On the Amplitude of External Perturbation and Chaos via Devil’s Staircase
- Stability of Attractors -

Sadataka Furui
Graduate School of Teikyo University
2-17-12 Toyosatodai, Utsunomiya, 320-0003 Japan *

Tomoyuki Takano
IIX inc, Location 7F Daiichi Life Ins. Bld. Annex,
2-7-12 Nishi-Gotanda, Shinagawa-ku, Tokyo 141-0031, Japan †

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Abstract

We made a chaotic memristic circuit proposed by Chua and Muthuswamy-Chua, and analyzed the behavior of the voltage of the capacitor, electric current in the inductor and the voltage of the memristor by adding an external sinusoidal oscillation of a type $\gamma \omega \cos \omega t$ to $\dot{y}(t) \simeq \dot{i}_L(t)$, when the $\dot{x}(t) \simeq \dot{v}_C(t)$ is expressed by $y(t)/C$, and studied Devil’s staircase route to chaos.

We compared the frequency of the driving oscillation $f_s$ and the frequency of the response $f_d$ in the window and assigned $W = f_s/f_d$ to each window. When capacitor $C = 1$, we observe stable attractors of Farey sequences $W = \{1/2, 2/3, 3/4, 4/5, \ldots, 14/15, 1\}$, which can be interpreted as hidden attractors, while when $C = 1.2$, we observe disturbed attractors.

Frequency dependent stability of chaotic circuits due to octonions is discussed.

Keywords: Devil’s staircase, Chaos, Octonions, Chua’s circuit, Memristor.

*E-mail address: furui@umb.teikyo-u.ac.jp
†E-mail address: takano_t@iix.co.jp
1 Introduction

In 1993, Hayes, Grebogi and Ott[1] discussed an electronic circuit in a chaotic regime, using the characterization of chaos as deterministic noise defined in [2]. Typical evolution equation of the physical system is characterized by

\[ \dot{x}(t) = F_\mu(x(t)), \]  

(1)

where \( x \) is a set of coordinates in \( R^m \) and \( F_\mu(x) \) determines the nonlinear time evolution. The nature of asymptotic motion depends upon the parameter \( \mu \) and points where the change of asymptotic regime happens are called bifurcation points. The asymptotic motion follows an attractor in phase space, but its structure is not well understood, and unexpected stable attractors or chaotic attractors are observed. Chaos is characterized by sensitivity to the initial condition, or the dependence on the difference of orbits

\[ d(t) = |s'(t) - s(t)| \sim e^{\lambda_1 t}. \]  

(2)

The parameter \( \lambda_1 \) is called the largest Lyapunov coefficient.

How to measure the strangeness and the dimension of attractors was discussed by [3]. In 1984, [4] showed that there are strange attractors whose Lyapunov exponents are negative. They showed that some chaotic attractors become nonstrange by certain weak perturbations, and the structure of Strange Nonchaotic Attractor (SNA) is rather complicated.

When a compact set \( S \) is covered by \( N(r) \) cubes of edge length \( r \), and each cube \( i \) is associated with the probability \( p_i(r) \), we define the capacity dimension \( D_0 \) as

\[ D_0 = \lim_{r \to 0} \frac{N(r)}{|\log r|}, \]

and the information dimension \( D_1 \) as

\[ D_1 = \lim_{r \to 0} \frac{H(r)}{\log r}, \]

where

\[ H(r) = \sum_{i=1}^{N(r)} p_i(r) \log p_i(r). \]

A differential equation of a pendulum

\[ \frac{d^2 \phi(t)}{dt^2} + \nu \frac{d\phi(t)}{dt} + g \sin \phi(t) = F + G \sin(\omega_1 t + \alpha_1) + H \sin(\omega_2 t + \alpha_2), \]  

(3)

was studied in [5], where \( \omega_1 \) and \( \omega_2 \) are incommensurate. They defined \( \omega = \omega_1/\omega_2 \) and considered a discrete map

\[ \phi_{n+1} = F(\phi_n, \theta_n), \]  

(4)

\[ \theta_{n+1} = [\theta_n + 2\pi \omega], \]  

(5)
where a square bracket indicates modulo 2π. They conjectured that $D_0 = 2$ and $D_1 = 1$. The structure of the attractor of dynamical systems whose Lyapunov exponents are non positive and which is defined by the two variables $(x, \theta)$ was studied by \cite{6,7,8,9}. The conjecture on $D_0$ and $D_1$ were confirmed in \cite{10}. Parameters that distinguish stable chaotic configurations and disturbed chaotic configurations are important.

In 1971, Chua \cite{11} proposed that in addition to resistor, inductor and conductor, there is an interesting non-linear circuit element which is called memristor, a combination of memory and resistor. Chua and Kang \cite{12} represented the dynamical system, which is called memristive system, by

$$
\dot{x} = f(x, u, t),
\ y = g(x, u, t)u,
$$

where $u$ and $y$ denote the input and the output of the system, respectively, and $x$ denotes the state of the system.

In 2010, \cite{13} showed that one can produce a non linear memristor, and combined with inductor, capacitor and opamp, presented chaotic behaviors. The chaotic behavior of memristive system was analyzed in \cite{14,15,16} and \cite{17}.

Chaos in electronic circuits that consists of capacitors, inductors and opamp was studied in 1986 \cite{18}. When an external current source is added to the Chua’s circuit, a Farey sequence period adding law was observed in \cite{20}. Analyses of driven Chua’s circuit were done in \cite{23,24} and in \cite{25}. We analyzed the chaotic behaviors of Chua’s circuit by adding an external current in the system \cite{28}. In this paper, we add an external oscillation of current to the memristic system, which we call Muthuswamy-Chua’s circuit (MC circuit) and study changes in the chaotic behaviors due to an addition of driving currents.

Electrons are described by spinors and in the spinor theory of Cartan \cite{30}, self-dual vector field $X_i(i = 1, 2, 3, 4)$ is produced from spinors as a Plücker coordinate. Spinors are described by quaternions $\mathcal{H}$ and bispinors that consist of two quaternions with a new imaginary unit $l$ composes an octonion,

$$
\mathcal{O} = \mathcal{H} + l\mathcal{H},
$$

which has the automorphism: $G_2$ group. There are 24 dimensional bases in the $G_2$ group, and it has the triality symmetry. We study the electromagnetic field of the memristor using the octonion bases \cite{31,34,35,36}.

The paper is organized as follows. In Sect.2, we present chaos in Chua’s circuit, and in Sect.3 we present Chaos in Memristic circuit. In Sect.4, Strange nonchaotic attractor in driven memristor is studied, and in Sect.5, the resonance frequency of the driven MC circuit is analyzed by using octonion basis. Difference of stable strange chaotic attractors and disturbed strange chaotic attractor are discussed in Sect.6.

### 2 Chaos in Chua’s circuit

Chua’s circuit consists of an autonomous circuit which contains a three-segments piecewise-linear resistor, two capacitors, one inductor and a variable resistor \cite{18}. The equation of the
circuit is described by the coupled differential equation
\[
\begin{align*}
C_1 \frac{dv_{C1}}{dt} &= G(v_{C2} - v_{C1}) - \tilde{g}(v_{C1}, m_0, m_1), \\
C_2 \frac{dv_{C2}}{dt} &= G(v_{C1} - v_{C2}) + i_L, \\
L \frac{di_L}{dt} &= -v_{C2}.
\end{align*}
\] (7)

The three-segment piecewise-linear resistor which constitutes the non-linear element is characterized by
\[
\tilde{g}(v_{C1}, m_0, m_1) = (m_1 - m_0)(|v_{C1} + B_p| - |v_{C1} - B_p|)/2 + m_0 v_{C1},
\]
where $B_p$ is chosen to be 1V, $m_0$ is the slope (mA/V) outside $|v_{C1}| > B_p$, and $m_1$ is the slope inside $-B_p < v_{C1} < B_p$.

Here $v_{C1}$ and $v_{C2}$ are the voltages of the two capacitors (in V), $i_L$ is the current that flows in the inductor (in A), $C_1$ and $C_2$ are capacitance (in F), $L$ is inductance (in H), and $G$ is the conductance of the variable resistor (in $\Omega^{-1}$).

The function $\tilde{g}(v_{C1}, m_0, m_1)$ can be regarded as an active resistor. If it is locally passive, the circuit is tame, but when it is locally active, it keeps supplying power to the external circuit. The chaotic behavior is expected to be due to the power dissipated in the passive element. A difference from the van der Pol circuit is that the passive area and the active area are not separated by the strange attractor, but they are entangled. The orbit in the $v_{C1} - v_{C2}$ plane shows a double scroll orbit or a hetero-clinic orbit (i.e. an orbit which starts from a fixed point to another fixed point). We assign $v_{C1}, v_{C2}$ and $i_L$, with a specific normalization, $x, y$ and $z$, respectively.

We did not control chaotic circuits by triggering simple electronic pulse as [19], when the orbit arrives in a certain region which we assigned 'e' in Fig.8 of [28] from which complication of the orbit starts. Main difference from [19] was that we have chosen a proper height of the pulse, so that the periodic trajectory was kept and we did not need to reset the current.

\[
\begin{align*}
\dot{x} &= \alpha(y - h(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y,
\end{align*}
\] (8)

where
\[
h(x) = \begin{cases} 
  m_1 x + \frac{1}{2}(m_0 - m_1)[|x + 1| - |x - 1|], & x \geq 1, \\
  m_1 x + (m_0 - m_1) |x|, & |x| \leq 1, \\
  m_1 x - (m_0 - m_1) x, & x \leq -1
\end{cases}
\]
is a piecewise linear equation.

We observed that periodic double scroll is running on heteroclinic orbits. A periodic double scroll can be moved to a homoclinic orbit (i.e. an orbit which has a single fixed point) by triggering a pulse when the orbit arrives at a certain region. A simulation of the Chua’s circuit was done in [23]. Sinusoidally driven Chua’s circuit was shown to have the Arnold’s tongue and devil’s staircase structure in the dynamical structure of the voltage [20], and the sinusoidally-driven double scroll Chua circuit was analyzed in [24].
Bifurcation and routes to chaos in other sinusoidally-driven extended Chua circuits was discussed in [25]. In 2010, [26] claimed that there appear hidden attractors in a $n$ dimensional system with continuous vector function $\Psi(x)$ ($\Psi(0) = 0$) and a constant $n \times n$ matrix $P$,

$$\frac{dx}{dt} = Px + \Psi(x).$$

To find a periodic solution close to the harmonic oscillation, they considered a matrix $K$, such that $P_0 = P + K$ has a pair of purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$) and other eigenvalues of $P_0$ have negative real parts. They introduced a finite sequence of continuous functions $\phi^0(x), \phi^1(x), \cdots, \phi^m(x)$, such that $\phi^0(x)$ is small and $\phi^m(x) = \phi(x)$. They took sufficiently long period $[0, T]$ and considered

$$\frac{dx}{dt} = P_0 x + \phi^0(x), \quad \frac{dx}{dt} = P_0 x + \phi^j(x), \quad (j = 1, \cdots, m)$$

with initial condition $x^j(0) = x^{j-1}(T)$. They showed that chaotic attractors can be produced by the perturbation. The review of chaotic hidden attractors in Chua’s circuit is given in [27]. Our result is a realization of the hidden periodic attractor by adding a specific $\phi^j(x)$ to the Chua’s circuit, and by keeping distance from unstable points.

3 Chaos in Muthuswamy-Chua’s circuits

In the study of chaotic behaviors in electronic circuits, Chua’s electronic devices have attracted constant research interests.

It was shown in [13] that a system with an inductor, capacitor and non-linear memristor can produce a chaotic circuit. The three-element circuit with the voltage across the capacitor $x(t) = v_C(t)$, the current through the inductor $y(t) = i_L(t)$ and the internal state of the memristor $z(t)$ satisfy the equation

$$\begin{cases} 
\dot{x} = \frac{y}{C}, \\
\dot{y} = \frac{1}{L}[x + \beta(z^2 - 1)y], \\
\dot{z} = -y - \alpha z + y z.
\end{cases} \quad (9)$$

We produced the similar circuit element as MC, measured the current through the inductor, and the voltage across the capacitor. We choose

$$\begin{align*}
C &= I_sC_nT_s, \\
L &= \frac{L_nT_s}{I_s}, \\
\beta &= \frac{\beta_{5k\text{pot}}}{R}, \\
\alpha &= \frac{1}{T_sC_f\alpha_{10k\text{pot}}},
\end{align*} \quad (10)$$

where $I_s = 10000$, $C_n = 1\text{nF}$, $T_s = 10^5$, $L_n = 360\text{mH}$, $R = 1\text{k\Omega}$, $C_f = 10\text{nF}$.
The chaotic attractor \( i_L(t) = y \) versus \( v_C(t) = x \) obtained by experiment is shown in Fig.1 Here we took \( \beta = 1.43 \).

When \( \dot{z} = 0 \), the eq. (9) becomes

\[
-\alpha z + (z - 1)y = 0 \quad \text{and} \quad z = \frac{y}{y - \alpha}.
\]

MC defined Memristor equation by replacing \( y \) by \( i_M \) and \( v_L + v_c = v_M \) as

\[
v_M(t) = \beta(z^2 - 1)i_M(t),
\]

\[
\dot{z} = i_M(t) - \alpha z - i_M(t)z, \quad (11)
\]

and by choosing \( \dot{z} = 0 \), the DC characteristic becomes

\[
v_M(t) = \beta(\frac{i_M(t)^2}{(i_M(t) + \alpha)^2} - 1)i_M(t). \quad (12)
\]

When \( C=1 \), \( L=3.3 \) and \( \alpha = 0.2 \) (Set \( C = 1 \)), and the DC characteristic is chosen, solution of the eq. (12) which yields Lorentz like attractor becomes Fig.2 and the DC \( v_M - i_M \) plot becomes Fig.3.

The bifurcation diagram and the Lyapunov exponent \( \lambda \) for the set \( C = 1 \) as a function of \( \beta \) are shown in Fig.4. The diagram shows that a bifurcation occurs at \( \beta = 1.23, 1.52 \) and 1.58 and two circuits, four circuits and eight circuits, respectively appear. In the region \( 1.5 < \beta < 1.8 \) the Lyapunov exponent \( \lambda \) is positive.

The Fig.6 shows that, when \( C = 1.2 \) the Lyapunov exponent is positive in the region \( 1.3 < \beta < 1.7 \), and the oscillation is chaotic.

Ginoux, Letelier and Chua [16] calculated the flow curvature manifold of the memristor. Our result is consistent with their results.
4 Strange nonchaotic attractor in driven Muthuswamy-Chua’s circuit

To the system of Muthuswamy-Chua, we added external input voltage $F = \gamma \sin \omega t$ and considered the system of the equation

\[
\begin{align*}
\dot{x} &= \frac{y}{C}, \\
\dot{y} &= \frac{1}{L}[x + \beta(z^2 - 1)y + \gamma \sin \omega t], \\
\dot{z} &= -y - \alpha z + yz \\
\end{align*}
\]

Strange nonchaotic attractor in periodically driven systems were investigated in [29] and it was argued that without external oscillation, i.e. in autonomous systems it is difficult to find a SNA, but in periodically driven systems, SNAs had been observed. We want to investigate details of chaos and strange attractors in driven MC circuits. When $\gamma = 0.01$ i.e. very small, dependence of $y$ as a function of $\omega$ is chaotic. In Fig.6 we present the bifurcation diagram of the system with $C = 1, L = 3.3, \alpha = 0.2, \beta = 0.5, \gamma = 0.01$. However, when we changed $\gamma = 0.2$, and modified $\omega$ from 0.01 to $\omega = 1$ we found a wide window of period 1 at around $\omega = 0.51$, as shown in Fig.7.

In the region $0.51 \leq \omega \leq 0.62$ the period of the external voltage agrees with the eigenperiod of the system

\[
\frac{1}{LC}T_s = \frac{1}{\sqrt{3.3}}T_s = 0.55T_s.
\]

The corresponding frequency is $f\omega T_s = 7.3$kHz (8.8kHz in the case of $C = 1$).

The bifurcation diagrams of the system with $C = 1, L = 3.3, \alpha = 0.2, \beta = 0.5, \gamma = 0.2$ (Set $C = 1$), in the range $0.28 \leq \omega \leq 0.44$ and $0.44 \leq \omega \leq 0.515$ are shown in Fig.8 and in Fig.9 respectively, together with the frequency of the response $f_d$. 

Figure 2: The attractor in the xyz-space of $C=1, L=3.3, \alpha = 0.2, \beta = 0.5$. Numerical simulation.

Figure 3: The DC $v_M - i_M$ plot at $C = 1, L = 3.3, \alpha = 0.2$. 
Figure 4: The bifurcation diagram and the Lyapunov exponent of Memristor. $C = 1, L = 3.3$.

Corresponding diagrams for $C = 1.2, L = 3.3, \alpha = 0.2, \beta = 0.5, \gamma = 0.2$ (Set $C = 1.2$), in the range $0.24 \leq \omega \leq 0.36$ and $0.36 \leq \omega \leq 0.46$ are shown in Fig.10 and in Fig.11 respectively.

We present the Lyapunov exponent $\lambda$ in Fig.12. When $C = 1.2$ and $\omega < 0.32$, $\lambda_1$ is almost 0 and negative at quasi periodic regions, or at nodes.

The frequency of the windows $\omega$ as a function of the number of nodes of the response for samples of $C = 1$ is shown in Fig.13. In the case of one node, we put the $\omega$ at $f_d = 8$, since when we put $\omega/2\pi f_d$ as a function of the number of node $f_d$, the $\omega$ of one node divided by 8 becomes close to that of node 7. The $\omega/n$ for samples of $C = 1.2$ are shown in Fig.14. It contains the series from $f_d = 2$ to 9, from 9 to 17, from 5 to 7 and from 8 to 11, which are shown by different colors. The single period window may be considered as 8 degenerate nodes. The energy $\omega/8$ of this node put at $n = 8$ is consistent with $\omega/2\pi f_d$ of $f_d = 7$, i.e. energy per node of $f_d = \text{Mod}(-1, 8) = 7$.

The matching $\omega/2\pi f_d$ in the case of $C = 1$ is about 10% lower than that of $C = 1.2$, and the width of the window of $C = 1$ is narrower than that of $C = 1.2$.

The attractor of $n = 1$ of samples of the set $C = 1.2$ at $\omega = 0.51$ is shown in Fig.15 and the time series of $x(t), y(t), z(t)$ are compared with $\gamma \sin \omega t$ in Fig.16. Since the frequency of the response and that of driving term are both equal to 1, the ratio of $W = f_s/f_d = 1/1$

At windows of nodes of $f_d = 2$ to $f_d = 9$ of the sets $C = 1.2$, we compared the time series $x(t)$ of the response and input $\gamma \sin \omega t$. The observed ratio $W = f_s/f_d$ turned out to be

$\{1, 2, 3, 4, 5, 6, 7, 8, 1\}$.
Figure 5: The bifurcation diagram and the Lyapunov exponent of Memristor. $C = 1.2, L = 3.3$.

The sequence follows the period adding law

$$\frac{q}{p} \rightarrow \frac{q+Q}{p+P} \rightarrow \frac{q+2Q}{p+2P} \rightarrow \cdots \rightarrow \text{chaos} \rightarrow \frac{Q}{P}$$

The time series $x(t)$ and $\gamma \sin \omega t$ at $\omega = 0.4$ allows an assignment of $W = 4/5$.

Between the window of $f_d = 2$ and $f_d = 3$, we found a window of $f_d = 5 = 2 + 3$, which appears from the Farey sum

$$\frac{1}{2} + \frac{2}{3} \rightarrow \frac{3}{5}$$

A comparison of the time series $x(t)$ and the corresponding $\gamma \sin \omega t$ at $\omega = 0.315$ allows the assignment $W = 3/5$. This new node of $W = 3/5$ and the node of $W = 1/2$ make a node of $f_d = 7 = 5 + 2$. A comparison of the time series $x(t)$ and the corresponding $\gamma \sin \omega t$ at $\omega = 0.300$ allows the assignment $W = 4/7$ as shown in Fig.17. The new $W$ again appears from the Farey sum

$$\frac{1}{2} + \frac{3}{5} \rightarrow \frac{4}{7}$$

The time series of $x(t)$ and $\gamma \sin \omega t$ of $W = 6/7$ are shown in Fig.18 for comparison.

A similar comparison of $f_d = 8$ data, $W = \frac{2}{3} + \frac{3}{5} \rightarrow \frac{5}{8}$ and $W = \frac{7}{8}$ are shown in Fig.19 and Fig.20, respectively.

We compared, also $f_d = 9$ data, $W = \frac{3}{4} + \frac{4}{5} \rightarrow \frac{7}{9}$ and $W = \frac{8}{9}$. 

9
Figure 6: The bifurcation diagram $C = 1, L = 3.3, \alpha = 0.2, \beta = 0.5, \gamma = 0.01.$

Figure 7: The bifurcation diagram $C = 1, L = 3.3, \alpha = 0.2, \beta = 0.5, \gamma = 0.2.$

The series 
\[
\left\{ \frac{3}{5}, \frac{4}{7}, \cdots, \frac{1}{2} \right\}
\]
follows the period adding law\cite{20} with $q = 3, Q = 1, p = 5, P = 2.$ The series 
\[
\left\{ \frac{5}{8}, \frac{7}{9}, \cdots, \frac{2}{1} \right\}
\]
follows the period adding law with $q = 5, Q = 2, p = 8, P = 1.$

There are other Farey sums 
\[
\frac{4}{5} + \frac{5}{6} \rightarrow \frac{9}{11}, \quad \text{upto} \quad \frac{7}{8} + \frac{8}{9} \rightarrow \frac{15}{17}
\]
from $\omega = 0.41$ until $\omega = 0.4375,$ which make the series of $W$
\[
\left\{ \frac{9}{11}, \frac{11}{13}, \frac{13}{15}, \frac{15}{17}, \frac{2}{1} \right\}.
\]

Near the region of $\omega = 0.45,$ there is a window which has 10 nodes from left cluster and 9 nodes from right cluster as shown in Fig[II]. We assign $W = \left\{ \frac{9}{9}, \frac{8}{8}, \frac{7}{7} \right\}$ at $\omega = 0.446, 0.450$ and $0.4515,$ respectively, and the Farey sums 
\[
\frac{8}{9} + \frac{9}{9} \rightarrow \frac{17}{18}, \quad \text{and} \quad \frac{9}{9} + \frac{8}{8} \rightarrow \frac{17}{17}
\]
at $\omega = 0.4425$ and $\omega = 0.448,$ respectively.

The bifurcation diagram of $C = 1.2$ shows that when $\omega = 0.3605,$ the system is disturbed. We studied Poincaré map of driven MC circuit at this $\omega = 0.3605$ by taking the Poincaré map on the $xy$ plane, $yz$ plane and $xz$ plane.
The vertical coordinate is \( y = i_L \) and the variable is \( \omega \).

Figure 8: The bifurcation diagram of the driven memristor. Set \( C = 1.0, 0.28 \leq \omega \leq 0.44 \).

Figure 9: The bifurcation diagram of the driven memristor. Set \( C = 1.0, 0.44 \leq \omega \leq 0.515 \).

The points are plotted, when \( \omega t \) satisfies \( \text{Mod}[\omega t, 2\pi] = 0 \). We calculated the Poincaré map of \( \omega = 0.3605 \) on \( xy, yz \) and \( xz \) plane, which are shown in Fig. 21, 22 and 23.

In the case of oscillation in the \( xy \) plane, we calculated \( \Theta_n = \tan^{-1}(y_n/x_n) \), where \( x_n \) and \( y_n \) are the \( x \) and \( y \) coordinates of the Poincaré map. The return map of \( \Theta_{n+1} \) v.s. \( \Theta_n \) simplifies the chaotic behaviors. It becomes a single line in the case of \( \omega < 0.26 \), four clusters when \( \omega = 0.3 \), and disturbed when \( \omega = 0.32 \). When \( \omega = 0.5 \), clusters appear again, and when \( \omega = 0.6 \), a single line appears again.

We observed that except near \( \omega = 0.36 \), \( \lambda \) is negative, and the oscillation is stable in most regions.

In the case of \( C = 1.2 \), we observed \( \text{Windows} = \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9} \} \), i.e. 8 levels were observed, (The level \( \frac{9}{10} \) is broken in the high-frequency part.) and above the level \( \frac{4}{5} \) the level \( \frac{7}{9} \) in the Farey sequence \( \{ \frac{3}{4}, \frac{7}{9}, \frac{4}{5} \} \) is made and the sequence \( \{ \frac{9}{11}, \cdots, \frac{15}{17} \} \) follows.

Below the level \( \frac{7}{9} \) the Farey sequences are different, and as the source of 8 levels in the \( C = 1.2 \) case, we imagine a combination of the input electron and the output electron both expressed by quaternions. Two quaternions can make an octonion which couples with 8 dimensional vector fields.

In the case of \( C = 1.0 \) and \( \gamma = 0.2 \), \( \omega \) of the node \( W = \frac{1}{1} \) is about 10% larger than that of \( C = 1.2 \), and the node \( n \) at highest \( \omega \) was assigned up to 15, in contrast to up to 8 in the case of \( C = 1.2 \).

The width of the windows is narrower, and the sequences of \( W \) consist of

\[
\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{10}{11}, \frac{11}{12}, \frac{12}{13}, \frac{13}{14}, \frac{14}{15}, \frac{15}{16} \}.
\]
The Farey sequences of $P \neq Q$ like $W = \{\frac{3}{5}, \frac{4}{7}\}$ and $W = \{\frac{5}{8}, \frac{7}{9}\}$, which were observed in $C = 1.2$, were not observed in $C = 1.0$.

Extending this analysis to the lower frequency region, we observed the devil’s staircase structure as observed in the perturbed Van der Pol system [32, 33] in the case of $C = 1$ as Fig.24, but in the case of $C = 1.2$, we observed disturbed oscillation as Fig.25.

The qualitative difference of the case of $C = 1$ and $C = 1.2$ can be explained by the hidden attractor[26, 27]. They showed that when there is a perturbation with amplitude small enough, the Poincaré map of $F$ of the set $\Omega$ of solutions of the differential equation into itself

$$F\Omega \subset \Omega$$

can introduce stable solutions. When $C = 1.2$ the $F\Omega$ goes outside $\Omega$ and disturbed oscillations dominate, but when $C = 1$, there appear solutions of $f_s < f_d$ and hidden attractors appear.

5 Conjecture on the frequency of the windows of driven Muthuswamy-Chua’s circuit

The four component Dirac’s spinor is a combination of two two-component spinors each transforms as a quaternion. É. Cartan[30] defined, using semi-spinors of an even number of indices

$$\xi_{even} := \xi_{12}, \xi_{23}, \xi_{34}, \xi_{13}, \xi_{24}, \xi_{14}, \xi_{1234}, \xi_0,$$

and semi-spinors of an odd number of indices

$$\xi_{odd} := \xi_1, \xi_2, \xi_3, \xi_4, \xi_{234}, \xi_{134}, \xi_{124}, \xi_{123}.$$
Figure 12: The Lyapunov exponent of memristor. Set $C = 1.2, 0.25 \leq \omega \leq 0.375$ (left) and $0.375 \leq \omega \leq 0.46$ (right). Two nodes near $\omega = 0.26$ of Fig.10 and one node near $\omega = 0.46$ of Fig.11 have corresponding peaks in Fig. 12, respectively, but the correspondence of 3 nodes near 0.34 of Fig.10 in Fig.12 is not clear.

for bases of spinors.

In octonion basis, we consider a fermion $\phi$ and its charge conjugate $C\phi$\cite{31,35}, which are described by

$$A = \xi_0 I + \xi_1 a_+ + \xi_2 a_- + \xi_3 a_3,$$

$$B = \xi_1234 I - \xi_234 a_+ - \xi_314 a_- - \xi_12 a_3.$$

and anti-fermion $\psi$ and its charge conjugate $C\psi$, which are described by

$$C = \xi_0 I + \xi_1 a_+ + \xi_2 a_- + \xi_3 a_3,$$

$$D = \xi_1234 I - \xi_234 a_+ - \xi_314 a_- - \xi_12 a_3.$$
The coupling of the spinor and vectors defined by Cartan is,

\[ \mathcal{F} = t^\phi CX\psi \]

\[ = x^1(-A_1D_4 - B_2D_3 + B_3D_2 + B_4C_1) \]

\[ + x^2(B_1D_3 - A_2D_4 - B_3D_1 + B_4C_2) \]

\[ + x^3(-B_1D_2 + B_2D_1 - A_3D_4 + B_4C_3) \]

\[ + x^4(-A_1D_1 - A_2D_2 - A_3D_3 + B_4C_4) \]

\[ + x^1'(B_1C_4 - A_2C_3 - A_3C_2 - A_4D_1) \]

\[ + x^2'(A_1C_3 + B_2C_4 - A_3C_1 - A_4D_2) \]

\[ + x^3'(A_1C_2 + B_2C_1 + B_3C_4 - A_4D_3) \]

\[ + x^4'(B_1C_1 - B_2C_2 - B_3C_3 + A_1D_4). \]
In our system, there are an input electron which is represented by the plane wave, and an output fermion. There are 4 possible choices of the input fermion wave functions $A, B, C, D$, and 2 possible types of exchanged photons i.e. left vertex and right vertex are the same $x_4$ or $x_4'$ exchange, and left vertex is $x_4$ and right vertex is $x_4'$ or vice versa, which we denote $x_4/x_4'$ exchange. Thus, there are 8 types of interactions.

The exchange of $x_4$ occurs between a spinor $(\vec{A}, B_4)$

$$\begin{pmatrix} B_4 + i A_3 & i A_1 - A_2 \\ i A_1 + A_2 & B_4 - i A_3 \end{pmatrix} = (\vec{A}, B_4),$$

and a spinor $(\vec{D}, C_4)$

$$\begin{pmatrix} C_4 + i D_3 & i D_1 - D_2 \\ i D_1 + D_2 & C_4 - i D_3 \end{pmatrix} = (\vec{D}, C_4),$$

is represented in Fig.26. Here, the wave function change from $A_4 \rightarrow B_4$, leads the energy change from $E$ to $-E$, but in the wavefunction, the change $e^{iEt}$ to $e^{-iEt}$, can be interpreted as taking a time reversed hole state.

The exchange of $x_4'$ between a time reversed hole $(\vec{B}, A_4)$

$$\begin{pmatrix} A_4 + i B_3 & i B_1 - B_2 \\ i B_1 + B_2 & A_4 - i B_3 \end{pmatrix} = (\vec{B}, A_4),$$

and a time reversed hole $(\vec{C}, D_4)$

$$\begin{pmatrix} D_4 + i C_3 & i C_1 - C_2 \\ i C_1 + C_2 & D_4 - i C_3 \end{pmatrix} = (\vec{C}, D_4),$$

is represented in Fig.27.
Since the vector particle is assumed to be self-dual, or propagation of $x_4$ is same as that of $x_4'$, the hole-particle interaction between $(\vec{C}, D_4)$ and $(\vec{A}, B_4)$ occurs as represented in Fig.28.

In our analysis of MC circuit, we defined the voltage across the capacitor $x_1 = v_C(t)$, the current through the inductor $x_2 = i_L(t)$ and the internal state of the memristor $x_3 = z(t)$.

From the frequency $\omega$ of the single period oscillation observed in the widest window of the bifurcation diagram, we calculate $\omega/8$ and compare the $\omega/n$ of oscillation of periodicity equals to $n$. We observed a three periodic oscillation around $\omega = 0.375$, $\omega/3 = 0.125$, a four periodic oscillation around $\omega = 0.416$, $\omega/4 = 0.104$, and a five periodic oscillation around $\omega = 0.444$, $\omega/5 = 0.088$. $\omega$ divided by a number of periods $n$ defined as $\omega/n$ is nearly linear function of $0.18 - 0.018n$, and $n$ runs from 2 to 15.

In the case of $C = 1.2$, we observe windows of periodicity $n$ from 2 to 9. The last window of periodicity 10 is broken at the high-frequency region. In the case of $C = 1.2$, we observed windows of $n = 2, \cdots, 9$, which have the smooth $n$ dependence of $\omega/n$.

The 2nd order hole-hole interaction occurs through exchange of $x_4'$, as shown in Fig.29. The 2nd order particle-particlele interaction occurs through exchange of $x_4$. 
The Poincaré map on the $xz$ plane. $\omega = 0.3605$.

The devil’s staircase for $C = 1$.

The devil’s staircase for $C = 1.2$.

The particle-hole interaction is possible through double vector $x_4/x_4'$ exchange, as shown in Fig. 30.

6 Discussion

We studied the sinusoidally driven memristor circuit. We observed chaotic strange attractors and different from the case of driven Chua’s circuit [20], we first observed period adding low as in LCR circuit [22]. In the case of $C = 1.2$, we observed a sequence of response frequency $f_d$ stepwise increasing, and the winding number $W = f_s/f_d$ shows a sequence $W = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{8}{9}, \frac{1}{1}\}$ and Farey sum sequences $W = \{\frac{3}{5}, \frac{4}{7}, \cdots, \frac{1}{2}\}$ and $\{\frac{5}{8}, \frac{7}{9}, \cdots, \frac{2}{1}\}$. It is interesting that in the analysis of non-linear circuit of [21], the number of response $P$ of the $v_C$, which corresponds to our $f_d$, was assigned as Level-2 devil’s staircase sequence from step 2, and 8 steps $P = 2, 3, \cdots, 9$ were considered as in our case of $C = 1.2$. The driving oscillation is $P = 1$, in their cases and their responses are not hidden oscillations.

In the case of $C = 1$, we find winding number’s sequences $W = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{14}{15}, \frac{1}{1}\}$, but
we do not find the Farey sum sequences observed in $C = 1.2$. When $\omega$ of $W = 1$ is divided by 8, it becomes close to $\omega$ of $W = \frac{6}{7}$ divided by 7.

In the Level-2 devil’s staircase sequence from step 3 to step 2 of [21], 7 steps $P = 3, 5, \cdots, 15$ were considered. We observed that above the frequency of $W = 1$ defined as $\omega_{\text{res}}$, the stability is independent of $C$ [37] as [21]. It means that the way to chaos via the devil’s staircase which is observed in the case of $C = 1$ is stable, but in a low frequency region it is disturbed in the case of $C = 1.2$.

Oscillation in the voltage of two capacitances $v_{C1}, v_{C2}$ and the current $i_L$ of an inductor in nonlinear coupled circuits can be analyzed using the quaternion bases. In MC circuits, octonion plays an important role in the matching of period $\omega/2\pi f_d$. Symmetry of electromagnetic field and that of probes must be studied together. We found disturbed strange attractors, or strange attractors whose frequency is unstable, in the driven MC circuit.

The Poincaré map of the memristor current obtained by choosing the plane defined by the condition $\text{Mod}[\omega t, 2\pi] = 0$ shows a relatively complicated structure at a certain frequency region. When the amplitude of the external oscillation is small, the frequency of strange attractor remains stable, but when it is large, the frequency at a certain region becomes disturbed. The condition depends upon the capacity of the condenser $C$.

We considered vector particle interactions with electron particles and holes which are represented by octonions. It is not evident that the electrons and positrons could be repre-
Figure 29: The double vector $x'_4$ exchange between hole-hole $({\vec{B}, A}_4), (\vec{C}, D_4)$.

Figure 30: The double vector $x_4/x_4'$ exchange between hole-particle $(\vec{B}, A_4), (\vec{A}, B_4)$.

Presented by octonions, instead of a combination of quaternions, but it is worth studying the possibility of representing quarks and anti-quarks by octonions [31, 34, 35, 36].

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