Dirac stars supported by nonlinear spinor fields

Vladimir Dzhunushaliev\textsuperscript{1,2,3,4,*} and Vladimir Folomeev\textsuperscript{1,3,4,†}

\textsuperscript{1} Institute of Experimental and Theoretical Physics,  
  Al-Farabi Kazakh National University, Almaty 050040, Kazakhstan  
\textsuperscript{2} Department of Theoretical and Nuclear Physics,  
  Al-Farabi Kazakh National University, Almaty 050040, Kazakhstan  
\textsuperscript{3} Institute of Physicotechnical Problems and Material Science of the NAS  
  of the Kyrgyz Republic, 263 a, Chui Street, Bishkek 720071, Kyrgyzstan  
\textsuperscript{4} Institut für Physik, Universität Oldenburg, Postfach 25 03  
  D-26111 Oldenburg, Germany

We study configurations consisting of a gravitating spinor field $\psi$ with a nonlinearity of the type $\lambda (\bar{\psi}\psi)^2$. To ensure spherical symmetry of the configurations, we use two spin-$\frac{1}{2}$ fields forming a spin singlet. For such systems, we find regular stationary asymptotically flat solutions describing compact objects. For negative values of the coupling constant $\lambda$, it is shown that even when $|\lambda| \ll 1$, one can obtain configurations with masses comparable to the Chandrasekhar mass. It enables us to speak of an astrophysical interpretation of the obtained systems, regarding them as Dirac stars.

PACS numbers: 04.40.Dg, 04.40.--b, 04.40.Nr

Keywords: Nonlinear spinor fields, regular solutions, compact gravitating configurations

I. INTRODUCTION

In recent decades, various fundamental fields have achieved widespread use in a variety of cosmological and astrophysical applications. In particular, this applies both to modeling the present accelerated expansion of the Universe \cite{1} and to describing its early inflationary stage \cite{2}. For this purpose, various scalar (boson) fields with spin 0 are most frequently employed. Such fields are also widely used in modeling compact astrophysical strongly gravitating objects – boson stars \cite{3}.

However, there may exist gravitating objects consisting of fields with nonzero spin. They may be systems supported by fields with integer spin: Yang-Mills configurations \cite{4} (consisting of massless vector fields) or Proca stars \cite{5} (consisting of massive vector fields). In the case of spin-$\frac{1}{2}$ fields, the literature in the field offers both gravitating configurations with noninteracting spinor fields \cite{6,7} and objects supported by nonlinear fields. In particular, nonlinear spinor fields have been used in obtaining cylindrically symmetric solutions in Ref. \cite{8} (string-like configurations) and in Ref. \cite{9} (wormhole solutions), and also in a cosmological context in Refs. \cite{10,11,12,13,14}, where the role of spinor fields in the evolution of anisotropic universes described by the Bianchi type I, III, V, VI, VI\textsubscript{0} models or of an isotropic Friedmann-Robertson-Walker universe is studied. In turn, for spherically symmetric systems, localized regular solutions have been found in Refs. \cite{15,16}. The papers \cite{17,18} study models of universe filled with tachyon and fermion fields interacting through the Yukawa scalar field. In Ref. \cite{19}, a topologically nontrivial solution with a spinor field within the Einstein-Dirac theory has been obtained.

Configurations consisting of spinor fields are prevented from collapsing under their own gravitational fields due to the Heisenberg uncertainty principle. The distinctive feature of such systems is that since the spin of a fermion has an intrinsic orientation in space, a system consisting of a single spinor particle cannot possess spherical symmetry. In order to ensure the latter, one can take two fermions having opposite spin, i.e., consider two spinor fields. For each of such spinors, the energy-momentum tensors will not be spherically symmetric (due to the existence of nondiagonal components), but their sum will give a tensor compatible with spherical symmetry of the spacetime (see below in Sec. \textsuperscript{I\textsubscript{I}}).

In the case of configurations supported by noninteracting spinor fields, their total mass $M \sim M^2_p/\mu$, where $\mu$ is the spinor field mass, and it is generally much smaller than the Chandrasekhar mass, $M_{\text{Ch}} \sim M^2_p/\mu^2$. In the present paper, we consider the case of nonlinear spinor fields and show that in this case there is a possibility of increasing the total mass considerably.

The paper is organized as follows. In Sec. \textsuperscript{I\textsubscript{I}} we present the general-relativistic equations for the systems under consideration. These equations are solved numerically in Sec. \textsuperscript{I\textsubscript{II}} for different values of the coupling constant $\lambda$, \textsuperscript{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}
and the possibility of obtaining configurations with astrophysical masses of the order of the Chandrasekhar mass is demonstrated, even in the case when $|\lambda| \ll 1$. Finally, in Sec. IV we summarize and discuss the obtained results.

II. FORMULATING THE PROBLEM AND GENERAL EQUATIONS

We consider compact gravitating configurations consisting of a spinor field and modeled within the framework of Einstein’s general relativity. The corresponding action for such a system can be represented in the form [the metric signature is $(+,−,−,−)$]

$$S = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R + S_{sp},$$

where $G$ is the Newtonian gravitational constant, $R$ – the scalar curvature, and $S_{sp}$ denotes the action of the spinor field. This action is obtained from the Lagrangian for the spinor field $\psi$ with the mass $\mu$,

$$L_{sp} = \frac{ihc}{2} \left( \bar{\psi} \gamma^\mu \psi_{;\mu} - \bar{\psi} \gamma^\mu \gamma^\alpha \psi \right) - \mu c^2 \bar{\psi} \psi - F(S),$$

which contains the covariant derivatives $\psi_{;\mu} = [\partial_\mu + \frac{i}{2} \gamma^\alpha \omega_{\alpha\mu}] \psi$, and $\gamma^a$ are the Dirac matrices in the standard representation in flat space [see, e.g., Ref. [20], formula (7.27)]. In turn, the Dirac matrices in curved space, $\gamma^\mu = e^a_\mu \gamma^a$, are obtained using the tetrad $e^a_\mu$, and $\omega_{ab\mu}$ is the spin connection [for its definition, see Ref. [20], formula (7.135)]. Finally, this Lagrangian contains an arbitrary nonlinear term $F(S)$, where the invariant $S$ can depend on $(\bar{\psi} \psi)$, $(\bar{\psi} \gamma^\mu \psi)$ or $(\bar{\psi} \gamma^\mu \gamma^\alpha \psi)$. Here, we will study the case of the simplest nonlinearity $F(S) \propto (\bar{\psi} \psi)^2$.

Varying the action (1) with respect to the metric and the spinor field, we derive the Einstein equations and the Dirac equation in curved spacetime:

$$R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R = \frac{8\pi G}{c^4} T^\nu_\mu,$$

$$ih\gamma^\mu \psi_{;\mu} - \mu c^2 \bar{\psi} \psi = 0,$$

$$ih\bar{\psi}_{;\mu} \gamma^\mu + \mu c^2 \bar{\psi} \psi = 0.$$  

The right-hand side of Eq. (3) contains the spinor field energy-momentum tensor $T^\nu_\mu$, which can be represented (already in a symmetric form) as

$$T^\nu_\mu = \frac{ihc}{4} \gamma^\mu^\rho \left[ \bar{\psi} \gamma_\rho \psi_{;\rho} + \bar{\psi} \gamma_\rho \psi_{;\mu} - \bar{\psi} \gamma_\rho \gamma^\nu \psi_{;\nu} + \bar{\psi} \gamma_{\rho \nu} \psi_{;\nu} \right] - \delta^\nu_\mu L_{sp}.$$  

Next, taking into account the Dirac equations (4) and (5), the Lagrangian (2) takes the form

$$L_{sp} = -F(S) + \frac{1}{2} \left( -\frac{\partial F}{\partial \bar{\psi}} + \frac{\partial F}{\partial \psi} \right).$$

For our purpose, we choose the nonlinear term appearing in this Lagrangian in a simple power-law form

$$F(S) = -\frac{k}{k+1} \lambda (\bar{\psi} \psi)^{k+1},$$

where $k, \lambda$ are some free parameters. Below, we take $k = 1$ to yield

$$F(S) = -\frac{\lambda}{2} (\bar{\psi} \psi)^2.$$  

[Concerning the physical meaning of the constant $\lambda$, see Eq. (12) below.]

Since in the present paper we consider only spherically symmetric configurations, it is convenient to choose the spacetime metric in the form

$$ds^2 = N(r)\sigma^2(r)d(x^0)^2 - \frac{dr^2}{N(r)} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).$$  

where \( N(r) = 1 - 2Gm(r)/(c^2r) \), and the function \( m(r) \) corresponds to the current mass of the configuration enclosed by a sphere with circumferential radius \( r \); \( x^0 = ct \) is the time coordinate.

In order to describe the spinor field, it is necessary to choose the corresponding ansatz for \( \psi \) compatible with the spherically symmetric line element \( \text{(10)} \). This can be done as follows (see, e.g., Refs. 7, 21, 22):

\[
\psi^T = 2 e^{i\frac{\omega}{r}} \left\{ \begin{array}{c} (0) \\ (-g) \\ (g) \end{array} \right\} (i f \sin \theta e^{-i\varphi} - if \cos \theta) + (i f \sin \theta e^{i\varphi} - if \cos \theta),
\]

where \( E \) is the energy associated with oscillations of the spinor field, \( f(r) \) and \( g(r) \) are two real functions. This ansatz ensures that the spacetime of the system under consideration remains static. Here, each row: (a) describes a fermion with spin 1/2, and these two fermions have opposite spins; (b) corresponds to the spinor ansatz of Ref. \( 6 \); (c) is related to the spinor ansatz of Ref. \( 7 \) by the expression \( \psi^{[7]} = S\psi_i \), where \( i = 1, 2 \) is the row number from \( \text{(10)} \). The matrix \( S \) is given by the expression

\[
S = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} e^{i(\varphi/2)} & -ie^{-i(\varphi/2)} & 0 \\ e^{-i(\varphi/2)} & ie^{-i(\varphi/2)} & 0 \\ 0 & 0 & e^{-i(\varphi/2)} \\ 0 & 0 & e^{i(\varphi/2)} \end{array} \right).
\]

Thus the ansatz \( \text{(10)} \) describes two Dirac fields, and for each of them the energy-momentum tensor is not spherically symmetric, but their sum yields a spherically symmetric energy-momentum tensor.

To see the physical meaning of the arbitrary constant \( \lambda \) appearing in \( \text{(3)} \), let us write down the energy density \( \varepsilon \) of the spinor field \( (\text{(9)}) \) component of the energy-momentum tensor \( \text{(8)} \), with account taken of Eqs. \( \text{(9)} - \text{(11)} \):

\[
\varepsilon \equiv T_{0}^{0} = \frac{8E}{\sigma \sqrt{N}} \left( f^2 + g^2 \right) + 32\lambda \left( f^2 - g^2 \right)^2.
\]

It is seen from this expression that \( \lambda > 0 \) corresponds to the case of repulsion and \( \lambda < 0 \) – to the case of attraction.

Now, substituting the ansatz \( \text{(10)} \) [by multiplying each row of \( \text{(10)} \) by the matrix \( \text{(11)} \) and the metric \( \text{(9)} \) into the field equations \( \text{(3)} \) and \( \text{(4)} \)], we have

\[
\begin{align*}
\tilde{f}' + \left( \frac{N'}{4N} + \frac{\sigma'}{2\sigma} + \frac{1}{x} \left( 1 + \frac{1}{\sqrt{N}} \right) \right) \tilde{f} + \left( \frac{1}{\sqrt{N}} - \frac{E}{\sigma N} + 8\lambda \frac{f^2 - g^2}{\sqrt{N}} \right) \tilde{g} &= 0, \\
\tilde{g}' + \left( \frac{N'}{4N} + \frac{\sigma'}{2\sigma} + \frac{1}{x} \left( 1 - \frac{1}{\sqrt{N}} \right) \right) \tilde{g} + \left( \frac{1}{\sqrt{N}} + \frac{E}{\sigma N} + 8\lambda \frac{f^2 - g^2}{\sqrt{N}} \right) \tilde{f} &= 0, \\
\tilde{m}' &= 8x^2 \left[ \frac{E^2}{\sigma N} \frac{f^2 + g^2}{\sigma N} + 4\lambda \left( f^2 - g^2 \right)^2 \right], \\
\frac{\sigma'}{\sigma} &= \frac{8x}{\sqrt{N}} \left[ \frac{E^2}{\sigma N} \frac{f^2 + g^2}{\sigma N} + \frac{E}{\sigma N} \frac{f^2 - g^2}{\sigma N} \right],
\end{align*}
\]

where the prime denotes differentiation with respect to the radial coordinate. Here, Eqs. \( \text{(15)} \) and \( \text{(16)} \) are the \((\text{0})\) and \((\text{1})\) components of the Einstein equations, respectively. The above equations are written in terms of the following dimensionless variables:

\[
x = r/\lambda_c, \quad \tilde{E} = \frac{E}{\mu c^2}, \quad \tilde{f}, \tilde{g} = \sqrt{4\pi \lambda^3/3} \frac{\mu}{M_p} f, g, \quad \tilde{m} = \frac{\mu}{M_p^2} m, \quad \tilde{\lambda} = \frac{1}{4\pi \lambda^2 \mu c^2} \left( \frac{M_p}{\mu} \right)^2 \lambda,
\]

where \( M_p \) is the Planck mass and \( \lambda_c = \hbar/\mu c \) is the Compton length; \( N = 1 - 2\tilde{m}/x \). Note here that using the Dirac equations \( \text{(13)} \) and \( \text{(14)} \) one can eliminate the derivatives of \( \tilde{f} \) and \( \tilde{g} \) from the right-hand side of Eq. \( \text{(15)} \).

For numerical integration of the above equations, we take the following boundary conditions in the vicinity of the center:

\[
\tilde{g} \approx \tilde{g}_c + \frac{1}{2} \tilde{g}_2 x^2, \quad \tilde{f} \approx \tilde{f}_1 x, \quad \sigma \approx \sigma_c + \frac{1}{2} \sigma_2 x^2, \quad \tilde{m} \approx \frac{1}{6} \tilde{m}_3 x^3,
\]

where the index “c” denotes central values of the corresponding variables. The expansion coefficients \( \tilde{f}_1, \tilde{m}_3, \sigma_2, \tilde{g}_2 \) can be found from the set of equations \( \text{(13)} - \text{(16)} \). In turn, the expansion coefficients \( \sigma_c, \tilde{g}_c \), and also the parameter \( \tilde{E} \), are arbitrary. Their values are chosen in such a way that we have regular and asymptotically flat solutions with the functions \( N(x \to \infty), \sigma(x \to \infty) \to 1 \). In this case the asymptotic value of the function \( \tilde{m} \) will correspond to the ADM mass of the configurations under consideration.
FIG. 1: Dimensionless Dirac-star total mass $\bar{M}$ as a function of the energy $\bar{E}$ for $\bar{\lambda} = -100, -50, -20, 0$, and 20. The bold dots mark the positions of the configurations for which the graphs of Fig. 3 are plotted.

FIG. 2: Maximum Dirac-star masses as a function of $|\bar{\lambda}|$. The solid curve corresponds to the asymptotic relation (18).

III. NUMERICAL SOLUTIONS

Integration of the equations (13)-(16) is performed from the center of the configuration (at $x \approx 0$), where a particular value of $\bar{g}_c$ corresponding to the central density of the spinor field is specified, to some boundary point where the functions $\bar{g}, \bar{f}$ and their derivatives go to zero. Since with increasing distance the spinor fields decay exponentially fast as $\bar{g}, \bar{f} \sim e^{-\sqrt{1-\bar{E}^2}x}$, this point can be approximately regarded as some effective radius $x_{\text{eff}}$ of the configurations under investigation (by analogy with the case of boson stars [3]). Depending on the value of the central density of the spinor field, $x_{\text{eff}}$ is of the order of several hundreds for $\bar{g}_c \approx 0$, and it decreases down to $x_{\text{eff}} \sim 10$ for $\bar{g}_c \sim 1$, i.e., as the central density increases, the characteristic sizes of the configurations under consideration decrease. In turn, the energy of oscillations $\bar{E}$, starting from the value $\bar{E} \approx 1$ for $\bar{g}_c \approx 0$, at first decreases as $\bar{g}_c$ increases, and then can start growing again. This is illustrated in Fig. 1 where the dependences of the Dirac-star total mass $\bar{M}$ on $\bar{E}$ are shown for different values of the coupling constant $\bar{\lambda}$.

In plotting the above dependencies we have kept track of the sign of the binding energy (B.E.), which is defined as the difference between the energy of $N_f$ free particles, $\bar{E}_f = N_f \mu c^2$, and the total energy of the system, $\bar{E}_t = M c^2$, i.e., B.E. = $\bar{E}_f - \bar{E}_t$. Here, the total particle number $N_f$ is equal to the Noether charge $Q$ of the system, which is defined via the timelike component of the 4-current $j^\alpha = \bar{\psi} \gamma^\alpha \psi$ as $Q = \int j^t dV$, where the integration is performed over the element of spatial volume $dV = 4\pi N^{-1/2} r^2 dr$. In the dimensionless variables (17), we then have

$$N_f = Q = 8 \left( \frac{M_p}{\mu} \right)^2 \int_0^\infty \frac{\bar{f}^2 + \bar{g}^2}{\sqrt{N}} x^2 dx.$$ 

A necessary, albeit not sufficient, condition for energy stability is the positiveness of the binding energy. Therefore, since configurations with negative B.E. are certainly unstable, the graphs in Fig. 1 are plotted only to $\bar{E}$ for which the B.E. becomes equal to 0 (except the case of $\lambda = -100$ where the procedure of obtaining solutions is very difficult technically, and we could find them only to B.E. $\approx 0.21$; this corresponds to the very left point in the graph).

It is seen from Fig. 1 that for all $\lambda$ there is a maximum of the mass at some value of $\bar{E}$ (or $\bar{g}_c$). Such a behavior of the curves resembles the behavior of the corresponding dependencies “mass – central density (energy of oscillations)” for boson stars supported by a complex scalar field (see, e.g., Refs. [7, 24, 25]). In the case of boson stars, the presence of such a maximum corresponds to the boundary between configurations which are stable or unstable against linear perturbations [25]. Naively, one might expect that for the Dirac stars a similar situation will occur. But this issue requires special studies.
FIG. 3: Spinor fields \(g\) and \(f\) as functions of dimensionless radius \(x\) for \(\lambda = -100\) and \(\lambda = 0\). The dashed line shows the solution to Eqs. (19)-(21) with \(E/\sigma\) from the exact \(g_c\) = 0.0275, \(\lambda = -100\) model, scaled to \(\lambda = -100\).

FIG. 4: Dimensionless Dirac-star total mass \(\bar{M}\) as a function of \(E/\sigma\) for the limiting configurations described by Eqs. (19)-(21).

A. The limiting configurations for \(|\lambda| \gg 1\)

It was shown in Ref. [7] that the maximum mass of Dirac stars supported by a noninteracting spinor field is \(M_{\text{max}} \approx 0.709 M_p^2/\mu\). For the mass of a spinor field \(\mu \sim 1\) GeV, it gives the total mass \(M \sim 10^{14}\) g, i.e., the stars with small masses and radii \(R \sim 10\) fm. (Then, by analogy to mini-boson stars [26], one can speak of mini-Dirac stars.) In turn, one can see from the results obtained above that the use of positive values of the coupling constant \(\lambda\) leads to decreasing the maximum mass. In this connection, from the point of view of possible astrophysical applications, it seems more interesting to use negative values of \(\lambda\). Numerical calculations indicate that as \(|\lambda|\) increases, there is a considerable growth in maximum masses of the configurations under consideration (see Fig. 1). For clarity, in Fig. 2, we have plotted the dependence of the maximum mass \(M_{\text{max}}\) as a function of \(|\lambda|\). In this figure, the solid line corresponds to the interpolation formula

\[
M_{\text{max}} \approx 0.415 \sqrt{|\lambda| M_p^2/\mu},
\]

which holds asymptotically for \(|\lambda| \gg 1\).

We have found that in the case of the spinor systems considered here, as in the case of boson stars of Ref. [24], the large-|\lambda| configurations have the structure that differs significantly from that of the small-|\lambda| systems. These distinctions are illustrated in Fig. 3 which shows the spinor field distributions along the radius for the cases of \(\lambda = 0\) and \(\lambda = -100\) (for the case of \(\lambda = -100\), we take the configuration with a maximum mass marked by a bold dot in Fig. 1). It is seen from this figure that in both cases the main contribution to the energy density (and correspondingly to the mass) is given by the function \(\bar{g}\). This function decays exponentially to zero in a characteristic length \(\sim 1/\mu\) for small |\lambda|, but for large |\lambda| is characterized by relatively slow decline out to radii \(\sim 2|\lambda|^{1/2}/\mu\) with exponential decay only at larger radii (cf. Ref. [24]). For this reason, the majority of the mass of the large-|\lambda| systems is concentrated in the region of slow decline, which becomes increasingly dominant as |\lambda| increases.

As in the case of boson stars of Ref. [24], such a behavior of the spinor fields enables one to introduce an alternative nondimensionalization of Eqs. (13)-(16) valid at large \(|\lambda|\): \(\bar{g}_*, \bar{f}_* = |\lambda|^{1/2} \bar{g}, \bar{f}, \bar{m}_* = |\lambda|^{-1/2} \bar{m}, \text{ and } \bar{x}_* = |\lambda|^{-1/2} \bar{x}\). Using these new variables and taking into account that the leading term in Eq. (14) is the third term \((...)\bar{g}\), this equation yields (in the approximation of \(\bar{f} \ll \bar{g}\))

\[
\bar{g}_* = \sqrt{\frac{1}{8} \left(1 - \frac{E}{\sigma \sqrt{N}}\right)}.
\]
one can see the presence of a maximum of the mass, parameter \( \bar{\lambda} \) these limiting equations to determine the rescaled total mass \( \bar{M} \) except the behavior at large radii (see Fig. 3). Since \( \bar{\lambda} \) does not appear explicitly in Eqs. (20) and (21), one can use these limiting equations to determine the rescaled total mass \( \bar{M} = M / (|\lambda|^{1/2} M_p^2 / \mu) \) as a function of the single free parameter \( \bar{E} / \sigma_c \). The corresponding results of numerical solution of Eqs. (19)-(21) are given in Fig. 4, from which one can see the presence of a maximum of the mass,

\[
\bar{M}_{\text{max}} \approx 0.46 \sqrt{|\lambda| M_p^2 / \mu}.
\]

The magnitude of the numerical coefficient appearing here differs from that given in Eq. (18) by \( \sim 10\% \), that is obviously related to the fact that we do not take into account the influence of the function \( f \) on the approximate solutions. Despite this, one can believe that the above approximation agrees well with the exact solution. Notice also that the calculations indicate that these approximate solutions describe fairly well only systems located near the maximum of the mass, and the deviations from the exact solutions become stronger, the further away we are from the maximum.

As in the case of boson stars of Ref. [24], for the Dirac star the ground state of the spinor field is not a zero-energy state (because of self-gravity). Moreover, at large \( |\lambda| \), the spinor field is spread over a relatively large length scale \( |\lambda|^{1/2} \mu^{-1} \gg \mu^{-1} \); this enables one to neglect locally the derivatives of \( \bar{g} \) and \( \bar{f} \). This allows the possibility of, firstly, obtaining the solution of the equation for the spinor field [19] in the form of (19) when one can neglect the influence of the function \( \bar{f} \) and its derivative. Secondly, in the approximation of neglecting the derivatives, one can introduce an effective equation of state. To do this, let us use the components \( T^0_0 = \varepsilon \) and \( T^i_0 = -p_r \) of the energy-momentum tensor (20), where \( \varepsilon \) is the effective energy density of the spinor fluid and \( p_r \) is its radial pressure:

\[
\bar{\varepsilon} \equiv \frac{\varepsilon}{\gamma} = \frac{\bar{E}}{\sigma \sqrt{N}} (\bar{f}^2 + \bar{g}^2) + 32 \bar{\lambda} (\bar{f}^2 - \bar{g}^2)^2,
\]

\[
\bar{p}_r \equiv \frac{p_r}{\gamma} = \frac{\bar{E}}{\sigma \sqrt{N}} (\bar{g} \bar{f}' - \bar{f} \bar{g}') - 32 \bar{\lambda} (\bar{f}^2 - \bar{g}^2)^2,
\]

with \( \gamma = e^2 M_p^2 / (4 \pi \mu \lambda^3) \). (Note that, as in the case of boson stars of Ref. [24], the radial, \( p_r \), and tangential, \( p_t = -T^0_2 \), components of pressure for the Dirac star are not equal to each other.) For \( |\lambda| \gg 1 \), in the approximation used here, one can obtain

\[
\bar{\varepsilon}_* \equiv |\lambda| \bar{\varepsilon} = 8 \bar{g}^2 \left( \frac{\bar{E}}{\sigma \sqrt{N}} + 4 \bar{g}^2 \right), \quad \bar{p}_{r*} \equiv |\lambda| \bar{p}_r = 32 \bar{g}^4.
\]

Taking into account the expression (19) for \( \bar{g}_* \) and eliminating from these relations \( \bar{E} / (\sigma \sqrt{N}) \), one can derive the following effective equation of state:

\[
\bar{p}_{r*} = \frac{1}{9} \left( 1 + 3 \bar{\varepsilon}_* \pm \sqrt{1 + 6 \bar{\varepsilon}_*} \right).
\]

The dimensionless quantities appearing here are related to the dimensional energy density and pressure in the following manner: \( \bar{p}_{r*}, \bar{\varepsilon}_* = (p_r, \varepsilon) / \varepsilon_0 \), where \( \varepsilon_0 = (\mu c^2)^2 / |\lambda| \). Then the relations (18) and (22), using the expression for \( \lambda \) from Eq. (17), are equivalent to the statement that \( M^{\text{max}} \sim M^3 / \varepsilon_0 \) for a fluid star with an equation of state of the form of Eq. (25) (cf. Ref. [24] where a similar expression has been obtained for boson stars). In the case of boson stars, such a limiting transition from a scalar field configuration to a fluid system when the coupling constant tends to infinity enables one to assume that stable configurations can occur. In fact, both the systems supported by a relativistic fluid [27] and the configurations consisting of a complex scalar field [25], located to the left of the first peak in the mass in the “mass–central density” diagram, are stable against linear perturbations. It seems reasonable to suppose that the same stability criterium may be applied for the spinor field configurations considered in the present paper. However, this question requires special studies, for example, by analogy with Ref. [6].
IV. CONCLUSIONS AND DISCUSSION

The paper studies compact strongly gravitating configurations supported by nonlinear spin-$\frac{1}{2}$ fields. The use of two spinor fields having opposite spins enabled us to get a diagonal energy-momentum tensor suitable for a description of spherically symmetric systems. Consistent with this, we have found localized regular zero-node asymptotically flat solutions for explicitly time-dependent spinor fields, oscillating with a frequency $E/\hbar$. It was shown that for all values of the energy $E$ and of the coupling constant $\lambda$ considered here these solutions describe configurations possessing a positive ADM mass. This enables one to use such solutions for a description of compact gravitating objects (Dirac stars).

From the results obtained earlier for Dirac stars without nonlinearity, it follows that for typical values of the spinor field mass total masses of such configurations are extremely small (see, e.g., Ref. [7]). Here we show that the presence of nonlinearity of spinor fields can alter the situation drastically. In the simplest case the nonlinearity can be chosen in a quadratic form of the type $\lambda(\bar{\psi}\psi)^2$. Then families of gravitational equilibria may be parameterized by the single dimensionless quantity $\tilde{\lambda} = \lambda M^2 c^4 / 4\pi\hbar^3$. Consistent with the dimensions of $|\lambda|$ = erg cm$^3$, one can assume that its characteristic value is $\lambda \sim \tilde{\lambda} \mu c^2 \lambda^3$, where the dimensionless quantity $\tilde{\lambda} \sim 1$. Then the obtained dependence of the maximum mass of the systems under consideration on $|\lambda|$ in the limit $|\lambda| \gg 1$ [see Eq. (13)] can be represented as

$$M^{\text{max}} \approx 0.415 \sqrt{|\lambda| M^2 / \mu} = (0.19 \text{ GeV}^2) M_\odot \sqrt{|\lambda| / \mu^2}.$$ 

This mass is comparable to the Chandrasekhar mass for the typical mass of a fermion $\mu \sim 1$ GeV, for which the dimensional coupling constant $\lambda \ll 1$. In this respect the behavior of the dependence of the maximum mass of the Dirac stars on the coupling constant is similar to that of boson stars of Refs. 24, 25.

Note that, in the absence of gravity, a nonlinear spinor field has been investigated in Ref. 26, relating to the problem of the quantization of an electron. In Refs. 30, 31, it was shown that the corresponding nonlinear Dirac equation has regular solutions with finite energy (also without gravity). This permits us to assume that in our case a nonlinear spinor field can approximately describe fermions (or quarks) which are in some quantum state where they can be approximately described by some collective wave function obeying a nonlinear Dirac equation. A similar situation occurs in considering bosons in a Bose-Einstein condensate described by the Gross-Pitaevski equation, and also in describing Cooper pairs in a superconductor by means of the Ginzburg-Landau equation.

In conclusion, we would like to briefly address the question of stability of the configurations under consideration. Similarly to models of neutron and boson stars, which can be parameterized by their central densities, one can consider a 1-parameter family of Dirac stars described by the central value of the spinor field $g_c$. The total mass is then a function of this parameter, and for any value of the coupling constant $\lambda$, there exists a first peak in the mass (a local maximum). In the case of neutron and boson stars, a transition through this local maximum indicates an onset of instability against perturbations which compress the entire star as a whole. One can naively expect that in the case of Dirac stars a similar situation will also take place. However, this requires special consideration by investigating the stability of spinor field configurations against, for instance, linear perturbations.

Acknowledgments

The authors gratefully acknowledge support provided by Grant No. BR05236494 in Fundamental Research in Natural Sciences by the Ministry of Education and Science of the Republic of Kazakhstan. We are grateful to the Research Group Linkage Programme of the Alexander von Humboldt Foundation for the support of this research and also would like to thank the Carl von Ossietzky University of Oldenburg for hospitality while this work was carried out. We also wish to thank E. Radu for a fruitful discussion of the problem statement and of the obtained results.

[1] L. Amendola and S. Tsujikawa, Dark energy: theory and observations (Cambridge University Press, Cambridge, England, 2010).
[2] A. D. Linde, Lect. Notes Phys. 738, 1 (2008) arXiv:0705.0164 [hep-th]]
[3] P. E. Schunck and E. W. Mielke, Class. Quant. Grav. 20, R301 (2003).
[4] R. Bartnik and J. Mckinnon, Phys. Rev. Lett. 61, 141 (1988).
[5] R. Brito, V. Cardoso, C. A. R. Herdeiro, and E. Radu, Phys. Lett. B 752, 291 (2016), arXiv:1508.08352 [gr-qc]]
[6] F. Finster, J. Smoller, and S. T. Yau, Phys. Rev. D 59, 104020 (1999) [gr-qc/9801079].
[7] C. A. R. Herdeiro, A. M. Pombo, and E. Radu, Phys. Lett. B 773, 654 (2017) arXiv:1708.05674 [gr-qc]].
[8] K. A. Bronnikov, E. N. Chudaeva, and G. N. Shikin, Gen. Rel. Grav. 36, 1537 (2004).
[9] K. A. Bronnikov and J. P. S. Lemos, Phys. Rev. D 79, 104019 (2009) [arXiv:0902.2360 [gr-qc]].
[10] B. Saha, Astrophys. Space Sci. 357, no. 1, 28 (2015) [arXiv:1409.4993 [gr-qc]].
[11] B. Saha, Eur. Phys. J. Plus 130, no. 10, 208 (2015) [arXiv:1504.03883 [gr-qc]].
[12] B. Saha, Eur. Phys. J. Plus 131, no. 5, 170 (2016) [arXiv:1507.03847 [gr-qc]].
[13] B. Saha, Int. J. Theor. Phys. 55, 2259 (2016) [arXiv:1603.04623 [gr-qc]].
[14] B. Saha, Eur. Phys. J. Plus 131, no. 7, 242 (2016) [arXiv:1606.00214 [gr-qc]].
[15] V. G. Krechet and I. V. Sinilshchikova, Russ. Phys. J. 57, no. 7, 870 (2014).
[16] V. Adanhounme, A. Adomou, F. P. Codo and M. N. Hounkonnou, J. Mod. Phys. 3, 935 (2012) [arXiv:1211.3388 [math-ph]].
[17] M. O. Ribas, F. P. Devecchi, and G. M. Kremer, Europhys. Lett. 93, 19002 (2011) [arXiv:1012.5557 [gr-qc]].
[18] M. O. Ribas, F. P. Devecchi, and G. M. Kremer, Mod. Phys. Lett. A 31, no. 06, 1650039 (2016) [arXiv:1602.06874 [gr-qc]].
[19] V. Dzhunushaliev, Grav. Cosmol. 24, no. 3, 267 (2018).
[20] I. Lawrie, A unified grand tour of theoretical physics (Institute of Physics Publishing, Bristol and Philadelphia, 2002).
[21] X. z. Li, K. l. Wang, and J. z. Zhang, Nuovo Cim. A 75, 87 (1983).
[22] K. L. Wang and J. Z. Zhang, Nuovo Cim. A 86, 32 (1985).
[23] Yu. Obukhov, “Dirac operator in spherical coordinates”, unpublished.
[24] M. Colpi, S. L. Shapiro, and I. Wasserman, Phys. Rev. Lett. 57, 2485 (1986).
[25] M. Gleiser and R. Watkins, Nucl. Phys. B319, 733 (1989).
[26] T. D. Lee and Y. Pang, Nucl. Phys. B 315, 477 (1989).
[27] R. Tooper, Astrophys. J. 142, 1541 (1965).
[28] E. W. Mielke and F. E. Schunck, Nucl. Phys. B 564, 185 (2000) [gr-qc/0001061].
[29] W. Heisenberg, Introduction to the unified field theory of elementary particles. (Max-Planck Institut für Physik und Astrophysik, Interscience Publishers, London, New York, Sydney, 1966).
[30] R. Finkelstein, R. LeLevier, and M. Ruderman, Phys. Rev. 83, 326 (1951).
[31] R. Finkelstein, C. Froindal, and P. Kaus, Phys. Rev. 103, 1571 (1956).