Nonlinear interplay of Alfvén instabilities and energetic particles in tokamaks

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Abstract

The confinement of energetic particles (EPs) is crucial in the efficient heating of tokamak plasmas. Plasma instabilities such as Alfvén eigenmodes (AEs) can redistribute the EP population, making the plasma heating less effective and leading to additional loads on the walls. The nonlinear dynamics of toroidicity induced AEs (TAEs) is investigated by means of the global gyrokinetic particle-in-cell code ORB5, within the NEMORB project. The nonperturbative nonlinear interplay of TAEs and EPs due to the wave–particle nonlinearity is studied. In particular, we focus on the linear modification of the frequency, growth rate and radial structure of the TAE, caused by the nonlinear evolution of the EP distribution function. For the ITPA benchmark case, we find that the frequency increases when the growth rate decreases, and the mode shrinks radially. The theoretical interpretation is given in terms of a nonperturbative nonlinear evolution of the AE in relation to the Alfvén continuum.

Keywords: energetic particles, Alfvén instabilities, gyrokinetics, particle-in-cell

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the main heating mechanisms for tokamak plasmas is the injection of beams of energetic particles (EPs), whose task is the thermalization and transfer of energy to the bulk plasma. The process of thermalization is not immediate, and during the time that the EPs circulate inside the tokamak, they can cause plasma waves to be unstable via resonant interaction. Shear Alfvén waves (SAWs) are among the most unstable because the characteristic Alfvén velocity \( v_A = B/\sqrt{4\pi \rho} \) (\( B \) is the equilibrium magnetic field and \( \rho \) the mass density of the plasma) is comparable with that of the EPs. In addition, SAW group velocity is directed along the magnetic field line and, therefore, EPs can stay in resonance and effectively exchange energy with the wave [1, 2].

SAWs in a nonuniform equilibrium experience continuum damping [3, 4] because of the formation of singular structures where the SAW continuum is resonantly excited. Two types of shear Alfvén modes exist in tokamak plasmas: energetic-particle continuum modes (EPMs) [5, 6], with frequency determined by the EP characteristic frequencies, and discrete Alfvén eigenmodes (AEs), with frequencies inside SAW continuum gaps [7]. EPMs can become unstable if the drive exceeds a threshold determined by the continuum damping absorption; AEs, on the other hand, are practically unaffected by continuum damping [1–4], and therefore are generally more unstable. A unified approach for weakly and strongly driven AEs and EPMs has been derived recently, based on a generalized fishbone-like dispersion relation (GFLDR), which helps in extracting the underlying physics of the numerical simulations [8].

The class of AEs of interest in this paper is named toroidicity-induced AEs (TAEs), whose frequency lies inside the continuum gap formed by the toroidal curvature [7, 9, 10]. The linear and nonlinear dynamics of TAEs have been the task of several numerical benchmarks, like the International Tokamak Physics Activity (ITPA), comparing the results of several gyrokinetic (GK), gyrofluid, and hybrid GK-MHD codes [11]. Significant progress in the understanding of the wave–particle nonlinear interaction of TAEs has recently been achieved by means of Hamiltonian-mapping techniques [12].

The numerical tool adopted for the studies presented here is the nonlinear gyrokinetic particle-in-cell (PIC) code ORB5 [13], which now includes all extensions (including the
electromagnetic and multi-species extensions) of the NEMORB project [14–17]. The GK model contains the treatment of the resonances of ions and electrons, which is neglected by fluid models. The Lagrangian formulation that is used is based on the GK Vlasov–Maxwell equations of Sugama, Brizard and Hahm [18, 19]. The conservation theorems for the energy and momentum in the GK framework [20] are automatically satisfied due to the method of derivation of the GK Vlasov–Maxwell equations from a discretized Lagrangian [15]. As a consequence, this model can be adopted in principle for rigorous nonlinear electromagnetic simulations of global instabilities in the presence of EPs and turbulence, where all nonlinearities are treated on the same footing in a self-consistent way.

The linear dynamics of the ITPA-TAE case has been investigated with ORB5 and described in [21] (as was also done, for example, with the linear GK code GYGGLES in [22]). In particular, the radial structure of the mode has been found to depend not only on the interaction with the EP population, but on the position of the frequency with respect to the gap in the continuum. In the linear regime, the dependence of the AE mode on the position of the frequency, growth rate and radial structure is investigated here.

The paper is organized as follows. Section 2 is devoted to the description of the model. The equilibrium is given in section 3. The linear dynamics depending on the position of the frequency in the continuum spectrum is recalled in section 4. The nonlinear modification of the frequency, growth rate and radial structure is investigated in section 5. Finally, a summary of the results is given in section 6.

2. The model

The gyrokinetic Lagrangian (see [15] and references therein) is the starting point for the derivation of the model equations of ORB5:

$$\mathcal{H}_2 = \frac{e^2}{2mc^2} (J_0 A_{||})^2 - \frac{mc^2}{2B^2} |\nabla \phi|^2$$  \hspace{1cm} (4)

Here $f$ and $f_{eq}$ are the total and equilibrium (i.e. independent of time) distribution functions, the integrals are over the phase space volume, with $dV$ being the real-space infinitesimal and $dW = (2\pi/m^2)B_{||} d\mu_1 d\mu_2$ the velocity-space infinitesimal, and $\alpha$ is the gyroangle. The phase space coordinates are $Z = (R, p_{||}, \mu)$, i.e. respectively the gyrocenter position, canonical parallel momentum $p_1 = mU + (e/c)J_0 A_\parallel$, and magnetic momentum $\mu = mv_\parallel^2/(2B)$. $U$ is the parallel velocity: $U = \partial \mathcal{H}/\partial p_\parallel$. Dotted variables are meant to be subjected to time derivative. The Jacobian $B_{||}^*$ is given by the parallel component of $B^* = B + (c/e)\mu_\parallel \nabla \times b$, where $B$ and $b$ are the equilibrium magnetic field and magnetic unit vector. The time-dependent fields are named $\phi$ and $A_\parallel$, and they are respectively the perturbed scalar potential and the parallel component of the perturbed vector potential. In our notation, on the other hand, $A$ is the equilibrium vector potential. The summation is over all species present in the plasma, and the gyroaverage operator is labeled here by $J_0$. The gyroaverage operator reduces to the Bessel function if we transform into Fourier space. No finite Larmor radius (FLR) effects are retained in the simulations shown in this paper, consistent with [21]. This means that the Bessel functions are calculated for the vanishing argument for all species: $J_0 = 1$. FLR has been shown to have a stabilizing effect on TAEs (see, for example, [11]). The investigation of the FLR effects with ORB5 will be done in a separate paper.

The model equations of ORB5 are the gyrocenter trajectories, and the two equations for the fields [15]. They are derived by imposing the minimal action principle with respect to the phase space coordinates $(R, p_{||}, \mu)$, and to $\phi$ and $A_\parallel$.

The gyrocenter trajectories are:

$$\frac{\text{d} R}{\text{d} t} = \frac{1}{m} \left( \frac{e}{c} A_\parallel + p_{||} b \right) + \frac{mc}{e} \mu \alpha$$

$$\frac{\text{d} p_{||}}{\text{d} t} = -B_{||}^* \cdot \left[ \mu \nabla B + e \nabla J_0 \left( \phi - \frac{p_1}{mc} A_\parallel \right) \right]$$ \hspace{1cm} (5)

The GK Poisson equation is:

$$- \nabla \cdot \frac{mc^2}{B^2} \int \frac{dW_{eq}}{d\phi} \nabla \phi = \sum_{sp} \int \frac{dW_e}{d\phi} J_0 f$$ \hspace{1cm} (7)

The Ampère equation is:

$$\sum_{sp} \int \frac{dW_e}{d\phi} \left( \frac{e}{mc} f_0 - \frac{e^2}{m^2 c^2} A_{||} \right) + \frac{1}{4\pi} \nabla^2 A_\parallel = 0$$ \hspace{1cm} (8)

Here the form $J_0 = 1$ is given for simplicity, also in view of the comparison with MHD codes. For more complicated models, see [15, 16]. In the simulations shown in this paper, a Fourier filter in mode numbers is applied, keeping only $n = 6$ and $m = 9, 10, 11, 12$. Dirichlet boundary conditions are imposed on the potentials. Lost particles at the external boundary are reinserted in order to satisfy the conservation of mass.
The equilibrium of the International Tokamak Physics Activity (ITPA) TAE case [11] is considered in this paper, as in [21, 22]. The major radius and minor radius are $R = 10$ m and $a = 1$ m. The toroidal magnetic field is $B_0 = 3$ T, and the safety factor is $q(r) = 1.71 + 0.16(r/a)^2$. The bulk ion is chosen to be hydrogen. The ion and electron average densities are $n_i = n_e = 2 \cdot 10^{20}$ m$^{-3}$. In the case of a nonuniform profile of an additional species (for example, of the EP), the bulk ion and electron profiles are corrected in order to satisfy quasi-neutrality, as described below. The bulk ion and electron temperature is $T_i = T_e = 1$ keV. The corresponding bulk ion cyclotron frequency is $\Omega_i = 2.87 \cdot 10^8$ rad s$^{-1}$, the thermal ion velocity is $v_{th,i} = \sqrt{T_i/m_i} = 3.095 \cdot 10^3$ m s$^{-1}$ and the sound Larmor radius is $\rho_i = \sqrt{T_i/m_i}/\Omega_i = 1.078 \cdot 10^{-3}$ m.

The electron beta on axis is $\beta_e = 8\pi n_e T_e/B_0^2 = 8.955 \cdot 10^{-4}$ and the Alfvén velocity on axis is $v_A = 1.46 \cdot 10^7$ m s$^{-1}$.

Given these parameters, we can calculate the continuous spectrum near the toroidicity induced gap of the TAE with toroidal mode number $n = 6$ (see figure 1). The $X$-point where the two cylinder continuum branches with $m = 10$, 11 cross (neglecting toroidicity and compressibility) is located at $\omega_{\text{UCAP}} = 0.5 \, v_A/qR = 4.17 \cdot 10^8$ rad s$^{-1} = 1.454 \cdot 10^{-3} \Omega_i$ (where $q = 1.75$). The continuum accumulation points can be calculated using the approximated formula given in [21] (derived after [7, 9]), which is valid for small values of the inverse aspect ratio, and slightly underestimates the gap width. We obtain that the lower continuum accumulation point (LCAP) calculated with compressibility effects, is located at $\omega_{\text{LCAP}} \simeq 4.4 \cdot 10^8$ rad s$^{-1}$ = 0.492 $v_A/qR$ and the upper continuum accumulation point (UCAP) at $\omega_{\text{UCAP}} \simeq 4.4 \cdot 10^5$ rad s$^{-1}$ = 0.53 $v_A/qR$. The upshift of the frequency due to compressibility has been estimated by using the beta-induced Alfvén eigenmode CAP calculated in [31]: $\omega(T) = (\omega_{\gamma=0}^2 + \omega_{\text{BAE-CAP}}^2)^{1/2}$, with $\omega_{\text{BAE-CAP}} = 8 \cdot 10^5$ rad s$^{-1}$ = 0.1 $v_A/qR$. In the proximity of the TAE frequency, this gives an upshift of about $0.9 \cdot 10^8$ rad s$^{-1}$ = 0.01 $v_A/qR$ (neglected in the calculation of the CAPs in [22]).

A Maxwellian distribution function in velocity-space is considered for a population of deuterium, playing the role of the EP species. The EP averaged concentration of a reference case is $n_{\text{EP}}/n_e = 0.004$ with radial profile given by:

$$n_{\text{EP}}(s)/n_{\text{EP}}(s_0) = \exp\left[\Delta \kappa_u \tanh((s - s_0)/\Delta)\right]$$

with $s_0 = 0.5$, $\Delta = 0.2$, and $\kappa_u = 3.333$. The radial coordinate here is $s = \sqrt{\psi/\psi_{\text{edge}}} \simeq r/a$, with $\psi$ being the poloidal magnetic equilibrium flux. When the EP ion species is added, the total number of electrons in the volume is kept the same, and the bulk ion average concentration is diminished in order for the total number of ions (bulk plus energetic) to equal the total number of electrons. The bulk ions profile is kept flat, as in the absence of
EP. The electron profile is shaped (see figure 1) in the simulations used in this paper, in order to satisfy $n_e(r) = \sum_i Z_i n_i(r)$ (whereas the electron profile was kept flat in [21]). The reference EP temperature in this paper is $T_{EP} = 400$ keV (whereas most of the TAE simulations in [21] used $T_{EP} = 500$ keV). Simulations with $m_i/m_e = 200$ are considered in this paper (where the ion mass is realistic and the electron mass is chosen to be higher than the realistic one). Convergence tests are given in appendix A.

4. Linear dynamics

In this section, we investigate numerically the dynamics of linear simulations, namely simulations where all three species follow unperturbed trajectories. When an initial perturbation is allowed to evolve in time in the absence of EP, a TAE is observed as a discrete global mode with frequency lying within the gap. The measured frequency is:

$$\omega = \nu_b/\sqrt{qR}.$$ 

4.1. Dependence on the EP concentration

Two temperatures of the EP populations have been considered: $T_{EP}/T_e = 200$ and $T_{EP}/T_e = 400$. In the former case, with increasing EP concentration, we observe a decreasing value of the frequency, going below the LCAP. On the contrary, in the latter case, the frequency slowly increases with EP concentration. Modes with frequency inside the continuum gap created by toroidicity, are practically unaffected by continuum.
damping. On the other hand, modes with frequency belonging to the continuous spectrum are affected by continuum damping and need a stronger drive to become unstable. The former take the name of TAEs, whereas the latter are named EPMs [5, 6].

Both cases show a clearly linearly increasing value of growth rate with the EP concentration. The damping is measured as $\gamma (n_{\text{EP}} \to 0) \sim -10^{-3} v_A / q R$, and is consistent with the analytical estimations of the electron Landau damping given in [9, 10]. The electron Landau damping is estimated to be the dominant damping mechanism in this regime (see also [26]).

When considering modes with frequency inside the toroidicity induced gap (like, for example, for $T_{\text{EP}} / T_e = 400$), the structure of the TAE in the poloidal plane does not show any ‘boomerang’ (a.k.a. ‘croissant’) shape, because of the weak interaction with the continuum, consistent with what was shown in [21]. When increasing the EP concentration, we observe that the radial width increases (see figure 3) as the region where the drive overcomes the damping increases. Both modes with $T_{\text{EP}} / T_e = 200$ and $T_{\text{EP}} / T_e = 400$ show a poloidal Fourier decomposition dominated by $m = 10$ and $m = 11$. Scalings similar to those obtained in [21], namely without imposing the quasineutrality at the initial time, are shown in appendix B.

### 4.2. Dependence on the EP temperature

By considering $n_{\text{EP}} / n_e = 0.0031$ and increasing the EP temperature, we note that at low temperatures the frequency

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**Figure 4.** Frequency (left) and growth rate (right) dependence on EP temperature. The frequencies of the lower and upper CAPs with compressibility effects are also plotted as magenta and green dashed lines. The analytical predictions for the growth rate are also shown.

**Figure 5.** Structure of the scalar potential $\phi$ in the poloidal plane, for $T_{\text{EP}} / T_e = 200$ (left) and $T_{\text{EP}} / T_e = 600$ (right). Here $n_{\text{EP}} / n_e = 0.0031$, and $\kappa_n = 3.333$. 
decreases, reaches the minimum of $\omega \simeq 0.477 \nu_R / qR$ at $T_{\text{EP}}/T_e > 200$ (consistent with [22]), then grows linearly and enters the continuum gap, and the growth rate increases, overcoming the instability threshold at $T_{\text{EP}}/T_e \simeq 50$, and then reaches a maximum value of $\gamma \simeq 0.06 \nu_R / qR$ at $T_{\text{EP}}/T_e \simeq 600$ (consistent with [22], see figure 4). A detailed investigation into the resonances in the linear and nonlinear phase is outside the scope of this paper, and the implementation of specific diagnostics in phase space in ORB5 is in progress.

At low EP temperatures, the structure in the poloidal plane has a characteristic boomerang shape which reflects the fact that the frequency is well below the LCAP (see [21]). On the other hand, for higher temperatures, the frequency is in the gap and the boomerang shape disappears (see figure 5).

4.3. Dependence on the EP density gradient

With increasing EP density gradient, the frequency and the growth rate are observed to increase linearly (see figure 6). The linear dependence of the frequency and growth rate on the EP density gradient is very similar to the dependence on the EP density (consistent with [9]).

Regarding the mode structure, when keeping the local concentration of EP constant and increasing the EP density...
In this section, we consider a typical simulation with nonlinear wave–particle interaction. In a PIC code, this means that the bulk ions and electrons follow unperturbed trajectories, whereas the EP follow perturbed trajectories. We focus here on the nonlinear frequency and structure modification. The investigation of the dependence of the saturation levels over \( n_{\text{EP}} \) is discussed in [26], where the results of the GK PIC codes ORB5 and EUTERPE are shown.

5. Nonlinear dynamics and quasilinear analysis

The evolution of the mode amplitude of a TAE and of the EP profile is described here. The EP concentration is \( n_{\text{EP}}/n_e = 0.004 \) and the EP temperature is \( T_{\text{EP}}/T_e = 400 \). The initial EP normalized gradient at \( s = 0.5 \) is \( n_v = 3.33 \). After a linear phase \( t < 100 qR/\nu_\lambda \) the mode amplitude, measured as the maximum of the scalar potential \( \phi \) in the poloidal plane, is observed to enter a ‘drift-phase’, characterized by a slower subexponential growth \( 100 qR/\nu_\lambda < t < 150 qR/\nu_\lambda \), and then a saturation \( t > 150 qR/\nu_\lambda \) (see figure 8). A comparison with the case with \( T_{\text{EP}}/T_e = 200 \) is shown in appendix C.

The EP profile is observed to start redistributing during the drift phase. Due to the global extension of the considered TAE mode, the EP redistribution takes a large portion of the radial domain. During this quite radially uniform redistribution, no steepened gradients are found at the edge of the TAE radial location. A comparison of the EP redistribution of ORB5 and EUTERPE is shown in [26].

5.1. Nonlinear mode amplitude and EP profile

The frequency of the TAE is observed to go up in values, starting from \( \omega(t = 0) \simeq 0.5 \nu_\lambda/qR \), which is the linear value, and rising in the drift phase up to values of \( \omega(t = 200 qR/\nu_\lambda) \simeq 0.54 \nu_\lambda/qR \). This value of the saturation is above the UCAP calculated according to the approximated formula of [21], which underestimates the gap width. More generally, we can state that this saturated frequency is in the proximity of the UCAP. For comparison, the case with \( T_{\text{EP}}/T_e = 200 \) shows a frequency starting at \( \omega(t = 0) \simeq 0.47 \nu_\lambda/qR \), which is the linear value, and rising in the drift phase up to values of \( \omega \simeq 0.52 \nu_\lambda/qR \). In summary, the frequency is observed to ‘chirp’ upwards for all cases of interest, independent of the EP temperature, and saturate around the upper CAP. This nonlinear frequency modification occurs mainly in the ‘drift phase’.

The growth rate reduces drastically in the drift phase from the linear value of \( \gamma(t = 0) \simeq 0.08 \nu_\lambda/qR \) to zero. Note that the decrease in the growth rate and mode saturation cannot be explained as simply due to the EP radial redistribution. In fact, as shown in figure 8, the EP density gradient is at saturation between \( t \simeq 150qR/\nu_\lambda \) and \( t \simeq 200qR/\nu_\lambda \), around a value of \( -\nu_\nu \simeq 1.8 \), but the linear investigation shown in section 4.3 has proved that for those values of density gradient, the mode is still linearly unstable. Therefore, we deduce that the radial profile redistribution is not the sole reason for the modification of the growth rate. The flattening of the EP distribution function in \( v \)-space along the TAE resonance condition is thought to be the other main reason for saturation, consistent with [12]. This will be investigated in detail in a future work with a specific diagnostics, which is under construction.

5.2. Nonlinear frequency and growth rate

The frequency of the EP is discussed in appendix C. A comparison with the case with \( T_{\text{EP}}/T_e = 200 \) shows a frequency starting at \( \omega(t = 0) \simeq 0.47 \nu_\lambda/qR \), which is the linear value, and rising in the drift phase up to values of \( \omega \simeq 0.52 \nu_\lambda/qR \). In summary, the frequency is observed to ‘chirp’ upwards for all cases of interest, independent of the EP temperature, and saturate around the upper CAP. This nonlinear frequency modification occurs mainly in the ‘drift phase’.

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5.3. Nonlinear mode structure

During the drift phase, the radial width of the mode becomes smaller (see figures 10 and 11). In order to measure the mode...
width of the scalar potential, we take a cut at \( q = 0 \), and at the instant of the transit of the peak of intensity. The radial width is defined by selecting the two radial locations where the scalar potential has half the intensity of the peak. During the saturation phase, no difference is found in the main mode structure, except for small perturbations being observed radially. A comparison with the case with \( T_{\text{EP}} / T_e = 200 \) is shown in appendix C.

Regarding modes whose frequency lies outside the continuum gap (like in the case of \( T_{\text{EP}} / T_e = 200 \)), a different shape of the poloidal section is observed, as described in section 4.2 for the linear phase, with a body and two wings, similar to a ‘boomerang’ [21]. During the drift phase, the radial width of the boomerang body becomes smaller, the wings extension does not change, and the wings tilt. During the saturation phase, the loss of the wings generated by the EP is observed. The difference in the mode structure is accompanied by a qualitatively different mode width scaling with the linear drive, as described in section 5.4. A detailed Fourier decomposition of the mode structure in the poloidal angle is outside the scope of this paper, but it has been described in [26], where it was proved not to have big qualitative modifications during the nonlinear phase. We confirm that the main qualitative modification occurs in the radial direction.

5.4. Dependence of nonlinear mode width on linear drive

The mode width has been found to vary during the drift phase of the NL evolution, and shown in section 5.3 for \( n_{\text{EP}} / n_e = 0.004, T_{\text{EP}} / T_e = 400 \). Here, we quantify the importance of this NL structure modification for varying the intensity of the drive. The linear mode width has been qualitatively shown to increase with EP concentration (see section 4.1) and the quantitative comparison of the linear and NL mode width for difference EP concentration is now shown in figure 12. The mode width here is calculated as the radial width where the amplitude of the scalar potential is half the value of the peak. At two different EP temperatures, the scaling is found to be different. In particular, for \( T_{\text{EP}} / T_e = 200 \), the nonlinear shrinking is observed to scale linearly with the growth rate, whereas for \( T_{\text{EP}} / T_e = 400 \), a fit with higher power law is found. The difference in the power law between the case at \( T_{\text{EP}} / T_e = 200 \) and the case at \( T_{\text{EP}} / T_e = 400 \) comes from the different nature of the two modes: the mode with \( T_{\text{EP}} / T_e = 200 \) has frequency outside the TAE gap in the continuum (for all considered values of drives), and it has therefore the nature of an EPM, whereas the mode with \( T_{\text{EP}} / T_e = 400 \) has frequency well within the TAE gap (for all considered values of drives) and it has therefore the nature of a pure TAE (see figure 2). The coefficients of the
two scalings are estimated as:

$$w_{\text{LIN}}/w_{\text{NL}}(T_{\text{EP}}/T_e = 200) \approx 1 + 7.5 \cdot (\gamma/\omega)$$

(10)

for the case where $T_{\text{EP}}/T_e = 200$, and

$$w_{\text{LIN}}/w_{\text{NL}}(T_{\text{EP}}/T_e = 400) \approx 1 + 240 \cdot (\gamma/\omega)^3$$

(11)

for the case where $T_{\text{EP}}/T_e = 400$. These two scaling laws are shown in figure 12. According to these power laws, 10% of nonlinear modification is found for $\gamma/\omega \approx 0.01$ for the case where $T_{\text{EP}}/T_e = 200$, and for $\gamma/\omega \approx 0.07$ for the case with $T_{\text{EP}}/T_e = 400$.

5.5. Quasilinear analysis

The quasilinear analysis described here consists of investigating the linear effect of the nonlinearly modified EP distribution function by running nonlinear simulations with small initial perturbations, and loading an instantaneous snapshot of the nonlinearly modified EP distribution function taken from the simulation depicted in sections 5.1, 5.2, 5.3. In this way, the linear phase can be investigated at the beginning of the nonlinear simulation.

The measured frequencies and growth rates in the quasilinear simulations are found to approximate well the frequency and growth rate evolution of the nonlinear simulation (see figure 9). This confirms that the EP distribution function loaded at a particular instant of the nonlinear simulation contains all information for determining the TAE dynamics, as in the nonlinear simulations.

The mode structure diagnosed in the quasilinear simulations shows a good match with the mode structures
In summary, the nonlinear dynamics of the TAE considered in this paper, and described in sections 5.1, 5.2, 5.3, has been reproduced with quasilinear simulations, namely simulations where the linear effects of the nonlinearly modified EP distribution function is considered. This is consistent with the fact that wave–particle nonlinearity is only acting in the nonlinear simulations described in sections 5.1, 5.2, 5.3.

5.6. Theoretical interpretation

The nonlinear dynamics of the reference TAE shown in sections 5.1, 5.2, 5.3 has been described in terms of a frequency upshift and a radial shrinking occurring during the early nonlinear phase dubbed here as drift phase. The nonlinear dynamics is in general perturbative if the following inequality is satisfied [32]:

$$|\Delta \omega_{\text{EP}}| \ll |\Delta \omega_{\text{SAW}}|$$

where $\Delta \omega_{\text{EP}}$ is the nonlinear frequency modification and $\Delta \omega_{\text{SAW}}$ is the distance of the linear mode frequency with respect to the reference CAP. For this reference case, the inequality is not satisfied, because the two terms are of the same order of magnitude (see figure 9). This confirms that, for the selected case, the EP effects are highly nonperturbative. A strong nonlinear modification of the structure, is therefore a direct consequence.

A more detailed investigation of the nonlinear evolution of the frequency can also give information about the kind of nonlinear structure modification that should be expected. In fact, the frequency is observed to approach a CAP during the early nonlinear phase (see figure 9). If one drops the approximated formula calculated in [21] for formulas considering the effect of the finite inverse aspect ratio (like, for example, the one used in [22]), but still keeping the upshift due to the finite plasma compressibility, one can see that the frequency of the UCAP is slightly higher than the UCAP depicted in figure 9, and therefore closer to the saturated value of the nonlinear simulation. This means that, by approaching the continuum, the mode which is linearly a global AE, tends nonlinearly to a singular continuum mode [3, 4]. By ‘singular’ continuum mode here, we mean a mode with a structure presenting a radial logarithmic singularity, which can be straightforwardly calculated in a plasma with a sheared equilibrium magnetic field in slab geometry (see [3, 4]). This explains why the mode is observed to shrink radially during the early nonlinear phase. The interaction with the continuum is qualitatively different for EPMs with respect to AEs [2, 5, 6], and this yields a lower order polynomial dependence of the nonlinear frequency on the linear drive (see figure 12).

6. Conclusions

In this paper, an investigation into the nonlinear dynamics of the toroidicity-induced Alfvén eigenmode (TAE), in the regime of the International Tokamak Physics Activity (ITPA) [11], has been presented. The focus of this nonlinear study has been put on the wave–particle nonlinearity. The gyrokinetic particle-in-cell code ORB5, previously verified for TAEs in the linear regime [21], has been used for this nonlinear investigation. The relevance of such a study is linked to the importance of understanding the nonlinear interaction of global instabilities and energetic particles (EPs) in current tokamaks and future fusion reactors.

The nonlinear analysis of the mode amplitude has shown that a TAE evolves with an initial linear phase, then a sub-exponential drift phase, and then a saturation of the mode amplitude (see also [26]). In this paper, we have investigated in particular the nonlinear modification of the frequency, growth rate and mode structure. The frequency has been shown here to increase in time during the drift phase. The growth rate decreases during the drift phase and goes to zero when the mode saturates. The radial mode width also decreases in time. The reference TAE case has been found to fall in the highly nonperturbative regime. The radial shrinking has been explained in terms of the mode frequency approaching the continuum during the nonlinear phase. In particular, the radial shrinking results as the eigenfunction gets more singular, as the frequency approaches the upper continuum accumulation point. By measuring the evolution in time of the EP radial profile, we have observed that the density gradient becomes smaller in time.

A quasilinear analysis has been done, in order to shed light on the mechanism forming the basis of the nonlinear dynamics. With this aim, we have performed nonlinear simulations with a small initial field amplitude, and with the EP distribution function loaded from particular instants of the nonlinear simulation taken as a reference. The nonlinear dynamics has been correctly reproduced with this quasilinear
model. This confirms that the instantaneous dynamics in the nonlinear simulation is the linear effect of the nonlinearly modified EP distribution function, where the details in phase space are crucial.

The scaling of the nonlinear structure modification with the intensity of the linear drive has also been described, for two particular types of mode: modes inside the continuum gap, properly labelled as TAEs, and modes outside the continuum gap, more properly referred to as energetic-particle modes (EPMs). This has been achieved by considering the same equilibrium and initial mode, but driving it with two EP populations with different temperatures (namely $T_{EP}/T_e = 200$ for EPMs and $T_{EP}/T_e = 400$ for TAEs), in order to have different resonance frequency. The nonlinear radial shrinking of the EPMs has been found to scale linearly with the linear drive, whereas a higher-power polynomial function is necessary to describe the nonlinear shrinking of modes in the gap. A nonlinear radial shrink of 10% is found for linear growth rates above 7% for modes with $T_{EP}/T_e = 400$, and 1% for modes with $T_{EP}/T_e = 200$. One possible application of this result is helping to define the regime of applicability of models which aims to study the nonlinear wave–particle interaction of Alfvén modes with a prescribed, fixed mode structure (see for example [33, 34]). This would be crucial for both modes with characteristics of pure AEs, and of mixed characteristics of AEs/ EPMs. A dedicated investigation of this regime of applicability, with particular interest in realistic tokamak scenarios, will be done in a dedicated paper.

The implementation of diagnostics, in the sense of computational tools, such as those for the evolution of the EP distribution function in phase space, or for the energy exchange of the mode and the particles (like in [12]), is in progress. Such diagnostics will be used as a next step, to further understand the nonlinear evolution of the EP population in the cases described in this paper. Therefore, in this paper the emphasis has been on creating a robust basis of numerical results obtained with a recently developed gyrokinetic code, which serve as a fertile soil for a deeper theoretical understanding, to be reached in the near future with the help of such diagnostics.

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Appendix A. Linear convergence tests

When decreasing the electron mass from $m_e/m_i = 1/50$ to $m_e/m_i = 1/2000$, we observe that in the linear phase the frequency does not sensibly change within the error bar, and the growth rate converges above $m_i/m_e = 100$ (see figure A1). The same is found for the initial nonlinear phase (the ‘drift phase’), which is of interest in this paper. The deep nonlinear phase is outside the scope of this paper, and is studied elsewhere (see, for example, [26]). For this reason, in this paper, simulations with $m_i/m_e = 200$ are considered, which are numerically less demanding than simulations with realistic mass ratios.

Figure A1. Frequency (left) and growth rate (right) for $n_{EP}/n_e = 0.0031$, $T_{EP}/T_e = 400$, $\kappa_n = 3.333$. The frequency of the lower and upper CAPs is also shown, as dashed horizontal lines.
Appendix B. Linear scalings with flat i/e initial profiles

In order to bridge a gap with previous work [21], we show here the scalings of frequency and growth rates, when the equilibrium profiles of the bulk ions and electrons are initialized as flat. The quasineutrality here is not imposed at $t = 0$, but it is achieved automatically by the code which redistributes the profiles in the first time steps. In this case, with increasing EP concentration, we observe a linearly decreasing value of frequency (entering the continuum below the LCAP) and a linearly increasing value of growth rate. This is a regime closer to that considered in [9], whose analytical prediction of the growth rate gives a good match with the scaling obtained numerically (see figure B1, where the theoretical damping is neglected).

Due to the low frequency, with respect to the LCAP, the structure of the TAE shows an evident ‘boomerang’ [21] (or ‘croissant’) shape. When increasing the EP concentration, we observe that both the radial width of the body and the wings extension increase (see figure B2).

Appendix C. Nonlinear evolution of a TAE with $T_{\text{EP}}/T_e = 200$

The nonlinear evolution of a TAE with $T_{\text{EP}}/T_e = 200$ is depicted here for comparison with the case with $T_{\text{EP}}/T_e = 400,$
Figure C1. Maximum of the scalar potential measured in the poloidal plane (left) and EP density gradient at $s = 0.5$ (right), for a nonlinear simulation.

Figure C2. Zoom of the scalar potential near $s = 0.5$, $\theta = 0$, depicted here at four different times, for a nonlinear simulation with $n_{EP}/n_e = 0.004$, $T_{EP}/T_e = 200$. 
described in section 5. The evolution of the maximum value of the scalar potential measured in the poloidal plane at each instant, is shown in figure C1. In comparison with figure 8, one can note the longer times needed to reach saturation, and a lower saturation level. A detailed analysis of the saturation level has been made with ORBS and EUTERPE and described in [26].

In figure C2, the evolution of the mode structure near \(s = 0.5, \theta = 0\) is shown. One can see that the radial structure in the linear phase presents the characteristic boomerang shape typical of modes with frequency outside the TAE gap of the continuum. A first nonlinear modification is observed to be shrinking in the radial direction, which is followed by a loss of the ‘wings’ created by the interaction with the EP distribution. At longer times, no qualitative modification is observed.

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