The Erdős-Ko-Rado property for some 2-transitive groups

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Abstract

A subset of a group $G \leq \text{Sym}(n)$ is intersecting if for any pair of permutations $\pi, \sigma \in G$ there is an $i \in \{1, 2, \ldots, n\}$ such that $\pi(i) = \sigma(i)$. It has been shown, using an algebraic approach, that the largest intersecting sets in each of $\text{Sym}(n)$, $\text{Alt}(n)$ and $\text{PGL}(2,q)$ are exactly the cosets of the point-stabilizers. In this paper, we show how this method can be applied more generally to many 2-transitive groups. We then apply this method to the Mathieu groups and to all 2-transitive groups with degree no more than 20.

Keywords: permutation group, derangement graph, independent sets, Erdős-Ko-Rado property

1. Introduction

There have been several recent publications [2, 6, 12, 13, 16, 17, 19] that determine the maximum sets of elements from a permutation group such that any two permutations from the set both map at least one point to the same element. These results are considered to be versions of the Erdős-Ko-Rado (EKR) theorem [9] for permutations. In this paper, we focus on generalizing the algebraic method used in [12, 19] to prove the natural version of the EKR theorem for 2-transitive permutation groups.

The EKR theorem describes the size and structure of the largest collection of intersecting $k$-subsets of the set $[n] = \{1, 2, \ldots, n\}$. Provided that $n > 2k$, the largest such collection is comprised of all $k$-subsets that contain a common fixed element from $[n]$. This result has been extended to other objects for which there is a notion of intersection. For examples, an analogous result has been shown to hold for integer sequences [10], vector spaces over a finite field [14], partitions [18] and many other objects. Here we consider the extension to permutations.

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Let $G \leq \text{Sym}(n)$ be a permutation group with the natural action on the set $[n]$. Two permutations $\pi, \sigma \in G$ are said to intersect if $\pi \sigma^{-1}$ has a fixed point in $[n]$. In other words, $\sigma$ and $\pi$ do not intersect if $\pi \sigma^{-1}$ is a derangement. Throughout this paper, we denote the set of all derangement elements of group the $G$ by $D_G$. A subset $S \subseteq G$ is, called intersecting if any pair of its elements intersect. Clearly, the stabilizer of a point is an intersecting set in $G$; as is any coset of the stabilizer of a point. Note that such a coset is the collection of all permutations in the group that map $i$ to $j$ for some $i, j \in [n]$. In this paper we focus on 2-transitive groups and if $G$ is any 2-transitive subgroup of $\text{Sym}(n)$, then any stabilizer of a point has size $\frac{|G|}{n}$.

We say the group $G$ has the EKR property, if the size of any intersecting subset of $G$ is bounded above by the size of the largest point-stabilizer in $G$. Further, $G$ is said to have the strict EKR property if the only maximum intersecting subsets of $G$ are the cosets of the point-stabilizers. It is clear from the definition that if a group has the strict EKR property, then it will also have the EKR property.

It has been shown that the symmetric group has the strict EKR property [6, 12, 16], as does the alternating group [2, 15] and the group $\text{PGL}(2, q)$ [19]. In this paper we focus on an algebraic method that was used in [12, 2, 19] to show that these groups have the strict EKR property. First we show how this method is particularly effective for 2-transitive groups. In fact, we can prove a group has the strict EKR property using a few straight-forward calculations. As an example of this, we show that all the 2-transitive Mathieu groups have this property. We also consider all the 2-transitive groups with degree no more than 20; all of these have the EKR property and we show that several also have the strict EKR property.

2. Derangement Graphs

For a group $G$ we identify a graph such that the independent sets in this graph are intersecting sets of permutations from the group. This graph is called the derangement graph for $G$. The vertices of this graph are the elements in $G$ and two vertices are adjacent if and only if they are not intersecting. This graph is the Cayley graph on $G$ with the derangements from $G$, so the set $D_G$, as the connection set. We denote the derangement graph by $\Gamma_G = \Gamma(G, D_G)$. The degree of $\Gamma_G$ is $|D_G|$ and it is a vertex-transitive graph.

For any $i, j \in [n]$, we define the canonical independent sets to be the sets

$$S_{i,j} = \{ \pi \in G \mid \pi(i) = j \}.$$  

These are clearly independent sets in $\Gamma_G$ and they are the cosets of the point-stabilizers in $G$ under the natural action of $G$ on $[n]$. For each $i, j \in [n]$ it is easy to see that $|S_{i,j}| = \frac{|G|}{n}$. We will denote the characteristic vector of $S_{i,j}$ by $v_{i,j}$.

The group $G$ has the EKR property if

$$\alpha(\Gamma_G) = \frac{|G|}{n}.$$  

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Furthermore, a group has the strict EKR property if every independent set of maximum size is equal to $S_{i,j}$ for some $i$ and $j$.

For any group there is an association scheme called the conjugacy class scheme. The matrices $\{A_1, \ldots, A_k\}$ in the conjugacy class scheme of $G$ are the $|G| \times |G|$ matrices such that, for any $1 \leq i \leq k$, the entry $(g, h)$ of $A_i$ is 1 if $hg^{-1}$ belongs to the $i$-th conjugacy class, and 0 otherwise. The derangement graph for any group $G$ is a union of graphs in the conjugacy class scheme of $G$. (For more details on association schemes and the conjugacy class scheme see [4, Example 2.1 (2)].)

The derangement graph is a normal Cayley graph (meaning that the connection set is closed under conjugation). By the eigenvalues, eigenvectors and eigenspaces of a graph we mean the eigenvalues, eigenvectors and eigenspaces of the adjacency matrix of the graph. Using a result of Diaconis and Shahshahani [8], it is possible to calculate the eigenvalues of the derangement graph using the irreducible characters of the group (see [1, Chapter 4] for a detailed proof of this).

**Corollary 2.1.** The eigenvalues of the derangement graph $\Gamma_G$ are given by

$$\eta_\chi = \frac{1}{\chi(\text{id})} \sum_{x \in D_G} \chi(x),$$

where $\chi$ ranges over all the distinct irreducible characters of $G$. The multiplicity of the eigenvalue $\eta_\chi$ is equal to $\chi(\text{id})^2$. \(\square\)

For any group $G$ the largest eigenvalue of $\Gamma_G$ is $|D_G|$ and the all ones vector is an eigenvector; this is the eigenvalue given by the trivial character.

In this paper we focus on the standard character. The value of the standard character evaluated on any permutation is one less than the number of elements fixed by that permutation. This is indeed a character for any group, but for the 2-transitive groups it is an irreducible character (this follows by Burnside’s lemma, see [1, Chapter 3] for a detailed proof).

**Lemma 2.2.** Let $G$ be a 2-transitive subgroup of $\text{Sym}(n)$, then $-\frac{|D_G|}{n-1}$ is an eigenvalue of $\Gamma_G$.

**Proof.** By definition, the value of the standard character on the identity of $G$ is $n-1$ and its value on any derangement is $-1$. Applying Corollary 2.1 we have that the eigenvalue for the standard character, $\chi$, is

$$\eta_\chi = \frac{1}{\chi(\text{id})} \sum_{g \in D_G} \chi(g) = -\frac{|D_G|}{n-1}. \quad \square$$

This implies that the multiplicity of $-\frac{|D_G|}{n-1}$, as an eigenvalue, is at least $(n-1)^2$. If another irreducible character also gives this eigenvalue, then the multiplicity will be larger.
3. EKR property for 2-transitive groups

We focus on two methods that have been used to prove that a group has the EKR property. The first method uses a bound called the clique-coclique bound (coclique is another term for an independent set in a graph). The version we use here was originally proved by Delsarte [7]. Before stating this bound, we will need some notation.

Assume $A = \{A_0, A_1, \ldots, A_d\}$ is an association scheme on $v$ vertices and let $\{E_0, E_1, \ldots, E_d\}$ be the idempotents of this association scheme. (For a detailed discussion about association schemes, the reader may refer to [3] or [4].) For our purposes here, we only need to know that the matrices $A_i$ are simultaneously diagonalizable, and the $E_j$ are the projections to the common eigenspaces of the matrices in the association scheme. A graph in an association scheme $A$ is a graph whose adjacency matrix is one of the matrices in $A$.

For a set $S$ of vertices in a graph $X$, the characteristic vector of $S$ will be denoted by $v_S$ (the entries of $v_S$ are indexed by $V(X)$ and the $v$-entry is equal to 1 if $v \in S$ and 0 otherwise).

**Theorem 3.1.** Let $X$ be the union of some of the graphs in an association scheme $A$ on $v$ vertices. If $C$ is a clique and $S$ is an independent set in $X$, then

$$|C||S| \leq v.$$

If equality holds then

$$v_C^T E_j v_C \cdot v_S^T E_j v_S = 0, \quad \text{for all } j > 0. \Box$$

This gives a simple way to check if a group has the EKR property.

**Corollary 3.2.** If a 2-transitive group has a sharply-transitive set, then it has the EKR property.

**Proof.** A sharply-transitive set in a group $G \leq \text{Sym}(n)$ is a clique of size $n$ in $\Gamma_G$. By the ratio bound, the size of the largest independent set in $\Gamma_G$ is $\frac{|G|}{n}$. □

The clique-coclique bound actually holds for any vertex-transitive graph, but we are interested in the above version for association schemes because of the two following simple, but useful, corollaries which were proven in [12].

**Corollary 3.3.** Let $X$ be a union of graphs in an association scheme such that the clique-coclique bound holds with equality in $X$. Assume that $C$ is a maximum clique and $S$ is a maximum independent set in $X$. Then, for $j > 0$, at most one of the vectors $E_j v_C$ and $E_j v_S$ is not zero. □

In other words, provided that the clique-coclique bound holds with equality, for any module of $\Gamma_G$ (other than the trivial module) the projection of at most one of the vectors $v_C$ and $v_S$ will be non-zero, where $S$ is any maximum independent set and $C$ is any maximum clique.
For any group $G$ the clique-coclique bound (Theorem 3.1) applies to $\Gamma_G$ as $\Gamma_G$ is the union of graphs in the conjugacy class scheme. The idempotents of this scheme are $E_\chi$ where
\[(E_\chi)_{\pi,\sigma} = \frac{\chi(1)}{|G|}\chi(\pi^{-1}\sigma).\] (1)
where $\chi$ runs through the set of all irreducible characters of $G$. The vector space generated by the columns of $E_\chi$ is called the $\chi$-module of $\Gamma_G$.

For any character $\chi$ of $G$ and any subset $X$ of $G$ define
\[\chi(X) = \sum_{x \in X} \chi(x).\]

Using Corollary 3.3 and Equation (1) one observes the following.

**Corollary 3.4.** Assume the clique-coclique bound holds with equality for the graph $\Gamma_G$ and let $\chi$ be an irreducible character of $G$ that is not the trivial character. If there is a clique $C$ of maximum size in $\Gamma_G$ with $\chi(C) \neq 0$, then
\[E_\chi v_S = 0\]
for any maximum independent set $S$ of $\Gamma_G$. \hfill \Box

Let $G \leq \text{Sym}(n)$ and assume that for every irreducible representation $\chi$ of $G$, except the standard representation and the trivial representation, we can find a clique $C$ in $\Gamma_G$ of size $n$ such that $\chi(C) \neq 0$. Then the above corollary implies that the characteristic vector of any maximum independent set is in the span of the trivial module and the standard module.

The other method we use to show that a group has the EKR property is an eigenvalue bound called the ratio bound. The ratio bound is due to Delsarte who used a linear programming argument to prove this if for association schemes (see [20, Section 3.2]).

**Theorem 3.5.** Let $X$ be a $k$-regular graph on $n$ vertices with $\tau$ the least eigenvalue of $X$. For any independent set $S$ we have
\[|S| \leq \frac{n}{1 - \frac{k}{\tau}}.\]
Furthermore, the equality holds if and only if
\[A(X) \left(v_S - \frac{|S|}{n} \mathbf{1}\right) = \tau \left(v_S - \frac{|S|}{n} \mathbf{1}\right). \hfill \Box\]

There are large families of groups for which this bound can be used to show that group has the EKR property.

**Lemma 3.6.** Let $G$ be a 2-transitive group. If the eigenvalue arising from the standard representation of $G$ is the least eigenvalue of $\Gamma_G$, then $G$ has the EKR property.
Proof. Simply apply the eigenvalue from Lemma 2.2 in the ratio bound to get that
\[ \alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{|D_G|}{n}} = \frac{|G|}{n}. \]

The eigenvalue arising from the standard representation is the least eigenvalue for \( \Gamma_G \) for the following groups: Sym\((n)\) \[21\], PGL\((2, q)\), PSL\((2, q)\) \[19\], and all the 2-transitive Mathieu groups. Further, in the appendix we have a list of all the 2-transitive groups with degree no more than 20 for which this holds.

Note that, the second part of Theorem 3.5 states that if \(-\frac{|D_G|}{n-1}\) is the least eigenvalue, then the characteristic vector of any maximum independent set lies in the direct sum of the \(D_G\)-eigenspace and the \(-\frac{|D_G|}{n-1}\)-eigenspace of \(\Gamma_G\). If the standard representation is the only representation that gives the least eigenvalue of \(\Gamma_G\), then the characteristic vector of any independent set, when shifted by the all ones vector, lies in the standard module.

4. Strict EKR theorem for 2-transitive groups

In this section we describe a method used to show that a 2-transitive group has the strict EKR property. We call this the module method. This method has several components. First, the group must have the EKR property. Second, the characteristic vector for any maximum independent set must be in the sum of the standard module and the trivial module. We will show that the vectors \(v_{i,j}\) form a spanning set for the sum of these two modules; hence the characteristic vector of every maximum independent set is a linear combination of these vectors. Finally, if the only linear combination of these vectors that gives the characteristic vector of a maximum independent set is one of the vectors \(v_{i,j}\), then the strict EKR theorem holds for the group.

Before stating the conditions we need to check to show a 2-transitive group has the strict EKR property, we will need some technical lemmas. The first two give a subset of the vectors \(v_{i,j}\) form a basis for the standard module.

**Lemma 4.1.** Let \(G\) be a 2-transitive group and \(S_{i,j}\) the canonical independent sets of \(G\). Then \(v_{i,j} - \frac{1}{n} 1\) lies in the standard module.

**Proof.** Let \(\chi\) be the standard representation, we will show that \(E_\chi(v_{i,j} - \frac{1}{n} 1) = v_{i,j} - \frac{1}{n} 1\). Since \(G\) is 2-transitive we can assume with out loss of generality that \(i = j = n\). First note that

\[ E_\chi(v_{n,n} - \frac{1}{n} 1) = E_\chi(v_{n,n}). \]

Denote the row of \(E_\chi\) corresponding to \(\pi\) by \([E_\chi]_\pi\). If \(\pi(n) = n\) then

\[ [E_\chi]_\pi \cdot v_{n,n} = \frac{n-1}{|G|} \sum_{\sigma(n) = n} \chi(\pi^{-1}\sigma) = \frac{n-1}{|G|} \sum_{g(n) = n} \chi(g). \]
Use fix′ to denote the number of fixed points of an element from $G_n$ has on $[n - 1]$, then this is equal to
\[
\frac{n - 1}{|G|} \sum_{g \in G_n} \text{fix}'(g) = \frac{n - 1}{|G|} \left( \frac{|G|}{n} \right) = \frac{n - 1}{n}.
\]
The first equality holds by Burnside’s lemma and the fact that $G_n$ is transitive.

Note that since $\sum_{g \in G} \chi(g) = 0$ this implies that
\[
\sum_{i=1}^{n-1} \sum_{g(n)=i} \chi(g) = -\frac{|G|}{n},
\]
and since $G$ is 2-transitive, for any $i \neq n$
\[
\sum_{g(n)=i} \chi(g) = -\frac{|G|}{n(n-1)},
\]
We can apply this in the case that $\pi(n) \neq n$ to get that
\[
|E\chi|_{\pi} \cdot v_{n,n} = \frac{n - 1}{|G|} \sum_{\sigma(n)=i} \chi(\pi^{-1}\sigma) = \frac{n - 1}{|G|} \left( -\frac{|G|}{n(n-1)} \right) = -\frac{1}{n}.
\]

Lemma 4.2. Let $G$ be a 2-transitive group. The set
\[
B := \{v_{i,j} - \frac{1}{n} | i, j \in [n-1]\}
\]
is a basis for the standard module of $G$.

Proof. Let $V$ denote the standard module of $G$. According to Lemma 4.1 $B \subset V$ and since the dimension of $V$ is equal to $|B| = (n - 1)^2$, it suffices to show that $B$ is linearly independent. Note, also, that since $1$ is not in the span of $v_{i,j}$ for $i, j \in [n-1]$, it is enough to prove that the set $\{v_{i,j} | i, j \in [n-1]\}$ is linearly independent.

Define a matrix $L$ to have the vectors $v_{i,j}$, with $i, j \in [n-1]$, as its columns. Then the rows of $L$ are indexed by the elements of $G$ and the columns are indexed by the ordered pairs $(i,j)$, where $i, j \in [n-1]$; we will also assume that the ordered pairs are listed in lexicographic order. It is, then, easy to see that
\[
L^\top L = \frac{(n - 1)!}{2} I_{(n-1)^2} + \frac{(n-2)!}{2} (A(K_{n-1}) \otimes A(K_{n-1})),
\]
where $I_{(n-1)^2}$ is the $(n - 1) \times (n - 1)$ identity matrix, $A(K_{n-1})$ is the adjacency matrix of the complete graph $K_{n-1}$ and $\otimes$ is the tensor product. The distinct eigenvalues of $A(K_{n-1})$ are $-1$ and $n - 2$; thus the eigenvalues of $A(K_{n-1}) \otimes A(K_{n-1})$ are $-(n-2)$, $1$, $(n-2)^2$. This implies that the least eigenvalue of $L^\top L$ is
\[
\frac{(n - 1)!}{2} - \frac{(n-2)(n-2)!}{2} > 0.
\]
This proves that $L^\top L$ is non-singular and hence full rank. This, in turn, proves that $L$ has full rank and that $\{v_{i,j} \mid i, j \in [n-1]\}$ is linearly independent. 

Define the $|G| \times n^2$ matrix $H$ to be the matrix whose columns are the vectors $v_{i,j}$, for all $i, j \in [n]$. Note that since $H$ has constant row-sums, the vector 1 is in the column space of $H$. We denote by $H_{(i,j)}$ the column of $H$ indexed by the pair $(i, j)$, for any $i, j \in [n]$. Define the matrix $\overline{H}$ to be the matrix obtained from $H$ by deleting all the columns $H_{(i,n)}$ and $H_{(n,j)}$ for any $i, j \in [n-1]$. With a similar method as in the proof of [19, Proposition 10], we prove the following.

**Lemma 4.3.** The matrices $H$ and $\overline{H}$ have the same column space.

**Proof.** Obviously, the column space of $\overline{H}$ is a subspace of the column space of $H$; thus we only need to show that the vectors $H_{(i,n)}$ and $H_{(n,j)}$ are in the column space of $\overline{H}$, for any $i, j \in [n-1]$. Since $G$ is 2-transitive, it suffices to show this for $H_{(1,n)}$. Define the vectors $v$ and $w$ as follows:

\[
v := \sum_{i \neq 1,n} \sum_{j \neq n} H_{(i,j)} \quad \text{and} \quad w := (n-3) \sum_{j \neq n} H_{(1,j)} + H_{(n,n)}.
\]

The vectors $v$ and $w$ are in the column space of $\overline{H}$. It is easy to see that for any $\pi \in G$,

\[
v_\pi = \begin{cases} n - 2, & \text{if } \pi(1) = n; \\ n - 2, & \text{if } \pi(n) = n; \\ n - 3, & \text{otherwise}, \end{cases} \quad w_\pi = \begin{cases} 0, & \text{if } \pi(1) = n; \\ n - 2, & \text{if } \pi(n) = n; \\ n - 3, & \text{otherwise}. \end{cases}
\]

Thus

\[(v - w)_\pi = \begin{cases} n - 2, & \text{if } \pi(1) = n; \\ 0, & \text{if } \pi(n) = n; \\ 0, & \text{otherwise}, \end{cases}\]

which means that $(n - 2)H_{(1,n)} = v - w$. This completes the proof. 

If the columns of $\overline{H}$ are arranged so that the first $n$ columns correspond to the pairs $(i, i)$, for $i \in [n]$, and the rows are arranged so that the first row corresponds to the identity element, and the next $|D_G|$ rows correspond to the derangements of $G$, then $\overline{H}$ has the following block structure:

\[
\begin{bmatrix}
1 & 0 \\
0 & M \\
B & C
\end{bmatrix}.
\]

Note that the rows and columns of $M$ are indexed by the elements of $D_G$ and the pairs $(i, j)$ with $i, j \in [n-1]$ and $i \neq j$, respectively; thus $M$ is a $|D_G| \times (n-1)(n-2)$ matrix. Throughout the paper, we will refer to this matrix simply as “the matrix $M$ for $G$”.

The next proposition shows that the submatrix $B$ in $\overline{H}$ above contains an $n \times n$ identity matrix.
Proposition 4.4. Let $G \leq \text{Sym}(n)$ be 2-transitive. Then for any $x \in [n]$, there is an element in $G$ which has $x$ as its only fixed point.

Proof. Since $G$ is transitive, it suffices to show this for $x = n$. We need to show that the stabilizer of $n$ in $G$, denoted $G_n$, has a derangement. Suppose for every element $g \in G_n$, we have $|\text{fix}(g)| \geq 1$. This means that

$$\frac{1}{|G_n|} \sum_{g \in G_n} |\text{fix}(g)| \geq \frac{(n-1) + |G_n| - 1}{|G_n|} = \frac{(n-2 + |G_n|)}{|G_n|},$$

which is greater than 1, if $n > 2$. Hence by Burnside’s lemma, the number of orbits of the action of $G_n$ on $[n-1]$ is more than one which is a contradiction since $G_n$ acts transitively on $[n-1]$. Thus there must be a derangement in $G_n$. 

We are now ready to state the method that we call the module method which we use to prove that many 2-transitive groups have the strict EKR theorem.

Theorem 4.5. Let $G \leq \text{Sym}(n)$ be 2-transitive and assume the following conditions hold:

(a) $G$ has the EKR property;

(b) for any maximum intersecting set $S$ in $G$, the vector $v_S$ lies in the direct sum of the trivial and the standard modules of $G$; and

(c) the matrix $M$ for $G$ has full rank.

Then $G$ has the strict EKR property.

Proof. Since $G$ has the EKR property, the maximum size of an intersecting subset of $G$ is $|G|/n$, i.e. the size of a point-stabilizer. Suppose that $S$ is an intersecting set of maximum size. It is enough to show that $S = S_{i,j}$, for some $i, j \in [n]$. Without loss of generality, we may assume that $S$ includes the identity element. By assumption (b) and Lemma 4.2, $v_S$ is in the column space of $H$; thus according to Lemma 4.3, $v_S$ belongs to the column space of $\overline{H}$; therefore

$$\begin{bmatrix} 1 & 0 \\ 0 & M \\ B & C \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = v_S$$

for some vectors $z$ and $w$. Since the identity is in $S$, no elements from $\mathcal{D}_G$ are in $S$, the characteristic vector of $S$ has the form

$$v_S = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix}$$

for some vector 01-vector $t$. Thus we have

$$1^\top z = 1, \quad Mw = 0, \quad Bz + Cw = t.$$
Since $M$ has full rank, $w = 0$ and so $Bz = t$. Furthermore, according to Proposition 4.4 one can write

$$B = \begin{bmatrix} I_n \\ B' \end{bmatrix} \quad \text{and} \quad Bz = \begin{bmatrix} z \\ B'z \end{bmatrix}. $$

Since $Bz$ is equal to the 01-vector $t$, the vector $z$ must also be a 01-vector. But, on the other hand, $1^\top z = 1$, thus we conclude that exactly one of the entries of $z$ is equal to 1. This means that $v_S$ is the characteristic vector of the stabilizer of a point.

Through this paper we refer the conditions of Theorem 4.5 as conditions (a), (b) and (c), without reference to the theorem. We point out that the module condition (b) is the reason we call this method the module method.

This gives an algorithm for testing if a 2-transitive group has the EKR or strict EKR property (although it cannot determine if a group does not have the strict EKR property). First we calculate all the eigenvalues of $\Gamma_G$. If the standard representation gives the least eigenvalue, then the EKR property holds. If it is the only representation that gives the least eigenvalue, then the characteristic vector for any maximum independent set is in the standard module. If the matrix $M$ has full rank, then the group has the strict EKR property. If we can’t show that the standard representation gives the least eigenvalue, then we check if the derangement graph of the group has a clique of size $n$. If it does, then the group has the EKR property. If for each irreducible character (other than the trivial and the standard character) we can find a clique such that the projection to the corresponding module is non-zero, then we know that all characteristic vectors for maximum independent sets are in the direct sum of the trivial module and the standard module. Finally, we need to check that the matrix $M$ for $G$ has full rank. If it does, then the group has the strict EKR property.

This method has been applied to the symmetric group, alternating group and $\text{PGL}(2, q)$. In the next section we apply it to the Mathieu groups and all 2-transitive groups on 20 or fewer points.

5. EKR for the Mathieu groups

In this section, using the module method, we establish the strict EKR property for the 2-transitive Mathieu groups. Since the family of Mathieu groups is finite, the main approach of this problem uses a computer program to show that the conditions of Theorem 4.5 hold. All of these programs have been run in the GAP programming system [11].

Following the standard notation, we will denote the Mathieu group of degree $n$ by $M_n$. Note that $M_n \leq \text{Sym}(n)$ and we consider the natural action of $M_n$ on the set $[n]$, as usual. Table I lists some of the properties of the Mathieu groups which will be useful for our purpose (see [5] for more details).

A simple computer program confirms the following.
| Group | Order | Transitivity       | Simplicity |
|-------|-------|-------------------|------------|
| $M_{10}$ | 720   | sharply 3-transitive | not simple |
| $M_{11}$ | 7920  | sharply 4-transitive | simple     |
| $M_{12}$ | 95040 | sharply 5-transitive | simple     |
| $M_{21}$ | 20160 | 2-transitive       | simple     |
| $M_{22}$ | 443520 | 3-transitive       | simple     |
| $M_{23}$ | 10200960 | 4-transitive       | simple     |
| $M_{24}$ | 244823040 | 5-transitive       | simple     |

Table 1: Order and transitivity table for Mathieu groups

**Lemma 5.1.** Let $n \in \{10, 11, 12, 21, 22, 23, 24\}$. For each of the groups $M_n$ the least eigenvalue is $-\frac{D_{M_n}}{n-1}$ and the standard representation is the only representation with this eigenvalue.

This implies that condition (a) holds by the ratio bound. Furthermore, condition (b) also holds for all the Mathieu groups, since the standard module is the entire $-\frac{D_{M_n}}{n-1}$-eigenspace.

Finally, we need to confirm that condition (c) also holds for all the Mathieu groups. This requires showing that the matrix $M$ for each of the Mathieu groups has full rank. For small $n$ we can do this with a computer program.

**Lemma 5.2.** If $G = M_n$ and $n \in \{10, 11, 12, 21\}$, then the matrix $M$ for $G$ has full rank.

For $n = 22, 23, 24$ the Mathieu groups are too large to quickly check the rank of the matrix $M$ using a computer, so instead we determine the entries of the matrix $M^T M$. The entries of this matrix can be expressed as the linear combination of the identity matrix and the adjacency matrix of a graph that we define next.

For $n > 3$ define a graph $X_n$, which we call the pairs graph. For any $n > 3$, the vertices of $X_n$ are all the ordered pairs $(i, j)$, where $i, j \in [n-1]$ and $i \neq j$: the vertices $(i, j)$ and $(k, \ell)$ are adjacent in $X_n$ if and only if either $\{i, j\} \cap \{k, \ell\} = \emptyset$, $(i = \ell$ and $j \neq k)$ or $(i \neq \ell$ and $j = k)$. The graph $X_n$ is regular of valency $(n-2)(n-3)$. Note that the vertices of the pairs graph $X_n$ are the pairs from $[n-1]$. The next lemma has been proved, using a version of the ratio bound for cliques, in [2].

**Lemma 5.3.** For any $n > 3$, the least eigenvalue of the pairs graph $X_n$ is at least $-(n-3)$. $\square$

**Lemma 5.4.** The matrix $M$ for the group $M_{22}$ has full rank.

**Proof.** Let $C_{22}$ be one of the (two) conjugacy classes of $M_{22}$ whose elements are product of two disjoint 11-cycles. Set $N = M_{22}^T M_{C_{22}}$. Using a computer code we can establish

$$N = 1920 I + 96 A(X_n).$$
Lemma 5.3 shows that the least eigenvalue of $N$ is at least $1920 - 96(19) = 96$. This shows that $N$ is non-singular and we are done.

**Lemma 5.5.** The matrix $M$ for the group $M_n$ has full rank for $n \in \{23, 24\}$.

**Proof.** Let $C_{23}$ be one of the two conjugacy classes of $M_{23}$ of permutations that are 23-cycles and let $C_{24}$ be the only conjugacy class of $M_{24}$ whose elements are the product of two disjoint 12-cycles. Set $t_n = |C_n|$, for $n = 23, 24$. Assume $M_n^\dag$ to be the submatrix of $M$, with the rows labeled by $C_n$ and set $N_n = M_n^\dag M_n$, for $n = 23, 24$. We now calculate the entries of $N$. Since $M_n$ is 4-transitive, the entry $((a, b), (c, d))$ in $N_n$ depends only on the intersection of $\{a, b\}$ and $\{c, d\}$. To see this, consider the pairs $(a, b), (c, d)$ from $[n-1]$. If an element $\pi \in C_n$ maps $a \mapsto b$ and $c \mapsto d$, then for any pairs $(a', b'), (c', d')$ of elements of $[n-1]$, the permutation $g^{-1} \pi g \in C_n$ maps $a' \mapsto b'$ and $c' \mapsto d'$, where $g \in M_n$ is a permutation which maps $(a', b', c', d')$ to $(a, b, c, d)$. Therefore, we have

$$(N_n)_{(a,b),(c,d)} = \begin{cases} (N_n)_{(1,2),(1,2)}, & \text{if } (c, d) = (a, b); \\
(N_n)_{(1,2),(2,1)}, & \text{if } (c, d) = (b, a); \\
(N_n)_{(1,2),(2,3)}, & \text{if } a \neq d \text{ and } b = c; \\
(N_n)_{(1,2),(3,2)}, & \text{if } a = d \text{ and } b \neq c; \\
(N_n)_{(1,2),(3,4)}, & \text{if } a, b, c, d \text{ are distinct.} \
\end{cases} \quad (2)$$

Because of the 2-transitivity of $M_n$, we have $(N_n)_{(1,2),(1,2)} = \frac{t_n}{n-1}$ and using a simple computer code we can check that

$$\frac{(N_n)_{(1,2),(2,3)} = (N_n)_{(1,2),(3,4)}}{(n-1)(n-2)} = \frac{t_n}{(n-1)(n-2)}.$$ 

Also since elements of $C_n$ do not include a cycle of length 2 in their cycle decomposition, we have $(N_n)_{(1,2),(2,1)} = 0$. Thus we can re-write (2) as

$$(N_n)_{(a,b),(c,d)} = \begin{cases} \frac{t_n}{n-1}, & \text{if } (c, d) = (a, b); \\
0, & \text{if } (c, d) = (b, a); \\
\frac{t_n}{(n-1)(n-2)}, & \text{otherwise.} \end{cases}$$

This means that one can write

$$N_n = \frac{t_n}{n-1} I + \frac{t_n}{(n-1)(n-2)} A(X_n),$$

where $A(X_n)$ is the adjacency matrix of the pairs graph $X_n$. Then Lemma 5.3 shows that the least eigenvalue of $N_n$ is at least

$$\frac{t_n}{n-1} \left( 1 - \frac{n-3}{n-2} \right) > 0.$$ 

We conclude that $N_n$ and, consequently, the matrix $M$ are full rank. 

Putting these together we have the that all the Mathieu groups has the strict EKR property.

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Theorem 5.6. The Mathieu groups $M_n$, for $n \in \{10, 11, 12, 21, 22, 23, 24\}$, have the strict EKR property.

6. 2-transitive groups with small degree

In Appendix A we present Table A.2 in which we record the results of applying the module method to the 2-transitive permutation groups of degree at most 20. For each of these groups we first calculate the eigenvalues of the derangement graph. If the standard character gives the least eigenvalue, then we know by Theorem 3.5 that the group has the EKR property. If it is the only character that gives this eigenvalue, then condition (b) of 4.5 holds as well.

If the standard character does not produce the least eigenvalue, then we then check for sharply-transitive subgroups. If one exists, then according to Theorem 3.1 the group has the EKR property (we did not check for sharply transitive sets, so it is still possible that the group may have a clique of size $n$). If we do find such cliques, we then check the projections of these cliques to the different modules. If for every module, other than the trivial and the standard, we can find a clique of size $n$ whose projection to the module is non-zero, then condition (b) of 4.5 holds.

Finally, for all the groups we check if the matrix $M$ for $G$ has full rank (condition (c)). All these steps are implemented by a GAP program. Below we discuss two special cases.

If the eigenvalues of $\Gamma_G$ are $D_G$ and $-1$, then $\Gamma_G$ is the union of complete graphs. This means that the eigenvalue on the standard module is $-1$ or that

$$\frac{-D_G}{n-1} = -1.$$ 

From this we can conclude that the degree of $\Gamma_G$ is $n-1$ and that it is the union of $\frac{\mid G \mid}{n}$ copies of the complete graph on $n$ vertices. This means that $G$ has the EKR property, but, provided that $n > 3$, it does not have the strict EKR property. To see this, simply note that this graph has $\frac{n^{\frac{n(n-1)}{2}}}{n}$ independent sets of size $\frac{\mid G \mid}{n}$. Since $G$ is 2-transitive and, provided that $n > 3$, we have that $\frac{\mid G \mid}{n} > 2$ and there are more maximum independent sets than cosets of a point-stabilizer. If $n = 3$ then the only group to consider is Sym(3), which can easily been seen to have the strict EKR property (this case is so small that it is possible to list all the independent sets).

In [19] it is shown that the group PGL$(n, q)$ has the EKR property. This is due to the fact that these groups all contain a Singer cycle: these cycles form large cliques in $\Gamma_{\text{PGL}(n, q)}$, and the result follows from the clique-coclique bound. But, provided that $n > 2$, these groups never have the strict EKR property. This holds since the stabilizer of a hyperplane is an intersecting set with size equal to the stabilizer of a point.
7. Further Work

In the table in the appendix we have that every 2-transitive group with degree 20 or less has the EKR property. We would like to determine if every 2-transitive group has the EKR property.

We would also like to apply this method to other families of 2-transitive groups. For example, the group $\text{PSL}(2,q)$ has the EKR property and condition (b) holds (see [1] for details). Furthermore, a computer program has shown that for every $q \leq 30$, the matrix $M$ has full rank. But, we have been unable to prove that $M$ has full rank for every $q$.

The next step will be to test if the groups for which the matrix $M$ is not full rank have the strict EKR property. One approach is to search for the maximum independent sets in the derangement graphs. The problem of finding maximum independent sets is NP-hard so this would take a long time. Another approach would be to look for some “natural” independent sets. For example, the stabilizer of a set of points may form a maximum independent set, or perhaps the set of all permutations that fix at least two points from a set of three points would be the largest independent set.

The next open problem that we plan to work on is to determine if there is any structure to the maximum intersecting sets in groups that do not have the strict EKR property. For example, it has been conjectured [19] that the maximum independent set in $\Gamma_{\text{PGL}(3,q)}$ are either stabilizers of a point or the stabilizer of a hyperplane. It is not clear if the module method will be useful for this problem.

Our results for the subgroups with degree 16 are not as satisfying. For many of these groups the matrix $M$ does not have full rank, and we suspect that the strict EKR property does not hold. It would be interesting to look for non-canonical independent sets of maximum size in the derangement graphs of these groups. Also, we would like to know if it is possible to determine the structure of the derangement graph of the group $G \rtimes \mathbb{Z}_2$ when we know that structure of the derangement graph of $G$.

Finally, we plan to look more closely at the groups for which the matrix $M$ has full rank, but we have not yet been able to prove that the characteristic vector of every maximum independent set is in the sum of the trivial and standard module. In fact, for all of the 2-transitive groups we have considered, the maximum independent sets are either all in the standard module or the derangement graph is the union of complete graphs. It would be interesting to determine if there are groups for which this does not hold.

Appendix A. Module Method for Small Groups

In this appendix we present a table of our results from applying the the module method to all 2-transitive groups with degree at most 20.

This work was implemented by a program in GAP. Note that since all the groups $\text{Sym}(n)$ and $\text{Alt}(n)$ have the strict EKR property, they are excluded in the table. In the table we use the following terminology:
• **n**: degree of the group;

• **least**: a “Yes” in this column means that the least eigenvalue of the derangement graph is given by the standard character;

• **n-clique**: a “Yes” in this column means that our program has found a clique of size $n$ in $\Gamma_n$ (hence the clique-coclique bound holds with equality); the symbol “–” means that we don’t try to find a maximum clique, and the symbol “?” means that the program failed to find such a clique (but not that one does not exist!);

• **EKR**: a “Yes” in this column means that the group has the EKR property, i.e. condition (a) of the module method holds;

• **unique**: a “Yes” in this column means that the standard character is the only character giving the least eigenvalue; hence condition (b) of the module method holds; An “N/A” means that this condition is not applicable since the standard character does not give the least eigenvalue.

• **module by clique**: a “Yes” in this column means that using cliques of size $n$ and Corollary 3.4 we know that the characteristic vector of any maximum independent set of $\Gamma_G$ lies in the direct sum of the trivial and the standard characters of $G$; hence condition (b) of the module method holds; the symbol “–” means that we don’t verify this, and the symbol “?” means that the program could not find suitable cliques to prove that condition (b) holds;

• **rank**: a “Yes” in this column means that the matrix $M$ for the group $G$ has full rank, i.e. condition (c) of the module method holds;

• **strict**: a “Yes” in this column means that $G$ has the strict EKR property; the symbol “?” means that the program could not verify this. In all the cases where we have a “No”, either the derangement graph is the union of complete graphs or the group is $\text{PGL}(3, q)$ for some $q$;
Table A.2: EKR and strict EKR property for small 2-transitive groups

| n | Group | size | least n-clique | EKR | unique module by clique | rank | strict |
|---|-------|------|----------------|-----|-------------------------|------|--------|
| 5 | $\mathbb{Z}_5 \times \mathbb{Z}_4$ | 20 | Yes | – | Yes | Yes | – | No | No |
| 6 | PGL(2, 5) | 120 | Yes | Yes | Yes | No | – | Yes | Yes |
| 6 | Alt(5) | 60 | Yes | – | Yes | Yes | – | Yes | Yes |
| 7 | PGL(3, 2) | 168 | Yes | Yes | Yes | No | ? | No | No |
| 7 | $(\mathbb{Z}_7 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ | 42 | Yes | Yes | Yes | No | – | Yes | Yes |
| 8 | $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \text{PSL}(3, 2)$ | 1344 | Yes | – | Yes | Yes | – | Yes | Yes |
| 8 | PGL(2, 7) | 336 | Yes | Yes | Yes | No | – | Yes | Yes |
| 8 | $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7$ | 56 | Yes | – | Yes | Yes | – | No | No |
| 9 | PSL(2, 8) | 1512 | Yes | – | Yes | Yes | – | Yes | Yes |
| 9 | $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_4$ | 432 | Yes | Yes | Yes | No | – | Yes | Yes |
| 9 | $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ | 216 | No | Yes | Yes | N/A | ? | No | ? |
| 10 | PSL(2, 8) | 504 | Yes | – | Yes | Yes | – | Yes | Yes |
| 10 | $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ | 144 | No | Yes | Yes | N/A | ? | No | ? |
| 10 | $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ | 72 | Yes | – | Yes | Yes | – | No | No |
| 10 | $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Q}_8$ | 72 | Yes | – | Yes | Yes | – | No | No |
| 11 | $(\text{Alt}(6) \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ | 1440 | No | Yes | Yes | N/A | ? | Yes | ? |
| 11 | $\text{M}_{10}$ | 720 | Yes | – | Yes | Yes | – | Yes | Yes |
| 11 | Alt(6) | 720 | Yes | – | Yes | No | – | ? | Yes |
| 11 | PGL(2, 9) | 720 | Yes | Yes | Yes | No | – | Yes | Yes |
| 11 | $\text{Alt}(6)$ | 360 | Yes | – | Yes | Yes | – | Yes | Yes |
| 11 | $\text{M}_{11}$ | 7920 | Yes | – | Yes | Yes | – | Yes | Yes |
| 11 | PSL(2, 11) | 660 | Yes | – | Yes | Yes | – | No | ? |
| 11 | $(\mathbb{Z}_{11} \times \mathbb{Z}_5) \rtimes \mathbb{Z}_2$ | 110 | Yes | – | Yes | Yes | – | No | No |
| 12 | $\text{M}_{12}$ | 95040 | Yes | – | Yes | Yes | – | Yes | Yes |
Table A.2 – continued from previous page

| $n$ | Group | size | least $n$-clique | EKR | unique module by clique | rank | strict |
|-----|-------|------|------------------|-----|------------------------|------|--------|
| 12  | $M_{11}$ | 7920 | Yes             | Yes | Yes                    | –    | Yes    |
| 12  | PGL(2, 11) | 1320 | Yes             | Yes | No                     | –    | Yes    |
| 12  | PGL(2, 11) | 660  | –               | Yes | Yes                    | –    | Yes    |
| 13  | PSL(3, 3) | 5616 | Yes             | Yes | Yes                    | –    | Yes    |
| 13  | $(Z_{13} \times Z_4) \times Z_3$ | 156 | Yes             | Yes | Yes                    | No   | No     |
| 14  | PGL(2, 13) | 2184 | Yes             | Yes | No                     | –    | Yes    |
| 14  | PSL(2, 13) | 1092 | Yes             | Yes | No                     | –    | Yes    |
| 15  | $A_8$ | 20160 | Yes             | Yes | Yes                    | No   | ?      |
| 15  | $A_7$ | 2520  | Yes             | Yes | Yes                    | –    | No     |
| 16  | $(Z_2 \times Z_2 \times Z_2 \times Z_2) \times A_8$ | 322560 | Yes             | Yes | Yes                    | –    | Yes    |
| 16  | $((Z_2 \times Z_2 \times Z_2 \times Z_2) \times A_6) \times Z_2$ | 11520 | No             | Yes | Yes                    | N/A  | Yes    |
| 16  | $(((Z_2 \times Z_2 \times Z_2) \times A_5) \times Z_3) \times Z_2$ | 5760 | No             | Yes | Yes                    | N/A  | Yes    |
| 16  | $((Z_2 \times Z_2 \times Z_2 \times Z_2) \times A_5) \times Z_3$ | 2880 | Yes             | Yes | Yes                    | No   | ?      |
| 16  | $(Z_2 \times Z_2 \times Z_2 \times Z_2) \times A_7$ | 40320 | Yes             | Yes | Yes                    | Yes  | Yes    |
| 16  | $(Z_2 \times Z_2 \times Z_2 \times Z_2) \times A_6$ | 5760 | Yes             | Yes | Yes                    | Yes  | Yes    |
| 17  | $((Z_2 \times Z_2 \times Z_2) \times A_5) \times Z_2$ | 1920 | No             | Yes | Yes                    | N/A  | Yes    |
| 17  | $((Z_2 \times Z_2 \times Z_2) \times A_5) \times Z_2$ | 960  | No             | Yes | Yes                    | N/A  | Yes    |
| 17  | $((Z_2 \times Z_2 \times Z_2 \times Z_2) \times A_5) \times Z_3$ | 960 | No             | Yes | Yes                    | N/A  | Yes    |
| 17  | $((Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2) \times Z_3) \times Z_2$ | 480 | No             | Yes | Yes                    | N/A  | ?      |
| 17  | $((Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2) \times Z_3) \times Z_2$ | 240 | Yes             | –   | Yes                    | Yes  | No     |
| 18  | $PSL(2, 16) \times Z_4$ | 16320 | Yes             | Yes | Yes                    | –    | Yes    |
| 18  | $PGL(2, 16)$ | 8160 | Yes             | Yes | Yes                    | –    | Yes    |
| 17  | $PSL(2, 16)$ | 4080 | Yes             | Yes | Yes                    | –    | Yes    |
| 17  | $Z_{17} \times Z_{16}$ | 272 | Yes             | Yes | Yes                    | –    | No     |
| 18  | $PGL(2, 17)$ | 4896 | Yes             | Yes | No                     | –    | Yes    |
| 18  | $PSL(2, 17)$ | 2448 | Yes             | Yes | Yes                    | –    | Yes    |
| 19  | $(Z_{19} \times Z_2) \times Z_2$ | 342 | Yes             | Yes | Yes                    | –    | No     |

Continued on next page
| n  | Group         | size | least | n-clique | EKR | unique | module by clique | rank | strict |
|----|---------------|------|-------|----------|-----|--------|------------------|------|--------|
| 20 | PGL(2, 19)    | 6840 | Yes   | –        | Yes | No     | –                | Yes  | Yes    |
| 20 | PSL(2, 19)    | 3420 | Yes   | –        | Yes | Yes    | –                | Yes  | Yes    |
References

[1] Bahman Ahmadi. *Maximum Intersecting Families of Permutations*. University of Regina, 2013. Ph.D. thesis.

[2] Bahman Ahmadi and Karen Meagher. A new proof for the Erdős-Ko-Rado Theorem for the alternating group. *ArXiv e-prints*, February 2013.

[3] R. A. Bailey. *Association Schemes: Designed Experiments, Algebra and Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[4] Eiichi Bannai and Tatsuro Ito. *Algebraic Combinatorics. I*. The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, 1984.

[5] Peter J. Cameron. *Permutation Groups*, volume 45. Cambridge University Press, 1999.

[6] Peter J. Cameron and Cheng Yeaw Ku. Intersecting families of permutations. *European J. Combin.*, 24(7):881–890, 2003.

[7] Philippe Delsarte. *An Algebraic Approach to the Association Schemes of Coding Theory*. Philips Research Reports: Supplements. N.V. Philips’ Gloeilampenfabrieken, 1973.

[8] Persi Diaconis and Mehrdad Shahshahani. Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete*, 57(2):159–179, 1981.

[9] Paul Erdős, Chao Ko, and Richard Rado. Intersection theorems for systems of finite sets. *The Quarterly Journal of Mathematics*, 12(1):313–320, 1961.

[10] Peter Frankl and Norihide Tokushige. The Erdős-Ko-Rado theorem for integer sequences. *Combinatorica*, 19:55–63, 1999.

[11] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.6.3*, 2013.

[12] Chris Godsil and Karen Meagher. A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations. *European J. Combin.*, 30(2):404–414, 2009.

[13] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[14] W. N. Hsieh. Intersection theorems for systems of finite vector spaces. *Discrete Math.*, 12:1–16, 1975.

[15] Cheng Yeaw Ku and Tony W. H. Wong. Intersecting families in the alternating group and direct product of symmetric groups. *Electr. J. Comb.*, 14(1), 2007.
[16] Benoit Larose and Claudia Malvenuto. Stable sets of maximal size in Kneser-type graphs. *European J. of Combin.*, 25(5):657 – 673, 2004.

[17] Li Wang. Erdős-Ko-Rado theorem for irreducible imprimitive reflection groups. *Front. Math. China*, 7(1):125 – 144, 2012.

[18] Karen Meagher and Lucia Moura. Erdős-Ko-Rado theorems for uniform set-partition systems. *Electron. J. Combin.*, 12:Research Paper 40, 12 pp. (electronic), 2005.

[19] Karen Meagher and Pablo Spiga. An Erdős-Ko-Rado theorem for the derangement graph of PGL(2, q) acting on the projective line. *Journal of Combinatorial Theory, Series A*, 118(2):532–544, 2011.

[20] Michael W. Newman. *Independent Sets and Eigenspaces*. University of Waterloo, 2004. Ph.D. thesis.

[21] Paul Renteln. On the spectrum of the derangement graph. *Electron. J. Combin.*, 14(1):Research Paper 82, 17 pp. (electronic), 2007.