Dynamically enhancing qubit-oscillator interactions with anti-squeezing

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The interaction strength of an oscillator to a qubit grows with the oscillator’s vacuum field fluctuations. The well known degenerate parametric oscillator has revived interest in the regime of strongly detuned squeezing, where its eigenstates are squeezed Fock states. Owing to these amplified field fluctuations, it was recently proposed that squeezing this oscillator would dynamically boost its coupling to a qubit. In a superconducting circuit experiment, we observe a two-fold increase in the dispersive interaction between a qubit and an oscillator at 5.5 dB of squeezing, demonstrating in-situ dynamical control of qubit-oscillator interactions. This work initiates the experimental coupling of oscillators of squeezed photons to qubits, and cautiously motivates their dissemination in experimental platforms seeking enhanced interactions.

Introduction: The magnitude of an electromagnetic oscillator’s vacuum field fluctuations sets the scale for its coupling strength to a qubit [1]. The value of these fluctuations is directly related to the mode’s impedance, and is therefore set by design. For example, the larger the mode impedance, the stronger its electrical field will fluctuate, thus enhancing the coupling to the charge degree of freedom of a qubit [2, 3]. Conversely, the lower the mode impedance, the stronger its magnetic field will fluctuate, thus enhancing the coupling to a spin [4]. Despite this design flexibility, some qubits remain difficult to couple to [5, 6]. Recently, Leroux, Govia and Clerk [7] have proposed to boost these fluctuations dynamically. This would enable in-situ enhancement of qubit-oscillator interactions, with far reaching applications such as pushing weakly coupled systems into the strong coupling regime [2, 3, 8], exploring the exotic ultra-strong regime [9], and observing dynamically-activated quantum phase transitions [10].

The proposal [7] considers a ubiquitous system in quantum optics: the degenerate parametric oscillator (DPO), albeit operated in a new regime (Fig. 1a). In the usual regime, widely employed for quantum-limited amplification [11–15], a pump modulates the oscillator frequency at twice its resonance, thus inducing resonant squeezing. Instead, in the new regime of interest, the pump is far detuned from the parametric resonance [16]. This added detuning renders the system Hamiltonian diagonalizable by a Bogoliubov transformation, and is therefore referred to as a Bogoliubov oscillator (BO) [17]. Unlike a regular harmonic oscillator whose eigenstates are circular Fock states, the eigenstates of a BO are squeezed Fock states. Their amplified fluctuations in the anti-squeezed quadrature are the root cause for enhancing qubit-BO interactions. Although many proposals have praised the interests of BOs for enhanced coupling [18, 19], wideband quantum limited amplification [17] and quantum transduction [20], their experimental implementation has remained unexplored.

In this experiment, we have observed that squeezing a BO amplifies its dispersive coupling to a qubit. We measure a two-fold increase in the dispersive interaction strength at 5.5 dB of squeezing. Moreover, we demonstrate amplification that evades the gain-bandwidth constraint [17]. We demonstrate these phenomena in a circuit quantum electrodynamics (cQED) architecture (Fig. 1b), where a rich toolbox of nonlinear dipoles is available to couple low loss modes [21]. While a regular transmon plays the role of the qubit [22], implementing a strongly detuned squeezing Hamiltonian without activating spurious nonlinear processes is a technical challenge [13, 14, 23]. To this end, we implement the BO by strongly pumping a resonator interrupted by a superconducting nonlinear asymmetric inductive element (SNAIL), capable of inducing significant squeezing while maintaining a vanishingly small Kerr non-linearity [24].

Theory: A SNAIL-resonator with bare frequency ω0, pumped at a frequency ωp, detuned from the degenerate parametric resonance 2ω0, emulates the DPO model in a frame rotating at half the pump frequency (Appendix B 1). It is described by the following Hamiltonian and Lindblad operator:

\[ H_{\text{ph}} = \hat{\delta}_{a} \hat{a}^\dagger \hat{a} - \frac{\lambda}{2} (\hat{a}^2 + \hat{a}^2) , \quad L_{\text{ph}} = \sqrt{\kappa} \hat{a} , \]

where \( \hat{a} \) is a bosonic annihilation operator, \( \kappa \) is the dissipation rate, \( \lambda \) is the amplitude of the two-photon pump, and \( \delta_{a} \) is the detuning between the oscillator and half the pump frequency. This system is widely operated in the resonant squeezing regime \( \delta_{a} = 0 \) and \( \lambda < \kappa/2 \), for near-quantum limited amplification and squeezed radiation generation [11–15]. Interestingly, at \( \delta_{a} = 0 \), the dynamics would be unstable in absence of dissipation.

Instead, we focus on the detuned squeezing regime \( \kappa/2 \ll \lambda < |\delta_{a}| \). Introducing the squeezing parameter

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FIG. 1. Principle of the experiment. (a) A cavity oscillator (blue mirrors) is driven by a two-photon pump with amplitude $\lambda$ and detuning $\delta_a$ (gold double arrow) while dissipating at rate $\kappa$ (blue arrow). Its eigenstates (blue levels), that are effectively thermally occupied (full circles), consist of squeezed Fock states $\ket{n_a}$ (right insets: Wigners at $S = e^{2r} = 6$ dB). This results in an enhanced coupling $g \cosh r$ (gold blue exchange arrows) to a qubit (red levels) and induces qubit dephasing (gold fuzz). (b) Superconducting circuit layout with a diagonal break for compactness. A quarter wavelength phasing (gold fuzz). (a) Superconducting circuit layout with a diagonal break for compactness. A quarter wavelength phasing (gold fuzz).

The Hamiltonian and loss operator (1) rewrite:

$$
H_{\text{ph}}/\hbar = \Omega_a[r]\alpha^\dagger\alpha,
L_{\text{ph}} = \sqrt{\kappa} \left( \alpha \cosh r + \alpha^\dagger \sinh r \right),
$$

where $\Omega_a[r] = \delta_a / \cosh 2r$, and the Lindblad operator $L_{\text{ph}}$.

This results in an enhanced coupling $g \cosh r$ (gold blue exchange arrows) to a qubit (red levels) and induces qubit dephasing (gold fuzz). (b) Superconducting circuit layout with a diagonal break for compactness. A quarter wavelength phasing (gold fuzz).

Instead, the dispersive regime is well adapted to measuring the qubit-BO coupling through qubit spectroscopy [31–33], even in the presence of a non-vanishing bath occupation [34–36]. We place ourselves in the dispersive regime where $ge^{\gamma} \ll \kappa, \gamma_1, \gamma_\phi$, when divided by $|\delta_q \pm \Omega_a[r]|$, are of order $\eta \ll 1$. Note that we require the qubit to be detuned from both the BO resonance and its mirror idler frequency. Retaining up to second order terms in $\eta$, the loss operators remain unchanged and Hamiltonian (3) is diagonalized as (Appendix B 3):

$$
H_{q,\text{ph}}/\hbar = \Omega_a[r]\alpha^\dagger\alpha + \left[ \delta_q + \chi[r] \left( \alpha^\dagger \alpha + \frac{1}{2} \right) \right] \sigma_z^2 / 2,
$$

where:

$$
\chi[r] = \frac{2g^2}{\delta_q - \Omega_a[r]} \cosh^2 r + \frac{2g^2}{\delta_q + \Omega_a[r]} \sinh^2 r.
$$
The enhanced dispersive coupling is immediately visible in Eq. (5). The first term has the familiar form of a dispersive interaction with a modified detuning and a $g$ coupling enhanced by $\cosh r$, while the second term emerges from the presence of the mirror idler resonance. Observing the dispersive coupling enhancement is the main goal of this experiment.

The Bogoliubov oscillator: Parametric oscillators have long been employed to amplify signals for qubit readout and generate squeezed radiation [11–14, 29, 30]. In our three-wave mixing device, we observe amplification by setting the pump frequency at the parametric resonance $\omega_p = 2\omega_a$ (Fig. 2a). By increasing the pump power close to the parametric instability $\lambda \leq \kappa/2$, we observe up to 12 dB of gain. We enter the regime of the BO by detuning the pump away from the parametric resonance $\omega_p = 2\omega_a - 2\delta_a$, where the detuning verifies $|\delta_a| \gg \kappa_a/2$. When $\delta_a/2\pi = \pm 30$ MHz (Fig. 2b), as we increase the pump power, the oscillator resonance shifts down from $\omega_a$ to $\omega_p/2$, following the theoretical prediction $\omega_p/2 + \Omega_a[r]$ where $\Omega_a[r] = \delta_a/\cosh 2r$. Moreover, this resonator of squeezed photons responds to regular plane waves at a mirror frequency $\omega_p/2 - \Omega_a[r]$. This idler peak merges into the signal peak when $\lambda \geq \sqrt{\delta_a^2 - \kappa^2/4}$ [17], which we refer to as the coalescence regime. Symmetrically, for $\delta_a/2\pi = -30$ MHz (Fig. 2c), the oscillator resonance shifts up from $\omega_a$ to $\omega_p/2$. This symmetric behavior differs from the response of a Kerr oscillator to a detuned pump, where the sign of the Kerr sets the direction of the shift, independently of the pump frequency.

The results of Fig. 2 demonstrate that $\lambda$ – the only fit parameter relating data and theory – is reliably identified at every pump power, thereby fully characterizing the BO oscillator.

A striking feature appears in the amplitude response...
of the oscillator, where gain is observed at both signal and idler frequencies. Indeed, in the resonant regime \( \delta_a = 0 \), the 3 dB amplification bandwidth \( \Delta_{3dB} \) reduces with gain \( G \) according to the gain-bandwidth product constraint \( \Delta_{3dB} G = \kappa \) (Fig. 3). In contrast in the detuned regime, following either the signal or idler peak, we observe a constant amplification bandwidth, independently of the gain. As \( \lambda \) approaches \( |\delta_a| \), the two peaks merge and the amplification bandwidth more than doubles. This amplifier, praised for evading the fundamental gain-bandwidth constraint, has been coined the Bogoliubov amplifier [17].

**Qubit spectroscopy in the presence of squeezed photons:**

After having characterized the BO, we now turn to the impact of its amplified fluctuations on the qubit. The oscillator, whose eigenstates are now squeezed Fock states, is expected to strongly affect the qubit spectrum both dispersively and dissipatively (Appendix B 4). In the weak dispersive regime \( \chi[r] \ll \kappa \), we relate the qubit frequency shift \( \Delta \omega_q \) and induced dephasing \( \Delta \gamma_\phi \) to the mean and correlation function of the BO number operator. We find:

\[
\Delta \omega_q = \chi \left[ r \right] \left[ \frac{1}{2} + \langle \alpha^\dagger \alpha \rangle \left[ r \right] \right] - \frac{1}{2} \chi[0], \\
\Delta \gamma_\phi = \frac{\chi^2}{\kappa} \left[ 1 + \langle \alpha^\dagger \alpha \rangle \left[ r \right] \right] \langle \alpha^\dagger \alpha \rangle \left[ r \right],
\]

where \( \chi[r] \) is the modified interaction strength given by Eq. (5), and \( \langle \alpha^\dagger \alpha \rangle \left[ r \right] \) is the mean occupancy of the BO mode. These equations are derived for a two level system and are adapted for a transmon to fit our data (Appendix F). The frequency shift \( \Delta \omega_q \) can be decomposed in two parts. First, a photon number independent term \( \frac{1}{2} \chi \left[ r \right] \), which is reminiscent of the Lamb shift experienced by an atom immersed in the vacuum fluctuations of an electromagnetic mode. Second, a photon number dependent term \( \chi \left[ r \right] \left[ \alpha^\dagger \alpha \right] \left[ r \right] \), which is reminiscent of the AC-Stark effect. The term \( \chi[0]/2 \) is subtracted since the frequency shift is referenced to the absence of pump (\( r = 0 \)). Interestingly, the expression of the induced dephasing \( \Delta \gamma_\phi \) is akin to the dephasing of a qubit dispersively coupled to a mode of thermal occupation \( \langle \alpha^\dagger \alpha \rangle \left[ r \right] \) [37]. This reveals that the qubit experiences the squeezed bath populating higher Bogoliubov energy levels, as a thermal bath. In principle, this induced decoherence, flagged by [28], could be cancelled by injecting conversely squeezed radiation while preserving the interaction enhancement [7].

The qubit-oscillator detuning is set to \( (\delta_q - \delta_a) / 2\pi = -100 \) MHz, thus placing the system in the weak dispersive regime \( \chi \left[ r \right] = 0 \), \( 2\pi = -250 \) kHz (Appendix D 4). For various pump detunings \( \delta_a / 2\pi \in \{0, \pm20, \pm30, \pm40\} \) MHz, and pump amplitudes inducing up to 8 dB of squeezing \( S \), we acquire the qubit reflection spectrum through its dedicated port (Fig. 4). From each spectrum, we extract the frequency shift \( \Delta \omega_q \) and linewidth broadening \( 2\Delta \gamma_\phi \) referenced to \( S = 0 \) dB (pump off). For \( \delta_a = 0 \), the balance of resonant two-photon pumping and dissipation stabilizes a squeezed steady-state. At maximal steady-state squeezing, the oscillator mean occupancy is found to be of less than 2 photons (Appendix C). Hence, the variations of \( \Delta \omega_q \) and \( \Delta \gamma_\phi \) are consistent with a constant dispersive interaction strength (see Fig. 4 left). This is in stark contrast with the case \( |\delta_a| \gg \kappa / 2 \), where the two-photon pump is balanced, not by dissipation, but by the detuning \( \delta_a \). Three notable features are visible in Fig. 4 right. First, for each detuning, as the pump amplitude approaches the instability point \( \lambda = |\delta_a| \) where the squeezing parameter \( r = \frac{1}{2} \tanh^{-1} \lambda / |\delta_a| \) diverges, we observe rapidly increasing frequency shifts and line broadenings. Second, the symmetry between positive and negative detunings is broken. Indeed, the BO frequency shifts towards the qubit for \( \delta_a > 0 \) and away from the qubit for \( \delta_a < 0 \). Interestingly, despite this asymmetry, the qubit frequency shifts down with increasing \( \lambda \), regardless of the sign of \( \delta_a \), showing that the dominant effect at play is the BO enhanced fluctuations, and not a trivial modulation of the BO-qubit detuning. Finally, the magnitudes of the qubit spectral shift and broadening are large. At maximal squeezing, the qubit frequency shifts by at least 4 times the bare qubit-BO dispersive coupling. Such large shifts cannot be explained by an unchanged interaction strength and a simple increase in the BO population. In-
Indeed, we estimate \(\langle n_1 n_2 \rangle [\ell] \leq 1.2\) over this entire dataset, thus hinting towards a significant enhancement of the qubit-BO interaction strength.

*Enhancing the dispersive interaction via anti-squeezing:* We measure the dispersive coupling \(\chi [r]\) by adapting the procedure of Refs. 32, 33 (Fig. 5). For each pump detuning \(\delta_p\) and amplitude \(\lambda\), we apply a weak drive tone on the BO at its frequency \(\Omega_p/\pi + \Omega_1 [r]\).

The drive power is quantified in units of photon number \(\bar{n}_d\) that would be injected in the oscillator in the absence of squeezing (Appendix D 4). For each value of \(\bar{n}_d\), a qubit reflection spectrum is acquired. Two effects are observed: a frequency shift \(\Delta \omega_a [r, \bar{n}_d]\) and a linewidth broadening \(\Delta \gamma_a [r, \bar{n}_d]\) referenced to \(\bar{n}_d = 0\).

Adapting the procedure that resulted in Eq. (6) in the presence of the drive far from coalescence \(|2\Omega_1 [r]| \gg \kappa/2\), we find \(\Delta \omega_a [r, \bar{n}_d] = \chi [r] \bar{n}_d \cosh^2 r\) and \(\Delta \gamma_a [r, \bar{n}_d] = (2\chi^2 [r] / \kappa) (1 + 2 \sinh^2 r) \bar{n}_d \cosh^2 r\).

The frequency shift resembles the usual AC-Stark shift albeit the extra term \(\cosh^2 r\) that indicates the enhanced coupling of the BO to the incident drive. On the other hand, the linewidth broadening has a form corresponding to the induced dephasing by an amplified coherent drive \(\bar{n}_d \cosh^2 r\), superimposed to the effective thermal occupation \(\sinh^2 r\) (Appendix B 4). We fit the measured frequency shifts and linewidth broadenings to these derived expressions, keeping only \(\chi [r]\) as a free parameter, and report the results in Fig. 5.

The left panel of Fig. 5 displays a control experiment at \(\delta_a = 0\) that shows no enhancement in \(\chi\) as expected by theory (Appendix E). The right panel displays \(\chi\) versus \(\lambda\) in the BO regime \(|\delta_a| \gg \kappa/2\). As previously observed in Fig. 4, the symmetry between positive and negative detunings is broken. This is expected since two different effects contribute to the variation of \(\chi\) with squeezing. First, the enhanced fluctuations of the BO result in an enhanced interaction strength, revealed by the \(\cosh^2 r, \sinh^2 r\) factors in Eq. (5). This effect is independent of the sign of \(\delta_a\). Second, as the BO is squeezed, its frequency \(\Omega_a [r]\) varies thus modifying the qubit-BO detuning. It is this effect that depends on the sign of \(\delta_a\). For positive pump detunings (empty symbols), the BO shifts towards the qubit so the two contributions add, resulting in a significant increase in \(\chi\). We measure up to a two-fold increase in \(\chi\) for \(\delta_a/2\pi = \pm 20\) MHz, from \(\chi/2\pi = -250\) kHz to \(\chi/2\pi = -510\) kHz at \(\lambda/2\pi = 17\) MHz corresponding to \(S = 5.5\) dB of squeezing. Only 15% of this increase is attributed to the reduced qubit-BO detuning. The converse is true for negative pump detunings (full symbols): the BO moves away from the qubit. Remarkably, the effect of enhanced fluctuations outweighs the effect of increased detuning, resulting in a measurable, yet modest, increase in \(\chi\) even for negative detunings. The matching of theory to data noticeably degrades at large \(|\delta_a|\), possibly due to the narrowing proximity of the idler peak to the qubit.

**Conclusion:** In conclusion, we have observed a two-fold increase in the dispersive interaction between a qubit and a BO at 5.5 dB of squeezing. A word of caution is however necessary. The BO, through its amplified field fluctuations, couples more strongly not only to the qubit but to all coupled modes, including the environment. This in turn, induces decoherence on the qubit, as warned by Ref. 28 and observed in this experiment. Future experiments could evade this induced decoherence by conversely squeezing the environment, as proposed by Ref. 7. Our experiment elucidates the non-trivial impact of squeezing a BO on both qubit coupling and induced noise. This opens the door to a realm of applications for BOs, including improved qubit readout, fast two-qubit gates, enhanced interactions to weakly coupled systems 7, 18, 19, quantum transduction 20, and squeezing induced quantum phase transitions 10.

**Author contributions** M.V, T.K and Z.L conceived the experiment. M.V designed the sample with guidance from W.C.S. M.V fabricated and measured the sample. M.V and Z.L analyzed the data and wrote the manuscript with input from all authors. M.V, A.P and Z.L derived the theory with support from P.C-I, A.S and M.M. Experimental support was provided by W.C.S, A.B, M.D and T.K.
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Appendix A: Sample and Setup

1. Circuit Implementation

We implement a BO coupled to a qubit in a cQED coplanar waveguide architecture (Fig. 1b). The oscillator is fabricated from a quarter wavelength superconducting resonator shunted to ground through a SNAIL element [24]. This element consists of three large Josephson junctions in parallel with a small one, forming a loop threaded by magnetic flux. The SNAIL endows the resonator with non-linearity that has a vanishing Kerr at a well chosen flux, while maintaining a significant three-wave mixing term [23]. This choice of nonlinear element was essential to implement Hamiltonian (1) with minimal parasitic terms (Appendix B 1). An inductive coupler channels both direct current (DC) for flux biasing, and radio-frequency (RF) probe and pump tones [39, 40]. At the Kerr-free point, the oscillator frequency is $\omega_a/2\pi = 6.940 \text{ GHz}$, and its dissipation rate $\kappa/2\pi = 8.7 \text{ MHz}$, largely dominated by coupling to the transmission line (Appendix C 1). The resonator is capacitively coupled to a flux tunable transmon. The transmon is coupled to a transmission line for direct reflection spectroscopy, inducing a total linewidth $\gamma/2\pi = 9.4 \text{ MHz}$ at $\omega_a/2\pi = 6.840 \text{ GHz}$ (Appendix D 3). Since the transmon anharmonicity $E_c/\hbar = 114 \text{ MHz}$ is much larger than $\gamma$, its two lowest energy eigenstates implement the qubit (Appendix D 2). The resonant coupling strength is $g/2\pi = 4.9 \text{ MHz}$ (Appendix D 3).

2. Fabrication

The sample is made out of a 280 $\mu$m thick intrinsic silicon chip, sputtered with 100 nm of niobium. A first laser lithography step patterns the large features of the circuit on S1805 resist. It is revealed in MF319, and subsequently etched with SF$_6$. The Al/AlOx/Al Josephson junctions are fabricated during a second step of electronic lithography, using a Dolan bridge technique on a bilayer of MMA/MAA and PMMA. After reveal in a 1:3 H$_2$O/IPA solution at 6 $^\circ$C for 90 s followed by 10 s in IPA, the chip is loaded in a Plassys evaporator. A 2 min argon milling cleaning is implemented to ensure good electrical contact between the two metallic layers. Then the chip is evaporated with a 35 nm thick layer of Aluminium with an angle of -30$^\circ$, followed by 5 min of oxidation in 5 mbar of pure oxygen, and the evaporation of 100 nm of Aluminium with a +30$^\circ$ angle. After lift-off, the chip is baked at 200$^\circ$C for 1 h. The resulting junctions are of three types as summarized in table I. The SQUID embedded in the transmon features a big junction in parallel with a tiny one, while the SNAIL embedded in the resonator features three big junctions in parallel with a small one (see Fig. 1).

3. Wiring

The sample is mounted in a microwave sample holder which was designed in-house, and tested to be free of spurious electromagnetic modes up to 15 GHz. It is then mounted at the base plate of a Bluefors LD250 and cooled down to 15 mK. The wiring diagram is detailed on Figure 6. We use the four channels of a Keysight PNA N5222A to measure the reflection spectra of the resonator and...
transmon ports, denoted $\Gamma_a$ and $\Gamma_q$ respectively. Two DC current sources Yokogawa GS200 are used to bias the flux loops of the SNAIL and the SQUID with fluxes $\Phi_a$ and $\Phi_q$. The Traveling Wave Parametric Amplifier (TWPA) provided by the group of W. Oliver at Lincoln Labs is powered by a R&S SGS100A. It amplifies the transmon signal by about 20 dB away from its dispersive feature at 6.0 GHz. The tone that pumps the SNAIL is provided by an Agilent Technologies E8257D and travels to the SQUID with a pass-band from 11 GHz to 24 GHz. This pump drive line, but features a high-rejection high-pass filter, includes a smaller amount of flat attenuation than the drive line. In order to maximize the amount of pump power reaching the sample around the parameter defined by $\tanh^{2} \lambda = 7 \text{GHz}$, we designed a dedicated microwave line for the pump. It displays around 26 dB less attenuation than the drive line and travels to the resonator VNA drive line, or through a distinct one. In order to maximize the amount of pump power reaching the sample around the parametric resonance $\omega_p/2\pi \approx 14 \text{GHz}$, without giving up on line attenuation at the resonance frequency $\omega_p/2\pi \approx 7 \text{GHz}$, we designed a dedicated microwave line for the pump. It includes a smaller amount of flat attenuation than the drive line, but features a high-rejection high-pass filter, with a pass-band from 11 GHz to 24 GHz. This pump line displays around 26 dB less attenuation than the drive line above 11 GHz, while maintaining sufficient attenuation around the oscillator frequency. These two orders of magnitude were crucial to approach instability in the BO regime $\lambda \sim \delta_a \gg \kappa/2$, without heating up the cryostat. Finally a R&S SMB100A provides the coherent drive on the resonator injecting photons to calibrate the dispersive interaction strength. All instruments are referenced to a Stanford Research Systems FS725 Rubidium clock.

### Appendix B: Theory of the Dispersive Interaction of a Qubit and Squeezed Photons

In this appendix, we derive the pieces of theory relevant to the understanding of the interaction of a qubit with squeezed photons. To begin with, we will show how a SNAIL-resonator can emulate a Bogoliubov oscillator (BO) which hosts squeezed photons as eigenstates, and we will follow the procedure of Ref. [41] to derive the master equation and input-output relation for such excitations. Then we will derive a perturbative Hamiltonian capturing the interaction between a qubit and squeezed photons. Finally we will compute the relevant observables to the description of the spectral properties of a qubit interacting with an oscillator filled of squeezed photons, whether it is driven or not. Extension of these results to a transmon are presented in Appendix F.

#### 1. Squeezed Photons in a SNAIL-resonator

First we consider a SNAIL-resonator biased at its Kerr-free flux point and strongly driven, or ”pumped”, close to the parametric resonance. At this specific operating point, it can be minimally described by an anharmonic oscillator with bare frequency $\omega_a$ dressed by a third-order nonlinearity $g_3$, such that its Hamiltonian writes [23]:

$$\mathcal{H}_a/h = \omega_a a^\dagger a + g_3(a + a^\dagger)^3 - \frac{\varepsilon_p}{\omega} \cos \omega_p t (a + a^\dagger), \quad (B1)$$

where $g_3 \ll \omega_a$. As customary for driven systems, we displace operator $a$ by its mean value $a \rightarrow a + \xi(t)$, where $\xi$ is a complex time-dependent parameter verifying $\xi = -i\omega_a \xi - (\kappa/2) \xi + i\varepsilon_p \cos(\omega_p t)$ [13]. At times $t \gg \kappa/\omega_a$ and in the regime where $\kappa \ll |\omega_a \pm \omega_p|$, we find:

$$\xi(t) \approx \frac{\varepsilon_p/2}{\omega_a - \omega_p} e^{-i\omega_p t} + \frac{\varepsilon_p/2}{\omega_a + \omega_p} e^{i\omega_p t}. \quad (B2)$$

Further going to a frame rotating at the half the pump frequency through the unitary $U_\omega = \exp(i\omega_p t a^\dagger a/2)$, resulting in $a \rightarrow a e^{-i\omega_p t/2}$, the transformed Hamiltonian exactly writes:

$$\mathcal{H}_a^{\xi,\omega}/h = \delta_a a^\dagger a + g_3\left( a e^{-i\omega_p t/2} + a^\dagger e^{i\omega_p t/2} - \Pi e^{-i\omega_p t} - \Pi^* e^{i\omega_p t} \right)^3, \quad (B3)$$

where $\delta_a = \omega_a - \omega_p/2$. We place ourselves in the regime where $|\delta_a| \ll \omega_a$, hence $\Pi \approx \varepsilon_p/3\delta_a$. We next perform the rotating wave approximation (RWA), and define the time-averaged photon Hamiltonian as $\mathcal{H}_{ph} \equiv \mathcal{H}_a^{\xi,\omega}/[42]$. We find:

$$\mathcal{H}_{ph}/h = \delta_a a^\dagger a - \frac{\lambda}{2}(a^2 + a^\dagger^2), \quad (B4)$$

where $\lambda \approx 2g_3\varepsilon_p/\omega_a$ is the two-photon pump amplitude. Thus a SNAIL-resonator pumped near the parametric resonance $2\omega_a$ emulates a degenerate parametric oscillator (DPO) [16]. The validity of the RWA is granted by $g_3\ll\varepsilon_p$. In the case where $\lambda < |\delta_a|$, the latter Hamiltonian can be diagonalized by means of a Bogoliubov transformation using the canonical basis $\alpha = a \cosh r - a^\dagger \sinh r$ where $r$ is the squeezing parameter defined by $\tanh 2r = \lambda/|\delta_a|$. This approach is equivalent to transforming the Hamiltonian through the squeezing unitary $U_\omega = e^{r/2(a^2 - a^\dagger^2)}$ by noting that $\alpha = U_\omega^a a U_\omega$. In this new basis the Hamiltonian (B4) writes:

$$\mathcal{H}_{ph}/h = \Omega_a [r] a^\dagger a \alpha, \quad (B5)$$

where $\Omega_a [r] = \delta_a / \cosh 2r$. One can also show that $\Omega_a [r] = \sqrt{\delta_a^2 - \lambda^2}$. When $\lambda = 0$, the Hamiltonian $\mathcal{H}_{ph}$ is that of a simple harmonic oscillator, and its eigenstates are Fock states $\{|n_a\}$ with eigenenergies $n_a\delta_a$, where $n_a$ are integers. Instead, when the two-photon pump is applied, the eigenstates are squeezed Fock states $\{U_\omega^a |n_a\}$ with eigenenergies $n_a\Omega_a [r]$ (see Fig. 1).

| Junction type | Big | Small | Tiny |
|--------------|-----|-------|------|
| Surface [µm²] | 2.10 | 0.14  | 0.08 |
| Inductance [nH] | 0.19 | 2.57  | 4.48 |

TABLE I. Characteristics of the three junction types, as measured on test structures fabricated on the same chip.
2. Input-Output Theory for Squeezed Photons

The resonator drive is applied through a coupled feedline hosting a continuum of modes \{c(\omega)\}_\omega which will ultimately interact with mode \(a\). It can be thought of as set of harmonic oscillators at all possible frequencies \(\omega \in [0, +\infty)\) described by the Hamiltonian:

\[ \mathcal{H}_{\text{bath}} = \int d\omega' \hbar \kappa c(\omega')^\dagger c(\omega'), \] where \([c(\omega), c(\omega')^\dagger] = \delta(\omega - \omega').\]

In order to account for the evolution of the SNAIL-resonator opened to its environment, the dynamics of the total system \{resonator + bath\} needs to be addressed through \(\mathcal{H}_{\text{tot}} = \mathcal{H}_a + \mathcal{H}_{\text{bath}} + \mathcal{H}_{\text{int}}\) where:

\[ \mathcal{H}_{\text{int}}/\hbar = \sqrt{\frac{\kappa}{2\pi}} \int d\omega' \left\{ \cosh r \left[ a c(\omega')^\dagger + a^\dagger c(\omega')^\dagger \right] + \sinh r \left[ a c(\omega') + a^\dagger c(\omega')^\dagger \right] \right\}. \]

such that each mode \(c(\omega)\) is coupled to the cavity at a rate \(\kappa\). The latter expression assumes the Markov approximation which neglects the frequency dependence of the coupling constant: \(\kappa(\omega) \approx \kappa\). This approximation is well verified in our experiment since the impedance of the transmission line is almost flat over the frequency window (of order \(\kappa\)) sampled by the resonator.

We follow the same treatment for the total Hamiltonian as in the previous subsection. While the bath part is trivially modified, the transformed interaction part writes in the Bogoliubov basis:

\[ \mathcal{H}_{\text{int}}^\omega/\hbar = \frac{\sqrt{\kappa}}{2\pi} \int d\omega' \left\{ \cosh r \left[ \alpha c(\omega')^\dagger + a^\dagger c(\omega')^\dagger \right] + \sinh r \left[ \alpha c(\omega') + a^\dagger c(\omega')^\dagger \right] \right\}. \]

At this stage the RWA is valid as long as: \(\kappa e^2 \ll \omega_p\), a regime safely maintained for all squeezing values. Then we can write the equations of motion for the Heisenberg operators \(\alpha(t)\) and \(c(\omega')(t)\):

\[ \partial_t \alpha(t) = -i\Omega_a[\sigma_a] \alpha(t) \]

\[ -i\sqrt{\frac{\kappa}{2\pi}} \int d\omega' c(\omega') \cosh r + c(\omega')^\dagger \sinh r, \]

\[ \partial_t c(\omega') = -i\omega' c(\omega') \]

\[ -i\sqrt{\frac{\kappa}{2\pi}} \left( \alpha(t) \cosh r + \alpha^\dagger(t)^\dagger \sinh r \right), \]

where the explicit time-dependence of the operator \(c(\omega)(t)\) has been omitted. Integrating equation (B8b) from a past reference time \(t_0\) until the experiment time \(t\), and defining the input field operator as \(a_{\text{in}}(t) = (-i/\sqrt{2\pi}) \int d\omega' c(\omega')(t_0) e^{-i\omega'(t-t_0)},\) we can rewrite equation (B8b) as:

\[ \partial_t \alpha(t) = -i\Omega_a[\sigma_a] \alpha(t) - \frac{\kappa}{2} \alpha(t) \]

\[ + \sqrt{\kappa} \left( a_{\text{in}}(t) \cosh r - a_{\text{in}}(t)^\dagger \sinh r \right). \]

Equation (B8b) could also have been integrated from a future time \(t_1\) until the experiment time \(t\), defining the output filed operator:

\[ a_{\text{out}}(t) = (i/\sqrt{2\pi}) \int d\omega' c(\omega')(t_1) e^{-i\omega'(t-t_1)}. \] The input and output fields satisfy the closure relation:

\[ a_{\text{out}}(t) + a_{\text{in}}(t) = \sqrt{\kappa} \left( \alpha(t) \cosh r + \alpha(t)^\dagger \sinh r \right). \]

The input and output fields have zero mean, and obey the commutation relations:

\[ [a_{\text{in}}(t), a_{\text{in}}(t')^\dagger] = \delta(t-t'), \]

\[ [a_{\text{out}}(t), a_{\text{in}}(t')] = 0 \] (same for \(a_{\text{out}}(t)\)).

The temperature of the environment is defined through the thermal occupancy \(\bar{n}_\text{th} = (1 + \bar{n})/2\). In the case where the oscillator is driven with a coherent tone of amplitude \(\varepsilon_d\) at a frequency \(\omega_d\), the input operator needs to be displaced by a classical contribution:

\[ a_{\text{in}}(t) = i\varepsilon_d/(\sqrt{2\pi}) \delta(t-t'). \]

Together, the quantum Langevin equation (B9) and the input-output relation (B10) fully capture the dynamics of the squeezed photons in contact with their environment. It is here described in terms of the incoming and outgoing fields of the bare mode \(a\), which correspond to the physical port used to drive and read-out the BO. Interestingly the decay rate of these squeezed photons does not change with squeezing, which in turn does not limit the Gain-Bandwidth product to a constant value when the BO is operated as an amplifier (see Fig. 3).

3. Dispersive Transformation

The coupling of the BO with a qubit is now addressed. Following the main text, the qubit with frequency \(\omega_q\) is introduced through the Pauli operators \((\sigma_a, \sigma_x, \sigma_+).\) In a rotating frame at frequency \(\omega_p/2\) for both modes, and assuming the BO-qubit coupling to be small \((g \ll \omega_q, \omega_d)\), the system can be described by a Jaynes-Cummings Hamiltonian augmented by a squeezing term:

\[ \mathcal{H}_{q-ph}/\hbar = \delta_q \sigma_a a^\dagger - \frac{\lambda}{2} \left( a^2 + a^\dagger)^2 + \delta_q \sigma_x^2/2 \]

\[ + g (a^\dagger \sigma_+ + a^\dagger \sigma_-), \]

where \(\delta_q = \omega_q - \omega_p/2\). Continuing with a diagonalization of the oscillator-only part of the Hamiltonian, we find in the Bogoliubov basis:

\[ \mathcal{H}_{q-ph}/\hbar = \Omega_x[\sigma_a] \alpha^\dagger \alpha + \delta_q \sigma_x^2/2 + g \cosh r \left( \sigma_+ + \alpha^\dagger \sigma_- \right) + g \sinh r \left( \sigma_- + \alpha^\dagger \sigma_+ \right). \]

While Ref. [7] focused on the resonant limit \(\delta_q \approx \Omega_x[\sigma_a]\), we place ourselves in the dispersive regime. Owing to the presence of the pump mixing signal and idler photons, the dispersive interaction to a BO is restricted to the regime where:

\[ \Delta[\sigma] = \delta_q - \Omega_x[\sigma_a] \gg g e^r, \]

\[ \Sigma[\sigma] = \delta_q + \Omega_x[\sigma_a] \gg g e^r. \]
Not only the qubit needs to be far from resonance with the BO signal frequency, but also with the mirror idler one. Therefore, we use a Schrieffer-Wolff (SW) transformation to write the Hamiltonian in a basis that decouples the qubit and the Bogoliubov mode at ﬁrst order in coupling over the detunings. The generator of this transformation writes:

\[ S = \frac{g \cosh r}{\Delta[r]} \alpha \sigma_+ - \frac{g \sinh r}{\Sigma[r]} \alpha \sigma_- - \text{h.c.} \] (B14)

In the transformed basis \( \alpha \to e^{-S} \alpha e^S \), \( \sigma_z \to e^{-S} \sigma_z e^S \), the Hamiltonian writes at second order in \( g/\Delta[r], g/\Sigma[r] \):

\[ \mathcal{H}_{\text{q-ph}}/\hbar = \Omega \alpha^\dagger \alpha + \delta_n \sigma_z + \chi[r] \left( \alpha^\dagger \alpha + \frac{1}{2} \right) \sigma_z + \chi[a][r] \left( \alpha^2 + \alpha^\dagger 2 \right) \sigma_z / 2, \] (B15)

where the dispersive interaction strength \( \chi[r] \) and its anomalous counterpart \( \chi_{a}[r] \) write:

\[ \chi[r] = \frac{2g^2 \cosh^2 r}{\delta q - \Omega_{a}[r]} + \frac{2g^2 \sinh^2 r}{\delta q + \Omega_{a}[r]}, \] (B16a)

\[ \chi_{a}[r] = \frac{g^2 \sinh 2r}{\delta q} - \frac{\delta_{\alpha}^2}{\delta q - \Omega_{a}^2[r]}, \] (B16b)

Further assuming \( \chi_{a}[r] \ll |\Omega_{a}[r]| \), Hamiltonian (B15) can be approximated by its secular part, which yields Eq. (4) of the main text.

Similarly the loss operators are dressed by the SW transformation. Introducing the dimensionless parameter \( \eta \) such that: \( \gamma \equiv \kappa, \Gamma_1, \Gamma_\phi < \eta \times \min(|\Delta[r]|, |\Sigma[r]|) \), and under the assumption that \( \eta \ll 1 \), these composite loss operators can be split into indistinguishable channels up to second order in \( \eta \):

\[ L_\alpha = \sqrt{\kappa} \left( \cosh r \alpha + \sinh r \alpha^\dagger \right), \] (B17a)

\[ L_{\alpha}^- = \sqrt{\kappa} \left( \frac{g \cosh^2 r}{\Delta[r]} - \frac{g \sinh^2 r}{\Sigma[r]} \right) \sigma_- , \] (B17b)

\[ L^+_\alpha = \sqrt{\kappa} \left( \frac{g \cosh r}{\Delta[r]} - \frac{g \sinh r}{\Sigma[r]} \right) \alpha^\dagger , \] (B17c)

\[ L_- = \sqrt{\Gamma_1} \sigma_- , \] (B17d)

\[ L^\phi_\phi = \sqrt{\Gamma_\phi} \left( \frac{g \cosh r}{\Delta[r]} + \frac{g \sinh r}{\Sigma[r]} \alpha^\dagger \right) \sigma_z , \] (B17e)

\[ L_\phi = \sqrt{\Gamma_\phi} \sigma_z , \] (B17f)

\[ L^+_\phi = \sqrt{\Gamma_\phi} \left( \frac{2g \cosh r}{\Delta[r]} \alpha + \frac{2g \sinh r}{\Sigma[r]} \alpha^\dagger \right) \sigma_+ , \] (B17g)

\[ L^-_\phi = \sqrt{\Gamma_\phi} \left( \frac{2g \cosh r}{\Delta[r]} \alpha^\dagger + \frac{2g \sinh r}{\Sigma[r]} \alpha \right) \sigma_- . \] (B17h)

Among these loss operators we identify the Purcell relaxation of the qubit through the Bogoliubov mode (B17b), and its induced excitation counterpart (B17c), along with the cavity dressed dephasing (B17e), excitation (B17g) and relaxation (B17h). When entering a Lindblad master equation, the amplitudes of the aforementioned processes are of order \( 3/\eta \). Hence at second order, only the bare loss operators (B17a), (B17d), (B17f) need to be considered.

Finally, it is instructive to look at the dispersive interaction strength in the limit \( |\deltaq| \gg |\alpha| \). In this regime, the renormalization of the oscillator frequency is negligible when compared to the qubit-oscillator detuning, such that \( \Delta[r] \approx \Sigma[r] \approx \delta_q \). Denoting the bare interaction parameter \( \chi_0 \equiv g^2/\delta q \), the enhanced dispersive interaction strength reads \( \chi[r] \approx \chi_0 \cosh 2r \).

4. Spectroscopy of a Qubit Interacting with Squeezed Photons

We consider a BO continuously squeezed, and coherently driven at its renormalized frequency. At long times, the occupation of the BO converges towards a mean-photon number \( \tilde{n}_\alpha \), and ﬂuctuates by \( \delta n_\alpha(t) \). The statistical properties of the BO occupancy reﬂect both the effects of the squeezing and the coherent drive. Computing the impact of this mixed statistics on a dispersively coupled qubit is the topic of this part [31, 33].

A qubit initialized in a coherent superposition of its basis states at a time \( t_0 \) will pickup a relative phase according to its dispersive interaction with the BO (see Eq. (4)). After an interaction time \( t \), we write this phase:

\[ \varphi(t) \equiv \bar{\varphi} + \delta \varphi(t). \] (B18)

which displays the Lamb-shifted qubit detuning \( \delta_q + \chi[r]/2 \), and the AC-Stark contribution \( \chi[r]\tilde{n}_\alpha \). The ﬂuctuating part reads:

\[ \delta \varphi(t) = \chi \int_{t_0}^{t_0 + t} d\tau \delta n_\alpha(\tau) , \] (B19)

and its randomness is at the heart of the dephasing mechanism. As the BO excitations are short-lived compared to the typical qubit-BO interaction time \( \kappa \gg \chi[r] \), \( \delta \varphi \) can be thought of as a sum of independent random variables. Hence the central limit theorem applies, and \( \delta \varphi \) follows a Gaussian distribution. Since \( \delta n_\alpha \) has zero mean, so does \( \delta \varphi \). The induced dephasing by the BO on the qubit \( \Delta \gamma_{\phi} \) is commonly deﬁned as \( e^{-\Delta \gamma_{\phi} t} \equiv \langle e^{i \delta \varphi(t)} \rangle \), where \( \langle \cdot \rangle \) refers to the average over multiple noise realizations (statistical ensemble average). Owing to the previously detailed statistics of \( \delta \varphi \), we ﬁnd that \( \langle e^{i \delta \varphi(t)} \rangle = e^{-\frac{1}{2} \langle (\delta \varphi)^2 \rangle} \), so that the induced dephasing reads:

\[ \Delta \gamma_{\phi} = \frac{\chi_0^2[r]}{2 t} \int_{t_0}^{t_0 + t} dt_1 dt_2 \delta n_\alpha(t_2) \delta n_\alpha(t_1) , \] (B20)

\[ \langle \delta n_\alpha(t_2) \delta n_\alpha(t_1) \rangle = \langle n_\alpha(t_2) - \tilde{\alpha}(n_\alpha(t_1) - \tilde{\alpha}) \rangle , \]
where \( \mathbf{n}_c(t) = \alpha(t)\dagger\alpha(t) \). To lowest order in \( \chi/\kappa \), the average \( \langle \cdot \rangle \) denotes the expectation value of the uncoupled system. Elucidating the dispersive and dissipative effects of the BO on the qubit amounts to solving the quantum Langevin equation (B9), and computing the mean-photon number at long times \( \bar{n}_\alpha = \langle \alpha\dagger\alpha \rangle \), and the correlation function \( C(t_1,t_2) = \langle \delta n_\alpha(t_2)\delta n_\alpha(t_1) \rangle = \langle n_\alpha(t_2)n_\alpha(t_1) \rangle - \bar{n}_\alpha^2 \).

In the presence of a coherent drive of amplitude \( \epsilon_d \) at the BO resonance, Eq. (B9) is most conveniently solved in a displaced frame. Specifically we write \( \alpha(t) = \alpha(t) + d(t) \), where \( \alpha(t) \) solves the classical part of Eq. (B9), and \( d(t) \) its quantum part. The displaced Bogoliubov operator \( d \) follows the same commutation relations as the original one. We find:

\[
\alpha(t) = \alpha(t_0)e^{-(i\Omega_a+\kappa/2)(t-t_0)} - i\int_{t_0}^{t}d\tau \left\{ \frac{\epsilon_d}{2} e^{-i\Omega_a\tau} \cosh r + \frac{\epsilon_d}{2} e^{i\Omega_a\tau} \sinh r \right\} e^{-(i\Omega_a+\kappa/2)(t-\tau)},
\]

\[
d(t) = d(t_0)e^{-(i\Omega_a+\kappa/2)(t-t_0)} + \sqrt{\kappa} \int_{t_0}^{t}d\tau \left\{ a_{\text{in}}(\tau) \cosh r - a_{\text{in}}(\tau) \right\} e^{-(i\Omega_a+\kappa/2)(t-\tau)}. \tag{B21a}
\]

Thus the mean-photon number can be readily computed as \( \bar{n}_\alpha = |\alpha(t)|^2 + \langle d\dagger(t)d(t) \rangle \). Moreover, owing to the quadratic nature of the system-bath Hamiltonian, we can use Wick’s theorem to compute the correlation function:

\[
C(t_1,t_2) = \langle d_i\dagger d_j\rangle \langle d_j d_i \rangle + \langle d_i\dagger d_j\rangle \langle d_j\dagger d_i \rangle \\
+ \alpha_i\dagger\alpha_j(\langle d_i d_j \rangle + \langle d_j d_i \rangle) \tag{B22}
\]

where \( d_i = d(t_i) \) and \( \alpha_i = \alpha(t_i) \). We then focus at the long time limit \( t \geq t_0 \gg \kappa^{-1} \) for which the BO converges to a limit cycle with amplitude:

\[
\alpha(t) = -i\frac{\epsilon_d}{\kappa} \cosh r e^{-i\Omega_a t} - i\frac{\epsilon_d}{\kappa} \sinh r e^{i\Omega_a t}. \tag{B23}
\]

In the limit \( |2\Omega_a r| \gg \kappa/2 \) (far from coalescence), the classical part of the Bogoliubov mode reduces to an amplified coherent signal \( \alpha(t) \approx -ie^{-i\Omega_a t}(\epsilon^+_d/\kappa) \cosh r \). Indeed, in that regime, the BO induces negligible mixing between the signal and idler components of the drive.

Next we turn to the statistical properties of the quantum part in the long time limit \( t_1,t_2 \geq t_0 \gg \kappa^{-1} \):

\[
\langle d\dagger(t_2)d(t_1) \rangle = (\sinh^2 r + \bar{n}_{\text{th}} + 2\bar{n}_{\text{th}} \sinh^2 r) e^{i\Omega_a(t_2-t_1)} e^{-\kappa|t_2-t_1|/2}, \tag{B24a}
\]

\[
\langle d(t_2)d\dagger(t_1) \rangle = (1 + \sinh^2 r + \bar{n}_{\text{th}} + 2\bar{n}_{\text{th}} \sinh^2 r) e^{-i\Omega_a(t_2-t_1)} e^{-\kappa|t_2-t_1|/2}, \tag{B24b}
\]

\[
\langle d(t_2)d(t_2) \rangle = i\frac{\kappa}{4\Omega_a} \frac{\sinh 2r}{1 - i\kappa/2\Omega_a} (1 + 2\bar{n}_{\text{th}}) e^{-(i\Omega_a+\kappa/2)|t_2-t_1|}. \tag{B24c}
\]

First, we focus on the situation where the environment is held in vacuum: \( \bar{n}_{\text{th}} = 0 \). At zeroth order in \( \eta = \kappa/4|\Omega_a[\tau]| \), the anomalous correlator (B24c) vanishes, and the displaced Bogoliubov mode \( d \) resembles a thermal field with occupancy \( \sinh^2 r \). At first order in \( \eta \) we find \( \bar{n}_\alpha = \bar{n}_d \cosh^2 r + \sinh^2 r \), where \( \bar{n}_d = |\epsilon_d^+/|/\kappa^2 \) is the number of circulating photons that the coherent drive would maintain in the oscillator in the absence of squeezing. Far from coalescence, the mean occupation of the BO results from the sum of the amplified drive and the effective thermal population. Second, we look at the correlation function, either for a squeezed oscillator in contact with vacuum (up to first order in \( \eta \)), or for a regular oscillator in contact with a hot environment:

\[
C(t_1,t_2 \mid \bar{n}_{\text{th}} = 0) \approx \sinh^2 r (1 + \sinh^2 r) e^{-\kappa|t_2-t_1|} + \bar{n}_d \cosh^2 r (1 + 2 \sinh^2 r) e^{-\kappa|t_2-t_1|/2}, \tag{B25a}
\]

\[
C(t_1,t_2 \mid \bar{n}_{\text{th}} = \bar{n}_{\text{th}}) = \bar{n}_{\text{th}} (1 + \bar{n}_{\text{th}}) e^{-\kappa|t_2-t_1|} + \bar{n}_d (1 + 2 \bar{n}_{\text{th}}) e^{-\kappa|t_2-t_1|/2}. \tag{B25b}
\]

Note that the correlation function (B25a) also features oscillatory terms proportional to \( \text{Re}(\eta e^{2i\Omega_a t_1 - \kappa|t_2-t_1|/2}) \).
that we omitted here, anticipating on the averaging performed when computing the induced dephasing. Comparing these two correlation functions lets us confirm the resemblance of a BO with a hot oscillator with thermal occupancy \( \sinh^2 r \).

Finally, we can write the frequency shift of a qubit dispersively coupled to a driven BO far from coalescence as \( \Delta \omega_q = \Delta \omega_q[r] + \Delta \omega_q[r, \bar{n}_d] \) where:

\[
\Delta \omega_q[r] = \chi[r] \left( \frac{1}{2} + \sinh^2 r \right), \tag{B26a}
\]

\[
\Delta \omega_q[r, \bar{n}_d] = \chi[r] \bar{n}_d \cosh^2 r. \tag{B26b}
\]

The first contribution amounts to a modified Lamb shift accounting for the equivalent thermal occupation of the BO. The second contribution is an AC-Stark shift accounting for the amplification of the input drive by the BO anti-squeezing. Similarly the induced dephasing reads \( \Delta \gamma_\phi = \Delta \gamma_\phi[r] + \Delta \gamma_\phi[r, \bar{n}_d] \) where:

\[
\Delta \gamma_\phi[r] = \frac{\chi^2[r]}{\kappa} \sinh^2 r \left( 1 + \sinh^2 r \right), \tag{B27a}
\]

\[
\Delta \gamma_\phi[r, \bar{n}_d] = \frac{2\chi^2[r]}{\kappa} \left( 1 + 2 \sinh^2 r \right) \bar{n}_d \cosh^2 r. \tag{B27b}
\]

We can map the first term to the characteristic dephasing of a qubit dispersively coupled to a hot oscillator [37]. The second term features the induced dephasing of a qubit measured by an amplified coherent drive on the oscillator, plus a cross term related to the equivalent BO thermal population. These equations are derived for a two-level system, and are adapted for a transmon in Appendix F.

**Appendix C: Degenerate Parametric Oscillator Calibrations**

In this appendix, we present the calibration of the Kerr-free flux point of the SNAIL-resonator, necessary to operate it as a DPO. Then we turn to the description of its microwave response, and show how we can use it to calibrate the two-photon pump amplitude, whether the two-photon pump frequency matches the degenerate parametric resonance or not. Finally we discuss the various definitions of squeezing, whether it is enforced via a detuned pump or not.

1. **Kerr-Free Flux Point of a SNAIL-Resonator**

Following [43], a SNAIL-resonator is most generally described by the Hamiltonian:

\[
\mathcal{H}_a/\hbar = \omega_a(\Phi_a) a^\dagger a + \sum_{m \geq 3} g_m(\Phi_a) (a + a^\dagger)^m, \tag{C1}
\]

where \( g_m(\Phi_a) \) is the \( m \)th-order nonlinearity inherited from the SNAIL potential energy, depending on the flux \( \Phi_a \) threading its loop. The fourth-order term of this expansion contributes to the Kerr nonlinearity of the oscillator. Owing to the specific choice of SNAIL parameters (see Table I), the Kerr amplitude vanishes at a given flux point [24]. We identify this specific flux point by performing a Kerr spectroscopy of the oscillator (Fig. 7). At each flux bias, we set a microwave drive 300 MHz above resonance populating the oscillator with increasing power (bottom to top, curves offset for clarity) in units of circulating photon number \( \bar{n}_d \) indicated on the right. Each panel corresponds to a flux point where the Kerr non-linearity is negative (I), close to zero (II) and positive (III). Fitted response (full lines) are overlaid to the data (open circles).
FIG. 8. Calibration of the two-photon pump when δa = 0. (a) Complex response Γa of a weak reflected signal on the oscillator port for increasing steady-state squeezing (color). (b) Reflection gain magnitude in decibels (y-axis) versus detuning of the probe tone with half the pump frequency (x-axis) for increasing steady-state squeezing (color). The data in (a,b) (open circles) are fitted to Eq. (C3) (solid lines). (c) Two-photon pump amplitude (y-axis, left) resulting from the fits in (a,b) versus the square root of the microwave pump power at 300 K (x-axis), applied through the low-power port. The maximum power reflection gain (y-axis, right) is deduced from Eq. (C4), and the steady-state squeezing (colorbar, common to (a) and (b)) is deduced from Eq. (C7). The dashed line is a linear extrapolation of the fit before saturation. The colored arrow indicates the maximum steady-state squeezing in decibels before saturation. The shaded area marks the instability region where λ > κ/2.

2. Microwave Response and Two-Photon Pump Calibration

Following Appendix B 2, the QLE for the bare oscillator Heisenberg operator a(t) writes at the Kerr-free flux point:

$$\partial_t a(t) = \frac{i}{\hbar} [\mathcal{H}_{ph}, a(t)] - \frac{\kappa}{2} a(t) + \sqrt{\kappa} a_{in}(t),$$

where $\mathcal{H}_{ph}$ was defined in Eq. (B4), and $\kappa$ is the coupling rate of the oscillator to its feedline. Since the oscillator is overcoupled to its feedline, no other dissipation channel needs to be included. As we measure in reflection, the input-output relation reads $a_{out}(t) + a_{in}(t) = \sqrt{\kappa} a(t)$. We compute the complex output and express in the following form:

$$a_{out}[\omega] = \Gamma_a[\omega] a_{in}[\omega] + \Gamma_i[\omega] a_{in}[\omega],$$

where $\Gamma_a[\omega]$ is the complex signal gain response, and $\Gamma_i[\omega]$ is the complex idler gain response. We find:

$$\Gamma_a[\omega] = -1 + \frac{\kappa^2/2 - i\kappa(\omega + \delta_a)}{\kappa^2/4 + \delta_a^2 - \omega^2 - i\kappa \omega}. \tag{C3}$$

Note that this computation is carried out in the rotating frame, hence $\omega$ is the deviation from half the pump frequency. In practice we measure $\Gamma_a[\omega]$, by acquiring two PNA traces (Appendix A 3). The first one probes the resonator under the specified pumping conditions. The second one probes the same frequency window, with the pump off and after flux tuning the resonator out of the frequency window. We divide the first trace by this second reference trace to recover $\Gamma_a[\omega]$. In that respect, $|\Gamma_a[\omega]|^2$ represents the frequency dependent power reflection gain of the system. Its maximum defines the gain $G \equiv \max_\omega |\Gamma_a[\omega]|^2$.

When $|\delta_a| < \kappa/2$, the reflection gain is maximized at half the pump frequency, i.e. $\omega = 0$, to a value:

$$G = 1 + \frac{\kappa^2 \lambda^2}{(\kappa^2/4 + \delta_a^2 - \lambda^2)^2}. \tag{C4}$$

For a given microwave pump power, the reflection gain features two local maxima when $\lambda < \lambda_{\text{co}} = \sqrt{\delta_a^2 - \kappa^2/4}$. These two peaks merge in the coalescence regime $\lambda \geq \lambda_{\text{co}}$ into a single one with maximum gain given by Eq. (C4). Note that $\lambda_{\text{co}}$ differs from the critical amplitude for which the gain diverges $\lambda_{\text{crit}} = \sqrt{\delta_a^2 + \kappa^2/4}$. As previously, the two-photon pump amplitude $\lambda$ is the only free parameter when fitting the data to Eq. (C3). The calibration results are presented in Fig. 9(a) for $\delta_a/2\pi \in \{\pm20, \pm30, \pm40\}$ MHz. We start by setting the flux at the Kerr-free point, however, when the pump is activated, the Kerr is dressed and may deviate from zero. In Fig. 9(b) we detail the procedure of adjusting the flux in order to reduce this dynamical Kerr effect. We display the phase response of the oscillator in the presence of an increasing pump power at $\omega_p = 2\omega_a^0 - 2\delta_a$ for $|\delta_a|/2\pi = 30$ MHz. Whether $\delta_a$ is negative or positive, the critical value $\lambda_c$ is not reached for the same critical power $P_{\text{crit}}$. Indeed, when $\delta_a > 0$, a positive dynamical Kerr accelerates the collapse of the oscillator signal and idler peaks. Conversely when $\delta_a < 0$, it slows down this process. The critical values $\lambda_c$ for each sign of the detuning only match when this spurious dynamical Kerr effect becomes negligible. When $|\delta_a|/2\pi = 30$ MHz and the oscillator initially sits at the Kerr-free flux point, the dynamical Kerr $\lambda_{\text{dyn}}$ is found to be positive (Fig. 9(b) top panels). Tweeking the flux bias towards higher frequencies, the two pictures can be symmetrized (middle panels), or bent in the other direction.
FIG. 9. Calibration of the two-photon pump when \( \Delta_a \geq \kappa/2 \). (a) Two-photon pump amplitude (y-axis) fitted at each pump power (x-axis) when the detuning is negative (full dots) or positive (open dots). The table summarizes the bare frequencies \( \omega_a^* \) of the oscillator used for the calibrations, as the ones that symmetrize the responses to a microwave pump at frequencies \( \pm|\delta_a| \). (b) Calibration of the dynamical Kerr-free point \( \Phi^*_a \) when \( |\delta_a|/2\pi = 30 \) MHz. From bottom to top the flux threading the SNAIL loop is increased, setting the bare oscillator frequency to 6.963 GHz (bottom), 6.949 GHz (middle) and 6.941 GHz (top). Left column: reflection phase response (color) versus input signal frequency (x-axis) and applied pump power at 300 K (y-axis) when the detuning between the bare cavity and half the pump frequency is \( \delta_a = -30 \) MHz. Center column: same as left for \( \delta_a/2\pi = +30 \) MHz. The colorbar (in radians) is indicated in the bottom panel. Right column: Two-photon pump amplitude (y-axis) fitted at each pump power (x-axis) when the detuning is negative (full dots) or positive (open dots). The right and center plots of the middle panels are nearly symmetric, as demonstrated by the matching of the two types of fit on the right plot: this defines the dynamical Kerr-free point.

(b) Calibration of the two-photon pump when \( \Delta_a \geq \kappa/2 \). (a) Two-photon pump amplitude (y-axis) fitted at each pump power (x-axis) when the detuning is negative (full dots) or positive (open dots). The table summarizes the bare frequencies \( \omega_a^* \) of the oscillator used for the calibrations, as the ones that symmetrize the responses to a microwave pump at frequencies \( \pm|\delta_a| \). (b) Calibration of the dynamical Kerr-free point \( \Phi^*_a \) when \( |\delta_a|/2\pi = 30 \) MHz. From bottom to top the flux threading the SNAIL loop is increased, setting the bare oscillator frequency to 6.963 GHz (bottom), 6.949 GHz (middle) and 6.941 GHz (top). Left column: reflection phase response (color) versus input signal frequency (x-axis) and applied pump power at 300 K (y-axis) when the detuning between the bare cavity and half the pump frequency is \( \delta_a = -30 \) MHz. Center column: same as left for \( \delta_a/2\pi = +30 \) MHz. The colorbar (in radians) is indicated in the bottom panel. Right column: Two-photon pump amplitude (y-axis) fitted at each pump power (x-axis) when the detuning is negative (full dots) or positive (open dots). The right and center plots of the middle panels are nearly symmetric, as demonstrated by the matching of the two types of fit on the right plot: this defines the dynamical Kerr-free point.

3. Steady-State Squeezing

In the detuned case, the Bogoliubov transformation that diagonalizes Hamiltonian (B4) defines a squeezing amplitude \( S = e^{2\tau} \) that quantifies the anisotropy of the BO eigenstates (see Fig. 1). An arbitrarily large squeezing will result in anti-squeezed fluctuations in one quadrature and conversely squeezed fluctuations in the other, with no saturation. Noting that \( \forall x \in (-1, 1), \tanh^{-1}(x) = \frac{1}{2}(\ln(1 + x) - \ln(1 - x)) \), it is instructive to write the squeezing amplitude as:

\[
S = \sqrt{\frac{\delta_a + \lambda}{\delta_a - \lambda}}. \tag{C5}
\]

Yet, this Bogoliubov transformation is only valid in the detuned case. In the resonant case, the two photon pump is no longer balanced by the detuning, but rather by dissipation. The oscillator reaches a steady state, and from its quadrature statistical fluctuations we may define a squeezing parameter \([44]\). Steady-state observables can be computed analytically by solving the Lindblad master equation:

\[
\dot{\rho} = -\frac{i}{\hbar} \{H_{ph}, \rho\} + \kappa D(\alpha)\rho. \tag{C7}
\]

Defining the oscillator quadratures as \( X_{\theta} = (a e^{-i\theta} + a^\dagger e^{i\theta})/2 \) and \( P_{\theta} = (ae^{-i\theta} - a^\dagger e^{i\theta})/2i \), we find for \( \delta_a = 0 \):

\[
\langle a^4 \rangle = \frac{1}{2} \frac{\lambda^2}{\kappa^2/4 - \lambda^2}; \tag{C6a}
\]

\[
\langle a^2 \rangle = \frac{1}{2} \frac{i\lambda e/2}{\kappa^2/4 - \lambda^2}; \tag{C6b}
\]

\[
\langle X_{\theta}^2 \rangle = \frac{1}{4} \frac{\kappa^2/4 + \lambda(\kappa/2) \sin 2\theta}{\kappa^2/4 - \lambda^2}; \tag{C6c}
\]

\[
\langle P_{\theta}^2 \rangle = \frac{1}{4} \frac{\kappa^2/4 + \lambda(\kappa/2) \sin 2\theta}{\kappa^2/4 - \lambda^2}; \tag{C6d}
\]

where \( \langle \cdot \rangle = \text{Tr}(\rho_{\infty}) \) and \( \dot{\rho}_{\infty} = 0 \). When \( \lambda = 0 \) we recover the isotropic vacuum field fluctuation: \( \forall \theta, \langle X_{\theta}^2 \rangle = \langle X_{\theta}^2 \rangle_{\text{vac}} = 1/4 \) (same for \( P_{\theta} \)). For \( \lambda > 0 \), the steady-state is anti-squeezed along \( X_{\pi/4} \), and squeezed along \( P_{\pi/4} \) though it saturates to half the amplitude of vacuum field fluctuations \([45]\). The steady-state squee-
ing is defined as [21]:
\[
S_\infty \equiv \frac{\langle X_q^2/4 \rangle}{\langle X_{\text{vac}}^2 \rangle} = \frac{\kappa/2}{\kappa/2 - \lambda}.
\] (C7)

This is the metric that we chose to compare the resonant case with the eigenstate squeezing of the BO. One can show that in the large gain limit \( S_\infty \sim \sqrt{G} \), as illustrated on Fig. 8. Also, combining Eqs. (C4) (at \( \delta_a = 0 \)) and (C6a), we find the useful relation \( \langle a|a \rangle = (\sqrt{G} - 1)/4 \). Thus, regarding Fig. 4, we can estimate the mean occupancy of the resonantly squeezed oscillator at \( S_\infty = 8 \) dB to be of approximately 1.3 photons.

**Appendix D: Transmon Characteristics and Calibrations**

The transmon is a superconducting qubit design featuring a Josephson element with energy \( E_J \), shunted by a large capacitor with charging energy \( E_C \), in the regime where \( E_J \gg E_C \). It is well described by the three-level Hamiltonian:
\[
\mathcal{H}_t = \sum_{i \in \{g,e,f\}} \hbar \omega_i |i \rangle \langle i |,
\] (D1)

where \( |g\rangle, |e\rangle, |f\rangle \) denote its three lowest energy states. The \( |g\rangle \) and \( |e\rangle \) states define the qubit states, with transition frequency \( \omega_q \equiv \omega_e - \omega_g = \frac{1}{\pi} (\sqrt{8E_J E_C} - E_C) \). Single-photon excitations to the \( |f\rangle \) state are detuned from the qubit transition by the anharmonicity \( \chi_q \approx -E_C/\hbar \) such that: \( \omega_{ef} = \omega_q + \chi_q \).

1. **Single-Tone Spectroscopy**

In the present experiment, the transmon Josephson element is a SQUID. Controlling the flux \( \Phi_q \) threading the SQUID loop lets us tune the transmon resonant frequency. Moreover, the transmon is strongly coupled to a microwave feedline. Photon leakage through this port dominates over every other relaxation channel. This feature was chosen to mimick the small relaxation time of typical mesoscopic qubits, and also to let us record the reflection spectrum of the transmon directly, without relying on an extra readout mode (see Fig. 10). Specifically, in the case where the transmon mode is populated with much less than 1 photon, the complex amplitude of a weak reflected signal on its input port writes:
\[
\Gamma_q(\omega) = -1 + \frac{\gamma_1}{\gamma_1/2 - i(\omega - \omega_q)},
\] (D2)

where \( \gamma_1 \) is the qubit relaxation rate, dominated by the coupling to its feedline, and \( \gamma_t = \gamma_1 + 2\gamma_\phi \) is the total linewidth of the transmon spectral line. Pure dephasing acts at a rate \( \gamma_\phi \). The latter equation describes a circular trajectory in the complex plane, symmetric about the real axis, with an accumulation point \( \Gamma_\infty = -1 \). In principle, the reflection spectroscopy of such a system can distinguish the coupling rate to its feedline (here, \( \gamma_1 \)) from the other contributions to the total linewidth (here, \( 2\gamma_\phi \)). Using Eq. (D2) to fit the data presented in Fig. 10 (bottom panels, fit not shown) would yield \( \gamma_1/2\pi = 5.0 \) MHz and \( \gamma_\phi/2\pi = 2.2 \) MHz, thus placing the system in the overcoupled regime \( \gamma_t > \gamma_\phi/2 \). However, in this very regime, fitting both rates is prone to errors due to imperfections of the experimental setup [46]. These imperfections can lead to deviations from the canonical spectroscopic response, such as tilted circles in the complex plane. It turns out that such tilts are present in the data. As a consequence, we renounce on fitting \( \gamma_1 \) and \( \gamma_\phi \) separately. Rather, we employ a fit function representing circles with any orientation in the complex plane, thus sensitive to \( \gamma_t \) only (see Fig. 10 bottom panels, blue lines). This procedure lets us fit the total linewidth of the transmon line reliably and accurately.
with the resonant case (see Fig. 12). Setting the oscillator to its Kerr-free flux point, we record its reflection spectrum as the transmon frequency is swept across. From input-output theory we expect the following response:

\[ \Gamma_a(\omega) = -1 + \left( \frac{\kappa}{2} - i(\omega - \omega_a) + \frac{g^2}{2} - i(\omega - \omega_q) \right) \]

where \( g \) is the resonant coupling amplitude. Having previously calibrated the decay rates of the transmon (\( \gamma_q/2\pi = 8.0 \text{ MHz} \)) and the oscillator (\( \kappa/2\pi = 8.7 \text{ MHz} \)), the recorded map can be fitted using \( g \) as the only fitting parameter. When the transmon and the oscillator are on resonance, the oscillator spectrum displays a partially resolved splitting. Indeed, the coupling amplitude \( g/2\pi = 6.1 \text{ MHz} \) is smaller than the decay rates of both modes, thus placing the system just below the strong resonant coupling regime.

4. Transmon-Oscillator Dispersive Coupling and Photon Number Calibration

Next we turn to the characterization of the coupling in the dispersive limit: \( |\omega_q - \omega_d| \gg g \). Following Ref. [32, 33], the dispersive interaction can be revealed through the measurement of the AC-Stark shift \( \Delta \omega_q \) and induced dephasing \( \Delta \gamma_d \) of the qubit upon coherent driving of the oscillator. Specifically, a coherent drive at frequency \( \omega_d \) and power \( P_{\text{drive}} \) stabilizes a coherent field \( \alpha_{g,e} \) in the

\[ \chi_{q}(\omega,q) = \frac{\kappa}{\gamma_q} \left( \frac{1}{\gamma_q} + \frac{1}{\gamma_a} \right) \]

\[ \Gamma_a(\omega) = -1 + \left( \frac{\kappa}{2} - i(\omega - \omega_a) + \frac{g^2}{2} - i(\omega - \omega_q) \right) \]

2. Two-Tone Spectroscopy

So far, the anharmonicity of the transmon has been disregarded. Unlike the previous discussion, driving the transmon with higher powers unravels its multi-level structure. We reveal transmon states beyond the qubit manifold by performing a two-tone spectroscopy, saturating the g-e transition with a resonant microwave drive, and then probing the transmon with a weak tone (see Fig. 11). Due to the finite occupation of the \( |e\rangle \) state provided by the saturation drive, the e-f transition can be revealed by the weak tone. Note that the spectroscopic tone is about 5000 times less powerful than the saturation one. We repeat the experiment at multiple flux points, thus varying the qubit frequency. The fitted anharmonicity fluctuates around \(-100 \text{ MHz}\), the value predicted by electromagnetic simulations of the transmon design.

3. Transmon-Oscillator Resonant Coupling

Making the most of the wide tunability range of both the transmon and oscillator frequencies, we can study their interaction in different detuning regimes. We begin

\[ \chi_{\text{osc}}(\omega) = \frac{-\gamma_q}{\gamma_a} - \frac{\gamma_a}{\gamma_q} \]

\[ \chi_{\text{osc}}(\omega) = \frac{-\gamma_q}{\gamma_a} - \frac{\gamma_a}{\gamma_q} \]

\[ \chi_{\text{osc}}(\omega) = \frac{-\gamma_q}{\gamma_a} - \frac{\gamma_a}{\gamma_q} \]

\[ \chi_{\text{osc}}(\omega) = \frac{-\gamma_q}{\gamma_a} - \frac{\gamma_a}{\gamma_q} \]

\[ \chi_{\text{osc}}(\omega) = \frac{-\gamma_q}{\gamma_a} - \frac{\gamma_a}{\gamma_q} \]
cavity, whether the qubit is in $|g\rangle$ or $|e\rangle$, such that:

$$\alpha_{g,e} = -\frac{i\kappa}{2} + i(\omega_a - \omega_d + \chi/2) \sqrt{\frac{P_{\text{drive}}}{P_0}},$$

where $P_0$ is the drive power maintaining one photon in the oscillator (regardless of the qubit state since $\chi \ll \kappa$). Subsequently, the finite occupation of the oscillator shifts the qubit frequency by $\Delta\omega_q = \text{Re}(\chi\alpha^*_e\alpha_e)$, where $\chi$ is the dispersive interaction amplitude. Moreover, the occupation number of the coherent field follows Poisson statistics, leading to an induced dephasing of the qubit: $\Delta\gamma_\phi = -\text{Im}(\chi\alpha^*_e\alpha_e)$. In the weak-dispersive limit $\chi \ll \kappa$ and resonant driving, these formula simplify to: $\Delta\omega_q = \chi\bar{n}_d$ and $\Delta\gamma_\phi = 2\chi^2\bar{n}_d/\kappa$, where $\bar{n}_d = P_{\text{drive}}/P_0$ is the mean photon number injected by the coherent drive in the oscillator. Both the dispersive coupling $\chi$ and the photon number calibration $P_0$ can be extracted from the joint fitting of the AC-Stark shift and induced dephasing with the applied drive power.

This procedure is presented in Fig. 13 for multiple qubit frequencies $\omega_q$, while the oscillator sits at its Kerr-free point and is driven at resonance $\omega_d = \omega_q$. For each value of $\omega_q$, we measure the qubit AC-Stark shift $\Delta\omega_q$ and measurement induced dephasing $\Delta\gamma_\phi$ as a function of the drive power $P_{\text{drive}}$ on the oscillator. We fit this entire data set to the above formula keeping as free parameters: $\chi$ at every qubit frequency, and a single power calibration $P_0$. We find $P_0 = 8.1 \pm 0.3$ nW. Also, the evolution of $\chi$ with qubit-oscillator detuning clearly displays the straddling regime: when the oscillator frequency lies between the $g-e$ transitions, virtual transitions to the $|f\rangle$ state strongly affects the dispersive interaction strength [22].

Finally, we fit the extracted $\chi$ versus $\omega_q$ to the analytical result accounting for the transmon $|f\rangle$ state:

$$\chi = \frac{2g^2}{\omega_q - \omega_a} \frac{x_q}{\omega_q - \omega_a + \chi_q}.$$  \hspace{1cm} (D5)

Keeping as free parameters $g$ and $x_q$, we find $g/2\pi = 4.9$ MHz and $E_C/\hbar = -\chi_q/2\pi = 114$ MHz, which are close to the values extracted from the anti-crossing and two-tone spectroscopy described in previous sections.

**Appendix E: Dispersive Interaction of a Qubit and a Resonantly Squeezed Oscillator**

In this appendix, we review the modification of the qubit spectral properties when the SNAIL-resonator is pumped at the degenerate parametric resonance. While the evolution of the measurement induced dephasing as a function of the cavity gain was covered extensively in Ref. [38, 47], analysis of the concurrent frequency shift and thus the dispersive interaction strenght was not addressed.

Starting from the system Hamiltonian (B11), we can write a SW transformation that leaves invariant the bare
the qubit-oscillator detuning. This will be the regime of interest for the remainder of this appendix. Introducing the dimensionless parameter \( \eta \) such that: \( q, \kappa < \eta < \kappa \), and under the assumption that \( \eta \ll 1 \), Hamiltonian (B11) reads in the transformed basis \( a \rightarrow e^{-S} a e^S \), \( \sigma_z \rightarrow e^{-S} \sigma_z e^S \) up to second order in \( \eta \):

\[
\mathcal{H}_{q-ph}/\hbar = -\frac{\lambda}{2} (a^2 + a^4) + \left[ \frac{\delta_q + \chi_0}{2} \right] \sigma_z + \text{h.c.}
\]

where \( \chi_0 = 2g^2/\delta_q \) is the bare dispersive interaction parameter. Corrections to \( \chi_0 \) occur at order four in \( \eta \).

Like in the previous part, \( \chi_0 \) can be inferred from the joint measurement of the AC-Stark shift and linewidth broadening of the qubit in the presence of a microwave drive, resonant with the oscillator. Following the derivation of Appendix B 4, the dressing of the qubit spectral features are deduced from the steady-state properties of the oscillator, to lowest order in \( \chi/\kappa \). The oscillator dynamics is governed by the Lindblad master equation \( \dot{\rho} = -\frac{\Gamma}{2} [\mathcal{H}_{ph} + \mathcal{H}_{drive}, \rho] + \kappa \mathcal{D}(a^\dagger a) \rho \), where in the rotating frame \( \mathcal{H}_{drive}/\hbar = (\varepsilon_d/2)a^\dagger + (\varepsilon_d^*/2)a \). Since \( \delta_q = 0 \), the drive frequency is commensurate with the pump frequency, and their relative phase is expected to modify the system response [38]. The drive complex amplitude is defined as \( \varepsilon_d = |\varepsilon_d| e^{i(\theta - \pi/4)} \), so that when \( \theta = 0 \) the in-phase component of the drive lies along the squeezed quadrature of the oscillator (Appendix C3).

Defining the mean occupation of the oscillator in the steady-state \( \bar{n}_q = \text{Tr}(a^\dagger a \rho_\infty) \) with \( \partial_t \rho_\infty = 0 \), the qubit frequency shift reads \( \Delta \omega_q \equiv \chi_0 \bar{n}_q = \Delta \omega_{q0}[\lambda] + \Delta \omega_q[\bar{n}_q] \) where:

\[
\Delta \omega_{q0}[\lambda] = \frac{\lambda^2/4}{\kappa^2 + \lambda^2} \chi_0, \\
\Delta \omega_q[\bar{n}_q] = \frac{\kappa^2}{4} \frac{\lambda^2}{\kappa^2 + \lambda^2} \bar{n}_q \chi_0.
\]

While this change of frame was used in the BO regime \( \kappa/2 \ll \lambda < |\delta_q| \) in Ref. [28], here we focus on the usual amplifier regime \( |\delta_q| < \kappa/2 \). Note that when \( \delta_q = 0 \), the qubit frequency in the rotating frame corresponds to oscillator part, including the two-photon pump. Its generator reads:

\[
S = \frac{g}{\delta_q - \delta_a} - \frac{1}{\lambda^2} a \sigma_+ - \text{h.c.} + \frac{\lambda g}{\delta_q - \delta_a} - \frac{1}{\lambda^2} a \sigma_- - \text{h.c.}.
\]

FIG. 14. Dispersive interaction of the qubit with a resonantly squeezed oscillator \( \delta_q = 0 \). (a) Top: AC-Stark shift \( \Delta \omega_q[\lambda, \bar{n}_q] \) (y-axis) versus drive phase (x-axis) for various oscillator steady-state squeezing (color) and a fixed drive amplitude. Bottom: same for linewidth broadening \( 2 \Delta \gamma \) (y-axis). Data points (stars) are extracted in a procedure akin to Fig. 13. Solid lines are fits to Eq. (E1) and Eq. (4) of Ref. [38], yielding \( \chi_0 \) for each steady-state squeezing value. (b) Dispersive interaction strength (y-axis) versus steady-state squeezing (x-axis bottom and color) and amplifier gain (x-axis top). The solid line marks the theoretical value that is expected to be independent of squeezing up to second order in \( \kappa/\delta_q \).

\( g, \kappa < \eta \times |\delta_q| \), and under the assumption that \( \eta \ll 1 \), Hamiltonian (B11) reads in the transformed basis \( a \rightarrow e^{-S} a e^S \), \( \sigma_z \rightarrow e^{-S} \sigma_z e^S \) up to second order in \( \eta \):

\[
\mathcal{H}_{q-ph}/\hbar = -\frac{\lambda}{2} (a^2 + a^4) + \left[ \frac{\delta_q + \chi_0}{2} \right] \sigma_z + \text{h.c.}
\]
Appendix F: Dispersive Interaction of a Transmon and Squeezed Photons

In this appendix we extend the results of Appendix B obtained for a two-level system to the higher energy levels of the transmon, beyond the qubit manifold. Following Ref. [22], the transmon-BO Hamiltonian reads in the Bogoliubov basis and under the RWA:

\[ \mathcal{H}_{\text{t-ph}} / \hbar = \Omega_{a}[r] \alpha^\dagger \alpha + \sum_{k} \delta_{k} |k \rangle \langle k | \]
\[ + \sum_{k} g_{k,k+1} \cosh r \ ( \alpha |k + 1 \rangle \langle k | + \text{h.c.} ) \]  
\[ + \sum_{k} g_{k,k+1} \sinh r \ ( \alpha |k \rangle \langle k + 1 | + \text{h.c.} ) \]  

(F1)

where \( \delta_{k} = \omega_{k} - k \times \frac{\pi}{2} \) and \( g_{k,k+1} \approx \sqrt{\frac{k+1}{\gamma}} \). We neglect multi-photon transitions in the transmon spectrum. Indeed, despite the multi-level structure of the transmon, in the limit \( E_{f} \gg E_{C} \), selection rules forbid photo-assisted transitions between non-neighboring energy levels. Moving on with the dispersive transformation, the generator of the SW unitary reads:

\[ \mathcal{S} = \sum_{k} \frac{g_{k,k+1} \cosh r}{\delta_{k,k+1} - \Omega_{a}[r]} \ ( \alpha |k + 1 \rangle \langle k | - \text{h.c.} ) \]
\[ - \sum_{k} \frac{g_{k,k+1} \sinh r}{\delta_{k,k+1} + \Omega_{a}[r]} \ ( \alpha |k \rangle \langle k + 1 | - \text{h.c.} ) \]  

(F2)

where \( \delta_{k,k+1} = \delta_{k+1} - \delta_{k} \). We introduce a dimensionless parameter \( \eta \) such that: \( \forall k, g_{k,k+1} e^{r} < \eta \times \min(\delta_{k,k+1} \pm \Omega_{a}[r]) \). This dispersive transformation requires that all the allowed transmon transitions are detuned from the BO signal and idler frequencies. This regime is safely maintained in our experiment. In the transformed basis \( \alpha \rightarrow e^{-\mathcal{S}} \alpha e^{\mathcal{S}} \), and \( \forall k, |k \rangle \rightarrow e^{-\mathcal{S}} |k \rangle \), the restriction of Hamiltonian (F1) to the two lowest energy transmon levels reads at second-order in \( \eta \):

\[ \mathcal{H}_{\text{t-ph}} / \hbar = \left( \Omega_{a}[r] + \Omega^{(2)}_{a}[r] \right) \alpha^\dagger \alpha + \left( \delta_{q} + \delta^{(2)}_{q} [r] \right) \sigma_{z} / 2 \]
\[ + \chi_{t}[r] \alpha^\dagger \alpha \sigma_{z} / 2 \]  

(F3)

where we introduced the qubit-manifold spin operator \( \sigma_{z} = |e \rangle \langle e | - |g \rangle \langle g | \), and:

\[ \Omega^{(2)}_{a}[r] = - \frac{g^{2} \cosh^{2} r}{\Delta[r]} - \frac{g^{2} \sinh^{2} r}{\Sigma[r]} \]  
\[ \delta^{(2)}_{q}[r] = \frac{g^{2} \cosh^{2} r}{\Delta[r]} + \frac{g^{2} \sinh^{2} r}{\Sigma[r]} \left( \chi_{q} = \Sigma [r] \right) \]  
\[ \chi_{t}[r] = \frac{2g^{2} \chi_{q}}{\Delta[r]} \left( \chi_{q} + \Delta[r] \right) \cosh^{2} r \]
\[ + \frac{2g^{2} \chi_{q}}{\Sigma[r]} \left( \chi_{q} + \Sigma[r] \right) \sinh^{2} r \]  

(F4a)

(F4b)

(F4c)

One can check that in the limit \( |\chi_{q}| \gg |\Delta[r]|, |\Sigma[r]| \) we recover \( \delta_{q}^{(2)}[r] \approx \chi_{t}[r] / 2 \), as customary for a dispersively coupled qubit.

---

FIG. 15. Dispersive interaction of the transmon with the BO when \( \delta_{a} / 2 \pi = 20 \text{ MHz} \). (a) Top: Amplitude (y-axis) of a weak reflected signal with frequency \( \omega_{\text{probe}} \) (x-axis) on the transmon port in the presence of a resonant microwave drive on the BO injecting an increasing number of photons (color), and BO squeezing of 1.9 dB (left), 3.6 dB (center) and 5.3 dB (right) with \( \delta_{a} > 0 \). The induced dephasing and AC-Stark shift on the transmon are extracted by fitting the complex data (open symbols) to circles in the complex plane (solid lines). Insets: zoom on the resonance as indicated by the grey box. Middle: AC-Stark shift of the transmon \( \Delta \omega_{q}[r, \bar{n}_{a}] \) (y-axis) versus number of injected photons (x-axis, color). Bottom: same for linewidth broadening \( 2 \Delta \gamma_{q}[r, \bar{n}_{a}] \). Linear fits (black lines) of the AC-Stark shift (Eq. (F6b)) and the induced dephasing (Eq. (F6b)) versus calibrated injected photon number yield the dispersive interaction amplitude \( \chi \) for each BO squeezing. (b) Left: dispersive interaction strength (y-axis) versus BO squeezing (x-axis) for \( \delta_{a} > 0 \). The full data record (lightgrey squares, 72 points) is coarse-grained (black squares), and compared with the analytical result with no fit parameters (black line, Eq. (F4c)). Right: same for \( \delta_{a} < 0 \).
Similarly the linewidth broadening of the transmon qubit coupled to a driven BO reads $\Delta \omega_{q} = \Delta \omega_{q}[r] + \Delta \omega_{q}[\bar{r}, \bar{n}_{d}]$ where:

$$\Delta \omega_{q}[r] = \delta_{r}^{(2)}[r] + \chi_{t}[r] \sin^{2} r,$$  \hspace{1cm} (F5a)

$$\Delta \omega_{q}[\bar{r}, \bar{n}_{d}] = \chi_{t}[r] \bar{n}_{d} \cosh^{2} r.$$ \hspace{1cm} (F5b)

Finally, the frequency shift of a transmon dispersively coupled to a driven BO reads $\Delta \omega_{q} = \Delta \omega_{q}[r] + \Delta \omega_{q}[\bar{r}, \bar{n}_{d}]$ where:

$$\Delta \gamma_{\phi} = \frac{\chi^{2}[r]}{\kappa} \sinh^{2} r \left(1 + \sinh^{2} r\right),$$ \hspace{1cm} (F6a)

$$\Delta \gamma_{\phi}[\bar{r}, \bar{n}_{d}] = \frac{2\chi^{2}[r]}{\kappa} \left(1 + 2 \sinh^{2} r\right) \bar{n}_{d} \cosh^{2} r.$$ \hspace{1cm} (F6b)

These are the transmon version of Eqs. (B26) and (B27), used to fit the data in Figs. 4 and 5.

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