MOMENT MAP, CONVEX FUNCTION AND EXTREMAL POINT

KING-LEUNG LEE, JACOB STURM, AND XIAOWEI WANG

Abstract. The moment map $\mu$ is a central concept in the study of Hamiltonian actions of compact Lie groups $K$ on symplectic manifolds. In this short note, we propose a theory of moment maps coupled with an $Ad_K$-invariant convex function $f$ on $\mathfrak{t}^*$, the dual of Lie algebra of $K$, and study the properties of the critical point of $f \circ \mu$. Our motivation comes from Donaldson [Don17] which is an example of infinite dimensional version of our setting. As an application, we interpret Kähler-Ricci solitons as a special case of the generalized extremal metric.

Contents

1. Introduction 1
2. Convex function on $\mathfrak{t}^*$ 2
3. Moment maps and convex functions 3
  3.1. Calabi-Matsushima decomposition 4
  3.2. $\mu$-invariant and extremal vector fields 6
4. Applications in Kähler geometry 10
  4.1. $\text{Ham}(X,\omega_X)$-action on $\mathcal{J}(X,\omega_X)$ 10
  4.2. Functional on $\mathfrak{ham}(X,\omega_X)$. 14
  4.3. Kähler-Ricci soliton. 14
4.4. Tian-Zhu’s generalized Futaki-invariant 15
5. Appendix 17
  5.1. Extending $Ad_K$-invariant functions 17
  5.2. Identification of $\text{PSH}(X,\omega_X) = \text{Ham}(X,\omega_X)/\text{Ham}(X,\omega_X)$ 17
  5.3. Legendre Transform 18
References 19

1. Introduction

Let $(Z,\omega,I)$ be a Kähler manifold equipped with Kähler form $\omega$ and complex structure $I$. Suppose $(Z,\omega)$ admits a Hamiltonian action of a compact Lie group $K$, that is, suppose there exists a moment map

$$\mu : Z \to \mathfrak{t}^* = \text{Lie}(K)^*$$

satisfying

\begin{align*}
\langle d\mu, \eta \rangle_{\text{Lie}} &= \omega(\cdot, \sigma_z(\eta)) \quad \text{for all } \eta \in \mathfrak{t}, \\
\mu(k \cdot z) &= Ad_k^\ast \mu(z) \quad \text{for all } k \in K
\end{align*}

where $\langle \cdot, \cdot \rangle_{\text{Lie}}$ is the canonical pairing and $\sigma_z : \mathfrak{t} \to T_z Z$ denotes the infinitesimal action of $K$. We assume further that the $K$-action on $Z$ can be complexified to a $G$-action with $K \subseteq G$ and $\mathfrak{g} := \text{Lie}(G) = \mathfrak{t} \otimes \mathbb{C}$, i.e. $G$ is the reductive group complexifying $K$. Let $f$ be a smooth, strictly convex $Ad_K$-invariant $\mathbb{R}$ valued function on $\mathfrak{t}^*$. Our first main result is the following generalization of Matsushima-Calabi decomposition (which is the case $f = |\mu|^2$):

**Theorem 1.1 (Theorem 3.3).** Suppose that $z$ is a critical point (necessarily a strict minimum) of $f \circ \mu$, i.e. suppose $df|_{\mu(z)} \in \mathfrak{t}_z$ where $\mathfrak{t}_z$ is the stabilizer of $z$. Then we have decomposition:

$$\mathfrak{g}_z = \bigoplus_{\lambda \leq 0} \mathfrak{g}_\lambda \text{ where } \mathfrak{g}_\lambda := \left\{ \xi \in \mathfrak{g} \mid \text{ ad}_{\sqrt{-1}\xi} df|_{\mu(z)} \xi = \lambda \xi \right\}.$$  

In particular, $\mathfrak{g}_0 \subset \mathfrak{g}_z$ is the reductive part of $\mathfrak{g}_z$, and $df|_{\mu(z)} \in \mathfrak{z}(\mathfrak{g}_0)$.

Next we define (cf. Theorem 3.9) the “$\mu$-invariant” and prove that its non-vanishing is an obstruction to the existence of a critical point. When applied to the infinite dimensional setting of Fano manifolds in [Don17], the $\mu$-invariant is the well known Tian-Zhu [TZ02] generalization of Futaki’s invariant to the setting of Kähler-Ricci solitons.

Date: August 9, 2022.
To explain in more detail one of the motivations of this work, let \((X,\omega_X)\) be a symplectic manifold and 
\[ J_{\text{int}}(X,\omega_X) := \{ J \in \Gamma(\text{End}(TX)) \mid J^2 = -I, \omega_X(J,\cdot) = \omega_X(\cdot,\cdot), \omega_X(J,\cdot) > 0 \text{ and } N_\omega(J) = 0 \} \]
with \(N_\omega(J)\) being the Nijenhuis tensor associated to \(J\). Then \(J_{\text{int}}(X,\omega)\) is the space of integrable \(\omega_X\)-compatible almost complex structures on \((X,\omega_X)\), which is an infinite-dimensional Kähler manifold if equipped with the Berdntsson Kähler form \(\langle \cdot,\cdot \rangle\) defined in [Don17, Theorem 1]. Note, this is different from the Kähler form in [Don97]. By applying the above Theorem to the Kähler manifold \((Z,\omega) := (J_{\text{int}},\langle \cdot,\cdot \rangle)\), we obtain a generalization of Calabi's decomposition Theorem [TZ00, Theorem A] first discovered by Tian and Zhu (revisited in the work of Nakamura [Naka,Nakb]). Furthermore, Tian-Zhu's generalized Futaki invariant is a special case of the \(\mu\)-invariants we introduce in Theorem 3.9.

Acknowledgments. The work of the last author was partially supported by a Collaboration Grants for Mathematicians from Simons Foundation:631318 and NSF:DMS-1609335.

2. Convex function on \(\mathfrak{t}^*\)

Let \(G\) be a reductive linear algebraic group, \(K \subseteq G\) be a fixed maximal compact subgroup, and \(\mathfrak{t} = \text{Lie}(K)\). For \(\eta \in \mathfrak{t}\) and \(\xi^* \in \mathfrak{t}^*\) we write 
\[ \eta(\xi^*) := \langle \xi^*,\eta \rangle = \xi^*(\eta) \text{ with } \langle \cdot,\cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R} \text{ be the natural pairing.} \]
Then if \(k \in K\) we have 
\[ \langle \xi^*,\eta \rangle = \langle \text{Ad}_k^*\xi^*,\text{Ad}_k\eta \rangle. \]
Let \(f : \mathfrak{t}^* \to \mathbb{R}\) be a strictly convex function which is \(\text{Ad}_k^*\)-invariant, that is 
\[ 0 < \langle \cdot,\nabla^2 f\mid_{\alpha^*}(\cdot) \rangle : \text{Sym}^2\mathfrak{t}^* \to \mathbb{R} \]
and 
\[ f \circ \text{Ad}_k^* = f \text{ for all } k \in K. \]

Definition 2.1. Fix \(\alpha^* \in \mathfrak{t}^*\) then 
\[ df_{\alpha^*} \in \mathfrak{t}. \]
We define an isomorphism:
\[ \begin{pmatrix} \nabla^2 f\mid_{\alpha^*} \end{pmatrix} : \mathfrak{t}^* \xrightarrow{\xi^*} \mathfrak{t} \]
\[ \xi^* \mapsto \xi := \nabla^2 f\mid_{\alpha^*}(\xi^*) \]
The convexity of \(f\) in (2.1) allows us to define a positive definite inner product on \(\mathfrak{t}^*\) for every \(\alpha^* \in \mathfrak{t}^*\) as follows.
\[ (\langle \eta^*,\xi^* \rangle)_{\alpha^*} := \frac{\partial^2}{\partial s \partial t} \bigg|_{s=t=0} f(\alpha^* + t\xi^* + s\eta^*) = \nabla^2 f\mid_{\alpha^*}(\xi^*)(\eta^*) = \langle \eta^*,\nabla^2 f\mid_{\alpha^*}(\xi^*) \rangle \]

Proposition 2.2. For \(k \in K\), and \(\alpha^*,\beta^*,\gamma^* \in \mathfrak{t}^*\) and \(\xi,\eta \in \mathfrak{t}\) we have

- \[ df\mid_{\gamma^*}(\beta^*) = df\mid_{\text{Ad}_k^*\gamma^*}(\text{Ad}_k^*\beta^*) \in \mathbb{R} \quad \text{or equivalently } \text{Ad}_k(df\mid_{\gamma^*}) = df\mid_{\text{Ad}_k^*\gamma^*} \in \mathfrak{t}. \]
- \[ (\nabla^2 f\mid_{\text{Ad}_k^*\alpha^*}) \circ \text{Ad}_k^*\beta^* = \text{Ad}_k \circ (\nabla^2 f\mid_{\alpha^*})(\beta^*). \]
- \[ \langle \langle \alpha^*,\beta^* \rangle \rangle_{\gamma^*} = \langle \langle \text{Ad}_k^*\alpha^*,\text{Ad}_k^*\beta^* \rangle \rangle_{\text{Ad}_k^*\gamma^*} \]
- \[ \text{ad}_\gamma(df\mid_{\gamma^*}) = \nabla^2 f\mid_{\gamma^*}(\text{ad}_\gamma(\gamma^*)) \in \mathfrak{t} \quad \text{or equivalently } (\nabla^2 f\mid_{\cdot \gamma^*})^{-1}\text{ad}_\gamma(df\mid_{\gamma^*}) = \text{ad}_\gamma^*(\gamma^*) \in \mathfrak{t}^* \]

In particular, if we assume further that \(\nabla^2 f\) is invertible then by letting \(\gamma^* = \mu(z),\eta = df\mid_{\mu(z)}\), one obtains
\[ \text{ad}_{df\mid_{\mu(z)}}(\mu(z)) = (\nabla^2 f\mid_{\mu(z)})^{-1}\text{ad}_{df\mid_{\mu(z)}}(df\mid_{\mu(z)}) = 0, \forall z \in \mathbb{Z}. \]
- Suppose \(\text{ad}_\gamma^* = 0\), then we have
\[ \nabla^2 f\mid_{\gamma^*} \circ \text{ad}_\gamma^* = \text{ad}_\gamma \circ \nabla^2 f\mid_{\gamma^*} : \mathfrak{t}^* \to \mathfrak{t}, \text{ hence } \nabla^2 f\mid_{\mu} \circ \text{ad}_{df\mid_{\mu}} = \text{ad}_{df\mid_{\mu}} \circ \nabla^2 f\mid_{\mu} \text{ by (2.9).} \]
Proof. To prove (2.5), we apply (2.2) to obtain
\[ \left. \frac{df}{dt} \right|_{t=0} (\gamma^* + t \cdot \beta^*) = \frac{df}{dt} \bigg|_{t=0} (Ad_k\gamma^* + t \cdot Ad_k\beta^*) = \left. \frac{df}{dt} \right|_{t=0} (Ad_k\gamma^*(Ad_k\beta^*)). \]
For (2.6), by applying (2.5) we obtain: for all \( k \in K \)
\[ (\nabla^2 f|_{Ad_k^{-1} \alpha^*} : \alpha^* : Ad_k \beta^*) = \left. \frac{d^2}{dt^2} \right|_{t=0} \frac{df}{dt} \bigg|_{t=0} (Ad_k(\alpha^* + t \beta^*) = \left. \frac{d^2}{dt^2} \right|_{t=0} (Ad_k(\alpha^* + t \beta^*)) = \left. \frac{d^2}{dt^2} \right|_{t=0} (Ad_k(\alpha^* + t \beta^*)). \]
The proof of (2.7) is similar, by definition (2.4)
\[ \langle Ad_k^* \alpha^*, Ad_k^* \beta^* \rangle = \left. \frac{\partial^2}{\partial s \partial t} \right|_{(s,t) = (0,0)} f(Ad_k^* \gamma^* + s \cdot Ad_k^* \alpha^* + t \cdot Ad_k^* \beta^*) \]
and now we apply (2.2) again to remove the \( Ad_k^* \) from the last expression.
To prove (2.8) we first rewrite (1) as
\[ \left. \frac{df}{dt} \right|_{\alpha^*} = \left. \frac{df}{dt} \right|_{Ad_k^{-1} \alpha^*} \]
(2.11)
\[ Ad_k(\alpha^*) = \left. \frac{df}{dt} \right|_{Ad_k^{-1} \alpha^*} \in \mathfrak{t} \]
Now we substitute \( k = \exp(\mathbf{t} \cdot \gamma) \) so \( Ad_k^* \gamma^* = \gamma^* + \mathbf{t} \cdot \mathbf{ad}_k^*(\gamma^*) + O(t^2) \) and differentiate:
\[ \mathbf{ad}_k^*(\alpha^*) = \left. \frac{d}{dt} \right|_{t=0} Ad_k(\alpha^*) = \left. \frac{d}{dt} \right|_{t=0} (Ad_k(\alpha^* + t \gamma^*)) = \nabla^2 f|_{\gamma^*} \left( (\mathbf{ad}_k^*(\gamma^*)) \right). \]
To prove (2.10), we substitute \( k = \exp(\mathbf{t} \cdot \gamma) \) in (2.7). Then by our assumption \( \mathbf{ad}_k^*(\gamma^*) = 0 \implies Ad_k^* \gamma^* = \gamma^* \) we have
\[ \langle \mathbf{\alpha}^*, \mathbf{\beta}^* \rangle = \langle \mathbf{Ad}_k^* \mathbf{\alpha}^*, \mathbf{Ad}_k^* \mathbf{\beta}^* \rangle \]
Now differentiating with respect to \( t \) gives
\[ 0 = \langle \mathbf{ad}_k^*(\alpha^*), \beta^* \rangle + \langle \mathbf{\alpha}^*, \mathbf{ad}_k^*(\beta^*) \rangle = \langle \mathbf{ad}_k^*(\alpha^*), \nabla^2 f|_{\gamma^*} \rangle + \langle \mathbf{\alpha}^*, \nabla^2 f|_{\gamma^*} \rangle \]
\[ = -\langle \mathbf{\alpha}^*, \mathbf{ad}_k^*(\nabla^2 f|_{\gamma^*}) \rangle + \langle \mathbf{\alpha}^*, \nabla^2 f|_{\gamma^*} \rangle \]
\[ = \langle \mathbf{\alpha}^*, \mathbf{ad}_k^*(\nabla^2 f|_{\gamma^*}) \rangle. \]
3. Moment maps and convex functions
Now let \((Z, \omega, J)\) be a Kähler manifold with Kähler form \( \omega \) and complex structure \( J \in \text{End}(TZ) \). The Riemannian metric is given as:
\[ \langle u, v \rangle_{TZ} := \omega(u, Jv) \text{ and } \langle Ju, v \rangle_{TZ} = \omega(u, v). \]
Let \( K \) be a compact Lie group acting on \( Z \) isometrically with \( \mu : Z \to \mathfrak{t}^* \) being a \( K \)-equivariant moment map, that is, for all \( z \in Z, \mathbf{V} \in T_z Z \) and \( \eta \in \mathfrak{t} \) we have
\[ \mu(k \cdot z) = Ad_k^*(\mu(z)) \]
(3.12)
\[ d\mu_z(v) = \langle Jv, \sigma_z(\eta) \rangle_{T_z Z} = \langle Jv, \sigma_z(\eta) \rangle_{T_z Z} \]
(3.13)
Proposition 3.1. Let \( z \in Z \) and \( \xi \in \mathfrak{t} \). Then
\[ d\mu_z(\sigma_z(\xi)) = \mathbf{ad}_k^*(\mu(z)) \]
(3.14)
In particular,
(1) If \( \delta \in \mathfrak{t}_z \) then \( \mathbf{ad}_k^*(\delta) = 0 \).
(2) If \( \eta \in \mathfrak{t}^* \) then (3.13) with \( v = \sigma(\xi) \) implies
\[ d\mu_z(\sigma_z(\xi), \eta)_{T_z Z} = -\langle J\sigma(\xi), \mu(\xi) \rangle_{T_z Z} = \langle \nabla^2 f|_\mu(\xi^*), \eta \rangle := \nabla^2 f|_\mu(\xi^*). \]
(3.15)
In particular, \( \mathbf{ad}_k^*(\mu(z)) : \mathfrak{t}^* \to \mathfrak{t}^* \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle \) as it was so on the right hand side of the above equation.
Proof. For the first equality let $k = \exp(t \cdot \xi)$ and compute
\[
\text{ad}_k^\ast(\mu(z)) = \frac{d}{dt} \bigg|_{t=0} \text{Ad}_k^\ast(\mu(z)) = \frac{d}{dt} \bigg|_{t=0} (\mu(k \cdot z)) = d\mu_z(\sigma_z(\xi))
\]
The second identity is (2.8) in Proposition 2.2. For the last identity, we have Notice that we have
\[
\langle \mu, \text{ad}_{df}|_{\mu(z)}(\xi^*) \rangle_{\mu(z)} = 0 \quad \forall \xi^* \in \text{Sym}^2 \mathfrak{t}^* \quad \forall \mu \in \mathfrak{g}
\]
and if we assume further that $f(0) = 0$ then we will have
\[
\frac{d}{dt} \left( -\log \left| \frac{\mu(z(t))}{|z|} \right| \right) = -\langle \mu(z(t)), df|_{\mu(z(t))} \rangle_{\mu(z(t))} 
\]
which already appeared in [Don17].

**Corollary 3.2.** Let $z(t) := g(t) \cdot z$ with $g(t) \in G$ satisfying:
\[
g \ni \frac{dg(t)}{dt}g^{-1}(t) = \sqrt{-1} \cdot df|_{\mu(z(t))}\in \sqrt{-1} \mathfrak{t}^*.
\]
If we write $g(t) = e^{\sqrt{-1}t\xi_t}$ with $\xi_t \in \mathfrak{t}$ then
\[
\frac{dg(t)}{dt}g^{-1}(t) = \sqrt{-1} \xi_t \iff z(t) = g(t) \cdot z = I \circ \sigma_z(0)(\xi_t) = \frac{dz(t)}{dt} \bigg|_{t=0} = -\sigma_z(0)(df|_{\mu(z(0))})
\]
That is, the negative gradient flow of $f \circ \mu$.
Moreover, we have
\[
\frac{d}{dt} \left( -\log \left| \frac{\mu(z(t))}{|z|} \right| \right) = -\langle \mu(z(t)), df|_{\mu(z(t))} \rangle_{\mu(z(t))}
\]
and if we assume further that $0 < \langle \cdot, \nabla^2 f|_{\alpha^*}(\cdot) \rangle : \text{Sym}^2 \mathfrak{t}^* \to \mathbb{R}$ for all $\alpha^* \in \mathfrak{t}^*$ and $f(0) = 0$ then we will have
\[
\frac{d}{dt} \left( -\log \left| \frac{\mu(z(t))}{|z|} \right| \right) = -\langle \mu(z(t)), df|_{\mu(z(t))} \rangle_{\mu(z(t))} \leq -f(\mu(z(t))).
\]
which already appeared in [Don17].

**3.1. Calabi-Matsushima decomposition.** In this section, prove a generalized version of classical Calabi-Matsushima decomposition [Calabi]. Let $g = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ and
\[
\mathfrak{g}_\xi = \{\xi_1 + \sqrt{-1} \xi_2 \in \mathfrak{g} : \sigma(\xi_1) + J_\sigma(\xi_2) = 0\} \subset \mathfrak{g}
\]

**Theorem 3.3.** Assume that $z \in Z$ is a critical point of $f \circ \mu$, which will be called an $f$-extremal point. Then

1. $df|_{\mu(z)} \in \mathfrak{t}_z = \{\xi \in \mathfrak{t} : \sigma(\xi) = 0\}.$
2. $\text{ad}_{df|_{\mu(z)}}^\ast(\mu(z)) = 0.$
3. We have
\[
\mathfrak{g}_z = \bigoplus_{\lambda \leq 0} \mathfrak{g}_\lambda \text{ where } \mathfrak{g}_\lambda = \{\xi \in \mathfrak{g}_z : \sqrt{-1} \text{ad}_{df|_{\mu(z)}}^\ast(\xi) = \lambda \xi\}
\]
4. In particular, we have $\mathfrak{g}_0 = \mathfrak{t}_z \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathfrak{g}_{\text{red}} := \mathfrak{g}/\mathfrak{g}_{\text{solv}}$ where $\mathfrak{g}_{\text{solv}} \subseteq \mathfrak{g}$ is the maximal solvable subalgebra.

\[\text{For } f \in C^2(\mathfrak{t}^*), \xi^* \in \mathfrak{t}^* \text{ with } \nabla^2 f \geq 0, \text{ we have}
\]
\[
f(\xi^*) - f(0) = \int_0^1 \frac{d}{dt} (f(t\xi^*)) dt = \int_0^1 (\xi^*, df|_{t\xi^*}) dt 
\]
\[
\left( \frac{d}{dt} (\xi^*, df|_{t\xi^*}) = (\xi^*, \nabla^2 f|_{t\xi^*}(\xi^*)) \right)_{t \geq 0} \geq 0 
\]
from which we deduce
\[
f(\mu(z)) = f(\mu(z)) - f(0) \leq (\mu(z), df|_{\mu(z)})
\]
Proof. To prove (1) we assume $z$ is a critical point of $f \circ \mu$ so $d(f \circ \mu)\xi(v) = 0$ for all tangent vectors $v \in T_z Z$. Thus
\[
df|_{\mu(z)}(d\mu_z(v)) = \langle d\mu_z(v), df|_{\mu(z)} \rangle = 0 \text{ for all } v \in T_z Z.
\]
Now (3.13) implies $\langle t v, \sigma_z(d\mu_z) \rangle = 0$ for all $v$, hence $\sigma_z(df|_{\mu(z)}) = 0$.

Part (2) follows from Part (1) of Proposition 3.14.

To prove (3), we let $k = \exp(t \cdot df|_{\mu(z)})$. Then (1) implies $k \cdot z = z$ so $\mu(z) = A^* k \mu(z)$ by (3.12). Thus if we substitute $k$ and $\alpha^* = \mu(z)$ into part (2) of Proposition 2.2, we have
\[
\langle \xi^*, \eta^* \rangle_{\mu(z)} = \langle \alpha^*, \alpha^* \rangle_{\mu(z)}
\]
and differentiate with respect to $t$ we get
\[
\langle \alpha^*df|_{\mu(z)} \xi^*, \eta^* \rangle_{\mu(z)} + \langle \xi^*, \alpha^*df|_{\mu(z)} \eta^* \rangle_{\mu(z)} = 0
\]
or equivalently, combining Proposition 2.2 part (4) with the identification:
\[
\nabla^2 f|_{\alpha^*} : \xi^* \mapsto \xi := \nabla^2 f|_{\alpha^*}(\xi^*)
\]
we obtain
\[
\langle \alpha^*df|_{\mu(z)} \xi^*, \eta \rangle_{\mu(z)} + \langle \xi, \alpha^*df|_{\mu(z)} \eta \rangle_{\mu(z)} = 0 \text{ for all } \xi, \eta \in \mathfrak{t}.
\]
Thus $\langle \cdot, \cdot \rangle_{\mu(z)}$ extends uniquely to a Hermitian inner product on $\mathfrak{g} := \mathfrak{t} \otimes \mathbb{C}$ for which the linear operator
\[
\sqrt{-\text{Ad}df|_{\mu(z)}} : \mathfrak{g} \rightarrow \mathfrak{g}
\]
is Hermitian.

In particular, all of its eigenvalues are real numbers. Moreover, $df|_{\mu(z)} \in \mathfrak{k}_z$ so $\sqrt{-\text{Ad}df|_{\mu(z)}} \in \mathfrak{g}_z$. To prove (3), we need the following:

Lemma 3.4. Let $z$ be a critical point of $f \circ \mu$ and assume
\[
\sqrt{-\text{Ad}df|_{\mu(z)}} \xi = \lambda \xi
\]
for some $\xi \in \mathfrak{g}_z$. Then $\lambda \leq 0$.

Proof. Write $\xi = \xi_1 + \sqrt{-1}\xi_2$ with $\xi_1, \xi_2 \in \mathfrak{t}$. Then
\[
\lambda \xi_1 + \lambda \sqrt{-1} \xi_2 = \lambda \xi = \sqrt{-\text{Ad}df|_{\mu(z)}}(\xi_1 + \sqrt{-1} \xi_2) \implies \begin{cases} \text{Ad}df|_{\mu(z)} \xi_1 = \lambda \xi_2 \\ \text{Ad}df|_{\mu(z)} \xi_2 = -\lambda \xi_1. \end{cases}
\]
Now (3.15) implies
\[
\lambda \langle \xi_2, \xi_2 \rangle_{\nabla^2 f|_{\mu(z)}} = \langle \text{Ad}df|_{\mu(z)}(\xi_1), \xi_2 \rangle_{\nabla^2 f|_{\mu(z)}} \overset{(3.15)}{=} -\langle J \sigma(\xi_1), \sigma(\xi_2) \rangle_T Z = -\langle \sigma(\xi_2), \sigma(\xi_2) \rangle_T Z = -|\sigma(\xi_2)|^2_T Z
\]
where in the last equality the fact that $\xi_2 \in \mathfrak{g}_z$ so $\sigma(\xi_1) + J \sigma(\xi_2) = 0$ is used. Thus $\lambda \leq 0$ with equality $|\sigma(\xi_2)|_T Z = |\sigma(\xi_1)|_T Z = 0$, i.e. if and only if $\xi_2 \in \mathfrak{t}_z \otimes \mathbb{C} \subset \mathfrak{g}_z$.

Part (3) and (4) follows from the Lemma above, our proof is thus completed.

Example 3.5. Notice that the convexity of $f$ is necessary in Theorem 3.3 above. Let us consider $K \times K$-action on $Z \times Z$ with
\[
\begin{array}{cccc}
\mathfrak{t}^* & \otimes \mathfrak{t}^* \\
(\alpha^*, \beta^*) & \mapsto & \mathbb{R} & \mapsto & |\alpha^*|^2 - |\beta^*|^2
\end{array}
\]
Let $z_0 \in Z$ be a extremal point, i.e. $\mu_Z(z_0) \in \mathfrak{k}_{z_0}$. Then $(z_0, z_0) \in Z \times Z$ is a $f$-extremal point as
\[
\begin{array}{c}
\sqrt{-\text{Ad}df|_{\mu_Z}(z_0)}(\xi_0) = \left[ \sqrt{-1}(\mu_Z(z_0), -\mu_Z(z_0)), (\xi_0) \right] = \lambda \cdot (\xi, 0) \\
\sqrt{-\text{Ad}df|_{\mu_Z}(z_0)}(0, \xi_0) = \left[ \sqrt{-1}(\mu_Z(z_0), -\mu_Z(z_0)), (0, \xi) \right] = -\lambda \cdot (0, \xi)
\end{array}
\]
that is
\[
\sqrt{-\text{Ad}df|_{\mu_Z}(z_0)} : \mathfrak{g}_{z_0} \oplus \mathfrak{g}_{z_0} \rightarrow \mathfrak{g}_{z_0} \oplus \mathfrak{g}_{z_0}
\]
is indefinite contrast to Theorem 3.3 where $f$ is assumed to be convex.
3.2. $\mu$-invariant and extremal vector fields. In this subsection, we introduce an obstruction for the existence of an $f$-extremal point (defined in Theorem 3.3) and introduce the extremal vector field for those $z \in Z$ when the $\dim_{\mathbb{R}} T_z$ is maximal (cf. Definition 3.18).

Lemma 3.6. Consider the $G$-action on the polynomial ring $\mathbb{C}[g^*]$ induced by the $Ad_G^*$-action on $g^*$, we have $\mathbb{C}[g^*]^G \cong \mathbb{C}[\xi^*]^W$ with $W$ being the Weyl group of $G$. In particular,

(1) any $f \in \mathbb{R}[\xi^*]^{W} = R[\xi^*]^{K}$ is a restriction of a unique $f \in \mathbb{C}[g^*]^G$. (cf. Reference can be found here.)

(2) \[
\nabla|_{\xi^*} : \mathbb{C}[g^*] \rightarrow \mathbb{C}[\xi^*]^{\mathbb{R}} \rightarrow \mathbb{C}[\xi^*]^{\mathbb{R}} \rightarrow \nabla|_{\xi^*} \in T_{\xi^*} \mathbb{C}[g^*] \equiv \mathbb{C}
\]

satisfying

\[
\nabla|_{\xi^*} = Ad_{\gamma}(\nabla|_{\xi^*}) \in \mathbb{C}, \ \forall \gamma \in G.
\]

Moreover, for $\xi_0 \neq \xi_1 \in \xi^*$ satisfying $\nabla^2 f|_{\xi_0} \geq 0$ in the sense of (2.1) with $\xi_0 \xi_1 \subset \xi^*$ denoting the line joining $\xi_0$ and $\xi_1$.

Then \[
\langle \xi_1 - \xi_0, \nabla f|_{\xi_0} \rangle - \langle \xi_1 - \xi_0, \nabla f|_{\xi_1} \rangle > 0
\]

with $= \text{if and only if} \ nabla^2 f|_{\xi_0} \equiv 0$. In particular, if one assume that \[
\nabla^2 f|_{\xi_0} > 0
\]

Then $df|_{\xi_0} \rightarrow Ad_{\xi} \xi^*$ is one-to-one for any $\xi \in G$.

Proof. By our assumption that $G$ is reductive, we conclude part (1) from the Harish-Chandra isomorphism and Proposition 2.2-(1). In particular, we have isomorphism of the GIT quotients $g^* // G \cong \xi^* // W$.

Next we prove the last part. The proof of identity (3.19) is the same as the part (1) of Proposition 2.2. Let $\xi_0 \neq \xi_1 \in \xi^*$ such that $\nabla^2 f|_{\xi_0} = \nabla^2 f|_{\xi_1}$ and $\xi_0 = (1 - t)\xi_0 + t\xi_1 \in \xi^*$ then we have

\[
\langle \xi_1 - \xi_0, \nabla f|_{\xi_0} \rangle - \langle \xi_1 - \xi_0, \nabla f|_{\xi_1} \rangle = \int_0^1 \frac{d}{dt}(\xi_1 - \xi_0, \nabla f|_{\xi_1}) dt = \int_0^1 (\xi_1 - \xi_0, \nabla^2 f|_{\xi_1}(\xi_1 - \xi_0)) dt > 0.
\]

Hence, our claim follows from (3.19).

Example 3.7. Let $g = gl(n)$ and $g > t = u(n)$. Then $f(\xi) = \text{Tr}(\xi^k)$, $\xi \in g$ extends $f(\xi) = |\xi|^{k/2}$, $\forall \xi \in u = \{\xi \in g|\xi = -\xi^*\}$. And $df|_{\xi} = k\xi^{k-1} \in g(n)$.

We recall the definition of the moment map:

\[
\mu_{z}(v) = \omega(v, \sigma_z(\xi)) = \langle Jv, \sigma_z(\xi) \rangle_{T_zZ} \text{ for all } \xi \in \xi \text{ and } v \in T_zZ.
\]

In the presence of a complex structure, by comparing the $(0, 1)-(1, 0)$- components

\[
(\xi, v)^{(1, 0)} := \frac{1}{2}(1 - \sqrt{-1}J)v, \text{ and } \partial f(v) := \frac{1}{2}(df(v) + \sqrt{-1}d((J \cdot v)),
\]

we obtain for all $\xi \in \xi$ and $v \in T_z^C Z.

(22)

\[
\langle \partial f|_{\xi}(v), \xi \rangle = \langle (d - \sqrt{-1}J \circ d)\mu_z(v), \xi \rangle = \langle d\mu|_{\xi}(v), \xi \rangle + \langle \sqrt{-1}d\mu|_{\xi}(J \cdot v), \xi \rangle
\]

Now we claim that (3.22) holds for all $\xi \in \xi \otimes_{\mathbb{R}} \mathbb{C} = g$, but this follows from the fact that the holomorphic $G$-action on $Z$ naturally extends the $\mathbb{R}$ linear morphism $\sigma_z : \xi \rightarrow T_zZ$ to a $\mathbb{C}$-linear morphism $\sigma_z : g \rightarrow T^C Z$.

This establishes (3.22).

Lemma 3.8. Let $C \subset Z$ be any complex subvariety and $\{\xi(z)\} : C \rightarrow g$ being holomorphic function satisfying $\xi(z) \in g_z$. Then $\mu(z) = (\xi(z))$ is a holomorphic function in $z \in C$.

\[\text{More explicitly, it suffices to prove it holds if } \xi \in \xi \text{ is replaced by } \sqrt{-1}\xi. \text{ For this, we multiply both sides of (3.21) by } \sqrt{-1}. \text{ This has the effect of replacing } \xi \text{ by } \sqrt{-1}\xi \text{ on the left. On the right, we get } \sqrt{-1} \cdot (\sigma|_{\xi(\xi)})^{1,0} = \sqrt{-1} \cdot \frac{1}{2} (1 - \sqrt{-1}J) \sigma|_{\xi(\xi)} = \frac{1}{2} (1 - \sqrt{-1}J) \sigma|_{\xi(\xi)} = \frac{1}{2} (1 - \sqrt{-1}J) \sigma|_{\xi(\xi)} (\sigma|_{\xi(\xi)})^{1,0} \text{ i.e. } \sigma \text{ is a } \mathbb{C}\text{-linear map.} \]
Proof. It follows from (3.22) and the assumption that \( \hat{\partial} \xi = 0 \) and \( \xi(z) \in \mathfrak{g}_z \) that
\[
\hat{\partial} \langle \mu, \xi \rangle |_{z} = \hat{\partial} \langle \mu, \xi \rangle |_{z} + \langle \mu, \hat{\partial} \xi \rangle |_{z} = 0.
\]
\[\text{□}\]

Theorem 3.9 \((\mu\text{-invariant})\). Let \( \xi \in \mathfrak{t}_z \), \( \eta \in \mathfrak{g}_z \) and \( \phi : g \rightarrow \mathbb{R} \) satisfying:
\[
(3.23) \quad \phi(\text{Ad}_g \xi) = \phi(\xi) \text{ for all } g \in G.
\]
Then
1. \( \langle d\phi|_{\text{Ad}_g \xi}, \text{Ad}_g \eta \rangle \) is independent of \( g \in G \).
2. The function \( \chi : G \rightarrow \mathbb{C} \) defined via
\[
(3.24) \quad \chi(g) := \langle \mu(g \cdot z), d\phi|_{\text{Ad}_g \xi}, \text{Ad}_g \eta \rangle = \langle \mu(g \cdot z), \text{Ad}_g \eta \rangle - \langle d\phi|_{\text{Ad}_g \xi}, \text{Ad}_g \eta \rangle
\]
is constant.

Proof. The first part follows from our assumption \( \phi(\text{Ad}_g \xi) = \phi(\xi) \) and the proof of Proposition 2.2 (1).

For the second part, the independence of \( g \in G \) for the first term in (3.24) was proved in [Wan04, Proposition 6]. We give more conceptual proof as follows. By the Ad\(_K\)-equivariance of \( \mu \), one obtains
\[
\chi|_{G \cdot z} \text{ is constant function.}
\]
On the other hand, Lemma 3.8 implies that \( \chi|_{G \cdot z} \) is holomorphic as \( \text{Ad}_g(\xi) \) varies holomorphically along \( G \cdot z \subset Z \). So \( \chi|_{G \cdot z} \) must be a constant function since \( \chi|_{K \cdot z} \) is and \( K \cdot z \subset G \cdot z \) is a totally real subvariety.

For the second term in (3.24), the independence of \( g \in G \) follows from part (1) of Lemma 3.6 and the fact
\[
(3.25) \quad \langle d\phi|_{\text{Ad}_g \xi}, \text{Ad}_g \eta \rangle \overset{\chi}{=} \langle \text{Ad}_g^*\phi(\xi), \text{Ad}_g \eta \rangle = \langle d\phi(\xi), \eta \rangle.
\]
\[\text{□}\]

Remark 3.10. Notice that no convexity is assumed for \( \phi \). This is particular the case in [Ham02, HamL12]’s work on g-soliton, that is well-defindefness of Futaki-invariants needs no convexity.

Example 3.11. Let \( \chi : g \rightarrow \mathbb{C} \) be Lie algebra morphism \(^3\) and \( \psi \in \mathbb{C}[g] \) (or even more generally, \( \psi \) is an analytic function defined over \( g \)) with \( (\mathbb{C}[g], [\cdot, \cdot]) \) being equipped with a natural polynomial Poisson algebra structure induced from \( (g, [\cdot, \cdot]) \). Then \( \phi := \psi \circ \chi \) satisfies the assumption (3.23) as
\[
\frac{d}{dt} \bigg|_{t=0} \phi(\text{Ad}_{e^{t\eta}} \xi) = \chi \left( \psi'(\xi) \cdot [\eta, \xi] \right) = \chi \left( (\psi(\xi)), \xi \right) = 0.
\]

Definition 3.12. Let \( f : \mathfrak{t}^* \rightarrow \mathbb{R} \) be a convex function and for any \( \xi \in \mathfrak{t} \) let
\[
\hat{f}(\xi) := \sup_{\eta \in \mathfrak{t}^*} \left( \eta^* (\xi) - f(\eta^* ) \right) : \mathfrak{t} \to \mathbb{R}, \quad \xi \in \mathfrak{t}
\]
be the Legendre transform of \( f \), and \( \forall \eta \in g \) we write \( (\eta f)(\xi) := (\eta f)|_{\xi} \) by regarding \( \eta \in g = T\mathfrak{g} \) as a vector field on \( g \).

Corollary 3.13. Let \( f \in \mathbb{R}[\mathfrak{t}^*]^W = \mathbb{R}[\mathfrak{t}^*] \hat{\text{Ad}}_K^* \) satisfying \( \nabla^2 f|_{\mathfrak{t}^*} > 0 \) in the sense of (2.1). Then for \( \xi \in \mathfrak{t}_z \) and \( \eta \in \mathfrak{g}_z \) the function
\[
(3.26) \quad F(\eta) := \langle \mu(g \cdot z), \text{Ad}_g \eta \rangle - \langle \mu(g \cdot z), \text{Ad}_g \eta \rangle - \langle d\hat{f}|_{\text{Ad}_g \xi}, \text{Ad}_g \eta \rangle
\]
is independent of \( g \in G \), where \( d\hat{f} = (df)^{-1} \) is defined via: \( g^* \ni d\hat{f}|_{\xi} = (df)^{-1}(\xi) := \eta^* \leftrightarrow df|_{\eta^*} = \xi \in \mathfrak{g} \).

Proof. By Chevalley isomorphism, we know that every \( f \in \mathbb{R}[\mathfrak{t}^*]^W = \mathbb{R}[\mathfrak{t}^*] \hat{\text{Ad}}_K^* \) is the restriction of a function which by abusing of notation still denoted by \( f \in \mathbb{C}[\mathfrak{g}^*] \hat{\text{Ad}}_G^* \). Hence assumption (3.23) is met and our claim follows from Theorem 3.9. \[\text{□}\]

Remark 3.14. Indeed, the Chevalley type isomorphism holds in more general sense, which was included in Appendix 5.1.

Definition 3.15. We define the \( \mu \)-invariant for \( (z, \xi) \in Z \times \mathfrak{t}_z \) as:
\[
\langle \mu(z), d\hat{f}|_{\xi} \rangle = \langle \mu(z), \eta \rangle - \langle \eta f|_{\xi} \rangle \text{ for } \eta \in \mathfrak{g}_z.
\]
In particular, if \( z \in Z \) is a critical point of \( f \circ \mu \) then
\[
df|_{\mu(z)} = \xi \equiv d\hat{f}|_{\xi} = \mu(z) \text{ and hence } \langle \mu(z), d\hat{f}|_{\xi} \rangle = 0, \quad \forall \eta \in \mathfrak{t}_z.
\]
\(^3\)In particular, it descends to a linear morphism \( g/[g, g] \rightarrow \mathbb{C} \).
Example 3.16. Let \( Z = \mathbb{F}^n \) and \( \mu(z) = \sqrt{-1} \frac{z^*}{|z|^2} \) then for any \( A \in \text{GL}(n+1) \) and \( \xi \in \mathfrak{u}(n+1)_z \) then we have \( \xi \cdot z = \lambda \cdot z \) and

\[
\langle \mu(A \cdot z), \text{Ad}_A \xi \rangle = \text{Tr} \left( \frac{A \xi A^{-1}}{|A|^2} \right) = \frac{\text{Tr}(Az^*A^* \cdot A \xi A^{-1})}{|A|^2}
\]

\[
= \frac{\text{Tr}(z^*A^* \cdot A \xi)}{|A|^2}
\]

\[
\left( : \xi \cdot z = \lambda z \right) = \lambda \cdot \frac{\text{Tr}(z^*A^* \cdot A z)}{|A|^2} = \lambda.
\]

Lemma 3.17. For any \( z \in Z \), there is at most one \( \xi_f \in \mathfrak{t}_z \) when exists solving

\[
(3.27) \quad \langle \mu(z) - df(\xi_f, \eta) \rangle = 0, \quad \forall \eta \in \mathfrak{t}_z
\]

Proof. Let \( \xi_i \in \mathfrak{t}_z, \quad i = 0, 1 \) and \( \xi = (1-t) \xi_0 + t \xi_1 \in \mathfrak{t}_z \). Then we have

\[
\langle (df)^{-1}(\xi_1), \xi_1 - \xi_0 \rangle - \langle (df)^{-1}(\xi_0), \xi_1 - \xi_0 \rangle = \int_0^1 \frac{d}{dt} \langle (df)^{-1}(\xi_t), \xi_t - \xi_0 \rangle \cdot dt
\]

Differentiating \( df|_{df^{-1}(\xi_1)} = \xi_t \) implies:

\[
\nabla f|_{df^{-1}(\xi_1)} \left( \frac{d}{dt} \langle df^{-1}(\xi_t) \rangle \right) = \frac{d\xi_t}{dt} = \xi_1 - \xi_0
\]

from which we deduce

\[
\int_0^1 \frac{d}{dt} \langle (df)^{-1}(\xi_t), \xi_t - \xi_0 \rangle \cdot dt = \int_0^1 \langle (\nabla f)|_{df^{-1}(\xi_1)} \rangle^{-1} \cdot (\xi_1 - \xi_0, \xi_1 - \xi_0) \cdot dt
\]

\[
\text{(2.3)} \quad > \quad 0
\]

as \( (\nabla f)^{-1} > 0 \) by our assumption (3.20), from which we deduce that extremal vector \( \xi_z \) must be unique by letting \( \eta = \xi_0 - \xi_1 \in \mathfrak{t}_z \).

The final conclusion follows from our assumption by letting \( \xi = \xi_f \in \mathfrak{t}_z \) be the unique solution to

\[
(df)^{-1}(\xi) - \pi_{\mathfrak{t}^*_z} \mu(z) \in \mathfrak{t}_z^* := \{ \xi \in \mathfrak{t}_z^* \mid \langle \xi, \eta \rangle = 0, \quad \forall \eta \in \mathfrak{t}_z \} \subset \mathfrak{t}_z^*.
\]

\[\square\]

Definition 3.18. Let \( z \in Z \) and \( f \) satisfies the convexity assumption (3.20) and

\[
\text{(3.28)} \quad \dim \mathfrak{m} \cap \mathfrak{g}_z = \dim \mathfrak{t}_z = \dim \mathfrak{g}_z/(\mathfrak{g}_z)_{\text{Solv}}.
\]

that is, \( \dim \mathfrak{t}_z \) achieves maximal dimension among \( z' \in G \cdot z \), and

- The composition of morphism

\[
\hat{\mathfrak{t}}(\mathfrak{t}_z) \xrightarrow{df-(df)^{-1}} \mathfrak{t}^* \xrightarrow{\pi \mathfrak{t}^*_z} \hat{\mathfrak{t}}(\mathfrak{t}_z)_z^*
\]

is surjective, where \( \hat{\mathfrak{t}}(\mathfrak{t}_z) \subset \mathfrak{t}_z \) is the center of \( \mathfrak{t}_z \) and \( \pi \mathfrak{t}^*_z : \mathfrak{t}^* \rightarrow \hat{\mathfrak{t}}(\mathfrak{t}_z)_z^* \) is the projection induced by the inclusion \( \hat{\mathfrak{t}}(\mathfrak{t}_z) \subset \mathfrak{t} \).

Let \( \xi_f \in \hat{\mathfrak{t}}(\mathfrak{t}_z) \) be the unique solution (whose existence is guaranteed by Lemma 3.17)

\[
(3.29) \quad \langle \mu(z) - df(\xi_f, \eta) \rangle = 0, \quad \forall \eta \in \mathfrak{t}_z.
\]

We call \( \xi_f \) the \( f \)-extremal vector field at \( z \) inspired by the work of [FM95]. In particular, \( z \) is \( f \)-extremal when \( df|_{\mu(z)} = \xi_f \).

\[\vspace{1em}\]

\[\text{This condition is necessary thanks to Theorem 3.3, where} \ (\mathfrak{g}_z)_{\text{Solv}} \subset \mathfrak{g}_z \text{ is the maximal solvable subalgebra.}\]
Remark 3.19. Notice that the (3.29) condition implies that $\xi_f \in \mathfrak{y}(\mathfrak{t}_z) \subset \mathfrak{t}_z$, the center of $\mathfrak{t}_z$. To see that, one notice that for all $\eta \in \mathfrak{t}_z$ and $k \in K_z$, then

$$0 = \langle Ad_k \mu(z) - Ad_k df(\xi), Ad_k \eta \rangle = \langle \mu(z) - df|_{Ad_k \xi}, Ad_k \eta \rangle$$

Since $Ad_k : \mathfrak{t}_z \to \mathfrak{t}_z$ is an isomorphism, we obtain $Ad_k \xi = \xi$ by the uniqueness of extremal vector field (cf. Lemma 3.17), hence $\xi_f \in \mathfrak{y}(\mathfrak{t}_z)$ as we claimed.

Our next result, stating that if $\xi_f(z)$ is the $f$-extremal vector field at $z$ then for any $z' \in G \cdot z$ satisfying the condition (3.28), there is a unique $\xi_f(z')$ canonically related to $\xi_f(z)$. More precisely, we have the following:

Corollary 3.20. Let $z \in Z$ be a point satisfying (3.28) and $\xi_f(z) \in \mathfrak{t}_z$ is the $f$-extremal vector field at $z$. For any $z' = g \cdot z \in Z$, $g \in G$ at which (3.28) is satisfied, there is a $g' \in G_{g \cdot z}$ such that

$$\xi_f(z') = \xi_f(g \cdot z) = Ad_{g' \cdot g} \xi_f(z) \in \mathfrak{t}_{g \cdot z} = (Ad_g \mathfrak{g}_z) \cap \mathfrak{t}$$

is the $f$-extremal vector field at $z' = g \cdot z \in Z$.  

Proof. By our assumption (3.28)

$$\dim_C(\mathfrak{t}_z := \mathfrak{g}_z \cap \mathfrak{t}) = \dim_C \left( \mathfrak{g}_{g \cdot z} := (Ad_g \mathfrak{g}_z) \cap \mathfrak{t} \right) = \dim_C \mathfrak{g}_z/(\mathfrak{g}_z)_{\text{Solv}}$$

both $Ad_g(\mathfrak{t}_z = \mathfrak{g}_z \cap \mathfrak{t}), \mathfrak{g}_z \cap \mathfrak{t} \subset \mathfrak{g}_{g \cdot z}$ are the Lie algebras of maximal compact subgroups of $G_z$. In particular, they are conjugate to each other inside $\mathfrak{g}_z$ by the Cartan-Iwasawa-Malcev Theorem, hence

$$(3.30) \quad \exists g' \in G_{g \cdot z}, \text{ such that } Ad_{g'} \circ Ad_g(\mathfrak{t}_z) = \mathfrak{t}_{g \cdot z} \subset \mathfrak{g}_{g \cdot z}$$

since $\mathfrak{t}_{g \cdot z} = Ad_g \mathfrak{g}_z \cap \mathfrak{t} = \mathfrak{g}_g \cap \mathfrak{t}$. By Theorem 3.9 and our assumption $\xi_f(z)$ being the $f$-extremal vector field at $z$, $\forall \eta \in \mathfrak{t}_z$ we have

$$0 = \langle \mu(z) - d\hat{f}|_{\xi_f(z)}(\eta), \eta \rangle = \langle \mu((g' \circ g) \cdot z) - d\hat{f}|_{Ad_{g'} \circ g \circ \xi_f(z)}(\eta), Ad_{g'} \circ g \circ \eta \rangle$$

as $Ad_{g'} \circ g \circ \eta \in \mathfrak{t}_{g \cdot z}$, $Ad_{g'} \circ g \circ \eta \in \mathfrak{t}_{g \cdot z}$ by (3.30). By the uniqueness $\xi_f$ in Definition 3.18, one obtains

$$\xi_f(z') = \xi_f(g \cdot z) = Ad_{g' \circ g} \xi_f(z)$$

which depends only on $g \cdot z \in Z$ but not on $g, g'$.

\[ \square \]

Example 3.21. Let $\left( Z = (\mathbb{P}^1)^d, \omega_Z = \sum_{i=1}^d \pi_i \omega_{\mathbb{P}^1} \right)$ and $K = SU(2) < SL(2)$ and let $z_t(d') := d' \{ [1, 1] \} + (d - d') \{ [0, 1] \} \in Z$, then for $d' > 0$ we have

$$\mathfrak{t}_{z_t'(d')} = \begin{cases} \mathfrak{u}(1) = \sqrt{-1} \mathfrak{t} R = \mathfrak{r} \cdot \xi_0 & t = 0 \\
0 & t \neq 0 \end{cases} \quad \text{and} \quad \xi_f = \begin{cases} \xi_0 & t = 0 \\
\hat{\eta} & t \neq 0 \end{cases} \quad \dim \mathfrak{t}_{z_t} = 1$$

even though $z_t(d'/2)$ are polystable.

Example 3.22. Let $(\bullet, \bullet)$ denote the Killing form on $\mathfrak{t}$, which induces an isomorphism of $\mathfrak{t} \to \mathfrak{t}^*$, hence equips $\mathfrak{t}^*$ with an inner product $| \bullet, \bullet |_{\mathfrak{t}^*}$. If $f = | \bullet, \bullet |_{\mathfrak{t}^*}$, then the extremal vector field is the unique vector $\xi_f \in \mathfrak{t}_z$ satisfying

$$\langle \xi_f, \eta \rangle_{\mathfrak{t}} = \langle \mu(z), \eta \rangle_{\text{Lie}} \quad \text{for all } \eta \in \mathfrak{t}_z.$$

Notice that $f(\xi_f^*) = | \xi_f^* |_{\mathfrak{t}^*}^2 = -\text{Tr}((\xi_f^*)^2) \in \mathbb{R}[\mathfrak{t}^*]^W$, thanks to the fact

$$\mathfrak{t} := \{ \xi \in g \mid \xi = -\theta(\xi) \} \subset g$$

with $\theta : g \to g$ being the Cartan involution for $\mathfrak{t}$. Hence, $| \xi_f^* |_{\mathfrak{t}^*}^2 \in \mathbb{R}[\mathfrak{t}^*]^W$ is the restriction of $\text{Tr}((\xi_f^*)^2) \in C[\mathfrak{t}^*]^{Ad_G}$ to $\mathfrak{t}$.

\[ ^5 \text{Although } g, g' \text{ are not necessarily unique, } \xi_f(z') \text{ is uniquely determined by } z' \in Z. \text{ One should also notice that } g' \text{ seems to be necessary as } Ad_g \xi_f(z) \text{ might not be in } \mathfrak{t}. \]
4. Applications in Kähler geometry

Now we apply the theory we developed in the previous section to Kähler geometry. Let $(X,\omega_X)$ be a symplectic manifold with a pre-quantum complex line bundle $(L,h,\nabla) \to X$ equipped with a Hermitian metric $h$ together with a $h$-compatible connection $\nabla$ such that its curvature form satisfies: $\text{Ric}(\nabla) = \omega_X$. Let

\[ J_{\text{int}}(X,\omega_X) := \{ J \in \Gamma(\text{End}(TX)) \mid J^2 = -I, \omega_X(J \cdot , J \cdot ) = \omega_X(\cdot , \cdot ), \omega_X(J \cdot , J \cdot ) > 0 \text{ and } N_{\omega_X}(J) = 0 \} \]

with $N_{\omega}$ denoting the Nijenhuis tensor associated to $J$. So $J_{\text{int}}(X,\omega)$ is the space of integrable $\omega_X$-compatible almost complex structures on $(X,\omega_X)$, which is an infinite-dimensional Kähler manifold if equipped with two Kähler forms

- Donaldson-Fujiki’s Kähler form $(J_{\text{int}},\Omega,\mathcal{J})$ introduced in [Fuj90] and [Don97].
- the Berndtsson’s Kähler form $(J_{\text{int}},\langle \cdot , \cdot \rangle)$ defined in [Don17, Theorem 1], which is different from the one defined in [Don97].

Let $\text{Ham}(X,\omega)$ be the group of Hamiltonian diffeomorphism, thanks to the work of [Don97] and [Don17], both cases admit a moment map. We will apply the theory developed in the previous sections to the Kähler manifold $(Z,\omega) := (J_{\text{int}},\langle \cdot , \cdot \rangle)$.

4.1. $\text{Ham}(X,\omega_X)$-action on $J_{\text{int}}(X,\omega_X)$.

Definition 4.1. A vector field $\xi \in \Gamma(TX)$ is Hamiltonian with respect to $\omega_X$, if there is a function $\theta_{\xi,\omega_X} \in C^\infty(X)$ unique up to adding a constant satisfying $d\theta_{\xi,\omega_X}(\cdot) := \omega(\xi,\cdot)$, and $\theta_{\xi,\omega_X}$ is called a Hamiltonian of $\xi$. Let

\[ \mathfrak{h}\mathfrak{m}(X,\omega_X) := \{ \xi \in \Gamma(TX) \mid \xi \text{ is Hamiltonian} \} = \left\{ \theta \in C^\infty(X) \middle| \left[ \theta, \omega_X \right] = 0 \right\} \]

denotes the Lie algebra of the Hamiltonian diffeomorphism $\text{Ham}(X,\omega_X) \subset \text{Diff}(X)$.

To simplify the notation, from now on we will write $\omega = \omega_X$ if no confusion is caused. For any $g \in \text{Diff}(X)$ and $\xi \in \mathfrak{h}\mathfrak{m}(X,\omega_X)$, we have

\[ \omega\left( g \cdot \left( \xi_{g^{-1}(x)} \right) \cdot x \right) = \omega\left( \xi_{g^{-1}(x)} \cdot x \right) \]

\[ = (g \cdot \omega)\left( \xi_{g^{-1}(x)} \cdot \left( \left( g^{-1} \right)^* \cdot x \right) \right) \]

\[ = g^* \omega\left( \xi_{g^{-1}(x)} \cdot \left( \left( g^{-1} \right)^* \cdot x \right) \right) \]

\[ = d\theta_{\xi,g^*}\omega\left( \left( \left( g^{-1} \right)^* \cdot x \right) \cdot \right) \]

from which we obtain an action

\[ (g,\xi) \mapsto \mathfrak{h}\mathfrak{m}(X,\omega_X) \to C^\infty(X) \]

\[ \text{Ad}_g \xi := g_* \left( \xi_{g^{-1}(x)} \right) \]

\[ \text{Ad}_g \theta_{\xi,\omega} := \theta_{\text{Ad}_g \xi,\omega} = (g^* \cdot \theta_{\xi,\omega}) \]

Clearly we have

\[ \text{Ad}_g \{ \theta, \theta' \} := \{ \text{Ad}_g \theta, \text{Ad}_g \theta' \} \]

as $(C^\infty(X),\{ \cdot, \cdot \}) \cong (\mathfrak{h}\mathfrak{m},\{ \cdot, \cdot \}) \subset (\text{Diff},\{ \cdot, \cdot \})$ is a Lie subalgebra. As a consequence, we have

Lemma 4.2. With the notation introduced above, we have for $g \in \text{Diff}(X)$

\[ \theta_{\text{Ad}_g \xi,\omega} = \theta_{\xi,g^*\omega} \circ g^{-1} \text{ or equivalently } \theta_{\xi,g^*\omega} = \theta_{\text{Ad}_g \xi,\omega} \circ g = g^* \theta_{\xi,\omega}. \]

Definition 4.3. Let $J \in J_{\text{int}}(X,\omega_X)$ and $\xi \in \text{aut}(X,J)$. We define the complexified $\text{Ham}(X,\omega_X)$-orbit of $J$ in $J_{\text{int}}(X,\omega_X)$ to be:

\[ \text{Ham}^C(X,\omega_X) \cdot J := \bigcup_{g_x} \left\{ J_g := g_x \cdot J \right\} \]

\[ \text{Ad}_g \xi := g_* \left( \xi_{g^{-1}(x)} \right) \text{ satisfying:} \]

\[ \frac{dg_x}{ds} = L_{g_x \theta} + J_{g_x \theta} \]

\[ g_0 = \text{id}. \]
where $v_{\phi_i}, i = 1, 2$ are the Hamiltonian vector fields for $\phi_i:\{\begin{array}{l}
d\phi_i = \omega(\nu_{\phi_i}, \cdot) \\
\phi_i \in C^\infty(X, R)
\end{array}\}$. Thus for $s \in [0, 1]$, we have

$$\omega_s := g_s^*\omega \text{ satisfying } \dot{\omega}_s \in \text{Im} \sqrt{-1} \partial J_s \bar{\partial} J_s \text{ and } L_{\xi_s} J_s = 0 \text{ for } \xi_s := (g_s^{-1})_*\xi.$$  

**Remark 4.4.** Notice that the Definition of $\text{Ham}^C(X, \omega_X) \cdot J$ above agree with Donaldson’s description in [Don97] (cf. Appendix 5.2). 

**Definition 4.5.** Let

$$\omega_s := \omega + \sqrt{-1} \partial J_s \bar{\partial} J_s \omega_s = \omega + \frac{\sqrt{-1}}{2} d \circ (d - \sqrt{-1} J \circ d) \phi_s = \omega + \frac{1}{2} d \circ J \circ d \phi_s, \ s \in [0, 1]$$

be a smooth family of Kähler form on $X$ parametrized by $s \in [0, 1]$ and let $f_s^* \omega_s = \omega$ with $\{f_s\}_{s \in [0, 1]} \subset \text{Diff}(X)$ be the family of diffeomorphism obtained via Moser’s trick: that is, if we let $f_s$ obtained by integrating the vector field $v_s(x) := \frac{d}{dt} f_s(x\circ s)$, defined via the following:

$$0 = \frac{d(f_s^* \omega_s)}{ds} = f_s^* (L_{v_s} \omega_s) + f_s^* (\omega_s) = f_s^* (L_{v_s} \omega_s) + f_s^* \left( \frac{1}{2} J \circ d \phi_s \right)$$

(4.35) $$\implies v_s := \frac{1}{2} J v_{\phi_s} \omega_s - v_{\phi_s} \omega_s$$

with

$$-\omega_s(v_{\phi_s} \omega_s, \cdot) = -d\omega_s = v_{\omega_s, \cdot} + \frac{1}{2} J \circ d \phi_s = v_{s, \omega_s} - \frac{1}{2} d \phi_s (J\bullet)$$

\begin{align*}
\left( J \circ d \phi_s = d \phi_s (J^{-1} \bullet) = -d \phi_s (J\bullet) \right) &= \omega_s (v_s, \cdot) - \frac{1}{2} \omega_s (v_{\phi_s} \omega_s, J\bullet) = \omega_s (v_s, \bullet) + \frac{1}{2} \omega_s (J v_{\phi_s} \omega_s, \bullet)
\end{align*}

Let $\xi^C := - \sqrt{-1} J \xi \in \Gamma(T^{0,1}X)$ (i.e. $J \xi^C = - \sqrt{-1} \xi^C$) be a holomorphic vector field satisfying:

$$L_{\xi^C} J = L_{\xi^C} = L_{\xi^C} J = L_{2\xi} J = 0$$

and we define $\theta_{\xi^C, \omega} \in \theta_{\xi^C, \omega} \in \text{Ham}^C(X, \omega_X) \subset C^\infty(X, C)^+$ to be function ( unique up to a constant ) solving:

$$\omega(\xi^C, \bullet) = \partial \theta_{\xi^C, \omega} (\bullet).$$

**Remark 4.6.** If $\xi \in \Gamma(TX)$ is a real Hamiltonian vector field satisfying: $\omega(\xi, \bullet) = d\theta_{\xi, \omega} (\bullet)$, then

$$\omega(\xi^C, \bullet) = \omega \left( \frac{\xi - \sqrt{-1} J \xi}{2}, \bullet \right) = \frac{1}{2} \left( d\theta_{\xi, \omega} (\bullet) - \sqrt{-1} \omega (J \xi, \bullet) \right)$$

$$= \frac{1}{2} \left( d\theta_{\xi, \omega} (\bullet) + \sqrt{-1} \omega (\xi, J \bullet) \right) = \frac{1}{2} \left( d\theta_{\xi, \omega} (\bullet) + \sqrt{-1} d\theta_{\xi, \omega} (J \bullet) \right)$$

$$\left( J \circ d \theta_{\xi, \omega} (\bullet) = d \theta_{\xi, \omega} (J^{-1} \bullet) \right) = \frac{1}{2} \left( d - \sqrt{-1} J \circ d \right) \theta_{\xi, \omega} (\bullet) = \partial J \theta_{\xi, \omega} (\bullet) := \partial J \theta_{\xi^C, \omega} (\bullet).$$

In order to fit the $\text{Ham}(X, \omega_X)$-action on $\text{Ham}(X, \omega_X)$ into the story we developed in Section 3, e.g. Corollary 3.13, we need a form of Chevalley isomorphism or a natural extension of $\text{Ad}_{\text{Ham}(X, \omega_X)}$-invariant function $\text{Ham}(X, \omega_X) \to \mathbb{R}$ to a $\text{Ad}_{\text{Ham}(X, \omega_X)}$-invariant function $\text{Ham}^C(X, \omega_X) \to \mathbb{C}$.

Unfortunately, it is not possible to have such a statement. Even though we have defined $\text{Ham}^C(X, \omega_X) < \text{Diff}(X)$, it is NOT a group complexifying the group $\text{Ham}(X, \omega_X)$ in the usual sense. Luckily, we do have a weaker version of Chevalley type result taking form in Lemma 4.7 combined which with Example 3.11 gives precisely the Tian-Zhu's generalized Futaki invariant.

\[\text{Since for any } v \in \Gamma(TX), \text{ we have}\]

\[\text{Lemma 4.7} \quad (L_{\xi^C} J_\xi)_v = \left( (g_{\xi^C}^{-1})_* (v_{g_\xi^C}) \right)
\]

\[\text{Example 3.11} \quad \text{In particular, we have } \xi \in \text{Ham}(X, \omega) + J \cdot \text{Ham}(X, \omega).\]
Lemma 4.7. Fix a $J \in \mathcal{J}_{\text{hol}}(X, \omega = \omega_X)$, let $\xi^C = \xi_1 + \sqrt{-1} \xi_2 \in \text{ham}^C(X, \omega)$ be a $J$-holomorphic vector field satisfying $\omega_s(\xi^C, \cdot) = \partial J \theta_{\xi^C,\omega} \cdot \omega_s$ with $\omega_s = \omega + \sqrt{-1} \partial J \partial J \phi_s$ and $\theta_{\xi^C,\omega}$ be the unique $\mathbb{C}$-valued function normalized via:

\begin{equation}
\int_X \theta_{\xi^C,\omega} \omega^n_s = 0.
\end{equation}

Let $\{f_s := g_s^{-1}\} \subset \text{Diff}(X)$ be defined in Definition 4.5. Then

1. $\theta_{\xi^C,\omega_s} = \theta_{\xi^C,\omega} + \sqrt{-1} \xi^C \phi_s$
2. \[
\frac{d}{ds} \left( g_s^{-1} \cdot \theta_{\xi^C,\omega} \right) = - \left\{ f_s^* \theta_{\xi^C,\omega} + f_s^* \left( \psi_s + \frac{\sqrt{-1}}{2} \phi_s \right) \right\}, \quad f_s^* \omega_s = \omega.
\]

In particular, we have $f_s^* \theta_{\xi^C,\omega_s} \in \text{ham}(X, \omega_X)$, i.e.

\[ \int_X (f_s^* \theta_{\xi^C,\omega}) \omega_X^n = 0. \]

Proof. Let us introduce

\begin{equation}
\mathbf{L} \xi^C J : = (d \circ \xi^C + \xi^C \circ d) \circ J = \left( d \circ \xi + \xi \circ d + \sqrt{-1} (d \circ (J \xi^C) + (J \xi^C) \circ d) \right) \circ J = L_{\xi^C} J + \sqrt{-1} L_{J \xi^C} J = L_{\xi^C} J - \sqrt{-1} (L_{\xi^C} J) \circ J = 0
\end{equation}

where the last identity follows from [Tia00, Claim on page 36] and our assumption $L_{\xi} J = 0$. Moreover, one has

$\mathbf{L} \xi^C \circ d = d \circ \mathbf{L} \xi^C$

\[
\omega_s(\xi^C, \cdot) = \partial J \theta_{\xi^C,\omega} + \frac{1}{2} \xi^C \circ d \circ (J \circ d \phi_s)
\]

\[
= \partial J \theta_{\xi^C,\omega} - \frac{1}{2} \partial J \xi \circ d \circ (J \circ d \phi_s) + \frac{1}{2} L_{\xi^C}(J \circ d \phi_s)
\]

\[
(J \circ d \phi_s(\cdot)) = d \phi_s(J^{-1}(\cdot))
\]

\[
(\xi \circ d \phi_s(\cdot)) = \partial J \theta_{\xi^C,\omega} + \frac{1}{2} \left( \xi \circ d \circ (d \phi_s(J \xi^C)) + (\mathbf{L} \xi^C \circ J) \circ d \phi_s + J \circ d (L_{\xi^C} \phi_s) \right)
\]

\[
= \partial J \xi \circ d \circ (d \circ d \phi_s(J \xi^C)) + (\mathbf{L} \xi^C \circ J) \circ d \phi_s + J \circ d (L_{\xi^C} \phi_s)
\]

\[
\omega_s(\xi^C, \cdot) = \partial J \theta_{\xi^C,\omega} + \frac{1}{2} d \circ \left( d - \sqrt{-1} J \circ d \right) (\xi^C \phi_s) + \frac{1}{2} L_{\xi^C} J \circ d \phi_s
\]

\begin{equation}
= \partial J \left( \theta_{\xi^C,\omega} + \sqrt{-1} (\xi^C \phi_s) \right) + \frac{1}{2} (\mathbf{L}_{\xi^C} J) \circ d \phi_s
\end{equation}

The first part follows from the fact:

\[
\int_X (\theta_{\xi^C,\omega} + \sqrt{-1} (\xi^C \phi_s)) \omega_s^n = \int_X n \cdot (\theta_{\xi^C,\omega} + \sqrt{-1} (\xi^C \phi_s)) \cdot \sqrt{-1} \partial J \partial J \phi_s \wedge \omega_s^{n-1} + \sqrt{-1} (\xi^C \phi_s) \omega_s^n
\]

\[
= \sqrt{-1} \int_X (n \cdot \partial J (\theta_{\xi^C,\omega} + \sqrt{-1} (\xi^C \phi_s)) \wedge \partial J \phi_s \wedge \omega_s^{n-1} + (\xi^C \phi_s) \omega_s^n)
\]

\[
= \sqrt{-1} \int_X (\phi_s \wedge (\xi^C,\omega_s^n) + (\xi^C \phi_s) \omega_s^n) = \sqrt{-1} \int_X (\xi^C \partial J \phi_s \wedge \omega_s^n + (\xi^C \phi_s) \omega_s^n) = 0.
\]
Now for the second part, we have
\[
\frac{d}{ds} \left( f_s^\star \theta_{\xi^C, \omega_s} \right) = f_s^\star \left( L_{\phi_s} \theta_{\xi^C, \omega_s} \right) + f_s^\star \left( \frac{d\theta_{\xi^C, \omega_s}}{ds} \right)
\]
(4.35)
\[
= -f_s^\star \left( \frac{1}{2} [v_{\phi_s \omega_s} + v_\phi, \theta_{\xi^C, \omega_s}] \right) + f_s^\star \left( \frac{d\theta_{\xi^C, \omega_s}}{ds} \right)
\]
(4.38)
\[
= -f_s^\star \left( \frac{1}{2} [v_{\phi_s \omega_s} + v_\phi, \theta_{\xi^C, \omega_s}] \right) + \sqrt{-1} f_s^\star (\xi(\phi_s))
\]
\[
= -f_s^\star \left( \frac{1}{2} [v_{\phi_s \omega_s} + v_\phi, \theta_{\xi^C, \omega_s}] \right) + f_s^\star (\xi(\phi_s)) - f_s^\star (d\theta_{\xi^C, \omega_s}(v_{\phi_s \omega_s}))
\]
(4.39)
\[
= -f_s^\star \theta_{\xi^C, \omega_s} \left( \psi_s + \sqrt{-1} \phi_s \right)
\]
thanks to the following:
\[-(\partial_\xi \theta_{\xi^C, \omega_s} + \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s}) + 2(J\xi^C)(\phi_s)
\]  
\[-(\partial_\xi \theta_{\xi^C, \omega_s} + \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s}) + 2d\phi_s(J\xi^C)
\]  
\[-(\partial_\xi \theta_{\xi^C, \omega_s} + \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s}) + 2\omega_s(\xi, Jv_{\phi_s \omega_s})
\]  
\[-(\partial_\xi \theta_{\xi^C, \omega_s} + \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s})
\]  
\[-(\partial_\xi \theta_{\xi^C, \omega_s} + \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s}) - (\partial_\xi \theta_{\xi^C, \omega_s} - \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s})
\]  
\[-(\partial_\xi \theta_{\xi^C, \omega_s} - \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s})
\]  
\[-(\partial_\xi \theta_{\xi^C, \omega_s} - \bar{\partial}_\xi \theta_{\xi^C, \omega_s})(Jv_{\phi_s \omega_s}) = \sqrt{-1} J \circ d \theta_{\xi^C, \omega_s}(Jv_{\phi_s \omega_s}) = -\sqrt{-1} d\theta_{\xi^C, \omega_s}(v_{\phi_s \omega_s})
\]  
(4.40)

Remark 4.8. If $\xi^C = \xi^{1,0}$ with $\xi$ being a Killing vector field then we have $\theta_{\xi^C, \omega_s} = \theta_{\xi, \omega_s}$ thanks to Remark 4.6 and $\{\theta_{\xi^C, \omega_s}, \psi_s\} = \xi(\phi_s) = 0$. This together with (4.31) and the fact: $d((g_s^{-1})^\star \theta_{\xi^C, \omega}) \in \omega_s$ imply
\[
\frac{d}{ds} \left( g_s^{-1}(\xi) \right) = g_s \left( \left( L_{\phi_s \omega_s} \xi \right) - L_{\xi^C}(J \cdot v_{\phi_s \omega_s}) \right) = g_s \left( \left( -L_{\xi^C} \cdot v_{\phi_s \omega_s} - J \cdot L_{\xi^C} v_{\phi_s \omega_s} \right) \right)
\]  
\[
\left( \because L_{\xi^C} = 0 \right) = g_s \left( \left( -J \cdot L_{\xi^C} v_{\phi_s \omega_s} \right) \right) = 0. \quad (\because \xi(\phi_s) = 0)
\]

Corollary 4.9. Fix a $F(t) \in C[\Omega]$ or $C^\infty(\Omega)$, let $\{f_s = g_s^{-1}\} \subset \text{Diff}_0(X)$, $J \in \mathcal{J}_{\text{int}}(X, \omega = \omega_X)$, $\xi^C, \eta^C \in H^0(T_X^{1,0} \Omega)$, i.e. $L_{\xi^C} J = 0$, $L_{\eta^C} J = 0$ and $\theta_{\xi^C, \omega_s}, \theta_{\eta^C, \omega_s}$ defined in Lemma 4.7. Then

1. $\int_X F(\theta_{\xi^C, \omega_s}) \omega^n_s$ is independent of $s$.
2. If we write $\theta_{\eta^C, \omega_s} := (g_s^{-1})^\star \theta_{\eta^C, \omega}$, then
\[
\int_X \frac{d}{ds} \left|_{s=0} \right. F(\theta_{\xi^C, \omega_s}) \omega^n_s = \int_X \theta_{\eta^C, \omega} \cdot F(\theta_{\xi^C, \omega}) \omega^n_s.
\]
is independent of $s$.

Proof. Let $\phi_s^C := \psi_s + \sqrt{-1} \phi_s$, then the first part follows:
\[
\frac{d}{ds} \left( \int_X F(\theta_{\xi^C, \omega_s}) \omega^n_s \right) = \frac{d}{ds} \left( \int_X F(\theta_{\xi^C, \omega_s}) g_s^\omega \omega^n \right) = \frac{d}{ds} \left( \int_X F((g_s^{-1})^\star \theta_{\xi^C, \omega_s}) \omega^n \right)
\]
(4.41)
\[
= \int_X F' \left( f_s^\star \theta_{\xi^C, \omega_s} \right) \left( -f_s^\star \theta_{\xi^C, \omega_s} \right) \omega^n = \int_X F' \left( f_s^\star \psi_s^C \right) \omega^n = \int_X \left( dF(\theta_{\xi^C, \omega_s}) \wedge d(\phi_s^C) \right) \omega^n = 0.
\]
Let us introduce function $\text{K}.$

**Lemma 4.11.** Consider the Example 5.4, with $F(t) = e^t - t$ then we have

$$\frac{d}{dt} F(\psi_{\epsilon}^c, \phi_{\epsilon}^c) = \frac{d}{dt} \left( e^\epsilon \psi_{\epsilon}^c + \epsilon \phi_{\epsilon}^c - e^\epsilon \psi_{\epsilon}^c - \epsilon \phi_{\epsilon}^c \right) = \left( \psi_{\epsilon}^c (e^\epsilon \phi_{\epsilon}^c - 1) \right).$$

**4.2. Functional on $\text{ham}(X, \omega_X).$** In this section, we study an $\infty$-dimensional example of Section 3. We introduce a special type of $\text{Ad}_F(\text{Ham}(X, \omega_X))$-invariant functional on $\text{ham}(X, \omega_X).$

**Lemma 4.11.** Let $F(x) \in \mathbb{R}[x]$ and we define functional

$$\text{D} = D_F: \begin{array}{c}
\text{ham}(X, \omega_X)^* = \wedge^0_X^n \\
\mapsto \begin{array}{c}
\mathbb{R} \\
\int_X F(u) \omega^n X
\end{array}
\end{array}.$$

Then

$$dD \mid_{\omega^n / n!} : \begin{array}{c}
\text{ham}(X, \omega_X)^* = \wedge^0_X^n \\
\mapsto \begin{array}{c}
\text{ham}(X, \omega_X) \\
F'(u) - \frac{1}{V} \int_X F''(u) \omega^n X /
\end{array}
\end{array}.$$

$$\nabla^2 D \mid_{\omega^n / n!} : \begin{array}{c}
\text{ham}(X, \omega_X)^* = \wedge^0_X^n \\
\mapsto \begin{array}{c}
\text{ham}(X, \omega_X) \\
F''(u) \cdot \delta u - \frac{1}{V} \int_X F''(u) \cdot \delta u \omega^n X /
\end{array}
\end{array}.$$

where we use identification $\text{ham}(X, \omega_X)^* = \text{ham}(X, \omega_X) \cdot \omega^n X.$

**4.3. Kähler-Ricci soliton.** In this subsection, following [Don17] we will interpret Kähler-Ricci soliton equation as a D-extremal point (in the sense of Section 3) in $J_{\text{int}}(X, \omega_X)$ with respect to the $\text{Ham}(X, \omega_X)$-action.

To achieve that, let us recall that $(L, h, \nabla) \to (X, \omega_X = \text{Ric}(\nabla))$ is a symplectic manifold equipped with a pre-quantum line bundle $(L, h)$ with Hermitian metric $h$ with $\omega = \text{Ric}(h).$ Assume further that $c_1(L) = c_1(X, \omega_X)$, then for a fixed $J \in J_{\text{int}}(X, \omega_X)$ the space of

$$\ker(d \nabla^V) := \{ \alpha \in \Gamma(\wedge^*_X^n(\omega_X) \otimes L) \mid d \nabla \alpha = 0 \} \subset \Gamma(\wedge^*_X^{n,0}(X) \otimes L)$$

is of dim $= 1$ and $\ker(d \nabla^V) = \mathbb{C} \cdot \alpha$. In particular, $N := \ker(d \nabla^V) \to J_{\text{int}}(X, \omega_X)$ is a line bundle over $J_{\text{int}}(X, \omega_X),$

$$N \searrow \pi_J(K_X) \otimes L \searrow \Gamma(\wedge^*_X^{n,0}(X) \otimes L) \text{ with } X := X \times J_{\text{int}}(X, \omega_X) \leftarrow (X, J).$$

For a local section $\alpha \in \Gamma(N)$ near $J \in J_{\text{int}}(X, \omega_X)$, Donaldson [Don17, Theorem 1] defines the quadratic pairing:

$$\langle \beta, \gamma \rangle_J := -\langle \beta, \gamma \rangle_J + \frac{\langle \alpha, \beta \rangle \langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle} \quad \text{with } \beta := \delta \alpha \text{ and } \langle \beta, \beta \rangle_J := \int_X \beta \wedge h \beta \text{ for } \beta \in \wedge^*_X^{n,0}(X) \otimes L$$

which descends to a $\text{Ham}(X, \omega_X)$-invariant functional on $J_{\text{int}}(X, \omega_X)$, called Berndtsson metric by Donaldson. Moreover, it was proved in [Don17, Proposition 1] that the moment map for $\text{Ham}(X, \omega_X)$-action on $(Z, \omega) = (J_{\text{int}}(X, \omega_X), \langle \cdot, \cdot \rangle_J)$ is

$$\mu(J) = \mu(\langle \alpha(J) \rangle) := \frac{\langle \alpha(J) \wedge h \alpha(J) \rangle}{\langle \alpha(J), \alpha(J) \rangle} \omega^n X / V \cdot n! \in \text{ham}(X, \omega_X)^* \text{ with } V = \int_X \omega^n X / n! \text{ and } \alpha \in N \cdot J \subset \Gamma(\wedge^*_X^{n,0}(X) \otimes L).$$

Let us introduce function $\phi = \phi_J \in C^\infty(X)$ via:

$$\Omega(J) := \frac{\alpha(J) \wedge h \alpha(J)}{\langle \alpha(J), \alpha(J) \rangle} \omega^n X / V \cdot n! \text{ with normalization } \frac{1}{V} \int_X (e^{\phi_J} - 1) \omega_X^n / n! = 0$$

and the second part follows from the fact:

$$\frac{d}{ds} \left( \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} F'(\psi_{\epsilon}^c, \phi_{\epsilon}^c) \right) = \frac{d}{ds} \left( \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} F'(\psi_{\epsilon}^c, \phi_{\epsilon}^c) + \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} \frac{dF'}{ds} \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} \right) = \left\{ \psi_{\epsilon}^c, \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} \right\} F'(\psi_{\epsilon}^c, \phi_{\epsilon}^c) + \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} \left\{ \psi_{\epsilon}^c, F'(\psi_{\epsilon}^c, \phi_{\epsilon}^c) \right\} = \left\{ \psi_{\epsilon}^c, \theta_{\psi_{\epsilon}^c, \phi_{\epsilon}^c} \cdot F'(\psi_{\epsilon}^c, \phi_{\epsilon}^c) \right\}. \square$$.  

then 
\[ \mu(J) = \frac{\alpha(J) \wedge \delta(J)}{[\alpha(J), \alpha(J)]} \cdot \frac{\omega^n}{V \cdot n!} = (e^\phi - 1) \frac{\omega^n}{V \cdot n!} \text{ and } \text{Ric}(\Omega(J)) = \omega_X \]
where we regard $\Omega(J)$ is a metric on the canonical line bundle $\wedge^\theta_lX^*(X) \to X$. Let $\xi$ be the Hamiltonian Killing vector field and let 
\[ d\theta_\xi = \omega_X(\xi, \cdot) \text{ satisfying } \int_X (e^\theta_\xi - 1) \frac{\omega^n}{V \cdot n!} = 0, \]
then associated Kähler-Ricci soliton equation with Kähler-Ricci soliton vector field $\xi$ is:
\[ (4.47) \quad \frac{\alpha(J) \wedge \delta(J)}{[\alpha(J), \alpha(J)]} = \Omega(J) = \frac{e^\theta_\xi \cdot \omega^n}{V \cdot n!} \begin{cases} \phi_J = \theta_\xi; \\ \int_X (e^\theta_\xi - 1) \frac{\omega^n}{V \cdot n!} = \int_X (e^\theta_\xi - 1) \frac{\omega^n}{V \cdot n!} = 0. \end{cases} \]
Following (3.16), the negative gradient flow for the functional $D_F : \text{Ham}(X, \omega_X)^* \to \mathbb{R}$ is given by 
\[ (4.49) \quad \phi - \frac{1}{V} \int_X \varphi(X) \omega_X^{n-1} = dD_F(J) = F'(\mu(J)) - \frac{1}{V} \int_X F'(\mu(J)) \omega_X^{n-1} = \phi_J - \frac{1}{V} \int_X \phi_J \omega_X^{n-1} \in \text{Ham}(X, \omega_X) \]
where $\varphi \in \text{PSH}(X, \omega_X)$. On the other hand, the Kähler Ricci flow is given by 
\[ \sqrt{-1} \partial_J \partial_J \phi = \omega = \sqrt{-1} \partial_J \partial_J \log \frac{\Omega(J)}{\omega_X^{n-1}/n!} = \sqrt{-1} \partial_J \partial_J \log \left( \frac{\Omega(J)}{\omega_X^{n-1}/n!} - \frac{1}{V} + \frac{1}{V} \right) \cdot V \]
comparing with (4.49), we deduce that $F'(u) = \log(Vu + 1)$ with $u = \frac{\mu(J)}{\omega_X^{n-1}/n!}$ hence 
\[ F'(u) = \frac{1}{u + 1/V} \text{ and } F(u) = (u + 1/V) \log(Vu + 1) - u + \text{const}. \]
By applying Theorem 3.3 to $(Z, \omega) = (\mathcal{J}_{\text{aut}}(X, \omega_X), \langle \cdot, \cdot \rangle|_\mathcal{J})$ equipped with $\text{Ham}(X, \omega_X)$-invariant convex functional $D_F$ we recover [TZ00, Theorem A] as follows

**Corollary 4.12.** Let us continue with the notation introduced as above and assume further that $(X, \omega_X, J)$ is a Fano manifold admitting a Kähler-Ricci soliton with the Kähler-Ricci soliton vector field, equivalently, the D-extremal vector field, $\xi \in \text{aut} := \text{aut}(X, \omega_X, J)$. Then we have the Calabi decomposition:
\[ \text{aut} = \text{aut}_0 \oplus \bigcup_{\lambda > 0} \text{aut}_\lambda \text{ with } \text{aut}_\lambda := \left\{ \eta^C \in \text{aut} \left| [\xi, \eta^C] = \lambda \eta^C \right. \right\}. \]

4.4. Tian-Zhu’s generalized Futaki-invariant. Next we discuss how the generalized Futaki invariant firstly introduced in [TZ00] fits into our framework developed in Section 3. Let us assume $(X, \omega_X, J)$ is a Fano manifold as before. To apply Theorem 3.9 and Example 3.11 to the situation here, we need to establish the $\text{Ad}_{\text{Ham}}(X, \omega_X)^*$-invariance of 
\[ (4.50) \quad \langle dD_F^{-1}(\theta^C_{\xi^C}), \theta^C_{\eta^C} \rangle \text{ for } \theta_{\xi^C}, \theta_{\eta^C} \in \text{aut}(X, \omega_X, J) \subset \text{Ham}(X, \omega_X) \text{ at } J \in \mathcal{J}(X, \omega_X). \]
For this, we have the following statement parallel to Corollary 3.13:

**Proposition 4.13.** Suppose $F \in C^\infty(\mathbb{R})$ is convex with $\hat{F} \in C^\infty(\text{Im}F', \mathbb{R})$ being the Legendre transform. Let $D$ be defined as in Lemma 4.11 such that its Legendre transform satisfies(cf. Section 5.3):
\[ \hat{D}_F(\phi) = D_\hat{F}(\phi) : C^\infty(X) \longrightarrow \mathbb{R}, \int_X \hat{F}(\phi) \frac{\omega_X^n}{n!} \]
Then 
\[ (1) \text{ The negative gradient flow equation is given by } \dot{\phi} = dD_\hat{F}(\phi). \]

\[ \text{The normalization of } \theta_\xi \text{ is chosen so that there is no extra constant in the Kähler-Ricci soliton } \phi = \theta_\xi. \]
For any $\xi, \eta \in \mathfrak{aut}(X, \omega, J)$
\[
\langle d\mathcal{D}(\theta_{\xi}), \theta_{\eta}\rangle = \langle d\mathcal{D}(\theta_{\xi}), \theta_{\eta}\rangle
\]
is constant along the Ham$^C$($X, \omega$)-orbit of $J \in \mathcal{J}(X, \omega_X)$. In particular, the $\mu$-invariant in this case is given by $\langle d\mathcal{D}(\theta_{\xi}) - \mu(J), \theta_{\eta}\rangle$.

Proof. By Proposition 5.3, we have
\[
\langle d\mathcal{D}(\theta_{\xi}), \theta_{\eta}\rangle = \langle d\mathcal{D}(\theta_{\xi}), \theta_{\eta}\rangle = \frac{1}{V} \int_X \widehat{\alpha}(\theta_{\xi}) \cdot \theta_{\eta} \frac{\omega^n}{n!}
\]
Then our claim follows from Corollary 4.9 and the assumption that $\theta_{\xi}, \theta_{\eta} \in \mathfrak{aut}(X, \omega_X, J)$.

\[\square\]

Remark 4.14. Note that $\mathcal{D}$ is not a Diff($X$)-invariant functional.  

To recover Tian-Zhu’s generalized Futaki-invariant, let us fix some notation:
\[
\text{Ric}(\omega_X) - \omega_X = \sqrt{-1} \partial \bar{\partial} \phi \text{ with normalization } \frac{1}{V} \int_X (e^\phi - 1) \frac{\omega^n}{n!} = 0.
\]
For any $\xi, \eta \in \mathfrak{aut}(X, \omega_X, J)$, we define normalizations:
\[
(4.52)
\]
\[
d\theta_{\eta}(\bullet) = \hat{d}\theta_{\eta}(\bullet) = \partial C_{\eta}\theta_{\eta}(\bullet) = \omega_X(\eta, \bullet) \text{ with normalization } 0 = \left\{ \begin{array}{lcl}
\int_X \theta_{\eta} \frac{\omega^n}{n!} & = & \int_X (e^\phi - 1) \frac{\omega^n}{n!} \quad (\text{ cf. [TZ90, Section 2]})
\end{array}\right.
\]

Proposition 4.15. The Tian-Zhu generalized invariant is given by
\[
\int_X \frac{\tilde{\theta}_{\eta} \cdot \omega^n}{n!} = \left\{ \begin{array}{lcl}
\mu(J) & := & \frac{1}{V} (e^\phi - 1) \frac{\omega^n}{n!} \quad (\text{ cf. 4.47})
\end{array}\right.
\]
\[
d\mathcal{D}(\theta_{\xi}) := \frac{1}{V} (e^\phi - 1) \frac{\omega^n}{n!} \quad (\text{ cf. Example 5.4 })
\]

Proof. By Definition, we have $\tilde{\theta}_{\eta} = \theta_{\eta} - C_{\eta}$. Using
\[
0 = \int_X \tilde{\theta}_{\eta} \cdot e^\phi \frac{\omega^n}{n!} = \int_X (\theta_{\eta} - C_{\eta}) \left( \mu(J) + \frac{\omega^n}{n!} \right)
\]
we obtain that $\tilde{\theta}_{\eta} = \theta_{\eta} - (\mu(J), \theta_{\eta})$. Let us introduce $\hat{C}$ via:
\[
\int_X e^\hat{C} \frac{\omega^n}{n!} = \int_X e^\phi \cdot C_{\eta} \frac{\omega^n}{n!} \text{ with } \frac{1}{V} \int_X (e^\phi - 1) \frac{\omega^n}{n!} = 0 \text{ then } e^\hat{C} = \int_X e^\phi \cdot C_{\eta} \frac{\omega^n}{n!} / \frac{1}{V} \int_X e^\phi \cdot \omega_X
\]

For any $\phi \in \text{Diff}(X)$ the functional
\[
\phi^*(\omega \cdot \frac{\omega^n}{n!}) \quad \rightarrow \quad \int_X F(\phi^*(\omega \cdot \frac{\phi^*\omega^n}{n!}))) \frac{\omega^n}{n!}
\]
one has
\[
\int_X F(\phi^*(\omega \cdot \frac{\phi^*\omega^n}{n!}))) \frac{\omega^n}{n!} \neq \int_X F(\phi^*(\omega \cdot \frac{\phi^*\omega^n}{n!})))
\]
for general $\phi \in \text{Diff}(X)$.
Then
\[
\frac{1}{V} \int_X \tilde{\eta}_t e^{\theta X} \frac{\omega_X}{n!} = \frac{1}{V} \int_X (\theta_t - (\mu(J), \theta_t)) e^{\theta X} \frac{\omega_X}{n!} = e^C \frac{1}{V} \int_X (\theta_t - (\mu(J), \theta_t)) \left( e^{\theta X} - 1 \right) \frac{\omega_X}{n!}.
\]
\[
(1) \int_X (e^{\theta X} - 1) \frac{\omega_X}{n!} = 0
\]
\[
(2) \int_X \theta X \omega_X = 0
\]
Notice that the normalizations of \( \frac{1}{V} \int_X (e^{\theta X} - 1) \frac{\omega_X}{n!} = 0 \) and \( \int_X \theta X \omega_X = 0 \) imply
\[
d \tilde{D}_F(\tilde{\lambda}) = e^{\theta X} \frac{\omega_X}{n!} - \frac{1}{V} \int_X e^{\theta X} \frac{\omega_X}{n!}.
\]
\[
\Box
\]

5. Appendix

5.1. Extending \( \text{Ad}_K \)-invariant functions. As we mentioned in Remark 3.14, the \( \text{Ad}_G \)-extension of \( f \in C^\infty(\mathfrak{g})^{\text{Ad}_K} \) is non-unique and exists in quite general sense contrast to the canonical Chevalley isomorphism.

**Proposition 5.1.** Let \( f \in C^\infty(\mathfrak{g})^{\text{Ad}_K} \) be a restriction of a function, which by abusing of notation still denoted by \( f \in C^\infty(\mathfrak{g})^{\text{Ad}_G} \).

**Proof.** For simplicity, we will specialize to the case that \( K = U(n) \), \( G = GL(n, \mathbb{C}) \) and \( f : \mathfrak{t} \to \mathbb{R} \) a smooth \( \text{Ad}_K \)-invariant function. Then \( f \) is the restriction of a smooth \( \text{Ad}_G \)-invariant function \( F : \mathfrak{g}^* \to \mathbb{R} \).

**Lemma 5.2.** \( A, B \in \mathfrak{t} \) and assume that \( A = g B g^{-1} \) for some \( g \in G \). Then \( A = k B k^{-1} \) for some \( k \in K \).

Let \( f : \mathfrak{t} \to \mathbb{R} \) a smooth \( \text{Ad}_K \)-invariant function. Define a smooth function \( \tilde{f} : (\sqrt{1-c^2})^n \to \mathbb{R} \) by the formula \( \tilde{f}(\lambda_1, ..., \lambda_n) = f(\xi) \) where \( \xi \in \mathfrak{t} \) is any element whose characteristic polynomial is \( \prod_{j=1}^n (x - \lambda_j) \). This is well defined since Lemma 5.2 implies that if \( \xi, \xi' \in \mathfrak{t} \) have the same characteristic polynomial, then they are conjugate by an element in \( K \) and \( f \) is \( \text{Ad}_K \)-invariant.

The function \( \tilde{f} \) is smooth and invariant under the permutation group \( S_n \). Let \( \tilde{F} : \mathbb{C}^n \to \mathbb{R} \) be any smooth \( \mathfrak{S}_n \)-invariant extension of \( f \), for example, \( \tilde{F}(\lambda_1, ..., \lambda_n) = \tilde{f}(\sqrt{1-c^2}) \). Let us define an extension of \( f \), which by abusing of notation still denoted by \( f(\xi) := F(\lambda_1, ..., \lambda_n) : g \to \mathbb{R} \) where \( \prod_{j=1}^n (x - \lambda_j) \) is the characteristic polynomial of \( \xi \in \mathfrak{g} \). Note \( F \) is well defined since \( \tilde{F} \) is invariant under the symmetric group \( \mathfrak{S}_n \). Moreover \( f|_{\mathfrak{g}} \) is \( \text{Ad}_G \)-invariant since all the elements in an \( \text{Ad}_G \) orbit have the same characteristic polynomial and it agrees with \( f \) on \( \mathfrak{t} \) by construction. \( \Box \)

5.2. Identification of \( \text{PSH}(X, \omega_X) = \text{Ham}(X, \omega_X)^C / \text{Ham}(X, \omega_X) \). Let \( \omega_s := \omega_{\phi_s} = g_{\theta_s}^{-1} \omega = g_{\theta_s} \omega \), and we define map

\[
\nu : \begin{array}{ccc}
\text{PSH}(X, \omega_X, J) & \longrightarrow & \mathcal{J} \text{int} \left( g_{\phi_s}^{-1} \right)^* J
\end{array}
\]

where

\[
\text{PSH}(X, \omega_X, J) := \left\{ \phi \mid \omega_{\phi} := \omega + \sqrt{-1} \theta \phi > 0, \int_X \omega^n = 0 \right\}
\]

and

\[
\text{PSH}(X, \omega_X, J) := \left\{ (\phi, g_{\phi}) \in \text{PSH}(X, \omega_X) \times \text{Diff}(X) \mid g_{\phi}^* \omega = \omega_{\phi} \right\} \subset \text{PSH}(X, \omega_X, J) \times \text{Diff}(X).
\]

Then we have that \( \text{Im} \nu = \text{Ham}^C(X, \omega_X) / \text{Ham}(X, \omega_X) \). To see this one notice that, if we let \( J := g^{-1} \)

\[
\frac{d}{ds} \omega = \frac{d}{ds} (g_{\phi}^{-1})^* \omega = f_s^* L_{g_{\phi}} \omega + \sqrt{-1} f_s^* \partial_{\partial g} g_{\phi} = f_s^* \partial_{\partial g} (f_s \cdot \phi) + f_s^* \partial_{\partial g} \phi
\]

with

\[
dJ := J \circ d \text{ hence } \partial = \frac{d + \sqrt{-1} dJ}{2} \text{ and } \partial = \frac{d - \sqrt{-1} dJ}{2}
\]
0 = \frac{d}{ds}(f^*_s \omega_s) = f^*_s L_{f_s} \omega_s + \sqrt{-1} f^*_s \partial J \partial \phi_s = f^*_s d(\omega_s \bullet) + f^*_s dJ \phi_s = d\left( f^*_s (\omega_s \bullet) + f^*_s (J \circ d \phi_s) \right)

by solving \( f^*_s (\omega_s \bullet) + f^*_s (d \phi_s) = 0 \). For \( v \in T_x X \), we have

\[ \omega_s(f^*_s|_{f_s(x)})(f_s(v)) \bigg|_{f_s(x)} = \omega_s(J \circ d(f^*_s \phi)|_{f_s(x)})(f_s(v)) \bigg|_{f_s(x)} \]

and

\[ f^*_s(J \circ d(f^*_s \phi)|_{f_s(x)})(v) \bigg|_{f_s(x)} = d \phi_s|_{f_s(x)} \left( J|_{f_s(x)} \left( f_s(v) \right) \right) \]

hence

\[ \omega \left( f^*_s \right) \left( J \circ d(f^*_s \phi)|_{f_s(x)} \right) \bigg|_{f_s(x)} = \left( J_s \circ d(f^*_s \phi) \right) \bigg|_{f_s(x)} \]

from which we obtain

\[ \text{diff}(X) \ni \left( f^*_s \right) \left( J \circ d(f^*_s \phi) \right) \bigg|_{f_s(x)} = \left( J_s \circ d(f^*_s \phi) \right) \bigg|_{f_s(x)} \in J_s \circ \text{ham}(X, \omega_X) \].

5.3. Legendre Transform. Let \( F : \mathbb{R} \ni U \rightarrow \mathbb{R} \) be a convex function and \( D_F : C^\infty(X, U) \rightarrow \mathbb{R} \) be a functional defined by

\[ D_F(\varphi) := \int_X F(\varphi(x))d\nu_X \]

for a fixed volume form \( d\nu_X \).

**Proposition 5.3.** Let \( P := \{ t \in \mathbb{R} | t = f'(x) \text{ for some } x \in X \} = \text{Im}(dF) \) for any \( \psi \in C^\infty(X, P) \). Then the Legendre transform \( \tilde{D}_F : C^\infty(X, P) \rightarrow \mathbb{R} \) is given by

\[ \tilde{D}_F(\psi) = \int_X \tilde{F}(\psi(x))d\nu_X \]

with \( \tilde{F} \) being the Legendre transform of \( f \) corresponding to the pairing

\[ \langle \varphi, \psi \rangle := \int_X \varphi(x)\psi(x)d\nu_X \].

**Proof.** Fix \( \psi \), then we have

\[ \tilde{D}_F(\psi) := \sup_{\varphi \in C^\infty(X, \mathbb{R})} \{ \langle \varphi, \psi \rangle - D_F(\varphi) \} \]

\[ = \sup_{\varphi \in C^\infty(X, U)} \left( \int_X \varphi(x)\psi(x)d\nu_X - \int_X F(\varphi(x))d\nu_X \right) \]

\[ = \sup_{\varphi \in C^\infty(X, U)} \int_X \left( \varphi(x)\psi(x) - F(\varphi(x)) \right)d\nu_X \]

\[ \leq \int_X \left( \sup_{u \in U} \left( a \cdot \psi(x) - F(u) \right) \right)d\nu_X \]

\[ = \int_X \tilde{F}(\psi(x))d\nu_X \].

We also need to show that there is a \( \varphi \in C^\infty(X, U) \) satisfying:

\[ \int_X (\varphi(x)\psi(x) - F(\varphi(x)))d\nu_X \geq \int_X \tilde{F}(\psi(x))d\nu_X \]

For that, by our assumption \( \psi \in C^\infty(X, P) \), there is a unique \( p_x \) satisfying \( \psi(x) = F'(p_x) \). We define \( \varphi_c(x) = p_x \), we only need to show that \( \varphi_c(x) \) is continuous. However, \( F \) is convex and smooth.
implies \((dF)^{-1}\) exists and smooth. Hence \(\varphi(x) = (dF)^{-1}(\psi(x))\) which is smooth, and for any \(x\), \(\bar{F}(\psi(x)) = \varphi(x)\psi(x) - F(\varphi(x))\), which implies the inequality is equality. \(\square\)

**Example 5.4.** \(F(u) = (u + 1/V) \log(u + 1/V) - (u + 1/V) + 1\) then \(\bar{F}(p) = e^p - p\). Then the functions

\[
F'(u) = \log(u + 1/V) \quad \text{with} \quad \Im(F') = (-\infty, +\infty), \quad \Dom(F') = (-1/V, +\infty)
\]

\[
\bar{F}'(p) = e^p - 1/V \quad \text{with} \quad \Im(\bar{F}') = (-1/V, +\infty), \quad \Dom(\bar{F}') = (-\infty, +\infty)
\]

are inverse to each other (cf. (4.42)). By Proposition 5.3, one has

\[
\bar{D}_F = D_{\bar{F}}
\]

as we know from (4.47) that \(u = \frac{\mu(j)}{\omega^n/n!} > \frac{1}{V} \) from which we deduce that the corresponding \(\mu\)-invariant in Theorem 3.9 is given by:

\[
\langle (dD_F)^{-1}(\theta_j), \eta \rangle = \langle dD_{\bar{F}}(\theta_j), \theta_\eta \rangle = \int_X \hat{F}'(\theta_j) \cdot \theta_\eta \frac{\omega^n}{n!} = \int_X \frac{e^{\theta_j} - 1}{V} \cdot \theta_\eta \frac{\omega^n}{n!} = \frac{1}{V} \int_X \left( \frac{1}{V} \int_X e^{\theta_j} \frac{\omega^n}{n!} \right) \theta_\eta \frac{\omega^n}{n!}.
\]

**References**

[Cal82] Eugenio Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR645743

[Cal85] , *Extremal Kähler metrics. II*, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 95–114. MR876039

[Don97] S. K. Donaldson, *Remarks on gauge theory, complex geometry and 4-manifold topology*, Fields Medalists’ lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, pp. 384–403. MR1622931

[Don17] , *The Dirac functional, Berveaudon convexity and moment maps*, Geometry, analysis and probability, Progr. Math., vol. 310, Birkhäuser/Springer, Cham, 2017, pp. 57–67. MR3821922

[Fuj90] Akira Fujiki, *The moduli spaces and Kähler metrics of polarized algebraic varieties*, Sugaku 42 (1990), no. 3, 231–243 (Japanese). MR1073369

[FM95] Akito Futaki and Toshiki Mabuchi, *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. 301 (1995), no. 2, 199–210. DOI 10.1007/BF01446626. MR1314584

[Naka] Satoshi Nakamura, *Hessian of the Ricci Calabi functional*, arXiv:1803.02431.

[Nakb] , *H-functional and Matsushima type decomposition theorem*, arXiv:1905.05526.

[Tao00] Gang Tian, *Canonical metrics in Kähler geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2000. Notes taken by Moise Akveld. MR1787650

[TZ02] Gang Tian and Xiaohua Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comment. Math. Helv. 77 (2002), no. 2, 297–325. DOI 10.1007/s00014-002-8341-3. MR1915043

[TZ00] , *Uniqueness of Kähler-Ricci solitons*, Acta Math. 184 (2000), no. 2, 271–305. DOI 10.1007/BF02392630. MR1768112

[Wan04] Xiaowei Wang, *Moment map, Futaki invariant and stability of projective manifolds*, Comm. Anal. Geom. 12 (2004), no. 5, 1009–1037. MR2103309

Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain

Email address: king.lee@uam.es

Department of Mathematics and Computer Science, Rutgers University, Newark NJ 07102-1222, USA

Email address: sturm@rutgers.edu

Department of Mathematics and Computer Science, Rutgers University, Newark NJ 07102-1222, USA

Email address: xiaowwan@rutgers.edu