A Uniqueness Theorem on Inverse Spectral Problems for the Sturm–Liouville Differential Operators on Time Scales

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Abstract. In the paper, Sturm–Liouville differential operators on time scales consisting of a finite number of isolated points and segments are considered. Such operators unify differential and difference operators. We obtain properties of their spectral characteristics including asymptotic formulae for eigenvalues and weight numbers. Uniqueness theorem is proved for recovering the operators from the spectral characteristics.

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1. Introduction

Time scale theory unifies discrete and continuous calculus. It has important applications in natural sciences, engineering, economics and in other fields; for examples see [1–5]. Models of processes in these cases include differential equations on a time scale, i.e. closed subset of the real line. Various aspects of differential equations on time scales including boundary value problems were considered in [1–3,6–13].

In this paper, we study inverse spectral problems for the Sturm–Liouville operator on time scales. Such problems consist in recovering operators from given spectral characteristics. For the classical Sturm–Liouville operators on an interval, inverse problems have been studied fairly completely; the classical results can be found in [14–16]. However, nowadays there are only few works on inverse problem theory for differential operators on time scales because the
statement and the study of inverse spectral problems essentially depend on the structure of the time scale. In particular, in [6] an Ambarzumian type theorem was obtained for Sturm–Liouville operators on time scales.

We consider bounded time scales \( T \) consisting of \( N < \infty \) segments and \( M < \infty \) isolated points:

\[
T = \bigcup_{l=1}^{N+M} [a_l, b_l], \quad a_{l-1} \leq b_{l-1} < a_l \leq b_l, \quad l = 2, N + M, \quad a_l < b_l \text{ iff } l \in \{ l_k \}_{k=1}^N,
\]

(1)

where \( l_k \) denotes the indice corresponding to the \( k \)-th segment. The case \( N = 1, M = 0 \) corresponds to the classical Sturm–Liouville operator.

If \( T \) consists only of isolated points, i.e. \( N = 0 \), we have a difference operator. Inverse spectral problems for the difference operators were studied in [17–21] and other works. In [17] the coefficients of finite discrete Sturm–Liouville type boundary value problem are recovered from the spectrum and the set of normalization constants or from two spectra. The works [18,19] are devoted to the discrete analogues of inverse scattering problems on semiaxis and the whole axis. Yurko [20] studied the so-called operators of triangular structure, which generalize the difference ones, and proved the uniqueness theorem and obtained the algorithm for recovery from the Weyl matrix. In [21] the uniqueness theorem for the inverse problem from the eigenvalues and the weight numbers of Sturm–Liouville type difference operator on a finite set of integers is proved. Let us note that this result is the particular case of Theorem 5 in the present paper. Moreover, some numerical methods for solving inverse problems for ordinary differential operators are based on their approximations by difference operators (see [22] and references therein).

The paper is organized as follows. The Sturm–Liouville operator on the time scale \( T \) is introduced in Sect. 2. We study the following its spectral characteristics: the spectra of two boundary value problems with one common boundary condition, the weight numbers and the Weyl function. In Sect. 3, we establish their asymptotical behavior (Theorems 1–4). In Sect. 4, we study three inverse problems of recovering the potential of the Sturm–Liouville operator from the given Weyl function, the two spectra or the spectrum along with the weight numbers. The uniqueness theorem for these inverse problems is proved, see Theorem 5. We also offer Algorithm 1, which allows one to recover the potential of the difference Sturm–Liouville operator (i.e. when \( N = 0 \)).

2. Sturm–Liouville Operators on Time Scales

For convenience of the reader here we provide necessary notions of the time scale theory (see [1,2] for more details). Let \( T \) be an arbitrary closed subset of \( \mathbb{R} \), which we refer to as the time scale. We define the so-called jump functions...
\(\sigma\) and \(\sigma_-\) on \(T\) in the following way:

\[
\sigma(x) = \begin{cases} 
\inf\{s \in T : s > x\}, & x \neq \max T, \\
\max T, & x = \max T,
\end{cases}
\]

\[
\sigma_-(x) = \begin{cases} 
\sup\{s \in T : s < x\}, & x \neq \min T, \\
\min T, & x = \min T.
\end{cases}
\]

A point \(x \in T\) is called left-dense, left-isolated, right-dense and right-isolated, if \(\sigma_-(x) = x\), \(\sigma_-(x) < x\), \(\sigma(x) = x\) and \(\sigma(x) > x\), respectively. If \(\sigma_-(x) < x < \sigma(x)\), then \(x\) is called isolated; if \(\sigma_-(x) = x = \sigma(x)\), then \(x\) is called dense.

Denote \(T^0 := T \setminus \{\max T\}\), if \(\max T\) is left-isolated, and \(T^0 := T\), otherwise. We also denote by \(C(B)\) the class of functions continuous on the subset \(B \subseteq T\).

A function \(f\) on \(T\) is called \(\Delta\)-differentiable at \(t \in T^0\), if for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|
\]

for all \(s \in (t - \delta, t + \delta) \cap T\). The value \(f^\Delta(t)\) is called the \(\Delta\)-derivative of the function \(f\) at the point \(t\).

The following proposition gives conditions of \(\Delta\)-differentiability at points of different types.

**Proposition 1.**  
(1) If \(f(t)\) is \(\Delta\)-differentiable at \(t\), then \(f(t)\) is continuous in \(t\).

(2) Let \(t \in T\) be a right-isolated point. Then \(f\) is \(\Delta\)-differentiable at \(t\), if and only if \(f\) is continuous in \(t\). In this case we have

\[f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.
\]

(3) Let \(t \in T\) be a right-dense point. Then \(f\) is \(\Delta\)-differentiable at \(t\), if and only if there exists the limit

\[\lim_{s \to t, s \in T} \frac{f(t) - f(s)}{t - s} = f^\Delta(t).
\]

In particular, if \((t - \varepsilon, t + \varepsilon) \subseteq T\) for some \(\varepsilon > 0\), then \(f\) is \(\Delta\)-differentiable at \(t\), if and only if \(f\) is differentiable at \(t\). In this case the equality \(f^\Delta(t) = f'(t)\) holds.

We also introduce derivatives of the higher order \(n \geq 2\). Let the \((n-1)\)-th \(\Delta\)-derivative \(f^{\Delta^n-1}\) of \(f\) be defined on \(T^{0^n-1}\), where \(a^n = \underbrace{a \ldots a}_n\) for any symbol \(a\). If \(f^{\Delta^n-1}\), in turn, is \(\Delta\)-differentiable on \(T^{0^n} := (T^{0^n-1})^0\), then \(f^{\Delta^n} := (f^{\Delta^{n-1}})^\Delta\) is called the \(n\)-th \(\Delta\)-derivative of \(f\) on \(T^{0^n}\). For \(n \geq 1\) we also denote by \(C^n(T)\) the class of functions \(f\) for which there exists the \(n\)-th \(\Delta\)-derivative \(f^{\Delta^n}\) and \(f^{\Delta^n} \in C(T^{0^n})\). From now on, \(f^{\Delta^n}(x_1, \ldots, x_n)\)
denotes the \( \nu \)-th \( \Delta \)-derivative of the function \( f(x_1, \ldots, x_n) \) with respect to the first argument, and \( f^{(\nu)}(x_1, \ldots, x_n) \) denotes its classical \( \nu \)-th derivative with respect to the first argument.

A function \( F(t) \) is called antiderivative of \( f(t) \), if there exists \( F^\Delta(t) = f(t) \)
for all \( t \in T^0 \). In [1, Section 1.4] it was established that any function from \( C(T^0) \)
has antiderivatives, which differ by constant. For any \( a, b \in T \), the formula
\[
\int_a^b f(t) \, \Delta t := F(b) - F(a)
\]
defines the definite \( \Delta \)-integral of a function \( f(t) \) on \( T \cap [a, b] \).

Consider the following Sturm–Liouville equation on \( T : \)
\[
\ell y := -y^\Delta(x) + q(x)y(\sigma(x)) = \lambda y(x), \quad x \in T^{0^2}.
\] (2)

Here \( \lambda \) is the spectral parameter, and \( q(x) \in C(T^{0^2}) \) is a real-valued function. A function \( y \) is called a solution of equation (2), if \( y \in C^2(T) \) and equality (2) is fulfilled.

For definiteness, we restrict ourselves to time scales \( T \) of the form (1). We consider \( N > 0 \) or \( M \geq 3 \), otherwise Eq. (2) degenerates. Let us additionally assume that \( q \in W^1_2[a_k, b_k], \ k = 1, N \). Note that the last condition is equivalent to the belongingness of \( q \) to the corresponding Sobolev-type space on \( T \) (see [7]).

For \( T \) of the form (1) some concepts of time scale theory can be clarified. In particular, if \( a, b \in \bigcup_{k=1}^{N+M} \{a_k, b_k\}, \ a \leq b \), then by additivity of \( \Delta \)-integral and [1, Theorem 1.79]
\[
\int_a^b f(t) \, \Delta t = \sum_{k: \ b_k \in [a, b]} f(b_k)(a_{k+1} - b_k) + \sum_{k: \ a_k < b_k \leq b} \int_{a_k}^{b_k} f(t) \, dt.
\] (3)

If \( y \) satisfies (2), then \( \Delta \)-derivative of the function \( y \) on \( T \) can be represented as
\[
y^\Delta(b_l) = \frac{y(a_{l+1}) - y(b_l)}{a_{l+1} - b_l}, \ l = 1, N + M - 1;
\]
\[
y^\Delta(x) = y'(x), \ x \in [a_k, b_k], \ k = 1, N,
\] (4)

where \( y' \) denotes the classical derivative of \( y \). By virtue of (4), Eq. (2) is equivalent to the system of \( N \) Sturm–Liouville equations on the intervals:
\[
- y''(x_k) + q(x_k)y(x_k) = \lambda y(x_k), \quad x_k \in (a_k, b_k), \ k = 1, N,
\] (5)

along with the relations
\[
y^\Delta(b_l) = \frac{1}{a_{l+1} - b_l} (y^\Delta(a_{l+1}) - y^\Delta(b_l)) = (g(b_l) - \lambda)y(a_{l+1}), \ l \in S,
\] (6)

where
\[
S := \{l: 1 \leq l \leq N + M - 1 - \mu_1\},
\]
\[
\mu_1 := \delta(a_{N+M}, b_{N+M}), \quad \delta(k, n) := \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}
\]

According to (4), relations (6) are equivalent to the following jump conditions:
\[
\begin{align*}
y(a_{l+1}) &= \alpha_{11}^l(\lambda)y(b_l) + \alpha_{12}^l(\lambda)y_b^l(b_l), \quad l = 1, N + M - 1, \\
y^\Delta(a_{l+1}) &= \alpha_{21}^l(\lambda)y(b_l) + \alpha_{22}^l(\lambda)y^\Delta(b_l), \quad l \in S,
\end{align*}
\]
(7)

where
\[
\begin{align*}
\alpha_{11}^l(\lambda) &= 1, \\
\alpha_{12}^l(\lambda) &= a_{l+1} - b_l, \\
\alpha_{21}^l(\lambda) &= (a_{l+1} - b_l)(q(b_l) - \lambda), \\
\alpha_{22}^l(\lambda) &= 1 + (a_{l+1} - b_l)^2(q(b_l) - \lambda).
\end{align*}
\]

Thus, Eq. (2) on \( T \) is equivalent to the system of equations (5) along with the jump conditions (7). We arrange the coefficients of the jump conditions into the matrices
\[
\alpha^l(\lambda) := \begin{pmatrix} \alpha_{11}^l(\lambda) & \alpha_{12}^l(\lambda) \\ \alpha_{21}^l(\lambda) & \alpha_{22}^l(\lambda) \end{pmatrix}, \quad l \in S;
\]
\[
\alpha^{N+M-1}(\lambda) := (\alpha_{11}^{N+M-1}(\lambda), \alpha_{12}^{N+M-1}(\lambda)), \quad N + M - 1 \notin S.
\]

Denote by \( L_j \) the boundary value problem for Eq. (2) on \( T \) with the boundary conditions
\[
y^\Delta(a_1) = y(b_{N+M}) = 0, \quad j = 0, 1.
\]
Let \( S(x, \lambda) \) and \( C(x, \lambda) \) be solutions of Eq. (2) on \( T \) satisfying the initial conditions
\[
S^\Delta(a_1, \lambda) = C(a_1, \lambda) = 1, \quad S(a_1, \lambda) = C^\Delta(a_1, \lambda) = 0.
\]
(8)

For each fixed \( x \), the functions \( S(x, \lambda) \) and \( C(x, \lambda) \) are entire in \( \lambda \) of order 1/2. We introduce the entire functions
\[
\Theta_0(\lambda) := S(b_{N+M}, \lambda), \quad \Theta_1(\lambda) := C(b_{N+M}, \lambda).
\]

For \( j = 0, 1 \) eigenvalues \( \{\lambda_{nj}\}_{n \geq 1} \) of the boundary value problem \( L_j \) coincide with zeros of the entire function \( \Theta_j(\lambda) \), which is called the characteristic function for \( L_j \). We provide several examples of the characteristic functions for various time scales and \( q \equiv 0 \). Let us agree that \( \lambda = \rho^2 \).

**Example 1.** Let \( T = \{t\}_{t=0}^3 \), i.e. \( N = 0, M = 4 \). Then \( q(t) \) is defined in \( t = 0 \) and \( t = 1 \). Using (7), one can compute the characteristic functions
\[
\Theta_0(\lambda) = \lambda^2 - 4\lambda + 3, \quad \Theta_1(\lambda) = \lambda^2 - 3\lambda + 1.
\]

Thus, we have \( \lambda_{10} = 1, \lambda_{20} = 3, \lambda_{11} = \frac{3 - \sqrt{5}}{2}, \lambda_{21} = \frac{3 + \sqrt{5}}{2} \).

**Example 2.** Consider \( T = [0, 1] \), i.e. \( N = 1, M = 0 \). In this case (2) coincides with the classical Sturm-Liouville equation, and
\[
\Theta_0(\lambda) = \frac{\sin \rho}{\rho}, \quad \Theta_1(\lambda) = \cos \rho, \quad \lambda_{nj} = \pi \left(n - \frac{j}{2}\right), \quad n \geq 1, \quad j = 0, 1.
\]
Example 3. Consider $T = [0,1] \cup [2,3]$, i.e. $N = 2$, $M = 0$. Then
\[
\Theta_0(\lambda) = \cos^2 \rho + \frac{2 - \lambda}{\rho} \cos \rho \sin \rho - \sin^2 \rho,
\]
\[
\Theta_1(\lambda) = (\lambda - 1) \sin^2 \rho + \cos^2 \rho - 2 \rho \sin \rho \cos \rho.
\]
With the standard method involving Rouche’s theorem it can be established that
\[
\{\lambda_{nj}\}_{n \geq 1} = \left\{ (\pi n + o(1))^2 \right\}_{n \geq 0} \bigcup \left\{ \left( \pi \left( n - \frac{1-j}{2} \right) + o(1) \right)^2 \right\}_{n \geq 1-j}, \quad j = 0, 1.
\]

In these examples we can conclude that both spectra are finite if and only if $N = 0$. In the last case each one contains $M - 2$ numbers. This observation remains true in the general case. In the next section we prove it and obtain the asymptotic formulae when $N > 0$.

To obtain the other necessary properties of the eigenvalues, we introduce the notion of the Wronskian-type determinant $W(\varphi, \psi) := \varphi(t) \psi(t) - \varphi^\Delta(t) \psi(t)$, where $\varphi(t)$ and $\psi(t)$ are solutions of equation \((4)\). By virtue of Theorem 3.13 in [1], we have $W(\varphi, \psi) \equiv \text{const}$ on $T^0$.

**Proposition 2.**
1. The sequences $\{\lambda_{n0}\}_{n \geq 1}$ and $\{\lambda_{n1}\}_{n \geq 1}$ have no common elements.
2. All zeros of $\Theta_j(\lambda)$, $j = 0, 1$, are real and simple.

**Proof.**
1. It is obvious that $W(C, S) = 1$ for all $t \in T^0$ and $\lambda \in \mathbb{C}$. Suppose that $S(b_{N+M}, \lambda_0) = C(b_{N+M}, \lambda_0) = 0$ for some $\lambda_0$. If $a_{N+M} < b_{N+M}$, then $\{C(t, \lambda_0), S(t, \lambda_0)\}, t \in (a_{N+M}, b_{N+M})$, is a fundamental system of solutions of the $N$-th Eq. \((5)\) in $\lambda = \lambda_0$, and we arrive at the contradiction. In the case when $a_{N+M} = b_{N+M}$, from \((7)\) follows the linear dependence of the vectors
\[
(S(b_{N+M-1}, \lambda_0), C(b_{N+M-1}, \lambda_0))^T, \quad (S^\Delta(b_{N+M-1}, \lambda_0), C^\Delta(b_{N+M-1}, \lambda_0))^T,
\]
where $T$ is the transposition sign. The latter contradicts to $W(C, S) = 1$.

2. Consider the case $j = 1$. Let $\Theta_1(\lambda_0) = 0$ and $t \in T^0\!^2$. From \((2)\) for $y = C(t, \lambda)$ and for $y = C(t, \lambda_0)$ one can obtain
\[-C^\Delta(t, \lambda) C(\sigma(t), \lambda_0) + C^\Delta(t, \lambda_0) C(\sigma(t), \lambda) = (\lambda - \lambda_0) C(\sigma(t), \lambda) C(\sigma(t), \lambda_0).
\]
The relation \((f(t)g(t))^\Delta = f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t)\) yields that
\[-C^\Delta(t, \lambda) C(t, \lambda_0) + C^\Delta(t, \lambda_0) C(t, \lambda)\]^\Delta = (\lambda - \lambda_0) C(\sigma(t), \lambda) C(\sigma(t), \lambda_0).
\]
Denote $t_r := \max T^0$. Note that $t_r = b_{N+M}$ when $b_{N+M}$ is left-dense and $t_r = b_{N+M-1}$ in the opposite case. Integrating both sides of the previous relation and using the initial conditions \((8)\), we get
\[
(\lambda - \lambda_0) \int_{a_1}^{t_r} C(\sigma(t), \lambda) C(\sigma(t), \lambda_0) \Delta t = C^\Delta(t_r, \lambda_0) C(t_r, \lambda) - C(t_r, \lambda_0) C^\Delta(t_r, \lambda).
\]
Due to real-valuedness of \( q \), since \( \lambda_0 \) is an eigenvalue, the number \( \overline{\lambda_0} \) is also an eigenvalue with the eigenfunction \( C(t, \overline{\lambda_0}) = \overline{C(t, \lambda_0)} \). Substituting \( \lambda = \overline{\lambda_0} \) into (9), we have

\[
-2 \text{Im} \lambda_0 \int_{a_1}^{t_r} |C(\sigma(t), \lambda_0)|^2 \Delta t = C^\Delta(t_r, \lambda_0)\overline{C(t_r, \lambda_0)} - C(t_r, \lambda_0)\overline{C^\Delta(t_r, \lambda_0)}.
\]

Then the following relation is obvious in the case \( t_r = b_{N+M} \):

\[
-2 \text{Im} \lambda_0 \int_{a_1}^{t_r} |C(\sigma(t), \lambda_0)|^2 \Delta t = 0. \tag{10}
\]

The formula is also valid when \( t_r \neq b_{N+M} \) since \( C^\Delta(t_r, \lambda_0) = \frac{C(b_{N+M}, \lambda_0) - C(t_r, \lambda_0)}{b_{N+M} - t_r} \) in this case.

From (3) it follows that

\[
\int_{a_1}^{t_r} |C(\sigma(t), \lambda_0)|^2 \Delta t = \sum_{k: b_k < t_r} (a_{k+1} - b_k)|C(a_{k+1}, \lambda_0)|^2
+ \sum_{k=1}^{N} \int_{a_k}^{b_k} |C(t, \lambda_0)|^2 dt > 0. \tag{11}
\]

Indeed, in the case when \( a_1 < b_1 \) the function \( C(x, \lambda_0) \) is non-zero one on \([a_1, b_1]\) and \( \int_{a_1}^{b_1} |C(t, \lambda_0)|^2 dt > 0 \); when \( a_1 = b_1 \) the relation \( C(a_1, \lambda_0) = C(a_1, \lambda_0) + C^\Delta(a_1, \lambda_0)(a_2 - a_1) = 1 \) is fullfiled, and since \( a_1 = b_1 < t_r \), we have

\[
\sum_{k: b_k < t_r} (a_{k+1} - b_k)|C(a_{k+1}, \lambda_0)|^2 \geq a_2 - a_1 > 0.
\]

From (10) and (11) we conclude that \( \text{Im} \lambda_0 = 0 \). Further, from (9), real-valuedness of \( C(t, \lambda_0) \) and the equality \( \Theta_1(\lambda_0) = 0 \) we get in both cases \( a_{N+M} < b_{N+M} \) and \( a_{N+M} = b_{N+M} \) that

\[
\int_{a_1}^{t_r} |C(\sigma(t), \lambda_0)|^2 \Delta t = C^\Delta(t_r, \lambda_0) \Theta_1^\prime(\lambda_0).
\]

From the equality \( W(C, S) = 1 \) it follows that

\[
\int_{a_1}^{t_r} |C(\sigma(t), \lambda_0)|^2 \Delta t = -\frac{\Theta_1^\prime(\lambda_0)}{\Theta_0(\lambda_0)}. \tag{12}
\]

Thus, (11) yields the simplicity of \( \lambda_0 \) as the zero of \( \Theta_1(\lambda) \). The case \( j = 0 \) can be treated analogously.

Let \( \Phi(x, \lambda) \), \( x \in T \), be a solution of equation (2) satisfying the boundary conditions

\[
\Phi^\Delta(a_1, \lambda) = 1, \quad \Phi(b_{N+M}, \lambda) = 0.
\]

We call \( M(\lambda) := \Phi(a_1, \lambda) \) the Weyl function, which generalizes the classical Weyl function.

It is obvious that

\[
\Phi(x, \lambda) = S(x, \lambda) + M(\lambda)C(x, \lambda), \tag{13}
\]
\[ M(\lambda) = -\frac{\Theta_0(\lambda)}{\Theta_1(\lambda)}, \]

Put
\[ \alpha_n := \text{Res}_{\lambda=\lambda_n} M(\lambda) = -\frac{\Theta_0(\lambda_n)}{\Theta_1'(\lambda_n)}, \quad n \geq 1. \]

We call \( \alpha_n \) weight numbers. The numbers \( 1/\alpha_n \) generalize the classical weight numbers for the Sturm–Liouville operator. From (11) and (12) for \( \lambda_0 \in \{\lambda_n\}_{n \geq 1} \) it follows that \( \alpha_n > 0 \) for all \( n \).

The Weyl function \( M(\lambda) \), the spectra \( \{\lambda_n\}_{n \geq 1}, j = 0, 1 \), and the weight numbers \( \{\alpha_n\}_{n \geq 1} \) are called spectral characteristics. In the next section we establish their properties including asymptotic formulae.

3. Properties of the Spectral Characteristics

Let us put \( d_k := b_{l_k} - a_{l_k}, \quad k = \overline{1,N} \), where \( l_k \) are determined in (1). Without loss of generality, we assume that \( l_k < l_{k+1}, \quad k = \overline{1,N-1} \). Denote also \( l_0 := 1, \quad l_{N+1} := N + M, \quad \mu_0 := \delta(a_1, b_1) \)

\[ \beta^l(\lambda) = \begin{cases} \left( \frac{\beta_{11}(\lambda)}{\beta_{21}(\lambda)}, \frac{\beta_{12}(\lambda)}{\beta_{22}(\lambda)} \right), & l = \overline{1,l_N-1}, \\ \left( \frac{\beta_{11}(\lambda)}{\beta_{21}(\lambda)}, \frac{\beta_{12}(\lambda)}{\beta_{22}(\lambda)} \right), & l = \overline{l_N,l_{N+1}-\mu_1}, \end{cases} \]

where \( \beta^l(\lambda) \) are determined for \( k = \overline{1,N+\mu_1} \) and \( s = \overline{1,l_k-l_{k-1}} \) as follows:

\[ \beta^{l-k-s}(\lambda) := \alpha^{l_k-1}(\lambda) \ldots \alpha^{l_k-s}(\lambda); \quad \beta^{l_N}(\lambda) := (1,0), \quad l_N = N + M. \]

By virtue of (7), we have

\[ (y(a_{l_k}), y^\Delta(a_{l_k}))^T = \beta^{l-k-s}(\lambda)(y(b_{l_k-s}), y^\Delta(b_{l_k-s}))^T, \quad k = \overline{1,N}, \quad s = \overline{1,l_k-l_{k-1}}, \]

\[ (y(a_{l_{N+1}}), y^\Delta(a_{l_{N+1}}))^T = \beta^{l_{N+1}-s}(\lambda)(y(b_{l_{N+1}-s}), y^\Delta(b_{l_{N+1}-s}))^T, \quad s = \overline{1,l_{N+1}-l_N}. \]

Further we establish asymptotic formulae for the elements of \( \beta^l(\lambda) \).

**Lemma 1.** For \( k = \overline{1,N+\mu_1}, \quad s = \overline{1,l_k-l_{k-1}} \) the following asymptotic formulae are fulfilled:

\[ \beta^{l_k-s}_{ij}(\lambda) = a^{l_k-s}_{ij}(\lambda^{s-2+i} + b^{l_k-s}_{ij}(\lambda^{s-3+i} + O(\lambda^{s-4+i})), \]

\[ i = \overline{1,2-\delta(k,N+1)}, \quad j = 1,2, \]

\[ a^{l_k-s}_{ij} = (-1)^{s-2+i}(a_{l_k-s+1} - b_{l_k-s})^2 - 2(a_{l_k} - b_{l_k-1})^{i-2} \prod_{l=l_k-s}^{l_k-1} (a_{l+1} - b_l)^2, \]
Thus, by induction (15)–(17) prove for

\[
T
\]

\[
\text{directly. Let (15)–(17) be fulfilled for some }
\]

\[
\text{Fix any }
\]

\[
\text{Let us split }
\]

\[
\sum_{l} q(b_l)
\]

\[
(17)
\]

**Proof.** Fix any \( k \in \frac{1}{N} + \frac{1}{\mu_1} \). In the case \( s = 1 \) formulae (15)–(17) are checked directly. Let (15)–(17) be fulfilled for some \( s = \mu \in [1, l_k - l_{k-1}] \). Then representations (16) and (17) in \( s = \mu \) yield (16) and (17) in \( s = \mu + 1 \). Thus, by induction (15)–(17) is proved for \( s = 1, l_k - l_{k-1} \).

Let us split \( T \) into the union of the sets

\[
T_m := \bigcup_{k=m}^{N+M} \{ a_k, b_k \}, \quad T_{m,0} := \bigcup_{k=1}^{m-1} \{ a_k, b_k \}, \quad m = 1, N + M - \mu_1,
\]

and consider the solutions \( S_m(x, \lambda), C_m(x, \lambda) \) of Sturm–Liouville equation (2) on \( T_m \) satisfying the initial conditions

\[
S_m(a_m, \lambda) = S_m^\Delta(a_m, \lambda) - 1 = C_m(a_m, \lambda) - 1 = C_m^\Delta(a_m, \lambda) = 0.
\]

If \( T_m \cap T_{m,0} = \emptyset \), then the functions \( S_m \) and \( C_m \) are completely determined by these initial conditions.

The functions \( S_{l_k}(x + a_l_k, \lambda), C_{l_k}(x + a_l_k, \lambda) \), \( x \in [0, d_k], k = 1, N \), can be obtained as the solutions of the following integral equations:

\[
\begin{aligned}
S_{l_k}(x + a_l_k, \lambda) &= \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho (x - t)}{\rho} S_{l_k}(t + a_l_k, \lambda) q_k(t) \, dt, \\
C_{l_k}(x + a_l_k, \lambda) &= \cos \rho x + \int_0^x \frac{\sin \rho (x - t)}{\rho} C_{l_k}(t + a_l_k, \lambda) q_k(t) \, dt,
\end{aligned}
\]

(18)

where \( q_k(x) := q(a_l_k + x), x \in [0, d_k] \). Obviously, \( q_k \in W^1_2[0, d_k] \). Substituting the standard asymptotic formulae for \( S_{l_k}(x + a_l_k, \lambda) \) and \( C_{l_k}(x + a_l_k, \lambda) \) (see [16, Sect.1.1]) into (18), for \( x = d_k \) we obtain

\[
S_{l_k}(b_l_k, \lambda) = \frac{\sin \rho d_k}{\rho} \left[ 1 + \frac{A_{k1}}{\rho^2} \right] - \frac{\cos \rho d_k}{\rho^2} \omega_k
\]
\[-\frac{1}{4\rho^3} \int_0^{d_{k}} q'_k(t) \sin \rho(2t - d_{k}) \, dt + \frac{O(e^{\tau \rho |d_{k}|})}{\rho^4}, \tag{19}\]

\[S'_l (b_{k}, \lambda) = \cos \rho d_{k} \left[ 1 + \frac{A_{k2}}{\rho^2} \right] + \frac{\sin \rho d_{k}}{\rho} \omega_{k} \]

\[+ \frac{1}{4\rho^2} \int_0^{d_{k}} q'_k(t) \cos \rho(2t - d_{k}) \, dt + \frac{O(e^{\tau \rho |d_{k}|})}{\rho^3}, \tag{20}\]

\[C_{l} (b_{k}, \lambda) = \cos \rho d_{k} \left[ 1 + \frac{A_{k3}}{\rho^2} \right] + \frac{\sin \rho d_{k}}{\rho} \omega_{k} \]

\[\frac{1}{4\rho^2} \int_0^{d_{k}} q'_k(t) \cos \rho(2t - d_{k}) \, dt + \frac{O(e^{\tau \rho |d_{k}|})}{\rho^3}, \tag{21}\]

\[C'_l (b_{k}, \lambda) = -\rho \sin \rho d_{k} \left[ 1 + \frac{A_{k4}}{\rho^2} \right] + \cos \rho d_{k} \omega_{k} \]

\[-\frac{1}{4\rho} \int_0^{d_{k}} q'_k(t) \sin \rho(2t - d_{k}) \, dt + \frac{O(e^{\tau \rho |d_{k}|})}{\rho^2}. \tag{22}\]

Here

\[\tau := \text{Im} \rho, \quad \omega_{k} := \frac{1}{2} \int_0^{d_{k}} q_k(t) \, dt,\]

\[\tilde{A}_{ki} := \frac{(-1)^{[(i-1)/2]} q_k(0)}{4} + \frac{(-1)^{i-1} q_k(d_{k})}{4} - \frac{\omega_{k}^2}{2}, \tag{23}\]

\[i = 1, 4, k = 1, N, \text{ and } [x] \text{ denotes the integer part of } x.\]

Denote \(D_0^m (\lambda) := S_m (b_{N+M}, \lambda), \quad D_1^m (\lambda) := C_m (b_{N+M}, \lambda), \quad m = 1, N + M - \mu_1.\) In particular, \(\Theta_j (\lambda) = D_j^1 (\lambda), \quad j = 0, 1.\) We also introduce the functions \(\Phi_m (x, \lambda), x \in T_m,\) which are solutions of equation (2), \(m = 1, N + M - \mu_1,\) satisfying the boundary conditions \(\Phi^m_m (a_m, \lambda) = 1, \quad \Phi_m (b_{N+M}, \lambda) = 0.\) One can obtain the following formulae, which are analogues of (13), (14):

\[\Phi_m (x, \lambda) = S_m (x, \lambda) + M_m (\lambda) C_m (x, \lambda), \tag{24}\]

where

\[M_m (\lambda) = -\frac{D_0^m (\lambda)}{D_1^m (\lambda)}. \tag{25}\]

**Lemma 2.** For \(j = 0, 1\) the following representations hold:

\[D_j^l (\lambda) = \beta_{1,2-j}^l (\lambda), \quad l = 1, N + 1, N + M - 1, \tag{26}\]

\[D_j^{l-s} (\lambda) = \rho^{\mu_1-1} \beta_{2-s,2-j}^l (\lambda) \prod_{i=1}^{N-1} \beta_{22}^i (\lambda) \beta_{1,1+\mu_1}^l (\lambda) \left( \prod_{l=k}^{N} g_l (\rho) + \frac{O(e^{\tau |\gamma_k|})}{\rho^3} \right), \tag{27}\]

\[s = 1, l_k - l_{k-1} - 1 + \delta (k, 1) \mu_0,\]
\[ \mathcal{D}^{l_k}_j(\lambda) = (-1)^j(1-\delta_k) \rho^{\mu_1} \prod_{i=k}^{N-1} \beta^{l_i}_{22}(\lambda) \beta^{l_i}_{1,1+\mu_1}(\lambda) \times \left( \prod_{l=k+1}^{N} g_l(\rho) \nu_{k-j}(\rho) + O(\varepsilon^{\frac{\|\gamma_k\|}{\rho^2}}) \right), \]  

where \( k = 1, N \). For these \( k \) and \( j = 0, 1 \) we denoted \( \gamma_k := \sum_{l=k}^{N} d_l \), \( \delta_k := \delta(l_k, N + M) \),

\[ g_k(\rho) := v_{k0}(\rho) + (-1)^{\delta_k} \frac{v_{k1}(\rho)}{\rho(a_{lk} - b_{lk-1})}, \quad l_k > 1, \]

\[ v_{k,j}(\rho) := f_{kj}(\rho) \left( 1 + \frac{A_{kj}}{\rho^2} \right) + f_{k,1-j}(\rho) \frac{c_k(-1)^{j+\delta_k}}{\rho} \]

\[ + \frac{(-1)^{\delta_k}}{4\rho^2} \int_0^{d_k} f_{kj}((2t/d_k - 1)\rho) q_k(t) dt, \]

where

\[ f_{k0}(x) := \begin{cases} \sin d_k x, & \delta_k = 1, \\ \cos d_k x, & \delta_k = 0, \end{cases} \quad f_{k1}(x) := \begin{cases} \cos d_k x, & \delta_k = 1, \\ \sin d_k x, & \delta_k = 0, \end{cases} \]

\( c_k \) and \( A_{kj} \) are certain constants, which can be expressed from \( q_k \) :

\[ c_k := \begin{cases} \omega_k, & \delta_k = 1, \\ \omega_k + \frac{1}{a_{lk+1} - b_{lk}}, & \delta_k = 0, \end{cases} \quad A_{kj} := \begin{cases} \frac{A_k}{a_{lk+1} - b_{lk}}, & \delta_k = 1, \\ \frac{A_k}{a_{lk+1} - b_{lk}}, & \delta_k = 0, \end{cases} \]

while the constants \( A_{ki} \) are given in (23).

**Proof.** We will prove these formulae by induction. For \( k = N + M - \mu_1 \) the formulae (26) or (28) is fulfilled: (26) follows from the jump conditions (7) while (28) follows from (19), (21). Let \( \mathcal{D}^{m+1}_j(\lambda) \) be given by formulae (28) for some \( l_k = m + 1 \). We consider two possible cases. First, let \( l_i = l_k - 1, \quad i = k - 1 > 0 \). Using (7) we expand \( Y_0 := S_m \) and \( Y_1 := C_m \) with respect to the system \( \{C_{m+1}, S_{m+1}\} \) on \( T_{m+1} \):

\[ \mathcal{D}^{m}_j(\lambda) = (\alpha^{m}_{11}(\lambda) Y_j(b_m, \lambda) + \alpha^{m}_{12}(\lambda) Y'_j(b_m, \lambda)) \mathcal{D}^{m+1}_j(\lambda) \]

\[ + (\alpha^{m}_{21}(\lambda) Y_j(b_m, \lambda) + \alpha^{m}_{22}(\lambda) Y'_j(b_m, \lambda)) \mathcal{D}^{m+1}_0(\lambda), \quad j = 0, 1. \]  

From (19)–(22) and the definition of \( \beta^j(\lambda) \) it follows that

\[ \alpha^{m}_{21}(\lambda) Y_j(b_m, \lambda) + \alpha^{m}_{22}(\lambda) Y'_j(b_m, \lambda) = (-1)^j \rho^j \beta^{m}_{22}(\lambda) \left( v_{k-1,j}(\rho) + O(\varepsilon^{\frac{\|\gamma_{k-1}\|}{\rho^2}}) \right), \]

\[ \alpha^{m}_{11}(\lambda) Y_j(b_m, \lambda) + \alpha^{m}_{12}(\lambda) Y'_j(b_m, \lambda) = (-1)^j \rho^j \beta^{m}_{12}(\lambda) \left( v_{k-1,j}(\rho) + O(\varepsilon^{\frac{\|\gamma_{k-1}\|}{\rho^2}}) \right) \]

\[ = (-1)^{j+1} \rho^j \beta^{m}_{22}(\lambda) \left( \frac{\lambda^{-1} v_{k-1,j}(\rho)}{a_{lk} - b_{lk-1}} + O(\varepsilon^{\frac{\|\gamma_{k-1}\|}{\rho^2}}) \right). \]  

(31)
These relations with (30) and the induction assumption (28) give formulae (28) for \( k = i \).

Second, let \( a_n = b_n \). Expanding \( S_m, C_m \) with respect to the system \( \{S_{m+1}, C_{m+1}\} \) on \( T_{m+1} \), we get

\[
D_j^n(\lambda) = \alpha_{1,2-j}^m(\lambda)D_1^{m+1}(\lambda) + \alpha_{2,2-j}^m(\lambda)D_0^{m+1}(\lambda), \quad j = 0, 1. \tag{32}
\]

Bracing \( \alpha_{2,2-j}^m(\lambda) \) with (15) and applying the induction assumption, we prove (27) in \( s = 1 \).

The other cases are operated with the same technique. \( \Box \)

Lemma 2 yields that

\[
\Theta_j(\lambda) = \left\{ F_j(\lambda) + O \left( \exp(\gamma_1|\tau|)\lambda^{N+M-2+j(1-\mu_0)/2-\mu_1/2} \right) \right\}, \quad N > 0, \quad j = 0, 1,
\]

\[
\beta_{1,2-j}(\lambda), \quad N = 0, \quad j = 0, 1, \tag{33}
\]

where

\[
F_j(\lambda) := (-1)^j(1-\delta_1)(1-\mu_0)\rho^{\mu_1+j(1-\mu_0)-1} \prod_{k=1-\mu_0}^{N-1} \beta_{2,2-j(0,k)}^k(\lambda) \beta_{1,1+\mu_1}(\lambda)
\]

\[
\times \prod_{k=2}^{N} f_{k0}(\rho) f_{1,(1-\mu_0)j}(\rho). \tag{34}
\]

By the standard method involving Rouche’s theorem [16], from (33) the following structure of the spectra can be established.

**Theorem 1.** Each spectrum consists of \( N + 1 \) parts:

\[
\{\lambda_{nj}\}_{n \geq 1} = \Lambda_j \bigcup \left( \bigcup_{k=1}^{N} \{ (\rho_{nj}^{(k)})^2 \}_{n \geq 1} \right), \quad j = 0, 1,
\]

where \( \Lambda_j \) contains \( N + M + j(1-\mu_0)\text{sign}(N-1+\mu_1)-\mu_1-1 \) elements and for the subsequences \( \{ (\rho_{nj}^{(k)})^2 \}_{n \geq 1} \) the following asymptotic formulae are fulfilled:

\[
\rho_{nj}^{(k)} = \pi \frac{n - \delta_j^{(1,k)}}{d_k} + o(1), \quad \delta_0^j := \frac{1}{2} \delta_j(0, 0), \quad \delta_1^j := \frac{1}{2} - \delta_0^j, \quad 1 \leq k \leq N. \tag{35}
\]

The main parts of eigenvalues’ roots in (35) from different subsequences can occur arbitrarily close to each other, which causes the difficulty in the further refinement of these asymptotic formulae. To overcome it, we make the following additional assumption:

\[
d_k = rx_k, \quad x_k \in \mathbb{Q}, \quad k = 1, N, \text{ for some } r > 0, \tag{36}
\]

which means commensurability of the segments. Analogous commensurability assumptions appear also in other situations, e.g. for studying spectral properties of differential operators on geometrical graphs (see, e.g., [23]). Assumption (36) is needed for Theorems 2–4 and is not used anywhere else. It yields that
for any fixed $s, k \in \mathbb{N}$ and $j, \nu \in \{0, 1\}$ for all $l, n \in \mathbb{N}$ we have the following alternatives:

$$\frac{d_k}{d_s} = \frac{l - \delta^j_k}{n - \delta^\nu_s} \quad \text{or} \quad \left| \frac{d_k}{d_s} - \frac{l - \delta^j_k}{n - \delta^\nu_s} \right| \geq (Cn)^{-1},$$

where and in the sequel $C$ denotes different sufficiently large constants. Then we have

$$\left| f_{kj} \left( \pi n - \delta^\nu_s \frac{n}{d_s} \right) \right| = 0 \quad \text{or} \quad C > \left| f_{kj} \left( \pi n - \delta^\nu_s \frac{n}{d_s} \right) \right| > C^{-1}. \quad (37)$$

Denote by $\eta_{kj}(\rho)$ the multiplicity of $\rho$ as a zero of the function

$$\prod_{l=0}^{N} f_{lj}(\rho), \quad k = 1, N.$$ 

For briefness denote different sequences from $l^2$ by one and the same symbol $\{\kappa_n\}_{n \geq 1}$. We also use $\{\kappa_{n}(z)\}_{n \geq 1}$ to designate different sequences of functions which are continuous in some disk $|z| \leq R$ with

$$\left\{ \max_{|z| \leq R} |\kappa_{n}(z)| \right\}_{n \geq 1} \in l^2.$$

The following theorem refines formulae (35) under the additional condition (36).

**Theorem 2.** If (36) is fulfilled, for the subsequences $\{(\rho_{n_j}^{(k)})^2\}_{n_j \geq 1}$ we have

$$\rho_{n_j}^{(k)} = \frac{n_1 n_2}{d_k} + O \left( \frac{1}{n} \right), \quad k = 1, N, \quad n \in \mathbb{N}. \quad (38)$$

**Proof.** We plan to use the formulae of Lemma 2. In the case $N = 1$ the computations are analogous to the classical case of the Sturm–Liouville equation on interval since the problem of close eigenvalues’ roots does not arise. Therefore, we consider only the case $N > 1$.

For definiteness we consider

$$\rho_{n_j}^{(N)} = K_{n_j} + z_{n_j} := \frac{n - \delta^0}{d_N} + z_{n_j}, \quad z_{n_j} = o(1), \quad n \to \infty.$$ 

We substitute $\rho = \rho_{n_j}^{(N)}$ into $\mathcal{D}_{\nu}^{l}(\rho^2), \nu = 0, 1, l = 1, N + M - \mu_1$. The following formulae can be proved by induction:

$$\mathcal{D}_{\nu}^{l} \left( \left( \rho_{n_j}^{(N)} \right)^2 \right) = \left( \rho_{n_j}^{(N)} \right)^{\mu_1 + \nu - 1} \beta_{1,1+\mu_1}^{l-N} \prod_{i=k}^{N-1} \beta_{i+2}^{l_i} \left( \left( \rho_{n_j}^{(N)} \right)^2 \right)$$

$$\times \left( c_{N+j}^{k\nu, n_\nu(K_{n_j})} + \sum_{l=0}^{\eta_{n_\nu(K_{n_j})}} \eta_{n_\nu(K_{n_j})}^{-1} z_{n_j}^l O \left( n^{l-\eta_{n_\nu(K_{n_j})}} \right) \right), \quad (39)$$
\[ \mathcal{D}_\nu^{k-s} \left( \left( \rho_{n_j}^{(N)} \right)^2 \right) = \left( \rho_{n_j}^{(N)} \right)^{\mu_1-1} \beta_{1,1+\mu_1}^{l_N} \left( \left( \rho_{n_j}^{(N)} \right)^2 \right) \]
\[ \times \prod_{i=k}^{N-1} \beta_{22}^{l_i} \left( \left( \rho_{n_j}^{(N)} \right)^2 \right)^{\beta_{2,2-j}^{l_k-s}} \left( \rho_{n_j}^{(N)} \right)^2 \]
\[ \times \left( \mathcal{C}_{n_j}^{\kappa_0} \sum_{\eta} \eta_{k_0}^{(K_{n_j})-1} + \sum_{l=0}^{\eta} z_{n_j}^l O \left( n^{l-\eta_{k_0}(K_{n_j})} \right) \right), \]

where \( |c_{n_j}^{k-1}| \geq C - \eta_{k_0}(K_{n_j}) \) for sufficiently large \( n, s = 1, l_k - l_{k-1} - 1 + \delta(k, 1) \mu_0 \). Their proof is conducted according the scheme of the one of Lemma 2; for formula (39) one should consider two cases \( f_{k\nu}(K_{n_j}) = 0 \) and \( C > |f_{k\nu}(K_{n_j})| > C^{-1} \) due to (37). In the first case \( v_{k\nu}(\rho_{n_j}^{(N)}) = (\pm d_k + o(1))z_{n_j} + O(n^{-1}) \) and the degree \( \eta_{k\nu}(K_{n_j}) = \eta_{k+1,0}(K_{n_j}) + 1 \), in the second case \( C > |v_{k\nu}(\rho_{n_j}^{(N)})| > C^{-1} \) and \( \eta_{k\nu}(K_{n_j}) = \eta_{k+1,0}(K_{n_j}) \).

We note that \( \mathcal{D}_2^{l}(\rho_{n_j}^{(N)}2) = 0 \). Thus, from (39) for \( \mu_0 = 0 \) or from (40) for \( \mu_0 = 1 \) we have \( |z_{n_j}|^N \leq C^N \sum_{l=0}^{N} |z_{n_j}|^l O(n^{l-N}) \), which yields
\[
|y_{n_j}|^N \leq C^{N+1} \sum_{l=0}^{N-1} |y_{n_j}|^l
\]
for \( y_{n_j} = nz_{n_j} \) and sufficiently large \( C \). From the last inequality it follows that \( y_{n_j} = O(1) \). Indeed, if \( |y_{n_j}| > 2 \), we can estimate
\[
|y_{n_j}| \leq C^{N+1} \sum_{l=0}^{N-1} |y_{n_j}|^{l+1-N} \leq 2C^{N+1},
\]
and, hence, \( \{y_{n_j}\}_{n=1}^{\infty} \) is bounded.

Thus, we proved (38) for \( k = N \). The other formulae can be proved analogously.

Asymptotic formulae (38) can be refined as well. Namely, the following theorem holds.

**Theorem 3.** Denote
\[
z_k := \frac{1}{\pi} \left( c_k + \sum_{l=\max(1,l_k)}^{l_k-1} (a_{l+1} - b_l)^{-1} \right), \quad k = 1, N,
\]
where \( c_k \) are defined in Lemma 2. If (36) holds, then
\[
\rho_{n_j}^{(k)} = \frac{\pi(n - \delta_k^{1, l_k})}{d_k} + \frac{z_k}{n - \delta_k^{1, l_k} n} + r_{n_j}^{(k)} = o(1), \quad k = 1, N, \quad n \in \mathbb{N}.
\]
Provided all \( z_k/d_k, k = 1, N, \) are distinct, we have \( r_{n_j}^{(k)} = \kappa_n/n \).

(41)
Proof. By the same reason as for the proof of Theorem 2, we consider only the case $N > 1$. Any point in the vicinity of $\rho_{n,j}^{(N)}$ can be represented in the form

$$\rho_{n,j}^{(N)}(z) = \frac{n - \delta_0^N}{d_N} + \frac{z}{n - \delta_0^N} =: K_{n,j} + \frac{z}{n - \delta_0^N} , \quad |z| \leq C. $$

Substituting $\rho = \rho_{n,j}^{(N)}(z)$ into the $\mathcal{D}_{\nu}^{k}((\rho^{(N)}_{n,j}(z))^2)$, $\nu = 0, 1, l = 1, N + M - \mu_1$, analogously to (39) and (40) one can prove that

$$\mathcal{D}_{\nu}^{k}((\rho^{(N)}_{n,j}(z))^2) = (-1)^{\nu(1-\delta_k)} \rho_{n,j}^{(N)}(z)^{\nu+1} \prod_{i=k}^{N-1} \beta_{22}^{i}((\rho^{(N)}_{n,j}(z))^2) \beta_{1,1+\mu_1}^l((\rho^{(N)}_{n,j}(z))^2) \times \left(\prod_{l=k+1}^{N} g_l(\rho_{n,j}^{(N)}(z))v_{k\nu}(\rho_{n,j}^{(N)}(z)) + \frac{\kappa_n(z)}{K_{n,j}^{[\eta_{0}\nu_{(K_{n,j})}]+1}}\right), \quad k = 0, N , \quad (42)$$

$$\mathcal{D}_{\nu}^{k-s}((\rho^{(N)}_{n,j}(z))^2) = \rho_{n,j}^{(N)}(z)^{-s} \prod_{i=k}^{N-1} \beta_{2,2-s}^{i}((\rho^{(N)}_{n,j}(z))^2) \beta_{1,1+\mu_1}^l((\rho^{(N)}_{n,j}(z))^2) \times \left(\prod_{l=k}^{N} g_l(\rho_{n,j}^{(N)}(z)) + \frac{\kappa_n(z)}{K_{n,j}^{[\eta_{0}\mu_0(K_{n,j})]+1}}\right), \quad k = 0, N , \quad s = 1, l_k - l_{k-1} - 1 + \delta(k, 1)\mu_0. \quad (43)$$

For the proof it is sufficient to obtain the analogue of the second relation in (31) with $\kappa_n(z)$ instead of $O(e^{\tau|d_k|})$ in the case when $f_{k\nu}(K_{n,j}) = 0$ using Lemma 1; the other computations are similar to the proof of Theorem 2.

Denote

$$I := \left\{ m: f_{m0}(K_{n,j}) = 0, m > 1 \right\} \cup \left\{ 1 \right\}, \quad f_{1, j(1-\mu_0)}(K_{n,j}) = 0, \quad \left\{ m: f_{m0}(K_{n,j}) = 0, m > 1 \right\}, \quad f_{1, j(1-\mu_0)}(K_{n,j}) \neq 0. $$

Consider $\mu_0 = 0$ and (42) for $k = 1$ (for $\mu_0 = 1$ one uses (43)). Using the condition (37) we obtain

$$\Theta_j \left( (\rho_{n,j}^{(N)}(z))^2 \right) = \mathcal{D}_{\nu}^{1} \left( (\rho_{n,j}^{(N)}(z))^2 \right)$$

$$= \left(\rho_{n,j}^{(N)}(z)^{\nu+1} \beta_{1,1+\mu_1}^l((\rho_{n,j}^{(N)}(z))^2) \prod_{i=1}^{N-1} \beta_{22}^{i}((\rho_{n,j}^{(N)}(z))^2) \times \left(\prod_{m \in I} \left(\frac{d_m}{n - \delta_0^N} - \frac{d_N z_m}{n - \delta_0^N} \right) + \frac{\kappa_n(z)}{(n - \delta_0^N)_{n_j(K_{n,j})+1}}\right), \quad (44)$$

where $|C_{n,j}| \geq C_{m_j(K_{n,j})}^{-N}$. With Rouche’s theorem we obtain that $\Theta_j((\rho_{n,j}^{(N)}(z))^2)$ has $\eta_{1,j(1-\mu_0)}(K_{n,j})$ zeros $\rho_{n,j}^{(N)}(z) = \frac{n - \delta_0^N}{d_N} + \frac{z}{n - \delta_0^N}$ with $z = d_N z_m/d_m + o(1), m \in I$. This means that $\Theta_j(\rho^2)$ has the following $\eta_{1,j(1-\mu_0)}(K_{n,j})$ zeros which are close to $K_{n,j}$:
\[
\pi \frac{n - \delta_N^0}{d_N} + \frac{d_N z_m + o(1)}{d_m(n - \delta_N^0)} = \pi \frac{l - \delta_m^j(l,m)}{l - \delta_m^j(l,m)}, \quad m \in I,
\]
such that \(d_N/d_m = (n - \delta_N^0)/(l - \delta_m^j(l,m))\) for some \(l \in \mathbb{N}\). In particular, we have (41) when \(m = N \in I\).

Assume the numbers \(z_k/d_k, k = 1, N\), are distinct. Then using (44) in the vicinity \(|z - z_N| < \delta\) for a sufficiently small \(\delta\) it is easy to prove (41) for \(k = N\) with \(r_{nj}^{(k)} = \kappa_n/n\).

The other formulae can be proved by the same way. \(\square\)

Let us obtain asymptotic formulae for the weight numbers. For them one can prove the analogues of Theorems 1–3. However, for briefness we provide only formulae under the conditions of Theorem 3.

**Theorem 4.** The sequence \(\{\alpha_n\}_{n \geq 1}\) consists of \(N + 1\) parts:

\[
\{\alpha_n\}_{n \geq 1} = A \bigcup \left( \bigcup_{k=1}^{N} \{\alpha_n^k\}_{n \geq 1} \right), \quad \alpha_n^k := \text{Res}_{\lambda = (\rho_n^{(k)})^2} M(\lambda),
\]

\[
A := \left\{ \text{Res } M(\lambda) : z \in \Lambda_1 \right\}.
\]

If (36) is fulfilled and all \(z_k/d_k, k = 1, N\), are distinct, the following asymptotic formulae hold:

\[
\alpha_n^k = \begin{cases} 
2 \frac{\kappa_n}{d_1} \left(1 + \frac{\kappa_n}{n}\right), & k = 1, \mu_0 = 0, \\
\kappa_n \frac{\alpha_n^k}{n}, & \text{all the other cases.} 
\end{cases}
\]

**Proof.** First of all, we note that (45) follows from Theorem 1.

Let us prove (46). Consider the case \(k = 1, \mu_0 = 0, N > 1\). Then \(\delta_1^0 = 0, \delta_1 = 0, z_1 = c_1, f_{11}(x) = \sin d_1 x, f_{10}(x) = \cos d_1 x\). Let \(\delta > 0\) be a sufficiently small number such that \(2\delta < |z_l/d_l - z_1/d_1|, l = 2, N\). Denote

\[
\rho_n(z) := \pi \frac{n}{d_1} + \frac{z_1 + z}{n} =: K_n + \frac{z_1 + z}{n}, \quad |z| \leq \delta.
\]

By Cauchy’s residue theorem we obtain

\[
\alpha_n^1 = \frac{1}{2\pi i} \int_{|z| = \delta} \frac{2\rho_n(z)}{n} M(\rho_n^2(z)) \, dz
\]

for sufficiently large \(n\). Further, for \(|z| \leq \delta\) by (42) we get

\[
M(\rho_n^2(z)) = \frac{\prod_{l=2}^{N} g_l(\rho_n(z))v_{10}(\rho_n(z)) + \frac{\kappa_n(z)}{n^\eta_{10}(K_{n1})+1}}{\rho_n(z) \left(\prod_{l=2}^{N} g_l(\rho_n(z))v_{11}(\rho_n(z)) + \frac{\kappa_n(z)}{n^\eta_{11}(K_{n1})+1}\right)}.
\]
Since \( \left| \prod_{i=2}^{N} g_i(\rho_n(z)) \right| \geq C^{-1} n^{-\eta_1(K_{n1})+1} \) on \( |z| = \delta \) and \( |\eta_{10}(K_{n1}) - \eta_1(K_{n1})| \leq 1 \), we have

\[
M(\rho_n^2(z)) = \frac{v_{10}(\rho_n(z)) + \kappa_n(z)}{\rho_n(z) \left( v_{11}(\rho_n(z)) + \frac{\kappa_n(z)}{n} \right)}.
\]

Substituting Taylor’s formulae of sin and cos into (29), we write

\[
v_{11}(\rho_n(z)) = \left( -1 \right)^n d_1 \left( z + \frac{\kappa_n(z)}{n} \right), \quad v_{10}(\rho_n(z)) = \left( -1 \right)^n \left( 1 + \frac{\kappa_n(z)}{n} \right).
\]

Using (47) and the subsequent formulae, we get

\[
a_n^1 = \frac{2}{d_1 2\pi i} \int_{|z|=\delta} \frac{1 + \frac{\kappa_n(z)}{n}}{z + \frac{\kappa_n(z)}{n}} \, dz = \frac{2}{d_1 2\pi i} \int_{|z|=\delta} \left( \frac{1}{z} + \frac{\kappa_n(z)}{n} \right) \, dz.
\]

From this equation we obtain (46) for \( k = 1, \mu_0 = 0 \).

The other cases can be operated analogously. \(\square\)

Now let us study the asymptotical behavior of the functions \( C_{l_k}(x, \lambda) \) and \( \Phi_{l_k}(x, \lambda) \) in the case \( N > 0, k = 1, N \). For our purposes it is sufficient to consider \( \rho \in \Omega_{\delta} := \{ z : \arg z \in [\delta, \pi - \delta], \; |z| \geq \delta \}, \; \delta > 0, \) and \( x \in (a_{l_k}, b_{l_k}) \).

From (21)–(22) it follows that

\[
C_{l_k}^{(\nu)}(x + a_{l_k}, \lambda) = \frac{(-i \rho)^{\nu}}{2} \exp(-i \rho x)[1], \quad x \in (0, d_k), \; \rho \in \Omega_{\delta}, \; \nu = 0, 1.
\]

Using the standard approach (see, for example, [13]) one can prove the following formulae:

\[
\Phi_{l_k}^{(\nu)}(x + a_{l_k}, \lambda) = (i \rho)^{\nu - 1} \exp(i \rho x)[1], \quad x \in [0, d_k), \; \rho \in \Omega_{\delta}, \; \nu = 0, 1. \tag{49}
\]

### 4. Inverse Problems

Consider the following three inverse problems.

**Inverse problem 1.** Given \( M(\lambda) \), find \( q(x) \) on \( T^{0^2} \).

**Inverse problem 2.** Given \( \{\lambda_{nj}\}_{n \geq 1}, \; j = 0, 1 \), find \( q(x) \) on \( T^{0^2} \).

**Inverse problem 3.** Given \( \{\lambda_{n1}\}_{n \geq 1}, \{\alpha_{n}\}_{n \geq 1} \), find \( q(x) \) on \( T^{0^2} \).

First, we show that these inverse problems are equalent, i.e. their input data uniquely determine each other. Since \( \Theta_0(\lambda) \) and \( \Theta_1(\lambda) \) have no common zeros, \( \{\lambda_{n0}\}_{n \geq 1} \) and \( \{\lambda_{n1}\}_{n \geq 1} \) are determined as zeros and poles of the Weyl function. Conversely, Hadamard’s factorization theorem gives

\[
\Theta_j(\lambda) = C_j p_j(\lambda), \quad p_j(\lambda) = \lambda^{s_j} \prod_{\lambda_{nj} \neq 0} \left( 1 - \frac{\lambda}{\lambda_{nj}} \right), \quad j = 0, 1,
\]
where $C_j$ is a non-zero complex constants, while $s_j$ is the multiplicity of the zero eigenvalue in the spectrum $\{\lambda_{n_j}\}_{n \geq 1}$.

By virtue of (33), the following limits exist:

$$\lim_{\lambda \to i \infty} \frac{\Theta_j(\lambda)}{F_j(\lambda)} = 1, \quad j = 0, 1,$$

where the function $F_j(\lambda)$ is given by (34). Hence, we have

$$C_j = \lim_{\lambda \to i \infty} \frac{F_j(\lambda)}{p_j(\lambda)},$$

and the characteristic functions $\Theta_j(\lambda)$ are uniquely determined by their zeros $\{\lambda_{n_j}\}_{n \geq 1}$. Taking into account formula (14) we conclude that two spectra uniquely determine the Weyl function as well.

Using Lemmas 1 and 2, one can prove by technique analogous to [16] that the weight numbers and the poles uniquely determine the Weyl function by the formula

$$M(\lambda) = \begin{cases} -\mu_0(a_2 - \delta(N + M, 1) - a_1) + \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda - \lambda_{n_1}}, & N > 0, \\ -(a_2 - a_1) + \sum_{n=1}^{M-2} \frac{\alpha_n}{\lambda - \lambda_{n_1}}, & N = 0. \end{cases}$$

Thus, given the input data of one inverse problem, we can recover them of any other one. Moreover, both characteristic functions are determined by specifying the input data of any Inverse problem 1–3.

Further, using the ideas of the method of spectral mappings [16] we prove the uniqueness theorem for the solutions of the inverse problems. For this purpose together with the boundary value problem $L_0$ we consider a problem $\tilde{L}_0$ of the same form but with another potential $\tilde{q}$. In this section we agree that if a certain symbol $\gamma$ denotes an object related to $L_0$, then this symbol with tilde $\tilde{\gamma}$ will denote the analogous object related to $\tilde{L}_0$.

**Theorem 5.** If one of the following conditions is fulfilled, then $q = \tilde{q}$ on $T_0^2$:

1. $M(\lambda) = \tilde{M}(\lambda)$;
2. $\{\lambda_{n_j}\}_{n \geq 1} = \{\tilde{\lambda}_{n_j}\}_{n \geq 1}$, $j = 0, 1$;
3. $\{\lambda_{n_1}\}_{n \geq 1} = \{\tilde{\lambda}_{n_1}\}_{n \geq 1}$ and $\{\alpha_n\}_{n \geq 1} = \{\tilde{\alpha}_n\}_{n \geq 1}$.

Thus, specification of the spectral data of any type uniquely determines the potential $q$.

**Proof.** I. At first, fix $m \in \overline{1, N + M}$ such that $[a_m, b_m] \subseteq T_0^2$ and suppose that $D_j^m(\lambda) \equiv \tilde{D}_j^m(\lambda)$, $j = 0, 1$. It follows from (25) that $M_m(\lambda) \equiv \tilde{M}_m(\lambda)$. Let us prove that $q$ and $\tilde{q}$ coincide on $[a_m, b_m]$. 


First, consider the case $a_m < b_m$. For $x \in (a_m, b_m)$ we define the functions

$$P_j(x, \lambda) = (-1)^j(\Phi_m(x, \lambda)\tilde{C}_m^{(2-j)}(x, \lambda) - \tilde{\Phi}_m^{(2-j)}(x, \lambda)C_m(x, \lambda)), \quad j = 1, 2.$$  

By virtue of the relation $C_m(x, \lambda)\Phi_m(x, \lambda) - C_m'(x, \lambda)\Phi_m(x, \lambda) \equiv 1$, we have

$$P_1(x, \lambda)\tilde{C}_m(x, \lambda) + P_2(x, \lambda)\tilde{C}_m'(x, \lambda) = C_m(x, \lambda). \quad (50)$$

It also follows from (48), (49) that for each fixed $x \in (a_m, b_m)$

$$P_1(x, \lambda) = 1 + O\left(\frac{1}{\rho}\right), \quad P_2(x, \lambda) = O\left(\frac{1}{\rho^2}\right), \quad \rho \to \infty, \quad \rho \in \Omega_{\delta}. \quad (51)$$

On the other hand, using (24) and the coincidence of the Weyl functions, we get

$$P_j(x, \lambda) = (-1)^j(S_m(x, \lambda)\tilde{C}_m^{(2-j)}(x, \lambda) - \tilde{S}_m^{(2-j)}(x, \lambda)C_m(x, \lambda)), \quad j = 1, 2,$$

and consequently, for each fixed $x \in (a_m, b_m)$, the functions $P_1(x, \lambda)$ and $P_2(x, \lambda)$ are entire in $\lambda$ of order $1/2$. By the Phragmén–Lindelöf theorem and Liouville’s theorem, asymptotics (51) imply $P_1(x, \lambda) \equiv 1$ and $P_2(x, \lambda) \equiv 0$, which along with (50) give $C_m(x, \lambda) = \tilde{C}_m(x, \lambda)$ for $x \in (a_m, b_m)$ and, by continuity, for $x \in [a_m, b_m]$. Then $q(x) = \tilde{q}(x)$ for $x \in [a_m, b_m]$.

Now let $a_m = b_m$. The relation $a_m \in T_0^2$ means $m < N + M$ and $m < N + M - 1$ if $\mu_1 = 1$. If we prove that $D_j^m(\lambda)$, $j = 0, 1$, uniquely determine $q(a_m)$, this will yield $q(a_m) = \tilde{q}(a_m)$. Solving system (32) with respect to $D_0^{m+1}(\lambda)$ and $D_1^{m+1}(\lambda)$, we get

$$D_0^{m+1}(\lambda) = \alpha_{11}^m(\lambda)D_0^m(\lambda) - \alpha_{12}^m(\lambda)D_1^m(\lambda) = D_0^m(\lambda) - (a_m + 1 - b_m)D_1^m(\lambda),$$

$$D_1^{m+1}(\lambda) = \alpha_{22}^m(\lambda)D_1^m(\lambda) - \alpha_{21}^m(\lambda)D_0^m(\lambda). \quad (53)$$

Thus, by (52) the function $D_0^{m+1}(\lambda)$ can be computed. Let us write the asymptotic formulae for $D_0^m(\lambda)/D_0^{m+1}(\lambda)$, $\rho \in \Omega_{\delta}$, $\rho \to \infty$. There are two possible cases:

Case 1. $a_{m+1} < b_{m+1}$. Then $l_k = m + 1$ for some $k \in \overline{1, N}$. Use (27) for $l_k - s = m$ and (28) for $l_k = m + 1$:

$$\frac{D_0^m(\lambda)}{D_0^{m+1}(\lambda)} = \alpha_{22}^m(\lambda)\frac{g_k(\rho) + o(\exp(|\tau|d_k)\rho^{-2})}{v_{k0}(\rho) + o(\exp(|\tau|d_k)\rho^{-2})}.$$  

Dividing the numerator and the denominator on $f_{k0}(\rho)$ and using the estimate $|f_{k0}(\rho)| \geq C^{-1}e^{\epsilon|\tau|d_k}$ for $\rho \in \Omega_{\delta}$, we get by (29) that

$$\frac{D_0^m(\lambda)}{D_0^{m+1}(\lambda)} = \alpha_{22}^m(\lambda)\frac{1 + P_1(\rho) + P_2(\rho) + o(\rho^{-2})}{1 + P_1(\rho) + o(\rho^{-2})}, \quad (54)$$

where

$$P_1(\rho) := (-1)^{\delta_1}f_{k1}(\rho) + \frac{A_{k0}}{\rho}.$$
which are less than \( m \).

From this formula one can compute the quantity \( q(a_m) \).

Case 2. \( a_{m+1} = b_{m+1} \). Let \( N_m \) be the number of the indices in \( \{ b_s \}_{s=1}^N \) which are less than \( m \). Then the functions \( D_0^m(\lambda) \) and \( D_0^{m+1}(\lambda) \) are given by (26) or by (27) depending on whether \( N - N_m \) is zero or not respectively. Then

\[
\frac{D_0^m(\lambda)}{D_0^{m+1}(\lambda)} = (a_{m+1} - a_m)^2 (q(a_m) - \lambda) - (a_{m+1} - a_m) \rho i + 1 + o(1).
\]

From this formula one can compute the quantity \( q(a_m) \).

Applying the formulæ of Lemma 1, we obtain

\[
\frac{D_0^m(\lambda)}{D_0^{m+1}(\lambda)} = (a_{m+1} - a_m)^2 \left( -\lambda + \frac{1}{(a_{m+1} - a_m)^2} \frac{1}{(a_{m+1} - a_m)(a_{m+2} - a_m)} + q(a_m) \right) + o(1).
\]

(55)

With this relation \( q(a_m) \) can be computed.

II. Let us prove by induction that the spectral data uniquely determine the potential \( q(x) \). From the assumption of the theorem we have \( D_j^1(\lambda) = \tilde{D}_j^1(\lambda), \ j = 0, 1 \), and, by part I, \( q \equiv \tilde{q} \) on \( [a_1, b_1] \).

Let \( m \in \mathbb{N}, \ N = M - 1 \) be such that \( [a_{m+1}, b_{m+1}] \subset T^{02} \) and

\[
D_j^m(\lambda) \equiv \tilde{D}_j^m(\lambda), \ j = 0, 1, \ q(x) = \tilde{q}(x), \ x \in T_{m+1,0}.
\]

(56)

In the case \( a_{m+1} = b_{m+1} \) we obtain \( D_j^{m+1}(\lambda) \equiv \tilde{D}_j^{m+1}(\lambda), \ j = 0, 1, \) from formulæ (52) and (53). In the case \( a_{m+1} < b_{m+1} \) the functions \( D_0^{m+1}(\lambda) \) and \( D_1^{m+1}(\lambda) \) are solutions of non-degenerate system (30). Moreover, by (56) the functions \( \tilde{D}_0^{m+1}(\lambda) \) and \( \tilde{D}_1^{m+1}(\lambda) \) are solutions of the same system, which yields \( D_j^{m+1}(\lambda) \equiv \tilde{D}_j^{m+1}(\lambda), \ j = 0, 1 \). By virtue of part I, we conclude that \( q \equiv \tilde{q} \) on \( [a_{m+1}, b_{m+1}] \), and the theorem is proved by induction.
Developing the ideas of the method of spectral mapping [16], one can obtain the algorithm for the recovery of the potential. Here we restrict ourself the case of $N = 0$ (i.e. the case of the difference Sturm–Liouville operator) since it is sufficient to have the computations of Theorem 5 for the algorithm.

**Algorithm 1.** Let the functions $D^j_j(\lambda) = \Theta_j(\lambda)$, $j = 0, 1$, be given. To recover $q(a_l)$, $l = 1, M - 2$, for $m = 1, M - 2$ do the following:

1. Construct the function $D^m_0(\lambda)$ using (52) and the known functions $D^m_0(\lambda), D^m_1(\lambda)$.
2. Find $q(a_m)$ from the relation (55).
3. If $m < M - 2$, construct the function $D^m_1(\lambda)$ using (53) and the found value $q(a_m)$.

**Example 4.** Let us consider the time scale $T$ and the characteristic functions $\Theta_j(\lambda)$, $j = 0, 1$, from Example 1 and apply Algorithm 1. First, we compute

$$D^2_0(\lambda) = \Theta_0(\lambda) - \Theta_1(\lambda) = 2 - \lambda, \quad \frac{\Theta_0(\lambda)}{D^2_0(\lambda)} = -\lambda + 2 + \frac{1}{\lambda - 2}.$$  

By formula (55), $2 = 2 + q(0)$ and $q(0) = 0$. With (53) we find $D^2_1(\lambda) = (1 - \lambda)D^m_1(\lambda) + \lambda D^m_0(\lambda) = 1 - \lambda$.

Further, $D^3_0(\lambda) = D^2_0(\lambda) - D^2_1(\lambda) = 1$ and $D^2_0(\lambda)/D^2_0(\lambda) = 2 - \lambda$. Applying (55), we conclude that $2 = 2 + q(1)$ and $q(1) = 0$. Thus, the potential $q$ is recovered.

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**References**

[1] Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. Birkhäuser, Boston (2001)

[2] Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)

[3] Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math. 18, 18–56 (1990)

[4] Atici, F.M., Biles, D.C., Lebedinsky, A.: An application of time scales to economics. Math. Comput. Model. 43, 718–726 (2006)

[5] Prasad, K., Md, K.: Stability of positive almost periodic solutions for a fishing model with multiple time varying variable delays on time scales. Bull. Int. Math. Virtual Inst. 9, 521–533 (2019)

[6] Ozkan, A.S.: Ambarzumyan-type theorems on a time scale. J. Inverse Ill Posed Probl. 26(5), 633–637 (2018)
[7] Rynne, B.P.: L2 spaces and boundary value problems on time-scales. J. Math. Anal. Appl. 328, 1217–1236 (2007)
[8] Agarwal, R.P., Bohner, M., O’Regan, D.: Time scale boundary value problems on infinite intervals. J. Comput. Appl. Math. 141, 27–34 (2002)
[9] Eckhardt, J., Teschl, G.: Sturm–Liouville operators on time scales. J. Differ. Equ. Appl. 18(11), 1875–1887 (2012)
[10] Amster, P., De Nápoli, P., Pinasco, J.P.: Eigenvalue distribution of second-order dynamic equations on time scales considered as fractals. J. Math. Anal. Appl. 343, 573–584 (2008)
[11] Zhang, Y., Ma, L.: Solvability of Sturm–Liouville problems on time scales at resonance. J. Comput. Appl. Math. 233, 1785–1797 (2010)
[12] Ozkan, A.S., Adalar, I.: Half-inverse Sturm–Liouville problem on a time scale. Inverse Probl. (2019). https://doi.org/10.1088/1361-6420/ab2a21
[13] Yurko, V.: Inverse problems for Sturm–Liouville differential operators on closed sets. Tamkang J. Math. 50(3), 199–206 (2019)
[14] Marchenko, V.A.: Sturm–Liouville Operators and Their Applications, Naukova Dumka, Kiev, 1977. English transl, Birkhäuser (1986)
[15] Levitan, B.M.: Inverse Sturm–Liouville Problems, Nauka, Moscow, 1984; Engl. transl., VNU Sci. Press, Utrecht (1987)
[16] Freiling, G., Yurko, V.A.: Inverse Sturm–Liouville Problems and Their Applications. NOVA Science Publishers, New York (2001)
[17] Atkinson, F.: Discrete and Continuous Boundary Problems. Academic Press, New York (1964)
[18] Guseinov, G.S.: Determination of an infinite non-self-adjoint Jacobi matrix from its generalized spectral function. Mat. Zametki 23(2), 237–248 (1978)
[19] Guseinov, G.S., Tuncay, H.: On the inverse scattering problem for a discrete one-dimensional Schrödinger equation. Commun. Fac. Sci. Univ. Ank. Ser. A1(44), 95–102 (1995)
[20] Yurko, V.A.: An inverse problem for operators of a triangular structure. Results Math. 30, 346–373 (1996)
[21] Bohner, M., Kemaloğlu, H.: Inverse problems for Sturm–Liouville difference equations. Filomat 30, 1297–1304 (2016)
[22] Ignatiev, M., Yurko, V.: Numerical methods for solving Inverse Sturm–Liouville problems. Results Math. 52, 63–74 (2008)
[23] Bondarenko, N.P.: An inverse problem for Sturm–Liouville operators on trees with partial information given on the potentials. Math. Methods Appl. Sci. 42, 1512–1528 (2019)

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