Representation of the Resonances
of a
Relativistic Quantum Field Theoretical Model
in
Lax-Phillips Scattering Theory
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Abstract: We apply the quantum Lax-Phillips scattering theory to a relativistically
covariant quantum field theoretical form of the (soluble) Lee model. We construct the
translation representations with the help of the wave operators, and show that the resulting
Lax-Phillips $S$-matrix is an inner function (the Lax-Phillips theory is essentially a theory of
translation invariant subspaces). We then discuss the non-relativistic limit of this theory,
and show that the resulting kinematic relations coincide with the conditions required for
the Galilean description of a decaying system.

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1. Introduction.

The theory of Lax and Phillips\(^1\) (1967), originally developed for the description of resonances in electromagnetic or acoustic scattering phenomena, has been used as a framework for the construction of a description of irreversible resonant phenomena in the quantum theory\(^2\)–\(^5\) (which we will refer to as the quantum Lax-Phillips theory). This leads to a time evolution of resonant states which is of semigroup type, i.e., essentially exponential decay. Semigroup evolution is necessarily a property of irreversible processes\(^6\). It appears experimentally that elementary particle decay, to a high degree of accuracy, follows a semigroup law, and hence such processes seem to be irreversible.

The theory of Weisskopf and Wigner\(^7\), which is based on the definition of the survival amplitude of the initial state \(\phi\) (associated with the unstable system) as the scalar product of that state with the unitarily evolved state,

\[
(\phi, e^{-iHt}\phi)
\]

cannot have exact exponential behavior\(^8\). One can easily generalize this construction to the problem of more than one resonance\(^9,10\). If \(P\) is the projection operator into the subspace of initial states (\(N\)-dimensional for \(N\) resonances), the reduced evolution operator is given by

\[
P e^{-iHt} P.
\]

This operator cannot be an element of a semigroup.\(^8\)

Experiments on the decay of the neutral \(K\)-meson system\(^11\) show clearly that the phenomenological description of Lee, Oehme and Yang\(^12\), and Wu and Yang\(^13\), by means of a 2 \(\times\) 2 effective Hamiltonian which corresponds to an exact semigroup evolution of the unstable system, provides a very accurate description of the data. Since the Wigner-Weisskopf theory cannot provide a semigroup evolution law\(^8\), the effective 2 \(\times\) 2 Hamiltonian cannot emerge in the framework of this theory. Furthermore, it has been shown, using estimates based on the quantum mechanical Lee-Friedrichs model\(^14\), that the experimental results appear to rule out the application of the Wigner-Weisskopf theory to the decay of the neutral \(K\)-meson system.

While an exponential decay law can be derived explicitly in terms of a Gel’fand triple\(^15\), the representation of the resonant state in this framework is in a Banach space which does not, in general, coincide with a quantum mechanical Hilbert space; it does not have the properties of a Hilbert space, such as scalar products and the possibility of calculating expectation values. One cannot compute physical properties other than the lifetimes in this way.

The quantum Lax-Phillips theory provides the possibility of constructing a fundamental theoretical description of the resonant system which has exact semigroup evolution, and represents the resonance as a state in a Hilbert space. The Lax-Phillips theory, based on certain classes of functions with half-line support properties, necessarily deals with families of Hilbert spaces of Hardy class functions. We conjecture that the Gel’fand triples constructed on such spaces for the description of resonant states can be imbedded in the large Hilbert space of the Lax-Phillips theory.

Progress has recently been made on the application of stochastic methods\(^16\) to construct a generalization of the Schrödinger evolution which describes collapse of the wave
function during the measurement process, and may indeed provide a framework for general irreversible processes, such as the particle decay problem with which we shall be concerned here. The work of Parthsarathy and Hudson, imbedding such processes in a Hilbert space (in which martingales may be represented as semigroups) leads us to conjecture a close relation with the Lax-Phillips theory.

These conjectures, if realized, would establish the Lax Phillips theory as a unifying framework for several approaches to the description of irreversible processes. We confine ourselves in the present work to a discussion of the quantum Lax-Phillips theory.

In the following, we describe briefly the structure of the quantum Lax-Phillips theory, and give some physical interpretation for the states of the Lax-Phillips Hilbert space.

The Lax-Phillips theory is defined in a Hilbert space $H$ of states which contains two distinguished subspaces, $D_\pm$, called “outgoing” and “incoming”. There is a unitary evolution law which we denote by $U(\tau)$, for which these subspaces are invariant in the following sense:

$$U(\tau)D_+ \subset D_+ \quad \tau \geq 0$$
$$U(\tau)D_- \subset D_- \quad \tau \leq 0 \quad (1.2)$$

The translates of $D_\pm$ under $U(\tau)$ are dense, i.e.,

$$\bigcup_\tau U(\tau)D_\pm = \overline{H} \quad (1.3)$$

and the asymptotic property

$$\bigcap_\tau U(\tau)D_\pm = \emptyset \quad (1.4)$$

is assumed. It follows from these properties that

$$Z(\tau) = P_+ U(\tau) P_- \quad (1.5)$$

where $P_\pm$ are projections into the subspaces orthogonal to $D_\pm$, is a strongly contractive semigroup, i.e.,

$$Z(\tau_1)Z(\tau_2) = Z(\tau_1 + \tau_2) \quad (1.6)$$

for $\tau_1, \tau_2$ positive, and $\| Z(\tau) \| \rightarrow 0$ for $\tau \rightarrow 0$. It follows from (1.2) that $Z(\tau)$ takes the subspace $K$, the orthogonal complement of $D_\pm$ in $\overline{H}$ (associated with the resonances in the Lax-Phillips theory), into itself, i.e.,

$$Z(\tau) = P_K U(\tau) P_K \quad (1.7)$$

The relation (1.7) is of the same structure as (1.1'); there is, as we shall see in the following, an essential difference in the way that the subspaces associated with resonances are defined. The argument that (1.1') cannot form a semigroup is not valid for (1.7).

There is a theorem of Sinai which affirms that a Hilbert space with the properties that there are distinguished subspaces satisfying, with a given law of evolution $U(\tau)$, the properties (1.2), (1.3), (1.4) has a foliation into a one-parameter (which we shall denote as...
s) family of isomorphic Hilbert spaces, which are called auxiliary Hilbert spaces, $\mathcal{H}_s$ for which

$$\mathcal{H} = \int \oplus \mathcal{H}_s.$$  \hspace{1cm} (1.8)

Representing these spaces in terms of square-integrable functions, we define the norm in the direct integral space (we use Lebesgue measure) as

$$\|f\|^2 = \int_{-\infty}^{\infty} ds \|f_s\|^2_{\mathcal{H}},$$ \hspace{1cm} (1.9)

where $f \in \mathcal{H}$ represents a vector in $\mathcal{H}$ in terms of a function in the $L^2$ function space $\mathcal{H} = L^2(-\infty, \infty; H)$; $f_s$ is an element of $\mathcal{H}_s$, the $L^2$ function space (which we shall call the auxiliary space) representing $\mathcal{H}_s$ for any $s$ [we shall not add in what follows a subscript to the norm or scalar product symbols for scalar products of elements of the auxiliary Hilbert space associated to a point $s$ on the foliation axis].

The Sinai theorem furthermore asserts that there are representations for which the action of the full evolution group $U(\tau)$ on $L^2(-\infty, \infty; H)$ is translation by $\tau$ units. Given $D_\pm$ (the subspaces of $L^2$ functions representing $D_\pm$), there is such a representation, called the incoming representation\(^1\), for which the set of all functions in $D_-$ have support in $(-\infty, 0)$ and constitute the subspace $L^2(-\infty, 0; H)$ of $L^2(-\infty, \infty; H)$; there is another representation, called the outgoing representation, for which functions in $D_+$ have support in $(0, \infty)$ and constitute the subspace $L^2(0, \infty; H)$ of $L^2(-\infty, \infty; H)$. The fact that $Z(\tau)$ in Eq. (1.7) is a semigroup is a consequence of the definition of the subspaces $D_\pm$ in terms of support properties on intervals along the foliation axis in the outgoing and incoming translation representations respectively. The non self-adjoint character of the generator of the semigroup $Z(\tau)$ is a consequence of this structure.

Lax and Phillips\(^1\) show that there are unitary operators $W_\pm$, called wave operators, which map elements in $\mathcal{H}$, respectively, to these representations. They define an $S$-matrix,

$$S = W_+ W_-^{-1}$$ \hspace{1cm} (1.10)

which connects the incoming to the outgoing representations; it is unitary, commutes with translations, and maps $L^2(-\infty, 0; H)$ into itself (invariance of a subspace of Hardy class functions). Since $S$ commutes with translations, it is diagonal in Fourier (spectral) representation. As pointed out by Lax and Phillips\(^1\), according to a special case of a theorem of Fourès and Segal\(^19\), an operator with these properties can be represented as a multiplicative operator-valued function $S(\sigma)$ which maps $\mathcal{H}$ into $\mathcal{H}$, and satisfies the following conditions:

(a) $S(\sigma)$ is the boundary value of an operator—valued function $S(z)$ analytic for $\text{Im} z > 0$.

(b) $\|S(z)\| \leq 1$ for all $z$ with $\text{Im} z > 0$.

(c) $S(\sigma)$ is unitary for almost all real $\sigma$.  

\hspace{1cm} 4
An operator with these properties is known as an inner function; such operators arise in the study of shift invariant subspaces, the essential mathematical content of the Lax-Phillips theory. The singularities of this $S$-matrix, in what we shall define as the spectral representation (defined in terms of the Fourier transform on the foliation variable $s$), coincide with the spectrum of the generator of the semigroup characterizing the evolution of the unstable system.

In the framework of quantum theory, one may identify the Hilbert space $\mathcal{H}$ with a space of physical states, and the variable $\tau$ with the laboratory time (the semigroup evolution is observed in the laboratory according to this time). The representation of this space in terms of the foliated $L^2$ space $\mathcal{H}$ provides a natural probabilistic interpretation for the auxiliary spaces associated with each value of the foliation variable $s$, i.e., the quantity $\|f_s\|^2$ corresponds to the probability density for the system to be found in the neighborhood of $s$. For example, consider an operator $A$ defined on $\mathcal{H}$ which acts pointwise, i.e., contains no shift along the foliation. Such an operator can be represented as a direct integral

$$ A = \int_\oplus A_s. \quad (1.11) $$

It produces a map of the auxiliary space $\mathcal{H}$ into $\mathcal{H}$ for each value of $s$, and thus, if it is self-adjoint, $A_s$ may act as an observable in a quantum theory associated to the point $s$.

The expectation value of $A_s$ in a state in this Hilbert space defined by the vector $\psi$, the component of $\psi \in \mathcal{H}$ in the auxiliary space at $s$, is

$$ \langle A_s \rangle_s = \frac{\langle \psi_s, A_s \psi_s \rangle}{\|\psi_s\|^2} \quad (1.12) $$

. Taking into account the a priori probability density $\|\psi_s\|^2$ that the system is found at this point on the foliation axis, we see that the expectation value of $A$ in $\mathcal{H}$ is

$$ \langle A \rangle = \int ds \langle A_s \rangle_s \|\psi_s\|^2 = \int ds \langle \psi_s, A_s \psi_s \rangle, \quad (1.13) $$

the direct integral representation of $(\psi, A\psi)$.

As we have remarked above, in the translation representations for $U(\tau)$ the foliation variable $s$ is shifted (this shift, for sufficiently large $|\tau|$, induces the transition of the state into the subspaces $\mathcal{D}_\pm$). It follows that $s$ may be identified as an intrinsic time associated with the evolution of the state; since it is a variable of the measure space of the Hilbert space $\mathcal{H}$, this quantity itself has the meaning of a quantum variable.

We are presented here with the notion of a virtual history. To understand this idea, suppose that at a given time $\tau_0$, the function which represents the state has some distribution $\|\psi_s^{\tau_0}\|^2$. This distribution provides an a priori probability that the system would be found at time $s$ (not necessarily equal to $\tau_0$), if the experiment were performed at time $s$ corresponding to $\tau = s$ on the laboratory clock. The state of the system therefore contains information on the structure of the history of the system as it is inferred at $\tau_0$.

We shall assume the existence of a unitary evolution on the Hilbert space $\mathcal{H}$, and that for

$$ U(\tau) = e^{-iK\tau}, \quad (1.14) $$

5
the generator \( K \) can be decomposed as

\[
K = K_0 + V
\]

in terms of an unperturbed operator \( K_0 \) with spectrum \((-\infty, \infty)\) and a perturbation \( V \), under which this spectrum is stable. We shall, furthermore, assume that wave operators exist, defined on some dense set, as

\[
\Omega_+ = \lim_{\tau \to -\infty} e^{iK\tau} e^{-iK_0\tau}.
\]

In the soluble model that we shall treat as an example here, the existence of the wave operators is assured.

With the help of the wave operators, we can define translation representations for \( U(\tau) \). The translation representation for \( K_0 \) is defined by the property

\[
0 \langle s, \alpha | e^{-iK_0\tau} f \rangle = 0 \langle s - \tau, \alpha | f \rangle,
\]

where \( \alpha \) corresponds to a label for the basis of the auxiliary space. Noting that

\[
K\Omega_\pm = \Omega_\pm K_0
\]

we see that

\[
\text{out}_\text{in} \langle s, \alpha | e^{-iK\tau} f \rangle = \text{out}_\text{in} \langle s - \tau, \alpha | f \rangle,
\]

where

\[
\text{out}_\text{in} \langle s, \alpha | f \rangle = 0 \langle s, \alpha | \Omega_\pm^\dagger f \rangle
\]

It will be convenient to work in terms of the Fourier transform of the \( \text{in} \) and \( \text{out} \) translation representations; we shall call these the \( \text{in} \) and \( \text{out} \) spectral representations, \( i.e., \)

\[
\text{out}_\text{in} \langle \sigma, \alpha | f \rangle = \int_{-\infty}^{\infty} e^{-\sigma s} \text{out}_\text{in} \langle s, \alpha | f \rangle.
\]

In these representations, (1.20) is

\[
\text{out}_\text{in} \langle \sigma, \alpha | f \rangle = 0 \langle \sigma, \alpha | \Omega_\pm^\dagger f \rangle
\]

and (1.19) becomes

\[
\text{out}_\text{in} \langle \sigma, \alpha | e^{-iK_0\tau} f \rangle = e^{-i\sigma \tau} \text{out}_\text{in} \langle \sigma, \alpha | f \rangle.
\]

Eq. (1.17) becomes, under Fourier transform

\[
0 \langle \sigma, \alpha | e^{-iK_0\tau} f \rangle = e^{-i\sigma \tau} 0 \langle \sigma, \alpha | f \rangle.
\]

For \( f \) in the domain of \( K_0 \), (1.23) implies that

\[
0 \langle \sigma, \alpha | K_0 f \rangle = \sigma_0 \langle \sigma, \alpha | f \rangle.
\]
With the solution of (1.25), and the wave operators, the in and out spectral representations of a vector $f$ can be constructed from (1.24).

We are now in a position to construct the subspaces $D_{\pm}$, which are not given, a priori, in the Lax-Phillips quantum theory. Identifying out$(s, \alpha|f)$ with the outgoing translation representation, we shall define $D_{+}$ as the set of functions with support in $(0, \infty)$ in this representation. Similarly, identifying in$(s, \alpha|f)$ with the incoming translation representation, we shall define $D_{-}$ as the set of functions with support in $(-\infty, 0)$ in this representation. The corresponding elements of $\mathcal{H}$ constitute the subspaces $D_{\pm}$. By construction, $D_{\pm}$ have the required invariance properties under the action of $U(\tau)$.

The outgoing spectral representation of a vector $g \in \mathcal{H}$ is

$$
_{\text{out}}\langle \sigma \alpha | g \rangle = \langle \sigma \alpha | \Omega_{+}^{-1} g \rangle = \int d\sigma' \sum_{\alpha'} \langle \sigma \alpha | S | \sigma' \alpha' \rangle_{0} \langle \sigma' \alpha' | \Omega_{-}^{-1} g \rangle
$$

where we call

$$
S = \Omega_{+}^{-1} \Omega_{-}
$$

the quantum Lax-Phillips $S$-operator. We see that the kernel $\langle \sigma \alpha | S | \sigma' \alpha' \rangle_{0}$ maps the incoming to outgoing spectral representations. Since $S$ commutes with $K_{0}$, it follows that

$$
\langle \sigma \alpha | S | \sigma' \alpha' \rangle_{0} = \delta(\sigma - \sigma') S^{\alpha \alpha'}(\sigma)
$$

(1.28)

It follows from (1.16) and (1.22), in the standard way, that

$$
\langle \sigma \alpha | S | \sigma' \alpha' \rangle_{0} = \lim_{\epsilon \to 0} \delta(\sigma - \sigma') \{ \delta^{\alpha \alpha'} - 2\pi i_{0} \langle \sigma \alpha | T(\sigma + i \epsilon) | \sigma' \alpha' \rangle_{0} \},
$$

(1.29)

where

$$
T(z) = V + VG(z)V = V + VG_{0}(z)T(z).
$$

(1.30)

We remark that, by this construction, $S^{\alpha \alpha'}(\sigma)$ is analytic in the upper half plane in $\sigma$. The Lax-Phillips $S$-matrix is given by the inverse Fourier transform,

$$
S = \{ \langle \sigma \alpha | S | \sigma' \alpha' \rangle_{0} \};
$$

(1.31)

this operator clearly commutes with translations.

From (1.29) it follows that the property (a) above is true. Property (c), unitarity for real $\sigma$, is equivalent to asymptotic completeness, a property which is stronger than the existence of wave operators. For the relativistic Lee model, which we shall treat in this paper, this condition is satisfied. In the model that we shall study here, we shall see that there is a wide class of potentials $V$ for which the operator $S(\sigma)$ satisfies the property (b) specified above.

In the next section, we review briefly the structure of the relativistic Lee model, and construct explicitly the Lax-Phillips spectral representations and $S$-matrix. Introducing
auxiliary space variables, we then characterize the properties of the finite rank Lee model potential which assure that the \(S\)-matrix is an inner function, \textit{i.e.}, that property \((b)\) listed above is satisfied.

2. Relativistic Lee-Friedrichs Model

In this section, we define the relativistic Lee model\(^{22}\) in terms of bosonic quantum fields on spacetime \((x \equiv x^\mu)\). These fields evolve with an invariant evolution parameter\(^{23}\) \(\tau\) (which we identify here with the evolution parameter of the Lax-Phillips theory discussed above); at equal \(\tau\), they satisfy the commutation relations (with \(\psi^\dagger\) as the canonical conjugate field to \(\psi\); the fields \(\psi\), which satisfy first order evolution equations as for nonrelativistic Schrödinger fields, are just annihilation operators)

\[
[\psi_\tau (x), \psi^\dagger_\tau (x')] = \delta^4(x - x').
\]  

(2.1)

We remark that Antoniou, \textit{et al}\(^{24}\), have constructed a relativistic Lee model of a somewhat different type; their field equation is second order in the derivative with respect to the variable conjugate to the mass.

In momentum space, for which

\[
\psi_\tau (p) = \frac{1}{(2\pi)^2} \int d^4xe^{-ip\mu x^\mu} \psi_\tau (x),
\]  

(2.2)

this relation becomes

\[
[\psi_\tau (p), \psi^\dagger_\tau (p')] = \delta^4(p - p').
\]  

(2.3)

The manifestly covariant spacetime structure of these fields is admissible when \(E, p\) are not \textit{a priori} constrained by a sharp mass-shell relation. In the mass-shell limit (for which the variations in \(m^2\) defined by \(E^2 - p^2\) are small), multiplying both sides of (2.3) by \(\Delta E = \Delta m^2 / 2E\), one obtains the usual commutation relations for on-shell fields,

\[
[\tilde{\psi}_\tau (p), \tilde{\psi}^\dagger_\tau (p')] = 2E\delta^3(p - p'),
\]  

(2.4)

where \(\tilde{\psi}(p) = \sqrt{\Delta m^2}\psi(p)\). The generator of evolution

\[
K = K_0 + V
\]  

(2.5)

for which the Heisenberg picture fields are

\[
\psi_\tau (p) = e^{iK_\tau} \psi_0 (p) e^{-iK_\tau}
\]  

(2.6)

is given, in this model, as (we write \(p^2 = p_\mu p^\mu, \ k^2 = k_\mu k^\mu\) in the following)

\[
K_0 = \int d^4p \left\{ \frac{p^2}{2M_V} b^\dagger (p) b(p) + \frac{p^2}{2M_N} a^\dagger_N (p) a_N (p) \right\} + \int d^4k \frac{k^2}{2M_\theta} a^\dagger_\theta (k) a_\theta (k)
\]  

(2.7)
and

\[ V = \int d^4p \int d^4k (f(k) b^\dagger(p) a_N(p-k) a_\theta(k) + f^*(k) b(p) a_N^\dagger(p-k) a_\theta^\dagger(k)), \]  
(2.8)

describing the process \( V \leftrightarrow N + \theta \). The fields \( b(p) \), \( a_N(p) \) and \( a_\theta \) are annihilation operators for the \( V \), \( N \), and \( \theta \) particles, respectively and \( M_v, M_N \) and \( M_\theta \) are the mass parameters for the fields\(^{22}\).

The operators

\[ Q_1 = \int d^4p [b^\dagger(p) b(p) + a_N^\dagger(p) a_N(p)] \]  
(2.9)

\[ Q_2 = \int d^4p [a_N^\dagger(p) a_N(p) - a_\theta^\dagger(p) a_\theta(p)] \]

are conserved, enabling us to decompose the Fock space to sectors. We shall study the problem in the lowest sector \( Q_1 = 1, Q_2 = 0 \), for which there is just one \( V \) or one \( N \) and one \( \theta \). In this sector the generator of evolution \( K \) can be rewritten in the form

\[ K = \int d^4p K^p = \int d^4p (K^p_0 + V^p) \]

where

\[ K^p_0 = \frac{p^2}{2M_v} b^\dagger(p) b(p) + \int d^4k \left( \frac{(p-k)^2}{2M_N} + \frac{k^2}{2M_\theta} \right) a_N^\dagger(p-k) a_\theta^\dagger(k) a_\theta(k) a_N(p-k) \]

and

\[ V^p = \int d^4k \left( f(k) b^\dagger(p) a_N(p-k) a_\theta(k) + f^*(k) b(p) a_N^\dagger(p-k) a_\theta^\dagger(k) \right) \]

In this form it is clear that both \( K \) and \( K_0 \) have a direct integral structure. This implies a similar structure for the wave operators \( \Omega_{\pm} \) and the possibility of defining restricted wave operators \( \Omega^p_{\pm} \) for each value of \( p \). From the expression for \( K^p_0 \) we see that \( |V(p)\rangle = b^\dagger(p)|0\rangle \) can be regarded as a discrete eigenstate of \( K^p_0 \) and, therefore, is annihilated by \( \Omega^p_{\pm} \). This, in turn, implies that \( \Omega^p_{\pm}|V(p)\rangle = 0 \) for every \( p \).

We give the explicit solution in the following\(^{25}\). The \( S \) matrix takes the simple form

\[ \langle 0|\sigma'\alpha'|S|\sigma\alpha\rangle_0 = \delta(\sigma - \sigma') \left\{ 1 - 2\pi i \int d^4p \frac{(|n\rangle_{p,\sigma})^{\alpha'} (|n\rangle_{p,\sigma})^{\alpha^*}}{h(p, \sigma + i\epsilon)} \right\}, \]  
(2.9)

where

\[ h(p, z) = z - \frac{p^2}{2M_v} - \int d^4k \frac{|f(k)|^2}{z - (p-k)^2 - \frac{k^2}{2M_N} - \frac{k^2}{2M_\theta}} \]  
(2.10)

is the well-known denominator function of the Lee-Friedrichs model, and

\[ (|n\rangle_{p,\sigma})^{\alpha} = \int d^4k f^*(k) \langle N(p-k)\theta(k)|\sigma\alpha(N\theta)\rangle_0^* \]  
(2.11)
On the pole (which occurs in the \(N, \theta\) channel), the numerator projects out the auxiliary space vector corresponding to the resonant pole. Note that the complex pole is in \(\sigma \rightarrow z\), not in \(p^\mu\), so the energy momentum remains real. We prove \(^{25}\) (a theorem on Nevanlinna class functions) that
\[
P_{n,P}(\sigma) \equiv |n\rangle_{\sigma,P} \langle n|_{\sigma,P} = |n\rangle_P \langle n|,
\]
(2.12)

independent of \(\sigma\). For each \(p^\mu\), a little algebra gives (for the diagonal)
\[
S_p(\sigma) = 1 - |n\rangle_{pp} \langle n| + \frac{h(p, \sigma - i\epsilon)}{h(p, \sigma + i\epsilon)} |n\rangle_{pp} \langle n|,
\]
(2.13)

where the last term contains a projection to the subspace \(\mathcal{K}\).

Analyticity arguments give
\[
\frac{h(p, \sigma - i\epsilon)}{h(p, \sigma + i\epsilon)} = \frac{\sigma - \mu_p^*}{\sigma - \mu_p} e^{if_p(\sigma)}.
\]
(2.14)

If there is no singularity at \(\infty\) (an exponentially bounded singularity would correspond, in the Lax-Phillips terminology, to a “trivial inner factor”), we obtain the simple form
\[
S_p(\sigma) = 1 - |n\rangle_{pp} \langle n| + \frac{\sigma - \mu_p^*}{\sigma - \mu_p} |n\rangle_{pp} \langle n|.
\]
(2.15)

The resonant wave function is then given explicitly as a vector in the Lax-Phillips Hilbert space as
\[
\langle \sigma \rho' \alpha | R \rangle_p = \delta^4(\rho - \rho') 2i(\Im \mu_p) \frac{|n\rangle_p \langle n|}{\sigma - \mu_p}.
\]
(2.16)

There is no simpler non-trivial Lax-Phillips structure; the Lee-Friedrichs model therefore provides the simplest example\(^{26}\) of a Lax-Phillips scattering theory with a resonance.

3. Nonrelativistic limit

The structure of the Galilean group requires that the nonrelativistic limit contains elementary objects of definite mass. The condition
\[
E - Mc^2 = \varepsilon < \infty
\]
(3.1)
as \(c \rightarrow \infty\) has been found\(^{27}\) to be an effective method for taking this limit. For example, for the free particle, the Hamiltonian
\[
K_0 = \frac{p^\mu p_\mu}{2M} = -\frac{1}{Mc^2} (\varepsilon + Mc^2)^2 + \frac{P^2}{2M}
\]
(3.2)
\[
= \frac{P^2}{2M} - \varepsilon + \text{const}
\]
implies that
\[
\frac{dt}{d\tau} = \frac{\partial K_0}{\partial E} = \frac{\partial K_0}{\partial \varepsilon} = 1,
\]
and hence in this limit \( t = \tau \) (up to a constant). If there are many particles, \( K_0 \) has the form of a sum over such terms, and all of the \( \{t_i\} \) can be put into correspondence with \( \tau \), the Newtonian time. In this way, in refs. (27), one shows that the relativistic micronanical ensemble goes over to the nonrelativistic form, and quantum mechanical wave packets, relativistically spread in the \( t \) direction, contract to have support in the neighborhood of \( t \sim \tau \).

Examining the momentum conservation law for the \( VN\theta \) vertex,
\[
p_V^\mu = p_\theta^\mu + p_N^\mu,
\]
which must be valid by translation invariance of the whole system, we see that
\[
\varepsilon_V + M_Vc^2 = \varepsilon_\theta + M_\theta c^2 + \varepsilon_N + M_Nc^2.
\]
Since \( \varepsilon_V, \varepsilon_\theta, \varepsilon_N \) are bounded as \( c \to \infty \), it follows that
\[
M_V = M_\theta + M_N,
\]
the Galilean mass conservation law, and
\[
\varepsilon_V = \varepsilon_\theta + \varepsilon_N,
\]
a conservation law for the mass fluctuation residues in the Galilean limit.

The relativistic mass inequality for a decaying particle (the mass of the decaying system must be equal or greater than the masses of the decay products), in the form required by the Stueckelberg unperturbed Hamiltonian structure, can be written as
\[
-\frac{p_V^\mu p_V\mu}{2M_V} = -\frac{p^2}{2M_V} + \frac{(\varepsilon + M_Vc^2)^2}{2M_Vc^2} \geq -\frac{(p-k)^2}{2M_N} + \frac{(\varepsilon_N + M_Nc^2)^2}{2M_Nc^2} - \frac{k^2}{2M_\theta} + \frac{(\varepsilon_\theta + M_\theta c^2)^2}{2M_\theta c^2}.
\]
The Galilean mass terms and linear \( \varepsilon \) terms cancel, and for \( (\varepsilon/c)^2 \to 0 \), one obtains
\[
\frac{(p-k)^2}{2M_V} + \frac{k^2}{2M_\theta} \geq \frac{p^2}{2M_V},
\]
a property that should be satisfied for a decaying particle (the kinetic energy of the particles in the final state is greater or equal to that of the initial particle).
The spectral representation $|\sigma\alpha\rangle$ for the problem is represented in the direct sum of the sectors $\{\langle p_V, \beta | \sigma\alpha \rangle_0, \langle p_N, p_\theta, \beta | \sigma\alpha \rangle_0 \}$ ($p_V \equiv p, p_\theta \equiv k$), solutions of

$$\{ - \frac{p^2}{2M_V} + \frac{(\epsilon_V + M_Vc^2)^2}{2M_Vc^2} \} \langle p_V, \beta | \sigma\alpha \rangle_0 = \sigma \langle p_V, \beta | \sigma\alpha \rangle_0, \quad (3.9)$$

and

$$\{ - \frac{p^2}{2M_N} + \frac{(\epsilon_N + M_Nc^2)^2}{2M_Nc^2} \} \langle p_N, k, \beta | \sigma\alpha \rangle_0 = \sigma \langle p_N, k, \beta | \sigma\alpha \rangle_0. \quad (3.10)$$

Since $[E, t] = i\hbar$, it is also true that $[\epsilon, t] = i\hbar$, i.e., $\epsilon \to i\partial_t$, as in the nonrelativistic model of Flesia and Piron\(^2\). Note that in the limit $c \to \infty$, the same (infinite) constant must be subtracted from $\sigma$ in both equations. In the nonrelativistic model of Flesia and Piron\(^2\), the structure discussed by Howland\(^28\), often utilized in dealing with time dependent Hamiltonian theories, was used:

$$K_0 = E + H_0$$
$$K = E + H = K_0 + V \quad (3.11)$$

The model representation\(^29\) is defined by\(^5\)

$$0\langle s, \alpha | K_0 | t\beta \rangle_m = -i\partial_s \langle s, \alpha | t\beta \rangle_m \quad = \sum_{\beta'} (i\partial_t \delta_{\beta\beta'} + H_0^{\beta\beta'} \langle s\alpha | t\beta' \rangle_m), \quad (3.12)$$

where, generally,

$$0\langle s, \alpha | K_0 | f \rangle = \sum_{\beta'} (-i\partial_t \delta_{\beta\beta'} + H_0^{\beta\beta'} m \langle t\beta' | f \rangle). \quad (3.13)$$

The relation (3.12), in the form

$$-i(\partial_s + \partial_t)0\langle s\alpha | t\beta \rangle_m = m \sum_{\beta'} H_0^{\beta\beta'} \langle s\alpha | t\beta' \rangle_m, \quad (3.14)$$

is seen to determine the $s + t$ dependence of the transformation function, but the $s - t$ dependence remains arbitrary. Unitarity restricts its form; it can be shown that\(^5\)

$$S^{\beta\beta'}(\sigma) = \sum_{\alpha'\alpha} U^{\alpha\beta*}(\sigma) (S^{aux})^{\alpha\alpha'} U^{\alpha'\beta'}(\sigma),$$

so that the Lax-Phillips $S$-matrix is related to the $S$-matrix of the problem $H = H_0 + V$ in the auxiliary space by a unitary transformaiton with some similarity to dilation analytic methods\(^30\).

The non-relativistic limit of the relativistic results (3.9) and (3.10 contain terms of order $1/c^2$ which remove the arbitrariness of description of the transformation; its consequences for the nonrelativistic problem are under investigation.
4. Conclusions

We have studied the application of Lax-Phillips quantum theory to a soluble relativistic quantum field theoretical model. In this model, we obtain the Lax-Phillips $S$-matrix explicitly as an inner function (the Lax-Phillips structure is defined pointwise on a foliation over the total energy-momentum of the system). The structure of the Lee model $S$-matrix has a term with factorized numerator, corresponding to the transition matrix element of the interaction, and denominator $h(p, \sigma)$ which contains the zero inducing the resonance pole. The numerator factors can be identified as a vector field over the complex $\sigma$ plane. Foliating the $S$-matrix over the total energy momentum $p$, it takes on the form of a projection into the space complementary to the discrete subspace of the rank one potential of the model (for each point $\sigma, p$), plus an scalar inner function on the discrete subspace. The vector field on the complex extension of the spectral representation (on the singular point, it corresponds to the projection into the resonant eigenstate), is independent of the spectral parameter up to a scalar multiplicative function. It then follows that the projection is in fact independent of $\sigma$. This result leads to the conclusion that the properties of the $S$-matrix are essentially derived from the properties of a scalar inner function.

This inner function consists of, in general, a rational factor, which contains all of the zeros and poles, and a singular factor (constructed with singular measure). If the singular factor is exponentially bounded, it is, in the terminology of Lax and Phillips\textsuperscript{1}, a trivial inner function. The application of this inner function does not change the resonance structure, but the functional form of the eigenfunctions and scattering states may be altered.

We then studied the rational case, the simplest possible model for a non-trivial Lax-Phillips theory, for which the inner function reduces to just the ratio $(\sigma - \mu_p^*)/(\sigma - \mu_p)$. We therefore see, conversely, that the simplest model for a non-trivial Lax-Phillips theory corresponds to a rank one Lee model\textsuperscript{26}.

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