Restoration and Dynamical Breakdown of the $\phi \to -\phi$ Symmetry in the (1+1)-dimensional Massive sine-Gordon Field Theory

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Within the framework of the Gaussian wave-functional approach, we investigate the influences of quantum and finite-temperature effects on the $Z_2$-symmetry($\phi \to -\phi$) of the (1+1)-dimensional massive sine-Gordon field theory. It is explicitly demonstrated that by quantum effects the $Z_2$-symmetry can be restored in one region of the parameter space and dynamically spontaneously broken in another region. Moreover, a finite-temperature effect can further restore the $Z_2$-symmetry only.

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I. INTRODUCTION

The massive sine-Gordon field theory (MSGFT) [1] is an interesting and important one in quantum field theory, particle physics and condensed matter physics. It is a simple generalization of the massless sine-Gordon field theory (SGFT) [2] which received an extensive investigations [3,4], with a vacuum angle $\theta$ added in the argument of the cosine and a mass term $m_0^2\phi^2$ added in the Lagrangian. Nevertheless, different from the SGFT, the Lagrangian of MSGFT is no longer invariant under the transformation of the field $\phi \to (\phi + \frac{2n\pi}{\beta})$, and at the classical level, the symmetry $\phi \to -\phi$ ($Z_2$-symmetry) can be spontaneously broken, which is an important phenomenon for quantum field theory and particle physics [5]. Furthermore, the 1 + 1-dimensional (2D) MSGFT is equivalent to many other important models, such as the massive Schwinger model at a special coupling strength and on the zero-charge sector [6], the massive Schwinger-Thirring model on the zero-charge sector [7,8], the two-dimensional (2D) lattice Abelian Higgs model [9], the 2D neutral Yukawa gas [10], and so on. Also, the 2D MSGFT is useful in the investigations of fluid membranes [11] and vortices in 2D superconductors [12]. Therefore, it is of general importance to explore various properties of the MSGFT.

In fact, many investigations of the 2D MSGFT have existed. Early in 1970s, the 2D MSGFT was analyzed within the framework of constructive quantum field theory. The existence of the 2D MSGFT was proved for a certain region of the parameter space [13,14], and Fröhlich et al also showed the nontriviality of the scattering, the existence of the one boson states (Fermion-anti-Fermion bound state), the zero-charge of the physical states in the 2D MSGFT, and so on [15]. Recently, in order to reveal the phase structure of the 2D Abelian Higgs model at a special coupling strength and on the zero-charge sector [16], the massive Schwinger-Thirring model on the zero-charge sector [17], the two-dimensional (2D) lattice Abelian Higgs model [18], the 2D neutral Yukawa gas [19], and so on. Also, the 2D MSGFT is useful in the investigations of fluid membranes [20] and vortices in 2D superconductors [21]. Therefore, it is of general importance to explore various properties of the MSGFT.

Nevertheless, we feel that for the 2D MSGFT there are still many problems worth investigating. For instance, the spontaneous breakdown of $Z_2$-symmetry ($Z_2$SSB) is such an interesting phenomenon. As was stated in the first paragraph, $Z_2$-symmetry of the classical vacuum of the 2D MSGFT is spontaneously broken [15,16,21]. The $Z_2$SSB of the classical vacuum means that the classical potential has a minimum, for example, at two points in the field space $\phi = \pm \phi_1$ (here, $\phi_1$ is positive), or say, is two-fold degenerate (infinitely degenerate for the SGFT), and accordingly the classical vacuum is occasionally located at one of the two points $\phi = \pm \phi_1$ and consequently $Z_2$-symmetry is broken. Similarly, at a quantum level, when effective potential for a quantum field theory is, for example, two-fold degenerate, the quantum vacuum is spontaneously located at a non-zero field point, and consequently $Z_2$-symmetry is also spontaneously broken. In the 1970’s, Ref. [16] inferred that for the 2D MSGFT the $Z_2$SSB at a classical level would

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be usually maintained at a quantum level, and Ref. 12 pointed out qualitatively, based on a semiclassical calculation, that for the bosonic equivalent to the massive Schwinger model $Z_2$-symmetry suffers spontaneous breakdown in some case. Besides, Ref. 13 pointed out that the $Z_2$-symmetry in the 2D MSGFT is presumably dynamically broken in some case. Recently, we also demonstrated explicitly the existence of the $Z_2$-SSB phenomenon within the framework of the GWFA 21. However, as compared to what occurs at the classical level, the influence of quantum effects on the $Z_2$-SSB phenomenon is not clear. Usually, quantum effects can alter the $Z_2$-symmetry of the vacuum, and accordingly turn a symmetric vacuum at the classical level into an asymmetric one at the quantum level, or vice versa, an asymmetric vacuum at the classical level into a symmetric one at the quantum level. The latter is called the restoration of $Z_2$-symmetry 22,36 (1996), and the former is called the dynamical breakdown of $Z_2$-symmetry ($Z_2$DSB) in this paper. By dynamical breakdown 6, we mean that it is just by quantum effects that the $Z_2$-symmetry enjoyed by the classical vacuum is spontaneously broken at a quantum level. When $Z_2$DSB occurs, we also say the $Z_2$-symmetry is dynamically spontaneously broken, which is really involved in the type of the phenomenon occurred in the pure $\lambda\phi^4$ model 24. For a field theory, the occurrences or disappearances of the $Z_2$-symmetry restoration and $Z_2$DSB can be seen through a comparison between the two regions of the parameter space where the $Z_2$SSB occurs respectively at the classical and quantum levels. If there is a parameter region where the classical vacuum is asymmetric and the quantum vacuum symmetric, then in that region, the symmetry breakdown phenomenon is compressed and $Z_2$-symmetry is restored by quantum effects. If there is a parameter region where the classical vacuum is symmetrical and the quantum vacuum asymmetrical, then in that region, the symmetry breakdown phenomenon is enhanced and $Z_2$-symmetry is dynamically spontaneously broken by quantum effects. Obviously, Fig. 1 in Ref. 21 suggested the occurrence of the $Z_2$-symmetry restoration, but failed to give such a parameter region where the $Z_2$-symmetry restoration phenomenon occurs. Furthermore, we donot know if the $Z_2$DSB phenomenon really occurs in the 2D MSGFT. This paper will address the above problems. Within the framework of the GWFA, we shall continue to investigate the 2D MSGFT with zero vacuum angle, explicitly give the region of the parameter space where the $Z_2$-symmetry is restored, discuss $Z_2$DSB, and analyze the influence of a finite temperature effect on them.

Both the $Z_2$-symmetry restoration and $Z_2$DSB are interesting and important. They are the inverse of each other, and can correspond to some phase transitions which occur at inverse directions. Also they mirror some aspects of quantum effects. Besides, the $Z_2$-symmetry restoration is the reverse of the fate of the fase vacuum 2. Early in 1981, Rajaraman and Lakshmi proposed at the first time the concept of symmetry restoration and demonstrated explicitly the restoration of the $Z_2$-symmetry in a special 2D $\phi^4$ model 22, with the loop-expansion effective-potential method 24. Later, a re-investigation of the same model appealing to the same method revealed the dependence of the symmetry-restoring result upon the renormalization condition 26. Later again, a Monte Carlo numerical study about another slightly different 2D $\phi^6$ theory with three degenerate vacua showed that the quantum corrections make the vacuum $Z_2$-symmetrical 27. Recently, we re-investigated the above two $\phi^6$ models with the GWFA and demonstrated well the restoration of the $Z_2$symmetry 28. Although the above two models can display the restoration of the $Z_2$-symmetry, they (in the continuum, not the lattice case) are intrinsically dependent upon the renormalized condition 20,25,29, and the model in Ref. 27 is little typical because its classical vacuum is three-fold degenerate. The $\lambda\phi^4$ model is a good one for displaying the $Z_2$-symmetry restoration, and was investigated in various dimensions through the GWFA 30. In this paper, one will see that the MSGFT is also a good field-theoretic example for displaying the $Z_2$-symmetry restoration. As for the dynamical breakdown of the $Z_2$-symmetry, its existence was shown only in the pure $\lambda\phi^4$ model 22, and, to our knowledge, no other investigations demonstrated the occurrence of the $Z_2$DSB in any other scalar field theories. In this paper, we shall show that the 2D MSGFT also suffers $Z_2$DSB.

As was mentioned above, for the 2D MSGFT, quantum effects can extend and also shrink the parameter region where the classical vacuum is $Z_2$-symmetrical. Then, when a finite temperature is introduced into the 2D MSGFT, will the parameter regions of the $Z_2$-symmetry restoration and $Z_2$DSB by quantum effects be extended or shrunk by the finite temperature effect? This is an interesting problem. When the $Z_2$-symmetry restoration region is extended, or the $Z_2$DSB region is shrunk, we say the $Z_2$-symmetry is restored by a finite temperature effect 31. This phenomenon is important in particle physics, cosmology and condensed matter physics (31,32,33) (1996). In the present paper, using the GWFA in thermofield dynamics 34,33,35,36, we shall consider the influence of a finite temperature on the $Z_2$-symmetry for the 2D MSGFT with zero vacuum angle. One will see that for the 2D MSGFT,
a finite temperature effect can further restore the $Z_2$-symmetry and would prevent quantum effects from dynamically breaking the $Z_2$-symmetry.

For both the zero- and finite-temperature cases, this paper will make use of the GWFA for investigations. Although it is difficult to control the accuracy of the GWFA, the GWFA has succeeded in extracting the non-perturbative information of many field theoretical models\([37,21]\). So, we feel that for the $Z_2$-symmetry restoration and $Z_2$DSB phenomena in the 2d MSGFT, the quantitative result of the GWFA is necessary and useful, at least, can provide a basis and reference for further investigations.

This paper is organized as follows. Sect.II will discuss the classical vacuum of the 2D MSGFT so as to lay the foundation for studying the $Z_2$-symmetry restoraton and $Z_2$DSB phenomena. In Sect.III, we shall give the finite temperature Gaussian effective potential (FTGEP), the zero-temperature limit of which is just the Gaussian effective potential (GEP) of quantum field theory. The restoration and dynamical breakdown of the $Z_2$-symmetry by quantum effects will be investigated in Sect.IV. In Sect.V we shall consider the influence of a finite temperature effect on the zero-temperature results, and Sect. VI will conclude this paper with a brief discussion.

**II. CLASSICAL VACUUM**

In this section, we give a discussion about the classical aspects of the 2D MSGFT for the convenience of latter analyses. Although the analyses in the present section is simple, and some results are straightforward or were mentioned in Ref.\([21]\), they are the basis of the investigation in this paper.

For the (1+1)-D MSGFT with zero vacuum angle $\theta = 0$, the Lagrangian is\([3]\]

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi_x \partial^\mu \phi_x - \frac{1}{2} \phi_x^2 - \frac{m^2}{\beta^2} (1 - \cos(\beta \phi_x)) \equiv \frac{1}{2} \partial_\mu \phi_x \partial^\mu \phi_x - U(\phi_x),
\]

with $\phi_x = \phi(x)$, where $m_0$ and $m$ are in mass dimension and the dimensionless $\beta$ is the coupling parameter. In the case of $m_0 = 0$, Eq.(1) describes the SGFT, and when $\beta^2 \to 0$, Eq.(1) describes a free theory with the squared mass $(m_0^2 + m^2)$ if $m_0^2 + m^2 > 0$. Because $|\cos(\beta \phi_x)| \leq 1$ (hereafter $| \cdots |$ represents absolute value), we have to maintain a positive $m_0^2$ in Eq.(1) for avoiding an unbounded-below vacuum\([4]\). Similar to SGFT\([3]\), it is always viable to have $\beta > 0$ because the sign of $\phi$ is free to be redefined. Nevertheless, different from the SGFT, both the positive and the negative $m^2$ should be considered in Eq.(1) because the physics of the negative is not equivalent to the one of the positive.

Evidently, the Lagrangian Eq.(1) is invariant under the transformation of $\phi \to -\phi$. Let us discuss if the $Z_2$-symmetry of the Lagrangian is enjoyed by the classical vacuum. Consider the extremum condition of the classical potential $U(\phi) \left( \frac{dU(\phi)}{d\phi} = 0 \right)$

\[
\beta \phi + \frac{m^2}{m_0^2} \sin(\beta \phi) = 0, \tag{2}
\]

and the second derivative

\[
\frac{d^2 U(\phi)}{(d\phi)^2} = m_0^2 + m^2 \cos(\beta \phi). \tag{3}
\]

For any solution of Eq.(2) $\phi = \hat{\phi}$, the potential $U(\hat{\phi})$ can be expressed as

\[
U(\hat{\phi}) = \frac{2m^2}{\beta^2} \sin^2 \left( \frac{\beta \hat{\phi}}{2} \right) \left( 1 + \frac{m^2}{m_0^2} \cos^2 \left( \frac{\beta \hat{\phi}}{2} \right) \right). \tag{4}
\]

Obviously, when $m^2$ is positive, $U(\hat{\phi} = \hat{\phi}_1 \neq 0)$ is greater than the value $U(\hat{\phi} = 0) = 0$. When $m^2 < 0$, Eq.(4) can be solved with graphics, and Fig.1 solves graphically Eq.(2) with $\beta = 3.0$. In Fig.1, the one, three and seven

\[\text{Notice that the sign of the term } \frac{m^2}{\beta^2} \cos(\beta \phi_x) \text{ is the plus (+), which is identical with that of Eqs.(I.1) and (III.1) in Ref.}\ [5], \text{but contrary to that in Ref.}\ [6].
\[\text{We do not know if quantum effects make the 2D MSGF meaningful for a negative } m_0^2.\]
intersection points of the three lines I, II, and III with the sine curve are the solutions of Eq.(2) in the cases of $m_0^2 = -1.2, -0.5, -0.1$, respectively. From Fig.1, it can be seen that the more near the negative ratio $m_0^2/m^2$ is to zero, the greater the number of the solutions of Eq.(2) is. In general, for any given $m_0^2, m^2$ and $\beta$, Eq.(2) can have $(2N + 1)$ roots (here, $N$ is a non-negative integer). When $m^2 < 0$ and $m_0^2/m^2 < 1$, Eq.(2) has a unique zero root $\phi = 0$ because $\beta \phi > \sin(\beta \phi)$ for the case of $\phi > 0$. Thus, both for the case of $m^2 > 0$ and for the case of $m^2 < 0$ with $|m^2| < m_0^2$, the classical potential $U(\phi)$ is absolutely minimized at $\phi = 0$, and so the classical vacuum is symmetrical and possesses the same $Z_2$-symmetry with the Lagrangian. In the other case, i.e., for any negative $m^2$ with $|m^2| > m_0^2$, the situation is completely different, and there exist $2N$ non-zero roots of Eq.(2) $\phi = \pm \phi_i$ $(i = 1, 2, \ldots, N)$ besides the zero root $\phi = 0$, and $U(\phi)$ gets to a maximum at $\phi = 0$ because $\frac{d^2U(\phi)}{d\phi^2}$ of Eq.(3) is negative at $\phi = 0$. Then for the case of $m^2 < 0$ with $|m^2| > m_0^2$, the classical potential $U(\phi)$ must be minimized at $\phi = \pm \phi_1, \phi_3, \phi_5, \ldots$, and $Z_2$-symmetry of the classical vacuum is spontaneously broken. A further analysis indicates that the classical vacuum is located at $+\phi_1$ or $-\phi_1$ which are nearest to the zero root. The parameter space corresponding to the symmetric and the asymmetric phases of the classical vacuum is plotted in Fig.2. For any $\beta \neq 0$, on the parameter plane $m_0^2-m^2$, the boundary where the symmetric and the asymmetric vacua coexist corresponds to the ray, from the origin, with the slope $-1$, in the fourth quadrant of the parameter plane (Fig.2(a)), and the upper and lower regions of the boundary correspond to the $Z_2$-symmetric and asymmetric vacua, respectively. For any given $m_0^2 \neq 0$, on the parameter plane $m_2-\beta^2$, the boundary where the symmetric and the asymmetric vacua coexist is the line $m^2 = -m_0^2$ in the second quadrant of the parameter plane (Fig.2(b)), and the right region of the boundary corresponds to symmetric vacua, whereas the left region to asymmetric vacua. As an explicit illustration, the classical potentials at $\beta = 3.0$ and $m_0^2 = 0.6$ are depicted in Fig.3 for the cases of $m^2 = 6.0, -0.5, -3.0$, and $-6.0$, respectively. (In the numerical computation, unit mass is regarded as 1 and hence quantities concerned can be taken as dimensionless.)

When the classical vacuum is $Z_2$-symmetrical, the shape of the potential $U(\phi)$ is mainly a single well, and when the vacuum is asymmetrical, the shape of $U(\phi)$ is dominated by a double wells. These, according to the above simple analysis, are independent of the coupling parameter $\beta$. Moreover, the curvatures of the wells at their bottoms $\frac{d^2U(\phi)}{d\phi^2} \bigg|_{\phi = \pm \phi_1}$ are also independent of $\beta$, because for any given $m_0^2$ and $m^2$, any non-zero root of Eq.(2) $\phi_i$ is different for a different $\beta$ but the products $\beta \phi_i$s for all $i$ are the same with one another. Nevertheless, $U(\hat{\phi}_1)$ the depth of the $Z_2$-symmetry-broken wells with respect to $U(\hat{\phi} = 0)$ is deeper and deeper with the decrease of $\beta$.

In this section, we have given a wordy statement on the $Z_2$-symmetry of the classical vacuum. Next, we shall give the expression of the FTGEP so as to discuss the $Z_2$-symmetry of quantum vacuum.

### III. Finite Temperature Gaussian Effective Potential

The FTGEP is the effective potential of finite temperature field theory [33] obtained by the Gaussian approximation. There are several methods to calculate the FTGEP [33], and those methods are equivalent to one another. This section will calculate the FTGEP for the system Eq.(1) with the help of the GWFA in thermofield dynamics [35] (Roditi) [36].

Ref. [33] (Roditi) proposed the GWFA in thermofield dynamics for the quantum-mechanical $\lambda \phi^4$ model. In Ref. [33], we developed this method for a (D+1)-dimensional scalar field theory with any potential whose Fourier representation exists in the sense of tempered distributions [33]. According to the formulæ in Ref. [33], in order to calculate the FTGEP of the 2D MSGFT, it is enough to complete some ordinary integrals because the system Eq.(1) involves in the type of field systems in Ref. [33]. Hence, it is not necessary to give the derivation of the FTGEP of the 2D MSGFT in detail. Additionally, thermofield dynamics and its GWFA will be not introduced here, for they can be found in Ref. [33] and in Ref. [36] (C1 in Ref. [36] (page 744) should be $C_2$), respectively.

Now we shall give the FTGEP for the system Eq.(1). Substituting $U(\phi_x)$ of Eq.(1) into Eqs.(23) and (25) in Ref. [36] and performing some integrations, one can have the FTGEP of the 2D MSGFT at any temperature $T$:

$$V_T(\beta) = \frac{1}{2}[J_0(g) - J_0(M^2)] - \frac{1}{4}[u^2J_1(g) - M^2J_1(M^2)]$$

$$+ \frac{1}{4}m_0^2[J_1(g) - J_1(M^2)] + \frac{1}{2}m_0^2\phi^2$$

$$+ \frac{m^2}{\beta^2}[1 - \exp\left(-\frac{\beta^2}{4}[J_1(g) - J_1(M^2)]\cos(\beta\phi)\right]$$

$$- k_0T \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left[ \cosh^2(g(p, T)) \ln(\cosh^2(g(p, T))) - \sinh^2(g(p, T)) \ln(\sinh^2(g(p, T))) \right],$$

(5)
with the gap equation
\[ \mu^2(\varphi, T) = m_0^2 + m^2 \exp\{-\frac{\beta^2}{4}[J_1(g) - I_1(M^2)]\} \cos(\beta \varphi) \],

(6)

where \( k_0 \) represents the Boltzmann constant, \( M \) is normal-ordering mass (an arbitrary positive constant with mass dimension),

\[ I_n[y^2] = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{\sqrt{p^2 + y^2}}{(p^2 + y^2)^n} \]

and

\[ J_n(g) = \int_{-\infty}^{+\infty} \frac{dp\sqrt{p^2 + \mu^2(\varphi, T)}}{2\pi(p^2 + \mu^2(\varphi, T))^n} \]

with

\[ g(p, T) = \frac{1}{2} \ln \left( \frac{\exp\left(\frac{1}{2\pi T} \sqrt{p^2 + \mu^2(\varphi, T)}\right) + 1}{\exp\left(\frac{1}{2\pi T} \sqrt{p^2 + \mu^2(\varphi, T)}\right) - 1} \right). \]

Notice that in Eq.(5), \( \mu^2(\varphi, T) \) in \( J_n(g) \) should be replaced by \( \mu^2 \).

In the above, \( \varphi \) is the Gaussian-wave-functional expectation value of the field operator \[36\], can be any real constant, and the quantum vacuum is located at \( \varphi = \varphi_0 \) which is satisfied by the equation

\[ m_0^2 \varphi + \frac{m^2}{\beta} \exp\{-\frac{\beta^2}{4}[J_1(g) - I_1(M^2)]\} \sin(\beta \varphi) = 0 \]

(7)

and minimizes absolutely \( V_T(\varphi) \) with respect to \( \varphi \). Eq.(7) is the extreme condition for \( V_T(\varphi) \) and is obtained from the following equation

\[ \frac{dV_T(\varphi)}{d\varphi} = \int_{-\infty}^{+\infty} \frac{d\alpha}{2\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2}\right) U^{(1)}(\alpha) \left( \frac{\alpha}{2} \sqrt{J_1(g) - I_1(M^2)} + \varphi \right). \]

(8)

Here, \( U^{(1)}(y) \equiv \frac{dU(y)}{dy} \). If \( \varphi_0 \neq 0 \), then the symmetry \( \phi \rightarrow -\phi \) is spontaneously broken at the quantum level and a finite temperature.

The GWFA is a variational method. Because of the nature of the minimizing procedure, \( \mu \) in Eq.(5) should be chosen from the non-zero root of Eq.(6) and two end points of the range \( 0 < \mu < \infty \) so that \( V_T(\varphi) \) is an absolute minimum with respect to \( \mu \). Besides, sometimes the non-zero solution of Eq.(6) is multi-valued, and so in that case, the suitable root should be decided according to the stability condition (from Eq.(26) in Ref. [36])

\[ 1 - \frac{m^2\beta^2}{8} \exp\{-\frac{\beta^2}{4}[J_1(g) - I_1(M^2)]\} J_2(g) \cos(\beta \varphi) > 0. \]

(9)

Once \( \mu \) is determined, its value at \( \varphi_0 \) is the thermal mass of bosons Eq.(5) gives the FTGEP in the 2D MSGFT \[36\].

From Ref. [36], the above formulae contain no divergences, and hence no further renormalization procedures need to be performed. Thus, Eqs.(5),(6),(7) and the inequality Eq.(9) can correctly give the FTGEP for the 2D MSGFT. From the FTGEP, we can discuss the restoration and dynamical breakdown of the \( \mathbb{Z}_2 \)-symmetry. Next, we begin in the \( T = 0 \) case.

**IV. QUANTUM EFFECTS**

As was mentioned in Sect.I, for the system Eq.(1), with the aid of the GWFA, Ref. [21] showed the occurrence of the spontaneous \( \mathbb{Z}_2 \)-symmetry breakdown and gave the corresponding phase diagram of the parameter plane \( \beta^2-m_0^2 \). In this section, we shall discuss further the phases of the quantum vacuum on the parameter planes \( \beta^2-m^2 \) and \( m^2-m_0^2 \) so as to explicitly analyse the influence of quantum effects on the symmetry of vacuum.

\[ ^5 \text{It is difficult for Ref. [21] to explicitly discuss this influence because the parameter } m^2 \text{ was not considered in the numerical calculation there.} \]
When $T = 0$, Eqs. (5), (6), (7) and the inequality (9) are reduced to the potential

$$
\bar{V}_0(\varphi) = \frac{\mu^2 - M^2}{8\pi} + \frac{m_0^2 + m^2 - \mu^2}{\beta^2} - \frac{m_0^2}{8\pi} \ln\left(\frac{\mu^2}{M^2}\right) + \frac{1}{2}m_0^2\varphi^2,
$$

(10)

the gap equation

$$
\mu^2 = m_0^2 + m^2 \left(\frac{\mu^2}{M^2}\right)^{\frac{\varphi^2}{8\pi}} \cos(\beta \varphi),
$$

(11)

the extremum condition

$$
m_0^2\varphi + \frac{\mu^2 - m_0^2}{\beta} \tan(\beta \varphi) = 0
$$

(12)

and the stability condition

$$
1 - \beta^2 \frac{\mu^2 - m_0^2}{8\pi \mu^2} > 0,
$$

(13)

which are consistent with Eqs. (4), (5), (6) and the inequality Eq. (7) in Ref. [21], respectively. From the discussion in Ref. [21], the GEP $V_0$ is governed by the nonzero root of Eq. (11), which is also satisfied by the inequality, instead of the end points $\mu^2 = 0$ and $\mu^2 \to \infty$, and meanwhile, the coupling parameter $\beta^2$ is constrained to the range of $0 < \beta^2 < 8\pi$. (Note that in Eqs. (10), (12) and (13), we have used Eq. (11).)

Different from that choice in Ref. [21], now letting $M^2 = m_0^2$ and further defining the following dimensionless quantities

$$
\bar{V}_0(\varphi) \equiv \frac{V_{T=0}(\varphi)}{m_0^2}, \bar{\mu} \equiv \frac{\mu}{m_0}, \bar{m} \equiv \frac{m}{m_0},
$$

(14)

one can numerically calculate the GEP at any fixed $m_0^2$ from Eqs. (10)–(14), and accordingly analyse the phases of quantum vacuum on the parameter plane $m^2$-$\beta^2$. From the numerical results, the phase diagram of the quantum vacuum (a figure which indicates the regions on the parameter space where the corresponding vacua are $Z_2$-symmetrical and asymmetrical, respectively) is depicted in the parameter plane $m^2$-$\beta^2$ as Fig. 4. In this figure, the solid curve is the boundary of the symmetric and asymmetric phases where the vacuum can be located either at $\varphi = 0$ or at $\varphi \neq 0$, i.e., at which the GEP at $\varphi = 0$ is equal to the one at $\varphi = 0$. On the right of the boundary, the quantum vacuum is $Z_2$-symmetrical, whereas on the left, the $Z_2$-symmetry of quantum vacuum is spontaneously broken. The short-dashed line corresponds to the solid vertical line in Fig. 2(b). Thus, Fig. 4 indicates that the symmetry $\phi \to -\phi$ is restored by quantum effects in the domain I which is surrounded by the solid curve and the short-dashed line. Obviously, for the parameter $m^2$, there is a critical value of $\tilde{m}^2 \approx -1.68$, and when $m^2 < \tilde{m}^2$ no $Z_2$-symmetric vacuum can appear. For any fixed $\tilde{m}^2$ with $\tilde{m}^2 < \tilde{m} < 1.0$, there are two critical values of $\beta$, $\beta_1$ and $\beta_2$, at each of which the vacuum is either symmetrical or asymmetrical, and for any $\beta$ with $\beta_1 < \beta < \beta_2$, the vacuum is located at $\varphi = 0$. For an explicit illustration, we plot the GEP in Fig. 5 for the case of $\tilde{m}^2 = -1.4$. The curves II and IV correspond to the two critical cases. Thus, one has seen that at any fixed $m_0^2$, quantum effects can restore the $Z_2$-symmetry.

Perhaps one has noticed that there are only one critical value of $\beta$ for the $\beta$-$m_0^2$ phase diagram in Ref. [21], whereas now on the plane $\beta$-$\tilde{m}$ the two critical values $\beta_1$ and $\beta_2$ exist for any $\tilde{m}^2$ with $\tilde{m}^2 < \tilde{m}^2 < -1.0$. This difference between the $\beta$-$m^2$ and $\beta$-$m_0^2$ diagrams can be elucidated as per Fig. 6. In Fig. 6, the solid curve is the boundary between the symmetric and asymmetric phases on the parameter plane $\mu_0^2$-$\beta^2$, and the long-dashed boundary in the parameter plane $\beta^2$-$m_0^2$ of Fig. 1 in Ref. [21], and the short- or tiny-dashed curves I, II, III, IV and V are plotted from the expression of $\bar{\mu}_0^2$ for the given values of $\bar{m}$: $-2.0, \bar{m}^2 \approx -1.68, -1.3, -1.001$ and $-0.8$, respectively. Fig. 6 indicates that in the parameter plane $\mu_0^2$-$\beta^2$, any curve $\bar{m}^2$ = constant with $\bar{m}^2 < \bar{m}^2 < -1.0$, such as the curve III, intersects the solid curve at two points which correspond to $\beta_1$ and $\beta_2$.

In the above, the $Z_2$-DSB phenomenon is not displayed on the parameter plane $\tilde{m}^2$-$\beta$. Nevertheless, we dare not rashly have a conclusion that no $Z_2$-DSB can occur for the 2D MSGFT, because this theory has three parameters and the parameter plane $m_0^2$-$m^2$ is not considered still. Now, we turn on it. Letting the normal-ordering mass unit, and giving an explicit value of $\beta$, we can discuss the phase diagram of quantum vacuum on the parameter plane $m_0^2$-$m^2$. For the case of $\beta^2 = 3.0$, we numerically solved Eqs. (10)–(13) (when $M$ is unit, all quantities can be regarded as dimensionless ones), and the phase diagram on the parameter plane $m_0^2$-$m^2$ is plotted in Fig. 7 (only one part of the fourth quadrant is given). In Fig. 7, the solid curve is the boundary between the symmetric and asymmetric phases, and
the region above the solid curve corresponds to symmetric vacua and the region below the solid curve to asymmetric vacua. The short-dashed line in this figure is the boundary in Fig.2(a), and obviously the figure indicates that not only the $Z_2$-symmetry restoration occurs in the upper-left region I between the solid curve and the short-dashed line, but also the $Z_2$-symmetry is dynamically spontaneously broken in the lower-right region II between the solid curve and the short-dashed line. In order to give an evident demonstration for the restoration and dynamical breakdown of the $Z_2$-symmetry, the GEP is depicted in Fig.8 and Fig.9, respectively. Fig.8 is for the case of $m_0^2 = 1.5$, and the curves I, II and III correspond to $m^2 = -1.7, -2.1526852$ (the approximate critical value) and $-3.0$, respectively. In Fig.9, $m_0^2 = 8.0$, and the curves I, II and III correspond to $m^2 = -5.0, -6.3043705$ (the approximate critical value) and $-7.0$, respectively. The dynamical breakdown of the $Z_2$-symmetry in the 2D MSGFT is consistent with assertion of Ref. [4] (notice the third footnote. In the paragraph above Fig.2 of Ref. [21], the second sentence “Fröhlich [1] pointed out...” is misleading, and should be “Fröhlich [1] (the book) pointed out that for a sufficiently small $\tilde{m}_0^2$ there may be a phase transition.”). Additionally, we also consider the effect of the parameter $\beta^2$ on the phase diagram of the plane $m^2$-$m_0^2$, and the result indicates that the intersection point of the solid boundary with the short-dashed boundary is shifted up along the classical boundary with the increase of $\beta^2$. That is to say, when increasing the value of $\beta^2$, the $Z_2$-symmetry restoration domain I gets smaller and the $Z_2$-symmetry is dynamically spontaneously broken in the lower-right region II between the solid curve and the short-dashed line. In order to give an evident demonstration for the restoration and dynamical breakdown of the $Z_2$-symmetry in the 2D MSGFT. Next section, we shall analyse the FTGEP to discuss the influence of a finite temperature effect on the results in this section.

V. FINITE TEMPERATURE EFFECTS

Substituting the three equations between Eqs.(26) and (27) in Ref. [36] into Eqs.(5)—(7),(9) here, we have the FTGEP

$$V_T(\varphi) = \frac{\mu^2 - M^2}{8\pi} + \frac{1}{2} m_0^2 \varphi^2 - m_0^2 \frac{\mu^2}{8\pi} \ln(\frac{\mu^2}{M^2})$$

$$\frac{m^2}{\beta^2} \left[ 1 - \left( \frac{\mu^2}{M^2} \right)^2 \frac{1}{8\pi} \exp\{-\frac{1}{2} \beta^2 C_2\} \cos(\beta \varphi) \right] + \frac{1}{2} C_2 (m_0^2 - \mu^2) + k_b T K,$$

the gap equation

$$\mu^2 = m_0^2 + m^2 \left( \frac{\mu^2}{M^2} \right)^2 \frac{1}{8\pi} \exp\{-\frac{1}{2} \beta^2 C_2\} \cos(\beta \varphi),$$

the extremum condition

$$m_0^2 \varphi + \frac{m^2}{\beta} \left( \frac{\mu^2}{M^2} \right)^2 \frac{1}{8\pi} \exp\{-\frac{1}{2} \beta^2 C_2\} \sin(\beta \varphi) = 0$$

and the stability condition

$$1 - \frac{1}{8} m^2 \beta^2 \left( \frac{\mu^2}{M^2} \right)^2 \frac{1}{8\pi} \exp\{-\frac{1}{2} \beta^2 C_2\} J_2(g) \cos(\beta \varphi) > 0,$$

where

$$C_2 = \int_{-\infty}^{+\infty} \frac{dp}{2\pi \sqrt{p^2 + \mu^2}} \left\{ \exp\left\{ \frac{1}{k_b T} \sqrt{p^2 + \mu^2} \right\} - 1 \right\}$$

and

$$K = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \ln[1 - \exp\{-\frac{1}{k_b T} \sqrt{p^2 + \mu^2}\}].$$

Note that $\mu$ in Eqs.(16)—(18) represents $\mu(\varphi, T)$. From the formulae 3.337(1) and 8.526(1) in Ref. [22], one can have

$$C_2 = \frac{\gamma}{2\pi} + \frac{1}{2\pi} \ln\left( \frac{\mu}{4\pi k_b T} \right) + k_b T \frac{\mu}{2\mu} + \frac{1}{2\pi} \sum_{n=1}^{+\infty} \left\{ \frac{2\pi k_b T}{\sqrt{\mu^2 + 4\pi^2 k_b T^2 n^2}} - \frac{1}{n} \right\}.$$
with $\gamma$ the Euler’s constant. According to Sect.III, in order to calculate the FTGEP, we have to first choose $\mu$ from the root of the gap equation Eq.(16) and the end points of the region $0 \leq \mu < \infty$. For the end point $\mu \to 0$, the integral $K$ is finite, $C_2 \to \frac{1}{2\mu} \ln\left(\frac{2\mu}{\pi}\right) + \frac{1}{2\mu} \to \frac{1}{2\mu}$, and so $V_T \to -m_0^2 k_b T \to +\infty$. Therefore, we have to discard $\mu = 0$. As for the other end point $\mu \to \infty$, $K = 0$ and $C_2 = 0$, and hence

$$V_T(\phi) \to \frac{\mu^2}{8\pi} - \frac{m^2}{\beta^2} \left(\frac{\mu^2}{M^2}\right) \frac{1}{\kappa^2} \cos(\beta \varphi),$$

which is the same as that in the case of $T = 0$.\[21\] Thus, $\beta^2$ must be less than $8\pi$, and to calculate FTGEP, we should choose the nonzero root of Eq.(16) instead of the end points $\mu = 0$ and $\mu \to \infty$ (Of course, the root should be satisfied by the inequality Eq.(18)). The constraint of the parameters in the case of $T \neq 0$ is identical with that in the case of $T = 0$.

Now we can compute the FTGEP and analyse the symmetry of the vacuum at $T \neq 0$. Although the integrals $K$, $C_2$ and $J_2(g)$ have no simple analytical expressions, but it is easy to numerically compute them, and accordingly we can numerically solve the Eqs.(15)—(18). For the convenience of a comparison, we shall do as the last section and consider first the phase diagram on the parameter plane $m^2$-$\beta^2$ and then that on the plane $m_0^2$-$m^2$.

Taking $M = m_0$, using the definitions Eq.(14) and defining the dimensionless FTGEP

$$\tilde{V}_T(\phi) = \frac{V_T(\phi)}{m_0^2},$$

Eqs.(15)—(18) can be dimensionless, and we can numerically computed the FTGEP at a given temperature and discuss the symmetry of the vacuum on the parameter plane $\tilde{m}^2$-$\beta^2$. Fig.10 gives the numerical results at some temperatures. In Fig.10, the solid curves I,II and III are the boundaries between symmetric and asymmetric phases at $T = \frac{1}{6k_b}$, $\frac{1}{3k_b}$ and $\frac{1}{\kappa k_b}$, respectively, the shapes of which are similar to that in Fig.4, the short-dashed line corresponds to the classical boundary in Fig.2(b), and the curve I coincides almost with the boundary in Fig.4. At any temperature $T$, the vacuum is asymmetrical on the left of the solid curve of Fig.10, and symmetrical on the right. Fig.10 indicates that a finite temperature effect can further restore $Z_2$-symmetry of the quantum vacuum, and the $Z_2$-symmetry restoration domain is the domain between the curves II and I for the case of $\tilde{m}^2 = -1.4$ and $T = \frac{1}{\kappa k_b}$, the FTGEPs at $\beta = 4.71, 4.635, 3.0, 1.05$ and $0.85$ are plotted as the curves I,II,III,IV and V in Fig.11, respectively.

Now we are in a position to consider the parameter plane $m_0^2$-$m^2$. Letting $M$ unit, quantities in Eqs.(15)—(18) can be regarded as dimensionless, and from Eqs.(15)—(18), we can compute numerically the FTGEP at a given temperature and a given $\beta$ for discussing the phases of the vacuum on the plane $m_0^2$-$m^2$. Numerical analyses indicate that a finite temperature can restore further the $Z_2$-symmetry still, but cannot give rise to $Z_2$DSB. At $\beta = 3.0$, the phase diagrams for some temperatures are plotted in Fig.12, and the phase diagrams at any other $\beta$ are similar to Fig.12. In Fig.12, the solid curve is the boundary between the symmetric and asymmetric phases for $T = \frac{1}{6k_b}$, at any point of which $V_T(\phi = \varphi_0 = 0)$ is equal to $V_T(\phi = \varphi_0 \neq 0)$ and the vacuum is either symmetrical or asymmetrical. For the region above the boundary, the vacuum is symmetrical, whereas for the region below the boundary the vacuum is asymmetrical. The upper and lower tiny-dashed curves are the boundaries for $T = \frac{1}{6k_b}$ and $\frac{1}{\kappa k_b}$, respectively, and the upper tiny-dashed curve coincide almost with the corresponding boundary at $T = 0$, i.e., the solid curve in Fig.7. For a sufficient large $m_0^2$, the boundary at any $T \neq 0$ coincides presumably with the zero-temperature boundary \[19\]. Fig.12 indicates that the finite-temperature effect compresses the $Z_2$DSB domain at $T = 0$ and enlarge the $Z_2$-symmetry restoration domain at $T = 0$. Thus, the finite-temperature effect leads to no $Z_2$DSB but a further restoration of the $Z_2$-symmetry. It is evident from Fig.12 that the higher the temperature is, the larger the $Z_2$-symmetry restoration domain is. For an illustration of the phase diagram and the $Z_2$-symmetry restoration by the finite temperature effect, Fig.13 and Fig.14 give the FTGEPs at $\beta = 3.0$ for some cases. In Fig.13, the curves I, II and III are the FTGEPs at $T = \frac{1}{6k_b}$ and $m_0^2 = 1.5$ in the cases of the $m^2 = -1.7, -2.4821738$ and $3.0$, respectively, and meanwhile for the third case we also give the FTGEP at $T = \frac{1}{\kappa k_b}$ as the curve IV. In Fig.14, the curves I, II and III are the FTGEPs at $T = \frac{1}{\kappa k_b}$

\[6\]Up to $m_0^2 = 30.0$ the numerical computation showed such a coincidence. Here we use the word “presumably” because it is impossible to check numerically all values of $m_0^2$.\[21\]
and $m_0^2 = 8.0$ in the cases of $m^2 = -5.0$, $-6.391072$ and 6.7, respectively, and meanwhile for the third case we also give the FTGEP at $T = \frac{1}{m}$. It is noticed that the points \{ $m_0^2 = 1.5, m^2 = -2.4821738$ \} and \{ $m_0^2 = 8.0, m^2 = -6.391072$ \} on the parameter plane $m_0^2-m^2$ are located at the solid curve of Fig.12, and the points \{ $m_0^2 = 1.5, m^2 = -3.0$ \} and \{ $m_0^2 = 8.0, m^2 = -6.7$ \} on the parameter plane $m_0^2-m^2$ are located in the domain between the solid and the lower tiny-dashed curves of Fig.12. Thus Fig.13 and Fig.14 plainly display the above results.

VI. CONCLUSION

The $Z_2$-symmetry of the vacuum of the 2D MSGFT was investigated both in Ref. [21] and this paper. Ref. [21] gave the phase diagram on the parameter plane $m_0^2-\beta^2$ for the zero-temperature case and showed the existence of $Z_2$SSB, whereas the present paper further gave the phase diagrams on the parameter planes $m_0^2-\beta^2$ and $m_0^2-m^2$ both for $T = 0$ and $T \neq 0$ cases and exhibited the influences of quantum and finite temperature effects on the symmetry of the vacuum. From this paper, the 2D MSGFT at the quantum level can suffer not only the restoration of $Z_2$-symmetry but also $Z_2$-DSB, and a finite temperature effect can enlarge the $Z_2$-symmetry restoration phenomenon and compress the $Z_2$DSB, namely, a finite temperature effect can further restore the $Z_2$-symmetry only. Those conclusions in the present paper are interesting and enhance our understanding on the 2D MSGFT. As was pointed out in the introduction, the GWFA is believable, at least, qualitatively, albeit it is a simple approximation. In fact, our results have confirmed and realized the inferences or predictions in Refs. [15,16,17]. Although an investigation with the help of some better approximate methods [15] would perhaps have different numerical results about the 2D MSGFT, the results in this paper will provide a basis and a reference at least, for to our knowledge, no quantitative results about the restoration and dynamical breakdown of $Z_2$-symmetry in the 2D MSGFT existed in the literature.

Perhaps some results in this paper can be conceivable from quantum mechanics. Quantum effects altering the $Z_2$-symmetry of the vacuum consists in that the differences of classical potential at different well-bottoms (and/or field points) can be offset by the corresponding differences of quantum corrections of the potentials [22,23]. Obviously, occurrences of both the restoration and the dynamical breakdown of $Z_2$-symmetry have much to do with the depths and the bottom curvatures of the wells of the classical potential (from quantum mechanics [15], the quantum correction to the classical energy gets more remarkable with the increase of the curvature of the bottom). From Eqs.(3) and (4), the curvatures of the well-bottoms are governed by the parameters $m_0^2$ and $m^2$, and the depths of the wells by the parameters $\beta$, $m_0^2$ and $m^2$. Thus, the concrete results in this paper should be acceptable. In particular, some of them can be understood simply from the above statement. For example, one can have a straightforward understanding about the existence of $\beta_{c1}$ because from Sect.II the depth of the $Z_2$-symmetry-broken well gets deeper and deeper with the decrease of $\beta$.

Those figures about parameter planes and effective potentials in this paper indicated the existence of some phase transitions. In Ref. [21], we have discussed the phase transition related to the parameters $\beta$ and $m_0$, at $T = 0$, and our viewpoints about the phase transitions related to the present cases resemble those in Ref. [21]. So here we discuss them no longer. Nevertheless, we want to mention the meaning of $Z_2$-symmetry in the massive Schwinger model. From the results here, the restoration and dynamical breakdown of the $Z_2$-symmetry is relevant to the negative $m^2$. This reminds us that the system Eq.(1) with any negative $m^2$ and $\beta^2 = 4\pi$ is equivalent to the massive Schwinger model with $\pi$ vacuum angle and on the zero-charge sector [1]. It is known that the symmetry $\phi \rightarrow -\phi$ is charge conjugation in Fermi language, and the symmetric vacuum in the 2D MSGFT corresponds to the disappearance of half-asymptotic particles (quarks and antiquarks) in the massive Schwinger model and the asymmetric vacuum to the occurrence of half-asymptotic particles in the massive Schwinger model [13]. Hence, the restoration of $Z_2$-symmetry here implies a phase transition from a half-asymptotic-particle phase to a no-quark phase, and the $Z_2$DSB corresponds to an inverse phase transition, i.e., a phase transition from a no-quark phase to a half-asymptotic-particle phase. This is an interesting phenomenon and worthy of a detailed investigation.

Finally, we also to mention that it is very difficult to extract a non-perturbative information when a finite-temperature effect is introduced into a quantum field theory. Nevertheless, Ref. [35,36] and the present paper has indicated that the GWFA is a viable and effective tool for extracting the non-perturbative information of finite temperature field theories. With the aid of the GWFA, we have calculated the masses of the Schwinger boson and its two-particle bound states in Ref. [21]. From this paper and Ref. [35], after a finite temperature effect is introduced

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7The behaviours in quantum field theory have a bit of similarity to those in quantum mechanical case.

8In fact, when a statement in this paper is related to the comparison between the results here and the inference in Ref. [15], we have used this equivalence.
into the massive Schwinger model, it will be still possible to calculate the masses of the Schwinger boson and its bound states. We believe that the GWFA will become an important tool in finite temperature field theory.

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III are the boundaries between the symmetric and asymmetric phases for the temperatures are the other two boundaries in the cases of I and II are surrounded by the solid and short-dashed curves. Additionally, the upper and lower tiny-dashed curves is shown.

Fig.1 A graphical solution of Eq.(2) with $\beta = 3.0$. The lines I,II and III are $Y = \frac{m_0^2}{m^2} \beta \phi$ with $\frac{\mu_0^2}{m^2} = -1.2, -0.5$, and $-0.1$, respectively. This figure indicates that for the three cases there are 1, 3 and 7 roots of Eq.(2), respectively.

Fig.2 The parameter space of the classical 2D MSGFT corresponding to the symmetric and the asymmetric vacua. Fig.2(a) is on the the parameter plane $\{m_0, m^2\}$ at any given $\beta$, and Fig.2(b) is on the parameter plane $\{m^2, \beta^2\}$ at a given $m_0^2$.

Fig.3 Classical potentials at $\beta = 3.0$ and $m_0^2 = 0.6$ for some $m^2$s. The curves I, II, III and IV are for the cases of $m^2 = 6.0, -0.5, -3.0$ and $-6.0$, respectively.

Fig.4 The phase diagram on the parameter plane $\bar{m}^2 - \beta^2$ at $T = 0$, which is plotted from Eqs.(10)–(14). The short-dashed line corresponds to the boundary in Fig.2(b), and the domain between it and the solid curve (the domain I) is the $Z_2$-symmetry restoration region.

Fig.5 The GEP of the 2D MSGFT at points on the parameter plane $\bar{m}^2 - \beta^2$ in the case of $\bar{m}^2 = -1.4$. Only one-half of the symmetric GEP is shown. In this figure, the curves I, II, III, IV and V correspond to $\beta^2 = 4.6, \beta_{c2} \approx 4.4685, 3.0, \beta_{c1} \approx 1.8685$ and 1.3, respectively.

Fig.6 The elucidation of the existences of $\beta_{c1}$ and $\beta_{c2}$. The solid curve is transformed from the long-dashed curve in Fig.1 of Ref. [21]. The short- or tiny-dashed curves I,II,III, IV and V are plotted from the expression of $\mu_0^2$ for the given values of $\bar{m}$: $\bar{m} = -2.0, \bar{m}_c \approx -1.68, -1.3, -1.001$ and $-0.8$, respectively.

Fig.7 The phase diagram on the parameter plane $m_0^2 - m^2$ at $T = 0$, which is plotted from Eqs.(10)–(13). In this figure, the solid curve is the boundary for $\beta = 3.0$, the short-dashed line is the boundary in Fig.2(a), and the domains I and II are surrounded by the solid and short-dashed curves. Additionally, the upper and lower tiny-dashed curves are the other two boundaries in the cases of $\beta = 4.3$ and 2.4, respectively.

Fig.8 The GEP of the 2D MSGFT for the case of $m_0^2 = 1.5$ and $\beta = 3.0$. Only one-half of the symmetric potential is shown.

Fig.9 Similar to Fig.8 but for the case of $m_0^2 = 8.0$ and $\beta = 3.0$.

Fig.10 The $Z_2$-symmetry restoration on the parameter plane $\bar{m}^2 - \beta$ at finite temperatures. The solid curves I,II and III are the boundaries between the symmetric and asymmetric phases for the temperatures $T = 1/T_{hc}, 1/T_{hc}$ and $1/T_{hc}$, respectively. The curve I coincides almost with the solid curve in Fig.4, and the short-dashed line corresponds to the boundary at the classical case in Fig.2(b).
Fig. 11 The FTGEPs of the 2D MSGFT with $T = \frac{1}{2k_b}$ at points on the parameter plane $\tilde{m}^2 - \tilde{\beta}^2$. In this figure, $\tilde{m}^2 = -1.4$ and the points $\{-1.4, 4.635\}$ and $\{-1.4, 1.105\}$ are located at the curve II in Fig. 10.

Fig. 12 The restoration and dynamical breakdown of $Z_2$-symmetry on the parameter plane $m_0^2 - m^2$ for the 2D MSGFT at finite temperatures. The solid curve is the boundary between the symmetric and asymmetric phases for $T = \frac{1}{2k_b}$, and the upper and lower tiny-dashed curves are the boundaries for $T = \frac{1}{6k_b}$ and $\frac{1}{k_b}$. The three boundaries are for the case of $\beta = 3.0$. The short-dashed line is the boundary at the classical case in Fig. 2(a), and the upper tiny-dashed curve coincides almost with the solid boundary in Fig. 7. Besides, the domains I and II are those between the solid and short-dashed boundaries.

Fig. 13 The FTGEPs of the 2D MSGFT with finite temperatures at points on the parameter plane $m_0^2 - m^2$. All curves are plotted at $m_0^2 = 1.5$, the curves I, II and III are for $m^2 = -1.7, -2.4821738$ and $-3.0$ at $T = \frac{1}{2k_b}$, respectively, and the curve IV corresponds to same point as the curve III but at $T = \frac{1}{k_b}$. The point the curve II corresponds to is located at the solid boundary in Fig. 12.

Fig. 14 Similar to Fig. 13 but All curves are plotted at $m_0^2 = 8.0$. The curves I, II and III are for $m^2 = -5.0, -6.391072$ and $-6.7$ at $T = \frac{1}{2k_b}$, respectively, and the curve IV corresponds to same point as the curve III but at $T = \frac{1}{k_b}$. The point the curve II corresponds to is located at the solid boundary in Fig. 12.
