Anyons in discrete gauge theories with Chern-Simons terms

F.Alexander Bais*, Peter van Driel† and Mark de Wild Propitius‡

Instituut voor Theoretische Fysica
Valckenierstraat 65
1018XE Amsterdam

February 1992

Abstract

We study the effect of a Chern-Simons term in a theory with discrete gauge group \( H \), which in (2+1)-dimensional space-time describes (non-abelian) anyons. As in a previous paper [6], we emphasize the underlying algebraic structure, namely the Hopf algebra \( D(H) \). We argue on physical grounds that the addition of a Chern-Simons term in the action leads to a non-trivial 3-cocycle on \( D(H) \). Accordingly, the physically inequivalent models are labelled by the elements of the cohomology group \( H^3(H, U(1)) \). It depends periodically on the coefficient of the Chern-Simons term which model is realized. This establishes a relation with the discrete topological field theories of Dijkgraaf and Witten. Some representative examples are worked out explicitly.

Preprint ITFA-92-8 submitted to: Physics Letters B
1 Introduction

It is well established by now that quantum statistics for identical particles in (2+1)-dimensional space time can be quite exotic. The reason is that interchanges of identical particles in the plane are not organized by the symmetric group, but rather by the braid group \([1]\). The 1-dimensional unitary irreducible representations of the braid group are labelled by an angular parameter \(\Theta \in [0, 2\pi]\). Interchanging two identical particles in such a representation yields a phase \(\exp i\Theta\), and we see that in the plane we have the possibility of quantum statistics intermediate \([2]\) between the familiar examples bosons \(\Theta = 0\) and fermions \(\Theta = \pi\). Particles obeying intermediate quantum statistics have been named anyons \([3]\). Of course, particles with quantum statistical properties associated with higher dimensional unitary irreducible representations of the braid group are also conceivable. We call these non-abelian anyons.

Anyons and non-abelian anyons may appear as topological excitations in theories where a continuous gauge group \(G\) is spontaneously broken down to a discrete group \(H\). Their long range topological interactions are effectively described by a discrete \(H\) gauge theory \([4]\). The underlying algebraic structure of these theories strongly resembles that of conformal field theories, in particular holomorphic orbifold models \([5]\). In a previous paper \([6]\), we showed that this underlying structure is the quasi-triangular Hopf algebra \(D(H)\) \([7]\). This insight allowed us to give a complete description of the invariant couplings (i.e. fusion rules) and the two particle Aharonov-Bohm scattering (i.e. braiding properties) of these anyonic excitations. In the present paper we include a Chern-Simons term and analyse its physical consequences. What we will show - and work out explicitly for \(H \simeq \mathbb{Z}_N\) - is that the Chern-Simons term leads to a deformation of the algebra \(D(H)\) by a non-trivial 3-cocycle. This implies among others, that the dependence of the effective theory on the quantized Chern-Simons parameter \(\mu\) is periodic. This provides a correspondence with the discrete topological field theories introduced by Dijkgraaf and Witten \([9]\).

The paper is organized as follows. In section 2, we briefly introduce a \(G \simeq SU(2)\) gauge model with Chern-Simons term. Next we determine from purely physical arguments the fusion- and braiding properties of the superselection sectors for \(H \simeq \mathbb{Z}_N\). The resulting mathematical structure is precisely the quasi-Hopf algebra \(D^\omega(H)\), with \(\omega\) a 3-cocycle. This structure is discussed in section 3. In section 4 we treat some examples in detail, starting with \(D^\omega(\mathbb{Z}_N)\). We show that this algebra indeed gives the same fusion- and braiding properties as established in section 2 for a \(\mathbb{Z}_N\) gauge theory with Chern-Simons term. We also consider an example describing non-abelian anyons, namely \(H \simeq \overline{D}_2\). The paper closes with some concluding remarks.
2 The physical model

Although our analysis holds in general, we restrict our considerations to a model with gauge group $G \simeq SU(2)$ to make the discussion explicit. The starting point is the lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\rho\nu} F^{a\rho\nu} + (D_\rho \Phi)^* (D^\rho \Phi) - V(\Phi) + \frac{\mu}{4} \epsilon^{\kappa\sigma\rho} [F^a_{\kappa\sigma} A^a_{\rho} + \frac{2}{3} \epsilon^{abc} A^a_{\kappa} A^b_{\sigma} A^c_{\rho}], \quad (1)$$

We are working in (2+1)-dimensional space time, so greek indices run from 0 to 2. Latin indices label the three (hermitean) generators of $G \simeq SU(2)$. The covariant derivative takes the form $D_\rho \Phi = (\partial_\rho + i e A_\rho^a T^a) \Phi$, with the generators $T^a$ of $SU(2)$ in the representation of the Higgs field $\Phi$. The last term in $(1)$ is the Chern-Simons term with $\epsilon$ the completely anti-symmetric three dimensional Levi-Civita tensor. Requiring this theory to be gauge invariant at the quantum level, implies a quantization condition $(2)$ for the topological mass $\mu$

$$\mu = p e^2 / 4\pi \quad \text{with} \quad p \in \mathbb{Z}. \quad (2)$$

We set out to study the effect of the Chern-Simons term on the excitations in the discrete $H$ gauge theory obtained after breaking $G \simeq SU(2)$ down to the finite group $H$. It is illustrative to do this for a simple example, namely the discrete $\mathbb{Z}_N$ gauge theory that arises from the simple symmetry breaking scheme $SU(2) \to U(1) \to \mathbb{Z}_N$. Such a scheme can be realized by a proper choice of the Higgs field $\Phi$ and its potential $V(\Phi)$.

In the low energy $U(1)$ regime, the theory is governed by the effective lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^\rho_{\mu\nu} F^{\rho\mu\nu} + (D_\rho \Phi)^* (D^\rho \Phi) - V(\Phi) + \frac{\mu}{4} \epsilon^{\kappa\sigma\rho} F^a_{\kappa\sigma} A^a_\rho. \quad (3)$$

We have omitted the massive modes in this strictly abelian model, and more important the instantons labelled by $\pi_2(SU(2)/U(1)) \simeq \mathbb{Z}$. These instantons will play a profound role in the determination of the topological superselection sectors of the theory later on. Variation with respect to the vectorpotential $A^\sigma$, yields the field equation

$$\partial_\rho F^{\rho\sigma} + \frac{\mu}{2} \epsilon^{\sigma\tau\rho} F_{\tau\rho} = j^\sigma. \quad (4)$$

We want to consider all sectors of the theory. So from here on, we will take $j^\sigma$ to stand not just for the conserved current associated with the Higgs field $\Phi$, but also for contributions of possible other $SU(2)$ matter fields, which for notational convenience were not displayed in $(1)$. The different currents occurring in $(1)$ are separately conserved.
Integrating the zeroth component of (4) over two dimensional space, leads to the following relation between the corresponding conserved charges

\[ Q = q + \mu \phi, \]  
(5)

with \( Q = \int d^2 x \nabla \cdot \vec{E} \) the electric charge, \( q = \int d^2 x j^0 \) the global \( U(1) \) charge, and their difference \( q_{\text{ind}} = \mu \phi \) the charge induced by the Chern-Simons term. We denote the total magnetic flux by \( \phi \equiv \int d^2 x \epsilon^{ij} \partial_i A_j \).

In the \( U(1) \) phase electric charges are quantized in units \( e/2 \). Although there are no stable magnetic fluxes in this phase \( (\pi_1(SU(2)/U(1)) \simeq 0) \), there are instantons labelled by \( \pi_2(SU(2)/U(1)) \simeq \pi_2(S^2) \simeq \mathbb{Z} \). In 3 euclidean dimensions these instantons are monopoles carrying magnetic charge \( g = 4\pi k/e \) with \( k \in \mathbb{Z} \). Now suppose that in a next stage of symmetry breaking a component of the the Higgs field \( \Phi \) which carries a \( U(1) \) charge \( Ne/2 \) condenses. This gives rise to the subsequent symmetry breakdown \( U(1) \rightarrow Z_N \). In this \( Z_N \) phase we have magnetic flux excitations \( \phi = 4\pi m/Ne \) with \( m \in \pi_1(U(1)/Z_N) \simeq \mathbb{Z} \). In the present context this is not the whole story. The proper labelling of the magnetic flux sectors in the full theory is by \( \pi_1(SU(2)/Z_N) \simeq Z_N \). The apparent difference can be understood if the role of the instantons in the model is taken into account. In the \( Z_N \) phase these configurations describe tunneling events where magnetic flux excitations \( \phi = 4\pi/e \) decay into the vacuum. So in the \( SU(2) \rightarrow U(1) \rightarrow Z_N \) model where instantons are mandatory, flux is conserved modulo \( N \). In a strict \( G \simeq U(1) \rightarrow Z_N \) model, the flux is conserved (i.e. \( m \in \mathbb{Z} \)) if we do not put an instanton in the theory, or conserved modulo \( N \) if we do put an instanton in by hand. As we shall see later this choice also has an important bearing on the question whether the Chern-Simons parameter \( \mu \) has to be quantized in a \( G \simeq U(1) \) theory.

Let us now turn to the fate of the electric charges in the \( Z_N \) phase. As has been argued in \[ \mathbf{1} \], there are charges surviving the screening mechanism accompanying the breakdown of the continuous \( U(1) \) symmetry. The point is, the condensate of \( Ne/2 \) charges can only completely screen charges that are multiples of \( Ne/2 \). So the different \( Z_N \) charges are \( q = ne/2 \) with \( n \in \mathbb{Z} \) modulo \( N \). Since the gauge fields are massive, these charges do not carry long range Coulomb fields, and are unobservable classically. However, by means of Aharonov-Bohm scattering processes \[ \mathbf{1} \] with the magnetic fluxes, we are able to detect the \( Z_N \) charges quantum mechanically. The essential quantity featuring in the cross-sections for these scattering processes is the quantum phase picked up by such a charge after encircling a magnetic flux once. In the absence of a Chern-Simons term, this quantum phase generated by the coupling \(-j^\rho A_\rho\), takes the value

\[ e^{i(q_1 \phi_2 + q_2 \phi_1)} = e^{\frac{2\pi i}{N}(n_1 m_2 + n_2 m_1)}. \]  
(6)

Here we considered the process of carrying a flux-charge composite \((m_1, n_1) \equiv (\phi_1 = 4\pi m_1/Ne, q_1 = n_1 e/2)\) around a composite \((m_2, n_2)\). The phase \( (\mathbf{1}) \) indicating that these
dyonic composites behave as anyons, can be interpreted as the product of a $Z_N$ gauge transformation on the charge $n_1 \pmod{N}$ by an element $m_2$ and a transformation on the charge $n_2 \pmod{N}$ by an element $m_1$. In our model, the Aharonov-Bohm scattering processes are also the only way to detect the magnetic fluxes. Using (6), we therefore conclude that for $\mu = 0$ the flux $m$ is defined modulo $N$ as well. For a $Z_4$ gauge theory, for instance, these observations lead us to the spectrum depicted in the dashed box in Figure 1(a), where in this case the box has strictly periodic boundary conditions.

In the presence of a Chern-Simons term, the situation is slightly more involved. Instead of the ordinary coupling $-j^\rho A_\rho$, we find in (3) the coupling

$$\mathcal{L}_{\text{int}} = -(j^\rho - \frac{\mu}{4} \epsilon^{\rho\sigma\kappa} F_{\kappa\sigma}) A_\rho. \quad (7)$$

This means that instead of (8), the same process generates the phase

$$e^{i(\tilde{Q}_1 \phi_2 + \tilde{Q}_2 \phi_1)} = e^{i\frac{2\pi}{N}((n_1 + \frac{\mu}{2} m_1)(m_2 + \frac{\mu}{2} m_1)),} \quad (8)$$

where we introduced the charge

$$\tilde{Q} \equiv q + \frac{\mu}{2} \phi. \quad (9)$$

If we now, as before, interpret the total phase (8) as two gauge transformations, we see that the generator of the $Z_N$ transformations is the shifted charge $\tilde{Q}$. We will verify this later on by computing the Noether charge for the gauge transformations. We now argue that (8) allows for the conclusion that the spectrum of a $Z_N$ gauge theory in the presence of a Chern-Simons term can also be reduced to the box depicted in Figure 1(a), although the boundary condition in the $\phi$ direction is twisted. Consider the process in which we encircle a composite $(m_1 + m_2, n_1 + n_2)$ by an arbitrary composite $(m_3, n_3)$. The sum $m_1 + m_2$ does not necessarily lay between 0 and $N - 1$. Using the notation $[m_1 + m_2]$ for $(m_1 + m_2)$ modulo $N$, chosen between 0 and $N - 1$, we can rewrite the phase (8) as

$$e^{i(\tilde{Q}_3 \phi_1 + \tilde{Q}_1 \phi_2 + \tilde{Q}_2 \phi_3)} = e^{i(\tilde{Q}_3 \phi_1 + \tilde{Q}_2 \phi_3)}, \quad (10)$$

with the definitions

$$\tilde{Q}_{12} \equiv q_{12} + \frac{\mu}{2} \phi_{12} \quad (11)$$

$$\equiv (n_1 + n_2 + \frac{2p}{N}(m_1 + m_2 - [m_1 + m_2]) + \frac{p}{N}[m_1 + m_2]) \frac{e}{2}$$

$$\phi_{12} \equiv \frac{4\pi}{Ne}[m_1 + m_2]. \quad (12)$$
Figure 1: The spectrum of a $Z_4$ gauge theory. We depict the flux $\phi$ against the global $U(1)$ charge $q$, the Noether charge $\tilde{Q}$ and the electric charge $Q$ respectively. The Chern-Simons parameter $\mu$ is set to its minimal non-trivial value $\mu = e^2/4\pi$, i.e. $p = 1$. The identification of the encircled excitation with an excitation inside the dashed box is indicated with an arrow.

The equations (11) and (12) express the way charges and fluxes ‘add’, i.e. they specify the fusion rules for the excitations in our model. In terms of the quantum states $|m,n>$, these read

$$|m_1, n_1| \times |m_2, n_2| = |[m_1 + m_2], [n_1 + n_2 + \frac{2p}{N}(m_1 + m_2 - [m_1 + m_2])|.$$  \hspace{1cm} (13)

We conclude that the spectrum of a $Z_N$ gauge theory in the presence of a Chern-Simons term with $\mu$ quantized as (2), again boils down to the excitations in the dashed box in Figure 1(a). If $p \neq 0$ however, the modulo calculus to map excitations inside the box is not simply straight along the $\phi$ and $q$ axes. For example, the excitation $(4\pi/e, 0)$ obtained by fusing two fluxes $(2\pi/e, 0)$ is identified with $(0, e)$. In other words, the periodicity in the $\phi$ direction is twisted by $2p$ units, implying that if $p \neq 0$ the charge $q$ is no longer strictly conserved modulo $N$. The same is true for the charge $\tilde{Q}$, which is twisted by $p$ units as is clear from Figure 1(b). It is in fact the electric charge $Q$ defined in (3), which stands out in this respect. As Figure 1(c) indicates $\phi$ and $Q$ are independently conserved modulo $N$.

Note that the fusion rules (13) are periodic in $p$ with period $N/2$ for $N$ even. The braid properties (8) of the spectrum in contrast, are periodic in $p$ with period $N$. Thus there are only $N$ different $Z_N$ Chern-Simons gauge theories if $\mu$ is quantized as in (3).

At this point it is worthwhile to make a little digression and return to a question we alluded to before, namely whether the parameter $\mu$ should be quantized or not in a $G \simeq$
$U(1) \to Z_N$ theory. We can distinguish a few possibilities. Let us first assume that the theory contains instantons. Recall that these are the monopoles in 3 euclidean dimensions. Hence following Dirac’s argument [10], this implies that the charges $q$ are quantized, and as we argued before magnetic flux is conserved modulo $N$. If we now in addition assume that the Chern-Simons parameter $p$ is not quantized (i.e. $p \notin Z$), we run into a contradiction. As follows from Figure 1(a), a tunneling event by means of an instanton involving the magnetic flux $\phi = 4\pi/e$ would give a charge $2p$. This is inconsistent with the earlier observation that only (half) integral charges were present. The conclusion is that if instantons are present, then $\mu$ has to be quantized accordingly. Interestingly enough the (2+1)-dimensional version of Dirac’s quantization condition gives the correct $\mu$ quantization for general gauge groups. We now turn to the case where instantons are absent. The simplest situation arises if the charges $q$ are not quantized. Then flux is a $Z$-quantum number and $p$ need not be quantized. Finally, if charges are quantized $q = ne/2$ with $n \in Z$ and $p$ is irrational, then all fluxes $m \in Z$ are distinguishable. If $p = k/l$ with $k$ and $l$ relative primes however, then we find $m = m$ modulo $lN/$gcd$(k,N)$, where gcd$(k,N)$ denotes the greatest common divisor of $k$ and $N$.

We conclude this section with a direct computation of the Noether charge generating the residual $Z_N$ transformation in the presence of a Chern-Simons term. First we observe that the lagrangian (3) is not invariant under a gauge transformation

$$
\Phi(x) \mapsto e^{i\frac{Ne}{2}\Omega(x)} \Phi(x)
A_\rho(x) \mapsto A_\rho(x) - \partial_\rho \Omega(x),
$$

but rather changes by a total derivative

$$
\delta_\Omega L_{\text{eff}} = -\frac{\mu}{4} \delta_\sigma (\epsilon^{\sigma\rho\tau} F_{\rho\tau} \Omega).
$$

This term should be substracted from the usual Noether current

$$
\vec{J}_\Omega^n = -\frac{\partial L_{\text{eff}}}{\partial (\partial_\sigma \Phi)} \delta \Phi - \frac{\partial L_{\text{eff}}}{\partial (\partial_\sigma \Phi^\dagger)} \delta \Phi^\dagger + \frac{\partial L_{\text{eff}}}{\partial (\partial_\sigma A_\rho)} \delta A_\rho + \frac{\mu}{4} \epsilon^{\sigma\rho\tau} F_{\rho\tau} \Omega
= \left( j^\sigma - \frac{\mu}{4} \epsilon^{\sigma\rho\tau} F_{\rho\tau} \right) \Omega + \left( -F^{\sigma\rho} + \frac{\mu}{2} \epsilon^{\sigma\rho\tau} A_\tau \right) \partial_\rho \Omega.
$$

The corresponding Noether charge can be calculated either by partially integrating and subsequently using (4), or alternatively by taking $\Omega$ constant. We obtain the form

$$
\int d^2x \vec{J}_\Omega^0 = \Omega(\infty) \vec{Q}, \text{ with } \vec{Q} \text{ defined in (4).}
$$

From this exercise we learn that a gauge transformation $\Omega(\infty) = 4\pi l/Nc$ (with $l = 0, \ldots, N-1$) corresponding to the element $l$ of the residual symmetry group $\hat{Z}_N$, acts on the quantumstate $|m,n\rangle$ as

$$
U(l) |m,n\rangle = e^{i\frac{4\pi l}{Nc} \vec{Q}} |m,n\rangle = e^{i\frac{2\pi l(n+p\frac{m}{N})}{N}} |m,n\rangle.
$$
We see that the charge $\tilde{Q}$ that generates the residual gauge transformation (14) is indeed the same as the charge that enters the Aharonov-Bohm phase (8). The fact that the Noether charge $\tilde{Q}$, rather than the electric charge $Q$, features in the Aharonov-Bohm phase is well known. See for example [11].

The action (14) of the $\mathbb{Z}_N$ gauge group in the presence of a Chern-Simons term is rather unusual:

- The fractional spectrum of $\tilde{Q}$ indicates that the residual gauge transformations no longer form an ordinary $\mathbb{Z}_N$ representation. Indeed, performing two successive gauge transformations yields

$$U(k) \cdot U(l) = e^{i \frac{4\pi}{N} (k+l-[k+l])} \tilde{Q} U([l+k]) = e^{i \frac{2\pi}{N} m(k+l-[k+l])} U([l+k]).$$  (15)

Here we used the form that $\tilde{Q}$ takes for the excitations in our model, as we did in (14). It is easily verified that the additional phase in (15) is a 2-cocycle [12], so associativity is preserved. We conclude that the residual gauge transformations $U(l)$ constitute a projective- or ray representation of $\mathbb{Z}_N$ in the presence of a Chern-Simons term. These ray representations are trivial though; the extra phases can be eliminated by redefining the $U$’s.

- A similar observation can be made if we implement a residual gauge transformation (14) on a two-particle state

$$U(l)\left(|m_1, n_1 > |m_2, n_2 >\right) = e^{i \frac{4\pi}{N} \tilde{Q}_{12}} \left(|m_1, n_1 > |m_2, n_2 >\right)$$

$$= e^{i \frac{2\pi}{N} (m_1 + m_2 - [m_1 + m_2])} (U(l)|m_1, n_1 > U(l)|m_2, n_2 >).$$  (16)

This is obviously not the same as the ordinary product of the gauge transformation on the individual states $|m_1, n_1>$ and $|m_2, n_2>$. Stated mathematically: the Chern-Simons term also alters the co-multiplication by a 2-cocycle. These extra phases could have been eliminated by redefining the $U$’s as well. It is not possible however, to make the phases in (15) and (16) disappear simultaneously.

- There are now two isomorphic ways to describe three-particle states, namely $\left(|m_1, n_1 > |m_2, n_2 > |m_3, n_3 >\right)$ and $\left(|m_1, n_1 > |m_2, n_2 > |m_3, n_3 >\right)$. The isomorphism (expressing quasi-coassociativity) involves a 3-cocycle $\omega$. The 2-cocycles appearing in (15) and (16) stem from this 3-cocycle. All this will be argued in greater detail in the following sections.

The algebraic structure underlying a discrete $H$ gauge theory without a Chern-Simons term [6] is the Hopf algebra $D(H)$. In the following sections, we will show that for $H \simeq \mathbb{Z}_N$ the inclusion of a Chern-Simons term introduces the just mentioned non-trivial 3-cocycle.
ω on $D(Z_N)$. The generalisation of this procedure to $D(H)$ with non-abelian $H$, and thus the effect of a Chern-Simons term for non-abelian fluxes as introduced in [13], is straightforward.

3 The quasi-Hopf algebra $D^\omega(H)$

The distinguishing features of the quasi-Hopf algebra $D^\omega(H)$ are quasi-coassociativity and quasitriangularity. Quasi-coassociativity means that there exists an invertible element $\varphi \in D^\omega(H)^\otimes^3$, such that

$$(id \otimes \Delta)\Delta(a) = \varphi \cdot (\Delta \otimes id)\Delta(a) \cdot \varphi^{-1} \quad \forall a \in D^\omega(H). \quad (17)$$

If $(\Pi_i, V_i)$ denote representations of $D^\omega(H)$, then (17) establishes the equivalence of the representations $\Pi_1 \otimes (\Pi_2 \otimes \Pi_3)$ and $(\Pi_1 \otimes \Pi_2) \otimes \Pi_3$ by means of the non-trivial isomorphism

$$\Phi : (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3) \quad (18)$$

The fact that $D^\omega(H)$ is quasitriangular, implies something similar. It stands for the existence of an element $R \in D^\omega(H)^\otimes D^\omega(H)$, that satisfies among others

$$\Delta'(a) = R \cdot \Delta(a) \cdot R^{-1} \quad \forall a \in D^\omega(H), \quad (19)$$

where we defined $\Delta'(a) \equiv \sum b_i \otimes a_i$, if $\Delta(a) = \sum a_i \otimes b_i$. At the level of representations, relation (19) reflects the equivalence of $\Pi_1 \otimes \Pi_2$ and $\Pi_2 \otimes \Pi_1$. This equivalence is obtained through the non-trivial isomorphism $\Pi_1 \otimes \Pi_2(R)$ from $V_1 \otimes V_2$ into $V_1 \otimes V_2$, followed by the simple permutation

$$\sigma : \quad V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1. \quad (20)$$

We now turn to the explicit realization of these abstract notions in the algebra $D^\omega(H)$. This algebra is spanned by the basis $\{ g_{x,y} \}_{g,x \in H}$. In terms of these basis elements the multiplication and co-multiplication are given by

$$g_{x,y} \cdot h_{y,z} = \delta_{g,xh^{-1}} \theta_g(x,y) \cdot h_{x,z}, \quad \Delta(g_{x,y}) = \sum_{\{ h,k | hk = g \}} h_{x,y} \otimes k_{x,y} \gamma_x(h,k), \quad (21)$$

where the $U(1)$ valued functions $\theta_g$ and $\gamma_g$ equal 1 if $g$ or one of their variables is the unit $e$ of $H$. The elements $\varphi$ and $R$ take the form

$$\varphi = \sum_{g,h,k} \omega^{-1}(g,h,k) g_{e,e} \otimes h_{e,e} \otimes k_{e,e}, \quad R = \sum_{g,h} g_{e,e} \otimes h_{g,g}. \quad (22)$$
where $\omega$ is again $U(1)$ valued. Consistency of the use of $R$ and $\varphi$ in arbitrary tensor products implies

$$
\theta_g(x, y) = \frac{\omega(g, x, y) \omega(x, y, (xy)^{-1}gx)}{\omega(x, x^{-1}gx, y)} \tag{23}
$$

$$
\gamma_x(g, h) = \frac{\omega(g, h, x) \omega(x, x^{-1}gx, x^{-1}hx)}{\omega(g, x, x^{-1}hx)} \tag{24}
$$

where $\omega$ is a 3-cocycle

$$
\omega(g, h, k) \omega(g, hk, l) \omega(h, k, l) = \omega(gh, k, l) \omega(g, h, kl). \tag{25}
$$

Equation (25) determines $\omega$ uniquely up to the coboundary $\delta\beta$

$$
\omega(g, h, k) \mapsto \frac{\beta(g, h) \beta(gh, k)}{\beta(h, k) \beta(g, hk)} \omega(g, h, k). \tag{26}
$$

Equivalence classes of solutions of (25), and hence the different algebras $D^\omega(H)$, are labelled by the third cohomology group $H^3(H, U(1))$. The equivalence of $D^\omega(H)$ and $D^{\omega\delta\beta}(H)$ is argued as follows: If we accompany the transformation (26) on $D^\omega(H)$ by the basis transformation

$$
\delta \mapsto F \cdot \Delta(\delta) \cdot F^{-1}
$$

we can readily verify that the algebra $D^{\omega\delta\beta}(H)$ is obtained from $D^\omega(H)$ by a twist with the element $F = \sum_{g,h} \beta^{-1}(g,h) \delta \otimes \delta$.

$$
\Delta(\delta) \mapsto F \cdot \Delta(\delta) \cdot F^{-1} \tag{28}
$$

$$
R \mapsto F_{21} \cdot R \cdot F^{-1}
$$

$$
\varphi \mapsto (1 \otimes F) \cdot (id \otimes \Delta) (F) \cdot \varphi \cdot (\Delta \otimes id) (F^{-1}) \cdot (F^{-1} \otimes 1),
$$

with $F_{21} \equiv \sum_{g,h} \beta^{-1}(g,h) \delta \otimes \delta$. We will show shortly that this implies that the representation theory for the algebra $D^\omega(H)$ is equivalent to that for $D^{\omega\delta\beta}(H)$. There is therefore no way to distinguish these two algebras. It is noteworthy that you can always choose a twist (26), such that $\theta$ (23) and $\gamma$ (24) indeed equal 1 whenever one of the entries is the unit $e$ of $H$.

\[\text{(Of course, you can choose not to fix this 'gauge freedom'. In this case, the construction of the quasi-Hopf algebra $D^\omega(H)$ still goes along the same lines, with the sole difference that everywhere $\delta \mapsto \frac{1}{\sigma_3(e, e)} \delta$ has to be replaced by $\frac{1}{\sigma_3(e, e)} \delta$.)}\]
The representations of $D^\omega(H)$ can be found in much the same way as for $D(H)$. It is done by inducing the representations of centralizer subgroups as follows. Let $\{ A_i \}$ be the set of conjugacy classes of $H$ and introduce a fixed but arbitrary ordering $A_i = \{ A_{i1}, A_{i2}, \ldots, A_{ik} \}$. Let $A_N$ be the centralizer of $A_{i1}$ and $\{ A_{x1}, A_{x2}, \ldots, A_{xk} \}$ be a set of representatives of the equivalence classes of $G/A_N$, such that $A_{yi} = A_{xi} A_{yi}^{-1}$. Choose for convenience $A_{x1} = e$. Taking different sets of representatives yield representations that only differ by an unitary transformation. Consider the complex vectorspace representation given by the basis only differ by an unitary transformation. Consider the complex vectorspace representatives of the equivalence classes of $G/A_N$, with $θ$ associated to $A$. This can be directly verified by substitution of (23) and applying (25). Note that equation (30) requires that $θ_g$ is a solution of the second 'conjugated' cohomology group $H$ be the centralizer of $N$, and introduce a fixed but arbitrary ordering $A_i = \{ A_{i1}, A_{i2}, \ldots, A_{ik} \}$. Let $A_N$ be the centralizer of $A_{i1}$ and $\{ A_{x1}, A_{x2}, \ldots, A_{xk} \}$ be a set of representatives of the equivalence classes of $G/A_N$, such that $A_{yi} = A_{xi} A_{yi}^{-1}$. Choose for convenience $A_{x1} = e$. Taking different sets of representatives yield representations that only differ by an unitary transformation. Consider the complex vectorspace $V_A^\omega$ spanned by the basis $\{ | y_j, α_i > \}_{j=1,\ldots,\dim α}$. We denote the basis elements of the unitary irreducible representation $T$ of $A_N$ by $α_i$. This vectorspace carries a representation $Π_A^\omega$ of $D^\omega(H)$, given by

$$Π_A^\omega(\frac{g}{x})| A_{yi} α_i = δ_{g,x} x_{g_1} x^{-1} ε_g(x) |x^A y_i x^{-1}, Π( A_{x_k} x_{xi} ) α_i >,$$

with $A_{x_k}$ defined through $A_{yk} ≡ x^A y_k x^{-1}$. The new ingredient here is the phase $ε_g$ that is related to $θ_g$ by

$$θ_g(x, y) = \frac{ε_g(x) ε_{x^{-1} g_x}(y)}{ε_g(xy)},$$

(30)

to make (29) a representation.2 Associativity of the multiplication (21) guarantees that $θ_g$ is a solution of the second 'conjugated' cohomology group

$$θ_g(x, y)θ_g(xy, z) = θ_g(x, yz)θ_{x^{-1} g_x}(y, z).$$

(31)

This can be directly verified by substitution of (23) and applying (25). Note that equation (30) requires that $θ_g$ is in fact exact, i.e. an element of the trivial class of this cohomology group. It is not clear to us whether this requirement is met in general. However, in the examples we have studied so far, $θ$ indeed turned out to be exact.

We are now in a position to appreciate the equivalence of the representation theory of $D^\omega(H)$ and $D^\omega δ β(H)$, and in particular to interpret the twist freedom (26). At the level of a single representation (29) the twist (26, 27) remains unnoticed. Both the left hand- and the right hand side of (21) get multiplied by the phase $β(x, x^{-1} g x) / β(g, x)$, due to the way $ε$ and $δ$ transform. The transformation property $ε_g(x) ↔ ε_g(x) β(x, x^{-1} g x) / β(g, x)$ under a twist, is inferred from (23, 26, 30). At the level of the tensor product of two representations $Π_γ^C \otimes Π_δ^D$, the situation is slightly more involved. From (28), we see that twisting the algebra is the same as transforming the states in the tensor product $Π_γ^C \otimes Π_δ^D$ with $Π_γ^C \otimes Π_δ^F(F)$. On the basis vectors, this transformation has the effect

$$| C_{gi}, γ_i > | D_{gk}, δ_i > ↔ β(C_{gi}, D_{gk}) | C_{gi}, γ_i > | D_{gk}, δ_i >,$$

(32)

2The representation (29) is at variance with (3.3.1) in [dpr1]. We found it unavoidable to include the phase $ε$ to obtain true representations of $D^\omega(H)$.
i.e. these couplings get multiplied by a phase. This effect of the twist is physically unobservable, since quantum states are defined up to a phase.

**Fusion.** By means of Verlinde’s formula [16]

\[
N^{AB\gamma}_{\alpha\beta\delta} = \sum_{D,\delta} \frac{S^{AD}_{\alpha\beta} S^{BD}_{\beta\delta} (S^*)^{CD}_{\gamma\delta}}{S^{0\delta}_{\alpha\beta}}, \tag{33}
\]

the fusion rules for \(D^\alpha(H)\) can be obtained from the modular \(S\) matrix

\[
S^{AB}_{\alpha\beta} = \sum_{A_i \in \mathcal{A}, B_j \in \mathcal{B}} \alpha^* (A_i^{-1} B_j A_i) \beta^* (B_j^{-1} A_i B_j) \sigma(A_i | B_j), \tag{34}
\]

with \(\alpha(g) \equiv \text{Tr} \, \Phi(g)\) and \(\beta\) the twist independent phase \(\sigma(g|h) \equiv \epsilon(g|h)\epsilon_h(g)\).

**Braiding and Aharonov-Bohm scattering.** The operator \(\mathcal{R}\) that establishes a positively oriented interlacement of two excitations, is associated with the universal \(R\) matrix [22]. It is defined as

\[
\mathcal{R}^{AB}_{\alpha\beta} \equiv \sigma \circ (\Pi^A_\alpha \otimes \Pi^B_\beta)(R),
\]

with \(\sigma\) the permutation operator [24]. To be explicit, the braid operation \(\mathcal{R}\) on the state \(|g_i, \alpha_{i,j} > |B_{\gamma_k}, \beta_{\gamma_l} > \in |\mathcal{A}_\gamma, \alpha \triangleright \otimes |\mathcal{B}_\gamma, \beta \triangleright\rangle\) reads

\[
\mathcal{R}^{AB}_{\alpha\beta} |g_i, \alpha_{i,j} > |B_{\gamma_k}, \beta_{\gamma_l} > = \varepsilon_{\gamma m} (A_i | g_k) B_{\gamma m}, \beta (B_{\gamma m}^{-1} A_i B_{\gamma m}) \beta_{\gamma l} > |g_i, \alpha_{i,j} >, \tag{35}
\]

where \(B_{x_m}\) is defined through \(B_{\gamma m} \equiv A_i | g_k B_{\gamma m} A_i^{-1}\). For a non-trivial 3-cocycle \(\omega\), the braid operator \(\mathcal{R}\) does not satisfy the ordinary \(\sim\), but rather the quasi Yang-Baxter equation [14]

\[
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_1 = \mathcal{R}_2 \mathcal{R}_1 \mathcal{R}_2. \tag{36}
\]

\(\mathcal{R}_1\) acts on the three-particle states in \((V_1 \otimes V_2) \otimes V_3\) as \(\mathcal{R} \otimes 1\) and \(\mathcal{R}_2\) as \(\Phi^{-1} \cdot (1 \otimes \mathcal{R}) \cdot \Phi\), with \(\Phi\) the isomorphism defined in [18]. In passing, we mention that the braid operator is related to the modular \(S\) matrix : \(S = \frac{1}{\text{Tr}} \text{Tr} \, \mathcal{R}^2\).

The cross sections of elastic two-particle Aharonov-Bohm scattering are completely determined by the monodromy matrix \(\mathcal{R}^2\). The relationship can be expressed as follows [17]

\[
\frac{d\sigma}{d\phi} = \frac{1}{2\pi k \sin^2(\phi/2)} \left[1 - \text{Re} \, \langle \psi_{in} | \mathcal{R}^2 | \psi_{in} \rangle \right], \tag{37}
\]

with \(|\psi_{in}\rangle\) the incoming two-particle state, and \(k\) the relative wave vector.
4 Examples

An abelian example: cocycles on $D(Z_N)$. We turn to the example $H \simeq Z_N$, with multiplicative generator $g$, discussed in detail from the field theoretic point of view in section 2. Every element $g^m$ constitutes a conjugacy class and has $Z_N$ as its centralizer. We use $m$ to denote $g^m$ by $m$, so the group structure is presented by: $m_1 \cdot m_2 = [m_1 + m_2]$. Here $[m]$ means $m$ modulo $N$, chosen in the range $0, \ldots, N-1$. The $N$ irreducible representations $\Gamma$ of this group are all 1-dimensional, and are given by

$$\Gamma(m) = e^{i \frac{2\pi}{N} m}.$$  \hfill (38)

Since the group is abelian, the ‘conjugated’ 2-cocycle condition (31) coincides with the ordinary 2-cocycle condition. It is well-established (see for example [17]) that all even cohomology groups $H^2(Z_N, U(1))$ are trivial, while for odd cohomology groups we have $H^2(Z_N, U(1)) \simeq Z_N$. Thus $\theta_g \in H^2(Z_N, U(1))$ is trivial and can indeed be written as (30). An explicit realization of $\omega$ is given by

$$\omega(m_1, m_2, m_3) = e^{i \frac{2\pi}{N} m_1 (m_2 + m_3 - [m_2 + m_3])},$$  \hfill (39)

with $p \in [0, \ldots, N-1]$. From (39) together with (23, 24), we find

$$\theta_{m_1}(m_2, m_3) = \gamma_{m_1}(m_2, m_3) = \omega(m_1, m_2, m_3),$$  \hfill (40)

and consequently $\varepsilon_{m_1}(m_2) = \exp i \frac{2\pi}{N} m_1 m_2$ from (30). Thus the effect of a 3-cocycle $\omega$ on the representations (38) is precisely the additional phase occurring in (14)

$$\left. \begin{array}{l} m_1 \cr m_2 \end{array} \right| m, n > = \delta_{m_1, m} \ e^{i \frac{2\pi}{N} m_2 (n + p m_1)} \left. \begin{array}{l} m_2 \cr m, n > \end{array} \right|.$$  \hfill (41)

The 2-cocycle appearing in (14) is nothing but $\theta$, see (21, 30, 31). Implementation of the gauge transformation $\left. \begin{array}{l} m_1 \cr m_2 \end{array} \right|$ on a two particle state involves the co-multiplication (21). The extra phase $\gamma_g$ establishes the anticipated result given in (14), and the fusion rules obtained from (33) are the same as (13). The braid action $\mathcal{R}$ on the representations $|m_1, n_1 \otimes m_2, n_2 >$ is given by (35)

$$\mathcal{R} \left. \begin{array}{l} m_1, n_1 > \cr m_2, n_2 > \end{array} \right| = e^{i \frac{2\pi}{N} m_1 (n_2 + p m_2)} \left. \begin{array}{l} m_2, n_2 > \cr m_1, n_1 > \end{array} \right|.$$  \hfill (42)

We generate the phase (8) by repeating this operation, and obtain the following closed formula for the Aharonov-Bohm cross-sections (37)

$$\frac{d\sigma}{d\phi}[(m_1, n_1), (m_2, n_2)] = \frac{\sin^2 \frac{\pi}{N} (n_1 m_2 + n_2 m_1 + 2p m_1 m_2)}{2\pi k \sin^2(\phi/2)}.$$  \hfill (43)

13
Note that in this abelian model we find $\tilde{R}_2 = R_2$ because of the symmetry of $\omega$ in the last two entries: $\omega(m_1, m_2, m_3) = \omega(m_1, m_3, m_2)$, see (33). This implies that the quasi Yang-Baxter equation (33) projects down to the ordinary Yang-Baxter equation $R_1R_2R_1 = R_2R_1R_2$.

In conclusion, the algebraic framework $D^p(Z_N)$ yields exactly the same result as established in section 2. Stated differently, the introduction of a Chern-Simons term in our lagrangian (31) is equivalent to a deformation of the Hopf algebra $D(Z_N)$ by a non-trivial 3-cocycle. In the next example we study the effect of non-trivial 3-cocycles in the Hopf algebra $D(H)$ with non-abelian $H$. It is our conviction that in so doing we are merely studying the effect of adding a Chern-Simons term to a discrete non-abelian gauge theory.

A non-abelian example: cocycles on $D(\tilde{D}_N)$. We now consider the case of non-abelian $H$, describing non-abelian anyons. In contrast with the abelian example, the general solution of the 3-cocycle condition (23) for non-abelian groups is not known to us. We have however numerically evaluated \footnote{We thank Arjan van der Sijs and Alec Maassen van den Brink for substantial computational aid in the process of numerically solving cocycle conditions on various finite groups.} the 3-cocycle condition (23) for all groups up till order 23, and found among others

$$H^3(\tilde{D}_N, U(1)) \simeq Z_{4N}. \quad (44)$$

For comparison with the discussion in \footnote{We thank Arjan van der Sijs and Alec Maassen van den Brink for substantial computational aid in the process of numerically solving cocycle conditions on various finite groups.} we restrict our considerations to $D(\tilde{D}_2)$ with $H^3(\tilde{D}_2, U(1)) \simeq Z_8$. In other words, $\mu = \mu \text{ mod } 8\pi^2/4\pi$ and there are 8 inequivalent models. We have computed $\omega$ from (23) and found that $\theta_g$ defined through equation (23) is exact. The ambiguity in the solution of $\varepsilon$ from (23) is due to the 1-dimensional representations of $\tilde{D}_2$, which is the content of the first cohomology group $H^1(\tilde{D}_2, U(1)) \simeq Z_2 \times Z_2$. In Table 1 we have gathered the phases $\varepsilon_g(h)$ for a particular choice of 1-dimensional representations, and the twist freedom $\beta$.

The fusion algebra again only has half the period of the Chern-Simons parameter, i.e. there are only 4 different sets of fusion rules. As in the abelian example, we have to turn to the braid properties of the excitations to distinguish all 8 different Chern-Simons models. In \footnote{We thank Arjan van der Sijs and Alec Maassen van den Brink for substantial computational aid in the process of numerically solving cocycle conditions on various finite groups.}, we discussed the fusion rules in absence of a Chern-Simons term. Here, we only gather some salient fusion rules in Table 2 for the four inequivalent Chern-Simons models. We use the same notation as in \footnote{We thank Arjan van der Sijs and Alec Maassen van den Brink for substantial computational aid in the process of numerically solving cocycle conditions on various finite groups.}. The fusion rules (23) only involve $\varepsilon_g(h)$ in the twist independent combination $\sigma(g|h) = \varepsilon_g(h)\varepsilon_h(g)$ where $g$ and $h$ commute. So to acquire them we do not need the complete solution for $\varepsilon$ given in Table 1. In fact, if we are only interested in the fusion rules, a more economical way of calculating them is along the lines described in \footnote{We thank Arjan van der Sijs and Alec Maassen van den Brink for substantial computational aid in the process of numerically solving cocycle conditions on various finite groups.}, where the minimal set of defining equations for $\sigma(g|h)$ is derived.
The Aharonov-Bohm scattering of particles with fluxes that do not commute involves all $\varepsilon$’s. We illustrate this for the example in which a $\sigma_1^+$ excitation is scattered off a $\sigma_2^-$ excitation. The $\sigma_1^+$ excitation is associated with the conjugacy class $X_1 C = \{ X_1, \bar{X}_1 \}$. We choose the representatives as $X_1 x_1 = X_2 x_2 = e$, $X_1 x_2 = X_2$ and $X_2 x_2 = X_3$. Now suppose that this 2 particle system is described by the four component quantumstate

$$|\psi_{in}\rangle = (\cos \theta |X_1, 0\rangle_v + \sin \theta |\bar{X}_1, 0\rangle_v)(\cos \theta' |X_2, 2\rangle_v + \sin \theta' |\bar{X}_2, 2\rangle_v).$$

The monodromy operation $\mathcal{R}^2$ (33) on this state yields

$$\mathcal{R}^2|\psi_{in}\rangle = \varepsilon_{X_1}(X_2)\varepsilon_{\bar{X}_2}(X_1) \cos \theta \cos \theta'|X_1, 0\rangle_v |\bar{X}_2, -2\rangle_v +$$
$$\varepsilon_{X_1}(X_2)\varepsilon_{\bar{X}_2}(X_1) \cos \theta \sin \theta'|X_1, 0\rangle_v |X_2, -2\rangle_v +$$
$$\varepsilon_{X_1}(X_2)\varepsilon_{\bar{X}_2}(\bar{X}_1) \sin \theta \cos \theta'|X_1, 0\rangle_v |\bar{X}_2, -2\rangle_v +$$
$$\varepsilon_{X_1}(X_2)\varepsilon_{\bar{X}_2}(\bar{X}_1) \sin \theta \sin \theta'|X_1, 0\rangle_v |X_2, -2\rangle_v.$$
real part, then we obtain the following cross-section

\[
\frac{d\sigma}{d\varphi}(\sigma_1^+, \sigma_2^-) = \frac{1}{2\pi k \sin^2(\varphi/2)} \frac{1}{2} [1 + \cos \frac{\pi p}{8} \sin 2\theta \sin 2\theta'].
\]

5 Concluding remarks

We have found that the algebraic framework underlying discrete $H$ gauge theories with Chern-Simons term is the quasi Hopf algebra $D^\omega(H)$, i.e. the Chern-Simons term introduces a 3-cocycle $\omega$ on the Hopf algebra $D(H)$.

The framework obviously extends to some of the continuous gauge groups when we view them as limiting cases of discrete gauge groups. In particular the gauge group $U(1) \times Z_2$ that is relevant for the description of the Alice string corresponds to $D_\infty$. It would be interesting to review the discussions on the Alice string in our framework.

Our observations may be relevant in the setting of the fractional quantum Hall effect. In that respect it would be useful to know how to compute the anyonic wavefunctions using the connection with conformal blocks of the appropriate orbifold. This is subject of further scrutiny.

From the point of view of conformal field theory it is of interest to mention that the fusion rules of $D^\omega(\tilde{D}_2)$ for $p = 1$ coincide with the level 1 $SU(2)/D_2$-orbifold [3] after modding out the appropriate $Z_2$ generated by $\tilde{1}$ (see Table 2). Apparently, the algebraic structure of such non-holomorphic orbifolds is still determined by the ‘holomorphic’ Hopf algebra, be it deformed by a non-trivial 3-cocycle. To our knowledge, this has not been noticed before.

Acknowledgments

We wish to thank Arjan van der Sijs and Alec Maassen van den Brink for usefull discussions. This work has been partially supported by the Dutch Science Organization FOM/NWO.

References

[1] Y. Wu; Phys. Rev. Lett. 52 (1984) 2103.
[2] J.M. Leinaas and J. Myrheim; Nuovo Cimento 37B (1977) 1.
[3] F. Wilczek; Phys. Rev. Lett. 49 (1982) 957.
[4] M.G. Alford, J. March-Russel and F. Wilczek; Nucl. Phys. B337 (1990) 695.

[5] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde; Commun. Math. Phys. 123 (1989) 485.

[6] F.A. Bais, P. van Driel and M. de Wild Propitius; preprint ITFA-91-40 to be published in Phys. Lett. B.

[7] R. Dijkgraaf, V. Pasquier and P. Roche; Nucl. Phys. B (Proc. Suppl.) 18B (1990) 60.

[8] S. Deser, R. Jackiw and S. Templeton; Ann. Phys. [N.Y] 140 (1982) 372, J. Schonfeld: Nucl. Phys. B185 (1981)157.

[9] R. Dijkgraaf and E. Witten; Comm. Math. Phys. 129 (1990) 393.

[10] P.A.M. Dirac; Proc. R. Soc. London A133 (1931) 60.

[11] A.S. Goldhaber, R. Mackenzie and F. Wilczek; Mod. Phys. Lett. A4 (1989) 21, M. Mintchev and M. Rossi; Phys. Lett. B271 (1991) 187.

[12] R. Jackiw; ‘Quantum Field Theory and Quantumstatistics’, I.A. Batalin, C.J. Isham and G.A. Vilkovisky eds. (Adam Hilger, Bristol, UK, 1987).

[13] F.A. Bais; Nucl. Phys. B170 (FSI) (1980) 32, V. Poenaru and G. Toulouse; J. Phys. 38 (1977) 887.

[14] V.G. Drinfel’d; Problems of Modern Quantum Field Theory, Proceedings Alushta 1989, Research reports in physics, Springer Verlag Heidelberg (1989).

[15] Y. Aharonov and D. Bohm; Phys. Rev. 115 (1959) 485, E. Verlinde; Proceedings of the International Colloquium on Modern Quantum Field Theory, January 1990.

[16] E. Verlinde; Nucl. Phys. B300 (1988) 360.

[17] G. Moore and N. Seiberg; Comm. Math. Phys. 123 (1989) 177.