Nash Equilibrium Seeking for General Linear Systems With Disturbance Rejection

Xin Cai, Feng Xiao, Member, IEEE, Bo Wei, Mei Yu, and Fang Fang, Senior Member, IEEE

Abstract—This article explores aggregative games in a network of general linear systems subject to external disturbances. To deal with external disturbances, distributed strategy-updating rules based on the internal model are proposed for the case with perfect and imperfect information, respectively. Different from the existing algorithms based on gradient dynamics, by introducing the integral of the gradient of cost functions on the basis of the passivity theory, the rules are proposed to force the strategies of all agents to evolve to the Nash equilibrium, regardless of the effect of disturbances. The convergence of the two strategy-updating rules is analyzed via the Lyapunov stability theory, passivity theory, and singular perturbation theory. Simulations are performed to illustrate the effectiveness of the proposed methods.

Index Terms—Aggregative games, external disturbances, linear systems, Nash equilibrium (NE) seeking.

I. INTRODUCTION

In recent years, the game theory has been widely applied in engineering community, such as power grids [1], [2], [3], [4]; mobile sensor networks [5], [6]; and communication networks [7], [8]. A critical problem in the game theory is to seek Nash equilibrium (NE), which is the solution of the game theory corresponding to the desired target of multiagent systems.

The purpose of distributed NE seeking is to design a strategy-updating rule for agents to reach the NE of the game. According to the available information for agents, noncooperative games can be classified into the case with full/perfect information and the case with partial/imperfect information.

Manuscript received 15 April 2022; revised 25 June 2022; accepted 22 July 2022. Date of publication 19 August 2022; date of current version 20 July 2023. This work was supported in part by the National Natural Science Foundation of China (NSFC) under Grant 61873074 and Grant 61903140; in part by Beijing Natural Science Foundation under Grant 4222053; and in part by the Fundamental Research Funds for the Central Universities, China, under Grant 2020MS019. This article was recommended by Associate Editor P. Shi. (Corresponding author: Feng Xiao.)

Xin Cai is with the State Key Laboratory of Alternate Electrical Power System with Renewable Energy Sources and the School of Control and Computer Engineering, North China Electric Power University, Beijing 102206, China, and also with the School of Electrical Engineering, Xunjiang University, Urumqi 830047, China (e-mail: caixin_xd@126.com).

Feng Xiao is with the State Key Laboratory of Alternate Electrical Power System with Renewable Energy Sources and the School of Control and Computer Engineering, North China Electric Power University, Beijing 102206, China (e-mail: fengxiao@ncepu.edu.cn).

Bo Wei, Mei Yu, and Fang Fang are with the School of Control and Computer Engineering, North China Electric Power University, Beijing 102206, China (e-mail: bowei@ncepu.edu.cn; meiyu@ncepu.edu.cn; ffang@ncepu.edu.cn).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TCYB.2022.3195361.

Digital Object Identifier 10.1109/TCYB.2022.3195361

In the full/perfect information case, some research was conducted under the assumption that every agent can observe or receive the strategies of others [9], [10], [11], [12]. However, it is impractical for agents in large-scale networks to know actions of other agents due to agents’ limited observation and communication capabilities. Thus, there is an increasing interest in dealing with the issue of games with incomplete (imperfect) information. To overcome the difficulty that only partial information is available for agents, various consensus-based distributed discrete- and continuous-time NE seeking algorithms have been designed for n-person noncooperative games, aggregative games, two-network zero-sum games, and n-coalition noncooperative games [13], [14], [15], [16], [17], [18], [19], [20]. Here, we mainly focus on the continuous-time NE seeking algorithms for aggregative games.

Aggregative games, as a special class of noncooperative games with cost functions depending on each agent’s own strategy and an aggregation of all agents’ strategies, have attracted great attention in various fields, such as economic markets, communication networks, mechanical systems, and smart grids (see [19], [21], [22], [23], [24] and the references therein). A variety of distributed algorithms have been presented to seek NE of aggregative games for multiagent systems on fixed undirected graphs, time-varying graphs, directed graphs, or switching graphs [23], [25], [26], [27]. To reduce communication loads occurring in distributed continuous-time NE seeking algorithms, event-triggered and self-triggered communication schemes were presented in [24], [28], [29], [30], [31], and [32]. For aggregative games with nonsmooth cost functions, distributed subgradient algorithms were proposed in [33], [34], [35], and [36].

Note that a common assumption in the aforementioned literature is that agents’ strategies are steered by first-order systems. In view of the fact that physical systems generally have inherent complex dynamics in engineering scenarios, it is of practical interest to study aggregative games for multiagent systems with complex dynamics. Recent studies show distributed NE seeking algorithms designed for aggregative games with Euler–Lagrange systems [21]; double-integrator systems [29], [37], [38]; multi-integrator agents [8]; and nonlinear dynamic systems [39].

In [8], [21], [29], [37], [38], and [39], the presented results on the NE seeking for aggregative games with integrator-type agents primarily were based on gradient-based algorithms. However, approaches based on gradient dynamics cannot be directly used in general linear systems, because general linear systems with various linear relationships among states may...
not be reduced to integrator-type systems by the feedback linearization. Moreover, it is impractical to use the internal states of systems as the strategies in games. For complex physical systems, the outputs are more easily detected than the states of systems. To the best of our knowledge, how to regulate the outputs of general linear systems to reach the NE of aggregative games is still a problem to be solved.

Motivated by the competition among distributed energy resources in the electricity market [21], [22], in which the output power of each generation system is driven by the turbine-generator dynamics, the energy management problem can be formulated by an aggregative game with general linear systems. In addition, arising from the environment or communication, disturbances and noises are ubiquitous in practical dynamical systems. Therefore, we consider an aggregative game with disturbed general linear systems in this article. First, in the case of the perfect information, a strategy-updating rule, combined with state feedback, gradient information, and the estimation of disturbances, is developed for agents with inherent dynamics modeled by general linear systems. Then, by introducing a dynamic average consensus estimator, a distributed strategy-updating rule is presented for the game with imperfect information. Different from bounded disturbances considered in [40], a class of deterministic disturbances generated by exosystems is considered in this article. Based on the internal model, an observer is designed to estimate the disturbances. Compared with the existing distributed seeking algorithms presented for aggregative games with integrator-type agents, our proposed method can steer strategies of algorithms presented for aggregative games with integrator-disturbances. Compared with the existing distributed seeking algorithms (see [8], [21], [37], [39], [45]) are not applicable to the studied linear systems. To overcome this difficulty, the integrals of cost functions’ gradients over time are considered in this article. Based on the internal model, an observer is designed to estimate the disturbances. Compared with the existing distributed seeking algorithms presented for aggregative games with integrator-type agents, our proposed method can steer strategies of agents with more complex dynamics to the NE. The main contributions of this article are given as follows.

1) We study an aggregative game of a multiagent system with general linear dynamics. Aggregative games are a kind of static noncooperative games, rather than dynamic noncooperative games studied in [41], [42], [43], and [44]. Different from the setups in which the internal states of agents are assumed to be strategies in games [8], [21], [37], [39], [45], the outputs of general linear systems are considered as the strategies of agents in this article.

2) A novel strategy-updating rule is proposed for general linear systems with disturbances. Since the system structure of general linear systems is more complex than that of integrator-type dynamical systems, the existing gradient-based NE seeking algorithms (see [8], [21], [37], [39], [45] and the references therein) are not applicable to the studied linear systems. To overcome this difficulty, the integrals of cost functions’ gradients over time are introduced in the strategy-updating rule to force agents’ outputs to arrive at the NE of the aggregative game.

3) Different from the Lyapunov stability theory used to analyze the convergence of NE seeking algorithms in the existing literature, the passivity theory plays a critical role in the design and analysis of the proposed strategy-updating rules.

The organization of this paper is given as follows. In Section II, the considered problem is formulated. In Section III, two strategy-updating rules are designed and the convergence is analyzed. In Section IV, simulation examples are presented. The conclusions and future topics are stated in Section V.

**Notations:** $R$ and $R^n$ denote the real numbers set and the $n$-dimensional Euclidean space, respectively. Given a vector $x \in R^n$, $\|x\|$ is the Euclidean norm. $\otimes$ denotes the Kronecker product. $A^T$ and $\|A\|$ are the transpose and the spectral norm of matrix $A$, respectively. Denote $\text{col}(x_1, \ldots, x_d) = [x_1^T, \ldots, x_d^T]^T$. Given matrices $A_1, \ldots, A_n, \text{blk}(A_1, \ldots, A_n)$ denotes the block-diagonal matrix with $A_i$ on the diagonal. $I_n$ is the $n \times n$ identity matrix. $0$ denotes a zero matrix with appropriate dimensions. $1_n$ and $0_n$ are $n$-dimensional column vectors consisting of all 1s and 0s, respectively.

### II. PROBLEM FORMULATION

Consider an aggregative game $G = (\mathcal{I}, \Omega, J)$ with $N$ agents indexed in $\mathcal{I} = \{1, \ldots, N\}$. $\Omega = \Omega_1 \times \cdots \times \Omega_N \subset R^N$, where $\Omega_i \subset R$ is the strategy set of agent $i \in \mathcal{I}$, $J = (J_1, \ldots, J_N)$, where $J_i(y_i, \sigma(y_i)) : \Omega_i \times R \rightarrow R$ is agent $i$’s cost function depending on agent $i$’s strategy $y_i \in \Omega_i$ and an aggregate function $\sigma(y) : \Omega \rightarrow R$. Define $\sigma(y) = \sum_{i=1}^N \psi_i(y_i)$ with $\psi_i(y_i) : \Omega_i \rightarrow R i \in \mathcal{I}$, $y = [y_1, \ldots, y_N]^T$ denotes the strategy profile of game $G$, $y_i = [y_{i-1}, y_{i+1}, \ldots, y_N]^T$ is the strategy vector of agents except agent $i$. In game $G$, the aim of each agent is to control its strategy $y_i$ to minimize its cost function $J_i$ for given $y_{i-}$, that is, $\min_{y_{i-} \in \Omega_i} J_i(y_i, \sigma(y_i)), i \in \mathcal{I}$.

Here, assume that the outputs of agents are their strategies in game $G$ and are driven by their inherent dynamics, modeled by the following single-input–single-output (SISO) linear system:

$$
\begin{align}
\dot{y}_i &= A_i y_i + B_i (u_i + d_i) \\
y_i &= C_i y_i
\end{align}
$$

In (1), $x_i \in R^n$, $u_i \in R$, and $y_i \in R$ are agent $i$’s (in $\mathcal{I}$) state, control input, and output (strategy), respectively. $A_i$, $B_i$, and $C_i$ are constant matrices with appropriate dimensions. Note that we focus on the case of 1-D output for the ease of exposition. However, the forthcoming results also hold for higher dimensions. Assume that $(A_i, B_i)$ is stabilizable and $(A_i, C_i)$ is observable. $d_i \in R$ is the local disturbance generated by the exosystem

$$
\begin{align}
\dot{v}_i &= S_i v_i \\
d_i &= U_i v_i
\end{align}
$$

where $v_i \in R^d$ is the internal state of exosystem (2). The matrix pair $(S_i, U_i)$ is observable. Suppose that all the eigenvalues of $S_i$, $i \in \mathcal{I}$ are distinctively lying on the imaginary axis. It means that disturbance $d_i$ is bounded. Here, we deal with the effects of a class of deterministic disturbances on system (1). Therefore, it is assumed that agent $i$ knows $S_i$ and $U_i$, but is ignorant of internal state $v_i$, for all $i \in \mathcal{I}$.

**Definition 1 (NE, [46, Definition 3.7]):** For aggregative game $G = (\mathcal{I}, \Omega, J)$, $y^*$ is a pure strategy NE if for each agent $i \in \mathcal{I}$

$$
y_i^* = \arg \min_{y_i \in \Omega_i} J_i(y_i, \sigma(y_1^*, \ldots, y_{i-1}^*, y_{i+1}^*, \ldots, y_N^*))
$$

At the NE point, no agent will unilaterally change its strategy for the less cost. The following two assumptions are widely used in [21], [22], [24], and [47].
Assumption 1: For all \( i \in \mathcal{I} \), the cost function \( J_i(y_i, \sigma(y)) \) is continuously differentiable in its arguments and convex in \( y_i \) for every fixed \( y_{-i} \).

For agent \( i \), \( \nabla_y J_i(y_i, \sigma(y)) \) is the gradient of cost function \( J_i \) with respect to strategy \( y_i \). Denote \( \phi(y) = [y_1, J_1(y_1, \sigma(y)), \ldots, y_N J_N(y)]^T \). By Corollary 4.2 in [46] and [47], under Assumption 1, the aggregative game has a unique NE \( y^* \in \Omega \) satisfying

\[
\phi(y^*) = 0_N.
\] (3)

Assumption 2: \( \phi: \Omega \to R^N \) is strongly monotonic, that is, \( (y - y')^T(\phi(y) - \phi(y')) \geq \mu \|y - y'\|^2 \), for all \( y, y' \in \Omega \) and \( \mu > 0 \).

Under Assumption 2, the aggregative game has a unique (pure) NE [48, Th. 3].

Remark 1: The Pareto optimization can minimize a weighted sum of cost functions of all agents. In other words, the Pareto optimum is a solution of \( \min_{y \in \Omega} \sum_{i=1}^N \alpha_i J_i(y) \) with \( \alpha_i > 0 \) and \( \sum_{i=1}^N \alpha_i = 1 \). Compared with the definition of NE, the Pareto optimum characterizes a global performance. The method proposed in this article is employed to steer agents’ strategies to the unique NE of the aggregative game, which is not Pareto optimal. The analysis of the Pareto optimality is shown in the simulation.

Problem: The objective of this article is to design strategy-updating rules for every agent with dynamics (1) to achieve the NE of aggregative game \( G \).

III. MAIN RESULTS

In this section, for the perfect information games, a strategy-updating rule is first developed for the agents with dynamics (1). Then, we consider the case with imperfect (incomplete) information and propose an improved strategy-updating rule. Moreover, the convergence of two strategy-updating rules is analyzed.

A. Strategy-Updating Rule With Perfect Information

In the game with perfect information, each agent can obtain the strategies of other agents, which can be considered as a special case of the complete communication graph. For agents disturbed by external disturbances, combined with the internal model, gradient, and its integral, a strategy-updating rule is proposed for agents to update strategies. It is shown in Fig. 1.

Let \( e_i = -\nabla_y J_i(y_i, \sigma(y)) \) be the gradient of the agent \( i \)'s cost function. \( \gamma_i \) denote an auxiliary variable with \( \dot{\gamma}_i = e_i \), and \( \hat{d}_i \) be the observation of disturbances. The strategy-updating rule for agent \( i \in \mathcal{I} \) is designed as follows:

\[
\begin{align*}
\dot{x}_i &= (A_i - B_i K_i) x_i + B_i \left( k_P e_i + k_h \gamma_i - \hat{d}_i + d_i \right) \\
y_i &= C_i x_i
\end{align*}
\] (4)

where \( K_i \in R^{l \times n} \) such that \( (A_i - B_i K_i) \) is Hurwitz, and \( k_P \) and \( k_h \) are positive constants.

To cope with external disturbances, the observer of disturbances is designed as follows based on the idea of the internal model:

\[
\begin{align*}
\dot{\hat{d}}_i &= U_i(z_i + k_o \hat{B}^T_i x_i) \\
\dot{z}_i &= S_i z_i - k_o \hat{B}^T_i B_i (k_P e_i + k_h \gamma_i) \\
&+ (S_i k_o \hat{B}^T_i - k_o \hat{B}^T_i A_i + k_o \hat{B}^T_i B_i K_i) x_i
\end{align*}
\] (5)

where \( z_i \) is the internal state and \( k_o, \hat{B}_i \in R^{d} \) is the observer gain matrix and satisfies that \( (S_i - k_o \hat{B}^T_i B_i U_i) \) is Hurwitz.

In order to analyze the properties of systems (4) and (5), let \( x_i = \left[ \begin{array}{c} x_i \\ y_i \end{array} \right] \) be an augmented state of the system, \( \rho_i = y_i - (z_i + k_o \hat{B}^T_i x_i) \) be the observation error, and \( H_i = A_i - B_i K_i \). The closed-loop systems (4) and (5) can be rewritten by

\[
\begin{align*}
\dot{\chi}_i &= A_i \chi_i + B_i e_i + D_i \rho_i \\
y_i &= C_i \chi_i \\
\dot{\rho}_i &= (S_i - k_o \hat{B}^T_i B_i U_i) \rho_i
\end{align*}
\] (6)

where

\[
A_i = \begin{bmatrix} H_i & B_i k_h \\ 0 & 1 \end{bmatrix}, B_i = \begin{bmatrix} B_i k_P \\ 1 \end{bmatrix}, C_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, D_i = \begin{bmatrix} B_i U_i \\ 0 \end{bmatrix}
\] for \( e_i \in R \) is the control input, and \( \rho_i \in R^d \) is the external disturbances. We will analyze the convergence of system (6) to show the effectiveness of the proposed method in the sequel.

According to the definition of NE (see Definition 1), we know that the NE describes a global property of game \( G \). To analyze the property and convergence of all agents’ strategies, it is necessary to construct a dynamical system composed of \( N \) systems described by (6). Denote \( \chi = \text{col}(\chi_1, \ldots, \chi_N), e = [e_1, \ldots, e_N]^T, \rho = \text{col}(\rho_1, \ldots, \rho_N), \) and \( y = [y_1, \ldots, y_N]^T \).

Then, the overall dynamical system can be expressed by

\[
\dot{\chi} = A \chi + B e + D \rho \\
y = C \chi \\
\dot{\rho} = (S - K \hat{B}^T B U) \rho
\] (7)

where \( A = \text{blk}[A_1, \ldots, A_N], B = \text{blk}[B_1, \ldots, B_N], C = \text{blk}[C_1, \ldots, C_N], D = \text{blk}[D_1, \ldots, D_N], S = \text{blk}[S_1, \ldots, S_N], B = \text{blk}[B_1, \ldots, B_N], U = \text{blk}[U_1, \ldots, U_N], \) and \( K_o = \text{blk}[k_o, \ldots, k_o] \). In (7), \( \chi, e, y, \) and \( \rho \) can be regarded as the state, the control input, the output, and the external disturbance, respectively. From the perspective of the control theory, the relationship between the equilibrium point of (7) and the NE of aggregative game \( G = (\mathcal{I}, \Omega, \mathcal{J}) \) will be analyzed in the forthcoming Lemma 1. When \( e = 0 \), system (7) can be regarded as a cascade system. The input-to-state stability of the cascade system will be analyzed to show the convergence of all agents’ strategies (see Theorem 1).
Lemma 1: Under Assumption 1, if \((\chi^*, 0_{Nq})\) is an equilibrium point of (7), \(y^*\) is an NE of aggregative game \(G = (\Omega, \Sigma, J)\). Conversely, if \(y^*\) is an NE of the aggregative game, there exists \(\chi^* \in R^{n(n+1)}\) such that \((\chi^*, 0_{Nq})\) is an equilibrium point of (7).

Proof (Sufficiency): If \((\chi^*, \rho^*)\) is the equilibrium point of system (7), it satisfies
\[
0_{N(n+1)} = A\chi^* + Be^* + DP\rho^*, \tag{8a}
\]
\[
0_{Nq} = (S - K_rB^TBU)\rho^*. \tag{8b}
\]
Denote \(x = \text{col}(x_1, \ldots, x_N)\) and \(y = [\gamma_1, \ldots, \gamma_N]^T\). By a simple derivation, (8a) yields that
\[
0_{Nn} = Hx^* + B(K_pe^* + K_r\gamma^*) + BU\rho^*, \tag{9a}
\]
\[
0_N = e^* = -\phi(y^*) \tag{9b}
\]
where \(H = \text{blk}(H_1, \ldots, H_N), K_p = \text{diag}(k_p, \ldots, k_{p_N})\), and \(K_r = \text{diag}(k_r, \ldots, k_{r_N})\). Since \((S - K_rB^TBU)\) is Hurwitz, it is derived from (8b) that \(\rho^* = 0_{Nq}\). Thus, it is derived from (9a) that \(Hx^* + BK_r\gamma^* = 0_{Nq}\). From system (7), it follows that \(y^* = C\chi^*\). According to (9b), \(\phi(y^*) = 0_{Nq}\), which satisfies condition (3). So, we have the conclusion that \(y^*\) is the NE of game \(G = (\Omega, \Sigma, J)\).

Necessary: If \(y^*\) is an NE of aggregative game \(G = (\Omega, \Sigma, J)\), we have \(\nabla_y f_i(y^*, \sigma(y^*)) = 0\) \(\forall i \in \Omega\), and \(\phi(y^*) = 0_{Nq}\) such that (9b) is satisfied. Then, there exists \(x_i^* \in R^n\) such that \(\gamma_i^* = C_i x_i^*\) for all \(i \in \Omega\). Since \((S - K_rB^TBU)\) is Hurwitz, \(\rho^* = 0_{Nq}\) and (8b) is satisfied. With \(\gamma_i^*\), \(C_i x_i^*\) acts as a state, input, and output, respectively. All poles of transfer function matrix \(G(s) = C(sI - A)^{-1}B\) have nonpositive real parts, which can be easily derived from (6). Thus, by Lemma 3 in the Appendix, system (11) is passive. There exists a storage function \(V_1 = (1/2)\tilde{\chi}^T\tilde{P}\tilde{\chi}\), where \(\tilde{P} = \text{blk}(P_1, \ldots, P_N)\) with \(P_i = P_i^T > 0, \forall i \in \Omega\). The derivation of \(V_1\) along the solutions of (11) is
\[
\dot{V}_1 = \frac{1}{2}\tilde{\chi}^T(PA + A^T\rho)\tilde{\chi} + \tilde{\chi}^T\tilde{P}\tilde{\chi}\tilde{e}
\]
\[
= -\frac{1}{2}\tilde{\chi}^TQ\tilde{\chi} + \tilde{\chi}^TC\tilde{e}
\]
\[
\leq -\tilde{\chi}^T\tilde{y}
\]
\[
= (e^* - e^*)^T\tilde{y}
\]
\[
= -\phi(y^*) = (y^* - y^*)^T(y^* - y^*)
\]
\[
\leq -\mu\\|\tilde{y}\|^2 \leq -\mu\|\tilde{\chi}\|^2
\]
where \(\mu\) is the least positive eigenvalue of \(C^T\tilde{C}\), \(Q = \text{blk}(Q_1, \ldots, Q_N)\) satisfying \(PA + A^T\rho = -Q^T\tilde{C}\) and \(PB = C^T\tilde{C}\) in Lemma 3 in the Appendix. The second inequality derives from the strong monotonicity of function \(\phi\) in Assumption 2. It indicates that system (11) can converge globally exponentially to the origin.

Next, we consider cascade system (10). Then, the derivative of \(V_1\) along the solutions of (10) is
\[
\dot{V}_1 = \tilde{\chi}^T(PA + A^T\rho)\tilde{\chi} + \tilde{\chi}^T\tilde{P}\tilde{\chi}\tilde{e}
\]
\[
\leq e^T\tilde{y} + \tilde{\chi}^T(PD\rho)
\]
\[
\leq -\mu\|\tilde{\chi}\|^2 + \|\tilde{\chi}\|\|PD\|\|\rho\|
\]
\[
\leq -\mu(1-b)c\|\tilde{\chi}\|^2 + \|\tilde{\chi}\|\|PD\|\|\rho\|
\]
where \(0 < b < 1\). Then, for \(\|\tilde{\chi}\| \geq (\|PD\|/\|\mu_b\|)\|\rho\|\), \(\dot{V}_1 \leq -\mu(1-b)c\|\tilde{\chi}\|^2\) by [49, Th. 4.19], the \(\tilde{\chi}\)-subsystem in (10) is input-to-state stable. Since \((S - K_rB^TBU)\) is Hurwitz, \(\rho\) globally asymptotically converges to the origin. Then, the origin of cascade system (10) is globally asymptotically stable by [49, Lemma 4.7]. Therefore, \(\chi\) and \(y\) of system (7) globally asymptotically converge to \(\chi^*\) and \(y^*\), respectively, that is, all agents’ strategies, updated by strategy-updating rule (4), asymptotically reach the NE of game \(G\).

B. Strategy-Updating Rule With Imperfect Information

In this section, we consider that each agent cannot obtain all agents’ actions in a large-scale network. So, they need to estimate aggregator \(\sigma\) through communication with their neighbors over an undirected and connected graph \(\tilde{G}\). The Laplacian matrix of graph \(\tilde{G}\) is defined by \(L\). For more details
about undirected graphs, please refer to [50]. Denote agent $i$’s estimation of aggregator $\sigma$ by $\eta_i \in R, \forall i \in I$. To prepare for our development, define the map $F_i : \Omega_i \times R \rightarrow R$ as

$$F_i(y_i, \eta_i) = \left( \nabla_y J_i(y_i, \sigma) + \sigma \nabla_y J_i(y_i, \sigma) \right)|_{\sigma = \eta_i}, \forall i \in I. \quad (12)$$

Furthermore, denote $F(y, \eta) = [F_1(y_1, \eta_1), \ldots, F_N(y_N, \eta_N)]^T$ with $\eta = [\eta_1, \ldots, \eta_N]^T$. When estimation $\eta_i = \sigma, \forall i \in I$, it follows from (12) that $F(y, \sigma) = \phi(y)$.

Assumption 3: $F(y, \eta)$ is Lipschitz continuous in $\eta \in R^N$ for every $y \in \Omega$, that is, there exists a positive constant $\theta$ such that $||F(y, \eta) - F(y, \eta')|| \leq \theta ||\eta - \eta'||, \forall \eta, \eta' \in R^N$.

Based on the strategy-updating rule (4) and observer (5), we employ a dynamic consensus protocol such that $\eta_i = \eta_j$ for all $i, j \in I$, before all agents reach the NE. Let $e_i^j = -F_i(y_i, \eta_i)$ denote the gradient of agent $i$’s cost function coupled with aggregator estimation $\eta_i$, which is different from the case of perfect information. Denote $\dot{y}_i^j = e_i^j$. The strategy-updating rule for agent $i \in I$ is designed by

$$\dot{x}_i = H_k \eta_i + B_k e_i^j + k_k y^i_j + B_k U_k \rho_i$$
$$y_i = C_k \eta_i$$

where disturbance observation error $\rho_i$ is given by

$$\dot{\rho}_i = (S_k - K_o B_k^T B_k) \rho_i.$$  

According to the dynamic consensus protocol proposed in [51], the estimation of aggregator $\sigma$ is expressed by

$$\delta \dot{\eta}_i = \eta_i - \sum_{j \in N_i} (\eta_i - \eta_j) - \sum_{j \in N_j} (\omega_i - \omega_j) + N\phi_i(\eta_i)$$

$$\delta \dot{\omega}_i = \sum_{j \in N_i} (\eta_i - \eta_j)$$

where $\omega_i$ is an auxiliary variable and $\delta$ is a small positive constant. Because the aggregate function $\sigma$ is time varying. In order to track the changing aggregator, $\delta$ is needed to make protocol (15) be a fast subsystem tracking the aggregator in the slow subsystem.

Let $e^j = [e_i^j, \ldots, e_N^j]^T, \eta = [\eta_1, \ldots, \eta_N]^T$, and $\omega = [\omega_1, \ldots, \omega_N]^T$. Based on the analysis in the previous section, (13)–(15) can be described by

$$\dot{\chi} = A_k \chi + B_k e^j + D \rho$$
$$\dot{y} = C_k \chi$$
$$\dot{\rho} = (S - K_o B_k^T B_k) \rho$$

$$\begin{bmatrix} \dot{\eta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -I - L \\ L \end{bmatrix} \begin{bmatrix} \eta \\ \omega \end{bmatrix} + \begin{bmatrix} N \end{bmatrix}.$$

In close-loop system (16), $(\eta, \omega)$ is a fast subsystem, and $(\chi, \rho)$ is a slow subsystem. For a connected and undirected graph $G$ and by using [52, Th. 5], we find that $\eta(t)$ first converges exponentially to equilibrium point $\sum_{i=1}^{N} \phi_i(y_i) I_N$ with slowly changing $y$. Then, $(\chi, \rho)$ asymptotically converges to its equilibrium point $(\chi^*, 0_N)$.

Theorem 2: Suppose Assumptions 1 and 3 and the conditions in Theorem 1 hold. There exists a positive constant $\delta^*$ such that for every $0 < \delta < \delta^*$, $\eta(t)$ exponentially converges to $\sum_{i=1}^{N} \phi_i(y_i) I_N$. Agents with dynamics (1) follow strategy-updating rule (13). Then, the strategies of all agents globally

asymptotically converge to the unique NE $y^*$ of aggregative game $G = (I, \Omega, \cdot)$ with imperfect information.

Proof: Denote $\tilde{\chi} = \chi - \chi^*, \tilde{y} = y - y^*, \tilde{\epsilon} = e^j - e^j$, $\tilde{\eta} = \eta - \eta^*$, and $\tilde{\omega} = \omega - \omega^*$. Then, system (16) can be rewritten by

$$\dot{\tilde{\chi}} = A \tilde{\chi} + B \tilde{\epsilon} + D \rho$$
$$\dot{\tilde{y}} = C \tilde{\chi}$$
$$\dot{\tilde{\rho}} = (S - K_o B^T B) \tilde{\rho}$$

$$\begin{bmatrix} \dot{\tilde{\eta}} \\ \dot{\tilde{\omega}} \end{bmatrix} = \begin{bmatrix} -I - L \\ L \end{bmatrix} \begin{bmatrix} \tilde{\eta} \\ \tilde{\omega} \end{bmatrix}$$

where $\tilde{\eta}$ and $\tilde{\omega}$ are the quasi-steady states of $\eta$ and $\omega$ for fixed $y$, respectively. According to the stability analysis of the singular perturbation system, the proof of Theorem 2 has following three steps.

1) Quasi-steady-State Analysis: Set $\delta = 0$ such that $\eta$ and $\omega$ are at the quasi-steady states, that is, $\eta = \eta^* = \sum_{i=1}^{N} \phi_i(y_i)$ and $\omega = \omega^*$. In Theorem 1, (17) can be reduced to (10). According to the analysis in Theorem 1, the reduced system is globally asymptotically stable at $(\chi^*, 0_N)$.

2) Boundary-Layer Analysis: Let $\tau = t/\delta$, and $\delta = 0$ to freeze $y$. The boundary-layer system in (17) is rewritten in $\tau$-time scale by

$$\begin{bmatrix} \frac{d\tilde{\eta}}{d\tau} \\ \frac{d\tilde{\omega}}{d\tau} \end{bmatrix} = \begin{bmatrix} -I - L \\ L \end{bmatrix} \begin{bmatrix} \tilde{\eta} \\ \tilde{\omega} \end{bmatrix}.$$

The following analysis method is similar to [51] and [53]. Let $r$ be an $N$-dimensional column vector such that $r^T L = 0$. $R = [R, r]$ denotes an $N \times N$ orthonormal matrix. Then, let $\hat{\omega} = \hat{\omega}^0 + \hat{\omega}^e$, which is decomposed into an $(N - 1)$-dimensional column vector $\hat{\omega}^e$ and a scalar $\hat{\omega}^0$. Because $\hat{\omega}^0$ does not interact with other states in the boundary-layer system, (18) can be written by

$$\begin{bmatrix} \frac{d\tilde{\eta}}{d\tau} \\ \frac{d\tilde{\omega}}{d\tau} \end{bmatrix} = \begin{bmatrix} -I - L \\ L \hat{R} \end{bmatrix} \begin{bmatrix} \tilde{\eta} \\ \tilde{\omega} \end{bmatrix}.$$

For a connected and undirected graph $G$ and from [53, Lemma 2.2], $[I - L - L \hat{R}]$ is Hurwitz. $(\tilde{\eta}, \tilde{\omega})$ can exponentially converge to the origin, uniformly in $(\tau, t)$. Based on the converse Lyapunov theorem in [49, Th. 4.14], there exists a Lyapunov function $V_2$ for the boundary-layer system (18). It satisfies $c_1 ||\tilde{\eta}||^2 + ||\tilde{\omega}||^2 \leq V_2 \leq c_2 (||\tilde{\eta}||^2 + ||\tilde{\omega}||^2)$ and $\dot{V}_2 \leq -c_3 (||\tilde{\eta}||^2 + ||\tilde{\omega}||^2)$ for some positive constants $c_1, c_2$, and $c_3$.

3) Comprehensive Analysis: Since $y$ is not a true constant parameter, we have to keep track of the effect of the interaction between the slow and fast dynamics. Now, consider a composite Lyapunov function candidate $V = (1 - \varepsilon)V_1 + \varepsilon V_2$. $0 < \varepsilon < 1$. The derivative of $V$ along the trajectories of system (17) is

$$\dot{V} = (1 - \varepsilon) \dot{V}_1 + \varepsilon \left( \frac{1 - \varepsilon}{\delta} - 1 \right) \dot{V}_2 + (1 - \varepsilon) \frac{\partial V_1}{\partial \tilde{\chi}} (\tilde{\chi}^*_{\tilde{\eta}} - \tilde{\chi}^*_{\tilde{\eta}})$$

$$\leq -(1 - \mu) c_1 (\tilde{\eta}^2 + \tilde{\omega}^2) \leq -c_1 (\tilde{\eta}^2 + \tilde{\omega}^2)$$

$$+ (1 - \varepsilon) \tilde{V}_t^T P \tilde{B} (\tilde{e}^j_{\tilde{\eta}} - \tilde{e}^j_{\tilde{\eta}})$$
\[ = -(1 - \varepsilon)\mu(1 - b)c\|\dot{\bar{x}}\|^2 - \varepsilon \left( \frac{1}{\delta} - 1 \right) c_3 (\|\tilde{\bar{y}}\|^2 + \|\tilde{o}\|^2)
+ (1 - \varepsilon)\tilde{\bar{x}}^T PB (-F(\tilde{\bar{y}} + y*, \tilde{\bar{y}} + \tilde{o} + F(\tilde{\bar{y}} + y*, \tilde{\bar{y}}))
\leq -(1 - \varepsilon)\mu(1 - b)c\|\dot{\bar{x}}\|^2 - \varepsilon \left( \frac{1}{\delta} - 1 \right) c_3 (\|\tilde{\bar{y}}\|^2 + \|\tilde{o}\|^2)
+ (1 - \varepsilon)\|\tilde{\bar{x}}\| \|PB\| \|F(\tilde{\bar{y}} + y*, \tilde{\bar{y}} + \tilde{o}) - F(\tilde{\bar{y}} + y*, \tilde{\bar{y}}))
\leq -(1 - \varepsilon)\mu(1 - b)c\|\dot{\bar{x}}\|^2 - \varepsilon \left( \frac{1}{\delta} - 1 \right) c_3 (\|\tilde{\bar{y}}\|^2 + \|\tilde{o}\|^2)
+ \theta (1 - \varepsilon)\|PB\| \|\tilde{\bar{x}}\| \|\tilde{o}\|
= -(1 - \varepsilon)\|\tilde{\bar{x}}\| \Lambda \|\tilde{o}\| - \varepsilon \left( \frac{1}{\delta} - 1 \right) c_3 \|\tilde{o}\|^2\]

where
\[
\Lambda = \begin{bmatrix}
\mu(1 - b)c & -\frac{1}{\varepsilon} \theta \|PB\| \\
-\frac{1}{\varepsilon} \theta \|PB\| & \frac{(1 - \delta)}{(1 - \varepsilon)\delta} c_3
\end{bmatrix}.
\]

The sufficient condition for \(\dot{V} < 0\) is that \(\Lambda\) is a positive-definite matrix, that is, there exists a positive constant \(\delta^*\), for all \(0 < \delta < \delta^*\), \((\tilde{x}, \tilde{o})\) asymptotically converges to the origin of (17) [49, Th. 11.3]. It indicates that system (16) is stable asymptotically at NE \(y^*\).

\textbf{Remark 2:} In recent work [39], aggregative games of multiple complex dynamic systems with external disturbances were studied. However, the devised algorithm was hard to be applied directly here due to the following three aspects. First, the dynamics of agents considered in [39] are nonlinear and can be simplified by integrator-type dynamics, while general linear systems are considered in this article. Second, the external disturbances considered in [39] are generated by unknown systems, while a class of deterministic disturbances is considered in this article, which can be compensated by observers. Third, the dynamic average consensus protocol employed in [39] requires that initial states be located at the origin, while the consensus protocol proposed in this article does not require any initial conditions.

\textbf{Remark 3:} By the idea of the singular perturbation method in [49, Ch. 11], the strategy-updating rule designed in this section can be formulated as a singular perturbation model. To make (15) be a fast system, parameter \(\delta\) is designed in (15). It is convenient to analyze the performance of the network of general linear systems by designing a small parameter \(\delta\). Furthermore, fast system (15) is easily realized by the embedded technology. The parameter may be obtained in a distributed way, which is a problem to be solved in the future.

IV. SIMULATIONS

In this section, we will present two examples to verify the effectiveness of the proposed strategy-updating rules for all agents in aggregative games.

\textbf{A. Network of Double-Integrator Agents}

Consider a multiagent system with five double-integrator agents described by (1), where \(A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\), \(B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), and \(C = \begin{bmatrix} 1 & 0 \end{bmatrix}\). The communication graph is depicted in Fig. 2(a).

Agent \(i\)'s cost function is given by
\[
J_i(y_i, \sigma) = (y_i - b_i)^2 - (c_0 - C\sigma)y_i
\tag{19}
\]
\[
J_i(P_i, \sigma) = c_i(P_i) - p(\sigma)P_i
\tag{20}
\]
where \([b_1, \ldots, b_9]^T = [50, 55, 60, 65, 70]^T\), \(c_0 = 5\), and \(C = 0.02\). According to the conditions stated in Theorem 1, we select \(K_i = [1, 10]^T\), \(k_{p_1} = 12\), \(k_{p_2} = 2\), and \(k_{o_1} = [12, 15, 12, 15, 12]^T\), \(\forall i \in \mathcal{I}\). Besides, set \(\delta = 0.1\) in the game with incomplete information. Disturbance \(d_i\) is a sinusoidal signal. It is clear that cost functions (19) and (20) satisfy the conditions in Assumptions 1 and 2.

It is calculated that the NE is \(y^* = [50, 54.95, 59.9, 64.85, 69.8]^T\). The sum of the five agents' costs at the NE is 296.58. Note that the Pareto optimum is \(y' = [46.82, 51.82, 56.82, 61.82, 66.82]^T\), at which the total cost is 143.13. It is obvious that the obtained NE is not the Pareto optimum. The simulation results of strategy-updating rules (4) and (13) for the double-integrator agents with perfect and imperfect information are shown in Figs. 3 and 4, respectively. It is seen that the multiagent system with dynamics (1) under the two strategy-updating rules achieves the NE asymptotically. Moreover, the strategy-updating rule (13) in the case of imperfect information has the same convergence performance as that with perfect information.

\textbf{B. Generation Systems in Electricity Markets}

In the electricity market, power plants as participants compete with each other to obtain generation index which minimize their own costs. It can be described by an aggregative game. We consider a network of six generation systems communicating with each other via an undirected and connected graph as shown in Fig. 2(b). The cost function of generation system \(i\) is described by

\[
\text{Fig. 2. Communication graphs for the two examples. (a) Example A. (b) Example B.}
\]

\[
\text{Fig. 3. Outputs of double-integrator agents updated by (4).}
\]

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Table I

| Description of Variables and Parameters Appearing in the Turbine-Generator Model |
|----------------------------------------------------------------------------------|
| **State variables**                                                               |
| $P_i$  | output power, in p.u.               |
| $X_{ei}$  | valve opening, in p.u.              |
| $w_i$  | relative speed, in rad/s            |
| **Parameters**                                                                   |
| $T_{mi}$  | time constants of machine's turbine, in s |
| $K_{mi}$  | gain of machine's turbine           |
| $T_{ei}$  | time constants of speed governor, in s |
| $K_{ei}$  | gain of speed governor              |
| $R_i$  | regulation constant of machine's turbine, in p.u. |
| $w_0$  | synchronous machine speed, in rad/s |
| $D_i$  | per unit damping constant           |
| $H_i$  | inertia constant, in s              |
| $u_i$  | control input of generator system, in p.u. |

where $P_i \in R$ is the output power of generation system $i$, in p.u., $P_i = [P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_N]^T$. $c(P_i)$ is the generation cost defined by $c(P_i) = \alpha_i + \beta_i P_i + \xi_i P_i^2$. $p(\sigma)$ is electricity price given by $p(\sigma) = (p_0 - a \sigma)$, $\alpha_i, \beta_i$, and $\xi_i$ are characteristics of generation system $i$, $p_0$ and $a$ are constants, and $\sigma = \sum_{i=1}^N P_i$ denotes the aggregator.

Without regard to the mechanical and electromagnetic loss, the classical dynamical model of the $i$th turbine-generator is governed by (refer to [54] and [55])

$$
\begin{bmatrix}
\dot{P}_i \\
\dot{X}_{ei} \\
\dot{w}_i
\end{bmatrix}
= 
\begin{bmatrix}
-1/T_{mi} & K_{mi}/T_{mi} & 0 \\
0 & -1/T_{ei} & 0 \\
0 & 0 & -1/T_{ei}
\end{bmatrix}
\begin{bmatrix}
P_i \\
X_{ei} \\
w_i
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
(u_i + d_i)
$$

where the notation for the generator system model is given in Table I. The dynamics of the turbine-generator system consist of the mechanical equation described by $w$, the turbine dynamics described by $P_i$, and the turbine valve control equation described by $X_{ei}$ [54], [55]. The persistent disturbance is modeled by $d_i = m_i \sin(2\pi ft)$, where $m_i$ is an unknown amplitude and $f = 500$ Hz. The parameters of the six generation systems are brought from [21] and [56] and given in Table II. The parameters in strategy-updating rules (4) and (13) are given by $\delta = 0.1$, $k_{P_i} = k_{I_i} = 1$, $K_i = [1, 1.2, 0.8, 1.1, 0.9, 1]$, and $k_{oi} = [4, 4, 4, 4, 4, 8; 4, 4, 4, 4, 4, 8]^T$, which satisfy the conditions in Theorems 1 and 2.

In the perfect information case, the regulation process of output powers of the six generators by strategy-updating rule (4) is shown in Fig. 5. It is easily seen that the strategies of the six generators can arrive at the NE of the aggregative game. In addition, taken output $P_1$ of the 1st generator as an example, the influence of parameters $k_{P_1}$ and $k_{I_1}$ on the convergence of strategy-updating rule (4) is shown in Figs. 6 and 7, respectively. In Fig. 6, with fixed $k_{I_1}$, the overshoot increases as $k_{P_1}$ increases. Similarly, it is seen from Fig. 7 that the overshoot increases as $k_{I_1}$ increases with fixed $k_{P_1}$. The evolution of the six generators’ strategies updated by rule (13)
reach a consensus. Figs. 8 and 9 illustrate that the two-time
that the aggregator estimations of generator systems rapidly
NE by only local interactions with neighbors. Fig. 9 shows
be seen that the strategies of the six generators can reach the
in the imperfect information setting is shown in Fig. 8. We
can see that the strategies of the six generators can reach the
Fig. 9 shows that the aggregator estimations of generator systems rapidly
Fig. 8 and 9 illustrate that the two-time scale designed in the proposed rule (13) can facilitate strategy
V. CONCLUSION
Aggregate games of multiple general linear systems sub-
ject to persistence disturbances were considered in this article. Based on the internal model, a distributed observer has

been proposed for the agents to reject disturbances. Besides, the integral of cost functions’ gradient was introduced in
strategy-updating rules to overcome the difficulty that grad-
ient dynamics cannot be directly applied to general linear
systems. The convergence of the proposed rules was ana-
yzed via the passivity theory, singular perturbation analysis,
and Lyapunov stability theory. All agents’ strategies, updated
by the two rules, can asymptotically converge to the NE,
which was verified by simulation examples. Some future
research could be done along the following two directions:
1) the practical dynamic systems, such as hybrid systems [57],
switched systems [58], could be considered in noncooper-
ative games and 2) taken the time-varying demand in the
economic dispatch of smart grids into consideration, (time-
V. CONCLUSION
Aggregate games of multiple general linear systems sub-
ject to persistence disturbances were considered in this article. Based on the internal model, a distributed observer has

APPENDIX
PASSIVE LINEAR SYSTEM
Consider a linear time-invariant system
\[
\dot{x} = Ax + Bu 
\]
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \), and \( y \in \mathbb{R} \) are the state, control input, and output, respectively. \( (A, B) \) is controllable and \( (A, C) \) is observable. The transfer function of system (21) is denoted by \( G(s) = C(sI - A)^{-1}B \). If all poles of \( G(s) \) have nonpositive real parts, \( G(s) \) is called positive real [49, Definition 6.4]. System (21) is said to be passive if there exists a continuous differentiable positive semidefinite function \( V(x) \) such that \( \dot{V} = (\partial V / \partial x)(Ax + Bu) \leq u^T y \), \( \forall (x, u) \in \mathbb{R}^n \times \mathbb{R} \).

Lemma 2 [49, Lemma 6.2]: \( G(s) \) is positive real if and only if there exist matrices \( P = P^T \geq 0 \) and \( Q \) such that \( PA + A^T P = -QQ^T \) and \( PB = C^T \).

Lemma 3 [49, Lemma 6.4]: The lineare time-invariant min-
imal realization (21) with \( G(s) \) is passive if \( G(s) \) is positive real.

REFERENCES
[1] L. Chen, N. Liu, S. Yu, and Y. Xu, “A stochastic game approach for
distributed voltage regulation among autonomous PV prosumers,” IEEE
Trans. Power Syst., vol. 37, no. 1, pp. 776–787, Jan. 2022.
[2] C. Mu, K. Wang, Z. Ni, and C. Sun, “Cooperative differential game-
based optimal control and its application to power systems,” IEEE Trans.
Ind. Informat., vol. 16, no. 8, pp. 5169–5179, Aug. 2020.
[3] F. Fang, Z. Zhu, S. Jin, and S. Hu, “Two-layer game theoretic microgrid
capacity optimization considering uncertainty of renewable energy,” IEEE
Syst. J., vol. 15, no. 3, pp. 3620–3627, Sep. 2021.
[4] Z. Wang et al., “Distributed generalized Nash equilibrium seeking for
energy sharing games in prosumers,” IEEE Trans. Power Syst., vol. 36,
no. 5, pp. 3973–3986, Sep. 2021.
[5] M. S. Stanković, K. H. Johansson, and D. M. Stipanović, “Distributed
seeking of Nash equilibria with applications to mobile sensor networks,” IEEE
Trans. Autom. Control, vol. 57, no. 4, pp. 904–919, Apr. 2012.
[6] B. Huang, Z. Meng, and F. Chen, “Distributed nonlinear place-
ment for a class of multicluster Euler-Lagrange systems,” IEEE
Trans. Syst., Man, Cybern., Syst., early access, Feb. 15, 2022.
[7] K. Fan, B. Feng, X. Zhang, and Q. Zhang, “Network selection based
on evolutionary game and deep reinforcement learning in space-air-
ground integrated network,” IEEE Trans. Netw. Sci. Eng., vol. 9, no. 3,
pp. 1802–1812, May/Jun. 2022.
[8] A. R. Romano and L. Pavel, “Dynamic NE seeking for multi-integrator networked agents with disturbance rejection,” IEEE Trans. Control Netw. Syst., vol. 7, no. 1, pp. 129–139, Mar. 2020.

[9] S. Li and T. Başar, “Distributed algorithms for the computation of noncooperative equilibria,” Automatica, vol. 23, no. 4, pp. 523–533, Jul. 1987.

[10] J. Shi, C. Liu, and G. Arslan, “Dynamic fictitious play, dynamic gradient play, and distributed convergence to Nash equilibria,” IEEE Trans. Autom. Control, vol. 60, no. 3, pp. 312–327, Mar. 2005.

[11] S. Grammatico, “Dynamic control of agents playing aggregative games with coupling constraints,” IEEE Trans. Autom. Control, vol. 62, no. 9, pp. 4537–4548, Sep. 2017.

[12] C. De Persis and S. Grammatico, “Continuous-time integral dynamics for a class of aggregative games with coupling constraints,” IEEE Trans. Autom. Control, vol. 65, no. 5, pp. 2171–2196, May 2020.

[13] R. Zhu, J. Zhang, K. You, and T. Başar, “Asynchronous networked aggregative games,” Automatica, vol. 136, Feb. 2022, Art. no. 11054.3.

[14] S. Huang, J. Lei, and Y. Hong, “A linearly convergent distributed Nash equilibrium seeking algorithm for aggregative games,” IEEE Trans. Autom. Control, early access, Feb. 24, 2022, doi: 10.1109/TAC.2022.3154536.

[15] B. Gharesifard and J. Cortés, “Distributed convergence to Nash equilibria in two-network zero-sum games,” Automatica, vol. 49, no. 6, pp. 1683–1692, Jun. 2013.

[16] C.-X. Shi and G. Hu, and S. Xu, “An extremum seeking-based approach for Nash equilibrium seeking in N-cluster noncooperative games,” Automatica, vol. 114, Apr. 2020, Art. no. 108815.

[17] M. Ye, G. Hu, L. Xie, and S. Xu, “Differentially private distributed Nash equilibrium seeking for aggregative games,” IEEE Trans. Autom. Control, vol. 67, no. 5, pp. 2451–2458, May 2022.

[18] B. Huang, C. Yang, Z. Meng, F. Chen, and W. Ren, “Distributed nonlinear placement for multi-cluster systems: A time-varying Nash equilibrium-seeking approach,” IEEE Trans. Cybern., early access, Jun. 30, 2021, doi: 10.1109/TBCY.2021.3085583.

[19] X. Cai, F. Xiao, and B. Wei, “Distributed generalized Nash equilibrium seeking for noncooperative games with unknown cost functions,” Int. J. Robust Nonlinear Control, early access, Aug. 5, 2022, doi: 10.1002/rnc.6314.

[20] X. Cai, F. Xiao, and B. Wei, “Resilient Nash equilibrium seeking in multiagent games under false data injection attacks,” IEEE Trans. Syst. Man, Cybern., Syst., early access, Jun. 14, 2022, doi: 10.1109/TSMC.2022.3180006.

[21] Z. Deng and S. Liang, “Distributed algorithms for aggregative games of multiple heterogeneous Euler–Lagrange systems,” Automatica, vol. 99, pp. 246–252, Jan. 2019.

[22] Z. Deng and X. Nian, “Distributed algorithm design for aggregative games of disturbed multiagent systems over weight-balanced digraphs,” Int. J. Robust Nonlinear Control, vol. 28, no. 17, pp. 5344–5357, Nov. 2018.

[23] Z. Deng and X. Nian, “Distributed generalized Nash equilibrium seeking algorithm design for aggregative games over weight-balanced digraphs,” IEEE Trans. Neural Netw. Learn. Syst., vol. 30, no. 3, pp. 695–706, Mar. 2019.

[24] C.-X. Shi and G.-H. Yang, “Distributed Nash equilibrium computation in aggregative games: An event-triggered algorithm,” Inf. Sci., vol. 489, pp. 289–302, Jul. 2019.

[25] S. Liang, P. Yi, and Y. Hong, “Distributed Nash equilibrium seeking for aggregative games with coupled constraints,” Automatica, vol. 85, pp. 179–185, Nov. 2017.

[26] M. Ye and G. Hu, “Distributed Nash equilibrium seeking in multiagent games under switching communication topologies,” IEEE Trans. Cybern., vol. 48, no. 11, pp. 3308–3327, Nov. 2018.

[27] Y. Zhu, W. Yu, G. Wen, and G. Chen, “Distributed Nash equilibrium seeking in an aggregative game on a directed graph,” IEEE Trans. Autom. Control, vol. 66, no. 6, pp. 2746–2753, Jun. 2021.

[28] X. Cai, F. Xiao, and B. Wei, “A distributed strategy-updating rule with event-triggered communication for noncooperative games,” in Proc. 39th Conf. Decis. Control, Dec. 2020, pp. 4747–4752.

[29] X. Cai, F. Xiao, and B. Wei, “A distributed event-triggered generalized Nash equilibrium seeking algorithm,” in Proc. 40th Conf. Decis. Control, Jul. 2021, pp. 5252–5257.

[30] A. Cortés and S. Martinez, “Self-triggered best-response dynamics for continuous games,” IEEE Trans. Autom. Control, vol. 60, no. 4, pp. 1115–1131, Apr. 2015.

[31] F. Xiao, Y. Shi, and W. Ren, “Robustness analysis of asynchronous sampled-data multiagent networks with time-varying delays,” IEEE Trans. Autom. Control, vol. 63, no. 7, pp. 2145–2152, Jul. 2018.
[57] Y. Zheng, J. Ma, and L. Wang, “Consensus of hybrid multi-agent systems,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 4, pp. 1359–1365, Apr. 2018.

[58] T. Huang and Y. Sun, “Finite-time stability of switched linear time-delay systems based on time-dependent Lyapunov functions,” *IEEE Access*, vol. 8, pp. 41551–41556, 2020.

Xin Cai received the B.S. degree in automation from North China Electric Power University, Baoding, China, in 2012, and the M.S. degree in control science and engineering from Xinjiang University, Urumqi, China, in 2015. She is currently pursuing the Ph.D. degree in control science and engineering with North China Electric Power University, Beijing. She has been a Lecturer with the School of Electrical Engineering, Xinjiang University since 2015. Her research interests are in the fields of coordination of multiagent systems, distributed algorithms, game theory, and optimization.

Feng Xiao (Member, IEEE) received the B.S. and M.S. degrees in mathematics from Inner Mongolia University, Hohhot, China, in 2001 and 2004, respectively, and the Ph.D. degree in systems and control from Peking University, Beijing, China, in 2008. He became a Faculty Member with the School of Automation, Beijing Institute of Technology, Beijing, in 2008. From June 2010 to May 2013, he worked as a Postdoctoral Fellow with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada. From January 2016 to January 2017, he was a Visiting Professor with the Department of Mechanical Engineering, University of Victoria, Victoria, BC, Canada. He was also a Professor with the Harbin Institute of Technology, Harbin, China, and is currently a Professor with the School of Control and Computer Engineering, North China Electric Power University, Beijing. His current research interests include group intelligence, coordination control, and networked systems.

Dr. Xiao was a recipient of the Izaak Walton Killam Postdoctoral Fellowship and the Dorothy J. Killam Memorial Postdoctoral Fellow Prize at the University of Alberta in 2010, and was a recipient of the Program for New Century Excellent Talents in University, China, and the Excellent Young Scientists Fund by NSFC, China.

Bo Wei received the B.S. degree in mathematics from the Hubei University for Nationalities, Enshi, China, in 2011, the M.S. degree in mathematics from China Three Gorges University, Yichang, China, in 2014, and the Ph.D. degree in control science and engineering from the Harbin Institute of Technology, Harbin, China, in 2019. He is currently an Associate Professor with the School of Control and Computer Engineering, North China Electric Power University, Beijing, China. His research interests include coordination of multiagent systems, event-triggered systems, and hybrid dynamical systems.

Mei Yu received the M.S. degree in automation from Qufu Normal University, Qufu, China, in 2002, and the Ph.D. degree from Peking University, Beijing, China, in 2005. She has been an Associate Professor with the School of Control and Computer Engineering, North China Electric Power University, Beijing, since 2009. Her research interests include new energy grid connection modeling and analysis, multiagent systems, and networked control systems.

Fang Fang (Senior Member, IEEE) received the M.S. degree in control theory and engineering from North China Electric Power University (Baoding Campus), Baoding, China, in 2001, and the Ph.D. degree in thermal power engineering from North China Electric Power University, Beijing, China, in 2005. He is currently a Professor of Control Science and Engineering and the Dean of the School of Control and Computer Engineering with North China Electric Power University. He has authored or coauthored more than 60 high-level publications, and headed more than 25 research projects or industrial projects. He is leading the Interagency Cooperation Projects of the National Key Research and Development Programs Project and the National Natural Science Foundation Project of China. His current research interests include modeling and control of power generation units, configuration and operation of integrated energy systems, and optimal dispatching of virtual power plants. Prof. Fang is a Council Member of the Chinese Association for Automation, and a Senior Member of the Chinese Society for Electrical Engineering and Chinese Association for Artificial Intelligence.