FLAT MITTAG-LEFFLER MODULES
OVER COUNTABLE RINGS

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ABSTRACT. We show that over any ring, the double Ext-orthogonal class to all flat Mittag-Leffler modules contains all countable direct limits of flat Mittag-Leffler modules. If the ring is countable, then the double orthogonal class consists precisely of all flat modules, and we deduce, using a recent result of Šaroch and Trlifaj, that the class of flat Mittag-Leffler modules is not precov-erring in Mod-$R$ unless $R$ is right perfect.

INTRODUCTION

The notion of a Mittag-Leffler module was introduced by Raynaud and Gruson [10], who used the concept to prove a conjecture due to Grothendieck that the projectivity of infinitely generated modules over commutative rings is a local property. This is a crucial step for defining and working with infinitely generated vector bundles, as considered by Drinfeld in [3], to which we also refer for more explanation.

The main step behind this geometrically motivated result is a completely general characterization of projective modules over any (in general non-commutative) ring $R$. Namely, one can show (consult [3]) that an $R$-module is projective if and only if $M$ satisfies the following three conditions:

1. $M$ is flat,
2. $M$ is Mittag-Leffler,
3. $M$ is a direct sum of countably generated modules.

As mentioned by Drinfeld, the proof of projectivity of a given module might be non-constructive even in very simple cases, because it requires the Axiom of Choice. This applies for instance to the ring $\mathbb{Q}$ of rational numbers and the $\mathbb{Q}$-module $\mathbb{R}$ of real numbers. The main trouble there is condition (3). Thus, one might consider replacing projective modules by flat Mittag-Leffler modules (these are called “projective modules with a human face” in a preliminary version of [3]).
However, a surprising result in [5, §5] indicates that if one is interested in homological algebra, this might not be a good idea at all. Namely, the class of flat Mittag-Leffler abelian groups does not provide for precovers (sometimes also called right approximations). In the present paper, we use recent results due to Šaroch and Trlifaj [11] to show this is a much more general phenomenon and applies to many geometrically interesting examples. Namely, we prove in Theorem 6 that the class of flat Mittag-Leffler \( R \)-modules over a countable ring \( R \) is precovering if and only if \( R \) is a right perfect ring. Note that in that case the classes of projective modules, flat Mittag-Leffler modules and flat modules coincide, so the flat Mittag-Leffler precovers are just the projective ones.

1. Preliminaries

In this paper, \( R \) will always be an associative, not necessarily commutative, ring with a unit. If not specified otherwise, a module will stand for a right \( R \)-module.

We will denote by \( D \) the class of all modules which are flat and satisfy the Mittag-Leffler condition in the sense of [10, 8]:

**Definition 1.** \( M \) is called a *Mittag-Leffler* module if the canonical morphism \( \rho : M \otimes R \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes R Q_i \) is injective for each family \((Q_i | i \in I)_{\text{left}} R\)-modules.

A crucial closure property of the class \( D \) has been obtained in [8]:

**Proposition 2** ([8, Proposition 2.2]). Let \( R \) be a ring and let \((F_i, u_{ji} : F_i \rightarrow F_j)\) be a direct system of modules from \( D \) indexed by \((I, \leq)\). Assume that for each increasing chain \((i_n | n < \omega)\) in \( I \), the module \( \lim_{\longrightarrow} F_i \) belongs to \( D \). Then \( M = \lim_{\longrightarrow} F_i \) belongs to \( D \).

Let us look more closely at countable chains of modules and their limits. Recall that given a sequence of morphisms

\[
F_0 \xrightarrow{u_0} F_1 \xrightarrow{u_1} F_2 \xrightarrow{u_2} F_3 \rightarrow \ldots
\]

we have a short exact sequence

\[
\eta : 0 \rightarrow \bigoplus_{n<\omega} F_n \xrightarrow{\varphi} \bigoplus_{n<\omega} F_n \rightarrow \lim_{\longrightarrow} F_n \rightarrow 0,
\]

such that \( \varphi \) is defined by \( \varphi \iota_n = \iota_n - \iota_{n+1}u_n \), where \( \iota_n : F_n \rightarrow \bigoplus F_m \) are the canonical inclusions. Note the following simple fact:

**Lemma 3.** Given a chain \( F_0 \xrightarrow{u_0} F_1 \xrightarrow{u_1} F_2 \rightarrow \ldots \) of morphisms as above and a number \( n_0 < \omega \), the middle term of \((*)\) decomposes as

\[
\bigoplus_{n<\omega} F_n = \varphi \left( \bigoplus_{m<n_0} F_m \right) \oplus \left( \bigoplus_{m\geq n_0} F_m \right).
\]

**Proof.** Note that the module \( \varphi(\bigoplus_{m<n_0} F_m) \) is generated by elements of the form

\[
(0, \ldots, 0, x_m, -u_m(x_m), 0 \ldots) \in \bigoplus_{n<\omega} F_n,
\]
where \( m < n_0 \) and \( x_m \in F_m \). It follows easily that one can uniquely express each \( y = (y_n) \in \bigoplus \nsubseteq F_n \) as \( y = z + w \), where \( z \in \varphi(\bigoplus_{m<n_0} F_m) \) and \( w \in \bigoplus_{m\geq n_0} F_m \). Namely, we take \( z = (y_0, \ldots, y_{n_0-1}, y_0', 0, 0, \ldots) \) with \( -y_0' = u_{n_0-1}(y_{n_0-1}) + u_{n_0-1}u_{n_0-2}(y_{n_0-2}) + \cdots + u_{n_0-1}u_{n_0-2} \cdots u_1 u_0(y_0) \) and \( w = y - z \in \bigoplus_{m\geq n_0} F_m \). □

We will also need a few simple results concerning infinite combinatorics, starting with a well-known lemma.

**Lemma 4.** For any cardinal \( \mu \) there is a cardinal \( \lambda \geq \mu \) such that \( \lambda^{\aleph_0} = 2^\lambda \).

**Proof.** We refer for instance to [7] Lemma 3.1. For the reader’s convenience, we recall how to construct \( \lambda \). We put \( \mu_0 = \mu \) and for each \( n < \omega \) inductively construct \( \mu_{n+1} = 2^{\mu_n} \). Then \( \lambda = \sup_{n<\omega} \mu_n \) has the required property; see for example [9] p. 50, fact (6.21)]. □

The next lemma deals with a construction of a large family of “almost disjoint” maps \( f : \omega \to \lambda \). The result is well known in the literature and it has many different proofs. We refer for instance to [2] Lemma 2.3] or [4] Proposition II.5.5).

**Lemma 5.** Let \( \lambda \) be an infinite cardinal. Then there is a subset \( J \subseteq \lambda^\omega \) of cardinality \( \lambda^{\aleph_0} \) such that for any pair of distinct maps \( f, g : \omega \to \lambda \) of \( J \), the set formed by the \( x \in \omega \) on which the values \( f(x) \) and \( g(x) \) coincide is a finite initial segment of \( \omega \).

**Proof.** Consider the tree \( T \) of the finite sequences of elements of \( \lambda \); i.e. \( T = \{ t : n \to \lambda \mid n < \omega \} \). Since \( \lambda \) is infinite, we have \( |T| = \bigcup_{n<\omega} \lambda^n | = \lambda \), so there is a bijection \( b : T \to \lambda \). For every map \( f : \omega \to \lambda \) denote by \( A_f : \omega \to T \) the induced map which sends \( n < \omega \) to \( f \mid n \in T \). Clearly, if \( f, g \) are two different maps in \( \lambda^\omega \), the values of \( A_f \) and \( A_g \) coincide only on a finite initial segment of \( \omega \). Now we can put \( J = \{ b \circ A_f \mid f \in \lambda^\omega \} \). □

**2. Main result**

Now we are in a position to state our main result, which is inspired by [8] §5]. It will be proved by using a cardinal argument similar to the one in [2] Proposition 2.5]. Note that the result sharpens [11] Theorem 3.9] by removing the additional set-theoretical assumption of the Singular Cardinal Hypothesis and also [8] Corollaries 7.6 and 7.7] by removing the assumption that \( \mathcal{D} \) is closed under products.

Regarding the notation and terminology, given a class \( \mathcal{C} \subseteq \text{Mod-}R \), we put \( \mathcal{C}^\perp = \{ M \in \text{Mod-}R \mid \text{Ext}^1_R(\mathcal{C}, M) = 0 \} \) and dually \( ^\perp \mathcal{C} = \{ M \in \text{Mod-}R \mid \text{Ext}^1_R(M, \mathcal{C}) = 0 \} \). Recall that a module is called cotorsion if it cannot be non-trivially extended by a flat module.

We recall also the notion of a precover, or sometimes called a right approximation. If \( \mathcal{X} \) is any class of modules and \( M \in \text{Mod-}R \), a homomorphism \( f : X \to M \) is called an \( \mathcal{X}\)-precover of \( M \) if \( X \in \mathcal{X} \) and for every homomorphism \( f' \in \text{Hom}_R(\mathcal{X}', M) \) with \( \mathcal{X}' \in \mathcal{X} \) there exists a homomorphism \( g : \mathcal{X}' \to X \) such that \( f' = fg \). The class \( \mathcal{X} \) is called precovering if each \( M \in \text{Mod-}R \) admits an \( \mathcal{X} \)-precover.
**Theorem 6.** Let $R$ be a ring and let $\mathcal{D}$ be the class of all flat Mittag-Leffler right $R$-modules. Given any countable chain

$$ F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow \ldots $$

of morphisms such that $F_n \in \mathcal{D}$ for all $n < \omega$, we have $\lim_n F_n \in \mathcal{D}$. If, moreover, $R$ is a countable ring, then the following hold:

1. $\mathcal{D}^\perp$ is precisely the class of all cotorsion modules.
2. $\mathcal{D}$ is a precovering class in $\text{Mod}-R$ if and only if $R$ is right perfect.

**Proof.** Assume we have a countable direct system $(F_n, u_n)$ as above, put $F = \lim_n F_n$, and fix a module $C \in \mathcal{D}^\perp$. We must prove that $\text{Ext}^1_R(F, C) = 0$.

Let us fix an infinite cardinal $\lambda$, depending on $C$, such that we have $\lambda \geq |\text{Hom}_R(F_n, C)|$ for each $n < \omega$ and $\lambda^\aleph_0 = 2^\lambda$; we can do this using Lemma 4. Applying Lemma 4 we find a subset $J \subseteq \lambda^\omega$ of cardinality $2^\lambda$ such that the values of each pair $f, g : \omega \rightarrow \lambda$ of distinct elements of $J$ coincide only on a finite initial segment of $\omega$. We claim that there is a short exact sequence of the form

$$ 0 \rightarrow P \rightarrow E \rightarrow F(2^\lambda) \rightarrow 0 $$

such that $E \in \mathcal{D}$ and $|\text{Hom}_R(P, C)| \leq 2^\lambda$.

Let us construct such a sequence. First, denote for each $\alpha < \lambda$ by $F_{n, \alpha}$ a copy of $F_n$, and by $P$ the direct sum $\bigoplus_{n, \alpha} F_{n, \alpha}$ taken over all pairs $(n, \alpha)$ such that $n < \omega$ and $\alpha = f(n)$ for some $f \in J$. Note that $P$ is a summand in $\bigoplus_{n < \omega} F_n^{(\lambda)}$, so we have

$$ |\text{Hom}_R(P, C)| \leq |\text{Hom}_R(\bigoplus F_n^{(\lambda)}, C)| \leq \prod_{n < \omega} |\text{Hom}_R(F_n, C)|^\lambda \leq \lambda^{\omega \times \lambda} = 2^\lambda. $$

Next, we will construct $E$. Given $f \in J$, let

$$ \iota_f : \bigoplus_{n < \omega} F_n \rightarrow P $$

be the split inclusion which sends each $F_n$ to $F_{n, f(n)}$. Using the earlier short exact sequence $(*)$, we can extend $P$ by $F$ via the following pushout diagram:

$$
\begin{array}{c}
0 \rightarrow \bigoplus_{n < \omega} F_n \rightarrow \bigoplus_{n < \omega} F_n \rightarrow F \rightarrow 0 \\
\eta \downarrow \quad \iota_f \downarrow \quad \phi_f \downarrow \\
0 \rightarrow P \rightarrow E_f \rightarrow F \rightarrow 0.
\end{array}
$$

(\Delta)

Now, we can put these extensions for all $f \in J$ together. Namely, let $\sigma : P^{(J)} \rightarrow P$ be the summing map and consider the pushout diagram:

$$
\begin{array}{c}
\bigoplus \varepsilon_f : 0 \rightarrow P^{(J)} \rightarrow \bigoplus_{f \in J} E_f \rightarrow F^{(J)} \rightarrow 0 \\
\varepsilon \downarrow \quad \pi \downarrow \\
0 \rightarrow P \rightarrow E \rightarrow F^{(J)} \rightarrow 0.
\end{array}
$$

For each $g \in J$, the composition of the canonical inclusion $\nu_g : E_g \rightarrow \bigoplus_{f \in J} E_f$ with the morphism $\pi$ yields a monomorphism $E_g \rightarrow E$. In fact, if $y \in E_g$ is such that $\pi \nu_g(y) = 0$, then $\rho(\nu_g(y)) = 0$; hence the exact sequence $\varepsilon_g$ gives that $y$ is in the image of $P$ and the composition of the canonical embedding $\mu_g : P \rightarrow P^{(J)}$ with the morphism $\sigma$ is a monomorphism. From now on we shall without loss of generality view these monomorphisms $E_g \rightarrow E$ as inclusions.
To prove the existence of \((\dagger)\), it suffices to show that \(E \in \mathcal{D}\) in \(\varepsilon\). To this end, denote for any subset \(S \subseteq J\) by \(M_S\) the module
\[
M_S = \sum_{f \in S} \Im \vartheta_f \quad (\subseteq E; \text{ see diagram } (\Delta)).
\]

Then the family \((M_S \mid S \subseteq J \& |S| \leq \aleph_0)\) with obvious inclusions forms a direct system and we claim that its union is the whole of \(E\). Indeed, it is straightforward to check, using diagram \((\Delta)\) and the construction of the embeddings \(E_g \subseteq E\), that
\[
E = P + \sum_{f \in J} \Im \vartheta_f.
\]

Further, the left-hand square of diagram \((\Delta)\) is a pull-back, which implies \(P \cap \Im \vartheta_f = \Im \iota_f\) and
\[
P \cap \sum_{f \in J} \Im \vartheta_f \supseteq \sum_{f \in J} \Im \iota_f = P.
\]

Thus, \(E = \sum_{f \in J} \Im \vartheta_f\) and the claim is proved.

Moreover, the union of any chain \(M_{S_0} \subseteq M_{S_1} \subseteq M_{S_2} \subseteq \ldots\) from the direct system belongs to the direct system again. Therefore, if we prove that \(M_S \in \mathcal{D}\) for each countable \(S \subseteq J\), it will follow from Proposition \(2\) that \(E \in \mathcal{D}\). Our task is then reduced to proving the following lemma:

**Lemma 7.** With the notation as above, the following hold:

1. Given \(S \subseteq T \subseteq J\) with \(S\) and \(T\) finite and such that \(|T| = |S| + 1\), the inclusion \(M_S \subseteq M_T\) splits and there is \(n_0 < \omega\) such that \(M_T/M_S \cong \bigoplus_{m \geq n_0} F_m\).
2. Given a countable subset \(S \subseteq J\), the module \(M_S\) is isomorphic to a countable direct sum with each summand isomorphic to some \(F_n, n < \omega\). In particular, \(M_S \in \mathcal{D}\).

**Proof.** Let us focus on (1) since (2) is an immediate consequence. Denote by \(f : \omega \to \lambda\) the single element of \(T \setminus S\), and let \(n_0 < \omega\) be the smallest number such that \(f(n_0) \neq g(n_0)\) for each \(g \in S\).

We claim that the following are satisfied by the construction:
\[
M_S \cap \Im \vartheta_f = \iota_f \left( \bigoplus_{m < n_0} F_m \right) = \vartheta_f \circ \varphi \left( \bigoplus_{m < n_0} F_m \right).
\]

The second equality holds simply because \(\vartheta_f \circ \varphi = \iota_f\) by diagram \((\Delta)\). For the first, note that \(M_S \cap \Im \vartheta_f\), as a submodule of \(E\), is contained in \(P\). Since \(P \cap \Im \vartheta_f = \Im \iota_f\), we have
\[
M_S \cap \Im \vartheta_f = \left( \sum_{g \in S} \Im \vartheta_g \right) \cap \Im \iota_f = \left( \sum_{g \in S} \Im \iota_g \right) \cap \Im \iota_f = \iota_f \left( \bigoplus_{m < n_0} F_m \right),
\]
by the construction of \(P\). This proves the claim.

Invoking Lemma \(3\) we further deduce that
\[
\Im \vartheta_f = \vartheta_f \left( \bigoplus_{m < n_0} F_m \right) \oplus \vartheta_f \left( \bigoplus_{m \geq n_0} F_m \right).
\]
In particular, the inclusion $M_S \cap \text{Im} \vartheta_f \subseteq \text{Im} \vartheta_f$ splits and so does the inclusion $M_S \subseteq M_S + \text{Im} \vartheta_f = M_T$. Moreover, we have the isomorphisms
\[ M_T/M_S = (M_S + \text{Im} \vartheta_f)/M_S \cong \text{Im} \vartheta_f/(M_S \cap \text{Im} \vartheta_f) \cong \bigoplus_{m \geq n_0} F_m, \]
which finishes the proof of the lemma. \hfill \Box

Having established the existence of (†) such that $E \in \mathcal{D}$ and $|\text{Hom}_R(P, C)| \leq 2^\lambda$, let us apply $\text{Hom}_R(-, C)$ to (†). Since $C \in \mathcal{D}^\perp$ by assumption, we get an exact sequence
\[ \text{Hom}_R(P, C) \to \text{Ext}^1_R(F^{(2^\lambda)}, C) \to 0. \]
Suppose now that $\text{Ext}^1_R(F, C) \neq 0$. Then we would have $|\text{Ext}^1_R(F^{(2^\lambda)}, C)| \geq 2^{2^\lambda}$, which would contradict the fact that $|\text{Hom}_R(P, C)| \leq 2^\lambda$. Hence $\text{Ext}^1_R(F, C) = 0$ as desired.

To finish the proof of Theorem 6, suppose $R$ is a countable ring. Since each $F \in \mathcal{D}$ is flat, $\mathcal{D}^\perp$ contains all cotorsion modules. On the other hand, if $C$ is not cotorsion, there is a countable flat module $F$ such that $\text{Ext}^1_R(F, C) \neq 0$; see for instance [6] Theorems 4.1.1 and 3.2.9. By the first part of Theorem 6 we know that $F \in 1(\mathcal{D}^\perp)$, so $C \notin \mathcal{D}^\perp$. Hence $\mathcal{D}^\perp$ consists precisely of cotorsion modules.

The fact that $\mathcal{D}$ is not precoversing unless $R$ is right perfect (and $\mathcal{D}$ is then the class of projective modules) follows directly from [11] Theorem 3.10. This finishes the proof of Theorem 6. \hfill \Box

Remark 8. The proof of Theorem 6 is to some extent constructive. Namely, if $R$ is a countable ring and $C$ is a module which is not cotorsion, the theorem gives us a recipe for how to construct $E \in \mathcal{D}$ such that $\text{Ext}^1_R(E, C) \neq 0$, and it allows us to estimate the size of $E$ based on the size of $C$. Note that if $R$ is non-perfect, the size of $E$ must grow with the size of $C$. This is because for any set $S \subseteq \mathcal{D}$, we have $1(S^\perp) \subseteq \mathcal{D} \subseteq \text{Flat-}R$ by [1] Proposition 1.9 and [6] Corollary 3.2.3.

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