Phenomenological improvement of the linear-σ model in the large-$N_c$ limit

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The linear-σ model has been widely used to describe the chiral phase transition. Numerically, the critical temperature $T_c$ of the chiral phase transition is in agreement with other effective theories of QCD. However, in the large-$N_c$ limit $T_c$ scales as $\sqrt{N_c}$ which is not in line with the NJL model and with basic expectations of QCD, according to which $T_c$ is –just as the deconfinement phase transition– $N_c$-independent. This mismatch can be corrected by a phenomenologically motivated temperature dependent parameter.

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1. Introduction

The Quantum Chromodynamics (QCD) at finite temperature and finite density is a central topic in high energy physics. For small temperatures and densities the quark and gluon degrees of freedom are confined in hadrons. It is expected that there exists a region in the QCD phase diagram where quarks and gluons behave like a plasma (deconfinement) [1, 2]. The QCD Lagrangian does not allow to directly calculate the confinement/deconfinement phase transition or the related chiral phase transition. The latter is mathematically well defined in the limit of zero quark masses, in which the QCD-Lagrangian is invariant under chiral symmetry transformation and the chiral condensate is an exact order parameter [3, 4, 5]. Two effective models, the Nambu Jona-Lasino model (NJL) [6, 7, 8, 9] and the linear-σ model [10, 11, 12], have been often used to study the properties of the chiral phase transition.

Beside these phenomenological approaches to QCD there is also the large-$N_c$ approximation [13, 14]. The number of color degrees of freedom in

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the QCD Lagrangian is three, but a theory with an infinite large number of color degrees of freedom shows a behavior similar to the one of a theory with three colors. In the large-$N_c$ limit the gauge symmetry of the QCD is changed from $SU(3)$ to $SU(N_c \gg 3)$. Enlarging the number of colors also leads to a modified QCD coupling $g_{QCD}$ in a way that for $N_c \to \infty$ the product $g_{QCD}^2 N_c$ remains constant. Quarks and gluons are still present but gluons dominate the behavior of the theory. For low temperatures there still exists an confined phase, where the degrees of freedom are mesons and baryons [15]. For high enough $T$ it is believed that the theory is deconfined. Although also in the large-$N_c$ limit the theory is not solvable, it is significant simpler because only planar diagram survive.

2. Linear-$\sigma$ model in the large-$N_c$ limit

The linear-$\sigma$ model [10, 11, 12], is an effective theory which is able to describe the mass splitting of the pions and the sigma via spontaneous symmetry breaking. The model is built with terms which are invariant under chiral symmetry transformation. In the vacuum the chiral symmetry is spontaneously broken and the pions emerge as Goldstone bosons. In the original form there is no explicit $N_c$ dependency. From former studies one knows that the quark-antiquark meson masses are independent of the number of colors, but the coupling of three mesons is suppressed by a factor of $1/\sqrt{N_c}$ and the four mesons coupling by a factor of $1/N_c$. [14]. These scaling properties can be implemented by redefining the meson four point interaction $\lambda \to 3\lambda/4N_c$, while the parameter $\mu$ is not affected in the large-$N_c$ limit: $\mu \to \mu$. The Lagrangian of the $\sigma$-model as function of $N_c$ reads:

$$L_{\sigma}(N_c) = \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2} \mu^2 \Phi^2 - \frac{\lambda}{4} \frac{3}{N_c} \Phi^4 ,$$

(1)

where the scaler field $\sigma$ and the pseudoscalar pion triplet $\vec{\pi}$ are described by $\Phi^t = (\sigma, \vec{\pi})$. For $\mu^2 > 0$ the chiral condensate is $\varphi_0 = \varphi(T = 0) = \mu \sqrt{N_c/3\lambda} = \sqrt{N_c/3} f_\pi$. The only scale in the Lagrangian is $f_\pi$, to be identified with the pion decay constant. Note that the chiral condensate scales with $\varphi_0 \propto \sqrt{N_c}$. The tree-level masses are not effected by the $N_c$-scaling and read $m_{\sigma}^2 = 3\lambda f_{\pi}^2 - \mu^2$, $m_{\pi}^2 = 0$.

The behavior at finite temperature can be analyzed using the Cornwall-Jackiw-Tomboulis (CJT) formalism [16]. The gap equation for the chiral condensate is

$$0 = \varphi(T) \left( \frac{3}{N_c} \lambda \varphi(T)^2 - \mu^2 + \frac{9}{N_c} \lambda \int G_\sigma + \frac{9}{N_c} \lambda \int G_\pi \right) .$$

(2)

The full propagators $G_\sigma$ and $G_\pi$ have the form:
\begin{align}
G_i &= \int_0^\infty dk \frac{k^2}{2\pi^2} \frac{1}{\sqrt{k^2 + m_i^2}} \left[ \exp \left( \frac{\sqrt{k^2 + m_i^2}}{T} \right) - 1 \right]^{-1}.
\end{align}

The critical temperature \( T_c \) is defined as the temperature where the condensate exactly vanishes \( \varphi(T_c) = 0 \). This leads to the following \( N_c \) dependent scaling of the critical temperature:

\begin{align}
T_c(N_c) &= \sqrt{2} f_\pi \sqrt{\frac{N_c}{3}} \propto N_c^{1/2}.
\end{align}

For the case \( N_c = 3 \) obtains the known result \( T_c = \sqrt{2} f_\pi \), but it also clearly emerges that \( T_c \) increases with \( N_c \). This means that for \( N_c \gg 3 \) the chiral phase transition will not take place. The result is general for all hadronic models which do not include color degrees of freedom or temperature dependent parameters. This result, first noticed in Ref. [17], contradicts the results found in the NJL-model [7] where the critical temperature \( T_c \) remains constant in the large-\( N_c \) limit, and with the fact that the related confinement/deconfinement phase transition is expected to be proportional to \( \Lambda_{QCD} \), which is a large-\( N_c \) independent quantity.

### 3. Phenomenological modification of the linear-\( \sigma \) model

In order to solve this discrepancy a phenomenological approach is proposed. In Refs. [18, 19] it is argued that the \( T^2 \) scaling of order parameters is general. A phenomenological way to take this property into account is to make the parameter \( \mu^2 \) temperature dependent:

\begin{align}
\mu^2 \to \mu(T)^2 &= \mu^2 \left( 1 - \frac{T^2}{T_0^2} \right).
\end{align}

The parameter \( T_0 \) is a new temperature scale and should be of the order of \( \Lambda_{QCD} \). The Eq. (5) modifies the gap equation (2) and leads to a different critical temperature:

\begin{align}
T_c(N_c) &= T_d \left( 1 + \frac{T_d^2}{2 f_\pi^2} \frac{3}{N_c} \right)^{-1/2} \propto N_c^0.
\end{align}

The critical temperature is constant in the limit \( N_c \to \infty \).

Now we turn back to the \( N_c = 3 \) and study the case where the explicit symmetry breaking term \( \epsilon \sigma \) is present. The complete Lagrangian including a second temperature scale reads:
Fig. 1. Finite temperature behavior of the masses. The dashed line represents the mass of the pions and the continuous line mass of the $\sigma$. Above a critical temperature of $T_c \approx 200$ MeV the masses become degenerated.

$$L_{\sigma}(T_0) = \frac{1}{2}(\partial_{\mu}\Phi)^2 + \frac{1}{2}\mu^2 \left(1 - \frac{T^2}{T_0^2}\right)\Phi^2 - \frac{\lambda}{4}\Phi^4 + \epsilon\sigma.$$  

(7)

All parameters are fixed via the masses and the pion decay constant ($\epsilon = f_\pi M_\sigma^2$, $\lambda = (M_\sigma^2 - M_\pi^2)/(2f_\pi^2)$, $\mu^2 = (M_\sigma^2 - 3M_\pi^2)/2$). The quantity $T_0$ is set to a value of $T_0 = 0.27$ GeV. The vacuum masses are chosen to be the following: the mass of the $\sigma$-field is $M_\sigma = 1.2$ GeV (for the discussion of the value of the $\sigma$ mass in the vacuum, see Refs [20] and refs therein), of the $\pi$-field is $M_\pi = 0.135$ GeV and the value for the pion decay constant is $f_\pi = 92.4$ MeV.

The finite temperature behavior for the masses, see Fig. 1, is similar to the one with no temperature dependent parameters. Until the critical temperature $T_c$ is reached the temperature dependency of the pion mass as the $\sigma$ mass varies slowly. Close to $T_c$ the mass of the $\sigma$ drops and slightly above $T_c$ the mass becomes degenerated with the pion mass. At high temperature both masses rise linearly.

Beside these similarities there are two remarkable properties that differ. First, the order of the phase transition is changed from a first order to a crossover phase transition. Second, the critical temperature is lowered to $T_c \approx 200$ MeV. Both phenomena can be seen in Fig. 2., where the case $N_c \to \infty$ is shown: in this limit the chiral symmetry is restored through the new temperature scale $T_0$ and not via mesonic loops.
Fig. 2. The chiral condensate for different number of colors and temperature scale $T_0$. Continuous line: $N_c = 3$ and $T_0 = 270 \text{ MeV}$, dotted line: $N_c = 3$ and $T_0 \to \infty$, dashed line: $N_c \to \infty$ and $T_0 = 270 \text{ MeV}$.

4. Conclusions

In this work the mismatch between the NJL-model and purely hadronic-models in the large-$N_c$ limit has been studied. We have found that the linear-$\sigma$ model implies a scaling of $T_c$ which is at odd with the NJL model and basic expectations [3].

In order to solve this issue we have introduced a phenomenologically motivated temperature dependent parameter. As a result the critical temperature remains constant in the large-$N_c$ limit. Moreover, for $N_c = 3$ the critical temperature is lowered to $T_c \approx 200 \text{ MeV}$, a value which is in line with recent model and lattice results on the chiral phase transition.

Future studies should go beyond the simple phenomenological Ansatz presented in this work and include, for instance, the coupling of hadrons to the Polyakov loop [2, 21]. Preliminary results [22] show that this approach also leads to the correct large-$N_c$ scaling of the critical temperature $T_c \sim N_c^0$.

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