ON A FAMILY OF DIFFERENTIAL-REFLECTION OPERATORS: INTERTWINING OPERATORS, AND FOURIER TRANSFORM OF RAPIDLY DECREASING FUNCTIONS

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Abstract. We introduce a family of differential-reflection operators \( \Lambda_{A,\varepsilon} \) acting on smooth functions defined on \( \mathbb{R} \). Here \( A \) is a Sturm-Liouville function with additional hypotheses and \( \varepsilon \in \mathbb{R} \). For special pairs \((A,\varepsilon)\), we recover Dunkl’s, Heckman’s and Cherednik’s operators (in one dimension). The spectral problem for the operators \( \Lambda_{A,\varepsilon} \) is studied. In particular, we obtain suitable growth estimates for the eigenfunctions of \( \Lambda_{A,\varepsilon} \).

As the operators \( \Lambda_{A,\varepsilon} \) are mixture of \( d/dx \) and reflection operators, we prove the existence of an intertwining operator \( V_{A,\varepsilon} \) between \( \Lambda_{A,\varepsilon} \) and the usual derivative. The positivity of \( V_{A,\varepsilon} \) is also established.

Via the eigenfunctions of \( \Lambda_{A,\varepsilon} \), we introduce a generalized Fourier transform \( \mathcal{F}_{A,\varepsilon} \).

An \( L^p \)-harmonic analysis for \( \mathcal{F}_{A,\varepsilon} \) is developed when \( 0 < p \leq \frac{1}{\varepsilon} \) and \(-1 \leq \varepsilon \leq 1\). In particular, an \( L^p \)-Schwartz space isomorphism theorem for \( \mathcal{F}_{A,\varepsilon} \) is proved.

1. Introduction

Dunkl’s ascertainment in the late eighties of the operators that now bear his name is one of the most significant developments in the theory of special functions associated with root systems [20]. Some early work in this area was done by Koornwinder [36]. A lot of the motivation for the subject comes from analysis on symmetric spaces. In the one-variable cases, spherical functions on Riemannian symmetric spaces can be written as special functions depending on parameters which assume only special discrete values. The case of more general parameter values yields special functions associated with root systems.

In [20] Dunkl generalized the operator \( d/dx \) to a mixture of a differential and a reflection operators (in one dimension):

\[
D_\alpha f(x) = f'(x) + \frac{2\alpha + 1}{x} \left( \frac{f(x) - f(-x)}{2} \right), \quad \alpha > -1/2.
\]

(1.1)

By the specialization \( \alpha = \frac{1}{2}d - 1 \) with \( d \in \mathbb{N}_{\geq 2} \), the operator \( D_\alpha^2 \) coincides on even functions with the radial part of the Laplace operator on the flat symmetric space \( M(d)/SO(d) \), where \( M(d) \) is the motion group of \( \mathbb{R}^d \). Important work in the analysis of Dunkl operators has been done by several authors (see [22–24,32,42,45–47,55]; this list is far from being complete).

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Some years after, Heckman \[33\] wrote down a trigonometric variant of the Dunkl operators \((1.1)\) (in one dimension):

\[
H_{\alpha,\beta}f(x) = f'(x) + (2\alpha + 1)\coth x + (2\beta + 1)\tanh x \left(\frac{f(x) - f(-x)}{2}\right),
\]

where \(\alpha \geq \beta \geq -1/2\) and \(\alpha \neq -1/2\). Heckman’s operators play a key role in proving the existence of the nowadays called Opdam’s shift operators. For \(\alpha = \frac{r}{2}(p - 1)\) and \(\beta = \frac{1}{2}(q - 1)\) with \(p \geq q > 0\), the restriction of \(H_{\alpha,\beta}^2\) to even functions coincides with the radial part of the Laplace-Beltrami operator on Riemannian symmetric spaces of the non-compact type and of real rank one. Significant results in the analysis of Heckman operators have been obtained by several authors (see for instance \([9,33-35,40,41]\)).

Next, in \([13]\) Cherednik made a slight but significant variation of Heckman’s operator. He put (in one dimension)

\[
\tilde{H}_{\alpha,\beta}f(x) = f'(x) + (2\alpha + 1)\coth x + (2\beta + 1)\tanh x \left(\frac{f(x) - f(-x)}{2}\right) - \varrho f(-x), \tag{1.2}
\]

where \(\alpha \geq \beta \geq -1/2, \alpha \neq -1/2,\) and \(\varrho = \alpha + \beta + 1\). It is known by now that harmonic analysis associated with \(\tilde{H}_{\alpha,\beta}\) has a considerable technical difficulties to be overcome compare to harmonic analysis for Heckman’s operator \(H_{\alpha,\beta}\) (see for instance \([2,14,43,48]\)).

The growing interest on these differential-reflection operators comes from their relevance for generalizing harmonic analysis on Riemannian symmetric spaces, and from their importance for developing new topics in mathematical physics and probability (see for instance \([3,25,28,29,47,48]\)).

In the present paper we consider some aspects of harmonic analysis associated with the following family of \((A,\varepsilon)\)-operators

\[
\Lambda_{A,\varepsilon}f(x) = f'(x) + \frac{A'(x)}{A(x)} \left(\frac{f(x) - f(-x)}{2}\right) - \varepsilon \varrho f(-x),
\]

where \(A\) is so-called a Chébli function on \(\mathbb{R}\) (i.e. \(A\) is a continuous \(\mathbb{R}^+\)-valued function on \(\mathbb{R}\) satisfying certain regularity and convexity hypotheses), \(\varrho\) is the index of \(A\), and \(\varepsilon \in \mathbb{R}\). We note that \(\varrho \geq 0\). The function \(A\) and the real number \(\varepsilon\) are the deformations parameters giving back the above three cases (as special examples) when:

\begin{itemize}
  \item[(1)] \(A(x) = A_\alpha(x) = |x|^{2\alpha+1}\) and \(\varepsilon\) arbitrary (Dunkl’s operators),
  \item[(2)] \(A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1}\) and \(\varepsilon = 0\) (Heckman’s operators),
  \item[(3)] \(A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1}\) and \(\varepsilon = 1\) (Cherednik’s operators).
\end{itemize}

This paper consists of three parts. In the first part we consider the spectral problem for this family of \((A,\varepsilon)\)-operators. More precisely, let \(\lambda \in \mathbb{C}\) and consider the equation

\[
\Lambda_{A,\varepsilon}f(x) = i\lambda f(x), \tag{1.3}
\]

where \(f : \mathbb{R} \to \mathbb{C}\). We prove that there exists a unique solution \(\Psi_{A,\varepsilon}(\lambda, \cdot)\) of \((1.3)\) satisfying \(\Psi_{A,\varepsilon}(\lambda, 0) = 1\). Moreover, under the assumption \(-1 \leq \varepsilon \leq 1\), we establish in Theorems \([4,2]\) and \([4,3]\) suitable estimates for the growth of the eigenfunction \(\Psi_{A,\varepsilon}(\lambda, x)\) and of its partial derivatives. Our first step is Theorem \([4,1]\) where we prove that \(\Psi_{A,\varepsilon}(\lambda, \cdot) > 0\)
whenever $\lambda \in \mathbb{R}$. These estimates are the key tools for developing $L^p$-harmonic analysis associated with the $(A,\varepsilon)$-operators (see Sections 8 and 9).

We note that $\Psi_{A,\varepsilon}$ reduces to the Dunkl kernel in the $(A_d,\varepsilon)$-case [21, 44]; to the Heckman kernel in the $(A_{d,\beta},0)$-case [2, 35]; and to the Cherednik kernel (or Opdam’s kernel) in the $(A_{d,\beta},1)$-case [2, 43].

In the second part of this paper we study the existence and the positivity of an intertwining operator between $\Lambda_{A,\varepsilon}$ and the ordinary derivative. We prove that there exists a unique isomorphism $V_{A,\varepsilon} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ satisfying $\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \frac{d}{dx}$, with $V_{A,\varepsilon}f(0) = f(0)$ (see Theorem 6.3). The construction of $V_{A,\varepsilon}$ involves Delsarte type operators [37, 49].

The intertwining operator $V_{A,\varepsilon}$ plays a crucial role for developing Fourier analysis associated with the $(A,\varepsilon)$-operators. In particular, it allows to write the eigenfunction $\Psi_{A,\varepsilon}$ as

$$\Psi_{A,\varepsilon}(\lambda, x) = V_{A,\varepsilon}(e^{i\lambda \cdot}) (x),$$ (1.4)

which gives a link between the Fourier transform with kernel $\Psi_{A,\varepsilon}$ (say $\mathcal{F}_{A,\varepsilon}$) and the Euclidean Fourier transform. This alliance between $\mathcal{F}_{A,\varepsilon}$ and the Euclidean Fourier transform will be a crucial trick to overcome difficulties in several places.

Another important result concerning the intertwining operator $V_{A,\varepsilon}$ is that the latter is of positive type in the sense that, if $f \geq 0$ then $V_{A,\varepsilon}f \geq 0$ (see Theorem 7.1). The major technical step in the proof of Theorem 7.1 is the positivity of $V_{A,\varepsilon}(h_t(u, \cdot))(x)$, where $h_t(u, v)$ denotes the Euclidean heat kernel at time $t > 0$. For $\varepsilon = 0$ and $1$, this result can be found in [53] and [54]. We pin down that the positivity of $V_{A,\varepsilon}$ played a fundamental role in [8] in establishing an analogue of Beurling’s theorem, and its relatives such as theorems of type Gelfand-Shilov, Morgan’s, Hardy’s, and Cowling-Price in the setting of this paper.

In the particular case where $A = A_d$ (see (1)), the intertwining operator $V_{A,\varepsilon}$ reduces to the Dunkl intertwining operator in one dimension (see for instance [45, 46]).

The third part of this paper is concerned with a development of the $L^p$-harmonic analysis for a Fourier transform $\mathcal{F}_{A,\varepsilon}$ when $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon}}$ and $-1 \leq \varepsilon \leq 1$. Here

$$\mathcal{F}_{A,\varepsilon}f(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx$$

for $f \in L^1(\mathbb{R}, A(x)dx)$.

Using the estimates for the growth of $\Psi_{A,\varepsilon}(\lambda, x)$ we get the holomorphic properties of $\mathcal{F}_{A,\varepsilon}$ on $L^p(\mathbb{R}, A(x)dx)$. A Riemann-Lebesgue lemma is also obtained for $1 \leq p < \frac{2}{1+\sqrt{1-\varepsilon}}$.

We then turn our attention to an $L^p$-Schwartz space isomorphism theorem for $\mathcal{F}_{A,\varepsilon}$. In [31] Harish-Chandra proved an $L^2$-Schwartz space isomorphism for the spherical Fourier transform on non-compact Riemannian symmetric spaces. This result was extended to $L^p$-Schwartz spaces with $0 < p < 2$ by Trombi and Varadarajan [50] (see also [18, 26, 27]). In the early nineties, Anker gave a new and simple proof of their result, based on the Paley-Wiener theorem for the spherical Fourier transform on Riemannian symmetric spaces [1]. Recently, Anker’s method was used in [39] to prove
an $L^p$-Schwartz space isomorphism theorem for the Heckman-Opdam hypergeometric functions. Our Approach is inspired from Anker’s paper [loc. cit.]. More precisely, for $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, put

$$ \mathbb{C}_{p,\varepsilon} := \{ \lambda \in \mathbb{C} \mid \text{Im } \lambda \leq \varepsilon \left( (2/p) - 1 - \sqrt{1-\varepsilon^2} \right) \}.$$  

Denote by $\mathcal{S}(\mathbb{R})$ the $L^p$-Schwartz space on $\mathbb{R}$, and by $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ the Schwartz space on the tube domain $\mathbb{C}_{p,\varepsilon}$. We prove that $\mathcal{F}_{A,\varepsilon}$ is a topological isomorphism between $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ (see Theorem 8.12).

We close the third part of this paper by establishing a result in connection with pointwise multipliers of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$. More precisely, for arbitrary $\alpha \geq 0$, a function $\psi$ defined on the tube domain $\mathbb{C}_\alpha := \{ \lambda \in \mathbb{C} \mid |\text{Im } \lambda| \leq \alpha \}$ is called a pointwise multiplier of $\mathcal{S}(\mathbb{C}_\alpha)$ if the mapping $\phi \mapsto \psi\phi$ is continuous from $\mathcal{S}(\mathbb{C}_\alpha)$ into itself. In [4] Betancor et al. characterize the set of pointwise multipliers of the Schwartz spaces $\mathcal{S}(\mathbb{C}_\alpha)$.

Under the assumptions $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho = 0$, and $\frac{2}{1+\sqrt{1-\varepsilon^2}} \leq p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho > 0$, we prove that if $T$ is in the dual space $\mathcal{S}'(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$ such that $\psi := \mathcal{F}_{A,\varepsilon}(T)$ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$, then for any $s \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and continuous functions $f_m$ defined on $\mathbb{R}$, $m = 0, 1, \ldots, \ell$, such that

$$ T = \sum_{m=0}^{\ell} \Lambda_{A,\varepsilon}^m f_m $$

and, for every such $m$,

$$ \sup_{x \in \mathbb{R}} (|x|^s + 1)^{\frac{2}{p} - \sqrt{1-\varepsilon^2}} \varrho^{|\alpha|} |f_m(x)| < \infty. $$

The organization of this paper is as follows: In Section 2 we recapitulate some definitions and basic notations, as well as some results from literature. In Sections 3 and 4 we study the main properties of the eigenfunction $\Psi_{A,\varepsilon}$. In particular, we obtain estimates for the growth of $\Psi_{A,\varepsilon}$ and of its partial derivatives. A Laplace type representation of the eigenfunction $\Psi_{A,\varepsilon}$ is derived in Section 5. Sections 6 and 7 are devoted to the existence and to the positivity of the intertwining operator $V_{A,\varepsilon}$ between $\Lambda_{A,\varepsilon}$ and the ordinary derivative. In Section 8 we develop the $L^p$-harmonic analysis for the Fourier transform $\mathcal{F}_{A,\varepsilon}$, where we mainly prove an $L^p$-Schwartz space isomorphism theorem for $\mathcal{F}_{A,\varepsilon}$. Finally, in Section 9 we characterize the distributions $T \in \mathcal{S}'(\mathbb{R})$ so that $\mathcal{F}_{A,\varepsilon}(T)$ is a pointwise multiplier of the Schwartz space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$.

2. Background

In this introductory section we present results from [15, 17, 51, 52]. See also [5-7]. Throughout this paper we will denote by $A$ a function on $\mathbb{R}$ satisfying the following hypotheses:

(H1) $A(x) = |x|^{|\alpha|+1} B(x)$, where $\alpha > -\frac{1}{2}$ and $B$ is any even, positive and smooth function on $\mathbb{R}$ with $B(0) = 1$.

(H2) $A$ is increasing and unbounded on $\mathbb{R}_+$. 


Lemma 2.2. Let $|\phi| := \lim_{x \to +\infty} A'(x)/A(x) \geq 0$ exists.

Such a function $A$ is called a Chébli function. From (H1) it follows that

$$\frac{A'(x)}{A(x)} = \frac{2\alpha + 1}{x} + C(x), \quad x \neq 0,$$

where $C := B'/B$ is an odd and smooth function on $\mathbb{R}$.

Let $\Delta_{\lambda}$, or simply $\Delta$, be the following second order differential operator

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.$$

For $\mu \in \mathbb{C}$, we consider the Cauchy problem

$$\begin{align*}
\Delta f(x) &= -(\mu^2 + c^2)f(x) \\
f(0) &= 1, \quad f'(0) = 0.
\end{align*}$$

In [17] the author proved that the system (2.3) admits a unique solution $\varphi_\mu$. For every $\mu \in \mathbb{C}$, the solution $\varphi_\mu$ is an even smooth function on $\mathbb{R}$ and the map $\mu \mapsto \varphi_\mu(x)$ is analytic. The following Laplace type representation of $\varphi_\mu$ can be found in [17] (see also [51]).

Lemma 2.1. For every $x \in \mathbb{R}^*$ there exists a probability measure $\nu_x$ on $\mathbb{R}$ supported in $[-|x|, |x|]$ such that for all $\mu \in \mathbb{C}$

$$\varphi_\mu(x) = \int_{-|x|}^{|x|} e^{t(\mu+\nu)} \nu_x(dt).$$

Also, for $x \in \mathbb{R}^*$, there is a non-negative even continuous function $K(|x|, \cdot)$ supported in $[-|x|, |x|]$ such that for all $\mu \in \mathbb{C}$

$$\varphi_\mu(x) = \int_0^{2|x|} K(|x|, t) \cos(\mu t) dt. \quad (2.4)$$

The following estimates of the eigenfunctions $\varphi_\mu$ can be found in [7, 15, 17, 52].

Lemma 2.2. Let $\mu \in \mathbb{C}$ such that $|\text{Im } \mu| \leq \varnothing$. Then

1) $\varphi_{\pm \varnothing}(x) = 1$.
2) $\varphi_{-\mu}(x) = \varphi_\mu(x)$.
3) $|\varphi_\mu(x)| \leq 1$.
4) $|\varphi_\mu(x)| \leq e^{\text{Im } \mu |x|} \varphi_0(x)$.
5) $|\varphi_\mu(x)| \leq c (\varrho^2 + |\mu|^2) e^{\text{Im } \mu |x|} \varphi_0(x)$.
6) $e^{-\varrho |x|} \leq \varphi_0(x) \leq c(|x| + 1) e^{-\varrho |x|}$.

The Chébli transform of $f \in L^1(\mathbb{R}^+, A(x)dx)$ is given by

$$\mathscr{F}_\Delta(f)(\mu) := \int_{\mathbb{R}^+} f(x) \varphi_\mu(x) A(x)dx. \quad (2.5)$$

The following Plancherel and inversion formulas for $\mathscr{F}_\Delta$ are proved in [17].
Theorem 2.3. There exists a unique positive measure $\pi$ with support $\mathbb{R}_+$ such that $\mathcal{F}_\Delta$ induces an isometric isomorphism from $L^2(\mathbb{R}_+, A(x)dx)$ onto $L^2(\mathbb{R}_+, \pi(d\mu))$, and for any $f \in L^1(\mathbb{R}_+, A(x)dx) \cap L^2(\mathbb{R}_+, A(x)dx)$ we have

$$\int_{\mathbb{R}_+} |f(x)|^2 A(x) dx = \int_{\mathbb{R}_+} |\mathcal{F}_\Delta(f)(\mu)|^2 \pi(d\mu).$$

The inverse transform is given by

$$\mathcal{F}_\Delta^{-1} g(x) = \int_{\mathbb{R}_+} g(\mu) \varphi_\mu(x) \pi(d\mu). \quad (2.6)$$

To have a nice behavior for the Plancherel measure $\pi$ we must add a further (growth) restriction on the function $A$. Following [51], we will assume that $A'/A$ satisfies the following additional hypothesis:

(H4) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty)$ (for some $x_0 > 0$),

$$\frac{A'(x)}{A(x)} = \begin{cases} 
2 \phi + e^{-\delta x} D(x) & \text{if } \phi > 0, \\
2 \alpha + \frac{1}{x} + e^{-\delta x} D(x) & \text{if } \phi = 0,
\end{cases} \quad (2.7)$$

with $D$ being a smooth function bounded together with its derivatives.

In these circumstances the Plancherel measure $\pi$ is absolutely continuous with respect to the Lebesgue measure and has density $|c(\mu)|^{-2}$ where $c$ is continuous function on $\mathbb{R}_+$ and zero free on $\mathbb{R}_+^*$ (see [6]). Moreover, by [52 Proposition 6.1.12 and Corollary 6.1.5] (see also [10]), for $\mu \in \mathbb{C}$ we have

(i) If $\varrho \geq 0$ and $\alpha > -1/2$, then $|c(\mu)|^{-2} \sim |\mu|^{2\alpha+1}$ whenever $|\mu| \gg 1$.

(ii) If $\varrho > 0$ and $\alpha > -1/2$, then $|c(\mu)|^{-2} \sim |\mu|^2$ whenever $|\mu| \ll 1$.

(iii) If $\varrho = 0$ and $\alpha > 0$, then $|c(\mu)|^{-2} \sim |\mu|^{2\alpha+1}$ whenever $|\mu| \ll 1$.

In the literature, the function $c$ is called Harish-Chandra’s function of the operator $\Delta$. We refer to [11] for more details on the $c$-function.

Henceforth we will assume that Chébli’s function $A$ satisfies the additional hypothesis (H4). It follows that for $|x|$ is large enough:

(i) $A(x) = O(e^{2\delta x})$ for $\varrho > 0$.

(ii) $A(x) = O(|x|^{2\alpha+1})$ for $\varrho = 0$.

We close this section by giving some basic results of (the analogue of) the Abel transform associated with the second order differential operator $\Delta$.

Denote by $\mathcal{S}_e(\mathbb{R})$ the space of even and compactly supported functions in $C^\infty(\mathbb{R})$. In [51] the author has proved that the Abel transform defined on $\mathcal{S}_e(\mathbb{R})$ by

$$\mathcal{A} f(y) = \frac{1}{2} \int_{|x|>|y|} K(|x|, y) f(x) A(x) dx \quad (2.8)$$

is an automorphism of $\mathcal{S}_e(\mathbb{R})$ and satisfying

$$\mathcal{A} \circ (\Delta + q^2) = \frac{d^2}{dx^2} \circ \mathcal{A}. \quad (2.9)$$
Furthermore, on $\mathcal{D}(\mathbb{R})$, we have
\[
\mathcal{F}_\Lambda = \mathcal{F}_{\text{euc}} \circ \mathcal{A},
\]  
(2.10)
where $\mathcal{F}_{\text{euc}}$ is the Euclidean Fourier transform.

3. A FAMILY OF DIFFERENTIAL-REFLECTION OPERATORS

For $\varepsilon \in \mathbb{R}$ we consider the following differential-reflection operators
\[
\Lambda_{\Lambda,\varepsilon} f(x) = f'(x) + \frac{A'(x)}{A(x)} \left( \frac{f(x) - f(-x)}{2} \right) - \varepsilon \partial f(-x).
\]  
(3.1)

In view of (2.1) and the hypothesis (H4) on $A'/A$, the space $\mathcal{D}(\mathbb{R})$ (of smooth functions with compact support on $\mathbb{R}$) and the space $\mathcal{S}(\mathbb{R})$ (of Schwartz functions on $\mathbb{R}$) are invariant under the action of $\Lambda_{\Lambda,\varepsilon}$.

Let $S$ denote the symmetry $(S f)(x) := f(-x)$. The following lemma is needed later. The easy proof is left to the reader.

**Lemma 3.1.** Let $f \in C^\infty(\mathbb{R})$ such that $\sup_{t \in \mathbb{R}} (1 + |x|) e^{\varepsilon |x|} |f^{(r)}(x)| < \infty$ for every $r, t \in \mathbb{N}$ and for some $2Q \leq s < \infty$, and let $g \in C^\infty(\mathbb{R})$ such that $g$ and all its derivatives are at most of polynomial growth. Then
\[
\int_{\mathbb{R}} \Lambda_{\Lambda,\varepsilon} f(x) g(x) A(x) dx = - \int_{\mathbb{R}} f(x) (\Lambda_{\Lambda,\varepsilon} + 2\varepsilon \partial S) g(x) A(x) dx.
\]

Let $\lambda \in \mathbb{C}$ and consider the initial data problem
\[
\Lambda_{\Lambda,\varepsilon} f(x) = i\lambda f(x), \quad f(0) = 1,
\]  
(3.2)
where $f : \mathbb{R} \to \mathbb{C}$. We have the following statement.

**Theorem 3.2.** Let $\lambda \in \mathbb{C}$. There exists a unique solution $\Psi_{\Lambda,\varepsilon}(\lambda, \cdot)$ to the problem (3.2). Further, for every $x \in \mathbb{R}$, the function $\lambda \mapsto \Psi_{\Lambda,\varepsilon}(\lambda, x)$ is analytic on $\mathbb{C}$. More explicitly:

(i) For $i\lambda \neq \varepsilon Q$,
\[
\Psi_{\Lambda,\varepsilon}(\lambda, x) = \varphi_\mu(x) + \frac{1}{i\lambda - \varepsilon Q} \varphi'_\mu(x),
\]  
(3.3)
where
\[
\mu^2_\varepsilon := \lambda^2 + (\varepsilon^2 - 1)Q^2.
\]  
(3.4)
We may rewrite the solution (3.3) as
\[
\Psi_{\Lambda,\varepsilon}(\lambda, x) = \varphi_\mu(x) + (i\lambda + \varepsilon Q) \frac{\sg(x)}{A(x)} \int_0^{i|\lambda|} \varphi'_\mu(t) A(t) dt.
\]  
(3.5)

(ii) For $i\lambda = \varepsilon Q$,
\[
\Psi_{\Lambda,\varepsilon}(\lambda, x) = 1 + 2\varepsilon Q \frac{\sg(x)}{A(x)} \int_0^{i|\lambda|} A(t) dt.
\]  
(3.6)
Proof. Assume first that $i \lambda \neq \varepsilon g$. After the formula (3.5) is established the restriction on $\lambda$ can be dropped by analytic continuation. Write $f$ as the superposition $f = f_e + f_o$ of an even function $f_e$ and an odd function $f_o$. Then, the problem (3.2) is equivalent to the following system:

\[
\begin{array}{c}
\left\{ \begin{array}{l}
f'_e(x) + \frac{A'(x)}{A(x)} f_o(x) = (i \lambda + \varepsilon g) f_e(x), \\
f'_o(x) = (i \lambda - \varepsilon g) f_o(x), \\
f_e(0) = 1, \ f_o(0) = 0.
\end{array} \right.
\end{array}
\]

(3.7a)

(3.7b)

(3.7c)

Combining the two equations above yields

\[ f''e(x) + \frac{A'(x)}{A(x)} f'_e(x) = -(\lambda^2 + \varepsilon^2 g^2) f_e(x). \]

That is

\[ \Delta f_e(x) = -\left( \lambda^2 + (\varepsilon^2 - 1)g^2 + g^2 \right) f_e(x). \]

Since $f_e(0) = 1$, the uniqueness of the solution to the Cauchy problem (2.3) gives

\[ f_e(x) = \varphi_{\mu_e}(x), \]

which, in part, explains the uniqueness of the desired solution. Now, from (3.7b) we obtain

\[ f_o(x) = \frac{1}{i \lambda - \varepsilon g} \varphi'_{\mu_e}(x). \]

Consequently

\[ \Psi_{A,e}(\lambda, x) = \varphi_{\mu_e}(x) + \frac{1}{i \lambda - \varepsilon g} \varphi'_{\mu_e}(x). \]

(3.8)

Further we have

\[ \varphi'_{\mu_e}(x) = -(\mu_e^2 + g^2) \frac{sg(x)}{A(x)} \int_0^{|x|} \varphi_{\mu_e}(t) A(t) dt, \]

(3.9)

which is a consequence of the following known formula for even functions

\[ g'(x) = \frac{sg(x)}{A(x)} \int_0^{|x|} \Delta g(t) A(t) dt \]

(3.10)

and the fact that $\varphi_{\mu}$ satisfies (2.3). Hence, we can rewrite the solution (3.8) as

\[ \Psi_{A,e}(\lambda, x) = \varphi_{\mu_e}(x) + (i \lambda + \varepsilon g) \frac{sg(x)}{A(x)} \int_0^{|x|} \varphi'_{\mu_e}(t) A(t) dt. \]

(3.11)

Since the function $\mu \mapsto \varphi_{\mu}(x)$ is holomorphic for all $\mu \in \mathbb{C}$, it follows from (3.11) that for every $x \in \mathbb{R}$, the map $\lambda \mapsto \Psi_{A,e}(\lambda, x)$ is analytic on $\mathbb{C}$. $\square$
4. Growth of the Eigenfunctions

The eigenfunction $\Psi_{A,\varepsilon}$ is of particular interest as it gives rise to an associated integral transform on $\mathbb{R}$ which generalizes the Euclidean Fourier transform in a natural way (see Section 8). Its definition and essential properties rely on suitable growth estimates of $\Psi_{A,\varepsilon}$. The following positivity result is the basic ingredient in obtaining these estimates.

**Theorem 4.1.** Assume that $-1 \leq \varepsilon \leq 1$. For all $\lambda \in i\mathbb{R}$, the function $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is real and strictly positive.

**Proof.** If we take complex conjugates in (3.2), we see that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ and $\Psi_{A,\varepsilon}(\lambda, \cdot)$ satisfy the same system (3.2). Since $\Psi_{A,\varepsilon}(\lambda, 0) = 1$, the uniqueness part in Theorem 3.2 shows that $\Psi_{A,\varepsilon}(\lambda, x) = \Psi_{A,\varepsilon}(\lambda, x)$ for all $x \in \mathbb{R}$.

Assume that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is not strictly positive. Since $\Psi_{A,\varepsilon}(\lambda, 0) = 1 > 0$, it follows that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ vanishes. Let $x_0$ be a zero of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ so that

$$|x_0| = \inf \{|x| : \Psi_{A,\varepsilon}(\lambda, x) = 0\}.$$  

Since $\Psi_{A,\varepsilon}(\lambda, 0) = 1$ we have $\Psi_{A,\varepsilon}(\lambda, x) \geq 0$ on $[-|x_0|, |x_0|]$. In particular $\Psi_{A,\varepsilon}(\lambda, -x_0) \geq 0$. We claim that

$$\begin{cases}
\Psi_{A,\varepsilon}(\lambda, x_0) = 0, \\
\Psi_{A,\varepsilon}(\lambda, -x_0) = 0.
\end{cases} \tag{4.1}$$

To prove (4.1), let us first assume that $x_0 > 0$. Then $\Psi_{A,\varepsilon}'(\lambda, x_0) \leq 0$. Moreover,

$$\Psi_{A,\varepsilon}'(\lambda, x) = -\frac{A'(x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x)) + \varepsilon \lambda \Psi_{A,\varepsilon}(\lambda, x) + i\varepsilon \Psi_{A,\varepsilon}(\lambda, x). \tag{4.2a}$$

From (4.3) it follows that $\Psi_{A,\varepsilon}'(\lambda, x_0)$ is positive. This is due to the fact that $\varepsilon \geq 1$ and the fact that $A'/2A$ is a decreasing function on $\mathbb{R}^+$ and $\lim_{x \to +\infty} A'(x)/2A(x) = 0$. We deduce that $\Psi_{A,\varepsilon}'(\lambda, x_0) = 0$, and therefore, from (4.3), $\Psi_{A,\varepsilon}(\lambda, -x_0) = 0$.

Now, let us assume that $x_0 < 0$. Then $\Psi_{A,\varepsilon}'(\lambda, x_0) \geq 0$. Moreover, for $x_0 < 0$, equation (4.3) implies $\Psi_{A,\varepsilon}'(\lambda, x_0) \leq 0$. This is due to $\varepsilon \leq 1$ and to assumptions on $A'/2A$. Then, as above, we conclude that $\Psi_{A,\varepsilon}'(\lambda, x_0) = 0$, and once again appealing to (4.3) we have $\Psi_{A,\varepsilon}(\lambda, -x_0) = 0$. This finishes the proof of the claim (4.1).

Starting this time from $\Psi_{A,\varepsilon}(\lambda, -x_0) = 0$ and proceeding analogously as in the case $\Psi_{A,\varepsilon}(\lambda, x_0) = 0$, we conclude that

$$\begin{cases}
\Psi_{A,\varepsilon}'(\lambda, -x_0) = 0, \\
\Psi_{A,\varepsilon}(\lambda, x_0) = 0.
\end{cases} \tag{4.2b}$$

In summary, $\Psi_{A,\varepsilon}(\lambda, \pm x_0) = 0$ and $\Psi_{A,\varepsilon}'(\lambda, \pm x_0) = 0$. Differentiating (4.2b), we see that the second derivative of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ vanishes at $\pm x_0$. Repeating the same argument over and over again to get $\Psi_{A,\varepsilon}^{(k)}(\lambda, \pm x_0) = 0$ for all $k \in \mathbb{N}$. Since $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is a real analytic
function, we deduce that \( \Psi_{A,\varepsilon}(\lambda, x) = 0 \) for all \( x \in \mathbb{R} \). This contradicts \( \Psi_{\varepsilon}(\lambda, 0) = 1 \). Thus, either \( \Psi_{A,\varepsilon}(\lambda, x) \) is strictly positive for all \( x \), or it is strictly negative for all \( x \). But since \( \Psi_{A,\varepsilon}(\lambda, 0) = 1 \), it must be \( \Psi_{A,\varepsilon}(\lambda, x) > 0 \) for all \( x \in \mathbb{R} \). \( \square \)

The following theorem contains important estimates for the growth of the eigenfunction \( \Psi_{A,\varepsilon} \).

**Theorem 4.2.** Suppose that \(-1 \leq \varepsilon \leq 1 \) and \( x \in \mathbb{R} \). Then:

1) For real \( \lambda \) we have \( |\Psi_{A,\varepsilon}(\lambda, x)| \leq \sqrt{2} \).
2) For \( \lambda = a + ib \in \mathbb{C} \) we have \( |\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(ib, x) \).
3) For \( \lambda = ib \in i\mathbb{R} \) we have \( \Psi_{A,\varepsilon}(ib, x) \leq \Psi_{A,\varepsilon}(0, x) e^{\|b\|} \).
4) For \( \lambda = 0 \) we distinguish the following two cases:
   a) For \( \varepsilon = 0 \), we have \( \Psi_{A,\varepsilon}(0, x) = 1 \).
   b) For \( \varepsilon \neq 0 \), there is a constant \( c_{\varepsilon} > 0 \) such that \( \Psi_{A,\varepsilon}(0, x) \leq c_{\varepsilon}(|x| + 1)e^{-\varepsilon(1 - \sqrt{|\varepsilon^2|}|x|)} \).

**Proof.**

1) Assume that \( \lambda \in \mathbb{R} \). Since \( \Psi_{A,\varepsilon}(\lambda, x) \) is a solution of the problem (3.2), we deduce that

\[
\Psi'_{A,\varepsilon}(\lambda, x) = -\frac{A'(x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x)) + \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, -x) + i\lambda\Psi_{A,\varepsilon}(\lambda, x). \tag{4.4}
\]

Thus \( \Psi_{A,\varepsilon}(\lambda, -x) \) satisfies the following equation

\[
[\Psi_{A,\varepsilon}(\lambda, -x)]' = \frac{A'(-x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, -x) - \Psi_{A,\varepsilon}(\lambda, x)) - \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, x) - i\lambda\Psi_{A,\varepsilon}(\lambda, -x). \tag{4.5}
\]

If we take complex conjugates in (4.5), we obtain

\[
[\Psi_{A,\varepsilon}(\lambda, -x)]' = \frac{A'(-x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, -x) - \Psi_{A,\varepsilon}(\lambda, x)) - \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, x) + i\lambda\Psi_{A,\varepsilon}(\lambda, -x).
\]

Hence

\[
[|\Psi_{A,\varepsilon}(\lambda, -x)|^2]' = [\Psi_{A,\varepsilon}(\lambda, -x)]' \Psi_{A,\varepsilon}(\lambda, -x) + [\Psi_{A,\varepsilon}(\lambda, -x)]' \Psi_{A,\varepsilon}(\lambda, -x)
\]

\[
= \frac{A'(-x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, -x) - \Psi_{A,\varepsilon}(\lambda, x))\Psi_{A,\varepsilon}(\lambda, -x) - \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, x)(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x))
\]+ \frac{A'(-x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, -x) - \Psi_{A,\varepsilon}(\lambda, x))\Psi_{A,\varepsilon}(\lambda, -x) - \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, x)(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x)).
\]

Similarly we have

\[
[|\Psi_{A,\varepsilon}(\lambda, x)|^2]' = [\Psi_{A,\varepsilon}(\lambda, x)]' \Psi_{A,\varepsilon}(\lambda, x) + [\Psi_{A,\varepsilon}(\lambda, x)]' \Psi_{A,\varepsilon}(\lambda, x)
\]

\[
= -\frac{A'(x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x))\Psi_{A,\varepsilon}(\lambda, x) + \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, -x)(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x))
\]- \frac{A'(x)}{2A(x)}(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x))\Psi_{A,\varepsilon}(\lambda, x) + \varepsilon\gamma\Psi_{A,\varepsilon}(\lambda, -x)(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x)).
\]
Using the fact that \( A' \) is an odd function we obtain
\[
\frac{[|\Psi_{A,e}(\lambda,-x)|^2]'}{[|\Psi_{A,e}(\lambda,x)|^2]'} = \left( \Psi_{A,e}(\lambda,-x) - \Psi_{A,e}(\lambda,x) \right) \left( -\frac{A'(x)}{2A(x)} \Psi_{A,e}(\lambda,-x) + \frac{A'(x)}{2A(x)} \Psi_{A,e}(\lambda,x) \right) + \left( \Psi_{A,e}(\lambda,-x) - \Psi_{A,e}(\lambda,x) \right) \left( -\frac{A'(x)}{2A(x)} \Psi_{A,e}(\lambda,-x) + \frac{A'(x)}{2A(x)} \Psi_{A,e}(\lambda,x) \right)
\]
\[
= -\frac{A'(x)}{A(x)} \left| \Psi_{A,e}(\lambda,-x) - \Psi_{A,e}(\lambda,x) \right|^2.
\]
Since \( A'(x)/A(x) \geq 2 \) for all \( x \in \mathbb{R}_+ \), it follows that
\[
[|\Psi_{A,e}(\lambda,-x)|^2]'+[|\Psi_{A,e}(\lambda,x)|^2]' \leq 0, \quad \forall x \in \mathbb{R}_+.
\]
This implies
\[
|\Psi_{A,e}(\lambda,-x)|^2 + |\Psi_{A,e}(\lambda,x)|^2 \leq |\Psi_{A,e}(\lambda,0)|^2 + |\Psi_{A,e}(\lambda,0)|^2 = 2, \quad \forall x \in \mathbb{R}_+.
\]
As a consequence
\[
|\Psi_{A,e}(\lambda,-x)| \leq \sqrt{2} \quad \text{and} \quad |\Psi_{A,e}(\lambda,x)| \leq \sqrt{2}, \quad \forall x \in \mathbb{R}_+.
\]
This finishes the proof of the first statement.

2) For \( \lambda = a + ib \in \mathbb{C} \) we define the function \( Q_{e,1} \) by
\[
Q_{e,1}(x) = \frac{\Psi_{A,e}(\lambda,x)}{\Psi_{A,e}(ib,x)}.
\]
By Theorem 4.1 the function \( Q_{e,1} \) is well defined. Since \( \Psi_{A,e}(\lambda, x) \) satisfies the differential-reflection equation (4.4), it follows that
\[
\frac{\Psi'_{A,e}(\lambda, x)}{\Psi_{A,e}(ib, x)} = \frac{A'(x)}{2A(x)} (Q_{e,1}(x) - Q_{e,1}(-x)) \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} + \varepsilon Q_{e,1}(-x) \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} + i\lambda Q_{e,1}(x).
\]
We now take the derivative of \( Q_{e,1} \):
\[
Q'_{e,1}(x) = \frac{\Psi'_{A,e}(\lambda, x)}{\Psi_{A,e}(ib, x)} - Q_{e,1}(x) \frac{\Psi'_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)}
\]
\[
= -\frac{A'(x)}{2A(x)} Q_{e,1}(x) + \frac{A'(x)}{2A(x)} Q_{e,1}(-x) \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} + \varepsilon Q_{e,1}(-x) \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} - Q_{e,1}(x) \left( -\frac{A'(x)}{2A(x)} \right) \left( 1 - \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} \right) + \varepsilon Q_{e,1}(-x) \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} - b) + i\lambda Q_{e,1}(x)
\]
\[
= \left( \frac{A'(x)}{2A(x)} + \varepsilon \right) (Q_{e,1}(-x) - Q_{e,1}(x)) \frac{\Psi_{A,e}(ib, -x)}{\Psi_{A,e}(ib, x)} + (i\lambda + b) Q_{e,1}(x).
\]
Hence,
\[
\begin{align*}
\{Q_{e,i}(x)\} & = Q_{e,i}(x)Q_{e,i}(x) + Q_{e,i}(x)\overline{Q_{e,i}(x)} \\
& = 2 \text{Re} \left[ Q_{e,i}^\prime(x)Q_{e,i}(x) \right] \\
& = 2 \text{Re} \left[ \left( \frac{A'(x)}{2A(x)} + \varepsilon \varepsilon \right) \left( Q_{e,i}(-x)Q_{e,i}(x) - |Q_{e,i}(x)|^2 \right) \frac{\Psi_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)} + (i\varepsilon + b)|Q_{e,i}(x)|^2 \right] \\
& = 2 \text{Re} \left[ \left( \frac{A'(x)}{2A(x)} + \varepsilon \varepsilon \right) \left( Q_{e,i}(-x)Q_{e,i}(x) - |Q_{e,i}(x)|^2 \right) \frac{\Psi_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)} \right] \\
& = -2 \left( \frac{A'(x)}{2A(x)} + \varepsilon \varepsilon \right) \left( |Q_{e,i}(x)|^2 \right) - \text{Re} \left[ Q_{e,i}(-x)Q_{e,i}(x) \right] \frac{\Psi_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)}.
\end{align*}
\]
Similarly we have
\[
\begin{align*}
\{Q_{e,i}(-x)\} & = Q_{e,i}(-x)\overline{Q_{e,i}(x)} - Q_{e,i}(-x)Q_{e,i}^\prime(-x) \\
& = -2 \text{Re} \left[ Q_{e,i}^\prime(-x)Q_{e,i}(x) \right] \\
& = -2 \left( \frac{A'(x)}{2A(x)} - \varepsilon \varepsilon \right) \left( |Q_{e,i}(-x)|^2 \right) - \text{Re} \left[ Q_{e,i}(x)Q_{e,i}^\prime(-x) \right] \frac{\Psi_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)}.
\end{align*}
\]
From (4.6) and (4.7) it follows that for every \( x \in \mathbb{R}_+ \)
\[
\begin{align*}
\{Q_{e,i}(x)\} & \leq -2 \left( \frac{A'(x)}{2A(x)} + \varepsilon \varepsilon \right) \left( |Q_{e,i}(x)| - |Q_{e,i}(-x)| \right) \frac{\Psi_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)}, \\
\{Q_{e,i}(-x)\} & \leq -2 \left( \frac{A'(x)}{2A(x)} - \varepsilon \varepsilon \right) \left( |Q_{e,i}(-x)| - |Q_{e,i}(x)| \right) \frac{\Psi_{A,e}(ib, x)}{\Psi_{A,e}(ib, x)}.
\end{align*}
\]
Thus we can conclude that
\[
\{Q_{e,i}(x)\} \leq 0 \quad \text{if} \quad |Q_{e,i}(x)| \geq |Q_{e,i}(-x)|,
\]
and
\[
\{Q_{e,i}(-x)\} \leq 0 \quad \text{if} \quad |Q_{e,i}(-x)| \geq |Q_{e,i}(x)|.
\]
As a real analytic function of \( x \), \( |Q_{e,i}(x)|^2 \) and \( |Q_{e,i}(-x)|^2 \) coincide either everywhere or on a discrete subset of \( \mathbb{R} \) with no accumulation point. In the first case, \( |Q_{e,i}(x)|^2 = |Q_{e,i}(-x)|^2 \) is a decreasing function of \( x \in \mathbb{R}_+ \). In the second case, for \( x \in \mathbb{R}_+ \), let
\[
M(x) := \max \{ |Q_{e,i}(x)|^2, |Q_{e,i}(-x)|^2 \}.
\]
If \( |Q_{e,i}(x)| > |Q_{e,i}(-x)| \), then \( M(x) = |Q_{e,i}(x)|^2 \) and \( M'(x) = \{ |Q_{e,i}(x)|^2 \}^\prime < 0 \). If \( |Q_{e,i}(x)| < |Q_{e,i}(-x)| \), then \( M(x) = |Q_{e,i}(-x)|^2 \) and \( M'(x) = \{ |Q_{e,i}(-x)|^2 \}^\prime < 0 \). If \( |Q_{e,i}(x)| = |Q_{e,i}(-x)| \) for some \( x \in \mathbb{R}_+ \), then \( M \) has left and right derivatives at \( x \),
which are non-positive. Thus $M$ is decreasing on $\mathbb{R}_+$. In conclusion, for every $x \in \mathbb{R}_+$, $|Q_{e,\lambda}(x)|^2 \leq M(0) = 1$ and $|Q_{e,\lambda}(-x)|^2 \leq M(0) = 1$. That is for every $x \in \mathbb{R}$, we have $|Q_{e,\lambda}(x)| \leq |Q_{e,\lambda}(0)| = 1$. This finishes the proof of the second statement.

3) We proceed analogously to the function $Q_{e,\lambda}$ above by considering the function

$$R_{e,b}(x) := \frac{\Psi_{A,e}(ib, x)e^{-ib|x|}}{\Psi_{A,e}(0, x)}.$$ 

4) The fact that $\Psi_{A,0}(0, x) = 1$ follows immediately from (3.6). Assume that $\varepsilon \neq 0$. In this case

$$\Psi_{A,e}(0, x) = \varphi_{\mu_e}(x) - \frac{1}{\varepsilon_o} \varphi'_{\mu_e}(x),$$

where $\mu_e$ satisfies $(\mu_e)^2 = (e^2 - 1)|\eta|^2$. Since $|\eta| \leq 1$, it follows from Lemma 2.2 (4), 5), 6) that there exists a positive constant $c_\varepsilon$ such that

$$\Psi_{A,e}(0, x) \leq c_\varepsilon (|x| + 1)e^{-\varepsilon(1 - \sqrt{1-|\eta|^2})|x|}.$$ 

Henceforth, we will assume that $-1 \leq \varepsilon \leq 1$. Beside the growth estimates above, we will include estimates for the partial derivatives of $\Psi_{A,e}$. We remind the reader that

$$\varphi_{\mu_e}(x) = 1$$

with $\varphi_{\mu_e}(x) = 1$ and

$$\varphi_{i\sqrt{1-\varepsilon^2}}(x) \leq c(|x| + 1)e^{-\varepsilon(1 - \sqrt{1-|\eta|^2})|x|},$$

(see Lemma 2.2).

**Theorem 4.3.** 1) Assume that $\lambda \in \mathbb{C}$ and $|x| \geq x_0$ with $x_0 > 0$. Given $N \in \mathbb{N}$, there is a positive constant $c$ such that

$$\left| \frac{\partial^N}{\partial \lambda^N} \Psi_{A,e}(\lambda, x) \right| \leq c(|x| + 1)^N e^{\varepsilon|m,| |x|} \varphi_{i\sqrt{1-\varepsilon^2}}(x). \tag{4.8}$$

2) Assume that $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. Given $M \in \mathbb{N}$, there is a positive constant $c$ such that

$$\left| \frac{\partial^M}{\partial \lambda^M} \Psi_{A,e}(\lambda, x) \right| \leq c|x|^M e^{\varepsilon|m,| |x|} \varphi_{i\sqrt{1-\varepsilon^2}}(x). \tag{4.9}$$

**Proof.** 1) If $N = 0$ this is nothing but Theorem 4.2 (2), 3) and 4). So assume $N \geq 1$. On the one hand, $\Psi_{A,e}(\lambda, x)$ satisfies the following equation

$$\Psi'_{A,e}(\lambda, x) = -\frac{A'(x)}{2A(x)} \left( \Psi_{A,e}(\lambda, x) - \Psi_{A,e}(\lambda, -x) \right) + \varepsilon_0 \Psi_{A,e}(\lambda, -x) + i\lambda \Psi_{A,e}(\lambda, x).$$

This allows us to express the derivatives of $\Psi_{A,e}(\lambda, \cdot)$ in terms of lower order derivatives. On the other hand, since $A'(2A)$ satisfies the hypothesis (H4), it follows that there exists a positive constant $C$ such that

$$\left| \frac{A'(x)}{2A(x)} \right|^{(N)} \leq C, \quad \forall |x| \geq x_0 \text{ with } x_0 > 0.$$
Now the estimate (4.8) can be proved by induction on $N$.

2) Recall that the mapping $\lambda \mapsto \Psi_{A,\epsilon}(\lambda, x)$ is entire, for every $x \in \mathbb{R}$, and that
\[
|\Psi_{A,\epsilon}(\lambda, x)| \leq ce^{\lambda |x|} \varphi_{1/\sqrt{1-\epsilon^2}}(x) \tag{4.10}
\]
for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. If $M = 0$ this is just (4.10). So assume $M > 0$. If $x = 0$, the statement follows from Liouville’s theorem. If $x \neq 0$, apply Cauchy’s integral formula for $\Psi_{A,\epsilon}(\lambda, x)$ over a circle with radius proportional to $\frac{1}{M}$, centered at $\lambda$ in the complex plane. 

5. A Laplace type representation of the eigenfunctions

In this section we will show that $\Psi_{A,\epsilon}(\lambda, \cdot)$ can be expressed as the Laplace transform of a compactly supported function. In the literature this is the so-called Mehler’s type formula.

Denote by $C_c^\infty(\mathbb{R})$ the space of even functions in $C_c^\infty(\mathbb{R})$. For $f \in C_c^\infty(\mathbb{R})$ we set
\[
\mathcal{E}_\epsilon f(x) := f(x) - \frac{\varrho_\epsilon |x|}{2} \int_{|y|<|x|} f(y) \frac{J_1(\varrho_\epsilon \sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dy, \tag{5.1}
\]
where $J_1$ is the Bessel function of the first kind, and
\[
\varrho_\epsilon := \sqrt{1-\epsilon^2} \varrho. \tag{5.2}
\]
If $\epsilon = \pm 1$, then $\varrho_{\pm 1} = 0$, and therefore $\mathcal{E}_{\pm 1} = \text{id}$. The following statement is nothing but a reformulation of Proposition 2.1 in [49]. See also Theorem 5.1 in [37].

**Proposition 5.1.** The transform integral $\mathcal{E}_\epsilon$ is an automorphism of $C_c^\infty(\mathbb{R})$ satisfying
\[
\begin{cases}
\frac{d^2}{dx^2} \circ \mathcal{E}_\epsilon = \mathcal{E}_\epsilon \circ \left( \frac{d^2}{dx^2} - \mathcal{E}_\epsilon^2 \right), \\
\mathcal{E}_\epsilon f(0) = f(0). \tag{5.3}
\end{cases}
\]

The transform inverse $\mathcal{E}_\epsilon^{-1}$ is given by
\[
\mathcal{E}_\epsilon^{-1} f(x) = f(x) + \frac{\varrho_\epsilon |x|}{2} \int_{|y|>|x|} f(y) \frac{I_1(\varrho_\epsilon \sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dy, \tag{5.4}
\]
where $I_1$ is the modified Bessel function of the first kind.

Let $\mathcal{D}_c(\mathbb{R})$ be the space of even functions in $\mathcal{D}(\mathbb{R})$. For $g \in \mathcal{D}_c(\mathbb{R})$ put
\[
^t \mathcal{E}_\epsilon g(y) = g(y) - \frac{\varrho_\epsilon}{2} \int_{|x|>|y|} |x| g(x) \frac{J_1(\varrho_\epsilon \sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dx, \tag{5.5}
\]
where $\varrho_\epsilon$ is as in (5.2). We may rewrite $^t \mathcal{E}_\epsilon g$ as
\[
^t \mathcal{E}_\epsilon g(y) = - \int_{|x|<|y|} g'(x) J_0(\varrho_\epsilon \sqrt{x^2-y^2}) dx. \tag{5.6}
\]
Below we will show that $D(e(R))$ is stable by $tE$. Thus, one may check that for all $f \in C^\infty_e(R)$ and all $g \in D(e(R))$,
\[
\int_R E_\varepsilon f(x) g(x) dx = \int_R f(y) tE_\varepsilon g(y) dy.
\]

**Theorem 5.2.** The integral transform $tE_\varepsilon$ is an automorphism of $D(e(R))$ satisfying
\[
tE_\varepsilon^2 = \left(\frac{d^2}{dx^2} - \varrho_\varepsilon^2\right) tE_\varepsilon.
\]  
(5.7)

The transform inverse $tE_\varepsilon^{-1}$ is given by
\[
tE_\varepsilon^{-1} g(y) = g(y) + \frac{\varrho_\varepsilon}{2} \int_{|x|>|y|} |x| g(x) \frac{J_0(\varrho_\varepsilon \sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dx,
\]  
(5.8)

which we may rewrite it as
\[
tE_\varepsilon^{-1} g(y) = - \int_{|y|}^{\infty} g'(x) J_0(\varrho_\varepsilon \sqrt{x^2-y^2}) dx.
\]  
(5.9)

**Proof.** It is clear that $tE_\varepsilon g$ is an even function whenever $g$ is even. A direct calculation gives the intertwining property (5.7), which we may rewrite it as
\[
D^2 \circ tE_\varepsilon = tE_\varepsilon \circ (D^2 + \varrho_\varepsilon^2),
\]  
where $D := \frac{d}{dx}$. Thus, for all $N \in \mathbb{N}$ and for all $y \in \mathbb{R}^+$, we have
\[
D^{2N} \circ tE_\varepsilon g(y) = tE_\varepsilon \circ (D^2 + \varrho_\varepsilon^2)^N g(y)
\]
\[
= - \int_{|y|}^{\infty} J_0(\varrho_\varepsilon \sqrt{x^2-y^2}) D(D^2 + \varrho_\varepsilon^2)^N g(x) dx.
\]

Using the well know fact that $|J_0(r)| \leq 1$ for all $r \in \mathbb{R}^+$, it follows that if $\text{supp}(g) \subset [-a,a]$, then there exists a constant $c$ such that
\[
\sup_{y \in [-a,a]} |D^{2N} \circ tE_\varepsilon g(y)| \leq c \sup_{x \in [-a,a]} |D^M g(x)| < \infty,
\]
for some positive integer $M$. Thus, the space $D_e(R)$ is stable by $tE_\varepsilon$.

We now prove that the transform inverse of $tE_\varepsilon$ is given by (5.8). Recall that we may rewrite $tE_\varepsilon$ as
\[
tE_\varepsilon g(y) = - \int_{|y|}^{\infty} g'(x) J_0(\varrho_\varepsilon \sqrt{x^2-y^2}) dx.
\]

We may also rewrite the “potential” transform inverse as
\[
tE_\varepsilon^{-1} g(y) = - \int_{|y|}^{\infty} g'(x) I_0(\varrho_\varepsilon \sqrt{x^2-y^2}) dx.
\]
We will assume that \( y > 0 \). Then

\[
{\mathcal{E}}_e({\mathcal{E}}_e^{-1} g)(y) = -\int_{x > y} \{ {\mathcal{E}}_e^{-1} g(x) \}' J_0(Q_e \sqrt{x^2 - y^2}) \, dx
\]

\[
= \int_{x > y} \{ \int_{s > x} g'(s) I_0(Q_e \sqrt{s^2 - x^2}) ds \} J_0(Q_e \sqrt{x^2 - y^2}) \, dx
\]

\[
= -\int_{x > y} g'(x) J_0(Q_e \sqrt{x^2 - y^2}) \, dx
\]

\[
+ \int_{x > y} \{ \int_{s > x} g'(s) \partial_s I_0(Q_e \sqrt{s^2 - x^2}) ds \} J_0(Q_e \sqrt{x^2 - y^2}) \, dx.
\]

Integration by parts implies

\[
\int_{s > x} g'(s) \partial_s I_0(Q_e \sqrt{s^2 - x^2}) \, ds = \frac{Q_e^2}{2} x g(x) - \int_{s > x} g(s) \partial_s \partial_s I_0(Q_e \sqrt{s^2 - x^2}) \, ds.
\]

Above we have used the fact that \( I_0'(z) = I_1(z) \) and that the function \((\frac{z}{2})^{-\nu} I_\nu(z)\) is normalized at 0 by 1. Thus,

\[
{\mathcal{E}}_e({\mathcal{E}}_e^{-1} g)(y) = -\int_{x > y} g'(x) J_0(Q_e \sqrt{x^2 - y^2}) \, dx + \frac{Q_e^2}{2} \int_{x > y} x g(x) J_0(Q_e \sqrt{x^2 - y^2}) \, dx
\]

\[
- \int_{x > y} g(s) \{ \int_{y}^{s} J_0(Q_e \sqrt{s^2 - x^2}) \partial_s \partial_s I_0(Q_e \sqrt{s^2 - x^2}) \, dx \} \, ds.
\]

Next, we will compute the integral within brackets on the right hand side of the identity above. On the one hand, since \( I'_0(z) = I_1(z) \), we have

\[
\partial_s \partial_s I_0(Q_e \sqrt{s^2 - x^2}) = -Q_e x \partial_s \left( (s^2 - x^2)^{-1/2} I_1(Q_e \sqrt{s^2 - x^2}) \right)
\]

\[
= Q_e \frac{x s}{(s^2 - x^2)^{1/2}} I_1(Q_e \sqrt{s^2 - x^2}) - Q_e^2 \frac{x s}{(s^2 - x^2)^{3/2}} I'_1(Q_e \sqrt{s^2 - x^2})
\]

\[
= -Q_e^2 \frac{x s}{(s^2 - x^2)^{3/2}} I_2(Q_e \sqrt{s^2 - x^2}).
\]

Above we have used the well known differentiation identity \( I'_\nu(z) = I_{\nu+1}(z) + \frac{z}{\nu} I_\nu(z) \). On the other hand, using the following integral formula (see [30] formula (1), page 725)

\[
\int_0^\infty (a^2 - x^2)^{-\nu/2 - 1} I_\nu(a \sqrt{a^2 - x^2}) \, dx = \frac{\left(\frac{\nu}{2}\right)^\nu \Gamma\left(\frac{\nu+1}{2}\right)}{2 \Gamma\left(\frac{\nu+2}{2}\right)} J_\nu(a), \quad \text{Re} \, \nu > \text{Re} \, \mu > -1
\]
we have
\[
\begin{align*}
\int_y^x J_0(q_e \sqrt{x^2 - y^2}) \partial_x \partial_y J_0(q_e \sqrt{s^2 - x^2}) dx &= - \frac{q_e^2}{2} \int_y^x \frac{x}{s^2 - x^2} J_0(q_e \sqrt{x^2 - y^2}) J_2(q_e \sqrt{s^2 - x^2}) dx \\
&= - \frac{q_e^2}{2} s J_2(q_e \sqrt{s^2 - y^2}) \\
&= - q_e \frac{s}{\sqrt{s^2 - y^2}} J_1(q_e \sqrt{s^2 - y^2}) + \frac{q_e^2}{2} s J_0(q_e \sqrt{s^2 - y^2}) \\
&= \partial_x \left( J_0(q_e \sqrt{s^2 - y^2}) \right) + \frac{q_e^2}{2} s J_0(q_e \sqrt{s^2 - y^2}).
\end{align*}
\]
Above we have used the recurrence relation $J_{r+1}(z) + J_{r-1}(z) = \frac{2z}{z} J_r(z)$ and the fact that $J'_0(z) = -J_1(z)$. Consequently,
\[
\begin{align*}
1 & \mathcal{E}_e^{-1}(\mathcal{E}_e g)(y) = \int_{x>y} g'(x) J_0(q_e \sqrt{x^2 - y^2}) dx + \frac{q_e^2}{2} \int_{x>y} x g(x) J_0(q_e \sqrt{x^2 - y^2}) dx \\
&\quad - \int_{x>y} g(x) \partial_x \left( J_0(q_e \sqrt{x^2 - y^2}) \right) dx - \frac{q_e^2}{2} \int_{x>y} x g(x) J_0(q_e \sqrt{x^2 - y^2}) dx \\
&= g(y).
\end{align*}
\]
Similarly one proves that $1 \mathcal{E}_e^{-1}(\mathcal{E}_e g) = g$. \hfill \Box

Recall from Lemma 2.1 that for every $\mu \in \mathbb{C}$, the eigenfunction $\varphi_\mu$ has the following Laplace type representation
\[
\varphi_\mu(x) = \int_0^{x|} K(|x|, y) \cos(\mu y) dy, \quad x \in \mathbb{R}^+,
\]  
(5.10)
where $K(|x|, \cdot)$ is a non-negative even continuous function supported in $[-|x|, |x|]$. The following alternative Laplace type representation of $\varphi_\mu$ is needed for later use.

For $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$ put
\[
K_e(|x|, y) := 1 \mathcal{E}_e^{-1} K(|x|, \cdot)(y).
\]  
(5.11)

Observe that $K_e(x, \cdot)$ is even, continuous and supported in $[-|x|, |x|]$. We note that if $\varepsilon = \pm 1$, then the transformation $\mathcal{E}_{\varepsilon 1} = \text{id}$, and therefore $K_{\varepsilon 1}(x, y) = K(|x|, y)$.

**Lemma 5.3.** Let $\lambda \in \mathbb{C}$. The integral representation (5.10) can be rewritten as
\[
\varphi_\mu(x) = \int_0^{x|} K_e(x, y) \cos(\lambda y) dy,
\]  
where the relationship between $\mu_e$ and $\lambda$ is $\mu_e^2 = \lambda^2 + (\varepsilon^2 - 1)q_e^2$. \hfill \begin{center} \textbf{17} \end{center}
Proof. The cases $\varepsilon = \pm 1$ are trivial since $\mathcal{E}_{\pm 1} = \text{id}$. So assume $\varepsilon \neq \pm 1$. Observe that we may rewrite $\mathcal{E}_{\varepsilon}$ as

$$\mathcal{E}_{\varepsilon}(f)(y) := \left( y \int_{0}^{1} f(\tau y) J_{0}(\eta_{\varepsilon} y \sqrt{1 - \tau^{2}}) d\tau \right)'$$

for $y > 0$. Using the following integral formula (see [30, formula (7), page 722])

$$\int_{0}^{a} \cos(b t) J_{0}(b \sqrt{a^{2} - t^{2}}) dt = \frac{\sin(a \sqrt{b^{2} + c^{2}})}{\sqrt{b^{2} + c^{2}}}, \quad b > 0$$

we obtain

$$\mathcal{E}_{\varepsilon}(\cos(\mu \cdot))(y) = \left( y \int_{0}^{1} \cos(\mu \tau y) J_{0}(\eta_{\varepsilon} y \sqrt{1 - \tau^{2}}) d\tau \right)'$$

$$= \left( \frac{\sin(y \sqrt{\mu^{2} + \eta_{\varepsilon}^{2}})}{\sqrt{\mu^{2} + \eta_{\varepsilon}^{2}}} \right)'$$

$$= \cos \left( \left( 1 - \varepsilon^{2} \eta_{\varepsilon}^{2} + \mu^{2} \right)^{1/2} y \right).$$

Thus

$$\mathcal{E}_{\varepsilon}^{-1}(\cos(\lambda \cdot))(y) = \cos(\mu \cdot y).$$

Hence the integral representation (5.10) becomes

$$\varphi_{\mu_{\varepsilon}}(x) = \int_{0}^{1} K(|x|, y) \mathcal{E}_{\varepsilon}^{-1}(\cos(\lambda \cdot))(y) dy$$

$$= \int_{0}^{1} \mathcal{E}_{\varepsilon}^{-1}(\cos(\lambda \cdot))(y) \cos(\lambda y) dy.$$

We now establish a Laplace type representation of the eigenfunction $\Psi_{A,\varepsilon}(\lambda, \cdot)$. Henceforth we will use the following notation

$$G_{\varepsilon}(x, y) := \int_{|t|}^{x} K_{\varepsilon}(t, y) A(t) dt. \quad (5.12)$$

The function $G_{\varepsilon}(x, \cdot)$ is even, continuous on its support $[-|x|, |x|]$ and of class $C^{1}$ on $]-|x|, |x|[$ (see e.g. [38, Lemma 2.8]).

**Theorem 5.4.** For all $\lambda \in \mathbb{C}$ the function $\Psi_{A,\varepsilon}(\lambda, \cdot) : \mathbb{R}^{*} \to \mathbb{C}$ is the Laplace transform of a compactly supported function. More precisely,

$$\Psi_{A,\varepsilon}(\lambda, x) = \int_{|t|<|x|} \mathcal{K}_{\varepsilon}(x, y) e^{\lambda y} dy, \quad \forall x \in \mathbb{R}^{*},$$

where

$$\mathcal{K}_{\varepsilon}(x, y) := \frac{K_{\varepsilon}(x, y)}{2} + \varepsilon \frac{sg(x)}{2A(x)} G_{\varepsilon}(x, y) - \frac{sg(x)}{2A(x)} \partial_{y} G_{\varepsilon}(x, y). \quad (5.13)$$
Proof. By invoking the identity \((3.5)\) and Lemma \([5,3]\) in the second equality below we have
\[
\Psi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_{\varepsilon}}(x) + (i\lambda + \varepsilon \phi) \frac{\text{sg}(x)}{A(x)} \int_0^{\lvert x \rvert} \varphi_{\mu_{\varepsilon}}(t)A(t)dt
\]
\[
= \int_0^{\lvert x \rvert} K_\varepsilon(x, y) \cos(\lambda y)dy + \varepsilon \phi \frac{\text{sg}(x)}{A(x)} \int_0^{\lvert x \rvert} \{ \int_0^{\lvert y \rvert} K_\varepsilon(t, y) \cos(\lambda y)dy \} A(t)dt
\]
\[
+ i \frac{\text{sg}(x)}{A(x)} \int_0^{\lvert x \rvert} \{ \int_0^{\lvert y \rvert} K_\varepsilon(t, y) \lambda \cos(\lambda y)dy \} A(t)dt
\]
\[
= \int_0^{\lvert x \rvert} K_\varepsilon(x, y) \cos(\lambda y)dy + \varepsilon \phi \frac{\text{sg}(x)}{A(x)} \int_0^{\lvert x \rvert} \cos(\lambda y) \{ \int_0^{\lvert y \rvert} K_\varepsilon(t, y)A(t)dt \} dy
\]
\[
+ i \frac{\text{sg}(x)}{A(x)} \int_0^{\lvert x \rvert} (\sin(\lambda y)') \{ \int_0^{\lvert y \rvert} K_\varepsilon(t, y)A(t)dt \} dy
\]
\[
= \int_{-\infty}^{\lvert x \rvert} K_\varepsilon(x, y) \cos(\lambda y)dy + \varepsilon \phi \frac{\text{sg}(x)}{2A(x)} G_\varepsilon(x, y) - \frac{\text{sg}(x)}{2A(x)} dy G_\varepsilon(x, y) e^{\lambda y} dy.
\]

\(\square\)

6. The existence of an intertwining operator

This section is concerned with the existence of an intertwining operator between \(\Lambda_{A,\varepsilon}\) and the ordinary derivative \(d/dx\).

Recall from \((2.8)\) the definition of the Abel transform of a function \(f \in \mathcal{D}_c(\mathbb{R})\) (the space of even and smooth functions with compact support on \(\mathbb{R}\)),
\[
\mathcal{A} f(y) = \frac{1}{2} \int_{|x|>|y|} K(|x|, y)f(x)A(x)dx, \quad y \in \mathbb{R}
\]
where \(K(|x|, y)\) is as in \((2.4)\). It is natural to define for smooth even functions the dual transform \(1^* \mathcal{A}\) of \(\mathcal{A}\) in the following sense
\[
\int_{\mathbb{R}} f(y)1^* \mathcal{A} g(y)dy = \int_{\mathbb{R}} 1^* \mathcal{A} f(x)g(x)A(x)dx.
\]

In \([51]\) the author showed that
\[
1^* \mathcal{A} f(x) = \frac{1}{2} \int_{|x|-|y|} K(|x|, u)f(u)du.
\]

Further, by \([51]\) Theorem 5.1, the transform \(1^* \mathcal{A}\) is an automorphism of \(C^\infty_c(\mathbb{R})\) (the space of even and smooth functions on \(\mathbb{R}\)) satisfying
\[
(\Delta + \varepsilon^2) \circ 1^* \mathcal{A} = 1^* \mathcal{A} \circ d^2/dx^2, \quad (6.1)
\]
where \(\Delta\) is the operator \((2.2)\).

For \(-1 \leq \varepsilon \leq 1\) we define the integral transform \(\mathcal{A}_\varepsilon\) on \(\mathcal{D}_c(\mathbb{R})\) by
\[
\mathcal{A}_\varepsilon g(y) = \frac{1}{2} \int_{|x|-|y|} K_\varepsilon(x, y)g(x)A(x)dx,
\]
where the kernel $K_\varepsilon$ is as in (5.11). We note that for $\varepsilon = \pm 1$ the transform $A_\varepsilon$ reduces to the Abel transform $A$. Thus we may think of $A_\varepsilon$ as a deformation of the transform $A$. Let $\mathcal{A}_\varepsilon$ be the linear mapping of $C^\infty_c(\mathbb{R})$ so that
\[ \int_{\mathbb{R}} f(y) \mathcal{A}_\varepsilon g(y) dy = \int_{\mathbb{R}} \mathcal{A}_\varepsilon f(x) g(x) A(x) dx, \]
for $f \in C^\infty_c(\mathbb{R})$ and $g \in D_c(\mathbb{R})$. Then
\[ \mathcal{A}_\varepsilon f(x) = \frac{1}{2} \int_{|y|<|x|} K_\varepsilon(x, y) f(y) dy. \]
Notice that for $f \in C^\infty_c(\mathbb{R})$ and $g \in D_c(\mathbb{R})$, the functions $\mathcal{A}_\varepsilon f$ and $\mathcal{A}_\varepsilon g$ belong respectively to $C^\infty_c(\mathbb{R})$ and $D_c(\mathbb{R})$. Moreover,
\[ \mathcal{A}_\varepsilon = t_{\mathcal{E}_{\varepsilon^{-1}}} \circ A, \quad (6.2) \]
and
\[ \mathcal{A}_\varepsilon = \mathcal{A} \circ \mathcal{E}_{\varepsilon^{-1}}. \quad (6.3) \]

**Corollary 6.1.** Let $D$ be the ordinary derivative and let $\Delta$ be the operator (2.2). Then for all $\varepsilon \in \mathbb{R}$ we have:

1. $\mathcal{A}_\varepsilon \circ (\Delta + \varepsilon^2) = (D^2 + \varepsilon^2) \circ \mathcal{A}_\varepsilon$, where $\varepsilon^2 = (1 - \varepsilon^2) \varepsilon^2$.
2. $(\Delta + \varepsilon^2) \circ \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon \circ (D^2 + \varepsilon^2)$.

**Proof.** The first statement is an immediate consequence of (2.9) and (5.7). The second transmutation property follows from (5.3) and (6.1). \qed

Recall that the space of smooth functions $C^\infty(\mathbb{R})$ equipped with the topology of compact convergence for all derivatives is a Fréchet space. For $f \in C^\infty(\mathbb{R})$ we define $V_{\mathcal{A}_\varepsilon} f$ by
\[ V_{\mathcal{A}_\varepsilon} f(x) = \begin{cases} \int_{|y|<|x|} \mathbb{K}_\varepsilon(x, y) f(y) dy, & x \neq 0 \\ f(0), & x = 0 \end{cases} \quad (6.4) \]
where the kernel $\mathbb{K}_\varepsilon(x, y)$ is as in (5.13). Observe that
\[ \Psi_{\mathcal{A}_\varepsilon}(\lambda, x) = V_{\mathcal{A}_\varepsilon}(e^{i\lambda \cdot}) (x). \quad (6.5) \]

**Lemma 6.2.** The operator $V_{\mathcal{A}_\varepsilon}$ can be expressed as
\[ V_{\mathcal{A}_\varepsilon} f(x) = \left( \text{id} + \varepsilon \mathcal{M} \right) \mathcal{A}_\varepsilon f(x) + \left( \varepsilon^2 \mathcal{M} \right) \mathcal{A}_\varepsilon (I f_0) (x), \quad (6.6) \]
where
\[ \mathcal{M} h(x) := \frac{\text{sg}(x)}{A(x)} \int_0^{|x|} h(t) A(t) dt, \]
and
\[ Ih(x) := \int_0^x h(t) dt. \quad (6.7) \]
Proof. As usual, we write \( f \) as the superposition \( f = f_e + f_o \) of an even function \( f_e \) and an odd function \( f_o \). On the one hand, we have
\[
V_{A,e}f_e(x) = \int_{-|x|}^{x} \frac{K_e(x,y)}{2} f_e(y)dy + \varepsilon \frac{sg(x)}{2A(x)} \int_{-|x|}^{|x|} G_e(x,y)f_e(y)dy
\]
\[
= \int_{0}^{x} K_e(x,y)f_e(y)dy + \varepsilon \frac{sg(x)}{A(x)} \int_{0}^{|x|} G_e(x,y)f_e(y)dy
\]
\[
= \int_{0}^{x} \mathcal{A}_e f_e(x) + \varepsilon \mathcal{Q} \circ \mathcal{A}_e f_e(x).
\]
On the other hand,
\[
V_{A,o}f_o(x) = -\frac{sg(x)}{A(x)} \int_{0}^{x} f_o(y)\partial_y G_e(x,y)dy.
\]
We claim that
\[
-\frac{sg(x)}{A(x)} \int_{0}^{x} f_o(y)\partial_y G_e(x,y)dy = \left( \varepsilon^2 \mathcal{Q} \mathcal{M} + \frac{d}{dx}\right) \mathcal{A}_e f_o(x),
\]
where \( f_o \) is as in \( (6.7) \). Indeed, by invoking formula \( (3.10) \) in the first equality below and the transmutation property in Corollary \( 6.1.3 \) in the second equality below we have
\[
\frac{d}{dx} \mathcal{A}_e f_o(x) = \frac{sg(x)}{A(x)} \int_{0}^{x} \Delta \mathcal{A}_e f_o(s)A(s)ds
\]
\[
= \frac{sg(x)}{A(x)} \int_{0}^{x} \mathcal{A}_e \left( \frac{d^2}{dx^2} - \varepsilon^2 \mathcal{Q}^2 \right) f_o(s)A(s)ds
\]
\[
= \frac{sg(x)}{A(x)} \int_{0}^{x} \mathcal{A}_e \left( \frac{d^2}{dx^2} - \varepsilon^2 \mathcal{Q}^2 \frac{sg(x)}{A(x)} \right) \mathcal{A}_e f_o(s)A(s)ds
\]
\[
= \frac{sg(x)}{A(x)} \int_{0}^{x} \left\{ \int_{0}^{u} K_e(s,u)f'_o(u)du \right\} A(s)ds - \varepsilon^2 \mathcal{Q}^2 \mathcal{M} \circ \mathcal{A}_e f_o(x)
\]
\[
= \frac{sg(x)}{A(x)} \int_{0}^{x} f'_o(u) \left\{ \int_{u}^{x} K_e(s,u)A(s)ds \right\} du - \varepsilon^2 \mathcal{Q}^2 \mathcal{M} \circ \mathcal{A}_e f_o(x)
\]
\[
= -\frac{sg(x)}{A(x)} \int_{0}^{x} f_o(u)\partial_y G_e(x,u)du - \varepsilon^2 \mathcal{Q}^2 \mathcal{M} \circ \mathcal{A}_e f_o(x).
\]
This concludes the proof of claim \( (6.8) \), and therefore the proof of the Lemma \( 6.2 \). \( \square \)

**Theorem 6.3.** The operator \( V_{A,e} \) is the unique automorphism of \( C^\infty(\mathbb{R}) \) such that
\[
\Lambda_{A,e} \circ V_{A,e} = V_{A,e} \circ \frac{d}{dx},
\]
where \( \Lambda_{A,e} \) is the differential-reflection operator \( (3.1) \).

**Proof.** For the proof of this theorem it is more convenient to rewrite \( V_{A,e}f_o \) in \( (6.6) \) as
\[
V_{A,e}f_o(x) = \mathcal{M} \circ \mathcal{A}_e f'_o(x).
\]
Indeed,
\[ V_{A,e}f_o(x) = \frac{d}{dx} A_e(If_o)(x) + \epsilon^2 \mathcal{C} \mathcal{A}_e(If_o)(x) \]
\[ = \mathcal{M} \mathcal{A}_e(If_o)(x) + \epsilon^2 \mathcal{C} \mathcal{A}_e(If_o)(x) \]
\[ = \mathcal{M} (\mathcal{A} + \epsilon^2 \mathcal{C}) A_e(If_o)(x) \]
\[ = \mathcal{M} A_e((If_o)')(x) \]
\[ = \mathcal{M} A_e(f'_o)(x). \]

Let \( C^\infty_{c}(\mathbb{R}) \) and \( C^\infty_{o}(\mathbb{R}) \) be the subspaces of even and odd functions in \( C^\infty(\mathbb{R}) \), respectively. Firstly, the operator \( d/dx \) is one to one from \( C^\infty_{o}(\mathbb{R}) \) onto \( C^\infty_{c}(\mathbb{R}) \), and \( d/dx \circ I = I \circ d/dx = id \). Secondly, The transform \( \mathcal{M} \) is an isomorphism from \( C^\infty_{c}(\mathbb{R}) \) to \( C^\infty_{o}(\mathbb{R}) \) and its inverse is given by
\[ \mathcal{M}^{-1} = \frac{d}{dx} + \frac{A'(x)}{A(x)} \text{id}. \] (6.11)

Thus, from (6.6) and (6.10) it follows that \( V_{A,e} \) is an automorphism of \( C^\infty(\mathbb{R}) \). We now prove the transmutation property (6.9).

By (6.10) we have
\[ \Lambda_{A,e}(V_{A,e}f_o) = \Lambda_{A,e}(\mathcal{M} A_e(f'_o)) \]
\[ = (\text{id} + \epsilon \mathcal{Q} \mathcal{M}) A_e(f'_o). \]

Above we have used the fact that
\[ \Lambda_{A,e} \circ \mathcal{M} = \text{id} + \epsilon \mathcal{Q} \mathcal{M}. \]

Moreover, one can check that
\[ \Lambda_{A,e}(V_{A,e}f_o) = \Lambda_{A,e}(A_e f_e + \epsilon \mathcal{Q} A_e f_e) \]
\[ = \frac{d}{dx} A_e f_e - \epsilon \mathcal{Q} A_e f_e + \epsilon \mathcal{Q} \left( \frac{d}{dx} + \frac{A'(x)}{A(x)} \right) A_e f_e + \epsilon^2 \mathcal{Q} \mathcal{C} A_e f_e \]
\[ = \left( \frac{d}{dx} + \epsilon^2 \mathcal{Q} \right) A_e f_e. \]

Above we have used (6.11). In summary,
\[ \Lambda_{A,e}(V_{A,e}f_o) = \left( \frac{d}{dx} + \epsilon^2 \mathcal{Q} \right) A_e f_e + (\text{id} + \epsilon \mathcal{Q} \mathcal{M}) A_e(f'_o). \] (6.12)

Now, by invoking the expression (6.6) of the operator \( V_{A,e} \) we get
\[ V_{A,e}(f'_o) = (\epsilon^2 \mathcal{Q} + \frac{d}{dx})^1 A_e f_e, \]
and
\[ V_{A,e}(f'_o) = (\text{id} + \epsilon \mathcal{Q}) A_e(f'_o). \]

That is
\[ V_{A,e}(f') = (\epsilon^2 \mathcal{Q} + \frac{d}{dx})^1 A_e f_e + (\text{id} + \epsilon \mathcal{Q}) A_e(f'_o). \]

This compares well with (6.12).
Proof. and every odd function operator intertwining property
Theorem 6.5. That is 
Lemma 6.4. where 
Starting from the expression (6.6) of 
first equality below, we obtain 
The uniqueness of 
On the space 
This finishes the proof of Lemma 6.4. □

The reader will have no trouble verifying that for every even function 
\( = \int R g(x)A(x)dx \), \( g(x)A(x)A(x)dx \) and that 
\( V_{A,e}f(0) = f(0) \).

On the space \( D(\mathbb{R}) \) of smooth functions with compact support, we consider the dual operator \( ^1V_{A,e} \) of \( V_{A,e} \) in the sense that

\[
\int_{\mathbb{R}} V_{A,e}f(x)g(x)A(x)dx = \int_{\mathbb{R}} f(y)^1V_{A,e}g(y)dy.
\] (6.13)

That is

\[
^1V_{A,e}g(y) = \int_{|y|>|t|} R_{y}(x,y)g(x)A(x)dx.
\] (6.14)

**Lemma 6.4.** The dual operator \( ^1V_{A,e} \) can be expressed as

\[
^1V_{A,e}g(y) = \mathcal{A}_{g}g_{e}(y) - \left( \varepsilon_{g} - \frac{d}{dx} \right) \mathcal{A}_{e}(Jg_{o})(y),
\] (6.15)

where

\[
Jh(x) := \int_{-\infty}^{x} h(t)dt.
\]

**Proof.** The reader will have no trouble verifying that for every even function \( f \in C^{\infty}(\mathbb{R}) \) and every odd function \( g \in D(\mathbb{R}) \)

\[
\int_{\mathbb{R}} \mathcal{M} f(x)g(x)A(x)dx = - \int_{\mathbb{R}} f(x)Jg(x)A(x)dx.
\]

Starting from the expression (6.6) of \( V_{A,e} \) in Lemma 6.2, and by invoking (6.10) in the first equality below, we obtain

\[
\int_{\mathbb{R}} V_{A,e}f(x)g(x)A(x)dx
\]

\[
= \int_{\mathbb{R}} \{ \mathcal{A}_{e}f_{e}(x)g_{e}(x) + \varepsilon_{g} \mathcal{M} \mathcal{A}_{e}f_{e}(x)g_{o}(x) + \mathcal{M} \mathcal{A}_{e}f_{o}(x)g_{o}(x) \} A(x)dx
\]

\[
= \int_{\mathbb{R}} \{ f_{e}(x) \mathcal{A}_{e}g_{e}(x) - \varepsilon_{g} \mathcal{A}_{e}f_{e}(x)Jg_{o}(x)A(x) - \mathcal{A}_{e}(f_{o}'(x))Jg_{o}(x)A(x) \} dx
\]

\[
= \int_{\mathbb{R}} \{ f_{e}(x) \mathcal{A}_{e}g_{e}(x) - \varepsilon_{g} f_{e}(x) \mathcal{A}_{e}Jg_{o}(x) - f_{o}'(x) \mathcal{A}_{e}Jg_{o}(x) \} dx
\]

\[
= \int_{\mathbb{R}} \{ f(x) \mathcal{A}_{e}g_{e}(x) - \varepsilon_{g} f(x) \mathcal{A}_{e}Jg_{o}(x) + f_{o}(x) \frac{d}{dx} \mathcal{A}_{e}Jg_{o}(x) \} dx
\]

\[
= \int_{\mathbb{R}} f(x) \{ \mathcal{A}_{e}g_{e}(x) - \varepsilon_{g} \mathcal{A}_{e}Jg_{o}(x) + \frac{d}{dx} \mathcal{A}_{e}Jg_{o}(x) \} dx.
\]

This finishes the proof of Lemma 6.4. □

The operator \( ^1V_{A,e} \) satisfies the following additional property.

**Theorem 6.5.** The operator \( ^1V_{A,e} \) is the unique automorphism of \( D(\mathbb{R}) \) satisfying the intertwining property

\[
\frac{d}{dx} \circ ^1V_{A,e} = ^1V_{A,e} \circ (A_{A,e} + 2\varepsilon_{g}S),
\]
where $S$ denotes the symmetry $(Sf)(x) := f(-x)$.

**Proof.** The statement follows immediately from below:

$$
\int_{\mathbb{R}} \frac{d}{dx} V_{A,\varepsilon} f(x) g(x) dx = - \int_{\mathbb{R}} f(x) V_{A,\varepsilon} g'(x) A(x) dx
= - \int_{\mathbb{R}} f(x) \Lambda_{A,\varepsilon} V_{A,\varepsilon} g(x) A(x) dx
= \int_{\mathbb{R}} (\Lambda_{A,\varepsilon} + 2\varepsilon S) f(x) V_{A,\varepsilon} g(x) A(x) dx
= \int_{\mathbb{R}} V_{A,\varepsilon} (\Lambda_{A,\varepsilon} + 2\varepsilon S) f(x) g(x) dx.
$$

Above we have used Lemma [3.1].

7. The positivity of the intertwining operator

We shall say that a linear operator $L$ on $\mathcal{D}(\mathbb{R})$ is positive, if $L$ leaves the positive cone $\mathcal{D}(\mathbb{R})^+ := \{ f \in \mathcal{D}(\mathbb{R}) : f(x) \geq 0 \text{ for all } x \in \mathbb{R} \}$ invariant. The following statement is the central result of this section.

**Theorem 7.1.** For $-1 \leq \varepsilon \leq 1$, the intertwining operator $V_{A,\varepsilon}$ is positive.

For $\varepsilon = 0$ and $1$, Theorem 7.1 is known (cf. [53] and [54]). However, the case $-1 \leq \varepsilon \leq 1$ has some technical difficulties to be overcome compared to $\varepsilon = 0$ and $1$, as $\varepsilon$ could be positive as well as negative.

The proof of the above theorem affords several steps, the crucial one being the positivity of $V_{A,\varepsilon}(p_s(u, \cdot))(x)$ for every $s > 0$ and $u, x \in \mathbb{R}$, where

$$
p_s(u, v) := \frac{e^{-\frac{(u-v)^2}{4s}}}{2\sqrt{\pi}s}
$$
denotes the Euclidean heat kernel.

For simplicity we will write $W_{\varepsilon}(s; u, x)$ instead of $V_{A,\varepsilon}(p_s(u, \cdot))(x)$. Below we list some properties of $W_{\varepsilon}(s; u, x)$.

**Lemma 7.2.** For every $s > 0$ and $u, x \in \mathbb{R}$, we have

1) $W_{\varepsilon}(s; u, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_{A,\varepsilon}(-\lambda, x) e^{-s\lambda^2} e^{i\lambda u} d\lambda$.

2) The function $(u, x) \mapsto W_{\varepsilon}(s; u, x)$ is of class $C^1$ on $\mathbb{R}^2$.

3) $(\Lambda_{A,\varepsilon} + \partial_u) W_{\varepsilon}(s; u, x) = 0$.

4) $\lim_{\|\varepsilon x\| \to +\infty} W_{\varepsilon}(s; u, x) = 0$.

**Proof.** 1) For $x = 0$, we have $W_{\varepsilon}(s; u, 0) = p_s(u, 0) = \frac{e^{-\frac{\varepsilon^2 su^2}{2\sqrt{s}}}}{\sqrt{\sqrt{s} \pi}}$. Thus, for $x = 0$, the statement follows from the well known fact

$$
\int_{\mathbb{R}} e^{-x^2} e^{ix\lambda} d\lambda = \sqrt{\frac{\pi}{s}} e^{\frac{s^2}{4}}.
$$
For $x \neq 0$, using again (7.1) together with the Laplace type representation (6.4) of $V_{A,e}$, we have

$$W_e(s; u, x) = \frac{1}{2\pi} \int_{|y|}^{[x]} K_{e}(x, y) \left( \int_{\mathbb{R}} e^{-ix^2} e^{i\lambda(u-y)} d\lambda \right) dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{|y|}^{[x]} K_{e}(x, y) e^{-iy \lambda} dy \right) e^{-ix^2} e^{i\lambda u} d\lambda$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_{A,e}(-\lambda, x) e^{-ix^2} e^{i\lambda u} d\lambda.$$

2) For $|x| \geq x_0$ with $x_0 > 0$, the statement follows from 1) and the growth estimate of $|\partial_x \Psi_{A,e}(\lambda, x)|$ (see Theorem 4.3). Assume that $|x| \leq x_0$. Using the fact that

$$|\varphi'(x)| \leq c |\mu^2 + \rho^2| (|x| + 1) |x| e^{r(|\mu| + |\rho|)|x|},$$

and that

$$|\varphi''(x)| \leq c |\mu^2 + \rho^2| (|x| + 1)^2 e^{r(|\mu| + |\rho|)|x|},$$

for all $\mu \in \mathbb{C}$ and $x \in \mathbb{R}$ (cf. [52] Proposition 6.1.5), we deduce from (3.3) that

$$|\partial_x \Psi_{A,e}(\lambda, x)| \leq c (|\lambda| + 1)^2 (|x| + 1)^2 e^{r(|\mu| + |\rho|)|x|},$$

where $\mu^2 = \lambda^2 - (1 - \epsilon^2)q^2$. It follows that in both cases $\lambda^2 - (1 - \epsilon^2)q^2 \geq 0$, we have $|\partial_x \Psi_{A,e}(\lambda, x)| \leq c (|\lambda| + 1)^2$ for all $|x| \leq x_0$.

3) In view of 1), the present statement is easy to check.

4) For $x = 0$, $W_e(s; u, 0) = p_e(u, 0) = \frac{e^{-\epsilon^2 x^2}}{2\sqrt{\pi}} \rightarrow 0$ as $||u, x|| \rightarrow \infty$.

For $x \neq 0$, using 1) and the growth property of the eigenfunction $\Psi_{A,e}$ in Theorem 4.2.4, we get

$$\begin{align*} |W_e(s; u, x)| &\leq c_e (1 + |x|) e^{-\epsilon(1 - \sqrt{1 - \epsilon^2})|x|}. \end{align*}$$

Now, the statement follows by means of polar coordinates. \hfill \Box

The following lemma is also needed.

**Lemma 7.3.** Writing $W_e$ as $W_e(s; u, x) = W_{e}(s; u, x) + W_{e}^{\text{odd}}(s; u, x)$, where $W_{e}$ (resp. $W_{e}^{\text{odd}}$) denotes the even (resp. the odd) part of $W_{e}$ with respect to $x$, we have $W_{e}(s; u, x) > 0$.

**Proof.** Using Lemma 7.2.1 together with the expression (3.5) of the eigenfunction $\Psi_{A,e}$, we have

$$W_{e}(s; u, x)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{e}(x) + (-i\lambda + \epsilon \rho) \frac{sg(x)}{A(x)} \int_{0}^{[x]} \varphi_{e}(z) A(z) dz e^{-ix^2} e^{i\lambda u} d\lambda$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{e}(x) e^{-ix^2} e^{i\lambda u} d\lambda + \frac{sg(x)}{2\pi A(x)} \int_{\mathbb{R}} \int_{0}^{[x]} \varphi_{e}(z) A(z) dz (-i\lambda + \epsilon \rho) e^{-ix^2} e^{i\lambda u} d\lambda$$

$$= W_{e}^{\text{even}}(s; u, x) + W_{e}^{\text{odd}}(s; u, x).$$
Next we shall prove that $W^\text{even}_e(s; u, x) > 0$. By Lemma 5.3, we have
\[
\varphi_{\mu_1}(x) = \int_0^{s^{|x|}} K_e(x, r) \cos(\lambda r) dr,
\]
where the kernel $K_e(x, s)$ is nonnegative. Thus,
\[
W^\text{even}_e(s; u, x) = \frac{1}{\pi} \int_0^{+\infty} e^{-|x|^2} \cos(\lambda u) \left( \int_0^{s^{|x|}} K_e(x, r) \cos(\lambda r) dr \right) d\lambda
\]
\[
= \frac{1}{\pi} \int_0^{s^{|x|}} K_e(x, r) \left( \int_0^{+\infty} e^{-|x|^2} \cos(\lambda u) \cos(\lambda r) d\lambda \right) dr
\]
\[
= \frac{1}{4 \sqrt{\pi s}} \int_0^{s^{|x|}} K_e(x, r) \left( e^{-\frac{|x|\lambda}{\pi}} + e^{-\frac{|x|\lambda}{4\pi}} \right) dr.
\]
Using the fact that $r \mapsto K_e(x, r)$ is even, we deduce that
\[
W^\text{even}_e(s; u, x) = \frac{1}{4 \sqrt{\pi s}} \int_{-|x|}^{s^{|x|}} K_e(x, r) dr
\]
\[
\geq \frac{e^{-|u|^{2}/4s}}{4 \sqrt{\pi s}} \int_{-|x|}^{s^{|x|}} K_e(x, r) dr
\]
\[
= \frac{e^{-|u|^{2}/4s}}{2 \sqrt{\pi s}} \varphi_{\psi_{1\varphi_{-},e}}(x) > 0.
\]
\[\Box\]

Now we come to the crucial step in the proof of Theorem 7.1.

**Theorem 7.4.** For every $s > 0$ and $u, x \in \mathbb{R}$, we have
\[
W_e(s; u, x) \geq 0.
\]

**Proof.** For $(u, x) \in \mathbb{R} \times \{0\}$, we have
\[
W_e(s; u, x) = 0.
\]
For $s > 0$ and $(u, x) \in (\mathbb{R} \times \{0\})^c$, we will assume that $W_e(s; u, x)$ is not always nonnegative. Since $W_e(s; u, 0) > 0$ and $\lim_{||u, x|| \to +\infty} W_e(s; u, x) = 0$ (see Lemma 7.2.3), then the above assumption implies that the function $(u, x) \mapsto W_e(s; u, x)$ admits an absolute minimum $(u_0, x_0) \in (\mathbb{R} \times \{0\})^c$ such that $W_e(s; u_0, x_0) < 0$. In particular, $W^\text{odd}_e(s; u_0, x_0) = (W_e(s; u_0, x_0) - W_e(s; u_0, -x_0))/2 \leq 0$. We claim that
\[
W^\text{odd}_e(s; u_0, x_0) < 0. \tag{7.2}
\]
Indeed, if $W^\text{odd}_e(s; u_0, x_0) = 0$, then $W_e(s; u_0, x_0) = W^\text{even}_e(s; u_0, x_0)$, which is impossible since $W_e(s; u_0, x_0) < 0$ while $W^\text{even}_e(s; u_0, x_0) > 0$ (see Lemma 7.3).
On the other hand, using the fact that \((u_0, x_0)\) is an absolute minimum, we have

\[
(L_{A,e} + \partial_n)W_e(s; u_0, x_0) = \left( \frac{A'(x_0)}{A(x_0)} + 2 \epsilon \varrho \right) W^{\text{odd}}_e(s; u_0, x_0) - \epsilon \varrho W_e(s; u_0, x_0) \quad (7.3)
\]

\[
= -\epsilon \varrho W^{\text{even}}_e(s; u_0, x_0) + \left( \frac{A'(x_0)}{A(x_0)} + \epsilon \varrho \right) W^{\text{odd}}_e(s; u_0, x_0). \quad (7.4)
\]

Recall that our assumption is that \(W_e(s; u, x)\) is not always non-negative for all \(s > 0\) and \((u, x) \in (\mathbb{R} \times \{0\})^c\). We shall use Lemma 7.2.2 to prove that this assumption fails.

**case 1:** For \(\varrho = 0\), the identity (7.3) becomes

\[
(L_{A,e} + \partial_n)W_e(s; u_0, x_0) = \frac{A'(x_0)}{A(x_0)} W^{\text{odd}}_e(s; u_0, x_0).
\]

Now, Lemma 7.2.2 and the inequality (7.2) imply that \((A'/A)(x_0) = 0\), which is not true in the light of the hypotheses (H2) and (H4) on \(A'/A\) with \(\varrho = 0\).

**case 2:** Let \(\varrho > 0\) and \(\epsilon = 0\). As in the previous case, Lemma 7.2.2 and the inequality (7.2) imply that \((A'/A)(x_0) = 0\). However, by the hypothesis (H2) on \(A'/A\), we have \(A'/A(x) \gtrless \pm 2 \varrho \gtrless 0\) for all \(x \gtrless 0\). Hence our assumption does not hold true.

**case 3:** Let \(\varrho > 0\) and \(\epsilon > 0\).

**subcase 3.1:** Assume that \(x_0 > 0\). As \(W^{\text{even}}_e(s; u_0, x_0) > 0\) and \(W^{\text{odd}}_e(s; u_0, x_0) < 0\), it follows from (7.4) that \((L_{A,e} + \partial_n)W_e(s; u_0, x_0) < 0\), which is absurd by Lemma 7.2.2.

**subcase 3.2:** Assume that \(x_0 < 0\). We pin down that

\[
\frac{A'(x_0)}{A(x_0)} + 2 \epsilon \varrho \leq -2(1 - \epsilon) \varrho \leq 0.
\]

By Lemma 7.2.2, we have

\[
(L_{A,e} + \partial_n + \epsilon \varrho \text{id})W_e(s; u_0, x_0) = \epsilon \varrho W_e(s; u_0, x_0) < 0, \quad (7.6)
\]

while, by (7.3), (7.2) and (7.5),

\[
(L_{A,e} + \partial_n + \epsilon \varrho \text{id})W_e(s; u_0, x_0) = \left( \frac{A'(x_0)}{A(x_0)} + 2 \epsilon \varrho \right) W^{\text{odd}}_e(s; u_0, x_0) \geq 0,
\]

which contradicts the inequality (7.6).

**case 4:** Let \(\varrho > 0\) and \(\epsilon < 0\).

**subcase 4.1:** Assume that \(x_0 > 0\). Note that

\[
\frac{A'(x_0)}{A(x_0)} + 2 \epsilon \varrho \geq 2(1 + \epsilon) \varrho \geq 0.
\]

Hence, the identities (7.3), (7.2) and (7.7) imply that \((L_{A,e} + \partial_n)W_e(s; u_0, x_0) < 0\), which is absurd by Lemma 7.2.2.

**subcase 4.2:** Assume that \(x_0 < 0\). On the one hand, by Lemma 7.2.2, we have

\[
(L_{A,e} + \partial_n - \epsilon \varrho \text{id})W_e(s; u_0, x_0) = -\epsilon \varrho W_e(s; u_0, x_0) < 0. \quad (7.8)
\]

On the other hand, since \(x_0 < 0\) we have \(A'/A(x_0) < 0\). Thus, by (7.3) and (7.2),

\[
(L_{A,e} + \partial_n - \epsilon \varrho \text{id})W_e(s; u_0, x_0) = -2 \epsilon \varrho W^{\text{even}}_e(s; u_0, x_0) + \frac{A'(x_0)}{A(x_0)} W^{\text{odd}}_e(s; u_0, x_0) > 0,
\]
Theorem 8.1. \[\text{operator} 1\] (Inversion formula) \[\text{that} (\text{Plancherel formula}) \]

Proof of Theorem 7.1. Let \(f\) be a positive function in \(\mathcal{D}(\mathbb{R})\). Proving that \(V_{A,\varepsilon}(f) \geq 0\) is equivalent to showing that \(V_{A,\varepsilon}^0(f) \geq 0\) (see (6.4) and (6.14)). By (6.13) we have

\[\int_{\mathbb{R}} f(x) V_{A,\varepsilon}(p_s(u, .))(x) A(x) dx = \int_{\mathbb{R}} V_{A,\varepsilon}^0 f(x) p_s(x, u) dx = (V_{A,\varepsilon}^0 f) \ast \text{euc} q_s(u),\]

where \(q_s(r) := \frac{e^{-r^2/4s}}{2\sqrt{\pi}s}\). Since \(f \geq 0\) and \(V_{A,\varepsilon}(p_s(u, .))(x) = W_{s}(s; u, x) \geq 0\), it follows that \((V_{A,\varepsilon}^0 f) \ast \text{euc} q_s(u) \geq 0\) for all \(s > 0\) and \(u, x \in \mathbb{R}\). Thus

\[0 \leq \lim_{s \to 0} (V_{A,\varepsilon}^0 f) \ast \text{euc} q_s(u) = V_{A,\varepsilon}^0 f(u).\]

8. Fourier transform of \(L^p\)-Schwartz spaces

Assume that \(-1 \leq \varepsilon \leq 1\). For \(f \in L^1(\mathbb{R}, A(x) dx)\) put

\[\mathcal{F}_{A,\varepsilon} f(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{A,\varepsilon}(\lambda, -x) A(x) dx.\]  \(\text{(8.1)}\)

To state its alleged inverse transform, let us introduce the following Plancherel measure

\[\pi_{\varepsilon}(d\lambda) = \frac{|\lambda|}{\sqrt{1 - (1 - \varepsilon^2)q^2}} |c(\sqrt{1 - (1 - \varepsilon^2)q^2})| \mathbf{1}_{\mathbb{R} \setminus (\sqrt{1 - \varepsilon^2}e, \sqrt{1 - \varepsilon^2}q)}(\lambda) d\lambda,\]  \(\text{(8.2)}\)

where \(c\) is the Harish-Chandra’s function associated with the second order differential operator \(\Delta\) (see Section 2 for more details on the \(c\)-function). Below \(f^*(x) := f(-x)\).

**Theorem 8.1.** Let \(f\) be a smooth function with compact support on \(\mathbb{R}\). Then

1) (Inversion formula)

\[f(x) = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon} f(\lambda) \Psi_{A,\varepsilon}(\lambda, x) (1 - \frac{\varepsilon q}{i\lambda}) \pi_{\varepsilon}(d\lambda).\]  \(\text{(8.3)}\)

2) (Plancherel formula)

\[\int_{\mathbb{R}} |f(x)|^2 A(x) dx = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon} f(\lambda) \mathcal{F}_{A,\varepsilon} d\lambda (1 - \frac{\varepsilon q}{i\lambda}) \pi_{\varepsilon}(d\lambda).\]  \(\text{(8.4)}\)

We may rewrite (8.4) for two smooth and compactly supported functions \(f\) and \(g\) as

\[\int_{\mathbb{R}} f(x) g(-x) A(x) dx = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon} f(\lambda) \mathcal{F}_{A,\varepsilon}^* g(\lambda) (1 - \frac{\varepsilon q}{i\lambda}) \pi_{\varepsilon}(d\lambda).\]  \(\text{(8.5)}\)
Proof. For the sake of completeness, we provide a detailed proof.

1) Below $Jh(x) := \int_{-\infty}^{x} h(t)dt$. Using the superposition (3.3) of the eigenfunction $\Psi_{\varepsilon}(\lambda, x)$, we obtain

$$\mathcal{F}_{\lambda}\phi(\lambda) = 2\mathcal{F}_{\lambda}(f_{\varepsilon})(\mu_{\varepsilon}) + 2(i\lambda + \varepsilon \phi)\mathcal{F}_{\lambda}(J f_{\varepsilon})(\mu_{\varepsilon}),$$

where $F_{\Delta}$ is as in (2.5). By the inversion formula (2.6) for the Chébli transform $\mathcal{F}_{\lambda}$ we deduce that

$$f(x) = \int_{\mathbb{R}^{+}} \left\{ \mathcal{F}_{\lambda}(f_{\varepsilon})(\mu_{\varepsilon})\phi_{\mu_{\varepsilon}}(x) + \mathcal{F}_{\lambda}(J f_{\varepsilon})(\mu_{\varepsilon})\phi'_{\mu_{\varepsilon}}(x) \right\} \pi(d\mu_{\varepsilon}). \quad (8.6)$$

Now, let us express $\phi_{\mu_{\varepsilon}}$ and $\phi'_{\mu_{\varepsilon}}$ in terms of $\Psi_{A,\varepsilon}$ as follows

$$\phi_{\mu_{\varepsilon}}(x) = \frac{1}{2}(\Psi_{A,\varepsilon}(-\lambda, x) + \Psi_{A,\varepsilon}(-\lambda, x)), \quad \phi'_{\mu_{\varepsilon}}(x) = \frac{i\lambda + \varepsilon \phi}{2}(\Psi_{A,\varepsilon}(-\lambda, -x) - \Psi_{A,\varepsilon}(-\lambda, x)).$$

Consequently, formula (8.6) becomes

$$f(x) = \frac{1}{2} \int_{\mathbb{R}^{+}} \Psi_{A,\varepsilon}(-\lambda, x) \left\{ \mathcal{F}_{\lambda}(f_{\varepsilon})(\mu_{\varepsilon}) + (i\lambda + \varepsilon \phi)\mathcal{F}_{\lambda}(J f_{\varepsilon})(\mu_{\varepsilon}) \right\} \pi(d\mu_{\varepsilon})$$

$$+ \frac{1}{2} \int_{\mathbb{R}^{+}} \Psi_{A,\varepsilon}(-\lambda, x) \left\{ \mathcal{F}_{\lambda}(f_{\varepsilon})(\mu_{\varepsilon}) - (i\lambda + \varepsilon \phi)\mathcal{F}_{\lambda}(J f_{\varepsilon})(\mu_{\varepsilon}) \right\} \pi(d\mu_{\varepsilon})$$

$$= \frac{1}{4} \int_{\mathbb{R}^{+}} \left\{ \Psi_{A,\varepsilon}(-\lambda, -x)\mathcal{F}_{A,\varepsilon}(f)(\lambda) + \Psi_{A,\varepsilon}(-\lambda, x)\mathcal{F}_{A,\varepsilon}(\tilde{f})(\lambda) \right\} \pi(d\mu_{\varepsilon}). \quad (8.7)$$

Further, it is easy to check that

$$\Psi_{A,\varepsilon}(\lambda, x) = \left(1 + \frac{\phi \varepsilon}{i\lambda}\right)\Psi_{A,\varepsilon}(-\lambda, -x) - \frac{\phi \varepsilon}{i\lambda} \Psi_{A,\varepsilon}(\lambda, -x), \quad (8.8)$$

and therefore

$$\mathcal{F}_{A,\varepsilon}(\tilde{f})(\lambda) = \left(1 + \frac{\phi \varepsilon}{i\lambda}\right)\mathcal{F}_{A,\varepsilon}(f)(-\lambda) - \frac{\phi \varepsilon}{i\lambda} \mathcal{F}_{A,\varepsilon}(f)(\lambda). \quad (8.9)$$

In view of (8.8) and (8.9) we obtain

$$\int_{\mathbb{R}^{+}} \mathcal{F}_{A,\varepsilon}(f)(\lambda)\Psi_{A,\varepsilon}(-\lambda, -x) \pi(d\mu_{\varepsilon}) = \int_{\mathbb{R}^{+}} \Psi_{A,\varepsilon}(\lambda, x)\mathcal{F}_{A,\varepsilon}(f)(\lambda) \left(1 - \frac{\phi \varepsilon}{i\lambda}\right) \pi(d\mu_{\varepsilon})$$

$$+ \int_{\mathbb{R}^{+}} \Psi_{A,\varepsilon}(-\lambda, x)\mathcal{F}_{A,\varepsilon}(f)(\lambda) \left(\frac{\phi \varepsilon}{i\lambda}\right) \pi(d\mu_{\varepsilon}), \quad (8.10)$$

and

$$\int_{\mathbb{R}^{+}} \mathcal{F}_{A,\varepsilon}(\tilde{f})(\lambda)\Psi_{A,\varepsilon}(-\lambda, x) \pi(d\mu_{\varepsilon}) = \int_{\mathbb{R}^{+}} \Psi_{A,\varepsilon}(-\lambda, x)\mathcal{F}_{A,\varepsilon}(f)(-\lambda) \left(1 + \frac{\phi \varepsilon}{i\lambda}\right) \pi(d\mu_{\varepsilon})$$

$$+ \int_{\mathbb{R}^{+}} \Psi_{A,\varepsilon}(-\lambda, x)\mathcal{F}_{A,\varepsilon}(f)(\lambda) \left(-\frac{\phi \varepsilon}{i\lambda}\right) \pi(d\mu_{\varepsilon}). \quad (8.11)$$

By substituting (8.10) and (8.11) into (8.7), we get the inversion formula (8.3).

2) On the one hand, using the fact that $\Psi_{A,\varepsilon}(\lambda, x) = \Psi_{A,\varepsilon}(-\lambda, x)$ for $\lambda \in \mathbb{R}$, we have

$$\mathcal{F}_{A,\varepsilon}(\tilde{g})(\lambda) = \int_{\mathbb{R}} \tilde{g}(x)\Psi_{\varepsilon}(-\lambda, x)A(x)dx.$$
Applying the identity (8.7) for \( f \) implies
\[
\int_{\mathbb{R}} f(x)g(x)A(x)dx = \frac{1}{4} \int_{\mathbb{R}^+} \left\{ \mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(g)(\lambda)} + \mathcal{F}_{A,\varepsilon}(\hat{f})(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{g})(\lambda)} \right\} \pi_{\varepsilon}(d\lambda).
\] (8.12)

On the other hand, from (8.8) it follows that
\[
\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda) = \left(1 + \frac{\varepsilon_0}{i\lambda}\right)\mathcal{F}_{A,\varepsilon}(f)(\lambda) - \frac{\varepsilon_0}{i\lambda}\mathcal{F}_{A,\varepsilon}(f)(-\lambda).
\] (8.13)

Hence
\[
\left(1 - \frac{\varepsilon_0}{i\lambda}\right)\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)}
= \left(1 + \frac{\varepsilon_0^2}{\lambda^2}\right)|\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2 - \frac{\varepsilon_0}{i\lambda}\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)}.
\] (8.14)

Now let us rewrite (8.13) as
\[
\mathcal{F}_{A,\varepsilon}(f)(-\lambda) = \frac{i\lambda}{i\lambda - \varepsilon_0}\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(\lambda)} + \frac{\varepsilon_0}{-i\lambda + \varepsilon_0}\overline{\mathcal{F}_{A,\varepsilon}(f)(\lambda)}.
\]

Hence
\[
\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)} = \frac{i\lambda}{i\lambda - \varepsilon_0}\overline{\mathcal{F}_{A,\varepsilon}(f)(\lambda)}\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(\lambda)} + \frac{\varepsilon_0}{-i\lambda + \varepsilon_0}|\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2,
\]
which implies
\[
\left(1 - \frac{\varepsilon_0}{i\lambda}\right)\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)}
= -\left(\frac{\varepsilon_0}{i\lambda}\right)\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(\lambda)} - \left(\frac{\varepsilon_0^2}{\lambda^2}\right)|\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2.
\] (8.15)

Thus, (8.14) becomes
\[
\left(1 - \frac{\varepsilon_0}{i\lambda}\right)\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)} = |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2 - \frac{\varepsilon_0}{i\lambda}\mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)}.
\] (8.16)

This is the key identity towards the Plancherel formula (8.4). One more thing, from (8.16) we also have
\[
\left(1 - \frac{\varepsilon_0}{i\lambda}\right)\overline{\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)}\mathcal{F}_{A,\varepsilon}(f)(\lambda) = |\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)|^2 - \frac{\varepsilon_0}{i\lambda}\mathcal{F}_{A,\varepsilon}(\hat{f})(-\lambda)\mathcal{F}_{A,\varepsilon}(f)(-\lambda).
\] (8.17)

Indeed, we obtain (8.17) in three steps:

1. replace \( f \) by \( \hat{f} \) in (8.16).
2. substitute \( \lambda \) by \( -\lambda \) in the resulting identity from step 1.
3. take the complex conjugates in the resulting identity from step 2.
By putting the pieces together we arrive at
\[
\int_{\mathbb{R}} \mathcal{F}_{A,e}(f)(\lambda) \overline{\mathcal{F}_{A,e}(f)(-\lambda)} \left(1 - \frac{\varepsilon \Omega}{\lambda^2}\right) \pi_e(d\lambda) = 
\int_{\mathbb{R}^+} |\mathcal{F}_{A,e}(f)(\lambda)|^2 \pi_e(d\lambda) - \int_{\mathbb{R}^+} \mathcal{F}_{A,e}(f)(\lambda) \overline{\mathcal{F}_{A,e}(f)(\lambda)} \left(\frac{\varepsilon \Omega}{\lambda^2}\right) \pi_e(d\lambda)
+ \int_{\mathbb{R}^-} |\mathcal{F}_{A,e}(f)(-\lambda)|^2 \pi_e(d\lambda) - \int_{\mathbb{R}^-} \mathcal{F}_{A,e}(f)(-\lambda) \overline{\mathcal{F}_{A,e}(f)(-\lambda)} \left(\frac{\varepsilon \Omega}{\lambda^2}\right) \pi_e(d\lambda)
= \int_{\mathbb{R}^+} \left(|\mathcal{F}_{A,e}(f)(\lambda)|^2 + |\mathcal{F}_{A,e}(f)(\lambda)|^2\right) \pi_e(d\lambda),
\]
which compares very well with \(4\|f\|_{L^2(\mathbb{R}, A(x)dx)}\) (see (8.12)). \(\square\)

Remarks 8.2. 1) For \(\varepsilon = 1\), the Plancherel formula (8.4) corrects Theorem 5.13 in [12] (stated without a proof).

2) For \(\varepsilon = 0\) we can prove the following stronger versions of the inversion and the Plancherel formulas:

(i) If \(f \in L^1(\mathbb{R}, A(x)dx)\) and \(\mathcal{F}_{A,0}(f) \in L^1(\mathbb{R}, \pi_0(d\lambda))\) then
\[
f(x) = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,0}(f)(\lambda) \Psi_{A,0}(\lambda, x) \pi_0(d\lambda) \quad \text{almost everywhere.}
\]

(ii) If \(f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)\), then \(\mathcal{F}_{A,0}f \in L^2(\mathbb{R}, \pi_0(d\lambda))\) and \(\|\mathcal{F}_{A,0}f\|_{L^2} = 2\|f\|_{L^2}\).

(iii) There exists a unique isometry on \(L^2(\mathbb{R}, A(x)dx)\) that coincides with \((1/2)\mathcal{F}_{A,0}\) on \(L^1 \cap L^2(\mathbb{R}, A(x)dx)\).

The following lemma is needed in the proof of the Paley-Wiener theorem below.

Lemma 8.3. For \(R > 0\), denote by \(\mathcal{D}_R(\mathbb{R})\) the space of smooth functions with support inside \([-R, R]\). Then, \(f \in \mathcal{D}_R(\mathbb{R})\) if and only if \(1V_{A,e}f \in \mathcal{D}_R(\mathbb{R})\).

Proof. The direct statement follows from (6.14). The converse direction is more involved. On the one hand, one can prove that
\[
1V_{A,e}^{-1} g(y) = \mathcal{A}_{e}^{-1} g_{e}(y) + (\varepsilon \Omega + \frac{d}{dy}) \mathcal{A}_{e}^{-1} (Jg_o)(y),
\]
where \(Jh(x) := \int_{-\infty}^{x} h(t)dt\) and \(\mathcal{A}_{e}^{-1} = \mathcal{A}_{e}^{-1} \circ \mathcal{E}_e\) (see (6.3)). On the other hand, from (5.5) and [7] Lemma 4.10 it follows that if \(g_e \in \mathcal{D}_R(\mathbb{R})\) then \(\mathcal{A}_{e}^{-1} g_e \in \mathcal{D}_R(\mathbb{R})\). Further, one may check that \(g_o \in \mathcal{D}_R(\mathbb{R})\) if and only if \(Jg_o \in \mathcal{D}_R(\mathbb{R})\). Thus, as \(Jg_o\) is an even function, it follows from above that \(\mathcal{A}_{e}^{-1} (Jg_o) \in \mathcal{D}_R(\mathbb{R})\). \(\square\)

Let \(PW_{R}(\mathbb{C})\) be the space of entire functions \(h\) on \(\mathbb{C}\) which are of exponential type and rapidly decreasing, i.e.
\[
\exists R > 0, \forall t \in \mathbb{N}, \sup_{\lambda \in \mathbb{C}} (|\lambda| + 1)^{t} e^{-R|\text{Im} \lambda|} |h(\lambda)| < \infty.
\]

Theorem 8.4. Assume that \(-1 < \varepsilon \leq 1\). The Fourier transform \(\mathcal{F}_{A,e}\) is a linear isomorphism between \(\mathcal{D}_R(\mathbb{R})\) and the space of all entire functions \(h\) on \(\mathbb{C}\) satisfying (8.19).
Proof. The proof is standard. We shall only indicate how to proceed towards the statement. On the one hand, the Fourier transform $\mathcal{F}_{A,\varepsilon}(f) = \mathcal{F}_{\text{euc}} \circ V_{A,\varepsilon}(\hat{f})$, where $\mathcal{F}_{\text{euc}}$ is the Euclidean Fourier transform and $V_{A,\varepsilon}$ is the intertwining operator (6.14). Indeed, in view of Mehler’s type Theorem [5,4] we have

$$\mathcal{F}_{A,\varepsilon}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx = \int_{\mathbb{R}} \hat{f}(x)V_{A,\varepsilon}(\varepsilon^{1/2})(x)A(x)dx = \int_{\mathbb{R}} V_{A,\varepsilon}\hat{f}(x)\varepsilon^{1/2}dx.$$

Now, in view of Lemma 8.3, appealing to the Paley-Wiener theorem for the Euclidean Fourier transform $\mathcal{F}_{\text{euc}}$ we get the desired statement. 

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{3}{2 + \sqrt{1 - \varepsilon^2}}$, set $\theta_{p,\varepsilon} := \frac{2}{p} - 1 - \sqrt{1 - \varepsilon^2}$. Observe that $1 \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \leq 2$. We introduce the tube domain

$$\mathbb{C}_{p,\varepsilon} := \{ \lambda \in \mathbb{C} \mid |\Im \lambda| \leq \theta_{p,\varepsilon} \}.$$

For $\theta_{p,\varepsilon} = 0$ or $\varepsilon = 0$, the domain $\mathbb{C}_{p,\varepsilon}$ reduces to $\mathbb{R}$.

**Proposition 8.5.** For all $\lambda \in \mathbb{C}_{1,\varepsilon}$, the function $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is bounded for all $x \in \mathbb{R}$.

Proof. Let $R > 0$ be arbitrary but fixed and let $\mathbb{R}_{1,\varepsilon} := \{ \nu \in \mathbb{R} \mid |\nu| \leq \theta (1 - \sqrt{1 - \varepsilon^2}) \}$. Applying the maximum modulus principle together with the fact that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(i\im \lambda, x)$ in the domain $[-R, R] + i\mathbb{R}_{1,\varepsilon}$ implies that the maximum of $|\Psi_{A,\varepsilon}(\lambda, x)|$ is obtained when $\lambda = i\eta$ with $|\eta| = \theta (1 - \sqrt{1 - \varepsilon^2})$. Now, recall that $\Psi_{A,\varepsilon}(i\eta, x) + \Psi_{A,\varepsilon}(i\eta, -x) = 2\varphi_{\mu_\varepsilon}(x)$ when $\varepsilon \neq 0, \pm 1$, and $\Psi_{A,\varepsilon}(i\eta, x) + \Psi_{A,\varepsilon}(i\eta, -x) = 2$ when $\varepsilon = 0, \pm 1$. The parameter $\mu_\varepsilon$ satisfies $\mu_\varepsilon^2 = \lambda^2 - (1 - \varepsilon^2)\eta^2 = -\theta^2(1 - 2\sqrt{1 - \varepsilon^2}[1 - \sqrt{1 - \varepsilon^2}]) \leq 0$, and therefore $\mu_\varepsilon \in i\mathbb{R}$ with $|\mu_\varepsilon| \leq \theta$. Using the fact that $\Psi_{A,\varepsilon}(i\eta, x) > 0$ for all $x \in \mathbb{R}$, together with the fact that $\varphi_{\mu_\varepsilon}(x) \leq 1$ for $\mu_\varepsilon$ is as above (see Lemma 2.2.3), it follows that $\Psi_{A,\varepsilon}(i\eta, x) \leq 2$ for all $x \in \mathbb{R}$ and $-1 \leq \varepsilon \leq 1$. 

**Corollary 8.6.** Let $f \in L^1(\mathbb{R}, A(x)dx)$. Then the following holds.

1) The Fourier transform $\mathcal{F}_{A,\varepsilon}(f)(\lambda)$ is well defined for all $\lambda \in \mathbb{C}_{1,\varepsilon}$. Moreover,

$$|\mathcal{F}_{A,\varepsilon}(f)(\lambda)| \leq 2||f||_1, \quad \lambda \in \mathbb{C}_{1,\varepsilon}.$$

2) The function $\mathcal{F}_{A,\varepsilon}(f)$ is holomorphic on $\hat{\mathbb{C}}_{1,\varepsilon}$, the interior of $\mathbb{C}_{1,\varepsilon}$.

3) (Riemann-Lebesgue lemma)

$$\lim_{\lambda \in \mathbb{C}_{1,\varepsilon}, |\lambda| \to \infty} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| = 0. \quad (8.20)$$

Proof. The first two statements are direct consequences of Proposition 8.5, the fact that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is holomorphic in $\lambda$, and Morera’s theorem. For the Riemann-Lebesgue lemma, a classical proof for the Euclidean Fourier transform carries over. More precisely, assume that $f \in \mathcal{D}(\mathbb{R})$ (the space of smooth functions with compact support on
lemma 8.7. Let \( f \in L^p(\mathbb{R}, A(x)dx) \) with \( 1 < p \leq \frac{2}{1+\sqrt{1-\epsilon}} \). Then the following holds.

1) The Fourier transform \( \mathcal{F}_{A,e}(f)(\lambda) \) is well defined for all \( \lambda \in \hat{C}_{p,e}, \text{the interior of } C_{p,e}. \) Moreover,
\[
|\mathcal{F}_{A,e}(f)(\lambda)| \leq c\|f\|_p, \quad \lambda \in \hat{C}_{p,e}.
\]

2) The function \( \mathcal{F}_{A,e}(f) \) is holomorphic on \( \hat{C}_{p,e}. \)

3) (Riemann-Lebesgue lemma)
\[
\lim_{\lambda \in \hat{C}_{p,e}, |\lambda| \to \infty} |\mathcal{F}_{A,e}(f)(\lambda)| = 0.
\] (8.21)

**Proof.** The first two statements follow easily from the estimate
\[
\Psi_{A,e}(\lambda, x) \leq \begin{cases} 
 c(|x| + 1)e^{\text{Im } \lambda ||x||}e^{-\epsilon |x||1 - \sqrt{1-\epsilon}^q|} & \text{for } q > 0 \\
 c e^{\text{Im } \lambda ||x||} & \text{for } q = 0
\end{cases}
\]
the fact that \( A(x) \leq c|x|^p e^{2\eta|x|} \) (a consequence of the hypothesis (H4) on Chébli’s function \( A \)), the fact that \( \Psi_{A,e}(\lambda, \cdot) \) is holomorphic in \( \lambda \), and Morera’s theorem. The Riemann-Lebesgue lemma is established exactly as for (8.20) by approximating any function in \( L^p(\mathbb{R}, A(x)dx) \) by compactly supported smooth functions for all \( 1 \leq p < \infty. \)

**Theorem 8.8.** The Fourier transform \( \mathcal{F}_{A,e} \) is injective on \( L^p(\mathbb{R}, A(x)dx) \) for \( 1 \leq p \leq \frac{2}{1+\sqrt{1-\epsilon}}. \)

**Proof.** Take \( q \) such that \( p + q = pq. \) For \( f \in L^p(\mathbb{R}, A(x)dx) \) et \( g \in \mathcal{D}(\mathbb{R}) \) we have the inequalities
\[
|\langle f, g \rangle| := |\int_{\mathbb{R}} f(x)g(-x)A(x)dx| \leq \|f\|_{L^p} \|g\|_{L^q}
\]
and
\[
|\langle \mathcal{F}_{A,e}(f), \mathcal{F}_{A,e}(g) \rangle| := |\int_{\mathbb{R}} \mathcal{F}_{A,e}(f)(\lambda)\mathcal{F}_{A,e}(g)(\lambda)\left(1 - \frac{\epsilon Q}{iA}\right)\pi_e(\lambda)d\lambda|
\leq \|\mathcal{F}_{A,e}(f)\|_{L^q} \|\mathcal{F}_{A,e}(g)\|_{L^q} \leq c\|f\|_{L^p} \|\mathcal{F}_{A,e}(g)\|_{L^q}. \quad \text{(8.22)}
\]
Above we have used Corollary 8.6 and Lemma 8.7 to get (8.22). Therefore the mapping \( f \mapsto \langle f, g \rangle \) and \( f \mapsto \langle \mathcal{F}_{A,e}(f), \mathcal{F}_{A,e}(g) \rangle \) are continuous functionals on \( L^p(\mathbb{R}, A(x)dx). \) Now \( \langle f, g \rangle = \langle \mathcal{F}_{A,e}(f), \mathcal{F}_{A,e}(g) \rangle \) for all \( f \in \mathcal{D}(\mathbb{R}) \) and by continuity for all \( f \in L^p(\mathbb{R}, A(x)dx). \) Assume that \( f \in L^p(\mathbb{R}, A(x)dx) \) and that \( \mathcal{F}_{A,e}(f) = 0, \) then for all \( g \in \mathcal{D}(\mathbb{R}) \) we have \( \langle f, g \rangle = \langle \mathcal{F}_{A,e}(f), \mathcal{F}_{A,e}(g) \rangle = 0 \) and therefore \( f = 0. \)\( \square \)
Lemma 8.10. The Fourier transform 

\[
\text{Under the hypothesis (H4) on Chébli's function } A(x) \text{, there exists a } \beta > 0 \text{ such that for all } x, \quad A(x) \leq c|x|^\beta e^{2e|x|}.
\]

Proof. Let \( f \in \mathcal{S}_p(\mathbb{R}) \). For \( \lambda \in \mathbb{C}_{p,e} \), we have

\[
|\hat{f}(\lambda)| = \int_{\mathbb{R}} |f(x)| |\Psi_{A,e}(\lambda, -x)| A(x) dx
\]

\[
\leq \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-2/p} \varphi_0(x)^{2/p} \Psi_{A,e}(0, -x) e^{\text{Im} \lambda |x|} A(x) dx
\]

\[
\leq c_1 \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+1} e^{-2e|x|} A(x) dx.
\]

Under the hypothesis (H4) on Chébli’s function \( A(x) \), there exists a \( \beta > 0 \) such that

\[
A(x) \leq c|x|^\beta e^{2e|x|}.
\]
Hence,
\[
|F_{\Delta}(f)(\ell)| \leq c_2 \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-\frac{2}{p}} (|x| + 1)^{\frac{2}{p+1}} |x|^\ell dx < \infty.
\]
This proves that $F_{\Delta}(f)$ is well defined for all $f \in \mathcal{I}(\mathbb{R})$ when $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon}}$. Moreover, since the map $\lambda \mapsto \Psi_{\Delta}(\lambda, x)$ is holomorphic on $\mathbb{C}$, it follows that for all $f \in \mathcal{I}(\mathbb{R})$, the function $F_{\Delta}(f)$ is analytic in the interior of $\mathbb{C}_{p,\varepsilon}$, and continuous on $\mathbb{C}_{p,\varepsilon}$. Furthermore, by Theorem 4.3, we have
\[
|F_{\Delta}(f)(\ell)| \leq c_3 \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-\frac{2}{p}} (|x| + 1)^{\frac{2}{p+1}} |x|^\ell dx < \infty.
\]
Thus, all derivatives of $F_{\Delta}(f)$ extend continuously to $\mathbb{C}_{p,\varepsilon}$. Next, we will prove that given a continuous seminorm $\tau$ on $\mathcal{I}(\mathbb{C}_{p,\varepsilon})$, there exists a continuous seminorm $\sigma$ on $\mathcal{I}(\mathbb{R})$ such that
\[
\tau(F_{\Delta}(f)) \leq c_4 \sigma(f), \quad \forall f \in \mathcal{I}(\mathbb{R}).
\]
Note that the space $\mathcal{I}(\mathbb{C}_{p,\varepsilon})$ and its topology are also determined by the seminorms
\[
h \mapsto \overline{\tau}_{t,\ell}(h) := \sup_{\lambda \in \mathbb{C}_{p,\varepsilon}} \left| (\ell + 1)^{\ell} h(\lambda) \right|^{(t)}, \quad (8.26)
\]
where $t$ and $\ell$ are two arbitrary positive integers. By invoking Lemma 5.1 we have for $r \in \mathbb{N}$,
\[
(i\lambda)^r F_{\Delta}(f)(\lambda) = (i\lambda)^r \int_{\mathbb{R}} f(x) \Psi_{\Delta}(\lambda, x) A(x) dx
\]
\[
= \int_{\mathbb{R}} f(x) \Lambda_{\Delta}^r \Psi_{\Delta}(\lambda, x) A(x) dx
\]
\[
= (-1)^r \int_{\mathbb{R}} (\Lambda_{\Delta} + 2\varepsilon S)^r f(x) \Psi_{\Delta}(\lambda, x) A(x) dx
\]
\[
= \int_{\mathbb{R}} \Lambda_{\Delta}^r f(-x) \Psi_{\Delta}(\lambda, x) A(x) dx
\]
\[
= F_{\Delta}(\Lambda_{\Delta}^r f)(\lambda),
\]
where $S$ denotes the symmetry $Sf(x) = f(-x)$. Above we have used $(\Lambda_{\Delta} + 2\varepsilon S)^r \circ S = (-1)^r S \circ \Lambda_{\Delta}^r$. Thus
\[
\left( (i\lambda)^r F_{\Delta}(f)(\lambda) \right)^{(t)} = \int_{\mathbb{R}} \Lambda_{\Delta}^r f(x) \partial_1^t \Psi_{\Delta}(\lambda, -x) A(x) dx.
\]
On the one hand, using Theorem 4.3.2 we obtain
\[
\left| (\ii)^{\ell} \mathcal{F}_{A,\varepsilon}(f) (\lambda) \right|^{(I)} \leq c_5 \int_{\mathbb{R}} |\Lambda'_{\varepsilon}f(x)| \ (|x| + 1)^\ell \varphi_{1/\sqrt{1 - \varepsilon}} (x) e^{\ii \text{Im} x|\lambda|} A(x) dx
\]
\[= c_5 \int_{|x| \leq a} |\Lambda'_{\varepsilon}f(x)| \ (|x| + 1)^\ell \varphi_{1/\sqrt{1 - \varepsilon}} (x) e^{\ii \text{Im} x|\lambda|} A(x) dx + c_5 \int_{|x| > a} |\Lambda'_{\varepsilon}f(x)| \ (|x| + 1)^\ell \varphi_{1/\sqrt{1 - \varepsilon}} (x) e^{\ii \text{Im} x|\lambda|} A(x) dx \leq c_6 \int_{|x| \leq a} |\Lambda'_{\varepsilon}f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+\ell+1} e^{-2\varepsilon|x|} A(x) dx + c_6 \int_{|x| > a} |\Lambda'_{\varepsilon}f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+\ell+1} e^{-2\varepsilon|x|} A(x) dx.
\]
On the other hand, by mimicking the proof of [17, Lemma 4.18] we have:

(i) For $|x| \leq a$,
\[
|\Lambda'_{\varepsilon}f(x)| \leq c_1 \left( \sum_{i=0}^{r} |f^{(i)}(x)| + \sum_{i=0}^{r-1} |f^{(i)}(-x)| + \sum_{i=0}^{r} \sum_{m=0}^{n_r} |f^{(i)}(\xi_m)| \right),
\]
where $\xi_m = \xi_m(x, r) \in ]-|x|, |x|[].$

(ii) For $|x| > a$,
\[
|\Lambda'_{\varepsilon}f(x)| \leq c_2 \left( \sum_{i=0}^{r} |f^{(i)}(x)| + \sum_{i=0}^{r-1} |f^{(i)}(-x)| \right).
\]

The estimate
\[
\tau(\mathcal{F}_{A,\varepsilon}(f)) \leq c_8 \sum_{\text{finite}} \sigma(f), \quad \forall f \in \mathcal{S}_p(\mathbb{R})
\]
is now a matter of putting the pieces together.

The injectivity of the transform $\mathcal{F}_{A,\varepsilon}$ on $\mathcal{S}_p(\mathbb{R})$ is clear, by the fact that $\mathcal{F}_{A,\varepsilon}$ is injective on $L^q(\mathbb{R}, A(x) dx)$ for $1 \leq q \leq \frac{2}{1+\varepsilon}$ (see Theorem 8.8) and the fact that $\mathcal{S}_p(\mathbb{R})$ is a dense subspace of $L^q(\mathbb{R}, A(x) dx)$ for all $q < \infty$ so that $p \leq q$.

This concludes the proof of Lemma 8.10. \hfill \square

**Lemma 8.11.** Let $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\varepsilon}$. The inverse Fourier transform $\mathcal{F}_{A,\varepsilon}^{-1} : \mathcal{S}_p(\mathbb{C}) \rightarrow \mathcal{S}(\mathbb{R})$ given by
\[
\mathcal{F}_{A,\varepsilon}^{-1} h(x) = \frac{1}{4} \int_{\mathbb{R}} h(\lambda) \Psi_{A,\varepsilon}(\lambda, x) \left( 1 - \frac{\varepsilon \lambda}{iA} \right) \pi_{\varepsilon}(d\lambda)
\]
is continuous for the topologies induced by $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ and $\mathcal{S}_p(\mathbb{R})$.

**Proof.** Let $f \in \mathcal{S}(\mathbb{R})$ and let $h \in \mathcal{S}_p(\mathbb{C})$ so that $f = \mathcal{F}_{A,\varepsilon}^{-1}(h)$. Given a seminorm $\sigma$ on $\mathcal{S}_p(\mathbb{R})$ we should find a continuous seminorm $\tau$ on $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ such that $\sigma(f) \leq c \tau(h)$.
Denote by $g$ the image of $h$ by the inverse Euclidean Fourier transform $\mathcal{F}_{euc}^{-1}$. Making use of the Paley-Wiener Theorem \cite{6.4} for $\mathcal{F}_{A,d}$ and the classical Paley-Wiener theorem for $\mathcal{F}_{euc}$, we have the following support conservation property: $\text{supp}(f) \subseteq I_R := [-R,R] \leftrightarrow \text{supp}(g) \subseteq I_R$.

For $f \in \mathbb{N}_{\geq 1}$, let $\omega_j \in C^\infty(\mathbb{R})$ with $\omega_j = 0$ on $I_{j-1}$ and $\omega_j = 1$ outside of $I_j$. Assume that $\omega_j$ and all its derivatives are bounded, uniformly in $j$. We will write $g_j = \omega_j g$, and define $h_j := \mathcal{F}_{euc}(g_j)$ and $f_j := \mathcal{F}_{A,d}^{-1}(h_j)$. Note that $g_j = g$ outside $I_j$. Hence, by the above support property, $f_j = f$ outside $I_j$. We shall estimate the function

$$x \mapsto (|x| + 1)^t \varphi_0(x)^{-2/p} |f_j^{(k)}(x)|$$

on $I_{j+1} \setminus I_j$ with $j \in \mathbb{N}_{\geq 1}$. Recall that $f_j = f$ on $I_{j+1} \setminus I_j$. In view of Theorem 4.3 we have

$$|f_j^{(k)}(x)| \leq \int \frac{1 - \frac{\epsilon_0}{i \lambda}}{\lambda} \pi_{\epsilon}(d\lambda)$$

where

$$\left| 1 - \frac{\epsilon_0}{i \lambda} \right| \pi_{\epsilon}(d\lambda) = \frac{1}{\sqrt{\Delta^2 + \epsilon_0^2 \Delta^2}} \left[ \frac{1}{\sqrt{\Delta^2 - (1 - \epsilon_0^2)\Delta^2}} \right] \mathbb{1}_{-\sqrt{1-\epsilon_0^2}, \sqrt{1-\epsilon_0^2}}(\lambda) d\lambda.$$

By knowing about the asymptotic behavior of the $c$-function (see Section 2), one comes to

$$|f_j^{(k)}(x)| \leq c_1 \varphi_1 \sqrt{\Delta^2 - (1 - \epsilon_0^2)\Delta^2} \tau_{\epsilon,0}^{(0)}(h_j),$$

for some integer $t_1 > 0$. It follows that

$$\sup_{x \in I_{j+1} \setminus I_j} (|x| + 1)^t \varphi_0(x)^{-2/p} |f_j^{(k)}(x)| \leq c_2^{-1} j^{t_1 + 1} \epsilon_0^{j^{t_1 + 1} + \sqrt{1-\epsilon_0^2}} \tau_{\epsilon,0}^{(0)}(h_j).$$

Recall that the two seminorms $\tau_{t,\epsilon}^{(\theta,\rho,\alpha)}$ (see (8.25)) and $\tau_{t,\epsilon}^{(\theta,\rho,\alpha)}$ (see (8.26)) are equivalent on $\mathcal{S}(\mathbb{R}_{+},\epsilon)$. Since $h_j = \mathcal{F}_{euc}(g_j)$, it follows that

$$(1 + \lambda)^n h_j(\lambda) = \sum_{\ell=0}^{n} \binom{n}{\ell} \lambda^{\ell} \mathcal{F}_{euc}(g_j)(\lambda).$$

Thus

$$\tau_{\epsilon,0}^{(0)}(h_j) \leq \sum_{\ell=0}^{n} \binom{n}{\ell} \int \left| g_j^{(\ell)}(y) \right| dy$$

$$ \leq c_3 \sum_{\ell=0}^{n} \sup_{y \in \mathbb{R}} \left| g_j^{(\ell)}(y) \right|$$

$$ = c_3 \sum_{\ell=0}^{n} \sup_{w \in \mathbb{R}} \sup_{y \in \mathbb{R}} \left| g_j^{(\ell)}(wy) \right|.$$
Now one uses the Leibniz rule to compute the derivatives of $g_j = \omega_j g$. Since $\omega_j = 0$ on $I_{j-1}$ and is bounded, together with all its derivatives uniformly in $j$, then we have

$$\tilde{\tau}^{(0)}_{t_i,0}(h_j) \leq c_4 \sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R} \setminus I_{j-1}} (y + 1)^2 |g^{(\ell)}(wy)|.$$ 

Hence

$$j^{s+1} e^{\rho_i(j+1-i)} \tilde{\tau}^{(0)}_{t_i,0}(h_j) \leq c_5 \sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R} \setminus I_{j-1}} (y + 1)^{s+3} e^{\rho_i(j+1-i)} |g^{(\ell)}(wy)|.$$ 

Recall that $g(x) = \mathcal{F}^{-1}_{\text{eu}}(h)(x)$, where $\mathcal{F}_{\text{eu}}$ is the Euclidean Fourier transform and $h \in PW(\mathbb{C})$. By Cauchy’s integral theorem, it is known that

$$p(u) e^{\rho u} g^{(\ell)}(u) = \text{cst} \int_{\mathbb{R}} p(i\partial_{\rho}) [(i\lambda - \alpha)^{\ell} h(\lambda + i\alpha)] e^{i\rho u} d\lambda,$$

for any polynomial $p \in \mathbb{R}[u]$. Hence,

$$\sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R} \setminus I_{j-1}} (y + 1)^{s+3} e^{\rho_i(j+1-i)} |g^{(\ell)}(wy)|$$

$$\leq c_6 \sum_{\ell=0}^{t_1} \sup_{|\text{Im} \lambda| \leq \omega} |(\lambda + 1)^{s+1} |h^{(\ell)}(\lambda)|$$

$$= c_6 \sum_{\ell=0}^{t_1} \tilde{\tau}^{(0)}_{t_2,\ell}(h),$$

for some integer $t_2 > 0$.

It remains for us to estimate the function

$$x \mapsto (|x| + 1)^s \varphi_0(x)^{-2/p} |f^{(k)}(x)|$$

on $I_1 = [-1, 1]$. First, it is not hard to prove that for $|x| \leq 1$, there is a positive constant $c$ and an integer $m_k \geq 1$ such that

$$\left| \frac{\partial^k}{\partial x^k} \Psi_{A,\varepsilon}(\lambda, x) \right| \leq c \frac{(|\lambda| + 1)^{m_k}}{|i\lambda - \varepsilon Q|} \varphi_0(x) \quad (8.27)$$

for $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \sqrt{1 - \varepsilon^2 Q}$. Now, arguing as above, we have

$$|f^{(k)}(x)| \leq c_1 \varphi_0(x) \int_{\mathbb{R}} |h(\lambda)| \frac{(|\lambda| + 1)^{m_k}}{|i\lambda - \varepsilon Q|} \left| 1 - \frac{\varepsilon Q}{i\lambda} \right| \pi_\varepsilon(d\lambda).$$

Since $I_1$ is compact, it follows that

$$\sup_{x \in I_1} \left( |x| + 1 \right)^s \varphi_0(x)^{-2/p} |f^{(k)}(x)| \leq c_2 \int_{\mathbb{R}} |h(\lambda)| \frac{(|\lambda| + 1)^{m_k}}{|i\lambda - \varepsilon Q|} \left| 1 - \frac{\varepsilon Q}{i\lambda} \right| \pi_\varepsilon(d\lambda)$$

$$\leq c_3 \tilde{\tau}^{(0)}_{t_3,0}(h),$$

for some integer $t_3 > 0$.

This finishes the proof of Lemma 8.11. □
In summary, we have proved:

**Theorem 8.12.** Let \(-1 \leq \varepsilon \leq 1\) and \(0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}\). Then the Fourier transform \(\mathcal{F}_{A,\varepsilon}\) is a topological isomorphism between \(\mathcal{S}_p(\mathbb{R})\) and \(\mathcal{S}(\mathbb{C}_p,\varepsilon)\).

9. **Pointwise Multipliers**

For \(-1 \leq \varepsilon \leq 1\) and \(0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}\), denote by \(\mathcal{S}_p'(\mathbb{R})\) and by \(\mathcal{S}'(\mathbb{C}_p,\varepsilon)\) the topological dual spaces of \(\mathcal{S}_p(\mathbb{R})\) and \(\mathcal{S}(\mathbb{C}_p,\varepsilon)\), respectively.

Let \(f\) be a Lebesgue measurable function on \(\mathbb{R}\) such that

\[
\int_{\mathbb{R}} |f(x)|\varphi_0(x)^{2/p}(|x| + 1)^{-\ell} A(x) dx < \infty
\]

for some \(\ell \in \mathbb{N}\). Then the functional \(T_f\) defined on \(\mathcal{S}_p(\mathbb{R})\) by

\[
\langle T_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(-x)A(x) dx, \quad \phi \in \mathcal{S}_p(\mathbb{R})
\]

is in \(\mathcal{S}_p'(\mathbb{R})\). Indeed,

\[
|\langle T_f, \phi \rangle| \leq c_{\ell,0}^{(p)}(\phi) \int_{\mathbb{R}} |f(x)|\varphi_0(x)^{2/p}(|x| + 1)^{-\ell} A(x) dx < \infty.
\]

Further, since \(p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \leq 2\), the Schwartz space \(\mathcal{S}_p(\mathbb{R})\) can be seen as a subspace of \(\mathcal{S}_p'(\mathbb{R})\) by identifying \(f \in \mathcal{S}_p(\mathbb{R})\) with \(T_f \in \mathcal{S}_p'(\mathbb{R})\).

Now let \(h\) be a measurable function on \(\mathbb{R}\) such that

\[
\int_{\mathbb{R}} |h(\lambda)(|\lambda| + 1)^{-\ell}| \left| 1 - \frac{\varepsilon \lambda}{|\lambda|} \right| \pi_{\varepsilon}(d\lambda) < \infty
\]

for some \(\ell \in \mathbb{N}\). Here \(\pi_{\varepsilon}(d\lambda)\) denotes the Plancherel measure \((8.2)\),

\[
\pi_{\varepsilon}(d\lambda) = \frac{|\lambda|}{\sqrt{1^2 - (1 - \varepsilon^2)\varrho^2}} \left[ c(\sqrt{1^2 - (1 - \varepsilon^2)\varrho^2})^2 \right]^{-\frac{1}{2}} \mathbb{1}_{[-1 - \varepsilon\varrho, 1 - \varepsilon\varrho]}(\lambda) d\lambda,
\]

where \(c\) is the Harish-Chandra’s function associated with the operator \(\Delta\) (see Section 2).

Then the functional \(\mathcal{T}_h\) defined on \(\mathcal{S}(\mathbb{C}_p,\varepsilon)\) by

\[
\langle \mathcal{T}_h, \psi \rangle = \int_{\mathbb{R}} h(\lambda)\psi(\lambda) \left(1 - \frac{\varepsilon \varrho}{|\lambda|} \right) \pi_{\varepsilon}(d\lambda), \quad \psi \in \mathcal{S}(\mathbb{C}_p,\varepsilon)
\]

is in the dual space \(\mathcal{S}'(\mathbb{C}_p,\varepsilon)\). In fact,

\[
|\langle \mathcal{T}_h, \psi \rangle| \leq c \tau_{0,0}^{(p)}(\psi) \int_{\mathbb{R}} |h(\lambda)(|\lambda| + 1)^{-\ell}| \left| 1 - \frac{\varepsilon \varrho}{|\lambda|} \right| \pi_{\varepsilon}(d\lambda) < \infty.
\]

Moreover, since \(|c(\mu)|^{-2} \sim |\mu|^{2\alpha+1}\) for \(|\mu|\) large (with \(\alpha > -1/2\)) and

\[
|c(\mu)|^{-2} \sim \begin{cases} |\mu|^2 & \text{for } |\mu| \ll 1 \text{ and } \varrho > 0, \\ |\mu|^{2\alpha+1} & \text{for } |\mu| \ll 1 \text{ and } \varrho = 0, \end{cases}
\]

it follows that the Schwartz space \(\mathcal{S}(\mathbb{C}_p,\varepsilon)\) can be identified with a subspace of \(\mathcal{S}'(\mathbb{C}_p,\varepsilon)\).
For $T$ in $\mathcal{S}'(\mathbb{R})$, we define the distributional Fourier transform $\mathcal{F}_{A,e}(T)$ of $T$ on $\mathcal{S}(\mathbb{R}^d) = \mathcal{F}_{A,e}(\mathcal{S}(\mathbb{R}))$ by
\[
\langle \mathcal{F}_{A,e}(T), \mathcal{F}_{A,e}(\phi) \rangle = \langle T, \phi \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}).
\] (9.1)

That is,
\[
\langle \mathcal{F}_{A,e}(T), \psi \rangle = \langle T, \mathcal{F}_{A,e}^{-1}(\psi) \rangle, \quad \psi \in \mathcal{S}(\mathbb{R}).
\]

This definition is an extension of the Fourier transform on $\mathcal{S}(\mathbb{R})$ whenever $\mathcal{S}(\mathbb{R})$. Indeed, let $f \in \mathcal{S}(\mathbb{R})$ with $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}} \leq 2$. Applying Fubini’s theorem, then, for every $\phi \in \mathcal{S}(\mathbb{R})$, we have
\[
\langle \mathcal{F}_{A,e}(f), \mathcal{F}_{A,e}(\phi) \rangle = \int_{\mathbb{R}} \mathcal{F}_{A,e}(f)(\lambda) \mathcal{F}_{A,e}(\phi)(\lambda) \left( 1 - \frac{\varepsilon Q}{|\lambda|} \right) \pi_e(d\lambda) = \int_{\mathbb{R}} f(x) \left( \int_{\mathbb{R}} \mathcal{F}_{A,e}(\phi)(\lambda) \Psi_{A,e}(\lambda, -x) \left( 1 - \frac{\varepsilon Q}{|\lambda|} \right) \pi_e(d\lambda) \right) A(x) dx
\]
\[
= \int_{\mathbb{R}} f(x) \phi(-x) A(x) dx
\]
\[
= \langle T_f, \phi \rangle.
\]

Hence $\mathcal{F}_{A,e}(T_f) = \mathcal{F}_{A,e}(f)$.

A function $\psi$ defined on $\mathbb{C}_{p,e}$ is called a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,e})$ if the mapping $\phi \mapsto \psi\phi$ is continuous from $\mathcal{S}(\mathbb{C}_{p,e})$ into itself. The following statement comes from [4, Proposition 3.2], with changes appropriate to our setting.

**Lemma 9.1.** Let $\psi$ be a function defined on $\mathbb{C}_{p,e}$. Then, $\psi$ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,e})$ if and only if $\psi$ satisfies the following three conditions:

(i) $\psi$ is holomorphic in the interior of $\mathbb{C}_{p,e}$.

(ii) For every $t \in \mathbb{N}$, the derivatives $\psi^{(t)}$ extend continuously to $\mathbb{C}_{p,e}$.

(iii) For every $t \in \mathbb{N}$, there exists $n_t \in \mathbb{N}$, such that
\[
\sup_{\lambda \in \mathbb{C}_{p,e}} (|\lambda| + 1)^{-n_t} |\psi^{(t)}(\lambda)| < \infty.
\] (9.2)

**Theorem 9.2.** Suppose that $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho = 0$, and $\frac{2}{2+\sqrt{1-\varepsilon^2}} \leq p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho > 0$. If $T \in \mathcal{S}'(\mathbb{R})$ such that $\psi := \mathcal{F}_{A,e}(T)$ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,e})$, then for any $s \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and continuous functions $f_m$ defined on $\mathbb{R}$, $m = 0, 1, \ldots, \ell$, such that
\[
T = \sum_{m=0}^{\ell} \Lambda_{A,e}^m f_m
\]
and, for every such $m$,
\[
\sup_{x \in \mathbb{R}} (|x| + 1)^s |\varphi_0(x)\varrho^{\frac{2}{p} + \sqrt{1-\varepsilon^2}}| |f_m(x)| < \infty.
\] (9.3)

Here $\Lambda_{A,e}$ is the differential-reflection operator (3.1).
Proof. It is assumed that \( \psi = \mathcal{F}_{A,e}(T) \) is a pointwise multiplier of \( \mathcal{J}(\mathbb{C}_{p,e}) \). Then by Lemma 9.1, for all \( t \in \mathbb{N} \) there is an integer \( n_t \in \mathbb{N} \) such that
\[
\sup_{\lambda \in \mathbb{C}_{p,e}} (|t| + 1)^{-n_t} |\psi^{(t)}(\lambda)| < \infty.
\] (9.4)

Fix \( s \in \mathbb{N} \) and consider an integer \( \ell \) that will be later specified. Define the function \( \kappa \) on \( \mathbb{C}_{p,e} \) by
\[
\kappa(\lambda) = (i\lambda + \varrho + 1)^{-\ell} \psi(\lambda).
\]

In view of our assumption on \( p \), the function \( \kappa \) satisfies the first and the second conditions in the definition of the space \( \mathcal{J}(\mathbb{C}_{p,e}) \). Further, since \( |\Psi_{A,e}(\lambda, x)| \leq \sqrt{2} \) for all \( \lambda \in \mathbb{R} \), we have
\[
|\mathcal{F}_{A,e}^{-1}(\kappa)(x)| := \left| \int_{\mathbb{R}} \kappa(\lambda) \Psi_{A,e}(\lambda, x) \left(1 - \frac{\varepsilon_0}{i\lambda} \right) \pi_e(d\lambda) \right| \leq c_1 \int_{\mathbb{R}} \left| \kappa(\lambda) \right| \left| 1 - \frac{\varepsilon_0}{i\lambda} \right| \pi_e(d\lambda),
\]
where
\[
\left| 1 - \frac{\varepsilon_0}{i\lambda} \right| \pi_e(d\lambda) = \frac{\sqrt{\lambda^2 + \varepsilon_0^2}}{\sqrt{(1 + \varepsilon_0^2)}^2} \frac{1}{\sqrt{(1 + \varepsilon_0^2)^2}} \int_{\{x : \lambda \in \mathbb{R}\} - \sqrt{1 - \varepsilon_0} \sqrt{1 - \varepsilon_0} \pi_e(\lambda)} d\lambda.
\]

Thus, in view of the estimate (9.4) and the behavior of \( |c(\mu)|^{-2} \) for small and large \( |\mu| \), it follows that \( \mathcal{F}_{A,e}^{-1}(\kappa)(x) \) exists for all \( x \in \mathbb{R} \) provided that \( \ell > n_0 + 2\alpha + 2 \). Moreover, for all \( \phi \in \mathcal{J}(\mathbb{R}) \), Fubini’s theorem leads to
\[
\int_{\mathbb{R}} \phi(-x) \mathcal{F}_{A,e}^{-1}(\kappa)(x) A(x) dx = c_1 \int_{\mathbb{R}} \phi(-x) \left( \int_{\mathbb{R}} \kappa(\lambda) \Psi_{A,e}(\lambda, x) \left(1 - \frac{\varepsilon_0}{i\lambda} \right) \pi_e(d\lambda) \right) A(x) dx
\]
\[
= c_1 \int_{\mathbb{R}} \kappa(\lambda) \left( \int_{\mathbb{R}} \phi(-x) \Psi_{A,e}(\lambda, x) A(x) dx \right) \left(1 - \frac{\varepsilon_0}{i\lambda} \right) \pi_e(d\lambda)
\]
\[
= c_1 \int_{\mathbb{R}} \kappa(\lambda) \mathcal{F}_{A,e}(\phi)(\lambda) \left(1 - \frac{\varepsilon_0}{i\lambda} \right) \pi_e(d\lambda).
\]

It follows that the inverse Fourier transform \( \mathcal{F}_{A,e}^{-1}(\kappa) \) of \( \kappa \) as an element of \( \mathcal{J}(\mathbb{C}_{p,e}) \) concurs with the classical Fourier transform of \( \kappa \). Further
\[
T = \mathcal{F}_{A,e}^{-1}((i\lambda + \varrho + 1)^{\ell} \kappa) = \sum_{m=0}^{\ell} \binom{\ell}{m} (\varrho + 1)^{\ell-m} \Lambda_{A,e}^{m} \mathcal{F}_{A,e}^{-1}(\kappa) = \sum_{m=0}^{\ell} \Lambda_{A,e}^{m} f_m.
\]

It remains for us to show that, given \( s \in \mathbb{N} \), the functions \( f_m \) satisfy (9.3), provided that \( \ell \) is large enough. To do so, we will use a similar approach to that in the proof of Lemma 8.11.

Denote by \( \xi := \mathcal{F}_{A,e}^{-1}(\kappa) \) and by \( g := \mathcal{F}_{euc}^{-1}(\kappa) \), where \( \mathcal{F}_{euc} \) denotes the Euclidean Fourier transform. Observe that if \( \ell \) is large enough, then \( g \) is well defined. For \( j \in \mathbb{N}_{\geq 1} \), let \( \omega_j \in C^\infty(\mathbb{R}) \) such that \( \omega_j = 0 \) on \( I_{j-1} := [-j+1, j-1] \) and \( \omega_j = 1 \) outside of \( I_j \). We shall assume that \( \omega_j \) together with all its derivatives are bounded, uniformly in \( j \).

We set \( g_j := \omega_j g \), and define \( \kappa_j := \mathcal{F}_{euc}(g_j) \) and \( \xi_j := \mathcal{F}_{A,e}^{-1}(\kappa_j) \). Since \( \omega_j = 1 \) outside of \( I_j \), it follows that \( g_j - g = 0 \) outside of \( I_j \). That is \( \text{supp}(g_j - g) \subset I_j \). Using the support
conservation property from the proof of Lemma 8.11, we deduce that $\xi$ may differ from $\xi_j$ only inside $I_j$. Now, we will estimate the function

$$x \mapsto (|x| + 1)^t \varphi_0(x)^{-\frac{1}{p} + \frac{2}{p} \sqrt{1 - \epsilon^2}} \xi(x),$$

(9.5)

first on $I_1$ and next on $I_{j+1} \setminus I_j$ for $j \in \mathbb{N}_{\geq 1}$.

We claim that $|\Psi_{A\varepsilon}(\lambda, x)| \leq c_2(|\lambda| + 1)\varphi_0(x)$ for $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \sqrt{1 - \epsilon^2}$. Indeed, as $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \sqrt{1 - \epsilon^2}$, it follows that $\mu_\varepsilon \in \mathbb{R}$. Thus, the claim follows from the superposition (3.3) of $\Psi_{A\varepsilon}(\lambda, x)$ and the facts that $|\varphi_{\mu_\varepsilon}(x)| \leq \varphi_0(x)$ and $|\varphi_{\mu_\varepsilon}(x)| \leq c (\mu_\varepsilon^2 + \hat{g}^2)\varphi_0(x)$ (see Lemma 2.24 and 2.25).

From the claim above we have

$$|\xi(x)| \leq c_3 \int_\mathbb{R} |\varphi_0(x)| |\Psi_{A\varepsilon}(\lambda, x)| \left| 1 - \frac{\varepsilon_0}{\lambda} \right| \pi_x(d\lambda)$$
$$\leq c_4 \varphi_0(x) \int_\mathbb{R} |\varphi_0(x)| |\Psi_{A\varepsilon}(\lambda, x)| \left| 1 - \frac{\varepsilon_0}{\lambda} \right| \pi_x(d\lambda).$$

Since $I_1$ is compact, we deduce that for every $s \in \mathbb{N}$

$$\sup_{x \in I_1} (|x| + 1)^s \varphi_0(x)^{-\frac{1}{p} + \frac{2}{p} \sqrt{1 - \epsilon^2}} |\xi(x)| < \infty$$

whenever $\ell > n_0 + 2t + 3$. Here the parameter $n_0$ comes from (9.4). Now we consider the estimate of the function (9.5) on $I_{j+1} \setminus I_j$ for $j \in \mathbb{N}_{\geq 1}$. Recall that $\xi = \xi_j$ outside of $I_j$.

Arguing as above, we obtain

$$|\xi_j(x)| \leq c_5 \varphi_0(x) \sup_{\lambda \in \mathbb{R} \setminus [-\sqrt{1 - \epsilon^2}, \sqrt{1 - \epsilon^2}]} |(\lambda + 1)^t \kappa_j(\lambda)|$$

for some integer $t_j > 2t + 3$. It follows that

$$\sup_{x \in \mathbb{R} \setminus [-\sqrt{1 - \epsilon^2}, \sqrt{1 - \epsilon^2}]} |(\lambda + 1)^t \kappa_j(\lambda)|.$$
In fact, starting from $g = \mathcal{F}_{\text{eu}}^{-1}(\kappa)$, we obtain

$$g^{(q)}(x) = c \int_{\mathbb{R}} \kappa(\ell)(i\ell)^q e^{i\ell x} d\ell. \quad (9.9)$$

Thus, if $\ell > n_0 + t_1 + 1$ then by Riemann–Lebesgue lemma for the Euclidean Fourier transform, $g^{(q)}(x) \to 0$ as $|x| \to \infty$. Thus (9.8) holds true. Now, in view of (9.8) we may rewrite (9.7) as

$$(\ell + 1)^{i} \kappa_j(\ell) = \sum_{r=0}^{n_1} \sum_{q=0}^{t_1} c_{q,r} \int_{\mathbb{R}} g^{(q)}(x) \omega_j^{(r-q)}(x) e^{i\ell x} dx.$$ \[(\ell + 1)^{i} \kappa_j(\ell) = \sum_{r=0}^{n_1} \sum_{q=0}^{t_1} c_{q,r} \int_{\mathbb{R}} g^{(q)}(x) \omega_j^{(r-q)}(x) e^{i\ell x} dx. \quad (9.7)\]

Recall that the function $\omega_j$ vanishes on $I_{j-1}$ and is bounded, together with all its derivatives, uniformly in $j$. Therefore,

$$|(\ell + 1)^{i} \kappa_j(\ell)| \leq c \sum_{q=0}^{t_1} \int_{\mathbb{R}\setminus I_{j-1}} |g^{(q)}(x)| dx$$

$$\leq c \sum_{q=0}^{t_1} \sup_{x \in \mathbb{R}\setminus I_{j-1}} (|x| + 1)^2 |g^{(q)}(x)|.$$ \[(9.6)\]

This finishes the proof of our claim (9.6).

It follows that

$$\int_{\mathbb{R}} e^{(\frac{2}{\lambda} - 1 - \frac{t}{\lambda} - x)} \psi_j \sup_{x \in \mathbb{R}\setminus I_{j-1}} |(\ell + 1)^{i} \kappa_j(\ell)|$$

$$\leq c \sum_{q=0}^{t_1} \sup_{x \in \mathbb{R}\setminus I_{j-1}} (x + 1)^{i+2} e^{(\frac{2}{\lambda} - 1 - \frac{t}{\lambda} - x)} |\psi_j^{(q)}(x)|.$$ \[(9.11)\]

Next we shall prove that the right hand is finite. Assume first that $\rho = 0$. By (9.9) we have

$$(x + 1)^{i+2} \psi_j^{(q)}(w x) = \sum_{r=0}^{s+2} c_{q,r} \int_{\mathbb{R}} \kappa(\ell) \lambda^q \partial^{(r)}_x e^{i\lambda x} d\ell. \quad (9.10)$$

We claim that

$$(\kappa(\ell) \lambda^q)^{(r)} \to 0 \quad \text{as} \quad |\ell| \to +\infty \quad (9.11)$$

provided that $\ell$ is large enough. Indeed, this claim follows immediately from the fact that

$$(\kappa(\ell) \lambda^q)^{(r)} = \sum_{a=0}^{r} \sum_{b=0}^{a} c_{a,b} \lambda^{q-r+a} \kappa^{(a)}(\ell) \quad (\text{with } r - a \leq q)$$

$$= \sum_{a=0}^{r} \sum_{b=0}^{a} c_{a,b} \lambda^{q-r+a} (i\lambda + 1)^{-\ell+a+b} \psi^{(b)}(\lambda), \quad (9.12)$$

together with the fact that $\psi$ satisfies (9.4). Thus, by (9.11) we may rewrite (9.10) as

$$(x + 1)^{i+2} \psi_j^{(q)}(w x) = \sum_{r=0}^{s+2} c_{q,r} \int_{\mathbb{R}} (\kappa(\ell) \lambda^q)^{(r)} e^{i\lambda x} d\ell. \quad (9.13)$$
Using again the fact that $\psi$ satisfies (9.4) together with the double sum (9.12), it follows from (9.13) that for $\varrho = 0$

$$\sup_{w \in \{\pm 1\}} \sup_{x \in \mathbb{R}^+ \setminus I_{j-1}} (x + 1)^{s+2} |g^{(q)}(wx)| < \infty$$

provided that $\ell$ is large enough.

Now assume that $\varrho > 0$. Since $g = F^{-1}_{\text{eu}}(\kappa)$ and $\kappa$ is holomorphic in the interior of $\mathbb{C}_{p,\varepsilon}$, Cauchy’s integral theorem gives

$$p(u) e^{iu} g^{(q)}(u) = \text{cst} \int_{\mathbb{R}} p(i \partial_\lambda) \left((i \lambda - \alpha)\kappa(\lambda + i \alpha)\right) e^{i\lambda u} d\lambda,$$

with $p(x) = (x + 1)^{s+2}$ and $\alpha = \left(\frac{2}{p} - 1 - \sqrt{1 - \varepsilon^2}\right) \varrho$. The same argument as above implies that

$$\sup_{w \in \{\pm 1\}} \sup_{x \in \mathbb{R}^+ \setminus I_{j-1}} (x + 1)^{s+2} e^{\left(\frac{2}{p} - 1 - \sqrt{1 - \varepsilon^2}\right) \varrho} |g^{(q)}(wx)| < \infty$$

provided that $\ell$ is large enough.

Putting the pieces together we conclude that

$$\sup_{x \in I_{j+1} \setminus I_j} (|x| + 1)^{s} \varphi_0(x)^{-\frac{2}{p} + \sqrt{1 - \varepsilon^2}} |\xi_j(x)| < \infty$$

for $\ell$ large enough. □

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