb{E}[S_t e^{(\ell-k)S_t + kX_t}] < +\infty$ depending on whether $\Xi = +\infty$ or $\Xi < +\infty$.

Let us prove that $\varphi'_-(\lambda) \leq 1$ when $\Xi = +\infty$. This means that $\varphi'_-(\lambda) = 1 = \varphi'_+(\lambda)$ if $\Xi = +\infty$, so that the function $\varphi$ is differentiable at $\lambda$, and hence throughout $\mathbb{R}$. For every $k \in (0, \lambda)$ we have by concavity $\varphi(\lambda) \leq \varphi(k) + \varphi'(k)(\lambda - k)$, so that

$$\frac{\varphi(k) - \varphi(\lambda)}{k - \lambda} \leq \varphi'(k) = -\frac{\mathbb{E}[X_t e^{\varphi(k)S_t + kX_t}]}{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]}.$$  \hfill (G.4)

Pick $\epsilon \in (0, 1)$ and $s > 0$. The bound (G.4) gives for all $k \in (0, \lambda)$

$$\varphi(k) - \varphi(\lambda) \leq \frac{\mathbb{E}[X_t e^{\varphi(k)S_t + kX_t}]}{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]} - \frac{\mathbb{E}[X_t e^{\varphi(k)S_t + kX_t}]}{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]} \mathbb{I}_{\{X_t > (1-\epsilon)s\}}$$

$$\leq - (1 - \epsilon) \frac{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]}{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]} \mathbb{I}_{\{X_t > (1-\epsilon)s\}}$$

$$= (1 - \epsilon) \left\{ -1 + \frac{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]}{\mathbb{E}[S_t e^{\varphi(k)S_t + kX_t}]} \mathbb{I}_{\{X_t \leq (1-\epsilon)s\}} \right\}$$

Recalling that $\varphi(k) \leq \ell - k$ for $k \geq 0$ we see that $S_t e^{\varphi(k)S_t + (1-\epsilon)s} \leq S_t e^{(\ell-k)S_t} \leq S_t e^{(\ell-\epsilon/2)S_t}$ for $k \in (\lambda/2, \lambda)$ with $\mathbb{E}[S_t e^{(\ell-\epsilon/2)S_t}] < +\infty$. Then, by sending $k$ to $\lambda$, the dominated convergence theorem shows that

$$\varphi'_-(\lambda) \leq (1 - \epsilon) \left\{ -1 + \frac{\mathbb{E}[S_t e^{(\ell-\lambda)S_t}]}{\mathbb{E}[S_t e^{(\ell-\lambda)S_t}]} \mathbb{I}_{\{X_t \leq s\}} \right\}. $$
The monotone convergence theorem yields \( \lim_{t \to +\infty} E[S_t e^{(t-\lambda)S_t + \lambda X_t} 1_{\{S_t \leq 0\}}] = +\infty \) if \( \Xi = +\infty \). Thus, by sending \( s \) to infinity we realize that
\[
\varphi_-'(\lambda) \leq -1 + \epsilon,
\]
and the arbitrariness of \( \epsilon \) implies \( \varphi_-'(\lambda) \leq -1 \).

To conclude the proof of part (b) of the lemma it remains to verify that
\[
\varphi_-'(\lambda) = \lim_{k \to \lambda} \varphi'(k) = -\frac{E[X_t e^{(\lambda)S_t + \lambda X_t}]}{E[S_t e^{(\lambda)S_t + \lambda X_t}]}
\]
when \( \Xi < +\infty \). The equality \( \varphi_-'(\lambda) = \lim_{k \to \lambda} \varphi'(k) \) is a general property of concave functions (see [45], theorem 24.1). The limit is an application of the dominated convergence theorem to (G.3). If fact, since \( X_t \leq S_t \), for all \( k \in (\lambda/2, \lambda) \) we have
\[
S_t e^{\varphi(k)S_t + kX_t} \leq S_t e^{(\varphi(k)-\varphi)(S_t + kX_t)} = S_t e^{\varphi(S_t + kX_t)} \leq S_t e^{(\varphi(\lambda/2)X_t)} = S_t e^{(\varphi(\lambda/2)S_t + (\lambda/2)X_t)}
\]
with \( E[S_t e^{(\varphi(\lambda/2)S_t + (\lambda/2)X_t)}] < +\infty \) when \( \Xi < +\infty \).

Appendix H. Proof of lemma 4.3

Part (c) of the lemma is trivial. In order to address parts (a) and (b) consider the function \( \Phi \) that maps the pair \( (\zeta, k) \in \mathbb{R}^2 \) in
\[
\Phi(\zeta, k) := E[e^{\zeta S_t + kX_t}] = E[e^{\zeta X_t + kS_t^2}].
\]
We have \( \varphi(k) = \sup \{ \zeta \in \mathbb{R} : \Phi(\zeta, k) \leq 1 \} \) with the manifest symmetry \( \varphi(-k) = \varphi(k) \). If \( r = +\infty \), then \( \Phi \) is finite and analytic throughout \( \mathbb{R}^2 \). If \( 0 < r < +\infty \), then \( \Phi \) is finite and analytic in the region of pairs \( (\zeta, k) \in \mathbb{R}^2 \) with \( |k| < \sqrt{6r} \), whereas \( \Phi(\zeta, k) = +\infty \) for \( |k| > \sqrt{6r} \).

H.1. Proof of part (a)

When \( r = +\infty \), \( \varphi(k) \) is for each \( k \in \mathbb{R} \) the unique real number \( \zeta \) that solves the equation \( \Phi(\zeta, k) = 1 \). It turns out that \( \varphi \) is analytic throughout \( \mathbb{R} \) by the analytic implicit function theorem. As \( 1 = \Phi(\varphi(k), k) = E[e^{\varphi(k)S_t + kS_t^2}] \), we find for all \( k \)
\[
\varphi'(k) = \frac{\partial \Phi}{\partial \varphi}(\varphi(k), k) = \frac{k E[S_t e^{\varphi(k)S_t + kS_t^2}]}{3 E[S_t e^{\varphi(k)S_t + kS_t^2}]}.
\]
Let us show by contradiction that \( \lim_{k \to +\infty} \varphi'(k) = -\infty \). The limit \( \lim_{k \to -\infty} \varphi'(k) = +\infty \) will follow by the symmetry of \( \varphi \). Assume that \( \eta := \lim_{k \to +\infty} \varphi'(k) > -\infty \). By concavity we have \( 0 = \varphi'(0) \geq \varphi'(k) \geq \eta \) and \( 0 = \varphi(0) \geq \varphi(k) \geq \varphi'(k) k \), so that \( \varphi(k) \geq \eta k \) for \( k > 0 \). We find \( \varphi(k) s + (1/6) k^2 s^3 \geq \eta ks + (1/6) k^2 s^3 \geq \sqrt{12|\eta|} k^3 \) for \( k > 0 \) and \( s > \sqrt{12|\eta|}/k \). Thus, for all \( k > 0 \)
\[
1 = E[e^{\varphi(k)S_t + kS_t^2}] \geq e^{\sqrt{12|\eta|} k^3} P[S_t > \sqrt{12|\eta|}/k],
\]
which is a contradiction since \( \lim_{k \to +\infty} P[S_t > \sqrt{12|\eta|}/k] = 1 \).

H.2. Proof of part (b)

If \( 0 < r < +\infty \), then \( \varphi(k) \) for \( |k| < \sqrt{6r} \) is the unique real number \( \zeta \) that satisfies \( \Phi(\zeta, k) = 1 \), whereas \( \varphi(k) = -\infty \) for \( |k| > \sqrt{6r} \). The function \( \varphi \) turns out to be analytic on the open interval.
We study the derivatives of $\varphi$ when the boundary points $-\sqrt{6r}$ and $\sqrt{6r}$ are approached. Thanks to the symmetry of $\varphi$, it is sufficient to investigate the limit $\lim_{k \to \sqrt{6r}} \varphi'(k)$, which exists by concavity.

Concavity of $\varphi$ gives $k \varphi'(k) \leq \varphi(k)$ for all $k \in (0, \sqrt{6r})$ as $\varphi(0) = 0$. Concavity and upper semicontinuity of $\varphi$ imply $\lim_{k \to \sqrt{6r}} \varphi(k) = \varphi(\sqrt{6r}) = -\xi$ (see [45], corollary 7.5.1). Thus, $\lim_{k \to \sqrt{6r}} \varphi'(k) = -\infty$ when $\xi = -\infty$.

Let us show that $\lim_{k \to \sqrt{6r}} \varphi'(k) = -\infty$ even if $\xi > -\infty$ and $\Lambda < 1$. To begin with, we claim that $\lim_{k \to \sqrt{6r}} \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}] = +\infty$ in this case. If fact, under the assumption $\xi > -\infty$, the equality $e^x \geq 1 + x$ for all $x$ gives for $k \in (0, \sqrt{6r})$

$$\Lambda := \mathbb{E}\left[e^{\xi S_1 + k^3 S_1^3}\right] \geq \mathbb{E}\left[e^{\varphi(k)S_1 + k^3 S_1^3 - \xi \varphi(k)}\right] \geq \mathbb{E}\left[e^{\varphi(k)S_1 + k^3 S_1^3}\right] + \left\{ \xi - \varphi(k) \right\} \mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}\right] \geq 1 + \left\{ \xi - \varphi(k) \right\} \mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}\right].$$

If $\Lambda < 1$, then $\varphi(k) - \xi > 0$ and

$$\mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}\right] \geq \frac{1 - \Lambda}{\varphi(k) - \xi}.$$ 

By sending $k$ to $\sqrt{6r}$ from below we get $\lim_{k \to \sqrt{6r}} \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}] = +\infty$ since we have seen that $\lim_{k \to \sqrt{6r}} \varphi(k) = \xi$. At this point, we pick $s > 0$ and use the fact that $1 \leq s^{-3} S_1^3$ when $S_1 > s$ to state for $k \in (0, \sqrt{6r})$ the bound

$$\varphi'(k) = -\frac{k}{3} \mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}\right] \leq -\frac{k}{3} \mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3} \mathbb{1}_{\{S_1 \leq s\}} + \mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3} \mathbb{1}_{\{S_1 > s\}}\right]\right].$$

By sending $k$ to $\sqrt{6r}$ we obtain $\lim_{k \to \sqrt{6r}} \varphi'(k) \leq -\sqrt{2r/3} s^2$ as a consequence of the fact that $\lim_{k \to \sqrt{6r}} \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}] = +\infty$ because $\lim_{k \to \sqrt{6r}} \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^3 S_1^3}] = +\infty$. The arbitrariness of $s$ implies $\lim_{k \to \sqrt{6r}} \varphi'(k) = -\infty$.

The case $\xi > -\infty$, $\Lambda = 1$, and $\Xi := \mathbb{E}[S_1^3 e^{S_1 + r S_1^3}] = +\infty$ is immediate. Since the function that maps $x > 0$ in $x/(a + x)$ is non-decreasing for any $a \geq 0$, starting from (H.1) we find for every $s > 0$ and $\sigma > 0$

$$\varphi'(k) \leq -\frac{k}{3} \mathbb{E}\left[S_1 e^{\varphi(k)S_1 + k^3 S_1^3} \mathbb{1}_{\{S_1 \leq s\}} + s^{-2} \mathbb{E}\left[S_1^3 e^{\varphi(k)S_1 + k^3 S_1^3} \mathbb{1}_{\{S_1 \leq s\}}\right]\right].$$


Thanks to the constraints on $S_1$, the dominated convergence theorem applies and gives
\[
\lim_{k \to \infty} \varphi'(k) \leq -\sqrt{\frac{2r}{3}} \mathbb{E}[S_1 e^{S_1 + \xi} \mathbb{I}_{\{S_1 \leq \sigma\}}]. \tag{H.2}
\]
As $\lim_{r \to +\infty} \mathbb{E}[S_1 e^{S_1 + \xi} \mathbb{I}_{\{S_1 \leq \sigma\}}] = \Xi = +\infty$, by sending $\sigma$ to infinity we realize that $\lim_{k \to \infty} \varphi'(k) \leq -\sqrt{\frac{2r}{3} \sigma}$. As before, the arbitrariness of $\sigma$ implies $\lim_{k \to \infty} \varphi'(k) = -\infty$.

Finally, let us discuss the case $\xi > -\infty$, $\Lambda = 1$, and $\Xi < +\infty$. By sending first $\sigma$ to infinity and then $s$ to infinity in (H.2) we find under the hypothesis $\Xi < +\infty$
\[
\lim_{k \to \infty} \varphi'(k) \leq -\sqrt{\frac{2r}{3}} \mathbb{E}[S_1 e^{S_1 + \xi}] \mathbb{I}_{\{S_1 \leq \sigma\}}.
\]

Let us demonstrate the opposite bound. Once again, we use the inequality $e^x \geq 1 + x$ valid for all $x \in \mathbb{R}$ to write down for $k \in (0, \sqrt{6r})$ the bound
\[
1 = \mathbb{E}[e^{\xi(k+S_1)}] = \mathbb{E}[e^{\xi S_1 + \sqrt{6r}X_1} e^{(\varphi(k)-\xi)S_1+(k-\sqrt{6r})X_1}]
\geq \Lambda + \{\varphi(k) - \xi\} \mathbb{E}[S_1 e^{S_1 + \sqrt{6r}X_1}]
+ \{k - \sqrt{6r}\} \mathbb{E}[X_1 e^{S_1 + \sqrt{6r}X_1}]
= 1 + \{\varphi(k) - \xi\} \mathbb{E}[S_1 e^{S_1 + \sqrt{6r}X_1}]
+ \{k - \sqrt{6r}\} \mathbb{E}[X_1 e^{S_1 + \sqrt{6r}X_1}]. \tag{H.3}
\]

Concavity gives $\xi := \varphi(\sqrt{6r}) \leq \varphi(k) + \varphi'(k)(\sqrt{6r} - k)$ for $k \in (0, \sqrt{6r})$. By combining this bound with (H.3) we realize that for $k \in (0, \sqrt{6r})$
\[
\varphi'(k) \geq -\frac{\mathbb{E}[X_1 e^{S_1 + \sqrt{6r}X_1}]}{\mathbb{E}[S_1 e^{S_1 + \sqrt{6r}X_1}]},
\]
so that
\[
\lim_{k \to \infty} \varphi'(k) \geq -\sqrt{\frac{2r}{3}} \frac{\mathbb{E}[S_1 e^{S_1 + \xi} \mathbb{I}_{\{S_1 \leq \sigma\}}]}{\mathbb{E}[S_1 e^{S_1 + \sqrt{6r}X_1}]} = -\sqrt{\frac{2r}{3}} \frac{\mathbb{E}[S_1 e^{S_1 + \xi} \mathbb{I}_{\{S_1 \leq \sigma\}}]}{\mathbb{E}[S_1 e^{S_1 + \sqrt{6r}X_1}]}.
\]

**Appendix I. Proof of Lemma 4.4**

Fubini’s theorem gives for each $k \geq 0$
\[
\mathbb{E}[e^k \int_0^1 |B_{1,\tau}| d\tau] = \mathbb{E}\left[1 + \int_0^{+\infty} ke^{k\tau} \mathbb{I}_{\{\int_0^\tau |B_{1,\tau}| d\tau > x\}} d\tau\right]
= 1 + \int_0^{+\infty} ke^{k\tau} \mathbb{P}\left[\int_0^1 |B_{1,\tau}| d\tau > x\right] dx. \tag{I.1}
\]

On the other hand, it is known [61] that $\lim_{x \to +\infty} xe^{\frac{kx^2}{12}} \mathbb{P}\left[\int_0^1 |B_{1,\tau}| d\tau > x\right] = \sqrt{2/3\pi}$, so that
\[
\mathbb{P}\left[\int_0^1 |B_{1,\tau}| d\tau > x\right] \leq C \frac{e^{-\pi x^2}}{1 + x} \tag{I.2}
\]
for all $x > 0$ with some constant $C > 0$. By combining (I.1) with (I.2) one can prove through simple manipulations that there exists a constant $L > 0$ such that for all $k \geq 0$
\[ e^{k^2} = \mathbb{E}\left[e^{k\int_0^1 |B_{1,\tau}| \, d\tau}\right] \leq \mathbb{E}\left[e^{k\int_0^1 \nu \, d\tau}\right] \leq L e^{k^2}. \]

The equality on the left follows from the fact that \( \sqrt{3} \int_0^1 B_{1,\tau} \, d\tau \) is distributed as a standard Gaussian variable.

**Appendix J. Proof of lemma 4.5**

Let \( P := \mathbb{P}[S_1 \in \cdot \cdot \cdot] \) be the probability measure induced by \( S_1 \) on \([0, +\infty)\) and consider the function \( \Phi \) that maps the pair \((\zeta, \kappa) \in \mathbb{R}^2\) in

\[
\Phi(\zeta, \kappa) := \mathbb{E}\left[e^{\zeta S_1 + \kappa X_0}\right] = \int_{[0, +\infty)} e^{\zeta r} \mathbb{E}\left[e^{\kappa \int_0^r |B_{1,\tau}| \, d\tau}\right] P(dr)
= \int_{[0, +\infty)} e^{\zeta r} \mathbb{E}\left[e^{\kappa^{3/2} \int_0^r |B_{1,\tau}| \, d\tau}\right] P(dr).
\]

(J.1)

The mapping \( \Phi \) gives \( \varphi \) through the formula \( \varphi(k) = \sup\{ \zeta \in \mathbb{R} : \Phi(\zeta, k) \leq 1 \} \). To begin with, we identify the effective domain of \( \Phi \).

Formula (4.3) entails that there exists a constant \( C > 0 \) such that \( \mathbb{E}[e^{-\nu_1 \int_0^1 |B_{1,\tau}| \, d\tau}] \leq Ce^{-\nu_1 x_6/3} \) for all \( s \geq 0 \). It follows that for every \( \zeta \in \mathbb{R} \) and \( k < 0 \)

\[
\Phi(\zeta, k) \leq C \mathbb{E}\left[e^{(\zeta - \nu_1)|k|^{2/3} S_1}\right].
\]

(J.2)

At the same time, (4.3) states that there exists \( x_6 > 0 \) such that \( \mathbb{E}[e^{-\nu_1 \int_0^1 |B_{1,\tau}| \, d\tau}] \geq e^{-\nu_1 x_6/3} \) for all \( s > x_6 \). Thus, for every \( \zeta \in \mathbb{R} \) and \( k < 0 \) we also have

\[
\Phi(\zeta, k) \geq \mathbb{E}\left[e^{(\zeta - \nu_1)|k|^{2/3} S_1}\mathbb{I}_{\{ |k| S_1^{3/2} > x_6 \}}\right].
\]

(J.3)

Bound (J.2) shows that \( \Phi \) is finite and analytic in the region of pairs \((\zeta, k)\) such that \( \zeta < \ell + \nu_1 |k|^{2/3} \) and \( k < 0 \). Bound (J.3) implies that \( \Phi(\zeta, k) = +\infty \) if \( \ell < +\infty \), \( k < 0 \), and \( \zeta > \nu_1 |k|^{2/3} + \ell \).

To complete the picture, we observe that lemma 4.4 gives for all \( \zeta \in \mathbb{R} \) and \( k \geq 0 \)

\[
\mathbb{E}\left[e^{\zeta S_1 + \kappa X_0}\right] \leq \Phi(\zeta, k) \leq L \mathbb{E}\left[e^{\zeta S_1 + \kappa X_0}\right]
\]

with some positive constant \( L \). Bearing in mind that \( \ell = +\infty \) when \( r > 0 \), these bounds tell us that \( \Phi \) is finite and analytic in the region of pairs \((\zeta, k) \in \mathbb{R}^2\) such that \( k < \sqrt{6r} \), whereas \( \Phi(\zeta, k) = +\infty \) if \( r < +\infty \) and \( k > \sqrt{6r} \).

In the sequel we shall need the estimate

\[
\mathbb{E}\left[X_1 e^{\zeta S_1 + \kappa X_1}\right] \leq L \mathbb{E}\left[S_1^{3/2}(1 + k\Delta_S^{3/2})e^{\zeta S_1 + \kappa X_1}\right],
\]

which holds for all \( \zeta \in \mathbb{R} \) and \( k \geq 0 \). To verify this bound, we appeal to lemma 4.4 to get for any \( s \geq 0 \) and \( \lambda > 0 \)

\[
\mathbb{E}\left[\int_0^1 |B_{1,\tau}| \, d\tau e^{\zeta \int_0^\tau |B_{1,\tau}| \, d\tau}\right] \leq \mathbb{E}\left[\frac{e^{(\zeta + \lambda) \int_0^1 |B_{1,\tau}| \, d\tau}}{\lambda}\right] \leq \frac{L e^{(\zeta + \lambda)^2}}{\lambda}.
\]

The choice \( \lambda = 2/(1 + s) \) yields

\[
\mathbb{E}\left[\int_0^1 |B_{1,\tau}| \, d\tau e^{\zeta \int_0^\tau |B_{1,\tau}| \, d\tau}\right] \leq L(1 + s)e^{k^2}.
\]
This way, for every $\zeta \in \mathbb{R}$ and $k \geq 0$ we find

$$
E[X_1 e^{(S_1 + kX_1)}] = \int_{[0, +\infty)} s^{3/2} e^{ks} E\left[ \int_0^1 |B_{1, \tau}| \, d\tau \, e^{k \tau/2} |B_{1, \tau}| \, d\tau \right] P(ds)
\leq L \int_{[0, +\infty)} s^{3/2} (1 + ks^{3/2}) e^{ks + \frac{1}{2}k^2} \, d\tau \, d\tau \, P(ds)
= LE^{1/2}(1 + ks^{3/2}) e^{kS_1 + \frac{1}{2}k^2}.
$$

In order to prove part (a), (b), and (c) of the lemma it is convenient to study the properties of $\varphi$ on the negative semiaxis first.

### J.1. The function $\varphi$ on $(-\infty, 0)$

Bound (J.3) and the monotone convergence theorem show that $\lim_{\zeta \uparrow \ell + \nu_1 |k|^{2/3}} \Phi(\zeta, k) = +\infty$ for $k < 0$ if either $\ell = +\infty$ or $\ell < +\infty$ and $E[e^{S_1}] = +\infty$. Assume that $\Phi(\ell + \nu_1 |k|^{2/3}, k) > 1$ for all $k < 0$ when $\ell < +\infty$ and $E[e^{S_1}] < +\infty$, as in part (c) of the lemma. Then, irrespective of the values of $\ell$ and $E[e^{S_1}]$ we have $\lim_{\zeta \uparrow \ell + \nu_1 |k|^{2/3}} \Phi(\zeta, k) > 1$ for $k < 0$. As a consequence, for each $k < 0$, $\varphi(k)$ is the unique real number $\zeta$ that solves the equation $\Phi(\zeta, k) = 1$. It turns out that $\varphi$ is analytic on $(-\infty, 0)$ by the analytic implicit function theorem. The identity $E[e^{\varphi(\zeta)S_1 + kX_1}] = \Phi(\varphi(k), k) = 1$ yields for any $k < 0$

$$
\varphi'(k) = -\frac{\partial^2}{\partial \zeta^2} (\varphi(k), k) = -\frac{E[X_1 e^{\varphi(k)S_1 + kX_1}]}{E[S_1 e^{\varphi(k)S_1 + kX_1}]}.
$$

We point out that $\varphi(k) < \ell + \nu_1 |k|^{2/3}$ for all $k < 0$.

Since $X_1$ is non-negative we have $\lim_{k \to -\infty} \varphi'(k) \leq 0$, where the limit exists by concavity. Let us prove that $\lim_{k \to -\infty} \varphi'(k) = 0$. The concavity of $\varphi$ yields $0 = \varphi(0) \leq \varphi(k) - \varphi'(k)k$ for every $k < 0$. Thus, we find $\lim_{k \to -\infty} \varphi'(k) = 0$ if we demonstrate that $\limsup_{k \to -\infty} \varphi(k)/k \geq 0$. The latter is immediate when $\ell < +\infty$ as $\varphi(k) < \ell + \nu_1 |k|^{2/3}$ for all $k < 0$. To address the case $\ell = +\infty$, let $s_0 > 0$ be such that $E[e^{-s_0 |B_{i, \tau}| \, d\tau}] \geq e^{-\nu_1 s_0^{2/3}}$ for all $s > s_0$ and recall that there exists a number $\delta > 0$ such that $P[S_1 > \delta] > 0$. Pick $k < 0$ such that $|k|^{3/2} > s_0$. If $s > \delta$, then $|k|^{3/2} > s_0$, and we find

$$
1 = E[e^{\varphi(k)S_1 + kX_1}] \geq \int_{[0, +\infty)} \int_{[0, +\infty)} e^{-s_0 |k|^{3/2} j_0 \nu_1 \tau} \, d\tau \, d\tau \, P(ds)
\geq \int_{[0, +\infty)} e^{\varphi(k) - \nu_1 |k|^{2/3}} P(ds)
= E[e^{\varphi(k) - \nu_1 |k|^{2/3}}] P[S_1 > \delta].
$$

Jensen’s inequality allows us to conclude that

$$
1 \geq E[e^{\varphi(k) - \nu_1 |k|^{2/3}}] P[S_1 > \delta] \geq e^{\varphi(k) - \nu_1 |k|^{2/3}} E[S_1 | S_1 > \delta] P[S_1 > \delta].
$$

Notice that $E[S_1 | S_1 > \delta] < +\infty$ since $\ell = +\infty$. Thus, for any $k < 0$ such that $|k|^{3/2} > s_0$ we have

$$
\varphi(k) \leq \nu_1 |k|^{2/3} - \ln \frac{E[S_1 | S_1 > \delta]}{P[S_1 > \delta]},
$$

which implies $\limsup_{k \to -\infty} \varphi(k)/k \geq 0$.  


J.2. Proof of part (a)

If \( r = +\infty \), then \( \Phi \) is analytic throughout \( \mathbb{R}^2 \). In this case, \( \varphi(k) \) is for each \( k \in \mathbb{R} \) the unique real number \( \zeta \) that solves the equation \( \Phi(\zeta, k) = 1 \). It turns out that \( \varphi \) is analytic throughout \( \mathbb{R} \) by the analytic implicit function theorem. As \( 1 = \Phi(\varphi(k), k) = \mathbb{E}[e^{\varphi(k)S_1 + kX_1}] \), we find for all \( k \)

\[
\varphi'(k) = - \frac{\partial \Phi}{\partial \zeta}(\varphi(k), k) = - \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1 + kX_1}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1 + kX_1}]}. 
\]

We know from the previous section that \( \lim_{k \to -\infty} \varphi'(k) = 0 \), and one can show that \( \lim_{k \to +\infty} \varphi'(k) = -\infty \) by means of the same arguments we used for the area. Indeed, (J.4) gives \( 1 = \Phi(\varphi(k), k) \geq \mathbb{E}[e^{\varphi(k)S_1 + k^2X_1}] \) for any \( k > 0 \).

J.3. Proof of part (b)

Assume that \( 0 < r < +\infty \). In this case, the function \( \Phi \) is analytic in the region of pairs \((\zeta, k) \in \mathbb{R}^2 \) with \( k < \sqrt{6r} \), whereas \( \Phi(\zeta, k) = +\infty \) if \( k > \sqrt{6r} \). As a consequence, \( \varphi(k) \) for \( k < \sqrt{6r} \) is the unique real number \( \zeta \) that satisfies \( \Phi(\zeta, k) = 1 \), whereas \( \varphi(k) = -\infty \) for \( k > \sqrt{6r} \). The function \( \varphi \) turns out to be analytic on \((-\infty, \sqrt{6r})\) by the analytic implicit function theorem. For all \( k \in (-\infty, \sqrt{6r}) \) we have \( \mathbb{E}[e^{\varphi(k)S_1 + kX_1}] = \Phi(\varphi(k), k) = 1 \) and

\[
\varphi'(k) = - \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1 + kX_1}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1 + kX_1}]}.
\]

In section J.1 we proved that \( \lim_{k \to -\infty} \varphi'(k) = 0 \). We now investigate the derivatives of \( \varphi \) when the boundary point \( \sqrt{6r} \) is approached. Basically, the arguments are the ones we used for the area, and the limit \( \lim_{k \to \sqrt{6r}} \varphi'(k) = -\infty \) is proved exactly in the same way when \( \xi = -\infty \). Let us show that \( \lim_{k \to \sqrt{6r}} \varphi'(k) = -\infty \) even if \( \xi > -\infty \) and \( \Lambda < 1 \) or \( \xi < -\infty \) and \( \Lambda = 1 \) and \( \Xi = +\infty \). Recalling that \( \sqrt{3} \int_0^1 B_1, \tau \, d\tau \) is distributed as a Gaussian variable with mean 0 and variance 1, we can write down for \( k \in (0, \sqrt{6r}) \) the bound

\[
\frac{k^3}{3} \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^2X_1}] = \int_{(0, +\infty)} k^3 e^{\varphi(k)s + \frac{k^2}{2}s^2} P(ds) 
= \int_{(0, +\infty)} \mathbb{E}[S_1 e^{\varphi(k)s + k^2/2s} B_{1, \tau}^s] P(ds) 
\leq \int_{(0, +\infty)} \mathbb{E}[|B_{1, \tau}|] P(ds) 
= \mathbb{E}[X_1 e^{\varphi(k)S_1 + kX_1}].
\]

At the same time, lemma 4.4 yields

\[
\mathbb{E}[S_1 e^{\varphi(k)S_1 + kX_1}] = \int_{(0, +\infty)} se^{\varphi(k)s} \mathbb{E}[e^{k^{3/2}/2 B_{1, \tau}}] P(ds) 
\leq L \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^2X_1}] .
\]

It follows that for every \( k \in (0, \sqrt{6r}) \)

\[
\varphi'(k) = - \frac{\mathbb{E}[X_1 e^{\varphi(k)S_1 + kX_1}]}{\mathbb{E}[S_1 e^{\varphi(k)S_1 + kX_1}]} \leq - \frac{k}{3L \mathbb{E}[S_1 e^{\varphi(k)S_1 + k^2X_1}]}.
\]
Starting from this bound, one can demonstrate that \( \lim_{k \to \sqrt{6r}} \varphi'(k) = -\infty \) if \( \xi > -\infty \) and \( \Lambda < 1 \) or \( \xi > -\infty \) and \( \Lambda = 1 \) and \( \Xi = +\infty \) by resorting to the strategies we devised for the area. It is only needed to check that \( \lim_{k \to \sqrt{6r}} \mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + \frac{k}{6} k^2 S_1^2} \right] = +\infty \) when \( \xi > -\infty \) and \( \Lambda < 1 \).

Actually, the inequality \( e^x \geq 1 + x \) valid for all \( x \) gives for \( k \in (0, \sqrt{6r}) \)

\[
\Lambda := \mathbb{E} \left[ e^{\xi S_1 + \sqrt{6r} X_1} \right] \geq \mathbb{E} \left[ e^{\varphi(k) S_1 + k X_1} (\xi - \varphi(k)) S_1 \right] \\
\geq \mathbb{E} \left[ e^{\varphi(k) S_1 + k X_1} + \{ \xi - \varphi(k) \} \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}] \right] \\
= 1 + \{ \xi - \varphi(k) \} \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}].
\]

If \( \Lambda < 1 \), then \( \varphi(k) - \xi > 0 \) and

\[
\frac{1}{\varphi(k) - \xi} \leq \mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + k X_1} \right] \leq L \mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + \frac{k}{6} k^2 S_1^2} \right],
\]

the last bound being (1.5). Since \( \lim_{k \to \sqrt{6r}} \varphi(k) = \xi \) as \( -\varphi \) is convex and lower semicontinuous (see [45], corollary 7.5.1), from here we get \( \lim_{k \to \sqrt{6r}} \mathbb{E} [S_1 e^{\varphi(k) S_1 + \frac{k}{6} k^2 S_1^2}] = +\infty \).

Finally, let us show that if \( \xi > -\infty \) and \( \Lambda = 1 \) and \( \Xi < +\infty \), then

\[
\lim_{k \to \sqrt{6r}} \varphi'(k) = - \frac{\mathbb{E} [X_1 e^{\xi S_1 + \sqrt{6r} X_1}] \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]}{\mathbb{E} [S_1 e^{\varphi(k) S_1 + \sqrt{6r} X_1}]}.
\]

We stress that \( \mathbb{E} [X_1 e^{\xi S_1 + \sqrt{6r} X_1}] < +\infty \) when \( \Xi := \mathbb{E} [S_1^3 e^{\xi S_1 + r S_1^2}] < +\infty \). In fact, we have seen at the beginning that \( \mathbb{E} [X_1 e^{\xi S_1 + k X_1}] \leq L \mathbb{E} [S_1^{3/2} (1 + k S_1^{3/2}) e^{\xi S_1 + \frac{k}{6} k^2 S_1^2}] \) for all \( \xi \in \mathbb{R} \) and \( k \geq 0 \) with some positive constant \( L \). As for the area, for all \( k \in (0, \sqrt{6r}) \) we have

\[
\varphi'(k) \geq - \frac{\mathbb{E} [X_1 e^{\xi S_1 + \sqrt{6r} X_1}] \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]}{\mathbb{E} [S_1 e^{\varphi(k) S_1 + \sqrt{6r} X_1}]},
\]

The problem is to prove that

\[
\lim_{k \to \sqrt{6r}} \varphi'(k) \leq - \frac{\mathbb{E} [X_1 e^{\xi S_1 + \sqrt{6r} X_1}] \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]}{\mathbb{E} [S_1 e^{\varphi(k) S_1 + \sqrt{6r} X_1}]}. \tag{J.6}
\]

The limit exist by concavity and \( \Lambda > 0 \) and observe that

\[
\varphi'(k) = - \frac{\mathbb{E} [X_1 e^{\varphi(k) S_1 + k X_1}] \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]}{\mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]} \\
\leq - \frac{\mathbb{E} [X_1 e^{\varphi(k) S_1 + k X_1}] \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]}{\mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}] + \lambda^{-1} \mathbb{E} [X_1 e^{\varphi(k) S_1 + k X_1}]} \tag{J.7}
\]

We shall use the fact that \( \xi \leq \varphi(k) \leq 0 \) for \( k \in (0, \sqrt{6r}) \) as \( X_1 \geq 0 \). Since the function that maps \( x \geq 0 \) in \( x/(a + x) \) is non-decreasing for any \( a \geq 0 \) we can change (J.7) with

\[
\varphi'(k) \leq - \frac{\mathbb{E} [X_1 e^{\xi S_1 + k X_1}] \mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}]}{\mathbb{E} [S_1 e^{\varphi(k) S_1 + k X_1}] + \lambda^{-1} \mathbb{E} [X_1 e^{\xi S_1 + k X_1}]}.
\]

On the other hand, the conditions \( \sqrt{\lambda} I_{\left\{ S_1 \leq \lambda \right\}} \leq \lambda \) and \( \int_0^1 [B_{1, r}] d\tau \leq \lambda \) imply \( S_1 \leq 1 \), so that

\[
\mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + k X_1} I_{\left\{ X_1 \leq \lambda S_1 \right\}} \right] = \mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + k X_1} I_{\left\{ X_1 \leq \lambda S_1 \right\}} \right] \\
\leq E_\Lambda(k) + \epsilon_\Lambda
\]
with
\[ E_\lambda(k) := E \left[ S_1 e^{\varphi(k) S_1 + kX^2/2} \mathbb{1}_{[B_1, +]} |dr \right] \]
and
\[ \epsilon_\lambda := E \left[ e^{\sqrt{\frac{r}{2}} \lambda} \mathbb{1}_{[B_1, +]} |dr \right] \).

We stress that \( \lim_{\lambda \to +\infty} \epsilon_\lambda = 0 \) by the dominated convergence theorem. By putting the pieces together we find
\[ \varphi'(k) \leq - \frac{E[X_1 e^{\xi S_1 + 4X_1}]}{E_\lambda(k) + \epsilon_\lambda + \lambda^{-1}} \).

At this point, we notice that the dominated convergence theorem gives \( \lim_{k \to +\infty} E_\lambda(k) = E_\lambda(\sqrt{6r}) \) since \( E[e^{\xi S_1^2}] < +\infty \) for all \( \xi \in \mathbb{R} \) due to the fact that \( r > 0 \). The monotone convergence theorem yields \( \lim_{k \to +\infty} E[X_1 e^{\xi S_1 + 4X_1}] = E[X_1 e^{\xi S_1 + \sqrt{6r}X_1}] \). It follows that
\[ \lim_{k \to +\infty} \frac{\varphi'(k)}{E_\lambda(\sqrt{6r}) + \epsilon_\lambda + \lambda^{-1}} \leq \frac{E[X_1 e^{\xi S_1 + \sqrt{6r}X_1}]}{E[X_1 e^{\xi S_1 + \sqrt{6r}X_1}]}. \]

This bound demonstrates (J.6) by sending \( \lambda \) to infinity as \( \lim_{\lambda \to +\infty} \epsilon_\lambda = 0 \) and
\[ E_\lambda(\sqrt{6r}) = E \left[ S_1 e^{\xi S_1 + \sqrt{6r}X_1} \mathbb{1}_{[B_1, +]} |dr \right] = E \left[ S_1 e^{\xi S_1 + \sqrt{6r}X_1} \right]. \]

J.4. Proof of part (c)

In the light of section J.1, it remains to verify that
\[ \lim_{k \to 0} \varphi'(k) = \left\{ \begin{array}{ll}
- \frac{E[X_1]}{E[S_1]} & \text{if } E[S_1^2] > 0, \\
- \sqrt{\frac{E[S_1^2]}{E[S_1]}} & \text{if } E[S_1^2] = 0.
\end{array} \right. \]

The case \( \ell > 0 \) is immediate. Assume that \( \ell > 0 \) and fix a positive number \( \zeta < \ell \). Since \( \lim_{k \to 0} \varphi(k) = 0 \) as \( -\varphi \) is convex and lower semicontinuous (see [45], corollary 7.5.1), there exists \( \delta > 0 \) such that \( \varphi(k) \leq \zeta \) for \( k \in (-\delta, 0) \). Then, \( (X_1 \vee S_1)e^{\varphi(k) S_1 + kX_1} \leq (X_1 \vee S_1)e^{\xi S_1} \) for \( k \in (-\delta, 0) \) with \( E[(X_1 \vee S_1)e^{\xi S_1}] < +\infty \) as \( \zeta < \ell \). The dominated convergence theorem gives
\[ \lim_{k \to 0} \varphi'(k) = - \lim_{k \to 0} \frac{E[X_1 e^{\varphi(k) S_1 + kX_1}]}{E[X_1 e^{\varphi(k) S_1 + kX_1}]} = E[X_1]. \]

To address the case \( \ell = 0 \) and \( E[S_1^2] < +\infty \), we notice at first that the identity \( E[e^{\varphi(S_1 + kX_1)}] = 1 \) for \( k < 0 \), combined with the inequality \( e^x \geq 1 + x \) valid for all \( x \in \mathbb{R} \), gives \( \varphi(k)E[S_1] + kE[X_1] \leq 0 \) with \( E[X_1] = \sqrt{8/\pi} E[S_1^2] \). On the other hand, the concavity of \( \varphi \) shows that \( 0 = \varphi(0) \leq \varphi(k) - \varphi'(k)k \) for \( k < 0 \). Thus, \( \varphi(k) \geq -E[X_1]/E[S_1] \) for all \( k < 0 \). Let us prove that \( \lim_{k \to 0} \varphi'(k) \leq -E[X_1]/E[S_1] \). Since for every \( k < 0 \)
\[ \varphi'(k) = - \frac{E[X_1 e^{\varphi(k) S_1 + kX_1}]}{E[S_1 e^{\varphi(k) S_1 + kX_1}]}, \]
as \( \varphi(k) \geq 0 \) and since \( \lim_{k \to 0} E[X_1 e^{kX_1}] = E[X_1] \) by the monotone convergence theorem, we find \( \lim_{k \to 0} \varphi'(k) \leq -E[X_1]/E[S_1] \) if \( \lim_{k \to 0} E[S_1 e^{\varphi(k) S_1 + kX_1}] = E[S_1] \). The monotone convergence theorem gives \( \lim_{k \to 0} E[e^{-|k| S_1^2/2} \mathbb{1}_{[B_1, +]} |dr] = 1 \) for all \( s > 0 \). At the same time, for \( k < 0 \) we have
the two bounds \( \varphi(k) \leq \nu_1 |k|^{2/3} \) as \( \ell = 0 \) and \( \mathbb{E} |e^{-|k|^{1/2} \int_0^1 [B_t, \tau]}| dt | \leq C e^{-\nu_1 |k|^{2/3}} \) for any \( s \geq 0 \). The latter allow us to invoke the dominated convergence theorem to state that
\[
\lim_{k \to 0} \mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + kX_1} \right] = \int_{[0, +\infty)} \lim_{k \to 0} \mathbb{E} \left[ e^{-|k|^{1/2} \int_0^1 [B_t, \tau]} P(dx) \right] = \int_{[0, +\infty)} sP(ds) = \mathbb{E}[S_1].
\]

Finally, we discuss the case \( \mathbb{E}[S_1^{3/2}] = +\infty \). Pick \( \lambda > 0 \) and observe that, similarly to (7.1), for all \( k < 0 \)
\[
\varphi'(k) = - \frac{\mathbb{E}[X_1 e^{\varphi(k) S_1 + kX_1}]}{\mathbb{E}[X_1 e^{\varphi(k) S_1 + kX_1}]} \leq - \frac{\mathbb{E}[X_1 e^{\varphi(k) S_1 + kX_1}]}{\mathbb{E}[S_1 e^{\varphi(k) S_1 + kX_1} P(X_1 \leq \lambda S_1)] + \lambda^{-1} \mathbb{E}[X_1 e^{\varphi(k) S_1 + kX_1}]}.
\]

Since \( \varphi(k) \geq 0 \) for \( k < 0 \) and since the function that maps \( x > 0 \) in \( \frac{x^2}{a+x} \) is non-decreasing for any \( a \geq 0 \), we find
\[
\varphi'(k) \leq - \frac{\mathbb{E}[X_1 e^{kX_1}]}{\mathbb{E}[S_1 e^{\varphi(k) S_1 + kX_1} P(X_1 \leq \lambda S_1)] + \lambda^{-1} \mathbb{E}[X_1 e^{kX_1}]}.
\]

Now we make use of the known fact [61] that there exists a constant \( D > 0 \) such that for all \( x \geq 0 \)
\[
P \left( \int_0^1 |B_t, \tau| dt \leq x \right) \leq De^{-\frac{4x^3}{3\pi^2}}.
\]

Then, for all \( k < 0 \) in a neighborhood of the origin we have \( \varphi(k) \leq 2\nu_1^3 / 27 \lambda^2 \) as \( \lim_{k \to 0} \varphi(k) = 0 \) and
\[
\mathbb{E} \left[ S_1 e^{\varphi(k) S_1 + kX_1} P(X_1 \leq \lambda S_1) \right] \leq \mathbb{E} \left[ S_1 e^{\varphi(k) S_1} P \left( \sqrt{\frac{1}{2}} \int_0^1 [B_t, \tau] dt \leq \lambda \right) \right]
\leq D \mathbb{E} \left[ S_1 e^{\varphi(k) - \frac{4\nu_1^3}{3\pi^2} S_1} \right] \leq D \mathbb{E} \left[ S_1 e^{-\frac{2\nu_1^3}{3\pi^2} S_1} \right] < +\infty.
\]

In conclusion, for every \( k < 0 \) in a neighborhood of the origin
\[
\varphi'(k) \leq - \frac{\mathbb{E}[X_1 e^{kX_1}]}{D \mathbb{E}[S_1 e^{-\frac{2\nu_1^3}{3\pi^2} S_1}]} + \lambda^{-1} \mathbb{E}[X_1 e^{kX_1}]. \quad (J.8)
\]

The monotone convergence theorem yields \( \lim_{k \to 0} \mathbb{E}[X_1 e^{kX_1}] = \mathbb{E}[X_1] = \sqrt{8/9\pi} \mathbb{E}[S_1^{3/2}] = +\infty \). Thus, by sending \( k \to 0 \) from below in (J.8) we get \( \lim_{k \to 0} \varphi'(k) \leq -\lambda \). The arbitrariness of \( \lambda \) implies \( \lim_{k \to 0} \varphi'(k) = -\infty \).

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