ON THE COHOMOLOGY OF TAUPOLOGICAL BUNDLES
OVER THE QUOT SCHEME OF CURVES

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ABSTRACT. We consider tautological bundles and their exterior and symmetric powers over the Quot scheme of zero dimensional quotients over the projective line. We prove several results regarding the vanishing of their higher cohomology, and we describe the spaces of global sections via tautological constructions. To this end, we make use of the embedding of the Quot scheme into the product of two Grassmannians, which over the projective line was studied by Strømme as an explicit local complete intersection. This allows us to construct resolutions with vanishing cohomology for the tautological bundles and their exterior and symmetric powers.

1. Introduction

1.1. Tautological vector bundles. Let $C$ be a smooth complex projective curve. Let $\text{Quot}_C(\mathbb{C}^N, n)$ denote the Quot scheme parameterizing rank 0 degree $n$ quotients of the trivial bundle of rank $N$ over $C$:

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \to Q \to 0, \quad \text{rank } Q = 0, \quad \text{deg } Q = n.$$ 

For any line bundle $L \to C$, there is an induced tautological rank $n$ vector bundle over $\text{Quot}_C(\mathbb{C}^N, n)$ given by

$$L[n] = \pi_*(p^*L \otimes Q).$$

(1.1.1)

Here

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O} \to Q \to 0$$

denotes the universal exact sequence over $C \times \text{Quot}_C(\mathbb{C}^N, n)$, and $p$ and $\pi$ are the two projections over the factors.

We consider the Schur functors associated to the tautological bundles $L[n]$. In this context, the holomorphic Euler characteristics of $\bigwedge^k L[n]$, $\text{Sym}^k L[n]$ and $\left( \bigwedge^k L[n] \right)^\vee$ were recently calculated in [OS]. Lifting the numerical results of [OS] to the individual cohomology groups is a natural question.

In genus 0, we show the following three results.

Theorem 1.1.2. For all line bundles $L \to \mathbb{P}^1$ with $\text{deg } L \geq n$, we have

$$H^0\left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \bigwedge^k L[n] \right) \cong \bigwedge^k H^0(L^\oplus N)$$
and the higher cohomology vanishes
\[ H^i \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \bigwedge^k L^{[n]} \right) = 0, \quad i > 0. \]

**Theorem 1.1.3.** For all line bundles \( L \to \mathbb{P}^1 \) with \( \deg L \geq n \geq k \), we have
\[ H^0 \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \text{Sym}^k L^{[n]} \right) \cong \text{Sym}^k H^0(L^\otimes N) \]
and the higher cohomology vanishes
\[ H^i \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \text{Sym}^k L^{[n]} \right) = 0, \quad i > 0. \]

**Theorem 1.1.4.** For all line bundles \( L, M \to \mathbb{P}^1 \) with \( \deg M \geq n \) and \( 0 \leq \deg M - \deg L \leq 1 \), for all \( k_1, \ldots, k_r \geq 0 \), not all zero, and \( 1 \leq r \leq N - 1 \), we have
\[ H^i \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \left( \bigwedge^{k_1} L^{[n]} \right)^\vee \otimes \left( \bigwedge^{k_2} M^{[n]} \right)^\vee \otimes \cdots \otimes \left( \bigwedge^{k_r} M^{[n]} \right)^\vee \right) = 0, \quad i \geq 0. \]

We expect that the vanishing of higher cohomology in Theorems 1.1.2 and 1.1.3 holds whenever \( \deg L \geq -1 \). We note that for the determinant line bundles \( \bigwedge^n L^{[n]} \), the bound obtained in our theorems improves the bound derived via the usual vanishing theorems, namely \( \deg L \geq Nn - N - n \). For instance, Kodaira vanishing can easily be applied here taking into account the description of the ample cone in [Str] and a standard calculation of the canonical bundle via Grothendieck–Riemann–Roch.\(^1\)

While the above theorems concern genus 0, we also obtain the following corollary in arbitrary genus. Let \( y \) be a variable. Setting
\[ \bigwedge_y V := \sum_k y^k \bigwedge V, \quad \text{Sym}_y V := \sum_k y^k \text{Sym}^k V, \]
the result below recovers Theorem 1, a special case of Theorem 2, and Theorem 4 in [OS].

**Corollary 1.1.5.** Let \( L, M_1, \ldots, M_r \to C \) be line bundles over a smooth projective curve, where \( 1 \leq r \leq N - 1 \). Then
\begin{align*}
(1.1.5a) & \sum_{n=0}^{\infty} q^n \chi \left( \text{Quot}_C(\mathbb{C}^N, n), \bigwedge_y L^{[n]} \right) = (1 - q)^{-\chi(O_C)} (1 + qy)^{N\chi(L)}, \\
(1.1.5b) & \sum_{n=0}^{\infty} q^n \chi \left( \text{Quot}_C(\mathbb{C}^N, n), \bigotimes_{i=1}^r \left( \bigwedge_{y_i} M_i^{[n]} \right)^\vee \right) = (1 - q)^{-\chi(O_C)}.
\end{align*}
Furthermore, in genus 0, we have
\[ (1.1.5c) \sum_{n \geq k} q^n y^k \chi \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \text{Sym}^k L^{[n]} \right) = (1 - q)^{-1} (1 - qy)^{-N\chi(L)}. \]

\(^1\)In arbitrary genus, the explicit bound \( \deg L \geq 2g(N + n + 1) + n(N + 1/N) \) ensures the vanishing of higher cohomology of the determinant line bundles \( \bigwedge^n L^{[n]} \). This can be proven substituting the ample cone calculation in [Str] with the positivity results for the tautological bundles obtained in [O, Theorem 21]. We do not record the derivation here since this would take us too far afield. The bound is certainly not optimal.
We will show how to derive Corollary 1.1.5 from Theorems 1.1.2, 1.1.3 and 1.1.4 in Section 4.1.

Formulas (1.1.5a), (1.1.5b) and (1.1.5c) were previously established in [OS] based on reduction to genus 0 using universality statements as in [EGL, OS, St], and equivariant torus localization in genus 0. The localization calculation is however combinatorially involved and relies on several mysterious simplifications. In the present paper, Theorems 1.1.2, 1.1.3 and 1.1.4 reflect an efficient and more conceptual approach to the full cohomology. They replace the localization calculation with a geometric argument.

The key idea is to use the twofold Grothendieck embedding of the Quot scheme into a product of Grassmannians,

\[ \iota : \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \hookrightarrow G_1 \times G_2, \]

so that the image of \( \iota \) is the zero locus of a regular section \( \sigma \) of an explicit vector bundle

\[ \mathcal{E} \to G_1 \times G_2. \]

This embedding was considered and studied in detail by Strømme [Str] and is specific to the Quot scheme over the projective line.

The Koszul resolution for \( \sigma \) induces resolutions

\[ \cdots \to R_2 \to R_1 \to R_0 \to \iota_* \mathcal{F} \to 0 \]

for each one of the tautological bundles \( \mathcal{F} \) appearing in Theorems 1.1.2, 1.1.3 and 1.1.4. Crucially, we use the Borel–Weil–Bott theorem and combinatorial arguments involving the Littlewood–Richardson rule to show that the terms \( R_j \) of these resolutions have vanishing cohomology for all \( j \geq 1 \). This allows us to control the cohomology of the tautological bundles and establish our results.

There are no obstacles to apply the same reasoning relatively to any \( \mathbb{P}^1 \)-bundle over an arbitrary base.

It was remarked in [OS] that the above results have an intriguing analogy with the expressions for the Euler characteristics of tautological bundles over the Hilbert scheme of points computed for instance in [D, Sc1, Sc2]; see also [A, K], among others. The approach in [Sc1, Sc2] also rests on constructing explicit resolutions, with the Bridgeland–King–Reid correspondence playing a central role in this context.

It is easy to see how the sections of the bundles \( \bigwedge^k L[n] \) and \( \text{Sym}^k L[n] \) over \( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \) arise geometrically. Indeed, from the universal quotient

\[ \mathbb{C}^N \otimes \mathcal{O} \to \mathcal{Q} \to 0 \]

over \( \mathbb{P}^1 \times \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \), after tensoring by \( L \) and pushing forward, we immediately obtain a map

\[ H^0(L^{\oplus N}) \otimes \mathcal{O}_{\text{Quot}} \to L[n]. \]
Taking exterior and symmetric powers, and taking cohomology, we obtain morphisms

\[(1.1.6) \quad \Phi_k : \bigwedge^k H^0(L^\oplus N) \to H^0 \left( \bigwedge^k L^{[n]} \right), \quad \Psi_k : \text{Sym}^k H^0(L^\oplus N) \to H^0 \left( \text{Sym}^k L^{[n]} \right). \]

Our proofs will show that $\Phi_k$ and $\Psi_k$ are isomorphisms when $\text{deg} L \geq n$, with the further assumption $n \geq k$ imposed for symmetric powers. Thus all sections of $\bigwedge^k L^{[n]}$ and $\text{Sym}^k L^{[n]}$ are obtained via tautological constructions, while the higher cohomology vanishes. From this point of view, Theorems 1.1.2, 1.1.3, and 1.1.4 provide a geometric interpretation of the formulas in [OS].

In higher genus, we have the following expectation stated in [OS, Question 20]:

\[(1.1.7) \quad H^\bullet \left( \text{Quot}_{P^1}(C^N, n), \bigwedge^k L^{[n]} \right) \cong \bigwedge^k H^\bullet (L^\oplus N) \otimes \text{Sym}^{n-k} H^\bullet (O_C). \]

Here, the exterior and symmetric powers are understood in the graded sense.

In genus 0, our theorems for $\text{deg} L \geq n$ are consistent with this expectation. We offer additional modest evidence for $k = 1$. Specifically, we prove

**Corollary 1.1.8.** For all line bundles $L \to \mathbb{P}^1$, we have

\[H^i \left( \text{Quot}_{\mathbb{P}^1}(C^N, n), L^{[n]} \right) = 0, \quad i \geq 2.\]

### 1.2. Higher rank.

In higher rank, the $K$-theoretic invariants of the Quot scheme are largely unexplored. Letting $\text{Quot}_{\mathbb{P}^1}(C^N, n, r)$ denote the Quot scheme of quotients of rank $r$ and degree $n$ of the rank $N$ trivial bundle, and letting $L^{[n]}$ be defined as the higher rank analogue of equation (1.1.1) above, we propose

**Conjecture 1.2.1.** Let $n = (N - r)a + b$ with $0 \leq b < N - r$. Then for all line bundles $L \to \mathbb{P}^1$, we have

\[\chi \left( \text{Quot}_{\mathbb{P}^1}(C^N, n, r), \bigwedge^k L^{[n]} \right) = \binom{N \chi(L)}{k}\]

for all $k \leq n + r(a+1)$.

For symmetric powers, we state the following

**Conjecture 1.2.2.** Let $n = (N - r)a + b$ with $0 \leq b < N - r$. Then for all line bundles $L \to \mathbb{P}^1$, we have

\[\chi \left( \text{Quot}_{\mathbb{P}^1}(C^N, n, r), \text{Sym}^k L^{[n]} \right) = \binom{N \chi(L) + k - 1}{k}\]

for all $k \leq n + r(a+1)$.

Finally, for the dualized exterior powers, we have
Conjecture 1.2.3. Let $r > 0$ and write $n = ar + b$ with $0 \leq b < r$. Let $1 \leq m \leq N - r - 1$ and $k_1, \ldots, k_m$ be nonnegative integers with $0 < k_1 + \cdots + k_m \leq n + (N - r)(a + 1)$. Then for all line bundles $L_1, \ldots, L_m \to \mathbb{P}^1$, we have
\[
\chi \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r), \left( \bigwedge^{k_1} L_1^{[n]} \right)^\vee \otimes \cdots \otimes \left( \bigwedge^{k_m} L_m^{[n]} \right)^\vee \right) = 0.
\]

When $r = 0$, Conjectures 1.2.1 and 1.2.2 recover Theorems 1.1.2 and 1.1.3, while the case $r = N - 1$ can be verified by hand since the Quot scheme is a projective space. For all three conjectures, we checked the answer by computer in several other cases. The bound on $k$ appears to be be sharp.

At the level of cohomology, we expect that Theorem 1.1.2, 1.1.3 and 1.1.4 extend to higher rank quotients, subject to appropriate bounds on $k$. While Strømme’s construction is valid for quotients of arbitrary rank, the Borel–Weil–Bott arguments require new ideas.

The answers predicted by the conjectures stabilize as $n$ becomes large with respect to $N, k, r$. Equivalently, for each fixed $k$, the generating series
\[
\sum_{n=0}^{\infty} q^n \chi \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r), \bigwedge^{k} L^{[n]} \right), \quad \sum_{n=0}^{\infty} q^n \chi \left( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n, r), \text{Sym}^k L^{[n]} \right)
\]
are given by rational functions with (simple) pole only at $q = 1$. It is natural to wonder whether this statement is correct for all partitions and for the associated Schur functors of $L^{[n]}$.

1.3. Plan of the paper. We review Strømme’s embedding, construct the resolutions of the tautological bundles, and establish Theorems 1.1.2, 1.1.3 and 1.1.4 in Section 2. This relies on the Borel–Weil–Bott analysis of the resolutions which is carried out in Section 3. Corollaries 1.1.5 and 1.1.8 are proved in Section 4.

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2. Grassmannian embedding of the Quot scheme and resolutions

2.1. Strømme’s embedding. We begin by describing Strømme’s construction which exhibits the Quot scheme over $\mathbb{P}^1$ as the zero locus of a regular section of a vector bundle over the product of two Grassmannians [Str].

For each integer $m \geq n$, the embedding takes the form
\[
\iota_m : \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \hookrightarrow G(V_{m-1}, n) \times G(V_m, n),
\]
where
\[
G_1 = G(V_{m-1}, n), \quad G_2 = G(V_m, n)
\]
are the Grassmannians of \( n \)-dimensional quotients of two vector spaces of dimensions \( Nm \) and \( N(m + 1) \) respectively. We identify

\[
V_{m-1} = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m - 1)^{\oplus N}), \quad V_m = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)^{\oplus N})
\]

and we write \( W = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \). Thus, there is a morphism

\[
V_{m-1} \otimes W \to V_m.
\]

Explicitly, \( \iota_m \) is the product of two Grothendieck embeddings. When \( m \geq n \), each short exact sequence in the Quot scheme

\[
0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1} \to Q \to 0,
\]

yields two exact sequences of vector spaces

\[
0 \to H^0(S(m-1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m-1)^{\oplus N}) \to H^0(Q(m-1)) \to 0,
\]

\[
0 \to H^0(S(m)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m)^{\oplus N}) \to H^0(Q(m)) \to 0.
\]

Then \( \iota_m \) is given by the assignment

\[
[0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_{\mathbb{P}^1} \to Q \to 0] \mapsto \left[ V_{m-1} \to H^0(Q(m-1)) \right] \times \left[ V_m \to H^0(Q(m)) \right].
\]

To describe the equations cutting out \( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \) in \( G_1 \times G_2 \), write

\[
\mathcal{A}_1, \mathcal{B}_1 \to G_1, \quad \mathcal{A}_2, \mathcal{B}_2 \to G_2
\]

for the tautological subbundles and the quotient bundles over the two Grassmannians. Let \( \text{pr}_1 \) and \( \text{pr}_2 \) be the two projections on \( G_1 \times G_2 \). The sheaf

\[
E = \text{pr}_1^* \mathcal{A}_1^\vee \otimes W^\vee \otimes \text{pr}_2^* \mathcal{B}_2 \to G_1 \times G_2
\]

admits a natural section \( \sigma \) induced by the composition

\[
\sigma: \text{pr}_1^* \mathcal{A}_1 \otimes W \to V_{m-1} \otimes W \otimes \mathcal{O}_{G_1 \times G_2} \to V_m \otimes \mathcal{O}_{G_1 \times G_2} \to \text{pr}_2^* \mathcal{B}_2.
\]

Strømme shows that the section \( \sigma \) is regular and vanishes exactly along \( \text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n) \), see [Str, §4].

2.2. Resolutions. As a result of the above discussion, the section \( \sigma \) induces a Koszul resolution

\[
\cdots \to \bigwedge^2 E^\vee \to \bigwedge^1 E^\vee \to \mathcal{O}_{G_1 \times G_2} 
\to \mathcal{O}_{\text{Quot}} \to 0.
\]

Note that if \( \deg L = m \), by the definition of the embedding \( \iota_m \) we have

\[
L^n = \iota_m^* \text{pr}_2^* \mathcal{B}_2.
\]

Hence, tensoring (2.2.1) with \( \text{pr}_2^* \bigwedge^k \mathcal{B}_2 \), we obtain the resolution

\[
\cdots \to \bigwedge^2 E^\vee \otimes \text{pr}_2^* \bigwedge^k \mathcal{B}_2 \to \bigwedge^1 E^\vee \otimes \text{pr}_2^* \bigwedge^k \mathcal{B}_2 \to \text{pr}_2^* \bigwedge^k \mathcal{B}_2 \to (\iota_m)^* \left( \bigwedge^k L^n \right) \to 0
\]
over \(G_1 \times G_2\). We set
\[
\mathcal{V}_\ell = \bigwedge^\ell E^\vee \otimes \text{pr}_2^* B_2, \quad \ell \geq 0,
\]
and will show

**Proposition 2.2.2.** In the resolution
\[
\cdots \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{V}_0 \to (\iota_m)_*(\bigwedge^k L^{[n]}) \to 0,
\]
the sheaves \(\mathcal{V}_\ell\) have no cohomology for \(\ell \geq 1\), while the sheaf \(\mathcal{V}_0\) has no higher cohomology.

For the symmetric products, we similarly define
\[
\mathcal{W}_\ell = \bigwedge^\ell E^\vee \otimes \text{pr}_2^* \text{Sym}^k B_2,
\]
and have an analogous result.

**Proposition 2.2.3.** In the resolution
\[
\cdots \to \mathcal{W}_2 \to \mathcal{W}_1 \to \mathcal{W}_0 \to (\iota_m)_*(\text{Sym}^k L^{[n]}) \to 0,
\]
the sheaves \(\mathcal{W}_\ell\) have no cohomology if \(\ell \geq 1\) and \(\deg L \geq n \geq k\), while \(\mathcal{W}_0\) has no higher cohomology.

A further analysis is needed for Theorem 1.1.4. When \(\deg M = m \geq n\) and \(\deg L = m - 1\), we have
\[
L^{[n]} = i^*_m \text{pr}_1^* B_1, \quad M^{[n]} = i^*_m \text{pr}_2^* B_2.
\]
Thus for the bundle
\[
\mathcal{F} = \left( \bigwedge^{k_1} L^{[n]} \right)^\vee \otimes \left( \bigwedge^{k_2} M^{[n]} \right)^\vee \cdots \otimes \left( \bigwedge^{k_r} M^{[n]} \right)^\vee
\]
which appears in Theorem 1.1.4, we obtain a resolution
\[
\cdots \to \mathcal{U}_2 \to \mathcal{U}_1 \to \mathcal{U}_0 \to (\iota_m)_* \mathcal{F} \to 0,
\]
where
\[
\mathcal{U}_\ell = \bigwedge^\ell E^\vee \otimes \left( \text{pr}_2^* \bigwedge^{k_1} B_2^\vee \otimes \text{pr}_2^* \bigwedge^{k_2} B_2^\vee \cdots \otimes \text{pr}_2^* \bigwedge^{k_r} B_2^\vee \right).
\]

Theorem 1.1.4 also covers the case \(L = M\). In this situation, the resolution needs to be slightly modified so that
\[
\mathcal{U}_\ell = \bigwedge^\ell E^\vee \otimes \left( \text{pr}_2^* \bigwedge^{k_1} B_2^\vee \otimes \text{pr}_2^* \bigwedge^{k_2} B_2^\vee \cdots \otimes \text{pr}_2^* \bigwedge^{k_r} B_2^\vee \right).
\]

**Proposition 2.2.4.** For \(k_1, \ldots, k_r\) not all zero, the cohomology of \(\mathcal{U}_\ell\) vanishes for all \(\ell \geq 0\).
2.2.1. **The main theorems.** Before turning our attention to the proofs of these three propositions, we note that our main Theorems 1.1.2, 1.1.3, and 1.1.4 follow immediately from them.

For Theorem 1.1.2, we use Proposition 2.2.2. The associated spectral sequence shows that the higher cohomology of $\bigwedge^k L^{[n]}$ vanishes, while in degree zero we have

$$H^0\left(\text{Quot}_{P^1}(\mathbb{C}^N, n), \bigwedge^k L^{[n]}\right) = H^0(G_1 \times G_2, \mathcal{V}_0).$$

Recalling that $\mathcal{V}_0 = \text{pr}_2^* \bigwedge^k B_2$, we compute

$$H^0(G_1 \times G_2, \mathcal{V}_0) = H^0\left(G_2, \bigwedge^k B_2\right) = \bigwedge^k V_m = \bigwedge^k H^0(\mathbb{C}^N \otimes O_{P^1}(m)) = \bigwedge^k H^0(L^{\otimes N}),$$

as needed.

For Theorem 1.1.3, we make use of Proposition 2.2.3. The only difference is that in degree 0, the initial term $\mathcal{W}_0 = \text{pr}_2^* \text{Sym}^k B_2$ has sections

$$H^0(G_1 \times G_2, \mathcal{W}_0) = H^0(G_2, \text{Sym}^k B_2) = \text{Sym}^k V_m = \text{Sym}^k H^0(\mathbb{C}^N \otimes O_{P^1}(m)) = \text{Sym}^k H^0(L^{\otimes N}).$$

Finally, for Theorem 1.1.4, the argument uses Proposition 2.2.4. This time around, the cohomology vanishes for all terms of the resolution and in all degrees.

2.3. **Analysis of the resolutions.** We now turn to Propositions 2.2.2, 2.2.3, and 2.2.4 and deduce them from the Grassmannian vanishing results of Section 3. We begin by making the terms of the resolutions more explicit.

2.3.1. **Partitions and Cauchy’s formula.** We use standard terminology on partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$. We set

$$|\lambda| = \sum_{i=1}^r \lambda_i,$$

and we let $\lambda^\dagger$ be the transpose partition obtained by exchanging the rows and columns of the Young diagram of $\lambda$.

Throughout, we will always assume that our base scheme is defined over a field of characteristic 0.

For each partition $\lambda$, we let $S_\lambda$ denote the associated Schur functor. For a partition $\lambda$ and any vector bundle $V \rightarrow Y$ over a base $Y$, there is an associated vector bundle $S_\lambda(V) \rightarrow Y$. The cases $\lambda = (1^k)$ and $\lambda = (k)$ correspond to the $k$th exterior and $k$th symmetric powers, respectively.

The vector bundles $S_\lambda(V) \rightarrow Y$ are also defined when $\lambda$ is not a partition but rather an arbitrary dominant weight $\lambda_1 \geq \cdots \geq \lambda_r$, where $r = \text{rank}(V)$, and we now allow the entries to be negative. We have

$$S_{-\lambda}(V) = S_\lambda(V^\vee),$$
where $-\lambda$ denotes the sequence $-\lambda_r \geq \cdots \geq -\lambda_1$. In addition, 
\[ S_\lambda(V) \otimes \det V = S_{\lambda+(1')} (V). \]

If $V, W \to Y$ are two vector bundles, Cauchy’s identity 
\[ \wedge^\ell (V \otimes W) = \bigoplus_{|\lambda| = \ell} S_{\lambda^\dagger} (V) \otimes S_\lambda(W) \]
holds, where the sum is over all partitions $\lambda$ of size $\ell$. We only need to consider those partitions $\lambda$ with at most $\text{rank}(W)$ rows and at most $\text{rank}(V)$ columns since the term is 0 otherwise. Applying this formula to the bundle $E$ whose section cuts out the Quot scheme, we obtain
\[ (2.3.1a) \quad \wedge^\ell E = \bigoplus_{|\lambda| = \ell} S_{\lambda^\dagger} (A_1) \boxtimes (S_\lambda (B_2^\vee \oplus B_2^\vee) \otimes \wedge^k B_2). \]

Here, $\lambda$ is a partition with at most $2n$ rows and the number of columns at most equal to
\[ (2.3.1b) \quad \text{rank}(A_1) = \dim V_{m-1} - n = \dim V_m - N - n \leq \dim V_m - n - 1. \]

In the discussion below, the abbreviation $d = \dim V_m$ will often be used.

Using (2.3.1a), we immediately obtain the expressions
\[ (2.3.2a) \quad \mathcal{V}_\ell = \bigoplus_{|\lambda| = \ell} S_{\lambda^\dagger} (A_1) \boxtimes \left( S_\lambda (B_2^\vee \oplus B_2^\vee) \otimes \wedge^k B_2 \right), \]
and
\[ (2.3.2b) \quad \mathcal{W}_\ell = \bigoplus_{|\lambda| = \ell} S_{\lambda^\dagger} (A_1) \boxtimes (S_\lambda (B_2^\vee \oplus B_2^\vee) \otimes \text{Sym}^k B_2). \]

Finally, we have
\[ (2.3.2c) \quad \mathcal{U}_\ell = \bigoplus_{|\lambda| = \ell} S_{\lambda^\dagger} (A_1) \boxtimes \left( S_\lambda (B_2^\vee \oplus B_2^\vee) \otimes \wedge^{k_1} B_2^\vee \otimes \wedge^{k_2} B_2^\vee \otimes \cdots \otimes \wedge^{k_r} B_2^\vee \right) \]
or
\[ (2.3.2d) \quad \mathcal{U}_\ell = \bigoplus_{|\lambda| = \ell} \left( S_{\lambda^\dagger} (A_1) \otimes \wedge^{k_1} B_2^\vee \right) \boxtimes \left( S_\lambda (B_2^\vee \oplus B_2^\vee) \otimes \wedge^{k_2} B_2^\vee \otimes \cdots \otimes \wedge^{k_r} B_2^\vee \right), \]

corresponding to the two cases of Theorem 1.1.4.

2.3.2. The Borel–Weil–Bott theorem. To compute the cohomology of the bundles $\mathcal{V}_\ell$, $\mathcal{W}_\ell$ and $\mathcal{U}_\ell$, we use the Borel–Weil–Bott theorem [B]. For integers $0 < n < d$, let $G = G(d, n)$ denote the Grassmannian of $n$-dimensional quotients of a $d$-dimensional vector space, and let $A, B \to G$ denote the tautological subbundle and quotient. For a partition 
$\mu = (\mu_1, \ldots, \mu_{d-n})$
with $d - n$ rows, we form the string
\[ \rho + (0, \mu) = (d - 1, d - 2, \ldots, 1, 0) + (0, \ldots, 0, \mu_1, \ldots, \mu_{d-n}). \]

**Theorem 2.3.3** (Borel–Weil–Bott). The bundle $S_\mu(A)$ has at most one non-zero cohomology group. Furthermore, if the string $\rho + \mu$ contains repetitions, then all cohomology groups of $S_\mu(A)$ vanish.

This formulation can be found in [W, Corollary 4.1.9]. We note from it that $S_\mu(A)$ has no cohomology provided there exists $j$ such that
\[ (2.3.4a) \quad j \leq \mu_j \leq n + j - 1. \]

Let us record the “dual” rephrasing of condition (2.3.4a) which is also useful here. For a partition $\nu$ with $n$ rows, the bundle $S_\nu(B^\vee)$ over the Grassmannian $G(d, n)$ has no cohomology provided that there exists $j$ such that
\[ (2.3.4b) \quad j \leq \nu_j \leq d - n + j - 1. \]

In Proposition 2.2.4, we also consider the bundle
\[ S_\mu(A) \otimes \bigwedge^k B^\vee = S_{-\mu}(A^\vee) \otimes S_{(1^k)}B^\vee \]
for $0 \leq k \leq n$. Again by the Borel–Weil–Bott theorem [W, Corollary 4.1.9], all cohomology vanishes provided that the string
\[ (d - 1, \ldots, 0) + (-\mu_{d-n}, \ldots, -\mu_1, \ldots, 1, 0, \ldots, 0) \]
has repetitions. That happens when there exists $j$ such that
\[ (2.3.5) \quad j - 1 \leq \mu_j \leq n + j - 1 \text{ and } \mu_j \neq j + k - 1. \]
Of course, the case $k = 0$ recovers (2.3.4a).

2.3.3. **Indices of partitions.** The following definition is not standard but is crucial for our arguments.

**Definition 2.3.6.** Let $n$ be a non-negative integer. Let $\lambda \neq 0$ be a partition satisfying the following condition

\[ (\ast) \quad \text{for all } j, \text{ the number of boxes in the } j\text{th column of } \lambda \text{ is either } < j \text{ or } \geq n + j. \]

Let $i$ denote the largest index $j$ such that the $j$th column has $\geq n + j$ boxes. We refer to $i$ as the $n$-index of $\lambda$. If $\lambda$ does not satisfy (\ast), we leave the $n$-index undefined. \qed

It may help to visualize partitions $\lambda$ of $n$-index $i$. There are $i$ “long” columns with at least $n + i$ boxes, while the remaining columns are “short” containing at most $i$ boxes. In
The following variation is needed for Proposition 2.2.4 and is connected to condition (2.3.5) above.

**Definition 2.3.8.** Let \(0 \leq k \leq n\) be integers. Let \(\lambda \neq 0, \lambda \neq (1^k)\) be a partition satisfying the following condition

\((**\) for all \(j\), the number of boxes in the \(j\)th column of \(\lambda\) is either \(< j - 1\) or \(\geq n + j\) or equal to \(j + k - 1\).

Let \(i\) denote the largest index \(j\) such that the \(j\)th column has \(\geq n + j\) boxes. We refer to \(i\) as the \((k, n)\)-index of \(\lambda\), when defined. The case \(k = 0\) corresponds to the \(n\)-index defined above.

The partition \(\lambda = (1^k)\) is not considered here. The reason is that \(\lambda\) satisfies \((**\), yet no column has \(\geq n + 1\) boxes, so the index is undefined.

For a partition \(\lambda\) of \((k, n)\)-index \(i\), two shapes are possible:

(a) the partition \(\lambda\) has \(i\) “long” columns with \(\geq n + i\) boxes, and the remaining columns are “short” having \(\leq i - 1\) boxes.

(b) the partition \(\lambda\) has \(i\) “long” columns with \(\geq n + i\) boxes, the \((i + 1)\)st column has \(k + i\) boxes, and the remaining columns are “short” having \(\leq i\) boxes.

The partitions in Figure 2 satisfy (a) and (b) respectively. For case (b), the long and short columns are shown in dark gray and white, while the middle \((i + 1)\)st column is lighter gray. In both cases

\[
i \leq \lambda_{i+1} \leq i + 1, \ldots, i \leq \lambda_{i+n} \leq i + 1.
\]
Proof of Propositions 2.2.2 and 2.2.3. Recall from (2.3.2a) that

\[ V_\ell = \bigoplus_{|\lambda|=\ell} S_{\lambda'}(A_1) \boxtimes \left( S_\lambda(B_2^\vee \oplus B_2^\vee) \otimes \bigwedge^k B_2 \right). \]

When \( \ell \geq 1 \), we have \( \lambda \neq 0 \). For a partition \( \lambda \neq 0 \) appearing in the above sum, we distinguish two mutually exclusive situations:

(†) there exists \( j \) such that \( j \leq \lambda^1_1 \leq n + j - 1 \), or

(∗) for all \( j \), the number of boxes in the \( j \)th column of \( \lambda \) is either \( < j \) or \( \geq n + j \).

Of course, condition (∗) already appeared in Definition 2.3.6. In case (†), we noted in (2.3.4a) that \( S_{\lambda'}(A_1) \) has no cohomology. In case (∗), Proposition 3.1.2 in Section 3 below shows that

\[ S_\lambda(B_2^\vee \oplus B_2^\vee) \otimes \bigwedge^k B_2 \]

has no cohomology either. Consequently, \( V_\ell \) has no cohomology when \( \ell \geq 1 \), establishing Proposition 2.2.2.

Proposition 2.2.3 is proven in the same fashion. Indeed, by (2.3.2b) we have

\[ W_\ell = \bigoplus_{|\lambda|=\ell} S_{\lambda'}(A_1) \boxtimes \left( S_\lambda(B_2^\vee \oplus B_2^\vee) \otimes \text{Sym}^k B_2 \right). \]

This time, to deal with case (∗) we invoke Proposition 3.1.3.

\[ \square \]

Proof of Proposition 2.2.4. We need to consider two cases corresponding to (2.3.2c) and (2.3.2d). Assume first

\[ U_\ell = \bigoplus_{|\lambda|=\ell} S_{\lambda'}(A_1) \boxtimes \left( S_\lambda(B_2^\vee \oplus B_2^\vee) \otimes \bigwedge^k B_2 \right). \]

When \( \ell = 0 \), we must have \( \lambda = 0 \), and the cohomology of

\[ \bigwedge^{k_1} B_2^\vee \otimes \bigwedge^{k_2} B_2^\vee \otimes \cdots \otimes \bigwedge^{k_r} B_2^\vee \]

vanishes when \( (k_1, \ldots, k_r) \) are not all zero. This is well-known, but a quick argument is as follows. By the Pieri rule, each tensor product corresponds to adding boxes to the previous partitions, no two boxes in the same row. Thus, each summand of the tensor product is of the form \( S_\nu(B_2^\vee) \) where \( \nu \) has at most \( r \) columns. Furthermore, \( \nu \neq 0 \) since \( k_1 + \cdots + k_r > 0 \). In particular,

\[ 1 \leq \nu_1 \leq r \leq N - 1 \leq \dim V_m - n \]

for \( m \geq n \). Thus, condition (2.3.4b) is satisfied with \( j = 1 \).

When \( \ell \neq 0 \), we have \( \lambda \neq 0 \). Again, we distinguish the same two cases (†) and (∗), both leading to cohomology vanishing. For case (∗), we invoke Proposition 3.1.4 (1). The partition \( \lambda \) was seen in (2.3.1b) to have at most

\[ \dim V_m - N - n \leq \dim V_m - (r + 1) - n \]
columns for $r \leq N - 1$, so the hypothesis of the proposition applies.

We also need to inspect

$$
U_{\ell} = \bigoplus_{|\lambda| = \ell} \left( S_{\lambda^t} (A_1) \otimes \bigwedge^{k_1} B_1^\vee \right) \otimes \left( S_{\lambda}(B_2^\vee \oplus B_2') \otimes \bigwedge^{k_2} B_2' \otimes \cdots \otimes \bigwedge^{k_r} B_2' \right).
$$

The case $\ell = 0$ is similar, but a few modifications are necessary when $\ell \neq 0$. When $\lambda^t$ satisfies (2.3.5) (for $k = k_1$), the first factor $S_{\lambda^t} (A_1) \otimes \bigwedge^{k_1} B_1^\vee$ has no cohomology. Otherwise, $\lambda$ satisfies condition (**). Vanishing is clear when $\lambda = (1^{k_1})$ since in this case $S_{\lambda}$ is an exterior power, and tensor products of exterior powers have been analyzed above. Otherwise, letting $i$ denote the $(k_1, n)$-index of $\lambda$, the second factor

$$
S_{\lambda}(B_2^\vee \oplus B_2') \otimes \bigwedge^{k_2} B_2' \otimes \cdots \otimes \bigwedge^{k_r} B_2'
$$

has no cohomology by Proposition 3.1.4.

\[ \square \]

3. Cohomology vanishing on the Grassmannian

3.1. Overview. We establish the vanishing results which played a crucial role in the proofs of Propositions 2.2.2, 2.2.3, and 2.2.4.

We continue to write $G = G(d, n)$ for the Grassmannian of $n$-dimensional quotients of a $d$-dimensional vector space, and $B \to G$ for the tautological rank $n$ quotient.

Recall from Section 2.3.2 that $S_{\delta}(B^\vee)$ has no cohomology provided that there exists $j$ such that

\[ j \leq \delta_j \leq d - n + j - 1. \]

Corresponding to each of the three resolutions $\mathcal{V}_\bullet$, $\mathcal{W}_\bullet$, $\mathcal{U}_\bullet$, we have

**Proposition 3.1.2.** Let $\lambda \neq 0$ be a partition that fits in the $(2n) \times (d - n - 1)$ rectangle and assume that $\lambda$ has $n$-index $i$. For every summand $S_{\delta}(B^\vee) \subset S_{\lambda}(B^\vee \oplus B^\vee) \otimes \bigwedge^*(B)$, the partition $\delta$ satisfies condition (3.1.1) with $j = i$.

In particular, all cohomology of $S_{\lambda}(B^\vee \oplus B^\vee) \otimes \bigwedge^*(B)$ vanishes.

**Proposition 3.1.3.** Let $\lambda \neq 0$ be a partition that fits in the $(2n) \times (d - n - 1)$ rectangle and assume that $\lambda$ has $n$-index $i$.

1. If $i < n$, then for every summand $S_{\delta}(B^\vee) \subset S_{\lambda}(B^\vee \oplus B^\vee) \otimes \text{Sym}^i(B)$, the partition $\delta$ satisfies condition (3.1.1) with $j = i$.

   In particular, all cohomology of $S_{\lambda}(B^\vee \oplus B^\vee) \otimes \text{Sym}^i(B)$ vanishes.

2. If $i = n$ and $k \leq n$, then for every summand $S_{\delta}(B^\vee) \subset S_{\lambda}(B^\vee \oplus B^\vee) \otimes \text{Sym}^k(B)$, the partition $\delta$ satisfies condition (3.1.1) with $j = i$.

   In particular, all cohomology of $S_{\lambda}(B^\vee \oplus B^\vee) \otimes \text{Sym}^k(B)$ vanishes.

**Proposition 3.1.4.** Let $\lambda \neq 0$ be a partition that fits in the $(2n) \times (d - n - r - 1)$ rectangle, for some $r \geq 0$. 

(1) Assume that \( \lambda \) has \( n \)-index \( i \). Let \( k_1, \ldots, k_r \) be nonnegative integers. Then every summand
\[
S_\delta(B^\vee) \subseteq S_\lambda(B^\vee \oplus B^\vee) \otimes \bigwedge^{k_1} B^\vee \otimes \cdots \otimes \bigwedge^{k_r} B^\vee
\]
satisfies condition (3.1.1) with \( j = i \).
In particular, all cohomology of \( S_\lambda(B^\vee \oplus B^\vee) \otimes \bigwedge^{k_1} B^\vee \otimes \cdots \otimes \bigwedge^{k_r} B^\vee \) vanishes.

(2) Assume \( \lambda \neq (1^k) \) has \((k, n)\)-index \( i \), for some \( 0 \leq k \leq n \). Let \( k_1, \ldots, k_{r-1} \) be nonnegative integers. Then every summand
\[
S_\delta(B^\vee) \subseteq S_\lambda(B^\vee \oplus B^\vee) \otimes \bigwedge^{k_1} B^\vee \otimes \cdots \otimes \bigwedge^{k_{r-1}} B^\vee
\]
satisfies condition (3.1.1) with \( j = i \).
In particular, all cohomology of \( S_\lambda(B^\vee \oplus B^\vee) \otimes \bigwedge^{k_1} B^\vee \otimes \cdots \otimes \bigwedge^{k_{r-1}} B^\vee \) vanishes.

3.2. Littlewood–Richardson coefficients. For the proofs of the above propositions, we need a few preliminaries about the Littlewood–Richardson coefficients. The material below is well-known, but to establish the notation, we recall several definitions and basic facts, some of which can be found in [SS, §§2, 3].

For two partitions \( \alpha \) and \( \beta \) of the same size \( |\alpha| = |\beta| \), write \( \alpha \geq \beta \) (\( \alpha \) dominates \( \beta \)) if, for all \( m \), we have
\[
\alpha_1 + \cdots + \alpha_m \geq \beta_1 + \cdots + \beta_m.
\]
Note that \( \alpha \geq \beta \) if and only if \( \alpha^\dagger \leq \beta^\dagger \).

Given partitions \( \alpha, \beta, \gamma \), let \( c^\gamma_{\alpha,\beta} \) denote the Littlewood–Richardson coefficient, which is the multiplicity of the Schur functor \( S_\gamma \) in the tensor product \( S_\alpha \otimes S_\beta \).

The coefficient \( c^\gamma_{\alpha,\beta} \) counts the number of Littlewood–Richardson tableaux. These are fillings of the skew tableau of shape \( \gamma/\alpha \) with content \( \beta \) (i.e., for all \( i \), the label \( i \) appears exactly \( \beta_i \) times) such that the following two properties hold:

- (Semistandard) In each row, the entries are weakly increasing from left to right, and in each column, the entries are strictly increasing from top to bottom.
- (Lattice word property) Let \( w \) be the word (called reading word) obtained by reading the entries in each row from right to left, starting with the top row and going down. For each \( i \) and \( m \), let \( w_i(m) \) be the number of times that \( i \) appears in the first \( m \) entries of \( w \). Then for all \( m \) and \( i \), we have \( w_i(m) \geq w_{i+1}(m) \).

We collect a few facts about these coefficients in the next result.

**Proposition 3.2.1.** (1) For any complex vector bundles \( V, W \), we have
\[
S_\gamma(V \oplus W) \cong \bigoplus_{\alpha, \beta} (S_\alpha(V) \otimes S_\beta(W))^\otimes c^\gamma_{\alpha,\beta}
\]
where the sum is over all partitions \( \alpha, \beta \).

(2) If \( c^\gamma_{\alpha,\beta} \neq 0 \), then \( |\gamma| = |\alpha| + |\beta| \).
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If $c_{\gamma,\alpha,\beta} \neq 0$, then $\gamma$ contains both $\alpha$ and $\beta$, i.e., $\gamma_i \geq \max(\alpha_i, \beta_i)$ for all $i$.

In a Littlewood–Richardson tableau of shape $\gamma/\alpha$ and type $\beta$, all occurrences of the number $i$ must appear in rows $i$ and later.

As a consequence, if $c_{\gamma,\alpha,\beta} \neq 0$, then $\alpha + \beta$ dominates $\gamma$, i.e., for all $m$, we have

$$\sum_{i=1}^{m} (\alpha_i + \beta_i) \geq \sum_{i=1}^{m} \gamma_i.$$ 

If $c_{\gamma,\alpha,\beta} \neq 0$, then $\gamma$ dominates $\alpha \cup \beta$ (this is the partition obtained from all of the rows of $\alpha$ and $\beta$ placed one after the other according to their lengths).

Proof. (1) see [SS, (4.5)] for a derivation.

(2) and (3) are clear from the interpretation in terms of tableaux.

(4) We prove this by induction on $i$. If $i = 1$, there is nothing to show. Now suppose the statement is true for $i$. Suppose that there is a Littlewood–Richardson tableau in which $i+1$ appears in the first $i$ rows. Let $w$ be the reading word of this tableau. By the lattice word property, this instance of $i+1$ cannot appear in a row before the earliest (relative to $w$) instance of $i$, so it must appear in row $i$, and it must appear to the left of $i$ in the tableau. However, this violates the semistandard condition.

(5) This is a consequence of (4) since $\alpha \cup \beta = (\alpha^{\dagger} + \beta^{\dagger})^{\dagger}$ and $c_{\lambda,\alpha,\beta} \neq 0$. □

3.3. Lemmas. To carry out the proofs of Propositions 3.1.2, 3.1.3, and 3.1.4, we first establish a few supporting results.

First, by Proposition 3.2.1 (1), we have

$$S_{\lambda}(B^\vee \oplus B^\vee) \cong \bigoplus_{\alpha,\beta} (S_{\alpha}(B^\vee) \otimes S_{\beta}(B^\vee))^{\oplus c_{\alpha,\beta}^{\lambda}} \cong \bigoplus_{\alpha,\beta,\gamma} S_{\gamma}(B^\vee)^{\oplus c_{\alpha,\beta}^{\gamma}}.$$

Here, the number of rows of the partitions $\alpha, \beta, \gamma$ is less than or equal to $n$, while the number of rows in the partition $\lambda$ is less than or equal to $2n$.

We assume first that $\lambda$ has size $2n \times (d - n - 1)$ as needed in Propositions 3.1.2 and 3.1.3. Reserve $i$ to be the $n$-index of $\lambda$.

Pick a triple $\alpha, \beta, \gamma$ such that $S_{\gamma}(B^\vee) \neq 0$, and $c_{\alpha,\beta}^{\lambda} c_{\alpha,\beta}^{\gamma} \neq 0$. We will deduce a number of restrictions on the partitions $\alpha, \beta, \gamma$.

Lemma 3.3.2. We have $\alpha_i \geq i$.

Proof. Suppose that $\alpha_i < i$. Since $\lambda$ has $n$-index $i$, recall from (2.3.7) that $\lambda_{i+1} = \cdots = \lambda_{i+n} = i$ and thus $\lambda_i \geq i$. Then the $i$th column of the skew shape $\lambda/\alpha$ has at least $n+1$ boxes (in rows $i$ through $n+i$). But then any valid Littlewood–Richardson tableau of shape $\lambda/\alpha$ needs at least $n+1$ labels (because of the semistandard condition). This implies that $\beta_{n+1} > 0$, contradicting the fact that $\beta$ has at most $n$ rows. □

Lemma 3.3.3. We have

$$(\alpha_1 + \beta_1) + \cdots + (\alpha_i + \beta_i) \leq i(d - n + i - 1).$$
Proof. By Proposition 3.2.1(5), we know that $\lambda \geq \alpha \cup \beta$, so that
\begin{equation}
\lambda_1 + \cdots + \lambda_{2i} \geq (\alpha \cup \beta)_1 + \cdots + (\alpha \cup \beta)_{2i} \\
\geq (\alpha_1 + \beta_1) + \cdots + (\alpha_i + \beta_i).
\end{equation}
(3.3.3a)

Since $\lambda_j \leq d - n - 1$ for $j = 1, \ldots, i$ and $\lambda_j \leq i$ for $j = i+1, \ldots, 2i$, the lemma follows. \qed

Lemma 3.3.4. We have $i + 1 \leq \gamma_i \leq d - n + i - 1$.

Proof. We know that $\alpha_i \geq i$ from Lemma 3.3.2. If in fact $\alpha_i \geq i + 1$, then we can use Proposition 3.2.1(3) to conclude that $\gamma_i \geq i + 1$.

Otherwise, we have $\alpha_i = i$. Since $c_{\alpha_\beta}^{\lambda} \neq 0$, there is a Littlewood–Richardson tableau of shape $\lambda/\alpha$ and type $\beta$. Since $\lambda_{i+n} = i$, the $i$th column of $\lambda/\alpha$ has at least $i + n - \alpha_i^\dagger$ boxes, so that $\beta$ has at least $i + n - \alpha_i^\dagger$ rows (by the semistandard condition). Next, there is also a Littlewood–Richardson tableau of shape $\gamma/\alpha$ and type $\beta$.

Suppose that $\gamma_i = i$. Then the $i$th row of $\gamma/\alpha$ has no boxes.

From our previous observation, $\beta$ has at least $i + n - \alpha_i^\dagger$ rows. The integers in the interval $[i, i + n - \alpha_i^\dagger]$ cannot go in the first $i - 1$ rows of $\gamma/\alpha$ by Proposition 3.2.1(4), and cannot go in the $i$th row since it is empty. Thus, these numbers must go in rows $i + 1$ or higher. Again, since $\gamma_i = i$, they are also constrained to the first $i$ columns of $\gamma/\alpha$ as well. Now, in $\gamma/\alpha$, the $i$th column only has boxes in rows $\alpha_i^\dagger + 1, \ldots, n$, at most. Consequently, the labels $[i, i + n - \alpha_i^\dagger]$ can only be placed in rows $\alpha_i^\dagger + 1, \ldots, n$.

Suppose it is possible to do this. Consider the subdiagram of $\gamma/\alpha$ consisting of boxes that are filled with entries $\geq i$. If we subtract $i - 1$ from every entry, this is also a valid Littlewood–Richardson tableau since these numbers did not appear elsewhere in $\gamma/\alpha$ (so the lattice word property still holds). But then we have too many labels: in fact, at least $n - \alpha_i^\dagger + 1$ labels and only $n - \alpha_i^\dagger$ rows to put them into. We have a contradiction to Proposition 3.2.1(4) and hence $\gamma_i \geq i + 1$.

Finally, again by Proposition 3.2.1(4), we have $\alpha + \beta \geq \gamma$. Using Lemma 3.3.3, we obtain
\begin{equation}
i\gamma_i \leq \gamma_1 + \cdots + \gamma_i \leq i(d - n + i - 1),
\end{equation}
(3.3.4a)
and hence $\gamma_i \leq d - n + i - 1$. \qed

Lemma 3.3.5. \begin{enumerate} \item If $i < n$, then $\gamma_{i+1} \geq i$. \item If $i = n$, then $\gamma_n \geq 2n$. \end{enumerate}

Proof. Since $\lambda_{i+1} = i$, we have $\alpha_{i+1} \leq i$ by Proposition 3.2.1(3).

Let $c = i - \alpha_{i+1}$. Then $\lambda/\alpha$ contains the subrectangle occupying rows $i+1, \ldots, i+n$ and columns $\alpha_{i+1} + 1, \ldots, i$, which in particular has $n$ rows and $c$ columns. Since $c_{\alpha_\beta}^{\lambda} \neq 0$, we can

\[2\text{We could relax the condition that } \lambda \text{ is contained in the } (2n) \times (d - n - 1) \text{ rectangle here. It would suffice to know that } \lambda_1 + \cdots + \lambda_i \leq i(d - n - 1) + i - 1.\]
fill $\lambda/\alpha$ with content $\beta$. By examining the $n \times c$ subrectangle and using the semistandard property, we obtain $\beta_n \geq c$. Let $\beta'$ be the result of subtracting $c$ from all parts of $\beta$. Then

$$S_\beta(B^\vee) = (\det B^\vee)^{\otimes c} \otimes S_{\beta'}(B^\vee)$$

since $\text{rank}(B^\vee) = n$. Hence to compute $S_\alpha(B^\vee) \otimes S_\beta(B^\vee)$, we can first add $c$ to all values of $(\alpha_1, \ldots, \alpha_n)$ and then tensor with $S_{\beta'}(B^\vee)$.

In particular, if $i < n$, then $\gamma_{i+1} \geq \alpha_{i+1} + c = i$, again by Proposition 3.2.1(3). Otherwise, if $i = n$, since $\alpha_{n+1} = 0$ we find $c = n$. Using Lemma 3.3.2, we have $\alpha_n \geq n$, and thus $\gamma_n \geq \alpha_n + c \geq 2n$.

**3.4. Vanishing.** We continue to use the notation from the previous section.

**Proof of Proposition 3.1.2.** Consider the tensor product $S_{\gamma}(B^\vee) \otimes \wedge^k(B)$. First we use that

$$\wedge^k(B) = \det(B) \otimes \wedge^{n-k}B^\vee$$

and $\wedge^{n-k}B^\vee = S_{(n-k)}(B^\vee)$. The Pieri rule describes the outcome of tensoring with $\wedge^{n-k}B^\vee$. The results is a sum over partitions where we add $n - k$ boxes, no two in the same row. Tensoring with $\det(B)$ is the same as subtracting 1 from all entries. Therefore, for any summand $S_\delta B^\vee$ of this tensor product, we have $\gamma_i - 1 \leq \delta_i \leq \gamma_i$. Hence we conclude from Lemma 3.3.4 that

$$i \leq \delta_i \leq d - n + i - 1,$$

completing the argument.

**Proof of Proposition 3.1.3.** The Pieri rule applied to symmetric powers tells us that $S_\mu(B^\vee)$ is a summand of $S_\mu(B^\vee) \otimes \text{Sym}^k(B^\vee)$ if and only if $|\mu| = |\mu| + k$ and the interlacing property $\nu_j \geq \mu_j \geq \nu_{j+1}$ holds for all $j$. In fact, it makes no difference if some entries of $\nu$ and $\mu$ are negative since we can make them nonnegative by twisting by powers of $\det(B^\vee)$ and untwisting after.

If $S_\delta(B^\vee)$ is a summand of $S_\gamma(B^\vee) \otimes \text{Sym}^k(B)$, we obtain by dualizing that $S_{-\delta}(B^\vee)$ is a summand of $S_{-\gamma}(B^\vee) \otimes \text{Sym}^k(B^\vee)$. Thus, $|\gamma| = |\delta| + k$ and the interlacing property gives

$$\gamma_{j+1} \leq \delta_j \leq \gamma_j.$$

(In particular, if $S_\delta(B^\vee)$ is a summand of $S_\gamma(B^\vee) \otimes \text{Sym}^k(B)$, then $S_\gamma(B^\vee)$ is a summand of $S_\delta(B^\vee) \otimes \text{Sym}^k(B^\vee)$.)

Consider the $n$-index $i$ of $\lambda$. If $i < n$, then Lemma 3.3.5 tells us that $\gamma_{i+1} \geq i$. The interlacing property then forces $\delta_i \geq i$. If $i = n$, then $\gamma_n \geq 2n$. Since $\gamma$ is obtained from $\delta$ by adding $k$ boxes, we have

$$\delta_n \geq \gamma_n - k \geq 2n - k \geq n$$

since we assume that $k \leq n$.

In any case, under either assumption, we have shown that $\delta_i \geq i$ and also $\delta_i \leq \gamma_i \leq d - n + i - 1$ by Lemma 3.3.4. This is what we set out to prove.
Proof of Proposition 3.1.4. Assume now \( \lambda \) is of size \( 2n \times (d - n - r - 1) \). For case (1), for a partition of \( n \)-index \( i \), we have

\[
\lambda_j \leq d - n - r - 1 \text{ for } j \leq i, \quad \lambda_j \leq i \text{ for } i + 1 \leq j \leq 2i.
\]

Thus, by (3.3.3a) and (3.3.4a), we have

\[
i \gamma_i \leq \gamma_1 + \cdots + \gamma_i \leq \lambda_1 + \cdots + \lambda_{2i} \leq i(d - n - r - 1) + i \cdot i \implies \gamma_i \leq d - n - r - 1 + i.
\]

We also have \( \gamma_i \geq \alpha_i \geq i \) by Lemma 3.3.2. By repeated application of the Pieri rule, each summand

\[
S_\delta(B^\vee) \subseteq S_\gamma(B^\vee) \otimes \bigwedge^{k_1} B^\vee \otimes \bigwedge^{k_2} B^\vee \otimes \cdots \otimes \bigwedge^{k_r} B^\vee
\]

satisfies \( \gamma_j \leq \delta_j \leq \gamma_j + r \). The conclusion follows since

\[
i \leq \gamma_i \leq d - n - r - 1 + i \implies i \leq \delta_i \leq d - n + i - 1.
\]

The second case is entirely similar, but we record the details for completeness. If \( S_\gamma(B^\vee) \) is a summand of \( S_\lambda(B^\vee \oplus B^\vee) \), then by the same reasoning as in Lemma 3.3.2 we have \( i \leq \alpha_i \leq \gamma_i \). By (2.3.9)

\[
\lambda_1, \ldots, \lambda_i \leq d - n - r - 1, \quad \lambda_{i+1}, \ldots, \lambda_{2i} \leq i + 1.
\]

By (3.3.3a) and (3.3.4a), we obtain

\[
i \gamma_i \leq \gamma_1 + \cdots + \gamma_i \leq \lambda_1 + \cdots + \lambda_{2i} \leq i(d - n - r - 1) + i(i + 1) \implies \gamma_i \leq d - n - r + i.
\]

By repeated application of the Pieri rule, all summands \( S_\delta(B^\vee) \) of \( S_\gamma(B^\vee \otimes \bigwedge^{k_1} B^\vee \otimes \cdots \otimes \bigwedge^{k_r} B^\vee \) satisfy \( \gamma_i \leq \delta_i \leq \gamma_i + (r - 1) \). Since

\[
i \leq \gamma_i \leq d - n - r + i \implies i \leq \delta_i \leq d - n + i - 1.
\]

Therefore, condition (3.1.1) is satisfied for \( \delta \) and \( j = i \), and all cohomology vanishes. \( \square \)

4. Corollaries

4.1. Corollary 1.1.5 and universality. We explain the universality arguments needed to derive Corollary 1.1.5 from the genus 0 computations in Theorems 1.1.2, 1.1.3, and 1.1.4.

Regarding equation (1.1.5a), we have the factorization

\[
\sum_{n=0}^{\infty} q^n \chi\left(\text{Quot}_{\mathcal{O}}(\mathbb{C}^N, n), \bigwedge^y L^{[n]}\right) = A^{\chi(O_C)} \cdot B^{\chi(L)}
\]

where \( A, B \in 1 + q \mathbb{Q}[t]_t[q] \) are two universal power series whose coefficients may depend on \( N \) but not on the pair \((C, L)\). This factorization is by now a standard fact, see for instance [EGL, OS, St] for various incarnations of this statement. To establish (1.1.5a), we show

\[
A = (1 - q)^{-1}, \quad B = (1 + qy)^N.
\]
Specializing $C = \mathbb{P}^1$ and $\deg L = \ell \geq n$ in (4.1.1), and using Theorem 1.1.2 we obtain

$$[q^n] A \cdot B^{\ell+1} = \sum_{k=0}^{n} y^k \binom{N\chi(L)}{k},$$

where the brackets denote extracting the relevant coefficient in the $q$-expansion. By direct calculation, we also have

$$[q^n] (1 - q)^{-1} \cdot ((1 + qy)^N)^{\ell+1} = \sum_{k=0}^{n} y^k \binom{N\chi(L)}{k}.$$ 

It remains to explain that the coefficients $[q^n] A \cdot B^{\ell+1}$ for all $\ell \geq n$ determine the series $A, B$ at most uniquely. We argue inductively, each coefficient at a time. Explicitly, we write

$$A = 1 + a_1 q + a_2 q^2 + \cdots, \quad B = 1 + b_1 q + b_2 q^2 + \cdots.$$ 

Then

$$[q^n] A \cdot B^{\ell+1} = a_n + (\ell + 1)b_n + \text{lower order terms in } n.$$ 

The lower order terms are determined by the induction hypothesis. The inductive step follows since the principal terms $a_n + (\ell + 1)b_n$ for all $\ell \geq n$ determine $a_n, b_n$ at most uniquely.

For (1.1.5b) the argument is similar, using the factorization

$$\sum_{n=0}^{\infty} q^n \left( \text{Quot}_{C}(\mathbb{C}^N, n), \bigotimes_{i=1}^{r} (\bigwedge_{y_i} M_i^{[n]})^{\vee} \right) = A^{\chi(\mathcal{O}_C)} \cdot B_1^{\chi(M_1)} \cdots B_r^{\chi(M_r)}.$$ 

This time, we specialize $C = \mathbb{P}^1$, and

$$M_1 = M_2 = \cdots = M_r = M, \quad \deg M = m \geq n.$$ 

By Theorem 1.1.4, we have

$$[q^n] A \cdot (B_1 \cdots B_r)^{m+1} = 1 \text{ for all } m \geq n.$$ 

Indeed, the only non-zero contribution appears from the free term $y_1 = \cdots = y_r = 0$ and yields the answer $\chi(\text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n), \mathcal{O}) = 1$ since $\text{Quot}_{\mathbb{P}^1}(\mathbb{C}^N, n)$ is rational. By the above reasoning, the series

$$A = (1 - q)^{-1}, \quad B_1 \cdots B_r = 1$$

are uniquely determined. Next, we set

$$M_1 = L, \quad M_2 = \cdots = M_r = M,$$

where $\deg L = \deg M - 1 = m - 1 \geq n - 1$. This time around, Theorem 1.1.4 implies

$$[q^n] (A \cdot B_1^{-1}) \cdot (B_1 \cdots B_r)^{m+1} = 1 \implies A \cdot B_1^{-1} = (1 - q)^{-1}, \quad B_1 \cdots B_r = 1.$$ 

Therefore

$$A = (1 - q)^{-1}, \quad B_1 = \cdots = B_r = 1.$$
and (1.1.5b) follows from here.

Equation (1.1.5c) is immediate from Theorem 1.1.3 and no further argument is necessary. □

4.2. **Corollary 1.1.8.** We analyze the cohomology groups of $L^{[n]}$ for all line bundles $L \to \mathbb{P}^1$ using a few simple considerations. The corollary can also be derived by combining the methods of [BGS, Corollary 9.3] when adapted to the case of the projective line, followed by a calculation on the symmetric product.

Let $p \in \mathbb{P}^1$ and write $Q_p = Q|_p \times \text{Quot}$. The exact sequence

$$0 \to L(-p) \to L \to L_p \to 0$$

yields an exact sequence over Quot:

$$0 \to L(-p)^{[n]} \to L^{[n]} \to Q_p \to 0. \quad (4.2.1)$$

When $\deg L \geq n + 1$, the bundles $L^{[n]}$ and $L(-p)^{[n]}$ carry no higher cohomology by Theorems 1.1.2 or 1.1.3 for $k = 1$. Taking cohomology in (4.2.1), we obtain

$$H^i(Q_p) = 0, \quad i \geq 1. \quad (4.2.2)$$

We go back to (4.2.1) written for arbitrary $L$, not necessarily sufficiently positive. Considering cohomology again and using (4.2.2), we obtain

$$H^i(L^{[n]}) = H^i(L(-p)^{[n]}), \quad i \geq 2. \quad (4.2.3)$$

Since $H^i(L^{[n]}) = 0$ for $\deg L \geq n$ and $i \geq 2$, it follows from (4.2.3) that $H^i(L^{[n]}) = 0$ for all $i \geq 2$ and all $L$. □

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