Massive NGT and Spherically Symmetric Systems

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(January 5, 2022)

Abstract

The arguments leading to the introduction of the massive Nonsymmetric Gravitational action are reviewed [1,2], leading to an action that gives asymptotically well-behaved perturbations on GR backgrounds. Through the analysis of spherically symmetric perturbations about GR (Schwarzschild) and NGT (Wyman-type) static backgrounds, it is shown that spherically symmetric systems are not guaranteed to be static, and hence Birkhoff’s theorem is not valid in NGT. This implies that in general one must consider time dependent exteriors when looking at spherically symmetric systems in NGT. For the surviving monopole mode considered here there is no energy flux as it is short ranged by construction. Further work on the spherically symmetric case will be motivated through a discussion of the possibility that there remain additional modes that do not show up in weak field situations, but nonetheless exist in the full theory and may again result in bad global asymptotics. A presentation of the action and field equations in a general frame is given in the course of the paper, providing an alternative approach to dealing with the algebraic complications inherent in NGT, as well as offering a more general framework for discussing the physics of the antisymmetric sector.

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I. INTRODUCTION

In General Relativity (GR), just as in Maxwell’s electrodynamics, one finds that given a spherically symmetric system, there are no dynamical degrees of freedom in the theory. This is Birkhoff’s theorem, and implies that a time dependent source will not excite modes in the gravitational system, so that outside this source the system must be physically equivalent to the Schwarzschild solution (the non-trivial static spherically symmetric spacetime). It has been further established [3,4] that the solution is stable when perturbed, so that small deviations from spherical symmetry do not alter the large scale features of the spacetime, and systems that are only approximately spherically symmetric are therefore still very well modeled by the Schwarzschild solution. This establishes that the phenomenology of the Schwarzschild solution is physically relevant.

To see that Birkhoff’s theorem is not a generic feature of physical theories one need look no further than a scalar field. However the example that is important for this work is the (massive) Kalb-Ramond action [5] which (as will be demonstrated in Section II) has a single mode that is time dependent in general. As will be shown in Section II, the massive Nonsymmetric Gravitational Theory [1,2] (mNGT as opposed to the older versions of the theory, referred to as NGT, or massless NGT) becomes identically a massive Kalb-Ramond field with an additional curvature coupling term when considered as a perturbation about a Ricci-flat GR background, so the result that NGT has a monopole mode is not surprising. The mode considered here is short ranged, so that far enough away from the source one finds that the solution will be dominated by Schwarzschild behavior, and there is no energy flux. However, after demonstrating that spherically symmetric fields in the skew sector are not static in general, it will be shown in Section V that the symmetric sector will also no longer be static, through examination of a similar perturbation about the mNGT background discussed in Section V. This means that no static solutions can be considered rigorously as an exterior solution unless the solution is globally static (i.e. the interior is static as well).

The results in this paper are obtained through an examination of linearized perturbations, although the conclusions must hold in general as Birkhoff’s theorem would imply that these fields must be static as well. What cannot be examined in this fashion is whether in the full nonlinear theory more modes become excited. In particular, one will see in the case of a perturbation about a GR background (in Section II) that there are three propagating modes, even though the absence of gauge invariant kinetic terms in the full theory would suggest that all six modes in the antisymmetric metric could be independent degrees of freedom (as yet, the number of degrees of freedom in mNGT has not been rigorously established). Although this issue is not addressed directly in this work, the ability to recast the theory in a general basis given in Section V sets the stage for a complete analysis of the spherically symmetric system in NGT. Given that there is an additional mode in the general spherically symmetric system, how the fields may or may not approach an asymptotically flat spacetime should then be addressed, and also whether evolution singularities of the type discussed in [6] are encountered.
II. MASSIVE NGT

The original version of NGT \[7\] grew out of a re-interpretation of the Einstein-Straus \[8,9\] unified field theory as a purely gravitational system. The antisymmetric part of the metric and connection operationally produce different modes of parallel transport and index contraction \[10–14\], where the algebra is consistent with an enlargement of the tangent vector space to its hyperbolic complex extension \[12\]. It is important to note that the action cannot support the additional Bianchi identities and gauge invariance related to the extension of the tangent bundle, simply because the base manifold is locally diffeomorphic to \(\mathbb{R}^4\), and the variational principle is based on an integration over this real manifold. Any change of gauge that mixes real and hyperbolic complex covectors will cause the volume element to pick up a hyperbolic complex piece, and the action will no longer be real.

The hyperbolic complex structure is unnecessary for an operational discussion of the theory (although it may be relevant for a more fundamental discussion of its physical interpretation), and in this paper all quantities will be considered real, allowing antisymmetric contributions to the metric and connection coefficients. The dynamics of the theory will be determined from the first order action \((G = c = 1)\):

\[
S = \int d^4x \left\{ -g^{\mu\nu}R^{\text{NS}}_{\mu\nu} + g^{\mu\nu} \partial_\mu W_\nu + \frac{1}{2} \alpha g^{(\mu\nu)} W_\mu W_\nu + \frac{1}{4} m^2 g^{[\mu\nu]} g_{[\mu\nu]} \right\} + S_M, \tag{1}
\]

where:

\[
\frac{\delta S_M}{\delta g^{\mu\nu}} = T_{\mu\nu}, \tag{2}
\]

is the matter stress energy tensor that acts as a source in the gravitational field equations. The Ricci-like tensor for NGT appearing in (1) is written as:

\[
R^{\text{NS}}_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \frac{1}{2} (\partial_\nu \Gamma^\alpha_{\mu\alpha} + \partial_\mu \Gamma^\alpha_{\alpha\nu}) + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{(\alpha\beta)} - \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\mu\alpha}, \tag{3}
\]

and a mass term for \(g_{[\mu\nu]}\) has been included along with a term quadratic in \(W\) as new features of the action. As will become clear shortly, \(\alpha\) may be fixed uniquely by requiring good asymptotic behavior of perturbations about GR backgrounds. The new parameter in the massive action \((m)\) is an inverse length scale that must be constrained by experiment, and \(l\) is a Lagrange multiplier employed to enforce the vanishing of the trace of the antisymmetric part of the connection coefficients: \(\Gamma_\mu = \Gamma^\alpha_{[\mu\alpha]}\).

The field equations related to the metric compatibility conditions are derived through the variations:

\[
\frac{\delta S}{\delta l^\mu} = \Gamma_\mu = 0, \tag{4a}
\]

\[
\frac{\delta S}{\delta W_\mu} = \partial_\nu g^{\nu\mu} + \alpha g^{(\mu\nu)} W_\nu = 0, \tag{4b}
\]

\[
\frac{\delta S}{\delta \Gamma^\alpha_{\sigma\omega}} = \partial_\gamma g^{\sigma\omega} - g^{\sigma\omega} \Gamma^\alpha_{(\gamma\alpha)} + g^{\sigma\omega} \Gamma^\gamma_{\alpha\gamma} + g^{\sigma\alpha} \Gamma^\omega_{\gamma\alpha} - \frac{1}{2} \delta^\gamma_\sigma (\partial_\alpha g^{\sigma\alpha} + g^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} - l^\sigma) - \frac{1}{2} \delta^\alpha_\gamma (\partial_\omega g^{\alpha\omega} + g^{\alpha\beta} \Gamma^\omega_{\alpha\beta} + l^\omega). \tag{4c}
\]
Contracting on either index of (4c) and solving for the (anti-)symmetric parts of the
divergence of the densitized inverse metric results in the determination of the Lagrange multiplier
using (4b):
\[ l^{\sigma} = \frac{2}{3} \delta^{(\sigma)}_{\omega} W_{\omega}. \]
This also allows one to simplify the Kronecker-\(\delta\) terms, and
determine the compatibility conditions in undensitized form as:
\[ \partial_\gamma g_{\mu\nu} - g_{\mu\alpha} \Gamma^\alpha_{\nu\gamma} - g_{\alpha\nu} \Gamma^\alpha_{\mu\gamma} = \frac{2}{3} \alpha (g_{\mu[\gamma} g_{\alpha]\nu} + \frac{1}{2} g_{\mu\nu} g_{[\gamma\alpha]} g^{(\alpha\beta)} W_{\beta}), \tag{5} \]
where the inverse of the metric has been defined by
\[ g_{\mu\alpha} g^{\alpha\nu} = \delta^{\nu}_{\mu}, \]
which has been used in order to rewrite the compatibility conditions in terms of the components of the metric \( g_{\mu\nu} \).

The remaining field equations derived from the variation of the action with respect to
\( g_{\mu\nu} \) may be written as:
\[ R_{\mu\nu} := R^{\text{NS}}_{\mu\nu} + \partial_\mu W_{\nu} - \frac{1}{2} \alpha W_{\mu \nu} - \frac{1}{4} m^2 (g_{\mu\nu} - g_{\alpha\mu} g_{\beta\nu} g^{[\alpha\beta]} + \frac{1}{2} g_{\nu\mu} g^{[\alpha\beta]} g_{[\alpha\beta]})) = T_{\mu\nu} - \frac{1}{2} g_{\nu\mu} T, \tag{6} \]
where \( T = g^{\mu\nu} T_{\mu\nu} \), and the tensor \( R \) has been introduced in order to simplify the discussion
of the field equations. One may translate the conventions used here to those in [1] by taking
\( W \rightarrow -\frac{2}{3} W, T_{\mu\nu} \rightarrow -8\pi T_{\mu\nu}, \alpha \rightarrow -\frac{9}{4} \sigma \), and adjusting the definitions of the inverse metric:
\( g^{\mu\nu} \rightarrow g^{\mu\nu} \). To see the equivalence of the action, one further needs to rewrite \( \Gamma \) in terms of
the unconstrained \( W \) connection, and drop the contribution from the Lagrange multiplier \( l \).

The action for massless NGT is given by (1) with \( m = \alpha = 0 \). As will be demonstrated,
the new terms have been introduced in order to make all skew modes short ranged when
considering perturbations about GR backgrounds. One performs this expansion about a
symmetric, Ricci-flat background, where one assumes that all background curvatures fall
off at worst as \( 1/r \) as \( r \rightarrow \infty \). This allows one to talk sensibly of energy-momentum
and decompose fields via a spin projection, so that higher order poles and negative energy
(ghost) modes may be identified, as well as avoiding the full nonsymmetric structure of a
more general background that would make the analysis far more complicated. One considers
a perturbation of all quantities about a symmetric GR background as in [15]:
\[ g_{\mu\nu} \rightarrow g_{[\mu\nu]} + h_{\mu\nu}, \]
\[ \Gamma^\alpha_{\mu\nu} \rightarrow \{ \mu \} + \gamma^\alpha_{[\mu\nu]}, \tag{7} \]
where \( W, l \) and \( T \) are considered to be first order in the perturbation (as the background
is assumed to be Ricci-flat, there one has \( T_{\mu\nu} = 0 \)). As usual, indices will be ‘raised’ and
‘lowered’ by the symmetric background metric, and the covariant derivative \( \nabla \) is associated
with the background Christoffel symbols \( \{ \} \) determined from the background metric in the
usual manner. Corrections to the background curvatures, field equations and Lagrangian at
each order in the perturbation will be indicated by a superscript as: \( \partial R, 1R \ldots \).

The first order correction to the compatibility equation (3) can be solved explicitly for
\( \gamma \) to yield:
\[ \gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\nabla_\nu h_{[\beta\mu]} + \nabla_\mu h_{[\beta\nu]} - \nabla_{[\beta} h_{\mu]} + \frac{2}{3} \alpha \delta^\alpha_{[\mu} W_{\nu]} \), \tag{8} \]
and \( \gamma_\mu = \gamma^\alpha_{[\mu\alpha]} \) is seen to vanish by the linearization of the skew divergence equation:
\[ - \nabla_\nu h^{[\mu\nu]} = \alpha W_\mu. \tag{9} \]
In massless NGT \cite{10}, one had \((8,9)\) with \(\alpha = 0\), and hence there was no relation between the metric degrees of freedom and those of \(W\). The skew part of the linearization of equation (6) with \(m = 0\) as well as \(\alpha = 0\) became:

\[
\begin{align*}
\tilde{R}_{\mu\nu} &= \tilde{R}_{[\mu\nu]} + \partial_{[\mu}W_{\nu]} = \nabla_\alpha\gamma^\alpha_{[\mu\nu]} + \partial_{[\mu}W_{\nu]} = T_{[\mu\nu]} .
\end{align*}
\tag{10}
\]

The symmetric contribution, are the equations for a metric perturbation in GR \cite{16}, and will be ignored in the remainder of this section. Using (9) with \(\alpha = 0\), the assumption that the background is Ricci-flat \((\overset{0}{R}_{\mu\nu} = 0)\), and the commutation relation (for an arbitrary tensor \(B\)):

\[
\nabla_\alpha\nabla_\beta B_{\mu\nu} = -\frac{1}{2}(B_{\omega\nu}\overset{0}{R}_{\omega\mu\alpha\beta} + B_{\mu\omega}\overset{0}{R}_{\omega\nu\alpha\beta}),
\tag{11}
\]

(10) simplifies to:

\[
\begin{align*}

\nabla_\alpha\nabla_\alpha h_{[\mu\nu]} - 2\partial_{[\mu}W_{\nu]} - 4\overset{0}{R}_{\mu\alpha\beta}^\alpha\beta h_{[\alpha\beta]} = \nabla_\alpha F_{\mu\alpha\beta} - 2\partial_{[\mu}W_{\nu]} - 8\overset{0}{R}_{\mu\nu}^\alpha\beta h_{[\alpha\beta]} = -2T_{[\mu\nu]} .
\end{align*}
\tag{12}
\]

The second form is given in terms of the curl of the skew metric \((F_{\gamma\mu\nu} = \partial_{\gamma}h_{\mu\nu} + \partial_{\mu}h_{\gamma\nu} + \partial_{\nu}h_{\gamma\mu})\), in order to more easily demonstrate the result found previously by Damour, Deser, and McCarthy \cite{17,18} (using the fact that \(\nabla_\alpha\nabla_\nu F_{\mu\nu\gamma} = 0\) about a Ricci-flat background), that these antisymmetric perturbations will in general have bad asymptotic behavior. Although written as if the skew metric were a gauge field, the presence of the curvature coupling term implies that the associated gauge invariance is not present \cite{19}, and one is not allowed to make any choice of gauge in order to simplify this sector. One proceeds by taking the divergence of (12) and choosing the gauge \(\nabla_\alpha W_\alpha = 0\) (the theory had a \(U(1)\) invariance as \(W\) only appeared in the action in a curl) to find:

\[
\begin{align*}

\nabla_\alpha\nabla_\alpha W_\mu - 8\nabla_\nu\overset{0}{R}_{\mu\nu}^\alpha\beta h_{[\alpha\beta]} = -2\nabla_\nu T_{[\mu\nu]} .
\end{align*}
\tag{13}
\]

Notice that the background curvature acts as a source here, so that even if one postulates that the matter source is conserved, this curvature coupling (and in general other nonlinear terms) will still exist as a source, causing \(W\) to propagate and have asymptotic behavior consistent with a massless field \((\sim 1/r\) along the forward light cone). Using this asymptotic behavior to determine \(h\) from (12) results in a source with \(\sim 1/r\) behavior, causing the field \(h\) not to fall off as \(r \to \infty\) along the forward light cone. This analysis is correct since one has assumed that the background curvature falls off fast enough, and hence the potential term in (13) can be treated as a source, without changing the asymptotic behavior of the fields. It must be stressed that one is assuming that the background \(\textit{and} the radiative fields fall off at least as fast as } \sim 1/r\text{, and what has actually been derived here is a contradiction of this, since } h \text{ is driven to a constant and can no longer be considered as a perturbative mode.}

Any analysis of this sort also supposes that a solution of the linearized field equations does in fact correspond to an exact solution of the full nonlinear field equations. This is the case in GR \cite{16,20} at least for source free equations, but no such result exists yet for any of the models considered here. It is possible to take the stance that NGT is not linearization-stable, so that this sort of analysis necessarily produces spurious results that do not correspond to global solutions, but then one is denying the ability to do any sort of
perturbative analysis without the existence of an exact solution to back it up. Due to the scarcity of solutions, and the apparent existence of weak-field perturbative situations, this would seem an unreasonable position to adopt.

This result is not confined to curved backgrounds, and in fact the analysis about Minkowski space will serve to explicitly demonstrate the higher order pole leading to bad fall-off. Since this curvature coupling, and any nonlinear effects in general, will act as a nonconserved source term in the skew sector, the linearization that correctly represents the full nonlinear field equations in the asymptotic region will have a nonconserved source term. This is no more than the observation that once again, the full NGT action does not possess any form of additional gauge invariance in the skew sector. A gauge field coupled to a source (or matter) in a non-gauge invariant manner may have drastically different behavior than the empty space and apparently gauge invariant field equations, if indeed the action is consistent at all. Any analysis that attempts to determine the propagator or asymptotic behavior of the field must take the form of the source (or coupling to other fields) into account.

A trivial example of this is given by considering the Maxwell action. Coupling the usual gauge invariant kinetic terms to a nonconserved source gives an inconsistent set of field equations, and adding some sort of gauge fixing term will give a consistent set of equations, but the scalar ghost mode will be excited and depending on the gauge there may be higher order poles in the solution. The linearized field equations considered outside the source resemble those of the gauge invariant theory in a particular gauge, but treating them as such will not give asymptotic behavior that follows from coupling to the nonconserved source. The situation in NGT is more akin to enforcing the gauge condition in the action through the use of an auxiliary field as: \( b \partial_\mu A^\mu \) [21,22]. Source conservation and absence of ghosts relies on whether or not the scalar Lagrange multiplier field \( b \) has a source or not in the wave equation that determines it, and is thus a global question. Given that the source for \( A \) is not conserved, then \( b \) propagates and there are higher order poles in the solution for \( A \), leading to fields that do not fall off as \( r \to \infty \) along the forward light cone.

Considering the field equations for massless NGT linearized from (12) about Minkowski space:

\[
\begin{align*}
\Box h_{\mu \nu} - 2\partial_\mu W_\nu &= -2T_{[\mu \nu]}, \\
\partial_\nu h_{[\mu \nu]} &= 0,
\end{align*}
\]

(14a) (14b)

\( (\partial^\nu T_{[\nu \mu]} \neq 0) \), one may take a divergence (to find a wave equation for \( W \)) or a curl (to remove \( W \)) of the first of these, resulting in the unique consistent solution:

\[
\begin{align*}
h_{[\mu \nu]} &= -2\Box^{-1} \left[ T_{[\mu \nu]} + 2\Box^{-1} \partial^\alpha \partial_\mu T_{[\alpha \nu]} \right], \\
W_\mu &= -2\Box^{-1} \partial^\nu T_{[\mu \nu]}.
\end{align*}
\]

The presence of the higher order pole (and consequent bad fall-off) is now obvious from the presence of the \( \Box^{-2} \) term in the Green function solution, and is no more than the result of vanNieuwenhuizen [23] who showed that the only healthy quadratic actions built of antisymmetric tensor fields are the so-called Kalb-Ramond [5] (massless or massive) actions. One also notes that there are 5 modes here: 3 in \( h \), since 3 are determined algebraically by the second equation in (14), and 2 in \( W \) due to the previously mentioned \( U(1) \) gauge invariance [24]. If it is assumed that the (matter) source is conserved, the higher order poles
are removed at linear order, but show up in the second order correction to the fields, again causing a breakdown of the perturbative analysis.

This analysis correctly represents the asymptotic behavior of the fields \((W, h)\), and is equivalent to equation (18) of \([25]\), where the higher order pole resides in the projection operator: \(P(1^+)\). One also sees the true propagating nature of \(W\), and this is borne out by the analysis in \([26, 27]\) where there are five degrees of freedom evolving from each Cauchy surface, the extra two of which are associated with the field \(W\). That a Lagrange multiplier is propagating merely signifies that it is a determined multiplier, with its evolution derived from the field equations \([15]\) and not freely fixable as was done in \([28, 29]\) and in the next to last section of \([24]\) where \(ad \ hoc\) constraints were imposed on the linearized theory in order to obtain the dynamics of a Kalb-Ramond theory. That these constraints cannot exist is clear from the lack of gauge invariance in the full NGT action.

The result of vanNieuwenhuizen does however motivate a potential solution to this problem, since the massive Kalb-Ramond theory does not require a conserved current and yet has no ghost modes, higher order poles or tachyons. The additional terms in the action for mNGT \([1]\) are introduced in order to allow the linearized field equations of NGT to take on this form in the antisymmetric sector. These two terms play slightly different roles: the \(W^2\) term causes \(W\) to be determined in terms of metric functions directly (\(\alpha\) is fixed in order to find the correct form of the kinetic energy terms), and the mass term for \(g[\ ]\) makes the skew sector short-ranged, and ensures that the linearized field equations remain consistent when expanding about a flat background.

Thus mNGT should have a linearization about Minkowski space of the form:

\[
\partial^\alpha F_{\mu\nu\alpha} + m^2 h_{[\mu\nu]} = J_{[\mu\nu]}.
\] (16)

The solution to (16) can be found by taking a divergence and substituting back in to find:

\[
\partial^\nu h_{[\mu\nu]} = \frac{1}{m^2} \partial^\nu J_{[\mu\nu]}.
\] (17)

The higher order poles have disappeared, and it can be shown that the linearized Hamiltonian is weakly positive definite and that ghost modes are removed through the algebraic conditions that couple them locally to the source in (17). About a more general background one can allow a curvature coupling term, since it will not affect the behavior of the fields asymptotically once the background is assumed to fall off appropriately. Choosing the theory that results in this behavior in the linearized theory will fix \(\alpha\) uniquely.

Returning now to the field equations of mNGT expanded about a GR background following from (17), one finds:

\[
^1R_{\mu\nu} = \nabla \alpha \Gamma_{\mu\nu}^\alpha - \frac{1}{2} m^2 h_{[\mu\nu]} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T,
\] (18)

where the first order correction to the ‘Ricci’ tensor is given by:

\[
^1R_{\mu\nu} = \nabla \Gamma_{\mu\nu}^\alpha - \Gamma_{(\nu}^\alpha \Gamma_{\mu)}^\alpha.
\] (19)

Again ignoring the symmetric GR perturbations, the antisymmetric part of (18) is:
\[
\n\frac{\nabla \alpha \gamma_{[\mu \nu]} + \frac{1}{\alpha} \nabla_{[\mu} \nabla^{\alpha} h_{\alpha \mu]}}{\alpha} - \frac{1}{2} \frac{m^2 h_{[\mu \nu]}}{\alpha} = -\frac{1}{2} (\nabla^{\alpha} F_{\mu \alpha} + m^2 h_{[\mu \nu]}) - 2 \nabla^{\alpha} \nabla_{[\mu} h_{\alpha \nu]} + \frac{1}{\alpha} [1 + \frac{2}{3} \alpha] \nabla_{[\mu} \nabla^{\alpha} h_{\alpha \nu]} = T_{[\mu \nu]}.
\]

(20)

Requiring that this reduce to the massive Kalb-Ramond field equations (16) determines the (previously arbitrary) coupling: \( \alpha = 3/4 \). The last two terms can be reduced to a curvature term to give:

\[
\nabla^{\alpha} F_{\mu \alpha} + m^2 h_{[\mu \nu]} - 4^0 R_{[\mu \nu]} h_{[\alpha \beta]} = -2 T_{[\mu \nu]},
\]

(21)

so the skew sector perturbations are well-behaved when perturbing about any asymptotically flat GR background.

Expanding the action (1) to second order (ignoring surface terms) gives:

\[
2L = -2R - \frac{1}{2} h^1 R + h^{\mu \nu} R_{\mu \nu} + h^{\mu \nu} \partial_{[\mu} W_{\nu]} + \frac{1}{2} \alpha W^\mu W_\mu + l^\mu \Gamma_\mu - \frac{1}{4} m^2 h^{[\mu \nu]} h_{[\mu \nu]}
\]

(22)

and once compatibility is imposed, followed by the removal of \( W \), this becomes:

\[
2L = \frac{1}{12} F^{\mu \nu \gamma} F_{\mu \nu \gamma} - \frac{1}{4} m^2 h^{[\mu \nu]} h_{[\mu \nu]} - \nabla^{\gamma} h^{[\mu \nu]} \nabla_\gamma h_{[\mu \nu]} - \left( \frac{1}{2 \alpha} + \frac{1}{3} \right) \nabla_\nu h^{[\mu \nu]} \nabla^{\gamma} h_{[\mu \nu]}.
\]

(23)

Choosing \( \alpha = 3/4 \) results in kinetic terms identical to those of Kalb-Ramond theory on a GR background, giving the skew sector action:

\[
2L = \frac{1}{12} F^{\mu \nu \gamma} F_{\mu \nu \gamma} - \frac{1}{4} m^2 h^{[\mu \nu]} h_{[\mu \nu]} - h^{[\mu \nu]} h^{[\alpha \beta]} R_{\alpha \mu \beta \nu},
\]

(24)

which reproduces the linearized field equations (21). Thus the massive NGT action will be (1) with \( \alpha = 3/4 \) ([13]), giving the action (24) for perturbations about a GR background and guaranteeing good asymptotic behavior for these fields.

Although it has been established that the perturbation equations about a GR background are a consistent system resulting in good fall-off for the skew sector, it is not clear whether an asymptotic perturbation actually corresponds to a global solution (linearization stability). The (seemingly contrived) asymptotic limit of gauge invariant kinetic terms cannot be reflected in the full action, since there is no room for the additional gauge invariance in theories constructed from antisymmetric fields in this manner. This means that in general that one expects more (perhaps all 6) degrees of freedom in the skew sector evolving as degrees of freedom in a Cauchy analysis, whereas in any spacetime that has an asymptotically flat region only three will survive. This situation could be similar to that found in ([14], where vector fields were seen to increase their degrees of freedom when gravitational effects are taken into account. In order to obtain an asymptotically flat spacetime (with the reduced degrees of freedom of the vector fields) from physically reasonable initial data, the evolution equations were seen to have to encounter singularities. This is generally considered to be a sign of instability, and certainly not a desirable feature in any theory. Perturbations about NGT backgrounds should also be considered, since the physically interesting NGT solutions are most likely those that are not ‘close’ to a GR solution ([30]). It is hoped that a more complete analysis of the general spherically symmetric system should be able to say something about this issue, since it will certainly tell one how many degrees of freedom survive and how they couple to external fields, and hopefully something about how the system may or may not approach an asymptotically flat spacetime.
It is also true that the form of the action (1) is far from unique. In particular, one could replace the $W^2$ term with some combination of $W^2$ and $\frac{1}{\sqrt{-g}}g_{(\mu\nu)}\partial_\alpha[g^{[\mu\alpha]}\partial_\gamma[g^{[\nu\gamma]}]]$, giving the same perturbation equations (21), and resulting in an arbitrary coupling constant in the action. Further, since there is nothing preventing one from adding $\Gamma$ terms (they are tensors) or even infinite strings of terms of the form: $g_{(\alpha\beta)}g^{(\beta\gamma)}\ldots$ or $g_{[\alpha\beta]}g^{[\beta\gamma]}\ldots$ (which conveniently disappear in the asymptotic expansion), there is clearly an infinite number of actions that do this. These examples seem extremely unnatural and will not be considered further here, although the results of this paper would not change significantly for these more general actions.

III. SPHERICALLY SYMMETRIC PERTURBATION OF THE SCHWARZSCHILD SOLUTION IN A COORDINATE BASIS

The absence of a Birkhoff theorem may be derived from the perturbation equations (21) developed in Section II. In general the spherically symmetric fields in the skew sector will not be static, although the symmetric sector will remain static in the perturbation about the GR solution considered here. The background metric is Schwarzschild with (coordinate basis) metric written as: $g = \text{diag}(A(r), -1/A(r), -r^2, -r^2 \sin^2(\theta))$, where $A(r) = 1 - 2M_s/r$ and $M_s$ is the Schwarzschild mass parameter. The perturbation considered will be one that is spherically symmetric but not necessarily static. A killing vector analysis yields the general form of the spherically symmetric perturbation:

$$|h_{\mu\nu}| = \begin{pmatrix}
  h_{00}(t, r) & h_{01}(t, r) & h_{01}(t, r) + h_{[01]}(t, r) & 0 & 0 \\
  h_{(01)}(t, r) - h_{[01]}(t, r) & h_{11}(t, r) & 0 & 0 \\
  0 & 0 & h_{22}(t, r) & h_{[23]}(t, r) \sin(\theta) \\
  0 & 0 & -h_{[23]}(t, r) \sin(\theta) & h_{22}(t, r) \sin^2(\theta)
\end{pmatrix}.$$  (25)

Making a change of coordinates of the background geometry is equivalent to making a change of gauge on the perturbation: $\delta h = \mathcal{L}_\varepsilon[g]$, where $\varepsilon$ is the spherically symmetric vector gauge parameter generating diffeomorphisms between spherically symmetric spacetimes. This allows one to simplify the form of the perturbation by a suitable choice of gauge. Choosing the gauge parameter as:

$$\varepsilon^0 = -\int \left(\frac{h_{(01)}}{A(r)} - \frac{\partial_t[h_{22}]}{2rA(r)^2}\right) dr,$$  (26a)

$$\varepsilon^1 = \frac{h_{22}}{2r},$$  (26b)

removes the $\theta - \theta$, $\phi - \phi$, and symmetric $t - r$ perturbations altogether, and a remaining gauge transformation $\varepsilon^0 = \epsilon(t)$ allows one to remove an arbitrary function from the $t - t$ component of the form: $\delta h_{00} = 2A(r)\epsilon(t)$.

The field equations will be written without the source terms for simplicity although it is straightforward to include them and relate the constants of integration to properties of the source. First reviewing how the symmetric (in this case identically GR) perturbations become static, it is simplest to begin with the field equation: $\mathcal{R}_{(01)} = 0$, which implies:
\[ \partial_t [h_{11}(t, r)] = 0, \]  
(27)
immediately showing that \( h_{11} \) must be static. By considering \( \mathcal{R}_{22} = 0 \), it is determined to be:

\[ h_{11}(t, r) = -\frac{2\delta M_s}{rA^2}, \]  
(28)
where the integration constant has been combined with \( M_s \) and interpreted as a perturbation of the Schwarzschild mass parameter: \( \delta M_s \). Then one considers:

\[ g^{00} \mathcal{R}_{00} - g^{11} \mathcal{R}_{11} = 0, \]
leading to:

\[ h_{00}(t, r) = B(t)A - \frac{2\delta M_s}{r}, \]  
(29)
also giving a contribution arising from the perturbed mass parameter, as well as an arbitrary function of time as an integration constant, removable by the remaining choice of gauge noted above with: \( \epsilon(t) = -B(t)/2 \). Thus one has that the symmetric perturbations are static and interpretable as being due to a small change in the total energy of the system: \( \delta M_s \).

In the skew sector, the \( t - r \) field equation gives:

\[ \mathcal{R}_{[01]} = \frac{1}{2}(2A'' - m^2)h_{[01]} = 0, \]  
(30)
from which one must conclude that \( h_{[01]} \) vanishes outside the source. (Primes will denote the derivative of a function of one variable where convenient.) In massless Kalb-Ramond theory, this is the surviving spherically symmetric ghost mode which in that case is pure gauge. When a mass term is added, although these modes are now no longer pure gauge, they do not propagate since they are locally coupled to the source. It is these modes that one eventually must worry about, since in the full theory they may play a nontrivial dynamical role. In the \( \theta - \phi \) sector:

\[ \mathcal{R}_{[23]} = -\frac{1}{2} \left\{ \frac{1}{A} \partial_t^2 [h_{[23]}] - A \partial_r^2 [h_{[23]}] - (A' - \frac{2A}{r}) \partial_r [h_{[23]}] + \left( 4 \frac{r^2}{r^2} (1 - A) + m^2 \right) h_{[23]} \right\} \sin(\theta) = 0. \]  
(31)

In order to derive the asymptotic form of this perturbation, it is convenient to define \( h_{[23]} = rf(t, r) \), leading to:

\[ \frac{1}{A} \partial_t^2 [f] - A \partial_r^2 [f] - A' \partial_r [f] + (\frac{3A'}{r} + \frac{2A}{r^2} + m^2) f = 0. \]  
(32)
Introducing the coordinate:

\[ r* = \int \frac{dr}{A(r)} = r + 2M_s \ln(\frac{r}{2M_s} - 1), \]  
(33)
one obtains (after multiplying by \( A \)) the partial differential equation for \( f \) in normal form:

\[ \partial_t^2 [f] - \partial_r^2 [f] + A (m^2 + \frac{2A}{r^2} + \frac{3A'}{r}) f = 0 \]  
(34)
where $r$ is considered as a function of $r^*$ as are $A(r)$ and $\partial_r A(r)$, and the perturbation $f = f(t, r^*)$. In this form it is obvious that (31) is a hyperbolic wave equation, and that the field $f$ is therefore nonlocally related to the source.

Using the fact that $1/r - 1/r^* \sim o(1/(r^*)^2)$ as $r^* \to \infty$, one keeps only the constant mass term asymptotically, as all other potential terms will be dominated by it. This leaves the massive scalar wave equation to determine the asymptotic form of the perturbation:

$$\partial_t^2 [f] - \partial_{r^*}^2 [f] + m^2 f \sim 0. \quad (35)$$

The static solution of this is easily seen to have the asymptotic form:

$$h_{[23]}(r) \sim F_0 \frac{r}{m} e^{-mr^*} \sim F_0 \frac{r}{m} e^{-mr} (\frac{r}{2M_s})^{-2mM_s}, \quad (36)$$

where a factor of $m$ has been introduced in order to make the constant $F_0$ dimensionless.

The general time dependent case may be handled by noting that the retarded and advanced Green functions for the massive scalar wave equation [31] ($x^2 = t^2 - \vec{x}^2$):

$$D_{ret,adv}(x) = D_{ret,adv}(t, r^*) = \frac{1}{2\pi} \theta(\pm x_0) \left[ \delta(x^2) - \frac{m \theta(x_0)}{2\sqrt{x^2}} J_1(m \sqrt{x^2}) \right], \quad (37)$$

depends only on $(t, r)$, and $r^* D_{ret,adv}(t, r^*)$ will solve (35). The asymptotic behavior of $h_{[23]}$ is then determined from:

$$h_{[23]} \sim r r^* D_{ret,adv}(t, r^*). \quad (38)$$

Note that the behavior on the light cone is determined from just the massless Green function $\delta(x^2)$ [32], and so it would appear that $h_{[23]}$ will behave as $r$ as $r \to \infty$ along the forward light cone. This is misleading, as it can be demonstrated explicitly [33] that for $C^\infty$ initial data with compact spatial support, a massive Klein-Gordon field is bounded everywhere by: $\phi \leq d(1+ |t|)^{-3/2}$, for some constant $d$, and therefore cannot radiate energy. This can also be understood by noting that because the field is massive, the effects propagating on the light cone must be fields of infinite energy, and given some physically reasonable source distribution, these infinite energy modes will not be excited.

The existence of time dependent solutions thus proves that Birkhoff theorem is not valid in mNGT, although the short-ranged nature of the skew sector implies that monopole radiation will not exist. The symmetric sector has remained static in this system, but as will be shown in Section IV, through a perturbation about an approximated mNGT solution, this will not be the case in general. The perturbation equations about a mNGT background have not been given in covariant form, primarily due to the complication involved (although it is possible in principle using a generalization of the inversion of the compatibility equation given in [34]). Instead the system may be developed in each case separately, and the analysis simplified by considering the field equations in a vierbein frame given in the next Section.

**IV. NONSYMMETRIC THEORIES IN A GENERAL FRAME**

The structure of the compatibility relations and field equations in nonsymmetric theories can be formulated in terms of components in a general moving frame (in the sense of
global section of the general linear frame bundle $GL\mathcal{M}$ of all linear frames over $\mathcal{M}$). The formalism given here is essentially a more systematic development of the approach in [33], and differs slightly from that of Hlavaty [34] in that the (in general nonsymmetric in a coordinate basis) connection coefficients have been split up into a connection that is torsion free, and another that is purely antisymmetric, instead of defining two types of covariant derivative, one associated with the NGT Christoffel symbols, and another that is in general non-symmetric and not in general torsion-free. The construction here has the advantage of only defining one covariant derivative, and the fact that it is torsion-free implies that the antisymmetric components in a general (non-coordinate) frame are related in the standard way to the structure constants. In a coordinate basis this is the usual split between the symmetric and antisymmetric components, however it is easily generalized to any basis by considering the antisymmetric components as a separate antisymmetric tensor, and the symmetric components as a torsion-free but generally non-compatible connection.

This provides a simple way to split the GR and NGT contribution in weak field situations, as well as generating computationally simpler systems to solve when inverting the compatibility relations. Note that although the formalism is developed for a general basis, the specialization to a vierbein basis (the reduction of $GL\mathcal{M}$ to $L\mathcal{M}$, the Lorentz frame bundle consisting of all Lorentz frames above $\mathcal{M}$) which will be utilized in the rest of this paper, is accomplished through the choice of the fiber metric as $g(\cdot) \rightarrow \eta$ above all points of the manifold. This is possible in NGT for the same reason that it is possible in GR: mathematically formulating a physical theory in a diffeomorphism invariant manner will always allow the introduction of these general linear frames. The reduction to Lorentz frames is also possible as one is assuming that the symmetric part of the metric that one is attempting to diagonalize is nondegenerate, allowing the reduction of the frame bundle. This construction will be of importance when considering the canonical analysis of NGT, as one would like to work in a surface compatible (generally non-coordinate) basis in order to avoid specialization to a particular choice of time parameter fixed by the foliation of the manifold, and is easily applied to other systems with a nonsymmetric metric and connection [29].

A. Metric, Compatibility and Curvature

The compatibility conditions in a coordinate basis (3) will be written for convenience as:

$$\partial_\gamma [g_{\mu\nu}] - g_{\mu\alpha} \Gamma^\alpha_{\nu\gamma} - g_{\alpha\nu} \Gamma^\alpha_{\gamma\mu} = -\Delta^0_{\gamma\mu\nu}$$

(39)

where $\Delta^0$ depends only on the metric or quantities directly derivable from it (and possibly other quantities, but for the purposes of this construction it does not depend on the connection coefficients). Parallel transport (and the related covariant derivative) will then be defined using just the symmetric part of the coordinate basis connection, and its action on the (coordinate) basis vectors is:

$$\nabla_{e_\alpha} [e]_\beta = \Gamma^\gamma_{(\alpha\beta)} e_\gamma, \quad \nabla_{e_\alpha} [\theta]_\gamma = -\Gamma^\gamma_{(\alpha\beta)} \theta^\beta,$$

(40)

and the connection is split into a symmetric connection and an antisymmetric tensor:

$$\Gamma^\gamma_{\mu\nu} \rightarrow \Gamma^\gamma_{(\mu\nu)} + \Lambda^\gamma_{[\mu\nu]}.$$

(41)
Thus $\Gamma$ will refer from this point onwards to the torsion-free (symmetric in a coordinate basis) part of the connection, and $\Lambda$ to the remaining tensor contribution. In this way, $\Gamma$ is a torsion-free (but non-compatible) covariant derivative since:

$$T^\gamma_{\mu\nu} = \theta^\gamma \left[ \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu - [e_\mu, e_\nu] \right] = 2\Gamma^\gamma_{\mu\nu} = 0. \quad (42)$$

The compatibility equation (39) then becomes:

$$\nabla_{e_\gamma} [g]_{\mu\nu} = e_\gamma [g_{\mu\nu}] - g_{\mu\alpha} \Gamma^\gamma_{\gamma\nu} - g_{\nu\alpha} \Gamma^\gamma_{\gamma\mu} = g_{\mu\alpha} \Lambda^\alpha_{\gamma\nu} + g_{\alpha\nu} \Lambda^\alpha_{\gamma\mu} - \Delta^\alpha_{\gamma\mu\nu}, \quad (43)$$

where the basis vectors are just directional derivatives along the coordinates: $e_\gamma [.] = \partial_\gamma [.]$.

With this definition of the covariant derivative and related connection coefficients, the geometric curvature is found as usual from:

$$R^\alpha_{\beta\mu\nu} = \theta^\alpha \left[ (\nabla_{e_\mu} \nabla_{e_\nu} - \nabla_{e_\nu} \nabla_{e_\mu} - \nabla_{[e_\mu, e_\nu]}) e_\beta \right]$$

$$= e_\mu \Gamma^\alpha_{\nu\beta} - e_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\gamma_{\nu\beta} \Gamma^\alpha_{\gamma\mu} - \Gamma^\gamma_{\mu\beta} \Gamma^\alpha_{\gamma\nu}, \quad (44)$$

and defining the two independent contractions:

$$R^1_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = e_\alpha \Gamma^\alpha_{\nu\mu} - e_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\gamma_{\nu\mu} \Gamma^\alpha_{\gamma\alpha} - \Gamma^\gamma_{\alpha\mu} \Gamma^\alpha_{\gamma\nu}, \quad (45a)$$

$$R^2_{\mu\nu} = R^\alpha_{\alpha\mu\nu} = e_\mu \Gamma^\alpha_{\nu\alpha} - e_\nu \Gamma^\alpha_{\mu\alpha}, \quad (45b)$$

The Ricci tensor will be defined as:

$$R_{\mu\nu} = R^1_{\mu\nu} - \frac{1}{2} R^2_{\mu\nu}$$

$$= e_\alpha \Gamma^\alpha_{\nu\mu} - \frac{1}{2} (e_\nu \Gamma^\alpha_{\alpha\mu} + e_\mu \Gamma^\alpha_{\alpha\nu}) + \Gamma^\gamma_{\nu\mu} \Gamma^\alpha_{\gamma\alpha} - \Gamma^\gamma_{\alpha\mu} \Gamma^\alpha_{\gamma\nu}, \quad (46)$$

This particular combination is symmetric, and obviously reduces to the GR Ricci tensor when the NGT antisymmetric terms vanish. Decomposition of (39) into $R_{\mu\nu}$ and another that depends on $\Lambda$ as: $R_{\mu\nu}^{NS} = R_{\mu\nu} + R^\Lambda_{\mu\nu}$ where $(\Lambda_{\mu} = \Lambda^\alpha_{\mu\alpha})$ gives:

$$R_{\mu\nu}^\Lambda = \nabla_{e_\nu} \Lambda^\alpha_{\mu\alpha} + \nabla_{e_\mu} \Lambda^\alpha_{\nu\alpha} + \Lambda^\alpha_{\mu\beta} \Lambda^\beta_{\nu\alpha}. \quad (47)$$

A more general basis is introduced at each point on the manifold through $e_A = E_A^\mu e_\mu$, where $E$ is locally an element of $Gl(4, R)$, and these bases are smoothly joined up to form sections of the tangent bundle $T(\mathcal{M})$. The general basis vectors are then given in terms of a coordinate basis through the vierbein-like quantities, which can be used to translate tensors from one choice of basis to the other:

$$e_A = E_A^\mu e_\mu, \quad E_A^\mu E_B^\nu g_{\mu\nu} = g_{AB}, \quad \text{etc..} \quad (48)$$

(In the usual orthonormal basis, one transforms the symmetric part of the metric to the Minkowski space metric, and the $E$'s provide the isomorphism between coordinate basis tensors and locally Lorentzian tensors.) The dual basis of $T^*(\mathcal{M})$ is introduced through the usual relation: $\theta^A [e_B] = \delta^A_B$, and the inverse of the vierbeins is defined through: $E_A^\mu E_A^\nu = \delta^\mu_\nu$. In this paper capital letters from the beginning of the alphabet: $A, B, C, \cdots$ will refer to components of the object decomposed in the general basis.

Parallel transport of the basis vectors now defines the generalized connection coefficients:
\[ \nabla_{e_A}[e]_B = \Gamma_{AB}^C e_C, \quad \nabla_{e_A}[\theta]^C = -\Gamma_{AB}^C \theta^B. \] (49)

The definition of the basis in (48) implies that it is no longer a coordinate basis in general, and hence the directional derivatives no longer necessarily commute, giving rise to the structure constants:

\[ [e_A, e_B] = C_{AB}^C e_C, \] (50a)
given by:

\[ C_{AB}^C = E^C_\nu (E_A^\mu \partial_\mu [E_B^\nu] - E_B^\mu \partial_\mu [E_A^\nu]), \] (50b)
calculated by noting that \( e_A[\cdot] = E_A^\mu \partial_\mu [\cdot] \). This also implies that a torsion-free connection will no longer be symmetric, and vanishing torsion now gives:

\[ T_{BC}^A = \theta^A [\nabla_{e_B} e_C - \nabla_{e_C} e_B - [e_B, e_C]] = 2\Gamma_{[BC]}^A - C_{BC}^A = 0, \] (51)

allowing one to determine the antisymmetric part as usual from the structure constants. This is the motivation for splitting up the connection in this way. Given some alternate split where \( \Gamma \) is not torsion free, one would have to distinguish between the effects of the general basis on the skew part of the connection coefficients, and that of the NGT effects (themselves tensors).

The compatibility condition (39) can now be written as:

\[ \nabla_{e_C}[g]_{AB} = g_{AD} \Lambda_{BC}^D + g_{DB} \Lambda_{CA}^D - \Delta^0_{CAB}, \] (52)

where since \( \Delta^0 \) and \( \Lambda \) are tensors, they are just redefined by multiplication by the appropriate combination of vierbeins. The symmetric part of this can now be solved for the symmetric part of \( \Gamma \) in terms of the antisymmetric part, the structure constants, and \( \Lambda \), to give:

\[ \Gamma_{C(AB)} = \frac{1}{2} \Delta_{C(AB)} - \Gamma_{A[BC]} + \Gamma_{B[CA]} - A^D_A \Lambda_{DBC} + A^D_B \Lambda_{DCA}, \] (53)

where the quantities:

\[ \Gamma_{ABC} = g_{(AD)} \Gamma_{BC}^D, \quad \Lambda_{ABC} = g_{(AD)} \Lambda_{BC}^D, \quad A^A_B = S^{(AC)} g_{[CB]}, \] (54)

have been defined for convenience, and \( S \) is the inverse of the symmetric part of the metric defined by: \( S^{(AB)} g_{(BC)} = \delta^A_C \). Also appearing is the symmetric part of:

\[ \Delta_{CAB} = e_B[g_{CA}] + e_A[g_{BC}] - e_C[g_{AB}] + \Delta^0_{BCA} + \Delta^0_{ABC} - \Delta^0_{CAB}. \] (55)

The antisymmetric part of the compatibility conditions can now be recast (using (53)) as 24 algebraic equations for \( \Lambda \):

\[ \Lambda_{CAB} - A^D_A A^E_B \Lambda_{ECD} - A^D_A A^E_C \Lambda_{EBD} + A^D_B A^E_A \Lambda_{EAD} + A^D_B A^E_C \Lambda_{EAD} = \Omega_{C[AB]}, \] (56)

where:

\[ \Omega_{C[AB]} = \frac{1}{2}(\Delta_{C[AB]} + A^D_B \Delta_{D(CA)} - A^D_A \Delta_{D(BC)}) \\
+ A^D_A (\Gamma_{B[CD]} + \Gamma_{C[BD]}) - A^D_B (\Gamma_{A[CD]} + \Gamma_{C[AD]}) - A^D_C \Gamma_{D[AB]} . \] (57)
The method for solving the compatibility conditions is to first determine the auxiliary quantities appearing in this relation: \((A, \Gamma, \Lambda, \Omega)\) in terms of the vierbeins and metric quantities, then solve for \(\Lambda\) through (56), determine \(\Gamma\) from (53), and then use \(S\) with \(\Gamma\) and \(\Delta\) to form \(\Gamma_{BC}\) and \(\Lambda_{AB}\). This may not seem like much of a simplification, but when specialized to a Lorentz frame, many of these quantities simplify considerably (as is the case in the Wyman sector in Section 5).

The curvature tensor \(R\) becomes:

\[
R_{ABCD} = \theta^A[(\nabla_\epsilon \epsilon_\phi - \nabla_\phi \epsilon_\epsilon - \nabla_[\epsilon_\phi,\epsilon_\delta])e_B] = \epsilon_C[\Gamma^A_{DB}] - \epsilon_D[\Gamma^A_{CB}] + \Gamma^E_{DB}\Gamma^A_{CE} - \Gamma^E_{CB}\Gamma^A_{DE} - C_{CD}E^A_{EB},
\]

and the contractions:

\[
R^1_{AB} = R^C_{ACB} = \epsilon_C[\Gamma^A_{BA}] - \epsilon_B[\Gamma^A_{CA}] + \Gamma^D_{BA}\Gamma^C_{CD} - \Gamma^D_{CA}\Gamma^C_{BD} - C_{CB}D^C_{DA}, \tag{59a}
\]

\[
R^2_{AB} = R^C_{CAB} = \epsilon_A[\Gamma^C_{BC}] - \epsilon_B[\Gamma^C_{AC}] - C_{AB}D^C_{DC}, \tag{59b}
\]

combine to give the Ricci tensor:

\[
R_{AB} = R^1_{AB} - \frac{1}{2}R^2_{AB} = \epsilon_C[\Gamma^C_{BA}] - \epsilon_B[\Gamma^C_{CA}] - \frac{1}{2}\epsilon_A[\Gamma^C_{BC}] + \frac{1}{2}\epsilon_B[\Gamma^C_{AC}]
\]

\[
+ \Gamma^D_{BA}\Gamma^C_{CD} - \Gamma^D_{CA}\Gamma^C_{BD} - C_{CB}D^C_{DA} + \frac{1}{2}C_{AB}D^C_{DC}
\]

\[
= \epsilon_C[\Gamma^C_{BA}] - \epsilon_B[\Gamma^C_{CA}] - \frac{1}{2}\epsilon_A[\Gamma^C_{BC}] + \frac{1}{2}\epsilon_B[\Gamma^C_{AC}]
\]

\[
+ \Gamma^D_{BA}\Gamma^C_{CD} - \Gamma^D_{CA}\Gamma^C_{BD} + \frac{1}{2}C_{AB}D^C_{DC}.
\]

In the split \(R^\text{NS}_{AB} = R_{AB} + R^A_{AB}\):

\[
R^A_{AB} = \nabla_\epsilon_C[\Lambda]^{C}_{AB} + \nabla_\epsilon_D[\Lambda]^{D}_{AB} + \Lambda_{AD}^{C}\Lambda_{BC}^{D}, \tag{61}
\]

as expected.

Since by construction \(\Gamma\) is a torsion free connection, and (38) is the standard curvature tensor constructed from it, one obtains the usual Bianchi identities (38) on the curvature tensor. One should note though that the connection is not compatible, and so the rotation coefficients are not antisymmetric. The relevant Ricci tensors are also not constructed in the same manner as in GR, so the implications of these identities are somewhat different. The first Bianchi identity gives the usual cyclic identity on the last three indices of (38), and leads to the result:

\[
R^1_{[AB]} = \frac{1}{2}R^2_{AB}, \tag{62}
\]

when one contracts on any lowered index. (This can also be proven directly using the Jacobi identity.) This tells us that the NGT Ricci tensor is symmetric \((R_{[AB]} = 0)\) in general, not just in a coordinate basis.

A detailed study of the contractions of the second Bianchi identity (the cyclic covariant derivative):

\[
\nabla_\epsilon_C[R]^{A}_{BDE} + \nabla_\epsilon_D[R]^{A}_{BEC} + \nabla_\epsilon_E[R]^{A}_{BCD} = 0, \tag{63}
\]

should result in a derivation of the equations of motion for matter fields (39,40) from the field equations.
B. The NGT Action and Field Equations in a General Basis

The translation of the field equations (4b,5,6) is accomplished through an almost straightforward substitution:

\[ \Lambda_A = 0, \]  
\[ \nabla_{e_B} [g]^{[AB]} = \alpha g^{(AB)} W_B, \]  
\[ R_{AB} := R_{NS}^{NS} + \nabla_{e_B} [W]_{AB} - \frac{1}{2} \alpha W_A W_B - \frac{1}{4} m^2 M_{AB} = 0, \]

where the density is \( \sqrt{-g} = \sqrt{-\det(g_{AB})} \), the mass tensor:

\[ M_{AB} = g_{[AB]} - g_{CA} g^{BD} g_{[CD]} + \frac{1}{2} g_{BA} g_{[CD]} g^{[CD]} \]

has been defined, and the tensor appearing in the compatibility equations is:

\[ \Delta^0_{CAB} = -\frac{2}{3} \alpha (g_{A[C} g_{D]} + \frac{1}{2} g_{AB} g^{[CD]}) g_{[DE]} W_E. \]

One must be careful to treat totally antisymmetric derivatives properly (the structure constants now come into the curl of a vector), and translate the metric density properly.

In order to define the action, one should note that the inverse of the metric is now:

\[ g_{AB} g^{BC} g_{BA} = \delta_A^C, \]

and the direct translation of the density results in: \( \sqrt{-g} \rightarrow \sqrt{-E g E^l} = E \sqrt{-\tilde{g}} \) where \( g = \det(g_{AB}) \) and \( E = \det(E^A_{\mu}) \). Then (1) is rewritten:

\[ S = \int_M d^4 x E \left\{ -g^{AB} R_{AB} + g^{AB} \nabla_{e_B} W]_{AB} + I^A \Lambda_A + \frac{1}{2} \alpha g^{(AB)} W_A W_B + \frac{1}{4} m^2 g^{[AB]} g_{[AB]} \right\}. \]

(Note that in a Lorentz basis, the inverse of the metric is not \( \eta_{\mu\nu} \).)

Deriving the equations of motion from this action should be approached with care. As it stands there are too many fields (the metric and the vierbeins share degrees of freedom) and one typically must choose either a coordinate basis (as in Section I), a Lorentz basis (so that all symmetric metric degrees of freedom are contained in the vierbeins), or a well-defined combination of the two. One must also realize that (67) as it stands assumes that the connection \( \Gamma \) is torsion-free \( a \ priori \), so that when varying the vierbein, \( \Gamma[\mu] \) must be varied as well. As an alternative, one may impose the torsion-free condition through additional Lagrange multiplier terms: \( L_{\mu} R_{[AB]}^{\mu} T_{\mu}^{[AB]} \) in the action, varying the full connection coefficients and vierbeins separately.

V. APPROXIMATION OF THE WYMAN SECTOR SOLUTION IN A VIERBEIN BASIS

In general, the spherically symmetric Killing vector analysis for a \((0,2)\) tensor gives both \( t - r \) and \( \theta - \phi \) skew components. However it is possible to show from the general spherically symmetric field equations that it is consistent to put either (or both) of these skew components to zero separately, since in either case one loses the corresponding field equation, and the system of equations remains consistent. Whether it is physically reasonable to do
this or not depends on the details of the matter coupling in the theory, and how it alters the global behavior of the skew sector. Here will be considered the field equations for what will be referred to as the Wyman sector \[41\] (keeping just the $\theta - \phi$ sector), although the asymptotics of the $t - r$ sector will be discussed briefly at the end of this section, where it will be argued that there are no static solutions with asymptotic behavior that is dominated by Schwarzschild (or equivalently, Newtonian) effects. This will allow an analysis of the perturbation equations for the spherically symmetric modes, in order to see the effects of the antisymmetric background.

In a coordinate basis, the Wyman metric looks like:

$$|g_{\mu\nu}| = \text{diag}\{\gamma(r), -\alpha(r), -r^2, -r^2\sin^2(\theta)\}, \quad g_{[23]} = f(r)\sin(\theta).$$  \hspace{1cm} (68)$$

(An appropriate coordinate system has been chosen in order to remove the symmetric $t - r$ metric component, and fix the $\theta - \theta$ component.) Introducing the usual choice of vierbein (using the functions defined by: $F = f(r)/r^2$, $E_0 = 1/\sqrt{\gamma(r)}$, $E_1 = 1/\sqrt{\alpha(r)}$):

$$E_A^\mu = \text{diag}\{E_0^0 = E_0, \quad E_1^1 = E_1, \quad E_2^2 = \frac{1}{r}, \quad E_3^3 = \frac{1}{r\sin(\theta)}\},$$  \hspace{1cm} (69)$$

the metric becomes:

$$|g_{(AB)}| = \eta_{AB}, \quad g_{[23]} = F,$$  \hspace{1cm} (70)$$

and the density $\sqrt{-g} = \sqrt{1 + F^2}$.

At this point one can invert the compatibility conditions and compute the field equations using the method of Section [IV], given in some detail in Appendix [A]. No attempt will be made here to solve the field equations exactly, although numerical evidence for the existence of an exact solution with asymptotic behavior that matches that given here has been found \[42\], ensuring that the approximations given come from a global solution. Instead, an approximation will be given that describes the asymptotic behavior of the exact solution. The idea will be to consider the skew sector as a small correction (of order some small dimensionless parameter $\kappa$, to be explicitly defined later) to the Schwarzschild solution far enough away from the source. This should be reasonable since one expects from the results of the perturbation in Section [III] that the skew sector will behave asymptotically as a decaying exponential, while the symmetric sector should behave as $\sim 1/r$, so that far enough away from the gravitational source the skew sector should be completely dominated by GR effects.

To lowest order in $\kappa$ (the skew sector) the work is already done, as the field equation for the skew function will be essentially the same as the static perturbation about a Schwarzschild background already considered in Section [III]. In the vierbein basis, this is derived as before from $\mathcal{R}_{[23]}$ [A8], and gives:

$$A\partial_r^2[F] + \left(A' + \frac{2A}{r}\right)\partial_r[F] - \frac{2}{r}(A' + \frac{A}{r})F - m^2F = 0,$$  \hspace{1cm} (71)$$

and it is trivial to see that when using: $F = f/r$, this reduces to the static limit of (B2), giving the asymptotic form for $F$:

$$F \sim F_0 e^{-mr}. \hspace{1cm} (72)$$
One now must consider how the presence of the skew sector affects the symmetric sector, particularly whether it really is a higher order effect. The asymptotic form of these corrections due to $F$ may be calculated by considering order $\kappa^2$ corrections to the vierbeins (order $\kappa$ terms will not depend on $F$, and so will be solely $\delta M_s$ corrections), calculated from the symmetric field equations with $F$ from (72) acting as a source. Writing the corrections to the vierbeins as:

$$E_0 \to E_0 + E_0^{(2)}$$

and the corrections to the field equations as $\mathcal{R}^{(2)}$, one calculates:

$$\mathcal{R}^{(2)}_{00} + \mathcal{R}^{(2)}_{11} = -\frac{2A}{r} \partial_r [\sqrt{A}E_0^{(2)} + \frac{E_1^{(2)}}{\sqrt{A}}] - AFF'' - \frac{3}{2} A(F')^2 - \frac{2A}{r} FF' = 0,$$

which, after translating it into a differential equation in $r^*$ and keeping only the asymptotically dominant terms, results in:

$$\partial_{r^*} [\sqrt{A}E_0^{(2)} + \frac{E_1^{(2)}}{\sqrt{A}}] \sim -\frac{5}{4} (F_0)^2 e^{-2mr^*}.$$

(74)

This integrates to give (the constant of integration is ignored as one could eliminate it through an appropriate choice of gauge as in Section III):

$$\sqrt{A}E_0^{(2)} + \frac{E_1^{(2)}}{\sqrt{A}} \sim \frac{5}{4} (F_0)^2 e^{-2mr^*}.$$

(75)

Considering next $\mathcal{R}^{(2)}_{33}$ and using (73) leads to the asymptotic equation:

$$\partial_{r^*} [r \sqrt{A}E_1^{(2)}] \sim -\frac{5}{4} (F_0)^2 e^{-2mr^*}.$$

(76)

The solution of this combined with the results of (73) gives (once again the constant of integration is ignored, this time as it would be interpretable as a perturbation of the mass parameter and not due to the effects of the skew sector):

$$E_1^{(2)} \sim \frac{5}{4} (F_0)^2 e^{-2mr^*}, \quad E_0^{(2)} \sim o(e^{-2mr^*})$$

(77)

where the dominant correction to the symmetric sector is $E_1^{(2)}$, and $E_0^{(2)}$ is down by $o(1/r^*)$.

It is not hard to see that these corrections are indeed of an order higher than the effects in the skew sector. Clearly one may define a small parameter $\kappa = F_0 \exp(-mr^*_0)$, where $r^*_0$ is chosen such that $F(r^*_0) \ll 1$, to define the small size of the skew sector when $r^* > r^*_0$.

The corrections to the symmetric sector are seen to be of order $\kappa^2$, and will therefore be neglected in the approximation of the background required in Section VI.

One may attempt to do the same sort of analysis keeping the $g_{01}$ component, however the linearized field equation implies immediately that the field must vanish (it is identical to (30)). Considering higher orders in the field in an attempt to generate a solution other than this trivial result, the third order correction gives (writing $W(r) = \sqrt{X(r)}$):

$$\mathcal{R}_{01} = -\frac{\sqrt{X}}{6} [A\partial_r^2[X] + (A' + \frac{2A}{r})\partial_r[X] + (\frac{12A}{r^2} + \frac{4A'}{r} + \frac{3}{2} m^2)X + \frac{12A'}{r} + 3m^2] = 0.$$

(78)
Writing $X = Y/r$ and transforming to the $r^*$ coordinate as before gives in canonical form:

$$\partial^2_{r^*}[Y] + A\left(\frac{12A}{r^2} + \frac{3A'}{r} + \frac{3}{2}m^2\right)Y + A\left(12A' + 3m^2r\right) = 0,$$

(79)

and keeping the dominant terms:

$$\partial^2_{r^*}[Y] + \frac{3}{2}m^2Y + 3m^2r^* = 0,$$

(80)

easily giving the asymptotic form of the solution:

$$W^2(r^*) = -2 + \frac{a}{r^*}\cos\left(\sqrt{\frac{3}{2}}mr^*\right) + \frac{b}{r^*}\sin\left(\sqrt{\frac{3}{2}}mr^*\right),$$

(81)

(where $(a, b)$ are arbitrary constants). The dominant part of this solution implies that $W$ is imaginary, and must be discarded. However one also sees that this solution is not in fact a small correction to the Schwarzschild metric asymptotically, and would have to be discarded for that reason alone. This is not surprising as one is trying to match a function that is small asymptotically (by hypothesis) to one that is constant, so keeping higher orders in $W$ will not change this. This implies that nontrivial static solutions that include this sector fail to be dominantly Schwarzschild for large $r$. This of course does not exclude solutions with asymptotic behavior that is of some other form, nor can one exclude the possibility that $W$ is nonvanishing only inside some finite radius.

VI. SPHERICALLY SYMMETRIC PERTURBATION ABOUT A WYMAN BACKGROUND

In an attempt to consider the perturbation equations for NGT about a general nonsymmetric background, one finds that the compatibility conditions prevent one from formulating the inversion in a useful form. This means that a fairly straightforward covariant formulation (like that given in Section 4) is not feasible, and instead one must treat each situation separately, in this case a spherically symmetric perturbation about the approximated mNGT Wyman solution given in the previous section. Here it is demonstrated that despite the remaining gauge freedom in the symmetric sector, both symmetric functions will in general pick up time dependence from the skew sector. Although this cross coupling is demonstrated explicitly in a perturbative scenario, it will certainly persist in a more general sense. The results here will show that the perturbations in the symmetric sector pick up time dependence that is algebraically determined by the skew function $F$, without themselves becoming independent degrees of freedom. The canonical analysis of the general spherically symmetric system will address rigorously how many degrees of freedom exist in each sector. If there are more in the nonperturbative theory, one can examine the dynamical approach to an asymptotically flat spacetime looking for possible singular behavior similar to that found in 4.

The perturbation of the Wyman metric (70) in a coordinate basis will look identical to (24) (using the gauge choice to simplify it as before). The background vierbeins will be the same as those in the Wyman solution (23), where now the perturbations of the vierbeins and skew metric functions are related to perturbations in the coordinate basis by:
\[
\delta W = \frac{h_{[01]}(t, r)}{\sqrt{\alpha(r)\gamma(r)}}, \quad \delta F = \frac{\delta h_{[23]}(t, r)}{r^2}, \quad \delta E_0 = -\frac{1}{2} \frac{h_{00}(t, r)}{\gamma(r)^{\frac{3}{2}}}, \quad \delta E_1 = -\frac{1}{2} \frac{h_{11}(t, r)}{\alpha(r)^{\frac{3}{2}}}.
\]  
(82)

In the vierbein basis the metric perturbation has nonvanishing components:
\[
h_{[01]} = \delta W, \quad h_{[23]} = \delta F.
\]  
(83)

The approximation of the background Wyman solution given in Section \[\text{V}\] greatly simplifies the algebra necessary to develop the perturbation given in Appendix \[\text{B}\]. Approximating the symmetric sector by the Schwarzschild solution and the antisymmetric sector by (72), first order in this static antisymmetric background is kept, as is the first order in the perturbations. As one shall see, this will be a reasonable approximation since it will be possible to keep the perturbations small compared to the background by an appropriate choice of integration constants (similarly to \(\delta M_s\) in the Schwarzschild case).

The field equation:
\[
\frac{1}{R_{[01]}} = 0,
\]  
yields precisely the same field equation as in the Schwarzschild case (30), allowing one to immediately set \(\delta W = 0\). The symmetric part:
\[
\frac{1}{R_{(01)}} = 0,
\]  
can be written as a total time derivative:
\[
\frac{2}{r\sqrt{A}}(\delta E_1 - \delta E) = -\frac{3}{2}F'\delta F - (1 - \frac{A'}{2A})F\delta F - F\partial_t[\delta F].
\]  
(85)

Now computing (\(\text{Tr}[\mathcal{R}_{AB}] := \mathcal{R}_{00} + \mathcal{R}_{11} + 2\mathcal{R}_{22}\)):
\[
\text{Tr}[\mathcal{R}_{AB}] = \frac{4}{r^2}\partial_t[r\sqrt{A}\delta E_1] - \frac{F}{A}\partial_t^2[\delta F] + 3AF\partial_t^2[\delta F]
\]  
\[
+ \left(\frac{8A}{r}F + 2AF' + 5AF''\right)\partial_r[\delta F] + (3AF'' + \frac{8A}{r}F' + 2AF' - m^2F)\delta F = 0,
\]  
(86)

and inserting (85) gives:
\[
\text{Tr}[\mathcal{R}_{AB}] = \frac{4}{r^2}\partial_t[r\sqrt{A}\delta E]
\]  
\[
- F \left(\frac{1}{A}\partial_t^2[\delta F] - A\partial_r^2[\delta F] - (A' + \frac{2A}{r})\partial_r[\delta F] + \left(\frac{2A'}{r} + \frac{2A}{r^2} + m^2\right)F\delta F\right) = 0.
\]  
(87)

Also,
\[
\frac{1}{R_{00}} + \frac{1}{R_{11}} = \frac{2A}{r}\partial_t[\sqrt{A}\delta E_0 + \frac{\delta E_1}{\sqrt{A}}]
\]  
\[
+ \frac{F}{A}\partial_t^2[\delta F] + AF\partial_r^2[\delta F] + A\left(\frac{2F}{r} + 3F'\right)\partial_r[\delta F] + A\left(\frac{2F'}{r} + F''\right)\delta F = 0,
\]  
(88)

gives the equality of spatial derivatives of \(\sqrt{A}\delta E_0\) and \(\delta E_1/\sqrt{A}\) up to order \(F\). This will be useful when considering:
\[ \mathcal{R}_{[23]} = \left( \frac{F}{r} - \frac{F'}{2} \right) A \partial_r \left[ \frac{\delta E_1}{\sqrt{A}} - \sqrt{A} \delta E_0 \right] - \left[ A F'' + \left( A' + \frac{2A}{r} \right) F' - \frac{2}{r} (A' + \frac{A}{r}) F \right] \frac{\delta E_1}{\sqrt{A}} \]
\[ + \frac{1}{2} \left( \frac{1}{A} \partial_r^2 [\delta F] - A \partial_{r r}^2 [\delta F] - \left( A' + \frac{2A}{r} \right) \partial_r [\delta F] + \left( \frac{2A'}{r} + \frac{2A}{r^2} + m^2 \right) \delta F \right) \]
\[ = \left( \frac{F}{r} - \frac{F'}{2} \right) A \partial_r \left[ \frac{\delta E_1}{\sqrt{A}} - \sqrt{A} \delta E_0 \right] - m^2 F \frac{\delta E_1}{\sqrt{A}} \]
\[ + \frac{1}{2} \left( \frac{1}{A} \partial_r^2 [\delta F] - A \partial_{r r}^2 [\delta F] - \left( A' + \frac{2A}{r} \right) \partial_r [\delta F] + \left( \frac{2A'}{r} + \frac{2A}{r^2} + m^2 \right) \delta F \right) = 0, \quad (89) \]

where use has been made of (71). One derives a simple field equation by inserting (89) in (87) and dropping the resulting terms that are of second order in the background skew field \( F \):

\[ \partial_r [r \sqrt{A} \delta E] = 0 \rightarrow \delta E = \frac{\delta M_s}{r \sqrt{A}}, \quad (90) \]

where the constant of integration has been identified with the GR-like perturbation of the Schwarzschild mass parameter.

Now (88) can be used to replace \( \delta E_0 \) with \( \delta E_1 \) at this order, and (85) to replace \( \delta E_1 \) with \( \delta E \) to find:

\[ \mathcal{R}_{[23]} = \frac{1}{2} \left\{ \frac{1}{A} \partial_r^2 [\delta F] - A \partial_{r r}^2 [\delta F] - \left( A' + \frac{2A}{r} \right) \partial_r [\delta F] + \left( \frac{2A'}{r} + \frac{2A}{r^2} + m^2 \right) \delta F \right\} \]
\[ - A (F' - \frac{2F}{r}) \partial_r \left[ \frac{\delta E}{\sqrt{A}} \right] - m^2 F \frac{\delta E}{\sqrt{A}} = 0, \quad (91) \]

and using (90) in this yields the wave equation for \( \delta F \):

\[ \frac{1}{A} \partial_r^2 [\delta F] - A \partial_{r r}^2 [\delta F] - \left( A' + \frac{2A}{r} \right) \partial_r [\delta F] + \left( m^2 + \frac{2A}{r^2} + \frac{2A'}{r} \right) \delta F \]
\[ = \frac{\delta M_s}{r A} [m^2 F + 4 \left( \frac{F}{r} - \frac{F'}{2} \right) (\frac{A}{r} + A')]. \quad (92) \]

Note that this is a static source and so will not in itself induce any wave solutions, but as before the effects of a matter source will show up asymptotically. The static part of the solution may be derived using the methods in Section III:

\[ \delta F = -2F_0 \frac{\delta M_s}{r} e^{-2mr} \ln \left( \frac{r}{2M_s} \right), \quad (93) \]

and is consistent with the static solution (72) derived about a Schwarzschild background with mass parameter \( M_s + \delta M_s \). Time dependent solutions are identical to those found from (71), and will induce time dependence in the symmetric sector through (83). Since \( \delta E_1 \) is related to \( \delta F \) locally, it is not an independent degree of freedom, and since the skew field is short-ranged, it will not radiate energy at infinity.

Using (92) and (71), one reduces (88) to an algebraic relation for \( \delta E_0 \):
\[
\frac{2A}{r} \partial_r \left[ \sqrt{A} \delta E_0 + \frac{\delta E}{\sqrt{A}} \right] + AF \partial_r^2 [\delta F] + \left( \frac{3A'}{2} F + \frac{2A}{r} F + \frac{A'}{2} F' \right) \partial_r [\delta F] \\
+ \left[ (A' + \frac{A}{2r}) F' - \left( \frac{(A')^2}{2A} + \frac{7A'}{2r} + \frac{3A}{r^2} + \frac{3}{4} m^2 F \right) \delta F \right] = 0.
\]

(94)

The solution for \( \delta E_0 \) can be written as:

\[
\sqrt{A} \delta E_0 = \sqrt{A} \tilde{E} - \frac{\delta E}{\sqrt{A}} + B(t)
\]

(95)

where \( B(t) \) is an arbitrary function of time (removable in the usual way using the remaining gauge transformation), the second term corresponds to the static Schwarzschild perturbation from Section III, and \( \tilde{E} \) solves the remainder of (94). Note that although not independent degrees of freedom, neither \( \delta E_0 \) nor \( \delta E_1 \) is static. This is in fact what one would expect when considering the effect of a spherically symmetric matter field to which Birkhoff’s theorem does not apply, on the GR background. The presence of the non-static field will induce time dependence in the gravitational fields, without exciting any independent modes. This is expected to continue to be the case in NGT: the general spherically symmetric system should only have degrees of freedom in the skew sector.

VII. CONCLUSIONS

The asymptotic behavior of the antisymmetric sector for the case of a static Wyman-type metric has been determined, and the corrections to the symmetric sector shown to be negligible provided one considers regions of spacetime far enough away from the gravitational source. It has also been determined that if one keeps the antisymmetric \( t \rightarrow r \) component, then one cannot have asymptotic behavior that is dominated by the Schwarzschild metric, and so it must be discarded. This analysis was facilitated by the introduction of a vierbein basis, although the formalism has been given for a general basis for completeness.

By considering a spherically symmetric perturbation of the Schwarzschild metric, it has been shown that NGT does not have a rigorous Birkhoff theorem as the antisymmetric sector will not remain static in general. (This has also been noted previously in a Unified Field Theory based on Lyra geometries [12].) Perturbing an approximate Wyman background in a vierbein basis has shown that the symmetric sector is also not static in general, although no additional modes become excited. This is important phenomenologically since one cannot consider the static solutions (Schwarzschild and Wyman) as the only spherically symmetric exterior solutions to the field equations, and one must therefore match an interior solution to a non-static exterior in general.

Perturbations of GR backgrounds have been shown to have good asymptotic behavior in general, since the ghost modes do not become excited and the remaining degrees of freedom are short ranged by construction. However this is not good enough since one expects that the physically interesting solutions to mNGT will not be the purely GR solutions, and one would therefore like to examine the behavior of perturbations on generic, asymptotically-flat, mNGT backgrounds. A covariant perturbative scheme, although possible in principle, would seem to be too complex to be of any practical value. Instead one may treat each case separately and consider the behavior of (perhaps several) modes about a particular
background, as was done here for the spherically symmetric perturbation about a Wyman background.

However this also may not be adequate to fully understand the dynamics of the skew sector in mNGT. The lack of additional gauge invariance in the skew sector may mean that there are more modes in the rigorous theory that will be seen in any sort of weak field, perturbative analysis. To determine whether or not this is the case will require a canonical analysis of the full theory. Partial information may be obtained by considering the full set of fields in a spherically symmetric system, and looking for global information about the behavior of the skew modes given a general coupling to external sources. This is not likely to be a tractable problem in a coordinate basis, and even in a Lorentz frame the field equations are not expected to be particularly enlightening, due to their complication alone. However the canonical analysis of this system will show which fields propagate in the general case, and allow one to get at the dynamics of the approach to an asymptotically well behaved spacetime.

ACKNOWLEDGMENTS

The author would like to thank the Natural Sciences and Engineering Research Council of Canada and the University of Toronto for funding during part of this work, the hospitality of the Department of Physics, Cave Hill Campus, University of West Indies, Barbados, and the Inter America Development Bank for supporting the stay in Barbados. Thanks also go to L. Demopoulos for suggestions, and J. W. Moffat, J. Légaré, P. Savaria, and N. Cornish for discussions related to this work.

APPENDIX A: WYMAN SECTOR FIELD EQUATIONS

Here the details of the calculation of the Wyman field equations are given, following the steps outlined in Section IV. Beginning with $A^A_B = \eta^{AC}g_{CB}$ from (54), one finds the remaining components:

$$A^2_3 = -A^3_2 = F.$$  \hfill (A1)

The inverse of the symmetric part of the metric is the Minkowski metric: $\eta$, whereas the full inverse metric is:

$$|g^{AB}| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{1+F^2} & -\frac{F}{1+F^2} \\ 0 & 0 & -\frac{F}{1+F^2} & -\frac{1}{1+F^2} \end{bmatrix},$$  \hfill (A2)

and is necessary in order to compute the mass tensor (54). The structure constants, and hence the skew components of the connection $\Gamma$, are then easily found to be:

$$\Gamma^0_{[10]} = \frac{1}{2}C^{0}_{10} = \frac{1}{2}E_1 \partial_r [\ln(E_0)],$$
\[ \Gamma_{[12]}^2 = \Gamma_{[13]}^3 = \frac{1}{2} C_{12}^2 = \frac{1}{2} C_{13}^3 = -\frac{1}{2r} E_1, \]
\[ \Gamma_{[23]}^3 = \frac{1}{2} C_{23}^3 = -\frac{1}{2r} \cot(\theta). \]  
(A3)

Then \( \Delta \) from (33) may be determined:

\[ -\Delta_{1[23]} = \Delta_{3[12]} = \Delta_{2[31]} = E_1 \partial_r [F], \]  
(A4)

and \( \Omega \) from (37):

\[ \Omega_{1[23]} = E_1 \left( \frac{F}{r} - \frac{1}{2} \partial_r [F] \right), \]
\[ \Omega_{3[12]} = \Omega_{2[31]} = \frac{1}{2} E_1 \partial_r [F]. \]  
(A5)

Now (50) is solved for \( \Lambda \) and (53) to calculate the connection components (\( \Gamma \)):

\[ \Lambda_{12}^3 = \Lambda_{31}^2 = -\frac{1}{2} \frac{E_1 F'}{1 + F^2}, \]
\[ \Lambda_{23}^1 = -\frac{E_1 F}{r} + \frac{1}{2} E_1 F' - \frac{1}{2} E_1 \partial_r [\ln(1 + F^2)], \]  
(A6)

and:

\[ \Gamma_{01}^0 = \Gamma_{00}^1 = -E_1 \partial_r [\ln(E_0)], \]
\[ \Gamma_{22}^1 = \Gamma_{33}^3 = -\frac{E_1}{r} - \frac{1}{2} E_1 \partial_r [\ln(1 + F^2)], \]
\[ \Gamma_{21}^2 = \Gamma_{31}^3 = \frac{E_1}{r} + \frac{1}{4} E_1 \partial_r [\ln(1 + F^2)], \]
\[ \Gamma_{12}^3 = \Gamma_{13}^3 = \frac{1}{4} E_1 \partial_r [\ln(1 + F^2)], \]
\[ \Gamma_{32}^3 = -\Gamma_{33}^3 = \frac{1}{r} \cot(\theta). \]  
(A7)

The field equations that remain are:

\[ R_{00} = E_1 \partial_r [\Gamma_{00}^1] + \Gamma_{00}^1 (\Gamma_{01}^0 + 2 \Gamma_{21}^2) + \frac{m^2}{4} \frac{F^2}{1 + F^2} = 0, \]
\[ R_{11} = -E_1 \partial_r [\Gamma_{01}^0] - 2 E_1 \partial_r [\Gamma_{21}^2] - (\Gamma_{01}^0)^2 - 2 (\Gamma_{21}^2)^2 - 2 (\Lambda_{12}^3)^2 - \frac{m^2}{4} \frac{F^2}{1 + F^2} = 0, \]
\[ R_{22} = R_{33} = E_1 \partial_r [\Lambda_{23}^1] + \frac{1}{r^2} + \Gamma_{22}^1 (\Gamma_{01}^0 + 4 \Gamma_{21}^2) - 2 \Lambda_{12}^3 \Lambda_{23}^1 + \frac{m^2}{4} \frac{F^2}{1 + F^2} = 0, \]
\[ R_{[23]} = E_1 \partial_r [\Lambda_{23}^1] + (\Gamma_{01}^0 - 4 \Gamma_{12}^2) \Lambda_{23}^1 + 2 \Gamma_{22}^1 \Lambda_{12}^3 - \frac{m^2}{4} (F + \frac{F}{1 + F^2}) = 0. \]  
(A8)

It can be shown that in the absence of any skew sector altogether, the Schwarzschild field equations are obtained, and in the absence of the mass term one has the Wyman Field equations [11].
APPENDIX B: PERTURBATION EQUATIONS

The background field will be that given in Appendix A, approximated by considering only first order contributions from the skew sector. The spherically symmetric perturbations about this background are given here in detail, keeping only first order in the background $F$, and setting $\alpha = 3/4$ throughout. One may begin by calculating first order corrections to metric quantities, first the density:

$$\delta \sqrt{-g} = \sqrt{-g} F \delta F,$$

and the inverse of the full metric is:

$$|\delta g^{AB}| = \begin{bmatrix} 0 & \delta W & 0 & 0 \\ -\delta W & 0 & 0 & 0 \\ 0 & 0 & 2F \delta F & -\delta F \\ 0 & 0 & \delta F & 2F \delta F \end{bmatrix}. \quad \text{(B2)}$$

The tensor $\delta A$ has remaining components:

$$\delta A_0^1 = \delta A_0^1 = \delta W, \quad \delta A_3^2 = -\delta A_2^3 = -\delta F; \quad \text{(B3)}$$

and the perturbation of the antisymmetric connection coefficients, derived from (50b):

$$\delta \Gamma_0[10] = \frac{1}{2} \delta C_{10}^0 = \frac{1}{2} \left( \delta E_1 \partial_r [\ln(E_0)] + \frac{E_1}{E_0} \partial_r [\delta E_0] - \frac{E_1}{E_0} \partial_r [\ln(E_0)] \delta E_0 \right),$$

$$\delta \Gamma_1[01] = \frac{1}{2} \delta C_{01}^1 = \frac{1}{2} \frac{E_0}{E_1} \partial_t [\delta E_1],$$

$$\delta \Gamma_2[12] = \delta \Gamma_3[13] = \frac{1}{2} \frac{1}{2} \delta C_{12}^2 = \frac{1}{2} \frac{1}{2} \delta C_{13}^3 = -\frac{1}{2} \frac{\delta E_1}{r}. \quad \text{(B4)}$$

The last equation of (64) is now solved to determine $\delta W$ in terms of metric functions to find:

$$\delta W_0 = \frac{4}{3} \frac{E_1}{\partial_r [\delta W] + \frac{2\delta W}{r}},$$

$$\delta W_1 = \frac{4}{3} \frac{E_0}{\partial_t [\delta W]},$$

$$\delta W_2 = \delta W_3 = 0. \quad \text{(B5)}$$

It is then fairly straightforward to calculate the remaining $\delta \Delta^0$:

$$\delta \Delta_0^{0[01]} = -\frac{1}{4} \delta W_1,$$

$$\delta \Delta_1^{0[01]} = -\frac{1}{4} \delta W_0,$$

$$\delta \Delta_2^{0[2a]} = \delta \Delta_3^{0[3a]} = -\frac{1}{4} \delta W_2,$$

$$\delta \Delta_2^{0[2a]} = -\delta \Delta_3^{0[2a]} = \frac{1}{4} F \delta W_a, \quad \text{(B6)}$$

where $a \in \{0, 1\}$ here and in the following. The remaining $\Delta$s are:
\[ \delta \Delta_{2(03)} = -\delta \Delta_{3(02)} = -\frac{1}{2} F \delta W_a, \]
\[ \delta \Delta_{0(10)} = 2 E_0 \partial_t [\delta W] - \frac{1}{2} \delta W_1, \]
\[ \delta \Delta_{1(10)} = 2 E_1 \partial_r [\delta W] - \frac{1}{2} \delta W_0, \]
\[ \delta \Delta_{2(2a)} = \delta \Delta_{3(3a)} = -\frac{1}{2} \delta W_a, \]
\[ \delta \Delta_{0(23)} = -E_0 \partial_t [\delta F], \]
\[ \delta \Delta_{1(23)} = -E_1 \partial_r [\delta F] - \delta E_1 \partial_r [F], \]
\[ \delta \Delta_{3(02)} = -\delta \Delta_{2(03)} = E_0 \partial_t [\delta F], \]
\[ \delta \Delta_{3(12)} = -\delta \Delta_{2(13)} = E_1 \partial_r [\delta F] + \delta E_1 \partial_r [F]. \]

Then \( \delta \Omega \) can be calculated:

\[ \delta \Omega_{0(10)} = E_0 \partial_t [\delta W] - \frac{1}{4} \delta W_1, \]
\[ \delta \Omega_{1(10)} = E_1 \partial_r [\delta W] - \frac{1}{4} \delta W_0, \]
\[ \delta \Omega_{2(02)} = \delta \Omega_{3(03)} = \frac{1}{4} \delta W_0 - \frac{E_1 \delta W}{r}, \]
\[ \delta \Omega_{2(12)} = \delta \Omega_{3(13)} = \frac{1}{4} \delta W_1, \]
\[ \delta \Omega_{0(23)} = -\frac{1}{2} E_0 \partial_r [\delta F], \]
\[ \delta \Omega_{1(23)} = -\frac{1}{2} E_1 \partial_r [\delta F] + \frac{F \delta E_1}{r} + \frac{E_1 \delta F}{r} - \frac{1}{2} \delta E_1 \partial_r [F], \]
\[ \delta \Omega_{3(02)} = -\delta \Omega_{2(03)} = \frac{1}{2} E_0 \partial_t [\delta F], \]
\[ \delta \Omega_{3(12)} = -\delta \Omega_{2(13)} = \frac{1}{2} E_1 \partial_r [\delta F] + \frac{1}{2} \delta E_1 \partial_r [F]. \]

One is now in a position to invert the compatibility conditions (56) to solve for the perturbations to the connection coefficients. First the corrections to \( \Lambda \):

\[ \delta \Lambda^0_{01} = -\frac{2}{3} E_0 \partial_t [\delta W], \]
\[ \delta \Lambda^1_{01} = -\frac{2}{3} E_1 \left( \frac{\delta W}{r} - \partial_r [\delta W] \right), \]
\[ \delta \Lambda^2_{20} = \delta \Lambda^3_{30} = -\frac{1}{3} E_1 \left( \frac{\delta W}{r} - \partial_r [\delta W] \right), \]
\[ \delta \Lambda^2_{21} = \delta \Lambda^3_{31} = \frac{1}{3} E_0 \partial_t [\delta W], \]
\[ \delta \Lambda^0_{23} = -\frac{1}{2} E_0 \partial_t [\delta F], \]
\[ \delta \Lambda^1_{23} = -\frac{E_1 \delta F}{r} - \frac{\delta E_1 F'}{r} + \frac{1}{2} E_1 \partial_r [\delta F], \]
\[ \delta \Lambda^2_{03} = \delta \Lambda^3_{20} = \frac{1}{2} E_0 \partial_t [\delta F], \]
\[ \delta \Lambda^2_{13} = \delta \Lambda^3_{21} = \frac{1}{2} E_1 \partial_r [\delta F] + \frac{1}{2} F' \delta E_1, \]

and then to \( \Gamma \):

\[ \delta \Gamma^0_{01} = \delta \Gamma^1_{00} = -\frac{E_1}{E_0} \partial_r [\delta E_0] - \delta E_1 \partial_r [\ln(E_0)] + \frac{E_1}{E_0} \delta E_0 \partial_r [\ln(E_0)], \]
\[ \delta \Gamma^0_{11} = \delta \Gamma^1_{10} = -\frac{E_0}{E_1} \partial_t [\delta E_1], \]
\[ \delta \Gamma^2_{03} = \delta \Gamma^3_{30} = -\delta \Gamma^3_{20} = -\frac{1}{2} E_1 F' \delta W, \]
\[ \begin{align*}
\delta \Gamma^0_{22} &= \delta \Gamma^0_{33} = 2 \delta \Gamma^2_{02} = 2 \delta \Gamma^2_{20} = 2 \delta \Gamma^3_{03} = 2 \delta \Gamma^3_{30} = E_0 F \partial_t [\delta F], \\
\delta \Gamma^1_{22} &= \delta \Gamma^1_{33} = -\frac{\delta E_1}{r} - E_1 F \partial_r [\delta F] - E_1 F' \delta F, \\
\delta \Gamma^2_{12} &= \delta \Gamma^2_{13} = \frac{1}{2} E_1 F \partial_r [\delta F] + \frac{1}{2} E_1 F' \delta F, \\
\delta \Gamma^2_{21} &= \delta \Gamma^3_{31} = \frac{\delta E_1}{r} + \frac{1}{2} E_1 F \partial_r [\delta F] + \frac{1}{2} E_1 F' \delta F. 
\end{align*} \]  

(B10)

The remaining field equations will be: \( {^1R}_{00}, {^1R}_{(01)}, {^1R}_{[01]}, {^1R}_{11}, {^1R}_{22} = {^1R}_{33} \) and \( {^1R}_{[23]} \). The relevant combinations will be quoted in Section [VI].
REFERENCES

[1] J. W. Moffat. Nonsymmetric gravitational theory. J. Math. Phys., 36, 3722–3732, 1995.
[2] J. W. Moffat. A new nonsymmetric gravitational theory. Phys. Lett. B, 355, 447–452, 1995.
[3] T. Regge and J. A. Wheeler. Stability of the Schwarzschild singularity. Phys. Rev. D, 108, 1063–1069, 1957.
[4] C. V. Vishveshwara. Stability of the Schwarzschild metric. Phys. Rev. D, 1, 2870–2879, 1970.
[5] M. Kalb and P. Ramond. Classical direct interstring action. Phys. Rev. D, 9, 2273–2284, 1974.
[6] J. A. Isenberg and J. M. Nestor. The effect of a gravitational interaction on classical fields: A Hamiltonian–Dirac analysis. Ann. Phys., 107, 56–81, 1977.
[7] J. W. Moffat. New theory of gravitation. Phys. Rev. D, 19, 3554–3558, 1979.
[8] A. Einstein. A generalization of the relativistic theory of gravitation. Ann. Math., 46, 578–584, 1945.
[9] A. Einstein and E. G. Straus. A generalization of the relativistic theory of gravitation, ii. Ann. Math., 47, 731–741, 1946.
[10] J. W. Moffat. Review of the nonsymmetric gravitational theory. In R. Mann and P. Wesson, editors, Gravitation: A Banff Summer Institute, New Jersey, 1991. World Scientific.
[11] Albert Einstein. The Meaning of Relativity. Princeton University Press, Princeton, New Jersey, 5 edition, 1974.
[12] G. Kunstatter and R. Yates. The geometrical structure of a complexified theory of gravitation. J. Phys. A, 14, 847–854, 1981.
[13] R. B. Mann. New ghost-free extensions of general relativity. Class. Quantum Grav., 6, 41–57, 1989.
[14] G. Kunstatter, J. W. Moffat, and J. Malzan. Geometrical interpretation of a generalized theory of gravitation. J. Math. Phys., 24, 886–889, 1983.
[15] P. F. Kelly. Expansions of non-symmetric gravitational theories about a GR background. Class. Quantum Grav., 8, 1217–1229, 1991.
[16] Robert M. Wald. General Relativity. The University of Chicago Press, Chicago, 1984.
[17] T. Damour, S. Deser, and J. McCarthy. Theoretical problems in nonsymmetric gravitational theory. Phys. Rev. D, 45, R3289–R3291, 1992.
[18] T. Damour, S. Deser, and J. McCarthy. Nonsymmetric gravity theories: Inconsistencies and a cure. Phys. Rev. D, 47, 1541–1556, 1993.
[19] P. F. Kelly. Expansions of non-symmetric gravitational theories about a GR background. Class. Quantum Grav., 9, 1423, 1992. Erratum.
[20] A. E. Fischer and J. E. Marsden. The initial value problem and the dynamical formulation of general relativity. In S. W. Hawking and W. Israel, editors, General Relativity: An Einstein Centenary Survey, New York, 1979. Cambridge University Press.
[21] N. Nakanishi. Quantum electrodynamics in the general covariant gauge. Prog. theor. Phys., 38, 881–891, 1967.
[22] T. Goto and T. Obara. The canonical quantization of the free electromagnetic field in the landau gauge. Prog. Theor. Phys., 38, 871–880, 1967.
[23] P. vanNieuwenhuizen. On ghost-free tensor Lagrangians and linearized gravitation. Nucl. Phys. B, 60, 478–492, 1973.
[24] G. Kunstatter, H. P. Leivo, and P. Savaria. Dirac constraint analysis of a linearized theory of gravitation. *Class. Quantum Grav.*, 1, 7–13, 1984.

[25] R. B. Mann and J. W. Moffat. Ghost properties of generalized theories of gravitation. *Phys. Rev. D*, 26, 1858–1861, 1982.

[26] J. W. Moffat. A solution of the Cauchy initial value problem in the nonsymmetric theory of gravitation. *J. Math. Phys.*, 21, 1798–1801, 1980.

[27] J. C. McDow and J. W. Moffat. Consistency of the Cauchy initial value problem in a nonsymmetric theory of gravitation. *J. Math. Phys.*, 23, 634–636, 1982.

[28] J. W. Moffat. Gauge invariance and string interactions in a generalized theory of gravitation. *Phys. Rev. D*, 23, 2870–2874, 1981.

[29] R. B. Mann. Gravity, ghosts, and strings. *Can. J. Phys.*, 64, 589–594, 1985.

[30] N. J. Cornish and J. W. Moffat. A non-singular theory of gravity? *Phys. Lett. B*, 336, 337–342, 1994.

[31] N. N. Bogoliubov and D. V. Shirkov. *Introduction to the Theory of Quantized Fields*. Interscience Publishers inc., New York, 1959.

[32] C. Itzykson and J.-B. Zuber. *Quantum Field Theory*. McGraw-Hill Book Company, Toronto, 1980.

[33] M. Reed and B. Simon. *Scattering Theory*, volume III of *Methods of Modern Mathematical Physics*. Academic Press, Inc., New York, 1979.

[34] M. A. Tonnelat. *Einstein’s Theory of Unified Fields*. Gordon and Breach, Science Publishers, New York, 1982.

[35] Eugen Josef Vlachynsky. *Analytic Solutions in Moffat’s Nonsymmetric Generalized Theory of Gravitation*. PhD thesis, Department of Applied Mathematics, University of Sydney, 1988.

[36] V. Hlavatý. *Geometry of Einstein’s Unified Field Theory*. P. Noordhoff Ltd., Groningen, Holland, 1958.

[37] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleik. *Analysis, Manifolds and Physics*, volume 1. North Holland, New York, 1989.

[38] M. Nakahara. *Geometry, Topology and Physics*. Adam Hilger, New York, 1990.

[39] J. W. Moffat. Test-particle motion in the nonsymmetric gravitation theory. *Phys. Rev. D*, 35, 3733–3747, 1987.

[40] J. Légaré and J. W. Moffat. Field equations and conservation laws in the nonsymmetric gravitational theory. *Gen. Rel. Grav.*, 27, 761–775, 1995.

[41] M. Wyman. Unified field theory. *Can. J. Math.*, 2, 427–439, 1950.

[42] N. J. Cornish. The nonsingular Schwarzschild-like solution to massive nonsymmetric gravity. *UTPT-94-37*, 1994.

[43] Kenneth A. Dunn. *Birkhoff’s Theorem and Equations of Motion in a Unified Field Theory*. PhD thesis, Department of Mathematics, University of Toronto, 1971.