Anti-concentration of polynomials: dimension-free covariance bounds and decay of Fourier coefficients

Itay Glazer  
Northwestern University  

Dan Mikulincer *  
Weizmann Institute of Science  

July 15, 2022

Abstract

We study random variables of the form $f(X)$, when $f$ is a degree $d$ polynomial, and $X$ is a random vector on $\mathbb{R}^n$, motivated towards a deeper understanding of the covariance structure of $X^\otimes d$. For applications, the main interest is to bound $\text{Var}(f(X))$ from below, assuming a suitable normalization on the coefficients of $f$. Our first result applies when $X$ has independent coordinates, and we establish dimension-free bounds. We also show that the assumption of independence can be relaxed and that our bounds carry over to uniform measures on isotropic $L_p$ balls. Moreover, in the case of the Euclidean ball, we provide an orthogonal decomposition of $\text{Cov}(X^\otimes d)$. Finally, we utilize the connection between anti-concentration and decay of Fourier coefficients to prove a high-dimensional analogue of the van der Corput lemma, thus partially answering a question posed by Carbery and Wright.

1 Introduction

Let $X \sim \mu$ be a random vector in $\mathbb{R}^n$. Fix $d \in \mathbb{N}$ and consider the tensor power $X^\otimes d$, which is a random vector in $(\mathbb{R}^n)^\otimes d$. The main motivation for this present study came from trying to understand the spectrum of the covariance matrix, $\text{Cov}(X^\otimes d)$. This question has lately gained interest in the study of central limit theorems for tensor powers ([13, 28, 41, 44, 45]) with connections to random geometric graphs ([10–12, 38] and universality of neural networks ([26]).

Specifically, we are interested in identifying regimes where the smallest, non-trivial, eigenvalue of $\text{Cov}(X^\otimes d)$ can be bounded from below in a dimension-free way. We remark that corresponding bounds for the largest eigenvalue can be proved in a straightforward manner using standard concentration techniques, and that, typically, one cannot expect to obtain dimension-free bounds (see [26, Lemma 4] and the remark that follows).

Observe that $\text{Cov}(X^\otimes d)$ is an $n^d \times n^d$ matrix, which is necessarily singular due to symmetries. Thus, we slightly abuse notations and consider $X^\otimes d$ as a random element in $\text{Sym}_d(\mathbb{R}^n)$, the subspace of symmetric tensors. Note that even if $\text{Cov}(X)$ is simple, say if $X$ is isotropic, $\text{Cov}(X^\otimes d)$ can be quite complicated because of the introduced dependencies.

*DM is partially supported by a European Research Council grant no. 803084
To set the stage for our results, we now rephrase the problem as a question about anti-concentration of polynomials. Introduce the multi-indices \((I_1, \ldots, I_n) = I \in \mathbb{N}^n\), for which we use the standard multi-index notation. For \((x_1, \ldots, x_n) = x \in \mathbb{R}^n\),

\[ |I| = \sum_{i=1}^{n} I_i \text{ and } x^I = \prod_{i=1}^{n} x_i^{I_i}. \]

We fix a standard orthonormal basis for \(\text{Sym}_d(\mathbb{R}^n)\), indexed by the multi-indices, \(\{e_I\}_{|I|=d}\). To bound the eigenvalues of \(\text{Cov}(X \otimes d)\) from below it will be enough to show that if \(v \in \text{Sym}_d(\mathbb{R}^n)\) is a unit vector, then \(\text{Var}(\langle v, X \otimes d \rangle)\) is large. Write

\[ v = \sum_{|I|=d} v_I e_I \text{ with } \sum_{|I|=d} v_I^2 = 1. \]

Let us define the homogeneous degree \(d\) polynomial \(f : \mathbb{R}^n \to \mathbb{R}\), by

\[ f(x) = \sum_{|I|=d} v_I x^I = \langle x^d, v \rangle. \]

Hence, \(\text{Var}(\langle v, X \otimes d \rangle) = \text{Var}(f(X))\). From this perspective, our original question reduces to showing that if \(f\) is a homogeneous polynomial, such that the square of its coefficients sums to 1, then \(f(X)\) cannot be too concentrated around its expectation.

The phenomenon of anti-concentration is further manifested through sublevel set and Fourier estimates. A polynomial \(f(X)\) which is not too concentrated around any point is expected to have a low probability of being contained in a small interval in \(\mathbb{R}\), and to have fast decay of Fourier coefficients (see the discussion in Section 2.3). We explore all of the above in this paper.

Our main results are summarized below:

- We show that if \(\mu\) is a product measure, one can bound \(\text{Var}(f(X))\) in a way that depends only on the degree \(d\) and the marginal of the measure \(\mu\). Moreover, the bound is uniform over isotropic log-concave measures. The result also applies to non-homogeneous polynomials, under some appropriate assumption concerning the coefficients of \(f\).

- To allow some form of dependence, we also consider the case where \(\mu\) is the uniform measure on an isotropic \(L_p\) ball and obtain corresponding results.

- In case \(X\) is uniformly distributed on the isotropic Euclidean ball, or more generally, when \(X\) is radially symmetric, we completely characterize the spectrum of \(\text{Cov}(X \otimes d)\) and express the eigenvectors in terms of the spherical harmonics.

- When specializing to log-concave measures, we also establish sublevel estimates. Namely, not only is \(\text{Var}(f(X))\) large, but for \(\epsilon > 0\), one can control,

\[ \mathbb{P}(\{|f(X)| \leq \epsilon\}). \]

- We apply our results to log-concave product measures and derive a dimension-free multivariate analogue of the classical van der Corput lemma for polynomials (cf. [14, Section 7]). Informally, let \(f\) be a polynomial of degree \(d\) that has at least one large coefficient, which corresponds to a monomial of degree \(d\). Then, if \(\mu\) is a log-concave product measure, the Fourier coefficients of \(f \ast \mu\) decay rapidly. When considering the cube, this gives a partial answer to a question asked by Carbery and Wright in [15].
Acknowledgments: We are indebted to an anonymous referee for spotting a mistake in Lemma 8 in an earlier version, as well as for many useful comments.

2 Main results and related work

Before stating our main results let us first introduce some notation and definitions.

2.1 Definitions, notation and conventions

If \( f : \mathbb{R}^n \to \mathbb{R} \), \( f(x) = \sum_{|I| \leq d} \alpha_I x^I \) is a degree \( d \) polynomial, we define its \( d \)-level content as,
\[
\text{coeff}_d(f) := \sqrt{\sum_{|I| = d} \alpha_I^2}.
\]

As will become apparent, \( \text{coeff}_d(f) \) may serve as a scale parameter to measure the variance of the push-forward measure \( f_\ast \mu \).

If \( \nu \) is a measure on \( \mathbb{R} \), we will denote by \( \nu \otimes^n \) its \( n \)-fold tensor product, which is a product measure on \( \mathbb{R}^n \). We say that a measure on \( \mathbb{R}^n \) is isotropic if it is centered and its covariance matrix is the identity. If \( \mu \) is of the form \( \mu = e^{-\varphi(x)} dx \) for some convex function \( \varphi \), we will say that \( \mu \) is log-concave.

As a convention, an absolute constant will be denoted by \( C \), \( C' \), etc. A constant depending on a given data will be denoted using subscript, e.g. \( C_d \) (resp. \( C_{d,n} \)) is a constant depending only on \( d \) (resp. \( d \) and \( n \)). Still, when formulating the main results, to maximize clarity, we will state the precise dependence of the constants on the data.

2.2 Variance bounds for polynomials

Our first main result deals with product measures.

**Theorem 1.** Let \( \mu \) be a centered measure on \( \mathbb{R} \) whose support is infinite and let \( d \in \mathbb{N} \). Then:

1. There exists a constant \( C_{\mu,d} \), which depends on \( \mu \) and \( d \) only, such that for every \( n \in \mathbb{N} \) and every polynomial \( f : \mathbb{R}^n \to \mathbb{R} \) of degree \( d \),
   \[
   \text{Var}_{\mu \otimes^n}(f) \geq C_{\mu,d} \cdot \text{coeff}_d^2(f).
   \]

2. If \( \mu \) is also log-concave and isotropic, one may always take \( C_{\mu,d} = \frac{1}{2^d} \).

The requirement that \( \mu \) has infinite support is necessary here. Otherwise, one can always choose a polynomial \( f \) of degree large enough, so that \( f \) vanishes on the support of \( \mu \otimes^n \). In which case \( \text{Var}_{\mu \otimes^n}(f) = 0 \). Theorem 1 includes in it the standard Gaussian, which was considered before in [26, Lemma 5]. We recover this result and actually improve upon the stated constant.

Most of the work on normal approximations for tensor powers revolved around product measures (see [13, 28]). In this case, Theorem 1 gives a complete dimension-free picture. Still, the question is also interesting for measures that do not have a product structure. Let us point out that our proof of Theorem 1 goes through an orthogonal decomposition of \( L^2(\mu \otimes^n) \), which relies on a particular form taken by orthogonal polynomials of measures on the real line. Hence,
it is adapted to product measures, and we are not able to apply it in the general case (however, see [5], for some examples of high-dimensional orthonormal polynomials, where our method could prove useful).

To address the point raised above, we identify one class of non-product measures where we can derive similar results, the uniform measures on isotropic \( L_p \) balls. For \( p \geq 1 \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) define its \( p \)-norm, by \( \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \) and define the unit ball of this norm, \( B_{p,n} := \{ x \in \mathbb{R}^n : \|x\|_p \leq 1 \} \).

An isotropic \( L_p \) ball is a re-normalization \( \tilde{B}_{p,n} = z_{p,n} B_{p,n} \), such that the measure \( \text{Uniform}(\tilde{B}_{p,n}) \) is isotropic.

The uniform measure on \( \tilde{B}_{p,n} \) is reminiscent of a product measure. Specifically, it is a well known fact that if \( Y \) is a random vector in \( \mathbb{R}^n \) with a product density, proportional to \( e^{-\|z\|_p^p} \), then if \( U \sim \text{Uniform}([0,1]) \) is independent from \( Y \), we have that \( U^{\frac{1}{p}} \frac{1}{\|Y\|_p} \) is uniformly distributed on \( B_{p,n} \) (see [50]). Coupling this with the previous theorem we then obtain.

**Theorem 2.** Let \( p \geq 1 \), and let \( \mu = \text{Uniform}(\tilde{B}_{p,n}) \). Fix \( d \in \mathbb{N} \), then there exists a constant \( C_d > 0 \), which depends only on \( d \), such that if \( f : \mathbb{R}^n \to \mathbb{R} \) is a degree \( d \) homogeneous polynomial,

\[
\mathbb{E}_\mu[f^2] \geq C_d \text{coeff}_d^2(f).
\]

The constant \( C_d \) in Theorem 2 is explicit. Since it has a somewhat complicated expression, we chose to present it this way. Whether the same conclusion holds for general isotropic log-concave measures, maybe with suitable symmetries, is an interesting question that is left open.

In contrast to Theorem 1, Theorem 2 is restricted to homogeneous polynomials and only deals with the second moment, as opposed to the variance. As it turns out, this is a necessity, as illustrated by the following example.

**Example 3.** Suppose that \( X_n \sim \text{Uniform}(\tilde{B}_{p,n}) \) for \( p \) an even natural number, and consider the following polynomial \( f_n(x) = \frac{1}{\sqrt{n}} \left( \|x\|_p^p - \mathbb{E} \left[ \|X_n\|_p^p \right] \right) \) of degree \( p \). Then \( \text{coeff}_p(f) = 1 \). However, an easy calculation (see Section 6.2.1) shows,

\[
\text{Var} \left( \frac{1}{\sqrt{n}} \|X_n\|_p^p \right) = \mathbb{E} \left[ f_n^2(X_n) \right] \xrightarrow{n \to \infty} 0. \tag{1}
\]

One may wonder whether polynomials satisfying (1) are abundant, or whether it is some pathological example. When \( X \sim \text{Uniform}(\tilde{B}_{2,n}) \), the following proposition shows that the latter holds, i.e. the polynomial \( \frac{1}{\sqrt{n}} \|x\|_2^2 \) is essentially the only bad example. We do this by providing a complete description of the eigenvalues and eigenvectors of the matrix \( \text{Cov} \left( X_{\otimes d} \right) \) in terms of spherical harmonics. For a more complete picture we refer to Section 6.

**Proposition 1** (see Corollary 9). Let \( X_n \sim \text{Uniform}(\tilde{B}_{2,n}) \). Write \( \lambda_1 \leq \lambda_2 \leq \ldots \) for the eigenvalues of the matrix \( \text{Cov} \left( X_{\otimes d} \right) \), in increasing order. Then the following hold:

1. “pathological spectral gap”: If \( d = 2 \), then

\[
\lambda_1 = \frac{4}{n+4} = O(n^{-1}),
\]

has multiplicity one, with eigenvector \( \frac{1}{\sqrt{n}} \|x\|_2^2 \), and the rest of the eigenvalues are bounded from below by \( \frac{2}{d} \).
2. For $d \geq 3$ we have a uniform lower bound on the eigenvalues

$$\lambda_i \geq \frac{1}{(d+1)!},$$

for all $n$. If $n \geq d$, then the $\lambda_1$-eigenspace is spanned by monomials of the form $x_{i_1} \cdots x_{i_d}$ with $i_1 < \cdots < i_d$.

We remark that the lower bound in Item (2) can be further improved (see Remark (10)), and in fact $\lim_{n \to \infty} \lambda_1 = 1$ whenever $d \geq 3$.

Sub-level set estimates: Anti-concentration of polynomials with log-concave variables is a well studied topic with many known results, most notably the work of Carbery and Wright ([15]), but see also [43]), which also established reverse Hölder inequalities. However, the results listed above are, in some sense, of a different flavor.

In brief, (see Theorem 6 below for exact formulation), the Carbery-Wright inequality says that if $X$ is log-concave and $f$ is a degree $d$ polynomial, then for every $\varepsilon > 0$,

$$\mathbb{P} (|f(X)| \leq \varepsilon) \lesssim \frac{\varepsilon^{\frac{d}{2}}}{\mathbb{E} [|f(X)|^2]^{\frac{1}{2}}},$$

In other words, the inequality says something about sublevel sets of the form $\{x \in \mathbb{R}^n : |f(x)| \leq \varepsilon\}$ under a moment assumption.

In the same context, our result can roughly be stated as: if the coefficients of $f$ are large, then $f(x)$ is not too concentrated around its mean, in the sense that the variance is large.

While the results are not implied by nor imply one another, they turn out to be complementary. By combining our results we then obtain the following corollary, which is essentially a sublevel estimate, where the moment assumption is replaced by an assumption on the coefficients.

**Corollary 4.** Let $\mu$ be a log-concave measure on $\mathbb{R}^n$ with $X \sim \mu$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d$. Fix $\varepsilon > 0$,

1. If $\mu = \nu^{\otimes n}$ is an isotropic, log-concave product measure, then there exists a universal constant $C > 0$, such that for any $y \in \mathbb{R}$,

$$\mathbb{P} (|f(X) - y| \leq \varepsilon) \leq C d \left( \frac{\varepsilon}{\text{coeff}_d(f)} \right)^{\frac{1}{d}}.$$

2. If $\mu = \text{Uniform}(\tilde{B}_{p,n})$, for some $p \geq 1$ and $f$ is homogeneous,

$$\mathbb{P} (|f(X)| \leq \varepsilon) \leq C_d \left( \frac{\varepsilon}{\text{coeff}_d(f)} \right)^{\frac{1}{d}},$$

where $C_d > 0$ is a constant which depends only on $d$.

Note that in Item (2), we provide estimates only for balls around 0. Similarly as in the discussion after Theorem 2, by taking $f_n(x) = \frac{1}{\sqrt{n}}\|x\|_p^n$ and $X_n \sim \text{Uniform}(\tilde{B}_{p,n})$ one can see there is no hope for uniform estimates for sets of the form $\{|f(X) - y| \leq \varepsilon\}$. 


There are some other works which considered anti-concentration of polynomials under an assumption on the coefficients ([21, 22, 40, 47]), mostly as part of the Littlewood-Offord theory, which first introduced the problem for linear maps. However, previous results were constrained to multi-linear polynomial with some combinatorial properties. Another related paper is [27], where dimension-dependent results were obtained in a similar setting to the one considered here. We also mention the work of Paouris ([46], see also [35]), which derived a similar result for the push-forward of general log-concave measures under linear maps.

### 2.3 Decay of Fourier coefficients

Given a measure \( \mu \) on \( \mathbb{R}^n \) and a polynomial \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), anti-concentration results (as in Corollary 4) can be rephrased by saying that the density of the pushforward measure \( f_* \mu \) does not explode too quickly around its singular values. This explosion rate is controlled by the rate of decay of the Fourier coefficients \( t \mapsto \mathcal{F}(f_* \mu)(t) \) of \( f_* \mu \). In fact, using standard Fourier analysis, one can show that in order to prove anti-concentration inequalities as in Corollary 4, it is enough to give an upper bound of the form \( |\mathcal{F}(f_* \mu)(t)| < C_d \cdot |t|^{-\frac{n}{2}} \) (for \( \text{coeff}_d(f) = 1 \)).

In this work, we take the other direction and use our anti-concentration results to obtain improved bounds on the decay of Fourier coefficients. This reasoning is not new. Indeed, in the case \( n = 1 \), one of the earliest results, dating back to the 1920’s, is the classical van der Corput lemma connecting between derivatives of a function \( f \) to the decay of its Fourier coefficients.

**Lemma 1** ([51, Proposition 2]). Let \( f \) be a smooth function on \( \mathbb{R} \), and let \( \mu = \rho(x) dx \) be a measure on \( \mathbb{R} \). Fix \( a < b \) and suppose that \( k \in \mathbb{N} \) is such that \( f^{(k)} \geq 1 \) for every \( x \in (a, b) \). Then, if either of the following conditions holds,

\[
\begin{align*}
\text{•} & \quad k \geq 2, \\
\text{•} & \quad k = 1, \text{ and } f' \text{ is monotonic},
\end{align*}
\]

there exists a constant \( C_k \), which depends only on \( k \), such that,

\[
|\mathcal{F}(f_* \mu_{|(a,b)})(t)| := \left| \int_a^b e^{itf(x)} d\mu(x) \right| \leq C_k |t|^{-\frac{n}{2}} \left( \rho(b) + \int_a^b |\rho'(x)| \, dx \right). \tag{2}
\]

In [14], a multivariate analogue of the van der Corput lemma was obtained. In particular, given a degree \( d \) polynomial \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), and \( \mu = \text{Uniform}([-1, 1]^n) \), if \( \partial^I f \big|_{[-1, 1]^n} \geq 1 \), where \( \partial^I = \partial x_1^{I_1} \ldots \partial x_n^{I_n} \), for some \( |I| = d \), then,

\[
|\mathcal{F}(f_* \mu)(t)| < C_{d,n} |t|^{-\frac{n}{2}}, \tag{3}
\]

where \( C_{d,n} \) depends on \( n \) and \( d \) (see [14, Theorem 7.2]).

Other than that, there have been many works on generalizing the van der Corput lemma to higher dimensions, and by now there are plenty of results for different classes of functions and domains (see for example [16, 17, 30, 49]). However, to the best of our knowledge, none of these results include dimension-free estimates. In [15], Carbery and Wright asked whether the constant in (3) can be replaced by a dimension-free constant, while only assuming \( \|f\|_1 \geq C_d \) and \( \int_{[-1,1]^n} f = 0 \). Since our Theorem 1 is inherently dimension-free we are able to prove the first dimension-free bound, which applies to a large class of measures. In particular, when specializing to the cube, this gives a partial answer to their question.
Theorem 5. Let $\nu^{\otimes n}$ be an isotropic log-concave product measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d$, $f(x) = \sum_{|I|\leq d} \alpha_I x^I$. Denote $M_d(f) = \max\{|\alpha_I| : |I| = d\}$. Then, for every $t \in \mathbb{R}$,
\[
\left| \int_{\mathbb{R}^n} e^{itf(x)} d\nu^{\otimes n}(x) \right| \leq \frac{Cd}{(M_d(f)|t|)^{\frac{d}{2}}},
\]
for some universal constant $C > 0$.

Theorem 5 gives a positive answer to the question posed in [15], under the assumption that $M_d(f) \geq C_d'$. This is a stronger requirement than the one in Theorem 1 which only requires that $\text{coeff}_d(f) \geq C_d'$, and both are stronger than the condition than $||f||_1 \geq C_d$ (by Theorem 1). We do not know whether this is actually necessary but point out that a recent analogous result for the Gaussian measures obtained precisely the same dependence on the quantity $M_d(f)$ (see [37, Corollary 4.1]).

2.4 Further discussion and future directions

In this paper we study the pushforward $f_*\mu$ of a well-behaved measure $\mu$ under polynomial maps $f : \mathbb{R}^n \to \mathbb{R}$ of bounded degree. We focus on the regime where $n$ is arbitrarily large, motivated by questions in high-dimensional geometry. There are a few interesting variants which are worth mentioning. For simplicity of presentation, we assume $f$ is a homogeneous polynomial of degree $d$, but the discussion below easily generalizes to all polynomials.

First of all, one can also consider regimes of bounded complexity (i.e. $n, d$ and $\mu$ are fixed), and try to obtain more refined estimates than the ones afforded in the asymptotic realm. When $f$ and $\mu$ are fixed, it is known (see [32], as well as [3, Parts II,III] and the references within) that the explosion rate of $\frac{df}{dx}$ and the decay of Fourier coefficients of $f_*\mu$ are both controlled by the singularities of $f$. The behavior of these singularities can be quantified by the so-called log-canonical threshold of $f$, or $\text{lct}(f)$, so that bad singularities correspond to low values of the lct (see [36, 42] for a definition and an overview on the log-canonical threshold). In this case, when $\varepsilon \ll 1$, the term $\varepsilon^q$ in Corollary 4 may actually be replaced by $\varepsilon^{\text{lct}(f)}$. Moreover, one always has $\text{lct}(f) \geq 1/d$. This suggests that, while being tight, the Carbery-Wright inequality is somewhat pessimistic, and could be improved for specific polynomial mappings.

With this in hand, it is still a non-trivial task to obtain effective sublevel and Fourier estimates in terms of the lct, which are uniform on reasonable complexity classes of $\mu$, and with $\deg(f)$ bounded. One can further consider the case of polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^m$, for $m > 1$. Here, $\text{lct}(f)$ still controls the explosion rate of $f_*\mu$ but does not control $\mathcal{F}(f_*\mu)$ anymore. Concrete uniform bounds will be of interest.

Secondly, instead of working over $\mathbb{R}$, one can work with any local field $F$. For $p$-adic fields, the study of $f_*\mu$, for suitable $\mu$, is of arithmetic nature; for example, one can take the collection of normalized Haar measures $\mu_{\mathbb{Z}_p^n}$ on $\mathbb{Z}_p^n$ (the ring of $p$-adic integers), which can be thought of as a $p$-adic analogue of $\mathcal{B}_{2,n}$ or the normalized Gaussian. For simplicity, assume that $f$ has coefficients in $\mathbb{Z}$. Then, for each $k \in \mathbb{N}$, we have
\[
P\left(|f(X)|_p \leq p^{-k}\right) = \#\left\{a \in (\mathbb{Z}/p^k\mathbb{Z})^n : f(a) = 0 \mod p^k\right\}
\]
(4)
(here $| \cdot |_p$ stands for the $p$-adic absolute value). Thus, sublevel set estimates translate in the $p$-adic world into estimates on the number of solutions of congruences of $f$ modulo $p^k$. This is
a fundamental question in number theory which is strongly related to Igusa’s local Zeta function (see e.g. [23–25, 31]). For a fixed $f$, sharp sublevel set estimates can be given (see [31], and the discussion after [53, Corollary 2.9]); there exists a constant $C_{f,p} > 0$ such that for all $k \in \mathbb{N}$,

$$
\mathbb{P} \left( |f(X)|_p \leq p^{-k} \right) < C_{f,p} \cdot k^{n-1} p^{-k \text{lct}(f)}.
$$

In [53, Corollary 2.9] and [29, Theorem 8.18], sublevel set estimates were given for polynomial maps $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ for $m \geq 1$, and more recently, sharp and field independent estimates were given in [20, Theorem 4.12].

In a similar fashion, the study of $F(f^* \mu)$, translates in the $p$-adic world to the study of exponential sums, which goes back to Gauss. The Fourier coefficients are essentially of the following form:

$$
\frac{1}{p^{kn}} \sum_{x \in (\mathbb{Z}/p^k \mathbb{Z})^n} \exp \left( \frac{2\pi i f(x)}{p^k} \right).
$$

Igusa showed [32] that (5) can be bounded from above by $C_{f,p} \cdot k^{n-1} \cdot p^{-k \text{lct}(f)}$, and further conjectured that $C_{f,p}$ can be replaced by $C_f$. This was recently proved in [18, Theorem 1.5]. Moreover, $p$-adic analogues of the van der Corput lemma were given in [19, 48].

It will be interesting to find sublevel and Fourier estimates in the $p$-adic case, when the complexity is unbounded. This, along with the variants presented above will be studied in a sequel to this paper.

### 3 Orthogonal polynomials

For the rest of this section we fix a centered measure $\mu$ on $\mathbb{R}$, such that for every $d \in \mathbb{N}$, $\int \mathbb{R} x^d d\mu(x) < \infty$ and such that $\mu$ is supported on infinitely many points. We associate to $\mu$ a sequence of orthonormal polynomials $\{p_d\}_{d=0}^{\infty}$ satisfying,

$$
\langle p_d, p_{d'} \rangle_{L^2(\mu)} := \int_{\mathbb{R}} p_d(x)p_{d'}(x)d\mu(x) = \delta_{d,d'}.
$$

Such a sequence may be obtained by applying the Gram-Schmidt algorithm to the set $\{1, x, x^2, \ldots \}$, with respect to the standard inner product on $L^2(\mu)$. Remark that, by definition, if $f \in L^2(\mu)$ then

$$
f = \sum_{d=0}^{\infty} \langle f, p_d \rangle_{L^2(\mu)} p_d,
$$

where the equality is to be understood in $L^2(\mu)$, and where

$$
\langle f, p_d \rangle_{L^2(\mu)} = \int_{\mathbb{R}} f(x)p_d(x)d\mu(x).
$$

Observe that $p_0 \equiv 1$ and that since $\mu$ is centered, $p_1(x) \propto x$. Moreover, it is easy to see that for every $d \in \mathbb{N}$, $p_d$ is a polynomial of degree $d$. The reader is referred to [52] for more details pertaining to orthogonal polynomials. We will mostly be interested in the following simple representation which is a consequence of the Gram-Schmidt process.
Inside the Hilbert space $L^2(\mu)$, for $k \in \mathbb{N}$, define $Q_k : L^2(\mu) \to L^2(\mu)$ as the orthogonal projection onto the closed subspace $\text{span}\{1, x, x^2, \ldots, x^k\}$. Then,

$$p_d = \frac{1}{c_{\mu,d}} (x^d - Q_{d-1}x^d), \quad (7)$$

where the constant $c_{\mu,d} := \left( E_\mu \left[ (x^d - Q_{d-1}x^d)^2 \right] \right)^{\frac{1}{2}}$ ensures that $E_\mu[p_d^2] = 1$. Note that $c_{\mu,d} > 0$. Indeed, since $\mu$ is not supported on a finite number of points, $x^d \notin \text{span}\{1, x, \ldots, x^{d-1}\}$.

We now show that monomials have tractable expansions with respect to the above orthogonal polynomials.

**Lemma 2.** Fix $d \in \mathbb{N}$,

1. For any $k > d$, $\langle p_k(x), x^d \rangle_{L^2(\mu)} = 0$.
2. $\langle p_d(x), x^d \rangle_{L^2(\mu)} = c_{\mu,d}$, where $c_{\mu,d}$ is as defined by (7).

**Proof.** Item (1) is a direct consequence of the Gram-Schmidt process. For Item (2), note that since $Q_{d-1}$ is an orthogonal projection, we have:

$$c_{\mu,d} = \frac{1}{c_{\mu,d}} \left( E_\mu \left[ (x^d - Q_{d-1}x^d)^2 \right] \right)^{\frac{1}{2}} = \frac{1}{c_{\mu,d}} \left( E_\mu \left[ x^{2d} \right] - E_\mu \left[ (Q_{d-1}x^d)^2 \right] \right),$$

which concludes the proof. \(\square\)

We next bound the constant $c_{\mu,d}$ from below. We start with the case of the cube and then use it to prove a general bound for isotropic log-concave measures.

**Lemma 3.** Suppose that $\mu = \text{Uniform}([-1, 1])$. Then,

$$c_{\mu,d} = \langle x^d, p_d(x) \rangle_{L^2(\mu)} \geq \frac{1}{2d}.$$

**Proof.** In this case, the sequence $p_d$ is given by the Legendre polynomials, and we have the following representation (see [52, Chapter 4]):

$$p_d(x) = \frac{\sqrt{2d + 1}}{2d!} \frac{\partial^d}{\partial x^d} (x^2 - 1)^d.$$  

A direct calculation involving a $d$-fold integration by parts (see e.g. [1, Section 15]) gives,

$$\langle x^d, p_d(x) \rangle_{L^2(\mu)} = \frac{1}{\sqrt{2d + 1}} \frac{2^d(d!)^2}{(2d)!} \geq \frac{1}{2d}.$$

\(\square\)

**Lemma 4.** Let $\mu$ be a log-concave and isotropic measure on $\mathbb{R}$. Then $c_{\mu,d} \geq \frac{1}{9 \cdot 188}$. 

9
Proof. Write \( f := x_d - Q_{d-1} x_d \) and \( \mu = g(x) dx \). Since \( \mu \) is log-concave and isotropic, it follows e.g. from [39, Lemma 5.5 and Theorem 5.14] that \( g(x) \geq \frac{1}{16} \) for all \( x \in [-\frac{1}{2}, \frac{1}{2}] \). Hence, we get:

\[
\mathbb{E}_\mu[f^2] \geq \frac{1}{16} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)^2 \, dx = \frac{1}{72} \int_{-1}^{1} \tilde{f}(t)^2 \, dt,
\]

where \( \tilde{f}(t) := f(\frac{1}{\sqrt{d}} t) \) with \( \text{coeff}_d(\tilde{f}) = 9^{-d} \). Let us write \( h_d(x) := \sqrt{2d+1} \frac{\partial^d}{\partial x^d} (x^2 - 1)^d \), for the Legendre polynomial of degree \( d \), as in the proof of Lemma 3. So, from (6),

\[
\frac{1}{2} \int_{-1}^{1} \tilde{f}(t)^2 \, dt \geq \left( \frac{1}{2} \int_{-1}^{1} \tilde{f}(x) h_d(x) \, dx \right)^2.
\]

By first applying Item (1) of Lemma 2 and then Lemma 3, we get,

\[
\left( \frac{1}{2} \int_{-1}^{1} \tilde{f}(x) h_d(x) \, dx \right)^2 = \left( \frac{1}{2} \int_{-1}^{1} \frac{1}{g_d} x^d h_d(x) \, dx \right)^2 \geq \frac{1}{4d^2} \frac{1}{4d}.
\]

The claim follows.

4 Anti-concentration of polynomials

4.1 Product measures - Proof of Theorem 1

We now consider \( \mathbb{R}^n \) equipped with a product measure \( \mu^\otimes n \), where \( \mu \) is some measure on \( \mathbb{R} \). Suppose that \( \{p_d\}_{d=0}^\infty \) is the sequence of orthonormal polynomials, with respect to \( \mu \), as constructed in (7).

To find an orthogonal decomposition of \( L^2(\mu^\otimes n) \), for a multi-index \( I = (I_1, \ldots, I_n) \in \mathbb{N}^n \) we define the multivariate polynomial,

\[
p_I(x) := \prod_{i=1}^n p_{I_i}(x_i).
\]

Since \( L^2(\mu^\otimes n) = L^2(\mu)^\otimes n \) we have that the set \( \{p_I\}_{I \in \mathbb{N}^n} \) is a complete orthonormal system in \( L^2(\mu^\otimes n) \). Our next step is to show that for degree \( d \) polynomials, the inner product with \( p_I \) depends only on the coefficient of \( x^I \), as long as \( |I| = d \).

Lemma 5. Fix \( d \in \mathbb{N} \) and let \( q(x) = \sum_{i=1}^d \sum_{|I|=i} \alpha_I x^I \) be a degree \( d \) polynomial in \( \mathbb{R}^n \). Then, for any \( J \in \mathbb{N}^n \) with \( |J| = d \),

\[
\langle q(x), p_J(x) \rangle_{L^2(\mu^\otimes n)} = \alpha_J \prod_{i=1}^n c_{\mu, I_i},
\]

where \( c_{\mu, I_i} \) is as in (7).
Proof. Clearly, we have,

\[ \langle q(x), p_J(x) \rangle_{L^2(\mu^\otimes n)} = \sum_{I \in \mathbb{N}^n} \sum_{|I| = i} \alpha_I \langle x^I, p_J(x) \rangle_{L^2(\mu^\otimes n)}. \]

We first claim that if \( I \neq J \) with \(|I| \leq d\), then \( \langle x^I, p_J(x) \rangle_{L^2(\mu^\otimes n)} = 0 \). Indeed, since \(|J| = d\), necessarily, there exists some \( j \in [n] \) such that \( J_j > I_j \). We now use the product structure to write,

\[ \langle x^I, p_J(x) \rangle_{L^2(\mu^\otimes n)} = \prod_{i=1}^n \langle x^I_i, p_{J_i}(x_i) \rangle_{L^2(\mu)} = 0. \]

The second equality follows from Lemma 2 which implies \( \langle x^J_j, p_{J_j}(x_j) \rangle_{L^2(\mu)} = 0 \). So,

\[ \langle q(x), p_J(x) \rangle_{L^2(\mu^\otimes n)} = \alpha_J \langle x^J, p_J(x) \rangle_{L^2(\mu^\otimes n)} = \alpha_J \prod_{i=1}^n \langle x^J_i, p_{J_i}(x_i) \rangle_{L^2(\mu)} = \alpha_J \prod_{i=1}^n c_{\mu, J_i}, \]

where we have used Lemma 2 for the last equality.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. From (6), we have,

\[ \text{Var}_{\mu^\otimes n}(f) = \sum_{I \in \mathbb{N}^n, |I| \neq 0} \langle f(x), p_I(x) \rangle_{L^2(\mu^\otimes n)}^2 \geq \sum_{I \in \mathbb{N}^n, |I| = d} \langle f(x), p_I(x) \rangle_{L^2(\mu^\otimes n)}^2 = \sum_{I \in \mathbb{N}^n, |I| = d} \alpha_I^2 \prod_{i=1}^n c_{\mu, I_i}^2, \]

where the second equality is Lemma 5. Since \( c_{\mu, I_i} > 0 \) there exists a constant \( C_{\mu, d} \) such that for any \( I \in \mathbb{N}^n \) with \(|I| = d\), \( \prod_{i=1}^n c_{\mu, I_i}^2 \geq C_{\mu, d} \). This concludes Item (1).

Item (2) is now a direct consequence of (9) and Lemma 4. Indeed,

\[ \text{Var}_{\mu^\otimes n}(f) \geq \sum_{I \in \mathbb{N}^n, |I| = d} \alpha_I^2 \prod_{i=1}^n \left( \frac{1}{9} \cdot \frac{1}{18^d} \right)^2 \geq \sum_{I \in \mathbb{N}^n, |I| = d} \alpha_I^2 \left( \frac{1}{9^d} \cdot \frac{1}{18^d} \right)^2 \geq \text{coeff}^2_{\mu}(f) \cdot \frac{1}{21^d}. \]

4.2 A sublevel estimate for log-concave product measures

The aim of this subsection is to show that, when specializing Theorem 1 to the case of log-concave measures, we can translate our variance estimates into estimates on small balls probabilities, or sublevel estimates. This is essentially the first Item of Corollary 4.

Our main tool for this is the celebrated inequality of Carbery-Wright, which we state in the form suited to our needs.

Theorem 6. ([15, Theorem 8]) Let \( \mu \) be a log-concave measure on \( \mathbb{R}^n \) and let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a polynomial of degree \( d \). Then, if \( X \sim \mu \), for every \( \varepsilon > 0 \),

\[ \mathbb{P}(\{|f(X)| \leq \varepsilon\}) \leq C_d \frac{\varepsilon^{\frac{d}{2}}}{\mathbb{E}[f(X)^2]^{\frac{1}{2}}} . \]
Thus, the Carbery-Wright inequality says that an estimate for the sublevel sets of a polynomial $f$ may be obtained by bounding the second moment of $f$, which is precisely the content of Theorem 1.

**Proof of Items 1 in Corollary 4.** Fix $y \in \mathbb{R}$ and define the polynomial $f_y(x) = f(x) - y$. It is clear that $\text{coeff}_d(f_y) = \text{coeff}_d(f)$. Combining this fact with Theorem 1, we deduce,

$$E[f_y(X)^2] \geq \text{Var}(f_y(X)) \geq \frac{1}{2^{15d}}\text{coeff}_d^2(f).$$

Now, Theorem 6 implies,

$$P(|f(X) - y| \leq \varepsilon) = P(|f_y(X)| \leq \varepsilon) \leq C'd \frac{\varepsilon^{\frac{k}{p}}}{E[f_y(X)^2]^\frac{1}{p}} \leq C'd \left(\frac{\varepsilon}{\text{coeff}_d(f)}\right)^{\frac{k}{p}},$$

for some constant $C' > 0$.

4.3 Anti concentration on $L_p$ balls - Proof of Theorem 2

In this subsection we fix some $p \geq 1$ and the measure $\mu$ on $\mathbb{R}^n$, with density $\frac{1}{(2\Gamma(\frac{1}{p}))^n} e^{-\|x\|^p_p} dx$, where $\Gamma$ stands for the Gamma function. Observe that $\mu$ is a log-concave product measure. Recall that,

$$B_{p,n} = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\},$$

is the unit ball with respect to the norm $\|\cdot\|_p$ and that if $Z = (Z_1, \ldots, Z_n) \sim \mu$ and $U \sim \text{Uniform}([0,1])$ is independent from $Z$, then

$$X = U^\frac{k}{p} \frac{Z}{\|Z\|_p},$$

(10)

is uniformly distributed on $B_{p,n}$ (see [50]). In other words, to generate $X$, one can first generate $Z$ and normalize by $\|Z\|_p$ to obtain something which is distributed according to the normalized cone measure on the boundary of $B_{p,n}$. To get a random vector uniformly distributed on $B_{p,n}$ all that is left is to choose a random scale according to $U^\frac{k}{p}$.

Before proceeding, we need the following technical lemma.

**Lemma 6.** Let $p \geq 1$, and $Z, \mu$ as above. Then, for any $k > -n$,

$$E[\|Z\|^{k}_{p}] = \frac{\Gamma\left(\frac{n+k}{p}\right)}{\Gamma\left(\frac{n}{p}\right)}.$$

Moreover, if $n > k^2$ and $k \geq 2$, then

$$\frac{1}{20} p^{\frac{k}{p}} n^{-\frac{k}{p}} \leq E\left[\frac{1}{\|Z\|_{p}^{k}}\right] = \frac{\Gamma\left(\frac{2k}{p}\right)}{\Gamma\left(\frac{m}{p}\right)} \leq 25 p^{\frac{k}{p}} n^{-\frac{k}{p}}.$$
Proof. Note that for any function \( h : \mathbb{R}_0^d \rightarrow \mathbb{R} \), we have the change of coordinates formula:

\[
    \int_{\mathbb{R}^n} h(||x||_p^d) dx = \frac{2^n \Gamma \left( \frac{1}{p} \right)^n}{p^{n-1} \Gamma \left( \frac{n}{p} \right)} \int_0^\infty r^{n-1} h(r) dr.
\]  

(11)

The pre-factor can be verified by integrating against the density of \( \mu \) (see also [6]). The identity in (11) immediately implies:

\[
    \mathbb{E} \left[ \frac{1}{||Z||_p^k} \right] = \int_{\mathbb{R}^n} ||x||_p^k d\mu(x) = \frac{1}{\left( \frac{2}{p} \Gamma \left( \frac{1}{p} \right) \right)^n} \int_{\mathbb{R}^n} ||x||_p^k e^{-||x||_p^p} dx = \frac{p}{\Gamma \left( \frac{n}{p} \right)} \int_0^\infty |n+k-1| e^{-r^p} dr
\]

\[
    = \frac{1}{\Gamma \left( \frac{n}{p} \right)} \int_0^\infty t^{\frac{n+k}{p}-1} e^{-t} dt = \frac{\Gamma \left( \frac{n+k}{p} \right)}{\Gamma \left( \frac{n}{p} \right)}.
\]

Now, suppose that \( k \geq 2 \) and \( n > k^2 \). To estimate \( \mathbb{E} \left[ \frac{1}{||Z||_p^k} \right] \), we first consider the case \( n < p \). For this, we use Wendel’s inequality for ratios of Gamma functions [33], to deduce,

\[
    \frac{k^p n^{-\frac{1}{p}}}{n^p} \leq \left( \frac{n-k}{p} \right)^{-\frac{k}{p}} \leq \frac{\Gamma \left( \frac{n-k}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} \leq \frac{p}{n-k} \cdot \left( \frac{n}{p} \right)^{1-\frac{k}{p}} \leq 2p^\frac{k}{n} n^{-\frac{1}{p}}.
\]

When \( n \geq p \), we use Stirling’s approximation for the Gamma function ([34]), and the inequality \((1 - \frac{1}{x})^x < \frac{1}{e} < (1 - \frac{1}{x})^{x-1}\) for \( x > 1 \), to deduce:

\[
    \frac{\Gamma \left( \frac{n-k}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} \leq \left( \frac{n-k}{p} \right)^{-\frac{k}{p}} \left( \frac{p}{n-k} \right)^{\frac{k}{p}} e^{\frac{1}{12(n-k)}} \leq 2e^{\frac{k}{p}} \left( 1 - \frac{k}{n} \right)^{\frac{p}{p}} e^{\frac{k}{p}} \leq 4 \left( 1 - \frac{k}{n} \right)^{\frac{p}{p}} n^{-\frac{k}{p}} \leq 4(2e)^{\frac{k}{p}} p^\frac{k}{p} n^{-\frac{k}{p}} \leq 25 p^\frac{k}{n} n^{-\frac{k}{p}}.
\]

To get a corresponding bound in the other direction, we similarly have:

\[
    \frac{\Gamma \left( \frac{n-k}{p} \right)}{\Gamma \left( \frac{n}{p} \right)} \geq e^{-\frac{k}{12n}} \left( 1 - \frac{k}{n} \right)^{\frac{p}{p}} e^{\frac{k}{p}} \geq e^{-\frac{k}{12}} \cdot \left( 1 - \frac{k}{n} \right)^{\frac{p}{p}} \geq \frac{p^\frac{k}{p}}{2} \left( 1 - \frac{1}{k} \right)^{\frac{p}{p}} n^{-\frac{k}{p}} \geq \frac{1}{20 p^\frac{k}{n} n^{-\frac{k}{p}}}.
\]

Combining the above displays finishes the proof.

We now prove the main result of this section, a lower bound for the second moment of a homogeneous polynomial over the unit \( L_p \) ball. The main theorem will follow by appropriately re-scaling \( B_{p,n} \) to be isotropic.

**Lemma 7.** Let the above notations prevail and let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a homogeneous polynomial of degree \( d \). Then as long as \( n > 16d^2 \),

\[
    \mathbb{E}[f^2(X)] \geq C_d \cdot n^{-\frac{2d}{p}} \text{coeff}_d^2(f).
\]
Proof. With the above notations, let us estimate

\[ \mathbb{E} \left[ f^2(X) \right] = \mathbb{E} \left[ f^2 \left( U \frac{Z}{\|Z\|_p} \right) \right] = \mathbb{E} \left[ U_{\frac{2d}{n}} \mathbb{E} \left[ f^2(Z) \mathbb{1}_{\|Z\|_{2d}^p} \right] \right] = \frac{1}{\frac{n}{2d} + 1} \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} f^2(Z) \right]. \]

The first equality is (10), the second is the homogeneity of \( f \) and the third follows by independence of \( U \). Fix \( \delta > 0 \) and define the set,

\[ A_\delta = \{ x \in \mathbb{R}^n : f^2(x) > \delta \}. \]

So, we have

\[ \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} f^2(Z) \right] \geq \delta \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \mathbb{1}_{\mathbb{1} \{ Z \in A_\delta \}} \right]. \quad (12) \]

Moreover, by Cauchy-Schwartz,

\[ \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \mathbb{1}_{\{ Z \notin A_\delta \}} \right] \leq \sqrt{\mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \right] \mathbb{P}(Z \notin A_\delta)}. \]

Since we have assumed \( n > 16d^2 \), we can apply the second part of Lemma 6 twice, for \( k = 2d \) and \( k = 4d \). Thus,

\[ \mathbb{E} \left[ f^2(X) \right] \geq \frac{\delta}{\frac{2d}{n} + 1} \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \mathbb{1}_{\mathbb{1} \{ Z \in A_\delta \}} \right] \]

\[ = \frac{\delta}{\frac{2d}{n} + 1} \left( \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \right] - \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \mathbb{1}_{\mathbb{1} \{ Z \notin A_\delta \}} \right] \right) \]

\[ \geq \frac{\delta}{\frac{2d}{n} + 1} \left( \mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \right] - \sqrt{\mathbb{E} \left[ \frac{1}{\|Z\|_{2d}^p} \right] \mathbb{P}(Z \notin A_\delta)} \right) \]

\[ \geq \frac{\delta p^{\frac{2d}{n}}}{20(\frac{2d}{n} + 1)n^{\frac{2d}{p}}} \left( 1 - 100 \sqrt{\mathbb{P}(Z \notin A_\delta)} \right). \quad (13) \]

We turn to estimate \( \mathbb{P}(Z \notin A_\delta) \). Applying Lemma 6 for a single coordinate, with \( k = 2 \), shows

\[ \mathbb{E} \left[ Z_1^2 \right] = \frac{\Gamma \left( \frac{2}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \geq \frac{1}{4}, \]

where the inequality follows from Wendel’s inequality, [33]. Since \( Z \) is also log-concave, we may invoke Item 1 of Corollary 4. So,

\[ \mathbb{P}(Z \notin A_\delta) = \mathbb{P}(f^2(Z) \leq \delta) = \mathbb{P} \left( |f(Z)| \leq \sqrt{\delta} \right) \leq C_d \left( \frac{\delta}{\text{coeff}^2_d(f)} \right)^{\frac{1}{2p}}. \]

Let us choose

\[ \delta = \frac{\text{coeff}^2_d(f)}{(10^5 C_d)^{2d}} \]

and plug it into (13). As long as \( 2d < n \), we obtain,

\[ \mathbb{E}[f^2(X)] \geq \frac{\delta p^{\frac{2d}{n}}}{80 n - \frac{2d}{p}} \geq \frac{\delta}{80 n - \frac{2d}{p}}. \]

\( \square \)
Theorem 2 is now an immediate consequence.

**Proof of Theorem 2.** Let $X = (X_1, \ldots, X_n) \sim \text{Uniform}(B_{p,n})$ and let $z_{p,n} = \mathbb{E} [X_i^2]^{-\frac{1}{2}}$ be such that $z_{p,n}X$ is isotropic, that is, $z_{p,n}X \sim \text{Uniform}(\tilde{B}_{p,n})$. It follows e.g. from [6, Theorem 7], that $z_{p,n} \geq C \cdot n^\frac{1}{p}$, for an absolute constant $C > 0$. If $n > 16d^2$ then our claim follows by Lemma 7 and homogeneity.

$$E[f^2(z_{p,n}X)] = z_{p,n}^2E[f^2(X)] \geq C_{d} \frac{C_{d} \text{coeff}_d^2(f)}{n^\frac{1}{p}} = C_{d} \text{coeff}_d^2(f).$$

When $n \leq 16d^2$, we can use the fact that $\tilde{B}_{p,n}$ contains a cube of length uniformly bounded from below by a constant depending on $d$. Our claim then follows from Theorem 1. The proof is complete.

We may now also prove Item (2) of Corollary 4.

**Proof of Item 2 in Corollary 4.** The proof is essentially identical to the case of product measures. If $X \sim \text{Uniform}(\tilde{B}_{p,n})$, from Theorem 6,

$$\mathbb{P}(|f(X)| \leq \varepsilon) \leq C_{d} \frac{\varepsilon^\frac{1}{p}}{E[f(X)^2]^\frac{1}{2p}} \leq C_{d} \left( \frac{\varepsilon}{\text{coeff}_d(f)} \right)^{\frac{1}{p}},$$

where the second inequality is Theorem 2.

## 5 Dimension-free van der Corput estimates

Fix a measure $\mu$ on $\mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. The aim of this section is to bound the following quantity from above:

$$\left| \int_{\mathbb{R}^n} e^{itf(x)} d\mu(x) \right|.$$

In other words, if $f_* \mu$ is the push-forward of the measure $\mu$, we wish to study the rate of decay of the Fourier coefficients of $f_* \mu$. We first prove a variant of Lemma 1 for polynomials and isotropic log-concave measures on the real line.

**Lemma 8.** Let $\mu$ be an isotropic log-concave measure on $\mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$ a polynomial of degree $d$ and $k \in [1, d]$, an integer. Then, for every $t \in \mathbb{R}$,

$$\left| \int_{\{x \in \mathbb{R} : |f^{(k)}(x)| \geq 1\}} e^{itf(x)} d\mu(x) \right| \leq C \cdot dk |t|^{-\frac{1}{k}},$$

for some universal constant $C > 0$. 

15
Proof. We start by observing that since \( \mu \) is log-concave its density \( \rho \) is uni-modal. So, there exists a point \( x_0 \in \mathbb{R} \), such that \( \rho \) is non-decreasing up to \( x_0 \) and non-increasing from \( x_0 \). This immediately implies \( \int_a^b |\rho(x)| \, dx \leq 2 \sup_{x \in \mathbb{R}} \rho(x) \), for every interval \([a, b] \subseteq \mathbb{R}\). Furthermore, since \( \mu \) is isotropic, by [39, Lemma 5.5], \( \sup_{x \in \mathbb{R}} \rho(x) \leq 1 \).

Let \( \beta > 0 \) be a real number equal to 1 if \( k = 1 \), and to be determined later for \( k \geq 2 \), and define the sets,

\[
A_1 = \{ x \in \mathbb{R} : |f'(x)| \geq \beta \text{ and } |f^{(k)}(x)| \geq 1 \}, \\
A_2 = \{ x \in \mathbb{R} : |f'(x)| < \beta \text{ and } |f^{(k)}(x)| \geq 1 \}.
\]

We decompose the integral on these sets to obtain,

\[
\left| \int_{\{x \in \mathbb{R} : |f^{(k)}(x)| \geq 1\}} e^{uf(x)} \, d\mu(x) \right| \leq \int_{A_1} e^{uf(x)} \, d\mu(x) + \int_{A_2} e^{uf(x)} \, d\mu(x).
\]

Since \( f^{(k)} \) is a polynomial of degree less than \( d \), its derivative may change signs at most \( d \) times. So, \( \{ x \in \mathbb{R} : |f^{(k)}(x)| \geq 1 \} \) can be decomposed as a union of \( M \) pairwise disjoint intervals, with \( M \leq d \). Explicitly, we have the following identity,

\[
\{ x \in \mathbb{R} : |f^{(k)}(x)| \geq 1 \} = \bigcup_{i=1}^M [a_i, b_i],
\]

where on each interval either \( f^{(k)}(x) \geq 1 \), or \( f^{(k)}(x) \leq -1 \). For the region \( A_2 \), since \( \sup_{x \in \mathbb{R}} \rho(x) \leq 1 \), we get,

\[
\left| \int_{A_2} e^{uf(x)} \, d\mu(x) \right| \leq \int_{A_2} \rho(x) \, dx \leq \text{Vol}(A_2).
\]

For each \( 1 \leq i \leq M \), the set \( A_2 \cap [a_i, b_i] \) is a sublevel set of \( f' \) restricted to the region \([a_i, b_i]\). When \( k = 1 \), \( \text{Vol}(A_2) = 0 \), by our choice of \( \beta \), and we need only consider \( A_1 \). If \( k \geq 2 \), we invoke the sublevel estimate in [14, Proposition 2.1] \footnote{Proposition 2.1 of [14] is stated for functions which are defined on the entire real line, but the proof works for functions defined on any interval (see Section 2 of [14]).}, on each interval \([a_i, b_i]\) separately, and sum the corresponding volumes to obtain,

\[
\text{Vol}(A_2) = \sum_{i=1}^M \text{Vol}(A_2 \cap [a_i, b_i]) \leq \sum_{i=1}^M Ck\beta^{\frac{1}{k-1}} \leq Cdk\beta^{\frac{1}{k-1}}.
\]

To handle \( A_1 \), we use the fact that both \( f' \) and \( f'' \) are polynomials of degree less than \( d \). Thus, a similar reasoning to the one used before allows us to refine the partition in (14) into no more than \( 4d \) intervals, with the property that on each interval \( f' \) is monotone, and either \( f'(x) \geq \beta \), or \( f'(x) \leq -\beta \), or \( |f'(x)| \leq \beta \). In particular, we can write \( A_1 \) as a disjoint union of \( M' \leq 4d \) intervals taken from this refined partition

\[
A_1 = \bigcup_{i=1}^{M'} [c_i, d_i],
\]
such that on each interval $[c_i, d_i]$, either $f^{(k)}(x) \geq 1$, or $f^{(k)}(x) \leq -1$, and moreover $f'$ is monotone.

For each $1 \leq i \leq M'$, we integrate by parts, and use the bounds $\rho(x) \leq 1$ and $|f'(x)| \geq \beta$, to obtain:

\[
\left| \int_{c_i}^{d_i} e^{\mu f(x)} d\mu(x) \right| = \left| \int_{c_i}^{d_i} tf'(x) e^{\mu f(x)} \rho(x) dx \right| \leq \left| \left( e^{\mu f(x)} \frac{\rho(x)}{tf'(x)} \right) \right|_{c_i}^{d_i} + \left| \int_{c_i}^{d_i} e^{\mu f(x)} \left( \frac{\rho(x)}{tf'(x)} \right)' dx \right|
\]

\[
\leq \left| \frac{\rho(d_i)}{tf'(d_i)} + \frac{\rho(c_i)}{tf'(c_i)} \right| + \left| \int_{c_i}^{d_i} \rho(x) \left( \frac{1}{tf'(x)} \right)' dx \right| + \left| \int_{c_i}^{d_i} |\rho'(x)| \frac{1}{|t||f'(x)|} dx \right|
\]

\[
\leq \frac{2}{|t|\beta} + \frac{1}{|t|} \left| \left( \frac{1}{f'(x)} \right)' \right|_{c_i}^{d_i} + \frac{1}{|t|\beta} \left| \rho'(x) \right|_{c_i}^{d_i}
\]

\[
\leq \frac{4}{|t|\beta} + \frac{1}{|t|} \left| \left( \frac{1}{f'(x)} \right)' \right|_{c_i}^{d_i} \leq \frac{4}{|t|\beta} + \frac{1}{|t|} \left( \frac{1}{|f'(d_i)|} + \frac{1}{|f'(c_i)|} \right) \leq \frac{6}{|t|\beta}.
\]

When moving between the third and fourth lines we have used the fact that $f'(x)$ is monotone on $[c_i, d_i]$. Summing over all intervals $[c_i, d_i]$, we get

\[
\left| \int_{A_1} e^{\mu f(x)} d\mu(x) \right| \leq \sum_{i=1}^{M'} \int_{c_i}^{d_i} e^{\mu f(x)} d\mu(x) \leq M' \frac{6}{|t|\beta} \leq \frac{24d}{|t|\beta}.
\]

Coupling this with (15), we obtain,

\[
\left| \int_{\mathbb{R} : f^{(k)}(x) \geq 1} e^{\mu f(x)} d\mu(x) \right| \leq \frac{24d}{|t|\beta} + Cdk\beta^{\frac{1}{\epsilon}} \leq \frac{24dk}{|t|\beta} + Cdk\beta^{\frac{1}{\epsilon}}.
\]

To conclude the proof we take $\beta = \frac{1}{|t|^x}$. \hfill \qedsymbol

Our result for log-concave product measures is a consequence of the one-dimensional estimate coupled with the anti-concentration result, proven in Section 4.

**Proof of Theorem 5.** Let $\mu = \nu^{\otimes n}$ be an isotropic log-concave product measure on $\mathbb{R}^n$. For convenience we denote,

\[
J(t) := \left| \int_{\mathbb{R}^n} e^{tf(x)} d\mu(x) \right|.
\]

Now, let $I \in \mathbb{N}^n$ with $|I| = d$, be such that $M_d(f) = \alpha_I$ and fix some $\epsilon > 0$, to be determined later. We write $I = (\tilde{I}, I_n)$, where $\tilde{I}$ is a multi-index on $n-1$ indices. Without loss of generality, we may assume that $I_n \geq 1$. Define the set,

\[
A := \left\{ x \in \mathbb{R}^n : \left| \frac{\partial^{I_n}}{\partial x^{I_n}} f(x) \right| \geq \epsilon \right\}.
\]
If $\bar{A} := \mathbb{R}^n \setminus A$, then,

$$J(t) \leq \left| \int_A e^{itf(x)} d\mu(x) \right| + \left| \int_{\bar{A}} e^{itf(x)} d\mu(x) \right|. \quad (16)$$

We estimate each term separately. First, observe that $\frac{\partial I_n}{\partial x_n} f$ is a polynomial of degree at most $d - I_n$ and, clearly $\text{coeff}_{d - I_n} \left( \frac{\partial I_n}{\partial x_n} f \right) \geq I_n! M_d(f)$. Hence, by applying Corollary 4 to $\frac{\partial I_n}{\partial x_n} f$, one has,

$$\int_{\bar{A}} e^{itf(x)} d\mu(x) \leq \mathbb{P}\left( \left| \frac{\partial I_n}{\partial x_n} f(X) \right| \leq \varepsilon \right) \leq C d \left( \frac{\varepsilon}{I_n! M_d(f)} \right)^{\frac{1}{d - I_n}}. \quad (17)$$

To deal with the first term in (16), write $x = (\bar{x}, x_n)$, and note that,

$$\int_A e^{itf(x)} d\nu^{\otimes n}(x) \leq \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{itf(\bar{x}, x_n)} 1_A d\nu(\bar{x}) \, d\nu^{\otimes n-1}(x)$$

$$= \int_{\mathbb{R}^n} \int_{\{x_n : \frac{\partial f}{\partial x_n}(x_n) \geq \varepsilon\}} e^{itf_{\bar{x}}(x_n)} d\nu(x_n) \, d\nu^{\otimes n-1}(x),$$

where $f_{\bar{x}}(x_n) := f(\bar{x}, x_n)$. We invoke Lemma 8, with $k = I_n$, on the polynomial $\frac{1}{\varepsilon} f_{\bar{x}}$, which yields,

$$\int_{\{x_n : \frac{\partial f}{\partial x_n}(x_n) \geq \varepsilon\}} e^{itf_{\bar{x}}(x_n)} d\nu(x_n) \leq \int_{\{x_n : \frac{\partial f}{\partial x_n}(x_n) \geq 1\}} e^{it\frac{1}{\varepsilon} f_{\bar{x}}(x_n)} d\nu(x_n) \leq \frac{C' d I_n}{(|t| \varepsilon)^{\frac{1}{I_n}}} \leq \frac{e C' d}{(|t| \varepsilon)^{\frac{1}{I_n}}} = \frac{e C' d}{(|t| \varepsilon)^{\frac{1}{I_n}}} \leq \frac{e C' d}{(|t| \frac{1}{I_n})^{\frac{1}{I_n}}},$$

for some constant $C' > 0$, where in the last inequality we have used $k \leq e(k!)^{\frac{1}{I_n}}$. We have thus established,

$$|J(t)| \leq C d \left( \frac{\varepsilon}{I_n! M_d(f)} \right)^{\frac{1}{d - I_n}} + \frac{e C' d}{(|t| \frac{1}{I_n})^{\frac{1}{I_n}}}.$$

Choose $\varepsilon = I_n! M_d(f) \frac{1}{|t|} \cdot \frac{1}{\frac{1}{|t|^{\frac{1}{I_n}}}}$, to get,

$$|J(t)| \leq \frac{(C + e C') d}{(M_d(f) |t|)^{\frac{1}{I_n}}},$$

as required. \[\square\]
6 Spectrum of the covariance matrix for tensor powers- the case of the Euclidean ball

The goal of this section is to compute the spectrum of $\text{Cov}(X^\otimes d)$, where $X \sim \text{Uniform}(\hat{B}_2,n)$. Since $p = 2$ in this subsection, we simply write $B_n$ (resp. $\hat{B}_n$) instead of $B_{2,n}$ (resp. $\hat{B}_{2,n}$), and set $\mu = \text{Uniform}(\hat{B}_n)$. We further denote by $R_n$ the radius of $\hat{B}_n$.

Recall that $X^\otimes d$ is a random vector in $\text{Sym}_d(\mathbb{R}^n)$, which we identify with $\mathcal{P}_d(\mathbb{R}^n)$, the space of all real-valued homogeneous polynomials of degree $d$. Taking the inner product $\langle \sum a_I x^I, \sum b_J x^J \rangle := \sum_I a_I b_I$, with $\{x^I\} | I | = d$ as an orthonormal basis, one can represent $\text{Cov}(X^\otimes d)$ by the matrix $C = \{C_{I,J}\}_{|I|,|J|=d}$, where

$$C_{I,J} = \mathbb{E}_\mu [x^{I+J}] - \mathbb{E}_\mu [x^I] \mathbb{E} [x^J].$$

As it turns out, it will be more convenient to work with a different inner product, whose naturality will be apparent soon.

**Definition 7** (see [7]). Let $f = \sum_I a_I x^I$ and $g = \sum_J b_J x^J$ be in $\mathcal{P}_d(\mathbb{R}^n)$. Let $D_f$ be the partial differential operator $\sum_I a_I \partial^I$, where $\partial^I = \partial x_1^{I_1} \cdots \partial x_n^{I_n}$. The **Bombieri inner product** is defined as follows:

$$\langle f, g \rangle_B := D_f(g) = \sum_I I! \cdot a_I b_I,$$

where $I! := I_1! \cdots I_n!$. We define the corresponding **Bombieri norm**:

$$\|f\|_B = \sqrt{\sum_I I! \cdot a_I^2}.$$

Let us record one important observation, which will be used later on. Given $f \in \mathcal{P}_{d-q}(\mathbb{R}^n)$, $h \in \mathcal{P}_q(\mathbb{R}^n)$ and $g \in \mathcal{P}_d(\mathbb{R}^n)$, we have the following identity (see e.g. [8, Lemma 11]),

$$\langle h f, g \rangle_B := D_h f(g) = D_f(D_h(g)) = \langle f, D_h(g) \rangle_B. \tag{18}$$

To see the connection with the matrix $C$, write

$$\widetilde{C} := \left\{ \widetilde{C}_{I,J} \right\}_{|I|,|J|=d}; \text{ where } \widetilde{C}_{I,J} = \frac{\mathbb{E}_\mu [x^{I+J}] - \mathbb{E}_\mu [x^I] \mathbb{E} [x^J]}{I!}.$$

Then for every $f = \sum_I a_I x^I$ and $g = \sum_J b_J x^J$ in $\mathcal{P}_d(\mathbb{R}^n)$, one has

$$\langle \widetilde{C} f, g \rangle_B = \left\langle \sum_I \left( \sum_J a_J \widetilde{C}_{I,J} \right) x^I, \sum_J b_J x^J \right\rangle_B = \sum_I I! b_J a_I \widetilde{C}_{I,J} = \langle C f, g \rangle_{L^2(\mu)} - \langle f, 1 \rangle_{L^2(\mu)} \langle g, 1 \rangle_{L^2(\mu)}. \tag{19}$$

Note that $\widetilde{C} := D \cdot C$, where $D$ is the diagonal matrix, $D_{I,J} = \frac{1}{I!}$. Also, while $\widetilde{C}$ is not symmetric, it is self-adjoint with respect to the Bombieri inner product.

For an $N \times N$-matrix $M$ with non-negative eigenvalues, we denote by $0 \leq \lambda_1(M) \leq \cdots \leq \lambda_N(M)$, its eigenvalues in increasing order. The main result of this section is a complete characterization of $\{\lambda_i(\widetilde{C})\}$, along with their corresponding eigenspaces (Theorem 8). We then use the connection between $\widetilde{C}$ and $C$, to deduce information about the spectrum of $C$ (Corollary 9). Since the matrix $\widetilde{C}$ depends on the parameters $n$ and $d$, the same is also true for the quantities $\lambda_i(\widetilde{C})$. In the sequel, we suppress this dependence to simplify the notation.
6.1 Spherical harmonics

Before we state the main result, we need to collect a few basic facts about spherical harmonics. We refer to [4, Chapter 5] and [2, Chapter 2] for more details.

We write \( \mathcal{H}_d(\mathbb{R}^n) \) for the subspace of \( \mathcal{P}_d(\mathbb{R}^n) \) consisting of all degree \( d \) homogeneous harmonic polynomials on \( \mathbb{R}^n \), and \( \mathcal{H}_d(S^n) \) for its restriction to the unit sphere \( S^n \). Let \( \mu_{S^n} \) be the unique \( \text{SO}_n(\mathbb{R}) \)-invariant probability measure on the \( n-1 \)-dimensional sphere \( S^n \). Denote by \( L^2(S^n) \) the space of \( L^2 \)-integrable real valued functions on the sphere, with the inner product

\[
\langle f, g \rangle_{L^2(S^n)} = \int_{S^n} f \cdot g \, d\mu_{S^n}.
\]

It turns out that the inner products \( \langle \cdot, \cdot \rangle_{L^2(\mu)} \), \( \langle \cdot, \cdot \rangle_{L^2(S^n)} \) and \( \langle \cdot, \cdot \rangle_B \) are comparable on the subspace of \( d \)-harmonic polynomials.

**Lemma 9.** Let \( f, g \in \mathcal{H}_d(\mathbb{R}^n) \). Then we have:

\[
\langle f, g \rangle_{L^2(\mu)} = \gamma_{d,n} \cdot \langle f, g \rangle_B \quad \text{and} \quad \langle f, g \rangle_{L^2(\mu)} = \frac{n}{n + 2d} R_n^{2d} \langle f, g \rangle_{L^2(S^n)},
\]

where

\[
\gamma_{d,n} = \frac{1}{n(n+2)\ldots(n+2d-2)}.
\]

**Proof.** The fact that \( \langle f, g \rangle_{L^2(\mu)} = \gamma_{d,n} \cdot \langle f, g \rangle_B \) follows e.g. from [4, Theorem 5.14]. For the second claim, recall that the isotropic ball \( B_n \) has radius \( R_n \) and volume \( V_n = R_n^n \cdot \text{Vol}(B_n) \).

Writing \( d\sigma \) for the surface measure on \( S^n \) (so that \( d\sigma = n \cdot \text{Vol}(B_n) \mu_{S^n} \)), one has

\[
\langle f, g \rangle_{L^2(\mu)} = \int_{B_n} f(x)g(x) \, d\mu(x) = \frac{1}{R_n^n \cdot \text{Vol}(B_n)} \int_{S^n} \int_0^{R_n} r^{2d+n-1} \left( \int_{S^n} f(x) \cdot g(x) \, d\sigma(x) \right) \, dr
dr
\]

\[
= \frac{n}{n + 2d} R_n^{2d} \left( \int_{S^n} f(x) \cdot g(x) \, d\mu_{S^n}(x) \right) \, dr = \frac{n}{n + 2d} R_n^{2d} \langle f, g \rangle_{L^2(S^n)}.
\]

\[\blacksquare\]

**Lemma 10** (see e.g. [4, Theorem 5.12] and [2, Theorem 2.1.1]).

1. The Hilbert space \( L^2(S^n) \) can be decomposed into a direct sum \( L^2(S^n) = \bigoplus_{l \in \mathbb{N}} \mathcal{H}_l(S^n) \), where \( \mathcal{H}_l(S^n) \) is orthogonal to \( \mathcal{H}_m(S^n) \) for every \( m \neq l \).

2. For each \( d \geq 2 \), we have an \( \langle \cdot, \cdot \rangle_B \)-orthogonal decomposition

\[
\mathcal{P}_d(\mathbb{R}^n) = \mathcal{H}_d(\mathbb{R}^n) \oplus \|x\|^2 \mathcal{H}_{d-2}(\mathbb{R}^n) \oplus \cdots \oplus \|x\|^2 \left[ \frac{d}{2} \right] \mathcal{H}_{d-2} \left[ \frac{d}{2} \right] (\mathbb{R}^n).
\]

6.2 Calculation of the spectrum

We are now ready to state the main theorem which describes the spectrum of \( \tilde{C} \), and in fact shows that the decomposition in (20) is an eigenspace decomposition.

**Theorem 8.** Each subspace \( \|x\|^2 \mathcal{H}_{d-2}(\mathbb{R}^n) \) of \( \mathcal{P}_d(\mathbb{R}^n) \), with \( i \in \{0, \ldots, \left[ \frac{d}{2} \right] \} \), is a \( \tilde{C} \)-eigenspace with eigenvalue \( \eta_i \), where

\[
\eta_i = R_n^{2d} \frac{n}{n + 2d} \cdot \frac{1}{2^i i! n(n+2)\ldots(n+2d-2i-2)}.
\]
if $i < \frac{d}{2}$, and whenever $d$ is even,

$$
\eta^i_n = R^{2d}_n \frac{d^2}{2^d \left( \frac{d}{2} \right)! (n + 2d) (n + d) \prod_{j=0}^{\frac{d}{2} - 1} (d + n - 2j)}.
$$

In particular, the multiplicity $\text{mult}(\eta_i)$ of the eigenvalue $\eta_i$ is equal to the dimension of $\mathcal{H}_{d-2i}(\mathbb{R}^n)$:

$$
\text{mult}(\eta_i) = \left( \frac{n + d - 2i - 1}{n - 1} \right) - \left( \frac{n + d - 2i - 3}{n - 1} \right).
$$

**Proof.** Write $\mathcal{P}_d(\mathbb{R}^n) = \bigoplus_{i=0}^{\lceil \frac{d}{2} \rceil} W_i$, where $W_i := \|x\|^{2i} \mathcal{H}_{d-2i}(\mathbb{R}^n)$. First note that for every $f \in W_i, g \in W_j$ with $i \neq j$, we have,

$$
\langle \tilde{C}f, g \rangle_B = \langle f, g \rangle_{L^2(\mu)} = R^{2d}_n \frac{n}{n + 2d} \langle f \rangle_{L^2(\mathbb{R}^n)} = 0.
$$

The first equality is (19), the second is Lemma 9 and the third follows from the first item of Lemma 10. We see that $\tilde{C}(W_i)$ is orthogonal to $W_j$ for all $j \neq i$ and therefore $\tilde{C}(W_i) = W_i$. Furthermore, the same reasoning shows that if $f \in W_0$, then

$$
\langle \tilde{C}f, \rangle_B = \langle f, 1 \rangle^2_{L^2(\mu)} = R^{2d}_n \frac{n}{n + 2d} \langle f \rangle_{L^2(\mathbb{R}^n)} = 0.
$$

Similarly, any $f \in W_i$ can be written as $f = \|x\|^{2i} \cdot g(x)$ with $g(x) = \sum a_j x^j \in \mathcal{H}_{d-2i}(\mathbb{R}^n)$. Now, if $\Delta$ stands for the Laplacian, it is straightforward to verify (e.g. [4, 4.5 and 5.22]) that $\Delta^{2i}(\|x\|^{2i} g(x)) = b_i g(x)$, where

$$
b_i := 2^i! \prod_{j=1}^{i} (n + 2d - 2j - 2i). \tag{21}
$$

Moreover, by (18), $\Delta^{2i}$ is the conjugate of multiplication by $\|x\|^{2i}$ with respect to the Bombieri inner product. Thus,

$$
\langle f, f \rangle_B = \langle \|x\|^{2i} g(x), \|x\|^{2i} g(x) \rangle_B = \langle g(x), \Delta^{2i}(\|x\|^{2i} g(x)) \rangle_B = b_i \langle g, g \rangle_B. \tag{22}
$$

Using (22) we obtain, for $i < \frac{d}{2}$:

$$
\langle \tilde{C}f, f \rangle_B = \langle f, 1 \rangle^2_{L^2(\mu)} = R^{2d}_n \frac{n}{n + 2d} \cdot \langle f, g \rangle_{L^2(\mathbb{R}^n)}
$$

$$
= R^{2d}_n \frac{n}{n + 2d} \cdot \frac{1}{\gamma_{d-2i,n}(g, g) B}
$$

$$
= R^{2d}_n \frac{n}{n + 2d} \cdot \frac{1}{\gamma_{d-2i,n}(g, g) B}
$$

$$
= R^{2d}_n \frac{n}{n + 2d} \cdot \frac{1}{2^i! \prod_{j=1}^{i} (n + 2d - 2j - 2i)}. \tag{f, f}_B.
$$

Finally, for $2i = d$ (and $d$ even), using Lemma 9 we have

$$
\langle \tilde{C} \|x\|^d, \|x\|^d \rangle_B = \langle \|x\|^d, \|x\|^d \rangle_{L^2(\mu)} - \langle \|x\|^d, 1 \rangle^2_{L^2(\mu)}
$$

$$
= \frac{n}{n + 2d} R^{2d}_n - \left( \frac{n}{n + d} R^n_d \right)^2
$$

$$
= R^{2d}_n \left( \frac{n}{n + 2d} - \frac{n^2}{n^2 + 2dn + d^2} \right)
$$

$$
= R^{2d}_n \frac{d^2 n}{(n + 2d) (n + d)^2}.
$$

21
Note that
\[ \langle \|x\|^d, \|x\|^d \rangle_B = b_{d/2} = 2^{d/2} \left( \frac{d}{2} \right)! \prod_{j=1}^{d/2} (d + n - 2j), \]
so
\[ \langle \tilde{C} \|x\|^d, \|x\|^d \rangle_B = \frac{R_n^{2d}}{b_{d/2} (n + 2d) (n + d)} \langle \|x\|^d, \|x\|^d \rangle_B, \]
as required.

To put everything together, we have shown that every \( W_i \) is a \( \tilde{C} \)-invariant subspace and that the Rayleigh quotient \( \langle f, f \rangle_B \) is constant on \( W_i \). Since \( \tilde{C} \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle_B \) we can conclude that it is a constant multiple of the identity on \( W_i \) and the claim follows. \( \square \)

Since \( \tilde{B}_n \) is isotropic, it is well known that \( R_n = \sqrt{n + 2} \). Let us now understand the quantities \( \eta_i \) better. If \( i < \frac{d}{2} \), we have,
\[
n_i \geq \eta_i \geq \frac{n^d}{n + 2d} \cdot \frac{1}{2^i!(n + 2)...(n + (2d - 2i))} = \frac{n^i}{(1 + \frac{2d}{n})^i \cdot 2^i!(1 + \frac{2}{n})...(1 + \frac{2d - 2i}{n})} \geq \frac{n^i}{(d + 1)!}. \tag{23}\]

For the first inequality, we have used the definition of \( \eta_i \), according to which, as long as \( n \geq 2 \),
\[
\eta_i = R_n^{2d} \cdot \frac{n^i}{(n + 2d)!} \leq \frac{(n + 2)^d}{(n + 2) d^i} \leq \left( \frac{n}{2} + 1 \right)^i \leq n^i. \]

For the last inequality in (23), we used the elementary estimate \( \binom{d}{i} \geq \left( \frac{d}{i} \right)^i \), which implies \( \frac{d^i}{n^i (d-i)!} = \binom{d}{i} \geq 2^i \), whenever \( i < \frac{d}{2} \). Combining (23) with a similar calculation for \( i = \frac{d}{2} \), when \( d \) is even, one has:
\[
\eta_{\frac{d}{2}} = \Theta_d \left( \frac{d}{2} - 2 \right) \text{ and } \eta_i = \Theta_d \left( n^i \right), \tag{24}\]
where \( \Theta_d \) means we omit constants which depend only on \( d \).

Thus, when \( d \geq 3 \), Theorem 8 and the discussion above give the following dimension-free bound:
\[
\lambda_1(\tilde{C}) = \eta_0 \geq \frac{1}{(d + 1)!}. \tag{25}\]

If, on the other hand, \( d = 2 \), then the smallest eigenvalue is
\[
\lambda_1(\tilde{C}) = \eta_1 = (n + 2)^2 \frac{4}{2(n + 4)(n + 2)^2} = \frac{2}{n + 4}, \tag{26}\]
and is of multiplicity one, with eigenvector \( \sum_{i=1}^{n} x_i^2 \). Indeed, we have
\[
\lambda_2(\tilde{C}) = \eta_0 = \frac{n + 2}{n + 4} > \lambda_1(\tilde{C}) \tag{27}\]

Since \( C \) is a product of \( \tilde{C} \) with the diagonal matrix \( D^{-1} \) (with \( D_{1,1}^{-1} = I \)) we can now use the spectrum of \( \tilde{C} \) to deduce information on the spectrum of \( C \).
Corollary 9. Write $\lambda_1(C) \leq \cdots \leq \lambda_N(C)$ for the spectrum of $C$, with $N = \left( d + n - 1 \atop d \right)$. Then, the following estimates hold:

1. For all $i$, we have
   
   $d! \lambda_i(\tilde{C}) \geq \lambda_i(C) \geq \lambda_i(\tilde{C})$, 
   
   where $\lambda_1(\tilde{C}) \leq \cdots \leq \lambda_N(\tilde{C})$ are explicitly given by Theorem 8.

2. “pathological spectral gap”: If $d = 2$, $n \geq 3$ then the smallest eigenvalue $\lambda_1(C)$ has multiplicity one, with eigenvector $\sum_{i=1}^{n} x_i^2$, and

   $\lambda_1(C) = \frac{4}{n+4} = O(n^{-1}).$

   The rest of the eigenvalues are bounded from below by $\frac{5}{7}$.

3. For $d \geq 3$ we have a uniform lower bound on the eigenvalues

   $\lambda_i(C) \geq \frac{1}{(d+1)!},$  \quad (28)

   for all $n$. If $n \geq d$, then the $\lambda_1(C)$-eigenspace is spanned by monomials of the form $x_{i_1} \cdots x_{i_d}$ with $i_1 < \cdots < i_d$. Moreover, we have

   $\lim_{n \to \infty} \lambda_1(C) = \lim_{n \to \infty} \lambda_1(\tilde{C}) = 1.$

Proof. Note that $\tilde{C} = D \cdot C$ is a product of two positive definite matrices. Since $D$ is a diagonal matrix with diagonal entries in the range $\left[ \frac{d}{n}, 1 \right]$, it follows e.g. by [54, Theorem 3] that:

   $d! \lambda_i(\tilde{C}) \geq \lambda_i(C) \geq \lambda_i(\tilde{C}),$

which is Item (1). Item (2) now follows from (26) and (27).

If $n \geq d \geq 3$, it is easy to verify that $\eta_{i+1} \geq \eta_i$ for all $0 \leq i < \left\lfloor \frac{d}{2} \right\rfloor$. Item (1) implies that $\lambda_1(C) \geq \lambda_1(\tilde{C}) = \eta_0$. On the other hand, monomials $x^I$ of the form $x_{i_1} \cdots x_{i_d}$ with $i_1 < \cdots < i_d$ satisfy $\langle x^I, x^I \rangle = \langle x^I, x^I \rangle_B$ and they are harmonic, so

   $Cx^I = D^{-1} \tilde{C} x^I = D^{-1} \eta_0 x^I = \eta_0 x^I.$

This shows that $\lambda_1(C) = \lambda_1(\tilde{C}) = \eta_0$. Finally, we have:

   $\lim_{n \to \infty} \lambda_1(C) = \lim_{n \to \infty} \lambda_1(\tilde{C}) = \lim_{n \to \infty} \frac{(n+2)^d}{(n+2)(n+2d)(n+2d-2)} = 1$

which finishes Item (3).

Remark 10. The lower bound in (28) can be improved, by considering more refined estimates on the possible entries of $D$ and on $\eta_i$, for small values on $n$. Since we already know that $\lim_{n \to \infty} \lambda_1(C) = \lim_{n \to \infty} \lambda_1(\tilde{C}) = 1$, we chose to ignore this low-dimensional issue, and to keep the (slightly non-optimal) current bound.
6.2.1 Further discussion

We conclude the section with a discussion on the asymptotic behavior of spectrum of $C$ as well as on the general case of radial measures.

Partition of the spectrum into different asymptotic scales

By combining Theorem 8, (24) and Item (1) of Corollary 9, we see that the eigenvalues $\lambda_i(C)$ can be partitioned into subsets $A_0, ..., A_{\lceil d^2 - 1 \rceil}$ with respect to different asymptotic behaviors. The subset $A_j$ consists of eigenvalues which are of magnitude $\sim n^j$ (up to a constants depending only on $d$). For $d = 2$ there is an additional eigenvalue $\lambda_1(C) \sim n^{-1}$ which belongs to a unique asymptotic scale $A_{-1}$.

One may wonder whether this phenomenon can be generalized to other families of measures with some form of symmetry. Namely, how general is the situation where all eigenvalues of $\text{Cov}(X^{\otimes d})$ converge to a discrete set of asymptotic scales as $n$ grows? In particular, does it hold for the uniform measure on $L_p$ balls?

Let us consider the case when $X \sim \text{Uniform}(\tilde{B}_{p,n})$ for $p$ an even natural number. Write $R_{n,p}$ for the radius of $\tilde{B}_{p,n}$. Using a coordinate change, as in (11), it can be seen that the polynomial $f = \frac{1}{\sqrt{n}} \|x\|_p^p$ satisfies

$$\frac{\langle Cf, f \rangle}{\langle f, f \rangle} = \text{Var} \left( \frac{1}{\sqrt{n}} \|X\|_p^p \right) = \frac{1}{n} \left( \mathbb{E} (\|X\|_p^{2p}) - \mathbb{E} (\|X\|_p^p)^2 \right)
= \frac{R_{n,p}^2}{n} \left( \frac{n}{n + 2p} - \left( \frac{n}{n + p} \right)^2 \right)
= \frac{R_{n,p}^2}{n + 2p} \left( \frac{p^2}{(n + 2p)(n + p)^2} \right) = \Theta_p \left( n^{-1} \right).$$

In particular, we see that the eigenvalues of $\text{Cov}(X^{\otimes p})$, are not bounded from below, in a way reminiscent of the Euclidean case.

Radial measures

The results of this section generalize to radial measures of the form $\frac{du}{dx} = \rho(\|x\|_2)$, for some $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. Indeed, the only difference lies at Lemma 9, where now we will have,

$$\langle f, g \rangle_{L^2(\mu)} = \beta_{\mu,d} \langle f, g \rangle_{L^2(\mathbb{R}^n)},$$

with,

$$\beta_{\mu,d} := n \text{Vol}(B_n) \int_0^\infty r^{n+2d-1} \rho(r)dr$$

($\beta_{\mu,d} = \frac{n}{n+2d} R_{n,d}^d$ in the case of the isotropic Euclidean ball). In particular, the matrix $\tilde{C}$ has the same eigenspace decomposition as in Theorem 8, with eigenvalues $\eta_{\mu,i} = \eta_i \cdot \beta_{\mu,d} \cdot \frac{n+2d}{n R_{n,d}^d}$, $i < \frac{d}{2}$, and $\eta_{\mu,d/2} = \beta_{\mu,d} - R_{n,d}^d/b_{d/2}$, with $b_{d/2}$ as defined in (21). Consequently, Items (1) and (3) of Corollary 9 hold for radial measures as well, with slightly different lower bounds. Item (2), i.e. the “pathological spectral gap” phenomenon, is true only for certain classes of measures, and
depends on $\beta_{\mu,d}$. For example, for $\gamma_n$, the standard Gaussian in $\mathbb{R}^n$, a calculation shows,

$$\beta_{\gamma_n,d} = \frac{n\text{Vol}(B_n)}{\sqrt{2\pi}} \int_0^\infty r^{2n+2d-1}e^{-r^2/2}dr = \frac{2^dn\text{Vol}(B_n)}{2\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2} + d\right) = 2^d\frac{\Gamma\left(\frac{n}{2} + d\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Note that for $d = 2$, we have

$$\eta_{\gamma_n,1} = \frac{2}{n} \left( \frac{\Gamma\left(\frac{n}{2} + 2\right)}{\Gamma\left(\frac{n}{2}\right)} - \frac{\Gamma\left(\frac{n}{2} + 1\right)^2}{\Gamma\left(\frac{n}{2}\right)^2} \right) = \frac{1}{2n} \left(n(n+2) - n^2\right) = 1.$$

So, there is no pathological spectral gap. Thus, we can see that, in contrast to the Euclidean ball, the spectrum $\eta_{\gamma_n,i}$ is bounded uniformly from below, which is consistent with Theorem 1.

In fact, among all log-concave and isotropic radial measures, the Euclidean ball is the extremal case, for which the pathological eigenvector $\|x\|_2^2$ has the smallest eigenvalue. This is related to the thin-shell phenomenon, which states that every log-concave and isotropic measure should be well concentrated around a Euclidean sphere. A lower bound for thin-shell was proven in [9, Theorem 2], where it was shown that $\text{Var}\left(\frac{1}{\sqrt{n}}\|X\|_2\right) \geq \frac{1}{n+4}$ for every isotropic and log-concave $X$ in $\mathbb{R}^n$, satisfying a certain monotonicity assumption. As we have seen above, the minimum is attained when $X \sim \text{Uniform}(B_{2,n})$.

References

[1] George B. Arfken and Hans J. Weber. *Mathematical methods for physicists*. Harcourt/Academic Press, Burlington, MA, fifth edition, 2001.

[2] David H. Armitage and Stephen J. Gardiner. *Classical potential theory*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001.

[3] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988. Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi.

[4] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[5] Dominique Bakry, Stepan Orevkov, and Marguerite Zani. Orthogonal polynomials and diffusion operators. *arXiv preprint arXiv:1309.5632*, 2013.

[6] Franck Barthe, Olivier Guédon, Shahar Mendelson, and Assaf Naor. A probabilistic approach to the geometry of the $l_p^n$-ball. *Ann. Probab.*, 33(2):480–513, 2005.

[7] Bernard Beauzamy, Enrico Bombieri, Per Enflo, and Hugh L. Montgomery. Products of polynomials in many variables. *J. Number Theory*, 36(2):219–245, 1990.

[8] Bernard Beauzamy and Jérôme Dégot. Differential identities. *Trans. Amer. Math. Soc.*, 347(7):2607–2619, 1995.

[9] S. G. Bobkov and A. Koldobsky. On the central limit property of convex bodies. In *Geometric aspects of functional analysis*, volume 1807 of *Lecture Notes in Math.*, pages 44–52. Springer, Berlin, 2003.
[10] Matthew Brennan, Guy Bresler, and Brice Huang. De Finetti-style results for Wishart matrices: Combinatorial structure and phase transitions. arXiv preprint arXiv:2103.14011, 2021.

[11] Matthew Brennan, Guy Bresler, and Dheeraj Nagaraj. Phase transitions for detecting latent geometry in random graphs. Probab. Theory Related Fields, 178(3-4):1215–1289, 2020.

[12] Sébastien Bubeck, Jian Ding, Ronen Eldan, and Miklós Z. Rácz. Testing for high-dimensional geometry in random graphs. Random Structures Algorithms, 49(3):503–532, 2016.

[13] Sébastien Bubeck and Shirshendu Ganguly. Entropic CLT and phase transition in high-dimensional Wishart matrices. Int. Math. Res. Not. IMRN, (2):588–606, 2018.

[14] Anthony Carbery, Michael Christ, and James Wright. Multidimensional van der Corput and sublevel set estimates. J. Amer. Math. Soc., 12(4):981–1015, 1999.

[15] Anthony Carbery and James Wright. Distributional and $L^q$ norm inequalities for polynomials over convex bodies in $\mathbb{R}^n$. Math. Res. Lett., 8(3):233–248, 2001.

[16] Anthony Carbery and James Wright. What is van der Corput’s lemma in higher dimensions? In Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), number Vol. Extra, pages 13–26, 2002.

[17] Michael Christ, Xiaochun Li, Terence Tao, and Christoph Thiele. On multilinear oscillatory integrals, nonsingular and singular. Duke Math. J., 130(2):321–351, 2005.

[18] R. Cluckers, M. Mustaţă, and K. H. Nguyen. Igusa’s conjecture for exponential sums: optimal estimates for nonrational singularities. Forum Math. Pi, 7:e3, 28, 2019.

[19] Raf Cluckers. Analytic van der Corput lemma for $p$-adic and $\mathbb{F}_q((t))$ oscillatory integrals, singular Fourier transforms, and restriction theorems. Expo. Math., 29(4):371–386, 2011.

[20] Raf Cluckers, Itay Glazer, and Yotam I Hendel. A number theoretic characterization of $E$-smooth and (FRS) morphisms: estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points. arXiv preprint arXiv:2103.00282, 2021.

[21] Kevin P. Costello. Bilinear and quadratic variants on the Littlewood-Offord problem. Israel J. Math., 194(1):359–394, 2013.

[22] Kevin P. Costello, Terence Tao, and Van Vu. Random symmetric matrices are almost surely nonsingular. Duke Math. J., 135(2):395–413, 2006.

[23] Jan Denef. Report on Igusa’s local zeta function. Number 201-203, pages Exp. No. 741, 359–386 (1992). 1991. Séminaire Bourbaki, Vol. 1990/91.

[24] Jan Denef and François Loeser. Motivic Igusa zeta functions. J. Algebraic Geom., 7(3):505–537, 1998.

[25] Marcus du Sautoy and Fritz Grunewald. Analytic properties of zeta functions and subgroup growth. Ann. of Math. (2), 152(3):793–833, 2000.
[26] Ronen Eldan, Dan Mikulincer, and Tselil Schramm. Non-asymptotic approximations of neural networks by Gaussian processes. In Mikhail Belkin and Samory Kpotufe, editors, *Proceedings of Thirty Fourth Conference on Learning Theory*, volume 134 of *Proceedings of Machine Learning Research*, pages 1754–1775. PMLR, 15–19 Aug 2021.

[27] Matt Emschwiller, David Gamarnik, Eren C Kızıldağ, and Ilias Zadik. Neural networks and polynomial regression. demystifying the overparametrization phenomena. *arXiv preprint arXiv:2003.10523*, 2020.

[28] Xiao Fang and Yuta Koike. New error bounds in multivariate normal approximations via exchangeable pairs with applications to Wishart matrices and fourth moment theorems. *to appear in Ann. Appl. Probab.*, 2020.

[29] Itay Glazer and Yotam I Hendel. On singularity properties of word maps and applications to probabilistic Waring type problems. *arXiv preprint arXiv:1912.12556*, 2019.

[30] Philip T. Gressman and Lechao Xiao. Maximal decay inequalities for trilinear oscillatory integrals of convolution type. *J. Funct. Anal.*, 271(12):3695–3726, 2016.

[31] J. Igusa. Complex powers and asymptotic expansions. I. Functions of certain types. *J. Reine Angew. Math.*, 268/269:110–130, 1974. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II.

[32] J. Igusa. *Forms of higher degree*, volume 59 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[33] G. J. O. Jameson. Inequalities for gamma function ratios. *Amer. Math. Monthly*, 120(10):936–940, 2013.

[34] G. J. O. Jameson. A simple proof of Stirling’s formula for the gamma function. *Math. Gaz.*, 99(544):68–74, 2015.

[35] B. Klartag and R. Vershynin. Small ball probability and Dvoretzky’s theorem. *Israel J. Math.*, 157:193–207, 2007.

[36] János Kollár. Which powers of holomorphic functions are integrable? *arXiv preprint arXiv:0805.0756*, 2008.

[37] Egor Kosov. Distributions of polynomials in Gaussian random variables under structural constraints. *arXiv preprint arXiv:2007.12742*, 2020.

[38] Suqi Liu and Miklos Z Racz. Phase transition in noisy high-dimensional random geometric graphs. *arXiv preprint arXiv:2103.15249*, 2021.

[39] László Lovász and Santosh Vempala. The geometry of logconcave functions and sampling algorithms. *Random Structures Algorithms*, 30(3):307–358, 2007.

[40] Raghu Meka, Oanh Nguyen, and Van Vu. Anti-concentration for polynomials of independent random variables. *Theory Comput.*, 12:Paper No. 11, 16, 2016.

[41] Dan Mikulincer. A CLT in Stein’s Distance for Generalized Wishart Matrices and Higher-Order Tensors. *International Mathematics Research Notices*, 01 2021. rnaa336.
[42] M. Mustaţă. IMPANGA lecture notes on log canonical thresholds. In Contributions to algebraic geometry, EMS Ser. Congr. Rep., pages 407–442. Eur. Math. Soc., Zürich, 2012. Notes by Tomasz Szemberg.

[43] F. Nazarov, M. Sodin, and A. Volberg. The geometric Kannan-Lovász-Simonovits lemma, dimension-free estimates for the distribution of the values of polynomials, and the distribution of the zeros of random analytic functions. Algebra i Analiz, 14(2):214–234, 2002.

[44] Ivan Nourdin and Fei Pu. Gaussian fluctuation for gaussian Wishart matrices of overall correlation. arXiv preprint arXiv:2103.16630, 2021.

[45] Ivan Nourdin and Guangqu Zheng. Asymptotic behavior of large gaussian correlated Wishart matrices. arXiv preprint arXiv:1804.06220, 2018.

[46] Grigoris Paouris. Small ball probability estimates for log-concave measures. Trans. Amer. Math. Soc., 364(1):287–308, 2012.

[47] Alexander Razborov and Emanuele Viola. Real advantage. ACM Trans. Comput. Theory, 5(4):Art. 17, 8, 2013.

[48] Keith M. Rogers. A van der Corput lemma for the $p$-adic numbers. Proc. Amer. Math. Soc., 133(12):3525–3534, 2005.

[49] Michael Ruzhansky. Multidimensional decay in the van der Corput lemma. Studia Math., 208(1):1–10, 2012.

[50] G. Schechtman and J. Zinn. On the volume of the intersection of two $L^p$ balls. Proc. Amer. Math. Soc., 110(1):217–224, 1990.

[51] Elias M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[52] Gábor Szegő. Orthogonal polynomials. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[53] Willem Veys and W. A. Zúñiga Galindo. Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra. Trans. Amer. Math. Soc., 360(4):2205–2227, 2008.

[54] Bo Ying Wang and Fu Zhen Zhang. Some inequalities for the eigenvalues of the product of positive semidefinite Hermitian matrices. Linear Algebra Appl., 160:113–118, 1992.