Special polynomials associated with the $K_2$ hierarchy

Nikolai A. Kudryashov

Department of Applied Mathematics
Moscow Engineering and Physics Institute
(State University)
31 Kashirskoe Shosse, 115409, Moscow,
Russian Federation

Abstract

New special polynomials associated with the rational solutions of analogue to the Painlevé hierarchies are introduced. The Hirota relations for these special polynomials are found. Differential - difference hierarchies for finding special polynomials are presented. These formulae allow us to search the special polynomials associated with the hierarchy studied. The ordinary differential hierarchy for the Yablonskii - Vorob’ev polynomials is given.

Keywords: Special polynomials, the Painlevé equation, Special solutions, Differential - difference equations

PACS: 02.30.Hq - Ordinary differential equations

1 Introduction

In this paper our interest is in special polynomials associated with rational solutions of the hierarchy [1–3]

$$\left( \frac{d}{dz} + w \right) H_N \left[ w_z - \frac{1}{2} w^2 \right] - z w - \beta = 0$$

(1.1)

Here the operator $H_N$ is determined by the following recursion formula [1–4]

$$H_{N+2} = J[v] \Omega[v] H_N$$

(1.2)
under the conditions
\[ H_0[v] = 1, \quad H_1[v] = v_{zz} + 4 v^2, \] (1.3)
where the operators \( \Omega[v] \) and \( J[v] \) take the form [1–4]
\[ \Omega = D^3 + 2 v D + v_z, \quad D = \frac{d}{dz} \] (1.4)
\[ J = D^3 + 3 (vD + D v) + 2 \left( D^2 v D^{-1} + D^{-1} v D^2 \right) + \]
\[ + 8 \left( v^2 D^{-1} + D^{-1} v^2 \right), \quad D^{-1} = \int dz \] (1.5)

Hierarchy (1.1) is important because the special solutions of the Fordy-Gibbons equation [5], the Caudrey-Dodd-Gibbon hierarchy (Savada-Kotera hierarchy) [6, 7] and the Kaup-Kupershmidt hierarchy [8] can be expressed through solutions of the hierarchy (1.1). Hierarchy (1.1) is called the \( K_2 \) hierarchy in [2] taking into consideration jubilee year in 2000 by Kovalevskaya and Kruskal. Apparently hierarchy (1.1) defines new transcendental functions like the Painlevé equations do.

Hierarchy (1.1) can be written in the form as well
\[ -\frac{1}{2} \left( \frac{d}{dz} - 2 w \right) G_N \left[ -2 w_z - 2 w^2 \right] - z \, w - \beta = 0 \] (1.6)
The recursion relations \( G_N \) is determined by the operator [1–4]
\[ G_{N+2} = J_1[u] \, \Omega[u] \, G_N \] (1.7)
under the conditions
\[ G_0[u] = 1, \quad G_1[u] = u_{zz} + \frac{1}{4} u^2 \] (1.8)
The operator \( J_1[u] \) takes the form
\[ J_1 = D^3 + \frac{1}{2} \left( D^2 u D^{-1} + D^{-1} u D^2 \right) + \]
\[ + \frac{1}{8} \left( u^2 D^{-1} + D^{-1} u^2 \right) \] (1.9)

Hierarchy (1.1) is similar in appearance to the second Painlevé hierarchy but hierarchy (1.1) is distinguished from the \( P_2 \) hierarchy by the operator \( H_N \) in (1.1) [9–11]. Special polynomials associated with the rational solutions of the \( P_2 \) hierarchy were considered in papers [10,11]. Recently it was shown [12]
that the rational solutions of hierarchy (1.1) at $N = 1$ can be found using the special polynomials $Q_{n}^{(1)}(z)$ and $R_{n}^{(1)}(z)$ by the formulae

$$w(z; \beta_{n}^{(1)}) = (-1)^{n} \frac{d}{dz} \ln \frac{Q_{n-1}^{(1)}}{Q_{n}^{(1)}}, \quad w(z; \beta_{n}^{(2)}) = (-1)^{n-1} \frac{d}{dz} \ln \frac{R_{n-1}^{(1)}}{R_{n}^{(1)}},$$  \hspace{1cm} (1.10)$$

These special polynomials $Q_{n}^{(1)}(z)$ and $R_{n}^{(1)}(z)$ were obtained in [12] taking into consideration the power expansions near infinity [11, 13, 14]. This approach is not simple for calculating the special polynomials. This raises the question of whether the recursion formulae for finding the special polynomials $Q_{n}^{(1)}(z)$ and $R_{n}^{(1)}(z)$.

The aim of this paper is to introduce the special polynomials $Q_{n}^{(N)}(z)$ and $R_{n}^{(N)}(z)$ associated with the rational solutions of hierarchy (1.1) and to derive the recursion formulae for finding the special polynomials $Q_{n}^{(N)}(z)$ and $R_{n}^{(N)}(z)$.

The main result of this paper is the differential - difference hierarchies

$$P_{n+1} P_{n-1} = P_{n} F_{N} \left[ A \frac{d^{2}}{dz^{2}} \ln P_{n} \right] \hspace{1cm} (1.11)$$

for the polynomials $P_{n}(z) \equiv Q_{n}^{(N)}(z)$ (and $P_{n}(z) \equiv R_{n}^{(N)}(z)$). Here $A$ is a constant, $F_{N}$ is an operator which depends on the operators $H_{n}[v]$ or $G_{n}[u]$. This formula allows us to look for special polynomials associated with rational solutions of hierarchy (1.1) using $P_{0}(z) = Q_{0}^{(N)}(z) = R_{0}^{(N)}(z) = 1$, $P_{1}(z) = Q_{1}^{(N)} = z$ and $P_{1}(z) = R_{1}^{(N)}(z) = z^{2}$.

This paper is organized as follows. The general properties of the hierarchy considered are presented in section 2. Some relations for the special polynomials $Q_{n}^{(N)}(z)$ and $R_{n}^{(N)}(z)$ and the differential - difference hierarchies for finding polynomials $Q_{n}^{(N)}(z)$ and $R_{n}^{(N)}(z)$ are given in section 3. The special polynomials $Q_{n}^{(N)}(z)$ and $R_{n}^{(N)}(z)$ associated with the rational solutions of the first, the second and the third members of hierarchy (1.1) are introduced in sections 4, 5 and 6.

## 2 Some properties of hierarchy (1.1)

Let us briefly review some facts concerning equation (1.1) needed later. Suppose $w(z) \equiv w(z; \beta)$ is a solution of (1.1).

Assuming

$$w(z) = \frac{\varphi_{zz}}{\varphi_{z}} \hspace{1cm} (2.1)$$
and taking into account
\[ w_z - \frac{1}{2} w^2 = \{ \varphi; z \} = \frac{\varphi_{zzz}}{\varphi_{zz}} - \frac{3\varphi_{zz}^2}{2\varphi_z^2} \]  \hspace{1cm} (2.2)
we have the equation in the form [2, 3]
\[ H_N \{ \{ \varphi; z \} \} - z - (\beta - 1) \frac{\varphi}{\varphi_z} = 0 \]  \hspace{1cm} (2.3)

Using
\[ w(z) = -\frac{\psi_{zz}}{2\psi_z} \]  \hspace{1cm} (2.4)
and taking into consideration
\[ -2w_z - 2w^2 = \{ \psi; z \} = \frac{\psi_{zzz}}{\psi_{zz}} - \frac{3\psi_{zz}^2}{2\psi_z^2} \]  \hspace{1cm} (2.5)
we obtain the equation in the form
\[ G_N \{ \{ \psi; z \} \} - z - (\beta - 1) \frac{\psi}{\psi_z} = 0 \]  \hspace{1cm} (2.6)

The Backlund transformations for solutions of hierarchy (1.1) can be written in the form [2, 3]
\[ w(z, 2 - \beta) = w(z, \beta) - \frac{2\beta - 2}{H_N \left[ w_z - \frac{1}{2} w^2 \right] - z} \]  \hspace{1cm} (2.7)
and
\[ w(z, -1 - \beta) = w(z, \beta) - \frac{2\beta + 1}{G_N \left[ -2w_z - 2w^2 \right] - z} \]  \hspace{1cm} (2.8)
These transformations allow us to find the rational solutions of hierarchy (1.1) [2, 3]. These formulae can be presented in the form
\[ G_N \left[ -2w_z - 2w^2 \right] - z = -\frac{2\beta + 1}{y(z; -1 - \beta) - y(z; \beta)} \]  \hspace{1cm} (2.9)
and
\[ H_N \left[ w_z - \frac{1}{2} w^2 \right] - z = -\frac{2(\beta - 1)}{y(z; 2 - \beta) - y(z; \beta)} \]  \hspace{1cm} (2.10)

The rational solutions of hierarchy (1.1) are classified in the following theorem.
Theorem 2.1. Equations (1.1) possesses rational solutions if and only if 
\( \beta \in \mathbb{Z}/\{1 \pm 3k, \ k \in \mathbb{N} \cup 0\} \). They are unique and have the form

\[
\begin{align*}
  w(z; \beta_n^{(1)}) &= (-1)^n \frac{d}{dz} \ln \left( \frac{Q_{n-1}^{(N)}}{Q_n^{(N)}} \right), \\
  w(z; \beta_n^{(2)}) &= (-1)^{n-1} \frac{d}{dz} \ln \left( \frac{R_{n-1}^{(N)}}{R_n^{(N)}} \right),
\end{align*}
\]

where \( Q_n^{(N)}(z) \) and \( R_n^{(N)}(z) \) are polynomials, \( n \in \mathbb{N} \) and

\[
\begin{align*}
  \beta_n^{(1)} &= (-1)^n \left( 3 \left\lfloor \frac{n+1}{2} \right\rfloor - 1 + (-1)^n \right), \\
  \beta_n^{(2)} &= (-1)^{n+1} \left( 3 \left\lfloor \frac{n}{2} \right\rfloor + 1 + (-1)^{n+1} \right)
\end{align*}
\]

with \( \lfloor x \rfloor \) denoting the integer part of \( x \). The only remaining rational solution is the trivial solution \( w(z; 0) = 0 \).

Proof. This theorem can be proved by the analogy with corresponding theorem of the work [12]. The rational solutions of (1.1) can be described with the help of two families of polynomials. The polynomials \( \{Q_n^{(N)}(z)\} \) we call the first family and \( \{R_n^{(N)}(z)\} \) the second. By \( p_n^{(1)} (p_n^{(2)}) \) denote the degree of \( Q_n^{(N)}(z) \) (\( R_n^{(N)}(z) \)). The special polynomials \( Q_n^{(N)}(z) \) and \( R_n^{(N)}(z) \) can be defined as monic polynomials and each polynomial can be presented in the form

\[
\begin{align*}
  Q_n^{(N)}(z) &= \sum_{k=0}^{p_n^{(1)}} A_{n,k}^{(1,N)} z^{p_n^{(1)}-k}, & A_{n,0}^{(1,N)} &= 1, \\
  R_n^{(N)}(z) &= \sum_{k=0}^{p_n^{(2)}} A_{n,k}^{(2,N)} z^{p_n^{(2)}-k}, & A_{n,0}^{(2,N)} &= 1.
\end{align*}
\]

The first non-trivial solutions of (1.1) are \( w(z; -1) = 1/z \) and \( w(z; 2) = -2/z \). Hence it can be set \( Q_0^{(N)}(z) = R_0^{(N)}(z) = 1, Q_1^{(N)}(z) = z, R_1^{(N)}(z) = z^2 \).

We also observe that degree of polynomials can be rewritten in terms of \( \beta_n^{(j)} \)

\[
\begin{align*}
  p_n^{(1)} &= \sum_{i=1}^{n} |\beta_i^{(1)}| = \frac{k(k+1)}{2} - \frac{1}{2} \left\lfloor \frac{k+1}{3} \right\rfloor - \frac{3}{2} \left\lfloor \frac{k+1}{3} \right\rfloor^2, & k &\overset{\text{def}}{=} |\beta_n^{(1)}|, \\
  p_n^{(2)} &= \sum_{i=1}^{n} |\beta_i^{(2)}| = \frac{k(k+1)}{2} + \frac{1}{2} \left\lfloor \frac{k+2}{3} \right\rfloor - \frac{3}{2} \left\lfloor \frac{k+2}{3} \right\rfloor^2, & k &\overset{\text{def}}{=} |\beta_n^{(2)}|.
\end{align*}
\]
The first few $\beta_n^{(j)}$ and $p_n^{(j)}$ are given in Table 2.1.

Table 2.1: Values of $\beta_n^{(j)}$ and $p_n^{(j)}$ ($j = 1, 2$).

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\beta_n^{(1)}$ | -1  | 3   | -4  | 6   | -7  | 9   | -10 | 12  | -13 | 15  | -16 | 18  |
| $p_n^{(1)}$     | 1   | 4   | 8   | 14  | 21  | 30  | 40  | 52  | 65  | 80  | 96  | 114 |
| $\beta_n^{(2)}$ | 2   | -3  | 5   | -6  | 8   | -9  | 11  | -12 | 14  | -15 | 17  | -18 |
| $p_n^{(2)}$     | 2   | 5   | 10  | 16  | 24  | 33  | 44  | 56  | 70  | 85  | 102 | 120 |

3 Hirota relations and differential - difference hierarchies for special polynomials

In this section we derive the Hirota relations and the differential - difference hierarchies for special polynomials $Q_n^{(N)}(z)$ and $R_n^{(N)}(z)$. Later in this section when it does cause any contradiction the index $N$ will be omitted.

Theorem 3.1. Special polynomials $Q_n(z)$ and $R_n(z)$ satisfy the following Hirota relations

$$D_z Q_{2m+2} \bullet Q_{2m} = (4 + 6m) Q_{2m+1}^{\frac{1}{2}} Q_{2m+2}^{\frac{1}{2}}, \quad (m = 0, 1, \ldots) \quad (3.1)$$

$$D_z Q_{2m+1} \bullet Q_{2m-1} = (1 + 6m) Q_{2m}^{2}, \quad (m = 1, 2, \ldots) \quad (3.2)$$

$$D_z R_{2m+2} \bullet R_{2m} = (5 + 6m) R_{2m+1}^{2}, \quad (m = 0, 1, \ldots) \quad (3.3)$$

$$D_z R_{2m+1} \bullet R_{2m-1} = (2 + 6m) R_{2m-1}^{\frac{1}{2}} R_{2m}^{\frac{1}{2}} R_{2m+1}^{\frac{1}{2}}, \quad (m = 1, 2, \ldots) \quad (3.4)$$

where $D_z$ is the Hirota operator defined by

$$D_z f(z) \bullet g(z) = \left[ \left( \frac{d}{dz_1} - \frac{d}{dz_2} \right) f(z_1) g(z_2) \right]_{z_1 = z_2 = z} \quad (3.5)$$

Proof. Without loss of generality let us prove formulae (3.1) and (3.2). Comparison of equations (2.3) and (2.9), (2.6) and (2.10) yields

$$\varphi = \left( \frac{Q_{2m+2}}{Q_{2m}} \right)^{\frac{1}{2}}, \quad \psi = \frac{Q_{2m+1}}{Q_{2m-1}} \quad (3.6)$$
Solutions of equations (2.3) and (2.10) can be found taking into account the following recursion formulae [2–4]

\[ \psi_z = \frac{\varphi^4}{\varphi_z^2}, \quad \varphi_z = \frac{\psi}{\psi_z} \]  \hspace{1cm} (3.7)

Substituting (3.6) into the second equation (3.7) yields

\[ (Q_{2m+2,z} Q_{2m} - Q_{2m+2} Q_{2m})^2 (Q_{2m+1,z} Q_{2m-1} - Q_{2m+1} Q_{2m-1}) = 4 Q_{2m}^3 Q_{2m+1}^2 Q_{2m+2} \]  \hspace{1cm} (3.8)

Assuming

\[ \varphi = \left( \frac{Q_{2m}}{Q_{2m-2}} \right)^{\frac{1}{2}}, \quad \psi = \frac{Q_{2m+1}}{Q_{2m-1}} \]  \hspace{1cm} (3.9)

in the first equation (3.7) we get

\[ (Q_{2m+1,z} Q_{2m+1} - Q_{2m+1} Q_{2m-1}) (Q_{2m,z} Q_{2m-2} - Q_{2m} Q_{2m-2})^2 = 4 Q_{2m-2}^2 Q_{2m-1}^2 Q_{2m}^3 \]  \hspace{1cm} (3.10)

From (3.8) and (3.10) we have

\[ \frac{(Q_{2m+2,z} Q_{2m} - Q_{2m+2} Q_{2m})^2}{Q_{2m}^2 Q_{2m+1}^2 Q_{2m+2}} = \frac{(Q_{2m,z} Q_{2m-2} - Q_{2m} Q_{2m-2,z})^2}{Q_{2m-2}^2 Q_{2m-1}^2 Q_{2m}} \]  \hspace{1cm} (m = 1, 2, ...)

(3.11)

By the induction taking into account \( m = 1 \) and \( Q_2(z) = (z^2 + C_1)^2 \) (where \( C_1 \) is constant) in the right hand side (3.11) we get

\[ (Q_{2,z} Q_0 - Q_2 Q_{0,z})^2 = 16 Q_0^2 Q_1^2 Q_2 \]  \hspace{1cm} (3.12)

Suppose we have at \( m = k \)

\[ (Q_{2k,z} Q_{2k-2} - Q_{2k} Q_{2k-2,z})^2 = F_1(k) Q_{2k-2} Q_{2k-1}^2 Q_{2k} \]  \hspace{1cm} (3.13)

then from (3.11) we obtain the equality at \( m = k + 1 \)

\[ (Q_{2k+2,z} Q_{2k} - Q_{2k+2} Q_{2k,z})^2 = F_1(k) Q_{2k} Q_{2k+1}^2 Q_{2k+2} \]  \hspace{1cm} (3.14)

Taking into account equation (3.14) at \( k = m \) we find from (3.10)

\[ Q_{2m+1,z} Q_{2m-1} - Q_{2m+1} Q_{2m-1,z} = F_2(m) Q_{2m}^2 \]  \hspace{1cm} (3.15)
We determine constants $F_1(m)$ and $F_2(m)$ taking into account unique polynomials. Finally the relations for $Q_n(z)$ and for $R_n(z)$ can be written in the form

\[ Q_{2m+2z} Q_{2m} - Q_{2m+2} Q_{2m,z} = (4 + 6m) Q_{2m}^{\frac{1}{2}} Q_{2m+1} Q_{2m+2}^{\frac{1}{2}}, \quad (m = 0, 1, ... \) \tag{3.16} \]

\[ Q_{2m+1,z} Q_{2m-1} - Q_{2m+1} Q_{2m-1,z} = (1 + 6m) Q_{2m}^{2}, \quad (m = 1, 2, ...) \tag{3.17} \]

\[ R_{2m+2,z} R_{2m} - R_{2m+2} R_{2m,z} = (5 + 6m) R_{2m+1}^{2}, \quad (m = 0, 1, ...) \tag{3.18} \]

\[ R_{2m+1,z} R_{2m-1} - R_{2m+1} R_{2m-1,z} = (2 + 6m) R_{2m-1}^{2} R_{2m} R_{2m+1}^{2}, \quad (m = 1, 2, ...) \tag{3.19} \]

Now if we write equations (3.16), (3.17), (3.18) and (3.19) using the Hirota operator we get equations (3.1), (3.2), (3.3) and (3.4).

**Theorem 3.2.** The following relations are valid for the special polynomials $Q_n(z)$ and $R_n(z)$

\[ \frac{d^2}{dz^2} \ln (Q_{2m}) + \frac{D_z^2 Q_{2m} \cdot Q_{2m-1}}{Q_{2m} Q_{2m-1}} = 0, \quad (m = 1, 2, ...) \tag{3.20} \]

\[ \frac{d^2}{dz^2} \ln (Q_{2m}) + \frac{D_z^2 Q_{2m} \cdot Q_{2m+1}}{Q_{2m+2} Q_{2m+1}} = 0, \quad (m = 0, 1, ...) \tag{3.21} \]

\[ \frac{d^2}{dz^2} \ln R_{2m+1} + \frac{D_z^2 R_{2m} \cdot R_{2m+1}}{R_{2m} R_{2m+1}} = 0, \quad (m = 0, 1, ...) \tag{3.22} \]

\[ \frac{d^2}{dz^2} \ln R_{2m+1} + \frac{D_z^2 R_{2m+1} \cdot R_{2m+2}}{R_{2m+1} R_{2m+2}} = 0, \quad (m = 0, 1, ...) \tag{3.23} \]

where $D_z$ is the Hirota operator defined by (3.5).

**Proof.** To prove theorem (3.2) we use the relations for solutions $w(z; \beta)$, $w(z; 2 - \beta)$ and $w(z; -1 - \beta)$ in the form [3]

\[ w_z(z; \beta) - \frac{1}{2} w^2(z; \beta) = w_z(z; 2 - \beta) - \frac{1}{2} w^2(z; 2 - \beta) \tag{3.24} \]

\[ w_z(z; \beta) + w^2(z; \beta) = w_z(z; -1 - \beta) + w^2(z; -1 - \beta) \tag{3.25} \]
Assuming

\[ w(z; \beta) = \frac{Q_{2m,z}}{Q_{2m}} + \frac{Q_{2m+1,z}}{Q_{2m+1}}, \quad w(z; 2 - \beta) = \frac{Q_{2m+1,z}}{Q_{2m+1}} + \frac{Q_{2m+2,z}}{Q_{2m+2}} \]  

(3.26)

in equality (3.24) we have

\[ 2 \frac{Q_{2m+2,zz}}{Q_{2m+2}} - \frac{Q_{2m+2,z}}{Q_{2m+2}} - 2 \frac{Q_{2m,zz}}{Q_{2m}} + \frac{Q_{2m,z}}{Q_{2m}} = 0 \]  

(3.27)

From (3.27) we obtain

\[ \frac{d^2}{dz^2} \ln Q_{2m+2} + \frac{D_z^2 Q_{2m+1} \cdot Q_{2m+2}}{Q_{2m+1} Q_{2m+2}} = \frac{d^2}{dz^2} \ln Q_{2m} + \frac{D_z^2 Q_{2m+1} \cdot Q_{2m}}{Q_{2m+1} Q_{2m}} \]  

(3.28)

Substituting

\[ w(z; \beta) = \frac{Q_{2m-1,z}}{Q_{2m-1}} - \frac{Q_{2m,z}}{Q_{2m}}, \quad w(z; -1 - \beta) = -\frac{Q_{2m,z}}{Q_{2m}} - \frac{Q_{2m+1,z}}{Q_{2m+1}} \]  

(3.29)

in equality (3.25) yields

\[ \frac{Q_{2m-1,z,z}}{Q_{2m-1}} - \frac{2 Q_{2m,z} Q_{2m-1,z}}{Q_{2m} Q_{2m-1}} = \frac{Q_{2m+1,zz}}{Q_{2m+1}} - \frac{2 Q_{2m,z} Q_{2m+1,z}}{Q_{2m} Q_{2m+1}} \]  

(3.30)

Adding \( \frac{Q_{2m,zz}}{Q_{2m}} \) to both parts of equality (3.30) we get

\[ \frac{D_z^2 Q_{2m-1} \cdot Q_{2m}}{Q_{2m-1} Q_{2m}} = \frac{D_z^2 Q_{2m} \cdot Q_{2m+1}}{Q_{2m} Q_{2m+1}} \]  

(3.31)

Adding \( \frac{2 Q_{2m,zz}}{Q_{2m}} - \frac{Q_{2m,zz}}{Q_{2m}} \) to both parts of equality (3.30) yields

\[ \frac{d^2}{dz^2} \ln Q_{2m} + \frac{D_z^2 Q_{2m} \cdot Q_{2m-1}}{Q_{2m} Q_{2m-1}} = \frac{d^2}{dz^2} \ln Q_{2m} + \frac{D_z^2 Q_{2m} \cdot Q_{2m+1}}{Q_{2m} Q_{2m+1}} \]  

(3.32)

We have

\[ \frac{d^2}{dz^2} \ln Q_0 + \frac{D_z^2 Q_0 \cdot Q_1}{Q_0 Q_1} = 0 \]  

(3.33)

By the induction assuming at \( m = k \) we have

\[ \frac{d^2}{dz^2} \ln Q_{2k} + \frac{D_z^2 Q_{2k} \cdot Q_{2k+1}}{Q_{2k} Q_{2k+1}} = 0 \]  

(3.34)
then we obtain from (3.28)

$$\frac{d^2}{dz^2} \ln Q_{2k+2} + \frac{D^2 Q_{2k+2} \cdot Q_{2k+1}}{Q_{2k+2} Q_{2k+1}} = 0$$  \hspace{1cm} (3.35)$$

From (3.32) we have

$$\frac{d^2}{dz^2} \ln Q_{2k+2} + \frac{D^2 Q_{2k+2} \cdot Q_{2k+3}}{Q_{2k+2} Q_{2k+3}} = 0$$  \hspace{1cm} (3.36)$$

This completes the proof. □

**Theorem 3.3.** The following differential - difference hierarchies are valid for the special polynomials $Q_n(z)$ and $R_n(z)$

$$Q_{2m+2} Q_{2m} = Q_{2m+1} \left( H_N \left[ \frac{3}{2} \frac{d^2}{dz^2} \ln(Q_{2m+1}) \right] - z \right)^2, \quad (m = 0, 1, \ldots) \quad (3.37)$$

$$Q_{2m+1} Q_{2m-1} = -Q_{2m}^2 \left( G_N \left[ 6 \frac{d^2}{dz^2} \ln(Q_{2m}) \right] - z \right), \quad (m = 1, 2, \ldots) \quad (3.38)$$

$$R_{2m+2} R_{2m} = -R_{2m+1}^2 \left( G_N \left[ 6 \frac{d^2}{dz^2} \ln(R_{2m+1}) \right] - z \right), \quad (m = 0, 1, \ldots) \quad (3.39)$$

$$R_{2m+1} R_{2m-1} = R_{2m}^2 \left( H_N \left[ \frac{3}{2} \frac{d^2}{dz^2} \ln(R_{2m}) \right] - z \right)^2, \quad (m = 1, 2, \ldots) \quad (3.40)$$

**Proof.** Without loss of generality let us obtain the formula (3.37). From (3.20) we have the equality

$$\frac{Q_{2m,z,z}}{Q_{2m}} = \frac{Q_{2m,z}^2}{2 Q_{2m}} - \frac{Q_{2m,z} Q_{2m+1,z}}{Q_{2m} Q_{2m+1}} = \frac{Q_{2m+1,z,z}}{2 Q_{2m+1}}$$  \hspace{1cm} (3.41)$$

Assuming

$$y(z; \beta_n) = -\frac{d}{dz} \ln \frac{Q_{2m}}{Q_{2m+1}}$$  \hspace{1cm} (3.42)$$

we get

$$y_z - \frac{1}{2} y^2 = -\frac{Q_{2m,z,z}}{Q_{2m}} + \frac{Q_{2m,z}^2}{2 Q_{2m}} + \frac{Q_{2m,z} Q_{2m+1,z}}{Q_{2m} Q_{2m+1}} + \frac{Q_{2m+1,z,z}}{Q_{2m+1}} - \frac{3 Q_{2m+1,z,z}}{2 Q_{2m+1}} = \frac{3}{2} \frac{d^2}{dz^2} \ln Q_{2m+1}$$  \hspace{1cm} (3.43)$$
Taking into account (3.1) we obtain from equation (2.10)

\[
H_N \left[ \frac{3}{2} \frac{d^2}{dz^2} \ln Q_{2m+1} \right] - z = \frac{Q_{2m}^{2m} Q_{2m+2}^{2m+2}}{Q_{2m+1}^{2m+1}}
\]  

(3.44)

This gives equality (3.37).

The differential - difference hierarchies (3.37), (3.38), (3.39) and (3.40) are useful for finding the special polynomials \(Q_n(z)\) and \(R_n(z)\) associated with rational solutions of hierarchy (1.1).

4 Special polynomials associated with the first member of hierarchy (1.1)

The expressions \(H_1[v]\) and \(G_1[u]\) take the form

\[
H_1[v] = v_{zz} + 4 v^2, \quad G_1[u] = u_{zz} + \frac{1}{4} u^2,
\]  

(4.1)

Substituting (4.1) into hierarchies (1.1) and (1.6) at \(N = 1\) we have the fourth order equation [15–19]

\[
w_{zzzz} + 5 w_z w_{zz} - 5 w^2 w_{zz} - 5 w w_z^2 + w^5 - z w - \beta_1 = 0
\]  

(4.2)

Assuming \(Q_0^{(1)}(z) = R_0^{(1)} = 1, Q_1^{(1)} = z, R_1^{(1)} = z^2\) and using the differential - difference hierarchies (3.37), (3.38), (3.39) and (3.40) we find the special polynomials \(Q_n^{(1)}(z)\) and \(R_n^{(1)}(z)\) at \(n \geq 2\). The first nine special polynomials \(Q_n^{(1)}(z)\) and \(R_n^{(1)}(z)\) are given in Tables (4.1) and (4.2).

Recently Clarkson and Mansfield [10] studied the structure of the roots of the Yablonskii - Vorob’ev polynomials and showed that they have a highly regular pattern which is very symmetric and structured. In Figure 1 and Figure 2 the locations of the roots for the polynomials \(Q_{11}^{(1)}(z) = 0\) and \(R_{11}^{(1)}(z) = 0\) are plotted. From these plots we observe that the roots of the polynomials form approximately regular pentagons. In the case of the roots for the polynomials \(Q_{11}^{(1)}(z) = 0\) we have the cutting angle but for the polynomials \(R_{11}^{(1)}(z) = 0\) we get the acute angle. From plots we can see that the roots of the polynomials form a highly symmetric structures.
Table 4.1: Polynomials $Q_n^{(1)}(z)$

$Q_0^{(1)} = 1,$
$Q_1^{(1)} = z,$
$Q_2^{(1)} = z^4,$
$Q_3^{(1)} = z^8,$
$Q_4^{(1)} = z^4(5^5 - 504)^2,$
$Q_5^{(1)} = z(z^{20} - 3276 z^{15} + 6604416 z^{10} + 3328625664 z^5 - 119830523904),$
$Q_6^{(1)} = (z^{15} - 6552 z^{10} - 13208832 z^5 - 951035904)^2,$
$Q_7^{(1)} = z^{40} - 29952 z^{35} + 203793408 z^{30} + 3066139754496 z^{25} +
+5234197284126720 z^{20} + 36006491762989203456 z^{15} -
-3574462636834928197632 z^{10} - 7206116675859215246426112 z^5 +
+12971010016546565874435670016,$
$Q_8^{(1)} = z^2(z^{25} - 37440 z^{20} - 179262720 z^{15} - 4698117365760 z^{10} -
-1500270490124550144)^2,$
$Q_9^{(1)} = z^{65} - 142272 z^{60} + 5244715008 z^{55} + 18447301656576 z^{50} +
+5585422603926896640 z^{45} - 228285442646619148288 z^{40} +
+1478238865378129843895402496 z^{35} +
+22449112629907483670818980888576 z^{30} -
-40520916747771106259841854742724608 z^{25} +
+5403734062860426731139723747278192640 z^{20} -
-588158921353048128175262650355386563689664 z^{15} -
-3874752265000443047392623159427256373215232 z^{10} +
+117172508495415739775315292434108023272602861568 z^5 +
+1406070101944988877303783509209296279271234338816

Using the special polynomials $Q_n^{(1)}(z)$ and $R_n^{(1)}(z)$ we find the rational solutions of equation (4.2) by means of formulae (1.10).

$$w(z; -1) = \frac{1}{z}, \quad w(z; 3) = \frac{3}{z}, \quad w(z; -4) = \frac{4}{z},$$

$$w(z; 6) = -\frac{6(z^5 + 336)}{z(z^5 - 504)}.$$
Table 4.2: Polynomials $R_n^{(1)}(z)$

\[
\begin{align*}
R_0^{(1)} &= 1, \\
R_1^{(1)} &= z^2, \\
R_2^{(1)} &= z^5 + 36, \\
R_3^{(1)} &= (z^5 - 144)^2, \\
R_4^{(1)} &= z(z^{15} - 1152z^{10} + 1824768z^5 + 131383296), \\
R_5^{(1)} &= z^4(z^{10} - 3168z^5 - 3193344)^2, \\
R_6^{(1)} &= z^8(z^{25} - 15840z^{20} + 63866880z^{15} + 708155965440z^{10} + 192217762806726656), \\
R_7^{(1)} &= z^4(z^{20} - 22176z^{15} - 95001984z^{10} - 902898855936z^5 + 303374015594496)^2, \\
R_8^{(1)} &= z(z^{55} - 88704z^{50} + 1900039680z^{45} + 21067639971840z^{40} + 1029196347904327680z^{35} - 3888565614158008025088z^{30} + 119982378242306659615506432z^{25} + 745508187107335699834161070080z^{20} - 865913009382826100820052971356160z^{15} + 1723009449942845244977210857913057280z^{10} + 603307224662092676093915178711788814336z^5 - 14479373391890224226253964289082931544064), \\
R_9^{(1)} &= z(z^{35} - 94248z^{30} - 95001984z^{25} - 60569464919040z^{20} - 1037918350852669440z^{15} + 4274562936151633821696z^{10} + 6463139159461270338404352z^5 + 310230679654140976243408896)^2 \end{align*}
\]

We have obtained solutions (4.3) using the special polynomials $Q_n^{(1)}(z)$ from table (4.1) and we find solutions (4.4) taking into account the special polynomials $R_n^{(1)}(z)$ from table (5.2).

\[
\begin{align*}
w(z; 2) &= -\frac{2}{z}, \\
w(z; -3) &= \frac{3(z^5 - 24)}{z(z^5 + 36)}, \\
w(z; 5) &= -\frac{5z^4(z^5 + 216)}{(z^5 + 36)(z^5 - 144)}, \\
w(z; -6) &= \frac{6(z^{20} - 576z^{15} - 912384z^{10} - 459841536z^5 - 3153199104)}{z(z^5 - 144)(z^{15} - 1152z^{10} + 1824768z^5 + 131383296)} \tag{4.4}
\end{align*}
\]
5  Special polynomials associated with the second member of hierarchy \((1.1)\)

From \((1.2)\) and \((1.7)\) we have

\[ H_2[v] = v_{zzzz} + 12 v v_{zz} + 6 v_z^2 + \frac{32}{3} v^3, \]  \hspace{1cm} (5.1)

\[ G_2[u] = u_{zzzz} + \frac{3}{2} u u_{zz} + \frac{3}{4} u_z^2 + \frac{1}{6} u^3; \] \hspace{1cm} (5.2)

Substituting (5.1) and (5.2) into (1.1) and (1.6) at \(n = 2\) yields the sixth - order equation

\[ w_{zzzzzz} + 7 w_z w_{zzzz} - 20 w_z^2 w_{zz} - 21 w w_{zzz}^2 + 14 w_z w w_{zzzz} - 7 w^3 w_{zzzz} - \frac{28}{3} w w_z^2 + 14 w^4 w_{zz} + 28 w^3 w_z^2 - 28 w w_z w w_{zzzz} - 14 w^2 w_z w_{zz} - \frac{4}{3} w^7 - z w - \beta_2 = 0 \] \hspace{1cm} (5.3)

Taking \(Q^{(2)}_0(z) = R^{(2)}_0 = 1, \ Q^{(2)}_1(z) = z, \ R^{(2)}_1 = z^2\) and using the differential - difference hierarchies (3.37), (3.38), (3.39) and (3.40) we get the special
polynomials $Q^{(2)}_n(z)$ and $R^{(2)}_n(z)$ at $n \geq 2$. The first nine special polynomials are given in Tables (5.1) and (5.2).

In Figure 3 and Figure 4 the locations of the roots for the polynomials $Q^{(2)}_{11}(z) = 0$ and $R^{(2)}_{11}(z) = 0$ are plotted. From these plots we observe that the roots of the polynomials form approximately regular heptagon.

Using the special polynomials $Q^{(2)}_n(z)$ and $R^{(2)}_n(z)$ we get the rational solutions of equation (5.3) in the form

$$w(z; -1) = \frac{1}{z}, \quad w(z; 3) = -\frac{3}{z}, \quad w(z; -4) = \frac{4 (z^7 + 1296)}{z (z^7 - 1728)},$$

$$w(z; 6) = -\frac{6 (z^{14} - 9504 z^7 + 1244160)}{(z^7 - 1728) z (z^7 + 4320)},$$

$$w(z; -7) = \frac{7 (z^{21} + 12960 z^{14} + 2799360000 z^7 - 15721205760000) z^6}{(z^7 + 4320) (z^{21} + 43200 z^{14} - 24261120000 z^7 + 2620200960000)},$$

Figure 2: Roots of polynomial $R^{(1)}_{11}(z) = 0$
Table 5.1: Polynomials $Q^{(2)}_n(z)$

\[
\begin{align*}
Q^{(2)}_0 &= 1, \\
Q^{(2)}_1 &= z, \\
Q^{(2)}_2 &= z^4, \\
Q^{(2)}_3 &= z (z^7 - 1728), \\
Q^{(2)}_4 &= (z^7 + 4320)^2, \\
Q^{(2)}_5 &= z^{21} + 43200 z^{14} - 2426112000 z^7 + 2620209600000, \\
Q^{(2)}_6 &= z^2 (z^{14} + 280800 z^7 + 4852240000)^2, \\
Q^{(2)}_7 &= z^5 (z^{35} + 1684800 z^{28} + 1610938368000 z^{21} - 6322020876288000000 z^{14} - \ldots - 1652309762644377600000000 z^7 + 8039191269849484492800000000), \\
Q^{(2)}_8 &= z^{10} (z^{21} + 3369600 z^{14} - 2396998656000 z^7 + 4556215045324800000000), \\
Q^{(2)}_9 &= z^{16} (z^{49} + 15724800 z^{42} + 67115962368000 z^{35} - 3421303297671168000000 z^{28} - \ldots - 136126583654974488576000000 z^{21} - 10091226998243644807525171200000000 z^{14} + \ldots + 1218293191730724544899136238911488000000000000).
\end{align*}
\]

\[
\begin{align*}
w(z; 2) &= -\frac{2}{z}, & w(z; -3) &= \frac{3}{z}, & w(z; 5) &= -\frac{5}{z}, \\
w(z; -6) &= \frac{6}{z}, & w(z; 8) &= -\frac{8 (z^7 - 71280)}{z (z^7 + 95040)}.
\end{align*}
\]

\(w(z; 2) = \frac{2}{z}, \quad w(z; 3) = \frac{5}{z}, \quad w(z; 5) = -\frac{5}{z}, \quad w(z; 6) = \frac{6}{z}, \quad w(z; 8) = \frac{8 (z^7 - 71280)}{z (z^7 + 95040)}
\]

We have found solutions (5.4) using the special polynomials $Q^{(2)}_n(z)$ from table (5.1) and we obtained solutions (5.5) taking into account the special polynomials $R^{(2)}_n(z)$ from table (5.2).

6 Special polynomials associated with the third member of hierarchy (1.1)

Using (1.3) and (1.8) we have from (1.2) and (1.7)

\[
H_3[v] = v_{zzzzzzzz} + 20 v v_{zzzzzz} + 60 v_z v_{zzzz} + 134 v_{zz} v_{zzzz} + 136 v^2 v_{zzzz} + 84 v_z^2 v_{zz} + 544 v v_z v_{zz} + 408 v v_z^2 v_z + 396 v_z^2 v_z + \frac{1120}{3} v^3 v_{zz} + 560 v^2 v_z^2 + \frac{256}{3} v^5.
\]
Table 5.2: Polynomials $R_n^{(2)}(z)$

$R_0^{(2)} = 1$,  
$R_1^{(2)} = z^2$,  
$R_2^{(2)} = z^5$,  
$R_3^{(2)} = z^{10}$,  
$R_4^{(2)} = z^{16}$,  
$R_5^{(2)} = z^{10}(z^7 + 95040)^2$,  
$R_6^{(2)} = z^8(z^{25} - 15840z^{20} + 6386680z^{15} + 70815965440z^{10} + 192217762806726656)$,  
$R_7^{(2)} = z^{21}(z^{21} + 1615680z^{14} - 383885568000z^7 + 33167713075200000)^2$,  
$R_8^{(2)} = z^{56} + 8078400z^{49} + 21497591808000z^{42} - 3946957855948800000z^{35} + 3188510434349678592000000z^{28} - 9839758265150932098780000000000z^{21} + 30109660107783765528202444800000000000z^{14} + 81760631332679116451438867251200000000000z^7 + 4415074091964672288377698315648000000000000000$,  
$R_9^{(2)} = (z^{35} + 12117600z^{28} - 35317472256000z^{21} + 97264318593024000000z^{14} + 8011467393870200832000000z^7 − 173047695707596337971200000000000)^2$.

$G_3[u] = u u u u u + \frac{7}{2} u u u u + \frac{21}{2} u u u u u + \frac{37}{2} u u u u + 4 u^2 u u u + \frac{25}{4} u u u u u u + 12 u u^2 u + \frac{33}{2} u^2 u^2 + \frac{25}{12} u^3 u z + \frac{25}{8} u^2 u z^2 + \frac{1}{12} u^5$ \hspace{1cm} (6.2)

Substituting (6.1) and (6.2) into equations (1.1) and (1.6) we find the tenth - order equation at $N = 3$ (This equation is presented in the appendix A).

Assuming $Q_0^{(3)}(z) = R_0^{(3)} = 1$, $Q_1^{(3)} = z$, $R_1^{(3)} = z^2$ and using the differential - difference hierarchy (3.37), (3.38), (3.39) and (3.40) we get the special polynomials $Q_n^{(3)}(z)$ and $R_n^{(3)}(z)$ at $n \geq 2$. The first nine special polynomials are given in Tables (6.1) and (6.2).

In Figure 5 and Figure 6 the locations of the roots for the polynomials $Q_{11}^{(3)}(z) = 0$ and $R_{11}^{(3)}(z) = 0$ are plotted. From these plots we observe that the roots of the polynomials form approximately regular polygons with eleven angles. Studying other plots we observe the symmetric plots for other roots of the polynomials $Q_n^{(3)}(z) = 0$ and $R_n^{(3)}(z) = 0$.  

17
Figure 3: Roots of polynomial $Q^{(2)}_{11}(z) = 0$

Using the special polynomials $Q^{(3)}_n(z)$ and $R^{(3)}_n(z)$ we get the rational solutions $w(z; \beta)$ of the tenth-order equation in the form

$$
\begin{align*}
  w(z; -1) &= \frac{1}{z}, & w(z; 3) &= -\frac{3}{z}, & w(z; -4) &= \frac{4}{z}, & w(z; 6) &= -\frac{6}{z} \\
  w(z; -7) &= \frac{7}{z}, & w(z; 9) &= -\frac{9}{z}, & w(z; -10) &= \frac{10}{z}, & w(z; 12) &= -\frac{12(z^{11} - 17926272000)}{z(z^{11} + 21511526400)},
\end{align*}
$$

(6.3)
Figure 4: Roots of polynomial $R_{11}^{(2)}(z) = 0$

\[ w(z; 2) = -\frac{2}{z}, \quad w(z; -3) = \frac{3}{z}, \quad w(z; 5) = -\frac{5}{z}, \]

\[ w(z; -6) = \frac{6(z^{11} - 20736000)}{z(z^{11} + 24883200)}, \quad (6.4) \]

\[ w(z; 8) = -\frac{8(z^{22} + 135302400z^{11} + 406332702720000)}{z(z^{11} + 24883200)(z^{11} - 43545600)} \]

We have got solutions (6.3) using the special polynomials $Q_n^{(3)}(z)$ from table (6.1) and we found solutions (6.4) taking into account the special polynomials $R_n^{(3)}(z)$ from table (6.2).

7 Conclusion

We have introduced the special polynomials associated with the rational solutions of hierarchy (1.1). This hierarchy arises if we look for the special solutions taking into account the scaling reductions for the Fordy - Gibbons
Table 6.1: Polynomials $Q^{(3)}_n(z)$

| $Q^{(3)}_0$ | 1 |
|-------------|---|
| $Q^{(3)}_1$ | $z$ |
| $Q^{(3)}_2$ | $z^4$ |
| $Q^{(3)}_3$ | $z^8$ |
| $Q^{(3)}_4$ | $z^{14}$ |
| $Q^{(3)}_5$ | $z^{21}$ |
| $Q^{(3)}_6$ | $z^{30}$ |
| $Q^{(3)}_7$ | $z^{40}$ |
| $Q^{(3)}_8$ | $z^{30} \left(z^{11} + 21511526400\right)^2$ |
| $Q^{(3)}_9$ | $z^{21} \left(z^{44} + 153653760000 \cdot z^{33} + 231372884028948000000 \cdot z^{22} - 124429597575821589793996800000000 \cdot z^{11} - 281754797178280224267740286812160000000000 \right)$ |

We hope that the recursion relations (3.1), (3.2), (3.3), (3.4), (3.20), (3.21), (3.22), (3.23), and the differential - difference hierarchies (3.37), (3.38), (3.39) and (3.40) will be useful in proving various properties for the special polynomials associated with hierarchy (1.1).

8 Acknowledgments

This work was supported by the International Science and Technology Center under Project B 1213.
Table 6.2: Polynomials $R_n^{(3)} (z)$

\[
R_0^{(3)} = 1, \\
R_1^{(3)} = z^2, \\
R_2^{(3)} = z^5, \\
R_3^{(3)} = z^{10}, \\
R_4^{(3)} = z^{16}, \\
R_5^{(3)} = z^2 \left( z^{11} - 43545600 \right)^2, \\
R_6^{(3)} = z^{33} - 609638400 z^{22} - 90260037697536000 z^{11} - 49130343719522795520000, \\
R_7^{(3)} = (z^{22} + 518192640 z^{11} - 112825047121920000)^2, \\
R_8^{(3)} = z \left( z^{55} + 41455411200 z^{44} + 4280582287805644800000 z^{33} - 44082692205779023508275200000000 z^{22} - 104521852719951396510535778304000000000 z^{11} - 758577798300375451931486464619110400000000000000 \right), \\
R_9^{(3)} = z^4 \left( z^{33} + 207277056000 z^{22} - 4632032309590425600000 z^{11} + 437026689971085146849280000000000000 \right)^2
\]

References

[1] N.A. Kudryashov, Two hierarchies of ordinary differential equations and their properties, Phys. Lett. A 252 (1999) 173 - 179

[2] N.A. Kudryashov, Double Backlund transformations and special integrals for the K II hierarchy, Phys. Lett. A 273 (2000)194 - 202

[3] N.A. Kudryashov, Analytical theory of nonlinear differential equations, Institute of Computer Investigations, Moscow-Izhevsk, (2004), 360 p. (in Russian)

[4] J. Weiss, On classes of integrable systems and the painleve property, J. Math. Phys., 25(1) (1984) 13-24

[5] A.P. Fordy, J. Gibbons, Some remarkable nonlinear transformations, Phys. Lett. A 75, (1980), 325-325

[6] P.J. Caudrey, R.K. Dodd, J.D. Gibbon, Proc. Roy Sci. Lond. A 351, 1 (1976), 407

[7] K. Sawada, T. Kotera, A method for finding N-soliton of the KdV equation and the KdV - like equation, Prog. Theor. Ohys., 51 (1974), 1355 - 1367
Figure 5: Roots of polynomial $Q_{11}^{(3)}(z)$

[8] *D.J. Kaup*, On the inverse scattering problem for cubic eigenvalue problems of the class $\Psi_{xxx} + 6Q\Psi_x + 6R\Psi = 0$, Stud. Appl. Math., 62 (1980), 189 - 216

[9] *N.A. Kudryashov*, The first Painleve and the second Painleve equations of higher order and some relations between them, Physics Letters A, v.224, (1997), 353 - 360

[10] *P.A. Clarkson, E.L. Mansfield*, The second Painleve equation, its hierarchy and associated special polynomials, Nonlinearity 16 (2003) R1

[11] *M.V. Demina, N.A. Kudryashov*, The Yablonskii - Vorob’ev polynomials for the second Painleve hierarchy, Chaos, Solitons and Fractals, 32(2), (2007), 526 -537

[12] *N.A. Kudryashov, M.V. Demina*, Special polynomials associated with the fourth order analogue to the Painlevé equations, Physics Letters A, (2007) doi:10.1016/j.physleta.2006.10.102

[13] *M.V. Demina, N.A. Kudryashov*, Power and non - power expansions of the solutions for the fourth - order analogue to the second Painleve equation, Chaos, Solitons and Fractals, 32(1), (2007), 124 -144
Figure 6: Roots of polynomial $R_{11}^{(3)}(z)$

[14] N.A. Kudryashov, O.Yu. Efimova, Power expansions for solution of the fourth - order analog to the first Painleve equation, Chaos, Solitons and Fractals, 30(1), (2006), 110 -124

[15] Andrew N.W. Hone, Non - autonomous Henon - Heiles systems, Physica D 1998; 118: 1 - 16

[16] N.A. Kudryashov, Transcendents defined by nonlinear fourth - order ordinary differential equations, J. Phys. A.: Math. Gen. 32 (1999) 999 -1013

[17] F. Jrad, U. Mugan, Non - polynomial fourth order equations which pass the Painleve test, Zeitschrift fur Naturforschung A, 60A (2005), 387 - 400

[18] V.I. Gromak, On the fourth - order nonlinear differential equations with the Painleve property, Differential equations, 42(8) (2006), 1017 - 1026

[19] C.M. Cosgrove, Higher - order Painleve equations in the Polynomial class II: Bureau Symbol P1, Studies in Applied Mathematics, 116, (2006), 321 - 413
[20] A.I. Yablonskii, On rational solutions of the second Painleve equations, Vesti Akad. Nauk BSSR, Ser. Fiz. Tkh. Nauk, 3 (1959), 30 - 35 (in Russian)

[21] A.P. Vorob’ev, On rational solutions of the second Painleve equations, Differential equations 1 (1965) 79 - 81 (in Russian)

[22] P.A. Clarkson, Remarks on the Yablonskii - Vorob’ev Polynomials, Physics Letters A, 319, (2003) 137 - 144

[23] P.A. Clarkson, Painleve equations - Nonlinear Special Functions, (2004) Report of IMC, 15 december 2004

[24] P.A. Clarkson, The third Painleve equation and associated special polynomials, J. Phys. A: Math. Gen. (2003) 9507 - 9532

[25] K. Okamoto, Studies on the Painleve equations III, Math. Ann. 275 (1986) 221 - 255

[26] H. Umemura, H. Watanabe, Solutions of the second and fourth Painleve equations, I, Nagoya Math. J. Vol. 148 (1997) 151 - 198
A  The third member of hierarchy (A.1)

\[ w_{10z} + 11 w_z w_{8z} - 11 w^2 w_{8z} + 44 w_{2z} w_{7z} - 88 w_z w_{7z} + \]
\[ +110 w_{3z} w_{6z} - 242 w w_{2z} w_{6z} - 66 w^2 w_z w_{6z} - 198 w_z^2 w_{6z} + \]
\[ +44 w^4 w_{6z} + 176 w_{4z} w_{5z} - 418 w w_{3z} w_{5z} - 990 w_z w_{2z} w_{5z} - \]
\[ -188 w^2 w_{2z} w_{5z} - 396 w w_z^2 w_{5z} + 528 w^3 w_z w_{5z} - 253 w w^4 w_{5z} - \]
\[ -330 w^2 w_{3z} w_{4z} - 1518 w_z w_{3z} w_{4z} - 1166 w_{2z}^2 w_{4z} + 1100 w^3 w_{2z} w_{4z} - \]
\[ -1540 w w_z w_{2z} w_{4z} - \frac{2090}{3} w_z^3 w_{4z} + 2310 w^2 w_z^2 w_{4z} + 110 w^4 w_z w_{4z} - \]
\[ -\frac{242}{3} w^6 w_{4z} - 1540 w_{2z} w_{3z}^2 - 880 w w_z w_{3z}^2 + 660 w^3 w_{3z}^2 - \]
\[ -1100 w w_{2z}^2 w_{3z} - 3300 w_z^2 w_{2z} w_{3z} + 220 w^4 w_{2z} w_{3z} + 880 w^3 w_z^2 w_{3z} + \]
\[ +7920 w^2 w_z w_{2z} w_{3z} + 3960 w w_z^3 w_{3z} - 968 w^5 w_z w_{3z} - 1430 w_z w_{2z}^3 + \]
\[ +1870 w^2 w_{2z}^3 + 1320 w^3 w_z w_{2z}^2 + 8250 w w_z^2 w_{2z}^2 - 726 w^5 w_{2z}^2 - \]
\[ -4620 w^4 w_{2z} w_z^2 + \frac{9460}{3} w z w_z^3 w_{2z} - \frac{220}{3} w^6 w_z w_{2z} + \frac{7040}{3} w_z w_{2z}^4 + \]
\[ + \frac{220}{3} w^8 w_{2z} + \frac{2200}{3} w w_z^5 - \frac{440}{3} w^5 w_z^3 + \frac{880}{3} w^7 w_z^2 - \]
\[ -2200 w^3 w_z^4 - \frac{8}{3} w^{11} - z w - \beta_3 = 0, \quad w_z = \frac{d^n w}{dz^n} \]