An iterative domain decomposition method for free boundary problems with nonlinear flux jump constraint

Juan Galvis · H. M. Versieux

Abstract In this paper we design an iterative domain decomposition method for free boundary problems with nonlinear flux jump condition. The proposed scheme requires, in each iteration, the approximation of the flux on both sides of the free interface. We present a finite element implementation of our method. The numerical implementation uses harmonically deformed triangulations to inexpensively generate finite element meshes in subdomains. We apply our method to a simplified model for jet flows in pipes and to a magnetohydrodynamics model. Finally, we present numerical examples illustrating the robustness and convergence of our scheme.

Keywords Iterative scheme · Free boundary problems · Domain decomposition method · Jet flow

Mathematics Subject Classification 65N30 · 65N55

1 Introduction

In this paper, we propose a numerical iterative method for approximating the solutions of free boundary problems in two dimensions. Our iterative method for free boundary problems is based on domain decomposition and damped Newton's method ideas. In general terms, free boundary problems seek to determine an unknown function \( u \) with some prescribed conditions on an unknown interior interface, or exterior boundary, or (sub)domain. In many applications, the value of \( u \) on the free interface is prescribed and it is required that \( u \) satisfies a condition involving (both sides) derivatives of \( u \) on the interface. We mention jump conditions.
of Stefan, Bernoulli and Gibbs–Thomson type, among others. There is considerable literature of iterative methods for these types of free boundary problems; see for instance Bouchon et al. (2005), Eppler and Harbrecht (2006), Flucher and Rumpf (1997), Kärkkäinen and Tiihonen (1999), Kuster et al. (2007), van der Zee et al. (2010), Zhang and Babuška (1996) and references therein. In particular, numerical finite elements methods have been proposed to solve Stefan-like free boundary problems (including time-dependent problems) and some other similar phase transition problems; see for instance Barrett and Elliott (1985), Chen et al. (2000), Nochetto et al. (1991a, b), Saavedra and Scott (1991). These methods use a variational formulation of their original problem. Level set approach for Stefan problems were also proposed in Chen et al. (1997) and references therein.

The free boundary conditions considered here, up to our knowledge, have not been extensively studied numerically. We are particularly interested in free boundary problems where the unknown functions satisfy a nonlinear jump constraint across the free interface. More precisely, let \( \Omega \subset \mathbb{R}^2 \), assume \( \partial \Omega \) is locally Lipschitz, and \( \partial \Omega = \Sigma^+ \cup \Sigma^- \), where \( \Sigma^\pm \) are open and connected sets. Also, let \( g \in L^1_{\text{loc}}(\Omega) \), with \( |\nabla g| \in L^2(\Omega) \) such that \( g|_{\Sigma^+} > 0 \) and \( g|_{\Sigma^-} < 0 \), and let \( \lambda \in \mathbb{R} \). We want to find a function \( u \in L^1_{\text{loc}}(\Omega) \), with \( |\nabla u| \in L^2(\Omega) \) and a free interface \( \Gamma \) (dividing \( \Omega \) in two subdomains \( \Omega^+ = \{ u > 0 \} \) and \( \Omega^- = \{ u < 0 \} \)) such that \( u \) and \( \Gamma \) satisfy the following subdomain equations and boundary condition,

\[
-\Delta u = 0 \quad \text{in} \quad \Omega^+, \tag{1}
\]
\[
-\Delta u = 0 \quad \text{in} \quad \Omega^-, \tag{2}
\]
\[
u = g \quad \text{on} \quad \Sigma^+ \cup \Sigma^- \tag{3}
\]

and free interface condition

\[
|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda \quad \text{on} \quad \Gamma. \tag{4}
\]

Here, \( u^\pm = u|_{\Omega^\pm} \) (\( u^\pm \) denote the restriction of the solution \( u \) on \( \Omega^\pm \)), and its derivatives are interpreted as side limits. The interface condition is imposed in a weak sense; see Alt et al. (1984) for more details and a proof of the well posedness of this problem.

We are not aware of a numerical method to solve problem (1)–(4). The finite element methods mentioned earlier to handle Stefan, Bernoulli and similar free boundary conditions are based on variational formulations. They do not seem to be easily extended to handle the nonlinear free boundary constraint (4). Also, Bernoulli-type free boundary problems (when one of the phases is a constant function) seems to be easier to handle numerically. In the later case, using the fact that the tangential derivative on the free interface is zero and that the flux sign can be a priori determined, the interface condition reduces to a linear condition of the form \( \partial_\eta u = \lambda \) where \( \partial_\eta \) is the normal derivative on the free interface.

We have two main applications in mind: (1) a jet flow model studied by Alt et al. (1984, 1985) and (2) a free boundary problem arising in magnetohydrodynamics studied in Friedman and Liu (1995) and Kang et al. (1997). These applications are simplified mathematical versions of complicated flow models and focus on the main modeling aspects. Despite the mathematical simplifications, in either case, the resulting models are still complex requiring a numerical approximation of their solutions. Also, the method presented here can be easily extended to handle different free boundary problems such as the stationary solutions of the Stefan’s problem and other similar problems.

The iterative method proposed in this paper for problem (1)–(4) is based on the following ideas. Assume the solution \( u \) is sufficiently regular (for instance, \( u|_{\Omega^\pm} \in C^1(\Omega^\pm) \) or \( u|_{\Omega^\pm} \in W^2(\Omega^\pm) \)). Since \( u = 0 \) on \( \Gamma, |\nabla u^+| = |\partial_\eta u^+| \) on \( \Gamma \), where \( \eta \) is the outer normal vector of the region defined by the support of \( u^+ \). Hence, the free boundary condition (4) reads

\[
|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda \quad \text{on} \quad \Gamma.
\]
Next, assume we have an approximation $\tilde{\Gamma}$ of $\Gamma$ dividing $\Omega$ in two different regions $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$. We also assume that $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$ are connected subdomains, and $\partial \tilde{\Omega}^+ \cap \partial \tilde{\Omega} = \Sigma^+$ and $\partial \tilde{\Omega}^- \cap \partial \tilde{\Omega} = \Sigma^-$. To construct an approximation $\tilde{u}$ of $u$, we can solve the Dirichlet problems (1)–(3) in the approximated subdomains with homogeneous Dirichlet boundary condition on the approximated free interface $\tilde{\Gamma}$. The solution of these two independent problems give $\tilde{u}^+$ and $\tilde{u}^-$. We observe that we do not expect the function $\tilde{u}$ to satisfy condition (5), since $\tilde{\Gamma}$ is only an approximation of $\Gamma$. Finally, we update the approximation of the free boundary using the quantity $\sigma = |\partial_n \tilde{u}^+(x)|^2 - |\partial_n \tilde{u}^-(x)|^2 - \lambda$ and a perturbation of $\tilde{\Gamma}$ in its normal direction $\eta(x)$. More specifically, we locally move $\tilde{\Gamma}$ in the direction of $\eta(x)$ by a magnitude $\tau \sigma$ where $\tau$ is a positive damping parameter. Once the new approximation of $\Gamma$ is obtained, we restart this procedure.

We also observe that our method can handle more general problems. For instance, the same ideas apply to the following abstract free boundary problem. Let $\mathcal{L}_+$ and $\mathcal{L}_-$ represent two second-order elliptic operators. Assume $\Omega \subset \mathbb{R}^n$, and let $\phi$ and $\psi : \mathbb{R} \to \mathbb{R}$ be two increasing functions. Consider the problem of finding $u$ and a free interface $\Gamma$ such that

$$
\begin{align*}
\mathcal{L}_+ u &= 0 \quad \text{in } \Omega^+ = \{u > 0\}, \\
\mathcal{L}_- u &= 0 \quad \text{in } \Omega^- = \{u < 0\}, \\
\phi(\partial_n u^+) - \psi(\partial_n u^-) &= \lambda(x) \quad \text{on the free boundary } \Gamma = \{u = 0\} \\
u &= g \quad \text{on } \partial \Omega.
\end{align*}
$$

(6)

Here $\partial_n u^\pm$ represents the outward normal derivative with respect to $\Omega^\pm$. To our knowledge there is no rigorous studies of such general class of problems. We mention that this formulation includes the stationary solutions of two-phase Stefan problem (see Kamenomostskaja 1961; Friedman 1968; Rubenstein 1971; Augusto Visintin 2008 and references therein) and other problems involving nonlinear free boundary conditions (see Leito and Queiroz 2012).

The rest of the paper is organized as follows. In Sect. 2, we describe our iterative scheme. Section 3 describes the finite element implementation of our method. In Sect. 4 we present numerical experiments for the jet flow model proposed by Alt, Caffarelli and Friedman. Numerical results for the magnetohydrodynamics problem studied in Friedman and Liu (1995); Kang et al. (1997) are presented in Sect. 5. In Sect. 6, we study numerically some convergence properties of our scheme. Finally, we present our conclusions and comments in Sect. 7.

2 Description of the iterative method for the free interface

Assume $\Gamma_n$ is an approximation of the free boundary $\Gamma$ (4) dividing the domain $\Omega$ into two subdomains, $\Omega^+_n$ (enclosed by $\Sigma^+ \cup \Gamma_n$) and $\Omega^-_n$ (enclosed by $\Sigma^- \cup \Gamma_n$). We define the $n$th approximation of $u$ as follows. In $\Omega^+_n$, the function $u_n$ solves

$$
\begin{align*}
-\Delta u_n &= 0 \quad \text{in } \Omega^+_n, \\
u_n &= g \quad \text{on } \Sigma^+, \\
u_n &= 0 \quad \text{on } \Gamma_n.
\end{align*}
$$

(7)

In $\Omega^-_n$, the function $u_n$ solves

$$
\begin{align*}
-\Delta u_n &= 0 \quad \text{in } \Omega^-_n, \\
u_n &= g \quad \text{on } \Sigma^-, \\
u_n &= 0 \quad \text{on } \Gamma_n.
\end{align*}
$$

(8)
The main idea to define the updated approximation of the free boundary $\Gamma_{n+1}$ is very simple. First, we define

$$\sigma_n = (\partial_\eta u^+_n)^2 - (\partial_\eta u^-_n)^2 - \lambda.$$  \hfill (9)

Here, we use the notation $\partial_\eta u^\pm$ as the outward normal derivative of $u^\pm_n$ with respect to the region $\Omega^\pm_n$. Next, if for instance $\sigma_n(x) > 0$ for some point $x \in \Gamma_n$, then we would like to locally update $\Gamma_n$ such that $\sigma_n(x)$ is closer to zero. This can be done by decreasing the flux of $u_n$ in $\Omega^+_n$ and/or increasing the flux of $u_n$ in $\Omega^-_n$ in a neighborhood of that point. We expect to obtain this by locally moving the free interface $\Gamma_n$ in the normal direction outward to $\Omega^+_n$.

We define the new approximation of the free interface by

$$\Gamma_{n+1} = \{ x + \tau \sigma_n \hat{n}_{\Gamma_n}; \ \text{with} \ x \in \Gamma_n \}. \hfill (10)$$

Here, $\tau$ is a small positive parameter, and $\hat{n}_{\Gamma_n}$ represents the unitary normal vector on $\Gamma_n$ outward to $\Omega^+_n$. In our numerics, we impose this last equation in the vertices of our triangulation; see Sect. 3 for more details.

The motivation behind this update procedure comes from the simplified one dimensional problem. Assume $\Omega = (0, 1)$, $u(0) = 1$ and $u(1) = -1$, and let $a \in (0, 1)$ represent the free interface $\Gamma$. Suppose that $\bar{a}$ approximates $a$, and let $\bar{u}$ be harmonic in $(0, \bar{a})$ and $(\bar{a}, 1)$, respectively. Hence, $\hat{u}$ is linear in $(0, \bar{a})$ and $(\bar{a}, 1)$. If we move $\bar{a}$ to the right (in the outward normal direction of $\hat{\Omega}^+ = (0, \bar{a})$), we are at the same time decreasing $|\partial_\eta \bar{u}^+| = 1/\bar{a}$ and increasing $|\partial_\eta \bar{u}^-| = 1/(1 - \bar{a})$. The situation is considerably more complex in two dimensions. Formally speaking, in some particular situations one could expect to see a similar behavior. For instance, if the updated interface approximation is such that $\Omega^+_n \subset \Omega_{n+1}^+$, the harmonic function $u^+_{n+1}$ would have more area to decrease from $g$ on $\Sigma^+$ to zero on $\Gamma_{n+1}$ when compared with $u^+_n$. Hence, we could expect that $|\partial_\eta u^+_{n+1}| \leq |\partial_\eta u^+_n|$ (similarly, $|\partial_\eta u^-_{n+1}| \geq |\partial_\eta u^-_{n+1}|$). We observe that although we do not rigorously justify this fact in two dimensions, our numerical results suggest the convergence of the numerical scheme.

Finally, we observe that there are several ways to define $\Gamma_0$ dividing the domain $\Omega$ into two parts as desired. For instance, we can take $\Gamma_0$ as the zero level set of any regular extension of the boundary data $g$.

**Remark 1** We note that we need only an approximation of $\sigma_n$ (which requires only approximation of the flux). This is important in case $u$ is not regular enough to allow the computation of the square of the flux.

### 3 Finite element implementation

Now we describe the finite element implementation of our iterative method. In each iteration we have to approximate the solutions of problems (7) and (8) as well as $\sigma$ in (9).

Let $n \in \mathbb{N}$ be our iteration parameter, $\mathcal{T}^h_n$ be a triangulation of $\Omega$ with nodes $\{x_{n,j}^h\}_{j=1}^{N_v}$ and edges $\{e_{n,\ell}\}_{\ell=1}^{N_e}$, and let $\Gamma_n^h$ be an approximation of the free boundary $\Gamma$, such that $\Gamma_n^h \subset \bigcup_{\ell=1}^{N_e} e_{n,\ell}$. Here, we also assume $\Gamma_n^h$ divides $\Omega$ into two subdomains, $\Omega^+_{n,h}$ (enclosed by $\Sigma^+ \cup \Gamma^+_n$) and $\Omega^-_{n,h}$ (enclosed by $\Sigma^- \cup \Gamma^-_n$). Set $V^{+,h}_0 = \{ v \in \mathcal{P}^1(\mathcal{T}^h_n, \Omega^+_{n,h}); \ v = 0 \text{ on } \partial \Omega^+_{n,h} \}$, where $\mathcal{P}^1(\mathcal{T}^h_n, \Omega^+_{n,h})$ represents the set of continuous piecewise linear functions on $\mathcal{T}^h_n$ (the space $V^{+,h}_0$ is defined similarly). Let $V_i \subset \{1, \ldots, N_v\}$ represent the set of indices of the vertices in $\Gamma^h_n$ ($j \in V_i$ if and only if $x_{n,j}^h \in \Gamma^h_n$). Also, let $V^+_b$ and $V^-_b$ represent the set of
indexes of the vertices in $\Sigma^+$ and $\Sigma^-$, respectively. The $n$th approximation of the solution $u$ of (1)–(4), denoted by $u_n^h$, solves the finite element problems,

$$\begin{align*}
\int_{\Omega^+_n} \nabla u_n^h \nabla z^h \, dx &= 0 \quad \text{for all } z^h \in V_0^{+,h} \\
u_n^h(x_{n,j}^h) &= g(x_{n,j}^h) \quad \text{for } j \in V_b^+ \\
u_n^h(x_{n,j}^h) &= 0 \quad \text{for } j \in V_i,
\end{align*}$$

(11)

and

$$\begin{align*}
\int_{\Omega^-_n} \nabla u_n^h \nabla z^h \, dx &= 0 \quad \text{for all } z^h \in V_0^{-,h} \\
u_n^h(x_{n,j}^h) &= g(x_{n,j}^h) \quad \text{for } j \in V_b^- \\
u_n^h(x_{n,j}^h) &= 0 \quad \text{for } j \in V_i.
\end{align*}$$

(12)

We define $\rho_{n,h}^+$ as an appropriate piecewise linear approximation of the flux of $u_n^h|_{\Omega^+_n}$ across $\Gamma_n^h$; see Appendix. Analogously, we define $\rho_{n,h}^-$ as the discrete flux of $u_n^h|_{\Omega^-_n}$ across $\Gamma_n^h$. We note that

$$\rho_{n,h}^+ = \sum_{x_{n,j}^h \in \Gamma_n^h} \alpha_{n,j}^+ \psi_j \quad \text{and} \quad \rho_{n,h}^- = \sum_{x_{n,j}^h \in \Gamma_n^h} \alpha_{n,j}^- \psi_j,$$

where the function $\psi_j$ represents a basis for the space $P^1(T_n, \Omega) \cap H^1_0(\Omega)$ restricted to $\Gamma_n^h$.

Define

$$\sigma_n^h(x) = \sum_{x_{n,j}^h \in \Gamma_n^h} \left( (\alpha_{n,j}^+)^2 - (\alpha_{n,j}^-)^2 - \lambda^2 \right) \psi_j(x).$$

(13)

Next, we obtain our new approximation of the free interface. We need to address two issues when implementing the update proposed in (10). First, since $\Gamma_n^h$ is piecewise linear, its normal vector $\tilde{n}_{\Gamma_n^h}(x)$ is not well defined when $x$ is a vertex of $T_n^h$. Different strategies can be used to handle this problem; for instance, we can define the normal vector as the average of the two adjacent normal vectors of $x$; or we can interpolate the vertices of $\Gamma_n^h$ by a smooth curve and define the normal vector of $\Gamma_n^h$ at $x$ as the normal vector of the smooth interpolation of $\Gamma_n^h$. In our numerics, we implemented the first strategy. Second, we need to ensure that the updated approximation of the free interface $\Gamma_{n+1}^h$ is represented by a union of edges of the new triangulation $T_{n+1}^h$. We now describe how we address this second issue.

We apply Eq. (10) on the vertices of $\Gamma_n^h$ to obtain our new approximation $\Gamma_{n+1}^h$. More precisely, set $\tilde{n}_{\Gamma_n^h}^+$ as an approximation of the unitary normal vector of $\Gamma_n^h$ outward to $\Omega_n^+$. We obtain $T_{n+1}^h$ from $T_n^h$ using a harmonic extension of the displacement $\tau \sigma_n^h \tilde{n}_{\Gamma_n^h}^+$ as follows. First, we introduce the vector function $\tilde{\omega}_n^h = (\omega_1^h, \omega_2^h)$ where each component satisfies

$$\begin{align*}
\int_{\Omega^+_n} \nabla \omega_j^h \nabla z^h \, dx &= 0 \quad \text{for all } z^h \in P^1(T_n, \Omega^+_n), \\
u_j^h(x_{n,j}^h) &= 0 \quad \text{for all } j \in V_b^+ \\
u_j^h(x_{n,j}^h) &= \tau \sigma_n^h(x_{n,j}^h)(\tilde{n}_{\Gamma_n^h}(x_{n,j}^h)) \cdot \vec{e}_j \quad \text{for all } j \in V_i.
\end{align*}$$

(14)
and
\[
\begin{align*}
\int_{\Omega_n^h} \nabla \omega_j^h \nabla z^h \, dx &= 0 & \text{for all } z^h \in \mathcal{P}^1(T_n, \Omega_n^h) \\
w_j^h(x_{n,j}^h) &= 0 & \text{for all } j \in V_b^- \\
w_j^h(x_{n,j}^h) &= \tau \sigma_n^h(x_{n,j}^h)(\tilde{\eta}^\Gamma_n(x_{n,j}^h)) \cdot \tilde{e}_l & \text{for all } j \in V_i,
\end{align*}
\]
where \( \tilde{e}_1 = (1, 0) \) and \( \tilde{e}_2 = (0, 1) \). We observe that \( \sigma_n^h(x_{n,j}^h) = 0 \) if \( x_{n,j}^h \in \Gamma_n^h \cap \partial \Omega \). Then we define the nodes of the new triangulation
\[
x_{n+1,j}^h = x_{n,j}^h + \tilde{\omega}(x_{n,j}^h).
\]
The edges and triangles structures of \( T_{n+1}^h \) are inherited directly from \( T_n^h \). More precisely, edges are always straight line segments, and \( T_{n+1}^h \) has an edge connecting two nodes \( x_{n+1,j}^h \) and \( x_{n+1,k}^h \), \( j \neq k \in \{1, \ldots, N\} \) if and only if \( T_n^h \) has an edge connecting nodes \( x_{n,j}^h \) and \( x_{n,k}^h \). We now define the new approximation of the free interface. First, we observe that
\[
x_{n+1,j}^h = \{ x_{n,j} + \tau \sigma_n^h(x_{n,j}^h)(\tilde{\eta}^\Gamma_n(x_{n,j}^h)) \} \quad \text{for } j \in V_i.
\]
Let \( \{ e_{n+1,\ell} \}_{\ell=1}^{N_e} \) be the set of edges of \( T_{n+1}^h \), and let \( E_i \subset \{1, \ldots, N\} \) be the set of indexes of edges connecting nodes \( x_{n+1,j}^h \) and \( x_{n+1,k}^h \), with \( j, k \in V_i \). We define
\[
\Gamma_{n+1}^h = \cup_{\ell \in E_i} e_{n+1,\ell}.
\]
Finally, we observe that the initial triangulation \( T_0 \) can be defined as any regular triangulation of \( \Omega \) containing vertices on the initial approximation \( \Gamma_0 \) of \( \Gamma \).

### 3.1 The parameter \( \tau \)

We now elaborate on the main issues driving the choice of the damping parameter \( \tau \). We already argued that moving the actual approximation of the free interface \( \Gamma_n^h \) in the direction of \( \sigma_n^h \tilde{\eta}^\Gamma_n^h \) is motivated by the a priori knowledge of the increase or decrease of the solution’s gradient. Then, the main role of the damping parameter \( \tau \) is to avoid degeneracy of the next approximation of the free interface \( \Gamma_{n+1}^h \) and the new triangulation \( T_{n+1}^h \). We observe that the total displacement from the vertices of \( \Gamma_n^h \) to the vertices of \( \Gamma_{n+1}^h \), that is, the quantity \( |\tau \sigma_n^h| \) should be chosen such that:

- The number of elements of the new triangulation is the same as in the triangulation from the previous step.
- The updated triangulation \( T_{n+1}^h \) defined from (17) should be regular.
- The new free interface approximation \( \Gamma_{n+1}^h \) does not have self-intersections.

These requirements can always be achieved if \( \tau \) is small enough (due to the compactness of \( T_n^h \)). Indeed, the harmonic extension dislocation update of the triangulation is bounded by \( \max |\tau \sigma_n^h| \). Hence, it suffices to choose \( \tau \) such that \( \max |\tau \sigma_n^h| \) is smaller than a third of the minimal distance among the vertices of the \( T_n^h \). In our numerical experiments, the free interface is rather smooth. This allows us to choose \( \tau \) constant independent of the iteration counter \( n \). This choice has the advantage of yielding a less expensive iteration.
We consider the following model of two planar flows along an infinite pipe with one free interface studied in Alt et al. (1984), and apply our method to approximate its solution in a truncated pipe. Let \( u \) denote the stream function associated with the irrotational flow of two ideal fluids. The regions occupied by each different fluid are represented by the support of \( u^+ \) and \( u^- \), where \( u^+ (u^-) \) denotes the positive (negative) part of \( u \). Let \( N_1 : \mathbb{R} \rightarrow (c_1, c_2) \), with \( 0 < c_1 < c_2 \) be a continuous and piecewise \( C^2 \) function, satisfying \( \lim_{y \to \infty} N_1(y) = B \), \( \int_0^\infty (N_1(y) - B)^2 \, dy < \infty \), and \( \int_0^\infty N_1'(y)^2 \, dy < \infty \). The Alt et al. model assumes that the two fluids occupy an infinity semi-strip region enclosed by the graph \( \{(N_1(y), y); \ y > a, \ a < 0\} \), and the lines \( y = a \) and \( x = -1 \). The fluids enter the region at the boundary \( \{y = a\} \), and the two fluids are separated from each other in \( \{y < 0\} \) by a given continuous and piecewise \( C^2 \) curve \( N_2 : [a, 0] \rightarrow (-1 + \delta, c_2) \), satisfying \( \text{dist}(N_2, N_1) > 0 \). A special truncated case of this configuration is shown in Fig. 1 (left). The problem consists in finding the free boundary separating the two fluids in the region \( y > 0 \), assuming each flow has constant speed when \( y \to \infty \). More specifically, we look for \( u \) and \( \lambda \) satisfying

\[
\Delta u = 0 \quad \text{in each fluid}
\]

\[
|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda \quad \text{on the free boundary separating the two fluids}
\]

\[
u = Q \quad \text{on } \{x = -1\} \text{ and } u = -1 \quad \text{on } N_1
\]

\[
u = g \quad \text{on } [-1, N_1(a)] \times \{y = a\}
\]

where

\[
\left\{ \begin{align*}
\lambda &= 1/(1 + b)^2 - Q^2/(B - b)^2, \\
&\text{and the free boundary}
\end{align*} \right.
\]

approximates the point \( (b, 0) \) when \( y \to \infty \).

Here, the function \( g \in C^1 \) is monotone decreasing and

\[
0 \leq g(x) \leq Q \quad \text{for } x < N_2(a),
\]

\[
-1 \leq g(x) \leq 0 \quad \text{for } x > N_2(a),
\]

\[
g(-1) = Q \quad \text{and } g(N_1(a)) = -1.
\]

![Fig. 1 Simple vertical pipe configuration](image-url)
Existence and uniqueness of the solution for this problem was studied in Alt et al. (1985), where it was proved that minimizers of an appropriate functional are weak solutions of the problem (19).

We approximate the solution of the above problem working with a truncated domain. We now refer to the problem configuration in Fig. 1. Given a positive constant $Q$, and functions $N : [R^-, 0] \to (-1, 1)$ and $g : [-1, 1] \to \mathbb{R}$, we want to find $u : (-1, 1) \times (R^-, R^+) \to \mathbb{R}$ and free interface $\Gamma$ represented by

$$\Gamma = \{(y, f_\Gamma(y)); \text{ with } 0 \leq y \leq R^+\}$$

where $f_\Gamma : [0, R^+] \to \mathbb{R}$ is such that $f_\Gamma(0) = N(0)$ (see Fig. 1, right picture). The function $u$ and the free interface $\Gamma$ satisfy

$$\Delta u = 0 \text{ in } \Omega^- \quad \text{and} \quad \Delta u = 0 \text{ in } \Omega^+$$

(21)

where

$$\Omega^+ = \{-1 < x < N(y), R^- < y \leq 0\} \cup \{-1 < x < f_\Gamma(y), 0 \leq y < R^-\}$$

$$\Omega^- = \{N(y) < x < 1, R^- < y \leq 0\} \cup \{f_\Gamma(y) < x < 1, 0 \leq y < R^-\}.$$  

The function $u$ has to satisfy the following known given data:

$$u(N(y), y) = 0, \quad R^- < y < 0; \quad u(x, R^-) = g(x), \quad -1 < x < 1; \quad \frac{\partial u}{\partial y}(x, R^+) = 0, \quad -1 < x < 1;$$

$$u(-1, y) = Q, \quad R^- < y < R^+; \quad u(1, y) = -1, \quad R^- < y < R^+;$$

(22) (23) (24) (25) (26)

and the following conditions on the free interface:

$$u = 0, \quad \text{on } \Gamma \quad (\text{or } u(f_\Gamma(y), y) = 0, 0 \leq y < R^+)$$

$$\frac{\partial u}{\partial y}(y) = 0 \quad \text{on } \Gamma \quad (\text{or } u(f_\Gamma(y), y) = 0, 0 \leq y < R^+)$$

(27) (28)

where $\lambda$ is given by

$$\lambda = \left(\frac{1}{1+b}\right)^2 - \left(\frac{Q}{2-b}\right)^2.$$ (29)

We note that the boundary condition on the top of the domain (see Fig. 1) is the homogeneous Neumann boundary condition; hence, the free boundary is not fixed at the top. That is, the value $f_\Gamma(R^+)$ is not prescribed.

For each value of $b \in (-1, 1)$, we can compute $\lambda$ through (29) and use our method to find an approximation of $u = u^b$ and $\Gamma = \Gamma^b$ (represented by $f_\Gamma = f_\Gamma^b$) that solve (21)–(29). Since the free boundary must have the vertical line $x = b$ as an asymptote, a feasible approximation of the free interface $\Gamma$ is obtained if $b = b^*$ where

$$b^* = f_\Gamma^b(R^+)$$ (30)

This is compatible with the asymptote condition $\lim_{y \to -\infty} f_\Gamma(y) = b^*$.

Next, we use a bisection algorithm (applied to the function $(-1, 1) \ni b \mapsto f_\Gamma^b(R^+) \in (-1, 1)$) to find the correct value of $b^*$ such that (30) is satisfied. In the two examples presented next, we run our method until $\|\sigma^h_n\|_{L^\infty} < \text{tol} = 10^{-6}$. 
An iterative domain decomposition method

Fig. 2 Free interface that solves (21)–(29) and (30) with \( N(y) = 0.5|y|/R^- \), \( g(x) = (0.5 - x)/1.5 \) if \(-1 < x < 0.5\), \( g(x) = 5(0.5 - x)/0.5 \) if \(0.5 < x < 1\), and \( b^* = 1/3\). Initial configuration (left), final configuration (middle) and a zoom around \((0, 0)\) showing the free interface and the final mesh (right).

The first example considers the nozzle represented by \( N(y) = 0.5|y|/R^- \). The data on the bottom is given by \( h(x) = (0.5 - x)/1.5 \) if \(-1 < x < 0.5\) and \( h(x) = 5(0.5 - x)/0.5 \) if \(0.5 < x < 1\). We obtain \( b^* = 1/3\). The initial free boundary approximation \( \Gamma_0 \) is the straight line from \((0, 0)\) to \((0, R^+)\). The resulting free boundary is displayed in Fig. 2.

In the second example of jet flow problem, we consider the nozzle represented by \( N(y) = 0.5|y|/R^- \) with \( Q = 1 \) and the Dirichlet data on the bottom side given by \( h(x) = (0.5 - x)/1.5 \) if \(-1 < x < 0.5\) and \( h(x) = 5(0.5 - x)/0.5 \) if \(0.5 < x < 1\). We obtained \( b^* = 1\). The initial free boundary approximation \( \Gamma_0 \) is the straight line from \((0, 0)\) to \((0, R^+)\). The resulting free boundary and numerical solution (with constant lines—stream lines) are displayed in Fig. 3.

5 Application to a free boundary problem arising in magnetohydrodynamics

In this section we apply our methodology to a plasma problem studied in Friedman and Liu (1995) and Kang et al. (1997). Here, we are interested in modeling the plasma confined in a Tokamak machine. More specifically, given \( \Omega \subset \mathbb{R}^2 \) and the positive constants \( \gamma \) and \( \lambda \), the plasma problem is to find \( u \), a closed curve \( \Gamma \) lying in \( \Omega \) and a positive constant \( \beta \) such that

\[
\begin{align*}
-\Delta u &= \beta u & & \text{in } \Omega^- = \text{int}\{x \in \Omega, u(x) \leq 0\}, \\
\int_{\Omega^-} u^2 &= 1
\end{align*}
\]

\[
\begin{align*}
-\Delta u &= 0 & & \text{in } \Omega^+ = \{x \in \Omega, u(x) > 0\}, \\
u &= \gamma & & \text{on } \partial \Omega, \\
u &= 0 & & \text{on } \Gamma \text{ and } \Gamma = \partial \Omega^-, \\
|\partial_\nu u^+|^2 - |\partial_\nu u^-|^2 &= \lambda & & \text{on } \Gamma.
\end{align*}
\]

(31)

Here, the plasma is enclosed by the curve \( \Gamma \), and the complement of this region with respect to \( \Omega \) is a vacuum. The function \( u \) represents a flux function associated with the magnetic induction \( \vec{B} \), satisfying \( \vec{B} = (u_{x_2}, -u_{x_1}, 0) \).
Fig. 3 The free interface Π that solves (21)–(29) and (30) with \( N(y) = 0.5(|y|/R)^{0.25}, g(x) = (0.5-x)/1.5 \) if \(-1 < x < 0.5\), \( g(x) = (0.5-x)/0.5 \) if \(0.5 < x < 1\), \( Q = 1 \) and \( b^* = 1 \).

Fig. 4 Free interface that solves (31) with \( \Omega = \{(x, y) : x^2 + y^2 < 1\}, \gamma = 1 \) and \( \lambda = 2^2 - 1^2 = 3 \). Initial configuration: an ellipse centered at \((1/5, 1/5)\) and with axis 1/3 and 1/2 (left). Final configuration (right).

In this problem, the free boundary is a closed curve separating the domain into two connected components, as shown in Friedman and Liu (1995). The adaptation of our scheme to treat this problem is straightforward. In our experiments we used the stopping criterium \( \|\sigma_n^h\|_{L^\infty} < \text{tol} = 10^{-6} \) in our iteration.

In the first example, we consider the case of \( \Omega \) being a ball with center \((0, 0)\) and radius 1. We choose \( \gamma = 1 \) and \( \lambda = 2^2 - 1^2 = 3 \). The initial approximation of the free boundary is an ellipse centered at \((1/5, 1/5)\) and with axis 1/3 and 1/2. The resulting configuration is depicted in Fig. 4 and \( \beta = 13.6727 \). Similar results are presented in Fig. 5 where the
initial approximation is a smoothly perturbed ellipse. We observe that the final shape of the free boundary approximates a circular region. This coincide with the results in Kang et al. (1997) where the authors proved that for the domain \( \Omega \) being the unit circle, the resulting free boundary is circular and centered at \((0, 0)\).

The second example considers the configuration described in Fig. 6. The domain \( \Omega \) corresponds to a circle from which it has been cut off the region \( \{y < -2/3\} \) and the intersection with the circle with center \((5/3, 0)\) and radius 1. In this example, we use \( \lambda = 5^2 - 1^2 = 4 \). The initial approximation of the free boundary is a ball with center \((0, 0)\) and radius \(1/3\). The resulting free boundary is presented in Fig. 6 (center) and the solution is plotted in Fig. 6 (right). The computed value of \( \beta = 13.7034 \).

### 6 Additional numerical examples illustrating the convergence of the method

In this section we present some representative numerical examples. We also use \( \|\sigma_n^h\|_{L^\infty} < \text{tol} = 10^{-6} \) as the iteration stopping criterion.

#### 6.1 A known free interface and error decay

We consider a problem (1)–(4) with a known exact solution, which allows us to measure the accuracy of our method. The domain is \( \Omega = [0, 1] \times [0, 1], \lambda = -1 \), and the boundary data are given by \( g(x, y) = 2 \min\{x - 0.5, 0\} + \max\{x - 0.5, 0\} \). We note that \( u(x, y) = 2 \min\{x - 0.5, 0\} + \max\{x - 0.5, 0\} \) is a solution of the problem. For this exact solution, the free interface is the straight vertical line from \((0.5, 0)\) to \((0.5, 1)\).

We apply our method with the initial approximation of the free boundary given by \( \Gamma_0 = \{(x, y); y = 0.5 + 0.1 \sin(2\pi y)\} \) and the parameter \( \tau = 10^{-4} \). We present the initial and final subdomain configuration in Fig. 7. In Fig. 8 we present the \( L^\infty \) norm of \( \log(|\sigma_n^h|) \) in (13) along the number of iterations \( n \). We observe a decay of the value \( |\sigma_n^h| \) faster than...
Fig. 6: Free interface that solves (31) with depicted domain $\Omega$, $\gamma = 1$ and $\lambda = 5^2 - 1^2 = 4$. Initial configuration: the circle with center $(0,0)$ and radius $1/3$. Final configuration (left). Solution (right).
Fig. 7 Results for the test problem in Sect. 6.1. Initial configuration (left), Final configuration (right)
Fig. 8 Result for the test problem in Sect. 6.1. The initial approximation is given by \( \Gamma_0 = \{(x, y); y = 0.5 + 0.1 \sin(2\pi y)\} \) and the parameter \( \tau = 10^{-4} \). We plot \( \log(\|\sigma^h\|_\infty) \) over the iterations required to achieve \( \|\sigma^h\|_\infty < 10^{-4} \) for three different meshes (left). The corresponding solution error is (in log scale) in \( |\cdot|_{H^1(D)} \) and \( L^\infty(D) \) norms.
Fig. 9 Results for the test problem in Sect. 6.2 that involves heterogeneous coefficients. Initial configuration (left), Final configuration (right).
Fig. 10 Results for the test problem in Sect. 6.2 that involves heterogeneous coefficients. Here, we zoom the free interface in Fig. 9 (blue dotted line marked with blue circles) and we add the free interface approximation obtained after one level of refinement (red solid line marked with squares) (color figure online)

\( O(e^{-0.004n}) \). This example also shows that our stopping criteria are effective in the sense that at the last iteration, we see that the \( L^\infty \) norm is already in stagnated plateau for the corresponding mesh size.

6.2 An example with heterogeneous coefficients

In this subsection we treat a special case of the more general problem (6). More precisely,

\[
\begin{aligned}
- \nabla \cdot (a_1 \nabla u^+) &= 0 & \text{in } \{u > 0\} \\
- \nabla \cdot (a_2 \nabla u^-) &= 0 & \text{in } \{u < 0\} \\
a_1 (\partial_y u^+)^2 - a_2 (\partial_y u^-)^2 &= \lambda & \text{on } \Gamma = \{u = 0\} \\
u &= g & \text{on } \partial \Omega.
\end{aligned}
\]  

(32)

Here, we consider \( \Omega = \{(x, y) : x^2 + y^2 < 1\} \) and the coefficient(s)

\[ a_1(x, y) = 1 + 100(y + 1)^2 \quad \text{and} \quad a_2(x, y) = 1 + 100(y - 1)^2. \]

The Dirichlet data around the circle is given by \( g(x, y) = x \) and we use \( \lambda = -1 \). The initial approximation of the free boundary is the straight line \( \Gamma_0 = \{(0, y), 0 \leq y \leq 1\} \). We run our method with \( \tau = 10^{-5} \). We show the resulting free boundary in Fig. 9. Additionally, in Fig. 10 we zoom in the free interface and compare it with the approximation obtained after one level of refinement is performed. Figure 10 clearly illustrates the convergence of the method.
7 Conclusions and comments

We have proposed a simple iterative method to handle free boundary problems involving nonlinear flux conditions. It is important to note that the numerical treatment of nonlinear flux conditions on the free interface have not been extensively studied in the literature. This is the case despite the fact that the mathematical analysis of simple models with nonlinear flux conditions on the free interface have been carried out by Caffarelli and co-authors a couple of decades ago. The proposed method is a simple domain decomposition method with inexpensive iterations. As a consequence it can be used for the better understanding of simplified models of complex flow problems. We present numerical results showing that our iterative method is effective and performs well in several applications where nonlinear flux jump constraint drives the free interface behavior.

Acknowledgments The authors are thankful to Prof. Eduardo Teixeira for bringing this problem to our attention. H.M.V. was partially supported by FAPERJ Grants E-26/102.965/2011 and E-26/111.416 /2010.

Appendix: An approximation of the flux

Given a free interface approximation $\Gamma^h$, we consider the approximation of the flux of $u^h$ [the solution of problem (11)] on $\Omega^+_h$.

Denote by $A = [a_{ij}]$ the Neumann finite element matrix defined by

$$a_{ij} = \int_{\Omega_h^+} a_1 \nabla \phi_i \cdot \nabla \phi_j \, dx$$

where $\{\phi_i\}$ are the usual hat basis function of the space $P^1(T_h^+, \Omega_h^+)$. We classify the nodes in interior nodes I, boundary nodes $\Sigma^+$ and interface nodes $\Gamma$. This classification gives the following block structure of the matrix $A$:

$$A = \begin{pmatrix} A_{II} & A_{I\Sigma^+} & A_{I\Gamma} \\ A_{I\Sigma^+}^T & A_{\Sigma^+\Sigma^+} & A_{\Sigma^+\Gamma} \\ A_{I\Gamma}^T & A_{\Sigma^+\Gamma}^T & A_{\Gamma\Gamma} \end{pmatrix}.$$

The solution of (11) is given by:

$$u^h = \begin{pmatrix} u^h_I \\ u^h_{\Sigma^+} \\ u^h_{\Gamma} \end{pmatrix} = \begin{pmatrix} A_{II}^{-1}(A_{I\Sigma^+} g^h) \\ g^h \\ 0 \end{pmatrix}.$$

We define $\mu$ by

$$\mu = A_{I\Gamma}^T u_I = A_{I\Gamma}^T A_{II}^{-1} A_{I\Sigma^+} g^h.$$

Let $N_\Gamma$ be the number of vertices of $T^h$ on $\Gamma^h$. We note that using basic finite element analysis $\mu = (\mu_i) \in \mathbb{R}^{N_\Gamma}$ with

$$\mu_i = \int_{\Omega_h^+} (a_1 \nabla u^h) \cdot \nabla \phi_{\ell_i} \, dx = \int_{\Gamma^h} (a_1 \nabla u^h) \cdot \eta_{\Gamma^h} \phi_{\ell_i} \, ds.$$

Here, given $i \in \{1, \ldots, N_\Gamma\}$, $\ell_i$ represents the index of the a node of $T^h$ belonging to $\Gamma^h$. 

\[ \text{ Springer \, SimAC} \]
We use $\mu$ to obtain a piecewise linear approximation of the flux $\nabla u^h \cdot \eta_{\Gamma^h}$. Since $u^h = 0$ on $\Gamma^h$, for each edge of $e_k$ of $\Gamma^h$ we have

$$\nabla u^h|_{e_k} = \partial \eta_k u \eta_k$$

where $\eta_k$ represents the normal vector to edge $e_k$ pointing in the outward direction of $\Omega^+_h$.

Hence,

$$\mu_i = \int_{\Gamma^h} (\eta_{\Gamma^h}^T a_i \eta_{\Gamma^h}) \partial \eta_{\Gamma^h} u \phi_i^h \ ds.$$

(33)

We define $\lambda^h_1$ the piecewise linear approximation of $\partial \eta_{\Gamma^h} u$ as follows. First, we observe that $\lambda^h_1 \in \text{span}\{\phi_i, \Gamma^h\}_{1 \leq i \leq N_{\Gamma h}}$. Next, we introduce the matrix $Q = [q_{ij}] \in \mathbb{R}^{N_{\Gamma h} \times N_{\Gamma h}}$ with

$$q_{ij} = \int_{\Gamma^h} (\eta_{\Gamma^h}^T a_i \eta_{\Gamma^h}) \phi_i \phi_j \ ds.$$

Finally, based on relation (33) we define

$$\lambda^h_1 = \sum_{i=1}^{N_{\Gamma h}} \alpha_i \phi_i \mid_{\Gamma^h}$$

(34)

where $\alpha = (\alpha_i)$ is the solution of

$$Q\alpha = \mu.$$

In a similar way, we define $\lambda^h_2$, the approximation of of the flux on $\Gamma^h$, of the solution of (12).

**Remark 2** A more regular approximation of the flux can be done in practice. For instance, we could obtain $\alpha$ as the solution of the following problem:

$$(Q + \epsilon D)\alpha = \mu.$$

where $D$ is diffusion of operator on $\Gamma^h$ and $\epsilon$ is a regularization parameter.

**References**

Alt HW, Caffarelli LA, Friedman A (1984) Jets with two fluids. i. one free boundary. Indiana Univ Math J 33(2):213–247

Alt HW, Caffarelli LA, Friedman A (1984) Variational problems with two phases and their free boundaries. Trans Am Math Soc 282(2):431–461

Alt HW, Caffarelli LA, Friedman A (1985) Abrupt and smooth separation of free boundaries in flow problems. Ann Scuola Norm Sup Pisa Cl Sci (4) 12(1):137–172

Barrett JW, Elliott CM (1985) Fixed mesh finite element approximations to a free boundary problem for an elliptic equation with an oblique derivative boundary condition. Comput Math Appl 11(4):335–345

Bouchon F, Clain S, Touzani R (2005) Numerical solution of the free boundary Bernoulli problem using a level set formulation. Comput Methods Appl Mech Eng 194(36–38):3934–3948

Chen S, Merriman B, Osher S, Smereka P (1997) A simple level set method for solving Stefan problems. J Comput Phys 135(1):8–29

Chen Z, Shih T, Yue X (2000) Numerical methods for Stefan problems with prescribed convection and nonlinear flux.IMA J Numer Anal 20(1):81–98

Eppler K, Harbrecht H (2006) Efficient treatment of stationary free boundary problems. Appl Numer Math 56(10–11):1326–1339

Flucher M, Rumpf M (1997) Bernoulli’s free-boundary problem, qualitative theory and numerical approximation. J Reine Angew Math 486:165–204

Springer
Friedman A (1968) The Stefan problem in several space variables. Trans Am Math Soc 133:51–87
Friedman A, Liu Y (1995) A free boundary problem arising in magnetohydrodynamic system. Ann Scuola
Norm Sup Pisa Cl Sci (4) 22(3):375–448
Kamenomostskaja SL (1961) On Stefan’s problem. Mat Sb (NS), 53 (95):489–514
Kang KK, Lee JY, Seo JK (1997) Identification of a free boundary arising in a magnetohydrodynamics system.
Inverse Prob 13(5):1301–1309
Kärkkäinen KT, Tiihonen T (1999) Free surfaces: shape sensitivity analysis and numerical methods. Int J
Numer Methods Eng 44(8):1079–1098
Kuster CM, Gremaud PA, Touzani R (2007) Fast numerical methods for Bernoulli free boundary problems.
SIAM J Sci Comput 29(2):622–634
Nochetto RH, Paolini M, Verdi C (1991a) An adaptive finite element method for two-phase Stefan problems
in two space dimensions. I. Stability and error estimates. Math Comp 57(195):73–108, S1–S11
Nochetto RH, Paolini M, Verdi C (1991b) An adaptive finite element method for two-phase Stefan problems in
two space dimensions. II. Implementation and numerical experiments. SIAM J Sci Stat Comput 12(5):1207–
1244
Rubenstein LI (1971) The Stefan problem. American Mathematical Society, Providence. Translated from the
Russian by A. D. Solomon. Translations of mathematical monographs, vol 27
Saavedra P, Scott LR (1991) Variational formulation of a model free-boundary problem. Math Comp 57(196):451–475
Teixeira EV, Leito R, de Queiroz OS (2012) Regularity for degenerate two-phase free boundary problems.
eprint. arXiv:1202.5264
Visintin A (2008) Introduction to Stefan-type problems. In: Handbook of differential equations: evolutionary
equations, vol IV. Elsevier/North-Holland, Amsterdam, p 377–484
van der Zee KG, van Brummelen EH, de Borst R (2010) Goal-oriented error estimation and adaptivity for
free-boundary problems: the domain-map linearization approach. SIAM J Sci Comput 32(2):1064–1092
Zhang Z, Babuška I (1996) A numerical method for steady state free boundary problems. SIAM J Numer Anal
33(6):2184–2214