FLAT COSET DECOMPOSITIONS OF SKEW LATTICES

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Abstract. Skew lattices are non-commutative generalizations of lattices, and the cosets represent the building blocks that skew lattices are built of. As by Leech’s Second Decomposition Theorem any skew lattice embeds into a direct product of a left-handed skew lattice by a right-handed one, it is natural to consider the so called flat coset decompositions, i.e. decompositions of a skew lattice into right and left cosets, thus finding the smallest atoms that compose the structure.

INTRODUCTION

Skew lattices can be understood either as non-commutative generalizations of lattices or as double bands, where by a band we refer to a semigroup of idempotents. Although skew lattices were introduced already by Jordan in [12], and studied later by Cornish [1], the systematical study of the modern skew lattice theory began in 1989 by Leech’s paper [15], where Leech’s First and Second Decomposition Theorems were proven, revealing the structure of a skew lattice, see Section 2 for exact statements of the theorems. Both decomposition theorems were motivated by their analogues in the theory of bands.

However, in addition to the outer structure that is revealed by the two decomposition theorem, skew lattices also possess an interesting inner structure, the so called coset structure. Already in the 1989 foundation paper [15] certain aspects of the coset structure of a skew lattice were analyzed, however it was fully explored in [17] where Leech studied what he referred to as the global geometry of skew lattices. The coset structure was later used in [6] and [7] to characterize certain sub-varieties of skew lattices, and in [14] for the purpose of studying the distributivity of skew lattices, an approach proposed in [23].

By Leech’s First Decomposition Theorem a skew lattice is a lattice of its Green’s \( D \)-classes. The coset structure provides us with an introspective of how different \( D \)-classes are “glued” together into a lattice thus providing important additional information. Given...
a pair of comparable \( D \)-classes each of the two classes induces a partition of the other class, and the blocks of these partitions are called cosets. The internal structure of a skew lattice is described by the coset decompositions introduced in \[17\] that reveal the interplay of a pair of comparable \( D \)-classes.

A skew lattice is called right-handed if Green’s relation \( L \) is trivial, and it is called left-handed if Green’s relation \( R \) is trivial, see Section 2 for precise definitions. By Leech’s Second Decomposition Theorem any skew lattice \( S \) embeds into the direct product of a right-handed (called the right factor of \( S \)) and a left-handed skew lattice (called the left factor of \( S \)). A skew lattice is called flat if it is either right- or left-handed. Flat skew lattices thus form examples of skew lattice that are general enough to reveal structural properties of skew lattices. More precisely, it was proven in \[11\] that a skew lattice satisfies any identity or equational implication satisfied by both its left factor and its right factor. The flat orders that we study in Section 3 were introduced in \[23\], and were motivated by Michael Kinyon’s talk that was held at the Workshop on Algebra and Logic, University of Lisbon, 2009.

In Section 4 we explore the flat coset structure of a skew lattice and its connections to the study of the coset laws for important skew lattice properties, such as symmetry, cancellation and normality. This approach enables us to classify certain varieties of skew lattices. The results of Section 4 were motivated by the earlier studies of \[17\], \[5\], \[21\], \[7\], \[22\] and \[23\]. We demonstrate the impact of the study of the flat coset structure in the case of skew lattices of matrices, following the work of \[8\], \[9\] and \[4\].

Basic knowledge on semigroup theory and lattice theory can be found in \[20\] and \[10\], respectively.

1. Preliminaires

1.1. Order structure. A skew lattice is an algebra \((S; \lor, \land)\), where \(\land\) and \(\lor\) are idempotent and associative binary operations, such that the absorption laws \(x \land (x \lor y) = x = (y \lor x) \land x\) and \(x \lor (x \land y) = x = (y \land x) \lor x\) are satisfied. A band is a semigroup of idempotents, while a semilattice is a commutative band. When \(S\) is a commutative semigroup, the set \(E(S)\) of all idempotents in \(S\) is a semilattice under the semigroup multiplication. When \(S\) is not commutative, \(E(S)\) needs not be closed under multiplication \[11\]. Recall that a band is right [left] regular if it satisfies the identity \(xyx = yx \ [xyx = xy]\). A band is regular if it satisfies \(xyxx = xyxx\), and is rectangular if it satisfies \(xyxx = xyxx\). Skew lattices can be seen as double regular bands as the band reducts \((S, \land)\) and \((S, \lor)\) are regular. A skew lattice \(S\) is rectangular if and only if \(x \land y = y \lor x\) holds or equivalently if its band reducts \((S, \land)\) and \((S, \lor)\) are rectangular. Moreover, given sets \(L\) and \(R\) the direct product \(L \times R\) forms a rectangular skew lattice under the operations \((x, y) \lor (x', y') = (x', y)\) and \((x, y) \land (x', y') = (x, y')\), and any rectangular skew lattice is isomorphic to \(L \times R\) for some sets \(L, R\), cf. \[15\].

The following technical result was proven in \[7\]. We shall make frequent use of this result in the remainder of the paper.
Lemma 1.1.1. [22] Let $S$ be a skew lattice and let $x_1, x_2, u, v$ in $S$ be such that $u \preceq x_i \preceq v$ for $i \in \{1, 2\}$. Then

$$x_1 \land v \land x_2 = x_1 \land x_2 \text{ and } x_1 \lor u \lor x_2 = x_1 \lor x_2.$$  

Green’s relations are five equivalence relations characterizing the elements of a semigroup in terms of the principal ideals they generate. When $S$ is a semigroup we set $S^1 = S$ if $S$ has an identity element, and $S^1 = S \cup \{1\}$ otherwise, with 1 being an element disjoint from $S$, and the operation is extended to $S^1$ to set $x \cdot 1 = x = 1 \cdot x$ for all $x \in S^1$. Given $x, y \in S$ the Green’s relations $L, R$ and $J$ are defined by:

$$xRy \iff xS^1 = yS^1; \quad xLy \iff S^1x = S^1y; \quad xJy \iff S^1xS^1 = S^1yS^1.$$  

Finally, Green’s relations $D$ and $H$ are defined by $D = R \lor L (= R \circ L = L \circ R)$ and $H = R \land L (= R \cap L)$. As $R$- and $L$-classes are contained in $D$-classes, and an $R$-class intersects an $L$-class in an $H$-class, $D$-classes are often visualized as ‘eggboxes’ with $R$-classes being represented by rows, $L$-classes by columns and $H$-classes by individual cells of the eggbox, cf. Figure 1 below.

If $S$ is a band then $D = J$ and $H = \Delta_S = \{(x, x) : x \in S\}$. Moreover, in the case of a band the definitions of Green’s relations simplify as follows: $xRy$ iff $xy = y$ and $yx = x$; $xLy$ iff $xy = x$ and $yx = y$; $xJy$ iff $xxy = x$ and $yxy = y$. If $S$ is a skew lattice we define the Green’s relations on $S$ to be the Green’s relations on the band reduct $(S, \land)$. It follows that $xRy$ iff $x \land y = y$ and $y \land x = x$, or dually $x \lor y = x$ and $y \lor x = y$; $xLy$ iff $x \lor y = y$ and $y \lor x = x$, or dually $x \land y = y$ and $y \land x = x$; $xJy$ iff $x \land y \land x = x$ and $x \lor y \lor x = x$. Right-handed skew lattices are the skew lattices for which $R = D$, while left-handed skew lattices are determined by $L = D$ [18].

Given a skew lattice $S$ the natural partial order $\preceq$ is defined by $x \preceq y$ if $x \land y = y = y \land x$ (or dually $x \lor y = x = y \lor x$), and the natural preorder $\precsim$ is defined by $x \precsim y$ if $y \lor x \land y = y$, or dually $x \lor y \lor x = x$. Observe that $xJy$ iff $x \preceq y$ and $y \preceq x$. Relation $D$ is often referred to as the natural equivalence.

Theorem 1.1.2 ([15] Leech’s First Decomposition Theorem). Let $S$ be a skew lattice. Then $D$ is a congruence, $S/D$ is the maximal lattice image of $S$ and $D$-classes of are maximal rectangular sub-skew lattices of $S$.

1.2. Rectangular structure. A primitive skew lattice is a skew lattice that has exactly two (comparable) $D$-classes. A skew diamond $\{J > A, B > M\}$ is a skew lattice that has two incomparable $D$-classes, $A$ and $B$, their join class $J = A \lor B$ and their meet class $M = A \land B$. Given a skew lattice with comparable $D$-classes $X > Y$ and $x \in X$ there exists $y \in Y$ such that $x \geq y$, and dually given $y \in Y$ there exists $x \in X$ such that $x \geq y$. If $A$ and $B$ are (incomparable) $D$-classes, $J = A \lor B$ and $M = A \land B$ then for every $a \in A$ there exists $b \in B$ such that $a \lor b = b \lor a$ in $J$ and $a \land b = b \land a$ in $M$. Moreover, in a skew diamond $\{J > A, B > M\}$ we obtain $J = \{a \lor b : a \in A, b \in B \}$ and $M = \{a \land b : a \in A, b \in B \}$ (cf. [15]).
In Figure 1 we see the ‘eggbox’ corresponding to a rectangular skew lattice, i.e., a single $D$-class. As relation $H$ is trivial in skew lattices (and bands), each cell is a singleton and hence represents an individual element of a rectangular skew lattice. Given $x$ and $y$, $x \land y$ is the unique element in the row of $x$ and the column of $y$, while $x \lor y$ is the unique element in the column of $x$ and the row of $y$. The eggbox corresponding to a $D$-class of a flat skew lattice $S$ is either a single row (if $S$ right-handed) or a single column (if $S$ left-handed).

**Theorem 1.2.1** ([15] Leech’s Second Decomposition Theorem). Given a skew lattice $S$ relations $L$ and $R$ are always congruences. Moreover, $S/L$ is the maximal right-handed image of $S$, $S/R$ is the maximal left-handed image of $S$; and the natural projections $S \rightarrow S/L$ and $S \rightarrow S/R$ together yield an isomorphism of $S$ with the fibered product $S/R \times S/L = \{ (x, y) : x \in S/R, y \in S/L, p(x) = q(y) \}$ where $p : S/L \rightarrow S/D$ and $q : S/R \rightarrow S/D$ are natural homomorphisms.

Any sub-lattice $T$ of $S$ intersects each $D$-class of $S$ in at most one point. If $T$ meets each $D$-class of $S$ in exactly one point, then $T$ is called a **lattice section** of $S$. As such, it is a maximal sub-lattice that is also an internal copy inside $S$ of the maximal lattice image $S/D$ [17]. A lattice section of $S$ (if it exists) is therefore isomorphic to $S/D$. As any skew lattice $S$ is embedded in the product $S/R \times S/L$, joint properties of $S/R$ and $S/L$ are often passed on to $S$, and conversely. In particular, $S/R$ and $S/L$ belong to a variety $V$ if and only if $S$ does [5].

**Theorem 1.2.2.** [3] A skew lattice $S$ has a lattice section $S_0$ if and only if one of the following equivalent statements holds:

(i) $S$ has a left handed sub-skew lattice $S'$ and a right handed sub-skew lattice $S''$ both of which meet every $D$-class in $S$.

(ii) Sub-skew lattices $S_L$ and $S_R$ exist whose intersection with any $D$-class is an $L$-class [respectively, $R$-class] of $S$.

When the conditions of Theorem 1.2.2 hold, with $S_L$ and $S_R$ as given in (ii), then, the natural epimorphisms $S \rightarrow S/R$ and $S \rightarrow S/L$ induce, under restriction, isomorphisms of $S_L$ with $S/R$ and $S_R$ with $S/L$. Every $x \in S$ factors uniquely as $x = x' \land x''$ with $x' \in S_L \cap D_x$, $x'' \in S_R \cap D_x$. Under this decomposition, the operations are defined.
componentwise, that is, \((x' \land x'') \land (y' \land y'') = (x' \land y') \land (x'' \land y'')\) and \((x' \lor x'') \lor (y' \lor y'') = (x' \lor y') \lor (x'' \lor y'')\). The functions \(\pi_L : S \to S_L\) and \(\pi_R : S \to S_R\) defined by \(\pi_L(x) = x'\) and \(\pi_R(x) = x''\) are retractions of \(S\) upon \(S_L\) and \(S_R\), respectively, and the commuting composite \(\pi_L \pi_R = \pi_R \pi_L\) is a retraction of \(S\) upon \(S_0\). Moreover, \(\ker(\pi_L) = R\), \(\ker(\pi_R) = L\) and \(\ker(\pi_L \pi_R) = D\). Furthermore, \(S\) is the internal fibred product of \(S_L\) and \(S_R\), the internal left and internal right factors of \(S\), respectively, describing the Inner Kimura Decomposition introduced in \([3]\). We shall call flat skew lattices to \(S/R\) and \(S/L\).

1.3. Coset structure. Consider a skew lattice \(S\) with \(D\)-classes \(A > B\). Given \(b \in B\), the subset \(A \land b \land A = \{a \land b \land a \mid a \in A\}\) of \(B\) is said to be a coset of \(A\) in \(B\) (or an \(A\)-coset in \(B\)). Similarly, a coset of \(B\) in \(A\) (or a \(B\)-coset in \(A\)) is any subset \(B \lor a \lor B = \{b \lor a \lor b \mid b \in B\}\) of \(A\), for a fixed \(a \in A\). On the other hand, given \(a \in A\), the image set of \(a\) in \(B\) is the set \(a \land B \land a = \{a \land b \land a \mid b \in B\} = \{b \in B \mid b < a\}\). Dually, given \(b \in B\) the set \(b \lor A \lor b = \{a \in A : b < a\}\) is the image set of \(b\) in \(A\).

Theorem 1.3.1. \([7]\) Let \(S\) be a skew lattice with comparable \(D\)-classes \(A > B\). Then, \(B\) is partitioned by the cosets of \(A\) in \(B\) and the image set of any element \(a \in A\) is a transversal of the cosets of \(A\) in \(B\); dual remarks hold for any \(b \in B\) and the cosets of \(B\) in \(A\) that determine a partition of \(A\). Moreover, any coset \(B \lor a \lor B\) of \(B\) in \(A\) is isomorphic to any coset \(A \land b \land A\) of \(A\) in \(B\) under a natural bijection \(\varphi\) defined implicitly for any \(a \in A\) and \(b \in B\) by: \(x \in B \lor a \lor B\) corresponds to \(y \in A \land b \land A\) if and only if \(x \geq y\). Furthermore, the operations \(\land\) and \(\lor\) on \(A \lor B\) are determined jointly by the coset bijections and the rectangular structure of each \(D\)-class.

Let \(S\) be a skew lattice with comparable \(D\)-classes \(A > B\) and let \(y, y' \in B\). Then \(A \land y \land A = A \land y' \land A\) iff for all \(x \in A\) the equality \(x \land y \land x = x \land y' \land x\) holds. Dual results hold, having a similar statement (cf. \([7]\)). All cosets and image sets are rectangular sub skew lattices. Furthermore, all coset bijections are isomorphisms between cosets.

A skew lattice is said to be symmetric if for all \(x, y \in S\), \(x \land y = y \land x\) holds if and only if \(x \lor y = y \lor x\) holds. \(S\) is called upper symmetric if \(x \land y = y \land x\) implies \(x \lor y = y \lor x\); and \(S\) is called lower symmetric if \(x \lor y = y \lor x\) implies \(x \land y = y \land x\). Symmetric skew lattices with countably finite \(D\)-classes always have lattice sections (cf. \([13]\)). Finally, a skew lattice \(S\) is called cancellative if for all \(x, y, z \in S\), \(z \lor x = z \lor y\) and \(z \land x = z \land y\) impy \(x = y\), and \(x \lor z = y \lor z\) and \(x \land z = y \land z\) imply \(x = y\). Cancellative skew lattices are always symmetric, see \([5]\).

Theorem 1.3.2. \([7]\) Let \(S\) be a skew lattice. Then \(S\) is symmetric if and only if for all skew diamonds \(\{J > A, B > M\}\) and all \(m \in M, m' \in M, j, j' \in J\) then the following equivalences hold:

(i) \(J \land m \land J = J \land m' \land J\) if and only if \(A \land m \land A = A \land m' \land A\) and \(B \land m \land B = B \land m' \land B\), and

(ii) \(M \lor j \lor M = M \lor j' \lor M\) if and only if \(A \lor j \lor A = A \lor j' \lor A\) and \(B \lor j \lor B = B \lor j' \lor B\).
Remark 1.3.3. In [12] examples are given that show the independence between (i) and (ii) above. Furthermore, the skew lattices satisfying (i) correspond to the lower symmetric skew lattices, while the skew lattices satisfying (ii) correspond to the upper symmetric skew lattices as discussed in the proof of Theorem 3.5 in the paper [17], and expressed below:

Theorem 1.3.4. [17] Let $S$ be a skew lattice. Then, $S$ is lower symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $m \in M$, $(A \wedge m \wedge A) \cap (B \wedge m \wedge B) \subseteq J \wedge m \wedge J$. Dually, $S$ is upper symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $j \in J$, $(A \vee j \vee A) \cap (B \vee j \vee B) \subseteq M \vee j \vee M$. Moreover, the following hold:

(i) $S$ is lower symmetric if and only if given any skew diamond $\{ J > A, B > M \}$ in $S$ and any $m, m' \in M$, $J \wedge m \wedge J = J \wedge m' \wedge J \iff (A \wedge m \wedge A = A \wedge m' \wedge A$ and $B \wedge m \wedge B = B \wedge m' \wedge B)$;

(ii) $S$ is upper symmetric if and only if given any skew diamond $\{ J > A, B > M \}$ in $S$ and any $j, j' \in J$, $M \vee j \vee M = M \vee j' \vee M \iff (A \vee j \vee A = A \vee j' \vee A$ and $B \vee j \vee B = B \vee j' \vee B)$.

Theorem 1.3.5. [7] Let $S$ be a quasi-distributive, symmetric skew lattice. Then $S$ is cancellative iff one (and hence both) of the following equivalent statements hold:

(i) given any skew diamond $\{ J > A, B > M \}$ in $S$ and any $x, x' \in A$, $M \vee x \vee M = M \vee x' \vee M$ holds if and only if $B \vee x \vee B = B \vee x' \vee B$ holds;

(ii) given any skew diamond $\{ J > A, B > M \}$ in $S$ and any $x, x' \in A$, $B \wedge x \wedge B = B \wedge x' \wedge B$ holds if and only if $J \wedge x \wedge J = J \wedge x' \wedge J$ holds.

Recall from [12] that a skew lattice $S$ is right cancellative if for all $x, y, z \in S$ the pair of equalities $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$ implies $x = y$. Left cancellative skew lattices are defined dually. A skew lattice is simply cancellative if for all $x, y, z \in S$ the pair of equalities $x \vee z \vee x = y \vee z \vee y$ and $x \wedge z \wedge x = y \wedge z \wedge y$ implies $x = y$. Clearly, cancellative skew lattices are the ones that are simultaneously right cancellative and left cancellative. If $S$ is symmetric then right cancellation is equivalent to left cancellation and thus coincides with (full) cancellation. Moreover, a skew lattice is right cancellative skew lattices if and only if it is simply cancellative and simultaneously right upper symmetric and left lower symmetric. Dually, a right cancellative skew lattice is a simply cancellative skew lattice that is simultaneously left lower symmetric and right upper symmetric (cf. [12]).

Recall that a skew lattice is said to be normal if it satisfies the identity $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$ and, dually, it is named conormal if it satisfies $x \vee y \vee z \vee w = x \vee z \vee y \vee w$, cf. [16].

Theorem 1.3.6. [22] Let $S$ be a skew lattice. Then $S$ is normal iff for each pair of comparable $\mathcal{D}$-classes $A > B$ in $S$, the class $B$ is an entire coset of $A$ in $B$. That is,

$$A \wedge x \wedge A = A \wedge x' \wedge A$$

holds for all $x, x' \in B$. Dually, $S$ is conormal iff $B \vee x \vee B = B \vee x' \vee B$ holds for all pairs of comparable $\mathcal{D}$-classes $A > B$ in $S$ and all $x, x' \in A$. 
2. Flat order structure

2.1. Right and left preorders. In the following paragraphs we will discuss two flat preorders. They are the right and left weak versions of the natural order in a skew lattice and relate with the natural preorder also there defined. They are essential for the understanding of the flat coset structure defined later in this paper.

Definition 2.1.1. Let $S$ be a skew lattice and consider in it the following relations:

- $x \leq_L y$ if $x = x \land y$, or dually $y = x \lor y$;
- $x \leq_R y$ if $x = y \land x$, or dually $y = y \lor x$.

The relation $\leq_L$ is a preorder and $x \leq_L y$ together with $y \leq_L x$ is equivalent to $x \leq_L y$. A similar remark holds for $\leq_R$.

Remark 2.1.2. Clearly, $x \leq y$ implies both $x \leq_L y$ and $x \leq_R y$, and either of the two further implies $x \leq y$. Hence, $\leq$ (seen as a subset of $S \times S$) is contained both in $\leq_L$ and $\leq_R$. Moreover, both orders $\leq_L$, $\leq_R$ are contained in $\leq$. Thus, the four preorders are related as expressed by the following diagram:

\[
\begin{array}{ccc}
  x & \leq & y \\
 \downarrow & & \downarrow \\
 x \leq_R y & \quad & x \leq_L y \\
 \downarrow & & \downarrow \\
 x \leq y & & x \leq y \\
\end{array}
\]

The preorder $\leq_L$ [$\leq_R$] is a partial order in any right [left]-handed skew lattice since in this case it coincides with the natural partial order $\leq$. We call flat preorder to any right or left preorder. The following results capture the impact of the right order structure of skew lattices, in analogy to results established for the natural order. Dual results hold for a left order structure.

Lemma 2.1.3. Let $S$ be a skew lattice and let $x, y \in S$. Then $x \leq_R y$ together with $x \leq_L y$ implies $x = y$. Dually, $x \leq_L y$ together with $x \leq_R y$ implies $x = y$.

Proof. The proof is direct, since $x \leq_R y$ implies $y = x \land y$, while $x \leq_L y$ implies $x \land y = x$. The dual statement has a similar proof. \qed

Lemma 2.1.4. Let $S$ be a skew lattice and $x, y \in S$. Then:

1. $x \leq_R y$ if and only if $x \in y \land S$;
2. $x \leq_L y$ if and only if $x \in S \land y$.

Proof. 1. Let $x \leq_R y$. Then $x = y \land x \in y \land S$. Conversely, assume $x \in y \land S$. Then $x = y \land u$ for some $u \in S$. Thus $y \land x = y \land y = x$ which implies $x \leq_R y$. A similar argumentation verifies 2. \qed
Proposition 2.1.5. Given any element $y$ in a skew lattice $S$, the sets $y \wedge S$ and $S \wedge y$ are subalgebras of $S$.

Proof. Take $u, v \in S \wedge y$. By Lemma 2.1.4, the elements $u$ and $v$ are of the form $u = y \wedge u$ and $v = y \wedge v$. Then

$$y \wedge u \wedge v = u \wedge v$$

and thus $u \wedge v \in y \wedge S$. Using Lemma 2.1.4 again in order to prove that also $u \vee v \in y \wedge S$ we need to show that $u \vee v \leq_R y$ which is equivalent to $y = y \vee u \vee v$. This is indeed the case as:

$$y \vee u \vee v = y \vee (y \wedge u) \vee (y \wedge v)$$

which equals $y$ by absorption. \hfill \qed

Lemma 2.1.6. Let $A$ and $B$ be comparable $R$-classes in a skew lattice $S$ such that $A \geq B$, and let $x \in A$, $y \in B$. Then:

(i) For each $a \in A$ there exists $b \in R_y$ such that $a \geq_L b$, and dually for each $b \in B$ there exists $a \in R_x$ such that $a \geq_L b$.

(ii) If $a \in A$ and $b \in B$ satisfy $a \geq_L b$ then $a \geq_L u$, for all $u \in L_b$; and dually, $v \geq_L b$, for all $v \in L_a$.

Similar remarks hold regarding the preorder $\geq_R$.

Proof. To prove the first statement of (i) let $a \in A$ and take $b = y \wedge a$. Then $b \wedge a = y \wedge a \wedge a = y \wedge a = b$ and thus $b \leq_L a$. On the other hand, $y \wedge b = y \wedge y \wedge a = y \wedge a = b$ and $b \wedge y = y \wedge a \wedge y = y$ which proves $b \in R_y$. To show (ii) let $a \in A$ and $b, u \in B$ such that $a \geq_L b$ and $u \in L_b$. Then $u = u \wedge b = u \wedge b \wedge a = u \wedge a$, due to the hypothesis. Hence, $a \geq_L u$. The dual statements have analogous proofs. \hfill \qed

Remark 2.1.7. In general, it is possible that $a \in A$ and $b, b' \in B$ exist such that $b \neq b'$, $a \geq_L b$ and $a \geq_L b'$. Later, in the following section, we shall see that, if $a \in A$ and $b \in B$ satisfy $a \geq_L b$ then there is a partition determined by $R_a$ on $R_b$ where $b = b'$ for all $b' \in R_b$ such that $b$ and $b'$ are in the same block and $a \geq_L b'$; and dually, if $a \in A$ and $b \in B$ are such that $a \geq_L b$, $R_b$ determines a partition of $R_a$ where $a = a'$ for all $a' \in R_a$ such that $a$ and $a'$ are in the same block and $a' \geq_L b$.

2.2. Right and left center. The notion of center is of great importance in the study of any algebra, relating with the variety of normal skew lattices. In the following we shall present the properties of elements in the right and in the left center. We will also show the natural relation of those flat centers with the full center.

Definition 2.2.1. Arbitrary elements $x, y$ of a skew lattice $S$ are said to meet-commute if $x \wedge y = y \wedge x$; and, dually, they join-commute if $x \vee y = y \vee x$. An element of $S$ is called a central element if it both meet- and join-commutes with all elements of the skew lattice. By a result of Leech [15] an element $x$ is central if and only if the $D$-class $D_x$ is trivial. The set of all central elements is called the center of a skew lattice and it forms a lattice. We shall denote the center of a skew lattice $S$ by $Z(S)$. We say that an element $x$ in a skew lattice $S$ is right-central if $R_x$ is trivial; and $x$ is left-central if $L_x$ is trivial. The following theorem yields several useful characterizations of left- [right-]central elements.
Theorem 2.2.2. Let $S$ be a skew lattice and $a \in S$. The following statements are equivalent:

(i) The element $a$ is right-central.
(ii) For all $b \in S$, $b \lor a = a \lor b \lor a$.
(iii) For all $b \in S$, $a \lor b = b \lor a \lor b$.
(iv) For all $b \in S$, $a \land b = a \land b \land a$.
(v) For all $b \in S$, $b \land a = b \land a \land b$.

Dually, the following statements are equivalent:

(vi) The element $a$ is left-central.
(vii) For all $b \in S$, $a \lor b = a \land b \lor a$.
(viii) For all $b \in S$, $b \land a = b \lor a \land b$.
(ix) For all $b \in S$, $a \land b = b \land a \land b$.
(x) For all $b \in S$, $b \land a = a \land b \land a$.

Proof. A direct calculation verifies the equivalences (ii) $\iff$ (iii) and (iv) $\iff$ (v). Let us now prove the equivalence (i) $\iff$ (ii). First assume that (i) holds and let $b \in S$. Assume without loss of generality that $D_{b \lor a} \neq D_a$. By Lemma 2.1.6 for the comparable $D$-classes $D_{b \lor a} > D_a$ there exists $u \in R_a$ such that $u \leq_L b \lor a$. But, by the assumption, $u \in R_a$ implies $u = a$. Hence $a \leq_L b \lor a$ and thus $a \lor b \lor a = b \lor a$. To prove the converse, assume that (ii) holds and let $u \in R_a$ be arbitrary. Then $u = u \lor a$ and $a = a \lor u$, which by (ii) and the fact that $a \not\in R u$ implies:

$$u = u \lor a = a \lor u \lor a = a.$$ 

That proves (i). The equivalence (i) $\iff$ (iv) is proved in a similar fashion, and dual argumentation verifies the equivalence of conditions (vi)–(x). □

Definition 2.2.3. The set of all right-central elements in $S$ is called the right-center of $S$ and will be denoted by $Z_R(S)$; while the set of all left-central elements in $S$ is called the left-center of $S$ and will be denoted by $Z_L(S)$.

Proposition 2.2.4. The right-center $Z_R(S)$ is a left-handed skew lattice and the left-center $Z_L(S)$ is a right-handed skew lattice. Moreover, if $S$ is right-handed then $Z_R(S) = Z(S)$ and $Z_L(S) = S$; if $S$ is left-handed then $Z_R(S) = S$ and $Z_L(S) = Z(S)$. Furthermore,

$$Z(S) = Z_L(S) \cap Z_R(S).$$

Proof. Let $x, y \in Z_R(S)$. To see that $Z_R(S)$ is closed under the operations $\land$ and $\lor$ we need to show that $x \land y$ and $x \lor y$ are right-central. We shall make use of Theorem 2.2.2 (iv) in order to prove that $x \land y$ is right-central. Let $b \in S$ be arbitrary. Then, using that both $x$ and $y$ are right-central we get

$$x \land y \land b \land x \land y = x \land y \land b \land y = x \land y \land b,$$

and it follows by Theorem 2.2.2 (iv) that $x \land y$ is right-central. We prove the right-centralness of $x \lor y$ in a similar fashion, using Theorem 2.2.2 (iii). Hence $Z_R(S)$ is a skew lattice. The left-handedness of $Z_R(S)$ follows immediately from Theorem 2.2.2 The
assertion for \( Z_L(S) \) is proved dually. Next, assume that \( S \) is right-handed. Then \( \mathcal{D} = \mathcal{R} \) and thus \( \mathcal{R}_x = \{ x \} \) is equivalent to \( \mathcal{D}_x = \{ x \} \), which proves \( Z_R(S) = Z(S) \). Moreover, in this case all elements \( a \in S \) satisfy the condition (ix) of Theorem 2.2.2 and thus \( Z_L(S) = S \). Again, the assertion about left-handed skew lattices is proved dually. Finally, \( x \in Z_L(S) \cap Z_R(S) \) iff \( \mathcal{R}_x = \{ x \} = \mathcal{L}_x \) iff \( \mathcal{D}_x = \{ x \} \) iff \( x \in Z(S) \).

2.3. A classification for symmetry. The variety of symmetric skew lattices is of great importance for the study of skew lattices. We will look at right [left] upper [lower] symmetric skew lattices and discuss several characterizations of these by identities.

**Definition 2.3.1.** We say that a skew lattice \( S \) is right symmetric if its right factor \( S/\mathcal{L} \) is symmetric, and we say that \( S \) is left symmetric if its left factor \( S/\mathcal{R} \) is symmetric. Moreover, \( S \) is right upper symmetric if \( S/\mathcal{L} \) is upper symmetric; \( S \) is left upper symmetric if \( S/\mathcal{R} \) is upper symmetric; \( S \) is right lower symmetric if \( S/\mathcal{L} \) is lower symmetric; and \( S \) is left lower symmetric if \( S/\mathcal{R} \) is lower symmetric.

**Theorem 2.3.2.** ([5]) Given a skew lattice \( S \):

i) \( S \) is right upper symmetric iff it satisfies the identity \( x \lor y \lor x = (y \land x) \lor y \lor x \).

ii) \( S \) is left upper symmetric iff it satisfies the identity \( x \lor y \lor x = x \lor y \lor (x \land y) \).

iii) \( S \) is right lower symmetric iff it satisfies the identity \( x \land y \land x = x \land y \land (x \lor y) \).

iv) \( S \) is left lower symmetric iff it satisfies the identity \( x \land y \land x = (y \lor x) \land y \land x \).

**Definition 2.3.3.** A skew lattice is upper symmetric if it is simultaneously right and left upper symmetric, and it is lower symmetric if it is simultaneously right and left lower symmetric. A skew lattice is both upper and lower symmetric if and only if it is symmetric (cf. [5]).

**Proposition 2.3.4.** Let \( S \) be a skew lattice. Then:

i) \( S \) is right upper symmetric iff it satisfies \( x \lor y \lor x = (x \land y \land x) \lor y \lor x \).

ii) \( S \) is left upper symmetric iff it satisfies \( x \lor y \lor x = x \lor y \lor (x \land y \land x) \).

iii) \( S \) is right lower symmetric iff it satisfies \( x \land y \land x = x \land y \land (x \lor y \lor x) \).

iv) \( S \) is left lower symmetric iff it satisfies \( x \land y \land x = (x \lor y \lor x) \land y \land x \).

**Proof.** We prove (i) as the proofs of (ii)–(iv) are similar. Assume that \( S \) is right upper symmetric and let \( x, y \in S \). Then absorption implies \( x \lor y \lor x = (x \land y \land x) \lor (x \lor y \lor x) \). Using this and right upper symmetry yields

\[
x \lor y \lor x = (x \land y \land x) \lor (x \lor y \lor x) = (x \land y \land x) \lor (y \land x) \lor y \lor x = (x \land y \land x) \lor y \lor x,
\]

where the final equality follows by Lemma 1.1.1. To prove the converse, assume that \( S \) satisfies \( x \lor y \lor x = (x \land y \land x) \lor y \lor x \). Then

\[
x \lor y \lor x = (x \land y \land x) \lor y \lor x = (x \land y \land x) \lor (y \land x) \lor y \lor x = (y \land x) \lor y \lor x,
\]

where the second equality follows by Lemma 1.1.1 and the final one by absorption. \( \square \)
Definition 2.3.5. Observe, that given elements \( x, y \) in a skew lattice \( S \), \( x \lor y = y \lor x \lor y \) is equivalent to \( y \lor x = x \lor y \lor x \), because of the idempotency of \( \lor \). We say that \( x \) and \( y \) right-join commute if \( x \lor y = y \lor x \lor y \), and we say that they right-meet commute if \( x \land y = x \land y \lor x \). Similarly, \( x \) and \( y \) are said to left-join commute if \( x \lor y = x \lor y \lor x \), and left-meet commute if \( x \land y = y \land x \land y \).

Proposition 2.3.6. Let \( S \) be a skew lattice. Then:

(i) \( S \) being right upper symmetric is equivalent to \( \forall x, y \in S: (x \text{ and } y \text{ right-meet commute implies } x \text{ and } y \text{ right-join commute}) \).

(ii) \( S \) being left upper symmetric is equivalent to \( \forall x, y \in S: (x \text{ and } y \text{ left-meet commute implies } x \text{ and } y \text{ left-join commute}) \).

(iii) \( S \) being right lower symmetric is equivalent to \( \forall x, y \in S: (x \text{ and } y \text{ right-join commute implies } x \text{ and } y \text{ right-meet commute}) \).

(iv) \( S \) being left lower symmetric is equivalent to \( \forall x, y \in S: (x \text{ and } y \text{ left-join commute implies } x \text{ and } y \text{ left-meet commute}) \).

Proof. We prove (i); the other proofs are similar. Assume that \( S \) is right upper symmetric and let \( x, y \) be right-meet commuting elements. Hence \( y \land x \land y = y \land x \) and thus using Proposition 2.3.4 we obtain

\[
y \lor x \lor y = (y \land x \land y) \lor x \lor y = (y \land x) \lor x \lor y = x \lor y
\]

by absorption, which proves that \( x \) and \( y \) also right join commute. To prove the converse, assume that for all pairs of elements in \( S \) right meet commutativity implies right join commutativity. Take any \( x, y \in S \) and set \( a = x \), \( b = (x \land y \land x) \lor y \). We claim that \( a \) and \( b \) right meet commute. We have

\[
a \land b = x \land ((x \land y \land x) \lor y).
\]

Using Lemma 1.1.1 we can insert \( y \land x \) in the middle of the above expression to obtain

\[
a \land b = x \land (y \land x) \land ((x \land y \land x) \lor y)
\]

which equals \( x \land y \land x \) by absorption. But then also

\[
a \land b \land a = (x \land y \land x) \land x = x \land y \land x,
\]

which proves that \( a \) and \( b \) right meet commute. The assumption now implies that \( a \) and \( b \) right join commute, i.e.

\[
(x \land y \land x) \lor y \lor x = x \lor (x \land y \land x) \lor y \lor x = x \lor y \lor x,
\]

where the latter equality follows by absorption. Right upper symmetry now follows by Proposition 2.3.4. \( \square \)

Corollary 2.3.7. Let \( S \) be a skew lattice. Then:

(i) \( S \) being right symmetric is equivalent to \( \forall x, y \in S: (x \text{ and } y \text{ right-join commute iff } x \text{ and } y \text{ right-meet commute}) \).

(ii) \( S \) being left symmetric is equivalent to \( \forall x, y \in S: (x \text{ and } y \text{ left-join commute iff } x \text{ and } y \text{ left-meet commute}) \).
3. Flat coset structure

3.1. On left and right cosets. We are now ready to describe the coset structure of a skew lattice. To do so, we shall introduce flat cosets and construct a decomposition theorem as Theorem 3.1.6, the flat version of Leech’s coset decomposition Theorem 1.3.1. We shall also clarify the relation of flat cosets with cosets on flat skew lattices.

**Definition 3.1.1.** A right coset of $A$ in $B$ is any set of the form $b \wedge A$, where $b \in B$. Similarly, a right coset of $B$ in $A$ is any set of the form $B \vee a$ for $a \in A$. Given $b \in B$, the right image set of $b$ in $A$ is the set $b \vee A = \{a \in A : b \geq_L a\}$. Dually, given any $a \in A$, the right image set of $a$ in $B$ is the set $B \wedge a = \{b \wedge a : b \in B\}$. Left cosets and left image sets are defined analogously. We say that a coset is flat whenever it is a right coset or a left coset.

**Proposition 3.1.2.** Let $S$ be a skew lattice with comparable $D$-classes $A > B$. Then, $b \vee A = \{a \in A : a \geq_L b\}$ and $B \wedge a = \{b \in B : a \geq_L b\}$. Similar remarks hold for left image sets.

**Proof.** Let $x \in A$ and consider $b \vee x \in b \vee A \subseteq A$. By absorption, $b \wedge (b \vee x) = b$ so that $b \vee x \geq_L b$. Conversely, let $x \in A$ such that $x \geq_L b$, that is, $b \vee x = x$. Then, $x \in b \vee A$. The proof of the dual statement is analogous. □

**Remark 3.1.3.** Every right coset of $A$ in $B$ is a subset of the corresponding (full) coset of $A$ in $B$. Moreover, the (full) image set of $a$ in $B$ is a subset of the corresponding right image set of $a$ in $B$ as $a \wedge B \subseteq B \wedge a$. Dual remarks hold for left cosets and left image sets, and analogue statements are true regarding cosets and image sets in $A$.

In fact, given any $x \in S$ the following four sets are (subset) related as expressed by the following diagram, where the arrows $\rightarrow$ represent the set inclusion $\subseteq$:

\[
\begin{array}{ccc}
A \wedge x \wedge A & \rightarrow & A \wedge x \\
\downarrow & & \downarrow \\
{\{x\}} & \rightarrow & x \wedge A \\
\end{array}
\]

If $S$ is a left-handed skew lattice then the right cosets are trivial and the left cosets equal the (full) cosets. Similarly, in right-handed skew lattices right cosets coincide with (full) cosets while the left cosets are trivial.

**Definition 3.1.4.** The right meet operation, denoted by $\wedge_R$, is defined in a skew lattice $S$ by $x \wedge_R y = y \wedge x \wedge y$. Dually, the right join operation, denoted by $\vee_R$, is defined in a skew lattice $S$ by $x \vee_R y = x \vee y \vee x$. Left meet and join operations are defined similarly and denoted by $\wedge_L$ and $\vee_L$, respectively.
Lemma 3.1.5. If \((S; \wedge, \lor)\) is a skew lattice then \((S; \wedge_R, \lor_R)\) is a right-handed skew lattice and \((S; \wedge_L, \lor_L)\) is a left-handed skew lattice. If \(S\) is right- [left-]handed then operations \(\wedge_R [\wedge_L]\) and \(\lor_R [\lor_L]\) coincide with \(\wedge\) and \(\lor\), respectively.

Theorem 3.1.6. Let \(S\) be a skew lattice with comparable \(\mathcal{D}\)-classes \(A > B\). Then:

(i) The right cosets of \(A\) in \(B\) form a partition of \(B\). Moreover, given \(b \in B\) we have \(b \in b \wedge A\), and \(x \in b \wedge A\) is equivalent to \(x \wedge A = b \wedge A\). Moreover, the partition of \(B\) by right cosets of \(A\) in \(B\) refines the partition by cosets of \(A\) in \(B\).

(ii) The right image set of any element \(a \in A\) in \(B\) forms a transversal of the family of all right cosets of \(A\) in \(B\). All right image sets are equipotent.

(iii) Given any \(a \in A\) and \(b \in B\) there exists a bijection \(\varphi_{a,b}^R : B \lor a \to b \wedge A\) that maps \(x\) to \(y\) if and only if \(x \geq_L y\).

(iv) The right meet and right join operations on \(A \cup B\) are determined by the right coset bijections.

Similar remarks describe the interplay between the left cosets.

Proof. (i) Take \(b \in B\) and let \(a \in A\) be arbitrary. Then \(b = b \wedge (b \lor a) \in b \wedge A\) and thus \(\bigcup \{b \wedge A : b \in B\} = B\). Moreover, if \(x \in b \wedge A\) then \(x = b \wedge a'\) for some \(a' \in A\) so that \(x \wedge A = b \wedge a' \wedge A = b \wedge A\), where the latter equality follows by Lemma 1.1.3. To see that the partition by right cosets refines the coset partition, observe that \(b \wedge A \subseteq A \wedge b \wedge A\) as \(b = (a \lor b) \wedge b\) for any \(a \in A\). Thus \(A \wedge b \wedge A = \bigcup \{x \wedge A : x \in A \wedge b \wedge A\}\).

(ii) Let \(a\) and consider the right image set \(B \lor a\). For all \(b \in B\) the intersection of \(B \lor a\) by \(b \wedge A\) is the singleton \(\{b \wedge a\}\), and the cardinality of \(B \lor a\) equals the cardinality of the set of all the cosets of \(A\) in \(B\).

(iii) Define the map \(\varphi_{a,b}^R : B \lor a \to b \wedge A\) by \(x \mapsto b \wedge x\). Note that given \(x \in B \lor a\) we indeed have \(b \wedge x \leq_L x\), so that \(\varphi_{a,b}^R(x)\) is the unique \(y \in b \wedge A\) s.t. \(y \leq_L x\). We claim that \(\varphi_{a,b}^R\) is a bijection with the inverse \(\psi_{b,a}^R : b \wedge A \to B \lor a\) defined by \(y \mapsto y \lor a\). Take \(x \in B \lor a\). Then \(a \lor x = a\) and thus \(\psi_{b,a}^R(\varphi_{a,b}^R(x)) = (b \lor x) \lor a = (b \lor x) \lor a \lor x = (b \lor x) \lor x = x\), where we used Lemma 1.1.1 and absorption. Similarly, we prove \(\varphi_{a,b}^R(\psi_{b,a}^R(y)) = y\) for all \(y \in b \wedge A\).

(iv) Take \(a \in A\) and \(b \in B\). Then \(a \lor_R b = (b \lor a) \wedge b\) with \(b' = b \wedge a = \varphi_{a,b}^R(b)\). Similarly, \(b \lor_R a = b \lor (a \wedge b)\) with \(a' = a \wedge b = (\varphi_{a,b}^R)^{-1}(b)\). Analogously, \(\lor_R\) can also be described by flat coset bijections.

Definition 3.1.7. We refer to the bijections \(\varphi_{a,b}^R\) from Theorem 3.1.6 as to right coset bijections. Left coset bijections are defined dually.

Remark 3.1.8. The right [left] cosets and all [right] left image sets are rectangular sub skew lattices. Moreover, right [left] coset bijections are isomorphisms between the corresponding right [left] cosets. To see this, take comparable \(\mathcal{D}\)-classes \(A > B\) and \(a \in A, b \in B\). Let \(\varphi_{a,b}^R\) be the bijection from \(B \lor a\) to \(b \wedge A\) defined by \(\varphi_{a,b}^R(x) = b \wedge x\), as in Theorem 3.1.6.
For \(x, y \in B \lor a\) we obtain
\[
\varphi_{a,b}^R(x) \land \varphi_{a,b}^R(y) = b \land x \land b \land y = b \land x \land b \land x \land y \quad \text{by Lemma 1.1.1}
\]
\[
= b \land x \land y \quad \text{by idempotency}
\]
\[
= \varphi_{a,b}^R(x \land y),
\]
and \(\varphi_{a,b}^R(x) \lor \varphi_{a,b}^R(y) = \varphi_{a,b}^R(y) \land \varphi_{a,b}^R(x) = \varphi_{a,b}^R(y \land x) = \varphi_{a,b}^R(x \lor y)\) due to the rectangularity of \(A\) and \(B\).

**Proposition 3.1.9.** Let \(S\) be a skew lattice with comparable \(D\)-classes \(A > B\) and let \(y, y' \in B\). The following are equivalent:

(i) \(y \land A = y' \land A\);

(ii) \(y \land x = y' \land x\), for all \(x \in A\);

(iii) \(y \land x = y' \land x\), for some \(x \in A\).

**Proof.** As (ii) \(\Rightarrow\) (iii) is immediate it suffices to show (iii) \(\Rightarrow\) (i) \(\Rightarrow\) (ii). Assume that (iii) holds and take any \(a \in A\). By (iii) there exists \(x \in A\) s.t. \(y \land x = y' \land x\). Using Lemma 1.1.1 we get
\[
y \land a = y \land x \land a = y' \land x \land a = y' \land a
\]
and thus \(y \land A \subseteq y' \land A\). Likewise, \(y' \land A \subseteq y \land A\) and (i) follows. Finally, assume that (i) holds and take any \(x \in A\). By the assumption, \(y \in y' \land A\) and thus \(y = y' \land a\) for some \(a \in A\). Hence \(y \land x = y' \land a \land x = y' \land x\), where the final equality follows by Lemma 1.1.1. \(\square\)

Let \(S\) be a skew lattice and let
\[
\varphi : S \rightarrow S/\mathcal{R} \times S/\mathcal{D}/S/\mathcal{L}
\]
\[
x \mapsto (x_L, x_R)
\]
be the isomorphism from Theorem 1.2.1. Given a \(D\)-class \(D\) in \(S\) denote \(D_L = \{x_L \mid (x_L, x_R) = \varphi(x)\text{ for some } x \in D\}\) and \(D_R = \{x_R \mid (x_L, x_R) = \varphi(x)\text{ for some } x \in D\}\). The following lemma is a direct consequence of Theorem 1.2.1.

**Lemma 3.1.10.** Let \(x, y \in A\) and \(u, v \in B\). Then:

(i) \(A \land x \land A = A \land y \land A\) if and only if \(A_L \land x_L \land A_L = A_L \land y_L \land A_L\) and \(A_R \land x_R \land A_R = A_R \land y_R \land A_R\);

(ii) \(B \lor u \lor B = B \lor v \lor B\) if and only if \(B_L \lor u_L \lor B_L = B_L \lor v_L \lor B_L\) and \(B_R \lor u_R \lor B_R = B_R \lor v_R \lor B_R\).

Propositions 3.1.11 and Corollary 3.1.12 that follow describe the relation between the left [right] cosets and the full cosets of a skew lattice.

**Proposition 3.1.11.** Let \(S\) be a skew lattice and let \(A > B\) be \(D\)-classes as above. Given \(x, y \in B\) and \(u, v \in A\) the following hold:

(i) \(x \land A = y \land A\) if and only if \(x_L = y_L\) and \(x_R \land A_R = y_R \land A_R\);

(ii) \(A \land x = A \land y\) if and only if \(x_R = y_R\) and \(A_L \land x_L = A_L \land y_L\);

(iii) \(B \lor u = B \lor v\) if and only if \(u_L = v_L\) and \(B_R \lor u_R = B_R \lor v_R\);
The following result is a direct consequence of Lemma \ref{lem:3.1.10} and Proposition \ref{prop:3.1.11}.

**Corollary 3.1.12.** Let $S$ be a skew lattice with comparable $D$-classes $A > B$ and $x, y \in B$, $u, v \in A$. Then:

(i) $x \land A = y \land A$ if and only if $A \land x \land A = A \land y \land A$ and $x \mathcal{R} y$;
(ii) $A \land x = A \land y$ if and only if $A \land x \land A = A \land y \land A$ and $x \mathcal{L} y$;
(iii) $B \lor u = B \lor v$ if and only if $B \lor u \lor B = B \lor v \lor B$ and $u \mathcal{R} v$;
(iv) $u \lor B = v \lor B$ if and only if $B \lor u \lor B = B \lor v \lor B$ and $u \mathcal{L} v$.

**Remark 3.1.13.** The three results above permit us a better understanding on the relation between flat cosets and full cadets, a matter that we will continue to discuss right after the subsequent example. In particular, Proposition \ref{prop:3.1.11} expresses that all flat cosets correspondent to certain cosets in the flat lattice. This correspondence is natural and it can be clarified with the example of the next section.

### 3.2. Example on matrices in a ring

Let $R$ be a ring and $E(R)$ the set of all idempotent elements in $R$. Set $x \land y = xy$ and $x \lor y = x \circ y = x + y - xy$. If $S \subseteq E(R)$ is closed under both $\cdot$ and $\circ$ then $(S; \cdot, \circ)$ is a skew lattice. By a skew lattice in a ring $R$ we mean a set $S \subseteq E(R)$ that is closed under both multiplication and $\nabla$, defined by,

$$x \nabla y = (x \circ y)^2 = x + y + yx - xyx - yxy,$$

and forms a skew lattice for the two operations. In particular, we have to make sure that $\nabla$ is associative in $S$. Given a multiplicative band $B$ in a ring $R$ the relation between $\circ$ and $\nabla$ is given by $e \nabla f = (e \circ f)^2$ for all $e, f \in B$. In the case of right-handed skew lattices the nabla operation reduces to the circle operation. In the remainder of this subsection we shall assume that $R$ is a fixed ring and $S$ is a right-handed skew lattice in $R$. Recall that in the right-handed case $\nabla$ reduces to the circle operation. If $S$ has two comparable $D$-classes $A > B$ then given $a \in A$ and $b \in B$, $bA = \{ ba : a \in A \}$ is a coset of $A$ in $B$ and $B \circ a = \{ b + a - ba : b \in B \}$ is a coset of $B$ in $A$ (cf. \cite{2}).

The standard form for pure bands in matrix rings that was developed by Fillmore et al. in \cite{8} and \cite{9}. Based on this form, Cvetko-Vah described in \cite{4} the standard form for right-handed skew lattices in $M_n(F)$. Let $F$ be a field of characteristic different than 2 and $S \subseteq M_n(F)$ a primitive skew lattice with two comparable $D$-classes $A > B$. Then a basis for $F^n$ exists such that in this basis both $A$ and $B$ contain a diagonal matrix, the two diagonal matrices in $S$ form a lattice, and given any matrices $a \in A$, $b \in B$ and $b$
have block forms:
\[
a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ a_{31} & a_{32} & a_{31}a_{13} + a_{32}a_{23} \end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix} I & b_{12} & b_{13} \\ b_{21} & b_{21}b_{12} & b_{21}b_{13} \\ b_{31} & b_{31}b_{12} & b_{31}b_{13} \end{bmatrix}
\]

Denote the diagonal matrices in \( A \) and \( B \) by \( a_0 \) and \( b_0 \), respectively. If \( S \) is right-handed then \( aa_0 = a_0 \) and \( bb_0 = b_0 \) which implies \( a_{31} = a_{32} = 0 = b_{21} = b_{31} \). Thus \( a \) and \( b \) have block forms:
\[
a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

On the other hand, if \( S \) is left-handed then \( a_{13} = a_{23} = 0 = b_{12} = b_{13} \) and thus \( a \) and \( b \) have block forms:
\[
a = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix} I & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{bmatrix}
\]

Let \( S \) be right-handed. Given matrices \( a, a' \in A \) we obtain:
\[
B \circ a = B \circ a' \Leftrightarrow b_0 \circ a = b_0 \circ a' \Leftrightarrow a_{23} = a'_{23},
\]
and given \( b, b' \in B \) we obtain:
\[
bA = b'A \Leftrightarrow ba_0 = b'a_0 \Leftrightarrow b_{12} = b'_{12}.
\]

Similarly, if \( S \) is left-handed we obtain:
\[
a \circ B = a' \circ B \iff a_{32} = a'_{32} \quad \text{and} \quad Ab = Ab' \iff b_{21} = b'_{21}.
\]

Let \( S \subseteq M_n(F) \) be a primitive skew lattice with comparable \( D \)-classes \( A > B \). As primitive skew lattices have lattice sections, we can set \( S_R = \{ a_R : a \in S \} \) as in Theorem 1.2.2. Then \( S_R \) is a set of upper triangular matrices such that \( a_R \in A_R \) and \( b_R \in B_R \) have block forms:
\[
a_R = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix}
\quad \text{and} \quad
b_R = \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Similarly, we set \( S_L = \{ a_L : a \in S \} \), which is the set of lower triangular matrices such that \( a_L \in A_L \) and \( b_L \in B_L \) have block forms:
\[
a_L = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}
\quad \text{and} \quad
b_L = \begin{bmatrix} I & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{bmatrix}
\]

Then
\[
a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ a_{31} & a_{32} & a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix} = a_La_R
\]
From the above equivalences we can thus observe that being in the same flat coset is a lattice decomposition by right cosets. We aim to determine a complete description, achieved

\[ b = \begin{bmatrix} I & b_{12} & b_{13} \\ b_{21} & b_{21}b_{12} & b_{21}b_{13} \\ b_{31} & b_{31}b_{12} & b_{31}b_{13} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{bmatrix} = b_Lb_R \]

Remark 3.2.1. The following description clarifies what being in the same coset means in the matrix case. Let \( S \) be a skew lattice in \( M_n(F) \), \( A > B \) comparable \( \mathcal{D} \)-classes in \( S \), \( x, y \in B \) and \( u, v \in A \). Then by Lemma 3.1.10:

(i) \( Ax = Ay \) if and only if \( x_{21} = y_{21} \) and \( x_{12} = y_{12} \); 
(ii) \( B \nabla u \nabla B = B \nabla v \nabla B \) if and only if \( u_{32} = v_{32} \) and \( u_{23} = v_{23} \).

Similarly, Proposition 3.1.1 implies:

(i) \( xA = yA \) if and only if \( x_{21} = y_{21} \), \( x_{31} = y_{31} \) and \( x_{12} = y_{12} \); 
(ii) \( Ax = Ay \) if and only if \( x_{21} = y_{21} \), \( x_{12} = y_{12} \) and \( x_{13} = y_{13} \); 
(iii) \( B \nabla u = B \nabla v \) if and only if \( u_{31} = v_{31} \), \( u_{32} = v_{32} \) and \( u_{23} = v_{23} \); 
(iv) \( u \nabla B = v \nabla B \) if and only if \( u_{32} = v_{32} \), \( u_{13} = v_{13} \) and \( u_{23} = v_{23} \).

From the above equivalences we can thus observe that being in the same flat coset is a relation determined by the equalities \( x_{31} = y_{31} \) and \( x_{13} = y_{13} \) in the lower coset case, or \( u_{32} = v_{32} \) and \( u_{23} = v_{23} \) in the upper coset case. This gives us a description extending the one given in [4].

3.3. Flat coset decomposition. In the following results we further discuss this skew lattice decomposition by right cosets. We aim to determine a complete description, achieved towards the end of this section.

Proposition 3.3.1. Let \( A > B \) be comparable \( \mathcal{D} \)-classes in a skew lattice \( S \) and \( x, y \in A \), \( x, x' \in B \). The intersection \((x \wedge A) \cap (A \wedge x')\) is nonempty if and only if \( A \wedge x \wedge A = A \wedge x' \wedge A \). Dually, \((y \vee B) \cap (B \vee y')\) is nonempty if and only if \( B \vee y \vee B = B \vee y' \vee B \). Furthermore, whenever the intersections are nonempty we have:

\[ (x \wedge A) \cap (A \wedge x') = \{x \wedge x'\} \quad \text{and} \quad (y \vee B) \cap (B \vee y') = \{y \vee y'\}. \]

Proof. If \((x \wedge A) \cap (A \wedge x') \neq \emptyset\) then there exists \( u \in B \) s.t. \( u \wedge A = x \wedge A \) and \( A \wedge u = A \wedge x' \). It follows that \( A \wedge x \wedge A = A \wedge u \wedge A = A \wedge x' \wedge A \). On the other hand, assume that \( A \wedge x \wedge A = A \wedge x' \wedge A \). There exist \( a, a' \in A \) such that \( x' = a \wedge x \wedge a \) and \( x = a' \wedge x' \wedge a' \). Thus \( x \wedge x' = x \wedge a \wedge x \wedge a = x \wedge a \in x \wedge A \) and \( x \wedge x' = a' \wedge x' \wedge a' \wedge x' = a' \wedge x' \in A \wedge x' \). This proves that \( x \wedge x' \in (x \wedge A) \cap (A \wedge x') \). Moreover, take any \( y \in (x \wedge A) \cap (A \wedge x') \). There exist \( u, v \in A \) such that \( x \wedge u = y = v \wedge x' \). Lemma 3.1.1 together with \( u \wedge v \geq x, x' \) implies \( y = x \wedge u \wedge v \wedge x' = x \wedge x' \). The proof regarding the intersection \((y \vee B) \cap (B \vee y')\) is similar.

Remark 3.3.2. Corollary 3.1.12 together with Proposition 3.3.1 implies that each coset can be viewed as a rectangular with rows being right cosets and columns being left cosets. This will be made precise by Theorem 3.3.6 that follows after the following result of a more technical nature.

Proposition 3.3.3. Let \( S \) be a skew lattice with comparable \( \mathcal{D} \)-classes \( A > B \) and \( x, y \in B \). The following statements are equivalent:

and
\((i)\) \(A \land x \land A = A \land y \land A;\)
\((ii)\) \(A \land x \land y = A \land y\) and \(x \land A = x \land y \land A;\)
\((iii)\) \(y \land x \land A = y \land A\) and \(A \land x = A \land y \land x.\)

Dual results hold for \(B\)-cosets in \(A.\)

Proof. We shall only prove the equivalence between \((i)\) and \((ii)\) as the equivalence between \((i)\) and \((iii)\) then follows by analogy. Assume \((i)\) and take any \(a \in A.\) Then \(a \land x \land a = a \land y \land a\) and thus \(a \land x \land a \land y = a \land y \land a \land y.\) Using idempotency and Lemma 1.1.1 the latter is equivalent to \(a \land x \land y = a \land y\) and \(A \land x \land y = A \land y\) follows. Similarly we show that \(x \land A = x \land y \land A.\) To prove the converse, assume \((ii)\). Then \(A \land x \land A = A \land (x \land y \land A) = (A \land x \land y) \land A = A \land y \land A.\)

Remark 3.3.4. Proposition 3.3.3 yields that given comparable \(D\)-classes \(A \gg B\) and \(x, y \in B,\) \(A \land x \land A = A \land y \land A\) is equivalent to the existence of a linking element \(b \in A \land x \land A\) such that \(A \land b = A \land y\) and \(x \land A = b \land A\) (with \(b = x \land y\)). Another link of the kind can be found in a dual way by choosing an element \(c \in A \land x \land A\) such that \(c \land A = y \land A\) and \(A \land x = A \land c\) (with \(c = y \land x\)). The dual statement to this also provides the linking elements regarding cosets of \(B\) in \(A.\)

Lemma 3.3.5. Let \(S\) be a skew lattice with comparable \(D\)-classes \(A \gg B.\) Then, for all \(b \in B,\) \((A \land b) \times (b \land A)\) is a rectangular sub skew lattice of \(B\) under the operations \((x, y) \lor (x', y') = (x', y')\) and \((x, y) \land (x', y') = (x, y').\) Similar remarks hold for \(B\)-cosets in \(A.\)

Theorem 3.3.6. Let \(S\) be a skew lattice with two comparable \(D\)-classes \(A \gg B.\) Then, for all \(x \in B,\) there exists an isomorphism \(\delta_{A \land x \land A} : A \land x \land A \to (A \land x) \times (x \land A).\) Dually, for all \(y \in A,\) there exists an isomorphism \(\delta_{B \lor y \lor B} : B \lor y \lor B \to (y \lor B) \times (B \lor y).\)

Proof. Let \(x \in B\) and consider the map \(\delta_{A \land x \land A}\) from \(A \land x \land A\) to \((A \land x) \times (x \land A)\) that maps \(z\) to \((z \land x, x \land z).\) As an element \(z \in A \land x \land A\) is of the form \(z = a \land x \land a\) for some \(a \in A,\) it follows that \((z \land x, x \land z) = (a \land x, x \land a)\) and the map \(\delta_{A \land x \land A}\) is well defined. To see that \(\delta_{A \land x \land A}\) is injective take \(z, z' \in A \land x \land A\) such that \((z \land x, x \land z) = (z' \land x, x \land z').\) Then \(z \land x = z' \land x\) and \(x \land z = x \land z'\) so that \(z = z \land x \land z = z' \land x \land z = z' \land x \land z'.\) In order to prove surjectivity of \(\delta_{A \land x \land A}\) let \((y, y') \in (A \land x) \times (x \land A)\) be arbitrary. There exist \(a, a' \in A\) such that \(y = a \land x\) and \(y' = x \land a'.\) Then \(y \lor y' = a \land x \land a'\) and we will prove that \((y, y') = \delta_{A \land x \land A}(y \land y').\) By the regularity of \(\land\) and Remark 3.1.3 we obtain:

\[
\delta_{A \land x \land A}(y \land y') = (y \land y' \land x, x \land y \land y') = (a \land x \land a' \land x, x \land a \land x \land a') = (a \land x, x \land a') = (y, y').
\]

Therefore \(\delta_{A \land x \land A}\) is a bijection. It remains to prove that \(\delta_{A \land x \land A}\) is a homomorphism. To see this, let \(z, z' \in A \land x \land A\) be arbitrary. As \(x, z\) and \(z'\) all lie in the rectangular band \(B\)
Then the following is a commutative diagram of skew lattice isomorphisms:

\[ \delta_{u \land x \land u}(z \land z') = (z \land z' \land x, x \land z \land z') = (z \land x, x \land z') = (z \land x, x \land z) \land (z' \land x, x \land z') = \delta_{A \land x \land A}(z) \land \delta_{A \land x \land A}(z'). \]

On the other hand, due to the the rectangularity of \( A \land x \land A \) and \( (A \land x) \times (x \land A) \), \( \delta_{A \land x \land A}(z \lor z') = \delta_{A \land x \land A}(z' \lor z) = \delta_{A \land x \land A}(z') \land \delta_{A \land x \land A}(z) = \delta_{A \land x \land A}(z) \lor \delta_{A \land x \land A}(z') \). The proof that \( (y \lor B) \times (B \lor y) \) and \( B \lor y \lor B \) are isomorphic is derived in a dual fashion using the map \( \delta_{B \lor y \lor B} : B \lor y \lor B \to (y \lor B) \times (B \lor y) \) defined by \( u \mapsto (y \lor u, u \lor y) \).

**Definition 3.3.7.** Whenever \( A, B, C \) and \( D \) are sets and \( f : A \to B \) and \( g : C \to D \) are maps we denote by \( f \times g \) the map from \( A \times C \) to \( B \times D \) that assigns to each pair \((x, x') \in A \times C\) the pair \((f(x), g(x'))\).

**Corollary 3.3.8.** Let \( S \) be a primitive skew lattice with two comparable \( D \)-classes \( A > B \). Given \( a \in A \) and \( b \in B \) consider the maps:

\[
\varphi_{a,b} : B \lor a \lor B \to A \land b \land A \quad \varphi_{a,b}^L : a \lor B \to A \land b \quad \varphi_{a,b}^R : B \lor a \to b \land A
data x \mapsto x \land b \land x \quad x \mapsto x \land b \quad x \mapsto b \land x.
\]

Then the following is a commutative diagram of skew lattice isomorphisms:

\[
\begin{array}{ccc}
B \lor a \lor B & \xrightarrow{\delta_{B \lor a \lor B}} & (a \lor B) \times (B \lor a) \\
\varphi_{a,b} & \downarrow & \varphi_{a,b}^L \times \varphi_{a,b}^R \\
A \land b \land A & \xrightarrow{\delta_{A \land b \land A}} & (A \land b) \times (b \land A)
\end{array}
\]

**Proof.** It remains to prove that the above diagram is commutative. To see this let \( x \in B \lor a \lor B \) be arbitrary. Then

\[
(\varphi_{a,b}^L \times \varphi_{a,b}^R)(\delta_{B \lor a \lor B}(x)) = (\varphi_{a,b}^L \times \varphi_{a,b}^R)(a \lor x, x \lor a) = ((a \lor x) \land b, b \land (x \lor a)),
\]

while

\[
\delta_{A \land b \land A}(\varphi_{a,b}(x)) = \delta_{A \land b \land A}(x \land b \land x) = (x \land b, b \land x).
\]

We need to prove that \((a \lor x) \land b = x \land b \) and \(b \land (x \lor a) = b \land x \). Both \((a \lor x) \land b \) and \(b \land x \) lie in the left coset \( A \land b \). By Theorem 3.1.6 in order to prove that they are equal it suffices to show that they are both \( \leq_R \) a common \( u \in a \lor B \). As \( x \in B \lor a \lor B \) it follows that \( u = a \lor x \in a \lor B \); we claim that \( u \) has the desired property. Indeed:

\[
u \land (a \lor x) \land b = (a \lor x) \land b \Rightarrow (a \lor x) \land b \leq_R u
\]

and

\[
u \land x \land b = (a \lor x) \land x \land b = x \land b \Rightarrow x \land b \leq_R u.
\]

A similar argument verifies that \( b \land (x \lor a) = b \land x \).
4. Flat Coset Laws

4.1. Coset laws for symmetry. The following results show the impact of the flat coset decomposition on the coset laws for symmetric skew lattices, for cancellative skew lattices and for normal skew lattices. We will start with the case of symmetric skew lattices regarding the classification presented earlier in Subsection 2.3.

Theorem 4.1.1. Let $S$ be a skew lattice. Then:

(a) $S$ is right lower symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $m, m' \in M$, $(m \land J = m' \land J \iff m \land A = m' \land A \land m \land B = m' \land B)$;

(b) $S$ is right upper symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $j, j' \in J$, $(M \lor j = M \lor j' \iff A \lor j = A \lor j' \land B \lor j = B \lor j')$;

(c) $S$ is left lower symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $m, m' \in M$, $(J \land m = J \land m' \iff A \land m = A \land m' \land B \land m = B \land m')$;

(d) $S$ is left upper symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $j, j' \in J$, $(j \lor M = j' \lor M \iff j \lor A = j' \lor A \land j \lor B = j' \lor B)$.

Proof. We shall prove (a) as (b)–(d) are proven in a similar fashion. By definition, $S$ is right lower symmetric if and only if $S/L$ is lower symmetric. By Theorem 1.3.2 (i) and Remark 1.3.3, $S/L$ is lower symmetric if and only if given any skew diamond $\{ J > A, B > M \}$ is $S$ and any $m, m' \in M$:

$$m_R \land J_R = m'_R \land J_R \iff (m_R \land A_R = m'_R \land A_R \land m_R \land B_R = m'_R \land B_R).$$

It follows from Proposition 3.1.11 (i) that for $D \in \{ A, B, J \}$,

$$m \land D = m' \land D \iff (m_L = m'_L \land m_R \land D_R = m'_R \land D_R).$$

Thus $S$ is right lower symmetric if and only if given any skew diamond $\{ J > A, B > M \}$ is $S$ and any $m, m' \in M$:

$$m \land J = m' \land J \iff (m_L = m'_L \land m_R \land J_R = m'_R \land J_R).$$

By the above, the condition (2) is equivalent to

$$m \land J = m' \land J \iff (m_L = m'_L \land m_R \land A_R = m'_R \land A_R \land m_R \land B_R = m'_R \land B_R),$$

which is by (1) further equivalent to

$$m \land J = m' \land J \iff (m \land A = m \land A \land m \land B = m' \land B).$$

$\square$

Corollary 4.1.2. Let $S$ be a skew lattice. Then $S$ is symmetric if and only if for all skew diamonds $\{ J > A, B > M \}$ and all $m \in M, j \in J$ the following hold:

$$m \land J = (m \land A) \cap (m \land B) \land M \lor j = (A \lor j) \cap (B \lor j),$$

$$J \land m = (A \land m) \cap (B \land m) \land j \lor M = (j \lor A) \cap (j \lor B).$$
4.2. Coset laws for normality. We shall now turn our attention to normal skew lattices and corresponding coset laws. The relation with quasi normality shall also be discussed.

Proposition 4.2.1. Let $S$ be a skew lattice. Then $S$ is normal iff for each comparable pair of $D$-classes $A > B$ in $S$ and all $x, x' \in B$ the following pair of implications hold:

(i) if $xLx'$ then $A \land x = A \land x'$;
(ii) if $xRx'$ then $x \land A = x' \land A$.

Dually, $S$ is conormal iff for all comparable pairs of $D$-classes $A > B$ in $S$ and for all $x, x' \in A$ the following pair of implications hold:

(iii) if $xRx'$ then $x \lor B = x' \lor B$;
(iv) if $xLx'$ then $B \lor x = B \lor x'$.

Proof. First assume that $S$ is normal. If $xLx'$ then by Corollary 3.1.12 and Theorem 1.3.6 we obtain:

$$A \land x = (A \land x \land A) \cap L_x = (A \land x' \land A) \cap L_{x'} = A \land x'$$

which proves (i). The proof for (ii) is similar.

Conversely, assume that (i) and (ii) hold, and let $A > B$ be comparable $D$-classes in $S$. Take $x, x' \in B$. As $B$ is rectangular there exists $z \in B$ such that $xRz$ and $zLx'$. By the assumption we have $x \land A = z \land A$ and $A \land z = A \land x'$, and thus $A \land x \land A = A \land z \land A = A \land x' \land A$. Hence $S$ is a normal skew lattice by Theorem 1.3.6. The statement regarding conormal skew lattices has a similar proof.

Remark 4.2.2. Recall that a skew lattice is left [right] normal if it satisfies $x \land y \land z = x \land z \land y$ [or $y \land z \land x = z \land y \land x$, respectively]. Right [left] normality implies normality. Due to this, right [left] normal skew lattices satisfy the conditions (i) and (ii) of Proposition 4.2.1. A counterexample to the converse of the implications in that conditions is any normal skew lattice that is not right normal.

Remark 4.2.3. Roughly speaking, Proposition 4.2.1 tells us that a skew lattice $S$ is normal iff for each comparable pair of $D$-classes $A > B$ in $S$, $B$ is the entire left coset of $A$ in $B$ (corresponding to the condition (i), not allowing any pair of left cosets one above the other) and, simultaneously, $B$ is the entire right coset of $A$ in $B$ (corresponding to the condition (ii), not allowing any pair of right cosets side by side).

Definition 4.2.4. A skew lattice is right quasi normal (RQN) if it satisfies the identity $y \land x \land a = y \land a \land x \land a$, and it is left quasi normal (LQN) if it satisfies the identity $a \land x \land y = a \land x \land a \land y$. Equivalently, right [left] quasi normal skew lattices are the ones for which $(S; \land)$ is a right [left] quasi normal band. These bands are defined in [10]. Dual definitions determine [left] right quasi normal skew lattices. The following results provide us with useful characterizations of such algebras.

Proposition 4.2.5. A skew lattice $S$ is right quasi normal if and only if for all $y \in S$ the factor algebra $(y \land S)/R$ is a lattice, i.e., $(y \land S) \cap R_x = \{x\}$ for all $x \in y \land S$. Dually, $S$ is left quasi normal if and only if for all $y \in S$ the factor algebra $(S \land y)/L$ is a lattice, i.e., $(S \land y) \cap L_x = \{x\}$, for all $x \in S \land y$. 
Hence, Proposition 4.2.5 implies that \( S \) is right quasi normal and let \( y \in S \), \( x, x' \in y \wedge S \) be such that \( x \leq x' \). Then \( x = y \wedge x \) and \( x' = y \wedge x' \) by Lemma 2.1.4 Thus:

\[
x = y \wedge x = y \wedge x \wedge x' = y \wedge x' \wedge x = y \wedge x = x',
\]

where the second and forth equality follow by \( x \leq x' \), and the third equality follows by right quasi normality.

Conversely, assume that \( (y \wedge S)/R \) is a lattice for all \( y \), and take arbitrary \( x, y, a \in S \). Consider \( y \wedge S \) that is a subalgebra by Proposition 2.1.5. By regularity we have:

\[
(y \wedge x \wedge a) \wedge (y \wedge a \wedge x \wedge a) = (y \wedge x \wedge a) \wedge (y \wedge x \wedge a) = y \wedge x \wedge a
\]

and

\[
(y \wedge a \wedge x \wedge a) \wedge (y \wedge x \wedge a) = (y \wedge a \wedge x \wedge a) \wedge (y \wedge a \wedge x \wedge a) = y \wedge a \wedge x \wedge a.
\]

Thus \( (y \wedge a \wedge x \wedge a) L(y \wedge x \wedge a) \). However, as by the assumption all \( L \)-classes of \( y \wedge S \) are trivial, \( y \wedge x \wedge a = y \wedge a \wedge x \wedge a \) follows. The proof of the dual statement is similar. \( \Box \)

The next result relates Propositions 4.2.1 and 4.2.6 giving us a characterization for left [right] quasi normal skew lattices of coset nature.

**Proposition 4.2.6.** Let \( S \) be a skew lattice. Then,

(i) \( S \) is left quasi normal if and only if for all comparable \( D \)-classes \( A > B \) in \( S \) and \( x, x' \in B \) such that \( x \leq x' \), then \( x \wedge A = x' \wedge A \).

(ii) \( S \) is right quasi normal if and only if for all comparable \( D \)-classes \( A > B \) in \( S \) and \( x, x' \in B \) such that \( x \leq x' \), then \( A \wedge x = A \wedge x' \).

Dual results hold for conormality.

**Proof.** Let \( a \in A \) and \( x, x' \in B \) such that \( x \leq x' \). Due to the hypothesis and the fact that \( x \wedge a = x' \wedge a = x' \wedge x'' \wedge a = x' \wedge a \), so that \( x \wedge A = x' \wedge A \) as required by (i). Conversely, let \( x \in y \wedge S \) and consider \( A = D_y \). Let \( x' \in S \) such that \( x \leq x' \). Then \( x' = x \wedge x' = y \wedge x \wedge x' = y \wedge x' \in y \wedge S \). Then, the hypothesis implies that \( A \wedge y \wedge x = A \wedge y \wedge x' \). As \( y \in A \) then Proposition 3.1.9 implies that

\[
x = y \wedge x = y \wedge y \wedge x = y \wedge y \wedge x' = y \wedge x' = x'.
\]

Hence, Proposition 4.2.6 implies that \( S \) is right quasi normal. The proof of (ii) is similar. \( \Box \)

The following result is a consequence of Propositions 4.2.1 and 4.2.6. In fact, it also follows the research made for bands of semigroups in [19] when considering the reducts \( (S; \wedge) \) and \( (S; \vee) \) of a skew lattice \( S \).

**Corollary 4.2.7.** Let \( S \) be a skew lattice. Then, \( S \) is normal if and only if \( S \) is simultaneously right quasi normal and left quasi normal. Dually, \( S \) is conormal if and only if \( S \) is simultaneously right quasi conormal and left quasi conormal.
4.3. Coset laws for cancellation. In the remainder of the paper we will give a further insight to the flat coset decomposition of cancellative skew lattices for which the lattice image is distributive, and therefore the ones permitting the coset laws established in [7].

Remark 4.3.1. Recall that given a skew diamond \{ J > A, B > M \} and elements \( x, x' \in A \), the equality \( M \lor x \lor M = M \lor x' \lor M \) always implies \( B \lor x \lor B = B \lor x' \lor B \). Likewise, the equality \( J \land x \land J = J \land x' \land J \) implies \( B \land x \land B = B \land x' \land B \). Proposition 4.3.2 below is a flat version of this result.

Proposition 4.3.2. Let \( S \) be a skew lattice and \{ \( J > A, B > M \) \} a skew diamond in \( S \). Given any \( x, x' \in A \) the following hold:

(i) if \( M \lor x = M \lor x' \) then \( B \lor x = B \lor x' \);

(ii) if \( x \land J = x' \land J \) then \( x \land B = x' \land B \).

Similar remarks hold regarding left cosets.

Proof. We will prove (i) having in mind that (ii) follows by a dual argument. Let \( x, x' \in A \) and assume that \( M \lor x = M \lor x' \). Proposition [3.1.9] implies the existence of \( m \in M \) such that \( m \lor x = m \lor x' \). Let \( b \in B \) be such that \( m \leq_L b \). Then,

\[
\begin{align*}
b \lor x &= b \lor m \lor x \\
&= b \lor m \lor x' \\
&= b \lor x'.
\end{align*}
\]

Theorem [3.1.9] then implies \( B \lor x = B \lor x' \).

Proposition 4.3.3. Let \( S \) be a skew lattice such that \( S/D \) is a distributive lattice. Then given any skew diamond \{ \( J > A, B > M \) \} in \( S \) and any \( x, x' \in A \) the following equivalences hold:

(i) \( (M \lor x \lor M = M \lor x' \lor M \iff B \lor x \lor B = B \lor x' \lor B) \) if and only if \( (M \lor x = M \lor x' \iff B \lor x = B \lor x' \) and \( x \lor M = x' \lor M \iff x \lor B = x' \lor B) \);

(ii) \( (B \land x \land B = B \land x' \land B \iff J \land x \land J = J \land x' \land J) \) if and only if \( (x \land B = x' \land B \iff x \land J = x' \land J) \).

Proof. We will only show (i) as (ii) has an analogous proof. By Proposition [3.3.2] and the comment above it, all the direct implications of the considered equivalences always hold. So, only the converse implications need to be addressed. Let \{ \( J > A, B > M \) \} be a skew diamond in \( S \) and \( x, x' \in A \). First assume that \( M \lor x \lor M = M \lor x' \lor M \iff B \lor x \lor B = B \lor x' \lor B \) holds. If \( B \lor x = B \lor x' \) then Corollary [3.1.12] implies \( B \lor x \lor B = B \lor x' \lor B \) and \( x \not\in x' \). Hence \( M \lor x \lor M = M \lor x' \lor M \) and \( x \not\in x' \) by the assumption, and thus \( M \lor x = M \lor x' \) follows by Corollary [3.1.12].

Conversely, assume that both \( M \lor x = M \lor x' \iff B \lor x = B \lor x' \) and \( x \lor M = x' \lor M \iff x \lor B = x' \lor B \) hold. If \( B \lor x \lor B = B \lor x' \lor B \) then by Proposition [3.3.3] there exists \( y \in B \lor x \lor B \) such that \( B \lor y = B \lor x \) and \( y \lor B = x' \lor B \). Proposition [4.3.2] then implies \( M \lor y = M \lor x \) and \( y \lor M = x' \lor M \). Thus \( M \lor x \lor M = M \lor y \lor M = M \lor x' \lor M \) follows. \( \square \)
**Definition 4.3.4.** A skew lattice is upper cancellative if it is upper symmetric and simply cancellative. Dually, a skew lattice is lower cancellative if it is lower symmetric and simply cancellative.

**Proposition 4.3.5.** Let $S$ be a skew lattice such that $S/D$ is a distributive lattice.

(i) if $S$ is lower symmetric then $S$ is lower cancellative if and only if $M \lor x \lor M = M \lor x' \lor M \iff B \lor x \lor B = B \lor x' \lor B$ holds for all skew diamonds $\{ J > A, B > M \}$ in $S$ and all $x, x' \in A$.

(ii) if $S$ is upper symmetric then $S$ is upper cancellative if and only if given any skew diamond $\{ J > A, B > M \}$ in $S$ and any $x, x' \in A$, $B \land x \land B = B \land x' \land B \iff J \land x \land J = J \land x' \land J$ holds.

**Proof.** We will now prove (i). The proof of (ii) is similar.

Let $\{ J > A, B > M \}$ be a skew diamond in $S$. By Remark 4.3.1 the implication $M \lor x \lor M = M \lor x' \lor M \implies B \lor x \lor B = B \lor x' \lor B$ always holds. So, let $x, x' \in A$ be such that $B \lor x \lor B = B \lor x' \lor B$ and suppose that $M \lor x \lor M \neq M \lor x' \lor M$. Let $m_0 \in M$. Consider $u = m_0 \lor x \lor m_0$ and $v = m_0 \lor x' \lor m_0$. There exists $b_0 \in B$ such that $b_0 > m_0$. Then, $b_0 \lor u \lor b_0 = b_0 \lor x \lor b_0 = b_0 \lor x' \lor b_0 = b_0 \lor v \lor b_0$, where the second equality is due to the assumption that $B \lor x \lor B = B \lor x' \lor B$, and thus $u < b_0 \lor v \lor b_0$ and $v < b_0 \lor v \lor b_0$. Therefore $m_0 < u, v, b_0 < b_0 \lor u \lor b_0$ determine a copy of $NC_5$ and hence contradicts the assumption that $S$ is simply cancellative.

Conversely, if $S$ is not lower cancellative (ie. it is not simply cancellative, since it is lower symmetric by the assumption), then by a result of [5] $S$ contains a subalgebra $S'$ isomorphic to $NC_5$, given by the diagram below. (In $NC_5$ operations on $x_1$ and $x_2$ can be defined in two ways: for $i, j \in \{1, 2\}$ either $x_1 \land x_j = x_j$ and $x_i \lor x_j = x_i$ which yields a right-handed structure, or $x_1 \land x_j = x_i$ and $x_i \lor x_j = x_j$ yielding a left-handed structure.) Let $A, B, M$ and $J$ denote the $D$-classes of elements $x_1, y, u$ and $v$ in $S$, respectively.

```
       u
      /\
     /  \
x1 ------ x2  y
      \  /
     \//
      \v
```

Since $x_1$ and $x_2$ are both contained in the image of $u$ in $A$, they cannot lie in the same coset of $M$ in $A$. On the other hand, $B \lor x_1 \lor B$ and $B \lor x_2 \lor B$ both contain $v$ and hence coincide by Theorem 1.3.1. □

**Proposition 4.3.6.** Let $S$ be a symmetric skew lattice such that $S/D$ is a distributive lattice. Then, the following statements are equivalent:

(i) $S/R$ is cancellative;

(ii) given any skew diamond $\{ J > A, B > M \}$ in $S$ and any $x, x' \in A$, $M \lor x = M \lor x'$ holds if and only if $B \lor x = B \lor x'$ holds;
(iii) given any skew diamond \( \{ J > A, B > M \} \) in \( S \) and any \( x, x' \in A, x \wedge B = x' \wedge B \) holds if and only if \( x \wedge J = x' \wedge J \) holds.

A dual result holds regarding right cosets in the skew lattice \( S \).

Proof. We will show that \((i) \iff (ii)\) using the characterization of Theorem 1.3.5. The equivalence \((i) \iff (iii)\) is proved similarly.

Let \( \{ J > A, B > M \} \) be a skew diamond in \( S \). Assume that \( S/\mathcal{R} \) is cancellative. Due to Lemma 4.3.2 we need only to show that \( B \lor x = B \lor x' \) implies \( M \lor x = M \lor x' \), for all \( x, x' \in A \). As \( S/\mathcal{R} \) is cancellative and left-handed, all the cosets in \( S/\mathcal{R} \) are left cosets and thus

\[
M_L \lor x_L = M_L \lor x'_L \iff B_L \lor x_L = B_L \lor x'_L.
\]

Let \( x, x' \in A \) such that \( B \lor x = B \lor x' \). Then, Proposition 3.1.11 implies that

\[
(3) \quad B \lor x = B \lor x' \Rightarrow x_R = y_R \quad \text{and} \quad B_L \lor x_L = B_L \lor x'_L
\]

\[
(4) \quad \Rightarrow x_R = y_R \quad \text{and} \quad M_L \lor x_L = M_L \lor x'_L
\]

\[
(5) \quad \Rightarrow M \lor x = M \lor x'.
\]

Conversely, assume that \( M \lor x = M \lor x' \) if and only if \( B \lor x = B \lor x' \), for all skew diamonds \( \{ J > A, B > M \} \) in \( S \) and all \( x, x' \in A \). Then, \( M_L \lor x_L = M_L \lor x'_L \) if and only if \( B_L \lor x_L = B_L \lor x'_L \), for all skew diamonds \( \{ J > A, B > M \} \) in \( S/\mathcal{R} \) and all \( x_L, x'_L \in A_L \). As \( S/\mathcal{R} \) is a left-handed skew lattice, all its cosets are left cosets and, therefore, \( S/\mathcal{R} \) is cancellative due to Theorem 1.3.5. \( \square \)

Proposition 4.3.6 above leads us to define the following notions which are not to be confused with left and right cancellation as defined in the preliminary section.

**Definition 4.3.7.** A left-coset cancellative skew lattice is a skew lattice \( S \) such that \( S/\mathcal{R} \) is cancellative. Dually, a right-coset cancellative skew lattice is a skew lattice \( S \) such that \( S/\mathcal{L} \) is cancellative. Due to \cite{3} both of these classes of algebras constitute varieties.

As it was proved in \cite{3} that a skew lattice \( S \) satisfies any identity that is satisfied by both its left factor \( S/\mathcal{R} \) and its right factor \( S/\mathcal{L} \), the following result is a direct consequence of the definitions.

**Corollary 4.3.8.** A skew lattice is cancellative if and only if it is both right-coset cancellative and left-coset cancellative.

The result of Proposition 4.3.6 provides us with a deeper insight on the coset structure of cancellative skew lattices and new subclasses determined by the corresponding laws for flat cosets. These achievements close the section and the paper. Several aspects of research on the combinatorial consequences of such coset decomposition can be found in \cite{23}. Furthermore, the impact of the flat coset structure in other coset laws regarding strictly categorical or distributive skew lattices as in \cite{22}, \cite{13} or \cite{14} are a matter of research that we will address to in the future.
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