INDEPENDENT SET IN CATEGORICAL PRODUCTS OF COGRAPHS AND SPLITGRAPHS

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Abstract. We show that there are polynomial-time algorithms to compute maximum independent sets in the categorical products of two cographs and two splitgraphs. We show that the ultimate categorical independence ratio is computable in polynomial time for cographs.

1 Introduction

Let \(G\) and \(H\) be two graphs. The categorical product also goes under the name of tensor product, or direct product, or Kronecker product, and even more names have been given to it. It is defined as follows. It is a graph, denoted as \(G \times H\). Its vertices are the ordered pairs \((g, h)\) where \(g \in V(G)\) and \(h \in V(H)\). Two of its vertices, say \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent if

\[
\{ g_1, g_2 \} \in E(G) \quad \text{and} \quad \{ h_1, h_2 \} \in E(H).
\]

One of the reasons for its popularity is Hedetniemi’s conjecture, which is now more than 40 years old [11, 25, 23, 34].

Conjecture 1. For any two graphs \(G\) and \(H\)

\[
\chi(G \times H) = \min \{ \chi(G), \chi(H) \}.
\]

It is easy to see that the right-hand side is an upperbound. Namely, if \(f\) is a vertex coloring of \(G\) then one can color \(G \times H\) by defining a coloring \(f'\) as follows

\[
f'((g, h)) = f(g), \quad \text{for all } g \in V(G) \text{ and } h \in V(H).
\]

Recently, it was shown that the fractional version of Hedetniemi’s conjecture is true [35].
When $G$ and $H$ are perfect then Hedetniemi’s conjecture is true. Namely, let $K$ be a clique of cardinality at most

$$|K| \leq \min \{ \omega(G), \omega(H) \}.$$  

It is easy to check that $G \times H$ has a clique of cardinality $|K|$. One obtains an ‘elegant’ proof, via homomorphisms, as follows. By assumption, there exist homomorphisms $K \to G$ and $K \to H$. This implies that there also is a homomorphism $K \to G \times H$ (see, e.g., [10, 12]). (Actually, if $W$, $P$ and $Q$ are any graphs, then there exist homomorphisms $W \to P$ and $W \to Q$ if and only if there exists a homomorphism $W \to P \times Q$.) In other words [10, Observation 5.1],

$$\omega(G \times H) \geq \min \{ \omega(G), \omega(H) \}.$$  

Since $G$ and $H$ are perfect, $\omega(G) = \chi(G)$ and $\omega(H) = \chi(H)$. This proves the claim, since

$$\chi(G \times H) \geq \omega(G \times H) \geq \min \{ \omega(G), \omega(H) \} = \min \{ \chi(G), \chi(H) \} \geq \chi(G \times H).$$  

(1)

Much less is known about the independence number of $G \times H$. It is easy to see that

$$\alpha(G \times H) \geq \max \{ \alpha(G) \cdot |V(H)|, \alpha(H) \cdot |V(G)| \}.$$  

(2)

But this lowerbound can be arbitrarily bad, even for cographs [15]. For any graph $G$ and any natural number $k$ there exists a cograph $H$ such that

$$\alpha(G \times H) \geq k + L(G, H),$$

where $L(G, H)$ is the lowerbound expressed in (2). When $G$ and $H$ are vertex transitive then equality holds in (2) [33].

**Definition 1.** A graph is a cograph if it has no induced $P_4$, i.e., a path with four vertices.

Cographs are characterized by the property that every induced subgraph $H$ satisfies one of

(a) $H$ has only one vertex, or
(b) $H$ is disconnected, or
(c) $H$ is disconnected.

It follows that cographs can be represented by a cotree. This is pair $(T, f)$ where $T$ is a rooted tree and $f$ is a 1-1 map from the vertices of $G$ to the leaves of $T$. Each internal node of $T$, including the root, is labeled as $\otimes$ or $\oplus$. When the label is $\oplus$ then the subgraph $H$, induced by the vertices in the leaves, is disconnected. Each child of the node represents one component. When the node is labeled as
then the complement of the induced subgraph $H$ is disconnected. In that case, each component of the complement is represented by one child of the node.

When $G$ is a cograph then a cotree for $G$ can be obtained in linear time.

Cographs are perfect, see, eg, [17] Section 3.3. When $G$ and $H$ are cographs then $G \times H$ is not necessarily perfect. For example, when $G$ is the paw, ie, $G \simeq K_1 \otimes (K_2 \oplus K_1)$ then $G \times K_3$ contains an induced $C_5$ [22]. Ravindra and Parthasarathy characterize the pairs $G$ and $H$ for which $G \times H$ is perfect [22, Theorem 3.2].

2 Independence in categorical products of cographs

It is well-known that $G \times H$ is connected if and only if both $G$ and $H$ are connected and at least one of them is not bipartite [31]. When $G$ and $H$ are connected and bipartite, then $G \times H$ consists of two components. In that case, two vertices $(g_1, h_1)$ and $(g_2, h_2)$ belong to the same component if the distances $d_G(g_1, g_2)$ and $d_H(h_1, h_2)$ have the same parity.

Definition 2. The rook’s graph $R(m, n)$ is the linegraph of the complete bipartite graph $K_{m,n}$.

The rook’s graph $R(m, n)$ has as its vertices the vertices of the grid, $(i, j)$, with $1 \leq i \leq m$ and $1 \leq j \leq n$. Two vertices are adjacent if they are in the same row or column of the grid. The rook’s graph is perfect, since all linegraphs of bipartite graphs are perfect (see, eg, [17]). By the perfect graph theorem, also the complement of rook’s graph is perfect [19].

Lemma 1. Let $m, n \in \mathbb{N}$. Then

$$K_m \times K_n \simeq \bar{R},$$

where $\bar{R}$ is the complement of the rook’s graph $R = R(m, n)$.

Proof. Two vertices $(i, j)$ and $(i', j')$ are adjacent in $K_m \times K_n$ when $i \neq i'$ and $j \neq j'$. That is, they are adjacent when they are not in the same row or column of the $m \times n$ grid. Thus, $K_m \times K_n$ is the complement of the rook’s graph $R(m, n)$.

Lemma 2. Let $G$ and $H$ be complete multipartite. Then $G \times H$ is perfect.

Proof. Ravindra and Parthasarathy prove that $G \times H$ is perfect if and only if either

(a) $G$ or $H$ is bipartite, or
(b) Neither $G$ nor $H$ contains an induced odd cycle of length at least 5 nor an induced paw.
Since \( G \) and \( H \) are perfect, they do not contain an odd hole \( 3 \). Furthermore, the complement of \( G \) and \( H \) is a union of cliques, and so the complements are \( P_3 \)-free. The complement of a paw is \( K_1 \oplus P_3 \) and so it has an induced \( P_3 \). This proves the claim. 

Let \( G \) and \( H \) be complete multipartite. Let \( G \) be the join of \( m \) independent sets, say with \( p_1, \ldots, p_m \) vertices, and let \( H \) be the join of \( n \) independent sets, say with \( q_1, \ldots, q_n \) vertices. We shortly describe how \( G \times H \) is obtained from the complement of the rook’s graph \( R(m,n) \). We call the structure a generalized rook’s graph.

Each vertex \((i, j)\) in \( R(m, n) \) is replaced by an independent set \( I(i, j) \) of cardinality \( p_i \cdot q_j \). Denote the vertices of this independent set as 
\[
(i_s, j_t) \quad \text{where } 1 \leq s \leq p_i \text{ and } 1 \leq t \leq q_j.
\]

Two vertices \((i_s, j_t)\) and \((i'_s, j_t)\) are adjacent and these types of row- and column-adjacencies are the only adjacencies in this generalized rook’s graph. The graph \( G \times H \) is obtained from the partial complement of the generalized rook’s graph.

**Lemma 3.** Let \( G \) and \( H \) be complete multipartite graphs. Then
\[
\alpha(G \times H) = \kappa(G \times H) = \max \{ \alpha(G) \cdot |V(H)|, \alpha(H) \cdot |V(G)| \}.
\]

**Proof.** Two vertices \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent if \( g_1 \) and \( g_2 \) are not in a common independent set in \( G \) and \( h_1 \) and \( h_2 \) are not in a common independent set in \( H \).

Let \( \Omega \) be a maximum independent set of \( G \). Then
\[
\{ (g, h) \mid g \in \Omega \text{ and } h \in V(H) \}
\]
is an independent set in \( G \times H \). We show that all maximal independent sets are of this form or of the symmetric form with \( G \) and \( H \) exchanged.

Consider the complement of the rook’s graph. Any independent set must have all its vertices in one row or in one column. This shows that every maximal independent set in \( G \times H \) is a generalized row or column in the rook’s graph. Since the graphs are perfect, the number of cliques in a clique cover of \( G \times H \) equals \( \alpha(G \times H) \).

**Remark 1.** Notice that complete multipartite graphs are not vertex transitive, unless all independent sets have the same cardinality.
Lemma 4. Let $G$ and $H$ be cographs and assume that $G$ is disconnected. Say that $G = G_1 \oplus G_2$. Then
\[
\alpha(G \times H) = \alpha(G_1 \times H) + \alpha(G_2 \times H).
\]
Proof. By definition of the categorical product, no vertex of $V(G_1) \times V(H)$ is adjacent to any vertex of $V(G_2) \times V(H)$. $\square$

Lemma 5. Let $G$ and $H$ be connected cographs. Say $G = G_1 \otimes G_2$ and $H = H_1 \otimes H_2$. Then
\[
\alpha(G \times H) = \min \{ \alpha(G_1 \times H), \alpha(G_2 \times H), \alpha(G \times H_1), \alpha(G \times H_2) \}.
\]
Proof. Every vertex of $V(G_1) \times V(H_1)$ is adjacent to every vertex of $V(G_2) \times V(H_2)$ and, likewise, every vertex of $V(G_1) \times V(H_2)$ is adjacent to every vertex of $V(G_2) \times V(H_1)$. This proves the claim. $\square$

Theorem 1. There exists a polynomial-time algorithm which computes $\alpha(G \times H)$ when $G$ and $H$ are cographs.
Proof. By Lemmas 4 and 5. $\square$

3 Splitgraphs

Földes and Hammer introduced splitgraphs [6]. We refer to [7, Chapter 6] and [21] for some background information on this class of graphs.

Definition 3. A graph $G$ is a splitgraph if there is a partition $\{S, C\}$ of its vertices such that $G[S]$ is a clique and $G[S]$ is an independent set.

Theorem 2. Let $G$ and $H$ be splitgraphs. There exists a polynomial-time algorithm to compute the independence number of $G \times H$.
Proof. Let $\{S_1, C_1\}$ and $\{S_2, C_2\}$ be the partition of $V(G)$ and $V(H)$, respectively, into independent sets and cliques. Let $c_i = |C_i|$ and $s_i = |S_i|$ for $i \in \{1, 2\}$. The vertices of $C_1 \times C_2$ form a rook’s graph.

We consider three cases. First consider the maximum independent sets without any vertex of $V(C_1) \times V(C_2)$. Notice that the subgraph of $G \times H$ induced by the vertices of $V(S_1) \times V(C_2) \cup V(C_1) \times V(S_2) \cup V(S_1) \times V(S_2)$ is bipartite. A maximum independent set in a bipartite graph can be computed in polynomial time.

Consider maximum independent sets that contain exactly one vertex $(c_1, c_2)$ of $V(C_1) \times V(C_2)$. The maximum independent set of this type can be computed as
follows. Consider the bipartite graph of the previous case and remove the neighbors of \((c_1, c_2)\) from this graph. The remaining graph is bipartite. Maximizing over all pairs \((c_1, c_2)\) gives the maximum independent set of this type.

Consider maximum independent sets that contain at least two vertices of the rook’s graph \(V(C_1) \times V(C_2)\). Then the two vertices must be in one row or in one column of the grid, since otherwise they are adjacent. Let the vertices of the independent set be contained in row \(c_1 \in V(C_1)\). Then the vertices of \(V(S_1) \times V(C_2)\) of the independent set are contained in

\[
W = \{ (s_1, c_2) \mid s_1 \notin N_G(c_1) \text{ and } c_2 \in C_2 \}.
\]

Consider the bipartite graph with one color class defined as the following set of vertices

\[
\{ (c_1, s_2) \mid c_1 \in C_1 \text{ and } s_2 \in S_2 \} \cup \{ (s_1, s_2) \mid s_1 \in V(S_1) \text{ and } s_2 \in V(S_2) \},
\]

and the other color class defined as

\[
W \cup \{ (c_1, c_2) \mid c_2 \in C_2 \}.
\]

Since this graph is bipartite, the maximum independent set of this type can be computed in polynomial time by maximizing over the rows \(c_1 \in C_1\) and columns \(c_2 \in C_2\).

This proves the theorem. \(\square\)

4 Tensor capacity

In this section we consider the powers of a graph under the categorical product.

**Definition 4.** The independence ratio of a graph \(G\) is defined as

\[
i(G) = \frac{\alpha(G)}{|V(G)|}. \tag{4}
\]

For background information on the related Hall-ratio we refer to \[24,26,29,30\].

By (2) for any two graphs \(G\) and \(H\) we have

\[
i(G \times H) \geq \max \{ i(G), i(H) \}. \tag{5}
\]

It follows that \(i(G^k)\) is non-decreasing. Also, it is bounded from above by 1 and so the limit when \(k \to \infty\) exists. This limit was introduced in \[2\] as the ‘ultimate categorical independence ratio.’ See also \[19,13,20\]. For simplicity we call it the tensor capacity of a graph. Alon and Lubetzky, and also Tóth claim that computing the tensor capacity is NP-complete, but neither provides a proof \[120,28,30\].
Definition 5. Let $G$ be a graph. The tensor capacity of $G$ is
\[
\Theta^T(G) = \lim_{k \to \infty} i(G^k).
\] (6)

Brown et al. [2, Theorem 3.3] obtain the following lowerbound for the tensor capacity.
\[
\Theta^T(G) \geq a(G) \text{ where } a(G) = \max_{I \text{ an independent set}} \frac{|I|}{|I| + |N(I)|}.
\] (7)

It is related to the binding number $b(G)$ of the graph $G$. Actually, the binding number is less than 1 if and only if $a(G) > \frac{1}{2}$. In that case, the binding number is realized by an independent set and it is equal to $b(G) = \frac{1 - a(G)}{a(G)}$ [16,28]. The binding number is computable in polynomial time [5,16,32]. See also Corollary 1 below.

The following proposition was proved in [2].

**Proposition 1.** If $i(G) > \frac{1}{2}$ then $\Theta^T(G) = 1$.

Therefore, a better lowerbound for $\Theta^T(G)$ is provided by
\[
\Theta^T(G) \geq a^*(G) = \begin{cases} 
a(G) & \text{if } a(G) \leq \frac{1}{2} \\
1 & \text{if } a(G) > \frac{1}{2}.
\end{cases}
\] (8)

Definition 6. Let $G = (V, E)$ be a graph. A fractional matching is a function $f : E \to \mathbb{R}^+$, which assigns a non-negative real number to each edge, such that for every vertex $x$
\[
\sum_{e \ni x} f(e) \leq 1.
\]

A fractional matching $f$ is perfect if it achieves the maximum
\[
f(E) = \sum_{e \in E} f(e) = \frac{|V|}{2}.
\]

Alon and Lubetzky proved the following theorem in [1] (see also [16]).

**Theorem 3.** For every graph $G$
\[
\Theta^T(G) = 1 \iff a^*(G) = 1 \iff G \text{ has no fractional perfect matching}.
\] (9)

**Corollary 1.** There exists a polynomial-time algorithm to decide whether
\[
\Theta^T(G) = 1 \text{ or } \Theta^T(G) \leq \frac{1}{2}.
\]
The following theorem was raised as a question by Alon and Lubetzky in [1][20]. The theorem was proved by Ágnes Tóth [28].

**Theorem 4.** For every graph $G$

$$
\Theta^T(G) = a^*(G).
$$

Equivalently, every graph $G$ satisfies

$$
a^*(G^2) = a^*(G). \quad (10)
$$

Tóth proves that

if $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$ then $a(G \times H) \leq \max \{ a(G), a(H) \}$. \quad (11)

Actually, Tóth shows that, if $I$ is an independent set in $G \times H$ then

$$
|N_{G \times H}(I)| \geq |I| \cdot \min \{ b(G), b(H) \}.
$$

From this, Theorem 4 easily follows. As a corollary (see [1][20][28]) one obtains that, for any two graphs $G$ and $H$

$$
i(G \times H) \leq \max \{ a^*(G), a^*(H) \}.
$$

Tóth also proves the following theorem in [28]. This was conjectured by Brown et al. [2].

**Theorem 5.** For any two graphs $G$ and $H$,

$$
\Theta^T(G \oplus H) = \max \{ \Theta^T(G), \Theta^T(H) \}. \quad (12)
$$

Notice that the analogue of this statement, with $a^*$ instead of $\Theta^T$, is straightforward. The theorem follows from (11) via the following lemma. This lemma was proved by Alon and Lubetzky in [1].

**Lemma 6.** For any two graphs $G$ and $H$,

$$
\Theta^T(G \oplus H) = \Theta^T(G \times H). \quad (13)
$$

Tóth proves the following theorem in [27, Corollary 3]. This is proved as a corollary of a theorem which says that, if for all $x \in V$, $d(x) \geq n - \alpha(G)$, and if $i(G) \leq \frac{1}{2}$, then for all $k \in \mathbb{N}$,

$$
i(G^k) = i(G).
$$

**Theorem 6.** Let $G$ be a complete multipartite graph. Let $\alpha$ be the size of the largest partite class of $G$. Then

$$
\Theta^T(G) = \begin{cases} 
  \frac{\alpha}{n} & \text{if } \alpha \leq \frac{n}{2} \\
  1 & \text{otherwise.}
\end{cases}
$$
For cographs we obtain the following theorem.

**Theorem 7.** There exists a polynomial-time algorithm to compute the tensor capacity for cographs.

**Proof.** By Theorem 4 it is sufficient to compute \( a(G) \), as defined in (7).

Consider a cotree for \( G \). For each node the algorithm computes a table. The table contains numbers \( \ell(k) \), for \( k \in \mathbb{N} \), where

\[
\ell(k) = \min \{ |N(I)| \mid I \text{ is an independent set with } |I| = k \}.
\]

Notice that \( a(G) \) can be obtained from the table at the root node via

\[
a(G) = \max_k \frac{k}{k + \ell(k)}.
\]

Assume \( G \) is the union of two cographs \( G_1 \oplus G_2 \). An independent set \( I \) is the union of two independent sets \( I_1 \) in \( G_1 \) and \( I_2 \) in \( G_2 \). Let the table entries for \( G_1 \) and \( G_2 \) be denoted by the functions \( \ell_1 \) and \( \ell_2 \). Then

\[
\ell(k) = \min \{ \ell_1(k_1) + \ell_2(k_2) \mid k_1 + k_2 = k \}.
\]

Assume that \( G \) is the join of two cographs, say \( G = G_1 \otimes G_2 \). An independent set in \( G \) can have vertices in at most one of \( G_1 \) and \( G_2 \). Therefore,

\[
\ell(k) = \min \{ \ell_1(k) + |V(G_2)|, \ell_2(k) + |V(G_1)| \}.
\]

This proves the theorem. \( \square \)

**Remark 2.** The tensor capacity is computable in polynomial time for many other classes of graphs via similar methods [18].

### 5 Concluding remarks

It would be interesting to know whether the tensor capacity for splitgraphs is computable in polynomial time. Also, is the independence number for the product of three splitgraphs, \( G_1 \times G_2 \times G_3 \) NP-complete?

### References

1. Alon, N. and E. Lubetzky, Independent sets in tensor graph powers, *Journal of Graph Theory* **54** (2007), pp. 73–87.
2. Brown, J., R. Nowakowski and D. Rall, The ultimate categorical independence ratio of a graph, *SIAM Journal on Discrete Mathematics* **9** (1996), pp. 290–300.
3. Chudnovsky, M., N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, *Annals of Mathematics* **164** (2006), pp. 51–229.
4. Corneil, D., H. Lerchs and L. Stewart-Burlingham, Complement reducible graphs, *Discrete Applied Mathematics* 3 (1981), pp. 163–174.

5. Cunningham, W., Computing the binding number of a graph, *Discrete Applied Mathematics* 27 (1990), pp. 283–285.

6. Földes, S. and P. Hammer, Split graphs, *Proceedings of the 8th South-Eastern Conference on Combinatorics, Graph Theory and Computing* (1977), pp. 311–315.

7. Golumbic, M., *Algorithmic graph theory and perfect graphs* Elsevier, Annals of Discrete Mathematics 57, Amsterdam, 2004.

8. Grötschel, M., L. Lovász and A. Schrijver, Relaxations of vertex packing, *Journal of Combinatorial Theory, Series B* 40 (1986), pp. 330–343.

9. Hahn, G., P. Hell and S. Poljak, On the ultimate independence ratio of a graph, *European Journal of Combinatorics* 16 (1995), pp. 253–261.

10. Hahn, G. and C. Tardif, Graph homomorphisms: structure and symmetry. In: (G. Hahn and G. Sabidussi eds.) *Graph symmetry – algebraic methods and applications*, NATO ASI Series C: Mathematical and Physical Sciences, Vol. 497, Kluwer, 1997, pp. 107–166.

11. Hedetniemi, S., Homomorphisms of graphs and automata. Technical report 03105-44-T, University of Michigan, 1966.

12. Hell, P. and J. Nešetřil, *Graphs and homomorphisms*, Oxford University Press, 2004.

13. Hell, P., X. Yu and H. Zhou, Independence ratios of graph powers, *Discrete Mathematics* 27 (1994), pp. 213–220.

14. Imrich, W. and S. Klavžar, *Product graphs: structure and recognition*, John Wiley & Sons, New York, USA, 2000.

15. Jha, P. and S. Klavžar, Independence in direct-product graphs, *Ars Combinatoria* 50 (1998).

16. Kloks, T., C. Lee and J. Liu, Stickiness, edge-thickness, and clique-thickness in graphs, *Journal of Information Science and Engineering* 20 (2004), pp. 207–217.

17. Kloks, T. and Y. Wang, *Advances in graph algorithms*. Manuscript 2013.

18. Kratsch, D., T. Kloks and H. Müller, Measuring the vulnerability for classes of intersection graphs, *Discrete Applied Mathematics* 77 (1997), pp. 259–270.

19. Lovász, L., Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics* 2 (1972), pp. 253–267.

20. Lubetzky, E., *Graph powers and related extremal problems*, PhD Thesis, Tel Aviv University, Israel, 2007.

21. Merris, R., Split graphs, *European Journal of Combinatorics* 24 (2003), pp. 413–430.

22. Ravinda, G. and K. Parthasarathy, Perfect product graphs, *Discrete Mathematics* 20 (1977), pp. 177–186.

23. Sauer, N., Hedetniemi’s conjecture – a survey, *Discrete Mathematics* 229 (2001), pp. 261–292.

24. Simonyi, G., Asymptotic values of the Hall-ratio for graph powers, *Discrete Mathematics* 306 (2006), pp. 2593–2601.

25. Tardif, C., Hedetniemi’s conjecture, 40 years later, *Graph theory notes of New York New York Academy of Sciences, LIV* (2008), pp. 46–57.

26. Tóth, Á., *Asymptotic values of graph parameters*, Diploma Thesis, Department of Computer Science and Information Theory, Faculty of Electrical Engineering and Informatics, Budapest University of Technology and Economics, Hungary, 2008.

27. Tóth, Á., The ultimate categorical independence ratio of complete multipartite graphs, *SIAM Journal on Discrete Mathematics* 23 (2009), pp. 1900–1904.

28. Tóth, Á., Answer to a question of Alon and Lubetzky about the ultimate categorical independence ratio. Manuscript on arXiv:1112.6172v1, 2011.
29. Tóth, Á., On the ultimate direct Hall-ratio. Manuscript 2011.
30. Tóth, Á., Colouring problems related to graph products and coverings, PhD Thesis, Department of Computer Science and Information Theory, Faculty of Electrical Engineering and Informatics, Budapest University of Technology and Economics, Hungary, 2012.
31. Weichsel, P., The Kronecker product of graphs, Proceedings of the American Mathematical Society 13 (1962), pp. 47–52.
32. Woodall, D., The binding number of a graph and its Anderson number, Journal of Combinatorial Theory, Series B 15 (1973), pp. 225–255.
33. Zhang, H., Independent sets in direct products of vertex-transitive graphs, Journal of Combinatorial Theory, Series B 102 (2012), pp. 832–838.
34. Zhu, X., A survey of Hedetniemi’s conjecture, Taiwanese Journal of Mathematics 2 (1998), pp. 1–24.
35. Zhu, X., The fractional version of Hedetniemi’s conjecture is true, European Journal of Combinatorics 32 (2011), pp. 1168–1175.