A variational principle for nonpotential perturbations of gradient flows of nonconvex energies

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Abstract

We investigate a variational approach to nonpotential perturbations of gradient flows of nonconvex energies in Hilbert spaces. We prove existence of solutions to elliptic-in-time regularizations of gradient flows by combining the minimization of a parameter-dependent functional over entire trajectories and a fixed-point argument. These regularized solutions converge up to subsequence to solutions of the gradient flow as the regularization parameter goes to zero. Applications of the abstract theory to nonlinear reaction-diffusion systems are presented.

1 Introduction

This work is concerned with a nonpotential perturbation of a gradient flow driven by a possibly nonconvex energy \( \phi : H \to (-\infty, +\infty] \), namely

\[
\begin{align*}
    u' + D\phi(u) &\ni f(u) \quad \text{a.e. in } (0, T), \\
    u(0) &= u_0.
\end{align*}
\]

Here \( H \) is a real Hilbert space, \( u' \) denotes the time derivative of \( u \), and we assume that the energy \( \phi \) can be decomposed as

\[
\phi = \varphi_1 - \varphi_2, \quad \varphi_1, \varphi_2 : H \to (-\infty, +\infty],
\]

where \( \varphi_1, \varphi_2 \) are proper, bounded from below, and lower semicontinuous (l.s.c.) functionals. The symbol \( D\phi \) represents some suitably-defined gradient of the functional \( \phi \) (see below), \( f : H \to H \) is an (at most) linearly growing and continuous function, and \( u_0 \in D(\phi) := \{ u \in H : \phi(u) < +\infty \} \). Note that we are not assuming here \( f(u) = D\phi(u) \) for some \( F : H \to \mathbb{R} \). In particular, the perturbation term \( f \) is nonpotential. Including a

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\end{itemize}
nonpotential term allows us to apply our theory to systems of differential equations, see Section 6.

Gradient flows arise ubiquitously in connection with dissipative evolution and correspond to problem (1.1) for \( f = 0 \), namely,

\[
\begin{align*}
    u' + \text{D} \phi(u) &\ni 0 \quad \text{a.e. in} \ (0, T), \\
    u(0) = u_0.
\end{align*}
\]

Together with its nonpotential perturbation (1.1), the latter describes a variety of dissipative evolution situations and it is therefore crucially relevant in applications.

A recent variational approach to dissipative problems is the so-called weighted energy-dissipation (WED) procedure. This consists in defining an energy-dissipation functional \( I_\varepsilon \) over entire trajectories which depends on a parameter \( \varepsilon \) and prove that its minimizers converge to solutions to the target problem for \( \varepsilon \to 0 \). Such a global-in-time variational approach to dissipative problems is interesting, since it paves the way to the application of tools and techniques of the calculus of variation (e.g. Direct Method, relaxation, \( \Gamma \)-convergence). Moreover, the WED procedure brings also a new tool to check qualitative properties of solutions and comparison principles for dissipative problems. A detailed discussion of this application will appear in a forthcoming paper. In addition, the minimization problem features, typically, more regular solutions. This is indeed the case here, as the Euler-Lagrange system associated with the minimization of the WED functional corresponds to an elliptic-in-time regularization of the gradient flow problem. The elliptic-regularization approach to evolution equations has to be traced back at least to [Li] and [Ol], see also [Li-Ma]. A first occurrence of the WED functional approach is in [Hi] and [Hi]. Later, the WED formalism has been reconsidered by Mielke and Ortiz [Mi-Or] for rate-independent equations. The gradient flow case with \( \lambda \)-convex potentials has been studied by Mielke and Stefanelli [Mi-St]. The extension to the genuinely non-convex energy case is due to Akagi and Stefanelli [Ak-St]. Finally, [Ak-St2] and [Ak-St3] are concerned with the WED functional for doubly nonlinear problems.

This note extends the WED variational approach to the nonpotential case. In particular, the results from [Mi-St] and [Ak-St] will be recovered. In addition, our new technique will allow the application of the method to systems of gradient flows, e.g., of reaction diffusion equations coupled via the reaction terms. We remark that existence results for the Cauchy problem (1.1)-(1.2) have already been proved in [At-Da] and [Ol]. The main result of this work is that solutions to (1.1)-(1.2) can be obtained as limits of solutions to an elliptic regularization of (1.1)-(1.2), which is tackled by combining a fixed-point argument and a variational technique. Having already observed that \( f(u) \neq \text{D}F(u) \) for all \( F : H \to \mathbb{R} \), we intend to use here a variational technique in order to solve a problem which has no variational nature. We do this by combining the WED approach with a fixed-point argument. More precisely, we define an operator \( S : L^2(0, T; H) \to L^2(0, T; H) \) through the minimization of the WED-type functional by letting

\[
S : v \mapsto u = \arg \min_w I_{\varepsilon,v}(w),
\]

\[
I_{\varepsilon,v}(w) = \int_0^T \exp(-t/\varepsilon) \left( \frac{\varepsilon}{2} |w'|^2 + \phi(w) - (f(v), w) \right) dt,
\]
and we check that $S$ has a fixed point which satisfies an elliptic-in-time regularization of equation (1.3) (cf. Theorem 4 and Theorem 3):

$$-\varepsilon u'' + u' + \partial \phi (u) \ni f(u) \quad \text{a.e. in } (0, T),$$

$$u(0) = u_0, \quad u'(T) = 0.$$  (1.5)

Then, by passing to the limit $\varepsilon \to 0$ we recover a solution to equation (1.1).

In Section 2 we enlist the assumptions which are assumed throughout the paper and we state our main results. We first prove our results in the simpler case of a convex potential $\phi$ in Section 3. In Section 4 we prove the results in full generality, namely we deal with the case of nonconvex energies $\phi$. Section 5 illustrates how to generalize our results to the case of less regular initial data. Finally, we present applications of our abstract theory to reaction-diffusion systems in Section 6.

## 2 Assumptions and main results

We enlist here the assumptions which are considered throughout the paper. Let $H$ be a real Hilbert space with scalar product $(\cdot, \cdot)$ and norm $| \cdot |$. Let the function $f : H \to H$ be continuous and sublinear, namely

$$|f(u)| \leq C_1 (1 + |u|)$$  (2.1)

for all $u \in H$ and some positive constant $C_1$. We assume that the functional $\phi$ can be decomposed as $\phi = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 : H \to [0, +\infty]$ are bounded from below, proper, l.s.c., and convex functionals. Furthermore, we assume $D(\varphi_1) \subset D(\varphi_2), D(\partial \varphi_1) \subset D(\partial \varphi_2)$, and that there exist constants $k_1, k_2 \in [0, 1)$, $C_2 > 0$, and a non-decreasing function $\ell : \mathbb{R} \to [0, +\infty)$ such that

$$\varphi_2(u) \leq k_1 \varphi_1(u) + C_2$$  (2.2)

for all $u \in D(\varphi_1)$ and

$$|\xi|^2 \leq k_2 (\partial \varphi_1(u))^\circ \|\xi\|^2 + \ell(|u|)(\varphi_1(u) + 1)$$  (2.3)

for all $u \in D(\partial \varphi_2)$ and $\xi \in \partial \varphi_2(u)$. Here $\partial \varphi_1, \partial \varphi_2$ denote the subdifferentials of $\varphi_1$ and $\varphi_2$ respectively and $(\partial \varphi_1(u))^\circ$ the element of $\partial \varphi_1(u)$ with minimal norm. Moreover, let $(X, | \cdot |_X)$ be a Banach space compactly embedded in $H$ such that

$$\varphi_1(u) \geq c_X |u|_X^2 - C_3$$  (2.4)

for all $u \in D(\phi)$ and some strictly positive constants $c_X$ and $C_3$.

**Remark 1** We remark that these assumptions are standard and general enough to include a variety of different problems (cf., e.g., [Ak-St, Ot, Ot2] and Section 6).

We are interested in problem

$$u' + \partial \varphi_1(u) - \partial \varphi_2(u) \ni f(u) \quad \text{a.e. in } (0, T),$$

$$u(0) = u_0.$$  (2.5-6)

Strong solutions to problem (2.5)-(2.6) are defined as follows.
**Definition 2 (Strong solution)** Let the above assumptions be satisfied and \( u_0 \in D(\varphi_1) \). Then, \( u \in H^1(0,T;H) \) is a strong solution to (2.5)-(2.6) if \( u(t) \in D(\partial \varphi_1) \) for a.e. \( t \in (0,T) \) and it satisfies

\[
\begin{align*}
    u' + \xi &= f(u) + \eta \quad \text{a.e. in } (0,T), \\
    \xi &\in \partial \varphi_1(u) \quad \text{a.e. in } (0,T), \\
    \eta &\in \partial \varphi_2(u) \quad \text{a.e. in } (0,T), \\
    u(0) &= u_0,
\end{align*}
\]

for given \( \xi, \eta \in L^2(0,T,H) \).

The main result of this work is the following theorem whose proof is detailed in Section 4.

**Theorem 3 (Elliptic regularization)** Let \( u_0 \in D(\partial \varphi_1) \). Then, the regularized problem

\[
\begin{align*}
    -\varepsilon u''_{\varepsilon} + u'_{\varepsilon} + \xi_{\varepsilon} &= f(u_{\varepsilon}) + \eta_{\varepsilon} \quad \text{a.e. in } (0,T), \\
    \xi_{\varepsilon} &\in \partial \varphi_1(u_{\varepsilon}) \quad \text{a.e. in } (0,T), \\
    \eta_{\varepsilon} &\in \partial \varphi_2(u_{\varepsilon}) \quad \text{a.e. in } (0,T), \\
    u_{\varepsilon}(0) &= u_0, \quad u_{\varepsilon}'(T) = 0
\end{align*}
\]

admits (at least) a solution \( u_{\varepsilon} \in H^2(0,T;H) \) for \( \varepsilon > 0 \) small enough. Furthermore, there exist a sequence \( \varepsilon_n \to 0 \) such that \( u_{\varepsilon_n} \to u \) weakly in \( H^1(0,T;H) \) and strongly in \( C([0,T];H) \) and \( u \) is a strong solution of (2.5)-(2.6).

Theorem 3 extends the former analysis from [Ak-St] and from [Mi-St], as our theory applies to the nonpotential perturbations. And it is worth mentioning that solutions to both problem (2.5)-(2.6) and problem (2.11)-(2.14) might be nonunique. Even in the case \( f = 0 \), we provide an alternative proof of the results in [Ak-St]. Note however that assumption (2.4) is not required in [Mi-St] and it is replaced by a weaker one in [Ak-St], namely the exponent 2 in (2.4) is replaced by \( p \geq 1 \). On the other hand, (2.4) is necessary in order to apply the Gronwall Lemma 7 which is one of the main technical tool of this work. Additionally, we can prove similar results also in the case \( u_0 \in D(\varphi_1) \) (cf. Section 5). More precisely, we approximate \( u_0 \in D(\varphi_1) \) by a sequence \( u_{0\varepsilon} \in D(\partial \varphi_1) \), we solve equation (2.11)-(2.13) coupled with \( u_{\varepsilon}(0) = u_{0\varepsilon} \) and \( u_{\varepsilon}'(T) = 0 \) for all \( \varepsilon \) small enough, and we pass to the limit \( \varepsilon \to 0 \).

### 3 Convex energy

Before moving to the proof of the main result in full generality, let us present the argument in the simpler case of convex energy. In particular, throughout this section we assume \( \phi \) to be convex, namely \( \varphi_2 = 0 \) (i.e., \( \phi = \varphi_1 \)). Problem (2.5)-(2.6) then reads

\[
\begin{align*}
    u' + \xi &= f(u) \quad \text{a.e. in } (0,T), \\
    \xi &\in \partial \phi(u) \quad \text{a.e. in } (0,T), \\
    u(0) &= u_0.
\end{align*}
\]
As we mentioned in the introduction \( f(u) \neq \partial F(u) \) for any \( F : H \to \mathbb{R} \). As a consequence, system \( (3.1) - (3.3) \), as well as its elliptic-in-time regularization, cannot be seen as the Euler-Lagrange system corresponding to a minimization problem. The strategy to overcome this obstruction is to combine the WED approach with a fixed-point procedure.

Let us consider the map \( S : L^2(0,T;H) \to L^2(0,T;H) \), given by \( S : v \mapsto u \) where \( u \) is the global minimizer of the functional \( I_{\varepsilon,v} \) defined by

\[
I_{\varepsilon,v}(u) = \int_0^T \exp(-t/\varepsilon) \left( \frac{\varepsilon}{2} |u'|^2 + \phi(u) - (f(v), u) \right) dt
\]

over the convex set \( K(u_0) := \{ u \in H^1(0,T;H) : u(0) = u_0 \} \). The main result of this section is the following.

**Theorem 4 (Convex case)** Let assumption of Theorem 3 be satisfied with \( \phi = \varphi_1 \). Then, for all \( \varepsilon \) small enough, the map \( S \) has at least one fixed point \( u_\varepsilon = S(u_\varepsilon) \). This satisfies the regularized system

\[
-\varepsilon u''_\varepsilon + u'_\varepsilon + \xi - f(u_\varepsilon) = 0 \quad \text{a.e. in } (0,T),
\]

\[
\xi_\varepsilon \in \partial \phi(u_\varepsilon) \quad \text{a.e. in } (0,T),
\]

\[
u'_\varepsilon(T) = 0,
\]

\[
u_\varepsilon(0) = u_0,
\]

along with the solution \( \xi_\varepsilon \in L^2(0,T;H) \). Moreover, the solution(s) to the regularized system \( (3.5) - (3.8) \) converge(s) (up to subsequences) to (one of) the solution(s) to the gradient flow problem \( (3.1) - (3.3) \) weakly in \( H^1(0,T;H) \) and strongly in \( C([0,T];H) \) for \( \varepsilon \to 0 \).

### 3.1 Preliminary results

In order to prove Theorem 4 we collect some preliminary results.

For all \( v \in L^2(0,T;H) \) and for \( \varepsilon \) small enough it is proved in [Mi-St] that there exists a unique minimizer \( u \in K(u_0) \) for the functional \( I_{\varepsilon,v} \) defined by (3.4). In particular, existence is trivial for every \( \varepsilon \), while the uniqueness follows from uniform convexity for \( \varepsilon \) small enough, independently of \( v \). Moreover, \( u \) is one of the possibly many solutions to the regularized problem:

\[
-\varepsilon u'' + u' + \xi - f(u) = 0 \quad \text{a.e. in } (0,T),
\]

\[
\xi \in \partial \phi(u) \quad \text{a.e. in } (0,T),
\]

\[
\varepsilon u'(T) = 0,
\]

\[
u(0) = u_0.
\]

Using the maximal regularity estimate, derived in [Mi-St, Lemma 4.1], we have

\[
\varepsilon^2 \|u''\|_{L^2(0,T;H)}^2 + \|u'\|_{L^2(0,T;H)}^2 + \|\xi\|_{L^2(0,T;H)}^2 + \phi(u(T)) \leq C + \|f(v)\|_{L^2(0,T;H)}^2
\]

and hence

\[
\|u\|_{L^2(0,T;H)}^2 \leq C + C \|f(v)\|_{L^2(0,T;H)}^2,
\]

where \( C \) denotes a positive constant depending on \( |(\partial \varphi_1(u_\varepsilon))'| \). This ensures that the map \( S \) is well-defined.
3.2 Proof of Theorem 4

The proof of the first part of the theorem follows from an application of the Schaefer fixed-point Theorem 8 in the Appendix. More precisely, we check that the map $S$ satisfies the assumptions of Theorem 8 and hence we prove existence of a fixed point for $S$. In what follows the symbol $C$ denotes a positive constant possibly depending on $T, u_0, \phi$, but not on $\varepsilon$ which may vary even within the same line.

The map $S : L^2(0, T; H) \to L^2(0, T; H)$ is continuous. Let $v_1, v_2 \in L^2(0, T; H)$ be given and denote by $u_1$ and $u_2$ the unique minimizers of $I_{\varepsilon,v_1}$ and $I_{\varepsilon,v_2}$ respectively. Then, by computing the difference between the two corresponding regularized equations, choosing $w = u_1 - u_2$ as test function and integrating over $[0, t]$ for $t \in (0, T)$, we get

$$-\varepsilon(w'(t), w(t)) + \varepsilon \int_0^t |w'|^2 + \int_0^t (\xi_1 - \xi_2, w) + \frac{1}{2} |w(t)|^2$$

$$\leq \int_0^t (f(v_1) - f(v_2), w).$$

As $\phi$ is convex, the term $\int_0^t (\xi_1 - \xi_2, w)$ is nonnegative. Hence,

$$-\varepsilon(w'(t), w(t)) + \varepsilon \int_0^t |w'|^2 + \frac{1}{2} |w(t)|^2 \leq \int_0^t (f(v_1) - f(v_2), w)$$

$$\leq \frac{1}{2} \int_0^t |f(v_1) - f(v_2)|^2 + \frac{1}{2} \int_0^t |w|^2. \quad (3.15)$$

By applying the Gronwall Lemma 7 from Appendix, we have

$$\frac{1}{2} |w(t)|^2 \leq \varepsilon(w'(t), w(t)) + C \int_0^t |f(v_1) - f(v_2)|^2$$

$$+ C \int_0^t \varepsilon(w', w) + Ct \int_0^t |f(v_1) - f(v_2)|^2$$

$$\leq C \int_0^t |f(v_1) - f(v_2)|^2 + \varepsilon(w'(t), w(t)) + \varepsilon C \int_0^t (w', w). \quad (3.16)$$

Substituting the latter into relation (3.15), choosing $t = T$, and recalling that $\varepsilon w'(T) = 0$, we get

$$\varepsilon \int_0^T |w'|^2 + \frac{1}{2} |w(T)|^2 \leq C \int_0^T |f(v_1) - f(v_2)|^2$$

$$+ \varepsilon C \int_0^T |(w'(t), w(t))| + \varepsilon C \int_0^T \int_0^t |(w'(t), w(t))|$$

$$\leq C \int_0^T |f(v_1) - f(v_2)|^2 + \frac{\varepsilon}{2} \int_0^T |w'|^2 + \varepsilon C \int_0^T |w|^2. \quad (3.17)$$

Integrating (3.15) over $[0, T]$ and adding it to (3.17), we obtain

$$- \varepsilon \int_0^T (w', w) + \varepsilon \int_0^T \int_0^t |w'|^2 + \frac{1}{2} \int_0^T |w|^2 + \varepsilon \int_0^T |w'|^2 + \frac{|w(T)|^2}{2}$$

$$\leq C \int_0^T |f(v_1) - f(v_2)|^2 + \frac{\varepsilon}{2} \int_0^T |w'|^2 + C \int_0^T |w|^2.$$
By using once again estimate (3.16), we conclude that

\[
\frac{1 - \varepsilon}{2} |w(T)|^2 + \frac{1}{2} \int_0^T |w|^2 + \frac{\varepsilon}{2} \int_0^T |w'|^2 \\
\leq C \int_0^T |f(v_1) - f(v_2)|^2 + C \int_0^T |w|^2 \\
\leq C \int_0^T |f(v_1) - f(v_2)|^2 + \varepsilon C \int_0^T (w'(t), w(t)) \\
+ \varepsilon C \int_0^T \int_0^t (w'(t), w(t)) \\
\leq C \int_0^T |f(v_1) - f(v_2)|^2 + \frac{\varepsilon}{2} |w(T)|^2 \\
+ \varepsilon C \int_0^T |w|^2.
\]

Thus, for \( \varepsilon \) small enough, namely \( \varepsilon \leq \min\{(1 + C)^{-1}, (2C)^{-1}\} \), we have

\[
\frac{\varepsilon}{2} \int_0^T |w'|^2 + \frac{1}{4} \int_0^T |w|^2 \leq C \int_0^T |f(v_1) - f(v_2)|^2.
\]

Since \( f \) is continuous and linearly bounded, if \( v_1 - v_2 \to 0 \) in \( L^2(0, T; H) \) then, \( f(v_1) - f(v_2) \to 0 \) in \( L^2(0, T; H) \) and \( w \to 0 \) in \( H^1(0, T; H) \). This proves the continuity of \( S : L^2(0, T; H) \to H^1(0, T; H) \) and hence of \( S : L^2(0, T; H) \to L^2(0, T; H) \).

**Compactness.** We now prove that the map \( S \) is compact. Using the maximal regularity estimate (3.13) and the linear growth of \( f \), we get

\[
\varepsilon^2 \|u''\|^2_{L^2(0,T;H)} + \|u'\|^2_{L^2(0,T;H)} + \|\xi\|^2_{L^2(0,T;H)} + \phi(u(T)) \leq C + \|f(v)\|^2_{L^2(0,T;H)} \\
\leq C + C \|v\|^2_{L^2(0,T;H)}
\]

Take now \( v \in B \), where \( B \subset L^2(0, T; H) \) is a bounded set. Then,

\[
\|u'\|^2_{L^2(0,T;H)} \leq C = C(B)
\]

and, recalling that \( u(0) = u_0 \), we have that \( \|u\|^2_{L^2(0,T;H)} \leq C \). Testing equation (3.9), with \( u' \) and integrating first on \( [0, t] \) and then on \( [0, T] \), we get

\[
-\frac{\varepsilon}{2} \int_0^T |u|^2 + \frac{\varepsilon T}{2} |u(0)|^2 + \int_0^T \int_0^t |u'|^2 - T \phi(u_0) + \int_0^T \phi(u) \leq T \int_0^T |f(v)||u'|
\]

\[
\leq C + C \|v\|^2_{L^2(0,T;H)}.
\]

Thus, by using assumption (2.4),

\[
c \int_0^T |u|^2_X \leq C + \int_0^T \phi(u) \leq C.
\]

As \( L^2(0, T; X) \cap H^1(0, T; H) \) is compactly embedded in \( L^2(0, T; H) \), by the Aubin-Lions Lemma [Sl Thm. 3], the map \( S \) is compact.
Boundedness of \( A := \{ v \in L^2(0,T;H) : v = \alpha S(v) \text{ for } \alpha \in [0,1] \} \). In order to apply the Schaefer fixed-point Theorem, we are left to prove that \( A \) is bounded. First note that \( A = \{ 0 \} \cup \{ v \in L^2(0,T;H) : v/\alpha = S(v) \text{ for } \alpha \in (0,1] \} \). Thus, \( A \) is bounded if and only if \( \{ v \in L^2(0,T;H) : v/\alpha = S(v) \text{ for } \alpha \in (0,1] \} \) is bounded. We now prove that \( \tilde{A} := \{ u \in L^2(0,T;H) : u = S(\alpha u) \text{ for } \alpha \in (0,1] \} \) is bounded. This yields \( A \) bounded.

Let \( u \in \tilde{A} \). Then, there exists \( \alpha \in (0,1] \) and \( \xi \in L^2(0,T;H) \) such that \( u \) solves

\[
-\varepsilon u'' + u' + \xi - f(\alpha u) = 0 \quad \text{a.e. in } (0,T),
\]

\[
\xi \in \partial \phi(u) \quad \text{a.e. in } (0,T),
\]

\[
u'(T) = 0,
\]

\[
u(0) = u_0.
\]

Testing this equation with \( u' \) and integrating over \( (0,t) \), we get

\[
-\varepsilon \int_0^t (u'', u') + \int_0^t |u'|^2 + \phi(u(t)) - \phi(u_0) = \int_0^t (f(\alpha u), u').
\]

Hence, recalling assumptions (2.4) and (2.1), one has

\[
-\frac{\varepsilon}{2} |u'(t)|^2 + \frac{\varepsilon}{2} |u'(0)|^2 + \frac{1}{2} \int_0^t |u'|^2 + \frac{c_x}{2} |u(t)|^2_X - \phi(u_0) \leq C + C\alpha^2 \int_0^t |u|^2
\]

which, recalling that \( \alpha \leq 1 \) and \( |\cdot| \leq C|\cdot|_X \), yields

\[
\frac{1}{2} \int_0^t |u''|^2 + \frac{1}{2} |u(t)|^2 \leq C + C \int_0^t |u|^2 + \frac{\varepsilon C}{2} |u'(t)|^2.
\]

Applying the Gronwall Lemma from Appendix, we get

\[
|u(t)|^2 \leq C + \frac{\varepsilon C}{2} |u'(t)|^2 + C \int_0^t \left( C + \frac{\varepsilon C}{2} |u'(s)|^2 \right) \exp(C(t-s))\,ds
\]

\[
\leq C + \frac{\varepsilon C}{2} |u'(t)|^2 + \varepsilon C \exp(TC) \int_0^t |u'|^2
\]

\[
\leq C + \frac{\varepsilon C}{2} |u'(t)|^2 + \varepsilon C \int_0^t |u'|^2.
\]

Integrating (3.20) over \([0,T]\) and adding (3.20) to it along with the choice \( t = T \), one gets

\[
\frac{1}{2} \int_0^T \int_0^t |u''|^2 + \frac{1}{2} \int_0^T |u|^2 + \frac{1}{2} \int_0^T |u'|^2 + \frac{1}{2} |u(T)|^2
\]

\[
\leq C + C \int_0^T \int_0^t |u|^2 + \varepsilon C \int_0^T |u'|^2 + C \int_0^T |u|^2
\]

and hence, thanks to estimate (3.21),

\[
\frac{1}{2} \int_0^T |u|^2 + \left( \frac{1}{2} - C\varepsilon \right) \int_0^T |u'|^2 \leq C + C(1+T) \int_0^T |u|^2
\]

\[
\leq C + C\varepsilon(1+T) \int_0^T |u'|^2.
\]
For all $\varepsilon$ small enough, we have that
\[ \|u\|_{H^1(0,T;H)} \leq C, \tag{3.22} \]
where $C$ does not depend on $\varepsilon$ nor $\alpha$.

As a consequence of Theorem 8, $S$ has a fixed point $u \in L^2(0,T;H)$, $u = S(u)$. This solves the regularized equation (3.5) and
\[ u = \arg\min_{w \in K(u_0)} \int_0^T \exp\left(-t/\varepsilon\right) \left( \frac{\varepsilon}{2} |w'|^2 + \phi(w) - (f(u),w) \right). \]

The causal limit. The crucial issue for the WED theory is the so-called causal limit, namely the convergence of the WED minimizers as $\varepsilon \to 0$.

Let $u_\varepsilon$ be (one of) the solution(s) to the Euler-Lagrange system. By testing regularized equation (3.5) with $u'_\varepsilon$ and repeating the argument presented above with $\alpha = 1$ (cf. (3.18) and (3.22)), we obtain that
\[ \|u_\varepsilon\|_{H^1(0,T;H)} \leq C, \tag{3.23} \]
where $C$ does not depend on $\varepsilon$. Thanks to the maximal regularity estimate (3.13) and of the sublinear growth of $f$, we have that
\[ \varepsilon^2 \|u''_\varepsilon\|_{L^2(0,T;H)}^2 + \|u'_\varepsilon\|_{L^2(0,T;H)}^2 + \|\xi_\varepsilon\|_{L^2(0,T;H)}^2 + \phi(u_\varepsilon(T)) \leq C + \|f(u_\varepsilon)\|_{L^2(0,T;H)}^2 \leq C. \tag{3.24} \]

Furthermore, integrating (3.19) over $[0,T]$ (with $\alpha = 1$), we deduce
\[ \|u_\varepsilon\|_{L^2(0,T;X)} \leq C. \]

As a consequence of these uniform estimates and of the compact embedding $H^1(0,T;H) \cap L^2(0,T;X) \hookrightarrow L^2(0,T;H)$ there exist (not relabeled) subsequences $u_\varepsilon$ and $\xi_\varepsilon$ such that
\[ u_\varepsilon \to u \quad \text{weakly in } H^1(0,T;H) \text{ and in } L^2(0,T;X), \tag{3.25} \]
\[ u_\varepsilon \to u \quad \text{strongly in } C([0,T];H) \text{ and in } L^2(0,T;H), \tag{3.26} \]
\[ \xi_\varepsilon \to \xi \quad \text{weakly in } L^2(0,T;H). \tag{3.27} \]

The demiclosedness of maximal monotone operators \cite{Br3} entails that $\xi \in \partial\phi(u)$ a.e. in $(0,T)$. The trajectory $u$ is hence the strong solution to (3.1)-(3.3). This concludes the proof of Theorem 4.

Explicit convergence rate. Under additional assumptions on $f$, we can obtain an estimate for the convergence rate of $\|u_\varepsilon - u\|_{C([0,T];H)}$. In particular, let $f_i : H \to H$, $i = 1,2$, be such that
\[ f = f_1 + f_2, \]
\[ f_1 \text{ is Lipschitz continuous and} \]
\[ -f_2 \text{ is monotone.} \tag{3.28} \]
By testing the difference between the regularized equation and the gradient flow equation with \( w = u - u_\varepsilon \) and using the convexity of \( \phi \), we get (cf. \[Mi-St\])

\[
\varepsilon \int_0^t |w'|^2 + \frac{1}{4} |w(t)|^2 \leq |u_0 - u_{0\varepsilon}|^2 + \varepsilon \int_0^t |u'|^2 + \varepsilon^2 |u'_\varepsilon(t)|^2 \\
+ \frac{\varepsilon^2}{2} |u'_\varepsilon(0)|^2 + \int_0^t (f(u) - f(u_\varepsilon), u - u_\varepsilon) \\
\leq C\varepsilon + \int_0^t (f(u) - f(u_\varepsilon), u - u_\varepsilon) \\
\leq C\varepsilon + L \int_0^t |w|^2, 
\]

where \( L \) is the Lipschitz constant of \( f_1 \). Applying the Gronwall Lemma, we obtain

\[
\varepsilon \int_0^t |w'|^2 + |w(t)|^2 \leq C \varepsilon 
\]

which entails

\[
\|u_\varepsilon - u\|_{C([0,T];H)} \leq C \varepsilon^{1/2}.
\]

### 4 Nonconvex energies

We now come to the proof of Theorem 3, namely we consider \( \phi = \varphi_1 - \varphi_2 \) nonconvex. This forces us to introduce a further approximation which will be then removed before taking the causal limit \( \varepsilon \to 0 \). In particular, we regularize the problem for all \( \lambda > 0 \) by replacing \( \varphi_2 \) with its Moreau-Yosida regularization \[Br3\]:

\[
\varphi_2^\lambda(u) = \inf_{v \in H} \left( \frac{1}{2\lambda} |u - v|^2 + \varphi_2(v) \right) = \frac{1}{2\lambda} |u - J_\lambda u|^2 + \varphi_2(J_\lambda u) \quad \text{for all } u \in H,
\]

where \( J_\lambda u \) denotes the resolvent for \( \partial \varphi_2 \). It is well known that

\[
\varphi_2^\lambda \in C^{1,1}, \\
\varphi_2^\lambda(u) \leq \varphi_2(u) \quad \text{for all } u \in D(\varphi_2), \\
D\varphi_2^\lambda(u) = \partial \varphi_2(J_\lambda u) \quad \text{for all } u \in H, \\
|D\varphi_2^\lambda(u)| \leq |\eta| \quad \text{for all } [u, \eta] \in \partial \varphi_2.
\]

Here \( D\varphi_2^\lambda \) denotes the Fréchet derivative of \( \varphi_2^\lambda \). In particular, \( D\varphi_2^\lambda : H \to H \) is (single-valued and) Lipschitz continuous. Hence, \( g = f + D\varphi_2^\lambda \) satisfies assumption (2.1). Thus, Theorem 4 ensures the existence of (at least) a solution \( u_\varepsilon,\lambda \) to

\[
-\varepsilon u''_{\varepsilon,\lambda} + u'_{\varepsilon,\lambda} + \xi_{\varepsilon,\lambda} = f(u_{\varepsilon,\lambda}) + D\varphi_2^\lambda(u_{\varepsilon,\lambda}) \quad \text{a.e. in } (0, T), \\
\xi_{\varepsilon,\lambda} \in \partial \varphi_1(u_{\varepsilon,\lambda}) \quad \text{a.e. in } (0, T), \\
u_{\varepsilon,\lambda}(0) = u_0, \quad u'_{\varepsilon,\lambda}(T) = 0.
\]
We now derive estimates on \( u_{\varepsilon,\lambda} \) which are uniform with respect to \( \lambda \) (as well as \( \varepsilon \)) in order to pass to the limit. Henceforth the symbol \( C \) will be independent of \( \lambda \) as well. Testing (4.15) by \( u'_{\varepsilon,\lambda} \) and integrating over \([0, t]\), we obtain
\[
-\varepsilon \int_0^t \left( u''_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda} \right) + \int_0^t \left| u'_{\varepsilon,\lambda} \right|^2 + \varphi_1(u_{\varepsilon,\lambda}(t)) - \varphi_2^\lambda(u_{\varepsilon,\lambda}(t)) = \int_0^t \left( f(u_{\varepsilon,\lambda}), u'_{\varepsilon,\lambda} \right) + \varphi_1(u_0) - \varphi_2^\lambda(u_0).
\]
Hence, using assumption (2.2) and inequality (4.2), we get
\[
-\varepsilon \int_0^t \left( u''_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda} \right) + \int_0^t \left| u'_{\varepsilon,\lambda} \right|^2 + (1-k_1)\varphi_1(u_{\varepsilon,\lambda}(t)) \leq \int_0^t \left( f(u_{\varepsilon,\lambda}), u'_{\varepsilon,\lambda} \right) + C. \tag{4.8}
\]
Arguing as in Section 3.2, we obtain
\[
\| u_{\varepsilon,\lambda} \|_{H^1(0, T; H)} \leq C. \tag{4.9}
\]
Integrating relation (4.8) over \([0, T]\) and using the above estimates, we additionally find
\[
\int_0^T \varphi_1(u_{\varepsilon,\lambda}) \leq C. \tag{4.10}
\]
Thus, thanks to assumption (2.4), we deduce
\[
\| u_{\varepsilon,\lambda} \|_{L^2(0, T; X)} \leq C,
\]
\[
\| u_{\varepsilon,\lambda} \|_{C([0, T]; H)} \leq C.
\]
Applying, the maximal regularity estimate (3.13), we have
\[
\varepsilon^2 \left\| u''_{\varepsilon,\lambda} \right\|_{L^2(0, T; H)} + \left\| u'_{\varepsilon,\lambda} \right\|_{L^2(0, T; H)} + \left\| \xi_{\varepsilon,\lambda} \right\|_{L^2(0, T; H)} + \varphi_1(u_{\varepsilon,\lambda}(T)) \leq \left\| f(u_{\varepsilon,\lambda}) + D\varphi_2^\lambda(u_{\varepsilon,\lambda}) \right\|_{L^2(0, T; H)}.
\]
Thanks to (4.4) and assumption (2.3), we estimate
\[
| f(u_{\varepsilon,\lambda}) + D\varphi_2^\lambda(u_{\varepsilon,\lambda}) |^2 \leq (1+\delta)k_2|\xi_{\varepsilon,\lambda}|^2 + C_\delta |f(u_{\varepsilon,\lambda})|^2
\]
for every \( \delta > 0 \) and some \( C_\delta \). Thus, as \( k_2 < 1 \), choosing \( \delta \) sufficiently small,
\[
\varepsilon^2 \left\| u''_{\varepsilon,\lambda} \right\|_{L^2(0, T; H)} + \left\| u'_{\varepsilon,\lambda} \right\|_{L^2(0, T; H)} + \left\| \xi_{\varepsilon,\lambda} \right\|_{L^2(0, T; H)} + \varphi_1(u_{\varepsilon,\lambda}(T)) \leq C, \tag{4.11}
\]
\[
\varepsilon \left\| u'_{\varepsilon,\lambda} \right\|_{C([0, T]; H)} \leq C. \tag{4.12}
\]
As a consequence of assumptions (2.2), (2.3) and of the above estimates, we get
\[
\int_0^T \varphi_2(u_{\varepsilon,\lambda}) + \int_0^T |D\varphi_2^\lambda(u_{\varepsilon,\lambda})|^2 \leq C.
\]
We deal first with the passage to the limit for \( \lambda \to 0 \) for \( \varepsilon \) fixed. Owing to the obtained uniform estimates, up to some not relabeled subsequence, we have
\[
\begin{align*}
 u_{\varepsilon,\lambda} &\to u_\varepsilon \quad \text{weakly in } H^2(0, T; H) \text{ and in } L^2(0, T; X), \\
 \xi_{\varepsilon,\lambda} &\to \xi_\varepsilon \quad \text{weakly in } L^2(0, T; H), \\
 D\varphi_2^\lambda(u_{\varepsilon,\lambda}) &\to \eta_\varepsilon \quad \text{weakly in } L^2(0, T; H), \\
 u_{\varepsilon,\lambda} &\to u_\varepsilon \quad \text{strongly in } C([0, T]; H).
\end{align*}
\]
Using continuity of $f$, we obtain

$$f(u_{\varepsilon, \lambda}) \to f(u_\varepsilon) \quad \text{strongly in } L^2(0, T; H).$$

As a consequence of the demiclosedness of maximal monotone operators we conclude that $\xi_\varepsilon \in \partial \varphi_1(u_\varepsilon)$ almost everywhere. The inclusion $\eta_\varepsilon \in \partial \varphi_2(u_\varepsilon)$ follows then by the standard monotonicity argument [Ba, Sec. 1.2]. As $H = H^*$ is compactly embedded in $X^*$ we have the following convergence result (again for a not-relabeled subsequence)

$$u'_{\varepsilon, \lambda} \to u'_\varepsilon \quad \text{strongly in } C([0, T]; X^*).$$

In particular, $u'_\varepsilon(T) = 0$ and $u_\varepsilon$ solves equation (2.11). In addition, the sequence $u_\varepsilon$ satisfies the estimates (4.9)-(4.12) and

$$\int_0^T \varphi_2(u_\varepsilon) + \int_0^T |\eta_\varepsilon|^2 \leq C.$$

Let us now consider the causal limit $\varepsilon \to 0$. By taking (not relabeled) subsequences one has

$$u_\varepsilon \to u \quad \text{weakly in } H^1(0, T; H) \text{ and in } L^2(0, T; X),$$

$$u_\varepsilon \to u \quad \text{strongly in } C \left([0, T]; H\right),$$

$$\xi_\varepsilon \to \xi \quad \text{weakly in } L^2(0, T; H),$$

$$\eta_\varepsilon \to \eta \quad \text{weakly in } L^2(0, T; H),$$

$$f(u_\varepsilon) \to f(u) \quad \text{strongly in } L^2(0, T; H).$$

By the demiclosedness of maximal monotone operators, one concludes $\xi(t) \in \partial \varphi_1(u(t))$ and $\eta(t) \in \partial \varphi_2(u(t))$ for almost every $t \in (0, T)$. Hence, $u$ solves equation (2.7) and the assertion of Theorem 3 follows.

## 5 More general initial data

The results of Theorem 3 are also valid under weaker assumptions on the initial datum $u_0$. Aiming at clarity, we first illustrate the case of a convex energy. We use here the notation of Section 3 and follow closely the argument in [Mi-St, Secs. 2.5-6]. From [Br1, Br2] we define the interpolation set $D_{r,p}$ as

$$D_{r,p} = \left\{ u \in \overline{D(\partial \phi)} : \varepsilon \mapsto \varepsilon^{-r}|u - J_\varepsilon u| \in L^p \left(0, 1, \varepsilon^{-1}d\varepsilon\right) \right\},$$

where $J_\varepsilon = (id + \varepsilon \partial \phi)^{-1}$ is the standard resolvent operator. We recall the following properties from [Br1, Br2]

$$u_0 \in D_{r,p} \text{ iff } \exists \varepsilon \in [0, 1] \mapsto v(\varepsilon) : v \in W^{1,1}_{\text{loc}}(0, 1], \text{ continuous in } [0, 1], \quad v(0) = u_0,$$

$$v(\varepsilon) \in D(\partial \phi) \text{ a.e. and } \varepsilon^{1-r}(|\partial \phi(v(\varepsilon))| + |v'(\varepsilon)|) \in L^p(0, 1, \varepsilon^{-1}d\varepsilon).$$

$$D(\partial \phi) \subset D(\phi) = D_{1/2,2} \subset D_{1/2,\infty} \subset D_{r,\infty} \text{ for } r \in (0, 1/2).$$

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Let now \( u_0 \in D_{r,\infty} \) for \( r \in (0,1/2] \) and the sequence \( u_{0\varepsilon} \in D(\partial \varphi) \) be such that \( u_{0\varepsilon} \to u_0 \) strongly in \( H \) and

\[
\varepsilon^{-r}|u_{0\varepsilon} - u_0| + \varepsilon^{1-r}|(\partial \varphi(u_{0\varepsilon}))^0| \leq C.
\]

Arguing as in Section 3 it is possible to prove existence of a solution \( u_\varepsilon \) to the regularized problem

\[
-\varepsilon u''_\varepsilon + u'_\varepsilon + \xi_\varepsilon - f(u_\varepsilon) = 0 \quad \text{a.e. in } (0,T),
\]

\[
\xi_\varepsilon \in \partial \varphi(u_\varepsilon) \quad \text{a.e. in } (0,T),
\]

\[
u'_\varepsilon(T) = 0,
\]

\[
u_\varepsilon(0) = u_{0\varepsilon}.
\]

Estimate (3.23) reads in this case

\[
\varepsilon^2 \left\| u''_\varepsilon \right\|^2_{L^2(0,T;H)} + \| u'_\varepsilon \|^2_{L^2(0,T;H)} + \| \xi_\varepsilon \|^2_{L^2(0,T;H)} + \phi(u_\varepsilon(T)) \leq C + \| f(u_\varepsilon) \|^2 \leq C\varepsilon^{2r-1}.
\]

If \( u_0 \in D(\varphi) = D_{1/2,2} \) then \( r = 1/2 \) and the estimate suffices to pass to the limit. By assuming (3.27), we can argue as in (3.28)-(3.29) and obtain

\[
\frac{\varepsilon}{2} \int_0^t |u' - u'_\varepsilon|^2 + \frac{1}{4} |u(t) - u_\varepsilon(t)|^2 \leq C \left( |u_0 - u_{0\varepsilon}|^2 + \varepsilon \int_0^t |u'|^2 + \varepsilon^2 |u'_\varepsilon(t)|^2 + \frac{\varepsilon^2}{2} |u'_\varepsilon(0)|^2 \right)
\]

\[
\leq C\varepsilon^{2r}.
\]

Thus, uniform convergence holds for all \( r \in (0,1/2] \).

We deal now with the nonconvex energy case. Let \( \varphi_1, \varphi_2, f \) satisfy assumptions of Theorem 4, define

\[
D_{r,p}(\varphi_1) = \{ u \in \overline{D(\partial \varphi_1)} : \varepsilon \mapsto \varepsilon^{-r}|u - J_\varepsilon u| \in L^p(0,1,\varepsilon^{-1}d\varepsilon) \},
\]

\[
J_\varepsilon = (id + \varepsilon \partial \varphi_1)^{-1}.
\]

Assume \( u_0 \in D_{r,\infty}(\varphi_1) \) for \( r \in (0,1/2] \) so that there exists a sequence \( u_{0\varepsilon} \in D(\partial \varphi_1) \subset D(\partial \varphi_2) \) such that \( u_{0\varepsilon} \to u_0 \) strongly in \( H \),

\[
\varepsilon^{-r}|u_{0\varepsilon} - u_0| + \varepsilon^{1-r}|(\partial \varphi_1(u_{0\varepsilon}))^0| \leq C,
\]

and \( \varepsilon^{1-r}|\partial \varphi_2(u_{0\varepsilon})| \leq C \) (by using assumption (2.3)). This is enough to combine the uniform estimates of Section 4 with the approximation of the initial datum \( u_{0\varepsilon} \in D_{r,p}(\varphi_1) \) and extend the results of Theorem 4 to the case \( u_0 \in D_{r,\infty}(\varphi_1) \).

### 6 Applications

Our results yield a generalization to the nonpotential perturbation case of the theory in \( \text{Ak-St} \) and \( \text{Mi-St} \). Our analysis applies to most of the examples described in Section 6 of \( \text{Ak-St} \) and Section 7 of \( \text{Mi-St} \), e.g., quasilinear parabolic PDEs, the Allen-Cahn equation, the sublinear heat equation. Moreover, the occurrence of a nonpotential term allows us to apply the abstract theory to systems, in particular to reaction-diffusion and nonlinear diffusion systems.
6.1 Reaction-diffusion systems

Consider the system

\[ u_t = D_1 \Delta u + f_1(u, v) \quad \text{in} \quad \Omega \times (0, T), \quad (6.1) \]
\[ v_t = D_2 \Delta v + f_2(u, v) \quad \text{in} \quad \Omega \times (0, T), \quad (6.2) \]
\[ \partial_n u = \partial_n v = 0 \quad \text{on} \quad \partial \Omega \times (0, T), \quad (6.3) \]

where \( \Omega \) is a bounded subset of \( \mathbb{R}^d \) with sufficiently smooth boundary \( \partial \Omega \) and \( n \) denotes the unit outward normal vector on \( \partial \Omega \). Assume \( D_1, D_2 > 0 \) and

\[ f(u, v) = \begin{pmatrix} f_1(u, v) \\ f_2(u, v) \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

to be a linearly bounded continuous function. System (6.1)-(6.3) arises in a variety of different situations. The choice

\[ f_1(u, v) = A u \left( 1 - \frac{u}{K} \right) - \frac{B uv}{1 + Eu}, \quad (6.4) \]
\[ f_2(u, v) = C uv \frac{1 + Eu}{1 + Eu} - Dv, \quad (6.5) \]

where \( K > 0 \) and \( A, B, C, D, E \geq 0 \) models a diffusive prey-predator system (cf., e.g., [Mu, Du1, Du2]). In this context, \( u \) represents the number of preys and \( v \) the number of predators, \( K \) is the so-called capacity of the environment for the prey and \( D_1 > 0 \) and \( D_2 > 0 \) are the corresponding diffusion coefficients. Usually one is interested in solutions \((u, v)\) such that \( u \in [0, K] \) and \( v > 0 \). This implies that (6.4)-(6.5) can be equivalently rewritten as

\[ f_1(u, v) = A U \left( 1 - \frac{U}{K} \right) - \frac{B U V}{1 + EU}, \quad (6.6) \]
\[ f_2(u, v) = C U V \frac{1 + EU}{1 + EU} - DV, \quad (6.7) \]

where

\[ U = (\min\{u, K\})^+, \quad V = (v)^+. \]

By choosing a different form of \( f \), the system relates to pattern formation in animal coating (cf. [Mu, Mu2]). The reaction term takes here the form

\[ f_1(u, v) = \alpha - u - h(u, v), \quad (6.8) \]
\[ f_2(u, v) = \gamma (\beta - v) - h(u, v), \quad (6.9) \]
\[ h(u, v) = \rho uv \frac{1 + u + \delta u^2}{1 + u + \delta u^2}, \quad (6.10) \]

where \( \alpha, \beta, \gamma, \delta, \rho \) are positive constants. As \( u \) and \( v \) represent concentrations and \( u, v > 0 \), we can conveniently rewrite (6.10) as

\[ h(u, v) = \frac{\rho uv}{1 + (u)^+ + \delta u^2}. \quad (6.11) \]
Yet another example of choice of $f$ of application interest is

\begin{align*}
    f_1(u, v) &= p - u g(v), \quad \text{(6.12)} \\
    f_2(u, v) &= k(u g(v) - v), \quad \text{(6.13)}
\end{align*}

which is related to combustion. Here $p$ and $k$ are positive constants and $g(v) = \exp(v/(1 + \delta v))$ with $\delta > 0$. The latter choices are known as Scott-Wang-Showalter model [SWS]. Here $u$ denotes the concentration of an intermediate chemical species and $v$ is the temperature.

Note that the reaction terms corresponding to any of the choices (6.6)-(6.7), (6.8)-(6.9) together with (6.11), or (6.12)-(6.13) are continuous and satisfy assumption (2.1). We are hence in the position of applying our abstract theory to all these systems.

At first we rewrite system (6.1)-(6.3) as

\begin{align*}
    \left( \begin{array}{c}
    u_t \\
    v_t 
    \end{array} \right) + \partial \phi \left( \begin{array}{c}
    u \\
    v 
    \end{array} \right) \ni \tilde{f} \left( \begin{array}{c}
    u \\
    v 
    \end{array} \right) \quad \text{in} \quad (0, T), \quad \text{(6.14)}
\end{align*}

where

\[
    \phi \left( \begin{array}{c}
    u \\
    v 
    \end{array} \right) = \begin{cases}
    \frac{1}{2} \int_{\Omega} D_1 |\nabla u|^2 + D_2 |\nabla v|^2 + |u|^2 + |v|^2 & \text{if } u \in D \text{ and } v \in D, \\
    +\infty & \text{else,}
    \end{cases}
\]

\[D = \{ u \in H^2(\Omega) : \partial_n u = 0 \text{ on } \partial \Omega \}, \quad H = L^2(\Omega), \quad X = H^1(\Omega),\]

\[\tilde{f} \left( \begin{array}{c}
    u \\
    v 
    \end{array} \right) = f \left( \begin{array}{c}
    u \\
    v 
    \end{array} \right) + \left( \begin{array}{c}
    u \\
    v 
    \end{array} \right) \cdot \]

It is straightforward to check that $\phi$ and $\tilde{f}$ satisfy the assumptions of Theorem 3. We hence have the following.

**Theorem 5** Let $u_0, v_0 \in D$. Then, for every $T > 0$ and for $\varepsilon = \varepsilon(T) > 0$ sufficiently small, the system

\begin{align*}
    -\varepsilon u_{tt} + u_t &= D_1 \Delta u + f_1(u, v) \quad \text{in} \ \Omega \times (0, T), \\
    -\varepsilon v_{tt} + v_t &= D_2 \Delta v + f_2(u, v) \quad \text{in} \ \Omega \times (0, T), \\
    \partial_n u &= \partial_n v = 0 \quad \text{on} \ \partial \Omega \times (0, T), \\
    u(0) &= u_0, \quad v(0) = v_0 \quad \text{in} \ \Omega, \\
    \varepsilon u'(T) &= 0, \varepsilon v'(T) = 0
\end{align*}

admits at least a solution $(u_\varepsilon, v_\varepsilon) \in H^2(0, T; (L^2(\Omega))^2) \cap L^2(0, T; (H^1(\Omega))^2)$. Moreover, $u_\varepsilon \to u$ and $v_\varepsilon \to v$ weakly in $H^1(0, T; L^2(\Omega))$ and strongly in $C([0, T]; L^2(\Omega))$ where $(u, v)$ is a solution to system (6.1)-(6.3).
6.2 Nonlinear diffusion

We can also apply our abstract results to systems of nonlinear reaction-diffusion equations of the following type

\begin{align*}
  u_t &= D_1 \Delta_p u + |u|^{m-2}u - |u|^{q-2}u + f_1(u, v) & \text{in } \Omega \times (0, T), \\
  v_t &= D_2 \Delta_p v + |v|^{m-2}v - |v|^{q-2}v + f_2(u, v) & \text{in } \Omega \times (0, T), \\
  \partial_n u &= \partial_n v = 0 & \text{on } \partial \Omega \times (0, T),
\end{align*}

(6.15) (6.16) (6.17)

where \( 1 < q < m < +\infty, 1 < p < +\infty, \) and \( \Delta_p \) is the so-called \( p \)-Laplacian given by

\[ \Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u). \]

In order to write system (6.15)-(6.17) to the abstract setting, we define \( H = L^2(\Omega), \) \( X = D(\varphi_1) = W^{1,p}(\Omega) \cap L^m(\Omega), D(\varphi_2) = L^q(\Omega), \)

\[ \varphi_1(u, v) = \begin{cases} 
  \int_\Omega \frac{D_1}{p}|\nabla u|^p + \frac{D_2}{p}|\nabla v|^p + \frac{1}{m}|u|^m + \frac{1}{m}|v|^m & \text{if } u \in D(\varphi_1) \text{ and } v \in D(\varphi_1), \\
  +\infty & \text{else},
\end{cases} \]

and

\[ \varphi_2(u, v) = \begin{cases} 
  \frac{1}{q} \int_\Omega |u|^q + |v|^q & \text{if } u \in D(\varphi_2) \text{ and } v \in D(\varphi_2), \\
  +\infty & \text{else}.
\end{cases} \]

Moreover, we assume

\[ f(u, v) = \left( \frac{f_1(u, v)}{f_2(u, v)} \right) : \mathbb{R}^2 \to \mathbb{R}^2 \]

to be linearly bounded and continuous. It can be easily checked that assumptions of Theorem 3 are satisfied (cf. Section 6.1 of [Ak-St]) and we hence conclude the following.

**Theorem 6** Let \( u_0, v_0 \in D(\varphi_1). \) Then, for every \( T > 0 \) and for \( \varepsilon = \varepsilon(T) > 0 \) sufficiently small, the system

\begin{align*}
  -\varepsilon u_{tt} + u_t &= D_1 \Delta_p u + |u|^{m-2}u - |u|^{q-2}u + f_1(u, v) & \text{in } \Omega \times (0, T), \\
  -\varepsilon v_{tt} + v_t &= D_2 \Delta_p v + |v|^{m-2}v - |v|^{q-2}v + f_2(u, v) & \text{in } \Omega \times (0, T), \\
  \partial_n u &= \partial_n v = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) &= u_0, \quad v(0) = v_0 & \text{in } \Omega, \\
  \varepsilon u'(T) &= 0, \quad \varepsilon v'(T) = 0
\end{align*}

admits at least a solution

\[ (u_\varepsilon, v_\varepsilon) \in H^2(0, T; (L^2(\Omega))^2) \cap L^p(0, T; (W^{1,p}(\Omega))^2) \cap L^m(0, T; (L^m(\Omega))^2). \]

Moreover, \( u_\varepsilon \to u \) and \( v_\varepsilon \to v \) weakly in \( H^1(0, T; L^2(\Omega)) \) and strongly in \( C([0, T]; L^2(\Omega)) \) where \( (u, v) \) is a solution to system (6.15)-(6.17).

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7 Appendix

We collect here two tools for the Reader’s convenience.

Lemma 7 (Gronwall lemma) Let \( \alpha, u \in L^1(0,T) \) and \( B > 0 \). Assume

\[
    u(t) \leq \alpha(t) + \int_0^t Bu(s)ds \quad \text{for a.e. } t \in (0,T). \tag{7.1}
\]

Then,

\[
    u(t) \leq \alpha(t) + \int_0^t B\alpha(s) \exp(B(t-s))ds. \tag{7.2}
\]

Proof. Define \( v(t) = \exp(-Bt) \int_0^t Bu(s)ds \). Then, \( v \in W^{1,1}(0,T) \), \( v(0) = 0 \) and

\[
    v'(t) = B \exp(-Bt) \left( u(t) - \int_0^t Bu(s)ds \right) \leq B \exp(-Bt)\alpha(t) \quad \text{for a.a. } t \in (0,T).
\]

Thus, by integrating over \((0,t)\) we get

\[
    \exp(-Bt) \int_0^t Bu(s)ds = v(t) \leq \int_0^t B \exp(-Bs)\alpha(s)ds
\]

yielding

\[
    \int_0^t Bu(s)ds \leq \int_0^t B \exp(B(t-s))\alpha(s)ds. \tag{7.3}
\]

By substituting \((7.3)\) into \((7.1)\) we get \((7.2)\). •

Theorem 8 (Schaefer fixed-point Theorem [Ev, Thm. 4, Ch. 9]) Let \( X \) be a Banach space, \( S : X \to X \) be continuous and compact, and

\[
    \bigcup_{\alpha \in [0,1]} \{ u \in X : u = \alpha S(u) \}
\]

be bounded. Then, \( S \) has a fixed point.

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