A New Non-Commutative Field Theory

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Abstract: In this note we investigate a new type of non-commutative field theory based on a constant skew-symmetric three-form parameter. In 3 + 1 dimensions such a three-form parameter can be viewed as a short-distance regulator which nevertheless preserves spatial-rotation and at long range preserves Lorentz invariance approximately. For a scalar field theory with quartic self-interaction we obtain drastically improved ultra-violet behavior of the diagrams, due to the oscillatory dependence of the interaction vertex on the momenta. The radiative corrections to the coupling are rendered finite already at the one-loop level. The key finding of this paper is that what appears as the reemergence of UV divergences as IR singularity in $p \to 0$ limit, must be interpreted simply as the logarithmic running of the coupling. Thus at low energies the theory is virtually indistinguishable from the standard theory. Conversely at high energies the diagram converges exponentially fast, the running of the coupling stops and the theory avoids developing the Landau pole. Bare coupling defined at high energy can be kept small, and in this sense the theory is similar to asymptotically free theories.

Keywords: Non-commutative field theory, sun-product.
1. Introduction

Non-commutative theories have been extensively studied recently in connection with string theory \[1, 2\]. Non-commutative geometry and field theory naturally arises as a limit of string theory in the presence of the anti-symmetric two-form NS-NS field $B_{\mu\nu}$. Such field theories have been also studied in their own right with an emphasis on the possibility of viewing the non-commutativity as a short-distance regulator \[3, 4, 5\]. Studies of perturbative dynamics have revealed the so-called UV-IR connection \[6, 7\], i.e. the strong IR effects appearing due to integrating out the high momentum modes.

The author was originally motivated by the intriguing problem of finding a generalization of non-commutative field theory in the case that the B-field is linear, i.e. its field strength $H = dB$ is constant. Unfortunately the string sigma model is no longer soluble exactly in this case and so far little progress has been made attacking the problem directly.

We propose in Section 2 an ad-hoc construct for a non-local field theory with a three-form skew-symmetric parameter. Formally the development follows closely that of the conventional non-commutative theory, but in several important physical aspects the results deviate significantly.

Renormalizability of the theory is checked up to two loop order in Sections 3, 4. The four-point (coupling renormalization) amplitude is finite already at the one-loop order. We conjecture that the four-point function diagrams are finite at arbitrary loop-order after taking into account the renormalization of divergent two-point subdiagrams.

In conventional non-commutative theory the planar diagrams get overall phases \[8\] that do not depend on the internal momenta, the loop integrals being exactly the same as in the corresponding diagrams of the commutative theory. This results in non-local counterterms which nonetheless are of the same form as the original lagrangian. By contrast, in the present theory the divergent parts of the diagram do not depend on the quantum parameter $\theta$, thus the necessary counterterms are local. These arise only for the case when the terms in the original lagrangian were local as well.

An important property of the theory is that the diagrams which are rendered finite by the non-commutative cutoff are nonetheless singular as the spatial part of external momentum tends to zero. We interpret this in terms of the conventional renormalization group approach as the logarithmic running of the coupling constant. The relation between the bare coupling at high momenta and the apparent physical coupling at low momenta is the same as in conventional $\phi^4$ field theory. At high energies convergence of the diagram is exponential, so corrections to the bare coupling are small. This means that unlike conventional field theory the running of the coupling eventually stops, allowing to keep the bare
coupling small and avoiding the Landau pole.

2. The ⊠-Product

We would like to construct a quantum space equipped with an anti-symmetric three-form fuzziness parameter. Define a non-local ⊠-product\(^1\) of three functions, while leaving the binary product intact,

\[
(f \otimes g \otimes h) \big|_{x_0} = e^{i \theta_{\mu
u\rho} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \frac{\partial}{\partial z_\rho}} f(x) g(y) h(z) \big|_{x=y=z=x_0} \tag{2.1}
\]

Here \(\theta_{\mu\nu\rho}\) is a completely anti-symmetric three-form in \(d\) dimensions, however we restrict ourselves to the case \(d = 4\) throughout the paper. In four spacetime dimensions such a form has exactly four independent components: \(\theta_{123}, \theta_{012}, \theta_{013}, \theta_{023}\). Provided that the vector dual to our three-form is time-like, \(i.e.\ \theta^2_{123} > \theta^2_{012} + \theta^2_{013} + \theta^2_{023}\), one can always choose a coordinate system in which only \(\theta_{123}\) is nonzero. Thus without loss of generality we set \(\theta_{ijk} = \theta \epsilon_{ijk}\) - implying the absence of time-like components and consequently time derivatives in the definition of the sun-product (2.1). This will nevertheless allow for a construction of a Galilean theory: spatial rotation and translation invariance are preserved.

The \(\theta\) parameter has mass dimension \([-3]\), therefore we introduce a convenient mass scale for the fuzziness of space \(\mu = \theta^{-1/3}\). Naively, experiments in the infrared \(p \ll \mu\) should not be able to discern the quantum structure of the underlying space, because the sun-product (2.1) goes over to ordinary product in this limit. In reality non-commutative field theory exhibits strong IR effects \([6, 7, 8]\), and we find that the present theory is no exception.

We note also that the underlying quantum structure of the space is not clear as yet, in the sense of the equivalence between the Weyl operator ordering on the quantum space and the Moyal star-product on the commutative space. It is tempting to speculate that the algebra of the coordinates may be obtained by considering the sun-product of the coordinates themselves. In this way we arrive at the following anomalous Jacobi-like identity,

\[
x_1[x_2, x_3] + x_2[x_3, x_1] + x_3[x_1, x_2] = i \theta \tag{2.2}
\]

We do not make any use of this and leave the investigation of this quantum space to the future. In this paper we concentrate instead on the perturbative properties of the quantum field theory on this space.

\(^1\)Pronounced as “sun-product”.
3. The Interaction Vertex

Consider the quantum field theory of one real scalar with quartic self-interaction:

\[ \mathcal{L} = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} (\phi \otimes \phi \otimes \phi \phi) \right]. \quad (3.1) \]

It is clear that the kinetic and mass terms are not modified, for the same reasons as in the standard non-commutative theory. However, the interaction term is understood as a non-local momentum-dependent four-point vertex (Figure 1). Had we attempted a different prescription for the interaction from the one above, it would be reduced to the commutative case. For example, \( \phi \otimes \phi \otimes (\phi^2) \) is equivalent to \( \phi^4 \) due to conservation of momentum.

In momentum representation the interaction vertex is obtained by symmetrization with respect to permutations of the external legs

\[ V^{(4)} = \lambda \frac{1}{4!} \sum_{a \neq b \neq c} \exp(i \theta \epsilon_{ijk} p_a^i p_b^j p_c^k) = \lambda \cos(\theta p_1^1 p_2^2 p_3^3). \quad (3.2) \]

Because of momentum conservation all twenty-four terms arising from reordering have the same magnitude, twelve occurring with plus sign and twelve with a minus, for example \( p_1^1 p_2^2 p_3^3 = -p_1^2 p_3^2 p_4^3 \). Moreover, it is possible to give the following geometrical interpretation of this phase (Figure 2). Construct a tetrahedron with the four (spatial) momenta being the vectors normal to the faces of the tetrahedron and equal to the area of the corresponding face. The phase in (3.2) is equal exactly to the square of the volume of the tetrahedron (see caption to Figure 2).

The non-locality and non-renormalizability introduced through the higher-derivative interactions both in standard non-commutative theory and in our sun-product theory is potentially troublesome, in that the \( \theta \)-expanded interactions are non-renormalizable but the very special exponential structure prevents the appearance of most problems generically associated with such non-renormalizable interactions.

Note that \( |V^{(4)}| \leq \lambda \) : the vertex is bounded from above by its standard value. Therefore, the convergence of all diagrams is improved, or at worst unchanged. This may enable us to use the standard theorems about renormalizability at higher loop order and the disentanglement of overlapping divergences \[10\]. One must worry that the necessary counterterms may not be of the same non-local form as in the original lagrangian, however it turns out that only local counterterms for the two-point function appear. We illustrate these issues in Sections 4, 5 by computing the divergent and finite parts of the one- and two-loop diagrams.
Figure 2: Given four arbitrary vectors that add up to nil, one can always construct a tetrahedron with the four vectors being the area-normals of the faces, \( \vec{p}_1 = \frac{1}{2} \vec{l}_1 \times \vec{l}_2 \) etc... Compute: 
\[ 8 \vec{p}_1 \cdot (\vec{p}_2 \times \vec{p}_3) = [\vec{l}_1 \times \vec{l}_2] \cdot [\vec{l}_2 \times \vec{l}_3] \times [\vec{l}_3 \times \vec{l}_1] = (\vec{l}_2 : [\vec{l}_3 \times \vec{l}_1])^2, \]
and this quantity is equal to the volume of the tetrahedron squared.

Figure 3: The one-loop mass renormalization diagram: the sun-phase vanishes due to the collinearity of any three of the legs at the vertex. Conventional regularization is necessary to cancel the quadratic divergence.

4. One-loop Renormalization

Let us see what happens in this theory with the one-loop renormalization of mass. Since any three legs of the vertex in the diagram (Figure 3) have linearly dependent momenta, being \( p, k, -p, -k \), the phase factor is trivial. This means that the diagram is still quadratically divergent. Perhaps this is a good thing, as the non-locality of the theory does not give rise to new infrared singularities in the propagator (at least in this order)

\[
i \Gamma^{(2)} = \lambda \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} = \frac{\lambda}{32\pi^2} \left( \Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right) \]  \hspace{1cm} (4.1)

Thus the counterterms necessary to cancel the divergence in this one-loop mass renormalization diagram are indeed of the same form as the lagrangian we started with. It turns out that at two-loop order (see Section 5) the mass renormalization diagram contains both convergent and divergent contributions. The divergent part is of the same form as the original lagrangian and is local, while only the finite part contains nontrivial dependence on the quantum parameter \( \theta \).
Figure 4: The one-loop coupling renormalization diagram: the sun-phase does not vanish at either of the two vertices for generic values of momenta, being $k \cdot (q \times p)$ and $k \cdot (r \times s)$ respectively. The contribution of this diagram is finite.

Presently we consider the one-loop radiative correction to the coupling (Figure 4). Momenta in bold represent spatial vectors

$$i \Gamma^{(4)} = \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\cos \theta k \cdot (p \times q)}{k^2 + m^2} \frac{\cos \theta r \cdot (s \times k)}{(p + q + k)^2 + m^2}$$

$$= \frac{\lambda^2}{32\pi^2} \int_0^\infty d\alpha d\beta \frac{1}{(\alpha + \beta)^2} e^{-(\alpha + \beta)m^2} e^{-\frac{\alpha \beta}{\alpha + \beta}(p + q)^2} e^{-\frac{\alpha \beta}{\alpha + \beta}(p \times q + r \times s)^2} + [r \leftrightarrow s] \quad (4.2)$$

Upon introducing the convenient variables $z = \alpha + \beta$ and $u = \alpha / z$, obtain

$$i \Gamma^{(4)} = \frac{\lambda^2}{32\pi^2} \int_0^\infty z^{-1} dz \int_0^{z^{-1}} du e^{-zm^2} e^{-zu(1-u)(p+q)^2} e^{-\frac{u^2}{(p \times q + r \times s)^2}} + [r \leftrightarrow s]. \quad (4.3)$$

This is now identical to the standard result, with the expression $\Lambda_{eff}^2 = \theta^2(p \times q \pm r \times s)^2$ playing the role of the cutoff length scale. The integral in (4.3) would have been logarithmically divergent near $z \rightarrow 0$ without this cutoff. Thus, for sufficiently large $\Lambda_{eff}^2$, the $z$ integral can be approximated by its leading logarithmic divergence,

$$i \Gamma^{(4)} = \frac{\lambda^2}{32\pi^2} \left[ \ln \frac{(p \times q + r \times s)^2}{\mu^4} + [r \leftrightarrow s] + \int_0^1 du \ln \frac{m^2 + (p + q)^2 u(1-u)}{\mu^2} \right] \quad (4.4)$$

For small values of $\Lambda_{eff}^2$, the integral converges much faster, in fact exponentially.

At this point we recognize the reemergence of the UV-IR mixing of [6, 7]. The amplitude (4.4) is divergent for some accidental values of non-zero momenta, but more importantly it is singular as $p \rightarrow 0$. In Section 6 we show that this does not lead to genuine IR singularity because the effective action remains finite.

In conventional field theory the logarithmic divergence in the four-point function is interpreted in terms of renormalization of the coupling [10, 11]. We would like to make the connection between our theory and the standard one. For that, we should consider the first term in (4.4) as the correction to the effective coupling at low energy. Taking into account the additional contributions in the $t, u$ channels,

$$\lambda_{eff} = \lambda - \frac{3\lambda^2}{8\pi^2} \ln \frac{|p|}{\mu} \quad \text{for small } p \quad (4.5)$$
For large $p$ the behavior of the integral is different, and $\lambda_{\text{eff}}$ is exponentially close to $\lambda$. In this way, the running of the coupling stops once momenta reach the fuzziness scale $p >> \mu$. Thus in this theory there is no Landau pole singularity. However, there is instead the possibility of destabilization at nonperturbatively low momenta $|p| \sim \mu \exp(-1/\lambda)$. We fully expect that higher loop diagrams will remove this apparent pathology, changing the behavior at such low momenta to

$$\lambda_{\text{eff}} = \frac{\lambda}{1 + \frac{3\lambda}{8\pi^2} \ln \frac{|p|}{\mu}}$$

as is the modification of the one-loop result inferred from standard renormalization group approach [10, 11].

5. Two-loop Mass Renormalization

Two-loop renormalization in non-commutative field theory was considered in detail in [12, 13, 14]. We consider the two-loop order primarily because at one-loop order the two-point function does not receive non-trivial $\theta$-dependent corrections.

In order to obtain radiative corrections to the mass at two loop order (Figure 5) we should compute the double integral

$$\lambda^2 \int \int d^4 k_1 \ d^4 k_2 \ \frac{\cos \theta \ \vec{p} \cdot (\vec{k}_1 \times \vec{k}_2)}{(2\pi)^4 (2\pi)^4 (k_1^2 + m^2)(k_2^2 + m^2)} \frac{\cos \theta \ \vec{p} \cdot (\vec{k}_1 \times \vec{k}_2)}{(p + k_1 + k_2)^2 + m^2}.$$ (5.1)

This integral has overall quadratic divergence by power counting, which is not completely cured by the appearance of the phases. The product of the two cosines has a zero-frequency component as well as a $2 \theta \ \vec{p} \cdot (\vec{k}_1 \times \vec{k}_2)$ component. Such zero-frequency components do not seem to persist at higher loop order. We therefore conjecture that the two-point function at three and higher loop order are finite after the renormalization of divergent one- and two-loop subdiagrams.

The finite part of the integral can be computed as usual with the introduction of Schwinger parameters $\alpha, \beta, \gamma$ and performing the gaussian integrations over momenta

$$\int_0^\infty \ \int_0^\infty \frac{e^{-(\alpha + \beta + \gamma) m^2} e^{-\frac{\alpha \beta \gamma}{\alpha + \beta + \gamma} p^2}}{(\alpha + \beta + \gamma + \alpha \gamma)(\alpha + \beta + \gamma + \alpha \gamma + \theta^2 p^2)} d\alpha d\beta d\gamma = \frac{\mathcal{F}(m^2 \theta |\vec{p}|, \theta^2 |\vec{p}|)}{\theta |\vec{p}|}$$ (5.2)

where $\mathcal{F}$ is a function weakly dependent on its dimensionless arguments.

Again, the result is analogous to the standard one, if we identify $\theta |\vec{p}|$ with the effective cutoff. Aside from the leading $1/\theta |\vec{p}|$ behaviour there is also subleading term $\sim p^2 \ln p^2 \theta |\vec{p}|$ corresponding to the necessary wavefunction renormalization at two loops in ordinary field theory. Again, the divergence of this diagram, which is normally manifest in the ultra-violet region, here reappears in the infrared: $\vec{p} \to 0$. 

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Figure 5: Two-loop mass renormalization diagram. The vertex phase is nontrivial, but is the same for both vertices. This leads to both a finite and an infinite contribution. Since the infinite contribution is $\theta$ independent, the infinity can be cancelled by a local counterterm.

6. Effective Action

The interaction of UV and IR in non-commutative field theories leads to troublesome IR singularities. It has been suggested that these IR effects can be explained through the appearance of closed string modes in string theory. Another possibility might have been to introduce extra IR cutoffs. In \cite{8} a suggestion was made to solve this problem through a non-local field redefinition, however the field transformation is potentially singular, and in any case the problem reappears at the second loop order. Therefore we are compelled to investigate this issue in the present theory.

The appropriate tool is to check whether the effective action is finite, after the removal of the UV cutoff $\Lambda$. The quadratic one-loop effective action in our theory is

$$\Gamma^{(2)}_{\text{eff}} = \int d^4 p \phi(p) \phi(-p) \left[ p^2 + m^2 + \frac{\lambda}{32\pi^2} \left( \Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right) \right] \quad (6.1)$$

The divergent part as $\Lambda \to \infty$ is removed as usual by reabsorbing into the definition of the mass.

The one-loop effective four-point function is read off from (4.4)

$$i \Gamma^{(4)}_{\text{eff}} = \frac{\lambda^2}{32\pi^2} \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 p_4 \phi(p_1) \phi(p_2) \phi(p_3) \phi(p_4) \delta^4(p_1 + p_2 + p_3 + p_4)$$

$$\left[ \ln \frac{(p_1 \times p_2 + p_3 \times p_4)^2}{\mu^4} + \int_0^1 d\alpha \ln \frac{(p_1 + p_2 + p_3 + p_4)^2}{\mu^2} \frac{\alpha(1 - \alpha)}{(1 - \alpha)} \right] \quad (6.2)$$

Despite the fact that there is a logarithmic divergence in the integrand near $p \to 0$, this is more than compensated by the measure in the integral. The IR singularity in the two-loop two-point function is potentially more troublesome, since it is not a logarithmic one as above, but is a pole as $p \to 0$.

$$\Gamma^{(2)\, \text{loop}}_{\text{two}} = \int dp_\nu d^3 p \phi(p) \phi(-p) \left[ p^2 + M^2 + \frac{\lambda^2}{\theta(p) \cdot p} \mathcal{F}(m^2 \theta|p|, p^2 \theta|p|) \right] \quad (6.3)$$

The crucial difference with standard non-commutative theory is that the effective cutoff appearing above involves momenta along all three spatial directions, while in star-product...
theory only momenta in the non-commutative plane appear. Thus, the integral above is actually convergent near $p \to 0$. The theory is free of the IR singularity.

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