ABSTRACT

We derive a charged black hole solution in four dimensions described by $SL(2, R) \times SU(2) \times U(1)/U(1)^2$ WZW coset model. Using the algebraic Hamiltonian method we calculate the corresponding solution that is exact to all orders in $\frac{1}{k}$. It is shown that unlike the 2D black hole, the singularity remains also in the exact solution, and moreover, in some range of the gauge parameter the space-time does not fulfil the cosmic censor conjecture, i.e. we find a naked singularity outside the black hole. Exact dual models are derived as well, one of them describes a 4D space-time with a naked singularity. Using the algebraic Hamiltonian approach we also find the exact to all orders $O(d, d)$ transformation of the metric and the dilaton field for general WZW coset models and show the correction with respect to the transformations in one loop order.

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1. Introduction

Since the pioneering paper of Witten on the two dimensional black hole\[1\], black holes have vigorously been studied in string theory\[2\]. Although we do not have an indication whether string theory is the theory of nature, it is still very helpful to investigate quantum aspects of black holes in this framework.

Toward a more realistic string theory, we want to investigate black holes in four dimensions. All the vacuum solutions of the Einstein’s equations automatically fulfill the condition for vanishing of the beta-function to one loop order. However, they are not guarantied to correspond to conformal field theories (CFT). The Schwarzschild, Nordstrom-Reissner and the Kerr solutions are particular examples. We have very limited methods to obtain backgrounds that correspond to CFTs: The principal one is to use WZW or WZW coset models. Up to now, none of these black hole solutions was shown to correspond to a WZW coset model (the Schwarzschild solution probably cannot correspond to ungauged WZW model since the WZW background has an antisymmetric tensor. If it correspond to gauged WZW, we could at most hope to obtain it after conformal rescaling). All the theories that were classified to correspond to 4D Lorentzian metric\[3\] \[4\] do not correspond to any of the above vacuum solutions. A classical solution of dilatonic charged black hole in 4D was derived in \[5\] \[6\]. It was shown that the presence of the dilaton field changes the causal structure of the black hole and leads to curvature singularity at finite radii. The black hole in \[5\] was extensively studied in connection with extremal dilatonic black hole (i.e.when the inner and the outer horizons coincide). It was argued\[7\] \[8\] that such a black hole behaves like an elementary particle as its spectrum of excitations has an energy gap. Recently it was shown\[9\] that at certain limiting cases of the solution in\[5\] (in which the asymptotic two sphere has a finite radius), these solutions correspond to exact string solutions. In these limiting cases the solutions become a simple product of the 2D black hole and a non- singular CFT on the 2-sphere. Other conformal solutions of 4D charged black holes were derived in \[10\] \[11\].
The general procedure to derive sigma models that correspond to gauged WZW models is to parameterize a group and derive its WZW coset models by integrating out the gauge fields. This procedure is obviously correct only to one loop order. The exact background has $\mathcal{O}(\frac{1}{k})$ corrections with respect to the semiclassical solution, where $k$ is the level. Up to now, two methods were suggested to derive the exact to all orders background. The first is the algebraic Hamiltonian method\[12\][13], where one parameterizes the group, writes the Casimir operators for the zero modes of the Virasoro generators $L_0, \bar{L}_0$ and compares to the Laplacian in curved space. With this approach one can derive the exact metric and dilaton field. The other method\[14\][15] is a direct field theoretical approach, based on replacing the classical WZW action by the exact effective one and then eliminating the gauge fields, keeping only the local terms. Both methods coincide, and in the case of Witten’s 2D black hole yield a background that was confirmed to satisfy the beta-function equations at least up to the fourth order in $\alpha'\[16\][17]$. The exact “black hole” was shown to have no singularity\[18\] although there is an event horizon: In the exact case a new Euclidean region appears between the singularity and the black hole interior and the boundary between the Lorentzian and the Euclidean regions is a coordinate singularity, which turns out to be a surface of time reflection symmetry in an extended space-time. One could conjecture that this is how string theory resolves the problem of space-time singularities in general (and in black holes in particular), namely, singularities in one loop order solutions disappear when introducing all higher orders corrections. But this is not the case as we show in this paper. Another example where this conjecture fails was considered in\[19\]. The exact metric that correspond to the semiclassical cosmological model in\[20\] was shown to have a singularity.

The main motivation of this paper is to investigate a solution of charged black hole in four dimensions and study the higher orders corrections to the metric. Our solution describes an axisymmetric 4D black hole which is not asymptotically flat, and carries both electric and axionic charges but no magnetic charge. We shall see that the space-time structure in the exact metric depends strongly on the gauge
parameter, unlike in the semiclassical limit. In all cases the exact metric remains singular, and moreover, some additional naked singularities can appear. In one case we will find that the 4D semiclassical metric remains exact to all orders.

The second issue we are interested in is to study duality in the algebraic Hamiltonian approach and find all the exact dual models to our black hole solution. We will see that this approach is natural to derive the exact $O(d,d)$ transformations which relate all the dual models. In particular we find one dual model which describes a naked singularity in space-time (without a black hole).

$O(d,d)$ symmetries became very popular recently as a helpful tool to derive semiclassical solutions from known sigma models that correspond to CFT’s. These are symmetries of the background that appear when the background is independent of $d$ of the target space coordinates. The corresponding duality transformations of the space-time metric, the antisymmetric tensor and the dilaton field are known to one loop order. In this paper we shall obtain the exact to all orders transformations of the metric and the dilaton field in general and show all the higher order corrections to the semiclassical transformations.†

The paper is organized as follows: In section 2 we derive the charged black hole solution as an $SL(2,R) \times SU(2) \times U(1)/U(1)^2$ WZW coset model. This is obtained by integrating out the gauge fields in the gauged sigma model action. In section 3 we derive the exact solution that correspond to the semiclassical solution of section 2 and analyse it with respect to the semiclassical limit. Here we show that the exact metric remains singular. Moreover, the space-time described by the exact metric depends drastically on the relation between the gauge parameter and the levels. In some range of the gauge parameter the exact solution describes a charged black hole, while in some other ranges naked singularities appear on cone surfaces or on an infinite string, which cross the event horizon, so strictly speaking the solution does not describe a black hole and the space-time becomes

† In the case of $SL(2,R)/U(1)$ and $SU(2)/U(1)$ there is a regularization scheme where the semiclassical background receives no higher order corrections in $\alpha'^{(27)}$. In such a case the semiclassical $O(d,d)$ symmetry transformations are also exact.
non-physical. In section 4 we derive an expression for the metric, the space-time gauge fields and the dilaton field for all the dual models. These are related by the exact $O(3, 3)$ symmetry. In particular we derive a dual model which describes a naked singularity in 4D spacetime. In section 5 we obtain the exact abelian $O(d, d)$ symmetry transformations of the metric and the dilaton field for general WZW coset models. We show that when writing the inverse exact metric as composed of the one loop order part plus the $O(\frac{1}{k})$ corrections, the former part transforms exactly as in the one loop order $O(d) \times O(d)$ transformations while the latter part is unchanged. Therefore, knowing both the antisymmetric tensor to one loop order and the exact metric is enough to obtain all the exact dual models. Section 6 is reserved for summary and discussion.

2. Four Dimensional Charged Black Hole Solution

In this section we shall construct CFTs derived from $SL(2, R)k_1 \times SU(2)k_2 \times U(1)/U(1)^2$ WZW coset model which describe charged black hole in four dimensions in the closed bosonic string theory. The model we describe here is based on our previous works [11, 10]. It can be embedded also in the framework of superstring theories or the heterotic strings (i.e. starting with $N = 1$ or $N = \frac{1}{2}$ supersymmetric WZW). We shall concentrate here only in the closed bosonic strings. The central charge of our model is $c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 + 2} - 1$. In order that this model describes the complete space-time we need to have either $c = 26$ in the bosonic strings or $c = 15$ in the $N = 1$ superstrings. Alternatively we can describe our space time as a tensor product $M^4 \times K$ where $M$ is the four dimensional Lorentzian space-time and $K$ is some internal space, represented by another CFT[30] so that the total central charge is 26 (or 15 in the supersymmetric case). In this case our model can also be regarded as a Kaluza Klein model, with one compactified dimension that is part of $K$. In both pictures, for any integer $k_2$ we can find the appropriate $k_1$.

† In[29] and in [20] related coset models were derived, leading to other 4D cosmological solutions.
In this section we shall obtain the background by integrating out the gauge fields. Therefore the model is correct to one loop order (a semiclassical solution). It is clear that this model will have corrections of order $\mathcal{O}(\frac{1}{k_1}), \mathcal{O}(\frac{1}{k_2})$ in the space-time metric, the antisymmetric tensor, the gauge fields and in the dilaton field. A priori, only when $k_1, k_2 \to \infty$ this model can be regarded as exact to all orders. (In the $N = 1$ supersymmetric case we expect it to be exact to all orders in $\frac{1}{k}$ also for finite $k$, based on [17]).

To describe closed bosonic strings which have space-time gauge fields in their massless spectrum $^{[31]} [32] [33]$ we use the fields $X^\mu$ which are the space time coordinates (in our case $\mu = 0, ..., 3$) and compactified free bosonic fields $X^a$ which realizes the Kac-Moody currents of the gauge group $\tilde{G}$, with $a = 1, ..., \dim \tilde{G}$. In our model we seek $U(1)$ space-time gauge fields, thus have one compactified field which we denote by $Z$. The sigma model action which we will derived correspond to

$$S = \frac{1}{2\pi} \int d^2\sigma (G_{\mu\nu}(X) + B_{\mu\nu}(X)) \partial_+ X^\mu \partial_- X^\nu + \partial_- Z \partial_+ Z + A_\mu(X) \partial_+ X^\mu - \frac{1}{8\pi} \int d^2\sqrt{h} R^{(2)} \Phi(X)$$

(2.1)

where $G_{\mu\nu}$ is the space-time metric, $B_{\mu\nu}$ is the antisymmetric tensor, $A_\mu$ is the background space-time gauge field (the electromagnetic vector), $h$ is the determinant of the world sheet metric, $R^{(2)}$ is the curvature of the worldsheet and $\Phi$ is the dilaton field. (Notice that in the heterotic strings we fermionize the bosonic field $Z$ which will contribute only to the right moving sector). The $U(1)$ symmetry transformation that corresponds to this action is

$$\delta Z(\sigma) = f(X(\sigma))$$

$$\delta A_\mu(X) = -2 \partial_\mu f(X(\sigma))$$

$$\delta G_{\mu\nu} = \delta B_{\mu\nu} = A_\mu(X) \partial_\nu f(X)$$

(2.2)

with $f$ an infinitesimal function.
The WZW action [34][35] for a group $G$ is
\[ S_0(g) = \frac{-k}{4\pi} \int \Sigma d^2 \sigma Tr(g^{-1} \partial_+ g g^{-1} \partial_- g) - \Gamma \]  
where $g$ is an element of the group $G$ and $\Gamma$ is the Wess Zumino term
\[ \Gamma = \frac{ik}{12\pi} \int B Tr(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg) \]

$B$ is the manifold whose boundary is a Riemann surface $\Sigma$. We use Lorentzian metric on the worldsheet $\Sigma$.

Consider the group $G = SL(2, R) \times SU(2) \times U(1)$. We denote the group elements $g$ by the direct product $(h_1, h_2, e^{iX})$, where $h_1 \in SL(2, R)$, $h_2 \in SU(2)$ and $X$ is a free compactified $U(1)$ field ($i.e.$ $X \sim X + 2\pi R$, where $R$ is the radius of compactification). The ungauged action is $S_0(g) = S_0(h_1) + S_0(h_2) + S_0(X)$. We denote the level of the $SL(2, R)$ WZW by $k_1$ and that of $SU(2)$ by $k_2$. In order that the action is uniquely defined the level of the compact group $SU(2)$ should be integer [5]. Now we parameterize the group elements of $SL(2, R)$ and $SU(2)$ by
\[ h_1 = \exp(\frac{t_L}{2} \sigma_3) \exp(r \sigma_1) \exp(\frac{t_R}{2} \sigma_3) \]
\[ h_2 = \exp(i \frac{\phi_L}{2} \sigma_3) \exp(i \theta \sigma_1) \exp(i \frac{\phi_R}{2} \sigma_3) \]

where $\sigma_i$ are the Pauli matrices. Expressed in terms of these coordinates, the ungauged action is
\[ S_0 = \frac{k_1}{8\pi} \int d^2 \sigma (4\partial_+ r \partial_- r + \partial_+ t_L \partial_- t_L + \partial_+ t_R \partial_- t_R + 2 \cosh 2r \partial_+ t_R \partial_- t_L) \]
\[ + \frac{k_2}{8\pi} \int d^2 \sigma (4\partial_+ \theta \partial_- \theta + \partial_+ \phi_L \partial_- \phi_L + \partial_+ \phi_R \partial_- \phi_R + 2 \cos 2\theta \partial_+ \phi_R \partial_- \phi_L) \]
\[ + \frac{1}{4\pi} \int d^2 \sigma \partial_+ X \partial_- X \]

and $k_1, k_2$ are taken to be positive. To obtain the $SL(2, R)$ WZW we have sub-
stituted \( h_1 \) and took \(-k_1\), since we want the \( SL(2, R) \) manifold to have signature \((-+ +)\) and not \((+--)\). Next we wish to gauge a diagonal \( U(1)^2 \) subgroup. The standard way to gauge \( H_L \times H_R \) subgroup of \( G_L \times G_R \) in WZW action\(^{[30]}\) is by replacing derivatives by covariant derivatives. The gauged action is

\[
S(A, B, g) = S_0(g) + \frac{k}{2\pi} \int d^2\sigma \text{tr}(A_- g^{-1} \partial_+ g - B_+ \partial_- g g^{-1} + B_+ g A_- g^{-1})
\]

\[
-\frac{k}{4\pi} \int d^2\sigma \text{tr}(A_+ A_- + B_+ B_-)
\]

where the symmetry transformation is \( \delta g = v g - g u \), \( \delta A_i = -D_i u \), \( \delta B_i = -D_i v \).

However the WZ term \( \Gamma(g) \) has a gauge invariant extension only if one restricts to an “anomaly-free” subgroup of \( G_L \times G_R \). Denote the generators of \( H_R \) and \( H_L \) by \( T_{a,L} \) and \( T_{a,R} \). The anomaly free condition is the following\(^{[37]}\):

\[
\text{tr} T_{a,L} T_{b,L} = \text{tr} T_{a,R} T_{b,R} \quad \text{for } a, b = 1, \ldots, \text{dim}H
\]

(\( \text{tr} \) is the trace on the \( G_L \times G_R \) Lie algebra. When \( G \) is a product of groups \( G_i \) with levels \( k_i \) this reads \( \text{tr} = \Sigma k_i \text{tr}_i \) where \( \text{tr}_i \) is the trace in the representations of the Lie algebra of the group \( G_i \)). We shall gauge a (axial) \( U(1)^2 \) subgroup of \( SL(2, R) \times SU(2) \times U(1) \), generated by the following infinitesimal gauge transformations:

\[
\begin{align*}
\delta h_1 &= (\epsilon_1 \sin \psi \sin \alpha + \epsilon_2 \cos \alpha) \frac{\sigma_3}{2} h_1 + h_1 \frac{\sigma_3}{2} (\epsilon_1 \sin \varphi \sin \beta + \epsilon_2 \cos \beta) \\
\delta h_2 &= \sqrt{\frac{k_1}{k_2}} (\epsilon_1 \sin \psi \cos \alpha - \epsilon_2 \sin \alpha) \frac{i \sigma_3}{2} h_2 + \sqrt{\frac{k_1}{k_2}} h_2 \frac{i \sigma_3}{2} (-\epsilon_1 \sin \varphi \cos \beta + \epsilon_2 \sin \beta) \\
\delta e^{iX} &= i \sqrt{\frac{k_1}{2}} \epsilon_1 (\cos \psi e^{iX} + e^{iX} \cos \varphi)
\end{align*}
\]

where \( \epsilon_1, \epsilon_2 \) are infinitesimal and \( \alpha, \beta, \psi, \varphi \) are arbitrary. (Notice that since in the \( SL(2, R) \) WZW we had to take \(-k_1\) for \( h_1 \), here the anomaly free condition is

\[\dagger\] In the axial gauging it is more common to transform \( A \rightarrow -A \) so that \( A, B \) have the same gauge transformation.
\(-k_1 \text{tr}_{SL(2,R)} + k_2 \text{tr}_{SU(2)} + \text{tr}_{U(1)}\). To gauge the above symmetry we introduce two abelian gauge fields \(A_1, A_2\) that transform as

\[
\delta A_{1,i} = -\partial_i \epsilon_1 \quad ; \quad \delta A_{2,i} = -\partial_i \epsilon_2 \quad (2.10)
\]

The full gauged action is the following:

\[
S(A_1, A_2, g) = S_0(g) + \frac{k_1}{4\pi} \int d^2\sigma (\sin \psi \sin \alpha A_{1+} + \cos \alpha A_{2+}) (\partial_- t_R + \cosh 2r \partial_- t_L)
\]

\[
+ (\sin \varphi \sin \beta A_{1-} + \cos \beta A_{2-}) (\partial_+ t_L + \cosh 2r \partial_+ t_R)
\]

\[
+ \frac{\sqrt{k_1k_2}}{4\pi} \int d^2\sigma (\sin \psi \cos \alpha A_{1+} - \sin \alpha A_{2+}) (\partial_- \phi_R + \cos 2\theta \partial_- \phi_L)
\]

\[
+ (-\sin \varphi \cos \beta A_{1-} + \sin \beta A_{2-}) (\partial_+ \phi_L + \cos 2\theta \partial_+ \phi_R)
\]

\[
+ \frac{\sqrt{k_1/2}}{2\pi} \int d^2\sigma (\cos \psi A_{1+} \partial_- X + \cos \varphi A_{1-} \partial_+ X)
\]

\[
+ \frac{k_1}{4\pi} \int d^2\sigma (\sin \psi \sin \varphi \sin \alpha \sin \beta A_{1+} A_{1-} + \sin \psi \sin \alpha \cos \beta A_{1+} A_{2-}
\]

\[
+ \sin \varphi \sin \beta \cos \alpha A_{2+} A_{1-} + \cos \alpha \cos \beta A_{2+} A_{2-}) \cosh 2r
\]

\[
+ (-\sin \psi \sin \varphi \cos \alpha \cos \beta A_{1+} A_{1-} + \sin \psi \cos \alpha \sin \beta A_{1+} A_{2-}
\]

\[
+ \sin \varphi \sin \alpha \cos \beta A_{2+} A_{1-} - \sin \alpha \sin \beta A_{2+} A_{2-}) \cos 2\theta
\]

\[
+ \frac{k_1}{4\pi} \int d^2\sigma (A_{1+} A_{1-} (\cos \varphi \cos \psi + 1) + A_{2+} A_{2-}) \quad (2.11)
\]

It is easy to check that the gauged action preserves the two \(U(1)\) local symmetries in (2.9),(2.10). (This is another way to see that the gauged action is anomaly free.)
Now we pick two gauge conditions. We take

\[ \phi \equiv \phi_R = -\phi_L, \quad t \equiv t_R = -t_L \]  \hspace{1cm} (2.12)

Among the various (dual) solutions that are obtained with all the parameters, our black hole solution is obtained by taking

\[ \varphi = 0 \]  \hspace{1cm} (2.13)

It must be emphasized that once \( \sin \varphi = 0 \) we cannot take also \( \sin \psi = 0 \), since the gauge fixing (2.12) will not be valid. As we see later, this is the reason why this model cannot describe a background without electromagnetic fields.

Finally, we are integrating out the gauge fields \( A_1, A_2 \) and obtain the following action:

\[ I = \int D[r, t, \theta, \phi] e^{S_{BH}} \det \left[ -\frac{2\pi^2}{k_1 \Delta} \right] \]  \hspace{1cm} (2.14)

where

\[ \Delta = \cos \alpha \cos \beta \cosh 2r - \sin \alpha \sin \beta \cos 2\theta + 1 \]  \hspace{1cm} (2.15)

and

\[ S_{BH} = \frac{k_2}{2\pi} \int d^2\sigma (\partial_+ \theta \partial_- \theta + \frac{k_1}{k_2} \partial_+ r \partial_- r \]

\[ -\frac{k_1 \sinh^2 r (1 + \cos(\alpha + \beta) + 2 \sin \alpha \sin \beta \sin^2 \theta)}{k_2 \Delta} \partial_+ t \partial_- t \]

\[ + \frac{\sin^2 \theta (2 \cos \alpha \cos \beta \cosh^2 r - \cos(\alpha - \beta) + 1)}{\Delta} \partial_+ \phi \partial_- \phi \]

\[ + 2 \sqrt{\frac{k_1}{k_2}} \frac{\sinh^2 r \sin^2 \theta}{\Delta} (\cos \beta \sin \alpha \partial_+ t \partial_- \phi - \cos \alpha \sin \beta \partial_- t \partial_+ \phi) \]

\[ - \frac{\sqrt{2k_1}}{k_2} \tan \left( \frac{\psi}{2} \right) \frac{\sinh^2 r (2 \sin \beta \sin^2 \theta + (\sin \alpha - \sin \beta))}{\Delta} \partial_+ t \partial_- X \]
\[-\sqrt{\frac{2}{k_2}} \tan \left( \frac{\psi}{2} \right) \sin^2 \theta \frac{(2 \cos \beta \cosh^2 r + (\cos \alpha - \cos \beta))}{\Delta} \partial_+ \phi \partial_- X \]

\[+ \frac{1}{4\pi} \tan^2 \left( \frac{\psi}{2} \right) \int d^2 \sigma \partial_+ X \partial_- X \quad (2.16)\]

The determinant to one loop order is calculated by following Buscher, and gives rise to the dilaton term:

\[\det \frac{-\pi^2}{k_1 \Delta} = \exp \left( -\frac{1}{8\pi} \int d^2 \sigma \sqrt{h} R^{(2)} \log \Delta + a \right) \quad (2.17)\]

where \(a\) is an arbitrary constant.

Now we choose \(\alpha = \beta\) and denote

\[Q = \tan \alpha \quad (2.18)\]

We further absorb \(\sqrt{k_1} \) in \(t\) and absorb \(\sqrt{\frac{2}{k_2}} \tan \left( \frac{\psi}{2} \right) \) in \(X\). (recall that we had to restrict to \(\sin \psi \neq 0\), otherwise our gauge fixing is not valid. The rescaling of \(X\) is equivalent to restricting \(\tan \left( \frac{\psi}{2} \right) = \sqrt{\frac{k_1}{k_2}}\).) Finally, we redefine the field \(r\) as \(\hat{r} = \cosh^2 r\). Identifying the sigma model (2.16) with the string action (2.1) we readily see that the gauged action describes (to one loop order) the following background (we omitted the hat from \(r\) and have an overall factor of \(k_2\)):

\[ds^2 = -\frac{(r - 1)(1 + Q^2 \sin^2 \theta)}{r + Q^2 \sin^2 \theta} dt^2 + \frac{k_1}{k_2 r (r - 1)} dr^2 + \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta} d\phi^2 + d\theta^2 \quad (2.19)\]

the antisymmetric tensor (the "axion field") which has only the \(t, \phi\) component

\[B_{t\phi} = 2Q \frac{(r - 1) \sin^2 \theta}{r + Q^2 \sin^2 \theta} \]

the electromagnetic vector potential

\[A_t = Q \sqrt{Q^2 + \frac{(r - 1) \sin^2 \theta}{r + Q^2 \sin^2 \theta}} \]

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\[ A_{\phi} = \sqrt{Q^2 + 1} \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta} \]  

(2.20)

and the dilaton field is

\[ \Phi = \ln(r + Q^2 \sin^2 \theta) + a \]  

(2.21)

In principle, the remaining part of the metric \( G_{XX} = \frac{1}{4} \) gives rise to an additional scalar field (that has only zero mode in our case), associated with the following vertex operator \(^{[41]}\)

\[ \partial X \bar{\partial} X e^{i(-K_t t + K_r r + K_\theta \theta + K_\phi \phi)} \]  

(2.22)

This space-time metric (2.19) describes an axisymmetric black hole in four dimensions: the sphere \( r = 1 \) is the event horizon and \( r + Q^2 \sin^2 \theta = 0 \) is the singularity, hidden inside the horizon. If one were to interpret \( r \) as representing the radius in polar coordinates, the fact that there is a singularity at the origin, \( r = 0 \), only for the angular value \( \theta = 0, \pi \) appears puzzling. If we define the metric on the manifold \( R^4 \) with the origin removed, we then have incomplete geodesics (such as those on the plane \( \theta = \pi/2 \)) which terminates at \( r = 0 \) but along which the curvature remains finite. In fact, this space-time is extendible, and by defining the coordinate \( \tilde{r} = r + Q^2 \) we see that the singularity has a topology of \( S^2 \times R \), that is a sphere cross time . (In the Kerr solution the singularity is at \( r^2 + a \cos^2 \theta = 0 \), with \( a \) being the total angular momentum divided by the mass \(^{[42]}\). In that solution there is a ring singularity (with topology of \( S^1 \times R \)) in the \( z = 0 \) plane.) The above singularity is seen by calculating all the scalar curvatures of the metric (Riemann curvature, Ricci curvature, scalar curvature) which all blow up only at \( r + Q^2 \sin^2 \theta = 0 \). The expressions for \( R_{\mu \nu} \) and the scalar curvature \( R^\mu_\mu \) are given in the appendix. The classical charged black hole solution, described by the Nordstrom-Reissner metric has both inner and outer event horizons. Our dilatonic charged black hole can thus be regarded as an extremal black hole.
The metric (2.19) is not asymptotically flat. In cartesian coordinates, at $r \to \infty$ the metric approaches (for $k_1 = k_2$)

$$dS^2 = -(1 + Q^2 \frac{x^2 + y^2}{x^2 + y^2 + z^2})dt^2 + \frac{dx^2 + dy^2 + dz^2}{x^2 + y^2 + z^2}$$

(2.23)

In other words, it describes a distribution of matter all over the space-time. For $Q = 0$ our metric coincides with that in the extremal black hole solution in[5].

The metric $G_{\mu\nu}$ in (2.19) was read directly from the sigma-model action (2.16). The Einstein metric is obtained by conformal rescaling of the sigma-model metric by the dilaton field. In our notation $G^E_{\mu\nu} = e^\Phi G_{\mu\nu}$, where $\Phi$ is given in (2.21). Notice that for $Q = 0$ and $k_1 = k_2$ the Einstein metric after making coordinate transformation $r \to r^2$ is the following:

$$dS^2 = -r^2 (1 - \frac{1}{r^2})dt^2 + (1 - \frac{1}{r^2})^{-1}dr^2 + r^2 (\sin^2 \theta d\phi^2 + d\theta^2)$$

(2.24)

and the black hole has a magnetic field only.

We can calculate all the curvature tensors and scalars in the Einstein metric and see that there remains a singularity only at $r + Q^2 \sin^2 \theta = 0$. (For the formulas of the transformations of the curvature tensors and scalar under conformal rescaling see e.g. [42].) Thus the Einstein metric describes a black hole as well.

From the explicit expression for the electromagnetic vector field we obtain the electromagnetic tensor $F_{\mu\nu}$, defined by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Hence, the electric ($F_{0,\mu}$) and the magnetic ($F_{i,\mu}$) fields are:

$$E_r = Q \sqrt{Q^2 + 1} \frac{(1 + Q^2 \sin^2 \theta) \sin^2 \theta}{(r + Q^2 \sin^2 \theta)^2}$$

(2.25)

$$E_\theta = Q \sqrt{Q^2 + 1} \frac{(r - 1)r \sin 2\theta}{(r + Q^2 \sin^2 \theta)^2}$$

(2.26)

$$B_r = \sqrt{Q^2 + 1} \frac{r^2 \sin 2\theta}{(r + Q^2 \sin^2 \theta)^2}$$

(2.27)
\[ B_\theta = \sqrt{Q^2 + 1} \frac{Q^2 \sin^4 \theta}{(r + Q^2 \sin^2 \theta)^2} \]  

(2.28)

If we observe the action (2.16) we see that vector potential is multiplied by \( \tan(\frac{\psi}{2}) \), which we have absorbed in \( X \).\(^\dagger\) Only when \( \tan(\frac{\psi}{2}) = 0 \) the black hole has no electromagnetic field. However \( \sin \psi = 0 \) invalidates our gauge fixing (2.12) (and globally reduces the 5D sigma model action to a 4D one).

Now, the effective action [32] of this theory is obtained in the Kaluza-Klein fashion \[^{[43]}\]\ as a dimensional reduction from the five-dimensional effective action. Denote the scalar field which was introduced as \( G_{XX} = e^\varphi \). Then the effective action can be written in the following way (we drop the volume element due to the integration over the fifth dimension and use \( \nabla \varphi = 0 \))

\[ S_{eff} = \int d^4x \sqrt{-g} e^{\varphi + \frac{1}{2} \varphi} (R + (\nabla \varphi)^2 - e^{-\varphi} \frac{1}{4} F^2 - \frac{1}{12} H^2) \]  

(2.29)

where \( g \) is the four dimensional Lorentzian metric, \( F = dA \) and \( H = dB \). From here we see that the total electric charge is

\[ q_E = \int e^{\varphi - \frac{1}{2} \varphi} \ast F d^2S \]  

(2.30)

where the integral is over a 2-sphere at infinity (\( \ast F \) is \( F \)-dual). Thus

\[ q_E = 4\pi Q \sqrt{Q^2 + 1} \exp(a) \int_0^\pi d\theta \sin^3 \theta \sqrt{1 + Q^2 \sin^2 \theta} \]  

(2.31)

The magnetic charge vanishes since

\[ q_M = \int F d^2S = 0 \]  

(2.32)

\( \dagger \) We could alternatively absorb \( \sqrt{Q^2 + 1} \) in \( X \) and then the vector potential is

\[ A_t = Q \tan(\frac{\psi}{2}) \frac{(r - 1) \sin^2 \theta}{r + Q^2 \sin^2 \theta} \quad A_\phi = \tan(\frac{\psi}{2}) \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta} \quad \text{and} \quad G_{XX} = \tan(\frac{\psi}{2})/(Q^2 + 1). \]
Two axionic charges are associated with the action (2.29). The first one is

$$q_{ax} = \int H d^3 S = (16\pi/3)Q$$

(2.33)

The other one, which vanishes in our solution is

$$\tilde{q}_{ax} = \int F \wedge F = 0$$

(2.34)

(although locally $F \wedge F \neq 0$). Thus, this black hole carries both electric and axionic charges but has no magnetic charge. On the other hand, it is well known that the equations of motion are invariant under the duality transformation $F \rightarrow * F$ [5]. Hence, magnetically charged black hole solution may as well be obtained as an equivalent string theory.

3. The Exact Metric in The Algebraic Hamiltonian Approach

In the previous section we have derived a semiclassical background by integrating out the gauge fields in the WZW coset model. The exact to all orders background has $O(\frac{1}{k})$ corrections, so that in the limit $k_1, k_2 \rightarrow \infty$ it reduces to the semiclassical one. Getting the precise corrections has a special interest: The semiclassical metric describes a singularity hidden by the event horizon, but this is not necessarily a property of the exact metric. The issue we want to learn is how the space-time structure of the semiclassical background changes when introducing all the higher order corrections. To find the exact metric that corresponds to the solution in section 2 we can use the algebraic Hamiltonian approach for cosets $G/H$. This method was derived in[12, 13] and we first briefly describe it. (For a review see [44].) We shall concentrate on the closed bosonic strings only.

Consider the Tachyon state $T$, which is the ground state of the string theory. We denote by $J^G_a, J^H_i$ the currents of the group $G$ and its subgroup $H$, respectively ($a = 1, ..., \dim G, i = 1, ..., \dim H$) and $J^G_{a,n}, J^H_{i,n}$ are their “Fourier” components in
the Kac-Moody algebra. \( L_0 \) is the zero generator of the Virasoro algebra. Then the following conditions should be satisfied:

\[
(L_0 + \bar{L}_0 - 2)T = 0 ; \quad (J^H_0 + \bar{J}^H_0)T = 0 ; \quad J^G_n T = \bar{J}^G_n T = 0, \quad n \geq 1 \tag{3.1}
\]

Here

\[
L_0 = \frac{\Delta_G}{k - \tilde{c}_G} - \frac{\Delta_H}{k - \tilde{c}_H} \quad \quad \bar{L}_0 = \frac{\bar{\Delta}_G}{k - \tilde{c}_G} - \frac{\bar{\Delta}_H}{k - \tilde{c}_H} \tag{3.2}
\]

where \( \Delta_G, \Delta_H \) are the Casimir operators in \( G \) and in \( H \), i.e. \( \Delta_G = J^G \cdot J^G \), \( \bar{\Delta}_G = \bar{J}^G \cdot \bar{J}^G \), \( \Delta_H = J^H \cdot J^H \), \( \bar{\Delta}_H = \bar{J}^H \cdot \bar{J}^H \), and \( \tilde{c}_G, \tilde{c}_H \) are the coexter of \( G, H \) respectively. The second condition in (3.1) is a remnant of the gauge invariance \( T(h_L gh_R^{-1}) = T(g) \) which demands that the tachyon is a singlet under the action of the subgroup \( H \).

In the algebraic Hamiltonian approach we parameterize the group elements of \( G \) by \( X_\mu, \mu = 1,\ldots,N = \dim G \) and express the currents in terms of first order differential operators of \( X_\mu \) which satisfy the Lie algebra of the group. Then we need to define gauge invariant coordinates \( \tilde{X}_\mu, \mu = 1,\ldots,D = \dim G - \dim H \) and write the Casimir operators in terms of \( \tilde{X}_\mu \). As is well known, the effective action for the Tachyon is

\[
S(T) = \int d^D X \sqrt{-G} e^\Phi (G^{\mu \nu} \partial_\mu T \partial_\nu T - V(T)) \tag{3.3}
\]

where \( \Phi \) is the dilaton field and \( V(T) \) is the Tachyon potential. On the other hand, since the Tachyon is completely defined through the action of the zero modes, its action is equivalent to

\[
S(T) = \int d^D X \sqrt{-G} e^\Phi (THT - V(T)) \tag{3.4}
\]

where \( H = L_0 + \bar{L}_0 \) is the Hamiltonian. Comparing (3.3) and (3.4), expressed in terms of \( \tilde{X}_\mu \), we obtain

\[
L_0 + \bar{L}_0 = -e^{-\Phi} \frac{1}{\sqrt{-G}} \partial_\mu (e^\Phi \sqrt{-G} G^{\mu \nu} \partial_\nu) \tag{3.5}
\]

from which we find the exact metric and the exact dilaton field.
Now we return to the gauged action described in section 2. We recall that the background electromagnetic vector $A_t, A_\phi$ in (2.20) can be written as $G_t X + B_t X, G_\phi X + B_\phi X$ respectively. Thus we are analyzing a $5 \times 5$ metric. In the previous section, the group elements of $SL(2, R)$ and $SU(2)$ were

$$h_1 = e^{\frac{1}{2} t L} e^{t \sigma_3} e^{r \sigma_1} e^{\frac{1}{2} t R \sigma_3}$$

and

$$h_2 = e^{\frac{1}{2} \phi L} e^{i \theta \sigma_1} e^{\frac{1}{2} \phi R \sigma_3}$$

respectively, and the gauge transformation (2.9) amounted to shifting $t_L, t_R, \phi_L, \phi_R$ and $X$ only. To match these coordinates we define the following first order differential generators which satisfy the Lie algebra of $SL(2, R)$ and $SU(2)$. Here $J_a$ are the generators of $SL(2, R)$ and $I_a$ are the generators of $SU(2)$. ($J_3$ correspond to $-i \frac{\partial}{\partial \phi}$ and $I_3$ correspond to $\frac{\partial}{\partial \varphi}$.)

$$J_3 = i \partial_t L \quad ; \quad J_3 = i \partial_t R$$

$$J_{\pm} = i e^{\pm t L} \left( \frac{1}{2} \partial_r \pm \frac{1}{\sinh 2r} (\partial_{t_L} - \cosh 2r \partial_{t_R}) \right)$$

$$J_{\pm} = i e^{\pm t R} \left( \frac{1}{2} \partial_r \pm \frac{1}{\sinh 2r} (\partial_{t_L} - \cosh 2r \partial_{t_R}) \right)$$  \hfill (3.6)

$$I_3 = i \partial \phi_L \quad ; \quad I_3 = i \partial \phi_R$$

$$I_{\pm} = \pm e^{\mp i \phi R} \left( \frac{1}{2} \partial_\theta \pm \frac{i}{\sin 2\theta} (\partial_{\phi_R} - \cos 2\theta \partial_{\phi_L}) \right)$$

$$I_{\pm} = \pm e^{\mp i \phi L} \left( \frac{1}{2} \partial_\theta \pm \frac{i}{\sin 2\theta} (\partial_{\phi_L} - \cos 2\theta \partial_{\phi_R}) \right)$$  \hfill (3.7)

and we define the generator of the $U(1)$ group by

$$K = \bar{K} = i \partial Y$$

In the coset model which described the charged black hole solution (2.19) the $U(1)^2$ gauged subgroup was generated according to (2.9) with $\varphi = 0$ and $\alpha = \beta$. So in
terms of the above differential operators the gauged currents are

\[ J_1 = \sin \psi \sin \alpha J_3 + \sqrt{\frac{k_1}{k_2}} \sin \psi \cos \alpha I_3 + \sqrt{k_1} \cos \psi K \]

\[ J_1 = \sqrt{k_1} K \]

\[ J_2 = \cos \alpha J_3 - \sqrt{\frac{k_1}{k_2}} \sin \alpha I_3 \]

\[ J_2 = \cos \alpha J_3 + \sqrt{\frac{k_1}{k_2}} \sin \alpha I_3 \]

(3.8)

The central charge of \( J_3 \) is \( k_1 \) and the central charge of \( I_3 \) is \( k_2 \), therefore the central charge of \( J_1 \) is \( k_1(\sin^2 \psi (\sin^2 \alpha + \cos^2 \alpha) + \cos^2 \psi) = k_1 \) and the central charge of \( J_2 \) is \( k_1(\cos^2 \alpha + \sin^2 \alpha) = k_1 \). Similarly the central charge of \( J_1, J_2 \) is also \( k_1 \). In the gauged model we have

\[ L_0 = \frac{\Delta_{SL(2,R)}}{k_1 - 2} + \frac{\Delta_{SU(2)}}{k_2 + 2} - \partial_r^2 - \frac{J_1}{k_1} - \frac{J_2}{k_1} \]

(3.9)

\[ L_0 = \frac{\bar{\Delta}_{SL(2,R)}}{k_1 - 2} + \frac{\bar{\Delta}_{SU(2)}}{k_2 + 2} - \bar{\partial}_r^2 - \frac{\bar{J}_1}{k_1} - \frac{\bar{J}_2}{k_1} \]

(3.10)

where

\[ \Delta_{SL(2,R)} = \bar{\Delta}_{SL(2,R)} = -\frac{1}{4} \partial_r^2 - \frac{1}{2} \coth 2r \partial_r \]

\[ + \frac{1}{\sinh^2 2r} (\partial_{t_L}^2 - 2 \cosh 2r \partial_{t_L} \partial_{t_R} + \partial_{t_R}^2) \]

(3.11)

\[ \Delta_{SU(2)} = \bar{\Delta}_{SU(2)} = -\frac{1}{4} \partial_\theta^2 - \frac{1}{2} \cot 2\theta \partial_\theta \]

\[ - \frac{1}{\sin^2 2\theta} (\partial_{\phi_L}^2 - 2 \cos 2\theta \partial_{\phi_L} \partial_{\phi_R} + \partial_{\phi_R}^2) \]

(3.12)

It is easy to see that these Casimir operators produce the ungauged action in (2.6).
Now we need to find three independent coordinates $t, \phi, X$ which are linear combinations of $t_L, t_R, \phi_L, \phi_R, Y$ and are gauge invariant, i.e.

$$(\mathcal{J}_1 + \bar{\mathcal{J}}_1)(t, \phi, Z) = 0 \ ; \ (\mathcal{J}_2 + \bar{\mathcal{J}}_2)(t, \phi, Z) = 0 \quad (3.13)$$

(namely, it vanishes for each one of the coordinates separately. In the vector gauging we should replace the $+$ sign by a $-$ sign.) Thus if the Tachyon is $T(t, \phi, Z)$ it satisfies the second condition in (3.1). (This is like picking a gauge fixing in the gauged action.) The exact metric is obtained by substituting $t, \phi, Z$ in $L_0 + \bar{L}_0$ by using the chain rule. The inverse of the exact metric is obtained from those terms with quadratic derivatives. Since $r, \theta$ are unchanged, $G_{rr} = 2(k_1 - 2)$ and $G_{\theta\theta} = 2(k_2 + 2)$.

Obviously, if $t, \phi, X$ fulfil (3.13), then any non-vanishing linear combinations of them are appropriate as well. Different choices of $(t, \phi, X)$ yield different (dual) metrics, which are related by similarity transformations. We shall return to discuss related subjects in sections 4,5. In this section, however, we seek the exact metric that correspond to our solution in section 2. A priory, it is not trivial to guess the appropriate combinations. Therefore we use the following method. First we shall calculate the inverse metric of the semiclassical model in section 2. Since we know that the exact metric has only $O(\frac{1}{k})$ corrections, we then easily find the right combinations. The inverse metric of the \textit{semiclassical} model is the following: (we suppress $k_1, k_2$ factors that were absorbed in $t, \phi$ and $X$. These factors will come out from the gauge invariant conditions and we shall absorb them in the coordinates at the end)

$$G^{tt} = -\coth^2 r + \tan^2 \alpha \tan^2 \theta$$

$$G^{t\phi} = -\frac{1}{\cos^2 \alpha} \tan \alpha \tan^2 \theta$$

$$G^{tX} = \frac{\tan \alpha}{\cos \alpha} \tan^2 \theta$$
\[ G^{\phi X} = - \frac{1}{\cos^3 \alpha} (\tan^2 \theta + \cos^2 \alpha) \]

\[ G^{\phi \phi} = \frac{\tan^2 \alpha}{\cosh^2 r} - \frac{1}{\cos^4 \alpha} \frac{2(1 + \cos^2 \alpha) \sin^2 \alpha \cos 2\theta + (1 + \cos^2 \alpha)^2 + \sin^4 \alpha}{\sin^2 2\theta} \]

\[ G^{X \phi} = - \tan \alpha \frac{1}{\cos^2 \alpha} \]

\[ G^{XX} = \frac{1}{\cos^2 \alpha} (\tan^2 \theta + \cos^2 \alpha) \quad (3.14) \]

From these expressions we get the right gauge invariant combinations:

\[ t = t_L - t_R - \tan \alpha \sqrt{\frac{k_2}{k_1}} (\phi_L + \phi_R) \quad (3.15) \]

\[ \phi = \tan \alpha (t_L + t_R) - \frac{1}{\cos^2 \alpha} \sqrt{\frac{k_2}{k_1}} ((1 + \cos^2 \alpha) \phi_R + \sin^2 \alpha \phi_L) \quad (3.16) \]

\[ X = \sqrt{\frac{k_2}{k_1}} \frac{1}{\cos \alpha} (\phi_L + \phi_R) - \frac{\sin \psi}{\sqrt{k_1 (\cos \psi + 1)}} Y \quad (3.17) \]

Notice that since \( \tan(\frac{\psi}{2}) \) was absorbed in \( X \) in (2.19)(2.20) and \( \psi \) disappeared from the action, we have defined gauge invariant coordinates so that the exact metric will be independent of \( \psi \). Now we calculate the exact inverse metric, Using the chain rule. We obtain the following metric:

\[ G^{tt} = -\frac{1}{k_1 - 2} (\coth^2 r - \frac{2}{k_1}) + \frac{k_2}{(k_2 + 2)k_1} \tan^2 \alpha (\tan^2 \theta - \frac{2}{k_2}) \]

\[ G^{t\phi} = - \frac{k_2}{(k_2 + 2)k_1 \cos^2 \alpha} \tan \alpha (\tan^2 \theta - \frac{2}{k_2}) \]

\[ G^{tX} = \frac{k_2}{(k_2 + 2)k_1 \cos \alpha} (\tan^2 \theta - \frac{2}{k_2}) \]
\[
G^{\phi X} = - \frac{k_2}{(k_2 + 2)k_1 \cos^3 \alpha} \left( \tan^2 \theta + \cos^2 \alpha - \frac{2}{k_2} \sin^2 \alpha \right)
\]

\[
G^{\phi \phi} = \frac{1}{k_1 - 2 \cosh^2 r} - \frac{1}{k_1 \cos^2 \alpha}
\]

\[
\frac{k_2}{(k_2 + 2)k_1 \cos^4 \alpha} \left[ -2(1 + \cos^2 \alpha) \sin^2 \alpha \cos 2\theta + (1 + \cos^2 \alpha)^2 + \sin^4 \alpha \right]
\]

\[
G^{XX} = \frac{k_2}{(k_2 + 2)k_1 \cos^2 \alpha} \left( \tan^2 \theta + \cos^2 \alpha - \frac{2}{k_2} \sin^2 \alpha \right)
\]

and the dilaton field is

\[
\Phi = -\frac{1}{2} \ln(\Sigma_1 \Sigma_2)
\]

where

\[
\Sigma_1 = \cosh^2 r + \frac{k_1(k_2 + 2)}{k_2(k_1 - 2)} \tan^2 \alpha \sin^2 \theta
\]

\[
\Sigma_2 = \cosh^2 r + \frac{k_1(k_2 + 2)}{k_2(k_1 - 2)} \tan^2 \alpha \sin^2 \theta + \frac{2k_1}{(k_1 - 2)k_2} \left( \frac{k_2}{k_1} - \tan^2 \alpha \right)
\]

The final step is to calculate the exact metric from its inverse. Then we take a pre-factor \(2(k_2 + 2)\) (as we had in the semiclassical solution), absorb \(\sqrt{\frac{k_1 - 2}{2k_2(k_2 + 2)}}\) in \(t\), \(\sqrt{\frac{k_1 - 2}{2(k_2 + 2)}}\) in \(\phi\) and \(\sqrt{\frac{k_1 - 2}{2(k_2 + 2)}}\) in \(X\) and redefine \(\hat{r} = \cosh^2 r\). Thus, we obtain the following exact four dimensional metric (we omitted the hat from \(r\)):

\[
dS^2 = -\frac{(r - 1)(1 + C + Q^2 \sin^2 \theta)}{r + Q^2 \sin^2 \theta + C} dt^2 + \frac{(k_1 - 2)}{(k_2 + 2)r(r - 1)} dr^2
\]

\[
+ \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta} d\phi^2 + d\theta^2
\]

\[
A_t = \frac{k_2}{k_2 + 2} Q \sqrt{Q^2 + \frac{k_1(k_2 + 2)}{(k_1 - 2)k_2} \left( \frac{k_2}{k_1} (r - 1)(\sin^2 \theta(1 + \frac{2}{k_3}) - \frac{2}{k_3}) \right)}
\]
\[ A_\phi = \sqrt{Q^2 + \frac{k_1(k_2 + 2)}{(k_1 - 2)k_2} \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta}} \quad (3.24) \]

\[ \Phi = -\frac{1}{2} \ln((r + Q^2 \sin^2 \theta)(r + Q^2 \sin^2 \theta + C)) \quad (3.25) \]

where

\[ Q^2 = \frac{(k_2 + 2)k_1}{(k_1 - 2)k_2} \tan^2 \alpha \quad (3.26) \]

and

\[ C = \frac{2}{k_1 - 2}(1 - \frac{k_1}{k_2} \tan^2 \alpha) \quad (3.27) \]

(Notice that when \( k_1 < 2 \) we should take \(|k_1 - 2| \) in \( C, Q \) since we have absorbed \( \sqrt{k_1 - 2} \) in \( t,X \). Thus \( Q^2 \geq 0 \).) It is easy to see that for \( k_1, k_2 \to \infty \) the exact solution is precisely the semiclassical one, however, for finite \( k_1, k_2 \) the space-time might change drastically. In the appendix we have given the expressions for the Ricci tensor and the scalar curvature of this metric. It can be seen that \( r = 1 \) is the event horizon, as in the semiclassical solution. The metric is singular in three cases: (i) when \( \Sigma_1 = r + Q^2 \sin^2 \theta = 0 \).

(ii) when \( \Sigma_2 = r + Q^2 \sin^2 \theta + C = 0 \).

(iii) when \( 1 + C + Q^2 \sin^2 \theta = 0 \).

(Notice that \( C + Q^2 = \frac{1}{k_1 - 2}(k_1 \tan^2 \alpha + 2) \))

(a) For \( C > -1 \) (\( \tan^2 \alpha < \frac{k_2}{2} \)): The singularity is hidden by the event horizon. In this case the solution describes a black hole.

(b) For \( C = -1 \): The black hole singularity extends up to the horizon. In addition, there exist a naked \textit{string} singularity (at \( \sin \theta = 0 \)) that crosses the event horizon.

(c) For \( C < -1 \): The black hole singularity extends outside the event horizon and becomes a naked singularity. In addition, there is a singularity on two cone surfaces (\( \theta = \arcsin(\sqrt{1+e}) \) and \( \theta = \pi - \arcsin(\sqrt{1+e}) \)) which cross the event horizon and become naked.

Hence, we reach the following conclusion: Unlike in the 2D black hole case, the exact metric is singular (for any choice of the gauge parameter \( \alpha \)), and furthermore,
for a certain range of the gauge parameter (where \( \tan^2 \alpha \geq \frac{k_2}{k_1} \)) the semiclassical action describes a black hole while the exact one describes a non-physical spacetime.

Finally, when \( \tan^2 \alpha = \frac{k_2}{k_1} \) the semiclassical metric and dilaton and the exact metric and dilaton are identical, up to shifting \( k_1 \rightarrow k_1 - 2 \) and \( k_2 \rightarrow k_2 + 2 \). It is not possible to derive the antisymmetric tensor by the algebraic Hamiltonian approach. However, in the sigma model where we have integrated out the gauge fields, the antisymmetric tensor has components \( B_{X\phi} = G_{X\phi} \), \( B_{Xt} = G_{Xt} \) and
\[
\sqrt{\frac{Q^2 + 1}{k_1}} B_{t\phi} = G_{zt}.
\]
We conjecture that because of the construction of the sigma model, the first two equalities remain also in the exact solution and the last one is corrected by a \( C \) dependence (like \( A_t \)), so that when \( C = 0 \) only \( A_t, B_{t\phi}, B_{tz} \) have \( \mathcal{O}(\frac{1}{k}) \) corrections.

4. Exact Dual Models

In section 3 we have used the algebraic Hamiltonian approach to calculate the exact metric and dilaton field that correspond to our black hole solution in section 2. This means the following: We have used specific generators for the \( U(1)^2 \) gauged group (specific gauging) that matched the gauging in the WZW sigma model and for these generators we have used specific gauge invariant coordinates that matched the classical solution. In order to get all the dual metrics we should consider all the different anomaly free gaugings and all different gauge invariant combinations for each gauging. It is easy to see that different gauge invariant coordinates for one particular gauging correspond to a constant coordinate transformation. The aim of this section is to derive a formula for all the dual metrics. We will see that all the dual models are related by \( O(3, 3) \) symmetry transformations, of which the semiclassical limit is well known.

In the model we were using in section 3 we gauged a \( U(1)^2 \) subgroup whose generators correspond to the transformations(2.9). Instead of looking for all other anomaly free generators we shall use the following method: Consider the \( L_0 + \bar{L}_0 \)
operator we have used for the gauged model in (3.9)(3.10)(3.8). This can be written as

\[ L_0 + \bar{L}_0 = 2 \frac{\Delta_{SL(2,R)}}{k_1 - 2} + 2 \frac{\Delta_{SU(2)}}{k_2 + 2} - 2 \partial_t^2 - \frac{1}{k_1} \sum_{i=1,2} (\mathcal{F}_i^2 + \bar{\mathcal{F}}_i^2) \]

\[ = (\partial_t, \partial_{\phi L}, \partial_Y)(G^{LL} - \mathcal{F}^L)(\partial_t L, \partial_{\phi L}, \partial_Y) + (\partial_t, \partial_{\phi R}, \partial_Y)G^{LR}(\partial_t R, \partial_{\phi R}) \]

\[ + (\partial_t, \partial_{\phi R})(G^{RR} - \mathcal{F}^R)(\partial_t R, \partial_{\phi R}) - \frac{\partial_{\phi}^2 - 2 \csc 2r \partial_{\phi} - \frac{\partial_{\phi}^2 - 2 \cot 2\theta \partial_{\phi}}{2(k_1 + 2)}} \]

where \( G^{LL}, G^{LR}, G^{RR} \) are obtained from the Casimir operators of the group \( SL(2, R) \times SU(2) \times U(1) \) and \( \mathcal{F}^L, \mathcal{F}^R \) are obtained from \( \frac{1}{k_1} \sum_{i=1,2} (\mathcal{F}_i^2 + \bar{\mathcal{F}}_i^2) \). (Here \( G^{LR} \) is a \((3 \times 2)\) matrix with zeros in the last line.) Denote

\[ \begin{pmatrix} t \\ \phi \\ X \end{pmatrix} = A \begin{pmatrix} t_L \\ \phi_L \\ Y \end{pmatrix} + \tilde{B} \begin{pmatrix} t_R \\ \phi_R \end{pmatrix} \]

where \( A, \tilde{B} \) are two \( 3 \times 3 \) and \( 3 \times 2 \) matrices, respectively, obtained from (3.15)-(3.17). The inverse metric is of course block diagonal and in the block of \( t, \phi, X \) it is

\[ G^{-1} = -A^T G^{LL} A - A^T G^{LR} \tilde{B} - \tilde{B}^T G^{RR} \tilde{B} + A^T \mathcal{F}^L A + \tilde{B}^T \mathcal{F}^R \tilde{B} \]

(4.3)

For any constant matrices \( O_1, \tilde{O}_2 \) which are \( O(3) \) and \( O(2) \) matrices, respectively, the transformation

\[ \begin{pmatrix} t_L \\ \phi_L \\ X \end{pmatrix} \rightarrow O_1 \begin{pmatrix} t_L \\ \phi_L \\ X \end{pmatrix} ; \quad \begin{pmatrix} t_R \\ \phi_R \end{pmatrix} \rightarrow \tilde{O}_2 \begin{pmatrix} t_R \\ \phi_R \end{pmatrix} \]

(4.4)

\[ G^{LL} \rightarrow O_1 G^{LL} O_1^T \quad ; \quad G^{LR} \rightarrow O_1 G^{LR} \tilde{O}_2^T \quad ; \quad G^{RR} \rightarrow \tilde{O}_2 G^{RR} \tilde{O}_2^T \]

(4.5)

leaves the Casimir operators of the ungauged model unchanged. We can now
express the ungauged model with the rotated coordinates \( t'_L, t'_R, \phi'_L, \phi'_R, X' \) (rotate the Casimir operator of the ungauged group) while gauging the subgroup generated with the same generators (not rotate the generators), e.g. \( \mathcal{J}_1 = i (\sin \psi \sin \alpha \partial_{\phi'_L} + \sqrt{k_1/k_2} \sin \psi \cos \alpha \partial_{\phi'_L} + \sqrt{k_1} \partial_{t'}), \) etc’. This is of course still an anomaly free gauging. Thus \( A, B \) and \( \mathcal{J}^{LL}, \mathcal{J}^{RR} \) are unchanged and we get dual models with

\[
G^{-1} = -A^T O_1 G^{LL} O_1^T A - A^T O_1 G^{LR} \tilde{O}_2^T \tilde{B} - \tilde{B}^T \tilde{O}_2 G^{RR} \tilde{O}_2^T \tilde{B} + A^T \mathcal{J}^{LL} A + \tilde{B}^T \mathcal{J}^{RR} \tilde{B}
\]

(4.6)

However, with this method we can find only dual models that are related by \( O(3) \times O(2) \) duality, while we expect our model to possess \( O(3) \times O(3) \) symmetry (since the background is independent of the 3 coordinates \( t, \phi, X \)). In order to see the full symmetry we need to use an equivalent model. We consider the model \( SL(2, R) \times SU(2) \times U(1)^2/U(1)^3 \). The two \( U(1) \) groups are defined by \( X_L, X_R \) so that in the notations of section 3 we have the currents

\[
K = i \partial_{X_L} \quad \tilde{K} = i \partial_{X_R}
\]

and \( \partial^2_Y \) is replaced by \( \partial^2_{X_L} + \partial^2_{X_R} \) in the operator \( L_0 + \tilde{L}_0 \) of the ungauged model used in section 3. We need to define three left and right generators that fulfil the anomaly free condition (2.8). We shall choose the generators that produce exactly the same metric we derived in the \( SL(2, R) \times SU(2) \times U(1)/U(1)^2 \) model and then show how to derive all other dual models. We write the group elements as \( g = diag(h_1, h_2, e^{iX_L}, e^{iX_R}) \), where as in section 2 \( h_1 \in SL(2, R), \ h_2 \in SU(2) \). Now we gauge the \( U(1)^3 \) currents that correspond to the following generators:

\[
T_{1,L} = \frac{1}{\sqrt{2}} diag(\sin \alpha \sigma_3^L/2, \sqrt{k_1/k_2} \cos \alpha \sigma_3^L/2, i \sqrt{k_1/2}, 0) \quad \text{and} \quad T_{1,R} = diag(0, 0, 0, i \sqrt{k_1/2})
\]

\[
T_{2,L} = diag(\cos \alpha \sigma_3^L/2, -\sqrt{k_1/k_2} \sin \alpha \sigma_3^L/2, 0, 0)
\]
\[ T_{2,R} = \text{diag}(\cos \frac{\alpha}{2}, \sqrt{\frac{k_1}{k_2}} \sin \frac{i\alpha}{2}, 0, 0) \]

\[ T_{3,L} = \text{diag}(0, 0, i\sqrt{\frac{k_1}{2}}, 0) \]

\[ T_{3,R} = \frac{1}{\sqrt{2}} \text{diag}(\sin \frac{\alpha}{2}, -\sqrt{\frac{k_1}{k_2}} \cos \frac{i\alpha}{2}, 0, i\sqrt{\frac{k_1}{2}}) \]  \hspace{1cm} (4.7)

In the axial gauging this corresponds to the three constraints:

\[ 0 = J_1 + \bar{J}_1 = \frac{1}{\sqrt{2}}(\sin \alpha J_3 + \sqrt{\frac{k_1}{k_2}} \cos \alpha I_3 + \sqrt{k_1} K) + \sqrt{k_1} \bar{K} \]

\[ 0 = J_2 + \bar{J}_2 = \cos \alpha J_3 - \sqrt{\frac{k_1}{k_2}} \sin \alpha I_3 + \cos \alpha \bar{J}_3 + \sqrt{\frac{k_1}{k_2}} \sin \alpha \bar{I}_3 \]

\[ 0 = J_3 + \bar{J}_3 = \sqrt{k_1} K + \frac{1}{\sqrt{2}}(\sin \alpha \bar{J}_3 - \sqrt{\frac{k_1}{k_2}} \cos \alpha \bar{I}_3 + \sqrt{k_1} \bar{K}) \]  \hspace{1cm} (4.8)

We take the gauge invariant coordinates that match those we used in (3.17), so that we only replace the dependence on \( Y \) by a dependence on \( X^L, X^R \).

\[ t = t_L - t_R - \tan \alpha \sqrt{\frac{k_2}{k_1}} (\phi_L + \phi_R) \]  \hspace{1cm} (4.9)

\[ \phi = \tan \alpha(t_L + t_R) - \frac{1}{\cos^2 \alpha} \sqrt{\frac{k_2}{k_1}} ((1 + \cos^2 \alpha) \phi_R + \sin^2 \alpha \phi_L) - \frac{4}{\sqrt{2k_1} \cos \alpha} (X_L - \frac{1}{\sqrt{2}} X_R) \]  \hspace{1cm} (4.10)

\[ X = \sqrt{\frac{k_2}{k_1} \cos \alpha} (\phi_L + \phi_R) + \frac{1}{\sqrt{k_1(\sqrt{2} - 1)}} (X_L - X_R) \]  \hspace{1cm} (4.11)

which we shall write as

\[
\begin{pmatrix}
t \\
\phi \\
X
\end{pmatrix} =
A
\begin{pmatrix}
t_L \\
\phi_L \\
X_L
\end{pmatrix} +
B
\begin{pmatrix}
t_R \\
\phi_R \\
X_R
\end{pmatrix}
\]  \hspace{1cm} (4.12)

and \( A, B \) are the two corresponding 3 \( \times \) 3 matrices. It is easy to see that this model
yields precisely the metric we found in section 3: Using the constraints \( \mathcal{F}_i^2 = \mathcal{F}_i^2 \) it can be seen that when substituting \( t, \phi, X \) all the contributions from \( \partial_X^L \) and \( \partial_X^R \) cancel out, as in the case in section 3 where all derivatives \( \partial_Y \) cancelled out.

At this stage we can readily derive all the exact dual models. Here we shall write in details. First we write \( L_0 + \bar{L}_0 \) in the following way:

\[
L_0 + \bar{L}_0 = (\partial_{tL}, \partial_{\phi L}, \partial_{X L})(G^{LL} - \mathcal{F}^{LL}) \begin{pmatrix} \partial_{tL} \\ \partial_{\phi L} \\ \partial_{X L} \end{pmatrix} + (\partial_{tL}, \partial_{\phi L}, \partial_{X L})G^{LR} \begin{pmatrix} \partial_{tR} \\ \partial_{\phi R} \\ \partial_{X R} \end{pmatrix}
\]

\[
+ (\partial_{tR}, \partial_{\phi R}, \partial_{X R})(G^{RR} - \mathcal{F}^{RR}) \begin{pmatrix} \partial_{tR} \\ \partial_{\phi R} \\ \partial_{X R} \end{pmatrix} - \frac{\partial_t^2 + 2 \coth 2r \partial_r}{2(k_1 - 2)} - \frac{\partial_\theta^2 + 2 \cot 2\theta \partial_\theta}{2(k_2 + 2)}
\]

(4.13)

where \( G^{LL}, G^{LR}, G^{RR} \) correspond to the Casimir operator of the ungauged model and \( \mathcal{F}^{LL}, \mathcal{F}^{RR} \) correspond to the gauged \( U(1)^3 \) currents.

\[
G^{RR} = G^{LL} = \begin{pmatrix} \frac{2}{(k_1 - 2) \sinh^2(2r)} \\ \frac{-2}{(k_2 + 2) \sin^2(2\theta)} \\ -2 \end{pmatrix}
\]

(4.14)

\[
G^{LR} = \begin{pmatrix} -\frac{4 \cosh(2r)}{(k_1 - 2) \sinh^2(2r)} \\ \frac{4 \cos(2\theta)}{(k_2 + 2) \sin^2(2\theta)} \\ 0 \end{pmatrix}
\]

(4.14)

\[
\mathcal{F}^{LL} = -\frac{1}{2k_1} \begin{pmatrix} \cos^2 \alpha + 1 & -\frac{\sqrt{k_1}}{\sqrt{k_2}} \sin \alpha \cos \alpha & \sqrt{k_1} \sin \alpha \\ -\frac{\sqrt{k_1}}{\sqrt{k_2}} \sin \alpha \cos \alpha & \frac{k_1}{k_2} (\sin^2 \alpha + 1) & \frac{k_1}{\sqrt{k_2}} \cos \alpha \\ \sqrt{k_1} \sin \alpha & \frac{k_1}{\sqrt{k_2}} \cos \alpha & 3k_1 \end{pmatrix}
\]

(4.15)

\[
\mathcal{F}^{RR} = -\frac{1}{2k_1} \begin{pmatrix} \cos^2 \alpha + 1 & \frac{\sqrt{k_1}}{\sqrt{k_2}} \sin \alpha \cos \alpha & \sqrt{k_1} \sin \alpha \\ \frac{\sqrt{k_1}}{\sqrt{k_2}} \sin \alpha \cos \alpha & \frac{k_1}{k_2} (\sin^2 \alpha + 1) & -\frac{k_1}{\sqrt{k_2}} \cos \alpha \\ \sqrt{k_1} \sin \alpha & -\frac{k_1}{\sqrt{k_2}} \cos \alpha & 3k_1 \end{pmatrix}
\]

(4.16)

Now the Casimir operator of the ungauged group is invariant under the \( O(3) \times O(3) \).
transformation

\[
\begin{pmatrix}
    t_L \\
    \phi_L \\
    X_L
\end{pmatrix} \rightarrow O_1 \begin{pmatrix}
    t_L \\
    \phi_L \\
    X_L
\end{pmatrix} ; \quad \begin{pmatrix}
    t_R \\
    \phi_R \\
    X_R
\end{pmatrix} \rightarrow O_2 \begin{pmatrix}
    t_R \\
    \phi_R \\
    X_R
\end{pmatrix}
\]

\[G^{LL} \rightarrow O_1 G^{LL} O_1^T ; \quad G^{LR} \rightarrow O_1 G^{LR} O_2^T ; \quad G^{RR} \rightarrow O_2 G^{RR} O_2^T \quad (4.17)\]

where \(O_1\) and \(O_2\) are two constant \(O(3)\) matrices. From now we just repeat the steps from (4.3) to (4.6), i.e. rotate \(G^{LL}, G^{LR}, G^{RR}\) while gauging the anomaly free subgroup generated by (4.7). Therefore we get an expression for the \(t, \phi, X\) components of the metric in all the dual models:

\[G^{-1} = -A^T O_1 G^{LL} O_1^T A - A^T O_1 G^{LR} O_2^T B - B^T O_2 G^{RR} O_2^T B + 2 A^T J^{LL} A \quad (4.18)\]

where we used \(A^T J^{LL} A = B^T J^{RR} B\). The other generator of the \(O(3,3)\) symmetry are: coordinate transformations \((t, \phi, X) \rightarrow (t, \phi, X) C^T\) where \(C\) is a constant \(GL(3, R)\) matrix- this amounts to transforming \(G \rightarrow C^T G C\), and a constant shift of the antisymmetric tensor. (Notice that by a similarity transformation one can diagonalize \(A^T J^{LL} A\) and \(B^T J^{RR} B\) to become \(-\frac{\hat{c}}{k_1 I}, \) where \(I\) is the unit matrix.) Thus we extended the \(O(d,d)\) symmetry to the exact case.

In particular, we can obtain the axial-vector duality. This duality was investigated in the sigma model of \(U(1)^d\) gauged WZW in [40, 28]. As mentioned before, in the algebraic Hamiltonian approach the gauge invariance conditions for abelian gauging are

\[\mathcal{J}_i \pm \bar{\mathcal{J}}_i = 0\]

where the + sign correspond to the axial gauging and the − sign to the vector gauging. In particular one can interchange axial-vector gauging by taking \(O_1 = -O_2 = \mathbf{1}\) in (4.18), where \(I\) is the unit matrix.
In the rest of this section we shall examine the vector gauging that correspond to the generators we used in section 3 for the axial gauging (i.e. use the same currents $J_1, J_2, \bar{J}_1, \bar{J}_2$ in (3.8)). These two CFT’s are completely equivalent [28]. As is easily seen, the $\frac{1}{k_1}$ term in $L_0 + \bar{L}_0$ is the same as in the axial gauging. The exact inverse metric in the vector gauging is

\[
G^{tt} = -\frac{1}{k_1 - 2}(\tanh^2 r - \frac{2}{k_1}) + \frac{k_2}{(k_2 + 2)k_1} \tan^2 \alpha (\cot^2 \theta - \frac{2}{k_2})
\]

\[
G^{t\phi} = -\frac{k_2}{(k_2 + 2)k_1} \frac{1}{\cos^2 \alpha} \tan \alpha (\cot^2 \theta - \frac{2}{k_2})
\]

\[
G^{tX} = \frac{k_2}{(k_2 + 2)k_1} \frac{\tan \alpha}{\cos \alpha} (\cot^2 \theta - \frac{2}{k_2})
\]

\[
G^{\phi X} = -\frac{k_2}{(k_2 + 2)k_1} \frac{1}{\cos^3 \alpha} (\cot^2 \theta + \cos^2 \alpha - \frac{2}{k_2} \sin^2 \alpha)
\]

\[
G^{\phi \phi} = \frac{1}{k_1 - 2} \frac{\tan^2 \alpha}{\sinh^2 r} - \frac{1}{k_1 \cos^2 \alpha}
\]

\[
G^{XX} = \frac{k_2}{(k_2 + 2)k_1} \frac{1}{\cos^4 \alpha} \frac{2(1 + \cos^2 \alpha) \sin^2 \theta \cos 2\theta + (1 + \cos^2 \alpha)^2 + \sin^4 \alpha}{\sin^2 2\theta}
\]

and the dilaton field is

\[
\Phi = -\frac{1}{2} \ln(\Sigma_1 \Sigma_2)
\]

where

\[
\Sigma_1 = \sinh^2 r + \frac{k_1(k_2 + 2)}{k_2(k_1 - 2)} \tan^2 \alpha \cos^2 \theta
\]

\[
\Sigma_2 = \sinh^2 r + \frac{k_1(k_2 + 2)}{k_2(k_1 - 2)} \tan^2 \alpha \cos^2 \theta + \frac{2k_1}{(k_1 - 2)k_2} \left(\frac{k_2}{k_1} - \tan^2 \alpha\right)
\]

We see that the only difference between this solution and the axially gauged solution in section 3 (3.18)(3.19) is a replacement $\cos \theta \leftrightarrow \sin \theta$, $\cosh r \leftrightarrow \sinh r$. 29
Therefore, the models that correspond to the axial and the vector gaugings are self dual: One can transform from each other by a shift $\theta \rightarrow \theta + \frac{\pi}{2}$, $r \rightarrow r + i\frac{\pi}{2}$.

Finally, we shift $\theta$ to $\theta + \pi/2$ and redefine $\sinh^2 r \rightarrow r$. Then the metric with a pre-factor $2(k_2 + 2)$ (and with the appropriate absorption of constants in $t, \phi, X$) is

\[
\begin{align*}
    dS^2 &= -\frac{(r + 1)(1 + C + Q^2 \sin^2 \theta)}{r + Q^2 \sin^2 \theta + C} dt^2 + \frac{k_1 - 2}{k_2 + 2} \frac{dr^2}{r(r + 1)} \\
    &\quad + \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta} d\phi^2 + d\theta^2 
\end{align*}
\]

where $C, Q$ are defined in (3.27)(3.26), and the electromagnetic vector has the $t, \phi$ components. For $C \geq 0$ this metric describes a naked singularity at $r + Q^2 \sin^2 \theta = 0$. (When $C = 0$ the exact metric is the same as the semiclassical one.) For $0 > C > -1$ there is a naked singularity at $r + Q^2 \sin^2 \theta + C = 0$. For $C = -1$ there exist additional naked string singularity (at $\sin \theta = 0$). This singularity does not exist in the semiclassical limit. For $C < -1$ there is, in addition, a new naked singularity at the two cone surfaces $\theta_1 = \arcsin(\sqrt{-1 - C}/Q)$ and $\theta_2 = \pi - \theta_1$.

5. Exact $O(d, d)$ transformations of the metric and the dilaton

The $O(d, d)$ symmetry (duality) appears when the background in independent of $d$ of the $D$ space-time coordinates. It can be seen at the classical level that there is a symmetry transformation that can be applied on the background $G_{\mu\nu}$, $B_{\mu\nu}$, accompanied by a transformation of the dilaton field, that leaves the one loop effective action unchanged[21, 22]. The symmetry transformation can be derived also by gauging a $U(1)^d$ subgroup in a sigma-model with $D + 2d$ target space dimensions with $2d$ Killing vectors[26] (i.e. the ungauged background is independent of the $2d$ coordinates from which we gauge out $d$) and also by means of string field theory[23, 24]. The latter two approaches gives the one loop order duality transformations based on conformal field theories. Here we shall interpret the
sigma-model approach in [26] to the exact action by the algebraic Hamiltonian approach.

Consider a $D$ dimensional background that is independent of $d$ coordinates which we denote by $Y_i, i = 1, \ldots, d$ and the rest of the coordinates are denoted by $X_\mu, \mu = 1, \ldots, D - d$. In this section we shall consider the $O(d, d)$ transformations for the case when the target space metric satisfy $G_{i\mu} = 0$. The generalization to the case $G_{i\mu}(X) \neq 0$ can be established as well\textsuperscript{[45]}. We shall denote $G_{ij}$ by $G$ and $G_{\mu\nu}$ by $\tilde{G}$.

We shall consider a group $G$ WZW model with level $k$ that is described by the following sigma model:

$$S = \frac{k}{8\pi} \int d^2 \sigma (\tilde{G}_{\mu\nu}(X) \partial_\mu X^\nu \partial_- X^- + \partial_+ \theta^i_1 \partial_- \theta^i_1 + \partial_+ \theta^i_2 \partial_- \theta^i_2 + 2E_{ij}(X) \partial_+ \theta^i_1 \partial_- \theta^j_2$$

(5.1)

The action is described by a target space with $D + d$ dimensions with $X_\mu, \mu = 1, \ldots, D - d$, and $\theta^i_1, \theta^i_2, i = 1, \ldots, d$. Now we want to gauge the $U(1)^d_L \times U(1)^d_R$ subgroup, that correspond to the holomorphic conserved currents

$$J^i = \partial_+ \theta^i_2 + E_{ji} \partial_+ \theta^j_1$$

$$\bar{J}^j = \partial_- \theta^j_1 + E_{ij} \partial_- \theta^i_2$$

(5.2)

Let us represent this model by the algebraic Hamiltonian approach. The ungauged WZW is exact (up to a shift $k \rightarrow k - \tilde{c}_G$). Reading the casimir operator from the ungauged model (5.1)it can be written as

$$-\Delta = -\bar{\Delta} = K^\mu(X) \partial_\mu X^\nu + F^{\mu\nu}(X) \partial_\mu X_\nu + (1 - EE^T)_{ij}^{-1} \partial_\theta^i_1 \partial_\theta^j_1$$

$$+(1 - E^T E)^{-1}_{ij} \partial_\theta^i_2 \partial_\theta^j_2) - 2(E(1 - E^T E)^{-1})_{ij} \partial_\theta^i_1 \partial_\theta^j_2$$

(5.3)

where I is the $(d \times d)$ unit matrix. We parameterize the $U(1)$ gauged currents by
the commuting generators
\[ J_i = i \partial_{\theta^i_1} ; \quad \tilde{J}_i = i \partial_{\theta^i_2} \]  \hspace{1cm} (5.4)

and take the gauged currents to be \( \mathcal{J}_i = J_i \) and \( \tilde{\mathcal{J}}_i = \tilde{J}_i \). Obviously they correspond to an anomaly free gauging. The coset model \( G/U(1)^d \) correspond to

\[ L_0 + \tilde{L}_0 = \frac{2\Delta}{k - \tilde{c}_G} - \frac{1}{k} \sum_{i=1}^{d} (\mathcal{J}_i^2 + \tilde{\mathcal{J}}_i^2) \]  \hspace{1cm} (5.5)

We shall use the axial gauging. Define the gauge invariant coordinates

\[ Y^i = \theta^i_1 + \theta^i_2 \]  \hspace{1cm} (5.6)

Substituting \( Y^i \) in (5.5) we obtain the following \( (D \times D) \) metric of the coset model:

\[ G^{-1}_{G/H} = \begin{pmatrix} \tilde{G}^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix} \]

where

\[ G^{-1} = \frac{2}{k - \tilde{c}_G}[(E^{-1} - E^T)^{-1}(1 + \frac{1}{2}(E^{-1} + E^T)) + (E^T^{-1} - E)^{-1}(1 + \frac{1}{2}(E^{-1} + E)) + \frac{\tilde{c}_G}{k} \mathbf{1}] \]  \hspace{1cm} (5.7)

and \( \tilde{G} \) is unchanged \( (i.e.\) obtained from \( F^{-1} \). Now, the Casimir operator of the (ungauged) group \( G(5.3) \) (alternatively, the ungauged action (5.1)) is invariant under the transformations

\[ \theta_1 \rightarrow O_1 \theta_1 ; \quad \theta_2 \rightarrow O_2 \theta_2 ; \quad E \rightarrow O_1 E O_2^T \quad (E^T \rightarrow O_2 E^T O_1^T) \]  \hspace{1cm} (5.8)

where \( O_1, O_2 \) are two constant \( O(d) \) matrices. This transformation is just a duality transformation, however, if the coordinates \( \theta^i_1, \theta^i_2 \) are compactified, one should
restrict to $O(d,\mathbb{Z})$ matrices in order to preserve the periodicity. In the latter case, if we take general $O(d,\mathbb{R})$ matrices the action is still conformal but not equivalent to the original one. As we did in section 4, we rotate the coordinates $\theta_1, \theta_2$ independently, then express the Casimir operator of $G$ in terms of the rotated coordinates, but unchange the generators of the gauged subgroup (namely, $\mathcal{J}_i = i\partial_{\theta_i}$ and $\mathcal{J}_i = i\partial_{\theta_i}$). Thus the only change in $G^{-1}$ is $E \rightarrow O_1 E O_2^T \left( E^T \rightarrow O_2 E^T O_1^T \right)$. Hence, we obtain the exact transformation of the metric through the transformation of $E$. Of course this $O(d) \times O(d)$ duality transformation is accompanied by a transformation of the exact antisymmetric tensor which we know only to one loop order. The transformation of the dilaton term can be found from the first order differential operators which are not changed under the $O(d) \times O(d)$ duality. Therefore it is easy to see that $e^\Phi \sqrt{G}$ must be invariant under the duality transformation, i.e.

$$\Phi' = \Phi + \frac{1}{2} \ln \left( \frac{\det G}{\det G'} \right) \quad (5.9)$$

where $G'$ is the transformed metric. This is the same transformation as in the semiclassical limit. The fact that $e^\Phi \sqrt{G}$ is independent of $k$ was pointed out in [13]. The matrix $E$ is general. Starting with an exact metric that correspond to some matrix $E$ implies that there is a larger conformal theory from which the coset model can be obtained (since $S_{G/H} = S_G - S_H$). Thus the procedure is general for all models where the metric $G_{\mu\nu}$ is independent of $Y^i$ and $G_{i\mu} = 0$.

Obviously, one could choose other invariant coordinates $\tilde{Y}_i = C_{ij} Y_i$ in (5.6), where $C$ is a $GL(d,\mathbb{R})$ matrix. Then $G^{-1} \rightarrow C^T G^{-1} C$ so the higher orders corrections to the inverse metric is not necessarily diagonal (like the metric we derived in section 3). But given an exact metric, one can diagonalize the $\frac{\delta^2}{\delta k^2}$ correction by a constant coordinate transformation. The important point is that this term- the $\frac{1}{k}$ correction of the inverse metric with respect to the one loop order- is unchanged under the $O(d) \times O(d)$ duality.

We return now to the notation we were using in section 4, but write the $k$ dependence explicitly. In general, when gauging a $U(1)^d$ subgroup in an action
with 2d isometries, with $G_{i\mu} = 0$, taking the $d$ gauge invariant coordinates as the vector $Y = A\theta^L + B\theta^R$ (where $A, B$ are $(d \times d)$ matrices), the exact inverse metric for the $Y_i$ components is

$$G^{-1} = -\frac{1}{k - \tilde{c}_G} (A^T G^{LL} A + A^T G^{LR} B + B^T G^{RR} B) + \frac{1}{k} (A^T J^{LL} A + B^T J^{RR} B)$$

$$= -\frac{1}{k - \tilde{c}_G} [A^T (G^{LL} - J^{LL}) A + A^T G^{LR} B + B^T (G^{RR} - J^{RR}) B] + \frac{\tilde{c}_G}{k} (A^T J^{LL} A + B^T J^{RR} B)$$

$$= \frac{1}{k - \tilde{c}_G} (G_{\text{classical}}^{-1} - \frac{2\tilde{c}_G}{k} A^T J^{LL} A)$$ (5.10)

where $G^{LL}, G^{LR}, G^{RR}$ correspond to the Casimir operator of the ungauged group and $J^{LL}, J^{RR}$ correspond to the gauged currents. We used $A^T J^{LL} A = B^T J^{RR} B$ and took the classical metric with a pre-factor $k$. (The classical metric is obtained by plugging $\tilde{c}_G = \tilde{c}_H = 0$.) Under the $O(d) \times O(d)$ duality only $G^{LL}, G^{LR}, G^{RR}$ change. Thus only the semiclassical part of the inverse metric changes. Moreover, all the semiclassical backgrounds which are obtained by $O(d) \times O(d)$ transformations can be obtained also by different gaugings of the ungauged action[26] (i.e. picking different generators for the gauged subgroup). So one can apply the one loop transformation on the classical part of the inverse metric while leaving the $\frac{1}{k}$ correction unchanged and get exact $O(d) \times O(d)$ dual models. Writing

$$G^{-1}_{\text{exact}} = \frac{1}{k - \tilde{c}_G} (G^{-1}_{\text{classical}} + 2\tilde{c}_G C^T C)$$ (5.11)

under the exact $O(d) \times O(d)$ duality the transformation is

$$G'_{\text{exact}}^{-1} = \frac{1}{k - \tilde{c}_G} (G'_{\text{classical}}^{-1} + 2\tilde{c}_G C'^T C)$$ (5.12)

$$\Phi'_{\text{exact}} = \Phi_{\text{exact}} + \frac{1}{2} \ln \left( \frac{\det G_{\text{exact}}}{\det G'_{\text{exact}}} \right)$$ (5.13)

where $G'_{\text{classical}}$ is the dual classical metric. We see that in order to find exact dual models we do not need to have the exact antisymmetric tensor! Denote the
one loop order limit of $G, B$ by $\hat{G}, \hat{B}$ then the general $O(d) \times O(d)$ transformation to one loop order is \[23\]

$$
G_{\text{classical}}^{-1} = \frac{1}{4} \left( (O_1 + O_2)(\hat{G}^{-1} - \hat{B}\hat{G}^{-1}\hat{B})(O_1 - O_2)^T - (O_1 + O_2)(\hat{G}^{-1}\hat{B}(O_1 - O_2)^T + (O_1 - O_2)\hat{B}\hat{G}^{-1}(O_1 + O_2)) \right)
$$

(5.14)

which we now substitute in (5.12).

The rest of the $O(d,d)$ generators apply as in the semiclassical limit. These are $G_{\text{exact}} \rightarrow C^T G_{\text{exact}} C$, where $C$ is a $GL(d,R)$ constant matrix, and constant shifts of the antisymmetric tensor.

Finally, consider a group $G$ WZW model with level $k$, which has a $U(1)^d$ global symmetry (i.e. the background is independent of $d$ coordinates). In order to obtain the $O(d) \times O(d)$ duality one has to use an equivalent model $G \times U(1)^d/U(1)^d_k$. The WZW is exact up to a shift $k \rightarrow k - \tilde{c}_G$, but the $O(d) \times O(d)$ duality introduces the $\frac{1}{k}$ corrections in the dual models.

6. Summary

In this paper we have derived a charged black hole solution in four dimensions based on $SL(2,R) \times SU(2) \times U(1)/U(1)^2$ WZW coset model. We compared the semiclassical solution, obtained by integrating out the gauge fields in the sigma model, to the exact to all orders solution obtained by the algebraic Hamiltonian approach. We have seen that the space-time singularity exists also in the exact solution. Moreover, the structure of the space-time described by the exact metric depends strongly on the gauge parameter, unlike in the semiclassical limit. According to the value of the gauge parameter, we have seen that a naked string singularity or a surface (membrane) singularity could exist and the black hole singularity can extends outside the horizon. The exact vector dual model was derived explicitly as well. In the semiclassical limit there is a naked singularity with a
topology $S^2 \times R^1$. In the exact solution the space time structure depends on the
gauge parameter, so that there might appear additional naked (string, membrane)
singularities which do not show in the semiclassical limit.

We have seen that the algebraic Hamiltonian approach is useful to study duality
of metrics. In particular we were able to determine how the exact metric and
dilaton transform under the $O(d, d)$ duality and discovered that the $O(\frac{1}{k})$
corrections to the inverse metric (with respect to the semiclassical inverse metric) are
invariant under the $O(d) \times O(d)$ duality and only the ”semiclassical” part of it
transforms. (The semiclassical part transforms as in the one loop order transform-
mations). Therefore, although the algebraic Hamiltonian approach has a major
disadvantage of not being useful to calculate the antisymmetric tensor, knowing
the antisymmetric tensor to one loop order only is enough to obtain the metrics
and the dilaton in all the exact $O(d, d)$ dual model.

APPENDIX

This appendix contains the expressions for the Ricci tensor and the scalar
curvature that correspond to our charged black hole solution. In section 2 we have
derived a semiclassical solution that corresponds to the following metric (before
the coordinate transformation on $r$ in (2.19))

$$dS^2 = -\frac{\sinh^2 r (1 + Q^2 \sin^2 \theta)}{\cosh^2 r + Q^2 \sin^2 \theta} dt^2 + \frac{k_1}{k_2} dr^2 + \frac{\cosh r \sin^2 \theta}{\cosh^2 r + Q^2 \sin^2 \theta} d\phi^2 + d\theta^2 \quad (A.1)$$

with $Q^2 = \tan^2 \alpha$. The corresponding exact metric was derived in section 3 (again,
before transforming $r$ in (3.22))

$$dS^2 = -\frac{\sinh^2 r (1 + C + Q^2 \sin^2 \theta)}{\cosh^2 r + Q^2 \sin^2 \theta + C} dt^2 + \frac{(k_1 - 2)}{(k_2 + 2)} dr^2$$

$$+ \frac{\cosh^2 \sin^2 \theta}{\cosh^2 r + Q^2 \sin^2 \theta} d\phi^2 + d\theta^2 \quad (A.2)$$
with
\[ Q^2 = \frac{(k_2 + 2)k_1}{(k_1 - 2)k_2} \tan^2 \alpha \] (A.3)
and
\[ C = \frac{2}{k_1 - 2} \left(1 - \frac{k_1}{k_2} \tan^2 \alpha \right) \] (A.4)

We shall write both metrics as follows:
\[ dS^2 = -\sinh^2 r (B + Q^2 \sin^2 \theta) \frac{dt^2}{\Sigma_1} + dr^2 + \cosh^2 r \sin^2 \theta \frac{d\phi^2}{\Sigma_1} + d\theta^2 \] (A.5)

where \( \Sigma_1 = \cosh^2 r + Q^2 \sin^2 \theta \) and \( \Sigma_2 = \cosh^2 r + Q^2 \sin^2 \theta + C \). Eventually we can use the limit \( C = 0 \) \( (B = 1) \) and \( a = \frac{k_1}{k_2} \) for the semiclassical metric. The Ricci tensor is the following:

\[ R_{rr} = (B + Q^2 \sin^2 \theta) \left(\frac{1}{\Sigma_2} + \frac{3 \cosh^2 r}{\Sigma_2^2} \right) + Q^2 \sin^2 \theta \left(\frac{1}{\Sigma_1} + \frac{3 \sinh^2 r}{\Sigma_1^2} \right) \] (A.6)

\[ R_{tt} = \frac{\sinh^2 r}{\Sigma_2} \left(\frac{1}{a} \left(B + Q^2 \sin^2 \theta\right) - \frac{3 \cosh^2 r}{\Sigma_2} \right) + Q^2 \sin^2 \theta - \frac{\sinh^2 r}{\Sigma_1} \left(\frac{3 \sinh^2 r}{\Sigma_1^2} \right) \] (A.7)

\[ R_{\phi\phi} = \frac{\sin^2 \theta \cosh^2 r}{\Sigma_1^2} \left(\frac{4Q^2}{\Sigma_1} \left(\frac{1}{a} \sin^2 \theta \sinh^2 r + \cosh^2 r \cos^2 \theta\right) - \frac{Q^2}{\Sigma_2} \left(\frac{1}{a} \sin^2 \theta \right) \right) \] (A.8)

\[ R_{\theta\theta} = \frac{\cosh^2 r}{\Sigma_1} + \frac{3Q^2 \cosh^2 r \cos^2 \theta}{\Sigma_1^2} - \frac{Q^2 \sinh^2 r}{(B + Q^2 \sin^2 \theta)^2 \Sigma_2} \left(B \cos 2\theta - Q^2 \sin^2 \theta\right) \] (A.9)

Finally, we give the expression for the scalar curvature \( R_{\mu}^{\mu} \). Here we use the
coordinate transformation \( \cosh^2 r \to r \)

\[
R^\mu_\nu = \frac{-Q^2 \frac{1}{a} \sin^2 \theta + r(1 + Q^2 - \frac{1}{a} Q^2 \sin^2 \theta)}{\Sigma_1} - \frac{B + Q^2 \sin^2 \theta - 1}{a \Sigma_2}
\]

\[
+ \frac{7Q^2 \left( \frac{1}{a} (r-1) \sin^2 \theta + r \cos^2 \theta \right)}{\Sigma_1^2} + \frac{3\frac{1}{a} r (B + Q^2 \sin^2 \theta + 1)}{\Sigma_2^2}
\]

\[
- \frac{Q^2 (r-1) \cos 2\theta}{(B + Q^2 \sin^2 \theta) \Sigma_2} + \frac{Q^2 (r-1)(\frac{1}{4} \sin^2 2\theta - B \cos 2\theta + Q^2 \sin^2 \theta)}{(B + Q^2 \sin^2 \theta)^2 \Sigma_2}
\]

\[
+ \frac{3Q^4 (r-1) \sin^2 2\theta}{(B + Q^2 \sin^2 \theta) \Sigma_2^2} - \frac{\frac{1}{4} Q^4 (r-1)^2 \sin^2 2\theta}{(B + Q^2 \sin^2 \theta)^2 \Sigma_2^2}
\]

\[
- \frac{Q^2}{\Sigma_1 \Sigma_2} \left( \frac{1}{a} \sin^2 \theta (B + Q^2 \sin^2 \theta + 1) + 2 \frac{r (r-1) \cos^2 \theta}{B + Q^2 \sin^2 \theta} \right) \quad (A.10)
\]

Carefully taking the limits, it is easily seen that the curvature blows up in three cases: (i) \( r + Q^2 \sin^2 \theta = 0 \) (ii) \( r + Q^2 \sin^2 \theta + C = 0 \) and (iii) \( 1 + C + Q^2 \sin^2 \theta = 0 \).

In the semiclassical limit, since \( C = 0 \) the only singularity is at \( r + Q^2 \sin^2 \theta = 0 \). However, in the exact solution \( C \) can take any value. In particular, for \( C \leq -1 \) there is an additional singularity at \( \sin \theta = \frac{-(C+1)}{Q} \). For \( C < -1 \) this singularity is on cone surfaces and for \( C = -1 \) it is a string singularity.
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