ON THE CIRCUMRADIUS OF A SPECIAL CLASS OF n-SIMPLICIES

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ABSTRACT. An n-simplex is called circumscriptible (or edge-incentric) if there is a sphere tangent to all its n(n+1)/2 edges. We obtain a closed formula for the radius of the circumscribed sphere of the circumscriptible n-simplex, and also prove a double inequality involving the circumradius and the edge-inradius of such simplices. Among this inequality settles affirmatively a part of a problem posed by the authors.

1. INTRODUCTION

A (non-degenerate) n-simplex Ω = [A₀, A₁, ..., Aₙ], n ≥ 1, is defined as the convex hull of n + 1 affinely independent points (or position vectors) A₀, A₁, ..., Aₙ in Euclidean n-space. The points A₀, A₁, ..., Aₙ are the vertices of Ω, and the line segments aᵢⱼ joining two different vertices Aᵢ and Aⱼ are its edges.

Every n-simplex has a circumscribed sphere passing through its n + 1 vertices and an inscribed sphere tangent to each of its n + 1 facets. For the circumradius R and the inradius r, we have the celebrated Euler’s inequality as follows

\[ R \geq nr. \tag{1.1} \]

An n-simplex is circumscriptible (or edge-incentric) if there is a sphere tangent to all its n(n+1)/2 edges. Considering such a simplex, we call this the edge-tangent sphere of the n-simplex, and note ρ as the edge-inradius of this sphere. Of course, not every n-simplex (n ≥ 3) has an edge-tangent sphere. However, we have the following sufficient and necessary condition given by Lin and Zhu [3] (see also Hajja [1, p. 242, Theorem 4.1]).

Theorem 1.1. The n-simplex Ω has an edge-tangent sphere if and only if there exist (i.e., so-called the balloon radii) xᵢ > 0 with 0 ≤ i ≤ n satisfying aᵢⱼ = xᵢ + xⱼ for 0 ≤ i < j ≤ n or

\[ xᵢ = \frac{1}{n(n-1)} \left( \sum_{i=0}^{n} aᵢⱼ - \sum_{0 \leq i < j \leq n} aᵢⱼ \right). \]

In 2006, Hajja [1] derived many geometrical properties of the circumscriptible n-simplex. He also proved a closed formula involving the edge-inradius (i.e., the radius of the edge-tangent sphere):

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**Theorem 1.2.** ([1, p. 249, Theorm 7.2 (d)]) The edge-inradius $\rho$ of $\Omega$ is given by

$$\rho^2 = \frac{2(n-1)}{\left(\sum_{i=0}^{n} \frac{1}{x_i}\right)^2 - \left(n-1\right)\sum_{i=0}^{n} \frac{1}{x_i^2}}.$$  \hspace{1cm} (1.2)

The original for Theorem 1.2 is based on the following generalized formula of the edge-inradius of a circumscribable $n$-simplex in terms of its edge-lengths given by Ivanoff [2], Lin and Zhu [3].

**Theorem 1.3.** Given a circumscribable $n$-simplex $\Omega$, we have

$$\rho^2 = -\frac{|A|}{2|A_1|},$$

where

$$A = \begin{pmatrix} -2x_0^2 & 2x_0x_1 & \cdots & 2x_0x_n \\ 2x_0x_1 & -2x_1^2 & \cdots & 2x_1x_n \\ \vdots & \vdots & \ddots & \vdots \\ 2x_0x_n & 2x_1x_n & \cdots & -2x_n^2 \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & A & \vdots \\ 1 & \cdots & \cdots & \cdots \end{pmatrix}.$$}

For the circumradius $R$ of a circumscribable $n$-simplex $\Omega$, in 2007, an interesting problem stems naturally from the above investigations and the following Theorem 1.4 by Hajja [1, p. 261]: Finding a closed formula for the circumradius of a circumscribable $n$-simplex in terms of its balloon radii $x_i$ with $0 \leq i \leq n$ as similarly (1.2).

**Theorem 1.4.** (see [2] and also [5]) For a circumscribable $n$-simplex $\Omega$, then we have

$$R^2 = -\frac{|D|}{2|D_1|},$$

where

$$D = \begin{pmatrix} 0 & (x_1 + x_0)^2 & (x_2 + x_0)^2 & \cdots & (x_n + x_0)^2 \\ (x_0 + x_1)^2 & 0 & (x_2 + x_1)^2 & \cdots & (x_n + x_1)^2 \\ (x_0 + x_2)^2 & (x_1 + x_2)^2 & 0 & \cdots & (x_n + x_2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (x_0 + x_n)^2 & (x_1 + x_n)^2 & (x_2 + x_n)^2 & \cdots & 0 \end{pmatrix},$$

and

$$D_1 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & D & \vdots \\ 1 & \cdots & \cdots & \cdots \end{pmatrix}.$$}

A double inequality for the radius of the circumscribable tetrahedron $\Omega = A_0A_1A_2A_3A_4$ and sharpening Euler’s inequality (1.1) is proved in [1] and [6]

$$R \geq \sqrt{3}\rho \geq 3r. \hspace{1cm} (1.3)$$
As a generalization of inequality (1.3), Wu and Zhang [6] posed an analogous problem for the circumscribable $n$-simplex.

**Problem 1.1.** In a circumscribable $n$-simplex $\Omega$, prove or disprove that

$$R \geq \sqrt{\frac{2n}{n-1}} \rho \geq nr.$$  

(1.4)

Recently, Wu et al. [7] proved the right hand of double inequality (1.4). In this paper, we will give a closed formula for the circumradius of a circumscribable $n$-simplex in terms of its balloon radii $x_i$ with $0 \leq i \leq n$ as similarly (1.2), and settle the left hand of double inequality (1.4) affirmatively.

2. Main Results

**Theorem 2.1.** The radius $R$ of the circumscribed sphere of a circumscribable $n$-simplex $\Omega$ is given by

$$\left(\frac{R}{\rho}\right)^2 = \frac{[MP - (n-1)(n-3)]^2 - [M^2 - (n-1)N][P^2 - (n-1)Q]}{16(n-1)^2},$$

(2.1)

and

$$R^2 = \frac{[MP - (n-1)(n-3)]^2 - [M^2 - (n-1)N][P^2 - (n-1)Q]}{8(n-1)[P^2 - (n-1)Q]},$$

(2.2)

where

$$M = \sum_{i=0}^{n} x_i, \quad N = \sum_{i=0}^{n} x_i^2, \quad P = \sum_{i=0}^{n} \frac{1}{x_i}, \quad \text{and} \quad Q = \sum_{i=0}^{n} \frac{1}{x_i^2}.$$  

(2.3)

**Remark 2.1.** When $n = 2$, then, in the triangle, we have the well known formula

$$R = \frac{(x_0 + x_1)(x_1 + x_2)(x_2 + x_0)}{4\sqrt{x_0x_1x_2(x_0 + x_1 + x_2)}} = \frac{a_{01}a_{12}a_{02}}{\sqrt{(a_{01} + a_{12} + a_{02})(a_{01} + a_{12} - a_{02})(a_{12} + a_{02} - a_{01})(a_{01} + a_{02} - a_{12})}}.$$  

**Remark 2.2.** In 2007, Hajja [1, p. 261] said: “the questions regarding non-regular edge-incentric $d$-simplices in which the circumcenter and the incenter coincide were not considered. We expect these questions to be rather difficult, since we were unable to find a closed formula for the circumradius of an edge-incentric $d$-simplex in terms of its balloon radii. Such a formula in the form of a quotient of two determinants is given in [2].”

For the given formula (2.2), in our private communication, Hajja also said: “I am really very impressed that you succeeded in finding a closed formula for $R$ — I have tried to do so last year but never was able to. Actually, I asked a colleague in Germany for help but he failed too”.

**Theorem 2.2.** For circumscribable $n$-simplex $\Omega$, we have

$$0 \leq R^2 - \frac{2n}{n-1} \rho^2 \leq (n+1)^2 |OG|^2,$$

(2.4)

where $O$ and $G$ are the circumcenter and the centroid of the circumscribable $n$-simplex $\Omega$, respectively.

**Remark 2.3.** The left hand of double inequality (2.4) is just the left hand of double inequality (1.4).
3. Preliminary Results

Throughout this section, let $A$, $A_1$, $D$, $D_1$ and $x_i$ for $0 \leq i \leq n$ be defined by the above section, $V$ be the volume of $\Omega$, and

$$B_1 = \begin{pmatrix} x_3^2 & 1 \\ x_1^2 & 1 \\ x_2^2 & 1 \\ \vdots \\ x_n^2 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \end{pmatrix}.$$  \(\text{(3.1)}\)

**Lemma 3.1.** \((\text{[1] Lemma 1})\) We have

$$|A| = (-1)^n (n-1) 2^{2n+1} \left( \prod_{i=0}^{n} x_i \right)^2,$$  \(\text{(3.2)}\)

and

$$A^{-1} = \begin{pmatrix} 2 - n & \frac{1}{4n-4} & \frac{1}{4n-4} & \cdots & \frac{1}{4n-4} \\ \frac{1}{2n} & \frac{1}{4n-4} & \cdots & \frac{1}{4n-4} \\ \frac{1}{2n} & \frac{1}{4n-4} & \cdots & \frac{1}{4n-4} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{2n} & \frac{1}{4n-4} & \cdots & \frac{1}{4n-4} \end{pmatrix}.$$  \(\text{(3.3)}\)

**Proof.** It is clear that

$$|A| = 2^{n+1} \left( \prod_{i=0}^{n} x_i \right)^2 \begin{vmatrix} -1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & -1 \end{vmatrix}$$

$$= 2^{n+1} \left( \prod_{i=0}^{n} x_i \right)^2 \begin{vmatrix} 2 & -2 & 0 & \cdots & 0 \\ 0 & 2 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 0 & 0 & \cdots & -2 \end{vmatrix}$$

$$= 2^{n+1} \left( \prod_{i=0}^{n} x_i \right)^2 \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{vmatrix}$$

$$= 2^{n+1} \left( \prod_{i=0}^{n} x_i \right)^2 \cdot (-1)^n (n-1) 2^n$$

$$= (-1)^n (n-1) 2^{2n+1} \left( \prod_{i=0}^{n} x_i \right)^2.$$
Let us to compute $A^{-1}$. We know that the adjoint of $A$ is

$$A^* = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1,n+1} \\
A_{21} & A_{22} & \cdots & A_{2,n+1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n+1,1} & A_{n+1,2} & \cdots & A_{n+1,n+1}
\end{pmatrix},$$

where $A_{ij}$ is the cofactor of the element $2x_{i-1}x_{j-1}(i \neq j)$ or $-2x_{i-1}^2(i = j)$ of $A$.

From the process of computing $|A|$ above, for $0 \leq j \leq n$, it is easily to obtain that

$$A_{jj} = (-1)^{n-1}(n-2)2^{2n-1} \left( \prod_{i=1, i \neq j}^{n} x_i \right)^2 = \frac{2-n}{4(n-1)} \cdot \frac{1}{x_j^2} \cdot |A|.$$

Now, we compute $A_{ij}$ for $0 \leq i < j \leq n$ because of $A_{ij} = A_{ji}$ with $A = A^T$. That is

$$A_{ij} = (-1)^{i+j} \begin{pmatrix}
-2x_0^2 & 2x_1x_0 & \cdots & 2x_{j-2}x_0 & 2x_jx_0 & \cdots & 2x_nx_0 \\
2x_0x_1 & -2x_1^2 & \cdots & 2x_{j-2}x_1 & 2x_jx_1 & \cdots & 2x_nx_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2x_0x_{n-2} & 2x_{n-2}x_{n-2} & \cdots & 2x_{j-2}x_{n-2} & 2x_jx_{n-2} & \cdots & 2x_nx_{n-2} \\
2x_0x_i & 2x_1x_i & \cdots & 2x_{j-2}x_i & 2x_jx_i & \cdots & 2x_nx_i \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2x_0x_n & 2x_1x_n & \cdots & 2x_{j-2}x_n & 2x_jx_n & \cdots & -2x_n^2
\end{pmatrix}.$$
\[
= (-1)^{2j+1} 2^n (-2)^{n-1} \left( \prod_{i=0}^{n} x_i \right)^2
\]

\[
= (-1)^n 2^{2n-1} \left( \prod_{i=0}^{n} x_i \right)^2 = \frac{1}{4(n-1)} \cdot \frac{1}{x_{i-1}x_{j-1}} \cdot |A|,
\]

where
\[
C_1 = \begin{pmatrix}
1 & -2 & 1 & \cdots & 1 \\
1 & 1 & -2 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & -2 \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}_{(j-i) \times (j-i)},
\]

and
\[
C_2 = \begin{pmatrix}
0 & -2 & 0 & \cdots & 0 \\
0 & 0 & -2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -2 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}_{(j-i) \times (j-i)}.
\]

It follows that (3.2) is true from the above \( A^* \), \( A_{ij} \) and

\[
A^{-1} = \frac{1}{|A|} A^*.
\]

**Remark 3.1.** Wu et al. [7] directly gave \( A^{-1} \) without computing process, and we here give the complete proof of this lemma. And the proof of (3.1) is more simple than Hajja [1, p. 250].

**Lemma 3.2.** For \( x_i > 0 \) with \( 0 \leq i \leq n \),

\[
|D| = (-1)^n 2^{2n-3} \left( \prod_{i=0}^{n} x_i \right)^2 \cdot \left\{ [MP - (n-1)(n-3)]^2 - [M^2 - (n-1)N][P^2 - (n-1)Q] \right\},
\]

where \( M, N, P, \) and \( Q \) are given by (2.3).

We shall give two proofs of Lemma 3.2 as follows.

**Proof 1.** Denoting the \( j \)-th column by \( P_j \) and the \( j \)-th row by \( Q_j \), we perform the following operations on \( |D| \):

1. We accession a new row \( Q_1 = (1 \ x_0^2 \ x_1^2 \ x_2^2 \ \cdots \ x_n^2) \).
2. We subtract \( Q_1 \) from \( Q_{j+1} \) for \( j = 1, \cdots, n+1 \).
3. We accession a new column \( P_1 = (1 \ 0 \ x_0^2 \ x_1^2 \ x_2^2 \ \cdots \ x_n^2) \).
4. We subtract \( P_1 \) from \( P_{j+2} \) for \( j = 1, \cdots, n+1 \).
5. We divide \( Q_{j+2} \) and \( P_{j+2} \) by taking appropriate common factor \( x_{j-1} \) for \( j = 1, \cdots, n+1 \).
6. We add \( P_{j+3} \) from \( P_3 \) for \( j = 1, \cdots, n \) and divide \( P_3 \) by taking a common factor \( n-1 \).
(7) We subtract $P_3$ from $P_{j+3}$ for $j = 1, \ldots, n$.

(8) We add $\frac{1}{4} \left( \frac{1}{n-1} - \frac{1}{x_j} \right) Q_{j+3}$ from $Q_1$ and $\frac{1}{4} \left( x_j - \frac{1}{n-1} M \right) Q_{j+3}$ from $Q_2$

for $j = 1, 2, \ldots, n$, and also add $\frac{1}{2} \left( \frac{1}{n-1} - \frac{1}{x_0} \right) Q_3$ from $Q_1$ and $\frac{1}{2} \left( x_0 - \frac{1}{n-1} M \right) Q_3$

from $Q_2$.

It is clear to see that

$$|D| = \begin{vmatrix}
0 & (x_1 + x_0)^2 & (x_2 + x_0)^2 & \cdots & (x_n + x_0)^2 \\
(x_0 + x_1)^2 & 0 & (x_2 + x_1)^2 & \cdots & (x_n + x_1)^2 \\
(x_0 + x_2)^2 & (x_1 + x_2)^2 & 0 & \cdots & (x_n + x_2)^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(x_0 + x_n)^2 & (x_1 + x_n)^2 & (x_2 + x_n)^2 & \cdots & 0
\end{vmatrix}$$

$$= \begin{vmatrix}
1 & x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\
-1 & -x_0^2 & x_0^2 + 2x_0x_1 & x_0^2 + 2x_0x_2 & \cdots & x_0^2 + 2x_0x_n \\
-1 & x_1^2 + 2x_0x_1 & -x_1^2 & x_1^2 + 2x_1x_2 & \cdots & x_1^2 + 2x_1x_n \\
-1 & x_2^2 + 2x_0x_2 & x_2^2 + 2x_1x_2 & -x_2^2 & \cdots & x_2^2 + 2x_2x_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & x_n^2 + 2x_0x_n & x_n^2 + 2x_1x_n & x_n^2 + 2x_2x_n & \cdots & -x_n^2
\end{vmatrix}$$

$$= \begin{vmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\
x_0^2 & -1 & -x_0^2 & x_0^2 + 2x_0x_1 & x_0^2 + 2x_0x_2 & \cdots & x_0^2 + 2x_0x_n \\
x_1^2 & -1 & x_1^2 + 2x_0x_1 & -x_1^2 & x_1^2 + 2x_1x_2 & \cdots & x_1^2 + 2x_1x_n \\
x_2^2 & -1 & x_2^2 + 2x_0x_2 & x_2^2 + 2x_1x_2 & -x_2^2 & \cdots & x_2^2 + 2x_2x_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_n^2 & -1 & x_n^2 + 2x_0x_n & x_n^2 + 2x_1x_n & x_n^2 + 2x_2x_n & \cdots & -x_n^2
\end{vmatrix}$$

$$= \begin{vmatrix}
1 & 0 & -1 & -1 & \cdots & -1 \\
0 & 1 & x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\
x_0^2 & -1 & -2x_0^2 & 2x_0x_1 & 2x_0x_2 & \cdots & 2x_0x_n \\
x_1^2 & -1 & 2x_0x_1 & -2x_1^2 & 2x_1x_2 & \cdots & 2x_1x_n \\
x_2^2 & -1 & 2x_0x_2 & 2x_1x_2 & -2x_2^2 & \cdots & 2x_2x_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_n^2 & -1 & 2x_0x_n & 2x_1x_n & 2x_2x_n & \cdots & -2x_n^2
\end{vmatrix}$$

$$= \prod_{i=0}^{n} x_i^2$$
\[ 1 \quad 0 \quad -\frac{1}{n-1}P \quad -\frac{1}{n-1}\frac{1}{x_1} \quad \frac{1}{2} \quad \ldots \quad \frac{1}{2} \\
0 \quad 1 \quad \frac{n-1}{2} \quad x_1 \quad x_2 \quad \ldots \quad x_n \]
\[
\begin{array}{cccccc}
x_0 & -\frac{x_0}{x_1} & 2 & 2 & 2 & \ldots & 2 \\
x_1 & -\frac{1}{x_2} & 2 & -2 & 2 & \ldots & 2 \\
x_2 & -\frac{1}{x_2} & 2 & 2 & -2 & \ldots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_n & -\frac{1}{x_n} & 2 & 2 & 2 & \ldots & -2 \\
\end{array}
\]
\[
= (n-1) \prod_{i=0}^{n} x_i^2  \\
= (n-1) \prod_{i=0}^{n} x_i^2 \\
= (n-1) \prod_{i=0}^{n} x_i^2  \\
= (n-1)^2 \frac{2^{n-3}}{n-1} \left( \prod_{i=0}^{n} x_i \right)^2 \cdot (X_1^2 - X_2 X_3).
\]

where
\[
X_1 = MP - (n-1)(n-3),
\]
\[
X_2 = P^2 - (n-1)Q,
\]
\[
X_3 = M^2 - (n-1)N,
\]

and \(M, N, P, Q\) are given by \(2.3\). \(\square\)

**Proof 2.** It is easily to find that
\[
\begin{vmatrix}
A & B_1 \\
-B_2 & E_2
\end{vmatrix} = \begin{vmatrix}
A & B_1 \\
0 & E_2 + B_2 A^{-1} B_1
\end{vmatrix} = |A| \cdot |E_2 + B_2 A^{-1} B_1|, \quad (3.4)
\]
and
\[
\begin{vmatrix}
A & B_1 \\
-B_2 & E_2
\end{vmatrix} = \begin{vmatrix}
A & B_1 B_2 \\
0 & E_2
\end{vmatrix} = |A + B_1 B_2|, \quad (3.5)
\]
where \(E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\).

By means of \(3.2\), we obtain
\[
E_2 + B_2 A^{-1} B_1 = \begin{pmatrix}
\frac{1}{4(n-1)} X_1 \\
\frac{1}{4(n-1)} X_3
\end{pmatrix} \begin{pmatrix}
\frac{1}{4(n-1)} X_2 \\
\frac{1}{4(n-1)} X_1
\end{pmatrix}. \quad (3.6)
\]
From (3.1) and (3.4)–(3.6), it is deduced that

\[ |D| = |A + B_1B_2| = |A| \cdot |E_2 + B_2A^{-1}B_1| 
= (-1)^n \frac{2^{2n-3}}{n-1} \left( \prod_{i=0}^{n} x_i \right)^2 (X_1^2 - X_2X_3). \]

where \( X_1, X_2 \) and \( X_3 \) are given by (3.3).

This evidently completes the proof of Lemma 3.2. □

Lemma 3.3. ([5, Corollary 2, p. 96]) For \( n \)-simplex \( \Omega \), we have

\[ (n!)^2 V^2 R^2 = -\det \left( -\frac{1}{2} a_{ij}^2 \right) = \frac{(-1)^n |D|}{2n+1}. \]

Lemma 3.4. ([3, Corollary 1]) Given a circumscribable \( n \)-simplex \( \Omega \), we have

\[ (n!)^2 V^2 \rho^2 = 2^n (n-1) \left( \prod_{i=0}^{n} x_i \right)^2. \]

Lemma 3.5. ([5, (3.5.11), p. 112]) Let \( O \) and \( G \) are the circumcenter and the centroid of the \( n \)-simplex \( \Omega \), respectively. Then we have

\[ |OG|^2 = R^2 - \frac{1}{(n+1)^2} \sum_{0 \leq i<j \leq n} a_{ij}^2. \]

4. THE PROOF OF THEOREM 2.1

Proof. This follow straightforwardly from Theorem II and Lemmas 3.2, 3.3, 3.4 by standard arguments. □

5. THE PROOF OF THEOREM 2.2

Proof. We will prove Theorem 2.2 with two steps.

(i) Firstly, we prove the left hand of inequality (2.4).

Let \( M, N, P, \) and \( Q \) are given by (2.3). By using the well-known power mean inequality and Cauchy inequality, then we have \( N \geq \frac{M^2}{n+1}, Q \geq \frac{P^2}{n+1} \), and \( MP \geq (n+1)^2 \). Further considering \( P^2 - (n-1)Q > 0 \) follows that

\[ [M^2 - (n-1)N][P^2 - (n-1)Q] \leq \left[ M^2 - \frac{n-1}{n+1} M^2 \right] \left[ P^2 - \frac{n-1}{n+1} P^2 \right] = \frac{4}{(n+1)^2} M^2 P^2. \] (5.1)

From (5.1), \( 1 - \frac{4}{(n+1)^2} > 0 \) and the function

\[ y = \left[ 1 - \frac{4}{(n+1)^2} \right] \left[ x - \frac{(n-3)(n+1)^2}{n+3} \right]^2 - \frac{4(n-1)(n-3)^2}{n+3} \]
is increasing on interval $\left(\frac{(n-3)(n+1)^2}{n+3}, +\infty\right)$, and $MP \geq (n+1)^2 \geq \frac{(n-3)(n+1)^2}{n+3}$ for $n \geq 2$, we obtain

$$[MP - (n-1)(n-3)]^2 - [M^2 - (n-1)N][P^2 - (n-1)Q]$$

$$\geq [MP - (n-1)(n-3)]^2 - \frac{4}{(n+1)^2} M^2 P^2$$

$$= \left[1 - \frac{4}{(n+1)^2}\right] \left[MP - \frac{(n-3)(n+1)^2}{n+3}\right]^2 - \frac{4(n-1)(n-3)^2}{n+3}$$

$$= 32(n-1).$$

According to (2.1) and (5.2), we get

$$\left(\frac{R}{\rho}\right)^2 \geq \frac{32n(n-1)}{16(n-1)^2} = \frac{2n}{n-1}.$$ 

It is clear to show that the left of inequality (2.4) holds.

(ii) Secondly, with Lemma 3.5, the right hand of inequality (2.4) is

$$(n+1)^2 R^2 - \sum_{0 \leq i < j \leq n} a_{ij}^2 \geq R^2 - \frac{2n}{n-1} \rho^2.$$ 

(5.3)

For $n = 3$, the required result is proved in [4].

Now we prove that inequality (5.3) holds when $n \geq 4$. Obviously, from Theorems 1.2–2.1, inequality (5.3) is equivalent to

$$n(n+2) R^2 + \frac{2n}{n-1} \rho^2 \geq \sum_{0 \leq i < j \leq n} a_{ij}^2,$$

or

$$n(n+2) \cdot \frac{[MP - (n-1)(n-3)]^2 - [M^2 - (n-1)N][P^2 - (n-1)Q]}{P^2 - (n-1)Q}$$

$$+ \frac{4n}{P^2 - (n-1)Q} \geq M^2 + (n-1)N,$$

that is

$$n(n+2)[MP - (n-1)(n-3)]^2 + 32n(n-1)$$

$$\geq [(n^2 + 10n - 8)M^2 - (n-1)(n-2)(n-4)N][P^2 - (n-1)Q],$$

(5.4)

where $M, N, P,$ and $Q$ are given by (2.3).

From the proof of Theorem 2.1, we have $N \geq \frac{M^2}{n+1}$ and $Q \geq \frac{P^2}{n+1}$. Hence, in order to prove inequality (5.4), we only need to prove the following inequality

$$n(n+2)[MP - (n-1)(n-3)]^2 + 32n(n-1) \geq \frac{12n(3n-2)}{(n+1)^2} M^2 P^2$$
or
\[(n^2 + 5n - 26) \left[ MP - \frac{(n + 2)(n - 3)(n + 1)^2}{n^2 + 5n - 26} \right]^2 - \frac{4(3n - 10)^2(n + 1)^4}{n^2 + 5n - 26} \geq 0. \tag{5.5}\]

When \(n \geq 4(n \in \mathbb{N})\), it’s clear that \((n + 1)^2 > \frac{(n+2)(n-3)(n+1)^2}{n^2 + 5n - 26}\), and the function
\[f(x) = (n^2 + 5n - 26) \left[ x - \frac{(n + 2)(n - 3)(n + 1)^2}{n^2 + 5n - 26} \right]^2 - \frac{4(3n - 10)^2(n + 1)^4}{n^2 + 5n - 26}\]
is increasing on interval \([ (n+1)^2, +\infty) \). Thus, from \(MP \geq (n + 1)^2\), we get \(f(MP) \geq f((n + 1)^2) = 0\).

It is just as inequality \((5.5)\). Further, inequality \((5.4)\) or \((5.3)\) holds.

The proof of Theorem 2.2 is thus completed.

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