THE TRACIAL HAHN-BANACH THEOREM, POLAR DUALS, MATRIX CONVEX SETS, AND PROJECTIONS OF FREE SPECTRAHEDRA

J. WILLIAM HELTON\textsuperscript{1}, IGOR KLEP\textsuperscript{2}, AND SCOTT MCCULLOUGH\textsuperscript{3}

Abstract. This article investigates matrix convex sets. It introduces tracial analogs, which we call contractively tracial convex sets. Critical in both contexts are completely positive (cp) maps. While unital cp maps tie into matrix convex sets, trace preserving cp (CPTP) maps tie into contractively tracial sets. CPTP maps are sometimes called quantum channels and are central to quantum information theory.

Free convexity is intimately connected with Linear Matrix Inequalities (LMIs) \( L(x) = A_0 + A_1 x_1 + \cdots + A_g x_g \succeq 0 \) and their matrix convex solution sets \( \{ X : L(X) \succeq 0 \} \), called free spectrahedra. The classical Effros-Winkler theorem states that matrix convex sets are solution sets of LMIs with operator coefficients. Indeed, this is the Hahn-Banach theorem for matrix convex sets. We give the analogous theorem for contractively tracial sets.

The projection of a free spectrahedron in \( g + h \) variables to \( g \) variables is a matrix convex set called a free spectrahedrop. As a class, free spectrahedrops are more general than free spectrahedra, but at the same time more tractable than general matrix convex sets. Moreover, many matrix convex sets can be approximated from above by free spectrahedrops. We give a number of results on free spectrahedrops and their polar duals. For example, the free polar dual of a free spectrahedrop is a free spectrahedrop. We also give a Positivstellensatz for free polynomials which are positive on a free spectrahedrop.

1. Introduction

This article investigates matrix convex sets from the perspective of the emerging areas of free real algebraic geometry and free analysis [Voi04, Voi10, KVV+, MS11, Pop10, AM+, BB07, dOHMP09, HKM13b, PNA10]. It also introduces tracial analogs, which we...
call contractively tracial convex sets, of matrix convex sets appropriate for the quantum channel and quantum operation interpolation problems. Matrix convex sets arise naturally in a number of contexts, including engineering systems theory, operator spaces, systems and algebras and are inextricably linked to unital completely positive (ucp) maps [SIG97, Arv72, Pau02, Far00, HMPV09]. On the other hand, completely positive trace preserving (CPTP) maps are central to quantum information theory [NC10, JKPP11]. Hence there is an inherent similarity between matrix convex sets and structures naturally occurring in quantum information theory.

Given positive integers $d$ and $n$, let $\mathbb{S}_n^g$ denote the set of $g$-tuples $X = (X_1, \ldots, X_g)$ of real $n \times n$ symmetric matrices and let $\mathbb{S}^g$ denote the sequence $(\mathbb{S}_n^g)$. A subset $\Gamma \subseteq \mathbb{S}^g$ is a sequence $\Gamma = (\Gamma(n))_n$ such that $\Gamma(n) \subseteq \mathbb{S}_n^g$. A matrix convex set is a subset $\Gamma \subseteq \mathbb{S}^g$ which is closed with respect to direct sums and (simultaneous) conjugation by isometries. Closed under direct sums means if $X \in \Gamma(n)$ and $Y \in \Gamma(m)$, then

$$X \oplus Y := \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \in \Gamma(n + m).$$

Likewise, closed under conjugation by isometries means if $X \in \Gamma(n)$ and $V$ is an $n \times m$ isometry, then

$$V^*XV := (V^*X_1V, \ldots, V^*X_gV) \in \Gamma(m).$$

The simplest examples of matrix convex sets arise as solution sets of linear matrix inequalities (LMIs). The use of LMIs is a major advance in systems engineering in the past two decades [SIG97]. Furthermore, LMIs underlie the theory of semidefinite programming, itself a recent major innovation in convex optimization [Nem06].

Matrix convex sets determined by an LMI are based on a free analog of an affine linear functional, often called a linear pencil, which we now describe. Given a positive integer $d$ and symmetric $d \times d$ matrices, $A_j$, let

$$L(x) = A_0 + \sum_{j=1}^g A_j x_j.$$  

In the case that $A_0 = I_d$, the linear pencil $L$ is monic. Replacing $x \in \mathbb{R}^g$ with a tuple $X = (X_1, \ldots, X_g)$ of $n \times n$ symmetric matrices and letting $W \otimes Z$ denote the Kronecker product of matrices leads to the evaluation of the free affine linear functional,

(1.1) $$L(X) = A_0 \otimes I_n + \sum A_j \otimes X_j.$$  

The inequality $L(X) \succeq 0$ is a free linear matrix inequality (free LMI). The solution set $\Gamma$ of this free LMI is the sequence of sets

$$\Gamma(n) = \{ X \in \mathbb{S}_n^g : L(X) \succeq 0 \}$$
and is known as a **free spectrahedron** (or **free LMI domain**). It is easy to see that $\Gamma$ is a matrix convex set.

By the Effros-Winkler matricial Hahn-Banach Separation Theorem [EW97], (up to a technical hypothesis) every matrix convex set is the solution set of $L(X) \succeq 0$, as in Equation (1.1), of some monic linear pencil provided the $A_j$ are allowed to be self-adjoint operators on a (common) Hilbert space. Equivalently, every matrix convex set is a (perhaps infinite) intersection of free spectrahedra. Thus, being a spectrahedron imposes a strict finiteness condition on a matrix convex set.

In between (closed) matrix convex sets and spectrahedra, lie the class of domains we call spectrahedrops. These are coordinate projections of free spectrahedra. In formulas, given a linear pencil,

$$L(x, y) = A_0 + \sum_{j=1}^g A_j x_j + \sum_{k=1}^h B_k y_k,$$

the sequence $\Delta = (\Delta(n))_n$ of sets

$$\Delta(n) = \{ X \in S^g_n : \exists Y \in S^h_n \text{ such that } L(X, Y) \succeq 0 \}$$

is called a **free spectrahedrop**.

In applications, one is presented with a convex set, but would like, for optimization purposes say, to know if it is a spectrahedron or a spectrahedrop. In other cases, one is presented with an algebraically defined set which is not necessarily convex and it is then natural to consider the relaxation obtained by replacing the set with its matrix convex hull or an approximation thereof. Thus, it is of interest to know when the convex hull of a set is a spectrahedron or perhaps a spectrahedrop. An approach to these problems, approximating from above by spectrahedrops, was pursued in the article [HKM+]. Here we develop the duality approach. Typically the second polar dual of a set is its closed matrix convex hull.

### 1.1. Results on Polar Duals and Free Spectrahedrops

Spectrahedrops form a much bigger class of matrix convex sets than free spectrahedra, but at the same time they are more tractable than general matrix convex sets. Moreover, many matrix convex sets can be approximated from above by free spectrahedrops [HKM+].

We list here our main results on free spectrahedrops and polar duals. For the reader unfamiliar with the terminology, the definitions not already introduced can be found in Section 2 with the exception of free polar dual whose definition appears in Subsection 4.2.

1. A perfect free Positivstellensatz (Theorem 5.1) for any symmetric free polynomial $p$ on a free spectrahedron $\Delta$ as in (1.2). It says that $p(X)$ is positive semidefinite for
all $X \in \Delta$ iff $p$ has the form

$$p(x) = f(x)^* f(x) + \sum_{\ell} q_{\ell}(x)^* L(x, y) q_{\ell}(x)$$

where $f$ and $q_{\ell}$ are vectors with polynomial entries. If the degree of $p$ is less than or equal to $2r + 1$, then $f$ and $q_{\ell}$ have degree no greater than $r$;

(2) The free polar dual of a free spectrahedron is a free spectrahedron (Theorem 4.15 and Corollary 4.22);

(3) The matrix convex hull of a union of finitely many bounded free spectrahedrons is a bounded free spectrahedron (Proposition 4.23);

(4) A matrix convex set is, in a canonical sense, generated by a finite set (equivalently a single point) iff it is the polar dual of a free spectrahedron (Theorem 4.9).

1.2. Results on Interpolation of cp Maps and Quantum Channels. A completely positive (cp) map $M_n \to M_m$ which is trace preserving is called a quantum channel, and a cp map which is trace non-increasing for positive semidefinite arguments, is a quantum operation. These maps figure prominently in quantum information theory [NC10].

The cp interpolation problem is formulated as follows. Given $A \in S^n_g$ and $B \in S^m_g$, does

$$B_\ell = \Phi(A_\ell) \quad \text{for} \quad \ell = 1, \ldots, g$$

for some cp map $\Phi : M_n \to M_m$? One can require further that $\Phi$ be unital, a quantum channel or a quantum operation. Imposing either of the latter two constraints pertains to quantum information theory [Ha11, Kle07, NCSB], where one is interested in quantum channels (resp., quantum operations) that send a prescribed set of quantum states into another set of quantum states.

A byproduct of the methods used in this paper and in [HKM13a] are solutions to cp interpolation problems in the form of an algorithm presented as Theorem 3.6 in Subsection 3.2. Ambrozie and Gheondea [AG+] solved these interpolation problems with LMI algorithms. While equivalent to theirs, our solutions are formulated as concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver.

1.3. Free Tracial Hahn-Banach Theorem. Matrix convex sets are closely connected with ranges of unital cp maps. Indeed, given a tuple $A \in S_m^g$, the matrix convex hull of the set \{A\} is the sequence of sets

$$\left( \{ B_j = \Phi(A_j) \text{ for some ucp map } \Phi : M_m \to M_n \} \right)_{n}.$$

From the point of view of quantum information theory it is natural to consider hulls of ranges of quantum operations. We say that $\mathcal{Y} \subseteq S^g$ is contractively tracial if $Y \in \mathcal{Y}(m)$ and if
\( C_\ell \) are \( n \times m \) matrices such that
\[
\sum C_\ell^* C_\ell \preceq I_m.
\]
then \( \sum C_j Y C_j^* \in \mathcal{Y}(n) \). It is clear that intersections of contractively tracial sets are again contractively tracial, giving rise, in the usual way, to the notion of contractive tracial hulls, denoted \( \text{cthull} \). For a tuple \( A \),
\[
\text{cthull}(A) = \{ B : \Phi(A) = B \text{ for some quantum operation } \Phi \}.
\]
While the unital and quantum interpolation problems have very similar formulations, contractive tracial hulls possess far less structure than matrix convex hulls. Indeed, as is easily seen, contractive tracial hulls need not be levelwise convex or closed with respect to direct sums. However they do have a few good properties which we develop in Section 6.

Section 7 contains notions of free spectrahedra and corresponding Hahn-Banach type separation theorems tailored to the tracial setting. To understand convex contractively tracial sets, given \( B \in \mathbb{S}_n^g \), let \( \mathcal{H}_B = (\mathcal{H}_B(m))_m \) denote the sequence of sets
\[
\mathcal{H}_B(m) = \{ Y \in \mathbb{S}_m^g : \exists T \succeq 0, \text{ tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \}.
\]
We call \( \mathcal{H}_B \) a tracial spectrahedron. (Note that \( \mathcal{H}_B \) is not closed under direct sums, and thus it is not a matrix convex set.) These \( \mathcal{H}_B \) are all contractively tracial and levelwise convex. Indeed for such structural reasons, and in view of the tracial Hahn-Banach separation theorem immediately below, we believe these to be the natural analogs of free spectrahedra in the tracial context.

**Theorem 1.1** (cf. Theorem 7.6). If \( \mathcal{Y} \subseteq \mathbb{S}_m^g \) is contractively tracial, levelwise convex and closed, and if \( Z \in \mathbb{S}_m^g \) is not in \( \mathcal{Y}(m) \), then there exists a \( B \in \mathbb{S}_m^g \) such that \( \mathcal{Y} \subseteq \mathcal{H}_B(m) \), but \( Z \notin \mathcal{H}_B(m) \).

Because of the asymmetry between \( B \) and \( Y \) in the definition of \( \mathcal{H}_B \), there is a second type of tracial spectrahedron. Given \( Y \in \mathbb{S}_n^g \), we define the (opp-)tracial spectrahedron as the sequence \( \mathcal{H}_Y^{\text{opp}} = (\mathcal{H}_Y^{\text{opp}}(m))_m \) by
\[
\mathcal{H}_Y^{\text{opp}}(m) = \{ B \in \mathbb{S}_m^g : \exists T \succeq 0, \text{ tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \}.
\]
Proposition 7.12 computes the hulls resulting from the two different double duals determined by the two notions of tracial spectrahedron.


1.4. **Reader's guide.** The paper is organized as follows. Section 2 introduces terminology and notation used throughout the paper. Section 3 solves the cp interpolation problem, and includes a background section on cp maps. Section 4 contains our main results on polar duals, free spectrahedra and free spectrahedrops. We show that a matrix convex set is
finitely generated iff it is the polar dual of a free spectrahedron (Theorem 4.9). Furthermore, we prove that the polar dual of a free spectrahedron is again a free spectrahedron (Theorem 4.15). We conclude this section by reinterpreting the unital cp interpolation problem with the help of free polar duals in Subsection 4.6. Section 5 contains the “perfect” Convex Positivstellensatz for free polynomials positive semidefinite on free spectrahedrons. The proof depends upon the results of Section 4. In Section 6 we introduce tracial sets and hulls and discuss their connections with the quantum interpolation problems. Finally, Section 7 introduces tracial spectrahedra and proves a Hahn-Banach separation theorem in the tracial context, see Theorem 7.6. This theorem is then used to suggest corresponding notions of duality.

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2. Preliminaries

This section introduces terminology and presents preliminaries on free polynomials, free sets and free convexity needed in the sequel.

2.1. Free Sets. Fix a positive integer $g$. For a positive integer $n$, let $\mathbb{S}_n^g$ denote the set of $g$-tuples of real symmetric $n \times n$ matrices and let $\mathbb{S}^g$ denote the sequence $(\mathbb{S}_n^g)_{n \in \mathbb{N}}$. A subset $\Gamma$ of $\mathbb{S}_n^g$ is a sequence $\Gamma = (\Gamma(n))_{n \in \mathbb{N}}$ where $\Gamma(n) \subseteq \mathbb{S}_n^g$ for each $n$. The subset $\Gamma$ is closed with respect to direct sums if $A = (A_1, \ldots, A_g) \in \Gamma(n)$ and $B = (B_1, \ldots, B_g) \in \Gamma(m)$ implies

$$A \oplus B := \left( \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_g & 0 \\ 0 & B_g \end{pmatrix} \right) \in \Gamma(n + m).$$

It is closed with respect to (simultaneous) unitary conjugation if for each $n$, each $A \in \Gamma(n)$ and each $n \times n$ unitary matrix $U$,

$$U^* A U = (U^* A_1 U, \ldots, U^* A_g U) \in \Gamma(n).$$

The set $\Gamma$ is a free set if it is closed with respect to direct sums and simultaneous unitary conjugation. We refer the reader to [Voi04, Voi10, KVV+, MS11, Pop10, AM+, BB07] for a systematic study of free sets and free function theory. We call a free set $\Gamma$ (uniformly) bounded if there is a $C \in \mathbb{R}_{>0}$ such that $C - \sum X_j^2 \geq 0$ for all $X \in \Gamma$.

2.2. Free Polynomials. One natural way that free sets arise is as the nonnegativity set of a free polynomial. Given a positive integer $\ell$ and $\nu$, let $\mathbb{R}^{\ell \times \nu}$ denote the collection of $\ell \times \nu$
matrices. Given freely noncommuting variables \( x = (x_1, \ldots, x_g) \) let \( \langle x \rangle \) denote the words in \( x \). An expression of the form,

\[
P = \sum_{w \in \langle x \rangle} B_w w \in \mathbb{R}^{\ell \times \nu} \langle x \rangle,
\]

where \( B_w \in \mathbb{R}^{\ell \times \nu} \), and the sum is finite, is a free (noncommutative) matrix-valued polynomial. The collection of all \( \ell \times \nu \)-valued free polynomials is denoted \( \mathbb{R}^{\ell \times \nu} \langle x \rangle \) and \( \mathbb{R} \langle x \rangle \) denotes the set of scalar-valued free polynomials. We use \( \mathbb{R}^{\ell \times \nu} \langle x \rangle_k \) to denote free polynomials of degree \( \leq k \). Here the degree of a word is its length.

There is a natural involution \( \ast \) on \( \langle x \rangle \) which reverses the order of a word. This involution extends to \( \mathbb{R}^{\ell \times \nu} \langle x \rangle \) by

\[
P^\ast = \sum_{w} B_w^\ast w^\ast \in \mathbb{R}^{\mu \times \ell} \langle x \rangle.
\]

If \( \mu = \ell \) and \( P^\ast = P \), then \( P \) is symmetric. A free polynomial is evaluated at an \( X \in S_n^g \) by

\[
P(X) = \sum_{w \in \langle x \rangle} B_w \otimes w(X) \in \mathbb{R}^{\ell n \times \mu n},
\]

where \( \otimes \) denotes the (Kronecker) tensor product. Note that if \( P \in \mathbb{R}^{\ell \times \ell} \langle x \rangle \) is symmetric, and \( X \in S_n^g \), then \( P(X) \in \mathbb{R}^{\ell n \times \ell n} \) is a symmetric matrix.

**2.3. Free Semialgebraic Sets.** The nonnegativity set \( \mathcal{D}_P \) of a symmetric free polynomial is the sequence of sets \( \mathcal{D}_P(n) \) where, for a positive integer \( n \),

\[
\mathcal{D}_P(n) = \{ X \in S_n^g : P(X) \succeq 0 \}.
\]

It is readily checked that \( \mathcal{D}_P \) is a free set. By analogy with (commutative) real algebraic geometry, we call \( \mathcal{D}_P \) a basic free semialgebraic set. Often it is assumed that \( P(0) \succ 0 \). The free set \( \mathcal{D}_P \) has the additional property that it is closed with respect to restriction to reducing subspaces; that is, if \( X \in \mathcal{D}_P(n) \) and \( \mathcal{H} \subseteq \mathbb{R}^n \) is an invariant (reducing) subspace for \( X \) of dimension \( m \), then that \( X \) restricted to \( \mathcal{H} \) is in \( \mathcal{D}_P(m) \).

**2.4. Free Convexity.** A set \( \Gamma = (\Gamma(n))_n \subseteq S^g \) is matrix convex or free convex if it is closed under direct sums and (simultaneous) isometric conjugation; i.e., if for each \( m \leq n \), each \( A = (A_1, \ldots, A_g) \in \Gamma(n) \), and each isometry \( V : \mathbb{R}^m \to \mathbb{R}^n \),

\[
V^* AV := (V^* A_1 V, \ldots, V^* A_g V) \in \Gamma(m).
\]

In particular, a matrix convex set is a free set.

In the case that \( \Gamma \) is matrix convex, it is easy to show that \( \Gamma \) is levelwise convex in the sense that each \( \Gamma(n) \) is itself convex. More generally, if \( A^\ell = (A_1^\ell, \ldots, A_g^\ell) \) are in \( \Gamma(n_\ell) \)
for $1 \leq \ell \leq k$, then $A = \bigoplus_{\ell=1}^{k} A^\ell \in \Gamma(n)$, where $n = \sum \ell n_\ell$. Hence, if $V_\ell$ are $n_\ell \times m$ matrices (for some $m$) $\sum_{\ell=1}^{k} V_\ell^* V_\ell = I_m$ so that

$$V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{pmatrix}$$

is an isometry, then

$$V^* A V = \sum_{\ell=1}^{k} V_\ell^* A^\ell V_\ell \in \Gamma(m).$$

A sum as in Equation (2.3) is a **matrix convex combination** of the $g$-tuples $\{A^\ell : \ell = 1, \ldots, k\}$.

**Lemma 2.1** ([HKM+, Lemma 2.3]). Suppose $\Gamma$ is a free subset of $S^g$.

(1) If $\Gamma$ is closed with respect to restriction to reducing subspaces, then the following are equivalent:
   (i) $\Gamma$ is matrix convex;
   (ii) each $\Gamma(n)$ is convex in the classical sense of taking scalar convex combinations.

(2) If $\Gamma$ is (nonempty and) matrix convex, then $0 \in \Gamma(1)$ iff $\Gamma$ is **closed with respect to (simultaneous) conjugation by contractions**.

### 2.5. Linear Pencils and Free Convex Sets.

Classical convex sets in $\mathbb{R}^g$ are defined as intersections of half-spaces and are thus described by linear functionals. Analogously, matrix convex sets are defined by linear pencils; cf. [EW97, HM12]. This section presents basic facts about linear pencils and associated matrix convex sets.

#### 2.5.1. Linear Pencils.

Given $k \times k$ matrices $A = (A_0, \ldots, A_g) \in S^{g+1}_k$, let

$$L_A(x) = A_0 + \sum_{j=1}^{g} A_j x_j \in S_k \langle x \rangle$$

denote the corresponding **(affine) linear pencil** of size $k$. In the case that $A_0 = 0$; i.e., $A = (A_1, \ldots, A_g) \in S^g_k$, let

$$A_A(x) = \sum_{j=1}^{g} A_j x_j$$

denote the corresponding **homogeneous (truly) linear pencil** and

$$\mathfrak{L}_A = I - A_A$$
the associated **monic linear pencil**.

A linear pencil \( L \) can of course be evaluated at a point \( x \in \mathbb{R}^g \) in the obvious way, producing the Linear Matrix Inequality (LMI), \( L(x) \succeq 0 \). The solution set \( \mathcal{D}_L(1) \) to this inequality is known as a **spectrahedron** or **LMI domain** and is obviously a convex semialgebraic set.

### 2.5.2. Free Spectrahedra.

The pencil \( L_A \) is a free polynomial with matrix coefficients, so is naturally evaluated on \( X \in S_g^m \) using (Kronecker’s) tensor product

\[
L_A(X) := A_0 \otimes I + \sum_{j=1}^g A_j \otimes X_j.
\]

The free semialgebraic set \( \mathcal{D}_{L_A} \) is easily seen to be matrix convex. We will refer to \( \mathcal{D}_{L_A} \) as a **free spectrahedron** or **free LMI domain** and say that a free set \( \Gamma \) is **freely LMI representable** if there is a linear pencil \( L \) such that \( \Gamma = \mathcal{D}_L \). In particular, if \( \Gamma \) is freely LMI representable with a monic \( \mathcal{L}_A \), then 0 is in the interior of \( \Gamma(1) \).

The following is a special case and variant (see [HM12, §6]) of a theorem due to Effros and Winkler [EW97]. Given a free set \( \Gamma \), if 0 \( \in \Gamma(1) \), then 0 \( \in \Gamma(n) \) for each \( n \). In this case we will write 0 \( \in \Gamma \).

**Theorem 2.2.** If \( \mathcal{C} = (\mathcal{C}(n))_{n \in \mathbb{N}} \subseteq S_g^n \) is a closed matrix convex set containing 0 and \( Y \in S_m^n \) is not in \( \mathcal{C}(m) \), then there is a monic linear pencil \( \mathcal{L} \) of size \( m \) such that \( \mathcal{L}(X) \succeq 0 \) for all \( X \in \mathcal{C} \), but \( \mathcal{L}(Y) \not\succeq 0 \).

By the following result from [HM12], linear matrix inequalities account for matrix convexity of free semialgebraic sets.

**Theorem 2.3.** Fix \( p \) a symmetric matrix polynomial. If \( p(0) \succ 0 \) and the strict positivity set \( \mathcal{P}_p = \{ X : p(X) \succ 0 \} \) of \( p \) is bounded, then \( \mathcal{P}_p \) is matrix convex iff if is freely LMI representable with a monic pencil.

### 2.6. The LMI Domination Problem.

Given two linear pencils \( L_A \) and \( L_B \), we say \( L_A \) dominates \( L_B \) if

\[
\mathcal{D}_{L_B} \subseteq \mathcal{D}_{L_A}.
\]

**Remark 2.4.** An important special case of the commutative (scalar) monic LMI domination problem \( \mathcal{D}_L(1) \subseteq \mathcal{D}_A(1) \) (the so-called matrix cube problem) was studied by Ben-Tal and Nemirovskii [BN02] and shown to be NP hard; see also [Nem06, KTT13]. □

The papers [HKM13a, HKM12] contain computer algorithms which, given \( A \) and \( B \), produce an LMI which is feasible iff domination occurs for monic pencils \( \mathcal{L}_A \) and \( \mathcal{L}_B \); we
refer to [KS13, HKN14] for extensions to non-monic pencils whose spectrahedra might have empty interior. The algorithms are based upon Positivstellensätze for monic linear pencils; that is, algebraic certificates for one LMI to dominate another [HKM13a, HKM12]. The first result is from [HKM13a] and yields a little cleaner statement than the second, but at the expense of a boundedness assumption.

Suppose $L = L_A$ is a monic linear pencil, and the subspace $\mathcal{H} \subseteq \mathbb{R}^k$ is reducing for $L$; i.e., $A_j \mathcal{H} \subseteq \mathcal{H}$ for all $j$. Since each $A_j$ is symmetric, it also follows that $A_j \mathcal{H}^\perp \subseteq \mathcal{H}^\perp$. Hence, with respect to the decomposition $\mathbb{R}^k = \mathcal{H} \oplus \mathcal{H}^\perp$, the pencil $L$ decomposes as

$$L = \tilde{L} \oplus \tilde{L}^\perp = \begin{pmatrix} \tilde{L} & 0 \\ 0 & \tilde{L}^\perp \end{pmatrix}, \quad \text{where} \quad \tilde{L} = I + \sum_{j=1}^g \tilde{A}_j x_j,$$

and $\tilde{A}_j$ is the restriction of $A_j$ to $\mathcal{H}$. (The pencil $\tilde{L}^\perp$ is defined similarly.) If $\mathcal{H}$ has dimension $\ell$, then by identifying $\mathcal{H}$ with $\mathbb{R}^\ell$, the pencil $\tilde{L}$ is a monic linear pencil of size $\ell$. We say that $\tilde{L}$ is a subpencil of $L$. If $\mathcal{D}_{\mathcal{L}'} = \mathcal{D}_{\mathcal{L}}$, then $\mathcal{L}'$ is a defining pencil for $\mathcal{D}_{\mathcal{L}}$. If no proper subpencil of $\mathcal{L}$ is a defining pencil for $\mathcal{D}_{\mathcal{L}}$, then $\mathcal{L}$ is a minimal pencil. Equivalently [HKM13a, §3.3], $\mathcal{L}$ is of minimal size in the sense that no pencil of smaller size defines $\mathcal{D}_{\mathcal{L}}$.

**Theorem 2.5.** Let $\mathcal{L}_A$, $\mathcal{L}_B$ be given monic pencils. If $\mathcal{D}_{\mathcal{L}_B}$ is bounded, then $\mathcal{D}_{\mathcal{L}_B} \subseteq \mathcal{D}_{\mathcal{L}_A}$ iff

$$A = V^*(I_{\mu} \otimes B)V$$

for some isometry $V$ and $\mu \in \mathbb{N}$.

Moreover, if $\mathcal{L}_A$ and $\mathcal{L}_B$ are minimal for the sets $\mathcal{D}_{\mathcal{L}_A}$ and $\mathcal{D}_{\mathcal{L}_B}$ respectively and if $\mathcal{D}_{\mathcal{L}_A} = \mathcal{D}_{\mathcal{L}_B}$, then $A$ and $B$ have the same size and there is a unitary $U$ such that $B = U^* AU$. If $\mathcal{L}_A$, though possibly not $\mathcal{L}_B$, is minimal, then $\mathcal{D}_{\mathcal{L}_A} = \mathcal{D}_{\mathcal{L}_B}$ iff $B = U^* AU \oplus C$ where $\mathcal{D}_{\mathcal{L}_C} \supseteq \mathcal{D}_{\mathcal{L}_A}$.

In case $\mathcal{D}_{\mathcal{L}_A}$ is not bounded, the linear Positivstellensatz 2.5 fails (see Example 2.6) and we must revert to the Convex Positivstellensatz [HKM12], see Theorem 2.7 below.

**Example 2.6.** Let $g = 1$, and consider $\mathcal{L}_A(x) = 1 + x$, $\mathcal{L}_B(x) = 1 + 2x$. Then

$$\mathcal{D}_{\mathcal{L}_B} = \left\{ X : X \succeq -\frac{1}{2} \right\} \subseteq \mathcal{D}_{\mathcal{L}_A} = \left\{ X : X \succeq -1 \right\}.$$

It is clear that (2.9) fails for this pair of linear pencils. In fact this example is representative in the sense that if $\mathcal{L}_B$ is a monic linear pencil and $\mathcal{D}_{\mathcal{L}_B}$ is unbounded, then there is a monic linear pencil $\mathcal{L}_A$ with $\mathcal{D}_{\mathcal{L}_B} \subseteq \mathcal{D}_{\mathcal{L}_A}$ such that the conclusion of Theorem 2.5 fails. □
2.7. A Convex Positivstellensatz. Positivstellensätze are pillars of real algebraic geometry [BCR98]. We next recall the Positivstellensatz for a free polynomial \( p \) to be nonnegative on a free spectrahedron \( D_L \) from [HKM12]. It is “perfect” in the sense that \( p \) is only assumed to be nonnegative on \( D_L \), and we obtain degree bounds on the scale of \( \deg(p)/2 \) for the polynomials involved in the positivity certificate. In Section 5, we will extend this Positivstellensatz to free spectrahedrops (i.e., projections of free spectrahedra), see Theorem 5.1.

**Theorem 2.7.** Suppose \( \mathcal{L} \) is a monic linear pencil. Then a matrix polynomial \( p \) is positive semidefinite on \( D_{\mathcal{L}} \) iff it has a weighted sum of squares representation with optimal degree bounds:

\[
p = s^*s + \sum_{j} \text{finite} f_j^*\mathcal{L}f_j,
\]

where \( s, f_j \) are matrix polynomials of degree no greater than \( \frac{\deg(p)}{2} \).

In particular, if \( \mathcal{L}_A, \mathcal{L}_B \) are monic linear pencils, then \( D_{\mathcal{L}_B} \subseteq D_{\mathcal{L}_A} \) iff there exists a positive integer \( \mu \) and a contraction \( V \) such that

\[
\mu \sum_{j=1}^{\mu} V_j^*BV_j = V^*(I_{\mu} \otimes B)V.
\]

**Proof.** The first statement is [HKM12, Theorem 1.1]. Applying this to the LMI inclusion problem, we see \( D_{\mathcal{L}_B} \subseteq D_{\mathcal{L}_A} \) is equivalent to

\[
\mathcal{L}_A(x) = S^*S + \sum_{j=1}^{\mu} V_j^*\mathcal{L}_B(x)V_j
\]

for some matrices \( S, V_j \); i.e.,

\[
I = S^*S + \sum_{j} V_j^*V_j = S^*S + V^*V
\]

\[
A = \sum_{j=1}^{\mu} V_j^*BV_j = V^*(I_{\mu} \otimes B)V,
\]

where \( V \) is the block column matrix of the \( V_j \). Equation (2.12) simply says that \( V \) is a contraction, and (2.13) is (2.11).

3. Completely Positive Interpolation

In this section we present solutions to three completely positive (cp) interpolation problems. The unital cp interpolation problem comes from trying to understand matrix convex
sets which arise in convex optimization. Namely, it pertains to the basic problem of determining inclusion of finitely generated matrix convex sets: to each unital cp map one can associate an inclusion of free spectrahedra $D_{L^A} \subseteq D_{L^B}$, and vice-versa (cf. [HKM13a]). This interpolation result plays an important role in the proof of the main result on the polar dual of a free spectrahedron, Theorem 4.15, via its appearance in the proof of Proposition 4.19.

On the other hand, the trace preserving (resp., trace non-increasing) cp interpolation problem arises in quantum information theory, where one is interested in quantum channels (resp., quantum operations) that send a prescribed (finite) set of quantum states into another set of quantum states [Ha11, Kle07, NCSB].

One solution to these interpolation problems is an algorithm presented as Theorem 3.6 in Subsection 3.2 which rephrases the interpolation problems as concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. Alternative algorithms, based in part on the results of Section 4, appear in Subsection 4.6. In Section 6 we investigate quantum channels and quantum operations with emphasis on convexity properties of their ranges.

3.1. Completely Positive Maps over the Reals. This subsection collects basic facts about completely positive (cp) maps $\phi : S \to M_d$, where $S$ is a subspace of $M_n$ closed under transpose (see for instance [Pau02]) containing a positive definite matrix. We call such an $S$ a operator system. Because the results in the literature are for complex scalars and here the setting is real scalars, we provide a few details and proofs.

Given a subspace $S$ of $M_n$ closed under transpose, and a linear map $\phi : S \to M_d$ define its ampliation $\phi_\ell : M_\ell(S) \to M_\ell(M_d)$ by applying $\phi$ entrywise,

$$\phi_\ell(S_{j,k}) = \left(\phi(S_{j,k})\right).$$

The map $\phi$ symmetric if $\phi(S^*) = (\phi(S))^*$ and it is completely positive if each $\phi_\ell$ is positive in the sense that if $S \in M_\ell(S)$ is positive semidefinite, then so is $\phi_\ell(S) \in M_\ell(M_d)$. In what follows, often $S$ is a subspace of $S_n$ (and is thus automatically closed under the transpose operation).

The Choi matrix of a mapping $\phi : M_n \to M_d$ is the $n \times n$ block matrix with $d \times d$ matrix entries given by

$$(C_\phi)_{i,j} = (\phi(E_{i,j}))_{i,j}.$$  

On the other hand, a matrix $C = (C_{i,j}) \in M_n(M_d)$ determines a mapping $\phi_C : M_n \to M_d$ by $\phi_C(E_{i,j}) = C_{i,j} \in M_d$. A matrix $C$ is a Choi matrix for $\phi : S \to M_d$, if the mapping $\phi_C$ agrees with $\phi$ on $S$.

Theorem 3.1. For $\phi : M_n \to M_d$, the following are equivalent:
(a) \( \phi \) is completely positive;
(b) the Choi matrix \( C_\phi \) is positive semidefinite.

Suppose \( S \subseteq M_n \) is an operator system. For a symmetric \( \phi : S \rightarrow M_d \), the following are equivalent:

(i) \( \phi \) is completely positive;
(ii) \( \phi_d \) is positive;
(iii) there exists a completely positive mapping \( \Phi : M_n \rightarrow M_d \) extending \( \phi \);
(iv) there is a positive semidefinite Choi matrix for \( \phi \);
(v) there exists \( n \times d \) matrices \( V_1, \ldots, V_{nd} \) such that

\[
\phi(A) = \sum V_j^* AV_j.
\]

Finally, if \( S \) is any subspace of \( M_n \) a mapping \( \phi : S \rightarrow M_d \) has a completely positive extension \( \Phi : M_n \rightarrow M_d \) iff \( \phi \) has a positive semidefinite Choi matrix.

Remark 3.2. In the setting of real scalars, positivity of \( \phi : S \rightarrow M_d \) is not sufficient to imply that \( \phi \) is symmetric. Indeed, a skew symmetric matrix in \( S \) is not in the (real of course) span of the symmetric matrices. By contrast, over the complex numbers, if \( A^* = -A \), then \( iA \) is self-adjoint. For a concrete example, let \( S \) denote those \( 2 \times 2 \) matrices constant on the diagonal. The mapping \( \phi : S \rightarrow M_2 \) defined by \( \phi(I) = I \),

\[
\phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}
\]

is positive, but not symmetric. \( \square \)

Lemma 3.3. The cp mapping \( \phi : M_n \rightarrow M_d \) as in (3.1) is

(a) unital (that is, \( \phi(I_n) = I_d \)) iff
\[
\sum_j V_j^* V_j = I;
\]
(b) trace preserving iff
\[
\sum_j V_j V_j^* = I;
\]
(c) trace non-increasing for positive semidefinite matrices (i.e., \( \text{tr}(\phi(P)) \leq \text{tr}(P) \) for all positive semidefinite \( P \)) iff
\[
\sum_j V_j V_j^* \preceq I.
\]
Proof. We prove (c) and leave items (a) and (b) as an easy exercise for the reader. For $A \in M_n$, we have

$$\text{tr}(\phi(A)) = \sum_j \text{tr}(V_j^*AV_j) = \text{tr}(A \sum_j V_jV_j^*).$$

Hence the trace non-increasing property for $\phi$ is equivalent to

$$\text{tr}(P(I - \sum_j V_jV_j^*)) \geq 0$$

for all positive semidefinite $P$, i.e., $I - \sum_j V_jV_j^* \succeq 0$. \hfill \blacksquare

**Proposition 3.4.** The linear mapping $\phi : M_n \to M_d$ is

(a) unital (that is, $\phi(I_n) = I_d$) iff its Choi matrix $C$ satisfies

$$(3.5) \quad \sum_{j=1}^{n} C_{j,j} = I;$$

(b) trace preserving iff its Choi matrix $C$ satisfies

$$(3.6) \quad (\text{tr}(C_{i,j}))_{i,j=1}^{n} = I_n;$$

(c) trace non-increasing for positive semidefinite matrices (i.e., $\text{tr}(\phi(P)) \leq \text{tr}(P)$ for all positive semidefinite $P$) iff

$$(3.7) \quad (\text{tr}(C_{i,j}))_{i,j} \preceq I_n,$$

where $C$ is the Choi matrix for $\phi$.

**Proof.** Statement (a) follows from

$$\phi(I_n) = \phi\left(\sum_{j=1}^{n} E_{j,j}\right) = \sum_{j=1}^{n} C_{j,j},$$

where $C$ is the Choi matrix for $\phi$. Here $E_{i,j}$ denote the matrix units,

For (b), let $X = \sum_{i,j=1}^{n} \alpha_{i,j} E_{i,j}$. Then

$$\text{tr}(X) = \sum_{i=1}^{n} \alpha_{i,i}$$

$$\text{tr}(\phi(X)) = \sum_{i,j=1}^{n} \alpha_{i,j} \text{tr}(C_{i,j}).$$

Since $\text{tr}(\phi(X)) = \text{tr}(X)$ for all $X$, this linear system yields $\text{tr}(C_{i,j}) = \delta_{i,j}$ for all $i, j$.

Finally, for statement (c), if $\phi$ is trace non-increasing, choosing $X = xx^*$ a rank one matrix, $X = (x_ix_j)$, we find that

$$\sum x_ix_j \text{tr}(C_{i,j}) = \text{tr}(\phi(X)) \leq \text{tr}(X) = \sum x_i^2.$$
Hence $I - (\text{tr}(C_{i,j})) \succeq 0$. Conversely, if $I - (\text{tr}(C_{i,j})) \succeq 0$, then for any positive semidefinite rank one matrix $X$, the computation above shows that $\text{tr}(\phi(X)) \leq \text{tr}(X)$. Finally, use the fact that any positive semidefinite matrix is a sum of rank one positive semidefinite matrices to complete the proof.

**Example 3.5.** Here is an example of a trace preserving cp map $\phi : \mathcal{S} \to M_2$, where $\mathcal{S} \subseteq M_2$ is an operator system, that does not admit an extension to a trace non-increasing cp map $\phi : M_2 \to M_2$. This is in contrast to the classical Arveson extension theorem [Arv69] which says that any ucp map extends to the full algebra.

Let $\mathcal{S} = \text{span}\{I_2, E_{1,2}, E_{2,1}\}$,

$$V = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix},$$

and consider the cp map $\phi : \mathcal{S} \to M_2$,

$$\phi(A) = V^*AV \quad \text{for } A \in \mathcal{S}.$$  

We have

$$\phi(I_2) = V^*V = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \quad \phi(E_{1,2}) = \frac{\sqrt{3}}{2}E_{1,2}, \quad \phi(E_{2,1}) = \frac{\sqrt{3}}{2}E_{2,1},$$

so $\phi$ is trace preserving on $\mathcal{S}$.

Now let us consider a cp extension (still denoted by $\phi$) of $\phi$ to $M_2$. Letting

$$\phi(E_{1,1}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

the Choi matrix for $\phi$ is

$$C' = \begin{pmatrix} a & b & 0 & \frac{\sqrt{3}}{2} \\ b & c & 0 & 0 \\ 0 & 0 & \frac{1}{2} - a & -b \\ \frac{\sqrt{3}}{2} & 0 & -b & \frac{3}{2} - c \end{pmatrix} \succeq 0.$$

Supposing $\phi : M_2 \to M_2$ is trace non-increasing,

$$1 = \text{tr}(E_{1,1}) \geq \text{tr}(\phi(E_{1,1})) = a + c$$

$$1 = \text{tr}(E_{2,2}) \geq \text{tr}(\phi(E_{2,2})) = 2 - a - c,$$

whence $a + c = 1$. Since $C$ is positive semidefinite, the nonnegativity of the diagonal of $C$ now gives us

$$0 \leq a \leq \frac{1}{2}.$$
But then the $2 \times 2$ minor
\[
\begin{pmatrix}
a & \sqrt{3} \\
\frac{\sqrt{3}}{2} & \frac{1}{2} + a
\end{pmatrix}
\]
is not positive semidefinite, a contradiction. \hfill \Box

3.2. Quantum Interpolation Problem: Algorithms. The cp interpolation problem is formulated as follows. Given $A^1 \in S_n^g$ and given $A^2$ in $S_m^g$, does
\begin{equation}
A^2_\ell = \Phi(A^1_\ell) \quad \text{for} \quad \ell = 1, \ldots, g
\end{equation}
for some cp map $\Phi : M_n \to M_m$? One can require further that
\begin{enumerate}
\item $\Phi$ be \textbf{unital}, or
\item $\Phi$ be \textbf{trace preserving}, or
\item $\Phi$ be \textbf{trace non-increasing} in the sense that $\text{tr}(\Phi(P)) \leq \text{tr}(P)$ for positive semidefinite $P$.
\end{enumerate}

A cp map $M_n \to M_m$ which is trace preserving is called a \textbf{quantum channel}, and a cp map which is trace non-increasing is a \textbf{quantum operation} (though many use quantum operation and quantum channel synonymously to mean trace preserving and completely positive). In particular, a quantum channel can be thought of as a quantum operation which always succeeds; whereas a quantum operation $\Phi$ succeeds with probability $\text{tr}(\Phi(P))$. As will be seen in Sections 6 and 7, from the point of view of convexity and convex hulls, quantum operations are natural.

Our solution to these interpolation problems is an algorithm (equivalent to one stated quite differently, earlier in [AG+]). We now present the algorithm as a theorem.

**Theorem 3.6.** Given symmetric matrices: $A^1_\ell \in S_n$ and $A^2_\ell \in S_m$ for $\ell = 1, \ldots, g$. Let $\alpha_{p,q}^\ell$ denote the $(p,q)$ entry of $A^1_\ell$.

There exists a cp map $\Phi : M_n \to M_m$ that solves the interpolation problem
\[\Phi(A^1_\ell) = A^2_\ell, \quad \ell = 1, \ldots, g\]
iff the following feasibility semidefinite programming problem has a solution:
\begin{equation}
(C_{p,q})_{p,q=1}^n := C \succeq 0, \quad \forall \ell = 1, \ldots, g : \sum_{p,q}^n \alpha_{p,q}^\ell C_{p,q} = A^2_\ell,
\end{equation}
for the unknown $mn \times mn$ symmetric matrix $C = (C_{p,q})_{p,q=1}^n$ consisting of $m \times m$ blocks $C_{p,q}$. Furthermore,
(1) the map $\Phi$ is unital iff in addition to (3.9) we have

\[ \sum_{p=1}^{n} C_{p,p} = I_m; \]  

(2) the map $\Phi$ is a quantum channel iff in addition to (3.9) we have

\[ (\text{tr}(C_{p,q}))_{p,q} = I_n. \]  

(3) the map $\Phi$ is a quantum operation iff in addition to (3.9) we have

\[ (\text{tr}(C_{p,q}))_{p,q} \preceq I_n. \]  

In each case the constraints on $C$ are LMIs, and the set of solutions $C$ constitute a bounded spectrahedron.

**Remark 3.7.** In the unital case the obtained spectrahedron is free. Namely, for fixed $A^1 \in S_n^o$, the sequence of solution sets to (3.9) and (3.10) parametrized over $m$ is a free spectrahedron. In the two quantum cases, for each $m$, the solutions $D(m)$ at level $m$ form a spectrahedron, but the sequence $D = (D(m))_m$ is in general not a free spectrahedron since it fails to respect direct sums. □

**Proof.** This interpolation result is a consequence of Theorem 3.1. Let $S$ denote the span of $\{A^1\}$ and $\phi$ the mapping from $S$ to $M_m$ defined by $\phi(A^1) = A^2$. This mapping has a completely positive extension $\Phi : M_n \to M_m$ iff it has a positive semidefinite Choi matrix. The conditions on $C$ evidently are exactly those needed to say that $C$ is a positive semidefinite Choi matrix for $\phi$.

The additional conditions in (3.10) and (3.11) (i.e., $\phi(I_n) = I_m$ and trace preservation) are clearly linear, so produce a spectrahedron in $R^{mn \times mn}$. Both spectrahedra are bounded. Indeed, in each case we have $C_{p,p} \preceq I_m$, so $C \preceq I_{mn}$. Likewise, the additional condition in (3.12) is an LMI constraint, producing a bounded spectrahedron.

We note that cp maps between subspaces of matrix algebras in the absence of positive definite elements were treated in [HKN14, Section 8].

## 4. Free Spectrahedrops and Polar Duals

This section starts by introducing free spectrahedrops as coordinate projections of spectrahedra. We then continue with a review of free polar duals [EW97] and their basic properties before turning to two main results, stated now without technical hypotheses. Firstly, a free convex set is, in a canonical sense, generated by a finite set (equivalently a single point) iff it is the polar dual of a free spectrahedron (Theorem 4.9). Secondly, the polar dual of a free spectrahedron is again a free spectrahedron (Theorem 4.15).
4.1. Projections of Free Spectrahedra: Free Spectrahedrops. Let $L$ be a linear pencil in the variables $(x_1, \ldots, x_g; y_1, \ldots, y_h)$. Thus, for some $d$ and $d \times d$ symmetric matrices $D, \Omega_1, \ldots, \Omega_g, \Gamma_1, \ldots, \Gamma_h$,

\begin{equation}
L(x, y) = D + \sum_{j=1}^{g} \Omega_j x_j + \sum_{\ell=1}^{h} \Gamma_\ell y_\ell.
\end{equation}

The set

$$\text{proj}_x D_L(1) = \{x \in \mathbb{R}^g : \exists y \in \mathbb{R}^h \text{ such that } L(x, y) \succeq 0\}$$

is known as a spectrahedral shadow or a semidefinite programming (SDP) representable set \cite{BPR13} and the representation afforded by $L$ is an SDP representation. SDP representable sets are evidently convex and lie in a middle ground between LMI representable sets and general convex sets. They play an important role in convex optimization \cite{Nem06}. In the case that $S \subseteq \mathbb{R}^g$ is closed semialgebraic and with some mild additional hypothesis, it is proved in \cite{HN10} based upon the Lasserre–Parrilo construction \cite{Las09, Par06} that the convex hull of $S$ is SDP representable.

Given a linear pencil $L$, let $\text{proj}_x D_L = (\text{proj}_x D_L(n))_n$ denote the free set

$$\text{proj}_x D_L(n) = \{X \in S^g_n : \exists Y \in S^h_n \text{ such that } L(X, Y) \succeq 0\}.$$ 

We will call a set of the form $\text{proj}_x D_L$ a free spectrahedron. We call $D_L$ an LMI lift of $\text{proj}_x D_L$. Thus a free spectrahedron is a coordinate projection of a free spectrahedron. Clearly, free spectrahedrops are matrix convex. In particular, they are closed with respect to restrictions to reducing subspaces.

**Lemma 4.1** ([HKM+, §4.1]). If $K = \text{proj}_x D_L$ is a free spectrahedrop containing $0 \in \mathbb{R}^g$ in the interior of $K(1)$, then there exists a monic linear pencil $\mathcal{L}(x, y)$ such that

\begin{equation}
K = \text{proj}_x D_\mathcal{L} = \{X \in S^g_n : \exists Y \in S^h_n : \mathcal{L}(X, Y) \succeq 0\}.
\end{equation}

If, in addition, $D_L$ is bounded, then we may further ensure $D_\mathcal{L}$ is bounded.

It turns out that if the free spectrahedrop $K$ is closed and bounded, and contains 0 in its interior, then there is a monic linear pencil $\mathcal{L}$ such that $D_\mathcal{L}$ is bounded and $K = \text{proj}_x D_\mathcal{L}$. See Theorem 4.15.

**Example 4.2.** Consider

\begin{equation}
p = 1 - x_1^2 - x_2^4.
\end{equation}

In this case $p$ is symmetric with $p(0) = 1 > 0$. 
The free semialgebraic set \( D_p \) is called the real \textbf{bent free TV screen}, or (bent) TV screen for short. While \( D_p(1) \) is convex, it is known that \( D_p \) is not matrix convex, see [DHM07] or [BPR13, Chapter 8]. Indeed, already \( D_p(2) \) is not a convex set.

A non-convex 2-dimensional slice of \( D_p(2) \).

That the set \( D_p(1) \) is a spectrahedral shadow is well known. Indeed, letting

\[
L(x_1, x_2, y) = \begin{pmatrix}
1 & 0 & x_1 \\
0 & 1 & y \\
x_1 & y & 1
\end{pmatrix} \oplus \begin{pmatrix}
1 & x_2 \\
x_2 & y
\end{pmatrix},
\]

it is readily checked that proj\(_x\) \( D_L(1) = D_p(1) \). Further, Lemma 4.1 implies that \( L \) can be replaced by a monic linear pencil \( L \). An explicit construction of such an \( L \) can be found in [HKM+, §7.1]. We remark that proj\(_x\) \( D_L \) strictly contains the matrix convex hull of \( D_p \), cf. [HKM+, §7.1].
4.2. Basics of Polar Duals. By precise analogy with the classical $\mathbb{R}^g$ notion, the free polar dual $K^\circ = (K^\circ(n))_n$ of a free set $K \subseteq \mathbb{S}^g$ is

$$\mathcal{K}^\circ(n) := \{ A \in \mathbb{S}_n^g : \mathcal{L}_A(X) = I \otimes I - \sum_j^g A_j \otimes X_j \succeq 0 \text{ for all } X \in K \}.$$  

Example 4.3. Given $\varepsilon > 0$, consider the free $\varepsilon$ ball centered at 0,

$$\mathcal{N}_\varepsilon := \{ X \in \mathbb{S}^g : \| X \| \leq \varepsilon \} = \left\{ X : \varepsilon^2 I \succeq \sum_j X_j^2 \right\}.$$ 

Its polar dual is the free $\frac{1}{\varepsilon}$ ball centered at 0,

$$\mathcal{N}^\circ = \mathcal{N}_{\frac{1}{\varepsilon}},$$

and is thus bounded. \hfill \Box

We say that 0 is in the interior of the subset $\Gamma \subseteq \mathbb{S}^g$ if $\Gamma$ contains some free $\varepsilon$ ball centered at 0.

Lemma 4.4. Suppose $K \subseteq \mathbb{S}^g$ is matrix convex. Then the following are equivalent:

(i) $0 \in \mathbb{R}^g$ is in the interior of $\mathcal{K}(1)$;
(ii) $0 \in \mathbb{S}_n^g$ is in the interior of $\mathcal{K}(n)$ for some $n$;
(iii) $0 \in \mathbb{S}_n^g$ is in the interior of $\mathcal{K}(n)$ for all $n$;
(iv) $0$ is in the interior of $\mathcal{K}$.

Proof. It is clear that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). Assume (ii) holds. There is an $\varepsilon > 0$ with $\mathcal{N}_\varepsilon(n) \subseteq \mathcal{K}(n)$. Since $\mathcal{K}$ is closed with respect to restriction to reducing subspaces, and

$$\mathcal{N}_\varepsilon(1) \oplus \cdots \oplus \mathcal{N}_\varepsilon(1) \subseteq \mathcal{N}_\varepsilon(n),$$

we see $\mathcal{N}_\varepsilon(1) \subseteq \mathcal{K}(1)$, i.e., (i) holds.

Now suppose (i) holds, i.e., $\mathcal{N}_\varepsilon(1) \subseteq \mathcal{K}(1)$ for some $\varepsilon > 0$. We claim that $\mathcal{N}_{\varepsilon/g^2} \subseteq \mathcal{K}$. Let $X \in \mathcal{N}_{\varepsilon/g^2}$ be arbitrary. It is clear that

$$\left[ -\frac{\varepsilon}{g} , \frac{\varepsilon}{g} \right]^g \subseteq \mathcal{K}(1),$$

hence $[-\varepsilon/g, \varepsilon/g]^g \otimes I_n \subseteq \mathcal{K}(n)$. Since each $X_j$ has norm $\leq \varepsilon/g^2$, matrix convexity of $\mathcal{K}$ implies that

$$(0, \ldots, 0, gX_j, 0, \ldots, 0) \in \mathcal{K}$$

and thus

$$X = \frac{1}{g} ((gX_1, 0, \ldots, 0) + \cdots + (0, \ldots, 0, gX_g)) \in \mathcal{K}.$$

\hfill \blacksquare
For the readers’ convenience, the following proposition lists some properties of $\mathcal{K}^\circ$. The bipolar result of Item 6 is due to [EW97]. Given $\Gamma_\alpha$, a collection of matrix convex sets, it is readily verified that $\Gamma = (\Gamma(n))_n$ defined by $\Gamma(n) = \bigcap_\alpha \Gamma_\alpha(n)$ is again matrix convex. Likewise, if $\Gamma$ is matrix convex, then so is its closure $\Gamma = (\Gamma(n))_n$. Given a subset $\mathcal{K}$ of $\mathbb{S}^g$, let $\text{co}^{\text{mat}}\mathcal{K}$ denote the intersection of all matrix convex sets containing $\mathcal{K}$. Thus, $\text{co}^{\text{mat}}\mathcal{K}$ is the smallest matrix convex set containing $\mathcal{K}$. Likewise, $\overline{\text{co}}^{\text{mat}}\mathcal{K} = \text{co}^{\text{mat}}\mathcal{K}$ is the smallest closed matrix convex set containing $\mathcal{K}$. Details, and an alternate characterization of the matrix convex hull of a free set $\mathcal{K}$, can be found in [HKM+].

**Proposition 4.5.** Suppose $\mathcal{K} \subseteq \mathbb{S}^g$.

1. $\mathcal{K}^\circ$ is a closed matrix convex set containing 0;
2. if 0 is in the interior of $\mathcal{K}$, then $\mathcal{K}^\circ$ is bounded;
3. $\mathcal{K}(n) \subseteq \mathcal{K}^\circ(n)$ for all $n$; that is, $\mathcal{K} \subseteq \mathcal{K}^\circ$;
4. $\mathcal{K}$ is bounded iff 0 is in the interior of $\mathcal{K}^\circ$;
5. if there is an $m$ such that 0 $\in \mathcal{K}(m)$, then $\mathcal{K}^\circ = \overline{\text{co}}^{\text{mat}}\mathcal{K}$;
6. if $\mathcal{K}$ is a closed matrix convex set containing 0, then $\mathcal{K} = \mathcal{K}^\circ$; and
7. if $\mathcal{K}$ is matrix convex, then $\mathcal{K}(1)^\circ = \mathcal{K}^\circ(1)$.

**Proof.** Matrix convexity in (1) is straightforward.

If $\mathcal{K}$ has 0 in its interior, then there is a small free neighborhood $\mathcal{N}_\varepsilon$ of 0 inside $\mathcal{K}$. Hence $\mathcal{K}^\circ \subseteq \mathcal{N}_\varepsilon = \mathcal{N}_{1/\varepsilon}$ is bounded.

Item (3) is a tautology. Indeed, if $X \in \mathcal{K}(n)$, then we want to show $\mathcal{L}_X(A) \succeq 0$ whenever $\mathcal{L}_A(Y) \succeq 0$ for all $Y$ in $\mathcal{K}$. But this follows simply from the fact that $\mathcal{L}_X(A)$ and $\mathcal{L}_A(X)$ are unitarily equivalent.

If $\mathcal{K}$ is bounded, then it is evident that 0 is in the interior of $\mathcal{K}^\circ$. If 0 is in the interior of $\mathcal{K}^\circ$, then, by item (2), $\mathcal{K}^\circ$ is bounded. By item (3), $\mathcal{K} \subseteq \mathcal{K}^\circ$ and thus $\mathcal{K}$ is bounded.

To prove (5), first note that 0 $\in \overline{\text{co}}^{\text{mat}}\mathcal{K}(m)$ and since $\overline{\text{co}}^{\text{mat}}\mathcal{K}(m)$ is matrix convex, 0 $\in \overline{\text{co}}^{\text{mat}}\mathcal{K}(1)$. Now suppose $W \not\in \overline{\text{co}}^{\text{mat}}\mathcal{K}$. The Effros-Winkler matricial Hahn-Banach Theorem 2.2 produces a monic linear pencil $\mathcal{L}_A$ (with the size of $A$ no larger than the size of $W$) separating $W$ from $\overline{\text{co}}^{\text{mat}}\mathcal{K}$; that is, $\mathcal{L}_A(W) \not\succeq 0$ and $\mathcal{L}_A(X) \succeq 0$ for $X \in \text{co}^{\text{mat}}\mathcal{K}$. Hence $A \in \mathcal{K}^\circ$. Using the unitary equivalence of $\mathcal{L}_W(A)$ and $\mathcal{L}_A(W)$ it follows that $\mathcal{L}_W(A) \not\succeq 0$, and thus $W \not\in \mathcal{K}^\circ$. Thus, $\mathcal{K}^\circ \subseteq \overline{\text{co}}^{\text{mat}}\mathcal{K}$. The reverse inclusion follows from item (3).

Finally, suppose $\mathcal{K}$ is matrix convex and $y \in \mathcal{K}(1)^\circ$. Thus, $\sum y_j x_j = \langle y, x \rangle \leq 1$ for all $x \in \mathcal{K}(1)$. Given $X \in \mathcal{K}(m)$ and a unit vector $v \in \mathbb{R}^m$, since $v^*Xv \in \mathcal{K}(1)$, we have

$$1 \geq \sum y_j v^*X_j v.$$
Hence,
\[ v^*(I - \sum y_j X_j)v \geq 0 \]
for all unit vectors \( v \). So \( y \in K^o(1) \). The reverse inclusion is immediate. ■

**Corollary 4.6.** If \( K \subseteq S^g \), then \( K^{oo} = \mathcal{cmat}(K \cup \{0\}) \). Here \( 0 \in \mathbb{R}^g \).

**Proof.** Note that \( K^o = (K \cup \{0\})^o \) and hence,
\[ K^{oo} = (K \cup \{0\})^{oo} \]
By item (5) of Proposition 4.5,
\[ \mathcal{cmat}(K \cup \{0\}) = (K \cup \{0\})^{oo} \]

**Lemma 4.7.** Suppose \( K \subseteq S^{g+h} \), and consider its image \( \text{proj} K \subseteq S^g \) under the projection \( \text{proj} : S^{g+h} \rightarrow S^g \). A tuple \( A \in S^g \) is in \( (\text{proj} K)^o \) iff \( (A, 0) \in K^o \).

**Proof.** Note that \( A \in (\text{proj} K)^o \) iff for all \( X \in \text{proj} K \) we have \( \mathfrak{L}_A(X) \geq 0 \) iff \( \mathfrak{L}_{(A,0)}(X,Y) \geq 0 \) for all \( X \in \text{proj} K \) and all \( Y \in S^h \) iff \( \mathfrak{L}_{(A,0)}(X,Y) \geq 0 \) for all \( (X,Y) \in K \) iff \( (A, 0) \in K^o \). ■

Now we give an example in a classical commutative situation. We refer the reader to [BPR13] for background on classical convex algebraic geometry.

**Example 4.8.** Let us find the scalar sublevel set \( D^o_p(1) \) of the polar dual of the bent TV screen \( D_p = \{ (X,Y) : 1 - X^2 - Y^4 \geq 0 \} \). We note that \( D^o_p(1) \) coincides with the classical polar dual of \( D_p(1) \); this follows from Proposition 4.5, cf. [HKM+, Example 4.7].

We first find the boundary \( \partial D^o_p(1) \) using Lagrange multipliers. Consider a linear function \( 1 - (c_1 x + c_2 y) \) that is nonnegative but not strictly positive on \( D^o_p(1) \) and its values on (the boundary of) \( D_p(1) \). The KKT conditions are
\[ 1 - x^2 - y^4 = 0, \quad c_1 = 2\lambda x, \quad c_2 = 4\lambda y^3, \quad 1 = c_1 x + c_2 y. \]
Eliminating \( x, y, \lambda \) leads to the following formula relating \( c_1, c_2 \):
\[ q(c_1, c_2) := -16c_1^8 + 48c_1^6 - 48c_1^4c_2^2 + 16c_1^2 - 20c_1^2c_2^4 - c_2^8 + c_2^4 = 0. \]
Thus the boundary of \( D^o_p(1) \) is contained in the zero set of \( q \). Since \( q \) is irreducible, \( \partial D^o_p(1) \) in fact equals the zero set of \( q \). In particular, \( D^o_p(1) = \{ (x, y) \in \mathbb{R}^2 : q(x, y) \geq 0 \} \) is not a spectrahedron, since it fails the line test in [HV07].
4.3. Polar Duals of Free Spectrahedra. The next theorem completely characterizes finitely generated matrix convex sets $\mathcal{K}$ containing 0 in their interior. Namely, such sets are exactly polar duals of bounded free spectrahedra.

**Theorem 4.9.** Suppose $\mathcal{K}$ is a closed matrix convex set with 0 in its interior. If there is an $\Omega \in \mathcal{K}$ such that for each $X \in \mathcal{K}$ there is a $\mu \in \mathbb{N}$ and an isometry $V$ such that

$$X_j = V^*(I_\mu \otimes \Omega_j)V,$$

then

$$\mathcal{K}^\circ = \mathcal{D}_\mathcal{L}_\Omega,$$

where $\mathcal{L}_\Omega$ is the monic linear pencil $\mathcal{L}_\Omega(x) = I - \sum \Omega_jx_j$.

Conversely, if there is an $\Omega$ such that Equation (4.6) holds, then $\Omega \in \mathcal{K}$ and for each $X \in \mathcal{K}$ there is an isometry $V$ such that Equation (4.5) holds.

A variant of Theorem 4.9 in which the condition that 0 is in the interior of $\mathcal{K}$ is replaced by the weaker hypothesis that 0 is merely in $\mathcal{K}$ and of course with a slightly weaker conclusion, is stated as a separate result, Proposition 4.13 below.

**Example 4.10.** Recall the free bent TV screen is the nonnegativity set $\mathcal{D}_p$ for the polynomial $p = 1 - x^2 - y^4$. Let $\mathcal{K}$ denote the closed matrix convex hull of $\mathcal{D}_p$. Then $\mathcal{K}(1) = \mathcal{D}_p(1)$ and hence, by Proposition 4.5 and Example 4.8, $\mathcal{D}_p(1) = \mathcal{D}_p(1)^\circ$ is not a spectrahedron. Hence, $\mathcal{D}_p^\circ$ is not a free spectrahedron. In particular, $\mathcal{K}$ cannot be represented by a single $\Omega$ as in Theorem 4.9.
Lemma 4.11. Suppose $\Omega \in S_0^d$ and consider the monic linear pencil $\mathfrak{L}_\Omega = I - \sum \Omega_j x_j$.

(1) If $\mathcal{D}_{\mathfrak{L}_\Omega}$ is bounded, then $X \in S'$ is in $\mathcal{D}_{\mathfrak{L}_\Omega}$ iff there is an isometry $V$ such that Equation (4.5) holds.

(2) Let $\Omega' = \Omega \oplus 0$ where $0 \in S_0^d$. Then $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$ iff there is an isometry $V$ such that

$$X_j = V^*(I \otimes \Omega'_j)V.$$

Remark 4.12. As an alternate of (2), $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$ iff there exists a contraction $V$ such that Equation (4.5) holds. \hfill \Box

Proof. Note that $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$ iff $\mathcal{D}_{\mathfrak{L}_\Omega} \subseteq \mathcal{D}_{\mathfrak{L}_X}$. Thus if $\mathcal{D}_{\mathfrak{L}_\Omega}$ is bounded, then the result follows from Theorem 2.5. On the other hand, if $X$ has the representation of Equation (4.5), then evidently $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$.

If $\mathcal{D}_{\mathfrak{L}_\Omega}$ is not necessarily bounded and $X \in \mathcal{D}_{\mathfrak{L}_\Omega} (m)$, then, by Theorem 2.7,

$$X = \sum_{j=1}^\mu V_j^* \Omega V_j,$$

for some $\mu$ and operators $V_j : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$I - \sum V_j^* V_j \succeq 0.$$

There is a $\nu > \mu$ and $m \times n$ matrices $V_{\mu+1}, \ldots, V_{\nu}$ such that

$$\sum_{j=1}^\nu V_j^* V_j = I.$$

For $1 \leq j \leq \mu$, let

$$W_j = \begin{pmatrix} V_j \\ 0 \end{pmatrix}$$

and similarly for $\mu < j \leq \nu$, let $W_j = \begin{pmatrix} 0 & V_j^* \end{pmatrix}^*$. With this choice of $W$, note that

$$\sum W_j^* W_j = I_m.$$

If $X$ has the representation as in Equation (4.7) and $\mathfrak{L}_\Omega(Y) \succeq 0$, then

$$\mathfrak{L}_X(Y) = \sum_j (W_j \otimes I)^* \mathfrak{L}_{\Omega'}(Y)(W_j \otimes I).$$

On the other hand,

$$\mathfrak{L}_{\Omega'}(Y) = \mathfrak{L}_\Omega(Y) \oplus I \succeq 0.$$

Hence $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$. \hfill \blacksquare
Proof of Theorem 4.9. Suppose first (4.6) holds for some $\Omega \in \mathbb{S}_n^+$. Since $\mathcal{D}_\Sigma = \mathcal{K}$ and evidently $\Omega \in \mathcal{D}_\Sigma$, it follows that $\Omega \in \mathcal{K}$. Since 0 is assumed to be in the interior of $\mathcal{K}$, its polar dual $\mathcal{K}^\circ = \mathcal{D}_\Sigma$ is bounded by Proposition 4.5. Thus, if $X \in \mathcal{K} = \mathcal{D}_\Sigma$, then by Lemma 4.11, $X$ has a representation as in Equation (4.5).

Conversely, assume that $\Omega \in \mathcal{K}$ has the property that any $X \in \mathcal{K}$ can be represented as in (4.5). Consider the matrix convex set

$$\Gamma = \{ V^*(I_\mu \otimes \Omega)V : \mu \in \mathbb{N}, V^*V = I \}.$$

Since $\Omega \in \mathcal{K}$, it follows that $\Gamma \subseteq \mathcal{K}$. On the other hand, the hypothesis is that $\mathcal{K} \subseteq \Gamma$. Hence $\mathcal{K} = \Gamma$. Now, for $L_X$ a monic linear pencil, $L_X(\Omega) \succeq 0$ iff $L_X(\Omega') \succeq 0$ iff $L_X(\Omega') \succeq 0$. On the other hand, $L_X(\Omega)$ is unitarily equivalent to $L_\Omega(X)$. Thus $X \in \mathcal{K}^\circ$ iff $X \in \mathcal{D}_\Sigma$.

Proposition 4.13. Suppose $\mathcal{K}$ is a closed matrix convex set containing 0. If there is a $\Omega \in \mathcal{K}$ such that for each $X \in \mathcal{K}$ there is a $\mu \in \mathbb{N}$ and an isometry $V$ such that

(4.8) 
$$X_j = V^*(I_\mu \otimes \Omega_j)V,$$

then

(4.9) 
$$\mathcal{K}^\circ = \mathcal{D}_\Sigma,$$

where $\mathcal{L}_\Omega$ is the monic linear pencil $\mathcal{L}_\Omega(x) = I - \sum \Omega_j x_j$. Here $\Omega' = \Omega \oplus 0$ as in Lemma 4.11.

Conversely, if there is an $\Omega$ such that Equation (4.9) holds, then $\Omega \in \mathcal{K}$ and for each $X \in \mathcal{K}$ there is an isometry $V$ such that Equation (4.8) holds.

Proof. Suppose first (4.9) holds for some $\Omega \in \mathbb{S}_n^+$. Since $\mathcal{D}_\Sigma = \mathcal{K}$ and evidently $\Omega \in \mathcal{D}_\Sigma$, it follows that $\Omega \in \mathcal{K}$. By Lemma 4.11, if $X \in \mathcal{K} = \mathcal{D}_\Sigma$, then $X$ has a representation as in Equation (4.8).

Conversely, assume that $\Omega$ has the property that any $X \in \mathcal{K}$ can be represented as in (4.8). Consider the matrix convex set

$$\Gamma = \{ V^*(I_\mu \otimes \Omega')V : \mu \in \mathbb{N}, V^*V = I \}.$$

Since 0, $\Omega \in \mathcal{K}$, it follows that $\Omega' = \Omega \oplus 0 \in \mathcal{K}$ and thus $\Gamma \subseteq \mathcal{K}$. On the other hand, the hypothesis is that $\mathcal{K} \subseteq \Gamma$. Hence $\mathcal{K} = \Gamma$. Now, for $\mathcal{L}_X$ a monic linear pencil, $\mathcal{L}_X(\Omega) \succeq 0$ iff $\mathcal{L}_X(\Omega') \succeq 0$ iff

$$\mathcal{L}_X(\Omega) \succeq 0 \iff \mathcal{L}_X(\Omega') \succeq 0.$$
over all choices of $\mu$ and isometries $V$. Thus, $X \in \mathcal{K}$ iff $\mathcal{L}_X(\Omega) \succeq 0$. On the other hand, $\mathcal{L}_X(\Omega)$ is unitarily equivalent to $\mathcal{L}_\Omega(X)$. Thus $X \in \mathcal{K}$ iff $X \in \mathcal{D}_\varepsilon_0$.

**Remark 4.14.**

(1) For perspective, in the classical (not free) situation when $g = 2$, it is known that $\mathcal{K} \subseteq \mathbb{R}^2$ has an LMI representation iff $\mathcal{K}^\circ$ is a numerical range [Hen10, HS12]. It is well known that the polar dual of a spectrahedron is not necessarily a spectrahedron. This is the case even in $\mathbb{R}^g$, cf. [BPR13, Section 5] or Example 4.8.

(2) In the commutative case the polar dual of a spectrahedron (more generally, of a spectrahedral shadow) is a spectrahedral shadow, see [GN11] or [BPR13, Chapter 5].

(3) It turns out that the $\Omega$ in Theorem 4.9 can be taken to be an extreme point of $\mathcal{K}$ in a very strong free sense. We refer to [Far00, Kls+, WW99] for more on matrix extreme points.

4.4. The Polar Dual of a Free Spectrahedron is a Free Spectrahedron. This subsection contains a duality result for free spectrahedrons (Theorem 4.15) along with a few corollaries.

It can happen that $\mathcal{D}_\varepsilon$ is not a bounded, but the projection $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_\varepsilon$ is. Corollary 4.17 says that a free spectrahedron is closed and bounded if and only if it is the projection of some bounded free spectrahedron. For expository purposes, it is convenient to introduce the following terminology. A free spectrahedron $\mathcal{K}$ is called **stratospherically bounded** if there is a linear pencil $\mathcal{L}$ such that $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_\mathcal{L}$, and $\mathcal{D}_\varepsilon$ is bounded.

**Theorem 4.15.** Suppose $\mathcal{K}$ is a closed matrix convex set containing 0.

(1) If $\mathcal{K}$ is a free spectrahedron and 0 is in the interior of $\mathcal{K}$, then $\mathcal{K}^\circ$ is a stratospherically bounded free spectrahedron.

(2) If $\mathcal{K}^\circ$ is a free spectrahedron containing 0 in its interior, then $\mathcal{K}$ is a stratospherically bounded free spectrahedron.

In particular, if $\mathcal{K}$ is a bounded free spectrahedron with 0 in its interior, then both $\mathcal{K}$ and $\mathcal{K}^\circ$ are stratospherically bounded free spectrahedrons (with 0 in their interiors).

Before presenting the proof of the theorem we state a few corollaries and Proposition 4.19 needed in the proof.

**Corollary 4.16.** Given $\Omega \in \mathbb{S}_d$, let $\mathcal{L}_\Omega$ denote the corresponding monic linear pencil. The free set $\mathcal{D}_\varepsilon_{\mathcal{L}_\Omega}$ is a stratospherically bounded free spectrahedron.

**Proof.** The set $\mathcal{D}_\varepsilon_{\mathcal{L}_\Omega}$ is (trivially) a free spectrahedron with 0 in its interior. Thus, by Theorem 4.15, $\mathcal{D}_\varepsilon_{\mathcal{L}_\Omega}$ is a stratospherically bounded free spectrahedron. \[\square\]
Corollary 4.17. A free spectrahedron $\mathcal{K} \subseteq \mathbb{S}^g$ is closed and bounded iff it is stratospherically bounded.

Proof. Implication ($\Leftarrow$) is obvious. ($\Rightarrow$) Let us first reduce to the case where $\mathcal{K}(1)$ has nonempty interior. If $\mathcal{K}(1)$ has empty interior, then it is contained in a proper affine hyperplane $\{\ell = 0\}$ of $\mathbb{R}^g$. Here $\ell$ is an affine linear functional. In this case we can solve for one of the variables thereby reducing the codimension of $\mathcal{K}(1)$. (Note that $\ell = 0$ on $\mathcal{K}(1)$ implies $\ell = 0$ on $\mathcal{K}$, cf. [HKM+, Lemma 3.3].)

Now let $\hat{x} \in \mathbb{R}^g$ be an interior point of $\mathcal{K}(1)$. Consider the translation

$$ (4.10) \quad \tilde{\mathcal{K}} = \mathcal{K} - \hat{x} = \bigcup_{n \in \mathbb{N}} \{X - \hat{x}I_n : X \in \mathcal{K}(n)\}. $$

Clearly, $\tilde{\mathcal{K}}$ is a bounded free spectrahedron with 0 in its interior. Hence by Theorem 4.15, it is stratospherically bounded. Translating back, we see $\mathcal{K}$ is a stratospherically bounded free spectrahedron. \hfill \blacksquare

Example 4.18. Each stratospherically bounded free spectrahedron is closed, since it is the projection of a (levelwise) compact spectrahedron. Hence a bounded free spectrahedron $\mathcal{K}$ will not be stratospherically bounded if it is not closed. For a concrete example, consider the linear pencil

$$ L(x, y) = \begin{pmatrix} 2 - x & 1 \\ 1 & 2 - y \end{pmatrix} \oplus \begin{pmatrix} 2 + x \end{pmatrix}, $$

and let $\mathcal{K} = \text{proj}_x \mathcal{D}_L$. Then

$$ \mathcal{K} = \{X \in \mathbb{S} : -2 \preceq X \prec 2\} $$

is bounded but not closed. \hfill \square

Proposition 4.19. Given $\Omega \in \mathbb{S}^g_d$ and $\Gamma \in \mathbb{S}^h_d$, the sequence $\mathcal{K} = (\mathcal{K}(n))_n$, $\mathcal{K}(n) = \{A \in \mathbb{S}^g_d : A = V^*(I_\mu \otimes \Omega)V, \ 0 = V^*(I_\mu \otimes \Gamma)V \text{ for some isometry } V \text{ and } \mu \leq nd\}$, is a stratospherically bounded free spectrahedron.

Let $\mathcal{L}(\Omega, \Gamma)$ denote the monic linear pencil corresponding to $(\Omega, \Gamma)$. The free set

$$ \mathcal{C} = \{A : (A, 0) \in \mathcal{D}_{\mathcal{L}(\Omega, \Gamma)}^o\} $$

is a stratospherically bounded free spectrahedron.

Proof. Let $\mathcal{S}$ denote the span of $\{I, \Omega_1, \ldots, \Omega_g, \ \Gamma_1, \ldots, \Gamma_h\}$ Thus $\mathcal{S}$ is an operator system in $M_d$ (the fact that $I \in \mathcal{S}$ implies $\mathcal{S}$ contains a positive definite element). Let

$$ \phi : \mathcal{S} \to M_n $$
denote the mapping determined by
\[ I \mapsto I, \quad \Omega_j \mapsto A_j, \quad \text{and} \quad \Gamma_\ell \mapsto 0. \]
Observe that, by Theorem 3.1, \( A \in \mathcal{K}(n) \) iff \( \phi \) has a completely positive extension \( \Phi : M_d \to M_n \). Theorem 3.6 expresses existence of such a \( \Phi \) as a (unital) cp interpolation problem in terms of a free spectrahedron. For the reader’s convenience we write out this critical LMI explicitly. Let \( \omega_{pq}^j \) denote the \((p, q)\)-entry of \( \Omega_j \) and \( \gamma_{pq}^\ell \) the \((p, q)\)-entry of \( \Gamma_\ell \).

Now \( A \) is in \( \mathcal{K}(n) \) iff there exists \( n \times n \) matrices \( C_{p,q} \) satisfying
\begin{enumerate}[(i)]
\item \( \sum_{p,q=1}^d E_{p,q} \otimes C_{p,q} \succeq 0; \)
\item \( \sum_{p=1}^d C_{p,p} = I_n; \)
\item \( \sum_{p,q=1}^d \omega_{pq}^\ell C_{p,q} = A_\ell \) for \( \ell = 1, \ldots, g \); and
\item \( \sum_{p,q=1}^d \gamma_{pq}^\ell C_{p,q} = 0 \) for \( \ell = 1, \ldots, h. \)
\end{enumerate}
Thus, \( \mathcal{K} \) is the projection of an explicitly constructed free spectrahedron. Moreover, items (i) and (ii) together imply \( 0 \preceq c_{p,p} \preceq I \). It now follows that \( \|c_{p,q}\| \leq 1 \) for all \( p, q \). Thus, this free spectrahedron is bounded. Hence, \( \mathcal{K} \) is a stratospherically bounded free spectrahedron.

Let \( \Omega' = \Omega \oplus 0 \) and \( \Gamma' = \Gamma \oplus 0 \) where \( 0 \in \mathbb{S}^d_+ \). Note that
\[ \mathcal{D}_{\mathcal{L}(\Omega,\Gamma)} = \mathcal{D}_{\mathcal{L}(\Omega',\Gamma')}. \]
By Lemma 4.11, \( (A,0) \in \mathcal{D}_{\mathcal{L}(\Omega,\Gamma)} \) iff
\[ A \in \left\{ B : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ such that } B = V^* (I_\mu \otimes \Omega') V, \ 0 = V^* (I_\mu \otimes \Gamma') V \right\}. \]
By the first part of the proposition (applied to the tuple \((\Omega',\Gamma')\)), it follows that \( \mathcal{C} \) is a stratospherically bounded free spectrahedron.

We are now ready to give the proof of Theorem 4.15.

**Proof of Theorem 4.15.** Suppose \( \mathcal{K} \) is a free spectrahedron with 0 in its interior. By Lemma 4.1, there exists \((\Omega,\Gamma)\), a pair of tuples of matrices, such that
\[ \mathcal{K} = \left\{ X : \exists Y \text{ such that } (X,Y) \in \mathcal{D}_{\mathcal{L}(\Omega,\Gamma)} \right\} = \text{proj}_x \mathcal{D}_{\mathcal{L}(\Omega,\Gamma)}, \]
where \( \mathcal{L}_{(\Omega,\Gamma)}(x,y) \) is the monic linear pencil associated to \((\Omega,\Gamma)\).

Observe, \( A \in \mathcal{K}^o \) iff for each \( X \in \mathcal{K} \),
\[ \mathcal{L}_A(X) \succeq 0. \]
Thus, \( A \in \mathcal{K}^o \) iff
\[ \mathcal{L}_{(A,0)}(X,Y) \succeq 0 \]
for all \((X, Y) \in \mathcal{D}_{\mathcal{L}(\Omega, \Gamma)}\) iff
\[
(A, 0) \in \mathcal{D}_{\mathcal{L}(\Omega, \Gamma)}^c.
\]
Summarizing, \(A \in \mathcal{K}^c\) iff \((A, 0) \in \mathcal{D}_{\mathcal{L}(\Omega, \Gamma)}^c\). Thus, by the second part of Proposition 4.19, \(\mathcal{K}^c\) is a stratospherically bounded free spectrahedron.

Because \(\mathcal{K}\) contains 0 and is a closed matrix convex set, \(\mathcal{K}^{cc} = \mathcal{K}\) by Proposition 4.5. Thus, if \(\mathcal{K}^c\) is a free spectrahedron with 0 in its interior, then, by what has already been proved, \(\mathcal{K}^{cc} = \mathcal{K}\) is a stratospherically bounded free spectrahedron.

Finally, if \(\mathcal{K}\) is a bounded free spectrahedron with 0 in its interior, then \(\mathcal{K}^{cc}\) contains 0 in its interior and is a stratospherically bounded free spectrahedron. Hence, \(\mathcal{K} = \mathcal{K}^{cc}\) is also a stratospherically bounded free spectrahedron. 

Note that the polar dual of a free spectrahedron is a matrix convex set generated by a singleton (Theorem 4.9) and is a free spectrahedron by the above corollary.

The following result follows from the proofs of Theorem 4.15 and Proposition 4.19.

**Corollary 4.20.** Let \(\mathcal{L}\) denote the monic linear pencil associated with \((\Omega, \Gamma)\). If \(\mathcal{D}_{\mathcal{L}}\) is bounded, then the polar dual of \(\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathcal{L}}\) is the free set given by
\[
\mathcal{K}^c(n) = \{ A \in \mathcal{S}_n^g : (A, 0) \in \mathcal{D}_{\mathcal{L}}^c \} = \{ A \in \mathcal{S}_n^g : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ s.t. } A = V^*(I_{\mu} \otimes \Omega)V, \ 0 = V^*(I_{\mu} \otimes \Gamma)V \}.
\]
Whether or not \(\mathcal{D}_{\mathcal{L}}\) is bounded, the polar dual of \(\mathcal{K} = \text{proj}_x \mathcal{D}_{\mathcal{L}}\) is the free set
\[
\mathcal{K}^c(n) = \{ A \in \mathcal{S}_n^g : (A, 0) \in \mathcal{D}_{\mathcal{L}}^c \} = \{ A \in \mathcal{S}_n^g : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ s.t. } A = V^*(I_{\mu} \otimes \Omega')V, \ 0 = V^*(I_{\mu} \otimes \Gamma')V \},
\]
where \(\Omega' = \Omega \oplus 0\) and \(\Gamma' = \Gamma \oplus 0\), as in Lemma 4.11.

**Proof.** From the proof of Theorem 4.15, \(\mathcal{K}^c = \{ A : (A, 0) \in \mathcal{D}_{\mathcal{L}}^c \}\). From the proof of Proposition 4.19, if \(\mathcal{D}_{\mathcal{L}}\) is bounded, then, writing \(\mathcal{L} = \mathcal{L}_\Delta\),
\[
\mathcal{D}_{\mathcal{L}}^c = \{ X : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ such that } X = V^*(I_{\mu} \otimes \Delta)V \}.
\]
In any case (whether or not \(\mathcal{D}_{\mathcal{L}}\) is bounded),
\[
\mathcal{D}_{\mathcal{L}}^c = \{ X : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ such that } X = V^*(I_{\mu} \otimes \Delta')V \}.
\]

To each subset \(\Gamma \subseteq \mathcal{S}^g\) we associate its interior \(\text{int} \Gamma = (\text{int} \Gamma(n))_{n \in \mathbb{N}},\) where \(\text{int} \Gamma(n)\) denotes the interior of \(\Gamma(n)\) in the Euclidean space \(\mathcal{S}^g_n\). We say \(\Gamma\) has nonempty interior if there is \(n\) with \(\text{int} \Gamma(n) \neq \emptyset\).

**Corollary 4.21.** If \(\mathcal{K} \subseteq \mathcal{S}^g\) is a bounded free spectrahedron, then \(\overline{\mathcal{K}}\) is a free spectrahedron.
Proof. As in the proof of Corollary 4.17, we may assume the interior of $K$ is nonempty. This implies there is a $\hat{x} \in \mathbb{R}^g$ in the interior of $K(1)$. Consider the translation $\tilde{K} = K - \hat{x}$ as in (4.10). This is a free spectrahedron containing 0 in its interior. Hence its closure $\overline{\tilde{K}} = \tilde{K}^\circ$ is a free spectrahedron by Theorem 4.15. Thus so is $\overline{K} = \tilde{K} + \hat{x}$. 

Corollary 4.22. If $K \subseteq S^g$ is a free spectrahedron with nonempty interior, then $K^\circ$ is a free spectrahedron.

Proof. We are assuming $K = \text{proj } D_{L_A}$ and apply Lemma 4.7 which says that $K^\circ = \{ B \in S^g : (B,0) \in D_{L_A}^g \}$. So if we prove $D_{L_A}^g$ is a free spectrahedron, then

$$K^\circ = \text{proj} \left( D_{L_A}^g \cap (S^g \otimes \{0\}^k) \right)$$

is the intersection of two free spectrahedrons, so is a free spectrahedron. Thus without loss of generality we may take $K = D_{L_A}$ and proceed. We will demonstrate the corollary holds in this case as a consequence of the Convex Positivstellensatz, Theorem 2.7.

Suppose $\hat{x} \in \mathbb{R}^g$ is in the interior of $D_{L_A}$. Without loss of generality we may assume $L_0 = L_A(\hat{x}) \succ 0$, cf. [HV07]. Define the monic linear pencil

$$\mathcal{L}(y) = L_0^{-\frac{1}{2}} L(y + \hat{x}) L_0^{-\frac{1}{2}} = I + \sum_{j=1}^g L_0^{-\frac{1}{2}} A_j L_0^{-\frac{1}{2}} y_j.$$ 

By definition, a tuple $\Omega \in S^g$ is in $D_{L_A}^g$ iff $D_{L_A} \subseteq D_{S^g}$. Equivalently, with

$$D_{\mathcal{L}} \subseteq D_L.$$ 

By Theorem 2.7, there is a $S \succeq 0$ and matrices $V_k$ with

$$L(y) = S + \sum_k V_k^* \mathcal{L}(y) V_k.$$ 

That is,

$$(I + \sum_{j=1}^g \Omega_j \hat{x}_j) \succeq \sum_k V_k^* V_k, \quad \text{and} \quad \Omega_j = \sum_k V_k^* L_0^{-\frac{1}{2}} A_j L_0^{-\frac{1}{2}} V_k, \quad j = 1, \ldots, g.$$
Equivalently, there is a completely positive mapping $\Phi$ satisfying
\begin{align}
\Phi(L_0^{-\frac{1}{2}}A_jL_0^{-\frac{1}{2}}) &= \Omega_j, \quad j = 1, \ldots, g \\
\Phi(I) &\preceq I + \sum_j \Omega_j \hat{x}_j.
\end{align}

As in Theorem 3.6 we now employ the Choi matrix $C$. Conditions (4.12) translate into linear constraints on the block entries $C_{ij}$ of $C$. Similarly, (4.13) transforms into an LMI constraint on the entries of $C$. Thus $C$ provides a free spectrahedral lift of $D^o_{\mathcal{L}_A}$.

**4.5. The Free Convex Hull of a Union.** In this subsection we prove that the convex hull of a union of free spectrahedrons is again a free spectrahedron.

**Proposition 4.23.** If $S_1, \ldots, S_t \subseteq \mathbb{S}^g$ are stratospherically bounded free spectrahedrons, each of which contains 0 in its interior, then $\overline{\text{co mat}}(S_1 \cup \cdots \cup S_t)$ is a stratospherically bounded free spectrahedron with 0 in its interior.

**Proof.** Let $\mathcal{K} = S_1 \cup \cdots \cup S_t$. Then
$$\mathcal{K}^o = S_1^o \cap \cdots \cap S_t^o.$$ Since each $S_j$ is a stratospherically bounded free spectrahedron with 0 in interior, the same holds true for $S_j^o$ by Theorem 4.15. It is clear that these properties are preserved under a finite intersection, so $\mathcal{K}^o$ is again a stratospherically bounded free spectrahedron with 0 in its interior. Hence
$$(\mathcal{K}^o)^o = \overline{\text{co mat}} \mathcal{K} \quad (\text{by Lemma 4.5})$$


is a stratospherically bounded free spectrahedron with 0 in its interior by Theorem 4.15. \] \]

**4.6. Alternative Unital cp Interpolation Algorithms.** In Subsection 3.2 we gave LMIs equivalent to the trace preserving, trace non-increasing and unital cp interpolation problem. Now we give two seemingly disparate, but in fact fundamentally similar, viewpoints, based on the theory of polar duals of matrix convex sets, to converting the unital problem to LMIs.

**4.6.1. A Duality Approach.** Given $A, B \in \mathbb{S}^g$.

(1) $B \in \text{co mat}(A)$ iff $\text{co mat}(B) \subseteq \text{co mat}(A)$ iff $\text{co mat}(A)^o \subseteq \text{co mat}(B)^o$.

(2) By Theorem 4.9, $\text{co mat}(A)^o = D_{\mathcal{L}_A}$ and $\text{co mat}(B)^o = D_{\mathcal{L}_B}$.

(3) Thus we have a standard LMI inclusion problem
$$D_{\mathcal{L}_A} \subseteq D_{\mathcal{L}_B}.$$

(4) The paper [HKM13a] gives an LMI whose feasibility is equivalent to this inclusion; cf. [HKN14].
4.6.2. **Free Spectrahedrop Approach.** Given $A, B \in \mathbb{S}^g$.

(1) By Proposition 4.19 the free set $\text{co}^\text{mat}(A)$ is a free spectrahedrop with a bounded LMI lift of the form

$$L(x, y) = E_0 + \sum_{j=1}^{g} E_j x_j + \sum_{k=1}^{h} F_k y_k.$$ 

(2) By the definition of a free spectrahedrop, $B \in \text{co}^\text{mat}(A)$ iff the LMI $L(B, Y) \succeq 0$ has a solution $Y$.

## 5. Positivstellensatz for Free Spectrahedrops

This section focuses on polynomials positive on a free spectrahedrop, extending our Positivstellensatz for free polynomials positive on free spectrahedra, Theorem 2.7, to a Convex Positivstellensatz for free spectrahedrops, Theorem 5.1.

Let $\mathcal{L}$ denote a monic linear pencil of size $d$,

$$\mathcal{L}(x, y) = I + \sum_{j=1}^{g} \Omega_j x_j + \sum_{k=1}^{h} \Gamma_k y_k,$$

and let $\mathcal{K} = \text{proj}_x \mathcal{D}_\mathcal{L}$. For positive integers $\mu$ and $r$ we define the **truncated quadratic module** in $\mathbb{R}^{\mu \times \mu}(x)$ associated to $\mathcal{L}$ and $\mathcal{K}$ by

$$M_\mu^r(\mathcal{L}) = \left\{ \sum_{\ell} q_{\ell}^* \mathcal{L} q_{\ell} + \sigma : q_{\ell} \in \mathbb{R}^{d \times \mu}(x)^r, \sigma \in \Sigma_\mu^r(x), \sum_{\ell} q_{\ell}^* \Gamma_k q_{\ell} = 0 \text{ for all } k \right\}.$$ 

Here $\Sigma_\mu^r = \Sigma_\mu^r(x)$ denotes the set of all sums of hermitian squares $h^*h$ for $h \in \mathbb{R}^{\mu \times \mu}(x)^r$. It is easy to see $M_\mu^r(\mathcal{L}) = \bigcup_{r \in \mathbb{N}} M_\mu^r(\mathcal{L})^r$ is a quadratic module in $\mathbb{R}^{\mu \times \mu}(x)$.

The main result of this section is the following Positivstellensatz:

**Theorem 5.1.** A polynomial $p \in \mathbb{R}^{\mu \times \mu}(x)^{2r+1}$ is positive semidefinite on $\mathcal{K}$ iff $p \in M_\mu^r(\mathcal{L})^r$.

**Remark 5.2.** Several remarks are in order.

(1) In case there are no $y$-variables in $\mathcal{L}$, Theorem 5.1 reduces to the Convex Positivstellensatz of [HKM12].

(2) If $r = 0$, i.e., $p$ is linear, then Theorem 5.1 reduces to Corollary 4.20.

(3) A Positivstellensatz for commutative polynomials strictly positive on spectrahedrops was established by Gouveia and Netzer in [GN11]. A major distinction is that the degrees of the $q_i$ and $\sigma$ in the commutative theorem behave very badly.

(4) Observe that $\mathcal{K}$ is in general not closed. Thus Theorem 5.1 yields a “perfect” Positivstellensatz for certain non-closed sets. □
5.1. **Proof of Theorem 5.1.** We begin with some auxiliary results.

**Proposition 5.3.** With $\mathcal{L}$ a monic linear pencil as in (5.1), $M_z^\mu(\mathcal{L})_r$ is a closed convex cone in $\mathbb{R}^{n\times\mu}(x)_{2r+1}$.

The convex cone property is obvious. For the proof of that this cone is closed, it is convenient to introduce a norm compatible with $\mathcal{L}$.

Given $\varepsilon > 0$, let $B_\varepsilon^g := \{ X \in S^g_n : \| X \| \leq \varepsilon \}$. There is an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, if $(X,Y) \in S^g_{r+1}$ and $\| (X,Y) \| \leq \varepsilon$, then $\mathcal{L}(X,Y) \geq \frac{1}{2}$. In particular, $B_{\varepsilon}^{g+h} \subseteq D_\mathcal{L}$. Using this $\varepsilon$ we norm matrix polynomials in $g+h$ variables by

$$
\| p(x,y) \| := \max \left\{ \| p(X,Y) \| : (X,Y) \in B_{\varepsilon}^{g+h} \right\}.
$$

(Let us point out that on the right-hand side of (5.3) the maximum is attained. This follows from the fact that the bounded free semialgebraic set $B_{\varepsilon}^{g+h}$ levelwise compact and matrix convex; see [HM04, Section 2.3] for details). Note that if $f \in \mathbb{R}^{d\times\mu}(x)_\beta$ and if $\| f(x)^* \mathcal{L}(x,y) f(x) \| \leq N^2$, then $\| f^* f \| \leq 2N^2$.

**Proof of Proposition 5.3.** Suppose $\{ p_n \}$ is a sequence from $M_z^\mu(\mathcal{L})_r$ which converges to some $p \in \mathbb{R}^{d\times\mu}(x)$ of degree at most $2r+1$. By Caratheodory’s convex hull theorem (see e.g. [Bar02, Theorem I.2.3]), there is an $M$ such that for each $n$ there exist matrix-valued polynomials $r_{n,i} \in \mathbb{R}^{\mu\times\mu}(x)_r$ and $t_{n,i} \in \mathbb{R}^{d\times\mu}(x)_r$ such that

$$
p_n = \sum_{i=1}^M r_{n,i}^* r_{n,i} + \sum_{i=1}^M t_{n,i}^* \mathcal{L}(x,y) t_{n,i}.
$$

Since $\| p_n \| \leq N^2$, it follows that $\| r_{n,i} \| \leq N$ and likewise $\| t_{n,i}^* \mathcal{L}(x,y) t_{n,i} \| \leq N^2$. In view of the remarks preceding the proof, we obtain $\| t_{n,i} \| \leq \sqrt{2}N$ for all $i, n$. Hence for each $i$, the sequences $(r_{n,i})$ and $(t_{n,i})$ are bounded in $n$. They thus have convergent subsequences. Passing to one of these subsequential limits finishes the proof.

Next is a variant of the Gelfand-Naimark-Segal (GNS) construction.

**Proposition 5.4.** If $\lambda : \mathbb{R}^{\nu\times\nu}(x)_{2k+2} \to \mathbb{R}$ is a linear functional which is nonnegative on $\Sigma_k^{\nu+1}$ and positive on $\Sigma_k^\nu \setminus \{0\}$, then there exists a tuple $X = (X_1, \ldots, X_\nu)$ of symmetric operators on a Hilbert space $\mathcal{X}$ of dimension at most $\nu \sigma_{\#}(k) = \nu \dim \mathbb{R}(x)_k$ and a vector $\gamma \in \mathcal{X}^{\oplus\nu}$ such that

$$
\lambda(f) = \langle f(X) \gamma, \gamma \rangle
$$

(5.4)
for all \( f \in \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+1} \), where \( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathcal{X} \). Further, if \( \lambda \) is nonnegative on \( M^*_x(\mathfrak{L})_k \), then \( X \) is in the closure \( \overline{K} \) of the free spectrahedron \( K \) coming from \( \mathfrak{L} \).

Conversely, if \( X = (X_1, \ldots, X_g) \) is a tuple of symmetric operators on a Hilbert space \( \mathcal{X} \) of dimension \( N \), the vector \( \gamma \in \mathcal{X}^{\oplus \nu} \), and \( k \) is a positive integer, then the linear functional \( \lambda : \mathbb{R}^{\nu \times \nu} \langle x \rangle_{2k+1} \to \mathbb{R} \) defined by

\[
\lambda(f) = \langle f(X)\gamma, \gamma \rangle
\]

is nonnegative on \( \Sigma^*_k \). Further, if \( X \in K \), then \( \lambda \) is nonnegative also on \( M^*_x(\mathfrak{L})_k \).

Proof. The first part of the forward direction is standard, see e.g. [HKM12, Proposition 2.5]. In the course of the proof one constructs \( X_j \) as the operators of multiplication by \( x_j \) on a Hilbert space \( X \), which, as a set, is \( \mathbb{R}^{\langle x \rangle_k \times \nu} \) (the set of row vectors of length \( \nu \) whose entries are polynomials of degree at most \( k \)). The vector space \( X^{\oplus \nu} \) in which \( \gamma \) lies is \( \mathbb{R}^{\langle x \rangle_k \times \nu} \) and \( \gamma \) can be thought of as the identity matrix in \( \mathbb{R}^{\langle x \rangle_k \times \nu} \). Indeed, the (column) vector \( \gamma \) has \( j \)-th entry the row vector with \( j \)-th entry the empty set (which plays the role of multiplicative identity) and zeros elsewhere.

In particular, for \( p \in \mathcal{X} = \mathbb{R}^{\langle x \rangle_k \times \nu} \), we have \( p = p(X)\gamma \). Let \( \sigma \) denote the dimension of \( \mathcal{X} \) (which turns out to be \( \nu^2 \) times the dimension of \( \mathbb{R}^{\langle x \rangle_k} \)).

We next assume that \( \lambda \) is nonnegative on \( M^*_x(\mathfrak{L})_k \) and claim that then \( X \in \overline{K} \). Assume otherwise. Then, as \( \overline{K} \) is closed matrix convex (and \( K \) contains 0 since \( \mathfrak{L} \) is monic), the matricial Hahn-Banach Theorem 2.2 applies: there is a monic linear pencil \( \mathfrak{L} \Lambda \) of size \( \sigma \) such that \( \mathfrak{L} \Lambda |_{K} \succeq 0 \) and \( \mathfrak{L} \Lambda (X) \not\succeq 0 \). In particular, \( D_{\mathfrak{L} \Lambda} \supseteq K \), whence

\[
D_{\mathfrak{L} \Lambda}^o \subseteq K^o.
\]

By Corollary 4.20,

\[
K^o(n) = \left\{ A \in S^n_\mathbb{R} : \exists \mu \in \mathbb{N} \exists \text{isometry } V : \sum_{j=1}^{\mu} V_j^* \Gamma V_j = 0, \sum_{j=1}^{\mu} V_j^* \Omega V_j = A \right\}.
\]

Since \( \Lambda \in K^o \), there is an isometry \( W \) with

\[
\sum_{j=1}^{\eta} W_j^* \Gamma W_j = 0, \quad \sum_{j=1}^{\eta} W_j^* \Omega W_j = \Lambda.
\]

Here, \( W = \text{col}(W_1, \ldots, W_\eta) \) for some \( \eta \), and \( W_j \in \mathbb{R}^{d \times \sigma} \).

Since \( \mathfrak{L} \Lambda (X) \not\succeq 0 \), there is \( u \in \mathbb{R}^{\sigma} \otimes \mathcal{X} \) with

\[
u^* \Lambda(X) u < 0.
\]

Let

\[
u = \sum_i e_i \otimes v_i,
\]
where $e_i \in \mathbb{R}^r$ are the standard basis vectors, and $v_i \in \mathcal{X}$. By the construction of $X$ and $\gamma$, there is a polynomial $p_i \in \mathbb{R}^{d \times \nu}$ with $v_i = p_i(X)\gamma$. Now (5.6) can be written as follows:

$$0 > u^* \mathcal{L}_\Lambda(X)u = (\sum_i e_i \otimes v_i)^* \mathcal{L}_\Lambda(X)(\sum_j e_j \otimes v_j)$$

(5.7)

$$= \sum_{i,j,\ell} (e_i \otimes v_i)^* (W_\ell \otimes I)^* \mathcal{L}(X,Y)(W_\ell \otimes I)(e_j \otimes v_j)$$

$$= \sum_{i,j,\ell} (W_\ell e_i \otimes p_i(X)\gamma)^* \mathcal{L}(X,Y)(W_\ell e_j \otimes p_j(X)\gamma).$$

Letting $\vec{p}_\ell(x) = \sum_j W_\ell e_j \otimes p_j(x) \in \mathbb{R}^{d \times \nu}(x)_k$, (5.7) is further equivalent to

(5.8) $$0 > \sum_\ell (\vec{p}_\ell(X)\gamma)^* \mathcal{L}(X,Y)(\vec{p}_\ell(X)\gamma) = \lambda(q),$$

where $q = \sum_\ell \vec{p}_\ell(x)^* \mathcal{L}(x,y)\vec{p}_\ell(x)$ is a matrix polynomial only in $x$ by (5.5), and thus $q \in M^\mu_x(\mathcal{L})_k$. But now (5.8) contradicts the nonnegativity of $\lambda$ on $M^\mu_x(\mathcal{L})_k$.

The converse is obvious.

**Proof of Theorem 5.1.** Assume $p|_\mathcal{K} \succeq 0$ and $p \not\in M^\mu_x(\mathcal{L})_r$. By the scalar Hahn-Banach theorem and Proposition 5.3, there is a strictly separating positive linear functional $\lambda : \mathbb{R}^{d \times \nu}(x)_2 \rightarrow \mathbb{R}$ nonnegative on $M^\mu_x(\mathcal{L})_r$. We extend $\lambda$ to a linear functional (still called $\lambda$) on $\mathbb{R}^{d \times \nu}(x)_2$ by mapping

$$E_{ij} \otimes u^* v \mapsto \begin{cases} 0 & \text{if } i \neq j \text{ or } u \neq v \\ C & \text{otherwise,} \end{cases}$$

where $i, j = 1, \ldots, \mu$, and $u, v \in \langle x \rangle$ are of length $r + 1$. For $C > 0$ large enough, this $\lambda$ will be nonnegative on $\Sigma^\mu_{r+1}$. Perturbing $\lambda$ if necessary, we may further assume $\lambda$ is strictly positive on $\Sigma^\mu_r \setminus \{0\}$. Now applying Proposition 5.4 yields a matrix tuple $X \in \mathcal{K}$ and a vector $\gamma$ satisfying (5.4) (with $k = r$). But then

$$0 > \lambda(p) = \langle p(X)\gamma, \gamma \rangle \geq 0,$$

a contradiction.

---

6. Tracial Sets

While this paper's original motivation arose from considerations of free optimization as it appears in linear systems theory, determining the matrix convex hull of a free set has an analog in quantum information theory, see [LP11]. In free optimization, the relevant maps are completely positive and *unital* (ucp). In quantum information theory, the relevant maps are completely positive and *trace preserving* (CPTP) or *trace non-increasing*. This
section begins by recalling the two quantum interpolation problems from Subsection 3.2 before reformulating these problem in terms of tracial hulls. Corresponding duality results are the topic of the next section.

Recall a quantum channel is a cp map $\Phi$ from $M_n$ to $M_k$ which is trace preserving, 
$$\text{tr}(\Phi(X)) = \text{tr}(X).$$

The dual $\Phi'$ of $\Phi$ is the mapping from $M_k$ to $M_n$ defined by
$$\text{tr}(\Phi(X)Y^*) = \text{tr}(X\Phi'(Y)^*).$$

**Lemma 6.1** ([LP11, Proposition 1.2]). $\Phi'$ is a quantum channel cp iff $\Phi$ is unital cp.

Recall the cp interpolation problem from Subsection 3.2. It asks, given $A \in S_g^n$ and given $B$ in $S_g^m$: does $B_j = \Phi(A_j)$ for $j = 1, \ldots, g$ for some unital cp map $\Phi : M_n \to M_m$? The set of solutions $B$ for a given $A$ is the matrix convex hull of $A$. The version which arises in quantum information theory [Ha11, Kle07, NCSB] replaces unital with trace preserving (resp. trace non-increasing). Namely, does $B_j = \Phi(A_j)$ for $j = 1, \ldots, g$ for some trace preserving (resp. trace non-increasing) cp map $\Phi : M_n \to M_m$? The set of all solutions $B$ for a given $A$ is the tracial hull of $A$. Thus,

$$\text{thull}(A) = \{ B : \Phi(A) = B \text{ for some trace preserving cp map } \Phi \}.$$  

We define the contractive tracial hull of a tuple $A$ by

$$\text{cthull}(A) = \{ B : \Phi(A) = B \text{ for some cp trace non-increasing map } \Phi \}.$$  

The article [LP11] determines when $B \in \text{thull}(A)$ for $g = 1$ (see Section 3.2). For any $g \geq 0$ the paper [AG+, Section 3] converts this problem to an LMI suitable for semidefinite programming; see Theorem 3.6 here for a similar result.

While the unital and trace preserving (or trace non-increasing) interpolation problems have very similar formulations, tracial hulls possess far less structure than matrix convex hulls. Indeed, as is easily seen, tracial hulls need not be convex (levelwise) and contractive tracial hulls need not be closed with respect to direct sums. Tracial hulls are studied in Subsection 6.1, and contractive tracial hulls in Subsection 6.2. Section 7 contains “tracial” notions of half-space and corresponding Hahn-Banach type separation theorems.

6.1. **Tracial Sets and Hulls.** A set $\mathcal{Y} \subseteq S^g$ is tracial if $Y \in \mathcal{Y}(n)$ and if $C_\ell$ are $m \times n$ matrices such that

$$\sum C_\ell^*C_\ell = I_n,$$

(6.3)
then $\sum C_j Y C_j^* \in \mathcal{Y}(m)$. The **tracial hull** of a subset $\mathcal{S} \subseteq \mathbb{S}^g$ is the smallest tracial set containing $\mathcal{S}$, denoted $\text{thull}(\mathcal{S})$. Note that, in the case that $\mathcal{S}$ is a singleton, this definition is consistent with the definition afforded by Equation (6.1).

The following lemma is an easy consequence of a theorem of Choi, stated in [Pau02, Proposition 4.7]. It caps the number of terms needed in a convex combination to represent a given matrix tuple $Z$ in the tracial hull of $T$, hence is an analog of Caratheodory’s convex hull theorem (see e.g. [Bar02, Theorem I.2.3]).

**Lemma 6.2.** Suppose $T \in \mathbb{S}^g_n$ and $C_1, \ldots, C_N$ are $m \times n$ matrices making $\sum C_i^* C_i = I_n$. If $Z = \sum_{\ell=1}^N C_{\ell} T C_{\ell}^*$, then there exists $m \times n$ matrices $V_1, \ldots, V_{mn}$ such that $\sum V_{\ell}^* V_{\ell} = I_n$ and

$$Z = \sum_{\ell=1}^{mn} V_{\ell} T V_{\ell}^*.$$ 

**Proof.** The mapping $\Phi : M_n \to M_m$ defined by

$$\Phi(X) = \sum_{\ell=1}^N C_{\ell} X C_{\ell}^*$$ 

is completely positive. Hence, by [Pau02, Proposition 4.7], there exist (at most) $nm$ matrices $V_j : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\Phi(X) = \sum_{\ell=1}^{mn} V_{\ell} X V_{\ell}^*.$$ 

In particular,

$$Z = \Phi(T) = \sum_{\ell=1}^{mn} V_{\ell} T V_{\ell}^*.$$ 

Further, for all $m \times m$ matrices $X$,

$$\text{tr}(X) = \text{tr} \left( X \sum_{\ell=1}^N C_{\ell} X C_{\ell}^* \right) = \text{tr} \left( \sum_{\ell=1}^N C_{\ell} X C_{\ell}^* \right) = \text{tr} \left( \Phi(X) \right)$$

$$= \text{tr} \left( \sum_{\ell=1}^{mn} V_{\ell} X V_{\ell}^* \right) = \text{tr} \left( X \sum_{\ell=1}^{mn} V_{\ell}^* V_{\ell} \right).$$

It follows that $\sum V_{\ell}^* V_{\ell} = I$. 

**Lemma 6.3.** For $\mathcal{S} = \{T\}$ a singleton,

$$\text{thull}(\{T\}) = \{ \sum_{\ell=1}^N C_{\ell} T C_{\ell}^* : \sum_{\ell=1}^N C_{\ell}^* C_{\ell} = I \}.$$ 

Moreover, this set is closed (levelwise).

The tracial hull of a subset $\mathcal{S} \subseteq \mathbb{S}^g$ is

$$\text{thull}(\mathcal{S}) = \left\{ \sum_{\ell=1}^N C_{\ell} T C_{\ell}^* : \sum_{\ell=1}^N C_{\ell}^* C_{\ell} = I, T \in \mathcal{S} \right\} = \bigcup_{T \in \mathcal{S}} \text{thull}(\{T\}).$$

If $\mathcal{S}$ is a finite set, then the tracial hull of $\mathcal{S}$ is closed.
Proof. The first statement follows from the observation that \( \{ \sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I \} \) is tracial.

To prove the moreover, suppose \( T \) has size \( n \) and suppose \( Z^k \) is a sequence from \( \mathcal{Y}(m) \). By Lemma 6.2 for each \( k \) there exists \( nm \) matrices \( V_{k,\ell} \) of size \( n \times m \) such that \( Z^k = \sum_{\ell} V_{k,\ell} TV_{k,\ell}^* \) and each \( V_{k,\ell} \) is a contraction. Hence, by passing to a subsequence, we can assume, that for each fixed \( \ell \), the sequence \( (V_{k,\ell})_k \) converges to some \( W_\ell \). Hence \( Z^k \) converges to \( Z = \sum_\ell W_\ell TW_\ell^* \). Also, since \( \sum_\ell V_{k,\ell}^* V_{k,\ell} = I \) for each \( k \), we have \( \sum_\ell W_\ell^* W_\ell = I \), whence \( Z \in \mathcal{Y}(m) \).

To prove the second statement, let \( S \subseteq S^g \) be given. Evidently,
\[
S \subseteq \bigcup_{T \in S} \text{thull}(\{ T \}) \subseteq \text{thull}(S).
\]
Hence it suffices to show that \( \bigcup_{T \in S} \text{thull}(\{ T \}) \) itself is tracially convex. To this end, suppose \( X \in \bigcup_{T \in S} \text{thull}(\{ T \}) \) and \( C_1, \ldots, C_N \) with \( \sum C_\ell^* C_\ell = I \) are given (and of the appropriate sizes). There is a \( S \in S \) such that \( X \in \text{thull}(\{ S \}) \). Hence, by the first part of the lemma, \( \sum C_\ell XC_\ell^* \in \text{thull}(\{ S \}) \subseteq \bigcup_{T \in S} \text{thull}(\{ T \}) \) and the desired conclusion follows.

The final statement of the lemma follows by combining its first two assertions and using the fact that the closure of a finite union is the finite union of the closures.

6.2. Contractively Tracial Sets and Hulls. A set \( Y \subseteq S^g \) is contractively tracial if \( Y \in \mathcal{Y}(m) \) and if \( C_\ell \) are \( n \times m \) matrices such that
\[
\sum C_\ell^* C_\ell \preceq I_m,
\]
then \( \sum C_j Y C_j^* \in \mathcal{Y}(n) \). Note that, in this case, \( Y \) is closed under unitary conjugation and compression to subspaces, but not necessarily direct sums. It is clear that intersections of contractively tracial sets are again contractively tracial.

The contractive tracial hull of a set \( S \), defined as the smallest contractively tracial set containing \( S \), is consistent with our earlier one in terms of cp maps in the case that \( S \) is a singleton.

Lemma 6.4. The contractive tracial hull of a subset \( S \subseteq S^g \) is
\[
\text{chtull}(S) = \left\{ \sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell \preceq I, T \in S \right\} = \bigcup_{T \in S} \text{chtull}(\{ T \}).
\]
If \( S \) is a finite set, then the contractive tracial hull of \( S \) is closed.

Proof. Proof is the same as for Lemma 6.3, so is omitted.
6.2.1. **Convexity and Contractive Tracial Sets.** Tracial and contractively tracial sets are not necessarily convex, as Example 6.5 below shows, and they are not necessarily free sets because they may not respect direct sums. The relation between these two failings are explained in this subsection.

**Example 6.5.** Tracial and contractively tracial hulls need not be convex (levelwise) as this example shows. Consider

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
\]

To show that \( D = \frac{1}{2}(A + B) \) is not in \( \text{thull}(\{A,B\}) \), suppose there exists \( 2 \times 2 \) matrices \( V_1, \ldots, V_m \) such that \( \sum V_j^*V_j = I \) and

\[
\sum V_jAV_j^* = D.
\]

On the one hand, the trace of \( D \) is zero, on the other hand, \( \sum V_jAV_j^* \) has trace 1. Hence \( D \) is not in the tracial hull of \( A \). A similar argument shows that \( D \) is not in the tracial hull of \( B \). Hence by Lemma 6.3, \( D \notin \text{thull}(\{A,B\}) \).

Now consider the tuples \( A = (A_1, A_2) \) and \( B = (B_1, B_2) \) defined by,

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -B_2, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -B_1.
\]

In this case \( D = \frac{1}{2}(A + B) \) is,

\[
D = (D_1, D_2) = \frac{1}{2} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right).
\]

Suppose \( \sum C_j^*C_j \leq I \). Let

\[
F_k = \sum C_jA_kC_j^*
\]

and note \( \text{tr}(F_k) \geq 0 \). On the other hand, \( \text{tr}(D_k) = 0 \). Hence, if \( F_k = D_k \), then

\[
0 = \text{tr}(F_k) = \sum \text{tr}(C_jA_kC_j^*) \geq 0.
\]

But then, for each \( j \),

\[
0 = \text{tr}((C_j(A_1 + A_2)C_j^*) = \text{tr}(C_j^*C_j^*).
\]

It follows that \( C_j = 0 \) for each \( j \) and thus \( F_k = 0 \), a contradiction. Thus, \( D \) is not in the contractive tracial hull of \( A \) and by symmetry it is not in the contractive tracial hull of \( B \). By Lemma 6.4, \( D \) is not in the contractive tracial hull generated by \( \{A, B\} \). \( \Box \)
6.2.2. Tracial Sets and Convexity. Recall, a subset $Y$ of $S^g$ is levelwise convex if each $Y(n)$ is convex (in the usual sense as a subset of $S^g_n$). Say that $Y$ is closed with respect to convex direct sums if given $\ell$ and $Y^1, \ldots, Y^\ell \in Y$ and given $\lambda_1, \ldots, \lambda_\ell \geq 0$ with $\sum \lambda_j \leq 1$,
\[ \bigoplus_j \lambda_j Y^j \in Y. \]

**Lemma 6.6.** If $Y$ is contractively tracial, then $Y$ is levelwise convex iff $Y$ is closed with respect to convex direct sums.

**Proof.** Suppose each $Y(m)$ is convex. Given $Y^j \in Y(m_j)$ for $1 \leq j \leq \ell$, let $m = \sum m_j$. Consider, the block operator column $W_j$ embedding $\mathbb{R}^{m_j}$ into $\mathbb{R}^m = \bigoplus_j \mathbb{R}^{m_j}$. Note that $W_j^* W_j = I_{m_j}$ and thus contractively tracial implies $W_j Y^j W_j^* \in Y(m_j)$. Hence, given $\lambda_j \geq 0$ with $\sum \lambda_j = 1$, convexity of $Y(m)$ (in the ordinary sense), implies
\[ \bigoplus_j \lambda_j Y^j = \sum \lambda_j Y^j W_j Y^j W_j^* \in Y(m). \]

To prove the converse, suppose $Y^j \in Y(n)$ and $m = \ell n$. In this case, $\sum W_j W_j^* = I_n$ and hence tracial implies,
\[ \sum W_j^* \left( \bigoplus_j \lambda_j Y^j \right) W_j = \sum \lambda_j Y^j \in Y(n). \]

6.3. Classical Duals of Free Convex Hulls and of Tracial Hulls. This subsection gives properties of the classical polar dual of matrix convex hulls and tracial hulls.

**Lemma 6.7.** Suppose $A \in S^g_n$.

(i) $\text{co}^\text{mat}(A)^{oc} = \{ Y : \text{thull}(Y) \subseteq A^{oc} \}$;

(ii) $\text{thull}(A)^{oc} = \{ Y : A^{oc} \supseteq \text{co}^\text{mat}(Y) \}$; and

(iii) $\text{thull}(B) \subseteq \text{thull}(A)$ iff $A^{oc} \supseteq \text{co}^\text{mat}(Y)$ implies $B^{oc} \supseteq \text{co}^\text{mat}(Y)$.

**Proof.** The first formula:
\[
\text{co}^\text{mat}(A)^{oc} = \{ Y : 1 - \text{tr}(\sum_j V_j^* A V_j Y) \geq 0, \sum_j V_j^* V_j = I \}
= \{ Y : 1 - \text{tr}(A \sum_j V_j Y V_j^*) \geq 0, \sum_j V_j^* V_j = I \}
= \{ Y : 1 - \text{tr}(AG) \geq 0, G \in \text{thull}(Y) \}
= \{ Y : A^{oc} \supseteq \text{thull}(Y) \}.
\]
The second formula:
\[
\text{thull}(A)^{oc} = \{ Y : 1 - \text{tr}\left( \sum_j V_j^* A V_j Y \right) \geq 0, \sum_j V_j V_j^* = I \}
\]
\[
= \{ Y : 1 - \text{tr}(A \sum_j V_j Y V_j^*) \geq 0, \sum_j V_j V_j^* = I \}
\]
\[
= \{ Y : A^{oc} \supseteq \text{co}^\text{mat}(Y) \}.
\]
The third formula: \(\text{thull}(B) \subseteq \text{thull}(A)\) iff \(\text{thull}(B)^{oc} \supseteq \text{thull}(A)^{oc}\) iff
\[
\{ Y : B^{oc} \supseteq \text{co}^\text{mat}(Y) \} \supseteq \{ Y : A^{oc} \supseteq \text{co}^\text{mat}(Y) \},
\]
iff \(A^{oc} \supseteq \text{co}^\text{mat}(Y)\) and \(B^{oc} \supseteq \text{co}^\text{mat}(Y)\).

7. Tracial Spectrahedra and an Effros-Winkler Separation Theorem

Classically, convex sets are delineated by half-spaces. In this section a notion of half-space suitable in the tracial context – we call them tracial spectrahedra – are introduced. Subsection 7.3 contains a free Hahn-Banach separation theorem for tracial spectrahedra. The section concludes with applications of this Hahn-Banach theorem. Subsection 7.4 suggests several notions of duality based on the tracial separation theorem from Subsection 7.3. Subsection 7.5 studies free (convex) cones.

7.1. Tracial Spectrahedra. Polar duality considerations in the trace non-increasing context lead naturally to inequalities of the type,
\[
I \otimes T - \sum_{j=1}^g B_j \otimes Y_j \succeq 0,
\]
for tuples \(B, Y \in S^g\) and a positive semidefinite matrix \(T\) with trace at most one. Two notions, in a sense dual to one another, of half-space are obtained by fixing either \(B\) or \(Y\).

Given \(B \in S^g_k\), let
\[
\mathcal{H}_B = \bigcup_{m \in \mathbb{N}} \{ Y \in S^g_m : \exists T \succeq 0, \text{tr}(T) \leq 1, \ I \otimes T - \sum B_j \otimes Y_j \succeq 0 \}
\]
\[
= \bigcup_{m \in \mathbb{N}} \{ Y \in S^g_m : \exists T \succeq 0, \text{tr}(T) = 1, \ I \otimes T - \sum B_j \otimes Y_j \succeq 0 \}.
\]
Sets of the form \(\mathcal{H}_B\) will be called tracial spectrahedra. Tracial spectrahedra obtained by fixing \(Y\) appear in Subsubsection 7.4.2.

Proposition 7.1. Let \(B \in S^g_k\) be given.
(a) The set \(\mathcal{H}_B\) is contractively tracial;
(b) For each \(m\), the set \(\mathcal{H}_B(m)\) is convex; and
(c) For each $m$, the set $\mathcal{H}_B(m)$ is closed.

In summary, $\mathcal{H}_B$ is levelwise compact and closed, and is contractively tracial.

**Remark 7.2.** Of course $\mathcal{H}_B$ is not a free set since, in particular, it is not closed with respect to direct sums.

**Proof.** Suppose $Y \in \mathcal{H}_B(m)$ and $C_\ell$ satisfying Equation (6.4) are given. There is an $m \times m$ positive semidefinite matrix $T$ with trace at most one such that

$$I \otimes T - \sum_j B_j \otimes Y_j \succeq 0.$$ 

It follows that

$$0 \leq I \otimes \sum_\ell C_\ell TC_\ell^* - \sum_j B_j \otimes \sum_\ell C_\ell Y_j C_\ell^*.$$ 

Note that $T' = \sum_\ell C_\ell TC_\ell^*$ is positive semidefinite and has trace at most one. Moreover,

$$\text{tr}(T') = \text{tr}(T \sum_\ell C_\ell^* C_\ell) = \text{tr}(T^\frac{1}{2} C_\ell^* C_\ell T^\frac{1}{2}) \leq \text{tr}(T) \leq 1.$$ 

Hence $\sum C_\ell Y_j C_\ell^* \in \mathcal{Y}(n)$ and item (a) of the proposition is proved.

To prove item (b), suppose both $Y^1$ and $Y^2$ are in $\mathcal{H}_B$. To each there is an associated positive semidefinite matrix of trace at most one, say $T_1$ and $T_2$. If $0 \leq s_1, s_2 \leq 1$ and $s_1 + s_2 = 1$, then $T = \sum s_\ell T_\ell$ is positive semidefinite and has trace at most one. Moreover, with $Y = \sum s_j Y_j$,

$$I \otimes T - \sum_j B_j \otimes \left( \sum_\ell s_\ell \sum_j Y_j^\ell \right) = \sum_\ell s_\ell \left( I \otimes T_\ell - \sum_j B_j \otimes Y_j^\ell \right) \succeq 0.$$ 

To prove (c), suppose the sequence $(Y^k)_k$ from $\mathcal{H}_B(m)$ converges to $Y \in \mathcal{S}_m^\circ$. For each $k$ there is a positive semidefinite matrix $T^k$ of trace at most one such that

$$I \otimes T^k - \Lambda_B(Y^k) \succeq 0.$$ 

Choose a convergent subsequence of the $T^k$ with limit $T$. Then this $T$ witnesses that $Y \in \mathcal{H}_B(m)$. 

To proceed toward the separation theorem we start with some preliminaries.

**7.2. An Auxiliary Result.** Given a positive integer $n$, let $\mathcal{T}_n$ denote the positive semidefinite $n \times n$ matrices (with real entries) of trace one. Each $T \in \mathcal{T}_n$ corresponds to a state on $M_n$, the $n \times n$ matrices, via the trace,

$$M_n \ni A \mapsto \text{tr}(AT).$$

Conversely, to each state $\varphi$ on $M_n$ we can assign a matrix $T$ such that $\varphi$ is the map (7.1). Note that $\mathcal{T}_n$ is a convex, compact subset of $\mathcal{S}_n$, the symmetric $n \times n$ matrices.
The following lemma is a version of [EW97, Lemma 5.2]. An affine linear mapping $f : \mathbb{S}_n \rightarrow \mathbb{R}$ is a function of the form $f(x) = a_f + \lambda_f(x)$, where $\lambda_f$ is linear and $a_f \in \mathbb{R}$.

**Lemma 7.3.** Suppose $\mathcal{F}$ is a convex set of affine linear mappings $f : \mathbb{S}_n \rightarrow \mathbb{R}$. If for each $f \in \mathcal{F}$ there is a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, then there is a $\mathfrak{T} \in \mathcal{T}_n$ such that $f(\mathfrak{T}) \geq 0$ for every $f \in \mathcal{F}$.

**Proof.** For $f \in \mathcal{F}$, let

$$B_f = \{T \in \mathcal{T}_n : f(T) \geq 0\} \subseteq \mathcal{T}_n.$$  

By hypothesis each $B_f$ is non-empty and it suffices to prove that

$$\bigcap_{f \in \mathcal{F}} B_f \neq \emptyset.$$  

Since each $B_f$ is compact, it suffices to prove that the collection $\{B_f : f \in \mathcal{F}\}$ has the finite intersection property. Accordingly, let $f_1, \ldots, f_m \in \mathcal{F}$ be given. Arguing by contradiction, suppose

$$\bigcap_{j=1}^{m} B_{f_j} = \emptyset.$$  

Define $F : \mathbb{S}_n \rightarrow \mathbb{R}^m$ by

$$F(T) = (f_1(T), \ldots, f_m(T)).$$  

Then $F(\mathcal{T}_n)$ is both convex and compact because $\mathcal{T}_n$ is both convex and compact and each $f_j$, and hence $F$, is affine linear. Moreover, $F(\mathcal{T}_n)$ does not intersect

$$\mathbb{R}_{\geq 0}^m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_j \geq 0 \text{ for each } j\}.$$  

Hence there is a linear functional $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$\lambda(F(\mathcal{T}_n)) < 0 \quad \text{and} \quad \lambda(\mathbb{R}_{\geq 0}^m) \geq 0.$$  

There exists $\lambda_j$ such that $\lambda(x) = \sum \lambda_j x_j$. Since $\lambda(\mathbb{R}_{\geq 0}^m) \geq 0$ it follows that each $\lambda_j \geq 0$ and since $\lambda \neq 0$, for at least one $k$, $\lambda_k > 0$. Without loss of generality, it may be assumed that $\sum \lambda_j = 1$. Let

$$f = \sum \lambda_j f_j.$$  

Since $\mathcal{F}$ is convex, it follows that $f \in \mathcal{F}$. On the other hand, $f(T) = \lambda(F(T))$. Hence if $T \in \mathcal{T}_n$, then $f(T) < 0$. Thus, for this $f$ there does not exist a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, a contradiction which completes the proof.
7.3. A Tracial Spectrahedron Separating Theorem. The following lemma is proved by a variant of the Effros-Winkler construction of separating LMIs (i.e., the matricial Hahn-Banach Theorem) in the theory of matrix convex sets. Linear functionals \( \lambda : \mathbb{S}_n^g \to \mathbb{R} \) are in one-one correspondence with elements \( B \in \mathbb{S}_n^g \) via the pairing,

\[
\lambda(X) = \text{tr} \left( \sum B_j X_j \right), \quad X = (X_1, \ldots, X_g).
\]

Write \( \lambda_B \) for this \( \lambda \). To avoid confusion with the free polar duals appearing earlier in this article, let \( U^\circ \) denote the conventional polar dual of a subset \( U \subseteq \mathbb{S}_n^g \). Thus,

\[
U^\circ = \{ B \in \mathbb{S}_n^g : \lambda_B(X) \leq 1 \text{ for all } X \in U \}.
\]

Lemma 7.4. Fix positive integers \( m, n, \) and suppose that \( S \) is a nonempty subset of \( \mathbb{S}_m^g \). Let \( U \) denote the subset of \( \mathbb{S}_n^g \) consisting of all tuples of the form

\[
\sum_{\ell=1}^\mu C_{\ell} Y_{\ell} C_{\ell}^*,
\]

where each \( C_{\ell} \) is \( n \times m \), each \( Y_{\ell} \in S \) and \( \sum C_{\ell}^* C_{\ell} \preceq I \). If \( B \in \mathbb{S}_n^g \) is in the conventional polar dual of \( U \), then there exists a positive semidefinite \( m \times m \) matrix \( T \) with trace at most one such that

\[
I \otimes T - \sum B_j \otimes Y_j \succeq 0
\]

for every \( Y \in S \).

Proof. Given \( C_{\ell} \) and \( Y_{\ell} \) as in the statement of the lemma, define \( f_{C,Y} : \mathbb{S}_m^g \to \mathbb{R} \) by

\[
f_{C,Y}(X) = \text{tr} \left( \sum C_{\ell} X C_{\ell}^* \right) - \lambda_B \left( \sum C_{\ell} Y C_{\ell}^* \right).
\]

To show that the set \( \{ f_{C,Y} : C, Y \} \) is convex, suppose, for \( 1 \leq s \leq N, C^s = (C_1^s, \ldots, C_{\mu_s}^s) \) is a tuple of \( n \times m \) matrices and likewise \( Y_{s,\ell} \) for \( 1 \leq s \leq N \) and \( 1 \leq \ell \leq \mu_s \) are from \( S \) and and \( \lambda_1, \ldots, \lambda_N \) are positive numbers with \( \sum \lambda_s = 1 \). In this case,

\[
\sum \lambda_s f_{C^s,Y_{s,\ell}} = f_{C,Y}
\]

for

\[
C = \left( \frac{1}{\sqrt{\lambda_s}} C_{s,\ell}^s \right)_{s,\ell}, \quad Y = \left( Y_{s,\ell} \right)_{s,\ell}.
\]

Given \( C_1, \ldots, C_{\mu} \) are \( n \times m \) matrices and \( Y_1, \ldots, Y_{\mu} \), let \( D = \sum C_{\ell}^* C_{\ell} \). Suppose \( D \) has norm one. There is a unit vector \( \gamma \) such that \( ||D\gamma|| = ||D|| = 1 \). Choose \( T = \gamma \gamma^* \). Thus \( T \in \mathcal{T}_m \). Moreover,

\[
\text{tr} \left( \sum C_{\ell} T C_{\ell}^* \right) = \text{tr}(TD) = \langle D\gamma, \gamma \rangle = 1.
\]

Thus, using the assumption that \( B \) is in \( U^\circ \),

\[
f_{C,Y}(T) = 1 - \lambda_B \left( \sum C_{\ell} Y_{\ell} C_{\ell}^* \right) \geq 0.
\]
If $D$ is not of norm one, a simple scaling argument gives the same conclusion; that is, 

$$f_{C,Y}(T) \geq 0.$$ 

By Lemma 7.3, it follows that there is a $\mathfrak{T} \in \mathcal{T}_m$ such that 

$$\text{tr} \left( \sum C \mathfrak{T} C^* \right) - \lambda_B \left( \sum C^* Y C \right) \geq 0,$$ 

regardless of the norm of $\sum C^*_e C_e$. 

Now the aim is to show that 

$$\Delta := I \otimes \mathfrak{T} - \sum_j B_j \otimes Y_j \succeq 0$$ 

for every $Y \in \mathcal{S}$. Accordingly, let $Y \in \mathcal{S}$ and $\gamma = \sum e_s \otimes \gamma_s \in \mathbb{R}^n \otimes \mathbb{R}^m$ be given. Compute, 

$$\langle \Delta \gamma, \gamma \rangle = \sum_s \langle \mathfrak{T} \gamma_s, \gamma_s \rangle - \sum_j \sum_{s,t} \langle B_j \gamma_s, \gamma_t \rangle.$$ 

Now let $\Gamma^*$ denote the matrix with $s$-th column $\gamma_s$. Hence $\Gamma$ is $n \times m$. Compute, 

$$\lambda_B(\Gamma Y \Gamma^*) = \text{tr} \left( \sum B_j (\Gamma Y_j \Gamma^*) \right) = \sum_j \sum_{s,t} \langle B_j \gamma_s, \gamma_t \rangle.$$ 

Similarly, 

$$\text{tr}(\Gamma \mathfrak{T} \Gamma^*) = \sum_s \langle \mathfrak{T} \gamma_s, \gamma_s \rangle.$$ 

Thus, using the inequality (7.2), 

$$\langle \Delta \gamma, \gamma \rangle = \text{tr}(\Gamma \mathfrak{T} \Gamma^*) - \lambda_B(\Gamma Y \Gamma^*) \geq 0.$$ 

It is in this last step that the contractively tracial, not just tracial is needed, so that it is not necessary for $\Gamma^* \Gamma$ to be a multiple of the identity. 

**Proposition 7.5.** If $\mathcal{Y} \subseteq \mathbb{S}_n^g$ is contractively tracial and if $B \in \mathbb{S}_n^g$ is in the conventional polar dual $\mathcal{Y}(n)^{\text{oc}}$ of $\mathcal{Y}(n)$, then $\mathcal{Y} \subseteq \mathcal{F}_B$. 

**Proof.** Suppose $\mathcal{Y}$ is contractively tracial and $Y \in \mathcal{Y}(m)$. Letting $\mathcal{S} = \{ Y \}$ in Lemma 7.4, it follows that there is a $T$ such that 

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$ 

Thus, $Y \in \mathcal{F}_B$ and the proof is complete. 

We are now ready to state the separation result for closed levelwise convex tracial sets. 

**Theorem 7.6.**
(i) If $Y \subseteq S^g$ is contractively tracial, levelwise convex, and if $Z \in S^g_m$ is not in the closure of $Y(m)$, then there exists a $B \in S^g_m$ such that $Y \subseteq \mathcal{H}_B$, but $Z \notin \mathcal{H}_B$. Hence,

$$\mathcal{Y} = \bigcap\{\mathcal{H}_B : \mathcal{H}_B \supseteq Y\} = \bigcap_{n \in \mathbb{N}} \mathcal{H}_B.$$

(ii) The levelwise closed convex contractively tracial hull of a subset $Y$ of $S^g$ is

$$\bigcap\{\mathcal{H}_B : \mathcal{H}_B \supseteq \mathcal{Y}\}.$$

**Proof.** To prove item (i), suppose $Z \in S^g_m$ but $Z \notin Y(m)$. Since $Y$ is levelwise convex, there is $\lambda_B$ such that $\lambda_B(Y) \leq 1$ for all $Y \in Y(m)$, but $\lambda_B(Z) > 1$ by the usual Hahn-Banach separation theorem for closed convex sets. Thus $B$ is in the conventional polar dual of $Y(m)^\circ$. From Proposition 7.5, $Y \subseteq \mathcal{H}_B$.

On the other hand, if $T \in T_m$ and $\{e_1, \ldots, e_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then, with $e = \sum e_s \otimes e_s \in \mathbb{R}^m \otimes \mathbb{R}^m$,

$$\langle (I \otimes T - \sum B_j \otimes Z_j)e, e \rangle = \text{tr}(T) - \text{tr} \left( \sum B_j Z_j \right) = 1 - \lambda_B(Z) < 0.$$

Hence $Z \notin \mathcal{H}_B$ and the conclusion follows.

To prove item (ii), first note that, letting $I$ denote the intersection of the $\mathcal{H}_B$ which contain $Y$, that $Y \subseteq I$. Since the intersection of tracial spectrahedra is levelwise closed and convex, and contractively tracial, the levelwise closed convex tracial hull $\mathcal{H}$ of $Y$ is also contained in $I$. On the other hand, from (i),

$$\mathcal{H} = \bigcap\{\mathcal{H}_B : \mathcal{H}_B \supseteq \mathcal{H}\} \supseteq I \supseteq \mathcal{H}.$$

**Remark 7.7.** The contractive tracial hull of a point. Fix a $Y \in S^g$ and let $\mathcal{Y}$ denote its contractive tracial hull,

$$\mathcal{Y} = \{ \sum V_j Y V_j^* : \sum V_j^* V_j \preceq I \}.$$

Evidently each $Y(m)$ (taking $V_j : \mathbb{R}^n \to \mathbb{R}^m$) is a convex set. From Lemma 6.4, $\mathcal{Y}$ is closed. Hence Theorem 7.6 applies and gives a duality description of $\mathcal{Y}$. Namely, $\tilde{Y}$ is in the contractive tracial hull $\mathcal{Y}$ iff for each $B$ for which there exists a positive semidefinite $T$ of trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0,$$

there exists a positive semidefinite $\tilde{T}$ of trace at most one such that

$$I \otimes \tilde{T} - \sum B_j \otimes \tilde{Y}_j \succeq 0.$$

**7.4. Tracial Polar Duals.** We now introduce two natural notions of polar duals based on the tracial spectrahedra. Rather than exhaustively studying these duals, we list a few properties to illustrate the possibilities.
7.4.1. *Ex Situ Tracial Dual*. Suppose $\mathcal{K} \subseteq \mathbb{S}^g$. Let $\hat{\mathcal{K}}$ denote its *ex situ tracial dual* defined by

$$(7.3) \quad \hat{\mathcal{K}} = \bigcap_{B \in \mathcal{K}} \mathfrak{S}_B.$$ 

Thus,

$$(7.4) \quad \hat{\mathcal{K}}(n) = \left\{ Y \in \mathbb{S}_n^g : \forall B \in \mathcal{K} \exists T \succeq 0 \text{ such that } \text{tr}(T) \leq 1 \text{ and } I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}.$$ 

**Proposition 7.8.** If $\mathcal{K}$ is matrix convex and each $\mathcal{K}^o(n)$ is bounded (equivalently, $\mathcal{K}(1)$ contains $0$ in its interior), then

(i) $\hat{\mathcal{K}}(n) = \left\{ Y \in \mathbb{S}_n^g : \exists T \succeq 0, \text{ such that } \text{tr}(T) \leq 1 \text{ and } \forall B \in \mathcal{K}, I \otimes T - \sum B_j \otimes Y_j \succeq 0 \right\}$;

(ii) $\hat{\mathcal{K}}(n) = \{ SMS : M \in \mathcal{K}^o(n), S \succeq 0, \text{ tr}(S^2) \leq 1 \}$.

**Proof.** Suppose $\mathcal{K}$ is matrix convex. To prove item (i), let $Y \in \hat{\mathcal{K}}(n)$ be given. For each $B$, let $\mathcal{T}_B = \{ T \in \mathcal{T}_n : I \otimes T - \sum B_j \otimes Y_j \succeq 0 \}$. Thus, the hypothesis that $Y \in \hat{\mathcal{K}}(n)$ is equivalent to assuming that for every $B$ in $\mathcal{K}$, the set $\mathcal{T}_B$ is nonempty.

That $\mathcal{T}_B$ is compact will be verified by showing it satisfies the finite intersection property. Now given $B_1, \ldots, B_\ell \in \mathcal{K}$, let $B = \bigoplus_k B_k \in \mathcal{K}$. Since $B \in \mathcal{K}$, there is a $T$ such that

$$\bigoplus_k \left( I \otimes T - \sum B_j \otimes Y_j \right) = I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$ 

Hence $T \in \bigcap_{k=1}^\ell \mathcal{T}_{B_k}$. It follows that the collection $\{ \mathcal{T}_B : B \in \mathcal{K} \}$ has the finite intersection property and hence there is a $T \in \bigcap_{B \in \mathcal{K}} \mathcal{T}_B$ and the forward inclusion in item (i) follows. The reverse inclusion holds whether or not $\mathcal{K}$ is matrix convex.

To prove item (ii), suppose $Y \in \hat{\mathcal{K}}(n)$. Thus, by what has already been proved, there is a positive semidefinite matrix $S$ such that $\text{tr}(S^2) \leq 1$ and

$$(7.5) \quad I \otimes S^2 - \sum B_j \otimes Y_j \succeq 0,$$ 

for all $B \in \mathcal{K}$. For positive integers $k$, let $S_k^+$ denote the inverse of $S + \frac{1}{k}$. Multiplying (7.5) on the left and on the right by $I \otimes S_k^+$ yields

$$I \otimes P - \sum B_j \otimes S_k^+ Y_j S_k^+ \succeq 0,$$ 

where $P$ is the projection onto the range of $S$. It follows that $M_k = S_k^+ Y S_k^+ \in \mathcal{K}^o(n)$. Since $\mathcal{K}^o(n)$ is bounded (by assumption) and closed, it is compact and consequently a subsequence of $(M_k)_k$ converges to some $M \in \mathcal{K}^o(n)$. Hence, $Y = SMS$.

Reversing the argument above shows, if $M \in \mathcal{K}^o(n)$ and $S$ is positive semidefinite with $\text{tr}(S^2) \leq 1$, then $Y = SMS \in \hat{\mathcal{K}}(n)$ and the proof is complete. ■
Proposition 7.9. The ex situ tracial dual $\hat{\mathcal{K}}$ of a free spectrahedron $\mathcal{K} = \mathcal{D}_\mathcal{C}_\mathcal{O}$ is exactly the set

$$\left\{ \sum_\ell C_\ell^* \Omega C_\ell : \text{tr}\left( \sum_\ell C_\ell^* C_\ell \right) \leq 1 \right\}.$$

Proof. Suppose $Y$ is in the ex situ tracial dual. By Proposition 7.8, there is a positive semidefinite matrix $S$ with $\text{tr}(S^2) \leq 1$ and an $M \in \mathcal{K}^\circ$ such that $Y = SMS$. Since $M \in \mathcal{K}^\circ$, by Remark 4.12 (following Corollary 4.20) there is a positive integer $\mu$ and a contraction $V$ such that

$$M = V^* (I_\mu \otimes \Omega) V = \sum_k V_k^* \Omega V_k.$$

Hence,

$$Y = \sum_k SV_k^* \Omega V_k S.$$

Finally,

$$\text{tr}\left( \sum_k SV_k^* V_k S \right) \leq \text{tr}(S^2) \leq 1.$$

Conversely suppose $\text{tr}(\sum_\ell C_\ell^* C_\ell) \leq 1$ and $Y = \sum_\ell C_\ell^* \Omega C_\ell$. Let $T = \sum_\ell C_\ell^* C_\ell$ and note that for $B \in \mathcal{K}$,

$$I \otimes T - \sum_j B_j \otimes Y_j = \sum_\ell C_\ell^* (I \otimes I - \sum_j B_j \otimes \Omega_j) C_\ell \succeq 0.$$ 

7.4.2. In Situ Tracial Dual. Given a free set $\mathcal{K} \subseteq \mathcal{S}^g$, we can define another dual set, $\mathcal{K}^\circ = (\mathcal{K}^\circ(m))_m$ by

$$\mathcal{K}^\circ(m) = \{ B \in \mathcal{S}^g_m : \mathcal{K} \subseteq \mathcal{J}_B \}.$$ 

Equivalently,

$$\mathcal{K}^\circ(m) = \{ B \in \mathcal{S}^g_m : \forall Y \in \mathcal{K} \exists T \succeq 0, \text{ such that } \text{tr}(T) \leq 1 \text{ and } I \otimes T - \sum_j B_j \otimes Y_j \succeq 0 \}.$$ 

Each $\mathcal{K}^\circ(m)$ is levelwise convex. Moreover, if $B \in \mathcal{K}^\circ$ and $V^* V \preceq I$, then $V^* B V \in \mathcal{K}^\circ$. On the other hand, there is no reason to expect that $\mathcal{K}^\circ$ is closed with respect to direct sums. Hence it need not be matrix convex.

A subset $\mathcal{Y}$ of $\mathcal{S}^g$ is contractively convex if $Y \in \mathcal{Y}$ and if $\sum_j C_j^* C_j \preceq I$, then $\sum_j C_j^* Y C_j \in \mathcal{Y}$. In general, contractively convex sets need not be levelwise convex:

Example 7.10. The $2 \times 2$ matrices $A, B$ from Example 6.5. The smallest contractively convex set containing $A, B$ is the levelwise closed set

$$\mathcal{Y} = \left\{ \sum_j C_j^* A C_j : \sum_j C_j^* C_j \preceq I \right\} \cup \left\{ \sum_j D_j^* B D_j : \sum_j D_j^* D_j \preceq I \right\}.$$ 

Each matrix in $\mathcal{Y}$ is either positive semidefinite or negative semidefinite, so $\frac{1}{2}(A+B) \notin \mathcal{Y}$. \qed
Proposition 7.11. The set $K^\circ$ is contractively convex.

Proof. Suppose $B \in K^\circ(m)$. Let $n \times m$ matrices $C_1, \ldots, C_\ell$ such that $\sum C_k^* C_k \preceq I$ be given and consider the $n \times n$ matrix $D = \sum C_k B C_k^*$.

Given $Y \in K(p)$, there exists a positive semidefinite $p \times p$ matrix $T$ of trace at most one such that $I \otimes T - \sum B_j \otimes Y_j \succeq 0$.

Thus,

$$I \otimes T - \sum_{j=1}^g D_j \otimes Y_j = I \otimes T - \sum_{j=1}^g \sum_k C_k^* B_j C_k \otimes Y_j$$

$$= (I - \sum_k C_k^* C_k) \otimes T + \sum_k (C_k \otimes I)(I \otimes T - \sum_j B_j \otimes Y_j)(C_k \otimes I) \succeq 0.$$

Hence $D \in K^\circ$ and the proof is complete. \hfill \blacksquare

The contractive convex hull of $Y$ is the smallest levelwise closed set containing $Y$ which is contractively convex. The following proposition finds the two hulls defined by applying the two notions of tracial polar duals introduced above.

Proposition 7.12. For $K \subseteq S^g$, the set $\hat{(K^\circ)}$ is the levelwise closed convex contractively tracial hull of $K$. Similarly, $\hat{(K)^b}$ is the levelwise closed contractively convex hull of $K$.

The proof of the second statement rests on the following companion to Lemma 7.4. Given a tuple $Y$, we introduce

$$S_Y^{\text{opp}} = \{B : \exists T \succeq 0 \text{ such that } \text{tr}(T) \leq 1, I \otimes T - \sum B_j \otimes Y_j \succeq 0\},$$

the (opp-)tracial spectrahedron.

Lemma 7.13. Fix positive integers $m, n$, and suppose that $S$ is a nonempty subset of $S^g_n$. Let $U$ denote the subset of $S^g_m$ consisting of all tuples of the form

$$\sum_{\ell=1}^\mu C_\ell^* B^\ell C_\ell,$$

where each $C_\ell$ is $n \times m$, each $B^\ell \in S$ and $\sum C_\ell^* C_\ell \preceq I$.

(1) If $Y \in S^g_m$ is in the conventional polar dual of $U$, then there exists a positive semidefinite $m \times m$ matrix $T$ with trace at most one such that $I \otimes T - \sum B_j \otimes Y_j \succeq 0$ for every $B \in S$. 


(2) The tracial spectrahedra $\mathcal{H}_Y^{\text{opp}}$ are closed and contractively convex.

(3) If $K \subseteq S^g$ is contractively convex and if $Y \in S_m$ is in the conventional polar dual $K(m)^{oc}$ of $K(m)$, then $K \subseteq H^{\text{opp}}_Y$.

(4) If $K \subseteq S^g$ is levelwise closed and convex, and contractively convex, then
$$K = \bigcap_n \{ H^{\text{opp}}_Y : H^{\text{opp}}_Y \supseteq K \} = \bigcap_n \{ Y : Y \in K(n)^{\circ} \}. $$

(5) The levelwise closed and convex contractively convex hull of $K \subseteq S^g$ is $\bigcap \{ H^{\text{opp}}_Y : H^{\text{opp}}_Y \supseteq K \}$.

(6) For $K \subseteq S^g$, we have $Y \in \hat{K}(n)$ if and only if $K \subseteq H^{\text{opp}}_Y$.

Proof. Let $G$ denote the collection of tuples $(C, B)$ where $C = (C_\ell)_1^\mu$ and $B = (B_\ell)_1^\mu$ as in the statement of the lemma. Given $(C, B) \in G$, define $f_{C, B} : S_m \to \mathbb{R}$ by
$$f_{C, B}(X) = \text{tr} \left( \sum \C_\ell X C_\ell^* \right) - \lambda_Y \left( \sum \C_\ell B_\ell C_\ell^* \right).$$

Note that the assumption that $Y$ is in the polar dual of $T$ says $1 \geq \lambda_Y (\sum_\ell C_\ell B_\ell C_\ell^*)$.

To show that the set $\mathcal{F} = \{ f_{C, B} : (C, B) \in G \}$ is convex, suppose, for $1 \leq s \leq N$, $C^s = (C_1^s, \ldots, C_{\mu_s}^s)$ is a tuple of $n \times m$ matrices and likewise $B^s$, for $1 \leq s \leq N$ and $1 \leq j \leq \mu_s$, are from $S$ and $\lambda_1, \ldots, \lambda_N$ are positive numbers with $\sum \lambda_s = 1$. In this case,
$$\sum \lambda_s f_{C^s, B^s} = f_{C, B}$$

for $C = \left( \frac{1}{\sqrt{\lambda_s}} C_\ell^s \right)_s^\ell$ and $B = \left( \frac{1}{\sqrt{\lambda_s}} B_\ell^s \right)_s^\ell$.

Given $(C, B) \in G$, let $D = \sum \C_\ell C_\ell^*$. Suppose $D$ has norm one. There is a unit vector $\gamma$ such that $\|D\gamma\| = \|D\| = 1$. Choose $T = \gamma \gamma^*$. Thus $T \in T_m$. Moreover,
$$\text{tr} \left( \sum \C_\ell T C_\ell^* \right) = \text{tr} (TD) = \langle D\gamma, \gamma \rangle = 1.$$  

Thus,
$$f_{C, B}(T) = 1 - \lambda_Y \left( \sum \C_\ell B_\ell C_\ell^* \right) \geq 0.$$  

If $D$ doesn’t have norm one, a simple scaling argument gives the same conclusion; that is,
$$f_{C, B}(T) \geq 0.$$  

By Lemma 7.3, it follows that there is a $\Xi \in T_m$ such that $f_C(\Xi) \geq 0$ for all $C$ and $Y$; i.e.,

\begin{equation}
\text{tr} \left( \sum \C_\ell \Xi C_\ell^* \right) - \lambda_Y \left( \sum \C_\ell B_\ell C_\ell^* \right) \geq 0,
\end{equation}

regardless of the norm of $\sum \C_\ell C_\ell^*$.  

Now the aim is to show that
\[ \Delta := I \otimes \mathbf{T} - \sum_j B_j \otimes Y_j \succeq 0 \]
for every \( B \in \mathcal{S} \). Accordingly, let \( B \in \mathcal{S} \) and \( \gamma = \sum s \otimes \gamma_s \in \mathbb{R}^n \otimes \mathbb{R}^m \) be given. Compute,
\[ \langle \Delta \gamma, \gamma \rangle = \sum_s \langle \mathbf{T} \gamma_s, \gamma_s \rangle - \sum_j \sum_{s,t} (B_j)_{s,t} \langle Y_j \gamma_s, \gamma_t \rangle. \]
Now let \( \Gamma^* \) denote the matrix with \( s \)-th column \( \gamma_s \). Hence \( \Gamma \) is \( n \times m \). Compute,
\[ \lambda_Y(\Gamma^* \Gamma^*) = \sum_j \text{tr}((\Gamma^* B_j \Gamma) Y_j) \]
\[ = \sum_j \text{tr}(B_j (\Gamma Y_j \Gamma^*)) \]
\[ = \sum_j \sum_{s,t} (B_j)_{s,t} \langle Y_j \gamma_s, \gamma_t \rangle. \]
Similarly,
\[ \text{tr}(\Gamma^* \mathbf{T} \Gamma^*) = \sum_s \langle \mathbf{T} \gamma_s, \gamma_s \rangle. \]
Thus, using the inequality (7.6), observe that
\[ \langle \Delta \gamma, \gamma \rangle = \text{tr}(\Gamma^* \mathbf{T} \Gamma^*) - \lambda_Y(\Gamma^* B \Gamma) \geq 0 \]
to complete the proof of item (1).

The proof of item (2) follows an argument given in the proof of Proposition 7.11 and is omitted.

To prove (3), suppose that \( Y \in \mathcal{K}(m)^{\text{oc}} \). Given \( B \in \mathcal{K}(n) \), an application of the first part of the lemma with \( \mathcal{S} = \{B\} \) produces an \( m \times m \) positive semidefinite matrix \( \mathbf{T} \) with \( \text{tr}(\mathbf{T}) \leq 1 \) such that \( I \otimes \mathbf{T} - \sum B_j \otimes Y_j \succeq 0 \). Hence, \( B \in \mathfrak{N}^{\text{opp}}_Y \).

Moving on to item (4). From (3), if \( Y \in \mathcal{K}(m)^{\text{oc}} \), then \( \mathcal{K} \subseteq \mathfrak{N}^{\text{opp}}_Y \). On the other hand, if \( Y \in \mathcal{S}_m^n \) and \( \mathcal{K} \subseteq \mathfrak{N}^{\text{opp}}_Y \), then, for \( B \in \mathcal{K}(m) \),
\[ I \otimes \mathbf{T} - \sum B_j \otimes Y_j \succeq 0 \]
for some positive semidefinite \( \mathbf{T} \) with trace at most one. In particular, with \( e = \sum_{s=1}^m e_s \otimes e_s \in \mathbb{R}^m \otimes \mathbb{R}^m \),
\[ 0 \leq \langle I \otimes \mathbf{T} - \sum B_j \otimes Y_j e, e \rangle = \text{tr}(\mathbf{T}) - \lambda_Y(B). \]
Hence \( Y \in \mathcal{K}(m)^{\text{oc}} \). Continuing with the proof of (4), from item (3), we have
\[ \mathcal{K} \subseteq \bigcap_n \bigcap \{ \mathfrak{N}^{\text{opp}}_Y : Y \in \mathcal{K}(n)^{\text{oc}} \}. \]
To establish the reverse inclusion, suppose that $C$ is not in $\mathcal{K}(m)$. Since $\mathcal{K}(m)$ is assumed closed and convex, there exists a $Y \in \mathcal{K}(m)^{\circ}$, the conventional polar dual of $\mathcal{K}(m)$ (so that $\lambda_Y(\mathcal{K}(m)) \leq 1$) with $\lambda_Y(C) > 1$. In particular, $\mathcal{K} \subseteq \mathfrak{F}_Y^{\text{opp}}$. On the other hand, if $T$ is $m \times m$ and positive semidefinite with trace at most one, then with $e = \sum e_s \otimes e_s$, 

$$
\langle (I \otimes T - \sum C_j \otimes Y_j)e, e \rangle = \text{tr}(T) - \sum \text{tr}(C_j Y_j) = \text{tr}(T) - \lambda_Y(C) < 0.
$$

Hence, $C \notin \mathfrak{F}_Y^{\text{opp}}$.

To prove item (5), let $\mathcal{H}$ denote the contractively convex hull of $\mathcal{K}$. Let also $\mathcal{I}$ denote the intersection of the tracial spectrahedra $\mathfrak{F}_Y^{\text{opp}}$ such that $\mathcal{K} \subseteq \mathfrak{F}_Y^{\text{opp}}$. Evidently $\mathcal{H} \subseteq \mathcal{I}$. On the other hand, using item (4), 

$$
\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{I} \subseteq \bigcap \{ \mathfrak{F}_Y^{\text{opp}} : \mathfrak{F}_Y^{\text{opp}} \supseteq \mathcal{H} \} = \mathcal{H}.
$$

Finally, for item (6), first suppose $Y \in \hat{\mathcal{K}}(n)$. By definition, for each $B \in \mathcal{K}$ there is a positive semidefinite $T$ of trace at most one such that $I \otimes T - \sum B_j \otimes Y_j \succeq 0$. Hence $\mathcal{K} \subseteq \mathfrak{F}_Y^{\text{opp}}$. Conversely, if $B \in \mathfrak{F}_Y^{\text{opp}}$, then $I \otimes T - \sum B_j \otimes Y_j \succeq 0$ for some positive semidefinite $T$ of trace at most one depending on $B$. Thus, if $\mathcal{K} \subseteq \mathfrak{F}_Y^{\text{opp}}$, then $Y \in \hat{\mathcal{K}}(n)$. 

**Proof of Proposition 7.12.** Since 

$$
(\hat{\mathcal{K}})^{\circ} = \bigcap_{B \in \mathcal{K}^{\circ}} \hat{\mathfrak{F}}_B = \bigcap_{\mathcal{K} \subseteq \hat{\mathfrak{F}}_B} \hat{\mathfrak{F}}_B,
$$

item (ii) of Theorem 7.6 gives the conclusion of the first part of the proposition.

Likewise, 

$$
(\hat{\mathcal{K}})^{\circ} = \{ B : \hat{\mathcal{K}} \subseteq \hat{\mathfrak{F}}_B \} = \bigcap_{Y \in \hat{\mathcal{K}}} \mathfrak{F}_Y^{\text{opp}} = \bigcap \{ \mathfrak{F}_Y^{\text{opp}} : \mathfrak{F}_Y^{\text{opp}} \supseteq \mathcal{K} \}
$$

and the term on the right hand side is, by Lemma 7.13, the closed contractive convex hull of $\mathcal{K}$.

**7.5. Matrix Convex Tracial Sets and Free Cones.** In this subsection we introduce and study properties of free (convex) cones.

A set $\mathcal{S}$ in $\mathbb{S}^g$ is a **free cone** if for all positive integers $m, n, \ell$, tuples $T \in \mathcal{S}(n)$ and $n \times m$ matrices $C_1, \ldots, C_\ell$, the tuple $\sum C_i^* T C_i$ is in $\mathcal{S}(m)$. The set $\mathcal{S}$ is a **free convex cone** if for all positive integers $m, n, \ell$, tuples $T^1, \ldots, T^\ell \in \mathcal{S}(n)$ and $n \times m$ matrices $C_1, \ldots, C_\ell$, the tuple $\sum_i C_i^* T_i C_i$ lies in $\mathcal{S}(m)$. Finally, a subset $\mathcal{Y}$ of $\mathbb{S}^g$ is a **contractively tracial convex set** if $\mathcal{Y}$ is contractively tracial and given positive integers $m, n, \mu$ and $Y^1, \ldots, Y^\mu \in \mathcal{Y}(m)$
and $n \times m$ matrices $C_1, \ldots, C_\mu$ with $\sum C_j^* C_j \preceq I$, the tuple 

$$\sum C_j Y^j C_j^*$$

lies in $\mathcal{Y}(n)$. This condition is an analog to matrix convexity of a set containing 0 which we studied earlier in this paper. Surprisingly, we have:

**Proposition 7.14.** Every contractively tracial convex set is a free convex cone.

For the proof of this proposition we introduce an auxiliary notion and then give a lemma. A subset $\mathcal{Y}$ of $\mathbb{S}^g$ is **closed with respect to identical direct sums** if for each $Y \in \mathcal{Y}$ and positive integer $\ell$, the tuple $I_\ell \otimes Y$ is in $\mathcal{Y}$.

**Lemma 7.15.** Suppose $\mathcal{Y} \subseteq \mathbb{S}^g$.

1. If $\mathcal{Y}$ is contractively tracial and closed with respect to identical direct sums, then $\mathcal{Y}$ is a free cone.

2. If $\mathcal{Y}$ is contractively tracial and closed with respect to direct sums, then $\mathcal{Y}$ is a free convex cone.

3. If $\mathcal{Y}$ is a tracial set containing 0 which is levelwise convex and closed with respect to identical direct sums, then each $\mathcal{Y}(m)$ is a cone in the ordinary sense.

**Proof.** To prove the first statement, let $Y \in \mathcal{Y}(n)$ and a positive integer $\ell$ be given. Let $V_k$ denote the block $1 \times \ell$ row matrices with $m \times n$ matrix entries with $I_n$ in the $k$-th position and 0 elsewhere, for $k = 1, \ldots, \ell$. It follows that $\sum V_k^* V_k = I$. Since also $I_\ell \otimes Y$ is in $\mathcal{Y}$ and $\mathcal{Y}$ is tracial,

$$\sum V_k (Y \otimes I_\ell) V_k^* = kY \in \mathcal{Y}(n).$$

Now let positive integers $m$ and $\ell$ and $m \times n$ matrices $C_1, \ldots, C_\ell$ and $Y^1, \ldots, Y^\ell \in \mathcal{Y}(n)$ be given. Choose a positive integer $k$ such that each $D_j = \frac{C_j}{\sqrt{k}}$ has norm at most one. Consider $M_j$ equal the block $1 \times \ell$ row matrix with $m \times n$ entries with $D_j$ in the $j$-th position and 0 elsewhere, for $j = 1, \ldots, \ell$. It follows that

$$(7.7) \quad \sum_j M_j^* M_j = \text{diag}(D_1^* D_1, \ldots, D_\ell^* D_\ell) \preceq I.$$ 

Since $\mathcal{Y}$ is tracial, and assuming either $Y^j = Y^k$ for all $j,k$ and $\mathcal{Y}$ is closed under identical direct sums or assuming that $\mathcal{Y}$ is closed under direct sums, $\bigoplus_{j=1}^\ell Y^j$ is in $\mathcal{Y}$ and hence,

$$(7.8) \quad \sum_j M_j (\bigoplus_{j=1}^\ell Y^j) M_j^* = k \sum_j D_j Y^j D_j^* = \sum C_j Y^n C_j^* \in \mathcal{Y}(n).$$

Thus, in the first case $\mathcal{Y}$ is a free cone and in the second a free convex cone.
To prove the third statement, note that the argument used to prove the first part of the lemma shows, if \( \mathcal{Y} \) is a tracial set that is closed with respect to identical direct sums and if each \( C_j = I \), then \( \ell \mathcal{Y} = \sum C_j (Y \otimes I_\ell) C_j^* \) is in \( \mathcal{Y}(n) \). If \( \mathcal{Y} \) is levelwise convex, since also \( 0 \in \mathcal{Y}(n) \), it follows that \( \mathcal{Y}(n) \) is a convex cone. \( \blacksquare \)

**Proof of Proposition 7.14.** Fix positive integers \( n \) and \( \nu \). Let \( Y^1, \ldots, Y^\nu \in \mathcal{Y}(n) \) be given. Let \( C_\ell \) denote the inclusion of \( \mathbb{R}^n \) as the \( \ell \)-th coordinate in \( \mathbb{R}^{n\nu} = \bigoplus_{i=1}^\nu \mathbb{R}^n \). In particular, \( C_\ell^* C_\ell = I_n \) and hence, \( Z_\ell = C_\ell Y_\ell C_\ell^* \in \mathcal{Y}(n\nu) \) (based only on \( \mathcal{Y} \) being a tracial set). Now let \( V_\ell \) denote the block \( \nu \times \nu \) matrix with \( n \times n \) entries with \( I_n \) in the \( (\ell, \ell) \) position and zeros \((n \times n \) matrices) elsewhere. Note that \( \sum V_\ell^* V_\ell = I_{n\nu} \). Hence,

\[
\sum V_\ell Z_\ell V_\ell^* = \text{diag} \left( Y^1, Y^2, \ldots, Y^\nu \right) \in \mathcal{Y}(n\nu).
\]

Thus \( \mathcal{Y} \) is closed with to identical direct sums. By the second part of Lemma 7.15, \( \mathcal{Y} \) is a free convex cone. \( \blacksquare \)

**Remark 7.16.** If \( \mathcal{Y} \subseteq \mathbb{S}_n^g \) is a cone and if \( B \in \mathbb{S}_n^g \) is in the polar dual of the set \( \mathcal{U} \) consisting of tuples \( \sum C_j Y^j C_j^* \) for \( Y^j \in \mathcal{Y} \) and \( C_j \) such that \( \sum C_j^* C_j \preceq I \), then

\[
\sum B_j \otimes Y_j \preceq 0
\]

for all \( Y \in \mathcal{Y}(m) \). In particular, the polar dual \( \mathcal{B} = \mathcal{Y}^\circ \) of a cone \( \mathcal{Y} \) is a free convex cone.

To prove this assertion, pick \( B \in \mathbb{S}_n^g \) in the polar dual of \( \mathcal{U} \). Fix a positive integer \( m \). By Lemma 7.4, there exists a positive semidefinite \( T \) with trace at most one such that

\[
I \otimes T - \sum B_j \otimes Y_j \succeq 0
\]

for all \( Y \in \mathcal{Y}(m) \). Since \( \mathcal{Y}(m) \) is a cone, \( I \otimes T - \sum B_j \otimes t^2 Y_j \succeq 0 \) for all real \( t \) and hence

\[
- \sum B_j \otimes Y_j \succeq 0.
\]

It follows that

\[
(7.9) \quad - \sum C^* B_j C \otimes Y_j \succeq 0
\]

for any \( C \). The conventional polar dual of a set is convex, which implies convex combinations with various \( C_j \) in (7.9) are in \( \mathcal{B} \). Hence \( \mathcal{B} \) is a free convex cone. \( \square \)
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J. William Helton, Department of Mathematics, University of California, San Diego
E-mail address: helton@math.ucsd.edu

Igor Klep, Department of Mathematics, The University of Auckland, New Zealand
E-mail address: igor.klep@auckland.ac.nz

Scott McCullough, Department of Mathematics, University of Florida, Gainesville
E-mail address: sam@math.ufl.edu