Determining measurement errors: where the good meet the bad

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Recently there have been claims of experimental violations of Heisenberg’s error-disturbance relation [Rozema \textit{et al}, PRL 109, 200404 (2012), Erhard \textit{et al}, Nature Phys. 8, 185 (2012)]. These claims were contradicted by a proof of a form of such relations [Busch \textit{et al}, PRL 111, 160405 (2013)], showing that the validity of any formalization of Heisenberg’s principle depends on the choice of appropriate measures of error and disturbance. While the error quantity underlying the experiments was shown to be conceptually problematic in general, it turns out to coincide, in the qubit experiments in question, with the alternative error measure used by Busch \textit{et al}. Here we analyze this coincidence and discover a “strong measurement” technique, hitherto deemed unavailable, for determining the disturbance measure. We show that this procedure reflects directly the operational meaning of the disturbance measure, in contrast to the “weak-value” method used in the experiments of Rozema \textit{et al} and several other groups. The latter is thus not fundamental for an understanding of the quantum mechanical error-disturbance trade-off. The “strong” scheme is also shown to be applicable to a wider class of joint measurement scenarios.

Recent claims by Ozawa and others of a violation of Heisenberg’s error-disturbance relation, or the more general joint measurement error trade-off – both theoretical (e.g. [1]) and experimental [2-4] – are based on the observation that the product of measurement errors in a joint measurement need not be bounded by the noncommutativity of the quantities under consideration. While this realization is found to be true for arbitrary state-dependent error measures, the measures, usually denoted by $\varepsilon$ for error and $\eta$ for disturbance, adopted by these researchers (and others) are not all that good: they have only a limited domain of applicability in which their operational significance is warranted [5]. Outside this domain these quantities turn bad in that they fail to be faithful measures of error or disturbances. Interestingly, in the case of qubit observables as they appear in the experiments in question, $\varepsilon$ and $\eta$ are well-behaved, and it has been shown in [6] that they then do agree with the alternative, “good” measures of error and disturbance introduced by Busch \textit{et al} in [7]. The significance of this coincidence has not yet been fully elucidated, but it does show that the quantities $\varepsilon, \eta$ may sometimes be useful for error estimation, after all.

A scheme for measuring the quantities $\varepsilon, \eta$ was proposed by Lund and Wiseman [8] via reconstruction formulas that render them as 	extit{weak values}. Implementations of this method were realized for $\eta$ in the case of qubit observables in [2,3,4,8]. Curiously, not much was made of the fact that the coupling strength parameter for the weak measurement applied dropped out of the calculations before the weak limit was taken. It was also not noted that in the qubit experiments in question, the quantities $\varepsilon, \eta$ do in fact have a quite more direct probabilistic interpretation than is usually thought. Here we revisit the experimental scheme and investigate the significance of these two observations.

We are thus led to a change of perspective – away from the focus on weak values to considering the 	extit{strong-measurement} limit of the scheme. This yields at once both a much simpler way of evaluating $\eta$ and a more direct operational interpretation of this quantity. We also show that this alternative method is rather more generally applicable to typical joint measurement schemes.

The measures of error and disturbance in question, based on the notion of 	extit{measurement noise}, show a formal resemblance with the classic formula for the root-mean-square (rms) error. We briefly review their definitions. Suppose we have an approximate or accurate measurement scheme for an observable $A$, with probe readout observable $P$, initial probe state $\phi$ and unitary coupling $U$, and we are considering the disturbance to a second observable $B$ in the initial state $\psi$; then

$$
\varepsilon(A)^2 := \langle \psi \otimes \phi | (U^\dagger (I \otimes P) U - A \otimes I) \psi \otimes \phi \rangle,
$$

$$
\eta(B)^2 := \langle \psi \otimes \phi | (U^\dagger (B \otimes I) U - B \otimes I) \psi \otimes \phi \rangle.
$$

The immediate physical meaning of, say, the second quantity is that of an expectation value of the hermitian operator that is the square of the difference between the operator $B$ and its Heisenberg-evolved version. In the traditional observable-as-operator perspective this looks like a root-mean-square (rms) deviation of the observables. However, in order to experimentally determine the rms deviation of the values of two observables, these observables need to be jointly measurable, hence commuting, whereas there is no reason why this should be the case here. Similar comments apply to the first quantity. Hence these definitions can only have their intended operational content in very limited circumstances. These very circumstances – the commutativity of the target and approximating observables – are in fact given in the experiments to be discussed here. Even then, as we will show below, the quantities $\varepsilon(A), \eta(B)$ will at best be error and disturbance estimates.

It is known that a sequence of an approximate measurement of $A$ followed by an accurate measurement of $B$
constitutes a joint measurement of two unsharp observables \(C\) and \(D\), represented by positive operator valued measures (in short POVMs or POMs). The error of the \(A\) measurement is reflected in the difference between the \(A\) and \(C\) distributions, and likewise the disturbance of \(B\) is manifest in the difference between the \(B\) and \(D\) distributions. This shows that disturbance is itself captured as a form of approximation error in a joint measurement, and error-disturbance relations are seen to be a special form of joint measurement error relations (e.g., [1, 7]).

The above quantities can be written in terms of the object system alone, e.g., (assuming a discrete-valued observable system, e.g., \(\{1, 7\}\)).

is manifest in the difference between the \(\eta\) and \(\eta\) distribution if the operators \(\rho\) and \(\sigma\) with \(\rho \leq \sigma\) and positive operators \(\rho\) become problematical [7].

The map \((b_k, b_l) \rightarrow \text{Re tr}[\rho b_k D_l]\) is a probability distribution if the operators \(b_k\) and \(D_l\) commute. But this will not be the case in general, and then the “rms” interpretation of \(\eta\) becomes problematical [3].

Lund and Wiseman 11 proposed to consider the quantities \(\text{Re tr}[\rho b_k D_l]\) as “weak valued probabilities”, which led them to rewrite (1) formally as

\[
\eta(B)^2 = \sum_{\delta b} (\delta b)^2 P_{WV}(\delta b),
\]

with \(P_{WV}(\delta b) = \sum_{k,l:b_k=b_l+\delta b} P_{WV}(b_k, b_l).
\]

This equation is then taken at face value in [10] and also [2], as if it had an immediate operational meaning. Yet, however suggestive the form of the above expression for \(\eta(B)\) may be, since in general the so-called weak-valued probabilities are not probabilities at all, there is in general no operational significance to calling them a rms deviation.

As will be seen below, the qubit measurement model proposed by Lund and Wiseman for an experimental determination of \(\eta(B)\) does fall into the class of schemes where the commutativity of \(B\) with the approximator \(D\) is given. In this model (Fig. 1) an initial approximate (or weak) measurement of the qubit observable \(B = X\)

is done, with strength \(2\gamma^2 - 1\). This is then followed by an approximate measurement of \(Z\) on the resulting state, with strength \(\cos 2\theta\). Finally there is an accurate \(X\) measurement (denoted \(X_f\)). The initial and final \(X\) measurements are intended to provide information about the disturbance of \(X\) by the approximate \(Z\) measurement. The probe and measurement system performing the first \(X\) measurement and the approximate \(Z\) measurement are again qubit observables, and their readout observables are \(Z_p\) and \(Z_m\), respectively.

The scheme thus realises a joint (sequential) measurement of three \(\pm 1\) valued observables, with probabilities

\[
P_{k,\ell,n} := P(Z_p = k, X_f = \ell, Z_m = n), \quad k, \ell, n \in \{+, -\}.
\]

The calculation of these probabilities and the associated POVMs are given in the Supplemental Material [11] (see also [12]).

It is important to note that the weak-valued probabilities \(P_{WV}(b_k, b_l)\) required for \(P_{WV}(\delta b)\) do not coincide with the operational joint probabilities \(P(Z_p = k, X_f = \ell)\) of the proposed experiment. Indeed, the reconstruction of the value of \(\eta(B)\) from these operational joint probabilities is rather indirect in the proposed setup. Therefore the rms interpretation for \(\eta(B)\) does not apply in this weak measurement scheme.

\[
\gamma|0\rangle + \gamma'|1\rangle
\]

\[
\alpha|0\rangle + \beta|1\rangle
\]

\[
\cos \theta|0\rangle + \sin \theta|1\rangle
\]

\[
Z_p
\]

\[
Z_m
\]

FIG. 1. Model implementation of a determination of \(\eta(X)\). The top and bottom wires represent the probe and measuring system while the middle wire corresponds to the observed qubit. As shown in the text, the value of \(\eta(X)\) can be extracted from the joint distribution of the initial and final \(X\) measurements, obtained by reading the outputs \(Z_p\) and \(X_f\).

As observed in [10], the “weak-valued” probabilities \(P_{WV}(\delta b) = P_{WV}(\pm 2)\) can be expressed in terms of the operational joint probabilities \(\sum_n P_{k,\ell,n} = P(Z_p = k, X_f = \ell)\) in a rather involved way as follows [11].

\[
2P_{WV}(\delta X = \pm 2) = 2P_{WV}(X_i = \mp 1|X_f = \pm 1)P(X_f = \pm 1)
\]

\[
= P(X_f = \pm 1) \mp \frac{P(Z_p = 1|X_f = \pm 1) - P(Z_p = -1|X_f = \pm 1)}{2\gamma^2 - 1} P(X_f = \pm 1)
\]

\[
= P(Z_p = 1, X_f = \pm 1) + P(Z_p = -1, X_f = \pm 1) \mp \frac{P(Z_p = 1, X_f = \pm 1) - P(Z_p = -1, X_f = \pm 1)}{2\gamma^2 - 1}.
\]

Using the explicit expressions for these joint probabilities and inserting the resulting values of \(P_{WV}(\delta X = \pm 2)\) into
yield the same result for all values of the strength parameter, $2\gamma^2 - 1$:

$$\eta(X) = \sqrt{2} |\cos \theta - \sin \theta|,$$

There is no need to perform the limit to vanishing strength, $\gamma \to 1/\sqrt{2}$ for the determination of $\eta(X)$. (To be sure, the limit is required in the experiments for the purpose of having a particular state pass practically undisturbed, thereby enabling a test of Branciard’s inequality at saturation threshold \cite{X}.)

We can therefore use the freedom of choice of initial interaction strength to explore what happens if we set $\gamma$ to maximum strength, $\gamma = 1$, so that we are in fact performing a sharp X measurement. Thus we are led to taking seriously the fact that here the numbers $\text{tr} [\rho B_k D_\ell]$ are bona fide probabilities: they are simply the operational joint probabilities of the outcomes of the initial and final X measurements, and directly yield the probabilities for these values differing by $\delta X = \pm 2$:

$$P_{WV} (\delta X = \pm 2) = P(\mathcal{Z}_p = \mp 1, \mathcal{X}_f = \pm 1).$$

Correspondingly, the value of the disturbance quantity $\eta(X)$ is given by the actual squared difference of the two measurements:

$$4P(\mathcal{Z}_p = +1, \mathcal{X}_f = -1) + 4P(\mathcal{Z}_p = -1, \mathcal{X}_f = +1) \equiv \eta(X)^2.$$ 

This surprising result becomes understandable when one considers that the $\mathcal{X}_f$ measurement can be viewed as an approximate repetition of the initial sharp measurement of $\mathcal{X}$; the measurement of $\mathcal{X}_f$ is distorted into a measurement of a POVM $D$ that is also compatible with approximate $Z$ measurement performed between the $X$ measurements. In this case $\mathcal{X}_f$, or rather $D$, acts as a smeared version of $X$ on the initial state, and thus commutes with the initial sharp measurement \cite{X}. When the approximating observable commutes with the target, the noise based disturbance measure $\eta(X)$ has a clear and well defined meaning \cite{X}.

It is therefore the strong measurement limit that provides a direct operational scheme for determining $\eta(X)$. This quantity is the rms deviation of the values of two $X$ measurements performed on the same system before and after the $Z$ measurement. It is important to note that this interpretation works under the assumption that the initial $X$ measurement is of the Lüders type, which projects into $X$ eigenstates. In this case, the quantities $\text{tr} [\rho B_k D_\ell]$ appearing in \cite{X} represent exactly the joint probabilities for the initial and final $X$ measurements. In particular, the final marginal of the scheme of Fig. \cite{X} is not affected by the presence of the initial $X$ measurement \cite{X}. It is easy to check that other ways of making the initial $X$ measurement may lead to additional distortions of the final $X$ measurement, represented by positive operators other than $D_\pm$. This shows that the quantity $\eta(X)$ describes not merely a difference between $X$ and its distortion (or approximation) $D$, but reflects also the specific method of determining the rms deviation. Put differently, $\eta(X)$ depends on the joint probabilities associated with that method.

In contract, the disturbance measure used by Busch et al., $\Delta(X, D)$, is based on a comparison of the $X$ distribution measured directly on a state $\rho$ (for one ensemble) and the $X$ distribution measured on the states emerging from a $Z$ measurement (for another ensemble). It is instructive to elaborate a little further on this crucial difference between the concepts. To this end, we briefly recall the definition of the $\Delta$ measure. For any two (discrete) probability distributions $p: x_k \mapsto p_k$, $q: y_\ell \mapsto q_\ell$, a coupling is defined to be a joint probability distribution $\gamma: (x_k, y_\ell) \mapsto \gamma_{k\ell}$ with $p$ and $q$ as the Cartesian marginals. The set of couplings between $p$ and $q$ will be denoted $\Gamma(p, q)$. Then, the (Wasserstein) 2-distance \cite{X} of $p$ and $q$ is defined as

$$D_2(p, q) = \inf_{\gamma \in \Gamma(p, q)} \mathcal{D}_2^2(p, q) = \inf_{\gamma \in \Gamma(p, q)} \left( \sum |x_k - y_\ell|^2 \gamma_{k\ell} \right)^{1/2}.$$ 

This is a distance between probability measures due to the choice of the minimizing joint probability.

We can now define the (Wasserstein) 2-distance between two observables, say $b_k \mapsto B_k$ and $b_\ell \mapsto D_\ell$, using the notation $p^B_\rho$, $p^D_\rho$ for their probability distributions with respect to the state $\rho$:

$$\Delta_2(B, D) := \sup_{\rho} D_2(p^B_\rho, p^D_\rho).$$

This distance between the observable can be determined from the statistics $p^B_\rho$, $p^D_\rho$, obtained in separate runs of $B$ and $D$ measurements on different ensembles of systems prepared in the same state $\rho$. The method does not depend on whether or not $B$ and $D$ commute. In the commutative case, since $\eta(B)$ is obtained from a particular coupling of the distributions of $B$ and $D$, it is always true that

$$\eta(B) \geq \Delta_2(p^B_\rho, p^D_\rho).$$ 

so that this latter (metric) quantity is a more refined assessment of the difference between the distributions. However, it is useful to know that $\eta(B)$ can be taken as a disturbance estimate, giving an upper bound to $\Delta_2(p^B_\rho, p^D_\rho)$.

In the present case,

$$B_\pm = \frac{1}{2} (I \pm X), \quad D_\pm = \frac{1}{2} (I \pm \sin(2\theta)X),$$

and as shown in \cite{X}, then

$$\Delta_2(B, D)^2 = 2|b - d| = 2(1 - \sin(2\theta)) = \eta(X)^2.$$ 

The formal difference between $\eta$ and $\Delta$ lies in the fact that for $\eta(X)$, a specific coupling is chosen (namely,
\( \gamma_{k\ell} = \text{tr} [\rho B_k D_{\ell}] \), while for \( D_2(p_B^p, p_B^D) \), one must find the minimising \( \gamma \). In the present experiment, it happens to be the case that \( \eta(X) \) is state-independent and coincides with \( D_2(p_B^p, p_B^D) \) if \( \rho \) is taken to be one of the eigenstates of \( X \). This quantity finally turns out also to be equal to the worst case value \( \Delta(B, D) \). In this way, the experiment under consideration provides a direct operational determination of the disturbance estimate \( \eta \), which coincides with \( \Delta \) in this case.

The alternative, “strong measurement”, perspective on the disturbance measure \( \eta \) presented in the above model can be generalized to a rather wider class of sequential joint measurement scenarios. Let \( B \) be a sharp observable with values \( b_k \) and spectral projections \( B_k \). Suppose an approximate measurement of \( A \) represented by POVM \( C \) is followed by a sharp measurement of observable \( B \). This sequential scheme defines a joint measurement of \( A \) and some POVM \( D \), which is an approximation of \( B \). Assume that the disturbance is benign, in the sense that the \( D_\ell \) commute with the \( B_k \), which occurs typically with \( D \) being a smearing of \( B \) by means of a stochastic matrix \( \{\lambda_{m}\} \), i.e., \( D_\ell = \sum_m \lambda_{m} B_m \). Now assume that the measurement of \( C \) is preceded by a projective measurement of \( B \). It follows that the operational joint probabilities are

\[
P(B_i = b_k, B_j = b_\ell, C = c_\alpha) = \text{tr} \left[ T_n^C (B_k \rho B_k) B_i \right]
= \text{tr} \left[ B_k \rho B_k (T_n^C)^*(B_\ell) \right].
\]

Here \( n \rightarrow T_n^C \) denotes the instrument associated with \( C \), giving the state change conditional on the outcome \( n \), and \( (T_n^C)^* \) is the dual of the operation \( T_n^C \). Disregarding the outcomes of the \( C \) measurement and noting that \( D_\ell = \sum_n (T_n^C)^*(B_\ell) \), we obtain the marginal probability

\[
P(B_i = b_k, B_j = b_\ell) = \text{tr} [B_k \rho B_k D_\ell] = \text{tr} [\rho B_k D_\ell] \equiv P_{WV}(b_k, b_\ell),
\]

since \( B_k \) commutes with \( D_\ell \). Therefore,

\[
\eta(B)^2 = \sum_{k,\ell} (b_k - b_\ell)^2 P_{WV}(b_k, b_\ell)
\]

\[
= \sum_{k,\ell} (b_k - b_\ell)^2 P(B_i = b_k, B_j = b_\ell).
\]

To summarize, the fact that the weak-value based reconstruction of \( \eta(B) \) does not depend on the specific value of the coupling strength of the initial measurement has led us to consider the strong measurement limit, which turns out, perhaps surprisingly, to provide a direct operational method of determining \( \eta(X) \) that was considered unavailable previously [10]. In doing so, we have utilized the fact that in the present experiment, the distorted observable \( D \) and its undisturbed version \( B = X \) do actually commute, so that the “weak valued probability” \( P_{WV}(X_i, X_f) \) is in fact a proper, operational joint probability. This scheme generalizes to all compatible pairs of observables \( C, D \) realized in a sequential measurement of first measuring \( C \) as an approximation to \( A \) and then \( B \) sharply, rendering the second marginal \( D \). Preceding this sequence with a sharp, projective measurement of \( B \) leads to a direct operational formula for the disturbance \( \eta(B) \). This works whenever the disturbance the \( C \) measurement exerts on \( B \) is such that the distorted observable \( D \) still commutes with \( B \). In that case, as noted in [7], \( \eta(B) \) is an operationally well defined error estimate and provides, in particular, a useful upper bound for the tighter, metric error measure \( \Delta(B, D) \).

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SUPPLEMENTAL MATERIAL TO: “DETERMINING MEASUREMENT ERRORS: WHERE THE GOOD MEET THE BAD”

The experimental setup displayed in Figure 1 in the main text (reproduced below) consists of a three-qubit system, the object in initial state \( \alpha|0\rangle + \beta|1\rangle \), a “weak measurement probe (\( p \))” initially in state \( \gamma|0\rangle + \gamma'|1\rangle \), and the apparatus \( m \) with initial state \( \cos \theta|0\rangle + \sin \theta|1\rangle \), all in their respective 2-dimensional Hilbert spaces \( \mathcal{H}, \mathcal{H}_p \) and \( \mathcal{H}_m \), respectively.

In the scenario when the disturbance measure for the observable \( X \) is to be determined, the initial approximate \( X \) measurement is enacted by first applying a Hadamard gate on the object system \( \mathcal{H} \), followed by a \( C_{NOT} \) gate acting on \( \mathcal{H}_p \), controlled on \( \mathcal{H} \), and finally with another Hadamard gate performed on \( \mathcal{H} \). This is followed by the device whose disturbance is being measured, wherein a \( C_{NOT} \) gate acts on \( \mathcal{H}_m \), again controlled on \( \mathcal{H} \). Sharp \( Z \) measurements are then performed on \( \mathcal{H}_p \) and \( \mathcal{H}_m \), (denoted \( Z_p \) and \( Z_m \) respectively), along with a sharp \( X \) measurement \( (X_f) \) on \( \mathcal{H} \):

The state of the object and weak probe combined \( |\psi_1\rangle \), after the the initial interaction is then given by:

\[
|\psi_1\rangle = (I \otimes H)C_{NOT}(I \otimes H)(\gamma|0\rangle + \gamma'|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)
\]
\[
= \frac{1}{\sqrt{2}}(I \otimes H)C_{NOT}((\gamma|0\rangle + \gamma'|1\rangle) \otimes ((\alpha + \beta)|0\rangle + (\alpha - \beta)|1\rangle)
\]
\[
= \frac{1}{\sqrt{2}}((\gamma|0\rangle + \gamma'|1\rangle) \otimes (\alpha + \beta)|0\rangle + (\gamma'|0\rangle + \gamma|1\rangle) \otimes (\alpha - \beta)|1\rangle)
\]
\[
= \frac{1}{2}[(\gamma(\alpha + \beta) + \gamma'(\alpha - \beta))|0\rangle \otimes |0\rangle + (\gamma'(\alpha + \beta) + \gamma(\alpha - \beta))|1\rangle \otimes |0\rangle
\]
\[
+ (\gamma(\alpha + \beta) - \gamma'(\alpha - \beta))|0\rangle \otimes |1\rangle + (\gamma'(\alpha + \beta) - \gamma(\alpha - \beta))|1\rangle \otimes |1\rangle]
\]
\[
= |p_0\rangle \otimes |0\rangle + |p_1\rangle \otimes |1\rangle,
\]

where

\[
|p_0\rangle = \frac{1}{2}(\gamma(\alpha + \beta) + \gamma'(\alpha - \beta))|0\rangle + (\gamma'(\alpha + \beta) + \gamma(\alpha - \beta))|1\rangle
\]
\[
|p_1\rangle = \frac{1}{2}(\gamma(\alpha + \beta) - \gamma'(\alpha - \beta))|0\rangle + (\gamma'(\alpha + \beta) - \gamma(\alpha - \beta))|1\rangle.
\]

The state of the whole system after the measuring device, \( |\psi_f\rangle \) is then

\[
|\psi_f\rangle = (I \otimes C_{NOT})(|p_0\rangle \otimes |0\rangle + |p_1\rangle |1\rangle) \otimes (\cos \theta|0\rangle + \sin \theta|1\rangle)
\]
\[
= |p_0\rangle \otimes |0\rangle \otimes (\cos \theta|0\rangle + \sin \theta|1\rangle) + |p_1\rangle \otimes |1\rangle \otimes (\sin \theta|0\rangle + \cos \theta|1\rangle)
\]
\[
= |p_0\rangle \otimes |0\rangle \otimes |m_0\rangle + |p_1\rangle \otimes |1\rangle \otimes |m_1\rangle,
\]

with

\[
|m_0\rangle = \cos \theta|0\rangle + \sin \theta|1\rangle
\]
\[
|m_1\rangle = \sin \theta|0\rangle + \cos \theta|1\rangle.
\]
Now writing \( |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and \(|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \), the eigenstates of \( X \), we have

\[
\psi_f = \frac{1}{\sqrt{2}} (|p_0\rangle \otimes |+\rangle \otimes |m_0\rangle + |p_1\rangle \otimes |+\rangle \otimes |m_1\rangle + |p_0\rangle \otimes |-\rangle \otimes |m_0\rangle - |p_1\rangle \otimes |-\rangle \otimes |m_1\rangle)
\]

\[
= \frac{1}{2\sqrt{2}} \left[ \left( |\gamma(\alpha + \beta) + \gamma'(\alpha - \beta)\rangle \cos \theta + (\gamma(\alpha + \beta) - \gamma'(\alpha - \beta)) \sin \theta \right) |0\rangle \otimes |+\rangle \otimes |0\rangle \\
+ \left( |\gamma(\alpha + \beta) + \gamma'(\alpha - \beta)| \sin \theta + |\gamma(\alpha + \beta) - \gamma'(\alpha - \beta)) \cos \theta \right) |0\rangle \otimes |+\rangle \otimes |1\rangle \\
+ \left( |\gamma'(\alpha + \beta) + |\gamma(\alpha - \beta)\rangle \cos \theta + |\gamma'(\alpha + \beta) - \gamma(\alpha - \beta)) \sin \theta \right) |1\rangle \otimes |+\rangle \otimes |0\rangle \\
+ \left( |\gamma'(\alpha + \beta) + |\gamma(\alpha - \beta)\rangle \cos \theta + |\gamma'(\alpha + \beta) - \gamma(\alpha - \beta)) \sin \theta \right) |0\rangle \otimes |+\rangle \otimes |1\rangle \\
+ \left( |\gamma'(\alpha + \beta) + |\gamma(\alpha - \beta)\rangle \cos \theta + |\gamma'(\alpha + \beta) - \gamma(\alpha - \beta)) \sin \theta \right) |0\rangle \otimes |-\rangle \otimes |0\rangle \\
+ \left( |\gamma'(\alpha + \beta) + |\gamma(\alpha - \beta)\rangle \cos \theta + |\gamma'(\alpha + \beta) - \gamma(\alpha - \beta)) \sin \theta \right) |0\rangle \otimes |-\rangle \otimes |1\rangle \\
+ \left( |\gamma'(\alpha + \beta) + |\gamma(\alpha - \beta)\rangle \cos \theta + |\gamma'(\alpha + \beta) - \gamma(\alpha - \beta)) \sin \theta \right) |1\rangle \otimes |-\rangle \otimes |0\rangle \\
+ \left( |\gamma'(\alpha + \beta) + |\gamma(\alpha - \beta)\rangle \cos \theta + |\gamma'(\alpha + \beta) - \gamma(\alpha - \beta)) \sin \theta \right) |1\rangle \otimes |-\rangle \otimes |1\rangle. 
\]

From here the probabilities of the various outcomes can be read off; writing, say \( P_{++} \) for the probability \( P(Z_{p} = +1, X_{f} = -1, Z_{m} = +1) \), we have:

\[
8P_{++} = 1 + (2\gamma^2 - 1)(\alpha^\beta + \overline{\alpha} \overline{\beta}) + \sin(2\theta)(2\gamma^2 - 1) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) + 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{+-} = 1 + (2\gamma^2 - 1)(\alpha^\beta + \overline{\alpha} \overline{\beta}) + \sin(2\theta)(2\gamma^2 - 1) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) - 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{-+} = 1 + (1 - 2\gamma^2)(\alpha^\beta + \overline{\alpha} \overline{\beta}) + \sin(2\theta)(1 - 2\gamma^2) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) + 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{--} = 1 + (1 - 2\gamma^2)(\alpha^\beta + \overline{\alpha} \overline{\beta}) - \sin(2\theta)(1 - 2\gamma^2) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) - 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{++} = 1 + (2\gamma^2 - 1)(\alpha^\beta + \overline{\alpha} \overline{\beta}) - \sin(2\theta)(2\gamma^2 - 1) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) + 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{+-} = 1 + (2\gamma^2 - 1)(\alpha^\beta + \overline{\alpha} \overline{\beta}) - \sin(2\theta)(2\gamma^2 - 1) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) - 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{-+} = 1 + (1 - 2\gamma^2)(\alpha^\beta + \overline{\alpha} \overline{\beta}) - \sin(2\theta)(1 - 2\gamma^2) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) + 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

\[
8P_{--} = 1 + (1 - 2\gamma^2)(\alpha^\beta + \overline{\alpha} \overline{\beta}) - \sin(2\theta)(1 - 2\gamma^2) + (\alpha^\beta + \overline{\alpha} \overline{\beta}) - 2\gamma'(|\alpha|^2 - |\beta|^2) \cos(2\theta)
\]

This gives the respective 8-outcome POVM with positive operators \( E_{klm} \) on the target system:

\[
8E_{++} = (1 + \sin(2\theta))(2\gamma^2 - 1)I + (2\gamma^2 - 1 + \sin(2\theta))X + 2\gamma' \cos(2\theta)Z
\]

\[
8E_{+-} = (1 + \sin(2\theta))(2\gamma^2 - 1)I + (2\gamma^2 - 1 + \sin(2\theta))X - 2\gamma' \cos(2\theta)Z
\]

\[
8E_{-+} = (1 + \sin(2\theta))(1 - 2\gamma^2)I + (1 - 2\gamma^2 + \sin(2\theta))X + 2\gamma' \cos(2\theta)Z
\]

\[
8E_{--} = (1 + \sin(2\theta))(1 - 2\gamma^2)I + (1 - 2\gamma^2 + \sin(2\theta))X - 2\gamma' \cos(2\theta)Z
\]

\[
8E_{++} = (1 - \sin(2\theta))(2\gamma^2 - 1)I + (2\gamma^2 - 1 - \sin(2\theta))X + 2\gamma' \cos(2\theta)Z
\]

\[
8E_{+-} = (1 - \sin(2\theta))(2\gamma^2 - 1)I + (2\gamma^2 - 1 - \sin(2\theta))X - 2\gamma' \cos(2\theta)Z
\]

\[
8E_{-+} = (1 - \sin(2\theta))(1 - 2\gamma^2)I + (1 - 2\gamma^2 - \sin(2\theta))X + 2\gamma' \cos(2\theta)Z
\]

\[
8E_{--} = (1 - \sin(2\theta))(1 - 2\gamma^2)I + (1 - 2\gamma^2 - \sin(2\theta))X - 2\gamma' \cos(2\theta)Z
\]

From here we can read off the actual (marginal) 2-outcome POVMs that are being measured on the system at the three stages. Firstly the \( Z \) measurement defines the positive operators \( B_k = \sum_{tm} E_{ktm} \) representing the initial weak \( X \) measurement:

\[
B_+ = \frac{1}{2} \left[ I + (2\gamma^2 - 1)X \right]
\]

\[
B_- = \frac{1}{2} \left[ I - (2\gamma^2 - 1)X \right],
\]

the final sharp \( X_f \) corresponds to measuring the POVM \( D_t = \sum_{km} E_{ktm} \):

\[
D_+ = \frac{1}{2} \left[ I + \sin(2\theta)X \right]
\]

\[
D_- = \frac{1}{2} \left[ I - \sin(2\theta)X \right],
\]
and the observable actually being measured by the measurement device whose disturbance power is being assessed is
\[ C_m = \sum_{k\ell} E_{k\ell}. \]

We also note down the POVM, \( F_{k\ell} = \sum_m E_{k\ell m}, \) representing the joint measurement of the initial weak \( X \) observable and the final \( X_f \) measurement, which is used to calculate the disturbance quantity:

\[
\begin{align*}
F_{++} &= \frac{1}{4} \left[ \left( 1 + \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} + (2\gamma^2 - 1 + \sin(2\theta))X \right] \\
F_{+-} &= \frac{1}{4} \left[ \left( 1 - \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} - (2\gamma^2 - 1 - \sin(2\theta))X \right] \\
F_{-+} &= \frac{1}{4} \left[ \left( 1 - \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} + (2\gamma^2 - 1 - \sin(2\theta))X \right] \\
F_{--} &= \frac{1}{4} \left[ \left( 1 + \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} - (2\gamma^2 - 1 + \sin(2\theta))X \right].
\end{align*}
\]

The associated operational joint probabilities in the state \( \alpha|0\rangle + \beta|1\rangle \) are (putting \( \langle X \rangle = \alpha\bar{\beta} + \bar{\alpha}\beta \)):

\[
\begin{align*}
P(Z_p = +1, X_f = +1) &= \frac{1}{4} \left[ \left( 1 + \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} + (2\gamma^2 - 1 + \sin(2\theta))\langle X \rangle \right] \\
P(Z_p = +1, X_f = -1) &= \frac{1}{4} \left[ \left( 1 - \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} - (2\gamma^2 - 1 - \sin(2\theta))\langle X \rangle \right] \\
P(Z_p = -1, X_f = +1) &= \frac{1}{4} \left[ \left( 1 - \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} + (2\gamma^2 - 1 - \sin(2\theta))\langle X \rangle \right] \\
P(Z_p = -1, X_f = -1) &= \frac{1}{4} \left[ \left( 1 + \sin(2\theta)(2\gamma^2 - 1) \right) \mathbb{1} - (2\gamma^2 - 1 + \sin(2\theta))\langle X \rangle \right].
\end{align*}
\]

With these expressions it is straightforward to verify Eq. (3) of the main text,

\[
2P_{WV}(\delta X = \pm 2) = 2P_{WV}(X_i = \mp 1, X_f = \pm 1)P(X_f = \pm 1)
\]

\[
= P(Z_p = 1, X_f = \pm 1) + P(Z_p = -1, X_f = \pm 1) + \frac{P(Z_p = 1, X_f = \pm 1) - P(Z_p = -1, X_f = \pm 1)}{2\gamma^2 - 1}.
\]

The last expression, which can be directly evaluated using the above probabilities, is to be compared with the weak-valued probability on the left hand side:

\[
P_{WV}(\delta X = \pm 2) = P_{WV}(X_i = \mp 1, X_f = \pm 1) = \left( \frac{1}{2} \mp X \right) \frac{1}{2} \left( \mathbb{1} \mp 2 \sin(2\theta)X \right)
\]

\[
= \frac{1}{2} (1 - \sin(2\theta)) \frac{1}{2} (1 \mp \langle X \rangle).
\]

We observe that these weak-valued joint probabilities do not coincide with the operational probabilities, \( P(Z_p = \mp 1, X_f = \pm 1), \) except in the strong measurement case, \( \gamma = 1. \)

Finally we verify the strong measurement realization of \( \eta(X) \).

\[
4P(Z_p = +1, X_f = -1) + 4P(Z_p = -1, X_f = +1) = 2 - 2\sin(2\theta)(2\gamma^2 - 1).
\]

Note that this is already state-independent. On putting \( \gamma = 1, \) we finally obtain

\[
4P(Z_p = +1, X_f = -1) + 4P(Z_p = -1, X_f = +1) = 2 - 2\sin(2\theta) = \eta(X)^2.
\]