Gauss-Seidel Method with Oblique Direction

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Abstract

In this paper, a Gauss-Seidel method with oblique direction (GSO) is proposed for finding the least-squares solution to a system of linear equations, where the coefficient matrix may be full rank or rank deficient and the system is overdetermined or underdetermined. Through this method, the number of iteration steps and running time can be reduced to a greater extent to find the least-squares solution, especially when the columns of matrix A are close to linear correlation. It is theoretically proved that GSO method converges to the least-squares solution. At the same time, a randomized version–randomized Gauss-Seidel method with oblique direction (RGSO) is established, and its convergence is proved. Theoretical proof and numerical results show that the GSO method and the RGSO method are more efficient than the coordinate descent (CD) method and the randomized coordinate descent (RCD) method.

Key words: linear least-squares problem, oblique direction, coordinate descent method, randomization, convergence property.

1 Introduction

Consider a linear least-squares problem

$$\arg \min_{x \in \mathbb{R}^n} \|Ax - b\|^2,$$ (1)

where $b \in \mathbb{R}^m$ is a real $m$ dimensional vector, and the columns of coefficient matrix $A \in \mathbb{R}^{m \times n}$ are non-zero, which doesn't lose the generality of matrix $A$. Here and in the sequel, $\|\cdot\|$ indicates the Euclidean norm of a vector. When $A$ is full column rank, (1) has a unique solution $x^* = A^+ b = (A^T A)^{-1} A^T b$, where $A^T$ and $A^+$ are the Moore-Penrose pseudoinverse [3] and the transpose of $A$, respectively. One of the iteration methods that can be used to solve (1) economically and effectively is the coordinate descent (CD) method [10], which is also obtained by applying the classical Gauss-Seidel iteration method to the following normal equation (see [22])

$$A^T Ax = A^T b.$$ (2)

In solving (1), the CD method has a long history of development, and is widely used in various fields, such as machine learning [7], biological feature selection [6], tomography [5, 29], and so on. Inspired by the randomized coordinate descent (RCD) method proposed by Leventhal and Lewis [10] and its linear convergence rate analyzed theoretically [12], a lot of related work such as the
randomized block versions [11, 13, 21] and greedy randomized versions of the CD method [11, 30] have been developed and studied. For more information about a variety of randomized versions of the coordinate descent method, see [17, 23, 28] and the references therein. These methods mentioned above are based on the CD method, and can be extended to Kaczmarz-type methods. The recent work of Kaczmarz-type method can be referred to [26, 25, 15, 16, 24, 20]. Inspired by the above work, we propose a new descent direction based on the construction idea of the CD method, which is formed by the weighting of two coordinate vectors. Based on this, we propose a Gauss-Seidel method with oblique direction (GSO) and construct the randomized version–randomized Gauss-Seidel method with oblique direction (RGSO), and analyze the convergence properties of the two methods.

Regarding our proposed methods–the GSO method and the RGSO method, we emphasize the efficiency when the columns of matrix $A$ are close to linear correlation. In [14], it is mentioned that when the rows of matrix $A$ are close to linear correlation, the convergence speed of the K method and the randomized Kaczmarz method [27] decrease significantly. Inspired by the above phenomena, we experimented the convergence performance of the CD method and the RCD method when the columns of matrix $A$ are close to linear correlation and it is found through experiments that the theoretical convergence speed and experimental convergence speed of the CD method and the RCD method will be greatly reduced. The exponential convergence in expectation of the RCD method is as follows:

$$E_k \delta(x^{(k+1)}) \leq \left(1 - \frac{1}{\kappa^2(A)}\right) \delta(x^{(k)}),$$

(3)

where $\delta(x) = F(x) - \min F$, $F(x) = \|Ax - b\|^2$. Here and in the sequel, $\|A\|_2 = \max \|Ax\|$, $\|A\|_F$, $\kappa_2(A) = \|A\|_F \cdot \|A^T\|_2$ are used to denote Euclidean norm, Frobenius norm and the scaled condition number of the matrix $A$, respectively. The subgraph (a) in Figure 1 shows that when the column of matrix $A$ is closer to the linear correlation, $\kappa^2(A)$ will become larger, which further reduce the convergence rate of the RCD method. The subgraph (b) in Figure 1 illustrates the sensitivity of the CD method and the RCD method to linear correlation column of $A$. This further illustrates the necessity of solving this type of problem, and the GSO method and the RGSO method we proposed can be used effectively to solve that one. For the initial data setting, explanation of the experiment and the GSO method and the RGSO method we proposed, please refer to Section 4 in this paper.

In this paper, $\langle \cdot, \cdot \rangle$ stands for the scalar product, and we indicate by $e_i$ the column vector with 1 at the ith position and 0 elsewhere. In addition, for a given matrix $G = (g_{ij}) \in R^{m \times n}$, $g_i$, and $\sigma_{\min}(G)$, are used to denote its ith row, jth column and the smallest nonzero singular value of $G$ respectively. Given a symmetric positive semidefinite matrix $G \in R^{m \times n}$, for any vector $x \in R^n$ we define the corresponding seminorm as $\|x\|_G = \sqrt{x^T G x}$. Let $E_k$ denote the expected value conditional on the first $k$ iterations, that is,

$$E_k[\cdot] = E[\cdot|j_0, j_1, \ldots, j_{k-1}],$$

where $j_s(s = 0, 1, \ldots, k-1)$ is the column chosen at the sth iteration.

The organization of this paper is as follows. In Section 2, we introduce the CD method and its construction idea. In Section 3, we propose the GSO method naturally and get its randomized version–RGSO method, and prove the convergence of the two methods. In Section 4, some numerical examples are provided to illustrate the effectiveness of our new methods. Finally, some brief concluding remarks are described in Section 5.
Figure 1: Matrix $A$ is generated by the $\text{rand}$ function in the interval $[c, 1]$. (a): $\kappa^2_f(A)$ of matrix $A$ changes with $c$. (b): When the system is consistent, the number of iterations for the CD method and the RCD method to converge with the change of $c$, where the maximum number of iterations is limited to 800,000.

2 Coordinate Descent Method

Consider a linear system

$$\tilde{A}x = b,$$

where the coefficient matrix $\tilde{A} \in \mathbb{R}^{n \times n}$ is a positive-definite matrix, and $b \in \mathbb{R}^n$ is a real $m$ dimensional vector. $\hat{x}^* = \tilde{A}^{-1}b$ is the unique solution of (4). In this case, solving (4) is equivalent to the following strict convex quadratic minimization problem

$$f(x) = \frac{1}{2}x^T \tilde{A}x - b^Tx.$$

From [10], the next iteration point $x^{(k+1)}$ is the solution to $\min_{t \in \mathbb{R}} f(x^{(k)} + td)$, i.e.

$$x^{(k+1)} = x^{(k)} + \frac{(b - \tilde{A}x^{(k)})^Td}{d^T \tilde{A}d} d,$$

where $d$ is a nonzero direction, and $x^{(k)}$ is a current iteration point. It is easily proved that

$$f(x^{(k+1)}) - f(\hat{x}^*) = \frac{1}{2}||x^{(k+1)} - \hat{x}^*||_A^2 = \frac{1}{2}||x^{(k)} - \hat{x}^*||_A^2 - \frac{(b - \tilde{A}x^{(k)})^Td)^2}{2d^T \tilde{A}d}.$$ (6)

One natural choice of a set of easily computable search directions is to choose $d$ by successively cycling through the set of canonical unit vectors $\{e_1, \ldots, e_n\}$, where $e_i \in \mathbb{R}^n, i = 1, \ldots, n$. When $A \in \mathbb{R}^{m \times n}$ is full column rank, we can apply (2) to (5) to get:

$$x^{(k+1)} = x^{(k)} + \frac{r^{(k)} A_i}{||A_i||^2} e_i,$$

where $i = \text{mod}(k, n) + 1$. This is the iterative formula of CD method, also known as Gauss-Seidel method. This method is linearly convergent but with a rate not easily expressible in terms of typical matrix quantities. See [4, 8, 19]. The CD method can only ensure that one entry of $A^T r$ is 0 in
each iteration, i.e. \( A^T r^{(k)} = 0 \) (\( i = \text{mod}(k, n) + 1 \)). In the next chapter, we propose a new oblique direction \( d \) for (5), which is the weight of the two coordinate vectors, and use the same idea to get a new method—the GSO method. The GSO method can ensure that the two entries of \( A^T r \) are 0 in each iteration, thereby accelerating the convergence.

**Remark 1.** When \( \tilde{A} \) is positive semidefinite matrix, (4) may not have a unique solution, replace \( \tilde{x}^* \) with any least-squares solution. (5), (6) still hold, if \( d^T \tilde{A} d \neq 0 \).

**Remark 2.** The Kaczmarz method can be regarded as a special case of (5) under a different regularizing linear system

\[
AA^T y = b, \quad x = A^T y,
\]

when \( d \) is selected cyclically through the set of canonical unit vectors \( \{e_1, ..., e_m\} \), where \( e_i \in \mathbb{R}^m \), \( i = 1, 2, \cdots, m \).

### 3 Gauss-Seidel Method with Oblique Direction and its Randomized Version

#### 3.1 Gauss-Seidel Method with Oblique Direction

We propose a similar \( d \), that is \( d = e_{i_k} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} e_{i_k} \), where \( e_i \in \mathbb{R}^n \), \( i = 1, 2, \cdots, n \). Using (2) and (5), we get

\[
x^{(k+1)} = x^{(k)} + \frac{(A^T b - A^T Ax^{(k)})^T (e_{i_k} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} e_{i_k})}{||A(e_{i_k} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} e_{i_k})||^2} (e_{i_k} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} e_{i_k}) = x^{(k)} + \frac{A^T r^{(k)} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} A^T r^{(k)}}{||A_{i_k}||^2 - \frac{\langle A_{i_k}, A_{i_k} \rangle^2}{||A_{i_k}||^2}} (e_{i_k} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} e_{i_k}).
\]

Now we prove that \( A^T r^{(k)} = 0 \).

\[
A^T r^{(k)} = \langle A_{i_k}, b - Ax^{(k)} \rangle = \langle A_{i_k}, r^{(k-1)} \rangle - \langle A_{i_k}, r^{(k-1)} \rangle + \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} A^T r^{(k-1)} = \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} A^T r^{(k-1)}, \quad k = 2, 3, ...
\]

We only need to guarantee \( A^T r^{(1)} = 0 \), so we need to take the simplest coordinate descent projection as the first step. It becomes

\[
x^{(k+1)} = x^{(k)} + \frac{A^T r^{(k)}}{||A_{i_k}||^2 - \frac{\langle A_{i_k}, A_{i_k} \rangle^2}{||A_{i_k}||^2}} (e_{i_k} - \frac{\langle A_{i_k}, A_{i_k} \rangle}{||A_{i_k}||^2} e_{i_k}).
\]

The algorithm is described in Algorithm 1. Without losing generality, we assume that all columns of \( A \) are not zero vectors.
Algorithm 1 Gauss-Seidel method with Oblique Projection (GSO)

Require: $A \in R^{m \times n}$, $b \in R^m$, $x^{(0)} \in R^n$, $K$, $\epsilon > 0$
1: For $i = 1 : n$, $N(i) = \|A_i\|^2$
2: Compute $r^{(0)} = b - Ax^{(0)}$, $a_0 = \frac{\langle A_i, r^{(0)} \rangle}{N(i)}$, $x^{(1)} = x^{(0)} + a_0 e_1$, $r^{(1)} = r^{(0)} - a_0 A_1$, and set $i_{k+1} = 1$
3: for $k = 1, 2, \cdots, K - 1$ do
4: Set $i_k = i_{k+1}$ and choose a new $i_{k+1}$: $i_{k+1} = mod(k, n) + 1$
5: Compute $G_k = (A_{i_k} A_{i_{k+1}})$ and $g_k = N(i_{k+1}) - \frac{G_k}{N(i_k)} G_k$
6: if $g_{i_k} > \epsilon$ then
7: Compute $\alpha_k = \frac{\langle A_{i_{k+1}} r^{(i_k)} \rangle}{g_{i_k}}$ and $\beta_k = -\frac{G_k}{N(i_k)} \alpha_k$
8: Compute $x^{(k+1)} = x^{(k)} + \alpha_k e_{i_{k+1}} + \beta_k e_{i_k}$, and $r^{(k+1)} = r^{(k)} - \alpha_k A_{i_{k+1}} - \beta_k A_{i_k}$
9: end if
10: end for
11: Output $x^{(K)}$

It’s easy to get
$$A_{i_k}^T r^{(k)} = A_{i_k}^T (r^{(k-1)} - \alpha_k A_{i_k} - \beta_k^{-1} A_{i_{k-1}})$$
$$= A_{i_k}^T r^{(k-1)} - \frac{\langle A_{i_k} r^{(k-1)} \rangle}{g_{i_k}} A_{i_k} + \frac{\langle A_{i_{k-1}} A_{i_k} \rangle A_{i_{k-1}}}{\|A_{i_{k-1}}\|^2 g_{i_k}} A_{i_{k-1}}$$
$$= 0. \quad k = 2, 3, \cdots$$

The last equality holds due to $A_{i_k}^T r^{(k)} = 0$, $k = 1, 2, \cdots$. Before giving the proof of the convergence of the GSO method, we redefine the iteration point. For $x^{(0)} \in R^n$ as initial approximation, we define $x^{(0,0)}, x^{(0,1)}, \ldots, x^{(0,n)} \in R^n$ by

$$
\begin{align*}
x^{(0,0)} &= x^{(0)} + A_{i_1}^T (b - Ax^{(0)}) e_1, \\
x^{(0,1)} &= x^{(0,0)} + A_{i_2}^T (b - Ax^{(0,0)}) e_2 - \frac{\langle A_{i_2} A_{i_1} \rangle}{\|A_{i_1}\|^2} e_1, \\
x^{(0,2)} &= x^{(0,1)} + A_{i_3}^T (b - Ax^{(0,0)}) e_3 - \frac{\langle A_{i_3} A_{i_2} \rangle}{\|A_{i_2}\|^2} e_2, \\
&\cdots \\
x^{(0,n-1)} &= x^{(0,n-2)} + A_{i_n}^T (b - Ax^{(0,n-2)}) e_n - \frac{\langle A_{i_n} A_{i_{n-1}} \rangle}{\|A_{i_{n-1}}\|^2} e_{n-1}, \\
x^{(0,n)} &= x^{(0,n-1)} + A_{i_1}^T (b - Ax^{(0,n-1)}) e_1 - \frac{\langle A_{i_1} A_{i_0} \rangle}{\|A_{i_0}\|^2} e_0.
\end{align*}
$$

(9)

For convenience, denote $A_{i_{n+1}} = A_1$, $b_{n+1} = b_1$. When the iteration point $x^{(p,n)} \forall p \geq 0$ is given, the iteration points are obtained continuously by the following formula

$$
\begin{align*}
\text{for} \quad i = 1 : n \\
x^{(p+1,i)} &= x^{(p+1,i-1)} + A_{i_1}^T (b - Ax^{(p+1,i-1)}) e_i - \frac{\langle A_{i_1} A_{i_0} \rangle}{\|A_{i_0}\|^2} e_1.
\end{align*}
$$

(10)

where $x^{(p+1,0)} = x^{(p,n)}$. Then, we can easily obtain that $x^{(k+1)} = x^{(p,i)}$, and $A_{i_k}^T r^{(k)} = A_{i+1}^T r^{(p,i)} = 0$, if $k = p \cdot n + i, 0 \leq i < n$.

The convergence of the GSO is provided as follows.
The residuals satisfy
and get

Apply \((\ref{4})\) to \((\ref{9})\), we get

Apply \((\ref{2})\) to \((\ref{6})\), we get

The residuals satisfy

Taking the limit of \(p\) on both sides of the above equation, we get

Using the above equation and \((\ref{14})\), we can easily deduce that

Because the sequence \(\{\||x^{(p,i)} - \tilde{x}\|_{\|A\|}\}_{p=0, i=0}^{\infty, n-1}\) is bounded, we obtain

According to \((\ref{16})\) we get that the sequence \(\{Ax^{(p,0)}\}_{p=0}^{\infty}\) is bounded, thus there exists a convergent subsequence \(\{Ax^{(p,0)}\}_{j=1}^{\infty}\), let’s denote it as

\[
\lim_{j \to \infty} Ax^{(p,0)} = \tilde{b}.
\]
From (9)-(10), we get
\[ x^{(p+1)} = x^{(p,0)} - \frac{A_i^T (b - Ax^{(p,0)})}{\|A_i\|^2} \left( e_{2} - \frac{\langle A_2, A_1 \rangle}{\|A_1\|^2} e_1 \right), \quad \forall \; j > 0. \]

By multiplying the both sides of the above equation left by matrix $A$ and using (14), we can get that
\[ \lim_{j \to \infty} Ax^{(p,1)} = \hat{b}. \]

With the same way we obtain
\[ \lim_{j \to \infty} Ax^{(p,i)} = \hat{b}, \quad \forall \; i = 0, 1, \cdots, n - 1. \]

Then, from (15) we get for any $i = 1, \cdots, n - 1$,
\[ \lim_{j \to \infty} A^T r^{(p,i)} = A^T (b - \hat{b}) = 0. \]

From (15) and the above equation, we get
\[ \lim_{j \to \infty} \|x^{(p,i)} - \tilde{x}\|_{A'_{iA}} = 0, \quad \forall \; i = 0, 1, \cdots, n - 1 \]

Hence,
\[ \lim_{j \to \infty} \|x^{(p,i)} - \tilde{x}\|_{A'_{iA}} = 0, \quad \forall \; i = 0, 1, \cdots, n - 1 \]
then (11) holds.

**Remark 3.** When $g_{i_k} = 0$, $A_{i_{k+1}}$ is parallel to $A_{i_k}$, i.e. $\exists \lambda > 0$, s.t. $A_{i_k} = \lambda A_{i_{k+1}}$. According to the above derivation, the GSO method is used to solve (1.2) which is consistent, so the following equation holds:
\[ A_{i_k}^T b = \lambda A_{i_{k+1}}^T b, \]
which means for (2) the $i_k$th equation: $\langle A_{i_k},Ax \rangle = A_{i_k}^T b$, and the $i_{k+1}$th equation: $\langle A_{i_{k+1}},Ax \rangle = A_{i_{k+1}}^T b$ are coincident, and we can skip this step without affecting the final calculation to obtain the least-squares solution. When $g_{i_k}$ is too small, it is easy to produce large errors in the process of numerical operation, and we can regard it as the same situation as $g_{i_k} = 0$ and skip this step.

**Remark 4.** By the GSO method, we have: $\|x^{(k+1)} - \tilde{x}\|_{A'_{iA}}^2 = \|x^{(k)} - \tilde{x}\|_{A'_{iA}}^2 - \frac{(A_{i_{k+1}}^T r_i)^2}{g_{i_k}}$, where $g_{i_k} = \|A_{i_{k+1}}\|^2 - \frac{\langle A_{i_{k+1}}, A_i \rangle^2}{\|A_i\|^2}$. But the CD method holds: $\|x^{(k+1)} - \tilde{x}\|_{A'_{iA}}^2 = \|x^{(k)} - \tilde{x}\|_{A'_{iA}}^2 - \frac{(A_i^T r_i)^2}{g_{i_k}}$. So the GSO method is faster than the CD method unless $\langle A_{i_k}, A_{i_{k+1}} \rangle = 0$. When $\langle A_{i_k}, A_{i_{k+1}} \rangle = 0$, the convergence rate of the GSO method is the same as that of the CD method. This means that when the coefficient matrix $A$ is a column orthogonal matrix, the GSO method degenerates to the CD method.

**Remark 5.** The GSO method needs $8m + 5$ floating-point operations per step, and the CD method needs $4m + 1$ floating-point operations per step.

**Remark 6.** When the matrix $A$ is full column rank, let $x^*$ be the unique least-squares solution of (1.1), the sequence $\{x^{(k)}\}_{k=1}^\infty$ generated by the GSO method holds: $\lim_{k \to \infty} \|x^{(k)} - x^*\|_{A'_{iA}} = 0$, that is, $\lim_{k \to \infty} \|A(x^{(k)} - x^*)\|^2 = 0$. Therefore,
\[ \lim_{k \to \infty} \|x^{(k)} - x^*\|^2 = 0. \]
Example 1. Consider the following systems of linear equations

\[
\begin{align*}
5x_1 + 45x_2 &= 50, \\
9x_1 + 80x_2 &= 89, \\
\end{align*}
\]

(18)

\[
\begin{align*}
x_1 + 11x_2 &= 12, \\
-2x_1 - 21x_2 &= -23, \\
3x_1 + 32x_2 &= 35
\end{align*}
\]

(19)

and

\[
\begin{align*}
x_1 + 9x_2 &= 0, \\
4x_1 + 36x_2 &= 42.5, \\
13x_1 + 118x_2 &= 131,
\end{align*}
\]

(20)

(18) is square and consistent, (19) is overdetermined and inconsistent, and (20) is overdetermined and inconsistent. Vector \(x^* = (1, 1)^T\) is the unique solution to the above (18) and (19), is the unique least-squares solution to (20). It can be found that the column vectors of these systems are close to linearly correlated. Numerical experiments show that they take 650259, 137317, 3053153 steps respectively for the CD method to be applied to the above systems to reach the relative solution error.

3.2 Randomized Gauss-Seidel Method with Oblique Direction

If the columns whose residual entries are not 0 in algorithm [1] are selected uniformly and randomly, we get a randomized Gauss-Seidel method with oblique direction (RGSO) and its convergence as follows.

Algorithm 2 Randomized Gauss-Seidel Method with Oblique Direction (RGSO)

Require: \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, x^{(0)} \in \mathbb{R}^n, K, \varepsilon > 0\)

1: For \(i = 1 : n\), \(N(i) = ||A_i||^2\)
2: Randomly select \(i_1\), and compute \(r^{(0)} = b - Ax^{(0)}, \alpha_0 = \frac{\langle A_i, r^{(0)} \rangle}{||A_i||^2}\), and \(x^{(1)} = x^{(0)} + \alpha_0 e_i\)
3: Randomly select \(i_2 \neq i_1\), and compute \(r^{(1)} = r^{(0)} - \alpha_0 A_{i_1}\), \(\alpha_1 = \frac{\langle A_i, r^{(1)} \rangle}{||A_i||^2 \alpha_0^2}\), and \(\beta_1 = \frac{|\langle A_i, A_{i_2} \rangle|}{||A_i||^2 \alpha_1}\)
4: Compute \(x^{(2)} = x^{(1)} + \alpha_1 e_{i_2} + \beta_1 e_{i_1}\), and \(r^{(2)} = r^{(1)} - \alpha_1 A_{i_2} - \beta_1 A_{i_1}\)
5: for \(k = 2, 3, \cdots, K - 1\) do
6: Randomly select \(i_{k+1} (i_{k+1} \neq i_k, i_{k-1})\)
7: Compute \(G_{i_k} = \langle A_{i_k}, A_{i_{k+1}} \rangle\) and \(g_{i_k} = N(i_{k+1}) - \frac{G_{i_k}}{N(i_k)} G_{i_k}\)
8: if \(g_{i_k} > \varepsilon\) then
9: Compute \(\alpha_k = \frac{\langle A_{i_k}, r^{(k)} \rangle}{g_{i_k}}\) and \(\beta_k = -\frac{G_{i_k}}{N(i_k)} \alpha_k\)
10: Compute \(x^{(k+1)} = x^{(k)} + \alpha_k e_{i_{k+1}} + \beta_k e_{i_k}\), and \(r^{(k+1)} = r^{(k)} - \alpha_k A_{i_{k+1}} - \beta_k A_{i_k}\)
11: end if
12: end for
13: Output \(x^{(K)}\)

Lemma 1. Consider (1), where the coefficient \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\) is a given vector, and \(\hat{x}\) is any solution to (1), then we obtain the bound on the following expected conditional on the first \(k (k \geq 2)\) iteration
In particular, if $A$ has full column rank, we have the equivalent property

$$E_k \frac{(A_T^i r^{(k)})^2}{g_{ik}} \geq \frac{1}{n-2} \frac{\sigma^2_{\min}(A)}{||A||^2 - \sigma^2_{\min}(A)}.$$  

Proof. For the RGSO method, it is easy to get that $A_T^i r^{(k)} = 0$ ($k = 1, 2, \ldots$) and $A_T^{i+1} r^{(k)} = 0$ ($k = 2, 3, \ldots$) are still valid.

$$E_k \frac{(A_T^i r^{(k)})^2}{g_{ik}} = \frac{1}{n-2} \sum_{s=1, s \neq i, i+1}^{n} \frac{(A_T^i r^{(k)})^2}{||A_s||^2 - \frac{\langle A_s A_s^T \rangle}{||A_s||^2}} \geq \frac{1}{n-2} \sum_{s=1, s \neq i, i+1}^{n} \left( \frac{||A_s^T A(\tilde{x} - x^{(k)})||^2}{||A_s||^2 - \frac{\langle A_s A_s^T \rangle}{||A_s||^2}} \right) \geq \frac{1}{n-2} \frac{\sigma^2_{\min}(A)}{||A||^2 - \sigma^2_{\min}(A)}, \quad k = 2, 3, \ldots$$

The first inequality uses the conclusion of $|b_1| + |b_2| \geq |b_1 + b_2|$ (if $|a_1| > 0$, $|a_2| > 0$), and the second one uses the conclusion of $||A^T z||_2^2 \geq \sigma^2_{\min}(A)||z||_2^2$, if $z \in R(A)$. 

**Theorem 2.** Consider $[\mathbf{1}]$, where the coefficient $A \in R^{m \times n}$, $b \in R^m$ is a given vector, and $\tilde{x}$ is any least-squares solution of $[\mathbf{1}]$. Let $x^{(0)} \in R^n$ be an arbitrary initial approximation, and define the least-squares residual and error by

$$F(x) = ||Ax - b||^2,$$

$$\delta(x) = F(x) - \min F,$$

then the RGSO method is linearly convergent in expectation to a solution in $[\mathbf{1}]$. For each iteration $k = 2, 3, \ldots$,

$$E_k \delta(x^{(k+1)}) \leq \left( 1 - \frac{1}{n-2}(k^2(A) - 1) \right) \delta(x^{(k)}).$$

In particular, if $A$ has full column rank, we have the equivalent property

$$E_k \left[ \|x^{(k+1)} - x^*\|_{A^T A}^2 \right] \leq \left( 1 - \frac{1}{n-2}(k^2(A) - 1) \right) \|x^{(k)} - x^*\|_{A^T A}^2,$$

where $x^* = A^T b = (A^T A)^{-1}A^T b$ is the unique least-squares solution.

Proof. It is easy to prove that

$$F(x) - F(\tilde{x}) = ||x - \tilde{x}||_{A^T A}^2 = \delta(x).$$
Applying (2) to (6) with $d = e_i + x_k - (k_i, r_k)$, we get that

$$F(x^{(k+1)}) - F(x) = ||x^{(k+1)} - \hat{x}||^2_{A^T A}$$

$$= ||x^{(k)} - \hat{x}||^2_{A^T A} - \frac{(A^T_{ik}, r^{(k)})^2}{g_{ik}}.$$ 

Making conditional expectation on both sides, and applying Lemma 1, we get

$$E_k[F(x^{(k+1)}) - F(x)] = ||x^{(k)} - \hat{x}||^2_{A^T A} - E_k\left[\frac{(A^T_{ik}, r^{(k)})^2}{g_{ik}}\right]$$

$$\leq ||x^{(k)} - \hat{x}||^2_{A^T A} - \frac{\sigma^2_{\min}(A)||x - x^{(k)}||^2_{A^T A}}{(n - 2)(||A||^2_2 - \sigma^2_{\min}(A))},$$

that is

$$E_k\delta(x^{(k+1)}) \leq \left(1 - \frac{\sigma^2_{\min}(A)}{(n - 2)(||A||^2_2 - \sigma^2_{\min}(A))}\right)\delta(x^{(k)})$$

$$= \left(1 - \frac{1}{(n - 2)(k^2_p(A) - 1)}\right)\delta(x^{(k)}).$$

If $A$ has full column rank, the solution in (1) is unique and the $\hat{x} = x^*$. Thus, we get

$$E_k[||x^{(k+1)} - x^*||^2_{A^T A}] \leq \left(1 - \frac{1}{(n - 2)(k^2_p(A) - 1)}\right)||x^{(k)} - x^*||^2_{A^T A}.$$ 

\[\square\]

**Remark 7.** In particular, after unitizing the columns of matrix $A$, we can get from Lemma 1

$$E_k\left[A^T_{ik} r^{(k)}\right]^2 = \frac{1}{n - 2} \sum_{j=1}^{n} \frac{||A_i||^2_2}{||A_j||^2_2 - ||A_i||^2_2} \frac{(A^T_{ij}, r^{(k)})^2}{g_{ij}}$$

$$\geq \frac{\sigma^2_{\min}(A)||x - x^{(k)}||^2_{A^T A}}{(1 - \gamma_k^2)(||A||^2_2 - 2)}$$

where $\gamma_k = \min_{s \neq i, j, k} |(A_s, A_k)|$. Then we get from Theorem 1

$$E_k\delta(x^{(k+1)}) \leq \left(1 - \frac{\sigma^2_{\min}(A)}{(1 - \gamma_k^2)(||A||^2_2 - 2)}\right)\delta(x^{(k)}).$$

Comparing the above equation with (6), we can get that under the condition of column unitization, the RGSO method is theoretically faster than the RCD method. Note that by Remark 3, we can avoid the occurrence of $\gamma_{ik} = 1$. 

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4 Numerical Experiments

In this section, some numerical examples are provided to illustrate the effectiveness of the coordinate descent (CD) method, the Gauss-Seidel method with oblique direction (GSO), the randomized coordinate descent (RCD) method (with uniform probability) and the randomized Gauss-Seidel method with oblique direction (RGSO) for solving \( A^T A x = A^T b \). All experiments are carried out using MATLAB (version R2019b) on a personal computer with 1.60 GHz central processing unit (Intel(R) Core(TM) i5-10210U CPU), 8.00 GB memory, and Windows operating system (64 bit Windows 10).

Obtained from [18], the least-squares solution set for (1) is

\[
LSS(A; b) = S(A, b_A) = \{ P_{N(A)}(x^{(0)}) + x_{LS}, x^{(0)} \in \mathbb{R}^n \},
\]

where \( LSS(A; b) \) is the set of all least solutions to (1), and \( x_{LS} \) is the unique least-squares solution of minimal Euclidean norm. For the consistent case \( b \in R(A), LSS(A; b) \) will be denoted by \( S(A; b) \). If \( b = b_A + b_A^* \), with

\[
b_A = P_{\mathbb{R}(A)}(b), \quad b_A^* = P_{N(A^T)}(b),
\]

where \( P \) denotes the orthogonal projection onto the vector subspace \( S \) of some \( \mathbb{R}^n \). From Theorem 1 and Theorem 2, we can know that the sequence \( \|x^{(k)} - \hat{x}\|_{N^A}^2 \) generated by the GSO method and the RGSO method converges to 0. Due to

\[
\|x^{(k)} - \hat{x}\|_{N^A}^2 = \|A(x^{(k)} - \hat{x})\|^2 = \|b_A + b_A^* - r^{(k)} - b_A\|^2 = \|b_A^* - r^{(k)}\|^2,
\]

where \( b_A^* \) can be known in the experimental hypothesis, and \( r^{(k)} \) is calculated in the iterative process, we can propose a iteration termination rule: The methods are terminated once residual relative error (RRE), defined by

\[
RRE = \frac{\|b_A^* - r^{(k)}\|^2}{\|b\|^2}
\]

at the current iterate \( x^{(k)} \), satisfies \( RRE < \frac{1}{2} \times 10^{-6} \) or the maximum iteration steps 500,000 being reached. If the number of iteration steps exceeds 500,000, it is denoted as "-". IT and CPU are the medians of the required iterations steps and the elapsed CPU times with respect to 50 times repeated runs of the corresponding method. To give an intuitive demonstration of the advantage, we define the speed-up as follows:

\[
speed-up^1 = \frac{\text{CPU of CD}}{\text{CPU of GSO}}, \quad \text{speed-up}^2 = \frac{\text{CPU of RGS}}{\text{CPU of RGSO}}
\]

In our implementations, all iterations are started from the initial guess \( x_0 = \text{zeros}(n, 1) \). First, set a least-squares solution \( \hat{x} \), which is generated by using the MATLAB function rand. Then set \( b_A = A\hat{x} \). When linear system is consistent, \( b_A^* = 0, b = b_A \), else \( b_A^* \in \text{null}(A^T), b = b_A + b_A^* \). When the column of the coefficient matrix \( A \) is full rank, the methods can converge to the only least-squares solution \( x^* \) under the premise of convergence.

4.1 Experiments for Random Matrix Collection in [0, 1]

The random matrix collection in [0, 1] is randomly generated by using the MATLAB function \( \text{rand} \), and the numerical results are reported in Tables 1-9. According to the characteristics of the matrix generated by MATLAB function \( \text{rand} \), Table 1 to Table 3, Table 4 to Table 6, Table 7 to Table 9.
are the experiments respectively for the overdetermined consistent linear systems, overdetermined inconsistent linear systems, and underdetermined consistent linear systems. In Table 1 to Table 6, under the premise of convergence, all methods can find the unique least-squares solution \( x^* = (A^T A)^{-1} A^T b \). In Table 7 to Table 9, all methods can find the least-squares solution under the premise of convergence, but they can’t be sure to find the same least-squares solution.

From these tables, we see that the GSO method and the RGSO method are more outstanding than the CD method and the RCD method respectively in terms of both IT and CPU with significant speed-up, regardless of whether the corresponding linear system is consistent or inconsistent. We can observe that in Tables 1-6, for the overdetermined linear systems, whether it is consistent or inconsistent, CPU and IT of all methods increase with the increase of \( n \), and the CD method is extremely sensitive to the increase of \( n \). When \( n \) increases to 100, it stops because it exceeds the maximum number of iterations. In Tables 7-9, for the underdetermined consistent linear system, CPU and IT of all methods increase with the increase of \( m \).

### Table 1: IT and CPU of CD, RCD, GSO and RGSO for \( m \times n \) matrices \( A \) with \( n = 50 \) and different \( m \) when the overdetermined linear system is consistent

| \( m \times n \) | 1000 × 50 | 2000 × 50 | 3000 × 50 | 4000 × 50 | 5000 × 50 |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| CD              | IT 73004  | 74672     | 74335     | 74608     | 74520     |
|                 | CPU 0.1605| 0.3082    | 0.5200    | 0.9833    | 1.3256    |
| GSO             | IT 11110  | 11081     | 10915     | 10951     | 10934     |
|                 | CPU 0.0379| 0.0711    | 0.1224    | 0.2412    | 0.3244    |
| speed-up        | 4.23      | 4.33      | 4.25      | 4.08      | 4.09      |
| RCD             | IT 1733   | 1596      | 1505      | 1583      | 1522      |
|                 | CPU 0.0125| 0.0151    | 0.0196    | 0.0322    | 0.0416    |
| RGSO            | IT 778    | 752       | 789       | 700       | 685       |
|                 | CPU 0.0070| 0.0086    | 0.0145    | 0.0210    | 0.0267    |
| speed-up        | 2.17      | 1.75      | 1.36      | 1.53      | 1.56      |

### Table 2: IT and CPU of CD, RCD, GSO and RGSO for \( m \times n \) matrices \( A \) with \( n = 100 \) and different \( m \) when the overdetermined linear system is consistent

| \( m \times n \) | 1000 × 100 | 2000 × 100 | 3000 × 100 | 4000 × 100 | 5000 × 100 |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| CD              | IT -      | -         | -         | -         | -         |
|                 | CPU -      | -         | -         | -         | -         |
| GSO             | IT 84810  | 81595     | 80120     | 80630     | 79131     |
|                 | CPU 0.2945| 0.5315    | 0.9227    | 1.7860    | 2.6375    |
| speed-up        | -         | -         | -         | -         | -         |
| RCD             | IT 3909   | 3304      | 3564      | 3391      | 3187      |
|                 | CPU 0.0278| 0.0318    | 0.0475    | 0.0719    | 0.0957    |
| RGSO            | IT 1567   | 1598      | 1486      | 1751      | 1432      |
|                 | CPU 0.0148| 0.0204    | 0.0264    | 0.0546    | 0.0631    |
| speed-up        | 1.88      | 1.56      | 1.80      | 1.32      | 1.52      |

### 4.2 Experiments for Random Matrix Collection in \([c, 1]\)

From example 1, it can be observed that when the columns of the matrix are nearly linear correlation, the GSO method can find the objective solution of the equation with less iteration steps and running time than the CD method. In order to verify this phenomenon, we construct several 3000 × 50 and 1000 × 3000 matrices \( A \), which entries is independent identically distributed uniform
Table 3: IT and CPU of CD, RCD, GSO and RGSO for \(m \times n\) matrices \(A\) with \(n = 150\) and different \(m\) when the overdetermined linear system is consistent

| \(m \times n\) | 1000 \(\times\) 150 | 2000 \(\times\) 150 | 3000 \(\times\) 150 | 4000 \(\times\) 150 | 5000 \(\times\) 150 |
|---------------|----------------|----------------|----------------|----------------|----------------|
| **CD**        | IT             | -             | -             | -             | -             |
|               | CPU            | 0.1591        | 0.3004        | 0.5266        | 1.0081        | 1.4170        |
| **GSO**       | IT             | 11124         | 10955         | 10875         | 10984         | 10910         |
|               | CPU            | 0.0442        | 0.0716        | 0.1337        | 0.2411        | 0.3376        |
| speed-up\(^1\) | -              | -             | -             | -             | -             | -             |
| **RCD**       | IT             | 1736          | 1786          | 1706          | 1599          | 1514          |
|               | CPU            | 0.0129        | 0.0164        | 0.0244        | 0.0338        | 0.0414        |
| speed-up\(^2\) | 1.96           | 1.60          | 1.59          | 1.48          | 1.43          | -             |

Table 4: IT and CPU of CD, RCD, GSO and RGSO for \(m \times n\) matrices \(A\) with \(n = 50\) and different \(m\) when the overdetermined linear system is inconsistent

| \(m \times n\) | 1000 \(\times\) 50  | 2000 \(\times\) 50  | 3000 \(\times\) 50  | 4000 \(\times\) 50  | 5000 \(\times\) 50  |
|---------------|----------------|----------------|----------------|----------------|----------------|
| **CD**        | IT             | 73331          | 73895          | 73910          | 74810          | 74606          |
|               | CPU            | 0.1591         | 0.3004         | 0.5266         | 1.0081         | 1.4170         |
| **GSO**       | IT             | 11124          | 10955          | 10875          | 10984          | 10910          |
|               | CPU            | 0.0442         | 0.0716         | 0.1337         | 0.2411         | 0.3376         |
| speed-up\(^1\) | -              | -              | -              | -              | -              | -              |
| **RCD**       | IT             | 1736           | 1786           | 1706           | 1599           | 1514           |
|               | CPU            | 0.0129         | 0.0164         | 0.0244         | 0.0338         | 0.0414         |
| speed-up\(^2\) | 1.91           | 1.88           | 1.72           | 1.52           | 1.26           | -              |

Table 5: IT and CPU of CD, RCD, GSO and RGSO for \(m \times n\) matrices \(A\) with \(n = 100\) and different \(m\) when the overdetermined linear system is inconsistent

| \(m \times n\) | 1000 \(\times\) 100  | 2000 \(\times\) 100  | 3000 \(\times\) 100  | 4000 \(\times\) 100  | 5000 \(\times\) 100  |
|---------------|----------------|----------------|----------------|----------------|----------------|
| **CD**        | IT             | 84415          | 84104          | 80361          | 79462          | 79572          |
|               | CPU            | 0.2829         | 0.5457         | 0.9160         | 1.7187         | 2.5587         |
| speed-up\(^1\) | -              | -              | -              | -              | -              | -              |
| **RCD**       | IT             | 3973           | 3511           | 3599           | 3092           | 3221           |
|               | CPU            | 0.0305         | 0.0329         | 0.0473         | 0.0615         | 0.0943         |
| speed-up\(^2\) | 2.14           | 1.62           | 1.70           | 1.44           | 1.42           | -              |
Table 6: IT and CPU of CD, RCD, GSO and RGSO for $m \times n$ matrices $A$ with $n = 150$ and different $m$ when the overdetermined linear system is inconsistent

| $m \times n$ | 1000 x 150 | 2000 x 150 | 3000 x 150 | 4000 x 150 | 5000 x 150 |
|--------------|-------------|-------------|-------------|-------------|-------------|
| CD IT CPU    | - - - - -   | - - - - -   | - - - - -   | - - - - -   | - - - - -   |
| GSO IT CPU   | 288578 1.0080 | 267841 0.0478 | 265105 0.0016 | 262289 0.0247 | 258320 0.0047 |
| speed-up     | 1 - - - - - | 1 - - - - - | 1 - - - - - | 1 - - - - - | 1 - - - - - |
| RCD IT CPU   | 6799 0.0478 | 5690 0.0520 | 5340 0.0690 | 4860 0.0977 | 4979 0.1390 |
| RGSO IT CPU  | 2834 0.0047 | 2472 0.0300 | 2463 0.0467 | 2475 0.0739 | 2368 0.0979 |
| speed-up     | 2 1.94 1.73 | 1.48 1.32 | 1.42 1.32 | 1.32 1.29 | 1.29 1.28 |

Table 7: IT and CPU of CD, RCD, GSO and RGSO for $m \times n$ matrices $A$ with $n = 1000$ and different $m$ when the underdetermined linear system is consistent

| $m \times n$ | 100 x 1000 | 200 x 1000 | 300 x 1000 | 400 x 1000 | 500 x 1000 |
|--------------|-------------|-------------|-------------|-------------|-------------|
| CD IT CPU    | 3805 0.0025 | 11193 0.0089 | 22638 0.0215 | 43868 0.0499 | 82643 0.1102 |
| GSO IT CPU   | 1621 0.0016 | 3544 0.0044 | 6824 0.0111 | 12339 0.0224 | 24149 0.0507 |
| speed-up     | 1 1.52 2.01 | 1.93 2.23 | 1.73 2.17 | 2.05 2.17 | 2.17 2.17 |
| RCD IT CPU   | 4113 0.0210 | 10926 0.0593 | 21267 0.1151 | 39220 0.2219 | 70545 0.4207 |
| RGSO IT CPU  | 1876 0.0102 | 4152 0.0253 | 7985 0.0497 | 13158 0.0877 | 24441 0.1680 |
| speed-up     | 2 2.05 2.34 | 2.31 2.53 | 2.50 2.50 | 2.50 2.50 | 2.50 2.50 |

Table 8: IT and CPU of CD, RCD, GSO and RGSO for $m \times n$ matrices $A$ with $n = 2000$ and different $m$ when the underdetermined linear system is consistent

| $m \times n$ | 100 x 2000 | 200 x 2000 | 300 x 2000 | 400 x 2000 | 500 x 2000 |
|--------------|-------------|-------------|-------------|-------------|-------------|
| CD IT CPU    | 3285 0.0029 | 7790 0.0071 | 13913 0.0160 | 21575 0.0314 | 32445 0.0487 |
| GSO IT CPU   | 1622 0.0022 | 3324 0.0095 | 5027 0.0156 | 7079 0.0231 | 9858 0.0231 |
| speed-up     | 1 1.35 1.39 | 1.68 2.01 | 2.11 2.11 | 2.11 2.11 | 2.11 2.11 |
| RCD IT CPU   | 3636 0.0195 | 8113 0.0488 | 14382 0.0828 | 21696 0.1320 | 31904 0.1988 |
| RGSO IT CPU  | 1741 0.0114 | 3580 0.0235 | 5892 0.0393 | 8343 0.0598 | 1745 0.0908 |
| speed-up     | 1.71 2.08 2.11 | 2.21 2.21 | 2.19 2.19 | 2.21 2.21 | 2.19 2.19 |
Table 9: IT and CPU of CD, RCD, GSO and RGSO for $m \times n$ matrices A with $n = 3000$ and different $m$ when the underdetermined linear system is consistent

| $m \times n$ | $100 \times 3000$ | $200 \times 3000$ | $300 \times 3000$ | $400 \times 3000$ | $500 \times 3000$ |
|-------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| CD IT       | 3215              | 6924              | 11717             | 17393             | 25296             |
| CPU         | 0.0029            | 0.0069            | 0.0138            | 0.0238            | 0.0384            |
| GSO IT      | 1624              | 3207              | 4940              | 6679              | 8683              |
| CPU         | 0.0025            | 0.0051            | 0.0099            | 0.0146            | 0.0214            |
| speed-up$^1$| 1.19              | 1.35              | 1.39              | 1.63              | 1.79              |
| RCD IT      | 3475              | 7633              | 12272             | 18248             | 25115             |
| CPU         | 0.0193            | 0.0427            | 0.0714            | 0.1104            | 0.1587            |
| RGSO IT     | 1686              | 3499              | 5491              | 7537              | 9966              |
| CPU         | 0.0107            | 0.0225            | 0.0377            | 0.0532            | 0.0724            |
| speed-up$^2$| 1.80              | 1.90              | 1.89              | 2.08              | 2.19              |

random variables on some interval $[c,1]$. When the value of $c$ is close to 1, the column vectors of matrix $A$ are closer to linear correlation. Note that there is nothing special about this interval, and other intervals yield the same results when the interval length remains the same.

From Table 10 to Table 12 it can be seen that no matter whether the system is consistent or inconsistent, overdetermined or underdetermined, with $c$ getting closer to 1, the CD and the RCD method have a significant increase in the number of iterations, and the speed-up$^1$ and the speed-up$^2$ also increase greatly. In Table 10 and Table 11 when $c$ increases to 0.45, the number of iterations of the CD method exceeds the maximum number of iterations. In Table 12, when $c$ increases to 0.6, the number of iterations of the CD method and RCD method exceeds the maximum number of iterations.

In this group of experiments, it can be observed that when the columns of the matrix are close to linear correlation, the GSO method and the RGSO method can find the least-squares solution more quickly than the CD method and the RCD method.

Table 10: IT and CPU of CD, RCD, GSO and RGSO for $A \in \mathbb{R}^{3000 \times 50}$ with different $c$ when the overdetermined linear system is consistent

| $c$  | 0.15 | 0.30 | 0.45 | 0.60 | 0.75 | 0.90 |
|------|------|------|------|------|------|------|
| CD IT| 141636 | 273589 | - | - | - | - |
| CPU  | 0.9638 | 1.8351 | - | - | - | - |
| GSO IT| 12201 | 12979 | 12763 | 11814 | 10126 | 7017 |
| CPU  | 0.1575 | 0.1625 | 0.1583 | 0.1519 | 0.1261 | 0.0862 |
| speed-up$^1$| 6.12 | 11.30 | - | - | - | - |
| RCD IT| 2196 | 3850 | 6828 | 13978 | 36858 | 216260 |
| CPU  | 0.0278 | 0.0483 | 0.0851 | 0.1752 | 0.4506 | 2.6451 |
| RGSO IT| 749 | 757 | 650 | 696 | 572 | 421 |
| CPU  | 0.0145 | 0.0145 | 0.0124 | 0.0132 | 0.0111 | 0.0079 |
| speed-up$^2$| 1.92 | 3.33 | 6.87 | 13.22 | 40.68 | 336.90 |

5 Conclusion

A new extension of the CD method and its randomized version, called the GSO method and the RGSO method, are proposed for solving the linear least-squares problem. The GSO method is deduced to be convergent, and an estimate of the convergence rate of the RGSO method is obtained. The GSO method and the RGSO method are proved to converge faster than the CD method and
Table 11: IT and CPU of CD, RCD, GSO and RGSO for $A \in \mathbb{R}^{3000 \times 50}$ with different $c$ when the over-determined linear system is inconsistent

| c    | 0.15  | 0.30  | 0.45  | 0.60  | 0.75  | 0.90  |
|------|-------|-------|-------|-------|-------|-------|
| CD   | IT    |       |       |       |       |       |
|      |       | 140044| 270445| -     | -     | -     |
|      | CPU   | 0.9483| 1.8366| -     | -     | -     |
| GSO  | IT    |       |       |       |       |       |
|      |       | 12075 | 12910 | 12678 | 11689 | 10118 | 7112  |
|      | CPU   | 0.1602| 0.1623| 0.1598| 0.1519| 0.1284| 0.0882|
| speed-up$^1$ |   | 5.92  | 11.32 | -     | -     | -     |
| RCD  | IT    |       |       |       |       |       |
|      |       | 2227  | 3864  | 6493  | 14256 | 37734 | 20942 |
|      | CPU   | 0.0301| 0.0479| 0.0825| 0.1783| 0.4645| 2.5826|
| speed-up$^2$ |   | 1.91  | 3.32  | 5.39  | 13.58 | 38.52 | 292.21|

Table 12: IT and CPU of CD, RCD, GSO and RGSO for $A \in \mathbb{R}^{1000 \times 3000}$ with different $c$ when the underdetermined linear system is consistent

| c    | 0.15  | 0.30  | 0.45  | 0.60  | 0.75  | 0.90  |
|------|-------|-------|-------|-------|-------|-------|
| CD   | IT    |       |       |       |       |       |
|      |       | 143373| 246942| 441147| -     | -     | -     |
|      | CPU   | 0.3344| 0.5818| 1.0359| -     | -     | -     |
| GSO  | IT    |       |       |       |       |       |
|      |       | 24358 | 23888 | 22310 | 19485 | 16795 | 11509 |
|      | CPU   | 0.1072| 0.1048| 0.0987| 0.0878| 0.0748| 0.0525|
| speed-up$^1$ |   | 3.12  | 5.55  | 10.49 | -     | -     | -     |
| RCD  | IT    |       |       |       |       |       |
|      |       | 122119| 194440| 346301| -     | -     | -     |
|      | CPU   | 0.9120| 1.4251| 2.5529| -     | -     | -     |
| RGSO | IT    |       |       |       |       |       |
|      |       | 28166 | 24936 | 24201 | 22318 | 18433 | 13717 |
|      | CPU   | 0.2711| 0.2450| 0.2318| 0.2082| 0.1813| 0.1306|
| speed-up$^2$ |   | 3.36  | 5.82  | 11.01 | -     | -     | -     |

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the RCD method, respectively. Numerical experiments show the effectiveness of the two methods, especially when the columns of coefficient matrix $A$ are close to linear correlation.

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