Scattering by a long-range potential

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The phenomenon of wave tails has attracted much attention over the years from both physicists and mathematicians. However, our understanding of this fascinating phenomenon is not complete yet. In particular, most former studies of the tail phenomenon have focused on scattering potentials which approach zero asymptotically \((x \to \infty)\) faster than \(x^{-2}\). It is well-known that for these (rapidly decaying) scattering potentials the late-time tails are determined by the first Born approximation and are therefore linear in the amplitudes of the scattering potentials (there are, however, some exceptional cases in which the first Born approximation vanishes and one has to consider higher orders of the scattering problem). In the present study we analyze in detail the late-time dynamics of the Klein-Gordon wave equation with a \((\text{slowly decaying})\) Coulomb-like scattering potential: 

\[ V(x) = \alpha/x. \]

In particular, we write down an explicit solution (that is, an exact analytic solution which is not based on the first Born approximation) for this scattering problem. It is found that the asymptotic \((t \to \infty)\) late-time behavior of the fields depends non-linearly on the amplitude \(\alpha\) of the scattering potential. This non-linear dependence on the amplitude of the scattering potential reflects the fact that the late-time dynamics associated with this slowly decaying scattering potential is dominated by \textit{multiple} scattering from asymptotically far regions.

I. INTRODUCTION

It is well-known that waves and fields propagating under the influence of a scattering potential do not cut off sharply after the passage of the primary pulse. Instead, in the presence of a scattering potential, propagating waves tend to die off gently, leaving behind “tails” which decay to zero only asymptotically. The precise description of these late-time tails is an important subject in the scattering theory of waves and fields.

Late-time decaying tails arise naturally in a variety of physical situations. For instance, the seminal work of Price [1] has revealed that the late-time dynamics of massless fields propagating in (curved) black-hole spacetimes are characterized by inverse power-law decaying tails. The phenomenon of wave tails in curved spacetimes has since been studied by many researchers, see [2] and references therein.

The physically most interesting mechanism for the production of asymptotically \((t \to \infty)\) decaying tails is the backscattering of the waves by an effective scattering potential at asymptotically \((x \to \infty)\) far regions [1,3]. In particular, the propagation of waves and fields is often governed by a Klein-Gordon wave equation of the form

\[ \left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V(x) \right] \Psi(x,t) = 0, \]

where \(V(x)\) is an effective scattering potential [It is well-known that the wave equation \([4]\) with \(V \equiv 0\) describes a tail-free propagation of the fields].

In a brilliant work Ching et. al. [2] provided a simple and intuitive explanation for the tail phenomena: consider a wave from a source point \(y\) and an observer which is located at a fixed spatial point \(x_{\text{observer}}\). The asymptotic \((t \to \infty)\) late-time tail observed at the point \(x_{\text{observer}}\) is a consequence of the waves first propagating to a distant point \(x' > y, x_{\text{observer}}\), being backscattered by the scattering potential \(V(x')\) at \(x'\), and then returning to the location \(x_{\text{observer}}\) of the observer at a time \(t \approx (x' - y) + (x' - x_{\text{observer}}) \approx 2x'\). Note that according to this heuristic picture, the scattering amplitude is expected to be proportional to (linear in) \(V(x' \approx t/2)\).

Despite of the numerous works dedicated to the study of asymptotically decaying wave tails (see [2] and references therein), our understanding of this fascinating phenomenon is not complete yet. In particular, it is worth noting that most previous studies of the tail phenomenon have focused on scattering potentials which approach zero asymptotically \((x \to \infty)\) faster than \(x^{-2}\). It was shown in [2,6] that for these (rapidly decaying) scattering potentials the late-time tails can be determined by the first Born approximation. Accordingly, the leading-order late-time tails associated with these rapidly decaying scattering potentials are linear in the amplitudes of the potentials [2,7].

Before proceeding further, it is important to mention that there are some exceptional cases in which the first Born approximation vanishes and one has to consider higher orders of the scattering problem [2]. That happens for a 3+1 dimensions scattering potential of the form \(V(x) \sim x^{-\alpha}\), when \(\alpha\) is an odd integer in the range \(0 \leq \alpha - 3 < 2l\) and \(l\) is the multipole index of the mode. This exceptional behavior also happens for the spherically symmetric mode in odd 2\(l + 3 \geq 5\) spatial dimensions. (In fact, it can be shown that the wave equation for the \(l\)-th multipole in 3+1 dimensions and the spherically symmetric wave equation in odd \(d = 2l + 3\) spatial dimensions are two faces of the \textit{same} mathematical problem.)
It is worth emphasizing that much less is known about the tail phenomenon associated with slowly decaying scattering potentials – scattering potentials which approach zero asymptotically \((x \to \infty)\) slower than \(x^{-2}\). In this respect it is worth mentioning the work of Ching et. al. [5], who analyzed the particular case of an inverse square law scattering (repulsive) potential of the form
\[
V(x) = \frac{\nu}{x^2} , \tag{2}
\]
with \(\nu > 0\). This scattering potential is actually the boundary between the well-studied family of rapidly-decaying potentials (scattering potentials which approach zero asymptotically faster than \(x^{-2}\)) [3, 8] and the poorly-explored family of slowly-decaying potentials (scattering potentials which approach zero asymptotically slower than \(x^{-2}\)).

It was found in [3] that the ‘marginally’-decaying [8] scattering potential [2] produces a late-time tail of the form [9]
\[
\Psi(t \gg x) \sim A(\nu)t^{-(2\nu+2)} . \tag{3}
\]
The temporal dependence [3] of the late-time tail is obviously not linear in the scattering potential [2] [In particular, the amplitude \(A(\nu)\) of the tail [3] was found in [2] to depend non-linearly on the amplitude \(\nu(\nu + 1)\) of the scattering potential [2]]. This implies that, for this marginally-decaying scattering potential, the first Born approximation fails to describe the correct late-time behavior of the fields.

To the best of our knowledge, there are no results in the literature for the tail phenomenon associated with slowly-decaying scattering potentials – long-range potentials that approach zero asymptotically \((x \to \infty)\) slower than \(x^{-2}\).

The main goal of the present study is to extend our knowledge about the tail phenomenon for slowly decaying scattering potentials. As explained above, the results of [3] for the marginally-decaying case [2] indicate that the first Born approximation may fail to describe the correct late-time tails associated with slowly decaying scattering potentials. Thus, the analytical approximations used in [2, 8] to analyze the late-time tails associated with rapidly-decaying potentials [10] may not be valid in the regime of the slowly-decaying scattering potentials.

One is therefore forced to go beyond the first Born approximation when studying the late-time wave dynamics associated with slowly decaying potentials. This fact prevents one from reaching general conclusions about these slowly-decaying late-time tails [11]. Instead, we are forced to solve explicitly \(\text{case-by-case}\) the Klein-Gordon wave equation [13], specifying in each case the explicit form of the slowly-decaying scattering potential \(V(x)\).

In the present study we shall analyze the late-time dynamics of the Klein-Gordon wave equation [13] with a slowly-decaying Coulomb-like scattering potential
\[
V(x) = \frac{\alpha}{x} , \tag{4}
\]
where \(x \in [0, \infty]\). Here \(\alpha > 0\) is the amplitude of the scattering potential [12]. Fortunately, as we shall show below, one can write down an \textit{explicit} solution (that is, an exact analytic solution which is not based on the first Born approximation) for the Klein-Gordon wave equation [13] with the Coulomb-like scattering potential [4]. This fact will allow us to study analytically the asymptotic late-time dynamics of the wave fields in the presence of this slowly-decaying scattering potential.

II. FORMALISM

The temporal evolution of a wave-field whose dynamics is governed by the Klein-Gordon wave equation [13] is given by [2, 12, 13]
\[
\Psi(x, t > 0) = \int [G(x, y; t)\Psi(y, 0) + \dot{G}(x, y; t)\Psi(y, 0)] dy , \tag{5}
\]
where the (retarded) Green’s function \(G(x, y; t)\) satisfies the relation
\[
\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V(x)\right]G(x, y; t) = \delta(t)\delta(x - y) . \tag{6}
\]
The causality requirement dictates the initial condition \(G(x, y; t < 0) = 0\). The problem now becomes to find the explicit form of the Green’s function \(G(x, y; t)\); the temporal evolution of the fields for any initial data can then be obtained by performing the spatial integration in the characteristic equation [3].

In order to find the Green’s function \(G(x, y; t)\) we first use the Fourier transform
\[
\check{G}(x, y; \omega) = \int_{0}^{\infty} G(x, y; t)e^{i\omega t} dt . \tag{7}
\]
The Fourier transform \(\check{G}(x, y; \omega)\) satisfies the Schrödinger-like wave equation [2, 13, 15]
\[
\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right]\check{G}(x, y; \omega) = -\delta(x - y) \tag{8}
\]
and is analytic in the upper half \(\omega\)-plane. The time-dependent Green’s function, \(G(x, y; t)\), can be obtained by the inversion integral [2, 13, 15]
\[
G(x, y; t) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \check{G}(x, y; \omega)e^{-\omega t} d\omega , \tag{9}
\]
where \(\epsilon\) is some positive constant.

The Green’s function \(\check{G}(x, y, \omega)\) can be expressed in terms of two linearly independent solutions of the homogeneous Schrödinger-like wave equation [2, 13, 15]
\[
\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right]\psi_i(x, \omega) = 0 \quad ; \quad i = 1, 2 . \tag{10}
\]
The two basic solutions \( \{ \bar{\psi}_1, \bar{\psi}_2 \} \) which are required in order to build the Green’s function \( \bar{G}(x, y; \omega) \) are defined by the two boundary conditions \([5, 13–15]\): the first solution \( \bar{\psi}_1(x; \omega) \) corresponds to a vanishing field at \( x = 0 \) where the scattering potential is infinitely repulsive, whereas the second solution \( \bar{\psi}_2(x; \omega) \) corresponds to purely outgoing waves at \( x \to \infty \) [see Eqs. (19) and (20) below].

Using the two characteristic solutions \( \{ \bar{\psi}_1, \bar{\psi}_2 \} \), the Green’s function \( \bar{G}(x, y; \omega) \) can be written in the form \([5, 14, 16]\)

\[
\bar{G}(x, y; \omega) = \frac{1}{W(\omega)} \left\{ \begin{array}{ll}
\bar{\psi}_1(x; \omega)\bar{\psi}_2(y; \omega) & \text{for } x < y ; \\
\bar{\psi}_1(y; \omega)\bar{\psi}_2(x; \omega) & \text{for } x > y ,
\end{array} \right.
\]

where

\[
W(\omega) = W(\bar{\psi}_1, \bar{\psi}_2) = \bar{\psi}_1\bar{\psi}_{2,x} - \bar{\psi}_1,x\bar{\psi}_2
\]  

is the \( x \)-independent Wronskian of the system.

It proofs useful to bend the integration contour in Eq. \([9]\) into the lower-half of the complex \( \omega \)-plane \([13–15]\). It was shown by Leaver \([14]\) that in the complex-frequency picture the late-time behavior of the field (the asymptotic tail) can be associated with the existence of a branch cut in the Green’s function \( \bar{G}(x, y; \omega) \) of the Klein-Gordon wave equation [see, in particular, Eq. \([25]\) below]. This cut is usually placed along the negative imaginary \( \omega \)-axis. The asymptotic late-time tail arises from the integral of \( \bar{G}(x, y; \omega) \) around the branch cut \([14]\). In particular, taking cognizance of Eqs. \([9] \) and \([11]\) one finds that the branch cut contribution to the Green’s function \( G(x, y; \omega) \) is given by \([5, 14, 16]\)

\[
G^C(x, y; \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{\bar{\psi}_1(y; \omega e^{2\pi i})\bar{\psi}_2(x; \omega e^{2\pi i})}{W(\omega e^{2\pi i})} - \frac{\bar{\psi}_1(y; \omega)\bar{\psi}_2(x; \omega)}{W(\omega)} \right] e^{-i\omega t} d\omega .
\]

### III. THE EXPLICIT (ANALYTIC) SOLUTION

We shall now evaluate the integral \([13]\) for the branch cut contribution to the Green’s function \( G(x, y; \omega) \). Substituting the slowly-decaying scattering potential \([41]\) into the characteristic equation \([10]\), one obtains the Schrödinger-like wave equation

\[
\left( \frac{d^2}{dx^2} + \omega^2 - \frac{\alpha}{x} \right) \bar{\psi}(x; \omega) = 0 .
\]

Defining

\[
z \equiv -2i\omega x ,
\]

we can write Eq. \([14]\) in the form

\[
\left( \frac{d^2}{dz^2} - \frac{1}{4} - \frac{i\alpha/2\omega}{z} \right) \bar{\psi}(z) = 0 .
\]

Equation \([10]\) is the familiar Whittaker differential equation \([17]\). Its two basic solutions which are required in order to build the Green’s function are (see Eqs. 13.3.2 and 13.3.33 of \([17]\))

\[
\bar{\psi}_1(z) = A e^{-\frac{1}{2}z} M(1 + i\alpha/2\omega, 2, z) \quad (17)
\]

and

\[
\bar{\psi}_2(z) = B e^{-\frac{1}{2}z} U(1 + i\alpha/2\omega, 2, z) ,
\]

where \( M(a, b, z) \) and \( U(a, b, z) \) are the confluent hypergeometric functions \([17]\), and \( A \) and \( B \) are normalization constants.

Using equations 13.5.5 and 13.1.8 of \([17]\), one finds the asymptotic behaviors

\[
\bar{\psi}_1(x; \omega) \sim x \quad \text{as} \quad x \to 0 ,
\]

and

\[
\bar{\psi}_2(x; \omega) \sim -i\alpha/2\omega e^{i\omega x} \quad \text{as} \quad x \to \infty .
\]

Thus, the characteristic solution \( \bar{\psi}_1(x; \omega) \) describes a vanishing field at \( x = 0 \) where the scattering potential is infinitely repulsive, whereas the characteristic solution \( \bar{\psi}_2(x; \omega) \) describes outgoing waves at \( x \to \infty \).

In addition, using Eq. 13.1.22 of \([17]\), one finds that the \( x \)-independent Wronskian \( W(\bar{\psi}_1, \bar{\psi}_2) \) of the system is given by

\[
W(\omega) = \frac{2i\omega AB}{\Gamma(1 + i\alpha/2\omega)} .
\]

Taking cognizance of Eqs. \([21] \)–\([24]\) one finds \([18]\)

\[
\bar{\psi}_1(z e^{2\pi i}) = \bar{\psi}_1(z) .
\]

and (see Eq. 13.1.6 of \([17]\))

\[
\bar{\psi}_2(z e^{2\pi i}) = \bar{\psi}_2(z) + \frac{B}{A} \frac{2\pi i}{\Gamma(i\alpha/2\omega)} \bar{\psi}_1(z) .
\]

From Eqs. \([22]\) and \([23]\) one finds the simple relation

\[
W(\omega e^{2\pi i}) = W(\omega) .
\]

For the time-dependent Green’s function, where \( \omega \equiv i\omega \).

It is worth emphasizing that the expression \([26]\) for the branch cut contribution to the Green’s function is exact for all times. We shall now focus on the asymptotic \((t \to \infty)\) late-time behavior of the scattered fields.
IV. THE ASYMPTOTIC LATE-TIME TAIL

In the asymptotic \( t \to \infty \) limit the effective contribution to the integral in (26) comes from \( \omega \)-values in the regime \( \omega = O(1) \). This observation is attributed to the presence of the rapidly decreasing term \( e^{-\omega t} \) which, in the \( t \to \infty \) asymptotic limit, limits the effective contribution to the integral (26) to small \( \omega \)-values in the regime \( 0 \leq \omega t \lesssim 1 \), that is to small \( \omega \)-values with \( \omega_{\text{max}} = O(1) \).

This mathematical observation has a simple physical interpretation: it is well-known \[5, 6, 14, 15\] that the asymptotic late-time \(( t \to \infty )\) dynamics is determined by the backscattering of the field from asymptotically far \(( x \to \infty )\) regions. Thus, the late-time dynamics of the field is dominated by the low-frequency contribution to the Green’s function because only low frequencies can be backscattered by the small scattering potential at spatial infinity [remember that \( V(x \to \infty) = 0 \)].

Since the late-time asymptotic \(( t \gg x, y)\) behavior of the field is dominated by the low-frequency contribution to the Green’s function (frequencies with \( \omega x \ll 1 \) and \( \omega y \ll 1 \)), one may use Eq. 13.3.1 of \[17\] to write

\[
\tilde{\psi}_1(-2\omega x) \sim -2A\omega \sqrt{x/\alpha} I_1(2\sqrt{\alpha x}),
\]

where \( I_1(z) \) is the modified Bessel function \[17\]. Substituting (27) into (26), one obtains

\[
G_C(x, y; t) = -\sqrt{xy} I_1(2\sqrt{\alpha x}) I_1(2\sqrt{\alpha y}) \int_0^\infty e^{-\omega t} d\omega .
\]

Performing the integration in (28), we finally find

\[
G_C(x, y; t \to \infty) = -\sqrt{xy} I_1(2\sqrt{\alpha x}) I_1(2\sqrt{\alpha y}) \frac{1}{t}.
\]

for the asymptotic late-time behavior of the Green’s function \[19\]. The temporal evolution of the field, \( \Psi(x, t \to \infty) \), can no longer be obtained by substituting the expression (29) for the time-dependent Green’s function into the characteristic equation (19).

We learn from Eq. (29) that the asymptotic \(( t \gg x_{\text{observer}})\) behavior of the field is not linear in the amplitude \( \alpha \) of the Coulomb-like scattering potential \[4\]. Thus, the late-time behavior associated with the slowly decaying scattering potential \[4\] is determined by multiple scattering from asymptotically far regions. This confirms our previous expectation that, for slowly decaying scattering potentials (scattering potentials which approach zero slower than \( x^{-2} \)), the first Born approximation fails to describe the correct late-time behavior of the fields \[20\].

V. THE NEAR_REGION

The expression (29) for the asymptotic behavior of the time-dependent Green’s function can be simplified if we assume that the observer is situated in the near-region

\[
x_{\text{observer}} \ll \alpha^{-1}.
\]

In this regime one may use Eq. 9.6.7 of \[17\] to simplify the Green’s function (29):

\[
G_C(x, y; t \to \infty) = -xy \frac{\alpha}{t} \quad \text{for} \quad x \ll \alpha^{-1}.
\]

We learn from Eq. (31) that the asymptotic dynamics of the field in the region \( x_{\text{observer}} \ll \alpha^{-1} \) is determined by the first Born approximation. That is, in the regime (30) the leading-order late-time tail is linear in the amplitude of the scattering potential: \( \Psi(x, t \to \infty) \propto V(t/2) \) \[21\].

VI. SUMMARY

The time evolution of wave fields governed by the Klein-Gordon wave equation (1) with the (slowly-decaying) Coulomb-like scattering potential \[4\] was investigated. It was shown that one can write down an explicit solution (that is, an exact analytic solution which is not based on the first Born approximation) for this scattering problem. This fact allowed us to study analytically the asymptotic late-time dynamics of the fields.

The asymptotic \(( t \to \infty )\) behavior of the fields was found to depend non-linearly on the amplitude \( \alpha \) of the scattering potential \[4\]. This non-linear dependence on the amplitude of the scattering potential reflects the fact that the late-time dynamics of the wave equation (1) in the presence of the slowly-decaying scattering potential \[4\] is dominated by multiple scattering from asymptotically far regions.

Finally, we have shown that the late-time behavior of the fields in the near-region \( x_{\text{observer}} \ll \alpha^{-1} \) is determined by the first Born approximation. That is, in this regime the leading-order late-time tail is linear in the amplitude \( \alpha \) of the scattering potential.

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It is worth emphasizing that these analytical approximations are based on the first Born approximation \( \frac{\alpha}{2\omega} \ll 1 \) for the tail phenomenon which is valid for all scattering potentials that decay asymptotically faster than \( x^{-2} \). It is important to emphasize that there are exceptional cases in which the first Born approximation vanishes \( \frac{\alpha}{2\omega} \). In these cases one has to consider higher orders of the scattering problem. That happens for a 3+1 dimensions scattering potential of the form \( V(x) \sim x^{-\alpha} \), where \( \alpha \) is an odd integer in the range \( 0 \leq \alpha < 2l \) and \( l \) is the multipole index of the mode. This exceptional behavior also happens for the spherically symmetric mode in odd \( 2l + 3 \geq 5 \) spatial dimensions. Unfortunately, such a generic analysis cannot be provided for slowly-decaying scattering potentials; for this family of scattering potentials one is forced to go beyond the first Born approximation when studying the scattering of the waves.

Note that the amplitude \( \alpha \) has the dimensions of length\(^{-1}\).

We have not found in the physical (nor in the mathematical) literature the sub-leading correction terms to Eq. 13.3.1 of [17] for the asymptotic form of \( M(a, b, \bar{z}/a) \) with \( \alpha \rightarrow \infty \). However, numerically we find that for \( b = 2 \) the leading-order correction to the formula 13.3.1 of [17] is well described by a factor \( (1 + \beta \cdot \bar{z}/a) \) on the r.h.s of this equation with \( \beta \sim 0.36 \). Thus, an improved approximation for \( \psi_1(-2\pi x) \) can be written as [see Eqs. (15), (17), and 13.3.1 of [17] ]:

\[
\psi_1(-2\pi x) \sim -2\pi x e^{\bar{x}x}(\bar{x}x - 2\pi x)^{-1/2} f_1(2\bar{x}x - 2\pi x)(1 - 2\bar{x}x).
\]

Expanding this expression to second order in \( \bar{x} \) one finds:

\[
\psi_1(-2\pi x) \sim -2\pi x \sqrt{\bar{x}x} f_1(2\sqrt{\bar{x}x} - 2\pi x)(1 - 2\sqrt{\bar{x}x}).
\]

where \( f(x; \omega) \equiv x(1 - \beta^2) - 2\sqrt{x} \bar{x}/f_1(2\sqrt{\bar{x}x})/f_1(2\sqrt{\bar{x}x}). \) (Here we have used Eq. 9.6.28 of [17].) Thus, the leading-order correction to Eq. (29) is described by the factor \( 1 + \omega f(x; \omega) + f(y; \omega) \) on the r.h.s of this equation. Performing the integration in (28), one finds that the leading-order correction to the Green’s function (29) is described by the factor \( 1 + f(x; \omega) + f(y; \omega) \) on the r.h.s of this equation. As expected, this correction factor approaches 1 in the asymptotic late-time limit \( t \rightarrow \infty \).

It is worth emphasizing again that in the framework of the first Born approximation, the amplitude to be backscattered, and thus also the late-time tail itself, are linear in the effective scattering potential.

Using Eq. 9.6.10 of [17], one finds that the leading-order correction to Eq. (61) is given by a factor \( 1 + a^2(x^2 + y^2)/2 \) on the r.h.s of this equation. Note that this correction factor is non-linear in the amplitude \( \alpha \) of the scattering potential and, as expected, it approaches 1 in the regime \( \omega a_0 \rightarrow \infty \).