On the 4-adic complexity of the two-prime quaternary generator

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Abstract

R. Hofer and A. Winterhof proved that the 2-adic complexity of the two-prime (binary) generator of period $pq$ with two odd primes $p \neq q$ is close to its period and it can attain the maximum in many cases. When the two-prime generator is applied to producing quaternary sequences, we need to determine the 4-adic complexity. It is proved that there are only two possible values of the 4-adic complexity for the two-prime quaternary generator, which are at least $pq - 1 - \max\{\log_4(pq^2), \log_4(p^2q)\}$. Examples for primes $p$ and $q$ with $5 \leq p, q < 10000$ illustrate that the 4-adic complexity only takes one value larger than $pq - \max\{\log_4(p), \log_4(q)\}$, which is close to its period. So it is good enough to resist the attack of the rational approximation algorithm.

Keywords Cryptography · Feedback with carry shift registers · Two-prime generators · Quaternary sequences · 4-Adic complexity

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1 Introduction

Feedback with carry shift registers (FCSRs), which were invented by M. Goresky and A. Klapper in 1990’s, play an important role in spread-spectrum multiple-access communication and cryptography for the design of pseudo-random sequences. FCSRs can be implemented in hardware for high speed calculation and they have an algebraic theory parallel to that of linear feedback shift registers (LFSRs). We refer the reader to the monograph [4] for the theory on FCSRs.

Let \( m \geq 2 \) and \( s_\infty = (s_0, s_1, \ldots, s_{T-1}) \) be a \( T \)-periodic \( m \)-ary sequence over \( \mathbb{Z}_m \), the integer residue ring modulo \( m \). The shortest length (denoted by \( \Phi_m(s_\infty) \)) of an FCSR that can generate \( s_\infty \) is called the \( m \)-adic complexity, which is one of the most important cryptographic measures. We have

\[
\Phi_m(s_\infty) = \left\lfloor \log_m \left( \frac{m^T - 1}{\gcd(S(m), m^T - 1)} \right) \right\rfloor, \tag{1}
\]

where \( S(X) = \sum_{i=0}^{T-1} s_i X^i \in \mathbb{Z}[X] \) and \( \left\lfloor z \right\rfloor \) is the smallest integer that is equal to or greater than \( z \). It is desirable that the \( m \)-adic complexity of an \( m \)-ary sequence is as large as possible. Any \( m \)-ary sequence with small \( m \)-adic complexity is not suitable for cryptographic applications. Clearly if \( \gcd(S(m), m^T - 1) = 1 \), the \( m \)-adic complexity achieves the maximum.

In most references, the 2-adic complexity of binary sequences (i.e., when \( m = 2 \)) was discussed. In particular, a new way was developed by Xiong et al. [20] to determine the 2-adic complexity by computing Gauss periods. It is very efficient for binary sequences with optimal autocorrelation including Legendre sequence, Hall’s sextic residue sequences and two-prime sequences, see [6, 20]. After that, the 2-adic complexity of (generalized) cyclotomic sequences derived from (generalized) cyclotomic residue classes in \( \mathbb{Z}_N \), the integer residue ring modulo \( N \), has been highly paid attention by researchers, see [2, 5, 6, 13, 15–17, 19, 20, 22–26].

For the case \( m > 2 \), one can find many observations about the \( m \)-adic complexity of \( m \)-ary sequences, see [4, 10, 11, 21, 27]. However, it seems that the study of \( m \)-adic complexity of \( m \)-ary sequences has not been fully developed.

In particular, from an engineering standpoint, the preferred alphabet sizes are \( m = 2, 4, 8, \ldots \), because of the compatibility with the binary \( \{0, 1\} \) nature of data representation in electronic hardware. So it is of interest to consider the case of \( m = 4 \), i.e., quaternary sequences, see [12]. Very recently, S. Qiang et al consider the 4-adic complexity of the quaternary cyclotomic sequences with period \( 2p \) [14]. Partially motivated by [14], we will discuss the 4-adic complexity of the two-prime quaternary sequences investigated in our earlier work [1, 3]. Note that, it seems that for quaternary sequences we cannot use the sequence autocorrelation function to estimate the \( m \)-adic complexity, as in [5, 6, 16]. And it is quite difficult to use Gauss periods here, as in [14]. However, it is interesting that the Hall polynomials can help us to deal with this issue.

Let \( p \) and \( q \) be two distinct odd primes with \( \gcd(p - 1, q - 1) = 4 \) and \( e = (p - 1)(q - 1)/4 \). We denote by \( \mathbb{Z}_{pq}^* \) the multiplicative group of \( \mathbb{Z}_{pq} \) the ring of
Let $g$ be a common primitive root of $p$ and $q$ [18]. By the Chinese Remainder Theorem the multiplicative order of $g$ modulo $pq$ is $e$. There also exists an integer $h$ satisfying

$$h \equiv g \pmod{p}, \quad h \equiv 1 \pmod{q}.$$ 

Below we always fix the definitions of $g$ and $h$.

Define the generalized cyclotomic classes of order 4 modulo $pq$ as

$$D_i = \{g^sh^i \pmod{pq} : s = 0, 1, \ldots, e - 1\}, \quad 0 \leq i < 4,$$

and we have

$$\mathbb{Z}^*_pq = D_0 \cup D_1 \cup D_2 \cup D_3.$$ 

We see that each $D_i$ has the cardinality $(p - 1)(q - 1)/4$ for $0 \leq i < 4$. We note that $h^4 \in D_0$, since otherwise, we write $h^4 \equiv g^sh^i \pmod{pq}$ for some $0 \leq s < e$ and $1 \leq i < 4$ and get $g^{e-s}h^{4-i} = 1 \in D_0$, a contradiction.

We also define

$$P = \{p, 2p, \ldots, (q - 1)p\}, \quad Q = \{q, 2q, \ldots, (p - 1)q\}, \quad R = \{0\}.$$ 

Then we define the quaternary sequence $e_\infty = (e_0, e_1, \ldots, e_{pq-1})$ over $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ of length $pq$ by

$$e_u = \begin{cases} 2, & \text{if } u \pmod{pq} \in Q \cup R, \\ 0, & \text{if } u \pmod{pq} \in P, \\ i, & \text{if } u \pmod{pq} \in D_i, \quad i = 0, 1, 2, 3. \end{cases} \quad (2)$$

The linear complexity and trace representation of $e_\infty$ have been considered in [1, 3]. Here we will determine the 4-adic complexity of $e_\infty$ using the standard formula Eq. (1) when $m = 4$. We organize the work as follows: we firstly give the main result in Sect. 2, then we discuss the properties of Hall polynomials and other necessary statements to finish the proof of main theorem in Sect. 3. In Sect. 4 we mention the symmetric 4-adic complexity. We draw a conclusion in the fifth section.

### 2 4-Adic complexity: main contribution

Our main contribution is described in the following theorem.

**Theorem 1** Let $p$ and $q$ be two distinct odd primes with $\gcd(p - 1, q - 1) = 4$. Let $e_\infty$ be a quaternary sequence of length $pq$ defined in (2). If $r_1 = \gcd(p + 3, 4^q - 1)$ and $r_2 = \gcd(q - 1, (4^p - 1)/3)$, then the 4-adic complexity $\Phi_4(e_\infty)$ of $e_\infty$ satisfies

$$\Phi_4(e_\infty) \in \left\lceil \log_4 \left(\frac{4^{pq} - 1}{\max(r_1, r_2)}\right) \right\rceil, \left\lceil \log_4 \left(\frac{4^{pq} - 1}{(2pq + 1) \cdot \max(r_1, r_2)}\right) \right\rceil.$$
Table 1 Examples

| $p$   | $q$   | $r_1$ | $r_2$ | $\Phi_4(e^\infty)$ |
|-------|-------|-------|-------|--------------------|
| 41, 173, 349, 569 | 5     | 11    | 1     | $pq - 1$           |
| 617, 1237, 1609   | 5     | 31    | 1     | $pq - 2$           |
| 1361             | 5     | 341   | 1     | $pq - 4$           |
| 233              | 29    | 59    | 1     | $pq - 2$           |
| 5                | 89, 353, 397, 617, 1321 | 1 | 11   | $pq - 1$           |
| 5                | 1117, 1489, 1613 | 1 | 31   | $pq - 2$           |
| 5                | 2729  | 1     | 341   | $pq - 4$           |

If $p \equiv q + 4 \pmod{8}$; and

$$\Phi_4(e^\infty) \in \left\{ \left[ \log_4\left(\frac{4^{pq} - 1}{\max(r_1, r_2)}\right) \right], \left[ \log_4\left(\frac{4^{pq} - 1}{(6pq + 1) \cdot \max(r_1, r_2)}\right) \right] \right\}$$

if $p \equiv q \equiv 5 \pmod{8}$.

We remark that either $r_1 = 1$ or $r_2 = 1$. In fact, let $r_1 > 1$ and $r_2 > 1$. Assume that $\bar{r}_1$ is a prime divisor of $r_1$, then $4^q \equiv 1 \pmod{\bar{r}_1}$. Hence $q$ divides $\bar{r}_1 - 1$ and $\bar{r}_1 = 1 + mq$ for $m \in \mathbb{Z}$. Since $\bar{r}_1$ and $q$ are odd, it follows that $m$ is even and $\bar{r}_1 \geq 1 + 2q$. On the other hand, since $\bar{r}_1$ divides $p + 3$, it follows that $1 + 2q \leq \bar{r}_1 < p + 3$ and hence $p > 2q - 2$. Similarly, assume that $\bar{r}_2$ is a prime divisor of $r_2$, we will get $1 + 2p \leq \bar{r}_2 < q - 1$ and hence $p < (q - 2)/2$. A contradiction. We illustrate Theorem 1 by some examples in Table 1.

If we suppose $p < q$. We find that $\gcd(q - 1, 4^p - 1) \leq (q - 1)/4$ and

$$(6pq + 1) \max(r_1, r_2) \leq (6pq + 1)(q - 1)/4 < 3pq^2/2.$$  

Then by (1) we obtain that $\Phi_4(e^\infty) \geq pq - 1 - \log_4(pq^2)$. Similarly, if $p > q$, we have $\Phi_4(e^\infty) \geq pq - 1 - \log_4(p^2q)$.

We will use the Hall polynomial for the studying of 4-adic complexity of above mentioned sequences. Their properties will be studied in the next section.

3 Auxiliary results and proof of main contribution

To prove Theorem 1, we only need to determine the value of $\gcd(E(4), 4^{pq} - 1)$ by (1), where

$$E(X) = \sum_{0 \leq u < pq} e_u X^u \in \mathbb{Z}[X], \ e_u \text{ is in (2)}.$$  

Here and hereafter, the polynomials are with integer coefficients.
Since \( \gcd(4^p - 1, 4^q - 1) = 3 \) for two distinct odd primes \( p \) and \( q \), one can write
\[
4^{pq} - 1 = (4^q - 1) \cdot \frac{4^p - 1}{3} \cdot \frac{3(4^{pq} - 1)}{(4^p - 1)(4^q - 1)},
\]
from which we turn to consider
\[
\gcd(E(4), 4^q - 1), \gcd\left( E(4), \frac{4^p - 1}{3} \right) \quad \text{and} \quad \gcd\left( E(4), \frac{3(4^{pq} - 1)}{(4^p - 1)(4^q - 1)} \right).
\]

Then we will use the following lemma to finish the proof of Theorem 1.

**Lemma 1** Let \( p \) and \( q \) be two distinct odd primes with \( \gcd(p - 1, q - 1) = 4 \).

(i) We have \( \gcd(E(4), 4^q - 1) = \gcd(4^q - 1, p + 3) \).

(ii) We have \( \gcd(E(4), \frac{4^p - 1}{3}) = \gcd(q - 1, (4^p - 1)/3) \).

(iii) Let \( d = \gcd\left( E(4), \frac{3(4^{pq} - 1)}{(4^p - 1)(4^q - 1)} \right) \). If \( d > 1 \), then \( d \) is a prime and
\[
d = \begin{cases} 
2pq + 1, & \text{if } p \equiv q + 4 \pmod{8}, \\
6pq + 1, & \text{if } q \equiv p + 5 \pmod{8}.
\end{cases}
\]

For our purpose, we need to consider the Hall polynomial \( H_j(X) \in \mathbb{Z}[X] \) of \( D_j \), the generalized cyclotomic classes modulo \( pq \) defined in Sect. 1, where
\[
H_j(X) = \sum_{u \in D_j} X^u, \quad j = 0, 1, 2, 3.
\]
It is clear for \( 0 \leq j, k < 4 \)
\[
H_j(X^u) \equiv H_{j+k}(X) \pmod{X^{pq} - 1} \quad \text{for } u \in D_k,
\]
where \( j + k \) in the subscript of \( H \) is taken modulo 4.

Let
\[
U(X) = H_1(X) + 2H_2(X) + 3H_3(X).
\]
Then we have
\[
E(X) = U(X) + 2 \sum_{0 \leq u < p} X^{uq},
\]
and hence
\[
E(4) = U(4) + 2 \sum_{0 \leq u < p} 4^{uq} = U(4) + \frac{2(4^{pq} - 1)}{4q - 1}.
\]

\[1\] Let \( \ell = \gcd(4^p - 1, 4^q - 1) \). Then there exist \( k, l \) such that \( kp + lq = 1 \) and \( 4^{kp+lq} \equiv 1 \pmod{\ell} \), i.e., \( 4 \equiv 1 \pmod{\ell} \) and \( \ell = 3 \).
For each fixed $u$

According to \[8, Lemma 3.3\], we have

\[ U(4^{h^2}) \equiv H_3(4) + 2H_0(4) + 3H_1(4) \pmod{4pq - 1}. \]

3.1 Product of $U(4)$ and $U(4^{h^2})$ modulo $4pq - 1$

According to (3), we have since $h^2 \in D_2$

\[ U(4^{h^2}) \equiv H_3(4) + 2H_0(4) + 3H_1(4) \pmod{4pq - 1}. \]

To compute $U(4) \cdot U(4^{h^2})$ modulo $4pq - 1$, we only need to study the products

\[ H_l(4)H_{l+k}(4) = \sum_{x \in D_l, y \in D_{l+k}} 4^{x+y}, \quad 0 \leq l, k < 4. \]

**Lemma 2** Let $p$ and $q$ be two distinct odd primes with $\gcd(p - 1, q - 1) = 4$.

(i) The number of solutions $(x, y)$ of congruence

\[ x + y \equiv 0 \pmod{pq}, \quad x \in D_l, \; y \in D_{l+k} \]

is given by $\frac{(p-1)(q-1)}{4}$ for all $0 \leq l < 4$ if $k = 0$ and $\frac{(p-1)(q-1)}{16}$ is odd or $k = 2$ and $\frac{(p-1)(q-1)}{16}$ is even, and otherwise the number of solutions of this congruence is equal to 0.

(ii) For each fixed $u \in \{1, 2, \ldots, q - 1\}$, the number of solutions $(x, y)$ of congruence

\[ x + y \equiv up \pmod{pq}, \quad x \in D_l, \; y \in D_{l+k} \]

is given by $\frac{(p-1)(q-5)}{16}$ for all $0 \leq l < 4$ if $k = 0$ and $\frac{(p-1)(q-1)}{16}$ is odd or $k = 2$ and $\frac{(p-1)(q-1)}{16}$ is even, and otherwise the number of solutions of this congruence is equal to $\frac{(p-1)(q-1)}{16}$.

(iii) For each fixed $v \in \{1, 2, \ldots, p - 1\}$, the number of solutions $(x, y)$ of congruence

\[ x + y \equiv vq \pmod{pq}, \quad x \in D_l, \; y \in D_{l+k} \]

is given by $\frac{(p-5)(q-1)}{16}$ for all $0 \leq l < 4$ if $k = 0$ and $\frac{(p-1)(q-1)}{16}$ is odd or $k = 2$ and $\frac{(p-1)(q-1)}{16}$ is even, and otherwise the number of solutions of this congruence is equal to $\frac{(p-1)(q-1)}{16}$.

**Proof** According to \[8, Lemma 3.3\], we have $-1 \in D_0$ when $(p-1)(q-1)/16$ is odd, and $-1 \in D_2$ when $(p-1)(q-1)/16$ is even.

(i) For $x \in D_l, y \in D_{l+k}$, we see that $x + y \equiv 0 \pmod{pq}$ iff $y \equiv -x \pmod{pq}$ iff $-1 \in D_k$. So when $-1 \in D_0 \cup D_2$, $(x, -x)$ are solutions of $x + y \equiv 0 \pmod{pq}$ for each $x \in D_l$, then the number of solutions is $|D_l| = \frac{(p-1)(q-1)}{4}$.\[\square\] Springer
(ii) If \(-1 \in D_0\), then the number of solutions of congruences \(x + y \equiv up \pmod{pq}\), \(x + y \not\equiv 0 \pmod{q}\) is equal to the number of solutions of \(x - y \equiv up \pmod{pq}\). Thus, the statement of this lemma for \(k = 0\) follows from [18, Lemma 4] and for \(k \neq 0\) from [18, Lemma 2], respectively.

If \(-1 \in D_2\), then the number of solutions of congruences \(x + y \equiv up \pmod{pq}\), \(x + y \not\equiv 0 \pmod{q}\) is equal to the number of solutions of \(x - z \equiv up \pmod{pq}\) for \(x \in D_1, z \in D_{l+k+2}\). So similarly, the statement of this lemma for \(k = 2\) follows from [18, Lemma 4] and from [18, Lemma 2] for otherwise.

(iii) We can prove (iii) in the same way as in (ii).

\[
(i, j)_4 = |(1 + D_i) \cap D_j|, \quad 0 \leq i, j < 4.
\]

The number of solutions \((x, y)\) of congruence

\[
1 + x \equiv y \pmod{pq}, \quad x \in D_i, \quad y \in D_j
\]

has been deeply studied in the literature, see e.g. [18]. The symbol \((i, j)_4\), called the cyclotomic numbers of order four modulo \(pq\), is used to denote the number of solutions, that is,

\[
(i, j)_4 = |(1 + D_i) \cap D_j|, \quad 0 \leq i, j < 4.
\]

It is noted that, there always exist integers \(a\) and \(b\) such that \(pq = a^2 + 4b^2\) and \(a \equiv 1 \pmod{4}\) for odd primes \(p\) and \(q\) with \(\gcd(p - 1, q - 1) = 4\). Indeed, by [9, Proposition 8.3.1] and its explanations (after [9, Proposition 8.3.1]) there are only four integer pairs \((\pm x_1, \pm y_1)\) such that \(p = x_1^2 + 4y_1^2\). It is clear \(x_1\) is odd due to odd \(p\), so either \(x_1 \equiv 1 \pmod{4}\) or \(x_1 \equiv 3 \pmod{4}\). If \(x_1 \equiv 3 \pmod{4}\), we see that \(-x_1 \equiv 1 \pmod{4}\), then we always have a solution for example \((x_1, y_1)\) such that \(x_1 \equiv 1 \pmod{4}\). Similarly we have integer pair \((x_2, y_2)\) such that \(q = x_2^2 + 4y_2^2\) and \(x_2 \equiv 1 \pmod{4}\). Hence we get

\[
pq = (x_1^2 + 4y_1^2)(x_2^2 + 4y_2^2) = (x_1x_2 + 4y_1y_2)^2 + 4(x_1y_2 - x_2y_1)^2 = (x_1x_2 - 4y_1y_2)^2 + 4(x_1y_2 + x_2y_1)^2.
\]

That is, we have two decompositions \(pq = a_1^2 + 4b_1^2 = a_2^2 + 4b_2^2\) with \(a_1 \equiv a_2 \equiv 1 \pmod{4}\) and \(a_1 \not\equiv \pm a_2\). For example, if \(p = 5\) and \(q = 41\), then we have \(pq = 205 = 13^2 + 4 \times 3^2 = (-3)^2 + 4 \times 7^2\) with \(a_1 = 13\) and \(a_2 = -3\).

**Lemma 3** [18] Let \(p\) and \(q\) be two distinct odd primes with \(\gcd(p - 1, q - 1) = 4\). Let

\[
M = \frac{(p-2)(q-2)-1}{4}.
\]

One can choose suitable integers \(a\) and \(b\) with \(pq = a^2 + 4b^2\), \(a \equiv 1 \pmod{4}\) and
such that the cyclotomic numbers are given in Table 2 for all \(0 \leq i, j < 4\) if \(\frac{(p-1)(q-1)}{16}\) is even, and in Table 3 otherwise.

**Lemma 4** Let \(p\) and \(q\) be two distinct odd primes with \(\gcd(p-1, q-1) = 4\). Let \(d > 1\) be a divisor of \(\frac{3(4pq-1)}{(4p-1)(4q-1)}\). For \(0 \leq l, k < 4\), we have

\[
    H_l(4) \cdot H_{l+k}(4) \equiv \sum_{f=0}^{3} (k, f)_{4} H_{f+l}(4) + \Delta \pmod{d},
\]

where \(\Delta = \frac{(p+1)(q+1)-4}{8}\) if \(k = 0\) and \(\frac{(p-1)(q-1)}{8}\) is odd or \(k = 2\) and \(\frac{(p-1)(q-1)}{16}\) is even, and otherwise \(\Delta = -\frac{(p-1)(q-1)}{8}\).

**Proof** By definition we have

\[
    H_l(4) \cdot H_{l+k}(4) = \sum_{x \in D_l, y \in D_{l+k}} 4^{x+y}.
\]
Firstly, we compute

\[ \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x+y} \equiv \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x(y^{-1}+1)} \pmod{4^{pq} - 1}, \]

where \( x^{-1} \) is an inverse of \( x \) modulo \( pq \). Since \( yx^{-1} \pmod{pq} \in D_k, x + y \not\equiv 0 \pmod{p} \) and \( x + y \not\equiv 0 \pmod{q} \), it follows that

\[ \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x(y^{-1}+1)} \equiv \sum_{x \in D_l, w \in D_k \atop w+1 \equiv 0 \pmod{pq}} 4^{x(w+1)} \]

\[ \equiv \sum_{f=0}^{3} \sum_{w \in D_k \atop w+1 \in D_f} \sum_{x \in D_l} 4^{x(w+1)} \equiv \sum_{f=0}^{3} \sum_{w \in D_k \atop w+1 \in D_f} H_{f+l}(4) \pmod{4^{pq} - 1}. \]

The last ‘\( \equiv \)’ comes from (3). Now since \( (D_k + 1) \cap D_f \models (k, f)_4 \), we obtain

\[ \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x+y} \equiv \sum_{f=0}^{3} (k, f)_4 H_{f+l}(4) \pmod{4^{pq} - 1}. \]

Secondly, we calculate

\[ \Gamma = \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x+y} + \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{P}, x+y \not\equiv 0 \pmod{Q}} 4^{x+y} + \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{Q}, x+y \not\equiv 0 \pmod{P}} 4^{x+y}. \]

We consider the case when \( k = 0 \) and \( (p-1)(q-1)/16 \) is odd or \( k = 2 \) and \( (p-1)(q-1)/16 \) is even. In this case, by Lemma 2 (i) we get

\[ \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x+y} \equiv \frac{(p-1)(q-1)}{4} \pmod{4^{pq} - 1}. \]

By Lemma 2 (ii), we get

\[ \sum_{x \in D_l, y \in D_{l+k}, \atop x+y \equiv 0 \pmod{pq}} 4^{x+y} \equiv \sum_{\ell=1}^{q-1} \frac{(p-1)(q-5)}{4^{\ell p}} \equiv \frac{(p-1)(q-5)}{4^{p-1}} \cdot \left( \frac{4^{pq} - 1}{4^{p-1} - 1} \right) \pmod{4^{pq} - 1}. \]
Similarly, by Lemma 2 (iii), we get
\[
\sum_{x \in D_{i}, y \in D_{i+k}, x+y \notin P} 4^{x+y} \equiv \frac{(q-1)(p-5)}{16} \cdot \left(\frac{4^{pq} - 1}{4^{q} - 1} - 1\right) \pmod{4^{pq} - 1}.
\]

Then, we have
\[
\Gamma \equiv \frac{(p-1)(q-1)}{4} - \frac{(p-1)(q-5)}{16} - \frac{(q-1)(p-5)}{16} \equiv \Delta \pmod{d}.
\]

Putting everything together, we prove the desired result. For other cases, it can be done in the same way.

Now we prove a result of the product of $U(4)$ and $U(4^{b^2})$ modulo $d$. □

Lemma 5 Let $p$ and $q$ be two distinct odd primes with $\gcd(p - 1, q - 1) = 4$. Let $d > 1$ be a divisor of $\frac{3(4^{pq} - 1)}{4^{p-1} - 1}$. We have

\[
4U(4)U(4^{b^2}) \equiv \begin{cases} -2(4b + 3)\mathcal{H} + 5pq + 9, & \text{if } \frac{(p-1)(q-1)}{16} \text{ is even,} \\ -2(4b + 3)\mathcal{H} - 3pq + 9, & \text{if } \frac{(p-1)(q-1)}{16} \text{ is odd,} \end{cases} \pmod{d},
\]

where $\mathcal{H} = H_0(4) + H_2(4) - H_1(4) - H_3(4)$ and $b$ is defined in Lemma 3.

Proof Since $U(4) = H_1(4) + 2H_2(4) + 3H_3(4)$ and $U(4^{b^2}) \equiv H_5(4) + 2H_0(4) + 3H_1(4) \pmod{4^{pq} - 1}$, we can write

\[
U(4) \cdot U(4^{b^2}) \equiv \sum_{0 \leq i, j < 4} a_{ij} H_i(4) \cdot H_j(4) \pmod{4^{pq} - 1}
\]

for some (unique) integers $a_{ij}$ as coefficients.

Let $(p-1)(q-1)/16$ be even. By Lemma 4, we re-write

\[
U(4) \cdot U(4^{b^2}) \equiv L_0(H_0(4) + H_2(4)) + L_1(H_1(4) + H_3(4)) + L_3 \pmod{d},
\]

where

\[
\begin{align*}
L_0 &= 3(0, 1) + 3(0, 3) + 2(1, 0) + 2(1, 2) + 6(1, 3) + 4(2, 0) + 10(2, 3) + 6(3, 0), \\
L_1 &= 3(0, 0) + 3(0, 2) + 6(1, 0) + 2(1, 1) + 2(1, 3) + 10(2, 0) + 4(2, 1) + 6(3, 1), \\
L_3 &= -(2pq + 9p + 9q - 16)/2.
\end{align*}
\]

After some tedious computations of $L_0, L_1$ by Lemma 3 (with $M = \frac{(p-2)(q-2)-1}{4}$), we derive

\[
U(4) \cdot U(4^{b^2}) \equiv (-2b + 9M + 2)(H_0(4) + H_2(4)) + (2b + 9M + 5)(H_1(4) + H_3(4)) + L_3 \pmod{d},
\]

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which leads to

\[ 2U(4) \cdot U(4^{h^2}) \equiv (-4b - 3)(H_0(4) + H_2(4)) + (4b + 3)(H_1(4) + H_3(4)) \]
\[ (18M + 7)(H_0(4) + H_2(4) + H_1(4) + H_3(4)) + 2L_3 \pmod{d}. \]

Since \( H_0(4) + H_2(4) + H_1(4) + H_3(4) \equiv 1 \pmod{d} \) and

\[ 18M + 7 + 2L_3 = 18 \frac{(p - 2)(q - 2) - 1}{4} + 7 + (-2pq + 9p + 9q - 16) \]
\[ = \frac{5pq + 9}{2}, \]

we obtain

\[ 4U(4) \cdot U(4^{h^2}) \equiv -2(4b + 3)(H_0(4) + H_2(4) - H_1(4) - H_3(4)) \]
\[ + 5pq + 9 \pmod{d}. \]

We finish the proof for the first case. For the case when \((p - 1)(q - 1)/16\) is odd, it can be done in the same way. \(\Box\)

The proof of the following lemma can be easily carried out by computation as in Lemma 5, we give this here for the completeness.

**Lemma 6** Let \( p \) and \( q \) be two distinct odd primes with \( \gcd(p - 1, q - 1) = 4 \). Let \( d > 1 \) be a divisor of \( \frac{3(4pq - 1)}{(4p - 1)(4q - 1)} \). For \( \mathcal{H} = H_0(4) + H_2(4) - H_1(4) - H_3(4) \), we have

\[ \mathcal{H}^2 \equiv pq \pmod{d}. \]

**Proof** Since \( H_0(4) + H_2(4) + H_1(4) + H_3(4) \equiv 1 \pmod{d} \), we obtain

\[ \mathcal{H}^2 \equiv (2H_0(4) + 2H_2(4) - 1)^2 \]
\[ \equiv 4H_0^2(4) + 4H_2^2(4) + 8H_0(4)H_2(4) - 4H_0(4) - 4H_2(4) + 1 \pmod{d}. \]

Let \((p - 1)(q - 1)/16\) be even. Using Lemmas 3 and 4, we get

\[ \mathcal{H}^2 \equiv M_0H_0(4) + M_1H_1(4) + M_2H_2(4) + M_3H_3(4) + M_4 \pmod{d}, \]

where

\[ \begin{align*}
M_0 &= 4(0, 0)_4 + 4(0, 2)_4 + 8(2, 0)_4 - 4, \\
M_1 &= 4(0, 1)_4 + 4(0, 3)_4 + 8(2, 1)_4, \\
M_2 &= 4(0, 0)_4 + 4(0, 2)_4 + 8(2, 2)_4 - 4, \\
M_3 &= 4(0, 1)_4 + 4(0, 3)_4 + 8(2, 3)_4, \\
M_4 &= 2p + 2q - 4.
\end{align*} \]
After computing $M_i$ by formulae for cyclotomic numbers, we have

$$M_0 = M_1 = M_2 = M_3 = 4M,$$

and hence

$$\mathcal{H}^2 \equiv 4M \left( H_0(4) + H_2(4) + H_1(4) + H_3(4) \right) + 2p + 2q - 3 \equiv 4M + 2p + 2q - 3 \equiv pq \pmod{d}.$$ 

For the case when $(p - 1)(q - 1)/16$ is odd, it can be done in the same way. □

### 3.2 Proof of Lemma 1

It is easy to see that for any polynomial $G(X) \in \mathbb{Z}[X]$, $G(4) \equiv G(1) \pmod{3}$, so we get

$$E(4) = H_1(4) + 2H_2(4) + 3H_3(4) + 2 \sum_{i=0}^{p-1} 4^i q \equiv H_1(1) + 2H_2(1) + 3H_3(1) + 2 \sum_{i=0}^{p-1} 1^i q \equiv 6(p - 1)(q - 1)/4 + 2p \equiv 2p \pmod{3},$$

and hence $3 \nmid E(4)$. Now we prove (i) and (ii).

According to [1, Lemma 2], we see that $D_j \mod q = \{1, 2, \ldots, q - 1\}$ and when $s$ ranges over $\{0, 1, \ldots, e - 1\}$, $g^s h^j \pmod{q}$ takes on each element of $\{1, 2, \ldots, q - 1\}$ exactly $(p - 1)/4$ times, hence for $0 \leq j < 4$ we get

$$H_j(4) \equiv \frac{p - 1}{4} \left( 4 + \cdots + 4^{q-1} \right) \equiv \frac{p - 1}{4} \cdot \left( \frac{4^q - 1}{4 - 1} - 1 \right) \pmod{4^q - 1}.$$

Then together with $4^i q \equiv 1 \pmod{4^q - 1}$ for all $i \in \mathbb{N}$, we obtain

$$E(4) \equiv 6 \cdot \frac{p - 1}{4} \cdot \left( \frac{4^q - 1}{4 - 1} - 1 \right) + 2p \equiv \frac{p + 3}{2} \pmod{4^q - 1}.$$

Thus, due to $3 \nmid E(4)$ and $3 \nmid p + 3$, we derive

$$\gcd(E(4), 4^q - 1) = \gcd(4^q - 1, \frac{p + 3}{2}) = \gcd(4^q - 1, p + 3).$$

Similarly, we have

$$H_j(4) \equiv \frac{q - 1}{4} \left( 4 + \cdots + 4^{p-1} \right) \equiv \frac{q - 1}{4} \cdot \left( \frac{4^p - 1}{4 - 1} - 1 \right) \pmod{4^p - 1}.$$
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and

\[ E(4) \equiv - \frac{6(q - 1)}{4} + \frac{2(4pq - 1)}{4^q - 1} \equiv - \frac{3(q - 1)}{2} \quad \text{(mod } (4^p - 1)/3) \text{).} \]

Thus, due to the fact\(^2\) that \(3^2 \nmid (4^p - 1)\) and \(3 \nmid E(4)\), we derive

\[ \gcd(E(4), \gcd(\frac{4p - 1}{3})) = \gcd(4^p - 1, q - 1) \]

if \(q \not\equiv 1 \pmod{3}\), and otherwise

\[ \gcd(E(4), 4^p - 1) = \gcd(4^p - 1, (q - 1))/3. \]

Thus, \(\gcd(E(4), \frac{4p - 1}{3}) = r_2\).

Now we turn to prove (iii).

Suppose \(d > 1\). Note that \(d\) is odd with \(3 \nmid d\) since \(3 \nmid E(4)\). We now assume that \(d_0 > 3\) is an odd prime with \(d_0 | d\). Below we will discuss the possible value of \(d_0\).

We first show \(4^p \not\equiv 1 \pmod{d_0}\) and \(4^q \not\equiv 1 \pmod{d_0}\) to get the possible form of \(d_0\). Since \(d_0\) is a divisor of \(\frac{3(4pq - 1)}{(4^p - 1)(4^q - 1)}\), we see that \(d_0\) is a divisor of

\[ \frac{3(4pq - 1)}{4^p - 1} = 3(1 + 4^p + \cdots + 4^{(q-1)p}), \]

from which we derive

\[ 1 + 4^p + \cdots + 4^{(q-1)p} \equiv 0 \pmod{d_0}. \]

If \(4^p \equiv 1 \pmod{d_0}\), that is \(d_0 | (4^p - 1)\), we get from above

\[ 0 \equiv 1 + 4^p + \cdots + 4^{(q-1)p} \equiv q \pmod{d_0}. \]

So we have \(d_0 = q\) since \(d_0 > 3\) is a prime. Together with \(d_0 | E(4)\) and \(d_0 | (4^p - 1)\), we derive by (ii) of this lemma,

\[ q | \gcd(E(4), 4^p - 1) \Rightarrow q | (q - 1), \]

a contradiction. Similarly one can prove \(4^q \not\equiv 1 \pmod{d_0}\). From discussions above, we also see that \(p \nmid d_0\) and \(q \nmid d_0\). Therefore, the congruences

\[ \begin{cases} 4^{pq} \equiv 1 \pmod{d_0}, \\ 4^p \not\equiv 1 \pmod{d_0}, \\ 4^q \not\equiv 1 \pmod{d_0}, \end{cases} \quad (5) \]

\(\text{Let } 3^2 \nmid (4^p - 1). \text{ That is } 4^p \equiv 1 \pmod{9}. \text{ Then } p \text{ divides the value of the Euler’s totient function } \phi(9) = 6 \text{ and we have a contradiction since } p > 3.\)
tell us that $m = pq$ is the smallest integer such that $4^m \equiv 1 \pmod{d_0}$. Then we have $pq \mid (d_0 - 1)$ since $4^{d_0 - 1} \equiv 1 \pmod{d_0}$ by the Fermat Little Theorem. Then $d_0$ is of the form

$$d_0 = 1 + 2\lambda pq, \text{ for some } 0 < \lambda \in \mathbb{Z}.$$  

We note that either $p \equiv q + 4 \pmod{8}$ or $p \equiv q \equiv 5 \pmod{8}$, since $\gcd(p - 1, q - 1) = 4$. We have $d \mid U(4)$ by (4). Now we consider the following two cases.

- Suppose $p \equiv q + 4 \pmod{8}$.
  In this case we see that $(p - 1)(q - 1)/16$ is even. Then since $d \mid U(4)$, by Lemma 5 we have

$$-2(4b + 3)H + 5pq + 9 \equiv 0 \pmod{d},$$

that is,

$$2(4b + 3)H \equiv 5pq + 9 \pmod{d}.$$  

Using Lemma 6, we obtain

$$4(4b + 3)^2 pq \equiv 25p^2 q^2 + 90pq + 81 \pmod{d}. \quad (6)$$

Then from (6), we obtain

$$- 4(4b + 3)^2 + 25pq + 90 - 162\lambda \equiv 0 \pmod{d_0}, \quad (7)$$

where we use $1 \equiv -2\lambda pq \pmod{d_0}$.

Since the left hand side of (7) is odd, it can be written as

$$-4(4b + 3)^2 + 25pq + 90 - 162\lambda = \delta(1 + 2\lambda pq) \quad (8)$$

with odd $\delta = 8\mu + 3$ for some integer $\mu$.  

Below we consider the possible values of $\mu$ and $\lambda$ from (8).

1. If $\mu < 0$, since $pq = a^2 + 4b^2$ and

$$25pq + 90 - 4(4b + 3)^2 \geq 36b^2 - 96b + 79 > 0,$$

it follows that $-162\lambda < (3 + 8\mu)(1 + 2\lambda pq)$ or

$$162\lambda > (-3 - 8\mu)(1 + 2\lambda pq) \geq 5 + 10\lambda pq > 10\lambda pq.$$

3 If $\delta$ is of the form $8\mu + 1, 8\mu + 5$ or $8\mu + 7$, we have from (8)

$$-4(4b + 3)^2 + 25pq + 90 - 162\lambda \not\equiv \delta(1 + 2\lambda pq) \pmod{8}.$$
We get $10pq < 162$, and hence $p = 3$ and $q = 5$ (or $p = 5$ and $q = 3$). This is a contradiction to $\gcd(p - 1, q - 1) = 4$.

(2) If $\mu = 0$, then by (8) we get that

$$-4(4b + 3)^2 + 25pq + 90 - 162\lambda = 3(1 + 2\lambda pq), \quad (9)$$

and hence $25pq > 6\lambda pq$. So $\lambda \in \{1, 2, 3, 4\}$.

Let $\lambda = 1$ or $\lambda = 4$. By (9), we obtain $-b^2 + pq \equiv 0 \pmod{3}$. Since $p > 3$ and $q > 3$, it follows that $b \not\equiv 0 \pmod{3}$. Hence $b^2 \equiv 1 \pmod{3}$ and $pq \equiv b^2 \equiv 1 \pmod{3}$. This leads to $1 + 2\lambda pq \equiv 0 \pmod{3}$, a contradiction to that $d_0 = 1 + 2\lambda pq$ is a prime.

Let $\lambda = 2$. We have $d_0 = 1 + 4pq$ and $d_0 \equiv 5 \pmod{8}$ since $p \equiv q + 4 \pmod{8}$. By condition $d_0$ is a prime. Then there exist a primitive root $\theta$ modulo $d_0$ and the order of $\theta$ modulo $d_0$ is equal to $\varphi(d_0) = d_0 - 1 = 4pq$, i.e., $4pq$ is the smallest integer such that $\theta^{4pq} \equiv 1 \pmod{d_0}$. Write $2 \equiv \theta^m \pmod{d_0}$ for some integer $m$, we have

$$1 \equiv 4^{pq} \equiv 2^{2pq} \equiv \theta^{2mpq} \pmod{d_0},$$

since $d_0$ is a divisor of $4^{pq} - 1$. Hence $2mpq$ is divided by $4pq$, which indicates that $m$ is even and 2 is a quadratic residue modulo $d_0$. We obtain a contradiction since 2 is a quadratic residue modulo $d_0$ iff $d_0 \equiv \pm 1 \pmod{8}$.

Let $\lambda = 3$. In this case we have by (9)

$$-4(4b + 3)^2 + 25pq + 90 - 486 = 3(1 + 6pq),$$

that is, $-4(4b+3)^2+7pq-399 = 0$, which leads to $-4(4b+3)^2 \equiv 0 \pmod{7}$. So we write $4b + 3 = 7w$ for some $w \in \mathbb{Z}$ and get $-4 \cdot 7w^2 + pq - 57 = 0$. Hence $pq \equiv 1 \pmod{7}$ and $d_0 = 1 + 6pq \equiv 0 \pmod{7}$, which contradicts to $d_0$ being prime.

(3) If $\mu > 0$, then by (8) we get that

$$25pq > (3 + 8\mu)(1 + 2\lambda pq),$$

from which we derive $\mu = \lambda = 1$. Then $d_0 = 1 + 2pq$ (which is a prime) and $4(4b + 3)^2 = 3pq - 83$ by (8) again.

From discussions above, we conclude that $d$ is a power of prime $d_0$ with $d_0 = 1 + 2pq$. If we assume $d_0^2 \mid d$, putting (6) with $4(4b + 3)^2 = 3pq - 83$ together, we get

$$(3pq - 83)pq \equiv 25p^2q^2 + 90pq + 81 \pmod{d_0^2},$$

that is,

$$22p^2q^2 + 173pq + 81 \equiv 0 \pmod{4p^2q^2 + 4pq + 1},$$
this is impossible.
So, we prove that \( \gcd \left( E(4), \frac{3(4pq-1)}{(4^4-1)(4^4-1)} \right) \) equals either 1 or 1 + 2pq, where 1 + 2pq is a prime number.

• Suppose \( p \equiv q \equiv 5 \pmod{8} \). The proof is similar to above. We give a sketch of proof for convenience of the reader.

In this case \((p - 1)(q - 1)/16\) is odd. Since \( d \mid U(4) \) and according to Lemma 5, we have

\[-2(4b + 3)\mathcal{H} - 3pq + 9 \equiv 0 \pmod{d}.\]

Fatherly, using Lemma 6 we have

\[4(4b + 3)^2 pq \equiv (-3pq + 9)^2 \equiv 9p^2q^2 - 54pq + 81 \pmod{d}. \tag{10}\]

Since \( d_0 \mid d \), we have from (10)

\[-4(4b + 3)^2 + 9pq - 54 - 162\lambda = \delta \cdot d_0 = \delta (1 + 2\lambda pq) \tag{11}\]

with odd \( \delta = 8\mu + 7 \) for some integer \( \mu \).\(^4\) Below we consider the possible values of \( \mu \) and \( \lambda \) from (11).

(1) If \( \mu \geq 0 \) then by (11) we get \( 9pq > 14pq \). It is impossible.

(2) If \( \mu < -1 \) then by (11) we obtain that

\[-4(4b + 3)^2 + 9pq - 54 - 162\lambda \leq -9 - 18\lambda pq, \]

that is,

\[-4(4b + 3)^2 + 9pq - 45 \leq 162\lambda - 18\lambda pq. \]

Since \( pq \geq 65 \), it follows that \( 162\lambda - 3\lambda pq < 0 \), which leads to

\[-4(4b + 3)^2 + 9pq - 45 \leq -15\lambda pq \leq -15pq. \]

Then

\[24(1 + 4b^2) \leq 24pq \leq 4(4b + 3)^2 + 45. \tag{12}\]

\(^4\) If \( \delta \) is of the form \( 8\mu + 1, 8\mu + 3 \) or \( 8\mu + 5 \), we have from (11)

\[-4(4b + 3)^2 + 9pq - 54 - 162\lambda \not\equiv \delta (1 + 2\lambda pq) \pmod{8}. \]
In this case, we see that \(32b^2 - 96b - 57 \leq 0\). According to [8] \(b\) is even, hence only \(b = 2\) satisfies the last inequality. Then by (12) we get

\[
24pq \leq 4 \cdot 11^2 + 25 = 529, \quad \text{i.e.,} \quad pq < 22.
\]

No such \(p\) and \(q\) exist.

(3) If \(\mu = -1\) then by (11) we have

\[
-4(4b + 3)^2 + 9pq - 54 - 162\lambda = -(1 + 2\lambda pq),
\]

from which \(b \not\equiv 0 \pmod{3}\), since otherwise \(d_0 = 1 + 2\lambda pq \equiv 0 \pmod{3}\) from (13), this means that \(d_0\) is not a prime. So from (13) again, we have \(1 \equiv b^2 \equiv 1 + 2\lambda pq \pmod{3}\). Then we derive \(2\lambda pq \equiv 0 \pmod{3}\) and hence \(\lambda \equiv 0 \pmod{3}\), i.e., \(\lambda = 3, 6, 9, \ldots\).

It is easy to prove that the equality (13) is not true for \(pq = 65\).

Let \(\lambda \geq 6\). In this case for \(pq > 65\) we see that

\[
-4(4b + 3)^2 + 9pq - 54 = -\lambda(1 + 2pq - 162) \leq -6(1 + 2pq - 162).
\]

Then

\[
21(1 + 4b^2) \leq 21pq \leq 4(4b + 3)^2 + 1020.
\]

Hence \(20b^2 - 96b - 1035 \leq 0\), from which we get \(b = -4, -2, 2, 4\) since \(b\) is even and \(b \not\equiv 0 \pmod{3}\). For these values of \(b\) by (14) we get \(21pq \leq 2464\) or \(pq < 118\). No such \(p\) and \(q\) exist.

Let \(\lambda = 3\). Then \(d_0 = 1 + 6pq\) and \(4(4b + 3)^2 = 15pq - 539\) by (13).

From discussions above, we conclude that \(d\) is a power of prime \(d_0\) with \(d_0 = 1 + 6pq\). We can show that \((1 + 6pq)^2\) does not divide \(d\) in the same way as before.

So, we prove that \(\gcd\left(\mathcal{E}(4), \frac{3(4pq - 1)}{(4p - 1)(4q - 1)}\right)\) equals either 1 or \(1 + 6pq\), where \(1 + 6pq\) is a prime number.

This completes the proof of Lemma 1. \(\square\)

### 3.3 Proof of Theorem 1

From

\[
\gcd\left(4^q - 1, \frac{4^p - 1}{3}\right) = 1,
\]

we see that \(\gcd\left(\mathcal{E}(4), 4^q - 1\right)\) and \(\gcd\left(\mathcal{E}(4), \frac{4^p - 1}{3}\right)\) do not share the common divisor of larger than one. That is, any divisor larger than one of \(\gcd\left(\mathcal{E}(4), 4^q - 1\right)\) does not divide \(\gcd\left(\mathcal{E}(4), \frac{4^p - 1}{3}\right)\), and vice versa.

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From (5), we see that
\[ \gcd\left(\frac{4^p - 1}{3}, \gcd\left(\frac{3(4^pq - 1)}{(4^p - 1)(4^q - 1)}\right)\right) = 1. \]

Since otherwise, we assume that \( d_0 \) is a prime with \( d_0 \mid \gcd(E(4), 4^p - 1) \) and \( d_0 \mid \gcd\left(E(4), \frac{3(4^pq - 1)}{(4^p - 1)(4^q - 1)}\right) \). Then we obtain \( d_0 \mid (4^p - 1) \) and \( d_0 \mid (4^pq - 1) \), which contradicts to (5). Similarly, we have
\[ \gcd\left(\frac{4^q - 1}{3}, \gcd\left(\frac{3(4^pq - 1)}{(4^p - 1)(4^q - 1)}\right)\right) = 1. \]

So we derive
\[ \gcd(E(4), 4^pq - 1) = \gcd(E(4), 4^q - 1) \cdot \gcd\left(E(4), \frac{4^p - 1}{3}\right) \cdot \gcd\left(E(4), \frac{3(4^pq - 1)}{(4^p - 1)(4^q - 1)}\right). \]

Then Theorem 1 is proved by Lemma 1 in terms of (1).

4 Symmetric 4-adic complexity

According to the discussion of Hu and Feng [7], it is interesting to determine the 2-adic complexity of \( \tilde{s}^\infty \), where \( s^\infty = (s_{T-1}, s_{T-2}, \ldots, s_0) \) is taken from the binary sequence \( s^\infty \) of period \( T \) in inverse order. They defined
\[ \Phi_2(s^\infty) = \min\left(\Phi_2(s^\infty), \Phi_2(3s^\infty)\right), \]
which is called the symmetric 2-adic complexity of \( s^\infty \). They claimed that the symmetric 2-adic complexity is better than 2-adic complexity in measuring the security for binary sequences.

Therefore, it is interesting to consider \( \Phi_4(\tilde{e}^\infty) \) for \( \tilde{e}^\infty = (e_{pq-1}, e_{pq-2}, \ldots, e_0) \), where \( e_u \) is defined in (2). The generating polynomial of \( \tilde{e}^\infty \) is
\[ \tilde{E}(X) = \sum_{i=1}^{pq} e_{pq-i} X^{i-1}. \]

In this case, we have the following statement.

Lemma 7 Let \( e^\infty \) be defined as in (2) and \( \tilde{e}^\infty = (e_{pq-1}, \ldots, e_1, e_0) \). Then

(i) \( 4\tilde{E}(4) \equiv E(4) \mod 4^{pq} - 1 \) if \( \frac{(p-1)(q-1)}{16} \) is odd;
(ii) \( 4\tilde{E}(4) \equiv U(4h^2) + 2 \sum_{u=0}^{p-1} 4^{uq} \mod 4^{pq} - 1 \) if \( \frac{(p-1)(q-1)}{16} \) is even, where \( U(X) = H_1(X) + 2H_2(X) + 3H_3(X) \).
Proof By definition of $\tilde{e}^\infty$ we see that $\tilde{E}(4) = \sum_{i=1}^{pq} e_{pq-i}4^{i-1}$. Hence

$$4\tilde{E}(4) = \sum_{i=1}^{pq} e_{pq-i}4^i = \sum_{i=0}^{pq-1} e_{pq-i}4^i + e_04^pq - e_{pq}. \quad (15)$$

It is clear that $e_{pq-i} = e_i$ for $i \in P \cup Q \cup R$.

Let $i \in \mathbb{Z}^*_{pq}$. As noted above we have $-1 \in D_0$ if $(p-1)(q-1)/16$ is odd and $-1 \in D_2$ if $(p-1)(q-1)/16$ is even, respectively. So, $e_{-i} = j$ iff $i \in D_j$ in the first case and $e_{-i} = j$ iff $i \in D_{(j+2) \mod 4}$ in the second case. Thus, this statement follows from the definition of $U(X)$, (3) and (15).

According to Lemmas 5 and 7, we get the same result for $\Phi_4(\tilde{e}^\infty)$ of $\tilde{e}^\infty$ as that of $e^\infty$ in Theorem 1. We state below:

**Theorem 2** Let $p$ and $q$ be two distinct odd primes with $\gcd(p-1, q-1) = 4$. Let $e^\infty$ be a quaternary sequence of length $pq$ defined in (2) and $\tilde{e}^\infty = (e_{pq-1}, e_{pq-2}, \ldots, e_0)$. Then we have

$$\Phi_4(\tilde{e}^\infty) = \Phi_4(e^\infty),$$

and hence the symmetric 4-adic complexity $\overline{\Phi}_4(e^\infty)$ of $e^\infty$ satisfies

$$\overline{\Phi}_4(e^\infty) = \Phi_4(\tilde{e}^\infty) = \Phi_4(e^\infty).$$

5 Final remarks and conclusions

Sequences generated by FCSRs share many important properties by LFSR sequences. In this work, we continued to study quaternary sequences of period $pq$ generated by the two-prime generator. We determined the possible values of the 4-adic complexity, which is larger than $pq - \log_4(pq^2)$ for $p < q$. Our study is based on the properties of the Hall polynomials with respect to Whiteman generalized cyclotomic classes modulo $pq$. Our result showed that the 4-adic complexity of these sequences is obviously large enough to resist against the Rational Approximation Algorithm for FCSR.

With the help of computer program we verified that neither $(2pq + 1) \nmid E(4)$ nor $(6pq + 1) \nmid E(4)$ for all primes $p, q$ such that $5 \leq p, q < 10,000$ and $pq \leq 424,733$. So we conjecture that

$$\Phi_4(e^\infty) = \left\lceil \log_4 \left( \frac{4pq - 1}{\max(r_1, r_2)} \right) \right\rceil,$$

for all odd primes $p$ and $q$ with $\gcd(p-1, q-1) = 4$.

Finally we have to remark that, for most $u$ with $\gcd(u, pq) = 1$, we always have

$$e_u - e_{u+p} - e_{u+q} + e_{u+p+q} = 0.$$
Thus the correlation of order 4 is large, which leads to a lack of practical applications. However, we hope the method in this work might be helpful for other study.

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