Arithmetic Riemann-Roch isomorphism, Chern-Simons invariant and Liouville action

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Abstract

Using the arithmetic Schottky uniformization theory, we show the arithmeticity of $PSL_2(\mathbb{C})$ Chern-Simons invariant, and give an explicit formula of the arithmetic Riemann-Roch isomorphism for algebraic curves in terms of this invariant. As its application, we determine the unknown constant of the holomorphic factorization formula of determinant of Laplacians on Riemann surfaces.

1. Introduction

Arakelov theory, Chern-Simons theory and Liouville theory have different origins and important roles in many areas of mathematics containing arithmetic (algebraic) geometry, differential geometry, global analysis and mathematical physics. The aim of this paper is to show that a combination of the 3 theories contributes to these advances. More precisely, we consider together the subjects:

- the arithmetic Riemann-Roch theorem for algebraic curves (cf. [D, GS]),
- the $PSL_2(\mathbb{C})$ Chern-Simons theory for hyperbolic 3-manifolds with boundary (cf. [GM, MP]),
- the holomorphic factorization formula of determinant of Laplacians on Riemann surfaces in terms of the classical Liouville action (cf. [Z1, Z2, MT]).

As a result of this consideration, we have the following results:

- arithmeticity of the $PSL_2(\mathbb{C})$ Chern-Simons invariant (cf. Corollary 4.3),
- an explicit formula of the arithmetic Riemann-Roch isomorphism under the trivialization by the arithmetic $PSL_2(\mathbb{C})$ Chern-Simons invariant and Eichler cohomology (cf. Theorem 5.2),
- determination of the unknown constant in the factorization formula of Zograf and McIntyre-Takhtajan (cf. Theorems 5.3 and 5.4).
We will review the main idea of this paper. The arithmetic Riemann-Roch theorem is an advanced, i.e., metrized version of the Grothendieck Riemann-Roch theorem, and is especially applied to Diophantine problem. For a family of algebraic curves, these theorems gives an isometric isomorphism (up to a nonzero constant)

$$\lambda_k^{\otimes 12} \cong \kappa^{\otimes d_k}; \quad d_k := 6k^2 - 6k + 1,$$

where $\lambda_k$ denotes the $k$th tautological line bundle with Quillen metric, and $\kappa$ denotes the Deligne pairing of the relative canonical sheaf with itself. The Liouville (field) theory gives rise to the Liouville action which is a functional on the space of conformal metrics on Riemann surfaces whose critical points correspond to hyperbolic metrics. Then the holomorphic factorization formulae relate the classical Liouville action with the determinants of Laplacians in terms of Zograf-McIntyre-Takhtajan’s infinite products which are extensions of Ramanujan’s delta function. Notice that these determinants corresponds to the Quillen metric on $\lambda_k$, and as is shown by Aldrovandi [A], the classical Liouville action gives the natural metric on $\kappa$. Therefore, one can suppose that the arithmetic Riemann-Roch isometry is equivalent to the holomorphic factorization formula.

In order to make this equivalence more complete, we use the theories of arithmetic Schottky uniformization [I2] and of Chern-Simons line bundles [GM]. In [I2], one has a higher genus version of the Tate curve which becomes Schottky uniformized Riemann surfaces over $\mathbb{C}$ and gives local coordinates on the moduli of algebraic curves. In [GM], it is shown that the $PSL_2(\mathbb{C})$ Chern-Simons invariants of Schottky uniformized 3-manifolds give rise to a hermitian line bundle, called the Chern-Simons line bundle, $\mathcal{L}$ which is isometrically isomorphic to $\lambda_1^{\otimes (-6)}$ on the Schottky space. We will show that $\mathcal{L}^{\otimes 2}$ is actually defined on the moduli space of Riemann surfaces, and that one has an isometric isomorphism (up to a nonzero constant)

$$\lambda_1^{\otimes 12} \cong \mathcal{L}^{\otimes (-2)}.$$

By the above isomorphisms, we have an identification

$$\mathcal{L}^{\otimes 2} \cong \kappa^{\otimes (-1)}$$

as line bundles over the Deligne-Mumford compactification of the moduli space. Then by the arithmetic Schottky uniformization theory, one can modify the exponential of the $PSL_2(\mathbb{C})$ Chern-Simons invariant such as to have a universal expression as a power series over $\mathbb{Z}$. This arithmetic Chern-Simons invariant
gives a (local) canonical trivialization of $\kappa$, and a result of [I3] implies that the Eichler cohomology groups of Schottky groups gives a canonical trivialization of $\lambda_k$. As the consequence of the above consideration, under these trivializations, the arithmetic Riemann-Roch isomorphism is explicitly expressed as Zograf-McIntyre-Takhtajan’s infinite products. This explicit formula also can determine the constant in the holomorphic factorization formula.

We give some comments on related works. In [MP, Theorem 1.1], McIntyre and Park express the $PSL_2(\mathbb{C})$ Chern-Simons invariants of Schottky uniformized 3-manifolds in terms of the Bergman tau function (cf. [KK1]) and Zograf’s function. Since this tau function has the modular property (cf. [KK2, (3.15)]), McIntyre-Park’s formula seems to represent the isomorphism $\lambda_1^{\otimes 24} \cong \mathcal{L}^{\otimes (-4)}$. Another relationship between the $PSL_2(\mathbb{C})$ Chern-Simons invariants and Deligne pairings are given by Morishita and Terashima [MoT] for 3-manifolds obtained as knot complements.

2. The Chern-Simons line bundle

2.1. $PSL_2(\mathbb{C})$ Chern-Simons invariant.

In [CS], Chern and Simons studied secondary characteristic classes of connections on principal bundles, and their work has been developed to the (called) Chern-Simons theory. For a Riemannian oriented 3-manifold $X$ with metric $g$ and an orthonormal frame $S$ over $X$, the Chern-Simons invariant $CS(g,S)$ is defined as the integral

$$\frac{1}{16\pi^2} \int_X S^* \left( \text{Tr} \left( \omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) \right),$$

where $\omega$ denotes the Levi-Civita connection form for $g$. In [RSW], Ramadas, Singer and Weitsman extended the Chern-Simons theory to $SU(2)$ flat connections on compact 3-manifolds with boundary. In this case, $CS(g,S)$ depends on the boundary value of $S$, and $e^{2\pi \sqrt{-1}CS}$ gives a section of a hermitian holomorphic line bundle over the moduli space of flat $SU(2)$ connections on the boundary surface. They also proved that this bundle is isomorphic to the determinant line bundle given by Quillen [Q].

One of other important contributions in the Chern-Simons theory is the $PSL_2(\mathbb{C})$ Chern-Simons invariant introduced by Yoshida [Y]. Guillarmou and Moroianu [GM], McIntyre and Park [MP] extended this invariant for nonclosed
3-manifolds, and they studied the relationship between the associated hermitian holomorphic line bundle and Quillen’s determinant line bundle in the Schottky setting. Recall that in [GM, Proposition 16] and [MP, 4.5], for each Schottky uniformized 3-manifold \( X \) with hyperbolic metric \( g \), the \( PSL_2(\mathbb{C}) \) Chern-Simons invariant \( CS^{PSL_2(\mathbb{C})} \) is defined and expressed as

\[
CS^{PSL_2(\mathbb{C})}(g, S) = -\frac{\sqrt{-1}}{2\pi^2} \text{Vol}_R(X) + \frac{\sqrt{-1}}{2\pi} \chi(\partial X) + CS(g, S).
\]

Here \( \text{Vol}_R(X) \) denotes the renormalized volume of \( X \), and \( \chi(\partial X) \) denotes the Euler characteristic of the boundary of \( X \).

### 2.2. Chern-Simons line bundle.

We define the Chern-Simons line bundle \( L_{\mathfrak{g}} \) over the Schottky space \( \mathfrak{g}_g \) of degree \( g \) following Freed [F], Ramadas-Singer-Weitsman [RSW] and especially Guillarmou and Moroianu [GM].

Schottky groups of degree \( g \) are free groups generated by \( \gamma_1, \ldots, \gamma_g \in PSL_2(\mathbb{C}) \) which map Jordan curves \( C_1, \ldots, C_g \subset \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \) to other Jordan curves \( C_{-1}, \ldots, C_{-g} \subset \mathbb{P}^1(\mathbb{C}) \) (with orientation reversed). Each element \( \gamma \in \Gamma - \{1\} \) is conjugated in \( PSL_2(\mathbb{C}) \) to \( z \mapsto q_{\gamma}z \) for some \( q_{\gamma} \in \mathbb{C} \times \) with \( |q_{\gamma}| < 1 \), called the multiplier of \( \gamma \). Therefore,

\[
\frac{\gamma(z) - a_{\gamma}}{\gamma(z) - b_{\gamma}} = q_{\gamma} \frac{z - a_{\gamma}}{z - b_{\gamma}}
\]

for some element \( a_{\gamma}, b_{\gamma} \) of \( \mathbb{P}^1(\mathbb{C}) \) called the attractive, repulsive fixed points of \( \gamma \) respectively. Then the discontinuity set \( \Omega_{\Gamma} \subset \mathbb{P}^1(\mathbb{C}) \) under the action of \( \Gamma \) has a fundamental domain \( D \) which is given by the complement of the union of the interiors of \( C_i \) (\( i = \pm 1, \ldots, \pm g \)). The quotient space \( \Omega_{\Gamma}/\Gamma \) of \( \Omega_{\Gamma} \) by \( \Gamma \) is a (compact) Riemann surface of genus \( g \) which we denote by \( R_{\Gamma} \). Furthermore, by a result of Koebe, every Riemann surface of genus \( g \) can be represented in this manner. A Schottky group \( \Gamma \) is marked if its free generators \( \gamma_1, \ldots, \gamma_g \) are fixed, and a marked Schottky group \( (\Gamma; \gamma_1, \ldots, \gamma_g) \) is normalized if \( a_{\gamma_1} = 0, b_{\gamma_1} = \infty \) and \( a_{\gamma_2} = 1 \). By definition, the Schottky space \( \mathfrak{g}_g \) of degree \( g \) is the space of marked Schottky groups of degree \( g \) modulo conjugation in \( PSL_2(\mathbb{C}) \) which becomes the space of normalized Schottky groups of degree \( g \) if \( g > 1 \). Then \( \mathfrak{g}_g \) is a covering space of the moduli space of Riemann surfaces of genus \( g \).

For each Schottky group \( \Gamma \subset PSL_2(\mathbb{C}) \), denote by \( X = X_{\Gamma} = \mathbb{H}^3/\Gamma \) the associated hyperbolic 3-manifold, where \( \mathbb{H}^3 \) denotes the 3-dimensional hyperbolic
space. Then the boundary of \(X\) becomes a Riemann surface \(R = R_\Gamma\) which is Schottky uniformized by \(\Gamma\). Denote by \(F(X)\) the frame bundle of \(X\), and by \(C^\infty_{\text{ext}}(R)\) the space of sections in \(C^\infty(R)\) which are extendible to \(\overline{X} = X \cup R\). Then we consider the map \(c_\Gamma : C^\infty(R, F(X)) \times C^\infty_{\text{ext}}(R, SO(3)) \to \mathbb{C}\) defined as

\[
c_\Gamma (\hat{S}, a) := \exp \left( \frac{2\pi \sqrt{-1}}{16\pi^2} \left( \int_R \text{Tr}(\hat{\omega} \wedge da \wedge a^{-1}) + \int_X \frac{1}{3} \text{Tr} ((\hat{a}^{-1}d\hat{a})^3) \right) \right),
\]

where \(\hat{\omega}\) is the connection form of the Levi-Civita connection of the associated metric on \(\overline{X}\) in any extension of \(\hat{S}\) to \(\overline{X}\), and \(\hat{a}\) is any smooth extension of \(a\) on \(\overline{X}\). Note that any \(a \in C^\infty(R, SO(3))\) can be extended to some \(\tilde{a}\) on \(\overline{X}\), and by [GM, Lemma 21], \(c_\Gamma (\hat{S}, a)\) depends only on \(\Gamma, \hat{S}\) and \(a\). Then \(c_\Gamma\) is well-defined and satisfies the cocycle condition

\[
c_\Gamma (\hat{S}, ab) = c_\Gamma (\hat{S}, a) \cdot c_\Gamma (\hat{S}a, b).
\]

The complex vector space \(L_\Gamma\) is defined as the space of complex-valued functions \(f\) on \(C^\infty_{\text{ext}}(R, F(X))\) which satisfy

\[
f (\hat{S}a) = c_\Gamma (\hat{S}, a) \cdot f (\hat{S}) \quad (a \in C^\infty_{\text{ext}}(R, SO(3))).
\]

Since any element in \(L_\Gamma\) is determined by its value on any frame extendible to \(\overline{X}\) by this condition, one can define the Chern-Simons line bundle over \(\mathfrak{g}_g\) as

\[
\mathcal{L}_{\mathfrak{g}_g} := \bigsqcup_{\Gamma \in \mathfrak{g}_g} L_\Gamma.
\]

Furthermore, the cocycle is of absolute value 1, and hence there exists a canonical hermitian metric \(\langle \cdot, \cdot \rangle_{\text{CS}}\) on \(\mathcal{L}_{\mathfrak{g}_g}\) defined as

\[
\langle f_1, f_2 \rangle_{\text{CS}} := f_1 (\hat{S}) \overline{f_2 (\hat{S})}
\]

for two sections \(f_1, f_2\) of \(\mathcal{L}_{\mathfrak{g}_g}\) and \(\hat{S} \in C^\infty(R, F(X))\).

**2.3. An isomorphism with the determinant line bundle.**

We review a result of [GM] on an explicit isomorphism between the Chern-Simons line bundle with connection and hermitian structure and the determinant line bundle on the Schottky space.
Let $R$ be a Riemann surface of genus $g > 0$, and \{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\} be a set of standard generators of $\pi_1(R, x_0)$ for some $x_0 \in R$ satisfying
\[(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}) \cdots (\alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}) = 1.\]
Then one can take a marked Schottky group $(\Gamma; \gamma_1, ..., \gamma_g)$ such that $R = R_\Gamma$ and that each $C_k$ is homotopic to $\alpha_k$. Therefore, there is uniquely a basis $\varphi_1, ..., \varphi_g$ of holomorphic 1-forms such that $\int_{\alpha_j} \varphi_i$ is equal to Kronecker’s delta $\delta_{ij}$, and then the period matrix $\tau = \left(\int_{\beta_j} \varphi_i\right)$ becomes a symmetric matrix whose imaginary part is positive definite. For each Schottky group $\Gamma$ of rank $g$, the differential operator $\partial_\Gamma : C^\infty(R) \to C^\infty(R, \Lambda^{1,0}(R))$ for the complex structure on $R$ is Fredholm on Sobolev spaces, and $\partial_\Gamma$ ($\Gamma \in \mathfrak{S}_g$) give a family of operators parametrized by points on $\mathfrak{S}_g$. Therefore, as in [Q], its determinant line bundle $\det(\partial)$ of $\partial_\Gamma$ is defined as the line
\[
\det(\partial)_\Gamma := \Lambda^g(\text{coker}(\partial_\Gamma)).
\]
Since
\[
\text{coker}(\partial_\Gamma) = \ker \left(\overline{\partial_\Gamma} : C^\infty(R, \Lambda^{1,0}) \to C^\infty(R, \Lambda^2(R))\right) =: H^{0,1}(R_\Gamma)
\]
becomes the space of holomorphic 1-forms on $R = R_\Gamma$, the line bundle $\det(\partial)$ over $\mathfrak{S}_g$ is a holomorphic line bundle, called the Hodge line bundle, with a holomorphic canonical section
\[
\varphi := \varphi_1 \wedge \cdots \wedge \varphi_g.
\]
For each Riemann surface $R_\Gamma$, let $h$ be the associated hyperbolic metric, and $\det' \Delta_h$ be the (regularized) determinant of its Laplacian defined by Ray-Singer [RS]. Then the hermitian Quillen metric on $\det(\partial)$ is defined as
\[
\|\varphi\|_Q^2 := \frac{\|\varphi\|_h^2}{\det' \Delta_h} = \frac{\det \text{Im}(\tau)}{\det' \Delta_h}
\]
at $\Gamma \in \mathfrak{S}_g$, where $\|\varphi\|_h$ is the hermitian product on $\Lambda^g(\text{coker}(\partial_\Gamma))$ induced by $h$. Therefore, there is the unique hermitian connection $\nabla^{\det}$ associated to the holomorphic structure on $\det(\partial)$ and the hermitian norm $\|\varphi\|_Q$.

To describe the relationship between the Chern-Simons line bundle and the determinant line bundle, we use a formula proved by Zograf [Z1, Z2] whose generalization by McIntyre-Takhtajan [MT] will be reviewed below. Denote by
$S_L : \mathcal{G}_g \to \mathbb{R}$ the classical Liouville action which is explicitly described by Takhtajan and Zograf [ZT] as

$$S_L = \frac{\sqrt{-1}}{2} \int \int_D \left( \left| \frac{\partial \log \rho}{\partial z} \right|^2 + \rho \right) dz \wedge d\overline{z} + \frac{\sqrt{-1}}{2} \sum_{k=2}^g \oint_{C_k} \left( \log \rho - \frac{1}{2} \log |\gamma_k'| \right)^2 \left( \frac{\gamma_k''}{\gamma_k'} dz - \frac{\gamma_k'}{\gamma_k''} d\overline{z} \right) + 4\pi \sum_{k=2}^g \log |c(\gamma_k)|^2,$$

where $\rho(z)|dz|^2$ denotes the pullback of the hyperbolic metric on $R_{\Gamma}$, and $c(\gamma) = c$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

**Theorem 2.1 (Zograf [Z1, Z2]).**

(1) There exists a holomorphic function $F_1 : \mathcal{G}_g \to \mathbb{C}$ such that

$$\frac{\det' \Delta_h}{\det \text{Im}(\tau)} = c_g \exp \left( - \frac{S_L}{12\pi} \right) |F_1(\Gamma)|^2,$$

where $c_g$ is a nonzero constant depending only on $g$.

(2) If the dimension $\delta_{\Gamma}$ of limit set of $\Gamma$ satisfies $\delta_{\Gamma} < 1$, then $F_1(\Gamma)$ has the following absolutely convergent product:

$$F_1(\Gamma) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left( 1 - q^{1+m}_{\gamma} \right),$$

where $q_{\gamma}$ denotes the multiplier of $\gamma \in \Gamma$ and $\{\gamma\}$ runs over all distinct primitive conjugacy classes in $\Gamma - \{1\}$.

By a result of Krasnov [Kr] (see also [TT, (1.13)]), if $\Gamma$ is a Schottky group of degree $g$ then for $X = \mathbb{H}^3/\Gamma$ and $R = R_{\Gamma}$,

$$\text{Vol}_R(X) = \frac{1}{4} S_L - \frac{\pi}{2} \chi(R) = \frac{1}{4} S_L + \pi(g-1),$$

and hence one has:
Corollary 2.2. Let the notation be as above. Then
\[
\frac{\det' \Delta_h}{\det \text{Im}(\tau)} = c_g \cdot e^{(1-g)/3} \exp \left( \frac{\text{Vol}_R(X)}{3\pi} \right) |F_1(\Gamma)|^2.
\]

Theorem 2.3 (Guillarmou and Moroianu [GM, Theorem 43]). There exists a connection \( \nabla^L \) and a hermitian metric \( \| \cdot \|_L \) on the Chern-Simons line bundle \( L = L_{S_g} \) on \( S_g \) such that \( L \otimes (-1)^{S_g} \) is isomorphic to \( (\det \partial) \otimes 6 \) when equipped with their connections and hermitian products induced by those of \( (L_{S_g}, \nabla^L, \| \cdot \|_L) \) and \( (\det \partial, \nabla^{\det}, \| \cdot \|_Q) \). More precisely, there is an explicit isometric isomorphism of holomorphic hermitian line bundles given by
\[
\left( \sqrt{c_g \cdot e^{1-g}} \cdot F_1 \varphi \right)^{\otimes 6} \mapsto e^{-2\pi \sqrt{-1} \text{CS}_{PSL_2(C)}},
\]
where \( F_1 \) is the holomorphic function on \( S_g \), \( \varphi \) is the canonical section of \( \det \partial \) as above.

3. Chern-Simons line bundle on the moduli of curves

3.1. Asymptotic of renormalized volumes.

For each \( 0 \leq i \leq [g/2] \), let \( t_i \) be a degeneration parameter of a family of stable (algebraic) curves of of genus \( g \) such that the degeneration under \( t_i = 0 \) is desingularized to a stable curve of genus \( g - 1 \) if \( i = 0 \), and to 2 stable curves of genus \( i \) and \( g - i \) if \( i > 0 \).

Proposition 3.1. For a family of Schottky uniformized 3-manifolds \( X \) with degenerating boundary \( \partial X \) of genus \( g \), one has
\[
\exp \left( \text{Vol}_R(X)/\pi \right) \sim |t_i|^{1/2} \quad (t_i \to 0)
\]
which means that
\[
\lim_{t_i \to 0} \exp \left( \text{Vol}_R(X)/\pi \right) \cdot |t_i|^{-1/2}
\]
exists and is not equal to 0.

Proof. As is mentioned above, \( \text{Vol}_R(X) = -S_L/4 + \pi(g - 1) \). Therefore, if \( i = 0 \), then the assertion follows from [Z1, Theorem 2.4]. Assume \( i > 0 \), and consider a family of Schottky groups \( \Gamma \) given by
\[
\Gamma = \langle \gamma_1, \ldots, \gamma_i, \mu^{-1}\gamma_{i+1}\mu, \ldots, \mu^{-1}\gamma_g\mu \rangle,
\]
where \( \mu \in PSL_2(\mathbb{C}) \) has multiplier \( t_i \). Then under \( t_i \to 0 \), one has \( F_1(\Gamma) \to F_1((\gamma_1, \ldots, \gamma_i)) \), and hence \( F_1(\Gamma) \sim 1 (t_i \to 0) \). Therefore, by Theorem 2.1,

\[
\exp\left( \frac{\text{Vol}_R(X)}{\pi} \right) = e^{g-1} \cdot \exp\left( -\frac{S_L}{(4\pi)} \right) \sim \|\varphi\|_Q^{-6} (t_i \to 0),
\]

and hence the assertion follows from the asymptotic of \( \|\varphi\|_Q \) shown in [Fr, Corollary 5.8]. □

3.2. Chern-Simons line bundle on the moduli of curves.

Let \( M_g \) denote the moduli stack over \( \mathbb{Z} \) of proper smooth algebraic curves of genus \( g \). Then the associated complex orbifold \( M_g(\mathbb{C}) \) becomes the moduli space of Riemann surfaces of genus \( g \). Let \( \overline{M}_g \) denote the Deligne-Mumford compactification of \( M_g \) as the moduli stack of stable curves of genus \( g \). Then the complement \( \partial M_g = \overline{M}_g - M_g \) is the union of normal crossing divisors \( D_i \) defined as \( t_i = 0 \ (0 \leq i \leq [g/2]) \). For the universal stable curve \( \pi : C_g \to \overline{M}_g \), the Hodge line bundle \( \lambda_1 \) is defined as \( \det\left( \pi_* \left( \omega_{C_g/\overline{M}_g} \right) \right) \), where \( \omega_{C_g/\overline{M}_g} \) denotes the dualizing sheaf on \( C_g \) over \( \overline{M}_g \).

**Theorem 3.2.** The bundle \( L^{\otimes 2}_{\mathfrak{S}_g} \) can be descended to a line bundle on \( M_g(\mathbb{C}) \) which we denote by \( L_{\mathcal{M}_g(\mathbb{C})} \). Furthermore, there exists an isomorphism

\[
\lambda_1^{\otimes 12} \sim L_{\mathcal{M}_g(\mathbb{C})}^{\otimes (-1)}
\]

which is also an isometry up to a nonzero constant.

**Proof.** By [GM, Theorem 43], \( (F_1(\varphi)^6 \cdot e^{2\pi \sqrt{-1} CS_{PSL_2(\mathbb{C})}}) \) is a flat section of

\[
(\lambda_1)^{\otimes 6} \otimes L_{\mathfrak{S}_g}
\]

which becomes a trivial bundle on \( \mathfrak{S}_g \) with trivial connection. By [GM, Proposition 16],

\[
CS_{PSL_2(\mathbb{C})}(g, S) = -\frac{\sqrt{-1}}{2\pi^2} \text{Vol}_R(X) + \frac{\sqrt{-1}}{2\pi} \chi(\partial X) + CS(g, S),
\]

and hence

\[
\left| \exp(-2\pi \sqrt{-1} CS_{PSL_2(\mathbb{C})}) \right| = e^{-\text{Vol}_R(X)/\pi} \cdot e^{\chi(\partial X)} = e^{S_L/(4\pi)} \cdot e^{3(1-g)}.\]
By Theorem 2.3,
\[ \Phi = (F_1 \varphi)^{12} \cdot \exp \left( 2\pi \sqrt{-1} \text{CS}^{PSL_2(\mathbb{C})} \right)^2 \]
gives a parallel section of \( \lambda_1^{\otimes 12} \otimes \mathcal{L}_{\mathcal{E}_g}^{\otimes 2} \) with canonical connection. Since \( \mathcal{G}_g \) is a unramified covering of \( \mathcal{M}_g(\mathbb{C}) \), \( \Phi \) gives a local system on \( \mathcal{M}_g(\mathbb{C}) \) with coefficients in \( \mathbb{C} \), and one has the associated monodromy representation \( \pi_1(\mathcal{M}_g) \to \mathbb{C}^\times \).

Then by the above asymptotic,
\[ |F_1|^{12} \cdot \left| \exp \left( 2\pi \sqrt{-1} \text{CS}^{PSL_2(\mathbb{C})} \right) \right|^2 \sim |t_i| (t_i \to 0), \]
and hence a small loop in \( \mathcal{M}_g(\mathbb{C}) \) around \( t_i = 0 \) has the trivial holonomy. Therefore, this monodromy representation is trivial. Since \( \lambda_1 \) comes from the line bundle denoted by the same notation on \( \mathcal{M}_g \), \( \mathcal{L}_{\mathcal{E}_g}^{\otimes 2} \) can be descended to a hermitian holomorphic line bundle on \( \mathcal{M}_g(\mathbb{C}) \) which we denote by \( \mathcal{L}_{\mathcal{M}_g}(\mathbb{C}) \) such that \( \Phi \) gives an isometric isomorphism \( \lambda_1^{\otimes 12} \simeq \mathcal{L}_{\mathcal{M}_g}(\mathbb{C})^{(-1)} \) up to a nonzero constant. □

**Corollary 3.3.** There exists a natural extension \( \mathcal{L}_{\overline{\mathcal{M}}_g}(\mathbb{C}) \) of \( \mathcal{L}_{\mathcal{M}_g}(\mathbb{C}) \) as a line bundle on \( \overline{\mathcal{M}}_g(\mathbb{C}) \). Furthermore,
\[ \lambda_1^{\otimes 12} \simeq \mathcal{L}_{\overline{\mathcal{M}}_g}(\mathbb{C})^{(-1)} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(\mathcal{M}_g) \]
which associates \( (F_1 \varphi)^{12} \) with \( \exp \left( 2\pi \sqrt{-1} \text{CS}^{PSL_2(\mathbb{C})} \right)^{-2} \) up to a nonzero constant.

**Proof.** This follows immediately from Theorem 3.2 and its proof. □.

### 4. Arithmetic Chern-Simons invariant

#### 4.1. Arithmetic Riemann-Roch theorem.

For a positive integer \( k \), put \( d_k = 6k^2 - 6k + 1 \), and define the \( k \)th tautological line bundle on \( \overline{\mathcal{M}}_g \) as
\[ \lambda_k := \det \left( \pi_* \left( \omega_{\overline{\mathcal{M}}_g/\mathcal{M}_g}^{\otimes k} \right) \right) \]
which is a metrized line bundle on \( \mathcal{M}_g \) equipped with Quillen metric. Denote by \( \kappa_g \) the hermitian line bundle on \( \overline{\mathcal{M}}_g \) defined as the following Deligne’s pairing:
\[ \kappa_g := \left\langle \omega_{\overline{\mathcal{M}}_g}, \omega_{\overline{\mathcal{M}}_g}/\mathcal{M}_g \right\rangle. \]
Further, denote by \( a(g) \) the Deligne constant \((1 - g) (24\zeta'(−1) - 1)\), where \( \zeta'(−1) \) denotes the derivative of Riemann’s zeta function \( \zeta \) at \(-1\).

**Theorem 4.1 (arithmetic Riemann-Roch theorem [D, GS]).** There exists a unique (up to a sign) isomorphism

\[
\lambda_k^\otimes \cong k_{\hat{dr}_k} \otimes \mathcal{O}_{\mathcal{M}_g} (\partial \mathcal{M}_g) \cdot e^{a(g)}
\]

between the line bundles over \( \overline{\mathcal{M}}_g \) which is an isometry between the line bundles over \( \mathcal{M}_g(\mathbb{C}) \) for these hermitian structure.

### 4.2. Arithmetic Schottky uniformization.

Arithmetic Schottky uniformization theory [I2] constructs a higher genus version of the Tate curve, and its 1-forms and periods. We review this theory for the special case concerned with universal deformations of irreducible degenerate curves.

Denote by \( \Delta \) the graph with one vertex and \( g \) loops. Let \( x_{\pm 1}, \ldots, x_{\pm g}, y_1, \ldots, y_g \) be variables, and put

\[
A = \mathbb{Z} \left[ \frac{1}{x_i - x_m} : (k, l, m \in \{\pm 1, \ldots, \pm g\}, l \neq m) \right], \\
A_\Delta = A[[y_1, \ldots, y_g]], \\
B_\Delta = A_\Delta [1/y_i (1 \leq i \leq g)].
\]

Then it is shown in [I2, Section 3] that there exists a stable curve \( C_\Delta \) of genus \( g \) over \( A_\Delta \) which satisfies the followings:

- \( C_\Delta \) is a universal deformation of the universal degenerate curve with dual graph \( \Delta \) which is obtained from \( \mathbb{P}^1_A \) by identifying \( x_i \) and \( x_{-i} (1 \leq i \leq g) \). The ideal of \( A_\Delta \) generated by \( y_1, \ldots, y_g \) corresponds to the closed substack \( \partial \mathcal{M}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g \) of \( \overline{\mathcal{M}}_g \) via the morphism \( \text{Spec}(A_\Delta) \rightarrow \overline{\mathcal{M}}_g \) associated with \( C_\Delta \).

- \( C_\Delta \) is smooth over \( B_\Delta \), and is Mumford uniformized (cf. [Mu]) by the subgroup \( \Gamma_\Delta \) of \( PGL_2(B_\Delta) \) with \( g \) generators

\[
\phi_i = \left( \begin{array}{cc} x_i & x_{-i} \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} y_i & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} x_i & x_{-i} \\ 1 & 1 \end{array} \right)^{-1} \mod (B_\Delta) \quad (1 \leq i \leq g).
\]

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Furthermore, $C_\Delta$ has the following universality: for a complete integrally closed noetherian local ring $R$ with quotient field $K$ and a Mumford curve $C$ over $K$ such that $\Delta$ is the dual graph of its degenerate reduction, there is a ring homomorphism $A_\Delta \to R$ which gives rise to $C_\Delta \otimes_{A_\Delta} K \cong C$.

- By substituting $\alpha_{\pm i} \in \mathbb{C}$ to $x_{\pm i}$ and $q_i \in \mathbb{C}^\times$ to $y_i$ ($1 \leq i \leq g$), $C_\Delta$ becomes the Riemann surface Schottky uniformized by $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ if $\alpha_{\pm i}$ are mutually different and $q_i$ are sufficiently small.

Actually, $C_\Delta$ is constructed in [I2] as the quotient of a certain subspace of $\mathbb{P}_{B_\Delta}^1$ by the action of $\Gamma$ using the theory of formal schemes. Furthermore, as is shown in [MD] and [I1, Section 3], the normalized holomorphic 1-forms $\omega_i$ ($1 \leq i \leq g$) on Schottky uniformized Riemann surfaces $X_\Gamma$ have the universal expression

$$
\sum_{\phi \in \Gamma_\Delta/(\phi_i)} \left( \frac{1}{z - \phi(x_i)} - \frac{1}{z - \phi(x_{-i})} \right) = \left( \frac{1}{z - x_{i}} - \frac{1}{z - x_{-i}} \right) + \sum_{\phi \in \Gamma_\Delta/(\phi_i) - \{1\}} \left( \frac{1}{z - \phi(x_i)} - \frac{1}{z - \phi(x_{-i})} \right) + \cdots
$$

$$
\in A_\Delta \left[ \prod_{k=1}^{g} \frac{1}{(z - x_k)(z - x_{-k})} \right]
$$

which make a basis of regular 1-forms on $C_\Delta$. We denote this basis on $C_\Delta$ by the same symbol $\{\omega_i\}$, and put $\omega = \omega_1 \wedge \cdots \wedge \omega_g$.

Let $(\Gamma; \gamma_1, ..., \gamma_g)$ be a Schottky group, and for $i = 1, ..., g$, put $\gamma_{-i} = \gamma_i^{-1}$. Then by Proposition 1.3 of [I2] and its proof, if $\gamma \in \Gamma$ has the reduced expression $\gamma_{\sigma(1)} \cdots \gamma_{\sigma(l)}$ ($\sigma(i) \in \{-1, ..., \pm g\}$) such that $\sigma(1) \neq -\sigma(l)$, then its multiplier $q_\gamma$ has universal expression as an element of $A_\Delta$ which is divisible by $y_{\sigma(1)} \cdots y_{\sigma(l)}$. Therefore, Zograf’s function $F_1$ has a universal expression as an invertible element of $A_\Delta$.

In the case when we consider the Schottky space $\mathfrak{S}_g$ of degree $g > 1$ as the moduli space of normalized Schottky groups, we assume that the above $\phi_1, ..., \phi_g$ are normalized by considering $x_1, x_{-1}$ as $0, \infty$ respectively, namely,

$$
\phi_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & y_1 \end{array} \right) \mod (B_\Delta^\times),
$$

and by putting $x_2 = 1$. Then as is shown in [I2, 1.1], the associated generalized Tate curve $C_\Delta$ is defined over $A'_\Delta = A'[[y_1, ..., y_g]]$, where $A'$ is obtained from $A$.
by deleting $x_{-1}$ and putting $x_1 = 0$, $x_2 = 1$. Therefore, one has the associated morphism $\text{Spec} (A_{\Delta}') \to \overline{\mathcal{M}}_g$.

4.3. Arithmeticity of Chern-Simons invariant.

**Theorem 4.2.** There exists an isomorphism

$$
\mathcal{L}_{\overline{\mathcal{M}}_g(\mathbb{C})} \cong \kappa_g^{\otimes (-1)}
$$

between the line bundles over $\overline{\mathcal{M}}_g(\mathbb{C})$ which is isometric over $\mathcal{M}_g(\mathbb{C})$.

**Proof.** The existence follows from Corollary 3.3 and the arithmetic Riemann-Roch theorem in the case when $k = 1$. The uniqueness follows from that $\overline{\mathcal{M}}_g(\mathbb{C})$ is compact. □

An element of $A_{\Delta}$ and $A_{\Delta}'$ is called primitive if this is not congruent to 0 modulo any prime.

**Corollary 4.3.** There exists a unique (up to a sign) primitive element of $A_{\Delta}$ which gives a universal expression of

$$
\exp \left( 4\pi \sqrt{-1} \text{CS}_{PSL_2(\mathbb{C})} \right)
$$

times a certain nonzero constant.

**Proof.** By the arithmetic Schottky uniformization theory, $\omega = (2\pi \sqrt{-1})^g \varphi$ gives a trivialization of $\lambda_1$ over $\text{Spec} (A_{\Delta})$. Hence by Corollary 3.3 and Theorem 4.2, a certain multiple of the holomorphic function

$$
F_1^{12} \cdot \exp \left( 4\pi \sqrt{-1} \text{CS}_{PSL_2(\mathbb{C})} \right)
$$

on $\mathcal{G}_g$ gives a trivialization over $\text{Spec} (A_{\Delta})$ of the $\mathbb{Z}$-structure on

$$
\mathcal{L}_{\overline{\mathcal{M}}_g(\mathbb{C})} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g} (-\partial \mathcal{M}_g)
$$
given by Theorem 4.2. Therefore, the assertion follows from that $F_1$ is given as an invertible element of $A_{\Delta}$. □

**Definition.** We denote this element of $A_{\Delta}$ by ACS$_g$, and call it the arithmetic universal Chern-Simons invariant.
5. Explicit arithmetic Riemann-Roch isomorphism

5.1. Holomorphic factorization formula.
Assume that $g > 1$, and take an integer $k > 1$. Let $(\Gamma; \gamma_1, \ldots, \gamma_g)$ be a marked normalized Schottky group, and $\mathbb{C}[z]_{2k-2}$ be the $\mathbb{C}$-vector space of polynomials $f = f(z)$ of $z$ with degree $\leq 2k - 2$ on which $\Gamma$ acts as

$$\gamma(f)(z) = f(\gamma(z)) \cdot \gamma'(z)^{1-k} \quad (\gamma \in \Gamma, \ f \in \mathbb{C}[z]_{2k-2}).$$

Take $\xi_{1,k-1}, \xi_{2,1}, \ldots, \xi_{2,2k-2}, \xi_{i,0}, \ldots, \xi_{i,2k-2} \ (3 \leq i \leq g)$ as elements of the Eichler cohomology group $H^1(\Gamma, \mathbb{C}[z]_{2k-2})$ of $\Gamma$ which are uniquely determined by the condition:

$$\xi_{i,j}(\gamma_l) = \begin{cases} \delta tl(z-1)j, & (i = 2), \\
\delta ilzj, & (i \neq 2) \end{cases}$$

for $1 \leq l \leq g$. Then it is shown in [MT, Section 4] that

$$\Psi_{g,k}(\varphi, \xi) := \frac{1}{2\pi \sqrt{-1}} \sum_{i=1}^{g} \oint_{C_i} \varphi \cdot \xi(\gamma_l)dz$$

for $\varphi \in H^0 \left( X_\Gamma, \Omega^k_{X_\Gamma} \right), \xi \in H^1(\Gamma, \mathbb{C}[z]_{2k-2})$ is a non-degenerate pairing on

$$H^0 \left( X_\Gamma, \Omega^k_{X_\Gamma} \right) \times H^1(\Gamma, \mathbb{C}[z]_{2k-2}).$$

Then there exists a basis

$$\{\varphi_{1,k-1}, \varphi_{2,1}, \ldots, \varphi_{2,2k-2}, \varphi_{i,0}, \ldots, \varphi_{i,2k-2} (3 \leq i \leq g)\}$$

which we call the normalized basis of $H^0 \left( X_\Gamma, \Omega^k_{X_\Gamma} \right)$ such that $\Psi_{g,k}(\varphi_{i,j}, \xi_{l,m}) = \delta_{il} \cdot \delta_{jm}$.

**Remark.** Since $-\pi \cdot \Psi_{g,k}$ is the pairing given in [MT, (4.1)],

$$\left\{ -\frac{\varphi_{1,k-1}}{\pi}, \frac{\varphi_{2,1}}{\pi}, \ldots, -\frac{\varphi_{j,2k-2}}{\pi}, \ldots, -\frac{\varphi_{j,2k-2}}{\pi} \ (3 \leq j \leq g) \right\}$$

is the natural basis for $n$-differentials defined in [MT].

In what follows, put

$$\{\varphi_{1,1}, \varphi_{2,1}, \ldots, \varphi_{2,2k-1}, \varphi_{j,0}, \ldots, \varphi_{j,2k-2} (3 \leq j \leq g)\},$$

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and \( \varphi = \varphi_1 \wedge \cdots \wedge \varphi_{(2k-1)(g-1)}. \)

**Theorem 5.1 (McIntyre-Takhtajan [MT, Theorem 2]).** Assume that \( k > 1 \).

1. There exists a holomorphic function \( F_k \) on \( S_g \) which gives an isometry between \( \lambda_k \) on \( M_g(\mathbb{C}) \) with Quillen metric and the holomorphic line bundle on \( M_g(\mathbb{C}) \) determined by the hermitian metric \( \exp \left( \frac{S_L}{12\pi} \right)^{d_k} \). More precisely, there exists a positive real number \( c_{g,k} \) depending only on \( g \) and \( k \) such that

\[
\exp \left( \frac{S_L}{12\pi} \right)^{d_k} = c_{g,k} |F_k|^2 \|\varphi\|_Q^2,
\]

where \( \|\cdot\|_Q \) denotes the Quillen metric.

2. On the whole Schottky space \( S_g \) classifying marked normalized Schottky groups \( (\Gamma; \gamma_1, \ldots, \gamma_g) \), \( F_k \) is given by the absolutely convergent infinite product

\[
(1 - q_{\gamma_1})^2 \cdots (1 - q_{\gamma_1}^k)^2 (1 - q_{\gamma_2}^{k-1}) \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_{\gamma}^{k+m}),
\]

where \( \{\gamma\} \) runs over primitive conjugacy classes in \( \Gamma - \{1\} \).

**Remark.** As is seen in 4.2, \( F_k \) has a universal expression as an element of \( A_{\Delta'} \) which we denote by the same symbol.

**5.2. Explicit Riemann-Roch isomorphism.**

**Theorem 5.2.** The isomorphism

\[
\mathcal{O}_{M_g}(\partial M_g) \sim \lambda_k^{12} \otimes \kappa_g^{(-d_k)}
\]

given by Theorem 4.1 maps \( 1 \) to \( \pm (F_k \varphi)^{12} \cdot (\text{ACS}_g)^{d_k} \).

**Proof.** First, we prove the assertion when \( k = 1 \). By the definition of ACS\(_g\), the image of \( 1 \) is a constant multiple of \( (F_1 \omega)^{12} \cdot \text{ACS}_g \), and by the arithmetic Schottky uniformization theory, the both are primitive. Therefore, they are equal up to a sign. Second, we assume that \( k > 1 \). Since the norm of ACS\(_g\) is a constant multiple of \( \exp \left( -\frac{S_L}{(2\pi)} \right) \), Theorem 5.1 implies that the image of \( 1 \) is primitive and a certain multiple of \( (F_k \varphi)^{12} \cdot (\text{ACS}_g)^{d_k} \). As is shown in [I3, Theorem 6.1], \( \varphi \) gives a trivialization of \( \lambda_k \) over the generalized Tate curve \( C_\Delta \), and hence by the
arithmetic Schottky uniformization theory, \((F_k \varphi)^{12} \cdot (\text{ACS}_g)^{d_k}\) is also primitive. This completes the proof. □

5.3. Constant in the factorization formula.

**Theorem 5.3.** Let \(c_g\) be the constant given in Theorem 2.1. Then

\[
c_g = (2\pi)^{2g} \cdot \exp \left( \frac{(g - 1) \left(24\zeta'_Q(-1) + 1\right)}{6} \right).
\]

**Proof.** Represent \(\Omega_{\mathcal{C}_g/\mathcal{M}_g} \big/ \mathcal{M}_g\) as \(\mathcal{M}_g(\mathcal{D})\) and \(\mathcal{M}_g(\mathcal{D}')\) for the natural \(\mathbb{Z}\)-structure on \(\mathcal{M}_g\) such that the supports of positive divisors \(\mathcal{D}\) and \(\mathcal{D}'\) are disjoint. Then for the sections 1 of \(\mathcal{M}_g(\mathcal{D})\) and of \(\mathcal{M}_g(\mathcal{D}')\), \(\langle 1, 1 \rangle\) gives a (Zariski) local section of the Deligne pairing \(\langle \Omega_{\mathcal{C}_g/\mathcal{M}_g}, \Omega_{\mathcal{C}_g/\mathcal{M}_g} \rangle\) which is not congruent to 0 modulo any prime. Denote by \(S_A[\log \rho]\) the Liouville action defined in [A, Definition 4.1]. Then by the Gauss-Bonnet theorem, \(S_L = 2\pi S_A[\log \rho]\) if \(\rho\) is the hyperbolic metric with constant curvature \(-1\). Therefore, by [A, Corollary 5.2],

\[
\exp \left( \frac{S_L}{2\pi} \right) = \|\langle T_X, T_X \rangle\| \cdot e^{2g-2} = \|\langle 1, 1 \rangle\| \cdot e^{2g-2}.
\]

Let \(\omega_i (1 \leq i \leq g)\) be as in 4.2 which becomes the normalized basis of holomorphic 1-forms on Schottky uniformized Riemann surfaces. Then \(\int_{\omega_i} \omega_j = 2\pi \sqrt{-1} \delta_{ij}\), and hence

\[
\frac{\det \text{Im}(\tau)}{\det' \Delta_h} = \frac{\| (\omega_1 \wedge \cdots \wedge \omega_g) / (2\pi \sqrt{-1})^g \|_H^2}{\det' \Delta_h} = \frac{\| \omega_1 \wedge \cdots \wedge \omega_g \|_Q^2}{(2\pi)^{2g}}.
\]

Therefore, by the arithmetic Riemann-Roch theorem,

\[
\left( \frac{\det \text{Im}(\tau)}{\det' \Delta_h} \right)^6 = \| \lambda_1 \|_Q^{12} \cdot (2\pi)^{-12g}
\]

\[
= \|\langle T_X, T_X \rangle\| \cdot |F_1|^{-12} \cdot e^a(g) \cdot (2\pi)^{-12g}
\]

\[
= \exp \left( \frac{S_L}{2\pi} \right) \cdot |F_1|^{-12} \cdot \exp \left( (1 - g) \left(24\zeta'_Q(-1) + 1\right) \right) \cdot (2\pi)^{-12g}.
\]

This implies

\[
c_g = (2\pi)^{2g} \cdot \exp \left( \frac{(g - 1) \left(24\zeta'_Q(-1) + 1\right)}{6} \right)
\]

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which completes the proof. □

**Theorem 5.4.** The constant $c_{g,k}$ in Theorem 5.1 is determined as

$$c_{g,k} = \exp \left( \frac{(g - 1) \left( 24 \zeta'_Q(-1) + 2d_k - 1 \right)}{6} \right).$$

**Proof.** By the arithmetic Riemann-Roch theorem and Theorem 5.3,

$$\|\varphi\|_Q^{12} = \|\langle T_X, T_X \rangle\|^{d_k} \cdot |F_k|^{-12} \cdot e^{a(g)}$$

$$= \exp \left( \frac{S}{2\pi} \right)^{d_k} \cdot |F_k|^{-12} \cdot \exp \left( (1 - g) \left( 24 \zeta'_Q(-1) + 2d_k - 1 \right) \right)$$

which completes the proof. □

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