Quark mass dependent collective excitations and quark number susceptibilities within the hard thermal loop approximation

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We calculate all those QCD $N$-point functions which are relevant for three-loop QCD thermodynamics calculation with finite quark masses within the hard thermal loop approximation. Using the effective quark propagator, we also calculate second order quark and baryon number susceptibilities within the hard thermal loop approximation and compare the results with available lattice data.

I. INTRODUCTION

It is an experimental fact that colored quarks and gluons are confined to hadrons by the strong interaction. The theory, which correctly describes this interaction, is known as Quantum Chromodynamics (QCD). In extreme conditions, such as at very high temperatures and/or densities, hadronic states are believed to undergo a (partially crossover) transition into a deconfined state of quarks and gluons, known as Quark-Gluon Plasma (QGP). Such extreme conditions exist in the very early universe and can also be generated in ultrarelativistic heavy-ion collision experiments at the Relativistic Heavy Ion Collider (RHIC), the Large Hadron Collider (LHC) and in the future also at the Facility for Antiproton and Ion Research (FAIR). In addition, the cores of the densest astrophysical objects in existence, neutron stars, may contain cold deconfined matter, commonly referred to as quark matter. The reason why deconfined matter is expected to be encountered at high energy densities is related to the asymptotic freedom of QCD, i.e., the fact that the value of the strong coupling constant decreases logarithmically as a function of the energy scale. Proceeding to higher energies, nonperturbative effects are also expected to diminish in importance, and calculations based on a weak coupling expansion should eventually become feasible at such extreme conditions. This is very important for a successful quantitative description of the system especially at nonzero chemical potentials, as no nonperturbative first principles method applicable to finite-density QCD exists due to the Sign Problem of lattice QCD. Unfortunately, at energy densities of phenomenological relevance for most practical applications, the value of the strong coupling constant is not small, and it in fact turns out that e.g., for bulk thermodynamic quantities a strict expansion in the QCD coupling constant converges only at astronomically high temperatures and chemical potentials. The source of this problem has been readily identified as the infrared sector of the theory, i.e., contributions from soft gluonic momenta of the order of the Debye mass or smaller. This suggests that to improve the situation, one needs a way of reorganizing the perturbative series that allows for the soft contributions to be included in the result in a physically consistent way. To date, two successful variations of resummed perturbation theory have been introduced as remedy, namely Hard Thermal Loop perturbation theory (HTLpt) [1–7] and Dimensionally Reduced (DR) effective field theory [8, 9], of which the latter is only applicable to high temperatures but the former in principle covers also the cold and dense part of the QCD phase diagram. Both approaches have been shown to lead to results which are in quantitative agreement with lattice QCD in the high-temperature and zero (or small-) density limit, down to temperatures a few times the pseudocritical temperature of the deconfinement transition.

In nearly all thermodynamic calculations applying perturbation theory, either the temperature or the chemical potentials are assumed to be the dominant energy scale in the system and in particular much larger than the QCD scale or any quark masses. It is, however, questionable, whether the latter is necessarily a good approximation at the lowest temperatures and densities where the HTLpt results are typically applied; the strange quark mass is after all of the order of 100 MeV, which is certainly not negligible at the temperatures reached in the RHIC and LHC heavy ion experiments. There have been very few high-order perturbative calculations incorporating finite quark masses, and in the context of high temperatures, the state-of-the-art result is from a two-loop perturbative calculation performed long ago [10] in a renormalization scheme for quark masses. Later, two-loop perturbative calculation for the thermodynamical quantities has been extended in Refs. [11, 12] using the MS renormalization scheme. Additionally, the current state-of-the-art result for the QCD pressure at zero temperature and finite quark mass is from a three-loop bare perturbation theory [13]. On the other hand, finite quark masses have not been considered at all in any resummed perturbative framework for thermodynamic calculations. Collective excitations of heavy-fermion have been studied long ago in Ref. [14] calculating heavy-fermion propagator. Later, in Ref. [15], the authors have studied quark mass dependent thermal excitations calculating both quark and gluon propagators. In this article we calculate all the QCD $N$-point functions, such as gluon propagator, quark propagator, three- and four-point quark-gluon vertices considering HTL approximation. These $N$-point functions are relevant for higher order thermodynamics calculation within the HTLpt framework. In

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this direction, we calculate quark number susceptibility (QNS) and baryon number susceptibility (BNS) using two-loop $\Phi$-derivable self-consistent approximation within HTL scheme, as described in [16, 17] for massless quarks.

The paper is organized as follows. In Sec. II we calculate the HTL effective gluon-propagator and gluon dispersion relations including finite quark masses. In Sec. III we calculate the HTL effective quark-propagator with finite quark mass. In Sec. IV, we discuss about the three- and four-point quark-gluon vertices with finite quark mass. In the next section (Sec. V), we calculate the second-order light quark- as well as strange quark-number susceptibility which lead to the computation of the second order BNS. In Sec. VI, we conclude our results.

II. GLUON PROPAGATOR

One loop Feynman diagram for the gluon self-energy consisting the quark loop is illustrated in the Fig. (1).

![Gluon self-energy diagram](image)

The frequency sum for the first term in Eqs. (4) and (5) can be evaluated easily as

$$\Pi^f_{\mu\nu}(P,m) = \frac{4ig^2}{2} \sum_f \int \frac{d^4K}{(2\pi)^4} \left( 2k_{f}^2 - K^2 + m_f^2 \right) \Delta(K) \Delta(Q)$$

and

$$\Pi^f_{\mu\nu}(P,m) \approx 2ig^2 \sum_f \int \frac{d^4K}{(2\pi)^4} \left[ \Delta(K) + 2 \left( k^2 + m_f^2 \right) \Delta(K) \Delta(Q) \right].$$

Similarly, the trace of the gluon self-energy tensor is

$$\Pi^f_{\mu\nu}(P,m) \approx \frac{-4ig^2}{2} \sum_f \int \frac{d^4K}{(2\pi)^4} \left[ \Delta(K) - m_f^2 \Delta(K) \Delta(Q) \right].$$

The frequency sum for the first term in Eqs. (4) and (5) can be evaluated easily as

$$i \int \frac{d^4K}{(2\pi)^4} \Delta(K)$$

$$= -T \sum_n \int \frac{d^4k}{(2\pi)^3} \frac{1}{k^2 - k^2 - m_f^2}$$

$$= - \int k^2 dk n'_f(E_k) + n_f(E_k) \frac{2E_k}{2E_k},$$

where

$$E_k = \sqrt{k^2 + m_f^2},$$

$$n'_f(E_k) = \frac{1}{\exp[\beta(E_k - \mu_f)] + 1}.$$
Using the relations in Eqs. (9)-(11), the following Matsubara sum can be performed as

\[
T \sum_{k_0} \tilde{\Delta}(K)\tilde{\Delta}(P-K) = \int_0^\beta d\tau e^{ip_0\tau} \Delta_F(\tau, E_k)\Delta_F(\tau, E_{pk}) = -\sum_{s_1, s_2 = \pm 1} \frac{s_1 s_2}{4E_k E_{pk}} \left[ 1 - n^+_F(s_1 E_k) - n^-_F(s_2 E_{pk}) \right]
\]

\[
= 1 \frac{1}{4E_k E_{pk}} \left[ n^+_F(E_k) - n^+_F(E_{pk}) \right] \frac{p_0 + E_k - E_{pk}}{p_0 - E_k + E_{pk}} - \frac{n^-_F(E_k) - n^-_F(E_{pk})}{p_0 - E_k + E_{pk}} + \frac{1}{p_0 - E_k - E_{pk}} \right].
\] (12)

where

\[
E_k = \sqrt{k^2 + m_f^2}, \quad E_{pk} = \sqrt{|p - k|^2 + m_f^2} \quad (13)
\]

In HTL approximation,

\[
E_{pk} = \sqrt{|p - k|^2 + m_f^2} \approx E_k - v \hat{k} \cdot p,
\]

\[
n^+_F(E_{pk}) \approx n^+_F(E_k) - \hat{k} \cdot p \frac{dn^+_F(E_k)}{dk}, \quad (15)
\]

with \(v = k/E_k\).

Using the HTL approximations (Eq. (15)), the Matsubara sum in Eq. (12) can be written as

\[
T \sum_{k_0} \tilde{\Delta}(K)\tilde{\Delta}(P-K) = \frac{1}{4E_k^2} \left[ n^+_F(E_k) + n^-_F(E_k) \right]
\]

\[
+ \frac{dn^+_F(E_k)}{dk} \left[ \frac{\hat{k} \cdot p}{p_0 + v \hat{k} \cdot p} - \frac{dn^-_F(E_k)}{dk} \left[ \frac{\hat{k} \cdot p}{p_0 - v \hat{k} \cdot p} \right] \right].
\] (16)

So, Eq. (4) becomes

\[
\Pi^{f\mu}(p_0, p, m) = 2ig^2 \sum_f \int \frac{d^4K}{(2\pi)^4} \left[ \tilde{\Delta}(K) + 2E_k^2 \tilde{\Delta}(K)\tilde{\Delta}(Q) \right]
\]

\[
= -4ig^2 \sum_f \int \frac{k^2 dk d\Omega}{4\pi^2} \left[ \tilde{\Delta}(K) + \frac{\hat{k} \cdot p}{p_0 + v \hat{k} \cdot p} \right]
\]

\[
\times \left[ 1 - \frac{p_0}{2pv} \log \frac{p_0 + pv}{p_0 - pv} \right].
\] (17)

Eq. (17) can also be simplified as

\[
\Pi^{f\mu}(p_0, p, m) = \frac{g^2}{2\pi^2} \sum_f \int \frac{k^2 dk}{E_k} \left[ n^+_F(E_k) + n^-_F(E_k) \right]
\]

\[
\times \left[ 1 + \frac{m_f^2}{2E_k^2} \right].
\] (18)

Similarly, the trace of the gluon self-energy tensor in Eq. (5) can be written using the Matsubara sums in Eqs. (6) and (12) as

\[
\Pi^{\mu\mu}(p_0, p, m) = \frac{g^2}{2\pi^2} \sum_f \int \frac{d^4K}{(2\pi)^4} \left[ \tilde{\Delta}(K) - m_f^2 \tilde{\Delta}(K)\tilde{\Delta}(Q) \right]
\]

\[
= \frac{g^2}{2\pi^2} \sum_f \int \frac{k^2 dk d\Omega}{4\pi^2} \left[ n^+_F(E_k) + n^-_F(E_k) \right]
\]

\[
\times \left[ 1 + \frac{m_f^2}{2E_k^2} \right].
\] (19)

Eq. (19) can also be simplified as

\[
\Pi^{\mu\mu}(p_0, p, m) = \frac{g^2}{2\pi^2} \sum_f \int \frac{k^2 dk}{E_k} \left[ n^+_F(E_k) + n^-_F(E_k) \right]
\]

\[
\times \left[ 1 + \frac{m_f^2}{2E_k^2} \right].
\] (20)

Pure Yang-Mills contributions to the gluon self-energy do not get affected by the inclusion of finite quark masses. So, the time-time and \(\mu\mu\) component of the total gluon self-energy tensor from fermion as well as pure Yang-Mills diagrams is

\[
\Pi^{\mu\mu}(p_0, p, m) = \frac{g^2 T^2 C_A}{3} \left[ 1 - \frac{p_0}{2p} \log \frac{p_0 + p}{p_0 - p} \right].
\]
\[ + \frac{g^2}{2 \pi^2} \sum_j \int \frac{k^2 dk}{E_k} \left[ n_F^+(E_k) + n_F^-(E_k) \right] \]
\[ \times \left[ 1 + \frac{p_0^2 - p^2}{p_0^2 - p^2 v^2} - \frac{p_0}{pv} \log \frac{p_0 + pv}{p_0 - pv} \right], \quad (21) \]
and
\[ \Pi_{\mu}(p_0, p, m) = \frac{g^2 T^2 C_A}{3} + \frac{g^2}{2 \pi^2} \sum_j \int \frac{k^2 dk}{E_k} \left[ n_F^+(E_k) + n_F^-(E_k) \right] \]
\[ \times \left[ 1 + \frac{m_f^2}{2E_k^2} \left( \frac{p_0^2 - p^2}{p_0^2 - p^2 v^2} \right) \right], \quad (22) \]

Note that, unlike massless case, gluonic and quark contributions can not combined together.

Now one-loop gluon self-energy can be decomposed in terms of two independent and mutually transverse second rank projection tensors as
\[ \Pi_{\mu\nu} = \Pi_T A_{\mu\nu} + \Pi_L B_{\mu\nu}, \quad (23) \]
with
\[ A_{\mu\nu} = g_{\mu\nu} - \frac{P_\mu P_\nu}{p^2} - B_{\mu\nu}, \quad (24) \]
\[ B_{\mu\nu} = -\frac{p^2}{p^2} \left( \mu_\mu - \frac{p_0 P_\mu}{p^2} \right) \left( \mu_\nu - \frac{p_0 P_\nu}{p^2} \right). \quad (25) \]

So, the longitudinal and transverse parts of gluon self-energy tensor can be obtained from Eqs. (21) and (22) as
\[ \Pi_L(p_0, p, m) = \frac{-p_0^2 - p^2}{p^2} \Pi_0(p_0, p, m) \]
\[ = -\frac{p_0^2 - p^2}{p^2} \left[ g^2 T^2 C_A \left( 1 - \frac{p_0}{2p} \log \frac{p_0 + p}{p_0 - p} \right) \right] \]
\[ + \frac{g^2}{2 \pi^2} \sum_j \int \frac{k^2 dk}{E_k} \left( n_F^+(E_k) + n_F^-(E_k) \right) \]
\[ \times \left[ 1 + \frac{p_0^2 - p^2}{p_0^2 - p^2 v^2} - \frac{p_0}{pv} \log \frac{p_0 + pv}{p_0 - pv} \right], \quad (26) \]
and
\[ \Pi_T(p_0, p, m) = \frac{1}{2} \left[ \Pi_{\mu}(p_0, p, m) - \Pi_L(p_0, p, m) \right] \]
\[ = \frac{g^2 T^2 C_A}{3} \left[ 1 + \frac{p_0^2 - p^2}{p^2} \left( 1 - \frac{p_0}{2p} \log \frac{p_0 + p}{p_0 - p} \right) \right] \]
\[ + \frac{g^2}{2 \pi^2} \sum_j \int \frac{k^2 dk}{E_k} \left( n_F^+(E_k) + n_F^-(E_k) \right) \]
\[ \times \left( \frac{2p_0^2}{p^2} - \frac{p_0^2 - p^2}{p^2} \frac{p_0}{pv} \log \frac{p_0 + pv}{p_0 - pv} \right). \quad (27) \]

### A. Debye mass

Debye mass in QCD is obtained as
\[ m_D^2 = \Pi_L(p_0 = 0, p \to 0, m). \quad (28) \]

At vanishing quark masses, Debye mass is
\[ m^2_D(T, \mu) = \frac{g^2 T^2}{3} \left[ C_A + \frac{N_f}{2} \left( 1 + \frac{3\mu^2}{\pi^2 T^2} \right) \right], \quad (29) \]

where \( \mu \) is common chemical potential for all the quark-flavors. Now, the Debye mass with finite quark masses can be obtained from Eq. (26) as
\[ m^2_D(T, \mu, m) = \Pi_L(p_0 = 0, p \to 0, m) \]
\[ = \frac{g^2 T^2 C_A}{3} + \frac{g^2}{2 \pi^2} \sum_{f=1}^{N_f} \int \frac{k dk}{v} \left( 1 + v^2 \right) \left[ n_F^+(E_k) + n_F^-(E_k) \right]. \quad (30) \]

Plasma frequency is the oscillation frequency for vanishing wave vectors, namely, spatially uniform oscillations [19] and can be expressed as
\[ \omega^2_p = \frac{g^2 T^2 C_A}{9} + \frac{g^2}{2 \pi^2} \sum_{f=1}^{N_f} \int \frac{dk dv}{v} \left( 1 - \frac{v^2}{3} \right) \left[ n_F^+(E_k) + n_F^-(E_k) \right]. \quad (31) \]

Unlike massless case, Debye mass and plasma frequency are not proportional to each other.

**FIG. 2.** Debye mass and plasma frequency plotted with temperature

In Fig. (2) we plot quark mass dependent Debye mass and plasma frequency, scaled with corresponding massless values, with temperature. In this plot and also in the rest of the plots, we use \( m_u = m_d = m_{ud} = 3.7 \text{ MeV} \) and \( m_s = 27m_{ud} \).
B. Dispersion relation

In-medium gluon propagator can be written as

$$D_{\mu\nu} = \frac{\xi p_\mu p_\nu}{P^4} + \frac{A_{\mu\nu}}{P^2 - \Pi_T} + \frac{B_{\mu\nu}}{P^2 - \Pi_L}$$

$$= \frac{\xi p_\mu p_\nu}{P^4} + \frac{A_{\mu\nu}}{\mathcal{D}_T} + \frac{B_{\mu\nu}}{\mathcal{D}_L}.$$ (32)

The zeros of the denominators give the dispersion laws as \(\omega_L\) and \(\omega_T\) as plotted in Fig. (3).

In Fig. 3 we plot longitudinal and transverse dispersion relations scaled with massless plasma frequency. At small momentum \((p \ll T)\), it is possible to find approximate analytic solution of \(\omega_L, T\) as

$$\omega_L^2 = \omega_p^2 + \frac{p^2}{\omega_p^2} a_L + \frac{p^4}{\omega_p^4} b_L + \mathcal{O}(p^6),$$ (33)

$$\omega_T^2 = \omega_p^2 + \frac{p^2}{\omega_p^2} a_T + \frac{p^4}{\omega_p^4} b_T + \mathcal{O}(p^6),$$ (34)

where

$$a_L = \frac{g^2 T^2 C_A}{15} + \frac{g^2}{2\pi^2} \sum_{f=1}^{N_f} N_f,$$

$$b_L = \frac{g^2 T^2 C_A}{21} - \frac{a^2_L}{\omega_p^4} + \frac{g^2}{2\pi^2} \sum_{f=1}^{N_f} N_f,$$

$$\times \int_0^\infty dk k^3 \left(1 - \frac{3v^2}{5}\right) \left[n^+_P(E_k) + n^-_P(E_k)\right].$$ (35)

$$a_T = \frac{\omega_p^2 + \frac{a_L}{3}}{T},$$ (37)

$$b_T = \frac{b_L}{5} - \frac{a_L}{3} + \frac{4 a^2_L}{45 \omega_p^2}.\tag{38}$$

From Figs. (2) and (3), we conclude that the finite mass effect to the gluon collective excitations is negligible and one may consider effective gluon propagator is same as in massless case to calculate various physical observables.

III. QUARK PROPAGATOR

Free massive quark propagator is

$$S_0(P) = \frac{i}{P - m_f}.$$ (39)

The inverse of the propagator in Eq. (39) can be written as

$$iS_0^{-1}(P) = \hat{P} - m_f.$$ (40)

Now, the inverse of in-medium quark propagator is

$$iS^{-1}(P) = iS_0^{-1}(P) - \Sigma(P),$$ (41)

where \(\Sigma(P)\) is the one-loop quark self-energy and can be decomposed as

$$\Sigma(P) = -a \hat{P} - b \hat{P} + c m_f,$$ (42)

The coefficients \(a, b, c\) can be obtained as

$$a = -\frac{1}{4p} \text{Tr}[\gamma \cdot \hat{P} \Sigma(P)],$$

$$b = -\frac{1}{4} \text{Tr}[\gamma_0 \Sigma(P)] + \frac{p_0}{4p} \text{Tr}[\gamma \cdot \hat{P} \Sigma(P)],$$

$$c = \frac{1}{4m_f} \text{Tr}[\Sigma(P)].$$ (43)

So, Eq. (41) becomes

$$iS^{-1}(P) = (1 + a) \hat{P} + b \hat{P} + (1 + c)m_f,$$

$$-iS(P) = \frac{(1 + a) \hat{P} + b \hat{P} + (1 + c)m_f}{D(p_0, P)},$$ (44)

where

$$D(p_0, P) = (1 + a)^2 P^2 + b^2 - (1 + c)^2 m_f^2 + 2b(1 + a)p_0$$

$$= D_+(p_0, P) D_-(p_0, P),$$ (45)

with

$$D_\pm(p_0, P) = (1 + a)p_0 + b \pm \sqrt{(1 + a)^2 P^2 + (1 + c)^2 m_f^2}.$$ (46)
Now, the numerator of Eq. (44) can be written as
\[
(1 + a)\mathbf{p} + b\mathbf{q} + (1 + c)m_f
\]
\[
= \frac{D_+(p_0, p)}{2} \left[ \gamma_0 + \frac{\gamma \cdot \mathbf{p}(1 + a)p + (1 + c)m_f}{\sqrt{(1 + a)^2 p^2 + (1 + c)^2 m_f^2}} \right] + \frac{D_-(p_0, p)}{2} \left[ \gamma_0 - \frac{\gamma \cdot \mathbf{p}(1 + a)p + (1 + c)m_f}{\sqrt{(1 + a)^2 p^2 + (1 + c)^2 m_f^2}} \right].
\]

So, Eq. (44) becomes
\[
-iS(P)
\]
\[
= \frac{1}{2D_+(p_0, p)} \left[ \gamma_0 - \frac{\gamma \cdot \mathbf{p}(1 + a)p + (1 + c)m_f}{\sqrt{(1 + a)^2 p^2 + (1 + c)^2 m_f^2}} \right] + \frac{1}{2D_-(p_0, p)} \left[ \gamma_0 + \frac{\gamma \cdot \mathbf{p}(1 + a)p + (1 + c)m_f}{\sqrt{(1 + a)^2 p^2 + (1 + c)^2 m_f^2}} \right].
\]

Zeros of the denominator \((D_\pm(p_0, p))\) of Eq. (48) give the dispersion relation. Now,
\[
iS^{-1}(P)
\]
\[
= (1 + a)\mathbf{p} + b\mathbf{q} - (1 + c)m_f
\]
\[
= [(1 + a)p_0 + b]\gamma_0 - (1 + a)\gamma \cdot \mathbf{p} p - (1 + c)m_f
\]
\[
= A_0(p_0, p)\gamma_0 - A_\gamma \gamma \cdot \mathbf{p} - A_m,
\]
with
\[
A_0(p_0, p) = (1 + a)p_0 + b,
\]
\[
A_\gamma(p_0, p) = (1 + a)p,
\]
\[
A_m(p_0, p) = (1 + c)m_f.
\]

Alternatively,
\[
D_\pm(p_0, p) = A_0(p_0, p) \pm \sqrt{A_0(p_0, p)^2 + A_m(p_0, p)^2}.
\]

Self-energy can also be written using Eqs. (42) and (50) as
\[
\Sigma(P) = -a\mathbf{p} - b\mathbf{q} + cm_f
\]
\[
= -(ap_0 + b)\gamma_0 + ap_\gamma \gamma \cdot \mathbf{p} + cm_f
\]
\[
= [p_0 - A_0(p_0, p)]\gamma_0 + [A_\gamma(p_0, p) - p]\gamma \cdot \mathbf{p} + [A_m(p_0, p) - m_f].
\]

Eq. (48) can also be written using Eq. (50) as
\[
-iS(P)
\]
\[
= \frac{1}{2D_+(p_0, p)} \left[ \gamma_0 - \frac{A_\gamma(p_0, p)\gamma \cdot \mathbf{p} + A_m(p_0, p)}{\sqrt{A_\gamma(p_0, p)^2 + A_m(p_0, p)^2}} \right] + \frac{1}{2D_-(p_0, p)} \left[ \gamma_0 + \frac{A_\gamma(p_0, p)\gamma \cdot \mathbf{p} + A_m(p_0, p)}{\sqrt{A_\gamma(p_0, p)^2 + A_m(p_0, p)^2}} \right].
\]

Now, the quark self-energy in Feynman gauge can be obtained from the Feynman diagram in Fig. (4)
\[
\left. \text{FIG. 4. Quark self-energy} \right|
\]
\[
\begin{array}{c}
P \\
\rightarrow \end{array} Q = \mathbf{p} \rightarrow P
\]
\]
\[
\left. \text{To compute the quark self-energy in Eq. (54), we need to compute the following Matsubara sums: } T \sum_{K} \mathbf{A}(P - K) \mathbf{A}(P) \text{ and } T \sum_k \mathbf{A}(P - K) \mathbf{A}(P). \text{ We evaluate them following the same procedure as in gluonic case using the mixed representation. Fermionic propagator } \mathbf{A}(P - K) \text{ is given in Eq. (10) and the gluonic propagator can be represented as } \mathbf{A}(P). \text{ Using Eqs. (10) and (55), the necessary Matsubara sum can be performed as} \right|
\]
\[
\begin{array}{c}
\Delta(K) = \frac{1}{k_0^2 - k^2} = \int_0^\beta d\tau e^{k\tau} \Delta_B(\tau, k),
\end{array}
\]
with
\[
\Delta_B(\tau, k) = \frac{1}{2k} \left[ n_B(k)e^{k\tau} + (1 + n_B(k))e^{-k\tau} \right].
\]
Within HTL approximation, the first two terms of Eq. (57) can be neglected and we can write the Matsubara sum as

\[ T \sum_{\mathbf{k}_0} \Delta(K) \Delta(P - K) \approx \frac{1}{4kE_{pk}} \left[ \frac{n_B(k) + n_{pF}(E_{pk})}{p_0 - k + E_{pk}} - \frac{n_B(k) + n_{pF}(E_{pk})}{p_0 + k - E_{pk}} \right], \]  

(58)

Following the similar procedure, we can calculate the second Matsubara sum as

\[ T \sum_{\mathbf{k}_0} k_0 \Delta(K) \Delta(P - K) \approx \frac{1}{4E_{pk}} \left[ \frac{n_B(k) + n_{pF}(E_{pk})}{p_0 - k + E_{pk}} + \frac{n_B(k) + n_{pF}(E_{pk})}{p_0 - k + E_{pk}} \right]. \]  

(59)

Using the approximations in Eq. (15), Eqs. (58) and (59) can be simplified as

\[ T \sum_{\mathbf{k}_0} \tilde{\Delta}(K) \tilde{\Delta}(P - K) \approx \frac{1}{4kE_{k}} \left[ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} - \frac{n_{pF}(E_k) + n_B(k)}{p_0 + k - E_k + vp \cdot \mathbf{k}} \right], \]  

(60)

and

\[ T \sum_{\mathbf{k}_0} k_0 \tilde{\Delta}(K) \tilde{\Delta}(P - K) \approx \frac{1}{4E_{k}} \left[ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} \right]. \]  

Using the Matsubara sums in Eqs. (60) and (61), the quark self-energy in Eq. (54) becomes

\[ \Sigma(P) = \frac{g^2 C_F}{4\pi^2} \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} \left[ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} \right] - \gamma \cdot \hat{k} \cdot k - 2m_f \]  

\[ \times \left\{ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} - \frac{n_{pF}(E_k) + n_B(k)}{p_0 + k - E_k + vp \cdot \mathbf{k}} \right\} \]  

\[ = \frac{g^2 C_F}{4\pi^2} \int \frac{k^2 dk}{E_k} \int \frac{d\Omega}{4\pi} \left[ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} \right] + \left( n_{pF}(E_k) + n_B(k) \right) \frac{k \hat{K} + 2m_f}{P^+ \cdot \hat{K}}, \]  

(62)

with

\[ \hat{p}^\pm \equiv (p_0^\pm, vp), \quad p_0^\pm = p_0 \pm (E_k - k), \quad \hat{K} \equiv (1, \hat{k}). \]  

(63)

Using Eqs. (43), (50) and (62), we can write

\[ A_0(p_0, p) = p_0 - \frac{1}{4} \text{Tr}[\gamma_0 \Sigma(P)] \]  

\[ = p_0 - 2g^2 C_F \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} \frac{1}{4E_{k}} \left\{ \frac{n_{pF}(E_k) + n_B(k)}{P^+ \cdot \hat{K}} + \frac{n_{pF}(E_k) + n_B(k)}{P^- \cdot \hat{K}} \right\} \]  

\[ = p_0 - \frac{g^2 C_F}{8\pi^2} \int k dk \frac{1}{p} \left\{ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} \log \frac{p_0^+ + vp}{p_0^- - vp} + \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k + vp \cdot \mathbf{k}} \log \frac{\tilde{p}_0^+ + vp}{\tilde{p}_0^- - vp} \right\}, \]  

(64)

\[ A_\sigma(p_0, p) = - \frac{1}{4} \text{Tr}[\gamma \cdot \hat{p} \Sigma(P)] \]  

\[ = p - 2g^2 C_F \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} \frac{1}{4E_{k}} \left\{ \frac{n_{pF}(E_k) + n_B(k)}{P^+ \cdot \hat{K}} + \frac{n_{pF}(E_k) + n_B(k)}{P^- \cdot \hat{K}} \right\} \]  

\[ = p + \frac{g^2 C_F}{4\pi^2} \int k dk \frac{1}{p} \left\{ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k + vp \cdot \mathbf{k}} \left\{ 1 - \frac{p_0^+}{2vp} \right\} \log \frac{p_0^+ + vp}{p_0^- - vp} \right\} + \left( n_{pF}(E_k) + n_B(k) \right) \left\{ 1 - \frac{\tilde{p}_0^+}{2vp} \log \frac{\tilde{p}_0^+ + vp}{\tilde{p}_0^- - vp} \right\}, \]  

(65)

\[ A_m(p_0, p) = m_f + \frac{1}{4} \text{Tr}[\Sigma(P)] \]  

\[ = m_f + 2g^2 C_F \int \frac{k^2 dk}{2\pi^2} \int \frac{d\Omega}{4\pi} \frac{m_f}{2kE_{k}} \left\{ \frac{n_{pF}(E_k) + n_B(k)}{P^+ \cdot \hat{K}} - \frac{n_{pF}(E_k) + n_B(k)}{P^- \cdot \hat{K}} \right\} \]  

\[ = m_f + \frac{g^2 C_F}{4\pi^2} \int k dk \frac{m_f}{p} \left\{ \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k - vp \cdot \mathbf{k}} \log \frac{\tilde{p}_0^+ + vp}{\tilde{p}_0^- - vp} - \frac{n_{pF}(E_k) + n_B(k)}{p_0 - k + E_k + vp \cdot \mathbf{k}} \log \frac{\tilde{p}_0^+ + vp}{\tilde{p}_0^- - vp} \right\}. \]  

(66)

Now, we can construct \( D_k(p_0, p) \) from Eq. (51) using Eqs. (64)-(66). We can get the quark dispersion relation...
at finite quark mass by solving $D_\pm(p_0, p) = 0$.

In Fig. (5) we plot dispersion relation for massive strange quark. The solid lines represent dispersion relations for massive quark whereas dashed lines represent corresponding massless modes. Note that unlike vanishing finite mass case, integration over loop momentum $k$ in Eqs. (64)-(66) can not be done analytically. As a consequence, one may use the expressions in Eqs. (64)-(66) only to calculate some quantities in one-loop order. For higher-loop computation, such as three-loop thermodynamics, one may not be able to proceed with general expressions of $A_0(p_0, p), A_s(p_0, p), A_m(p_0, p)$. Keeping that in mind, we can simplify the expressions of $A_0(p_0, p), A_s(p_0, p), A_m(p_0, p)$ at high temperature limit ($m_f \ll T$) as

$$A_0(p_0, p) \approx p_0 - \frac{m_q^2}{2p} \log \frac{p_0 + p}{p_0 - p},$$

$$A_s(p_0, p) \approx p_0 + \frac{m_q^2}{p} \left[ 1 - \frac{p_0}{2p} \log \frac{p_0 + p}{p_0 - p} \right],$$

$$A_0(p_0, p) \approx m_f,$$

where $m_q$ is the quark thermal mass and can be written as

$$m_q^2 = \frac{g^2 T^2 C_F}{8} \left( 1 + \frac{\mu^2}{\pi^2 T^2} \right).$$

In Fig. (6) we compare the full dispersion relation of massive $s$-quark along-with the dispersion relations that have been obtained considering ($m_f \ll T$). From Fig. (6) it is clear that the approximated results in Eq. (67) are good approximation at high temperature. For completeness, we discuss about three- and four-point quark-gluon vertex without high temperature limit ($m_f \ll T$) in the next section.

![Fig. 5. Dispersion plots for strange quark. The solid lines represent dispersion relations for massive quark whereas the dashed lines represent corresponding massless modes.](image)

![Fig. 6. Dispersion plots for strange quark. Dispersion relations from full quark propagator are compared with the approximated one in Eq. (67).](image)

**IV. THREE- AND FOUR-POINT QUARK-GLUON VERTICES**

The one-loop Feynman diagrams for the quark-gluon vertex is illustrated in Fig. (7).

![Fig. 7. Three-point quark-gluon vertex](image)

One-loop vertex correction from the two diagrams in Feynman gauge can be written as

$$ig \delta \Gamma_{\mu, ij}^{1, a}(P_1, P_2) = \int \frac{d^4 K}{(2\pi)^4} \left( ig \gamma^\alpha T_{ij}^{\alpha} \right) S_0'(P_2 - K) \left( ig \gamma^\mu T_{im}^{a} \right) S_0(\gamma^\alpha (P_1 - K)) \left( ig \gamma^\beta T_{mj}^{a} \right) \frac{-ig\alpha\beta \delta^{ij}}{K^2}$$

$$= -i(ig)^3 (T^\alpha T^\alpha T^\beta)_{ij} \int \frac{d^4 K}{(2\pi)^4} \gamma^\alpha S_0(P_2 - K) \gamma^\mu S_0(\gamma^\alpha P_1 - K) \gamma^\alpha \frac{1}{K^2}$$
and
\[\delta \Gamma_{\mu}^{a}(P_1, P_2) = -ig f_{abc} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(P_1 - P_2)^2} \gamma^{\alpha} S_0(P_1 - K) \gamma_{\beta} S_0(P_2 - K) \gamma_{\gamma} \frac{1}{(P_1 - K)^2(P_2 - K)^2}.\]

where
\[V_{\mu\alpha\beta} = g_{\mu\alpha} (2P_2 - 2P_1 + K)_{\beta} + g_{\alpha\beta} (P_1 - P_2 - 2K)_{\mu} + g_{\beta\mu} (P_1 - P_2 + K)_{\alpha}.\]

We can define one-loop vertex correction \(\delta \Gamma_{\mu;i}^{a}(P_1, P_2)\) in terms of vertex correction function \(\delta \Gamma^{a}(P_1, P_2)\) as
\[\delta \Gamma_{\mu;i}^{a}(P_1, P_2) = \delta \Gamma^{a}(P_1, P_2) T_{ij}.\]

One-loop vertex correction function \(\delta \Gamma_{\mu}^{a}(P_1, P_2)\) from the first diagram is
\[\delta \Gamma_{\mu}^{a}(P_1, P_2) = -ig^2 \left( C_F - \frac{C_A}{2} \right) \int \frac{d^4k}{(2\pi)^4} \gamma^{\alpha} S_0(P_2 - K) \gamma_{\mu} S_0(P_1 - K) \gamma_{\alpha} \frac{1}{K^2}.\]

The first term within the square bracket in Eq. (73) is non-leading in temperature and can be neglected using HTL approximation. The remaining contribution can be written as
\[\delta \Gamma_{\mu}^{a}(P_1, P_2) \approx -4ig^2 \left( C_F - \frac{C_A}{2} \right) T \sum_n \int \frac{d^3k}{(2\pi)^3} K_{\mu}(K + 2m_f) \Delta(P_1 - K) \Delta(P_2 - K) \Delta(K).\]

We need to calculate the following Matsubara sums:
\[X_0 = T \sum_n \Delta(K) \Delta(P_1 - K) \Delta(P_2 - K),\]
\[X_1 = T \sum_n k_0 \Delta(K) \Delta(P_1 - K) \Delta(P_2 - K),\]
\[X_2 = T \sum_n k_0^2 \Delta(K) \Delta(P_1 - K) \Delta(P_2 - K).\]

Now,
\[X_0 = T \sum_n \Delta(K) \Delta(P_1 - K) \Delta(P_2 - K)\]
\[= T \sum_{n,s_1,s_2} \frac{1}{8E_1E_2} \frac{s_1s_2}{(k_0 - sE)((p_{10} - k_0) - s_1E_1)(E_2 - p_{10} - k_0 - s_2E_2)}\]
\[= T \sum_{n,s_1,s_2} \frac{1}{8E_1E_2} \frac{s_1s_2}{(p_{10} - p_{20}) - s_1E_1 + s_2E_2} \left[ \frac{1}{k_0 - sE) + (p_{20} - k_0 - s_2E_2) - (k_0 - sE - (p_{20} - k_0 - s_1E_1)) \right]\]
\[= - T \sum_{n,s_1,s_2} \frac{1}{8E_1E_2} \frac{s_1s_2}{(p_{10} - p_{20}) - s_1E_1 + s_2E_2} \left[ \frac{1 + n_B(sE) - n_{\uparrow}(s_2E_2)}{p_{20} - sE - s_2E_2} - \frac{1 + n_B(sE) - n_{\uparrow}(s_1E_1)}{p_{10} - sE - s_1E_1} \right].\]
The Ward identity of the three-point quark-gluon vertex can be checked as

\begin{align}
\delta \Pi_{\mu}^{3}(P_{1}, P_{2}) & \\
& = 4g^2 (C_F - C_A/2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{8E_k^2} \left[ (n_B(k) + \frac{1}{2} \delta_B(E_k)) \left( \frac{K_{\mu}(k\tilde{K} + 2m_f)}{P_1^+ \cdot K} P_2^+ \cdot K \right) + (n_B(k) + \frac{1}{2} \delta_B(E_k)) \left( \frac{K_{\mu}(k\tilde{K} - 2m_f)}{P_1^- \cdot K} P_2^- \cdot K \right) \right],
\end{align}

Following the same procedure, the second diagram of three-point quark-gluon vertex can be written as

\begin{align}
\delta \Pi_{\mu}^{3}(P_{1}, P_{2}) & \\
& \approx 4g^2 C_F \int \frac{d^3k}{4\pi^2} \frac{1}{E_k} \int \frac{d\Omega}{4\pi} \left[ (n_B(k) + \frac{1}{2} \delta_B(E_k)) \left( K_{\mu}(k\tilde{K} + 2m_f) P_1^+ \cdot K P_2^+ \cdot K \right) + (n_B(k) + \frac{1}{2} \delta_B(E_k)) \left( K_{\mu}(k\tilde{K} - 2m_f) P_1^- \cdot K P_2^- \cdot K \right) \right].
\end{align}

Now, the Ward identity of the three-point quark-gluon vertex can be checked as

\begin{align}
(P_1 - P_2)^{a} \Gamma_{\mu}(P_{1}, P_{2}) & \\
& = \hat{P}_1 - \hat{P}_2 + \frac{g^2 C_F}{4\pi^2} \int \frac{k^2dk}{E_k} \int \frac{d\Omega}{4\pi} \left[ (n_B(k) + \frac{1}{2} \delta_B(E_k)) \left( P_1 \cdot \hat{K} - P_2 \cdot \hat{K} \right) \left( k\tilde{K} + 2m_f \right) \right].
\end{align}
\[ + \left( n_B(k) + n_F(E_{k}) \right) \frac{\left( P_1 \cdot \hat{K} - P_2 \cdot \hat{K} \right) \left( k \hat{K} - 2m_f \right)}{\left( P_1 - \hat{K} \right) \left( P_2 - \hat{K} \right)} \]

\[ \simeq \hat{P}_1 - \hat{P}_2 + \frac{g^2 C_F}{4\pi^2} \int \frac{dk}{E_k} \frac{d\Omega}{4\pi} \left( n_B(k) + n_F(E_k) \right) \frac{\left( \hat{P}_1^+ \cdot \hat{K} - \hat{P}_2^+ \cdot \hat{K} \right) \left( k \hat{K} + 2m_f \right)}{\left( \hat{P}_1^+ \cdot \hat{K} \right) \left( P_2^+ \cdot \hat{K} \right)} \]

\[ + \left( n_B(k) + n_F(E_k) \right) \frac{\left( \hat{P}_1^- \cdot \hat{K} - \hat{P}_2^- \cdot \hat{K} \right) \left( k \hat{K} - 2m_f \right)}{\left( \hat{P}_1^- \cdot \hat{K} \right) \left( P_2^- \cdot \hat{K} \right)} \]

\[ = \hat{P}_1 - m_f - \frac{g^2 C_F}{4\pi^2} \int \frac{dk}{E_k} \frac{d\Omega}{4\pi} \left( n_B(k) + n_F^+(E_k) \right) \frac{k \hat{K} + 2m_f}{P_1^+ \cdot \hat{K}} + \frac{n_B(k) + n_F^-(E_k) k \hat{K} - 2m_f}{P_1^- \cdot \hat{K}} - \hat{P}_2 + m_f + \frac{g^2 C_F}{4\pi^2} \int \frac{dk}{E_k} \frac{d\Omega}{4\pi} \left( n_B(k) + n_F^+(E_k) \right) \frac{k \hat{K} + 2m_f}{P_1^+ \cdot \hat{K}} + \frac{n_B(k) + n_F^-(E_k) k \hat{K} - 2m_f}{P_1^- \cdot \hat{K}} \]

\[ = iS^{-1}(P_1) - iS^{-1}(P_2). \tag{85} \]

Eq. (85) suggests that the three-point vertex satisfies the Ward identity with the quark-propagator. Note that without doing exact calculation, one can also predict form of the three-point vertex (Eq. (84)) using Ward identity. In the similar manner, one can also predict the form of four-point quark-gluon vertex from the Ward identity

\[ Q_i^\nu \Gamma_{\mu\nu}(P_1, P_2; Q_1) = \Gamma_{\mu}(P_1, P_2 - Q_1) - \Gamma_{\mu}(P_2, P_1 + Q_1), \tag{86} \]

without calculating the Feynman diagram, as

\[ \Gamma_{\mu\nu}(P_1, P_2; Q_1) = \frac{g^2 C_F}{4\pi^2} \int \frac{k^2 dk}{E_k^2} \int \frac{d\Omega}{4\pi} \frac{K_{\mu} \hat{K}_{\nu} \left( k \hat{K} + 2m_f \right) \left( \hat{P}_1^+ + Q_1 \right) \cdot \hat{K}}{\left( \hat{P}_1^+ \cdot \hat{K} \right) \left( \hat{P}_2^+ \cdot \hat{K} \right)} \left( \frac{1}{P_1^+ \cdot \hat{K}} + \frac{1}{P_2^+ \cdot \hat{K}} \right) \]

\[ + \frac{K_{\mu} \left( k \hat{K} - 2m_f \right) \left( \hat{P}_1^+ + Q_1 \right) \cdot \hat{K}}{\left( \hat{P}_1^+ \cdot \hat{K} \right) \left( \hat{P}_2^+ \cdot \hat{K} \right)} \left( \frac{1}{P_1^+ \cdot \hat{K}} + \frac{1}{P_2^+ \cdot \hat{K}} \right) \tag{87} \]

\[ \text{V. LEADING ORDER SUSCEPTIBILITIES} \]

The second-order QNS is the response of the quark number density with infinitesimal change in the quark chemical potential. Mathematically, the diagonal QNS can be written in terms of number density as

\[ \chi_{ff}(T) = \frac{\partial \mathcal{N}_f(T, \mu_f, m_f)}{\partial \mu_f} \bigg|_{\mu_f \to 0}, \tag{88} \]

where \( \mathcal{N}_f(T, \mu_f, m_f) \) represents the quark-number density for the quark flavor \( f \).

Now, the quark number density \( \mathcal{N}_f(T, \mu_f, m_f) \) in terms of the dressed fermion propagators \([16, 17]\) is given by

\[ \mathcal{N}_f(T, \mu_f, m_f) \]

\[ = -2N_c \int \frac{d^4 P}{(2\pi)^4} \partial_{\mu_f} \text{Tr} \left[ \text{Im log} \left( \gamma_0 iS^{-1}(P) \right) - \text{Im} (\gamma_0 \Sigma(P)) \text{Re} (iS(P)\gamma_0) \right] \]

\[ = -4N_c \int \frac{d^4 P}{(2\pi)^4} \partial_{\mu_f} \left[ \text{Im} \log D_+(p_0, p) + \text{Im} \log D_-(p_0, p) + \text{Im} [p_0 - A_0(p_0, p)] \text{Re} \left[ \frac{1}{D_+(p_0, p)} + \frac{1}{D_-(p_0, p)} \right] \right] \]

\[ - \text{Im} [p - A_+(p_0, p)] \text{Re} \left[ \frac{A_+(p_0, p)}{D_+(p_0, p)} - \frac{A_+(p_0, p)}{D_-(p_0, p)} \right] + \text{Im} [m_f - A_m(p_0, p)] \text{Re} \left[ \frac{A_+(p_0, p)}{D_+(p_0, p)} - \frac{A_+(p_0, p)}{D_-(p_0, p)} \right]. \tag{89} \]
So, the second-order diagonal QNS can be obtained as

$$\chi_{ff}(T) = -\lim_{\mu_f \to 0} 4N_c \int \frac{d^4 p}{(2\pi)^4} \frac{\partial^2 n_F^+}{\partial p_T^2}$$

\[
\times \left\{ \text{Im} \log D_+(p_0, p) + \text{Im} \log D_-(p_0, p) + \text{Im}[p_0 - A_0(p_0, p)] \text{Re} \left[ \frac{1}{D_+(p_0, p)} + \frac{1}{D_-(p_0, p)} \right] \\
- \text{Im}[p_0 - A_s(p_0, p)] \text{Re} \left[ \frac{A_s(p_0, p)}{D_+(p_0, p)} - \frac{A_s(p_0, p)}{D_-(p_0, p)} \right] + \text{Im}[m_f - A_m(p_0, p)] \text{Re} \left[ \frac{A_m(p_0, p)}{D_+(p_0, p)} - \frac{A_m(p_0, p)}{D_-(p_0, p)} \right] \right\}. \tag{90}
\]

Note that the $\mu_f$ derivative is only applied to the explicit chemical potential dependence as discussed in [16, 17]. Now, the first two terms within curly brackets in Eq. (90) give quasi-particle contribution to the QNS and can be obtained as

$$\chi_{ff}^{QP}(T, m_f) = \lim_{\mu_f \to 0} N_c \sum_f \int \frac{p^2 dp}{\pi^2} \frac{\partial^2}{\partial p_T^2} \left[ T \log \left( 1 + e^{-|\omega_+ + \mu_f|/T} \right) + T \log \frac{1 + e^{-|\omega_- + \mu_f|/T}}{1 + e^{-|\omega_- - \mu_f|/T}} \right]$$

\[
+ T \log \left( 1 + e^{-|\omega_+ - \mu_f|/T} \right) + T \log \frac{1 + e^{-|\omega_- - \mu_f|/T}}{1 + e^{-|\omega_- + \mu_f|/T}} \right]
= 2N_c \beta \int \frac{p^2 dp}{\pi^2} \left[ \frac{e^{\beta \omega_-}}{(e^{\beta \omega_-} + 1)^2} + \frac{e^{\beta \omega_+}}{(e^{\beta \omega_+} + 1)^2} - \frac{e^{\beta k}}{(e^{\beta k} + 1)^2} \right]. \tag{91}
\]

We identify the remaining contribution as the Landau-damping term that appears from the imaginary parts of the logarithmic terms within $A_0(p_0, p), A_s(p_0, p), A_m(p_0, p)$ and is obtained as

$$\chi_{ff}^{LD}(T, m_f) = -N_c \beta^2 \int \frac{p^2 dp}{\pi^2} \int_0^\infty d\omega \left[ \frac{1}{e^{\omega/T} - 1} \right]$$

\[
\times \left\{ \arg D_+(p_0, p) + \arg D_-(p_0, p) + \text{Im}[p_0 - A_0(p_0, p)] \text{Re} \left[ \frac{1}{D_+(p_0, p)} + \frac{1}{D_-(p_0, p)} \right] \\
+ \text{Im}[A_s(p_0, p) - p] \text{Re} \left[ \frac{A_s(p_0, p)}{\sqrt{A_s^2(p_0, p) + A_m^2(p_0, p)}} \left( \frac{1}{D_+(p_0, p)} - \frac{1}{D_-(p_0, p)} \right) \right] \\
- \text{Im}[A_m(p_0, p) - m_f] \text{Re} \left[ \frac{A_m(p_0, p)}{\sqrt{A_s^2(p_0, p) + A_m^2(p_0, p)}} \left( \frac{1}{D_+(p_0, p)} - \frac{1}{D_-(p_0, p)} \right) \right] \right\}. \tag{92}
\]

Total contribution can be written as a sum of QP and LD contribution as

$$\chi_{ff}(T, m_f) = \chi_{ff}^{QP}(T, m_f) + \chi_{ff}^{LD}(T, m_f). \tag{93}$$

Now, second-order BNS can be related with second-order QNSs as

$$\chi^B(T, \mathbf{m}) = \frac{1}{9} \left[ \chi_{uu} + \chi_{dd} + \chi_{ss} + 2\chi_{ud} + 2\chi_{ds} + 2\chi_{us} \right]. \tag{94}$$

At vanishing chemical potential, second-order off-diagonal susceptibilities are zero. Additionally, we consider the masses of $u$ and $d$ quarks as same. So, Eq. (94) becomes

$$\chi^B(T, \mathbf{m}) = \frac{1}{9} \left[ 2\chi_{uu} + \chi_{ss} \right]. \tag{95}$$

In Fig. (8), we compare the second-order light and strange quark susceptibilities (left panel) and also light quark susceptibility and three-times of baryon number susceptibility (right panel) with available lattice data from Wuppertal- Budapest group [20]. In this figure we use one-loop running coupling

$$\alpha_s = \frac{12\pi}{11N_c - 2N_f} \ln(A^2/A_{MS}^2), \tag{96}$$
where we use $\Lambda_{\text{MS}} = 176 \text{ MeV}$ [4]. The band in Fig. (8) indicates the sensitivity of susceptibilities with renormalization scale and we use the values of the renormalization scale $\pi T < \Lambda < 4\pi T$.

VI. CONCLUSION AND OUTLOOK

In this article we have calculated all the QCD $N$-point functions such as the quark and gluon propagators, three- and four-point quark-gluon vertices at finite quark mass within hard thermal loop approximation. Resulting $N$-point functions satisfy the Ward identity with corresponding $(N - 1)$-point functions. We have also calculated second-order quark and baryon number susceptibility within hard thermal loop approximation. In massless perturbative calculations one can not distinguish between second order quark and baryon number susceptibilities. Considering finite strange-quark mass, for the first time in any resummed perturbative framework, we are able to distinguish them within any (resummed)-perturbative treatment. The resulting quark and baryon number susceptibilities are $(5-10)\%$ lower than the WB group lattice data at high temperature. In the low-temperature region, we have gotten same trend as LQCD result that strange-quark number susceptibility and three-times of baryon-number susceptibility are lower than light-quark susceptibility. We have also found that due to the inclusion of finite quark mass, the renormalization dependent band gets reduced.

Looking to the future we want to extend the current calculation to the NNLO to study all the thermodynamical quantities. We expect that inclusion of finite quark masses to the NNLO thermodynamics will reduce the renormalization scale dependent band and will also improve the low temperature behavior of the existing massless results [4].

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