CONVERGENCE ANALYSES OF DAVIS–YIN SPLITTING VIA SCALED RELATIVE GRAPHS

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Abstract. Davis–Yin splitting (DYS) has found a wide range of applications in optimization, but its linear rates of convergence have not been studied extensively. The scaled relative graph (SRG) simplifies the convergence analysis of operator splitting methods by mapping the action of the operator to the complex plane, but the prior SRG theory did not fully apply to the DYS operator. In this work, we formalize an SRG theory for the DYS operator and use it to obtain tighter contraction factors.

Key words. Convex optimization, Splitting methods, Monotone operators, Euclidean geometry

MSC codes. 68Q25, 68R10, 68U05

1. Introduction. The Davis–Yin splitting (DYS) operator [18] encodes solutions of the three-operator monotone inclusion problem

\[
\text{find } x \in \mathcal{H} \in (\mathbb{A} + \mathbb{B} + \mathbb{C})x,
\]

where \( \mathcal{H} \) is a real Hilbert space, \( \mathbb{A} \) and \( \mathbb{B} \) are monotone, and \( \mathbb{C} \) is cocoercive, as fixed-points of the operator

\[
\mathbb{T} = \mathbb{I} - J_{\alpha \mathbb{B}} + J_{\alpha \mathbb{A}}(2J_{\alpha \mathbb{B}} - \mathbb{I} - \alpha CJ_{\alpha \mathbb{B}}),
\]

where \( \alpha > 0 \) and \( J_{\alpha \mathbb{A}} \) and \( J_{\alpha \mathbb{B}} \) are the resolvents of \( \mathbb{A} \) and \( \mathbb{B} \). DYS generalizes the forward-backward splitting (FBS) and Douglas–Rachford splitting (DRS) operators, and it has found a wide range of applications [7, 24, 44, 46–48] in optimization. However, linear convergence rates of DYS iterations have not been studied extensively.

On the other hand, the recently proposed scaled relative graph (SRG) [38] allows one to analyze the convergence of operator splitting methods by mapping the action of a multi-valued nonlinear operator to the extended complex plane, analogous to how the spectrum maps the action of a linear operator to the complex plane. Prior work used the SRG to analyze rates of convergence for FBS and DRS [38], but the prior SRG theory did not fully apply to the DYS operator.

In this paper, we analyze convergence rates of DYS iterations using the SRG. First, we extend the SRG theory to accommodate the DYS operator. We then establish linear rates of convergence of DYS by bounding the maximum modulus of the SRG on the complex plane. We furthermore provide a new result on averagedness coefficients of DYS.

1.1. Prior Works. Splitting methods for monotone inclusion problems is a powerful framework encompassing a wide range of convex optimization algorithms [2, 37, 40]. Classical splitting methods such as forward-backward splitting (FBS) [3, 6, 30], Douglas–Rachford splitting (DRS) [19, 28, 32], and alternating directions method of multipliers (ADMM) [21, 22], a derived splitting algorithm, have had an enormous range of applications. As a splitting scheme unifying FBS and DRS, Generalized forward-backward splitting (GFBS) [36] was first proposed, and Forward-Douglas–Rachford splitting (FDRS) [5] was then presented as an extension of GFBS.

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In 2017, Davis and Yin proposed what we call the Davis-Yin splitting (DYS) as a complete unification of FBS and DRS [18]. This breakthrough has found a wide range of new applications [7, 24, 44, 46–48] and many variants of DYS such as stochastic DYS [8, 34, 49–51], inexact DYS [52], adaptive DYS [35], inertial DYS [15], and primal-dual DYS [41] have been proposed.

However, the study of linear convergence rates of DYS has been limited. Bredies, Chenchene, Lorenz, and Naldi studied the convergence of DYS under a wider range of stepsizes [4] and Aragón-Artacho and Torregrosa-Belén also proved the convergence of the DYS iteration over a wider range of stepsizes [1]. Dao and Phan analyzed a modified version of DYS iterations [16], and Pedregosa analyzed sublinear convergence rates of DYS iterations [33], but they did not analyze linear rates of convergence. Condat and Richtárik analyzed linear convergence rates of a randomized version of Primal-Dual Davis–Yin (PDDY) [14]. Ryu, Taylor, Bergeling, and Giselsson [39] and Wang, Fazlyab, Chen, and Preciado [45] formulated the problem of computing tight contraction factors as SDPs using the performance estimation problem (PEP) and integral quadratic constraint (IQC) approaches, but these are numerical results, not analytic convergence proofs in the classical sense. To the best of our knowledge, the only prior works providing analytic linear convergence rates of DYS iterations are the arXiv version of the original DYS paper [18] and [14]. The latter work [14] considers randomized updates, which makes the setup distinct from ours.

On the other hand, the theory of scaled relative graph (SRG) by Ryu, Hannah, and Yin [38] allows one to analyze the convergence of operator splitting methods by mapping the action of a multi-valued nonlinear operator to the extended complex plane, analogous to how the spectrum maps the action of a linear operator to the complex plane. Follow-up work has extended the theory and applied it to analyze nonlinear operators: Huang, Ryu, and Yin characterized the SRG of normal matrices [25], Pates further characterized the SRG of linear operators using the Toeplitz-Hausdorff theorem [31], and Huang, Ryu, and Yin [26] established the tightness of the averagedness coefficient of the composition of averaged operators [29] and the DYS operator [18]. SRG has also found applications in control theory: Chaffey, Forni, and Rodolphe utilized the SRG to analyze input-output properties of feedback systems [10, 11], and Chaffey and Sepulchre furthermore used it as an experimental tool to determine properties of a given model [9, 12, 13].

1.2. Contribution and organization. Section 2 presents our first major contribution, formalizing an SRG theory for the DYS operator. This result reduces the problem of analyzing the DYS operator into bounding the maximum modulus of a subset of the complex plane. Section 3 presents our second major contribution, analyzing contraction factors of the DYS operator. These factors are tighter than those shown in [18] and are, to the best of our knowledge, the best known factors. Finally, Section 4 presents a minor contribution, a tight averagedness coefficient of the DYS operator under the assumption of strong monotonicity of $\mathbb{A}$ or $\mathbb{B}$ and Lipschitzness of $\mathbb{C}$. This is, to the best of our knowledge, the first averagedness result of DYS without cocoercivity of $\mathbb{C}$.

1.3. Preliminaries. Write $\mathcal{H}$ for a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Write $\mathbb{C} = \mathbb{C} \cup \{ \infty \}$ to denote the extended complex plane, and define $1/0 = \infty$ and $1/\infty = 0$ in $\mathbb{C}$. We use Minkowski-type set notations: for $\alpha \in \mathbb{C}$ and $W, Z \subseteq \mathbb{C}$, write

$$W \alpha = \alpha W = \{ \alpha w \mid w \in W \}, \quad W + \alpha = \alpha + W = \{ \alpha + w \mid w \in W \},$$
and
\[ W + Z = \{ w + z \mid w \in W, z \in Z \}, \quad WZ = \{ wz \mid w \in W, z \in Z \}. \]

For \( W \subset \mathbb{C} \), write
\[ |W| = \{ |w| \mid w \in W \}, \]
where \( |w| \) is absolute value (modulus) of a complex number \( w \). To clarify, \( |W| \) does not refer to the cardinality of \( W \). For \( W \subset \mathbb{C} \), write
\[ W^{-1} = \{ w^{-1} \mid w \in W \}. \]

For \( z \in \mathbb{C} \), write \( \text{Re}\, z \) and \( \text{Im}\, z \) for the real and imaginary part of \( z \), respectively, and \( \arg z \) for the argument of \( z \) that lies within \((-\pi, \pi]\).

**Geometry.** Given any \( x, y \in \mathcal{H} \), define
\[ \angle(x, y) = \begin{cases} \arccos \left( \frac{\langle x, y \rangle}{\|x\|\|y\|} \right) & \text{if } x \neq 0, y \neq 0 \\ 0 & \text{otherwise} \end{cases} \]
to be the angle between \( x \) and \( y \). Note that \( \angle(x, y) \in [0, \pi] \). The following spherical triangle inequality is the analog of the triangle inequality on spheres.

**FACT 1 (Fact 17 [38]).** Let \( \mathcal{H} \) be a real Hilbert space. Then,
\[ |\angle(x, y) - \angle(y, u)| \leq \angle(x, u) \leq \angle(x, y) + \angle(y, u), \quad \forall x, y, u \in \mathcal{H}. \]

We define notations for disks and circles on the complex plane. For \( z \in \mathbb{C} \) and \( r \in (0, \infty) \), define
\[ \text{Circ}(z, r) = \{ w \in \mathbb{C} \mid |w - z| = r \}, \quad \text{Disk}(z, r) = \{ w \in \mathbb{C} \mid |w - z| \leq r \}. \]

**Multi-valued operators.** In general, we follow standard convex operator theoretical notations as in [2, 40]. Write \( \mathbb{A} : \mathcal{H} \rightrightarrows \mathcal{H} \) to denote that \( \mathbb{A} \) is a multi-valued operator. Define \( \text{dom}\, \mathbb{A} = \{ x \in \mathcal{H} \mid \mathbb{A}x \neq \emptyset \} \). If all outputs of \( \mathbb{A} \) are singletons or the empty set, we say \( \mathbb{A} \) is single-valued. In this case, we identify \( \mathbb{A} \) with the function from \( \text{dom}\, \mathbb{A} \) to \( \mathcal{H} \). Define the graph of an operator \( \mathbb{A} \) as
\[ \text{graph}(\mathbb{A}) = \{ (x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in \mathbb{A}x \}. \]

For notational simplicity, we often do not distinguish an operator from its graph and, for example, write \( (x, u) \in \mathbb{A} \) to mean \( u \in \mathbb{A}x \). Define the inverse of \( \mathbb{A} \) as
\[ \mathbb{A}^{-1} = \{ (u, x) \mid (x, u) \in \mathbb{A} \}, \]
scalar multiplication with an operator as
\[ \alpha \mathbb{A} = \{ (x, \alpha u) \mid (x, u) \in \mathbb{A} \}, \]
the identity operator as
\[ \mathbb{I} = \{ (x, x) \mid x \in \mathcal{H} \}, \]
and
\[ \mathbb{A} + \alpha \mathbb{I} = \alpha \mathbb{I} + \mathbb{A} = \{ (x, \alpha x + u) \mid (x, u) \in \mathbb{A} \} \]
for any \( \alpha \in \mathbb{R} \). Define the resolvent of \( \mathbb{A} \) with stepsize \( \alpha > 0 \) as
\[ \mathbb{J}_\alpha \mathbb{A} = (\mathbb{I} + \alpha \mathbb{A})^{-1}. \]
Define addition and composition of operators \(\mathbf{A}: \mathcal{H} \rightrightarrows \mathcal{H}\) and \(\mathbf{B}: \mathcal{H} \rightrightarrows \mathcal{H}\) as
\[
\mathbf{A} + \mathbf{B} = \{(x, u + v) \mid (x, u) \in \mathbf{A}, (x, v) \in \mathbf{B}\},
\]
\[
\mathbf{AB} = \{(x, s) \mid \exists u \text{ such that } (x, u) \in \mathbf{B}, (u, s) \in \mathbf{A}\}.
\]

We call \(\mathcal{A}\) a class of operators if it is a set of operators. Operators in a same class need not be defined on the same Hilbert space. For an operator \(\mathbf{A}: \mathcal{H} \rightrightarrows \mathcal{H}\) and an operator class \(\mathcal{B}\), we define sum and product of them as
\[
\mathbf{A} + \mathcal{B} = \{\mathbf{A} + \mathbf{B} \mid \mathbf{B} \in \mathcal{B}, \mathbf{B}: \mathcal{H} \rightrightarrows \mathcal{H}\},
\]
\[
\mathbf{AB} = \{\mathbf{AB} \mid \mathbf{B} \in \mathcal{B}, \mathbf{B}: \mathcal{H} \rightrightarrows \mathcal{H}\}.
\]

Note that we restrict operands to be defined on the same domain Hilbert space \(\mathcal{H}\). For an operator class \(\mathcal{A}\) and an operator \(\mathbf{B}: \mathcal{H} \rightrightarrows \mathcal{H}\), define sum and product of them as
\[
\mathcal{A} + \mathbf{B} = \{\mathbf{A} + \mathbf{B} \mid \mathbf{B} \in \mathcal{B}, \mathbf{B}: \mathcal{H} \rightrightarrows \mathcal{H}, \mathcal{H} \text{ is a Hilbert space}\}.
\]

We define \(\mathcal{AB}\) similarly. For any \(\alpha \in \mathbb{R}\), define
\[
\alpha \mathcal{A} = \{\alpha \mathbf{A} \mid \mathbf{A} \in \mathcal{A}\}
\]
and
\[
\mathcal{A} + \alpha \mathbf{I} = \alpha \mathbf{I} + \mathcal{A} = \{\alpha \mathbf{I} + \mathbf{A} \mid \mathbf{A} \in \mathcal{A}\}.
\]

Define
\[
\mathcal{A}^{-1} = \{\mathbf{A}^{-1} \mid \mathbf{A} \in \mathcal{A}\}
\]
and \(\mathbf{J}_{\alpha \mathcal{A}} = (\mathbf{I} + \alpha \mathcal{A})^{-1}\) for \(\alpha > 0\).

We now define some special operator classes. Define the class of monotone operators as
\[
\mathcal{M} = \{\mathbf{A}: \mathcal{H} \rightrightarrows \mathcal{H} \mid \langle x - y, u - v \rangle \geq 0, \forall u \in \mathbf{Ax}, v \in \mathbf{Ay}, \mathcal{H} \text{ is a Hilbert space}\}.
\]

For \(\mu \in (0, \infty)\), define the class of \(\mu\)-strongly monotone operators as
\[
\mathcal{M}_\mu = \{\mathbf{A}: \mathcal{H} \rightrightarrows \mathcal{H} \mid \langle x - y, u - v \rangle \geq \mu \|x - y\|^2, \forall u \in \mathbf{Ax}, v \in \mathbf{Ay}, \mathcal{H} \text{ is a Hilbert space}\}.
\]

For \(L \in (0, \infty)\), define the class of \(L\)-Lipschitz operators as
\[
\mathcal{L}_L = \{\mathbf{A}: \text{dom } \mathbf{A} \to \mathcal{H} \mid \|u - v\|^2 \leq L^2\|x - y\|^2, \forall u \in \mathbf{Ax}, v \in \mathbf{Ay}, \text{dom } \mathbf{A} \subseteq \mathcal{H}, \mathcal{H} \text{ is a Hilbert space}\}.
\]

For \(\beta \in (0, \infty)\), define the class of \(\beta\)-cocoercive operators as
\[
\mathcal{C}_\beta = \{\mathbf{A}: \text{dom } \mathbf{A} \to \mathcal{H} \mid \langle x - y, u - v \rangle \geq \beta \|u - v\|^2, \forall u \in \mathbf{Ax}, v \in \mathbf{Ay}, \text{dom } \mathbf{A} \subseteq \mathcal{H}, \mathcal{H} \text{ is a Hilbert space}\}.
\]

For \(\theta \in (0, 1)\), define the class of \(\theta\)-averaged operators as
\[
\mathcal{N}_\theta = (1 - \theta)\mathbf{I} + \theta \mathcal{L}_1.
\]
Note that we automatically identify Lipschitz and cocoercive operators, as well as resolvents of monotone operators, as single-valued functions since they are necessarily single-valued. For notational convenience, we write

\[ M_0 = C_0 = M, \quad L_\infty = \{ A : H \ni H | H \text{ is a Hilbert space} \}. \]

To clarify,

\[ M_0 \neq \bigcup_{\mu > 0} M_\mu, \quad C_0 \neq \bigcup_{\beta > 0} C_\beta, \quad L_\infty \neq \bigcup_{L < \infty} L. \]

**Scaled relative graphs.** Define the SRG of an operator \( A : H \ni H \) as

\[
G(A) = \left\{ \frac{|u - v|}{|x - y|} \exp[\pm i\angle(u - v, x - y)] \mid u \in Ax, v \in Ay, x \neq y \right\}
\]

\[ \cup \{ \infty \} \text{ if } A \text{ is not single-valued}. \]

SRG depicts a change in output relative to the change in input of operators using complex numbers, analogous to how eigenvalues capture essential information about the action of linear operators. We define the SRG of a class of operators \( A \) as

\[ G(A) = \bigcup_{A \in A} G(A). \]

**Fact 2** (Proposition 1 [38]). Let \( \mu, \beta, L \in (0, \infty) \) and \( \theta \in (0, 1) \). The SRGs of \( L, N_\theta, M, M_\mu, \) and \( C_\beta \) are, respectively, given by

We say a class of operators \( A \) is SRG-full if

\[ A \in A \implies G(A) \subseteq G(A). \]

In essence, a class is SRG-full when it can be entirely characterized by its SRG.

**Fact 3** (Theorem 2, 3 [38]). Let \( \mu, \beta, L \in (0, \infty) \) and \( \theta \in (0, 1) \). Then, \( M, M_\mu, C_\beta, L, \) and \( N_\theta \) are SRG-full classes. In addition, their intersections are also SRG-full classes.
Fact 4 (Theorem 4, 5 [38]). If $A$ is a class of operators, then

$$G(\alpha A) = \alpha G(A), \quad G(I + A) = 1 + G(A), \quad G(A^{-1}) = G(A)^{-1}.$$  

where $\alpha$ is a nonzero real number. If $A$ is furthermore SRG-full, then $\alpha A, I + A,$ and $A^{-1}$ are SRG-full.

Note that Fact 4 implies that $J_{\alpha A} = (I + \alpha A)^{-1}$ is an SRG-full class for $\alpha > 0$ provided that $A$ is an SRG-full class.

Fact 5. If $A$ is an SRG-full class and

$$\frac{|u - v|}{|x - y|} \exp[i\angle(u - v, x - y)] \in G(A)$$

for any $u, v, x, y \in H$ with $x \neq y$, there exists a single-valued operator $A \in A$ such that

$$Ax = u, \quad Ay = v.$$

Proof. Define $A = \{(x, u), (y, v)\}$. Then,

$$G(A) = \frac{|u - v|}{|x - y|} \exp[\pm i\angle(u - v, x - y)] \subseteq G(A)$$

holds, so $A \in A$ by SRG-fullness.

Fact 6. Let $H$ be a real Hilbert space. If $A : H \rightarrow H$ is a single-valued operator such that $\text{dom}A$ is a singleton, $A$ is an element of any SRG-full class.

Proof. $G(A) = \emptyset$ is a subset of the SRG of any SRG-full class.

Arc property. Denote $\overline{z}$ the conjugate of $z \in \mathbb{C}$. Define the right-hand arc between $z, \overline{z} \in \mathbb{C}$ as

$$\text{Arc}^+(z, \overline{z}) = \left\{re^{i(1-2\theta)} \right| z = re^{i\varphi}, \varphi \in (-\pi, \pi], r \geq 0, \theta \in [0, 1] \right\}$$

and the left-hand arc between $z$ and $\overline{z}$ as

$$\text{Arc}^-(z, \overline{z}) = -\text{Arc}^+(z, \overline{z}).$$

Note that both $r$ and $\varphi$ used in defining the right-hand arc are uniquely determined unless $z = 0$, so the set defines an arc, not a cone. If an SRG-full class $A$ satisfies

$$\text{Arc}^+(z, \overline{z}) \subseteq G(A), \quad \forall z \in G(A) \setminus \{\infty\},$$

we say $A$ satisfies the right-arc property. Similarly, if

$$\text{Arc}^-(z, \overline{z}) \subseteq G(A), \quad \forall z \in G(A) \setminus \{\infty\},$$

holds, we say $A$ has the left-arc property. If an SRG-full class $A$ has either the right-arc property or the left-arc property, then it is said to have an arc property.

The maximum modulus theorem. Define the boundary of $A \subseteq \mathbb{C}$ as $\partial A = \text{cl}A \setminus \text{int}A$, where $\text{cl}A$ and $\text{int}A$ are the closure and interior of $A$, respectively.

Fact 7 (The maximum modulus theorem [42]). Let $D \subseteq \mathbb{C}$ be a compact set. If $h : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, then $\max_{z \in D} |h(z)|$ is attained at some point of $\partial D$. 

Analogously, define for operator classes \( A \) of single-valued operators. Assume \( G \) and \( A \) be the class of DYS operators for operator classes with stepsize \( \alpha \). 3.3 that we later prove, the fact that \( (2 \) such that \( \partial A \). In this section, we establish the correspondence between \( T_{A,B,C,\alpha,\lambda} \), \( C \), and \( \partial \) by Fact 7. Applying similar procedure with respect to other arguments, we obtain \( z_1^A \in \partial A \) such that

\[
|f(z_1^A, z_B^0, z_C^0)| = |f(z_A^0, z_B^0, z_C^0)|
\]

by Fact 7. Applying similar procedure with respect to other arguments, we obtain \( z_1^B \in \partial B \) and \( z_1^C \in \partial C \) such that

\[
|f(z_1^A, z_1^B, z_1^C)| = |f(z_A^0, z_B^0, z_C^0)|
\]
In Section 4, we use Theorem 2.1 with \( s \neq 0 \). In Section 3, we use Theorem 2.1 with \( s = 0 \) to obtain contraction factors of the DYS operator by bounding the maximum modulus of \( Z^{DYS}_{A,B,C,\alpha,\lambda} \) as described in the following corollary.

**Corollary 2.2.** Let \( A, B, C, \alpha, \) and \( \lambda \) be as in Theorem 2.1 and \( (2 - \lambda)I - \alpha C \) has an arc property. Then,

\[
\sup_{x,y \in \text{dom } T, x \neq y} \frac{\|Tx - Ty\|}{\|x - y\|} = \sup_{z \in Z^{DYS}_{A,B,C,\alpha,\lambda}} |z|.
\]

**Proof.** With \( s = 0 \) in Theorem 2.1, we have

\[
|G(T_{A,B,C,\alpha,\lambda})| = |Z^{DYS}_{A,B,C,\alpha,\lambda}|.
\]

and the left-hand side can be expressed as

\[
|G(T_{A,B,C,\alpha,\lambda})| = \left\{ \frac{\|Tx - Ty\|}{\|x - y\|} \mid T \in T_{A,B,C,\alpha,\lambda}, x, y \in \text{dom } T, x \neq y \right\}.
\]

Notably, Corollary 2.2 allows one to compute the tight contraction factor by calculating the maximum modulus of a set of complex numbers \( Z^{DYS}_{A,B,C,\alpha,\lambda} \) or, equivalently, by bounding the maximum of \( |\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)| \) subject to the constraints \( z_A \in G(J_{A,A}), z_B \in G(J_{A,B}), \) and \( z_C \in G(C) \) on the inputs to the complex polynomial \( \zeta_{DYS} \).

The arc property requirement on \( C \) holds under the standard assumption that \( C \) is cocoercive. Specifically, if \( \alpha, \beta_C \in (0, \infty) \) and \( C = C_{\beta_C} \), then \( \zeta((2 - \lambda)I - \alpha C) \) is a disk with its center on the real axis, thus an arc property is satisfied. However, the conclusion of Corollary 2.2 may fail if the arc property is not satisfied, as we demonstrate in the arXiv version of this paper \([27]\) by comparing the maximum modulus with tight factor obtained using the performance estimation problem (PEP) framework \([20,39,43]\).

When the arc property fails, we can consider an enlarged class \( C' \supset C \) such that \( (2 - \lambda)I - \alpha C' \) does satisfy an arc property.

**Corollary 2.3.** Let \( A, B, C, \alpha, \) and \( \lambda \) be as in Theorem 2.1 and let \( s = 0 \). Assume \( C \) is a class of single-valued operators, but do not assume that \( (2 - \lambda)I - \alpha C \) has an arc property. If there exists some SRG-full class \( C' \) of single-valued operators such that

\[
C' \supset C, \quad (2 - \lambda)I - \alpha C' \text{ has an arc property}
\]

then

\[
\sup_{x,y \in \text{dom } T, x \neq y} \frac{\|Tx - Ty\|}{\|x - y\|} \leq \sup_{z \in Z^{DYS}_{A,B,C',\alpha,\lambda}} |z|.
\]

(The right-hand side depends on \( C' \), rather than \( C \).)

**Proof.** Since \( C \subseteq C' \), we have \( T_{A,B,C,\alpha,\lambda} \subseteq T_{A,B,C',\alpha,\lambda} \). Therefore,

\[
\sup_{x,y \in \text{dom } T, x \neq y} \frac{\|Tx - Ty\|}{\|x - y\|} \leq \sup_{x,y \in \text{dom } T, x \neq y} \frac{\|Tx - Ty\|}{\|x - y\|} = \sup_{z \in Z^{DYS}_{A,B,C',\alpha,\lambda}} |z|,
\]

where the last equality follows from Corollary 2.2.
Note that we can apply Theorem 2.1, as well as Corollaries 2.2 and 2.3, for $A = M_{\mu_A} \cap C_{\beta_A} \cap L_{L_A}$, where $\mu_A \in [0, \infty)$, $\beta_A \in [0, \infty)$, and $L_A \in (0, \infty)$. (This includes classes like $M \cap L$ by setting $\mu_A = \beta_A = 0$.) To see this, observe that SRGs of $\mathcal{I} + \alpha M_{\mu_A}$, $\mathcal{I} + \alpha C_{\beta_A}$, and $\mathcal{I} + \alpha L_{L_A}$ are in form of a disk with its center being a nonnegative real number or $\{ z \in \mathbb{C} \mid \text{Re } z \geq a \}$ for some nonnegative $a$ and thus satisfy the right-arc property. Therefore, $\mathcal{I} + \alpha A$, as an intersection of them, is an SRG-full class of monotone operators satisfying the right-arc property.

The remainder of this section proves Theorem 2.1 with the following structure. Section 2.1 defines the quotient operator and characterizes some useful properties, Section 2.2 establishes an SRG product theorem and apply it to the quotient operator, and Section 2.3 proves Theorem 2.1.

2.1. Quotient operator. For $A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B : \mathcal{H} \rightrightarrows \mathcal{H}$, define the quotient operator of $A$ over $B$ as

$$
\frac{A}{B} = A/B = AB^{-1}.
$$

Correspondingly, for a class of operators $A$, define

$$
\frac{A}{B} = A/B = AB^{-1}.
$$

For a single-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$, define the selection of $A$ with respect to distinct $x, y \in \mathcal{H}$ as

$$
A_{x,y} = \begin{cases} 
\{ (x, Ax), (y, Ay) \} & \text{if } x, y \in \text{dom } A, \\
\{ (x, Ax) \} & \text{if } x \in \text{dom } A, y \notin \text{dom } A, \\
\{ (y, Ay) \} & \text{if } x \notin \text{dom } A, y \in \text{dom } A, \\
\emptyset & \text{otherwise.}
\end{cases}
$$

We clarify that $A_{x,y} = A/I_{x,y}$.

We point out some immediate consequences. Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be single-valued operators. Then, for distinct $x, y \in \mathcal{H}$,

$$
\mathcal{G} \left( \frac{A}{B} \right) = \begin{cases} 
\frac{\|Ax - Ay\|}{\|Bx - By\|} \exp[\pm i \angle (Ax - Ay, Bx - By)] & \text{if } Bx \neq By, \\
\infty & \text{if } Ax \neq Ay, Bx = By, \\
\emptyset & \text{otherwise.}
\end{cases}
$$

If $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a single-valued operator, then

$$
\bigcup_{x,y \in \mathcal{H}} \mathcal{G} \left( \frac{A}{I_{x,y}} \right) = \mathcal{G} (A).
$$

Informally, the following lemma says that for SRG-full classes, we can recover the entire SRG from a single $x, y$ pair.

**Lemma 2.4.** Let $\mathcal{H}$ be a real Hilbert space with $\dim \mathcal{H} \geq 2$. For any distinct $x, y \in \mathcal{H}$ and an SRG-full class $A$ consisting of single-valued operators,

$$
\mathcal{G} \left( A/I_{x,y} \right) = \mathcal{G} (A).
$$
Proof. For any $A : H \to H$ where $A \in A$, we have $G(A/I_{x,y}) \subseteq G(A)$ from (2.2), and $G(A/I_{x,y}) \subseteq G(A)$.

We now prove $G(A) \subseteq G(A/I_{x,y})$. Let $z_A \in G(A)$ with $\text{Im} z_A \geq 0$. Then, there exist $u, v \in H$ such that

$$z_A = \frac{\|u - v\|}{\|x - y\|} \exp[i\angle(u - v, x - y)],$$

since $\text{dim} H \geq 2$. By Fact 5, there exists $A \in A$ such that $A : H \to H$ and $Ax = u$, $Ay = v$. Hence, $z_A \in G(A/I_{x,y}) \subseteq G(A/I_{x,y})$. Since SRGs are symmetric about the real axis, we can apply the same reasoning for $z_A \in G(A)$ with $\text{Im} z_A < 0$. Hence, $G(A) \subseteq G(A/I_{x,y})$, and we conclude the proof. \(\square\)

We now introduce some rules of calculus with the quotient operators.

Lemma 2.5. (Calculus of quotient operators) Let $A, B, C, D : H \to H$ be single-valued operators and distinct $x, y \in H$. Then the following identities hold:

\[
\begin{aligned}
(2.3) & \quad G(A/B_{x,y}) = G(B/A_{x,y})^{-1} \\
(2.4) & \quad (A + tB)/B_{x,y} = A/B_{x,y} + tI \quad (t \in \mathbb{R}) \\
(2.5) & \quad (tA)/B_{x,y} = t(A/B_{x,y}) \quad (t \in \mathbb{R}) \\
(2.6) & \quad |G(A/C_{x,y})| = |G(A/B_{x,y})||G(B/C_{x,y})| \quad (Bx \neq By, Cx \neq Cy) \\
(2.7) & \quad G((A + D)/C_{x,y}) = G(A/C_{x,y}) \quad (Dx = Dy).
\end{aligned}
\]

Proof. In our proofs of (2.3), (2.4), and (2.5), we assume $x, y \in \text{dom } A \cap \text{dom } B$, as otherwise the statements are straightforwardly true due to (2.1). We assume similar conditions for (2.7) regarding domains of $A$ and $C$.

(2.3) If $Ax \neq Ay$ and $Bx \neq By$,

$$G(A/B_{x,y}) = \frac{\|Ax - Ay\|}{\|Bx - By\|} \exp[\pm i\angle(Ax - Ay, Bx - By)]$$

$$= \left(\frac{\|Bx - By\|}{\|Ax - Ay\|} \exp[\pm i\angle(Ax - Ay, Bx - By)]\right)^{-1} = G(B/A_{x,y})^{-1}.$$

If $Ax = Ay$ and $Bx \neq By$, then $G(A/B_{x,y}) = 0 = \infty^{-1} = G(B/A_{x,y})^{-1}$.

If $Ax \neq Ay$ and $Bx = By$, then $G(A/B_{x,y}) = \infty = 0^{-1} = G(B/A_{x,y})^{-1}$.

(2.4) Both sides evaluate to $Ax + tBx$ and $Ay + tBy$ for inputs $Bx$ and $By$, respectively. Otherwise, both sides evaluate to $\emptyset$.

(2.5) This follows immediately from our operator notation.

(2.6) This follows from

$$|G(A/C_{x,y})| = \frac{\|Ax - Ay\|}{\|Cx - Cy\|}$$

$$= \frac{\|Ax - Ay\|}{\|Bx - By\|} \frac{\|Bx - By\|}{\|Cx - Cy\|}$$

$$= |G(A/B_{x,y})||G(B/C_{x,y})|.$$

(2.7) Observe that

$$(A + D)x - (A + D)y = Ax - Ay,$$

which, together with (2.1), gives the desired result. \(\square\)
2.2. SRG product theorem. We first establish a product theorem for SRGs, which is of independent interest. This product theorem differs from [38, Theorem 7] in that one of operands is a single fixed operator, rather than an operator class. We then apply it to the quotient operator.

**Theorem 2.6 (SRG product theorem).** Let \( H \) be a real Hilbert space with \( \dim H \geq 2 \) and \( B : H \rightharpoonup H \) be a single-valued operator. Let \( A \) be a SRG-full class of single-valued operators satisfying an arc property, and assume \( \mathcal{G}(A) \neq \emptyset \). Then,

\[
\mathcal{G}(AB) = \mathcal{G}(A) \mathcal{G}(B).
\]

(Note, \( \infty \notin \mathcal{G}(A) \) since operators in \( A \) are single-valued.)

**Proof.** We first show \( \mathcal{G}(A) \mathcal{G}(B) \subseteq \mathcal{G}(AB) \). Let \( z_A \in \mathcal{G}(A) \) and \( z_B \in \mathcal{G}(B) \). If \( z_B = 0 \), there exists distinct \( x, y \in H \) such that \( Bx = By \). Take \( A = \{(Bx, 0)\} \). Then \( A \in A \) by Fact 6. This gives

\[
z_A z_B = 0 = \frac{\|ABx - ABy\|}{\|x - y\|} \exp[i\angle(ABx - ABy, x - y)] \in \mathcal{G}(AB).
\]

Otherwise, assume \( z_B \neq 0 \). Then there exists distinct \( x, y \in H \) such that \( Bx \neq By \) and

\[
z_B \in \frac{\|Bx - By\|}{\|x - y\|} \exp[\pm i\angle(Bx - By, x - y)].
\]

There also exists orthonormal vectors \( \{e_1, e_2\} \subset H \) such that

\[
e_1 = \frac{(x - y)/\|x - y\|}{\|Bx - By\|} = \|x - y\|(Re z_B) e_1 + (Im z_B) e_2.
\]

With the \( z_A \in \mathcal{G}(A) \), define

\[
u = \|x - y\|(Re(z_A z_B)) e_1 + (Im(z_A z_B)) e_2,
\]

\[
v = 0.
\]

The definitions of \( u \) and \( v \) imply

\[
z_A z_B \in \frac{\|u - v\|}{\|x - y\|} \exp[i\angle(u - v, x - y)]
\]

and

\[
\frac{\|u - v\|}{\|Bx - By\|} \exp[i\angle(u - v, Bx - By)] \in \{z_A, z_B\} \subseteq \mathcal{G}(A).
\]

By Fact 5, there exist \( \mathbb{A} \in A \) such that \( \mathbb{A} : H \rightharpoonup H \) and

\[
ABx = u, \quad ABy = v.
\]
We conclude

$$z_A z_B \in \frac{\|ABx - ABy\|}{\|x - y\|} \exp[i \angle(ABx - ABy, x - y)]$$

$$\subseteq \mathcal{G}(AB).$$

Next, we show $\mathcal{G}(AB) \subseteq \mathcal{G}(A) \mathcal{G}(B)$. Consider any $A : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $A \in \mathcal{A}$. We will show $\mathcal{G}(AB) \subseteq \mathcal{G}(A) \mathcal{G}(B)$, and, by (2.2), this is equivalent to

$$\mathcal{G}(AB/\mathbb{I}_{x,y}) \subseteq \mathcal{G}(A) \mathcal{G}(B)$$

for all distinct $x, y \in \text{dom } AB$.

Let $x, y \in \text{dom } AB$ be distinct. If $Bx = By$, then $0 \in \mathcal{G}(B)$ and $ABx = ABy$.

Thus,

$$\mathcal{G}(AB/\mathbb{I}_{x,y}) = \{0\} \subseteq \mathcal{G}(A) \mathcal{G}(B).$$

Next, consider the case $Bx \neq By$. Define

$$z_{AB} = \frac{\|ABx - ABy\|}{\|x - y\|} \exp[i \angle(ABx - ABy, x - y)]$$

so that $\mathcal{G}(AB/\mathbb{I}_{x,y}) = \{z_{AB}, \overline{z_{AB}}\}$. Furthermore, we write

$$\varphi_B = \angle(Bx - By, x - y),$$

$$\varphi_{AB/B} = \angle(ABx - ABy, Bx - By),$$

$$z_B = \frac{\|Bx - By\|}{\|x - y\|} \exp[i \varphi_B],$$

$$z_{AB/B} = \frac{\|ABx - ABy\|}{\|Bx - By\|} \exp[i \varphi_{AB/B}].$$

Observe that $z_{AB}$, $\overline{z_{AB}} \in \mathcal{G}(B)$ and $z_{AB/B} \in \mathcal{G}(A) \subseteq \mathcal{G}(A)$.

First, suppose $A$ has the right-arc property. By Fact 1,

$$\angle(ABx - ABy, x - y) \in \left[|\varphi_B - \varphi_{AB/B}|, \varphi_B + \varphi_{AB/B}\right].$$

Therefore,

$$z_{AB} \in \frac{\|ABx - ABy\|}{\|x - y\|} \exp[i \angle(ABx - ABy, x - y)]$$

$$= \frac{\|Bx - By\|}{\|x - y\|} \frac{\|ABx - ABy\|}{\|Bx - By\|} \exp[i \angle(ABx - ABy, x - y)]$$

$$\subseteq \frac{\|Bx - By\|}{\|x - y\|} \frac{\|ABx - ABy\|}{\|Bx - By\|} \exp[i \angle(ABx - ABy, x - y)]$$

$$= z_B \text{Arc}^+(z_{AB/B}, \overline{z_{AB/B}})$$

$$\subseteq \mathcal{G}(B) \mathcal{G}(A)$$

by the right-arc property of $A$. Furthermore,

$$z_{AB} \in \overline{z_B \text{Arc}^+(z_{AB/B}, \overline{z_{AB/B}})}$$

$$= \overline{z_B \text{Arc}^+(z_{AB/B}, \overline{z_{AB/B}})}$$

$$\subseteq \mathcal{G}(B) \mathcal{G}(A).$$
Thus, \( \mathcal{G}(\mathcal{A} \mathcal{B}/\mathcal{I}_{x,y}) = \{z_{AB}, \mathcal{A}\mathcal{B}\} \subseteq \mathcal{G}(\mathcal{A}) \mathcal{G}(\mathcal{B}) \).

If \( \mathcal{A} \) has the left-arc property, \( \mathcal{A} \) has the right-arc property. As \( -\mathcal{A} \in \mathcal{A} \),
\[
\mathcal{G}(\mathcal{A} \mathcal{B}/\mathcal{I}_{x,y}) = -\mathcal{G}(-\mathcal{A} \mathcal{B}/\mathcal{I}_{x,y}) \subseteq -\mathcal{G}(-\mathcal{A}) \mathcal{G}(\mathcal{B}) = \mathcal{G}(\mathcal{A}) \mathcal{G}(\mathcal{B})
\]
by Fact 4. Therefore, by (2.2), we conclude \( \mathcal{G}(\mathcal{A} \mathcal{B}) \subseteq \mathcal{G}(\mathcal{A}) \mathcal{G}(\mathcal{B}) \) when \( \mathcal{A} \) has an arc property.

\[ \square \]

**Corollary 2.7.** Let \( \mathcal{A} \) and \( \mathcal{H} \) be as in Theorem 2.6. Let \( \mathcal{B}: \mathcal{H} \rightarrow \mathcal{H} \) and \( \mathcal{C}: \mathcal{H} \rightarrow \mathcal{H} \) be single-valued operators. For \( cx \neq cy \),
\[
\bigcup_{\mathcal{A}: \mathcal{H} \rightarrow \mathcal{H}} \mathcal{G}\left(\frac{\mathcal{A} \mathcal{B}}{x,y}\right) = \mathcal{G}\left(\frac{\mathcal{A} \mathcal{B}}{x,y}\right) = \mathcal{G}(\mathcal{A}) \mathcal{G}\left(\frac{\mathcal{B}}{x,y}\right)
\]
holds.

**Proof.** The first identity holds by definition, and the second identity follows from Theorem 2.6 with replacing \( \mathcal{B} \) in the theorem statement to \( \mathcal{B}/x,y \).

\[ \square \]

**2.3. Proof of Theorem 2.1.** We have one final preliminary lemma to establish. The proof of Theorem 2.1 primarily focuses on operators on \( \mathcal{H} \) such that \( \dim \mathcal{H} \geq 2 \) (including the case \( \dim \mathcal{H} = \infty \)). The following lemma allows to contain the case \( \dim \mathcal{H} = 1 \).

**Lemma 2.8.** Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be SRG-full classes and \( \alpha, \lambda \in (0, \infty) \). Let \( \mathcal{H} \) be a real Hilbert space. Then,
\[
\bigcup_{\mathcal{A}, \mathcal{B}, \mathcal{C}: \mathcal{H} \rightarrow \mathcal{H}} \mathcal{G}(\mathcal{T}_{\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, \lambda}) \subseteq \bigcup_{\mathcal{A}', \mathcal{B}', \mathcal{C}' : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}} \mathcal{G}(\mathcal{T}_{\mathcal{A}', \mathcal{B}', \mathcal{C}', \alpha, \lambda}).
\]
To clarify, \( \mathcal{H} \times \mathcal{H} \) is a real Hilbert space with \( \dim(\mathcal{H} \times \mathcal{H}) = 2 \dim \mathcal{H} \).

**Proof.** For an operator \( \mathcal{A}: \mathcal{H} \rightarrow \mathcal{H} \), define the embedding \( \iota \) as
\[
\iota(\mathcal{A}): \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}
\]
\[
\iota(\mathcal{A}) = \{(x, 0), (u, 0) \mid (x, u) \in \mathcal{A}\},
\]
where \( 0 \in \mathcal{H} \). Intuitively speaking, \( \iota(\mathcal{A}) \) is essentially the same operator as \( \mathcal{A} \), with \( \text{graph}(\mathcal{A}) \subseteq \mathcal{H} \times \mathcal{H} \) embedded into \( (\mathcal{H} \times \{0\}) \times (\mathcal{H} \times \{0\}) \). One can observe that
\[
\iota(\mathcal{A} + \mathcal{B}) = \iota(\mathcal{A}) + \iota(\mathcal{B}), \quad \iota(\mathcal{A} \mathcal{B}) = \iota(\mathcal{A}) \iota(\mathcal{B}),
\]
\[
\iota(\alpha \mathcal{A}) = \alpha \iota(\mathcal{A}), \quad \iota(\mathcal{I} + \mathcal{A}) = \mathcal{I} + \iota(\mathcal{A}), \quad \iota(\mathcal{A}^{-1}) = \iota(\mathcal{A})^{-1},
\]
\[
\iota(\mathcal{J}_{\alpha}(\mathcal{A})) = \mathcal{J}_{\alpha}(\iota(\mathcal{A})), \quad \iota(\mathcal{T}_{\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, \lambda}) = \mathcal{T}_{\iota(\mathcal{A}), \iota(\mathcal{B}), \iota(\mathcal{C}), \alpha, \lambda}.
\]
We omit the proof of the first five statements for conciseness, which can be done by a simple algebra. The last two expressions are consequences of the former statements. In particular, observe that
\[
\iota(\mathcal{J}_{\alpha}(\mathcal{A})) = \iota((\mathcal{I} + \alpha \mathcal{A})^{-1}) = \iota((\mathcal{I} + \alpha \mathcal{A})^{-1} = (\mathcal{I} + \alpha \iota(\mathcal{A}))^{-1} = \mathcal{J}_{\alpha}(\iota(\mathcal{A})),
\]
and
\[
\iota(\mathcal{T}_{\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, \lambda}) = \iota((\mathcal{I} - \lambda \mathcal{J}_{\alpha}(\mathcal{B}) + \lambda \mathcal{J}_{\alpha}(\mathcal{A})(2\mathcal{J}_{\alpha}(\mathcal{B}) - \mathcal{I} - \alpha \mathcal{C}\mathcal{J}_{\alpha}(\mathcal{B})))
\]
\[
= \mathcal{I} - \lambda \iota(\mathcal{J}_{\alpha}(\mathcal{B})) + \lambda \iota(\mathcal{J}_{\alpha}(\mathcal{A}))(2\mathcal{J}_{\alpha}(\mathcal{B}) - \mathcal{I} - \alpha \iota(\mathcal{C})) \iota(\mathcal{J}_{\alpha}(\mathcal{B}))
\]
\[
= \mathcal{I} - \lambda \mathcal{J}_{\alpha}(\mathcal{B}) \mathcal{J}_{\alpha}(\mathcal{A})(2\mathcal{J}_{\alpha}(\mathcal{B}) - \mathcal{I} - \alpha \iota(\mathcal{C})) \mathcal{J}_{\alpha}(\mathcal{B})
\]
\[
= \mathcal{T}_{\iota(\mathcal{A}), \iota(\mathcal{B}), \iota(\mathcal{C}), \alpha, \lambda}.
\]
Furthermore, if $A \in \mathcal{A}$, then
\[ \mathcal{G}(\iota(A)) = \mathcal{G}(A) \subseteq \mathcal{G}(\mathcal{A}). \]
Therefore, if $\mathcal{A}$ is an SRG-full class, then
\[ A \in \mathcal{A} \Rightarrow \iota(A) \in \mathcal{A}. \]
The same argument holds for $B$ and $C$ as well. Therefore, if $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$,
\[ \mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) = \mathcal{G}(\iota(\mathcal{T}_{A,B,C,\alpha,\lambda})) = \mathcal{G}(\mathcal{T}_{\iota(A),\iota(B),\iota(C),\alpha,\lambda}) \]
and $\iota(A) \in \mathcal{A}$, $\iota(B) \in \mathcal{B}$, and $\iota(C) \in \mathcal{C}$, which concludes the proof. \hfill \Box

**Proof to Theorem 2.1.** Recall that our goal is to show
\[ |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| = |\mathcal{Z}^{\text{DYS}}_{A,B,C,\alpha,\lambda} - s| \]
given that $\mathcal{A}$ and $\mathcal{B}$ are SRG-full classes of monotone operators, $\mathbb{I} + \alpha\mathcal{A}$ has the right-arc property, and $\mathcal{C}$ is an SRG-full class of single-valued operators such that $(2 - \lambda(1 - s)^{-1})\mathbb{I} - \alpha\mathcal{C}$ has an arc property.

Then,
\[ |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| \]
\[ = \bigcup_{\mathcal{H} \subseteq \mathcal{K}} \bigcup_{\mathcal{H} \subseteq \mathcal{K}} |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| \]
\[ = \bigcup_{\mathcal{H} \subseteq \mathcal{K}} \bigcup_{\mathcal{H} \subseteq \mathcal{K}} |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| \]
\[ = \bigcup_{\mathcal{H} \subseteq \mathcal{K}} \bigcup_{\mathcal{H} \subseteq \mathcal{K}} |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| \]
\[ = \bigcup_{\mathcal{H} \subseteq \mathcal{K}} \bigcup_{\mathcal{H} \subseteq \mathcal{K}} |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| \]
where (a) follows from definition of $\mathcal{T}_{A,B,C,\alpha,\lambda}$, (b) follows from Lemma 2.8, (c) follows from (2.2), (d) will be established soon, and (e) follows from the fact that $|\mathcal{Z}^{\text{DYS}}_{A,B,C,\alpha,\lambda} - s|$ has no dependency on the bound union variables $\mathcal{H}$, $x$, and $y$. We establish (d) by showing
\[ \bigcup_{\mathcal{H} \subseteq \mathcal{K}} |\mathcal{G}(\mathcal{T}_{A,B,C,\alpha,\lambda}) - s| = |\mathcal{Z}^{\text{DYS}}_{A,B,C,\alpha,\lambda} - s| \]
for any Hilbert space $\mathcal{H}$ such that $\dim \mathcal{H} \geq 2$ and distinct $x, y \in \mathcal{H}$.

Now, let $s = 1 - \lambda t$. Note that $s \neq 1$ implies $t \neq 0$ and we have assumed that $(2 - t^{-1})\mathbb{I} - \alpha\mathcal{C}$ has an arc property. The assumption that $\mathbb{I} + \alpha\mathcal{A}$ has the right-arc property implies that $J_{\alpha\mathcal{A}}$ has the right-arc property. If $\mathbb{I} + \alpha\mathcal{A}$ satisfies the right-arc property and $z^{-1} \in \mathcal{G}(J_{\alpha\mathcal{A}})$, then $\text{Arc}^+(z, z) \subseteq \mathcal{G}(\mathbb{I} + \alpha\mathcal{A})$ and $(\text{Arc}^+(z, z))^{-1} = \text{Arc}^+(z^{-1}, z^{-1}) \subseteq \mathcal{G}(J_{\alpha\mathcal{A}}) = \mathcal{G}(\mathbb{I} + \alpha\mathcal{A})^{-1})$. So $J_{\alpha\mathcal{A}}$ has the right-arc property.
For (d), we have

\[
\bigcup_{B: \mathbb{H} \supseteq \mathbb{H}, A: \mathbb{H} \supseteq \mathbb{H}} \bigcup_{A \in A, C \in C} \bigg| \mathcal{G} \left( \frac{T_{A,B,C,\alpha,\lambda}}{\mathbb{I}_{x,y}} \right) - (1 - \lambda t) \bigg|
\]

(i) \quad \bigcup_{B: \mathbb{H} \supseteq \mathbb{H}} \bigcup_{A \in A} \bigg| \lambda \mathcal{G} \left( \frac{t(1 - J_{\alpha B} + J_{\alpha A}(2J_{\alpha B} - I - \alpha CJ_{\alpha B}))}{\mathbb{I}_{x,y}} \right) \bigg|

(ii) \quad \bigcup_{B: \mathbb{H} \supseteq \mathbb{H}} \bigcup_{z_B \in \mathcal{G}(J_{\alpha B}/\mathbb{I}_{x,y})} \lambda |t - z_B + \mathcal{G}(J_{\alpha A})(2z_B - 1 - \alpha \mathcal{G}(C) z_B)|

(iii) \quad \bigcup_{z_B \in \mathcal{G}(J_{\alpha B})} \lambda |t - z_B + \mathcal{G}(J_{\alpha A})(2z_B - 1 - \alpha \mathcal{G}(C) z_B)|

(iv) \quad |\mathcal{Z}^{\text{DYS}}_{A,B,C,\alpha,\lambda} - (1 - \lambda t)|.

Step (i) follows from

\[
\bigg| \mathcal{G} \left( \frac{T_{A,B,C,\alpha,\lambda}}{\mathbb{I}_{x,y}} \right) - (1 - \lambda t) \bigg|
\]

\[
= \bigg| \mathcal{G} \left( \frac{\lambda I - \lambda J_{\alpha B} + \lambda J_{\alpha A}(2J_{\alpha B} - I - \alpha CJ_{\alpha B})}{\mathbb{I}_{x,y}} \right) \bigg|
\]

\[
= \lambda \bigg| \mathcal{G} \left( \frac{t(1 - J_{\alpha B} + J_{\alpha A}(2J_{\alpha B} - I - \alpha CJ_{\alpha B}))}{\mathbb{I}_{x,y}} \right) \bigg|
\]

where we have used (2.4) and (2.5) of Lemma 2.5 and Fact 4. We discuss (ii) soon.

Step (iii) follows from

\[
\mathcal{G}(J_{\alpha B}) = \mathcal{G}(J_{\alpha B}/\mathbb{I}_{x,y}) = \bigcup_{B: \mathbb{H} \supseteq \mathbb{H}} \mathcal{G}(J_{\alpha B}/\mathbb{I}_{x,y}),
\]

where we have used Lemma 2.4. Step (iv) follows from the definition of $\mathcal{Z}^{\text{DYS}}_{A,B,C,\alpha,\lambda}$.

We now prove (ii) to conclude the proof. We do so by showing

\[
\bigcup_{A,C: \mathbb{H} \supseteq \mathbb{H}} \bigg| \mathcal{G} \left( \frac{t(1 - J_{\alpha B} + J_{\alpha A}(2J_{\alpha B} - I - \alpha CJ_{\alpha B}))}{\mathbb{I}_{x,y}} \right) \bigg|
\]

\[
= \bigcup_{z_B \in \mathcal{G}(J_{\alpha B}/\mathbb{I}_{x,y})} |t - z_B + \mathcal{G}(J_{\alpha A})(2z_B - 1 - \alpha \mathcal{G}(C) z_B)|.
\]

If $x \notin \text{dom} J_{\alpha B}$ or $y \notin \text{dom} J_{\alpha B}$, the both sides equal \emptyset. We now consider two nontrivial
cases. First, consider the case \((J_0 - tI)x \neq (J_0 - tI)y\). We have

\[
\bigcup_{A, C: H \supset H, \ A \in A, \ C \in C} \| G \left( (I - J_0B + J_0A(2J_0B - I - \alpha CJ_0B)) \right) \| 
\]

\[
= \bigcup_{C: H \supset H, A: H \supset H} \bigcup_{C \in C} \| G \left( (\frac{tI - J_0B + J_0A(2J_0B - I - \alpha CJ_0B)}{(J_0B - tI)x,y}) \right) \| G \left( \frac{J_0B - tI}{1_{x,y}} \right) 
\]

\[
= \bigcup_{C: H \supset H} \| G \left( (2J_0B - I - \alpha CJ_0B) \right) - 1 \| G \left( \frac{J_0B - tI}{1_{x,y}} \right) 
\]

where we used (2.4), (2.5), and (2.6) of Lemma 2.5 in (\(\forall\))'s and we used Corollary 2.7 and Fact 4 in (\(\forall\))'s together with the fact that \(J_0A\) and \((2 - t^{-1})I - \alpha C\) has an arc property. The second step is induced by Fact 4. For the second to last step, we utilized (2.3), (2.4), Fact 4, and the fact that \(G(J_0B/I_{x,y})\) is a set of either one real number or one complex conjugate pair. The last step follows from direct calculations.

Now, consider the case \((J_0B - tI)x = (J_0B - tI)y\). In this case, we have

\[
\bigcup_{A, C: H \supset H, \ A \in A, \ C \in C} \| G \left( (I - J_0B + J_0A(2J_0B - I - \alpha CJ_0B)) \right) \| 
\]

\[
= \bigcup_{C: H \supset H, A: H \supset H} \bigcup_{C \in C} \| G \left( J_0A(2J_0B - I - \alpha CJ_0B) \right) \| 
\]

\[
= \bigcup_{C: H \supset H} \| G \left( J_0A \right) G \left( \frac{2J_0B - I - \alpha CJ_0B}{1_{x,y}} \right) 
\]
\[ \sum_{c: H \ni \mathcal{H}} \left| \mathcal{G}(J_{\alpha,A}) \mathcal{G} \left( \frac{(2 - t^{-1})I - \alpha C}{1_{x,y}} \right) \right| \]

\[ = \sum_{z_B \in \mathcal{G}(J_{\alpha,B}/1_{x,y})} \left| \mathcal{G}(J_{\alpha,A}) \mathcal{G} \left( (2 - t^{-1})I - \alpha \mathcal{G}(C) \right) \right| \]

where we used (2.7) of Lemma 2.5 in (\( \diamond \))’s and Corollary 2.7 in (\( \bullet \))’s. The last step follows from \( \mathcal{G}(J_{\alpha,B}/1_{x,y}) = \{t\} \).

3. Contraction factors of DYS. Using the SRG theory of Section 2, we prove new contraction factors of DYS in three settings of interest. The three contraction factors of Theorems 3.1, 3.2, and 3.3 are strictly smaller (better) than the six contraction factors of [17, Theorem D.6] and are, to the best of our knowledge, the best known rates.

Theorem 3.1. Let \( 0 < \alpha < 4\beta_C < \infty \) and \( \lambda \in \left(0, 2 - \frac{\alpha}{\beta_C} \right) \). Let \( H \) be a real Hilbert space. Let \( A: H \ni H \) and \( B: H \ni H \) be monotone and \( C: H \ni H \) be \( \beta_C \)-cocoercive. If \( A \) is \( \mu_A \)-strongly monotone and \( L_A \)-Lipschitz for \( 0 < \mu_A \leq L_A < \infty \), then \( T_{A,B,C,\alpha,\lambda} \) is \( \rho \)-contractive with

\[ \rho = 1 - \frac{2\lambda}{4 - \alpha/\beta_C} + \lambda \sqrt{\frac{2}{4 - \alpha/\beta_C} \left( \frac{2}{4 - \alpha/\beta_C} - \frac{2\alpha\mu_A}{\alpha^2 L_A^2 + 2\alpha\mu_A + 1} \right)} \]

Symmetrically, if \( B \) is \( \mu_B \)-strongly monotone and \( L_B \)-Lipschitz for \( 0 < \mu_B \leq L_B < \infty \), then \( T_{A,B,C,\alpha,\lambda} \) is \( \rho \)-contractive with

\[ \rho = 1 - \frac{2\lambda}{4 - \alpha/\beta_C} + \lambda \sqrt{\frac{2}{4 - \alpha/\beta_C} \left( \frac{2}{4 - \alpha/\beta_C} - \frac{2\alpha\mu_B}{\alpha^2 L_B^2 + 2\alpha\mu_B + 1} \right)} \]

Theorem 3.2. Let \( 0 < \alpha < 4\beta_C < \infty \) and \( \lambda \in \left(0, 2 - \frac{\alpha}{\beta_C} \right) \). Let \( H \) be a real Hilbert space. Let \( A: H \ni H \) and \( B: H \ni H \) be monotone and \( C: H \ni H \) be \( \beta_C \)-cocoercive. If \( A \) is \( L_A \)-Lipschitz and \( B \) is \( \mu_B \)-strongly monotone for \( L_A, \mu_B \in (0, \infty) \), then \( T_{A,B,C,\alpha,\lambda} \) is \( \rho \)-contractive with

\[ \rho = \sqrt{1 - \min \left\{ \frac{2\alpha\mu_B \lambda(2 - \lambda)}{(1 + \alpha^2 L_A^2)(2 - \lambda + 2\alpha\mu_B)}, \frac{2\alpha\lambda(2 - \lambda)(\mu_B + L_A) + 2\alpha\mu_B L_A}{(1 + \alpha L_A)^2(2 - \lambda + 2\alpha\mu_B)} \right\}} \]

Symmetrically, if \( A \) is \( \mu_A \)-strongly monotone and \( B \) is \( L_B \)-Lipschitz for \( \mu_A, L_B \in (0, \infty) \), then \( T_{A,B,C,\alpha,\lambda} \) is \( \rho \)-contractive with

\[ \rho = \sqrt{1 - \min \left\{ \frac{2\alpha\mu_A \lambda(2 - \lambda)}{(1 + \alpha^2 L_B^2)(2 - \lambda + 2\alpha\mu_A)}, \frac{2\alpha\lambda(2 - \lambda)(\mu_A + L_B) + 2\alpha\mu_A L_B}{(1 + \alpha L_B)^2(2 - \lambda + 2\alpha\mu_A)} \right\}} \]

Theorem 3.3. Let \( 0 < \alpha < 4\beta_C < \infty \) and \( \lambda \in \left(0, 2 - \frac{\alpha}{\beta_C} \right) \). Let \( H \) be a real Hilbert space. Let \( A: H \ni H \) and \( B: H \ni H \) be monotone and \( C: H \ni H \) be \( \beta_C \)-cocoercive. If \( A \) is \( L_A \)-Lipschitz and \( C \) is \( \mu_C \)-strongly monotone for \( L_A \in (0, \infty) \) and
0 < \mu_C \leq \frac{1}{\beta C}, \text{ then } T_{A,B,C,\alpha,\lambda} \text{ is } \rho\text{-contractive with }

\rho = \sqrt{1 - 2\lambda \alpha \min \left\{ \frac{L_A + \mu_C \left(1 - \frac{\alpha}{2\beta C(2-\lambda)}\right)}{(1 + \alpha L_A)^2}, \frac{\mu_C \left(1 - \frac{\alpha}{2\beta C(2-\lambda)}\right)}{1 + \alpha^2 L_A^2} \right\}}.

Symmetrically, if B is \(L_B\)-Lipschitz and C is \(\mu_C\)-strongly monotone for \(L_B \in (0, \infty)\) and \(0 < \mu_C \leq \frac{1}{\beta C}\), then \(T_{A,B,C,\alpha,\lambda}\) is \(\rho\text{-contractive with }

\rho = \sqrt{1 - 2\lambda \alpha \min \left\{ \frac{L_B + \mu_C \left(1 - \frac{\alpha}{2\beta C(2-\lambda)}\right)}{(1 + \alpha L_B)^2}, \frac{\mu_C \left(1 - \frac{\alpha}{2\beta C(2-\lambda)}\right)}{1 + \alpha^2 L_B^2} \right\}}.

It is straightforward to verify that the values of \(\rho\) in Theorems 3.1, 3.2, and 3.3 indeed satisfy \(\rho < 1\), i.e., \(T_{A,B,C,\alpha,\lambda}\) becomes a strict contraction, under the stated range of parameters.

3.1. Proofs of Theorems 3.1, 3.2, and 3.3. Before going through proofs, we introduce two elementary Lemmas.

**Lemma 3.4.** For \(a, b, c, d \in [0, \infty)\),

\[
\left(\sqrt{ab} + \sqrt{cd}\right)^2 \leq (a + c)(b + d).
\]

**Proof.** This inequality is an instance of Cauchy–Schwarz. It can also be directly shown from

\[
\left(\sqrt{ab} + \sqrt{cd}\right)^2 = (a + c)(b + d) - \left(\sqrt{ad} - \sqrt{bc}\right)^2.
\]

**Lemma 3.5.** Consider a circle with center \(O\). Let \(Q \neq O\) be a fixed point. (\(Q\) may otherwise be inside, outside, or on the circle.) Let \(\phi \in [0, \pi]\), and let \(P\) be a point on the circle such that \(\angle POQ = \phi\). Then the length \(QP\) is an increasing function of \(\phi \in [0, \pi]\).

**Proof.** By the law of cosines,

\[
QP^2 = PO^2 + OQ^2 - 2PO OQ \cos \phi.
\]

**Proof of Theorem 3.1.** Consider the first statement, where \(A\) is \(\mu_A\)-strongly monotone and \(L_A\)-Lipschitz. Write

\[
A = M_{\mu_A} \cap L_{L_A}, \quad B = M, \quad C = C_{\beta C}.
\]

Then the first statement is equivalent to saying that \(T_{A,B,C,\alpha,\lambda}\) is \(\rho\)-Lipschitz, where \(\rho\) is as given. By Corollary 2.2 and Lemma 1.1, it is enough to show that \(\sup_{z \in \mathbb{R}^{DYS}_{A,B,C,\alpha,\lambda}} |z| \) is equal or smaller than \(\rho\), i.e., it is enough to show

\[
|\xi_{DYS}(z_A, z_B, z_C; \alpha, \lambda)| \leq \rho
\]
for any \( z_A \in \partial \mathcal{G}(\mathcal{J}_{\alpha A}), z_B \in \partial \mathcal{G}(\mathcal{J}_{\alpha B}) \) and \( z_C \in \partial \mathcal{G}(\mathcal{C}) \).

Let \( \theta = \frac{2}{\phi - \alpha/\beta c} \). Then, \( \alpha z_C \in \alpha \partial \mathcal{G}(\mathcal{C}) = \text{Circ}(2 - \theta^{-1}, 2 - \theta^{-1}) \) by Fact 2. So \( |2 - \theta^{-1} - \alpha z_C| = 2 - \theta^{-1} \). Therefore,

\[
|\zeta_{\text{DYS}}(z_A, z_B, z_C; \alpha, \lambda) - (1 - \lambda \theta)|^2 \\
= \lambda^2 |\theta^{-1}(z_A - \theta)(z_B - \theta) + (2 - \theta^{-1} - \alpha z_C)z_B|^2 \\
\leq \lambda^2 (\theta^{-1}|z_A - \theta||z_B - \theta| + |2 - \theta^{-1} - \alpha z_C||z_A||z_B|)^2 \\
= \lambda^2 (\theta^{-1}|z_A - \theta||z_B - \theta| + (2 - \theta^{-1})|z_A||z_B|)^2 \\
= \lambda^2 (\sqrt{\theta^{-1}|z_A - \theta|^2 \cdot \theta^{-1}|z_B - \theta|^2 + (2 - \theta^{-1})|z_A|^2 \cdot (2 - \theta^{-1})|z_B|^2})^2 \\
\leq \lambda^2 (\theta^{-1}|z_A - \theta|^2 + (2 - \theta^{-1})|z_A|^2)(\theta^{-1}|z_B - \theta|^2 + (2 - \theta^{-1})|z_B|^2),
\]

where the first inequality follows from the triangle inequality and the second is from Lemma 3.4. The second factor simplifies to

\[
\theta^{-1}|z_B - \theta|^2 + (2 - \theta^{-1})|z_B|^2 = 2 \left| z_B - \frac{1}{2} \right|^2 + \theta - \frac{1}{2} = \theta,
\]
since \( z_B \in \partial \mathcal{G}(\mathcal{J}_{\alpha B}) = \text{Circ}(\frac{1}{2}, \frac{1}{2}) \) by Fact 2 and 4.

We now bound the first factor. Similarly,

\[
\theta^{-1}|z_A - \theta|^2 + (2 - \theta^{-1})|z_A|^2 = 2 \left| z_A - \frac{1}{2} \right|^2 + \theta - \frac{1}{2},
\]

which is maximized (worst case) when \( |z_A - \frac{1}{2}| \) is maximized.

Let

\[
O_{\mu} = \frac{1}{2(1 + \alpha \mu_A)}, \quad O_L = \frac{1}{1 - \alpha^2 L_A^2}
\]

be the center of \( \partial \mathcal{G}(\mathcal{J}_{\alpha \mathcal{M}_\mu A}) \) and the center of \( \partial \mathcal{G}(\mathcal{J}_{\alpha \mathcal{L}_A}) \) (or \( \infty \) if \( \alpha L_A = 1 \)), respectively. Let

\[
P = z_A, \quad Q = \frac{1}{2}, \quad \{N, N'\} = \partial \mathcal{G}(\mathcal{J}_{\alpha \mathcal{L}_A}) \cap \partial \mathcal{G}(\mathcal{J}_{\alpha \mathcal{M}_\mu A}).
\]

We now characterize the maximum value of \( QP = |z_A - \frac{1}{2}| \) for each of three cases depicted in the figure below. (SRG of resolvent can be easily drawn by inversive geometry introduced in section 2 of [38].)

If \( \alpha L_A > 1 \), then \( \frac{1}{1 - \alpha L_A^2} < 0 < \frac{1}{2(1 + \alpha \mu_A)} < \frac{1}{2} \). So \( Q \) is on the right side of both centers of circles, \( O_L \) and \( O_{\mu} \). If \( \alpha L_A < 1 \), then \( \frac{1}{2(1 + \alpha \mu_A)} < \frac{1}{2} < \frac{1}{1 - \alpha^2 L_A^2} \). So \( Q \) resides between \( O_L \) and \( O_{\mu} \). In both cases, \( QP \) attains its maximum at \( P = N \) or \( N' \).
by Lemma 3.5. If $\alpha L_A = 1$, a similar reasoning shows that $\mathcal{Q} \mathcal{P}$ attains its maximum at $P = N$ or $N'$ by Lemma 3.5.

Thus,

$$\arg\max_{z_A \in \partial G(J_{\alpha A})} \left| z_A - \frac{1}{2} \right| = \{N, N'\} = \partial G(J_{\alpha L_A}) \cap \partial G(J_{\alpha M_{\mu A}}).$$

Since

$$\{N, N'\} = \frac{1}{\alpha^2 L_A^2 + 2\alpha \mu_A + 1} \left( \alpha \mu_A + 1 \pm i\alpha \sqrt{L_A^2 - \mu_A^2} \right),$$

and

$$\max_{z_A \in \partial G(J_{\alpha A})} \left| z_A - \frac{1}{2} \right|^2 = \left( \frac{\alpha \mu_A + 1}{\alpha^2 L_A^2 + 2\alpha \mu_A + 1} - \frac{1}{2} \right)^2 + \left( \frac{\alpha \sqrt{L_A^2 - \mu_A^2}}{\alpha^2 L_A^2 + 2\alpha \mu_A + 1} \right)^2$$

$$= \frac{1}{4} - \frac{\alpha \mu_A}{\alpha^2 L_A^2 + 2\alpha \mu_A + 1},$$

we bound the first factor as

$$\theta^{-1}|z_A - \theta|^2 + |2 - \theta^{-1} - \alpha z_C||z_A|^2 \leq \theta - \frac{2\alpha \mu_A}{\alpha^2 L_A^2 + 2\alpha \mu_A + 1}.$$

Finally, we conclude

$$|\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)| \leq |1 - \lambda \theta| + |\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda) - (1 - \lambda \theta)|$$

$$\leq 1 - \lambda \theta + \lambda \sqrt{\theta \left( \frac{2\alpha \mu_A}{\alpha^2 L_A^2 + 2\alpha \mu_A + 1} \right)}.$$

Note that $\lambda < 2 - \frac{\alpha}{2\beta C} = \theta^{-1}$. So $1 - \lambda \theta > 0$. This completes the proof for the first statement.

Since $\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)$ is symmetric with respect to $z_A$ and $z_B$, the second statement follows from the same argument.

*Proof of Theorem 3.2.* Consider the first statement, where $A$ is $L_A$-Lipschitz and $B$ is $\mu_B$-strongly monotone. Write

$$A = \mathcal{M} \cap L_{L_A}, \quad B = \mathcal{M}_{\mu_B}, \quad C = C_{\beta C}.$$

By the same reasoning as for Theorem 3.1, it suffices to show

$$|\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)| \leq \rho$$

for any $z_A \in \partial G(J_{\alpha A}), z_B \in \partial G(J_{\alpha B}),$ and $z_C \in \partial G(C)$.

Let

$$O_{\beta} = \frac{\alpha}{2\beta C}, \quad P = \alpha z_C, \quad Q = 2 - \lambda, \quad Z = 0,$$

as depicted in the following figure.
Observe that $\alpha z_C \in \text{Circ}\left(\frac{\alpha}{2z_C}, \frac{\alpha}{2z_C}\right)$. Since we assume $2 - \lambda > \frac{\alpha}{2z_C}$, $Q$ is on the right side of $O_\beta$. Thus, $|2 - \lambda - \alpha z_C| = QP \leq QZ = 2 - \lambda$ by Lemma 3.5. Then, we have

$$|\zeta_{\text{DYS}}(z_A, z_B, z_C; \alpha, \lambda)|^2 = \left| (1 - \lambda z_A)(1 - \lambda z_B) + \lambda(2 - \lambda - \alpha z_C)z_Az_B \right|^2$$

$$\leq \left| (1 - \lambda z_A)(1 - \lambda z_B) \right| + \lambda |2 - \lambda - \alpha z_C| |z_Az_B|^2$$

$$= \left| (1 - \lambda z_A)(1 - \lambda z_B) \right| + \lambda |2 - \lambda| |z_B|^2$$

for any $r > 0$, where the first inequality follows from the triangle inequality, the second is from $QP \leq QZ$, and the third is from Lemma 3.4. Set

$$r = \frac{2 - \lambda + 2\alpha \mu_B}{2 - \lambda}.$$ 

The second factor simplifies to

$$|1 - \lambda z_B|^2 + r\lambda(2 - \lambda)|z_B|^2 = 2\lambda(1 + \alpha \mu_B) \left| z_B - \frac{1}{2(1 + \alpha \mu_B)} \right|^2 + 1 - \frac{\lambda}{2(1 + \alpha \mu_B)} = 1$$

since $z_B \in \partial \mathcal{G}(J_{\alpha B}) = \text{Circ}\left(\frac{1}{2(1 + \alpha \mu_B)}, \frac{1}{2(1 + \alpha \mu_B)}\right)$. Therefore,

$$|\zeta_{\text{DYS}}(z_A, z_B, z_C; \alpha, \lambda)|^2$$

$$\leq |1 - \lambda z_A|^2 + r^{-1}\lambda(2 - \lambda) |z_A|^2$$

$$= \lambda(\lambda + (2 - \lambda)r^{-1}) \left| z_A - \frac{1}{\lambda + (2 - \lambda)r^{-1}} \right|^2 + 1 - \frac{\lambda}{\lambda + (2 - \lambda)r^{-1}},$$

which is maximized when $\left| z_A - \frac{1}{\lambda + (2 - \lambda)r^{-1}} \right|$ is maximized.

Let

$$O_0 = \frac{1}{2}, \quad O_L = \frac{1}{1 - \alpha^2 L_A^2}, \quad P = z_A, \quad Q = \frac{1}{\lambda + (2 - \lambda)r^{-1}},$$

$$M = \frac{1}{1 + \alpha L_A}, \quad \{N, N'\} = \partial \mathcal{G}(J_{\alpha L_A}) \cap \partial \mathcal{G}(J_{\alpha M}) = \frac{1}{1 + \alpha^2 L_A^2} \left( 1 \pm i\alpha L_A \right).$$
We now characterize the maximum value of $QP = |z_A - \frac{1}{z_A}|$. First, note that $Q$ is on the right side of $O_0$, since $r = \frac{2\lambda^{-1} + 2\alpha \mu B}{2 - \lambda} > 1$ and thus $Q = \frac{1}{z_A} > \frac{1}{2} = O_0$. Four cases in the following figure cover all possibilities (including the case $O_0 \leq Q \leq M$, in particular), and we can confirm that $QP$ attains its maximum at $P = M$ or $P \in \{N, N'\}$ in each case by applying Lemma 3.5 together with the fact that $Q$ is on the right side of $O_0$.

If $QP$ attains its maximum at $P = M$, i.e., $z_A = \frac{1}{1 + \alpha L_A}$,

$$|1 - \lambda z_A|^2 + r^{-1} \lambda (2 - \lambda)|z_A|^2 = 1 - \frac{2\alpha \lambda (\mu B(2 - \lambda) + L_A(2 - \lambda + 2\alpha \mu B))}{(1 + \alpha L_A)^2(2 - \lambda + 2\alpha \mu B)}.$$  

Otherwise, if $QP$ attains its maximum at $P \in \{N, N'\}$, i.e., $z_A \in \frac{1}{1 + \alpha^2 L_A}(1 \pm i\alpha L_A)$,

$$|1 - \lambda z_A|^2 + r^{-1} \lambda (2 - \lambda)|z_A|^2 = 1 - \frac{2\alpha \mu B(2 - \lambda + 2\alpha \mu B)}{(1 + \alpha^2 L_A)^2(2 - \lambda + 2\alpha \mu B)}.$$  

Therefore, 

$$|\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)|^2 \leq 1 - \min\left\{ \frac{2\alpha \mu B(2 - \lambda)}{(1 + \alpha^2 L_A)^2(2 - \lambda + 2\alpha \mu B)}, \frac{2\alpha \lambda((2 - \lambda)(\mu B + L_A) + 2\alpha \mu B L_A)}{(1 + \alpha L_A)^2(2 - \lambda + 2\alpha \mu B)} \right\}.$$

This completes the proof for the first statement.

Since $\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)$ is symmetric with respect to $z_A$ and $z_B$, the second statement follows from the same argument.
Proof of Theorem 3.3. First consider the first case, where $\mathbb{A}$ is $L_A$-Lipschitz and $\mathbb{C}$ is $\mu_C$-strongly monotone. Write

$$ \mathbb{A} = \mathbb{M} \cap L_{L_A}, \quad \mathbb{B} = \mathbb{M}, \quad \mathbb{C} = \mathbb{M}_{\mu_C} \cap C_{\beta_C}. $$

Let

$$ O = \frac{2 - \lambda}{\alpha}, \quad O_\beta = \frac{1}{2 \beta_C}, \quad \{ N, N' \} = \partial G(\mathbb{M}_{\mu_C}) \cap \partial G(C_{\beta_C}). $$

Define

$$ C' = \alpha^{-1}((2 - \lambda) I + L_R) $$

so that $(2 - \lambda) I - \alpha C' = L_R$ satisfies an arc property and set $R$ to the minimum value that satisfies $C \subset C'$. Observe that $G(C')$ is centered at $O$, which is on the right side of $O_\beta$ since $2 - \lambda > \frac{\alpha}{2 \beta_C}$. Therefore, we have

$$ R = \alpha \alpha = \sqrt{(2 - \lambda)(2 - \lambda - 2(1 - \eta)\alpha \mu_C)} $$

with $\eta = \frac{\alpha}{2 \beta_C(2 - \lambda)}$, as depicted in the following figure.

By Corollary 2.3, it suffices to show

$$ |\zeta_{\text{DYS}}(z_A, z_B, z_{C'}; \alpha, \lambda)| \leq \rho $$

for any $z_A \in \partial G(\mathbb{J}_{\alpha A})$, $z_B \in \partial G(\mathbb{J}_{\alpha B})$, and $z_{C'} \in \partial G(C')$. We have

$$ |\zeta_{\text{DYS}}(z_A, z_B, z_{C'}; \alpha, \lambda)|^2 
= |(1 - \lambda z_A)(1 - \lambda z_B) + \lambda(2 - \lambda - \alpha z_{C'}) z_A z_B|^2 
\leq |(1 - \lambda z_A)[1 - \lambda z_B] + \lambda[2 - \lambda - \alpha z_{C'}]|z_A||z_B|^2 
= |(1 - \lambda z_A)[1 - \lambda z_B] + \lambda R||z_A||z_B||^2 
= \left( \sqrt{|1 - \lambda z_A|^2} - 1 - \lambda z_B|^2 + \sqrt{(r^{-1}\lambda R||z_A||^2)} (r\lambda R||z_B||^2) \right)^2 
\leq (|1 - \lambda z_A|^2 + r^{-1}\lambda R||z_A||^2) (|1 - \lambda z_B|^2 + r\lambda R||z_B||^2) $$

for any $r > 0$, where first inequality follows from the triangle inequality, second equality follows from $z_{C'} \in \partial G(C') = \text{Circ} \left( \frac{2 - \lambda}{\alpha}, \frac{R}{\alpha} \right)$, and last inequality follows from Lemma 3.4. Set

$$ r = \frac{2 - \lambda}{R} $$

and the second factor simplifies to

$$ |1 - \lambda z_B|^2 + r\lambda R||z_B||^2 = 2\lambda \left| z_B - \frac{1}{2} \right|^2 + 1 - \frac{\lambda}{2} = 1 $$
as $z_B \in \partial G(J_{aB}) = \text{Circ} \left( \frac{1}{2}, \frac{1}{2} \right)$. Therefore,

$$|\zeta_{DYS}(z_A, z_B, z_C; \alpha, \lambda)|^2 \leq |1 - \lambda z_A|^2 + \frac{R^2}{2 - \lambda}|z_A|^2$$

$$= \left( \frac{\lambda^2 + \frac{R^2}{2 - \lambda}}{2 - \lambda} \right) |z_A| \geq \left( \frac{1}{\lambda + \frac{R^2}{2 - \lambda}} \right)^2 + 1 - \frac{\lambda}{\lambda + \frac{R^2}{2 - \lambda}},$$

which is maximized when $|z_A - \frac{1}{\lambda + \frac{R^2}{2 - \lambda}}|$ is maximized. By the same reasoning as that of Theorem 3.2, the maximum is attained at $z_A = \frac{1}{1 + \alpha^2 L_A^2} (1 \pm \iota \mu L_A)$ or $z_A = \frac{1}{1 + \alpha^2 L_A^2}$. (Since $A = M \cap L_A$ for both the current setup and that of Theorem 3.2, the illustration of $G(J_{aA})$ is identical and $Q = \frac{1}{1 + \alpha^2 L_A^2} > \frac{1}{2} = O_0$ implies $Q$ is on the right side of $O_0$). In the first case where $z_A = \frac{1}{1 + \alpha^2 L_A^2}$ is a maxima,

$$|1 - \lambda z_A|^2 + \lambda \frac{R^2}{2 - \lambda}|z_A|^2 = 1 + \frac{1}{(1 + \alpha L_A)^2} \left( -2 \lambda (1 + \alpha L_A) + \lambda^2 + \frac{\lambda R^2}{2 - \lambda} \right)$$

$$= 1 - \frac{2 \lambda \mu C (1 - \eta)}{(1 + \alpha L_A)^2}$$

which follows from the definition $R = \sqrt{(2 - \lambda)(2 - \lambda - 2(1 - \eta) \mu C)}$. In the second case where $z_A = \frac{1}{1 + \alpha^2 L_A^2} (1 \pm \iota \mu L_A)$ are maxima,

$$|1 - \lambda z_A|^2 + \lambda \frac{R^2}{2 - \lambda}|z_A|^2 = 1 + \frac{1}{1 + \alpha^2 L_A^2} \left( -2 \lambda + \lambda^2 + \frac{\lambda R^2}{2 - \lambda} \right)$$

$$= 1 - \frac{2 \lambda \mu C (1 - \eta) \mu C (1 + \alpha^2 L_A^2)}{1 + \alpha^2 L_A^2}.$$
Fig. 3.1. DYS contraction factors obtained by the previous work [17] (blue), our work (red), and the tight factor calculated via PEP [39] (black). For every case, we set \( \alpha = \lambda = 1 \) and operators in \( C \) to be \( \beta_C = 1 \)-cocoercive. We set \( A \) and \( B \) as 1-Lipschitz-monotone class in the plots above and below, respectively. We further impose the strong monotonicity for \( A \), \( B \), and \( C \) in each case and plot the contraction factors accordingly.

In Theorem D.6.3 and D.6.4 of [17], the contraction factors assuming conditions of our Theorem 3.2 (so \( A \) is \( \mu_A \)-strongly monotone and \( B \) is \( L_B \)-Lipschitz, and vice-versa) should be updated to

\[
\rho = \sqrt{1 - \frac{\lambda}{3} \min \left\{ \frac{2\alpha \mu_A}{(1 + \alpha L_B)^2}, \frac{\lambda}{1 + 2\alpha^2 L_B^2} \left( \frac{2 - \epsilon}{\lambda} - 1 \right) \right\}},
\]

\[
\rho' = \sqrt{1 - \frac{\lambda}{4} \min \left\{ \frac{2\alpha \mu_B}{1 + 2\alpha^2 L_A^2}, \frac{2\beta_C - \alpha/\epsilon}{\alpha}, \frac{\lambda}{1 + 2\alpha^2 L_A^2} \left( \frac{2 - \epsilon}{\lambda} - 1 \right) \right\}},
\]

respectively, where \( \epsilon \in \left( \frac{\alpha^2}{2\beta_C}, \min(1, 2 - \lambda) \right) \). In Theorem D.6.5 and D.6.6 of [17], the contraction factors assuming conditions of our Theorem 3.3 (so \( B \) is \( \mu_B \)-strongly monotone and \( C \) is \( L_C \)-Lipschitz, and \( A \) is \( \mu_A \)-strongly monotone and \( C \) is \( L_C \)-Lipschitz) should be updated to

\[
\rho = \sqrt{1 - \frac{\lambda}{4} \min \left\{ \frac{2\alpha \mu_C (1 - \eta)}{1 + 2\alpha^2 L_A^2}, \frac{2\eta \beta_C - \alpha/\epsilon}{\alpha}, \frac{\lambda}{1 + 2\alpha^2 L_A^2} \left( \frac{2 - \epsilon}{\lambda} - 1 \right) \right\}},
\]

\[
\rho' = \sqrt{1 - \frac{2\alpha \lambda \mu_C (1 - \eta)}{(1 + \alpha L_B)^2}},
\]

where \( \eta \in (0, 1) \) is a constant satisfying \( \eta > \frac{\alpha^2}{2\beta_C} \). \( \epsilon \) is as defined previously.

By simple calculation, we could verify that our six contraction factors are strictly better than those updated prior rates, respectively. The detailed calculations comparing our rates against the prior rates of [17] are carried out in the arXiv version of
reach a predetermined degree of error, we should consider expansion:

\[ \rho \]

Therefore, one can approximate a contraction factor in each case). The same applies to the contraction rates provided in the earlier work.

\[ (1 - \mathbb{B}) \]

by our results in two-operator splittings setups and how strong they are compared to splittings such as FBS and DRS, it is natural to ask what contraction rates are induced as it is proportional to the upper bound of the total number of iterations. Therefore, our contraction factors are a better proxy of the tight contraction factors than the earlier results. As depicted in Figure 3.1, the contraction factors we derived are much closer to the tight contraction factors evaluated by the PEP approach. In this sense, our contraction factors are always better; larger the value is, better our factor is. As presented in Table 3.1, this value is strictly larger than 1 in every case, meaning that our factors are always better than the prior factors. Indeed, these discrepancies are even higher than 4 in some cases, supporting significant improvement of our results in the sense we are aware of.

As DYS generalizes various two-operator splittings such as FBS and DRS, it is natural to ask what contraction rates are induced by our results in two-operator splitting setups and how strong they are compared to prior results. In particular, plugging \( \mathbb{B} = 0 \) into \( \mathcal{T}_{A,B,C,a,\lambda} \) gives \( \mathcal{T}_{A,a,C,a,\lambda} = (1 - \lambda)\mathbb{I} + \lambda \mathcal{J}_{aA}(\mathbb{I} - \alpha \mathcal{C}) \), an averaged version of the FBS operator. We consider a

\[
\begin{align*}
\mathcal{A} & \quad \mathcal{B} & \quad \mathcal{C} & \quad \frac{d\rho_{\text{ours}}}{d\mu} / \frac{d\rho_{\text{prev}}}{d\mu} |_{\mu=0} \\
\mathcal{M}_{\mu_a} \cap \mathcal{L}_{L_A} & \quad \mathcal{M} & \quad \mathcal{C}_{\beta_C} & \quad 3(1 + \alpha L_A)^2 / (1 + \alpha^2 L_A^2) \\
\mathcal{M} & \quad \mathcal{M}_{\mu_B} \cap \mathcal{L}_{L_B} & \quad \mathcal{C}_{\beta_C} & \quad (1 + \alpha L_B)^2 / (1 + \alpha^2 L_B^2) \\
\mathcal{L}_{L_A} & \quad \mathcal{M}_{\mu_B} & \quad \mathcal{C}_{\beta_C} & \quad 4(1 + 2\alpha^2 L_A^2) / (1 + \alpha^2 L_A^2) \\
\mathcal{M}_{\mu_B} & \quad \mathcal{L}_{L_B} & \quad \mathcal{C}_{\beta_C} & \quad 3(1 + \alpha L_B)^2 / (1 + \alpha^2 L_B^2) \\
\mathcal{L}_{L_A} & \quad \mathcal{M} & \quad \mathcal{M}_{\mu_C} \cap \mathcal{C}_{\beta_C} & \quad 4(1 + 2\alpha^2 L_A^2) / (1 + \alpha^2 L_A^2) \\
\mathcal{M} & \quad \mathcal{L}_{L_B} & \quad \mathcal{M}_{\mu_C} \cap \mathcal{C}_{\beta_C} & \quad (1 + \alpha L_B)^2 / (1 + \alpha^2 L_B^2)
\end{align*}
\]

this paper [27, Section 3.2]. In this section, we want to focus on the numerical results of this comparison with the worst-case rate obtained by the performance estimation problem (PEP) framework [20, 39, 43]. For a fair comparison with prior rates, we optimize the choice of \( \epsilon \) and \( \eta \) to minimize the factors above and compare with our results. As depicted in Figure 3.1, the contraction factors we derived are much closer to the tight contraction factors evaluated by the PEP approach. In this sense, our contraction factors are a better proxy of the tight contraction factors than the earlier results in [17]. We further investigate the advantages of our rates in assessing the computational cost, especially in an ill-conditioned case where the coefficient of the strong monotonicity \( \mu \) imposed to \( \mathcal{A}, \mathcal{B}, \) or \( \mathcal{C} \) is infinitesimal. For the six cases addressed in Theorems 3.1, 3.2, and 3.3, one can simply plug in \( \mu = 0 \) and confirm that the contraction factor \( \rho = 1 \) in such a case (\( \mu \) is one of the \( \mu_a, \mu_B, \) or \( \mu_C \) appearing in each case). The same applies to the contraction rates provided in the earlier work. Therefore, one can approximate a contraction factor \( \rho \) via the first-order Taylor series expansion: \( \rho \approx 1 + \mu (dp/d\mu|_{\mu=0}) \). If we want to evaluate the computational cost to reach a predetermined degree of error, we should consider

\[
- \frac{1}{\log \rho} \approx - \frac{1}{\log (1 + \mu (dp/d\mu|_{\mu=0}))} \approx - \frac{1}{\mu (dp/d\mu|_{\mu=0})}
\]

as it is proportional to the upper bound of the total number of iterations. Therefore, denoting the contraction factors in our and the previous work [17] by \( \rho_{\text{ours}} \) and \( \rho_{\text{prev}} \) respectively, \( \frac{d\rho_{\text{ours}}}{d\mu} / \frac{d\rho_{\text{prev}}}{d\mu} |_{\mu=0} \) would be an appropriate proxy to assess which of two are better; larger the value is, better our factor is. As presented in Table 3.1, this value is strictly larger than 1 in every case, meaning that our factors are always better than the prior factors. Indeed, these discrepancies are even higher than 4 in some cases, supporting significant improvement of our results in the sense we are aware of.

3.2.2. Contraction factors of FBS. As DYS generalizes various two-operator splittings such as FBS and DRS, it is natural to ask what contraction rates are induced by our results in two-operator splitting setups and how strong they are compared to prior results. In particular, plugging \( \mathbb{B} = 0 \) into \( \mathcal{T}_{A,B,C,a,\lambda} \) gives \( \mathcal{T}_{A,a,C,a,\lambda} = (1 - \lambda)\mathbb{I} + \lambda \mathcal{J}_{aA}(\mathbb{I} - \alpha \mathcal{C}) \), an averaged version of the FBS operator. We consider a
case where $A$ is $\mu_A$-strongly monotone and $C$ is $\beta_C$-cocoercive, which is also studied by Guo [23]. We first establish the contraction factor of FBS implied by Theorem 3.2 and compare it with the rate provided in [23].

Recall that the second instance of Theorem 3.2; since $B = 0$ is $L_B$-Lipschitz for any $L_B > 0$, the contraction factor stated in Theorem 3.2 is valid for any $L_B > 0$ in this case. Therefore, we can guarantee that $T_{A,0,C,0,\lambda}$ has a contraction factor smaller than or equal to the infimum of those factors. Therefore, $T_{A,0,C,0,\lambda}$ is $\rho_{ours}$-contractive where

$$\rho_{ours} = \sqrt{1 - \frac{2\alpha\mu_A\lambda(2 - \lambda)}{2 - \lambda + 2\alpha\mu_A}}.$$

The prior rate provided in Theorem 3.1 of [23] is translated

$$\rho_{prev} = \begin{cases} 1 - \lambda & \text{if } 0 < \lambda \leq \frac{8\alpha\beta_C^2 \mu_A + 4\beta_C(2\beta_C - \alpha)}{8\alpha^2 \beta_C^2 \mu_A + 4\beta_C(2\beta_C - \alpha)}, \\ 1 + \frac{\alpha}{\beta_C} & \text{if } \frac{8\alpha^2 \beta_C^2 \mu_A + 4\beta_C(2\beta_C - \alpha)}{8\alpha^2 \beta_C^2 \mu_A + 4\beta_C(2\beta_C - \alpha)} < \lambda < 2. \end{cases}$$

under $0 < \alpha < \min(2\beta_C, 2\beta_C(2 - \lambda)/\lambda)$. As shown in Figure 3.2, our rates are generally closer to the tight rates than the prior rates are, except for narrow ranges of $\lambda$.

4. Averagedness of DYS. In this section, we present a novel averagedness result of the DYS operator that does not require cocoercivity of $C$.

Theorem 4.1. Let $H$ be a real Hilbert space. Let $A: H \rightrightarrows H$, $B: H \rightrightarrows H$, and $C: H \rightrightarrows H$ be monotone operators. If $A$ is $\mu_A$-strongly monotone and $C$ is $L_C$-Lipschitz (so $C$ need not be cocoercive) with $\mu_A$, $L_C \in (0, \infty)$, and $\alpha \in (0, 2\mu_A/L_C^2)$, then $T_{A,B,C,\alpha,1}$ (so $\lambda = 1$) is $\frac{2}{1 - \alpha L_C/\mu_A}$-averaged. Symmetrically, if $B$ is $\mu_B$-strongly
monotone and \( C \) is \( L_C \)-Lipschitz with \( \mu_B, L_C \in (0, \infty) \), and \( \alpha \in (0, 2\mu_B/L_C^2) \), then \( T_{A,B,C,\alpha,1} \) is \( 2(2 - \theta^{-1})L_{C}/\mu_B \)-averaged.

Proof. Consider the first statement, where \( A \) is \( \mu_A \)-strongly monotone and \( C \) is \( L_C \)-Lipschitz. Write \( A = M_{A,\mu_A} \), \( B = M_\lambda \), and \( \mathcal{C} = \mathcal{M} \cap L_{C,\lambda} \). Let \( \theta = \frac{2}{2 - \theta^{-1}} \mu_C \). Then the statement is equivalent to saying that \( \mathcal{G}(T_{A,B,C,\alpha,1}) \subseteq \mathcal{G}(N_\theta) \).

We now follow a similar argument in the proof of Theorem 3.2. Define

\[
O = \frac{2 - \theta^{-1}}{\alpha}, \quad O_L = 0, \quad \{N, N'\} = \partial \mathcal{G}(M) \cap \partial \mathcal{G}(L_C),
\]

and

\[
C' = \alpha^{-1}((2 - \theta^{-1})I + L_R),
\]

to have \( (2 - \theta^{-1})I - \alpha C' = L_R \) satisfies an arc property and set \( R \) to the minimum value that renders \( C \subset C' \). Note that \( \mathcal{G}(C') \) is centered at \( O \), which is on the right side of \( O_L \) since \( 2 - \theta^{-1} > 0 \). Therefore, we have \( R = \alpha ON = \sqrt{(2 - \theta^{-1})^2 + \alpha^2 L_C^2} \) as depicted in the following figure.

Continuing on,

\[
|\mathcal{G}(T_{A,B,C,\alpha,1}) - (1 - \theta)| \subseteq |\mathcal{G}(T_{A,B,C',\alpha,1}) - (1 - \theta)|
= |Z_{A,B,C,\alpha,1}^{\text{DYS}}(1 - \theta)|,
\]

where the equality is a consequence of Theorem 2.1 with \( \lambda = 1 \) and \( s = 1 - \theta \). Now, it suffices to show \( \sup_{z \in \mathcal{D}_{A,B,C,\alpha,1}^{\text{DYS}}} |z - (1 - \theta)| \leq \theta \), or equivalently, by Lemma 1.1,

\[
|z_{A,B,C,\alpha,1}^{\text{DYS}}(z_A, z_B, z_C; \alpha,1) - (1 - \theta)| \leq \theta
\]

for any \( z_A \in \partial \mathcal{G}(J_{A,A}) \), \( z_B \in \partial \mathcal{G}(J_{A,B}) \), and \( z_C \in \partial \mathcal{G}(C') \).

With a similar argument as that for Theorem 3.1, we have

\[
|z_{A,B,C,\alpha,1}^{\text{DYS}}(z_A, z_B, z_C; \alpha,1) - (1 - \theta)|^2
= |\theta^{-1}(z_A - \theta)(z_B - \theta) + (2 - \theta^{-1} - \alpha z_C)z_Az_B|^2
\leq (\theta^{-1}|z_A - \theta||z_B - \theta| + |2 - \theta^{-1} - \alpha z_C||z_A||z_B|)^2
= (\sqrt{\theta^{-1}|z_A - \theta|^2} \cdot \theta^{-1}|z_B - \theta|^2
\]
\[+ \sqrt{|2 - \theta^{-1} - \alpha z_C|^2 (2 - \theta^{-1})^{-1} |z_A|^2 (2 - \theta^{-1}) |z_B|^2})^2
\leq \left(\theta^{-1}|z_A - \theta|^2 + |2 - \theta^{-1} - \alpha z_C|^2 (2 - \theta^{-1})^{-1} |z_A|^2\right)
\cdot \left(\theta^{-1}|z_B - \theta|^2 + (2 - \theta^{-1}) |z_B|^2\right),
\]
where the first inequality follows from the triangle inequality and the second is from Lemma 3.4.

The second factor simplifies to
\[ \theta^{-1}|z_B - \theta|^2 + (2 - \theta^{-1})|z_B|^2 = 2 \left| z_B - \frac{1}{2} \right|^2 + \theta - \frac{1}{2} = \theta \]
since \( z_B \in \partial G(J_\alpha B) = \text{Circ} \left( \frac{1}{2}, \frac{1}{2} \right) \).

The first factor simplifies to
\[ (\theta^{-1}|z_A - \theta|^2 + R^2(2 - \theta^{-1})^{-1}|z_A|^2) \]
\[ = (\theta^{-1} + R^2(2 - \theta^{-1})^{-1}) \left| z_A - \frac{1}{\theta^{-1} + R^2(2 - \theta^{-1})^{-1}} \right|^2 + \theta - \frac{1}{\theta^{-1} + R^2(2 - \theta^{-1})^{-1}} \]
\[ = (2(1 + \alpha \mu_A) \left| z_A - \frac{1}{2(1 + \alpha \mu_A)} \right|^2 + \theta - \frac{1}{2(1 + \alpha \mu_A)} \]
\[ = \theta, \]

where we used \( z_{C'} \in \partial G(C') = \text{Circ} \left( \frac{2 - \theta^{-1}}{\alpha}, \frac{R}{\alpha} \right) \) in (i), \( \theta^{-1} + R^2(2 - \theta^{-1})^{-1} = 2(1 + \alpha \mu_A) \) in (ii), and \( z_A \in \partial G(J_\alpha A) = \text{Circ} \left( \frac{1}{2(1 + \alpha \mu_A)}, \frac{1}{2(1 + \alpha \mu_A)} \right) \) in (iii). Thus, we conclude
\[ |\zeta_{\text{DYS}}(z_A, z_B, z_{C'}; \alpha, 1) - (1 - \theta)|^2 \leq \theta^2. \]

Since \( \zeta_{\text{DYS}}(z_A, z_B, z_{C'}; \alpha, \lambda) \) is symmetric with respect to \( z_A \) and \( z_B \), the second statement follows from the same argument.

The averagedness coefficient of Theorem 4.1 is, in fact, tight in the sense that it cannot be reduced (improved) without further assumptions. This can be proved by showing that there is a \( z \in \mathbb{Z}_{\text{DYS}}^{A,B,C,\alpha,1} \) such that \( z \neq 1 \) and \( z \in \partial G(N_0) \). However, we omit this proof for the sake of conciseness.

5. Conclusion. In this work, we extended the SRG theory to accommodate the DYS operator and used this theory to analyze the convergence of DYS iterations.

In our view, the SRG theory of Section 2 opens the door to many avenues of future work on splitting methods, and the analyses of Section 3 serve as demonstrations of such possibilities. One possible future direction is to further refine the analysis on the complex plane and obtain tighter contraction factors, as the contraction factors \( \rho \) presented in Section 3 are not tight in the sense that
\[ \rho > \sup_{T \in T_{A,B,C,\alpha,\lambda}} \frac{\|Tx - Ty\|}{\|x - y\|}. \]

Another direction is to characterize optimal or near-optimal choices of parameters \( \alpha \) and \( \lambda \) given a set of assumptions on the operators.

REFERENCES

[1] F. J. Aragón-Artacho and D. Torregrosa-Belén, A direct proof of convergence of Davis–Yin splitting algorithm allowing larger stepsizes, Set-Valued and Variational Analysis, 30 (2022), pp. 1–19.
[2] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2nd ed., 2017.

[3] A. Beck and M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM Journal on Imaging Sciences, 2 (2009), pp. 183–202.

[4] K. Bredees, E. Chenchene, D. A. Lorenz, and E. Naldi, *Degenerate preconditioned proximal point algorithms*, SIAM Journal on Optimization, 32 (2022), pp. 2376–2401.

[5] L. M. Briceno-Arias, *Forward-Douglas–Rachford splitting and forward-backward method for solving monotone inclusions*, Optimization, 64 (2015), pp. 1239–1261.

[6] R. E. Bruck, *On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space*, Journal of Mathematical Analysis and Applications, 61 (1977), pp. 159–164.

[7] J. A. Carrillo, K. Craig, L. Wang, and C. Wei, *Primal dual methods for Wasserstein gradient flows*, Foundations of Computational Mathematics, 22 (2022), pp. 389–443.

[8] V. Cevher, B. C. Vu, and A. Yurtsever, *Stochastic forward Douglas–Rachford splitting method for monotone inclusions*, in *Large-Scale and Distributed Optimization*, A. R. Pontus Giselsson, ed., Springer, 2018, pp. 149–179.

[9] T. Chaffey, *A rolled-off passivity theorem*, Systems & Control Letters, 162 (2022), p. 105198.

[10] T. Chaffey, F. Forni, and R. Sepulchre, *Scaled relative graphs for system analysis*, IEEE Conference on Decision and Control, (2021).

[11] T. Chaffey, F. Forni, and R. Sepulchre, *Graphical nonlinear system analysis*, IEEE Transactions on Automatic Control, (2023).

[12] T. Chaffey and A. Padoan, *Circuit model reduction with scaled relative graphs*, IEEE Conference on Decision and Control, (2022).

[13] T. Chaffey and R. Sepulchre, *Monotone one-port circuits*, IEEE Transactions on Automatic Control, (2023).

[14] L. Condat and P. Richtárik, *Randprox: Primal-dual optimization algorithms with randomized proximal updates*, International Conference on Learning Representations, (2023).

[15] F. Cui, Y. Tang, and Y. Yang, *An inertial three-operator splitting algorithm with applications to image inpainting*, Applied Set-Valued Analysis and Optimization, 1 (2019), pp. 113–134.

[16] M. N. Dao and H. M. Phan, *An adaptive splitting algorithm for the sum of two generalized monotone operators and one cocoercive operator*, Fixed Point Theory and Algorithms for Sciences and Engineering, 2021 (2021), pp. 1–19.

[17] D. Davis and W. Yin, *A three-operator splitting scheme and its optimization applications*, arXiv preprint arXiv:1504.01032, (2015).

[18] D. Davis and W. Yin, *A three-operator splitting scheme and its optimization applications*, Set-Valued and Variational Analysis, 25 (2017), pp. 829–858.

[19] J. Douglas and H. H. Rachford, *On the numerical solution of heat conduction problems in two and three space variables*, Transactions of the American Mathematical Society, 82 (1956), pp. 421–439.

[20] Y. Drori and M. Teboulle, *Performance of first-order methods for smooth convex minimization: a novel approach*, Mathematical Programming, 145 (2014), pp. 451–482.

[21] D. Gabay and B. Mercier, *A dual algorithm for the solution of nonlinear variational problems via finite element approximation*, Computers & Mathematics with Applications, 2 (1976), pp. 17–40.

[22] R. Glowinski and A. Marroco, *Sur l’approximation, par décléments finis d’ordre un, et la résolution, par pénalisation-dualité d’une classe de problèmes de dirichlet non linéaires*, ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, 9 (1975), pp. 41–76.

[23] K. Guo, *On the linear convergence rate of a relaxed forward-backward splitting method*, Optimization, 70 (2021), pp. 1161–1170.

[24] H. Heaton, D. McKenzie, Q. Li, S. W. Fung, S. Osher, and W. Yin, *Learn to predict equilibria via fixed point networks*, arXiv preprint arXiv:2106.00906, (2021).

[25] X. Huang, E. K. Ryu, and W. Yin, *Scaled relative graph of normal matrices*, arXiv preprint arXiv:2001.02061, (2019).

[26] X. Huang, E. K. Ryu, and W. Yin, *Tight coefficients of averaged operators via scaled relative graph*, Journal of Mathematical Analysis and Applications, 490 (2020), p. 124211.

[27] J. Lee, S. Yi, and E. K. Ryu, *Convergence analyses of Davis–Yin splitting via scaled relative gradient flows*, arXiv preprint arXiv:2207.04015, (2022).

[28] P.-L. Lions and B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM Journal on Numerical Analysis, 16 (1979), pp. 964–979.

[29] N. Ogura and I. Yamada, *Non-strictly convex minimization over the fixed point set of an asymptotically shrinking nonexpansive mapping*, Numerical Functional Analysis and Opti-
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[30] G. B. Passty, *Ergodic convergence to a zero of the sum of monotone operators in Hilbert space*, Journal of Mathematical Analysis and Applications, 72 (1979), pp. 383–390.

[31] R. Pates, *The scaled relative graph of a linear operator*, arXiv preprint arXiv:2106.05650, (2021).

[32] D. W. Peaceman and H. H. Rachford, Jr, *The numerical solution of parabolic and elliptic differential equations*, Journal of the Society for Industrial and Applied Mathematics, 3 (1955), pp. 28–41.

[33] F. Pedregosa, *On the convergence rate of the three operator splitting scheme*, arXiv preprint arXiv:1610.07830, (2016).

[34] F. Pedregosa, K. Fatras, and M. Casotto, *Proximal splitting meets variance reduction*, International Conference on Artificial Intelligence and Statistics, (2019).

[35] F. Pedregosa and G. Gidel, *Adaptive three operator splitting*, International Conference on Machine Learning, (2018).

[36] E. K. Ryu and W. Yin, *Large-Scale Convex Optimization: Algorithms & Analyses via Monotone Operators*, Cambridge University Press, 2022.

[37] E. M. Stein and R. Shakarchi, *Complex analysis*, vol. 2, Princeton University Press, 2010.