Supersymmetric DBI Equations in Diverse Dimensions from BRS Invariance of Pure Spinor Superstring

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Abstract

We examine the BRS invariance of the open pure spinor superstring in the presence of background superfields on a Dp-brane. It is shown that the BRS invariance leads not only to boundary conditions on the spacetime spinors, but also to supersymmetric DBI equations of motion for the background superfields on the Dp-brane. These DBI equations are consistent with the supersymmetric DBI equations for a D9-brane.

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1 Introduction

Dirac-Born-Infeld (DBI) theory is known as a non-linear generalization of Maxwell theory and may describe, along with the Wess-Zumino action, the low-energy effective dynamics on a single D-brane in string theory. The bosonic DBI action is derived from the world-sheet analysis of the bosonic open string [1]. A supersymmetric DBI action should be a part of the effective action on a D-brane in type II superstring theory. In the Ramond-Neveu-Schwarz (RNS) formulation, however, it is difficult to read off the target space geometry coupling to Ramond-Ramond fields, because space-time supersymmetry becomes manifest only after the GSO projection. So the RNS superstring has led to the only bosonic sector of the supersymmetric DBI action [2].

The Green-Schwarz (GS) formulation has an advantage in this direction. The Wess-Zumino term which ensures the $\kappa$-invariance of the world-volume action of a D-brane is constructed in [3]. In [4], the $\kappa$-symmetric approach, so called the superembedding formalism [5], is shown to lead to linearised supersymmetric DBI equations of motion for a D9-brane, which have the ten-dimensional $\mathcal{N} = 2$ supersymmetry. Furthermore, in [6], the classical $\kappa$-invariance of an open GS superstring in an abelian background is shown to imply that the
background fields should satisfy full non-linear equations of motion for a supersymmetric DBI action. Non-abelian extension of this formalism is discussed in [7] as the boundary fermion formalism where Chan-Patton factors describing coincident D-branes are replaced by boundary fermions\(^\ast\).

Unlike the formulations mentioned above, the pure spinor formulation [20] enables us to quantize a superstring in a super-Poincaré covariant manner. In this formulation, the \(\kappa\)-symmetry in the GS formulation is replaced with the BRS symmetry. It is shown in [21], correspondingly to the \(\kappa\)-symmetry analysis [6], that the classical BRS invariance of an open pure spinor superstring leads to supersymmetric DBI equations of motion on a D9-brane, which have the non-linear \(\mathcal{N} = 1\) supersymmetry as well as the manifest \(\mathcal{N} = 1\) supersymmetry. These equations precisely coincide with those obtained in the superembedding formalism [22]. Furthermore, the non-abelian extension of supersymmetric DBI equations is proposed. In [23] (see also [24] and [25]), D-brane boundary states are constructed in the pure spinor formulation. Especially, calculating the disk scattering amplitude suggests that the coupling of the boundary state to the background fields will reproduce the DBI kinetic term and the Wess-Zumino term of the D9-brane effective action. These achievements might imply the fact that the low-energy effective theory on the D9-brane is determined uniquely by the ten-dimensional \(\mathcal{N} = 2\) supersymmetry.

In this paper we will derive supersymmetric DBI equations of motion on a D\(_p\)-brane, as well as a D9-brane, from the BRS invariance of the open pure spinor superstring. Our approach is similar to that taken in [21] for a D9-brane. However the inclusion of Dirichlet components requires improvements which are not just a dimensional reduction of the case of a D9-brane. As in [21], we will provide two boundary terms, the counter term \(S_b\) for the \(\mathcal{N} = 1\) supersymmetry transformation of the world-sheet action \(S_0\) and the background superfield coupling \(V\) as a relevant extension of the pure spinor vertex operator. It is found that the contribution of Dirichlet components in them cannot be determined unless considering the BRS invariance. In [21], by using non-trivial boundary conditions given by the general variation of \(S_0 + S_b + V\), the BRS charge conservation leads to supersymmetric DBI equations for the D9-brane. On the other hand, we will show that supersymmetric DBI equations in diverse dimensions are extracted only from the BRS transformation of \(S_0 + S_b + V\) under a static-like gauge for the D-brane position.

This paper is organized as follows. In section 2, after introducing the type II pure spinor open superstring action, we construct the boundary term for the \(\mathcal{N} = 1\) supersymmetry

\(^\ast\)Other than this study, there have been many attempts to extend to the non-abelian DBI theory based on various approaches [8–19].
invariance of this action. In section 4.1, background superfield coupling is found by considering the modification of a vertex operator in the open pure spinor superstring. In section 4.2, we confirm that these background superfields satisfy supersymmetric DBI equations of motion. The last section is devoted to summary and discussions. In addition, we give a brief review of the covariant approach for the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in Appendix A. We will formulate a vertex operator in the open pure spinor superstring in Appendix B. We show that our result can be derived also from improving the method used in [21] to include Dirichlet components in Appendix C.

2 Open pure spinor superstring

The world-sheet action of the type II pure spinor open superstring [20] is given as

$$S_0 = \frac{1}{\pi \alpha'} \int \mathrm{d}z \mathrm{d}\bar{z} \left\{ \frac{1}{2} \partial x^m \bar{\partial} x_m + p_{\alpha} \bar{\partial} \theta^{\alpha} + \bar{p}_{\dot{\alpha}} \partial \bar{\theta}^{\dot{\alpha}} + \omega_{\alpha} \bar{\partial} \lambda^{\alpha} + \bar{\omega}_{\dot{\alpha}} \partial \bar{\lambda}^{\dot{\alpha}} \right\}, \quad (2.1)$$

where $x^m \ (m = 0, 1, \cdots, 9)$ is a ten-dimensional coordinate, $\theta^{\alpha}$ and $\bar{\theta}^{\dot{\alpha}} \ (\alpha = 1, \cdots, 16)$ are left- and right-moving ten-dimensional Majorana-Weyl spinors, respectively, and $\lambda^{\alpha}$ and $\bar{\lambda}^{\dot{\alpha}}$ are bosonic ghosts satisfying pure spinor constraints $\lambda \gamma^m \lambda = \bar{\lambda} \gamma^m \bar{\lambda} = 0$. The $(p_{\alpha}, \bar{p}_{\dot{\alpha}})$ and $(\omega_{\alpha}, \bar{\omega}_{\dot{\alpha}})$ are conjugate to $(\theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$ and $(\lambda^{\alpha}, \bar{\lambda}^{\dot{\alpha}})$, respectively. The world-sheet derivatives $\partial$ and $\bar{\partial}$ denote $\partial = \partial_{\tau} + \partial_{\sigma}$ and $\bar{\partial} = \partial_{\bar{\tau}} - \partial_{\bar{\sigma}}$, respectively. It implies $\mathrm{d}z \mathrm{d}\bar{z} = -\frac{1}{2} \mathrm{d}\tau \mathrm{d}\sigma$.

The action is invariant under the gauge transformations $\delta \lambda^{\alpha} = \Lambda^m (\gamma^m \lambda)^\alpha$ and $\delta \bar{\lambda}^{\dot{\alpha}} = \bar{\Lambda}^m (\gamma^m \bar{\lambda})_{\dot{\alpha}}$. We use $16 \times 16$ symmetric matrices $\gamma^m_{\alpha\beta}$ and $\gamma^{m\dot{\alpha}\dot{\beta}}$ which are off-diagonal blocks of the $32 \times 32$ gamma matrices and satisfy $\gamma^m_{\alpha\beta} \gamma^n_{\beta\gamma} + \gamma^n_{\dot{\alpha}\dot{\beta}} \gamma^m_{\dot{\beta}\gamma} = 2 \eta^{mn} \delta_{\alpha}^{\gamma}$. We frequently use the Fierz identity $\gamma^m_{\alpha\beta} \gamma^m_{\gamma\delta} = 0$.

The action (2.1) is invariant under the ten-dimensional $\mathcal{N} = 2$ supersymmetry transformations

$$
\delta \theta^{\alpha} = \epsilon^\alpha, \quad \delta \bar{\theta}^{\dot{\alpha}} = \tilde{\epsilon}^{\dot{\alpha}}, \quad \delta x^m = \frac{1}{2} \theta \gamma^m \epsilon + \frac{1}{2} \bar{\theta} \gamma^m \tilde{\epsilon},
$$

$$
\delta p_{\alpha} = \frac{1}{2} \partial x^m (\gamma^m_{\alpha} \epsilon) - \frac{1}{8} (\epsilon \gamma^m \theta) (\gamma^m \partial \theta)_{\alpha}, \quad \delta \bar{p}_{\dot{\alpha}} = \frac{1}{2} \bar{\partial} x^m (\gamma^m_{\dot{\alpha}} \tilde{\epsilon}) - \frac{1}{8} (\bar{\epsilon} \gamma^m \bar{\theta}) (\gamma^m \bar{\partial} \bar{\theta})_{\dot{\alpha}},
$$

where parameters $\epsilon$ and $\tilde{\epsilon}$ are ten-dimensional Majorana-Weyl spinors. For an open superstring, we are left with a surface term

$$
\delta_{\epsilon} S_0 = \frac{1}{2 \pi \alpha'} \int \mathrm{d}\tau \left\{ \frac{1}{2} \left( (\epsilon \gamma^m \theta - \tilde{\epsilon} \gamma^m \bar{\theta}) \dot{x}_m + \frac{1}{12} (\epsilon \gamma^m \theta)(\theta \gamma^m \dot{\theta}) - \frac{1}{12} (\tilde{\epsilon} \gamma^m \bar{\theta})(\bar{\theta} \gamma^m \bar{\dot{\theta}}) \right) \right\}, \quad (2.3)
$$

where “$|$” means “evaluated at the boundary” and we will omit it for brevity in the following. A dot on a field denotes the $\tau$-derivative of the field, while a prime does the $\sigma$-derivative. If
there are no background fields, the surface term (2.3) can be eliminated by imposing usual boundary conditions for Dp-branes.

\[ x^\mu = 0 \ , \ \dot{x}^i = 0 \ , \ \widehat{\theta} = \gamma^{1-p}\theta \ , \ \widehat{\lambda} = \gamma^{1-p}\lambda \ , \ (2.4) \]

and \( \mathcal{N} = 1 \) supersymmetry condition \( \widehat{\epsilon} = \gamma^{1-p}\epsilon \). These boundary conditions imply that \( p = \text{odd} \) for the type IIB string while \( p = \text{even} \) for the type IIA string. As usual, \( x^\mu (\mu = 0, \cdots, p) \) are Neumann coordinates, while \( x^i (i = p+1, \cdots, 9) \) are Dirichlet coordinates.

Instead of imposing boundary conditions, we will consider coupling to the background superfields preserving the \( \mathcal{N} = 1 \) supersymmetry specified by \( \widehat{\epsilon} = \gamma^{1-p}\epsilon \). To preserve \( \mathcal{N} = 1 \) supersymmetry, we must introduce a boundary term which eliminates (2.3).

### 3 \( \mathcal{N} = 1 \) supersymmetry and boundary term

Here we will introduce a boundary term \( S_b \) which leaves \( S_0 + S_b \) invariant under the \( \mathcal{N} = 1 \) supersymmetry up to equations of motion. For this purpose, it is convenient to introduce the following objects

\[
\begin{align*}
\theta^\pm_\alpha &= \frac{1}{\sqrt{2}} (\hat{\theta}^\pm + (\gamma^{1-p}\theta)^\alpha) \\
n_{\pm}^\pm &= \sqrt{2} \left( \hat{d}^\pm_\alpha + (\gamma^{1-p}d)^\alpha_\alpha \right) \\
\lambda^\pm_\alpha &= \frac{1}{\sqrt{2}} (\hat{\lambda}^\pm + (\gamma^{1-p}\lambda)^\alpha) \\
\omega^\pm_\alpha &= \sqrt{2} \left( \hat{\omega}^\pm_\alpha + (\gamma^{1-p}\omega)^\alpha_\alpha \right) 
\end{align*}
\]

where \( d_\alpha = \hat{p}_\alpha - \frac{1}{2} \partial x^m (\gamma_m \theta)^\alpha_\alpha - \frac{1}{4} (\theta \gamma^m \partial \theta) (\gamma^m \theta)^\alpha_\alpha \) and \( \hat{d}_\alpha = \hat{p}_\alpha - \frac{1}{2} \partial x^m (\gamma_m \hat{\theta})^\alpha_\alpha - \frac{1}{4} (\hat{\theta} \gamma^m \hat{\partial} \hat{\theta}) (\gamma^m \hat{\theta})^\alpha_\alpha \) are invariant under the \( \epsilon^- \) and \( \hat{\epsilon}^- \) supersymmetry in (2.2), respectively. By using these variables, the \( \mathcal{N} = 1 \) supersymmetry transformations specified by \( \hat{\epsilon} = \gamma^{1-p}\epsilon \) are represented as

\[
\begin{align*}
\delta_\eta \theta^\pm_+ &= \eta^\alpha_\pm \\
\delta_\eta \theta^\pm_- &= 0 \\
\delta_\eta x^\mu &= \frac{1}{2} \theta^\pm_+ \gamma^\mu \eta \\
\delta_\eta x^i &= \frac{1}{2} \theta^\pm_+ \gamma^i \eta \\
\delta_\eta \lambda^\pm_\alpha &= \delta_\eta \omega^\pm_\alpha = 0 
\end{align*}
\]

where we introduced \( \eta \) by \( \eta \equiv \frac{1}{\sqrt{2}} (\hat{\epsilon} + \gamma^{1-p}\epsilon) \). The \( \mathcal{N} = 1 \) supersymmetry transformation of \( S_0 \) is found to be

\[
\delta_\eta S_0 = -\frac{1}{2\pi a'} \int d\tau \left\{ \frac{1}{2} (\eta \gamma^\mu \theta^-) \dot{x}_\mu + \frac{1}{2} (\eta \gamma^i \theta^+ \theta) \dot{x}_i \\
+ \frac{1}{8} (\eta \gamma^m \theta^+ \theta^-) (\theta^- \gamma_m \theta^+ \theta^+ \theta) + \frac{1}{24} (\eta \gamma^m \theta^- \theta^- \gamma_m \hat{\theta}^+ \theta^+) \right\} ,
\]

\footnote{We must impose the same boundary condition on \( \theta \) and \( \lambda \) since BRS transformations relate them each other. These boundary conditions also eliminate the surface term which comes from the BRS transformation of the world-sheet action \( S_0 \). See [26,27] for related topics.}
where we have used the Fierz identity.

The boundary term \( S_b \) we found is

\[
S_b = \frac{1}{2\pi\alpha'} \int d\tau \left\{ \frac{1}{2} \Pi_+^\mu (\theta_+ + \dot{\theta}_+) - \frac{1}{2} y^i (\theta_+ + \gamma_i \dot{\theta}_+) - \frac{1}{8} (\theta_+ + \gamma^\mu \theta_-)(\theta_+ + \gamma_\mu \dot{\theta}_+) + \frac{1}{8} (\theta_+ + \gamma^\delta \theta_-)(\theta_+ + \gamma \dot{\theta}_+) \\
+ \frac{1}{24} (\theta_+ + \gamma^m \theta_-)(\theta_+ + \gamma^m \dot{\theta}_-) + \frac{1}{2} c_1 \Delta^+ \theta^\alpha + \frac{1}{2} c_2 \omega^+ \lambda^\alpha + y_i \tilde{\Pi}_+^i \right\},
\]

(3.4)

where \( c_1 \) and \( c_2 \) are constants. We have introduced the followings

\[
\Pi_+^\mu = \frac{1}{2} \left( \hat{\Pi}_+^\mu + \Pi_+^\mu \right) - \frac{1}{2} (\theta_- + \gamma^\mu \dot{\theta}_-), \quad \Pi_-^\mu = \frac{1}{2} \left( \hat{\Pi}_-^\mu + \Pi_-^\mu \right) - \frac{1}{2} (\theta_- + \gamma^\mu \dot{\theta}_-),
\]

\[
\hat{\Pi}_+^\mu = \frac{1}{2} \left( \hat{\Pi}_+^\mu - \Pi_+^\mu \right) - \frac{1}{2} (\theta_- + \gamma^\mu \dot{\theta}_-), \quad \hat{\Pi}_-^\mu = \frac{1}{2} \left( \hat{\Pi}_-^\mu - \Pi_-^\mu \right) - \frac{1}{2} (\theta_- + \gamma^\mu \dot{\theta}_-),
\]

(3.5)

\[
y^i = x^i + \frac{1}{2} (\theta_+ + \gamma^\mu \dot{\theta}_-),
\]

\[
\Delta^+_\alpha = d^+_\alpha + \frac{1}{2} (\gamma^\mu \theta_-) \alpha \left( \hat{\Pi}_+^\mu - \Pi_+^\mu \right) + \frac{1}{2} (\gamma^\delta \theta_-) \alpha \left( \hat{\Pi}_i^i - \Pi_i^i \right),
\]

where \( \Pi_+^\mu = \partial x^\mu + \frac{1}{2} \theta_i^\mu \partial \theta_i \) and \( \hat{\Pi}_+^\mu = \partial x^\mu + \frac{1}{2} \hat{\theta}_i^\mu \partial \hat{\theta}_i \) are \( \epsilon \)- and \( \hat{\epsilon} \)-supersymmetry invariants, respectively. Objects in (3.5) are invariant under the \( \mathcal{N} = 1 \) supersymmetry (up to equations of motion except for \( \Delta^+_\alpha \) and \( y^i \)). The equations of motion \( \partial \theta^\alpha = \partial \hat{\theta}^\alpha = 0 \) implies

\[
\theta'_+ = -\dot{\theta}_- \quad \text{and} \quad \theta'_- = -\dot{\theta}_+.
\]

(3.6)

Throughout this paper, we will use them only after transformations for supersymmetry and BRST symmetry. We note that the last three terms in (3.4) are invariant under the \( \mathcal{N} = 1 \) supersymmetry separately. This implies that they are not determined from the \( \mathcal{N} = 1 \) supersymmetry. It is worth noting that (3.4) cannot be extracted as a dimensional reduction of the one for the D9-brane.

### 3.1 BRS symmetry

We shall show that the last term \( y_i \tilde{\Pi}_+^i \) in (3.4) is required by the BRS invariance of \( S_0 + S_b \), when there is no background superfield coupling.

The action (2.1) is invariant under a pair of BRS variations, say \( \delta_1 \) and \( \delta_2 \). In the presence of the boundary, these BRS variation must satisfy \( \delta_1 = \delta_2 \) at the boundary. This implies
that the BRS transformations $\delta Q = \delta_1 + \delta_2$ remain unbroken in the presence of the boundary

$$
\delta_Q \theta^a = \lambda^a_\pm , \quad \delta_Q \lambda^a_\pm = 0 , \quad \delta_Q \omega^a_\pm = d^a_\pm ,
$$

$$
\delta_Q x^\mu = \frac{1}{2} \lambda^+ \gamma^\mu \theta^+ + \frac{1}{2} \lambda^- \gamma^\mu \theta^- , \quad \delta_Q x^i = \frac{1}{2} \lambda^+ \gamma^i \theta^+ + \frac{1}{2} \lambda^- \gamma^i \theta^- , \quad \delta_Q y^i = \lambda^+ \gamma^i \theta^- ,
$$

$$
\delta_Q \pi^\mu_+ = \lambda^+ \gamma^\mu \dot{\theta}^+ + \frac{1}{2} \lambda^- \gamma^\mu \theta^- , \quad \delta_Q \pi^i_+ = \lambda^+ \gamma^i \dot{\theta}^+ + \frac{1}{2} \lambda^- \gamma^i \theta^- ,
$$

$$
\delta_Q \pi^\mu_- = \lambda^+ \gamma^\mu \dot{\theta}^- + \frac{1}{2} \lambda^- \gamma^\mu \theta^- , \quad \delta_Q \pi^i_- = \lambda^+ \gamma^i \dot{\theta}^- + \frac{1}{2} \lambda^- \gamma^i \theta^- ,
$$

$$
\delta_Q \Delta^{+}_\alpha = -2(\gamma^\mu_+ \lambda^\mu_+) \pi^\mu_+ - 2(\gamma^i_+ \lambda^i_+) \pi^i_+ - (\gamma^\mu_- \lambda^\mu_-) \pi^\mu_- - (\gamma^i_- \lambda^i_-) \pi^i_- - (\gamma^m \lambda^- \dot{\theta}^-)(\theta^- \gamma^m \dot{\theta}^-) ,
$$

where equations of motion (3.6) are used after the BRS transformations. Again, we find the world-sheet action $S_0$ is BRS invariant $\delta_Q S_0 = 0$ up to a surface term, and satisfies

$$
\delta_Q (S_0 + S_b) = \frac{1}{2\pi \alpha'} \int d\tau \left\{ (1 - c_1) \pi^\mu_+ (\lambda^+ \gamma^\mu \theta^-) - \frac{1}{2} (c_1 + c_2) \pi^i_- (\lambda^- \gamma^i \theta^-) \right.
$$

$$
- \frac{1}{2} (c_1 + c_2) \pi^\mu_+ (\lambda^- \gamma^\mu \theta^-) + (1 - c_1) \pi^i_+ (\lambda^+ \gamma^i \theta^-)
$$

$$
+ \frac{1}{2} (c_2 - c_1) \Delta^{+}_\gamma \lambda^\mu \dot{\theta}^+ + \left( - \frac{1}{3} + \frac{c_1 - c_2}{4} \right) (\lambda_- \gamma^m \theta^-)(\theta^- \gamma^m \dot{\theta}^-)
$$

$$
+ \frac{1}{2} \left( \frac{1}{3} - c_1 \right) (\lambda^+ \gamma^m \theta^-)(\theta_- \gamma^m \dot{\theta}^-) \} .
$$

Let us assume that there are no background fields. In this case, the (3.8) must be eliminated by the usual boundary conditions $\theta^a_\infty = \lambda^a_\infty = 0$. It is obvious to see that these boundary conditions eliminate (3.8) as expected. It should be noted that this happens only when we include the term $y_i \pi^i_+$ in (3.4).

Finally we comment on $y_i$. Remarkably, we confirmed that $S_0 + S_b$ is independent of $y_i$. This strongly suggests that $y_i$ should represent the position of the D$p$-brane.

### 4 Supersymmetric DBI equations of motion

In this section, we will give the background coupling $V$ in terms of superfields on a D$p$-brane. Examining the BRS variation of $S_0 + S_b + V$, we obtain supersymmetric DBI equations of motion on the D$p$-brane.

#### 4.1 Background superfield coupling for D$p$-branes

In Appendix A, we define the ten-dimensional $\mathcal{N} = 1$ superfield $A_M = (A_m, A_\alpha)$. We introduce background superfields on a D$p$-brane as a dimensional reduction of $A_M$: $A_m =$
\( A_\mu(x^\mu, \theta_+) \) and \( A_\alpha = A_\alpha(x^\mu, \theta_+) \). Obviously they are invariant under the \( \mathcal{N} = 1 \) supersymmetry. Similarly we introduce \( \mathcal{W}^\alpha = \mathcal{W}^\alpha(x^\mu, \theta_+) \) and \( F_{mn} = F_{mn}(x^\mu, \theta_+) \). We use the ten-dimensional Majorana-Weyl spinor notation throughout this paper. This means that we are considering the DBI equations with 16 supersymmetries, for example \( \mathcal{N} = 4 \) supersymmetric DBI equations on a D3-brane.

The background coupling \( V \) used in [21] is regarded as an extension of the vertex operator of the open pure spinor superstring. We give a brief review of the vertex operator in Appendix B.

The background coupling \( V \) we introduce is

\[
V = \frac{1}{2\pi\alpha'} \int \! \! d\tau \left\{ \dot{\theta}_+^\alpha A_\alpha(x^\mu, \theta_+) + \Pi_+^\mu A_\mu(x^\mu, \theta_+) + \Pi_+^i A_i(x^\mu, \theta_+) \\
+ \frac{1}{2} \Delta_+^\alpha \mathcal{W}^\alpha(x^\mu, \theta_+) + \frac{1}{4} \mathcal{N}_+ \mathcal{F}(x^\mu, \theta_+) \right\}
\tag{4.1}
\]

where

\[
(\mathcal{N}_+)_{\alpha}^\beta = \frac{1}{2} \omega_+^+ \lambda_+^\beta , \\
\mathcal{F}_\alpha^\beta = \delta_\alpha^\beta \mathcal{F}^{(0)} + (\gamma^{m n})_{\alpha}^{\beta} \mathcal{F}^{(2)}_{m n} + (\gamma^{m n p q})_{\alpha}^{\beta} \mathcal{F}^{(4)}_{m n p q} .
\tag{4.2}
\]

Note that \( \mathcal{F}^{(0)}, \mathcal{F}^{(2)}_{m n} \) and \( \mathcal{F}^{(4)}_{m n p q} \) are some possible products of any number of vector field strengths \( F_{mn} \), which is consistent with analysis for D-brane boundary states [23] from the viewpoint of the pure spinor closed superstring. Needless to say, the \( V \) is invariant under the \( \mathcal{N} = 1 \) supersymmetry. Since we have made the factor \( 1/(2\pi\alpha') \) manifest in \( V \), dimensions of these superfields differ from conventional ones. In this sense, we assign dimensions to \([A_\alpha], [A_m], [W^\alpha] \) and \([F_{mn}] \) as \(-\frac{3}{2}, -1, -\frac{1}{2} \) and 0, respectively.

### 4.2 DBI equations from BRS symmetry

In this subsection, we will add the background superfield coupling \( V \) in (4.1) to the action \( S_0 + S_b \) and then require that the BRS variation \( \delta_Q(S_0 + S_b + V) \) vanishes. This requirement leads to boundary conditions on spacetime spinors and conditions on background superfields. The latter is found to be supersymmetric DBI equations of motion for them.

There are no more higher forms because of the property

\[
\omega_\alpha(\gamma^{m_1 \cdots m_{2k}})_{\beta}^{\alpha} \lambda^\beta = \pm \frac{1}{(10 - 2k)!} (-1)^{k+1} e^{m_1 \cdots m_{2k}} \omega_\alpha(\gamma^{n_1 \cdots n_{10 - 2k}})_{\beta}^{\alpha} \lambda^\beta ,
\tag{4.3}
\]

where the sign in the right hand side depends on the chirality of \( \lambda \).
We find that the BRS variation \( \delta Q \) may be expressed as

\[
\delta Q V = \frac{1}{2\pi\alpha'} \int d\tau \left\{ \Pi^\mu_+ \left[ -\lambda^\alpha_+ \partial_\mu A_\alpha + \lambda^\alpha_+ D_\alpha A_\mu + \frac{1}{2}(\lambda^- \gamma^\theta_-)(\partial_\mu A_\alpha - \partial_\alpha A_\mu) - (\lambda_+ \gamma_+ \mathcal{W}) \right] \\
+ \Pi^i_+ \left[ -\frac{1}{2}(\lambda^- \gamma_i \mathcal{W}) - \frac{1}{8}(\gamma_i \theta_-)_\alpha \lambda^\beta_+ \mathcal{F}_\beta^\alpha \right] \\
+ \tilde{\Pi}^\mu_+ \left[ -\frac{1}{2}(\lambda^- \gamma_\mu \mathcal{W}) - \frac{1}{8}(\gamma_\mu \theta_-)_\alpha \lambda^\beta_+ \mathcal{F}_\beta^\alpha \right] \\
+ \tilde{\Pi}^i_+ \left[ -\lambda^\alpha_+ \mathcal{D}_\alpha A_i + \frac{1}{2}(\lambda^- \gamma^\theta_- \mu) \partial_\mu A_i \right] \\
+ \frac{1}{2} \Delta_+^\beta \left[ -\lambda^\alpha_+ D_\alpha W^\beta - \frac{1}{2}(\lambda^- \gamma^\theta_- \mu) \partial_\mu W^\beta + \frac{1}{4} \lambda^\alpha_+ \mathcal{F}^\beta_\alpha \right] \\
+ \frac{1}{4} N_{+\gamma}^\beta \left[ \lambda^\alpha_+ D_\alpha \mathcal{F}^\beta_\gamma + \frac{1}{2}(\lambda^- \gamma^\theta_- \mu) \partial_\mu \mathcal{F}^\beta_\gamma \right] \\
+ \hat{\beta}_+^\beta \left[ -\lambda^\alpha_+ D_\alpha A_\beta - \lambda^\alpha_+ D_\alpha A_\beta - \frac{1}{2}(\lambda^- \gamma^\theta_- \mu) \partial_\mu A_\beta + (\gamma^m_\lambda \alpha_+ \beta) A_m \\
- \frac{1}{2}(\lambda^- \gamma^m_\lambda \beta) D_\beta A_m + \frac{1}{4}(\gamma^m_\lambda \alpha_+ \beta) \partial_\mu A_\beta + (\gamma^m_\alpha_+ \beta) \mathcal{F}^\gamma_\alpha \right]\right\}.
\]

(4.4)

Note that the supercovariant derivative on the Dp-brane is defined by

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha_+} + \frac{1}{2}(\gamma^\mu_\alpha_+ \theta_-) \partial_\mu .
\]

(4.5)

Gathering (3.8) and (4.4) together, we obtain the BRS variation of \( S_0 + S_b + V \) as

\[
\delta Q(S_0 + S_b + V) = \frac{1}{2\pi\alpha'} \int d\tau \left\{ \Pi^\mu_+ X_\mu + \Pi^i_+ X_i + \Pi_+^- Y_i + \Pi_+^- Y_\mu \\
- \frac{1}{2} \Delta^\beta_+ \Lambda_\beta + \frac{1}{4} N_{+\beta}^\alpha Z_\alpha^\beta + \hat{\beta}_+^\alpha \Theta_\alpha^+ + \hat{\theta}_+^\alpha \Theta_\alpha^- \right\}.
\]

(4.6)
where $X_m$, $Y_m$, $\Lambda^\beta$, $Z^\beta_\alpha$ and $\Theta^\pm_\alpha$ are given as follows

\[
X_m \equiv (1 - c_1)(\lambda_+^m \gamma m \theta_-) - \lambda_+^a \partial_m A_a + \lambda_+^a D_\alpha A_m - \frac{1}{2}(\lambda_-^m \gamma m \theta_-)(\partial_m A_n - \partial_n A_m) \\
- (\lambda_+^m \gamma m \mathcal{W}) ,
\]

(4.7)

\[
Y_m \equiv -\frac{1}{2}(c_1 + c_2)(\lambda_-^m \theta_-) - \frac{1}{2}(\lambda_-^m \gamma m \mathcal{W}) - \frac{1}{8}(\gamma m \theta_-)\alpha \lambda_+^\beta \mathcal{F}_\beta ,
\]

(4.8)

\[
\Lambda^\beta \equiv (c_1 - c_2)\lambda_+^\beta + \lambda_+^a D_\alpha \mathcal{W}^\beta + \frac{1}{2}(\lambda_-^\mu \gamma m \theta_-)\partial_\mu \mathcal{W}^\beta - \frac{1}{4}\lambda_+^\alpha \mathcal{F}_\alpha ,
\]

(4.9)

\[
Z^\beta_\alpha \equiv \lambda_+^\beta \gamma_\alpha \mathcal{F}_\alpha + \frac{1}{2}(\lambda_-^\mu \gamma m \theta_-)\partial_\mu \mathcal{F}_\alpha ,
\]

(4.10)

\[
\Theta^+_\alpha \equiv \left(-\frac{1}{3} + \frac{c_1 - c_2}{4}\right)(\gamma m \theta_-)\alpha (\lambda_-^m \gamma m \theta_-) - \lambda_+^\beta (D_\alpha A_\beta + D_\beta A_\alpha) - \frac{1}{2}(\lambda_-^m \gamma m \theta_-)\partial_\mu A_\mu \\
+ (\gamma m \lambda_+^m \mathcal{W}) - \frac{1}{2}(\lambda_-^m \gamma m \theta_-) D_\alpha A_m + \frac{1}{2}(\lambda_-^m \gamma m \theta_-) (\theta_- \lambda_-^m \mathcal{W}) \\
+ \frac{1}{4}(\gamma m \theta_-)\alpha (\lambda_-^m \gamma m \mathcal{W}) - \frac{1}{16}(\gamma m \theta_-)\alpha (\gamma m \theta_-) \lambda_+^\beta \gamma_\alpha \mathcal{F}_\beta ,
\]

(4.11)

\[
\Theta^-_\alpha \equiv \frac{1}{2}(c_1 - \frac{1}{3})(\gamma m \theta_-)\alpha (\lambda_-^m \gamma m \theta_-) - \frac{1}{2}(\lambda_-^m \gamma m \theta_-)(\gamma m \mathcal{W})\alpha .
\]

(4.12)

In the following, we will examine conditions that each term in (4.6) vanishes.

First of all, it is noted that we may fix degrees of freedom for the D-brane position by a *static-like* gauge

\[
y_i = -A_i .
\]

(4.13)

We will comment on this issue later.

To achieve our purpose, first, we focus on the term $\tilde{\Pi}_+^i X_i$ in (4.6) which takes the form

\[
\tilde{\Pi}_+^i \left[-\lambda_+^\alpha \gamma_{\alpha \beta} \left(c_1 \theta_-^\beta + \mathcal{W}^\beta\right) + \delta_Q(y_i + A_i)\right] .
\]

(4.14)

(4.13) suggests $\delta_Q(y_i + A_i) = 0$. In addition, we obtain the boundary condition on $\theta_-^\beta$ as

\[
\theta_-^\beta = -\frac{1}{c_1} \mathcal{W}^\beta .
\]

(4.15)

This eliminates $\theta_-^\alpha$ from (4.6) completely. Hereafter we understand $\theta_-^\alpha$ as (4.15). Note that (4.15) also leads to

\[
\dot{\theta}_-^\beta = -\frac{1}{c_1} \left(\Pi_+^\beta \partial_\mu \mathcal{W}^\beta + \dot{\gamma}_\gamma \partial_\gamma \mathcal{W}^\beta\right) .
\]

(4.16)

Secondly, the terms $\Pi_-^i Y_i$ and $\tilde{\Pi}_-^\mu Y_\mu$ reduce to

\[
\Pi_-^\mu \left[\left(c_2 \lambda_+^\alpha + \frac{1}{4}\lambda_+^\beta \mathcal{F}_\beta\right) \frac{1}{c_1}(\gamma m \mathcal{W})\alpha\right] ,
\]

(4.17)
and imply the boundary condition on $\lambda_-$

$$\lambda_\alpha = -\frac{1}{4c_2} \lambda_+ \mathcal{F}_\beta^\alpha .$$

This eliminates $\lambda^\alpha$ from (4.6) completely. Hereafter we understand $\lambda^\alpha_-$ as (4.18).

Here, it is better to comment on two consequences of the boundary conditions (4.15) and (4.18). First, consider the limit $\alpha' \to 0$. The limit $\alpha' \to 0$, after rescaling $A_\alpha \to (2\pi \alpha') A_\alpha$, $A_m \to (2\pi \alpha') A_m$, $W^\alpha \to (2\pi \alpha') W^\alpha$ and $F_{mn} \to (2\pi \alpha') F_{mn}$, turns the boundary conditions (4.15) and (4.18) to the usual boundary conditions $\theta^\alpha = \lambda^\alpha = 0$. The BRS invariance $\delta Q(S_0 + S_b + V) = 0$ then implies $\delta Q V = 0$, since $\delta Q(S_0 + S_b) = 0$ under these boundary conditions. We can show that $\delta Q V = 0$ with usual boundary conditions leads to the super Yang-Mills equations of motion (B.9), (B.10) and (B.11) as discussed in Appendix B. Next, we consider the BRS variation of the static-like gauge condition $\delta Q(y_i + A_i) = 0$. Under the boundary conditions (4.15) and (4.18), it reduces to the following equation

$$-D_\alpha A_i + \frac{1}{c_1} (\gamma_i W)_\alpha - \frac{1}{8c_1 c_2} \mathcal{F}_\alpha (\gamma^\mu W)_{\mu} \partial_{\mu} A_i = 0 .$$

This is one of the DBI equations.

Let us return to the subject. Thirdly, the term $\Delta^+ \Lambda^\beta$ in (4.6) is examined. We see that $\Lambda^\beta = 0$ reduces to

$$\frac{1}{c_1} D_\alpha W^\beta = \frac{1}{4c_2} \mathcal{F}_\alpha + \frac{1}{8c_1 c_2} \mathcal{F}_\alpha (\gamma^\mu W)_{\gamma} \partial_{\mu} W^\beta = 0 .$$

This is one of the DBI equations on a D$^p$-brane. This equation ensures that conditions (4.15) and (4.18) are consistent with BRS transformations $\delta_{Q} \theta^\alpha = \lambda^\alpha$ and $\delta_{Q} \lambda^\alpha = 0$.

As was done in [21], it is convenient to introduce a covariant derivative $\hat{D}_\alpha$ by

$$\hat{D}_\alpha \equiv D_\alpha + \frac{1}{8c_1 c_2} \mathcal{F}_\alpha (\gamma^\mu W)_{\gamma} \partial_{\mu} .$$

Applying it to $\frac{1}{c_1} W^\beta$, we obtain

$$\frac{1}{c_1} \hat{D}_\alpha W^\beta = \frac{1}{c_1} D_\alpha W^\beta + \frac{1}{8c_1 c_2} \mathcal{F}_\alpha (\gamma^\mu W)_{\gamma} \partial_{\mu} W^\beta = \frac{1}{4c_2} \mathcal{F}_\alpha ,$$

where in the last equality (4.20) is used. On the other hand, as (4.20) implies

$$\frac{1}{4c_2} \mathcal{F}_\alpha = \frac{1}{c_1} D_\alpha W^\gamma \left( \delta^\gamma_{\beta} - \frac{1}{2c_1} (\gamma^\mu W)_{\beta} \partial_{\mu} W^\gamma \right)^{-1} ,$$

$\hat{D}_\alpha$ is expressed as

$$\hat{D}_\alpha = D_\alpha + \frac{1}{2c_1} D_\alpha W^\beta (\delta^\beta_{\gamma} - \frac{1}{2c_1} (\gamma^\mu W)_{\gamma} \partial_{\mu} W^\beta)^{-1} (\gamma^\mu W)_{\gamma} \partial_{\mu} .$$
It follows that it satisfies the following anticommutation relation

\[ \{ \tilde{D}_\alpha, \tilde{D}_\beta \} = \left( \gamma^\mu_{\alpha\beta} + \frac{1}{16c_1^2} F^\gamma_{\alpha} F^\delta_{\beta} \gamma^\mu_{\gamma\delta} \right) \tilde{\partial}_\mu, \]

\[ \tilde{\partial}_\mu \equiv \partial_\mu + \frac{1}{2c_1^2} \partial_\mu \gamma^\alpha \left( \delta^\alpha - \frac{1}{2c_1^2} \gamma^\mu_{\beta\gamma} W^\gamma \partial_\mu W^\alpha \right) (\gamma^\rho W)_\beta \partial_\rho. \tag{4.25} \]

Fourthly, the term \((N_+)\lambda^\gamma Z^\alpha\) in (4.6) is examined. Using (4.22) and (4.25), it turns to

\[ \lambda^\alpha \lambda^\gamma \left( \frac{1}{c_1} \tilde{D}_\alpha F^\beta_{\gamma \gamma} \right) = \lambda^\alpha \lambda^\gamma \left( \frac{4}{c_1} \tilde{D}_\alpha \tilde{D}_\beta W^\beta \right) = \lambda^\alpha \lambda^\gamma \left( \gamma^\mu_{\alpha\gamma} + \frac{1}{16c_1^2} F^\delta_{\alpha} F^\gamma_{\beta} \gamma^\mu_{\gamma\delta} \right) \frac{2}{c_1} \tilde{\partial}_\mu W^\beta, \tag{4.26} \]

which vanishes due to the pure spinor constraint \(\lambda^\pm \gamma^\mu \lambda^\pm + \lambda^- \gamma^\mu \lambda^- = 0\).

Fifthly, we consider terms including \(\Pi_+^m\) in (4.6), \(\Pi_+^m X_\mu - \frac{1}{c_1} \Pi_+^m \partial_\mu W^\alpha \Theta^-_{\alpha}\), where the second term comes from (4.16). It is straightforward to see that it is eliminated by

\[ \partial_\mu A_\alpha - D_\alpha A_\mu + \frac{1}{c_1} (\gamma_\mu W)_\alpha \]

\[ + \frac{1}{6c_1^3} (\gamma_\mu W)_\alpha (W_{\gamma m} \partial_\mu W) + \frac{1}{8c_1 c_2} F^\beta_{\alpha} (\gamma_\mu W)_\beta (\partial_\mu A_n - \partial_n A_\mu) = 0, \tag{4.27} \]

which is one of the DBI equations. Combining it with (4.19), we obtain

\[ \partial_m A_\alpha - D_\alpha A_m + \frac{1}{c_1} (\gamma_m W)_\alpha \]

\[ + \frac{1}{6c_1^3} (\gamma_\mu W)_\alpha (W_{\gamma m} \partial_\mu W) + \frac{1}{8c_1 c_2} F^\beta_{\alpha} (\gamma_\mu W)_\beta (\partial_m A_n - \partial_n A_\mu) = 0. \tag{4.28} \]

Finally, we consider terms including \(\dot{\theta}^\alpha_+\) in (4.6), \(\dot{\theta}^\alpha_+ \Theta^-_\alpha - \frac{1}{c_1} \dot{\theta}^\beta_+ D_\beta W^\alpha \Theta^-_{\alpha}\), where the second term comes from (4.16). These terms are eliminated by

\[ - D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha\beta} A_m - \frac{1}{6c_1^3} (\gamma_\mu W)_\alpha (W_{\gamma m} D_\beta W) \]

\[ + \frac{1}{12c_1^2 c_2} F^\gamma_{\alpha} (\gamma_\mu W)_\beta (\gamma_\mu W)_\gamma - \frac{1}{8c_1^2 c_2} F^\gamma_{\alpha} (\gamma_\mu W)_\beta (\partial_m A_\beta + D_\beta A_m) = 0. \tag{4.29} \]

By eliminating \(\partial_m A_\beta + D_\beta A_m\) by (4.28), it reduces to

\[ - D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha\beta} A_m - \frac{1}{6c_1^3} (\gamma_\mu W)_\alpha (W_{\gamma m} D_\beta W) \]

\[ + \frac{1}{6c_1^2} (\gamma_\mu W)_\beta (\gamma_\mu W)_\gamma \left\{ - \frac{1}{4c_2} F^\gamma_{\alpha} + \frac{1}{8c_1^2 c_2} F^\gamma_{\alpha} (\gamma_\mu W)_\beta \partial_\mu W^\gamma \right\} \]

\[ + \frac{1}{64c_1^2 c_2} F^\gamma_{\alpha} F^\delta_{\beta} (\gamma_\mu W)_\gamma (\gamma_\mu W)_\delta (\partial_m A_n - \partial_n A_\gamma) = 0. \tag{4.30} \]

Finally substituting (4.20) into the expression in the curly braces in (4.30), we obtain

\[ - D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha\beta} A_m - \frac{1}{6c_1^3} (\gamma_\mu W)_\alpha (W_{\gamma m} D_\beta W) - \frac{1}{6c_1^3} (\gamma_\mu W)_\beta (W_{\gamma m} D_\alpha W) \]

\[ + \frac{1}{64c_1^2 c_2} F^\gamma_{\alpha} F^\delta_{\beta} (\gamma_\mu W)_\gamma (\gamma_\mu W)_\delta (\partial_m A_n - \partial_n A_\gamma) = 0. \tag{4.31} \]
As a result, we have obtained not only boundary conditions (4.15) and (4.18), but also independent equations for background superfields (4.28), (4.31) and (4.20) which eliminates (4.6). We note that \( c_1 \) and \( c_2 \) can be absorbed into redefinitions of \( W^\alpha \) and \( \tilde{F}^\beta_\alpha \) as \( \frac{1}{c_1} W^\alpha \rightarrow W^\alpha \) and \( \frac{1}{c_2} \tilde{F}_\alpha^\beta \rightarrow \tilde{F}_\alpha^\beta. \) So we will set \( c_1 = c_2 = 1 \) without loss of generality\(^5\).

Summarizing, we have obtained supersymmetric DBI equations of motion on a Dp-brane

\[
\partial_m A_\alpha - D_\alpha A_m + (\gamma_m W)_\alpha + \frac{1}{6} (\gamma^n W)_\alpha (W^n \partial_m W) + \frac{1}{8} \tilde{F}_\alpha^\beta (\gamma^n W)_\beta (\partial_m A_n - \partial_n A_m) = 0 ,
\]

\[
D_\alpha A_\beta + D_\beta A_\alpha - \gamma^m_{\alpha\beta} A_m + \frac{1}{6} (\gamma^n W)_\alpha (W^n \partial_m D_\beta W) + \frac{1}{6} (\gamma^n W)_\beta (W^n D_\alpha W) - \frac{1}{64} \tilde{F}_\gamma^\alpha \tilde{F}_\delta^\beta (\gamma^n W)_\gamma (\gamma^n W)_\delta (\partial_m A_n - \partial_n A_m) = 0 ,
\]

\[
D_\alpha W^\beta - \frac{1}{4} \tilde{F}_\alpha^\beta + \frac{1}{8} \tilde{F}_\gamma^\alpha (\gamma^\mu W)_\gamma \partial_\mu W^\beta = 0 .
\]

In the last equation, the index \( \mu \) may be replaced with \( m \) because \( \partial_i W^\beta = 0 \). Now it is manifest that our DBI equations on a Dp-brane can be expressed in a ten-dimensional covariant fashion. In other words, our result coincides with the dimensional reduction of those for a D9-brane, though the ten-dimensional covariance was absent in the beginning of our analysis.

## 5 Summary and discussions

We have examined the BRS invariance of the open pure spinor superstring in the presence of background superfields on a Dp-brane. It was shown that the BRS invariance leads not only to boundary conditions on the spacetime spinors, but also to supersymmetric DBI equations of motion for the background superfields on a Dp-brane. These DBI equations precisely coincide with those obtained by a dimensional reduction of the supersymmetric DBI equations for the abelian D9-brane given in [21, 22].

We have introduced the boundary term \( S_b \) and the background coupling \( V \). Both are determined by the BRS symmetry. In fact, \( S_b \) was shown to satisfy \( \delta_Q (S_0 + S_b) = 0 \), when we take the limit \( \alpha' \rightarrow 0 \) and turn off the background couplings. As for \( V \), we have shown that the conditions for \( \delta_Q (S_0 + S_b + V) = 0 \) reduce to the dimensional reduction of the super-Yang-Mills equations when \( \alpha' \rightarrow 0 \). In fact, taking the limit \( \alpha' \rightarrow 0 \), after rescaling \( A_\alpha \rightarrow (2\pi \alpha') A_\alpha, A_m \rightarrow (2\pi \alpha') A_m, W^\alpha \rightarrow (2\pi \alpha') W^\alpha \) and \( F_{mn} \rightarrow (2\pi \alpha') F_{mn}, \) the

\(^5\)If we construct the \( \kappa \)-invariant boundary term which cancels out an \( \mathcal{N} = 1 \) supersymmetry variation of the Green-Schwarz action and turn it into the BRS-invariant boundary term like (3.4) by the method used in [28] (see also the section 4.1 in [29]), it must be shown \( c_1 = c_2 = 1 \).
DBI equations (4.32)-(4.34) reduce to the super Yang-Mills equations of motion (B.9)-(B.11) with an appropriate dimensional reduction.

We note that the ten-dimensional Lorentz covariance is manifestly broken by the boundary term $S_b$ as well as the background coupling $V$. However the obtained DBI equations can be expressed in a covariant form. This implies that our result is consistent with that for a D9-brane.

We expect that we can extend our result so that the BRS invariance should lead to supersymmetric non-Abelian DBI equations of motion on a D$p$-brane. We would like to report this issue in the near future [30].

As an alternative to our study, non-abelian deformations of the maximally supersymmetric Yang-Mills theory can be specified based on spinorial cohomology [31], which may be closely related to the pure spinor fields in ten- and eleven-dimensional spacetime [32–34]. The structure of higher-derivative invariants in the maximally supersymmetric Yang-Mills theories are studied in [35]. Moreover, in [36, 37] the pure spinor superspace formalism is developed, which contains not only (minimal) pure spinor variables but also non-minimal pure spinor variables [38]. This enables us to construct the BRS invariant action for the ten-dimensional supersymmetric DBI theory. Recently, this off-shell action is studied further in [39, 40]. It is interesting to pursue these issues from the open string point of view.

On the other hand, the classical BRS invariance of a closed pure spinor superstring in a curved background is shown to imply that the background fields satisfy full non-linear equations of motion for the type II supergravity [41]. This is similar to the result for the classical $\kappa$-invariance of a closed Green-Schwarz superstring [42]. Moreover, recently in [43] the classical $\kappa$-invariance also leads to the generalized type II supergravity equations of motion\footnote{See [45] for further investigations based on double field theory.} whose solutions originally have found out in the context of integrable deformations of $AdS_5 \times S^5$ sigma models [44]. It is also interesting to consider whether the generalization of DBI equations can be derived analogously from the $\kappa$- or BRS-invariance of an open superstring.

An immediate task is to clarify contribution of the dilaton superfield to Bianchi identities. In that case we need to investigate closely the DBI equation corresponding to $J_{mna} = 0$ in the super Yang-Mills theory as we see in Appendix A. This equation is also useful to confirm that our result agrees with the one which comes from the bosonic part of the DBI action.

Finally, it is interesting for us to calculate quantum higher-derivative corrections to our result by analyzing the quantum BRS invariance of the open pure spinor superstring.
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Appendix

A Ten-dimensional $\mathcal{N} = 1$ super Yang-Mills space

We will review the ten-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory [46]. Introducing a superconnection one-form $A = E^M A_M$, where $E^M$ are supervielbein and $A_M = (A_m, A_\alpha)$ are superconnections, we define the gauge supercovariant derivative $\nabla_M$

$$\nabla_m = \partial_m + A_m , \quad \nabla_\alpha = D_\alpha + A_\alpha .$$

(A.1)

where $D_\alpha$ is the supercovariant derivative defined by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} (\gamma^m \theta)_\alpha \partial_m ,$$

(A.2)

which satisfies $\{ D_\alpha , D_\beta \} = \gamma_{\alpha\beta}^m \partial_m$. The field strengths $F_{MN}$ are defined by

$$[ \nabla_M , \nabla_N ] = T_{MN}^R \nabla_R + F_{MN} ,$$

(A.3)

where $T_{MN}^R$ are flat torsion tensors whose components are fixed to zero except for $T_{\alpha\beta}^m = \gamma_{\alpha\beta}^m$. According to this definition, these field strengths are invariant under the gauge transformations with a superfield parameter $\Omega$

$$\delta A_m = \partial_m \Omega , \quad \delta A_\alpha = D_\alpha \Omega .$$

(A.4)

For the on-shell super Yang-Mills theory, we might adopt a constraint [32] (see also [33])

$$F_{\alpha\beta} = 0 ,$$

(A.5)

which implies

$$D_\alpha A_\beta + D_\beta A_\alpha + \{ A_\alpha , A_\beta \} = \gamma_{\alpha\beta}^m A_m .$$

(A.6)
If we consider a dimensional reduction to four-dimensions, we see that this constraint reduces to the one in the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [47].

In the following, let us solve the Bianchi identities represented as

$$I_{MNR} = (-1)^{R(M+N)} \nabla_R F_{MN} - T_{MN}^S F_{SR}$$
$$+ (-1)^{M(R+N)} \nabla_N F_{RM} - (-1)^{R(M+N)} T_{RM}^S F_{SN}$$
$$+ \nabla_M F_{NR} - (-1)^{M(N+R)} T_{NR}^S F_{SM}.$$  \hspace{1cm} (A.7)

The first identity $I_{\alpha\beta\gamma} = 0$ implies

$$-\gamma^m_{\alpha\beta} F_{m\gamma} - \gamma^m_{\gamma\alpha} F_{m\beta} - \gamma^m_{\beta\gamma} F_{m\alpha} = 0.$$  \hspace{1cm} (A.8)

Thanks to the Fierz identity, we find that the field strength $F_{m\alpha}$ must take the form of

$$F_{m\alpha} = -\gamma_{m\alpha\beta} \mathcal{W}^\beta.$$  \hspace{1cm} (A.9)

In other words,

$$\partial_m A_\alpha - D_\alpha A_m + [A_m, A_\alpha] = -\gamma_{m\alpha\beta} \mathcal{W}^\beta.$$  \hspace{1cm} (A.10)

Next the second identity $I_{m\alpha\beta} = 0$ together (A.9) implies

$$\gamma_{m\alpha\beta} \nabla_\beta \mathcal{W}^\delta + \gamma_{m\beta\delta} \nabla_\alpha \mathcal{W}^\delta - \gamma_{n\alpha\beta} F_{nm} = 0.$$  \hspace{1cm} (A.11)

Multiplying this by $\gamma^{\alpha\beta}_p$, we find that

$$F_{mn} = \frac{1}{8} (\gamma_{mn})_\beta^{\alpha} \nabla_\beta \mathcal{W}^\alpha,$$  \hspace{1cm} (A.12)

which is equivalent to

$$\nabla_\alpha \mathcal{W}^\beta = -\frac{1}{4} (\gamma_{mn})_\alpha^{\beta} F_{mn}.$$  \hspace{1cm} (A.13)

The third identity $I_{mna} = 0$ implies

$$\nabla_\alpha F_{mn} = \gamma_{m\alpha\beta} \nabla_m \mathcal{W}^\beta - \gamma_{m\alpha\beta} \nabla_n \mathcal{W}^\beta.$$  \hspace{1cm} (A.14)

Taking (A.13) into account, (A.14) yields the result

$$\gamma_{m\alpha\beta} \nabla_m \mathcal{W}^\beta = 0.$$  \hspace{1cm} (A.15)

Furthermore, multiplying (A.15) by $\gamma^{n\alpha} \nabla_\gamma$ we find

$$\nabla^m F_{mn} = -\frac{1}{2} \gamma_{n\alpha\beta} \{ \mathcal{W}^\alpha, \mathcal{W}^\beta \}.$$  \hspace{1cm} (A.16)
The (A.16) and (A.15) imply the Maxwell equation for the gauge field \( \nabla_m f^{mn} = 0 \) and the Dirac equation for the gaugino \( \gamma^m_{\alpha\beta} \nabla_m \xi^\beta = 0 \), respectively.

Finally, the remaining identity \( I_{mnp} = 0 \) implies

\[
\nabla_m F_{np} + \nabla_n F_{pm} + \nabla_p F_{mn} = 0 ,
\]

(A.17)

and it suggests that \( F_{mn} \) is just the curl of a gauge field \( A_m \);

\[
F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] .
\]

(A.18)

The \( \theta \)-expansion of these superfields are studied in [48].

**B Massless vertex operator for pure spinor open superstring**

We present a review of the vertex operators in the open pure spinor superstring [20] (see also [29]). For simplicity, we focus on the left-moving sector only.

We consider a ghost number 1 massless vertex operator given by

\[
U = \lambda^\alpha A_\alpha (x, \theta) ,
\]

(B.1)

where \( A_\alpha (x, \theta) \) is a spinor superfield. The BRS transformation law is represented as

\[
Q x^m = \frac{1}{2} \lambda \gamma^m \theta , \quad Q \theta^\alpha = \lambda^\alpha , \quad Q d_\alpha = -\Pi^m (\gamma_m \lambda)_{\alpha} , \quad Q \lambda^\alpha = 0 , \quad Q \omega_\alpha = d_\alpha ,
\]

(B.2)

where \( Q \) denotes \( \delta_1 \) in section 3.1. Note that \( Q^2 \omega_\alpha = -\Pi^m (\gamma_m \lambda)_{\alpha} \) turns out the gauge transformation for \( \omega_\alpha \). Then cohomology condition, \( Q U = 0 \) up to the gauge transformation \( \delta U = Q \Omega \), implies

\[
D_\alpha (\gamma_{mnpr})^{\alpha\beta} A_\beta = 0 \quad \text{and} \quad \delta A_\alpha = D_\alpha \Omega ,
\]

(B.3)

where \( \Omega (x, \theta) \) is a gauge parameter and the derivative \( D_\alpha \) is given in (A.2).

To derive (B.3), we use the pure spinor constraint for the commutative bispinor \( \lambda \)

\[
\lambda^\alpha \lambda^\beta = \frac{1}{2^5 5!} \gamma_{mnpr}^{\alpha\beta} (\lambda \gamma_{\gamma \delta}^{mnpr} \lambda^\delta) .
\]

(B.4)

As a result, (B.3) is consistent with the super Yang-Mills equations of motion and the gauge transformations as we have seen in Appendix A.

Next, we derive an integrated vertex operator such as \( V = \int dz \mathcal{V} \). Recalling the RNS formulation, \( \mathcal{V} \) is given as the anticommutator of the unintegrated vertex operator \( U \) and
the $b$-ghost. However, in the pure spinor formulation, the reparametrization $b$-ghost is unclear without introducing the non-minimal part [38]. Fortunately, the above facts can be rephrased in terms of the BRS charge $Q$ as**

$$Q \mathcal{V} = \partial U \ .$$

We find the vertex operator $\mathcal{V}$ takes the form of

$$\mathcal{V} = \partial \theta^\alpha A_\alpha(x, \theta) + \Pi^m A_m(x, \theta) + d_\alpha W^\alpha(x, \theta) + \frac{1}{2} N^{mn} F_{mn}(x, \theta) \ , \quad (B.7)$$

where $N^{mn} = \frac{1}{2} \lambda^{\gamma} \gamma^{m} \omega$ is the ghost Lorentz current. Indeed, since

$$Q \mathcal{V} = \partial (\lambda^\alpha A_\alpha) + \lambda^\alpha \partial \theta^\beta (D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha \beta}^m A_m)$$

$$+ \lambda^\alpha \Pi^m (D_\alpha A_m - \partial_m A_\alpha - \gamma_{m \alpha \beta}^\gamma W^\gamma)$$

$$+ \lambda^\alpha d_\beta (-D_\alpha W^\beta + \frac{1}{4} (\gamma^{mn})^\beta_\alpha F_{mn}) + \frac{1}{2} \lambda^\alpha N^{mn} D_\alpha F_{mn} \ , \quad (B.8)$$

(B.7) implies the following equations

$$-D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha \beta}^m A_m = 0 \ , \quad (B.9)$$

$$D_\alpha A_m - \partial_m A_\alpha - \gamma_{m \alpha \beta}^\gamma W^\beta = 0 \ , \quad (B.10)$$

$$-D_\alpha W^\beta + \frac{1}{4} (\gamma^{mn})^\beta_\alpha F_{mn} = 0 \ , \quad (B.11)$$

$$\lambda^\alpha \lambda^\beta (\gamma^{mn})^\gamma_\beta D_\alpha F_{mn} = 0 \ . \quad (B.12)$$

The (B.9), (B.10) and (B.11) certainly correspond to the super-Yang-Mills equations (A.6), (A.10) and (A.13) in the abelian case, respectively. It follows that superfields $A_\alpha$ and $A_m$ are spinor and vector gauge fields in the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, and that $W^\alpha$ and $F_{mn}$ are spinor and vector field strengths for them. On the other hand, (B.12) vanishes by the pure spinor constraint

$$\lambda^\alpha \lambda^\beta (\gamma^{mn})^\gamma_\beta D_\alpha F_{mn} = 4 \lambda^\alpha \lambda^\beta D_\alpha D_\beta W^\gamma = 2 (\lambda^\gamma \lambda) \partial_m W^\gamma = 0 \ , \quad (B.13)$$

where (B.11) is used. If (B.9) is contracted with $(\gamma^{mnq})^\alpha_\beta$, we obtain the equation of motion for $A_\alpha$ in (B.3). Contraction of (B.9) with $\gamma_{n}^\alpha \beta$ also leads to

$$A_m = \frac{1}{8} \gamma_{n}^\alpha \beta D_\alpha A_\beta \ . \quad (B.14)$$

**The Jacobi identity implies**

$$Q \mathcal{V} = [Q, \{ \oint dz \ b, U \}] = - [U, \{ Q, \oint dz \ b \}] - [ \oint dz b, \{ U, Q \}] = \partial U \quad (B.5)$$

since $\{ Q, U \} = 0$, $\{ Q, b \} = T$ and $[ \oint dz T, U ] = \partial U$ for the conformal weight zero primary operator $U$.
Then the gauge transformation in (B.3) turns to $\delta A_m = \partial_m \Omega$. Similarly contracting (B.10) with $\gamma^{m\alpha\gamma}$ implies the equation for $\mathcal{W}^\alpha$

$$\mathcal{W}^\beta = \frac{1}{10} \gamma^{m\alpha\beta} (D_\alpha A_m - \partial_m A_\alpha), \quad (B.15)$$

and contracting (B.11) with $(\gamma^{pq})^\beta_\alpha$ implies the equation for $F_{mn}$

$$F_{mn} = \frac{1}{8} (\gamma_{mn})^\alpha_\beta D_\beta \mathcal{W}^\alpha. \quad (B.16)$$

Furthermore, utilizing (B.14), (B.10) and (B.16), we derive

$$\partial_{[m} A_{n]} = -\frac{1}{8} \gamma^{\alpha\beta}_{[m} D_\alpha (\partial_{n]} A_\beta) = -\frac{1}{8} \gamma^{\alpha\beta}_{[m} D_\alpha (D_\beta A_{n]} - (\gamma_{n]} \mathcal{W})_\beta)$$

$$\quad = \frac{1}{8} (\gamma_{mn})^\alpha_\beta D_\alpha \mathcal{W}^\beta = F_{mn}. \quad (B.17)$$

Besides, this equation together (B.10) implies

$$D_\alpha F_{mn} = \partial_{[m} D_{|\alpha|} A_{n]} = \partial_{[m} (\gamma_{n]} \mathcal{W})_\alpha. \quad (B.18)$$

(B.17) and (B.18) certainly correspond to remaining Bianchi identities (A.18) and (A.14) for the abelian case, respectively.

## C BRS charge conservation

We will derive the supersymmetric DBI equations by modifying the method used in [21] to include the Dirichlet components.

We require that the general variation $\delta(S_b + V)$ vanishes. This leads to boundary conditions in the presence of background superfields. Under these conditions, it is shown that the BRS charge conservation implies superfield equations for DBI fields.

Let us begin to examine a general variation of the world-sheet action $S_0$ in (2.1), its ten-dimensional $\mathcal{N} = 1$ supersymmetry counter-term $S_b$ in (3.4) and the background coupling $V$ in (4.1). We find that variations $\delta(S_0 + S_b)$ and $\delta V$ may be expressed as

$$\delta(S_0 + S_b) = \frac{1}{2\pi \alpha'} \int \mathcal{d}\tau \left\{ \delta \theta^+ \left[ \frac{1}{2} d_\alpha^\alpha + \Pi_+ (\gamma_\mu \theta_-)_\alpha + \Pi_+ (\gamma_\mu \theta_-)_\alpha - y_i (\gamma^i \hat{\theta}^i)_\alpha \right. 

\quad + \frac{1}{6} (\theta_- \gamma_m \hat{\theta}_-) (\gamma_m \theta_-)_\alpha \right] + \delta \lambda^\alpha \left[ \frac{1}{2} \left( 1 - c_1 \right) \Delta^+_\alpha - \frac{1}{6} (\theta_- \gamma_m \hat{\theta}_+) (\gamma_m \theta_-)_\alpha \right] 

\quad - \delta y^i_\mu \left[ \Pi_{\mu} - \frac{1}{2} (\theta_- \gamma_\mu \hat{\theta}_+) \right] + \delta \pi^i_+ y_i + \frac{1}{2} c_1 \delta \Delta^+_\alpha \theta^-_\alpha 

\quad + \frac{1}{2} (c_2 - 1) \omega^+_\alpha \delta \lambda^\alpha + \frac{1}{2} c_2 \delta \omega^+_\alpha \lambda^\alpha - \frac{1}{2} \omega^+ \delta \lambda^\alpha \right\}, \quad (C.1)$$

18
\[ \delta V = \frac{1}{2\pi\alpha'} \int d\tau \left\{ \delta \theta^\alpha_+ \left[ \dot{\theta}^\beta \left( \gamma^\alpha_{\beta\mu} A_\mu - D_\alpha A_\beta - D_\beta A_\alpha \right) + \Pi^\mu_+ (D_\alpha A_\mu - \partial_\mu A_\alpha) + \tilde{\Pi}^i_+ D_\alpha A_i \right] \\
- \frac{1}{2} \Delta^+_\beta D_\alpha W^\beta + \frac{1}{4} D_\alpha (N_+ F) \right\} + \delta y^\mu_+ \left[ \dot{\theta}^\alpha_+ (\partial_\mu A_\alpha - D_\alpha A_\mu) + \Pi^\nu_+ (\partial_\mu A_\nu - \partial_\nu A_\mu) \right. \\
+ \tilde{\Pi}^i_+ \partial_\mu A_i + \frac{1}{2} \Delta^+_\alpha \partial_\mu W^\alpha + \frac{1}{4} \partial_\mu (N_+ F) \right] + \delta \tilde{\Pi}^i_+ A_i \\
+ \frac{1}{2} \delta \Delta^+_\alpha W^\alpha + \frac{1}{8} \delta \lambda^\alpha_+ \mathcal{F}_\alpha^\beta \omega^\beta + \frac{1}{8} \lambda^\alpha_+ \mathcal{F}_\alpha^\beta \delta \omega^\beta \right\}, \tag{C.2} \]

where \( \delta y^\mu_+ \) defined by

\[ \delta y^\mu_+ = \delta x^\mu_+ + \frac{1}{2} \left( \theta^\alpha_+ \gamma^\mu \delta \theta^\alpha_+ \right), \tag{C.3} \]

is invariant under the \( \mathcal{N} = 1 \) supersymmetry. To lead the representation (C.2), we have used equations of motion (3.6). We also see that \( \delta \left( S_0 + S_b + V \right) / \delta y_i = 0 \) as mentioned in section 3.

To obtain boundary conditions from \( \delta (S_0 + S_b + V) = 0 \), first we focus on the terms with \( \delta \Delta^+_\alpha \) and \( \delta \omega^+_\alpha \), and derive

\[ \theta^-_\alpha = -\frac{1}{c_1} W^\alpha, \quad \lambda^-_\alpha = -\frac{1}{4c_2} \lambda^\beta_+ \mathcal{F}_\alpha^\beta. \tag{C.4} \]

They also lead to

\[ \dot{\theta}^-_\alpha = -\frac{1}{c_1} \left( \Pi^\mu_+ \partial_\mu W^\alpha + \dot{\theta}^\beta_+ D_\beta W^\alpha \right), \quad \dot{\theta}^-_\alpha = -\frac{1}{c_1} \left( \delta y^\mu_+ \partial_\mu W^\alpha + \delta \theta^\beta_+ D_\beta W^\alpha \right), \tag{C.5} \]

\[ \delta \lambda^-_\alpha = -\frac{1}{4c_2} \delta \lambda^\beta_+ \mathcal{F}_\alpha^\beta - \frac{1}{4c_2} \left( \lambda^\beta_+ \delta y^\mu_+ \partial_\mu \mathcal{F}_\alpha^\beta + \lambda^\beta_+ \delta \theta^\gamma_+ D_\gamma \mathcal{F}_\alpha^\beta \right). \]

Next, examining the terms with \( \delta \lambda^\alpha_+ \) in \( \delta (S_0 + S_b + V) \) we find

\[ \omega^-_\alpha = \frac{1}{4c_2} \omega^\beta_+ \mathcal{F}_\alpha^\beta. \tag{C.6} \]

Boundary conditions for \( \lambda^-_\alpha \) in (C.4) and \( \omega^-_\alpha \) in (C.6) are consistent with the ghost number charge conservation \( \lambda^\alpha \omega_\alpha \mid = \lambda^\alpha \tilde{\omega}_\alpha \mid \), where \( \mid \) means “evaluated at the boundary”. On the other hand, we can eliminate the terms with \( \delta \tilde{\Pi}^i_+ \) in \( \delta (S_0 + S_b + V) \) by fixing the static-like gauge (4.13). After substituting above conditions into \( \delta (S_0 + S_b + V) = 0 \), we examine the
terms with \( \delta y_\mu^\alpha \) and \( \delta \theta_\alpha^\mu \). They lead to complicated boundary conditions

\[
\Pi_{\alpha} = \Pi^\alpha_{\mu}(\partial_\mu A_\alpha - D_\alpha A_\mu + \frac{1}{2c_1}(\gamma_\mu W)_\alpha + \frac{1}{6c_1^3}(\gamma^m W)_\alpha (W^\gamma_m \partial_\mu W))
\]

\[
+ \Pi^\nu_{\mu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tilde{\Pi}^\mu_{\nu} \partial_\alpha A_\mu + \frac{1}{2c_1} \Delta^\mu_\alpha \partial_\mu W^\alpha + \frac{1}{4c_2} \partial_\mu (N_+ \mathcal{F}) ,
\]

(C.7)

\[
\frac{1}{2}d^-_\alpha = \hat{\theta}^\beta_\alpha \left( D_\alpha A_\beta + D_\beta A_\alpha - \gamma^m_\alpha_\beta A_m - \frac{1}{6c_1^3} (W^m \gamma_\beta D_\beta W) (\gamma^m_\alpha W) - \frac{1}{6c_1^3} (W^m \gamma_\alpha D_\alpha W) (\gamma^m_\beta W) \right)
\]

\[
+ \Pi^\nu_\mu (\partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{c_1} (\gamma_\mu W)_\alpha + \frac{1}{6c_1^3} (W^m \gamma_\alpha \partial_\mu W) (\gamma^m_\nu W))
\]

\[
+ \tilde{\Pi}^\mu_\nu \left( \frac{1}{c_1} (\gamma_\nu W)_\alpha - D_\alpha A_\nu \right) + \frac{1}{2c_1} \Delta^\mu_\beta D_\alpha W^\beta - \frac{1}{4c_2} D_\alpha (N_+ \mathcal{F}).
\]

(C.8)

The (C.7) is regarded as a modified Neumann boundary condition. Boundary conditions for \( \omega^-_\alpha \) in (C.6) and \( d^-_\alpha \) in (C.8) must be consistent with the BRS transformation \( \delta Q \omega^-_\alpha = d^-_\alpha \) up to the \( \Lambda \)-gauge transformation in section 2. In the following discussion, we will absorb \( c_1 \) and \( c_2 \) by rescaling \( W^\alpha \rightarrow c_1 W^\alpha \) and \( \mathcal{F}^\beta_\alpha \rightarrow c_2 \mathcal{F}^\beta_\alpha \).

To extract DBI equations, we impose the following relation for BRS currents

\[
\lambda^\alpha d_\alpha \bigg| = \tilde{\lambda}^\alpha \tilde{d}_\alpha \bigg|,
\]

(C.9)

which implies BRS charge conservation

\[
0 = \partial_\tau Q_{\text{total}} = \int d\sigma \partial_\tau (j^\tau_{\text{BRS}}) = \int d\sigma \partial_\sigma (j^\sigma_{\text{BRS}})
\]

\[
= \int d\sigma \partial_\sigma (j^\tau_{\text{BRS}} - \tilde{j}^\tau_{\text{BRS}}) = \left( \lambda^\alpha d_\alpha - \tilde{\lambda}^\alpha \tilde{d}_\alpha \right). \quad \text{(C.10)}
\]

Then we assume the Dirichlet boundary condition

\[
\Pi_{\alpha} = -\Pi^\mu_{\alpha}(\partial_\mu A_\alpha - \hat{\theta}^\alpha_\mu D_\alpha A_\mu + \frac{1}{2c_1} (\hat{\theta}^\alpha_\gamma W). \quad \text{(C.11)}
\]

This is parallel with the Neumann boundary condition in (C.7) and just the derivation of the static-like gauge (4.13) respect to the time-coordinate \( \tau \).

Under these boundary conditions (C.4), (C.7), (C.8) and (C.11), the BRS charge conservation (C.9) implies
\[ 0 = \lambda^\alpha \hat{d}_\alpha - \lambda^\alpha d_\alpha \]
\[ = \frac{1}{2} \lambda^\alpha d_\alpha + \frac{1}{2} \lambda^\alpha \Delta_\alpha^+ - \frac{1}{2} (\lambda - \gamma\mu\theta^-) \tilde{\Pi}^\alpha - \frac{1}{2} (\lambda - \gamma\iota\theta^-) \Pi^- - \frac{1}{4} (\lambda - \gamma^m \theta^-)(\theta - \gamma_m \dot{\theta}_+^+) \]
\[ = \lambda^\alpha \beta^\beta \left[ D_\alpha A_\beta + D_\beta A_\alpha - \gamma^m_{\alpha\beta} A_m - \frac{1}{6} (W_{\gamma^m D_\beta W})(\gamma_m W)_\alpha - \frac{1}{6} (W_{\gamma^m D_\alpha W})(\gamma_m W)_\beta \right. \]
\[ + \frac{1}{8} F^\alpha_\beta (\gamma^m W)_\gamma \left\{ \partial_m A_\beta - D_\beta A_m + (\gamma_m W)_\beta + \frac{1}{6} (\gamma^n W)_\beta (W_{\gamma_n \partial_m W}) \right\} \]
\[ + \lambda^\alpha \Pi^\mu_+ \left[ \partial_\mu A_\alpha - D_\alpha A_\mu + (\gamma_\mu W)_\alpha + \frac{1}{6} (W_{\gamma^m \partial_\mu W})(\gamma_m W)_\alpha + \frac{1}{8} F^\beta_\alpha (\gamma^n W)_\beta (\partial_\mu A_n - \partial_n A_\mu) \right] \]
\[ + \lambda^\alpha \tilde{\Pi}^i_+ \left[ - D_i A_\alpha + (\gamma_i W)_\alpha - \frac{1}{8} F^\beta_\alpha (\gamma^\mu W)_\beta \partial_\mu A_i \right] \]
\[ + \frac{1}{2} \lambda^\alpha \Delta_\beta^\beta \left[ D_\alpha W_\beta - \frac{1}{4} F^\beta_\alpha + \frac{1}{8} F^\gamma_\alpha (\gamma^\mu W)_\gamma \partial_\mu W_\beta \right] \]
\[ - \frac{1}{4} \lambda^\alpha N^\beta_+ \lambda^\gamma \left[ D_\alpha F^\gamma_\beta + \frac{1}{8} F^\delta_\alpha (\gamma^\mu W)_\delta \partial_\mu F^\gamma_\beta \right] \]  \quad \text{(C.12)}

Finally, we find that, to eliminate this expression, \((4.31), (4.27), (4.19), (4.20)\) and \((4.26)\) should be required, as expected. First four equations are supersymmetric DBI equations of motion on a Dp-brane, and last one is the pure spinor constraint.

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