Cohomology groups of homogeneous Poisson structures

Kentaro Mikami* Tadayoshi Mizutani†

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*Department of Computer Science and Engineering Akita University, partially supported by JSPS KAKENHI Grant Number JP26400063, JP23540067 and JP20540059.
†Professor Emeritus, Saitama University
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1 Introduction

Researches on the cohomology of formal Hamiltonian vector fields on symplectic \( 2n \)-planes have made much progress after the work of Gel’fand-Kalinin-Fuks ([2]). We see S. Metoki’s work [4] in which the author showed there exists a non-trivial relative cocycle in degree 9 in the case \( n = 1 \).

In 2009, D. Kotschick and S. Morita ([3]) gave some important contribution to this research area. One of the important notion adopted there was the notion of weight (Metoki says it as “type” in [4]). They decomposed the Gel’fand-Fuks cohomology groups according to the weights and investigated the multiplication of cocycles.

Also in a similar method, the structure of cochains were studied in the case of symplectic 2-plane in ([9]) and in ([6]), and further in the case of symplectic 4-plane in [8] or in the case of symplectic 6-plane in [5].

In this paper, we will study of Lie algebra cohomology groups of formal Hamiltonian vector fields of some kind of Poisson structures on \( \mathbb{R}^n \) including linear symplectic structures on \( \mathbb{R}^{2n} \) by the unified way introducing the notion of weight defined depending to homogeneity of Poisson structures. Using manipulation of Young diagrams, we get the recursive formulae for the Euler characteristics depending on the homogeneity and the weight (see Lemma 8.1). Moreover, in Theorem 8.2, we state that for each 1-homogeneous Poisson
structure, the Euler characteristics is always zero for any weight. And we will give some different view point for studying Poisson cohomology groups.

2 Preliminaries

2.1 Homology and cohomology of Lie algebra

We recall the definition of homology and cohomology groups of Lie algebras. Given a Lie algebra $g$, we can consider cohomology groups or homology groups of a Lie algebra $g$.

2.1.1 Cohomology groups

Let $C^m = \{ \sigma : g \times \cdots \times g \to \mathbb{R} | \text{skew-symmetric } m\text{-multi-linear} \} = \Lambda^m(g^*)$.

$$(d\sigma)(u_0, \ldots, u_m) = \sum_{0 \leq i < j \leq m} (-1)^{i+j} \sigma([u_i, u_j], \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots).$$

2.1.2 Homology groups

Let $C_m = \Lambda^m(g)$.

$$\partial(\Lambda^*(u_1, \ldots, u_m)) = \sum_{1 \leq i < j \leq m} (-1)^{i+j} \Lambda([u_i, u_j], \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots),$$

where the notation $\Lambda^*(u_1, \ldots, u_m)$ means the wedge product of $u_1, \ldots, u_m$, namely, $\Lambda^*(u_1, \ldots, u_m) = u_1 \wedge \cdots \wedge u_m$.

2.1.3 Duality

Hereafter, we only consider the trivial representation $(V, \phi)$ of Lie algebra, namely, $V = \mathbb{R}$ and $\phi$ is the zero map. Let $u_i$ be vectors and $a_i$ be covectors. The natural coupling of $u_1 \wedge \cdots \wedge u_k$ and $a_1 \wedge \cdots \wedge a_k$ is given by

$$\langle u_1 \wedge \cdots \wedge u_k, a_1 \wedge \cdots \wedge a_k \rangle = \sum_{\sigma} \text{sgn}(\sigma) \langle u_1, a_{\sigma(1)} \rangle \langle u_2, a_{\sigma(2)} \rangle \cdots \langle u_k, a_{\sigma(k)} \rangle$$

$$= \begin{vmatrix} \langle u_1, a_1 \rangle & \cdots & \langle u_1, a_k \rangle \\ \vdots & \vdots & \vdots \\ \langle u_k, a_1 \rangle & \cdots & \langle u_k, a_k \rangle \end{vmatrix}$$
When we are interested in the behavior of $u_1 \land u_2$, we have the following formula:

**Proposition 2.1**

$$\langle (u_1 \land u_2) \land \cdots \land u_k, a_1 \land \cdots \land a_k \rangle$$

$$= \sum_{i<j} (-1)^{i+j+1} \langle u_1 \land u_2, a_i \land a_j \rangle \langle u_3 \land \cdots \land u_k, a_1 \land \cdots \land \tilde{a}_i \land \cdots \land \tilde{a}_j \land a_k \rangle$$

**Proof:**

$$\text{LHS} = \sum_{\sigma} \text{sgn}(\sigma) \langle u_1, a_{\sigma(1)} \rangle \langle u_2, a_{\sigma(2)} \rangle \langle u_3, a_{\sigma(3)} \rangle \cdots \langle u_k, a_{\sigma(k)} \rangle$$

$$= \sum_{i<j} \sum_{\{\sigma(1), \sigma(2)\} = \{i, j\}} \text{sgn}(\sigma) \langle u_1, a_{\sigma(1)} \rangle \langle u_2, a_{\sigma(2)} \rangle \langle u_3, a_{\sigma(3)} \rangle \cdots \langle u_k, a_{\sigma(k)} \rangle$$

$$= \sum_{i<j} \sum_{\{\tau(3), \ldots, \tau(k)\} = \{i, j\}} \langle u_1, a_{\tau(1)} \rangle \langle u_2, a_{\tau(2)} \rangle \langle u_3, a_{\tau(3)} \rangle \cdots \langle u_k, a_{\tau(k)} \rangle$$

$$+ \sum_{i<j} \sum_{\{\tau(3), \ldots, \tau(k)\} = \{i, j\}} \langle u_1, a_{\tau(1)} \rangle \langle u_2, a_{\tau(2)} \rangle \langle u_3, a_{\tau(3)} \rangle \cdots \langle u_k, a_{\tau(k)} \rangle$$

$$= \sum_{i<j} \langle u_1, a_i \rangle \langle u_2, a_j \rangle - \langle u_1, a_2 \rangle \langle u_2, a_1 \rangle$$

$$\langle d(u), a_1 \land a_2 \rangle = -\langle u, [a_1, a_2] \rangle \quad \text{and} \quad \langle d(u), a_1 \land a_2 \rangle = \langle u, d^*(a_1 \land a_2) \rangle \quad \text{implies} \quad d^*(a_1 \land a_2) = -[a_1, a_2]$$

$$\langle d(u_1 \land u_2), a_1 \land a_2 \land a_3 \rangle = \langle u_1 \land u_2, d^*(a_1 \land a_2 \land a_3) \rangle$$

and

$$\langle d(u_1 \land u_2), a_1 \land a_2 \land a_3 \rangle = \langle d(u_1) \land u_2 - u_1 \land d(u_2), a_1 \land a_2 \land a_3 \rangle$$

$$= \sum_{i<j} \langle d(u_1), a_i \land a_j \rangle \langle u_2, a_k \rangle - \langle d(u_2), a_i \land a_j \rangle \langle u_1, a_k \rangle$$

$$= \sum_{i<j} \langle d(u_1, d^*(a_i \land a_j)) \langle u_2, a_k \rangle - \langle u_2, d^*(a_i \land a_j) \rangle \langle u_1, a_k \rangle$$

$$= \sum_{i<j} \langle u_1 \land u_2, d^*(a_i \land a_j) \land a_k \rangle$$

thus
\[ d^*(a_1 \land a_2 \land a_3) = \sum_{i<j} (-1)^{i+j+1} d^*(a_i \land a_j) \land a_k = \sum_{i<j} (-1)^{i+j}[a_i, a_j] \land a_k \]

In general,

\[ \langle d(u_1 \land \cdots \land u_k), a_1 \land \cdots \land a_{1+k} \rangle = \langle u_1 \land \cdots \land u_k, d^*(a_1 \land \cdots \land a_{1+k}) \rangle \]

\[ = \sum_i (-1)^{1+i} \langle d(u_1), \cdots \rangle \langle u_i, a_1 \land \cdots \land a_{1+k} \rangle \]

\[ = \sum_i (-1)^{1+i} \sum_{p<q} \langle d(u_1), \cdots \rangle \langle u_i, a_p \land a_q \rangle \langle \cdots \rangle \langle u_k, a_1 \land \cdots \land a_q \rangle \cdots \]

\[ = \sum_i \sum_{p<q} (-1)^{1+i+p+q} \langle u_1, \cdots \rangle \langle u_i, d^*(a_p \land a_q) \rangle \langle \cdots \rangle \langle u_k, a_1 \land \cdots \land a_q \rangle \cdots \]

Thus

\[ d^*(a_1 \land \cdots \land a_{1+k}) = \sum_{p<q} (-1)^{1+p+q} d^*(a_p \land a_q) \land a_1 \land \cdots \land \hat{a}_p \cdots \hat{a}_q \land \cdots \]

\[ = \sum_{p<q} (-1)^{p+q} [a_p, a_q] \land a_1 \land \cdots \land \hat{a}_p \cdots \hat{a}_q \land \cdots . \]

We get a more general formula of Proposition 2.1 as follows:

**Proposition 2.2**

\[ \langle (u_1 \land \cdots \land u_k) \land u_{k+1} \land \cdots \land u_n, a_1 \land \cdots \land a_n \rangle \]

\[ = \sum_{#M=k} (-1)^{|M|+k(k+1)/2} \langle u_1 \land \cdots \land u_k, a_M \rangle \langle u_{k+1} \land \cdots \land u_n, a_{\tilde{M}} \rangle \]

where \( M \) runs in the family of \( k \)-subsets of 1, 2, \ldots, \( n \). \( a_M = a_{m_1} \land \cdots \land a_{m_k} \) with \( M = \{ m_1 < m_2 < \cdots < m_k \} \) and \( |M| = \sum_{i=1}^{k} m_i \), and \( \tilde{M} = \{ 1, 2, \ldots, n \} \setminus M \).

When the chain and the cochain spaces are finite dimensional, then we can express each Betti number by the dimensions of the kernel spaces and the (co)chain spaces as follows:

**Cochain complex case:**

| \( \dim \) | \( C^{m-1} \rightarrow \) | \( C^m \rightarrow \) | \( C^{m+1} \rightarrow \) |
|---|---|---|---|
| \( \ker \) | \( d_{m-1} \) | \( d_m \) | \( d_{m+1} \) |
| \( \text{Betti} \) | \( k_m \) | \( k_{m+1} \) | \( k_{m+1} \) |

\[ k_m + k_{m-1} - d_{m-1} \]
Chain complex case:

|      | $\leftarrow C_{m-1}$ | $\leftarrow C_m$ | $\leftarrow C_{m+1}$ | $\leftarrow$ |
|------|-----------------------|------------------|-----------------------|-----------|
| dim  | $d_{m-1}$             | $d_m$            | $d_{m+1}$             |           |
| ker  | $\ell_{m-1}$         | $\ell_m$         | $\ell_{m+1}$         |           |
| Betti| $\ell_m + \ell_{m+1} - d_{m+1}$ |                |                       |           |

Now assume that $\dim C^m$ and $\dim C_m$ are finite dimensional and $\dim C^m = \dim C_m$ for each $m$. Since $\langle d_p \sigma, u \rangle = \langle \sigma, \partial_{p+1} u \rangle$, we have $(\text{Ker}d_p)^0 = \text{Im}(\partial_{p+1})$. Evaluating the dimensions of both sides, we have

$$\dim C^p - \dim \text{Ker}d_p = \dim C_{p+1} - \dim \text{Ker}\partial_{p+1}.$$  

Using the notations above, this is equivalent to

$$\dim C^p - k_p = \dim C_{p+1} - \ell_{p+1}.$$  

Thus, we claim that

$$\dim H_m = \ell_m + \ell_{m+1} - \dim C_{m+1}$$

$$= k_{m-1} - \dim C^{m-1} + \dim C_m$$

$$+ k_m - \dim C^m + \dim C_{m+1} - \dim C_{m+1}$$

$$= k_{m-1} + k_m - \dim C^{m-1} \quad \text{(if $\dim C^j = \dim C_j$)}$$

$$= \dim H^m.$$  

Thus, we have the following statement and we may choose cochain complex or chain complex if we are interested in only Betti numbers.

**Proposition 2.3** If the chain spaces and the cochain spaces are finite dimensional and $\dim C^m = \dim C_m$ for each $m$, then $\dim H^m = \dim H_m$ for each $m$.

### 2.2 Schouten bracket and Poisson structure

Let us recall the Schouten bracket on a $n$-dimensional smooth manifold $M$. Let $\Lambda^j T(M)$ be the space of $j$-vector fields on $M$. In particular, $\Lambda^1 T(M)$ is $\mathfrak{X}(M)$, the Lie algebra of smooth vector fields on $M$, and $\Lambda^0 T(M)$ is $C^\infty(M)$. $A = \sum_{j=0}^{n} \Lambda^j T(M)$ is the exterior algebra of multi-vector fields on $M$.  

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For $P \in \Lambda^p T(M)$ and $Q \in \Lambda^q T(M)$, the Schouten bracket $[P, Q]_S$ is defined to be an element in $\Lambda^{p+q-1} T(M)$. This bracket satisfies the following formulas.

$$[Q, P]_S = -(-1)^{(q+1)(p+1)} [P, Q]_S \quad \text{(symmetry)},$$

$$0 = \sum_{p,q,r} (-1)^{(p+1)(r+1)} [P, [Q, R]]_S \quad \text{(the Jacobi identity)},$$

$$[P, Q \wedge R]_S = [P, Q]_S \wedge R + (-1)^{(p+1)q} Q \wedge [P, R]_S,$$

$$[P \wedge Q, R]_S = P \wedge [Q, R]_S + (-1)^{q(r+1)} [P, R]_S \wedge Q,$$

another expression of Jacobi identity is the next

$$[P, [Q, R]]_S = [[[P, Q]_S, R]_S + (-1)^{(p+1)(q+1)} [Q, [P, R]]_S,$$

$$[[P, Q]_S, R]_S = [P, [Q, R]]_S + (-1)^{(q+1)(r+1)} [[P, R]_S, Q]_S.$$

Note that these formulae above are valid without any change for the case of $\mathbb{R}$-linear wedge product.

To define the usual Schouten bracket uniquely, we need also the following basic formulae concerning functions.

$$[X, Y]_S = \text{Jacobi-Lie bracket of } X \text{ and } Y,$$

$$[X, f]_S = \langle X, df \rangle.$$

For vector fields $X_1, \ldots, X_p$ and $Y_1, \ldots, Y_q$ the Schouten bracket of $X_1 \wedge \cdots \wedge X_p$ and $Y_1 \wedge \cdots \wedge Y_q$ is given by

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q]_S = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge (X_1 \wedge \cdots \wedge \hat{X}_i \cdots \wedge X_p) \wedge (Y_1 \wedge \cdots \wedge \hat{Y}_j \cdots Y_q).$$

Let $\pi$ be a 2-vector fields on $M$. $\pi$ is a Poisson structure if and only if $[\pi, \pi]_S = 0$. Locally, let $(x_1 \ldots x_n)$ be a local coordinates and

$$\pi = \frac{1}{2} \sum_{i,j} p_{ij} \partial_i \wedge \partial_j \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x_i} \text{ and } p_{ij} + p_{ji} = 0.$$

The Schouten bracket of $\pi$ and itself is calculated as follows

$$2[\pi, \pi]_S = \sum_{i,j} [\pi, p_{ij} \partial_i \wedge \partial_j]$$

$$= \sum_{i,j} ([\pi, p_{ij}] \wedge \partial_i \wedge \partial_j + p_{ij}[\pi, \partial_i]_S \wedge \partial_j - p_{ij} \partial_i \wedge [\pi, \partial_j]_S).$$
\[
4[\pi, \pi]_S = \sum_{k\ell} \sum_{i,j} (p_{k\ell} \, \partial_k \wedge \partial_{\ell} \, p_{ij} \wedge \partial_i \wedge \partial_j + p_{ij} \, [p_{k\ell} \, \partial_k \wedge \partial_{\ell} \, \partial_i \wedge \partial_j]_S \wedge \partial_j
- p_{ij} \, \partial_i \wedge [p_{k\ell} \, \partial_k \wedge \partial_{\ell} \, \partial_j]_S)
= \sum_{k\ell} \sum_{i,j} (p_{k\ell}(\partial_{\ell} p_{ij}) \, \partial_k \wedge \partial_i \wedge \partial_j - p_{k\ell}(\partial_k p_{ij}) \, \partial_{\ell} \wedge \partial_i \wedge \partial_j
- p_{ij} (\partial_i p_{k\ell}) \, \partial_k \wedge \partial_{\ell} \wedge \partial_j + p_{ij} (\partial_j p_{k\ell}) \, \partial_i \wedge \partial_{\ell} \wedge \partial_j)
- p_{ij} (\partial_i p_{k\ell}) \, \partial_k \wedge \partial_{\ell} \wedge \partial_j
- p_{ij} (\partial_j p_{k\ell}) \, \partial_i \wedge \partial_{\ell} \wedge \partial_k).
\]

The Poisson bracket of \( f \) and \( g \) is given by \( \{f, g\}_\pi = \pi(df, dg) \) so \( \{x_i, x_j\}_\pi = p_{ij} \) and

\[
\{\{x_i, x_j\}_\pi, x_k\}_\pi = p_{ij} \, \partial_k + \sum_{\lambda,\mu} \frac{\partial p_{ij}}{\partial x_{\lambda}} \, \partial_{\lambda} \partial_j \partial_k.
\]

\( \pi \) is Poisson if and only if Jacobi identity for the bracket holds, and it is equivalent to

\[
(2.1) \quad \sum_{i,j,k} \sum_{\lambda} p_{k\lambda} \frac{\partial p_{ij}}{\partial x_{\lambda}} = 0 \quad \text{(cyclic sum with respect to } i, j, k).\]

A given Poisson structure \( \pi \) on a manifold \( M \) yields two kinds of Lie algebras: one is the space of smooth functions \( C^\infty(M) \) on \( M \) with the Poisson bracket, the other is the space of Hamiltonian vector fields defined by \( \{f, \cdot\}_\pi = \pi(df, \cdot) \) with the Jacobi-Lie bracket. The relation of these Lie algebras is described by the following short exact sequence

\[
0 \to \text{Cent}_\pi(M) \to C^\infty(M) \to \{\{f, \cdot\}_\pi \in \mathfrak{x}(M)\} \to 0
\]

where \( \text{Cent}_\pi(M) = \{f \in C^\infty(M) \mid \{f, \cdot\}_\pi = 0\} \), the center of \( C^\infty(M) \), whose element is called a Casimir function. Obviously \( \text{Cent}_\pi(M) \) is a ideal of Lie algebra. Thus we have an isomorphism

\[
C^\infty(M)/\text{Cent}_\pi(M) \cong \{\{f, \cdot\}_\pi \in \mathfrak{x}(M)\}
\]

as Lie algebras. Roughly speaking, the space \( \text{Cent}_\pi(M) \) shows how far the Poisson structure of the manifold \( M \) is from symplectic structure, in the case of symplectic structure, \( \text{Cent}_\pi(M) \) is just the space of constant functions.

**Definition 2.1** We endow the space \( \mathbb{R}^n \) with the Cartesian coordinates \((x_1, \ldots, x_n)\). Then any 2-vector field
π is written as
\[ \pi = \frac{1}{2} \sum_{i,j} p_{ij}(x) \partial_i \wedge \partial_j \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x_i} \quad \text{and} \quad p_{ji}(x) = -p_{ij}(x). \]

Again, π is a Poisson structure if and only if the Schouten bracket \([\pi, \pi]_S\) vanishes or equivalently satisfy (2.1). We say that a Poisson structure π on \(\mathbb{R}^n\) is \(h\)-homogeneous when all the coefficients \(p_{ij}(x) = \{x_i, x_j\}_\pi\) are homogeneous polynomials of degree \(h\) in \(x_1, \ldots, x_n\).

The space of polynomials \(\mathbb{R}[x_1, \ldots, x_n]\) is a Lie sub-algebra with respect to the Poisson bracket defined by a \((h)\)-homogeneous Poisson structure and is a quotient Lie algebra of \(\mathbb{R}[x_1, \ldots, x_n]\) modulo Casimir polynomials as mentioned above. Thus, we may consider the two kinds of Lie algebra cohomology groups.

**Remark 2.1** There is a notion of subclass of Poisson structures, called exact Poisson structures or homogeneous Poisson structures on a manifold \(M\) satisfying the condition that the Poisson tensor \(\pi\) has a vector field \(X\) with \([X, \pi]_S = \pi\). As been discussed in [7], \(h\)-homogeneous Poisson structure on \(\mathbb{R}^n\) in our sense with \(h \neq 2\) is automatically exact or homogeneous, but when \(h = 2\) there are some examples 2-homogeneous in our sense but not exact nor homogeneous. It will be better to call our \(h\)-homogeneous Poisson structure by naive homogeneous Poisson structures, but sometimes call just homogeneous Poisson structures.

**Remark 2.2** It is well-known that a 1-homogeneous Poisson structure is nothing but a Lie Poisson structure and the space \(\mathbb{R}^n\) is the dual space of a Lie algebra and the Poisson bracket is the Lie algebra bracket defined on the space of linear functions on it, that is, the Lie algebra itself. In precise, let \(h\) be a finite dim Lie algebra. For \(F : h^* \rightarrow \mathbb{R}\), define \(\frac{\delta F}{\delta \mu} \in h^{**}\) by \(\frac{\delta F}{\delta \mu}(v) = \frac{d}{dt} F(\mu + tv)|_{t=0}\) and \(\{F, H\}(\mu) := \langle \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right], \mu \rangle\).

In particular, \(\{v, w\} = [v, w]\) for \(v, w \in h\).

### 2.3 Examples

On \(\mathbb{R}^n\) take a \(h\)-homogeneous 2-vector field \(\pi\). Then \(\pi\) is written as \(\pi = \frac{1}{2} \sum_{i,j} p_{ij}(x) \partial_i \wedge \partial_j\) where \(p_{ij}(x) + p_{ji}(x) = 0\) and \(p_{ij}(x)\) are \(h\)-homogeneous. Poisson condition for \(\pi\) is given globally by (2.1).

If \(n = 3\) then the condition is rather simple and is written as

\[ p_{12} \left( \frac{\partial p_{23}}{\partial x_2} - \frac{\partial p_{31}}{\partial x_1} \right) + p_{23} \left( \frac{\partial p_{31}}{\partial x_3} - \frac{\partial p_{12}}{\partial x_2} \right) + p_{31} \left( \frac{\partial p_{12}}{\partial x_1} - \frac{\partial p_{23}}{\partial x_3} \right) = 0 \]

for \(h\)-homogeneous polynomials \(\{p_{12}(x), p_{23}(x), p_{31}(x)\}\).
We will try to find some of homogeneous Poisson structures on $\mathbb{R}^3$. 2-vector fields on $\mathbb{R}^3$ are classified into 3 types by shape:

(1) $f \partial_i \wedge \partial_j$,

(2) $\partial_i \wedge (f \partial_j + g \partial_k)$ with $fg \neq 0$ and $\{i, j, k\} = \{1, 2, 3\}$,

(3) $\sum_{i=1}^{3} f_i \partial_{i+1} \wedge \partial_{i+2}$ with $f_1 f_2 f_3 \neq 0$ and $\partial_{i+3} = \partial_i$.

It is obvious 0-homogeneous 2-vector field $\pi_0 = \frac{1}{2} \sum_{ij} f_{ij} \partial_i \wedge \partial_j$ satisfies the Poisson condition automatically.

We examine the Poisson condition for $f \pi_0$. Our classical computation is the following: now $p_{ij} = f_{ij}$ and we check the left-hand-side of (2.1):

$$\sum_{ijk} \sum_{\lambda=1}^{n} p_{k\lambda} \frac{\partial p_{ij}}{\partial x_\lambda} = \sum_{ijk} \sum_{\lambda=1}^{n} f_{k\lambda} c_{ij} \frac{\partial f}{\partial x_\lambda} = \sum_{\lambda=1}^{n} \frac{\partial f}{\partial x_\lambda} \sum_{ijk} c_{ij} c_{k\lambda}.$$

When $n = 3$, (2.3) is 0 because $\sum_{123} c_{12} c_{3\lambda} = 0$ for each $\lambda = 1, 2, 3$, thus $f \pi_0$ is Poisson for any function $f$. Thus, 2-vector field of type (1) satisfies the Poisson condition automatically, so we may choose $h$-homogeneous polynomial $f$.

About type (2), we may assume $\partial_1 \wedge (f \partial_2 + g \partial_3)$. Then the Poisson condition is

$$\frac{\partial f}{\partial x_1} g - f \frac{\partial g}{\partial x_1} = 0.$$

If $f$ and $g$ satisfy (2.4) then the commonly multiplied $\phi f$ and $\phi g$ also satisfy (2.4) because

$$(\phi f)'(\phi g) - (\phi f)(\phi g)' = (\phi' f + \phi f')(\phi g) - (\phi f)(\phi' g + \phi g') = 0$$

where the dash of $f$, $f'$ means $\frac{\partial f}{\partial x_1}$.

If $f$ and $g$ are polynomials of the variables $x_2$ and $x_3$, then (2.4) holds and so $\phi \partial_1 \wedge (f \partial_2 + g \partial_3)$ give Poisson structures on $\mathbb{R}^3$. So, we have found examples of Poisson structures:

$$\phi^{[i]} \partial_1 \wedge (f^{[j]}(x_2, x_3) \partial_2 + g^{[j]}(x_2, x_3) \partial_3)$$

where $f^{[i]}$, $g^{[i]}$ and $\phi^{[i]}$ mean $i$-th homogeneous polynomials and $j > 0$.

As for the tensor of type (3), we have two kinds of examples of Poisson structures.
\[(3-1) \sum_{i=1}^{3} c_i x_i{}^p \partial_i \wedge x_{i+1}{}^p \partial_{i+1} \quad \text{(more general, } \frac{1}{2} \sum_{i=1}^{n} c_{ij} x_i{}^p \partial_i \wedge x_j{}^p \partial_j \text{ where } c_{ij} + c_{ji} = 0).\]

The reason is:

\[
[x_i{}^p \partial_i \wedge x_{i+1}{}^p \partial_{i+1}, x_j{}^p \partial_j \wedge x_{j+1}{}^p \partial_{j+1}]_S
\]
\[
= [x_i{}^p \partial_i, x_j{}^p \partial_j]_S \wedge x_{i+1}{}^p \partial_{i+1} + x_{j+1}{}^p \partial_{j+1} - [x_i{}^p \partial_i, x_{i+1}{}^p \partial_{i+1}]_S \wedge x_j{}^p \partial_j - x_j{}^p \partial_j
\]
\[
- [x_{i+1}{}^p \partial_{i+1}, x_j{}^p \partial_j]_S \wedge x_i{}^p \partial_i + x_{j+1}{}^p \partial_{j+1} + [x_{i+1}{}^p \partial_{i+1}, x_{j+1}{}^p \partial_{j+1}]_S \wedge x_j{}^p \partial_i - x_j{}^p \partial_i
\]

and

\[
[x_i{}^p \partial_i, x_j{}^p \partial_j]_S = x_j{}^p p x_j{}^p \partial_j \delta_i \partial_j - x_i{}^p p x_i{}^p \partial_i \delta_j \partial_i = \delta_{ij} p(x_i x_j)^{p-1}(x_i \partial_j - x_j \partial_i) = 0.
\]

\[(3-2) \sum_{i=1}^{3} c_i x_i{}^p \partial_{i+1} \wedge \partial_{i+2} \text{ where } p \text{ is a non-negative integer, } c_i \text{ are constant and } x_{i+3} = x_i.
\]

Reason is:

\[
[\sum_{i=1}^{3} c_i x_i{}^p \partial_{i+1} \wedge \partial_{i+2}, \sum_{j=1}^{3} c_j x_j{}^p \partial_{j+1} \wedge \partial_{j+2}]_S
\]
\[
= \sum_{i,j} c_i c_j ([x_i{}^p \partial_i, x_j{}^p \partial_j]_S \wedge \partial_{i+2} \wedge \partial_{j+2} - [x_i{}^p \partial_i, \partial_{j+2}]_S \wedge \partial_{i+2} \wedge x_j{}^p \partial_{j+1}
\]
\[
- [\partial_{i+2}, x_j{}^p \partial_{j+1}]_S \wedge \partial_{i+2} \wedge \partial_{j+2} + [\partial_{i+2}, \partial_{j+2}]_S \wedge \partial_i{}^p \partial_i + x_j{}^p \partial_{j+1})
\]
\[
= \sum_{i,j} c_i c_j ((x_i{}^p p x_j{}^p \partial_j \delta_{i+1} \partial_{i+1} - p x_j{}^p \partial_j \delta_{i+1} \partial_{i+1}) \wedge \partial_{i+2} \wedge \partial_{j+2}
\]
\[
+ p x_j{}^p \partial_j \delta_{i+2} \partial_{i+1} \wedge \partial_{i+2} \wedge \partial_{j+1} - p x_j{}^p \partial_j \delta_{i+2} \partial_{i+1} \wedge x_j{}^p \partial_{i+1} \wedge \partial_{j+2})
\]
\[
=p \sum_{i,j} c_i c_j (x_i x_j)^{p-1}(x_i \delta_{i+1} \partial_{i+1} \wedge \partial_{j+2} \wedge \partial_{j+2} - x_i \delta_{j+1} \partial_{i+1} \wedge \partial_{i+2} \wedge \partial_{j+2}
\]
\[
+ x_j \delta_{j+2} \partial_{i+1} \wedge \partial_{i+2} \wedge \partial_{j+1} - x_i \delta_{j+2} \partial_{i+1} \wedge \partial_{i+1} \wedge \partial_{j+1})
\]
\[
=p \sum_{j=i+1}^{n} c_i c_j (x_i x_j)^{p-1} x_i \partial_{i+2} \wedge \partial_{i+2} \wedge \partial_{i+3} - p \sum_{j=i+1}^{n-1} c_i c_j (x_i x_j)^{p-1} x_{i-1} \partial_{i+1} \wedge \partial_{i+2} \wedge \partial_{i+1}
\]
\[
+ p \sum_{j=i+2}^{n} c_i c_j (x_i x_j)^{p-1} x_i \partial_{i+3} \wedge \partial_{i+3} \wedge \partial_{i+4} - p \sum_{j=i+1}^{n-2} c_i c_j (x_i x_j)^{p-1} x_{i-2} \partial_{i+1} \wedge \partial_{i+2} \wedge \partial_{i-1} - p \sum_{j=i+3}^{n} c_i c_j (x_i x_j)^{p-1} x_i \partial_{i+4} \wedge \partial_{i+1} \wedge \partial_{i+4}
\]
\[
=0.
\]

We show a direct computation to find Poisson structures for $h = 1$. Take a general 1-homogeneous 2-vector.
If we transform some of the variables \(c_i\) by \(\{u_j\}\) as follows:

\[
c_1 - c_6 = u_1, \quad c_1 + c_6 = u_6, \quad c_2 - c_9 = u_2, \quad c_2 + c_9 = u_9, \quad c_5 - c_7 = u_5, \quad c_5 + c_7 = u_7.
\]
The Poisson condition (2.5) – (2.7) becomes

\[-2c_4u_2 + u_1u_7 + u_6u_5 = 0, \quad 2c_8u_1 + u_5u_9 - u_2u_7 = 0, \quad 2c_3u_5 - u_2u_6 + u_9u_1 = 0.\]

Solving these equations, we get 4 solutions:

1. \(u_6 = (2c_4u_2 - u_1u_7)/u_5, \quad u_9 = (-2c_8u_1 + u_2u_7)/u_5, \quad c_3 = (c_4u_2^2 - u_2u_1u_7 + c_8u_7^2)/u_5^2\)

   where \(u_1, u_2, u_5, u_7, c_4, c_8\) are free.

2. \(u_5 = 0, \quad u_7 = 2c_4u_2/u_1, \quad u_9 = u_2u_6/u_1, \quad c_8 = u_2^2c_4/u_1^2\)

   where \(u_1, u_2, u_6, c_3, c_4\) are free.

3. \(u_1 = 0, \quad u_2 = 0, \quad u_5 = 0\)

   where \(u_6, u_7, u_9, c_3, c_4, c_8\) are free.

4. \(u_1 = 0, \quad u_5 = 0, \quad u_6 = 0, \quad u_7 = 0, \quad c_4 = 0\)

   where \(u_2, u_9, c_3, c_8\) are free.

If \(n = 4\), then the condition becomes more complicated as we get 4 equations from (2.1).

Any constant 2-vector field \(\pi_0 = \frac{1}{2} \sum_{ij} c_{ij} \partial_i \land \partial_j\) is Poisson, and about \(f \pi_0\) if \(\pi_0 \land \pi_0 \neq 0\), i.e., symplectic then \(f \pi_0\) is Poisson only for constant function \(f\) and if \(\pi_0 \land \pi_0 = 0\), i.e., rank is 2 then \(f \pi_0\) is Poisson for any \(f\) from (2.3). Here, we used the relations \(\sum_{ij} c_{ij} c_{kl} = 0\) if \(\ell \in \{i, j, k\}\) and \(\sum_{ijk} c_{ij} c_{kl} \partial_i \land \cdots \land \partial_4 = \pm \pi_0 \land \pi_0\) if \(\{i, j, k, \ell\} = \{1, 2, 3, 4\}\). Thus, we see different situations even for 3 or 4 dimensional.

3 Polynomial algebra including constants

When a Poisson structure \(\pi\) is given on a manifold \(M\), we have a Lie algebra \((C^\infty(M), \{\cdot, \cdot\}_\pi)\) and we may consider Lie algebra cohomology groups in “primitive” sense as follows: the coboundary operator is a derivation of degree +1 and for each “\(k\)-cochain” \(\sigma\), \(d\sigma\) is defined by

\[(d\sigma)(f_0, \ldots, f_k) = \sum_{i < j} (-1)^{i+j}\sigma(\{f_i, f_j\}_\pi, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_k),\]

where \(\widehat{f_i}\) means omitting \(f_i\). Or, the boundary operator is given by

\[\partial (\land^\wedge(f_1, \ldots, f_k)) = \sum_{i < j} (-1)^{i+j}\land^\wedge(\{f_i, f_j\}_\pi, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_k).\]

But, it seems hard to handle “primitive (co)chain spaces” because the (co)chain complexes are huge spaces. Instead, in the following subsections we consider only homogeneous Poisson structures and polynomial functions and reduce the spaces by the notion of “weight”. Also, we study about Lie algebra (co)homology.
groups of formal Hamiltonian vector fields of the homogeneous Poisson structures. We first handle the algebra including constant polynomials, i.e.,

\begin{equation}
    \text{Poly}(\mathbb{R}^n) = \sum_{j=0}^{\infty} \mathfrak{s}_j, \quad \text{and the dual space is } \text{Poly}(\mathbb{R}^n)^* = \sum_{j=0}^{\infty} \overline{\mathfrak{s}}_j
\end{equation}

where $\mathfrak{s}_j$ is the space of $j$-th homogeneous polynomials and $\overline{\mathfrak{s}}_j$ is the dual space of $\mathfrak{s}_j$.

The dimension of $\overline{\mathfrak{s}}_j$ or $\mathfrak{s}_j$ is $(n-1+j)$, and $\dim \overline{\mathfrak{s}}_0 = \dim \mathfrak{s}_0 = 1$ in particular. The $m$-th cochain space or chain space of the Lie algebra $\text{Poly}(\mathbb{R}^n)$ by the Poisson bracket with polynomials in (3.1) are given by

\begin{align*}
    \Lambda^m(\text{Poly}(\mathbb{R}^n)^*) &= \sum_{0 \leq i_1 \leq \cdots \leq i_m} \overline{\mathfrak{s}}_{i_1} \wedge \cdots \wedge \overline{\mathfrak{s}}_{i_m} = \sum \Lambda^{k_0} \overline{\mathfrak{s}}_0 \otimes \Lambda^{k_1} \overline{\mathfrak{s}}_1 \otimes \cdots \otimes \Lambda^{k_\ell} \overline{\mathfrak{s}}_\ell \\
    \Lambda^m(\text{Poly}(\mathbb{R}^n)) &= \sum_{0 \leq i_1 \leq \cdots \leq i_m} \mathfrak{s}_{i_1} \wedge \cdots \wedge \mathfrak{s}_{i_m} = \sum \Lambda^{k_0} \mathfrak{s}_0 \otimes \Lambda^{k_1} \mathfrak{s}_1 \otimes \cdots \otimes \Lambda^{k_\ell} \mathfrak{s}_\ell
\end{align*}

where $\sum_{j=0}^{\infty} k_j = m$. When $k_j = 0$ for some $j$, then $\Lambda^{k_j} \overline{\mathfrak{s}}_j = \mathbb{R}$ and $\Lambda^{k_j} \overline{\mathfrak{s}}_j \otimes \Lambda^{k_\ell} \overline{\mathfrak{s}}_\ell = \Lambda^0 \overline{\mathfrak{s}}_j \otimes \Lambda^{k_\ell} \overline{\mathfrak{s}}_\ell = \Lambda^{k_\ell} \overline{\mathfrak{s}}_\ell$ and $\Lambda^{k_j} \mathfrak{s}_j \otimes \Lambda^{k_\ell} \mathfrak{s}_\ell = \Lambda^{k_\ell} \mathfrak{s}_\ell$ in the similar way. When $m = 0$, we have $\Lambda^0(\text{Poly}(\mathbb{R}^n)^*) = \mathbb{R}$ and $\Lambda^0(\text{Poly}(\mathbb{R}^n)) = \mathbb{R}$, where $\mathbb{R}$ is the (coefficient) field of algebra $\text{Poly}(\mathbb{R}^n)$. As a basis of $\mathfrak{s}_j$ we have \{x^A | A = (a_1, \ldots, a_n) \in \mathbb{N}^n \text{ and } |A| = a_1 + \cdots + a_n = j\}, and the dual basis \{z_A\} is given by

\begin{equation}
    z_A = \delta_0 \circ \left( \frac{1}{A!} \frac{\partial}{\partial x_1}^{a_1} \cdots \frac{\partial}{\partial x_n}^{a_n} \right) = \delta_0 \circ \left( \frac{1}{A!} \partial^A \right) \quad \text{with} \quad |A| = j
\end{equation}

where $A! = a_1! \cdots a_n!$ and $\delta_0$ is the Dirac delta function, which evaluates the target at the origin. Thus, $z_{(0,\ldots,0)} = \delta_0 \in \overline{\mathfrak{s}}_0$ is the dual basis of 1 $\in \mathfrak{s}_0 = \mathbb{R} = \{\text{constant polynomials}\}$.

### 3.1 Weight decomposition

**Definition 3.1** We introduce the notion of weight associated with the $h$-homogeneous Poisson structure on $\mathbb{R}^n$ as follows:

\begin{align*}
    \text{the weight of non-zero element of } \Lambda^{k_0} \overline{\mathfrak{s}}_0 \otimes \Lambda^{k_1} \overline{\mathfrak{s}}_1 \otimes \cdots \otimes \Lambda^{k_\ell} \overline{\mathfrak{s}}_\ell \text{ is } &\sum_{j=0}^{\ell} k_j (j - 2 + h), \quad (3.2) \\
    \text{and} \\
    \text{the weight of non-zero element of } \Lambda^{k_0} \mathfrak{s}_0 \otimes \Lambda^{k_1} \mathfrak{s}_1 \otimes \cdots \otimes \Lambda^{k_\ell} \mathfrak{s}_\ell \text{ is } &\sum_{j=0}^{\ell} k_j (j - 2 + h). \quad (3.3)
\end{align*}
Thus, the \( m \)-th cochain and chain space of the weight \( w \) are given by

\[
\Lambda^m_w(\text{Poly}(\mathbb{R}^n)^*) = \sum \Lambda^{k_0}_{i_0} \otimes \Lambda^{k_1}_{i_1} \otimes \cdots \otimes \Lambda^{k_{\ell}}_{i_{\ell}}
\]

(3.4)

\[
\Lambda^m_w(\text{Poly}(\mathbb{R}^n)) = \sum \Lambda^{k_0}_{0} \otimes \Lambda^{k_1}_{1} \otimes \cdots \otimes \Lambda^{k_{\ell}}_{\ell}
\]

(3.5)

with three conditions:

\[
\sum_{j=0}^{\infty} k_j = m ,
\]

(3.6)

\[
\sum_{j=0}^{\infty} k_j (j - 2 + h) = w ,
\]

(3.7)

and

\[
0 \leq k_j \leq \dim(\Xi_j) = \dim(\chi_j) = \left( n^{-1+j} \right) \text{ for } j = 0, 1, \ldots .
\]

(3.8)

### 3.2 Possibility of weight and range of degree

Using (3.6) and (3.7), \( m = 0 \) implies \( w = 0 \), in other words, \( \Lambda^0_w(\text{Poly}(\mathbb{R}^n)^*) = 0 \) if \( w \neq 0 \) and \( \Lambda^0_w(\text{Poly}(\mathbb{R}^n)^*) = \mathbb{R} \).

We prepare two equations, one is \((2 - h)\) times (3.6) :

\[
(2 - h) \sum_{j=0}^{\infty} k_j = (2 - h)m
\]

(3.9)

the other is a reform of (3.7):

\[
\sum_{j=0}^{\infty} jk_j = w + (2 - h)m
\]

(3.10)

(3.10) – (3.9) implies

\[
\sum_{j<2-h} (j - 2 + h)k_j + \sum_{j>2-h} (j - 2 + h)k_j = w .
\]

(3.11)

In (3.11), if \( h - 2 \geq 0 \) then it turns out \( w \geq 0 \). When \( h - 2 < 0 \), \( (3.11) \) implies

\[
0 \leq \sum_{j>2-h} (j - 2 + h)k_j = w - \sum_{j<2-h} (j - 2 + h)k_j = \begin{cases} w + k_0 & (h = 1) \\ w + 2k_0 + k_1 & (h = 0) \end{cases}
\]

(3.12)

(3.8) says \( 0 \leq k_0 \leq 1 \) and \( 0 \leq k_1 \leq n \), and so \( w \geq -1 \) when \( h = 1 \) and \( w \geq -(2 + n) \) when \( h = 0 \).
Subtracting \((3 - h)\) times (3.6) from (3.10), we have

\[
\sum_{j < 3-h} (j - 3 + h)k_j + \sum_{j > 3-h} (j - 3 + h)k_j = w - m
\]

\[
0 \leq \sum_{j > 3-h} (j - 3 + h)k_j = w - m - \sum_{j < 3-h} (j - 3 + h)k_j
\]

\[
m \leq w + \sum_{j < 3-h} (3 - h - j)k_j = \begin{cases} 
  w & h \geq 3 \\
  w + k_0 \leq w + 1 & h = 2 \\
  w + 2k_0 + k_1 \leq w + 2 + n & h = 1 \\
  w + 3k_0 + 2k_1 + k_2 \leq w + \frac{(n+2)(n+3)}{2} & h = 0
\end{cases}
\]

Subtracting \((1 - h)\) times (3.6) from (3.10), we have

\[
\sum_{j < 1-h} (j - 1 + h)k_j + \sum_{j > 1-h} (j - 1 + h)k_j = w + m
\]

\[
0 \leq \sum_{j > 1-h} (j - 1 + h)k_j = w + m - \sum_{j < 1-h} (j - 1 + h)k_j
\]

\[
-m \leq w + \sum_{j < 1-h} (1 - h - j)k_j = \begin{cases} 
  w & h \geq 1 \\
  w + k_0 \leq w + 1 & h = 0
\end{cases}
\]

We may summarize the discussion above as the following table.

| homogeneity | weight | range of m of m-th cochains |
|-------------|--------|-----------------------------|
| h = 0       | w \geq -2 - n | \max(0, -(w + 1)) \leq m \leq \frac{(n+2)(n+3)}{2} |
| h = 1       | w \geq -1   | \max(0, -w) \leq m \leq w + n + 2 |
| h = 2       | w \geq 0    | 0 \leq m \leq w + 1 |
| h \geq 3    | w \geq 0    | 0 \leq m \leq w |

**Example 3.1** When \(h = 0\) and \(w = -(2 + n)\), we determine the cochain spaces \(\Lambda_{(-2-n)}^n(\text{Poly}(\mathbb{R}^n)^*)\). In this case, (3.7) is

\[
-2 - n = w = \sum_j k_j(j - 2 + h) = \sum_j k_j(j - 2) = -2k_0 - k_1 + k_3 + 2k_4 + \cdots.
\]

\[
2k_0 - 2 + k_1 - n = k_3 + 2k_4 + \cdots.
\]

Since \(0 \leq k_0 \leq 1\) and \(0 \leq k_1 \leq n\), the last equation implies \(k_0 = 1\), \(k_1 = n\) and \(k_j = 0\) for \(j > 2\). Applying
these facts to (3.6), we have $k_2 = m - (1 + n)$. Thus

$$
\Lambda^m_{(-2-n)}(\text{Poly}(\mathbb{R}^n)^*) = \mathcal{E}_0 \otimes \Lambda^n \mathcal{E}_1 \otimes \Lambda^{m-(1+n)} \mathcal{E}_2.
$$

And, $0 \leq m - (1 + n) \leq (n + 1)n/2$, i.e., $1 + n \leq m \leq w + (n + 2)(n + 3)/2$.

When $w = -1 - n$, we have

$$(3.12) \quad 2k_0 - 1 + k_1 - n = k_3 + 2k_4 + \cdots.$$ 

Then $k_0 = 1$ and $k_1 = n$ or $k_1 = n - 1$. When $k_0 = 1$ and $k_1 = n$, $k_3 = 1$, $k_j = 0$ ($j > 3$) and using (3.6), $k_2 = m - 2 - n$.

When $k_0 = 1$ and $k_1 = n - 1$, $k_j = 0$ ($j > 2$) and $k_2 = m - n$ again using (3.6).

$$
\Lambda^m_{(-1-n)}(\text{Poly}(\mathbb{R}^n)^*) = \mathcal{E}_0 \otimes \left( \Lambda^n \mathcal{E}_1 \otimes \Lambda^{m-(2+n)} \mathcal{E}_2 \otimes \mathcal{E}_3 + \Lambda^{n-1} \mathcal{E}_1 \otimes \Lambda^{m-n} \mathcal{E}_2 \right).
$$

When $w = -n$, we have

$$(3.13) \quad 2k_0 + k_1 - n = k_3 + 2k_4 + \cdots.$$ 

If $k_0 = 0$ then $k_1 = n$, $k_j = 0$ ($j > 2$) and so (3.6) says $k_2 = m - n$.

If $k_0 = 1$, then $k_1 = n$, $n - 1$, or $n - 2$. If $k_0 = 1$, $k_1 = n - 2$, then $k_j = 0$ ($j > 2$) and using (3.6), $k_2 = m - n + 1$. If $k_0 = 1$, $k_1 = n - 1$, then $k_3 = 1$ $k_j = 0$ ($j > 3$) and using (3.6), $k_2 = m - n - 1$.

If $k_0 = 1$, $k_1 = n$, then $k_3 = 2$, $k_j = 0$ ($j > 3$) or $k_3 = 0$, $k_4 = 1$ $k_j = 0$ ($j > 4$). Using (3.6), $k_2 = m - n - 3$ in the former case and $k_2 = m - n - 2$ in the last case. Thus, we have

$$
\Lambda^m_{(-n)}(\text{Poly}(\mathbb{R}^n)^*) = \Lambda^n \mathcal{E}_1 \otimes \Lambda^{m-n} \mathcal{E}_2
$$

$$
\oplus \mathcal{E}_0 \otimes (\Lambda^{n-2} \mathcal{E}_1 \otimes \Lambda^{m-n+1} \mathcal{E}_2 \oplus \Lambda^{n-1} \mathcal{E}_1 \otimes \Lambda^{m-n-1} \mathcal{E}_2 \otimes \mathcal{E}_3
$$

$$
\oplus \Lambda^n \mathcal{E}_1 \otimes \Lambda^{m-n-3} \mathcal{E}_2 \otimes \Lambda^2 \mathcal{E}_3 \oplus \Lambda^n \mathcal{E}_1 \otimes \Lambda^{m-n-2} \mathcal{E}_2 \otimes \mathcal{E}_4).
$$

**Example 3.2** When $h = 1$ and $w = -1$, we determine the cochain spaces $\Lambda^m_{(-1)}(\text{Poly}(\mathbb{R}^n)^*)$. In this case, (3.7) is

$$
-1 = w = \sum_j k_j(j - 2 + h) = \sum_j k_j(j - 1) = -k_0 + k_2 + 2k_3 + \cdots.
$$

This implies $k_0 = 1$ and $k_j = 0$ for $j \geq 2$. Applying these facts to (3.6), we have $m = 1 + k_1$, i.e., $k_1 = m - 1$.
Thus $\Lambda_{(-1)}^m(\text{Poly}(\mathbb{R}^n)^*) = \overline{\mathcal{E}}_0 \otimes \Lambda^{m-1}\overline{\mathcal{E}}_1$. Thus, $0 \leq m - 1 \leq n$, i.e., $1 \leq m \leq w + n + 2$.

By the same way, we have

$$
\begin{align*}
\Lambda_0^m(\text{Poly}(\mathbb{R}^n)^*) &= \Lambda^m\overline{\mathcal{E}}_1 \oplus \overline{\mathcal{E}}_0 \otimes \Lambda^{m-2}\overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \\
\Lambda_1^m(\text{Poly}(\mathbb{R}^n)^*) &= \Lambda^{m-1}\overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \oplus \overline{\mathcal{E}}_0 \otimes \left(\Lambda^{m-2}\overline{\mathcal{E}}_3 \oplus \Lambda^{m-3}\overline{\mathcal{E}}_1 \otimes \Lambda^2\overline{\mathcal{E}}_2\right)
\end{align*}
$$

### 3.3 About $\overline{d}(\delta_0)$

Since the coboundary operator $\overline{d}$ is a derivation of degree 1, it is known that the behavior of $\overline{d}$ is under controlled by the action of $\overline{d}$ for each 1-cochain $\sigma$.

From the definition of coboundary operator of Lie algebra cohomology, we see that

$$
\overline{d}\sigma = \frac{1}{2} \sum (\overline{d}\sigma)(x^A, x^B)z_A \wedge z_B = -\frac{1}{2} \sum \langle \sigma, \{x^A, x^B\}_\pi \rangle z_A \wedge z_B,
$$

and, in particular

$$
(\overline{d}\delta_0)(x^A, x^B) = -\langle \delta_0, \{x^A, x^B\}_\pi \rangle = 0 \quad \text{if} \quad |A||B| = 0 \text{ or } |A| + |B| \neq 2 - h,
$$

where $h$ is the homogeneity of our Poisson structure. Thus, when $h > 0$, if $|A||B| \neq 0$ then $|A| + |B| \geq 2 > 2 - h$ and $\overline{d}(\delta_0) = 0$ holds as expected and we have $\overline{d}(\overline{\mathcal{E}}_0) = 0$.

Assume our Poisson structure $\pi$ is 0-homogeneous, given by $\{x^i, x^j\}_\pi = p_{i,j}$ (constant). Let us denote $\epsilon_i$ whose element is 1 if $i$-th position and 0 otherwise. Then

$$
(\overline{d}\delta_0)(x^i, x^j) = -\langle \delta_0, \{x^i, x^j\}_\pi \rangle = -p_{i,j},
$$

$$
\overline{d}\delta_0 = -\frac{1}{2} \sum_{i,j} p_{i,j} z_{\epsilon_i} \wedge z_{\epsilon_j} \equiv -\pi,
$$

$$
\overline{d}\pi = 0 \quad \text{from (3.17)}.
$$

**Remark 3.1** When $h = 0$, the weight of $\delta_0$ is $-2$ and the weight of $z_{\epsilon_i}$ is $-1$ and that of $\pi$ is $-2$.

**Remark 3.2** Preparing formulae of $\overline{d}z_{\epsilon_i}$, we reconfirm (3.18) by those properties of $\overline{d}z_{\epsilon_i}$:

$$
\overline{d}z_{\epsilon_i} = -\frac{1}{2} \sum_{A,B} \langle z_{\epsilon_i}, \{x^A, x^B\}_\pi \rangle z_A \wedge z_B = -\sum_{|A| = 1, |B| = 2} \langle z_{\epsilon_i}, \{x^A, x^B\}_\pi \rangle z_A \wedge z_B
$$

$$
= -\sum_{i,j} \langle z_{\epsilon_i}, \{x^i, x^{2j}\}_\pi \rangle z_{\epsilon_i} \wedge z_{2\epsilon j} - \frac{1}{2} \sum_{j \neq k} \langle z_{\epsilon_i}, \{x^{\epsilon_i + \epsilon_k}\}_\pi \rangle z_{\epsilon_i} \wedge z_{\epsilon_i + \epsilon_k}
$$

$$
= -\frac{1}{2} \sum_{i,j} \langle z_{\epsilon_i}, \{x^i, x^{2j}\}_\pi \rangle z_{\epsilon_i} \wedge z_{2\epsilon j} - \frac{1}{2} \sum_{i,j,k} \langle z_{\epsilon_i}, \{x^{\epsilon_i + \epsilon_j + \epsilon_k}\}_\pi \rangle z_{\epsilon_i} \wedge z_{\epsilon_i + \epsilon_j + \epsilon_k}
$$
\[=- \frac{1}{2} \sum_{i,j} \langle z_{e_i}, 2p_{i,j}x^{e_j} \rangle z_{e_i} \wedge z_{e_j} - \frac{1}{2} \sum_{i,j,k} \langle z_{e_i}, p_{i,k}x^{e_j} + p_{i,k}x^{e_k} \rangle z_{e_i} \wedge z_{e_j+e_k}\]

\[= - \sum_{i,j} p_{i,j}z_{e_i} \wedge z_{e_j} - \frac{1}{2} \sum_{i,j} p_{i,j}z_{e_i} \wedge z_{e_j+e_1} - \frac{1}{2} \sum_{i,k} p_{i,k}z_{e_i} \wedge z_{e_1+e_k}\]

\[= Q_\ell \wedge z_{2e_1} + \sum_j Q_j \wedge z_{e_1+e_j} \text{ where } Q_j = \sum_i p_{j,i}z_{e_i}.\]

\[2d\pi = d\left(\sum_{i,j} p_{i,j}z_{e_i} \wedge z_{e_j}\right) = \sum_{i,j} p_{i,j} \left((dz_{e_i}) \wedge z_{e_j} - z_{e_i} \wedge dz_{e_j}\right)\]

\[= 2 \sum_j Q_j \wedge dz_{e_j} = 2 \sum_j Q_j \wedge (Q_j \wedge z_{2e_j} + \sum_k Q_k \wedge z_{e_j+e_k})\]

\[= 2 \sum_j Q_j \wedge Q_k \wedge z_{e_j+e_k} = 0.\]

### 3.4 Cohomology and homology groups with respect to weight

**Proposition 3.1** The coboundary operator \(d\) on Poly(\(\mathbb{R}^n\)) preserves the weight, namely, \(d(\Lambda^m_w(\text{Poly}(\mathbb{R}^n)^*)) \subset \Lambda^{m+1}_w(\text{Poly}(\mathbb{R}^n)^*)\) holds. Thus we have the well-defined cohomology group for each weight;

\[
\overline{H}_w^m(\text{Poly}(\mathbb{R}^n)) := \text{Ker}(d) : \Lambda^m_w(\text{Poly}(\mathbb{R}^n)^*) \to \Lambda^{m+1}_w(\text{Poly}(\mathbb{R}^n)^*)) / d(\Lambda^{m-1}_w(\text{Poly}(\mathbb{R}^n)^*)\).

**Proof:** From the linearity of \(d\), it is enough only to check \(d(\sigma) \in \Lambda^{m+1}_w(\text{Poly}(\mathbb{R}^n)^*)\) for any generator \(\sigma = \sigma_1 \wedge \cdots \wedge \sigma_m \in \Lambda^m_w(\text{Poly}(\mathbb{R}^n)^*)\) where \(\sigma_i \in \overline{\mathbb{E}}_{\phi(i)}\) for \(i = 1, \ldots, m\). From the definition, \(w = \sum_{i=1}^m \text{wt}(\sigma_i)\) where we denote the weight of \(\sigma_i\) by \(\text{wt}(\sigma_i)\), namely \(\text{wt}(\sigma_i) = \phi(i) + h - 2\). If \(f \in \overline{\mathbb{E}}_a\) and \(g \in \overline{\mathbb{E}}_b\), then we have \(\{f, g\}_\pi \in \overline{\mathbb{E}}_{a+b+h-2}\) because the Poisson structure \(\pi\) is \(h\)-homogeneous. From (3.14) we see that \(d(\sigma_i) \in d(\overline{\mathbb{E}}_{\phi(i)}) \subset \sum_{a \leq b, a+b = \phi(i)-h+2} \overline{\mathbb{E}}_a \wedge \overline{\mathbb{E}}_b\), and we have

\[\text{wt}(d(\sigma_i)) = (a + h - 2) + (b + h - 2) = \phi(i) + h - 2 = \text{wt}(\sigma_i).\]

Thus,

\[\text{wt}(\sigma_1 \wedge \cdots \wedge \sigma_{i-1} \wedge d(\sigma_i) \wedge \sigma_{i+1} \wedge \cdots \wedge \sigma_m) = \text{wt}(\sigma_1) + \cdots + \text{wt}(\sigma_{i-1}) + \text{wt}(d(\sigma_i)) + \text{wt}(\sigma_{i+1}) + \cdots + \text{wt}(\sigma_m) = \text{wt}(\sigma) = w\]

and we conclude \(\text{wt}(d(\sigma)) = \text{wt}(\sigma)\).

Also, the weight is preserved by the boundary operator \(\partial\) and we have the following Proposition.

**Proposition 3.2** The boundary operator \(\partial\) on Poly(\(\mathbb{R}^n\)) preserves the weight, namely, \(\partial(\Lambda^m_w(\text{Poly}(\mathbb{R}^n))) \subset \)
Λ_{m-1}^{w}(Poly(\mathbb{R}^n)) holds. Thus we have the well-defined homology group for each weight;

\[ \overline{H}_{m,w}(Poly(\mathbb{R}^n)) := \text{Ker}(\partial : \Lambda_{w}^{m}(Poly(\mathbb{R}^n)) \to \Lambda_{w}^{m-1}(Poly(\mathbb{R}^n))/\partial(\Lambda_{w}^{m+1}(Poly(\mathbb{R}^n)))) . \]

### 4 Lie algebra (co)homology of Hamiltonian vector fields of polynomial potentials

For a general Poisson structure \( \pi \) on a manifold \( M \), the Lie algebra of Hamiltonian vector fields is identified with \( C^\infty(M)/\text{Cent}_\pi \) where \( \text{Cent}_\pi \) is the space of Casimir functions. The Lie bracket is given by

\[ [[f, g]] = \{[f, g]_\pi\} \text{ where } f, g \in C^\infty(M) \text{ and } [f] = f + \text{Cent}_\pi \in C^\infty(M)/\text{Cent}_\pi. \]

For a given homogeneous Poisson structure \( \pi \) of \( \text{Poly}(\mathbb{R}^n) \), the Lie algebra \( g \) of Hamiltonian vector fields with polynomial potentials is identified as \( g \cong \text{Poly}(\mathbb{R}^n)/\text{Cent}_\pi \), where \( \text{Cent}_\pi = \{f \in \text{Poly}(\mathbb{R}^n) \mid \{f, \cdot\}_\pi = 0\} \) is the space of Casimir polynomials. First we prepare a small lemma:

**Lemma 4.1** Let \( f \) be a Casimir polynomial and let \( f_j \) be the \( j \)-homogeneous part of \( f \), i.e., \( f_j \in \overline{S}_j \) and so \( f = \sum_j f_j \). Then each \( f_j \) are Casimir polynomials. (The converse is obviously true.)

The Lemma above implies \( \text{Cent}_\pi j = \{f_j \mid f \in \text{Cent}_\pi\} \). Since \( \overline{S}_0 \subset \text{Cent}_\pi \), we have

\[ g \cong \text{Poly}(\mathbb{R}^n)/\text{Cent}_\pi = \sum_{j=0}^{\infty} \overline{S}_j/\text{Cent}_\pi . \]

The dual space \( g^* \) is decomposed as \( g^* = \sum_{j=1}^{\infty} \overline{\mathcal{E}}_j \) where \( \overline{\mathcal{E}}_j = \{\sigma \in \overline{\mathcal{E}}_j \mid \langle \sigma, \text{Cent}_\pi \rangle = 0\} \). Thus, the \( m \)-th cochain or chain spaces with the weight \( w \) of \( g \) are given by

\[ C_m^{w}(\mathfrak{g}) = \sum_{j=1}^{\infty} \Lambda^{k_1} \overline{\mathcal{E}}_1 \otimes \Lambda^{k_1} \overline{\mathcal{E}}_2 \otimes \cdots \otimes \Lambda^{k_1} \overline{\mathcal{E}}_\ell \]

\[ C_{m,w}(\mathfrak{g}) = \sum_{j=1}^{\infty} \Lambda^{k_1} S_1 \otimes \Lambda^{k_1} S_2 \otimes \cdots \otimes \Lambda^{k_1} S_\ell \]

with (3.6) and (3.7). About dimensional restriction, we only say that

\[ 0 \leq k_j \leq \dim \overline{\mathcal{E}}_j = \dim \overline{\mathcal{E}}_j - \#\{\text{linear independent } j\text{-homogeneous Casimir polynomials}\} . \]

From the definition of coboundary operators, we see the next property: for each 1-cochain \( \sigma \) of \( \mathfrak{g} \),

\[ (d\sigma)([f], [g]) = -\langle \sigma, [\{f, g\}_\pi]\rangle = -\langle \sigma, \{f, g\}_\pi + \text{Cent}_\pi \rangle \]
\[ = -\langle \sigma, \{f, g\}_\pi \rangle = (\bar{d}\sigma)(f, g). \]

Using the derivation property of coboundary operator, we get

\[ (d\sigma)([f_0], [f_1], \ldots, [f_k]) = (\bar{d}\sigma)(f_0, f_1, \ldots, f_k) \text{ for } \sigma \in C^k. \]  

In the sense above, we may say that \( d \) is the restriction of \( \bar{d} \). Thus, we have the next Proposition.

**Proposition 4.1** \( d \) is the natural restriction of \( \bar{d} \) and we have the following commutative diagram

\[
\begin{array}{ccc}
C^m = \Lambda^m \mathfrak{g}^* & \subset & \Lambda^m (\text{Poly}(\mathbb{R}^n)^*) \\
\downarrow d & & \downarrow \bar{d} \\
C^{m+1} = \Lambda^{m+1} \mathfrak{g}^* & \subset & \Lambda^{m+1} (\text{Poly}(\mathbb{R}^n)^*)
\end{array}
\]

The coboundary operator \( d \) preserves the weight, namely, \( d(C^m_w) \subset C^{m+1}_w \) holds. Thus we have the cohomology group

\[ H^m_w(\mathfrak{g}) := \text{Ker}(d : C^m_w \to C^{m+1}_w)/d(C^{m-1}_w). \]

As a direct corollary of Proposition 4.1, we have

**Corollary 4.2** \( H^*_w(\mathfrak{g}) = \left( C^*_w \cap \text{ker}(\bar{d} : \Lambda^*_{\mathfrak{g}}(\text{Poly}(\mathbb{R}^n)^*), \Lambda^{*+1}_{\mathfrak{g}}(\text{Poly}(\mathbb{R}^n)^*)) \right) / d(C^{*-1}_w). \)

**Example 4.1** In the case \( \pi = \partial_1 \land \partial_2 \) on \( \mathbb{R}^3 \), i.e., when \( h = 0 \), we see the decomposition of \( \mathfrak{g} \). By direct calculation, we know \( \text{Cent}_\pi \cap \overline{S}_j = \{ \mathbb{R}x_3^j \} \), and

\[
\begin{align*}
\overline{S}_0/\text{Cent}_\pi = 0 & , \quad \overline{S}_1/\text{Cent}_\pi = \text{LSpan}(x_1, x_2) , \quad \overline{S}_2/\text{Cent}_\pi = \text{LSpan}(x_1^2, x_2^2, x_1 x_2, x_1 x_3, x_2 x_3) , \\
\overline{S}_j/\text{Cent}_\pi \equiv \overline{S}_j \backslash \{ \mathbb{R}x_3^j \} & \text{ (} j = 1, 2, \ldots \),
\end{align*}
\]

and so

\[ \overline{S}_j = \text{LSpan}(z_A \mid |A| = j \text{ and } a_3 \neq j). \]

### 4.1 Possibility of weight and range of degree in Hamiltonian case

By the same argument in § 3.2, we have

| homogeneity | weight | range of \( m \) of \( m \)-th cochains |
|-------------|--------|-----------------------------------|
| \( h = 0 \) | \( w \geq - \dim \overline{S}_1 \) | \( \max(0, -w) \leq m \leq w + 2 \dim \overline{S}_1 + \dim \overline{S}_2 \) |
| \( h = 1 \) | \( w \geq 0 \) | \( 0 \leq m \leq w + \dim \overline{S}_1 \) |
| \( h \geq 2 \) | \( w \geq (h - 1) \dim \overline{S}_1 \) | \( 0 \leq m \leq w \) |
where \( \dim \mathcal{S}_1 \leq n \) and \( \dim \mathcal{S}_2 \leq n(n + 1)/2 \), and depend on the existence of Casimir polynomials.

### 4.2 Finding a basis of \( \overline{S}_j/\text{Cent}_\pi j \) by Groebner basis theory

If we deal with the algebra of Hamiltonian vector fields, it seems hard to say which of homology or cohomology is easier, faster, lighter or better. We prepare the vector subspace \( \text{Cent}_\pi j \) of Casimir polynomials in \( \overline{S}_j \). Let \( f_1, \ldots, f_k \) be a basis of \( \text{Cent}_\pi j \). Since \( \mathcal{S}_j = \{ \sigma \in \overline{S}_j \mid \langle \sigma, \text{Cent}_\pi j \rangle = 0 \} \), take a generic element \( \sigma = \sum_A c_A z_A \in \overline{S}_j \) and solve the linear equations \( \sum_A c_A z_A, f_i \rangle = 0 \) for \( i = 1, \ldots, k \). Substituting the solution to \( \sigma \), we get a basis of \( \mathcal{S}_j \).

For \( S_j = \overline{S}_j/\text{Cent}_\pi j \), conceptually we know well about the structure of the quotient vector space. But it is not so easy to fix a basis of \( \overline{S}_j/\text{Cent}_\pi j \) in general. It is well-known that if there is a linear map \( P : \overline{S}_j \rightarrow \overline{S}_j \) with \( P^2 = P \) and \( P^{-1}(0) = \text{Cent}_\pi j \). Then the subspace \( P(\overline{S}_j) \subset \overline{S}_j \) is isomorphic with \( \overline{S}_j/\text{Cent}_\pi j \) by \( P(f) \mapsto f + \text{Cent}_\pi j \).

Now we make use of Groebner basis theory and observe that the normal form \( \phi \) with respect to some basis \( f_1, \ldots, f_k \) of \( \text{Cent}_\pi j \) satisfies the property above and is the projection we want.

#### 4.2.1 Remainder modulo Casimir polynomials

Let \( \overline{S}_j \) be the vector space of homogeneous \( j \)-polynomials of \( x_1, \ldots, x_n \), i.e., \( \overline{S}_j \) is generated by \( \{ x^A \mid |A| = j \} \). Let \( f_1, \ldots, f_k \) be a basis of \( \text{Cent}_\pi j \). To apply Groebner basis theory, we fix a term order \( x_1 > \cdots > x_n \), for instance, the graded reverse lexicographic order. By the notation \( \text{LM}(\cdot) \), we mean the leading monomial with respect to our term order. If \( g_1 \) and \( g_2 \) are linearly independent and \( \text{LM}(g_1) = \text{LM}(g_2) \), then \( h_1 = c_1 g_1 + c_2 g_2 \) and \( h_2 = c_2 g_1 - c_1 g_2 \) are again linearly independent, and \( \text{LM}(h_1) = \text{LM}(g_1) = \text{LM}(g_2) \) and \( \text{LM}(h_2) < \text{LM}(g_1) = \text{LM}(g_2) \), where \( g_1 = c_1 \text{LM}(g_1) + \cdots \) and \( g_2 = c_2 \text{LM}(g_2) + \cdots \). Thus, we may assume that \( \text{LM}(f_1) > \cdots > \text{LM}(f_k) \) and each \( f_i \) contains only \( \text{LM}(f_i) \) among \( \{ \text{LM}(f_i) \mid \ell = 1..k \} \) after changing a basis by a suitable linear transformation if necessary. This means we take the Groebner basis of \( \text{Cent}_\pi j \).

For each \( g \in \overline{S}_j \), we recall the division algorithm with respect to \( f_1, \ldots, f_k \):

\[
g = \sum_{i=1}^{k} a_i f_i + r
\]

with the next two conditions:

(a) if \( r \neq 0 \), each term of \( r \) is not multiple of \( \text{LM}(f_i) \) for each \( i \), and

(b) if \( a_i \neq 0 \) then \( \text{LM}(a_i f_i) \leq \text{LM}(g) \).
\( r \) is called the remainder or the normal form of \( g \) with respect to \( f_1, \ldots, f_k \).

In our case, the situation is very simple and we have the next Lemma:

**Lemma 4.2** The remainder \( r \) of a given \( j \)-homogeneous polynomial \( g \) with respect to \( f_1, \ldots, f_k \) is obtained as

\[
(4.6) \quad r = \sum_{x^A \notin \{\text{LM}(f_i) \mid i = 1..k \}} c_A x^A \quad \text{where } c_A \text{ are constants},
\]

and

\[
(4.7) \quad g = \sum_{i=1}^{k} a_i f_i + r \quad \text{where } a_i \text{ are constants}.
\]

The constants \( a_i \) and \( c_A \) with \( x^A \notin \{\text{LM}(f_i) \mid i = 1..k \} \) are uniquely determined by \( g \) (depending on the term order). If we denote the remainder \( r \) by \( \phi(g) \), then \( \phi^2 = \phi \) and \( \phi^{-1}(0) = \text{LSpan}(f_1, \ldots, f_k) \).

**Proof:** In our case, since \( g \) and \( f_i \) are homogeneous with the same degree \( j \), the condition (b) says \( a_i \) must be constant. Thus (4.5) yields that \( r \) is also the same \( j \)-homogeneous. The condition (a) yields that (4.6).

Since in the space \( \overline{S}_j \), \( \{\text{LM}(f_i) \mid i = 1..k \} \) and \( \{x^A \notin \{\text{LM}(f_i) \mid i = 1..k \} \} \) give a basis, and we see that

\[
g = \sum_i p_i \text{LM}(f_i) + \sum_{x^A \notin \{\text{LM}(f_i) \mid i = 1..k \}} b_B x^B = \sum_{i} a_i f_i + \sum_{x^A \notin \{\text{LM}(f_i) \mid i = 1..k \}} c_B x^B.
\]

Lemma 4.2 implies directly the next proposition.

**Proposition 4.3** For each \( f \in \overline{S}_j \), \( \phi(f) = 0 \) if and only if \( f \in \text{Cent}_\pi j \). Let \( \{x^A \mid |A| = j \} \) be the natural basis of \( \overline{S}_j \). Then \( \{x^A \in \overline{S}_j \mid \phi(x^A) = x^A \} \) is a basis of \( S_j = \phi(\overline{S}_j) \subset \overline{S}_j \). Also \( \{x^A \mid |A| = j \} \setminus \{\text{LM}(f_i) \mid f_i \in \text{Cent}_\pi j \} \) gives the same basis of \( S_j = \phi(\overline{S}_j) \subset \overline{S}_j \), where we assumed that \( \{f_i\} \) is a basis of \( \text{Cent}_\pi j \) with mutually distinct leading monomials \( \{\text{LM}(f_i)\} \).

For \( f \in S_i \) and \( g \in S_j \), the Lie bracket is given by \([f, g] = \phi([f, g]_\pi)\), where \( \phi \) is now the normal form with respect to \( \text{Cent}_\pi i + j - 2 + h \), here \( h \) is the homogeneity of the Poisson structure \( \pi \). Thus, the boundary operator \( \partial \) of \( g \) is given by

\[
(4.8) \quad \partial &\wedge(f_1, \ldots, f_k) = \sum_{i<j} (-1)^{i+j} \wedge(\phi([f_i, f_j]_\pi), \ldots, \overline{f_i}, \ldots, \overline{f_j}, \ldots)
\]

where \( f_i \in S_{r(i)} \subset \overline{S}_{r(i)} \).

**Remark 4.1** In the above, we do not miss \( \phi \)-projection. See (4.9) in the next example.
Example 4.2 Take a Lie Poisson structure \( sl(2) \) as a concrete example.

\[ \{ p, q \}_\pi = r, \quad \{ r, p \}_\pi = 2p, \quad \{ r, q \}_\pi = -2q. \]

Then we have

\[
\text{Cent}_\pi 2j - k = \begin{cases} 
\emptyset & \text{if } k = 1, \\
\mathbb{R} \cdot (4pq + r^2)^j & \text{if } k = 0.
\end{cases}
\]

Let \( x^A = p^{a_1} q^{a_2} r^{a_3} \) and \( z_A \) are the dual. Let \( \phi \) be the normal form \( \phi \) with respect to \( 4pq + r^2 \) under the term order \( p > q > r \). Then \( \phi(x^{1,1,0}) = -\frac{1}{4} x^{0,0,2} \) and \( \phi(x^A) = x^A \) for \( A \neq [1, 1, 0] \). We get a basis \( x^A \) \( (A \neq [1, 1, 0]) \) of \( S_2 = \phi(S_2) \). Since \( \text{Cent}_\pi 1 = 0 \), we see \( S_1 = \overline{S_1} \). Take \( f = x_1 \in S_1 \) and \( g = x_2 x_3 \in S_2 \), then

\[
(4.9) \quad \{ f, g \}_\pi = x_3^2 - 2x_1 x_2 \notin \phi(S_2), \quad \text{and} \quad \phi(\{ f, g \}_\pi) = \frac{3}{2} x_3^2 \in \phi(S_2).
\]

About \( \sum_{|A|=2} c_A z_A \in \mathcal{S}_2 \subset \overline{\mathcal{S}_2} \),

\[
0 = \langle \sum_{|A|=2} c_A z_A, 4pq + r^2 \rangle = 4c_{1,1,0} + c_{0,0,2}
\]

implies

\[
\sum_{|A|=2} c_A z_A = \sum_{a_1 \neq a_2} c_A z_A - \frac{c_{0,0,2}}{4} z_{1,1,0} + c_{0,0,2} z_{0,0,2} \quad \text{or} \quad \sum_{a_1 \neq a_2} c_A z_A + c_{1,1,0} z_{1,1,0} - 4c_{1,1,0} z_{0,0,2}.
\]

Thus, (natural) bases of \( \mathcal{S}_2 \) are given by

\[
z_A \ (a_1 \neq a_2) \quad \text{and} \quad -\frac{1}{4} z_{1,1,0} + z_{0,0,2} \quad \text{or} \quad z_A \ (a_1 = a_2) \quad \text{and} \quad z_{1,1,0} - 4z_{0,0,2}.
\]

4.2.2 Easy way to get the remainder

Even though the most polite way is using the normal form of Groebner basis theory, in our case, we may use the following formula due to Lemma 4.2:

\[
(4.10) \quad \phi(g) = g - \sum_i \frac{\partial^A g}{\partial^A_i f_i}
\]

where \( A_i \) is given by \( x^{A_i} = \text{LM}(f_i) \) and \( \partial^A = (\frac{\partial}{\partial x_1})^{a_1} \cdots (\frac{\partial}{\partial x_n})^{a_n} \). In practical calculation, (4.10) seems faster than the normal form.

Example 4.3 We deal with Example 4.2 with 3 variables \( x_1, x_2, x_3 \) with the graded reverse lexicographic
order as the term order. Consider 2-homogeneous space $S_2$ and the subspace $\{f_1 = 4x_1x_2 + x_3^2\}$. Now $\text{LM}(f_1) = x_1x_2$ and so the normal form of $g$ is given by
\[
\sum_{A \neq [1,1,0]} c_A x^A \text{ of }
g = a f_1 + \sum_{A \neq [1,1,0]} c_A x^A = 4a x^{[1,1,0]} + (a + c_{[0,0,2]}) x^{[0,0,2]} + \sum_{A \neq [1,1,0],[0,0,2]} c_A x^A.
\]
(4.11)

When $g = x_1x_2$, then (4.11) implies $4a = 1$, $c_A = 0$ ($A \neq [1,1,0],[0,0,2]$) and $a + c_{[0,0,2]} = 0$, thus $\phi(x_1x_2) = -\frac{1}{4}x_3^2$.

When $g = x_3^2$, then (4.11) implies $4a = 0$, $c_A = 0$ ($A \neq [1,1,0],[0,0,2]$) and $a + c_{[0,0,2]} = 1$, thus $\phi(x_3^2) = x_3^2$.

If we use the formula (4.10), $\phi(g)$ is given by
\[
\phi(g) = g - \frac{\partial_1 \partial_2 g}{4}(4x_1x_2 + x_3^2)
\]
and $\phi(x_1x_2) = x_1x_2 - \frac{1}{4}(4x_1x_2 + x_3^2) = -\frac{x_3^2}{4}$, $\phi(x_3^2) = x_3^2 - 0(4x_1x_2 + x_3^2) = x_3^2$, and so on.

Another easy way to get the remainder in our case is the following:

1. Take $\text{Cent}_j$. Let $\{f_\ell \mid \ell = 1..k\}$ be a basis with the properties described before.

2. Get the leading monomials of $\text{Cent}_j$, $\text{LMCasim} = \{\text{LM}(f_\ell) \mid \ell = 1..k\}$

   Now, $\phi(x^A) = x^A$ for $x^A \notin \text{LMCasim}$.

3. To get $\phi(\text{LM}(f_1))$, we use $\phi(f_1) = 0$. Since $\phi$ is $\mathbb{R}$-linear and satisfies the property above, we get the expression of $\phi(\text{LM}(f_1))$ by $\mathbb{R}$-linear combination of $x^A \notin \text{LMCasim}$.

**Example 4.4** Again, we deal with the Lie Poisson modeled by $\mathfrak{sl}(2)$ with 3 variables $x_1, x_2, x_3$ with the graded reverse lexicographic order as the term order. Consider 2-homogeneous space $\overline{S}_2$ and the Casimir subspace is given by $\{f_1 = 4x_1x_2 + x_3^2\}$. Now $\text{LM}(f_1) = x_1x_2$ and so $\phi(x^A) = x^A$ unless $a_1 = a_2 = 1$ where $\phi$ is the normal form with respect to the Casimir subspace. Since
\[
0 = \phi(f_1) = 4\phi(x_1x_2) + \phi(x_3^2) = 4\phi(x_1x_2) + x_3^2,
\]
we get $\phi(x_1x_2) = -\frac{1}{4}x_3^2$. 

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4.3 Fixing the dual basis of $\overline{S}_j/\text{Cent}_\pi j$ in $\Xi_j$

Let $\phi$ be the normal form with respect to $\text{Cent}_\pi j$ on $\overline{S}_j$. Using the natural basis $x^A$ with $|A| = j$ in $\overline{S}_j$, we consider $\tau = \sum_{|A| = j} \phi(x^A) \otimes z_A$. We have known that $\{x^B \in \overline{S}_j \mid \phi(x^B) = x^B\}$ is a basis of $\overline{S}_j/\text{Cent}_\pi j$ in Proposition 4.3, and the $x^B$-coefficient of $\tau$ with $\{x^B \in \overline{S}_j \mid \phi(x^B) = x^B\}$ form a basis of $\Xi_j$.

**Example 4.5** Again, we deal with Example 4.2 with 3 variables $x_1, x_2, x_3$ with the graded reverse lexicographic order as the term order. Consider 2-homogeneous space $\overline{S}_2$ and the subspace $\{f_1 = 4x_1x_2 + x_3^2\}$. Now $\text{LM}(f_1) = x_1x_2$ and the normal forms are given by $\phi(x_1x_2) = -\frac{1}{4}x_3^2$ and $\phi(x^B) = x^B$ for $|B| = 2$ and $B \neq [1, 1, 0]$. Then we have $\tau = \sum_{a_1 \neq a_2} x^A \otimes z_A + (-\frac{1}{4}x_3^2) \otimes z_{1,1,0} + (x_3^2) \otimes z_{0,0,2}$ and we get a basis of $\Xi_2$:

$$z_A (|A| = 2, a_1 \neq a_2), -\frac{1}{4}z_{1,1,0} + z_{0,0,2}.$$ 

**Example 4.6** Consider $\mathfrak{sl}(2)$ in $\mathbb{R}^4$, i.e., the Poisson bracket is given by

$$\{x_1, x_2\}_\pi = x_3 = -\{x_2, x_1\}_\pi, \{x_1, x_3\}_\pi = -2x_1 = -\{x_3, x_1\}_\pi, \{x_2, x_3\}_\pi = 2x_2 = -\{x_3, x_2\}_\pi,$$

$$\{x_1, x_4\}_\pi = \{x_2, x_3\}_\pi = 0 \quad (i = 1, 2, 3).$$

Then $\text{Cent}_\pi$ is rather complicated and generated by $f(4x_1x_2 + x_3^2)g(x_4)$ where $f, g$ are polynomials of one variable. Thus, $\text{Cent}_\pi j$ are given as

$$\text{Cent}_\pi 1 = [x_4], \text{Cent}_\pi 2 = [4x_1x_2 + x_3^2, x_4^2], \text{Cent}_\pi 3 = [(4x_1x_2 + x_3^2)x_4, x_4^3],$$

$$\text{Cent}_\pi 4 = [(4x_1x_2 + x_3^2)^2, (4x_1x_2 + x_3^2)x_4, x_4^4], \ldots$$

The leading monomials of $\text{Cent}_\pi j$ are

$$\text{LM}(\text{Cent}_\pi 1) = [x_4], \text{LM}(\text{Cent}_\pi 2) = [x_1x_2, x_4^2], \text{LM}(\text{Cent}_\pi 3) = [x_1x_2x_4, x_4^3],$$

$$\text{LM}(\text{Cent}_\pi 4) = [(x_1x_2)^2, (x_1x_2)x_4^2, x_4^4], \ldots$$

and so we have bases of $S_j$ as follows:

basis of $S_1 = [x_1, x_2, x_3]$, basis of $S_2 = [x^A \mid |A| = 2] \setminus [x_1x_2, x_4^2]$, 

basis of $S_3 = [x^A \mid |A| = 3] \setminus [x_1x_2x_4, x_4^3]$, 

basis of $S_4 = [x^A \mid |A| = 4] \setminus [(x_1x_2)^2, (x_1x_2)x_4^2, x_4^4], \ldots$
Concerning to find a basis of $\mathfrak{G}_j$, we need information about the normal form $\phi$ with respect to $\text{Cent}_\pi j$.

When $\mathfrak{G}_1$, we see that $\phi(x^i) = x^i$ for $i = 1, 2, 3$ and $\phi(x^{\ell}) = 0$. Thus, $\tau = \sum_{i=1}^{3} x^i \otimes z_i$ and we get a basis $\{z_i \mid i = 1, 2, 3\}$ on $\mathfrak{G}_1$.

When $\mathfrak{G}_2$, we get $\phi(x_1x_2) = -\frac{1}{4} x_3^2$, $\phi(x_1^2) = 0$ and the other monomials are invariant by $\phi$. Now

$$\tau = \sum_{|A|=2} \phi(x^A) \otimes z_A = \sum_{A \in \{1, 1, 0, 0\}, \{2, 0, 0, 2\}} x^A \otimes z_A + (\frac{1}{4} x_3^2) \otimes z_{[1,1,0,0]}$$

and so we get a basis of $\mathfrak{G}_2$

$$z_A (A \neq [1, 1, 0, 0], [0, 0, 2, 0], [0, 0, 0, 2]) \quad \text{and} \quad -\frac{1}{4} z_{[1,1,0,0]} + z_{[0,0,2,0]}.$$

By the same way, about $\mathfrak{G}_3$ we have a basis

$$z_A (A \neq [0, 0, 0, 3], [1, 1, 0, 1], [0, 0, 2, 1]) \quad \text{and} \quad -\frac{1}{4} z_{[1,1,0,1]} + z_{[0,0,2,1]}.$$

As a basis of $\mathfrak{G}_4$ we have

$$z_A (A \neq [0, 0, 0, 4], [0, 0, 2, 2], [1, 1, 0, 2], [0, 0, 4, 0], [1, 1, 2, 0], [2, 2, 0, 0])$$

and

$$\frac{1}{2} z_{[2,2,0,0]} + z_{[1,1,2,0]}, \quad -\frac{1}{4} z_{[1,1,0,2]} + z_{[0,0,2,2]}, \quad -\frac{1}{16} z_{[2,2,0,0]} + z_{[0,0,4,0]}.$$

**Remark 4.2** About some comparison of the effort of using our method or using the normal form in order to get the remainder, we refer to the subsection 6.3. There, we got data of elapsed time for concrete Poisson structures.

## 5 Subalgebra $\overline{\mathfrak{g}}$ excluding constant polynomials

$$\sum_{j=\ell}^{\infty} \overline{S}_j$$

is a subalgebra of $\text{Poly}(\mathbb{R}^n)$ for each $\ell \geq 1$ when the homogeneity of Poisson structure $\pi$ is $\geq 1$.

and when the homogeneity is $0$, $\sum_{j=\ell}^{\infty} \overline{S}_j$ is a subalgebra for $\ell \geq 2$. The notion of weight is valid on those subalgebras and we have the weight decomposition of cochain complex of $\overline{\mathfrak{g}} = \sum_{j=1}^{\infty} \overline{S}_j$ when $h > 0$ and $\overline{\mathfrak{g}} = \sum_{j=2}^{\infty} \overline{S}_j$ when $h = 0$. We denote by $\overline{C}_m^\ell$ the $m$-th cochain space with the weight $w$, namely

$$\overline{C}_m^\ell = \sum \Lambda^{k_\ell} \overline{\mathfrak{g}}_0 \otimes \Lambda^{k_1} \overline{\mathfrak{g}}_1 \otimes \cdots \otimes \Lambda^{k_{\ell}} \overline{\mathfrak{g}}_\ell$$

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with the condition (3.6), (3.7) and (3.8) with the additional restriction \( k_0 = k_1 = 0 \) if \( h = 0 \) and \( k_0 = 0 \) if \( h > 0 \).

### 5.1 Possibility of weight and range of degree when excluding constant polynomials

We follow the discussion in the subsection 3.2 with the restriction \( k_0 = 0 \) if \( h > 0 \) or \( k_0 = k_1 = 0 \) if \( h = 0 \), we see that

| homogeneity | weight | range of \( m \) of \( m \)-th cochains |
|--------------|--------|-----------------------------------------|
| \( h = 0 \)  | \( w \geq 0 \) | \( 0 \leq m \leq w + n(n + 1)/2 \) |
| \( h = 1 \)  | \( w \geq 0 \) | \( 0 \leq m \leq w + n \) |
| \( h > 1 \)  | \( w \geq 0 \) | \( 0 \leq m \leq w \) |

#### 5.2 \( \mathcal{g} \) and \( \text{Poly}(\mathbb{R}^n) \) when homogeneity \( h > 0 \)

The dimensional condition (3.8) yields \( k_0 = 0 \) or 1. Thus, we can decompose (3.4) as follows:

\[
\Lambda^m_w(\text{Poly}(\mathbb{R}^n)^*) = \sum_{k_0=0}^{\infty} \Lambda^{k_0} \overline{\mathcal{g}}_1 \otimes \cdots \otimes \Lambda^{k_0} \overline{\mathcal{g}}_\ell \oplus \sum_{k_0=1}^{\infty} \overline{\mathcal{g}}_0 \otimes \Lambda^{k_0} \overline{\mathcal{g}}_1 \otimes \cdots \otimes \Lambda^{k_0} \overline{\mathcal{g}}_\ell.
\]

The first term of the direct sum is the \( m \)-th cochain space \( \overline{\mathcal{C}}_w^m \) of \( \mathcal{g} \). About the second term, (3.6) implies \( \sum_{j=1}^{\infty} k_j = m - 1 \), and (3.7) implies \( \sum_{j=1}^{\infty} k_j(j - 2 + h) = w + (2 - h) \), thus, the second term coincides with \( \overline{\mathcal{C}}_0 \otimes \overline{\mathcal{C}}_{w+2-h}^{m-1} \), namely

\[
\Lambda^m_w(\text{Poly}(\mathbb{R}^n)^*) = \overline{\mathcal{C}}_w^m \oplus \overline{\mathcal{C}}_0 \otimes \overline{\mathcal{C}}_{w+2-h}^{m-1}.
\]

Similarly, we have

\[
\Lambda^m_w(\text{Poly}(\mathbb{R}^n)) = \overline{\mathcal{C}}_{m,w}^m \oplus \overline{\mathcal{C}}_0 \otimes \overline{\mathcal{C}}_{m-1,w+2-h}^{m-1}.
\]

Since we know that \( d(\overline{\mathcal{g}}_0) = 0 \) and \( \partial(\wedge^1(1, u_2, \ldots, u_m)) = -1 \wedge \partial(\wedge^1(u_2, \ldots, u_m)) \), we have

**Proposition 5.1** When \( h > 0 \), the followings hold

\[
\overline{\mathcal{H}}^m_w(\text{Poly}(\mathbb{R}^n)) \cong \overline{\mathcal{H}}^m_w(\mathcal{g}) \oplus \overline{\mathcal{H}}_{w+2-h}^{m-1}(\mathcal{g})
\]

\[
\overline{\mathcal{H}}_{m,w}(\text{Poly}(\mathbb{R}^n)) \cong \overline{\mathcal{H}}_{m,w}(\mathcal{g}) \oplus \overline{\mathcal{H}}_{m-1,w+2-h}(\mathcal{g})
\]

The proposition above says that it is enough to handle \( \overline{\mathcal{H}}^m_w(\mathcal{g}) \) to study \( \mathcal{H}^m_w(\text{Poly}(\mathbb{R}^n)) \). Thus, from now on we only deal with \( \mathcal{g} \) and the cochain complex \( \overline{\mathcal{C}}_w^m(\mathcal{g}) \) (sometimes simply denote \( \overline{\mathcal{C}}_w^m \)) or the chain complex
\( \overline{C}_{m,w}(\overline{g}) \) (or \( \overline{C}_{m,w} \)). We rewrite Proposition 4.1 by using the words of \( \overline{g} \) as follow:

**Proposition 5.2** When \( h > 0 \), \( d \) is the natural restriction of \( \overline{d} \) and we have the following commutative diagram

\[
\begin{array}{ccc}
C^m = \Lambda^m \overline{g}^* & \subset & \overline{C}^m = \Lambda^m \overline{g}^* \\
d \downarrow & & \downarrow \overline{d} \\
C^{m+1} = \Lambda^{m+1} \overline{g}^* & \subset & \overline{C}^{m+1} = \Lambda^{m+1} \overline{g}^*.
\end{array}
\]

(5.5)

The coboundary operator \( d \) preserves the weight, namely, \( d(C^m_w) \subset C^{m+1}_w \) holds. Thus we have the cohomology group

\[
H^m_w(\overline{g}) := \text{Ker}(d : C^m_w \to C^{m+1}_w) / d(C^{m-1}_w).
\]

As a direct corollary of Proposition above, we have

**Corollary 5.3** When \( h > 0 \), \( H^*_w(\overline{g}) = \left( C^*_w \cap \ker(\overline{d} : \overline{C^*_w} \to \overline{C^*_w}^{+1}) \right) / d(C^{-1}_w) \).

### 5.3 Young diagrams and cochain spaces

The \( m \)-th cochain space of weight \( w \)

\[
\overline{C}^m_w = \sum \Lambda^{k_1} \overline{g} \otimes \Lambda^{k_2} \overline{g} \otimes \Lambda^{k_3} \overline{g} \otimes \cdots
\]

satisfies the following three conditions (3.6), (3.7) and (3.8) with \( k_0 = 0 \).

Take a sequence \([k_1, k_2, \ldots]\) satisfying the three conditions above. Then there exists an \( \ell \) so that \( k_j = 0 \) (\( j > \ell \)) and we may write \([k_1, k_2, \ldots] = [k_1, k_2, \ldots, k_\ell]\) as a finite sequence. The conditions (3.6) and (3.7) for \([k_1, k_2, \ldots, k_\ell]\) are equivalent to (3.6) and (3.10) with \( k_0 = 0 \). As observed in [9], \( (\ell, \ldots, \ell, \underbrace{1, \ldots, 1}_{k_\ell \text{-times}}, \underbrace{1, \ldots, 1}_{k_1 \text{-times}}) \) is a Young diagram (in traditional expression) whose length \( m \) and consisting of \((w + (2 - h)m)\) cells from (3.6) and (3.10).
Conversely, for a given Young diagram $\lambda$ (in traditional), define

$$k_i := \# \{ j \mid \lambda_j = i \} = \text{the number of rows width } i \text{ in } \lambda.$$ 

Then $[k_1, k_2, \ldots]$ satisfies

$$k_1 + k_2 + \cdots + k_j + \cdots + k_\ell = \ell(\lambda) = \text{length of } \lambda,$$

and

$$k_1 + 2k_2 + \cdots + jk_j + \cdots + \ell k_\ell = |\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i = \text{the total area of } \lambda.$$ 

Thus, our cochain space $C^m_w$ corresponds to the Young diagrams of height $m$ and the weight $w + (2 - h)m$ under the dimensional condition.

### 5.4 Manipulation of Young diagrams

Let $\nabla^A_k$ be the set of Young diagrams whose total area (the number of cells) is $A$ and the height is $k$.

Especially, $\nabla^A_1 = \{ \square \ldots \square \square \}$ (the row of width $A$) and $\nabla^A_A = \{ \square \}$ (the column of height $A$). If $A \leq 0$ or $k \leq 0$ or $A < k$ then $\nabla^A_k = \emptyset$. Denoting the element of $\nabla^A_k$ by $T_A$, i.e., $\nabla^A_A = \{ T_A \}$, we have the following recursive formula;

$$\nabla^A_k = T_k \cdot (\nabla^0_{A-k} \sqcup \nabla^1_{A-k} \sqcup \cdots \sqcup \nabla^A_{A-k})$$

where “$\sqcup$” above means distributive concatenating operation of the tower $T_k$ and other Young diagrams. (In fact, the series of $\sqcup$ stops at min$(k, A-k)$, however, the above notation may not cause any confusion.) It is also convenient to regard $\nabla^0_A$ as the single set of the unital element and $\nabla^A_A$ ($A > 0$) or $\nabla^A_k$ ($A < k$ or $k < 0$) as the single set of the null element of the concatenation “$\sqcup$”. We see that $\nabla^A_1 = \{ T^1_A \}$. Using this operation, we can list up the elements of the set $\nabla^A_k$. For example, we have the following:

$$\nabla^5_3 = T_3 \cdot (\nabla^2_0 \sqcup \nabla^2_1 \sqcup \nabla^2_2 \sqcup \nabla^2_3) = T_3 \cdot (\nabla^2_0 \sqcup \nabla^2_2)$$

$$= T_3 \cdot T_1 \cdot (\nabla^1_0 \sqcup \nabla^1_1) \sqcup T_3 \cdot T_2 \cdot (\nabla^0_0 \sqcup \nabla^0_1 \sqcup \nabla^0_2)$$

$$= T_3 \cdot T_1 \cdot \nabla^1_1 \sqcup T_3 \cdot T_2 \cdot \nabla^0_0 = \{ T_3 \cdot T^2_1, T_3 \cdot T_2 \} = \{ \square \ldots \square \square \square \}. $$
If we decompose $A$ as $A = ak + b$ where $a > 0$ and $0 \leq b < k$, then we have

$$\nabla^A_k = T_k \bigcup_{j_a \leq \cdots \leq j_1 \leq k} T_{j_1} \cdots T_{j_{a-1}} \cdot \nabla^{A-k-\sum_{s=1}^{a-1} j_s}_{j_a}.$$  

In (5.6), we replace $A$ by $A + 1$ and $k$ by $k + 1$, and we have

$$\nabla^{A+1}_{k+1} = T_{k+1} \cdot (\nabla^{A-k}_0 \sqcup \nabla^{A-k}_1 \sqcup \cdots \sqcup \nabla^{A-k}_k \sqcup \nabla^{A-k}_{k+1}).$$

If we rewrite (5.6) formally

$$\nabla^{A-k}_0 \sqcup \nabla^{A-k}_1 \sqcup \cdots \sqcup \nabla^{A-k}_k = T_{k-1} \cdot \nabla^A_k$$

then we get another recursive formula;

(5.7)  

$$\nabla^{A+1}_{k+1} = T_{k+1} \cdot T_{k-1} \cdot \nabla^A_k \sqcup T_{k+1} \cdot \nabla^{A-k}_{k+1}) .$$

Denoting $T_{k-1} \cdot \nabla^A_k$ by $\hat{\nabla}^A_k$, we have another form

(5.8)  

$$\hat{\nabla}^{A+1}_{k+1} = \hat{\nabla}^A_k \sqcup T_{k+1} \cdot \hat{\nabla}^{A-k}_{k+1} .$$

$\hat{\nabla}^A_k$ satisfies

(5.9)  

$$\hat{\nabla}^k = \{id\} , \ \hat{\nabla}^k_1 = \{T_1^{k-1}\} , \ \hat{\nabla}^A_k = \{0\} \text{ if } A < k.$$  

We see how the formula (5.8) works on the same example:

$$\hat{\nabla}_3^5 = \hat{\nabla}_2^4 \sqcup T_3 \cdot \hat{\nabla}_3^3 = \hat{\nabla}_1^2 \sqcup T_2 \cdot \hat{\nabla}_2^2 = \{T_1^2, T_2\}$$

$$\nabla_3^5 = T_3 \cdot \hat{\nabla}_3^5 = T_3 \cdot \{T_1^2, T_2\} = \{T_3 T_1^2, T_3 T_2\} .$$

In (5.7), $T_{k+1} \cdot T_{k-1}^{-1}$ means adding one cell under the bottom of the left-most column of a Young diagram, we may another form of (5.7):

(5.10)  

$$\nabla^{A+1}_{k+1} = B \cdot \nabla^A_k \sqcup T_{k+1} \cdot \nabla^{A-k}_{k+1}$$

where $B$ is the operation adding one cell under the bottom of the left-most column of each Young diagram. Using (5.7), (5.8) or (5.10), we may write each Young diagram as $T_{\ell_1} \cdot T_{\ell_2} \cdots T_{\ell_{s-1}} \cdot T_{\ell_s}$ with $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_{s-1} \geq \ell_s > 0$. We may call it the decomposition by towers (or icicles) $\{T_j\}$.
For a given Young diagram \( \lambda \), to get its tower decomposition

\[
(5.11) \quad T_{\ell_1} \cdot T_{\ell_2} \cdots T_{\ell_{s-1}} \cdot T_{\ell_s}
\]

with \( \ell_1 \geq \ell_2 \geq \cdots \geq \ell_{s-1} \geq \ell_s > 0 \), is visually obvious, just slicing \( \lambda \) vertically. A mathematical formula is given by

\[
(5.12) \quad \ell_j = \#\{ i \mid \lambda_i - (j - 1) > 0 \} = \#\{ i \mid \lambda_i \geq j \} \quad (j = 1, \ldots, \lambda_1),
\]

and this says that the tower decomposition \((\ell_1, \ldots, \ell_s)\) is equal to the conjugate Young diagram of \( \lambda \).

**Lemma 5.1** The expression \([k_1, \ldots, k_p]\) in our notation of the Young diagram (5.11) is given by

\[
(5.13) \quad k_1 = \ell_1 - \ell_2, \quad k_2 = \ell_2 - \ell_3, \quad \ldots, \quad k_{s-1} = \ell_{s-1} - \ell_s, \quad k_s = \ell_s.
\]

**Proof:** Let us prepare a general Young diagram and slice it vertically. If we compare the height of the nearby two towers, then (5.13) is clear.

If you prefer a proof by induction, then follow the next: When \( s = 1 \), \( T_{\ell_1} \) is given by \( k_1 = \ell_1 \) and \( k_j = 0 \) for \( j > 1 \) and (5.13) holds.

When \( s = 2 \), \( T_{\ell_1} \cdot T_{\ell_2} \) is given by \( k_1 = \ell_1 - \ell_2 \), \( k_2 = \ell_2 \) and \( k_j = 0 \) for \( j > 2 \), and (5.13) holds.

Now assume (5.13) holds for general width \( s \) and consider width \( s + 1 \) YD \( T_{\ell_1} \cdots T_{\ell_s} T_{\ell_{s+1}} \). Let us take \( \lambda = T_{\ell_s} \cdots T_{\ell_2} T_{\ell_{s+1}} \). Then (5.13) says that

\[
k_1 = \ell_2 - \ell_3, \quad k_2 = \ell_3 - \ell_4, \quad \ldots, \quad k_{s-1} = \ell_{s-1} - \ell_s, \quad k_s = \ell_s + 1
\]

Then our notation of \( T_{\ell_s} \cdot \lambda \) is given by \( \bar{k}_1 = \ell_1 - \sum_{j=1}^{s} k_j = \ell_1 - \ell_2 \) and \( \bar{k}_j = k_{j-1} = \ell_j - \ell_{j+1} \) for \( j = 2, \ldots, s \) and \( \bar{k}_{s+1} = k_s = \ell_{s+1} \). Thus (5.13) holds for \( s + 1 \).

Solving (5.13) reversely, we have

\[
(5.14) \quad \ell_j = \sum_{i \geq j} k_i \quad (j = 1, 2, \ldots)
\]

**Corollary 5.4** If a Young diagram \( \lambda \) is denoted by \([k_1, \ldots, k_p]\), then \( T(m) \cdot \lambda \) is given by

\[
(5.15) \quad \bar{k}_1 = m - \sum_{j=1}^{p} k_j, \quad \bar{k}_2 = k_1, \quad \ldots, \quad \bar{k}_{p+1} = k_p
\]
**Proof:** Since \( \lambda = [k_1, \ldots, k_p] = T_{\ell_1} \cdots T_{\ell_p} \) with (5.14), \( T_m \cdot \lambda \) is given by \( T_m \cdot T_{\ell_1} \cdots T_{\ell_p} \). From (5.13), the notation is

\[
\bar{k}_1 = m - \ell_1 = m - \sum_{j \geq 1} k_j, \quad \bar{k}_2 = \ell_1 - \ell_2 - \sum_{j \geq 2} k_j = k_1,
\]

\[
\ldots, \quad \bar{k}_{p+1} = \ell_p - \ell_{p+1} - \sum_{j \geq p} k_j - \sum_{j \geq p+1} k_j = k_p.
\]

\[
\begin{align*}
\text{5.5} & \quad \textbf{Examples of decomposition of cochain complex according to weights} \\
& \quad \text{We will now show some examples of decomposition of a cochain complex according to weights.}
\end{align*}
\]

\[
\begin{align*}
\text{5.5.1} & \quad \text{case } h = 1 \\
& \quad \text{Since } h = 1, \text{ for a given weight } w, \text{ the } m\text{-th cochain space } C^m_w \text{ corresponds to the set } \nabla_{m}^{w+m} \text{ of Young diagrams of area } w + m \text{ and the height } m, \text{ and we have } m \leq w + n \text{ as in the table in § 3.2.}
\end{align*}
\]

If \( w = 0, \nabla_{m}^{w} = \{T_m\} \), this means \( k_1 = m \) and \( k_j = 0 (j > 1) \). Thus \( C^0_m = \Lambda^m \mathcal{S}_1 \).

If \( w = 1, \nabla_{m}^{1+m} = \{T_m \cdot T_1\} \), this means \( k_1 = m - 1, k_2 = 1 \) and \( k_j = 0 (j > 2) \). Thus \( C^1_m = \Lambda^{m-1} \mathcal{S}_1 \otimes \mathcal{S}_2 \).

If \( w = 2, \nabla_{m}^{2+m} = T_m \cdot (\nabla_1 \cup \nabla_2) = T_m \cdot \{T^1_2, T_1, T_2\} \), this means \( k_1 = m - 1, k_3 = 1 \) and \( k_j = 0 (j \neq 1, 3) \) or \( k_1 = m - 2, k_2 = 2 \) and \( k_j = 0 (j > 2) \). Thus \( C^2_m = \Lambda^{m-1} \mathcal{S}_1 \otimes \mathcal{S}_3 + \Lambda^{m-2} \mathcal{S}_1 \otimes \Lambda^2 \mathcal{S}_2 \).

If \( w = 3, \)

\[
\nabla_{m}^{3+m} = T_m \cdot (\nabla_1^3 \cup \nabla_2^3 \cup \nabla_3^3) = T_m \cdot \{T^3_1, T^2_2, T_1, T_3\} = \{T_m \cdot T^3_1, T_m \cdot T_2 \cdot T_1, T_m \cdot T_3\}
\]

this means \( k_1 = m - 1, k_4 = 1 \) and \( k_j = 0 (j \neq 1, 4) \), \( k_1 = m - 2, k_2 = 1, k_3 = 1 \) and \( k_j = 0 (j > 3) \) or \( k_1 = m - 3, k_2 = 3 \) and \( k_j = 0 (j > 2) \). Thus \( C^3_m = \Lambda^{m-1} \mathcal{S}_1 \otimes \mathcal{S}_4 + \Lambda^{m-2} \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \mathcal{S}_3 + \Lambda^{m-3} \mathcal{S}_1 \otimes \Lambda^3 \mathcal{S}_2 \).

Summarizing the expressions we got above,

\[
\begin{align*}
(5.16) & \quad C^0_m = \Lambda^m \mathcal{S}_1. \\
(5.17) & \quad C^1_m = \Lambda^{m-1} \mathcal{S}_1 \otimes \mathcal{S}_2, \\
(5.18) & \quad C^2_m = \Lambda^{m-1} \mathcal{S}_1 \otimes \mathcal{S}_3 + \Lambda^{m-2} \mathcal{S}_1 \otimes \Lambda^2 \mathcal{S}_2, \\
(5.19) & \quad C^3_m = \Lambda^{m-1} \mathcal{S}_1 \otimes \mathcal{S}_4 + \Lambda^{m-2} \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \mathcal{S}_3 + \Lambda^{m-3} \mathcal{S}_1 \otimes \Lambda^3 \mathcal{S}_2.
\end{align*}
\]

\[
\begin{align*}
5.5.2 & \quad \text{case } h = 1 \text{ and } n = 3 \\
& \quad \text{We restrict ourselves to } \mathbb{R}^3 \text{ and } h = 1. \text{ Then } m \leq w + n = w + 3 \text{ as in § 4.1 and we obtain the following}
\end{align*}
\]

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Example 5.1 (n=3, h=1)

\[ \overline{C}_0 = \Lambda^n \overline{\mathcal{E}}_1 \quad ((3 \choose m) \text{ dim}), \]
\[ \overline{C}_1 = \overline{\mathcal{E}}_2 \quad (6 \text{ dim}), \quad \overline{C}_1^2 = \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \quad (18 \text{ dim}), \quad \overline{C}_1^3 = \Lambda^2 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \quad (18 \text{ dim}), \]
\[ \overline{C}_1^4 = \Lambda^3 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \quad (6 \text{ dim}), \]
\[ \overline{C}_2 = \overline{\mathcal{E}}_3 \quad (10 \text{ dim}), \quad \overline{C}_2^2 = \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_3 \otimes \Lambda^2 \overline{\mathcal{E}}_2 \quad (45 \text{ dim}), \quad \overline{C}_2^3 = \Lambda^2 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_3 \otimes \overline{\mathcal{E}}_1 \otimes \Lambda^2 \overline{\mathcal{E}}_2 \quad (75 \text{ dim}), \]
\[ \overline{C}_2^4 = \Lambda^3 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_3 \otimes \Lambda^2 \overline{\mathcal{E}}_1 \otimes \Lambda^2 \overline{\mathcal{E}}_2 \quad (55 \text{ dim}), \quad \overline{C}_2^5 = \Lambda^3 \overline{\mathcal{E}}_1 \otimes \Lambda^2 \overline{\mathcal{E}}_2 \quad (15 \text{ dim}), \]
\[ \overline{C}_3 = \overline{\mathcal{E}}_4 \quad (15 \text{ dim}), \quad \overline{C}_3^2 = \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_4 \otimes \overline{\mathcal{E}}_2 \otimes \overline{\mathcal{E}}_3 \quad (105 \text{ dim}), \]
\[ \overline{C}_3^3 = \Lambda^2 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_4 \otimes \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \otimes \overline{\mathcal{E}}_3 \otimes \Lambda^3 \overline{\mathcal{E}}_2 \quad (245 \text{ dim}), \]
\[ \overline{C}_3^4 = \Lambda^3 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_4 \otimes \Lambda^2 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \otimes \overline{\mathcal{E}}_3 \otimes \overline{\mathcal{E}}_1 \otimes \Lambda^3 \overline{\mathcal{E}}_2 \quad (255 \text{ dim}), \]
\[ \overline{C}_3^5 = \Lambda^3 \overline{\mathcal{E}}_1 \otimes \overline{\mathcal{E}}_2 \otimes \overline{\mathcal{E}}_3 \otimes \Lambda^2 \overline{\mathcal{E}}_1 \otimes \Lambda^3 \overline{\mathcal{E}}_2 \quad (120 \text{ dim}), \quad \overline{C}_3^6 = \Lambda^3 \overline{\mathcal{E}}_1 \otimes \Lambda^3 \overline{\mathcal{E}}_2 \quad (20 \text{ dim}). \]

5.6 Concrete Poisson structure when \( n = 3 \) and \( h = 1 \):

We choose a specified Poisson structure, namely, we consider the Lie algebra \( \mathfrak{sl}(2) \) and consider the Lie Poisson structure. The Poisson bracket is defined by \( \{p, q\}_\pi = r, \{r, p\}_\pi = 2p, \{r, q\}_\pi = -2q \), where \( p, q, r \) are the coordinates in \( \mathbb{R}^3 \), namely,

\[
\{F, H\}_\pi = \begin{vmatrix} 2q & 2p & r \\ F_p & F_q & F_r \\ H_p & H_q & H_r \end{vmatrix} = \frac{\partial (2pq + r^2/2, F, H)}{\partial (p, q, r)}.
\]

We use the notations \( x^A = p^{a_1}q^{a_2}r^{a_3} \) for each triple \( A = (a_1, a_2, a_3) \) of non-negative integers. Then \( \{x^A \mid |A| := a_1 + a_2 + a_3 = k\} \) is a basis of \( \overline{\mathcal{S}}_k \) and the dual basis is given by \( \{z_A \mid |A| = k\} \). The coboundary operator \( \overline{d} \) for 1-cochains is defined as

\[
\overline{d}(z_C) = -\frac{1}{2} \sum_{A,B} \langle z_C, \{x^A, x^B\}_\pi \rangle z_A \wedge z_B.
\]

Since

\[
\{x^A, x^B\}_\pi = \begin{vmatrix} 2q & 2p & r \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \frac{x^{A+B}}{pq} = \frac{2pq}{b_1a_1} \begin{vmatrix} 2q & 2pq & r^2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\]

Since

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where $\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0), \epsilon_3 = (0, 0, 1)$, we have

\begin{equation}
\overline{d}(C) = -\sum_{A+B=C+\epsilon_3} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} z_A \wedge z_B - \frac{1}{2} \sum_{A+B=C+\epsilon_1+\epsilon_2-\epsilon_3} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} z_A \wedge z_B .
\end{equation}

For example, let $C = (1, 0, 0)$. In the first term of (5.20), $C + \epsilon_3 = (1, 0, 1)$ and so we find two summands corresponding to $A = (1, 0, 0), B = (0, 0, 1)$ or $A = (0, 0, 1), B = (1, 0, 0)$. In the second term, $C+\epsilon_1+\epsilon_2-\epsilon_3 = (2, 1, -1)$ and no summand corresponding to $B$ and $C$ with $B + C = (2, 1, -1)$. Thus, we get

\[
\overline{d}(z_{1,0,0}) = -(0 \ 1) z_{0,0,1} \wedge z_{1,0,0} - (0 \ 0) z_{0,0,0} \wedge z_{0,0,1} = -2z_{0,0,1} \wedge z_{1,0,0} .
\]

Similarly we get $\overline{d}(z_{0,1,0}) = 2z_{0,0,1} \wedge z_{0,1,0}$. For $C = (0, 0, 1)$, in the first term $C + \epsilon_3 = (0, 0, 2)$ and so $A = B = (0, 0, 1)$ and $z_A \wedge z_B = 0$. In the second term, since $A + B = (1, 1, 0)$ we have $A = (1, 0, 0)$ and $B = (0, 1, 0)$, or $A = (0, 1, 0)$ and have $B = (1, 0, 0)$, and so

\[
\overline{d}(z_{0,1,0}) = -\frac{1}{2} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} z_{0,1,0} \wedge z_{1,0,0} - \frac{1}{2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} z_{1,0,0} \wedge z_{0,1,0} = z_{0,1,0} \wedge z_{1,0,0} .
\]

As a summary, we obtain

\begin{align*}
\overline{d}(z_{1,0,0}) & = -2z_{0,0,1} \wedge z_{1,0,0} , \\
\overline{d}(z_{0,1,0}) & = 2z_{0,0,1} \wedge z_{0,1,0} , \\
\overline{d}(z_{0,0,1}) & = z_{0,1,0} \wedge z_{1,0,0} .
\end{align*}

By a similar argument, for $\overline{d}(z_A)$ for $|A| = 2$ we obtain the following result:

\begin{align*}
\overline{d}(z_{2,0,0}) & = -4z_{0,0,1} \wedge z_{2,0,0} + 2z_{1,0,0} \wedge z_{1,0,1} , \\
\overline{d}(z_{0,2,0}) & = 4z_{0,0,1} \wedge z_{0,2,0} - 2z_{0,1,0} \wedge z_{0,1,1} , \\
\overline{d}(z_{0,0,2}) & = z_{0,1,0} \wedge z_{1,0,1} + z_{0,1,1} \wedge z_{1,0,0} , \\
\overline{d}(z_{0,1,1}) & = 2z_{0,0,1} \wedge z_{0,1,1} + 4z_{0,0,2} \wedge z_{0,1,0} + z_{0,1,0} \wedge z_{1,1,0} + 2z_{0,2,0} \wedge z_{1,0,0} , \\
\overline{d}(z_{1,0,1}) & = -2z_{0,0,1} \wedge z_{1,0,1} - 4z_{0,0,2} \wedge z_{1,0,0} + 2z_{0,1,0} \wedge z_{2,0,0} - z_{1,0,0} \wedge z_{1,1,0} ,
\end{align*}

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Thus, we get 14 linear equations of 18 variables. Solving this equations, the kernel of \( \overline{d} : C_1^1 \rightarrow C_1^{n+1} \) is linearly spanned by
\[
π_{1,0} = 0 = c_{0,0,0}(−4z_{0,0,1} ∧ z_{0,0,0} + 2z_{1,0,0} ∧ z_{1,0,1}) + c_{0,0,2}(−4z_{0,0,1} ∧ z_{0,2,0} – 2z_{0,1,0} ∧ z_{0,1,1})
\]
\[
+ c_{0,0,2}(z_{0,1,0} ∧ z_{1,0,1} + z_{0,1,1} ∧ z_{1,0,0})
\]
\[
+ c_{0,1,1}(2z_{0,0,1} ∧ z_{0,1,1} + 4z_{0,0,2} ∧ z_{0,1,0} + z_{0,1,0} ∧ z_{1,1,0} + 2z_{0,2,0} ∧ z_{1,1,0})
\]
\[
+ c_{1,0,1}(−2z_{0,0,1} ∧ z_{0,1,0} + 2z_{0,0,2} ∧ z_{0,1,0} + 2z_{0,1,0} ∧ z_{0,0,0} − z_{1,0,0} ∧ z_{1,1,0})
\]
\[
+ c_{1,1,0}(−2z_{0,1,0} ∧ z_{1,0,1} − 2z_{0,1,1} ∧ z_{1,0,0})
\]

Taking the interior product by \( x^\iota (i = 1, 2, 3) \), we have
\[
4c_{1,0,1}z_{0,0,2} + (2c_{1,1,0} – c_{0,0,2})z_{0,1,1} − 2c_{0,0,0}z_{0,0,1} − 2c_{0,1,1}z_{0,2,0} + 2c_{0,0,0}z_{1,1,0} = 0,
\]
\[
− 4c_{0,0,1}z_{0,0,2} − 2c_{0,2,0}z_{0,1,1} + (−2c_{1,1,0} + c_{0,0,2})z_{1,0,1} + c_{0,1,1}z_{1,1,0} + 2c_{1,0,1}z_{2,0,0} = 0,
\]
\[
− 2c_{1,0,1}z_{1,0,1} + 2c_{1,1,1}z_{1,1,1} + 4c_{2,0,0}z_{0,0} − 4c_{2,0,0}z_{2,0,0} = 0.
\]

Thus, we get 14 linear equations of \( c_A \) (\( |A| = 2 \)) and solving them we see that \( c_{0,0,2} = 2c_{1,1,0} \) and \( c_A = 0 \) (\( A ≠ (0, 0, 2), (1, 1, 0) \) and \( |A| = 2 \)), i.e., \( π = c_{1,1,0}(z_{1,1,0} + 2z_{0,0,2}) \), namely, the kernel of \( \overline{d} : C_1^1 \rightarrow C_1^2 \) is spanned by \( z_{1,1,0} + 2z_{0,0,2} \), thereby the first Betti number is 1, and the rank of \( \overline{d} : C_1^1 \rightarrow C_1^2 \) is 5 because of \( \dim C_1^1 = \dim C_2^1 \).

Next, we consider the second cochain space. Take a 2-cochain \( π = \sum_{|α|=1,|A|=2} c_{α,A}z_{α} ∧ z_A \). Applying the interior products \( i_{x^\iota} \circ i_{x^\jmath} \circ i_{x^\kappa} \) with \( |α| = |β| = 1 \) and \( |A| = 2 \) to
\[
\sum_{|α|=1,|A|=2} c_{α,A} \overline{d}(z_{α} ∧ z_A) = 0,
\]
we get 17 linear equations of 18 variables. Solving this equations, the kernel of \( \overline{d} : C_1^2 \rightarrow C_1^3 \) is linearly spanned by the following 5 terms.
\[
− 2z_{0,0,1} ∧ z_{2,0,0} + z_{1,0,0} ∧ z_{1,0,1}, \quad z_{0,1,0} ∧ z_{0,1,1} − 2z_{0,0,1} ∧ z_{0,2,0}, \quad − z_{1,0,0} ∧ z_{0,1,1} + z_{0,1,0} ∧ z_{1,0,1},
\]
\[
z_{1,0,0} ∧ (−4z_{0,0,2} + z_{1,1,0}) − 2z_{0,1,0} ∧ z_{2,0,0} + 2z_{0,0,1} ∧ z_{1,1,0},
\]
\[
− 2z_{1,0,0} ∧ z_{0,2,0} + z_{0,1,0} ∧ (−4z_{0,0,2} ∧ z_{1,1,0}) + 2z_{0,0,1} ∧ z_{0,1,1}.
\]

Thus the kernel is 5-dimensional and the rank of \( \overline{d} : C_1^2 \rightarrow C_1^3 \) is 13 (= 18 – 5). Thereby, the second Betti number is 0. By the same method, the kernel of \( \overline{d} : C_1^3 \rightarrow C_1^4 \) is 13 dimensional and so the third Betti
number is 0 and the rank is 5.

Indeed, take an arbitrary 3-cochain \( \sigma = \sum_{\alpha, \beta, \gamma} c_{\alpha, \beta, \gamma} z_\alpha \wedge z_\beta \wedge z_\gamma \) \((|\alpha| = |\beta| = 1, |\gamma| = 2)\). Then we have

\[
\overrightarrow{d}(\sigma) = (c_{1,3,[0,1,1]} + c_{2,3,[1,0,1]} z_{1,0,0} \wedge z_{0,0,1} \wedge z_{0,1,0} \wedge (-4 z_{0,0,2} + z_{1,1,0})
+ (-2 c_{1,2,[0,1,1]} - 2 c_{2,1,3,[0,0,1]} - 2 c_{2,3,[1,0,2]} + c_{2,3,[0,0,2]}) (z_{1,0,0} \wedge z_{0,0,1} \wedge z_{0,1,0} \wedge z_{0,1,1})
+ (-4 c_{1,2,[0,2,0]} + 2 c_{2,3,[0,1,1]})(z_{1,0,0} \wedge z_{0,0,1} \wedge z_{0,1,0} \wedge z_{0,2,0})
+ (2 c_{1,2,[1,0,1]} - 2 c_{1,3,[1,0,0]} - 2 c_{2,3,[2,0,0]})(z_{1,0,0} \wedge z_{0,0,1} \wedge z_{0,1,0} \wedge z_{1,0,1})
+ (4 c_{1,2,[2,0,0]} + 2 c_{1,3,[1,0,1]})(z_{1,0,0} \wedge z_{0,0,1} \wedge z_{0,1,0} \wedge z_{2,0,0})
\).

Since \( \overline{C}_1^5 = (0) \), the kernel of \( \overrightarrow{d} : \overline{C}_1^1 \rightarrow \overline{C}_1^5 \) has the dimension \( \dim \overline{C}_1^1 = \dim (\Lambda^3 \overline{C}_1 \otimes \overline{C}_2) = 6 \), so

\[
\overline{H}_1^1 \equiv \text{LSpan}(z_{1,1,0}, z_{0,0,2}) / \text{LSpan}(z_{1,1,0} - 4 z_{0,0,2})
\]

and the fourth Betti number is 1. We summarize the discussion above into the table below.

| \( \text{wt}=1 \) | \( \overline{C}_1^1 \) | \( \overrightarrow{d} \) | \( \overline{C}_1^2 \) | \( \overrightarrow{d} \) | \( \overline{C}_1^3 \) | \( \overrightarrow{d} \) | \( \overline{C}_1^4 \) | \( \overrightarrow{d} \) | \( \overline{C}_1^5 \) | \( \overrightarrow{d} \) | \( \overline{C}_1^6 \) | \( \overrightarrow{d} \) | \( \overline{C}_1^7 \) | \( \overrightarrow{d} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \text{dim} \) | 6 | 18 | 18 | 6 | \( \text{Ker dim} \) | 1 | 5 | 13 | 6 | \( \text{rank} \) | 5 | 13 | 5 | 0 | \( \text{Betti} \) | 1 | 0 | 0 | 1 |

It is well-known that \( \sum_{m=0}^{w} (-1)^m \dim \overline{C}_w^m = \sum_{m=0}^{w} (-1)^m \dim \overline{H}_w^m \) holds and this number is called the Euler characteristic . In our case above, the number is 0. Later, we will show that this is true for 1-homogeneous Poisson structures in general. If we want to continue studying \( \overline{H}_w^m \) for this Poisson structure, we have to prepare \( d(z_A) \) further for \( |A| \leq w + 1 \).

Here we compute the Casimir polynomials: For \( F = \sum_{A} c_A p^{a_1} q^{a_2} r^{a_3} \) where \( A = (a_1, a_2, a_3) \) is a triple of non-negative integers, and \( c_A \) are constant, we see that

\[
\{r, F\}_\pi = \sum_{A} 2c_A (a_1 - a_2)p^{a_1} q^{a_2} r^{a_3}
\]

so any Casimir polynomial should be \( F = \sum c_{i,j,k} (pq)^i r^k \). By using the other condition \( \{p, F\}_\pi = 0 \), we see that

\[
F = \sum c_k (4pq + r^2)^k
\]
where $c_k$ are constant. Thus, $\bar{S}_{2k}$ contains a Casimir polynomial $(4pq + r^2)^k$ and $\bar{S}_{2k-1}$ does not contain any Casimir polynomials.

For the weight $=2$, all the Betti numbers are trivial, and for the weight $=3$ or $4$, we see non-trivial Betti numbers in the tables below:

| wt = 3 | $C_3^i \rightarrow C_3^{i+1} \rightarrow C_3^{i+2} \rightarrow C_3^{i+3} \rightarrow C_3^{i+4}$ |
|--------|-----------------------------------------------|
| dim    | 15, 105, 245, 255, 120, 20                   |
| rank   | 14, 91, 154, 100, 20, 0                      |
| dim(ker)| 1, 14, 91, 155, 100, 20                     |
| Betti num | 1, 0, 1, 0, 1, 0                            |

| wt = 4 | $C_4^i \rightarrow C_4^{i+1} \rightarrow C_4^{i+2} \rightarrow C_4^{i+3} \rightarrow C_4^{i+4}$ |
|--------|---------------------------------------------------------------|
| dim    | 21, 198, 618, 891, 630, 195, 15                              |
| rank   | 21, 176, 441, 450, 179, 15, 0                               |
| dim(ker)| 0, 22, 177, 441, 451, 180, 15                              |
| Betti num | 0, 1, 1, 1, 1, 1, 0                                 |

**Remark 5.1** Given a finite dimensional Lie algebra $\mathfrak{h}$, we have the so called Lie Poisson structure on $\mathfrak{h}^*$ as mentioned in Remark 2.2. As Lie algebras, $(\mathfrak{h}, [\cdot, \cdot])$ is a subalgebra of $(\sum_k \text{Sym}^k \mathfrak{h}, \{\cdot, \cdot\})$. The cochain complex of Lie algebra $\mathfrak{h}$ coincides with the weight zero cochain complex of $(\overline{C}_0, \overline{d})$ of Lie Poisson structure $\mathfrak{h}^*$. The Euler characteristic of Lie algebra cohomology groups of $\mathfrak{h}$ is zero and the Euler characteristic of $\overline{H}_0$ of 1-homogeneous Poisson structure with the weight 0 is also zero.

## 6 Contributions of Poisson structures

We would like to know the concrete behavior of $\overline{d}$ for each weight. Since $\overline{d}$ preserve weights, in the case of $h$-homogeneous structure, we have

$$\overline{d}(\overline{\Xi}_g) \subset \sum \overline{\Xi}_a \wedge \overline{\Xi}_b$$

where $g - 2 + h = (a - 2 + h) + (b - 2 + h)$, thus $a + b = g - h + 2$.  

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6.1 \( h = 1 \)

When \( h = 1 \), we see that
\[
\overline{d}(\Xi_1) \subset \Xi_1 \wedge \Xi_1, \quad \overline{d}(\Xi_2) \subset \Xi_1 \wedge \Xi_2, \quad \overline{d}(\Xi_3) \subset \Xi_1 \wedge \Xi_3 \oplus \Xi_2 \wedge \Xi_2.
\]

These come from \( \{\overline{S}_i, \overline{S}_j\}_\pi \subset \overline{S}_{i+j-1} \) in general. For instance,
\[
\{\overline{S}_1, \overline{S}_1\}_\pi \subset \overline{S}_1, \quad \{\overline{S}_1, \overline{S}_2\}_\pi \subset \overline{S}_2, \quad \{\overline{S}_1, \overline{S}_3\}_\pi \subset \overline{S}_3, \quad \{\overline{S}_2, \overline{S}_2\}_\pi \subset \overline{S}_3.
\]

6.1.1 \( \pi = x_1 \partial_2 \wedge \partial_3 + x_2 \partial_3 \wedge \partial_1 + x_3 \partial_1 \wedge \partial_2 \), i.e., Lie-Poisson of \( \mathfrak{so}(3) \)

As Lie algebras, \( \mathfrak{so}(3) \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) and we have some data of cohomology groups of lower weights as stated before.

6.1.2 \( \pi = x_3 \partial_1 \wedge \partial_2 \), i.e., Lie-Poisson of the Heisenberg Lie algebra

The Casimir polynomials are \( \{x_i^k\} (k = 1, 2, \ldots) \) and the cohomology groups of two kinds for lower weights are follows:

| wt = 0 | C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 | wt = 0 | C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 |
|--------|----------------|--------|----------------|
| dim    | 1  3  3  1     | dim    | 1  2  1  0     |
| dim(ker) | 1  2  3  1   | dim(ker) | 1  2  1  0    |
| Betti num | 1  2  2  1 | Betti num | 1  2  1  0    |

| wt = 1 | \bullet = 1 | \bullet = 1 | \bullet = 2 | \bullet = 3 | \bullet = 4 | \bullet = 1 | \bullet = 1 | \bullet = 2 | \bullet = 3 | \bullet = 4 | \bullet = 5 |
|--------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| dim of C_1 | 6  18  18  6  | dim of C_2 | 10  45  75  55  15  |
| dim(ker) | 3  10  13  6  | dim(ker) | 4  18  41  42  15  |
| Betti num | 3  7  5  1 | Betti num | 4  12  14  8  2 |
| dim of C_1 | 5  10  5  0 | dim of C_2 | 9  28  29  10  0 |
| dim(ker) | 3  7  5  0 | dim(ker) | 4  14  24  10  0 |
| Betti num | 3  5  2  0 | Betti num | 4  9  10  5  0 |

| wt = 3 | \bullet = 1 | \bullet = 1 | \bullet = 2 | \bullet = 3 | \bullet = 4 | \bullet = 5 | \bullet = 6 | \bullet = 7 |
|--------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| dim of C_3 | 15  105  245  255  120  20 | dim of C_4 | 21  198  618  891  630  195  15 |
| dim(ker) | 5  33  109  165  104  20 | dim(ker) | 6  49  221  477  464  180  15 |
| Betti num | 5  23  37  29  14  4 | Betti num | 6  34  72  80  50  14  0 |
| dim of C_3 | 14  73  114  65  10  0 | dim of C_4 | 20  146  322  291  100  5  0 |
| dim(ker) | 5  28  72  55  10  0 | dim(ker) | 6  43  160  204  95  5  0 |
| Betti num | 5  19  27  13  0  0 | Betti num | 6  29  57  42  8  0  0 |
6.1.3 Solvable but not nilpotent cases

The Lie algebra of upper triangle matrices of type (2,2) is an example of not nilpotent but solvable Lie algebra of dimension 3. Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We get $[A_1, A_3] = A_3$, $[A_2, A_3] = -A_3$ and the other brackets are trivial. The adjoint representation is given by

$$\text{ad}_{A_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{ad}_{A_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{ad}_{A_3} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix},$$

and those show the algebra is not nilpotent, but the derived algebra is 1-dimensional and we see that the algebra is solvable. We consider Lie Poisson manifold whose Casimir polynomials are $(x_1 + x_2)^k$ where we identify $A_i$ by $x_i$.

| $wt = 0$ | $C^i_0 \rightarrow C^i_0 \rightarrow C^i_0 \rightarrow C^i_3$ | $wt = 0$ | $C^i_0 \rightarrow C^i_0 \rightarrow C^i_0 \rightarrow C^i_3$ |
|---------|---------------------------------|---------|---------------------------------|
| dim     | 1 3 3 1                          | dim     | 1 2 1 0                          |
| dim(ker)| 1 2 2 1                          | dim(ker)| 1 1 1 0                          |
| Betti num| 1 2 1 0                          | Betti num| 1 1 0 0                          |

| $wt = 1$ | $C^i_1 \rightarrow C^i_1 \rightarrow C^i_1 \rightarrow C^i_1$ | $wt = 1$ | $C^i_1 \rightarrow C^i_1 \rightarrow C^i_1 \rightarrow C^i_1$ |
|---------|------------------------------------------------|---------|---------------------------------|
| dim     | 6 18 18 6                          | dim     | 5 10 5 0                          |
| dim(ker)| 3 9 12 6                          | dim(ker)| 2 5 5 0                          |
| Betti num| 3 6 3 0                          | Betti num| 2 2 0 0                          |

| $wt = 2$ | $\bullet = 1 \bullet = 2 \bullet = 3 \bullet = 4 \bullet = 5$ | $wt = 3$ | $\bullet = 1 \bullet = 2 \bullet = 3 \bullet = 4 \bullet = 5 \bullet = 6$ |
|---------|------------------------------------------------|---------|---------------------------------|
| dim of $C^i_2$ | 10 45 75 55 15 | dim of $C^i_3$ | 15 105 245 255 120 20 |
| dim(ker) | 4 17 38 40 15 | dim(ker) | 5 32 103 156 100 20 |
| Betti num | 4 11 10 3 0 | Betti num | 5 22 30 14 1 0 |
| dim of $C^i_2$ | 9 28 29 10 0 | dim of $C^i_3$ | 14 73 114 65 10 0 |
| dim(ker) | 3 10 19 10 0 | dim(ker) | 4 20 59 55 10 0 |
| Betti num | 3 4 1 0 0 | Betti num | 4 10 6 0 0 0 |
6.2 \( h = 2 \)

Even though the normal form of analytic Poisson structures are studied by J.F. Conn ([1]), it is not clear what is the typical 2-homogeneous Poisson structure in our context.

Here, we show some concrete examples: When \( h = 2 \), we see \( \{ \overline{S}_i, \overline{S}_j \} \pi \subset \overline{S}_{i+j} \) and so we have

\[
\overline{d}(\overline{e}_1) = (0), \quad \overline{d}(\overline{e}_2) \subset \Lambda^2 \overline{e}_1, \quad \overline{d}(\overline{e}_3) \subset \overline{e}_1 \otimes \overline{e}_2, \quad \overline{d}(\overline{e}_4) \subset \overline{e}_1 \otimes \overline{e}_3 \otimes \Lambda^2 \overline{e}_2, \\
\overline{d}(\overline{e}_5) \subset \overline{e}_1 \otimes \overline{e}_4 \otimes \overline{e}_2 \otimes \overline{e}_3.
\]

We deal with the following 3 cases of 2-homogeneous Poisson structures on \( \mathbb{R}^3 \).

- **case1**: \( \pi = \frac{1}{2}(x_1^2 \partial_2 \wedge \partial_3 + x_2^2 \partial_3 \wedge \partial_1 + x_3^2 \partial_1 \wedge \partial_2) \), Casimirs are \((x_1^3 + x_2^3 + x_3^3)^k\).
- **case2**: \( \pi = x_1 x_2 \partial_1 \wedge \partial_2 + x_2 x_3 \partial_2 \wedge \partial_3 + x_3 x_1 \partial_3 \wedge \partial_1 \), Casimirs are \((x_1 x_2 x_3)^k\).
- **case3**: \( \pi = x_1^2 \partial_2 \wedge \partial_3 + x_3 x_1 \partial_3 \wedge \partial_1 + x_1 x_2 \partial_1 \wedge \partial_2 \), Casimirs are \((x_1^2 + 2 x_2 x_3)^k\).

The cochain complex of weight 0 is trivial and only \( \overline{e}_1 \) for weight 1.

6.2.1 weight=2

The three 2-homogeneous Poisson structures have the same table when weight =2 as below.

\[
\begin{array}{c|cccccc}
\text{wt} = 2 & \bullet = 1 & \bullet = 2 & \bullet = 3 & \bullet = 4 & \bullet = 5 & \bullet = 6 & \bullet = 7 \\
\hline
\text{dim of } \overline{C}_4 & 21 & 198 & 618 & 691 & 630 & 195 & 15 \\
\text{dim(ker)} & 6 & 48 & 210 & 453 & 450 & 180 & 15 \\
\text{Betti num} & 6 & 33 & 60 & 45 & 12 & 0 & 0 \\
\hline
\text{dim of } \overline{C}_4^* & 20 & 146 & 322 & 291 & 100 & 5 & 0 \\
\text{dim(ker)} & 5 & 31 & 129 & 196 & 95 & 5 & 0 \\
\text{Betti num} & 5 & 16 & 14 & 3 & 0 & 0 & 0 \\
\end{array}
\]

However, \( H^*_2 \) differ as follows:
6.2.2 weight=3

| case1 | $C^1_2 \to C^2_2$ | case2 | $C^1_2 \to C^2_2$ | case3 | $C^1_2 \to C^2_2$ |
|-------|------------------|-------|------------------|-------|------------------|
| dim   | 6 3              | dim   | 6 3              | dim   | 5 3              |
| dim(ker) | 3 3               | dim(ker) | 3 3               | dim(ker) | 2 3               |
| Betti num | 3 0               | Betti num | 3 0               | Betti num | 2 0               |

\[ wt = 3 \]

| $C^1_3 \to C^2_3 \to C^3_3$ | dim | 10 18 1 |
|-------------------------------|-----|---------|
| case1 Betti num               | 2 9 0 |
| case2 Betti num               | 4 11 0 |
| case3 Betti num               | 5 12 0 |

6.2.3 weight=4

| case1 | $C^1_3 \to C^2_3 \to C^3_3$ | case2 | $C^1_3 \to C^2_3 \to C^3_3$ | case3 | $C^1_3 \to C^2_3 \to C^3_3$ |
|-------|------------------|-------|------------------|-------|------------------|
| dim   | 9 18 1           | dim   | 9 18 1           | dim   | 10 15 1          |
| Betti | 1 9 0            | Betti | 3 11 0           | Betti | 5 9 0            |

| $C^1_4 \to C^2_4 \to C^3_4$ | dim | 15 45 18 |
|-------------------------------|-----|---------|
| case1 Betti num               | 3 15 0 |
| case2 Betti num               | 3 15 0 |
| case3 Betti num               | 5 17 0 |

About $H^*_4$ we see that

| case1 | $C^1_4 \to C^2_4 \to C^3_4$ | case2 | $C^1_4 \to C^2_4 \to C^3_4$ | case3 | $C^1_4 \to C^2_4 \to C^3_4$ |
|-------|------------------|-------|------------------|-------|------------------|
| dim   | 15 42 18         | dim   | 15 42 18         | dim   | 14 40 15         |
| Betti | 3 12 0           | Betti | 3 12 0           | Betti | 4 15 0           |

6.3 Comparison of elapsed time

Here, we compare the elapsed time for concrete Lie Poisson structures by $\mathfrak{sl}(2, \mathbb{R})$ and 3-dim Heisenberg Lie algebra by Symbolic Calculus software, Maple. Two classes mean the polynomial algebra and the algebra of Hamiltonian vector fields of polynomials.

Polynomials
| weight | \(\text{HomoL} \) | \(\text{Cohom} \) | \(\text{HomoL} \) | \(\text{Cohom} \) |
|--------|----------------|----------------|----------------|----------------|
| 0      | 0.021          | 0.021          | 0.024          | 0.019          |
| 1      | 0.047          | 0.057          | 0.043          | 0.042          |
| 2      | 0.221          | 0.269          | 0.136          | 0.163          |
| 3      | 1.467          | 1.664          | 0.920          | 0.980          |
| 4      | 13.274         | 14.651         | 7.260          | 7.592          |

Hamiltonian

| weight | \(\text{HomoL} \) | \(\text{Cohom} \) | \(\text{HomoL} \) | \(\text{Cohom} \) |
|--------|----------------|----------------|----------------|----------------|
| 0      | 0.039(0.036)   | 0.022          | 0.023(0.032)   | 0.020          |
| 1      | 0.197(0.186)   | 0.062          | 0.086(0.080)   | 0.044          |
| 2      | 1.300(1.198)   | 0.285          | 0.429(0.353)   | 0.090          |
| 3      | 8.236(7.745)   | 1.512          | 2.335(1.865)   | 0.312          |
| 4      | 44.539(40.497) | 11.453         | 11.645(8.873)  | 1.552          |

The numbers in the parentheses are obtained by calculating the remainder by our method (4.10). Except only one case printed in red color, using (4.10) is faster than using the normal form. Even though, computing cohomology groups is faster than homology computing in Hamiltonian vector fields case.

7 Top Betti number

As before, let \(\mathfrak{g}\) be the Lie algebra defined by a non-trivial \(h\)-homogeneous Poisson structure on \(\mathbb{R}^n\). For a given weight \(w\), we have a sequence of cochain groups \(\{\overline{C}^m_w\}\). As remarked in Subsection 3.2, the sequence is finitely bounded, so let \(m_0(w) = \max\{m \mid \overline{C}^m_w \neq 0\}\). We call \(\overline{C}^{m_0(w)}_w\), \(\overline{H}^{m_0(w)}_w\) or \(\dim\overline{H}^{m_0(w)}_w\) as the top cochain group, the top cohomology group or the top Betti number.

7.1 Multi-index manipulation

As already seen in concrete examples, in order to deal with homogeneous polynomials of \(x_1, \ldots, x_n\), we use monomials as a basis, and we use multi-index notation. For each positive integer \(k\), let \(\mathbb{M}[k] = \{A \in \mathbb{N}^n \mid |A| = k\}\). We denote the dual basis of \(x^A\) by \(z_A\) or \(z^A_{|A|}\) emphasizing the degree of \(A\).

We put a total order in \(\mathbb{M}[k]\) and assign natural number \(j\) to \(A\) if \(A\) is the \(j\)-th element in \(\mathbb{M}[k]\) and denote this assignment by \(\text{od}(A) = j\). For \(\mathbb{M}[1]\), it is natural to define as follows: for \(E_j = (0, \ldots, 1, 0 \ldots)\), \(\text{od}(E_j) = j\).
For each positive $j$, we use the notation
\[
\begin{align*}
\zeta_{\text{Full}}^{(j)} &= z_j^{\od(1)} \land \cdots \land z_j^{\od(M[j])} & (\text{all } A \in M[j]), \\
\zeta_{\bullet < A}^{(j)} &= z_j^{\od(1)} \land \cdots \land z_j^{\od(A-1)} & (\text{all before } A), \\
\zeta_{\bullet > A}^{(j)} &= z_j^{\od(A+1)} \land \cdots \land z_j^{\od(M[j])} & (\text{all after } A), \\
\zeta_{< A \bullet}^{(j)} &= \zeta_{\bullet < A}^{(j)} \land \zeta_{\bullet > A}^{(j)} & (\text{all except } A),
\end{align*}
\]
in particular, if $A < B$
\[
\zeta_{A < B}^{(j)} = z_j^{\od(A)+1} \land \cdots \land z_j^{\od(B)-1} & (\text{between } A \text{ and } B).
\]

### 7.2 Case of polynomial algebra

In the concrete examples of 2-homogeneous Poisson structures in previous subsections, all the top Betti number is zero, but we have the next example of non-zero top Betti number.

**Example 7.1** Let us consider a small example, $n = 3$ and the 2-homogeneous Poisson bracket given by
\[
\{x_1, x_2\}_{\pi} = x_3^2, \quad \{x_1, x_3\}_{\pi} = 0, \quad \{x_2, x_3\}_{\pi} = 0
\]
and study of weight 2 cohomology groups. 1-cochain complex is $\mathbf{C}_1 = \mathbb{Z}_2 (6 \text{ dim})$ and 2-cochain complex is $\mathbf{C}_2 = \Lambda^2 \mathbb{Z}_1 (3 \text{ dim})$, and $\mathbf{C}_m = 0$ for $m > 2$. $d(\zeta^{(j)}) - 0 (j = 1, 2, 3)$ and
\[
d\left(\zeta^A_2\right) = -\sum_{i<j} \langle \zeta^A_2, \{x_i, x_j\}_{\pi}\rangle \zeta^j_1 \land \zeta^j_1 = -\langle \zeta^A_2, x_3^2\rangle \zeta^j_1 \land \zeta^j_1 = \begin{cases} -z^j_1 \land z^j_1 & \text{if } A = (0, 0, 2) \\ 0 & \text{otherwise} \end{cases}
\]

Thus, we have the table below left which tells that the top Betti number is not zero:

|       | $\mathbf{C}_1$ | $\mathbf{C}_2$ |
|-------|----------------|----------------|
| dim   | 6              | 3              |
| ker dim | 5              | 3              |
| Betti | 5              | 2              |

The table above right of $g = \bar{\pi}/\text{Cent}_r$ tells the top Betti number is zero in this case.

Each of our cochain group corresponds to a Young diagram with our dimensional condition and we have special Young diagram which satisfies the extremal dimensional condition in the sense that each index of the Young diagram is maximal. Namely, for each fixed $\ell$, it is the one given by $\text{YD}_0 = [md_j \mid j = 1 \ldots \ell]$,
where $m_d = \dim \mathfrak{g}_j = \binom{n-j}{j}$. The corresponding factor of the cochain complex is

$$
\Lambda^{m_d_1} \mathfrak{g}_1 \otimes \Lambda^{m_d_2} \mathfrak{g}_2 \otimes \cdots \otimes \Lambda^{m_d_\ell} \mathfrak{g}_\ell
$$

whose degree is

(7.1) \[ m_0 = \sum_{j=1}^{\ell} m_d_j \]

and the weight is

(7.2) \[ w_0 = \sum_{j=1}^{\ell} (j - 2 + h) m_d_j \]

from the definition, where $h$ the homogeneity of our Poisson bracket. Concerning to these weight $w_0$ and degree $m_0$, we have the following propositions.

**Proposition 7.1** For the given $w_0$ and $m_0$,

$$
C^m_{w_0} = (0) \quad \text{if} \quad m > m_0
$$

and

$$
C^{m_0}_{w_0} = \Lambda^{m_d_1} \mathfrak{g}_1 \otimes \Lambda^{m_d_2} \mathfrak{g}_2 \otimes \cdots \otimes \Lambda^{m_d_\ell} \mathfrak{g}_\ell \quad \text{(which is 1 dimensional)}.
$$

**Proof:** To show the first claim, suppose $C^m_{w_0} \neq 0$ for some $m > m_0$. Then there is a Young diagram $[k_i \mid j = 1 \ldots s]$ with

(7.3) \[ w_0 = \sum_{i=1}^{s} (i - 2 + h) k_i \quad \text{and the height is} \quad m = \sum_{i=1}^{s} k_i \]

and also satisfying the dimensional condition $0 \leq k_i \leq m_d_i$, where $m_d_i = \binom{n-i}{i}$. If $s \leq \ell$ then $m \leq m_0$ contradicting to the assumption, so we may assume $s > \ell$. Then, comparing the first equation of (7.3) and (7.2), we have

(7.4) \[ \sum_{i>\ell} (i - 2 + h) k_i = \sum_{j=1}^{\ell} (j - 2 + h)(m_d_j - k_j) . \]
On the other hand,

\[ 0 < \Delta = m - m_0 = \sum_{j=1}^{\ell} (k_j - m_j) + \sum_{i>\ell} k_i, \]

(7.5)

\[ \sum_{i>\ell} k_i = \Delta + \sum_{j=1}^{\ell} (m_j - k_j). \]

Extracting \((\ell - 2 + h)\) times (7.5) from (7.4), we have

\[ \sum_{i>\ell} (i-\ell)k_i = - (\ell - 2 + h)\Delta + \sum_{j=1}^{\ell} (j-\ell)(m_j - k_j). \]

The left hand side is positive and the right hand side is non-positive, this is impossible. Therefore, we conclude that \(C_{w_0}^m = 0\) for \(m > m_0\).

Starting from \(m = m_0\), we claim that there is no other Young diagram with the same weight and the same height, namely, . we conclude that \(s = \ell\) and \(k_j = m_j\) and so

\[ C_{w_0}^{m_0} = \Lambda^{m_1} \otimes \Lambda^{m_2} \otimes \cdots \otimes \Lambda^{m_\ell}. \]

\[ \text{Proposition 7.2} \]

We use the same values \(w_0\) and \(m_0\) for \(h\). When \(h = 1\) the Young diagram defined by \(k_1 = m_1 - 1, k_j = m_j (j = 2 \ldots \ell)\) is a factor, and the Young diagram defined by \(k_j = m_j (j = 1 \ldots \ell - 1), k_\ell = m_\ell - 2, k_{2\ell-1} = 1\) is another factor of \(C_{w_0}^{m_0-1}\). When \(h > 1\), if \(\ell > 1\) then \(k_1 = m_1 - 1, k_j = m_j (j = 2 \ldots \ell - 1), k_\ell = m_\ell - 1\) and \(k_{\ell+h-1} = 1\) else if \(\ell = 1\) then \(k_1 = m_1 - 2, k_h = 1\) is a factor of the direct sum of \(C_{w_0}^{m_0-1}\).

\[ \text{Proof:} \] It is just calculation that the Young diagrams in the Proposition satisfy the dimensional condition and its height is \(m_0 - 1\) and its weight is equal to \(w_0\). But a visual understanding is the following:

\[ h = 1 \text{ case 1} \]

\[ h = 1 \text{ case 2} \]
Our observation on the top Betti number is the next theorem.

**Theorem 7.1** Let $\mathfrak{g}$ be the Lie algebra $\mathbb{R}[x_1, \ldots, x_n]/\mathbb{R}$ defined by a non-trivial $h$-homogeneous Poisson structure on $\mathbb{R}^n$. For weight $w$ given by $w = \sum_{j=1}^{\ell} (j - 2 + h)(n - 1 + j)$ consider the top cochain group of dimension $m_0(w) = \sum_{j=1}^{\ell} (n - 1 + j)$. Then the Betti number $\dim H^{m_0(w)}_w = 0$ for each $h$.

**Proof:** We use the basis $\{z_A^A\}$ of $\mathfrak{g}$ where $A \subset \mathbb{N}^n$ with $|A| = j$, i.e., $A \in \mathfrak{g}[j]$.

First suppose $h = 1$. As we saw in Proposition 7.2, we have at least two factors of $C^{m_0-1}_w$ and one of them is $k_j = \text{md}_j$ for $j = 1 \ldots \ell - 1$, $k_\ell = \text{md}_\ell - 2$ and $k_{2\ell - 1} = 1$. Take a cochain

$$\sigma = z_1^{(1)} \wedge \cdots \wedge z_{\ell-1}^{(1)} \wedge (z_\ell^{(\ell)} \wedge z_A^{(\ell)} \wedge z_B^{(\ell)} \wedge z_C^{(\ell)}) \wedge z_{2\ell - 1}^{C}.$$

Homogeneity 1 implies $\overline{d}(\mathfrak{g}_1) \subset \Lambda^2 \mathfrak{g}_1$, and in general $\overline{d}(\mathfrak{g}_k) \subset \sum_{i+j=k+1} \mathfrak{g}_i \wedge \mathfrak{g}_j$, and the following holds

$$\overline{d}(z_1^{(1)}) = 0,$$

$$z_1^{(1)} \wedge \overline{d}(z_1^{(2)}) = 0,$$

$$\vdots$$

$$z_1^{(1)} \wedge \cdots \wedge z_{\ell-1}^{(1)} \wedge \overline{d}(z_\ell^{(\ell)} \wedge z_A^{(\ell)} \wedge z_B^{(\ell)} \wedge z_C^{(\ell)}) = 0.$$

Thus,

$$\overline{d}(\sigma) = \pm z_1^{(1)} \wedge \cdots \wedge z_{\ell-1}^{(1)} \wedge (z_\ell^{(\ell)} \wedge z_A^{(\ell)} \wedge z_B^{(\ell)} \wedge z_C^{(\ell)}) \wedge \overline{d}(z_{2\ell - 1}^{C})$$

$$= \pm z_1^{(1)} \wedge \cdots \wedge z_{\ell-1}^{(1)} \wedge (z_\ell^{(\ell)} \wedge z_A^{(\ell)} \wedge z_B^{(\ell)} \wedge z_C^{(\ell)}).$$
\[
\begin{align*}
&\sum_{|P|+|Q|=\ell+|C|, \text{od}P\leq\text{od}Q} \langle z^C_{2\ell-1}, \{x^P, x^Q\}_\pi \rangle z^P_P \wedge z^Q_Q \\
= \pm z^{(1)}_{\text{Full}} \wedge \cdots \wedge z^{(\ell)}_{\text{Full}} \langle z^C_{2\ell-1}, \{x^A, x^B\}_\pi \rangle .
\end{align*}
\]

Since the Poisson bracket is not trivial, there are some \(i, j\) such that \(\{x_i, x_j\}_\pi \neq 0\). Let us take \(A, B \in \mathfrak{W}[\ell]\) such that \(x^A = x^A_i\) and \(x^B = x^B_j\). Then \(\{x^A, x^B\}_\pi = \mathfrak{L}^2(x_i x_j)\mathfrak{L}^{\ell-1}(x_i, x_j) \neq 0\) and we can find some \(C \in \mathfrak{W}[2\ell - 1]\) satisfying \(\langle z^C, \{x^A, x^B\}_\pi \rangle \neq 0\). This means \(d(\sigma) \neq 0\) and also that \(d : \overline{C_w^{m_0(w)-1}} \to \overline{C_w^{m_0(w)}}\) is surjective in this case, thus \(\dim \overline{H_{m_0(w)}} = 0\).

Now assume that \(h > 1\). Then Proposition 7.1 says \(\overline{C_w^{m_0(w)}} = \prod_{j=1}^\ell \Lambda^\text{md}_j \overline{\mathbb{C}_j}\) and \(\dim \overline{C_w^{m_0(w)}} = 1\) and Proposition 7.2 says one factor of \(\overline{C_w^{m_0(w)-1}}\) is given by

\[(7.6) \quad k_1 = \text{md}_1 - 1, \quad k_2 = \text{md}_2, \quad \ldots, \quad k_{\ell-1} = \text{md}_{\ell-1}, \quad k_\ell = \text{md}_\ell - 1, \quad k_{\ell+h-1} = 1 .
\]

Since our Poisson bracket is non-trivial, for some \(i_0, j_0\) we may assume \(\{x_{i_0}, x_{j_0}\}_\pi \neq 0\).

If \(\ell > 1\) then we take a \((m_0(w) - 1)\)-cochain \(\sigma\) defined by

\[
\sigma = z^{(1)}_{\ll i_0} \wedge z^{(2)}_{\text{Full}} \wedge \cdots \wedge z^{(\ell-1)}_{\text{Full}} \wedge z^{(\ell)}_{\ll A} \wedge z^B_{\ll +h-1}
\]

else if \(\ell = 1\) then we take \(\sigma = z^{(1)}_{\ll i_0} \wedge z^{(1)}_{\ll j_0} \wedge z^{(1)}_{\ll j_0} \wedge z^B_{\ll h} .\)

By the homogeneity \(h > 1\), the coboundary operator \(d\) has the following properties

\[(7.7) \quad d(\overline{\mathbb{C}_i}) = 0 \quad (i < h) \quad \text{and} \quad d(\overline{\mathbb{C}_k}) \subset \sum_{i+j=k+2-h, i \leq j} \overline{\mathbb{C}_i} \wedge \overline{\mathbb{C}_j} .
\]

When \(\ell = 1\),

\[
d(\sigma) = \pm z^{(1)}_{\ll i_0} \wedge z^{(1)}_{\ll i_0} \wedge z^{(1)}_{\ll j_0} \wedge d(z^B_{\ll h})
\]

\[
= \pm z^{(1)}_{\ll i_0} \wedge z^{(1)}_{\ll i_0} \wedge z^{(1)}_{\ll j_0} \wedge \langle z^B_{\ll h}, \{x_i, x_j\}_\pi \rangle z_1^{\ell} \wedge z_1^{j}
\]

\[
= \pm z^{(1)}_{\text{Full}} \langle z^B_{\ll h}, \{x_{i_0}, x_{j_0}\}_\pi \rangle \neq 0 \quad \text{for some } B .
\]

When \(1 < \ell < h\) then it holds

\[
d(\sigma) = \pm z^{(1)}_{\ll i_0} \wedge z^{(1)}_{\ll i_0} \wedge \cdots \wedge z^{(\ell-1)}_{\ll A} \wedge d(z^B_{\ll +h-1})
\]

\[
= \pm z^{(1)}_{\ll i_0} \wedge z^{(2)}_{\text{Full}} \wedge \cdots \wedge z^{(\ell-1)}_{\text{Full}} \wedge z^{(\ell)}_{\ll A} \wedge \sum_{|P|+|Q|=\ell+1} \langle z^B_{\ll +h-1}, \{x^P, x^Q\}_\pi \rangle z_P \wedge z_Q
\]

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\[
= \pm z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(\ell-1)} \wedge z_{<\ell_0>}^{(\ell)} \wedge \sum_{|P|=1, |Q|=\ell} \langle z_{\ell+h-1}^B, \{x^P, x^Q\} \rangle z_1^P \wedge z_1^Q
\]

\[
= \pm \langle z_{\ell+h-1}^B, \{x_{\ell_0}, x_{\ell_0}^A\} \rangle z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge z_{\text{Full}}^{(3)} \wedge \cdots \wedge z_{\text{Full}}^{(\ell-1)} \wedge z_{<\ell_0>}^{(\ell)} .
\]

Since \( \{x_{\ell_0}, x_{\ell_0}^A\} = \ell x_{\ell-1} \{x_{\ell_0}, x_{\ell_0}^A\} \neq 0 \), there is some \( B \) with \( |B| = \ell + h - 1 \) such that \( \langle z_{\ell+h-1}^B, \{x_{\ell_0}, x_{\ell_0}^A\} \rangle \neq 0 \). Take \( A \) with \( |A| = A[j_0] = \ell \). Then \( \overline{d}(\sigma) \neq 0 \).

When \( \ell = h \),

\[
\overline{d}(\sigma) = \pm z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge z_{<\ell_0>}^{(h)} \wedge \overline{d}(z_{<\ell_0>}^{(h)}) \wedge z_{\ell+h-1}^B
\]

then, because of \( \overline{d}(z_{<\ell_0>}^{(h)}) \subset \overline{H}_1 \wedge \overline{H}_1 \wedge \Lambda^{md_{\ell}} \overline{H}_\ell \), we see

\[
= \pm z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{<\ell_0>}^{(h-1)} \wedge z_{<\ell_0>}^{(h)} \wedge \overline{d}(z_{\ell+h-1}^B)
= \pm 0 \quad \text{by the same argument when } \ell < h .
\]

When \( h < \ell \), we see that

\[
\overline{d}(\sigma) = \pm z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge z_{<\ell_0>}^{(h)} \wedge \overline{d}(z_{\text{Full}}^{(h)}) \wedge z_{\ell+h-1}^B
\]

and since

\[
\begin{align*}
z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge \overline{d}(z_{\text{Full}}^{(h)}) &= z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge \Lambda^{md_{\ell}} \overline{H}_1 \wedge \Lambda^{md_{\ell}} \overline{H}_h = 0 , \\
z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge z_{<\ell_0>}^{(h)} \wedge \overline{d}(z_{\text{Full}}^{(h)}) &= z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge z_{<\ell_0>}^{(h)} \wedge \overline{d}(z_{\text{Full}}^{(h)}) \wedge \Lambda^{md_{\ell}} \overline{H}_1 \wedge \Lambda^{md_{\ell}} \overline{H}_h = 0 , \\
\vdots
\end{align*}
\]

we have

\[
\overline{d}(\sigma) = \pm z_{<\ell_0>}^{(1)} \wedge z_{\text{Full}}^{(2)} \wedge \cdots \wedge z_{\text{Full}}^{(h-1)} \wedge z_{<\ell_0>}^{(h)} \wedge \overline{d}(z_{\ell+h-1}^B)
\]

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and $\overline{d}(\sigma) \neq 0$ again by the same argument when $\ell < h$.

\[
\overline{d}(\sigma) = \beta z^{(1)}_\text{Full} \wedge \cdots \wedge z^{(\ell)}_\text{Full} \neq 0
\]

this means $\overline{d} : \overline{C}^{m_0(w)-1}_w \rightarrow \overline{C}^{m_0(w)}_w$ is surjective and therefore $\overline{H}^{m_0(w)}_w = 0$.

### 7.3 Case of algebra of Hamiltonian vector fields

For a given non-trivial $h$-homogeneous Poisson structure on $\mathbb{R}^n$, let $\overline{g}$ be the polynomial algebra of $\mathbb{R}^n$ by the Poisson structure. And we can consider the subalgebra $g = \overline{g}/\text{Cent}_{\pi}$, which is modulo Casimir polynomials $\text{Cent}_{\pi}$. Now we have to handle $\dim \overline{\Xi}_j$ and so cochain complexes $\overline{C}^m_w$ carefully. Let us denote $\dim \overline{\Xi}_j$ by $\phi_j$ which is dependent on the Poisson structure. In general, $\phi_j = \dim \overline{\Xi}_j \leq \dim \overline{\Xi}_j = \text{md}_j = (n-1+j)$. Since the key discussion in the proofs of Proposition 7.1, 7.2 or Theorem 7.1 is to check the dimensional condition, only by replacing $\text{md}_j$ by $\phi_j$, we have analogous results of Proposition 7.1, 7.2 and Theorem 7.1 as follows:

Let

\[(7.8) \quad m_1 = \sum_{j=1}^{\ell} \phi_j\]

and

\[(7.9) \quad w_1 = \sum_{j=1}^{\ell} (j - 2 + h)\phi_j\]

where $h$ the homogeneity of our Poisson structure(tensor). Then we have

**Proposition 7.3** Let $w_1$ and $m_1$ be as above, then we have the following

\[\overline{C}^m_{w_1} = (0) \quad \text{if} \quad m > m_1\]

and

\[\overline{C}^{m_1}_{w_1} = \Lambda^{\phi_1}_{\overline{\Xi}_1} \otimes \Lambda^{\phi_2}_{\overline{\Xi}_2} \otimes \cdots \otimes \Lambda^{\phi_\ell}_{\overline{\Xi}_\ell} \quad \text{(which is 1 dimensional)}.

Also we have

**Proposition 7.4** For the same $h, w_1$ and $m_1$, when $h = 1$ the Young diagram defined by $k_1 = \phi_1 - 1$ and $k_j = \phi_j$ ($j = 2 \ldots \ell$) is a direct summand of $\overline{C}^{m_1-1}_{w_1}$. The Young diagram defined by $k_j = \phi_j$ ($j = 1 \ldots \ell - 1$), $k_\ell = \phi_\ell - 2$ and $k_{2\ell-1} = 1$ is another direct summand of $\overline{C}^{m_1-1}_{w_1}$. 

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When $h > 1$, if $\ell > 1$ then $k_1 = \phi_1 - 1$, $k_j = \phi_j$ ($j = 2 \ldots \ell - 1$), $k_\ell = \phi_\ell - 1$ and $k_{\ell+h-1} = 1$. If $h > 1$ and $\ell = 1$ then $k_1 = \phi_1 - 2$, $k_h = 1$ is a summand of the direct sum of $C_{m_1}^{w_1-1}$.

Combining the two Propositions above, we get the following theorem.

**Theorem 7.2** Let $g$ be the Lie algebra of polynomials defined by a non-trivial $h$-homogeneous Poisson structure on $\mathbb{R}^n$. For the weight $w$ given by $w = \sum_{j=1}^{\ell}(j - 2 + h)\phi_j$, the degree of the last cochain complex is $m_1(w) = \sum_{j=1}^{\ell}\phi_j$ and the Betti number $\dim H_{w_1}^{m_1(w)} = 0$ for each $h$.

8 **Euler characteristic of homogeneous Poisson structures**

We deal with Lie algebra cohomology groups of Lie algebra $g = \mathbb{R}[x_1, \ldots, x_n]/\mathbb{R}$ with a $h$-homogeneous Poisson structure on $\mathbb{R}^n$.

For given non-negative integers $w$ and $m$, $m$-th cochain space with the weight $w$ is given by

$$C_{w, m} := \sum \Lambda^{k_1} S_1 \otimes \Lambda^{k_2} S_2 \otimes \cdots \otimes \Lambda^{k_\ell} S_\ell$$

with the conditions

$$(8.1) \quad k_1 + k_2 + \cdots = m \quad \text{and} \quad \sum_{j=1}^{\ell}(j + h - 2)k_j = w.$$ 

In the subsection 5.3, we know the sequences $[k_1, k_2, \ldots]$ satisfying the above two conditions are equal to $\nabla_m^{w+(2-h)m}$ (= the set of Young diagrams with area $w + (2 - h)m$ and of length $m$). Since the Euler characteristic of $\{\nabla_{m, w}\}$ is equal to $\sum_{m} (-1)^m \dim C_{w, m}$, or that of $\{H_{m, w}\}$ is equal to $\sum_{m} (-1)^m \dim C_{w, m}$, we have to manipulate $\dim C_{w, m} = \sum_{\lambda \in \nabla_{m}^{w+(2-h)m}} \dim \lambda$. Let $\lambda \in \nabla_{m, w}$ and $\lambda = [k_1, \ldots, k_p]$ with $k_p > 0$ in our notation. The dimension of $\lambda$ as a direct summand of cochain space $C_{w, m}$ is

$$\dim \lambda = \binom{m_1}{k_1} \cdots \binom{m_p}{k_p}$$

where $m_j := \dim S_j = (n-1+j)$. Before applying recursive formula (5.10), we study about $B \cdot \lambda$ and $T_{m} \cdot \lambda$ for $\lambda \in \nabla_{m}^{w+(1-h)m}$. Since $B \cdot \lambda = [1 + k_1, k_2, \ldots, k_p]$,

$$\dim(B \cdot \lambda) = \binom{m_1}{1+k_1} \cdot \binom{m_2}{k_2} \cdots \binom{m_p}{k_p},$$
\[
\frac{\dim(B \cdot \lambda)}{\dim \lambda} = \frac{\binom{n}{1+k_1}}{\binom{n}{k_1}} = \frac{n - k_1}{1 + k_1} = \frac{1 + n}{1 + k_1} - 1,
\]

thus, we have

\begin{equation}
(8.2) \quad \dim(B \cdot \lambda) = (\frac{1 + n}{1 + k_1} - 1) \dim \lambda, \text{ where } \lambda = [k_1, k_2, \ldots] \text{ in our notation.}
\end{equation}

From Corollary 5.4, we see

\begin{equation}
(8.3) \quad \dim(T_m \cdot \lambda) = \left(\prod_{i=2}^{p+1} \binom{\text{md}_i}{0}\right) \prod_{i=2}^{p+1} \binom{\text{md}_i}{k_i}, \text{ where } k_i = k_{i-1}' + 1, \lambda = [k_1', k_2', \ldots, k_p'] \text{ in our notation.}
\end{equation}

Applying recursive formula (5.10) for \(\overline{C}_w^m = \nabla_{m-1}^{w+(2-h)m} \cup T_m \cdot \nabla_m^{w+(1-h)m} = B \cdot \overline{C}_{w+1-h}^{m-1} \cup T_m \cdot \nabla_m^{w+(1-h)m}\), we have

\begin{equation}
(8.4) \quad \overline{C}_w^m = B \cdot \nabla_{m-1}^{w+(2-h)m} \cup T_m \cdot \nabla_m^{w+(1-h)m} = B \cdot \overline{C}_{w+1-h}^{m-1} \cup T_m \cdot \nabla_m^{w+(1-h)m},
\end{equation}

where \(\nabla_m^{w+(1-h)m} = 0\) if \(w + (1 - h)m < m\). So, using (8.2) we see that

\[
\sum_m (-1)^m \dim \overline{C}_w^m = \sum_m (-1)^m \sum_{\lambda \in \overline{C}_{w+1-h}^{m-1}} \dim(B \cdot \lambda) + \sum_m (-1)^m \dim(T_m \cdot \nabla_m^{w+(1-h)m})
\]

\[
= \sum_m (-1)^m \sum_{\lambda = (k_1, k_2, \ldots) \in \overline{C}_{w+1-h}^{m-1}} (-1 + \frac{1 + n}{1 + k_1}) \dim \lambda + \sum_m (-1)^m \dim(T_m \cdot \nabla_m^{w+(1-h)m})
\]

\[
= \sum_m (-1)^{m-1} \dim \overline{C}_{w+1-h}^{m-1} + \sum_m (-1)^m \sum_{\lambda = (k_1, k_2, \ldots) \in \overline{C}_{w+1-h}^{m-1}} \frac{1 + n}{1 + k_1} \dim \lambda
\]

\[
+ \sum_m (-1)^m \dim(T_m \cdot \nabla_m^{w+(1-h)m}).
\]

Thus, we have the next lemma.

**Lemma 8.1** Let \(\overline{\chi}_w^h\) be the Euler characteristic of \(\overline{H}_w^1\) for \(h\)-homogeneous Poisson structure. Then we have the following recursive formula:

\begin{equation}
(8.5) \quad \overline{\chi}_w^h = \overline{\chi}_{w+1-h} + \sum_m (-1)^m \sum_{\lambda = (k_1, k_2, \ldots) \in \overline{C}_{w+1-h}^{m-1}} \frac{1 + n}{1 + k_1} \dim \lambda
\]

\[
+ \sum_m (-1)^m \sum_{\lambda = (k_1, k_2, \ldots) \in \nabla_m^{w+(1-h)m}} \prod_{i=2}^{p+1} \binom{\text{md}_i}{k_i}.
\]

**Remark 8.1** The above (8.5) shows \(h = 1\) is special and (8.5) tells us nothing about \(\overline{\chi}_w^h = 1\).

When \(h = 0\), we have a very clear property.
**Theorem 8.1** For each weight $w$, $\chi_{w}^{h=0} = \chi_{w+1}^{h=0}$ holds. In particular $\chi_{w}^{h=0} = 0$.

**Proof:** We simply show that the two terms in (8.5) of Lemma 8.1 are zero. $\lambda = (k_1, k_2, \ldots) \in C_{w+1}^{m-1}$ says that $\sum k_j = m - 1$ and $\sum (j - 2)k_j = w + 1$, so the first term is zero as below.

\[
\sum_{\lambda=(k_1,k_2,\ldots)\in C_{w+1}^{m-1}} (-1)^{1+\sum k_j} \frac{1+n}{1+k_1} \prod_{j}^{(md_j)} = \sum_{(j-2)k_j=w+1} (-1)^{1+\sum k_j} \frac{1+n}{1+k_1} \prod_{j}^{(md_j)} = 0.
\]

Concerning the second term, $\lambda = (k_1, k_2, \ldots) \in \nabla_{m}^{w+m}$ means $\sum k_j = m$ and $\sum jk_j = w + m = w + \sum k_j$, i.e., $\sum (j-1)k_j = w$. Thus,

\[
\sum_{m} (-1)^{m} \sum_{\lambda=(k_1,k_2,\ldots)\in \nabla_{m}^{w+m}} \prod_{i}^{(md_{i+1})} = \sum_{\sum (i-1)k_i=w} (-1)^{\sum k_i} \prod_{i}^{(md_{i+1})} = \sum_{k_1} (-1)^{k_1} \prod_{(i-1)k_i=w}^{(md_{i+1})} = 0.
\]

Here $k_1$ is an element of only Young diagram so not required the dimensional restriction.

Since we have known that $\chi_{w}^{h=0} = \chi_{w+1}^{h=0}$ for each weight $w$, we show $\chi_{w}^{h=0} = 0$ for some special $w$. Now we will show this at the minimum weight $n$ where the space dimension is $n$.

From the definition of $C_{m-n}^{-n} \{(k_1, k_2, \ldots)\}$ satisfies $\sum jk_j = m$ and $\sum (j-2)k_j = -n$, and $k_j \leq md_j = \binom{n-1+j}{n-1}$ for each $j$. We see directly that $k_1 = n$ and $k_j = 0$ for $j > 2$, and $k_2 = m - n$. Thus $C_{m-n}^m = \Lambda^n \nabla_1 \otimes \Lambda^{m-n} \nabla_2$.

Therefore,

\[
\sum_{m} (-1)^{m} \dim C_{m-n}^m = \sum_{m} (-1)^{m} \dim(\Lambda^n \nabla_1) \dim(\Lambda^{m-n} \nabla_2) = \sum_{m} (-1)^{m} \dim(\Lambda^m \nabla_0) = 0.
\]

**8.1 Our theorem and its proof**

In the previous section, we have dealt with some concrete examples and we expected the Euler characteristic of Lie algebra cohomology of Lie Poisson structure is zero. We have the following theorem for Lie Poisson structures.

**Theorem 8.2** On $\mathbb{R}^n$, consider a Poisson structure of homogeneity 1. Then for each given weight $w$, the alternating sum of $\dim C_w^m$ is 0, namely, the Euler characteristic of $\{H_w\}$ is 0. Also, the alternating sum of $\dim C_w^m$ is 0, namely, the Euler characteristic of $\{H_w^*\}$ is 0.
Proof: Since \( h = 1 \), \( m \)-th cochain space with weight \( w \) is \( \overline{C}^m_w = \nabla^{w+m}_m \). When \( w = 0 \), then \( \overline{C}^m_0 = \nabla^m_m = \{ T_m \} = \{ (k_1 = m) \} \) and \( \dim \overline{C}^m_0 = \dim \Lambda^n \mathbb{R} = \binom{n}{m} \) and so \( \sum_m (-1)^m \dim \overline{C}^m_0 = \sum_m (-1)^m \binom{n}{m} = 0 \).

When \( w > 0 \), applying (5.6) to \( \overline{C}^m_w \), we have

\[
\overline{C}^m_w = \nabla^{w+m}_m = T_m \cdot (\nabla_1^w + \cdots + \nabla_w^w).
\]

In Corollary 5.4, we see that if a Young diagram \( \lambda = [k_1, k_2, \ldots] \) then \( T_m \cdot \lambda = [m - \sum_i k_i, k_1, k_2, \ldots] \). Thus if \( \lambda \in \nabla^w_j \) then \( T_m \cdot \lambda = [m - j, k_1, k_2, \ldots] \) and

\[
\dim(T_m \cdot \lambda) = \binom{n}{m-j} \binom{\text{md}_j}{k_1} \binom{\text{md}_j}{k_2} \cdots.
\]

Therefore,

\[
\sum_m (-1)^m \dim \overline{C}^m_w = \sum_{m=1}^w \sum_{j=1}^m (-1)^m \dim(T_m \cdot \nabla^w_j) = \sum_{j=1}^w \sum_{\lambda \in \nabla^w_j} \sum_m (-1)^m \dim(T_m \cdot \lambda)
= \sum_{j=1}^w \sum_{\lambda=(k_1,\ldots,k_p) \in \nabla^w_j} \left( \sum_m (-1)^m \binom{n}{m-j} \binom{\text{md}_j}{k_1} \cdots \binom{\text{md}_j}{k_p} \right) = \sum_{j=1}^w \sum_{\lambda \in \nabla^w_j} 0 = 0.
\]

\[\blacksquare\]

8.2 Examples with \( h = 2 \)

Since the Euler characteristic is given by \( \sum_m (-1)^m \dim \overline{C}^m_w \), and each \( \overline{C}^m_w \) depends on \( n, w, m \) and only the homogeneity \( h \) of Poisson structures, if we pick up some \( h \)-homogeneous Poisson structure and calculate all the Betti numbers and get the Euler characteristic, then that Euler characteristic is common for the same \( n \) and \( h \).

Here, we handle homogeneity 2 cases and show that the Euler characteristic is not necessarily zero. Since the normal form of analytic Poisson structures are studied by J.F. Conn ([11]) and these are locally Lie Poisson structures, we have several cases of 2-homogeneous Poisson structures on \( \mathbb{R}^3 \).

- **case-1:** \( \pi = \frac{1}{2}(x_1^2 \partial_2 \wedge \partial_3 + x_2^2 \partial_3 \wedge \partial_1 + x_3^2 \partial_1 \wedge \partial_2) \),
- **case-2:** \( \pi = x_1 x_2 \partial_1 \wedge \partial_2 + x_2 x_3 \partial_2 \wedge \partial_3 + x_3 x_1 \partial_3 \wedge \partial_1 \),
- **case-3:** \( \pi = x_1^2 \partial_2 \wedge \partial_3 + x_3 x_1 \partial_3 \wedge \partial_1 + x_1 x_2 \partial_1 \wedge \partial_2 \).
Since \( h = 2 \), we see \( \{\tilde{S}_i, \tilde{S}_j\}_i \subset \tilde{S}_{i+j} \) and so we have

\[
\tilde{d}(\tilde{e}_1) = 0, \quad \tilde{d}(\tilde{e}_2) \subset \Lambda^2\tilde{e}_1, \quad \tilde{d}(\tilde{e}_3) \subset \tilde{e}_1 \otimes \tilde{e}_2, \quad \tilde{d}(\tilde{e}_4) \subset \tilde{e}_1 \otimes \tilde{e}_3 \otimes \Lambda^2\tilde{e}_2, \\
\tilde{d}(\tilde{e}_5) \subset \tilde{e}_1 \otimes \tilde{e}_4 \otimes \tilde{e}_2 \otimes \tilde{e}_3.
\]

The list below are the cochain complexes of \( n=3 \) (3 variables), homogeneity is 2. The subindex of \( \overline{C}_w \) is the weight.

\[
\begin{align*}
\overline{C}_1^1 &= \overline{e}_1 (3 \text{ dim}), \\
\overline{C}_2^1 &= \overline{e}_2 (6 \text{ dim}), \\
\overline{C}_2^2 &= \Lambda^2\overline{e}_1 (3 \text{ dim}), \\
\overline{C}_3^1 &= \overline{e}_3 (10 \text{ dim}), \\
\overline{C}_3^2 &= \overline{e}_1 \otimes \overline{e}_2 (18 \text{ dim}), \\
\overline{C}_3^3 &= \Lambda^3\overline{e}_1 (1 \text{ dim}), \\
\overline{C}_4^1 &= \overline{e}_4 (15 \text{ dim}), \\
\overline{C}_4^2 &= \overline{e}_1 \otimes \overline{e}_3 + \Lambda^2\overline{e}_2 (45 \text{ dim}), \\
\overline{C}_4^3 &= \Lambda^2\overline{e}_1 \otimes \overline{e}_2 (18 \text{ dim}), \\
\overline{C}_5^1 &= \overline{e}_5 (21 \text{ dim}), \\
\overline{C}_5^2 &= (\overline{e}_1 \otimes \overline{e}_4) + (\overline{e}_2 \otimes \overline{e}_3) (105 \text{ dim})
\end{align*}
\]

\[
\begin{align*}
\overline{C}_5^3 &= (\Lambda^2\overline{e}_1 \otimes \overline{e}_3) + (\overline{e}_1 \otimes \Lambda^2\overline{e}_2) (75 \text{ dim}), \\
\overline{C}_5^4 &= \Lambda^3\overline{e}_1 \otimes \overline{e}_2 (6 \text{ dim}), \\
\overline{C}_6^1 &= \overline{e}_6 (28 \text{ dim}), \\
\overline{C}_6^2 &= \overline{e}_1 \otimes \overline{e}_5 + \overline{e}_2 \otimes \overline{e}_4 + \Lambda^2\overline{e}_3 (198 \text{ dim}), \\
\overline{C}_6^3 &= \Lambda^2\overline{e}_1 \otimes \overline{e}_4 + \overline{e}_1 \otimes \overline{e}_2 \otimes \overline{e}_3 + \Lambda^3\overline{e}_2 (245 \text{ dim}), \\
\overline{C}_6^4 &= \Lambda^3\overline{e}_1 \otimes \overline{e}_3 + \Lambda^2\overline{e}_1 \otimes \Lambda^2\overline{e}_2 (55 \text{ dim}), \\
\overline{C}_7^1 &= \overline{e}_7 (36 \text{ dim}), \\
\overline{C}_7^2 &= \overline{e}_1 \otimes \overline{e}_6 + \overline{e}_2 \otimes \overline{e}_5 + \overline{e}_3 \otimes \overline{e}_4 (360 \text{ dim}), \\
\overline{C}_7^3 &= \Lambda^2\overline{e}_1 \otimes \overline{e}_5 + \overline{e}_1 \otimes \overline{e}_2 \otimes \overline{e}_4 + \overline{e}_1 \otimes \Lambda^2\overline{e}_3 + \Lambda^2\overline{e}_2 \otimes \overline{e}_3 (618 \text{ dim}), \\
\overline{C}_7^4 &= \Lambda^3\overline{e}_1 \otimes \overline{e}_4 + \Lambda^2\overline{e}_1 \otimes \overline{e}_2 \otimes \overline{e}_3 + \overline{e}_1 \otimes \Lambda^3\overline{e}_2 (255 \text{ dim}), \\
\overline{C}_7^5 &= \Lambda^3\overline{e}_1 \otimes \Lambda^2\overline{e}_2 (15 \text{ dim}).
\end{align*}
\]

From the list above, we see that the Euler characteristic for each weight varies as follows when \( n = 3 \):

| weight | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------|---|---|---|---|---|---|---|
| Euler number | -3 | -3 | 7 | 12 | 15 | -20 | -54 |

On the symplectic space \( \mathbb{R}^2 \), namely for homogeneity 0 non-degenerate Poisson structure, we know some subalgebras whose Euler characteristics are not necessarily zero (cf. [3], [9]).

### 8.3 Abstract way to get Euler characteristic for \( h = 2 \)

Since the cochain space \( \overline{C}_w^m \) with \( h \) corresponds to \( \nabla^{m+(2-h)m}_m \), when \( h = 2 \) the cochain space

\[
\overline{C}_w^m = \nabla^w_m \quad \text{for} \quad m = 0, \ldots, w.
\]
Concerning to $m = 0$, $\overline{C}_w^0 = 0$ when $w > 0$, and $\overline{C}_0^0 = \mathbb{R}$ as an unique exception.

### 8.3.1 $h = 2$ and $w = 0$

$w = 0$ is exceptional and $\overline{C}_0^0 = \mathbb{R}$, thus $\overline{X}_{w=0}^{h=2} = 1$.

### 8.3.2 $h = 2$ and $w = 1$

$\overline{C}_1^0 = 0$ and $\overline{C}_1^1 = \nabla_1^1 = \{T_1\} = \{(k_1 = 1)\}$. Thus $\overline{X}_{w=1}^{h=2} = -\dim \nabla_1 = -n$.

### 8.3.3 $h = 2$ and general $w$

From Lemma 8.1, if we put $h = 2$ then we have a relation between $\overline{X}_{[w]}$ and $\overline{X}_{[w-1]}$ as follows:

\[
(8.6) \quad \overline{X}_{w=2}^{h=2} = \overline{X}_{w-1}^{h=2} + \sum_{m=1}^{w} (-1)^m \left( \sum_{\lambda \in \nabla_{m-1}^1} \frac{1 + n}{1 + k_1} \dim \lambda \right) + \sum_{m=1}^{w} (-1)^m \left( \sum_{\lambda \in \nabla_{m-2}^1} \left( \sum_{k_1}^{(m_2)} \cdots \sum_{k_p}^{(m_{p+1})} \right) \right)
\]

\[w = 2:\]

2nd term of (8.6) = \[\sum_{m=1}^{2} (-1)^m \left( \sum_{\lambda \in \nabla_{m-1}^1} \frac{1 + n}{1 + k_1} \dim \lambda \right) = (-1)^2 \sum_{\lambda \in \nabla_1^1} \frac{1 + n}{1 + k_1} \dim \lambda = \frac{1 + n}{2}\]

3rd term of (8.6) = \[\sum_{m=1}^{2} (-1)^m \left( \sum_{\lambda \in \nabla_{m-2}^1} \left( \sum_{k_1}^{(m_2)} \cdots \sum_{k_p}^{(m_{p+1})} \right) \right) = (-1)^1 \sum_{\lambda \in \nabla_1^1} \left( \sum_{k_1}^{(m_2)} \cdots \sum_{k_p}^{(m_{p+1})} \right) = \frac{1 + n}{2} - (n+1)^2\]

Thus, $\overline{X}_2^{h=2} = \overline{X}_1^{h=2} + 2$nd term + 3rd term = $\overline{X}_1^{h=2} + \frac{1 + n}{2} - (n+1)^2 = \overline{X}_1^{h=2} = -n$.

\[w = 3:\]

2nd term of (8.6) = \[\sum_{m=1}^{3} (-1)^m \left( \sum_{\lambda \in \nabla_{m-1}^2} \frac{1 + n}{1 + k_1} \dim \lambda \right) = (-1)^2 \sum_{\lambda \in \nabla_1^2} \frac{1 + n}{1 + k_1} \dim \lambda = \frac{1 + n}{2} - \frac{1 + n}{0} \dim T_2 = (1 + n)\left(\frac{n+1}{2}\right) - \frac{1 + n}{\frac{n}{2}}\]

3rd term of (8.6) = \[\sum_{m=1}^{3} (-1)^m \left( \sum_{\lambda \in \nabla_{m-2}^2} \left( \sum_{k_1}^{(m_2)} \cdots \sum_{k_p}^{(m_{p+1})} \right) \right) = (-1)^1 \sum_{\lambda \in \nabla_1^2} \left( \sum_{k_1}^{(m_2)} \cdots \sum_{k_p}^{(m_{p+1})} \right) = -(n+1)^3\]

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Thus,

\[ \overline{X}^{h=2}_{3} = \overline{X}^{h=2}_{2} + 2\text{nd term} + 3\text{rd term} = -n + (1 + n)(\binom{n-1+2}{2} - \frac{1+n}{3}\binom{n}{2}) - \binom{n-1+3}{3} \]

\[ = -\frac{1}{6}n(n - 1)(n + 4). \]

\[ w = 4: \]

2nd term of (8.6) = \[ \sum_{m=1}^{4} (-1)^m \sum_{\lambda \in \mathcal{V}_{1,4-m}} \frac{1+n}{1+k_{1}} \dim \lambda \]

\[= (-1)^2 \sum_{\lambda \in \mathcal{V}_{1}} \frac{1+n}{1+k_{1}} \dim \lambda + (-1)^3 \sum_{\lambda \in \mathcal{V}_{2}} \frac{1+n}{1+k_{1}} \dim \lambda + (-1)^4 \sum_{\lambda \in \mathcal{V}_{3}} \frac{1+n}{1+k_{1}} \dim \lambda \]

\[= \frac{1+n}{1+0} \left( \dim \overline{X}_{3} \right) - \frac{1+n}{1+2} \dim \overline{X}_{1} \dim \overline{X}_{2} + \frac{1+n}{1+3} \left( \dim \overline{X}_{3} \right) \]

\[= (1+n)^{\binom{n-1+3}{3}} - \frac{1+n}{3} \left( \frac{n-1+1}{1} \right) \left( \frac{n-1+2}{2} \right) + \frac{1+n}{4} \left( \frac{n}{3} \right) \]

\[= - \frac{1}{24}n^4 + \frac{1}{12}n^3 + \frac{13}{24}n^2 + \frac{5}{12}n \]

3rd term of (8.6) = \[ \sum_{m=1}^{4} (-1)^m \sum_{\lambda \in \mathcal{V}_{1,4-m}} \binom{m_{d_{2}}}{k_{1}} \cdots \binom{m_{d_{p+1}}}{k_{p}} \]

\[= (-1)^1 \sum_{\lambda \in \mathcal{V}_{1}} \binom{m_{d_{2}}}{k_{1}} \cdots \binom{m_{d_{p+1}}}{k_{p}} + (-1)^2 \sum_{\lambda \in \mathcal{V}_{2}} \binom{m_{d_{2}}}{k_{1}} \cdots \binom{m_{d_{p+1}}}{k_{p}} \]

\[= - \binom{m_{d_{1}}}{1} + \binom{m_{d_{2}}}{2} = \frac{1}{12}n^4 - \frac{7}{12}n^2 - \frac{1}{2}n \]

Thus, \[ \overline{X}^{h=2}_{4} = \overline{X}^{h=2}_{3} + 2\text{nd term} + 3\text{rd term} = \frac{1}{24}n^4 + \frac{1}{4}n^3 + \frac{11}{24}n^2 - \frac{3}{4}n = \frac{1}{24}n(n - 1)(n^2 + 7n + 18). \]

Remark 8.2 It seems to be interesting to find the generating function of \( \{ \overline{X}^{h=2}_{w} \}_{w} \).

8.4 Euler characteristic for \( h = 3 \)

We put \( h = 3 \) in (8.5) of Lemma 8.1, we have a recursive formula for \( h = 3 \):

\[ \overline{X}^{h=3}_{w} = \overline{X}^{h=3}_{w-2} + \sum_{m} (-1)^m \sum_{\lambda = (k_{1}, k_{2}, \ldots) \in \mathcal{V}_{m-1}} \frac{1+n}{1+k_{1}} \dim \lambda \]

\[+ \sum_{m} (-1)^m \sum_{\lambda = (k_{1}, k_{2}, \ldots) \in \mathcal{V}_{m-2m}} \prod_{i} \binom{m_{d_{i+1}}}{k_{i}}. \]
We use the above recursive formula or original definition \( \dim C^m_w = \sum_{\lambda \in V^m_{m-m}} \dim \lambda \) and we see the Euler characteristic for some \( w \) as follows:

\[
\begin{align*}
\chi^{h=3}_0 &= 1, \\
\chi^{h=3}_1 &= 0, \\
\chi^{h=3}_2 &= -n, \\
\chi^{h=3}_3 &= -md_2, \\
\chi^{h=3}_4 &= -md_3 + \binom{n}{2}, \\
\chi^{h=3}_5 &= -md_4 + md_1 md_2, \\
\chi^{h=3}_6 &= -md_5 + md_1 md_3 + \binom{md_2}{2} - \binom{n}{3}.
\end{align*}
\]

Still we are interested in the generating function of \( \{\chi^{h=3}_w\}_w \).

## 9 Combinatorial approach to Poisson cohomology

In Poisson geometry, the Poisson cohomology group is well-known as follows.

**Definition 9.1** For each natural number \( m \), let us consider the vector space \( C^m = \Lambda^m T(M) \) of all \( m \)-vector fields on \( M \). Then we define a linear map \( d : C^m \to C^{m+1} \) by the Schouten bracket as \( d(U) = [\pi, U]_S \).

Then \( d \circ d = 0 \) follows due to the property \([\pi, \pi]_S = 0\) and the Jacobi identity of Schouten bracket. Thus, we have the Poisson cohomology group \( \{U \in \Lambda^m T(M) \mid [\pi, U]_S = 0\} / \{[\pi, W]_S \mid W \in \Lambda^{m-1} T(M)\} \).

Although the definition of Poisson cohomology is clear, calculation is not easy in general because cochain complexes are huge.

### 9.1 Poisson-like cohomology

Here, we discuss “Poisson-like” cohomology for a given homogeneous “Poisson-like” structure restricting cochain spaces to vector fields with polynomial coefficients, and also the notion of “weight” to reduce our discussion in finite dimensional vector spaces.

**Definition 9.2** Let \( X_{pol} \) be \( \{X \in \mathfrak{X}(\mathbb{R}^n) \mid \langle dx_j, X \rangle \text{ are polynomials for each } j\} \), and let \( X_\ell \) be \( \{X \in X_{pol} \mid \langle dx_j, X \rangle \text{ are } \ell\text{-homogeneous polynomials for each } j\} \) for each non-negative integer \( \ell \).

We see that \( X_{pol} = \bigoplus_\ell X_\ell \) as \( \mathbb{R} \)-vector space. Thus, the exterior 2-power of \( X_{pol} \) is

\[
\Lambda^2 X_{pol} = X_{pol} \wedge X_{pol} = \sum_{i \leq j} X_i \wedge X_j \quad \text{(direct sum)}
\]
as \( \mathbb{R} \)-modules, for instance.
Remark 9.1 We have $x_1 \partial_1 \in \mathfrak{x}_1$ and $x_1^2 \partial_1 \in \mathfrak{x}_2$. $(x_1 \partial_1) \wedge (x_1^2 \partial_1) \neq 0$ as $\mathbb{R}$-modules but as $C^\infty(\mathbb{R}^n)$-modules we see that $(x_1 \partial_1) \wedge (x_1^2 \partial_1) = x_1^3 \partial_1 \wedge \partial_1 = 0$.

For each natural number $m$, we consider $\Lambda^m \mathfrak{x}_{pol}$ and have natural $\mathbb{R}$-module decomposition

$$\Lambda^m \mathfrak{x}_{pol} = \sum_{m = k_0 + k_1 + \cdots} \Lambda^{k_0} \mathfrak{x}_0 \otimes \Lambda^{k_1} \mathfrak{x}_1 \otimes \cdots \Lambda^{k_\ell} \mathfrak{x}_\ell.$$ 

Since $\dim \mathfrak{x}_j = (n - 1 - j) n$, we have restrictions

$$0 \leq k_j \leq \dim \mathfrak{x}_j = (n - 1 + j) n.$$ 

Definition 9.3 Let us fix a non-negative integer $h$ (which plays a role of the homogeneity of homogeneous Poisson-like 2-vector later). We define the weight $w$ of a non-zero element of $\Lambda^{k_0} \mathfrak{x}_0 \otimes \Lambda^{k_1} \mathfrak{x}_1 \otimes \cdots \Lambda^{k_\ell} \mathfrak{x}_\ell$ to be

$$w = k_0 (0 + 1 - h) + k_1 (1 + 1 - h) + \cdots + k_\ell (\ell + 1 - h).$$ 

Definition 9.4 For each $m$ and $w$, define a vector subspace

$$C^m_w := \sum_{\text{"our cond"}} \Lambda^{k_0} \mathfrak{x}_0 \otimes \Lambda^{k_1} \mathfrak{x}_1 \otimes \cdots \Lambda^{k_\ell} \mathfrak{x}_\ell.$$ 

The "our cond" are (9.1), (9.2) and

$$k_0 + k_1 + \cdots + k_\ell = m.$$ 

Now, we restrict the Schouten bracket of $\oplus \Lambda^* T(\mathbb{R}^n)$ to $\oplus \Lambda^* \mathfrak{x}_{pol}$ and have a new bracket $[\cdot, \cdot]_\mathbb{R}$ and we call the $\mathbb{R}$-Schouten bracket.

Definition 9.5 The $\mathbb{R}$-Schouten bracket is characterized as follows (almost the same in the subsection 2.2):

For $P \in \Lambda^p \mathfrak{x}_{pol}$ and $Q \in \Lambda^q \mathfrak{x}_{pol}$, $[P, Q]_\mathbb{R} \in \Lambda^{p+q-1} \mathfrak{x}_{pol}$ holds, and

$$[Q, P]_\mathbb{R} = -(-1)^{(q+1)(p+1)} [P, Q]_\mathbb{R} \quad \text{(symmetry)},$$

$$0 = \sum_{p, q, r} (-1)^{(p+1)(r+1)} [P, [Q, R]_\mathbb{R}]_\mathbb{R} \quad \text{(the Jacobi identity)},$$

$$[P, Q \wedge R]_\mathbb{R} = [P, Q]_\mathbb{R} \wedge R + (-1)^{(p+1)q} Q \wedge [P, R]_\mathbb{R},$$

$$[P \wedge Q, R]_\mathbb{R} = P \wedge [Q, R]_\mathbb{R} + (-1)^{q(r+1)} [P, R]_\mathbb{R} \wedge Q,$$
another expression of Jacobi identity is the next

\[ [P, [Q, R]_\mathbb{R}]_\mathbb{R} = [[P, Q]_\mathbb{R}, R]_\mathbb{R} + (-1)^{(p+1)(q+1)}[Q, [P, R]_\mathbb{R}]_\mathbb{R} \]

\[ [[P, Q]_\mathbb{R}, R]_\mathbb{R} = [P, [Q, R]_\mathbb{R}]_\mathbb{R} + (-1)^{(q+1)(r+1)}[[P, R]_\mathbb{R}, Q]_\mathbb{R} \]

\[ [X, Y]_\mathbb{R} = \text{Jacobi-Lie bracket of } X \text{ and } Y \]

The \( \mathbb{R} \)-Schouten bracket also has an explicit expression given by

\[
(9.4) \quad [u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_q]_\mathbb{R} = \sum_{i,j} (-1)^{i+j} [u_i, v_j]_\mathbb{R} \wedge (u_1 \wedge \cdots \widehat{u_i} \cdots \wedge u_p) \wedge (v_1 \wedge \cdots \widehat{v_j} \cdots \wedge v_q)
\]

where \( u_i, v_j \in \mathfrak{x}_{pol} \) and \( \widehat{u_i} \) means omitting \( u_i \).

We have the same property that the ordinary Schouten bracket has.

**Proposition 9.1** Let \( \pi \in \Lambda^2 \mathfrak{x}_{pol} \) and \( P \in \Lambda^p \mathfrak{x}_{pol} \). Then

\[
(9.5) \quad 2[\pi, [\pi, P]_\mathbb{R}]_\mathbb{R} + [P, [\pi, \pi]_\mathbb{R}]_\mathbb{R} = 0
\]

holds and so if \( [\pi, \pi]_\mathbb{R} = 0 \) then \( [\pi, [\pi, -]_\mathbb{R}]_\mathbb{R} = 0 \) holds on \( \Lambda^* \mathfrak{x}_{pol} \).

**Definition 9.6** We call a 2-vector \( \pi \in \Lambda^2 \mathfrak{x}_{pol} \) is Poisson-like if \( \pi \) satisfies \( [\pi, \pi]_\mathbb{R} = 0 \).

A Poisson-like 2-vector \( \pi \) is \( h \)-homogeneous if \( \pi \in \bigoplus_{h=i+j, i \leq j} \mathfrak{x}_i \wedge \mathfrak{x}_j \).

**Proposition 9.2** Let \( \pi \) be a \( h \)-homogeneous Poisson-like 2-vector as in Definition 9.6. We see that \( [\pi, C^m_w]_\mathbb{R} \subset C^{m+1}_w \) and we get a sequence of cochain complexes: \( d : C^m_w \to C^{m+1}_w \), and the cohomology groups. We call them the Poisson-like cohomology groups of homogeneous Poisson-like 2-vector on \( \mathbb{R}^n \).

**Proof:** Since \( \pi \in \bigoplus_{h=i+j, i \leq j} \mathfrak{x}_i \wedge \mathfrak{x}_j \), we have

\[
[\pi, \mathfrak{x}_j]_\mathbb{R} \subset \bigoplus_{h=i+j, i \leq j} (\mathfrak{x}_i \wedge \mathfrak{x}_{j+i-1} \oplus \mathfrak{x}_{i+j-1} \wedge \mathfrak{x}_j)
\]

and the weight of \( \mathfrak{x}_i \wedge \mathfrak{x}_{j+i-1} \) is \( (i+1-h) + (j+1-1) = \ell + h \) and the weight of \( \mathfrak{x}_{i+j-1} \wedge \mathfrak{x}_j \) is \( (i+1-h) + (j+1-h) = \ell + 1 - h \). Thus, after applying \( d \), the degree changes to \( m+1 \) but the weight is invariant. \( \blacksquare \)

**Remark 9.2** We have two kinds of Schouten brackets, \([\cdot, \cdot]_S\) and \([\cdot, \cdot]_R\). Let \( \Phi : \oplus \Lambda^* \mathfrak{x}_{pol} \to \oplus \Lambda^* \mathfrak{x}_{pol} \) be the natural map relaxing the \( \mathbb{R} \)-linearity of the wedge product to the ordinary tensorial product. As commented in Remark 9.1, \( (x_1 \partial_1) \wedge (x_1^2 \partial_1) \neq 0 \) as \( \mathbb{R} \)-modules and \( \Phi \left( (x_1 \partial_1) \wedge (x_1^2 \partial_1) \right) = x_1^3 \partial_1 \wedge \partial_1 = 0. \)
From the definition of $[\cdot, \cdot]_\mathbb{R}$, we see quickly that $[\Phi(P), \Phi(Q)]_S = \Phi([P, Q]_\mathbb{R})$ for each $P, Q \in \bigoplus A^* \mathbb{R}$. Thus, for any Poisson-like 2-vector $\pi$, $\Phi(\pi)$ is a Poisson 2-tensor. But, the converse is not true. Namely, let $\overline{\pi}$ be an ordinary Poisson structure. There is no guarantee that each inverse element $u \in \Phi^{-1}(\overline{\pi})$ is Poisson-like. Let $\overline{\pi} = \partial_3 \partial_2 - 2 \partial_3 \partial_1 + \partial_3$, which is a Poisson structure due to $\mathfrak{sl}(2)$ we have already used. Let $\phi = \partial_1 \partial_2 (\partial_3) - 2 \partial_1 \partial_3 + 2 \partial_2 \partial_3$ so that $\Phi(\phi) = \overline{\pi}$. But, $[\phi, \phi]_\mathbb{R}/4$ is calculated to be non-zero as follows.

\[
\partial_1 \partial_2 (\partial_3) - 2 \partial_1 \partial_3 + 2 \partial_2 \partial_3 = 0.
\]

We rewrite (9.2) and (9.3) as follows:

(9.6) \[ 0 k_0 + 1 k_1 + \cdots + k_\ell = w + (h - 1) m , \]

(9.7) \[ k_1 + \cdots + k_\ell = m - k_0 . \]

The first equation means the total area and the second equation means the height of the Young diagram $(k_j \mid j = 1 \ldots \ell)$.

Hereafter, we will show some concrete examples and difference between this cohomology and the cohomologies in previous sections.

### 9.1.1 case $h=0$

(9.2) and (9.3) say that if the degree $m = 0$ then the weight $w = 0$, in other words, if the weight $w \neq 0$ then $C^w_w = 0$. We denote the left-hand-sides of (9.6) and (9.7) by $A$ and $H$ respectively, then using $h = 0$ we have $A = w - m, H = m - k_0$, and so $C^m_w = \sum H \Lambda^m x_0 \otimes \nabla(w - m, H)$, where we suppose $\nabla(0, 0)$ is the singleton of the trivial Young diagram but $\nabla(A, 0)$ with $A > 0$ is the empty set and do not sum up the terms containing these direct summands. We may regard $H$ as a parameter with the restrictions $H \leq w/2, m - n \leq H \leq m$ because by adding $m - H \geq 0, w - m \geq H$ we see $H \leq w/2$. Thus, when $w = 1$ we have

\[
C^m_1 = \sum_{H \leq 1/2} \Lambda^{m-H} x_0 \otimes \nabla(1 - m, H) = \Lambda^{m} x_0 \otimes \nabla(1 - m, 0) = \begin{cases} x_0 & \text{if } m = 1 , \\ \emptyset & \text{otherwise}. \end{cases}
\]

The Euler characteristic for $w = 1$, we may denote it by $\chi^{h=0}_{w=1} = -n$. 

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\[ C_m^2 = \sum_{H \leq 2/2} \Lambda^{m-H} x_0 \otimes \nabla(2-m,H) = \Lambda^{m-1} x_0 \otimes \nabla(2-m,1) + \Lambda^m x_0 \otimes \nabla(2-m,0) \]

\[
\begin{cases}
    x_1 & \text{if } m = 1, \\
    \Lambda^2 x_0 & \text{if } m = 2, \\
    \emptyset & \text{otherwise.}
\end{cases}
\]

\[ \chi_{h=0}^{w=2} = -n(n + 1)/2. \]

\[ C_m^3 = \Lambda^{m-1} x_0 \otimes \nabla(3-m,1) + \Lambda^m x_0 \otimes \nabla(3-m,0) = \begin{cases}
    x_2 & \text{if } m = 1, \\
    x_0 \otimes x_1 & \text{if } m = 2, \\
    \Lambda^3 x_0 & \text{if } m = 3, \\
    \emptyset & \text{otherwise.}
\end{cases} \]

\[ \chi_{h=0}^{w=3} = (n - 1)n(n + 1)/3. \]

\[ C_m^4 = \Lambda^m x_0 \otimes \nabla(4-m,0) + \Lambda^{m-1} x_0 \otimes \nabla(4-m,1) + \Lambda^{m-2} x_0 \otimes \nabla(4-m,2) \]

\[
\begin{cases}
    x_3 & \text{if } m = 1, \\
    x_0 \otimes x_2 + \Lambda^2 x_1 & \text{if } m = 2, \\
    \Lambda^2 x_0 \otimes x_1 & \text{if } m = 3, \\
    \Lambda^4 x_0 & \text{if } m = 4, \\
    \emptyset & \text{otherwise.}
\end{cases}
\]

\[ \chi_{h=0}^{w=4} = -(n - 3)(n - 1)n(n + 2)/8. \]

Since the 0-homogeneous Poisson-like 2-vectors are of the form \( \sum \partial_{2i-1} \wedge \partial_{2i} \) after a suitable change of coordinates, we take \( \pi = \partial_1 \wedge \partial_2 \) when \( n = 3 \). Then

\[ [\pi, w^A \partial_j]_\mathbb{R} = A_1 (w^{A-E_1} \partial_j) \wedge \partial_2 - A_2 (w^{A-E_2} \partial_j) \wedge \partial_1 \]

where \( A \in \mathfrak{m}[k] \) and \( j, k = 1, \ldots, n \). We get Betti numbers as follows.

| \( C_2^1 \rightarrow C_2^2 \) | \( C_2^1 \rightarrow C_2^3 \rightarrow C_2^3 \) | \( C_2^4 \rightarrow C_2^4 \rightarrow C_2^4 \) |
|---|---|---|
| dim | 6 | 9 | 3 |
| ker dim | 6 | 3 | 3 |
| Betti | 6 | 0 | 0 |

| \( C_3^1 \rightarrow C_3^2 \rightarrow C_3^3 \) | \( C_3^1 \rightarrow C_3^3 \rightarrow C_3^3 \) | \( C_3^1 \rightarrow C_3^3 \rightarrow C_3^3 \) |
|---|---|---|
| dim | 18 | 27 | 1 |
| ker dim | 3 | 26 | 1 |
| Betti | 3 | 11 | 0 |

| \( C_4^1 \rightarrow C_4^2 \rightarrow C_4^4 \) | \( C_4^1 \rightarrow C_4^3 \rightarrow C_4^3 \) | \( C_4^1 \rightarrow C_4^3 \rightarrow C_4^3 \) |
|---|---|---|
| dim | 30 | 90 | 27 |
| ker dim | 3 | 63 | 27 |
| Betti | 3 | 36 | 0 |

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9.1.2 case $h=1$

In this case, (9.6) says the weight $w$ is just the total area of Young diagram and (9.7) says its height is $m-k_0$. Thus, $m-k_0 \leq w$ and $m \leq w+n$ from (9.1).

If the weight $w=0$, then $k_j = 0$ ($j > 0$) and $k_0 = m$ and so

$$C^m_0 = \Lambda^m x_0 \quad \text{for} \quad m = 0, \ldots, n.$$ 

If the weight $w=1$, we know $\nabla(1, m-k_0) = \begin{cases} \emptyset & \text{if } m-k_0 \leq 0, \\ \{T_1\} & \text{if } m-k_0 = 1, \end{cases}$

we see $k_0 = m-1$, $k_1 = 1$ and $C^m_1 = \Lambda^{m-1} x_0 \otimes x_1$ for $m = 1, \ldots, n+1$.

In the same way, if when the weight $w=2$, we know $\nabla(2, m-k_0) = \begin{cases} \{T_2\} & \text{if } m-k_0 = 2, \\ \{T_1^2\} & \text{if } m-k_0 = 1, \end{cases}$ we see $(k_0 = m-1, k_1 = 1)$ or $(k_0 = m-2, k_1 = 2)$, and so

$$C^m_2 = \Lambda^{m-1} x_0 \otimes x_2 \oplus \Lambda^{m-2} x_0 \otimes \Lambda^2 x_1.$$ 

If the weight $w=3$, we know $\nabla(3, m-k_0) = \begin{cases} \{T_3\} & \text{if } m-k_0 = 3, \\ \{T_2 \cdot T_1\} & \text{if } m-k_0 = 2, \\ \{T_1^3\} & \text{if } m-k_0 = 1, \\ \emptyset & \text{otherwise}, \end{cases}$

we see $(k_0 = m-3, k_1 = 1)$, $(k_0 = m-2, k_1 = 1, k_2 = 1)$ or $(k_0 = m-1, k_3 = 1)$, thus we have

$$C^m_3 = \Lambda^{m-3} x_0 \otimes \Lambda^3 x_1 \oplus \Lambda^{m-2} x_0 \otimes x_1 \otimes x_2 \oplus \Lambda^{m-1} x_0 \otimes \Lambda^3 x_1.$$ 

Assume $n = 3$ now. Then we get

$$C^0_0 = \mathbb{R}, \quad C^1_0 = x_0, \quad C^2_0 = \Lambda^2 x_0, \quad C^3_0 = \Lambda^3 x_0,$$

$$C^1_1 = x_1, \quad C^2_1 = x_0 \otimes x_1, \quad C^3_1 = \Lambda^2 x_0 \otimes x_1, \quad C^4_1 = \Lambda^3 x_0 \otimes x_1,$$

$$C^2_2 = x_2, \quad C^3_2 = x_0 \otimes x_2 + \Lambda^2 x_1, \quad C^4_2 = \Lambda^2 x_0 \otimes x_2 + x_0 \otimes \Lambda^2 x_1,$$

$$C^3_3 = x_3, \quad C^4_3 = x_0 \otimes x_3 + x_1 \otimes x_2,$$

$$C^3_3 = \Lambda^2 x_0 \otimes x_3 + x_0 \otimes x_1 \otimes x_2 + \Lambda^3 x_1,$$

$$C^4_3 = \Lambda^3 x_0 \otimes x_3 + \Lambda^2 x_0 \otimes x_1 \otimes x_2 + x_0 \otimes \Lambda^3 x_1,$$

$$C^5_3 = \Lambda^3 x_0 \otimes x_1 \otimes x_2 + \Lambda^2 x_0 \otimes \Lambda^3 x_1, \quad C^6_3 = \Lambda^3 x_0 \otimes \Lambda^3 x_1.$$
Remark 9.3 From the above several examples, we expect the Euler characteristic is 0 when \( h = 1 \) likewise as the section 8.

In order to find concrete 1-homogeneous Poisson-like 2-vectors on \( \mathbb{R}^3 \), we prepare a candidate in general form, say

\[
u := \sum_{i,j,k} c_{i,E_j,k} \partial_i \wedge (w E_j \partial_k)\]

with the condition \([u, u] = 0\). Then we have a system of 2-homogeneous polynomials of \( c_{i,E_i,j} \).

It seems hard to know the whole solutions, but one of many solutions is

\[
\pi = (\partial_1 - \partial_3) \wedge (x_1 \partial_3 + x_3 \partial_3).
\]

We show the tables of Betti numbers of Poisson-like cohomologies for weight from 0 to 3.

| wt=0  | \( C_1^0 \to C_0^1 \to C_0^2 \) | wt=1  | \( C_1^1 \to C_1^2 \to C_1^3 \to C_1^4 \) |
|-------|---------------------------------|-------|---------------------------------|
| dim   | 3                               | dim   | 9                               |
| ker dim | 2                               | ker dim | 3                              |
| Betti | 2                               | Betti | 3                              |

| wt=2  | \( C_2^1 \to C_2^2 \to C_2^3 \to C_2^4 \) | |
|-------|---------------------------------|-----|
| dim   | 18                              | 18  |
| ker dim | 4                               | 4   |
| Betti | 4                               | 4   |

| wt=3  | \( C_3^1 \to C_3^2 \to C_3^3 \to C_3^4 \to C_3^5 \to C_3^6 \) |
|-------|---------------------------------|
| dim   | 30                              | 30  |
| ker dim | 3                               | 3   |
| Betti | 3                               | 3   |

9.1.3 case \( h=2 \)

In this subsection we deal with homogeneous Poisson-like 2-vectors with \( h = 2 \). Comparing (9.6) and (9.7), we see that \( -w \leq k_0 \) and so \( w \geq -n \).

If \( \ell > 1 \) in (9.6) or (9.7), then (9.6) – 2(9.7) implies \( m \leq w + 2k_0 + k_1 \leq w + 2n + n^2 \). If \( \ell = 1 \), then \( w = -k_0 \) and \( m = k_0 + k_1 \). If \( \ell = 0 \), \( m = k_0 \) and \( w = -k_0 \).

Finding \( C^m_w \) is equivalent to finding the Young diagrams of height \( m - k_0 \) and the area \( w + m \) for each \( k_0 \).

Since \( w \geq -n \), we put \( w = -n + j \) with non-negative integer \( j \). Then \( k_0 \geq n - j \). When \( j = 0 \), i.e., the weight \( w = -n \), we have \( k_0 = n \) and \( \nabla(-n + m, m - n) = \{ T_{m-n} \} \), this says \( k_1 = m - n \) and \( k_\ell = 0 \) for \( \ell > 1 \).

Thus, we get

\[
(9.8) \quad C^m_{-n} = \Lambda^n x_0 \otimes \Lambda^{m-n} x_1 .
\]

When \( j = 1 \), i.e., the weight \( w = -n + 1 \), we see that \( k_0 = n - 1 \) or \( k_0 = n \). If \( k_0 = n - 1 \), \( \nabla(-n + 1 + m, m - n + 1) = \{ T_{m-n+1} \} \), this says \( k_1 = m - n + 1 \) and \( k_\ell = 0 \) for \( \ell > 1 \). If \( k_0 = n \), then

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\[ \nabla(-n + 1 + m, m - n) = T_{m-n} \cdot \{\nabla(1, 1)\}, \] this says \( k_1 = m - n - 1, k_2 = 1, \) and \( k_{\ell} = 0 \) for \( \ell > 2. \) Thus,

(9.9) \[ C_{1-n}^m = \Lambda^{n-1}X_0 \otimes \Lambda^{m-n+1}X_1 \otimes \Lambda^nX_0 \otimes \Lambda^{m-n-1}X_1 \otimes X_2. \]

When \( j = 2, \) i.e., \( w = -n + 2, \) possibilities of \( k_0 \) are \( k_0 = n - 2, k_0 = n - 1 \) or \( k_0 = n. \) If \( k_0 = n - 2, \)

\[ \nabla(-n + m + 2, m - n + 2) = \{T_{m-n+2}\}, \] we see \( k_1 = m - n + 2, k_{\ell} = 0 \) (\( \ell > 1 \)). If \( k_0 = n - 1, \)

\[ \nabla(-n + m + 2, m - n + 1) = T_{m-n+1} \cdot \nabla(1, 1), \] i.e., \( k_1 = m - n, k_2 = 1, k_{\ell} = 0 \) (\( \ell > 2 \)). If \( k_0 = n, \)

\[ \nabla(-n + m + 2, m - n) = T_{m-n} \cdot (\nabla(2, 1) + \nabla(2, 2)) = T_{m-n} \cdot T_1^2 + T_{m-n} \cdot T_2, \] i.e., \( k_1 = m - n - 1, k_3 = 1, k_{\ell} = 0 \) (\( \ell \neq 1, 3 \)) or \( k_1 = m - n - 2, k_2 = 2, k_{\ell} = 0 \) (\( \ell > 2 \)). Combining those, we have

(9.10) \[ C_{2-n}^m = \Lambda^{n-2}X_0 \otimes \Lambda^{m-n+2}X_1 \otimes \Lambda^{n-1}X_0 \otimes \Lambda^{m-n}X_1 \otimes X_2 \]

\[ \oplus \Lambda^nX_0 \otimes \Lambda^{m-n-1}X_1 \otimes X_3 \oplus \Lambda^nX_0 \otimes \Lambda^{m-n-2}X_1 \otimes \Lambda^2X_2. \]

By the same discussion for \( j = 3, 4, \) we get

(9.11) \[ C_{3-n}^m = \Lambda^{n-3}X_0 \otimes \Lambda^{m-n+3}X_1 \oplus \Lambda^{n-2}X_0 \otimes \Lambda^{m-n+1}X_1 \otimes X_2 \]

\[ \oplus \Lambda^{n-1}X_0 \otimes (\Lambda^{m-n}X_1 \otimes X_3 \oplus \Lambda^{m-n-1}X_1 \otimes \Lambda^2X_2), \]

\[ \oplus \Lambda^nX_0 \otimes (\Lambda^{m-n-1}X_1 \otimes X_4 \oplus \Lambda^{m-n-2}X_1 \otimes X_2 \otimes X_3 \oplus \Lambda^{m-n-3}X_1 \oplus \Lambda^3X_2), \]

\[ C_{4-n}^m = \Lambda^{n-4}X_0 \otimes \Lambda^{m-n+4}X_1 \oplus \Lambda^{n-3}X_0 \otimes \Lambda^{m-n+2}X_1 \otimes X_2 \]

\[ \oplus \Lambda^{n-2}X_0 \otimes (\Lambda^{m-n+1}X_1 \otimes X_3 \oplus \Lambda^{m-n}X_1 \otimes \Lambda^2X_2), \]

\[ \oplus \Lambda^{n-1}X_0 \otimes (\Lambda^{m-n}X_1 \otimes X_4 \oplus \Lambda^{m-n-1}X_1 \otimes X_2 \otimes X_3 \oplus \Lambda^{m-n-2}X_1 \otimes \Lambda^3X_2), \]

\[ \oplus \Lambda^nX_0 \otimes (\Lambda^{m-n-1}X_1 \otimes X_5 \oplus \Lambda^{m-n-2}X_1 \otimes X_2 \otimes X_4 \]

\[ \oplus \Lambda^{m-n-2}X_1 \otimes \Lambda^2X_3 \otimes \Lambda^{m-n-3}X_1 \otimes \Lambda^2X_2 \otimes X_3 \oplus \Lambda^{m-n-4}X_1 \otimes \Lambda^4X_2). \]

Now assume \( n = 3. \) Then we have

(9.12) \[ C_{-3}^m = \Lambda^3X_0 \otimes \Lambda^{m-3}X_1, \]

\[ C_{-2}^m = \Lambda^2X_0 \otimes \Lambda^{m-2}X_1 + \Lambda^3X_0 \otimes \Lambda^{m-4}X_1 \otimes X_2, \]

\[ C_{-1}^m = \Lambda^3X_0 \otimes \Lambda^{m-1}X_1 + \Lambda^2X_0 \otimes \Lambda^{m-3}X_1 \otimes X_2 + \Lambda^3X_0 \otimes \Lambda^{m-4}X_1 \otimes X_3 + \Lambda^3X_0 \otimes \Lambda^{m-5}X_1 \otimes \Lambda^2X_2, \]

\[ C_{0}^m = \Lambda^mX_1 + \Lambda^2X_0 \otimes \Lambda^{m-2}X_1 \otimes X_2 + \Lambda^2X_0 \otimes \Lambda^{m-3}X_1 \otimes X_3 + \Lambda^2X_0 \otimes \Lambda^{m-4}X_1 \otimes \Lambda^2X_2 \]

\[ + \Lambda^3X_0 \otimes \Lambda^{m-4}X_1 \otimes X_4 + \Lambda^3X_0 \otimes \Lambda^{m-5}X_1 \otimes X_2 \otimes X_3 + \Lambda^3X_0 \otimes \Lambda^{m-6}X_1 \otimes \Lambda^3X_2, \]

\[ C_{1}^m = \Lambda^{m-1}X_1 \otimes X_2 + \Lambda^2X_0 \otimes (\Lambda^{m-2}X_1 \otimes X_3 + \Lambda^{m-3}X_1 \otimes \Lambda^2X_2). \]
\[ + \Lambda^2 x_0 \otimes (\Lambda^{-3} x_1 \otimes x_4 + \Lambda^{-4} x_1 \otimes x_2 \otimes x_3 + \Lambda^{-5} x_1 \otimes \Lambda^{3} x_2) + \Lambda^3 x_0 \otimes (\Lambda^{-4} x_1 \otimes x_5 + \Lambda^{-5} x_1 \otimes x_2 \otimes x_4 + \Lambda^{-5} x_1 \otimes \Lambda^{2} x_3 + \Lambda^{-6} x_1 \otimes \Lambda^{2} x_2 \otimes x_3 + \Lambda^{-7} x_1 \otimes \Lambda^{4} x_2) . \]

**Remark 9.4** In homogeneous Poisson case, we have Theorem 8.2 which says the Euler characteristic of 1-homogeneous Poisson structure is always zero. On the other hand, we have a concrete example of 2-homogeneous Poisson structure which Euler characteristic is not zero. Contrarily, in the case of homogeneous Poisson-like cohomology groups we expect all the Euler characteristic may be zero by looking at several concrete cochain complexes.

We take the following one as a 2-homogeneous Poisson-like 2-vector

\[ \pi = - \partial_3 \wedge (x_2 \partial_1) + (x_2 \partial_1) \wedge (x_2 \partial_2) + (x_3 \partial_1) \wedge (x_3 \partial_3) + \partial_2 \wedge (x_3^2 \partial_1) \]

(9.11) \[ - (x_2 \partial_2) \wedge (x_3 \partial_1) + (x_2 \partial_1) \wedge (x_3 \partial_3) - \partial_3 \wedge (x_3^2 \partial_1) + \partial_2 \wedge (x_2 x_3 \partial_1) . \]

We show some examples of Betti numbers of the cohomologies defined by the above \( \pi \).

| wt=−3 | \( C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \to C_{-3} \) |
|-------|-------------------------------------------------------------|
| dim   | 1 9 36 84 126 126 84 36 9 1 |
| ker dim | 0 1 8 28 56 70 56 28 8 1 |
| Betti | 0 0 0 0 0 0 0 0 0 0 |

| wt=−2 | \( C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \to C_{-2} \) |
|-------|-------------------------------------------------------------|
| dim   | 3 27 126 414 1024 1890 2520 2376 1539 651 162 18 |
| ker dim | 0 4 26 103 315 722 1183 1346 1032 507 144 18 |
| Betti | 0 1 3 3 4 11 15 9 2 0 0 0 |

| wt=−1 | \( C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \to C_{-1} \) |
|-------|-------------------------------------------------------------|
| dim   | 3 27 162 768 2745 7371 15084 23544 27621 23745 14418 5832 1407 153 |
| ker dim | 1 5 25 142 663 2228 5481 10124 13974 14039 9872 4579 1254 153 |
| Betti | 1 3 3 5 37 146 338 521 554 392 166 33 1 0 |

### 9.2 Poisson cohomology of polynomial modules

Since \( \Phi(\Lambda^m x_{pol}) = \mathbb{R}[x_1, \ldots, x_n] \otimes \Lambda^m x_0 \), we have a decomposition \( \Phi(\Lambda^m x_{pol}) = \oplus_p \Delta^m_p \) where the subspace \( \Delta^m_p \) is given by \( \Delta^m_p = p\)-polynomials \( \otimes \Lambda^m x_0 \).

**Definition 9.7** For a given non-negative integer \( h \), the weight of each non-zero element of \( \Delta^m_p \) is defined
as \( p - (h - 1)m \). We define the space of the elements of degree \( m \) and of weight \( w \), \( \overline{C}_w^m \), by

\[
(9.12) \quad \overline{C}_w^m = (w + (h - 1)m)\text{-polynomials} \otimes \Lambda^m X_0.
\]

We see easily the next Proposition.

**Proposition 9.3** If \( \pi \in \Delta_h^2 \), then \([\pi, \overline{C}_w^m]_S \subset \overline{C}_w^{m+1} \). Furthermore, if \([\pi, \pi]_S = 0\) then for each fixed weight \( w \), \( \lbrace \overline{C}_w^m \rbrace_m \) with \( u \rightarrow [\pi, u]_S \) forms a cochain complexes.

We may call the cohomology groups of the cochain complexes above as homogeneous Poisson polynomial cohomology groups.

Using \( \Phi : \oplus \Lambda^* X_{pol} \rightarrow \oplus \Lambda^* X_{pol} \) in Remark 9.2, we have a commutative diagram:

\[
\begin{array}{ccc}
\overline{C}_w^m & \xrightarrow{[\pi, \cdot]_S} & C_w^{m+1} \\
\Phi \downarrow & & \downarrow \Phi \\
\overline{C}_w^m & \xrightarrow{[\Phi(\pi), \cdot]_S} & C_w^{m+1}
\end{array}
\]

We remark that if \( m > n \) then \( \Phi(C_w^m) = 0 \) even though \( C_w^m \neq 0 \).

If \( h = 1 \) in (9.12), we have directly the next proposition.

**Proposition 9.4** On \( \mathbb{R}^n \), for each weight \( w \) and for each 1-homogeneous Poisson structure, the Euler characteristic of Poisson polynomial cohomology groups is always zero.

**Proof:** \( \dim \overline{C}_w^m = \dim(\text{\(w\)-polynomials}) \dim(\Lambda^m X_0) = \binom{n-1+w}{n-1} \binom{n}{m} \), and

\[
\sum_{m=0}^{n} (-1)^m \dim \overline{C}_w^m = \binom{n-1+w}{n-1} \sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0.
\]

The \( \Phi \)-image of the 2-homogeneous Poisson-like 2-vector (9.11) in the previous subsection, is just

\[
\overline{\pi} = (x_2^2 - x_3^2) \partial_1 \wedge \partial_2 + 2(x_2x_3 + x_3^2) \partial_1 \wedge \partial_3
\]

and satisfies \([\overline{\pi}, \overline{\pi}]_S = 0\), namely \( \overline{\pi} \) is a usual Poisson 2-vector field. In the following, we show several examples of the Poisson polynomial cohomology groups of \( \overline{\pi} \) on \( \mathbb{R}^3 \).

| wt=−3 | \( \overline{C}_{−1}^3 \) | wt=−2 | \( \overline{C}_{−2}^2 \rightarrow \overline{C}_{−2}^3 \) | wt=−1 | \( \overline{C}_{−1}^1 \rightarrow \overline{C}_{−1}^2 \rightarrow \overline{C}_{−1}^3 \) |
|------|-----------------|------|-----------------|------|----------------|
| dim 1 | \( \overline{C}_{−1}^0 \) | dim 3 | \( \overline{C}_{−2}^2 \) | dim 3 | \( \overline{C}_{−1}^1 \) |
| ker dim 1 | \( \overline{C}_{−1}^1 \) | ker dim 2 | \( \overline{C}_{−1}^2 \) | ker dim 1 | \( \overline{C}_{−1}^0 \) |
| Betti 1 | \( \overline{C}_{−1}^0 \) | Betti 2 | \( \overline{C}_{−1}^1 \) | Betti 1 | \( \overline{C}_{−1}^0 \) |
Remark 9.5 In the concrete examples above, the Euler characteristic of Poisson polynomial cohomology groups is zero except the case of weight is minimum and the cochain complex is single. And we expect that the Euler characteristic of the Poisson polynomial cohomology groups of 2-homogeneous Poisson structure may be zero in general. When the case of 3-homogeneous Poisson (only depends on homogeneity 3 but not depends on the structure itself) on \( \mathbb{R}^3 \), we have the distribution of the Euler characteristic below.

| wt=0 | \( C_0 \to C_0 \to C_0 \to C_0 \) | wt=1 | \( C_1 \to C_1 \to C_1 \to C_1 \) | wt=2 | \( C_2 \to C_2 \to C_2 \to C_2 \) |
|-------|-----------------------------------|-------|-----------------------------------|-------|-----------------------------------|
| dim   | 1 9 18 10                         | dim   | 3 18 30 15                        | dim   | 6 30 45 21                        |
| ker dim | 1 4 12 10                         | ker dim | 0 7 20 15                        | ker dim | 0 10 30 21                        |
| Betti | 1 4 7 4                           | Betti | 0 4 9 5                          | Betti | 0 4 10 6                          |

\[ h = 3, \text{wt} \mid -6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

Euler characteristic:

\[
\begin{array}{cccccccccccc}
-1 & -3 & -3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The results only depends on homogeneity 3 but not depends on the Poisson structure itself on \( \mathbb{R}^3 \). Still we expect the Euler characteristic may be zero for higher weights.

In fact, we have the following result including Proposition 9.4.

**Theorem 9.1** On \( \mathbb{R}^n \), for each \( h \)-homogeneous Poisson structure, the Euler characteristic of Poisson polynomial cohomology groups is always zero for each weight \( w \geq 1 - n \) when \( h > 0 \) and for each weight \( w \geq 1 \) when \( h = 0 \).

In order to prove the theorem above, we follow binomial expansion theorem twice.

Let \( h \) be non-negative integer and \( x \) be an indeterminate variable. Let us start from the binomial expansion

\[
(9.14) \quad ((x + 1)^{h-1} - 1)^n = \sum_m \binom{n}{m}(-1)^{n-m}(x + 1)^{(h-1)m}.
\]

Multiplying the above (9.14) by \( (x + 1)^{n-1+w} \), and expand as follows:

\[
(9.15) \quad (x + 1)^{n-1+w}((x + 1)^{h-1} - 1)^n = \sum_m \binom{n}{m}(-1)^{n-m}(x + 1)^{n-1+w}(x + 1)^{(h-1)m}
\]

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\[
\sum_m \binom{n}{m} (-1)^{n-m} (x + 1)^{n-1+w+(h-1)m} = \sum_m \binom{n}{m} (-1)^{n-m} \sum_k \binom{n-1+w+(h-1)m}{k} x^k \\
=(-1)^n \sum_k \sum_m (-1)^m \binom{n}{m} \binom{n-1+w+(h-1)m}{k} x^k .
\]

When \( h > 0 \), comparing the coefficients of \( x^{n-1} \) of the both sides, we conclude that if \( n - 1 + w \geq 0 \), then

\[(9.16) \quad \sum_m (-1)^m \binom{n-1+w+(h-1)m}{n-1} \binom{n}{m} = 0 \]

holds. When \( h = 0 \), we rewrite the left-hand-side of (9.15) and have \( (x + 1)^{1+w}(-x)^n \), so assuming \( -1 + w \geq 0 \), we have (9.16) for \( h = 0 \). ■

### 9.3 Discussion

We remark that \( w \) and \( m \) have to satisfy \( w + (h-1)m \geq 0 \) and \( 0 \leq m \leq n \).

If \( h = 1 \) in (9.12) then \( w \) and \( m \) are independent.

#### 9.3.1 h=0

Since \( w \) and \( m \) have to satisfy \( w + (h-1)m \geq 0 \) and \( 0 \leq m \leq n \), when \( h = 0 \) then \( m \) runs \( 0 \leq m \leq \min(n, w) \).

**w = 0, not satisfy w > 0:**

\[
\text{LHS of (9.16)} = \sum_{m=0}^{0} (-1)^m \binom{n-1+0-m}{n-1} \binom{n}{m} = \frac{(n-1)}{0} = 1
\]

**w = 1:**

\[
\text{LHS of (9.16)} = \sum_{m=0}^{1} (-1)^m \binom{n-1+1-m}{n-1} \binom{n}{m} \\
= \binom{n}{n-1} \binom{n}{0} - \binom{n-1}{n-1} \binom{n}{1} = \binom{n}{n-1} - \binom{n}{1} = 0
\]
w = 2:

\[
\text{LHS of (9.16)} = \sum_{m=0}^{2} (-1)^m \binom{n-1+2-m}{n-1} \binom{n}{m} = \binom{n+1}{n} - \binom{n}{n-1} + \binom{n-1}{2} = \binom{n+1}{2} - \binom{n}{1} + \binom{n}{2} = 0
\]

9.3.2 h=2

Since \(w\) and \(m\) have to satisfy \(w + (h-1)m \geq 0\) and \(0 \leq m \leq n\), when \(h = 2\) then \(m\) runs \(\max(0,-w) \leq m \leq n\).

n = 4 and \(w = -2\):

\[
\text{LHS of (9.16)} = \sum_{m=2}^{4} (-1)^m \binom{1+m}{3} \binom{4}{m} = \binom{3}{2} - \binom{4}{3} + \binom{5}{3} = \binom{4}{2} - 4^2 + \binom{5}{2} = 0
\]

n = 4 and \(w = -3\):

\[
\text{LHS of (9.16)} = \sum_{m=3}^{4} (-1)^m \binom{m-1}{3} \binom{4}{m} = -\binom{3}{3} + \binom{4}{3} + \binom{4}{4} = -\binom{4}{3} + \binom{4}{3} = 0
\]

n = 4 and \(w = -4\), not satisfy \(w \geq 1 - n\):

\[
\text{LHS of (9.16)} = \sum_{m=4}^{4} (-1)^m \binom{m-1}{3} \binom{4}{m} = -\binom{3}{3} + \binom{4}{4} = 1
\]

9.3.3 h=3

Since \(w\) and \(m\) have to satisfy \(w + (h-1)m \geq 0\) and \(0 \leq m \leq n\), when \(h = 3\) then \(m\) runs \(\max(0,-w/2) \leq m \leq n\).

n = 4 and \(w = -2\):

\[
\text{LHS of (9.16)} = \sum_{m=1}^{4} (-1)^m \binom{1+2m}{3} \binom{4}{m} = \binom{3}{1} + \binom{5}{3} - \binom{7}{3} + \binom{9}{3} = 0
\]

n = 4 and \(w = -3\):

\[
\text{LHS of (9.16)} = \sum_{m=2}^{4} (-1)^m \binom{2m}{3} \binom{4}{m} = \binom{4}{3} - \binom{6}{3} + \binom{8}{3} = 0
\]
n = 4 and w = -4 not satisfy \( w \geq 1 - n \):

\[
\text{LHS of (9.16)} = \sum_{m=2}^{4} (-1)^{m} \binom{2m-1}{3} \binom{4}{m} = \binom{3}{3} \cdot \binom{4}{2} - \binom{5}{3} \cdot \binom{4}{3} + \binom{7}{3} \cdot \binom{4}{4} = 1
\]

9.4 Stability of Poisson polynomial cohomology groups of special Lie Poisson structures on 3-space

Two Lie Poisson structures due to Heisenberg Lie algebra and \( \sp(\mathbb{R}^2) \) have special features.

9.4.1 Heisenberg case:

**Theorem 9.2** The dimension of the kernel subspaces and the Betti numbers of Heisenberg Lie Poisson structure are given by

| \( w > 0 \) | \( C_w^0 \rightarrow C_w^1 \rightarrow C_w^2 \rightarrow C_w^3 \) |
|---|---|---|---|
| \( \text{dim} \) | \( \binom{2+w}{2} \) | \( 3\binom{2+w}{2} \) | \( 3\binom{2+w}{2} \) | \( \binom{2+w}{w} \) |
| \( \text{ker dim} \) | 1 | \( \binom{3+w}{2} \) | \( w+3 \)(\( w+1 \)) | \( \binom{w+2}{2} \) |
| Betti | 1 | \( w+3 \) | \( 2w+3 \) | \( w+1 \) |
| \( \text{ker dim}(w = 0) \) | 1 | 2 | 3 | 1 |
| Betti \( (w = 0) \) | 1 | 2 | 2 | 1 |

Thus, we see that

\[
\dim \overline{H}_w^0 - \dim \overline{H}_w^1 = 0 \ (w > 0), \quad \dim \overline{H}_w^1 - \dim \overline{H}_w^2 = 1 \ (w > 0), \\
\dim \overline{H}_w^2 - \dim \overline{H}_w^3 = 2 \ (w > 0), \quad \dim \overline{H}_w^3 - \dim \overline{H}_w^4 = 1 \ (w > 0).
\]

**Proof:** 2-vector field is given by \( \{z, x\} = 0, \{z, y\} = 0, \{x, y\} = z \) i.e.,

\[
\pi = \frac{1}{2} \sum_{i,j} \{x_i, x_j\} \partial_{x_i} \wedge \partial_{x_j} = z\partial_x \wedge \partial_y
\]

and satisfies \([\pi, \pi]_S = 0\) and so a Poisson structure on \( \mathbb{R}^3 \).

\[
[\pi, x]_S = -z\partial_y, \quad [\pi, y]_S = z\partial_x, \quad [\pi, z]_S = 0, \\
[\pi, \partial_x]_S = 0, \quad [\pi, \partial_y]_S = 0, \quad [\pi, \partial_z]_S = -\partial_x \wedge \partial_y,
\]

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are the necessary information. We see that

\[ [\pi, \partial_x \wedge \partial y]_S = [\pi, \partial_x \wedge \partial z]_S = [\pi, \partial_y \wedge \partial z]_S = 0 \]

and

\[ [\pi, u^A]_S = -a_1 x^{a_1-1} y^{a_2} z^{a_3+1} \partial_y + a_2 x^{a_1} y^{a_2-1} z^{a_3+1} \partial_x \]

where \( u^A = x^{a_1} y^{a_2} z^{a_3} \) for \( A = [a_1, a_2, a_3] \in \mathbb{N}^3 \).

Now we assume \( w > 0 \). \( C_w^0 \):

\[
0 = [\pi, \sum_{|A|=w} c_A u^A]_S = \sum_{|A|=w} c_A(-a_1 x^{a_1-1} y^{a_2} z^{a_3+1} \partial_y + a_2 x^{a_1} y^{a_2-1} z^{a_3+1} \partial_x)
\]

implies

\[ \sum_{|A|=w} c_A a_1 x^{a_1-1} y^{a_2} z^{a_3+1} = 0 \] (9.17)

\[ \sum_{|A|=w} c_A a_2 x^{a_1} y^{a_2-1} z^{a_3+1} = 0 \] (9.18)

(9.17) says that if \( a_1 \neq 0 \) then \( c_A = 0 \) or (9.18) says that if \( a_2 \neq 0 \) then \( c_A = 0 \). Thus, the kernel subspace of \( C_w^0 \) is spanned by \( c^{|0,0,w|}_w \) and 1-dimensional.

About \( C_w^n \) (\( n = 3 \)), the whole space is the kernel subspace and \( \left( \begin{array}{c} n-1 \\ 1 \end{array} \right) \)-dimensional. The difference of the dimension of the kernel subspaces with the weight \( w \) and weight \( w + 1 \) is \( 1 + w \).

\( C_w^1 \):

\[
0 = [\pi, \sum_A (\alpha_A u^A \partial_x + \beta_A u^A \partial_y + \gamma_A u^A \partial_z)]_S \\
= \sum_A \alpha_A [\pi, u^A]_S \wedge \partial_x + \sum_A \beta_A [\pi, u^A]_S \wedge \partial_y + \sum_A \gamma_A [\pi, u^A]_S \wedge \partial_z - \sum_A \gamma_A u^A \partial_x \wedge \partial_y \\
= \sum_A \alpha_A a_1 x^{a_1-1} y^{a_2} z^{a_3+1} \partial_x \wedge \partial_y + \sum_A \beta_A a_2 x^{a_1} y^{a_2-1} z^{a_3+1} \partial_x \wedge \partial_y + \sum_A \gamma_A [\pi, u^A]_S \wedge \partial_z \\
- \sum_A \gamma_A u^A \partial_x \wedge \partial_y
\]

Thus, we have

\[ \sum_A \alpha_A a_1 x^{a_1-1} y^{a_2} z^{a_3+1} + \sum_A \beta_A a_2 x^{a_1} y^{a_2-1} z^{a_3+1} - \sum_A \gamma_A u^A = 0 , \] (9.19)
(9.20) \[ \sum_A \gamma_A a_1 x^{a_1-1} y^{a_2} z^{a_3+1} = 0 , \]

(9.21) \[ \sum_A \gamma_A a_2 x^{a_1} y^{a_2-1} z^{a_3+1} = 0 , \]

and (9.20) and (9.21) say that

(9.22) \[ \gamma_A = 0 \quad \text{unless} \quad A = [0,0,w] . \]

(9.19) and (9.22) imply that

\[ \sum_A \alpha_A a_1 x^{a_1-1} y^{a_2} z^{a_3+1} + \sum_A \beta_A a_2 x^{a_1} y^{a_2-1} z^{a_3+1} - \gamma_{[0,0,w]} z^w = 0 . \]

Comparing \( z^w \), we have

(9.23) \[ \alpha_{[1,0,w-1]} + \beta_{[0,1,w-1]} = \gamma_{[0,0,w]} . \]

Comparing \( z^j (j < w) \), we have \((w - j + 1)\) linear equations

\[ \alpha_{[1+p,w-j-p,j-1]}(1+p) + \beta_{[p,w-j+1-p,j-1]}(w-j+1-p) = 0 . \]

Thus, the dim of kernel subspace of \( C^1_w \) is

\[ 3\left( \frac{n-1+w}{n-1} \right) - \left( \frac{n-1+w}{n-1} - 1 \right) - \sum_{j=1}^{w}(w-j+1) = \frac{(w+2)(w+3)}{2} . \]

\( C^2_w \):

\[ 0 = \left[ \pi, \sum_A (\alpha_A u^A \partial_x \wedge \partial_y + \beta_A u^A \partial_x \wedge \partial_z + \gamma_A u^A \partial_y \wedge \partial_z) \right] \]

\[ = \sum_A [\pi, u^A] \wedge (\alpha_A \partial_x \wedge \partial_y + \beta_A \partial_x \wedge \partial_z + \gamma_A \partial_y \wedge \partial_z) \]

\[ = \sum_A (\beta a_1 x^{a_1-1} y^{a_2} z^{a_3+1} + \gamma a_2 x^{a_1} y^{a_2-1} z^{a_3+1}) \partial_x \wedge \partial_y \wedge \partial_z \]

By the almost same argument, we have linearly independent \((w-j)\) linear equations for \((0 \leq j < w)\). Thus, the dim of the kernel subspace of \( C^2_w \) is

\[ 3\left( \frac{n-1+w}{n-1} \right) - \sum_{j=0}^{w-1}(w-j) = (w+1)(w+3) . \]
Thus, we know the whole stuff and Betti # as the table in the theorem.

### 9.4.2 \( \text{sp}(\mathbb{R}^2) \) Poisson case:

**Theorem 9.3** The dimension of the kernel subspaces and the Betti numbers of \( \text{sp}(\mathbb{R}^2) \) Lie Poisson structure are given as follows: We have two kinds of lists of Betti numbers and they are modulo 2 periodic with respect to the weight \( w \).

| \( \dim \) | \( \overline{C}_w^0 \) | \( \overline{C}_w^1 \) | \( \overline{C}_w^2 \) | \( \overline{C}_w^3 \) |
|---|---|---|---|---|
| ker dim (\( w \) odd) | 0 | (\( \frac{2}{2} + w \)) | 2(\( \frac{2}{2} + w \)) | (\( \frac{w+2}{2} \)) |
| Betti (\( w \) odd) | 0 | 0 | 0 | 0 |
| ker dim (\( w \) even) | 1 | (\( \frac{2}{2} + w \)) - 1 | 2(\( \frac{2}{2} + w \)) + 1 | (\( \frac{w+2}{2} \)) |
| Betti (\( w \) even) | 1 | 0 | 0 | 1 |

**Proof:** 2-vector field given by \( \{ z, x \} = 2x, \{ z, y \} = -2y, \{ x, y \} = z \) is

\[
\pi = \frac{1}{2} \sum_{i,j} \{ x_i, x_j \} \partial_{x_i} \wedge \partial_{x_j} = 2x \partial_z \wedge \partial_x - 2y \partial_z \wedge \partial_y + z \partial_x \wedge \partial_y
\]

and satisfies \([ \pi, \pi ] = 0 \) and so a Poisson structure on \( \mathbb{R}^3 \).

\[
[\pi, x] = 2x \partial_z - z \partial_y, \quad [\pi, y] = -2y \partial_z + z \partial_y, \quad [\pi, z] = -2x \partial_x + 2y \partial_y,
\]

\[
[\pi, \partial_x] = -2 \partial_x \wedge \partial_x, \quad [\pi, \partial_y] = 2 \partial_z \wedge \partial_y, \quad [\pi, \partial_z] = -\partial_x \wedge \partial_y,
\]

are the necessary information to get Poisson polynomial cohomologies of our Poisson structure \( \pi \).

Since \( \overline{C}_w^m = w\text{-homog} \otimes \Lambda^m x_0, \overline{C}_0^m = \Lambda^m x_0 \) and

\[
0 = [\pi, \sum_j c_j \partial_{x_j}] = c_1(-2 \partial_x \wedge \partial_x) + c_2(2 \partial_z \wedge \partial_y) + c_3(-2 \partial_x \wedge \partial_y)
\]

implies \( c_1 = c_2 = c_3 = 0 \), i.e., \( \dim \text{ker subspace of } \overline{C}_0^1 = 0. \)

Since \( [\pi, \sum_j c_j \partial_{x_{j+1}} \wedge \partial_{x_{j+1}}] = 0 \), we see that \( \dim \text{ of the kernel subspace of } \overline{C}_0^j = 3 \), and \( \dim \text{ of the kernel subspace of } \overline{C}_0^3 = 1. \) We can continue this argument for weight 1,2, \ldots.

**Kernel subspace of 0-th cochain space:** About the kernel subspace of \( \overline{C}_w^0 \), we see that if the weight is odd then 0 and if the weight is even then 1 dimensional as follows. Take a cochain \( \sum_A c_A u_A \in \overline{C}_w^0 \) and

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suppose \( 0 = [\pi, \sum_A c_A u^A]_S \). Then we have

(9.24) \[ 2 \sum_A (a_1 - a_2) c_A u^A = 0 , \]

(9.25) \[ \sum_A c_A (z \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z}) u^A = 0 , \]

(9.26) \[ \sum_A c_A (z \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial z}) u^A = 0 . \]

(9.24) implies \((a_1 - a_2)c_A = 0\), so we see that if \(a_1 \neq a_2\) then \(c_A = 0\). Thus, we have to care only about \(c_{[i,i,w-2i]}\) which we denote by \(P_i\) for \(i\) from 0 to \(w\). Using (9.25), we have

\[
0 = \sum_{0 \leq i \leq w/2} P_i (z \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z}) x^i y^i z^{w-2i}
\]

\[
= \sum_{0 \leq i \leq w/2} i P_i x^i y^{i-1} z^{w-2i+1} - \sum_{0 \leq i < w/2} 2(w - 2i) P_i x^{i+1} y^i z^{w-2i-1}
\]

\[
= \sum_{0 \leq i \leq w/2-1} ((i + 1) P_{i+1} - 2(w - 2i) P_i) x^{i+1} y^i z^{w-2i-1} - \sum_{w/2-1 \leq i < w/2} 2(w - 2i) P_i x^{i+1} y^i z^{w-2i-1}
\]

If \(w\) is even, then the last term does not happen and we see that \((i + 1) P_{i+1} - 2(w - 2i) P_i = 0\) for \(i = 0, w/2 - 1\).

Thus, the freedom of \(c_A\) is 1. If \(w\) is odd, say \(w = 2\Omega + 1\), then the last term tells that \(P_{\Omega} = 0\) and also \((i + 1) P_{i+1} - 2(w - 2i) P_i = 0\) for \(i = 0, \Omega - 1\). Thus, \(P_i = 0\) for \(i = 0, \Omega\) and the kernel subspace is 0-dimensional. Namely, 0-th Betti number is 1 if \(w\) is even and 0 otherwise.

**Kernel subspace of 2-nd cochain space:** We try now 3-rd Betti number. Since the kernel subspace of \(C^3_w\) is itself, we search the kernel subspace of \(C^2_w\). Take a general 2-cochain \(\sum_A u^A (\alpha_A \partial_x \land \partial_y + \beta_A \partial_x \land \partial_z + \gamma_A \partial_y \land \partial_z)\) and take the Schouten bracket by \(\pi\). Then the coefficient of \(\partial_x \land \partial_y \land \partial_z\) is

(9.27) \[ \sum_A \left(2\alpha_A (a_1 - a_2) + \beta_A (z \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}) + \gamma_A (z \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial z})\right) u^A = 0 . \]

The coefficient of \(x^i y^i z^{w-2i}\) is

(9.28) \[ \beta_{[i+1,i]}(1 + i) - 2\beta_{[i,i-1]}(w - 2i + 1) + \gamma_{[i,1+i]}(1 + i) - 2\gamma_{[i-1,i]}(w - 2i + 1) = 0 \]

Using the notation \(Q_{[i,j]} = \beta_{[i,j]} + \gamma_{[j,i]}\), (9.28) is equivalent to

(9.29) \[ Q_{[1,0]} = 0. \]
Using (9.29) and (9.30), we have

$$Q_{[i+1,j]} = 0, \text{ i.e., } \beta_{[i+1,j]} = -\gamma_{[i,j+1]} \quad (0 \leq i \leq \frac{w-1}{2})$$

(9.31) says nothing when \(w\) is odd, and if \(w\) is even (9.31) is a part of (9.32). We summarize this observation as follows: if \(w = 2\Omega\) then we have linear independent \(\Omega\) relations and if \(w = 2\Omega + 1\) then we have \((\Omega + 1)\) linear independent relations.

Checking the coefficient of \(x^i y^j z^{w-i-j} (i \neq j)\), we have

\[-2\alpha_{[i,j]}(i-j) = \beta_{[i,j,i]}(1+i) - 2\beta_{[i,j,i,j]}(w - i - j + 1) + \gamma_{[i,j+1]}(1 + j) - 2\gamma_{[i,j+1]}(w - i - j + 1),\]

thus, we say the freedom of \(\alpha_A\) is \(\alpha_{[i,i]}\) with \(2i \leq w\) and the number is \(\Omega + 1\) even if \(w = 2\Omega\) or \(w = 2\Omega + 1\). We conclude that the kernel subspace of \(C^2_w\) is \((\Omega + 1) + \{(\frac{2+w}{2}) - \Omega^*\} + \{(\frac{2+w}{2})\}\) where

\[\Omega^* = \begin{cases} \Omega & \text{if } w = 2\Omega, \\ 1 + \Omega & \text{if } w = 2\Omega + 1, \end{cases}\]

and the third Betti number is given by

\[3(\frac{2+w}{2}) + (\Omega + 1) + \{(\frac{2+w}{2}) - \Omega^*\} + \{(\frac{2+w}{2})\} = \Omega + 1 - \Omega^*.\]

**Kernel subspace of 1-st cochain space:** Take a general 1-cochain \(\sum_A u^A(\alpha_A \partial_x + \beta_A \partial_y + \gamma_A \partial_z)\) and solve the equation of the Schouten bracket by \(\pi = 0\).

Then the coefficient of \(\partial_z \wedge \partial_x\) is

$$\sum_A \left(2(a_1 - a_2 - 1)\alpha_A - \gamma_A(-2x \frac{\partial}{\partial z} + z \frac{\partial}{\partial y})\right) u^A$$

(9.33)

and the coefficient of \(\partial_z \wedge \partial_y\) is

$$\sum_A \left(2(a_1 - a_2 + 1)\beta_A - \gamma_A(2y \frac{\partial}{\partial z} - z \frac{\partial}{\partial x})\right) u^A$$

(9.34)

and the coefficient of \(\partial_x \wedge \partial_y\) is

$$\sum_A \left(\beta_A(-2x \frac{\partial}{\partial z} + z \frac{\partial}{\partial y}) - \alpha_A(2y \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}) - \gamma_A\right) u^A.$$  

(9.35)
The coefficient of $x^iy^jz^{w-i-j}$ are

\begin{align}
\text{(9.36)} & \quad 2(i-j-1)\alpha_{[i,j]} + 2(w-i-j+1)\gamma_{[i-1,j]} -(j+1)\gamma_{[i+1,j]} = 0, \\
\text{(9.37)} & \quad 2(i-j+1)\beta_{[i,j]} - 2(w-i-j+1)\gamma_{[i,j-1]} + (i+1)\gamma_{[1,i+1]} = 0, \\
\text{(9.38)} & \quad \gamma_{[i,j]} = -2(w-i-j+1)\beta_{[i-1,j]} + (j+1)\beta_{[i,j+1]} \\
& \quad \quad \quad \quad - 2(w-i-j+1)\alpha_{[i,j-1]} + (i+1)\alpha_{[1,i+1]}.
\end{align}

In (9.36), if $i-j-1 \neq 0$ then $\alpha_{[i,j]}$ are determined by $\{\gamma_{[k,l]}\}$. Also in (9.37), if $i-j+1 \neq 0$ then $\beta_{[i,j]}$ are determined by $\{\gamma_{[k,l]}\}$.

If $i-j-1 = 0$ in (9.36) or if $i-j+1 = 0$ in (9.37), then we have the same relations

\begin{equation}
\text{(9.39)} \quad 2(w-2j)\gamma_{[j,j]} = (j+1)\gamma_{[j+1,j+1]}.
\end{equation}

We write the above recursive formula concretely as follows:

\[
2w\gamma_{[0,0]} = \gamma_{[1,1]}, \\
2(w-2)\gamma_{[1,1]} = 2\gamma_{[2,2]}, \\
\vdots \\
2(w-2\Omega+2)\gamma_{[\Omega-1,\Omega-1]} = \Omega\gamma_{[\Omega,\Omega]}, \\
2(w-2\Omega)\gamma_{[\Omega,\Omega]} = (\Omega+1)\gamma_{[\Omega+1,\Omega+1]} = 0.
\]

When $w = 2\Omega + 1$, the last equation above tells $\gamma_{[\Omega,\Omega]} = 0$ and we have $\gamma_{[j,j]} = 0$ for $j = 0..\Omega$. In (9.38), we put $i = j$ and have

\begin{equation}
\gamma_{[i,i]} = -2(w-2i+1)\beta_{[i-1,i]} + (i+1)\beta_{[i,i+1]} \\
- 2(w-2i+1)\alpha_{[i,i-1]} + (i+1)\alpha_{[1,i+1]} \\
= (i+1)(\beta_{[i,i+1]} + \alpha_{[1,i+1]}) - 2(w-2i+1)(\beta_{[i-1,i]} + \alpha_{[i,i-1]}).
\end{equation}

Denoting $\beta_{[i,i+1]} + \alpha_{[1,i+1]}$ by $R_{[i]}$, the above says that

\[
(i+1)R_{[i]} = (2w+2-4i)R_{[i-1]}.
\]

Since $R_{[0]} = 0$, we have $R_{[i]} = 0$ and so $\alpha_{[1,i+1]} + \beta_{[i,i+1]} = 0$ for each $i$. Now we count the freedom of $\alpha_{A}, $
and \( \gamma_A \). The freedom of \( \gamma_A \) is \((2^w + 1) - (1 + \Omega)\) because \( \gamma_{[i,i]} = 0 \). We see the freedom of \( \beta_A \) is \( 1 + \Omega \) by \( \beta_{[i,i+1]} \), and \( \alpha_A \) have no freedom because \( \alpha_{[1+i,i]} + \beta_{[i,i+1]} = 0 \) for each \( i \). Thus, the dimension of the kernel subspace of \( \overline{C}_w^1 \) is \((2^w + 1) - (1 + \Omega)\). The first Betti number is 0 because \((2^w + 1) - (1 + \Omega)\). Since we already know \( \dim \ker \) subspace of \( \overline{C}_w^2 \) is \( 2(2^w - 2) \) when \( w = 2\Omega + 1 \), the second Betti number is also 0.

When \( w = 2\Omega \) (even), the situation is slightly different. We will count the freedom of \( 3\Omega + 1 \) variables \( \{ \gamma_{[i,i]} \} (i = 0..\Omega), \{ \beta_{[i,i+1]} \} (i = 0..(\Omega - 1)) \) and \( \{ \alpha_{[i+1,i]} \} (i = 0..(\Omega - 1)) \). We have \((2\Omega + 1)\) linear equations (9.39) and (9.40) of those variables by using \( R_{[i]} := \beta_{[i,i+1]} + \alpha_{[1+i,i]} \) (where \( R_{[\Omega]} = 0 \)) as follows:

\[
\gamma_{[i,i]} + \phi(i)R_{[i-1]} - (i + 1)R_{[i]} = 0, \text{ where } \phi(i) := 2w + 2 - 4i
\]

and

\[
\psi(i)\gamma_{[i,i]} - (i + 1)\gamma_{[i+1,i+1]} = 0, \text{ where } \psi(i) := 2w - 4i.
\]

The matrix which express the linear equations is the following:

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \phi(1) & -2 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & -\Omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi(\Omega) \\
\psi(0) & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \psi(1) & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \psi(\Omega - 1) & -\Omega & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

where the \((1,1)\)-matrix is the identity matrix of size \( \Omega + 1 \), the \((2,2)\)-matrix is the zero matrix of size \( \Omega \), the \((1,2)\)-matrix is the \((\Omega + 1, \Omega)\) and the diagonal entries are \(-i\) and the below line consists of \( \phi(i) \), and the \((2,1)\)-matrix is the \((\Omega, \Omega + 1)\) and the diagonal entries are \( \psi(i - 1) \) and the above line consists of \(-i\).

We see that the rank of \( M \) is full \((2\Omega + 1)\) by the following Lemma. So, the freedom of those variables is \( 3\Omega + 1 - (2\Omega + 1) = \Omega \). Thus the dimension of the kernel subspace of \( \overline{C}_w^1 \) is \((2^w - (\Omega + 1)) + \Omega = 2^w - 1\) when \( w = 2\Omega \). This shows that the first Betti number is 0 and the second Betti number is also 0.

**Lemma 9.1**

\[
\det M = \Omega! \sum_{j=1}^{\Omega+1} \psi(0) \cdots \psi(j-2)\phi(j) \cdots \phi(\Omega)
\]

holds, where \( \phi(i) = 4\Omega + 2 - 4i \) and \( \psi(i) = 4\Omega - 4i \).
Remark 9.6 We already know the behavior of Betti numbers about \( SL(2) (A_1), SO(3) (B_1) \) or \( Sp(\mathbb{R}^2) (C_1) \). Here we calculate Betti numbers for lower weight of \( SO(4) (D_2) \) Lie Poisson structure:

Proof of Lemma: We follow the definition of determinant directly. Since the (2,2)-submatrix of \( M \) is zero matrix, among permutations of 1, 2, \ldots, \( 2\Omega + 1 \), the candidate \( \sigma \) which contribute the determinant of \( M \) must satisfy \( \sigma(i) \leq \Omega + 1 \) for each \( i > \Omega + 1 \) and so for some \( j \leq \Omega + 1, \sigma \) satisfies

\[
\sigma(\{\Omega + 2, \ldots, 2\Omega + 1\}) = \{1, \ldots, \widehat{j}, \ldots, \Omega + 1\}, \quad \sigma(\{1, \ldots, \Omega + 1\}) = \{j, \Omega + 2, \ldots, 2\Omega + 1\}.
\]

Furthermore, \( i \leq \sigma(1 + \Omega + i) \leq i + 1 \) and \( j \) is specified, then \( \sigma(1 + \Omega + i) = i \) for \( i < j \) and \( \sigma(1 + \Omega + i) = i + 1 \) for \( i \geq j \). For \( \sigma(i) (i \leq 1 + \Omega) \), effective possibilities are

\[
\begin{align*}
\sigma(1) &= 1 & \text{or} & \quad \sigma(1) &= 1 + \Omega + 1 \\
\sigma(2) &= 2 & \text{or} & \quad 1 + \Omega + 1 \leq \sigma(2) \leq 1 + \Omega + 2 \\
& \vdots \\
\sigma(\Omega) &= \Omega & \text{or} & \quad 2\Omega \leq \sigma(\Omega) \leq 1 + \Omega + \Omega \\
\sigma(\Omega + 1) &= \Omega + 1 & \text{or} & \quad 2\Omega + 1 = \sigma(\Omega + 1)
\end{align*}
\]

and if \( j \) is specified, then \( \sigma(j) = j \) holds necessarily and

\[
\sigma(1) = 1 + \Omega + 1, \ldots \sigma(j - 1) = 1 + \Omega + j - 1, \quad \sigma(j) = j, \quad \sigma(j + 1) = \Omega + j + 1, \ldots \quad \sigma(\Omega + 1) = 2\Omega + 1.
\]

Thus, when \( j \) is specified, the permutation \( \sigma \) is uniquely determined as

\[
\begin{pmatrix}
1 & \cdots & j - 1 & j + 1 & \cdots & \Omega + 1 & \Omega + 2 & \cdots & \Omega + j & \Omega + j + 1 & 2\Omega + 1 \\
\Omega + 2 & \Omega + j & \Omega + 1 + j & 2\Omega + 1 & 1 & j - 1 & j + 1 & \Omega + 1
\end{pmatrix}
\]

and is decomposed as \( (1, \Omega + 2) \cdots (j - 1, \Omega + j)(j + 1, \Omega + j + 1) \cdots (\Omega + 1, 2\Omega + 1) \) of \( \Omega \) transpositions and so the signature of \( \sigma \) is \((-1)^\Omega\). Therefore, we conclude

\[
\begin{align*}
\det M &= \sum_{j=1}^{\Omega+1} (1)(\Omega)(-1) \cdots (-j + 1)\phi(j) \cdots \phi(\Omega)\psi(0) \cdots \psi(j - 2)(-j)\psi(-j - 1) \cdots \psi(-\Omega) \\
&= \Omega! \sum_{j=1}^{\Omega+1} \psi(0) \cdots \psi(j - 2)\phi(j) \cdots \phi(\Omega).
\end{align*}
\]

\[\blacksquare\]

Remark 9.6 We already know the behavior of Betti numbers about \( SL(2) (A_1), SO(3) (B_1) \) or \( Sp(\mathbb{R}^2) (C_1) \). Here we calculate Betti numbers for lower weight of \( SO(4) (D_2) \) Lie Poisson structure:
Concerning $w$, we expect some mechanism, in some sense, combinations of patterns of $Sp(\mathbb{R}^2)$ and Heisenberg Lie algebra.

$SL(3) (A_2)$ is 8 dim and for the weight 0, 1, 2 we have

$$
\begin{array}{ccccccc}
\toprule
w\backslash m & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\midrule
0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 4 & 0 & 0 & 2 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 3 & 0 & 0 & 6 & 0 & 0 & 3 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bottomrule
\end{array}
$$

About $SO(5) (B_2)$ or $Sp(\mathbb{R}^4) (C_2)$:

$$
\begin{array}{cccccccc}
\toprule
w\backslash m & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\midrule
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
\bottomrule
\end{array}
$$

Looking at those examples, it seems to be interesting to investigate the Betti numbers associated with Lie Poisson structure of simple Lie algebras $A_n, B_n, C_n, \ldots$ and to understand periodicity or symmetries of their Betti numbers.

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