Higher level Zhu’s algebras and modules for vertex operator algebras

Katrina Barron, Nathan Vander Werf and Jinwei Yang

October 11, 2017

Abstract

Motivated by the study of indecomposable, nonsimple modules for a vertex operator algebra $V$, we study the relationship between various types of $V$-modules and modules for the higher level Zhu’s algebras for $V$, denoted $A_n(V)$, for $n \in \mathbb{N}$. We resolve some issues that arise in a few theorems previously presented in the literature, and give examples illustrating the need for certain modifications of previous statements. We establish that whether or not $A_{n-1}(V)$ is isomorphic to a direct summand of $A_n(V)$ affects the types of indecomposable $V$-modules which can be constructed by inducing from an $A_n(V)$-module, and we give some characterizations of the $V$-modules that can be constructed from such inducings. To illustrate these results, we give examples of $A_1(V)$ when $V$ is either the Heisenberg or the Virasoro vertex operator algebra. For these two examples, we show how the structure of $A_1(V)$ in relationship to $A_0(V)$ determines what types of indecomposable $V$-modules can be induced. We construct a family of indecomposable modules for the Virasoro that are logarithmic modules and are not highest weight modules.

1 Introduction

This paper was motivated by a desire to study indecomposable objects in certain module categories for a vertex operator algebra $V$ via modules for the higher level Zhu’s algebras for $V$, $A_n(V)$, for $n \in \mathbb{N}$. In [Z], Zhu introduced an associative algebra, which we denote by $A_0(V)$, for $V$ a vertex operator
algebra. This Zhu’s algebra has proved very useful in understanding the module structure of $V$ for certain classes of modules and certain types of vertex operator algebras. In particular, in the case that the vertex operator algebra is rational, Frenkel and Zhu ([FZ]) showed that there is a bijection between the module category for a vertex operator algebra and the module category for the Zhu’s algebra associated with this vertex operator algebra. Subsequently, Dong, Li and Mason ([DLM]) introduced higher level Zhu’s algebras, $A_n(V)$ for $n \in \mathbb{Z}_+$, and presented several statements that generalize the results of Frenkel and Zhu from the level zero algebras to these higher level algebras, particularly in the semi-simple setting, e.g. for rational vertex operator algebras.

For an irrational vertex operator algebra, instead of the irreducible modules, indecomposable modules are the fundamental objects in the module category. To date, it has proven difficult to find examples of vertex operator algebras that have certain nice finiteness properties but non semi-simple representation theory, i.e. so called $C_2$-cofinite irrational vertex operator algebras, which is the setting of logarithmic conformal field theory. And in general, in the indecomposable nonsimple setting, the correspondence between the category of such $V$-modules and the category of $A_n(V)$-modules is not well understood. In this paper we are able to obtain correspondences for certain module subcategories and illustrate that in certain settings the higher level Zhu’s algebras are effective tools for understanding and constructing $V$-modules in the non semi-simple case.

In this paper, we study two functors defined in [DLM], the functor $L_n$ from $\mathbb{N}_r$-gradable $V$-modules (also called admissible $V$-modules as in [DLM]) to $A_n(V)$-modules, and the functor $\Omega_n/\Omega_{n-1}$ from $A_n(V)$-modules to $\mathbb{N}_r$-gradable $V$-modules, and we investigate when the composition of these two functors is isomorphic to the identity morphism in various module categories. We show that there are some modifications and clarifications needed to some of the statements in [DLM] and give examples to show the necessity of these modifications and clarifications. We investigate the relationship between indecomposable modules for $A_n(V)$ and certain indecomposable modules for $V$. We present some sufficient conditions on both $V$-modules and $A_n(V)$-modules for the functors between the restricted module categories to be mutual inverses, and we investigate the question of what types of $V$-modules are constructed from the induction functor $L_n$ and how the structure of $A_n(V)$, in particular as regards $A_{n-1}(V)$ affects the structure of the types of indecomposable modules that can be induced by $L_n$, and for instance what size
of Jordan blocks for the $L(0)$ grading operator arise.

We present $A_1(V)$ for two vertex operator algebras: the generalized Verma module vertex operator algebras for the Heisenberg and Virasoro algebras. We describe the nature of the modules for these vertex operator algebras as regards the ring structure of $A_1(V)$, in particular in relationship to $A_0(V)$. We construct a family of indecomposable nonsimple modules for the Virasoro vertex operator algebra that are logarithmic modules and are not highest weight modules. We give concrete examples arising from the Virasoro vertex operator algebra motivating the need for the extra conditions we introduce as necessary conditions for the statements of some of the main theorems we prove in this paper to hold.

This paper is organized as follows, in Section 2, we give basic definitions including the definition of Zhu’s algebras $A_n(V)$, for $n \in \mathbb{N}$, the functor $\Omega_n$ from the category of $\mathbb{N}$-gradable $V$-modules to the category of $A_n(V)$-modules, and the functor $L_n$ from the category of $A_n(V)$-modules to the category of $\mathbb{N}$-gradable $V$-modules as defined in [DLM].

In Section 3, in the case when $U$ is an $A_n(V)$-module such that no nonzero submodule of $U$ factors through $A_{n-1}(V)$, we prove in Theorem 3.1 that $\Omega_n/\Omega_{n-1}(L_n(U)) \cong U$ as $A_n(V)$-modules. The italics in the previous statement give the extra condition necessary for this statement to hold in contrast to the statement in [DLM]. We prove Corollary 3.3 to Theorem 3.1 that shows this extra condition is not necessary in the case when $U$ is indecomposable and $A_n(V)$ decomposes into a direct sum of $A_{n-1}(V)$ and a direct sum complement. In Section 4, we give examples to illustrate both the need for the extra condition in Theorem 3.1 for the case of $n = 1$ and the Virasoro vertex operator algebra, and how this extra condition is not needed in, for instance, the case of the $n = 1$ and the Heisenberg vertex operator algebra, and how this is predicated on the relationship between $A_1(V)$ and $A_0(V)$ in these two examples.

In Section 3, we show that the functors $L_n$ and $\Omega_n/\Omega_{n-1}$ are inverse functors when restricted to the categories of simple $A_n(V)$-modules that do not factor through $A_{n-1}(V)$ and simple $\mathbb{N}$-gradable $V$-modules that are generated by their degree $n$ subspace (or equivalently, that have a nonzero degree $n$ subspace). This is a clarification of Theorem 4.9 of [DLM] where we add the extra condition in italics in the previous statement, for the result to hold. In Sections 4.2.2, we give an example involving a simple module for the Virasoro vertex operator algebra to show that this extra condition is necessary for the statement to hold.
In Section 3, we show that $L_n$ sends indecomposable $A_n(V)$-modules to indecomposable $\mathbb{N}$-gradable $V$-modules. We go on to identify a certain subcategory of $\mathbb{N}$-gradable $V$-modules on which $L_n \circ \Omega_n/\Omega_{n-1}$ is the identity functor, and we give further restricted categories of $\mathbb{N}$-gradable $V$-modules and $A_n(V)$-modules, respectively, on which the functors restricted to these subcategories are mutual inverses. More importantly, we identify some characteristics of the types of $V$-modules that can be induced through $L_n$ from an $A_n(V)$-module, for instance properties of the lowest $n$ and first $2n+1$ subspaces and where singular vectors must reside for an $\mathbb{N}$-gradable $V$-module to be induced from its degree $n$ subspace.

In Section 4 we construct some illustrative examples that give further insight into the relationship between the level one Zhu's algebra and the level zero Zhu's algebra and the impact that their ring structures have on their module categories. In particular, we construct an example in detail illustrating the need for our additional assumptions on the category of modules for the $n$ level Zhu's algebra, for certain functorial correspondences to hold.

The two examples we consider are the cases for the generalized Verma module vertex operator algebras for the Heisenberg and the Virasoro algebra. We construct indecomposable $\mathbb{N}$-gradable $V$-modules for each of these vertex operator algebras, $V$, by determining the level one Zhu's algebra associated with $V$, denoted $A_1(V)$. For the Heisenberg vertex operator algebra $A_1(V)$ is isomorphic to $\mathbb{C}[x] \oplus \mathbb{C}[x]$. Thus the irreducible (indecomposable) modules for $A_1(V)$ are irreducible (indecomposable) modules for either the zero level Zhu's algebra $A_0(V)$, which is $\mathbb{C}[\alpha(-1)] \cong \mathbb{C}[x]$, or its direct sum component (see Section 4.1 for details). In particular, there is no nontrivial module for $A_1(V)$ that does not factor through $A_0(V)$ in the case for when $V$ is the Heisenberg vertex operator algebra.

However, the level one Zhu's algebra for the Virasoro vertex operator algebra is isomorphic to $\mathbb{C}[x, y]/(xy)$ as an associative algebra. Since there is a node at $(0, 0)$ on the curve $xy = 0$, the level one Zhu's algebra provides indecomposable modules that do not factor through the level zero Zhu's algebra which is isomorphic to $\mathbb{C}[\omega] \cong \mathbb{C}[x]$. For this reason, we can construct explicit indecomposable modules for the Virasoro vertex operator algebra, that are not highest weight modules, from modules for $A_1(V)$ in this case. We show that in this case the structure of $A_1(V)$ versus $A_0(V)$ gives rise to a family of modules for each $k \in \mathbb{Z}_+$ that have the property that they are graded by the $L(0)$ operator into generalized eigenspaces comprised of Jordan Blocks of size $k$ and $k+1$. 
The indecomposable modules for the associative algebra \( \mathbb{C}[x, y]/(xy) \) were first studied and classified by Gelfand and Ponomarev \([\text{GP}]\) in order to study the representations of Lorentz groups. A study of indecomposable modules for \( \mathbb{C}[x, y]/(xy) \) can be found in \([\text{LS}]\), (cf. \([\text{L}], [\text{NR}], [\text{AL}]\)). In this paper, we present a particular class of modules of \( A_1(V) \) for \( V \) the Virasoro algebra, and in \([\text{BVY}]\) we further study the modules of \( A_1(V) \) and the resulting modules for the Virasoro vertex operator algebra.

2  The notion of Zhu’s algebra \( A_n(V) \) and the functors \( \Omega_n, \Pi_n, \) and \( L_n \)

Let \( V = (V, Y, 1, \omega) \) be a vertex operator algebra. In this section, following \([\text{DLM}]\), we recall the notion of the level \( n \) Zhu’s algebra for \( V \), denoted \( A_n(V) \), the functor \( \Omega_n \) from the category of \( \mathbb{N} \)-gradable \( V \)-modules to the category of \( A_n(V) \)-modules, and the functor \( L_n \) from the category of \( A_n(V) \)-modules to the category of \( \mathbb{N} \)-gradable \( V \)-modules. We recall several results of \([\text{DLM}]\), as well as introduce some notation and prove a lemma that will be useful in proving the main results of this paper.

First, we recall the definition of the level \( n \) Zhu’s algebra \( A_n(V) \), for \( n \in \mathbb{N} \), first introduced in \([\text{Z}]\) for \( n = 0 \), and then generalized to \( n > 0 \) in \([\text{DLM}]\).

**Definition 2.1 (\([\text{Z}], [\text{DLM}]\))** For \( n \in \mathbb{N} \), let \( O_n(V) \) be the subspace of \( V \) spanned by elements of the form

\[
u \circ_n v = \text{Res}_x \frac{(1 + x)^{wt u + n} Y(u, x)v}{x^{2n+2}},\]

for homogeneous \( u \in V \) and \( v \in V \), and by elements of the form \((L(-1) + L(0))v\) for \( v \in V \). The vector space \( A_n(V) \) is defined to be the quotient space \( V/O_n(V) \).

Then \( O_n(V) \) is a two-sided ideal of \( V \), and \( A_n(V) \) is an associative algebra under the multiplication \( \ast_n \) defined by

\[
u \ast_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \text{Res}_x \frac{(1 + x)^{wt u + n} Y(u, x)v}{x^{n+m+1}},\]

for \( v \in V \) and homogeneous \( u \in V \). Then extending linearly, \( u \ast_n v \) is defined for general \( u, v \in V \).

5
Remark 2.2 As noted in [DLM], we have \( O_n(V) \subset O_{n-1}(V) \) for \( n \in \mathbb{Z}_+ \), and thus there is a natural surjective algebra homomorphism from \( A_n(V) \) onto \( A_{n-1}(V) \) given by \( v + O_n(V) \mapsto v + O_{n-1}(V) \). If \( U \) is a module for \( A_n(V) \), then \( U \) is said to factor through \( A_{n-1}(V) \) if the kernel of the epimorphism from \( A_n(V) \) onto \( A_{n-1}(V) \), i.e. \( O_{n-1}(V) \), acts trivially on \( U \), giving \( U \) a well-defined \( A_{n-1}(V) \)-module structure.

Next we recall the definitions of various \( V \)-module structures. We assume the reader is familiar with the notion of weak \( V \)-module for a vertex operator algebra (cf. [LL]).

Definition 2.3 An \( \mathcal{N} \)-gradable weak \( V \)-module (also often called an admissible \( V \)-module as in [DLM]) \( W \) for a vertex operator algebra \( V \) is a weak \( V \)-module that is \( \mathcal{N} \)-gradable, \( W = \coprod_{k \in \mathbb{N}} W(k) \), with \( v_m W(k) \subset W(k + wt v - m - 1) \) for homogeneous \( v \in V \), \( m \in \mathbb{Z} \) and \( k \in \mathbb{N} \), and without loss of generality, we can assume \( W(0) \neq 0 \). We say elements of \( W(k) \) have degree \( k \in \mathbb{N} \).

An \( \mathcal{N} \)-gradable generalized weak \( V \)-module \( W \) is an \( \mathcal{N} \)-gradable weak \( V \)-module that admits a decomposition into generalized eigenspaces via the spectrum of \( L(0) = \omega_1 \) as follows: \( W = \coprod_{\lambda \in \mathbb{C}} W_\lambda \) where \( W_\lambda = \{ w \in W | (L(0) - \lambda Id_W)^j w = 0 \) for some \( j \in \mathbb{Z}_+ \} \), and in addition, \( W_{n+\lambda} = 0 \) for fixed \( \lambda \) and for all sufficiently small integers \( n \). We say elements of \( W_\lambda \) have weight \( \lambda \in \mathbb{C} \).

A generalized \( V \)-module \( W \) is an \( \mathcal{N} \)-gradable weak \( V \)-module where \( \dim W_\lambda \) is finite for each \( \lambda \in \mathbb{C} \).

An (ordinary) \( V \)-module is an \( \mathcal{N} \)-gradable generalized weak \( V \)-module such that the generalized eigenspaces \( W_\lambda \) are in fact eigenspaces, i.e. \( W_\lambda = \{ w \in W | L(0)w = \lambda w \} \).

We will often omit the term “weak” when referring to \( \mathcal{N} \)-gradable weak and \( \mathcal{N} \)-gradable generalized weak \( V \)-modules.

The term logarithmic is also often used in the literature to refer to \( \mathcal{N} \)-gradable weak generalized modules or generalized modules.

Remark 2.4 An \( \mathcal{N} \)-gradable \( V \)-module with \( W(k) \) of finite dimension for each \( k \in \mathbb{N} \) is not necessarily a generalized \( V \)-module since the generalized eigenspaces might not be finite dimensional.
We recall the functors $\Omega_n$ and $L_n$ for $n \in \mathbb{N}$ defined and studied in [DLM]. Let $W$ be an $\mathbb{N}$-gradable $V$-module, and let

$$\Omega_n(W) = \{ w \in W \mid v_i w = 0 \text{ if } wt(v_i) < -n \} \quad \text{for } v \in V \text{ of homogeneous weight}. \quad (1)$$

Then $\Omega_n(W)$ is an $A_n(V)$-module, via the action $[a] \mapsto o(a) = a_{wta-1}$ for $a \in V$ and $[a] = a + O_n(V)$.

**Remark 2.5** The functor $\Omega_n$ from $\mathbb{N}$-gradable $V$-modules to $A_n(V)$-modules we have defined here is the same functor defined in [DLM], but is not the functor called $\Omega_n$ in [V]; rather in [V] the functor denoted $\Omega_n$ is just the projection functor onto the $n$th graded subspace of $W$. We will denote this projection functor by $\Pi_n$; that is for $W$ an $\mathbb{N}$-gradable $V$-module, we have $\Pi_n(W) = W(n)$.

In order to define the functor $L_n$ from the category of $A_n(V)$-modules to the category of $\mathbb{N}$-gradable $V$-modules, we need several notions, including the notion of the universal enveloping algebra of $V$, which we now define.

Let

$$\hat{V} = \mathbb{C}[t, t^{-1}] \otimes V / DC[t, t^{-1}] \otimes V,$$ \quad (2)

where $D = \frac{d}{dt}\otimes 1 + 1 \otimes L(-1)$. For $v \in V$, let $v(m) = v \otimes t^m + DC[t, t^{-1}] \otimes V \in \hat{V}$. Then $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining the degree of $v(m)$ to be $wt(v) - m - 1$ for homogeneous $v \in V$, with bracket

$$[u(j), v(k)] = \sum_{i=0}^{\infty} \binom{j}{i} (u_i v)(j + k - i), \quad (3)$$

for $u, v \in V$, $j, k \in \mathbb{Z}$. Denote the homogeneous subspace of degree $m$ by $\hat{V}(m)$. In particular, the degree 0 space of $\hat{V}$, denoted by $\hat{V}(0)$, is a Lie subalgebra.

Denote by $U(\hat{V})$ the universal enveloping algebra of the Lie algebra $\hat{V}$. Then $U(\hat{V})$ has a natural $\mathbb{Z}$-grading induced from $\hat{V}$, and we denote by $U(\hat{V})_k$ the degree $k$ space with respect to this grading, for $k \in \mathbb{Z}$.

Given a weak $V$-module $W$, consider the following linear map

$$\varphi_W : U(\hat{V}) \longrightarrow \text{End}(W), \quad v(m) \mapsto v^W_m,$$

which we will often just denote by $\varphi$ if the module is $V$ itself or if $W$ is clear.

We shall need the following lemma in the proof of our main theorem:
Lemma 2.6 We have
\[ o(O_n(V)) \subseteq \prod_{i > n} \varphi(U(\hat{V})_i)\varphi(U(\hat{V})_{-i}). \] (5)

Proof. By the \( L(-1) \)-derivative property in \( V \), we have that \( o(L(-1)v) = -o(L(0)v) \), and thus \( o((L(-1) + L(0))v) = 0 \), showing (5) holds trivially for the elements of the form \((L(-1) + L(0))v\) in \( O_n(V) \).

Next, for homogeneous \( u, v \in V \), we have
\[
o \left( \text{Res}_x \frac{(1+x)^{\text{wt} u + n} Y(u, x)v}{x^{2n+2}} \right)
= \sum_{j \in \mathbb{N}} \binom{\text{wt} u + n}{j} o(u_{j-2n-2}v)
= \sum_{j \in \mathbb{N}} \binom{\text{wt} u + n}{j} \text{Res}_{x_1-x_2} \text{Res}_{x_1-x_2} Y(Y(u, x_1-x_2)v, x_2)
\]
\[
(x_1-x_2)^{j-2n-2} x_2^{\text{wt} u + \text{wt} v - j + 2n}
= \text{Res}_{x_1-x_2} \text{Res}_{x_1-x_2} Y(Y(u, x_1-x_2)v, x_2) \frac{x_1^{\text{wt} u + n} x_2^{\text{wt} v + n}}{(x_1-x_2)^{2n+2}}
= \text{Res}_{x_1} \text{Res}_{x_2} Y(u, x_1) Y(v, x_2) \frac{x_1^{\text{wt} u + n} x_2^{\text{wt} v + n}}{(x_1-x_2)^{2n+2}}
- \text{Res}_{x_1} \text{Res}_{x_2} Y(v, x_2) Y(u, x_1) \frac{x_1^{\text{wt} u + n} x_2^{\text{wt} v + n}}{(-x_2 + x_1)^{2n+2}}, \] (6)
where the last step follow from the Jacobi identity on \( V \).

The first term on the right hand side of (6) satisfies
\[
\text{Res}_{x_1} \text{Res}_{x_2} Y(u, x_1) Y(v, x_2) \frac{x_1^{\text{wt} u + n} x_2^{\text{wt} v + n}}{(x_1-x_2)^{2n+2}}
= \sum_{j \in \mathbb{N}} (-1)^j \binom{-2n-2}{j} \text{Res}_{x_1} \text{Res}_{x_2} Y(u, x_1) Y(v, x_2) x_1^{\text{wt} u - n - 2-j} x_2^{\text{wt} v + n + j}
= \sum_{j \in \mathbb{N}} (-1)^j \binom{-2n-2}{j} u^{\text{wt} u - n - 2-j} v^{\text{wt} v + n + j}
= \sum_{k \leq -n-1} (-1)^{n+k+1} \binom{-2n-2}{-n-k-1} u^{\text{wt} u + k+1} v^{\text{wt} v - k-1}
\subseteq \prod_{i > n} \varphi(U(\hat{V})_i)\varphi(U(\hat{V})_{-i}).
\]
Similarly, the second term of the right hand side of \((6)\) satisfies

\[
\text{Res}_{x_1} \text{Res}_{x_2} Y(v, x_2) Y(u, x_1) \frac{x_1^{w_{tu} + n} x_2^{w_{tu} + n}}{(-x_2 + x_1)^{2n+2}}
\]

\[
= \sum_{j \in \mathbb{N}} (-1)^j \binom{-2n-2}{j} \text{Res}_{x_1} \text{Res}_{x_2} Y(v, x_2) Y(u, x_1) x_1^{w_{tu} + n + j} x_2^{w_{tv} - n - j - 2}
\]

\[
= \sum_{j \in \mathbb{N}} (-1)^j \binom{-2n-2}{j} v^{w_{tv} - n - j} u^{w_{tu} + n + j}
\]

\[
= \sum_{k \leq -n-1} (-1)^{n+k+1} \left( \binom{-2n-2}{-n-k-1} u^{w_{tv} + k-1} v^{w_{tu} - k-1}
\right)
\]

\[
\in \prod_{i > n} \varphi(U(\hat{V})_i) \varphi(U(\hat{V})_{-i}).
\]

We can regard \(A_n(V)\) as a Lie algebra via the bracket \([u, v] = u \ast_n v - v \ast_n u\), and then the map \(v(\text{wt } v - 1) \mapsto v + O_n(V)\) is a well-defined Lie algebra epimorphism from \(\hat{V}(0)\) onto \(A_n(V)\).

Let \(U\) be an \(A_n(V)\)-module. Since \(A_n(V)\) is naturally a Lie algebra homomorphic image of \(\hat{V}(0)\), we can lift \(U\) to a module for the Lie algebra \(\hat{V}(0)\), and then to a module for \(P_n = \bigoplus_{p > n} \hat{V}(-p) \oplus \hat{V}(0) = \bigoplus_{p < -n} \hat{V}(p) \oplus \hat{V}(0)\) by letting \(\hat{V}(-p)\) act trivially for \(p \neq 0\). Define

\[
M_n(U) = \text{Ind}_{P_n}^{\hat{V}}(U) = U(\hat{V}) \otimes_{U(P_n)} U.
\]

We impose a grading on \(M_n(U)\) with respect to \(U, n\), and the \(\mathbb{Z}\)-grading on \(U(\hat{V})\), by letting \(U\) be degree \(n\), and letting \(M_n(U)(i)\) be the \(\mathbb{Z}\)-graded subspace of \(M_n(U)\) induced from \(\hat{V}\), i.e., \(M_n(U)(i) = U(\hat{V})_i \otimes_{U(P_n)} U\).

For \(v \in V\), define \(Y_{M_n(U)}(v, x) \in (\text{End}(M_n(U)))(x)\) by

\[
Y_{M_n(U)}(v, x) = \sum_{m \in \mathbb{Z}} v(m) x^{-m-1}.
\]

(7)

Let \(W_A\) be the subspace of \(M_n(U)\) spanned linearly by the coefficients of

\[
(x_0 + x_2)^{w_{tv} + n} Y_{M_n(U)}(v, x_0 + x_2) Y_{M_n(U)}(w, x_2) u
\]

\[- (x_2 + x_0)^{w_{tv} + n} Y_{M_n(U)}(Y(v, x_0) w, x_2) u \]

(8)
for \( v, w \in V \), with \( v \) homogeneous, and \( u \in U \). Set

\[
\overline{M}_n(U) = M_n(U)/\mathcal{U}(\hat{V})W_A.
\]

It is shown in \cite{DLM} that \( \overline{M}_n(U) = \bigoplus_{i \in \mathbb{N}} \overline{M}_n(U)(i) \) is an \( \mathbb{N} \)-gradable \( V \)-module with \( \overline{M}_n(U)(0) \neq 0 \) and \( \overline{M}_n(U)(n) \cong U \) as an \( A_n(V) \)-module.

It is also observed in \cite{DLM} that \( \overline{M}_n(U) \) satisfies the following universal property: For any weak \( V \)-module \( M \) and any \( A_n(V) \)-module homomorphism \( \phi : U \rightarrow \Omega_n(M) \), there exists a unique weak \( V \)-module homomorphism \( \Phi : M_n(U) \rightarrow M \), such that \( \Phi \circ \iota = \phi \) where \( \iota \) is the natural injection of \( U \) into \( \overline{M}_n(U) \). This follows from the fact that \( \overline{M}_n(U) \) is generated by \( U \) as a weak \( V \)-module.

Let \( U^* = \text{Hom}(U, \mathbb{C}) \). As in the construction in \cite{DLM}, we can extend \( U^* \) to \( M_n(U) \) by first an induction to \( M_n(U)(n) \) and then by letting \( U^* \) annihilate \( \bigoplus_{i \neq n} M_n(U)(i) \). In particular, we have that elements of \( M_n(U)(n) = \mathcal{U}(\hat{V})_0U \) are spanned by elements of the form

\[
op_i(a_1) \cdots \nop_s(a_s)U
\]

where \( s \in \mathbb{N} \), \( p_1 \geq \cdots \geq p_s \), \( p_1 + \cdots + p_s = 0 \), \( p_s \geq -n \), \( a_i \in V \) and \( \nop_i(a_i) = (a_i)(\text{wt} a_i - 1 - p_i) \). Then inducting on \( s \) by using Remark 3.3 in \cite{DLM} to reduce from length \( s \) vectors to length \( s - 1 \) vectors, we have a well-defined action of \( U^* \) on \( M_n(U)(n) \).

Set

\[
J = \{ v \in M_n(U) \mid \langle u', xv \rangle = 0 \text{ for all } u' \in U^*, x \in \mathcal{U}(\hat{V}) \}
\]

and

\[
L_n(U) = M_n(U)/J.
\]

**Remark 2.7** It is shown in \cite{DLM}, Propositions 4.3, 4.6 and 4.7, that \( L_n(U) \) is a well-defined \( \mathbb{N} \)-gradable \( V \)-module; in particular, it is shown that \( \mathcal{U}(\hat{V})W_A \subset J \), for \( W_A \) the subspace of \( M_n(U) \) spanned by the coefficients of \( S \), i.e., giving the associativity relations for the weak vertex operators on \( M_n(U) \).

## 3 Main results

We have the following theorem which is a necessary modification to what was presented as Theorem 4.2 in \cite{DLM}:
Theorem 3.1 Let $U$ be a nonzero $A_n(V)$-module, for $n \in \mathbb{N}$. Then $L_n(U)$ is an $\mathbb{N}$-gradable $V$-module. If we assume further, for $n > 0$, that there is no nonzero submodule of $U$ that can factor through $A_{n-1}(V)$, then $L_n(U)(0) \neq 0$, and $\Omega_n/\Omega_{n-1}(L_n(U)) \cong U$.

Proof. By Remark 2.7 $L_n(U)$ is a well-defined $\mathbb{N}$-gradable $V$-module and $\mathcal{U}(\hat{V})W_A \subset J$.

We have $U \subset M_n(U)(n) \subset M_n(U)$, and for $u \in U$, we have $\langle U^*, u \rangle = 0$ if and only if $u = 0$. Therefore $J \cap U = 0$, and thus $U$ is naturally isomorphic to a subspace of $L_n(U)$, namely $U + J$. We denote $u + J$ by $\bar{u}$ for $u \in U$.

Since $U$ is an $A_n(V)$-module, we have that $o(v) \cdot U = 0$ for all $v \in O_n(V)$ and thus $\bar{u} \in \Omega_n(L_n(U))$ for all $u \in U$. But $(U + J) \cap \Omega_{n-1}(L_n(U)) = \{u + J \mid u \in U \text{ and } v_i(u + J) = 0 \text{ if } \text{wt } v_i < -n + 1\}$ is annihilated by $\mathcal{U}(\hat{V})_i$ for $i < -n + 1$ and thus, by Lemma 2.6, is annihilated by the action of $O_{n-1}(V)$, implying that $(U + J) \cap \Omega_{n-1}(L_n(U))$ is a nontrivial $A_n(V)$-submodule of $U + J$ that factors through $A_{n-1}(V)$. This and the fact that $U \cap J = 0$, implies that $(U + J) \cap \Omega_{n-1}(L_n(U)) = 0$. Therefore there is an injection of $A_n(V)$-modules $\iota : U \hookrightarrow \Omega_n/\Omega_{n-1}(L_n(U))$ given by $u \mapsto \bar{u} + \Omega_{n-1}(L_n(U))$. In particular, this implies that $L_n(U)(0) \neq 0$ since $U$ is assumed to be nonzero.

Next we show the injection $\iota$ is also surjective, i.e., if $\bar{w} + \Omega_{n-1}(L_n(U)) \in \Omega_n/\Omega_{n-1}(L_n(U))$, then there exists $u \in U$ such that $\bar{u} = \bar{w} + v$ for $v \in \Omega_{n-1}(L_n(U))$, i.e. $\iota(u) = \bar{w} + \Omega_{n-1}(L_n(U))$. For convenience, we denote the coset $\bar{w} + \Omega_{n-1}(L_n(U))$ by $\hat{w}$.

Let $\hat{w} \in \Omega_n(L_n(U))$, and let $w \in M_n(U)$ be its preimage under the canonical projection.

If $\text{deg } w < n$, then for $v \in V$ with $\text{wt } v_i < -n + 1$, we have $\text{wt } v_i w < 0$ and thus $\hat{w} \in \Omega_{n-1}(L_n(U))$, and $\iota(0) = 0 = \hat{w}$.

For the case when $\text{deg } w = n$, again we recall that in [DLM] it is shown that $\mathcal{U}(\hat{V})W_A \subset J$ and thus there is a natural surjection

$$\pi : \overline{M}_n(U) = M_n(U) / \mathcal{U}(\hat{V})W_A \to L_n(U) = M_n(U) / J$$

$$w + \mathcal{U}(\hat{V})W_A \mapsto w + J.$$ 

This coupled with the fact that $\overline{M}_n(U)(n) \cong U$ implies that if $\text{deg } w = n$, then there exists a $u \in U$ such that $u + \mathcal{U}(\hat{V})W_A = w + \mathcal{U}(\hat{V})W_A \mapsto w + J = \hat{w} \in \Omega_n(L_n(U))$, proving that $\iota(u) = \hat{w} + \Omega_{n-1}(L_n(U))$. 

11
Finally if deg $w > n$, we will show that $\hat{w} = 0$ and thus $\iota(0) = \hat{w}$. Suppose not. If $\hat{w} \neq 0$, then in particular $w \notin J$, and thus there exists $x \in \mathcal{U}(V)_{n - \deg w}$ such that $\langle u', xw \rangle \neq 0$ for some $u' \in U^*$. Consider the $A_n(V)$-module generated by $xw \in M_n(U)$, defined as follows: For $v + O_n(V) \in A_n(V)$ and $u \in U$, let $(v + O_n(V)) \cdot u = v(wt v - 1)u$. Under this action, let $\mathcal{U} = A_n(V) \cdot xw$. By Lemma 2.6, $O_n(V) \cdot \mathcal{U} = \prod_{i > n} \varphi(\mathcal{U}(\hat{V}_i)) \varphi(\mathcal{U}(\hat{V}_{-i})) \cdot \mathcal{U} = \prod_{i > n} \mathcal{U}(\hat{V}_i) \mathcal{U}(\hat{V}_{-i}) \mathcal{U} = 0$. However this implies that $O_{n-1}(V) \cdot \mathcal{U} = \prod_{i > n-1} \mathcal{U}(\hat{V}_i) \mathcal{U}(\hat{V}_{-i}) \mathcal{U} = \mathcal{U}(\hat{V}_n) \mathcal{U}(\hat{V}_{-n}) \mathcal{U}$. But $\hat{w} \neq 0$ in $\Omega_n/\Omega_{n-1}(L_n(U))$ also implies, in particular, that $v_i(w + J) = 0$ if $wt v_i < -n$. Noting then that $\mathcal{U}(\hat{V}_{-n}) \cdot \mathcal{U} = \{ v_i o(v) xw \mid v_i \in \mathcal{U}(\hat{V}_{-n}), v \in A_n(V) \text{ with } wt o(v) = 0, \text{ and } wt x = n - wt w \}$, and thus we have that $wt v_i o(v) x = -wt w < -n$, we then must have that $\mathcal{U}(\hat{V}_{-n}) \cdot \mathcal{U} = 0$. This implies that $o(O_{n-1}(V)) \cdot \mathcal{U} = \mathcal{U}(\hat{V}_n) \mathcal{U}(\hat{V}_{-n}) \cdot \mathcal{U} = 0$, implying that $\mathcal{U} = A_n(V) \cdot xw$ is a submodule of $U$ that factors through $A_{n-1}(V)$. By assumption then, we must have that $xw = 0$, and thus $w \in J$, implying that $\iota(0) = \hat{0} = \hat{w}$.

Note that it is trivially true that $\Pi_n(L_n(U)) \cong U$ as was observed in [V] (where $\Pi_n$ in [V] is denoted by $\Omega_n$).

Theorem 4.2 in [DLM] only imposes the condition on $U$ that it be an $A_n(V)$-module that itself does not factor through $A_{n-1}(V)$, rather than the condition that no nonzero submodule of $U$ factor through $A_{n-1}(V)$. In Section 4.2.1, we give an example that shows that this extra condition is necessary. This example is based on a module for the Virasoro vertex operator algebra and the level one Zhu’s algebra. In Section 4.1, we show that this extra condition is not needed at level $n = 1$ for the Heisenberg vertex operator algebra. Furthermore in these examples, we observe why these cases are different in regards to the structure of $A_1(V)$ for the Heisenberg versus the Virasoro vertex operator algebras as regards the algebra $A_0(V)$.

One of the main reasons we are interested in Theorem 3.1 is what it implies for the question of when modules for the higher level Zhu’s algebras give rise to indecomposable nonsimple modules for $V$. In [V] it is claimed that if $A_n(V)$ is a finite dimensional semisimple algebra for all $n \in \mathbb{N}$, then $V$ is rational. But we are interested in the irrational case and in particular when $A_n(V)$ can be used to construct the indecomposable nonsimple modules for $V$ in the irrational setting.

To this end, we have the following two corollaries to Theorem 3.1.

**Corollary 3.2** Suppose that for some fixed $n \in \mathbb{Z}_+$, $A_n(V)$ has a direct
sum decomposition $A_n(V) \cong A_{n-1}(V) \oplus A'_n(V)$, for $A'_n(V)$ a direct sum complement to $A_{n-1}(V)$, and let $U$ be an $A_n(V)$ module. If $U$ is trivial as an $A_{n-1}(V)$-module, then $\Omega_n/\Omega_{n-1}(L_n(U)) \cong U$.

Proof. If $A_n(V) \cong A_{n-1}(V) \oplus A'_n(V)$, then any $A_n(V)$-module $U$ decomposes into $U = U_{n-1} \oplus U'$ where $U_{n-1}$ is an $A_{n-1}(V)$-module and $U'$ is an $A'_n(V)$-module. If $U$ has no nontrivial submodule that factors through $A_{n-1}(V)$ which is true if and only if $U_{n-1} = 0$, i.e. if and only if $U$ is trivial as an $A_{n-1}(V)$-module, then Theorem 3.1 implies $\Omega_n/\Omega_{n-1}(L_n(U)) \cong U$. \qed

An example of this setting is given in Section 4, below, namely that of the Heisenberg vertex operator algebra and the level one Zhu’s algebra.

**Corollary 3.3** Let $U$ be a nonzero indecomposable $A_n(V)$-module such that there is no nonzero submodule of $U$ that can factor through $A_{n-1}(V)$. Then $L_n(U)$ is a nonzero indecomposable $\mathbb{N}$-gradable $V$-module generated by its degree $n$ subspace, $L_n(U)(n) \cong U$, and satisfying

$$\Omega_n/\Omega_{n-1}(L_n(U)) \cong L_n(U)(n) \cong U$$

as $A_n(V)$-modules.

Furthermore if $U$ is a simple $A_n(V)$-module, then $L_n(U)$ is a simple $V$-module as well.

Proof. Suppose

$$L_n(U) = W_1 \oplus W_2,$$

where $W_1$ and $W_2$ are nonzero $\mathbb{N}$-gradable $V$-modules. Then by Theorem 3.1 and linearity, we have that

$$U \cong \Omega_n/\Omega_{n-1}(L_n(U)) = \Omega_n/\Omega_{n-1}(W_1 \oplus W_2) = \Omega_n/\Omega_{n-1}(W_1) \oplus \Omega_n/\Omega_{n-1}(W_2).$$

Since $U$ is indecomposable, $\Omega_n/\Omega_{n-1}(W_i) = 0$ for $i = 1$ or 2. Without loss of generality assume that $\Omega_n/\Omega_{n-1}(W_1) = 0$. Then we have in $L_n(U)$

$$0 \neq (U + J) \cap W_1(n) \subset (U + J) \cap \Omega_n(W_1) = (U + J) \cap \Omega_{n-1}(W_1),$$

which is an $A_{n-1}(V)$-module. Thus $\{u \in U \mid u + J \in W_1(n)\} \subset U$ is a nonzero submodule of $U$ that factors through $A_{n-1}(V)$, contradicting our assumption. Thus it follows that either $W_1$ or $W_2$ is zero, and $L_n(U)$ is indecomposable.
It is obvious that $L_n(U)$ is generated by $L_n(U)(n) \cong U$ and the fact that this is isomorphic as an $A_n(V)$-module to $\Omega_n/\Omega_{n-1}(L_n(U))$ follows directly from Theorem 3.1.

Finally we observe that the proof of Lemma 4.8 in [DLM] holds, proving the last statement.

**Definition 3.4** For $n \in \mathbb{N}$, denote by $\mathcal{A}_{n,n-1}$ the category of $A_n(V)$-modules such that there are no nonzero submodules that can factor through $A_{n-1}(V)$.

**Definition 3.5** For $n \in \mathbb{N}$, denote by $\mathcal{V}_n$ the category of weak $V$-modules whose objects $W$ satisfy: $W$ is $\mathbb{N}$-gradable with $W(0) \neq 0$; and $W$ is generated by $W(n)$.

With these definitions, we have that Theorem 3.1 shows that $\Omega_n/\Omega_{n-1} \circ L_n$ is the identity functor on the category $\mathcal{A}_{n,n-1}(V)$, and Corollary 3.3 shows that $L_n$ sends indecomposable objects in $\mathcal{A}_{n,n-1}(V)$ to indecomposable objects in $\mathcal{V}_n$. Natural questions that arise then are: What subcategory of $V$-modules does one need to restrict to so that these functors are inverses of each other on this restricted category? But more importantly, what can be said about the correspondences between the subcategories of simple versus indecomposable objects? Below we further investigate these questions.

**Remark 3.6** It is claimed in Theorem 4.9 in [DLM] that the functors $L_n$ and $\Omega_n/\Omega_{n-1}$ induce mutually inverse bijections between the isomorphism classes of simple objects in the category $\mathcal{A}_{n,n-1}$ and the subcategory of simple objects in the category of $\mathbb{N}$-gradable $V$-modules. However, in Section 4.2.2, we show that even in the case of simple objects, if the $\mathbb{N}$-gradable $V$-module is not generated by its degree $n$ subspace (e.g. if the degree $n$ subspace is zero), then this fails to hold. Hence the second condition in Definition 3.5 is a necessary extra condition for the categorical correspondence to hold, even in the simple case. The theorem below is a modification of Theorem 4.9 in [DLM] with this necessary added condition; see also [V].

**Theorem 3.7** $L_n$ and $\Omega_n/\Omega_{n-1}$ are equivalences when restricted to the full subcategories of completely reducible $A_n(V)$-modules whose irreducible components do not factor through $A_{n-1}(V)$ and completely reducible $\mathbb{N}$-gradable $V$-modules that are generated by their degree $n$ subspace (or equivalently, that have a nonzero degree $n$ subspace), respectively. In particular, $L_n$ and
\( \Omega_n/\Omega_{n-1} \) induce naturally inverse bijections on the isomorphism classes of simple objects in the category \( \mathcal{A}_{n,n-1} \) and isomorphism classes of simple objects in \( \mathcal{V}_n \).

**Remark 3.8** We note that Theorem 4.10 of [DLM] in the case of \( V \) rational (i.e., in the semisimple setting) also must be modified accordingly with the extra condition that the category of \( \mathbb{N} \)-gradable \( V \) modules must be restricted to the subcategory of modules generated by their degree \( n \) subspace in order for the statement of the theorem to hold; see also [V], where in the rational setting \( \Omega_n/\Omega_{n-1} \) can be replace by \( \Pi_n \) in many of the statements of [DLM].

We have the following result which gives a stronger statement than Corollary 3.3 and a way of constructing indecomposable \( \mathbb{N} \)-gradable \( V \)-modules which we employ in Section 4.2.1.

**Proposition 3.9** Let \( U \) be an indecomposable \( A_n(V) \)-module. Then \( L_n(U) \) is an indecomposable \( \mathbb{N} \)-gradable \( V \)-module generated by its degree \( n \) subspace.

Furthermore, if \( U \) is finite dimensional, then \( L_n(U) \) is an indecomposable \( \mathbb{N} \)-gradable generalized \( V \)-module.

**Proof.** Suppose \( L_n(U) = W_1 \oplus W_2 \), where \( W_1 \) and \( W_2 \) are nonzero \( \mathbb{N} \)-gradable submodules of \( L_n(U) \). Then \( U + J = L_n(U)(n) = W_1(n) \oplus W_2(n) \). Since \( U \) is indecomposable, we have that either \( U + J = W_1(n) \) or \( U + J = W_2(n) \). Without loss of generality, assume that \( U + J = W_2(n) \). Then \( W_1 \cap (U + J) = 0 \). Let \( \tilde{W}_1 \) be the preimage of \( W_1 \) in \( M_n(U) \). Then \( \tilde{W}_1 \cap U = 0 \), and hence \( \tilde{W}_1(n) \subset J \). Then since \( L_n(U) \) is generated by \( L_n(U)(n) \), we have that \( W_1 \) is generated by \( W_1(n) \), and thus we have \( \tilde{W}_1 \subset J \). Therefore \( W_1 = \tilde{W}_1/J \neq 0 \), contradicting the assumption that \( W_1 \) was nonzero.

Now assume that \( U \) is finite dimensional. Since \( L(0) = o(\omega) \) preserves \( U \), and \( U \) is finite dimensional, we have that \( U + J = L_n(U)(n) \) can be decomposed into a direct sum of generalized eigenspaces for \( L(0) \). However, since \( \omega \) is in the center of \( A_n(V) \), we have that the distinct generalized eigenspaces of \( U \) with respect to \( L(0) \) are distinct \( A_n(V) \)-submodules of \( U \). Therefore we must have that there exists \( \lambda \in \mathbb{C} \) and \( j \in \mathbb{Z}_+ \) such that in \( L_n(U) \), we have \( (L(0) - \lambda I_{U+j+1})^j(U + J) = 0 \).

Then \( L_n(U) \) has a \( \mathbb{C} \)-grading with respect to the eigenvalues of \( L(0) \) induced by \( M_n(U) \) and the eigenvalue \( \lambda \) of \( L(0) \) on \( U \), given by \( M_n(U)(i) = M_n(U)_{\lambda-n+i} \), proving that \( L_n(U) \) is an \( \mathbb{N} \)-gradable generalized \( V \)-module. ■
In general $\Omega_n/\Omega_{n-1}$ will not send indecomposable objects in $\mathcal{V}_n$ to indecomposable $A_n$-modules, or for that matter will $\Pi_n$. And so we have the following questions: If $W$ is an indecomposable object in $\mathcal{V}_n$, when are $W(n) = \Pi_n W$ and $\Omega_n/\Omega_{n-1}(W)$ indecomposable $A_n(V)$-modules? Furthermore, and more importantly for our purposes, what types of indecomposable modules can be constructed from the functor $L_n$? We begin to answer some of these questions below.

It is easy to see the following:

**Proposition 3.10** Let $W$ be an indecomposable object in $\mathcal{V}_n$. Then the $A_n(V)$-module $W(n)$ cannot be decomposed into a direct sum of subspaces $U_1$ and $U_2$ such that $\langle U_1 \rangle \cap \langle U_2 \rangle = 0$, where $\langle U_i \rangle$ denotes the $V$-submodule of $W$ generated by $U_i$, for $i = 1, 2$.

**Proof.** Otherwise, assume $W(n) = U_1 \oplus U_2$ such that $\langle U_1 \rangle \cap \langle U_2 \rangle = 0$. Since $W$ is generated by $W(n)$, we have $W = \langle U_1 \rangle \oplus \langle U_2 \rangle$, contradicting the assumption that $W$ was indecomposable. \hfill \Box

Regarding the question: When is $L_n(\Omega_n/\Omega_{n-1}(W)) \cong W$ for $W$ an $\mathbb{N}$-gradable $V$-module? It is clear that we must at least have that $W$ is an object in the category $\mathcal{V}_n$, i.e. $W$ must be generated by its degree $n$ subspace $W(n)$. However, in general we have that

$$\bigoplus_{k=0}^n W(k) \subset \Omega_n(W)$$

with equality holding if, for instance, $W$ is simple. In Section 4.2.1 we give an example, that of the Virasoro vertex operator algebra, to show that in the indecomposable case equality will not necessarily hold. But we do have the following sufficient criteria, where below we denote the cyclic submodule of $W$ generated by $w \in W$ by $Vw$:

**Theorem 3.11** Let $W$ be an $\mathbb{N}$-gradable $V$-module that is generated by $W(n)$ such that $\Omega_j(W) = \bigoplus_{k=0}^j W(k)$, for $j = n$ and $n-1$. Then $L_n(\Omega_n/\Omega_{n-1}(W))$ is naturally isomorphic to a quotient of $W$.

Furthermore, suppose $W$ also satisfies the property that for any $w \in W$, $w = 0$ if and only if $Vw \cap W(n) = 0$. Then

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong W.$$
Proof. Consider the $A_n(V)$-module injection from $W(n)$ into $\Omega_n(W) = \bigoplus_{k=0}^n W(k)$. Then by the universal property of $M_n(W(n))$, and since $W$ is generated by $W(n)$, there exists a unique $V$-module surjection

$$\Phi : M_n(W(n)) \rightarrow W$$

$$xw + U(\hat{V})W_A \mapsto \varphi(x)w$$

for $x \in U(\hat{V})$ and $w \in W(n)$, where $\varphi$ is defined in (I).

Letting $\mathcal{J} = J/U(\hat{V})W_A$, we have that $\mathcal{J}$ is naturally an $N$-gradable $V$-submodule of $M_n(W(n))$. Then certainly $\ker \Phi \subset \mathcal{J}$ and thus

$$L_n(\Omega_n/\Omega_{n-1}(W)) \cong L_n(W(n)) = M_n(W(n))/J \cong \overline{M}_n(W(n))/\mathcal{J}$$

is an $N$-gradable $V$-module quotient of $W \cong \overline{M}_n(W(n))/\ker \Phi$ by $\ker \Phi/\mathcal{J}$, proving the first paragraph of the theorem.

Now suppose that $Vw \cap W(n) = 0$ implies that $w = 0$ for $w \in W$. Let $\bar{w} \in J$ such that $\bar{w} = w + U(\hat{V})W_A$. Then $\langle \bar{w}', xw \rangle = 0$ for all $x \in U(\hat{V})$ and $w' \in W(n)^*$. Thus $V \cdot \Phi(\bar{w}) \cap W(n) = 0$, which implies that $\Phi(\bar{w}) = 0$. Therefore $\mathcal{J} \subset \ker \Phi$, implying $\mathcal{J} = \ker \Phi$, proving the second paragraph of the theorem.

Remark 3.12 Theorem 3.11 above gives some motivation and intuition about the subspace $J$ of $M_n(U)$ used to define $L_n(U)$. In fact, from Theorem 3.11 we see that $\mathcal{J} = J/U(\hat{V})W_A$ is the maximal submodule of $M_n(U)$ of the form $N/U(\hat{V})W_A$ such that $N \cap U = 0$.

We have the following corollary:

Corollary 3.13 Let $A_{n,n-1}^{Res}$ denote the subcategory of objects $U$ in $A_{n,n-1}$ that satisfy $\Omega_j(L_n(U)) = \bigoplus_{k=0}^j L_n(U)(k)$ for $j = n$ and $n-1$, and let $\mathcal{V}_n^{Res}$ denote the subcategory of objects $W$ in $\mathcal{V}_n$ that satisfy: $\Omega_j(W) = \bigoplus_{k=0}^j W(k)$ for $j = n$ and $n-1$; For any $w \in W$, $w = 0$ if and only if $Vw \cap W(n) = 0$; and $W(n)$ has no nonzero $A_n(V)$-submodule that is an $A_{n-1}(V)$-module.

Then the functors $\Omega_n/\Omega_{n-1}$ and $L_n$ are mutual inverses on the categories $\mathcal{V}_n^{Res}$ and $A_{n,n-1}^{Res}$, respectively. In particular, the categories $\mathcal{V}_n^{Res}$ and $A_{n,n-1}^{Res}$ are isomorphic.

Furthermore, the subcategory of simple objects in $\mathcal{V}_n^{Res}$ is isomorphic to the subcategory of simple objects in $A_{n,n-1}^{Res}$.
Proof. By Theorem 3.1, the functor \( \Omega_{n}/\Omega_{n-1} \) takes objects in \( V_{n}^{\text{Res}} \) to objects in \( A_{n,n-1}^{\text{Res}} \), and \( \Omega_{n}/\Omega_{n-1} \circ L_{n} \) is the identity on \( A_{n,n-1}^{\text{Res}} \).

By Theorem 3.1 and Remark 3.12, the functor \( L_{n} \) takes objects in \( A_{n,n-1}^{\text{Res}} \) to objects in \( V_{n}^{\text{Res}} \), and \( L_{n} \circ \Omega_{n}/\Omega_{n-1} \) is the identity on \( V_{n}^{\text{Res}} \).

The correspondence on the subcategories of simple objects follows from Theorem 3.7.

In particular, this illustrates that the categorical correspondence determined by the functors \( \Omega_{n}/\Omega_{n-1} \) and \( L_{n} \) restricts too subcategories of \( V_{n} \) and \( A_{n,n-1} \) that are too narrow to give a significant understanding of the relationship between indecomposable modules and higher level Zhu’s algebras as regards those modules that can be constructed via the functor \( L_{n} \), which is the setting that motivated this paper in the first place. That is, we are more interested in understanding the nature of the types of indecomposable \( V \)-modules that can be constructed from various classes of \( A_{n}(V) \)-modules through \( L_{n} \), and in fact the functor \( \Omega_{n}/\Omega_{n-1} \) is more useful in the indecomposable nonsimple setting as giving more information when it is not an inverse to \( L_{n} \). We study this issue further in [BVY], and in Section 4.2.1 below we give an example to illustrate the types of indecomposable modules one can construct through the functor \( L_{n} \) from indecomposable \( A_{n}(V) \)-modules.

We also observe the following:

**Proposition 3.14** Let \( U \) be an \( A_{n}(V) \)-module and \( W = L_{n}(U) \). Then \( \bigoplus_{j=0}^{n} W(j) \subset \Omega_{n}(W) \subset \bigoplus_{j=0}^{2n+1} W(j) \), and all singular vectors, \( \Omega_{0}(W) \), must in fact then be contained in \( \bigoplus_{j=0}^{n} W(j) \).

**Proof.** The first inclusion \( \bigoplus_{j=0}^{n} W(j) \subset \Omega_{n}(W) \) is obvious. Suppose \( w \in \Omega_{n}(W) \), but \( w \notin \bigoplus_{j=0}^{2n+1} W(j) \). Then since \( U = W(n) \), we have \( \varphi(U(\hat{V}))w \cap U = 0 \), and thus \( w \in J \), implying \( w = 0 \) in \( L_{n}(U) \). Therefore \( \Omega_{n}(W) \subset \bigoplus_{j=0}^{2n+1} W(j) \). Furthermore, any singular vector \( w \), i.e. any \( w \in \Omega_{0}(W) \), must in fact then be contained in \( \bigoplus_{j=0}^{n} W(j) \), otherwise if \( w \in \bigoplus_{j=n+1}^{2n+1} W(j) \) then again \( \varphi(U(\hat{V}))w \cap U = 0 \), implying \( w \in J \). Therefore \( \Omega_{0}(W) \subset \bigoplus_{j=0}^{n} W(j) \).

**Remark 3.15** Proposition 3.14 and the fact that if \( W = L_{n}(W(n)) \) for \( w \in W \), then \( Vw \cap W(n) = 0 \) if and only if \( w = 0 \), helps to characterize the modules in \( V_{n} \) that are in the image of the functor \( L_{n} \). That is if \( W =...
$L_n(W(n))$, then $W$ must satisfy the following:

(i) $W$ is generated by $W(n)$;
(ii) For $w \in W$, we have $Vw \cap W(n) = 0$ if and only if $w = 0$;
(iii) $\Omega_n(W) \subset \bigoplus_{j=0}^{2n+1} W(j)$ and $\Omega_0(W) \subset \bigoplus_{j=0}^{n} W(j)$.

In particular, we see that the higher level Zhu’s algebras, i.e., $A_n(V)$ for $n \geq 1$, can be used to construct indecomposable nonsimple modules for $V$ with Jordan blocks for the $L(0)$ operator of sizes $k$ through $k+n$, if $A_n(V)$ does not decompose into a direct sum with $A_{n-1}(V)$; see Corollary 3.2. We illustrate this below for $n = 1$ in Section 4.2.1.

4 Examples: Heisenberg and Virasoro vertex operator algebras

In [BVY], we determine the level one Zhu’s algebra for the generalized Verma module vertex operator algebra for the Heisenberg algebra and for the Virasoro vertex operator algebra. In [BVY], we then go on to give some results on the classification of certain modules for these vertex operator algebras using the structure of their level one Zhu’s algebras.

Here we recall from [FZ] and [W] (see also [BVY]), the level zero Zhu’s algebras and from [BVY] the level one Zhu’s algebras for these vertex operator algebras. We point out some distinctive features of these algebras which give interesting examples to illustrate aspects of Theorems 3.1 and 3.7, for instance how the nature of the embedding of the level zero Zhu’s algebras into the level one Zhu’s algebra affects the module structure, and how the need for assessing whether the degree one subspace of a module affects the functorial correspondence of the module categories for the Zhu’s algebras versus the vertex operator algebra.

4.1 The Heisenberg vertex operator algebra, and its level zero and level one Zhu’s algebras

Following, for example [LL], we denote by $\mathfrak{h}$ a one-dimensional abelian Lie algebra spanned by $\alpha$ with a bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle \alpha, \alpha \rangle = 1$, and by $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}k$
the affinization of $\mathfrak{h}$ with bracket relations

$$[a(m), b(n)] = m\langle a, b \rangle \delta_{m+n,0} k, \quad a, b \in \mathfrak{h}, \quad [k, a(m)] = 0,$$

where we define $a(m) = a \otimes t^m$ for $m \in \mathbb{Z}$ and $a \in \mathfrak{h}$.

Set

$$\hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes \mathbb{C}[t] \quad \text{and} \quad \hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}].$$

Then $\hat{\mathfrak{h}}^+$ and $\hat{\mathfrak{h}}^-$ are abelian subalgebras of $\hat{\mathfrak{g}}$. Consider the induced $\hat{\mathfrak{g}}$-module given by

$$M(1) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{h}) \otimes \mathfrak{h} \otimes \mathbb{C}[t]} \mathbb{C} \mathbf{1} \simeq S(\hat{\mathfrak{h}}^-) \quad (\text{linearly}),$$

where $U(\cdot)$ and $S(\cdot)$ denote the universal enveloping algebra and symmetric algebra, respectively, $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$ and $k$ acts as multiplication by $1$. Then $M(1)$ is a vertex operator algebra, often called the *Heisenberg vertex operator algebra*, or the *one free boson vertex operator algebra*.

The conformal element for $M(1)$ is given by $\omega = \frac{1}{2} \alpha(-1)^2 \mathbf{1}$, and any element of $M(1)$ can be expressed as a linear combination of elements of the form

$$\alpha(-k_1) \cdots \alpha(-k_j) \mathbf{1}, \quad \text{with} \quad k_1 \geq \cdots \geq k_j \geq 1. \quad (10)$$

It is known that $M(1)$ is simple and has infinitely many nonisomorphic irreducible modules, which can be easily classified (see [LL]). For every highest weight irreducible $M(1)$-module $W$ there exists $\lambda \in \mathbb{C}$ such that

$$W \cong M(1) \otimes_{\mathbb{C}} \Omega_\lambda,$$

where $\Omega_\lambda$ is the one-dimensional $\mathfrak{h}$-module such that $\alpha(0)$ acts as multiplication by $\lambda$. We denote these irreducible $M(1)$-modules by $M(1, \lambda) = M(1) \otimes_{\mathbb{C}} \Omega_\lambda$, and we let $v_\lambda \in \Omega_\lambda$ such that $\Omega_\lambda = \mathbb{C}v_\lambda$. Note that $M(1) = M(1,0)$.

Write $Y_{M(1,\lambda)}(v, x) = Y_\lambda(v, x)$ for the vertex operator corresponding to $v \in M(1)$ acting on the module $M(1, \lambda)$, and

$$Y_\lambda(\omega, x) = \sum_{n \in \mathbb{Z}} L_\lambda(n) x^{-n-2}.$$

In particular, $L_\lambda(0) v_\lambda = \frac{1}{2} \alpha(0)^2 v_\lambda = \frac{\lambda^2}{2} v_\lambda$. Thus $M(1, \lambda)$ is $\mathbb{N}$-gradable with

$$M(1, \lambda) = \coprod_{k \in \mathbb{N}} M(1, \lambda)_k.$$
and \( M(1, \lambda)_0 \neq 0 \) where \( M(1, \lambda)_k \) is the eigenspace of eigenvectors of weight \( k + \frac{\lambda^2}{2} \) with respect to \( L_\lambda(0) \), i.e. \( M(1, \lambda) \) is a generalized, and in fact an ordinary, \( M(1) \)-module.

We have the following level zero and level one Zhu’s algebras for \( M(1) \) (cf. [FZ], [BVY]):

\[
A_0(M(1)) \cong \mathbb{C}[x, y]/(y - x^2) \cong \mathbb{C}[x]
\]

(11)

under the identification

\[
\alpha(-1)1 + O_0(M(1)) \quad \longleftrightarrow \quad x + (p_0(x, y)),
\]

(12)

\[
\alpha(-1)^21 + O_0(M(1)) \quad \longleftrightarrow \quad y + (p_0(x, y)),
\]

(13)

where \( p_0(x, y) = y - x^2 \).

In addition, there is a bijection between isomorphism classes of irreducible \( N \)-gradable \( M(1) \)-modules and irreducible \( \mathbb{C}[x] \)-modules given by \( M(1, \lambda) \leftrightarrow \mathbb{C}[x]/(x - \lambda) \).

For the level one Zhu’s algebra, we have

\[
A_1(M(1)) \cong \mathbb{C}[x, y]/((y - x^2)(y - x^2 - 2))
\]

(14)

\[
\cong \mathbb{C}[x, y]/(y - x^2) \oplus \mathbb{C}[x, y]/(y - x^2 - 2)
\]

(15)

\[
\cong \mathbb{C}[x] \oplus \mathbb{C}[x] \cong A_0(M(1)) \oplus \mathbb{C}[x]
\]

(16)

under the identification

\[
\alpha(-1)1 + O_1(M(1)) \quad \longleftrightarrow \quad x + (p_0(x, y)p_1(x, y)),
\]

(17)

\[
\alpha(-1)^21 + O_1(M(1)) \quad \longleftrightarrow \quad y + (p_0(x, y)p_1(x, y)),
\]

(18)

where again \( p_0(x, y) = y - x^2 \) and in addition \( p_1(x, y) = y - x^2 - 2 \).

**Remark 4.1** In this case, since the ideals \( I_0 = (p_0(x, y)) \) and \( I_1 = (p_1(x, y)) \) are relatively prime, i.e. \( I_0 + I_1 = \mathbb{C}[x, y] \), we have that the level one Zhu’s algebra is naturally isomorphic to a direct sum of \( A_0(M(1)) \cong \mathbb{C}[x, y]/I_0 \) and its direct sum complement which is isomorphic to \( \mathbb{C}[x, y]/I_1 \).

Thus any indecomposable module \( U \) for \( A_1(M(1)) \) will either be an indecomposable module for \( A_0(M(1)) \) or an indecomposable module for its direct sum complement, \( \mathbb{C}[x, y]/I_1 \). That is we will either have that \( U \) itself will factor through \( A_0(M(1)) \) or only the zero submodule of \( U \) will factor through \( A_0(M(1)) \).

Therefore, any indecomposable \( U \) module for \( A_1(M(1)) \) which does not factor through \( A_0(M(1)) \), will satisfy

\[
U \cong \Omega_1/\Omega_0(L_1(U)).
\]
4.2 Virasoro vertex operator algebras, and their level zero and level one Zhu’s algebras

Let $\mathcal{L}$ be the Virasoro algebra with central charge $c$, that is, $\mathcal{L}$ is the vector space with basis $\{L_n \mid n \in \mathbb{Z}\} \cup \{c\}$ with bracket relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} c, \quad [c, L_m] = 0$$

for $m, n \in \mathbb{Z}$. Let $\mathcal{L}_{\geq 0}$ be the Lie subalgebra with basis $\{L_n \mid n \geq 0\} \cup \{c\}$, and let $\mathbb{C}_{c,h}$ be the 1-dimensional $\mathcal{L}_{\geq 0}$-module where $c$ acts as $c$ for some $c \in \mathbb{C}$, $L_0$ acts as $h$ for some $h \in \mathbb{C}$, and $L_n$ acts trivially for $n \geq 1$. Form the induced $\mathcal{L}$-module

$$M(c, h) = \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{L}_{\geq 0}} \mathbb{C}_{c,h}.$$ 

We shall write $L(n)$ for the operator on a Virasoro module corresponding to $L_n$, and $1_{c,h} = 1 \in \mathbb{C}_{c,h}$. Then

$$V_{Vir}(c, 0) = M(c, 0)/\langle L(-1)1_{c,0} \rangle$$

has a natural vertex operator algebra structure with vacuum vector $1 = 1_{c,0}$, and conformal element $\omega = L(-2)1_{c,0}$, satisfying $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$. In addition, for each $h \in \mathbb{C}$, we have that $M(c, h)$ is an ordinary $V_{Vir}(c, 0)$-module with $\mathbb{N}$-gradation

$$M(c, h) = \prod_{k \in \mathbb{N}} M(c, h)_k$$

where $M(c, h)_k$ is the $L(0)$-eigenspace with eigenvalue $h + k$. We say that $M(c, h)_k$ has degree $k$ and weight $h + k$.

We now fix $c \in \mathbb{C}$, and denote by $V_{Vir}$, the vertex operator algebra $V_{Vir}(c, 0)$.

From [W], and following [BVY], we have the following:

$$A_0(V_{Vir}) \cong \mathbb{C}[x, y]/(y - x^2 - 2x) \cong \mathbb{C}[x] \quad (19)$$

under the identification

$$L(-2)1 + O_0(V_{Vir}) \leftrightarrow x + (q_0(x, y)), \quad (20)$$

$$L(-2)^21 + O_0(V_{Vir}) \leftrightarrow y + (q_0(x, y)), \quad (21)$$

22
where \( q_0(x, y) = y - x^2 - 2x \).

In addition, there is a bijection between isomorphism classes of irreducible \( \mathbb{N} \)-gradable \( V_{Vir} \)-modules and irreducible \( \mathbb{C}[x] \)-modules given by \( L(c, \lambda) \leftrightarrow \mathbb{C}[x]/(x - \lambda) \) where \( T(c, \lambda) \) is the largest proper submodule of \( M(c, \lambda) \) and \( L(c, \lambda) = M(c, \lambda)/T(c, \lambda) \).

For the level one Zhu’s algebra, in [BVY] we show that

\[
A_1(V_{Vir}) \cong \mathbb{C}[x, y]/((y - x^2 - 2x)(y - x^2 - 6x + 4)) \quad (22)
\]

\[
\cong \mathbb{C}[\tilde{x}, \tilde{y}]/(\tilde{x}\tilde{y}) \quad (23)
\]

under the identification

\[
L(-2)1 + O_1(V_{Vir}) \leftrightarrow x + (q_0(x, y)q_1(x, y)),
\]

\[
L(-2)^21 + O_1(V_{Vir}) \leftrightarrow y + (q_0(x, y)q_1(x, y)),
\]

where \( q_0(x, y) = y - x^2 - 2x = \tilde{x} \) and in addition \( q_1(x, y) = y - x^2 - 6x + 4 = \tilde{y} \).

**Remark 4.2** In this case, the ideals \( J_0 = (q_0(x, y)) \) and \( J_1 = (q_1(x, y)) \) are not relatively prime since \( q_0(x, y) - q_1(x, y) = 4x - 4 \), i.e., \( q_0 \) and \( q_1 \) have the point of intersection \( (x, y) = (1, 3) \), and thus \( J_0 + J_1 \neq \mathbb{C}[x, y] \). Thus the level zero Zhu’s algebra is not isomorphic to a direct summand of the level one Zhu’s algebra. This results, as we shall see below, in several interesting examples illustrating the subtleties of the relationship between modules for the higher level Zhu’s algebras for \( V = V_{Vir} \) and \( \mathbb{N} \)-gradable \( V_{Vir} \)-modules.

**Remark 4.3** The result we obtain in [BVY] for \( A_1(V_{Vir}) \) differs from that presented in [V] as follows: In the notation of [V], letting \( L = L(0) \) and \( A = L(-1)L(1) \), then acting on a primary vector (i.e. a vector \( w \) for which \( L(n)w = 0 \) if \( n > 0 \)), we have that \( (y - x^2 - 2x)(y - x^2 - 6x + 4) \) acts (using the zero mode action as computed below in (27) and (28)) as \( 4(A^2 - 2AL + 2A) \). This implies that for such vectors \( A_1(V_{Vir}) \) acts as \( \mathbb{C}[A, L]/(A^2 - 2LA + 2A) \) and this algebra is almost the level one Zhu’s algebra for \( V_{Vir} \) given in [V] but still differs by a minus sign. Thus with the minus sign typo corrected, \( A_1(V_{Vir}) \) acts equivalently to the algebra given in [V] on \( \Omega_1(W) \) for an \( \mathbb{N} \)-gradable \( V \)-module \( W \), but will not in general act the same on \( \Omega_n(W) \) for \( n > 1 \). Since one of the important aspects of the information that the higher level Zhu’s algebras give is contained in the action of \( A_n(V) \) versus the action of \( A_{n-1}(V) \) through the natural epimorphism from \( A_n(V) \) to \( A_{n-1}(V) \), it is essential to have each \( A_n(V) \) realized as its full algebra \( V/O_n(V) \) rather than
as a realization of zero modes acting on $\Omega_n(V)$ so as to be able to compare, e.g. the action of $A_2(V_{vir})$ on an $A_2(V_{vir})$-module such as $\Omega_2(W)$ versus the action of $A_1(V_{vir})$ on the same module.

4.2.1 Virasoro Example 1: $\Omega_n/\Omega_{n-1}(L_n(U)) \ncong U$ since $U$ has a nonzero proper submodule that factors through $A_{n-1}(V)$

We will give a family of examples below to illustrate that, there are nontrivial examples of an indecomposable $U$ module that does not factor through $A_0(V_{vir})$, but a nontrivial submodule does, and then we have

$$\Omega_1/\Omega_0(L_1(U)) \ncong U.$$ 

In addition, we give other aspects of the structure of this family of indecomposable $V_{vir}$-modules, $\{W_k\}_{k \in \mathbb{Z}_+}$, that we construct. For instance, each $W_k$ for $k \in \mathbb{Z}_+$, has a Jordan block decomposition for $L(0)$ with Jordan blocks of size $k$ in the weight zero, and weight greater than one subspaces, and size $k + 1$ in the weight one and greater subspaces.

For $k \in \mathbb{Z}_+$, consider the indecomposable $A_1(V_{vir})$-module given by

$$U = \mathbb{C}[x, y]/((y - x^2 - 2x)^{k+1}, (y - x^2 - 6x + 4)).$$

Clearly $U$ is not a module for $A_0(V_{vir}) \cong \mathbb{C}[x, y]/(y - x^2 - 2x)$ and

$$U \cong \mathbb{C}[x]/(x - 1)^{k+1}.$$ 

Let $w$ be a lowest weight vector for the Virasoro algebra (i.e. $L(n)w = 0$ if $n > 0$) such that

$$L(0)^k w \neq 0; \quad L(0)^{k+1} w = 0.$$ 

Set

$$U' = \text{span}_\mathbb{C}\{L(-1)L(0)^i w \mid i = 0, \ldots, k\},$$ 

(where here we act by $L(-1)$ in order to make $U$ have degree one). We have the following lemma:

**Lemma 4.4** As $A_1(V_{vir})$-modules,

$$U \cong U'$$

under the homomorphism

$$f : U \rightarrow U'$$

\[ (x - 1)^i \mapsto L(-1)L(0)^i w, \] 

(26)
where $(x - 1)^i$ is the image of $(x - 1)^i$ in $U$ under the canonical projection and $i = 0, \ldots, k$.

**Proof.** Clearly the map $f$ is surjective, and since $U$ and $U'$ have the same dimension, $f$ is also injective. We need to show $f$ is an algebra homomorphism.

Denote by $[x]$ and $[y]$ be the image of $x$ and $y$ in $A_1(V_\text{vir})$ under the canonical projection and identification given by (24) and (25). Then $[x] = [L(-2)1]$ and $[y] = [L(-2)^21]$. Thus $[x]$ acts on the modules via

$$o(L(-2)1) = L(0).$$

To determine the action of $[y]$, recall the normal ordering notation

$$o_Y(u, x)Y(v, x)^o = \left( \sum_{m<0} u_n x^{-n-1} \right) Y(v, x) + Y(v, x) \sum_{m \geq 0} u_n x^{-n-1}.$$

Then

$$o(L(-2)^21) = (L(-2)^21)_{wt}L(-2)^21_{-1} = (L(-2)^21)_3$$
$$= \text{Res}_x x^3 o_Y(L(-2)1, x)Y(L(-2)1, x)^o$$
$$= \sum_{m<0} L(m-1)L(-m+1) + \sum_{m \geq 0} L(-m+1)L(m-1)$$
$$= L(1)L(-1) + L(0)^2 + L(-1)L(1) + 2 \sum_{i \geq 2} L(-i)L(i)$$
$$= [L(1), L(-1)] + L(0)^2 + 2 \sum_{i \geq 1} L(-i)L(i)$$
$$= 2L(0) + L(0)^2 + 2 \sum_{i \geq 1} L(-i)L(i).$$

Then we have

$$f([x] \cdot (x - 1)^i) = f([(x - 1)^{i+1} + (x - 1)^i])$$
$$= f((x - 1)^{i+1}) + f((x - 1)^i)$$
$$= L(-1)L(0)^{i+1} + L(-1)L(0)^i$$
$$= L(0)(L(-1)L(0)^i)$$
$$= o(L(-2)1)(L(-1)L(0)^i)$$
$$= [x] \cdot f((x - 1)^i),$$

25
and
\[
f([y] \cdot (x-1)^i) = f([x^2 + 6x - 4] \cdot (x-1)^i) \\
= f([(x-1)^2 + 8(x-1) + 3] \cdot (x-1)^i) \\
= f((x-1)^{i+2}) + 8f((x-1)^{i+1}) + 3f((x-1)^i) \\
= L(-1)L(0)^{i+2}w + 8L(-1)L(0)^{i+1}w + 3L(-1)L(0)^iw,
\]
while from (28), we have
\[
[y] \cdot f((x-1)^i) = (2L(0) + L(0)^2 + 2 \sum_{i \geq 1} L(-i)L(i)) \cdot L(-1)L(0)^iw \\
= L(-1)L(0)^{i+2}w + 8L(-1)L(0)^{i+1}w + 3L(-1)L(0)^iw \\
= f([y] \cdot (x-1)^i).
\]
Thus \(f\) is a \(A_n(V_{Vir})\)-module homomorphism, proving the lemma.

Now we see that in this case
\[
\Omega_1(L_1(U)) = L_1(U)(0) \oplus L_1(U)(1) \oplus \text{span}_C\{L(-1)^2L(0)^kw, L(-2)L(0)^kw\}
\]
and
\[
\Omega_0(L_1(U)) = L_1(U)(0) \oplus \text{span}_C\{L(-1)L(0)^kw\}.
\]
Since \(L(-1)L(0)^kw \in U\), by definition of \(J\), we have that \(L(-1)L(0)^kw \notin J\). In particular,
\[
\Omega_1/\Omega_0(L_1(U)) \cong U/\text{span}_C\{L(-1)L(0)^kw\} \\
\oplus \text{span}_C\{L(-1)^2L(0)^kw, L(-2)L(0)^kw\} \\
\not\cong U,
\]
giving a counter example to Theorem 4.2 in \(\text{[DLM]}\) and showing the necessity of the condition in our Theorem 3.1 that no nontrivial submodule of \(U\) factors through \(A_{n-1}(V)\). That is this case illustrates that since the submodule \(\text{span}_C\{L(-1)L(0)^kw\} \not\subset U\) is an \(A_0(V_{Vir})\)-module, the added condition in our Theorem 3.1 in comparison to Theorem 4.2 of \(\text{[DLM]}\) is indeed necessary for the statement to hold.
Note that in general here
\[
\Omega_m(L_1(U)) = \bigoplus_{j=0}^{m} L_1(U)(j) \oplus \text{span}_C \{ L(-s_1) \cdots L(-s_r)L(0)^k w \mid r \in \mathbb{Z}_+, s_1 \geq s_2 \geq \cdots \geq s_r \geq 1, s_1 + \cdots + s_r = m + 1 \},
\]
for \( m \in \mathbb{N} \),
\[
(L_1(U))(j) = \text{span}_C \{ L(-s_1) \cdots L(-s_r)L(-1)L(0)^i w \mid i = 0, \ldots, k, r \in \mathbb{N}, s_1 \geq s_2 \geq \cdots \geq s_r \geq 1, s_1 + \cdots + s_r + 1 = j \} \oplus \text{span}_C \{ L(-s_1) \cdots L(-s_r)L(0)^i w \mid i = 1, \ldots, k, r \in \mathbb{N}, s_1 \geq s_2 \geq \cdots \geq s_r \geq 1, s_1 + \cdots + s_r = j \}
\]
for \( j \in \mathbb{N} \), where the first direct summand, which occurs if \( j \neq 0 \), consists of Jordan blocks for \( L(0) \) of size \( k + 1 \), and the second direct summand, which occurs if \( j \neq 1 \), consists of Jordan blocks of size \( k \). For the first summand, there are \( p(j - 1) \) such blocks if \( j \neq 0 \), and for the second summand there are \( p(j) \) such blocks if \( j \neq 1 \), where \( p(j) \) denotes the number of partitions of \( j \).

Thus the generalized graded dimension of \( L_1(U) \) is given by
\[
q^{c/24} \left( k + (k + 1)q + \sum_{j=2}^{\infty} (kp(j) + (k + 1)p(j - 1))q^j \right)
= q^{c/24} \left( -kq + (k + (k + 1)q) \prod_{j=1}^{\infty} (1 - q^j)^{-1} \right). \quad (29)
\]

4.2.2 Virasoro Example 2: \( L_n(\Omega_n/\Omega_{n-1}(W)) \not\cong W \) since \( W \neq \langle W(n) \rangle \) even if \( W \) is simple

Now let \( c \neq 0 \), and let \( L(c, 0) \) be the unique simple minimal vertex operator algebra with central charge \( c \), up to isomorphism, i.e. \( L(c, 0) \) is isomorphic to the quotient of \( V_{Vir} = V_{Vir}(c, 0) \) by its largest proper ideal.

Let \( V = W = L(c, 0) \). Then \( W(0) = \mathbb{C}1 \), with \( 1 = 1_{(c,0)} \), and \( W(1) = 0 \).

Thus
\[
\Omega_0(W) = \mathbb{C}1 = W(0) = W(0) \oplus W(1) = \Omega_1(W).
\]

Therefore
\[
L_1(\Omega_1/\Omega_0(W)) = 0 \not\cong W.
\]
4.2.3 **Virasoro Example 3:** \( L_n(\Omega_n/\Omega_{n-1}(W)) \not\cong W \) since \( \Omega_n/\Omega_{n-1}(W) \neq W(n) \), an indecomposable, nonsimple case

Now let \( V = L(c, 0) \) as above and \( W = M(c, 0) \). Then

\[
\Omega_0(W) = \text{span}_C \{ 1, L(-1)1 \} = W(0) \oplus W(1)
\]

\[
\Omega_1(W) = \text{span}_C \{ 1, L(-1)1, L(-1)^21 \} = W(0) \oplus W(1) \oplus CL(-1)^21
\]

and

\[
\Omega_1/\Omega_0(W) \cong CL(-1)^21.
\]

Then we have

\[
L_1(\Omega_1/\Omega_0(W)) = L_1(CL(-1)^21) \cong M(c, -1) \not\cong M(c, 0) = W.
\]

Also note that

\[
L_1(\Pi_1(W)) = L_1(CL(-1)1) \cong \langle L(-1)1 \rangle \not\cong M(c, 0) = W.
\]

**REFERENCES**

[AL] D. Arnold and R. Laubenbacher, Finitely generated modules over pullback rings, *J. Alg.* 184 (1996), 304–332.

[BVY] K. Barron, N. Vander Werf and J. Yang, Level one Zhu’s algebras and certain modules for the Heisenberg and Virasoro vertex operator algebras, in preparation.

[DLM] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Alg.* 206 (1998), 67–98.

[FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* 66 (1992), 123–168.

[GP] I. M. Gelfand and V. A. Ponomarev, *Indecomposable representations of the Lorentz groups*, Russian Math. Surveys 23, 1968, 1–58.

[LS] R. Laubenbacher and B. Sturmfels, A normal form algorithm for modules over \( k[x, y]/\langle xy \rangle \), *J. Alg.* 184 (1996), 1001–1024.
[LL]  J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Prog. in Math., 227 (2004), Birkhäuser, Boston.

[L]  L. S. Levy, Mixed modules over $\mathbb{Z}G$, $G$ cyclic of prime order, and over related Dedekind pullbacks, *J. Alg.* 71 (1981), 62–114.

[NR]  L.A. Nazarova and A. V. Roiter, Finitely generated modules over a dyad of two local Dedekind rings, and finite groups with an Abelian normal divisor of index $p$, *Math. USSR-Izv* 3 (1969), 65–86.

[V]  J. van Ekeren, Higher level twisted Zhu algebras, *J. Math. Phys.* 52 052302 (2011), 36 pp.

[W]  W. Wang, Rationality of Virasoro vertex operator algebras, *Duke Math. J.* 71 (1993), 197–211.

[Z]  Y.-C. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* 9 (1996), 237–307.