Abstract. We prove the existence and give estimates of the fundamental solution (the heat kernel) for the equation \( \partial_t = \mathcal{L}^\kappa \) for non-symmetric non-local operators

\[
\mathcal{L}^\kappa f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - 1_{|z|<1} \langle z, \nabla f(x) \rangle) \kappa(x,z) J(z) \, dz,
\]

under broad assumptions on \( \kappa \) and \( J \). Of special interest is the case when the order of the operator \( \mathcal{L}^\kappa \) is smaller than or equal to 1. Our approach rests on imposing suitable cancellation conditions on the internal drift coefficient

\[
\int_{r \leq |z| < 1} z \kappa(x,z) J(z) \, dz, \quad 0 < r \leq 1,
\]

which allows us to handle the non-symmetry of \( z \mapsto \kappa(x,z) J(z) \). The results are new even for the 1-stable Lévy measure \( J(z) = |z|^{d-1} \).

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1. Introduction

In recent years, there has been a lot of interest in constructing semigroups for Lévy-type operators [41, 42, 14, 40, 27, 7, 8, 49, 6, 50, 63, 47, 38, 16, 15, 44]. Such operators arise naturally due to the Courrège-Waldenfels theorem [33, Theorem 4.5.21], [9, Theorem 2.21]. In general, they are not symmetric, so the \( L^2 \)-theory or Dirichlet forms and the corresponding \( L^2 \)-semigroups of operators does not apply in this context. We shall discuss operators of the form

\[
\mathcal{L}^\kappa f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - 1_{|z|<1} \langle z, \nabla f(x) \rangle) \kappa(x,z) J(z) \, dz,
\]

and allow for non-symmetric measures \( \kappa(x,z) J(z) \, dz \). The paper is a continuation of the research conducted in [27], where operators of the form (1) were considered under stronger conditions. We also improve and extend the results of [38], and part of those in [16] and [42].

We now introduce our setting, notation and motivations. Let \( d \in \mathbb{N} \) and \( \nu : [0, \infty) \to [0, \infty] \) be a non-increasing function \( (\nu \not\equiv 0) \) with

\[
\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(|x|) \, dx < \infty.
\]

We consider \( J : \mathbb{R}^d \to [0, \infty) \) such that for some \( c_1 \in [1, \infty) \) and all \( x \in \mathbb{R}^d \),

\[
c_1^{-1} \nu(|x|) \leq J(x) \leq c_1 \nu(|x|).
\]

Furthermore, suppose that \( \kappa(x,z) \) is a Borel function on \( \mathbb{R}^d \times \mathbb{R}^d \) such that

\[
0 < \kappa_0 \leq \kappa(x,z) \leq \kappa_1 < \infty,
\]

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for some numbers $\kappa_0$, $\kappa_1$, and there is $\beta \in (0, 1)$ and a number $\kappa_2 \geq 0$ with
\[ |\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^{\beta}. \]

The following concentration functions play a prominent role in the paper,
\[ h(r) := \int_{\mathbb{R}^d} \left( 1 + \frac{|x|^2}{r^2} \right) \nu(|x|) dx, \quad K(r) := r^{-2} \int_{|x|<r} |x|^2 \nu(|x|) dx, \quad r > 0. \]

We say that the weak scaling condition at the origin holds if there are $\alpha_h \in (0, 2]$ and $C_1 \in [1, \infty)$ such that
\[ h(r) \leq C_1 \lambda^{\alpha_h} h(\lambda r), \quad r, \lambda \in (0, 1). \]

In a similar fashion, we consider the existence of $\beta_h \in (0, 2]$ and $c_1 \in (0, 1]$ such that
\[ h(r) \geq c_h \lambda^{\beta_h} h(\lambda r), \quad r, \lambda \in (0, 1]. \]

We propose two more conditions on $\kappa(x, z)$ and $J(z)$. Suppose there are numbers $\kappa_3, \kappa_4 \geq 0$ such that
\[
\begin{align*}
\sup_{z \in \mathbb{R}^d} \int_{r \leq |z| < 1} z \kappa(x, z) J(z) dz &\leq \kappa_3 r h(r), \quad r \in (0, 1], \quad (7) \\
\int_{r \leq |z| < 1} z \left[ \kappa(x, z) - \kappa(y, z) \right] J(z) dz &\leq \kappa_4 |x - y|^\beta r h(r), \quad r \in (0, 1]. \quad (8)
\end{align*}
\]

We are ready to specify our framework. The dimension $d$ and the profile function $\nu$ are fixed. We will use alternatively two sets of assumptions.

(Q1): (2)–(5) hold, $\alpha_h = 1$; (7) and (8) hold;
(Q2): (2)–(6) hold, $0 < \alpha_h \leq \beta_h < 1$ and $1 - \alpha_h < \beta \wedge \alpha_h$; (7) and (8) hold.

For the sake of the discussion, we assume that (2)–(5) hold. We notify that the internal non-symmetry of the operator (1) may result in a non-zero internal drift coefficient
\[ \int_{\mathbb{R}^d} z \left( 1_{|z|<r} - 1_{|z|<1} \right) \kappa(x, z) J(z) dz. \]

Accordingly,
\[ L^c f(x) = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - 1_{|z|<r} \langle z, \nabla f(x) \rangle \right) \kappa(x, z) J(z) dz \\
+ \left( \int_{\mathbb{R}^d} z \left( 1_{|z|<r} - 1_{|z|<1} \right) \kappa(x, z) J(z) dz \right) \cdot \nabla f(x). \]

The influence of the internal drift term (9) may be different depending on the order of the operator $L^c$ measured by $\alpha_h$ and $\beta_h$ in (5) and (6). Below we will usually let $r = h^{-1}(1/t)$.

**Fact 1.1.** If (2)–(5) hold and $\alpha_h > 1$, then (7) and (8) hold.

The fact follows from Lemma 8.1. Thus, we have that if $\alpha_h > 1$ (the sub-critical case), then the inequalities (7) and (8) are automatically satisfied. It explains a posteriori the success of the analysis of the sub-critical non-symmetric case in [27], see also [42], [38], [16], [48]. On the other hand, if $\alpha_h = 1$ (the critical case) or $\alpha_h < 1$ (the super-critical case), the study of the operator (1) is harder and we need the conditions (7) and (8). This resolves a question about natural conditions for critical and sub-critical non-symmetric operators that lead to strong results, similar to those in [27], and which provide a thorough analysis of the operator, the fundamental solution for the corresponding parabolic equation, and the associated semigroup. We obtain such results if either of the set of assumptions (Q1) or (Q2) is satisfied, see Section 2.
that under the symmetry condition, i.e., when \( J(z) = J(-z) \) and \( \kappa(x, z) = \kappa(x, -z) \), \( x, z \in \mathbb{R}^d \), which is usually assumed in the literature, the problematic terms involving \( \nabla f \) disappear after rewriting the operator \( \mathcal{L}^\alpha \) as
\[
\frac{1}{2} \int_{\mathbb{R}^d} (f(x + z) + f(x - z) - 2f(x)) \kappa(x, z) J(z) \, dz .
\]
In fact, under the symmetry, we have
\[
\sup_{r \in (0, 1]} \sup_{x \in \mathbb{R}^d} \left| \int_{|z| < 1} z \kappa(x, z) J(z) \, dz \right| = 0 .
\] (10)
The condition (10) was used in a non-symmetric case in [38] and [16] for \( \nu(r) = r^{-d-1} \). The conditions (7) and (8) are much less restrictive.

**Example 1.** Let \( \nu(r) = r^{-d-1} \). Then (5) and (6) hold with \( \alpha_h = \beta_h = 1 \). The inequalities (7) and (8) are read as
\[
\sup_{x \in \mathbb{R}^d} \left| \int_{|z| < 1} z \kappa(x, z) J(z) \, dz \right| \leq c , \quad r \in (0, 1] ,
\]
\[
\int_{|z| < 1} z \left[ \kappa(x, z) - \kappa(y, z) \right] J(z) \, dz \leq c|x - y|^3 , \quad r \in (0, 1] .
\]
Hence, if \( J(z) \) and \( \kappa(x, z) \) are such that (2), (3), (4) and the above inequalities hold, then the assumptions (Q1) are satisfied. We also note that \( \int_{|z| < 1} |z| \nu(|z|) \, dz = c \log(1/r) \).

Due to our general setting, we can deal with other interesting operators.

**Example 2.** Let \( \nu(r) = r^{-d-1} \log(2 + 1/r) \). Then (5) holds with \( \alpha_h = 1 \), but not with any \( \alpha_h > 1 \). Furthermore, (6) holds for every \( \beta_h > 1 \), but not with \( \beta_h = 1 \). We also have that \( \nu(r) \) is comparable to \( r^{-d} h(r) \), see [27, Lemma 5.3 and 5.4]. Thus (7) and (8) allow, respectively, logarithmic growth as \( r \to 0 \) as follows
\[
\sup_{x \in \mathbb{R}^d} \left| \int_{|z| < 1} z \kappa(x, z) J(z) \, dz \right| \leq c \log(2 + 1/r) , \quad r \in (0, 1] ,
\]
\[
\int_{|z| < 1} z \left[ \kappa(x, z) - \kappa(y, z) \right] J(z) \, dz \leq c|x - y|^3 \log(2 + 1/r) , \quad r \in (0, 1] .
\]
Thus, if \( J(z) \) and \( \kappa(x, z) \) are such that (2), (3), (4) and the above inequalities hold, then the assumptions (Q1) are satisfied. Noteworthy, here \( \int_{|z| < 1} |z| \nu(|z|) \, dz \) is comparable to \( \log(2 + 1/r)^2 \) for small \( r \).

**Example 3.** Let \( \nu(r) = r^{-d-\alpha} \) and \( \alpha \in (1/2, 1) \). Then (5) and (6) hold with \( \alpha_h = \beta_h = \alpha \). Note that for \( J(z) \) and \( \kappa(x, z) \) to satisfy (Q2), we need the balance condition \( \alpha + \beta > 1 \) to hold. Furthermore, we have \( rh(r) = r^{1-\alpha} h(1) \), while \( \int_{|z| < 1} |z| \nu(|z|) \, dz \) is comparable to a positive constant for small \( r \).

Given a profile function \( \nu \), it is not hard to find \( J(z) \) and \( \kappa(x, z) \) such that (2)–(4) hold, and
\[
\sup_{x \in \mathbb{R}^d} \left| \int_{r \leq |z| < 1} z \kappa(x, z) J(z) \, dz \right| \geq c \int_{r \leq |z| < 1} |z| \nu(|z|) \, dz , \quad r \in (0, 1] ,
\]
for some \( c > 0 \). Therefore, in each of the above examples, such choice of \( J(z) \) and \( \kappa(x, z) \) is not admissible, because the condition (7) fails. In fact, they can be chosen so that (8) fails as well. Put differently, similarly to (10), the conditions (7) and (8) require certain cancellations to take place.
The success of our approach based on the usage of (7) and (8) suggests that also in other studies and contexts where a counterpart of (10) plays a role (see [13]) a relaxation of assumptions to proper counterparts of (7) and (8) should be possible.

As further applications, our results allow solving uniquely the martingale problem for the operator \( (\mathcal{L}^\kappa, C_0^\infty (\mathbb{R}^d)) \). They also have applications to the Kato class of the semigroup \((P_t^\kappa)_{t \geq 0}\) corresponding to \( \mathcal{L}^\kappa \), given in (17). For details see [27, Remarks 1.5 and 1.6].

Throughout the paper we make an effort to control how constants depend on the initial parameters of our model. The reason for that is twofold. First of all, it is necessary in the preliminaries, like Sections 4 and 5, to be able to execute the construction and to find the key properties of a candidate for the solution in Section 6. The second reason is more application-oriented: uniform results for families of operators or processes are desired in such areas as mixing property, multiscale models, homogenization, stationary distribution, see [1], [54], [60]. The operators we consider resemble those investigated in the study of mean field games [22].

The main tool used in this paper is the parametrix method, proposed by E. Levi [53] to solve elliptic Cauchy problems. It was successfully applied in the theory of partial differential equations [26], [58], [18], [23], with an overview in the monograph [24], as well as in the theory of pseudo-differential operators [21], [40], [42], [49], [63]. In particular, operators comparable in a sense to the fractional Laplacian were intensively studied by this method [19], [20], [46], [48], [21], also very recently [14], [38], [16], [50]. More detailed historical comments on the development of the method can be found in [24, Bibliographical Remarks] and in the introductions of [42] and [6].

Basically we follow the scheme of [27], which in turn extends and strengthens [40] and [14]. The results in the present paper are of the same type as in [27] with the main progress being the recognition and usage of the conditions (7) and (8).

Other related papers treat, for instance, (symmetric) singular Lévy measures [6], [50] or (symmetric) exponential Lévy measures [39]. We also list some papers that use different techniques to associate a semigroup with an operator by symbolic calculus [61], [31], [52], [30], [32], [34], [7], [8], Dirichlet forms [25], [10], [2], [11], [12] or perturbation series [57], [4], [36], [37], [35], [5]. For probabilistic methods and applications, we refer the reader to [17], [47], [56], [43], [51], [50].

As stated in the abstract, when the present paper was first made public on arXiv:1807.04257v1, the results were new even for the operators discussed in Example 1. Those operators are now included in the recent paper [45].

2. Main results

We start by giving an exact meaning to (1). We apply the operator (1), in a strong or weak sense, only when it is well defined according to the following definitions. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a Borel measurable function.

Definition 1 (Strong operator). We say that the operator \( \mathcal{L}^\kappa f \) is well defined if the gradient \( \nabla f(x) \) exists and the corresponding integral in (1) converges absolutely for every \( x \in \mathbb{R}^d \).

We denote by \( \mathcal{L}^{\kappa, \varepsilon} f \) the expression (1) with \( J(z) \) replaced by \( J(z)1_{|z| > \varepsilon} \), \( \varepsilon \in [0, 1] \).

Definition 2 (Weak operator). We let
\[
\mathcal{L}^{\kappa, 0+} f(x) := \lim_{\varepsilon \to 0^+} \mathcal{L}^{\kappa, \varepsilon} f(x),
\]
if the (strong) operators \( \mathcal{L}^{\kappa, \varepsilon} f \) are well defined for \( \varepsilon \in (0, 1] \), and the limit exists for every \( x \in \mathbb{R}^d \).

It is clear that the operator \( \mathcal{L}^{\kappa, 0+} \) is an extension of \( \mathcal{L}^{\kappa, 0} = \mathcal{L}^\kappa \), meaning that if \( \mathcal{L}^\kappa f \) is well defined, then so is \( \mathcal{L}^{\kappa, 0} f \) and \( \mathcal{L}^{\kappa, 0+} f = \mathcal{L}^\kappa f \). Therefore, it is desired to prove the existence of solutions to the equation \( \partial_t \mathcal{L}^\kappa = \mathcal{L}^{\kappa, 0+} \) and the uniqueness of a solution to \( \partial_t \mathcal{L}^\kappa = \mathcal{L}^{\kappa, 0+} \).
Here are our main results.

**Theorem 2.1.** Assume (Q1) or (Q2). Let $T > 0$. There is a unique function $p^\kappa(t, x, y)$ on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ such that

(i) For all $t \in (0, T]$, $x, y \in \mathbb{R}^d$, $x \neq y$,

$$\partial_t p^\kappa(t, x, y) = L_x^{\kappa, 0} p^\kappa(t, x, y).$$  \hfill (11)

(ii) The function $p^\kappa(t, x, y)$ is jointly continuous on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and for any $f \in C^\infty_c(\mathbb{R}^d)$,

$$\lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y) \, dy - f(x) \right| = 0.$$  \hfill (12)

(iii) For every $t_0 \in (0, T)$ there are $c > 0$ and $f_0 \in L^1(\mathbb{R}^d)$ such that for all $t \in (t_0, T]$,

$$|p^\kappa(t, x, y)| \leq c f_0(x - y),$$  \hfill (13)

and

$$|L_x^{\kappa, \varepsilon} p^\kappa(t, x, y)| \leq c, \quad \varepsilon \in (0, 1].$$  \hfill (14)

(iv) For every $t \in (0, T]$ there is $c > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|\nabla_x p^\kappa(t, x, y)| \leq c.$$  \hfill (15)

In the next theorem, we collect more qualitative properties of $p^\kappa(t, x, y)$. To this end, for $t > 0$ and $x \in \mathbb{R}^d$ we define the bound function,

$$\Upsilon_t(x) := \left( [h^{-1}(1/t)]^{-d} \wedge \frac{t K(|x|)}{|x|^d} \right).$$  \hfill (16)

It is an integrable function, which may provide sharp estimates for the heat kernel, extending the well known two-sided bounds $t^{-d/\alpha} \wedge t/|x|^{d+\alpha}$ of the fundamental solution to $\partial_t = \Delta^{\alpha/2}$, where $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is the fractional Laplacian, see [29, Theorem 1.1] as well as more detailed discussion provided in [29, Section 5]. Properties of the bound function can be found also in [27, Section 5].

**Theorem 2.2.** Assume (Q1) or (Q2). The following hold true.

1. (Non-negativity) The function $p^\kappa(t, x, y)$ is non-negative on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
2. (Conservativeness) For all $t > 0$, $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, y) \, dy = 1.$$

3. (Chapman-Kolmogorov equation) For all $s, t > 0$, $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, z) p^\kappa(s, z, y) \, dz = p^\kappa(t + s, x, y).$$

4. (Upper estimate) For every $T > 0$ there is $c > 0$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,

$$p^\kappa(t, x, y) \leq c \Upsilon_t(y - x).$$

5. (Fractional derivative) For every $T > 0$ there is $c > 0$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,

$$|L_x^{\kappa} p^\kappa(t, x, y)| \leq c t^{-1} \Upsilon_t(y - x).$$

6. (Gradient) For every $T > 0$ there is $c > 0$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,

$$|\nabla_x p^\kappa(t, x, y)| \leq c \left[ h^{-1}(1/t) \right]^{-1} \Upsilon_t(y - x).$$

7. (Continuity) The function $L_x^{\kappa} p^\kappa(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. 

\begin{align}
(8) \text{(Strong operator)} \quad & \text{For all } t > 0, x, y \in \mathbb{R}^d,
\partial_t p^\kappa(t, x, y) = \mathcal{L}_x^\kappa p^\kappa(t, x, y). \\
(9) \text{(Hölder continuity)} \quad & \text{For all } T > 0, \gamma \in [0, 1] \cap [0, \alpha_h), \text{ there is } c > 0 \text{ such that for all } t \in (0, T] \text{ and } x, x', y \in \mathbb{R}^d,
|p^\kappa(t, x, y) - p^\kappa(t, x', y)| \leq c(|x - x'|^\gamma + 1) \left[ h^{-1}(1/t) \right]^{-\gamma} (\Upsilon_t(y - x) + \Upsilon_t(y - x')). \\
(10) \text{(Hölder continuity)} \quad & \text{For all } T > 0, \gamma \in [0, \beta) \cap [0, \alpha_h), \text{ there is } c > 0 \text{ such that for all } t \in (0, T] \text{ and } x, y, y' \in \mathbb{R}^d,
|p^\kappa(t, x, y) - p^\kappa(t, x, y')| \leq c(|y - y'|^\gamma + 1) \left[ h^{-1}(1/t) \right]^{-\gamma} (\Upsilon_t(y - x) + \Upsilon_t(y' - x')).
\end{align}

The constants in (4) – (6) may be chosen to depend only on \(d, c, \kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \beta, \alpha_h, C_h, h, T\). The same holds for (9) and (10) but with additional dependence on \(\gamma\).

For \(t > 0\), we define

\[ P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y)f(y) \, dy, \quad x \in \mathbb{R}^d, \]

whenever the integral exists in the Lebesgue sense. We also put \(P_0^\kappa = \text{Id}\), the identity operator.

**Theorem 2.3.** Assume (Q1) or (Q2). The following hold true.

1. \((P_t^\kappa)_{t \geq 0}\) is an analytic strongly continuous positive contraction semigroup on \((C_0(\mathbb{R}^d), \| \cdot \|_\infty)\).
2. \((P_t^\kappa)_{t \geq 0}\) is an analytic strongly continuous semigroup on every \((L^p(\mathbb{R}^d), \| \cdot \|_p), p \in [1, \infty)\).
3. Let \((\mathcal{A}^\kappa, D(\mathcal{A}^\kappa))\) be the generator of \((P_t^\kappa)_{t \geq 0}\) on \((C_0(\mathbb{R}^d), \| \cdot \|_\infty)\). Then
   \begin{align*}
   & (a) \ C_0^2(\mathbb{R}^d) \subseteq D(\mathcal{A}^\kappa) \text{ and } \mathcal{A}^\kappa = \mathcal{L}_x^\kappa \text{ on } C_0^2(\mathbb{R}^d), \\
   & (b) \ (\mathcal{A}^\kappa, D(\mathcal{A}^\kappa)) \text{ is the closure of } (\mathcal{L}^\kappa, C_c^\infty(\mathbb{R}^d)), \\
   & (c) \text{ the function } x \mapsto p^\kappa(t, x, y) \text{ belongs to } D(\mathcal{A}^\kappa) \text{ for all } t > 0, y \in \mathbb{R}^d, \text{ and }
   \mathcal{A}_x^\kappa p^\kappa(t, x, y) = \mathcal{L}_x^\kappa p^\kappa(t, x, y) = \partial_t p^\kappa(t, x, y), \quad x \in \mathbb{R}^d.
   \end{align*}
4. Let \((\mathcal{A}^\kappa, D(\mathcal{A}^\kappa))\) be the generator of \((P_t^\kappa)_{t \geq 0}\) on \((L^p(\mathbb{R}^d), \| \cdot \|_p), p \in [1, \infty)\). Then
   \begin{align*}
   & (a) \ C_0^2(\mathbb{R}^d) \subseteq D(\mathcal{A}^\kappa) \text{ and } \mathcal{A}^\kappa = \mathcal{L}_x^\kappa \text{ on } C_0^2(\mathbb{R}^d), \\
   & (b) \ (\mathcal{A}^\kappa, D(\mathcal{A}^\kappa)) \text{ is the closure of } (\mathcal{L}^\kappa, C_c^\infty(\mathbb{R}^d)), \\
   & (c) \text{ the function } x \mapsto p^\kappa(t, x, y) \text{ belongs to } D(\mathcal{A}^\kappa) \text{ for all } t > 0, y \in \mathbb{R}^d, \text{ and in } L^p(\mathbb{R}^d), \\
   & \mathcal{A}^\kappa p^\kappa(t, \cdot, y) = \mathcal{L}^\kappa p^\kappa(t, \cdot, y) = \partial_t p^\kappa(t, \cdot, y).
   \end{align*}

Finally, we provide a lower bound for the heat kernel \(p^\kappa(t, x, y)\). It is quite typical that one first proves a lower bound in terms of \(h^{-1}\) and \(\nu\), as we do in the first two statements of Theorem 2.4, cf. [29, Remark 5.7 and Section 4], [28, Section 5]. In our setting, we have that \(\nu(r) \leq cr^{-d}K(r)\), see [29, Lemma 7.1], which indicates a possible difference between those lower bounds and the upper bound by \(\Upsilon_t(y - x)\). The converse inequality \(\nu(r) \geq cr^{-d}K(r)\) is well understood, see [29, Lemma 7.3], and leads to sharp two-sided bounds in the third statement of the theorem. For abbreviation, we write \(\varpi\) to denote the collection of \(d, c, \kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \beta, \alpha_h, C_h, h\).

**Theorem 2.4.** Assume (Q1) or (Q2). The following hold true.

1. There are \(T_0 = T_0(\nu, \varpi) > 0\) and \(c = c(\nu, \varpi) > 0\) such that for all \(t \in (0, T_0]\), \(x, y \in \mathbb{R}^d\),
\[ p^\kappa(t, x, y) \geq c \left[ h^{-1}(1/t) \right]^{-d} \nu((x - y)^2) \]
(18)
2. If additionally \(\nu\) is positive, then for every \(T > 0\) there is \(c = c(T, \nu, \varpi) > 0\) such that (18) holds for \(t \in (0, T]\) and \(x, y \in \mathbb{R}^d\).
(iii) If additionally there are $\tilde{\beta} \in [0, 2)$ and $\tilde{c} > 0$ such that $\tilde{c}\lambda^{d+\tilde{\beta}}\nu(\lambda r) \leq \nu(r)$, $\lambda \leq 1$, $r > 0$, then for every $T > 0$ there is $c = c(T, \nu, \tilde{c}, \tilde{\beta}, \bar{\omega}) > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$p^c(t, x, y) \geq c\Upsilon_t(y - x).$$

**Remark 2.5.** If (3), (4) hold, then $|\kappa(x, z) - \kappa(y, z)| \leq (2\kappa_1 \vee \kappa_2)|x - y|^{\beta_1}$ for every $\beta_1 \in [0, \beta]$. According to the parametrix method, the fundamental solution $p^c$ is expected to be given by

$$p^c(t, x, y) = p^{\bar{\kappa}}(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^{\bar{\kappa}}(t - s, x, z)q(s, z, y)\,dz\,ds,$$

where $q(t, x, y)$ solves the equation

$$q(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q(s, z, y)\,dz\,ds,$$

and

$$q_0(t, x, y) = (\mathcal{L}^{\bar{\kappa}}_x - \mathcal{L}^{\bar{\kappa}}_y)p^{\bar{\kappa}}(t, x, y).$$

Here $p^{\bar{\kappa}}$ is the heat kernel corresponding to the Lévy operator $\mathcal{L}^{\bar{\kappa}}$ obtained from the operator $\mathcal{L}^\kappa$ by freezing its coefficients: $\bar{\kappa}_w(z) = \kappa(w, z)$. In our setting, we draw the initial knowledge on $p^{\bar{\kappa}}$ from [29], which we then exploit in Sections 4 and 5 to establish further properties. Already in this preliminary part we essentially incorporate (7) and (8), which differs from [27]. We also see the effect of the internal drift and the fact that the order of the operator does not have to be strictly larger than one, e.g., Proposition 4.6, (46), Lemma 5.6. In Section 6.1 we carry out the construction of $q$ and so also $p^c$. In view of future developments, the following remark is notable.

**Remark 2.6.** We emphasize that the construction of $p^c$ is possible, and many preliminary facts hold true under a weaker assumption

(Q0): (2)–(5) hold, $\alpha_h \in (0, 1)$; (7) and (8) hold.

In particular, see Lemma 6.1, Theorem 6.2, Lemma 6.3, Proposition 6.15 and (29), (31).

The subsequent non-trivial step is to verify that $p^c$ is the actual solution. To this end, in Section 6.2 we need extra constraints which eventually result in (Q1) and (Q2), see for instance Lemma 6.10 and the comments preceding Lemma 6.5 and Lemma 6.8. In Section 6.3 we collect the initial properties of $p^c$. In Section 7 we establish a nonlocal maximum principle, analyze the semigroup $(P^c_t)_{t \geq 0}$, complement the fundamental properties of $p^c$, and prove Theorems 2.1–2.4. Section 8 contains auxiliary results. We give a final comment on the connection with [27].

**Remark 2.7.** The structure of the present paper is similar to that of [27] to keep the same train of thought, but also to facilitate the transition between the papers while comparing and identifying the corresponding results. The reason for doing the latter is that the proofs that are the same as in [27] are reduced to a minimum, we only list which facts are needed, occasionally give general ideas, and refer the reader to [27] for details. We deliberately focus on and explain those aspects that are different from [27]. We believe that such a presentation makes the content more comprehensible. In Lemma 7.10 we also give a correction of a part of the proof of [27, Lemma 4.10].

In what follows, the function $\nu$ and the constants $d$, $c_2$, $\kappa_0$, $\kappa_1$, $\kappa_2$, $\beta$, $\kappa_3$, $\kappa_4$, $\alpha_h$, $C_h$, $\beta_h$, $c_h$ can be regarded as fixed. Apart from Sections 7 and 8, we explicitly formulate all assumptions in lemmas, corollaries, propositions, and theorems. On the other hand, in Section 7 we assume that either (Q1) or (Q2) holds.
3. Notation

For the reader’s convenience, we collect inhere the notation repeatedly used in the paper. By $c(d, \ldots)$ we denote a positive number that depends only on the listed parameters $d, \ldots$. By $\sigma$ we represent the collection of $c_r, \kappa_0, \kappa_1, \kappa_3, \alpha_h, C_h, h$. Throughout the article, $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface measure of the unit sphere in $\mathbb{R}^d$. We use “:=” to denote the definition. As usual, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

The operator $\mathcal{L}^\kappa$ is given in (1). The functions $h(r), K(r)$, and $\Upsilon(x)$ were introduced in Section 1 and Section 2. Here is a glossary of symbols to be used (and explained) below. For

$$\mathcal{R}: \mathbb{R}^d \to [0, \infty),$$

we introduce the operator

$$\mathcal{L}^\mathcal{R}f(x) := \int_{\mathbb{R}^d} (f(x + z) - f(x) - 1_{|z|<1} \langle z, \nabla f(x) \rangle) \mathcal{R}(z)J(z)\,dz. \tag{20}$$

The corresponding heat kernel is denoted by

$$p^\mathcal{R}(t, x, y) = p^\mathcal{R}(t, y - x). \tag{21}$$

We let

$$\delta^\mathcal{R}_1(t, x, y; z) := p^\mathcal{R}(t, x + z, y) - p^\mathcal{R}(t, x, y) - 1_{|z|<r} \langle z, \nabla_x p^\mathcal{R}(t, x, y) \rangle, \tag{22}$$

and

$$\delta^\mathcal{R}_1(t, x, y; z) := \delta^\mathcal{R}_1(t, x, y; z).$$

Thus for $\mathcal{R}_1$ and $\mathcal{R}_2$ we have

$$\mathcal{L}^\mathcal{R}_1 p^\mathcal{R}_2(t, x, y) = \int_{\mathbb{R}^d} \delta^\mathcal{R}_2(t, x, y; z) \mathcal{R}_1(z)J(z)\,dz, \tag{23}$$

and

$$\mathcal{L}^\mathcal{R}_2 p^\mathcal{R}_1(t, x, y) = \int_{\mathbb{R}^d} \delta^\mathcal{R}_1(t, x, y; z) \mathcal{R}_1(z)J(z)\,dz + \left( \int_{\mathbb{R}^d} z\left(1_{|z|<r} - 1_{|z|<1}\right) \mathcal{R}_1(z)J(z)\,dz \right) \cdot \nabla_x p^\mathcal{R}_2(t, x, y). \tag{24}$$

Starting from Section 5, we shall use

$$\mathcal{R}_w(z) := \kappa(w, z), \tag{25}$$

which defines $\mathcal{L}^\mathcal{R}_w f(x), p^\mathcal{R}_w(t, x, y)$ and $\delta^\mathcal{R}_w(t, x, y; z)$. The main objects in the paper are

$$q_0(t, x, y) := \left( \mathcal{L}^\mathcal{R}_x - \mathcal{L}^\mathcal{R}_y \right) p^\mathcal{R}_y(t, x, y) = \int_{\mathbb{R}^d} \delta^\mathcal{R}_y(t, x, y; z) \left( \kappa(x, z) - \kappa(y, z) \right) J(z)\,dz, \tag{26}$$

$$q_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q_{n-1}(s, z, y)\,dz\,ds, \tag{27}$$

$$q(t, x, y) := \sum_{n=0}^\infty q_n(t, x, y), \tag{28}$$

and

$$p^\mathcal{R}_w(t, x, y) := p^\mathcal{R}_w(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^\mathcal{R}_w(t - s, x, z)q(s, z, y)\,dz\,ds. \tag{29}$$

The integral part of (29) is of special interest and to investigate its properties we introduce

$$\phi_q(t, x, s) := \int_{\mathbb{R}^d} p^\mathcal{R}_w(t - s, x, z)q(s, z, y)\,dz, \tag{30}$$
and
\[ \phi_y(t, x) := \int_0^t \phi_y(t, x, s) \, ds = \int_{\mathbb{R}^d} \int_0^t p^R_z(t-s, x, z) q(s, z, y) \, dz \, ds. \] (31)

Our estimates shall be frequently presented by means of
\[ \rho^2(t, x) := \left[ h^{-1}(1/t) \right]^\gamma (|x|^{\beta} \wedge 1) t^{-1} \mathcal{Y}_t(x). \] (32)

To shorten the notation in Section 4 we shall use the expressions
\[ \mathcal{F}_1 := \mathcal{Y}_t(y-x-z) 1_{|z| \geq h^{-1}(1/t)} + \left( \frac{|z|}{h^{-1}(1/t)} \right)^2 \mathcal{Y}_t(y-x), \]
\[ \mathcal{F}_2 := \mathcal{Y}_t(y-x-z) 1_{|z| < h^{-1}(1/t)} + \left( \frac{|z|}{h^{-1}(1/t)} \right) \mathcal{Y}_t(y-x). \]

Thus, \( \mathcal{F}_1 = \mathcal{F}_1(t, x, y; z) \) and \( \mathcal{F}_2 = \mathcal{F}_2(t, x, y; z) \). We shall also need the non-increasing function
\[ \Theta(t) := 1 + \ln \left( 1 \vee \left[ h^{-1}(1/t) \right]^{-1} \right), \quad t > 0. \]

We use the following function spaces: \( L^p(\mathbb{R}^d) \) denotes the Lebesgue space with \( p \in [1, \infty) \), \( C(D) \) are continuous functions on \( D \subseteq \mathbb{R}^n \), \( n \in \mathbb{N} \). Furthermore, \( C_b(\mathbb{R}^d) \), \( C_b(\mathbb{R}^d) \), \( C_c(\mathbb{R}^d) \) are subsets of \( C(\mathbb{R}^d) \) of functions that are bounded, vanish at infinity, and have compact support, respectively. We write \( f \in C^k(\mathbb{R}^d) \) if the function and all its derivatives up to (including if finite) order \( k \in \mathbb{N} \cup \{ \infty \} \) are elements of \( C(\mathbb{R}^d) \); we similarly understand \( C^k_b(\mathbb{R}^d) \), \( C^k_0(\mathbb{R}^d) \), \( C^k_c(\mathbb{R}^d) \). In particular, \( C^\infty_c(\mathbb{R}^d) \) are smooth functions with compact support. The set \( C^{k,\eta}(\mathbb{R}^d) \) consists of functions in \( C^k(\mathbb{R}^d) \) such that all the derivatives of order \( k \) are (uniformly) Hölder continuous with exponent \( 0 < \eta < 1 \); we similarly define \( C^{k,\eta}_b(\mathbb{R}^d) \), \( C^{k,\eta}_0(\mathbb{R}^d) \), \( C^{k,\eta}_c(\mathbb{R}^d) \).

4. Analysis of the heat kernel of \( L^R \)

In this section, we consider \( L^R \) given by (20) with \( J(z) \) satisfying (2) and (5), and a function \( \mathcal{R}(z) \) such that
\[ 0 < \kappa_0 \leq \mathcal{R}(z) \leq \kappa_1, \] (33)
and
\[ \int_{r \leq |z| < 1} z \mathcal{R}(z) J(z) \, dz \leq \kappa_3 r h(r), \quad r \in (0, 1]. \] (34)

The operator \( L^R f \) is well defined for functions \( f \in C^\infty_c(\mathbb{R}^d) \) and uniquely determines a Lévy process and its transition density \( p^R(t, x, y) \) as represented in (21). Then for all \( t > 0, x, y \in \mathbb{R}^d \),
\[ \partial_t p^R(t, x, y) = L^R p^R(t, x, y). \] (35)

For more information we refer the reader to [27, Section 6]; in particular, the condition [27, (96)] is satisfied, see (2), [27, (86)] and (5).

Clearly, (33) corresponds to (3), while (34) corresponds to (7). We want to emphasize the role of (7) and (34). In particular, (34) yields the following fundamental upper bound for \( p^R \) and its derivatives.

**Proposition 4.1.** Assume (2), (5), (33), (34). For every \( T \geq 0 \) and \( \beta \in \mathbb{N}_0^d \) there exists a constant \( c = c(d, T, \beta, \sigma) \) such that for all \( t \in (0, T] \), \( x, y \in \mathbb{R}^d \),
\[ |\partial_\beta^R p^R(t, x, y)| \leq c \left[ h^{-1}(1/t) \right]^{-|\beta|} \mathcal{X}_t(y-x). \]

**Proof.** The result follows from [29, Proposition 5.4 ii)] with \( r_*=1. \) \( \square \)

Here is a lower bound for \( p^R \).
Lemma 4.2. Assume (2), (5), (33), (34). For every $T, \theta > 0$ there exists a constant $\tilde{c} = \tilde{c}(d, T, \theta, \nu, \sigma)$ such that for all $t \in (0, T]$ and $|x - y| \leq \theta h^{-1}(1/t)$,

$$p^R(t, x, y) \geq \tilde{c} \left[ h^{-1}(1/t) \right]^{-d}.$$

Proof. We use [29, Corollary 5.5] with $x - y - tb_{\eta_0^{1/(1/\theta)}}$ in place of $x$, which is allowed since by (34) we have $|tb_{\eta_0^{1/(1/\theta)}}| \leq ah_0^{-1}(1/t)$ for $a = a(d, T, \sigma)$.

Proposition 4.1 enables analysis of the increments of the heat kernel.

Lemma 4.3. Assume (2), (5), (33), (34). For every $T > 0$ there exists a constant $c = c(d, T, \sigma)$ such that for all $r > 0$, $t \in (0, T]$, $x, x', y, z \in \mathbb{R}^d$ we have

$$|p^R(t, x + z, y) - p^R(t, x, y)| \leq c F_2(t, x, y; z),$$

$$|\nabla_x p^R(t, x + z, y) - \nabla_x p^R(t, x, y)| \leq c \left[ h^{-1}(1/t) \right]^{-1} F_2(t, x, y; z),$$

$$|\delta_{1,r}^R(t, x, y; z)| \leq c (F_1(t, x, y; z) 1_{|z| < r} + F_2(t, x, y; z) 1_{|z| \geq r}),$$

and whenever $|x' - x| < h^{-1}(1/t)$, then

$$|\delta_{1,r}^R(t, x', y; z) - \delta_{1,r}^R(t, x, y; z)| \leq c \left( \frac{|x' - x|}{h^{-1}(1/t)} \right) (F_1(t, x, y; z) 1_{|z| < r} + F_2(t, x, y; z) 1_{|z| \geq r}).$$

Proof. The inequalities follow from Proposition 4.1 and [27, Corollary 5.10], cf. [27, Lemma 2.3 – 2.8]. The idea of the proof is to represent the differences as integrals of derivatives in all cases when the absolute value of the argument increment is smaller than $h^{-1}(1/t)$.

Due to [27, Corollary 5.10], the inequality (36) can be written equivalently as

$$|p^R(t, x', y) - p^R(t, x, y)| \leq c \left( \frac{|x' - x|}{h^{-1}(1/t)} \wedge 1 \right) (\Upsilon_t(y - x') + \Upsilon_t(y - x)).$$

The form (36) is useful for estimating integrals, whereas (40) easily yields what follows.

Lemma 4.4. Assume (2), (5), (33), (34). For every $T > 0$ there exists a constant $c = c(d, T, \sigma)$ such that for all $t \in (0, T]$, $x, x', y, w \in \mathbb{R}^d$ and $\gamma \in [0, 1]$,

$$|p^R(t, x', y) - p^R(t, x, y)| \leq c |x - x'|^{\gamma} \wedge 1 \left[ h^{-1}(1/t) \right]^{-\gamma} (\Upsilon_t(y - x') + \Upsilon_t(y - x)).$$

In the next result we estimate $L_{x, t}^R p^R(t, x, y)$, crucially using (34).

Lemma 4.5. Assume (2), (5) and let $\tilde{R}_1, \tilde{R}_2$ satisfy (33), (34). For every $T > 0$ there exists a constant $c = c(d, T, \sigma)$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$ we have

$$\left| \int_{\mathbb{R}^d} \delta_{1,r}^R(t, x, y; z) \tilde{R}_1(z) J(z) dz \right| \leq c t^{-1} \Upsilon_t(y - x).$$

Proof. Let $I$ be the left hand side of (41). We note that the integral defining $I$ converges absolutely. Using (24) with $r = h^{-1}(1/t)$, and (38),

$$I \leq c \int_{|z| > h^{-1}(1/t)} F_2(t, x, y; z) \tilde{R}_1(z) J(z) dz + c \int_{|z| < h^{-1}(1/t)} F_1(t, x, y; z) \tilde{R}_1(z) J(z) dz$$

$$+ \int_{\mathbb{R}^d} z (1_{|y| < h^{-1}(1/t)} - 1_{|y| < 1}) \tilde{R}_1(z) J(z) dz \left| \nabla_x p^R(t, x, y) \right|.$$

By (34) and Proposition 4.1, the last term is bounded by $c t^{-1} \Upsilon_t(y - x)$. The same holds for the first two terms, because of (2), (33), [27, Lemma 5.1 (8) and 5.9].

In what follows, we shall see a difference in the estimates compared to [27]. The forthcoming result is an analogue of [27, Theorem 2.9] suitable for the present development.
Proposition 4.6. Assume (2), (5), (33), (34). For every $T > 0$, the inequalities

$$
\int_{\mathbb{R}^d} |\delta^R(t, x, y; z)| J(z) dz \leq c \vartheta(t) t^{-1} \Upsilon_t(y - x),
$$

$$
\int_{\mathbb{R}^d} |\delta^R(t, x', y; z) - \delta^R(t, x, y; z)| J(z) dz \leq c \left( \frac{|x'| - x}{h^{-1}(1/t)} \wedge 1 \right) \vartheta(t) t^{-1} (\Upsilon_t(y - x') + \Upsilon_t(y - x)),
$$

hold for all $t \in (0, T]$, $x, x', y \in \mathbb{R}^d$ with

(a) $\vartheta(t) = \Theta(t)$ and $c = c(d, T, \sigma)$ if $\alpha_h = 1$,
(b) $\vartheta(t) = t [h^{-1}(1/t)]^{-1}$ and $c = c(d, T, \sigma, \beta_h, c_h)$ if (6) holds for $0 < \alpha_h \leq \beta_h < 1$.

Proof. By (38) with $r = h^{-1}(1/t)$ we get

$$
\int_{\mathbb{R}^d} |\delta^R(t, x, y; z)| J(z) dz
\leq \int_{\mathbb{R}^d} |\delta^R_1(x, y; z)| J(z) dz + \int_{\mathbb{R}^d} |z| \left| 1_{|z| < r} - 1_{|z| < 1} \right| J(z) dz |\nabla_x P^R(t, x, y)|
\leq c \int_{|z| > h^{-1}(1/t)} \mathcal{F}_2(t, x, y; z) J(z) dz + c \int_{|z| < h^{-1}(1/t)} \mathcal{F}_1(t, x, y; z) J(z) dz
+ \int_{\mathbb{R}^d} |z| \left| 1_{|z| < h^{-1}(1/t)} - 1_{|z| < 1} \right| J(z) dz \left[ h^{-1}(1/t) \right]^{-1} \Upsilon_t(y - x).
$$

The first inequality in the statement follows from (2), [27, Lemma 5.1 (8) and 5.9] and Lemma 8.1. Now we prove the second inequality. If $|x' - x| \geq h^{-1}(1/t)$, then

$$
\int_{\mathbb{R}^d} (|\delta^R(t, x', y; z)| + |\delta^R(t, x, y; z)|) J(z) dz \leq c \vartheta(t) t^{-1} (\Upsilon_t(y - x') + \Upsilon_t(y - x)).
$$

If $|x' - x| < h^{-1}(1/t)$, we use (39), (37) and, again, [27, Lemma 5.1 and 5.9] and Lemma 8.1.

The next result is a tool to relate the heat kernels corresponding to two coefficients $\mathcal{R}_1, \mathcal{R}_2$.

Lemma 4.7. Assume (2), (5) and let (33), (34) hold for $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. For all $t > 0$, $x, y \in \mathbb{R}^d$ and $s \in (0, t)$,

$$
\frac{d}{ds} \int_{\mathbb{R}^d} p^{R_1}(s, x, z) p^{R_2}(t - s, z, y) dz
= \int_{\mathbb{R}^d} \mathcal{L}_x^{R_1} p^{R_1}(s, x, z) p^{R_2}(t - s, z, y) dz - \int_{\mathbb{R}^d} p^{R_1}(s, x, z) \mathcal{L}_z^{R_2} p^{R_2}(t - s, z, y) dz,
$$

and

$$
\int_{\mathbb{R}^d} \mathcal{L}_x^{R_3} p^{R_1}(s, x, z) p^{R_2}(t - s, z, y) dz = \int_{\mathbb{R}^d} p^{R_1}(s, x, z) \mathcal{L}_z^{R_3} p^{R_2}(t - s, z, y) dz.
$$

Proof. The proof is the same as in [27, Lemma 2.10]. The first part follows by the dominated convergence theorem, which justifies the differentiation under the integral sign, and then by applying (35). The second identity is obtained after changing the order of integration and integrating by parts, see (22) and (23). In both cases we use the fact that for all $0 < t_0 < T < \infty$ there exists a constant $c = c(d, T, t_0, \sigma)$ such that for all $t \in [t_0, T]$, $x, y \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} |\delta^{R_1}(t, x, y; z)| J(z) dz \leq c t^{-1} \Upsilon_t(y - x) \leq c t_0^{-1} \Upsilon_{t_0}(y - x),
$$

which is valid under the assumptions of the lemma, see (42).
5. Analysis of the heat kernel of $\mathcal{L}_x^{\kappa_0}$

In this section we work under (Q0). In particular, we consider $J(z)$ satisfying (2) and (5), and $\kappa(x, z)$ such that (3) and (7) hold. For a fixed $w \in \mathbb{R}^d$ we let $\Re_{w}(z) = \kappa(w, z)$, as in (25). Since (33) and (34) hold for $\Re_{w}$, the results of Section 4 remain in force. Like in Section 4 we let $p^{\Re_{w}}(t, x, y)$ be the heat kernel of $\mathcal{L}_x^{\kappa_0}$. This procedure is known as freezing coefficients of the operator $\mathcal{L}^\kappa$ given in (1).

First, we estimate $(\mathcal{L}_x^{\kappa_0'} - \mathcal{L}_x^{\kappa_0}) p^\kappa(t, x, y)$.

**Lemma 5.1.** Assume (Q0) and let (33), (34) hold for $\Re$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4)$ such that for all $t \in (0, T]$, $x, y, w, w' \in \mathbb{R}^d$ we have

$$\left| \int_{\Re} \delta^{\kappa}(t, x, y; z) (\kappa(w', z) - \kappa(w, z)) J(z) dz \right| \leq c \left( |w' - w|^\beta + 1 \right) t^{-1} \Upsilon_t(y - x).$$

**Proof.** If $|w' - w| \geq 1$ we apply (41). Let $I$ be the left hand side of (44) and $|w' - w| < 1$. We also note that the integral defining $I$ converges absolutely. Using (24) with $r = h^{-1}(1/t)$, and (38),

$$I \leq c \int_{|z| > h^{-1}(1/t)} \mathcal{F}_2(t, x, y; z) |\kappa(w', z) - \kappa(w, z)| J(z) dz$$

$$+ c \int_{|z| < h^{-1}(1/t)} \mathcal{F}_1(t, x, y; z) |\kappa(w', z) - \kappa(w, z)| J(z) dz$$

$$+ \left| \int_{\mathbb{R}^d} z (\mathbf{1}_{|z| < h^{-1}(1/t)} - \mathbf{1}_{|z| < 1}) (\kappa(w', z) - \kappa(w, z)) J(z) dz \right| |\nabla_x p^\kappa(t, x, y)|.$$

By (8) and Proposition 4.1 the last term is bounded by $|w' - w|^\beta t^{-1} \Upsilon_t(y - x)$. The same is true for the first two terms by (2), (4), [27, Lemma 5.1 (8) and 5.9].

Now we estimate $(\mathcal{L}_x^{\kappa_0'} - \mathcal{L}_x^{\kappa_0}) p^\kappa(t, x', y) - (\mathcal{L}_x^{\kappa_0'} - \mathcal{L}_x^{\kappa_0}) p^\kappa(t, x, y)$.

**Lemma 5.2.** Assume (Q0) and let (33), (34) hold for $\Re$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4)$ such that for all $t \in (0, T]$, $x, x', y, w, w' \in \mathbb{R}^d$,

$$\left| \int_{\mathbb{R}^d} (\delta^{\kappa}(t, x', y; z) - \delta^{\kappa}(t, x, y; z)) (\kappa(w', z) - \kappa(w, z)) J(z) dz \right|$$

$$\leq c \left( |x' - x| h^{-1}(1/t)^{-1} \right) (|w' - w|^\beta + 1) t^{-1} (\Upsilon_t(y - x') + \Upsilon_t(y - x)).$$

**Proof.** If $|x' - x| \geq h^{-1}(1/t)$ we apply (44). Let $I$ be the left hand side of (45) and $|x' - x| < h^{-1}(1/t)$. By (24) with $r = h^{-1}(1/t)$, and (39),

$$I \leq c \left( \frac{|x' - x|}{h^{-1}(1/t)} \right) \int_{|z| > h^{-1}(1/t)} \mathcal{F}_2(t, x, y; z) |\kappa(w', z) - \kappa(w, z)| J(z) dz$$

$$+ c \left( \frac{|x' - x|}{h^{-1}(1/t)} \right) \int_{|z| < h^{-1}(1/t)} \mathcal{F}_1(t, x, y; z) |\kappa(w', z) - \kappa(w, z)| J(z) dz$$

$$+ \left| \int_{\mathbb{R}^d} z (\mathbf{1}_{|z| < h^{-1}(1/t)} - \mathbf{1}_{|z| < 1}) (\kappa(w', z) - \kappa(w, z)) J(z) dz \right| |\nabla_x p^\kappa(t, x', y) - \nabla_x p^\kappa(t, x, y)|.$$

By (8) and (37), we bound the last expression by $(|w' - w|^\beta + 1)(|x' - x|/h^{-1}(1/t)) t^{-1} \Upsilon_t(y - x)$. For the first two terms we rely on (2), (4), [27, Lemma 5.1 (8) and 5.9].

$\square$
In Lemma 5.1 and 5.2 our assumptions (33) and (34) play an important role. They also influence Proposition 5.3 and other results. We note a difference in the estimates (46) in comparison to the corresponding bound in [27, Theorem 2.11].

In what follows we provide several results on the regularity of the heat kernel $p^w(t, x, y)$ in $w \in \mathbb{R}^d$.

**Proposition 5.3.** Assume (Q0). For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4)$ such that for all $t \in (0, T]$, $x, y, w, w' \in \mathbb{R}^d$,

$$|p^w(t, x, y) - p^w(t, x, y)| \leq c(|w' - w|^\beta \wedge 1) \Upsilon_t(y - x),$$

$$|\nabla_x p^w(t, x, y) - \nabla_x p^w(t, x, y)| \leq c(|w' - w|^\beta \wedge 1)[h^{-1}(1/t)]^{-1} \Upsilon_t(y - x),$$

$$|L_x^w p^w(t, x, y) - L_x^w p^w(t, x, y)| \leq c(|w' - w|^\beta \wedge 1)t^{-1} \Upsilon_t(y - x).$$

Moreover, for every $T > 0$, the inequality

$$\int_{\mathbb{R}^d} |\delta^w(t, x, y; z) - \delta^w(t, x, y; z)| J(z) dz \leq c(|w' - w|^\beta \wedge 1) \vartheta(t) t^{-1} \Upsilon_t(y - x),$$

holds for all $t \in (0, T]$, $x, y, w, w' \in \mathbb{R}^d$ with

(a) $\vartheta(t) = \Theta(t)$ and $c = c(d, T, \sigma, \kappa_2, \kappa_4)$ if $\alpha_h = 1$,

(b) $\vartheta(t) = t [h^{-1}(1/t)]^{-1}$ and $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_h, \alpha_h)$ if (6) holds for $0 < \alpha_h \leq \beta_h < 1$.

**Proof.** In what follows, we use [27, Corollary 5.14, Lemma 5.6] and the monotonicity of $h^{-1}$ without further comment. The proof resembles that of [27, Theorem 2.11], but in parts (ii), (iii) and (iv) different adjustments are needed to use our assumptions.

(i) Using Lemma 4.7, we get

$$p^w(t, x, y) - p^w(t, x, y) = \lim_{\varepsilon_1 \to 0^+} \int_{\varepsilon_1}^{t/2} \int_{\mathbb{R}^d} p^w(s, x, z) \left(L_x^w - L_x^w\right) p^w(t - s, z, y) dzds$$

$$+ \lim_{\varepsilon_2 \to 0^+} \int_{t/2}^{t - \varepsilon_2} \int_{\mathbb{R}^d} \left(L_x^w - L_x^w\right) p^w(t, s, z) p^w(s, t, y) dzds.$$

By Proposition 4.1 and (44),

$$\int_{\varepsilon}^{t/2} \int_{\mathbb{R}^d} p^w(s, x, z) \left|L_x^w - L_x^w\right| p^w(t - s, z, y) dzds$$

$$\leq c(|w' - w|^\beta \wedge 1) \int_{\varepsilon}^{t/2} \Upsilon_s(z - x) (t - s)^{-1} \Upsilon_{t-s}(y - z) dzds$$

$$\leq c(|w' - w|^\beta \wedge 1) \Upsilon_t(y - x) \int_{\varepsilon}^{t/2} t^{-1} ds.$$

Similarly,

$$\int_{t/2}^{t - \varepsilon} \int_{\mathbb{R}^d} \left|L_x^w - L_x^w\right| p^w(t, s, z) p^w(s, t, y) dzds \leq c(|w' - w|^\beta \wedge 1) \Upsilon_t(y - x).$$

(ii) Let $w_0 \in \mathbb{R}^d$ be fixed. Define $R(z) = (\kappa_0/(2\kappa_1))\kappa(w_0, z)$ and $\hat{R}_w(z) = R_w(z) - R(z)$. By the construction of the Lévy process, we have

$$p^w(t, x, y) = \int_{\mathbb{R}^d} p^w(t, x, \xi)p^{\hat{R}_w}(t, \xi, y) d\xi.$$

Then by (36) we can differentiate under the integral in (47). By Proposition 4.1 we get

$$|\nabla_x p^w(t, x, y) - \nabla_x p^w(t, x, y)| \leq \int_{\mathbb{R}^d} |\nabla_x p^w(t, x, \xi)| \left|p^{\hat{R}_w}(t, \xi, y) - p^{\hat{R}_w}(t, \xi, y)\right| d\xi.$$
\[ \leq c(|w' - w| \wedge 1) \left[ h^{-1}(1/t) \right]^{-1} \Upsilon_t(y - x). \]

(iii) By (47) we have
\[ \delta^\theta_{w'}(t, x, y; z) - \delta^\theta_w(t, x, y; z) = \int_{\mathbb{R}^d} \delta^\theta(t, x, \xi; z) \left( p^\theta_{w'}(t, \xi, y) - p^\theta_w(t, \xi, y) \right) d\xi. \]

Then by (41),
\[ |\mathcal{L}_{x}^{\theta} p^\theta_{w'}(t, x, y) - \mathcal{L}_{x}^{\theta} p^\theta_w(t, x, y)| \leq \int_{\mathbb{R}^d} |\mathcal{L}_{x}^{\theta} p^\theta(t, x, \xi)| \left| p^\theta_{w'}(t, \xi, y) - p^\theta_w(t, \xi, y) \right| d\xi \leq c(|w' - w| \wedge 1) t^{-1} \Upsilon_t(y - x). \]

(iv) By Proposition 4.6,
\[ \int_{\mathbb{R}^d} |\delta^\theta_{w'}(t, x, y; z) - \delta^\theta_w(t, x, y; z)| J(z) dz \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta^\theta(t, x, \xi; z)| J(z) dz \right) \left| p^\theta_{w'}(t, \xi, y) - p^\theta_w(t, \xi, y) \right| d\xi \leq c(|w' - w| \wedge 1) \int_{\mathbb{R}^d} \vartheta(t) t^{-1} \Upsilon_t(\xi - x) \Upsilon_t(\xi - y) d\xi \leq c(|w' - w| \wedge 1) \vartheta(t) t^{-1} \Upsilon_t(y - x). \]

\[ \Box \]

Our results mostly have the same form as those in [27], and similarly as in [27] we are able to deduce the joint continuity, the concentration of mass, and cancellations.

**Lemma 5.4.** Assume (Q0). The functions \( p^\theta_w(t, x, y) \) and \( \nabla_x p^\theta_w(t, x, y) \) are jointly continuous in \((t, x, y, w) \in (0, \infty) \times (\mathbb{R}^d)^3\). The function \( \mathcal{L}_{x}^{\theta} p^\theta_w(t, x, y) \) is jointly continuous in \((t, x, y, w, v) \in (0, \infty) \times (\mathbb{R}^d)^4\). Furthermore,
\[ \lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\theta_w(t, x, y) dy - 1 \right| = 0 \quad (48) \]

**Proof.** The result follows from Proposition 5.3, [27, Lemma 6.1], (38) and Lemma 8.3, cf. [27, Lemma 3.1, 3.2 and 3.4]. \[ \Box \]

**Lemma 5.5.** Assume (Q0). Let \( \beta_1 \in [0, \beta] \cap [0, \alpha_h) \). For every \( T > 0 \) there exists a constant \( c = c(T, \sigma, \kappa_2, \kappa_4, \beta_1) \) such that for all \( t \in (0, T] \), \( x \in \mathbb{R}^d \),
\[ \left| \int_{\mathbb{R}^d} \nabla_x p^\theta_w(t, x, y) dy \right| \leq c \left[ h^{-1}(1/t) \right]^{-1+\beta_1}. \]

**Proof.** The inequality stems from (36), Proposition 5.3 and Lemma 8.3, cf. [27, Lemma 3.4]. \[ \Box \]

Compared to [27], the two expressions in Lemma 5.6 and Lemma 5.7 need to be estimated separately.

**Lemma 5.6.** Assume (Q0). Let \( \beta_1 \in [0, \beta] \cap [0, \alpha_h) \). For every \( T > 0 \), the inequality
\[ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \delta^\theta(t, x, y; z) dy \right) J(z) dz \leq c \vartheta(t) t^{-1} \left[ h^{-1}(1/t) \right]^ {\beta_1}, \]
holds for all \( t \in (0, T] \), \( x \in \mathbb{R}^d \) with
\[ \text{(a) } \vartheta(t) = \Theta(t) \text{ and } c = c(T, \sigma, \kappa_2, \kappa_4, \beta_1) \text{ if } \alpha_h = 1, \]
(b) \( \vartheta(t) = t [h^{-1}(1/t)]^{-1} \) and \( c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \beta_h, c_h) \) if \( (6) \) holds for \( 0 < \alpha_h \leq \beta_h < 1 \).

**Proof.** We subtract zero and use \((46)\), to get

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta^{R_\gamma}(t, x, y; z) - \delta^{R_\gamma}(t, x, y; z) \, dy \, J(z) \, dz \leq c \int_{\mathbb{R}^d} \vartheta(t) \rho_0^{\beta_1}(t, x - y) \, dy.
\]

The result follows from Lemma 8.3(a). \(\square\)

**Lemma 5.7.** Assume \((Q0)\). Let \( \beta_1 \in [0, \beta] \cap [0, \alpha_h) \). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1) \) such that for all \( t \in (0, T] \), \( x \in \mathbb{R}^d \),

\[
\left| \int_{\mathbb{R}^d} \mathcal{L}_x^{R_\gamma} p^{R_\gamma}(t, x, y) \, dy \right| \leq c t^{-1} [h^{-1}(1/t)]^{\beta_1}.
\]

**Proof.** By Proposition 5.3, we have

\[
\left| \int_{\mathbb{R}^d} \mathcal{L}_x^{R_\gamma} p^{R_\gamma}(t, x, y) \, dy \right| = \left| \int_{\mathbb{R}^d} \left( \mathcal{L}_x^{R_\gamma} p^{R_\gamma}(t, x, y) - \mathcal{L}_x^{R_\gamma} p^{R_\gamma}(t, x, y) \right) \, dy \right| \leq c \int_{\mathbb{R}^d} \rho_0^{\beta_1}(t, x - y) \, dy.
\]

The result follows from Lemma 8.3(a). \(\square\)

6. LEVI’S CONSTRUCTION OF THE HEAT KERNEL

In this section, we focus on the objects introduced in Section 3 by the formulae \((26)-(31)\). The estimates are stated using a short notation proposed in \((32)\).

6.1. Construction of \( q(t, x, y) \).

**Lemma 6.1.** Assume \((Q0)\). For every \( T > 0 \) there exists a constant \( c = c(d, T, \sigma, \kappa_2, \kappa_4) \geq 1 \) such that for all \( \beta_1 \in [0, \beta], t \in (0, T] \) and \( x, x', y, y' \in \mathbb{R}^d \)

\[
|q_0(t, x, y)| \leq c \rho_0^{\beta_1}(t, y - x),
\]

and for every \( \gamma \in [0, \beta_1] \),

\[
|q_0(t, x, y) - q_0(t, x', y)| \leq c (|x - x'|^{\beta_1 - \gamma} \wedge 1) \left\{ \left( \rho_0^0 + \rho_0^{\beta_1} \right)(t, x - y) + \left( \rho_0^0 + \rho_0^{\beta_1} \right)(t, x' - y) \right\},
\]

and

\[
|q_0(t, x, y) - q_0(t, x, y')| \leq c (|y - y'|^{\beta_1 - \gamma} \wedge 1) \left\{ \left( \rho_0^0 + \rho_0^{\beta_1} \right)(t, x - y) + \left( \rho_0^0 + \rho_0^{\beta_1} \right)(t, x - y') \right\}.
\]

**Proof.** (i) \((49)\) follows from \((44)\).

(ii) For \( |x - x'| \geq 1 \) the inequality holds by \((49)\) and \([27, (92)]\):

\[
|q_0(t, x, y)| \leq c \rho_0^{\beta_1}(t, y - x) \leq c [h^{-1}(1/T) \vee 1]^{\beta_1 - \gamma} \rho_0^{\beta_1}(t, y - x).
\]

For \( 1 \geq |x - x'| \geq h^{-1}(1/t) \) the result follows from \((49)\) and

\[
|q_0(t, x, y)| \leq c \rho_0^{\beta_1}(t, y - x) = c [h^{-1}(1/t)]^{\beta_1 - \gamma} \rho_0^{\beta_1}(t, y - x) \leq c |x - x'|^{\beta_1 - \gamma} \rho_0^{\beta_1}(t, y - x).
\]

Now, \((44)\) and \((45)\) yield

\[
|q_0(t, x, y) - q_0(t, x', y)| = \left| \int_{\mathbb{R}^d} \delta^{R_\gamma}(t, x, y; z)(\kappa(x, z) - \kappa(y, z)) J(z) \, dz \right|
\]

\[ - \left| \int_{\mathbb{R}^d} \delta^{R_\gamma}(t, x', y; z)(\kappa(x', z) - \kappa(y, z)) J(z) \, dz \right|.
\]
\[
\begin{align*}
&\lesssim \left| \int_{\mathbb{R}^d} (\delta^{q_0}(t, x, y; z) - \delta^{q_0}(t, x', y; z)) (\kappa(x, z) - \kappa(y, z)) J(z)dz \right| \\
&\quad + \left| \int_{\mathbb{R}^d} (\delta^{q_0}(t, x', y; z)) (\kappa(x, z) - \kappa(x', z)) J(z)dz \right| \\
&\quad + c (|x - x'|^{\beta_1} \wedge 1) \int_{\mathbb{R}^d} |\delta^{q_0}(t, x', y; z)| J(z)dz \\
&\leq c (|x - y|^{\beta_1} \wedge 1) \left( \frac{|x - x'|}{h^{-1}(1/t)} \wedge 1 \right) (\rho^0_0(t, x - y) + \rho^0_0(t, x' - y)) + c (|x - x'|^{\beta_1} \wedge 1) \rho^0_0(t, x' - y).
\end{align*}
\]

Applying \(|x - y|^{\beta_1} \leq h^{-1}(1/t) \wedge 1\) we obtain
\[
|q_0(t, x, y) - q_0(t, x', y)| \leq c \left( \frac{|x - x'|}{h^{-1}(1/t)} \wedge 1 \right) (\rho^0_0(t, x - y) + \rho^0_0(t, x' - y)) \\
+ c (|x - x'|^{\beta_1} \wedge 1) \rho^0_0(t, x' - y).
\]

Thus, in the last case \(|x - x'| \leq h^{-1}(1/t) \wedge 1\) we have \(|x - x'|/h^{-1}(1/t) \leq |x - x'|^{\beta_1} \gamma [h^{-1}(1/t)]^{\gamma - \beta_1}\) and \(|x - x'|^{\beta_1} \leq |x - x'|^{\beta_1} \gamma [h^{-1}(1/t)]^{\gamma - \beta_1}\).

(iii) We treat the cases \(|y - y'| \geq 1\) and \(1 \geq |y - y'| \geq h^{-1}(1/t)\) like in part (ii). Note that by \(\delta^{q}(t, x, y; z) = \delta^{q}(t, y, -x, z); (44), (45)\) and Proposition 5.3,
\[
|q_0(t, x, y) - q_0(t, x, y')| \leq c \left( \frac{|y - y'|}{h^{-1}(1/t)} \wedge 1 \right) \rho^0_0(t, x - y) \\
+ c (|x - y'|^{\beta_1} \wedge 1) \left( \frac{|y - y'|}{h^{-1}(1/t)} \wedge 1 \right) (\rho^0_0(t, x - y) + \rho^0_0(t, x - y')) \\
+ c (|y - y'|^{\beta_1} \wedge 1) \rho^0_0(t, x - y').
\]

Applying \(|x - y'|^{\beta_1} \wedge 1 \leq (|x - y'|^{\beta_1} \wedge 1) + (|y - y'|^{\beta_1} \wedge 1)\) we obtain
\[
|q_0(t, x, y) - q_0(t, x, y')| \leq c \left( \frac{|y - y'|}{h^{-1}(1/t)} \wedge 1 \right) (\rho^0_0(t, x - y) + \rho^0_0(t, x - y')) \\
+ c (|y - y'|^{\beta_1} \wedge 1) (\rho^0_0(t, x - y) + \rho^0_0(t, x - y')).
\]

This proves (51) in the case \(|y - y'| \leq h^{-1}(1/t) \wedge 1\). \(\square\)

We thus estimated \(q_0\). The estimates are of the same form as in [27, Lemma 3.6]. Using Lemma 8.3 they propagate to functions \(q_n\) and \(q\) defined in (27) and (28), respectively, see the proof of Theorem 6.2. We also note that \(\beta_1\) is used in Lemma 6.1 merely for technical convenience, but becomes relevant when estimating \(q_n\) and \(q\).

We stress that among others, the inequality (54) plays a special role and is often used to improve the integrability or bounds of singular functions, sometimes along with cancellations like those proved in Lemma 5.5 or Lemma 5.7, see comments ahead of Lemma 6.6 and Lemma 6.8.
Theorem 6.2. Assume (Q0). The series in (28) is locally uniformly absolutely convergent on 
$(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and solves the integral equation
\[ q(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z) q(s, z, y) \, dz \, ds. \] (52)
Moreover, for every $T > 0$ and $\beta_1 \in (0, \beta] \cap (0, \alpha_h)$ there is a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1)$
such that on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,
\[ |q(t, x, y)| \leq c \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y), \] (53)
and for any $\gamma \in (0, \beta_1]$ there is $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \gamma)$ such that on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,
\[ |q(t, x, y) - q(t, x', y)| \leq c \left( |x - x'|^{\beta_1 - \gamma} \wedge 1 \right) \left\{ \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y) + \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x' - y) \right\}, \] (54)
and
\[ |q(t, x, y) - q(t, x', y')| \leq c \left( |y - y'|^{\beta_1 - \gamma} \wedge 1 \right) \left\{ \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y) + \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y') \right\}. \] (55)

Proof. The proof follows from Lemmas 6.1 and 8.3 – it is the same as for [27, Theorem 3.7].
\[ \square \]

6.2. Properties of $\phi_\gamma(t, x, s)$ and $\phi_\gamma(t, x)$. We shall prove estimates for the integral part of (29). We use the notation introduced in (30) and (31).

Lemma 6.3. Assume (Q0). Let $\beta_1 \in (0, \beta] \cap (0, \alpha_h)$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1)$ such that for all $t \in (0, T], x, y \in \mathbb{R}^d$,
\[ |\phi_\gamma(t, x)| \leq c t \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y). \]
For any $T > 0$ and $\gamma \in [0, 1] \cap [0, \alpha_h)$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \gamma)$ such that for all $t \in (0, T], x, x', y \in \mathbb{R}^d$,
\[ |\phi_\gamma(t, x) - \phi_\gamma(t, x')| \leq c \left( |x - x'|^{\gamma} \wedge 1 \right) t \left\{ \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y) + \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x' - y) \right\}. \]
For any $T > 0$ and $\gamma \in (0, \beta)$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \gamma)$ such that for all $t \in (0, T], x, y, y' \in \mathbb{R}^d$,
\[ |\phi_\gamma(t, x) - \phi_\gamma(t, x)| \leq c \left( |y - y'|^{\beta_1 - \gamma} \wedge 1 \right) t \left\{ \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y) + \left( \rho_0^\beta_1 + \rho_0^\beta_1 \right) (t, x - y') \right\}. \]

Proof. The proof follows from Lemma 4.4, Proposition 4.1, Theorem 6.2 and Lemma 8.3, and is the same as in [27, Lemma 3.8].
\[ \square \]

Lemma 6.4. Assume (Q0). The function $\phi_\gamma(t, x)$ is jointly continuous in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. The idea of the proof is the same as that of [27, Lemma 3.9] and relies on Proposition 4.1, (53), [27, (94)], Lemma 8.3 and 5.4, [27, Lemma 5.6 and 5.15].
\[ \square \]

From this moment on, the major effort is to obtain sufficient regularity of the integral part of (29). Recall that we need $\beta_1 < \alpha_h$ in order to apply Lemma 8.3. The additional condition $1 < \beta_1 + \alpha_h$, known in certain contexts as the balance condition, which shall appear below in our assumptions, makes it possible to differentiate (31), that is, to calculate and estimate its gradient, see Lemma 6.7. We need such a result if we want to apply either the strong or the weak operator (1) to the candidate of a solution defined by (29).
Lemma 6.5. Assume (Q0). For all $0 < s < t$, $x, y \in \mathbb{R}^d$,
\[
\nabla_x \phi_y(t, x, s) = \int_{\mathbb{R}^d} \nabla_x p^{R_x}(t - s, x, z) q(s, z, y) \, dz,
\]
(56)
\[\text{Lemma 6.6. Assume (Q0) and } 1 - \alpha_h < \beta \land \alpha_h. \text{ Let } \beta_1 \in (0, \beta] \cap (0, \alpha_h). \text{ For every } T > 0 \text{ there exists a constant } c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1) \text{ such that for all } t \in (0, T], \, x \in \mathbb{R}^d,
\]
\[
\int_{\mathbb{R}^d} \int_0^t |\nabla_x \phi_y(t, x, s)| \, ds \, dy \leq c \left[ h^{-1}(1/t) \right]^{-1 + \beta_1}.
\]
(57)
\text{Proof. We get (56) by (36), (53), Lemma 8.3, and the dominated convergence theorem. Now, by (30) and (56),}
\[
\mathcal{L}^{R_x} \phi_y(t, x, s) = \int_{\mathbb{R}^d} \mathcal{L}^{R_x} p^{R_x}(t - s, x, z) q(s, z, y) \, dz.
\]
(58)
\text{Finally, we use Fubini’s theorem justified by (43), (53) and Lemma 8.3(b).}
\]
\[\square\]

In the proof of the next result we recognize a typical \textit{modus operandi} when dealing with integrals of functions that at first glance seem to be too singular: we add and subtract $q(s, x, y)$, use its regularity (54) (which reduces part of the singularity) and profit from cancellations, this time from Lemma 5.5.

Lemma 6.6. Assume (Q0) and $1 - \alpha_h < \beta \land \alpha_h$. Let $\beta_1 \in (0, \beta] \cap (0, \alpha_h)$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1)$ such that for all $t \in (0, T]$, $x \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} \int_0^t |\nabla_x \phi_y(t, x, s)| \, ds \, dy \leq c \left[ h^{-1}(1/t) \right]^{-1 + \beta_1}.
\]
(59)
\text{Proof. By the monotonicity of } h^{-1}, \text{ if we prove the statement for some value of } \beta_1, \text{ then it also holds for smaller values. We assume that } 1 - \alpha_h < \beta_1 \text{ and we let } \gamma \in (0, \beta_1) \text{ satisfying } 1 - \alpha_h < \beta_1 - \gamma. \text{ By (56), Proposition 4.1, (54), Lemma 5.5 and (53),}
\]
\[
|\nabla_x \phi_y(t, x, s)| \leq \int_{\mathbb{R}^d} |\nabla_x p^{R_x}(t - s, x, z)| |q(s, z, y) - q(s, x, y)| \, dz
\]
\[+ \int_{\mathbb{R}^d} \nabla_x p^{R_x}(t - s, x, z) \, dz |q(s, x, y)|
\]
\[
\leq \int_{\mathbb{R}^d} (t - s) \rho_{-1}^{\beta_1 - \gamma} (t - s, x - z) \left( \rho_0^0 + \rho_0^{\beta_1} \right) (s, z - y) \, dz
\]
\[+ \int_{\mathbb{R}^d} (t - s) \rho_{-1}^{\beta_1 - \gamma} (t - s, x - z) \, dz \left( \rho_0^0 + \rho_0^{\beta_1} \right) (s, x - y)
\]
\[+ \left[ h^{-1}(1/(t - s)) \right]^{-1 + \beta_1} \left( \rho_0^{\beta_1} + \rho_0^0 \right) (s, x - y) .
\]
Finally, we integrate in $y$ over $\mathbb{R}^d$ using Lemma 8.3(a) and then in $s$ over $(0, t)$ using [27, Lemma 5.15]. Note that in the last step we integrate $[h^{-1}(1/(t - s))]^{-1 + \beta_1 - \gamma}$, which requires a condition $(-1 + \beta_1 - \gamma)/\alpha_h + 1 > 0$, equivalently $\alpha_h + \beta_1 > 1 + \gamma$, and is fulfilled thanks to our assumptions.
\[\square\]

Lemma 6.7. Assume (Q0) and $1 - \alpha_h < \beta \land \alpha_h$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta)$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,
\[
\nabla_x \phi_y(t, x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x p^{R_x}(t - s, x, z) q(s, z, y) \, dz \, ds,
\]
(60)
Proof. The proof is like in [27, Lemma 3.10] and rests on (56), Proposition 4.1, (53), Lemma 8.3, [27, (93), (94), Lemma 5.3 and 5.15, Proposition 5.8], (54), Lemma 5.5, and the fact that \( \alpha_h > 1/2 \).

So far, in Lemma 6.7 we managed to calculate and estimate the gradient of \( \phi_y(t, x) \), which is the integral part of (29). Now we shall treat in a similar fashion the operator (1) acting on \( \phi_y(t, x) \). The first step is to show that the operator can actually be applied (that the respective integrals converge) and to find a formula – Lemma 6.10. The second step is to prove the estimates in Lemma 6.13. To achieve that, for the first step, in Lemma 6.8 and 6.9, we prove auxiliary bounds justifying the use of Fubini’s theorem, however those technical results do not provide the desired estimates for the second step. The reason is that when using (54), due to the position of the absolute value, we cannot make use of additional cancellations and we merely rely on (42) and Lemma 5.6, which causes extra growth. Therefore, contrary to [27], we are forced to distinguish between the two steps. An improvement of the estimates, taking cancellations into account, is given in Lemma 6.11, Corollary 6.12 and Lemma 6.13.

In (a) below we address the critical case. In (b) we deal with the super-critical case.

**Lemma 6.8.** Assume (Q0). Let \( \beta_1 \in (0, \beta] \cap (0, \alpha_h) \). For all \( T > 0, \gamma \in (0, \beta_1] \) the inequalities

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta^{R_0}(t-s, x, z; w)||q(s, z, y)| \, dz \right) \kappa(x, w) J(w) \, dw \\
\leq c_1 \int_{\mathbb{R}^d} \vartheta(t-s) \rho_0^0(t-s, x-z) \left( \rho_0^\beta_1 + \rho_0^\gamma \right)(s, z-y) \, dz,
\]

\[
\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \delta^{R_0}(t-s, x, z; w)q(s, z, y) \, dz \right| \kappa(x, w) J(w) \, dw \leq c_2 (I_1 + I_2 + I_3),
\]

where

\[
I_1 + I_2 + I_3 := \int_{\mathbb{R}^d} \vartheta(t-s) \rho_0^{\beta_1-\gamma} (t-s, x-z) \left( \rho_0^0 + \rho_0^\beta_1 \right)(s, z-y) \, dz \\
+ \vartheta(t-s)(t-s)^{-1} \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-\gamma} \left( \rho_0^0 + \rho_0^\beta_1 \right)(s, x-y) \\
+ \vartheta(t-s)(t-s)^{-1} \left[ h^{-1}(1/(t-s)) \right]^{\beta_1} \left( \rho_0^0 + \rho_0^\beta_1 \right)(s, x-y),
\]

hold for all \( 0 < s < t \leq T, x, y \in \mathbb{R}^d \) with

(a) \( \vartheta(t) = \Theta(t) \) and \( c_1 = c_1(d, T, \sigma, \kappa_2, \kappa_4, \beta_1), c_2 = c_2(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \gamma) \) if \( \alpha_h = 1 \),

(b) \( \vartheta(t) = t \left[ h^{-1}(1/t) \right]^{-1} \) and \( c_1 = c_1(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \beta_h, c_h), c_2 = c_2(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \beta_h, c_h) \) if (6) holds for \( 0 < \alpha_h \leq \beta_h < 1 \).

**Proof.** The inequality (61) follows from (3), (42) and (53). Next, let \( I_0 \) be the left hand side of (62). By (54), (53), (3), (42), Lemma 5.6 and 8.3(a),

\[
I_0 \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\delta^{R_0}(t-s, x, z; w)||q(s, z, y) - q(s, x, y)| \, dz \kappa(x, w) J(w) \, dw \\
+ \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \delta^{R_0}(t-s, x, z; w) \, dz \right| \kappa(x, w) J(w) \, dw |q(s, x, y)| \\
\leq c \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta^{R_0}(t-s, x, z; w)| J(w) \, dw \right) \left( |x-z|^{\beta_1-\gamma} \wedge 1 \right) \left( \rho_0^0 + \rho_0^\beta_1 \right)(s, z-y) \, dz \\
+ c \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta^{R_0}(t-s, x, z; w)| J(w) \, dw \right) \left( |x-z|^{\beta_1-\gamma} \wedge 1 \right) dz \left( \rho_0^0 + \rho_0^\beta_1 \right)(s, x-y) \\
+ c \vartheta(t-s)(t-s)^{-1} \left[ h^{-1}(1/(t-s)) \right]^{\beta_1} \left( \rho_0^0 + \rho_0^\beta_1 \right)(s, x-y) \leq c(I_1 + I_2 + I_3).
\]
The inequality (62) looks a bit rough, but it is left in such form on purpose: it is used not only to prove Lemma 6.9 and Lemma 6.17, but also to shorten the argument in the proof of Lemma 6.13.

**Lemma 6.9.** Assume (Q0) and $1 - \alpha_h < \beta \wedge \alpha_h$. For any $\beta_1 \in (0, \beta]$ such that $1 - \alpha_h < \beta_1 < \alpha_h$ and $0 < \gamma_1 \leq \gamma_2 \leq \beta_1$ satisfying

$$1 - \alpha_h < \beta_1 - \gamma_1, \quad 2\beta_1 - \gamma_2 < \alpha_h,$$

the inequality

$$\int_{\mathbb{R}^d} \int_0^t \left| \int_{\mathbb{R}^d} \delta_\kappa(t - s, x, z; w)q(s, z, y)dz \right| ds \kappa(x, w)J(w)dw \leq c \vartheta(t) \left( \rho_0^{\beta_1} + \rho^{\beta_1 - \gamma_1} + \rho_0^{\beta_1 - \gamma_2} \right)(t, x - y),$$

holds for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$ with

(a) $\vartheta(t) = \Theta(t)$ and $c = c(d, T, \sigma, \kappa, \beta_1, \gamma_1, \gamma_2)$ if $\alpha_h = 1$,

(b) $\vartheta(t) = t \left[ h^{-1}(1/t) \right]^{-1}$ and $c = c(d, T, \sigma, \kappa, \beta_1, \gamma_1, \gamma_2, \beta_2, c_h)$ if (6) holds for $0 < \alpha_h \leq \beta_h < 1$.

**Proof.** Let $I_0$ be the left hand side of (62). In the two cases discussed below, we apply Lemma 8.3(b), the monotonicity of $h^{-1}$ and $\Theta$ (see also Lemma 8.2), and (A2) of [27, Lemma 5.3]. For $s \in (0, t/2)$ we use (61) to get

$$I_0 \leq c \vartheta(t - s) \left\{ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1} + (t - s)^{-1} \left[ h^{-1}(1/s) \right]^{\beta_1} + s^{-1} \left[ h^{-1}(1/s) \right]^{\beta_1} \right\} \rho_0^0(t, x - y) + (t - s)^{-1} \rho_0^{\beta_1}(t, x - y) \right\} \rho_0^0(t, x - y) + t^{-1} \rho_0^{\beta_1}(t, x - y).$$

For $s \in (t/2, t)$ we use (62) with $\gamma = \gamma_1$. While estimating the expression

$$\vartheta(t - s) \int_{\mathbb{R}^d} \rho_0^{\beta_1 - \gamma}(t - s, x - z)\rho_0^{\beta_1}(s, z - y)dz,$$

we use Lemma 8.3(b) with $n_1 = n_2 = 2\beta_1 - \gamma_2$, $m_1 = \beta_1 - \gamma_1$, $m_2 = \beta_1$ and later Lemma 8.2, so our assumptions concerning the choice of $\gamma_1$, $\gamma_2$ are used. More precisely, we have

$$I_1 \leq c \vartheta(t - s) \left\{ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma_1} \left[ h^{-1}(1/s) \right]^{\gamma_1} + s^{-1} \left[ h^{-1}(1/s) \right]^{\beta_1 + \gamma_1 - \gamma_2} \right\} \rho_0^0(t, x - y) + (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{2\beta_1 - \gamma_2} \left[ h^{-1}(1/s) \right]^{\gamma_1 - \beta_1} \rho_0^0(t, x - y) + c \vartheta(t - s)(t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma_1} \left[ h^{-1}(1/s) \right]^{\gamma_1 - \beta_1} \rho_0^0(t, x - y) + c \vartheta(t - s)s^{-1} \left[ h^{-1}(1/s) \right]^{\gamma_1} \rho_0^{\beta_1 - \gamma_1}(t, x - y) \leq c \vartheta(t - s) \left\{ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma_1} \left[ h^{-1}(1/t) \right]^{\gamma_1} + t^{-1} \left[ h^{-1}(1/t) \right]^{\beta_1 + \gamma_1 - \gamma_2} \right\} \rho_0^0(t, x - y) + (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{2\beta_1 - \gamma_2} \left[ h^{-1}(1/t) \right]^{\gamma_1 - \beta_1} \rho_0^0(t, x - y) + c \vartheta(t - s)(t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma_1} \left[ h^{-1}(1/t) \right]^{\gamma_1 - \beta_1} \rho_0^0(t, x - y) + c \vartheta(t - s)(t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\beta_1 - \gamma_1} \left[ h^{-1}(1/t) \right]^{\gamma_1 - \beta_1} \rho_0^0(t, x - y).$$
\[ \frac{c \vartheta(t-s) t^{-1} \left[ h^{-1}(1/t) \right]^{\gamma_1} \rho_0^{\beta_1-\gamma_1}(t,x) - y}. \]

Next, like above with [27, (94)],
\[ I_2 \leq c \vartheta(t-s)(t-s)^{-1} \left[ h^{-1}(1/t) \right]^{\beta_1-\gamma_1} \left( \rho_0^{\beta_1} + \rho_{\gamma_1-\beta_1} \right)(t,x-y). \]

Similarly, \[ I_3 \leq c \vartheta(t-s)(t-s)^{-1} \left[ h^{-1}(1/t) \right]^{\beta_1} \left( \rho_0^{\beta_1} + \rho_0^0 \right)(t,x-y). \] Finally, by [27, Lemma 5.15] and Lemma 8.2, and a fact that \( \alpha_h > 1/2 \),
\[ \int_0^t I_0 ds \leq c \vartheta(t)(\rho_0^{\beta_1} + \rho_{\gamma_1-\beta_1} + \rho_0^0)(t,x-y). \]

We can now successfully apply (1) to (31).

**Lemma 6.10.** Assume (Q1) or (Q2). We have for all \( t > 0, x, y \in \mathbb{R}^d \),
\[ \mathcal{L}_x^{R_s} \phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_x^{R_s} p^{R_s}(t-s,x,z) q(s,z,y) \, dz \, ds. \]

**Proof.** By (31) and (59) in the first equality, and Lemma 6.9 and (61) in the second (allowing us to change the order of integration twice) the proof is as follows
\[ \mathcal{L}_x^{R_s} \phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \delta^{R_s}(t-s,x,z;w) q(s,z,y) \, dz \, ds \right) \kappa(x,w) J(w) \, dw \]
\[ = \int_0^t \int_{\mathbb{R}^d} \left( \delta^{R_s}(t-s,x,z;w) \kappa(x,w) J(w) \right) q(s,z,y) \, dz \, ds. \]

We improve the estimates.

**Lemma 6.11.** Assume (Q0). Let \( \beta_1 \in (0,\beta] \cap (0,\alpha_h) \). For all \( T > 0, \gamma \in (0,\beta_1) \) there exist constants \( c_1 = c_1(d,T,\sigma,\kappa_2,\kappa_4,\beta_1) \) and \( c_2 = c_2(d,T,\sigma,\kappa_2,\kappa_4,\beta_1,\gamma) \) such that for all \( 0 < s < t \leq T, x, y \in \mathbb{R}^d \),
\[ \left| \mathcal{L}_x^{R_s} \phi_y(t,x,s) \right| \leq c_1 \int_{\mathbb{R}^d} \rho_0^{\beta_1}(t-s,x-z)(\rho_0^{\beta_1} + \rho_0^0)(s,z-y) \, dz, \quad (63) \]
\[ \left| \mathcal{L}_x^{R_s} \phi_y(t,x,s) \right| \leq c_2 (I_1 + I_2 + I_3), \quad (64) \]

where
\[ I_1 + I_2 + I_3 := \int_{\mathbb{R}^d} \rho_0^{\beta_1-\gamma}(t-s,x-z)(\rho_0^{\beta_1} + \rho_{\gamma-\beta_1})(s,z-y) \, dz \]
\[ + (t-s)^{-1} \left[ h^{-1}(1/(t-s)) \right]^{\beta_1-\gamma} (\rho_0^{\beta_1} + \rho_{\gamma-\beta_1})(s,x-y) \]
\[ + (t-s)^{-1} \left[ h^{-1}(1/(t-s)) \right]^{\beta_1} (\rho_0^{\beta_1} + \rho_0^0)(s,x-y). \]

**Proof.** The first inequality follows from (41) and (53). By (57), (54), (53), (41), Lemma 5.7 and 8.3(a),
\[ \left| \mathcal{L}_x^{R_s} \phi_y(t,x,s) \right| \leq \int_{\mathbb{R}^d} \left| \mathcal{L}_x^{R_s} p^{R_s}(t-s,x,z) \right| |q(s,z,y) - q(s,x,y)| \, dz \]
\[ + \left| \int_{\mathbb{R}^d} \mathcal{L}_x^{R_s} p^{R_s}(t-s,x,z) \, dz \right| |q(s,x,y)| \]
\[ \leq c \int_{\mathbb{R}^d} \rho_0^0(t-s, x-z) (|x-z|^\beta_1 \wedge 1) (\rho_0^0 + \rho_{\beta_1-\beta_1}^0)(s, z-y) \, dz \\
+ c \int_{\mathbb{R}^d} \rho_0^0(t-s, x-z) (|x-z|^\beta_1 \wedge 1) \, dz (\rho_0^0 + \rho_{\beta_1-\beta_1}^0)(s, x-y) \\
+ c (t-s)^{-1} [h^{-1}(1/(t-s))]^{\beta_1} (\rho_0^0 + \rho_{\beta_1}^0)(s, x-y) \leq c(I_1 + I_2 + I_3). \]

Here is a consequence of (63), (64), Lemma 8.3(a) and [27, Lemma 5.15].

**Corollary 6.12.** Assume (Q0) and \(1 - \alpha_h < \beta \wedge \alpha_h \). Let \(\beta_1 \in (0, \beta] \cap (0, \alpha_h)\). For every \(T > 0\) there exists a constant \(c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1)\) such that for all \(t \in (0, T]\), \(x \in \mathbb{R}^d\),

\[ \int_{\mathbb{R}^d} \int_0^t |\mathcal{L}_x^0 \phi_y(t, x, s)| \, ds \, dy \leq c t^{-1} [h^{-1}(1/t)]^{\beta_1}. \]

**Lemma 6.13.** Assume (Q0). Let \(\beta_1 \in (0, \beta] \cap (0, \alpha_h)\). For all \(T > 0\), \(0 < \gamma_1 \leq \gamma_2 \leq \beta_1\) satisfying

\[ 0 < \beta_1 - \gamma_1, \quad 2\beta_1 - \gamma_2 < \alpha_h, \]

there exists a constant \(c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1, \gamma_1, \gamma_2)\) such that for all \(t \in (0, T]\), \(x, y \in \mathbb{R}^d\),

\[ \int_0^t |\mathcal{L}_x^\gamma \phi_y(t, x, s)| \, ds \leq c (\rho_0^{\beta_1} + \rho_{\beta_1-\gamma_1}^0 + \rho_{\beta_1+\gamma_1-\gamma_2}^0)(t, x-y). \]

**Proof.** The proof goes by the same lines as the proof of Lemma 6.9 but with \(\vartheta\) replaced by 1, and Lemma 6.11 in place of Lemma 6.8.

**Lemma 6.14.** Assume (Q1) or (Q2). The function \(\mathcal{L}_x^\gamma \phi_y(t, x)\) is jointly continuous in \((t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\).

**Proof.** The proof is the same as in [27, Lemma 3.13] and requires Lemma 6.10, (57), (64), (41), (55), (53), [27, (94)], Lemma 5.15, Lemmas 8.3 and 5.4. \(\square\)

In the final results of that section, we prepare to calculate the time derivative of (29).

**Proposition 6.15.** Assume (Q0). For all \(t > 0\), \(x, y \in \mathbb{R}^d, x \neq y\), we have

\[ \phi_y(t, x) = \int_0^t \left( q(r, x, y) + \int_0^r \int_{\mathbb{R}^d} \mathcal{L}_x^\gamma p^0^\gamma(r-s, x, z) q(s, z, y) \, dz \, ds \right) \, dr. \]

**Proof.** The idea of the proof is that differentiating (31) in \(t > 0\), we expect to get

\[ \partial_t \phi_y(t, x) = q(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \partial_t \mathcal{L}_x^\gamma (t-s, x, z) q(s, z, y) \, dz \, ds. \]

Actually, we intend to prove the following integral counterpart,

\[ \phi_y(t, x) = \int_0^t \left( q(r, x, y) + \int_0^r \int_{\mathbb{R}^d} \partial_t \mathcal{L}_x^\gamma (r-s, x, z) q(s, z, y) \, dz \, ds \right) \, dr, \]

see (35). Therefore, the aim is to justify

\[ \int_0^t \int_{\mathbb{R}^d} \partial_t \mathcal{L}_x^\gamma (r-s, x, z) q(s, z, y) \, dz \, ds \, dr = \int_0^t \int_0^r \partial_s \phi_y(r, x, s) \, ds \, dr \\
= \int_0^t \int_0^{r(t)} \partial_s \phi_y(r, x, s) \, ds \, dr = \int_0^t \left( \phi_y(t, x) - \lim_{\varepsilon \to 0^+} \phi_y(s + \varepsilon, x, s) \right) \, ds \]

...
\[
= \phi_y(t, x) - \int_0^t \lim_{\varepsilon \to 0^+} \phi_y(s + \varepsilon, x, s) ds,
\]
and prove that \(\lim_{\varepsilon \to 0^+} \phi_y(s + \varepsilon, x, s) = q(s, x, y)\). Details are like in the proof of [27, Lemma 3.14]: we use (30), (35), (41), (53), (57), (65), (49), Lemma 8.3, [27, (92), (93), (94)], (48), (50), Proposition 4.1, [27, Lemma 5.6], [59, Theorem 7.21].

If we combine (52) and Lemma 6.10, then we can represent the integrand in the statement of Proposition 6.15 as \(q_0(s, x, y) + L^*_x \phi_y(s, x)\). Hence we conclude what follows.

**Corollary 6.16.** Assume (Q1) or (Q2). For all \(x, y \in \mathbb{R}^d, x \neq y\), the function \(\phi_y(t, x)\) is differentiable in \(t > 0\) and

\[
\partial_t \phi_y(t, x) = q_0(t, x, y) + L^*_x \phi_y(t, x).
\]

### 6.3. Properties of \(p^\kappa(t, x, y)\)
We collect what can already be said about \(p^\kappa(t, x, y)\).

**Lemma 6.17.** Assume (Q0) and \(1 - \alpha_h < \beta \wedge \alpha_h\). Let \(\beta_1 \in (0, \beta] \cap (0, \alpha_h)\). For every \(T > 0\) the inequalities

\[
\int_{\mathbb{R}^d} |\delta^\kappa(t, x, y; z)| \kappa(x, z) J(z) dz \leq c_1 \vartheta(t) \rho_0^0(t, x - y),
\]

(66)

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\delta^\kappa(t, x, y; z)| \kappa(x, z) J(z) dz \leq c_2 \vartheta(t) t^{-1} \left[ h^{-1}(1/t) \right]^{\beta_1},
\]

(67)

hold for all \(t \in (0, T]\), \(x, y \in \mathbb{R}^d\) with

(a) \(\vartheta(t) = \Theta(t)\) and \(c_1 = c_1(d, T, \sigma, \kappa_2, \kappa_4, \beta)\), \(c_2 = c_2(d, T, \sigma, \kappa_2, \kappa_4, \beta)\) if \(\alpha_h = 1\),

(b) \(\vartheta(t) = t [h^{-1}(1/t)]^{-1}\) and \(c_1 = c_1(d, T, \sigma, \kappa_2, \kappa_4, \beta, \beta_h, \alpha_h)\), \(c_2 = c_2(d, T, \sigma, \kappa_2, \kappa_4, \beta, \beta_h, \alpha_h)\)

if (6) holds for \(0 < \alpha_h \leq \beta_h < 1\).

**Proof.** By (29) and (59),

\[
\delta^\kappa(t, x, y; w) = \delta^\kappa(t, x, y; w) + \int_0^t \int_{\mathbb{R}^d} \delta^\kappa(t - s, x, z; w) q(s, z, y) dz ds.
\]

We deduce (66) from (42), Lemma 6.9, [27, (92), (93)]. The inequality (67) results from (62), [27, Lemma 5.15], Lemma 5.6, 8.3(a) and 8.2.

**Lemma 6.18.** Assume (Q0) and \(1 - \alpha_h < \beta \wedge \alpha_h\). Let \(\beta_1 \in (0, \beta] \cap (0, \alpha_h)\). For every \(T > 0\) there exists a constant \(c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta_1)\) such that for all \(t \in (0, T]\), \(x \in \mathbb{R}^d\),

\[
\left| \int_{\mathbb{R}^d} \nabla_x p^\kappa(t, x, y) dy \right| \leq c \left[ h^{-1}(1/t) \right]^{-1 + \beta_1}.
\]

**Proof.** We get the inequality from Lemma 5.5, (59), (56) and Lemma 6.6.

**Lemma 6.19.** Assume (Q1) or (Q2).

(a) The function \(p^\kappa(t, x, y)\) is jointly continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\).

(b) For every \(T > 0\) there is a constant \(c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta)\) such that for all \(t \in (0, T]\) and \(x, y \in \mathbb{R}^d\),

\[
|p^\kappa(t, x, y)| \leq c t \rho_0^0(t, x - y).
\]

(c) For all \(t > 0, x, y \in \mathbb{R}^d, x \neq y\),

\[
\partial_t p^\kappa(t, x, y) = L^*_x p^\kappa(t, x, y).
\]
(d) For every $T > 0$ there is a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta)$ such that for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}_x^\kappa p^\kappa(t, x, y)| \leq c_0(t, x - y),$$

and

$$|\nabla_x p^\kappa(t, x, y)| \leq c[h^{-1}(1/t)]^{-1} t\rho_0^\kappa(t, x - y).$$

(68)

(69)

(e) For all $T > 0$, $\gamma \in [0, 1] \cap [0, \alpha_h)$, there is a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta, \gamma)$ such that for all $t \in (0, T]$ and $x, x', y \in \mathbb{R}^d$,

$$|p^\kappa(t, x, y) - \rho^\kappa(t, x', y)| \leq c(|x - x'|^{\gamma} \wedge 1) t \left(\rho_{\gamma}^0(t, x - y) + \rho_{\gamma}^0(t, x') - y)\right).$$

For all $T > 0$, $\gamma \in [0, \beta) \cap [0, \alpha_h)$, there is a constant $c = c(d, T, \sigma, \kappa_2, \kappa_4, \beta, \gamma)$ such that for all $t \in (0, T]$ and $x, y, y' \in \mathbb{R}^d$,

$$|p^\kappa(t, x, y) - p^\kappa(t, x, y')| \leq c(|y - y'|^{\gamma} \wedge 1) t \left(\rho_{\gamma}^0(t, x - y) + \rho_{\gamma}^0(t, x - y')\right).$$

(69)

(f) The function $\mathcal{L}_x^\kappa p^\kappa(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. The statement of (a) follows from Lemmas 5.4 and 6.4. Part (b) is a result of Proposition 4.1 and Lemma 6.3. The equation in (c) is a consequence of (29), (35) and Corollary 6.16: $\partial_t p^\kappa(t, x, y) = \mathcal{L}_x^{\kappa,0} p^\kappa(t, x, y) + \mathcal{L}_x^{\kappa,0} \phi_0(t, x) = \mathcal{L}_x^{\kappa,0} p^\kappa(t, x, y)$. We get (68) by (29), (41), (65), [27, (92), (93)] (see also Lemma 6.10 and (57)). For the proof of (69) we use Proposition 4.1 and (60). The first inequality of part (e) follows from Lemmas 4.4 and 6.3, and [27, (92), (93)]. The same argument suffices for the second inequality of part (e) when supported by

$$|p^{\kappa}(t, x, y) - p^{\kappa}(t, x, y')| \leq |p^{\kappa}(t, -y, -x) - p^{\kappa}(t, -y', -x)| + |p^{\kappa}(t, x, y') - p^{\kappa}(t, x, y')|$$

and Proposition 5.3. Part (f) follows from Lemmas 5.4 and 6.14.

7. Main Results and Proofs

In the whole section, we assume that either (Q1) or (Q2) holds.

7.1. A nonlocal maximum principle. Recall that $\mathcal{L}^{\kappa,0} f := \lim_{\varepsilon \to 0^+} \mathcal{L}^{\kappa,\varepsilon} f$ is an extension of $\mathcal{L}^{\kappa} f := \mathcal{L}^{\kappa,0} f$. Moreover, the well-posedness of those operators requires the existence of the gradient $\nabla f$. The uniqueness of solutions to (71) stated in Corollary 7.2 will be used, for instance, in the next subsection. For the proofs of the following, see [27, Theorem 4.1].

Theorem 7.1. Let $T > 0$ and $u \in C([0, T] \times \mathbb{R}^d)$ be such that

$$\|u(t, \cdot) - u(0, \cdot)\|_\infty \xrightarrow{t \to 0^+} 0, \quad \sup_{t \in (0, T]} \|u(t, \cdot)1_{|\cdot| > r}\|_\infty \xrightarrow{r \to 0^+} 0.$$  

(70)

Assume that $u(t, x)$ satisfies the following equation: for all $(t, x) \in (0, T] \times \mathbb{R}^d$,

$$\partial_t u(t, x) = \mathcal{L}_x^{\kappa,0} u(t, x).$$

(71)

If $\sup_{x \in \mathbb{R}^d} u(0, x) \geq 0$, then for every $t \in (0, T]$,

$$\sup_{x \in \mathbb{R}^d} u(t, x) \leq \sup_{x \in \mathbb{R}^d} u(0, x).$$

(72)

Corollary 7.2. If $u_1, u_2 \in C([0, T] \times \mathbb{R}^d)$ satisfy (70), (71) and $u_1(0, x) = u_2(0, x)$, then $u_1 \equiv u_2$ on $[0, T] \times \mathbb{R}^d$. 

7.2. Properties of the semigroup $(P_t^\kappa)_{t \geq 0}$. Define

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y)f(y)dy.$$ 

We first collect some properties of $\Upsilon_t \ast f$.

**Remark 7.3.** We have $\Upsilon_t \ast f \in C_b(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Moreover, $\Upsilon_t \ast f \in C_0(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d) \cup C_0(\mathbb{R}^d)$, $p \in [1, \infty)$. Furthermore, there is $c = c(d)$ such that $\|\Upsilon_t \ast f\|_p \leq c\|f\|_p$ for all $t > 0$, $p \in [1, \infty]$. The above follows from $\Upsilon_t \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d)$ for every $q \in [1, \infty]$ (see [27, Lemma 5.6]), and from properties of the convolution.

**Lemma 7.4.** (a) We have $P_t^\kappa f \in C_b(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$. Moreover, $P_t^\kappa f \in C_0(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d) \cup C_0(\mathbb{R}^d)$, $p \in [1, \infty)$. For every $T > 0$ there exists a constant $c = c(d, T, \sigma, \kappa_4, \beta)$ such that for all $t \in (0, T]$ we get

$$\|P_t^\kappa f\|_p \leq c\|f\|_p.$$ 

(b) $P_t^\kappa : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$, $t > 0$, and for any bounded uniformly continuous function $f$,

$$\lim_{t \to 0^+} \|P_t^\kappa f - f\|_\infty = 0.$$ 

(c) $P_t^\kappa : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $t > 0$, $p \in [1, \infty)$, and for any $f \in L^p(\mathbb{R}^d)$,

$$\lim_{t \to 0^+} \|P_t^\kappa f - f\|_p = 0.$$ 

**Proof.** The proof is like that for [27, Lemma 4.4] and uses Remark 7.3, Lemma 6.19, 6.3, 8.3(a), (48), [27, Lemma 5.6] and Proposition 4.1. □

Our aim now is to prove Proposition 7.9 and Lemma 7.10. The first one provides an important link between $P_t^\kappa$ and the operator $L^\kappa$. The other complements the fundamental properties of $p^\kappa(t, x, y)$. They are both obtained by virtue of Corollary 7.2, but beforehand we have to make sure that certain functions satisfy (71). The necessary results are prepared in Lemma 7.5 – 7.8. In particular, we show that we can apply the operator $L^\kappa$ to $\int_0^t P_s^\kappa f(x) \, ds$.

The formula (74) below is the same as [27, (69)], however the proof given there fully relies on the condition $\alpha_h > 1$, while we assume that $\alpha_h \leq 1$. Here we obtain (74) under an additional restriction on $f$, which suffices for our purposes.

**Lemma 7.5.** For any $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$, we have for all $t > 0$, $x \in \mathbb{R}^d$,

$$\nabla_x P_t^\kappa f(x) = \int_{\mathbb{R}^d} \nabla_x p^\kappa(t, x, y)f(y)dy. \quad (73)$$

For any bounded (uniformly) H"older continuous function $f \in C_b^\alpha(\mathbb{R}^d)$, $1 - \alpha_h < \eta$, and all $t > 0$,

$$x \in \mathbb{R}^d,$$

$$\nabla_x \left( \int_0^t P_s^\kappa f(x) \, ds \right) = \int_0^t \nabla_x P_s^\kappa f(x) \, ds. \quad (74)$$

**Proof.** By (69) and [27, Corollary 5.10] for $|\varepsilon| < h^{-1}(1/t)$,

$$\frac{1}{\varepsilon} \left( p^\kappa(t, x + \varepsilon e_i, y) - p^\kappa(t, x, y) \right) \left| f(y) \right| \leq c \left[ h^{-1}(1/t) \right]^{-1} \Upsilon_t (x - y) |f(y)|.$$ 

The right hand side is integrable by Remark 7.3. We can use the dominated convergence theorem, which gives (73). For $f \in C_b^\alpha(\mathbb{R}^d)$ (we can assume that $\eta < \alpha_h$) we let $\bar{x} = x + \varepsilon \theta e_i$, and by (69), Lemma 6.18 and 8.3(a) we have

$$\left| \int_{\mathbb{R}^d} \frac{1}{\varepsilon} (p^\kappa(s, x + \varepsilon e_i, y) - p^\kappa(s, x, y)) f(y) \, dy \right| \leq \left| \int_{\mathbb{R}^d} \int_0^1 \partial_x p^\kappa(s, \bar{x}, y) \, d\theta f(y) \, dy \right|$$
Furthermore, for every \( T > u \), the equality follows from an application of Fubini’s theorem, justified by \([76]\). Proof. The proof is exactly like that of \([77]\), \([78]\), and all \( t > 0 \), \( x, t \in \mathbb{R}^d \),

\[
\mathcal{L}_x^s \mathcal{P}_t^sf(x) = \int_{\mathbb{R}^d} \mathcal{L}_x^s \mathcal{P}_t^sf(t, x, y)f(y)dy.
\]  

Furthermore, for every \( T > 0 \) there exists a constant \( c > 0 \) such that for all \( f \in L^p(\mathbb{R}^d) \), \( t \in (0, T) \),

\[
\|\mathcal{L}_x^s \mathcal{P}_t^sf\|_p \leq ct^{-1}\|f\|_p.
\]

Proof. By the definition and \([77]\),

\[
\mathcal{L}_x^s \mathcal{P}_t^sf(x) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \delta^s(t, x, y; z)f(y)dy \right) \kappa(x, z)J(z)dz.
\]

The inequality follows from an application of Fubini’s theorem, justified by \([66]\) and Remark 7.3. The inequality follows then from \([75]\), \([68]\), Remark 7.3.

Lemma 7.7. Let \( f \in C_0(\mathbb{R}^d) \). For \( t > 0 \), \( x \in \mathbb{R}^d \) we define \( u(t, x) = \mathcal{P}_t^sf(x) \) and \( u(0, x) = f(x) \). Then \( u \in C([0, T] \times \mathbb{R}^d) \), \([70]\) holds and \( \partial_t u(t, x) = \mathcal{L}_x^s u(t, x) \) for all \( t, T > 0 \), \( x \in \mathbb{R}^d \).

Proof. The proof is exactly like that of \([27\text{, Lemma 4.7}]\).
By (66), we get $I_1 \leq c \int_0^t \int_{\mathbb{R}^d} \vartheta(s) \rho_0^i(s,y-x) dy ds$, while by (67) $I_2 \leq c \int_0^t \vartheta(s) s^{-1} [h^{-1}(1/s)]^{\beta_1} ds$. The integrals are finite by [27, Lemma 5.15], Lemma 8.3(a) and 8.2. □

**Proposition 7.9.** For any $f \in C^2_0(\mathbb{R}^d)$ and all $t > 0$, $x \in \mathbb{R}^d$,

$$P_t^u f(x) - f(x) = \int_0^t P_s^u \mathcal{L}^\kappa f(x) \, ds.$$  \hspace{1cm} (79)

**Proof.** We outline the main steps, for details, see the proof of [27, Proposition 4.9].

(i) Note that $\mathcal{L}^\kappa f \in C_0(\mathbb{R}^d)$ for any $f \in C^2_0(\mathbb{R}^d)$.

(ii) Show that $\mathcal{L}^\kappa f \in C_0^\kappa(\mathbb{R}^d)$ for any $f \in C^{2,\kappa}_0(\mathbb{R}^d)$.

(iii) Show that (79) holds for any $f \in C^2_{0,\kappa}(\mathbb{R}^d)$ if $1 - \alpha_h < \eta \leq \beta$. It is achieved by using Corollary 7.2 to prove that the following functions are equal

$$u_1(t,x) = \begin{cases} P_t^u f(x), & t > 0, \\ f(x), & t = 0 \end{cases}, \quad u_2(t,x) = \begin{cases} f(x) + \int_0^t P_s^u \mathcal{L}^\kappa f(x) \, ds, & t > 0 \\ f(x), & t = 0 \end{cases}.$$  

Here we use Lemmas 7.7, 7.8 and [59, Theorem 7.21].

(iv) We extend (79) to $f \in C^2_0(\mathbb{R}^d)$ by approximating it with $f_n = (f \ast \phi_n) \cdot \varphi_n \in C^\infty_c(\mathbb{R}^d)$, where $\phi_n$ is a standard mollifier while $\varphi_n(x) = \varphi(x/n)$ for $\varphi \in C^\infty_c(\mathbb{R}^d)$ satisfying $\varphi(x) = 1$ if $|x| \leq 1$, and $\varphi(x) = 0$ if $|x| \geq 2$. □

**Lemma 7.10.** The function $p^\kappa(t,x,y)$ is non-negative, $\int_{\mathbb{R}^d} p^\kappa(t,x,y) dy = 1$ and $p^\kappa(t+s,x,y) = \int_{\mathbb{R}^d} p^\kappa(t,x,z) p^\kappa(s,z,y) dz$ for all $s,t > 0$, $x,y \in \mathbb{R}^d$.

**Proof.** The proof is like that of [27, Lemma 4.10] and we use Lemma 7.7, Theorem 7.1, Lemma 6.19, Corollary 7.2, Proposition 7.9. However, the proof of the convolution property in [27] contains a gap: at that stage it is not clear why the function $p^\kappa(t+s,x,y)$ should satisfy the equation (71) for all $x \in \mathbb{R}^d$. Here we present a correction that is valid for both papers.

Let $T, s > 0$ and $\varphi \in C^\infty_c(\mathbb{R}^d)$. For $t > 0$, $x \in \mathbb{R}^d$ define

$$u_1(t,x) = P_t^u f(x), \quad u_1(0,x) = f(x) = P_s^u \varphi(x),$$

and

$$u_2(t,x) = P_{t+s}^u \varphi(x), \quad u_2(0,x) = P_s^u \varphi(x).$$

By Lemma 7.4(b) $f \in C_0(\mathbb{R}^d)$ and thus by Lemma 7.7 $u_1$ satisfies the assumptions of Corollary 7.2. Now, since $\varphi$ has compact support by Lemma 6.19(a) we get $u_2 \in C([0,T] \times \mathbb{R}^d)$. We will use [27, (94)] several times in what follows. By Lemma 6.19(c) and (d),

$$\|u_2(t,\cdot) - u_2(0,\cdot)\|_\infty \leq \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |\mathcal{L}_x^\kappa p^\kappa(u+s,x,y)| du |\varphi(y)| dy$$

$$\leq ct \rho^0_h(s,0) \int_{\mathbb{R}^d} |\varphi(y)| dy \to 0, \quad \text{as} \ t \to 0^+.$$  

Furthermore, by Lemma 6.19(b)

$$\sup_{t \in [0,T]} \|u_2(t,\cdot) 1_{|\cdot| > r}\|_\infty \leq c(T+s) \sup_{x \in \mathbb{R}^d} \int_{|x| > r} \rho^0_h(s,x-y)|\varphi(y)| dy$$

$$= c(T/s + 1) \sup_{x \in \mathbb{R}^d} \int_{|x| > r} (\Upsilon_{t} \ast |\varphi|)(x) \to 0, \quad \text{as} \ r \to \infty,$$

because $\Upsilon_{t} \ast |\varphi| \in C_0(\mathbb{R}^d)$ (see Remark 7.3). Finally, by the mean value theorem, Lemma 6.19(c), (68) and the dominated convergence theorem $\partial_t u_2(t,x) = \int_{\mathbb{R}^d} \partial_t p^\kappa(t+s,x,y) \varphi(y) dy$. Then we
apply Lemma 6.19(c) and Lemma 7.6 to obtain $\partial_t u_2(t, x) = \mathcal{L}_x u(t, x)$. Therefore, by Corollary 7.2 $u_1 = P_t^\kappa P_x^\alpha \varphi = P_t^\kappa \varphi = u_2$. The convolution property now follows by Fubini theorem, the arbitrariness of $\varphi$ and by Lemma 6.19(a).

7.3. Proofs of Theorems 2.1–2.3. At this point we have all necessary tools to proceed exactly like in [27]. A slight difference in the proof of Theorem 2.4 is explained below.

Proof of Theorem 2.3. It is the same as in [27] and relies on Lemma 7.4, 7.10, Proposition 7.9, (73), (69), Remark 7.3, (66), (77), (3), (4), (75), Lemma 6.19(f) and (d), [27, Corollary 5.10], (76), [55, Chapter 1, Theorem 2.4(c) and 2.2], [55, Chapter 2, Theorem 5.2(d)].

Proof of Theorem 2.2. All the results are collected in Lemma 6.19 and 7.10, except for part (8), which is given in Theorem 2.3 part (3).

Proof of Theorem 2.1. The same as in [27].

Proof of Theorem 2.4. The argument is the same as in [27] except for the following modification of the prove of the estimate

$$
\sup_{|\xi| \leq 1/\sqrt{q}} |q(z, \xi)| \leq c_2 h(r).
$$

Since for $\varphi \in \mathbb{R}$ we have $|e^{i\varphi} - 1 - i\varphi| \leq |\varphi|^2$, by (7) we get

$$
|q(z, \xi)| \leq \int_{\mathbb{R}^d} |e^{i\varphi(w)} - 1 - i \varphi(w)| 1_{|w| < 1/|\xi|} |\kappa(x, w) J(w) dw |
$$

$$
+ |i \varphi(w) \int_{\mathbb{R}^d} w (1_{|w| < 1/|\xi|} - 1_{|w| < 1}) \kappa(x, w) J(w) dw |
$$

$$
\leq |\xi|^2 \int_{|w| < 1/|\xi|} |w|^2 \kappa(x, w) J(w) dw + \int_{|w| > 1/|\xi|} 2 \kappa(x, w) J(w) dw
$$

$$
+ |\xi| \int_{\mathbb{R}^d} w (1_{|w| < 1/|\xi|} - 1_{|w| < 1}) \kappa(x, w) J(w) dw \leq c h(1 \wedge 1/|\xi|).
$$

8. Appendix - Unimodal Lévy processes

Let $d \in \mathbb{N}$ and $\nu : [0, \infty) \rightarrow [0, \infty]$ be a non-increasing function satisfying

$$
\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(|x|) dx < \infty.
$$

For any such $\nu$, there exists a unique pure-jump isotropic unimodal Lévy process $X$ (see [3], [62]). We define $h(r), K(r)$ and $\Upsilon_t(x)$ as in the introduction. At this point we refer the reader to [27, Section 5] for various important properties of those functions. Following [27, Section 5] in the whole section we assume that $h(0^+) = \infty$. We consider the scaling conditions:

there are $\alpha_h \in (0, 2], C_h \in [1, \infty)$ and $\theta_h \in (0, \infty]$ such that

$$
h(r) \leq C_h \lambda^{\alpha_h} h(\lambda r), \quad \lambda \leq 1, r < \theta_h;
$$

there are $\beta_h \in (0, 2], c_h \in (0, 1]$ and $\theta_h \in (0, \infty]$ such that

$$
c_h \lambda^{\beta_h} h(\lambda r) \leq h(r), \quad \lambda \leq 1, r < \theta_h.
$$

The first and the latter inequality in the next lemma are taken from [27, Section 5]. We keep them here for easier reference.
Lemma 8.1. Let $h$ satisfy (80) with $\alpha_h > 1$, then
\[ \int_{r \leq |z| < \theta_h} |z|^\nu(|z|)dz \leq \frac{(d + 2)C_h}{\alpha_h - 1} rh(r), \quad r > 0. \]

Let $h$ satisfy (80) with $\alpha_h = 1$, then
\[ \int_{r \leq |z| < \theta_h} |z|^\nu(|z|)dz \leq [(d + 2)C_h] \ln(\theta_h/r) rh(r), \quad r > 0. \]

Let $h$ satisfy (81) with $\beta_h < 1$, then
\[ \int_{|z| < r} |z|^\nu(|z|)dz \leq \frac{d + 2}{ch(1 - \beta_h)} rh(r), \quad r < \theta_h. \]

**Proof.** Under (80) with $\alpha_h = 1$ we have
\[ \int_{r \leq |z| < \theta_h} |z|^\nu(|z|)dz \leq (d + 2) \int_r^{\theta_h} h(s)ds \leq (d + 2)C_h \int_r^{\theta_h} (r/s)h(r)ds, \]
which ends the proof. \square

Lemma 8.2. Assume (5). Let $k, l \geq 0$ and $\theta, \eta, \beta, \gamma \in \mathbb{R}$ satisfy $(\beta/2) \wedge (\beta/\alpha_h) + 1 - \theta > 0$, $(\gamma/2) \wedge (\gamma/\alpha_h) + 1 - \eta > 0$. For every $T > 0$ there exists a constant $c = c(\alpha_h, C_h, h^{-1}(1/T) \vee 1, \theta, \eta, \beta, \gamma, k, l)$ such that for all $t \in (0, T]$,
\[ \int_0^t [\Theta(u)]^l u^{-\eta} [h^{-1}(1/u)]^\gamma [\Theta(t - u)]^k (t - u)^{-\theta} [h^{-1}(1/(t - u))]^\beta du \leq c [\Theta(t)]^{l+k} t^{1-\eta-\theta} [h^{-1}(1/t)]^{\gamma+\beta}. \]

Furthermore, $\Theta(t/2) \leq c \Theta(t), t \in (0, T]$. **Proof.** The last part of the statement follows from [27, Lemma 5.3 and Remark 5.2]. Note that it suffices to consider the integral over $(0, t/2)$. Again by [27, Lemma 5.3 and Remark 5.2] we have for $c_0 = C_h[h^{-1}(1/T) \vee 1]^2$ and $s \in (0, 1),$
\[ [h^{-1}(t^{-1}s^{-1})]^{-1} \leq c_0 s^{-1/\alpha_h} [h^{-1}(1/t)]^{-1}, \quad h^{-1}(t^{-1}s^{-1}) \leq s^{1/2}h^{-1}(t^{-1}). \]

Thus for $u \in (0, t/2)$ we get
\[ [\Theta(t - u)]^k (t - u)^{-\theta} [h^{-1}(1/(t - u))]^\beta \leq c [\Theta(t)]^k t^{-\theta} [h^{-1}(1/t)]^\beta, \]
and we concentrate on
\[ \int_0^{t/2} [\Theta(u)]^l u^{-\eta} [h^{-1}(1/u)]^\gamma du \leq ct^{-\eta} [h^{-1}(1/t)]^\gamma \int_0^{1/2} [\Theta(ts)]^l s^{(\gamma/2)\wedge (\gamma/\alpha_h) - \eta} ds. \]

Furthermore, we have
\[ \int_0^{1/2} \left[ \ln \left( 1 + [h^{-1}(t^{-1}s^{-1})]^{-1} \right) \right]^{l} s^{(\gamma/2)\wedge (\gamma/\alpha_h) - \eta} ds \leq \int_0^{1/2} 2^{l} \left\{ \ln(c_0s^{-1}) \right\}^{l} + \left[ \ln \left( 1 + [h^{-1}(t^{-1})]^{-1} \right)^{l} \right] s^{(\gamma/2)\wedge (\gamma/\alpha_h) - \eta} ds \leq c [\Theta(t)]^{l}. \]

Finally,
\[ \int_0^{1/2} [\Theta(ts)]^l s^{(\gamma/2)\wedge (\gamma/\alpha_h) - \eta} ds \leq c [\Theta(t)]^{l}. \] \square
The next lemma is taken from [27, Section 5] and complemented with part (d). It is one of the most often used technical result in the paper. Let \( B(a, b) \) be the beta function, i.e., \( B(a, b) = \int_0^1 s^{a-1} (1 - s)^{b-1} ds \), \( a, b > 0 \).

**Lemma 8.3.** Assume (5) and let \( \beta_0 \in (0, \alpha_h \land 1) \).

(a) For every \( T > 0 \) there exists a constant \( c_1 = c_1(d, \beta_0, \alpha_h, C_h, h^{-1}(1/T) \lor 1) \) such that for all \( t \in (0, T] \) and \( \beta \in [0, \beta_0] \),

\[
\int_{\mathbb{R}^d} \rho^\beta_0(t, x) \, dx \leq c_1 t^{-1} \left[ h^{-1}(1/t) \right]^\beta.
\]

(b) For every \( T > 0 \) there exists a constant \( c_2 = c_2(d, \beta_0, \alpha_h, C_h, h^{-1}(1/T) \lor 1) \geq 1 \) such that for all \( \beta_1, \beta_2, n_1, n_2, m_1, m_2 \in [0, \beta_0] \) with \( n_1, n_2 \leq \beta_1 + \beta_2, m_1 \leq \beta_1, m_2 \leq \beta_2 \) and all \( 0 < s < t \leq T \), \( x \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \rho^\beta_0(t, x - z) \rho^\beta_0(s, z) \, dz \\
\leq c_2 \left[ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{n_1} + s^{-1} \left[ h^{-1}(1/s) \right]^{n_2} \right] \rho^0_0(t, x) \\
+ (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{m_1} \rho^\beta_0(t, x) + s^{-1} \left[ h^{-1}(1/s) \right]^{m_2} \rho^\beta_0(t, x).
\]

(c) Let \( T > 0 \). For all \( \gamma_1, \gamma_2 \in \mathbb{R}, \beta_1, \beta_2, n_1, n_2, m_1, m_2 \in [0, \beta_0] \) with \( n_1, n_2 \leq \beta_1 + \beta_2, m_1 \leq \beta_1, m_2 \leq \beta_2 \) and \( \theta, \eta \in [0, 1] \), satisfying

\[
(\gamma_1 + n_1 \land m_1)/2 \land (\gamma_1 + n_1 \land m_1)/\alpha_h + 1 - \theta > 0,
\]

\[
(\gamma_2 + n_2 \land m_2)/2 \land (\gamma_2 + n_2 \land m_2)/\alpha_h + 1 - \eta > 0,
\]

and all \( 0 < s < t \leq T \), \( x \in \mathbb{R}^d \), we have

\[
\int_0^t \int_{\mathbb{R}^d} (t - s)^{-1} \rho^\beta_0(t - s, x - z) \, dz \, ds \\
\leq c_3 t^{2 \land 2 \land \gamma_1 + n_1 \land m_1}/\alpha_h \land \left( (\gamma_2 + n_2 \land m_2)/\alpha_h \right) \land \left( \gamma_2 + n_2 \land m_2 \right), \quad (82)
\]

where \( c_3 = c_2 \left( C_h [h^{-1}(1/T) \lor 1]^2 \right)^{-\gamma_1 \land 0 + \gamma_2 \land 0}/\alpha_h B(k + 1 - \theta, l + 1 - \eta) \) and

\[
k = (\gamma_1 + n_1 \land m_1)/2 \land (\gamma_1 + n_1 \land m_1)/\alpha_h + 1 - \theta > 0,
\]

\[
(\gamma_2 + n_2 \land m_2)/2 \land (\gamma_2 + n_2 \land m_2)/\alpha_h + 1 - \eta > 0,
\]

and \( k, l \geq 0 \), \( \gamma_1, \gamma_2 \in \mathbb{R}, \beta_1, \beta_2, n_1, n_2, m_1, m_2 \in [0, \beta_0] \) with \( n_1, n_2 \leq \beta_1 + \beta_2, m_1 \leq \beta_1, m_2 \leq \beta_2 \) and \( \theta, \eta \in [0, 1] \), satisfying

\[
(\gamma_1 + n_1 \land m_1)/2 \land (\gamma_1 + n_1 \land m_1)/\alpha_h + 1 - \theta > 0,
\]

\[
(\gamma_2 + n_2 \land m_2)/2 \land (\gamma_2 + n_2 \land m_2)/\alpha_h + 1 - \eta > 0,
\]

and for all \( 0 < s < t \leq T \), \( x \in \mathbb{R}^d \), we have

\[
\int_0^t \int_{\mathbb{R}^d} (\Theta(t - s))^k (t - s)^{-1} \rho^\beta_0(t - s, x - z) \Theta(s) \, dz \, ds \\
\leq c_4 \Theta(t)^k \left( \gamma_1 \land 0 + \gamma_2 \land 0 \right)/\alpha_h \land \left( \gamma_2 + n_2 \land m_2 \right), \quad (83)
\]

where \( c_4 = c_4(d, \beta_0, \alpha_h, C_h, h^{-1}(1/T) \lor 1) \).

**Proof.** For the proof of part (d) we multiply the result of part (b) by

\[
[\Theta(t - s)]^k (t - s)^{-1} \left[ h^{-1}(1/(t - s)) \right]^{\gamma_1} \Theta(s)^{\gamma_2} (t - s)^{\gamma_2} h^{-1}(1/s)^{\gamma_2},
\]

and apply Lemma 8.2. \( \square \)
Remark 8.4. When using Lemma 8.3 without specifying the parameters, we apply the usual case, i.e., \( n_1 = n_2 = \beta_1 + \beta_2 \ (\leq \beta_0) \), \( m_1 = \beta_1 \), \( m_2 = \beta_2 \). Similarly, if only \( n_1, n_2 \) are specified, then \( m_1 = \beta_1 \), \( m_2 = \beta_2 \).

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