EMPIRICAL REGRESSION QUANTILE PROCESS WITH POSSIBLE APPLICATION TO RISK ANALYSIS

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Abstract. The processes of the averaged regression quantiles and of their modifications provide useful tools in the regression models when the covariates are not fully under our control. As an application we mention the probabilistic risk assessment in the situation when the return depends on some exogenous variables. The processes enable to evaluate the expected $\alpha$-shortfall ($0 \leq \alpha \leq 1$) and other measures of the risk, recently generally accepted in the financial literature, but also help to measure the risk in environment analysis and elsewhere.

1. Introduction

In everyday life and practice we encounter various risks, depending on various contributors. The risk contributors may be partially under our control, and information on them is important, because it helps to make good decisions about system design. This problem appears not only in the financial market, insurance and social statistics, but also in environment analysis dealing with exposures to toxic chemicals (coming from power plants, road vehicles, agriculture), and elsewhere; see [30] for an excellent review of such problems. Our aim is to analyze the risks with the aid of probabilistic risk assessment. In the literature were recently defined various coherent risk measures, some satisfying suitable axioms. We refer to [5], [6], [31], [41], [43], [36], [1], [12], [9], [37], [38], and to other papers cited in, for discussions and some projects. For possible applications in the insurance we refer to [10].

A generally accepted measure of the risk is the expected shortfall, based on quantiles of a portfolio return. Its properties were recently intensively studied. Acerbi and Tasche in [1] speak on "expected loss in the 100$\alpha$% worst cases", or shortly on "expected $\alpha$-shortfall", $0 < \alpha < 1$, which is defined as

$$-E\{Y|Y \leq F^{-1}(\alpha)\} = -\frac{1}{\alpha} \int_0^\alpha F^{-1}(u) du,$$

where $F$ is the distribution function of the asset $Y$. The quantity can be estimated by means of approximations of the quantile function $F^{-1}(u)$ by the sample quantiles.

The quantile regression is an important method for investigation of the risk of an asset in the situation that it depends on some exogenous variables. An averaged regression quantile, introduced in [22], or some of its modifications, serve as a convenient tool for the global risk measurement in such a situation, when the

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amount of covariates is not under our control. The typical model for the relation of the loss to the covariates is the regression model

\[
Y_{ni} = \beta_0 + X_{ni}^\top \beta + e_{ni}, \quad i = 1, \ldots, n
\]

where \(Y_{ni}, \ldots, Y_{nn}\) are observed responses, \(e_{n1}, \ldots, e_{nn}\) are independent model errors, possibly non-identically distributed with unknown distribution functions \(F_i\), \(i = 1, \ldots, n\). The covariates \(X_{ni} = (x_{i1}, \ldots, x_{ip})^\top\), \(i = 1, \ldots, n\) are random or nonrandom, and \(\beta^* = (\beta_0, \beta_1^\top) = (\beta_0, \beta_1, \ldots, \beta_p)^\top \in \mathcal{R}^{p+1}\) is an unknown parameter. For the sake of brevity, we also use the notation \(x^*_{ni} = (1, x_{i1}, \ldots, x_{ip})^\top\), \(i = 1, \ldots, n\).

An important tool in the risk analysis is the regression \(\alpha\)-quantile

\[
\hat{\beta}^*_n(\alpha) = \left(\hat{\beta}_{n0}(\alpha), (\hat{\beta}_n(\alpha))^\top\right) = \left(\hat{\beta}_{n0}(\alpha), \hat{\beta}_{n1}(\alpha), \ldots, \hat{\beta}_{np}(\alpha)\right)^\top.
\]

It is a \((p + 1)\)-dimensional vector defined as a minimizer

\[
\hat{\beta}^*_n(\alpha) = \arg \min_{b \in \mathcal{R}^{p+1}} \left\{ \sum_{i=1}^n \left[ \alpha(Y_i - x^*_i b) + (1 - \alpha)(Y_i - x^*_i b) \right] \right\}
\]

(1.3) where \(z^+ = \max(z, 0)\) and \(z^- = \max(-z, 0)\), \(z \in \mathcal{R}_1\).

The solution \(\hat{\beta}^*_n(\alpha) = (\hat{\beta}_{n0}(\alpha), (\hat{\beta}_n(\alpha))^\top\) minimizes the \((\alpha, 1 - \alpha)\) convex combination of residuals \((Y_i - x^*_i b)\) over \(b \in \mathcal{R}^{p+1}\), where the choice of \(\alpha\) depends on the balance between underestimating and overestimating the respective losses \(Y_i\). The increasing \(\alpha \nearrow 1\) reflects a greater concern about underestimating losses \(Y_i\), comparing to overestimating.

The methodology is based on the averaged regression \(\alpha\)-quantile, what is the following weighted mean of components of \(\hat{\beta}^*_n(\alpha)\), \(0 \leq \alpha \leq 1\):

\[
B_n(\alpha) = \mathbf{x}_n^\top \hat{\beta}^*_n(\alpha) = \hat{\beta}_{n0}(\alpha) + \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^p x_{ij} \hat{\beta}_j(\alpha), \quad \mathbf{x}_n^* = \frac{1}{n} \sum_{i=1}^n \mathbf{x}^*_i
\]

(1.4) In [22] it was shown that \(B_n(\alpha) - \beta_0 - \mathbf{x}_0^\top \beta\) is asymptotically equivalent to the \([na]\)-quantile \(e_{n,[na]}\) of the model errors, if they are identically distributed. Hence, \(B_n(\cdot)\) can help to make an inference on the expected \(\alpha\)-shortfall (1.1) even under the nuisance regression.

Besides \(B_n(\alpha)\), its various modifications can also be used, sometimes better comprehensible. The methods are nonparametric, thus applicable also to heavy-tailed and skewed distribution; notice that [8] speak about considerable improvement over normality, trailing to use different distributions. An extension to autoregressive models is possible and will be a subject of the further study; there the main tool will be the autoregression quantiles, introduced in [29], and their averaged versions. The autoregression quantile will reflect the value-at-risk, based on the past assets, while its averaged version will try to mask the past history.

The behavior of \(B_n(\alpha)\) with \(0 < \alpha < 1\) has been illustrated in [4] and [27], and summarized in [25]; here it is showed that \(B_n(\alpha)\) is nondecreasing step function of \(\alpha \in (0, 1)\). The extreme \(B_n(1)\) with \(\alpha = 1\) was studied in [19]. Notice that the upper bound of the number \(J_n\) of breakpoints of \(\hat{\beta}^*_n(\cdot)\) and also of \(B_n(\cdot)\) is \(\left(\frac{n}{p + 1}\right)\) = \(\mathcal{O}(n^{p+1})\). However, Portnoy in [34] showed that, under some condition on the
design matrix $X_n$, the number $J_n$ of breakpoints is of order $O_p(n \log n)$ as $n \to \infty$, and thus much smaller.

An alternative to the regression quantile is the two-step regression $\alpha$-quantile, introduced in [21]. Here the slope components $\beta$ are estimated by a specific rank-estimate $\tilde{\beta}_{nR}$, which is invariant to the shift in location. The intercept component is then estimated by the $\alpha$-quantile of residuals of $Y_i$’s from $\tilde{\beta}_{nR}$. The averaged two-step regression quantile $\tilde{B}_n(\alpha)$ is asymptotically equivalent to $\bar{B}_n(\alpha)$ under a wide choice of the R-estimators of the slopes. However, finite-sample behavior of $\tilde{B}_n(\alpha)$ generally differs from that of $\bar{B}_n(\alpha)$; is affected by the choice of R-estimator, but the main difference is that the number of breakpoints of $\tilde{B}_n(\alpha)$ exactly equals to $n$.

Being aware of various important applications of the problem, we shall study this situation in more detail. The averaged regression quantile $\bar{B}_n(\alpha)$ is monotone in $\alpha$, while the two-step averaged regression quantile $\tilde{B}_n(\alpha)$ can be made monotone by a suitable choice of R-estimator $\tilde{\beta}_{nR}$. Hence, we can consider their inversions, which in turn estimate the parent distribution $F$ of the model errors. As such they both provide a tool for an inference. The behavior of these processes and of their approximations is analyzed and numerically illustrated.

2. Behavior of $\tilde{B}_n(\alpha)$ over $\alpha \in (0, 1)$.

Let us first describe one possible form of the averaged regression quantile $\tilde{B}_n(\alpha)$ as a weighted mean of the basic components of vector $Y$. Consider again the minimization (1.2), fixed $\alpha \in [0, 1]$ fixed. This was treated in [20] as a special linear programming problem, and later on various modifications of this algorithm were developed. Its dual program is a parametric linear program, which can be written simply as

$$\begin{align*}
\text{maximize} \quad & Y_n^\top \hat{a}(\alpha) \\
\text{under} \quad & X_n^\top \hat{a}(\alpha) = (1 - \alpha)X_n^\top 1_n^\top \\
& \hat{a}(\alpha) \in [0, 1]^n, \ 0 \leq \alpha \leq 1
\end{align*}$$

(2.1)

where

$$X_n^* = \begin{bmatrix} x_{n1}^\top \\
\vdots \\
x_{n+p}^\top \end{bmatrix} \quad \text{is of order } n \times (p + 1).$$

(2.2)

The components of the optimal solution $\hat{a}(\alpha) = (\hat{a}_{n1}(\alpha), \ldots, \hat{a}_{nn}(\alpha))^\top$ of (2.1), called regression rank scores, were studied in [12], who showed that $\hat{a}_{ni}(\alpha)$ is a continuous, piecewise linear function of $\alpha \in [0, 1]$ and $\hat{a}_{ni}(0) = 1$, $\hat{a}_{ni}(1) = 0$, $i = 1, \ldots, n$. Moreover, $\hat{a}(\alpha)$ is invariant in the sense that it does not change if $Y$ is replaced with $Y + X_n^* b^*$, $\forall b^* \in R^{p+1}$ (see [12] for detail).

Let $\{x_{i1}^*, \ldots, x_{i_{p+1}}^*\}$ be the optimal base in (2.1) and let $\{Y_{i1}, \ldots, Y_{i_{p+1}}\}$ be the corresponding responses in model (1.2). Then $\tilde{B}_n(\alpha)$ equals to a weighted mean of $\{Y_{i1}, \ldots, Y_{i_{p+1}}\}$, with the weights based on the regressors. Indeed, we have a theorem

**Theorem 1.** Assume that the regression matrix (2.2) has full rank $p + 1$ and that the distribution functions $F_1, \ldots, F_n$ of model errors are continuous and increasing
in $(-\infty, \infty)$. Then with probability 1

$$B_n(\alpha) = \sum_{k=1}^{p+1} w_{k,\alpha} Y_{ik}, \quad \sum_{k=1}^{p+1} w_{k,\alpha} = 1$$

and

$$\hat{B}_n(\alpha) \leq B_n(1) < \max_{i \leq n} Y_i$$

where the vector $Y_n(1) = (Y_i, \ldots, Y_{i+p+1})^T$ corresponds to the optimal base of the linear program (2.1).

The vector $w_\alpha = (w_{1,\alpha}, \ldots, w_{p+1,\alpha})^T$ of coefficients equals to

$$w_\alpha = [n^{-1} 1_n^T X_n^*(X_{n1}^*)^{-1}]^T, \quad \text{while} \quad \sum_{k=1}^{p+1} w_{k,\alpha} = 1$$

where $X_{n1}^*$ is the submatrix of $X_n^*$ with the rows $x_{11}^*, \ldots, x_{p+1}^*$.

Proof. The regression quantile $\hat{\beta}_n^*(\alpha)$ is a step function of $\alpha \in (0, 1)$. If $\alpha$ is a continuity point of the regression quantile trajectory, then we have the following identity, proven in [22]:

$$\hat{B}_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} x_i^T \hat{\beta}_n^*(\alpha) = -\frac{1}{n} \sum_{i=1}^{n} Y_i \hat{a}_{mi}(\alpha)$$

where $\hat{a}_{mi}(\alpha) = \frac{\partial}{\partial \alpha} \hat{a}_{mi}(\alpha)$. Moreover, (2.1) implies

$$\sum_{i=1}^{n} \hat{a}_{mi}(\alpha) = -n$$

and

$$\sum_{i=1}^{n} x_{ij} \hat{a}_{mi}(\alpha) = -\sum_{i=1}^{n} x_{ij}, \quad 1 \leq j \leq p,$$

Notice that $\hat{a}_{mi}(\alpha) \neq 0$ iff $\alpha$ is the point of continuity of $\hat{\beta}_n^*(\cdot)$ and $Y_i = x_i^T \hat{\beta}_n^*(\alpha)$.

To every fixed continuity point $\alpha$ correspond exactly $p + 1$ such components, such that the corresponding $x_i^*$ belongs to the optimal base of program (2.1). Hence there exist coefficients $w_{k,\alpha}, k = 1, \ldots, p + 1$ such that

$$\hat{B}_n(\alpha) = -\frac{1}{n} \sum_{i=1}^{n} Y_i \hat{a}_{mi}(\alpha) = \sum_{k=1}^{p+1} w_{k,\alpha} Y_{ik}.$$

The equalities $Y_i = x_i^T \hat{\beta}_n^*(\alpha)$ hold just for $p + 1$ components of the optimal base $x_{11}^*, \ldots, x_{p+1}^*$. Let $X_{n1}^*$ be the submatrix of $X_n^*$ with the rows $x_{11}^*, \ldots, x_{p+1}^*$ and

$$\sum_{k=1}^{p+1} w_{k,\alpha} = 1.$$
Let us now consider $\tilde{B}_n(\alpha)$ as a process in $\alpha \in (0, 1)$. Assume that all model errors $e_{ni}$, $i = 1, \ldots, n$ are independent and equally distributed according to joint continuous increasing distribution function $F$. We are interested in the \textit{the average regression quantile process}

$$\bar{B}_n(\alpha) = \left\{ n^{1/2} \hat{\mathbf{x}}^\top \left( \hat{\beta}_n(\alpha) - \beta(\alpha) \right) : 0 < \alpha < 1 \right\}$$

where $\hat{\beta}(\alpha) = (F^{-1}(\alpha) + \beta_0, \beta_1, \ldots, \beta_p)^\top$ is the population counterpart of the regression quantile. As proven in [20], the process $\tilde{B}_n$ converges to a Gaussian process in the Skorokhod topology as $n \to \infty$, under mild conditions on $F$ and $\mathbf{X}_n$. More precisely,

$$W^* \Rightarrow (f(F^{-1}))^{-1} W^* \text{ as } n \to \infty$$

where $W^*$ is the Brownian bridge on (0,1).

However, we are rather interested in behavior of the process $\bar{B}_n$ under a finite number of observations. The trajectories of $\bar{B}_n$ are step functions, nondecreasing in $\alpha \in (0, 1)$, and they have finite numbers of discontinuities for each $n$. As shown in [7], if $\bar{B}_n(\alpha_1) = \bar{B}_n(\alpha_2)$ for $0 < \alpha_1 < \alpha_2 < 1$, then $\alpha_2 - \alpha_1 \leq \frac{p}{n+1}$ with probability 1, then the length of interval, on which is $\bar{B}_n(\alpha)$ constant, tends to 0 for $n \to \infty$ and fixed $p$. Let $0 < \alpha_1 < \ldots < \alpha_{J_n} < 1$ be the breakpoints of $\bar{B}_n(\alpha)$, $0 < \alpha < 1$, and $-\infty < Z_1 < \ldots < Z_{J_n+1} < \infty$ be the corresponding values of $\bar{B}_n(\alpha)$ between the breakpoints. Then we can consider the inversion $\hat{F}_n(z)$ of $\bar{B}_n(\alpha)$, namely

$$\hat{F}_n(z) = \inf\{ \alpha : \bar{B}_n(\alpha) \geq z \}, -\infty < z < \infty.$$

It is a bounded nondecreasing step function and, given $Y_1, \ldots, Y_n$ satisfying (1.2), $\hat{F}_n$ is a distribution function of a random variable attaining values $Z_1, \ldots, Z_{J_n+1}$ with probabilities equal to the spacings of $0, \alpha_1, \ldots, \alpha_{J_n}, 1$. The tightness of the empirical process $\hat{F}_n$ and its convergence to $F$ was studied in [33] under some specific conditions; $\hat{F}_n$ is recommended as an estimate of $F$, which would enable e.g. goodness-of-fit testing about $F$ in the presence of a nuisance regression.

3. Properties of the Averaged Two-Step Regression Quantile $\tilde{B}_n(\alpha)$

While the advantage of $\bar{B}_n(\alpha)$ is in its monotonicity, the inference based on the process $\bar{B}_n(\alpha)$ can be more comprehensible. Hence, we can consider the empirical process $\bar{B}_n(\alpha)$ based on two-step regression quantiles $\hat{\beta}_n(\alpha)$ as an alternative to $\hat{B}_n(\alpha)$. Both processes are asymptotically equivalent as $n \to \infty$.

The two-step regression $\alpha$-quantile treats the slope components $\beta$ and the intercept $\beta_0$ separately. The slope component part is an R-estimate $\hat{\beta}_{nR}$ of $\beta$. Its advantage is that it is invariant to the shift in location, hence independent of $\beta_0$. It starts with selection of a nondecreasing function $\varphi(u)$, $u \in (0,1)$, square-integrable on (0,1). Then we can consider two types of rank scores, generated by $\varphi$:

$$\mathcal{A}_n(i) = E\{ \varphi(U_{n,i}) \}, \ i = 1, \ldots, n$$

where $U_{n,1} \leq \ldots \leq U_{n,n}$ is the ordered random sample of size $n$ from the uniform (0,1) distribution.
Approximate scores:
(3.2)  
\[ A_n(i) = \begin{cases} 
\text{either} & 1 \ f_{i/n} \varphi(u)du, \\
\text{or} & \varphi\left(\frac{r}{n}\right), \quad i = 1, \ldots, n.
\end{cases} \]

The test criteria and estimates based on either of these scores are asymptotically equivalent as \( n \to \infty \); but the rank tests based on the exact scores are locally most powerful against pertinent alternatives under finite \( n \). The R-estimator \( \tilde{\beta}_{nR} \) of the slopes is a minimizer of the \( n \) measure of rank dispersion \( D_n(b) \):
(3.3)  
\[ \tilde{\beta}_{nR} = \arg\min_{b \in \mathbb{R}^p} D_n(b), \]

where
\[ D_n(b) = \sum_{i=1}^{n} (Y_i - x_i^T b) \ A_n(R_{ni}(Y_i - x_i^T b)), \quad b \in \mathbb{R}^p \]

and where \( A_n(\cdot) \) can be replaced with \( A(\cdot) \). Here \( R_{ni}(Y_i - x_i^T b) \) is the rank of the \( i \)-th residual, \( i = 1, \ldots, n \). The intercept component \( \tilde{\beta}_0(\alpha) \) pertaining to the two-step regression \( \alpha \)-quantile is defined as the \( \lfloor \alpha \rfloor \)-order statistic of the residuals \( Y_i - x_i^T \tilde{\beta}_{nR}, \ i = 1, \ldots, n \). The two-step \( \alpha \)-regression quantile is then the vector
(3.4)  
\[ \tilde{\beta}_n^*(\alpha) = \left( \begin{array}{c}
\tilde{\beta}_0(\alpha) \\
\tilde{\beta}_{nR}^*
\end{array} \right) \in \mathbb{R}^{p+1}. \]

The typical choice of \( \varphi \) is the following:
(3.5)  
\[ \varphi_\lambda(u) = \lambda - I[u < \lambda], \quad 0 < u < 1, \quad 0 < \lambda < 1 \]

combined with the approximate scores (ii) in (3.2). These scores were originated in [14]; he used the following scores [now known as Hájek’s rank scores]:
(3.6)  
\[ a_i(\lambda, b) = \begin{cases} 
0 & \ldots & R_{ni}(Y_i) < n\lambda \\
R_{ni}(Y_i) - n\lambda & \ldots & n\lambda \leq R_{ni}(Y_i) < n\lambda + 1 \\
1 & \ldots & n\lambda + 1 \leq R_{ni}(Y_i).
\end{cases} \]

The solutions of (3.3) are generally not uniquely determined. We can e.g. take the center of gravity of the set of all solutions; however, the asymptotic representations and distributions apply to any solution.

Define the averaged two-step regression \( \alpha \)-quantile \( \tilde{B}_n(\alpha) \) as
(3.7)  
\[ \tilde{B}_n(\alpha) = \tilde{x}_n^T \tilde{\beta}_n^*(\alpha). \]

By (3.3),
(3.8)  
\[ \tilde{B}_n(\alpha) = \left( Y_i - (x_i - \tilde{x}_n)^T \tilde{\beta}_{nR}^* \right)_{n: \lfloor \alpha \rfloor}, \]

hence it is equal to the \( \lfloor \alpha \rfloor \)-th order statistic of the residuals \( Y_i - (x_i - \tilde{x}_n)^T \tilde{\beta}_{nR}, \ i = 1, \ldots, n \). Then \( \tilde{B}_n(\alpha) \) is obviously scale equivariant and regression equivariant. [21] originally considered the two-step regression \( \alpha \)-quantile with \( \lambda = \alpha \) in (3.5) for each \( \alpha \in (0, 1) \) in the R-estimator of the slopes. The averaged two-step version corresponding to this choice is very close to \( \tilde{B}_n(\alpha) \), but for finite \( n \) it is generally not monotone in \( \alpha \). However, it suffices to consider \( \lambda \in (0, 1) \) fixed, independent of \( \alpha \); this makes \( \tilde{B}_n(\alpha) \) monotone in \( \alpha \), thus invertible and simpler. \( \tilde{B}_n(\alpha) \) is asymptotic equivalent to \( \tilde{B}_n(\alpha) \) under general conditions, hence also asymptotically equivalent to \( c_{n: \lfloor \alpha \rfloor} + \beta_0 + \tilde{x}_n^T \beta \). Hence \( \tilde{B}_n(\alpha) \) is a convenient tool for an inference under a
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nuisance regression. The asymptotic equivalence of $\tilde{B}_n(\alpha)$ and $\tilde{B}_n(\alpha)$ will be proven under the following mild conditions on $F$ and on $X_n = [x_{n1}, \ldots, x_{nn}]^\top$:

(A1): *Smoothness of $F$*: The errors $e_{ni}, i = 1, \ldots, n$ are independent and identically distributed. Their distribution function $F$ has an absolutely continuous density and positive and finite Fisher’s information:

$$0 < \mathcal{I}(f) = \int \left( \frac{f'(z)}{f(z)} \right)^2 dF < \infty.$$  

(A2): *Noether’s condition on regressors*:

$$\lim_{n \to \infty} Q_n = Q \quad \text{where} \quad Q_n = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)(x_i - \bar{x}_n)^\top$$

and $Q$ is positively definite $p \times p$ matrix; moreover,

$$\lim_{n \to \infty} \max_{1 \leq i \leq n} n^{-1}(x_i - \bar{x}_n)^\top Q_n^{-1}(x_i - \bar{x}_n) = 0.$$  

(A3): *Rate of regressors*:

$$\max_{1 \leq i \leq n} \|x_{ni} - \bar{x}_n\| = o(n^{1/4}) \quad \text{as} \quad n \to \infty.$$  

We shall prove the asymptotic equivalence for $R$-estimators based on score function $\varphi$; $0 < \lambda < 1$, because of its simplicity. However, an analogous proof applies to an $R$-estimator generated by any nondecreasing and square-integrable function $\varphi$.

**Theorem 2.** Let $\tilde{B}_n(\alpha) = \left(Y_i - (x_i - \bar{x}_n)^\top \beta_{nR}\right)_{n:[\alpha]}$ be two-step averaged $\alpha$-regression quantile (TARQ) in the model (1.2), with $R$-estimator $\tilde{\beta}_{nR}$ generated by $\varphi$ in (3.2), $\lambda \in (0,1)$ fixed. Then, under the conditions (A1)-(A3),

\begin{align*}
(3.9) & \quad n^{1/2} \left[ \tilde{B}_n(\alpha) - \beta_0 - \bar{x}_n \beta - e_{n:[\alpha]} \right] = o_p(1) \\
(3.10) & \quad n^{1/2} \left[ \tilde{B}_n(\alpha) - \tilde{B}_n(\alpha) \right] = o_p(1)
\end{align*}

as $n \to \infty$, uniformly over $\alpha \in (\varepsilon, 1 - \varepsilon) \subset (0,1)$, $\forall \varepsilon \in (0,\frac{1}{2})$.

**Proof.** Let us write

\begin{align*}
(3.11) \quad Y_i - (x_i - \bar{x}_n)^\top \tilde{\beta}_{nR} & = e_{ni} + \beta_0 + \bar{x}_n^\top \beta - (x_i - \bar{x}_n)^\top (\tilde{\beta}_{nR} - \beta), \quad i = 1, \ldots, n.
\end{align*}

We shall study the $[\alpha]$-quantile of variables

$r_{ni} = e_{ni} - (x_i - \bar{x}_n)^\top (\tilde{\beta}_{nR} - \beta) = Y_i - (x_i - \bar{x}_n)^\top \tilde{\beta}_{nR} - \beta_0 - \bar{x}_n^\top \beta, \quad i = 1, \ldots, n.$

Recall the Bahadur representation of sample $\alpha$-quantile $e_{n:[\alpha]}$ of $e_{n1}, \ldots, e_{nn}$:

\begin{align*}
(3.12) \quad n^{1/2} [e_{n:[\alpha]} - F^{-1}(\alpha)] & = n^{-1/2}[f(F^{-1}(\alpha))]^{-1} \sum_{i=1}^{n} \{\alpha - \mathcal{I}[e_{ni} < F^{-1}(\alpha)]\} + o(1)
\end{align*}
a.s. as \( n \to \infty \). Under conditions (A1)–(A3), the R-estimator \( \tilde{\beta}_{nR} \) admits the following asymptotic representation:

\[
(3.13) \quad n^{\frac{1}{2}}(\tilde{\beta}_{nR} - \beta)
= n^{-\frac{1}{2}}(f(F^{-1}(\lambda))^{-1}Q_n^{-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)(\lambda - I[\epsilon_{ni} < F^{-1}(\lambda)]) + o_p(n^{-1/4}),
\]

hence \( \|n^{\frac{1}{2}}(\tilde{\beta}_{nR} - \beta)\| = O_p(1) \). The details for (3.12) and (3.13) can be found in [32] and [33].

The [\( \lfloor n \rfloor \)] quantile \( \tilde{a}_n(\alpha) \) of \( r_{n1}, \ldots, r_{nn} \) is a solution of the minimization

\[
\tilde{a}_n(\alpha) = \arg \min_{a \in \mathbb{R}^d} \sum_{i=1}^{n} \rho \left( r_{ni} - a \right),
\]

where \( \rho(\alpha) = |\alpha|\{\alpha I[\alpha > 0] + (1-\alpha)I[\alpha < 0]\} \), \( \alpha \in \mathbb{R}^1 \). Denote as \( \psi_\alpha \) the right-hand derivative of \( \rho \), i.e., \( \psi_\alpha(\alpha) = \alpha - I[\alpha < 0] \), \( \alpha \in \mathbb{R} \). Using Lemma A.2 in [39], we can show that

\[
(3.14) \quad n^{-\frac{1}{2}} \sum_{i=1}^{n} \psi(r_{ni} - \tilde{a}_n(\alpha)) \to 0, \quad \text{i.e.}
\]

\[
-\sum_{i=1}^{n} \left( \alpha - I \left[ \epsilon_{ni} - (x_i - \bar{x}_n)^\top (\tilde{\beta}_{nR} - \beta) < \tilde{a}_n(\alpha) \right] \right) \to 0
\]

almost surely as \( n \to \infty \). Notice that \( \sum_{i=1}^{n}(x_i - \bar{x}_n) = 0 \); hence we conclude from [23], Lemma 5.5, that it holds

\[
(3.15) \quad \sup_{\|b\| \leq C} \left\{ n^{-\frac{1}{2}} \left| \sum_{i=1}^{n} I[\epsilon_{ni} - n^{-\frac{1}{2}}(x_i - \bar{x}_n)^\top b < \tilde{a}_n(\alpha)] \right| \right\} = o_p(1) \quad \text{as} \quad n \to \infty,
\]

for every \( C, \ 0 < C < \infty \). Inserting \( b \mapsto n^{\frac{1}{2}}(\tilde{\beta}_{nR} - \beta) = O_p(1) \) into (3.15), we obtain

\[
(3.16) \quad n^{-\frac{1}{2}} \left| \sum_{i=1}^{n} I \left[ \epsilon_{ni} - (x_i - \bar{x}_n)^\top (\tilde{\beta}_{nR} - \beta) < \tilde{a}_n(\alpha)] \right] - I[\epsilon_{ni} < \tilde{a}_n(\alpha)] \right| = o_p(1) \quad \text{as} \quad n \to \infty.
\]

Combining (3.12), (3.14), and (3.16), we conclude that \( n^{1/2}(\tilde{a}_n(\alpha) - e_{\lfloor n \rfloor \alpha}) = o_p(1) \) as \( n \to \infty \), hence

\[
(3.17) \quad n^{1/2} \left[ \tilde{B}_n(\alpha) - \beta_0 - \bar{x}_n^\top \beta - e_{\lfloor n \rfloor \alpha} \right] = o_p(1) \quad \text{as} \quad n \to \infty
\]

what gives (3.9). This together with Theorem 2 in [22] implies (3.10).

**Remark 1.** \( \tilde{\beta}_{nR} \) can be replaced by any \( \sqrt{n} \)-consistent R-estimator of \( \beta \). However, the score function of type \( \varphi_\lambda \) is more convenient for computation and hence more convenient for applications. Various choices of R-estimators are numerically compared in Section 4.
4. Computation and numerical illustrations

The simulation study describes the methods for computation of the proposed estimates and illustrates their properties. For the computation of the averaged regression quantile $\hat{B}_n(\alpha)$, the R package quantreg and its function rq() is used; it makes use of a variant of the simplex algorithm.

Concerning the two-step averaged regression quantile $\tilde{B}_n(\alpha)$, the most difficult is the first step - the computation of the R-estimator $\tilde{\beta}_{n_R}$ of the slopes. In the numerical illustration below the score function $\varphi_{\lambda}(u) = \lambda - I[u < \lambda]$ from (3.5) and the approximate scores (i) from (3.2) are applied. In this case the function rfit() from the R package Rfit could be directly used. For the rfit() function the score function corresponding to (3.5) and (i) of (3.2) has to be defined, i.e. at point $\lambda n \frac{i}{n + 1}$ attaining the value $\lceil \lambda n \rceil - 1 - \lambda(n - 1) = A_n(i)$. The function rfit() uses the minimization routine optim() which is a quasi-Newton optimizer. This method works well for simple linear regression model but is less precise in case of multiple regression. So, it is better to use the fact that when employing the score function (3.5) with $\lambda = \alpha$ and the approximate scores (i) from (3.2) the slope components of the regression $\alpha$-quantile and the two-step regression $\alpha$-quantile coincide, $\hat{\beta}_n(\alpha) = \tilde{\beta}_{n_R}$ for every fixed $\alpha \in (0, 1)$, see [20]. Therefore, the rq() function from the quantreg package is then used to find the exact solution $\tilde{\beta}_{n_R}$.

The averaged regression quantile $\bar{B}_n(\alpha)$ and the two-step averaged regression quantile $\tilde{B}_n(\alpha)$ (and their inversions) can be used as the estimates of the quantile function (and of the distribution function, respectively) of the model errors. The behavior of the proposed estimates is illustrated in the following simulation study.

The regression model

$$Y_{ni} = \beta_0 + x_{ni}^\top \beta + e_{ni}, \quad i = 1, \ldots, n$$

is simulated with the following parameters:

- sample size $n = 25$,
- $\beta_0 = 5$,
- $\beta = (\beta_1, \beta_2) = (-3, 2)$.

The columns of the regression matrix $(x_{11}, \ldots, x_{n1})^\top$ and $(x_{12}, \ldots, x_{n2})^\top$ are generated as two independent samples from the uniform distributions $U(0, 4)$ and $U(-4, 2)$, respectively, and are standardized so that $\sum_{i=1}^n x_{ij} = 0$, $j = 1, 2$. The errors $e_{ni}$ are generated from the standard normal, the standard Cauchy or the generalized extreme value (GEV) distribution with the shape parameter $k = -0.5$. For each case 10000 replications of the model were simulated and $\bar{B}_n(\alpha)$ and $\tilde{B}_n(\alpha)$ and their inversions were computed. For the two-step version $\tilde{B}_n(\alpha)$ the score-generating function (3.5) with fixed $\lambda = 0.5$ or 0.9 was used. For a comparison, the empirical quantile function of the errors $e_{ni}$ and its inversion were calculated as well. Empirical quantile estimates based on $\bar{B}_n$ and on $\tilde{B}_n$ were then calculated and plotted. Since the figures showing estimates of the true quantile functions and of the true distribution functions look very similar, up to the inversion, only the figures for the distribution functions are presented. The statistical software R was used for all calculations.

The Figures 1 - 3 show the empirical quantile estimates of the normal, Cauchy and GEV distribution functions. The approximation of the distribution functions appears to be very good. We notice that in the case of two-step regression quantile
Estimates based on $B_{\lambda} = 0.5$
Values of the distribution $f$.

Estimates based on $B_{\lambda} = 0.9$
Values of the distribution $f$.

Estimates based on $\tilde{B}$
Values of the distribution $f$.

Estimates based on EQF
Values of the distribution $f$.

\[ \tilde{B}_n(\alpha) \text{ with } \lambda \text{ fixed, the quality of the estimate is sensitive to the choice of } \lambda, \text{ especially for skewed or heavy-tailed distributions. The choice around } \lambda = 0.5 \text{ is generally recommended.} \]

5. Conclusion

The averaged regression quantile $\tilde{B}_n(\alpha)$ and its two-step modification $\tilde{B}_n(\alpha)$, $0 < \alpha < 1$ appear to be very convenient tools in the analysis of various functionals of the risk in the situation that the this depends on some exogenous variables, in the intensity that is not fully under our control. The choice of $\alpha \in (0, 1)$ provides a balance between the concerns about underestimating and overestimating the losses in the situation. The increasing $\alpha \to 1$ reflects a greater concern about underestimating the loss, comparing to overestimating. Both $B_n(\alpha)$ and $\tilde{B}_n(\alpha)$ can be advantageously used in estimating the expected shortfall (ES) and other modern measures of the risk, as well as estimating and testing other characteristics of the market or the everyday practice, based on the functionals of the quantiles.
Figure 2. Empirical quantile estimates of cauchy distribution function based on $\bar{B}$, $\bar{B}(0.5)$, $\bar{B}(0.9)$ and empirical quantile function (EQF) of errors.

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Estimates based on $B_{\lambda} = 0.5$

Estimates based on $B_{\lambda} = 0.9$

Estimates based on $B$

Estimates based on EQF

Figure 3. Empirical quantile estimates of GEV ($k = -0.5$) distribution function based on $B$, $B(0.5)$, $B(0.9)$ and empirical quantile function (EQF) of errors.

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