OKUTSU FRAMES OF IRREDUCIBLE POLYNOMIALS OVER HENSELIAN FIELDS

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Abstract. For a henselian valued field \((K, v)\) we establish a complete parallelism between the arithmetic properties of irreducible polynomials \(F \in K[x]\), encoded by their Okutsu frames, and the valuation-theoretic properties of their induced valuations \(v_F\) on \(K[x]\), encoded by their Mac Lane-Vaqué chains. This parallelism was only known for defectless irreducible polynomials.

Introduction

The pioneering work of Mac Lane on valuations on a polynomial ring [13], was inspired in a question of Ore about the design of an algorithm to compute prime ideal decomposition in number fields [22]. To solve this question, Mac Lane used the methods of [13] to develop a polynomial factorization algorithm over the completion \(K_v\) of any discrete rank-one valued field \((K, v)\) [14]. The algorithm finds key polynomials of certain valuations on \(K[x]\), as approximations to the irreducible factors in \(K_v[x]\) of any given separable polynomial in \(K[x]\).

Motivated by the computation of integral bases in finite extensions of local fields, Okutsu constructed similar approximations without using valuations on \(K[x]\), nor key polynomials [21]. Each irreducible polynomial \(F \in K_v[x]\) admits an Okutsu frame, which is a finite list of polynomials which are best possible approximations to \(F\) among all polynomials with degree smaller than a certain bound.

Fernández, Guàrdia, Montes and Nart showed that the approaches of Ore-Mac Lane and Okutsu are essentially equivalent in the discrete rank-one case [5, 6]. The techniques of Mac Lane in [13] were extended to arbitrary valued fields independently by Vaquié [23] and Herrera-Mahboub-Olalla-Spivakovsky [7, 8]. The extension of [14] to a polynomial factorization algorithm over arbitrary henselian fields is still an open problem.

Let \((K, v)\) be a henselian valued field of an arbitrary rank. Let \(\text{Irr}(K)\) be the set of all monic irreducible polynomials in \(K[x]\). Every \(F \in \text{Irr}(K)\) determines a valuation \(v_F\) on \(K[x]\) given by

\[ v_F(f) = v(f(\theta)), \quad \text{for all } f \in K[x], \]

where \(\theta\) is any root of \(F\) in an algebraic closure of \(K\).

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This valuation $v_F$ is the end node of some finite Mac Lane-Vaquié (MLV) chain of valuations on $K(x)$

$$
\mu_0 \rightarrow \mu_1 \rightarrow \cdots \rightarrow \mu_r \rightarrow v_F
$$

where $\mu_0$ admits key polynomials of degree one, and every step $\mu_i \rightarrow \mu_{i+1}$ may be either an ordinary or a limit augmentation.

We say that $F \in \text{Irr}(K)$ is defectless if the unique extension $w$ of $v$ to $K[x]/(F)$ satisfies $\deg(F) = e(w/v) f(w/v)$; or, equivalently, if the MLV chain of $v_F$ contains only ordinary augmentations \[24, 15\].

The results of \[5, 6\] were generalized in \[15\] to approximate defectless polynomials over arbitrary henselian fields. A certain Okutsu equivalence relation $\sim_{\text{ok}}$ was defined on the set $D_{\text{less}}$ of all defectless polynomials, so that the quotient set $\text{Irr}(K)/\sim_{\text{ok}}$ admits a parametrization by a Mac Lane space $\mathcal{M}$ described in terms of valuations on $K[x]$ \[15\] Thm. 5.14. On the other hand, a complete parallelism was established between the arithmetic properties of any $F \in D_{\text{less}}$, encoded by their Okutsu frames, and the valuation-theoretic properties of the valuation $v_F$, encoded by their MLV chains \[15\] Thms. 5.5, 5.6.

In this paper, we apply the methods of \[2\] Secs. 6,7 on valuative trees to generalize all these results to arbitrary irreducible polynomials in $K[x]$. The main obstacle is the presence of limit augmentations in the MLV chains of $v_F$. This implies that there are no longer “best possible” approximations to $F$ of a given bounded degree, so that Okutsu frames need to be reformulated.

The structure of the paper is as follows. Section 1 collects preliminary results on key polynomials, valuative trees and their tangent spaces. Let $\mathcal{T}_Q$ be the tree whose nodes are extensions of $v$ to valuations on $K[x]$ taking values in the divisible hull of the group $v(K^*)$. This tree admits a certain small-extensions closure $\mathcal{T}_Q \subset \mathcal{T}_{\text{sme}}$ whose tangent space plays a key role.

In Section 2, we generalize \[15\] Thm. 5.14. We introduce an Okutsu equivalence relation $\sim_{\text{ok}}$ on the set $L_{\text{fin}}(\mathcal{T}_Q)$ of finite leaves of the tree $\mathcal{T}_Q$. After a natural identification of $L_{\text{fin}}(\mathcal{T}_Q)$ with $\text{Irr}(K)$, the quotient set $\text{Irr}(K)/\sim_{\text{ok}}$ may be parametrized by the set $\mathcal{T}_{\text{prim}}$ of primitive tangent vectors in the tangent space of $\mathcal{T}_{\text{sme}}$.

In Section 3 we show that two irreducible polynomials are Okutsu equivalent if and only if they are “sufficiently close” with respect to the classical ultrametric topology induced by $v$. Also, we see that the subset of $\mathcal{T}_{\text{prim}}$ corresponding to defectless polynomials may be identified with the Mac Lane space $\mathcal{M}$.

In Section 4, an Okutsu frame of any $F \in \text{Irr}(K)$ is defined as a list of sets of polynomials: $[\Phi_0, \ldots, \Phi_r, \Phi_{r+1} = \{F\}]$. Each $\Phi_i$ contains monic polynomials $f \in K[x]$ of constant degree $m_i$, whose weighted values $v_F(f)/\deg(f)$ are cofinal in the set of all weighted $v_F$-values of polynomials of degree less than $m_{i+1}$. These degrees satisfy

$$1 = m_0 \mid m_1 \mid \cdots \mid m_r \mid m_{r+1} = \deg(F).$$

Theorems 4.4 and 4.5 show that the Okutsu frames of $F$ and the MLV chains of $v_F$ are essentially equivalent objects. As a consequence, we obtain still another interpretation of Okutsu frames: the set $\Phi_0 \cup \cdots \cup \Phi_r \cup \{F\}$ is a complete set of abstract key polynomials of $v_F$, in the terminology of \[3, 19\].
1. Valuative trees

In this section we recall some results on valuations on a polynomial ring, valuative trees and their tangent spaces.

Let \((K, v)\) be a valued field. Let \(k\) be the residue class field, \(\Gamma = v(K^*)\) the value group and \(\Gamma_Q = \Gamma \otimes \mathbb{Q}\) the divisible hull of \(\Gamma\). The \textit{rational rank} of \(\Gamma\) is defined as 
\[
\text{rr}(\Gamma) = \dim_{\mathbb{Q}} \Gamma_Q.
\]

Let \(\Gamma \hookrightarrow \Lambda\) be an extension of ordered abelian groups. We write simply \(\Lambda \infty\) instead of \(\Lambda \cup \{\infty\}\). Consider a valuation on the polynomial ring \(K[x]\)
\[
\mu : K[x] \longrightarrow \Lambda \infty
\]
whose restriction to \(K\) is \(v\).

Let \(p = \text{supp}(\mu) := \mu^{-1}(\infty) \subseteq \text{Spec}(K[x])\) be the support of \(\mu\). The valuation \(\mu\) induces in a natural way a valuation \(\overline{\mu}\) on the field of fractions of \(K[x]/p\), which is \(K(x)\) if \(p = 0\), or \(K[x]/p\) if \(p = fK[x]\) for some \(f \in \text{Irr}(K)\).

The residue field \(k_{\mu}\) of \(\mu\) is, by definition, the residue field of \(\overline{\mu}\). The field \(\kappa(\mu)\) of \textit{algebraic residues} of \(\mu\) is the relative algebraic closure of \(k\) in \(k_{\mu}\).

We say that \(\mu\) is \textit{commensurable} (over \(v\)) if \(\Gamma_{\mu}/\Gamma\) is a torsion group; or equivalently, \(\text{rr}(\Gamma_{\mu}/\Gamma) = 0\). In this case, there is a canonical embedding \(\Gamma_{\mu} \hookrightarrow \Gamma_Q\).

All valuations on \(K[x]\) satisfy \textit{Abhyankar’s inequality}
\[
\text{rr}(\Gamma_{\mu}/\Gamma) + \text{trdeg}(k_{\mu}/k) \leq 1.
\]
This yields a basic classification of these valuations in three families:

- \(\mu\) is \textit{valuation-algebraic} if \(\text{rr}(\Gamma_{\mu}/\Gamma) = \text{trdeg}(k_{\mu}/k) = 0\).
- \(\mu\) is incommensurable if \(\text{rr}(\Gamma_{\mu}/\Gamma) = 1\).
- \(\mu\) is \textit{residue transcendental} if \(\text{trdeg}(k_{\mu}/k) = 1\).

The valuations for which equality holds in Abhyankar’s inequality, corresponding to the two latter families, are said to be \textit{valuation-transcendental}.

All valuations with nontrivial support are valuation-algebraic, but the converse statement is false.

1.1. Key polynomials. For any \(\alpha \in \Gamma_{\mu}\), consider the abelian groups:
\[
P_{\alpha} = \{g \in K[x] \mid \mu(g) \geq \alpha\} \supset P_{\alpha}^+ = \{g \in K[x] \mid \mu(g) > \alpha\}.
\]
The \textit{graded algebra} of \(\mu\) is the integral domain:
\[
\mathcal{G}_\mu := \text{gr}_\mu(K[x]) = \bigoplus_{\alpha \in \Gamma_{\mu}} P_{\alpha} / P_{\alpha}^+.
\]

There is an \textit{initial coefficient} mapping \(\text{in}_\mu : K[x] \rightarrow \mathcal{G}_\mu\), given by \(\text{in}_\mu p = 0\) and
\[
\text{in}_\mu g = g + P_{\mu(g)}^+ \in P_{\mu(g)}/P_{\mu(g)}^+ , \quad \text{if } g \in K[x] \setminus p.
\]

The following definitions translate properties of the action of \(\mu\) on \(K[x]\) into algebraic relationships in the graded algebra \(\mathcal{G}_\mu\).

**Definition.** Let \(g, h \in K[x]\).

We say that \(g, h\) are \(\mu\)-\textit{equivalent}, and we write \(g \sim_\mu h\), if \(\text{in}_\mu g = \text{in}_\mu h\).

We say that \(g\) is \(\mu\)-\textit{divisible} by \(h\), and we write \(h \mid_\mu g\), if \(\text{in}_\mu h \mid \text{in}_\mu g\) in \(\mathcal{G}_\mu\).

We say that \(g\) is \(\mu\)-\textit{irreducible} if \(\text{in}_\mu g\) is a prime element; that is, it generates a nonzero homogeneous prime ideal.
We say that \( g \) is \( \mu \)-minimal if \( g \nmid \mu f \) for all nonzero \( f \in K[x] \) with \( \deg(f) < \deg(g) \).

For all \( g \in K[x] \setminus K \) we define the truncation \( \mu_g \) as follows:

\[
f = \sum_{0 \leq s} a_s g^s, \quad \deg(a_s) < \deg(g) \quad \Longrightarrow \quad \mu_g(f) = \min \{ \mu(a_s g^s) \mid 0 \leq s \} .
\]

This function \( \mu_g \) is not necessarily a valuation, but it is useful to characterize the \( \mu \)-minimality of \( g \). Let us recall [16, Prop. 2.3].

**Lemma 1.1.** A polynomial \( g \in K[x] \setminus K \) is \( \mu \)-minimal if and only if \( \mu_g = \mu \).

**Definition.** A (Mac Lane-Vaquie) key polynomial for \( \mu \) is a monic polynomial in \( K[x] \) which is simultaneously \( \mu \)-minimal and \( \mu \)-irreducible.

The set of key polynomials for \( \mu \) is denoted \( \text{KP}(\mu) \).

All \( \phi \in \text{KP}(\mu) \) are irreducible in \( K[x] \). A tangent direction of \( \mu \) is a \( \mu \)-equivalence class \( [\phi]_{\mu} \subset \text{KP}(\mu) \) determined by all key polynomials having the same initial coefficient in \( G_\mu \). We denote the set of all tangent directions of \( \mu \) by:

\[
\text{td}(\mu) = \text{KP}(\mu)/\sim_\mu.
\]

Since all polynomials in \( [\phi]_{\mu} \) have the same degree [16, Prop. 6.6], it makes sense to consider the degree \( \deg [\phi]_{\mu} \) of a tangent direction.

The basic families of valuations can be characterized as follows by their tangent directions [2, Thms. 1.2, 1.4].

- If \( \mu \) is valuation-algebraic, then \( \text{td}(\mu) = \emptyset \). That is, \( \text{KP}(\mu) = \emptyset \).
- If \( \mu \) is incommensurable, then \( \text{td}(\mu) \) is a one-element set.
- If \( \mu \) is residue transcendental, then \( \text{td}(\mu) \) is in bijection with \( \text{Irr}(\kappa(\mu)) \).

In the latter case, the bijection is determined by the choice of a key polynomial of minimal degree and a certain homogeneous unit in \( G_\mu \) [16, Sec. 6].

**Definition.** Suppose that \( \text{KP}(\mu) \neq \emptyset \) and take \( \phi \in \text{KP}(\mu) \) of minimal degree. The following data are independent of the choice of \( \phi \):

\[
\deg(\mu) = \deg(\phi), \quad \text{sv}(\mu) = \mu(\phi), \quad \text{wt}(\mu) = \text{sv}(\mu)/\deg(\mu).
\]

They are called the degree, the singular value and the weight of \( \mu \), respectively.

**Theorem 1.2.** [16, Thm. 3.9] If \( \text{KP}(\mu) \neq \emptyset \), then for all monic \( f \in K[x] \setminus K \), we have \( \mu(f)/\deg(f) \leq \text{wt}(\mu) \). Equality holds if and only if \( f \) is \( \mu \)-minimal.

1.2. **Tangent space of a valuative tree.** Let \( T = T(\Lambda) \) be the set of all valuations \( \mu : K[x] \rightarrow \Lambda \infty \), whose restriction to \( K \) is \( \nu \). This set admits a partial ordering. For \( \mu, \nu \in T \) we define

\[
\mu \leq \nu \iff \mu(f) \leq \nu(f), \quad \forall f \in K[x].
\]

As usual, we write \( \mu < \nu \) to indicate that \( \mu \leq \nu \) and \( \mu \neq \nu \).

This poset \( T \) has the structure of a tree. By this, we simply mean that the intervals

\[
(-\infty, \mu] := \{ \rho \in T \mid \rho \leq \mu \}
\]

are totally ordered for all \( \mu \in T \) [17, Thm. 2.4].

A node \( \mu \in T \) is a leaf if it is a maximal element with respect to the ordering \( \leq \). Otherwise, we say that \( \mu \) is an inner node.
Theorem 1.3. \cite[Thm. 2.3]{17} A node $\mu \in \mathcal{T}$ is a leaf if and only if $\text{KP}(\mu) = \emptyset$.

All valuations with nontrivial support are leaves of $\mathcal{T}$. We call them finite leaves. The leaves of $\mathcal{T}$ having trivial support are called infinite leaves.

A finite leaf $\nu \in \mathcal{T}$ has $\text{supp}(\nu) = F \text{K}[x]$ for some $F \in \text{Irr}(K)$. We define

$$\text{deg}(\nu) = \text{deg}(F), \quad sv(\nu) = \text{wt}(\nu) = \infty.$$ 

The following result recalls some fundamental properties of tangent directions. It follows from \cite[Thm. 1.15]{23}, \cite[Prop. 2.2]{17} and \cite[Prop. 2.5]{2}.

Lemma 1.4. Let $\mu < \nu$ be two nodes in $\mathcal{T}$. Let $t(\mu, \nu)$ be the set of all monic polynomials $\phi \in K[x]$ of minimal degree satisfying $\mu(\phi) < \nu(\phi)$.

(i) If $\nu$ is an inner node or a finite leaf, then $\text{deg}(\mu) \leq \text{deg}(\nu)$ and $\text{wt}(\mu) < \text{wt}(\nu)$.

(ii) The set $t(\mu, \nu)$ is a tangent direction of $\mu$. Moreover, for all $\phi \in t(\mu, \nu)$, $f \in K[x]$, the equality $\mu(f) = \nu(f)$ holds if and only if $\phi \mid_{\mu} f$.

(iii) If $\mu < \nu'$ for some $\nu' \in \mathcal{T}$, then $t(\mu, \nu) = t(\mu, \nu') \iff (\mu, \nu] \cap (\mu, \nu'] \neq \emptyset$.

Definition. The tangent space of $\mathcal{T}$ is the set

$$\mathcal{T}(\mathcal{T}) = \{(\mu, t) \mid \mu \text{ inner node in } \mathcal{T}, \, t \in \text{td}(\mu)\}.$$ 

1.3. Mac Lane–Vaquié chains. For any $\phi \in \text{KP}(\mu)$ and any $\gamma \in \Lambda_{\infty}$ such that $\mu(\phi) < \gamma$, we can construct the ordinary augmented valuation $\nu = [\mu; \phi, \gamma] \in \mathcal{T}$, defined in terms of $\phi$-expansions as

$$f = \sum_{0 \leq s} a_s \phi^s, \quad \text{deg}(a_s) < \text{deg}(\phi) \implies \nu(f) = \min_{0 \leq s} \{\mu(a_s) + s\gamma\}.$$ 

Note that $\nu(\phi) = \gamma$, $\mu < \nu$ and $t(\mu, \nu) = [\phi]_\mu$.

If $\gamma < \infty$, then $\phi$ is a key polynomial for $\nu$ of minimal degree \cite[Cor. 7.3]{16}.

Let $A = (\rho_i)_{i \in A} \subset \mathcal{T}$ be a totally ordered family not admitting a maximal element. Assume that $A$ is parametrized by a totally ordered set $A$ of indices such that the mapping $A \to A$ determined by $i \mapsto \rho_i$ is an isomorphism of ordered sets.

If $\text{deg}(\rho_i)$ is stable for all sufficiently large $i \in A$, we say that $A$ has stable degree, and we denote this stable degree by $\text{deg}(A)$.

We say that $f \in K[x]$ is $A$-stable if for some index $i \in A$, it satisfies

$$\rho_i(f) = \rho_j(f), \quad \text{for all } j > i.$$ 

We obtain a stability function $\rho_A$, defined on the set of all $A$-stable polynomials by $\rho_A(f) = \max\{\rho_i(f) \mid i \in A\}$.

A limit key polynomial for $A$ is a monic $A$-unstable polynomial of minimal degree. Let $\text{KP}_{\infty}(A)$ be the set of all these limit key polynomials. Since the product of stable polynomials is stable, all limit key polynomials are irreducible in $K[x]$.

The limit degree of $A$, denoted $\text{deg}_{\infty}(A)$, is the degree of any limit key polynomial. If $\text{KP}_{\infty}(A) = \emptyset$, we agree that $\text{deg}_{\infty}(A) = \infty$.

Definition. We say that $A$ is an essential continuous family of valuations in $\mathcal{T}$ if it has stable degree and $\text{deg}(A) < \text{deg}_{\infty}(A) < \infty$.

Take any limit key polynomial $\phi \in \text{KP}_{\infty}(A)$, and any $\gamma \in \Lambda_{\infty}$ such that $\rho_i(\phi) < \gamma$ for all $i \in A$. We define the limit augmentation $\nu = [A; \phi, \gamma] \in \mathcal{T}$ as the following
mapping, defined in terms of $\phi$-expansions:

$$f = \sum_{0 \leq s} a_s \phi^s \quad \deg(a_s) < \deg(\phi) \implies \nu(f) = \min_{0 \leq s} \{\rho_A(a_s) + s\gamma\}.$$  

Since $\deg(a_s) < \deg_\infty(A)$, all coefficients $a_s$ are $A$-stable.

Note that $\nu(\phi) = \gamma$ and $\rho_i < \nu$ for all $i \in A$. If $\gamma < \infty$, then $\phi$ is a key polynomial for $\nu$ of minimal degree \cite[Cor. 7.13]{ALBERICH}.

Consider a finite chain of valuations in $\mathcal{T}$

$$\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_r, \gamma_r} \mu_r = \nu$$

in which every node is an augmentation of the previous node, of one of the two types:

Ordinary augmentation: $\mu_{n+1} = [\mu_n; \phi_{n+1}, \gamma_{n+1}]$, for some $\phi_{n+1} \in \KP(\mu_n)$.

Limit augmentation: $\mu_{n+1} = [A_n; \phi_{n+1}, \gamma_{n+1}]$, for some $\phi_{n+1} \in \KP_\infty(A_n)$, where $A_n$ is an essential continuous family containing $\mu_n$ as its first valuation.

We always consider an implicit choice of a key polynomial $\phi_0 \in \KP(\mu_0)$ of minimal degree, and we denote $\gamma_0 = \mu_0(\phi_0)$.

Therefore, for all $n$ such that $\gamma_n < \infty$, the polynomial $\phi_n$ is a key polynomial for $\mu_n$ of minimal degree, and we have

$$m_n := \deg(\mu_n) = \deg(\phi_n), \quad \sv(\mu_n) = \mu_n(\phi_n) = \gamma_n.$$  

Definition. A chain of mixed augmentations as above is said to be a Mac Lane–Vaquié (MLV) chain if $\deg(\mu_0) = 1$ and every augmentation step satisfies:

- If $\mu_n \rightarrow \mu_{n+1}$ is ordinary, then $\deg(\mu_n) < \deg(\mu_{n+1})$.
- If $\mu_n \rightarrow \mu_{n+1}$ is limit, then $\deg(\mu_n) = \deg(A_n)$ and $\phi_n \notin t(\mu_n, \mu_{n+1})$.

In an MLV chain, all nodes $\mu_n$, with $n < r$, are residue transcendental valuations and satisfy $\phi_n \notin t(\mu_n, \mu_{n+1})$. In particular, $\nu(\phi_n) = \gamma_n$ for all $n$, by Lemma \cite{ALBERICH}.

Theorem 1.5. \cite[Thm. 4.3]{ALBERICH} Let $\nu \in \mathcal{T}$ be an inner node or a finite leaf. Then, $\nu$ is the end node of a finite MLV chain.

These valuations are “bien-spectifées” in Vaquié’s terminology.

The main advantage of imposing the technical condition of MLV chain is that the nodes of the chain are essentially unique \cite[Thm. 4.7]{ALBERICH}. Thus, we may read in the chain several data intrinsically associated to the valuation $\nu$.

Definition. The depth of $\nu$ is the length of any MLV chain with end node $\nu$. The limit-depth of $\nu$ is the number of limit augmentations in this MLV chain.

An inner node or a finite leaf $\nu \in \mathcal{T}$ is said to be an inductive valuation if it has limit-depth equal to zero.

The depth-zero valuations take the form $\nu = \omega_{a, \delta}$, for some $a \in K$ and $\delta \in \Lambda\infty$. They act as follows on $(x - a)$-expansions:

$$\nu \left( \sum_{0 \leq s} a_s (x - a)^s \right) = \min \left\{ v(a_s) + s\delta \mid 0 \leq s \right\}.$$  

Clearly, $\omega_{a, \infty}$ is a finite leave of $\mathcal{T}$ with support $(x - a)K[x]$, while for $\delta < \infty$ the valuation $\omega_{a, \delta}$ is an inner node admitting $x - a$ as a key polynomial. The valuation $\omega_{a, \delta}$ is commensurable if and only if $\delta \in \Gamma_{\infty}$. We clearly have

$$\omega_{a, \delta} \leq \omega_{b, \epsilon} \iff \delta \leq \min \{ v(b - a), \epsilon \}.$$
1.4. Parametrization of equivalence classes of extensions of $v$ to $K[x]$. Let $\Gamma \hookrightarrow \Lambda$ be an extension of ordered abelian groups, and let $\Delta \subset \Lambda$ be the relative divisible hull of $\Gamma$ in $\Lambda$. We say that the extension $\Gamma \hookrightarrow \Lambda$ is small if $\Lambda/\Delta$ is a cyclic group. For instance, for every valuation $\mu$ extending $v$ to $K[x]$, the extension $\Gamma \hookrightarrow \Gamma_{\mu}$ is small \cite{10} Theorem 1.5].

In \cite{11}, a totally ordered real vector space $R_{\text{sme}}$ was constructed, which contains all small extensions of $\Gamma$ up to $\Gamma$-equivalence. This object is canonical, depending only on the set of nonzero principal convex subgroups of $\Gamma$. However, the order-preserving embeddings $\Gamma \hookrightarrow R_{\text{sme}}$ are non-canonical, because they are obtained as an application of Hahn’s embedding theorem \cite{11} Section 4].

From now on, we fix any such embedding $\Gamma \hookrightarrow R_{\text{sme}}$ and we identify $\Gamma$ with its image in $R_{\text{sme}}$. By the universal property of $R_{\text{sme}}$, for any small extension $\Gamma \hookrightarrow \Lambda$, there exists an embedding $\Lambda \hookrightarrow R_{\text{sme}}$ fitting into a commutative diagram

$$
\Lambda \\
\uparrow \\
\Gamma \hookrightarrow R_{\text{sme}}.
$$

In particular, every extension of $v$ to $K[x]$ is equivalent to some extension of $v$ taking values in $R_{\text{sme}}$.

Consider two elements $x, y \in R_{\text{sme}}$ to be equivalent if there exists an isomorphism of ordered groups between the subgroups $\langle \Gamma, x \rangle$ and $\langle \Gamma, y \rangle$, which maps $x$ to $y$ and acts as the identity on $\Gamma$. There is a canonical set of representatives $\Gamma_{\text{sme}} \subset R_{\text{sme}}$ of this equivalence relation. We have $\Gamma \subset \Gamma_Q \subset \Gamma_{\text{sme}} \subset R_{\text{sme}}$.

For each $x \in \Gamma_{\text{sme}}$ let us denote

$$
(\Gamma_Q)_{\leq x} = \{ \gamma \in \Gamma_Q \mid \gamma \leq x \}, \quad (\Gamma_Q)_{\geq x} = \{ \gamma \in \Gamma_Q \mid \gamma \geq x \}
$$

The set $\Gamma_{\text{sme}}$ satisfies the following property, easily deduced from \cite{11} Lemma 5.4].

**Proposition 1.6.** The mapping $x \mapsto ((\Gamma_Q)_{\leq x}, (\Gamma_Q)_{\geq x})$ establishes an order-preserving isomorphism between $\Gamma_{\text{sme}}$ and the set of all quasicuts in $\Gamma_Q$.

A quasicut in $\Gamma_Q$ is a pair of subsets $D = (D^L, D^R)$ such that $D^L \leq D^R$ (every element in $D^L$ is less than or equal to every element in $D^R$), $D^L \cup D^R = \Gamma_Q$ and $D^L \cap D^R$ contains at most one element. The set of quasicuts admits a total ordering:

$$
(D^L, D^R) \leq (E^L, E^R) \iff D^L \subset E^L \text{ and } D^R \supset E^R.
$$

The following result follows immediately from Proposition 1.6.

**Corollary 1.7.** Every subset $S \subset \Gamma_Q$ admits a supremum in $\Gamma_{\text{sme}}$, which we simply denote by $\text{sup}(S)$. If $S$ does not contain a maximal element, then $\text{sup}(S) \notin \Gamma_Q$.

Indeed, if $S$ contains a maximal element $\gamma$, then $\text{sup}(S) = \gamma \in \Gamma_Q$. Otherwise, $\text{sup}(S)$ is the cut $(I, \Gamma_Q \setminus I)$, where $I$ is the initial segment of $\Gamma_Q$ generated by $S$. Since $I$ contains no maximal element, necessarily $I \neq (\Gamma_Q)_{\leq \gamma}$ for all $\gamma \in \Gamma_Q$.

Denote $T_Q = T(\Gamma_Q)$. Consider the intermediate tree $T_Q \subset T_{\text{sme}} \subset T(R_{\text{sme}})$, defined as follows

$$
T_{\text{sme}} = T_Q \cup \{ \rho \in T(R_{\text{sme}}) \mid \rho \text{ inner node with } \text{sv}(\rho) \in \Gamma_{\text{sme}} \}.
$$

Note that $T_Q$ and $T_{\text{sme}}$ have the same finite leaves.
The nodes of $\mathcal{T}_{\text{sme}}$ parametrize the equivalence classes of valuations on $K[x]$ whose restriction to $K$ is equivalent to $v$ \cite[Thm. 7.1]{2}. We need to recall some particular properties of the tree $\mathcal{T}_{\text{sme}}$ which will be useful in the sequel.

### 1.4.1. Inner depth-zero nodes

The inner depth-zero nodes in $\mathcal{T}_{\text{sme}}$ are of the form $\omega_{a,\gamma}$ for $a \in K$ and $\gamma \in \Gamma_{\text{sme}}$.

Let $-\infty = \min(\Gamma_{\text{sme}})$ be the absolute minimal element in $\Gamma_{\text{sme}}$, corresponding to the quasicut $(\emptyset, \Gamma_Q)$. By (1), we have

$$
\omega_{a,-\infty} = \omega_{b,-\infty} \leq \omega_{c,\gamma} \quad \text{for all } a, b, c \in K, \gamma \in \Gamma_{\text{sme}}.
$$

**Definition.** Let us denote by $\omega_{-\infty} = \omega_{a,-\infty}$ this minimal depth-zero valuation, which is independent of $a \in K$. By Theorem \cite[1.5]{1} and (2), $\omega_{-\infty}$ is an absolute minimal node of $\mathcal{T}_{\text{sme}}$. We say that $\omega_{-\infty}$ is the root node of $\mathcal{T}_{\text{sme}}$.

Since $\mathcal{T}_Q$ has no minimal node, the root node $\omega_{-\infty}$ must be incommensurable. Hence, it has a unique tangent direction; actually,

$$
\text{KP}(\omega_{-\infty}) = \{ x - a \mid a \in K \} = [x]_{\omega_{-\infty}}.
$$

All inner depth-zero nodes in $\mathcal{T}_{\text{sme}}$ are obtained as a single ordinary augmentation of the root node:

$$
\omega_{a,\gamma} = [\omega_{-\infty}; x - a, \gamma] \quad \text{for all } a \in K, \gamma \in \Gamma_{\text{sme}}, \gamma > -\infty.
$$

### 1.4.2. Limit augmentations

Let $\mathcal{A} = (\rho_i)_{i \in A}$ be an essential continuous family in $\mathcal{T}_Q$, and let $\phi \in \text{KP}_\infty(\mathcal{A})$ be a limit key polynomial. By Corollary \cite[1.7]{1}, there exists a minimal limit augmentation of $\mathcal{A}$ in $\mathcal{T}_{\text{sme}}$ with respect to $\phi$; namely

$$
\mu_{\mathcal{A}} := [\mathcal{A}; \phi, \gamma_{\mathcal{A}}], \quad \gamma_{\mathcal{A}} := \sup \{ \rho_i(\phi) \mid i \in A \} \in \Gamma_{\text{sme}}.
$$

Since $\mathcal{A}$ has no maximal element, the family $\{ \rho_i(\phi) \mid i \in A \}$ has no maximal element either. By Corollary \cite[1.7]{1}, $\gamma_{\mathcal{A}}$ does not belong to $\Gamma_Q$.

**Lemma 1.8.** \cite[2, Lem. 7.2]{2} *The value $\gamma_{\mathcal{A}} \in \Gamma_{\text{sme}} \setminus \Gamma_Q$ and the valuation $\mu_{\mathcal{A}} \in \mathcal{T}_{\text{sme}}$ are independent of the choice of the limit key polynomial $\phi$ in $\text{KP}_\infty(\mathcal{A})$.***

Since $\mu_{\mathcal{A}}$ is incommensurable, it has a unique tangent direction. Actually,

$$
\text{KP}(\mu_{\mathcal{A}}) = [\phi]_{\mu_{\mathcal{A}}} = \{ \phi + a \mid a \in K[x], \deg(a) < \deg(\phi), \rho_{\mathcal{A}}(a) > \gamma_{\mathcal{A}} \} = \text{KP}_\infty(\mathcal{A}).
$$

Also, all limit augmentations $[\mathcal{A}; \phi, \gamma]$ are ordinary augmentations of $\mu_{\mathcal{A}}$:

$$
[\mathcal{A}; \phi, \gamma] = [\mu_{\mathcal{A}}; \phi, \gamma] \quad \text{for all } \gamma \in \Gamma_{\text{sme}}, \gamma > \gamma_{\mathcal{A}}.
$$

### 1.4.3. Primitive nodes of $\mathcal{T}_{\text{sme}}$

For the ease of the reader we shall consider the depth-zero valuations as a special case of limit augmentations.

**Convention.** We admit the empty set $\mathcal{A} = \emptyset$ as an essential continuous family in $\mathcal{T}_Q$. We agree that this family has

$$
\gamma_{\mathcal{A}} = -\infty, \quad \mu_{\mathcal{A}} = \omega_{-\infty}, \quad \text{KP}_\infty(\mathcal{A}) = \text{KP}(\mu_{\mathcal{A}}) = \{ x - a \mid a \in K \}.
$$

Consider the subset $\text{KP}_{\text{str}}(\mu) \subset \text{KP}(\mu)$ of *strong* key polynomials, defined as

$$
\text{KP}_{\text{str}}(\mu) = \{ \phi \in \text{KP}(\mu) \mid \deg(\phi) > \deg(\mu) \}.
$$

If $\text{KP}_{\text{str}}(\mu) \neq \emptyset$, then $\mu$ admits more than one tangent direction; thus, it is necessarily a residue transcendental valuation. In particular, $\text{sv}(\mu) \in \Gamma_Q$ and $\mu \in \mathcal{T}_Q$. 
Definition. A limit-primitive node in $\mathcal{T}_{\text{sme}}$ is the inner limit node $\mu_A$ associated to an essential continuous family $A$ in $\mathcal{T}_Q$.

An ordinary-primitive node in $\mathcal{T}_{\text{sme}}$ is an inner node $\mu \in \mathcal{T}_Q$ such that $\mathcal{K}_{\text{str}}(\mu) \neq \emptyset$.

A primitive node in $\mathcal{T}_{\text{sme}}$ is a node which is either limit-primitive or ordinary-

primitive. Let us denote by $\text{Prim}(\mathcal{T}_{\text{sme}})$ the set of all primitive nodes.

For any inner node $\mu \in \mathcal{T}_{\text{sme}}$ and any $\phi \in \mathcal{K}(\mu)$, consider the set of all ordinary augmentations of $\mu$ with respect to $\phi$:

$$\mathcal{P}_\mu(\phi) = \{[\mu; \phi, \gamma] \mid \gamma \in \Gamma_{\text{sme}} \infty, \ \mu(\phi) < \gamma \leq \infty\} \subset \mathcal{T}_{\text{sme}}.$$

Definition. Let $\rho \in \mathcal{T}_{\text{sme}}$ be a primitive node. Then, we define

$$\mathcal{P}(\rho) = \begin{cases} \bigcup_{\phi \in \mathcal{K}_{\text{str}}(\rho)} \mathcal{P}_\rho(\phi), & \text{if } \rho \text{ is ordinary-primitive}, \\
\{\rho\} \cup \bigcup_{\phi \in \mathcal{K}(\rho)} \mathcal{P}_\rho(\phi), & \text{if } \rho \text{ is limit-primitive}.
\end{cases}$$

Theorem 1.9. [2, Thm. 7.3] Let $\nu \in \mathcal{T}_{\text{sme}}$ be either an inner node, or a finite leaf. There exists a unique primitive node $\rho \in \text{Prim}(\mathcal{T}_{\text{sme}})$ such that $\nu \in \mathcal{P}(\rho)$.

2. Okutsu equivalence of finite leaves

Let $(K, v)$ be a valued field. Consider an algebraic closure of $K$ and the corresponding separable closure: $K \subset K^\text{sep} \subset \overline{K}$. Let $\overline{v}$ be an extension of $v$ to $\overline{K}$, and let $(K^h, \overline{v})$ be the henselization of $(K, v)$ determined by $\overline{v}$. Thus, $K^h \subset K^\text{sep}$ is the fixed field of the decomposition group of $\overline{v}$ in the Galois group $\text{Gal}(K^\text{sep}/K)$.

In this section, we introduce a certain Okutsu equivalence $\sim_{\text{ok}}$ on the set $\mathcal{L}_{\text{fin}}(\mathcal{T}_Q)$ of finite leaves of $\mathcal{T}_Q$ and we obtain a natural parametrization of the quotient set $\mathcal{L}_{\text{fin}}(\mathcal{T}_Q)/\sim_{\text{ok}}$ by a certain subset of the tangent space of $\mathcal{T}_{\text{sme}}$. Also, we use the tree structure of $\mathcal{T}_Q$ to define an ultrametric topology on the set $\mathcal{L}_{\text{fin}}(\mathcal{T}_Q)$.

For any $\phi \in \text{Irr}(K)$, we denote by $K_{\phi}$ the simple field extension $K[x]/\phi K[x]$.

2.1. Finite leaves of $\mathcal{T}_Q$. Any $F \in \text{Irr}(K^h)$ determines a valuation $v_F$ on $K^h[x]$ defined as:

$$v_F(f) = v(f(\theta)) \quad \text{for all } f \in K^h[x],$$

where $\theta \in \overline{K}$ is a root of $F$. By the henselian property, the value $v(f(\theta))$ does not depend on the choice of $\theta$. The support of $v_F$ is the prime ideal $FK^h[x]$.

Let us denote by $w_F$ the restriction of $v_F$ to $K[x]$. The support of $w_F$ is the prime ideal $\phi K[x]$, where the “norm” polynomial $\phi = N(F) \in \text{Irr}(K)$ is uniquely determined by the equality $(FK^h[x]) \cap K[x] = \phi K[x]$. Let $\overline{w}_F$ be the valuation on $K_{\phi}$ naturally induced by $w_F$.

As a consequence of the results in [4, Sec. 17], the following mapping is bijective:

$$\text{Irr}(K^h) \longrightarrow \mathcal{L}_{\text{fin}}(\mathcal{T}_Q), \quad F \longmapsto w_F.$$

Let us now recall how to describe the extensions of $v$ to the field $K_{\phi}$, for an arbitrary $\phi \in \text{Irr}(K)$. Since $K^h/K$ is a separable extension, the factorization of $\phi$ into a product of monic irreducible polynomials in $K^h[x]$ takes the form

$$\phi = F_1 \cdots F_r,$$

with pairwise different $F_1, \ldots, F_r \in \text{Irr}(K^h)$, whose norm is $N(F_i) = \phi$ for all $i$.

Theorem 2.1. [18, Cor. 3.2] The extensions of $v$ to $K_{\phi}$ are precisely $\overline{w}_{F_1}, \ldots, \overline{w}_{F_r}$.
2.2. Primitive tangent space of $T_{\text{sme}}$ and finite leaves. By Theorem 1.9 each finite leaf $f \in \mathcal{L}_{\text{fin}}(T_Q)$ belongs to $T_{\text{sme}}$. We say that $\rho$ is the previous primitive node of $f$, and we denote it by $\rho = \rho(f) \in T_{\text{sme}}$.

Since $f$ is commensurable and has nontrivial support, we have $\rho(f) \neq f$. Indeed, the ordinary-primitive nodes have trivial support, while the limit-primitive nodes are incommensurable. Consider a finite MLV chain whose end node is $f$: $\mu_0 \xrightarrow{\phi_1} \mu_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{r-1}} \mu_r = f$.

If $f$ has depth zero, then $f = \mu_0 = \omega_{a,\infty}$ for some $a \in K$. Then,

$$\rho(f) = \omega_{-\infty}, \quad \deg(f) = \deg(\rho(f)) = 1.$$

Suppose that $f$ has a positive depth and the last augmentation $\mu_{r-1} \to f$ is ordinary; that is, $f = [\mu_{r-1}; \phi_r, \infty]$. By the definition of a MLV chain, $\phi_r \in \text{KP}(\mu_{r-1})$ and $\deg(f) = \deg(\phi_r) > \deg(\mu_{r-1})$; thus, $\mu_{r-1}$ is a ordinary-primitive node and

$$\rho(f) = \mu_{r-1}, \quad \deg(f) = \deg(\phi_r) > \deg(\rho(f)).$$

Suppose that $f$ has a positive depth and $\mu_{r-1} \to f$ is a limit augmentation; that is, $f = [A; \phi_r, \infty]$ for some essential continuous family $A$ in $T_Q$ whose first valuation is $\mu_{r-1}$. Then,

$$\rho(f) = \mu_A, \quad \deg(f) = \deg(\phi_r) = \deg(\rho(f)).$$

**Definition.** On the set $\mathcal{L}_{\text{fin}}(T_Q)$, we define the equivalence relation

$$f \sim \text{ok} f' \iff \rho(f) = \rho(f') \quad \text{and} \quad t(\rho(f), f) = t(\rho(f'), f').$$

In this case, we say that the finite leaves $f$ and $f'$ are Okutsu equivalent.

We denote by $[f]_{\text{ok}}$ the Okutsu equivalence class of $f$.

**Definition.** Let $T_{\text{prim}}$ be the subset of the tangent space of $T_{\text{sme}}$ consisting of all tangent vectors based on primitive nodes:

$$T_{\text{prim}} = \{ (\rho, t) \mid \rho \in \text{Prim}(T_{\text{sme}}), t \in \text{td}(\rho) \}.$$

The next result follows immediately from Theorem 1.9 and the definition of $\sim_{\text{ok}}$.

**Theorem 2.2.** There is a canonical bijective mapping

$$T_{\text{prim}} \to \mathcal{L}_{\text{fin}}(T_Q)/\sim_{\text{ok}}, \quad (\rho, [\phi, \rho]) \mapsto ([\rho; \phi, \infty])_{\text{ok}},$$

whose inverse mapping is: $[f]_{\text{ok}} \mapsto (\rho(f), t(\rho(f), f))$.

Combined with the identification $\mathcal{L}_{\text{fin}}(T_Q) = \text{Irr}(K)$ given by [3], this result generalizes [15, Thm. 5.14] to the most general case, where no assumption at all is made on the base valued field $(K, v)$.

Let us quote some basic properties of the equivalence relation $\sim_{\text{ok}}$.

**Lemma 2.3.** Let $f, f'$ be two finite leaves in $T_Q$, and let $\text{supp}(f) = \phi K[x]$.

(i) The polynomial $\phi$ belongs to $\text{KP}(\rho(f))$ and $t(\rho(f), f) = [\phi]_{\rho(f)}$.

(ii) Let $\rho$ be a primitive node. Then, $\rho = \rho(f)$ if and only if $\rho < f$, $\phi \in \text{KP}(\rho)$ and

$$\deg(\phi) > \deg(\rho), \quad \text{if} \ \rho \text{ is ordinary-primitive}.$$

(iii) If $\rho(f)$ is a limit-primitive node, then $f \sim_{\text{ok}} f'$ if and only if $\rho(f) = \rho(f')$.

(iv) If $f \sim_{\text{ok}} f'$, then $\deg(f) = \deg(f')$. 

Proof. By the definition of \( \rho(f) \), the valuation \( f = [\rho(f); \varphi, \gamma] \) is an ordinary augmentation of \( \rho(f) \), for some \( \varphi \in \mathcal{K}P(\rho(f)), \gamma \in \Gamma_{\text{sm}} \infty \). Since \( f \) has nontrivial support, we have necessarily \( \gamma = \infty \). Then, \( \varphi \in K[x] \) is a monic irreducible polynomial such that \( \text{supp}(f) = \varphi K[x] \). This implies \( \varphi = \phi \), and this proves (i).

Let \( \rho \) be a primitive node. If \( \rho = \rho(f) \), then \( \rho < f \) and \( \phi \in \mathcal{K}P(\rho) \) by (i). Also, (5) follows from (4).

Conversely, suppose that \( \rho < f \) and \( \phi \in \mathcal{K}P(\rho) \). Since \( \rho(\phi) < \infty = \varphi(\phi) \), Lemma 1.4 shows that \( t(\rho, f) = [\varphi]_\rho \) for some \( \varphi \in \mathcal{K}P(\rho) \) such that \( \varphi \mid_\rho \phi \). By [16] Prop. 6.6], we have \( \varphi \sim_\rho \phi \), so that \( t(\rho, f) = [\phi]_\rho \). This implies, \( [\rho; \phi, \infty] = f \) because both valuations coincide on \( \phi \)-expansions, again by Lemma 1.4.

If \( \rho \) is limit-primitive, or \( \rho \) is ordinary-primitive and (5) holds, then \( f \) belongs to \( \mathcal{P}(\rho) \) and this implies \( \rho = \rho(f) \) by Theorem 1.9. This ends the proof of (ii).

If \( \rho(f) \) is a limit-primitive node, then it has a unique tangent direction; thus, for all \( f' \) such that \( \rho(f) < f' \) we have necessarily \( t(\rho(f), f) = t(\rho(f), f') \). This proves (iii).

Finally, if \( f \sim_\text{ok} f' \), then (iv) follows from (i), because \( \text{deg}(f) = \text{deg} t(\rho(f), f) = \text{deg} t(\rho(f), f') = \text{deg}(f') \). \( \Box \)

2.3. Ultrametric topology on the set of finite leaves. Let us introduce an ultrametric topology on the set \( L_{\text{fin}}(T_Q) \).

Definition. For all \( f, g \in L_{\text{fin}}(T_Q) \), we define \( u(f, g) \) as follows:

\[
 u(f, g) = \begin{cases} 
 \text{wt}(f \wedge g), & \text{if } f \neq g, \\
 \infty, & \text{if } f = g,
\end{cases}
\]

where \( f \wedge g \) is the greatest common lower node in the tree \( T_Q \) [2, Sec. 5.6].

Lemma 2.4. The function \( u : L_{\text{fin}}(T_Q) \times L_{\text{fin}}(T_Q) \rightarrow \Gamma_Q \infty \) has the following properties:

(i) \( u(f, g) = \infty \iff f = g \).

(ii) \( u(f, g) \geq \min\{u(f, h), u(h, g)\} \) for all \( h \in L_{\text{fin}}(T_Q) \).

(iii) \( u(f, g) = u(g, f) \).

Proof. Conditions (i) and (iii) follow immediately from the definition of \( u \).

Let us prove (ii). If there is any coincidence between the three leaves \( f, g, h \), then the statement of (ii) is obvious. Let us suppose that these leaves are pairwise different.

Let \( \rho = h \wedge f, \mu = h \wedge g \). Since \( u(f, g) = u(g, f) \), we may assume that \( \rho \leq \mu \). The relative position of all these nodes in the tree \( T_Q \) is the following:

We have \( u(f, g) = u(f, h) = \text{wt}(\rho) \) and \( u(g, h) = \text{wt}(\mu) \). On the other hand, Lemma 1.4 shows that \( \text{wt}(\rho) < \text{wt}(\mu) \), if \( \rho < \mu \). Thus, condition (ii) holds.

If \( u(f, h) \neq u(h, g) \), then \( \rho < \mu \) and \( u(f, g) = \text{wt}(\rho) = \min\{u(f, h), u(h, g)\} \). The inequality \( u(f, g) > \min\{u(f, h), u(h, g)\} \) holds in the following situation:
As a consequence, the function \( u \) provides a structure of ultrametric space on the set \( \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) \), and a corresponding topology [9, Sec. 1.3]. A basis for the topology is formed by the balls

\[
B_{\gamma}(f) := \{ g \in \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) \mid u(f, g) > \gamma \}, \quad f, g \in \mathcal{L}_{\text{fin}}(\mathcal{T}_Q), \quad \gamma \in \Gamma_Q.
\]

In this topology, two finite leaves \( f, g \) are “close” if the value \( u(f, g) \) is “large”.

Finally, let us remark that two finite leaves in \( \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) \) are Okutsu equivalent if and only if they are “close enough” with respect to the ultrametric topology.

**Lemma 2.5.** Let \( f, g \in \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) \). Then, the following conditions are equivalent.

(i) \( f \sim_{\text{Ok}} g \),

(ii) \( \rho(f), \rho(g) < f \wedge g \),

(iii) \( u(f, g) > \max\{\text{wt}(\rho(f)), \text{wt}(\rho(g))\} \).

**Proof.** The equivalence between (i) and (ii) follows from Theorem 1.9. The equivalence between (ii) and (iii) follows from Lemma 1.4. \( \square \)

### 3. Okutsu equivalence in the henselian case

In this section, we suppose that \((K, v)\) is henselian. We fix an algebraic closure \( \overline{K} \) of \( K \) and we still denote by \( v \) the unique extension of \( v \) to \( \overline{K} \).

Recall the canonical bijection between \( \text{Irr}(K) \) and \( \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) \) given in (3):

\[
\text{Irr}(K) \longrightarrow \mathcal{L}_{\text{fin}}(\mathcal{T}_Q), \quad F \longmapsto v_F.
\]

Our first aim is to show that, under the identification \( \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) = \text{Irr}(K) \), the ultrametric topology on the set \( \mathcal{L}_{\text{fin}}(\mathcal{T}_Q) \) introduced in Section 2 coincides with the classical topology induced by \( v \) on \( \text{Irr}(K) \).

The following result was proved in [13] for defectless polynomials and extended to arbitrary irreducible polynomials in [18, Thm. 4.4].

**Theorem 3.1.** Let \( F \in \text{Irr}(K) \) and \( \phi \in \text{KP}(\mu) \) for some valuation \( \mu \) on \( K[x] \). Then,

\[
\phi \mid_{\mu} F \iff \mu < v_F \quad \text{and} \quad t(\mu, v_F) = [\phi]_{\mu}.
\]

In this case, \( F \sim_{\mu} \phi^\ell \) with \( \ell = \deg(F)/\deg(\phi) \).

**Corollary 3.2.** Let \( F \in \text{Irr}(K) \). Then, \( F \) is \( \mu \)-minimal for all valuations \( \mu < v_F \).

**Proof.** Let \( t(\mu, v_F) = [\phi]_{\mu} \). Since \( \mu(F) < \infty = v_F(F) \), Lemma 1.4 shows that \( \phi \mid_{\mu} F \). By Theorem 3.1, \( F \sim_{\mu} \phi^\ell \) with \( \ell = \deg(F)/\deg(\phi) \). Since \( \phi \) is \( \mu \)-minimal, Theorem 1.2 shows that

\[
\frac{\mu(F)}{\deg(F)} = \frac{\mu(\phi)}{\deg(\phi)} = \text{wt}(\mu).
\]

Hence, \( F \) is \( \mu \)-minimal, again by Theorem 1.2. \( \square \)
As another consequence of Theorem 3.1, the ultrametric distance \( u \) on \( \text{Irr}(K) = \mathcal{L}_{\text{fin}}(T_{\text{OKUTSU}}) \) is given by a classical formula.

**Corollary 3.3.** For all \( F, G \in \text{Irr}(K) \) we have
\[
    u(F, G) := u(v_F, v_G) = v_F(G) / \deg(G) = v(\text{Res}(F, G)),
\]
where \( \text{Res}(F, G) \) is the resultant of the two polynomials.

**Proof.** If \( F = G \), we have \( v_F(F) = \infty = u(F, F) \). If \( F \neq G \), then \( \mu = v_F \wedge v_G \) is an inner node of \( T_Q \) satisfying \( \mu < v_F \), \( \mu < v_G \) and \( t(\mu, v_F) \neq t(\mu, v_G) \).

Let \( t(\mu, v_F) = [\phi]_\mu \), \( t(\mu, v_G) = [\varphi]_\mu \), so that \( \phi \neq \mu \varphi \). By Theorem 3.1,
\[
    G \sim_\mu \varphi^\ell, \quad \ell = \deg(G) / \deg(\varphi).
\]

Hence, \( \phi \nmid_\mu G \) and this implies \( \mu(G) = v_F(G) \) by Lemma 1.1. On the other hand, \( G \) is \( \mu \)-minimal by Corollary 3.2. By Theorem 1.2,
\[
    \frac{v_F(G)}{\deg(G)} = \frac{\mu(G)}{\deg(G)} = \text{wt}(\mu) = u(F, G).
\]

The equality \( v_F(G) / \deg(G) = v(\text{Res}(F, G)) \) is well-known.

In particular, the ultrametric topology on \( \text{Irr}(K) \) coincides with the classical topology induced by the valuation \( v \).

**Definition.** We denote by \( \rho_F := \rho(\mathfrak{f}) \) the previous primitive node of the leaf \( \mathfrak{f} = v_F \).

**Corollary 3.4.** If \( F, G \in \text{Irr}(K) \) have the same degree, then the following conditions are equivalent.

(a) \( v_F \sim_{\text{OKUTSU}} v_G \).
(b) \( u(F, G) > \max\{\text{wt}(\rho_F), \text{wt}(\rho_G)\} \).
(c) \( F \sim_{\rho_F} G \).

**Proof.** Lemma 2.5 shows that conditions (a) and (b) are equivalent. Since \( \deg(F) = \deg(G) \), we have \( \deg(F - G) < \deg(F) \), so that
\[
    \rho_F(F - G) = v_F(F - G) = v_F(G).
\]

Thus, condition (c) is equivalent to \( v_F(G) > \rho_F(F) = \deg(F) \text{wt}(\rho_F) \), which is equivalent to (b) by Corollary 3.3 and the symmetry of the argument.

An obvious comparison of Corollary 3.4 with [15, Lem. 5.13] shows that the restriction of the equivalence relation \( \sim_{\text{OKUTSU}} \) to the subset \( \text{Dless}(K) \subset \text{Irr}(K) \) of defectless polynomials, coincides with the Okutsu equivalence defined in [15].

**Definition.** An \( F \in \text{Irr}(K) \) is said to be **defectless** if \( \deg(F) = e(\overline{\nu}_F / v)f(\overline{\nu}_F / v) \), where \( \overline{\nu}_F \) is the valuation on \( K_F \) induced by \( v_F \).

Vaquié characterized this property as follows [21, 17, Cor. 6.1].

**Theorem 3.5.** An \( F \in \text{Irr}(K) \) is defectless if and only if \( v_F \) is inductive.

Since \( v_F \) is an ordinary augmentation of its previous primitive node \( \rho_F \), we see that \( F \) is defectless if and only if \( \rho_F \) is inductive.

Therefore, after identifying \( \text{Irr}(K) = \mathcal{L}_{\text{fin}}(T_Q) \) through (3), Theorem 2.2 yields a bijection between \( \text{Dless}(K) / \sim_{\text{OKUTSU}} \) and the following subset of \( T^{\text{prim}} \):
\[
    T^{\text{ind}} := \{(\rho, t) \in T^{\text{prim}} \mid \rho \text{ is inductive}\}.
\]
This subset $\mathbb{T}^{\text{ind}}$ may be easily identified with the Mac Lane space $\mathbb{M}$ of [15]. Therefore, even in the henselian case, Theorem 2.2 extends [15, Thm. 5.14] to a parametrization of Okutsu equivalence classes of arbitrary irreducible polynomials by a certain subset of the tangent space of $T_{\text{sme}}$.

4. OKUTSU FRAMES OVER HENSELIAN FIELDS

We keep assuming that our valued field $(K,v)$ is henselian. Let us fix some $F \in \text{Irr}(K)$ of degree $n > 1$. We define the weight of a non-constant $g \in K[x]$ as

$$\text{wt}(g) := v_F(g)/\deg(g) \in \Gamma_Q.$$  

4.1. Definition of Okutsu frames. For every integer $1 < m \leq n$, consider the set

$$W_m(F) = \{\text{wt}(g) \mid g \in K[x] \text{ monic, } 0 < \deg(g) < m\} \subset \Gamma_Q.$$  

The polynomial $F$ is defectless if and only if all these subsets $W_m(F)$ contain a maximal value [24], [15, Thm. 5.7].

In Section 1.4, we defined a certain set $\Gamma_{\text{sme}}$ containing $\Gamma_Q$, and we showed the existence of the following supremums:

$$w_m(F) := \sup (W_m(F)) \in \Gamma_{\text{sme}}, \quad 1 < m \leq n.$$  

By Corollary 1.4, if $W_m(F)$ does not contain a maximal element, then $w_m(F) \notin \Gamma_Q$.

Lemma 4.1. Let $\theta \in \overline{K}$ be a root of $F$ in $\overline{K}$ and consider Krasner’s constant

$$\Omega(F) = \max \{v(\theta - \theta') \mid \theta' \text{ root of } F, \ \theta' \neq \theta\} \in \Gamma_Q.$$  

If $F$ is separable, then $w_n(F) \leq \Omega(F)$.

Proof. Suppose that $\text{wt}(g) > \Omega(F)$ for some monic $g \in K[x]$ such that $0 < \deg(g) < n$. Clearly, $\text{wt}(g) = v(g(\theta))/\deg(g)$ is the average of all $v(\theta - \alpha)$ for $\alpha$ running in the multiset $Z(g)$ of roots of $g$ in $\overline{K}$, counting multiplicities. Hence, we must have $v(\theta - \alpha) > \Omega(F)$ for some $\alpha \in Z(g)$.

By Krasner’s lemma, $\theta$ is purely inseparable over $K(\alpha)$. Since $\theta$ is separable over $K$, we must have $\theta \in K(\alpha)$ and this contradicts the fact that $\deg(g) < n$. \hfill $\Box$

If $F$ is inseparable and has defect, then the set $W_n(F)$ may be unbounded in $\Gamma_Q$ in which case, $w_n(F) = \max(\Gamma_{\text{sme}})$ corresponds to the quasicut $(\Gamma_Q, \theta)$.

Lemma 4.2. If there exists a maximal element in $W_n(F)$, then any monic $\phi \in K[x]$ of minimal degree satisfying $\text{wt}(\phi) = w_n(F)$ is irreducible in $K[x]$.

Proof. Suppose $\phi = ab$ for some monic $a, b \in K[x]$ with $\deg(a), \deg(b) < \deg(\phi)$. By the minimality of $\deg(\phi)$, we have $\text{wt}(a), \text{wt}(b) < w_n(F)$; thus,

$$w_n(F) = \text{wt}(\phi) = \frac{v_F(a) + v_F(b)}{\deg(\phi)} < \frac{\deg(a)w_n(F) + \deg(b)w_n(F)}{\deg(\phi)} = w_n(F),$$

which is a contradiction. \hfill $\Box$

Suppose that max($W_n(F)$) does not exist. For all weighted values $\beta \in W_n$, let $\deg(\beta)$ be the minimal $\ell \in \mathbb{N} \cap [1,n)$ such that there exists a monic $g \in K[x]$ of degree $\ell$ such that $\beta = \text{wt}(g)$. Consider the minimal $m \in \mathbb{N} \cap [1,n)$ such that there exists a totally ordered cofinal family of constant degree $m$:

$$B = (\beta_i)_{i \in B} \subset W_n(F), \quad \deg(\beta_i) = m, \ \forall \ i \in B.$$
We may assume that the set of indices $B$ is well-ordered and the mapping $i \mapsto \beta_i$ is an isomorphism of ordered sets between $B$ and our family $(\beta_i)_{i \in B}$.

For all $i \in B$, choose a monic $\chi_i \in K[x]$ of degree $m$ such that $\beta_i = \text{wt}(\chi_i)$.

Let $A \subset B$ be the subset of all indices $i \in B$ such that $\chi_i$ is irreducible. Then the subfamily $(\beta_i)_{i \in A}$ is a final segment of $B$.

Indeed, by the minimality of $m$, there exists $\beta_i \in B$ such that $\text{wt}(g) < \beta_i$ for all monic $g \in K[x]$ of degree less than $m$. Then, similar arguments to those used in the proof of Lemma 4.2 show that $\chi_j$ is irreducible for all $j \geq i$.

In particular, the family $(\beta_i)_{i \in A}$ is still cofinal in $W_n(F)$.

Consider the following set of monic irreducible polynomials of constant degree:

$$\Phi = \begin{cases} \{\chi_i \mid i \in A\}, & \text{if } \nexists \text{max}(W_m(F)), \\ \{\phi\}, & \text{wt}(\phi) = \text{max}(W_m(F)), \text{ otherwise.} \end{cases}$$

Let $m$ be the constant degree of the polynomials in the set $\Phi$. We may apply this construction to find a set $\Phi'$ of monic irreducible polynomials of constant degree optimizing the weighted values in $W_m(F)$. An iteration of this procedure leads to a finite sequence of such sets of polynomials

\begin{equation}
[\Phi_0, \Phi_1, \ldots, \Phi_r, \Phi_{r+1} = \{F\}],
\end{equation}

whose degrees grow strictly:

\begin{equation}
1 = m_0 < m_1 < \cdots < m_{r+1} = n, \quad m_\ell = \text{deg}(\Phi_\ell), \quad 0 \leq \ell \leq r + 1,
\end{equation}

and the following property is satisfied: for any index $0 \leq \ell \leq r$ and any monic polynomial $g \in K[x]$ with $0 < \text{deg}(g) < m_{\ell+1}$, there exists $\phi \in \Phi_\ell$ such that

\begin{equation}
\text{wt}(g) \leq \text{wt}(\phi).
\end{equation}

**Definition.** An *Okutsu frame* of $F$ is a list of monic irreducible polynomials as in $(7)$, having degrees as in $(8)$, and satisfying the fundamental property $(9)$.

Clearly, we may replace each $\Phi_\ell$ with a suitable subset so that the following additional properties are satisfied, for all $0 \leq \ell \leq r$:

(OF1) $\#\Phi_\ell = 1$ whenever $\text{max}(W_m(F))$ exists.

(OF2) If $\text{max}(W_m(F))$ does not exist, we may consider a total ordering on $\Phi_\ell$ determined by the action of $v_F$:

$$\phi < \phi' \iff v_F(\phi) < v_F(\phi').$$

(OF3) For all $\phi \in \Phi_\ell$, $\varphi \in \Phi_{\ell+1}$, we have $\text{wt}(\phi) < \text{wt}(\varphi)$.

From now on, we shall assume that our Okutsu frames satisfy these additional properties. Note that $w_{m_1}(F) < \cdots < w_{m_r}(F) = w_n(F)$ and

$$w_{m_{\ell+1}}(F) = \begin{cases} \text{wt}(\phi), & \text{if } \Phi_\ell = \{\phi\}, \\ \sup \{\text{wt}(\chi_i) \mid i \in A\}, & \text{if } \Phi_\ell = \{\chi_i \mid i \in A\}. \end{cases}$$
4.2. Okutsu frames and Mac Lane–Vaquié chains. In this section, we show that any MLV chain of \( v_F \) determines an Okutsu frame of \( F \).

**Lemma 4.3.** Let \([\Phi_0, \Phi_1, \ldots, \Phi_r, \{F\}]\) be an Okutsu frame of \( F \). For \( 0 \leq \ell \leq r \), let
\[
V_{m_{\ell}} = \{ f \in K[x] \text{ monic of degree } m_{\ell} \}.
\]

Then, there exists \( \max(W_{m_{\ell+1}}) \) if and only if there exists \( \max(V_{m_{\ell}}) \).

**Proof.** If \( \max(W_{m_{\ell+1}}) \) exists, then \( \Phi_\ell = \{ \phi \} \) with \( \max(W_{m_{\ell+1}}) = \wt(\phi) \), by the property (9). Obviously, \( \max(V_{m_{\ell}}) = v_F(\phi) \).

Suppose now that \( \max(V_{m_{\ell}}) = v_F(\varphi) \) for some monic \( \varphi \in K[x] \) of degree \( m_{\ell} \). By (9), for all monic \( f \in K[x] \) with \( \deg(f) < m_{\ell+1} \) we have \( v_F(f) \leq \wt(\phi) \), for some \( \phi \in \Phi_\ell \). Since \( v_F(\phi) \leq v_F(\varphi) \), we deduce that \( \wt(\varphi) = \max(W_{m_{\ell+1}}) \). \qed

**Theorem 4.4.** Let \( F \in \Irr(K) \) and consider a MLV chain of \( v_F \):
\[
\begin{align*}
\mu_0 &\rightarrow \phi_1, \\
\mu_1 &\rightarrow \phi_2, \\
&\vdots \rightarrow \phi_r, \\
\mu_r &\rightarrow v_F.
\end{align*}
\]

Take \( \Phi_{r+1} = \{ F \} \). For each \( 0 \leq \ell \leq r \), consider the following set of monic irreducible polynomials of constant degree:
\[
\Phi_\ell = \begin{cases} 
\{ \phi_\ell \}, & \text{if } \mu_{\ell+1} = [\mu_\ell; \phi_{\ell+1}, \gamma_{\ell+1}] \\
\{ \chi_i \mid i \in A_\ell \}, & \text{if } \mu_{\ell+1} = [A_\ell; \phi_{\ell+1}, \gamma_{\ell+1}] 
\end{cases}
\]
where \( A_\ell = (\rho_i)_{i \in A_\ell} \) with \( \rho_i = [\mu_\ell; \chi_i, v_F(\chi_i)] \) for all \( i \in A_\ell \). Then, \([\Phi_0, \ldots, \Phi_r, \Phi_{r+1}]\) is an Okutsu frame of \( F \).

**Proof.** Let us first prove that the fundamental property (9) holds for all \( 0 \leq \ell \leq r \).

Suppose that \( \mu_\ell \rightarrow \mu_{\ell+1} \) is an ordinary augmentation. By the definition of a MLV chain, we have \( m_\ell = \deg(\phi_\ell) < m_{\ell+1} = \deg(\phi_{\ell+1}) \). Hence, for all monic \( g \in K[x] \) of degree \( 0 < \deg(g) < m_{\ell+1} \), we have simultaneously \( \phi_{\ell+1} \leq_{\mu_\ell} \phi_\ell \) and \( \phi_{\ell+1} \leq_{\mu_\ell} g \). Since \( t(\mu_\ell, v_F) = [\phi_{\ell+1}]_{\mu_\ell} \), Lemma [1.4] shows that \( \mu_\ell(\phi_\ell) = v_F(\phi_\ell) \) and \( \mu_\ell(g) = v_F(g) \).

Now, since \( \phi_\ell \) is a key polynomial for \( \mu_\ell \), Theorem [1.2] implies
\[
\wt(g) = \frac{\mu_\ell(g)}{\deg(g)} \leq \frac{\mu_\ell(\phi_\ell)}{m_\ell} = \wt(\phi_\ell).
\]
This ends the proof concerning all ordinary augmentation steps.

Suppose that \( \mu_\ell \rightarrow \mu_{\ell+1} \) is a limit augmentation. We mimic the above arguments, just by replacing the pair \( \mu_\ell, \phi_\ell \) with the pair \( \rho_i, \chi_i \) for a sufficiently large \( i \in A_\ell \).

Let \( \mu_{A_\ell} = [A_\ell; \phi_{\ell+1}, \gamma_{A_\ell}] \) be the minimal limit augmentation of the essential continuous family \( A_\ell \). By the definition of a MLV chain, we have
\[
m_\ell = \deg(\chi_i) < m_{\ell+1} = \deg(\phi_{\ell+1}) \quad \text{for all } i \in A_\ell.
\]

Hence, for all monic \( g \in K[x] \) of degree \( 0 < \deg(g) < m_{\ell+1} \), we have simultaneously \( \phi_{\ell+1} \leq_{\mu_\ell} \chi_i \) and \( \phi_{\ell+1} \leq_{\mu_\ell} g \). Since \( t(\mu_{A_\ell}, v_F) = [\phi_{\ell+1}]_{\mu_{A_\ell}} \), Lemma [1.4] shows that
\[
\mu_{A_\ell}(g) = v_F(g), \quad \mu_{A_\ell}(\chi_i) = v_F(\chi_i) \quad \text{for all } i \in A_\ell.
\]
On the other hand, \( m_{\ell+1} = \deg_\infty(A_\ell) \) is the unstable degree of \( A_\ell \). Hence, \( g \) and \( \chi_i \) are \( A_\ell \)-stable. Take a sufficiently large \( i \) such that
\[
\rho_i(g) = \rho_{A_\ell}(g) = \mu_{A_\ell}(g) = v_F(g).
\]
By [2] Lem. 4.12], we may assume that $\chi_j \mid_{\rho_i} \chi_i$ for all $j > i$. This implies
\[ \rho_i(\chi_i) = \rho_j(\chi_i) = \rho_{A_i}(\chi_i) = \mu_{A_i}(\chi_i) = v_F(\chi_i), \]
as well. Since $\chi_i$ is a key polynomial for $\rho_i$, Theorem 1.2 implies
\[ \text{wt}(g) = \frac{\rho_i(g)}{\text{deg}(g)} \leq \frac{\rho_i(\chi_i)}{m_{\ell}} = \text{wt}(\chi_i). \]
This ends the proof of (9).

Finally, by [17 Thm. 4.7], the augmentation step $\mu_{\ell} \to \mu_{\ell+1}$ is ordinary if and only if the set $V_{m_{\ell}}$ contains a maximal element. By Lemma 4.3, $\#\Phi_{\ell} = 1$ if and only if $W_{m_{\ell+1}}$ contains a maximal element.

In particular, the length $r + 1$ of this Okutsu frame of $F$ is equal to the Mac Lane–Vaquié depth of $v_F$.

Conversely, all Okutsu frames of $F$ arise in this way.

**Theorem 4.5.** Let $[\Phi_0, \ldots, \Phi_r, \Phi_{r+1} = \{F\}]$ be an Okutsu frame of $F \in \text{Irr}(K)$. For all $0 \leq \ell \leq r + 1$, choose an arbitrary $\phi_\ell \in \Phi_\ell$ and denote $\gamma_\ell = v_F(\phi_\ell)$. Then, the truncation of $v_F$ by $\phi_\ell$ is a valuation $\mu_\ell$ fitting into a MLV chain of $v_F$:
\[ \mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{} \cdots \xrightarrow{} \phi_{\ell-1}, \gamma_{\ell-1} \mu_{\ell-1} \xrightarrow{} \phi_{\ell}, \gamma_{\ell} \mu_\ell \xrightarrow{F, \infty} \mu_{\ell+1} = v_F. \]

If $\Phi_\ell = \{\phi_\ell\}$, then $\mu_{\ell+1} = [\mu_\ell; \phi_{\ell+1}, \gamma_{\ell+1}]$ is an ordinary augmentation.

If $\Phi_\ell = \{\chi_i \mid i \in I_\ell\}$, then $\mu_{\ell+1} = [A_\ell; \phi_{\ell+1}, \gamma_{\ell+1}]$ is a limit augmentation with respect to the essential continuous family $A_\ell = \{\mu_\ell\} \cup (\rho_i)_{i \in A_\ell}$, where $\rho_i = [\mu_\ell; \chi_i, v_F(\chi_i)]$ and $A_\ell \subset I_\ell$ contains the indices $i$ such that $v_F(\chi_i) > \gamma_\ell$.

**Proof.** Let $\phi_0 = x - a$ for some $a \in K$. Since $\text{supp}(v_F) = FK[x]$ and $\text{deg}(F) > 1$, we have $\gamma_0 := v_F(\phi_0) < \infty$. Thus, the truncation of $v_F$ by $\phi_0$ is the depth-zero valuation $\mu_0 = \omega_{a, \gamma_0} \leq v_F$. Since $\mu_0$ has trivial support, we have $\mu_0 < v_F$ and $\phi_0$ is a key polynomial for $\mu_0$ of minimal degree. Also, the equality $\mu_0(\phi_0) = \gamma_0 = v_F(\phi_0)$ shows that $\phi_0 \not\in t(\mu_0, v_F)$.

Now, suppose that for some $0 \leq \ell \leq r$ we have constructed a MLV chain of the valuation $\mu_\ell$:
\[ \mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{} \cdots \xrightarrow{} \phi_{\ell}, \gamma_{\ell} \mu_\ell < v_F, \]
satisfying all conditions of the theorem, for all indices $0, \ldots, \ell$. Since $\mu_\ell(\phi_\ell) = \gamma_\ell = v_F(\phi_\ell)$, we have $\phi_\ell \not\in t(\mu_\ell, v_F)$.

Let us construct a further step of the MLV chain, satisfying the conditions of the theorem for the index $\ell + 1$ as well.

Denote $m_\ell = \text{deg}(\phi_\ell)$ for all $0 \leq \ell \leq r$. Suppose that $\Phi_\ell = \{\phi_\ell\}$. Since $\phi_\ell$ is a key polynomial for $\mu_\ell$ of minimal degree, Theorem 1.2 shows that
\[ \frac{\mu_\ell(\phi_{\ell+1})}{m_{\ell+1}} \leq \frac{\mu_\ell(\phi_\ell)}{m_\ell} = \text{wt}(\phi_\ell) < \text{wt}(\phi_{\ell+1}). \]
Hence, $\mu_\ell(\phi_{\ell+1}) < v_F(\phi_{\ell+1})$. We claim that $m_{\ell+1}$ is the least degree of a monic polynomial in $K[x]$ satisfying this inequality. Indeed, suppose that $g \in K[x]$ is a
monic polynomial of minimal degree such that $\mu_\ell(g) < v_F(g)$. By Lemma 1.4, $g$ is a key polynomial for $\mu_\ell$; hence, it is $\mu_\ell$-minimal and Theorem 1.2 shows that

$$\frac{\mu_\ell(g)}{\deg(g)} = \frac{\mu_\ell(\phi_\ell)}{m_\ell} = wt(\mu_\ell).$$

If $\deg(g) < m_{\ell+1}$, the fundamental property (9) would imply:

$$wt(g) \leq wt(\phi_\ell) = \frac{\mu_\ell(\phi_\ell)}{m_\ell} = \frac{\mu_\ell(g)}{\deg(g)},$$

contradicting our initial assumption.

Thus, $\phi_{\ell+1} \in K[x]$ is a monic polynomial of minimal degree satisfying $\mu_\ell(\phi_{\ell+1}) < v_F(\phi_{\ell+1})$. By Lemma 1.4, $\phi_{\ell+1}$ is a key polynomial for $\mu_\ell$. The ordinary augmentation

$$\mu_{\ell+1} := [\mu_\ell; \phi_{\ell+1}, \chi_{\ell+1}], \quad \chi_{\ell+1} = v_F(\phi_{\ell+1}),$$

satisfies $\mu_{\ell} < \mu_{\ell+1} \leq v_F$. If we add the augmentation step $\mu_\ell \rightarrow \mu_{\ell+1}$ to the previous MLV chain, we obtain a MLV chain of minimal degree. Since $\deg(\mu_{\ell+1}) = m_\ell < m_{\ell+1} = \deg(\mu_{\ell+1})$.

If $\ell = r$, then $\phi_{\ell+1} = F$ and $\chi_{\ell+1} = \infty$, so that $\mu_{\ell+1} = v_F$ and the theorem is proven.

If $\ell < r$, then $m_{\ell+1} < \deg(F)$ and this implies $\chi_{\ell+1} < \infty$. In this case, $\mu_{\ell+1}$ has trivial support and $\mu_{\ell+1} < v_F$. Also, $\phi_{\ell+1}$ is a key polynomial for $\mu_{\ell+1}$ of minimal degree. By Lemma 1.4 the truncation of $v_F$ by $\phi_{\ell+1}$ is equal to:

$$(v_f)_{\phi_{\ell+1}} = (\mu_{\ell+1})_{\phi_{\ell+1}} = \mu_{\ell+1}.$$ 

Finally, since $\mu_{\ell+1}(\phi_{\ell+1}) = \chi_{\ell+1} = v_F(\phi_{\ell+1})$, we have necessarily $\phi_{\ell+1} \notin t(\mu_{\ell+1}, v_F)$. This ends the recurrence step in the case $\Phi_\ell = \{\phi_\ell\}$.

Now, suppose that $\Phi_\ell = \{\chi_i \mid i \in I\}$ and let $i_0 \in I$ be the index for which $\phi_\ell = \chi_{i_0}$.

By our recurrence assumption, $\mu_\ell$ is a valuation and $\phi_\ell$ is a key polynomial for $\mu_\ell$ of minimal degree. Since $\deg t(\mu_\ell, v_F) \geq \deg(\mu_\ell) = m_\ell$, Lemma 1.4 shows that

$$\mu_\ell(a) = v_F(a) \quad \text{for all } a \in K[x], \text{ with } \deg(a) < m_\ell,$$

because $\phi_{1\mu_\ell}$ a for any $\phi \in t(\mu_\ell, v_F)$.

Denote $\beta_i = v_F(\chi_i)$ for all $i \in I$. By the additional property (i) of the Okutsu frame, we have $\beta_{i_0} < \beta_i$ for all $i_0 < i$ in $I$. Since all $\chi_i \in K[x]$ are monic polynomials of degree $m_\ell$, we deduce that

$$\mu_\ell(\chi_{i_0} - \chi_i) = v_F(\chi_{i_0} - \chi_i) = \beta_{i_0} = sv(\mu_\ell) \quad \text{for all } i_0 < i.$$ 

By [16] Prop. 6.3, all these $\chi_i \in \Phi_\ell$ are key polynomials for $\mu_\ell$ of degree $m_\ell$. Hence, we may consider the family of ordinary augmentations

$$\rho_i = [\mu_\ell; \chi_i; \beta_i] \quad \text{for all } i > i_0.$$ 

By comparing their action of $\chi_j$-expansions, we clearly have

$$\mu_\ell < \rho_i < \rho_j < v_F \quad \text{for all } i_0 < i < j \text{ in } I.$$ 

Denote $\rho_{i_0} := \mu_\ell$. For the totally ordered set of indices $A = I_{\geq i_0}$, we may consider a continuous family of valuations $A = (\rho_i)_{i \in A}$ of stable degree $m_\ell$.

Let us show that $\phi_{\ell+1}$ is $A$-unstable. For all $i \in A$, Theorem 1.2 shows that

$$\frac{\rho_i(\phi_{\ell+1})}{m_{\ell+1}} \leq sv(\rho_i) = \frac{\beta_i}{m_\ell} = \frac{\beta_{i_0}}{m_\ell} < wt(\phi_{\ell+1}),$$
the last inequality by the additional property (OF3) of the Okutsu frame. Hence, \( \rho_i(\phi_{\ell+1}) < v_F(\phi_{\ell+1}) \). By [17 Cor. 2.5], this implies
\[
\rho_i(\phi_{\ell+1}) < \rho_j(\phi_{\ell+1}) \quad \text{for all } i < j,
\]
so that \( \phi_{\ell+1} \) is \( \mathcal{A} \)-unstable.

Let us now show that \( m_{\ell+1} \) is the minimal degree of \( \mathcal{A} \)-unstability; that is, all monic \( g \in K[x] \) with \( \deg(g) < m_{\ell+1} \) are \( \mathcal{A} \)-stable. Since the product of \( \mathcal{A} \)-stable polynomials is \( \mathcal{A} \)-stable, we may assume that \( g \) is irreducible. By the fundamental property \( (9) \), there exists \( \chi_i \in \Phi_\ell \) such that
\[
\text{wt}(g) \leq \text{wt}(\chi_i) = \frac{\beta_i}{m_\ell} = \text{wt}(\rho_i).
\]

We claim that \( \rho_i(g) = \rho_j(g) \) for all \( j > i \). Indeed, suppose that \( \rho_i(g) < \rho_j(g) \leq v_F(g) \) for some \( j > i \). On one hand, from \( (10) \) we deduce
\[
\frac{\rho_i(g)}{\deg(g)} < \text{wt}(g) \leq \text{wt}(\rho_i),
\]
so that \( g \) is not \( \rho_i \)-minimal, by Theorem \( (12) \). This contradicts Corollary \( (3.2) \).

Therefore, \( \phi_{\ell+1} \) is a monic \( \mathcal{A} \)-unstable polynomial of minimal degree. In other words, \( \phi_{\ell+1} \in \text{KP}_\infty(\mathcal{A}) \). The limit augmentation
\[
\mu_{\ell+1} := [\mathcal{A}; \phi_{\ell+1}, \gamma_{\ell+1}], \quad \gamma_{\ell+1} = v_F(\phi_{\ell+1}),
\]
satisfies \( \mu_{\ell+1} \leq v_F \). Since \( \phi_\ell \not\in t(\mu_\ell, \mu_{\ell+1}) = t(\mu_\ell, v_F) \), if we add the augmentation step \( \mu_\ell \to \mu_{\ell+1} \) to the previous MLV chain, we obtain a MLV chain of \( \mu_{\ell+1} \).

If \( \ell = r \), we have \( \phi_{\ell+1} = F \) and \( \gamma_{\ell+1} = \infty \), so that \( \mu_{\ell+1} = v_F \) and the theorem would be proven.

If \( \ell < r \), then \( m_{\ell+1} < \deg(F) \) and this implies \( \gamma_{\ell+1} < \infty \). In this case, \( \mu_{\ell+1} \) has trivial support and \( \mu_{\ell+1} < v_F \). Also, \( \phi_{\ell+1} \) is a key polynomial for \( \mu_{\ell+1} \) of minimal degree. By Lemma \( (11) \) the truncation of \( v_F \) by \( \phi_{\ell+1} \) is equal to:
\[
(v_F)_{\phi_{\ell+1}} = (\mu_{\ell+1})_{\phi_{\ell+1}} = \mu_{\ell+1}.
\]
Finally, since \( \mu_{\ell+1}(\phi_{\ell+1}) = \gamma_{\ell+1} = v_F(\phi_{\ell+1}) \), we have necessarily \( \phi_{\ell+1} \not\in t(\mu_{\ell+1}, v_F) \).

This ends the recurrence step in the case \( \Phi_\ell = \{ \chi_i \mid i \in I \} \).

\[\Box\]

Corollary 4.6. Let \( [\Phi_0, \ldots, \Phi_r, \Phi_{r+1} = \{ F \}] \) be an Okutsu frame of some \( F \in \text{Irr}(K) \). Then, \( 1 = m_0 \mid m_1 \mid \cdots \mid m_r \mid \deg(F) \).

Proof. By Theorem \( (4.5) \) \( 1 = m_0, \ldots, m_{r+1} = \deg(F) \) are the degrees of a MLV chain of \( v_F \). Since \( (K, v) \) is henselian, all jumps \( m_{\ell+1}/m_\ell \) take integer values. Indeed, this follows from the results of Vaquié [24], described in [17 Sec. 6] as well.

\[\Box\]

4.3. Computation of ramification indices, residual degrees and defect. Since \( (K, v) \) is henselian, the valuation \( \emptyset_F \) is the only extension of \( v \) to the field extension \( K_F = K[x]/(F) \). By Ostrowski’s lemma, we have
\[
\deg(F) = e(F)f(F)d(F),
\]
where \( e(F) = e(\emptyset_F/v) \) is the ramification index, \( f(F) = f(\emptyset_F/v) \) the residual degree and \( d(F) = d(\emptyset_F/v) \) the defect of \( \emptyset_F/v \). Also, Ostrowski showed that \( d(F) \) is a power
of the exponent characteristic, defined as
\[ p = \begin{cases} \text{char}(k), & \text{if } \text{char}(k) > 0, \\ 1, & \text{if } \text{char}(k) = 0. \end{cases} \]

In [17, Sec.6], it is shown how to compute these invariants \(e(F), f(F), d(F)\) in terms of a MLV chain of \(v_F\). Hence, as a consequence of Theorem 4.5, we can compute them in terms of an Okutsu frame.

Let \([\Phi_0, \ldots, \Phi_r, \Phi_{r+1} = \{F\}]\) be an Okutsu frame of \(F \in \text{Irr}(K)\). Consider the MLV chain of \(v_F\) described in Theorem 4.5:
\[
\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \rightarrow \cdots \rightarrow \phi_{r-1}, \gamma_{r-1} \rightarrow \mu_{r-1} \xrightarrow{\phi_r, \gamma_r} \mu_r \xrightarrow{F, \infty} v_F.
\]
Recall that \(\gamma_\ell = v_F(\phi_\ell)\) for all \(\ell \geq 0\).

By the MLV condition, this chain induces a chain of abelian groups:
\[
\Gamma_{\mu_{\ell-1}} := \Gamma \supseteq \Gamma_{\mu_0} \supseteq \Gamma_{\mu_1} \supseteq \cdots \supseteq \Gamma_{\mu_r} = \Gamma v_F = \Gamma_{v_F},
\]
with \(\Gamma_{\mu_\ell} = \langle \Gamma, \gamma_0, \ldots, \gamma_\ell \rangle\) for all \(0 \leq \ell \leq r\) [17, Sec. 4.1]. Hence,
\[
e(F) = (\Gamma_{v_F} : \Gamma) = (\Gamma_{\mu_r} : \Gamma) = e_0 \cdots e_r,
\]
where \(e_\ell = (\Gamma_{\mu_\ell} : \Gamma_{\mu_{\ell-1}})\) for all \(0 \leq \ell \leq r\).

Thus, each index \(e_\ell\) can be computed as the least positive integer \(e\) such that
\[
e v_F(\phi_\ell) \in \langle \Gamma, v_F(\phi_0), \ldots, v_F(\phi_{\ell-1}) \rangle.
\]

On the other hand, Vaquié proved in [24] that \(d(F) = d_1 \cdots d_r\), where
\[
d_\ell = \begin{cases} 1, & \text{if } \mu_\ell \rightarrow \mu_{\ell+1} \text{ is an ordinary augmentation}, \\ m_{\ell+1}/m_\ell, & \text{if } \mu_\ell \rightarrow \mu_{\ell+1} \text{ is a limit augmentation}, \end{cases}
\]
for all \(0 \leq \ell \leq r\). Equivalently,
\[
d_\ell = 1 \iff \#\Phi_\ell = 1 \iff \exists \max(W_{m_{\ell+1}}).
\]
Thus, \(d(F)\) may be computed solely in terms of the Okutsu frame too.

Finally, \(f(F) = \deg(F)/e(F)d(F)\) is determined by \(e(F)\) and \(d(F)\).

4.4. Okutsu frames and abstract key polynomials. Abstract key polynomials were introduced by Herrera-Olalla-Spivakovsky as an alternative approach to the methods of Mac Lane and Vaquié, aiming at a thorough comprehension of the extensions to \(K[x]\) of arbitrary (not necessarily henselian) valuations on a field \(K\) [7, 8].

These polynomials were further studied by several authors and the comparison with Mac Lane-Vaquié key polynomials is by now fully understood [12, 3, 19, 1, 20].

Although they were classically defined only for valuations with trivial support, the paper [1] develops their properties for arbitrary valuations on \(K[x]\) too. Let us recall a concrete comparison result between the two sorts of “key polynomials”.

**Theorem 4.7.** [1, Thm. 2.21] Let \(\nu\) be a valuation on \(K[x]\) with nontrivial support. A monic polynomial \(Q \in K[x] \setminus K\) is an abstract key polynomial for \(\nu\) if and only if either \(\text{supp}(\nu) = QK[x]\), or the truncation \(\nu_Q\) is a valuation and \(Q\) is a (MLV) key polynomial of minimal degree for \(\nu_Q\).
A set $\Psi$ of abstract key polynomials for $\nu$ is said to be complete if for all non-constant $g \in K[x]$ there exists $Q \in \Psi$ such that
\begin{equation}
\deg(Q) \leq \deg(g) \quad \text{and} \quad \nu_Q(g) = \nu(g).
\end{equation}

As a consequence of Theorem 4.5 we derive another interpretation of abstract key polynomials.

**Theorem 4.8.** Suppose that $(K, \nu)$ is henselian and let $[\Phi_0, \ldots, \Phi_r, \Phi_{r+1} = \{F\}]$ be an Okutsu frame of some $F \in \text{Irr}(K)$. Then, the set $\Phi_0 \cup \cdots \cup \Phi_r \cup \{F\}$ is a complete set of abstract key polynomials for $v_F$.

**Proof.** By Theorems 4.5 and 4.7 all polynomials in the set $\Phi_0 \cup \cdots \cup \Phi_r \cup \{F\}$ are abstract key polynomials for $v_F$.

Take a monic $g \in K[x] \setminus K$. If $\deg(g) \geq \deg(F)$, then (11) is satisfied for $Q = F$. If $\deg(g) < \deg(F)$, then there exists $0 \leq \ell \leq r$ such that $m_\ell \leq \deg(g) < m_{\ell+1}$.

With the notation in Theorem 4.5 consider the MLV chain of $v_F$:
\[\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \longrightarrow \cdots \xrightarrow{\phi_{r-1}, \gamma_{r-1}} \mu_{r-1} \xrightarrow{\phi_r, \gamma_r} \mu_r \xrightarrow{F, \infty} \mu_{r+1} = v_F.\]

If $\Phi_\ell = \{\phi_\ell\}$, then $\mu_{\ell+1} = [\mu_\ell; \phi_{\ell+1}, \gamma_{\ell+1}]$ is an ordinary augmentation and $t(\mu_\ell, v_F) = t(\mu_\ell, v_{F_{\ell+1}}) = [\phi_{\ell+1}]_{\mu_\ell}$. Since $\deg(g) < \deg(\phi_{\ell+1})$, necessarily $\phi_{\ell+1} \nmid \mu_\ell$ and $\mu_\ell(g) \neq v_F(g)$ by Lemma 4.4. Thus, (11) is satisfied for $Q = \phi_\ell$.

If $\Phi_\ell = \{\chi_i \mid i \in I_\ell\}$, then $\mu_{\ell+1} = [\mathcal{A}_\ell; \phi_{\ell+1}, \gamma_{\ell+1}]$ is a limit augmentation and $\phi_{\ell+1}$ is a limit key polynomial for $\mathcal{A}_\ell$. Hence, $g$ is $\mathcal{A}_\ell$-stable and (11) is satisfied for $Q = \chi_i$, for a sufficiently large $i \in I_\ell$.

The converse statement follows easily from the definitions. Consider a complete set of abstract key polynomials for $\nu := v_F$,
\[\Psi = \Psi_{m_0} \cup \cdots \Psi_{m_r} \cup \Psi_{m_{r+1}} = \{F\}, \quad m_0 = 1 < m_1 < \cdots < m_r < \deg(F),\]
where all polynomials in $\Psi_{m_\ell}$ have degree $m_\ell$.

In order to show that the list $[\Psi_1, \ldots, \Psi_r, \{F\}]$ is an Okutsu frame of $F$, we need only to check that the property (9) is satisfied.

Take a monic $g \in K[x]$ such that $0 < \deg(g) < m_{\ell+1}$ for some $0 \leq \ell \leq r$. By the completeness of $\Psi$, there exists $P \in \Psi$ such that $\deg(P) \leq \deg(g)$ and $\nu_P(g) = \nu(g)$. Since $\deg(P) \leq m_\ell$, there exists $Q \in \Psi_{m_\ell}$ such that $\nu_P \leq \nu_Q \leq \nu$. Thus, $\nu_Q(g) = \nu(g)$ as well. By Theorem 4.7, $Q$ is a MLV key polynomial of minimal degree for $\nu_Q$. Thus, Theorem 4.2 shows that
\[\text{wt}(g) = \frac{\nu(g)}{\deg(g)} = \frac{\nu_Q(g)}{\deg(g)} \leq \frac{\nu_Q(Q)}{m_\ell} = \frac{\nu(Q)}{m_\ell} = \text{wt}(Q).\]

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