Convex and Non-convex Approaches for Statistical Inference with Noisy Labels

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Abstract

We study the problem of estimation and testing in logistic regression with class-conditional noise in the observed labels, which has an important implication in the Positive-Unlabeled (PU) learning setting. With the key observation that the label noise problem belongs to a special subclass of generalized linear models (GLM), we discuss convex and non-convex approaches that address this problem. A non-convex approach based on the maximum likelihood estimation produces an estimator with several optimal properties, but a convex approach has an obvious advantage in optimization. We demonstrate that in the low-dimensional setting, both estimators are consistent and asymptotically normal, where the asymptotic variance of the non-convex estimator is smaller than the convex counterpart. We also quantify the efficiency gap which provides insight into when the two methods are comparable. In the high-dimensional setting, we show that both estimation procedures achieve $\ell_2$-consistency at the minimax optimal $s \log p/n$ rates under mild conditions. Finally, we propose an inference procedure using a de-biasing approach. To the best of our knowledge, this is the first work providing point estimation guarantees and hypothesis testing results for GLMs with non-canonical link functions, which is of independent interest. We validate our theoretical findings through simulations and a real-data example.

keywords: generalized linear model, non-convexity, class-conditional label noise, PU-learning, regularization

1 Introduction

Label noise is a common phenomenon in a number of classification applications. For example, label noise occurs when humans are involved in labeling due to inattention or subjectivity (Ipeirotis et al., 2010; Smyth et al., 1995). Label noise can also come from bad data-entry (Sculley and Cormack, 2008) or is sometimes intentionally introduced to protect the privacy of a respondent (van den Hout and Peter G. M. van der Heijden, 2002). Consequently, it is important to investigate how to carry out valid statistical inference in the presence of label noise. In this paper, we study binary logistic
regression with label noise, where each observed response is generated by a random flip of the true original response.

This article addresses the estimation and testing problems of a binary logistic regression model where noise is present in responses. We assume a standard logistic model between the true binary responses and the known features, and a contamination process of the true labels. In particular, we assume that labels are corrupted with asymmetric probabilities based on their values, but those probabilities are not affected by the features. The goal is to estimate, and perform inferences on, the parameter in the logistic model, which parametrizes the relationship between the features and the true labels.

There is a substantial literature on the subject of learning with label noise data. Since the random classification noise model was first proposed in Angluin and Laird (1988), extensive studies have been conducted to develop algorithms for building a classifier that effectively separates true positive and negative samples from data with label noise, and to establish theoretical guarantees for the proposed classifiers (see e.g. Frénay and Verleysen (2014) for a comprehensive survey). On the other hand, there has been limited literature on the parametric estimation problem employing latent variable models (e.g. Magder and Hughes (1997), Hausman et al. (1998), and Bootkrajang and Kabán (2012)). In all these aforementioned works, either convergence to a local optimum was established without theoretical guarantees for the obtained estimators being provided, or the maximum likelihood estimator was considered in the theoretical analysis without discussion of the feasibility of obtaining such global optimum in a non-convex problem. In contrast, one of the contributions of our paper is that we demonstrate achieving the global maximum is possible with high probability.

1.1 Positive-Unlabeled learning

An important example of the label noise problem includes the Positive-Unlabeled (PU) learning problem, where labeled samples are known to be positive, but unlabeled samples may be either positive or negative. Positive-Unlabeled learning arises in many applications where obtaining negative responses is more costly or intractable. One concrete example arises from Deep-Mutational Scanning (DMS) data sets in biochemistry (Fowler and Fields, 2014), where a data set consists of functional (positive) variants of a protein, together with unknown functionality (unlabeled) variants from an initial library. In Section 7 we provide an evaluation of our approaches on such a data set.
Numerous other applications of PU-learning arise (see e.g. Liu et al. (2003); Yang et al. (2014); Elkan and Noto (2008)).

As in the case of the label noise problem, a large volume of literature is devoted to building a classifier separating underlying positive and negative samples using positive and unlabeled data, and establishing theoretical guarantees for the proposed classifiers (see e.g. Bekker and Davis (2018) for a comprehensive survey). In the related PU problem, there exists a line of work in the ecology literature on the statistical modeling of geographic distribution of species with presence-only data. Presence-only data is a kind of PU data since the inputs consist of records where a certain species has been found (positive samples), together with background sample points (unlabeled samples). Most of this literature focuses on modeling an occurrence rate, the number of expected observations per unit area. Since the occurrence rate concerns the presence of species in reference to the background, the analysis goal is different from modeling positive and negative responses. On the other hand, Ward et al. (2009) considered modeling of presence and absence of species. Treating an indicator of true positive and absence of species as a latent variable, the latent variable model was fitted via the EM algorithm. In high-dimensions, Song and Raskutti (2018) studied the estimation problem in PU-learning. They proposed an algorithm which converges to a stationary point of the objective function and provided a theoretical guarantee for the stationary point.

1.2 Our contributions

In this paper, we study the parametric estimation and testing problem given observations where labels are observed with noise. One of the consequences of the label noise is that the maximum likelihood objective yields a non-convex minimization problem (Magder and Hughes 1997; Bootkrajang and Kabán 2012; Song and Raskutti 2018). On the other hand, the surrogate loss based on an unbiased estimate of the original loss function leads to a convex minimization problem (Char-ganty and Liang 2014; Natarajan et al. 2018; Du Plessis et al. 2015). We propose and compare these two approaches in the classical regime and the high-dimensional regime, where the number of features $p$ is fixed or grows with $n$, potentially at a faster speed. In this paper, we make the following contributions:

- Theoretical guarantees for parameter estimates for both non-convex likelihood-based and convex surrogate approaches in the classical regime (Proposition 4.1 and 4.2). Our guarantee is for any local minimum, by establishing that the empirical likelihood function has actually
at most one stationary point with high probability (Proposition 4.3). In contrast, prior work either proves convergence to a local minimum or proves theory for the global minimizer without any guarantee of finding this point.

- Quantification of the efficiency gap of the two estimators based on the conditions of the design matrix $X$, which provides an insight into the performance of the convex versus non-convex estimators (Corollary 4.1).
- Mean-squared error guarantees and valid testing procedures in the high-dimensions, for the two estimators based on non-convex and convex approaches (Theorem 5.1 and 5.2). The error bounds match with the optimal $s \log p/n$ rates known as minimax optimal in the sparse regression literature (Raskutti et al., 2011). The testing procedure in high dimensions is based on de-biasing a penalized estimator and to the best of our knowledge, the first such theoretical analysis of testing procedures.

- A simulation study and a real data analysis to empirically support our theoretical findings.

Now we outline the remainder of the paper. We begin by discussing the set-up of the work in Section 2. In Section 3, we discuss how our noisy logistic regression model can be represented as a generalized linear model, and introduce the convex and non-convex approaches for parameter estimation. We establish point estimation guarantees and hypothesis testing in both low-dimensions (Section 4) and high-dimensions (Section 5). In Section 6 and 7, we apply convex and non-convex methods to synthetic and real data and compare the performance of the two estimators. Finally, we conclude the paper with remarks in Section 8.

## 2 Problem Setup

First we define the problem and introduce the major notation. We assume access to samples $(x_i, z_i)_{i=1}^n$ where $(z_i)_{i=1}^n$ are observed labels and $x_i \in \mathbb{R}^p$ is a $p$-dimensional feature vector such that $x_i = [1, x_{2i}, \ldots, x_{pi}]$. Each observed label $z_i$ is a corrupted version of a latent binary outcome $y_i$, where $y_i \sim p_{\beta_0}(y_i|x_i)$ is a true response with p.d.f is given by a logistic model,

$$p_{\beta}(y|x) = \exp(yx^T \beta - \log(1 + \exp(x^T \beta))),$$  

(1)
and $z_i$ is generated by flipping the value of $y_i$ randomly based on known noise rates $\rho_0$ and $\rho_1$, with

$$\rho_0 := P(z = 1|y = 0) \quad \text{and} \quad \rho_1 := P(z = 0|y = 1),$$

for $\rho_0 + \rho_1 < 1$. We assume that $z_i$ and $x_i$ are conditionally independent given the true response $y_i$. The goal is to estimate and perform inference on $\beta_0$.

The relationship between the conditional mean of $y$ and the conditional mean of $z$ can be obtained under the conditional independence assumption. By the factorization theorem, we have

$$E[z|x] = P(z = 1|y = 1)E[y|x] + P(z = 1|y = 0)(1 - E[y|x])$$

$$= (1 - \rho_1)E[y|x] + \rho_0(1 - E[y|x])$$

$$= (1 - \rho_1 - \rho_0)E[y|x] + \rho_0. \quad (2)$$

For the remainder of the paper, we let $P_{\beta}$ be the distribution of the data when $y|x \sim p_{\beta}(\cdot|x)$ in (1) and we will sometimes write $P(\cdot)$ for the probability distribution evaluated at the true parameter $\beta_0$, i.e. $P(\cdot) = P_{\beta_0} (\cdot)$.

**Connection to Positive-Unlabeled learning**

The set-up in the previous section has an important implication in Positive-Unlabeled (PU) learning. In PU learning, we learn a model with two sets of samples, where the first set consists of labeled and positive subjects and the second set consists of unlabeled subjects whose associated responses are unknown.

Two schemes are considered for PU-learning: the first scheme is a single training set scheme (Elkan and Noto, 2008) whose complete observations $(x_i, y_i, z_i)_{i=1}^n$ are from a single distribution and only $(x_i, z_i)_{i=1}^n$ are recorded. The second scheme is where observations in the positive and unlabeled set are drawn separately, with the unlabeled set drawn from the general population (Ward et al., 2009; Song and Raskutti, 2018). A subtle but important difference between the two schemes is that a sample from the first scheme has the same distribution as the joint distribution of the population but a sample from the second scheme does not. In the second scheme, positive subjects are over-represented in the data set, since the distribution of the unlabeled sample is the same as the population distribution and the labeled set consists of only positive subjects. Therefore, a case-control sampling model (McCullagh and Nelder, 1989) is necessary in the second scheme,
where different inclusion probabilities are allowed based on the value of the true responses.

We demonstrate how both PU schemes fit into the set-up of our label noise problem and also show how the error rates $\rho_1$ and $\rho_0$ are related with the number of labeled ($n_\ell$) and unlabeled samples ($n_u$), and the proportion of positives in the unlabeled set $\pi := P(y = 1|z = 0)$. We assume a parametric logistic model between $(x, y)$ as in (1). In both schemes, flipping probabilities from $y$ to $z$ do not depend on $x$. Also, $\rho_0 = P(z = 1|y = 0) = 0$ since all labeled elements ($z = 1$) are positive ($y = 1$) by the set-up of the PU problem. On the other hand, by Bayes’ theorem we have

$$
\rho_1 = \frac{P(y = 1|z = 0)P(z = 0)}{\pi P(z = 0)} = \frac{\pi n_u}{n_\ell + \pi n_u},
$$

where we use the definition $\pi := P(y = 1|z = 0)$ and $P(z = 0)/P(z = 1) \approx n_u/n_\ell$. Thus, the knowledge of $\pi$ practically amounts to knowing error rates ($\rho_0, \rho_1$) in PU-learning.

In the case-control sampling model, only selected subjects $(x_i, z_i, s_i = 1)_{i=1}^n$ are available in the data set where $s_i \in \{0, 1\}$ represents whether the $i$th subject is selected or not. It is a well-known result (e.g. McCullagh and Nelder (1989)) that case-control probabilities $P(y = 1|x, s = 1)$ differ from $P(y = 1|x)$ by the intercept, whose adjustment term is given by the log ratio of the different selection probabilities. More concretely, $p_\beta(y|x, s = 1)$ can be written as

$$
p_\beta(y|x, s = 1) = \exp\{y(x^\top \beta^\gamma) - \log(1 + \exp(x^\top \beta^\gamma))\},
$$

for $\beta^\gamma \in \mathbb{R}^p$ such that $\beta_1^\gamma = \beta_1 + \gamma$ and $\beta_j^\gamma = \beta_j$, $\forall j \geq 2$, and where $\gamma := \log(P(s = 1|y = 1)/P(s = 1|y = 0))$ is the log ratio of the different selection probabilities. The log ratio $\gamma$ can also be expressed as functions of $n_\ell$, $n_u$, and $\pi$. Specifically, $\gamma = \log(1 + n_\ell/\pi n_u)$ was derived in Ward et al. (2009).

We note that in both PU schemes the conditional distribution of $y$ follows a logistic model, with the parameter $\beta_0$ in the first scheme and $\beta_0^\gamma$ in the second scheme. Since our target of interest is the coefficients of the model and $\beta_{0j}^\gamma = \beta_{0j}^\gamma$, $\forall j \geq 2$, from this point on we will treat both sampling models the same. Specifically, we will omit conditioning on $s$ and dependence of $\gamma$ in $\beta_0^\gamma$, and we assume $y|x \sim p_{\beta_0}(y|x) = \exp\{y(x^\top \beta_0) - \log(1 + \exp(x^\top \beta_0))\}$ in both PU schemes.
3 Convex and non-convex approaches for inference

In this section, we briefly review generalized linear models (McCullagh and Nelder (1989)) and discuss how all models discussed above can be fitted into the generalized linear model (GLM) framework. Then we introduce two approaches to estimate the true parameter $\beta_0$, i.e. the parameter from which the data is generated. The first approach is to use a negative log-likelihood loss function, which is a non-convex function of $\beta$. In the second approach, we discuss how we can construct a convex surrogate function.

3.1 Generalized linear models (GLMs)

Generalized linear models (McCullagh and Nelder (1989)) are an extension of linear models, where a response $z \in \mathcal{Z}$ has a p.d.f of the form

$$p_\theta(z) = c(z) \exp(z\theta - A(\theta)),$$

(3)

for $\theta \in \mathbb{R}$ which can depend on $x$, and $A(\theta) = \log \int_{\mathcal{Z}} c(z) \exp(z\theta) dz$. The mean and variance of $z$ can be derived from (3):

$$E_\theta(z|x) = \mu = A'(\theta) \quad \text{and} \quad \operatorname{Var}_\theta(z|x) = A''(\theta) = V(\mu),$$

(4)

where the variance function $V$ is defined as $V := A'' \circ (A')^{-1}$ so that it is a function of $\mu$.

Another important component of the GLM is the link function $g$, which relates the linear predictor $x^T \beta$ to the mean of the response $\mu$ by $g(\mu) = x^T \beta$. By definition of $g$ and $\mu = A'(\theta)$, $\theta = (g \circ A')^{-1}(x^T \beta)$, and we can rewrite (3) in terms of the linear predictor $x^T \beta$ and the link function $g$. Therefore, the assumed distribution and the link function are two defining components of the GLM. We define the following:

**Definition 3.1 (GLM).** We say a sample $(x_i, z_i)_{i=1}^n$ is from a (GLM) with parameters $(A, g)$ if the p.d.f of $z$ has the form

$$p_\beta(z|x) = c(z) \exp(z h(x^T \beta) - A(h(x^T \beta))),$$

(5)

for some $c$ only depending on $z$ and $h := (A')^{-1} \circ g^{-1}$.
We require \( g \) to be strictly increasing so that responses are positively related with linear predictors. A GLM is called canonical if \( g = (A')^{-1} \) which implies \( h(\cdot) = I(\cdot) \), an identity function. Suppose a random variable \( y \) is from a canonical GLM \((A, (A')^{-1})\). Then we have

\[
p_\beta(y|\mathbf{x}) = c(y) \exp(y \mathbf{x}^\top \beta - A(\mathbf{x}^\top \beta)).
\]

For example, the logistic model (1) is an example of a canonical GLM.

As we will discuss shortly in more detail, the statistical models for noisy labels belong to a special class of non-canonical GLMs whose mean is linearly related to the mean \( A' \) of a canonical GLM. In this type of case, the link function \( g \) is determined by such linear relationship since the link function is the inverse of the mean, i.e. \( \mathbb{E}_\beta[z|x] = g^{-1}(\mathbf{x}^\top \beta) \). More concretely, suppose we have the following linear relationship

\[
\mathbb{E}_\beta[z|x] = aA'(\mathbf{x}^\top \beta) + b,
\]

for some \( a > 0 \), and \( b \geq 0 \). Then \( g \) has to satisfy the equation \( g(aA'(\mathbf{x}^\top \beta) + b) = \mathbf{x}^\top \beta \), i.e.

\[
g(t) = (A')^{-1} \left( \frac{t - b}{a} \right).
\]

Conversely, if \( g \) is taken to be as in (7), the linear relationship (6) is satisfied. We refer to this sub-class of GLM, where the link function \( g \) follows the form in (7), as (GLM-L) with parameters \((A, a, b)\).

### 3.2 Statistical models for noisy labels and GLMs

Now we relate the statistical models for noisy labels with the GLM framework. Since we have \( z \in \{0, 1\} \),

\[
p_\beta(z|\mathbf{x}) = (\mathbb{E}_\beta(z|\mathbf{x}))^z(1 - \mathbb{E}_\beta(z|\mathbf{x}))^{1-z}
\]

\[
= \exp \left( z \theta - \log(1 + e^\theta) \right)
\]
for \( \theta = \log \left( \frac{E_\beta(z|x)}{1 - E_\beta(z|x)} \right) \), and thus \( p_\beta(z|x) \) belongs to a GLM with \( A(t) = \log(1 + e^t) \). Also by (1) and (2), we have the representation

\[
E_\beta[z|x] = (1 - \rho_1 - \rho_0)E_\beta[y|x] + \rho_0
\]

\[
= (1 - \rho_1 - \rho_0) \frac{e^{x^\top \beta}}{1 + e^{x^\top \beta}} + \rho_0.
\]

From (8), we obtain the link function \( g_{LN}(\text{label-noise}) \) by solving \( g_{LN}(E_\beta[z|x]) = x^\top \beta \) for \( x^\top \beta \):

\[
g_{LN}(t) = \logit \left( \frac{t - \rho_0}{1 - \rho_1 - \rho_0} \right).
\]

Therefore \((x_i, z_i)_{i=1}^n\) belongs to (GLM-L) with \((\log(1 + \exp(\cdot)), (1 - \rho_1 - \rho_0), \rho_0)\).

In the subsequent analysis the variances of a clean label \( y \) and a noisy response \( z \) will play an important role. First we define mean functions \( \mu \) and \( \mu_z \) as \( \mu(t) := A'(t) \) and \( \mu_z(t) := A'(h_{LN}(t)) \), for \( A(\cdot) = \log(1 + \exp(\cdot)) \) and \( h_{LN} := (A')^{-1} \circ g_{LN}^{-1} \). In particular, we have \( E_\beta[z|x] = \mu_z(x^\top \beta) \) and \( E_\beta[y|x] = \mu(x^\top \beta) \). By the definition of \( V \) in (4), we have

\[
\text{Var}_\beta(z|x) = A''(h_{LN}(x^\top \beta)) = V(\mu_z(x^\top \beta))
\]

\[
\text{Var}_\beta(y|x) = A''(x^\top \beta) = V(\mu(x^\top \beta))
\]

where the last equality uses the fact that \((A')^{-1} \circ \mu = I\).

### 3.3 Non-convex approach using a negative log-likelihood loss

Given a sample \((x_i, z_i)_{i=1}^n\) from (GLM) with \((A, g)\), a natural approach for the estimation of \( \beta_0 \) is to take a likelihood-based approach due to the several optimality properties of a likelihood function. A negative log-likelihood loss can be obtained directly from (5) as

\[
\mathcal{L}_\ell^n(\beta) := \frac{1}{n} \sum_{i=1}^n A(h(x_i^\top \beta)) - z_i h(x_i^\top \beta) = \frac{1}{n} \sum_{i=1}^n \ell(x_i^\top \beta, z_i), \quad (10)
\]

where we define \( \ell(x^\top \beta, z) := A(h(x^\top \beta)) - zh(x^\top \beta) \). In general, the likelihood becomes a non-convex function of \( \beta \) unless \( g = (A')^{-1} \) i.e. \( g \) is canonical and \( h \) is an identity function.
The first and second derivative of the likelihood function are
\[ \nabla \mathcal{L}_n^\ell(\beta) = \frac{1}{n} \sum_{i=1}^{n} \ell'(x_i^\top \beta, z_i) x_i, \quad \nabla^2 \mathcal{L}_n^\ell(\beta) = \frac{1}{n} \sum_{i=1}^{n} \ell''(x_i^\top \beta, z_i) x_i x_i^\top, \]
where we write
\[ \ell'(t, z) = (A'(h(t)) - z) h'(t) \]
\[ \ell''(t, z) = A''(h(t)) h'(t)^2 + (A'(h(t)) - z) h''(t) \]
\[ := \rho_I(t) + \rho_R(t, z) \]
for \( \rho_I(t) := A''(h(t)) h'(t)^2 \) and \( \rho_R(t, z) := (A'(h(t)) - z) h''(t) \). Although \( \rho_I \geq 0 \), the sign of \( \rho_R \) is arbitrary, and thus \( \nabla^2 \mathcal{L}_n^\ell(\beta) \) is not necessarily a positive semi-definite matrix.

### 3.4 Construction of a convex surrogate loss

Next, we discuss an alternative approach involving a convex surrogate function when a sample is from a \((\text{GLM-L})\) model with parameters \((A, a, b)\). Essentially, we construct an unbiased estimator of a convex loss function with the same minimizer, which is a well-known idea in stochastic optimization \cite{nemirovski2009} and has also been investigated in the latent variable model literature \cite{loh2012, chaganty2014, natarajan2018}. More concretely, if the responses \((y_i)_{i=1}^n\) from a canonical GLM are available, we can minimize a convex loss \( \mathcal{L}_n^c(\beta) \) which we define as
\[ \mathcal{L}_n^c(\beta) := \frac{1}{n} \sum_{i=1}^{n} A(x_i^\top \beta) - y_i(x_i^\top \beta). \]

For example, we can take this convex approach if labels are not contaminated. Since \((y_i)_{i=1}^n\) are not available, we construct a surrogate function \( \mathcal{L}_n^s(\beta) \) by replacing \( z \) with a function output \( T(z) \) while keeping \( h(\cdot) = I(\cdot) \):
\[ \mathcal{L}_n^s(\beta) := \frac{1}{n} \sum_{i=1}^{n} A(x_i^\top \beta) - T(z_i)(x_i^\top \beta). \]

To obtain a consistent estimator, the function \( T \) needs to satisfy \( \mathbb{E}_\beta[T(z)|x] = A'(x^\top \beta) = \mathbb{E}_\beta[y|x] \). Such a function \( T \) is available by the \((\text{GLM-L})\) model class assumption. Specifically, we let \( T \) be
\( T(t) := (t - b)/a \) so that \( \mathbb{E}_{\beta}[T(z)|x] = \mathbb{E}_{\beta}[(z - b)/a] = A'(x^T\beta) \) by \([6]\). For a future reference we define

\[
\ell_s(x^T\beta, z) := A(x^T\beta) - T(z)(x^T\beta)
\]

so that \( \mathcal{L}^n_s(\beta) = n^{-1}\sum_{i=1}^n \ell_s(x_i^T\beta, z_i) \).

At any fixed parameter \( \beta \), the surrogate loss \([14]\) is an unbiased estimate of the loss \([13]\). We note

\[
\mathbb{E}_{\beta_0}[\mathcal{L}^s_n(\beta)] = \mathbb{E}_{\beta_0} \left[ \frac{1}{n} \sum_{i=1}^n A(x_i^T\beta) - \mathbb{E}_{\beta_0}[T(z_i)|x_i](x_i^T\beta) \right] = \mathbb{E}_{\beta_0} \left[ \frac{1}{n} \sum_{i=1}^n A(x_i^T\beta) - A'(x_i^T\beta_0)(x_i^T\beta) \right] = \mathbb{E}_{\beta_0}[\mathcal{L}^c_n(\beta)],
\]

where we use the law of iterative expectation and \( \mathbb{E}_{\beta_0}[y|x] = A'(x^T\beta_0) \).

### 3.5 Notation

Before proceeding, we pause to define some notation that will be useful in presenting our theoretical results. For \( v \in \mathbb{R}^p \), we denote the \( \ell_1 \), \( \ell_2 \), and \( \ell_\infty \) norms as

\[
\|v\|_1 := \sum_{i=1}^p |v_i|, \quad \|v\|_2 := \sqrt{v^Tv}, \quad \|v\|_\infty := \sup_{1 \leq j \leq p} |v_j|.
\]

Similarly, for a function \( f \), we define \( \|f\|_p := (\int |f(x)|^p dx)^{1/p} \) and \( \|f\|_\infty := \sup_x |f(x)| \). In the case of matrix norm, for \( A \in \mathbb{R}^{m \times n} \), we denote a Frobenius norm as

\[
\|A\|_F := \sqrt{\sum_{i,j} |A_{ij}|^2},
\]

an operator norm as \( \|A\|_2 := \sigma_{\text{max}}(A) \), and an element-wise max norm as \( \|A\|_{\text{max}} := \max_{i,j} |A_{ij}| \). We define a condition number of \( A \) as \( \kappa(A) := \sigma_{\text{max}}(A)/\sigma_{\text{min}}(A) \).

For a set \( S \), we use \( |S| \) to denote the cardinality of \( S \). For \( v \in \mathbb{R}^p \) and any subset \( S \subseteq \{1, \ldots, p\} \), \( v_S \in \mathbb{R}^{|S|} \) denotes the sub-vector of the vector \( v \) by selecting the components with indices in \( S \). Likewise for any matrix \( A \in \mathbb{R}^{m \times n} \), \( A_S \in \mathbb{R}^{m \times |S|} \) denotes a sub-matrix having columns in \( S \). For matrices \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{m \times n} \), we say \( A \succeq B \) if \( A - B \) is a positive semi-definite matrix and \( A \succ B \) if \( A - B \) is positive definite. Also we write \( C(A) \) to refer to a column space of \( A \). Also we use \( B_q(r; v) \) to denote a ball with radius \( r \) in the \( \ell_q \) norm centered at \( v \in \mathbb{R}^p \). If \( v = 0 \), we simply use \( B_q(r) \) to denote the ball.

For functions \( f \) and \( g \), we write \( f(n) = O(g(n)) \) if there exists a constant \( C > 0 \) such that \( f(n) \leq Cg(n) \), \( \forall n \), and \( f(n) \asymp g(n) \) if \( f(n) = O(g(n)) \) and \( g(n) = O(f(n)) \). Also for a random
variable $X_n$, we write $X_n = O_p(a_n)$ if $X_n/a_n$ is bounded in probability and $X_n = o_p(a_n)$ if $X_n/a_n$ converges to 0 in probability. Also for simplicity, we sometimes use $x^n_i$ to refer to the collection of random variables $(x_i)_{i=1}^n$. We write a.s. to denote ‘almost surely’, i.e. an event that occurs with probability 1. Also, for a sequence of events $(\mathcal{E}_n)_{n \geq 1}$, we say $\mathcal{E}_n$ holds with high probability (w.h.p) if $\mathbb{P}(\mathcal{E}_n) \xrightarrow{\text{n}} 1$.

4 Estimation and testing in the classical regime

In this section, we discuss the statistical properties of two estimators from convex and non-convex approaches in the classical regime where the number of features $p$ is fixed. In particular, we demonstrate that both approaches yield consistent estimators, but the estimator based on the non-convex approach has better efficiency than the convex counterpart in the large $n$ limit. Also, we quantify the efficiency gap between the two approaches and discuss when two approaches can be comparable.

4.1 Consistency and relative asymptotic efficiency

To avoid technical complications, we assume the parameter space for $\beta_0$ to be $B_2(r) := \{\beta \in \mathbb{R}^p; \|\beta\|_2 \leq r\}$ for some $r > 0$, a compact subspace of $\mathbb{R}^p$. We assume that the true parameter $\beta_0$ is an interior point of $B_2(r)$. We define a global minimizer of $\mathcal{L}_n^\ell(\beta)$ and $\mathcal{L}_n^s(\beta)$ as

$$\hat{\beta}_n^\ell \in \underset{\beta \in B_2(r)}{\arg\min} \mathcal{L}_n^\ell(\beta) \quad \text{and} \quad \hat{\beta}_n^s \in \underset{\beta \in B_2(r)}{\arg\min} \mathcal{L}_n^s(\beta).$$

(16)

By definition of $\mathcal{L}_n^\ell$ and $\mathcal{L}_n^s$, $\hat{\beta}_n^\ell$ is the solution of a non-convex optimization problem, whereas $\hat{\beta}_n^s$ is based on the convex problem. Clearly, it is not obvious whether it is feasible to obtain $\hat{\beta}_n^\ell$ in practice, since finding a global minimizer of a non-convex function is in general a challenging problem. However, obtaining a stationary point of $\mathcal{L}_n^\ell(\beta)$ is in fact enough when $n$ is sufficiently large, as we will demonstrate in Proposition 4.3 that in the classical regime, with high probability, $\mathcal{L}_n^\ell(\beta)$ has a unique stationary point (i.e. the global minimizer).

In the following Proposition 4.1 we show that both estimators are consistent for $\beta_0$ and also quantify their asymptotic efficiency. We first state the following minimum eigenvalue condition, which is a standard assumption when we consider the design conditioned on $(x_i)_{i=1}^n$ in the classical regime (e.g. Fahrmeir and Tutz (2001); Shao (2003)).
A1. There exist $c_\ell > 0$ and $c_X < \infty$ such that $\lambda_{\min}(n^{-1} \sum_{1 \leq i \leq n} x_i x_i^\top) \geq c_\ell$ and $\sup_{1 \leq i \leq n} \|x_i\|_\infty \leq c_X$, $\forall n$.

Proposition 4.1. (Fixed design) Suppose a sample $(x_i, z_i)_{i=1}^n$ is from a (GLM-L) with $\log(1 + \exp(\cdot)), (1 - \rho_1 - \rho_0)$ and $z_i \in \{0, 1\}$. Assume $\text{A1}$ and the classical regime where the number of features $p$ is fixed and the sample size $n \to \infty$. Then,

$$\sqrt{n}I_n^t(\beta_0)^{1/2}(\tilde{\beta}_t - \beta_0) \xrightarrow{d} N(0, I_p)$$
$$\sqrt{n}I_n^s(\beta_0)^{1/2}(\tilde{\beta}_s - \beta_0) \xrightarrow{d} N(0, I_p),$$

for positive definite matrices $I_n^t(\beta), I_n^s(\beta)$ defined as

$$I_n^t(\beta) := (1 - \rho_1 - \rho_0)^2 \cdot \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{V}(\mu(x_i^\top \beta))^2}{\mathcal{V}(\mu_z(x_i^\top \beta))} x_i x_i^\top,$$

$$I_n^s(\beta) := (1 - \rho_1 - \rho_0)^2 \cdot \left(\frac{1}{n} \sum_{i=1}^n \mathcal{V}(\mu(x_i^\top \beta)) x_i x_i^\top\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n \mathcal{V}(\mu_z(x_i^\top \beta)) x_i x_i^\top\right),$$

The proof essentially uses classical likelihood and generalized estimating equations theory and is provided in the Appendix A.1. One point that deserves special attention is the similarity between $I_n^t(\beta_0)$ and $I_n^s(\beta_0)$ in Proposition 4.1. In particular, if $\mathcal{V}(\mu_z(x_i^\top \beta)) \approx \mathcal{V}(\mu(x_i^\top \beta))$ for all $i$, the two information matrices will turn out to be very similar.

The following Corollary shows that $I_n^t(\beta_0) \succeq I_n^s(\beta_0)$ and quantifies the discrepancy between the two information matrices. First we define two weight matrices $W_y(\beta)$ and $W_z(\beta)$ as

$$W_y(\beta) := \text{diag}(\{\mathcal{V}(\mu(x_i^\top \beta))\}_{i=1}^n) \quad \text{and} \quad W_z(\beta) := \text{diag}(\{\mathcal{V}(\mu_z(x_i^\top \beta))\}_{i=1}^n),$$

whose diagonal entries consist of the conditional variances of $y_i$ and $z_i$ given $x_i$, respectively. We suppress the dependence on $\beta$ if $\beta = \beta_0$ and let $W_y := W_y(\beta_0)$ and $W_z := W_z(\beta_0)$ for ease of notation. Also, we define the gap $\delta(M, N)$ between two vector subspaces $M, N$ as (e.g. Kato (2013))

$$\delta(M, N) := \max\{\delta(M, N), \delta(N, M)\}, \quad \text{for} \quad \delta(M, N) := \sup_{u \in M, \|u\|_2 = 1} \inf_{v \in N} \|u - v\|_2. \quad (17)$$
Figure 1: The plot of the relative $\ell_2$ difference between $\mathcal{I}_n^{T}(\beta_0)$ and $\mathcal{I}_n^{a}(\beta_0)$ as a function of $\text{gap}^2 = \widehat{\delta}^2(\mathcal{C}(W_z^{-1}W_yX),\mathcal{C}(X))$. To generate design matrix $X$ with various gap values, each $x_i$ is sampled from an equal mixture of multivariate Gaussian distribution with different centers; see Section 6 for details. The results were averaged over 10000 repetitions at each center, and the bars denote one standard error.

The gap measures the distance between two subspaces, with $\widehat{\delta}(M,N) = 0$ if and only if $M = N$. Now we present the following Corollary.

**Corollary 4.1.** Assume the conditions as in Proposition 4.1. We have $\mathcal{I}_n^{T}(\beta_0) \succeq \mathcal{I}_n^{a}(\beta_0)$ and

$$
\|I_p - \mathcal{I}_n^{T}(\beta_0)^{-1/2} \mathcal{I}_n^{a}(\beta_0) \mathcal{I}_n^{T}(\beta_0)^{-1/2}\|_2 \leq c_n \widehat{\delta}^2(\mathcal{C}(W_z^{-1}W_yX),\mathcal{C}(X))
$$

(18)

where $c_n := \kappa(X^\top X/n)\kappa(W_y^2)\kappa(W_z^2)$ and $c_n = O(1)$. In particular, $\widehat{\beta}_s$ achieves asymptotic efficiency if $\mathcal{C}(W_z^{-1}W_yX) = \mathcal{C}(X)$.

The proof is provided in Appendix A.2. We note if $p = 1$, the relative $\ell_2$ difference equals to

$$
\|I_p - \mathcal{I}_n^{T}(\beta_0)^{-1/2} \mathcal{I}_n^{a}(\beta_0) \mathcal{I}_n^{T}(\beta_0)^{-1/2}\|_2 = 1 - \mathcal{I}_n^{T}(\beta_0)^{-1} \mathcal{I}_n^{a}(\beta_0)^{-1} = 1 - \text{ARE}(\widehat{\beta}_s,\widehat{\beta}_T; \beta_0)
$$

where $\text{ARE}(\widehat{\beta}_s,\widehat{\beta}_T; \beta_0)$ denotes the asymptotic relative efficiency of $\widehat{\beta}_s$ with respect to $\widehat{\beta}_T$. In general, we can find the direction $u$ such that $\|u\|_2 = 1$ and

$$
\|I_p - \mathcal{I}_n^{T}(\beta_0)^{-1/2} \mathcal{I}_n^{a}(\beta_0) \mathcal{I}_n^{T}(\beta_0)^{-1/2}\|_2 \geq 1 - \text{ARE}(u^\top \widehat{\beta}_s, u^\top \widehat{\beta}_T; \beta_0).
$$

The bound (18) shows that the relative $\ell_2$ difference between $\mathcal{I}_n^{T}(\beta_0)$ and $\mathcal{I}_n^{a}(\beta_0)$ depends on how dissimilar $W_z^{-1}W_y$ is from the identity matrix. We observe that $W_z^{-1}W_y$ is a diagonal matrix.
where the diagonal entries are ratios of the variances of \( z \) and \( y \), i.e.

\[
(W_z^{-1}W_y)_{ii} = \frac{\mathcal{V}(\mu(x_i^T \beta_0))}{\mathcal{V}(\mu_z(x_i^T \beta_0))}.
\]

In light of these observations, the inefficiency of a surrogate convex loss function can be understood as the result of sub-optimal weighting of the observations due to the mis-specification of the variance matrix for \( z \). In fact, in the special case of the intercept-only model, no covariate information is available for the optimal weighting of the observations. In this case, we have \( W_y = w_1I_n, W_z = w_2I_n \) for some \( w_1, w_2 > 0 \), and thus \( \mathcal{C}(W_z^{-1}W_yX) = \mathcal{C}(X) \) and the inequality (18) is sharp.

So far, we have considered the fixed design setting. Now we present the result equivalent to Proposition 4.1 in the random design. We assume that rows in the random design matrix satisfy a sub-gaussian tail condition. We define this sub-gaussian tail condition as follows:

**Definition 4.1 (sub-gaussian tail condition).** We say a random vector \( x \in \mathbb{R}^p \) satisfies the sub-gaussian tail condition with parameter \( K \) if

\[
\sup_{u \in \mathbb{R}^p; \|u\|_2 = 1} \mathbb{E}[\exp(u^T x)^2/K^2] \leq 2. \tag{19}
\]

For example, a random vector \( x \in \mathbb{R}^p \) with \( \mu_X = \|\mathbb{E}[x]\|_2 \) satisfies (19) with \( K = c_1\mu_X + c_2\sigma_X \) for some absolute constants \( c_1, c_2 > 0 \) if the centered vector \( x - \mathbb{E}[x] \) is sub-gaussian with parameter \( \sigma_X \), i.e.

\[
\sup_{u \in \mathbb{R}^p; \|u\|_2 = 1} \mathbb{E}[\exp(t(u^T x - \mathbb{E}[u^T x]))] \leq \exp(t^2\sigma_X^2/2), \forall t \in \mathbb{R}.
\]

We replace \( A1 \) with the following assumption:

**A1’. (Random design)** For a random feature vector \( x \in \mathbb{R}^p \), \( x \) satisfies the sub-gaussian tail condition with parameter \( K_X \) for a positive constant \( K_X < \infty \). There exists \( c_\ell > 0 \) such that \( \lambda_{\min}(\mathbb{E}[xx^T]) \geq c_\ell \).

**Proposition 4.2. (Random design)** Assume the conditions of Proposition 4.1 where \( A1 \) is replaced
Then, 
\[
\sqrt{n}(\hat{\beta}_\ell - \beta_0) \xrightarrow{d} N(0, I_\ell(\beta_0)^{-1}) \\
\sqrt{n}(\hat{\beta}_s - \beta_0) \xrightarrow{d} N(0, I_s(\beta_0)^{-1}),
\]
for \(I_\ell(\beta), I_s(\beta)\) defined as 
\[
I_\ell(\beta) := (1 - \rho_1 - \rho_0)^2 E_\beta \left( \frac{\mathcal{V}(\mu(x^\top \beta))^2}{\mathcal{V}(\mu_x(x^\top \beta))} xx^\top \right), \\
I_s(\beta) := (1 - \rho_1 - \rho_0)^2 E_\beta \left( \mathcal{V}(\mu(x^\top \beta)) xx^\top \right) E_\beta \left( \mathcal{V}(\mu_x(x^\top \beta)) xx^\top \right)^{-1} E_\beta \left( \mathcal{V}(\mu(x^\top \beta)) xx^\top \right).
\]
Also, \(I_\ell(\beta_0) \succeq I_s(\beta_0)\).

The result follows from classical M-estimation theory (see e.g. \textit{van der Vaart (1998)}). \(I_\ell(\beta_0) \succeq I_s(\beta_0)\) follows from Theorem 1 in \textit{Morton (1981)}.

The final result that we will present in this section is about the comparability between the global and local minimizer in the low-dimensional setting. So far, we have only considered the global minimizer of the empirical risk function \(\mathcal{L}_n^\ell(\beta)\) which is the MLE. However, since \(\mathcal{L}_n^\ell(\beta)\) is non-convex, obtaining the global minimizer \(\hat{\beta}_\ell\) is in general computationally intractable, and algorithms on the optimization of non-convex functions focus on finding a stationary point of the objective function.

The population risk, albeit non-convex, can be shown to be unimodal and also strongly convex around \(\beta_0\) in some GLMs. Often, fast probability tail decay of \(x\) and boundedness of the derivatives of the loss function allow enough concentration of the empirical risk function around the population counterpart that the empirical risk function has a unique stationary point, which in fact is the global minimum (Mei et al., 2018). We make the following assumption \(A_2\) about boundedness and smoothness of \(\ell\), for \(\ell\) defined in (10). In Corollary 4.2, we show that Assumption \(A_2\) is satisfied for the label noise model, (GLM) with parameters \((A, g)\).

**A2.** \(\ell''\) is Lipschitz w.r.t its first argument, i.e. \(|\ell''(a, t) - \ell''(a', t)| \leq L_\ell|a - a'|, \forall t.\) Furthermore, there exists \(C_\rho < \infty\) such that \(\max\{\|\ell\|_\infty, \|\ell\|_\infty, \|\rho_I\|_\infty, \|\rho_R\|_\infty\} \leq C_\rho.\)

**Proposition 4.3.** Suppose a sample is from a (GLM) with parameters \((A, g)\) and assume \(A_1\) and \(A_2\) Then, for any given \(\epsilon > 0\), there exists a unique stationary point of \(\mathcal{L}_n^\ell(\beta)\) in the interior of \(B_2(r)\) with probability at least \(1 - \epsilon\), given a sufficiently large \(n \geq C \log(1/\epsilon)p \log n\) where the
constant $C$ depends only on the model parameters in our assumptions. The unique stationary point is the global minimum of $L_n^\ell(\beta)$.

**Corollary 4.2.** Under Assumption $\text{A}^1$, the empirical log-likelihood for the label noise GLM has a unique stationary point, which is the MLE, with high probability.

Proofs of Proposition 4.3 and Corollary 4.2 are provided in Appendix A.3 and A.4.

## 5 Estimation and testing in the high-dimensional regime

### 5.1 $\ell_1$ and $\ell_2$ consistency

In many modern data sets, the number of the features $p$ may be comparable to sample size $n$, or may even be substantially larger ($p \gg n$). In this section, we discuss the estimation of $\beta_0$ in the high-dimensional regime. As in the previous section, we let the parameter space be $B_2(r)$ for some large enough $r$ such that the true parameter $\beta_0$ is an interior point of the parameter space. We propose two estimators $\hat{\beta}_H^\ell$, $\hat{\beta}_H^s$ as solutions of the following optimization problems

$$
\hat{\beta}_H^\ell \in \arg\min_{\beta \in B_1(R_n) \cap B_2(r_0; \beta_0)} L_n^\ell(\beta) + \lambda_\ell \|\beta\|_1 \quad \text{and} \quad \hat{\beta}_H^s \in \arg\min_{\beta \in B_2(r)} L_n^s(\beta) + \lambda_s \|\beta\|_1,
$$

(20)

where $L_n^\ell(\beta)$ is a non-convex negative log-likelihood loss and $L_n^s(\beta)$ is a convex surrogate loss. Due to the non-convexity in $L_n^\ell(\beta)$, we limit the feasible region in the first optimization problem to $B_2(r_0; \beta_0)$, a local neighborhood of $\beta_0$. An additional $\ell_1$ constraint $\|\beta\|_1 \leq R_n$ is imposed for theoretical convenience. Here $R_n$, $\lambda_\ell$ and $\lambda_s$ are tuning parameters which need to be chosen appropriately, and we will discuss their choices shortly. Finally, we note that in many cases, it is common to leave a finite number of coordinates unpenalized. An important special example is when the model includes an intercept feature. The theory that we develop in this section has a straightforward extension when the $\ell_1$ penalty is modified to exclude a finite subset of features.

In the low-dimensional setting, we established that the global minimizer can be obtained with high probability, but it is hard for a similar result to hold in the high-dimensional regime. Therefore, instead of $\hat{\beta}_H^\ell$ we make use of a stationary point and define $\tilde{\beta}_H^\ell$ to be a stationary point of the first optimization problem in (20).

We now study the statistical guarantees of the two estimators in the high-dimensional regime. First, we impose the standard sparsity assumption on $\beta_0$, $s_0 := \|\beta_0\|_0$. The core condition which
needs to be established is the restricted strong convexity (RSC) condition, the notion of which was first proposed by Negahban et al. (2012) for convex loss functions and extended for non-convex functions by Loh (2017); Loh and Wainwright (2015). Following the definition in Loh (2017), we define the (local) RSC condition as follows.

**Definition 5.1 (RSC condition).** We say $L_n$ satisfies a restricted strong convexity (RSC) condition with respect to $\beta_0$ with curvature $\alpha$, tolerance $\tau$, and radius $\delta > 0$ if there exist $\alpha > 0$, $\tau \geq 0$ such that

$$
\langle \nabla L_n(\beta) - \nabla L_n(\beta_0), \beta - \beta_0 \rangle \geq \alpha \|\beta - \beta_0\|^2_2 - \tau \frac{\log p}{n} \|\beta - \beta_0\|^2_1, \quad \forall \beta \in B_2(\delta; \beta_0).
$$

(21)

The main idea behind the definition of RSC is that it is the relaxed version of the strong convexity; when $\alpha > 0$, $\tau = 0$ and the inequality (21) holds for all $\beta$ and $\beta_0 \in \mathbb{R}^p$. Even if $L_n$ is convex, $L_n$ cannot be strongly convex in the high-dimensional regime due to the rank deficiency, which causes the curvature to vanish in some directions. The RSC condition guarantees that gradient information can still be exploited to direct the algorithm to the optimal point $\beta_0$ in the lack of strong convexity.

We discuss some conditions needed to establish the RSC condition for the negative log-likelihood loss $L_n^\ell(\beta)$. The first assumption concerns the boundedness of the signal.

**A3.** There exist $c_u, c_b < \infty$ such that $\max_{1 \leq i \leq n} |x_i^\top (\beta_0/\|\beta_0\|_2)| \leq c_u$ and $\|\beta_0\|_2 \leq c_b$, $\forall n$.

In words, we assume the size of $x$ projected onto $\beta_0$ as well as $\|\beta_0\|_2$ are bounded. Unlike for linear regressions, boundedness of linear signals is important for the recovery of $\beta_0$, since probabilities become trivial if linear predictors $|x_i^\top \beta_0|$ are unbounded. Comparable assumptions can be found in related literature (e.g. Van de Geer et al. (2014), Mei et al. (2018)).

Next, we discuss conditions on the curvature of the likelihood. Mainly, we need to ensure that in spite of the non-convexity, there is a sufficiently large curvature. We note that the second derivative of the likelihood, $\ell''(x_i^\top \beta, z_i)$, can be decomposed into two terms

$$
\ell''(x_i^\top \beta, z_i) = \rho_I(x_i^\top \beta) + (A'(h(x_i^\top \beta)) - z_i)h''(x_i^\top \beta)
$$

$$
= \rho_I(x_i^\top \beta) + \{A'(h(x_i^\top \beta)) - A'(h(x_i^\top \beta_0))\}h''(x_i^\top \beta) + \epsilon_i h''(x_i^\top \beta)
$$

for $\rho_I(t)$ defined in [12], and where we define $\epsilon_i := A'(h(x_i^\top \beta_0)) - z_i$. As $\mathbb{E}[\epsilon_i] = 0$, $\forall i$, the curvature
of $\mathcal{L}_n^\ell$ is determined by the first two terms. To guarantee a sufficiently positive curvature of the empirical likelihood function, we assume that there is a large enough region around $\beta_0$ with positive curvature, and the amount of non-convexity is limited.

A4. There exists $\tau > 0$ such that

$$\inf_{t_0:|t_0| \leq c_\beta \cdot u} \inf_{t:|t-t_0| \leq \tau} \rho_I(t) + \{A'(h(t)) - A'(h(t_0))\}''(t) \geq 2\alpha_\tau$$

for some $\alpha_\tau > 0$, with $c_\beta$ and $c_u$ defined as in A3. Moreover, there exists $0 \leq \kappa_C < \infty$ where

$$\inf_{t,t_0 \in \mathbb{R}} \{\rho_I(t) + \{A'(h(t)) - A'(h(t_0))\}''(t)\} \geq -\kappa_C.$$
The proof is deferred to the Appendix A.5. We also present an equivalent result for the convex surrogate loss when a sample is from a (GLM-L) model. We recall that the convex approach discussed in the previous section is available when a sample is from (GLM-L) model.

**Proposition 5.2.** Suppose a sample is from a (GLM-L) model with $(A,a,b)$ for $a > 0$ and $b \geq 0$, which satisfies the random design condition $A_1$ Also assume the high-dimensional regime as in the Proposition 5.1 and $\|\beta_0\|_2 = \mathcal{O}(1)$. Then there exist positive constants $\alpha_s$ and $\tau_s$ such that for $n \geq C \log(1/\epsilon)$, it holds with probability at least $1 - \epsilon$ that

\[
(\nabla \mathcal{L}^s_n(\beta) - \nabla \mathcal{L}^s_n(\beta_0))^\top (\beta - \beta_0) \geq \alpha_s \|\beta - \beta_0\|_2^2 - \tau_s \frac{\log p}{n} \|\beta - \beta_0\|_1^2, \quad \forall \beta \in \mathbb{B}_2(1; \beta_0),
\]

where the constant $C$ depends only on the model parameters.

The key observation to establish the RSC result (24) is that the form of $\mathcal{L}^s_n(\beta)$ coincides with the negative log-likelihood function of a generalized linear model with the canonical link. Although $P(T(z)|x)$ does not belong to the GLM family, the role of $T(z)$ is limited in establishing the restricted strong convexity, and the proof for the generalized linear model with the canonical link in Negahban et al. (2012) can be almost applied directly. More details are provided in Appendix A.6.

Any stationary point within the region where the RSC condition holds can be shown to be in the neighborhood of $\beta_0$ with decaying radius (Loh (2017)). Moreover, if the loss function is convex, it is sufficient that the RSC condition holds in the neighborhood of $\beta_0$, given sufficiently large $n$ (Negahban et al. (2012)). We state the following results regarding $\ell_1$ and $\ell_2$ error bounds, which can be essentially obtained by the application of Theorem 1 in Loh (2017) and Theorem 1 in Negahban et al. (2012). A minor complication is that a different RSC condition needs to be established to apply Theorem 1 in Negahban et al. (2012). However, it can be shown that the RSC condition 5.1 implies the RSC condition in Negahban et al. (2012). We defer the detailed discussion to the Appendix A.7.

**Theorem 5.1** ($\ell_1$ and $\ell_2$ error bound). Assume $\mathcal{L}_n^\ell$ and $\mathcal{L}_n^s$ satisfy the RSC conditions (23) and (24) and also assume the high-dimensional regime as in the Proposition 5.1. Suppose $R_n \geq \|\beta_0\|_1$ so that $\beta_0$ is feasible, and let $s_0 := \|\beta_0\|_0$.
1. If \( \lambda_{\ell} \geq 8 \max \{ \| \nabla \mathcal{L}_n^{\ell}(\beta_0) \|_\infty, \tau_{\ell} R_n (\log p/n) \} \), then,
\[
\| \tilde{\beta}_{\ell}^H - \beta_0 \|_2 \leq c_1 \frac{s_0 \lambda_{\ell}}{\alpha_{\ell}} \quad \text{and} \quad \| \tilde{\beta}_{\ell}^H - \beta_0 \|_1 \leq 4 c_1 \frac{s_0 \lambda_{\ell}}{\alpha_{\ell}}.
\] (25)

2. If \( \lambda_s \geq 2 \| \nabla \mathcal{L}_n^s(\beta_0) \|_\infty \) and \( n \geq (32 \tau_s / \alpha_s) s_0 \log p \), then
\[
\| \hat{\beta}_s^H - \beta_0 \|_2 \leq c_2 \frac{s_0 \lambda_s}{\alpha_s} \quad \text{and} \quad \| \hat{\beta}_s^H - \beta_0 \|_1 \leq 4 c_2 \frac{s_0 \lambda_s}{\alpha_s}.
\] (26)

Here \( c_1, c_2 > 0 \) are generic constants.

In particular, if \( \| \nabla \mathcal{L}_n^{\ell}(\beta_0) \|_\infty, \| \nabla \mathcal{L}_n^s(\beta_0) \|_\infty = O(\sqrt{\log p/n}) \) w.h.p, both estimators achieve the minimax-optimal error rates with the choices of \( \lambda_{\ell}, \lambda_s \propto \sqrt{\log p/n} \) and \( R_n \propto \sqrt{n / \log p} \). In the following Corollary 5.1, we summarize the results about error bounds for the noisy labels model.

**Corollary 5.1.** Suppose a sample \((x_i, z_i)_{i=1}^n\) is from a (GLM) with \((\log(1 + \exp(\cdot)), g_{LN})\) and \( z_i \in \{0, 1\} \). Assume the high-dimensional regime as in the Proposition 5.1, the random design condition \( A_1' \), and the boundedness of the signal \( A_3 \). Also suppose noise rates \( \rho_0 \) and \( \rho_1 \) are sufficiently small so that \( A_4 \) and \( A_5 \) hold, and the radius of the feasible region \( r_0 \) in (20) is chosen so that (22) holds. Then for the choices of \( \lambda_{\ell}, \lambda_s \propto \sqrt{\log p/n} \), it holds that
\[
\| \tilde{\beta}_{\ell}^H - \beta_0 \|_2 \leq c_1 \frac{s_0 \lambda_{\ell}}{\alpha_{\ell}} \quad \text{and} \quad \| \hat{\beta}_s^H - \beta_0 \|_2 \leq c_2 \frac{s_0 \lambda_s}{\alpha_s}
\]
with probability at least \( 1 - \epsilon \), given a sufficiently large sample size \( n \geq C(\log p \lor (1/\epsilon)^{1/2}) \), for a constant \( C \) which only depends on the model parameters.

Notably, both estimators achieve the same optimal rates although there could still be a constant gap between the two estimators due to the different multipliers. We compare the performance of the two estimators empirically in Section 6.

### 5.2 Hypothesis testing

Sparse estimators are known to have intractable limiting distributions even in the low-dimension regime (Knight and Fu (2000)). Nonetheless, it is of interest to quantify the uncertainty in the obtained estimators and test the significance of features. In this section, we discuss how we can carry out a test using the point estimates discussed in the previous section.
We take a de-biasing approach and obtain a one-step estimator whose direction is based on an estimating equation \( \psi_n(\beta) := \frac{1}{n} \sum_{i=1}^{n} \psi(x_i^\top \beta, z_i)x_i \). For a function \( \psi : \mathbb{R} \to \mathbb{R} \), we consider \( \psi \) satisfying the following two properties:

1. \( \psi \) has an expectation of zero at \( \beta = \beta_0 \):
   \[
   E_{\beta_0}[\psi(x^\top \beta_0, z)x] = 0,
   \tag{27}
   \]

2. The derivative of \( \psi \) with respect to its first argument can be decomposed into the sum of \( \psi'_R \) and \( \psi'_I \),
   \[
   \psi'(t, z) = \psi'_I(t) + \psi'_R(t, z)
   \tag{28}
   \]

   where \( \psi'_I \) and \( \psi'_R \) satisfy \( \psi'_I(t) > 0, \forall t \) and \( E_{\beta_0}[\psi'_R(x^\top \beta_0, z)] = 0 \).

Two particular choices of \( \psi \) that we will consider subsequently will be derivatives of the log-likelihood loss and the surrogate loss,

\[
\psi^\ell(x^\top \beta, z) := \ell'(x^\top \beta) = \{A'(h(x^\top \beta)) - z\}h'(x^\top \beta)
\]

\[
\psi^s(x^\top \beta, z) := \ell'_s(x^\top \beta, z) = A'(x^\top \beta) - T(z),
\]

where \( \ell'(t, z) \) is the derivative (with respect to a linear predictor) of the log-likelihood loss defined in \( (10) \), and \( \ell'_s(t, z) \) is the derivative of the surrogate loss defined in \( (15) \). Obviously, both \( \psi^\ell \) and \( \psi^s \) satisfy \( (27) \). Also, when \( \psi = \psi^\ell \), the choices of \( \psi'_I(t) = \rho_I(t) \) and \( \psi'_R(t, z) = \rho_R(t, z) \), for \( \rho_I \) and \( \rho_R \) are defined in \( (12) \), will satisfy \( (28) \). On the other hand, if \( \psi = \psi^s \), the choices of \( \psi'_I(t) = A''(t) \) and \( \psi'_R \equiv 0 \) will satisfy \( (28) \).

The derivative of the estimating equation plays an important role in determining the asymptotic variances of the generalized estimating equation (GEE) estimators \cite{Godambe1960}. We define an empirical Jacobian matrix \( \psi'_{I,n}(\beta) \) of \( E[\psi_n(\beta)] \) and the inverse of \( E[\psi'_{I,n}(\beta_0)] \) as

\[
\psi'_{I,n}(\beta) := \frac{1}{n} \sum_{i=1}^{n} \psi'_I(x_i^\top \beta)x_ix_i^\top \quad \text{and} \quad \Theta(\psi) := E[\psi'_I(x^\top \beta_0)x x^\top]^{-1},
\tag{29}
\]

We note that the minimum eigenvalue of \( E[\psi'_I(x^\top \beta_0)x x^\top] \) can be shown to be bounded above by a positive constant under our assumptions, so that \( \Theta(\psi) \) is well-defined (see Appendix A.9).
5.3 De-biasing

For an initial estimate $\hat{\beta}$, we define a de-biased estimator using $\psi$ as follows,

$$\hat{\beta}^{db}(\psi) := \hat{\beta} - \hat{\Theta}(\psi)\psi_n(\hat{\beta})$$

which is a one-step estimator starting at $\hat{\beta}$. Here a matrix $\hat{\Theta}(\psi)$ is an approximation of $\Theta(\psi)$.

We make the following assumption about the sparsity level of $\beta_0$ and $\Theta(\psi)$ similarly as in Geer et al. (2014). We define the column sparsity level of $\Theta(\psi)$ (except the diagonal entries) as $s := \max_{1 \leq j \leq p} \left| \Theta(\psi)_{j,j} \right|$, and recall the definition $s_0 := \|\beta_0\|_0$.

**A6.** $s_0 = o(\sqrt{n}/\log p)$, and $\|X\Theta(\psi)\|_\infty = O_p(1), \forall j$

Also we state conditions regarding the estimation equation $\psi$. In particular, we assume that $\psi$ and $\psi'$ are bounded and $\psi'$ is also Lipschitz continuous with respect to its first argument. Precisely,

**A7.** (Lipschitz continuity of $\psi'$ and boundedness of $\psi$ and $\psi'$) Both $\psi'_R$ and $\psi'_I$ are Lipschitz with respect to its first argument, i.e.

$$|\psi'_R(a,z) - \psi'_R(a',z)| \leq L_\psi |a - a'|, \forall z \quad \text{and} \quad |\psi'_I(a) - \psi'_I(a')| \leq L_\psi |a - a'|$$

In particular, $\psi'$ is Lipschitz with Lipschitz constant $2L_\psi$. Furthermore, there exists $C_\psi < \infty$ such that \(\max\{\|\psi\|_\infty, \|\psi'_R\|_\infty, \|\psi'_I\|_\infty\} \leq C_\psi\).

Now we state Theorem 5.2 which gives the asymptotic distributions of de-biased estimators.

**Theorem 5.2.** Assume the random design condition **A1**-**A6-A7**, and $\|\beta_0\|_2 = O(1)$. Suppose $\hat{\Theta}(\psi)$ is chosen to satisfy $\|\hat{\Theta}(\psi) - \Theta(\psi)\|_1 = o_p(1/\sqrt{\log p})$ and $\|\epsilon_j - \psi_{I,n}(\beta)\hat{\Theta}(\psi)\|_\infty = O_p(\sqrt{\log p/n})$, $\forall j$. For an initial estimate $\hat{\beta}$ satisfying $\ell_1$ and $\ell_2$ bounds $\|\hat{\beta} - \beta_0\|_1 = O_p(s_0\sqrt{\log p/n})$ and $\|\hat{\beta} - \beta_0\|_2 = O_p(s_0 \log p/n)$, and $\|\hat{\beta} - \beta_0\|_1/\|\hat{\beta} - \beta_0\|_2 = O(\sqrt{s_0})$ a.s., we have for any $j \in \{1, \ldots, p\}$,

$$\sqrt{n}(\hat{\beta}^{db}(\psi) - \beta_0)/\sigma(\psi)_j = Z_j + o_p(1)$$

for $Z_j$ which converges weakly to a $N(0,1)$ distribution and for

$$\sigma(\psi)_j := \sqrt{\Theta(\psi)_j} \mathbb{E}[\psi(x^\top \beta_0, z)^2 xx^\top] \Theta(\psi)_{j,j}.$$ 

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Moreover, if the bound in $\textbf{A6}$ and the conditions in the theorem statement regarding $\hat{\Theta}(\psi)_j$ hold uniformly in $j$, then the result also holds uniformly in $j$.

We note that obtaining $\hat{\Theta}(\psi)$ satisfying the conditions of Theorem 5.2 is possible by taking a similar approach as in Van de Geer et al. (2014) using node-wise regressions. In Appendix A.10, we provide more details about such construction. We also note that an initial estimate $\hat{\beta}$ which satisfies the following cone condition,

$$\|\hat{\beta} - \beta_0\|_1 \leq L \|\hat{\beta} - \beta_0\|_S$$

such that $|S| = s_0$, also satisfies the $\ell_1/\ell_2$ ratio condition of the error vector in Theorem 5.2 since $\|\hat{\beta} - \beta_0\|_1 \leq (L + 1)\sqrt{s_0}\|\hat{\beta} - \beta_0\|_2$. The proof of Theorem 5.2 is deferred to Appendix A.9. The main argument follows similar lines as in the proof of Theorem 3.1 in Van de Geer et al. (2014), with additional arguments to handle the potential non-monotonicity of $\psi$, which can arise from a non-convex loss function (due to a non-canonical GLM). Finally, we state the following Corollary 5.2 for the asymptotic distributions of the de-biased estimators for the label noise model.

**Corollary 5.2.** Suppose we have a sample $(x_i, z_i)_{i=1}^n$ from a GLM with parameters $(\log(1 + \exp(\cdot)), g_{LN})$ and $z_i \in \{0, 1\}$. We assume the conditions of Proposition 5.1. We also assume that $\hat{\Theta}(\psi^\ell)$ and $\hat{\Theta}(\psi^s)$ satisfy the conditions about $\hat{\Theta}(\psi)$ in Theorem 5.2 and $\textbf{A6}$ holds. We consider two de-biased estimators:

$$\hat{\beta}^\text{db}_{\ell} := \hat{\beta}^H_{\ell} - \hat{\Theta}(\psi^\ell)\psi^\ell_n(\hat{\beta}^H_{\ell})$$

and

$$\hat{\beta}^\text{db}_{s} := \hat{\beta}^H_{s} - \hat{\Theta}(\psi^s)\psi^s_n(\hat{\beta}^H_{s}).$$

We then have, for any $j \in \{1, \ldots, p\}$,

$$\sqrt{n}(\hat{\beta}^\text{db}_{\ell,j} - \beta_{0,j})/\sigma(\psi^\ell)_j = Z_j + o_p(1)$$

and

$$\sqrt{n}(\hat{\beta}^\text{db}_{s,j} - \beta_{0,j})/\sigma(\psi^s)_j = \tilde{Z}_j + o_p(1)$$

for $Z_j, \tilde{Z}_j$ which converge weakly to a $N(0, 1)$ distribution and for,

$$\sigma(\psi^\ell)_j = \sqrt{\mathcal{I}^\ell(\beta_0)^{-1}}_{jj} \quad \text{and} \quad \sigma(\psi^s)_j = \sqrt{\mathcal{I}^s(\beta_0)^{-1}}_{jj}$$

where $\mathcal{I}^\ell(\beta)$ and $\mathcal{I}^s(\beta)$ are defined in Proposition 4.2.

The conditions about $\psi^\ell$ and $\psi^s$ can be checked similarly as in the proof of Corollary 4.2. The rate conditions about the initial estimators can be checked by Corollary 5.1. Also, it is well known that both $\hat{\beta}^H_{\ell} - \beta_0$ and $\hat{\beta}^H_{s} - \beta_0$ belong to a cone $\{\Delta; \|\Delta_S\|_1 \leq 3\|\Delta_{S^c}\|_1\}$ where $S \subseteq \{1, \ldots, p\}$ is the
support of $\beta_0$. We note that these results are analogous to Proposition 4.2 in the low-dimensional setting once penalization and de-biasing are introduced.

6 Empirical study

In this section, we present results about the empirical behavior of the non-convex likelihood-based estimator and the convex surrogate estimator. Our focus in this section is two-fold. First, we study the relative efficiency of the two estimators when different design matrices are considered. In particular, we empirically demonstrate that the impact of design $X$ on the relative efficiency of the two estimators is well captured by the gap between $C(X)$ and $C(W^{-1}W_yX)$. Second, we study empirical performance of the two estimators in the low- and high-dimensional regimes, with and without regularization. In low-dimensions, the likelihood-based estimator is expected to perform better than the convex estimator. However, it is unclear whether this will continue to be true in high dimensions. Indeed, as we discuss hereafter, our simulation study shows that the convex estimator outperforms the likelihood-based estimator in sparse regimes, where signal strength is relatively low.

6.1 Methods

Based on the regime of each simulation, we obtain non-sparse estimates $\hat{\beta}_\ell$ and $\hat{\beta}_s$ from (16) in the low-dimensional regime or sparse estimates $\tilde{\beta}^H_\ell$ and $\tilde{\beta}^H_s$ from (20) in the high-dimensional regime. We recall that we define $\tilde{\beta}^H_\ell$ as a stationary point of the optimization problem in (20) due to the non-convexity of $\mathcal{L}^\ell_n(\beta)$. For non-convex problems, we initialize coefficients at the null model where $\beta = [0, \ldots, 0]^T$ if a problem is in the low-dimensional regime, and we use a local initialization using a convex estimate otherwise. To compare with the uncorrupted regime, the coefficient estimates $\hat{\beta}_{\text{ref}}$ and $\tilde{\beta}^H_{\text{ref}}$ are computed using logistic or $\ell_1$-penalized logistic regression on the un-corrupted data $(x_i, y_i)_{i=1}^n$.

In terms of optimization, we use the proximal gradient method combined with a back-tracking line search to solve optimization problems of (16) and (20). This approach guarantees that iterates converge to a stationary point of the objective function if the objective function is non-convex and converge to an optimum in the convex case (e.g. Chapter 10 in Beck (2017)). For $\hat{\beta}_{\text{ref}}$ and $\tilde{\beta}^H_{\text{ref}}$ we used the ‘glm()’ function from R base package and ‘glmnet()’ from R package glmnet respectively.
6.2 Impact of design

To study the relative efficiency of the two estimators in various designs, we fix dimensions \((n = 1000, p = 10)\) and consider a mixture of multivariate normal distributions with varying distances between the two mixture components. We will demonstrate that increase in distance between the means of the two mixture components leads to an increase in the gap between \(C(X)\) and the perturbed column space \(C(W^{-1}_z W_y X)\), and a larger gap between two subspaces is associated with greater efficiency differences in \(\tilde{\beta}_\ell\) and \(\tilde{\beta}_s\).

Now we describe our simulation set-up for this subsection. First, we generate a design matrix \(X = [x_1^\top, \ldots, x_n^\top]^\top\) by sampling each \(x_i\) from an equal mixture of multivariate Gaussian distribution centered at \(\mu_1 = (d, \ldots, d)\) and \(\mu_2 = (-d, \ldots, -d)\) with various \(d\) and covariance matrix \(\Sigma\) such that \(\Sigma_{ij} = 0.2|i-j|\). We let \(\beta_0 := [1/\sqrt{p}, \ldots, 1/\sqrt{p}]^\top\) so that \(\|\beta_0\|^2 = 1\). The true unobserved response \(y_i\) is drawn by \(y_i \sim \text{Ber}(p_{\beta_0}(x_i))\) where \(p_{\beta_0}(x_i) = (1 + \exp(-x_i^\top \beta_0))^{-1}\), and a noisy label \(z_i\) is obtained by flipping \(y_i\) based on noise rates \(\rho_0 = 10\%\) and \(\rho_1 = 5\%\). The range of \(d^2 = (0, \ldots, 2.5)\) is considered so that \(\text{dist}^2 := \|\mu_1 - \mu_2\|^2 = 4pd^2\) varies from 0 to 100. When \(\text{dist} = 0\), \(x_i\) is from single Gaussian distribution, i.e. \(x_i \sim N(0, \Sigma)\), \(\forall i\). For each \(d\), we repeat the experiment \(B = 10000\) times. At each \(d\) and iteration \(b = 1, \ldots, B\), we calculate

- relative \(\ell_2\) difference: \(\text{rd}(\mathcal{I}_{\ell}^f(\beta_0)_b, \mathcal{I}_{\ell}^s(\beta_0)_b) := \|I_p - \mathcal{I}_{\ell}^f(\beta_0)_b^{-1/2}\mathcal{I}_{\ell}^s(\beta_0)_b\mathcal{I}_{\ell}^f(\beta_0)_b^{-1/2}\|_2\),
- gap: \(\tilde{\delta}(\mathcal{C}(X_b), \mathcal{C}(W^{-1}_z W_y X_b)) = \|\mathcal{PC}(X_b) - \mathcal{PC}(W^{-1}_z W_y X_b)\|_2\) [Kato, 2013],
- mean squared errors: \(\text{mse}_b^\ell := \|\tilde{\beta}_\ell - \beta_0\|^2_2\) and \(\text{mse}_b^s := \|\tilde{\beta}_s - \beta_0\|^2_2\),
- asymptotic mean squared errors\(^\dagger\): \(\text{amse}_b^\ell := \text{tr}(\mathcal{I}_{\ell}(\beta_0)_b^{-1})/n\) and \(\text{amse}_b^s := \text{tr}(\mathcal{I}_{s}(\beta_0)_b^{-1})/n\),

where subscripts of \(b\) mean corresponding quantities are from the \(b\)th experiment. We summarize results by taking an average of \(B\) values.

To compare the efficiency of the two estimators, we calculate \(\tilde{r}_{\text{mse}}\), the ratio of estimated mean squared errors, and \(\tilde{r}_{\text{amse}}\), the ratio of asymptotic mean squared errors. More concretely, we let,

\[
\tilde{r}_{\text{mse}} := \frac{\text{mse}_b^\ell}{\text{mse}_b^s}, \quad \text{and} \quad \tilde{r}_{\text{amse}} := \frac{\text{amse}_b^\ell}{\text{amse}_b^s}.
\]

When \(n\) is sufficiently large, \(\tilde{r}_{\text{mse}}\) is expected to be close to \(\tilde{r}_{\text{amse}}\), and both to be close to \(r_{\text{amse}} := \lim_n E[\|\tilde{\beta}_\ell - \beta_0\|^2_2]/E[\|\tilde{\beta}_s - \beta_0\|^2_2]\). Note if the two estimators have the same efficiency, ratios will be close to 1. If the ratios are strictly less than 1, we can conclude that \(\tilde{\beta}_\ell\) is more efficient than \(\tilde{\beta}_s\).

\(^\dagger\) \(E[\|\tilde{\beta}_\ell - \beta_0\|^2_2] = \text{tr}(E(\tilde{\beta} - \beta_0)(\tilde{\beta} - \beta_0)^\top) \approx \text{tr}(\mathcal{I}_{\ell}(\beta_0)^{-1})/n\)
Figure 2: Ratios of mse and asymptotic mse and 1-relative $\ell_2$ difference with varying gap$^2$. Error bars refer to 1se.

Figure 3: Plot of the distance between the means of two mixture distributions vs. the gap between the two column spaces.

Figure 2 plots the ratios of the mean squared errors and asymptotic mean squared errors, as well as $1 - \text{rd}(I_n^+(\beta_0), I_n^-(\beta_0))$ with varying gap$^2$ values, i.e. $\tilde{\delta}^2(C(X), C(W_z^{-1}W_yX))$. We recall that $1 - \text{rd}(I_n^+(\beta_0), I_n^-(\beta_0)) = 1$ iff two estimators have the same asymptotic efficiency, i.e. $I_n^+(\beta_0) = I_n^-(\beta_0)$, and $1 - \text{rd}(I_n^+(\beta_0), I_n^-(\beta_0)) < 1$ if $I_n^+(\beta_0) > I_n^-(\beta_0)$. We see from Figure 2 that $1 - \text{rd}(I_n^+(\beta_0), I_n^-(\beta_0))$ linearly decreases with the gap$^2$, which aligns with the result of Corollary 4.1. Also, the efficiency of the surrogate estimator worsens compared to the likelihood-based estimator as the gap increases, but not in the linear fashion as in the case of $1 - \text{rd}(I_n^+(\beta_0), I_n^-(\beta_0))$. Unlike the relative $\ell_2$ difference where we associated the quantity with variance ratio of the two estimators with respect to a particular direction $u$, variance ratios in all directions are considered in $\hat{\gamma}_{\text{amse}}$ since $\hat{\gamma}_{\text{amse}} = \text{tr}(I_n^+(\beta_0)^{-1})/\text{tr}(I_n^-(\beta_0)^{-1})$. Figure 3 plots the gap $\tilde{\delta}(C(X), C(W_z^{-1}W_yX))$ as functions of dist = $\|\mu_1 - \mu_2\|_2$. We see that the gap between two subspaces increases as the distance between two mixture components increases.

6.3 Low-dimensional and high-dimensional setting

To study the empirical performance of non-convex and convex approaches in low and high-dimensional regimes, we consider the following two regimes: (i) fixed $p = 10$ and growing $n$; (ii) growing $(n, p)$ with $p = n$. Also, we consider two noise settings, where we use $\rho_1 = 5\%$ and $\rho_0 = 10\%$ for the first
noise setting (low-noise) and double the noise rates for the second noise level setting (high-noise).

A sample \( x_i \in \mathbb{R}^p \) is generated from multivariate gaussian distribution \( \mathcal{N}(0, \Sigma) \) where \( \Sigma \in \mathbb{R}^{p \times p} \) is given as \( \Sigma_{i,j} = C_{\Sigma}(0.2)^{|i-j|} \), where \( C_{\Sigma} \) is chosen so that \( \text{Var}(x_i^\top \beta_0) = 5 \). The sample size \( n \) varies from 1000 to 5000 where values in between are interpolated in a log scale. In both regimes, we first let 10 features be active (\( s = 10 \)) and true parameter be \( \beta_0 := [1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0] \). The true observed responses \( y_i \) and the noisy labels \( z_i \) are generated in the same way as in Section 6.2.

Each experiment in the low-dimensional regimes is repeated \( B = 300 \) times and \( B = 50 \) times in the high-dimensional regimes. The mean and standard errors of \( B \) trials are reported in Figure 4. Tuning parameter \( \lambda \) needs to be chosen for the high-dimensional estimators. We choose \( \lambda \) in each simulation based on the testing loss from 5-fold cross validation.

![Figure 4: Comparison of the log-likelihood and surrogate loss based estimators in the low and high-dimensional regimes. Reference loss (ref) refers to the logistic loss when clean data is available.](image)

Figure 4 shows the comparison results of the non-sparse and sparse estimators in both low and high-dimensional regimes. Not surprisingly, the likelihood-based estimator performs uniformly better than the convex estimator in the low-dimensional regime without any regularization in the both noise settings. The loss of efficiency by using a convex surrogate loss appears to be relatively small when the noise level is low. The performance of the surrogate estimator worsens when noise rates increase since the squared gap \( \hat{\delta}^2(C(X), C(W_z^{-1}W_y X)) \) increases, which agrees with Corollary 4.1. On the contrary, the convex surrogate estimator appears to perform uniformly better than the
likelihood-based loss in the high-dimensional setting.

It is well known that when no regularization is introduced, the likelihood function is the best function to optimize since the procedure results in the smallest asymptotic variance matrix (in regular problems). Its optimality (more precisely, the optimality of the score function) was also argued in classical estimating equation theory, where the score function is shown to be the best estimating equation function in the sense of minimizing the asymptotic variance \cite{Godambe1960}.

The surrogate loss \( L^s_n(\beta) \) has a stronger curvature than \( L^\ell_n(\beta) \); in fact the curvature of \( L^s_n(\beta) \) is the same as \( L^c_n(\beta) \), the logistic loss from clean data. However, \( \nabla L^s_n(\beta) \) has also a larger variance than \( \nabla L^\ell_n(\beta) \) due to noise in the responses, resulting in the larger asymptotic variance matrix. We conjecture that in a penalized problem, especially when signal is relatively small compared to noise, regularization plays a role in reducing the variability in \( \nabla L^s_n(\beta) \) which leads to the better performance of the convex estimator in some cases.

To test our conjecture, we carry out an additional set of simulation. The set-up of the simulation mainly follows the set-up in Section 6.2 except we choose dimensions \((n = 10000, p = 20)\), fix \( d = 3/\sqrt{p} \), and instead let the \( \ell_1/\ell_2 \) ratio of the true signal \( \beta_0 \) vary. More concretely, we fix \( \|\beta_0\|_2 = 1 \), and consider different sparsity levels \( s \) of \( \beta_0 = \left[1/\sqrt{s}, \ldots, 1/\sqrt{s}, 0, \ldots, 0 \right] \), so that \( \|\beta_0\|_1/\|\beta_0\|_2 \) varies from 1 to \( \sqrt{p} \). For each \( s = 1, \ldots, p \), we obtain both non-sparse and sparse estimates. In the case of sparse estimates, tuning is performed by minimizing test loss on a test set of size \( n \). At each \( s \), the experiment is repeated \( B = 10000 \) times. Similarly as in Section 6.2, we calculate the two mean squared ratios each for the non-sparse and sparse estimates,

\[
\hat{r}_{\text{mse}} := \frac{\text{mse}^{\ell}}{\text{mse}^s} \quad \text{and} \quad \hat{r}_{\text{smse}} := \frac{\text{smse}^{\ell}}{\text{smse}^s},
\]

where we recall the definitions of \( \text{mse}^{\ell}_b := \|\hat{\beta}_{\ell,b} - \beta_0\|^2_2 \) and \( \text{mse}^s_b := \|\hat{\beta}_{s,b} - \beta_0\|^2_2 \) and define \( \text{smse}^{\ell}_b := \|\hat{\beta}^H_{\ell,b} - \beta_0\|^2_2 \) and \( \text{smse}^s_b := \|\hat{\beta}^H_{s,b} - \beta_0\|^2_2 \).

As we can see from Figure 5, the likelihood-based estimator is always more efficient than the convex estimator in the case of non-sparse estimates. On the other hand, for the sparse estimators, the convex estimator outperforms the likelihood-based estimator when the \( \ell_1/\ell_2 \) ratio is small. As the \( \ell_1/\ell_2 \) ratio increases, we see from plot (b) that the amount of regularization decreases and the likelihood-based estimator starts to perform better than the convex estimator.
7 Application to beta-glucosidase protein data

In this section, we describe an application of our non-convex and convex methods to beta-glucosidase (BGL) protein sequence data. Beta-glucosidase is a key enzyme present in cellulase which converts cellobiose to glucose during cellulose hydrolysis. The BGL enzyme protein plays a significant role in bioethanol production (Singhania et al. (2013)). Due to its industrial importance, it is of great interest to understand the effects of mutations of the protein and design a protein with improved functionality.

The data set we analyze is a positive and unlabeled beta-glucosidase protein sequence data set generated in the Romero Lab (Romero et al. (2015)). Large-scale data were generated by deep mutational scanning (DMS) method, which applies the high-throughput screening method to sort out functional protein variants. Unlabeled sequences from the initial library whose associated functionality is unknown are obtained together with screened sequences to be positive. The data consists of $n_\ell = 2533388$ functional (positive label) and $n_u = 1500277$ unlabeled sequences.\footnote{The raw data is available in https://github.com/RomeroLab/seq-fcn-data.git} A sequence consists of 500 positions which takes one of 21 discrete values which correspond to 20 amino
acid letter codes plus an additional letter for the alignment gap. From an alternative experiment, the prevalence of functional sequences in the unlabeled data set is known to be 0.35. This data set is previously analyzed in Song and Raskutti (2018) where the likelihood based approach was taken with the $\ell_1$ penalization to obtain a sparse estimate. We obtained $p = 3097$ features using one-hot encoding of the sequences. Because each sequence contains only a few mutations, we obtained a sparse design matrix $X \in \mathbb{R}^{n \times p}$ by taking the amino acid levels in the WT sequence as the baseline levels. We note that the number of features ($p = 3097$) in the model is approximately one third of the number of maximum possible features ($10000 = 20 \times 500$). Since the sequences in the data set are local sequences around the wild-type sequence, some mutations were never observed.

We apply both convex and non-convex methods to the data set to estimate each mutation effect of the BGL sequence. Estimated coefficients are obtained by fitting the model using all samples. The model is then refitted using 90% of the randomly selected samples to compare prediction performance of the two methods on hold-out data sets. We use the area under the ROC curve (AUC) for the comparison of classification performances. However, for positive and unlabeled data, the ROC curve and AUC value calculated using the observed labels as the responses are biased for the ROC curve and AUC value for the unobserved true responses (Jain et al., 2017). Following the approach in Jain et al. (2017), we report the corrected ROC curve and AUC values.

![Graph](image.png)

(a) Estimated coefficients

(b) Corrected ROC curves

Figure 6: Plot of (a) the estimated coefficients and (b) the corrected ROC curves from the non-convex and convex approaches. In Figure (a), features are sorted based on the coefficients from the likelihood approach.
The results are provided in Figure 6. We observe that the two approaches produce similar coefficient estimates and comparable classification performance results. Both classifiers demonstrate good classification performance. The corrected AUC value is 0.7977 from the likelihood approach and 0.7989 from the surrogate approach, which is a significant improvement from AUC= 0.5 in the case of random classification.

8 Conclusion

We studied the binary regression problem in the presence of noise in labels in both the classic and high-dimension regimes. We demonstrated that the noisy label model belongs to a sub-class of generalized linear model family. We then discussed two approaches based on a convex surrogate loss for the sub-class of GLMs and a non-convex likelihood for the general class of GLMs. In the low-dimensional setting, the asymptotic distributions of the non-convex likelihood-based estimator and the convex surrogate estimator are derived. We also quantified the efficiency gap between the two approaches and argued that although the convex estimator is provably sub-optimal in terms of efficiency, the gap can be small in some applications. In the high-dimensional setting, we showed that both estimators, based on regularized non-convex and convex loss functions, achieve a minimax optimal $s \log p/n$ rate for the mean squared errors and derived the asymptotic distribution of the de-biased estimators which can be used for the hypothesis testing in a high-dimensional setting. We empirically demonstrated that both methods perform well in the simulation study and the real data analysis. In particular, although the estimator from the convex approach is sub-optimal in the low-dimensional regime, the efficiency gap between the two estimators is often small. Our empirical results suggest that in sparse regimes the convex surrogate estimator performs better than the likelihood-based estimator. It remains an open question to provide a theoretical justification for this claim.

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A Appendix

A.1 Proof of Proposition 4.1

We note that \( \hat{\beta}_\ell \) and \( \hat{\beta}_s \) are zeros of

\[
\nabla L_n^\ell(\beta) = \frac{1}{n} \sum_{i=1}^{n} (\mu(h_{LN}(x_i^\top \beta)) - z_i) h'_{LN}(x_i^\top \beta)x_i = \frac{1}{n} \sum_{i=1}^{n} \psi^\ell(\beta, (x_i, z_i))x_i \tag{31}
\]

\[
\nabla L_n^s(\beta) = \frac{1}{n} \sum_{i=1}^{n} (\mu(x_i^\top \beta) - T(z_i))x_i = \frac{1}{n} \sum_{i=1}^{n} \psi^s(\beta, (x_i, z_i))x_i, \tag{32}
\]

where we define

\[
\psi^\ell(\beta, (x, z)) := (\mu(h_{LN}(x^\top \beta)) - z) h'_{LN}(x^\top \beta)
\]

\[
\psi^s(\beta, (x, z)) := \mu(x^\top \beta) - T(z).
\]

For notational simplicity, in what follows we write \( \psi^{(i)}(\cdot) := \psi(\cdot, (x_i, z_i))x_i \) for any \( \psi \in \{\psi^\ell, \psi^s\} \). Also we define \( \psi_n := n^{-1} \sum_{i=1}^{n} \psi^{(i)} \).

Since \( \hat{\beta}_\ell \) and \( \hat{\beta}_s \) are the roots of (31) and (32) and both \( \psi_n^\ell \) and \( \psi_n^s \) converge a.s. to \( E_{\beta_0}[\psi_n^\ell(\beta)] \) and \( E_{\beta_0}[\psi_n^s(\beta)] \) which have a unique zero at \( \beta_0 \), both \( \hat{\beta}_\ell \) and \( \hat{\beta}_s \) are consistent for \( \beta_0 \) under the Assumption A1 (e.g. Proposition 5.5 in Shao (2003)).

Then by the second order Taylor expansion, we can establish the asymptotic normality of the two estimators (e.g. Theorem 5.14 in Shao (2003), Chapter 2.3.1 in Fahrmeir and Tutz (2001)).

We first define the inverse of the asymptotic variance using an estimating equation \( \psi \) as

\[
\mathcal{I}_n(\beta; \psi) := \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_\beta \left[ \frac{d}{d\beta} \psi^{(i)}(\beta) | x_i \right] \right) \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_\beta \left[ \psi^{(i)}(\beta) \psi^{(i)}(\beta)^\top | x_i \right] \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_\beta \left[ \frac{d}{d\beta} \psi^{(i)}(\beta) | x_i \right] \right). \tag{33}
\]

Let \( g_{CL}(\cdot) := \mu^{-1}(\cdot) \), which is a canonical link function. Recalling the fact \( h_{LN}(t) = g_{CL} \circ g_{LN}^{-1}(t) \) and \( g_{LN}^{-1}(t) = \mu_z(t) = (1 - \rho_1 - \rho_0)\mu(t) + \rho_0 \), we have,

\[
\psi^{\ell,(i)}(\beta) = (1 - \rho_1 - \rho_0) \left( \mu(h_{LN}(x_i^\top \beta)) - z_i \right) g'_{CL}(\mu_z(x_i^\top \beta)) \mu'(x_i^\top \beta)x_i \tag{34}
\]

\[
\psi^{s,(i)}(\beta) = \left( \mu(x_i^\top \beta) - T(z_i) \right) x_i. \tag{35}
\]
Also, \( \mu'(t) = \mathcal{V}(\mu(t)) \), since \( \mu'(t) = A''(g_{CL} \circ \mu(t)) = \mathcal{V}(\mu(t)) \) by the definition of \( \mathcal{V} \). Also \( g'_{CL}(t) = 1/\mathcal{V}(t) \) since \( g_{CL}(\mu(u)) = 1/\mu'(u) = 1/\mathcal{V}(u) \) by the chain rule. Plugging these expressions in (34), we obtain

\[
\psi_{\ell, (i)}(\beta) = (1 - \rho_1 - \rho_0) \left( \mu(h_{LN}(x_i^T \beta)) - z_i \right) \frac{\mathcal{V}(\mu(x_i^T \beta))}{\mathcal{V}(\mu_z(x_i^T \beta))} x_i.
\]

Then since \( \mathbb{E}_\beta[z_i | x_i] = \mu(h_{LN}(x_i^T \beta)) \), we get

\[
\mathbb{E}_\beta[\frac{d}{d\beta} \psi_{\ell, (i)}(\beta) | x_i] = A''(h_{LN}(x_i^T \beta)) \left( (1 - \rho_1 - \rho_0) \frac{\mathcal{V}(\mu(x_i^T \beta))}{\mathcal{V}(\mu_z(x_i^T \beta))} \right)^2 x_i x_i^T
\]

\[
= \mathcal{V}(\mu_z(x_i^T \beta)) \left( (1 - \rho_1 - \rho_0) \frac{\mathcal{V}(\mu(x_i^T \beta))}{\mathcal{V}(\mu_z(x_i^T \beta))} \right)^2 x_i x_i^T
\]

\[
= (1 - \rho_1 - \rho_0)^2 \frac{\mathcal{V}(\mu(x_i^T \beta))^2}{\mathcal{V}(\mu_z(x_i^T \beta))} x_i x_i^T.
\]

Also, since \( \psi_{\ell, (i)}(\beta) \) is a negative score function, the variance of the score function is the same as the expected negative derivative of the score function, i.e.

\[
\mathbb{E}_\beta[\psi_{\ell, (i)}(\beta) \psi_{\ell, (i)}(\beta)^T | x_i] = \mathbb{E}_\beta[\frac{d}{d\beta} \psi_{\ell, (i)}(\beta) | x_i].
\]

Then,

\[
\mathcal{I}_n(\beta; \psi_{\ell}) = (1 - \rho_1 - \rho_0)^2 \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{V}(\mu(x_i^T \beta))^2}{\mathcal{V}(\mu_z(x_i^T \beta))} x_i x_i^T.
\]

For the surrogate function, direct calculations give

\[
\mathbb{E}_\beta[\frac{d}{d\beta} \psi_{s, (i)}(\beta) | x_i] = A''(x_i^T \beta) x_i^T = \mathcal{V}(\mu(x_i^T \beta)) x_i^T
\]

\[
\mathbb{E}_\beta[\psi_{s, (i)}(\beta) \psi_{s, (i)}(\beta)^T | x_i] = \mathbb{E}_\beta[(\mu(x_i^T \beta) - T(z_i))^2 x_i^T] x_i = \text{Var}_\beta[T(z_i) | x_i] x_i x_i^T
\]

since \( \mathbb{E}_\beta[T(z_i) | x_i] = \mu(x_i^T \beta) \). Recalling the definition of \( T(t) = (t - \rho_0)/(1 - \rho_1 - \rho_0) \) and \( \mathcal{V} \),

\[
\text{Var}_\beta[T(z_i) | x_i] = (1 - \rho_1 - \rho_0)^{-2} \mathcal{V}(\mu_z(x_i^T \beta)).
\]

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Thus we have
\[ I_n^s(\beta; \psi^*) = \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{V}(\mu(x_i^\top \beta))x_i x_i^\top \right) \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{V}(\mu_z(x_i^\top \beta) \mathcal{P}(x_i^\top \beta) x_i x_i^\top \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{V}(\mu(x_i^\top \beta))x_i x_i^\top \right) \]
by applying (36) and (37) to (33).

### A.2 Proof of Corollary 4.1

First we show \( I_n^f(\beta_0) \succeq I_n^s(\beta_0) \). By the definition of \( W_y \) and \( W_z \), we have,
\[
I_n^f(\beta_0) = (1 - \rho_1 - \rho_0)^2 X^\top W_y W_z^{-1} W_y X/n
\]
\[
I_n^s(\beta_0) = (1 - \rho_1 - \rho_0)^2 X^\top W_y X (X^\top W_z X)^{-1} X^\top W_y X/n.
\]

Since the projection matrix \( \mathcal{P}_{W_z^{1/2}X} \) can be written as \( \mathcal{P}_{W_z^{1/2}X} = W_z^{1/2} X (X^\top W_z X)^{-1} X^\top W_z^{1/2} \), we have \( I_n^f(\beta_0) = (1 - \rho_1 - \rho_0)^2 X^\top W_y W_z^{-1/2} \mathcal{P}_{W_z^{1/2}X} W_z^{-1/2} W_y X/n \). Then for any \( v \in \mathbb{R}^n \),
\[
v^\top (I_n^f(\beta_0) - I_n^s(\beta_0))v = (1 - \rho_1 - \rho_0)^2 v^\top X^\top W_y W_z^{-1/2} (I_n - \mathcal{P}_{W_z^{1/2}X}) W_z^{-1/2} W_y X v/n
\]
\[
= (1 - \rho_1 - \rho_0)^2 \|(I_n - \mathcal{P}_{W_z^{1/2}X}) W_z^{-1/2} W_y X v\|_2^2 / n \geq 0 \tag{38}
\]
since \( I_n - \mathcal{P}_{W_z^{1/2}X} \) is idempotent. Thus, \( I_n^f(\beta_0) \succeq I_n^s(\beta_0) \).

Now we address the inequality (18). First, we have
\[
\|I_n^f(\beta_0)^{-1/2} (I_n^f(\beta_0) - I_n^s(\beta_0)) I_n^f(\beta_0)^{-1/2}\|_2 \leq \|I_n^f(\beta_0)^{-1/2}\|_2 \|I_n^f(\beta_0) - I_n^s(\beta_0)\|_2,
\]
and \( \|I_n^f(\beta_0)^{-1/2}\|_2^2 = \|I_n^f(\beta_0)^{-1}\|_2 = (1 - \rho_1 - \rho_0)^{-2} \sigma_{\min}(X^\top W_y W_z^{-1} W_y X/n)^{-1} \). Also, from (38),
\[
\|I_n^f(\beta_0) - I_n^s(\beta_0)\|_2 = (1 - \rho_1 - \rho_0)^2 \|(I_n - \mathcal{P}_{W_z^{1/2}X}) W_z^{-1/2} W_y X\|_2^2 / n.
\]

Let \( A := W_z^{-1/2} W_y X \). Then,
\[
\|I_n - \mathcal{P}_{W_z^{1/2}X}\|_2^2 = \sup_{u \in \mathbb{R}^n} \|I_n - \mathcal{P}_{W_z^{1/2}X} A u\|_2^2 / \|u\|_2^2 = \sup_{u \in \mathcal{C}(A)} \|I_n - \mathcal{P}_{W_z^{1/2}X} A u\|_2^2 / \|A^\top u\|_2^2
\]

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where $A^\dagger$ is a Moore-Penrose inverse of $A$. Since $\|A^\dagger u\|_2^2 \geq \|u\|_2^2/\sigma^2_{\text{max}}(A)$ for $u \in \mathcal{C}(A)$,

$$\|(I_n - \mathcal{P}_{W_z^{1/2}X})A\|_2^2 \leq \sigma^2_{\text{max}}(A) \sup_{u \in \mathcal{C}(A)} \frac{\|(I_n - \mathcal{P}_{W_z^{1/2}X})u\|_2^2}{\|u\|_2^2} = \sigma^2_{\text{max}}(A) \sup_{u \in \mathcal{C}(A), \|u\|_2=1} \|(I_n - \mathcal{P}_{W_z^{1/2}X})u\|_2^2.$$

Since $I_n - \mathcal{P}_{W_z^{1/2}X}$ is a projection operator onto the orthogonal space of $\mathcal{C}(W_z^{1/2}X)$, $\|(I_n - \mathcal{P}_{W_z^{1/2}X})u\|_2^2 = \inf_{v \in \mathcal{C}(W_z^{1/2}X)} \|u - v\|_2^2$. Therefore,

$$\|(I_n - \mathcal{P}_{W_z^{1/2}X})A\|_2^2 \leq \sigma^2_{\text{max}}(A) \sup_{u \in \mathcal{C}(A), \|u\|_2=1} \inf_{v \in \mathcal{C}(W_z^{1/2}X)} \|u - v\|_2^2 = \sigma^2_{\text{max}}(W_z^{-1/2}W_yX)\delta^2(\mathcal{C}(W_z^{-1/2}W_yX),\mathcal{C}(W_z^{1/2}X)).$$  \hspace{1cm} (39)

by the definition of the gap [17].

To proceed, we prove a lemma about the gap of two subspaces after linear transformation in relation to the original subspaces. Let $A(\mathcal{M}) := \{Av; v \in \mathcal{M}\}$, and note $\mathcal{C}(AX) = A(\mathcal{C}(X))$.

**Lemma A.1.** Let $\mathcal{M}, \mathcal{N} \subseteq \mathbb{R}^n$ be linear subspaces. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $A(\mathcal{M}) := \{Av; v \in \mathcal{M}\}$. Then

$$\delta(A(\mathcal{M}), A(\mathcal{N})) \leq \kappa(A)\delta(\mathcal{M}, \mathcal{N}),$$

where $\kappa(A) := \frac{\sigma_{\text{max}}(A)}{\sigma_{\text{min}}(A)}$ is a condition number of $A$.

**Proof.** By definition,

$$\delta(A(\mathcal{M}), A(\mathcal{N})) = \sup_{u \in A(\mathcal{M}), \|u\|_2=1} \inf_{v \in A(\mathcal{N})} \|u - v\|_2 = \sup_{u \in \mathcal{M}, \|Au\|_2=1} \inf_{v \in \mathcal{N}} \|Au - Av\|_2.$$

For any $v \in \mathbb{R}^n$, we have $\sigma_{\text{min}}(A)\|v\|_2 \leq \|Av\|_2 \leq \sigma_{\text{max}}(A)\|v\|_2$ with $\sigma_{\text{min}}(A) > 0$. Thus,

$$\sup_{u \in \mathcal{M}, \|Au\|_2=1} \inf_{v \in \mathcal{N}} \|Au - Av\|_2 \leq \sup_{u \in \mathcal{M}, \|u\|_2 \leq \sigma_{\text{min}}(A)} \sup_{v \in \mathcal{N}, \|v\|_2 \leq \sigma_{\text{max}}(A)} \inf_{u \in \mathcal{N}} \|u - v\|_2 \leq \frac{\sigma_{\text{max}}(A)}{\sigma_{\text{min}}(A)} \sup_{u \in \mathcal{M}, \|u\|_2 \leq \sigma_{\text{min}}(A)} \inf_{v \in \mathcal{N}} \|u - v\|_2$$

Thus,
where we use the fact for any $a \in \mathbb{R}$, $\inf_{v \in \mathcal{N}} \|u - v\|_2 = \inf_{v \in \mathcal{N}} \|u - av\|_2$ since if $v \in \mathcal{N}$ then $av \in \mathcal{N}$ by $\mathcal{N}$ being a linear subspace. We conclude by noting that $\sup_{u \in \mathcal{M}, \|u\|_2 \leq 1} \inf_{v \in \mathcal{N}} \|u - v\|_2 = \sup_{u \in \mathcal{M}, \|u\|_2 = 1} \inf_{v \in \mathcal{N}} \|u - v\|_2$.

By applying Lemma A.1 to (39), we have

$$\delta(\mathcal{C}(W_z^{-1/2}W_yX), \mathcal{C}(W_z^{1/2}X)) \leq \kappa(W_z^{1/2}) \delta(\mathcal{C}(W_z^{-1}W_yX), \mathcal{C}(X)).$$

Therefore,

$$\|\mathcal{I}_n^\ell(\beta_0)^{-1/2} (\mathcal{I}_n^\ell(\beta_0) - \mathcal{I}_n^\alpha(\beta_0)) \mathcal{I}_n^\ell(\beta_0)^{-1/2}\|_2 \leq \sigma_{\min} (X^TW_yW_z^{-1}W_yX/n)^{-1} \| (I_n - \mathcal{P}_W^{-1/2}X) W_z^{-1/2}W_yX \|^2_2/n$$

$$\leq \sigma_{\min} (X^TW_yW_z^{-1}W_yX/n)^{-1} \sigma_{\max}^2 (W_z^{-1/2}W_yX/\sqrt{n}) \kappa(W_z) \delta^2 (\mathcal{C}(W_z^{-1}W_yX), \mathcal{C}(X)).$$

Since $\sigma_{\min} (X^TW_yW_z^{-1}W_yX/n)^{-1} \sigma_{\max}^2 (W_z^{-1/2}W_yX/\sqrt{n}) \leq \kappa(X^TW/n) \kappa(W_y^2) \kappa(W_z)$,

$$\|\mathcal{I}_n^\ell(\beta_0)^{-1/2} (\mathcal{I}_n^\ell(\beta_0) - \mathcal{I}_n^\alpha(\beta_0)) \mathcal{I}_n^\ell(\beta_0)^{-1/2}\|_2 \leq \kappa(X^TW/n) \kappa(W_y^2) \kappa(W_z^2) \delta^2 (\mathcal{C}(W_z^{-1}W_yX), \mathcal{C}(X)).$$

Note by Assumption A[1] $\sup_i |x_i^\top \beta|$ can be bounded by the term independent of $n$ since $\sup_i |x_i^\top \beta| \leq \sup_i \|x_i\|_2 \|\beta\|_2 \leq rc_X \sqrt{p}$. It follows that $\kappa(W_y^2), \kappa(W_z^2) = O(1)$. Also $\lambda_{\max}(X^TW/n)$ is bounded by the term independent of $n$ since $p$ is fixed in the regime of interest and $\lambda_{\min}(X^TW/n) \geq c_\ell$ by Assumption A[III].

### A.3 Proof of Proposition 4.3

To show that there exists a unique stationary point in the interior of $\mathbb{B}_2(\epsilon)$ w.h.p, we show that there exists an $\ell_2$ ball of radius $\epsilon_0$ centered at $\beta_0$ in which $\mathcal{L}_n^{\ell}(\beta)$ is strongly convex w.h.p and has at least one local minimum, and no stationary point exists in $\mathbb{B}_2(\epsilon_0; \beta_0)^c := \{\beta; \|\beta - \beta_0\| \geq \epsilon_0\}$.

We use the following three lemmas to establish the result, whose proofs are provided at the end of this sub-section. The first lemma is about the gradient and Hessian of the population risk. The second and third lemma establish the uniform convergence of the empirical loss, gradient and Hessian to their population counterparts, respectively. We let $\mathcal{L}^{\ell}(\beta) := \mathbb{E}[\mathcal{L}_n^{\ell}(\beta)]$. 

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Lemma A.2. There exist an \( \epsilon_0 > 0 \) and a constant \( \gamma_\ell > 0 \) such that

\[
\inf_{\beta \in \mathbb{B}_2(\epsilon_0; \beta_0)} \| \nabla \mathcal{L}_n^\ell(\beta) \|_2 \wedge \inf_{\beta \in \mathbb{B}_2(\epsilon_0; \beta_0)} \lambda_{\min}(\nabla^2 \mathcal{L}_n^\ell(\beta)) \geq \gamma_\ell.
\]

Lemma A.3.

\[
P \left( \sup_{\beta \in \mathbb{B}_2(r)} \| \nabla \mathcal{L}_n^\ell(\beta) - \nabla \mathcal{L}^\ell(\beta) \|_2 \leq \tau \sqrt{\frac{Cp \log p}{n}} \right) \geq 1 - \delta
\]

where \( C = \log(1/\delta) \) and \( \tau = \tau_0 C_{\rho}(rK_X \vee 1) \) for an absolute constant \( \tau_0 > 0 \).

Lemma A.4 (Theorem 1 in Mei et al. (2018)). For \( n \geq C_p \log p \) where \( C = C_0 \cdot (\log(r\tau/\delta) \vee 1) \) for an absolute constant \( C_0 \) and \( \tau = K_X \max\{C_\rho, L_\ell^{1/3}\} \),

\[
P \left( \sup_{\beta \in \mathbb{B}_2(r)} \| \nabla^2 \mathcal{L}_n^\ell(\beta) - \nabla^2 \mathcal{L}^\ell(\beta) \|_2 \leq \tau^2 \sqrt{\frac{Cp \log p}{n}} \right) \geq 1 - \delta
\]

First we establish the result of Proposition 4.3 given Lemma A.2, A.3, and A.4. By Lemma A.3 and A.4 the following inequalities

\[
\sup_{\beta \in \mathbb{B}_2(r)} \| \nabla \mathcal{L}_n^\ell(\beta) - \nabla \mathcal{L}^\ell(\beta) \|_2 \vee \sup_{\beta \in \mathbb{B}_2(r)} \| \nabla^2 \mathcal{L}_n^\ell(\beta) - \nabla^2 \mathcal{L}^\ell(\beta) \|_2 \leq (\gamma_\ell/2) \wedge (\epsilon_\ell/4) \ni \epsilon_\ell/4
\]

hold with at least probability \( 1 - 3\delta \) given a sufficiently large sample size, for \( \gamma_\ell \) defined in Lemma A.2 \( \epsilon_L \) and \( \epsilon_g \) defined in (41) and (42), respectively. We show that on the event \( (40) \) there exists a unique global minimum inside \( \mathbb{B}_2(r) \).

First, on the event \( (40) \), we see that \( \mathcal{L}_n^\ell(\beta) \) is strongly convex over \( \mathbb{B}_2(\epsilon_0; \beta_0) \), since

\[
\inf_{\beta \in \mathbb{B}_2(\epsilon_0; \beta_0)} \lambda_{\min}(\nabla^2 \mathcal{L}_n^\ell(\beta)) \geq \inf_{\beta \in \mathbb{B}_2(\epsilon_0; \beta_0)} \lambda_{\min}(\nabla^2 \mathcal{L}^\ell(\beta)) - \sup_{\beta \in \mathbb{B}_2(\epsilon_0; \beta_0)} \| \nabla^2 \mathcal{L}_n^\ell(\beta) - \nabla^2 \mathcal{L}^\ell(\beta) \|_2 \geq \gamma_\ell/2.
\]

Then we argue that there exists a local minimum inside the ball \( \mathbb{B}_2(\epsilon_0; \beta_0) \). It is sufficient to show that there exists \( \beta \in \mathbb{B}_2(\epsilon_0; \beta_0) \setminus \partial \mathbb{B}_2(\epsilon_0; \beta_0) \) such that \( \mathcal{L}_n^\ell(\beta) < \inf_{\beta \in \partial \mathbb{B}_2(\epsilon_0; \beta_0)} \mathcal{L}_n^\ell(\beta) \). Take \( \beta = \beta_0 \).
Note there exists $\epsilon_L > 0$ such that

$$\inf_{\beta \in \partial B_2(\epsilon_0; \beta_0)} \mathcal{L}^\ell(\beta) - \mathcal{L}^\ell(\beta_0) = \epsilon_L, \quad (41)$$

since $\partial B_2(\epsilon_0; \beta_0)$ is compact and $A$ is a strictly convex function. Then on the event $(40)$,

$$\inf_{\beta \in \partial B_2(\epsilon_0; \beta_0)} \mathcal{L}^\ell_n(\beta) \geq \inf_{\beta \in \partial B_2(\epsilon_0; \beta_0)} \mathcal{L}^\ell(\beta) - \epsilon_L/4 \quad \text{and} \quad \mathcal{L}^\ell_n(\beta_0) \leq \mathcal{L}^\ell(\beta_0) + \epsilon_L/4.$$

Therefore,

$$\inf_{\beta \in \partial B_2(\epsilon_0; \beta_0)} \mathcal{L}^\ell_n(\beta) - \mathcal{L}^\ell_n(\beta_0) \geq \inf_{\beta \in \partial B_2(\epsilon_0; \beta_0)} \mathcal{L}^\ell(\beta) - \mathcal{L}^\ell(\beta_0) - \epsilon_L/2 \geq \epsilon_L/2,$$

where we use $(41)$ for the last inequality. Also, the empirical gradient does not vanish outside of $B_2(\epsilon_0; \beta_0)$, since

$$\inf_{\beta \in B_2(\epsilon_0; \beta_0)^c} \| \nabla \mathcal{L}^\ell_n(\beta) \|_2 \geq \inf_{\beta \in B_2(\epsilon_0; \beta_0)^c} \| \nabla \mathcal{L}^\ell(\beta) \|_2 - \sup_{\beta \in B_2(r)} \| \nabla \mathcal{L}^\ell_n(\beta) - \nabla \mathcal{L}^\ell(\beta) \|_2 \geq \gamma/2.$$

Finally, there exists no stationary point on the boundary of $B_2(r)$. Note there is no stationary point of $\mathcal{L}^\ell(\beta)$ on $\partial B_2(r)$ since $\langle \nabla \mathcal{L}^\ell(\beta), \beta_0 - \beta \rangle < 0$ for any $\beta \in \partial B_2(r)$. Since $\partial B_2(r)$ is compact, we have $\epsilon_g > 0$ such that

$$\sup_{\beta \in \partial B_2(r)} \langle \nabla \mathcal{L}^\ell(\beta), \beta_0 - \beta \rangle < -\epsilon_g. \quad (42)$$

Then for any $\beta \in \partial B_2(r)$,

$$\langle \nabla \mathcal{L}^\ell_n(\beta), \beta_0 - \beta \rangle = \langle \nabla \mathcal{L}^\ell(\beta), \beta_0 - \beta \rangle + \langle \nabla \mathcal{L}^\ell_n(\beta) - \nabla \mathcal{L}^\ell(\beta), \beta_0 - \beta \rangle \leq -\epsilon_g + \| \nabla \mathcal{L}^\ell_n(\beta) - \nabla \mathcal{L}^\ell(\beta) \|_2 \| \beta - \beta_0 \|_2 \leq -\epsilon_g/2.$$

Hence, on the event $(40)$, there exists a unique stationary point in $B_2(\epsilon_0; \beta_0) \subset B_2(r)$ which is a global minimum.

Now we turn to the proofs of three lemmas.
Proof of Lemma A.2. First, we lower bound the minimum eigenvalue of the Hessian.

\[
\inf_{u: \|u\|_2=1} u^T \nabla^2 \mathcal{L}^\ell(\beta) u = \inf_{u: \|u\|_2=1} \left( u^T \nabla^2 \mathcal{L}^\ell(\beta) u + u^T \left( \nabla^2 \mathcal{L}^\ell(\beta) - \nabla^2 \mathcal{L}^\ell(\beta_0) \right) u \right) \\
\geq \inf_{u: \|u\|_2=1} u^T \nabla^2 \mathcal{L}^\ell(\beta_0) u - \sup_{u: \|u\|_2=1} \left| u^T \left( \nabla^2 \mathcal{L}^\ell(\beta) - \nabla^2 \mathcal{L}^\ell(\beta_0) \right) u \right|
\]

At \( \beta = \beta_0 \),

\[
\inf_{u: \|u\|_2=1} u^T \nabla^2 \mathcal{L}^\ell(\beta_0) u = \mathbb{E}[\ell''(\mathbf{x}^T \beta_0, \mathbf{z})(\mathbf{x}^T u)^2] = \mathbb{E}[\rho_I(\mathbf{x}^T \beta_0)(\mathbf{x}^T u)^2]
\]

since \( \mathbb{E}[\rho_K(\mathbf{x}^T \beta_0, \mathbf{z})] = 0 \). Recalling the fact that \( \rho_I(t) = A''(h(t))h'(t)^2 \geq 0 \) for all \( t \), we have a lower bound

\[
\mathbb{E}[\rho_I(\mathbf{x}^T \beta_0)(\mathbf{x}^T u)^2] \geq \mathbb{E}[\rho_I(\mathbf{x}^T \beta_0)(\mathbf{x}^T u)^2 1\{ |x^T \beta_0| \leq \tau_c \}] \geq \inf_{|t| \leq \tau_c} \rho_I(t) \mathbb{E}[(x^T u)^2 1\{ |x^T \beta_0| \leq \tau_c \}]
\]

for any \( \tau_c > 0 \). We let \( \tau_c := \left( r^2 K^2_X \log \frac{16^2 K^4_X}{c_\ell^2} \right)^{1/2} \). Then by Cauchy-Schwarz, Assumption A1 and Lemma A.5

\[
\mathbb{E}[(x^T u)^2 1\{ |x^T \beta_0| \leq \tau_c \}] = \mathbb{E}[(x^T u)^2] - \mathbb{E}[(x^T u)^2 1\{ |x^T \beta_0| \geq \tau_c \}] \\
\geq c_\ell - \mathbb{E}[(x^T u)^4]^{1/2} \mathbb{P}(x^T \beta_0 \geq \tau_c)^{1/2} \geq c_\ell/2. \quad (43)
\]

Now we bound the difference term. Using Lipschitz assumption A2, we have

\[
\left| u^T \left( \nabla^2 \mathcal{L}^\ell(\beta) - \nabla^2 \mathcal{L}^\ell(\beta_0) \right) u \right| \leq \mathbb{E}[|\ell''(\mathbf{x}^T \beta, \mathbf{z}) - \ell''(\mathbf{x}^T \beta_0, \mathbf{z})| (\mathbf{x}^T u)^2] \\
\leq L_{I} \mathbb{E}[(x^T (\beta - \beta_0))(x^T u)^2].
\]

Then by Cauchy-Schwarz and sub-Gaussian moment property,

\[
L_{I} \mathbb{E}[(x^T (\beta - \beta_0))(x^T u)^2] \leq L_{I} \|\Delta_0\|_2 \mathbb{E}[(x^T \Delta_0/\|\Delta_0\|_2)^2]^{1/2} \mathbb{E}[(x^T u)^4]^{1/2} \leq 4\sqrt{2} K^3_X L_{I} \|\Delta_0\|_2 \quad (44)
\]

where \( \Delta_0 = \beta - \beta_0 \). Hence, combining (43), (44), we conclude for \( \beta \) such that \( \|\beta - \beta_0\|_2 \leq \epsilon_0 \), for
\[ \epsilon_0 := (\inf_{|t| \leq \tau_c} \rho_I(t) c_t) / (16 \sqrt{2} K_X^3 L), \]

\[ \inf_{u : \|u\|_2 = 1} \|u^\top \nabla^2 \mathcal{L}^\ell(\beta) u \| \geq \inf_{|t| \leq \tau_c} \rho_I(t) c_t / 4. \]

Now we address the lower bound of the gradient. Let \( \beta \in \mathbb{B}_2(\epsilon_0; \beta_0) \) be fixed.

\[
\langle \beta - \beta_0, \nabla \mathcal{L}^\ell(\beta) \rangle = E[\{ A'(h(x^\top \beta)) - A'(h(x^\top \beta_0)) \} h'(x^\top \beta) x^\top (\beta - \beta_0)] \\
= E[A''(h(x^\top \beta_i)) h'(x^\top \beta_i) h'(x^\top \beta)(x^\top (\beta - \beta_0))^2]
\]

for \( \beta_i = \beta_0 + v(\beta - \beta_0) \) where \( v \in [0, 1] \) by the mean value theorem.

Define an event \( \mathcal{E} := \{ |x^\top \beta_0| \leq \tau_c, |x^\top \Delta_0| \leq 2 \tau_c \} \) where \( \Delta_0 := \beta - \beta_0 \).

\[
\langle \beta - \beta_0, \nabla \mathcal{L}^\ell(\beta) \rangle \geq c_r E[(x^\top (\beta - \beta_0))^2 1_{\mathcal{E}}] \geq c_r c_t \| \Delta_0 \|^2 / 2
\]

for \( c_r := \left( \inf_{|t| \leq \tau_c} A''(h(t)) h'(t) \right) \left( \inf_{|t| \leq \tau_c} h'(t) \right) > 0 \), since

\[
E[(x^\top (\beta - \beta_0))^2 1_{\mathcal{E}}] = E[(x^\top (\beta - \beta_0))^2] - E[(x^\top (\beta - \beta_0))^2 1_{\mathcal{E}^c}] \geq \| \Delta_0 \|^2 / 2 \left( c_t E[(x^\top \Delta_0/\| \Delta_0 \|^2)]^{1/2} P(E^c)^{1/2} \right) \geq \| \Delta_0 \|^2 c_t / 2
\]

by Lemma A.5. We apply Cauchy-Schwarz inequality to (45) to obtain

\[
\| \nabla \mathcal{L}^\ell(\beta) \|_2 \geq c_t c_r \| \beta - \beta_0 \|_2 / 2 \geq c_t c_r \epsilon_0 / 2
\]

since \( \| \beta - \beta_0 \|_2 \geq \epsilon_0 \). Finally, take \( \gamma_\ell := c_t \left( \inf_{|t| \leq \tau_c} \rho_I(t) / 4 \land c_r \epsilon_0 / 2 \right) \) to conclude. \( \square \)

**Lemma A.5.** Let \( x \in \mathbb{R}^p \) be a random vector which satisfies the sub-gaussian tail condition with the parameter \( K_X \), and also let \( u_2, u_3 \in \mathbb{R}^p \) be non-random vectors such that \( \| u_1 \|_2 = 1, \| u_2 \|_2 \leq c_1 r, \) and \( \| u_3 \|_2 \leq c_2 r \) for some \( c_1, c_2 > 0 \). For \( \tau_c := \left( r^2 K_X^2 \log \frac{16^2 K_X^4}{c_t^4} \right)^{1/2} \), we have

\[
E[(x^\top u_1)^4]^{1/2} \{ P(|x^\top u_2| \geq c_1 \tau_c) + P(|x^\top u_3| \geq c_2 \tau_c) \}^{1/2} \leq c_t / 2
\]

**Proof.** Moment and tail properties of sub-Gaussian distribution give \( E[(x^\top u_1)^4]^{1/2} \leq 4 K_X^2 \) and
\( \mathbb{P}(|x^Tu_2| \geq c_1 \tau_c) + \mathbb{P}(|x^Tu_3| \geq c_2 \tau_c) \leq 4 \exp(-\tau_c^2/r^2 K^2_X) \). Then the choice of \( \tau_c \) gives the desirable bound.

**Proof of Lemma A.3.** By McDiarmid Inequality,

\[
\mathbb{P} \left( \sup_{\beta \in B_2(r)} |\mathcal{L}^\ell_n(\beta) - \mathcal{L}^\ell(\beta)| \leq \mathbb{E} \left[ \sup_{\beta \in B_2(r)} |\mathcal{L}^\ell_n(\beta) - \mathcal{L}^\ell(\beta)| \right] + t \right) \geq 1 - \exp(-t^2n/2C^2_\rho) \tag{46}
\]

by Assumption A.2. Then for \( t \geq \sqrt{2C_\rho \rho \log(1/\delta)} \), the probability of the LHS is bounded below by \( 1 - \delta \). Now, we bound the expectation term by the standard argument of symmetrization and concentration inequality.

\[
\mathbb{E} \left[ \sup_{\beta \in B_2(r)} |\mathcal{L}^\ell_n(\beta) - \mathcal{L}^\ell(\beta)| \right] = \mathbb{E} \left[ \sup_{\beta \in B_2(r)} \left| \frac{1}{n} \sum_{i=1}^n \ell(x_i^\top \beta, z_i) - \mathbb{E}[\ell(x_i^\top \beta, z_i)] \right| \right] 
\]

\[
\leq 2 \mathbb{E} \left[ \sup_{\beta \in B_2(r)} \left| \frac{1}{n} \sum_{i=1}^n \sigma_i \ell(x_i^\top \beta, z_i) \right| \right] 
\]

where we let \( (\sigma_i)_{i=1}^n \) be i.i.d. Rademacher variables independent from \((x_i, z_i)_{i=1}^n\). Since \(|\ell(t, z_i) - \ell(s, z_i)| \leq C_\rho |t - s| \) a.s. by Assumption A.2, contraction inequality gives

\[
\mathbb{E} \left[ \sup_{\beta \in B_2(r)} |\mathcal{L}^\ell_n(\beta) - \mathcal{L}^\ell(\beta)| \right] \leq 4C_\rho \mathbb{E} \left[ \sup_{\beta \in B_2(r)} \left| \frac{1}{n} \sum_{i=1}^n \sigma_i x_i^\top \beta \right| \right] 
\]

\[
\leq 4C_\rho \mathbb{E} \left[ \sup_{\beta \in B_2(r)} \|\beta\|_1 \left| \frac{1}{n} \sum_{i=1}^n \sigma_i x_i \right| \right] 
\]

\[
\leq 4rC_\rho \sqrt{p} \mathbb{E} \left[ \left| \frac{1}{n} \sum_{i=1}^n \sigma_i x_i \right| \right] .
\]

Since \( |\sigma_i| \leq 1 \) a.s. and \( \mathbb{E}[\sigma_i x_{ij}] = 0, \sigma_i x_{ij} \) is mean-zero \( cK_X \) sub-gaussian with some absolute constant \( c > 0, \forall i, j \). As \( (\sigma_i x_{ij})_{i=1}^n \) are independent for any \( j, \sum_{i=1}^n \sigma_i x_{ij}/n \) is also sub-gaussian with parameter \( cK_X/\sqrt{n} \). Therefore, by the bound on the maximum of sub-gaussian variables,

\[
\mathbb{E} \left[ \sup_{\beta \in B_2(r)} |\mathcal{L}^\ell_n(\beta) - \mathcal{L}^\ell(\beta)| \right] \leq c' r K_X C_\rho \sqrt{p \log p \over n} . \tag{47}
\]

where \( c' \) is an absolute constant. Combining (46) and (47), we obtain the desired inequality.

**Proof of Lemma A.4.** We verify assumptions 1-3 in [Mei et al. (2018)]. The first assumption is to
verify whether the gradient of the loss has a sub-Gaussian tail. The second assumption is to show that the Hessian evaluated on a unit vector is sub-Exponential. The third assumption is about the Lipschitz continuity of the Hessian. We mainly check whether quantities in interest satisfy a sub-gaussian/exponential moment bounds.

A1 For any $u \in \mathbb{R}^p$ such that $\|u\|_2 = 1$, $ \langle \ell'(x^T \beta, z)x, u \rangle$ is sub-gaussian since

$$\mathbb{E} (\langle \ell'(x^T \beta, z)x^T u \rangle)^{1/k} \leq \|\ell'\|_\infty \mathbb{E} \|x^T u\|^{1/k} \leq C_P K X \sqrt{k} \text{ for any } k \geq 1.$$

A2 Similarly for any $u \in \mathbb{R}^p$ such that $\|u\|_2 = 1$ $\langle u, \ell''(x^T \beta, z)x^T u \rangle$ is sub-exponential since

$$\mathbb{E} (\langle \ell''(x^T \beta, z)(x^T u)^2 \rangle)^{1/k} \leq 2C_P \mathbb{E} \|x^T u\|^{2k}^{1/k} \leq 4C_P K_X^2 k \text{ for any } k \geq 1.$$

A3 $\|\nabla^2 L'\|_2 = \sup_{u: \|u\|_2 = 1} \mathbb{E} [\rho_I(x^T \beta_0)(x^T u)^2] \leq 2C_P K_X^2$. Also from the Lipschitz continuity assumption of $\ell''$,

$$\mathbb{E} \left[ \sup_{\beta_1 \neq \beta_2} \frac{\|\nabla^2 L'_{\beta_1} - \nabla^2 L'_{\beta_2}\|_2}{\|\beta_1 - \beta_2\|_2} \right] = \mathbb{E} \left[ \sup_{\beta_1 \neq \beta_2, u: \|u\|_2 = 1} \frac{|\ell''(x^T \beta_1, z) - \ell''(x^T \beta_2, z)|(x^T u)^2}{\|\beta_1 - \beta_2\|_2} \right] \leq \sup_{\beta_1 \neq \beta_2, u: \|u\|_2 = 1} L_\ell \mathbb{E} \left[ \frac{|x^T \beta_1 - x^T \beta_2|(x^T u)^2}{\|\beta_1 - \beta_2\|_2} \right].$$

By Cauchy-Schwarz, $|x^T (\beta_1 - \beta_2)| \leq \|x\|_2 \|\beta_1 - \beta_2\|_2$ and $(x^T u)^2 \leq \|x\|_2^2$ since $\|u\|_2 = 1$. Thus

$$\mathbb{E} \left[ \sup_{\beta_1 \neq \beta_2} \frac{\|\nabla^2 L'_{\beta_1} - \nabla^2 L'_{\beta_2}\|_2}{\|\beta_1 - \beta_2\|_2} \right] \leq L_\ell \mathbb{E} \left[ \|x\|_2^3 \right] \leq 3^{3/2}L_\ell \rho^{3/2}K_X^2,$$

since $\mathbb{E}[\|x\|_2^3] = \mathbb{E}[(\sum_{i=1}^p x_i^2)^{3/2}] \leq p^{1/2}\mathbb{E}[(\sum_{i=1}^p |x_i|^3)] \leq 3^{3/2}p^{3/2}K_X^3$. \hfill \Box

### A.4 Proof of Corollary 4.2

We show that $A[2]$ is satisfied for a (GLM) with parameters $(\log(1 + \exp(\cdot)), g_{LN})$ with $z_i \in \{0, 1\}$. Then the result follows from the Proposition 4.3. From (11) and (12), we have

$$\ell'(t, z) = \left(A'(h_{LN}(t)) - z\right) h'_{LN}(t),$$

$$\ell''(t, z) = \rho_I(t) + \rho_R(t, z),$$

for $A(t) = \log(1 + \exp(t))$ and $\rho_I(t)$ and $\rho_R(t, z)$ such that

$$\rho_I(t) = A''(h_{LN}(t))h'_{LN}(t)^2, \quad \text{and} \quad \rho_R(t, z) = (A'(h_{LN}(t)) - z)h''_{LN}(t).$$
From Lemma [A.6] which is presented at the end of this subsection, \( \|h'_{LN}\|_\infty \leq 1 \) and \( \|h''_{LN}\|_\infty \leq 2 \).

Also \( A''(t) = e^t/(1+e^t)^2 \) is bounded by \( 1/4 \) and \( |A'(h_{LN}(t)) - z| \leq 1 \) for any \( t \) and \( z \in \{0, 1\} \), since \( 0 \leq A'(h_{LN}(t)) \leq 1, \forall t \). Thus

\[
|\ell'(t, z)| \leq 1, \quad |\rho_I(t)| \leq \frac{1}{4}, \quad \text{and} \quad |\rho_R(t, z)| \leq \|h'''_{LN}\|_\infty \leq 2, \quad \forall z \in \{0, 1\}, \forall t,
\]

and \( \max\{\|\ell''\|_\infty, \|\rho_I\|_\infty, \|\rho_R\|_\infty\} \) is bounded by 2.

To verify that \( \ell'' \) is \( L_{\ell} \)-Lipschitz where \( L_{\ell} \) does not depend on \( t \), it is sufficient to show that the gradients of \( \rho_I \) and \( \rho_R \) are bounded independent of \( t \). By calculation, we have

\[
\rho_I(t) = A''(h_{LN}(t))h'_{LN}(t)^3 + 2A''(h_{LN}(t))h'_{LN}(t)^2h''_{LN}(t)
\]

\[
\rho_R(t, z) = A''(h_{LN}(t))h'_{LN}(t)h''_{LN}(t) + \{A'(h_{LN}(t)) - z\}h'''_{LN}(t).
\]

We bound each term separately. As other terms can be bounded similarly other than the term involving \( A'' \), it is sufficient to show that \( A''(t) \) is bounded by an absolute constant. We have,

\[
|A''(t)| = \left| \frac{e^t}{(1+e^t)^2} - \frac{2e^{2t}}{(1+e^t)^3} \right| \leq \left| \frac{e^t}{(1+e^t)^2} \right| + \left| \frac{2e^t}{(1+e^t)^2} \left( \frac{e^t}{1+e^t} \right) \right| \leq \frac{1}{4} + \frac{1}{2} \leq 1.
\]

Finally, we present the Lemma about the boundedness of \( h'_{LN}, h''_{LN} \) and \( h'''_{LN} \).

**Lemma A.6.** For \( h_{LN} = (A')^{-1}g_{LN}^{-1} \), there exists \( C \leq 7 \) such that \( \max\{\|h'_{LN}\|_\infty, \|h''_{LN}\|_\infty, \|h'''_{LN}\|_\infty\} \leq C \).

**Proof.** From the definition of \( g_{LN} \) and \( h_{LN} \) in [9], we have

\[
h_{LN}(t) := \log \left( \frac{(1 - \rho - \rho_0)\mu(t) + \rho_0}{1 - (1 - \rho - \rho_0)\mu(t) - \rho_0} \right).
\]

Let \( a = 1 - \rho - \rho_0 \) and \( b = \rho_0 \). We have \( a\mu(t) + b \leq a + b < 1 \) and \( a\mu(t) \leq a\mu(t) + b \) for any \( t \). Then \( \forall t \),

\[
\frac{a\mu(t)}{a\mu(t) + b} \leq 1 \quad \text{and} \quad \frac{a(1 - \mu(t))}{1 - a\mu(t) - b} < 1
\]  \hspace{1cm} (48)
By definition of $h_{LN}(t) = \log(a\mu(t) + b) - \log(1 - a\mu(t) - b)$,

$$h'_{LN}(t) = \frac{d}{d\mu(t)} \log \left( \frac{a\mu(t) + b}{1 - (a\mu(t) + b)} \right) \frac{d\mu(t)}{dt} = \frac{a\mu(t)(1 - \mu(t))}{(a\mu(t) + b)(1 - a\mu(t) - b)} \leq 1$$

by the fact that

$$\frac{d\mu(t)}{dt} = A''(t) = \mu(t)(1 - \mu(t))$$

and the inequalities (48). In particular, $h'_{LN} \geq 0$ and $\|h'_{LN}\|_{\infty} \leq 1$.

Now we bound $h''_{LN}$. From elementary calculation, it can be shown that

$$h''_{LN}(t) = h'_{LN}(t)(1 - 2\mu(t)) - h'_{LN}(t)^2(1 - 2(a\mu(t) + b)),$$

$$h'''_{LN}(t) = h''_{LN}(t) \left\{ 1 - 2\mu(t) - 2h'_{LN}(t)(1 - 2(a\mu(t) + b)) \right\} - 2\mu(t)(1 - \mu(t))h'_{LN}(t)(1 - ah'_{LN}(t)).$$

In particular,

$$|h''_{LN}(t)| \leq h'_{LN}(t)|1 - 2\mu(t)| + h'_{LN}(t)^2|1 - 2(a\mu(t) + b)| \leq 2\|h'_{LN}\|_{\infty} \leq 2$$

since $\max_{0 \leq \mu \leq 1} |1 - 2\mu| = 1$ and $0 \leq \mu(t), a\mu(t) + b \leq 1$, for all $t$. Also,

$$|h'''_{LN}(t)| \leq |h''_{LN}(t)| \left\{ |1 - 2\mu(t)| + 2h'_{LN}(t)|1 - 2(a\mu(t) + b)| \right\} + 2\mu(t)(1 - \mu(t))h'_{LN}(t)(1 - ah'_{LN}(t)),$$

$$\leq 3\|h''_{LN}\|_{\infty} + \frac{1}{2}.$$  \hfill \Box

### A.5 Proof of Proposition 5.1

We show that $\langle \nabla L^t_n(\beta) - \nabla L^t_n(\beta_0), \Delta_0 \rangle$ is lower bounded by a positive plus a tolerance term, where the positive term is obtained from the region where the amount of non-convexity is limited, and the tolerance term scales with $\log p \|\Delta_0\|_1^2 / n$. The proofs use concentration results from the empirical processes literature, and use similar approaches as in Negahban et al. (2012) and Loh (2017) to establish $\ell_2$ error bounds, with modifications to accommodate generalized linear model settings where the function of interest is non-convex.
Lower-bound for \( \langle \nabla L_n^\ell(\beta) - \nabla L_n^\ell(\beta_0), \Delta_0 \rangle \)

Define \( \Delta_0 := \beta - \beta_0 \). First we show that

\[
\frac{\langle \nabla L_n^\ell(\beta) - \nabla L_n^\ell(\beta_0), \Delta_0 \rangle}{\|\Delta_0\|_2^2} \geq \frac{1}{n} \sum_{i=1}^{n} (\alpha_{\tau} + \kappa) \psi_{\tau}\|\Delta_0\|_2/r_0 (x_i^T \Delta_0 \phi_{\beta_0}(x_i, z_i)) - \kappa \left( \frac{x_i^T \Delta_0}{\|\Delta_0\|_2} \right)^2
\]

(49)

for \( \kappa := \kappa_C + \kappa_\varepsilon \), where \( \alpha_{\tau}, \kappa_C \), and \( \kappa_\varepsilon \) are defined in A4 and A5 and we define \( \psi_c(t) \) and \( \phi_{\beta_0}(x_i, z_i) \) as

\[
\psi_c(t) := t^2 \mathbb{1}\{|t| \leq c/2\} + (|t| - c)^2 \mathbb{1}\{c/2 \leq |t| \leq c\},
\]

\[
\phi_{\beta_0}(x_i, z_i) := \mathbb{1}\{|\varepsilon_i| \leq \frac{\alpha_{\tau}}{\|h''\|_\infty}\}.
\]

By definition,

\[
\langle \nabla L_n^\ell(\beta) - \nabla L_n^\ell(\beta_0), \Delta_0 \rangle = n^{-1} \sum_{i=1}^{n} \left( \ell'(x_i^T \beta, z_i) - \ell'(x_i^T \beta_0, z_i) \right)x_i^T \Delta_0
\]

\[
= n^{-1} \sum_{i=1}^{n} \ell''(x_i^T (\beta_0 + v_i \Delta_0), z_i)(x_i^T \Delta_0)^2
\]

for some \( v_i \in [0, 1] \) which only depends on \( x_i \). Let \( \beta_i := \beta_0 + v_i \Delta_0 \). Define an event

\[
\mathcal{E}_i := \{|x_i^T \Delta_0| \leq \frac{\tau\|\Delta_0\|_2}{r_0}, |\varepsilon_i| \leq \frac{\alpha_{\tau}}{\|h''\|_\infty}\},
\]

for \( \tau \) defined in A4. On \( \mathcal{E}_i \), we have by Assumption A3 and A4

\[
\ell''(x_i^T \beta_i, z_i) = \rho_I(x_i^T \beta_i) + \{A'(h(x_i^T \beta_i)) - A'(h(x_i^T \beta_0))\}h''(x_i^T \beta_i) + \varepsilon_i h''(x_i^T \beta_i)
\]

\[
\geq 2\alpha_{\tau} - |\varepsilon_i||h''\|_\infty \geq \alpha_{\tau},
\]

since \( |x_i^T \beta_i - x_i^T \beta_0| \leq |x_i^T \Delta_0| \leq \tau, |x_i^T \beta_0| \leq c_0c_b \), and \( \ell''(x_i^T \beta, z_i) \geq -(\kappa_C + \kappa_\varepsilon) = -\kappa \), for any \( i \), where we recall the definition \( \kappa := \kappa_C + \kappa_\varepsilon \). Then,
\[ \langle \nabla L_n^\ell (\beta) - \nabla L_n^\ell (\beta_0), \Delta_0 \rangle = n^{-1} \sum_{i=1}^{n} \ell''(x_i^T (\beta + v_i \Delta_0), z_i) (x_i^T \Delta_0)^2 \mathbb{1}_{\mathcal{E}_i} + \ell''(x_i^T (\beta_0 + v_i \Delta_0), z_i) (x_i^T \Delta_0)^2 \mathbb{1}_{\mathcal{E}_i} \]
\[ \geq n^{-1} \sum_{i=1}^{n} \alpha_r (x_i^T \Delta_0)^2 \mathbb{1}_{\mathcal{E}_i} - \kappa (x_i^T \Delta_0)^2 \mathbb{1}_{\mathcal{E}_i} \]
\[ \geq n^{-1} \sum_{i=1}^{n} (\alpha_r + \kappa) (x_i^T \Delta_0)^2 \mathbb{1}_{\mathcal{E}_i} - \kappa (x_i^T \Delta_0)^2. \]

Note for each \( i \),
\[ (x_i^T \Delta_0)^2 \mathbb{1}_{\mathcal{E}_i} \geq \psi_{\tau \| \Delta \|_2 / \tau_0} (x_i^T \Delta \phi_{\beta_0}(x_i, z_i)) \]
where we recall the definition \( \phi_{\beta_0}(x_i, z_i) := 1 \{|\epsilon_i| \leq \frac{\alpha_r}{\|h\|_{\infty}} \} \). Thus, we obtain (49). Therefore, to establish the result of Proposition [5.1], it suffices to establish the following inequality
\[ \frac{1}{n} \sum_{i=1}^{n} (\alpha_r + \kappa) \psi_{\tau \| \Delta \|_2 / \tau_0} (x_i^T \Delta \phi_{\beta_0}(x_i, z_i)) - \kappa (x_i^T \Delta)^2 \geq \alpha_\ell - \tau_\ell \left( \frac{\log p}{n} \right) \| \Delta \|_2^2, \] (50)
holds for all \( \Delta \in \mathbb{B}_2(r_0) \) with high probability for some \( \alpha_\ell, \tau_\ell > 0 \). We divide cases when \( \Delta \) belongs to the set \( \{ \Delta \in \mathbb{B}_2(r_0); \| \Delta \|_1 \leq c_0 \sqrt{\frac{n}{\log p}} \| \Delta \|_2 \} \) or not, for \( c_0 \) defined in (56).

**Case I: \( \| \Delta \|_1 \leq c_0 \sqrt{\frac{n}{\log p}} \| \Delta \|_2 \)**

Let \( A(\delta) := \{ \Delta \in \mathbb{B}_2(r_0); \| \Delta \|_1 \leq \delta \| \Delta \|_2 \} \). For any fixed \( \delta \) with \( 0 < \delta \leq c_0 \sqrt{\frac{n}{\log p}} \), we will first establish that
\[ \frac{1}{n} \sum_{i=1}^{n} (\alpha_r + \kappa) \psi_{\tau \| \Delta \|_2 / \tau_0} (x_i^T \Delta \phi_{\beta_0}(x_i, z_i)) - \kappa (x_i^T \Delta)^2 \geq \alpha_0 - \tau_0 \delta^2 \left( \frac{\log p}{n} \right) \] (51)
holds with high probability for some \( \alpha_0, \tau_0 > 0 \), for all \( \Delta \in A(\delta) \). To begin with, we first have
\[ \frac{1}{n} \sum_{i=1}^{n} (\alpha_r + \kappa) \psi_{\tau \| \Delta \|_2 / \tau_0} (x_i^T \Delta \phi_{\beta_0}(x_i, z_i)) - \kappa (x_i^T \Delta)^2 \]
\[ \geq \mathbb{E} \left[ (\alpha_r + \kappa) \psi_{\tau \| \Delta \|_2 / \tau_0} (x_i^T \Delta \phi_{\beta_0}(x_i, z_i)) - \kappa (x_i^T \Delta)^2 \right] - (\alpha_r + \kappa) U_1(\delta) - \kappa U_2(\delta) \] (52)
where we define

\[ U_1(\delta) := \sup_{\Delta \in \mathcal{A}(\delta)} \left( \frac{1}{n\|\Delta\|_2^2} \sum_{i=1}^n \psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i)) - \mathbb{E}[\psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i))] \right) \]

\[ U_2(\delta) := \sup_{\Delta \in \mathcal{A}(\delta)} \left( \frac{1}{n\|\Delta\|_2^2} \sum_{i=1}^n (x_i^\top \Delta)^2 - \mathbb{E}[(x_i^\top \Delta)^2] \right). \]

We will then use a peeling argument to prove a bound uniformly over all such \( \delta \).

**Case I - Step 1. Lower bound of the expectation**

First, we show for any \( i \),

\[ \mathbb{E} \left[ (\alpha_\tau + \kappa) \psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i)) - \kappa(x_i^\top \Delta)^2 \right] \geq \frac{\alpha_\tau c_\ell}{2} \|\Delta\|_2^2. \]  

(53)

Note,

\[ \mathbb{E} \left[ (\alpha_\tau + \kappa) \psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i)) - \kappa(x_i^\top \Delta)^2 \right] = \alpha_\tau \mathbb{E}[(x_i^\top \Delta)^2] - (\alpha_\tau + \kappa) \mathbb{E} \left[ (x_i^\top \Delta)^2 - \psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i)) \right]. \]

By \textbf{A11} \( \mathbb{E}[(x_i^\top \Delta)^2] \geq c_\ell \|\Delta\|_2^2 \). For the difference term,

\[ \mathbb{E} \left[ (x_i^\top \Delta)^2 - \psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i)) \right] \leq \mathbb{E}[(x_i^\top \Delta)^2]^{1/2} \left( \mathbb{P}(|x_i^\top \Delta| > \tau \|\Delta\|_2/r_0) + \mathbb{P}(|\epsilon_i| > \tau \|h''\|_\infty) \right)^{1/2} \leq 8K_X^2 \left( \exp \left( -\frac{\tau^2}{K_X^2 r_0^2} \right) + \frac{\kappa_\tau}{\alpha_\tau} \right)^{1/2} \|\Delta\|_2^2, \]

where for the last inequality we use the sub-gaussian tail probability property and Markov inequality as well as Assumption \textbf{A5} \text{ By assumption about the choice of } r_0 \text{ in (22),}

\[ 8K_X^2 \left( \exp \left( -\frac{\tau^2}{K_X^2 r_0^2} \right) + \frac{\kappa_\tau}{\alpha_\tau} \right)^{1/2} \leq \frac{\alpha_\tau c_\ell}{2(\alpha_\tau + \kappa)}. \]

In particular, \( (\alpha_\tau + \kappa) \mathbb{E} \left[ (x_i^\top \Delta)^2 - \psi_{\tau\|\Delta\|_2/r_0}(x_i^\top \Delta \phi_{\beta_0}(x_i, z_i)) \right] \leq (\alpha_\tau c_\ell/2) \|\Delta\|_2^2 \) and thus, we have \( \text{(53)}. \)
Case I - Step 2. Upper bound of $U_1(\delta)$

We note by construction $\psi_c$ is a $c$-Lipschitz function. By symmetrization and contraction theorem,

$$E[U_1(\delta)] \leq 2E \left[ \sup_{\Delta \in \mathbb{A}(\delta)} \left( \frac{1}{n\|\Delta\|_2^2} \left| \sum_{i=1}^n \sigma_i \psi_{r_0}(x_i^T \Delta \beta_0(x_i, z_i)) \right| \right) \right]$$

$$\leq 4E \left[ \sup_{\Delta \in \mathbb{A}(\delta)} \left( \frac{T}{r_0 n\|\Delta\|_2} \left| \sum_{i=1}^n \sigma_i x_i^T \Delta \beta_0(x_i, z_i) \right| \right) \right]$$

where $(\sigma_i)_{i=1}^n$ are Rademacher random variables independent from $(x_i, z_i)_{i=1}^n$. We use Holder’s inequality to get

$$E[U_1(\delta)] \leq \frac{4 \delta c}{r_0} E \left[ \left| \frac{1}{n} \sum_{i=1}^n \sigma_i \phi_{r_0}(x_i, z_i) x_i \right|_\infty \right]$$

where we use the fact that $\sup_{\Delta \in \mathbb{A}(\delta)} \|\Delta\|_1 \leq \delta \|\Delta\|_2$ by the definition of $\mathbb{A}(\delta)$. Now we bound the expectation using the maximal inequalities. We note, $|\sigma_i \phi_{r_0}(x_i, z_i)| \leq 1$ a.s. and $E[\sigma_i \phi_{r_0}(x_i, z_i) x_{ij}] = 0$ for all $i$ and $j$, thus $\sigma_i \phi_{r_0}(x_i, z_i) x_{ij}$ is mean-zero sub-gaussian whose parameter is a constant multiple of $K_X$. Then by the bound on the maximum of sub-gaussian variables,

$$E[U_1(\delta)] \leq \frac{c_0 \delta \tau K_X}{r_0} \sqrt{\frac{\log p}{n}},$$

where $c$ is an absolute constant. Finally, we apply the McDiarmid’s inequality to get probability tail bound. We have $\|\psi_{r_0}(\Delta)\|_2 \leq \tau^2 \|\Delta\|_2^2 / 4r_0^2 \leq \tau^2 / 4$. Thus, we get $P(U_1(\delta) \geq E[U_1(\delta)] + u) \leq \exp(-\frac{8nu^2}{\tau^4})$. Taking $u = \frac{\alpha \tau c_\ell}{12(\alpha + \kappa)} + \frac{c_0 \delta \tau K_X}{r_0} \sqrt{\frac{\log p}{n}}$, we obtain

$$P\left(U_1(\delta) \geq \frac{\alpha \tau c_\ell}{12(\alpha + \kappa)} + \frac{c' \delta \tau K_X}{r_0} \sqrt{\frac{\log p}{n}}\right) \leq \exp(-c'_1 n - c'_2 \delta^2 \log p), \quad (54)$$

where $c'$ is an absolute constant and $c'_1, c'_2$ are constants only depending on underlying model parameters $(\alpha, \kappa, c_\ell, K_X$ and $r_0)$. 

53
Case I - Step 3. Upper bound of $U_2(\delta)$

By definition,

$$U_2(\delta) \leq \sup_{\|\Delta\|_1 \leq \delta, \|\Delta\|_2} \left( \frac{1}{n\|\Delta\|_2^2} \left\| \sum_{i=1}^n (x_i^\top \Delta)^2 - \mathbb{E}[(x_i^\top \Delta)^2] \right\| \right)$$

$$= \sup_{\|v\|_1 \leq \delta, \|v\|_2 = 1} \left| v^\top \left( \sum_{i=1}^n \frac{x_i x_i^\top}{n} - \mathbb{E}(x_i x_i^\top) \right) v \right| .$$

To bound $U_2(\delta)$, we use the following result, which can be obtained by combining Lemma 12 and 15 in Loh and Wainwright (2012).

**Lemma A.7.** Suppose $x_i$ satisfies the sub-gaussian tail condition with the parameter $K_X$, for all $i = 1, \ldots, n$. For any $u > 0$, the following inequality holds with probability at least $1 - 2\exp(-c'nu(1 \wedge u)/2)$,

$$\sup_{\|v\|_2 \leq \sqrt{s(u)\|v\|_2}} \left| v^\top \left( \sum_{i=1}^n \frac{x_i x_i^\top}{n} - \mathbb{E}(x_i x_i^\top) \right) v \right| \leq 27uK_X^2 \|v\|_2^2$$  \hspace{1cm} (55)

where $s(u) := (c'n/4\log p)(u \wedge u^2)$ and $c'$ is a universal constant in Bernstein's inequality (see Corollary 2.8.3 in Vershynin (2018)), given a sufficient sample size $n \geq (4\log p/c')\max\{(u \wedge u^2), (u \wedge u^2)^{-1}\}$.

We take $u = u_0 := \alpha \tau c_\ell/(12 \cdot 27K_X^2 \kappa)$ and let $c_0$ as

$$c_0 := \sqrt{\frac{c'}{4}(u_0 \wedge u_0^2)} ,$$  \hspace{1cm} (56)

for $c'$ in Lemma A.7. Then $\delta^2 \leq c_0^2(n/\log p) = s(u_0)$ for $s(\cdot)$ defined in Lemma A.7. Then we apply Lemma A.7 to obtain

$$U_2(\delta) \leq \sup_{\|v\|_1 \leq \sqrt{s(u_0)\|v\|_2}} \left| v^\top \left( \sum_{i=1}^n \frac{x_i x_i^\top}{n} - \mathbb{E}(x_i x_i^\top) \right) v \right| \leq \frac{\alpha \tau c_\ell}{12\kappa}$$  \hspace{1cm} (57)

with probability at least $1 - 2\exp(-c''n)$ for $c'' = c'(u_0 \wedge u_0^2)/2$. 

54
Case I - Step 4. Combining bounds in Steps 1,2,3

Combining (53), (54) and (57) with (52), we have the inequality

\[
\frac{1}{n} \sum_{i=1}^{n} (\alpha_\tau + \kappa) \frac{\psi_r \Delta \phi_{\beta_0}(x_i, z_i)}{\|\Delta\|_2} - \kappa \frac{(x_i^T \Delta)^2}{\|\Delta\|_2} \geq \frac{\alpha_\tau c_\ell}{3} - c' \delta \left( \frac{(\alpha_\tau + \kappa) r K X}{r_0} \right) \sqrt{\frac{\log p}{n}}
\]

(58)

holds for all \( \Delta \in A(\delta) \) for any \( 0 < \delta \leq c_0 \sqrt{n / \log p} \), with probability at least \( 1 - c_1 \exp(-c_2 n - c_3 \delta^2 \log p) \) where \( c' \) is an absolute constant and \( c_1, c_2, \) and \( c_3 \) are constants which do not depend on \( n \) and \( p \). Finally, by applying the basic inequality \( 2ab \leq a^2 + b^2 \) to (58) with \( a = \sqrt{\frac{\nu}{6}} \) and \( b = \sqrt{c' \delta \left( \frac{(\alpha_\tau + \kappa) r K X}{r_0 \sqrt{\alpha_\tau c_\ell}} \right) \left( \frac{\log p}{n} \right)} \), we obtain the inequality

\[
\frac{1}{n} \sum_{i=1}^{n} (\alpha_\tau + \kappa) \frac{\psi_r \Delta \phi_{\beta_0}(x_i, z_i)}{\|\Delta\|_2} - \kappa \frac{(x_i^T \Delta)^2}{\|\Delta\|_2} \geq \frac{\alpha_\tau c_\ell}{4} - c'' \left( \frac{(\alpha_\tau + \kappa) r K X}{r_0 \sqrt{\alpha_\tau c_\ell}} \right)^2 \left( \frac{\log p}{n} \right),
\]

where \( c'' \) is an absolute constant. Thus the inequality (51) holds with \( \alpha_0 := \frac{\alpha_\tau c_\ell}{4} \) and \( \tau_0 := c'' \left( \frac{(\alpha_\tau + \kappa) r K X}{r_0 \sqrt{\alpha_\tau c_\ell}} \right)^2 \), with probability at least \( 1 - c_1 \exp(-c_2 n - c_3 \delta^2 \log p) \).

Case I - Step 5. Peeling argument for uniform result over \( \delta \)

We extend the inequality (51) with \( \delta = \|\Delta\|_1 / \|\Delta\|_2 \) for all \( \Delta \in \cup_{\delta \leq c_0 \sqrt{n / \log p}} A(\delta) \). Define \( g(\delta) := \tau_0 \delta^2 \left( \frac{\log p}{n} \right) \) and

\[
f_\Delta(x^n_1, z^n_1) := \frac{1}{n} \sum_{i=1}^{n} (\alpha_\tau + \kappa) \frac{\psi_r \Delta \phi_{\beta_0}(x_i, z_i)}{\|\Delta\|_2} - \kappa \frac{(x_i^T \Delta)^2}{\|\Delta\|_2}.
\]

Then by (51),

\[
P(\exists \Delta \in A(\delta); f_\Delta(x^n_1, z^n_1) < \alpha_0 - g(\delta)) \leq c_1 \exp(-c_2 n - c_3 \delta^2 \log p).
\]

We show that

\[
P(\exists \Delta \in \cup_{\delta \leq c_0 \sqrt{n / \log p}} A(\delta); f_\Delta(x^n_1, z^n_1) < \alpha_0 - 4g(\|\Delta\|_1 / \|\Delta\|_2)) \leq c'_1 \exp(-c'_2 n)
\]

(59)

for some \( c'_1 \) and \( c'_2 \) which do not depend on \( n \) and \( p \).
Let \( A_m := \{ \Delta \in B_2(r_0); 2^{m-1} \leq \| \Delta \|_1/\| \Delta \|_2 \leq 2^m \} \). \( \cup_{\delta \leq c_0 \sqrt{n/\log p}} A(\delta) \subseteq \bigcup_{m=1}^{M} A_m \) for \( M := \lceil \log_2 c_0 \sqrt{n/\log p} \rceil \). Thus by the union bound,

\[
P(\exists \Delta \in \cup_{\delta \leq c_0 \sqrt{n/\log p}} A(\delta); f_\Delta(x^n, z^n) < \alpha_0 - 4g(\| \Delta \|_1/\| \Delta \|_2))
\leq \sum_{m=1}^{M} P(\exists \Delta \in A_m; f_\Delta(x^n, z^n) < \alpha_0 - 4g(\| \Delta \|_1/\| \Delta \|_2))
\leq \sum_{m=1}^{M} P(\exists \Delta \in A(2^m); f_\Delta(x^n, z^n) < \alpha_0 - 4g(2^{m-1}))
\]

(60)

where the second inequality uses the fact \( g(\delta) \) is a monotone increasing function for \( \delta \geq 1 \) and \( A_m \subseteq A(2^m) \). Since \( g(2^{m-1}) = 4^{-1}g(2^m) \), we obtain

\[
P(\exists \Delta \in B_2(r_0); f_\Delta(x^n, z^n) < \alpha_0 - 4g(\| \Delta \|_1/\| \Delta \|_2))
\leq \sum_{m=1}^{M} P(\exists \Delta \in A(2^m); f_\Delta(x^n, z^n) < \alpha_0 - g(2^m))
\leq c_1 \exp(-c_2 n) \left( \sum_{m=1}^{\infty} \exp(-c_3 (2^m) \log p) \right)
\leq c'_1 \exp(-c_2 n)
\]

where the last inequality comes from the fact that the geometric sum converges.

**Case II**: \( \| \Delta \|_1 \geq c_0 \sqrt{n/\log p} \| \Delta \|_2 \)

We note that LHS of (50) is trivially lower-bounded by \(-\kappa \frac{1}{n} \sum_{i=1}^{n} \left( \frac{x_i^\top \Delta}{\| \Delta \|_2} \right)^2 \). Also from an application of Lemma A.7, we have

\[
\frac{1}{n} \sum_{i=1}^{n} (x_i^\top \Delta)^2 \leq \alpha' \| \Delta \|_2^2 + \tau' \frac{\log p}{n} \| \Delta \|_1^2, \quad \forall \Delta \in \mathbb{R}^p
\]

(61)

with probability at least \( 1 - 2\exp(-c'n) \) where \( \alpha', \tau', \) and \( c' \) are constants which only depend on model parameter \( K_X \) and not dimensions \((n, p)\). Let

\[
\gamma := \frac{1}{\kappa \tau' c_0^2} (\alpha_0 + \kappa \alpha').
\]
For any $\Delta$ such that $\|\Delta\|_1 \geq c_0 \sqrt{\frac{n}{\log p}} \|\Delta\|_2$, by definition of $\gamma$ we have,

$$(\alpha_0 + \kappa \alpha') = \gamma \kappa' c_0^2 \left( \frac{n}{\log p} \right) \left( \frac{\log p}{n} \right) \frac{\|\Delta\|_2}{\|\Delta\|_2} \leq \gamma \kappa' \frac{\|\Delta\|_2^2}{\|\Delta\|_2^2}.$$

Then, for all $\Delta \in \{\Delta; \|\Delta\|_2 \leq r_0, \|\Delta\|_1 \geq c_0 \sqrt{\frac{n}{\log p}} \|\Delta\|_2\}$,

$$-\kappa \sum_{i=1}^{n} \left( \frac{x_i^\top \Delta}{\|\Delta\|_2} \right)^2 \geq \alpha_0 - (\alpha_0 + \kappa \alpha') - \kappa' \frac{\|\Delta\|_2}{\|\Delta\|_2^2} \geq \alpha_0 - (1 + \gamma) \kappa \frac{\log p}{n} \frac{\|\Delta\|_1}{\|\Delta\|_2}$$

(62)

holds with the same high probability.

Finally, the result of Proposition 5.1 follows by combining (59) and (62) with $\alpha_{\ell} := \alpha_0$ and $\tau_{\ell} := 4\tau_0 \vee (1 + \gamma) \kappa \tau'$.

A.6 Proof of Proposition 5.2

$$\langle \nabla \mathcal{L}_n^s(\beta) - \nabla \mathcal{L}_n^s(\beta_0), \beta - \beta_0 \rangle = \frac{1}{n} \sum_{i=1}^{n} (\mu(x_i^\top \beta) - \mu(x_i^\top \beta_0)) x_i^\top (\beta - \beta_0)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mu' x_i^\top (\beta_0 + v x_i^\top (\beta - \beta_0)) (x_i^\top (\beta - \beta_0))^2$$

Then from the proof of Proposition 2 in [Negahban et al., 2012], there exist positive constants $\kappa_1$ and $\kappa_2$ such that

$$\langle \nabla \mathcal{L}_n^s(\beta) - \nabla \mathcal{L}_n^s(\beta_0), \beta - \beta_0 \rangle \geq \kappa_1 \|\Delta\|_2 \left( \|\Delta\|_2 - \kappa_2 \sqrt{\frac{\log p}{n}} \|\Delta\|_1 \right), \forall \beta \in \mathbb{B}_2(1; \beta_0)$$

with probability at least $1 - c_1 \exp(-c_2n)$, for some $c_1, c_2 > 0$. The result (24) follows from the basic arithmetic inequality $2ab \leq (a+b)^2$.

A.7 Proof of Theorem 5.1

The $\ell_1$ and $\ell_2$ error bounds of $\tilde{\beta}_\ell^H$ in (25) can be obtained as an immediate consequence of Theorem 1 in Loh (2017), since $\tilde{\beta}_\ell^H$ is guaranteed to be in the region where RSC condition holds by the
set-up of the problem. On the other hand, to obtain the error bounds of $\hat{\beta}_s^H$, we need to establish a different RSC condition, introduced by Negahban et al. (2012) as follows:

**Definition A.1** (restricted strong convexity in Negahban et al. (2012)). For a given set $S$, the loss function $L_n$ satisfies restricted strong convexity (RSC) with parameter $\alpha > 0$ if

$$L_n(\beta) - L_n(\beta_0) - \langle \nabla L_n(\beta_0), \beta - \beta_0 \rangle \geq \alpha \|\beta - \beta_0\|_2^2 \quad \text{for all } \beta - \beta_0 \in S.$$  

(63)

In the following Lemma A.8 we show that the RSC condition 5.1 implies the RSC condition in Negahban et al. (2012).

**Lemma A.8.** If $n \geq (32\tau s_0/\alpha) \log p$, then the RSC condition 5.1 implies (63) with parameter $\alpha/4$ and

$$S = \{\Delta \in \mathbb{R}^p; \|\Delta_S\|_1 \leq 3\|\Delta_S\|_1\} \cap \{\Delta \in \mathbb{R}^p; \|\Delta\|_2 \leq \delta\},$$

where $S \subseteq \{1, \ldots, p\}$ is the support of $\beta_0$ and $s_0 := |S|$.

Provided that Lemma A.8 is true and given the condition of $\lambda_s$ in Theorem 5.1, the $\ell_2$ error bound

$$\|\hat{\beta}_s^H - \beta_0\|_2 \leq \frac{8\sqrt{s_0} \lambda_s}{\alpha_s}$$

(64)

can be obtained by applying Theorem 1 in Negahban et al. (2012). Also it is well known that an error vector $\hat{\beta} - \beta_0$, where $\hat{\beta}$ is a solution of Lasso optimization problem, belongs to the cone $\{\Delta \in \mathbb{R}^p; \|\Delta_S\|_1 \leq 3\|\Delta_S\|_1\}$. Thus $\|\hat{\beta}_s^H - \beta_0\|_1 \leq 4\|\hat{\beta}_s^H - \beta_0\|_2 \leq 4\sqrt{s_0}\|\hat{\beta}_s^H - \beta_0\|_2$. Applying this inequality to (64) gives an $\ell_1$ bound. Now we present the proof of Lemma A.8.

**Proof of Lemma A.8.** For any $\beta$ such that $\beta - \beta_0 \in S$, we have,

$$L_n(\beta) = L_n(\beta_0) + \int \nabla L_n(\beta_0 + t(\beta - \beta_0))^T (\beta - \beta_0) dt$$

$$= L_n(\beta_0) + \nabla L_n(\beta_0)^T (\beta - \beta_0) + \int_0^1 t(\nabla L_n(\beta_0 + t(\beta - \beta_0)) - \nabla L_n(\beta_0))^T t(\beta - \beta_0) dt.$$  

(65)

By the RSC condition 5.1, for any $\beta \in B_2(\delta; \beta_0)$ it holds that

$$(\nabla L_n(\beta_0 + t(\beta - \beta_0)) - \nabla L_n(\beta_0))^T t(\beta - \beta_0) \geq t^2 \left( \alpha \|\beta - \beta_0\|_2^2 - \tau \left( \frac{\log p}{n} \right) \|\beta - \beta_0\|_1^2 \right).$$

(66)
Applying (66) to (65),
\[ \mathcal{L}_n(\beta) - \mathcal{L}_n(\beta_0) - \nabla \mathcal{L}_n(\beta_0) = \left( \alpha \left( \beta - \beta_0 \right) \right) - \tau \left( \frac{\log p}{n} \right) \left( \beta - \beta_0 \right) \right) \frac{dt}{t} \]
\[ = \frac{\alpha}{2} \left( \beta - \beta_0 \right) - \tau \left( \frac{\log p}{n} \right) \left( \beta - \beta_0 \right) \right) \frac{dt}{t} \]
Since \( \beta - \beta_0 \in \mathbb{S} \), \( \| \beta - \beta_0 \|_1 \leq 4\sqrt{s_0} \| \beta - \beta_0 \|_2 \). Therefore,
\[ \mathcal{L}_n(\beta) - \mathcal{L}_n(\beta_0) - \nabla \mathcal{L}_n(\beta_0) = \left( \alpha \left( \beta - \beta_0 \right) \right) - \tau \left( \frac{\log p}{n} \right) \left( \beta - \beta_0 \right) \right) \frac{dt}{t} \]
where the last inequality is from a given sample condition \( n \geq (32 \tau s_0 / \alpha) \log p \).

### A.8 Proof of Corollary 5.1

The Corollary 5.1 essentially follows from Proposition 5.1, Proposition 5.2, and Theorem 5.2. The main condition to verify is \( \| \nabla \mathcal{L}_n(\beta) \|_\infty, \| \nabla \mathcal{L}_n^s(\beta) \|_\infty = \mathcal{O}\left( \sqrt{\frac{\log p}{n}} \right) \) with high probability. To check this, we note that \( \| \nabla \mathcal{L}_n(\beta) \|_\infty \), for \( \mathcal{L}_n \in (\mathcal{L}_n^e, \mathcal{L}_n^s) \), has the form
\[ \| \nabla \mathcal{L}_n(\beta) \|_\infty = \max_{1 \leq j \leq p} \left| \frac{1}{n} \sum_{i=1}^{n} \xi_{ij} \right|, \]
where \( \xi_{ij} = \{(A'(h_{LN}(x_i^\top \beta_0)) - z_i)h'_{LN}(x_i^\top \beta_0) \}x_{ij} \) if \( \mathcal{L}_n = \mathcal{L}_n^e \), and \( \xi_{ij} = \{A'(x_i^\top \beta_0) - T(z_i) \}x_{ij} \) if \( \mathcal{L}_n = \mathcal{L}_n^s \). Also \( \mathbb{E}[\xi_{ij}] = 0, \forall i, j \) and \( (\xi_{ij})_{n=1}^{n} \) are independent for any \( j \in \{1, \ldots, p\} \).

From Lemma A.6, we have \( \| (A'(h_{LN}(x_i^\top \beta_0)) - z_i)h'_{LN}(x_i^\top \beta_0) \| \leq 1 \) and \( |A'(x_i^\top \beta_0) - T(z_i)| \leq 1 \) a.s. Thus \( \mathbb{E}[|\xi_{ij}|^{1/k}] \leq \mathbb{E}[|x_{ij}|^{1/k}] \leq \sqrt{k}K_X \) for any \( k \geq 1 \) by Assumption A1. In particular, \( \xi_{ij} \) is mean-zero sub-gaussian with parameter \( cK_X \) where \( c > 0 \) is an absolute constant, and \( \frac{1}{n} \sum_{i=1}^{n} \xi_{ij} \) is also sub-gaussian with parameter \( cK_X / \sqrt{n} \). By taking a union bound, for any \( t \geq 0 \) we have
\[ \mathbb{P} \left( \| \nabla \mathcal{L}_n(\beta) \|_\infty \geq t \sqrt{\frac{\log p}{n}} \right) \leq \exp(-t^2 \log p/c^2 K_X^2 + \log 2p) \]
Take \( t^2 = 4c^2 K_X^2 \). Then \( \| \nabla \mathcal{L}_n(\beta) \|_\infty \leq t \sqrt{\frac{\log p}{n}} \) with probability at least \( 1 - 1/p^2 \).
A.9 Proof of Theorem 5.2

First, for a given $\psi$, we let $\beta^\text{db} = \beta^\text{db}(\psi)$, $\hat{\Theta} = \hat{\Theta}(\psi)$, and $\Theta = \Theta(\psi)$ for ease of notation. For any fixed $j \in \{1, \ldots, p\}$, we have

$$
\hat{\beta}^\text{db} - \beta_{0j} = \hat{\beta}_j - \beta_{0j} - \hat{\Theta}_j^\top \left( \frac{1}{n} \sum_{i=1}^{n} \psi(x_i^\top \hat{\beta}, z_i) x_i \right).
$$

(67)

Let $\hat{\Delta} := \hat{\beta} - \beta_0$. By the Taylor expansion,

$$
\hat{\Theta}_j^\top \left( \frac{1}{n} \sum_{i=1}^{n} \psi(x_i^\top \hat{\beta}, z_i) x_i \right)
= n^{-1} \sum_{i=1}^{n} \left( \psi(x_i^\top \beta_0, z_i) + \psi'(v_i, z_i)(x_i^\top \hat{\beta} - x_i^\top \beta_0) \right) \hat{\Theta}_j^\top x_i
= n^{-1} \sum_{i=1}^{n} \left( \psi(x_i^\top \beta_0, z_i) + \psi'(x_i^\top \hat{\beta}, z_i)x_i^\top \hat{\Delta} + \{\psi'(v_i, z_i) - \psi'(x_i^\top \hat{\beta}, z_i)\}x_i^\top \hat{\Delta} \right) \hat{\Theta}_j^\top x_i
$$

for $v_i$ such that $|v_i - x_i^\top \hat{\beta}| \leq |x_i^\top (\hat{\beta} - \beta_0)|$.

First, we address the last term and show that it is $o_p(n^{-1/2})$.

$$
n^{-1} \sum_{i=1}^{n} \left\{ |\psi'(v_i, z_i) - \psi'(x_i^\top \hat{\beta}, z_i)| \right\} \lvert x_i^\top \hat{\Delta} \lvert \hat{\Theta}_j^\top x_i
\leq n^{-1} \sum_{i=1}^{n} |\psi'(v_i, z_i) - \psi'(x_i^\top \hat{\beta}, z_i)||x_i^\top \hat{\Delta}| |\hat{\Theta}_j^\top x_i| \tag{68}
$$

From $\text{A.7}$, $\psi'(t, z)$ is Lipschitz in $t$ with the Lipschitz constant $2L_\psi, \forall z$. Thus we have,

$$
|\psi'(v_i, z_i) - \psi'(x_i^\top \hat{\beta}, z_i)| \leq 2L_\psi |v_i - x_i^\top \hat{\beta}| \leq 2L_\psi |x_i^\top \beta_0 - x_i^\top \hat{\beta}|, \tag{69}
$$

and by combining (68), (69), we obtain

$$
\frac{1}{n} \sum_{i=1}^{n} (\psi'(v_i, z_i) - \psi'(x_i^\top \hat{\beta}, z_i)) x_i^\top \hat{\Delta} \hat{\Theta}_j^\top x_i \leq \frac{2L_\psi}{n} \sum_{i=1}^{n} (x_i^\top \hat{\Delta})^2 |\hat{\Theta}_j^\top x_i|
\leq \frac{2L_\psi}{n} \|X\hat{\Delta}\|_2^2 \max_{1 \leq i \leq n} |\hat{\Theta}_j^\top x_i|.
$$

Using the bound (61), $\frac{1}{n} \|X\hat{\Delta}\|_2^2 \leq \alpha' \|\hat{\Delta}\|_2^2 + \tau' \frac{\log p}{n} \|\hat{\Delta}\|_1^2$ w.h.p for $\alpha'$ and $\tau'$ which do not depend
on \((n, p)\) under \(\text{A1}^{\beta}\) and thus we have \(|X\hat{\Delta}|^2/n = O_p(s_0(\log p/n)) + O_p(s_0^2(\log p/n)^2) = o_p(n^{-1/2})\) by the rate assumption of \(s_0\) in \(\text{A6}\). Also,

\[
\max_{1 \leq i \leq n} |x_i^\top \hat{\Theta}_j| \leq \max_{1 \leq i \leq n} |x_i^\top (\hat{\Theta}_j - \Theta_j)| + \max_{1 \leq i \leq n} |x_i^\top \Theta_j| \leq \max_{1 \leq i \leq n} \|x_i\| \|\hat{\Theta}_j - \Theta_j\|_1 + \|X\Theta_j\|_\infty = O_p(1).
\]

It holds because \(\max_{i,j} |x_{ij}| = O_p(\sqrt{\log np}) = O_p(\sqrt{\log p})\) by \(p \gg n\) in the regime of interest and \(\text{A1}^\beta\) and \(\|\hat{\Theta}_j - \Theta_j\|_1 = o_p(\sqrt{1/\log p})\) from the assumption about \(\hat{\Theta}\), and \(\|X\Theta_j\|_\infty = O_p(1)\) from \(\text{A6}\). Therefore,

\[
\frac{1}{n} \|X\hat{\Delta}\|_2^2 \|X\hat{\Theta}_j\|_\infty = o_p(n^{-1/2}), \tag{70}
\]

and we have,

\[
\hat{\Theta}_j^\top \psi_n(\hat{\beta}) = \frac{1}{n} \sum_{i=1}^n \left( \psi(x_i^\top \hat{\beta}_0, z_i) + \psi'(x_i^\top \hat{\beta}, z_i)x_i^\top \hat{\Delta} \right) \hat{\Theta}_j^\top x_i + o_p(n^{-1/2}). \tag{71}
\]

Combining (67) with (71),

\[
\hat{\beta}_{j}^{db} - \beta_{0j} = \hat{\beta}_j - \beta_{0j} - n^{-1} \sum_{i=1}^n \left( \psi(x_i^\top \hat{\beta}_0, z_i) + \psi'(x_i^\top \hat{\beta}, z_i)x_i^\top \hat{\Delta} \right) \hat{\Theta}_j^\top x_i + o_p(n^{-1/2})
\]

\[
= e_j^\top \hat{\Delta} - \hat{\Theta}_j^\top \psi_n(\beta_0) - n^{-1} \sum_{i=1}^n \left( \psi'_I(x_i^\top \hat{\beta}) + \psi'_R(x_i^\top \hat{\beta}, z_i) \right) \hat{\Theta}_j^\top x_i x_i^\top \hat{\Delta} + o_p(n^{-1/2}),
\]

where we use the relationship \(\psi'(t, z) = \psi'_I(t) + \psi'_R(t, z)\) in (28). Recalling the definition \(\psi'_{I,n}(\beta) := n^{-1} \sum_{i=1}^n \psi'_I(x_i^\top \beta)x_i x_i^\top\),

\[
n^{-1} \sum_{i=1}^n \psi'_I(x_i^\top \hat{\beta}) \hat{\Theta}_j^\top x_i x_i^\top \hat{\Delta} = \hat{\Theta}_j^\top \left( n^{-1} \sum_{i=1}^n \psi'_I(x_i^\top \hat{\beta}) x_i x_i^\top \right) \Delta = \hat{\Theta}_j^\top \psi'_{I,n}(\hat{\beta}) \Delta
\]

thus we have

\[
\hat{\beta}_{j}^{db} - \beta_{0j} = -\hat{\Theta}_j^\top \psi_n(\beta_0) - n^{-1} \sum_{i=1}^n \psi'_R(x_i^\top \hat{\beta}, z_i) \hat{\Theta}_j^\top x_i x_i^\top \Delta + \hat{\Delta}^\top (e_j - \psi'_{I,n}(\hat{\beta}) \hat{\Theta}_j) + o_p(n^{-1/2}).
\]

We will show that the first term \(\sqrt{n} \hat{\Theta}_j^\top \psi_n(\beta_0)\) will converge to the normal distribution. Both remainder terms (Term-I and Term-II) need to be \(o_p(n^{-1/2})\). For the second remainder term (Term-
II), we have \( |\hat{\Delta}^T (e_j - \psi'_{\hat{\nu}, i, n}(\hat{\beta}) \hat{\Theta}_j)| \leq \|\hat{\Delta}\|_1 |e_j - \psi'_{\hat{\nu}, n}(\hat{\beta}) \hat{\Theta}_j\|_\infty = O_p(s_0 \sqrt{\log p/n})O_p(\sqrt{\log p/n}) = o_p(n^{-1/2}) \) by the rate condition A6 and the assumptions in the theorem. Now we address the first remainder term (Term-I):

\[
n^{-1} \sum_{i=1}^{n} \psi'_{R}(x_i^T \hat{\beta}, z_i) \hat{\Theta}_j x_i x_i^T \hat{\Delta} = n^{-1} \sum_{i=1}^{n} \left( \psi'_{R}(x_i^T \beta, z_i) + \left( \psi'_{R}(x_i^T \hat{\beta}, z_i) - \psi'_{R}(x_i^T \beta, z_i) \right) \right) \hat{\Theta}_j x_i x_i^T \hat{\Delta}. \tag{72}
\]

We need the following Lemma which establishes a kind of sparse eigenvalue condition.

**Lemma A.9.** Let \( E \in \mathbb{R}^{n \times n} \) be a random matrix which has a representation \( E = n^{-1} \sum_{i=1}^{n} e_i x_i x_i^T \), for random \( (e_i)_{i=1}^{n} \) such that \( \mathbb{E}[e_i | x_i] = 0 \) and \( |e_i| \leq c_e \) a.s., and \( x_i \) satisfies the sub-gaussian tail condition with the parameter \( K_X \) for all \( i \). Then for any \( s, s' \geq 1 \), if \( n \geq C(s + s') \log p \) for an absolute constant \( C \), there exist constants \( c_1, c_2 > 0 \) such that

\[
P \left( \sup_{u \in B_1(\sqrt{s}) \cap B_2(1)} |u^T E v| \geq c_1 n^{-1/2} \sqrt{(s + s') \log p} \right) \leq \frac{c_2}{p^{s+s'}}.
\]

The proof of the Lemma is presented at the end of this section. Now we apply Lemma A.9 to show that \( n^{-1} \sum_{i=1}^{n} \psi'_{R}(x_i^T \beta, z_i) \hat{\Theta}_j x_i x_i^T \hat{\Delta} \) is \( o_p(n^{-1/2}) \). We have,

\[
n^{-1} \sum_{i=1}^{n} \psi'_{R}(x_i^T \beta, z_i) \hat{\Theta}_j x_i x_i^T \hat{\Delta} = \hat{\Theta}_j \left( n^{-1} \sum_{i=1}^{n} \psi'_{R}(x_i^T \beta, z_i) x_i x_i^T \right) \hat{\Delta} = \hat{\Theta}_j E^R \hat{\Delta}
\]

where we define \( E^R := n^{-1} \sum_{i=1}^{n} \psi'_{R}(x_i^T \beta, z_i) x_i x_i^T \). From the condition of \( \hat{\beta} \) in Theorem 5.2 we have \( \hat{\Delta}/\|\hat{\Delta}\|_2 \in B_1(\sqrt{c s_0}) \cap B_2(1) \) for a constant \( c > 0 \). Also, \( \|\hat{\Theta}_j\|_1 \leq \|\hat{\Theta}_j - \Theta_j\|_1 + \|\Theta_j\|_1 \) and \( \|\hat{\Theta}_j\|_2 \geq \|\Theta_j\|_2 - \|\hat{\Theta}_j - \Theta_j\|_2 \leq \|\Theta_j\|_2 - \|\hat{\Theta}_j - \Theta_j\|_1 \). Define an event \( \mathcal{E}_n := \{\|\hat{\Theta}_j - \Theta_j\|_1 \leq 0.5\|\Theta_j\|_2\} \).

Then

\[
\frac{\|\hat{\Theta}_j\|_1}{\|\hat{\Theta}_j\|_2} \leq \frac{\|\Theta_j\|_1 + \|\hat{\Theta}_j - \Theta_j\|_1}{\|\Theta_j\|_2 - \|\hat{\Theta}_j - \Theta_j\|_1} \leq 3 \frac{\|\Theta_j\|_1}{\|\Theta_j\|_2}
\]

on \( \mathcal{E}_n \). We note that \( \Theta_j \) is at most \( s_s \) sparse vector, recalling the definition \( s_s := \max_{1 \leq j \leq p} \|\Theta_{j \cdot j}\| \). Also, \( \|\Theta\|_2 \geq 1 \), since \( \|\Theta\|_2 = \lambda_{\min}^{-1}(\mathbb{E}[\psi'_{j}(x^T \beta_0)x x^T]) \) and the minimum eigenvalue of \( \mathbb{E}[\psi'_{j}(x^T \beta_0)x x^T] \) can be shown to be bounded above and also bounded below by a positive constant. More concretely,
for any unit vector $u$,

$$u^T \mathbb{E}[\psi'_j(x^T \beta_0) xx^\top] u \geq \mathbb{E}[\psi'_j(x^T \beta_0)(x^T u)^2 I\{ |x^T \beta_0| \leq \tau_c \}] \geq \inf_{|t| \leq \tau_c} \psi'_j(t) c_\ell / 2,$$

and

$$u^T \mathbb{E}[\psi'_j(x^T \beta_0) xx^\top] u \leq C \psi \mathbb{E}[(x^T u)^2] \leq 2C \psi K_X^2$$

for $\tau_c := (2c_b^2 K_X^2 \log(16K_X^2/c_\ell))^1/2$ using Lemma A.5. Assumptions A1 and A7 where $c_b$ is a constant such that $\|\beta_0\|_2 \leq c_b$, which exists by the condition $\|\beta_0\|_2 = O(1)$.

Thus $\|\Theta_j\|_1 \leq \sqrt{s_0 / \tau_1} \|\Theta_j\|_2$, $\Theta_j / \|\Theta_j\|_2 \in \mathbb{B}_1(\sqrt{9(s_0 + 1)}) \cap \mathbb{B}_2(1)$ on $\mathcal{E}_n$. Also, we have $\mathbb{P}(\mathcal{E}_n) \rightarrow 1$ by $\|\Theta_j - \Theta_j\|_1 = o_p(1/\sqrt{\log p})$ and $\|\Theta_j\|_2 \approx 1$. Then on $\mathcal{E}_n$,

$$|\widehat{\Delta}^T E^R \Theta_j| \leq \|\widehat{\Delta}\|_2 \|\Theta_j\|_2 \sup_{u \in \mathbb{B}_1(\sqrt{s_0}) \cap \mathbb{B}_2(1), v \in \mathbb{B}_1(\sqrt{\tau_1}) \cap \mathbb{B}_2(1)} |u^T Ev|,$$

for $s = cs_0$ and $s' = 9(s_0 + 1)$. Since $\|\widehat{\Delta}\|_2 = O_p(\sqrt{s_0 \log p / n})$ and $\|\Theta_j\|_2 = O_p(1)$,

$$\|\widehat{\Delta}\|_2 \|\Theta_j\|_2 \sup_{u \in \mathbb{B}_1(\sqrt{s_0}) \cap \mathbb{B}_2(1), v \in \mathbb{B}_1(\sqrt{\tau_1}) \cap \mathbb{B}_2(1)} |u^T Ev| = O_p(\sqrt{s_0 \log p / n}) \cdot O_p(\sqrt{(s_0 + s_1) \log p / n}) = o_p(n^{-1/2})$$

on $\mathcal{E}_n$, where the last inequality is from the rate conditions $s_0, s_1 = o(\sqrt{n} / \log p)$ from A6. Since $\mathbb{P}(\mathcal{E}_n) \rightarrow 1$, we conclude $|\widehat{\Delta}^T E^R \Theta_j| = o_p(n^{-1/2})$.

For the second term in (72),

$$n^{-1} \sum_{i=1}^n \left| \psi'_R(x_i^T \hat{\beta}, z_i) - \psi'_R(x_i^T \beta_0, z_i) \right| |\hat{\Theta}_j^T x_i| |x_i^T \Delta| \leq L_\psi \|X \hat{\Theta}_j\|_\infty 1/n \|X \Delta\|^2_2$$

where we use A7 that $\psi'_R(t, z)$ is $L_\psi$-Lipschitz in $t$ for any $z$. Then from (70), we have that the second term is $o_p(n^{-1/2})$. Therefore, combining the results we obtain

$$n^{-1} \sum_{i=1}^n \psi'_R(x_i^T \hat{\beta}, z_i) \hat{\Theta}_j^T x_i |x_i^T \Delta| = o_p(n^{-1/2}).$$
So far, we have obtained,
\[ \hat{\beta}_{db}^{j} - \beta_{0j} = -\hat{\Theta}^{\top} \psi_{n}(\beta_{0}) + o_{p}(n^{-1/2}). \]

It remains to show that
\[ \sqrt{n} \hat{\Theta}^{\top} \psi_{n}(\beta_{0}) \]
\[ \sqrt{(\Theta^{\top} \mathbb{E}[\psi(x^{\top} \beta_{0}, z)xx^{\top}]\Theta)_{jj}} \]
\[ \overset{d}{\to} N(0, 1). \]

By CLT,
\[ \frac{1}{\sigma \sqrt{n}} \sum_{i=1}^{n} \psi(x_{i}^{\top} \beta_{0}, z_{i})x_{i}^{\top} \Theta_{j} \to N(0, 1) \]
where
\[ \sigma^{2} = \text{Var}(\psi(x^{\top} \beta_{0}, z)x^{\top} \Theta_{j}) = \mathbb{E}[\psi(x^{\top} \beta_{0}, z)^2 (x^{\top} \Theta_{j})^2] \]
since \( \mathbb{E}[\psi(x^{\top} \beta_{0}, z)] = 0 \) by (27). Thus it is sufficient to show \( \sqrt{n} \hat{\Theta}^{\top} \psi_{n}(\beta_{0}) = \sqrt{n} \Theta^{\top} \psi_{n}(\beta_{0}) + o_{p}(1) \) to conclude. Indeed, we have,
\[ |\sqrt{n}(\hat{\Theta}_{j} - \Theta_{j})^{\top} \psi_{n}(\beta_{0})| \leq \sqrt{n}\|\hat{\Theta}_{j} - \Theta_{j}\|_{1}\|\psi_{n}(\beta_{0})\|_{\infty} = o_{p}(1). \]

This holds because, by the condition of \( \hat{\Theta} \), \( \|\hat{\Theta}_{j} - \Theta_{j}\|_{1} = o_{p}(\sqrt{1/\log p}) \) and \( \|\psi_{n}(\beta_{0})\|_{\infty} = O_{p}(\sqrt{\log p/n}) \).

Recalling the definition of \( \psi_{n} \), we have,
\[ \|\psi_{n}(\beta_{0})\|_{\infty} = \max_{1 \leq j \leq p} \left| \frac{1}{n} \sum_{i=1}^{n} \psi(x_{i}^{\top} \beta_{0}, z_{i})x_{ij} \right|. \]

From A7 we have \( \|\psi\|_{\infty} \leq C_{\psi} \). Also \( \mathbb{E}[\psi(x_{i}^{\top} \beta_{0}, z_{i})x_{ij}] = 0 \) by (27). Thus \( n^{-1} \sum_{i=1}^{n} \psi(x_{i}^{\top} \beta_{0}, z_{i})x_{ij} \) is mean-zero sub-gaussian with a parameter \( cC_{\psi} K_{X} / \sqrt{n} \) for an absolute constant \( c > 0 \). Thus from the result on the maximum of sub-gaussian variables, \( \|\psi_{n}(\beta_{0})\|_{\infty} = O_{p}(\sqrt{\log p/n}) \).

Proof of Lemma A.9: First we establish the following inequality. For any \( s \geq 1 \), there exists \( c_{0} > 0 \)
which do not depend on dimensions \((n, p)\) such that
\[
\mathbb{P} \left( \sup_{u \in B_0(\tilde{s}) \cap B_2(1)} |u^\top Eu| \geq c_0 \sqrt{\frac{\log p}{n}} \right) \leq c_2/p^{\tilde{s}},
\]
holds where \(c_2\) is an absolute constant.

Since for any unit vector \(u \in \mathbb{R}^p\) and \(i\), \(\mathbb{E}[e_i(x_i^\top u)^2] = 0\) and \(\mathbb{E}[|e_i(x_i^\top u)^2|^{k}] \leq c_{e}E[(x_i^\top u)^{2k}]^{1/k} \leq 2c_{e}K_X^2k, \forall k \geq 1, e_i(x_i^\top u)^2\) is mean-zero sub-exponential whose parameter is \(cc_{e}K_X^2\) for an absolute constant \(c\). From Bernstein’s inequality, for every \(t \geq 0\), we have
\[
\mathbb{P}(|u^\top Eu| \geq tc_{e}K_X^2) \leq \exp(-c'_{e}n(t^2 \wedge t)),
\]
where \(c' > 0\) is an absolute constant. Note,
\[
B_0(\tilde{s}) \cap B_2(1) = \bigcup_{k=0}^{\tilde{s}} \{v \in B_2(1); \|v\|_0 = k \}
= \bigcup_{k=0}^{\tilde{s}} \bigcup_{S;|S|=k} \{v \in B_2(1); \text{supp}(v) = S \}.
\]
Taking a union bound,
\[
\mathbb{P}(\sup_{u \in B_0(\tilde{s}) \cap B_2(1)} |u^\top Eu| \geq tc_{e}K_X^2) \leq \sum_{k=0}^{\tilde{s}} \sum_{S;|S|=k} P(\|E_{S,S}\|_2 \geq tc_{e}K_X^2),
\]
where \(E_{S,S}\) is a sub-matrix of \(E\) supported on \(S\). Letting \(\mathcal{N}_\epsilon\) is an \(\epsilon\)-net of the sphere \(S^{\lfloor \tilde{s} \rfloor - 1}\), we have
\[
\|E_{S,S}\|_2 \leq \frac{1}{1 - 2\epsilon} \sup_{v \in \mathcal{N}_\epsilon} |v^\top E_{S,S}v|
\]
by the covering argument (e.g. Vershynin (2018)). Take \(\epsilon = 1/4\). Then,
\[
\mathbb{P}(\sup_{u \in B_0(\tilde{s}) \cap B_2(1)} |u^\top Eu| \geq tc_{e}K_X^2) \leq \sum_{k=0}^{\tilde{s}} \binom{p}{k} 9^k P(|v^\top E_{S,S}v| \geq tc_{e}K_X^2/2)
\leq 2 \exp(-c''_{e}n(t^2 \wedge t^2) + \tilde{s} \log(9p)),
\]

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where we use the bounds $|\mathcal{N}_{1/4}| \leq 9^{|S|}$, $(p_k^*) \leq p^k$, and (74), and $c''$ is a universal constant. Taking $t^2 = 2\hat{s}\log(9p)/c''n$, we have

$$
P(\sup_{u \in B_0(\hat{s}) \cap B_2(1)} |u^\top E u| \geq c_e K_X^2 \sqrt{2\hat{s}\log p\over n} ) \leq 2/(9p^\hat{s}),$$

given a sample size condition $n \geq 2\hat{s}\log 9p/c''$. The we obtain the inequality (73) with $c_0 = 4c_e K_X^2$ and $c_2 = 2/9$, where we use the inequality $p^5 \geq 9p$ for $p \geq 2$.

Now we show that on the event that

$$
\sup_{u \in B_1(\sqrt{s}) \cap B_2(1)} |u^\top E u| \leq \frac{c_0}{4}(s + s')^\frac{\log p}{n},
$$

we have

$$
\sup_{u \in B_1(\sqrt{s}) \cap B_2(1), v \in B_1(\sqrt{s'}) \cap B_2(1)} |u^\top E v| \leq c_1 \sqrt{(s + s') \log p \over n},
$$

where $c_1$ is a multiple of $c_0$.

From Lemma 11 in Loh and Wainwright (2012), we have

$$
B_1(\sqrt{s}) \cap B_2(1) \subseteq 3\text{conv}(B_0(s) \cap B_2(1))
$$

where conv$(D)$ denotes a convex hull of $D \subseteq \mathbb{R}^p$. Using this Lemma, for any $u \in B_1(\sqrt{s}) \cap B_2(1)$ and $v \in B_1(\sqrt{s'}) \cap B_2(1)$, we have the following representation

$$
u = \sum_{i=1}^p \alpha_i u_i \quad \text{and} \quad v = \sum_{j=1}^p \beta_j v_j
$$

for $\alpha_i \geq 0, \beta_j \geq 0, u_i, v_j$ such that $\sum_i \alpha_i = \sum_j \beta_j = 1$, $u_i \in B_0(s) \cap B_2(3)$ and $v_j \in B_0(s') \cap B_2(3)$, $\forall i, j$. Then,

$$
u^\top E \nu = (\sum_{i=1}^p \alpha_i u_i)^\top E (\sum_{j=1}^p \beta_j v_j) = \sum_{i,j} 9\alpha_i \beta_j u_i^\top E v_j
$$

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for $\tilde{u}_i, \tilde{v}_j \in \mathbb{B}_0(s + s') \cap \mathbb{B}_2(1) \forall i, j$. Since,

$$|\tilde{u}_i^T \tilde{v}_j| \leq \frac{1}{2} \left\{ |(\tilde{u}_i + \tilde{v}_j)^T E(\tilde{u}_i + \tilde{v}_j)| + |\tilde{u}_i^T E\tilde{u}_i| + |\tilde{v}_j^T E\tilde{v}_j| \right\},$$

by the basic inequality $2x^T y = (x + y)^T E(x + y) - x^T Ex - y^T Ey$ for any $x, y \in \mathbb{R}^p$, we have

$$\sup_{u \in \mathbb{B}_1(\sqrt{s}) \cap \mathbb{B}_2(1), \ \ v \in \mathbb{B}_1(\sqrt{s'}) \cap \mathbb{B}_2(1)} |u^T E v| \leq \sum_{i,j} \frac{27}{2} (\alpha_i \beta_j) \sup_{u \in \mathbb{B}_0(s + s') \cap \mathbb{B}_2(1)} |u^T E u|$$

$$\leq \sum_{i,j} (\alpha_i \beta_j) \left( \frac{27}{2} c_0 \sqrt{(s + s') \log p/n} \right) = c_1 \sqrt{(s + s') \log p/n}$$

where for the second inequality we use (75) and $c_1 = 27c_0/2$. Thus (76) holds with probability at least $1 - c_2/p^{s+s'}$.

(A.10) Construction of an approximate inverse of Fisher information matrix using node-wise regression

First we let $W(\beta) := \text{diag}\{\psi_i'(x_i^T \beta)\}_{i=1}^n$. We note the square root of $W(\beta)$ exists since $\psi_i(t) \geq 0$ for all $t$. Following the node-wise lasso construction in [Van de Geer et al. (2014)], we define

$$\hat{\gamma}_j := \arg \min_{\gamma \in \mathbb{R}^p} \frac{1}{2n} \|W(\hat{\beta})^{1/2} X_j - W(\hat{\beta})^{1/2} X_{-j} \gamma\|_2^2 + \lambda_j \|\gamma\|_1$$

$$\hat{\tau}_j^2 := \|W(\hat{\beta})^{1/2} X_j - W(\hat{\beta})^{1/2} X_{-j} \hat{\gamma}\|_2^2/n + \lambda_j \|\hat{\gamma}_j\|_1.$$

We construct $\hat{\Theta}(\psi)$ by taking $\hat{\Theta}(\psi)_j^T := \hat{\tau}_j^{-2}[-\hat{\gamma}_{j,1}, \ldots, 1, -\hat{\gamma}_{j,p}] \in \mathbb{R}^{1 \times p}$.

Lemma A.10 (Theorem 3.2 in [Van de Geer et al. (2014)]. Assume A1-A4, A6-A7 and $\lambda_j \asymp \sqrt{\log p/n}$ for all $j$. In addition we assume there exists $c_X > 0$ such that $\|x_i\|_\infty \leq c_X$ a.s. for all $i$.

Then for any $j \in \{1, \ldots, p\}$, we have

$$\|\hat{\Theta}_j(\psi) - \Theta_j(\psi)\|_1 = o_p(1/\sqrt{\log p}), \quad \|\hat{\Theta}_j(\psi) - \Theta_j(\psi)\|_2 = o_p(n^{-1/4}).$$

Proof. The result follows by checking the conditions of Theorem 3.2 in [Van de Geer et al. (2014)].