Gravitational memory for uniformly accelerated observers

Sanved Kolekar $^{1,2}$ * and Jorma Louko$^1$ †

$^1$ School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK

$^2$ UM-DAE Centre for Excellence in Basic Sciences, Mumbai 400098, India

April 14, 2017

Abstract

Recently, Hawking, Perry and Strominger described a physical process that implants supertranslational hair on a Schwarzschild black hole by an infalling matter shock wave without spherical symmetry. Using the BMS-type symmetries of the Rindler horizon, we present an analogous process that implants supertranslational hair on a Rindler horizon by a matter shock wave without planar symmetry, and we investigate the corresponding memory effect on the Rindler family of uniformly linearly accelerated observers. We assume each observer to remain linearly uniformly accelerated through the wave, in the sense of the curved spacetime generalisation of the Letaw-Frenet equations. Starting with a family of observers who follow the orbits of a single boost Killing vector before the wave, we find that after the wave has passed, each observer still follows the orbit of a boost Killing vector but this boost differs from trajectory to trajectory, and the trajectory-dependence carries a memory of the planar inhomogeneity of the wave. We anticipate this classical memory phenomenon to have a counterpart in Rindler space quantum field theory.

*sanved.kolekar@nottingham.ac.uk
†jorma.louko@nottingham.ac.uk
1 Introduction

Recently, Hawking, Perry and Strominger [1, 2] (HPS) have shown that a black hole in an asymptotically flat spacetime has an infinite collection of soft hairs corresponding to the infinite supertranslation symmetries of the flat spacetime at asymptotic infinity. These supertranslations are essentially diffeomorphisms on the spacetime which leave the asymptotic structure at null infinity intact and they belong to the Bondi-Metzner-Sachs (BMS) subgroup [3]. Classically, diffeomorphisms do not affect the vacuum associated with the phase space of the canonically conjugate variables of gravitational degrees of freedom. However, from a field theoretic perspective, it has been argued that the supertranslations act non-trivially on the degenerate vacua related to the infinite BMS asymptotic symmetries and are spontaneously broken accompanied by the creation/annihilation of Goldstone bosons, namely, the soft photons and soft gravitons. The results due to Christodoulou and Klainerman [4] on stability of Minkowski spacetime and asymptotic boundary conditions allow one to construct an infinite number of non-zero and finite conserved supertranslation and superrotation charges on past and future null infinity of asymptotically flat generic spacetimes. For the class of spacetimes containing a black hole in the interior as considered by HPS, it has been conjectured [1] that these conserved superrotation charges would require the outgoing Hawking quanta to have the required correlations needed to maintain the unitarity of the evaporation process. (Also see [5].)

A physical process version of implanting a soft hair was described in [2] consisting of the action of in-falling matter wave with a well defined non-zero and finite $T_{ab}$. A black hole can acquire or lose a supertranslation hair depending on the incoming or outgoing asymmetric shock wave. The resultant metric, after the wave, is related to the metric before the wave by a BMS supertranslation. These generate non-trivial time-translations on the null generators of the event horizon and could be responsible for the additional correlations in the outgoing Hawking quanta [6] thus acting like a gravitational memory on the horizon.

Hawking’s prediction of black hole radiation [7] relies on the semi-classical framework for gravity wherein only the matter fields are described in a quantum field theoretic setup and propagating on a classical background geometry. Within the same framework, however with a more practical approach, it is known that an Unruh-DeWitt detector coupled to the Hartle-Hawking state of the quantum field and positioned at a fixed radius outside the hole responds thermally [8, 9]. In particular, the response rate of the detector is of the Kubo-Martin-Schwinger form [10, 11, 12]. A uniformly linearly accelerated observer/trajectory plays a central role in these semi-classical analyses:
the thermal form of the Hawking radiation is a peculiarity associated only
with the uniformly linearly accelerated observers. A freely falling detector,
either radially in-falling or on a elliptical/circular orbit responds quite dif-
ferently albeit non-thermally [13].

An interesting question one would like to address is how does implanting
a black hole with supertranslation hair affect the thermal response of the
uniformly linearly accelerated detector. There are two aspects involved in
such an investigation. First, it is well known that passage of gravitational
radiation, in-falling or outgoing, results in a change in the mutual proper-
separation of geodesic observers at asymptotic infinity, an effect called as the
gravitational memory effect [14]. More recently, it has been shown [15] that
the memory effect for geodesic observers is equivalent to that of a diffeomor-
phism on the Schwarzschild metric belonging to the class of BMS supertrans-
lations at asymptotic infinity. On can expect the supertranslations to have
a distinguishable effect at a classical level on the congruence of uniformly
accelerated observers as well. The second aspect is to explore the effect on
quantum evolution of a scalar field propagating due to the presence of super-
translational diffeomorphisms. The former aspect of gravitational memory
for uniformly linearly accelerated observers in the case of a Schwarzschild
black hole and its Rindler analogue shall be our broad focus in the present
paper. We plan to address the related quantum aspects in a future paper
[16].

The direct analogue of black hole Hawking radiation in flat spacetime is
the Unruh effect [17, 18]. A uniformly linearly accelerated observer, moving
on an integral curve of a boost Killing vector, perceives the Minkowski vac-
um to be thermal with a temperature proportional to the magnitude of its
acceleration. In contrast to the black hole case, the mode solutions of the
quantum field in the Rindler spacetime are known in terms of well studied
special functions. The analytical tractability and conceptual similarity often
makes the latter a pre-exploratory basis for studying numerous black hole
effects, which we also exploit in this paper. We extend the physical process
of implanting a supertranslational hair described by HPS to the case of the
Rindler horizon and investigate their corresponding gravitational memory ef-
fects on uniformly accelerated observers. The horizon symmetries of the BMS
type for the Rindler spacetime have been found in [19, 20, 21] (for a related
discussion see [22]). In section 2, we briefly review a class of such Rindler
supertranslations and introduce an asymmetric matter shockwave impinging
on the Rindler horizon. In section 3, we propose a covariant way to define
uniformly linearly accelerated trajectory in curved spacetime. The motiva-
tion to do so is also discussed in the same section. In section 4, we analyse
the effect of implanting supertranslational hair on uniformly linearly acceler-
ated motion in the Rindler spacetime. Starting with a family of trajectories that follow the orbits of a single boost Killing vector before the wave, we find that after the wave has passed, each trajectory still follows the orbit of a boost Killing vector but this boost differs from trajectory to trajectory, and the trajectory-dependence carries a memory of the planar inhomogeneity of the wave. We further show that the effect of supertranslations on uniformly linearly accelerated observers in the Schwarzschild spacetime is even more drastic with the trajectory falling inside the black hole horizon for an ingoing shock-wave or the trajectory ejecting out to spatial infinity for a outgoing shockwave. Concluding remarks are collected in section 5.

The Minkowski metric is taken to have the mostly plus signature, and Roman indices run over all spacetime indices.

2 Implanting supertranslational hair to the Rindler horizon

In [2], HPS considered a shock-wave without spherical symmetry propagating on a Schwarzschild spacetime. The metric for the complete process of implanting the supertranslational hair is given by

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{2M}{r} - h(v - v_0)\frac{2\mu}{r} - h(v - v_0)\frac{MD^2C}{r^2}\right)dv^2 + 2dvdr \\
    &\quad - h(v - v_0)D_A\left(2C - \frac{4MC}{r} + D^2C\right)dvd\theta^A \\
    &\quad + \left(r^2\gamma_{AB} + h(v - v_0)2rD_AD_BC - h(v - v_0)r\gamma_{AB}D^2C\right)d\theta^A\theta^B
\end{align*}
\]

(2.1)

where the co-ordinates \((v, r, \Theta^A)\) are the Bondi co-ordinates, \(\gamma_{AB}d\theta^A\theta^B\) is the metric on the unit two-sphere, \(D^A\) is the covariant derivative on the unit two-sphere, \(h(v - v_0)\) is the Heavyside step function and the function \(C(\Theta)\) characterises the angular profile of the shock wave. The metric differs from a Schwarzschild metric of mass \(M > 0\) by

\[
h_{ab} = h(v - v_0)\left(\mathcal{L}_\Xi g_{ab} - \frac{2\mu}{r}\delta_a^v\delta_b^v\right)
\]

(2.2)

where \(\Xi^a = [\mathcal{C}, -D^2C/2, D^AC/r]\) is the BMS-type supertranslation vector, preserving the Bondi gauge \(g_{rr} = 0 = g_{rA}, \det(g_{AB}/r^2) = g(\Theta)\) and satisfying the asymptotic fall-off conditions required to preserve the asymptotic infinity.
The stress energy tensor of the shock wave is

\[
T_{vv} = \frac{1}{4\pi r^2} \left[ \mu + \frac{D^2(D^2 + 2)C}{4} - \frac{3MD^2C}{2r} \right] \delta(v - v_0)
\]

\[
T_{vA} = -\frac{3MD_A C}{8\pi r^2} \delta(v - v_0)
\]

Similar to the Schwarzschild case, we consider a physical process version of implanting supertranslation hair to the Rindler horizon. The metric of the Rindler spacetime before the shock wave can be expressed in the Bondi-type co-ordinates \((v, r, x, y)\) as

\[
ds^2 = -2\kappa rv^2 + 2dvdr + \delta_{AB}dx^A dx^B
\]

where \(\kappa > 0\) and the Rindler horizon is at \(r = 0\). The analogue of the BMS-type supertranslation for the Rindler horizon described in Bondi-type co-ordinates are given by corresponding supertranslation vector

\[
\Xi^a = \frac{1}{\kappa} [f(x, y), 0, -r \partial^A f(x, y)]
\]

which preserves the Bondi-type gauge chosen \(g_{rr} = 0, g_{vr} = 2\) and the structure of the horizon \(\mathcal{L}_x g_{rr} = 0 + \mathcal{O}(r), \mathcal{L}_x g_{Ar} = 0 + \mathcal{O}(r)\) \([19, 20, 21]\). Joining the metric in Eq. (2.4) to the supertranslated metric along a shock wave propagating at \(v = v_0\), we find that the perturbations to the Rindler metric \(h_{ab} = \mathcal{L}_x g_{ab}\) due to the shockwave are then given by

\[
\mathcal{L}_x g_{Av} = h(v - v_0)2r \partial_A f
\]

\[
\mathcal{L}_x g_{AB} = h(v - v_0)2 \left( \frac{r}{\kappa} \right) \partial_A \partial_B f
\]

where again \(h(v - v_0)\) is the Heavyside step function. Hence, the full metric describing the shockwave at \(v = v_0\) passing through the Rindler horizon at \(r = 0\) is

\[
ds^2 = -2\kappa rv^2 + 2dvdr + 4rh(v - v_0)\partial_A f dvdx^A
\]

\[
+ \left( \delta_{AB} + 2h(v - v_0)\frac{r}{\kappa} \partial_A \partial_B f \right) dx^A dx^B
\]

Working to linear order in \(f\), we find the linearised stress energy tensor of
the shockwave to be
\[ T_{vv} = -rh'\partial_A \partial^A f - \frac{r}{\kappa} h'' \partial_A \partial^A f \]
\[ T_{vr} = \frac{h'}{\kappa} \partial_A \partial^A f \]
\[ T_{vA} = -h' \partial_A f \]
\[ T_{AB} = \begin{cases} \frac{2h'}{\kappa} \partial_A \partial_B f & A = B \\ -2\frac{h'}{\kappa} \partial_A \partial_B f & A \neq B \end{cases} \]
where the primed indices denote differentiation with respect to \( v \). Note that \( T_{ab} \) is by construction covariantly conserved.

Similar to the Schwarzschild case, one could demand the surface stress energy tensor to have an additional \( \bar{\mu}h'(v - v_0) \) term, with \( \bar{\mu} > 0 \) a constant, in the null-null component \( T_{vv} \) such that
\[ T_{vv} = \bar{\mu}h' - rh'\partial_A \partial^A f - \frac{r}{\kappa} h'' \partial_A \partial^A f \] (2.10)
with all the remaining components of the surface stress energy tensor to be the same as in Eq. (2.9). The physical interpretation of \( \bar{\mu} \) is that of being the surface energy density of the shockwave. The corresponding metric perturbations in Eq. (2.7) to the Rindler metric then get modified to
\[ h_{Av} = h(v - v_0)2r \partial_A f \] (2.11)
\[ h_{AB} = \frac{\bar{\mu}}{\kappa} \delta_{AB}h(v - v_0)(e^{\kappa(v-v_0)} - 1) + h(v - v_0)2 \left( \frac{r}{\kappa} \right) \partial_A \partial_B f \] (2.12)
Thus, unlike in the Schwarzschild case wherein the surface energy density term \( \mu/4\pi r^2 \) leads to a perturbation in the \( g_{vv} \) component, it is the transverse part of the Rindler metric which gets perturbed due to the surface density \( \bar{\mu} \). We show in section 4 that this difference in the metric leads to a drastic difference in the effect on the uniformly linearly accelerated trajectories: the effect in Rindler is a trajectory-dependent Lorentz boost, while the effect in Schwarzschild is an instability that knocks the trajectory away from stationarity and, for \( \mu > 0 \), makes it fall into the black hole.

3 Letaw-Frenet equations in curved spacetime

To describe the memory effect for uniformly linearly accelerated observers, we need a covariant definition of such observers in a spacetime that is not necessarily flat. Below we motivate the need for such a construction.
In Minkowski spacetime, a uniformly accelerated trajectory may be defined as a timelike orbit of a Killing vector. These orbits were classified in terms of Lorentz-signature Frenet equations by Letaw [23], and summaries in terms of the geometry of the Killing vectors are given in [24, 25]. As each of the trajectories is an orbit of a one-parameter isometry group, the magnitude of the proper acceleration is constant along the trajectory. The uniformly accelerated trajectories are hence a special class among worldlines on which the proper acceleration four-vector has constant magnitude. Note that if just the magnitude of the proper acceleration is fixed to a constant $\alpha > 0$, the direction of the proper acceleration four-vector remains still freely specifiable on the $S^2$ of radius $\alpha$ in the hyperplane orthogonal to the four-velocity, at each point on the worldline.

Among the uniformly accelerated trajectories, the orbits of a boost are called uniformly linearly accelerated. In quantum field theory, a uniformly linearly accelerated observer reacts to the Minkowski vacuum as if it were a thermal state [18]; by contrast, the observer’s response under the other types of uniform acceleration is not expected to be thermal [23, 24, 25]. This special property of the boost can be attributed to the specific form of the entanglement between the field modes in the causally disconnected quadrants separated by the boost Killing horizon, and to the fact that each trajectory stays in one of the quadrants. In this paper we hence focus on observers of uniform linear acceleration.

To address uniformly linearly accelerated observers in the presence of matter shock waves, we do however need to generalise the notion of uniform linear acceleration to a spacetime that is not necessarily flat. We now proceed to do this.

Letaw [23] showed that the construction of the generalised Frenet equations in Minkowski spacetime can be utilised to define analogues of the scalar curvature, torsion and hypertorsion scalars in differential geometry for world lines in flat spacetime. In particular, the scalar curvature is the magnitude of the proper acceleration. The case of uniform linear acceleration then arises when the scalar curvature is fixed to a constant positive value while the torsion and hypertorsion scalars are taken to vanish.

To generalise the Letaw-Frenet construction to curved spacetime, we begin as in [23] by defining four unit vectors forming an orthogonal tetrad using the Gram-Schmidt orthogonalisation procedure. These are defined at each

---

1If the observer is direction-specific, thermality does however arise also for the accelerated trajectory constructed from a boost and a commuting spatial translation, but with an anisotropic temperature that contains a direction-dependent Doppler shift factor [26, 27].
point along the trajectory $x^a(\tau)$ of interest as

$$
\begin{align*}
V_0^i &= u^i = \frac{dx^i}{d\tau} \\
V_1^i &= \frac{a^i}{|a|} \\
V_2^i &= \frac{|a|^2 w^i - |a|^2 (w^k u_k) u^i - (w^k a_k) a^i}{N} \\
V_3^i &= \frac{-1}{\sqrt{6}} \epsilon^{ijkl} V_{0j} V_{1k} V_{2l}
\end{align*}
$$

(3.1)

where $a^i = w^j \nabla_j u^i$, $w^i = \nabla_j a^i$ and $N = |a| \left( |a|^2 w^k w^k - (a_k w^k)^2 + |a|^4 \right)^{1/2}$. Assuming $a^i \neq 0$ and $N \neq 0$, the four unit vectors of the tetrad by definition satisfy the following condition at the tangent space at each event along the trajectory

$$
V_{\alpha a} V_{\beta} = \eta_{\alpha \beta}
$$

(3.2)

where the Greek indices label the respective unit vector. The generalised Letaw-Frenet equations then are

$$
u^j \nabla_j V^i = K^\beta_{\alpha} V^i
$$

(3.3)

where

$$
K_{\alpha \beta} = \begin{pmatrix}
0 & -K(\tau) & 0 & 0 \\
K(\tau) & 0 & -T(\tau) & 0 \\
0 & T(\tau) & 0 & -\mathcal{V}(\tau) \\
0 & 0 & \mathcal{V}(\tau) & 0
\end{pmatrix}
$$

(3.4)

To arrive at (3.4), we may proceed as in flat spacetime [23], by taking the derivative of the orthogonality condition (3.2) along $u^i$, using (3.3) to deduce antisymmetry of $K_{\alpha \beta}$, and finally using (3.1) to deduce that the $\alpha^{th}$ row in $K_{\alpha \beta}$ can have non-zero entries only in the columns with $\beta \leq \alpha + 1$. The scalar quantities $K(\tau)$, $T(\tau)$ and $\mathcal{V}(\tau)$ can be straightforwardly identified as the analogues of the curvature scalar, torsion and the hypertorsion scalars respectively by simply constructing a local inertial frame around any event on the trajectory and matching the covariant scalars with those in the construction of Letaw’s Frenet equations in flat spacetime. Note that $K(\tau)$ is the magnitude of the proper acceleration, $K(\tau) = (u^i \nabla_j V_{0j}) V_1^i = |a|$. We may now define the curved spacetime analogue of uniform acceleration by requiring $K(\tau)$, $T(\tau)$ and $\mathcal{V}(\tau)$ to be independent of $\tau$. Uniform linear acceleration is defined as the special case in which $K$ is strictly positive and $T$ and $\mathcal{V}$ vanish. For uniform linear acceleration, the only non-trivial
Frenet equation is then the equation of motion for the normalised acceleration vector,

\[ u^j \nabla_j V_i^j = K_1^0 V_0^i \]

\[ \Rightarrow u^j \nabla_j a^i = w^i = |a|^2 u^i \quad (3.5) \]

The above equation was also obtained in [28] by generalising the differential-geometric characteristics of a rectangular hyperbola in Minkowski spacetime to curved spacetimes. As a technical caveat, we should note that the tetrad (3.1) is not well defined for uniform linear acceleration because the formula for \( V_2^2 \) takes the ambiguous form 0/0, using (3.5). This can be remedied by defining a binormal \( V_{2,3}^{ij} \) to the plane of \( V_{0}^i \) and \( V_{1}^i \) by

\[ V_{2,3}^{ij} = -\frac{1}{\sqrt{6}} \epsilon^{ijkl} V_{0k} V_{1l} \quad (3.6) \]

In the space spanned by this binormal, one can choose two unit vectors such that the orthonormality condition in Eq. (3.2) still holds. The analysis then proceeds as above and once again the only non-trivial Frenet equation is Eq. (3.5). The consistency of the setup can be verified by considering the change in the binormal \( V_{2,3}^{ij} \) along the trajectory,

\[ u^b \nabla_b V_{2,3}^{ij} = -\frac{1}{\sqrt{6}} \epsilon^{ijkl} (u^b \nabla_b V_{0k}) V_{1l} + \frac{1}{\sqrt{6}} \epsilon^{ijkl} V_{0k} (u^b \nabla_b V_{1l}) \]

\[ = 0 \quad (3.7) \]

where the first term vanishes since \( u^b \nabla_b V_{0k} \) is parallel to \( V_{1l} \) and the second term vanishes on using the constraint equation Eq. (3.5). The vanishing of the \( u^b \nabla_b V_{2,3}^{ij} \) confirms that Eq. (3.5) is consistent with the vanishing of the torsion and hypertorsion scalars as required. One can note that the explicit form of the unit vectors \( V_2^i \) and \( V_3^i \) in this case are not needed.

A heuristic way to arrive at the constraint equation Eq. (3.5) is as follows. In order to impose uniform linear acceleration, we demand that the acceleration vector \( a^i \) has a constant positive magnitude, and any change in \( a^i \) lies in the plane spanned by \( u^i \) and \( a^i \). The latter condition implies

\[ u^b \nabla_b a^i = w^i = p_1 u^i + p_2 a^i \quad (3.8) \]

where \( p_1 = -u_i w^i \) and \( p_2 = a_i w^i / |a|^2 \), using \( u_i a^i = 0 \). As \( |a| \) is constant, we have

\[ 0 = u^b \nabla_b (a^i a_i) = 2a_i w^i = 2p_2 |a|^2 \quad (3.9) \]
which implies $p_2 = 0$. As $u_ia^i = 0$, we have
\[ 0 = u^b \nabla_b (u^i a_i) = |a|^2 + u_i w^i \] (3.10)
which implies $p_1 = |a|^2$. Collecting, we obtain the constraint
\[ w^i - |a|^2 u^i = 0 \] (3.11)
which is identical to Eq. (3.5). Using Eqs. (3.8) and (3.11) and the constancy of $|a|$, we further see that all further of derivatives of $u^i$ lie in the plane spanned by $u^i$ and $a^i$.

4 The memory effect for accelerated observers

In this section, we investigate the gravitational memory effect of supertranslations on uniformly linearly accelerated trajectories. We consider the case of the Rindler and Schwarzschild black hole spacetimes respectively in the presence of supertranslational hair implanted by an asymmetric shock-wave as discussed in section 2. Starting with a family of uniformly linearly accelerated trajectories that follow the orbits of a single Killing vector before the shock wave, the task is to find what these trajectories have become after the wave has passed. We begin with Rindler spacetime case.

4.1 Rindler spacetime

We work in the Bondi-type gauge where the Rindler metric before the shockwave, $v < v_0$ can be expressed as
\[ ds^2 = -2\kappa r dv^2 + 2dvdr + \delta_{AB} dx^A dx^B \] (4.1)
The boost Killing vector in these co-ordinates is $\bar{\xi}^a = \kappa^{-1}(1, 0, 0, 0)$. Without loss of generality, let us consider a representative Rindler trajectory for $v < v_0$ with a world-line $x^i(\tau) = [\tau/\sqrt{2\kappa r_c}, r_c, x^A_c]$ where $\tau$ is the proper time along the trajectory and $r_c$, $x^A_c$ are the initial co-ordinates in the Bondi-type co-ordinates chosen. It is easy to check that the trajectory is linearly uniformly accelerated in the sense of Eq. (3.5), and $|a| = \kappa/\sqrt{2\kappa r_c}$.

Let an asymmetric shockwave at $v = v_0$ with the stress energy tensor (2.9) impinge on the Rindler horizon. Working to linear order in the perturbation, we can investigate separately the memory effect due to the surface energy density term $\bar{\mu}h(v - v_0)$ and the memory effect due to the supertranslation
perturbation terms. We first consider the case without the surface energy density term and set $\bar{\mu} = 0$. The resultant metric is

$$ds^2 = -2\kappa r dv^2 + 2dv dr + 4rh(v - v_0)\partial_A f dv dx^A + \left(\delta_{AB} + 2h(v - v_0)\frac{\kappa}{\bar{\mu}}\partial_A \partial_B f\right) dx^A dx^B$$

(4.2)

with the corresponding supertranslation vector

$$\Xi^a = \frac{1}{\kappa}[f(x, y), 0, -r \partial_A f(x, y)]$$

(4.3)

We assume that $h(v - v_0) = \lambda H(v - v_0)$ where $\lambda$ is a small dimensionless perturbative parameter and $H(v - v_0)$ is the Heavyside step function. To determine the trajectory on and after the shockwave for $v \geq v_0$, we make for the trajectory’s four-velocity the ansatz

$$u^a = \left[\frac{1}{\sqrt{2\kappa r}}, 0, \frac{\sqrt{2\kappa r}}{\kappa} \partial dx^A f\right]$$

(4.4)

where $\mathcal{E}(v)$ is of first order in $\lambda$ and must be determined from (3.5). The acceleration vector is

$$a_i = \left[0, \frac{\kappa}{2\kappa r}, \frac{1}{\kappa} \frac{dh}{dv} \partial_A f + \frac{1}{\kappa} \frac{dE}{dv} \partial_A f\right]$$

(4.5)

and it satisfies

$$a^2 = \frac{\kappa^2}{2\kappa r} + \mathcal{O}(\lambda^2)$$

(4.6)

This shows that the ansatz (4.4) is consistent with keeping the magnitude of the acceleration constant to linear order. What remains is to impose the constraint (3.5), which to linear order takes the form

$$0 = w_i - a^2 u_i = \left[0, 0, \frac{1}{\kappa \sqrt{2\kappa r}} \frac{d^2 h}{dv^2} + \frac{1}{\kappa \sqrt{2\kappa r}} \left(\frac{d^2 \mathcal{E}}{dv^2} - \kappa^2 \mathcal{E}\right) \partial_A f\right]$$

(4.7)

The equation for $\mathcal{E}(v)$ is hence

$$\frac{d^2 \mathcal{E}}{dv^2} - \kappa^2 \mathcal{E} = -\frac{d^2 h}{dv^2}$$

(4.8)

and matching to the orbits of the boost Killing vector $\bar{\xi}^a$ before the wave gives the initial condition $\mathcal{E}(v) = 0$ for $v < v_0$. The solution is

$$\mathcal{E}(v) = -h(v - v_0) \cosh[\kappa(v - v_0)]$$

(4.9)
The four-velocity vector of the trajectory is hence

\[ u^a = \left[ \frac{1}{\sqrt{2\kappa r}}, 0, -h(v - v_0) \cosh[\kappa(v - v_0)\sqrt{2\kappa r}] \right] \tag{4.10} \]

Integrating (4.10) to first order in \( \lambda \), we find that the trajectories are

\[ x^a(\tau) = \left[ \frac{\tau}{\sqrt{2\kappa r_c}}, r_c, x^A_c - h\left( \frac{\kappa(\tau - \tau_0)}{\sqrt{2\kappa r_c}} \right) \partial^A f(x^A_c)2r_c \sinh \left( \frac{\kappa(\tau - \tau_0)}{\sqrt{2\kappa r_c}} \right) \right] \tag{4.11} \]

where \( \tau_0 \) is the proper time at \( v = v_0 \).

For \( v < v_0 \), the trajectories (4.11) are by construction integral curves of the boost Killing vector \( \bar{\xi}^a = \kappa^{-1}(1, 0, 0, 0) \). What are these trajectories for \( v > v_0 \)?

For \( v > v_0 \), working to linear order in \( \lambda \), the perturbed metric is related to the Rindler metric by a diffeomorphism generated by the supertranslation vector \( \Xi^a (4.3) \). As \( L_{\Xi} \bar{\xi}^a = 0 \), it follows that \( \bar{\xi}^a \) is a boost Killing vector also for \( v > v_0 \). Assume now that \( f \) is generic. It is then immediate from (4.10) that the trajectories (4.11) are not orbits of \( \bar{\xi}^a \). Further, we have verified that a vector field parallel to the velocity field (4.10), of the form

\[ q^a = Q(r, x^A) \left[ 1, 0, E(v)2r \partial^A f \right], \tag{4.12} \]

satisfies Killing’s equation to linear order in \( \lambda \) only when \( E(v) = 0 \) and \( Q(r, x^A) \) is a constant. This means that the velocity field (4.10) is not parallel to a Killing vector, and the trajectories (4.11) do not constitute a family of integral curves of a Killing vector field.

However, the Letaw-Frenet construction at \( v > v_0 \) guarantees that each trajectory in the family (4.11) is the orbit of some boost Killing vector. This means that the Killing vector must differ from trajectory to trajectory. When the velocity vector in (4.10) is transformed to a set of standard Minkowski co-ordinates \( (T, X, Y^A) \), it takes the form

\[ U^a = \left[ \cosh \left( \frac{\kappa T}{\sqrt{2\kappa r_c}} - \frac{\log[\kappa r_c]}{2} \right), \sinh \left( \frac{\kappa T}{\sqrt{2\kappa r_c}} - \frac{\log[\kappa r_c]}{2} \right), h\left( \frac{\kappa(T - \tau_0)}{\sqrt{2\kappa r_c}} \right) \alpha^A \cosh \left( \frac{\kappa T}{\sqrt{2\kappa r_c}} - \kappa v_0 \right) \right] \tag{4.13} \]

where \( \alpha^A = \sqrt{2\kappa r_c} \partial^A f(x^A) \), and we have used (4.11) to express the velocity vector in terms of the proper time \( \tau \). From (4.13) we see that a trajectory
with given $r_c$, $x^A_c$ is an integral curve of a boost Killing vector that is obtained by applying to $\xi^a$ the Lorentz boost

$$
\Lambda^a_{\ b} = \begin{pmatrix}
1 & 0 & \alpha Y \cosh \beta & \alpha Z \cosh \beta \\
0 & 1 & -\alpha Y \sinh \beta & -\alpha Z \sinh \beta \\
\alpha Y \cosh \beta & \alpha Y \sinh \beta & 1 & 0 \\
\alpha Z \cosh \beta & \alpha Z \sinh \beta & 0 & 1
\end{pmatrix}
$$

(4.14)

where $\beta = (1/2) \log[\kappa r_c] - \kappa v_0$. Note that the magnitude and direction of the boost (4.14) depend on $r_c$ and $x^A_c$.

Collecting, we have shown that implanting a supertranslational hair on the Rindler horizon by our matter shock wave boosts a family of Rindler trajectories in a way that differs from trajectory to trajectory, and this trajectory-dependence and carries a memory of the planar inhomogeneity of the wave. This is the gravitational memory effect for uniformly linearly accelerated observers.

To end this subsection, we return to the case of positive $\bar{\mu}$. Working to linear order, we can set the perturbations due to the supertranslational terms to zero. The relevant metric in this case is

$$
\begin{align*}
 ds^2 &= -2\kappa r dv^2 + 2dvdr \\
 &\quad + \left( \delta_{AB} + \frac{\bar{\mu}}{\kappa} \delta_{AB} h(v - v_0) \left( e^{\kappa(v-v_0)} - 1 \right) \right) dx^A dx^B
\end{align*}
$$

(4.15)

We now find that the velocity vector field

$$
u^a = \left[ \frac{1}{\sqrt{2\kappa r}}, 0, 0, 0 \right]
$$

(4.16)

has $|\nu| = \kappa/\sqrt{2\kappa r}$ and satisfies the Letaw-Frenet constraint (3.5) for all $v$. For $v < v_0$, the trajectories are orbits of the boost Killing vector $\xi^a$. For $v > v_0$, the metric (4.15) is flat to linear order, and when the velocity vector (4.16) is transformed to a standard set of Minkowski co-ordinates $(T, X, Y^A)$, it takes the form

$$
U^a = \left[ \frac{X}{\sqrt{X^2 - T^2}}, \frac{T}{\sqrt{X^2 - T^2}}, \frac{\bar{\mu} e^{-\kappa v_0} x^A_c (T + X)}{2\sqrt{X^2 - T^2}} \right]
$$

(4.17)

where $X > |T|$. We see again that after the wave has passed, each trajectory is an integral curve of a boost, but the boost differs from trajectory to trajectory. The effect can be interpreted as a focusing due to the energy density in the wave.
4.2 Schwarzschild spacetime

We now turn to an infalling shockwave in the supertranslated Schwarzschild spacetime, and to its consequences for a family of trajectories that are static before the wave and are continued to the future of the wave as uniformly linearly accelerated trajectories. We proceed as in Rindler. Working in the Bondi gauge where the supertranslational hair implanting shockwave is infalling in the Schwarzschild black hole metric, the complete metric reads

\[ ds^2 = -\left(1 - \frac{2M}{r} - h(v-v_0)\frac{2\mu}{r} - h(v-v_0)\frac{MD^2C}{r^2}\right)dv^2 + 2dvdr \\
- h(v-v_0)D_A\left(2C - \frac{4MC}{r} + D^2C\right)dvd\Theta^A \\
+ \left(r^2\gamma_{AB} + h(v-v_0)2rD_AD_BC - h(v-v_0)r\gamma_{AB}D^2C\right)d\Theta^A\Theta^B \]

For notational simplicity, we write \( V = 1 - 2M/r \).

We consider trajectories that are static for \( v < v_0 \), \( x^i(\tau) = [\tau/\sqrt{V(r_c)}, r_c, \Theta^A_c] \), where \( \tau \) is the proper time along the trajectory and \( r_c, \Theta^A_c \) are the initial co-ordinates. These trajectories are uniformly linearly accelerated in the sense of section 3, and the magnitude of the acceleration is \( |a| = V'(r_c)/2\sqrt{V(r_c)} \). We continue the trajectories across and to the future of the shock wave by keeping them uniformly linearly accelerated.

We first consider the effect from the shockwave mass term \( \mu \), taking the perturbed metric to be

\[ ds^2 = -\left(1 - \frac{2M}{r} - h(v-v_0)\frac{2\mu}{r}\right)dv^2 + 2dvdr + r^2\gamma_{AB}d\Theta^A\Theta^B \]

and assuming \( \mu > 0 \). As the overall effect of \( \mu \) is to increase the mass of the black hole, we may anticipate the initially static trajectories to become unstable on crossing the shock wave and to fall into the black hole. We now verify that this is the case within the perturbative treatment. A nonperturbative treatment could be given by the methods of \[29, 30\].

Working to linear order in the perturbation, we assume that \( h(v-v_0) = \lambda H(v-v_0) \) where \( \lambda \) is a small dimensionless perturbative parameter and \( H(v-v_0) \) is the Heavyside step function. To find the trajectory for \( v \geq v_0 \), we assume \( r - 2M \gg 2\mu \) and seek the velocity vector field by the ansatz

\[ u^a = \left[1 + \frac{h(v-v_0)\mu}{rV} + \mathcal{E}(v), \mathcal{E}(v), 0, 0\right] \]
where $E(v)$ is to be determined. For the magnitude of the acceleration vector, we find

$$|a|^2 = \frac{V'^2}{4V} + \left(\frac{V''}{V}\right) \dot{E}' + \left(\frac{V'}{rV^2}\right) \mu h' + \left(\frac{V'}{r^2 V}\right) \mu h + \left(\frac{V'^2}{2rV^2}\right) \mu h \quad (4.21)$$

where the prime denotes differentiation with respect to the argument, that is, $V' = dV/dr$, $h' = dh/dv$ and $E' = dh/dv$. The constraint (3.5) gives

$$0 = w_i - a^2 u_i = \frac{1}{4V^{5/2} \gamma^2} \left[ 0, 4r \mu h'' + 2 \mu r h' V' + 4r^2 V E'' + 2 \xi r^2 V^2 E'' + 4 \mu V h' - r^2 V E V'' E, 0, 0 \right] \quad (4.22)$$

Differentiation of (4.21) with respect to $v$ shows that (4.22) implies constancy of $|a|$. The only equation that needs to be solved is hence (4.22).

Writing $E(v)$ in terms of $(1/\sqrt{V}) dr/dv$, (4.22) gives

$$\frac{d^2 r_c}{dv^2} - V(r_c) \left( |a|^2 - \frac{V''(r_c)}{2} \right) r_c = \frac{-\mu h'}{r_c} - \frac{\mu V(r_c) h}{r_c^2} - \frac{\mu V'(r_c) h}{2r_c} \quad (4.23)$$

where $r_c = r - r_e$. With the initial condition $r_c(v) = 0$ for $v < v_0$, the solution is

$$r = r_c - h(v - v_0) \frac{\mu}{r_c \beta} \sinh \left( \beta(v - v_0) \right)$$

$$-h(v - v_0) \left( \frac{\mu V}{r_c^2} + \frac{\mu V'}{2r_c} \right) \frac{2}{\beta^2} \sinh^2 \left( \frac{\beta}{2} (v - v_0) \right) \quad (4.24)$$

with $\beta^2 = V(r_c) (|a|^2 - V''(r_c)/2)$. The solution has exponential runaway and will eventually exit the regime in which the linearised treatment is valid, but the signs in (4.24) show that the trajectory will start to fall towards the black hole, as we anticipated.

When the supertranslation terms in the metric are added, the Rindler analysis suggests that the trajectories will carry a memory of the spherical anisotropy of the wave, and a generic trajectory will either fall into the black hole or escape to infinity.

5 Discussion

In this paper we have demonstrated and quantified a gravitational memory effect due to a matter shock wave that implants supertranslational hair on a Rindler horizon. We considered a family of observers who follow the integral curves of a Lorentz boost prior to the wave, and we assumed the observers
to continue as uniformly linearly accelerated across the wave, in the sense of a curved spacetime generalisation of the Letaw-Frenet uniform linear acceleration in flat spacetime [23]. After the wave has passed, we find that each observer still follows the orbit of a boost Killing vector, but this boost differs from trajectory to trajectory, and the trajectory-dependence carries a memory of the planar inhomogeneity of the wave. We also considered a matter shock wave that implants supertranslational hair on the Schwarzschild spacetime [2], showing that a similar memory effect on initially static uniformly linearly accelerated trajectories exists but involves an instability that makes the trajectories fall into the black hole or escape to the infinity.

In Schwarzschild, the linearised stress-energy tensor of the supertranslation - implementing shock wave involves a Dirac delta on on a null hypersurface [2]. In Rindler, by contrast, we found that the linearised stress-energy tensor of the supertranslation-implementing shock wave, in addition to a Dirac delta term, also involves a derivative of the Dirac delta on a null hypersurface. Studying the shock wave beyond the linearised theory [29, 30, 31] could hence be significantly more challenging in Rindler than in Schwarzschild.

While our discussion was classical, it is motivated by the potential of supertranslations as a solution to the black hole information paradox [1]. As the classical memory effect due to Rindler supertranslations involves a trajectory-dependent boost, the Killing horizons of the uniformly linearly accelerated trajectories in the future of the shock wave are boosted with respect to each other. In terms of spacetime regions separated by the Rindler horizons, some of the degrees of freedom that prior to the shock wave were inaccessible to a particular Rindler observer become hence accessible in the future of the shock wave, and vice versa. This leads us to anticipate that the classical memory effect due to the Rindler supertranslations has a counterpart in the thermal aspects of Rindler space quantum field theory, and we plan to address this effect in a future paper [16].

Acknowledgments

SK thanks the University of Nottingham for hospitality and the Department of Science and Technology, India for partial financial support. JL was supported in part by the Science and Technology Facilities Council (Theory Consolidated Grant ST/J000388/1).
References

[1] S. W. Hawking, M. J. Perry and A. Strominger, Phys. Rev. Lett. 116, 231301 (2016).

[2] S. W. Hawking, M. J. Perry and A. Strominger, “Superrotation Charge and Supertranslation Hair on Black Holes” (2016) arXiv:1611.09175.

[3] H. Bondi, M. G. J. van der Burg and A. Metzner, Proc. Roy. Soc. Lond. A 269, 21-52 (1962); R. Sachs, Proc. Roy. Soc. Lond. A 270, 103-126 (1962); R. Sachs, Phys. Rev. 128, 2851-2864 (1962).

[4] D. Christodoulou and S. Klainerman, The Global nonlinear stability of the Minkowski space, Princeton University Press, 1993.

[5] M. Mirbabayi and M. Porrati, Phys. Rev. Lett. 117, 211301 (2016).

[6] L. Mersini-Houghton and D. Morse, Hawking radiation conference, book of proceedings (2016), arXiv:1610.01501.

[7] S. W. Hawking, Nature 248, 30-31 (1974); S. W. Hawking, Commun. Math. Phys. 43, 199-220 (1975).

[8] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, Cambridge University Press, 1984.

[9] B. S. DeWitt, The Global Approach to Quantum Field Theory (Clarendon Press, Oxford, 2003).

[10] R. Kubo, J. Phys. Soc. Jap. 12, 570-586 (1957).

[11] P. C. Martin and J. S. Schwinger, Phys. Rev. 115, 1342-1373 (1959).

[12] R. Haag, N. M. Hugenholtz and M. Winnink, Commun. Math. Phys. 5, 215-236 (1967).

[13] K. K. Ng, L. Hodgkinson, J. Louko, R. B. Mann and E. Martín-Martínez, Phys. Rev. D 90, 064003 (2014); L. Hodgkinson, J. Louko and A. Ottewill, Phys. Rev. D 89, 104002 (2014).

[14] B. Zeldovich and A. G. Polnarev, Sov. Astron. Lett. 18, 17-23 (1974).

[15] A. Strominger and A. Zhiboedov, JHEP 01, 086 (2016); A. Strominger, JHEP 07, 152 (2014).

[16] S. Kolekar and J. Louko, in preparation.
[17] P. C. W. Davies, J. Phys. A 8, 609-616 (1975).

[18] W. G. Unruh, Phys. Rev. D 14, 870-892 (1976).

[19] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, Phys. Rev. Lett. 116, 091101 (2016), L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, JHEP 09, 100 (2016).

[20] C. Eling and Y. Oz, JHEP 07, 065 (2016).

[21] R. G. Cai, S. M. Ruan, Y. L. Zhang, JHEP 09, 163 (2016).

[22] M. Hotta, J. Trevison and K. Yamaguchi, Phys. Rev. D 94, 083001 (2016).

[23] J. Letaw, Phys. Rev. D 23, 1709-1714 (1981).

[24] L. Sriramkumar and T. Padmanabhan, Int. J. Mod. Phys. D 11, 1-34 (2002).

[25] J. Louko and A. Satz, Class. Quant. Grav. 23, 6321-6343 (2006).

[26] J. G. Russo and P. K. Townsend, J. Phys. Conf. Ser. 222, 012040 (2010) [Class. Quant. Grav. 27, 175005 (2010)].

[27] S. Kolekar and T. Padmanabhan, Phys. Rev. D 86, 104057 (2012).

[28] W. Rindler, Phys. Rev. 119, 2082-2089 (1960).

[29] C. Barrabes and W. Israel, Phys. Rev. D 43, 1129-1142 (1991).

[30] E. Poisson, A Relativists’s Toolkit, Cambridge University Press, 2004.

[31] R. P. Geroch and J. H. Traschen, Phys. Rev. D 36, 1017-1031 (1987).