ON THE MAXIMAL OPERATORS OF VILENKIN-FEJÉR MEANS ON HARDY SPACES

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ABSTRACT. The main aim of this paper is to prove that when $0 < p < 1/2$ the maximal operator $\sigma^*_w f := \sup_{n \in \mathbb{N}} |\sigma_n^w f| \log^2 (n + 1)$ is bounded from the martingale Hardy space $H_p$ to the space $L_p$, where $\sigma_n$ is $n$-th Fejér mean with respect to bounded Vilenkin system.

2000 Mathematics Subject Classification. 42C10.

Key words and phrases: Vilenkin system, Fejér means, martingale Hardy space.

1. INTRODUCTION

In one-dimensional case the weak type inequality for maximal operator of Fejér means for trigonometric system can be found in Zygmund [21], in Schipp [11] for Walsh system and in Pál, Simon [10] for bounded Vilenkin system. Fujii [4] and Simon [13] verified that the $\sigma^*_w$ is bounded from $H_1$ to $L_1$, where $\sigma^*_w$ denotes the maximal operator of Fejér means of Walsh-Fourier series. Weisz [18] generalized this result and proved the boundedness of $\sigma^*_w$ from the martingale Hardy space $H_p$ to the space $L_p$, for $1/2 < p \leq 1$.

Simon [12] gave a counterexample, which shows that boundedness of $\sigma^*_w$ does not hold for $0 < p < 1/2$. The counterexample for $\sigma^*_w$ when $p = 1/2$ is due to Goginava [7] (see also [3, 14]). In the endpoint case $p = 1/2$ two positive results were showed. Weisz [20] proved that $\sigma^*_w$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2, \infty}$. Goginava [6] proved that the maximal operator $\tilde{\sigma}^*_w$ defined by

$$\tilde{\sigma}^*_w f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n^w f|}{\log^2 (n + 1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, where $\sigma_n^w$ is $n$-th Fejér means of Walsh-Fourier series. He also proved, that for any nondecreasing function $\varphi : \mathbb{N} \to [1, \infty)$, satisfying the condition

$$\lim_{n \to \infty} \log^2 (n + 1) \frac{\log^2 (n + 1)}{\varphi (n)} = +\infty,$$

the maximal operator

$$\sup_{n \in \mathbb{N}} \frac{|\sigma_n^w f|}{\varphi (n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. 

For Walsh-Kaczmarz system analogical theorem was proved in [9] and for bounded Vilenkin system in [15].

The main aim of this paper is to prove that when $0 < p < 1/2$ the maximal operator

$$\sigma^*_p f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2}}$$

is bounded from the Hardy space $H_p$ to the space $L_p$ (see Theorem 1), where $\sigma_n$ is Fejér means of bounded Vilenkin-Fourier series.

We also prove that for any nondecreasing function $\varphi : \mathbb{N} \to [1, \infty)$, satisfying the condition

$$\lim_{n \to \infty} \frac{(n + 1)^{1/p - 2}}{\varphi(n)} = +\infty,$$

the maximal operator

$$\sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_p$ to the space $L_{p,\infty}$ when $0 < p < 1/2$. Actually, we prove a stronger result (see Theorem 2) than the unboundedness of the maximal operator $\sigma^*_p$ from the Hardy space $H_p$ to the spaces $L_{p,\infty}$.

In particular, we prove that under condition (2) there exists a martingale $f \in H_p$ ($0 < p < 1/2$) such that

$$\sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\varphi(n)}_{L_{p,\infty}} = \infty.$$

2. Definitions and Notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, ..., m_n, ...)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, ..., m_k - 1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$, as the complete direct product of the group $Z_{m_j}$, with the product of the discrete topologies of $Z_{m_j}$'s.

The direct product $\mu$ of the measures

$$\mu_k (\{j\}) := 1/m_k, \quad (j \in Z_{m_k}),$$

is the Haar measure on $G_m$, with $\mu (G_m) = 1$.

If $\sup_n m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded, then $G_m$ is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.
The elements of $G_m$ represented by sequences

\[ x := (x_0, x_1, ..., x_j, ...) \quad (x_k \in \mathbb{Z}_{m_k}). \]

It is easy to give a base, for the neighborhood of $x \in G_m$:

\[ I_0(x) := G_m, \]

\[ I_n(x) := \{ y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1} \}, \quad (n \in \mathbb{N}). \]

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I}_n := G_m \setminus I_n$.

Let

\[ e_n := (0, 0, ..., 0, x_n = 1, 0, ...) \in G_m, \quad (n \in \mathbb{N}). \]

Denote

\[ I_N^{k,l} := \begin{cases} 
I_N(0, ..., 0, x_k \neq 0, 0, ..., 0, x_l \neq 0, x_{l+1}, ..., x_{N-1}, ...), \\
\text{where } x_i \in \mathbb{Z}_{m_i}, \quad i \geq l + 1, \text{ for } k < l < N, \\
I_N(0, ..., 0, x_k \neq 0, 0, ..., x_{N-1} = 0, x_N, ...), \\
\text{where } x_i \in \mathbb{Z}_{m_i}, \quad i \geq N, \text{ for } l = N.
\end{cases} \]

and

\[ I_N^{k,\alpha,l,\beta} := I_N(0, ..., 0, x_k = \alpha, 0, ..., 0, x_l = \beta, x_{l+1}, ..., x_{N-1}), \quad k < l < N, \]

where $x_i \in \mathbb{Z}_{m_i}, \quad i \geq l + 1$.

It is evident

\[ I_N^{k,l} = \bigcup_{\alpha=1}^{m_k-1} \bigcup_{l=1}^{m_l-1} I_N^{k,\alpha,l,\beta} \]

and

\[ \overline{I}_N = \left( \bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \bigcup \left( \bigcup_{k=1}^{N-1} I_N^{k,N} \right). \]

If we define the so-called generalized number system, based on $m$ in the following way:

\[ M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in \mathbb{N}) \]

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in \mathbb{Z}_{m_j} \quad (j \in \mathbb{N})$ and only a finite number of $n_j$’s differ from zero. Let $|n| := \max \{ j \in \mathbb{N}, n_j \neq 0 \}$.

It is easy to show that

\[ \sum_{A=0}^{l} M_A \leq c M_l. \]

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesque space.

Next, we introduce on $G_m$ an ortonormal system which is called the Vilenkin system.
At first define the complex valued function \( r_k(x) : G_m \to C \), the generalized Rademacher functions as
\[
r_k(x) := \exp \left( \frac{2\pi i x_k}{m_k} \right), \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}) .
\]
Now define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) as:
\[
\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).
\]
Specifically, we call this system the Walsh one if \( m \equiv 2 \).
The Vilenkin system is ortonormal and complete in \( L^2(G_m) \) [1, 16].
Now we introduce analogues of the usual definitions in Fourier-analysis.
If \( f \in L^1(G_m) \) we can establish the the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system \( \psi \) in the usual manner:
\[
\hat{f}(n) := \int_{G_m} f \psi_n d\mu, \quad (n \in \mathbb{N}),
\]
\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}^+, \ S_0 f := 0),
\]
\[
\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f, \quad (n \in \mathbb{N}^+),
\]
\[
D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}^+),
\]
\[
K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k, \quad (n \in \mathbb{N}^+).
\]
Recall that
\[
D_{M_n}(x) = \begin{cases} 
M_n, & \text{if} \quad x \in I_n, \\
0, & \text{if} \quad x \notin I_n.
\end{cases}
\]
It is well-known that
\[
\sup_n \int_{G_m} |K_n(x)| d\mu(x) \leq c < \infty,
\]
and
\[
n |K_n(x)| \leq c \sum_{A=0}^{\lfloor n \rfloor} M_A |K_{M_A}(x)|.
\]
The norm (or quasinorm) of the space \( L^p(G_m) \) is defined by
\[
\|f\|_p := \left( \int_{G_m} |f(x)|^p d\mu(x) \right)^{1/p}, \quad (0 < p < \infty).
\]
The space \( L^p,\infty(G_m) \) consists of all measurable functions \( f \), for which
\[ \|f\|_{L_p(G_m)} := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty. \]

The \( \sigma \)-algebra generated by the intervals \( \{I_n(x) : x \in G_m\} \) will be denoted by \( F_n \ (n \in \mathbb{N}) \). Denote by \( f = (f^{(n)}, n \in \mathbb{N}) \) a martingale with respect to \( F_n \ (n \in \mathbb{N}) \) (for details see e.g. [17]). The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|. \]

In case \( f \in L_1 \), the maximal functions are also be given by

\[ f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|. \]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H_p \ (G_m) \) consist of all martingales for which

\[ \|f\|_{H_p} := \|f^*\|_p < \infty. \]

If \( f \in L_1 \), then it is easy to show that the sequence \( (S_{M_n}(f) : n \in \mathbb{N}) \) is a martingale. If \( f = (f^{(n)}, n \in \mathbb{N}) \) is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

\[ \hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \overline{\Psi}_i(x) d\mu(x). \]

The Vilenkin-Fourier coefficients of \( f \in L_1 \) are the same as those of the martingale \( (S_{M_n}(f) : n \in \mathbb{N}) \) obtained from \( f \).

For the martingale \( f \) we consider maximal operators

\[ \sigma^*_f := \sup_{n \in \mathbb{N}} |\sigma_nf|, \]
\[ \tilde{\sigma}^*_f := \sup_{n \in \mathbb{N}} \frac{|\sigma_nf|}{\log^2 (n + 1)}, \]
\[ \tilde{\sigma}_p^* f := \sup_{n \in \mathbb{N}} \frac{|\sigma_nf|}{(n + 1)^{1/p - 2}}. \]

A bounded measurable function \( a \) is \( p \)-atom, if there exist interval \( I \), such that

\[ a) \int_I ad\mu = 0, \]
\[ b) \|a\|_\infty \leq \mu(I)^{-1/p}, \]
\[ c) \text{supp}(a) \subset I. \]
3. Formulation of Main Results

Theorem 1. Let $0 < p < 1/2$. Then the maximal operator $\tilde{\sigma}_p^*$ is bounded from the Hardy martingale space $H_p(G_m)$ to the space $L_p(G_m)$.

Theorem 2. Let $\varphi : \mathbb{N} \to [1, \infty)$ be a nondecreasing function, satisfying the condition

\begin{equation}
\lim_{n \to \infty} \left( \frac{n + 1}{\varphi(n)} \right)^{1/p - 2} = +\infty,
\end{equation}

then

$$
\sup_{n \in \mathbb{N}} \left\| \sigma_n f \right\|_{L_p, \infty} = \infty.
$$

4. Auxiliary Propositions

Lemma 1. \cite{19} Suppose that an operator $T$ is sublinear and for some $0 < p \leq 1$

$$
\int_{I} |Ta|^p d\mu \leq c_p < \infty,
$$

for every $p$-atom $a$, where $I$ denote the support of the atom. If $T$ is bounded from $L_\infty$ to $L_\infty$, then

$$
\|Tf\|_{L_p(G_m)} \leq c_p \|f\|_{H_p(G_m)}.
$$

Lemma 2. \cite{2, 8} Let $2 < A \in \mathbb{N}_+$, $k \leq s < A$ and $q_A = M_{2A} + M_{2A-2} + \ldots + M_2 + M_0$. Then

$$
q_{A-1} |K_{q_{A-1}}(x)| \geq \frac{M_{2k}M_{2s}}{4},
$$

for

\begin{align*}
x & \in I_{2A} (0, \ldots, x_{2k} \neq 0, 0, \ldots, 0, x_{2s} \neq 0, x_{2s+1} \ldots x_{2A-1}), \\
k & = 0, 1, \ldots, A - 3, \quad s = k + 2, k + 3, \ldots, A - 1.
\end{align*}

Lemma 3. \cite{5} Let $A > t$, $t, A \in \mathbb{N}$, $z \in I_t \setminus I_{t+1}$. Then

$$
K_{MA}(z) = \begin{cases} 
0, & \text{if } z - ze_t \notin I_A, \\
\frac{M_t}{1-tt(I_z)}, & \text{if } z - ze_t \in I_A.
\end{cases}
$$

Lemma 4. Let $x \in I_N^{k,l}$, $k = 0, \ldots, N - 1$, $l = k + 1, \ldots, N$. Then
\[
\int_{I_N} |K_n(x-t)| \, d\mu(t) \leq \frac{c M_l M_k}{M_N^2}, \quad \text{when } n \geq M_N.
\]

**Proof.** Let \( x \in I_N^{k,a,l,\beta} \). Then applying Lemma 3 we have

\[ K_{M_A}(x) = 0, \quad \text{when } A > l. \]

Let \( k < A \leq l \). Then we get

\[ |K_{M_A}(x)| = \frac{M_k}{|1 - r_k(x)|} \leq \frac{m_k M_k}{2\pi \sigma}. \]

Let \( x \in I_N^{k,l} \), for \( 0 \leq k < l \leq N - 1 \) and \( t \in I_N \). Since \( x - t \in I_N^{k,l} \) and \( n \geq M_N \), combining (3), (5), (8) and (10) we obtain

\[ n |K_n(x)| \leq c \sum_{A=0}^{l} M_A M_k \leq c M_k M_l \]

and

\[ \int_{I_N} |K_n(x-t)| \, d\mu(t) \leq \frac{c M_k M_l}{M_N^2}. \]

Let \( x \in I_N^{k,N} \), then applying (8) we have

\[ \int_{I_N} n |K_n(x-t)| \, d\mu(t) \]

\[ \leq \sum_{|A|} M_A \int_{I_N} |K_{M_A}(x-t)| \, d\mu(t). \]

Let

\[
\begin{align*}
\{ & x = (0, \ldots, 0, x_k \neq 0, 0, \ldots, 0, x_N, x_{N+1}, x_q, \ldots, x_{|n|-1}, \ldots), \\
& t = (0, \ldots, 0, x_N, \ldots, x_{q-1}, t_q \neq x_q, t_{q+1}, \ldots, t_{|n|-1}, \ldots), \quad q = N, \ldots, |n| - 1.
\end{align*}
\]

Using Lemma 3 in (12) it is easy to show that

\[ \int_{I_N} |K_n(x-t)| \, d\mu(t) \]

\[ \leq \frac{c q - 1}{n} \sum_{A=0}^{M_k} M_A \int_{I_N} M_k \, d\mu(t) \leq \frac{c M_k M_q}{n M_N} \leq c M_k / M_N. \]

Let

\[
\begin{align*}
\{ & x = (0, \ldots, 0, x_m \neq 0, 0, \ldots, 0, x_N, x_{N+1}, x_q, \ldots, x_{|n|-1}, \ldots), \\
& t = (0, 0, \ldots, x_N, \ldots, x_{|n|-1}, \ldots).
\end{align*}
\]
If we apply Lemma 3 in (12) we obtain

\[ \int_{I_N} |K_n(x - t)| d\mu(t) \leq \frac{c}{n} \sum_{A=0}^{[n]-1} M_A \int_{I_N} K_n d\mu(t) \leq \frac{cM_k}{M_N}. \]

Combining (11), (13) and (14) we complete the proof of Lemma 4.

5. PROOFS OF THE THEOREMS

Proof of Theorem 1. By Lemma 1, the proof of Theorem 1 will be complete, if we show that

\[ \int_{I_N} \left( \sup_{n \in \mathbb{N}} |\sigma_n f| / (n + 1)^{1/p - 2} \right)^p d\mu \leq c < \infty, \]

for every p-atom \( a \), where \( I \) denotes the support of the atom. The boundedness of \( \sup_{n \in \mathbb{N}} |\sigma_n f| / (n + 1)^{1/p - 2} \) from \( L_\infty \) to \( L_\infty \) follows from (7).

Let \( a \) be an arbitrary p-atom, with support \( I \) and \( \mu(I) = M^{-1}_N \). We may assume that \( I = I_N \). It is easy to see that \( \sigma_n(a) = 0 \), when \( n \leq M_N \). Therefore, we can suppose that \( n > M_N \).

Since \( \|a\|_\infty \leq cM_N^{1/p} \) we can write

\[ \frac{|\sigma_n(a)|}{(n + 1)^{1/p - 2}} \leq \frac{1}{(n + 1)^{1/p - 2}} \int_{I_N} |a(t)| |K_n(x - t)| d\mu(t) \leq \frac{\|a\|_\infty}{(n + 1)^{1/p - 2}} \int_{I_N} |K_n(x - t)| d\mu(t) \leq \frac{cM_N^{1/p}}{(n + 1)^{1/p - 2}} \int_{I_N} |K_n(x - t)| d\mu(t). \]

Let \( x \in I_{N}^{k,l}, 0 \leq k < l \leq N \). From Lemma 4 we get

\[ \frac{|\sigma_n(a)|}{(n + 1)^{1/p - 2}} \leq \frac{cM_N^{1/p}}{M_N^{1/p - 2} M_N^2} = cM_{l}M_{k}. \]
Combining (14) and (15) we obtain

\[
\int_{I_N} |\sigma^* a(x)|^p \, d\mu(x) = \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0,j \in \{l+1, \ldots, N-1\}} \int_{I_N}^k |\sigma^* a(x)|^p \, d\mu(x)
\]

\[
+ \sum_{k=0}^{N-1} \int_{I_N}^k |\sigma^* a(x)|^p \, d\mu(x)
\]

\[
\leq c \sum_{k=0}^{N-1} \sum_{l=k+1}^{N-2} \frac{M_{l+1} \ldots M_{N-1}}{M_N} (M_l M_k)^p
\]

\[
+ c \sum_{k=0}^{N-1} \frac{1}{M_N} (M_N M_k)^p
\]

\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^p}{M_l} + c \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^{1-p}} = I + II.
\]

Then

\[
I = c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-2p}} \frac{(M_l M_k)^p}{M_l^{2p}}
\]

\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l^{1-2p}}
\]

\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{2l(1-2p)}
\]

\[
\leq c \sum_{k=0}^{N-2} \frac{1}{2k(1-2p)} < c < \infty.
\]

It is evident

\[
II \leq \frac{c}{M_N^{1-2p}} < c < \infty.
\]

Which complete the proof of Theorem 1.

**Proof of Theorem 2.** Let \(0 < p < 1/2\) and \(\{\lambda_k; \ k \in \mathbb{N}_+\}\) be an increasing sequence of the positive integers, such that

\[
\lim_{k \to \infty} \frac{\lambda_k^{1/p-2}}{\varphi(\lambda_k)} = \infty.
\]
It is evident that for every $\lambda_k$, there exists a positive integers $m'_k$, such that $q_{m'_k} < \lambda_k < cq_{m'_k}$. Since $\phi(n)$ is nondecreasing function, we have

\[
\lim_{k \to \infty} \frac{M_{2m'_k}^{1/p-2}}{\phi(q_{m'_k})} \geq c \lim_{k \to \infty} \frac{q_{m'_k}^{1/p-2}}{\phi(q_{m'_k})} = \lim_{k \to \infty} \frac{\lambda_k^{1/p-2}}{\phi(\lambda_k)} = \infty.
\]

Let $\{n_k; k \in \mathbb{N}_+\} \subset \{m'_k; k \in \mathbb{N}_+\}$ such that

\[
\lim_{k \to \infty} \frac{M_{2n_k}^{1/p-2}}{\phi(q_{n_k})} = \infty
\]

and

\[
f_{n_k}(x) = D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x), \quad n_k \geq 3.
\]

It is evident

\[
\tilde{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k+1} - 1, \\ 0, & \text{otherwise.} \end{cases}
\]

Then we can write

\[
S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), & \text{if } i = M_{2n_k}, \ldots, M_{2n_k+1} - 1, \\ f_{n_k}(x), & \text{if } i \geq M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases}
\]

From (16) we get

\[
\|f_{n_k}\|_{H_p}
= \left\| \sup_{n \in \mathbb{N}} S_{M_n} (f_{n_k}) \right\|_{L_p}
= \left\| D_{M_{2n_k+1}} - D_{M_{2n_k}} \right\|_{L_p}
= \left( \int_{I_{2n_k}} M_{2n_k}^p d\mu(x) + \int_{I_{2n_k+1}} (M_{2n_k+1} - M_{2n_k})^p d\mu(x) \right)^{1/p}
= \left( \frac{m_{2n_k} - 1}{M_{2n_k}^{2n_k+1}} M_{2n_k}^p + \frac{(m_{2n_k} - 1)^p}{M_{2n_k+1}^{2n_k}} \right)^{1/p}
\leq M_{2n_k}^{1-1/p}.
\]
By [17] we can write:

\[
\frac{\sigma_{q_{nk}} f_{nk} (x)}{\varphi (q_{nk})}
= \frac{1}{\varphi (q_{nk}) q_{nk}} \sum_{j=0}^{q_{nk}-1} S_j f_{nk} (x)
= \frac{1}{\varphi (q_{nk}) q_{nk}} \sum_{j=M_{2nk}}^{q_{nk}-1} S_j f_{nk} (x)
= \frac{1}{\varphi (q_{nk}) q_{nk}} \sum_{j=M_{2nk}}^{q_{nk}-1} \left( D_j (x) - D_{M_{2nk}} (x) \right)
= \frac{1}{\varphi (q_{nk}) q_{nk}} \sum_{j=0}^{q_{nk}-1} \left( D_{j+M_{2nk}} (x) - D_{M_{2nk}} (x) \right).
\]

Since

\[
D_{j+M_{2nk}} (x) - D_{M_{2nk}} (x) = \psi_{M_{2nk}} D_j, \quad j = 1, 2, \ldots, M_{2nk} - 1,
\]

we obtain

\[
\frac{\sigma_{q_{nk}} f_{nk} (x)}{\varphi (q_{nk})}
= \frac{1}{\varphi (q_{nk}) q_{nk}} \sum_{j=0}^{q_{nk}-1} D_j (x)
= \frac{1}{\varphi (q_{nk}) q_{nk}} \sum_{j=0}^{q_{nk}-1} K_{q_{nk}-1} (x).
\]

Let \( x \in I_{2^{2s}, 2l}^{2s, 2l} \). Using Lemma 2 we obtain

\[
\frac{\sigma_{q_{nk}} f_{nk} (x)}{\varphi (q_{nk})} \geq \frac{c M_{2s} M_{2l}}{M_{2nk} \varphi (q_{nk})}.
\]

Hence we can write:
From (18) and (19) we have

\[
\frac{c}{M_{2n_k} \varphi(q_{n_k})} \left( \mu \left\{ x \in G_m : \frac{\sigma_{q_{n_k}} f_{n_k}(x)}{\varphi(q_{n_k})} \geq \frac{c}{M_{2n_k} \varphi(q_{n_k})} \right\} \right)^{1/p} \geq \frac{c}{M_{2n_k} \varphi(q_{n_k})} M_{2n_k}^{1-1/p} \geq \frac{c}{M_{2n_k} \varphi(q_{n_k})} \to \infty, \quad \text{when } k \to \infty.
\]

Theorem 2 is proved.

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