Open String in Non-Relativistic Background

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Abstract: This note is devoted to the study of the open string description of Wilson loops and quarks in non-relativistic QFT.

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1. Introduction and Summary

Wilson loops are non-local gauge invariant operators in gauge theory in which the theory can be formulated. Mathematically we define a Wilson loop as the trace in an arbitrary representation $R$ of the gauge group $G$ of the holonomy matrix associated with parallel transport along a closed curve $C$ in space-time. Further, as it is well known from the times of birth of AdS/CFT correspondence the Wilson loops in $N = 4$ SYM theory can be calculated in dual description using macroscopic strings. This prescription is based on a picture of the fundamental string ending on the boundary of AdS along the path $C$ specified by the Wilson loop operator.

Recently the AdS/CFT correspondence has been generalized to the description of some non-relativistic strongly coupled conformal systems. Non-relativistic conformal symmetry contains the scaling transformation

$$x'^i = \lambda x^i, \quad t' = \lambda^z t,$$

where $z$ is a dynamical exponent. In case of $z = 2$ this symmetry is enhanced to Schrödinger symmetry. It is remarkable that it is possible to find the gravity dual of these non-relativistic field theories. The asymptotic metric in this case reads

$$ds^2 = \frac{R^2}{r^2}(-\frac{dt^2}{r^{2(z-1)}} + 2dt d\xi + (dx^i)^2) + \frac{R^2}{r^2}dr^2 + ds_M^2,$$

where $R$ is characteristic radius of space-time, $\xi$ is a compact light-like coordinate and where $ds_M^2$ is the metric of an appropriate compact manifold which allows to be a solution to the supergravity equations of motion. Since $\xi$ is compact the associated

\footnote{For review and extensive list of references, see [4, 5].}

\footnote{For another solutions that should be dual to non-relativistic field theories, see [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 61].}
quantum number is interpreted as the particle number. As we said above \( z \) that appears in
the scaling relation (1.1) is a critical exponent. The usual AdS case corresponds to \( z = 1 \).
On the other hand non-relativistic Dp-brane backgrounds are characterized by dynamical
exponents \( z = 2 \) \([21, 26]\).

The aim of this paper is to study some properties of non-relativistic \( d + 1 \) dimensional
field theories using the classical string solutions in \( d + 3 \) dimensional dual gravity background
\(^3\). We consider open string that moves with constant velocity along \( \xi \) dimension and that
also extends from the boundary of the space-time into the bulk, reaches its turning point
and then it goes back to the boundary. According to the bulk/boundary correspondence
the string action evaluated on this solution provides potential between quark anti-quark
pair. On the other hand the fact that the string moves along \( \xi \) direction makes the analysis
more interesting. Explicitly, we argue that open string that it stretched from the boundary
into the bulk of the space-time is formal solution of the string equations of motion since
the action evaluated on it is imaginary for sufficiently large \( r \). The similar situation occurs
in the context of the thermal CFT/AdS \([54, 56, 57, 59]\). We find that even in the zero
temperature non-relativistic QFT the static object that is described by stretched string
moving with constant velocity along \( \xi \) direction loses an energy in the process when the
energy and particle number are transferred from the background plasma into the end point
of the string and then it flows along the string up infinity. When we consider \( \cap \) shaped
string configuration that moves with constant velocity along \( \xi \)-direction we find that there
is no tail along \( \xi \) direction with agreement with the similar analysis performed in \([59]\) in
case of AdS/CFT correspondence.

We hope that our observation considering motion of the string along \( \xi \) direction in
non-relativistic background demonstrates that the non-relativistic version of AdS/CFT
correspondence is very rich and in some way enigmatic so that it deserves further study.
For example, it would be certainly very interesting to extend this analysis to the case of
thermal non-relativistic background.

This paper is organized as follows. In next section (2) we study the static open string
configuration in non-relativistic background that describes Wilson loop in non-relativistic
QFT. In section (3) we study the curved string in given background that moves with
constant velocity along \( \xi \) direction. Finally in appendix we present the Hamiltonian analysis
of the bosonic string in non-relativistic background.

2. Open String Description of Wilson Line in Non-relativistic Field Theory

We begin with the open string description of the Wilson loop in non-relativistic field theory.

Let us consider Nambu-Goto form of the string action

\[
S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det \mathbf{a}},
\]

where
\[
\mathbf{a}_{\alpha\beta} = \partial_{\alpha} x^M \partial_{\beta} x^N g_{MN}.
\]

\(^3\)For previous works along this issue, see \([45, 53]\).
We parameterize the string world-sheet with coordinates $\sigma^\alpha, \alpha, \beta = 0, 1, \sigma^0 = \tau, \sigma^1 = \sigma$. Further $x^M = (t, r, \xi, x^1, \ldots, x^d)$ parameterize the embedding of the string into non-relativistic background

$$
ds^2 = R^2 \left( -\frac{dt^2}{r^{2z}} + 2\frac{dt d\xi}{r^2} + \frac{dx^i dx_i}{r^2} + \frac{dr^2}{r^2} \right), \quad (2.3)$$

where $R$ is the characterized scale of the background and where $\xi$ is a compact light-like direction. Finally $x^i, i = 1, \ldots, d$ together with $t$ parameterize the boundary of the space-time (2.3) mapped at $r = 0$.

Our goal is to find solutions of the string equation of motion that provide the dual description of Wilson lines in the boundary non-relativistic quantum field theory. We also presume general time dependence of $\xi$. We start with the equations of motion for $x^M$ that follow from (2.1)

$$
\partial_\alpha [g_{MN} \partial_\beta x^N (a^{-1})^{\beta \alpha} \sqrt{-\det a}] - \frac{1}{2} \partial_M g_{KL} \partial_\alpha x^K \partial_\beta x^L (a^{-1})^{\beta \alpha} \sqrt{-\det a} = 0 .
$$

(2.4)

In order to find the static solution it is convenient to fix the world-sheet diffeomorphism by imposing the condition

$$
t = x^0 = \tau, \quad r = \sigma . \quad (2.5)
$$

In order to describe Wilson line in dual field theory with general time dependence of $\xi$ we consider following ansatz

$$
\xi = \xi(\tau), \quad x = x(\sigma) \quad (2.6)
$$

so that (2.5) together with (2.6) imply

$$
a_{\sigma \sigma} = \frac{R^2}{r^2} + \frac{R^2}{r^2} (\partial_\sigma x)^2 , \quad a_{\tau \tau} = -\frac{R^2}{r^{2z}} + 2 \frac{R^2}{r^2} \partial_\tau \xi . \quad (2.7)
$$

Then it is easy to see that the equation of motion for $\xi$ takes the form

$$
\partial_\tau [g_{\xi \xi} (a^{-1})^{\xi \xi} \sqrt{-\det a}] = 0
$$

(2.8)

that is solved when $\partial_\tau \xi = v_\xi = \text{const}$. Note also that for $\partial_\tau \xi = v_\xi \xi$ the equation of motion for $t$ is trivially satisfied. Finally we consider the equation of motion for $x$

$$
\partial_\alpha [g_{xx} \partial_\beta x (a^{-1})^{\beta \alpha} \sqrt{-\det a}] = \partial_\sigma \left[ g_{xx} \partial_\sigma x \sqrt{-\frac{a_{\sigma \sigma}}{a_{\tau \tau}}} \right] = 0
$$

(2.9)

that implies the differential equation for $x$ in the form

$$
\partial_\sigma x = \pm \frac{C \sqrt{g_{\tau \tau}}}{\sqrt{g_{xx} (-g_{tt} - 2g_{t\xi} v_\xi) - C^2 g_{xx}}} = \pm \frac{C r^{z+1}}{R^2 \sqrt{1 - 2v_\xi r^{2z-2} - C^2 r^{2z+2}}} ,
$$

(2.10)
where $C$ is an integration constant. Let us now analyze this differential equation. Clearly it possesses the solution $x = \kappa, C = 0$ that corresponds to the string that is extended from the boundary at $r = 0$ into the bulk space-time and that moves with constant velocity $v_\xi$ along $\xi$ direction. The configuration provides the dual description of an infinite long Wilson line extended in time direction. However we will argue below that even if this ansatz is the solution of the string equation of motion it cannot describe any physical configuration since the action evaluated on this ansatz is imaginary. This is a consequence of the non-zero motion of the string along $\xi$ direction. We present the resolution of this puzzle in the next section.

The second solution of the equation (2.10) corresponds to $\cap$ shaped string that extends from the point $x = -L/2$ at the boundary $r = 0$ into the bulk of the Schrödinger space time until it reaches its turning point $r_{\text{max}}$ at $x = 0$ and then it approaches the boundary at $x = L/2$. The turning point $r_{\text{max}}$ is the point when $\partial_x x = \infty$ and from (2.10) we see that it occurs when the denominator vanishes

$$1 - 2v_\xi r_{\text{max}}^{2z-2} - \frac{C^2}{R^4} r_{\text{max}}^{2z+2} = 0.$$  

(2.11)

The distance $L$ between two end points of the string is determined by the integral

$$L = 2 \int_0^{r_{\text{max}}} d\sigma \partial_x x = \frac{2C}{R^2} \int_0^{r_{\text{max}}} dr \frac{r^{z+1}}{\sqrt{1 - 2v_\xi r^{2z-2} - \frac{C^2}{R^4} r^{2z+2}}}.$$  

(2.12)

In principle this formula allows to express $r_{\text{max}}$ as a function of $L$. Then we can evaluate the action on given solution and find the potential $V(L)$ between quark and anti-quark-like objects that correspond to the end points of the string. However in order to find finite potential $V(L)$ we have to subtract the contribution from two infinite extended strings, following the standard recipe \cite{2, 3}. Implementing this procedure we find

$$S = -\int d\tau V(L) = -\frac{2}{2\pi\alpha'} \int d\tau \left[ \int_0^{r_{\text{max}}} d\sigma \sqrt{-\det a} + \int_0^\infty \sqrt{-\det a_{st}} \right] =$$

$$= -\frac{R^2}{\pi\alpha'} \int d\tau \left[ \int_0^{r_{\text{max}}} d\sigma \sqrt{1 - 2r^{2(z-1)}v_\xi} \left( \frac{1}{\sqrt{1 - 2v_\xi r^{2z-2} - \frac{C^2}{R^4} r^{2z+2}}} - 1 \right) - \right.$$

$$\left. - \int_0^\infty dr \sqrt{1 - 2r^{2(z-1)}v_\xi} r^{z+1} \right],$$  

(2.13)

where $\sqrt{-\det a_{st}} = \frac{1}{r^{z+1}} \sqrt{1 - 2r^{2(z-1)}v_\xi}$ is determinant evaluated on the solution of the equation of motion corresponding to the straight string $x = \text{const}$. However from (2.13) we immediately see that the action becomes imaginary for

$$r^{2(z-1)} > r^{2(z-1)}_* = \frac{1}{2v_\xi}$$  

(2.14)
and hence solution corresponding to straight string is unphysical. Note that the similar behavior was observed in case of AdS/CFT correspondence [54, 56, 57, 59]. On the other hand using $r_*$ we can rewrite the equation (2.11) in the form

$$1 - \left(\frac{r_{\text{max}}}{r_*}\right)^{2(z-1)} - \frac{C^2}{R^4} r_{\text{max}}^{2z+2} = 0$$

(2.15)

that implies that for $r_{\text{max}} < r_*$ and hence $\cap$ shaped string is well defined since the action evaluated on this solution is real. However as we argued above in order to regularize the divergences coming from the region around $r = 0$ we have to find physical open string configuration that describes single quark. We return to this problem in next section.

Let us now consider the case when $v_\xi = 0$ so that the equation (2.11) has the solution

$$r_{\text{max}} = \frac{R^{2z+1}}{C^{z+1}}.$$

(2.16)

It is convenient to introduce the variable $y = \frac{1}{r}$ so that $y_{\text{min}}^{z+1} = \frac{C}{R^z}$. Then (2.12) can be explicitly evaluated

$$L = 2 \int_{y_{\text{min}}}^{\infty} \frac{dy}{y^2} \frac{y_{\text{min}}^{z+1}}{\sqrt{y^{2(z+1)} - y_{\text{min}}^{2(z+1)}}} =$$

$$= \frac{2}{y_{\text{min}}} \int_{1}^{\infty} \frac{dk}{k^2 \sqrt{k^{2(z+1)} - 1}} = \frac{2}{y_{\text{min}}} \sqrt{\pi} \Gamma \left( \frac{z+2}{2z+2} \right) \frac{1}{\Gamma \left( \frac{1}{2z+2} \right)}.$$

(2.17)

This result allows us to express $y_{\text{min}}$ as a function of $L$. Further from (3.22) we find the quark anti-quark potential $V(L)$

$$S = -\int d\tau V(L) =$$

$$= -\frac{R^2}{\pi \alpha'} \int d\tau \left[ \int_0^{r_{\text{max}}} dr \frac{1}{r^{z+1}} \left( \frac{1}{\sqrt{1 - \left(\frac{r}{r_{\text{max}}}\right)^{2(z+1)}}} - 1 \right) - \frac{1}{r_{\text{max}}^{2z}} \right] =$$

$$= \int d\tau \frac{R^2}{\sqrt{\pi} \alpha' z} \left( \frac{2\sqrt{\pi}}{L} \right)^z \left( \frac{\Gamma \left( \frac{z+2}{2z+2} \right)}{\Gamma \left( \frac{1}{2z+2} \right)} \right)^{z+1}.$$

(2.18)

where in the final step of calculation we used (2.17). In fact, this result reproduces the calculation of the Wilson loop in the Lifshitz space-time that was performed in [37].

3. Curved Moving String

In the previous section we argued that straight string moving along $\xi$ direction with velocity $v_\xi > 0$ is not well defined since in this case the string action becomes imaginary for
\( r > r_\ast = \left(\frac{1}{2 \nu_\xi}\right)^{\frac{1}{1+z-1}} \). The similar situation occurs in case of the open string in thermal \( AdS_5 \) background \[54, 56, 57, 59\]. With analogy with this case we show that it is possible to find the solution of the string equation of motion that is real and that describes an infinite long open string that is stretched from the boundary at \( r = 0 \) into the bulk of the space-time. To do this we propose following ansatz

\[
\begin{align*}
v(\sigma, \tau) = \sigma, & \quad t = \tau, \quad x = \text{const}, \quad \xi = \xi = \nu_\xi \tau + y(\sigma) \quad (3.1)
\end{align*}
\]

so that

\[
\begin{align*}
a_{\tau\tau} = g_{\tau\tau} + 2 g_{\xi\tau} \nu_\xi \quad \text{or} \quad a_{\tau\sigma} = a_{\sigma\tau} = g_{\xi\sigma} \partial_\sigma y, \quad a_{\sigma\sigma} = g_{rr} \quad (3.2)
\end{align*}
\]

Let us now solve the equation of motion for this ansatz. First of all the equation of motion for \( t \) implies following differential equation for \( y \)

\[
(\partial_\sigma y) = \pm K \frac{\sqrt{-g_{rr}(g_{tt} + 2 g_{\xi\tau} \nu_\xi)}}{\sqrt{g_{t\xi}^2 v_\xi^2 - K^2 g_{t\xi}^2}} = \pm K \frac{1}{\sqrt{g_{t\xi}^2 v_\xi^2 - K^2}}. 
\]

In the same way the equation of motion for \( \xi \) implies

\[
\frac{g_{t\xi}^2 \partial_\sigma y}{\sqrt{-\det a}} = K_\xi, 
\]

where \( K_\xi \) is a constant. Solving this equation for \( \partial_\sigma y \) we find

\[
\partial_\sigma y = \pm K_\xi \frac{\sqrt{-g_{rr}(g_{tt} + 2 g_{\xi\tau} \nu_\xi)}}{\sqrt{g_{t\xi}^2 - K^2 g_{t\xi}^2}}. 
\]

Comparing this equation with \( (3.3) \) we find the relation between the integration constant \( K_\xi \) and \( K \) in the form \( K = K_\xi \nu_\xi \).

We see that there is a possibility that the denominator becomes imaginary. In order to avoid this possibility we have to demand that the denominator and numerator vanish at the same point. This happens at

\[
\frac{1}{v_\xi^{2z-2}} = 2 \nu_\xi, \quad K^2 = R^4 2^{\frac{2}{z-1}} \nu_\xi^2 \frac{2}{r^4}. 
\]

Then using \( (3.0) \) the equation \( (3.3) \) takes the form

\[
\partial_\sigma y = \pm \frac{K r^2}{R^2 v_\xi r^{z-1}} \frac{\sqrt{1 - 2 v_\xi r^{2z-2} - 2 z-2}}{\sqrt{1 - 2 z-2} v_\xi^2 r^{z-1} r^4} = \pm 2^{\frac{1}{z-1}} \nu_\xi^2 r^{3-z} \frac{\sqrt{1 - 2 v_\xi r^{2z-2}}}{\sqrt{1 - 2 z-1} v_\xi^2 r^{2} r^4}. 
\]

\[
(3.7)
\]

\( ^4 \)Similar analysis was performed in \[60\] for string in Lifshitz space-time.
Observe that for the most interesting case $z = 2$ this equation simplifies as

$$\partial_\sigma y = \pm 2v_\xi r \frac{\sqrt{1 - 2v_\xi r^2}}{\sqrt{1 - 2v^2 r^4}} = \pm \frac{2v_\xi r}{\sqrt{1 + 2v_\xi r^2}}$$

(3.8)

so that it can be be easily integrated with the result

$$y(r) = \pm \sqrt{1 + 2v_\xi r^2} \mp 1 ,$$

(3.9)

where the value of the integration constant was determined by the requirement that for $r = 0$ the time dependence of the end point of the string is $\xi = \tau v_\xi$.

Let us now return to general $z$ and define following currents

$$\pi^\alpha_t = \frac{\delta S}{\delta \partial_\alpha t} = -\frac{1}{2\pi\alpha'} g_{tN} \partial_\beta x^N (a^{-1})^{\beta\alpha} \sqrt{-\det a} ,$$

$$\pi^\alpha_i = \frac{\delta S}{\delta \partial_\alpha x^i} = -\frac{1}{2\pi\alpha'} g_{iN} \partial_\beta x^N (a^{-1})^{\beta\alpha} \sqrt{-\det a} ,$$

$$\pi^\alpha_\xi = \frac{\delta S}{\delta \partial_\alpha \xi} = -\frac{1}{2\pi\alpha'} g_{\xi N} \partial_\beta x^N (a^{-1})^{\beta\alpha} \sqrt{-\det a} ,$$

(3.10)

where $N = (t, x^i, \xi)$. Since the action does not explicitly depend on $t, \xi$ and $x^i$ it turns out that the currents (3.10) are conserved

$$\partial_\alpha \pi^\alpha_t = 0 , \quad \partial_\alpha \pi^\alpha_i = 0 , \quad \partial_\alpha \pi^\alpha_\xi = 0 .$$

(3.11)

In our case the current $\pi_\xi$ takes the form

$$\pi^\tau_\xi = \frac{1}{2\pi\alpha'} \frac{g_{t\xi} g_{rr}}{\sqrt{-\det a}} = \frac{1}{2\pi\alpha'} \frac{K_\xi}{\partial_\eta \xi} ,$$

$$\pi^\sigma_\xi = \frac{1}{2\pi\alpha'} \frac{g^2_{\xi t} v_\xi \partial_\sigma y}{\sqrt{-\det a}} = \frac{K}{2\pi\alpha'} .$$

(3.12)

For further purposes we also determine the spatial component of the current $\pi^\sigma_t$

$$\pi^\sigma_t = \frac{1}{2\pi\alpha'} \frac{g_{t\xi} \partial_\sigma y v_\xi}{\sqrt{-\det a}} = \frac{K_\xi v_\xi}{2\pi\alpha'} .$$

(3.13)

Note that in all these calculations we used

$$\sqrt{-\det a} = \frac{g^2_{t\xi} \partial_\sigma y}{K_\xi}$$

(3.14)

that follows from (3.4).
Now we give the physical interpretation of given solution. Since the open string obeys the Neumann boundary conditions we see that there should exist the transport of the energy and momenta from one end point of the string to the second one. Explicitly, let us define the total energy \( E \) and momentum \( P_\xi \) as

\[
E = -\int_0^\infty d\sigma \pi_\tau^\tau, \quad P_\xi = \int_0^\infty d\sigma \pi_\tau^\tau.
\]  

(3.15)

Then (3.11) imply

\[
\frac{dE}{dt} = -\int_0^\infty d\sigma \partial_\tau \pi_\tau^\tau = \int_0^\infty d\sigma \partial_\sigma \pi_\tau^\tau = \pi_\tau^\tau(\infty) - \pi_\tau^\tau(0).
\]

(3.16)

Using (3.13) we see that \( \pi_\sigma^\sigma(\infty) = \pi_\sigma^\sigma(0) \) and consequently the total change of the energy of the string is zero. On the other hand we can study the local gain and lose of the energy at the end point of the interval. In other words the string will either lost or gain an energy at \( r = 0 \) equal to

\[
\left. \frac{dE}{dt} \right|_{r=0} = -\pi_\tau^\tau(0) = -\frac{K}{2\pi\alpha'} = \mp \frac{1}{2\pi\alpha'} R^2 2^{1/3} v_\xi^{-1}.
\]

(3.17)

and at the same time it will either gain or lost the energy at \( r = \infty \). Further, the change of the momentum \( P_\xi \) is equal to

\[
\frac{dP_\xi}{dt} = \int_0^\infty d\sigma \partial_\tau \pi_\tau^\tau = \int_0^\infty d\sigma \partial_\sigma \pi_\tau^\tau = -\pi_\tau^\tau(\infty) + \pi_\tau^\tau(0)
\]

(3.18)

and it will either gain or lose momentum \( P_\xi \) at \( r = 0 \) given by the equation

\[
\left. \frac{dP_\xi}{dt} \right|_{r=0} = \pi_\xi^\xi(0) = -\frac{K}{2\pi\alpha'} = \mp \frac{1}{2\pi\alpha'} R^2 2^{1/3} v_\xi^{-1}.
\]

(3.19)

The concrete situation whether string gains or lose an energy is determined by the sign in front of the constant \( K \). Note that in the case of the calculation of drag force in the context of AdS/CFT correspondence authors gave strong physical arguments for the \(-\) sign in front of \( K \). However since we consider the background that is dual to quantum field theory at zero temperature there is no horizon and it is not completely clear which sign to choose.

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5Note that from the point of view of dual non-relativistic QFT \( P_\xi \) is interpreted as the operator of number of particles rather then operator of momenta \( P_\xi \).

6More precisely, if \( K \) were positive then the energy would flow down the string toward the horizon while if \( K \) were negative then the energy would flow upward from the horizon and the tail of string leads the quark. However it was argued in [54, 57] that the physical realistic solution is the one corresponding to the energy flow from the quark to the black hole horizon.
Further, the motion of string along $\xi$ dimension has different physical interpretation in the dual QFT than the motion along one of the boundary dimensions labelled by $x^i$. In fact, strong arguments were given in [22] for the interpretation of the total momentum along $\xi$ dimension as the quantum number that counts the number of particles in given theory. Then we suggest to interpret the previous result as the energy and particle flow along the string from its end point localized at the boundary $r = 0$ towards to the infinity. Note also that the energy and particle flow coincide. As we said above the boundary conditions also allow an existence of the second kind of string that describe the opposite situation when the string loses an energy and particle number that are injected into non-relativistic plasma. Finally we evaluate the Lagrangian density for given configuration. Using (3.14) we find

$$L = -\frac{1}{2\pi\alpha'} \frac{g_{\xi}^2 \partial_\sigma y}{K_\xi} = \pm \frac{1}{2\pi\alpha'} \frac{g_{\xi} \sqrt{-g_{rr}(g_{tt} + 2g_{t}\xi v_{\xi})}}{\sqrt{g_{t\xi}^2 - K_\xi^2}}.$$  

(3.20)

We see that for $r \to 0$ the Lagrangian density (3.20) approaches to

$$L = \mp \frac{1}{2\pi\alpha'} \frac{R^2}{r^{z+1}}$$  

(3.21)

and hence the action of the string diverges as $\mp \frac{1}{2\pi\alpha'} \frac{R^2}{2\epsilon}$ in the limit of $\epsilon \to 0$, where the origin of this divergence is the same as in AdS/CFT correspondence [4, 8]. Now due to the fact that (3.21) is real for all $r$ we see that this Lagrangian density can be imposed as the regularizator of $\cap$ shaped string. In other words the quark anti-quark potential is determined by following prescription

$$V(L) = \frac{2}{2\pi\alpha'} \left[ \int_0^{r_{\text{max}}} d\sigma \sqrt{-\det a} - \int_0^\infty \sqrt{-\det a_{\text{tailed}}} \right].$$  

(3.22)

4. Wilson Loop Along $x$ Direction and Tailed String

In this section return to the situation studied in the first section when we consider the open string description of the Wilson loop along $x$ direction. We also consider the situation when the string moves with the constant velocity $v_\xi$ along $\xi$ direction. Explicitly we consider following ansatz

$$\tau = t, \quad r = \sigma, \quad x = x(\sigma), \quad \xi = v_\xi \tau + y(\sigma)$$  

(4.1)

that leads to the following matrix components $a_{\alpha\beta}$

$$a_{\tau\tau} = g_{tt} + 2g_{t}\xi v_{\xi}, \quad a_{\tau\sigma} = a_{\sigma\tau} = g_{t}\xi \partial_\sigma y, \quad a_{\sigma\sigma} = g_{rr} + g_{xx}(\partial_\sigma x)^2.$$  

(4.2)
Let us now solve the equations of motion for $t, \xi$ and $x$. The equation of motion for $t$ and $\xi$ take the same form as in previous section and imply

$$\frac{g_\xi^2 v_\xi \partial_\sigma y}{\sqrt{-\det a}} = K, \quad \frac{g_\xi^2 \partial_\sigma y}{\sqrt{-\det a}} = K_\xi.$$  \hfill (4.3)

Finally the equation of motion for $x$ implies

$$\frac{g_{xx} \partial_\sigma x a_{rr}}{\sqrt{-\det a}} = K_x,$$  \hfill (4.4)

where $K_x$ is an integration constant.

We again see that the equation of motion for $t$ and $\xi$ are proportional so that it is sufficient to consider one of them, say the equation of motion for $\xi$. To proceed further we rewrite (4.3) and (4.4) into the form

$$(\partial_\sigma y)^2 g_\xi^2 (g_\xi^2 - K_\xi^2) + K_\xi^2 g_{xx} (\partial_\sigma x)^2 + K_x^2 a_{rr} g_{rr} = 0,$$

$$(g_{xx}^2 a_{rr} + K_x^2 a_{rr} g_{xx}) (\partial_\sigma x)^2 - K_x^2 g_\xi^2 (\partial_\sigma y)^2 + K_x^2 a_{rr} g_{rr} = 0.$$  \hfill (4.5)

Solving this system of equations with respect to $\partial_\sigma x, \partial_\sigma y$ we find

$$(\partial_\sigma y)^2 = -\frac{K_x^2 a_{rr} g_{rr} [a_{rr}^2 g_{xx} + a_{rr} K_x^2 - K_{rr}^2]}{a_{rr}^2 g_{xx} + K_x^2 a_{rr} |g_{\xi}^4 v_\xi^2 - g_{\xi}^2 K_\xi^2| + K_x^2 K_x^2 g_{\xi}^2},$$

$$(\partial_\sigma x)^2 = -\frac{1}{g_{xx} [a_{rr}^2 g_{xx} + K_x^2 a_{rr}] (g_{\xi}^4 v_\xi^2 - g_{\xi}^2 K_\xi^2) + K_x^2 K_x^2 g_{\xi}^2}.$$  \hfill (4.6)

We see that $\partial_\sigma y$ vanishes for $a_{rr} = 0$ at

$$\frac{1}{r_x^{2z-2}} = 2 v_\xi.$$  \hfill (4.7)

On the other the turning point of the Wilson loop defined by condition $\partial_\sigma x = \infty$ occurs when

$$[a_{rr}(r_{min}) g_{xx} + K_x^2 a_{rr}(r_{min})] (g_{\xi}^4 v_\xi^2 - K_\xi^2) = K_x^2.$$  \hfill (4.8)

Further, the consistency of the solution demands that $r_{min} < r_x$. Finally we have to demand that the projections of the two halves of the string onto the $\xi - x$ plane have to join smoothly so that we have to require that $\frac{\partial_x}{\partial y} = \frac{\partial_x}{\partial y} = \infty$. Then using (4.6) we obtain

$$\frac{\partial_\sigma x}{\partial_\sigma y} = \frac{g_\xi^2 v_\xi K_x}{K_\xi} \frac{1}{\sqrt{g_{xx} [a_{rr}^2 g_{xx} + a_{rr} K_x^2 - K_{rr}^2]}}.$$  \hfill (4.9)
We see that this expression diverges for $K_{\xi} = 0$ which however implies $\partial_\sigma y = 0$. In other words there is no tail in case of $\cap$ shaped string that moves around $\xi$ direction with the velocity $v_\xi$. This result is in agreement with the similar analysis performed in case of $\cap$ shaped string in AdS/CFT background [59].

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A. Wilson loops in Hamiltonian Formalism

In this appendix we develop the Hamiltonian formalism for the open string in non-relativistic background and reproduce the equations derived in the second section.

To begin with we consider the Nambu-Goto action for string in general background

$$ S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det \mathbf{a}} \quad (A.1) $$

so that the momenta conjugate to $x^M$ are equal to

$$ p_M = \frac{\delta S}{\delta \partial_\tau x^M} = \frac{1}{2\pi\alpha'} \frac{g_{MN}}{\sqrt{-\det \mathbf{a}}} \left( \partial_\tau x^N a_{\sigma\sigma} - \partial_\sigma x^N a_{\tau\sigma} \right). \quad (A.2) $$

Using this expression it is easy to derive following first class constraints

$$ \mathcal{H}_T \equiv p_M g^{MN} p_N + \frac{1}{(2\pi\alpha')^2} \partial_\sigma x^N g_{MN} \partial_\sigma x^N \approx 0 \quad (A.3) $$

and

$$ \mathcal{H}_S \equiv p_M \partial_\sigma x^M \approx 0. \quad (A.4) $$

Since the bare Hamiltonian density $\mathcal{H}_0 = \partial_\tau X^M p_M - \mathcal{L}$ vanishes for the Nambu-Goto string we find that the the total Hamiltonian density is sum of the constraints

$$ \mathcal{H}_T = N\mathcal{H}_T + N_S\mathcal{H}_S, \quad (A.5) $$

where $N, N_S$ are Lagrange multipliers. It is also important to stress that $\mathcal{H}_T, \mathcal{H}_S$ are the first class constraints and that the consistency of the time evolution of these constraints do not generate the secondary ones.

In order to study the static configuration of string it is convenient to fix the gauge symmetries generated by $\mathcal{H}_T, \mathcal{H}_S$. This can be done with the help of the gauge fixing functions

$$ G_H : x^0 - \tau = 0, \quad G_S : r - \sigma = 0 \quad (A.6) $$

The general analysis of Hamiltonian formulation of bosonic string in non-relativistic background was given in [53].
that have non-zero Poisson brackets with $\mathcal{H}_T, \mathcal{H}_S$ and consequently the set of constraints $\mathcal{H}_T, \mathcal{H}_S, G_T, G_S$ form the collection of the second class constraints. It is well known that these constrains strongly vanish and can be explicitly solved. In fact, solving the constraint $\mathcal{H}_S = 0$ for $p_r$ we find

$$p_r = -p_t \partial_\sigma x^i - p_\xi \partial_\sigma \xi$$

using the fact that $\partial_\sigma r = 1, \partial_\sigma t = 0$ as follows from (A.6). Then, inserting this result into $\mathcal{H}_T = 0$ and using (A.6) we can generally express $p_0$ as function of remaining canonical variable.

Let us now specialize to the case of string in non-relativistic background. The characteristic property of this background is the presence of non-zero metric component $g_{\xi \xi} \neq 0$. More precisely, let us consider the background (2.3) and determine corresponding inverse metric

$$g^{tt} = 0, \quad g^{\xi \xi} = \frac{1}{g_{\xi \xi}}, \quad g^{\xi x} = -\frac{g_{tt}}{g_{\xi \xi}}, \quad g^{rr} = \frac{1}{g_{rr}}, \quad g_{ij} = \frac{1}{g_{ii}} \delta_{ij}. \quad (A.8)$$

Using this form of the metric it is easy to solve the constraint $\mathcal{H}_T = 0$ for $p_0$ and we find

$$\mathcal{H}_{fix} = -p_0 = \frac{1}{2 g^{\xi \xi} p_\xi} \left( p_\xi g^{\xi \xi} p_\xi + (p_i \partial_\sigma x^i + p_\xi \partial_\sigma \xi) g^{rr}(p_j \partial_\sigma x^j + p_\xi \partial_\sigma \xi) + p_i g^{ij} p_j + \frac{1}{(2 \pi \alpha')^2} (g_{rr} + \partial_\sigma x^i g_{ij} \partial_\sigma x^j) \right),$$

where we used (2.3) and performed the standard identification between the Hamiltonian density $\mathcal{H}_{fix}$ of the gauge fixed theory and the momentum $p_0$.

Using $\mathcal{H}_{fix}$ it is easy to determine the equations of motion for $\xi$ and $x^i$

$$\partial_\tau \xi = \{ \xi, \mathcal{H}_{fix} \} =$$

$$-\frac{1}{2 g^{\xi \xi} p_\xi} \left( p_\xi g^{\xi \xi} p_\xi + (p_i \partial_\sigma x^i + p_\xi \partial_\sigma \xi) g^{rr}(p_j \partial_\sigma x^j + p_\xi \partial_\sigma \xi) + p_i g^{ij} p_j + \frac{1}{(2 \pi \alpha')^2} (g_{rr} + \partial_\sigma x^i g_{ij} \partial_\sigma x^j) \right) + g^{\xi \xi} \frac{g^{rr}}{g^{\xi \xi}} (p_j \partial_\sigma x^j + p_\xi \partial_\sigma \xi) \partial_\sigma \xi$$

and

$$\partial_\tau x^i = \{ x^i, \mathcal{H}_{fix} \} = \frac{1}{g^{\xi \xi} p_\xi} (\partial_\sigma x^i g^{rr}(p_j \partial_\sigma x^j + p_\xi \partial_\sigma \xi) + g^{ij} p_j).$$

Further the equations of motion for $p_\xi$ and $p_i$ take the form

$$\partial_\tau p_\xi = \{ p_\xi, \mathcal{H}_{fix} \} = \partial_\sigma \left[ \frac{g^{rr}}{g^{\xi \xi}} (p_j \partial_\sigma x^j + p_\xi \partial_\sigma \xi) \right],$$

$$\partial_\tau p_i = \{ p_i, \mathcal{H}_{fix} \} = \partial_\sigma \left[ \frac{g^{rr}}{g^{\xi \xi} p_\xi} p_i (p_j \partial_\sigma x^j + p_\xi \partial_\sigma \xi) + \frac{g_{ij} \partial_\sigma x^j}{g^{\xi \xi} (2 \pi \alpha')^2 p_\xi} \right].$$

(A.12)
In order to find the string configuration that moves with constant velocity along $\xi$ direction we consider following ansatz $x^i = x^i(\sigma)$, $\partial_\tau x = v_\xi$. Then (A.10) take the form

$$v_\xi = -\frac{g_{tt}}{2g_{t\xi}} - \frac{g_{t\xi}}{2(2\pi\alpha')^2p_\xi^2}(g_{rr} + g_{ij}\partial_\sigma x^i\partial_\sigma x^j)$$

(A.13)

that allows us to express $p_\xi$ as

$$p_\xi = \pm \frac{1}{2\pi\alpha'}\sqrt{-\frac{g_{t\xi}(g_{rr} + g_{ij}\partial_\sigma x^i\partial_\sigma x^j)}{2v_\xi + g_{tt}/g_{t\xi}}}.$$  

(A.14)

On the other hand the equation of motion for $x^i$ is automatically satisfied and the equation of motion for $p_\xi$ implies $\partial_\tau p_\xi = 0$. Since we also consider the static configuration we demand that $\partial_\tau p_i = 0$ so that the equation of motion for $p_i$ implies

$$\frac{g_{ij}g_{t\xi}\partial_\sigma x^j}{p_\xi} = C'$$

(A.15)

using the fact that $p_i = 0$. Then if we consider one mode $x \equiv x^1$ and insert (A.14) into (A.15) we find the differential equation for $x$

$$\partial_\sigma x = \pm \frac{C'}{2\pi\alpha'}\frac{\sqrt{g_{rr}}}{\sqrt{g_{xx}(-g_{tt} - 2v_\xi g_{t\xi}) - (\frac{C'}{2\pi\alpha'})^2g_{xx}}}$$

(A.16)

We see that this equation exactly reproduces the equation derived in section (3) after trivial identification of the constants $C$ and $C'$.

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