The spin–orbit resonances of the Solar system: A mathematical treatment matching physical data*

Francesco Antognini  
Departement Mathematik  
ETH-Zürich  
CH-8092 Zürich, Switzerland  
antognif@math.ethz.ch

Luca Biasco and Luigi Chierchia  
Dipartimento di Matematica e Fisica  
Università “Roma Tre”  
Largo S.L. Murialdo 1, I-00146 Roma (Italy)  
biasco,luigi@mat.uniroma3.it

Decemebrr 9, 2013

Abstract

In the mathematical framework of a restricted, slightly dissipative spin–orbit model, we prove the existence of periodic orbits for astronomical parameter values corresponding to all satellites of the Solar system observed in exact spin–orbit resonance.

Keywords: Periodic orbits. Celestial Mechanics. Spin–orbit resonances. Moons in the Solar systems. Mercury. Dissipative systems.

MSC2000 numbers: 70F15, 70F40, 70E20, 70G70, 70H09, 70H12, 34C23, 34C25, 34C60, 34D10

Contents

1 Introduction and results  2
2 Proof of the theorem  6
A Proof of Lemma 2.2  14
B Fourier coefficients of the Newtonian potential  16
C Small bodies  16

*Acknowledgments. We thank J. Castillo-Rogez, A. Celletti, M. Efroimsky and F. Nimmo for useful discussions. Partially supported by the MIUR grant “Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations” (PRIN2009).
1 Introduction and results

- **Satellites in spin–orbit resonance**

One of the many fascinating features of the Solar system is the presence of moons moving in a “synchronous” way around their planets, as experienced, for example, by earthlings looking always the same familiar face of their satellite. Indeed, eighteen moons of our Solar system move in a so-called 1:1 spin–orbit resonance: while performing a complete revolution on a (approximately) Keplerian ellipse around their principal body, they also complete a rotation around their spin axis (which is – again, approximately – perpendicular to revolution plane), in this way these moons always show the same side to their hosting planets.

The list of the eighteen moons is the following: Moon (Earth); Io, Europa, Ganymede, Callisto (Jupiter); Mimas, Enceladus, Tethys, Dione, Rhea, Titan, Iapetus, (Saturn); Ariel, Umbriel, Titania, Oberon, Miranda (Uranus); Charon (Pluto); minor bodies with mean radius smaller than 100 Km are not considered (see, however, Appendix C).

There is only one more occurrence of spin–orbit resonance in the Solar system: the strange case of the 3:2 resonance of Mercury around the Sun (i.e., Mercury rotates three times on its spin axis, while making two orbital revolutions around the Sun).

In this paper we discuss a mathematical theory, which is consistent with the existence of all spin–orbit resonances of the Solar system; in other words, we prove a theorem, in a framework of a well–known simple “restricted spin–orbit model”, establishing the existence of periodic orbits for parameter values corresponding to all the satellites (or Mercury) in our Solar system observed in spin–orbit resonance.

We remark that, in dealing with mathematical models trying to describe physical phenomena, one may be able to rigorously prove theorems only for parameter values, typically, quite smaller than the physical ones; on the other hand, for the true physical values, typically, one only obtains numerical evidence. In the present case, thanks to sharp estimates, we are able to fill such a gap and prove rigorous results for the real parameter values. Moreover, such results might also be an indication that the mathematical model adopted is quite effective in describing the physics.

- **The mathematical model**

We consider a simple – albeit non trivial – model in which the center of mass of the satellite moves on a given two–body Keplerian orbit focussed on a massive point (primary body) exerting gravitational attraction on the body of the satellite modeled by a triaxial ellipsoid with equatorial axes \(a \geq b > 0\) and polar axis \(c\); the spin polar axis is assumed to be perpendicular to the Keplerian orbit plane\(^1\); finally, we include also small dissipative effects (due to the possible internal non–rigid structures of the satellite), according to the “viscous–tidal model, with a linear dependence on the tidal frequency” ([6]): essentially, the dissipative term is given by the average over one revolution period of the so–called MacDonald’s torque [9]; compare [10].

For a discussion of this model, see [3]; for further references, see [7], [8], [12], and [4]; for a different (PDE) model, see [1].

The differential equation governing the motion of the satellite is then given by

\[
\ddot{x} + \eta (\dot{x} - \nu) + \varepsilon f(x, t) = 0 ,
\]  

where:

\(^1\) The largest relative inclination (of the spin axis on the orbital plane) is that of Iapetus \((8.298^\circ)\) followed by Mercury \((7^\circ)\), Moon \((5.145^\circ)\), Miranda \((4.338^\circ)\); all the other moons have an inclination of the order of one degree or less.
Figure 1: Triaxial satellite revolving on a rescaled Keplerian ellipse (equatorial section)

(a) $x$ is the angle (mod $2\pi$) formed by the direction of (say) the major equatorial axis of the satellite with the direction of the semi-major axis of the Keplerian ellipse plane; ‘dot’ represents derivative with respect to $t$ where $t$ (also defined mod $2\pi$) is the mean anomaly (i.e., the ellipse area between the semi-major axis and the orbital radius $\rho_e$ divided by the total area times $2\pi$) and $e$ is the eccentricity of the ellipse;

(b) the dissipation parameters $\eta = K\Omega_e$ and $\nu = \nu_e$ are real-analytic functions of the eccentricity $e$: $K \geq 0$ is a physical constant depending on the internal (non-rigid) structure of the satellite and\(^2\)

$$
\Omega_e \ := \ \left(1 + 3e^2 + \frac{3}{8}e^4\right) \frac{1}{(1 - e^2)^{3/2}},
$$

$$
N_e \ := \ \left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right) \frac{1}{(1 - e^2)^{6/5}},
$$

$$
\nu_e \ := \ \frac{N_e}{\Omega_e}.
$$

(c) the constant $\varepsilon$ measures the oblateness (or “equatorial ellipticity”) of the satellite and it is defined as $\varepsilon = \frac{B-A}{C}$, where $A \leq B$ and $C$ are the principal moments of inertia of the satellite ($C$ being referred to the polar axis);

(d) the function $f$ is the (“dimensionless”) Newtonian potential given by

$$
f(x, t) := -\frac{1}{2\rho_e(t)^3} \cos(2x - 2f_e(t)),
$$

where $\rho_e(t)$ and $f_e(t)$ are, respectively, the (normalized) orbital radius

$$
\rho_e(t) := 1 - e \cos(u_e(t))
$$

\(^2\)In [6] (see Eqns 2) $\Omega_e$ and $N_e$ are denoted, respectively, $\Omega(e)$ and $N(e)$, while, in [10], they are denoted, respectively, by $f_1(e)$ and $f_2(e)$. 

\[^2\] In [6] (see Eqns 2) $\Omega_e$ and $N_e$ are denoted, respectively, $\Omega(e)$ and $N(e)$, while, in [10], they are denoted, respectively, by $f_1(e)$ and $f_2(e)$. 

\[^3\] In [6] (see Eqns 2) $\Omega_e$ and $N_e$ are denoted, respectively, $\Omega(e)$ and $N(e)$, while, in [10], they are denoted, respectively, by $f_1(e)$ and $f_2(e)$. 

\[^4\] In [6] (see Eqns 2) $\Omega_e$ and $N_e$ are denoted, respectively, $\Omega(e)$ and $N(e)$, while, in [10], they are denoted, respectively, by $f_1(e)$ and $f_2(e)$.
and the polar angle (see Figure 1); the eccentric anomaly $u = u_e(t)$ is defined implicitly by the Kepler equation\(^4\)

$$t = u - e \sin(u).$$

Notice that the Newtonian potential $f(x,t)$ is a doubly-periodic function of $x$ and $t$, with periods $2\pi$.

**Remarks:**

(i) The principal moments of an ellipsoid of mass $m$ and with axes $a$, $b$ and $c$ are given by

$$A = \frac{1}{5}m(b^2 + c^2), \quad B = \frac{1}{5}m(a^2 + c^2), \quad C = \frac{1}{5}m(a^2 + b^2).$$

The oblateness $\varepsilon$ is then given by

$$\varepsilon = \frac{3}{2} \frac{B - A}{C} = \frac{3}{2} \frac{a^2 - b^2}{a^2 + b^2}. \quad (6)$$

(ii) There is no universally accepted determination of the internal rigidity constant $K$ for most satellites of the Solar system\(^5\). For the Moon and Mercury an accepted value is $\sim 10^{-8}$; see, e.g., [3]. However, for our analysis to hold it will be enough that $\eta \leq 0.008$ for the moons and $\eta \leq 0.001$ for Mercury.

The known physical parameter values of the eighteen moons of the Solar system needed for our analysis are reported in the following table\(^6\):

\(^3\)The analytic expression of the true anomaly in terms of the eccentric anomaly is given by $f_e(t) = 2 \arctan \left( \frac{1}{1 + \sqrt{1 + e^2 \tan^2 \left( \frac{u_e(t)}{2} \right)}} \right)$.

\(^4\)As well known (see [11]) $e = u_e(t)$ is, for every $t \in \mathbb{R}$, holomorphic for $|e| < r_*$, with $r_* := \max_{y \in \mathbb{R}} \frac{y}{\cosh(y)} = \frac{y_*}{\cosh(y_*)} = 0.6627434 \cdots$ and $y_* = 1.1996786 \cdots$.

\(^5\)See, however:
Lainey, V.; et al.. Strong Tidal Dissipation in Saturn and Constraints on Enceladus’ Thermal State from Astrometry. The Astrophysical Journal, Volume 752, Issue 1, article id. 14, 19 pp. (2012);

\(^6\)\(a \geq b\) denote the maximal and minimal observed equatorial radii, which, in our model, are assumed to be the axes of the ellipse modeling the equatorial section of the satellite; the dimensions of the polar radius are not relevant in our model, however, for all the cases considered in this paper it turns out to be always smaller or equal than the smallest equatorial radius.
| Principal body | Satellite | Eccentricity \( e \) \((km)\) | \( a \) \((km)\) | \( b \) \((km)\) | Oblateness \( \nu \) | \( \nu \) |
|----------------|-----------|-----------------|---------------|---------------|-----------------|-------|
| Earth Moon\(^{\dagger}\) | 0.0549 | 1740.19 | 1737.31 | 0.00248454179 | 1.018088056 |
| Jupiter Io\(^{\star}\) | 0.0041 | 1829.7 | 1819.2 | 0.00863266715 | 1.00010086 |
| Europa | 0.0994 | 1561.3 | 1560.3 | 0.00096104552 | 1.000530163 |
| Ganymede | 0.0111 | 2632.9 | 2629.5 | 0.0019382783 | 1.00000726 |
| Callisto | 0.0074 | 2411.8 | 2408.8 | 0.00186909769 | 1.000328561 |
| Saturn Mimas\(^{\star}\) | 0.0193 | 208.3 | 196.2 | 0.08966019091 | 1.002234993 |
| Enceladus\(^{\star}\) | 0.0047 | 257.2 | 251.2 | 0.03540026218 | 1.00013254 |
| Tethys\(^{\star}\) | 0.0001 | 538.7 | 527.0 | 0.03293212897 | 1.00000006 |
| Dione\(^{\star}\) | 0.0022 | 564.0 | 560.8 | 0.0853478156 | 1.00002904 |
| Rhea\(^{\star}\) | 0.001 | 766.8 | 761.8 | 0.098127957 | 1.00000006 |
| Titan\(^{\star}\) | 0.0288 | 2575.239 | 2574.932 | 0.0017882901 | 1.00497691 |
| Iapetus\(^{\star}\) | 0.0283 | 748.9 | 743.1 | 0.01166202215 | 1.004805592 |
| Uranus Ariel\(^{\star}\) | 0.0012 | 582.0 | 577.3 | 0.012162311957 | 1.000000864 |
| Umbriel\(^{\star}\) | 0.0039 | 587.5 | 581.9 | 0.01436601227 | 1.00009126 |
| Titania\(^{\star}\) | 0.0011 | 790.7 | 787.1 | 0.00684493838 | 1.000000726 |
| Oberon\(^{\star}\) | 0.0014 | 764.0 | 758.8 | 0.01024146739 | 1.00001176 |
| Miranda\(^{\star}\) | 0.0013 | 241.0 | 233.3 | 0.0489051956 | 1.00001014 |
| Pluto Charon\(^{\dagger}\) | 0.0022 | 605.0 | 602.2 | 0.00695821306 | 1.00002904 |

Table 1. Physical data of the moons in 1:1 spin–orbit resonance

\[^{\dagger}\text{Runcorn, S. K.; Hofmann, S. Proceedings from IAU Symposium no. 47, Dordrecht, Reidel (1972).}\]
\[^{\star}\text{Thomas, P. C.; et al. Icarus 135 (1998).}\]
\[^{\ddagger}\text{Thomas, P. C.; Icarus 73 (1988).}\]
\[^{\dagger}\text{Dougherty, M.K.; al. (eds.) DOI 10.1007/978-1-4020-9217-6_24, (2009).}\]
\[^{\dagger}\text{Iess, L.; et al. Science 327 (2010).}\]
\[^{\dagger}\text{Sicardy, B., et al. Nature 439 (2006)}\]

The corresponding data of Mercury are:

| Principal body | Satellite | Eccentricity \( e \) \((km)\) | \( a \) \((km)\) | \( b \) \((km)\) | Oblateness \( \nu \) | \( \nu \) |
|----------------|-----------|-----------------|---------------|---------------|-----------------|-------|
| Sun Mercury | 0.2056 | 2440.7 | 2439.7 | 0.0001470369 | 1.255835458 |

Table 2. Physical data of Mercury (3:2 spin–orbit resonance)

\[^{\dagger}\text{http://ssd.jpl.nasa.gov/?sat_phys_par} \text{and http://ssd.jpl.nasa.gov/?sat_elem}\]

\[^{\dagger}\text{Existence Theorem for Solar System spin–orbit resonances}\]

In this framework, a \( p q \) spin–orbit resonance (with \( p \) and \( q \) co–prime non–vanishing integers) is, by definition, a solution \( t \in \mathbb{R} \rightarrow x(t) \in \mathbb{R} \) of (1) such that

\[
x(t + 2\pi q) = x(t) + 2\pi p \quad (7)
\]
indeed, for such orbits, after \( q \) revolutions of the orbital radius, \( x \) has made \( p \) complete rotations\(^7\).

Our main result can, now, be stated as follows

**Theorem [moons]** The differential equation (1) (a) \( \div (d) \) admits spin–orbit resonances (7) with \( p = q = 1 \) provided \( e, \nu \) and \( \varepsilon \) are as in Table 1 and \( 0 \leq \eta \leq 0.008 \).

[Mercury] The differential equation (1) (a) \( \div (d) \) admits spin–orbit resonances (7) with \( p = 3 \) and \( q = 2 \) provided \( e, \nu \) and \( \varepsilon \) are as in Table 2 and \( 0 \leq \eta \leq 0.001 \).

In [2] (compare Theorem 1.2) existence of spin–orbit resonances with \( q = 1, 2, 4 \) and any \( p \) (co–prime with \( q \)) is proved\(^8\), while in [5] quasi–periodic solutions, corresponding to \( p/q \) irrational are studied in the same model. In [2] no explicit computations of constants (size of admissible \( \varepsilon \), size of admissible \( \eta \), ...) have been carried out.

The main point of this paper is to compute all constants explicitly in order to get nearly optimal estimates and include all cases of physical interest.

2 Proof of the theorem

**Step 1.** Reformulation of the problem of finding spin–orbit resonances

Let \( x(t) \) be a \( p:q \) spin–orbit resonance and let \( u(t) := x(qt) - pt - \xi \). Then, by (7) and choosing \( \xi \) suitably one sees immediately that \( u \) is \( 2\pi \)–periodic and satisfies the differential equation

\[
\frac{d^2}{dt^2} u(t) + \hat{\eta} (u'(t) - \hat{\nu}) + \hat{\varepsilon} f_x(\xi + pt + u(t), qt) = 0, \quad \langle u \rangle = 0,
\]

where \( \langle \cdot \rangle \) denotes the average over the period\(^9\) and

\[
\hat{\eta} := q\eta, \quad \hat{\nu} := q\nu - p, \quad \hat{\varepsilon} := q^2 \varepsilon.
\]

Separating the linear part from the non–linear one, we can rewrite (8) as follows: let

\[
\begin{cases}
Lu := u'' + \hat{\eta} u' \\
\Phi_\xi(u)(t) := \hat{\nu} - \hat{\varepsilon} f_x(\xi + pt + u(t), qt)
\end{cases}
\]

then, the differential equation in (8) is equivalent to

\[
Lu = \Phi_\xi(u).
\]

---

\(^7\)Of course, in physical space, \( x \) and \( t \) being angles, are defined modulus \( 2\pi \), but to keep track of the topology (windings and rotations) one needs to consider them in the universal cover \( \mathbb{R}/(2\pi\mathbb{Z}) \).

\(^8\)The procedure consisting in reducing the problem to a fixed point one containing parameters: the question is then solved by a Lyapunov–Schmidt or “range–bifurcation” decomposition. The “range equation” is solved by standard contraction mapping methods, but in order of the fixed point to correspond to a true solution of the original problem a compatibility (zero–mean) condition has to be satisfied (“the bifurcation equation”) and this is done exploiting a free parameter by means of a topological argument.

\(^9\)The parameter \( \xi \) is given by \( (1/2\pi) \int_0^{2\pi} (x(qt) - pt) dt \) and will be our “bifurcation parameter”.
Step 2. The Green operator $\mathcal{G} = L^{-1}$

Let $\mathcal{C}_k^{\text{per}}$ be the Banach space of $2\pi$–periodic $C^k(\mathbb{R})$ functions endowed with the $C^k$–norm; let $\mathcal{C}_k^{\text{per},0}$ be the closed subspace of $\mathcal{C}_k^{\text{per}}$ formed by functions with vanishing average over $[0, 2\pi]$; finally, denote by $\mathcal{B} := \mathcal{C}_0^{\text{per},0}$ the Banach space of $2\pi$–periodic continuous functions with zero average (endowed with the sup–norm).

The linear operator $L$ defined in (10) maps injectively $\mathcal{C}_2^{\text{per},0}$ onto $\mathcal{B}$; the inverse operator (the “Green operator”) $G = L^{-1}$ is a bounded linear isomorphism. Indeed, the following elementary Lemma holds:

**Lemma 2.1.** Let $\hat{\eta} < 2/\pi$. Then

$$\|G\|_{L(\mathcal{B},\mathcal{B})} \leq \left(1 + \frac{\hat{\eta}}{\pi} \left(1 - \frac{\hat{\eta}}{2}\right)^{-1}\right) \frac{\pi^2}{8}.$$ \hfill (11)

In particular, assuming

$$\hat{\eta} \leq \frac{\pi}{5} \left(\frac{10}{\pi^2} - 1\right),$$ \hfill (12)

one gets

$$\|G\|_{L(\mathcal{B},\mathcal{B})} \leq \frac{5}{4}.$$ \hfill (13)

The proof of the above lemma is based on the following elementary result, whose proof is given in Appendix A.

**Lemma 2.2.**

$$v \in \mathcal{C}_1^{\text{per},0} \implies \|v\|_{C^0} \leq \frac{\pi}{2} \|v'\|_{C^0}$$ \hfill (14)

$$v \in \mathcal{C}_2^{\text{per},0} \implies \|v\|_{C^0} \leq \frac{\pi^2}{8} \|v''\|_{C^0}$$ \hfill (15)

**Proof of Lemma 2.1** Given $g \in \mathcal{B}$ with $\|g\|_{C^0} = 1$ we have to prove that if $u \in \mathcal{C}_2^{\text{per},0}$ is the unique solution of $u'' + \hat{\eta} u' = g$ with $\langle u \rangle = 0$, then

$$\|u\|_{C^0} \leq \left(1 + \hat{\eta} \frac{\pi}{2} \left(1 - \frac{\hat{\eta}}{2}\right)^{-1}\right) \frac{\pi^2}{8}.$$ \hfill (16)

We note that, setting $v := u'$, we have that $v \in \mathcal{B}$ and $v' = -\hat{\eta} v + g$. Then we get

$$\|v\|_{C^0} \overset{(14)}{\leq} \frac{\pi}{2} \|v\|_{C^0} - \hat{\eta} v + g \|_{C^0} \leq \frac{\pi}{2} \langle \hat{\eta} v\rangle_{C^0} + 1,$$

which implies

$$\|u'\|_{C^0} = \|v\|_{C^0} \leq \left(1 - \frac{\pi}{2} \hat{\eta}\right)^{-1} \frac{\pi}{2}.\hfill (17)$$

---

10$\|v\|_{C^k} := \sup_{0 \leq j \leq k} \sup_{t \in \mathbb{R}} |D^j v(t)|.$

11$\|G\|_{L(\mathcal{B},\mathcal{B})} = \sup_{\|u\|_{C^0} = 1} \|G(u)\|_{C^0}.$

12It is easy to see that the estimates in Lemma 2.2 are sharp.
Since \( u'' = -\hat{\eta} u' + g \), we have

\[
\|u\|_{C^0} \leq \frac{\pi^2}{8} \| -\hat{\eta} u' + g\|_{C^0} \leq \frac{\pi^2}{8} (1 + \hat{\eta} \|u'\|_{C^0})
\]

and (16) follows by (17).

Step 3. Lyapunov–Schmidt decomposition

Solutions of (11) are recognized as fixed points of the operator \( G \circ \Phi_\xi \):

\[
u = G \circ \Phi_\xi(u),
\]

where \( \xi \) appears as a parameter.

To solve equation (18), we shall perform a Lyapunov–Schmidt decomposition. Let us denote by \( \hat{\Phi}_\xi \) : \( C^0_{\text{per}} \rightarrow B = C^0_{\text{per,0}} \) the operator

\[
\hat{\Phi}_\xi(u) := \frac{1}{\hat{\epsilon}} \left[ \Phi_\xi(u) - \langle \Phi_\xi(u) \rangle \right] = -f_x(\xi + pt + u(t), qt) + \phi_u(\xi);
\]

where

\[
\phi_u(\xi) := \frac{1}{2\pi} \int_0^{2\pi} f_x(\xi + pt + u(t;\xi), qt) dt .
\]

Then, equation (18) can be splitted into a “range equation”

\[
u = \hat{\epsilon} G \circ \hat{\Phi}_\xi(u),
\]

(19)

and a “bifurcation (or kernel) equation”

\[
\phi_u(\xi) = \frac{\hat{\eta}}{\hat{\epsilon}} \iff \langle \Phi_\xi(u;\xi) \rangle = 0.
\]

Remark 2.3. (i) If \((u, \xi) \in B \times [0, 2\pi]\) solves (21)&(22), then, \(x(t)\) solves (1).

(ii) \( \forall \xi \in [0, 2\pi], \Phi_\xi \in C^1(B, B) \); indeed, \( \forall(u, \xi) \in B \times [0, 2\pi] \),

\[
\|\hat{\Phi}_\xi(u)\|_{C^0} \leq 2 \sup_{\xi} |f_x| , \quad \|D_u \hat{\Phi}_\xi\|_{\mathcal{L}(B, B)} \leq 2 \sup_{\xi} |f_{xx}| .
\]

The usual way to proceed to solve (21)&(22) is the following:

1. for any \( \xi \in [0, 2\pi] \), find \( u = u(\cdot; \xi) \) solving (21);

2. insert \( u = u(\cdot, \xi) \) into the kernel equation (22) and determine \( \xi \in [0, 2\pi] \) so that (22) holds.
Step 4. Solving of the range equation (contracting map method)

For $\dot{\varepsilon}$ small the range equation is easily solved by standard contraction arguments. Let $R := \frac{5}{2} \varepsilon \sup_{T^2} |f_x|$ and let

\[
\begin{align*}
\mathbb{B}_R &:= \{ v \in \mathbb{B} : \|v\|_{C^0} \leq R \}, \\
\varphi : v \in \mathbb{B}_R &\rightarrow \varphi(v) := \dot{\varepsilon} \mathcal{G} \circ \hat{\Phi}_\xi(v).
\end{align*}
\]

(24)

Proposition 2.4. Assume that $\dot{\eta}$ satisfies (12) and that

\[
\frac{5}{2} \dot{\varepsilon} \sup_{T^2} |f_{xx}| < 1.
\]

(25)

Then, for every $\xi \in [0, 2\pi]$, there exists a unique $u := u(\cdot; \xi) \in \mathbb{B}_R$ such that $\varphi(u) = u$.

Proof. By (12) and (23) the map $\varphi$ in (24) maps $\mathbb{B}_R$ into itself and is a contraction with Lipschitz constant smaller than 1 by (25). The proof follows by the standard fixed point theorem.

Recalling (3), (4) and (9), the “range condition” (25) writes

\[
\varepsilon < \begin{cases} 
\frac{(1-\varepsilon)^3}{5}, & \text{if } (p, q) = (1, 1), \\
\frac{(1-\varepsilon)^3}{20}, & \text{if } (p, q) = (3, 2).
\end{cases}
\]

(26)

Step 5. Solving the bifurcation equation (22)

The function $\phi_u(\xi)$ in (20) can be written as

\[
\phi(\xi) = \phi^{(0)}(\xi) + \dot{\varepsilon} \check{\phi}^{(1)}_u(\xi; \dot{\varepsilon})
\]

(27)

with

\[
\phi^{(0)}(\xi) := \frac{1}{2\pi} \int_0^{2\pi} f_x(\xi + pt, qt) dt.
\]

(28)

By (24), for $\varepsilon$ satisfying (26),

\[
\sup_{\xi \in [0, 2\pi]} \left| \check{\phi}^{(1)}_u \right| \leq \frac{R \varepsilon}{\sup_{T^2} |f_{xx}|} \leq \frac{5}{2} \left( \sup_{T^2} |f_x| \right) \left( \sup_{T^2} |f_{xx}| \right).
\]

(29)

By (3), (4), for $\varepsilon$ satisfying (26), one finds immediately that

\[
\sup_{\xi \in [0, 2\pi]} \left| \check{\phi}^{(1)}_u \right| \leq M_1 := \frac{5}{(1-\varepsilon)^6}.
\]

(30)

Let us, now, have a closer look at the zero order part $\phi^{(0)}$. The Newtonian potential $f$ has the Fourier expansion

\[
f(x, t) = \sum_{j \in \mathbb{Z}, j \neq 0} \alpha_j \cos(2x - jt),
\]

(31)
where the Fourier coefficients $\alpha_j = \alpha_j(e)$ coincide with the Fourier coefficients of

$$G_e(t) := -\frac{e^{2i\mu(t)}}{2p_e(t)^3} = \sum_{j \in \mathbb{Z}, j \neq 0} \alpha_j \exp(ijt), \quad (32)$$

(see Appendix B). Thus,

$$f_x(\xi + pt, qt) = -2 \sum_{j \in \mathbb{Z}, j \neq 0} \alpha_j \sin(2\xi + (2p - jq)t)$$

and one finds:

$$\phi^{(0)}(\xi) = \begin{cases} 
-2\alpha_2 \sin(2\xi), & \text{if } (p, q) = (1, 1), \\
-2\alpha_3 \sin(2\xi), & \text{if } (p, q) = (3, 2).
\end{cases} \quad (33)$$

Define

$$a_{pq} := \begin{cases} 
2|\alpha_2| - \hat{\varepsilon}M_1, & \text{if } (p, q) = (1, 1), \\
2|\alpha_3| - \hat{\varepsilon}M_1, & \text{if } (p, q) = (3, 2).
\end{cases} \quad (34)$$

Then, from (27), (30), (33) and (34), it follows that $\phi([0, 2\pi])$ contains the interval $[-a_{pq}, a_{pq}]$, which is not empty provided (recall (9) and (30))

$$\varepsilon < \begin{cases} 
\frac{2(1-e)^6}{5}|\alpha_2(e)|, & \text{if } (p, q) = (1, 1), \\
\frac{(1-e)^6}{10}|\alpha_3(e)|, & \text{if } (p, q) = (3, 2).
\end{cases} \quad (35)$$

Therefore, we can conclude that the bifurcation equation (22) is solved if one assumes that $|\hat{\mu}/\varepsilon| \leq a_{pq}$, i.e. (recall again (9), (30) and (34)), if

$$\eta < \begin{cases} 
\frac{\varepsilon}{|\eta - 1|} \left(2|\alpha_2(e)| - \frac{5\varepsilon}{1-e}p\right), & \text{if } (p, q) = (1, 1), \\
\frac{2\varepsilon}{2p-3} \left(2|\alpha_3(e)| - \frac{20\varepsilon}{1-\varepsilon}p\right), & \text{if } (p, q) = (3, 2).
\end{cases} \quad (36)$$

We have proven the following:

**Proposition 1.** Let $(p, q) = (1, 1)$ or $(p, q) = (3, 2)$ and assume (12), (26), (35) and (36). Then, (1) admits $pq$ spin–orbit resonances $x(t)$ as in (7).

**Step 6. Lower bounds on $|\alpha_2(e)|$ and $|\alpha_3(e)|$**

In order to complete the proof of the Theorem, checking the conditions of Proposition 1 for the resonant satellites of the Solar system, we need to give lower bounds on the absolute values of the Fourier coefficients $\alpha_2(e)$ and $\alpha_3(e)$. To do this we will simply use Taylor formula to develop $\alpha_j(e)$ in power of $e$ up to a suitably large order\(^\text{13}\)

$$\alpha_j(e) = \sum_{k=0}^{h} a_j^{(k)} e^k + R_j^{(h)}(e) \quad (37)$$

\(^{13}\)We shall choose $h = 4$ for the 1:1 resonances and $h = 21$ for the 3:2 case of Mercury.
Lemma 2. Fix $0 < b < 1$. The solution $u_e(t)$ of the Kepler equation (5) is, for every $t \in \mathbb{R}$, holomorphic with respect to $e$ in the complex disk

$$|e| < e_* := \frac{b}{\cosh b}$$

and satisfies

$$\sup_{t \in \mathbb{R}} |u_e(t) - t| \leq b.$$ 

Moreover $\rho_e(t) = 1 - e \cos(u_e(t))$ satisfies

$$|\rho_e(t)| \geq 1 - b, \quad \forall t \in \mathbb{R}, \ |e| < e_*$$

and $G_e(t)$ (defined in (32)) satisfies

$$|G_e(t)| \leq \frac{2}{(1-b)^5} \left[ (1-e)(1+\cosh b)+1-b \right]^2, \quad \forall t \in \mathbb{R}, \ |e| < e_*.$$ 

Proof. Using that

$$\sup_{|\text{Im } z| < b} |\sin z| = \sup_{|\text{Im } z| < b} |\cos z| = \cosh b,$$

one sees that for $|e| < e_*$ the map $v \mapsto \chi_e(v)$ with $[\chi_e(v)](t) := e \sin (v(t) + t)$ is a contraction in the closed ball of radius $b$ in the space of continuous functions endowed with the sup-norm. Moreover, since $\chi_e(v)$ is holomorphic in $e$, the same holds for the fixed point $v_e(t)$ of $\chi_e$. The estimate in (39) follows by observing that $u_e(t) = v_e(t) + t$. Since by (39) we get

$$|\text{Im } (u_e(t))| \leq b, \quad \forall t \in \mathbb{R}, \ |e| < e_*,$$

estimate (40) follows by

$$|\rho_e(t)| \geq 1 - |e||\cos(u_e(t))| \geq 1 - e_* \cosh b = 1 - b.$$ 

Next, let $w_e(t) := \sqrt{\frac{1+e}{1-e}} \tan \left( \frac{u_e(t)}{2} \right)$ so that $f_e = 2 \arctan w_e$. Then,\footnote{Use $e^{2iz} = \frac{i-w}{w+i} = -\frac{(w-i)^2}{w^2+1}$ and $\tan^2(\alpha/2) = (1-\cos \alpha)/(1+\cos \alpha)$.}

$$|e^{2u_e(t)}| = \frac{|w - i|^4}{|w^2 + 1|^2} \leq \left( \frac{4}{|w^2 + 1|^2} + 2 \right)^2 = 4 \left( \frac{|1-e||1+\cos u_e|}{|1-e \cos u_e| + 1} \right)^2.$$ 

Then (41) follows by (40), (42) and (43).

Lemma 3. Let $R_j^{(h)}(e)$ be as in (37), $0 < b < 1$ and $0 < e < b/\cosh b$. Then,

$$|R_j^{(h)}(e)| \leq R_j^{(h)}(e; b)$$

with

$$R_j^{(h)}(e; b) := \frac{2}{(1-b)^5} \left( 1 + \frac{b}{\cosh b} - e \right)(1+\cosh b)+1-b \right)^2 \frac{e^{h+1}}{(e\cosh b - e)^{h+1}}.$$
Proof. For \( \mathbf{e}, \rho > 0 \) we set

\[
\begin{align*}
[0, \mathbf{e}]_{\rho} := \{ z \in \mathbb{C}, \text{ s.t. } z = z_1 + z_2, \ z_1 \in [0, \mathbf{e}], \ |z_2| < \rho \}.
\end{align*}
\]

Lemma 2 and standard (complex) Cauchy estimates imply, for \( 0 \leq s \leq 1 \),

\[
|D^{h+1}_s \alpha_j(s\mathbf{e})| \leq \frac{(h+1)!}{(e_* - \mathbf{e})^{h+1}} \sup_{[0, \mathbf{e}]_{e_* - \mathbf{e}}} |\alpha_j|
\]

and, therefore,

\[
|R_j^{(h)}(\mathbf{e})| \leq \frac{e^{h+1}}{(e_* - \mathbf{e})^{h+1}} \sup_{[0, \mathbf{e}]_{e_* - \mathbf{e}}} |\alpha_j|.
\]

By (41) we obtain

\[
\sup_{[0, \mathbf{e}]_{e_* - \mathbf{e}}} |\alpha_j| \leq \frac{2}{(1 - b)^2} ((1 + e_* - \mathbf{e})(1 + \cosh b) + 1 - b)^2
\]

from which, recalling (38), the lemma follows.

Now, in order to check the conditions of Proposition 1 we will expand \( \alpha_2 \) in power of \( \mathbf{e} \) up to order \( h = 4 \) and \( \alpha_3 \) up to order \( h = 21 \). Using the representation formula (53) for the \( \alpha_j \) given in Appendix B we find:

\[
\begin{align*}
\alpha_2(\mathbf{e}) &= - \frac{1}{2} + \frac{5}{4} \mathbf{e}^2 - \frac{13}{32} \mathbf{e}^4 + R_2^{(4)}(\mathbf{e}), \\
\alpha_3(\mathbf{e}) &= - \frac{7}{4} \mathbf{e} + \frac{123}{32} \mathbf{e}^3 - \frac{489}{256} \mathbf{e}^5 + \frac{1763}{4096} \mathbf{e}^7 - \frac{13527}{32768} \mathbf{e}^9 + \frac{180369}{1310720} \mathbf{e}^{11} \\
&+ \frac{5986093}{734003200} \mathbf{e}^{13} + \frac{24606987}{3355443200} \mathbf{e}^{15} + \frac{33790034193}{5261334937600} \mathbf{e}^{17} \\
&+ \frac{1193558821627}{210453397504000} \mathbf{e}^{19} + \frac{467145991400853}{9259494901760000} \mathbf{e}^{21} + R_3^{(21)}(\mathbf{e}).
\end{align*}
\]

In view of Lemma 3, we choose, respectively, \( b = 0.462678 \) and \( 15 \ b = 0.768368 \) to get lower bounds:

\[
\begin{align*}
|\alpha_2(\mathbf{e})| \geq & \left| \frac{1}{2} - \frac{5}{4} \mathbf{e}^2 + \frac{13}{32} \mathbf{e}^4 \right| - |R_2^{(4)}(\mathbf{e}; 0.462678)| \quad (44) \\
|\alpha_3(\mathbf{e})| \geq & \left| \sum_{k=1}^{21} \alpha_3^{(k)} \mathbf{e}^k \right| - |R_3^{(21)}(\mathbf{e}; 0.768368)|. \quad (45)
\end{align*}
\]

Step 7. Check of the conditions and conclusion of the proof

We are now ready to check all conditions of Proposition 1 with the parameters of the satellite in spin–orbit resonance given in Table 1 and 2.

In the following Table we report:

- in column 2: the lower bounds on \( |\alpha_q(\mathbf{e})| \) as obtained in Step 6 using (44) and (45) (with the eccentricities listed in Table 1 and 2);

\[15\text{The values for } b \text{ are rather arbitrary (as long as } 0 < b < 1\text{); our choice is made for optimizing the estimates.} \]
in column 3: the difference between the right hand side and the left hand side of the inequality\(^{16}\) (26);

in column 4: the difference between the right hand side and the left hand side of the inequality (35);

in column 5: the right hand side of the inequality (36), which is an upper bound for the admissible values of the dissipative parameter \(\eta\).

| Satellite | lower bound on \(|\alpha_q|\) | r.h.s. – l.h.s. of Eq. (26) | r.h.s. – l.h.s. of Eq. (35) | r.h.s. of Eq. (36) |
|-----------|-------------------------------|-----------------------------|-----------------------------|------------------|
| Moon      | 0.45475265                    | 0.1663508                   | 0.127144                    | 0.1225335        |
| Io        | 0.49997893                    | 0.1889174                   | 0.186489                    | 81.800325        |
| Europa    | 0.49988598                    | 0.1934518                   | 0.187978                    | 1.8031043        |
| Ganymede  | 0.49999849                    | 0.1974024                   | 0.196745                    | 264.3751         |
| Callisto  | 0.49993049                    | 0.1937258                   | 0.189389                    | 5.6260066        |
| Mimas     | 0.49938883                    | 0.0989819                   | 0.088051                    | 19.852395        |
| Enceladus | 0.49997228                    | 0.161793                    | 0.159015                    | 218.44519        |
| Tethys    | 0.49999999                    | 0.1670079                   | 0.166948                    | 458.43746        |
| Dione     | 0.49999395                    | 0.1901481                   | 0.188837                    | 281.18521        |
| Rhea      | 0.49999875                    | 0.1895878                   | 0.18899                     | 1554.7362        |
| Titan     | 0.49776167                    | 0.1830341                   | 0.166905                    | 0.0357326        |
| Iapetus   | 0.49790449                    | 0.171834                    | 0.155986                    | 2.2484865        |
| Ariel     | 0.4999982                     | 0.187186                    | 0.186401                    | 1321.448         |
| Umbriel   | 0.49999895                    | 0.1833031                   | 0.180992                    | 145.83674        |
| Titania   | 0.49999849                    | 0.1924958                   | 0.191838                    | 910.34423        |
| Oberon    | 0.49999755                    | 0.188917                    | 0.188081                    | 826.10305        |
| Miranda   | 0.49999789                    | 0.1505305                   | 0.149754                    | 3623.6286        |
| Charon    | 0.49999395                    | 0.1917247                   | 0.190414                    | 231.15781        |
| Mercury   | 0.27                          | 0.0244515                   | 0.006171                    | 0.0012363        |

Table 3. Check of the hypotheses of Proposition 1 for the satellites in spin–orbit resonance

The positive value reported in the third and fourth column means that the range condition (26) and the topological condition (35) are satisfied for all the moons in 1:1 resonance and for Mercury; the bifurcation condition (36) yields an upper bound on the admissible value for \(\eta\) (fifth column). Thus, \(\eta\) has to be smaller than the minimum between the value in the fifth column of Table 3 and the value in the right hand side of Eq. (12) (needed to give a bound on the Green operator): such minimum values is seen to be 0.008 \ldots for the moons in 1:1 resonance and 0.001 \ldots for Mercury.

The proof of the Theorem is complete. \[\square\]

\(^{16}\)Thus, the inequality is satisfied if the numerical value in the column is positive; the same applies to the 5th column.
A Proof of Lemma 2.2

Proof. We first prove (14). Up to a rescaling we can prove (14) assuming \( \|v\|_{C^0} = 1 \). Assume by contradiction that
\[
\|v\|_{C^0} =: c > \pi/2.
\]

Note that it is obvious that \( c \leq \pi \) since \( v \) has zero average and, therefore, it must vanish at some point. Up to a translation we can assume that \( |v| \) attains maximum at \(-c\). In case, multiplying by \(-1\), we can also assume that \(-c\) is a minimum namely
\[
\|v\|_{C^0} = c = -v(-c).
\]

Since \( \|v\|_{C^0} = 1 \) we get
\[
v(t) \leq -c + |t + c| \quad \forall t \in [-2c, 0]
\]
and, therefore,
\[
v(0) \leq 0, \quad v(-2c) \leq 0, \quad \int_{-2c}^{0} v \leq -c^2. \tag{46}
\]

Since \( \|v\|_{C^0} = 1 \) we also get
\[
v(t) \leq \pi - c - |t - \pi + c| \quad \forall t \in [0, 2\pi - 2c].
\]

Then
\[
\int_{0}^{2\pi - 2c} v \leq (\pi - c)^2.
\]

Combining with the last inequality in (46) we get
\[
\int_{-2c}^{2\pi - 2c} v \leq (\pi - c)^2 - c^2 = \pi(\pi - 2c) < 0,
\]
which contradicts the fact that \( v \) has zero average, proving (14)

We now prove (15). Up to a rescaling we can prove (15) assuming \( \|v''\|_{C^0} = 1 \). Assume by contradiction that
\[
\|v\|_{C^0} =: c > \pi^2/8. \tag{47}
\]

Up to a translation we can assume that \( |v| \) attains maximum at 0. In case, multiplying by \(-1\), we can also assume that \(-c\) is a minimum namely
\[
\|v\|_{C^0} = c = -v(0).
\]

Since \( \|v''\|_{C^0} = 1 \) we get
\[
v(t) \leq -c + t^2/2 \quad \forall t \in \mathbb{R}.
\]

Since \( v \) has zero average must exist
\[
t_1 \leq -\sqrt{2c}, \quad t_2 \geq \sqrt{2c}, \quad t_2 - t_1 < 2\pi, \quad \text{s.t.} \quad v(t_1) = v(t_2) = 0, \quad v(t) < 0 \quad \forall t \in (t_1, t_2). \tag{48}
\]

Moreover
\[
\int_{t_1}^{t_2} v \leq \int_{-\sqrt{2c}}^{\sqrt{2c}} -c + \frac{1}{2}t^2 = -\frac{2}{3}(2c)^{3/2}.
\]
Since \( v \) has zero average and is \( 2\pi \)-periodic
\[
\int_{t_2}^{2\pi + t_1} v = -\int_{t_1}^{t_2} v \geq \frac{2}{3}(2c)^{3/2} .
\] (49)

Set
\[ a := \pi + (t_1 - t_2)/2 \]
and note that
\[ 0 < a \leq \pi - \sqrt{2c} < \pi/2 , \quad a^2 < 2c \] (50)
by (48) and (47). Set
\[ u(t) := v(t + \pi + (t_1 + t_2)/2) . \]

Note that \( u \in B \cap C^2 \) and, by (48),
\[ \| u \|_{C^0} = c , \quad \| u'' \|_{C^0} = 1 , \quad u(-a) = u(a) = 0 , \quad \int_{-a}^{a} u = \int_{t_2}^{2\pi + t_1} v \geq \frac{2}{3}(2c)^{3/2} . \]

Consider now the even function
\[ w(t) := \frac{1}{2}(u(t) + u(-t)) . \]

Note that \( w \in B \cap C^2 \) and
\[ \| w \|_{C^0} \leq c , \quad \| w'' \|_{C^0} \leq 1 , \quad w(-a) = 0 , \quad \int_{-a}^{0} w = \frac{1}{2} \int_{-a}^{a} u \geq \frac{1}{3}(2c)^{3/2} . \] (51)

Set
\[ z(t) := c - \frac{c}{a^2} t^2 . \]

We claim that
\[ z(t) \geq w(t) \quad \forall -a \leq t \leq 0 . \] (52)

Then
\[ \int_{-a}^{0} w \leq \int_{-a}^{0} z = \frac{2}{3} c a^2 < \frac{1}{3}(2c)^{3/2} \leq \int_{-a}^{0} w , \]
which is a contradiction.

Let us prove the claim in (52). Note that \( z(-a) = w(-a) = 0 \). Assume by contradiction that there exists \( \bar{t} \in [-a, 0) \) such that
\[ z(\bar{t}) = w(\bar{t}) , \quad z(t) \geq w(t) \quad \forall t \in [-a, \bar{t}] , \quad z'(\bar{t}) \leq w'(\bar{t}) . \]

Then, since \( \| w'' \|_{C^0} \leq 1 \)
\[ w(t) \geq \frac{1}{2} c t^2 \]
\[ z(\bar{t}) + \frac{c}{a^2} (t - \bar{t})^2 = z(t) , \quad \forall t \in (\bar{t}, 0] . \]

Then
\[ w(0) > z(0) = c , \]
which contradicts the first inequality in (51). This completes the proof of (15).
B  Fourier coefficients of the Newtonian potential

Properties of the Fourier coefficients \( \alpha_j \) of the Newtonian potential \( f \), including Eq. \((32)\), have been discussed, e.g., in Appendix A of\(^{17}\) [2]. Here we provide a simple formula for the Fourier coefficients \( \alpha_j \) of the Newtonian potential \( f \) in \((3)\) (compare (d) of §1, and \((31)-(32)\)); namely we prove that

\[
\alpha_j = -\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{\rho^2(w^2 + 1)^2} \left[ (w^4 - 6w^2 + 1) c_j(u) - 4w(w^2 - 1) s_j(u) \right] du ,
\]

where \( w = w(u; e) := \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \), \( \rho = 1 - e \cos u \) and

\[
c_j(u) := \cos(ju - je \sin u), \quad s_j(u) := \sin(ju - je \sin u).
\]

\textbf{Proof.} If \( z = \arctan w \), then

\[
e^{2iz} = \frac{i - w}{w + i} = \frac{(w - i)^2}{w^2 + 1},
\]

so that if \( w_e(t) := w(u_e(t), e) \) one has \( f_e = 2 \arctan w_e \) and

\[
G_e = -\frac{1}{2\rho_e^3} \frac{(w_e - i)^2}{(w_e + i)^2} = -\frac{1}{2\rho_e^2} \frac{(w_e - i)^4}{(w_e^2 + 1)^2}
\]

\[
= -\frac{1}{2\rho_e^2} \frac{1}{(w_e^2 + 1)^2} \left( w_e^4 - 6w_e^2 + 1 - 4i w_e(w_e^2 - 1) \right).
\]

By parity properties, it is easy to see that the \( G_j \)'s are real, namely \( G_j = G_j \), so that

\[
\alpha_j = G_j = \frac{1}{2\pi} \int_0^{2\pi} G(t) e^{-ijt} dt = -\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i2f_e(t)-ijt}}{\rho_e(t)^3} dt
\]

\[
= -\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{\rho_e^3(w_e^2 + 1)^2} \left[ (w_e^4 - 6w_e^2 + 1) \cos(jt) - 4w_e(w_e^2 - 1) \sin(jt) \right] dt .
\]

Making the change of variable given by the Kepler equation \((5)\), i.e. integrating from \( t \) to \( u = u_e \) and setting \( u_e(t)' = \frac{1}{\rho_e(t)} \) one gets \((53)\). \( \blacksquare \)

C  Small bodies

In the Solar system besides the eighteen moons listed in Table 1 and Mercury there are other five minor bodies with mean radius smaller than 100 km observed in 1:1 spin–orbit resonance around their planets: Phobos and Deimos (Mars), Amalthea (Jupiter), Janus and Epimetheus (Saturn).

\(^{17}\)A factor \(-1/2\) is missing in the definition of \( G(t) \) given in [2], (iii) p. 4366 and, consequently, it has to be included at p. 4367 in line 6 (from above, counting also lines with formulas) in front of “Re”; in line 12, 17 and 18 the factor \( 1/(2\pi) \) has to be replaced by \(-1/(4\pi)\).
Besides being small, such bodies have also a quite irregular shape and only Janus and Epimetheus have a good equatorial symmetry.\(^\text{18}\) Indeed, for these two small moons (and only for them among the minor bodies), our theorem holds as shown by the data reported in the following table\(^\text{19}\):

| Satellite   | lower bound on \(|\alpha_q|\) | r.h.s. – l.h.s. of Eq. \((26)\) | r.h.s. – l.h.s. of Eq. \((35)\) | r.h.s. of Eq. \((36)\) |
|-------------|-------------------------------|---------------------------------|---------------------------------|-----------------|
| Janus       | 0.4999324                     | 0.1879319                       | 0.183652                        | 23.167321       |
| Epimetheus  | 0.49927518                    | 0.1699562                       | 0.158377                        | 6.3087689       |

Table 5. Check of the hypotheses of Proposition 1 for the small satellites in spin–orbit resonance

References

[1] Bambusi, D.; Haus, E.: *Asymptotic stability of synchronous orbits for a gravitating viscoelastic sphere*. Celestial Mech. Dynam. Astronom. 114, no. 3, 255–277 (2012).

[2] Biasco, L.; Chierchia, L.: *Low-order resonances in weakly dissipative spin-orbit models*. Journal of Differential Equations 246, 4345-4370 (2009).

[3] Celletti, A.: *Analysis of resonances in the spin-orbit problem in Celestial Mechanics: The synchronous resonance (Part I)*. J. Appl. Math. Phys. (ZAMP) 41, 174-204 (1990).

[4] Celletti, A.: *Stability and chaos in celestial mechanics*. In Stability and Chaos in Celestial Mechanics. Springer-Praxis, Providence, RI, 2010.

[5] Celletti, A; Chierchia, L.: *Quasi-periodic attractors in celestial mechanics* Arch. Rational Mech. Anal., 191, Issue 2, 311-345 (2009)

[6] Correia, A. C. M.; Laskar, J.: *Mercury’s capture into the 3/2 spin–orbit resonance as a result of its chaotic dynamics*, Nature 429 (24 June 2004), 848–850.

\(^{18}\)For pictures, see: http://photojournal.jpl.nasa.gov/catalog/PIA10369 (Phobos), http://photojournal.jpl.nasa.gov/catalog/PIA11826 (Deimos), http://photojournal.jpl.nasa.gov/catalog/PIA02532 (Amalthea), http://photojournal.jpl.nasa.gov/catalog/PIA12714 (Janus), http://photojournal.jpl.nasa.gov/catalog/PIA12700 (Epimetheus).

\(^{19}\)Positive values in the 3rd and 4th column and values less than 0.008 in the 5th column implies that the assumptions of Proposition 1 hold.
[7] Danby, J. M. A.: *Fundamentals of Celestial Mechanics*. Macmillan, New York 1962.

[8] Goldreich, P.; Peale, S.: *Spin-orbit coupling in the solar system*. Astronomical Journal 71, 425 (1967).

[9] MacDonald, G.J.F.: *Tidal friction*, Rev. Geophys. 2 (1964), 467–541.

[10] Peale, S.J.: *The free precession and libration of Mercury*, Icarus 178 (2005), 4–18.

[11] Wintner, A.: *The Analytic Foundations of Celestial Mechanics*. Princeton Univ. Press, Princeton, NJ, 1941.

[12] Wisdom, J.: *Rotational dynamics of irregularly shaped natural satellites*. Astron. J. 94, no. 5, 1350-1360 (1987).