Parallel Translations for a Left Invariant Spray

Ming Xu

Received: 9 November 2022 / Revised: 13 February 2023 / Accepted: 16 February 2023 / Published online: 9 March 2023 © The Author(s) under exclusive licence to Iranian Mathematical Society 2023

Abstract
In this paper, we study the left invariant spray geometry on a connected Lie group. Using the technique of invariant frames, we find the ordinary differential equations on the Lie algebra describing for a left invariant spray structure the linearly parallel translations along a geodesic and the nonlinearly parallel translations along a smooth curve. In these equations, the connection operator plays an important role. Using parallel translations, we provide alternative interpretations or proofs for some homogeneous curvature formulae. In particular, the Riemannian curvature appears in both a double Lie derivative along the spray vector and brackets between smooth vector fields induced by the connection operator. We propose two questions in left invariant spray geometry. One question generalizes Landsberg Problem in Finsler geometry, and the other concerns the restricted holonomy group.

Keywords Connection operator · Geodesic · Holonomy group · Invariant frame · Invariant spray structure · Parallel translation · Riemann curvature · Spray vector field

Mathematics Subject Classification 53B40 · 53C30 · 53C60

1 Introduction

Spray geometry concerns a spray structure $G$ on a smooth manifold $M$, which is a smooth vector field on $TM \setminus \{0\}$ with the standard local coordinate presentation $G = y^i \partial_{x^i} - 2G^i \partial_{y^i}$, where $G^i = G^i(x, y)$ is positive 2-homogeneous for the $y$-entry [15]. For example, when $G$ is induced by a Finsler metric $F$, then $G^i = 1/4g^{ij}([F^2]_{x^j}y^k - [F^2]_{x^j})$ [16]. Many notions in Finsler geometry, like (nonconstant) geodesic, Riemann
curvature, linear and nonlinear parallel translations, are only relevant to \( G \), i.e., they are originated from spray geometry [1]. See [5, 10, 11, 22] for some recent progress in this field.

In this paper, we discuss a special class of homogeneous spray structure. A spray structure \( G \) on a connected Lie group \( G \) is called left invariant, if it is preserved by all left translations [5, 18]. It appeared in [13, 14], where its inverse problem was concerned. Using a left invariant frame \( \{ \tilde{U}_i, \partial_{\tilde{U}_i}, \forall i \} \) on \( TG \) (see Sect. 2.2 below or Section 3.1 in [18]), a left invariant spray structure \( G \) can be presented as \( G = G_0 - H \), where \( G_0 = u^i \tilde{U}_i \) is the canonical bi-invariant spray structure, and \( H = H^i \partial_{\tilde{U}_i} \) is a left invariant vector field on \( TG \backslash 0 \) which is tangent to each \( T_g G \). The restriction \( \eta = H|_{T_e G \backslash \{0\}} \) is called the spray vector field associated to \( G \) [18]. This notion was first proposed by Huang in homogeneous Finsler geometry [2]. We usually present \( \eta \) as smooth function from \( g \backslash \{0\} \) to \( g \) (see Sect. 2.3 below).

The philosophy of homogeneous geometry implies that, to explore a left invariant \( G \), we only need to observe the interaction between the dynamical system of \( \eta \) and the Lie algebra structure of \( g \). Following this thought, we find homogeneous curvature formulae for \( G \) (see (3.13) below and Corollary 4.1 in [18]) which generalizes those of Huang [2, 3] in homogeneous Finsler geometry, and prove a correspondence between geodesics of \( G \) and integral curves of \(-\eta\) (see Theorem 3.2 below or Theorem D in [18]).

As a continuation of this exploration, we switch in this paper to linearly and nonlinearly parallel translations for \( G \). It turns out that the connection operator \( N : (g\backslash\{0\}) \times g \to g \) plays an important role. This notion was first defined for a homogeneous Finsler space by Huang, using fundamental tensor and Cartan tensor [2, 3]. He later pointed out another description (see (4) in [4]), which implies that it can be generalized to homogeneous spray geometry. In particular, for a left invariant spray structure \( G \) with the spray vector field \( \eta \), the connection operator is

\[
N(y, w) = \frac{1}{2} D\eta(y, w) - \frac{1}{2} [y, w]_g,
\]

in which \( D\eta(y, w) \) is the derivative of \( \eta : g\backslash\{0\} \to g \) at \( y \) in the direction of \( w \). Here \([\cdot, \cdot]_g\) is the Lie bracket of \( g = \text{Lie}(G) \). The more usual notation \([\cdot, \cdot] \) is reserved for the canonical bracket between two smooth vector fields (see Theorem 1.2 and Remark 1.3 below).

Firstly, we consider linearly parallel translations along a smooth curve on \((G, G)\) and prove

**Theorem 1.1** Let \( G \) be a connected Lie group endowed with a left invariant spray structure \( G \) with the spray vector field \( \eta \), \( c(t) \) a smooth curve on \((G, G)\) with nowhere-vanishing \( \dot{c}(t) \), and \( W(t) \) a vector field along \( c(t) \). Then \( W(t) \) is linearly parallel along \( c(t) \) if and only if \( w(t) = (L_{c(t)^{-1}})_* (W(t)) \in g \) is a solution of

\[
\frac{d}{dt} w(t) + N(y(t), w(t)) + [y(t), w(t)]_g = 0,
\]

in which \( y(t) = (L_{c(t)^{-1}})_* (\dot{c}(t)) \).

\[ Springer \]
As an application of Theorem 1.1, the connection operator and Riemann curvature operators for a left invariant $G$ can be alternatively described by

**Theorem 1.2** Let $c(t)$ be a geodesic on the connected Lie group $G$, for the left invariant spray structure $G$, and $y(t) = (L_{c(t)}^{-1})_* (\dot{c}(t))$ the corresponding integral curve for $-\eta$, where $\eta$ is the spray vector field associated with $G$. Denote by $w(t)$ the vector field along $y(t)$ which corresponds to a linearly parallel vector field $W(t)$ along $c(t)$, i.e., $w(t) = (L_{c(t)}^{-1})_* (W(t))$. Then $N(t) = N(y(t), w(t))$ and $R(t) = R_{y(t)}(w(t))$, where $N(\cdot, \cdot)$ is the connection operator and $R_{\cdot}$ is the Riemann curvature, are vector fields along $y(t)$ determined by

$$N(t) = -[\eta, w(t)] \quad \text{and} \quad R(t) = [\eta, N(t)] = -[\eta, [\eta, w(t)]].$$ (1.3)

**Remark 1.3** The vector fields $w(t)$ and $N(t)$ are smooth vector field along an integral of $-\eta$. So $[\eta, w(t)] = -[-\eta, w(t)]$ and $[\eta, N(t)] = -[-\eta, N(t)]$ in (1.3) are in deed Lie derivatives. Notice that they can be calculated as the canonical bracket between smooth vector fields. To be precise, let $X$ be a smooth vector field on $M$ and $Y(t)$ a smooth vector field along an integral curve $c(t)$ for $X$, then $[X, Y(t)]$ is a well defined smooth vector field along $c(t)$. When $c(t)$ is not constant, we may locally extend $Y(t)$ to a smooth vector field $Z$ on $M$, then $[X, Y(t)] = [X, Z]|_{c(t)}$ is independent of the extension. Using local coordinate, the bracket between $X = X^i \partial_x^i$ and $Y(t) = Y^i(t) \partial_x^i |_{c(t)}$ can be presented as

$$[X, Y(t)] = \left(\frac{dY^i(t)}{dt} \partial_x^i - Y^j(t) \frac{\partial}{\partial x^j} X^j\partial_x^i\right)|_{c(t)}.$$ (1.4)

Notice that when $c(t)$ is constant, (1.4) can still be used to calculate $[X, Y(t)]$, which is independent of the choice of local coordinate.

Theorem 1.1 (together with Theorem D in [18]) can provide shortcuts to other curvature formulae of $G$ as well. See Sect. 3.3 for some examples.

Nextly, we consider nonlinearly parallel translations along a smooth curve on $(G, G)$ and prove

**Theorem 1.4** Let $G$ be a connected Lie group endowed with a left invariant spray structure $G$, $c(t)$ a smooth curve on $G$ and $Y(t)$ a nowhere vanishing vector field along $c(t)$. Then $Y(t)$ is nonlinearly parallel along $c(t)$ iff $y(t) = (L_{c(t)}^{-1})_* (Y(t))$ is a solution of

$$\frac{d}{dt} y(t) + N(y(t), w(t)) = 0,$$ (1.5)

in which $w(t) = (L_{c(t)}^{-1})_* (\dot{c}(t))$.

When $w(t) \equiv w$ is constant, i.e., $c(t) = \exp t w$ is a one-parameter subgroup of $G$, Eq. (1.5) generates a one-parameter subgroup of diffeomorphisms on $g\setminus\{0\}$. So Theorem 1.4 has the following immediate consequence.

**Theorem 1.5** Let $G$ be a connected Lie group endowed with a left invariant spray structure $G$, and $c(t) = \exp tw$ for any $w \in g$ a one-parameter subgroup of $G$. 

 Springer
Denote by $P_{c(0),c(t):c}$ the nonlinear parallel translation along $c(\cdot)$ from $c(0)$ to $c(t)$. Then $\rho_t = (L_{c(t)^{-1}})_* \circ P_{c(0),c(t):c}$ is the one-parameter subgroup of diffeomorphisms generated by the smooth vector field $-N(\cdot, w)$ on $\mathfrak{g}\{0\}$.

For each $u \in \mathfrak{g}$, the connection operator $N$ and the Riemannian curvature $R$ for $(G, \mathcal{G})$ determines smooth vector fields $N^u(y) = N(y, u)$ and $R^u(y) = R_y(u)$ on $\mathfrak{g}\{0\}$.

**Theorem 1.6** Let $G$ be a left invariant spray structure on a Lie group $G$ with the connection operator $N$. Then for any $u, v \in \mathfrak{g}$, we have

$$\left[N^u, N^v\right] + N^{[u,v]} = \frac{1}{3} \left(D_u R^v - D_v R^u\right),$$

where the bracket between $N^u$ and $N^v$ is the canonical bracket between smooth vector fields on $\mathfrak{g}\{0\}$.

Theorem 1.6 indicates that the bracket $[N^u, N^v]$ contains curvature information of $(G, \mathcal{G})$. It can be viewed as an analog of the well known formula (see Remark 8.1.3 in [15])

$$[\delta x^i, \delta x^k] = -R^j_{kl} \partial_{x^k} = -\frac{1}{3} \left( \frac{\partial}{\partial y^i} R^i_{kl} - \frac{\partial}{\partial y^k} R^i_{jl}\right),$$

which alternatively determines the curvature. It suggests us to study the Lie algebra $\mathfrak{h}(G, \mathcal{G})$ that $N^v$ for all $v \in \mathfrak{g}$ generate, using the canonical bracket between two smooth vector fields on $\mathfrak{g}\{0\}$. By Theorem 1.6, the mapping $u \mapsto -N^u$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}(G, \mathcal{G})$ when $R \equiv 0$. So we have the following immediate corollary.

**Theorem 1.7** If the left invariant spray structure $\mathcal{G}$ on a Lie group $G$ has vanishing Riemann curvature everywhere, then $\mathfrak{h}(G, \mathcal{G})$ is a quotient Lie group of $\mathfrak{g}$, so it has a finite dimension.

The Lie algebra $\mathfrak{h}(G, \mathcal{G})$ contains all the information of the nonlinearly parallel translations on $(G, \mathcal{G})$. We propose two questions (see Questions 4.4, 4.5 in Sect. 4.3). The first question concerns if we always have $\dim \mathfrak{h}(G, \mathcal{G}) = +\infty$ when $\mathcal{G}$ is not affine. This question can be viewed as a generalization for the Landsberg Problem in Finsler geometry [12]. The second question concerns the relation between $\mathfrak{h}(G, \mathcal{G})$ and the (restricted) holonomy group of $(G, \mathcal{G})$. Notice that when $\mathcal{G}$ is induced by a left invariant Riemannian metric, we can often get $\mathfrak{h}(G, \mathcal{G}) = \text{Lie}(\text{Hol}_0(G, \mathcal{G}))$ [8, 9]. However, in Finsler or spray geometry, both $\mathfrak{h}(G, \mathcal{G})$ and $\text{Hol}_0(G, \mathcal{G})$ might have infinite dimensions [6], and then the second question becomes much harder.

Finally, we remark that this paper is the second one in the three sequel papers on homogeneous spray geometry. The main technique, i.e., the global invariant frames on a Lie group and its tangent bundle, is inherited from [18]. In this paper, we see that parallel translations are crucial for studying the geometry of a left invariant spray structure, where the connection operator plays important roles. This crucial observation helps us study invariant spray structures on more general smooth coset spaces [19].
This paper is organized as following. In Sect. 2, we summarize some necessary notions and techniques. In Sect. 3, we prove Theorems 1.1 and 1.2. In Sect. 4, we prove Theorem 1.4 and propose two questions.

2 Preliminaries

2.1 Spray Structure and Parallel Translation

In this subsection, we summarize some fundamental knowledge in [15] on spray geometry.

A spray structure on a smooth manifold $M$ is a smooth tangent vector field $G$ on the slit tangent bundle $TM \setminus 0$, which can be locally presented as

$$ G = y^i \partial_{y^i} - 2G^i_j \partial_{y^j} $$

for any standard local coordinate $(x^i, y^i)$, i.e., $x = (x^i) \in M$ and $y = y^i \partial_{x^i} \in T_x M$, such that each $G^i = G^i(x, y)$ is positive 2-homogeneous for the $y$-entry. A smooth curve $c(t)$ in $M$ with nonvanishing $\dot{c}(t)$ everywhere is called a geodesic for $G$, if its lifting $(c(t), \dot{c}(t))$ in $TM \setminus 0$ is an integral curve of $G$.

Following (2.6), we have the notations

$$ N^j_i = \frac{\partial G^i_j}{\partial y^j} \quad \text{and} \quad \delta x^i = \partial_{x^i} - N^i_j \partial_{y^j}. $$

Let $W(t) = z^i(t) \partial_{x^i}|_{c(t)}$ be a smooth vector field along the smooth curve $c(t)$ and we assume $\dot{c}(t)$ is nonvanishing everywhere. The linearly covariant derivative $D_{\dot{c}(t)} W(t)$ is the following smooth vector field along $c(t)$,

$$ D_{\dot{c}(t)} W(t) = \left( \frac{dz^i(t)}{dt} + z^j(t) N^i_j(c(t), \dot{c}(t)) \right) \partial_{x^i}|_{c(t)}.
$$

We say $W(t)$ is linearly parallel along $c(t)$ if $D_{\dot{c}(t)} W(t) \equiv 0$. For any initial value $w \in T_{c(t_0)} M$, there exists a unique linearly parallel $W(t)$ globally along $c(t)$. This fact follows from the existence and uniqueness theorem for ordinary differential equations with initial values, and the smoothness and positive 1-homogeneity of $N^j_i$.

Suppose $c(t)$ with $t \in [a, b]$, $c(a) = p$ and $c(b) = q$ is a smooth curve on $M$ with nonvanishing $\dot{c}(t)$ everywhere. Then the linear parallel translation $P^l_{p, q; c} : T_p M \to T_q M$ is defined by $P^l_{p, q; c}(w) = W(b)$, where $W(t)$ is the linearly parallel vector field along $c(t)$ satisfying $W(a) = w$.

Using the nonlinearly covariant derivative

$$ \tilde{D}_{\dot{c}(t)} W(t) = \left( \frac{dz^i(t)}{dt} + \dot{c}^j(t) N^i_j(c(t), W(t)) \right) \partial_{x^i}|_{c(t)}$$

for any nowhere vanishing $W(t) = z^i(t) \partial_{x^i}|_{c(t)}$ along $c(t)$, the nonlinearly parallel translation $P^nl_{p, q; c} : T_p M \setminus \{0\} \to T_q M \setminus \{0\}$ from $p = c(a)$ to $q = c(b)$ along a
smooth curve \( c(t) \) can be similarly defined. It can be alternatively described by the integral curves for the horizontal lifting of \( \dot{c}(t) \), i.e.,
\[
\tilde{c}(t) = c^i(t)\delta_x^i|_{T_{c(t)}M\setminus\{0\}} = \dot{c}^i(t)(\partial_x^i - N_j^i\partial_y^j)|_{T_{c(t)}M\setminus\{0\}}.
\]

To be precise, \( \tilde{c}(t) \) is viewed as a smooth tangent vector field on the submanifold \( S = \cup_s(T_{c(s)}M\setminus\{0\}) \). For any nowhere vanishing smooth vector field \( Y(t) \) along \( c(t) \), it is nonlinearily parallel, i.e., \( Y(t) = P_{\dot{c}(t),c(t);c}(Y(t_0)) \) for all values of \( t \), iff \( (c(t), Y(t)) \) is an integral curve of \( \tilde{c}(t) \) in \( S \).

### 2.2 Invariant Frames on a Lie Group

In this section, we summarize some notations in [18] for invariant frames.

Let \( G \) be a connected Lie group, \( L_g(g') = gg' \) and \( R_g(g') = g'g \) the left and right translations respectively. Denote by \( \mathfrak{g} = \text{Lie}(G) = T_eG \) the Lie algebra of \( G \) and by \([\cdot, \cdot]_G \) the Lie bracket on \( \mathfrak{g} \). We fix a basis \( \{e_1, \ldots, e_n\} \) of \( \mathfrak{g} \) with \([e_i, e_j]_G = c^k_{ij} e_k \).

For each \( 1 \leq i \leq n \), we have a left invariant tangent vector field \( U_i(g) = (L_g)_* (e_i) \), so \( \{U_i, \forall i\} \) is a left invariant frame on \( G \) satisfying \( [U_i, U_j] = c^k_{ij} U_k \). The complete lifting of \( U_i \) is denoted as \( \tilde{U}_i \), which is a smooth tangent vector field on \( TG \) (see Section 2.1 in [18]). Any tangent vector \( y \in T_gG \) can be uniquely presented as \( y = u^i U_i(g) \), which determines the smooth functions \( u^i \) on \( TG \). Denote by \( \partial_{u^i} \) the smooth vector fields on \( TG \) corresponding to the \( u^i \)-coordinates in each \( T_gG \), which are tangent to each \( T_gG \). It is easy to observe the left invariance of \( U_i \) and \( u^i \), so we call \( \{\tilde{U}_i, \partial_{u^i}, \forall i\} \) a left invariant frame on \( TG \).

The transformation between \( \{\tilde{U}_i, \partial_{u^i}, \forall i\} \) and \( \{\partial_{x^i}, \partial_{y^i}, \forall i\} \) for a standard local coordinate \( (x^i, y^i) \) is the following (see (2.2) and Lemma 2.1 in [18]),
\[
U_i = A^j_i \partial_{x^j}, \quad u^i = y^j B^i_j \quad \text{and} \quad \partial_{u^i} = A^j_i \partial_{y^j}.
\]
(2.8)
\[
\tilde{U}_i = A^j_i \partial_{x^j} + y^j \frac{\partial}{\partial x^j} A^k_j \partial_{x^k},
\]
(2.9)
where \( (A^j_i) = (A^j_i(x)) \) and \( (B^j_i) = (B^j_i(x)) = (A^j_i(x))^{-1} \) (i.e., \( A^j_i B^k_j = B^j_i A^k_j = \delta^k_j \)) are matrix valued functions which only depend on the \( x \)-entry. Notice that \( \partial_{x^i} \) in (2.8) and (2.9) are local tangent vector fields on \( G \), and their complete liftings to \( TG \) respectively. Comparing the coefficients of \( \partial_{x^k} \) in both sides of
\[
c^p_{qi} A^k_p \partial_{x^k} = c^p_{qi} U_p = [U_q, U_i] = \left( A^j_q \frac{\partial}{\partial x^j} A^k_j - A^j_i \frac{\partial}{\partial x^j} A^k_q \right) \partial_{x^k},
\]
we get (3.14) in [18])

**Lemma 2.1**
\[
A^j_q \frac{\partial}{\partial x^j} A^k_j - A^j_i \frac{\partial}{\partial x^j} A^k_q = c^p_{qi} A^k_p.
\]

Using the right invariant tangent vector fields \( V_i(g) = (R_g)_* (e_i) \) on \( G \) and the presentation \( y = v^i V_i(g) \in T_gG \), we can similarly get a right invariant frame \( \{\tilde{V}_i, \partial_{u^i}, \forall i\} \) on \( TG \).
To describe the transformation between \( \{ \tilde{U}_i, \partial_{u^i}, \forall i \} \) and \( \{ \tilde{V}_i, \partial_{v^i}, \forall i \} \), we denote by \( \phi_i^j \) and \( \psi_i^j \) the functions on \( G \) such that \( \text{Ad}(g)e_i = \phi_i^je_j \) and \( \text{Ad}(g^{-1})e_i = \psi_i^je_j \) (so we have \( \psi_i^j \phi_j^k = \phi_i^j \psi_k^j = \delta_i^k \)). Then at each \( g \in G \),

\[
U_i(g) = (L_g)_*(e_i) = (R_g)_*(R_{g^{-1}})_*(L_g)_*(e_i) = (R_g)_*(\text{Ad}(g)e_i) = (R_g)_*(\phi_i^j(g)e_j)
\]

so we have

\[
U_i = \phi_i^j V_j, \quad u^i = \psi_i^j v^j, \quad \partial_{u^i} = \phi_i^j \partial_{v^j}.
\]  

(2.10)

In [18], we have proved

Lemma 2.2 (1) \( \phi_i^j V_j \phi_i^k = c_{i_l}^l \phi_l^l \), (2) \( \tilde{U}_i = \phi_i^j \tilde{V}_j + c_{q_i}^l u^p \partial_{u^q} \).

We briefly recall its proof here. To prove (1), we observe

\[
V_j \phi_i^k e_k = \frac{d}{dt} (\text{Ad}(\exp te_j \cdot g)e_i) = \frac{d}{dt} \text{Ad}(\exp te_j)(\text{Ad}(g)e_i) = [e_j, \text{Ad}(g)e_i]_g,
\]

and

\[
\phi_i^j V_j \phi_i^k e_k = [\text{Ad}(g)e_i, \text{Ad}(g)e_i]_g = \text{Ad}(g)[e_i, e_i]_g = c_{i_l}^l \text{Ad}(g)e_j = c_{i_l}^l \phi_l^j e_k.
\]  

(2.11)

Then (1) follows after a comparison for the coefficients of \( e_k \) in (2.11). To prove (2), we apply Lemma 2.2 in [18] to the first equality in (2.10) and get

\[
\tilde{U}_i = \phi_i^j \tilde{V}_j + v^j V_j \phi_i^k \partial_{v^k} = \phi_i^j \tilde{V}_j + u^p \phi_p^j V_j \phi_i^k \partial_{u^k}.
\]

Then (2) follows after (1) and the third equality in (2.10) immediately.

### 2.3 Left Invariant Spray Structure on a Lie Group

In this subsection, we introduce the notions of invariant spray structure in [18].

Let \( G \) be a connected Lie group and \( \mathbf{G} \) a spray structure on \( G \). We call \( \mathbf{G} \) left invariant (or right invariant) if all left (or right respectively) translations preserve \( \mathbf{G} \). We call \( \mathbf{G} \) bi-invariant if it is both left and right invariant. Using the invariant frames \( \{ \tilde{U}_i, \partial_{u^i}, \forall i \} \) and \( \{ \tilde{V}_i, \partial_{v^i}, \forall i \} \) on \( TG \), the left and right invariances of \( \mathbf{G} \) can be equivalently described as

\[
[\tilde{V}_i, \mathbf{G}] = 0, \forall i \quad \text{and} \quad [\tilde{U}_i, \mathbf{G}] = 0, \forall i \quad \text{respectively}.
\]

The canonical bi-invariant spray structure \( \mathbf{G}_0 = u^i \tilde{U}_i = v^i \tilde{V}_i \) (see Theorem A in [18]) serves as the origin in the space of left invariant spray structures on \( G \). Any left
invariant spray structure $G$ on $G$ can be presented as $G = G_0 - H = G_0 - H^i \partial_{\mu^i}$. Here $H = H^i \partial_{\mu^i}$ is a left invariant smooth vector field on $TG \setminus 0$, and $H^i$ are left invariant smooth functions on $TG \setminus 0$ which are positive 2-homogeneous in each $T_\mu G$. We denote by $\eta$ the restriction of $H$ to $T_e G \setminus 0$ and call it the spray vector field associated with $G$. Usually $\eta$ is presented as a smooth map from $g \setminus 0$ to $g$, i.e., $\eta(y) = H^i(y, y) e_i$. The connection operator $N(\cdot, \cdot) : (g \setminus 0) \times g \to g$ is then defined by $N(y, w) = \frac{1}{2} D_\eta(y, w) - \frac{1}{2}[y, w]_g$, in which $D_\eta(y, w)$ is the derivative of the spray vector field $\eta : g \setminus 0 \to g$ at $y$ in the direction of $w$.

Using any standard local coordinate $(x^i, y^j)$, we can translate the left invariant spray structure $G = G_0 - H^i \partial_{\mu^i}$ back to its usual representation $G = y^i \partial_{x^i} - 2 G^j \partial_{y^j}$. The equalities in (2.8) and (2.9) enable us to present $N^k_j = \frac{\partial G^k_j}{\partial y^j}$ as follows (see (3.10) in [18]).

**Lemma 3.1**

\[ N^k_j = \frac{\partial G^k_j}{\partial y^j} = \frac{1}{2} A^k_i \frac{\partial}{\partial y^j} H^i - \frac{1}{2} u^i \frac{\partial}{\partial x^j} A^k_i - \frac{1}{2} y^j B^i_k \frac{\partial}{\partial x^j} A^i_k. \]

**3 Linear Parallel Translation Along a Geodesic**

**3.1 Proof of Theorem 1.1**

Suppose the connected Lie group $G$ is endowed with a left invariant spray structure $G = G_0 - H = u^i \tilde{U}_i - H^i \partial_{\mu^i}$ with the spray vector field $\eta : g \setminus 0 \to g$. Now we consider the linear parallel translation along a smooth curve $c(t)$ on $(G, G)$ with nowhere-vanishing $\dot{c}(t)$. We denote by $y(t) = (L_{c(t)^{-1}})_*(\dot{c}(t))$ a smooth curve on $g \setminus 0$. Using left translations, smooth vector fields along $c(t)$ and those along $y(t)$ can be one-to-one corresponded. Consider any smooth vector field $W(t) = w^j(t) U_j(c(t))$ along $c(t)$, then the corresponding $w(t) = (L_{c(t)^{-1}})_*(W(t))$ along $y(t)$ can be presented as $w(t) = w^j(t)e_i$.

**Lemma 3.1**

\[ D_{c(t)} W(t) = \left( \frac{dw^i(t)}{dt} + \frac{1}{2} w^j(t) \frac{\partial}{\partial u^j} H^i + \frac{1}{2} w^j(t) u^i(t) (\xi^j_k) \right) U_j(c(t)). \]

**Proof** In any standard local coordinate $(x^i, y^j)$, we have the presentations $c(t) = (c^i(t)), y^j(t) = y^j(c(t), \dot{c}(t)) = \dot{c}(t)$ and $W(t) = u^j(t) U_j(c(t)) = z^j(t) \partial_{x^j}|c(t)|$.

Using the notations in (2.8), i.e., $U_i = A_i^j \partial_{x^j}, u^i = y^j B^i_j$ and $(B^i_j(x)) = (A^j_i(x))^{-1}$, we also have $u^i(t) = y^j(t) B^i_j(c(t))$ and $z^j(t) = w^j(t) A^j_i(c(t))$. So at each point $x = c(t)$, the covariant derivative $D_{c(t)} W(t)$ (see (2.7)) can be calculated as following,

\[ D_{c(t)} W(t) = \left( \frac{dw^i(t)}{dt} + z^j(t) N^j_i(c(t), \dot{c}(t)) \right) \partial_{x^i} \]

\[ = \frac{\partial}{\partial t}(w^j(t) A^i_j) B^i_k U_k + z^j(t) \left( \frac{1}{2} A^j_i \frac{\partial}{\partial y^i} H^i - \frac{1}{2} u^i(t) \frac{\partial}{\partial x^j} A^i_j - \frac{1}{2} y^j(t) B^i_j \frac{\partial}{\partial x^j} A^i_k \right) \partial_{x^i} \]

\[ = \left( \frac{dw^i(t)}{dt} U_k + z^j(t) y^p(t) (B^i_j \frac{\partial}{\partial x^j} A^i_k) \partial_{x^i} \right) \]

\[ + \frac{1}{2} w^j(t) \frac{\partial}{\partial x^i} H^i U_i - \left( \frac{1}{2} z^j(t) y^p(t) B^i_j \frac{\partial}{\partial x^j} A^i_k + \frac{1}{2} z^j(t) y^p(t) B^i_j \frac{\partial}{\partial x^j} A^i_k \right) \partial_{x^i} \]

\[ = \left( \frac{dw^i(t)}{dt} + \frac{1}{2} w^j(t) \frac{\partial}{\partial x^i} H^i \right) U_k + \frac{1}{2} \left( z^j(t) y^p(t) B^i_j \frac{\partial}{\partial x^j} A^i_k - z^j(t) y^p(t) B^i_j \frac{\partial}{\partial x^j} A^i_k \right) \partial_{x^i}. \]
in which the second line uses Lemma 2.3 and the last line uses Lemma 2.1. □

Now we are ready to prove Theorem 1.1, which interprets Lemma 3.1 by left translations.

**Proof of Theorem 1.1** Lemma 3.1 indicates

\[ (L_{c(t)^{-1}})_*(D_{\hat{c}(t)}\omega(t)) = \left( \frac{d}{dt}w^j(t) + \frac{1}{2} w^j(t) \frac{\partial}{\partial u^j} H^k + \frac{1}{2} w^j(t) u^k(t) c^l_{kj} \right) e_l \]

\[ = \left( \frac{d}{dt}w^j(t) + \left( \frac{1}{2} w^j(t) \frac{\partial}{\partial u^j} H^l - \frac{1}{2} w^j(t) u^k(t) \right) c^l_{kj} \right) e_l + w^j(t) u^k(t) c^l_{kj} e_l \]

\[ = \frac{d}{dt}w(t) + N(y(t), w(t)) + [y(t), w(t)]_\mathfrak{g}, \]

where \( y(t) = u^l(t) e_l \) and \( w(t) = w^j(t) e_j \). So \( D_{\hat{c}(t)}\omega(t) \equiv 0 \) if and only if \( (L_{c(t)^{-1}})_*(D_{\hat{c}(t)}\omega(t)) \equiv 0 \), i.e., \( w(t) \) is a solution of \( \frac{d}{dt}w(t) + N(y(t), w(t)) + [y(t), w(t)]_\mathfrak{g} = 0 \). □

### 3.2 Proof of Theorem 1.2

The Riemann curvature formula for a left invariant spray structure \( \mathfrak{g} \) is given by Theorem C in [18]. In particular, when restricted to \( T_0 \mathfrak{g} \), the Riemann curvature operator \( R(\cdot) : (\mathfrak{g}\{0\}) \times \mathfrak{g} \to \mathfrak{g} \) satisfies (see Corollary 4.1 in [18]),

\[ R_y(w) = DN(\eta, y, w) - N(y, N(y, w)) + N(y, [y, w])_\mathfrak{g} - [y, N(y, w)]_\mathfrak{g}, \]

(3.13)

where \( DN(\eta, y, w) = \frac{d}{dt}|_{t=0}N(y + t\eta, w) \), i.e., it is the derivative of \( N(\cdot, w) \) at \( y \) in the direction of \( \eta(y) \). See Proposition 3.2 in [3] for (3.13) in homogeneous Finsler geometry, and Lemma 5.1 in [5] for another homogeneous Riemannian curvature formula when \( \eta(y) = 2P(y)y \) for some smooth positive 1-homogeneous function \( P(\cdot) \) on \( \mathfrak{g}\{0\} \).

Theorem D in [18] provides the following observation.

**Observation 1:** let \( \mathfrak{g} \) be a left invariant spray structure on the Lie group \( G \) with the associated spray vector field \( \eta \), then for any open interval \((a, b) \subset \mathbb{R} \) containing \( 0 \), there is a one-to-one correspondence between the following two sets:

1. The set of all curves \( c(t) \) on \( G \), with \( t \in (a, b) \) and \( c(0) = e \), which are geodesics for \( \mathfrak{g} \).
2. The set of all \( y(t) \) on \( \mathfrak{g}\{0\} \), with \( t \in (a, b) \), which are integral curves of \( -\eta \), and the correspondence is given by \( y(t) = (L_{c(t)^{-1}})_*(\hat{c}(t)) \).
Now suppose that $c(t)$ is a geodesic on $(G, G)$, where spray structure $G$ is left invariant. By Observation I, $y(t) = (L_{c(t)^{-1}})_\ast(\dot{c}(t))$ is an integral curve of $-\eta$ on $g\setminus\{0\}$. Let $W(t)$ be a linearly parallel vector field along $c(t)$ and denote by $w(t) = (L_{c(t)^{-1}})_\ast(\dot{c}(t))$ a smooth vector field along $y(t)$. By Theorem 1.1, we have $\frac{d}{dt} w(t) + N(y(t), w(t)) + [y(t), w(t)]_\mathfrak{g} = 0$. Now we prove Theorem 1.2, i.e., an alternative interpretation of (3.13).

**Proof of Theorem 1.2** By (1.4) in Remark 1.3, we have

$$[-\eta, w(t)] = \frac{d}{dt} w(t) + D\eta(y(t), w(t)) = \frac{d}{dt} w(t) + 2N(y(t), w(t)) + [y(t), w(t)]_\mathfrak{g},$$

in which $D\eta(y(t), w(t)) = \frac{d}{ds}|_{s=0}\eta(y(t)+sw(t)) = 2N(y(t), w(t)) + [y(t), w(t)]_\mathfrak{g}$. By Theorem 1.1, $\frac{d}{dt} w(t) + N(y(t), w(t)) + [y(t), w(t)] = 0$, so $N(t) = N(y(t), w(t)) = [-\eta, w(t)] = [-\eta, w(t)]$.

Using Theorem 1.1, Observation I and the linearity of $N(y, w)$ for the $w$-entry, we have

$$\frac{d}{dt} N(y(t), w(t)) = DN(-\eta, y(t), w(t)) + N(y(t), \frac{d}{dt} w(t))$$

$$= -DN(\eta, y(t), w(t)) + N(y(t), -N(y(t), w(t)))$$

$$= -DN(\eta, y(t), w(t)) - N(y(t), N(y(t), w(t)))$$

$$= -N(y(t), [y(t), w(t)]_\mathfrak{g}).$$

(3.14)

Here $DN(\eta, y(t), w(t))$ is the derivative of $N(\cdot, w(t))$ at $y(t)$ in the direction of $\eta(y(t))$, for each fixed value of $t$. So (1.4) and (3.14) imply

$$[\eta, N(t)] = -[-\eta, N(t)] = -\frac{d}{dt} N(y(t), w(t)) - D\eta(y(t), N(y(t), w(t)))$$

$$= DN(\eta, y(t), w(t)) + N(y(t), N(y(t), w(t))) + N(y(t), [y(t), w(t)]_\mathfrak{g})$$

$$= -(2N(y(t), N(y(t), w(t))) + [y(t), N(y(t), w(t))]_\mathfrak{g})$$

$$= DN(\eta, y(t), w(t)) - N(y(t), N(y(t), w(t))) + N(y(t), [y(t), w(t)]_\mathfrak{g})$$

$$= [y(t), N(y(t), w(t))]_\mathfrak{g}$$

$$= R_{y(t)}(w(t)) = R(t).$$

This ends the proof of Theorem 1.2. □

**3.3 Landsberg Curvature and S-Curvature for a Left Invariant Finsler Metric**

Theorem 1.1 and Observation I can be used to prove other curvature formulas for $(G, G)$ Here we take the Landsberg and S-curvature for a left invariant Finsler metric for example.

The Landsberg curvature $L$ for a Finsler metric $F$ can be calculated by

$$L_{\dot{c}(t)}(W(t), W(t), W(t)) = \frac{d}{dt} C_{\dot{c}(t)}(W(t), W(t), W(t)),$$

(3.15)
in which \( c(t) \) is a geodesic, \( W(t) \) is linearly parallel along \( c(t) \), and \( C(\cdot, \cdot, \cdot) \) is the Cartan tensor (see (7.16) in [16]). When \( F \) is left invariant, Theorem 1.1 and Observation I translate (3.15) to

\[
L_y(t)(w(t), w(t), w(t)) = \frac{d}{dt} C_y(t)(w(t), w(t), w(t))
\]

\[
= \left( \frac{d}{dt} C_y(t) \right)(w(t), w(t), w(t), \eta(y(t)))
\]

\[
+ 3C_y(t) \left( \frac{d}{dt} w(t), w(t), w(t) \right)
\]

\[
= -C_y(t)(w(t), w(t), w(t), \eta(y(t)))
\]

\[
- 3C_y(t)(N(y(t), w(t)) + [y(t), w(t)], w(t), w(t)),
\]

where \( y(t) = (L_{c(t)^{-1}})_*(\dot{c}(t)) \) is an integral curve of \(-\eta \) and \( w(t) = (L_{c(t)^{-1}})_*(W(t)) \) is a solution of (1.2). It verifies the Landsberg curvature formula in Proposition 6.1 in [2] for a left invariant Finsler metric, i.e.,

\[
L_y(w, w, w) = 3C_y(w, w, [w, y] - N(y, w)) - C_y(w, w, \eta(y)).
\]

The S-curvature for a Finsler metric \( F \) and a smooth measure \( d\mu \) can be calculated by

\[
S(c(t), \dot{c}(t)) = \frac{d}{dt} \ln \det \left( \langle W_i(t), W_j(t) \rangle_{\dot{c}(t)} \right) - \frac{d}{dt} \ln |\omega(W_1(t), \ldots, W_n(t))|,
\]

(3.16)
in which \( c(t) \) is a geodesic, \( \{W_1(t), \ldots, W_n(t)\} \) is any frame along \( c(t) \), and \( \langle \cdot, \cdot \rangle \) is the fundamental tensor of \( F \) [21]. When \( F \) is left invariant, \( d\mu = \omega \) is a left invariant volume form, and \( W_i(t) \) are taken to be linearly parallel along \( c(t) \) and orthonormal with respect to \( \langle \cdot, \cdot \rangle_{\dot{c}(t)} \), the first summand in the right of (3.16) vanishes, and Theorem 1.1 implies

\[
S(e, y(t)) = S(c(t), \dot{c}(t)) = -\frac{d}{dt} \ln |\omega(W_1(t), \ldots, W_n(t))|
\]

\[
= -\frac{d}{dt} \ln |\omega(w_1(t), \ldots, w_n(t))|
\]

\[
= -\frac{\omega(w_1(t), w_2(t), \ldots, w_n(t)) + \cdots + \omega(w_1(t), \ldots, w_{n-1}(t), \frac{d}{dt} w_n(t))}{\omega(w_1(t), \ldots, w_n(t))}
\]

\[
= -\sum_{i=1}^n \langle \frac{d}{dt} w_i(t), w_i(t) \rangle_{y(t)}
\]

\[
= \sum_{i=1}^n \langle N(y(t), w_i(t)) + [y(t), w_i(t)], w_i(t) \rangle_{y(t)}
\]

\[
= \text{Tr}_R N(y(t), \cdot) + \text{Tr}_R \text{ad}(y(t)).
\]

in which \( y(t) = (L_{c(t)^{-1}})_*(\dot{c}(t)) \) and \( w_i(t) = (L_{c(t)^{-1}})_*(W_i(t)) \) for each \( i \). It verifies the S-curvature formula in Proposition 6.1 of [2], i.e.,

\[
S(y) = \text{Tr}_R N(y, \cdot) + \text{Tr}_R \text{ad}(y).
\]
4 Nonlinear Parallel Translation Along a Smooth Curve

4.1 Proof of Theorem 1.4

Let $G$ be a connected Lie group endowed with a left invariant spray structure $G = G_0 - H = u^i \tilde{U}_i - \mathbf{H}^j \partial_{u^j}$ with the spray vector field $\eta$, and $c(t)$ with $t \in (a, b)$ a smooth curve on $G$ which is simple (i.e., it has no self intersection) and has nonvanishing $\dot{c}(t)$ for all values of $t$.

We denote by $\dot{c}(t) = w^i(t)U_i(c(t))$ the tangent vector field of $c(t)$, then its horizontal lifting $\overleftarrow{\mathcal{H}} \dot{c}(t) = w^i(t)\tilde{U}_i|_{T_c(t)G \setminus \{0\}}$ is a smooth tangent vector field on the imbedded submanifold $S = \cup_{t \in (a, b)} (T_c(t)G \setminus \{0\})$ of $TG \setminus \{0\}$. Lemma 3.2 in [18] provides a formula for $\tilde{U}_i|_{T_c(t)G \setminus \{0\}}$, i.e.,

**Lemma 4.1** The horizontal lifting of $U_q$ is $\tilde{U}_q^\mathcal{H} = \tilde{U}_q = \left(\frac{1}{2} \frac{\partial}{\partial u^i} H^i - \frac{1}{2} u^j c_i^j \right) \partial_{u^i}$.

Its proof uses (2.9), and a very similar calculation as (3.12).

Using Lemma 4.1 and (2) of lemma 2.2, we get the decomposition

$$\overleftarrow{\mathcal{H}} \dot{c}(t) = w^i(t)\tilde{U}_i|_{T_c(t)G \setminus \{0\}} = w^i(t)(\tilde{U}_i - \left(\frac{1}{2} \frac{\partial}{\partial u^i} H^j - \frac{1}{2} u^j c_i^j \right) \partial_{u^i})|_{T_c(t)G \setminus \{0\}}$$

at each $(c(t), y) \in S$ with $y = u^i U_i(c(t)) \in T_c(t)G \setminus \{0\}$. Both summands in the right side of (4.17) are smooth vector fields on $N$. In particular, the first one, $w^i(t)\phi_i^j(c(t)) \tilde{V}_j|_{T_c(t)G \setminus \{0\}}$ lifts $\dot{c}(t)$.

On $S$, we have the global coordinate $(t, u^1, \ldots, u^n)$ for $y = u^i U_i(c(t)) \in T_c(t)M \setminus \{0\}$, and the corresponding global frame $\{ \partial_t, \partial_{u^1}, \ldots, \partial_{u^n} \}$. Using this frame, $\overleftarrow{\mathcal{H}} \dot{c}(t)$ can be presented as

**Lemma 4.2** Using the global frame $\{ \partial_t, \partial_{u^1}, \ldots, \partial_{u^n} \}$ on $S$, we have

$$\overleftarrow{\mathcal{H}} \dot{c}(t) = \partial_t + \left(-\frac{1}{2} w^i \frac{\partial}{\partial u^i} H^j + \frac{1}{2} w^i(t) u^p \frac{\partial}{\partial u^i} c_p^j \right) \partial_{u^j},$$

for each $(c(t), y) \in S$ with $y = u^i U_i(c(t)) \in T_c(t)G \setminus \{0\}$.

**Proof** We only need to prove $\partial_t = w^i(t)\phi_i^j(c(t)) \tilde{V}_j|_{T_c(t)G \setminus \{0\}}$ on $N$. Notice that the smooth vector field on $N$ which lifts $\dot{c}(t)$ and keeps all $u^i$ invariant is unique. Obviously $\partial_t$ on $N$ is such a lifting. The left invariance of the $u^i$ implies $\tilde{V}_j u^i = 0, \forall i, j$. Together with (4.17), it implies $w^i(t)\phi_i^j(c(t)) \tilde{V}_j|_{T_c(t)G \setminus \{0\}}$ is also such a lifting. These two liftings must be the same.

**Proof of Theorem 1.4** Since Theorem 1.4 is a local result, we only need to prove it in the case that $c(t)$ is a simple smooth curve. We may further assume $c(t)$ is defined for

\[ \odot \text{ Springer} \]
\( t \in (a, b) \) and it has nonvanishing \( \dot{c}(t) \) everywhere, because continuity can help us with the rest.

Using the global coordinate \((t, u^1, \ldots, u^n)\) on \( S \), a curve \((c(t), Y(t))\) in \( S \) with \( Y(t) = u^1(t)U_1(c(t)) \in T_{c(t)}G\{0\} \) can be represented as \((t, u^1(t), \ldots, u^n(t))\). Then Lemma 4.2 indicates that \( Y(t) \) is nonlinearly parallel, i.e., \((c(t), Y(t))\) is an integral curve of \( \ddot{c}(t) \), iff

\[
\frac{d}{dt}u^j(t) = \frac{1}{2}u^j(t) \partial_{\rho_i}c_{\rho_i} - \frac{1}{2}u^j \frac{\partial}{\partial u^i} \hat{H}^j, \quad \forall j. \tag{4.18}
\]

Using left translations, i.e., \( y(t) = (L_{c(t)}^{-1})_* (Y(t)) = u^i(t)e_i \) and \( w(t) = (L_{c(t)}^{-1})_* (\dot{c}(t)) = w^i(t)e_i \), (4.18) is translated to \( \frac{d}{dt}y(t) + N(y(t), w(t)) = 0 \). This ends the proof. \( \square \)

### 4.2 Proof of Theorem 1.6

Suppose that \( G \) is a left invariant spray structure on the Lie group \( G \). Let \( N^y(y) = N(y, v) \) and \( R^y(y) = R_y(v) \) be the smooth vector fields induced by the connection operator \( N \) and the Riemannian curvature operator \( R \) of \((G, G)\) respectively. For the convenience of later calculation, we denote by \( D_{AB} \) the smooth vector field on \( g \) determined by the derivative of \( B \) in the direction of \( A \), i.e., \( (D_{AB})(y) = \frac{d}{dt} \big|_{t=0} B(y + tA(y)) \), where \( A \) and \( B \) are two smooth vector fields on \( g \{0\} \). Each vector \( v \in g \) can be naturally viewed as a constant vector field on \( g \), so \( N^u = N(y, u) = \frac{1}{2}D_u \eta - \frac{1}{2}[y, u]_g \). Similarly, \( R^u \) can also be reformulated as following.

**Lemma 4.3** Fixing \( u \in g \), then at each \( y \in g \{0\} \), we have

\[
R^u = \frac{1}{2} D_\eta D_u \eta - \frac{1}{4}[\eta(y), u]_g - \frac{1}{4} D_{D_u \eta} \eta + \frac{3}{4} D_{[y, u]_g} \eta - \frac{1}{4}[y, (D_u \eta)(y)]_g \\
- \frac{1}{4}[y, [y, u]_g]_g.
\]

**Proof** For any \( y \in g \{0\} \) and \( u \in g \), we have

\[
R_y(u) \\
= DN(\eta, y, u) - N(y, N(y, u)) + N(y, [y, u]_g) - [y, N(y, u)]_g \\
= D_\eta \left( \frac{1}{2} D_u \eta - \frac{1}{2}[y, u]_g \right) - \left( \frac{1}{2} D_{D_u \eta - [y, u]_g} \eta - \frac{1}{4}[y, \frac{1}{2}(D_u \eta)(y) - \frac{1}{2}[y, u]_g]_g \right) \\
+ \frac{1}{2} D_{[y, u]_g} \eta - \frac{1}{2}[y, [y, u]_g]_g - \frac{1}{2}[y, (D_u \eta)(y)]_g + \frac{1}{2}[y, [y, u]_g]_g \\
= \frac{1}{2} D_\eta D_u \eta - \frac{1}{2}[\eta(y), u]_g - \frac{1}{4} D_{D_u \eta} \eta + \frac{1}{4} D_{[y, u]_g} \eta + \frac{1}{4}[y, (D_u \eta)(y)]_g \\
- \frac{1}{4}[y, [y, u]_g]_g + \frac{1}{2} D_{[y, u]_g} \eta - \frac{1}{2}[y, [y, u]_g]_g - \frac{1}{2}[y, (D_u \eta)(y)]_g \\
+ \frac{1}{2}[y, [y, u]_g]_g \\
= \frac{1}{2} D_\eta D_u \eta - \frac{1}{4}[\eta(y), u]_g - \frac{1}{4} D_{D_u \eta} \eta + \frac{3}{4} D_{[y, u]_g} \eta - \frac{1}{4}[y, (D_u \eta)(y)]_g \\
- \frac{1}{4}[y, [y, u]_g]_g,
\]

which ends the prove. \( \square \)
Notice that two brackets will be used simultaneously below, i.e., \([\cdot, \cdot]_g\) for the Lie bracket of \(g\), and \([\cdot, \cdot]\) for the canonical bracket between smooth vector fields, with respect to the variable \(y \in g\setminus \{0\} \).

**Proof of Theorem 1.6** Lemma 4.3 provides that, at any \(y \in g\setminus \{0\},\)

\[
D_{Du\eta} \eta = 2D_\eta Du\eta + 3D_{[y,u]_g} \eta - [y, [y, u]_g]_g - 2[\eta(y), u]_g \\
- [y, (Du\eta)(y)]_g = 4R^u,
\]

so we have

\[
D_{Du\eta} D_v \eta = D_v D_{Du\eta} \eta - D_{D_v Du\eta} \eta \\
= D_v (2D_\eta Du\eta + 3D_{[y,u]_g} \eta - [y, [y, u]_g]_g - 2[\eta(y), u]_g \\
- [y, (Du\eta)(y)]_g - 4R^u) - D_{D_v Du\eta} \eta \\
= 2D_v Du\eta + 3D_v D_{[y,u]_g} \eta - [y, [v, u]_g]_g - [v, [y, u]_g]_g \\
- 2[(D_v \eta)(y), u]_g - [v, (Du\eta)(y)]_g - 4D_v R^u \\
- [y, (D_v Du\eta)(y)]_g - D_{D_v Du\eta} \eta + 2D_\eta Du \eta. \tag{4.19}
\]

Similarly, we have

\[
D_{D_v \eta} D_u \eta = 2D_{D_v \eta} D_u \eta + 3D_u D_{[y,v]_g} \eta - [y, [u, v]_g]_g - [u, [y, v]_g]_g \\
- 2[(D_u \eta)(y), v]_g - [u, (D_v \eta)(y)]_g - 4D_u R^v - [y, (D_u D_v \eta)(y)]_g \\
- D_{D_v D_u \eta} \eta + 2D_\eta D_u D_v \eta. \tag{4.20}
\]

Since \(D_u\) and \(D_v\) commute, (4.19) minus (4.20) provides

\[
[D_\eta, D_v \eta] = D_{D_\eta} D_v \eta - D_{D_v} D_\eta \eta \\
= D_v D_{[y,u]_g} \eta - D_u D_{[y,v]_g} \eta + [y, [u, v]_g]_g \\
+ [(D_u \eta)(y), v]_g - [(D_v \eta)(y), u]_g + \frac{4}{3} (D_u R^v - D_v R^u). \tag{4.21}
\]

Meanwhile, it is not hard to see that

\[
[D_\eta, [y, v]_g] = D_{[y, v]_g} [y, v]_g - [y, v]_g D_\eta \eta = [(D_\eta \eta)(y), v]_g - D_{[y,v]_g} D_\eta \eta. \tag{4.22}
\]
\[ [D_v \eta, [y, u]_g] = D_{D_v \eta} [y, u]_g - D_{[y, u]_g} D_v \eta = [(D_v \eta)(y), u]_g - D_{[y, u]_g} D_v \eta, \]
\[ ([y, u]_g, [y, v]_g] = D_{[y, u]_g} [y, v]_g - D_{[y, v]_g} [y, u]_g = [y, [u, v]_g]_g. \] (4.23)

Then the sum of (4.21), (4.23) and (4.24) minus (4.22) provides

\[ 4[N^u, N^v] = [D_u \eta - [y, u]_g, D_v \eta - [y, v]_g] = [D_u \eta, D_v \eta] + [D_v \eta, [y, u]_g] + [[y, u]_g, [y, v]_g] - [D_u \eta, [y, v]_g] = D_v D_{[y, u]_g} \eta - D_{[y, u]_g} D_v \eta - D_u D_{[y, v]_g} \eta + D_{[y, v]_g} D_u \eta + 2[y, [u, v]_g]_g + 4 \left( D_u R^v - D_v R^u \right) \]
\[ = -2D_{[u, v]_g} \eta + 2[y, [u, v]_g]_g + 4 \left( D_u R^v - D_v R^u \right) \]
\[ = -4N^{[u, v]}_g + 4 \left( D_u R^v - D_v R^u \right). \]

So we get \([N^u, N^v] + N^{[u, v]}_g = \frac{1}{3}(D_u R^v - D_v R^u),\) which ends the proof. \(\square\)

### 4.3 Two Questions Related to Landsberg Conjecture and Holonomy

Denote \(\mathfrak{g}(G, G)\) the Lie algebra generated by \(N^v\) for all \(v \in g\), using the canonical bracket between smooth vector fields on \(g \setminus \{0\}\). Theorems 1.4 and 1.5 imply that \(\mathfrak{g}(G, G)\) contains all information of nonlinear parallel translations, so it is worthy to be studied in homogeneous spray and Finsler geometries.

Here are some examples. When \(G = G_0 = u^i \tilde{U}_i\), we have \(\eta = 0\) and \(N(y, w) = \frac{1}{2}[y, w]\). It is easy to check \([-2N(\cdot, w_1), -2N(\cdot, w_2)] = -2N(\cdot, [w_1, w_2])\), so in this case \(\mathfrak{g} = \mathcal{N}\) is isomorphic to \(g / c(g)\). More generally, when the left invariant spray structure \(G\) is affine (see Definition 6.1.1 in [15]), the associated spray vector field \(\eta\) is quadratic. Then \(\mathfrak{g}\) is a finite dimensional subalgebra in \(\mathfrak{gl}(g, \mathbb{R})\). Theorem that \(\dim \mathfrak{g}(G, G) < +\infty\) when \((G, G)\) has vanishing

Above examples suggest we ask

**Question 4.4** Is there an example of left invariant spray structure \(G\) such that \(G\) is not affine and \(\dim \mathfrak{g}(G, G)\) is finite?

Finsler geometry provides another motivation for Question 4.4. In Finsler geometry, a metric is called a Berwald metric if its induced spray structure is affine [16], and it is called a Landsberg metric if all nonlinearly parallel translations are isometries for the Hessian metrics on the punctured tangent spaces [7]. Landsberg Problem asks if there exists a (regular) Landsberg metric which is not Berwald [12]. See [17, 20] and the references therein for some recent progress on this problem. Theorem 1.5 implies that, if \(G\) is induced by a left invariant Landsberg metric, then \(\mathfrak{g}\) is a Lie subalgebra in the space of all Killing vector fields for the Hessian metric of \(F(e, \cdot)\) on \(g \setminus \{0\}\), which must have a finite dimension. So Question 4.4 may be viewed as a generalization for Landsberg Problem in the left invariant spray geometry.
Another natural question for $\mathfrak{h}$ is the following.

**Question 4.5** What is the relation between the Lie algebra $\mathfrak{h}$ and the (restricted) holonomy group of $(G, G)$.

Here the (restricted) holonomy group of $(G, G)$ is defined similarly as in Riemannian or Finsler geometry, using the nonlinearly parallel translations. When the left invariant spray structure $G$ is induced by a Riemannian metric $F$ on the Lie group $G$, $\text{Hol}_0(G, G)$ is a compact Lie group. Comparing Lemma 2.2 in [9] and Definition 4 in [3], then we see that $N^u$ coincides with the linear operator $a_u$ in [8] for each $u \in \mathfrak{g}$. So in this case, Theorem 4.5 in [8] indicates $\mathfrak{h}(G, G)$ is the Lie algebra of the restricted holonomy group when either $G$ is compact or $(G, F)$ is irreducible with nonvanishing Ricci curvature.

However, when the left invariant spray structure $G$ is more generic, very likely both $\mathfrak{h}(G, G)$ and the holonomy group have infinite dimensions [6], making Question 4.5 is much harder in this situation.

**Acknowledgements** This paper is supported by Beijing Natural Science Foundation (1222003) and National Natural Science Foundation of China (11821101, 12131012). The author sincerely thanks the reviewers for their efforts and suggestions, and sincerely thanks Ming Li for helpful discussions.

**Declarations**

**Conflict of interest** On behalf of all authors, Ming Xu states that there is no conflict of interest.

**References**

1. Berwald, L.: Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. Math. Z. 25, 40–73 (1926)
2. Huang, L.: On the fundamental equations of homogeneous Finsler spaces. Differ. Geom. Appl. 40, 187–208 (2015)
3. Huang, L.: Ricci curvature of left invariant Finsler metrics on Lie groups. Israel J. Math. 207(2), 783–192 (2015)
4. Huang, L.: Flag curvatures of homogeneous Finsler spaces. Eur. J. Math. S.I. 3(4), 1000–1029 (2017)
5. Huang, L., Mo, X.: Inverse problem of left invariant sprays on Lie groups. Int. J. Math. (2021). https://doi.org/10.1142/S0129167X21500762
6. Hubicska, B., Matveev, V.S., Muzsnay, Z.: Almost all Finsler metrics have infinite dimensional holonomy group. J. Geom. Anal. 31, 6067–6079 (2021)
7. Ichijyō, Y.: On special Finsler connections with vanishing hv-curvature tensor. Tensor NS 32, 146–155 (1978)
8. Kostant, B.: Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold. Trans. Am. Math. Soc. 80(2), 528–542 (1955)
9. Kostant, B.: On holonomy and homogeneous spaces. Nagoya Math. J. 12, 31–54 (1957)
10. Li, Y., Mo, X., Yu, Y.: Inverse problem of sprays with scalar curvature. Int. J. Math. 30, 1950041 (2019)
11. Li, B., Shen, Z.: Sprays of isotropic curvature. Int. J. Math. 29(1), 1850003 (2018)
12. Matsumoto, M.: Remarks on Berwald and Landsberg spaces, Finsler geometry (Seattle, WA, 1995), Contemp. Math. 196, American Mathematical Society, Providence, pp. 79–82(1996)
13. Muzsnay, Z.: An invariant variational principle for canonical flows on Lie groups. J. Math. Phys. 46(11), 112902 (2005)
14. Muzsnay, Z., Thompson, G.: Inverse problem of the calculus of variations on Lie groups. Differ. Geom. Appl. 23, 257–281 (2005)
15. Shen, Z.: Differential Geometry of Spray and Finsler Spaces. Kluwer Academic Publishers, Philadelphia (2001)
16. Shen, Z.: Lectures on Finsler Geometry. World Scientific, Singapore (2001)
17. Tayebi, A., Najafi, B.: On homogeneous Landsberg surfaces. J. Geom. Phys. 168, 104314 (2021). https://doi.org/10.1016/j.geomphys.2021.104314
18. Xu, M.: Left invariant spray structure on Lie group. J. Lie Theory 32(1), 121–138 (2022)
19. Xu, M.: Submersion and homogeneous spray structure. J. Geom. Anal. 32(6), 172 (2022). https://doi.org/10.1007/s12220-022-00911-5
20. Xu, M., Matveev, V.S.: Proof of Laugwitz Conjecture and Landsberg Unicorn Conjecture for Minkowski norms with $SO(k) \times SO(n-k)$-symmetry. Can. J. Math. 74(5), 1486–1516 (2021)
21. Xu, M., Matveev, V.S., Yan, K., Zhang, S.: Some geometric correspondences for homothetic navigation. Publ. Math. Debrecen 97(3–4), 449–474 (2020)
22. Yang, G.: Some classes of sprays in projective spray geometry. Differ. Geom. Appl. 29, 601–614 (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.