Research Article

Optimal Strategies for an Ambiguity-Averse Insurer under a Jump-Diffusion Model and Defaultable Risk

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In this paper, we consider a robust optimal investment-reinsurance problem with a default risk. The ambiguity-averse insurer (AAI) may carry out transactions on a risk-free asset, a stock, and a defaultable corporate bond. The stock’s price is described by a jump-diffusion process, and both the jump intensity and the distribution of jump amplitude are uncertain, i.e., the jump is ambiguous. The AAI’s surplus process is assumed to follow an approximate diffusion process. In particular, the reinsurance premium is calculated according to the generalized mean-variance premium principle, and the reinsurance type has to follow a self-reinsurance function. In performing dynamic programming, both the predefault case and the postdefault case are analyzed, and the optimal strategies and the corresponding value functions are derived under the worst-case scenario. Moreover, we give a detailed proof of the verification theorem and give some special cases and numerical examples to illustrate our theoretical results.

1. Introduction

Investment is the most common way for the insurer to cope with the fierce competition in the insurance market and get higher returns, including risk-free investment (bank), risky investment (stock), and bond investment. The insurer can also transfer their risks by buying reinsurance. Therefore, the optimal investment-reinsurance problem of insurers has received extensive attention in the field of insurance and stochastic control. Browne [1] originally optimized the exponential utility of terminal wealth in order to obtain an optimal investment strategy for the insurer. Since then, a large number of works have been done concerning this topic (see Yang and Zhang [2], Bai and Guo [3], Huang et al. [4], Zhang et al. [5], Zeng et al. [6], Peng et al. [7], etc.).

For the stock’s price process, some scholars have paid attention to the jump risk, such as Yu et al. [8] and Zhang et al. [9]. This is because in the face of serious events (natural disasters and serious large-scale diseases), the stock price may jump to a new level. Therefore, it is not suitable for the stock price to be described by a geometric Brownian motion (GBM) with the constant appreciation rate and volatility. Moreover, the optimal investment-reinsurance problem under the jump-diffusion model has drawn much attention, for example, Li et al. [10], Cao [11], Lianget al. [12], Lin et al. [13], Yang and Zhang [2], and Zhao et al. [14].

Recently, the investors/insurers have an increasing interest in the default risk of corporate bonds with high yield. Default risk (credit risk) refers to the risk that the security issuer will not be able to repay the principal and interest at the maturity of the security, which makes the investor suffer losses. Therefore, it is the purpose of investors to reduce credit risk and obtain higher returns. In fact, several scholars have addressed the portfolio optimization on corporate bonds in the last several decades. Bielecki and Jang [15] studied an optimal allocation problem associated with
deflatable bond, and their goal was to maximize the expected utility of the terminal wealth. Bo et al. [16, 17] considered an investment-consumption problem for an investor who can invest in a deflatable market. For more results about default risk, see Capponi and Figueroa-López [18], Zhu et al. [19], Zhao et al. [20], and Deng et al. [21].

For the insurer, reinsurance is an important method to balance their risk and obtain higher profit via an optimal reinsurance strategy. The reinsurance includes the reinsurance premium and reinsurance type. In most of the above results, the reinsurance premium principle is calculated according to the expected value principle or variance principle, such as Liang and Bayraktar [22] and Sun et al. [23]. In previous conclusions, two types of reinsurance policies are most commonly studied in the literature. One policy is the proportional (quota-share) reinsurance (see Zhou et al. [24] and Shen and Zeng [25]), and the other policy is the excess-of-loss reinsurance (see [10, 14]). Recently, Zhang et al. [26] studied the optimal investment-insurance for insurers with the generalized mean-variance principle. In their article, a generalized mean-variance principle included two special cases: the expected value principle and the variance principle. The reinsurance policy was considered a self-reinsurance function, which included the proportional reinsurance and the excess-of-loss reinsurance.

Apart from the investment-reinsurance, recent advances are on the ambiguity aversion, the uncertainty associated with the model, and the risk aversion. In reality, it is a notorious fact that the return of risky assets is difficult to estimate accurately. Therefore, investors may consider some alternative models that are close to the estimated model to deal with portfolio selection in case of ambiguity. Anderson et al. [27] introduced the concept of ambiguity aversion and formulated a robust control problem for investors. Uppal and Wang [28] extended the results of Anderson et al. [27] under the model uncertainty robustness framework with different levels of ambiguity. For investors, Maenhout [29, 30] derived the closed-form solutions to robust optimal strategies by innovating a “homothetic robustness” framework. A lot of descendent researches of Maenhout [29, 30] concentrated on the influences of ambiguity in the field of finance or insurance, and the representative publications are Zhang and Siu [31], Yi et al. [32], Flor and Larsen [33], Pun and Wong [34], Zeng et al. [35], Zheng et al. [26], Zhang et al. [36], etc.

Moreover, the jump risk, especially that associated with disaster events, is more difficult to estimate accurately. So many scholars have paid attention to the ambiguity of jump risks. For the sake of explanation, the distribution of jump amplitude is assumed to be known and is restricted to be identical under the reference model’s measure \( P \) and the alternative measure \( P^H \), but the jump intensity is uncertain. This topic was studied by some scholars, for example, Branger and Larsen [37], Zeng et al. [6], Sun et al. [23], and Li et al. [10]. But in most instances, the distribution of jump amplitude is unknown. Jin et al. [38] considered the dynamic portfolio choice problem with ambiguous jump risks in a multidimensional jump-diffusion framework. In their results, both the jump amplitude distribution and the jump intensity were assumed to be uncertain.

For these reasons, we choose a jump-diffusion process to describe the price of the stock and consider the robust model to find an optimal strategy in this paper. Moreover, the AAI is allowed to buy reinsurance and allocate his/her wealth among a risk-free asset, a stock, and a deflatable corporate bond. According to Zhang et al. [5], we assume that the reinsurance premium is calculated about the generalized mean-variance principle, which is more general than that reported by Sun et al. [23] and Li et al. [10]. Specifically, in our model, the stock’s price changes dramatically, while the parameters of the underlying jump processes are difficult to estimate accurately. Therefore, we assume that the stock’s jump amplitude distribution and the jump intensity are uncertain, which are different from those reported by Sun et al. [23] and Li et al. [10]. The surplus of an AAI is described by an approximate diffusion process. In light of the principle of dynamic programming, the corresponding Hamilton–Jacobi–Bellman (HJB) equations are deduced for both the postdefault case and the predefault case. Using the variable change and variable separation techniques, we obtain the optimal reinsurance and investment strategies and the corresponding value functions. Our goal is to maximize the expected utility of terminal wealth under the worst-case scenario according to the max-min expected utility. Finally, we exemplify our deductions by some special cases and numerical cases, which verify our theoretical results.

Here, we arrange the remaining part of this paper as follows: Section 2 formulates the robust investment-reinsurance optimization regarding the default risk under the jump-diffusion model. In Section 3, we derive the closed-form expressions for the optimal strategies and the corresponding value functions under the predefault case and postdefault case, respectively. Section 4 provides a proof of the verification theorem. Section 5 provides some special cases. Numerical examples of our results are demonstrated in Section 6. Finally, conclusions are given in Section 7.

2. Model Formulation

In this article, we consider a complete probability space \((\Omega, \mathcal{F}, P)\). Let \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) be the right-continuous, \( P \)-complete filtration generated by two standard Brownian motions \([W_1(t)]\) and \([W_2(t)]\), a Poisson process \([N(t)]\), and two families of random variables \([Y_{i, j} \geq 1}]\) and \([Z_{i, i} \geq 1}]\). We assume that \([W_1(t)], [W_2(t)], [N(t)], [Y_{i, j}], [Z_{i, i}]\) are mutually independent. Let \( \mathcal{G}_t = (\mathcal{F}_t)_{t \geq 0} \) be the enlarged filtration of \( F \) and \( H \), i.e., \( \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^H \), where the filtration \( H = (\mathcal{F}_t^H)_{t \geq 0} \) is generated by a default process \([H(t)]\). We assume that every \( F \)-martingale is also a \( G \)-martingale. The probability measure \( P \) is the real-world probability measure, and \( Q \) is the risk-neutral measure.

2.1. The Financial Market. In this section, we consider a financial market consisting of three types of securities: a risk-free asset, a stock, and a deflatable corporate bond. The
price process of the risk-free asset under the measure P is described by
\[ dR(t) = rR(t)dt, \]
where \( r > 0 \) is the risk-free interest rate. The price process \( \{S(t)\}_{t \geq 0} \) of a stock is described by a jump-diffusion process:
\[ dS(t) = S(t)\left[\mu dt + \sigma dW_1(t) + d \sum_{i=1}^{N_1(t)} Y_i\right], \]
where \( \mu > 0 \) is the expected instantaneous rate of return of the stock; \( \sigma \) is a positive constant; \( \{N_1(t)\}_{t \in [0,T]} \) is a homogeneous Poisson process with intensity \( \lambda_1 \), representing the number of a stock price's jumps during the time interval \([0, t]\); and \( Y_i \) is the \( i \)-th jump amplitude of the stock's price, and \( Y_1, Y_2, \ldots \) are i.i.d. random variables with the common distribution function \( F_1(y) \), the first moment \( E[Y_1] = \mu_Y \), and the second moment \( E[Y_1^2] = \sigma_Y^2 \). We assume that \( P[Y_1 > 1 \text{ for all } i \geq 1] \) to ensure that the stock's price remains positive. Generally, the expected return of the stock is larger than the risk-free interest rate, so we assume that \( \mu + \lambda_Y \gamma_Y > r \).

Next, we consider that \( N(dt, dy) \) is a Poisson random measure on \( \mathbb{R} \times [0, T] \). We assume that \( \{N(t, A)\}_{t \in [0, T]} \) is a \((P, \mathbb{F})\)-martingale.

Next, we consider the price process of the defaultable corporate bond by the intensity-based approach. Let \( \Gamma \) be the time of default and \( \Gamma \) represent the first jump time of a Poisson process with constant intensity \( h^p > 0 \). A default indicator process is defined as \( H(t) = I_{\{t \leq \Gamma\}} \) for each \( t \geq 0 \), and the value of the corporate bond is assumed to be zero after default. Let \( \mathbb{H} \) be the filtration generated by the default process \( H(t) \) and augmented in the usual way. By definition, \( \Gamma \) is naturally an \( \mathbb{H} \)-stopping time and a \( \mathbb{G} \)-stopping time. Furthermore, the martingale default process is thus given by \( M^P(t) = H(t) - \int_0^t (1 - H(u))h^Pdu \), which is a \((P, \mathbb{G})\)-martingale. By Girsanov's theorem in Bielecki and Jang [15], under the chosen risk-neutral measure \( Q \), the arrival intensity of default is given by \( h^Q = h^P / \Delta \). We denote that \( 1/ \Delta \geq 1 \) is the default risk premium. We assume that there exists a defaultable zero coupon bond with a maturity date \( T_1 \), and the insurer can recover a fraction of the market value of the defaultable bond just prior to default. Now, for the positive interest rate \( r \), the price dynamics of the defaultable bond under \( P \) is (see Deng et al. [21])
\[ dp(t, T_1) = p(t, T_1) [r dt + (1 - H(t)) \delta (1 - \Delta) dt - (1 - H(t)) \xi dM^P(t)], \]
where \( \delta = \frac{\partial h^Q}{\partial t} \) represents the risk-neutral credit spread and \( 0 < \xi < 1 \) denotes the loss rate of the bond when a default occurs.

2.2. Dynamics of Surplus Process. The insurer's surplus process \( \{U_0(t)\}_{t \geq 0} \) is described by a jump-diffusion risk model:
\[ dU_0(t) = c dt + \sigma_d dW_2(t) - d \sum_{i=1}^{N_2(t)} Z_i, \]
where \( c \) is the premium rate and \( \sigma_d \geq 0 \) is a constant. \( \sum_{i=1}^{N_2(t)} Z_i \) represents the aggregate claim amount up to time \( t \), where \( \{N_2(t)\} \) is a homogeneous Poisson process with intensity \( \lambda_2 > 0 \), and the individual claim sizes \( Z_1, Z_2, \ldots \), independent of \( \{N_2(t)\} \), are i.i.d. positive random variables. The first moment \( E[Z_1] = \mu_Z \), and the second moment \( E[Z_1^2] = \sigma_Z^2 \). In addition, the insurance premium rate \( c \) under the expected value principle is given by \( c = \lambda_3 \mu_2 (1 + \theta_0) \), where \( \theta_0 > 0 \) is the relative safety loading of the insurer. Suppose the insurer wants to reduce his/her risk by purchasing reinsurance. If there is a claim \( Z_i \) at time \( t \), a proportion \( \mathcal{H}(Z) \) (self-reinsurance function, see Schmidli [39]) is paid by the insurer, and the rest \( Z_i - \mathcal{H}(Z_i) \) is paid by the reinsurer. \( c_\mathcal{H} \) is the premium rate of the reinsurer and is calculated according to the generalized mean-variance principle [5]. So, the premium rate of the insurer is
\[ c - c_\mathcal{H} = \lambda_3 [(\theta_0 - \eta) \mu_2 + (1 + \eta)(E[\mathcal{H}(Z)] - \theta_1 E(Z_i - \mathcal{H}(Z_i)^2)], \]
where \( \eta \geq 0 \) and \( \theta_1 \geq 0 \) is the relative safety loading of the reinsurer.

According to Grandell [40], the surplus process can be approximated by the following diffusion process:
\[ dU(t) = \lambda_3 [(\theta_0 - \eta) \mu_2 + \eta E[\mathcal{H}(Z)] - (1 + \eta) \theta_1 E(Z_i - \mathcal{H}(Z_i)^2)] dt + \sigma_d dW_2(t) \]
\[ + \sqrt{\lambda_3 E[\mathcal{H}(Z)_i^2]} dW_3(t). \]

While the self-reinsurance function \( \mathcal{H}(\cdot) \) may take various forms, Zhang et al. [5] supposed that
\[ \mathcal{H}(Z_i) = a (\eta + 2(1 + \eta) \theta_1 Z_i) \wedge Z_i, \quad t \in [0, T], \]
where \( a \in [0, (1/2)(1 + \eta) \theta_1] \). Next, we will only consider reinsurance strategies given by (8). In this case, since \( \mathcal{H}(Z_i) \) is uniquely characterized by the parameter \( a \), we also rewrite \( \mathcal{H}(Z_i) \) as \( \mathcal{H}(a, Z_i) \) to emphasize the dependence on \( a \) and call \( a \) as the insurer's reinsurance strategy. That is, \( \mathcal{H}(a, Z_i) = a (\eta + 2(1 + \eta) \theta_1 Z_i) \wedge Z_i \). At any time \( t \), with a larger \( a \), the insurer reduces expenses on reinsurance and pays a larger proportion of each claim by himself/herself. Specially, when \( a = 1/2(1 + \eta) \theta_1 \), \( \mathcal{H}(a, Z_i) = Z_i \), that is, the insurer pays all of the claims by himself/herself; when \( a = 0 \), he/she transfers all of the claims to the reinsurer according to Chen et al. [41]. Then, the surplus process of the insurer under the reinsurance \( \mathcal{H} \) at time \( t \) is given by
\[ \mathrm{d}U(t) = \lambda_2 \left[ \left( \theta_0 - \eta \right) \mu_Z + \eta E[H(a, Z_t) - (1 + \eta)\theta_1 I \right] \right] \mathrm{d}t + \sigma_0 \mathrm{d}W_2(t) + \sqrt{\lambda_2 E[H(a, Z_t)]^2} \mathrm{d}W_3(t). \]  

**Remark 1.** If \( \eta = 0 \), then \( H(Z_t) = 2a\theta_1 Z_t \) becomes a proportional reinsurance type (see also Zhou et al. [24] and Zheng et al. [26]); if \( \theta_1 = 0 \), then \( H(Z_t) = (a\eta) \wedge Z_t \) becomes an excess-of-loss reinsurance type, the reinsurance premium under the mean principle (see also Zhao et al. [14] and Li et al. [10]). Therefore, proportional reinsurance and excess-of-loss reinsurance are special cases of (8).

\[ \begin{align*}
\mathrm{d}X(t) &= \frac{X(t) - \pi_1(t) - \pi_2(t)}{R(t)} \mathrm{d}R(t) + \frac{\pi_1(t)}{S(t - t_1)} \mathrm{d}S(t) + \frac{\pi_2(t)}{p(t - T_1)} \mathrm{d}p(t, T_1) + \mathrm{d}U(t) \\
\end{align*} \]

Suppose that the insurer has an exponential utility function defined by

\[ U(x) = \frac{1}{\alpha} e^{-\alpha x}, \quad \alpha > 0. \]  

We denote the set of all admissible strategies by \( \Pi \). Then, we have the following definition for the set of admissible strategies.

**Definition 1.** A trading strategy \( \pi(t) = (\pi_1(t), \pi_2(t), a(t)) \) is said to be admissible if

(i) \( \pi(t) \) is \( G \)-progressively measurable

(ii) \( E^{\theta} \left[ \int_0^T (\pi_1(t) + \pi_2(t) + a^2(t)) \mathrm{d}t \right] < \infty \)

(iii) \( \forall (\pi(t), X^\pi(t)) \), the stochastic differential equation (10) has a pathwise unique solution \( X^\pi(t) \) with \( E^{\theta} [\exp(-\alpha X^\pi(t))] < \infty \), where \( B^\theta \) is the chosen measure to describe the worst case and will be shown later

However, the AAI wants to guard himself/herself against worst-case scenarios. We assume that the knowledge of the AAI about ambiguity is described by probability \( \mathcal{P} \), namely, the reference probability (or model). But, he/she is skeptical about this reference model and hopes to consider alternative models, which are defined as a class of 2.3. The Wealth Process. In this section, we assume that the insurer is allowed to invest all his/her surplus in the financial market defined above. The insurer’s trading strategy is \( \pi(t) = (\pi_1(t), \pi_2(t), a(t)) \), where \( \pi_1(t) \) is the total amount of wealth invested in the risky asset (a stock) at time \( t \), \( \pi_2(t) \) is the amount of wealth invested in the defaultable corporate bond, and \( a \) is the insurer’s reinsurance strategy. The remainder amount is invested in the risk-free asset. We assume that the corporate bond is not traded after default, and the investment horizon is \([0, T]\), where \( T < T_1 \). The reserve process subjected to this choice is denoted by \( X^\pi(T) \). Thus, the wealth process can be presented as follows:

\[ \begin{align*}
\mathrm{d}X^\pi(t) &= \frac{X(t) - \pi_1(t) - \pi_2(t)}{R(t)} \mathrm{d}R(t) + \frac{\pi_1(t)}{S(t - t_1)} \mathrm{d}S(t) + \frac{\pi_2(t)}{p(t - T_1)} \mathrm{d}p(t, T_1) + \mathrm{d}U(t) \\
\end{align*} \]

For an admissible control \( \pi(t) \) and an initial value \((x, h)\), we define the objective function as

\[ \sup_{\pi \in \Pi} E \left[ U(\{X^\pi(t)\}) \mid (X(t), H(t)) = (x, h) \right] = \sup_{\pi \in \Pi} \left[ \frac{1}{\alpha} e^{-\alpha x} \right] \left[ (X(t), H(t)) = (x, h) \right]. \]  

probability measures \( \mathcal{P}_\Lambda := \{ P^\phi \mid P^\phi \sim P \} \) equivalent to \( P \) (Anderson et al. [27] and Zeng et al. [6]). At first, we define a process \( \phi(t, y) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t), \phi(f, y)) \) such that

(1) \( \phi(t, y) \) is \( \mathcal{G}_t \)-measurable, for each \( t \in [0, T] \)

(2) \( \phi_4(t) = \phi_4(t, w) > 0 \) and \( \phi_5(t, y) = \phi_5(t, w) \phi(t, y, w) > 0 \) a.s. \( (t, y, w) \in [0, T] \times (-1, \infty) \times \Omega \)

(3) \( \int_0^T \phi_2(t, y) \mathrm{d}F_1(t, y) < \infty \) and \( \int_0^T \phi_3(t, y) \mathrm{d}F_1(t, y) \mathrm{d}F_2(t, y) < \infty \), P-a.s.

The alternative measures \( \phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t) \), and \( \phi(f, y) \) are positive stochastic processes. We write \( \Sigma \) for the space of all such processes \( \phi \). Note that \( P \) is the probability measure associated with the reference model. For every \( \phi \in \Sigma \), each probability measure \( P^\phi \in \mathcal{P} \) has a Radon–Nikodym derivative:

\[ \frac{dP^\phi}{dP} \mid_{\Sigma} := \Lambda^\phi(T), \]  

with respect to \( P \), where the process \( \Lambda^\phi(T) \) is modelled by the stochastic differential equation (see Jin et al. [38])
functions: \( X \)

\[
\pi_t = \int_0^t \sigma(X_u) dW_u + \int_0^t \eta(X_u) d\xi_u
\]

Note that \( \pi_t \) is not equal under \( P \) and \( P^\theta \). Under the probability measure \( P^\theta \), the jump intensity \( \lambda_1 \) and the density function \( dF_1(y) \) of the stock’s price process are changed into \( \phi_1(t)\lambda_1 \) and \( \phi(t, y)dF_1(y) \) in the alternative model. That is, \( \lambda^\theta_1(t) = \phi_1(t)\lambda_1 \), and \( \tilde{N}^\theta(t, y) := N(t, dy) - \phi_1(t)\phi(t, y)\lambda_1 dt dF_1(y) \) is a compensated Poisson random measure. For the default indicator process \( H(t) := H_{t\leq T} \), \( t \geq 0 \), the intensity \( h^\theta \) of the jumps becomes \( \phi_1(t)h^P \), and the jump size is always equal to 1, so the jump size distribution is identical under \( P \) and \( P^\theta \). For three standard Brownian motions, according to Girsanov’s theorem, \( dW_t^P = dW_t + \phi(t)dt \), \( i = 1, 2, 3 \). Thus, the wealth process under \( P^\theta \) becomes that:

\[
\begin{align*}
\Lambda^\theta(t) &:= \exp \left\{ \int_0^t \phi_1(u)dW_1(u) - \frac{1}{2} \int_0^t \phi_1^2(u)du - \int_0^t \phi_2(u)dW_2(u) - \frac{1}{2} \int_0^t \phi_2^2(u)du \\
&\quad - \int_0^t \phi_3(u)dW_3(u) - \frac{1}{2} \int_0^t \phi_3^2(u)du + \int_0^t \ln \phi_1(u)dH(u) + h^\theta \int_0^t (1 - \phi_4(u)) (1 - H(u)) du \right\} \tag{14}
\end{align*}
\]

with \( \Lambda^\theta(0) = 1 \), \( P \)-a.s. By the Itô differentiation rule, we get

\[
d\Lambda^\theta(t) = \Lambda^\theta(t) \left( - \phi_1(t)dW_1(t) - \phi_2(t)dW_2(t) \\
- \phi_3(t)dW_3(t) - (1 - \phi_4(t))dM^\theta(t) \right) \\
- \int_1^\infty (1 - \phi_5(t)\phi(t, y))N(dt, dy).
\]

Note that \( \phi_5(t) \) and \( \phi(t, y) \) are positive stochastic processes, and \( \phi(t, y) \) satisfies the following relationship:

\[
\int_1^\infty \phi(t, y)dF_1(y) = 1. \tag{16}
\]

The distribution of the stock’s jump amplitude and the jump intensity is ambiguous, so the density function

\[
dX^\theta(t) = \left\{ rX(t) + (\mu - r)\pi_1(t) + \pi_2(t)(1 - H(t))\delta + \lambda_2\left[ (\theta_0 - \eta)\mu_2 + \eta E\mathcal{H}(a, Z_i) - (1 + \eta)\theta_1E(Z_i - \mathcal{H}(a, Z_i)) \right] \\
- \phi_1(t)\pi_1(t)\sigma - \phi_3(t)\sqrt{\lambda_2E(\mathcal{H}(a, Z_i))^2} \right\} dt \\
- \sigma_0\phi_2(t) + \pi_1(t)\sigma dW_1^\theta(t) + \sigma_0dW_2^\theta(t) + \sqrt{\lambda_2E(\mathcal{H}(a, Z_i))^2} dW_3^\theta(t) \\
+ \pi_1(t) \int_1^\infty yN(dt, dy) - \pi_2(t)(1 - H(t-))\delta dH(t). \tag{17}
\]

To simplify further analysis, we define the following functions:

\[
g_0(x) = \frac{\eta x}{1 - 2(1 + \eta)\theta_1 x} \\
g(z) = \eta + 2(1 + \eta)\theta_1 z, \\
h_1(a) := \eta h_1(a) - (1 + \eta)\theta_1 h_2(a), \\
h_2(a) := E[\mathcal{H}(a, Z)] = 2 \int_0^a x \left( \int_0^\infty g_2^2(z) dz \right) dx, \tag{18}
\]

\[
h_3(a) := E[\mathcal{H}(a, Z)] = \int_0^a \left( \int_0^\infty g_2^2(z) dz \right) dx, \\
h_4(a) := E[(Z - \mathcal{H}(a, Z))^2] = 2 \int_a^{1/2(1 + \eta)\theta_1} \left( \int_0^\infty |z - xg(z)|g(z) dz \right) dx.
\]
Then, the dynamics of the wealth process under \( P^\phi \) is
\[
dX^\phi(t) = \left[ rX(t) + (\mu - r)\pi_1(t) + \pi_2(t)(1 - H(t))\delta + \lambda_2 (\phi_0 - \eta)\mu_2 + h_1(a)) - \phi_1(t)\pi_1(t)\sigma - \sigma_0\phi_2(t) - \phi_3(t)\right] dt \\
+ \pi_1(t)\sigma dW^p_1(t) + \sigma_0 dW^p_2(t) + \lambda_2 h_2(a)dP(t) + \pi_1(t) \int_{-1}^\infty yN(dt, dy) - \pi_2(t)(1 - H(t))\zeta dH(t).
\]

Next, we assume that the insurer determines a robust portfolio strategy which is the best choice in some worst-case models as Anderson et al. [42]. The insurer penalizes any deviation from this reference model and the penalty
\[
dX^\phi(t) = \left[ rX(t) + (\mu - r)\pi_1(t) + \pi_2(t)(1 - H(t))\delta + \lambda_2 (\phi_0 - \eta)\mu_2 + h_1(a)) - \phi_1(t)\pi_1(t)\sigma - \sigma_0\phi_2(t) - \phi_3(t)\right] dt \\
+ \pi_1(t)\sigma dW^p_1(t) + \sigma_0 dW^p_2(t) + \lambda_2 h_2(a)dP(t) + \pi_1(t) \int_{-1}^\infty yN(dt, dy) - \pi_2(t)(1 - H(t))\zeta dH(t).
\]

increases with this deviation. Then, we use relative entropy to measure the deviation between the reference measure \( P \) and an alternative measure \( P^\phi \). The increase in relative entropy from \( t \) to \( t + dt \) is shown by
\[
\frac{1}{2}\phi_1^2(t)dt + \frac{1}{2}\phi_2^2(t)dt + \frac{1}{2}\phi_3^2(t)dt + (\phi_4(t)\ln\phi_4(t) - \phi_4(t) + 1)h^p(1 - h)dt \\
+ \int_{-1}^\infty [\phi_5(t)\varphi(t, y)\ln(\phi_5(t)\varphi(t, y) - \phi_5(t)\varphi(t, y) + 1)]\lambda_1 dF_1(y)dt,
\]

with \( h \in [0, 1] \). The increase is caused by three diffusion components and two jump components.

On the basis of Branger and Larsen [37], which allows the insurer’s ambiguity aversion with respect to the diffusion risk and jump risk to differ from each other, we can modify problem (12) and define the value function as
\[
G(t, X^\phi(t, \psi(t)) = \frac{\phi_1^2(t)}{2\Psi_1(t, X^\phi(t), H(t))} + \frac{\phi_2^2(t)}{2\Psi_2(t, X^\phi(t), H(t))} + \frac{\phi_3^2(t)}{2\Psi_3(t, X^\phi(t), H(t))} \\
+ (\phi_4(t)\ln\phi_4(t) - \phi_4(t) + 1)h^p(1 - h) \\
+ \int_{-1}^\infty [\phi_5(t)\varphi(t, y)\ln(\phi_5(t)\varphi(t, y) - \phi_5(t)\varphi(t, y) + 1)]\lambda_1 dF_1(y)dt
\]

(22)

The five terms in (22) are scaled by \( \Psi_1 \geq 0, \Psi_2 \geq 0, \Psi_3 \geq 0, \Psi_4 \geq 0, \) and \( \Psi_5 \geq 0 \), which are state-dependent. We follow Maenhout [29] and set
\[
P_1(t, X, h) = -\frac{\beta_1}{aV(t, x, h)}, \\
P_2(t, X, h) = -\frac{\beta_2}{aV(t, x, h)}, \\
P_3(t, X, h) = -\frac{\beta_3}{aV(t, x, h)}, \\
P_4(t, X, h) = -\frac{\beta_4}{aV(t, x, h)}, \\
P_5(t, X, h) = -\frac{\beta_5}{aV(t, x, h)},
\]

(23)

where \( \beta_i \geq 0, i = 1, 2, 3, 4, 5, \) is the ambiguity aversion coefficient with respect to three diffusion risks and two jump risks. The larger the values of \( \Psi_1, \Psi_2, \Psi_3, \Psi_4, \) and \( \Psi_5 \) are, the less a given deviation from the reference model is penalized, the less the faith of the insurer in the reference model, and the more the worst-case model will deviate from the reference model.

3. The Main Result

In this section, the goal is to find the optimal allocation pair \((\pi_1(t), \pi_2(t), a(t))\) under the worst-case scenario. According to the dynamic programming principle, the HJB equation can be derived as (Anderson et al. [42])
\[ \begin{aligned}
\sup_{\pi \in \Pi} \inf_{\phi \in \Sigma} \left\{ \mathcal{A}^{\pi, \phi} V + G(t, X(t), \phi(t)) \right\} = 0, \quad (24) \end{aligned} \]

with the boundary condition \( V(T, x, h) = -(1/\alpha)e^{-\alpha x} \), where \( \mathcal{A}^{\pi, \phi} \) is the infinitesimal generator of \((21)\) under \( \mathcal{P}^{\phi} \) and is defined by

\[
\mathcal{A}^{\pi, \phi} V(t, x, h) = V_t + V_x \left\{ rx + (\mu - r) \pi_1(t) + (1 - h) \delta \pi_2(t) + \lambda_2 [ (\theta - \eta) \mu_Z + h_1(a) ] - \sigma \phi_1(t) \pi_1(t) - \sigma_0 \phi_2(t) - \phi_3(t) \sqrt{\lambda_2 h_2(a)} \right\} + \frac{1}{2} V_{xx} \left( \sigma^2 \pi_1^2(t) + \sigma^2_0 + \lambda_2 h_2(a) \right) + \lambda_1 E_{\mathcal{P}^\phi} [ V(t, x + \pi_1(t) Y, h) - V(t, x, h) ] + h^\phi (1 - h) V(t, x - \pi_2(t) \zeta, 1) - V(t, x, 0) \phi_4(t),
\]

where \( V_t, V_x, \) and \( V_{xx} \) represent the value function’s partial derivatives with respect to the corresponding variables. We split equation \((24)\) into two cases: the postdefault case \((h = 1)\) and the predefault case \((h = 0)\), and denote \( V \) as follows:

\[ \begin{aligned}
0 &= \sup_{\pi \in \Pi} \inf_{\phi \in \Sigma} \left\{ V_1(t, x, 1) + V_x(t, x, 1) \left\{ rx + (\mu - r) \pi_1(t) + \lambda_2 [ (\theta - \eta) \mu_Z + h_1(a) ] - \sigma \phi_1(t) \pi_1(t) - \sigma_0 \phi_2(t) - \phi_3(t) \sqrt{\lambda_2 h_2(a)} \right\} + \frac{1}{2} V_{xx}(t, x, 1) \left[ \sigma^2 \pi_1^2(t) + \sigma^2_0 + \lambda_2 h_2(a) \right] + \lambda_1 E_{\mathcal{P}^\phi} [ V(t, x + \pi_1(t) Y, 1) - V(t, x, 1) ] - \phi(t) a(t) V(t, x, 1) - \phi(0) a(t) V(t, x, 1) \\
&\quad - \phi_3(t) a(t) V(t, x, 1) - \phi_3(t) a(t) V(t, x, 1) \\
&\quad - \int_{-1}^{\infty} \left( \phi(t) a(t) V(t, x, 1) - \phi(t) a(t) V(t, x, 1) \right) \frac{dF(t)}{\beta} \right\},
\end{aligned} \]

\[ \begin{aligned}
0 &= \sup_{\pi \in \Pi} \inf_{\phi \in \Sigma} \left\{ V_1(t, x, 0) + V_x(t, x, 0) \left\{ rx + (\mu - r) \pi_1(t) + \delta \pi_2(t) + \lambda_2 [ (\theta - \eta) \mu_Z + h_1(a) ] - \sigma \phi_1(t) \pi_1(t) - \sigma_0 \phi_2(t) - \phi_3(t) \right\} \right. \\
&\quad \left. + \frac{1}{2} V_{xx}(t, x, 0) \left[ \sigma^2 \pi_1^2(t) + \sigma^2_0 + \lambda_2 h_2(a) \right] + \lambda_1 E_{\mathcal{P}^\phi} [ V(t, x + \pi_1(t) Y, 0) - V(t, x, 0) ] + h^\phi \phi_3(t) (V(t, x - \pi_2(t) \zeta, 1) \\
&\quad - V(t, x, 0)) - \phi(t) a(t) V(t, x, 0) - \phi(0) a(t) V(t, x, 0) \\
&\quad - \phi_3(t) a(t) V(t, x, 0) \\
&\quad - \int_{-1}^{\infty} \left( \phi(t) a(t) V(t, x, 0) - \phi(t) a(t) V(t, x, 0) \right) \frac{dF(t)}{\beta} \right\},
\end{aligned} \]

In the following two sections, we derive the optimal reinsurance and investment strategies and corresponding value functions in the postdefault case and predefault case, respectively.

3.1. The Postdefault Case. In this section, we will concentrate on the postdefault case, that is, HJB equation \((27)\) for \( V(t, x, 1) \), and we conjecture that the value function has the following form:
\[ V(t, x, 1) = \frac{1}{\alpha} \exp \left\{ -ax e^{(T-t)} \right\} g_1(t), \] (29)

where \( g_1(t) \) is a deterministic function, with \( g_1(T) = 1 \). We get

\[
\begin{align*}
V_t &= \left[ \frac{g_1(t)}{g_1(t)} + raxe^{(T-t)} \right] V(t, x, 1), \\
V_x &= -ae^{(T-t)} V(t, x, 1), \\
V_{xx} &= \alpha^2 e^{2r(T-t)} V(t, x, 1), \\
E[V(t, x + \pi_1(t)Y, 1) - V(t, x, 1)] &= \lambda_1 \\
\int_{-\infty}^{\infty} \left( e^{-\alpha \pi_1(t)e^{(T-t)}} - 1 \right) \phi_3(t) \varphi(t, y) dF_1(y) V(t, x, 1).
\end{align*}
\] (30)

Substituting these partial derivatives and \( E[\cdot] \) into equation (27), we get

\[
0 = \inf_{\phi \in \Phi} \sup_{t \in [0, T]} \left\{ \frac{g_1(t)}{g_1(t)} - ae^{(T-t)} \left[ (\mu - r) \pi_1(t) - \sigma_1 \pi_1(t) - \sigma_1 \phi_2(t) + \lambda_2 \left[ (\theta_0 - \eta) \mu_Z + h_1(a) \right] - \phi_3(t) \sqrt{\lambda_2 h_2(a)} \right] \\
+ \frac{1}{2} \alpha^2 e^{2r(t-t)} (\sigma^2_1 \pi_1(t) + \sigma^2_1) + \frac{1}{2} \alpha^2 e^{2r(T-t)} \lambda_2 h_2(a) \\
+ \lambda_1 \int_{-\infty}^{\infty} \left( e^{-\alpha \pi_1(t)e^{(T-t)}} - 1 \right) \phi_4(t) \varphi(t, y) dF_1(y) - \frac{\phi_1(t)}{2\beta_1} \alpha - \frac{\phi_2(t)}{2\beta_2} \alpha - \frac{\phi_3(t)}{2\beta_3} \right\}.
\] (31)

Fixing \( \pi \) and \( a \) and maximizing over \( \phi \) yield the following first-order condition for the minimum point \( \phi^* = (\phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*) \) (there is no ambiguity about the default jump risk after default):

\[
\begin{align*}
\phi_1^*(t) &= \beta_1 \sigma \pi_1(t) e^{(T-t)}, \\
\phi_2^*(t) &= \beta_2 \sigma \phi_1 e^{(T-t)}, \\
\phi_3^*(t) &= \beta_3 \sqrt{\lambda_2 h_2(a)} e^{(T-t)}, \\
\phi_4^*(t) &= \frac{\beta_3}{\alpha} \left( e^{-\alpha \pi_1(t)e^{(T-t)}} - 1 \right).
\end{align*}
\] (32)

Noting that \( E[\varphi(t, Y)] = 1 \) (formula (16)), we have

\[
\begin{align*}
\varphi^*(t, Y) &= \frac{1}{\phi_4^*(t)} \exp \left\{ \frac{\beta_3}{\alpha} \left( e^{-\alpha \pi_1(t)e^{(T-t)}} - 1 \right) \right\}, \\
\phi_4^*(t) &= E \left\{ \exp \left\{ \frac{\beta_3}{\alpha} \left( e^{-\alpha \pi_1(t)e^{(T-t)}} - 1 \right) \right\} \right\}.
\end{align*}
\] (33)

**Lemma 1.** For any \( t \in [0, T] \), the equation

\[
\pi_1(t) = \frac{e^{-r(T-t)}}{\sigma^2 (a + \beta_1)} \left[ \mu - r + \lambda_1 \int_{-\infty}^{\infty} ye^{-\alpha \pi_1(t)e^{(T-t)}} \exp \left\{ \frac{\beta_3}{\alpha} \left( e^{-\alpha \pi_1(t)e^{(T-t)}} - 1 \right) \right\} dF_1(y) \right],
\] (34)

has a unique positive solution \( \pi_1^*(t) \).
Proof. Suppose \( W (t, \pi_1) = \mu - r + \lambda_1 \int_{-1}^{0} y e^{-\alpha \pi_1 (t)} e^{(T-t)y} \exp \left\{ \frac{\beta}{\alpha} - \alpha \pi_1 (t) e^{(T-t)y} - 1 \right\} dF_1 (y) - \sigma^2 (\alpha + \beta_1) \pi_1 (t) \), \( e^{(T-t)} \), then we get

\[
\frac{\partial W (t, \pi_1)}{\partial \pi_1} = -\lambda_1 \alpha e^{(T-t)} \int_{-1}^{0} y^2 e^{-\alpha \pi_1 (t)} e^{(T-t)y} \exp \left\{ \frac{\beta}{\alpha} - \alpha \pi_1 (t) e^{(T-t)y} - 1 \right\} dF_1 (y) - \sigma^2 (\alpha + \beta_1) e^{(T-t)} - \lambda_1 \beta_2 e^{(T-t)} \int_{-1}^{0} y^2 e^{-2\alpha \pi_1 (t)} e^{(T-t)y} \exp \left\{ \frac{\beta}{\alpha} - \alpha \pi_1 (t) e^{(T-t)y} - 1 \right\} dF_1 (y) < 0,
\]

which implies that \( W (t, \pi_1) \) is a decreasing function w.r.t. \( \pi_1 \). Furthermore, we have \( W (t, 0) = \mu - r + \lambda_1 \int_{-1}^{0} y dF_1 (y) = \mu - r + \lambda_1 \mu_2 > 0 \). Also, we can find that if \( \pi_1 (t) > \mu - r + \lambda_1 \mu_2 / \sigma^2 (\alpha + \beta_1) e^{(T-t)} > 0 \), we have \( W (t, \pi_1) < 0 \).

Therefore, equation (34) has a unique positive root \( \pi^*_1 (t) \).

\[ \square \]

**Lemma 2.** For the functions \( h_i (a) \), \( i = 1, 2, 3, 4 \), defined above, we have the following relationship:

\[
h_1 (a) = -\frac{h'_1 (a)}{2} \left[ 2 (1 + \eta) \theta_1 - 1 / a \right],
\]

\[
h_1 (a) = -\frac{h'_1 (a)}{2} \left[ 2 (1 + \eta) \theta_1 - 1 / a \right] + \int_{0}^{a} h_1 (x) \frac{1}{2x^2} dx,
\]

\[
\left( 1 + \eta \right) \theta_1 h_4 (0) - \left( 1 + \eta \right) \theta_1 h_2 (a) + \int_{0}^{a} \left( \int_{\theta_1 (x)}^{\infty} g (z) dF_2 (z) \right) dx.
\]

Substituting \( \phi^* \) into (31), according to the first-order condition and Lemmas 1 and 2, we can obtain the maximum point \( \pi^*: (\pi^*_1, \pi^*_2, a^* ) \) given by

\[
\pi^*_1 (t) = \frac{e^{r (T-t)}}{a^2 (\alpha + \beta_1)} \left[ \mu - r + \lambda_1 \int_{-1}^{0} y e^{-\alpha \pi_1 (t)} e^{(T-t)y} \exp \left\{ \frac{\beta}{\alpha} - \alpha \pi_1 (t) e^{(T-t)y} - 1 \right\} dF_1 (y) \right],
\]

\[
\pi^*_2 (t) = 0,
\]

\[
a^* (t) = \frac{1}{2 (1 + \eta) \theta_1 + (\alpha + \beta_1) e^{(T-t)}}.
\]

Substituting \( \pi^* \) and \( \phi^* \) into (31), we get the equation

\[
0 = \frac{g'_1 (t)}{g_1 (t)} + \frac{\alpha + \beta_1}{2} a^2 e^{2r (T-t)} - \alpha \lambda_2 e^{r (T-t)} \left( \theta_0 - \eta \right) \mu_2 - \frac{\lambda_1 \alpha}{\beta_3} - \alpha e^{r (T-t)} (\mu - r) \pi^*_1 (t)
\]

\[
+ \frac{\alpha + \beta_1}{2} a^2 e^{2r (T-t)} \left( \pi^*_1 (t) \right)^2 + \frac{\lambda_1 \alpha}{\beta_3} \int_{-1}^{0} \exp \left\{ \frac{\beta}{\alpha} - \alpha \pi_1 (t) e^{(T-t)y} - 1 \right\} dF_1 (y)
\]

\[
- \alpha \lambda_2 e^{r (T-t)} \left[ h_1 (a^* (t)) - \frac{\alpha + \beta_3}{2} h_2 (a^* (t)) e^{r (T-t)} \right].
\]
Therefore, we can derive the following theorem.

**Theorem 1** (postdefault strategy). The robust optimal reinsurance and investment strategies for the period after default are given as

\[
\begin{align*}
\phi^*_1 (t) &= \beta_1 \sigma \pi_1 (t) e^{(T-t)}, \\
\phi^*_2 (t) &= \beta_2 \sigma e^{(T-t)}, \\
\phi^*_3 (t) &= \beta_3 \sqrt{\lambda_2} h_2 (a) e^{(T-t)}, \\
\varphi^* (t, y) &= \frac{1}{\phi_5^*} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-\alpha \pi_1 (t) e^{(T-t)} y} - 1 \right) \right\}, \\
\phi^*_4 (t) &= \mathbb{E} \left[ \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-\alpha \pi_1 (t) e^{(T-t)} y} - 1 \right) \right\} \right].
\end{align*}
\]

Furthermore, the postdefault value function is given by

\[
V (t, x, 1) = \frac{1}{\alpha} \exp \left\{ -\alpha e^{(T-t)} \right\} g_1 (t),
\]

where

\[
g_1 (t) = \exp \left\{ \int_t^T \left[ \frac{\alpha + \beta_2}{2} a \sigma^2 e^{2r(T-u)} - \alpha a_2 e^{(T-u)} (\theta_0 - \eta) \right] \frac{e^{r(T-u)}}{\alpha} \right\}
\]

\[
+ \frac{\lambda_3 \alpha}{\beta_5} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-\alpha \pi_1 (u) e^{(T-u)} y} - 1 \right) \right\} dF_1 (y) - \alpha a_2 e^{(T-u)} \left[ h_1 (a^* (u)) - \frac{\alpha + \beta_1}{2} a \sigma^2 e^{2r(T-u)} (\pi_1^* (u))^2 \right].
\]

3.2. The Predefault Case. In this section, we will concentrate on the predefault case, that is, HJB equation (28) for \( V (t, x, 0) \), and we conjecture that the value function has the following form:

\[
\begin{align*}
V_t (t, x, 0) &= \frac{g_1^* (t)}{g_2^* (t)} + r a \alpha e^{(T-t)} V (t, x, 0), \\
V_x (t, x, 0) &= -\alpha e^{(T-t)} V (t, x, 0), \\
V_{xx} (t, x, 0) &= a^2 e^{2r(T-t)} V (t, x, 0), \\
E [V (t, x + \pi_1 (t) Y, 0) - V (t, x, 0)] &= \lambda_1 \int_{-1}^{\infty} \left( e^{-\alpha \pi_1 (t) e^{(T-t)} y} - 1 \right) \phi_3 (t) \phi (t, y) dF_1 (y) V (t, x, 0), \\
V (t, x - \pi_2 (t) \zeta, 1) - V (t, x, 0) &= \left( \frac{g_1 (t)}{g_2 (t)} \exp \left\{ \alpha \zeta \pi_3 (t) e^{(T-t)} \right\} - 1 \right) V (t, x, 0).
\end{align*}
\]
Substituting these partial derivatives, $E[\cdot]$, and $V(1) - V(0)$ into equation (28), we get

$$0 = \inf_{\phi \in \mathbb{R}} \sup_{\pi \in \pi(t)} \left\{ \frac{g'_2(t)}{g_2(t)} - \alpha e^{r(T-t)} \left[ (\mu - r)\pi_1(t) + \pi_2(t)\delta - \sigma \phi_1(t)\pi_1(t) - \sigma \phi_2(t) + \lambda_2 (\theta_1 - \eta) \mu e + h_1(a) \right] \right. $$

$$+ \frac{1}{2} \alpha^2 e^{2r(T-t)} \left( \sigma^2 \pi_1^2(t) + \sigma^2 + \lambda_2 h_2(a) \right) - \frac{\phi_1^2(t)}{2\beta_1} \alpha $$

$$- \frac{\phi_2^2(t)}{2\beta_2} \alpha - \frac{\phi_1^2(t)}{2\beta_3} \alpha + \lambda_2 \int_{-1}^{\infty} \left( e^{-a\pi_1(t)e^{r(T-t)}} - 1 \right) \phi_3(t)\psi(t,y) dF_1(y) $$

$$+ \phi_3(t)\kappa \left( \frac{g_1(t)}{g_2(t)} \exp\left\{ a\pi_2(t)e^{r(T-t)} \right\} - 1 \right) - \frac{\phi_4(t)\ln \phi_4(t) - \phi_4(t) + 1}{\beta_4} \kappa $$

$$- \frac{\lambda_1}{\beta_5} \int_{-1}^{\infty} \left( \phi_3(t)\psi(t,y) \ln (\phi_3(t)\psi(t,y)) - \phi_5(t)\psi(t,y) + 1 \right) dF_1(y) \right\}.$$  

(44)

Fixing $\pi$ and maximizing over $\phi$ yield the following first-order condition for the minimum point $\phi^* = (\phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*, \phi_5^*)$:

$$\phi_1^* = \beta_1 \sigma \pi_1(t) e^{r(T-t)},$$

$$\phi_2^* = \beta_2 \sigma e^{r(T-t)},$$

$$\phi_3^* = \beta_3 \lambda_2 h_2(a) e^{r(T-t)},$$

$$\ln \phi_4^* = \frac{\beta_4}{\alpha} \left( \frac{g_1(t)}{g_2(t)} \exp\left\{ a\pi_2(t)e^{r(T-t)} \right\} - 1 \right),$$

$$\phi_5^* = \frac{\beta_5}{\alpha} \left( e^{-a\pi_1(t)e^{r(T-t)}} - 1 \right).$$  

(45)

Noting that $E[\psi(t,y)] = 1$, we have

$$\phi_1^* = \frac{1}{\phi_2^*} \exp\left( \frac{\beta_2}{\alpha} \left( e^{-a\pi_1(t)e^{r(T-t)}} - 1 \right) \right),$$

$$\phi_2^* = E \left[ \exp\left( \frac{\beta_2}{\alpha} \left( e^{-a\pi_1(t)e^{r(T-t)}} - 1 \right) \right) \right].$$

Substituting $\phi^*$ into (44), according to the first-order condition, we can obtain the maximum point $\pi^* := (\pi_1^*, \pi_2^*, \pi_3^*)$ given by

$$\pi_1^* = \frac{e^{-r(T-t)}}{\sigma^2 (\alpha + \beta_1)} \left( \mu - r + \lambda_1 \right) \int_{-1}^{\infty} ye^{-a\pi_1(t)e^{r(T-t)}} \exp\left( \frac{\beta_2}{\alpha} \left( e^{-a\pi_1(t)e^{r(T-t)}} - 1 \right) \right) dF_1(y),$$

$$\pi_2^* = \frac{e^{-r(T-t)}}{\alpha \pi_2^*} \left[ \ln \frac{g_2(t)}{\Delta \phi_3^*(t)g_1(t)} \right],$$

$$\alpha^* = \frac{1}{2(1 + \eta)\beta_1 + (\alpha + \beta_2)e^{r(T-t)}}.$$

(46)

(47)

Substituting $\pi^*$ into equation (45), we have

$$(ah^p/\beta_2)\phi_1^*(t) \ln \phi_1^*(t) + h^p \phi_1^*(t) - (\delta/\zeta) = 0,$$ and this equation has a unique positive root by the following lemma.

**Lemma 3.** Let $\bar{W}(t, \phi_4^*) = (ah^p/\beta_2)\phi_4^*(t) \ln \phi_4^*(t) + h^p \phi_4^*(t) - (\delta/\zeta)$, then $\bar{W}(t, \phi_4^*)$ has a unique positive root $\phi_4^*(t)$. 
Proof. It is similar to the proof of Proposition 4.2 in the study of Sun et al. [23]. So we omit it. According to (47) and (49), we obtain \( \alpha' = \alpha' \) and \( \pi_1' = \pi_1' \). Substituting \( \pi' \) and \( \phi' \) into (44), we get the equation

\[
0 = \frac{g'_1(t)}{g_2(t)} - \frac{\delta}{\pi} \ln g_2(t) + \frac{\alpha + \beta_2}{2} \lambda e^{2g(T-t)} - \alpha \lambda e^{e^{g(T-t)}} \cdot (\theta_0 - \eta) \mu Z - \alpha \pi' (t) \\
+ \frac{\alpha + \beta_1}{2} \lambda e^{2g(T-t)} (\pi_1' (t))^2 + \frac{\lambda_1 \alpha}{\beta_5} \int_{-1}^{\infty} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-2\pi'_1(t)e^{g(T-t)}} y - 1 \right) \right\} dF_1(y) \\
- \alpha \lambda e^{e^{g(T-t)}} \left[ h_1 (a' (t)) - \frac{\alpha + \beta_2}{2} h_2 (a' (t)) e^{2g(T-t)} \right] \\
+ \frac{\delta}{\pi} \left( \ln \Delta \phi_1' (t) + \ln g_1 (t) + \frac{\phi_1' (t) - 1}{\beta_4} a \Delta \right).
\]

(50)

Note that \( g_2 (t) > 0 \); in order to get the expression for \( g_2 (t) \) with the boundary condition \( g_2 (T) = 1 \), we try the following form of \( g_2 (t) = e^{\beta_1 t} \):

\[
0 = \frac{g'_2(t)}{g_2(t)} - \frac{\delta}{\pi} g_2(t) + \frac{\alpha + \beta_2}{2} \lambda e^{2g(T-t)} - \alpha \lambda e^{e^{g(T-t)}} (\theta_0 - \eta) \mu Z - \alpha \pi' (t) \\
+ \frac{\alpha + \beta_1}{2} \lambda e^{2g(T-t)} (\pi_1' (t))^2 + \frac{\lambda_1 \alpha}{\beta_5} \int_{-1}^{\infty} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-2\pi'_1(t)e^{g(T-t)}} y - 1 \right) \right\} dF_1(y) \\
- \alpha \lambda e^{e^{g(T-t)}} \left[ h_1 (a' (t)) - \frac{\alpha + \beta_2}{2} h_2 (a' (t)) e^{2g(T-t)} \right] \\
+ \frac{\delta}{\pi} \left( \ln \Delta \phi_1' (t) + \ln g_1 (t) + \frac{\phi_1' (t) - 1}{\beta_4} a \Delta \right).
\]

(51)

Therefore, we can derive the following theorem. \( \square \)

**Theorem 2** (predefault strategy). The robust optimal reinsurance and investment strategies for the period before default are given as

\[
\phi_1' (t) = \beta_1 \sigma \pi_1 (t) e^{2g(T-t)} = \phi_1' (t),
\]

\[
\phi_2' (t) = \beta_2 \sigma \pi_1 (t) e^{2g(T-t)} = \phi_2' (t),
\]

\[
\phi_3' (t) = \beta_3 \sqrt{\lambda Z_2} (a) e^{2g(T-t)} = \phi_3' (t),
\]

\[
\frac{a h^p_1}{\beta_4} \phi_1' (t) \ln \phi_1' (t) + \frac{h^p_1}{\beta_4} \phi_1' (t) - \frac{\delta}{\pi} = 0,
\]

\[
\phi' (t, y) = \frac{1}{\phi_1' (t)} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-2\pi'_1(t)e^{g(T-t)}} y - 1 \right) \right\} = \phi' (t, y),
\]

\[
\phi_5' (t) = E \left[ \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-2\pi'_1(t)e^{g(T-t)}} y - 1 \right) \right\} \right] = \phi_5' (t),
\]

\[
\pi_1' (t) = e^{\pi(T-t)} \ln \frac{g_2 (t)}{\alpha \pi_1 (t) g_1 (t)} \left[ \ln g_2 (t) + \frac{\phi_1' (t) - 1}{\beta_4} a \Delta \right].
\]

(52)

Furthermore, the predefault value function is given by

\[
V (t, x, 0) = -\frac{1}{\alpha} \exp \left\{ -\alpha x e^{g(T-t)} \right\} g_2 (t),
\]

where \( g_2 (t) = e^{\beta_1 t} \), in which

\[
\frac{g'_2(t)}{g_2(t)} = e^{\beta_1 t} \left[ \frac{\alpha + \beta_2}{2} \lambda e^{2g(T-t)} - \alpha \lambda e^{e^{g(T-t)}} \left( \theta_0 - \eta \right) \mu Z + \left( \mu - r \right) \pi_1 (u) \right] + \frac{\alpha + \beta_1}{2} \lambda e^{2g(T-t)} (\pi_1 (u))^2 \\
- \frac{\lambda_1 \alpha}{\beta_5} \int_{-1}^{\infty} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{-2\pi'_1(u)e^{g(T-t)}} y - 1 \right) \right\} dF_1 (y) - \frac{\alpha + \beta_2}{2} h_2 (a' (u)) e^{2g(T-t)} h_1 (a' (u)) - \frac{\alpha + \beta_2}{2} h_2 (a' (u)) e^{2g(T-t)} h_1 (a' (u)) \\
+ \frac{\delta}{\pi} \ln \Delta \phi_1' (u) + \ln g_1 (u) + \frac{\phi_1' (u) - 1}{\beta_4} a \Delta \right\} du.
\]

(54)
Next, putting the predefault and postdefault cases together, we have the following solution to the HJB equation (24) associated with the value function \( V(t,x,h) \): 
\[
V(t,x,h) = (1-h)V(t,x,0) + hV(t,x,1), \quad \text{where } h = 0 \text{ or } 1.
\]

(55)

Let us define the following processes which are candidate optimal strategies:
\[
\begin{align*}
\phi_1^*(t) &= \beta_1 \sigma_1(t)e^{\tau(T-t)}, \quad t \in [0,T], \\
\phi_2^*(t) &= \beta_2 \sigma_2(t)e^{\tau(T-t)}, \quad t \in [0,T], \\
\phi_3^*(t) &= \beta_3 \sqrt{\lambda_2 h_2(a)e^{\tau(T-t)}}, \quad t \in [0,T], \\
0 &= \frac{\alpha h_p^*}{\beta_4^*} \phi_4^*(t) \ln \phi_4^*(t) + h_p^* \phi_4^*(t) - \frac{\delta}{\zeta}, \quad t \in [0,T], \\
\psi^*(t,y) &= \frac{1}{\phi_5^*(t)} \exp \left[ \frac{\beta_5}{\alpha} \left( e^{\tau_{11}(t)e^{\tau_{11}(y)}-1} \right) \right], \quad t \in [0,T], \\
\psi_1^*(t) &= E \left[ \exp \left( \frac{\beta_1}{\alpha} \left( e^{\tau_{12}(t)e^{\tau_{12}(y)}-1} \right) \right) \right], \quad t \in [0,T], \\
\pi_1^*(t) &= \frac{e^{-\tau(T-t)}}{\alpha^2 (\alpha + \beta_1)} \left[ \mu - r + \lambda_1 \int_{t-\tau}^{\infty} ye^{\tau_{11}(t)e^{\tau_{11}(y)}} \right. \\
& \quad \left. \cdot \exp \left[ \frac{\beta_5}{\alpha} \left( e^{\tau_{11}(t)e^{\tau_{11}(y)}-1} \right) \right] dF_1(y), \right. \\
\pi_2^*(t) &= \left[ \frac{1}{\alpha^2} e^{-\tau(T-t)} \ln \frac{g_2(t)}{\Delta \phi_4^*(t) g_1(t)} \right], \quad t \in [0,T], \\
a^*(t) &= \frac{1}{2(1+\eta)} \theta_1 + (\alpha + \beta_3)e^{\tau(T-t)}, \quad t \in [0,T].
\end{align*}
\]

(56)

From the above expressions for \( \phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*, \phi_5^*, \) and \( \psi^* \), it is easy to verify that the expression \( \Delta \phi^*(t) \) with \( \phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*, \phi_5^*, \) and \( \psi^* \) instead of \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \) and \( \psi \) is indeed a \( \mathbb{P} \)-martingale, which ensures a well-defined \( \mathbb{P}^* \).

In the next section, we shall show that the above stochastic control policies are indeed the optimal strategies and that the value function \( \bar{V}(t,x,h) \) is unique.

### 4. Verification Theorem

In order to verify the candidate optimal strategies \( \pi^* \) and \( \phi^* \) are indeed optimal, and the candidate value function is (55), we give the verification theorem as follows.

**Proposition 1.** Let \( \Theta := (0,T) \times R \times [0,1] \) and \( \bar{\Theta} \) denote the closure of \( \Theta \). Suppose that there exists a function \( \bar{V} \in \mathbb{C}^{1,2,1}(\Theta) \cap \mathbb{C}(\bar{\Theta}) \) and a control \( (\phi^*, \pi^*) \in \Sigma \times \Pi \) such that

(1) \( \mathcal{A}^{\rho,\pi^*} \bar{V}(t,X^*(t),H(t)) + G(t,X^*(t),\phi(t)) \geq 0 \), for all \( \phi \in \Sigma \)

(2) \( \mathcal{A}^{\rho,\pi^*} \bar{V}(t,X^*(t),H(t)) + G(t,X^*(t),\phi^*(t)) \leq 0 \), for all \( \pi \in \Pi \)

(3) \( \mathcal{A}^{\rho,\pi^*} \bar{V}(t,X^*(t),H(t)) + G(t,X^*(t),\phi^*(t)) = 0 \)

(4) For all \( (\phi, \pi) \in \Sigma \times \Pi \), \( \lim_{\tau \to T} \bar{V}(t,X^*(t),H(t)) = U(X^*(T)) \)

(5) \( \{ \bar{V}(t,X^*(t),H(t)) \}_{t \in \mathbb{R}} \) and \( \{ G(t,X^*(t),\phi(t)) \}_{t \in \mathbb{R}} \) are uniformly integrable, where \( \mathbb{F} \) denotes the set of stopping times \( \tau \leq T \).

Then, \( \bar{V}(t,x,h) = V(t,x,h) \) and \( (\phi^*, \pi^*) \) are an optimal control.

**Proof.** According to \( \bar{V} \in \mathbb{C}^{1,2,1}(\Theta) \cap \mathbb{C}(\bar{\Theta}) \), choose \( (\phi, \pi) \in \Sigma \times \Pi \), by the definition of \( H(t) \),

\[
\bar{V}(t, X^*(t), H(t)) = \bar{V}(t, X^*(t), H(t)) - \int_t^T \frac{\partial}{\partial t} \bar{V}(t, X^*(t), H(t)) dt + \int_t^T \bar{V} \pi(t) dt + \int_t^T \bar{V} \phi(t) dt,
\]

and \( \bar{V}(t, X^*(t), H(t)) = \bar{V}(t, X^*(t), H(t)) + \int_t^T \bar{V} \pi(t) dt + \int_t^T \bar{V} \phi(t) dt \); therefore, the Itô formula can be applied to \( \bar{V}(t, X^*(t), H(t)) \),

\[
\begin{align*}
\int_t^T \bar{V} \pi_1(t) (u) \sigma dW_1^u(t) + \int_t^T \bar{V} \pi_2(t) dW_2^u(t) + \int_t^T \bar{V} \pi_3(t) \sqrt{\lambda_2 h_2(a)} dW_3^u(t) \\
+ \int_t^T \bar{V} \pi_4(t) \Sigma dW_4^u(t) + \int_t^T \bar{V} \pi_5(t) \chi dN(du, dy),
\end{align*}
\]

where \( \tau_n := T \wedge n \wedge \inf \{ u > t : |X^*(u)| \geq n \} \), and \( \bar{V}_x = \bar{V}_x(t,x,0) \) and \( a = a(t) \) for short. Because the continuous function \( \bar{V}_x \) is bounded on the set \( \{ t, \tau_n \} \times R \), we obtain the following estimate with an appropriate constant \( M > 0 \):

\[
\int_t^T \frac{1}{2} \bar{V}_x^2(t) \sigma^2 u^2 du \leq M \int_t^T \pi_2(t) u^2 du.
\]

(58)

Due to (ii) of Definition 1, it follows that

\[
E_{t,x,0}^{\rho^*} \left[ \int_t^T \frac{1}{2} \bar{V}_x^2(t) \sigma^2 u^2 du \right] < \infty.
\]

(59)

By the similar way as that in (59), we have

\[
E_{t,x,0}^{\rho^*} \left[ \int_t^T \frac{1}{2} \bar{V}_x^2(t) \sigma^2 du \right] < \infty,
\]

(60)

Therefore, taking expectations in (57) leads to
\( \tilde{V}(t, x, 0) = E_{t,x,0}^{p_\alpha} \left[ \tilde{V}(\tau_n, X^\alpha(t_n), 0) - \int_{t_n}^{T} e^{\phi u} \tilde{V}(u, X^\alpha(u), 0) \, du \right] . \) \hfill (61)

If we apply (61) to \((\phi, \pi^\alpha)\) with \(\phi \in \Sigma\) and use property 1, we get

\[ \tilde{V}(t, x, 0) \leq E_{t,x,0}^{p_\alpha} \left[ \tilde{V}(\tau_n, X^\pi(t_n), 0) + \int_{t_n}^{T} G(u, X^\pi(u), \phi(u)) \, du \right] . \] \hfill (62)

Letting \(n \to \infty\) and using properties 4 and 5, we have

\[ \tilde{V}(t, x, 0) \leq E_{t,x,0}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi(u)) \, du \right] . \] \hfill (63)

Since this holds for all \(\phi \in \Sigma\), we deduce that

\[ \tilde{V}(t, x, 0) \geq E_{t,x,0}^{p_\alpha} \left[ \tilde{V}(\tau_n, X^\alpha(t_n), 0) + \int_{t}^{T} G(u, X^\alpha(u), \phi^\ast(u)) \, du \right] . \] \hfill (66)

Letting \(n \to \infty\) and using properties 4 and 5, we have

\[ \tilde{V}(t, x, 0) \geq \inf_{\phi \in \Sigma} E_{t,x,0}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi^\ast(u)) \, du \right] \] \hfill (67)

Since this holds for all \(\pi \in \Pi\), we deduce that

\[ \tilde{V}(t, x, 0) \geq \sup_{\pi \in \Pi} \inf_{\phi \in \Sigma} E_{t,x,0}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi(u)) \, du \right] = V(t, x, 0) . \] \hfill (68)

According to (65) and (68), we get \(\tilde{V}(t, x, 0) = V(t, x, 0)\). Finally, we apply (61) to \((\phi^\ast, \pi^\ast)\) as the above process. Then,

\[ \tilde{V}(t, x, 0) \leq \inf_{\phi \in \Sigma} E_{t,x,0}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi^\ast(u)) \, du \right] . \] \hfill (64)

Hence,

\[ \tilde{V}(t, x, 0) \leq \sup_{\pi \in \Pi} \inf_{\phi \in \Sigma} E_{t,x,0}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi^\ast(u)) \, du \right] = V(t, x, 0) . \] \hfill (65)

Next, if we apply (61) to \((\phi^\ast, \pi)\) with \(\pi \in \Pi\) and use property 2, we get

\[ \tilde{V}(t, x, 0) = E_{t,x,0}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi^\ast(u)) \, du \right] = V(t, x, 0) . \] \hfill (69)

For \(h = 1\), we can obtain the following result by the similar method:

\[ \tilde{V}(t, x, 1) = E_{t,x,1}^{p_\alpha} \left[ U(X^\pi(T)) + \int_{t}^{T} G(u, X^\pi(u), \phi^\ast(u)) \, du \right] = V(t, x, 1) . \] \hfill (70)

Thus, \((\phi^\ast, \pi^\ast)\) is an optimal control strategy about (69)-(70), and \(\tilde{V}(t, x, h) = V(t, x, h)\).

Lemma 4. The following integral is finite:

\[ I(T) := E \left[ \exp \left\{ \int_{0}^{T} \left( \frac{\alpha}{2\beta_1} \phi_1^\ast(t)^2 + \frac{\alpha}{2\beta_2} \phi_2^\ast(t)^2 + \frac{\alpha}{2\beta_3} \phi_3^\ast(t)^2 + \frac{\alpha}{\beta_4} (t) \ln \phi_4^\ast(t) - \phi_4^\ast(t) + \frac{1}{h} \right) \right\} dt \right] < \infty . \] \hfill (71)
Proof. Putting $\phi^*$ and $\pi^*$ into (71), with an appropriate constant $M_1 > 0$, we have

$$I(T) = \mathcal{E}\left[ \exp \left\{ \int_0^T \frac{\alpha}{2} \sigma_0^2 \pi^*_1(t) + \beta_3 \sigma_2^2 \phi^*_2(t) + \frac{\phi^*_1(t)\ln \phi^*_1(t) - \phi^*_1(t) + 1}{\beta_4} \right\} + \frac{\lambda a}{\beta_5} \int_0^T (\phi^*_2(t)\psi^*(t,y) - \phi^*_2(t)\psi^*(t,y) + 1) dF_1(y) \right\} \right]\leq M_1 E\left[ \exp \left\{ \int_0^T \frac{\phi^*_1(t)\ln \phi^*_1(t) - \phi^*_1(t) + 1}{\beta_4} + \frac{\lambda a}{\beta_5} \int_0^T (\phi^*_2(t)\psi^*(t,y) - \phi^*_2(t)\psi^*(t,y) + 1) dF_1(y) \right\} dt \right] < \infty, \quad (72)$$

where the inequality is established because $\pi^*$ and $\phi^*$ are the deterministic and bound functions on $[0, T]$.

Proof. Substituting $(\pi^*, \phi^*)$ into (19), we have the wealth process under $(\pi^*, \phi^*)$:

$$E^{\pi^*, \phi^*} \left( \sup_{t \in [0, T]} |\bar{V}(t, X^{\pi^*}(t), H(t))|^4 \right) < \infty, \quad (73)$$

$$E^{\pi^*, \phi^*} \left( \sup_{t \in [0, T]} |G(t, X^{\pi^*}(t), \phi^*(t))|^2 \right) < \infty.$$  

$$X^{\pi^*}(t) = e^{rt}X^{\pi^*}(0) + \int_0^t e^{-r(u-t)} \left[ (\mu - r)\pi^*_1(u) + \pi^*_2(u)(1 - H(u))\delta + \lambda_2((\theta_0 - \eta)\mu_2 + h_1(a^*)) - \phi^*_1(u)\pi^*_1(u)\sigma - \sigma_0\phi^*_2(u) 
\right. $$

$$- \phi^*_1(u)\sqrt{h_2(a^*)} \right] du + \int_0^t e^{-r(u-t)} \pi^*_1(u)\sigma dW^1_{t, \phi^*}(u)$$

$$+ \int_0^t e^{-r(u-t)} \sigma_0 dW^2_{t, \phi^*}(u) + \int_0^t e^{-r(u-t)} \sqrt{h_2(a^*)} dW^3_{t, \phi^*}(u)$$

$$- \int_0^t e^{-r(u-t)} \pi^*_1(u)(1 - H(u-))\zeta dH(u) + \int_0^t \int_{-1}^1 e^{-r(u-t)} \pi^*_1(u)y\tilde{N}^y_{t, \phi^*}(du, dy). \quad (74)$$

For the candidate value function (21), we get

$$|\bar{V}(t, X^{\pi^*}(t), H(t))|^4 = \left| (1 - H(t))\bar{V}(t, X^{\pi^*}(t), 0) + H(t)\bar{V}(t, X^{\pi^*}(t), 1) \right|^4$$

$$\leq 4\left| \bar{V}(t, X^{\pi^*}(t), 0) \right|^4 + 4\left| \bar{V}(t, X^{\pi^*}(t), 1) \right|^4. \quad (75)$$

At first, we prove $E^{\pi^*, \phi^*} \left( \sup_{t \in [0, T]} |\bar{V}(t, X^{\pi^*}(t), 0)|^4 \right) < \infty$. $\pi^*$, $\phi^*$, and $g_2(t)$ are deterministic continuous functions and are bounded on $[0, T]$. Substituting (75) into (48), there are two constants $K_1$ and $K_2$ satisfying $0 < K_1 < K_2$ such that
\[
\| \hat{V}(t, X^\Pi(t), 0) \| ^4 = \frac{g_1(t)}{a^4} \exp \left\{ -4a \left[ e^{t} X^\Pi(0) + \int _{0}^{t} e^{-r(u-t)} \left( \mu + r \right) \pi _{1}^{u}(u) + \pi _{2}^{u}(u) (1 - H(u)) \delta + \lambda \left( \theta - \eta \right) \mu + h_{1}(a^*) \right) \right. \\
- \phi _{1}^{u}(u) \pi _{1}^{u}(u) \sigma - \phi _{2}^{u}(u) - \phi _{3}^{u}(u) \sqrt{\lambda \beta _{2}(a^*)} du + \int _{0}^{t} e^{-r(u-t)} \pi _{1}^{u}(u) \sigma dW_{1}^{\Pi}(u) \\
+ \int _{0}^{t} e^{-r(u-t)} \sigma _{0} dW_{2}^{\Pi}(u) + \int _{0}^{t} e^{-r(u-t)} \sqrt{\lambda \beta _{2}(a^*)} dW_{3}^{\Pi}(u) - \int _{0}^{t} e^{-r(u-t)} \pi _{2}^{u}(u) (1 - H(u-)) \xi dH(u) \\
+ \int _{0}^{t} \int _{-1}^{\infty} e^{-r(u-t)} \pi _{1}^{u}(u) y \tilde{N}^{\Pi} (du, dy) \right\} \\
\leq K_{1} \exp \left\{ -4a \left[ \int _{0}^{t} e^{-r(u-t)} \pi _{1}^{u}(u) \sigma dW_{1}^{\Pi}(u) + \int _{0}^{t} e^{-r(u-t)} \sigma _{0} dW_{2}^{\Pi}(u) + \int _{0}^{t} e^{-r(u-t)} \sqrt{\lambda \beta _{2}(a^*)} dW_{3}^{\Pi}(u) \\
- \int _{0}^{t} e^{-r(u-t)} \pi _{2}^{u}(u) (1 - H(u-)) \xi dH(u) + \int _{0}^{t} \int _{-1}^{\infty} e^{-r(u-t)} \pi _{1}^{u}(u) y \tilde{N}^{\Pi} (du, dy) \right\} \\
= K_{1} \exp \left\{ 4a \int _{0}^{t} e^{-r(u-t)} \pi _{1}^{u}(u) (1 - H(u-)) \xi dH(u) + 32a^2 \left[ \int _{0}^{t} e^{-2r(u-t)} \left( \pi _{1}^{u}(u) \sigma^2 + \sigma _{0}^2 + \lambda \beta _{2}(a^*) \right) du \\
- 32a^2 \int _{0}^{t} e^{-2r(u-t)} \pi _{1}^{u}(u) \sigma^2 du - 4a \int _{0}^{t} e^{-r(u-t)} \pi _{1}^{u}(u) \sigma dW_{1}^{\Pi}(u) - 32a^2 \int _{0}^{t} e^{-2r(u-t)} \sigma _{0}^2 du \\
- 4a \int _{0}^{t} e^{-r(u-t)} \sigma _{0} dW_{2}^{\Pi}(u) - 32a^2 \int _{0}^{t} e^{-2r(u-t)} \lambda \beta _{2}(a^*) du - 4a \int _{0}^{t} e^{-r(u-t)} \sqrt{\lambda \beta _{2}(a^*)} dW_{3}^{\Pi}(u) \\
- 4a \int _{0}^{t} \int _{-1}^{\infty} e^{-r(u-t)} \pi _{1}^{u}(u) y \tilde{N}^{\Pi} (du, dy) \right\} \\
\leq K_{2} \exp \left\{ -32a^2 \int _{0}^{t} e^{-2r(u-t)} \pi _{1}^{u}(u) \sigma^2 du - 4a \int _{0}^{t} e^{-r(u-t)} \pi _{1}^{u}(u) \sigma dW_{1}^{\Pi}(u) - 32a^2 \int _{0}^{t} e^{-2r(u-t)} \sigma _{0}^2 du \\
- 4a \int _{0}^{t} e^{-r(u-t)} \sigma _{0} dW_{2}^{\Pi}(u) - 32a^2 \int _{0}^{t} e^{-2r(u-t)} \lambda \beta _{2}(a^*) du - 4a \int _{0}^{t} e^{-r(u-t)} \sqrt{\lambda \beta _{2}(a^*)} dW_{3}^{\Pi}(u) \\
- 4a \int _{0}^{t} \int _{-1}^{\infty} e^{-r(u-t)} \pi _{1}^{u}(u) y \tilde{N}^{\Pi} (du, dy) \right\} \\
\leq K_{2} \exp \left\{ D_{1}(t) + D_{2}(t) + D_{3}(t) + D_{4}(t) \right\},
\]
(76)

where

\[
D_{1}(t) = -32a^2 \int _{0}^{t} e^{-2r(u-t)} \pi _{1}^{u}(u) \sigma^2 du - 4a \int _{0}^{t} e^{-r(u-t)} \pi _{1}^{u}(u) \sigma dW_{1}^{\Pi}(u), \\
D_{2}(t) = -32a^2 \int _{0}^{t} e^{-2r(u-t)} \sigma _{0}^2 du - 4a \int _{0}^{t} e^{-r(u-t)} \sigma _{0} dW_{2}^{\Pi}(u), \\
D_{3}(t) = -32a^2 \int _{0}^{t} e^{-2r(u-t)} \lambda \beta _{2}(a^*) du - 4a \int _{0}^{t} e^{-r(u-t)} \sqrt{\lambda \beta _{2}(a^*)} dW_{3}^{\Pi}(u), \\
D_{4}(t) = -4a \int _{0}^{t} \int _{-1}^{\infty} e^{-r(u-t)} \pi _{1}^{u}(u) y \tilde{N}^{\Pi} (du, dy),
\]
(77)
where $\pi^*$ and $\phi^*$ are deterministic continuous functions, which are bounded on $[0, T]$, and we get

\begin{equation}
E^{\pi^*} \left( \exp \{4D_1(t)\} \right) < E^{\pi^*} \left( \exp \left\{ -16\alpha \int_0^t \int_{-1}^{\infty} e^{-r(u-t)} \pi_1^* (u) y \bar{N}^{\pi^*} (du, dy) \right\} \right) < \infty. \tag{78}
\end{equation}

Applying Lemma 5 in Zeng and Taksar [43], we know that $\exp \{4D_1(t)\}$, $\exp \{4D_2(t)\}$, and $\exp \{4D_3(t)\}$ are martingales; then,

\begin{align}
E^{\pi^*} \left( \exp \{4D_1(t)\} \right) &< \infty, \tag{79} \\
E^{\pi^*} \left( \exp \{4D_2(t)\} \right) &< \infty. \tag{80}
\end{align}

\begin{align*}
E^{\pi^*} \left( \left| \bar{V} \left( t, X_{\pi^*} (t), 0 \right) \right|^4 \right) & \leq K_3 E^{\pi^*} \left[ \exp \{D_1(t) + D_2(t) + D_3(t) + D_4(t)\} \right] \\
& \leq K_3 \left( E^{\pi^*} \left[ \exp \{2D_3(t) + 2D_4(t)\} \right] E^{\pi^*} \left[ \exp \{2D_1(t) + 2D_2(t)\} \right] \right)^{1/2} \\
& \leq K_3 \left( E^{\pi^*} \left[ \exp \{4D_3(t)\} \right] E^{\pi^*} \left[ \exp \{4D_4(t)\} \right] \right) E^{\pi^*} \left[ \exp \{4D_2(t)\} \right] \left(1/2\right) < \infty. \tag{82}
\end{align*}

Consequently, $E^{\pi^*} \left( \sup_{t \in [0, T]} \left| \bar{V} \left( t, X_{\pi^*} (t), 0 \right) \right|^4 \right) < \infty$. Similarity, we can also prove

\begin{equation}
E^{\pi^*} \left( \sup_{t \in [0, T]} \left| \bar{V} \left( t, X_{\pi^*} (t), 1 \right) \right|^4 \right) < \infty. \tag{83}
\end{equation}

\begin{equation}
 f(t) = \frac{\alpha}{2\beta_1} \phi_1^* (t)^2 + \frac{\alpha}{2\beta_2} \phi_2^* (t)^2 + \frac{\alpha}{2\beta_3} \phi_3^* (t)^2 + \frac{\phi_4^* (t) \ln \phi_4^* (t) - \phi_4^* (t) + 1}{\beta_4} \text{h}^\alpha + \frac{\lambda \alpha}{\beta_5} \int_{-1}^{\infty} \left( \phi_5^* (t) \phi^* (t, y) \ln (\phi_5^* (t) \phi^* (t, y)) - \phi_5^* (t) \phi^* (t, y) + 1 \right) dF_4 (y), \tag{84}
\end{equation}

where $f(t)$ is obviously bounded, then we get

\begin{align}
E^{\pi^*} \left( \sup_{t \in [0, T]} \left| G \left( t, X_{\pi^*} (t), \phi^* (t) \right) \right|^2 \right) = E^{\pi^*} \left( \sup_{t \in [0, T]} \left| f(t) \right|^2 \right) \left( \bar{V} \left( t, X_{\pi^*} (t), h \right) \right) \left( \left( \bar{H} \left( t, X_{\pi^*} (t), h \right) \right) \right) \left(1/2\right) < \infty. \tag{85}
\end{align}

The first inequality follows from Cauchy–Schwarz inequality. The last inequality follows from (71). Lemma 5 is proved.

Based on the discussion above, the main theorem is summarized as follows.

**Theorem 3.** For the robust optimal control problem (21) with the exponential utility function (11), $\bar{V} \left( t, x, h \right)$ is the solution of (24) with the boundary condition $\bar{V} \left( T, x, h \right) = U(x)$, $(\pi^*, \phi^*)$ is an optimal strategy, and then $V \left( t, x, h \right) = \bar{V} \left( t, x, h \right)$ is the corresponding value function.
\textbf{Proof.} From Lemmas 1, 2, and 3 and Theorems 1 and 2, we can obtain properties 1–4 in Proposition 1. By Lemma 5, condition 5 in Proposition 1 also holds for $\bar{V}(t, x, h)$. From Proposition 1, we can obtain the result of Theorem 3, $(\pi^*, \phi^*)$ is an optimal strategy, and $V(t, x, h)$ is the corresponding value function. \hfill \square

\section{5. Some Special Cases}

In this section, we shall present some special cases of our results, such as the proportional reinsurance, the excess-of-loss reinsurance, and an ambiguity-neutral insurer of the insurance.

\subsection{5.1. The Proportional Reinsurance}

The parameter $\eta$ satisfies $\eta = 0$, and the insurer purchases reinsurance in the form of the proportional reinsurance, that is, $\mathcal{R}(a, Z) = 2a\theta_1 Z$. The result is shown by corollary as follows.

\textbf{Corollary 5.1.} If $\eta = 0$, then the value function $V(t, x, h) = (1-h)V(t, x, 0) + hV(t, x, 1), h = 0, 1$, and the optimal strategies $\pi^*$ and $\phi^*$ are given by

\begin{align*}
\phi^*_1 (t) &= \beta_1 \sigma_1 (t)e^{r(T-t)}, \quad t \in [0, T], \\
\phi^*_2 (t) &= \beta_2 \sigma e^{r(T-t)}, \quad t \in [0, T], \\
\phi^*_3 (t) &= 2\beta_3 \sqrt{\lambda} a^* (t)\theta_1 \sigma z e^{r(T-t)}, \\
0 &= \frac{ahp}{\beta_4} \phi^*_4 (t) \ln \phi^*_4 (t) + hp \phi^*_4 (t) - \frac{\delta}{\zeta} \quad t \in [0, T], \\
\phi^*_5 (t) &= \frac{1}{\phi^*_5 (t)} \exp \left[ \frac{\beta_2}{\alpha} \left( e^{-\alpha^* (t)e^{r(T-t)} y} - 1 \right) \right], \quad t \in [0, T], \\
\phi^*_6 (t) &= E \left[ \exp \left[ \frac{\beta_2}{\alpha} \left( e^{-\alpha^* (t)e^{r(T-t)} y} - 1 \right) \right] \right], \quad t \in [0, T], \\
\pi^*_1 (t) &= \frac{e^{-r(T-t)}}{\sigma^2 (\alpha + \beta_1)} \left[ -\mu + r + \lambda_1 \int_{-1}^{\infty} y e^{-\alpha^* (t)e^{r(T-t)} y} \exp \left[ \frac{\beta_2}{\alpha} \left( e^{-\alpha^* (t)e^{r(T-t)} y} - 1 \right) \right] dF_1 (y) \right], \quad t \in [0, T], \\
\pi^*_2 (t) &= \begin{cases} \\ \\
0, \\
\frac{1}{\alpha} \frac{g_2 (t)}{\Delta \phi^*_4 (t) g_1 (t)}, \quad t \in [0, T], \\
0, \quad t \in [T \wedge T], \\
\frac{1}{2\theta_1 + (\alpha + \beta_1) e^{r(T-t)}}, \quad t \in [0, T]. \\
\end{cases}
\end{align*}

The optimal value function $V(t, x, h)$ is as follows:

\begin{align*}
V(t, x, 1) &= \frac{1}{\alpha} \exp \left[ -\alpha x e^{r(T-t)} \right] g_1 (t), \text{ and } V(t, x, 0) = \frac{1}{\alpha} \exp \left[ -\alpha x e^{r(T-t)} \right] g_2 (t),
\end{align*}

where

\begin{align*}
g_1 (t) &= \exp \left[ \int_{t}^{T} \left[ \frac{\alpha + \beta_1}{2} e^{2r(T-u)} - \alpha \lambda_2 e^{r(T-u)} (\theta_0 - \eta) \mu_x - \frac{\lambda_1 \alpha}{\beta_0} e^{r(T-u)} (\mu - r) \pi^*_1 (u) + \frac{\alpha + \beta_1}{2} e^{2r(T-u)} (\pi^*_1 (u))^2 \\
&+ \frac{\lambda_1 \alpha}{\beta_0} \int_{-1}^{\infty} \exp \left[ \frac{\beta_2}{\alpha} \left( e^{-\alpha^* (u)e^{r(T-u)} y} - 1 \right) \right] dF_1 (y) - \alpha \lambda_2 e^{r(T-u)} \left[ -\theta_1 (1 - 2a^* (u)\theta_1) \sigma_2^2 - 2(\alpha + \beta_3) (a^* (u))^2 \sigma_2^2 e^{r(T-u)} \right] du \right] \\
&+ \frac{\lambda_1 \alpha}{\beta_0} \int_{-1}^{\infty} \exp \left[ \frac{\beta_2}{\alpha} \left( e^{-\alpha^* (u)e^{r(T-u)} y} - 1 \right) \right] dF_1 (y) - \alpha \lambda_2 e^{r(T-u)} \left[ -\theta_1 (1 - 2a^* (u)\theta_1) \sigma_2^2 - 2(\alpha + \beta_3) (a^* (u))^2 \sigma_2^2 e^{r(T-u)} \right] du \right],
\end{align*}

(88)
and \( g_2(t) = e^{\alpha t} \), in which

\[
\mathcal{V}_2(t) = e^{-\beta(T-t)} \int_0^T e^{\beta(u)} \left\{ \frac{\alpha + \beta_3}{2} \alpha \sigma_2 \epsilon^{2(\xi(T-u))} - \alpha \lambda \gamma \epsilon^\gamma (\theta_2 - \eta) \mu \epsilon - \alpha \epsilon^\gamma (\mu - \tau) \pi^3 (u) \right\} \]

\[
\quad + \frac{\alpha + \beta_1}{2} \alpha \sigma_2 \epsilon^{2(\xi(T-u))} \left( \phi_1^3(u) \right)^2 \left( \frac{\lambda_1}{\beta_5} + \frac{\lambda_1^2}{\beta_5^2} \right) \int_{\frac{\theta_1 \gamma}{\beta_4}}^\infty \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{\alpha \phi_1^3(u) \epsilon^{\alpha(\gamma - 1)}} - 1 \right) \right\} \]

\[
\quad \cdot \left( -\theta_1 (1 - 2a^* (u)) \theta_1 \right)^2 \sigma_2^2 - 2(\alpha + \beta_1) (a^* (u))^2 \phi_1^3(u) \epsilon^{\gamma} \epsilon^{(\xi(T-u))} \right\} \]

\[
\quad + \frac{\delta}{\xi} \left( \ln \Delta \phi_1^3 (u) + \ln g_1 (u) + \frac{\phi_1^3 (u) - 1}{\beta_4^3} \right) \]  

\[
\quad \mbox{du.} \tag{89}
\]

\textbf{Proof.} In this case, \( \mathcal{H} (a, Z) = 2a \theta_1 Z \), and then the functions in our results are as follows:

\[
\begin{align*}
\varphi_1 (a) &= \eta h_2 (a) - (1 + \eta) \theta_1 h_3 (a) = -\theta_1 (1 - 2a \theta_1)^2 a Z, \\
\varphi_2 (a) &= E[\mathcal{H} (a, Z)^2] = 4a^2 \sigma_2^2, \\
\varphi_3 (a) &= E[\mathcal{H} (a, Z)] = 2a \theta_1 \mu Z, \\
\varphi_4 (a) &= E[(Z - \mathcal{H} (a, Z))^2] = (1 - 2a \theta_1)^2 a Z. \\
\end{align*}
\]

(90)

According to Theorems 3.3 and 3.5, we can obtain the result. \( \square \)

\textbf{Remark 2.} If \( \eta = 0 \), and \( 2a \theta_1 = q \), then \( \mathcal{H} (Z) = 2a \theta_1 Z = qZ \), becomes a proportional reinsurance type. There are similar studies, for example, Zhou et al. [24], Zheng et al. [26], and Wang et al. [44].

\subsection*{5.2. The Excess-of-Loss Reinsurance}

The parameter \( \theta_1 \) satisfies \( \theta_1 = 0 \), and the insurer purchases reinsurance in the form of the excess-of-loss reinsurance, that is, \( \mathcal{H} (a, Z) = a \eta \wedge Z \). The result is shown by corollary as follows.

\textbf{Corollary 5.3.} If \( \theta_1 = 0 \), then the optimal value function \( V (t, x, h) = (1 - h) V (t, x, 0) + h V (t, x, 1), h = 0, 1 \), and the optimal strategies \( \pi^* \) and \( \phi^* \) are given by

\[
\begin{align*}
\phi^*_1 (t) &= \beta_1 \sigma \epsilon^{\gamma (T-t)}, \quad t \in [0, T], \\
\phi^*_2 (t) &= \beta_2 \sigma \epsilon^{\gamma (T-t)}, \quad t \in [0, T], \\
\phi^*_3 (t) &= \beta_3 \sqrt{\lambda_2 h_2 (a) \epsilon^{\gamma (T-t)}}, \\
0 &= \frac{\alpha h^p}{\beta_4} \phi^*_4 (t) \ln \phi^*_4 (t) + h^p \phi^*_4 (t) - \frac{\delta}{\xi}, \quad t \in [0, T], \\
\varphi^* (t, y) &= \frac{1}{\phi^*_4 (t)} \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{\alpha \phi_1^3 (t) \epsilon^{\alpha (\gamma - 1)}} - 1 \right) \right\}, \quad t \in [0, T], \\
\phi^*_5 (t) &= E \left\{ \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{\alpha \phi_1^3 (t) \epsilon^{\alpha (\gamma - 1)}} - 1 \right) \right\} \right\}, \quad t \in [0, T], \\
\pi^*_1 (t) &= \frac{e^{-\gamma (T-t)}}{\sigma^2 (\alpha + \beta_1)} \left[ \mu - r + \lambda_1 \right] \int_{\frac{\alpha \phi_1^3 (t) \epsilon^{\alpha (\gamma - 1)}}{\beta_4}}^\infty \exp \left\{ \frac{\beta_5}{\alpha} \left( e^{\alpha \phi_1^3 (t) \epsilon^{\alpha (\gamma - 1)}} - 1 \right) \right\} \]

\[
\quad \cdot \mbox{dF}_1 (y), \quad t \in [0, T], \\
\pi^*_2 (t) &= \left\{ \begin{array}{ll}
\frac{1}{\sigma} e^{-r (T-t)} \left[ \ln \frac{g_2 (t)}{\Delta \phi_4^* (t) g_1 (t)} \right], & t \in [0, \tau \wedge T], \\
0, & t \in [\tau \wedge T, T],
\end{array} \right. \\
\alpha^* (t) &= \frac{1}{(\alpha + \beta_3) \epsilon^{\gamma (T-t)}}, \quad t \in [0, T]
\end{align*}
\]
The optimal value function \( V(t, x, h) \) is as follows:

\[
V(t, x, 1) = \frac{1}{a} \exp\left\{ -ax e^{r(T-t)} \right\} g_1(t),
\]

\[
V(t, x, 0) = \frac{1}{a} \exp\left\{ -ax e^{r(T-t)} \right\} g_2(t),
\]

where

\[
g_1(t) = \exp\left\{ \int_t^T \left[ \frac{\alpha + \beta_2}{2} \alpha a_0^2 e^{2r(T-u)} - \alpha \lambda_2 e^{r(T-u)} (\theta_0 - \eta) \mu Z - \frac{\lambda_1}{\beta_5} \alpha - a e^{r(T-u)} (\mu - r) \pi^*_1 (u) + \frac{\alpha + \beta_1}{2} a^{2} e^{2r(T-u)} (\pi^*_1 (u))^2 \right] du \right\},
\]

and \( g_2(t) = e^{\Phi_1(t)} \), in which

\[
\Phi_2(t) = e^{-(b/c)(T-t)} \left[ \frac{\alpha + \beta_2}{2} \alpha a_0^2 e^{2r(T-u)} - \alpha \lambda_2 e^{r(T-u)} (\theta_0 - \eta) \mu Z - \frac{\lambda_1}{\beta_5} \alpha - a e^{r(T-u)} (\mu - r) \pi^*_1 (u) \right]
\]

\[
+ \frac{\alpha + \beta_1}{2} a^{2} e^{2r(T-u)} (\pi^*_1 (u))^2 - \frac{\lambda_1}{\beta_5} \alpha + \frac{\lambda_1}{\beta_5} \int_t^T \exp\left\{ \frac{\beta_3}{\beta_5} \left( e^{-\alpha \pi^*_1 (u)} e^{r(T-u)} - 1 \right) \right\} du,
\]

Proof. In this case, \( \mathcal{H}(a, Z) = a \eta \wedge Z \), and then the functions in our results are as follows:

\[
h_1(a) = \eta h_3(a) - (1 + \eta) \theta_1 h_4(a) = \eta \int_0^{a^\eta} (1 - F_2(z)) dz,
\]

\[
h_2(a) = E[\mathcal{H}(a, Z)^2] = 2 \int_0^{a^\eta} (1 - F_2(z)) dz,
\]

\[
h_3(a) = E[\mathcal{H}(a, Z)] = \int_0^{a^\eta} (1 - F_2(z)) dz.
\]

According to Theorems 1 and 2, we can obtain the result. \( \square \)

Remark 3. If \( \theta_1 = 0 \), then \( \mathcal{H}(Z) = (a \eta) \wedge Z_i \) becomes an excess-of-loss reinsurance type. Without regard for the default bond, let \( \eta = 1 \), then the reinsurance strategy becomes the result of Li et al. \[10\].

5.3. Ambiguity-Neutral Insurer (ANI) Case. If the insurer is an ambiguity-neutral insurer, then the aversion ambiguity coefficient \( \beta_i = 0 \), \( i = 1, 2, 3, 4, 5 \). In this case, for an admissible control \( (\tilde{a}(t), \pi_1(t), \pi_2(t)) \) and an initial value \( (x, h) \), the objective function is described by

\[
V(t, x, h) = \sup_{\pi \in \Pi} \left\{ \tilde{V}(t, x) + \tilde{V}_x (x) + \tilde{V}_h (h) (\tilde{V}(t, x, h) + h^\eta (1 - h) (\tilde{V}(t, x, \pi_2(t) \tilde{C}, 1) - \tilde{V}(t, x, 0))) \right\}.
\]
From the value functions
\[ V(t, x, 1) = \frac{1}{\alpha} \exp\left\{ -\alpha x e^{r(T-t)} \right\} g_{10}(t), \]
\[ V(t, x, 0) = \frac{1}{\alpha} \exp\left\{ -\alpha x e^{r(T-t)} \right\} g_{20}(t), \]
we can get the following candidate optimal strategies by the same way:
\[ \tilde{\pi}^*_2(t) = \frac{e^{-r(T-t)}}{\sigma^2 \alpha} \left[ \mu - r + \lambda_1 \left\{ e^{-\alpha \tilde{\pi}^*_2(t) e^{r(T-t)}} \right\} \right], \quad t \in [0, T], \]
\[ \tilde{\pi}^*_1(t) = \left\{ \begin{array}{ll}
\frac{1}{\alpha^2} e^{-r(T-t)} & \ln \frac{g_{20}(t)}{\Delta g_{10}(t)} \quad t \in [0, \tau \land T], \\
0 & t \in [\tau \land T, T],
\end{array} \right. \]
\[ \tilde{a}^*(t) = \frac{1}{2} \left( 1 + \eta \right) \theta_1 + \alpha e^{r(T-t)}, \quad t \in [0, T]. \]
where
\[ g_{10}(t) = \exp \left\{ \int_t^T \frac{\alpha^2}{2} e^{2r(T-u)} \left( \sigma^2 \left( \tilde{\pi}^*_1(u) \right)^2 + \sigma_0^2 + \lambda_2 h_2 \left( a^*(u) \right) \right) \\
- \alpha e^{r(T-u)} \left( \mu - r \right) \tilde{\pi}^*_1(u) + \lambda_2 \left( \left[ \left( \theta_0 - \eta \right) \mu_Z \right) \\
+ h_1 \left( a^*(u) \right) \right) + \lambda_1 \left( E \left( e^{-\alpha \tilde{\pi}^*_1(u) e^{r(T-t)}} \right) - 1 \right) \} du \right\}, \]
and
\[ g_{20}(t) = e^{-\left( \Delta \lambda \right) (T-t)} \int_0^T e^{-\left( \Delta \lambda \right) (T-u)} \left( \frac{\alpha^2}{2} e^{2r(T-u)} \left( \sigma^2 \left( \tilde{\pi}^*_1(u) \right)^2 \\
+ \sigma_0^2 + \lambda_2 h_2 \left( a^*(u) \right) \right) + \delta \left( \Delta - 1 - \frac{1}{\delta} \right) - \alpha e^{r(T-u)} \\
\cdot \left( \left( \mu - r \right) \tilde{\pi}^*_1(u) + \lambda_2 \left[ \left( \theta_0 - \eta \right) \mu_Z + h_1 \left( a^*(u) \right) \right) \right) \\
+ \lambda_1 \left( E \left( e^{-\alpha \tilde{\pi}^*_1(u) e^{r(T-t)}} \right) - 1 \right) \} du. \]

Remark 4. If all of the ambiguity aversion coefficients equal 0, i.e., \( \beta_i = 0, i = 1, 2, 3, 4, 5 \), our model reduces to an optimization problem for an ambiguity-neutral insurer (ANI). For the ANI, the optimization investment-reinsurance is researched by Cao [11], Yang and Zhang [2], etc.

### 6. Sensitivity Analysis

In this section, we will give several numerical examples to illustrate the influences of the parameters on the optimal strategies and the optimal value functions. Unless otherwise stated, the basic parameters are given in Table 1.

Some analyses of the optimal reinsurance strategy \( a^*(t) \) are shown in Figures 1–4. \( \beta_3 \) is the ambiguity aversion coefficient of the AAI. From Figure 1, it is found that \( \beta_3 \) affects the reinsurance strategy of the insurer. As \( \beta_3 \) increases, the insurer has lower risk exposure in the insurance market, so less amount of money will be paid to purchase reinsurance. \( \eta \) and \( \theta_1 \) are the relative safety loadings of the

| \( \mu \) | \( T \) | \( r \) | \( \alpha \) | \( \sigma_0 \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \theta_1 \) | \( \theta_0 \) | \( \delta \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 1 | 0.05 | 1 | 0.15 | 0.5 | 2 | 2 | 2 | 1 | 0.2 |

**Table 1: Model parameters.**

**Figure 1:** Effect of \( \beta_3 \) on \( a^*(t) \).

**Figure 2:** Effect of \( \eta \) on \( a^*(t) \).
Figure 3: Effect of $\theta_1$ on $a^*(t)$.

Figure 4: Effect of $\theta_1$ and $\beta_3$ on $a^*(t)$.

Figure 5: Effect of $\beta_5$ on $\pi_1^*(t)$.

Figure 6: Effect of $\beta_5$ on $\pi_1^*(t)$.

Figure 7: Effect of $\beta_1$ and the time $t$ on $\pi_1^*(t)$.

Figure 8: Effect of $\beta_5$ and the time $t$ on $\pi_1^*(t)$.
reinsurer. Figures 2 and 3 show the effects of $\eta$ or $\theta_1$ on the insurer’s reinsurance strategy. As $\eta$ or $\theta_1$ increases, the reinsurer pays more concern on his/her risk exposures and charges more for them. Consequently, the insurer decreases his/her demand for reinsurance and pays more claims by himself/herself. In Figure 4, we show the common effects of the safety loading $\theta_1$ and the ambiguity aversion coefficient $\beta_3$ on $\pi_1^*(t)$.

In Figures 5–8, we illustrate the impacts of the ambiguity aversion coefficients $\beta_1$ and $\beta_4$ on the stock strategy $\pi_2^*(t)$. From Figures 5 and 6, we find that the AAI will reduce the wealth invested on the stock, when there is a higher ambiguity aversion coefficient. From Figures 7 and 8, we find that as the time $t$ increases, the AAI increases the stock investment amount. These figures show that the robust optimal strategies can effectively reduce the sensitivity of $\pi_1^*(t)$ on the stock.

For the defaultable corporate bond, the value is assumed to be zero after default. In Figures 9–13, we show the numerical analysis of the defaultable corporate bond strategy $\pi_2^*(t)$ before default. $\beta_2$ and $\beta_4$ are the ambiguity aversion coefficients. As $\beta_4$ increases, the insurer will reduce the money on the defaultable corporate bond, but $\beta_2$ does not affect the investment, as shown in Figure 9. In Figure 10, the insurer will invest more amount of his/her money, if the defaultable corporate bond with a higher premium induces a
higher potential yield. Contrary to the default risk premium $1/\Delta$, the accession of $\zeta$ reduces the insurer’s investment on the defaultable bond, as shown in Figure 11. A higher loss rate $\zeta$ leads to a less recovery value, which implies a higher potential loss of the insurer. When $\eta = 0$, the reinsurance type is a proportional reinsurance, and when $\theta_1 = 0$, the reinsurance type becomes an excess-of-loss reinsurance. We show that the proportion reinsurance is always below the excess-of-loss reinsurance at the same time in Figure 12.

Figure 13 provides a full description of $\pi^*_2(t)$ with respect to $1/\Delta$ and $\zeta$ with two different reinsurance types, respectively. In Figure 14, we illustrate the predefault value function $V(0, x, 0)$ and the postdefault value function $V(0, x, 1)$ with respect to the initial wealth about a proportional reinsurance and an excess-of-loss reinsurance, respectively. We can see that the value functions of the insurer increase as the initial wealth increases and the predefault value function is always greater than or equal to the postdefault value function.

7. Conclusion

In this paper, we consider a robust optimal reinsurance-investment problem of an insurer under the generalized mean-variance premium principle and a defaultable market. The insurer can trade in a risk-free asset, a stock, and a defaultable corporate bond. The surplus of the AAI is described by an approximate diffusion process. The stock’s price process is described by a jump-diffusion model. Using
the dynamic programming approach, we study the pre-default case and the postdefault case, respectively, and derive the optimal strategies and the corresponding value functions under the worst-case scenario. We give some sensitivity analysis to illustrate our theoretical results. In future research, we will consider some complex models, such as the robust optimal reinsurance-investment problem of stochastic differential games.

**Data Availability**

All data generated or analyzed during this study are included in this article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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