1. Introduction

The discovery of a new class of materials by Shechtman et al. in 1984 [1], when they were studying the diffraction figures for an alloy of Aluminum and Manganese (which gave him the Nobel Prize in Chemistry of 2011 [2]), had started up a new and vibrant research area. At the first view, this system was defined as an intermediate structure between crystalline and amorphous solids, but today it is well recognized that quasicrystals are interpreted as a natural extension of the notion of a crystal to structures with quasiperiodic (QP), instead of periodic, arrangements of atoms [3, 4]. A more recent updated definition of quasicrystals with dimensionality $n$ ($n = 1, 2$ or $3$) is that they can also be defined as a projection of a periodic structure into a higher dimensional space $mD$, where $m > n$ [5]. The diffraction figure found by Schectman and collaborators has a long-range order but has no translational periodicity as the crystals, but rather the self-similarity property by scaling [1]. In the icosahedral and decagonal quasicrystals, the self-similarity is related to the golden ratio $(\frac{1 + \sqrt{5}}{2})$, so that the atoms are separated by distances which represent the Fibonacci sequence. These new materials have great potential for applications, as some researches show that they are rigid and brittle with unique transport characteristics [6] and have very low surface energies that make them good thermal insulators with photonic and thermoelectric properties [7–9]. Their spectra features a fractal structure and aspects of electronic [10, 11] and optical [10, 12, 13] localization. The growing studies on quasicrystalline materials made it possible to obtain systems with very thin layers arranged in a QP sequence [14] or wires with quantum wells of width around 7 nm [15]. Levine et al works [3], with the synthesis of a quasicrystal defined by the Fibonacci sequence inspired Merlin et al to create, in the laboratory, the first one-dimensional (1D) quasicrystal [16]. Since that, the way of studying
1D quasicrystal structures has become standard. In this procedure, we define two distinct building blocks, each of which carries the necessary physical information, and then they are arranged according to a particular sequence. For example, they can be described in terms of a series of generations that obey a relation of particular recursion [13]. The term quasicrystal is indeed limited to a subdivision of a larger group of deterministic aperiodic morphological sets. There are artificial systems which are part of other deterministic structures that cannot be given this nomenclature (for an up-to-date reference see [17]). Examples of aperiodic structures that differ from quasicrystals are systems that obey the Rudin–Shapiro and Thue–Morse [13] sequences. The researches have shown fractal properties in their spectra and the existence of a non-trivial phase transition, such as the metal-insulating phase [14, 18], only by adjusting some parameters of the generation sequence. Recently, it was reported that it is possible to have a topological phase in photonic quasicrystals (in 2D) without any magnetic field applied, but instead introducing an artificial gauge field via dynamic modulation [19]. The idea that photonic crystals could exhibit an analog like the quantum Hall edge states was initially proposed by Haldane and Raghu [20] in tri-dimensional photonic crystals.

On the other hand, phononic crystals are intensively studied as means to manipulate sound or elastic waves (for a review see [21]) in the same way like in photonic crystals. Following the same idea of Haldane and Raghu, researches have given attention to the search of the existence of topologically protected edge states [22, 23] in those systems, which could be beneficial for many practical applications[24–26]. The topological effects in the band structure in 1D phononic crystal can be characterized through topological invariants such as Berry phases [27, 28] and Zak phases [29–31]. Recently, it was shown that these edge modes exist in the band gap of 1D phononic crystals composed of two different crystals, with distinct topological properties [32]. It also has been found in other types of phononic crystals [33]. In this work, we investigate a system where it is possible to have these edge modes in 1D, i.e. we study the edge modes in 1D phononic quasicrystals.

Indeed, 1D quasicrystals were studied through the Harper model [34]. Many works on 1D quasicrystals showed that the localization property of the Harper model could be found in a quasicrystal through the Hamiltonian of Aubry–André, considering the potential incommensurable with the lattice parameter [35–37]. This system proved to present itself as a topological insulator which exhibits border states and non-trivial phases, experimentally verified in the works of Kraus et al [38], which used waveguides to obtain the frequency spectrum in a quasicrystal, indicating the existence of a photonic gap [39, 40]. The vast majority of published papers deal with superlattices, exhibiting a fragmented energy spectrum of the famous Hofstadter butterfly [41] at the electronic level, as well as for the optical case [42].

![Figure 1. Quasiperiodic lattice with the interaction of the potential of Aubry–André. Each atom is subject to a force constant $K_n$, incommensurate with the lattice parameter.](image)

Theoretical models for predicting the properties of QP systems have been of considerable interest to the scientific community, resulting in many theoretical and experimental studies [38, 43–46]. However, some properties, like edge modes and topological states, in 1D quasicrystals, remain unexplored. The localization of phonons in 1D lattices has already been studied for the Frankel–Kontorova model [47, 48] and for QP systems by the transfer matrix formalism [49, 50]. Some works compare the frequency spectrum with the energy bands obtained for a quasicrystal defined by the transfer matrix formalism [51, 52], in which the results retain fractality properties. However, these studies do not consider the effects of the initial phase $\phi$. Therefore, in this work, we have studied the frequency spectrum, and localization of phonons in a QP lattice through modulation of the Aubry–André [53] model in order to characterize these edge states as phononic topological states, and also we studied the analogous to metal-insulator phase transition for this system.

This paper is organized in the following way. In section 2, we present the theoretical model for the aperiodic (incommensurate) 1D system studied here. In section 3, we show our numerical results and discussion. First, we present a profile spectra similar to the Hofstadter butterfly [41]. After, we present the topological states of phonons, considering an equivalent Aubry–André hopping [53], indicating the presence of border states for phonons. Also, we study the phase transition through the inverse participation rate (IPR), where the system changes from extended to localized. Finally, in section 4, we present the conclusions of this paper.

### 2. Theoretical model

The simple model for the phonon system can be defined through the motion equation that represents the atoms as a spring-bound system with force constant $K_n$.

$$\omega^2 u_n = -K_{n+1} u_{n+1} - K_n u_{n-1} + V_n u_n.$$  \hspace{1cm} (1)

Here, $\omega$ matches the vibration frequency, $u_n$ are the individual displacements around the equilibrium position, and $V_n = K_{n+1} + K_n$.

Considering a system of $N$ atoms the motion equation has the following matrix form:
The force constant $K_n$ obeys a QP modulation, undergoing small changes dependent on each site, as shown in the figure 1. Each atom has a mass $m_n = 1.0$, connected by springs with force constant $K_n$, given by:

$$K_n = C(1 + \lambda \cos(2\pi bn + \phi)).$$  \hfill (3)

The cosine term of $K_n$ comes from the interaction with the external potential of amplitude $\lambda$ on the force constant $C$. The $\phi$ variable corresponds to the initial phase when $n = 0$, $b$ is the inverse of the period of the cosine function, and, it controls the periodicity of the modulation of our spring-bound model.

This type of system presents different results, depending on whether $b$ is rational or irrational [54]. Whenever $b$ is irrational, the spring modulation is incommensurate, resulting in a QP pattern. Specifically, if $b$ is equal to $(1 + \sqrt{5})/2$, we recover the so-called Fibonacci sequence [55].

From the eigenvalues of the motion equation (2) we can obtain the frequency spectrum, while the eigenvectors give us the individual displacements for each site. The nature of these displacements $u_n$, similar to what occurs with the wavefunction, can show itself as distributed over all sites or located in just a few ones [49]. If this location is concentrated at the edge of the system, these correspond to the topological states of the border.

The location of the displacements can be obtained by the inverse of the participation rate (IPR) [35]. The IPR of the eigenvector $k$ can be obtained by the following relation:

$$\text{IPR}_{(k)} = \frac{\sum_i |u_{k,i}|^4}{\left(\sum_i |u_{k,i}|^2\right)^2},$$  \hfill (4)

where the $i$ index represents the sum over all sites on the lattice. The IPR indicates the inverse of the number of occupied sites.
sites $L$, so when the oscillations are equally distributed, the IPR $\approx 1/L$, whereas on the opposite situation of extreme localization, we have only one site vibrating with the respective frequency, which results in IPR $\approx 1$.

The numerical results were obtained from the diagonalization of equation (2), with unitary values for both the mass ($m_a = 1.0$) and the force constant amplitude ($C = 1.0$). In this way, we can study the influence of modulation of the force constant $K_{nm}$ on the frequency spectrum of the phonons in an aperiodic 1D system.

From the diagonalization of the phonon system matrix (2), we found the frequency spectrum, in order to analyze the propagation of phonons in this QP media. In the works of You et al, this model was studied for the case in which the masses of successive atoms obey a Fibonacci sequence, using the formalism of the transfer matrix [52]. They demonstrated that the spectrum is truncated in a fractal to a larger amount of atoms, also evidenced in the works of Kohomoto et al [51] and in the works of Salazar et al [50], which proposed a modulation in the equation of motion. In our model, we used the computational package of Gnu Scientific Library (GSL) [56], implemented in C++ routines to find the properties of the phonon spectrum in QP media from a sine-type modulation in the force constant between neighboring atoms, with a varying $\phi$ phase.

### 3. Results and discussion

The phonon spectrum, ruled by the motion equation with force constant modeled in (3) presents a profile similar to the Hofstadter butterfly, when plotted as a function of the parameter $b$ [41].

The equation (3) depicts a spectrum of the quasicrystal when $b = (1 + \sqrt{5})/2 \approx 1.618$. In figure 2 we have the phonon spectrum for three different values of the potential amplitude $\lambda$ as a function of $b$. For $\lambda = 0.5$, in figure 2(a), the spectrum presents bands very close while for $\lambda = 1.0$, in figure 2(c), we have the parameter which represents a critical system [57]. For this value of $\lambda$ (figure 2(c)), the allowed frequencies are defined by several bands composed by increasingly narrow gaps located between the four larger gaps, characterizing a multifractal spectrum, as we will see further in figure 3. As $\lambda$ grows, there is a deformation between the gaps and the spectrum loses this characteristic, and we can see the deformation of the larger gaps for $\lambda = 3.0$ due to intense variations in the force constant (figure 2(b)).

On the other hand, the states that cross larger gaps are sensitive to the number of atoms in the lattice. Indeed, for $N = 100$ sites, we have obtained a spectrum with certain bands circumventing the larger gaps. In figure 3 we show the frequency spectrum as a function of $b$, for 400 sites and setting the phase $\phi = \pi/2$ to obtain higher definition in the frequency spectrum of phonons.

We can see that the figure (figure 3) is similar to the Hofstadter butterfly obtained for the electronic case of the Hamiltonian of Aubry–André [53, 58]. The variations of $b$ present a characteristic of self-similarity for the frequencies, maintaining the structure composed by the larger gaps and some crossed modes. In the highlighted region (figure 3(b)), we see that the three major gaps are replicated, and the frequency spectrum follows the same pattern (figure 3(c)). The limit for this replication is ruled by the precision of step, $b$, where we consider an increment of $10^{-3}$ for a fixed phase $\phi = \pi/2$ in the equation (3).

The phonon equivalent for the Hofstadter Butterfly is subject to a strong influence from the number of sites and the $\lambda$ parameter. In figure 4, we show the frequency modes around $b = (1 + \sqrt{5})/2 \approx 1.618$, for two different values of $N$ and $\lambda$. For $N = 100$ (figures 4(a) and (c)) the bands cross only the smaller gaps, regardless of the two values of $\lambda$, 0.5 and 1.0, while increasing the number of sites up to 206 (figures 4(b) and (d)) the bands are narrowed, and these states cross all gaps. When $\lambda = 1.0$ and $N = 100$ from figure 4(c) we verify the presence of forbidden bands for any approximation value for $b = (1 + \sqrt{5})/2$, within the analyzed range (1.615 up to 1.62), similar to what occurs in the electronic case, where they arise only for the finite system, and the origin of this effect is due to the conservation of the number of particles [58].

The $\phi$ phase also modifies the shape of the spectrum, as shown in figure 5(a), where we consider a lattice with 100 atoms and $\lambda = 0.5$. We can notice that the frequencies are

---

**Figure 3.** (a) Hofstadter butterfly for the frequency spectrum in a quasicrystal as a function of parameter $b$. We use $N = 400$, $\phi = \pi/2$ and $\lambda = 1.0$. (b) We highlight the region characterized by the dashed vertical line the frequencies in the region where $b \approx 0.618$. (c) To evidence the frequency replication pattern, we make an amplification in the dotted square in (b), where the spectrum repeats in a self-similar fashion.
distributed in four separate intervals with larger gaps. Only a few modes cross the forbidden frequency gaps, with an almost sinusoidal dispersion. Below of the main panel, we can see the displacements $u_n$ (equation (2)) against the index $n$, calculated for three case, characterized by the (red in color online version) cross in main panel on the figure 5, namely $u_1$ (figure 5(b)), $u_2$ (figure 5(c)) and to the right $u_3$ (figure 5(d)). We can see that $u_1$ and $u_3$ modes are strongly localized at the edges of the system, while the calculated mode for the center, labeled by $u_2$, it is extended mode through all sites. The modes $u_1$ and $u_3$ represent the topological states of phonons in this system. They are formed by states at the edge of the QP lattice, for a set of well-specified parameters $\phi$, $\lambda$ and $N$. On another hand, on the limit when $N$ goes to infinity, with $b$ incommensurate, the bands in the phonon spectrum showed in figure 5 should not depend on the phase $\phi$. In this case we should have a banded fragmented spectrum, like those in the [59], and the border states should disappear. In this case, we can solve this model using the transfer matrix model to find the eigenvalues of equation (1) [51].

The numerical precision for the inverse of the frequency ($b$) in the force constant $K_n$ considerably alters the allowed eigenvalues in the spectrum as a function of the potential phase, as shown in figure 6. In this way, we can see that the phonon states remain crossing the gaps; however, a translation occurs
in the modes, and also a gentle deformation can be observed. All the eigenvalues were obtained from the equation (2), and they were calculated for four different \( b \) approximations, for \( \phi \) values between 0 and \( 2\pi \). The border states that emerge in the larger gaps move, arising for different phase values.

The location of the frequency ranges within the upper gap depends heavily on the \( \lambda \) parameter, which represents the amplitude of the cosine modulation in equation (3). Clearly, from figure 2 we can see that there are two behaviors in the energy (frequency) spectra: banded (where the bands are well defined) or unbanded (where the states are very narrow sets so that it is impossible to define them as a band) spectrum, depending on the parameter \( \lambda \). This will play an essential role in the referred phase transition. In order to study this phase transition, in figure 7 we present the inverse of the participation rate for the frequency values located within the upper gap (frequency values greater than 1.7). We can see that, depending on the \( \lambda \) parameter, the IPR can present a phase transition in the displacements \( u_0 \) of the lattice. For values smaller than \( \lambda = 1.0 \) the displacements are scattered across all sites representing extended states of the system, but as we increase the value of \( \lambda \), the IPR shows an intense localization (high IPR), representing a transition in the system’s oscillations, from extended displacements to localized oscillations.

The location of the displacements in the phonon spectrum is also modified, as a function of the \( \phi \) phase, as we can see in figure 8, where we added the gray (color, in online version) scale for a system with 100 sites and \( \lambda = 0.5 \). The states that cross the second largest gap present a more intense localization between the gaps in this figure (higher IPR, darker color), while the remainder is fully extended (grayish, or red in color version, i.e. lower IPR; see figure 5). When we vary the \( \lambda \) parameter up to the critical value (\( \lambda = 1.0 \)), the set of extended bands are narrowed and the states that cross the larger gaps are

Figure 6. Frequencies as functions of the \( \phi \) phase, obtained for changes in the value of \( b \) in the third decimal place with \( N = 100 \) and \( \lambda = 0.5 \). (a) \( b = 1.615 \); (b) \( b = 1.617 \); (c) \( b = 1.618 \); (d) \( b = 1.620 \).
more localized, as seen in figure 9. The frequency range where the gaps exist are very close to that ones in figure 8, mainly in the upper gap, not altering, therefore, the frequency range for emergence of the topological states.

In figure 10, we present the frequency dispersion as a function of the amplitude parameter $\lambda$, with 100 atoms in the lattice, considering the initial phase $\phi = 0$. We can see that for values greater than $\lambda = 1.0$ the spectrum presents many narrow bands spread with higher band gaps, and for $\lambda < 1.0$, we have larger and well organized bands, characterizing $\lambda = 1.0$ as a critical value for a phase transition.

4. Conclusions

The system studied here consists of an adaptation of the Hamiltonian of Aubry–André to deal with the elementary vibrations of the unidimensional quasicrystalline lattice. The equation of the eigenvalues for our case consists of a coupled system, with the elementary oscillations in each site interacting by a force constant given in the coupling with the neighboring atoms. We have used the numerical diagonalization method to find the allowed frequencies (computational package of GSL). We have found that the frequency spectrum is modified according to the $\lambda$ interaction parameter, presenting the phonon equivalent for the Hofstadter Butterfly’s for $\lambda = 1.0$, keeping the symmetry and gaps close to the one obtained in the electronic case. Also, we have verified that the number of atoms in the network influences the number of gaps and edges modes that cross these gaps. For the very precise value of the $b$ parameter (around $b = 1.618$), and choosing a given $\lambda$, it is possible to control these frequency bands.

The interaction with this quasicrystalline potential can be modulated by the $\phi$ phase, causing the atoms on the edge of the lattice to move in a coordinated manner, leading to the emergence of topological states characterized by the Hofstadter Butterfly. The ability to control the frequency bands by varying the $\lambda$ parameter opens up possibilities for the design of new electronic devices with enhanced functionalities.
Finally, by studying the individual displacements $u_n$, it is possible to define an equivalent Chern number for our properties of a 2D photonic quasicrystal \cite{19}. Therefore, boundary states, which can manifest the same topological crystal, in order to show that is possible to have localized we have considered the second way in a 1D phononic quasicrystal \cite{38}, we have a phononic topological phase exhibiting the so-called 'topological states’. Indeed, the topological properties of the 1D quasicrystals can emerge in two ways: considering the existence of phase transitions when we have a continuously deforming between two topologically distinct quasicrystals or by the appearance of edge states which traverse the bulk gaps as a function of some controllable parameter (which in this case is the initial phase $\phi$). Specifically, we have considered the second way in a 1D phononic quasicrystal, in order to show that is possible to have localized boundary states, which can manifest the same topological properties of a 2D photonic quasicrystal \cite{19}. Therefore, it is possible to define an equivalent Chern number for our case, and to classify topologically the states of the system studied here \cite{38, 61}. We will consider this in further works. Finally, by studying the individual displacements $u_n$, for a given frequency, we demonstrate that it is possible to characterize an equivalent metal-insulator phase transition in this equivalent Aubry–André model (figure 7). We hope that the results presented here would stimulate other researchers to search these topological states in other equivalent systems, generating news applications, and future devices based on this fascinating theory.

Acknowledgments

We want to thank CNPq (Conselho Nacional de Desenvolvimento Científico e tecnológico) for the partial financing. M S Vasconcelos thanks the Department of Physics and Astronomy at the University of Western Ontario for their hospitality during his sabbatical as visiting professor where this study was finished. This research also was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) of Brazil (Finance Code 88881.172293/2018-01).

ORCID iDs

M S Vasconcelos https://orcid.org/0000-0002-5118-364X

References

[1] Sheehamn D, Blech I, Gratias D and Cahn J W 1984 Phys. Rev. Lett. 53 1951
[2] Smart A G 2011 Phys. Today 64 17
[3] Levine D and Steinhardt P J 1984 Phys. Rev. Lett. 53 2477
[4] Macié E 2014 ISRN Condens. Matter Phys. 2014 165943
[5] Valy Vardeny Z, Nahata A and Agrawal A 2013 Nat. Photon. 7 177
[6] Rechtsman M C, Jeong H C, Chaikin P M, Torquato S and Steinhardt P J 2008 Phys. Rev. Lett. 101 073902
[7] Sordelet D J and Dubois M J 1997 MRS Bull. 22 34
[8] Hippiert F, Brand R, Pellloth J and Calucyrauc Y 1994 J. Phys.: Condens. Matter 6 11189
[9] Dubois J M 2000 Mater. Sci. Eng. 294–6 4
[10] Vasconcelos M S, Mauriz P W, de Medeiros F F and Albuquerque E L 2007 Phys. Rev. B 76 165117
[11] Vasconcelos M, Azevedo D, Hadad A and Galvao D 2011 J. Phys.: Condens. Matter 23 405501
[12] Vasconcelos M S and Albuquerque E L 1998 Phys. Rev. B 57 2826
[13] Vasconcelos M S and Albuquerque E L 1999 Phys. Rev. B 59 11128
[14] Dal Negro L, Oton C J, Gaburro Z, Pavesi L, Johnson P, Lagendijk A, Righini R, Colocci M and Wiersma D S 2003 Phys. Rev. Lett. 90 055501
[15] Tanese D, Gurevich E, Baboux F, Jacqmin T, Lemaître A, Galopin E, Sagnes I, Amo A, Bloch J and Akkermans E 2014 Phys. Rev. Lett. 112 144604
[16] Merlin R, Bajema K, Clarke R, Juang F Y and Bhattacharya P K 1985 Phys. Rev. Lett. 55 1768
[17] Macié E 2017 Ann. Phys. Berl. 529 1700079
[18] Kyek A, Wagner F E, Palace P, Jianu A, Macovei D, Popescu R, Manaila R and Filotti G 2000 J. Alloys Compd. 313 13
[19] Bandres M A, Rechtsman M C and Segev M 2016 Phys. Rev. X 6 011016
[20] Haldane F D M and Raghu S 2008 Phys. Rev. Lett. 100 013904
[21] Hussein M I, Leamy M J and Ruzzene M 2014 Appl. Mech. Rev. 66 040802
[22] Wang P, Lu L and Bertoldi K 2015 Phys. Rev. Lett. 115 104302
[23] Peng Y G, Qin C Z, Zhao D G, Shen Y X, Xu Y Y, Bao M, Jia H and Zhu X F 2016 Nat. Commun. 7 13368
[24] Vila J, Pal R K and Ruzzene M 2017 Phys. Rev. B 96 134307
[25] Ni X, He C, Sun X C, ping Liu X, Lu M H, Feng L and Chen Y F 2015 New J. Phys. 17 053016
[26] Khanikaev A B, Fleury R, Mousavi S H and Alu A 2015 Nat. Commun. 6 8260
[27] Xiao D, Chang M C and Niu Q 2010 Rev. Mod. Phys. 82 1959
[28] Zak J 1989 Phys. Rev. Lett. 62 2747
[29] Atala M, Aidelbursburger M, Barreteiro J T, Abanid D, Kitagawa T, Demler E and Bloch I 2013 Nat. Phys. 9 795
[30] Yang Z, Gao F and Zhang B 2017 New J. Phys. 19 040802
[31] Yil J, Ruzzene M, Wen J, Yu D, Cai L and Yue L 2018 Sci. Rep. 8 6806
[32] Fleury R, Khanikaev A B and Alu A 2016 Nat. Commun. 7 11744
[33] Susstrunk R and Huber S D 2015 Science 349 47
[34] Anderson P W 1958 Phys. Rev. 109 1492
[35] Biddle J, Priour D J, Wang B and Das Sarma S 2011 Phys. Rev. B 83 075105
[36] Ganeshan S, Sun K and Das Sarma S 2013 Phys. Rev. Lett. 111 180403
[37] Rosas G and Pérez R 1998 Mater. Lett. 36 229
[38] Kraus Y E, Lahini Y, Ringel Z, Verbin M and Zilberberg O 2012 Phys. Rev. Lett. 109 106402
[39] Zoorob M E, Charlton M D, Parker G J, Baumerg J J and Nettl M C 2000 Nature 404 740
[40] Zhang X, Zhang Z Q and Chan C T 2001 Phys. Rev. B 63 081105
[41] Hofstadter D R 1976 Phys. Rev. B 14 2239
[42] Lang L J, Cai X and Chen S 2012 Phys. Rev. Lett. 108 220401
[43] Baboux F, Levy E, Lemaître A, Gómez C, Galopin E, Le Gratiet L, Sagnes I, Amo A, Bloch J and Akkermans E 2017 Phys. Rev. B 95 161114
[44] Wang R, Röntgen M, Morfonios C V, Pinheiro F A, Schmelcher P and Negro L D 2018 Opt. Lett. 43 1986
[45] Jeevan H and Ranganathan S 2004 J. Non-Cryst. Solids 334–5 184
[46] Man W, Megens M, Steinhardt P J and Chaikin P M 2003 Nature 436 993
[47] Frenkel Y I and Kontorova T 1938 Zh. Eksp. Teor. Fiz. 8 1340
[48] Aubry S and Daeron P L 1983 Physica D 8 381
[49] Burkov S E, Koltenbah B E C and Bruch L W 1996 Phys. Rev. B 53 14179
[50] Salazar F, Wang C, Gelover-Santiago A, Zentella-Dehesa A, Naumis G and Talamantes J 2003 J. Non-Cryst. Solids 329 167
[51] Kohmoto M and Banavar J R 1986 Phys. Rev. B 34 563
[52] You J Q, Yang Q B and Yan J R 1990 Phys. Rev. B 41 7491
[53] Aubry S and André G 1980 Ann. Ist. Phys. Soc. 3 33
[54] Vanossi A, Röder J, Bishop A R and Bortolani V 2000 Phys. Rev. E 63 017203
[55] Verbin M, Zilberberg O, Kraus Y E, Lahini Y and Silberberg Y 2013 Phys. Rev. Lett. 110 076403
[56] Gough B 2009 GNU Scientific Library Reference Manual 3rd revised edn (Boston, MA: Network Theory Ltd)
[57] Wilkinson M 1984 Proc. R. Soc. Lond. A 391 305
[58] Madsen K A, Bergholtz E J and Brouwer P W 2013 Phys. Rev. B 88 125118
[59] Luck J M and Petritis D 1986 J. Stat. Phys. 42 289
[60] Bahri Y, Vosk R, Altman E and Vishwanath A 2015 Nat. Commun. 6 7341
[61] Deymier P A, Runge K and Vasseur J O 2016 AIP Adv. 6 121801