EXPONENTIAL MIXING AND SMOOTH CLASSIFICATION OF COMMUTING EXPANDING MAPS

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ABSTRACT. We show that genuinely higher rank expanding actions of abelian semigroups on compact manifolds are $C^\infty$-conjugate to affine actions on infra-nilmanifolds. This is based on the classification of expanding diffeomorphisms up to Hölder conjugacy by Gromov and Shub, and is similar to recent work on smooth classification of higher rank Anosov actions on tori and nilmanifolds. To prove regularity of the conjugacy in the higher rank setting, we establish exponential mixing of solenoid actions induced from semigroup actions by nilmanifold endomorphisms, a result of independent interest. We then proceed similar to the case of higher rank Anosov actions.

1. INTRODUCTION

1.1. Smooth classification of higher rank expanding actions. Let $G$ be a connected and simply connected nilpotent Lie group. Let $\text{End}(G)$ and $\text{Aut}(G)$ denote the semigroup of endomorphisms of $G$ and group of automorphisms of $G$ respectively. Let $\Gamma \subset G$ be a discrete subgroup such that the quotient space $\Gamma \backslash G$ is compact. Then we call the compact manifold $\Gamma \backslash G$ a nilmanifold. A compact manifold $M$ is called an infra-nilmanifold if it admits a finite nilmanifold covering $\Gamma \backslash G$. If $A \in G \times \text{End}(G)$ satisfies that $A(\Gamma) \subset \Gamma$, then $A$ induces a smooth map on $\Gamma \backslash G$, a so-called affine nilendomorphism of $\Gamma \backslash G$. The $\text{End}(G)$ component of $A$ is called the linear part of $A$. A map on $M$ is called an affine infra-nilendomorphism if it lifts to an affine nilendomorphism of a finite nilmanifold covering $\Gamma \backslash G$.

Let $M$ be a compact smooth manifold endowed with a Riemannian metric $\| \cdot \|$. We say a smooth map

$$\tau : M \to M$$

is expanding if there exists a constant $c > 1$ such that for any $v$ in the tangent bundle $TM$ we have

$$\| D\tau(v) \| \geq c \| v \|.$$
It is natural to ask if one can classify all expanding maps, up to conjugacy. Based on the work of Shub [31], Gromov [9] found the best possible answer to this question: he proved that every expanding map on a compact manifold is topologically conjugate to an \textit{affine infra-nil endomorphism}. In other words, for a finite cover $\bar{M}$ of $M$, there exists a compact nilmanifold $\Gamma \sim G$, a homeomorphism $\phi: \bar{M} \to \Gamma \sim G$ and an affine nilendomorphism $A$ on $\Gamma \sim G$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\bar{M} & \xrightarrow{\phi} & \Gamma \sim G \\
\downarrow \tau & & \downarrow A \\
\bar{M} & \xrightarrow{\phi} & \Gamma \sim G.
\end{array}
$$

Moreover, $\phi$ is bi-Hölder. The linear part of $A = \phi \circ \tau \circ \phi^{-1}$ is given by the induced action of $\tau$ on the fundamental group $\Gamma$ of $M$. We remark that any expanding map has at least one fixed point $p \in M$ [30, Theorem 1] so that the induced map on $\pi_1(M, p)$ is well defined. Finally, if $M = \Gamma \sim G$ then $\phi$ can be chosen to be homotopic to the identity $\text{Id}$. We remark that Dekimpe clarified some of the algebraic issues with the notions of affine nilendomorphisms and the proof of the Gromov and Shub classification result [5].

We remark that in dimensions at least 5, passing to a finite cover $\bar{M}$ of $M$ if necessary, we can further assume that $\bar{M}$ is actually diffeomorphic to a nilmanifold $\Gamma \sim G$. Indeed, exotic differentiable structures on nilmanifolds always become standard on a finite cover, cf. the Appendix by J. Davis in [8].

In general, an expanding map is not $C^1$-conjugate to an affine infra-nilendomorphism. One can construct simple examples by perturbing a suitable affine example locally at a fixed point, changing the derivative at the fixed point. Furthermore, Farrell and Jones constructed expanding maps on tori with exotic differentiable structures [6]. For actions by semigroups $\mathbb{Z}^+_k$ of higher rank, i.e., $k \geq 2$, the situation changes dramatically. Note that more than higher rank of the acting semigroup is needed as one can always take product actions of individual non-algebraic expanding maps. We also have to avoid finite symmetries that disguise a product of rank one actions. Hence we make the following definition.

**Definition 1.1.** Let $\rho$ be a $C^\infty \mathbb{Z}^+_k$ ($k \geq 2$) action on a manifold $M$. We call $\rho$ \textbf{genuinely higher rank} if for all finite index sub-semigroups $Z$ of $\mathbb{Z}^k$, no continuous quotient of any finite extension of the $Z$-action factors through a finite extension of a $\mathbb{Z}_+$ action.

It is easy to show that after passing to a finite index sub-semigroup and a finite cover, a $\mathbb{Z}^+_k$ action $\rho$ with an expanding map is $C^0$ conjugate to an affine action $\rho_I$ on a nilmanifold via the conjugacy $\phi$ we get from a single expanding map, cf. Lemma 2.3. We will use genuinely higher rank in this paper to show that the Hölder conjugacy $\phi$ is actually $C^\infty$.

We summarize the main result of this paper as follows:
**Theorem 1.2.** Let \( \rho \) be a \( C^\infty Z^k_+ \) (\( k \geq 2 \)) action on a compact manifold \( M \). Suppose that \( \rho \) is genuinely higher rank and contains an expanding map \( \rho(a) \), for some \( a \in Z^k_+ \). Then \( M \) is diffeomorphic to an infra-nilmanifold \( \mathcal{M} \) and \( \rho \) is \( C^\infty \) conjugate to a \( Z^k_+ \) action \( \rho_1 \) on \( \mathcal{M} \) by affine nilendomorphisms.

**1.2. Related results.** Rigidity of higher rank actions on compact manifolds has been studied in different contexts. For Anosov actions, Rodriguez Hertz [26] classified \( Z^k \), \( k \geq 2 \), actions on tori containing one Anosov element and satisfying certain additional conditions. His work required that the rank \( k \) of the action is comparable to the dimension of the torus. Kalinin, Fisher and Spatzier [8] proved that if a \( Z^k \) action \( \alpha \) on a torus or nilmanifold is genuinely higher rank and contains “many” Anosov elements, then it is \( C^\infty \) conjugate to affine actions. Later Rodriguez Hertz and Wang [27] obtained the optimal result for Anosov actions on nilmanifolds by showing that existence of a single Anosov element implies existence of “many” Anosov elements. Here by saying “many” Anosov elements we mean that there is an Anosov element in each so-called Weyl chamber of Lyapunov exponents of \( \alpha \). We refer the reader to the introduction of [7] and to [32] for brief surveys of results and methods in the classification of higher rank Anosov actions.

**1.3. Exponential mixing of nilendomorphisms, expanding maps and their solenoids.** We will apply techniques similar to those in [8] and [27] to show the conjugacy \( \phi \) is \( C^\infty \). The first difficulty here is that these actions are not invertible, so the Weyl chambers of the Lyapunov exponents of \( \rho \) are only indirectly defined, and not in terms of the dynamical behavior (slow exponential growth) of actual elements close to the respective Weyl chamber walls. To overcome this difficulty, we shall extend the action \( \rho \) to the solenoid \( \mathcal{S}(M) \) of \( M \), defined below in detail in §2.4. Basically one wants to invert a covering map from a space to itself by considering the space of all possible orbits on which one has a topological inverse. As the future orbits are well defined this becomes the space of pasts. One can easily generalize this construction to semigroups generated by commuting covering maps. As discussed in [15], the solenoid has a completely algebraic description in terms of a \( p \)-adicification of the space. As it turns out, we will actually never need the original notion of the space of pasts and will work directly with the algebraic definition.

The notion of solenoid was used by Williams [36] to study expanding attractors, and also by Katok and Spatzier in [15] to prove measure rigidity statements. This makes \( \rho \) (and \( \rho_1 \)) a partially hyperbolic \( Z^k \) action on \( \mathcal{S}(M) \). According to [15], the Lyapunov exponents and Weyl chambers of both \( \rho \) and \( \rho_1 \) are well defined. This allows us to proceed as in the first step of the proof of global rigidity for Anosov actions.

The crucial next step in the proof of Theorem 1.2 is to prove an exponential mixing result for the action of \( \rho \) on the solenoid \( \mathcal{S}(M) \). This is the main novelty in this paper, and has independent interest. We actually prove such mixing quite generally for semigroups of endomorphisms of nilmanifolds. Indeed, passing to a finite cover, we may identify \( M \) as a nilmanifold \( \Gamma \sim G \).
By the theory of nilpotent Lie groups, there exists a nilpotent algebraic group \( N \) over \( \mathbb{Q} \) such that the nilpotent Lie group \( G = N(\mathbb{R}) \) and \( \Gamma = N(\mathbb{Z}) \). We will see later that the solenoid \( \mathcal{S}(M) \) of \( M \) can be identified with an \( S \)-adic nilmanifold \( N(\mathbb{Z}) \sim N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) \) for a finite subset \( S \) of primes. Let \( \mu \) denote the probability measure on \( \mathcal{S}(M) \) induced by the Haar measure on \( N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) \).

Note that \( \mu \) is a \( \mathbb{Z}^k \) action on \( \mathcal{S}(M) \) by nilendormorphisms. Since \( \mathcal{S}(M) \) exists a subgroup \( \rho \) of \( \mathbb{Z}^k \) such that the nilpotent Lie group \( G = N(\mathbb{R}) \times \mathcal{S}(M) \) denotes the space of \( \theta \)-Hölder functions defined on \( M \) and \( \mathcal{F}(M) \) respectively. Let \( \| \cdot \|_\theta \) denote the \( \theta \)-Hölder norm. Let \( \rho_1 \) denote a \( \mathbb{Z}^k \) action on \( \mathcal{S}(M) \) by nilendormorphisms. Suppose that \( \rho_1(a) \) acts ergodically on \( \mathcal{S}(M) \) for every \( a \in \mathbb{Z}^k \). Then there exists a constant \( \eta > 0 \) depending on \( \theta \), such that for any \( a \in \mathbb{Z}^k \), any \( f \in C^\theta(M) \), regarded as a function on \( \mathcal{S}(M) \), and any \( g \in C^\theta(\mathcal{S}(M)) \), we have

\[
(1.1) \quad \left| \int_{\mathcal{S}(M)} f(\rho_1(a)\bar{z})g(\bar{z})d\mu(\bar{z}) - \int_{\mathcal{S}(M)} f d\mu \int_{\mathcal{S}(M)} gd\hat{\mu} \right| \leq \frac{1}{\theta} \left\| f \right\|_\theta \left\| g \right\|_\theta.
\]

where \( \| \cdot \| \) denotes the supremum norm on \( \mathbb{Z}^k \).

In the case of \( \mathbb{Z}^k \) actions by ergodic automorphisms on nilmanifolds, exponential mixing was established by Gorodnik and Spatzier [10] and [11], based on the work of Green and Tao [12, 13]. In our case, the structure of \( \mathcal{S}(M) \) is essentially different from a real nilmanifold. Since \( \mathcal{S}(M) \) is an \( S \)-adic nilmanifold, \( p \)-adic analysis will play an important role in the argument. We will prove the theorem in §3.

In §3, we will show that if the action \( \rho_1 \) is genuinely higher rank, then there exists a subgroup \( \Sigma \subset \mathbb{Z}^k \) isomorphic to \( \mathbb{Z}^2 \) such that every element in \( \Sigma \) acts ergodically on \( \mathcal{S}(M) \). Therefore, under the genuinely higher rank hypothesis, we can choose a subgroup \( \Sigma \cong \mathbb{Z}^2 \) of \( \mathbb{Z}^k \) such that the above exponential mixing result holds for \( \Sigma \). Note that \( \rho \) is conjugate to \( \rho_1 \) via a Hölder homeomorphism \( \phi \), if Theorem 1.3 holds for \( \rho_1 \) and \( \mu \), then it will also hold for \( \rho \) and \( \tilde{\mu} := \phi^{-1} \rho \mu \phi \). Therefore, Theorem 1.3 implies the following corollary which is crucial to establish the smoothness of \( \phi \):

**Corollary 1.4.** Let \( \rho, M, \mathcal{S}(M), \tilde{\mu}, C^\theta(M), C^\theta(\mathcal{S}(M)) \) and \( \| \cdot \|_\theta \) be as above. Suppose the action \( \rho \) is genuinely higher rank. Let \( \mathcal{C} \) and \( \mathcal{C}_0 \) be two adjacent Weyl chambers. Then there exists a subgroup \( \Sigma \) of \( \mathbb{Z}^k \) isomorphic to \( \mathbb{Z}^2 \) and constants \( a_1 > 0 \) and \( \eta > 0 \), such that the following holds: \( \Sigma \) intersects both \( \mathcal{C} \) and \( \mathcal{C}_0 \) and for any \( f \in C^\theta(M) \), regarded as functions on \( \mathcal{S}(M) \), any \( g \in C^\theta(\mathcal{S}(M)) \) and any \( a \in \Sigma \),

\[
\left| \int_{\mathcal{S}(M)} f(\rho(a)\bar{z})g(\bar{z})d\tilde{\mu}(\bar{z}) - \int_{\mathcal{S}(M)} f d\tilde{\mu} \int_{\mathcal{S}(M)} gd\hat{\tilde{\mu}} \right| \leq a_1 e^{-\eta|a|} \left\| f \right\|_\theta \left\| g \right\|_\theta.
\]

Let us emphasize that our exponential mixing results are different from exponential mixing for just the semigroup. Indeed we can go to infinity in the solenoid in a variety of ways, e.g., by going back far in the past and returning.
to the present. Exponential mixing for just the future of an expanding map follows from the standard techniques of Markov sections and transfer operators. These techniques however are not able to handle our case. In addition, we allow for quite general semigroups of endomorphisms of nilmanifolds, not just expanding and hyperbolic ones. Quite generally, there are now several techniques available to prove exponential mixing: Fourier analysis, representation theory, Markov systems and transfer operators especially in combination with contact structures. However, for one reason or another, none of these work generally for semigroups of nilmanifold endomorphisms.

Finally, let us note three more corollaries of exponential mixing, similar to results in [10]. We refer there for a more extensive discussion of ideas and background. The proofs are identical, and we will not discuss them here in detail.

First consider a single ergodic nilendomorphism $\alpha$ on a nilmanifold $X$. For a function $f : X \to \mathbb{R}$, we set

$$S_n(f, x) = \sum_{i=0}^{n-1} f(\alpha^i(x)),$$

and for simplicity assume that $\int_X f d\mu = 0$.

One says that the sequence $f \circ \alpha^n$ satisfies the central limit theorem if for some $\sigma > 0$, $n^{-1/2} S_n(f, \cdot)$ converges in distribution to the normal law with mean 0 and variance $\sigma^2$. More generally, the sequence $f \circ \alpha^n$ satisfies the central limit theorem for subsequences if there exists $\sigma > 0$ such that for every increasing sequence of measurable functions $k_n(x)$ taking values in $\mathbb{N}$ such that for almost all $x$, $\lim_{n \to \infty} k_n(x)/n = c$ for some fixed constant $0 < c < \infty$, the sequence $n^{-1/2} S_{k_n}(f, \cdot)$ converges in distribution to the normal law with mean 0 and variance $\sigma^2/c$. We define $S_t(f, x)$ for all $t \geq 0$ by linear interpolation of its values at integral points. The sequence $f \circ \alpha^n$ satisfies the Donsker invariance principle if there exists $\sigma > 0$ such that the sequence of random functions $(n\sigma^2)^{-1/2} S_n(f, \cdot) \in C([0, 1])$ converges in distribution to the standard Brownian motion in $C([0, 1])$. The sequence $f \circ \alpha^n$ satisfies the Strassen invariance principle if there exists $\sigma > 0$ such that for almost every $x$, the sequence of functions $(2n\sigma^2 \log \log n)^{-1/2} S_{nt}(f, x)$ is relatively compact in $C([0, 1])$ and its limit set is precisely the set of absolutely continuous functions $g$ on $[0, 1]$ such that $g(0) = 0$ and $\int_0^1 g'(t)^2 \, dt \leq 1$. This is a strong version of the law of the iterated logarithm.

**Corollary 1.5.** Let $\alpha$ be an ergodic endomorphism of a compact nilmanifold $X$, and let $f$ be a Hölder function on $X$ which has zero integral.

1. If $f$ is not a measurable coboundary, then the sequence $\{f \circ \alpha^n\}$ satisfies the central limit theorem, the central limit theorem of subsequences, and the Donsker and Strassen invariance principles.

2. If $f$ is a measurable coboundary then $f$ is an $L^2$-coboundary. Equivalently, the variance $\sigma = 0$.

Livsic proved for Anosov diffeomorphisms that a measurable coboundary for a smooth function is automatically smooth. Veech discussed this issue for
ergodic toral automorphisms in [34] using sophisticated Fourier analysis. He also gave counter examples in the $C^1$-category. Gorodnik and Spatzier proved the nilautomorphism version of this result in [10].

**Corollary 1.6.** Let $M$ be as above, and let $\alpha : M \to M$ be a nilendomorphism, ergodic w.r.t. Haar measure. Suppose $f : M \to \mathbb{R}$ is a $C^\infty$ function, $g : M \to \mathbb{R}$ a measurable function such that $f = g - g \circ \alpha$. Then $g$ is $C^\infty$.

For the proof we just extend $f$ and $g$ to functions on the solenoid. Note that they are independent of the $p$-adic direction. Hence we can use the central limit theorem, Corollary 1.5, and exponential mixing as in [10].

In our last corollary, we consider genuinely higher rank actions. Again the proof is identical to [10].

**Corollary 1.7.** Let $\rho, M, \mathcal{S}(M), \tilde{\mu}, C^\theta(M)$, $C^\theta(S(M))$, and $\| \cdot \|_\theta$ be as above. Suppose the action $\rho$ is genuinely higher rank. Then any $C^\infty$ cocycle $\alpha : \mathbb{Z}_+^k \times M \to \mathbb{R}$ is $C^\infty$ cohomologous to a constant cocycle.

### 1.4. Organization of the paper.

Before §6, we always assume that $\dim M \geq 5$. In §2, we recall the result of Gromov and Shub on bi-Hölder conjugacies between expanding maps and their linearizations, reduce the main theorem to the case of actions Hölder conjugate to nilmanifold endomorphisms, recall the structures of solenoids, and discuss the Lyapunov exponents of the extended $\mathbb{Z}_+^k$-actions of $\rho$ and $\rho_1$ on the solenoid $\mathcal{S}(M)$. In §3, we will prove Theorem 1.3 and Corollary 1.4. In §4, we apply Corollary 1.4 and the techniques developed in [27] to show that every coarse Lyapunov distribution of $\rho$ admits a Hölder foliation with $C^\infty$ leaves. This result is crucial for applying the techniques developed in [8]. In §5, we combine the exponential mixing result, the result proved in §4 and the techniques developed in [8] to prove Theorem 1.2 when $\dim M \geq 5$. In §6, we discuss the case $\dim M \leq 4$.

**Notation 1.8.** In this paper, we will use the following conventions. For two quantities $A$ and $B$, $A \ll B$ means that there exists an absolute constant $C > 0$ such that $A \leq CB$. $A \gg B$ means that $B \ll A$. $A \asymp B$ means $A \ll B$ and $B \ll A$. $O(A)$ denotes a quantity $\asymp A$. Given a sequence of quantities $\{A_i : i \in \mathbb{N}\}$, another sequence $\{B_i : i \in \mathbb{N}\}$ is said to be of order $o(A_i)$ if $|B_i|/A_i \to 0$ as $i \to \infty$.

## 2. Preliminaries on expanding maps and solenoids

In this section we review and refine some background material on expanding maps and their conjugacies and centralizers. This allows us to reduce the main theorem to the case of actions Hölder conjugate to nilmanifold endomorphisms. We then define solenoid extensions and also Lyapunov exponents.

### 2.1. Gromov’s conjugacy theorem on expanding maps.

We recall the result of Shub [31] and Gromov [9]:

**Theorem 2.1** (see [31, Theorem 1] and [9, §1]). *Suppose $\tau : M \to M$ is an expanding $C^1$-map of a compact manifold $M$. Then there exist*
1. an infra-nilmanifold $\mathcal{M}$, finite covers $\overline{M}$ of $M$ and $\overline{\mathcal{M}}$ of $\mathcal{M}$ such that $\overline{\mathcal{M}} = \Gamma \sim G$, where $G$ denotes a simply connected nilpotent Lie group and $\Gamma$ denotes a lattice of $G$,

2. an expanding affine nilendomorphism $\tilde{\tau}_1 : \overline{M} \to \overline{\mathcal{M}}$ which covers an infra-nilendomorphism $\tau_1$ of $\mathcal{M}$,

3. a bi-Hölder homeomorphism $\phi : M \to \overline{M}$ such that $\tilde{\tau}_1 = \phi^{-1} \circ \tau_1 \circ \phi$ covers $\tau$ and descends to a homeomorphism $M \to \mathcal{M}$ intertwining $\tau$ and $\tau_1$. Moreover, the covering map $\tilde{\tau}_1$ of $\tau_1$ on $G$ is the automorphism of $G$ induced by the map $\tilde{\tau}_1^* : \Gamma \to \Gamma$.

2.2. Reduction to nilmanifolds. We discuss several basic properties of expanding maps and their commuting maps. We use these to reduce the proof of our main result to the case when the semigroup has a fixed point and when the linearization of the action of the semigroup action is on a nilmanifold rather than an infra-nilmanifold.

First we slightly generalize work of Walters from [35], cf. also [5].

**Proposition 2.2.** Suppose the semigroup $\mathbb{Z}_+^k$ acts by $\rho$ on a compact manifold $M$ with an expanding map $\rho(a)$. Then the action is bi-Hölder-conjugate to an action by affine infra-nilendomorphisms on an infra-nilmanifold $\mathcal{M}$.

**Proof.** By Theorem 2.1, $\rho(a)$ is $C^0$-conjugate to an infra-nilendomorphism $\alpha$ by a homeomorphism $\phi$. Any such conjugacy is bi-Hölder as is well-known. For $b \in \mathbb{Z}_+^k$, $\beta := \phi \circ \rho(b) \circ \phi^{-1}$ is a smooth map of $\mathcal{M}$ that commutes with $\alpha$. Let $\overline{\mathcal{M}}$ denote the finite nilmanifold cover of $\mathcal{M}$. Then $\alpha$ lifts to an affine endomorphism $A$ of $\overline{\mathcal{M}}$, and $\beta$ lifts to a homeomorphism $B$ such that $A^{-1}B^{-1}AB$ is an element of the holonomy of $\overline{\mathcal{M}}$ over $\mathcal{M}$ and thus an affine endomorphism. Thus $B$ and hence $\beta$ are affine by [35, Corollary 1].

**Lemma 2.3.** Let $\rho$ be a $\mathbb{Z}_+^k$ action on a compact manifold $M$ with an expanding map $\rho(a)$ which is Hölder conjugate to an affine action $\rho_1$ on an infra-nilmanifold $\mathcal{M}$. Then there is a sub-semigroup $\Sigma^+$ of finite index in $\mathbb{Z}_+^k$ which acts on a finite cover $\mathcal{M}$ of $\mathcal{M}$ by nilendomorphisms covering the restriction of the original action to $\Sigma^+$.

**Proof.** It suffices to prove that the linearization $\rho_1$ lifts since $\rho$ and $\rho_1$ are $C^0$-conjugate. By [30, Theorem 1], $\rho_1(a)$ has a fixed point $p \in \mathcal{M}$. Moreover, the set of fixed points of $\rho_1(a)$ is finite as $\rho_1(a)$ is expanding and $\mathcal{M}$ is compact. Hence there is a sub-semigroup $\Sigma^+$ of finite index in $\mathbb{Z}_+^k$ which also fixes $p$. As $\Sigma^+$ is finitely generated, we can find a finite cover $\mathcal{M}$ of $\mathcal{M}$ such that all elements of $\Sigma^+$ lift to $\overline{\mathcal{M}}$ as affine nilendomorphisms. Furthermore, as $\Sigma^+$ is finitely generated, we can pick lifts of generators of $\Sigma^+$ that all fix a given point $\overline{p}$ in the preimage of $p$. Since these lifts are determined by their derivative action at $\overline{p}$, we see that all the lifts of the generators of $\Sigma^+$ commute. Thus they define a lift of the action of $\Sigma^+$ to $\overline{\mathcal{M}}$ which covers the $\Sigma^+$ action on $\mathcal{M}$, as desired.

Under our higher rank assumptions on the semigroup actions, we will show that the covering map $\overline{\rho}$ of $\rho$ is $C^\infty$-conjugate to $\overline{\rho}_1$ by $\phi$. This implies that $\rho$ is
$C^\infty$-conjugate to $\rho_I$, as desired. Thus we can always work with the finite covers $\overline{M}$ and $\overline{\mathcal{H}}$ and actions $\overline{p}$ and $\overline{\rho}_I$ which have a common fixed point.

If the dimension $\dim(M) \geq 5$, we can make further reductions. Indeed, by Davis’ work on exotic differentiable structures on nilmanifolds [8, Theorem A.0.1, Appendix] and passing to a finite cover, we may assume the conjugacy is isotopic to a diffeomorphism $\psi: \overline{M} \to \Gamma \sim G$. Then we can conjugate $\rho$ by $\psi$ to a smooth action on the nilmanifold $\Gamma \sim G$. We will deal separately with the case $\dim(M) \leq 4$ in §6.

These reductions allows us to make the following hypotheses throughout except in §6.

2.3. Standing assumption. Henceforth, $M = \Gamma \sim G$ will denote a compact nilmanifold and $\rho$ will denote a $C^\infty$ genuinely higher rank action of a semigroup $\mathbb{Z}_+^k$ with $k \geq 2$ on $M$ such that

- $\rho(a)$ is an expanding map for some $a \in \mathbb{Z}_+^k$,
- $\rho$ is Hölder conjugate to an action of $\mathbb{Z}_+^k$ by affine nilendomorphisms on $M$,
- $\rho$ has a common fixed point.

Note that the linearization of $\rho$ is given by the induced action on the fundamental group $\Gamma$, thanks to the existence of a common fixed point.

2.4. Solenoids and extended actions. We will define the solenoid $\mathcal{S}(M)$ of $M$, extend $\rho$ and $\rho_I$ to $\mathbb{Z}^k$ actions on $\mathcal{S}(M)$, and define Lyapunov exponents of $\rho$ and $\rho_I$ on $\mathcal{S}(M)$, following [15].

For $a \in \mathbb{Z}^k$, let us denote by $\rho_*(a): N(\mathbb{Z}) \to N(\mathbb{Z})$ the induced map of $\rho(a)$ on the fundamental group of $M$. It is easy to see that the restriction of $\rho_I(a)$ on $N(\mathbb{Z})$ is just $\rho_*(a)$. Also note that $\rho_*(a)$ can be naturally extended to $N(\mathbb{Z}_p)$ for any prime $p$.

First recall Mal’cev’s theorem from the theory of nilpotent Lie groups (see [25, 12, 4], for example) that any lattice $\Gamma$ of a nilpotent Lie group $G$ must be arithmetic, i.e., there is a simply connected nilpotent algebraic group $N$ over $\mathbb{Q}$ such that $G = N(\mathbb{R})$ and $\Gamma = N(\mathbb{Z})$. Then $M = N(\mathbb{Z}) \sim N(\mathbb{R})$.

By [15], the abstract solenoid $\mathcal{S}(M, \rho)$ of $M$ is naturally defined as follows:

(2.1) $\mathcal{S}(M, \rho) := \{(z_n) \in M^{\mathbb{Z}_+^k} : z_{n+a} = \rho(a)z_n \text{ for all } a \in \mathbb{Z}_+^k\}$.

In other words, we attach each point on $M$ with all possible pasts with respect to all $a \in \mathbb{Z}_+^k$. On this space, one can easily define a $\mathbb{Z}_+^k$ action which extends the original $\mathbb{Z}_+^k$ action $\rho$ (see [15, §3] for details). The disadvantage is that it is hard to do concrete analysis and calculation with this definition. Therefore, we will give another definition and stick with it throughout the paper.

Given $a \in \mathbb{Z}_+^k$, $\rho(a)$ can be extended to a homeomorphism from $N(\mathbb{R})$ to itself (cf. [30] and [31]). For a fixed $z \in M$, the preimage of $z$ with respect to $\rho(a)$ is $\{N(\mathbb{Z})\rho^{-1}(a)(nz) : n \in N(\mathbb{Z})\}$. Therefore, to attach $z$ with a past with respect to $\rho(a)$ is the same as to attach $z$ with an element $n \in N(\mathbb{Z})$. Moreover, if $\rho^{-1}(a)(n_1^{-1}n_2) \in N(\mathbb{Z})$, then $\rho^{-1}(a)(n_1z) = \rho^{-1}(a)(n_2z)$. Taking this congruence
condition into account and passing to the inverse limit for all possible \( a \in \mathbb{Z}_p^k \), we will attach each \( z \in M \) with several \( p \)-adic components \( \xi_p \in N(\mathbb{Z}_p) \).

This discussion brings us the new definition of the solenoid \( \mathcal{S}(M) \) of \( M \):

**Definition 2.4** (see [15, Appendix]). Let

\[
\mathcal{S}'(M) := N(\mathbb{Z}) \sim \left( N(\mathbb{R}) \times \prod_{p \text{ prime}} N(\mathbb{Z}_p) \right),
\]

where \( N(\mathbb{Z}) \) acts on \( N(\mathbb{R}) \times \prod_{p \text{ prime}} N(\mathbb{Z}_p) \) diagonally, and in the product, \( p \) runs over all primes. For each prime number \( p \), we define

\[
M_p := \left\{ \nu \in N(\mathbb{Z}_p) : \| \rho_\nu(a) \|_p = \| \nu \|_p , \text{ for all } a \in \mathbb{Z}_p^k \right\},
\]

where \( \| \cdot \|_p \) denotes the \( p \)-adic norm. Let us define

\[
\mathfrak{T} M := \left( N(\mathbb{R}) \times \prod_{p \text{ prime}} N(\mathbb{Z}_p) / M_p \right) \quad \text{and} \quad \mathcal{S}(M) := N(\mathbb{Z}) \sim \mathfrak{T} M.
\]

Equip \( \mathcal{S}(M) \) with the product topology. We will denote the equivalence class of \( (z, (\xi_p)_{p \in S}) \in \mathfrak{T} M \) in \( \mathcal{S}(M) \) by \( [z, (\xi_p)_{p \in S}] \).

**Remark 2.5.**

1. \( M_p = N(\mathbb{Z}_p) \) for all but finitely many \( p \)'s. Therefore, there exists a finite set \( S \) of primes such that \( \mathcal{S}(M) = N(\mathbb{Z}) \sim \left( N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) / M_p \right) \).
2. Generalizing the argument from [15, Lemma 8.2], we see that every quotient \( N(\mathbb{Z}_p) / M_p \) is torsion free.
3. For each prime \( p \), we denote by \( \nu_p \) the Haar measure on \( N(\mathbb{Z}_p) \). By normalization, we assume that \( \nu_p(N(\mathbb{Z}_p)) = 1 \). Let \( \nu \) denote the Haar measure on \( N(\mathbb{R}) \) and also the induced measure on \( N(\mathbb{Z}) \sim N(\mathbb{R}) \). By normalization, we assume that \( \nu(N(\mathbb{Z}) \sim N(\mathbb{R})) = 1 \). Then the product measure \( \nu \times \prod_{p \in S} \nu_p \) induces a probability measure on the solenoid \( \mathcal{S}(M) \), which we denote by \( \mu \). Define \( \tilde{\nu} := \phi^{-1}_* (\nu) \) and \( \tilde{\mu} := \tilde{\nu} \times \prod_{p \in S} \nu_p \), then \( \tilde{\nu} \) is absolutely continuous with respect to \( \nu \) (since \( \phi \) is Hölder ), and \( \tilde{\mu} \) is preserved by the action of \( p \). Moreover, for any \( a \in \mathbb{Z}_p^k \), the action of \( \rho(a) \) is ergodic with respect to \( \tilde{\mu} \) if and only if \( \rho_\nu(a) \) is ergodic with respect to \( \mu \).
4. The definition above depends on the homotopy class of the action \( \rho \) as the \( M_p \)'s do. Since throughout this paper we fix the homotopy type, i.e., the induced action \( \rho_\nu \) on \( N(\mathbb{Z}) \), we may regard \( \mathcal{S}(M) \) as a fixed space.
5. We note that an infra-nilmanifold \( M \) can be regarded as a finite index factor of a homogeneous space \( N(\mathbb{Z}) \sim N(\mathbb{R}) \) where \( N \) denotes a nilpotent \( Q \)-group. Once we define the solenoid \( \mathcal{S}(N(\mathbb{Z}) \sim N(\mathbb{R})) \) of \( N(\mathbb{Z}) \sim N(\mathbb{R}) \), the solenoid of \( M \) is just the quotient of \( \mathcal{S}(N(\mathbb{Z}) \sim N(\mathbb{R})) \) by a finite group action. Thus solenoids for infra-nilmanifolds also have an explicit description.

For nilpotent algebraic group \( N \), we have the following version of Chinese remainder theorem:
**Lemma 2.6.** Given a finite subset of primes $S$, $\xi_p \in N(\mathbb{Z}_p)$ for $p \in S$ and $l_p \in \mathbb{Z}_+$ for $p \in S$, there exists $n \in N(\mathbb{Z})$ such that

$$n^{-1}\xi_p \equiv 0 \pmod{p^{l_p}}$$

for all $p \in S$.

**Proof.** We prove the statement by induction on the nilpotency degree of $N$.

If $N$ is abelian, this is just the Chinese remainder theorem as the action $n^{-1}\xi_p$ is a linear expression.

Suppose the statement holds if the nilpotency degree is $< d$. Now we assume that the nilpotency degree of $N$ is $d$. Take a nilpotent subgroup $N'$ of $N$ such that $N' \sim N$ is abelian and the nilpotency degree of $N'$ is $d-1$. Then considering the image of $\xi_p$ on $N'(\mathbb{Z}_p) \sim N(\mathbb{Z}_p)$, and applying the Chinese remainder theorem, we have that there exists $n_1 \in N(\mathbb{Z})$ such that

$$n_1^{-1}\xi_p \equiv 0 \pmod{p^{l_p}} \in N'(\mathbb{Z}_p) \sim N(\mathbb{Z}_p),$$

for all $p \in S$. Now by applying inductive hypothesis to $N'$, we have there exists $n_2 \in N'(\mathbb{Z})$ such that

$$n_2^{-1}n_1^{-1}\xi_p \equiv 0 \pmod{p^{l_p}},$$

for all $p \in S$. Then $n = n_1 n_2$ satisfies our condition.

This proves the lemma. $\square$

The informal discussion before Definition 2.4 may help with the next result and its proof.

**Proposition 2.7.** $\rho$ and $\rho_1$ can be extended to $\mathbb{Z}^k$ actions on $\mathcal{S}(M)$.

**Proof.** For $a \in \mathbb{Z}^k$, the action $\rho(a)$ on $\mathcal{S}(M)$ can be naturally defined as follows: for $\mathcal{Z} = [z,(\xi_p)_{p \in S}] \in \mathcal{S}(M)$,

$$\rho(a)(\mathcal{Z}) := [\rho(a)z,(\rho_*(a)\xi_p)_{p \in S}].$$

To extend $\rho$ to a $\mathbb{Z}^k$ action on $\mathcal{S}(M)$, it suffices to define the inverse of $\rho(a)$ for each $a \in \mathbb{Z}^k$. For $a \in \mathbb{Z}^k$, $\rho(a)$ can be extended to a homomorphism of the universal covering $N(\mathbb{R})$ of $M$ to itself (cf. [30] and [31]). Therefore $\rho^{-1}(a)$ is well defined on $N(\mathbb{R})$. Recall that $\rho_*(a)$ agrees with $\rho_1(a)$ when restricted to $N(\mathbb{Z})$. Then we define $\rho^{-1}(a)$ as follows: for $\mathcal{Z} = [z,(\xi_p)_{p \in S}] \in \mathcal{S}(M)$, we may pick $l_p$ for each $p \in S$ such that $\rho_*^{-1}(a)(N(p^{l_p} \mathbb{Z}_p)) \subset N(\mathbb{Z}_p)$. By Lemma 2.6, we can find $n \in N(\mathbb{Z})$ such that $n^{-1}\xi_p \equiv 0 \pmod{p^{l_p}}$. Let us write $[(z,(\xi_p)_{p \in S})] = (n^{-1}z,(n^{-1}\xi_p)_{p \in S})$, then $\rho_*^{-1}(a)$ is well defined on each $p$-adic component. Thus, we can define

$$\rho^{-1}(a)[(z,(\xi_p)_{p \in S})] := [(\rho^{-1}(a)(n^{-1}z),(\rho_*^{-1}(a)(n^{-1}\xi_p))_{p \in S})].$$

The same extension works for $\rho_1$ as well. $\square$

The conjugacy $\phi : M \to M$ can be extended to a homeomorphism $\phi : \mathcal{S}(M) \to \mathcal{S}(M)$ as follows. Recall that $\phi$ is homotopic to the identity and lifts to an $N(\mathbb{Z})$-equivariant map $\hat{\phi} : N(\mathbb{R}) \to N(\mathbb{R})$. Extend $\hat{\phi}$ to $\mathcal{S}M = N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p)/M_p$.
by crossing with the identity map on $\prod_{p \in S} N(\mathbb{Z}_p)/M_p$. It is easy to check that $\tilde{\phi} \times id$ commutes with $N(\mathbb{Z})$ acting on $\mathcal{F}M$. Hence the map descends to $\mathcal{F}(M)$.

On the real component, it is $\phi$, and on $p$-adic components, it is the identity map. It is easy to see that $\phi$ conjugates the extended actions $\rho$ and $\rho_1$ on $\mathcal{F}(M)$.

2.5. Lyapunov exponents and coarse Lyapunov decomposition. We need the following notation to define Lyapunov exponents of $\rho$ and $\rho_1$.

**Definition 2.8.** For $\mathcal{Z} = [(z, (z_p)_{p \in S})] \in \mathcal{F}(M)$ and an open neighborhood $U_z \subset \mathcal{F}(M)$ of $\mathcal{Z}$, let $M(\mathcal{Z})$ denote the connected component of $\mathcal{Z}$ in $U_z$. It is easy to see that $M(\mathcal{Z})$ is of the form $\{(y, (z_p)_{p \in S}) : y \in U \subset M\}$, where $U \subset M$ denotes an open neighborhood of $z$ in $M$. It is homeomorphic to an open set of $M$. Let us call $M(\mathcal{Z})$ a manifold slice passing through $\mathcal{Z}$.

The smoothness of a map defined on $\mathcal{F}(M)$ is defined as follows:

**Definition 2.9.** We say a continuous map defined on $\mathcal{F}(M)$ is $C^\infty$ if it is $C^\infty$ when restricted to every manifold slice.

**Remark 2.10.** It is easy to see that for each $a \in \mathbb{Z}^k$, $\rho(a)$ is $C^\infty$. Indeed, $\rho(a)$ maps every manifold slice $M(\mathcal{Z})$ to another manifold slice $M(\mathcal{Z})$, and when restricted to the manifold slice, $\rho(a)$ is smooth (because the map only depends on the real component). The same holds for $\rho_1$.

**Definition 2.11.** Since $\rho_1$ acts on $\mathcal{F}(M)$ by affine nilendomorphisms, it naturally induces a $\mathbb{Z}^k$ action on the nilpotent Lie group $N(\mathbb{R})$ by automorphisms, which we still denote by $\rho_1$. Let $D\rho_1$ denote the action on $n(\mathbb{R})$ induced by $\rho_1$. A character $\chi \in (\mathbb{R}^k)^*$ is called a real Lyapunov exponent of $\rho_1$ if the real Lyapunov subspace corresponding to $\chi$ defined as follows:

$$\sigma^\chi := \left\{ v \in n(\mathbb{R}) : \lim_{\|a\| \to \infty} \frac{\log \|D\rho_1(a)v\| - \chi(a)}{\|a\|} = 0 \right\}$$

is nontrivial.

Let $D\rho_1$ denote the action on $n(Q_p)$ induced by $\rho_1$. Note that $\rho$ induces the same action on $p$-adic components, so $D\rho_1$ is also the action induced by $\rho$. A character $\chi \in (\mathbb{R}^k)^*$ is called a $p$-adic Lyapunov exponent of $\rho_1$ (and also $\rho$) if the $p$-adic Lyapunov subspace corresponding to $\chi$ defined as follows:

$$\sigma^\chi := \left\{ v \in n(Q_p) : \lim_{\|a\| \to \infty} \frac{\log \|D\rho_1(a)v\|_p - \chi(a)}{\|a\|} = 0 \right\}$$

is nontrivial.

For $\mathcal{Z} \in \mathcal{F}(M)$, let $T_\mathcal{Z}(M)$ denote the tangent space of the manifold slice passing through $\mathcal{Z}$ based at $\mathcal{Z}$. A character $\chi \in (\mathbb{R}^k)^*$ is called a real Lyapunov exponent of $\rho$ if for $\mu$-a.e. $\mathcal{Z} \in \mathcal{F}(M)$, the real Lyapunov distribution corresponding to $\chi$ defined as follows:

$$E^\chi_\mathcal{Z} := \left\{ v \in T_\mathcal{Z}(M) : \lim_{\|a\| \to \infty} \frac{\log \|D\rho(a)(v)\| - \chi(a)}{\|a\|} = 0 \right\}$$

is nontrivial.
**Notation 2.12.** To distinguish the *Lyapunov exponents* of $\rho$ and $\rho_I$, later in this paper, we denote Lyapunov exponents of $\rho$ by $\chi, \chi_1, \chi_2, \ldots$, and denote Lyapunov exponents of $\rho_I$ by $\chi^l, \chi_1^l, \chi_2^l, \ldots$. Since $p$-adic Lyapunov exponents of $\rho$ and $\rho_I$ coincide, we do not distinguish the above two notions in $p$-adic directions. We say a Lyapunov exponent $\chi$ (or $\chi^l$) is of type $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$) if the corresponding Lyapunov distribution (or Lyapunov subspace) is in $n(\mathbb{K})$. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, let $T(\mathbb{K})$ denote the set of Lyapunov exponents of type $\mathbb{K}$. Note that $T(\mathbb{R})$ and $T(\mathbb{Q}_p)$ are disjoint.

**Remark 2.13.**

1. One can prove that (see [27])
   
   $$n(\mathbb{R}) = \bigoplus_{\chi^l \in T(\mathbb{R})} \sigma^{\chi^l}.$$

2. Since $\rho$ contains an expanding element, it is ergodic with respect to $\tilde{\mu}$. By Multiplicative Ergodic Theorem, there exist finitely many *Lyapunov exponents* $\chi$’s of type $\mathbb{R}$, a set of full $\tilde{\mu}$-measure set $\mathcal{P} \subset \mathcal{S}(M)$, and a $\rho$-invariant measurable splitting of the bundle $E(\mathcal{S}(M)) := \bigcup_{\mathbb{Z} \in \mathcal{S}(M)} T_\mathbb{Z}(M) = \bigoplus E^\chi$ over $\mathcal{P}$ such that for all $a \in \mathbb{Z}^k$ and $\nu \in E^\chi \sim \{0\}$,
   
   $$\lim_{n \to \infty} n^{-1} \log \left\| D\rho(na) \nu \right\| / \|\nu\| = \chi(a).$$

We refer to [14], [16], and [8] for details.

**Definition 2.14.** For a *Lyapunov exponent* $\chi^l$ of $\rho_I$, we define the *coarse Lyapunov subspace* associated with $\chi^l$ as follows:

$$\sigma^{[\chi^l]} := \bigoplus_{\chi^l = c \chi^l, c > 0} \sigma^{\chi^l}.$$ 

The corresponding decomposition

$$n(\mathbb{R}) = \bigoplus_{\chi^l \in T(\mathbb{R})} \sigma^{[\chi^l]}$$

is called the *coarse Lyapunov decomposition* of the real component of $\mathcal{S}(M)$.

Similarly, for a *Lyapunov exponent* $\chi$ of $\rho$, we define the *coarse Lyapunov distribution* associated with $\chi$ as follows:

$$E^{[\chi]} := \bigoplus_{\chi = c \chi, c > 0} E^\chi.$$ 

**Remark 2.15.** One can prove that $[\sigma^{\chi^l}, \sigma^{\chi^l}'] \subset \sigma^{\chi^l + \chi^l'}$ if $\chi^l$ and $\chi^l'$ are of the same type, cf. [27, Lemma 2.3]. Therefore, each $\sigma^{[\chi]}$ ($\chi$ can be real or $p$-adic) is a Lie subalgebra (of $n(\mathbb{R})$ or $n(\mathbb{Q}_p)$). Let $V^{[\chi]}$ denote the corresponding Lie subgroup (of $n(\mathbb{R})$ or $n(\mathbb{Q}_p)$), which will be called a *coarse Lyapunov subgroup*.

**Definition 2.16.** We define a *Weyl chamber* of $\rho$ (respectively $\rho_I$) to be a connected component of $\mathbb{R}^k \sim \bigcup_{\chi} \ker \chi$ (respectively $\mathbb{R}^k \sim \bigcup_{\chi} \ker \chi^l$), and a *real Weyl
chamber to be a connected component of \( \mathbb{R}^k \sim \cup_{\chi \in \mathcal{T}(\mathbb{R})} \ker \chi \) (respectively \( \mathbb{R}^k \sim \cup_{\chi' \in \mathcal{T}(\mathbb{R})} \ker \chi' \)).

Thanks to the existence of the conjugacy \( \phi \), we have the following correspondence between coarse Lyapunov distributions of \( \rho \) and coarse Lyapunov subgroups of \( \rho_1 \).

**Proposition 2.17.** The coarse Lyapunov distributions of \( \rho \) and the coarse Lyapunov subgroups of \( \rho_1 \) are in one-to-one correspondence with each other. A pair of corresponding coarse Lyapunov distribution and coarse Lyapunov subgroup have the same dimension and positively proportional coarse Lyapunov exponents. In consequence, \( \rho \) and \( \rho_1 \) have the same Weyl chambers and the same real Weyl chambers.

**Proof.** See [27, Lemma 4.9] or [8, Proposition 3.2].

2.6. The cohomological equation I. Recall that \( \rho(a) \) and \( \rho_1(a) \) are \( C^\infty \) for any \( a \in \mathbb{Z}^k \) (see Definition 2.9 and Remark 2.10).

Because the conjugacy \( \phi : \mathcal{S}(\mathbb{M}) \to \mathcal{S}(\mathbb{M}) \) is homotopic to \( \text{Id} \), for any \( a \in \mathbb{Z}^k \), \( \rho_1(a)^{-1} \rho(a) \) is homotopic to \( \text{Id} \). We write

\[
\rho_1(a)^{-1} \rho(a)(\mathbb{Z}) = \mathbb{Z} Q_a(\mathbb{Z}).
\]

where \( Q_a(\mathbb{Z}) \in N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) \). It is easy to see that \( Q_a \) is \( C^\infty \) since both \( \rho(a) \) and \( \rho_1(a) \) are. Since \( \rho(a) \) and \( \rho_1(a) \) are identical on \( p \)-adic components, \( Q_a(\mathbb{Z}) \) is only nontrivial on the real component, so it can be regarded as a \( C^\infty \) map \( Q_a : \mathcal{S}(\mathbb{M}) \to N(\mathbb{R}) \). In fact, for each \( p \in S \), let \( l_p(a) \geq 0 \) be an integer such that \( \rho_1(a)(N(p^{l_p(a)}(\mathbb{Z}_p))) \subset N(\mathbb{Z}_p) \), then if \( \mathbb{Z} = [(z, (\xi_p)_{p \in S})] \) and \( \mathbb{Z}' = [(z', (\xi'_p)_{p \in S})] \) satisfy that \( z = z' \) and \( \xi_p = \xi'_p \mod p^{l_p(a)} \) for each \( p \in S \), then \( Q_a(\mathbb{Z}) = Q_a(\mathbb{Z}') \).

In other words, \( Q_a \) can be treated as a \( C^\infty \) map defined on the finite cover \( N(n(\mathbb{Z})) \sim N(\mathbb{R}) \) of \( M \) where \( n = \prod_{p \in S} p^{l_p(a)} \). Writing \( \phi(\mathbb{Z}) = \mathbb{Z} h(\mathbb{Z}) \), then \( h(\mathbb{Z}) \) is also trivial on every \( p \)-adic component, and moreover, it only depends on the real component. Therefore, we can regard \( h \) as a map from \( M \) to \( N(\mathbb{R}) \). Since \( \phi \) conjugates \( \rho \) to \( \rho_1 \), we have that for any \( a \in \mathbb{Z}^k \) and any \( \mathbb{Z} \in \mathcal{S}(\mathbb{M}) \), the following holds:

\[
\mathbb{Z} h(\mathbb{Z}) = \phi(\mathbb{Z}) = \rho_1(a)^{-1} \phi(\rho(a) \mathbb{Z}) = \rho_1(a)^{-1} (\rho(a) \mathbb{Z} h(\rho(a) \mathbb{Z})) = \mathbb{Z} (Q_a(\mathbb{Z}) \rho_1(a)^{-1} (h(\rho(a) \mathbb{Z})).
\]

Let us first prove the following:

**Lemma 2.18.** If a map \( f : \mathcal{S}(\mathbb{M}) \to N(\mathbb{R}) \) is continuous and satisfies that \( \mathbb{Z} f(\mathbb{Z}) = \mathbb{Z} \) for all \( \mathbb{Z} \in \mathcal{S}(\mathbb{M}) \), then there exists \( \gamma_0 \in N(\mathbb{Z}) \cap Z(N(\mathbb{R})) \) (where \( Z(N(\mathbb{R})) \) denotes the center of \( N(\mathbb{R}) \)) such that \( f(\mathbb{Z}) = \gamma_0 \) for all \( \mathbb{Z} \in \mathcal{S}(\mathbb{M}) \).

**Proof.** In fact, \( f \) can be lifted to a continuous function \( f : N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) \to N(\mathbb{R}) \) such that \( f(\gamma g) = f(g) \) for all \( \gamma \in N(\mathbb{Z}) \). Then the condition of the claim implies that for all \( g \in N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) \), \( N(\mathbb{Z}) g f(g) = N(\mathbb{Z}) g \), or equivalently,
We claim that $gf(g)g^{-1} \in N(\mathbb{Z})$. Let $g$ vary in a manifold slice. Since $N(\mathbb{Z})$ is discrete and $gf(g)g^{-1}$ is continuous, we have that $gf(g)g^{-1}$ is constant in any manifold slice. Let us fix a manifold slice $\mathcal{U}$. Then there exists $\gamma_0 \in N(\mathbb{Z})$ such that $gf(g)g^{-1} = \gamma_0$ when $g \in \mathcal{U}$. Note that $f(g)$ and $gf(g)g^{-1}$ are locally constant in every $p$-adic component. Therefore, for each $p \in S$, there exists an integer $l_p > 0$ such that for any $g = (z, (\xi_p)_p) \in \mathcal{U}$ and $g' = (z, (\xi'_p)_p)$ such that $\xi'_p \xi_p^{-1} \in N(p^{l_p} \mathbb{Z}_p)$, we have that $f(g') = f(g)$ and $gf(g')(g')^{-1} = \gamma_0$. Let $P = \prod_{p \in S} p^{l_p}$.

We claim that $\gamma_0 \in Z(N(\mathbb{R}))$. In fact, for any $g = (z, (\xi_p)_p) \in \mathcal{U}$ and $n \in N(\mathbb{P}Z)$, let us denote $g' = (n^{-1} z, (\xi_p)_p)$. Then we have that $g' \in \mathcal{U}$. This implies that $g' f(g')(g')^{-1} = \gamma_0$, i.e., $n^{-1}zf(n^{-1} z, (\xi_p)_p)z^{-1}n = \gamma_0$. Note that

$$f(n^{-1} z, (\xi_p)_p) = f(z, (n\xi_p)_p) = f(z, (\xi_p)_p)$$

since $n \in N(p^{l_p} \mathbb{Z}_p)$ for any $p \in S$. Therefore, we have the following:

$$\gamma_0 = n^{-1} zf(n^{-1} z, (\xi_p)_p)z^{-1}n = n^{-1} zf(z, (\xi_p)_p)z^{-1}n = n^{-1} g f(g)g^{-1}n = n^{-1} \gamma_0n.$$

Thus $\gamma_0$ commutes with any $n \in N(\mathbb{P}Z)$ since $N(\mathbb{P}Z)$ is Zariski dense in $N(\mathbb{R})$, we have that $\gamma_0 \in Z(N(\mathbb{R}))$. Therefore we have that $f(g) = \gamma_0$ for any $g \in \mathcal{U}$. For any $n \in N(\mathbb{Z})$, we have that $n^{-1}zf(n^{-1} z, (\xi_p)_p)z^{-1}n = \gamma_0$ since $(n^{-1} z, (\xi_p)_p) \in \mathcal{U}$. This implies that $zf(n^{-1} z, (\xi_p)_p)z^{-1} = \gamma_0$ since $\gamma_0 \in Z(N(\mathbb{R}))$. Thus, we have that

$$\gamma_0 = zf(n^{-1} z, (\xi_p)_p)z^{-1} = zf(z, (n\xi_p)_p)z^{-1} = g(n)f(g(n))g(n)^{-1},$$

where $g(n) = (z, (n\xi_p)_p)$. Since $\{(n\xi_p)_p : n \in N(\mathbb{Z})\}$ is dense in $\Pi_{p \in S} N(\mathbb{Z}_p)$, we have that for any $g = (z, (\xi_p)_p) \in \mathcal{U}$ and any $((\xi'_p)_p) \in \Pi_{p \in S} N(\mathbb{Z}_p)$, $g' f(g')(g')^{-1} = \gamma_0$ where $g' = (z, (\xi'_p)_p)$. Since $gf(g)g^{-1}$ is constant on manifold slices and also the product of $p$-adic components, we have that $gf(g)g^{-1} = \gamma_0$ for any $g \in N(\mathbb{R}) \times \Pi_{p \in S} N(\mathbb{Z}_p)$. This implies that $f(g) = \gamma_0$ for all $g \in N(\mathbb{R}) \times \Pi_{p \in S} N(\mathbb{Z}_p)$ since $\gamma_0 \in Z(N(\mathbb{R}))$. □

In [27], the following lemma is proved. We follow its proof closely.

**Lemma 2.19** (See [27, Lemma 3.7]). There exists a $C^\infty$ map $Q'_a : \mathcal{I}(M) \to N(\mathbb{R})$ such that

$$h(\overline{z}) = Q'_a(\overline{z}) \rho_1(a)^{-1}(h(\rho(a)\overline{z})).$$

for all $\overline{z} \in \mathcal{I}(M)$.

**Proof.** From (2.2), we have that

$$\overline{z} = \overline{z}Q_a(\overline{z}) \rho_1(a)^{-1}(h(\rho(a)\overline{z}))h^{-1}(\overline{z}).$$

Let

$$f(\overline{z}) := Q_a(\overline{z}) \rho_1(a)^{-1}(h(\rho(a)\overline{z}))h^{-1}(\overline{z}).$$
Then $f : \mathcal{F}(M) \to N(\mathbb{R})$ is continuous. By Lemma 2.18, we conclude that

$$Q_a(z)\rho_l(a)^{-1}(h(\rho(a)z))h^{-1}(z) = \gamma_0$$

for some $\gamma_0 \in N(z) \cap Z(N(\mathbb{R}))$. Then $Q'_a(z) := \gamma_0^{-1}Q_a(z)$ is $C^\infty$ and satisfies

$$h(z) = Q'_a(z)\rho_l(a)^{-1}(h(\rho(a)z)).$$

This completes the proof. \hfill \square

**Definition 2.20.** Let $\mathcal{F}$ be a foliation of $S(M)$ with of $C^\infty$ leaves. We consider derivatives of order $k$ of functions or distributions $f$ along $\mathcal{F}$, which we denote by $\partial^k_{\mathcal{F}} f$. For $\theta \in (0,1)$, let $C^\infty_{\mathcal{F},\theta}$ denote the space of distributions $f$ on $M$ such that all partial derivatives of $f$ of any order along $\mathcal{F}$ exist as distributions on the space of $\theta$-Hölder functions. Let $C^\infty_{\mathcal{F},\theta}$ denote the space of $\theta$-Hölder functions $f$ on $M$ such that all partial derivatives of $f$ of any order along $\mathcal{F}$ are $\theta$-Hölder.

**Notation 2.21.** Throughout this paper, we will denote coarse Lyapunov subgroups corresponding to coarse Lyapunov exponents $[1^1], [1^1_1], [1^1_2], \ldots$ of $\rho_l$ by $V, V_1, V_2, \ldots$.

### 2.7. Outline of the proof

We briefly describe the basic idea (developed in [8] and [27]) to establish the smoothness of $h$.

We first establish the smoothness when $M$ is a torus.

To this end we first show that for every coarse Lyapunov exponent $[\chi]$ of $\rho$, the corresponding coarse Lyapunov distribution $E^{[\chi]}$ admits a Hölder foliation consisting of $C^\infty$ leaves. In order to show this, we make use of our exponential mixing result for solenoids (Corollary 1.4) proved in §3 and the techniques and results developed by Rodriguez Hertz and Wang [27].

Let $V$ be a coarse Lyapunov subgroup corresponding to $[\chi^1]$. We define $h_V(z)$ to be the projection of $h(z)$ on $V$. We want to show that for any coarse Lyapunov subgroup $V$, $h_V(z)$ is $C^\infty$.

From (2.2) we get the corresponding equation for $h_V(z)$:

$$h_V(z) = (Q'_a(z) + \rho_l(a)^{-1}h_V(\rho(a)z))_V$$

$$= (Q'_a(z))_V + \rho_l(a)^{-1}h_V(\rho(a)z),$$

where $(Q'_a(z))_V$ denotes the projection of $Q'_a(z)$ on $V$. By iterating (2.3), one finds the solution by the following formal series:

$$h_V = \sum_{i=0}^{\infty} \rho_l(a)^{-i}\Phi \circ \rho(a)^i,$$

where $\Phi(z) := (Q'_a(z))_V$. We will apply the exponential mixing result again to prove that for any coarse Lyapunov foliation $\mathcal{V}_1$, $h_V \in C^\infty_{\mathcal{V}_1,\theta}$. In the case of $\mathbb{Z}^k$ actions, exponential mixing was established by Fisher, Kalinin and Spatzier [8] for tori and by Gorodnik and Spatzier [10], [11] for nilmanifolds.

By variations of results of Rauch and Taylor in [27] proved by Fisher, Kalinin and Spatzier [8], and by Rodriguez Hertz and Wang [27], $h_V \in C^\infty_{\mathcal{V}_1,\theta}$ for all
possible coarse Lyapunov foliations $\mathcal{V}_i$ will imply that $h_V$ is $C^\infty$. Then it follows that $h$ is $C^\infty$ since $h$ can be written as the sum of $h_V$'s for all coarse Lyapunov subgroups $V$. This will prove the smoothness when $M$ is a torus.

For the general case, we follow the approach of Margulis and Qian [22]. We consider the derived series of $N$:

$$N = N_0 \supset N_1 \supset \cdots \supset N_{k-1} \supset N_k = \{0\},$$

and prove the smoothness of $h$ by induction on $k$.

3. EXPONENTIAL MIXING FOR EXTENDED $\mathbb{Z}^k$-ACTIONS ON SOLENOIDS

In this section, we will prove Theorem 1.3 and Corollary 1.4. As we discussed in §2, Corollary 1.4 is crucial to establish the smoothness of $\phi$.

3.1. Preparation for the proof. We need some preparation before proving the theorem.

**Definition 3.1.** Let us denote $[N, N]$ by $N'$. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$, let us define

$$\pi : N(\mathbb{K}) \to N'(\mathbb{K}) \sim N(\mathbb{K}) \cong \mathbb{K}^I$$

to be the canonical projection. Let

$$D\pi : n(\mathbb{K}) \to \mathbb{K}^I$$

denote the derived map of $\pi$ on the Lie algebras.

Let us define $M' := N'(\mathbb{Z}) \sim N'(\mathbb{R})$ and $M_0 := (N'(\mathbb{Z}) \sim N(\mathbb{Z})) \sim (N'(\mathbb{R}) \sim N(\mathbb{R}))$. Then $M_0$ is a torus and $M$ is a bundle over $M_0$ with $M'$ fibers. We call $M_0$ the maximal torus factor of $M$.

We will need the following effective equidistribution result for box maps on nilmanifolds.

**Theorem 3.2** (See [10, Theorem 2.1]). Let $w_1, w_2, \ldots, w_r$ be $r$ linearly independent vectors in the Lie algebra $n(\mathbb{R})$. Let $v \in n(\mathbb{R})$ be a fixed vector. We define the box map

$$\iota : B = [0, T_1] \times [0, T_2] \times \cdots \times [0, T_r] \to n(\mathbb{R})$$

as follows:

$$(t_1, t_2, \ldots, t_r) \in B \mapsto v + t_1 w_1 + t_2 w_2 + \cdots + t_r w_r.$$
Another important result we will need is the following effective equidistribution result for polynomial orbits on nilmanifolds, proved by Green and Tao [12].

**Theorem 3.3** (See [12, Theorem 8.6]). For \( N = (N_1, N_2, \ldots, N_r) \in \mathbb{Z}_+^r \), \([N]\) denotes the box set \([N_1] \times [N_2] \times \cdots \times [N_r] \), where \([N_i] := \{0, 1, \ldots, N_i\} \subset \mathbb{N}\). Let \( p : [N] \to N(\mathbb{R}) \) be a polynomial map. For \( i = 1, 2, \ldots, r \), let \( e_i \in [N] \) denote the vector with 1 on the \( i \)-th component and 0 on other components, and let \( \partial_i \pi(p(n)) := \pi(p(n)) - \pi(p(n - e_i)) \). Then there exist constants \( L_1, L_2 > 0 \) such that for every \( \delta > 0 \) and every \( f \in C^0(M) \), one of the following holds:

\[
\text{AA.1} \quad \left| \frac{1}{N_1 \cdots N_r} \sum_{n \in [N]} f(N(z)p(n)) - \int_M f(x) \, dx \right| \leq \delta \| f \|_0.
\]

\[
\text{AA.2} \quad \text{There exists } z \in Z^i \sim \{0\} \text{ such that } \| z \| \ll \delta^{-L_1} \text{ and } \text{dist}(\langle z, \partial_i \pi(p(n)) \rangle, Z) \ll \delta^{-L_2} / N_i
\]

for all \( i = 1, 2, \ldots, r \) and \( n \in [N] \). Here \( \text{dist}(x, Z) := \min_{z \in Z} ||x - z|| \).

**Remark 3.4.**

1. The statement in [12] is different from the above theorem. For example, the function is assumed to be Lipschitz continuous, and case AA.2 above is stated differently. The statement above follows the one stated and applied in [10]. To get the above modified version from the statement in [12], one needs to approximate Hölder functions by Lipschitz functions, keeping control of all the desired estimates. We refer to [10] for details.

2. The non-effective version of the equidistribution of polynomial orbits in nilmanifolds is proved by Leibman [17].

We introduce some notation.

**Notation 3.5.** The \( \mathbb{Q} \)-structure on \( N \) induces a \( \mathbb{Q} \)-structure on its Lie algebra \( \mathfrak{n} \). One can construct a basis, a so-called Mal’cev basis, \( \{e_1, e_2, \ldots, e_d\} \) of \( \mathfrak{n}(\mathbb{Q}) \) such that

\[ N(\mathbb{Z}) := \exp(\mathbb{Z} e_1 + \mathbb{Z} e_2 + \cdots + \mathbb{Z} e_d), \]

and

\[ F := \exp([0, 1] e_1) \exp([0, 1] e_2) \cdots \exp([0, 1] e_d) \]

is a fundamental domain for \( N(\mathbb{Z}) \sim N(\mathbb{R}) \).

For \( x \in M = N(\mathbb{Z}) \sim N(\mathbb{R}) \) and \( \epsilon > 0 \) small enough, \( U(x, \epsilon) \) denotes the box set \( N(\mathbb{Z}) \cdot \exp([0, \epsilon] e_1 + [0, \epsilon] e_2 + \cdots + [0, \epsilon] e_d) \subset N(\mathbb{Z}) \sim N(\mathbb{R}) \).

**Definition 3.6** (see [10]). For \( c, L > 0 \), we call \( w \in \mathbb{R}^l (c, L) \)-Diophantine if

\[ |\langle z, w \rangle| \geq c \| z \|^{-L} \]

for all \( z \in Z^i \sim \{0\} \).

The following lemma is proved in [10].

**Lemma 3.7** (see [10, Lemma 3.3]). Let \( V \subset \mathbb{R}^l \) be a subspace defined over \( \overline{\mathbb{Q}} \cap \mathbb{R} \) such that \( V \) is not contained in any proper subspace defined over \( \mathbb{Q} \). Then there exists \( w \in V \cap \overline{\mathbb{Q}} \) whose coordinates are real numbers linearly independent over \( \mathbb{Q} \).
For $p$-adic vector spaces, we have the following similar result:

**Lemma 3.8.** For any prime $p$, let $Q_p^a \subset Q_p$ denote the field of $p$-adic algebraic numbers. Let $V \subset Q_p^a$ be a subspace defined over $Q_p^a$ such that $V$ is not contained in any proper subspace defined over $Q$. Then there exists $w \in V \cap (Q_p^a)^l$ whose coordinates are linearly independent over $Q$.

**Proof.** The proof is the same as that of Lemma 3.7 (see [8, Lemma 3.3]). We will include the proof for completeness.

Since $V$ is defined over $Q_p^a$, we can choose a basis $\{v_i : i = 1, 2, \ldots, s\}$ of $V$ with coordinates in $Q_p^a$. Let $K \subset Q_p^a$ denote the field generated by their coordinates. Then $K$ is a finite extension of $Q$ in $Q_p$. Then we may choose $\alpha_1, \alpha_2, \ldots, \alpha_s \in Q_p^a$ which are linearly independent over $K$. Let $w := \sum_{i=1}^s \alpha_i v_i$. Let us denote $v_i = (u_{1i}, u_{2i}, \ldots, u_{li})$ for $i = 1, 2, \ldots, s$. Then for any $z = (z_1, z_2) \in Q^l$, we have that

$$\langle z, w \rangle = \sum_{i=1}^s z_i \sum_{j=1}^l \alpha_{ij} u_{ij} = \sum_{i=1}^s \sum_{j=1}^l z_i u_{ij} \alpha_i.$$

Since $\{\alpha_1, \ldots, \alpha_s\}$ are linearly independent over $K$, we have that $\langle z, w \rangle \neq 0$ unless

$$\sum_{j=1}^l z_j u_{ij} = 0,$$

i.e., $\langle z, w \rangle = 0$. Since we assume that $V$ is not in any proper $Q$-subspace, we have proved that the coordinates of $w$ are linearly independent over $Q$.

This completes the proof.

By [1, Theorem 7.3.2], a vector $w \in \mathbb{R}^l$ whose coordinates are algebraic numbers that are linearly independent over $Q$ is $(c, L)$-Diophantine for some $c, L > 0$.

For $p$-adic vectors, we have the same result:

**Lemma 3.9.** Let $w \in (Q_p^a)^l$ such that its coordinates are linearly independent over $Q$. Then $w$ is $(c, L)$-Diophantine for some constants $c, L > 0$.

**Proof.** Let $w = (w_1, w_2, \ldots, w_l)$. One can find a polynomial $P(x_1, x_2, \ldots, x_l)$ with integer coefficients such that the linear form $R(x_1, x_2, \ldots, x_l) := \sum_{j=1}^l w_j x_j$ is a factor of $P$ and $P(z'_1, z'_2, \ldots, z'_l) \neq 0$ for any nonzero integer vector $z' = (z'_1, z'_2, \ldots, z'_l)$ (just take the product of all Galois conjugates of the linear form $R(x_1, x_2, \ldots, x_l)$ and renormalize to make the coefficients integers). Let $K^l \subset Q_p^a$ denote the field generated by the coordinates of $w$. For any nonzero integer vector $z' = (z'_1, z'_2, \ldots, z'_l)$, we have that $P(z'_1, z'_2, \ldots, z'_l)$ is a nonzero integer. Moreover, since $P$ has coefficients in $\mathbb{Z}$, we have that $|P(z'_1, z'_2, \ldots, z'_l)| \leq c_1 \|z'\|^L$ for some constants $c_1, L > 0$ depending on $P$. Therefore

$$\|P(z'_1, z'_2, \ldots, z'_l)\|_p \geq |P(z'_1, z'_2, \ldots, z'_l)|^{1-L} \geq c_1^{-1} \|z'\|^{-L}.$$

Put $P = RQ$, then since $R$ and $P$ have coefficients in $K$, so does $Q$. It is easy to see that there exists a constant $c_2 > 0$, such that for any nonzero integer
vector $z' = (z'_1, z'_2, \ldots, z'_l)$, $\|Q(z'_1, z'_2, \ldots, z'_l)\|_p \leq c_2$ (in fact, $c_2$ is determined by the coefficients of $Q$). Therefore, for any $z' \in \mathbb{Z}^l \sim \{0\}$,
$$
\|(z', w)\|_p = \|R(z'_1, z'_2, \ldots, z'_l)\|_p
= \|P(z'_1, z'_2, \ldots, z'_l)\|_p \|Q^{-1}(z'_1, z'_2, \ldots, z'_l)\|_p
\geq c \|z'\|^{-L},
$$
where $c = c_1^{-1}c_2^{-1}$.

This completes the proof. \hfill \Box

Therefore, the vector $w$ we get from Lemma 3.7 (or Lemma 3.8) is $(c, L)$-Diophantine for some constants $c, L > 0$.

### 3.2. Ergodicity of the action.
Before proving Theorem 1.3, let us first study ergodicity of the $\mathbb{Z}^k$ action $\rho_l$ on the solenoid $\mathcal{F}(M)$. It is proved in [33] and [27] that a $\mathbb{Z}^k$ action on a nilmanifold $M$ by automorphisms is genuinely higher rank if and only if there exists a $\mathbb{Z}^2$ subgroup of $\mathbb{Z}^k$ all of whose nontrivial elements act ergodically. We will prove a similar result for $\mathbb{Z}^k$ actions on solenoids. Then we will prove Corollary 1.4 assuming Theorem 1.3.

**Definition 3.10.** We say $\rho_l : \mathbb{Z}_+^k \curvearrowright M$ is totally irreducible if for any sub-semi-group $\Gamma$ of $\mathbb{Z}_+^k$ of finite index, there are no proper $\Gamma$-invariant sub-nilmanifolds of $M$ with positive dimension.

**Lemma 3.11.** Let $\rho_l : \mathbb{Z}_+^k \curvearrowright M$ be an action by endomorphisms. Suppose $\rho_l$ is totally irreducible. We extend $\rho_l$ to a $\mathbb{Z}^k$ action on $\mathcal{F}(M)$, then every nontrivial $\rho_l(a)$ acts ergodically on $\mathcal{F}(M)$.

**Proof.** By Parry’s theorem [23], it suffices to show the statement when $M$ is a torus $\mathbb{Z}^{\dim M} \sim \mathbb{R}^{\dim M}$ (Although Parry’s theorem only takes care of nilmanifolds, its proof works for solenoids). Then every $\rho_l(a)$ can be extended to an action on $\mathbb{R}^{\dim M}$ and identified as an element in $\text{GL}(\dim M, \mathbb{Q})$.

We first claim that every $\rho_l(a)$ is semisimple. In fact, if $\rho_l(a)$ is not semisimple, then $\rho_l(a) = s(a)u(a) = u(a)s(a)$, where $s(a) \in \text{GL}(\dim M, \mathbb{Q})$ is semisimple and $u(a) \in \text{GL}(\dim M, \mathbb{Q})$ is unipotent. Then $\text{Fix}(u(a)) := \{v \in \mathbb{R}^{\dim M} : u(a)v = v\}$ is a nontrivial proper rational subspace of $\mathbb{R}^{\dim M}$ invariant under $\rho_l(\mathbb{Z}^k)$. Then $\text{Fix}(u(a))$ will define a proper $\rho_l(\mathbb{Z}^k)$ invariant subtorus of $M$ with positive dimension, which contradicts our assumption. This shows the claim.

We next claim that every nontrivial $\rho_l(a)$ does not admit any nontrivial proper invariant subtori of $M$. Otherwise, take a nontrivial proper $\rho_l(a)$ invariant subtorus with minimal dimension, say $M'$. Then $M'$ corresponds to a minimal $\rho_l(a)$-invariant rational subspace $V' \subset \mathbb{R}^{\dim M}$. For any $b \in \mathbb{Z}^k$, it is easy to see that $V' \cap \rho_l(b)V'$ is also a $\rho_l(a)$-invariant subspace defined over $\mathbb{Q}$. Since $V'$ is assumed to be minimal, we have $\rho_l(b)V' = V'$ or $\rho_l(b)V' \cap V' = \{0\}$. For the same reason, for any $b, b' \in \mathbb{Z}^k$, we have that $D\rho_l(b)V' = \rho_l(b)V'$ or $\rho_l(b)V' \cap \rho_l(b')V' = \{0\}$. Thus there are only finitely many possible Lie algebras $V''$ that $\rho_l(b)V'$ can be. This implies that there exists a subgroup $\Gamma$ of $\mathbb{Z}^k$ of finite index.
such that $V'$ is $\rho_1(\Gamma)$-invariant. This contradicts the assumption on irreducibility. Therefore, no $\rho_1(a)$ leaves any nontrivial proper subtori invariant.

Note that the dual space of $\mathcal{S}(\mathbb{Z}^{\dim M} \otimes \mathbb{R}^{\dim M})$ is $(\mathbb{Z}[\frac{1}{n}])^{\dim M}$ where $n = \prod_{s \in S} p$. For any nontrivial $\rho_1(a)$, if $\rho_1(a)$ is not ergodic, then there exists $z \in (\mathbb{Z}[\frac{1}{n}])^{\dim M}$ fixed by $\rho_1(a)$. This implies that $\rho_1(a)$ fixes a proper subtorus $M'$ of $M$, which contradicts the previous claim. 

**Lemma 3.12.** Let $\rho_1: \mathbb{Z}^k \to \mathcal{S}(M)$ be a genuinely higher rank action as above. Then there exists $a \in \mathbb{Z}^k$ such that $\rho_1(a)$ is ergodic.

**Proof.** For the same reason as above, we may assume that $M$ is a torus $\mathbb{Z}^{\dim M} \otimes \mathbb{R}^{\dim M}$. By [33, Corollary 6], we may assume that every $\rho_1(a) \in \text{GL}(\dim M, \mathbb{Q})$ is semisimple.

By passing to a subgroup of $\mathbb{Z}^k$ of finite index, we can decompose $M$ into almost direct product of $\rho_1$-invariant totally irreducible subtori:

$$M = M_1 \times \cdots \times M_s.$$ 

For each $i = 1, \ldots, s$, $\rho_1(a)$ acts either ergodically or trivially on $M_i$. Our goal is to find $a \in \mathbb{Z}^k$ such that $\rho_1(a)$ is not trivial on each $M_i$.

For contradiction, we suppose that every $\rho_1(a)$ acts trivially on some $M_i$. Define

$$\mathcal{F}_i := \{ a \in \mathbb{Z}^k : \rho_1(a) \text{ acts trivially on } M_i \},$$

then every $\mathcal{F}_i$ is a subgroup of $\mathbb{Z}^k$ and $\mathbb{Z}^k = \bigcup_{i=1}^s \mathcal{F}_i$. This implies that some $\mathcal{F}_i$ is a subgroup of $\mathbb{Z}^k$ of finite index, which contradicts our higher rank assumption (since the restriction of $\rho_1$ on $M_i$ is essentially trivial). 

**Definition 3.13.** We call an integer triple $(k, l, m) \in \mathbb{Z}^3$ primitive if $k, l, m$ do not have nontrivial common divisor.

**Lemma 3.14.** Let semisimple elements $A, B, C \in \text{GL}(d, \mathbb{Q})$ commute and act on $\mathcal{S}(\mathbb{Z}^d \otimes \mathbb{R}^d)$. Suppose for any $i, j \in \mathbb{Z}$, $A^i B^j$ is ergodic unless $i = j = 0$. Then there exist at most finitely many primitive triples $(k, l, m) \in \mathbb{Z}^3$ such that $A^k B^l C^m$ is not ergodic.

**Proof.** For $T \in \text{GL}(d, \mathbb{Q})$, define

$$\text{Fix}(T) := \{ v \in \mathbb{R}^d : Tv = v \}.$$ 

Let $S = A^k B^l C^m$ be a non-ergodic element. Then for some $r \in \mathbb{Z}$, $\text{Fix}(S^r)$ is nontrivial. Let $V = \text{Fix}(S^r)$. Then $V$ is rational and invariant under the action of $A, B$ and $C$.

If $V = \mathbb{R}^d$, then there exists only one primitive non-ergodic triple. In fact, for any primitive non-ergodic triple $(k_1, l_1, m_1)$, there exist $r_1 \in \mathbb{Z}$ and $v_1 \in \mathbb{R}^d \setminus \{0\}$, such that $A^{r_1 k_1} B^{r_1 l_1} C^{r_1 m_1} v_1 = v_1$. Since $A^k B^l C^m v_1 = v_1$ and $A^i B^j$ is ergodic for any $(i, j) \neq (0, 0)$, we have that $k_1 / k = l_1 / l = m_1 / m$.

If $V \neq \mathbb{R}^d$, then there exists a rational nontrivial $A, B, C$-invariant subspace $V'$ such that $\mathbb{R}^d = V \oplus V'$. $V$ and $V'$ correspond to nontrivial subtori $T$ and
we can pick the other generator of the form $mb$, $A^i B^j$ is ergodic on $\mathcal{I}(T)$ and $\mathcal{I}(T')$ for every $(i, j) \neq (0, 0)$ and $A^k B^l C^m$ is ergodic on $\mathcal{I}(\mathbb{Z}^d \sim \mathbb{R}^d)$ if and only if it is ergodic on both $\mathcal{I}(T)$ and $\mathcal{I}(T')$, we can complete the proof by induction.

**Proposition 3.15.** Let $\rho_1$ be a $\mathbb{Z}^k$ action on the solenoid $\mathcal{I}(M)$ of a nilmanifold $M$, extended from a $\mathbb{Z}^k_+$ action on $M$ by endomorphisms. If $\rho_1$ is genuinely higher rank, then there exists a subgroup $\Sigma$ of $\mathbb{Z}^k$ isomorphic to $\mathbb{Z}^2$ consisting of ergodic elements. In addition, given adjacent open Weyl chambers $\mathcal{C}$ and $\mathcal{C}_0$, we can find $\Sigma$ with generators in $\mathcal{C}$ and $\mathcal{C}_0$ respectively.

Proof. We will prove the following stronger statement: for any ergodic element $\rho_1(a)$, there exists a subgroup $\Sigma \cong \mathbb{Z}^2$ containing $a$ which consists of ergodic elements.

For the same reason as above, throughout this proof, we will assume that $M$ is a torus $\mathbb{Z}^{\dim M} \sim \mathbb{R}^{\dim M}$ and every $\rho_1(a) \in \text{GL}($dim $M, Q)$ is semisimple.

By passing to a subgroup of $\mathbb{Z}^k$ of finite index, we can decompose $M$ into almost direct product of $\rho_1$-invariant totally irreducible subtori:

$$M = M_1 \times \cdots \times M_s.$$ 

Let us prove the statement by induction on $s$.

When $s = 1$, the action $\rho_1$ is totally irreducible. Then the statement follows from Lemma 3.11. In fact, every nontrivial $\rho_1(a)$ is ergodic. Clearly we can choose $\Sigma$ with generators $a, b$ and $a \in \mathcal{C}_0$ and $b \in \mathcal{C}$ respectively.

Suppose the statement holds for $s - 1$; we shall prove the statement for $s$.

Let $M'_1 = M_2 \times \cdots \times M_s$. By Lemma 3.12, there exist ergodic elements. Take an ergodic element $a \in \mathbb{Z}^k$ with $a \in \mathcal{C}_0$. We want to show that there exists a subgroup $\Sigma \cong \mathbb{Z}^2$ generated by $a$ and some $b \in \mathcal{C}$ which consists of ergodic elements.

By the inductive assumption, there exist $b_1, b_2 \in \mathbb{Z}^k \cap \mathcal{C}$ such that the restriction of $\rho_1(a)$ and $\rho_1(b_1)$ to $\mathcal{I}(M_1)$ generate a $\mathbb{Z}^2$ action consisting of ergodic elements, and the restriction of $\rho_1(a)$ and $\rho_1(b_2)$ to $\mathcal{I}(M'_1)$ generate a $\mathbb{Z}^2$ action consisting of ergodic elements.

By Lemma 3.14 applied to $M_1$ and $M'_1$, there are at most finitely many primitive triples $(k, l, m) \in \mathbb{Z}^3$ such that the restriction of $\rho_1(ka + lb_1 + mb_2)$ onto $\mathcal{I}(M_1)$ or $\mathcal{I}(M'_1)$ is not ergodic. This implies that for all but finitely many primitive triples $(k, l, m) \in \mathbb{Z}^3$, $\rho_1(ka + lb_1 + mb_2)$ is ergodic on $\mathcal{I}(M)$.

Now consider $\mathbb{Z}^2$ subgroups of $\mathbb{Z}^3$ which contain $a$ as a generator. Then we can pick the other generator of the form $mb_1 + nb_2$ with $m, n \in \mathbb{Z}$. If some nontrivial $ka + l(mb_1 + nb_2)$ is not ergodic then its coefficients are a multiple of the finitely many primitive triples from above. In particular, $lm/n = m/n$ assumes one of finitely many values. Avoiding those gives us a $\mathbb{Z}^2$ which has only ergodic nontrivial elements, as desired. In addition, since $b_1$ and $b_2$ lie in $\mathcal{C}$, if $m, n$ are positive, then $mb_1 + nb_2$ belongs to $\mathcal{C}$ as well by convexity.

**Proof of Corollary 1.4 assuming Theorem 1.3.** By our assumption, $\rho_1$ is genuinely higher rank, then by Proposition 3.15, there exists a subgroup $\Sigma \cong \mathbb{Z}^2$ of $\mathbb{Z}^k$ such
that for every $a \in \Sigma$, $\rho_1(a)$ is ergodic. Then by Theorem 1.3, there exist constant $a_1 > 0$ and $\eta'_1 > 0$, such that for every $a \in \Sigma$, any $f \in C^0(M)$, considered as a function on $\mathcal{F}(M)$, and any $g \in C^0(\mathcal{F}(M))$, (1.1) holds. In addition, we can assume by Proposition 3.15 that $\Sigma$ intersects the adjacent Weyl chambers $\mathcal{C}$ and $\mathcal{C}_0$. Recall that $\rho_1$ and $\rho$ are conjugate via the bi-Hölder conjugacy $\phi$. Thus if the exponential mixing holds for $\rho_1$ and $\mu$, it also holds for $\rho$ and $\tilde{\mu} = \phi_*^{-1}(\mu)$. This proves Corollary 1.4. □

3.3. Maximal expanding factor. Assuming $\rho_1(a)$ is ergodic for every $a \neq 0 \in \mathbb{Z}^k$, we will study the maximal expanding factor of every $\rho_1(a)$.

**Lemma 3.16.** For $a \in \mathbb{Z}^k$, let $S(a) := \max_{x} \{ [\chi^i(a)] \}$ where $\chi^i$ runs over all Lyapunov exponents of $\rho_1$. Then

$$\inf_{a \in \mathbb{Z}^k} [S(a)] > 0.$$  

**Proof.** By passing from the action $\rho_1$ to its maximal torus factor $M_0$, we can reduce the proof to the case that $M$ is a torus $\mathbb{Z}^{\dim M} \simeq \mathbb{R}^{\dim M}$, again using Parry’s theorem [23] that a nilmanifold endomorphism is ergodic precisely when its projection to the maximal toral factor is ergodic. Then every $\rho_1(a), a \in \mathbb{Z}^k$, can be expressed as an element in $\text{GL}(\dim M, \mathbb{Q})$.

We first prove the statement assuming every $\rho_1(a), a \in \mathbb{Z}^k$, is semisimple.

For contradiction, suppose that there exists a sequence $\{a_r : r \in \mathbb{N}\}$ such that $\|a_r\| \to \infty$ as $r \to \infty$, and $S(a_r) \to 0$ as $r \to \infty$. Let $K \subset \overline{\mathbb{Q}}$ be a finite field extension of $\mathbb{Q}$ such that every $\rho_1(a_r)$ is diagonalizable in $\text{GL}(\dim M, K)$. So we can find a $g \in \text{GL}(\dim M, K)$ such that $g \rho_1(a_r) g^{-1}$ is a diagonal matrix in $\text{GL}(\dim M, K)$, denoted by $\text{diag}(a_r(i, i) : 1 \leq i \leq \dim M)$. Then $S(a_r) \to 0$ implies that for any prime ideal $\mathfrak{p}$ of the ring of algebraic integers $\mathcal{O}_K$, $|a_r(i, i)|_{\mathfrak{p}} \to 1$ as $r \to \infty$, for all $i = 1, 2, \ldots, \dim M$. Moreover, $|a_r(i, i)|_{\mathfrak{p}} \to 1$ as $r \to \infty$ for all $i = 1, 2, \ldots, \dim M$. Let $\mathcal{A}_K$ denote the adeles of $K$. Note that $K$ is discrete in $\mathcal{A}_K$, cf. [3, §2.14]. Thus there are only finitely many $u \in K$ with $|u|_v \leq 2$ for $v = \infty$ and $v = \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$. Therefore by passing to a subsequence of $\{a_r : r \in \mathbb{N}\}$, we have that for any $i = 1, \ldots, \dim M$, $a_r(i, i) = u_i$ for all $r > 0$ large enough. Then $|u_i|_v = 1$ for $v = \infty$ and $v = \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$. Moreover, for any $l \in \mathbb{N}$, we have that $|u_l|_v = 1$ for $v = \infty$ and $v = \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$. Thus we have that the set $\{u_l : l \in \mathbb{N}\}$ is finite since $K$ is discrete in $\mathcal{A}_K$. This implies that $u_l^{i_1} = 1$ for some integer $l_i > 0$. Let $U = \text{diag}(u_1, \ldots, u_{\dim M})$, then some power of $U$ is identity and $g \rho_1(a_r) g^{-1} = U$ for all large $r \in \mathbb{N}$. This implies right away that some finite power of $\rho_1(a_r)$ is the identity, and hence $\rho_1(a_r)$ is not ergodic, in contradiction to our hypothesis that all nontrivial elements are ergodic. This shows the statement assuming every $\rho_1(a_r)$ is semisimple.

Now we prove the statement in general. Suppose $\mathbb{Z}^k$ is generated by $a_1, \ldots, a_k$. Consider the Jordan decomposition of $\rho_1(a_1): \rho_1(a_1) = b_1 c_1$ with $b_1$ semisimple and $c_1$ unipotent. Since $c_1 \in \text{GL}(\dim M, \mathbb{Q})$, the eigenspace of $c_1$ with eigenvalue 1, which we denote by $W_1$, is nontrivial and defined over $\mathbb{Q}$. Also, $W_1$ is $\rho_1(\mathbb{Z}^k)$-invariant, and the restriction of $\rho_1(a_1)$ onto $W_1$ is semisimple. Repeating
this argument, we can find a sequence of rational $\rho_l(\mathbb{Z}^k)$-invariant subspaces $W_k \subset \cdots \subset W_2 \subset W_1$ such that for $i = 1, 2, \ldots, k$, the restriction of $\rho_l(a_i)$ on $W_i$ is semisimple. Then the restriction of $\rho_l(\mathbb{Z}^k)$ on $W_k$ is semisimple.

By the special case above, $\inf_{a \in Z^k \setminus \{0\}} \{S(a|W_k)\} > 0$. Then the statement follows since

$$\inf_{a \in Z^k \setminus \{0\}} \{S(a)\} \geq \inf_{a \in Z^k \setminus \{0\}} \{S(a|W_k)\}. \quad \square$$

**Lemma 3.17.** For all $a \in \mathbb{R}^k \setminus \{0\}$, we define $S(a) := \max_{i} |\chi^i(a)|$. Then $\sigma := \frac{1}{2} \inf_{a \in \mathbb{R}^k \setminus \{0\}} S(a) : a \in \mathbb{R}^k, \|a\| = 1$ is positive.

**Proof:** Let us first prove $S(a) > 0$ for every $a \in \mathbb{R}^k \setminus \{0\}$. Suppose $S(a) = 0$ for some $a \in \mathbb{R}^k \setminus \{0\}$. Since the line $\{ta : t \in \mathbb{R}\}$ comes arbitrarily close to integer points in $\mathbb{Z}^k$, we can find $t_1 \in \mathbb{R}$ and $a_1 \in \mathbb{Z}^k$ with $a_1 - t_1 a \to 0$ as $t \to \infty$. Since $S(t_1 a) = 0$, we have $S(a_i) \to 0$ as $i \to \infty$. This contradicts Lemma 3.16. This proves that $S(a) > 0$. Then the statement follows as $S$ is continuous.

**Remark 3.18.** Note that there are only finitely many Lyapunov exponents for $\rho_l$ and for any $a \in \mathbb{Z}^k$, $\Sigma^1 \chi^i(a) = 0$. Thus, the above lemma implies that there exists a constant $L' > 0$ such that for any $a \in \mathbb{Z}^k$, there exists a Lyapunov exponent $\chi^i(a)$ such that $\chi^i(a) \geq L'\|a\|$.  

3.4. *Proof of Theorem 1.3.* Now we are ready to prove Theorem 1.3.

We first prove the theorem for totally irreducible actions, and then deal with the general case.

**Totally irreducible case.** In this case we assume that the action $\rho_l : \mathbb{Z}^k \curvearrowright \mathcal{S}(\mathcal{M})$ is totally irreducible.

We deal with the following two cases separately:

**Case 1.** Let $L' > 0$ be the constant given in Remark 3.18. For all $a \in \mathbb{Z}^k$, there exists a real Lyapunov exponent $\chi^i(a)$ such that $\chi^i(a) \geq L'\|a\|$.

**Case 2.** Case 1 fails.

**Proof for Case 1.** For this case, the proof is more or less the same as that in [10]. Suppose $\chi^i_1(a) = \max_{i} \chi^i_1(a) : \chi^1 \in T(\mathbb{R})$. Recall that $\rho_l(a)$ is semisimple since $\rho_l$ is irreducible. Then, by our assumption, $\chi^i_1(a) \geq L'\|a\|$. Then $\sigma \chi^i_t \subset \mathfrak{h}(\mathbb{R})$ is the direct sum of the eigenspaces of $D\rho_l(a)$ with eigenvalues $\{\lambda_i(a) : i = 1, \ldots, r\}$ where $|\lambda_i(a)| = e^{\chi^i_1(a)}$ for every $i = 1, \ldots, r$. It is easy to see that $\sigma \chi^i_t$ is defined over $\mathbb{Q} \cap \mathbb{R}$. Thus $W' := D\sigma(\chi^i_t) \subset \mathbb{R}^k$ is also defined over $\mathbb{Q} \cap \mathbb{R}$. Then by Lemma 3.7 and the irreducibility assumption, there exists $w \in W'$ whose coordinates are algebraic numbers that are linearly independent over $\mathbb{Q}$. As discussed after Lemma 3.9, there exist constants $c, L > 0$ such that $|\langle w, z \rangle| \geq c\|z\|^{-L}$, for all $z \in \mathbb{Z}^l \setminus \{0\}$.

Let us fix a small constant $\epsilon > 0$. For each $p \in S$, we pick an integer $l_p(a) \geq 0$ such that $p^{-l_p(a)} \leq \epsilon$, and

$$\rho_l(a)(N(p^{l_p(a)}Z_p)) \subset N(Z_p).$$
We cut $\mathcal{F}(M)$ into disjoint small pieces along $p$-adic directions:

$$\mathcal{F}(M) = \bigcup_j B_j,$$

where

$$B_j := N(\mathbb{Z}) \sim \mathcal{F} \times \prod_{p \in S} \left( \xi_j(p)N(p^{l_j(a)}\mathbb{Z}_p) / M_p \right),$$

for a fixed fundamental domain $\mathcal{F} \subset N(\mathbb{R})$ of $N(\mathbb{Z}) \sim N(\mathbb{R})$ and some $\xi_j(p) \in N(\mathbb{Z}_p)$.

We fix a basis $\{w_1, \ldots, w_s\}$ of $\mathfrak{sl}_2$ and extend it to a fixed basis $\{w_1, \ldots, w_s, v_1, \ldots, v_s\}$ of $n(\mathbb{R})$. For $\varepsilon > 0$, we define

$$C(\varepsilon) := W(\varepsilon) + B(\varepsilon) \subset n(\mathbb{R}),$$

where $W(\varepsilon) := [-\varepsilon, \varepsilon]w_1 + \cdots + [-\varepsilon, \varepsilon]w_s$, and $B(\varepsilon) := [-\varepsilon, \varepsilon]v_1 + \cdots + [-\varepsilon, \varepsilon]v_s$. Then for $x \in N(\mathbb{Z}) \sim N(\mathbb{R})$ and $\varepsilon > 0$, we define

$$U(x, \varepsilon) := x exp(C(\varepsilon)) \subset N(\mathbb{Z}) \sim N(\mathbb{R}).$$

We then cut each $B_i$ into small pieces along the real component. In other words, we write

$$B_i = \bigcup_j B_{i,j},$$

where $B_{i,j} := U(x_i, \varepsilon) \times \prod_{p \in S} \xi_j(p)N(p^{l_j(a)}\mathbb{Z}_p) / M_p$, for some $x_i \in N(\mathbb{Z}) \sim N(\mathbb{R})$. Then

$$\int_{\mathcal{F}(M)} f(\rho_1(a)\overline{z})g(\overline{z})d\mu(\overline{z}) = \sum_{i,j} \int_{B_{i,j}} f(\rho_1(a)\overline{z})g(\overline{z})d\mu(\overline{z}).$$

It is easy to see that

$$\int_{B_{i,j}} f(\rho_1(a)\overline{z})g(\overline{z})d\mu(\overline{z}) = \left( g(\overline{z}_{i,j}) + O(\varepsilon^\theta \|g\|_\theta) \right) \int_{B_{i,j}} f(\rho_1(a)\overline{z})d\mu(\overline{z})$$

where $\overline{z}_{i,j} = (x_i, (\xi_j(p))_{p \in S}) \in B_{i,j}$. By Lemma 2.6, for each $j$, we may choose $n_j \in N(\mathbb{Z})$ such that $n_j^{-1}\xi_j(p) \in N(p^{l_j(a)}\mathbb{Z}_p)$ for all $p \in S$. Therefore,

$$B_{i,j} = U(n_j^{-1}x_i, \varepsilon) \times \prod_{p \in S} N(p^{l_j(a)}\mathbb{Z}_p) / M_p.$$

Since $\rho_1(a)(N(p^{l_j(a)}\mathbb{Z}_p)) \subset N(\mathbb{Z}_p)$ and since the value of the function $f$ only depends on its projection on $M$, we have that

$$\int_{B_{i,j}} f(\rho_1(a)\overline{z})d\mu(\overline{z}) = V \int_{U(n_j^{-1}x_i, \varepsilon)} f(\rho_1(a)x)d\nu(x),$$

where $V = \prod_{p \in S} v_p(N(p^{l_j(a)}\mathbb{Z}_p))$. Let $y_{i,j} := n_j^{-1}x_i$, then we have that

$$\int_{U(n_j^{-1}x_i, \varepsilon)} f(\rho_1(a)x)d\nu(x) = \int_{y_{i,j}} \int_{w \in W(\varepsilon)} f(y_{i,j}exp(D\rho_1(a)v + Dp_1(a)w))dw dv.$$
We want to show that for all \( v \in \mathfrak{n}(\mathbb{R}) \), the integral
\[
\frac{1}{\operatorname{Vol}(W(e))}\int_{w \in W(e)} f(y_{i,j} \exp(v + D\rho_1(a)w))dw
\]
estimates \( \int_{N(\mathbb{Z}) \sim N(\mathbb{R})} f(x)dv(x) \) with error \( O(e^{-\eta'L}\|f\|_\theta) \) for a constant \( \eta' > 0 \). Here \( \operatorname{Vol}(\cdot) \) denotes the volume with respect to the normalised Lebesgue measure \( dw \).

Let \( v \in \mathfrak{n}(\mathbb{R}) \) be fixed. We consider the box map
\[
i : [-\epsilon, \epsilon]^{s_1} \rightarrow \mathfrak{n}(\mathbb{R})
\]given by
\[
t = (t_1, \ldots, t_{s_1}) \in [-\epsilon, \epsilon]^{s_1} \mapsto v + t_1 w_1' + \cdots + t_{s_1} w_{s_1}',
\]where \( w_i' = D\rho_1(a)(w_i) \). Then it is easy to see that
\[
\frac{1}{\operatorname{Vol}(W(e))}\int_{w \in W(e)} f(y_{i,j} \exp(v + D\rho_1(a)w))dw = (2e)^{-s_1}\int_{[-\epsilon, \epsilon]^{s_1}} f(y_{i,j} \exp(i(t)))dt.
\]
Then the integral (3.2) is the integral of \( f \) along the box map \( i \) and based at the point \( y_{i,j} \). For contradiction, suppose (3.2) does not estimate \( \int_M f dv \) with error \( O(e^{-\eta'L}\|f\|_\theta) \). Then by Theorem 3.2, there exists \( z \in Z^l \sim \{0\} \) such that \( \|z\| \ll e^{L\eta'\|a\|} \) and
\[
|\langle z, D\pi(w_i') \rangle| \ll e^{L\eta'\|a\|} \epsilon.
\]
Recall that by irreducibility, all elements are semisimple, and hence \( \|w_i'\| = \|D\rho_1(a)w_i\| = e^{x_i'(a)}\|w_i\| \gg e^{L'\|a\|} \). Since \( \sigma x_i' \) is spanned by \( \{w_1', \ldots, w_{s_1}'\} \), we have that for all \( w \in \sigma x_i' \) with \( \|w\| = 1 \),
\[
|\langle z, D\pi(w) \rangle| \ll e^{(L_2\eta' - L')\|a\|} \epsilon.
\]
On the other hand, by Lemma 3.7, there exists \( w \in W \) such that
\[
|\langle D\pi(w), z \rangle| \gg \|z\|^{-L} \gg e^{-L_1L\eta'\|a\|}.
\]
Let \( \epsilon = e^{-L_1\|a\|} \) such that \( 0 < L_3 < L'/2 \). Then (3.3) and (3.4) will lead to a contradiction if \( L_3 + L_2\eta' - L' < -L_1L\eta' \). This shows that there exists constant \( \eta' > 0 \) such that, for all \( v \in \mathfrak{n}(\mathbb{R}) \), integral (3.2) estimates \( \int_M f dv \) with error \( O(e^{-\eta'L}\|f\|_\theta) \). This implies that
\[
\int_{U(y_{i,j}, \epsilon)} f(\rho_1(a)x)dv(x) = \int_{v \in \mathfrak{b}(\epsilon)} \operatorname{Vol}(W(e))\left( \int_M f dv + O(e^{-\eta'L}\|f\|_\theta) \right) dv
= \operatorname{Vol}(B(e))\operatorname{Vol}(W(e))\left( \int_M f dv + O(e^{-\eta'L}\|f\|_\theta) \right)
= v(U(y_{i,j}, \epsilon))\left( \int_M f dv + O(e^{-\eta'L}\|f\|_\theta) \right).
\]
Therefore,
\[
\int_{B_{i,j}} f(\rho_1(a)z) \, d\mu(z) = \prod_{p \in S} \nu_p(N(p^l_i(a)z_p)) \nu(U(y_{i,j},e)) \left( \int_{M} f \, dv + O(e^{-\eta\|a\||f\|_\theta}) \right)
\]
\[
= \prod_{p \in S} \nu_p(N(p^l_i(a)z_p)) \nu(U(x_{i,j},e)) \left( \int_{M} f \, dv + O(e^{-\eta\|a\||f\|_\theta}) \right)
\]
\[
= \mu(B_{i,j}) \left( \int_{M} f \, dv + O(e^{-\eta\|a\||f\|_\theta}) \right)
\]
\[
= \mu(B_{i,j}) \left( \int_{\mathcal{M}} f \, d\mu + O(e^{-\eta\|a\||f\|_\theta}) \right).
\]
Finally,
\[
\int_{\mathcal{M}} f(\rho_1(a)z) g(z) \, d\mu(z)
\]
\[
= \sum_{i,j} \int_{B_{i,j}} f(\rho_1(a)z) g(z) \, d\mu(z)
\]
\[
= \sum_{i,j} \left( g(z_{i,j}) + O(e^{d\|g\|_\theta}) \right) \int_{B_{i,j}} f(\rho_1(a)z) \, d\mu(z)
\]
\[
= \sum_{i,j} \left( g(z_{i,j}) + O(e^{d\|g\|_\theta}) \right) \mu(B_{i,j}) \left( \int_{\mathcal{M}} f \, d\mu + O(e^{-\eta\|a\||f\|_\theta}) \right)
\]
\[
= \int_{\mathcal{M}} f \, d\mu \sum_{i,j} g(z_{i,j}) \mu(B_{i,j}) + O(e^{d\|g\|_\theta}) \int_{\mathcal{M}} f \, d\mu \sum_{i,j} \mu(B_{i,j})
\]
\[
+ O(e^{-\eta\|a\||f\|_\theta}) \sum_{i,j} g(z_{i,j}) \mu(B_{i,j}) + O(e^{d\|g\|_\theta}) O(e^{-\eta\|a\||f\|_\theta}) \sum_{i,j} \mu(B_{i,j})
\]
\[
= \int_{\mathcal{M}} f \, d\mu \int_{\mathcal{M}} g \, d\mu + O(e^{d\|g\|_\theta}) + O(e^{d\|g\|_\theta}) \int_{\mathcal{M}} f \, d\mu
\]
\[
+ O(e^{-\eta\|a\||f\|_\theta}) \left( \int_{\mathcal{M}} g \, d\mu + O(e^{d\|g\|_\theta}) \right) + O(e^{d\|g\|_\theta} e^{-\eta\|a\||f\|_\theta} \|g\|_\theta).
\]
Since \(|\int_{\mathcal{M}} f \, d\mu| \leq \|f\|_\theta, |\int_{\mathcal{M}} g \, d\mu| \leq \|g\|_\theta\) and \(e = e^{-l_\sqrt{\|a\|}}\), we have there exists a constant \(\eta > 0\) such that
\[
\int_{\mathcal{M}} f(\rho_1(a)z) g(z) \, d\mu(z) = \int_{\mathcal{M}} f \, d\mu \int_{\mathcal{M}} g \, d\mu + O(e^{-\eta\|a\||f\|_\theta} \|g\|_\theta).
\]
This finishes the proof for Case 1. \(\square\)

**Proof for Case 2.** By Remark 3.18, there exists a Lyapunov exponent \(\chi_2^l \in T(Q_p)\) for some \(p \in S\) such that \(\chi_2^l(a) \geq L\|a\|.\) We may further assume that \(\chi_2^l(a) = \max_{\chi^l} \{\chi^l(a)\}.\) Let \(\sigma^{\chi_2^l} \subset n(Q_p)\) denote the generalized eigenspace of \(D\rho_1(a)\) with generalized eigenvalue \(e^{\chi_2^l(a)}\).

Let us fix a basis \(\{w_1, \ldots, w_{s_l}\}\) of \(\sigma^{\chi_2^l}\) and extend \(\{w_1, \ldots, w_{s_l}\}\) to a basis
\[
\{w_1, \ldots, w_{s_l}, v_1, \ldots, v_{s_2}\}
\]
of \( n(Q_p) \). Without loss of generality, we may assume \( \|w_i\|_p = 1 \) for \( i = 1, \ldots, s_1 \). Then \( \{D \rho_1(a)w_1, \ldots, D \rho_1(a)w_{s_1}\} \) is also a basis of \( \sigma^{k_1} \) and \( \|D \rho_1(a)w_i\|_p = e^{k_1(a)} \) for \( i = 1, 2, \ldots, s_1 \). Denote \( e^{k_1(a)} = p^h \), and \( D \rho_1(a)w_i = p^{-h}u_i \) for \( i = 1, \ldots, s_1 \). Then \( \{u_1, \ldots, u_{s_1}\} \) is a basis of \( \sigma^{k_1} \) and \( \|u_i\|_p = 1 \) for \( i = 1, \ldots, s_1 \).

Pick \( \delta > 0 \) small enough that \( \delta < e^{-L\|a\|} \) and the diameter of \( \rho_1(a)(U(x, \delta)) \) is less than \( e^{-L\|a\|/2} \) for all \( x \in M \). Let \( \epsilon = e^{-L\|a\|/2} \). For \( S \ni q \neq p \), let \( l_q(a) > 0 \) denote the smallest integer such that \( q^{-l_q(a)} \leq \epsilon \) and \( \rho_1(a)(N(q^{l_q(a)}Z_q)) \subset N(Z_q) \). Let \( l_p > 0 \) denote the smallest integer such that \( p^{-l_p} \leq \epsilon \). Let us cut \( \mathcal{F}(M) \) into small pieces as follows:

\[
\mathcal{F}(M) = \bigcup_j B_j,
\]

where \( B_j := N(Z) / \xi_j(p)N(p^hZ_p) / M_p \times U(x_j, \epsilon) \times \prod_{S \ni q \neq p} \xi_j(q)N(q^{l_q(a)}Z_q) / M_q \) for some \( x_j \in N(R) \), \( \xi_j(p) \in N(Z_p) \) and \( \xi_j(q) \in N(Z_q) \). Since \( e^{k_1(a)} = p^h \) and since \( \chi^1(a) \) is maximal among all \( \chi^1(a) \), we have that \( \rho_1(a)(N(p^hZ_p)) \subset N(Z_p) \).

For a positive integer \( h' \), define \( u_p(h') \subset n(Z_p) \) as

\[
u_p(h') := \left\{ t_1 w_1 + \cdots + t_{s_1} w_{s_1} + b_1 v_1 + \cdots + b_{s_2} v_{s_2} : \right\}
\]

\[
\|t_i\|_p \leq p^{-l_p} \text{ for } i = 1, \ldots, s_1 \text{, and } \|b_i\|_p \leq p^{-h'} \text{ for } i = 1, \ldots, s_2
\}

and define \( U_p(h') := \exp(u_p(h')) \subset N(Z_p) \).

Let us further cut each \( B_j \) along the \( p \)-adic direction:

\[
B_j = \bigcup_i B_{i,j},
\]

where \( B_{i,j} := N(Z) / \xi_{i,j}(p)U_p(h) / M_p \times U(x_j, \epsilon) \times \prod_{S \ni q \neq p} \xi_{i,j}(q)N(q^{l_q(a)}Z_q) / M_q \) for some \( \xi_{i,j}(p) \in N(Z_p) \). By Lemma 2.6, there exists \( n_{i,j} \in N(Z) \) such that \( n_{i,j}^{-1} \xi_{i,j}(p) \in U_p(h) \) and \( n_{i,j}^{-1} \xi_{i,j}(q) \in N(q^{l_q(a)}Z_q) \) for \( q \neq p \). Then

\[
B_{i,j} = N(Z) / U_p(h) / M_p \times U(y_{i,j}, \epsilon) \times \prod_{q \neq p} N(q^{l_q(a)}Z_q) / M_q,
\]

where \( y_{i,j} = n_{i,j}^{-1} x_j \). Then

\[
\int_{\mathcal{F}(M)} f(\rho_1(a)\overline{z}) g(\overline{z}) d\mu(\overline{z}) = \sum_{i,j} \int_{B_{i,j}} f(\rho_1(a)\overline{z}) g(\overline{z}) d\mu(\overline{z})
\]

\[
= \sum_{i,j} (g(\overline{z}_{i,j}) + O(e^{\|g\|_\theta})) \int_{B_{i,j}} f(\rho_1(a)\overline{z}) d\mu(\overline{z}),
\]

where \( \overline{z}_{i,j} = (x_j, \xi_{i,j}(p), (\xi_{i,j}(q))_{S \ni q \neq p}) \in B_{i,j} \).

By the argument in the proof for Case 1, to prove the exponential mixing result it suffices to show that

\[
\int_{B_{i,j}} f(\rho_1(a)\overline{z}) d\mu(\overline{z}) = \mu(B_{i,j}) \left( \int_{\mathcal{F}(M)} f d\mu + O(e^{-\eta^1(a)} \|f\|_\theta) \right),
\]
for a constant \( \eta' > 0 \). To estimate the above integral, we cut \( U_p(h) \) further as follows:

\[
U_p(h) = \bigcup_l \kappa_l(p) N(p^h \mathbb{Z}_p),
\]

where \( \kappa_l(p) \in U_p(h) \). According to this we can cut \( B_{i,j} \) into small pieces:

\[
B_{i,j} = \bigcup_l B_{i,j,l},
\]

where

\[
B_{i,j,l} := N(\mathbb{Z}) \sim \kappa_l(p) N(p^h \mathbb{Z}_p) / M_p \times U(y_{i,j}, e) \times \prod_{q \neq p} N(q^{l_q(a)} \mathbb{Z}_q) / M_q.
\]

Then

\[
\int_{B_{i,j,l}} f(\rho_l(a)z) \, d\mu(z) = \sum_l \int_{B_{i,j,l}} f(\rho_l(a)z) \, d\mu(z).
\]

Now let us look at \( \int_{B_{i,j,l}} f(\rho_l(a)z) \, d\mu(z) \) more carefully. First note that we can choose \( \kappa_l(p) \) to run over elements in

\[
\Delta(w_1, \ldots, w_{s_1}) := \{ \exp(t_1 w_1 + \cdots + t_{s_1} w_{s_1}) : t_i = 0, p^{l_p} \cdot 1, p^{l_p} \cdot 2, \ldots, p^{l_v(p-h-l_p - 1)} \}.
\]

By Lemma 2.6, for each \( w_i \), there exists \( w_i(h) \in \mathbb{N} \) such that \( w_i(h) \equiv w_i \) (mod \( p^h \)) and \( w_i(h) \equiv 0 \) (mod \( q^{l_q(a)} \)) for \( q \neq p \). Suppose \( \kappa_l(p) = \exp(t_1 w_1 + \cdots + t_{s_1} w_{s_1}) \), then direct calculation shows that

\[
\int_{B_{i,j,l}} f(\rho_l(a)z) \, d\mu(z) = V \int_{U(y_{i,j}, e)} f(\rho_l(a)(n_l^{-1}z)) \, d\nu(z),
\]

where \( V := v_p(N(p^h \mathbb{Z}_p)) \times \prod_{q \neq p} v_q(N(q^{l_q(a)} \mathbb{Z}_q)) \) and \( n_l := \exp(t_1 w_1 + \cdots + t_{s_1} w_{s_1}(h)) \). Let \( L' \) be the constant from Remark 3.18. Note that the diameter of \( \rho_l(a)U(y_{i,j}, e) \) is less than \( e^{-L' \|a\|} \), we have that

\[
\int_{B_{i,j,l}} f(\rho_l(a)z) \, d\mu(z) = V \nu(U(y_{i,j}, e)) \{ f(\rho_l(a)(n_l^{-1}y_{i,j})) + O(e^{-L' \|a\| \|f\|_\theta}) \}
\]

\[
= \mu(B_{i,j,l}) \{ f(\rho_l(a)(n_l^{-1}y_{i,j})) + O(e^{-L' \|a\| \|f\|_\theta}) \}.
\]

Therefore, to show the exponential mixing result, it suffices to show that the following summation

\[
\frac{1}{p^{(h-l_p)s_1}} \sum_{t_1, \ldots, t_{s_1}} f(\rho_l(a) \exp\left(- (t_1 p^{l_p} w_1(h) + \cdots + t_{s_1} p^{l_v} w_{s_1}(h)) y_{i,j}\right))
\]

estimates \( \int_{\mathcal{M}} f \, d\mu \) with error \( O(e^{-\eta' \|a\| \|f\|_\theta}) \). There is a factor of \( \frac{1}{p^{l_p \cdot \eta'}} \) in front of the summation because \( \mu(B_{i,j,l}) = \frac{\mu(B_{i,j,l})}{p^{l_p \cdot \eta'}} \).

First note that

\[
f\left(\rho_l(a) \exp\left(- (t_1 p^{l_p} w_1(h) + \cdots + t_{s_1} p^{l_v} w_{s_1}(h)) y_{i,j}\right) \right)
\]

\[
= f \left( \exp \left( - \sum_{j=1}^{s_1} t_j p^{l_v} D\rho_l(a) w_j(h) \right) z_{i,j} \right),
\]
where $z_{i,j} := \rho_i(a) y_{i,j}$. By previous discussion $D\rho_i(a) w_j = p^{-h} u_j$, for $j = 1, \ldots, s_1$. Let $u_j(h) \in n(\mathbb{Z})$ be such that $u_j(h) \equiv u_j \pmod{p^h}$ and $u_j(h) \equiv 0 \pmod{q^l_i(a)}$ for $S \ni q \neq p$. Then the difference between $\exp(-\sum_{j=1}^{s_1} t_j p^{l_i} D\rho_i(a) w_j(h)) z_{i,j}$ and $\exp(-p^{-h+l_p} \sum_{j=1}^{s_1} t_j u_j(h)) z_{i,j}$ is in $N(\mathbb{Z})$. Hence

$$f \left( \exp \left( -\sum_{j=1}^{s_1} t_j p^{l_i} D\rho_i(a) w_j(h) \right) z_{i,j} \right) = f \left( \exp \left( -p^{-h+l_p} \sum_{j=1}^{s_1} t_j u_j(h) \right) z_{i,j} \right).$$

Therefore we reduce our task to proving that

$$\frac{1}{p^{(h-l_p)s_1}} \sum_{t_1, \ldots, t_{s_1}} f \left( \exp \left( -p^{-h+l_p} \sum_{j=1}^{s_1} t_j u_j(h) \right) z_{i,j} \right) = \int_M f \, dv + O(e^{-\eta'[a]\|f\|_B}).$$

For $t = (t_1, \ldots, t_{s_1}) \in [0, p^{h-l_p} - 1]^{s_1}$, let us denote

$$P(t) := \exp \left( -p^{-h+l_p} \sum_{j=1}^{s_1} t_j u_j(h) \right) z_{i,j},$$

then it is easy to see that $P : [0, p^{h-l_p} - 1]^{s_1} \to N(\mathbb{R})$ is a polynomial map. Then by Theorem 3.3, either

$$\frac{1}{p^{(h-l_p)s_1}} \sum_{t \in [0, p^{h-l_p} - 1]^{s_1}} f(P(t)) = \int_M f \, dv + O(e^{-\eta'[a]\|f\|_B}),$$

which is case AA.1 in Theorem 3.3, or there exists $z \in \mathbb{Z}^l \sim \{0\}$ such that $\|z\| \ll e^{L^2 \eta'[a]}$ and

$$\text{dist}(\langle z, \partial_i \pi(P(t)) \rangle, \mathbb{Z}) \ll e^{L^2 \eta'[a]} / p^{h-l_p}$$

for all $i = 1, \ldots, s_1$ and $t \in [0, p^{h-l_p} - 1]^{s_1}$, which is case AA.2. Suppose the latter holds. Since $N'(Q_p) \sim N(Q_p) \cong Q_p^l$ is abelian, we may identify $Q_p^l$ with its Lie algebra. For $\nu \in n(Q_p)$, let $\tilde{\nu} \in \mathbb{Q}^l_p$ denote the projection of $\nu$ onto $Q_p^l$ under $D\pi$. Then it is easy to see that

$$\partial_i \pi(P(t)) = p^{-h+l_p} \tilde{u}_i(h)$$

for $i = 1, \ldots, s_1$. Then

$$\text{dist}(\langle z, \tilde{u}_i(h) / p^{h-l_p} \rangle, \mathbb{Z}) \ll e^{L^2 \eta'[a]} / p^{h-l_p},$$

for $i = 1, \ldots, s_1$. In other words, in $(\pmod{p^{h-l_p}})$ sense, $|\langle z, \tilde{u}_i(h) \rangle| \ll e^{L^2 \eta'[a]}$. Let $k$ denote a positive integer with $(k, p) = 1$. In the construction above, we can replace the basis $w_1, \ldots, w_{s_1}$ by $k w_1, \ldots, k w_{s_1}$ as the generalized eigenspace $\sigma^Y_i$ is invariant under multiplication by $k$. Note that multiplication by $k$ does not change the $p$-adic norm of any vector since $(k, p) = 1$. Tracking the argument, we eventually replace the polynomial $P(t)$ by $P^k(t) := \exp \left( -p^{-h+l_p} \sum_{j=1}^{s_1} t_j k u_j(h) \right) z_{i,j}$. By the same argument as for $P$, either the sum

$$\frac{1}{p^{(h-l_p)s_1}} \sum_{t \in [0, p^{h-l_p} - 1]^{s_1}} f(P(t)) = \int_M f \, dv + O(e^{-\eta'[a]\|f\|_B}),$$
and we are done, or there exists \( z(k) \in \mathbb{Z}^l \setminus \{0\} \) with \( \|z(k)\| < e^{L_1 \eta'} a \) and in (mod \( p^{h_l - l_p} \)) sense,

\[
\langle (z(k), k \tilde{u}_i(h)) \rangle < e^{L_2 \eta'} a
\]

for \( i = 1, \ldots, s_1 \). We choose \( \eta' \) small enough such that

\[
(e^{L_1 \eta'} a)^l \times (e^{L_2 \eta'} a)^{s_1} \leq (p^{h_l - l_p})^{1/8}.
\]

By the pigeonhole principle, there exist \( k_1, k_2 \in [-p^{(h_l - l_p)/4}, p^{(h_l - l_p)/4}] \) such that \( z(k_1) = z(k_2) \) and

\[
\langle z(k_1), k_1 \tilde{u}_i(h) \rangle \equiv \langle z(k_2), k_2 \tilde{u}_i(h) \rangle \mod p^{h_l - l_p}
\]

for \( i = 1, \ldots, s_1 \). Let us denote \( z(k_1) = z(k_2) \) by \( z \). Then \( \|z\| < e^{L_1 \eta'} a \) and

\[
\langle z, (k_1 - k_2) \tilde{u}_i(h) \rangle \equiv 0 \mod p^{h_l - l_p}
\]

for \( i = 1, \ldots, s_1 \). Since \( |k_1 - k_2| < 2p^{(h_l - l_p)/4} \), we have that \( (k_1 - k_2, p^{h_l - l_p}) \leq p^{(h_l - l_p)/4} \).

This shows that for some \( h' \geq 3(h - l_p)/4 \),

\[
\langle z, \tilde{u}_i(h) \rangle \equiv 0 \mod p^{h'}
\]

for \( i = 1, \ldots, s_1 \).

Let \( W \subset N'(Q_p) \sim N(Q_p) = Q_p^l \) denote the projection of \( \sigma^l \) on \( Q_p^l \) under \( \pi \). Apparently \( W \) is defined over \( Q_p^l \) and not defined over \( Q \). Let us choose a basis \( \{\tilde{u}_i : i = 1, 2, \ldots, s_1\} \) of \( W \) with coordinates in \( Q_p^l \). By Lemma 3.8, there exists \( \overline{w} = \sum_{i=1}^{s_1} \alpha_i \tilde{u}_i \in W \cap (Q_p^{al})^l \) whose coordinates are linearly independent over \( Q \). Moreover, multiplying by suitable powers of \( p \), we can choose \( \alpha_i \in \mathbb{Z}_p \) for \( i = 1, 2, \ldots, s_1 \). Then we have for any \( z' \in \mathbb{Z}^l \setminus \{0\} \), \( \langle z', \tilde{u}_i \rangle \neq 0 \).

On the one hand, since \( \|z, \tilde{w}\|_p \leq p^{-h'} \leq e^{-3L_1 |a|/8} \) for \( i = 1, 2, \ldots, s_1 \), we have that \( \| \langle z', \tilde{w}\rangle \|_p \leq p^{-h'} \) (by choosing each \( \alpha_i \) from \( \mathbb{Z}_p \)).

On the other hand, by Lemma 3.9, there exists a constant \( L > 0 \) such that for all \( z' \in \mathbb{Z}^l \setminus \{0\} \),

\[
\| \langle z', \tilde{w}\rangle \|_p \gg \| z' \|^{-L} \gg e^{-LL_1 \eta'} a
\]

The above two inequalities will lead to contradiction if \( LL_1 \eta' < 3L' / 8 \). This shows that

\[
\frac{1}{p^{(h_l - l_p)s_1}} \sum_{t \in [0, p^{h_l - l_p - 1}]} f(P(t)) = \int_M f dv + O(\|e^{-\eta'|a|} f\|_\theta),
\]

for a constant \( \eta' > 0 \). By the argument in the proof for Case 1, this finishes the proof of exponential mixing for Case 2.

This completes the proof of the totally irreducible part of Theorem 1.3. \( \square \)

In fact, using the same argument, one can prove a slightly stronger result:
\textbf{Proposition 3.19.} Under the totally irreducible assumption as above, let $\beta$ be an automorphism of $N$ defined over $\mathbb{Q}$ such that $\beta = \text{Id}$ on $N' \sim N$, then there exists a constant $\eta > 0$ such that
\[
\int_{\mathcal{F}(M)} f(\beta(\rho_1(a)z))g(z)\,d\mu(z) = \int_{\mathcal{F}(M)} f\,d\mu \int_{\mathcal{F}(M)} g\,d\mu + O(e^{-\eta||a||}||f||\theta\|g\|\theta),
\]
for any $f \in C^0(M)$ and any $g \in C^0(\mathcal{F}(M))$.

\textit{Proof.} From the proof above, we see that the basic scheme of the argument goes as follows: We first cut the whole space $\mathcal{F}(M)$ into small pieces along real and $p$-adic directions, and then apply Theorem 3.2 or Theorem 3.3 prove each small piece estimates the integral of the whole space with exponentially small error. The obstruction of the effective equidistribution is that $D\beta(\rho_1(a)(D\pi(\sigma^1)))$ lies in a rational linear subspace. Since $\beta$ acts trivially on $N' \sim N$, the proposition follows from the argument above. \hfill \Box

\textbf{General case.} We will prove the statement in general using induction on the dimension of $M$. By [10, Lemma 3.5], if the action $\rho_1$ on $M$ is not totally irreducible, then there exists a $\rho_1$-invariant nontrivial normal subgroup $N_1$ of $N$ defined over $\mathbb{Q}$ satisfying the following:

1. The restriction of $\rho_1$ on $N_1$ is totally irreducible.
2. $[N, N_1] \subset [N_1, N_1]$.

Then $Y := N(\mathbb{Z})N_1(\mathbb{R}) \sim N(\mathbb{R})$ and $Z := N_1(\mathbb{Z}) \sim N_1(\mathbb{R})$ are both compact nilmanifolds, and moreover, $M$ fibers over $Y$ with fibers isomorphic to $Z$. Let $\mu_Y$ and $\mu_Z$ denote the normalized measures on the solenoids $\mathcal{F}(Y)$ and $\mathcal{F}(Z)$ respectively, defined as the measure $\mu$ on $\mathcal{F}(M)$. Then for any continuous function $f$ defined on $\mathcal{F}(M)$, we have the following disintegration formula:
\[
\int_{\mathcal{F}(M)} f\,d\mu = \int_{\mathcal{F}(Y)} \int_{\mathcal{F}(Z)} f(z)\,d\mu_Z(z)\,d\mu_Y(z).
\]

Since $N_1$ is $\rho_1$-invariant, $\rho_1$ defines transformations of $Y$ and $Z$. Then
\[
\int_{\mathcal{F}(M)} f(\rho_1(a)\overline{x})g(\overline{x})\,d\mu(\overline{x}) = \int_{\mathcal{F}(Y)} \left( \int_{\mathcal{F}(Z)} f(\rho_1(a)(\overline{x})\rho_1(a)(\overline{y}))g(z)\,d\mu_Z(z) \right)\,d\mu_Y(\overline{y})
\]
\[
= \int_{\mathcal{F}} \left( \int_{\mathcal{F}(Z)} f(\rho_1(a)(\overline{x})\rho_1(a)(\overline{h}))g(\overline{x}\overline{h})\,d\mu_Z(\overline{x}) \right)\,dm_{\mathcal{F}}(\overline{h}),
\]

where $\mathcal{F} \subset N(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p)$ is a bounded fundamental domain for $\mathcal{F}(Y)$, and $m_{\mathcal{F}}$ denotes the measure on $\mathcal{F}$ induced by $\mu_Y$.

\textbf{Claim.} There exists a constant $\eta > 0$ such that for every $\overline{h} \in \mathcal{F}$,
\[
\int_{\mathcal{F}(Z)} f(\rho_1(a)(\overline{x})\rho_1(a)(\overline{h}))g(\overline{x}\overline{h})\,d\mu_Z(\overline{x})
\]
\[
= \int_{\mathcal{F}(Z)} f(\overline{x}\rho_1(a)(\overline{h}))\,d\mu_Z(\overline{x}) \int_{\mathcal{F}(Z)} g(\overline{x}\overline{h})\,d\mu_Z(\overline{x}) + O(e^{-\eta||a||}||f||\theta\|g\|\theta).}
\]
Proof of the claim. Let us write
\[ \rho_1(a)(\bar{h}) = \lambda \delta a \] with \( \alpha, \delta \in N_1(\mathbb{R}) \times \prod_{p \in S} N_1(\mathbb{Z}_p) \) and \( \lambda \in \mathbb{N} \).

Then
\[
\int_{\mathcal{F}(Z)} f(\rho_1(a)(Z)\rho_1(a)(\bar{h}))g(\bar{Z}\bar{h})d\mu_Z(Z) = \int_{\mathcal{F}(Z)} f(\beta(\rho_1(a)(Z))\delta a)g(\bar{Z}\bar{h})d\mu_Z(Z),
\]
where \( \beta \) denotes the transformation of \( \mathcal{F}(Z) \) induced by the automorphism \( m \rightarrow \lambda^{-1} m \lambda, m \in N_1(\mathbb{R}) \times \prod_{p \in S} N(\mathbb{Z}_p) \). Then obviously \( \beta \) is defined over \( \mathbb{Q} \). Since \([N, N_1] \subset [N_1, N_1] \), \( \beta \) acts trivially on \([N_1, N_1] \sim N_1 \). Let
\[ \phi_\theta(Z) := f(\bar{Z}\delta a) \text{ and } \phi_1(Z) := g(\bar{Z}\bar{h}) \text{ with } \bar{Z} \in \mathcal{F}(Z). \]

Then we have
\[ \| \phi_0 \|_\theta \ll \| f \|_\theta \text{ and } \| \phi_1 \|_\theta \ll \| g \|_\theta, \]
and
\[
\int_{\mathcal{F}(Z)} \phi_1 d\mu_Z = \int_{\mathcal{F}(Z)} f(\bar{Z}\rho_1(a)(\bar{h}))d\mu_Z(Z).
\]
Since every \( \rho_1(a) \) is still ergodic when restricted to \( \mathcal{F}(Z) \), we can apply Proposition 3.19 to conclude that there exists a constant \( \eta > 0 \) such that
\[
\int_{\mathcal{F}(Z)} f(\rho_1(a)(Z)\rho_1(a)(\bar{h}))g(\bar{Z}\bar{h})d\mu_Z(Z)
\]
\[ = \int_{\mathcal{F}(Z)} \phi_0(\beta(\rho_1(a)(Z)))\phi_1(Z)d\mu_Z(Z) \]
\[ = \int_{\mathcal{F}(Z)} \phi_0 d\mu_Z \int_{\mathcal{F}(Z)} \phi_1 d\mu_Z + O(\eta\|a\|_\theta \| \phi_0 \|_\theta \| \phi_1 \|_\theta) \]
\[ = \int_{\mathcal{F}(Z)} f(\bar{Z}\rho_1(a)(\bar{h}))d\mu_Z(Z) \int_{\mathcal{F}(Z)} g(\bar{Z}\bar{h})d\mu_Z(Z) + O(\eta\|a\|_\theta \| f \|_\theta \| g \|_\theta) \]
uniformly over \( \bar{h} \in \mathcal{F} \). This proves the claim. \( \square \)

Let us define \( \overline{f}(\overline{Y}) := \int_{\mathcal{F}(Z)} f(\bar{Z}\bar{Y})d\mu_Z(Z) \) and \( \overline{g}(\overline{Y}) := \int_{\mathcal{F}(Z)} g(\bar{Z}\bar{Y})d\mu_Z(Z) \) for \( \overline{Y} \in \mathcal{F}(Y) \). Then by the above claim, we conclude that
\[
\int_{\mathcal{F}(M)} f(\rho_1(a)(\bar{X}))g(\overline{X})d\mu(\bar{X}) = \int_{\mathcal{F}(Y)} \overline{f}(\rho_1(a)(\overline{Y}))\overline{g}(\overline{Y})d\mu_Y(\overline{Y}) + O(\eta\|a\|_\theta \| f \|_\theta \| g \|_\theta).
\]
Since \( \dim Y < \dim M \), the statement follows by induction.

4. Coarse Lyapunov foliations and smooth leaves

In this section, we will prove that for every coarse Lyapunov exponent \( [\chi] \), the corresponding coarse Lyapunov distribution \( E^{[\chi]} := \bigoplus_{\chi \in [\chi]} E^\chi \) admits a Hölder foliation with \( C^\infty \) leaves. We follow the argument developed by Rodriguez Hertz and Wang [27] with minor modification for our setup.

We first make the following definition:
Definition 4.1. For $a \in \mathbb{Z}^k$, we say $\rho(a)$ is uniformly hyperbolic if there exists a constant $\lambda > 1$ and $\rho$-invariant splitting of the bundle $\mathcal{E}(\mathcal{F}(M)) = \bigcup_{z \in \mathcal{F}(M)} T_z(M)$,

$\mathcal{E}(\mathcal{F}(M)) = \mathcal{E}^s_a \oplus \mathcal{E}^u_a$,

such that

$\|D\rho(a)(v)\| \leq \lambda^{-1}\|v\|$  for $v \in \mathcal{E}^s_a$

$\|D\rho(a)(v)\| \geq \lambda\|v\|$  for $v \in \mathcal{E}^u_a$.

Remark 4.2. When $\rho(a)$ is uniformly hyperbolic, both $\mathcal{E}^s_a$ and $\mathcal{E}^u_a$ admit Hölder foliations with $C^\infty$ leaves. We denote the corresponding foliations by $\mathcal{W}^s_a$ and $\mathcal{W}^u_a$, respectively. For $z \in \mathcal{F}(M)$ and $\square = s$ or $u$, we denote by $\mathcal{W}^\square_a(z)$ the leaf of $\mathcal{W}^\square_a$ passing through $z$.

The basic idea to establish the smoothness of leaves goes as follows: we start with a real Weyl chamber $\mathcal{C}_0 \subset \mathbb{R}^k$ with an element $a \in \mathbb{Z}^k \cap \mathcal{C}_0$ such that $\rho(a)$ is uniformly hyperbolic. We shall prove that any real Weyl chamber $\mathcal{C}$ adjacent to $\mathcal{C}_0$ also contains a uniformly hyperbolic element $a' \in \mathbb{Z}^k$. Therefore every real Weyl chamber of the action $\rho$ contains a uniformly hyperbolic element. For every coarse Lyapunov exponent $[\chi]$, we choose two adjacent Weyl chambers $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\ker \chi$ is the only Weyl chamber wall separating $\mathcal{C}_1$ and $\mathcal{C}_2$. Take $a_i \in \mathbb{Z}^k \cap \mathcal{C}_i$ such that $\rho(a_i)$ is uniformly hyperbolic. Without loss of generality, we may assume that $\chi(a_1) < 0$ and $\chi(a_2) > 0$. Then the intersection $\mathcal{W}^s_{a_1} \cap \mathcal{W}^u_{a_2}$ defines a Hölder foliation, and passing through every $z \in \mathcal{F}(M)$, the intersection $\mathcal{W}^s_{a_1}(z) \cap \mathcal{W}^u_{a_2}(z)$ is $C^\infty$. From our assumption it is easily seen that $\mathcal{W}^s_{a_1} \cap \mathcal{W}^u_{a_2}$ corresponds to the distribution $E^{[\chi]}$. This proves that $E^{[\chi]}$ admits a Hölder foliation with $C^\infty$ leaves. To show that $\mathcal{C}$ contains a uniformly hyperbolic element, we basically follow the argument by Rodriguez Hertz and Wang [27] with minor modifications.

4.1. Correspondence between foliations of $\rho$ and $\rho_1$. Let us fix a real Weyl chamber $\mathcal{C}_0$ and choose $a \in \mathbb{Z}^k \cap \mathcal{C}_0$ such that $\rho(a)$ is uniformly hyperbolic.

Recall that for each $z \in \mathcal{F}(M)$, $M(z)$ denotes a manifold slice passing through $z$. Then the leaves $\mathcal{W}^s_a(z)$ and $\mathcal{W}^u_a(z)$ are given by

$\mathcal{W}^s_a(z) = \left\{ \begin{array}{l} \bar{y} \in M(z) : \lim_{k \to +\infty} \text{dist}(\rho(a)^k(\bar{y}), \rho(a)^k(z)) = 0 \end{array} \right\}$,

and

$\mathcal{W}^u_a(z) = \left\{ \begin{array}{l} \bar{y} \in M(z) : \lim_{k \to +\infty} \text{dist}(\rho(a)^{-k}(\bar{y}), \rho(a)^{-k}(z)) = 0 \end{array} \right\}$.

Now, let us turn our attention to the affine action $\rho_1(a)$. Let $g^s_a(\mathbb{R})$ (and $g^u_a(\mathbb{R})$, respectively) denote the direct sum of $\sigma^{\chi l}$ such that $\chi l \in T(\mathbb{R})$, and $\chi l(a) < 0$ (and $\chi l(a) > 0$, respectively). From Remark 2.15, we can see that $g^s_a(\mathbb{R})$ and $g^u_a(\mathbb{R})$ are both Lie subalgebras of $n(\mathbb{R})$. Let $G^s_a$ and $G^u_a$ denote the corresponding Lie subgroups. Then it is easy to see that $g^s_a(\mathbb{R})$ and $g^u_a(\mathbb{R})$ correspond to the stable and unstable foliations of $M$ with respect to the action $\rho_1(a)$, and moreover, the stable (and unstable) leaf passing through any point $z \in \mathcal{F}(M)$ is $\mathbb{Z}G^s_a$ (and $\mathbb{Z}G^u_a$ respectively).
Because \( \phi \) conjugates \( \rho \) to \( \rho_1 \), we have that
\[
\phi(W_a^s(\mathbb{Z})) = \phi(\mathbb{Z})G_a^s \quad \text{and} \quad \phi(W_a^u(\mathbb{Z})) = \phi(\mathbb{Z})G_a^u.
\]
In particular, we have that \( \dim E_a^s = \dim g_a^s \) and \( \dim E_a^u = \dim g_a^u \). Therefore, for every real Lyapunov exponent \( \chi^l \) for \( \rho_1 \), \( \chi^l(a) \neq 0 \), and
\[
n(\mathbb{R}) = g_a^s(\mathbb{R}) \oplus g_a^u(\mathbb{R}).
\]
Let \( \mathcal{C} \) be a real Weyl chamber adjacent to \( \mathcal{C}_0 \). Let \( \ker \chi^l \) denote the real Weyl chamber wall separating \( \mathcal{C}_0 \) and \( \mathcal{C} \). Then by Remark 2.15, \( \sigma := \sigma^{[x]} \) is a Lie subalgebra of \( n(\mathbb{R}) \). Let \( V := V^{[x]} = \exp(\sigma^{[x]}) \) denote the corresponding Lie subgroup.

Without loss of generality, we may assume that \( \chi^l(a) < 0 \), then \( V \subset G_a^s \). Let us define the strong stable subspace of \( \rho_1(a) \) by
\[
g_a^{ss} := \bigoplus_{\sigma^{[x]} \neq \sigma} \sigma^{[x]}.
\]

**Lemma 4.3** (see [27, Lemma 3.1]).

1. \( g_a^{ss} \) is a Lie subalgebra.
2. \( \sigma \oplus g^u \) is a Lie subalgebra.
3. \( [\sigma, g_a^{ss}] \subset g_a^{ss} \).

Let \( G_a^{ss} := \exp g_a^{ss} \) be the Lie subgroup corresponding to \( g_a^{ss} \). On the level of Lie algebra, we have the following decomposition:
\[
n(\mathbb{R}) = g_a^u(\mathbb{R}) \oplus \sigma \oplus g_a^{ss}.
\]

On the level of Lie group, the following lemma is proved in [27]:

**Lemma 4.4** (see [27, Corollary 3.3]).

1. The multiplication map
\[
G_a^u \times V \times G_a^{ss} \rightarrow G
\]
\[
(g_u, g_v, g_{ss}) \rightarrow g_ug vg_{ss}
\]

is a \( C^\infty \) diffeomorphism.

2. The multiplication map
\[
V \times G_a^{ss} \rightarrow G_a^s
\]
\[
(g_v, g_{ss}) \rightarrow g_vg_{ss}
\]

is a \( C^\infty \) diffeomorphism.

**Remark 4.5.** In [27], the order of the multiplication is \( G_a^{ss} \times V \times G_a^u \rightarrow N(\mathbb{R}) \). The same proof works for our order here. We make this change in this paper because here \( M = N(\mathbb{Z}) \sim N(\mathbb{R}) \) while in [27] \( M = N(\mathbb{R})/N(\mathbb{Z}) \). Also, later in this section, we will make similar changes due to this reason.
From the above lemma, any \( g \in N(\mathbb{R}) \) can be uniquely written as \( g_u g_v g_{ss} \). Define \( g_s := g_v g_{ss} \). We call \( g_{ss} (g_v, g_u \text{ and } g_s \text{ respectively}) \) the projection of \( g \) onto \( G^s_a \) \((V, G^u_a \text{ and } G_s^a \text{ respectively})\).

Rodriguez Hertz and Wang [27] proved several results on the \( G^s_a \times V \times G^u_a \) coordinate of \( N(\mathbb{R}) \). We sum up them in the following proposition. We refer to [27] for proofs.

**Proposition 4.6.**

- For any given \( g \in N(\mathbb{R}) \), the restriction of the map \( h \mapsto (gh)_u \) to \( G^u_a \) is a \( C^\infty \) diffeomorphism from \( G^u_a \) to itself (cf. [27, Corollary 3.5]).
- For any \( g_1, g_2 \in N(\mathbb{R}) \), \((g_1 g_2)_V = (g_1(g_2)_u)_V(g_2)_V \). In particular, if \( g_2 \in G^s_a \), then \((g_1 g_2)_V = (g_1)_V(g_2)_V \) (cf. [27, Corollary 3.6]).

Let us recall an argument from [8] and [27] concerning the choice of \( a \in \mathbb{Z}^k \) and estimate of Lyapunov exponents.

Recall that \( a \in \mathcal{C}_0 \) where \( \mathcal{C}_0 \) denotes a Weyl chamber of the action \( \rho_1 \), \( \mathcal{C} \) denotes a Weyl chamber adjacent to \( \mathcal{C}_0 \), and \([\mathcal{H}^l] \) denotes the coarse Lyapunov exponent such that ker\( \chi \) separates \( \mathcal{C}_0 \) and \( \mathcal{C} \).

Recall that \( \Sigma \equiv \mathbb{Z}^2 \) denotes the subgroup of \( \mathbb{Z}^k \) given in Corollary 1.4. For any \( \xi > 0 \), we may choose \( b \in \Sigma \cap \mathcal{C} \) such that \(|\chi^l_1(b)| < \xi \|b\| \) for all \( \chi^l_1 \in [\mathcal{H}^l] \) or \([-\mathcal{H}^l] \). Then the restriction of \( \rho_1^{-1}(b) \) on \( V \) is contracting, and for any \( v \in \sigma \) and \( n \in \mathbb{N} \),

\[
\|D\rho_1(nb)v\| = O(e^{n\xi\|b\|}).
\]

For \( \xi > 0 \) small enough, we have that for any \( \chi^l_1 \) \( \notin [\mathcal{H}^l] \) or \([-\mathcal{H}^l] \), \( \chi^l_1(a) \) and \( \chi^l_2(b) \) have the same sign. Therefore, if \( a \in G^s_a \) and \( b \) have the same sign. Then the restriction of \( \rho_1^{-1}(b) \) on \( V \) is contracting, and for any \( v \in \sigma \) and \( n \in \mathbb{N} \),

\[
\|D\rho_1(nb)v\| = O(e^{n\xi\|b\|}).
\]

Since the foliation \( \mathcal{H}^s_a \) corresponds to \( G^s_a \) via \( \phi \), and \( \phi \) is \( \theta \)-Hölder, we conclude that for any \( w \in \mathcal{E}^s_a \), and any \( n \in \mathbb{N} \),

\[
\|D\rho(nb)w\| = O(e^{n\xi\|b\|})
\]

Summing up the argument above, we have the following proposition:

**Proposition 4.7 (see [27, Lemma 3.8]).** For any \( \xi > 0 \), there exists \( b \in \Sigma \setminus \{0\} \) such that the restriction of \( \rho_1^{-1}(b) \) on \( V \) is contracting, and for any \( w \in \mathcal{E}^s_a \) and any \( n \in \mathbb{N} \),

\[
\|D\rho(nb)w\| = O(e^{n\xi\|b\|}),
\]

in other words, \( \log\|D\rho(nb)\|_{\mathcal{E}^s_a} \leq n\xi\|b\| \).

### 4.2. The cohomological equation II.

Let \( h : \mathcal{H}(M) \rightarrow N(\mathbb{R}) \) be as defined in §2.6. Note that the value of \( h(\mathbb{Z}) \) only depends on the projection \( z \) of \( \mathbb{Z} \) on \( M \). Therefore, we may regard \( h \) as a \( C^\infty \) map from \( M \) to \( N(\mathbb{R}) \). Let \( h_u \) and \( h_V \) denote the projection of \( h \) on \( G^u_a \) and \( V \) respectively in \( G^s_a \times V \times G^u_a \) coordinate. We want to get the cohomological equation of \( h_V \).

Let us fix a constant \( \xi > 0 \). By Proposition 4.7, there exists \( b \in \Sigma \setminus \{0\} \) such that \( \rho_1^{-1}(b) \) contracts \( V \) and \( \|D\rho(nb)w\| = O(e^{n\xi\|b\|}) \) for all \( w \in \mathcal{E}^s_a \) and \( n \in \mathbb{N} \).

Applying Lemma 2.19 to \( b \), we have that

\[
h(\mathbb{Z}) = Q'_b(\mathbb{Z})\rho_1^{-1}(b)h(\rho(b)\mathbb{Z}),
\]
for some $C^\infty$ map $Q'_\theta : \mathcal{S}(M) \to N(\mathbb{R})$. By Prop. 4.6, $(g_1 g_2)_V = (g_1(g_2)_u)_V g_2)_V$. Projecting both sides of the equation above to $V$, we have

\begin{equation}
    h_V(\xi) = (Q'_\theta(\xi)\rho^{\theta^{-1}}(\xi)h_\xi(\rho(\xi)))_V \rho^{\theta^{-1}}(\xi)h_V(\rho(\xi)).
\end{equation}

**Notation 4.8.** We borrow the following notation from [27].

For $x, y \in \mathcal{S}(M)$ (or $N(\mathbb{R})$) and $\epsilon > 0$, let $B_\epsilon(x)$ denote the ball inside $M(x)$ (or $N(\mathbb{R})$) centered at $x$ with radius $\epsilon$. We denote by $B_\epsilon^E(f)$ the ball inside $F$ centered at $\xi$ with radius $\epsilon$.

Let us fix a constant $\delta > 0$ such that for $x, y \in \mathcal{S}(M)$ belonging to the same manifold slice and $\text{dist}(x, y) \leq \delta$, there exists a unique element $\xi(p(x, y)) \in B_\delta(\epsilon)$ with $y = x p(x, y)$. It is easy to see that the map $p : \{x, y\} \in \mathcal{S}(M) \times \mathcal{S}(M) : x$ and $y$ belong to the same manifold slice, and $\text{dist}(x, y) \leq \delta \to N(\mathbb{R})$ is $C^\infty$.

For $x, y \in \mathcal{S}(M)$ belonging to the same manifold slice and $\text{dist}(x, y) \leq \delta$, we write $\phi(x) = \phi(x) H(\xi)$ where $H(\xi) := h^{-1}(x)p(x, y)h(y)$. It is easy to see that the map

\begin{equation}
    (x, y) \to H(\xi)
\end{equation}

is $\theta$-Hölder in the pair $(x, y)$.

We first prove the following proposition:

**Proposition 4.9** (see [27, Corollary 3.14]).

\[ h_\xi \in C^{\infty, \theta}_{\mathcal{S}(M)} \]

**Proof.** The proof we present here follows the proof of [27, Lemma 3.13, Corollary 3.14] with minor modification.

We write $p^{-1}(\xi, y) h(x) = (p^{-1}(\xi, y) h(x))_u (p^{-1}(\xi, y) h(x))_s$, and we write $h(y) = h_\xi(y) h_s(y)$. Then

\[ H(\xi) = (p^{-1}(\xi, y) h(x))^{-1}_u (p^{-1}(\xi, y) h(x))^{-1}_u h_\xi(y) h_s(y). \]

$H(\xi) \in G_{a_u}$ if and only if $(p^{-1}(\xi, y) h(x))^{-1}_u h_\xi(y) = e$, i.e., $h_\xi(y) = (p^{-1}(\xi, y) h(x))_u$. Since $G_{a_u}$ corresponds to the foliation $\mathcal{W}^s_a$ via the conjugacy $\phi$. Therefore, near $\xi$ the leaf $\mathcal{W}^s_a(\xi)$ is defined by

\begin{equation}
    h_\xi(y) = (p^{-1}(\xi, y) h(x))_u.
\end{equation}

Since on $\mathcal{W}^s_a(s)$, $p(\xi, y)$ is $C^\infty$ and $h(x)$ is constant, we have that $h_\xi(y)$ is $C^\infty$ along $\mathcal{W}^s_a(\xi)$.

It remains to check that the partial derivatives along $\mathcal{W}^s_a$ vary Hölder continuously. We fix a neighborhood $\Omega$ of $\xi_0 \in \mathcal{S}(M)$ in the manifold slice $M(\xi_0)$ such that $\phi(\Omega)$ is of form $\phi(\xi_0) B_{\epsilon}(\xi_0) B_{\epsilon}(\xi_0)$. Then every $y \in \Omega$ can be projected to some $\xi = \xi(y) \in \Omega \cap \mathcal{W}^s_a(\xi_0)$ along $\mathcal{W}^s_a$. Since $\mathcal{W}^s_a$ is a Hölder foliation, we have that the map $\xi \to \xi(y)$ is Hölder in $\xi$. Thus the map $\xi \to h(\xi(y))$ is also Hölder. By (4.2), partial derivatives of $h_\xi(y)$ along $\mathcal{W}^s_a C^\infty$ depend on $\xi$ and $h(x)$, and thus are Hölder continuous in $\xi$.

This completes the proof.
Now let us get back to the cohomological equation (4.1). Let \( \tilde{h}_V := \log h_V \) and 
\[
\Psi := \log(Q'_b(\overline{z})\rho^{-1}_l(b)h_u(\rho(b)\overline{z}))_V .
\]
By Proposition 4.9, \( \Psi \in C^\infty_{W^u} \). Then (4.1) can be rewritten as 
\[
\exp \tilde{h}_V = \exp \Psi \exp(\rho^{-1}_l(b)\tilde{h}_V \circ \rho(b)).
\]
By Baker-Campbell-Hausdorff formula, we have that 
\[
\tilde{h}_V = \rho^{-1}_l(b)\tilde{h}_V \circ \rho(b) + \Psi + \frac{1}{2}[\Psi, \rho^{-1}_l(b)\tilde{h}_V \circ \rho(b)]
\]
(4.3) 
\[
- \frac{1}{12} [\rho^{-1}_l(b)\tilde{h}_V \circ \rho(b), [\Psi, \rho^{-1}_l(b)\tilde{h}_V \circ \rho(b)]] + \cdots
\]
where there are only finitely many terms on the right hand side since both sides of the equation belong to the Lie algebra \( \sigma \) of \( V \) which is nilpotent. Consider the derived series of \( \sigma \):
\[
\sigma = \sigma_0 \supset \sigma_1 \supset \cdots \supset \sigma_l = \{0\}.
\]
For \( i = 1, 2, \ldots, l \), let 
\[
\pi_i : \sigma \rightarrow \sigma_i \sim \sigma
\]
denote the canonical projection. Let \( \tilde{h}_i := \pi_i \circ \tilde{h}_V \) and \( \Psi_i := \pi_i \circ \Psi \).

Projecting the equation to \( \sigma_i \sim \sigma \), we will get the following linearized equation:
\[
\tilde{h}_1 = \rho^{-1}_l(b)\tilde{h}_1 \circ \rho(b) + \Psi_1.
\]
We first prove the following lemma on linearized cohomological equations:

**Proposition 4.10** (see [27, Proposition 3.15]). Let \( L \) be a vector space and \( B : L \rightarrow L \) be a linear isomorphism such that \( \|B^{-i}\| \) is uniformly bounded for all \( i \geq 0 \). For \( \xi > 0 \), let \( b \in \Sigma \sim \{0\} \) be the element given by Proposition 4.7. Suppose \( \psi : \mathcal{S}(M) \rightarrow L \) is in \( C^\infty_{W^u} \) and \( f : M \rightarrow L \) is Hölder continuous and solves the linear equation:
\[
f = B^{-1}f \circ \rho(b) + \psi,
\]
(4.5) 
then there exists \( \xi_0 > 0 \) such that for \( 0 < \xi \leq \xi_0 \), \( f \in C^\infty_{W^u} \).

**Proof.** We first claim that in order to show the lemma, it suffices to show the lemma assuming that the integral \( \int_{\mathcal{S}(M)} f \, d\tilde{\mu} = 0 \). In fact, let \( \overline{f} := \int_{\mathcal{S}(M)} f \, d\tilde{\mu} \) and let \( f_1 := f - \overline{f} \). Since \( \overline{f} \) is a constant function, to show \( f \in C^\infty_{W^u} \), it suffices to show that \( f_1 \in C^\infty_{W^u} \). We have that \( \int_{\mathcal{S}(M)} f_1 \, d\tilde{\mu} = 0 \) and \( f_1 \) satisfies the following equation:
\[
f_1 = B^{-1}f_1 \circ \rho(b) + \psi',
\]
where \( \psi' = \psi + B^{-1}\overline{f} \circ \rho(b) - \overline{f} \in C^\infty_{W^u} \). This proves the claim. Therefore we may assume that \( \int_{\mathcal{S}(M)} f \, d\tilde{\mu} = 0 \).
By iterating (4.5), we have that
\[ f = \sum_{j=0}^i B^{-i} \psi \circ \rho(jb) + B^{-i} f \circ \rho(ib). \]

We claim that
\[ f = \sum_{i=0}^\infty B^{-i} \psi \circ \rho(jb) \]
in the sense of distributions. To show this, it suffices to show that for any \( g \in C^\infty(M) \),
\[ \lim_{i \to \infty} \int_{\mathcal{F}(M)} B^{-i} \psi(\rho(ib)\overline{x}) g(\overline{x}) \text{d}\check{\mu}(\overline{x}) = 0. \]

In fact, since \( \check{\mu} \) is \( \rho \)-invariant and \( f \) has zero average with respect to \( \check{\mu} \), we have
\[ \int_{\mathcal{F}(M)} B^{-i} f \circ \rho(ib) \text{d}\check{\mu} = B^{-i} \int_{\mathcal{F}(M)} f \text{d}\check{\mu} = 0. \]

Since \( \int_{\mathcal{F}(M)} f \text{d}\check{\mu} = \int_{\mathcal{F}(M)} B^{-i} f \circ \rho(ib) \text{d}\check{\mu} + \int_{\mathcal{F}(M)} \psi \text{d}\check{\mu} \), we conclude \( \int_{\mathcal{F}(M)} \psi \text{d}\check{\mu} = 0. \)
Therefore, by Corollary 1.4, there exist constants \( C > 0 \) and \( \eta > 0 \) such that for all \( g \in C^\infty(M) \),
\[ \left\| \int_{\mathcal{F}(M)} B^{-i} \psi(\rho(ib)\overline{x}) g(\overline{x}) \text{d}\check{\mu} \right\| \leq C \| B^{-i} \| \| \psi \|_0 \| g \|_\theta e^{-\eta i \| b \|}. \]

Since \( \| B^{-i} \| \) is uniformly bounded, we have \( | \int_{\mathcal{F}(M)} B^{-i} \psi(\rho(ib)\overline{x}) g(\overline{x}) \text{d}\check{\mu} | \to 0 \) as \( i \to +\infty \), which proves the claim.

By [28, Theorem 1.1] and its variations proved by Fisher, Kalinin and Spatzier [8, Theorem 8.3.1] and Rodriguez Hertz and Wang [27, Theorem A.1], to show \( f \in C^\infty_\theta \) it suffices to show \( f \in C^\infty_\theta^k \), i.e., for all \( k \in \mathbb{N} \), \( \partial^k f \in (C^\theta(M))^* \).

Given \( \phi \in C^\theta(M) \), we have
\[ \langle \partial^k_\theta f, \phi \rangle = \sum_{i=0}^\infty \langle \partial^k_\theta (B^{-i} \psi \circ \rho(ib)), \phi \rangle. \]

Since every \( \partial^k_\theta (B^{-i} \psi \circ \rho(ib)) \in C^\infty_\theta \) is a Hölder continuous function, the term
\[ \langle \partial^k_\theta (B^{-i} \psi \circ \rho(ib)), \phi \rangle = \int_{\mathcal{F}(M)} \partial^k_\theta (B^{-i} \psi \circ \rho(ib)) \phi \text{d}\check{\mu}. \]

Fix a compactly supported positive \( C^\infty \) bump function \( \delta \) on \( \mathfrak{n}(\mathbb{R}) \) supported on a neighborhood around \( 0 \). For small \( \epsilon > 0 \), define on \( N(\mathbb{R}) \) a function
\[ \delta_\epsilon(x) = c_\epsilon \delta \left( \frac{\log x}{\epsilon} \right) \]
where \( c_\epsilon > 0 \) is chosen such that \( \int_{N(\mathbb{R})} \delta_\epsilon(x) \text{d}g = 1. \) Let \( \phi_\epsilon := \phi \ast \delta_\epsilon \). By standard facts on convolutions, we have the following hold:
1. \( \phi_\epsilon \) is \( C^\infty \).
2. \( \| \phi - \phi_\epsilon \|_\infty \leq a_0 \epsilon^\theta \| \phi \|_\theta \) for a constant \( a_0 > 0 \).
3. There exists a constant \( c_k > 0 \) such that \( \| \phi_\epsilon \|_{C^k} \leq c_k \epsilon^{-\dim M - k} \| \phi \|_\infty. \)
By Corollary 1.4 applied to \( \psi \) and \( \partial^k_{\mathcal{W}_d} \varphi_c \), we have

\[
|\langle \partial^k_{\mathcal{W}_d} (\psi \circ \rho(i b)), \varphi_c \rangle| = |\langle \psi \circ \rho(i b), \partial^k_{\mathcal{W}_d} \varphi_c \rangle| \\
\leq a_1 \| \psi \|_\theta \| \partial^k_{\mathcal{W}_d} \varphi_c \|_\theta e^{-\eta_i |b|} \\
\leq a_1 \| \psi \|_\theta \| \varphi_c \|_{C^{k+1}_i} e^{-\eta_i |b|} \\
\leq a_1 c_k \| \psi \|_\theta \| \varphi_c \|_\infty e^{-\dim M - k - 1} e^{-\eta_i |b|} \\
\leq C_1 e^{-\dim M - k - 1} e^{-\eta_i |b|} \| \varphi_c \|_\theta,
\]

(4.6)

where \( a_1 > 0 \) and \( \eta > 0 \) are constants from Corollary 1.4, and \( C_1 := a_1 c_k \| \psi \|_\theta \).

We also need to estimate \(|\langle \partial^k_{\mathcal{W}_d} (\psi \circ \rho(i a)), \varphi - \varphi_c \rangle|\):

\[
|\langle \partial^k_{\mathcal{W}_d} (\psi \circ \rho(i b)), \varphi - \varphi_c \rangle| \leq \| \partial^k_{\mathcal{W}_d} (\psi \circ \rho(i b)) \|_\infty \| \varphi - \varphi_c \|_\infty \\
\leq a_0 \| \partial^k_{\mathcal{W}_d} \psi \|_\infty \| \partial^k_{\mathcal{W}_d} \rho(i b) \|_\infty e^{\theta} \| \varphi \|_\theta.
\]

(4.7)

By [8, Lemma 3.6],

\[
\| \partial^k_{\mathcal{W}_d} \rho(i b) \|_\infty = O(\| D\rho(b) \|_{\mathcal{E}_d} \| i^k T \| \partial^k_{\mathcal{W}_d} \rho(b) \|_\infty^T),
\]

where \( T > 0 \) depends on \( k \) and \( \dim \mathcal{W}_d \). By Proposition 4.7, \( \| D\rho(b) \|_{\mathcal{E}_d} \| = O(e^{\xi |b|}) \).

Therefore, we have

\[
|\langle \partial^k_{\mathcal{W}_d} (\psi \circ \rho(i b)), \varphi - \varphi_c \rangle| \leq a_2 e^{\xi k |b|} i^T e^{\theta} \| \varphi \|_\theta \\
\leq a_2 e^{2 \xi k |b|} e^{\theta} \| \varphi \|_\theta
\]

(4.8)

for a constant \( a_2 > 0 \) depending on \( k, b, \psi \) and \( \dim \mathcal{W}_d \).

Let

\[
\epsilon = \exp \left( - \frac{i \| b \| (\eta + 2 k \xi)}{\dim M + k + 1 + \theta} \right),
\]

then \( e^{-\dim M - k - 1} e^{-\eta_i |b|} \) and \( e^{\theta} e^{2 \xi k |b|} \) are both equal to

\[
\exp \left( \frac{i \| b \| [2 \xi k (\dim M + k + 1) - \theta \eta]}{\theta + \dim M + k + 1} \right).
\]

For \( \xi \) small enough, we will have

\[
\eta_1 := - \frac{2 \xi k (\dim M + k + 1) - \theta \eta}{\theta + \dim M + k + 1} > 0.
\]

By the estimates above, we have that

\[
|\langle \partial^k_{\mathcal{W}_d} (\psi \circ \rho(i b)), \varphi \rangle| \leq a_3 e^{-\eta_1 |b|} \| \varphi \|_\theta
\]
for a constant $a_3 > 0$. Since by assumption there exists a constant $a_4 > 0$ such that $\|B^{-i}\| \leq a_4$ for all $i \in \mathbb{N}$, we will have that

$$|\langle \partial_{\#}^k f, \phi \rangle| \leq \sum_{i=0}^{\infty} |\langle \partial_{\#}^k (B^{-i} \psi \circ \rho (ib)), \phi \rangle|$$

$$\leq \sum_{i=0}^{\infty} \|B^{-i}\| |\langle \partial_{\#}^k (\psi \circ \rho (ib)), \phi \rangle|$$

$$\leq a_4 \sum_{i=0}^{\infty} |\langle \partial_{\#}^k (\psi \circ \rho (ib)), \phi \rangle|$$

$$\leq a_4a_3 \sum_{i=0}^{\infty} e^{-\eta_i |ib|} \|\phi\|_{\theta}$$

$$\leq a_5 \|\phi\|_{\theta},$$

where $a_5 = a_3a_4 \frac{1}{e^{-\eta_1 |b|}}$. This proves that $\partial_{\#}^k f \in (C^\theta (M))^\ast$. By our previous discussion, this completes the proof. \qed

**Proposition 4.11** (see [27, Lemma 3.20]). For $i = 0, 1, \ldots, l$, $\tilde{h}_i \in C_{\#}^{\infty, \theta}$. In particular, $\tilde{h}_V = \tilde{h}_1 \in C_{\#}^{\infty, \theta}$.

**Proof.** Let $\xi_0 > 0$ be the constant given by Proposition 4.10. For $\xi \in (0, \xi_0]$, let $b \in \Sigma \sim \{0\}$ be the element given by Proposition 4.7.

Let us prove this lemma by induction on $i$. For $i = 0$, the statement is trivial since $\sigma_0 = \sigma$. For $i \geq 1$, assume the lemma holds for all $j < i$. We want to show that the lemma holds for $i$.

Projecting (4.3), we have the following equation for $\tilde{h}_i$:

$$(4.10) \quad \tilde{h}_i = \rho_i^{-1}(b) \tilde{h}_i \circ \rho(b) + \Psi_i + \frac{1}{2} \Psi_i, \rho_i^{-1}(b) \tilde{h}_i \circ \rho(b) + \cdots.$$ 

Fix a subspace of $\mathfrak{z} < \sigma_i \sim \sigma$ such that $\mathfrak{z} \oplus (\sigma_i \sim \sigma_{i-1}) = \sigma_i \sim \sigma$. According to this decomposition we may write $\tilde{h}_i = \tilde{h}_3 + \tilde{h}_{3}^\perp$. Note that the canonical projection from $\mathfrak{z}$ to $\sigma_{i-1} \sim \sigma$ is a linear isomorphism. By the inductive hypothesis on $\tilde{h}_{i-1}$, we conclude that $\tilde{h}_3 \in C_{\#}^{\infty, \theta}$. Therefore $\rho_i^{-1}(b) \tilde{h}_3 \circ \rho(b) \in C_{\#}^{\infty, \theta}$.

By writing $\tilde{h}_i = \tilde{h}_3 + \tilde{h}_{3}^\perp$, we may write each higher order term in (4.10) as a Lie bracket monomial of $\rho_i^{-1}(b) \tilde{h}_3 \circ \rho(b)$, $\rho_i^{-1}(b) \tilde{h}_{3}^\perp \circ \rho(b)$ and $\Psi_i$. Because $\rho_i^{-1}(b) \tilde{h}_{3}^\perp \circ \rho(b) \in \sigma_i \sim \sigma_{i-1}$, every Lie bracket monomial containing $\rho_i^{-1}(b) \tilde{h}_{3}^\perp \circ \rho(b)$ vanishes. Note that $\rho_i^{-1}(b) \tilde{h}_3 \circ \rho(b)$ and $\Psi_i$ are both in $C_{\#}^{\infty, \theta}$, we have that the sum of higher order terms is in $C_{\#}^{\infty, \theta}$. Therefore

$$\tilde{h}_i = \rho_i^{-1}(b) \tilde{h}_i \circ \rho(b) + \tilde{\Psi}_i$$

where $\tilde{\Psi}_i = \Psi_i + \mid$ higher order terms $\mid$ is in $C_{\#}^{\infty, \theta}$. Let $B := \rho_1(b)$. Since $B^{-1}$ is contracting on $\sigma$, $\|B^{-i}\|$ is uniformly bounded when restricted to $\sigma_i \sim \sigma$. By Proposition 4.10, we conclude that $\tilde{h}_i \in C_{\#}^{\infty, \theta}$. \qed
For \( \overline{x} \in \mathcal{S}(M) \), let \( \mathcal{W}^{ss}_{a}(\overline{x}) \) denote the topological submanifold of \( M(\overline{x}) \) passing through \( \overline{x} \) defined by
\[
\mathcal{W}^{ss}_{a}(\overline{x}) := \phi^{-1}(\phi(\overline{x})G^{ss}_{a}).
\]
Obviously every \( \mathcal{W}^{ss}_{a}(\overline{x}) \) is contained in a \( \mathcal{W}^{s}_{a} \) leaf. Recall that
\[
H_{\overline{\tau}}(\overline{y}) = h^{-1}(\overline{x})p(\overline{x}, \overline{y})h(\overline{y}).
\]
The following proved in [27] gives the local description of \( \mathcal{W}^{ss}_{a}(\overline{x}) \):

**Lemma 4.12** (see [27, Lemma 4.1]). Inside \( \mathcal{W}^{s}_{a}(\overline{x}) \), \( \mathcal{W}^{ss}_{a}(\overline{x}) \) is locally defined by the equation
\[
(H_{\overline{\tau}}(\overline{y}))_{V} = e.
\]
Note that by Proposition 4.6,
\[
(H_{\overline{\tau}}(\overline{y}))_{V} = h_{V}(\overline{y})((h(\overline{x})p(\overline{x}, \overline{y})^{-1}))_{V}.
\]
By Proposition 4.11, \( h_{V}(\overline{y}) \) is \( C^\infty \) when restricted to \( \mathcal{W}^{s}_{a}(\overline{x}) \). Combined with
\[
((h(\overline{x})p(x, y)^{-1}))_{V}
\]
this implies that \( \overline{y} \mapsto H_{\overline{\tau}}(\overline{y}) \) is \( C^\infty \) in small neighborhoods of \( \overline{x} \) in \( \mathcal{W}^{s}_{a}(\overline{x}) \). Moreover, since partial derivatives of \( (H_{\overline{\tau}}(\overline{y}))_{V} \) along \( \mathcal{W}^{s}_{a}(\overline{x}) \) are polynomial combinations of \( \partial_{a}^{k} h_{V}(\overline{y}) \) and \( \partial_{a}^{k}((h(\overline{x})p(x, y)^{-1}))_{V} \), we conclude that all partial derivatives \( \partial_{a}^{k}(H_{\overline{\tau}}(\overline{y}))_{V} \) are \( \theta \)-Hölder continuous in \( \overline{x} \).

Our aim is to show that \( \mathcal{W}^{ss}_{a} \) defines a Hölder foliation with \( C^\infty \) leaves. By [27, Corollary 4.3], to show the smoothness of every \( \mathcal{W}^{ss}_{a}(\overline{x}) \), it suffices to show that for any \( \overline{x} \in \mathcal{S}(M) \), the map \( (H_{\overline{\tau}}(\overline{y}))_{V} \) is regular in \( \overline{y} \) at \( \overline{y} = \overline{x} \). We will modify the argument by Rodriguez Hertz and Wang to prove the result.

Let \( A \) be the set of points \( \overline{x} \in \mathcal{S}(M) \) where \( (H_{\overline{\tau}}(\overline{y}))_{V} \) is singular at \( \overline{x} \). We want to show that \( A \) is empty.

**Lemma 4.13** (see [27, Lemma 4.4]). \( A \) is closed and invariant under the \( \mathbb{Z}^{k} \)-action \( \rho \).

**Proof.** Since \( D_{\mathcal{W}^{ss}_{a}(\overline{x})}(H_{\overline{\tau}}(\overline{y}))_{V} \) depends continuously on \( \overline{x} \), and since being singular is a closed condition, we conclude that \( A \) is closed.

Let us show that \( A \) is \( \rho \)-invariant. Fix \( a' \in \mathbb{Z}^{k} \). For \( \overline{x} \in \mathcal{S}(M) \) and \( \overline{y} \in \mathcal{W}^{s}_{a}(\overline{x}) \), by the definition of \( H_{\overline{\tau}}(\overline{y}) \), we have
\[
\phi(\rho(a')\overline{x})H_{\rho(a')\overline{\tau}}(\rho(a')\overline{y}) = \phi(\rho(a')\overline{y}) = \rho_{1}(a')\phi(\overline{y})
\]
\[
= \rho_{1}(a')(\phi(\overline{x})H_{\overline{\tau}}(\overline{y}))
\]
\[
= \rho_{1}(a')\phi(\overline{x})\rho_{1}(a')H_{\overline{\tau}}(\overline{y})
\]
\[
= \phi(\rho(a')\overline{x})\rho_{1}(a')H_{\overline{\tau}}(\overline{y}).
\]
Since \( a' \) is fixed and both \( H_{\overline{\tau}}(\overline{y}) \) and \( H_{\rho(a')\overline{\tau}}(\rho(a')\overline{y}) \) are close to \( e \), we conclude that
\[
H_{\rho(a')\overline{\tau}}(\rho(a')\overline{y}) = \rho_{1}(a')H_{\overline{\tau}}(\overline{y}).
\]
By projecting the above equation to \( V \), we get that
\[
(H_{\rho(a')\overline{\tau}}(\rho(a')\overline{y}))_{V} = \rho_{1}(a')(H_{\overline{\tau}}(\overline{y}))_{V}.
\]
Since $\mathcal{W}_s^\varphi$ is $\rho$-invariant, we have that
\[
D_{\mathcal{W}_s^\varphi}|_{\mathcal{Y}=\rho(a')\mathcal{Y}}(H_{\rho(a')\mathcal{Y}})\varphi = D_{\rho_1(a')}|_{\mathcal{Y}}(D_{\mathcal{W}_s^\varphi}|_{\mathcal{Y}=\rho(a')\mathcal{Y}}(H_{\rho(a')\mathcal{Y}})) (D_{\rho(a')\mathcal{Y}} \rho(a')|_{\mathcal{Y}=\mathcal{Y}})^{-1}.
\]
Since $D_{\rho_1(a')}|_{\mathcal{Y}}$ and $D_{\rho(a')\mathcal{Y}} \rho(a')|_{\mathcal{Y}=\mathcal{Y}}$ are both regular, we conclude that
\[
D_{\mathcal{W}_s^\varphi}|_{\mathcal{Y}=\rho(a')\mathcal{Y}}(H_{\rho(a')\mathcal{Y}})\varphi
\]
is singular if and only if $D_{\mathcal{W}_s^\varphi}|_{\mathcal{Y}=\rho(a')\mathcal{Y}}(H_{\rho(a')\mathcal{Y}})$ is so. In other words, $\mathcal{X} \in A$ if and only if $\rho(a')\mathcal{X} \in A$. Since $a' \in \mathbb{Z}^k$ is chosen arbitrarily, we conclude that $A$ is $\rho$-invariant.

\begin{proposition}
$A = \emptyset$.
\end{proposition}

\textbf{Proof.} For contradiction, we assume that $A$ is not empty. Then $A$ supports an ergodic $\rho$-invariant probability measure $\mu$. By Oseledets’ multiplicative ergodic theorem adapted to $\mathbb{Z}^k$-actions (cf. [16, Proposition 2.1] and [27, Proposition 4.5]), there are finitely many linear functionals $\mathcal{X} \in (\mathbb{R}^k)^*$, a $\rho$-invariant subset $A' \subset A$ with $\mu(A') = 1$ and a $\rho$-invariant measurable splitting
\[
\mathcal{T}^{\mathcal{X}}(M) = \bigoplus_{\mathcal{X} \mu} E^\mathcal{X}_{\mu}(\mathcal{X})
\]
over $\mathcal{X} \in A'$ such that for all $a \in \mathbb{Z}^k$ and $\nu \in E^\mathcal{X}_{\mu}$,
\[
\lim_{k \to \infty} \frac{\log \|D\rho(ka)\nu\|}{k} = \overline{\lambda}(a).
\]
By Pesin’s strong stable manifold theorem (cf. [29], [26, Theorem 3.2] and [27, Lemma 4.6]), one can modify $A'$ such that for all $\mathcal{X} \in A'$ and $a \in \mathbb{Z}^k$, there are unique manifolds $\mathcal{W}_a^{s,\mu}(\mathcal{X})$ and $\mathcal{W}_a^{u,\mu}(\mathcal{X})$ respectively tangent to the stable and unstable distributions
\[
E^s_{a,\mu} := \bigoplus_{\mathcal{X}(a) < 0} E^\mathcal{X}_{\mu} \quad \text{and} \quad E^u_{a,\mu} := \bigoplus_{\mathcal{X}(a) > 0} E^\mathcal{X}_{\mu},
\]
moreover, near $\mathcal{X}$, $\mathcal{W}_a^{s,\mu}(\mathcal{X})$ is given by the set of $\mathcal{Y} \in M(\mathcal{X})$ (recall that $M(\mathcal{X})$ denotes a manifold slice passing through $\mathcal{X}$, see Definition 2.8) satisfying that $\text{dist}(\mathcal{Y}, \mathcal{X}) < \epsilon$ for some $\epsilon > 0$ depending on $\mathcal{X}$, and
\[
\limsup_{k \to \infty} \log \text{dist}(\rho(ka)\mathcal{Y}, \rho(ka)\mathcal{X}) \leq \max \{ \overline{\lambda}(a) : \overline{\lambda}(a) < 0 \}.
\]
Since $\phi$ is a bi-Hölder conjugacy between $\rho$ and $\rho_1$, it is easily seen that if $\overline{\lambda}(a) \neq 0$ for all Lyapunov functional $\overline{\lambda}$ for $\mu$, then for $\square = s, u$,
\[
\dim E^\square_{a,\mu} = \dim g^\square_{a},
\]
and
\[
\phi(\mathcal{W}_a^{\square,\mu}(\mathcal{X})) = \phi(\mathcal{X}) G^\square_{a},
\]
see [26, Proposition 3.1 and Corollary 3.3] and [27, Lemma 4.7] for details.

Let us define coarse Lyapunov distributions of $\mu$ as follows:
\begin{equation}
E^\mathcal{T}_{\mu} := \bigoplus_{\mathcal{T} = c\mathcal{X}, c > 0} E^\mathcal{X}_{\mu}.
\end{equation}
By [27, Lemma 4.9], the coarse Lyapunov subspaces in Definition 2.14 and the coarse Lyapunov distributions defined above are in one-to-one correspondence with each other. A pair of corresponding coarse Lyapunov subspace and coarse Lyapunov distribution have the same dimension and proportional coarse Lyapunov exponents.

According to the decomposition

$$g_a^s = \sigma \oplus g_a^{ss},$$

we decompose $E_{a,\overline{\mu}}^s$ as

$$E_{a,\overline{\mu}}^s = E_{a,\overline{\mu}}^V \oplus E_{a,\overline{\mu}}^{ss},$$

where $E_{a,\overline{\mu}}^V = E_{a,\overline{\mu}}^{\overline{x}_1}$ is the coarse Lyapunov distribution corresponding to $\sigma = \sigma_{\overline{x}_1}$, and $E_{a,\overline{\mu}}^{ss}$ is the direct sum of coarse Lyapunov distributions $E_{a,\overline{\mu}}^{\overline{x}_1}$ corresponding to $\sigma_{\overline{x}_1} \subset g_a^{ss}$. Moreover, there exists $\lambda > 0$ such that for any $\xi > 0$, we can choose $a \in \Sigma \cap C_0$ such that

$$\chi_{\overline{x}_1}(a) \in (\xi \|a\|, 0) \quad \text{if} \quad \sigma_{\overline{x}_1} \subset \sigma;$$

$$\chi_{\overline{x}_1}(a) < -\lambda \|a\| \quad \text{if} \quad \sigma_{\overline{x}_1} \subset g_a^{ss};$$

$$\chi_{\overline{x}_1}(a) > 0 \quad \text{if} \quad \sigma_{\overline{x}_1} \subset g_a^u.$$

Then by the correspondence, the Lyapunov exponents of $\rho(a)$ with respect to $E_{a,\overline{\mu}}^u$, $E_{a,\overline{\mu}}^V$, and $E_{a,\overline{\mu}}^{ss}$ are in the intervals $(0, \infty)$, $(-\kappa \eta \|a\|, 0)$ and $(-\infty, -\kappa^{-1} \lambda \|a\|)$, respectively, where $\kappa > 1$ denotes a constant determined by the Hölder index $\theta$ of the conjugacy $\phi$. Since the stable and unstable foliations remain the same after $a \in \Sigma \cap C_0$ is changed, we have that $\rho(a)$ is still uniformly hyperbolic.

We will now use the structure of stable manifolds in Pesin theory, as explained for example in the works of Ledrappier and Young [18, 19] or Ruelle [29]. The reader may note that the statements in [18, 19] are for diffeomorphisms of compact manifolds while our maps are homeomorphisms of solenoids which are smooth along the manifold slices. Still we have an invariant probability measure, and can study the derivative cocycle along the manifold slices which are bounded and in particular $L^1$. Then Oseledets’ Multiplicative Ergodic Theorem applies to give Lyapunov spaces for the derivative cocycle in the direction of the manifold slices and also stable and unstable manifolds (cf., e.g., [29, Theorem 1.6 and 5.1]).

Below, we will use several results from [19]. They depend on the existence of $(\epsilon, l)$-charts as defined in [19, Section 8.1]. Let us outline that we still have such charts in our non-manifold setting. We follow Ledrappier and Young who prove existence of these charts in [18, Appendix]. What is really needed is a full measure set $\Gamma$ and a measurable function $C : \Gamma \to [1, \infty]$ such that for the Lyapunov spaces $E_{\mu}^s(x)$ for the Lyapunov exponents $\tilde{\chi}$ as above we have the following.
For every $x \in \Gamma$ and some fixed $\varepsilon < \min \frac{|\tilde{\chi}(a)|}{100}$ for all non-zero Lyapunov exponents $\tilde{\chi}(a)$,

1. $C(x)^{-1}e^{(\tilde{\chi}(a) - \varepsilon) n} \leq \|D\rho(na)x\| \leq C(x)e^{(\tilde{\chi}(a) + \varepsilon) n}$ for all vectors $v \in E^{ss}_\mu(x)$.
2. $\|\sin \angle (E^{X_1}_\mu(x), E^{X_2}_\mu(x))\| \geq C(x)^{-1}$ if $\chi_1$ and $\chi_2$ are distinct Lyapunov exponents.
3. $C(\rho(\pm a)x) \leq e^{\varepsilon C(x)}$.

To prove these estimates we apply the Oseledets-Pesin Reduction Theorem [2, Theorem 5.10] together with [2, Lemma 5.13]. The latter gives us equation (3) above as the cocycle is integrable, and hence the coboundary in the Reduction Theorem is tempered.

Let $N$, $N^u$, $N^s$, $N^V$ and $N^{ss}$ denote the dimension of $M$, $E^u_{a,\bar{\mu}}$, $E^s_{a,\bar{\mu}}$, $E^V_{a,\bar{\mu}}$ and $E^{ss}_{a,\bar{\mu}}$, respectively. Let $B_{R^N}(r)$ denote the ball in $R^N$ centered at $0$ with radius $r$. Then for $\bar{\mu}$-almost every $z$, the Lyapunov decomposition at $z$:

$$T_{\bar{z}}(M) = E^u_{a,\bar{\mu}} \oplus E^V_{a,\bar{\mu}} \oplus E^{ss}_{a,\bar{\mu}}$$

can be locally foliated. To be specific, there exists a $\rho(a)$-invariant subset $A'' \subset A$ with $\bar{\mu}(A'') = 1$ and a measurable function $l : A'' \to (1, \infty)$ such that for $\bar{z} \in A''$, there is an embedding

$$\Phi_{\bar{z}} : B_{R^N}(l^{-1}(\bar{z})) \to M(\bar{z}),$$

satisfying several nice properties:

1. $\Phi_{\bar{z}}(0) = \bar{z}$, and $D|_{x=0}\Phi_{\bar{z}}$ sends the splitting $R^{N^u} \oplus R^{N^s} \oplus R^{N^V}$ to $E^{ss}_{a,\bar{\mu}} \oplus E^V_{a,\bar{\mu}} \oplus E^u_{a,\bar{\mu}}$.
2. Set $f_{\bar{z}} = \Phi^{-1}_{\rho(a)\bar{z}} \circ \rho(a) \circ \Phi_{\bar{z}}$ and $f_{\bar{z}}^{-1} = \Phi^{-1}_{\rho(-a)\bar{z}} \circ \rho(-a) \circ \Phi_{\bar{z}}$, then for $\square = u, V, ss$, and all non-zero vector $v \in R^N$,$$
\log \frac{\|D|_{x=0}f_{\bar{z}}(v)v\|}{\|v\|} \in (\lambda^\square_0 - \varepsilon, \lambda^\square_0 + \varepsilon),$$

where $\lambda^u_0$ and $\lambda^s_0$ denote respectively the smallest and largest Lyapunov exponents of $\rho(a)$ on $E^u_{a,\bar{\mu}}$ and the constant $\varepsilon > 0$ can be chosen arbitrarily small by modifying $A''$.
3. For $x, x' \in B_{R^N}(l(\bar{z})^{-1})$, $c < \frac{|x-x'|}{\dist(\Phi_{\bar{z}}, \Phi_{\bar{z}}(x))} < l(\bar{z})$.

The foliations $\mathcal{W}_a^{\square}$ (for $\square = u, s$) near $\bar{z}$ can be translated to the corresponding foliations on $R^N$ by $\Phi_{\bar{z}}^{-1}$. By [19, Lemma 8.2.3 & 8.2.5], there exists $\tau \in (0, 1/2)$, such that for all $\bar{y} \in \mathcal{W}_a^u(\bar{z}) \cap \Phi_{\bar{z}}(B_{R^N}(\tau l^{-1}(\bar{z})))$, the image of the strong stable foliation $\mathcal{W}_a^{ss}(\bar{y})$ passing through $\bar{y}$ under $\Phi_{\bar{z}}^{-1}$ is the graph of a map $g_{\bar{z}, \bar{y}} : \mathbb{R}^{N^u} \to \mathbb{R}^{N^V}$. This implies that $(H_{\bar{z}})_{\bar{y}} \circ \Phi_{\bar{z}}$ is constant along the graph of $g_{\bar{z}, \bar{y}}$ for all $\bar{y} \in B_{R^N}(l^{-1}(\bar{z}))$, cf. [27, Lemma 4.12]. Moreover, by [19, §8.3], for all $\bar{z} \in A''$, there exists a bi-Lipschitz homeomorphism

$$\pi_{\bar{z}} : \Phi_{\bar{z}}^{-1}(\mathcal{W}_a^u(\bar{z}) \cap \Phi_{\bar{z}}(B_{R^N}(\tau l^{-1}(\bar{z})))) \to U^N \subset \mathbb{R}^{N^V}$$

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such that for all \( \overline{y} \in W^{s}_a(\overline{z}) \cap \Phi_{\overline{z}}(B_{\mathbb{R}^N} (\tau I^1(\overline{z}))) \), \( \pi_{\overline{z}} \) maps the graph of \( g_{\overline{z}, \overline{y}} \) to a piece of a hyperplane parallel to \( \mathbb{R}^{N^s} \). Put \( P_{\overline{z}} := \Phi_{\overline{z}} \circ \pi_{\overline{z}}^{-1} \), then \( (H_{\overline{z}})_V \circ P_{\overline{z}} \) is constant along hyperplanes parallel to \( \mathbb{R}^{N^s} \).

Now we are ready to give the contradiction, cf. [27, Lemma 4.15 & 4.16].

On the one hand, we claim that for every \( \overline{z} \in A'' \), there exists a decreasing sequence of bounded open neighborhoods \( B_{k, \overline{z}} \subset W^{s}_a(\overline{z}) \) of \( \overline{z} \) such that

\[
\lim_{k \to \infty} \frac{\text{Vol}_{\phi(\overline{z})} G_{a} (\phi(B_{k, \overline{z}}))}{\text{Vol}_{W^{s}_a(\overline{z})} (B_{k, \overline{z}})} = 0,
\]

where \( \text{Vol}_{\phi(\overline{z})} G_{a} \) and \( \text{Vol}_{W^{s}_a(\overline{z})} \) denote the volume forms of the induced Riemannian metrics on \( \phi(\overline{z}) G_{a} \) and \( W^{s}_a(\overline{z}) \) respectively.

In fact, we may choose \( \delta_0 > 0 \) and a neighborhood \( B_{\overline{z}} \subset W^{s}_a(\overline{z}) \) such that

\[
B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_0) \subset P_{\overline{z}}^{-1} (B_{\overline{z}}).
\]

Fix a decreasing sequence \( \{\delta_k > 0 : k \in \mathbb{N}\} \) approaching 0 as \( k \to \infty \). For each \( k \in \mathbb{N} \), define

\[
B_{k, \overline{z}} := P_{\overline{z}} \left( B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k) \right).
\]

Since \( P_{\overline{z}} \) is bi-Lipschitz, to show the claim, it suffices to show that

\[
\lim_{k \to \infty} \frac{\text{Vol}_{\phi(\overline{z})} G_{a} (\phi \circ P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k)))}{\text{Vol}_{\mathbb{R}^{N^s}} (B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k))} = 0.
\]

The denominator is of order \( O(\delta_k^{N^V}) \). Let us analyze the numerator. Our aim is to show that the numerator is of order \( o(\delta_k^{N^V}) \) (cf. Notation 1.8). Since \( G^s \) is decomposed as \( G^{ss} \cdot V \) and since \( V \) normalizes \( G^{ss} \), we have that \( \text{dVol}_{G^s} = \text{dVol}_{G^{ss}} \cdot \text{dVol}_V \). It is easy to see that the \( G^{ss} \)-projection of \( \phi \circ P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k)) \) is uniformly bounded, so to show that

\[
\text{Vol}_{\phi(\overline{z})} G_{a} (\phi \circ P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k))) = o(\delta_k^{N^V}),
\]

it suffices to show that

\[
\text{Vol}_V \left( (H_{\overline{z}})_V \circ P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k)) \right) = o(\delta_k^{N^V}).
\]

We have seen that \( (H_{\overline{z}})_V \circ P_{\overline{z}} \) only depends on the second coordinate, so

\[
(H_{\overline{z}})_V \circ P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_0) \times B_{\mathbb{R}^V} (\delta_k)) = (H_{\overline{z}})_V \circ P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_k) \times B_{\mathbb{R}^V} (\delta_k)).
\]

Note that \( P_{\overline{z}} \) is Lipschitz, we have that

\[
P_{\overline{z}}(B_{\mathbb{R}^{N^s}} (\delta_k) \times B_{\mathbb{R}^V} (\delta_k)) \subset B_{\mathbb{R}^s}(\overline{z}, C\delta_k)
\]

for a constant \( C = C(\overline{z}) \), here \( B_{\mathbb{R}^s}(\overline{z}, r) \) denotes the ball in \( W^{s}_a(\overline{z}) \) centered at \( \overline{z} \) of radius \( r \). Then to prove the claim, it is enough to show that

\[
\text{Vol}_V \left( (H_{\overline{z}})_V (B_{\mathbb{R}^s}(\overline{z}, \delta)) \right) = o(\delta^{N^V}) \quad \text{as} \quad \delta \to 0.
\]

This is true since by our hypothesis, for \( \overline{z} \in A'' \subset A \), \( D_{\mathbb{R}^s}^{\mathbb{R}^s}(H_{\overline{z}}(\overline{y})) : \mathbb{R}^{s}(\overline{z}) \rightarrow \mathfrak{s} \) has rank less than \( N^V = \dim \mathfrak{s} \).
On the other hand, we claim that for every \( \overline{z} \in \mathcal{J}(M) \), there exists a positive continuous function \( J_{\overline{z}} \) such that

\[
\phi_* \text{dVol}_{W^{ss}_{\overline{z}}(\overline{z})} = J_{\overline{z}} \text{dVol}_{\phi(\overline{z})G_{\overline{z}}^s}.
\]

Note that this claim contradicts the previous one. Therefore, to complete the proof, it suffices to prove this claim.

Recall that \( \mu \) denotes the Haar measure on the solenoid \( \mathcal{J}(M) \) and \( \overline{\mu} := \phi_*^{-1} \mu \). Let \( \overline{\mu}_{\overline{z}} \) denote the conditional measure of \( \overline{\mu} \) along the stable leaf \( W^{s}_{\overline{z}}(\overline{z}) \). Then the Radon-Nykodim derivative of \( \text{d}\overline{\mu}_{\overline{z}} \) with respect to \( \text{dVol}_{W^{ss}_{\overline{z}}(\overline{z})} \) can be calculated as follows:

\[
(4.13) \quad \frac{\text{d}\overline{\mu}_{\overline{z}}}{\text{dVol}_{W^{ss}_{\overline{z}}(\overline{z})}}(\overline{y}) = r_{\overline{z}}(\overline{y}) := \prod_{k=0}^{\infty} J^s \rho(a)(\rho(ka)(\overline{y})) \prod_{k=0}^{\infty} J^s \rho(a)(\rho(ka)(\overline{z})) \tag{4.13},
\]

where \( J^s \rho(a) \) denotes the Jacobian of \( \rho(a) \) along the stable bundle. Since \( J^s \rho(a) \) is Hölder continuous, and \( \rho(ka) \) contracts the stable leaf \( W^{s}_{\overline{z}}(\overline{z}) \) exponentially with \( k \), we have that the infinite product (4.13) is uniformly convergent. Therefore \( r_{\overline{z}}(\overline{y}) \) is uniformly bounded for \( \overline{y} \) in a neighborhood \( B \subset W^{s}_{\overline{z}}(\overline{z}) \) of \( \overline{z} \). The same argument shows that \( r_{\overline{z}}(\overline{z}) \) is also uniformly bounded. Note that \( r_{\overline{z}}(\overline{z}) = r_{\overline{z}}^{-1}(\overline{y}) \), we conclude that \( r_{\overline{z}}(\overline{y}) \) is also bounded away from zero. Note that for almost every \( \overline{z} \) with respect to \( \mu \),

\[
\text{d}\overline{\mu}_{\overline{z}} = r_{\overline{z}} \text{dVol}_{W^{ss}_{\overline{z}}(\overline{z})},
\]

and

\[
\text{d}\phi_* \overline{\mu}_{\overline{z}} = \text{dVol}_{\phi(\overline{z})G_{\overline{z}}^s}.
\]

Define

\[
J_{\overline{z}}(\overline{y}) := \frac{1}{r_{\overline{z}}(\phi^{-1}(\overline{y}))},
\]

then it is easy to see that \( J_{\overline{z}} \) is a positive function and continuous in \( \overline{z} \), and we have that for almost every \( \overline{z} \) with respect to \( \mu \),

\[
\phi_* \text{dVol}_{W^{ss}_{\overline{z}}(\overline{z})} = \phi_* (r_{\overline{z}}^{-1} \text{d}\overline{\mu}_{\overline{z}}) = J_{\overline{z}} \phi_* \text{d}\overline{\mu}_{\overline{z}} = J_{\overline{z}} \text{dVol}_{\phi(\overline{z})G_{\overline{z}}^s}.
\]

Because both sides of the above equation are continuous in \( \overline{z} \) and agree on a \( \mu \)-full measure subset, we have that the equation holds for every \( \overline{z} \in \mathcal{J}(M) \). This proves the claim, and hence get the contradiction.

This completes the proof. \( \square \)

It immediately follows that

**Corollary 4.15.** \( W^{ss}_{\overline{z}} \) defines a Hölder foliation consisting of smooth leaves.

**Proof.** By Proposition 4.14, for every \( \overline{z} \in \mathcal{J}(M) \), the function \( (H_{\overline{z}})_V \) (restricted to \( W^{ss}_{\overline{z}}(\overline{z}) \)) is smooth and regular at \( \overline{z} \). Since \( W^{ss}_{\overline{z}}(\overline{z}) \) is locally defined by \( (H_{\overline{z}})_V = e \), we conclude that \( W^{ss}_{\overline{z}}(\overline{z}) \) is smooth.

This completes the proof of Proposition 4.14. \( \square \)

**Remark 4.16.** We call \( W^{ss}_{\overline{z}} \) the strongly stable foliation of \( \rho(a) \).

This result combined with a result of Mañé (see [20]) implies the following:
**Proposition 4.17.** For any Weyl chamber $C$ adjacent to $C_0$, and for any $a' \in C$, $\rho(a')$ is uniformly hyperbolic.

*Proof.* See the proof of [27, Proposition 4.17].

**Remark 4.18.** It follows from the proposition that for any Weyl chamber $C$ and any $a' \in C$, $\rho(a')$ is uniformly hyperbolic.

Combined with the discussion at the beginning of the section, Proposition 4.17 implies the following:

**Theorem 4.19.** For every coarse Lyapunov exponent $[\chi]$ of $\rho$, the corresponding coarse Lyapunov distribution $E^{[\chi]}$ admits a Hölder foliation consisting of $C^\infty$ leaves.

5. **Regularity of the Conjugacy**

In this section we will prove Theorem 1.2 when $\dim M \geq 5$. As we discussed in §2, we write $\phi(z) = z h(z)$. Then it suffices to show that $h$ is $C^\infty$.

We follow the process described in §2.7. Recall that we follow Notation 2.12 and Notation 2.21 to denote coarse Lyapunov exponents and coarse Lyapunov subgroups. By Theorem 4.19, every coarse Lyapunov exponent $[\chi]$ of $\rho$ admits a Hölder foliation with $C^\infty$ leaves. Let us call it the coarse Lyapunov foliation associated with $[\chi]$.

5.1. **Case of tori.** In this subsection, we assume that $M$ is a torus. By our discussion in §2.7, it is enough to show that

$$h_V = \sum_{i=0}^{\infty} \rho_1(a)^{-i} \Phi \circ \rho(a)^i$$

is $C^\infty$ for any coarse Lyapunov subgroup $V$ associated with a coarse Lyapunov exponent $[\chi]$. 

By [8, Corollary 8.4], to show $h_V$ is $C^\infty$, it suffices to show that $h_V \in C^\infty, V^*, \theta$ for every coarse Lyapunov foliation $V'$.

**Proposition 5.1.** For any coarse Lyapunov foliation $V'$, $h_V \in C^\infty, V^*, \theta$, i.e., for every $k \in \mathbb{N}$, $\partial_V^k h_V \in (C^\theta(M))^*$. 

*Proof.* Suppose $V'$ is associated with the coarse Lyapunov exponent $[\chi']$. For any fixed small constant $\xi > 0$, choose $a \in \Sigma$ such that $\chi'(a) > 0$ and $|\chi'(a)| < \xi \|a\|$ for all $\chi' \in [\chi']$. Then we have that

$$h_V = \sum_{i=0}^{\infty} \rho_1(a)^{-i} \Phi \circ \rho(a)^i,$$

where $\Phi = (Q'_V)_V$. Then repeating the argument in the proof of Proposition 4.10 to $V'$, one concludes that $h_V \in C^\infty, V^*, \theta$.

*Proof of Theorem 1.2 for the case of tori.* By [8, Corollary 8.4], Proposition 5.1 implies that $h_V \in C^{\infty}(M)$. This proves that $h$ is $C^\infty$ since $h$ is the sum of $h_V$’s where $V$ runs over all coarse Lyapunov subgroups of $\rho_1$. 

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5.2. **General Case.** Now we deal with the general case. As described in §2.7, we consider the derived series of \( \mathcal{N} \):
\[
\mathcal{N} = N_0 \supset N_1 \supset \cdots \supset N_{r-1} \supset N_r = \{0\}.
\]
Note that \( \rho_I \) preserves \( N_i \) for each \( i = 0, 1, \ldots, r \).

We will prove Theorem 1.2 by showing the following stronger statement.

**Proposition 5.2.** For \( i = 0, 1, \ldots, r \), let \( h : M \to N_I(\mathbb{R}) \) (regarded as a map defined on \( \mathcal{S}(M) \)) be a \( \theta \)-Hölder map. Suppose for all \( a \in \mathbb{Z}^k \), there exists a \( C^\infty \) map \( Q_a : \mathcal{S}(M) \to N_I(\mathbb{R}) \), such that
\[
h(\bar{z}) = Q_a(\bar{z})\rho_I^{-1}(a)h \circ \rho(a)(\bar{z}).
\]
Then \( h \) is \( C^\infty \).

**Proof of Theorem 1.2 from Proposition 5.2.** We conclude the proof by applying the proposition with \( i = 0 \).

**Proof of Proposition 5.2.** We will prove the statement by induction on \( i \).

When \( i = r \), the statement is trivial. Suppose the statement holds for \( i + 1 \). We need to prove the statement for \( i \).

Let \( n_i \) denote the Lie algebra of \( N_i \). Then \( n_i(\mathbb{R}) \) admits the following splitting
\[
n_i(\mathbb{R}) = n'_i \oplus n_{i+1}(\mathbb{R}),
\]
where \( n_{i+1} \) is the Lie algebra of \( N_{i+1} \) and \( n'_i \) is a subspace of \( n_i(\mathbb{R}) \). Then we write \( h = h_0h_1 \) where \( h_1 \in N_{i+1}(\mathbb{R}) \) and \( h_0 \in \exp n'_i \). To show \( h \) is \( C^\infty \), it suffices to show that both \( h_0 \) and \( h_1 \) are \( C^\infty \).

We first prove that \( h_0 \) is \( C^\infty \). Let \( G_i := N_{i+1} \sim N_i \) and \( \pi_i : N_i(\mathbb{R}) \to G_i(\mathbb{R}) = \mathbb{R}^{l_i} \) denote the projection from \( N_i(\mathbb{R}) \) to \( G_i(\mathbb{R}) \). Let \( \bar{h} = \pi_i \circ h \). Then \( h_0 \) is \( C^\infty \) if and only if \( \bar{h} \) is \( C^\infty \) since \( n'_i \) can be identified with the Lie algebra of \( G_i(\mathbb{R}) \). Let \( \bar{\rho}_I \) denote the induced action of \( \rho_I \) on \( \mathcal{S}(G_i(\mathbb{Z}) \sim G_i(\mathbb{R})) \). Then it is easy to see that \( \bar{h} \) satisfies
\[
\bar{h} = \bar{\rho}_I^{-1}(a)\bar{h} \circ \rho(a) + \bar{Q}_a,
\]
where \( \bar{Q}_a = \pi_i(Q_a) \). Then from our proof for the case of tori, we conclude that \( \bar{h} \) is \( C^\infty \), and thus \( h_0 \) is \( C^\infty \).

Let us prove that \( h_1 \) is \( C^\infty \). In fact, since \( h = h_0h_1 \), we have that for any \( a \in \mathbb{Z}^k \),
\[
h_0(\bar{z})h_1(\bar{z}) = Q_a(\bar{z})\rho_I^{-1}(a)(h_0 \circ \rho(a)(\bar{z})) \cdot \rho_I^{-1}(a)(h_1 \circ \rho(a)(\bar{z})).
\]
Therefore
\[
h_1(\bar{z}) = Q'_a(\bar{z})\rho_I^{-1}(a)(h_1 \circ \rho(a)(\bar{z})),
\]
where
\[
Q'_a(\bar{z}) = (h_0(\bar{z}))^{-1}Q_a(\bar{z})\rho_I^{-1}(a)(h_0 \circ \rho(a)(\bar{z})).
\]
Since \( \rho_I \) preserves \( N_{i+1} \), we have that \( h_1(\bar{z}) \) and \( \rho_I^{-1}(a)(h_1 \circ \rho(a)(\bar{z})) \) are both in \( N_{i+1}(\mathbb{R}) \). Therefore \( Q'_a(\bar{z}) \) is also in \( N_{i+1}(\mathbb{R}) \). Moreover \( Q_a \) is \( C^\infty \) since \( Q_a \) and \( h_0 \) are \( C^\infty \). By our inductive hypothesis, we conclude that \( h_1 \) is \( C^\infty \).

This completes the proof. \( \square \)
6. THE LOW DIMENSIONAL CASES

In this section we consider the case \( \dim M \leq 4 \). Note that in this case the fundamental group of \( M \) must be abelian, i.e., \( M \) is homeomorphic to a torus.

**Proof of Theorem 1.2 for \( \dim M \leq 4 \).** For \( \dim M \leq 3 \), \( M \) does not have any exotic differential structures (cf. [24] and [21]). Therefore, the argument for \( \dim M \geq 5 \) applies to this case.

For \( \dim M = 4 \), we have that the homeomorphism

\[ \phi \times \phi : M \times M \to M \times M \]

conjugates the action \( \rho \times \rho \) to \( \rho_1 \times \rho_1 \). It is easy to verify that \( \rho \times \rho \) has no rank-one factor. By Theorem 1.2, we conclude that \( \phi \times \phi \) is \( C^\infty \), since \( \dim(M \times M) \geq 5 \).

This happens only if \( \phi \) itself is \( C^\infty \).

This completes the proof. \( \square \)

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**References**

[1] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, New Mathematical Monographs, 4, Cambridge University Press, Cambridge, 2006.

[2] L. Barreira and Y. Pesin, *Smooth ergodic theory and nonuniformly hyperbolic dynamics*, in *Handbook of Dynamical Systems. Vol. 1B*, Elsevier B. V., Amsterdam, 2006, 57–263.

[3] J. W. S. Cassels and A. Fröhlich, *Algebraic Number Theory*, Academic Press, Thompson Book Co., 1967.

[4] L. J. Corwin and F. P. Greenleaf, *Representations of Nilpotent Lie Groups and their Applications. Part I. Basic Theory and Examples*, Cambridge Studies in Advanced Mathematics, 18, Cambridge University Press, Cambridge, 1990.

[5] K. Dekimpe, What an infra-nilmanifold endomorphism really should be..., *Topol. Methods Nonlinear Anal.*, 40 (2012), 111–136.

[6] F. T. Farrell and L. E. Jones, *Examples of expanding endomorphisms on exotic tori*, *Invent. Math.*, 45 (1978), 175–179.

[7] D. Fisher, B. Kalinin and R. Spatzier, *Totally nonsymplectic Anosov actions on tori and nilmanifolds*, *Geom. Topol.*, 15 (2011), 191–216.

[8] D. Fisher, B. Kalinin and R. Spatzier, *Global rigidity of higher rank Anosov actions on tori and nilmanifolds*, with an appendix by J. F. Davis, *J. Amer. Math. Soc.*, 26 (2013), 167–198.

[9] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.*, 53 (1981), 53–73.

[10] A. Gorodnik and R. Spatzier, *Exponential mixing of nilmanifold automorphisms*, *Journal d’Analyse Mathématique*, 123 (2014), 355–396.

[11] A. Gorodnik and R. Spatzier, *Mixing properties of commuting nilmanifold automorphisms*, *Acta Math.*, 215 (2015), 127–159.

[12] B. Green and T. Tao, *The quantitative behaviour of polynomial orbits on nilmanifolds*, *Ann. of Math.* (2), 175 (2012), 465–540.

[13] B. Green and T. Tao, *On the quantitative distribution of polynomial nilsequences–erratum*, *Ann. of Math.* (2), 179 (2014), 1175–1183.
[14] B. Kalinin and A. Katok, Invariant measures for actions of higher rank abelian groups, in Smooth Ergodic Theory and its Applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001, 593–637.

[15] A. Katok and R. J. Spatzier, Invariant measures for higher-rank hyperbolic abelian actions, Ergodic Theory and Dynamical Systems, 16 (1996), 751–778.

[16] B. Kalinin and V. Sadovskaya, Global rigidity for totally nonsymplectic Anosov $Z^k$ actions, Geom. Topol., 10 (2006), 929–954.

[17] A. Leibman, Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold, Ergodic Theory Dynam. Systems, 25 (2005), 201–213.

[18] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin’s entropy formula, Ann. of Math. (2), 122 (1985), 509–539.

[19] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension, Ann. of Math. (2), 122 (1985), 540–574.

[20] R. Mañé, Quasi-Anosov diffeomorphisms and hyperbolic manifolds, Trans. Amer. Math. Soc., 229 (1977), 351–370.

[21] E. E. Moise, Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung, Ann. of Math. (2), 56 (1952), 96–114.

[22] G. A. Margulis and N. Qian, Rigidity of weakly hyperbolic actions of higher real rank simple Lie groups and their lattices, Ergodic Theory Dynam. Systems, 21 (2001), 121–164.

[23] W. Parry, Ergodic properties of affine transformations and flows on nilmanifolds, American Journal of Mathematics, 91 (1969), 757–771.

[24] T. Radó, Über den Begriff der Riemannschen Fläche, Acta Litt. Sci. Szeged, 2 (1925), 101–121.

[25] M. S. Raghunathan, Discrete Subgroups of Lie Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, 2. Folge, 68, Springer-Verlag, Berlin Heidelberg, 1972.

[26] F. Rodriguez Hertz, Global rigidity of certain abelian actions by toral automorphisms, J. Mod. Dyn., 1 (2007), 425–442.

[27] F. Rodriguez Hertz and Z. Wang, Global rigidity of higher rank abelian Anosov algebraic actions, Invent. Math., 198 (2014), 165–209.

[28] J. Rauch and M. Taylor, Regularity of functions smooth along foliations, and elliptic regularity, Journal of Functional Analysis, 225 (2005), 74–93.

[29] D. Ruelle, Ergodic theory of differentiable dynamical systems, Inst. Hautes Études Sci. Publ. Math., 50 (1979), 27–58.

[30] M. Shub, Endomorphisms of compact differentiable manifolds, Amer. J. Math., 91 (1969), 175–199.

[31] M. Shub, Expanding maps, in Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, 273–276.

[32] R. Spatzier, On the work of Rodriguez Hertz on rigidity in dynamics, J. Mod. Dyn., 10 (2016), 191–207.

[33] A. N. Starkov, The first cohomology group, mixing, and minimal sets of the commutative group of algebraic actions on a torus, J. Math. Sci. (New York), 95 (1999), 2576–2582.

[34] W. A. Veech, Periodic points and invariant pseudomeasures for toral endomorphisms, Ergodic Theory Dynam. Systems, 6 (1986), 449–473.

[35] P. Walters, Conjugacy properties of affine transformations of nilmanifolds, Math. Systems Theory, 4 (1970), 327–333.

[36] R. F. Williams, Expanding attractors, Publications Mathématiques de l'IHÉS, 43 (1974), 169–203.

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