Analytical Description of the Earth Matter Effect on Neutrino Oscillations

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Abstract. We present a formalism that provides a precise description of 2ν oscillations in a medium with a symmetric but otherwise arbitrary density profile. The analytical expressions, derived in terms of the first and second order Magnus approximations of the evolution operator, prove to be remarkably simple and can be used in a wide interval of the neutrino energies. They incorporate in accurate way the matter effects on the flavor amplitudes for neutrinos crossing through the Earth. When applied to calculations in the GeV regime characteristic of atmospheric and accelerator neutrinos, this accuracy is complemented by a good reproduction of the position of the maxima in the corresponding transition probabilities.

1. Introduction
Experiments with neutrinos from different natural and artificial sources have provided us with very convincing evidence on the existence of neutrino oscillations. Experimental results can be accommodated within a scheme where at least two neutrinos are massive and lepton flavors are mixed. From the present data set, two neutrino mass-squared differences \( \delta m^2_{ij} \equiv m_i^2 - m_j^2 \) \((i, j = 1, 2, 3)\) and two mixing angles \( \theta_{ij} \) have been determined [1]: \( \delta m^2_{21} \approx 8.0 \times 10^{-5} \text{eV}^2, \theta_{12} \approx 35^\circ \) driving solar and reactor neutrino oscillations and \( \delta m^2_{32} \approx 2.5 \times 10^{-3} \text{eV}^2, \theta_{23} \approx 45^\circ \) that drives atmospheric and long baseline neutrino oscillations. The main goals of the next generation of neutrino experiments will be the determination of the absolute scale of neutrino mass, the third angle \( \theta_{13} \), and the CP-violating phase(s). The objectives also include the identification of neutrino mass hierarchy and precision measurements of the already known parameters.

The interpretation of the forthcoming results will be more involved and require more careful theoretical descriptions of neutrino oscillations that incorporate sub-leading processes. In particular, a detailed and comprehensive study of the matter effects on the flavor transformations for neutrinos propagating through the Earth will be needed. The problem has been investigated by direct numerical integration of the equation that governs flavor evolution in a medium. Yet, analytic calculations have been implemented to greatly simplify the numerical computations and also to gain a better understanding of the underlying physics. Many of these studies have been carried out under the assumption of one or several layers of constant density. Extensions for a varying density have been developed on the basis of the perturbation theory for oscillations, both in the low energy [2] and high energy regime [3, 4].
Here, we use the Magnus exponential expansion of the time-displacement operator $\mathcal{U}(t, t_0)$ in order to seek solutions to the problem of $2\nu$ oscillations in a medium with an arbitrary density profile, which is symmetric with respect to the middle point of the neutrino trajectory. The method is based on a formalism developed several years ago in order to incorporate non-adiabatic effects in the flavor transitions of neutrinos that propagates through a matter-enhanced oscillation region [5]. The main idea is to follow the evolution of the system in the adiabatic basis of the instantaneous energy eigenstates and incorporate the corrections to adiabaticity through the first and second order approximations in the Magnus expansion. Our procedure not only has a better agreement with numerical calculations than other analytical approaches, but is also valid in a wide energy interval, allowing for a unified description of the Earth effect on the oscillations of both low and high energy neutrinos.

2. Formalism

Typically, the quantity of interest is the probability $P_{\nu_e}$ of observing an electron neutrino at a distance $L \approx t_f - t_0$ from a source. If $|\nu(t_f)\rangle$ represents the neutrino state at time $t_f$, then

$$P_{\nu_e} = |\langle\nu_e|\nu(t_f)\rangle|^2 = |\langle\nu_e|\mathcal{U}(t_f, t_0)|\nu(t_0)\rangle|^2,$$

where $|\nu(t_0)\rangle$ denotes a certain initial state and $\mathcal{U}(t, t_0)$ is the evolution operator, which satisfies the Schrödinger-like equation [6]

$$i \frac{d\mathcal{U}}{dt}(t, t_0) = H(t)\mathcal{U}(t, t_0),$$

with the initial condition $\mathcal{U}(t_0, t_0) = 1$ ($\hbar = c = 1$).

We consider oscillations between two neutrino flavors, let say $\nu_e$ and $\nu_\alpha$. In the relativistic limit and after discarding an overall phase, the Hamiltonian of the system in the flavor basis $\{|\nu_e\rangle, |\nu_\alpha\rangle\}$ can be written as

$$H(t) = \frac{\Delta_0}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{V(t)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\theta$ is the mixing angle in vacuum and we have defined $\Delta_0 \equiv \delta m^2/2E$, with $E$ the neutrino energy and $\delta m^2$ the mass-squared difference relevant to the problem. The effect of the medium is accounted for by means of $V$, the difference of the potential energies $V_e$ and $V_\alpha$. In normal matter, to lowest order in the Fermi constant $G_F$, we have $V(t) = V_e(t) - V_\alpha(t) = \sqrt{2}G_F n_e(t)$, where $n_e(t)$ is the number density of electrons along the neutrino path.

The evolution operator in the flavor basis can be expressed as $\mathcal{U}(t_f, t_0) = U_m(t_f)\mathcal{U}^A(t_f, t_0)U_m^\dagger(t_0)$ in terms of the corresponding operator $\mathcal{U}^A(t, t_0)$ in the adiabatic basis of the (instantaneous) eigenstates $\{|\nu_1(t)\rangle, |\nu_2(t)\rangle\}$ of $H(t)$. Here,

$$U_m(t) = \begin{pmatrix} \cos \theta_m(t) & \sin \theta_m(t) \\ -\sin \theta_m(t) & \cos \theta_m(t) \end{pmatrix}$$

is the orthogonal transformation that, at each time, diagonalizes the matrix in Eq. (3). The mixing angle in matter $\theta_m(t)$ is given by $\sin 2\theta_m(t) = \Delta_0 \sin 2\theta/\Delta_m(t)$, where $\Delta_m(t) = \Delta_0 [(\epsilon(t) - \cos 2\theta)^2 + \sin^2 2\theta]^{1/2}$ stands for the difference between the energy eigenvalues and we have introduced the non-dimensional quantity $\epsilon(t) = V(t)/\Delta_0 = 2EV(t)/\delta m^2$.

The operator $\mathcal{U}^A$ can be factorized as $\mathcal{U}^A(t, t_0) = \mathcal{P}(t, t_0)\mathcal{G}^A(t, t_0)$, where

$$\mathcal{P}(t, t_0) = \begin{pmatrix} e^{-i\phi_{t_0-t}} & 0 \\ 0 & e^{i\phi_{t_0-t}} \end{pmatrix},$$
and $U^A_{\ell\ell'}$ is a $2 \times 2$ matrix that obeys Eq. (2) with the Hamiltonian

$$H^A_{\ell\ell'}(t) = i \frac{d\theta_m}{dt} \begin{pmatrix} 0 & -e^{-i\phi_{0\ell'-\ell} t} \\ e^{i\phi_{0\ell'-\ell} t} & 0 \end{pmatrix} .$$

(6)

In the previous equations $\phi_{x-y} = \int_x^y dt' \Delta_m(t')$.

In general, $U^A_{\ell\ell'}$ cannot be found exactly and one has to rest on some approximation to determine it. With this purpose, now we put $U^A_{\ell\ell'} = \exp \Omega$ and approximate $\Omega$ by means of the first two terms of its Magnus expansion [7]. Proceeding in this manner, after some algebraic manipulations we arrive at [8]:

$$U^A(t_f, t_0) \equiv \begin{pmatrix} \cos \xi - i \sin \xi & i \sin \xi e^{i\phi_{f-0}} \\ i \sin \xi & \cos \xi + i \sin \xi e^{-i\phi_{f-0}} \end{pmatrix},$$

(7)

where $\xi = [\xi_{(1)}^2 + \xi_{(2)}^2]^{1/2}$ and

$$\xi_{(1)} = 2 \int_{t_0}^{t_f} dt' \frac{d\theta_m}{dt'} \sin \phi_{f-\ell'},$$

(8)

$$\xi_{(2)} = \int_{t_0}^{t_f} dt' \int_{t_0}^{t'} dt'' \frac{d\theta_m}{dt'} \frac{d\theta_m}{dt''} \sin \phi_{\ell'-\ell''}.$$  

(9)

The above expression for $\xi_{(1)}$ was obtained by taking into account that $V(t) = V(2t - t_f)$ for a potential that, like in the Earth, is symmetric with respect to the middle point of the neutrino trajectory $\overline{t} = (t_f + t_0)/2$. In this situation $\theta_m(t_f) = \theta_m(t_0) = \theta^0_m$ and $U_m(t_f) = U_m(t_0)$. In what follows, we restrict ourselves to such a case, with $\theta^0_m$ being the angle evaluated at the surface of the medium.

From Eq. (7) we see that $U^A_{22} = U^A_{11}^{*}$ and $U^A_{21} = -U^A_{12}^{*}$, which are also satisfied by the matrix elements $U_{\ell\ell'}(\ell, \ell' = e, a)$. Such relations implies that our approximate solution for the time-displacement operator verifies the unitarity condition. This is an important quality of the Magnus expansion which remains true at higher order. Suppose that $|\nu(t_0)\rangle = \alpha|\nu_e\rangle + \beta|\nu_\alpha\rangle$, with $\alpha$ and $\beta$ non-negative (real) numbers satisfying $\alpha^2 + \beta^2 = 1$, then taking into account the relations between the $U_{\ell\ell'}(\ell, \ell' = e, a)$, we find

$$P_{\nu_e} = \alpha^2 + (\beta^2 - \alpha^2)(\text{Im}\, U_{ea})^2 + 2\alpha\beta \text{Im}\, U_{ee} \text{Im}\, U_{ea}.$$  

(10)

As we see, to this order, only the imaginary parts of the matrix elements of the evolution operator are relevant to the calculation of $P_{\nu_e}$. They can be computed in terms of the elements of matrix (7) by using of the relations $\text{Im}\, U_{ee} = \cos 2\theta^0_m \text{Im}\, U^A_{11} + \sin 2\theta^0_m \text{Im}\, U^A_{12}$ and $\text{Im}\, U_{ea} = \sin 2\theta^0_m \text{Im}\, U^A_{11} + \cos 2\theta^0_m \text{Im}\, U^A_{12}$.

### 3. High Energy Neutrinos

In this section we apply the present formalism to the oscillations of high-energy ($E \gtrsim 1$ GeV) neutrinos, which are of relevance for studies with atmospheric, accelerator, and cosmic neutrinos. If we assume that $\theta_{13}$ is not very small, then the quantity $\delta m^2_{21}/2E$ can be safely discarded in the equation governing the flavor evolution of a 3$\nu$-system [3]. In this case, the mixing angle $\theta_{12}$ does not play any role and the problem reduces to an effective one of two states $|\nu_e\rangle$ and $|\nu_\alpha\rangle = \sin \theta_{23} |\nu_\mu\rangle + \cos \theta_{23} |\nu_\tau\rangle$, where the matter oscillations are driven by the parameters $\delta m^2 = \delta m^2_{31}$ and $\theta = \theta_{31}$. 

3
We focus hereafter in the transition probabilities \( P(\nu_\mu \to \nu_e) = \sin^2 \theta_{23} P(\nu_\mu \to \nu_e) \) and \( P(\nu_\tau \to \nu_e) = \cos^2 \theta_{23} P(\nu_\mu \to \nu_e) \). Now, \( |\nu(t_0)| = |\nu_0| \) and from Eq. (10), with \( \alpha = 0 \) and \( \beta = 1 \), we get

\[
P(\nu_\alpha \to \nu_\epsilon) = (\text{Im}\, U_{\alpha\epsilon})^2 = (\cos 2\theta_m \text{Im}\, U_{12}^A - \sin 2\theta_m \text{Im}\, U_{11}^A)^2,
\]  

(11)

where according to Eq. (7)

\[
\text{Im}\, U_{11}^A = \cos \xi \sin \phi_{f-t_f} - \sin \xi \frac{\xi(2)}{\xi} \cos \phi_{f-t_f},
\]

\[
\text{Im}\, U_{12}^A = \sin \xi \frac{\xi(1)}{\xi}.
\]

(12)

In the adiabatic approximation \( U^A(t, t_0) \equiv P(t, t_0) \) and the corresponding expression for the transition probability (in a symmetric medium) becomes \( P(\nu_\alpha \to \nu_\epsilon) = \sin^2 2\theta_m \sin^2 \phi_{f-t_f} \). This result can be easily recovered from Eq. (11) by making \( d\theta_m/dt = 0 \), and therefore \( \xi(1) = \xi(2) = 0 \).

Suppose that \( V \gg \Delta_0 \), then \( \varepsilon \gg 1 \) and we can implement a perturbative expansion in \( 1/\varepsilon \) for a varying potential. Accordingly,

\[
2\theta_m \equiv \pi - \frac{1}{\varepsilon} \sin 2\theta
\]

(13)

and

\[
\xi(1) \equiv \Delta_0 \sin 2\theta \int_{t_f}^{t} dt' \cos \phi_{f-t'} + O\left(\frac{1}{\varepsilon}\right).
\]

(14)

Using the last two equations and keeping at most terms of \( O(1) \) in \( 1/\varepsilon \) (except in the phase \( \phi_{f-t'} \) ), Eq. (11) becomes

\[
P(\nu_\alpha \to \nu_\epsilon) = \left[ \sin \left( \Delta_0 \sin 2\theta \int_{t_f}^{t} dt' \cos \phi_{f-t'} \right) \right]^2
\]

(15)

in the first-order Magnus approximation (\( \xi(2) = 0 \)). It is pertinent to note that the expression

\[
P(\nu_\alpha \to \nu_\epsilon) = \Delta_0^2 \sin^2 2\theta \left[ \int_{t_f}^{t} dt' \cos \phi_{f-t'} \right]^2
\]

derived in Ref. [3] by using the standard perturbation series of the evolution operator, follows immediately from Eq. (15) when the sine function is replaced by its linear approximation.

The probability \( P(\nu_\alpha \to \nu_\epsilon) \) computed from the Magnus (Eq. (11)) and perturbative formulas are plotted in Fig. 1 as a function of \( E \). We use a simplified model of the electron density inside the Earth, the so called mantle-core-mantle [9]:

\[
n_e(r) = \begin{cases} 5.95 \text{ cm}^{-3}, & r \leq R_{\oplus}/2 \\ 2.48 \text{ cm}^{-3}, & R_{\oplus}/2 < r \leq R_{\oplus} \\ \end{cases}
\]

(16)

where \( R_{\oplus} \) denotes the Earth’s radius. In this case, the integrals in Eqs. (8) and (9) can be done explicitly. The results are

\[
\xi(1) = -2(\theta_m^C - \theta_m^M) \sin \phi_{f-t_{CM}},
\]

(17)

\[
\xi(2) = (\theta_m^C - \theta_m^M)^2 \sin 2\phi_{f-t_{CM}},
\]

(18)
where \( t_{CM} \) denotes the point in the neutrino trajectory corresponding to the interface between the core and the mantel. The quantities \( \theta_{CM}^c \) and \( \theta_{CM}^m \) are the values of the matter mixing angle in each of these regions. For comparison, we also plot \( P(\nu_a \rightarrow \nu_e) \) as given by the exact solution. We see that our result not only reproduces the overall behavior of the transition probability, but also works remarkably well in a quantitative sense. Moreover, it never yields values higher than one, a pathology presented by the perturbative expression as can be seen in the right panel of Fig. 1.

![Figure 1](image_url)

**Figure 1.** \( P(\nu_a \rightarrow \nu_e) \) as a function of the energy for a neutrino crossing the Earth passing by its center (left panel) and for a trajectory of Nadir angle \( \Theta \cong 26^\circ \) (right panel). The oscillation parameters are \( \delta m^2_{21} = 2.5 \times 10^{-3} \text{ eV}^2 \) and \( \theta_{13} = 10^\circ \). (a) corresponds to the perturbative approach (see Ref. [3]), (b) and (c) correspond to the Magnus approximation implemented in the adiabatic basis for the first and second order respectively, and (d) corresponds to the exact solution. Our approximation reproduces very well both, the value and position of the maxima of the exact calculation.

If the mixing angle \( \theta_{13} \) were vanishingly small or zero, then the problem is also described in terms of an effective two-state system \( \{|\nu_e\rangle, |\nu_b\rangle\} \), with \( |\nu_b\rangle = \cos \theta_{23} |\nu_\mu\rangle - \sin \theta_{23} |\nu_\tau\rangle \). In this case, the transition probabilities are

\[
P(\nu_\mu \rightarrow \nu_e) = \cos^2 \theta_{23} P(\nu_b \rightarrow \nu_e) \quad \text{and} \quad P(\nu_\tau \rightarrow \nu_e) = \sin^2 \theta_{23} P(\nu_b \rightarrow \nu_e),
\]

where \( P(\nu_b \rightarrow \nu_e) \) can be computed by the same expression given in Eq. (11), but with the oscillation parameters \( \delta m^2 = \delta m^2_{21} \) and \( \theta = \theta_{12} \). From the curves plotted in Fig. 2, it is again evident that the analytical expression derived by means of the adiabatic Magnus expansion gives the best approximation to the exact result.

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Figure 2. $P(\nu_b \to \nu_e)$ as a function of the energy for a neutrino crossing the Earth passing by its center (left panel) and for a trajectory of Nadir angle $\Theta \approx 26^\circ$ ($\cos \Theta = 0.9$) (right panel). The oscillation parameters are $\delta m^2_{21} = 8 \times 10^{-5}$ eV$^2$, $\tan^2 \theta_{12} = 0.4$, and $\theta_{13} = 0$. (a) corresponds to the perturbative approach, (b) and (c) correspond to the Magnus approximation implemented in the adiabatic basis for the first and second order respectively, and (d) corresponds to the exact solution.

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