Simple Algorithm for Factorized Dynamics of $g_n$-Automaton

Goro Hatayama*, Atsuo Kuniba*, and Taichiro Takagi†

Abstract

We present an elementary algorithm for the dynamics of recently introduced soliton cellular automata associated with quantum affine algebra $U_q(g_n)$ at $q = 0$. For $g_n = A^{(1)}_n$, the rule reproduces the ball-moving algorithm in Takahashi-Satsuma’s box-ball system. For non-exceptional $g_n$ other than $A^{(1)}_n$, it is described as a motion of particles and anti-particles which undergo pair-annihilation and creation through a neutral bound state. The algorithm is formulated without using representation theory nor crystal basis theory.

1 Introduction

The box-ball system invented by Takahashi and Satsuma [TS, T] is a remarkable example of one dimensional soliton cellular automata. It represents the dynamics of balls hopping along the array of boxes under a certain rule. The evolution equations have been studied extensively in [TTMS, TTM, TNS] by means of the ultradiscretization of soliton equations.

In [HKT1, FOY, HHIKTT], the box-ball system was identified with a solvable vertex model in statistical mechanics [B] in a low temperature limit. Mathematically it brought a crystal basis theory [K] into the game, and led to a generalization labeled with affine Lie algebras $g_n$. We call the resulting system the $g_n$-automaton. See [HKT1] for the construction for non-exceptional $g_n$, and [HKOT1] for the study of soliton scattering. The box-ball system corresponds to the $g_n = A^{(1)}_n$ case [FOY, HHIKTT].

The dynamics of the $g_n$-automaton is governed by the combinatorial $R$ [KMN] in the crystal theory, which specifies the local interaction of a box and a carrier in the language of the box-ball system. Except the $A^{(1)}_n$ case [NY], the complexity of the combinatorial $R$ [HKOT1, HKOT2] makes it tedious to compute the dynamics of the associated $g_n$-automaton. Fortunately the difficulty was overcome in [HKT2] for the automaton having a carrier with infinite capacity, where a factorization of the time evolution into simple Weyl group reflections was proved.

The aim of this paper is to translate the factorized dynamics [HKT2] stated in the crystal language into an analogue of the ball-moving algorithm, and thereby to describe the $g_n$-automaton in terms of a simple evolution rule. For simplicity we shall restrict ourselves to the case where all the boxes have minimal capacity. In the crystal formulation of the time evolution: $B_M \otimes (\cdots \otimes B_{m_i} \otimes B_{m_{i+1}} \otimes \cdots) \simeq (\cdots \otimes B_{m_i} \otimes B_{m_{i+1}} \otimes \cdots) \otimes B_M$, the situation corresponds to the choice $\forall m_i = 1$ and $M = \infty$. The factorized dynamics

*Institute of Physics, University of Tokyo, Komaba, Tokyo 153-8902, Japan
†Department of Applied Physics, National Defense Academy, Yokosuka 239-8866, Japan
for $A_1^{(1)}$-automaton reproduces the decomposition of the ball-moving algorithm in the box-ball system into a finer procedure where one only touches the balls with a fixed color. For the other $g_n$‘s in question, the dynamics is decomposed similarly, where each finer procedure (denoted by $K_j$ in Section 2) is now described as the motion of particles and anti-particles with a fixed color exhibiting pair-annihilation and creation through a neutral bound state.

In Section 3 we specify the set of local states and give the algorithm. In Section 4 we present examples. In Section 5 we briefly explain the relation of the algorithm in Section 2 and the result in [HKT2]. The key is the formula (3).

Although our notations originate in the crystal theory, the contents in Sections 2-3 use no result from it. Despite being just a translation of a more general result in [HKT2], we hope this paper possesses its own role to make the $g_n$-automaton more accessible and familiar to the people working on cellular automata and discrete integrable systems.

2 Algorithm

We formulate the $g_n$-automaton associated with the non-exceptional affine Lie algebra $g_n = A_1^{(1)}(n \geq 1), A_2^{(2)}(n \geq 3), A_2^{(2)}(n \geq 2), B_n^{(1)}(n \geq 3), C_n^{(1)}(n \geq 2), D_n^{(1)}(n \geq 4)$ and $D_n^{(2)}(n \geq 2)$. Actually the algorithm itself given below is meaningful formally for $A_3^{(2)}, B_2^{(1)}, B_2^{(1)}$ and $D_3^{(1)}$ as well. First we specify the set $B$ of local states and the sequence $j_1, \ldots, j_d \in B$ as follows.

| $g_n$       | $B$                       | $(j_d, \ldots, j_1)$       |
|------------|---------------------------|-----------------------------|
| $A_1^{(1)}$| $\{1, 2, \ldots, n+1\}$  | $(2, 3, \ldots, n+1)$       |
| $A_2^{(2)}$| $\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$ | $(2, 3, \ldots, n, -1, -n, \ldots, -3, -2)$ |
| $A_2^{(2)}$| $\{1, 2, \ldots, n, -n, \ldots, -2, -1, 0\}$ | $(2, 3, \ldots, n, -1, -n, \ldots, -3, -2, 0)$ |
| $B_n^{(1)}$| $\{1, 2, \ldots, n, 0, -n, \ldots, -2, -1\}$ | $(2, 3, \ldots, n, 0, -n, \ldots, -3, -2)$ |
| $C_n^{(1)}$| $\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$ | $(2, 3, \ldots, n, -1, -n, \ldots, -3, -2, -1)$ |
| $D_n^{(1)}$| $\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$ | $(2, 3, \ldots, n, -1, -n, \ldots, -3, -2)$ |
| $D_n^{(2)}$| $\{1, 2, \ldots, n, 0, -n, \ldots, -2, -1, 0\}$ | $(2, 3, \ldots, n, 0, -n, \ldots, -3, -2, 0)$ |

Here $\emptyset$ should be understood as a symbol and not an empty set. By the definition $d = n$ for $A_1^{(1)}$, $d = 2n - 2$ for $D_n^{(1)}$, $d = 2n - 1$ for $A_2^{(2)}$, $B_n^{(1)}$, $d = 2n$ for $A_2^{(2)}, C_n^{(1)}$ and $D_n^{(2)}$.

We let $W = \{ (\ldots, b_i, b_{i+1}, \ldots) \in \cdots \times B \times B \times \cdots \mid b_i = 1$ for $|i| \gg 1 \}$ be the set of states of the automaton at a fixed time. Setting $J = \{j_1, \ldots, j_d\}$, one finds

$$J = \begin{cases} B \setminus \{1, -1\} & \text{if } g_n = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, \\ B \setminus \{1\} & \text{otherwise} \end{cases}$$

2
For $j \in J$ (hence $j \neq 1$), we define

$$j^* = \begin{cases} j & \text{if } j \in \{0, -1, \emptyset\}, \\ -j & \text{if } j \in J \setminus \{0, -1, \emptyset\}. \end{cases} \quad (2)$$

The time evolution of the $g_n$-automaton $T : W \to W$ takes the factorized form

$$T = K_{j_4} \cdots K_{j_2} K_{j_1}. \quad (3)$$

To define the operator $K_j : W \to W$ ($j \in J$), one regards an element $(\ldots, b_i, b_{i+1}, \ldots)$ of $W$ as the one dimensional array of boxes containing $b_i$ in the $i$-th box. (The $(i+1)$-th box to the right of the $i$-th box.) In what follows we regard the box containing $1 \in B$ as an empty box. Thus, taking some $b (\neq 1) \in B$ away from a box means the change of the local states $b \to 1$. Similarly, putting $b (\neq 1) \in B$ into an empty box means the change $1 \to b$. Note that the boundary condition in (1) says that only finitely many boxes are non-empty. Under this convention the operator $K_j$ is defined by the following algorithm which consists of 4 Steps:

1. Replace each $-1$ by a pair $j, j^*$ within a box.
2. Pick the leftmost $j$ (if any) and move it to the nearest right box which is empty or containing just $j^*$. (Boxes involving the pair $j, j^*$ are prohibited as the destination.)
3. Repeat Step 2 for those $j$’s that are not yet moved until all of $j$’s are moved once.
4. Replace the pair $j, j^*$ within a box (if any) by $-1$.

**Remark 1.** For $g_n = A_n^{(1)}$ where no $-1$ is present in $B$ and $J$, Step 1 and 4 become void and the above procedure shrinks to the well known ball-moving algorithm in the box-ball system [I].

**Remark 2.** When $j \in \{-1, 0, \emptyset\}$, one can have two $j$’s within a box due to (2). As for those duplicated $j$’s, it does not matter which is left or right, but of course they should be distinguished according to whether they are moved or not yet moved. See $K_\emptyset$ in Example 3.

**Remark 3.** A little inspection shows that $K_{-1}$ actually admits a simpler description which is essentially the ball-moving algorithm in the box-ball system:

1’. Pick the leftmost $-1$ (if any) and move it to the nearest right empty box.
2’. Apply Step 1’ for those $-1$’s that are not yet moved (if any).
3’. Repeat Step 2’ until all the $-1$’s are moved once.

**Remark 4.** Consider the $g_n \neq A_n^{(1)}$ case and interpret the local states as follows:

| $B$ | Entry in the box |
|-----|------------------|
| 1   | empty            |
| $j \neq \pm 1$ | particle of color $j$ |
| $-1$ | bound state of $j$ and $j^*$ |

Since the set $B \setminus \{1, -1\}$ is invariant under the interchange $j \leftrightarrow j^*$, the state $j^*$ may be viewed as the anti-particle of $j$ and vice versa. In this sense the bound state $-1$ is neutral, and especially $0$ and $\emptyset$ represent the ‘self-neutral’ particles that can still form a bound state with another one. Under this interpretation, the above algorithm for $K_j$
describes the motion of right-moving particles of color $j$ seeking the empty box or a free partner, i.e., an anti-particle $j^*$ not yet paired with the other $j$'s, to form a neutral bound state within a box.

The algorithm for $K_j$ can also be stated in terms of local rules, which we shall now explain. For $j \in J$ we introduce a map $L_j : (\mathbb{Z}_{\geq 0}) \times B \to B \times (\mathbb{Z}_{\geq 0})$ as follows. Let the diagram $l \rightarrow_{\jmath} l'$ denote $L_j : (l, b) \mapsto (b', l')$. ($j$ is attached to the horizontal line.)

1. $j \notin \{1, 0, -1, \emptyset\}$ case. Assume $l \in \mathbb{Z}_{\geq 0}$, $b \in B \setminus \{j, -j, 1, -1\}$.

2. $j \in \{0, \emptyset\}$ case. Assume $l \in \mathbb{Z}_{\geq 0}$, $b \in B \setminus \{j, 1, -1\}$.

3. $j = -1$ case. Assume $l \in \mathbb{Z}_{\geq 0}$, $b \in B \setminus \{1, j\}$.

For $g_n = A_n^{(1)}$ we only have the third case with the $j$'s understood as $j \in \{2, 3, \ldots, n+1\}$. See also [HIK].

Given an automaton state $(\ldots, b_i, b_{i+1}, \ldots) \in W$, there exists an integer $m$ such that $b_{m'} = 1$ for all $m' < m$ owing to the boundary condition (1). Fix any such $m$. Then the operator $K_j : W \to W$ maps $(\ldots, b_i, b_{i+1}, \ldots)$ to $(\ldots, c_i, c_{i+1}, \ldots)$, where $c_{m'} = b_{m'}$ for all $m' < m$. The remaining $c_m, c_{m+1}, \ldots$ are determined by the composition of $L_j$'s as

$$
\begin{array}{ccccccc}
& & & b_m & b_{m+1} & \cdots \\
& & 0 & & & \\
& & c_m & c_{m+1} & \cdots
\end{array}
$$

It is easy to see that the result is independent of the choice of $m$. The nonnegative integers $l$ on the horizontal line stand for the number of color $j$ particles on the carrier. The diagrams for $L_j$ are viewed as the loading and unloading process of the color $j$ particles when the carrier proceeds to the right. They match the algorithm stated before. Perhaps the above formulation of $K_j$ using $L_j$ is easier to program.
3 Examples

Let us present examples of the factorized dynamics. We suppress the trivial left tail in the time evolution and only depict the part that corresponds to the composition of vertically.

Example 1. $g_n = D_4^{(1)}$. $T = K_2 K_3 K_4 K_{-3} K_{-2}$.

$$T: (-3, -2, 1, -2, 2, 3, 1, 1, 1, \ldots) \mapsto (1, -3, -2, 1, 3, -3, 3, 1, \ldots).$$

Example 2. $g_n = B_3^{(1)}$. $T = K_2 K_3 K_0 K_{-3} K_{-2}$.

$$T: (-1, -3, 0, 3, -2, 1, 1, 1, 1, 1, \ldots) \mapsto (1, 2, 1, 1, -3, -2, -2, 0, 3, 1, \ldots).$$
Example 3. \( g_n = A^{(2)}_4 \). \( T = K_2 K_{-1} K_{-2} K_\emptyset \).

Along the Steps 1-4 in Section 2, the action of \( K_\emptyset \) in the above goes as follows.

As cautioned in Remark 2, one must distinguish the moved \( \emptyset \)'s and those not yet moved. Here we have marked the moved ones with underlines.

4 Solitons

To save the space we shall write, for example, \( 2^{y_2} (-3)^{y_{-3}} \) to signify the configuration of the local states \( 2, \ldots, 2, -3, \ldots, -3 \) in a segment of an automaton state for nonnegative integers \( y_2 \) and \( y_{-3} \), etc. In the sequel we assume \( y_b \in \mathbb{Z}_{\geq 0} \) for any \( b \in B \). For each \( g_n \) consider the following configurations and define \( v \) from them.

\[ A^{(1)}_n : (n + 1)^{y_{n+1}} \ldots 3^{y_3} 2^{y_2}, \]
\[ v = \sum_{i=2}^{n+1} y_i. \]

\[ A^{(2)}_{2n-1} : (-2)^{y_{-2}} (-3)^{y_{-3}} \ldots (-n)^{y_{-n}} n^{y_n} \ldots 3^{y_3} 2^{y_2}, \]
\[ v = \sum_{i=2}^{n}(y_i + y_{-i}). \]
Relation to crystal theory

The factorized dynamics in this paper is a translation of a result in [HKT2]. By regarding the local states $1, \ldots, n$ as $T$, $\mu$, $B$ can be identified with the crystal $B_1$ as a set. In eq.(33) of [HKT2], take the integer $k$ as $k = n - 1$ for $D_n^{(1)}$ and $k = n$ for the other $g_n$'s. It leads to $a_k = 1$ (cf. Table I, II and eq.(11) therein) for all the $g_n$'s in question, which implies the boundary condition $\ldots 111 \ldots$. The time evolution in this case is given by $T = T_{k+\delta}$, where the data $i_{k+\delta}, \ldots, i_{k+1}$ in eq.(33) there reads

$$A_{2n}^{(2)} : (0)^{y_0} (-1)^{y_{-1}} (-2)^{y_{-2}} \ldots (-n)^{y_{-n}} n^{y_n} \ldots 2^{y_2} 1^{y_1}, \quad y_0 \in \{0, 1\}, \ y_1 = y_{-1},$$
$$v = y_0 + \sum_{i=1}^{n} (y_i + y_{-i}).$$

$$B_n^{(1)} : (-2)^{y_{-2}} (-3)^{y_{-3}} \ldots (-n)^{y_{-n}} 0^{y_0} n^{y_n} \ldots 3^{y_3} 2^{y_2}, \quad y_0 \in \{0, 1\},$$
$$v = y_0 + \sum_{i=1}^{n} (y_i + y_{-i}).$$

$$C_n^{(1)} : (-1)^{y_{-1}} (-2)^{y_{-2}} \ldots (-n)^{y_{-n}} n^{y_n} \ldots 2^{y_2} 1^{y_1}, \quad y_1 = y_{-1},$$
$$v = \sum_{i=1}^{n} (y_i + y_{-i}).$$

$$D_n^{(1)} : (-2)^{y_{-2}} (-3)^{y_{-3}} \ldots (-n)^{y_{-n}} n^{y_n} \ldots 3^{y_3} 2^{y_2}, \quad y_n y_{-n} = 0,$$
$$v = \sum_{i=2}^{n} (y_i + y_{-i}).$$

$$D_n^{(2)} : (0)^{y_0} (-1)^{y_{-1}} (-2)^{y_{-2}} \ldots (-n)^{y_{-n}} 0^{y_0} n^{y_n} \ldots 2^{y_2} 1^{y_1}, \ y_0, y_0 \in \{0, 1\}, \ y_1 = y_{-1},$$
$$v = y_0 + y_0 + \sum_{i=1}^{n} (y_i + y_{-i}).$$
Under the convention $T_k = id$, the operator $K_j$ in this paper has emerged from the formula

$$K_{jr} = T_{k+r}(T_{k+r-1})^{-1} \quad 1 \leq r \leq d. \quad (5)$$

We have verified that the right hand side yields the composition of $L_{jr}$’s described in Section 2 by an explicit calculation. Similar factorized dynamics can be formulated corresponding to the other boundary conditions than $1 \in B$ in (1) by choosing a different $k$ in the above. However such variations do not affect the qualitative feature of the automata.

Acknowledgements The authors thank K. Hikami, R. Inoue, M. Okado, T. Tokihiro and Y. Yamada for collaboration in their previous works.

References

[B] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London (1982).

[FOY] K. Fukuda, M. Okado, Y. Yamada, *Energy functions in box ball systems*, Int. J. Mod. Phys. A 15 (2000) 1379–1392.

[HHIKTT] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi and T. Tokihiro, *The $A^{(1)}_1$ Automata related to crystals of symmetric tensors*, J. Math. Phys. 42 (2001) 274–308.

[HKOT1] G. Hatayama, A. Kuniba, M. Okado and T. Takagi, *Combinatorial R matrices for a family of crystals: $C_{n}^{(1)}$ and $A_{2n-1}^{(2)}$ cases*, Progress in Mathematics Vol. 191 Physical Combinatorics (2000), Birkhäuser Boston, 105–139.

[HKOT2] G. Hatayama, A. Kuniba, M. Okado and T. Takagi, *Combinatorial R matrices for a family of crystals: $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ cases*, preprint math.QA/0012247.

[HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Scattering rules in soliton cellular automata associated with crystal bases*, preprint math.QA/0007173.

[HKT1] G. Hatayama, A. Kuniba, and T. Takagi, *Soliton cellular automata associated with crystal bases*, Nucl. Phys. B577[PM] (2000) 619–645.

[HKT2] G. Hatayama, A. Kuniba, and T. Takagi, *Factorization of combinatorial R matrices and associated cellular automata*, J. Stat. Phys. 102 (2001) 843–863.
[HIK] K. Hikami, R. Inoue and Y. Komori, *Crystallization of the Bogoyavlensky lattice*, J. Phys. Soc. Jpn. 68 (1999) 2234–2240.

[K] M. Kashiwara, *Crystalizing the $q$-analogue of universal enveloping algebras*, Commun. Math. Phys. 133 (1990) 249–260.

[KMN] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, *Affine crystals and vertex models*, Int. J. Mod. Phys. A 7 (suppl. 1A), (1992) 449-484.

[NY] A. Nakayashiki and Y. Yamada, *Kostka polynomials and energy functions in solvable lattice models*, Selecta Mathematica, New Ser. 3 (1997) 547-599.

[T] D. Takahashi, *On some soliton systems defined by using boxes and balls*, Proceedings of the International Symposium on Nonlinear Theory and Its Applications (NOLTA ’93), (1993) 555–558.

[TS] D. Takahashi and J. Satsuma, *A soliton cellular automaton*, J. Phys. Soc. Jpn. 59 (1990) 3514–3519.

[TNS] T. Tokihiro, A. Nagai and J. Satsuma, *Proof of solitonical nature of box and ball systems by means of inverse ultra-discretization*, Inverse Problems 15 (1999) 1639–1662.

[TTM] T. Tokihiro, D. Takahashi and J. Matsukidaira, *Box and ball system as a realization of ultradiscrete nonautonomous KP equation*, J. Phys. A 33 (2000) 607–619.

[TTMS] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, *From soliton equations to integrable cellular automata through a limiting procedure*, Phys. Rev. Lett. 76, (1996) 3247–3250.