On Baer Invariants of Pairs of Groups

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Abstract

In this paper, we use the theory of simplicial groups to develop
the Schur multiplier of a pair of groups \((G, N)\) to the Baer invariant
of it, \(\mathcal{V}M(G, N)\), with respect to an arbitrary variety \(\mathcal{V}\). Moreover,
we present among other things some behaviors of Baer invariants of
a pair of groups with respect to the free product and the direct limit.
Finally we prove that the nilpotent multiplier of a pair of groups does
commute with the free product of finite groups of mutually coprime
orders.

Key Words: Baer invariant; Pair of groups; Simplicial groups.

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1 Introduction and motivation

Schur (1904), introduced the Schur multiplier of a group $G$, $M(G)$, by Pro-
jective representations of $G$ which the second integral homology group of $G$.
The second homology plays the special role, this led H. Hopf (1942), to find
an effective method for calculating it. Hopf’s integral homology formula is
identical to the Schur multiplier of a finitely presented group. R. Baer (1945)
using the variety of groups, generalized the notion of the Schur multiplier of
a group $G$ to the Baer invariant of it with respect to a variety $\mathcal{V}$, $\mathcal{V}M(G)$.

In the present section, we outline the topological interpretations of the
Schur multiplier and the Schur multiplier of a pair of groups, and a mo-
tivation to define the Baer invariant of a pair of groups with topological
approach. Proofs, requiring the theory of simplicial groups, deferred to Sec-
tion 2. Section 3 is devoted to the results that obtained from the long exact
sequence of the Baer invariant of a pair of groups. Also we show that our
definition of the Baer invariant of a pair of groups is a vast generalization of
the Baer invariant of a special pair of groups which was defined by Moghad-
dam, Salemkar and Saany (2007). At the end of Section 3, we show that
the Baer invariant of a pair of groups commutes with the direct limit. Also
we obtain a long exact sequence that contains the Schur multiplier and the
2-nilpotent multiplier of a pair of groups. Computation of the Baer invariant
of a pair of free products of groups is given in Section 4. Also in this section,
we present an explicit formula for the 2-nilpotent multiplier of a pair of free
products of groups.

As convention, throughout the article we use $\mathcal{V}$ as an arbitrary variety
of groups defined by a set of laws $V$. We note that for any group $G$ one
can construct functorially a free simplicial group $K$, called free simplicial
resolution of $G$, whose $\pi_0(K) \cong G$, $\pi_m(K) \cong 0$ for $m \geq 1$, with $K_m$ is free
group (see Duskin, 1975). Given a functor $T : \text{Groups} \to \text{Groups}$ we
define left derived functors as

\[ L^T_m(G) = \pi_m(T(K)), \ m \geq 0. \]

The groups \( L^T_m(G) \) are independent of the choice of the free simplicial resolution. For more details see for instance Inassaridze (1974).

The topological interpretation of the c-nilpotent multiplier of a group arose in the work of Burns and Ellis (1997), on the 2-nilpotent multiplier of the free product of groups. Burns and Ellis (1997) observed that there are natural isomorphisms

\[
\begin{align*}
\frac{G}{\gamma_{c+1}(G)} & \cong L^\tau_0(G) \cong \pi_0\left( \frac{K}{\gamma_{c+1}(K)} \right), \\
M^{(c)}(G) & \cong L^\tau_1(G) \cong \pi_1\left( \frac{K}{\gamma_{c+1}(K)} \right),
\end{align*}
\]

where \( K \) is a free simplicial resolution of \( G \) and \( \tau_c \) is the functor that sends \( G \) to \( G/\gamma_{c+1}(G) \). In Franco (1998), described the Baer invariant of a group \( G \) in topological language as follows:

\[
\begin{align*}
\frac{G}{V(G)} & \cong L^\tau_0(G) \cong \pi_0\left( \frac{K}{V(K)} \right), \\
VM(G) & \cong L^\tau_1(G) \cong \pi_1\left( \frac{K}{V(K)} \right),
\end{align*}
\]

where \( K \) is a free simplicial resolution of \( G \) and \( \tau_V \) is the functor that sends \( G \) to \( G/V(G) \).

A group \( G \) with a normal subgroup \( N \) denoted by \((G, N)\) is called a homomorphism of pairs \((G_1, N_1) \to (G_2, N_2)\) is a group homomorphism \( G_1 \to G_2 \) that sends \( N_1 \) into \( N_2 \).

Ellis (1998), introduced the Schur multiplier of a pair \((G, N)\) as a functorial abelian group \( M(G, N) \) whose feature is the following natural exact sequence

\[
\begin{align*}
\cdots \to M(G, N) \to M(G) \to M\left( \frac{G}{N} \right) \\
& \to \frac{N}{[N, G]} \to (G)^{ab} \to \left( \frac{G}{N} \right)^{ab} \to 0.
\end{align*}
\]
The natural epimorphism \( G \to G/N \) implies the following exact sequence of free simplicial groups

\[
1 \to \ker(\alpha) \to K \xrightarrow{\alpha} L \to 1,
\]

where \( K \) and \( L \) are free simplicial resolution of \( G \) and \( G/N \), respectively (see Franco, 1998). The short exact sequence of simplicial groups

\[
1 \to \ker\left( \frac{\alpha}{V(\alpha)} \right) \to \frac{K}{V(K)} \xrightarrow{\alpha/\Delta} \frac{L}{V(L)} \to 1 \quad (1.2)
\]
gives rise a long exact sequence of homotopy groups as follows:

\[
\cdots \to \pi_1(\ker\left( \frac{\alpha}{V(\alpha)} \right)) \to \pi_1\left( \frac{K}{V(K)} \right) \to \pi_1\left( \frac{L}{V(L)} \right) \\
\to \pi_0(\ker\left( \frac{\alpha}{V(\alpha)} \right)) \to \pi_0\left( \frac{K}{V(K)} \right) \to \pi_0\left( \frac{L}{V(L)} \right) \to 0.
\]

Franco (1998), proved that \( \pi_0(\ker\left( \frac{\alpha}{V(\alpha)} \right)) \cong \frac{N}{[NV^*G]} \). Using some isomorphisms we can rewrite the above long exact sequence as follows:

\[
\cdots \to \pi_1(\ker\left( \frac{\alpha}{V(\alpha)} \right)) \to VM(G) \to VM(G/N) \\
\to \frac{N}{[NV^*G]} \to \frac{G}{V(G)} \to \frac{G/N}{V(G/N)} \to 0.
\]  

Indeed Franco obtained the Fröhlich long exact sequence independent of the method of Fröhlich (1963).

Now, we define the Baer invariant of a pair of groups \((G, N)\) as follows:

\[
VM(G, N) = \pi_1\left( \ker\left( \frac{\alpha}{V(\alpha)} \right) \right).
\]

2 Preliminaries and notation

In this section we recall some basic notations and properties of simplicial groups which will be needed in the sequel. We refer the reader to Curtis (1971) or Georss and Jardine (1999) for further details.
Definition 2.1. A simplicial set $K$ is a sequence of sets $K_0, K_1, K_2, \ldots$ together with maps $d_i : K_n \to K_{n-1}$ (faces) and $s_i : K_n \to K_{n+1}$ (degeneracies), for each $0 \leq i \leq n$, such that the following conditions hold:

$$
\begin{align*}
   d_jd_i &= d_{i-1}d_j \quad \text{for } j < i \\
   s_js_i &= s_{i+1}s_j \quad \text{for } j \leq i \\
   d_js_i &= \begin{cases} 
   s_{i-1}d_j & \text{for } j < i; \\
   \text{identity} & \text{for } j = i, i+1; \\
   s_id_{j-1} & \text{for } j > i + 1.
   \end{cases}
\end{align*}
$$

A simplicial map $f : K \to L$ is a sequence of functions $f_n : K_n \to L_n$, with the following commutative diagram

$$
\begin{array}{ccc}
   K_{n+1} & \xleftarrow{s_i} & K_n & \xrightarrow{d_i} & K_{n-1} \\
   f_{n+1} \downarrow & & f_n \downarrow & & f_{n-1} \downarrow \\
   L_{n+1} & \xleftarrow{s_i} & L_n & \xrightarrow{d_i} & L_{n-1}
\end{array}
$$

Like topological spaces, the homotopy groups of simplicial sets is defined. The category of simplicial sets and topological spaces can be related by two functors as follows:

- The geometric realization, $| - |$, is the functor from the category of simplicial sets to the category of CW complexes.
- The singular simplicial, $S_*(-)$, is the functor from the category of topological spaces to the category of simplicial sets.

A simplicial set $K$ is called a simplicial group if each $K_i$ is group and all faces and degeneracies are homomorphisms. There is a basic property of simplicial groups which due to Moore (1954–55) its homotopy groups $\pi_*(G.)$ can be obtained as the 'homology of' a certain chain complex $(NG, \partial)$. 

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Definition 2.2. If $K$ is a simplicial group, then the Moore complex $(NK, \partial)$ of $K$ is the (nonabelian) chain complex defined by $(NK)_n = \cap_{i=0}^{n-1} \text{Kerd}_i$ with $\partial_n : NK_n \to NK_{n-1}$ which is the restriction of $d_n$.

A simplicial group $K$ is said to be free if each $K_n$ is a free group and degeneracy homomorphisms $s_i$'s send the free basis of $K_n$ into the free basis of $K_{n+1}$.

Theorem 2.3. (see Curtis, (1971)).

1. For every simplicial group $K$, the homotopy group $\pi_n(K)$ is abelian even for $n = 1$.

2. Every epimorphism between simplicial groups is a fibration.

3. Let $K_*$ be a simplicial group, then $\pi_*(K) \cong H_*(NK_*)$.

4. $H_n(N(K \otimes L)) \cong H_n(N(K) \otimes N(L))$.

3 Some properties of the Baer invariant of a pair of groups

In this section we study some behaviors of the Baer invariant of a pair of groups. Let $f : (G_1, N_1) \to (G_2, N_2)$ be a homomorphism of pairs of groups, then functorial property of free simplicial resolution yields the following diagram of free simplicial groups:

$$
\begin{array}{ccc}
\ker(\alpha_1) & \longrightarrow & K_1 \longrightarrow \alpha_1 L_1 \\
\downarrow & & \downarrow \\
\ker(\alpha_2) & \longrightarrow & K_2 \longrightarrow \alpha_2 L_2 \\
\end{array}
$$

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where $K_i$ and $L_i$ are the corresponding free simplicial resolution of $G_i, G_i/N_i$, respectively. Therefore we have the following commutative diagram

\[
\begin{array}{c}
\ker \alpha_1/V(\alpha_1) \xrightarrow{\gamma} K_1/V(K_1) \xrightarrow{\alpha_1/V(\alpha_1)} L_1/V(L_1) \\
\ker \alpha_2/V(\alpha_2) \xrightarrow{\alpha_2/V(\alpha_2)} K_2/V(K_2) \xrightarrow{\gamma} L_2/V(L_2)
\end{array}
\]

By the above diagram we have $\pi_1(\gamma) : \pi_1\left(\ker \alpha_1/V(\alpha_1)\right) \to \pi_1\left(\ker \alpha_2/V(\alpha_2)\right)$.

Indeed, we can state the following theorem.

**Theorem 3.1.** The Baer invariant of a pair of groups is a functor from the category of pairs of groups to the category of abelian groups.

The long exact sequence of (1.3) implies the following theorems.

**Theorem 3.2.** The Baer invariant of a group is a special case of the Baer invariant a pair of groups i.e. $VM(G, G) \cong VM(G)$. Thus for a cyclic group $C$ and a free group $F$ we have $VM(C, C) = 1 = VM(F, F)$. Also $VM(G, 1)$ is a trivial group.

**Theorem 3.3.** Let $G$ be the semi-direct product of $N$ by $Q$. Then $VM(G, N) \cong \ker \left(VM(G) \to VM(G/N)\right)$ and $VM(G) \cong VM(G, N) \oplus VM(Q)$.

**Proof.** The hypothesis implies that the exact sequence (1.2) splits and hence the result holds. \hfill \Box

The above theorem shows that if $G$ is the semi-direct product of $N$ by $Q$, then the Baer invariant of $(G, N)$ can be described in presentation of groups as follows.

**Corollary 3.4.** Let $G \cong F/R$ be a free presentation of $G$ and $N \cong S/R$ be a normal subgroup of $G$ which has a complement in $G$, then

$$VM(G, N) \cong \frac{R \cap \left[SV^*F\right]}{[RV^*F]}.$$
Note that the above corollary shows that our definition of the Baer invariant of a pair of groups is a vast generalization of the one by Moghaddam, Salemkar and Saany (2007).

**Theorem 3.5.** Suppose that $M$ and $N$ are two subgroups of a group $G$ such that $M \cong MN$, then there exists the following isomorphism

$$\forall M(MN, N) \cong \forall M(M, M \cap N).$$

**Proof.** By the second isomorphism theorem we have $\frac{MN}{N} \cong \frac{M}{M \cap N}$. Let $K$ and $L$ be the free simplicial groups corresponding to $MN$ and $MN/N$, respectively. Because of the functorial property of free simplicial resolution corresponding to each group, we conclude that $K$ and $L$ are also simplicial groups corresponding to $M$ and $\frac{M}{M \cap N}$, respectively. Hence by the definition the result holds. \hfill \Box

Using the exact sequence (1.3) and the structure of its sixth term given by Eckmann, Hilton, and Stammbach (1972) and Lue (1976), when $N$ is a central and an $N_c$-central subgroup of $G$, respectively, we have the following theorem.

**Theorem 3.6.** Let $N$ be a central subgroup of $G$ then $M(G, N) \cong G^{ab} \otimes N$. Also, if $N$ is an $N_c$-central subgroup of $G$, then $M^{(c)}(G, N) \cong N \otimes \frac{G}{\gamma_c(G)} \otimes \cdots \otimes \frac{G}{\gamma_c(G)}$ with $c$ copies of $\frac{G}{\gamma_c(G)}$.

**Theorem 3.7.** Let $\{(G_i, N_i)\}_{i \in I}$ be a given direct system of pairs of groups with the directed index set $I$, then $\forall M(\lim_{\to} G_i, \lim_{\to} N_i) \cong \lim_{\to} \forall M(G_i, N_i)$.

**Proof.** For any $i \in I$, let $K_i$ and $L_i$ be the corresponding free simplicial resolutions of $G_i$ and $G_i/N_i$, respectively. Assume that $\alpha_i : K_i \to L_i$ is the corresponding epimorphism of simplicial groups. Thus we can consider the following exact sequence of simplicial groups.

$$1 \to \ker(\frac{\alpha_i}{V(\alpha_i)}) \to \frac{K_i}{V(K_i)} \xrightarrow{\alpha_i} \frac{L_i}{V(L_i)} \to 1.$$
Vasagh, Mirebrahimi and Mashayekhy proved that $\lim \pi_n(K_i) \cong \pi_n(\lim K_i)$, where $K_i$ is a simplicial group. Hence $\lim K_i$ and $\lim L_i$ are simplicial groups corresponding to $\lim G_i$ and $\lim G_i/N_i$, respectively, and we have the following exact sequence:

$$1 \to \ker \left( \frac{\lim (\alpha_i)}{V(\lim (\alpha_i))} \right) \to \frac{\lim (K_i)}{V(\lim (K_i))} \xrightarrow{\lim (\alpha_i)} \frac{\lim (L_i)}{V(\lim (L_i))} \to 1$$

Since the functor $- / V(-)$ has right adjoint, $\lim (\frac{K_i}{V(K_i)}) \cong \frac{\lim K_i}{V(\lim K_i)}$. The fact that direct limit preserve the exact sequence yields the following commutative diagrams:

$$\begin{array}{cccc}
1 & \to & \ker \left( \frac{\lim (\alpha_i)}{V(\lim (\alpha_i))} \right) & \to \\
\downarrow & & \downarrow & \downarrow \\
1 & \to & \lim \left( \frac{\alpha_i}{V(\alpha_i)} \right) & \to \\
\end{array}$$

Five Lemma implies that $\lim (\ker(\frac{\alpha_i}{V(\alpha_i)})) \cong \ker(\lim (\frac{\alpha_i}{V(\alpha_i)}))$. Also the homotopy groups of simplicial groups commute with direct limits (See Vasagh, Mirebrahimi and Mashayekhy), hence we have

$$\mathcal{V}M(\lim G_i, \lim N_i) \cong \pi_1 \left( \ker \lim \left( \frac{\alpha_i}{V(\alpha_i)} \right) \right) \cong \pi_1 \left( \lim \left( \ker \left( \frac{\alpha_i}{V(\alpha_i)} \right) \right) \right) \cong \lim \mathcal{V}M(G_i, N_i).$$

Let $G$ be a group and $N$ be a normal subgroup of it, and consider $K_i$ and $L_i$ as the free simplicial resolutions corresponding to $G$ and $G/N$, respectively. The simplicial epimorphism $\alpha : K_i \to L_i$ gives rise epimorphisms $\alpha_n : K_i/\gamma_n(K_i) \to L_i/\gamma_n(L_i)$ and $\beta_n : \gamma_n(K_i)/\gamma_{n+1}(K_i) \to \gamma_n(L_i)/\gamma_{n+1}(L_i)$.
which induce the following commutative diagram:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & \ker(\beta_n) & \gamma_n(K) \\
\downarrow & \downarrow & \downarrow \\
1 & \ker(\alpha_{n+1}) & \gamma_{n+1}(K) \\
\downarrow & \downarrow & \downarrow \\
1 & \ker(\alpha_n) & \gamma_n(L) \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1 \\
\end{array}
\]

(3.1)

Since \(\beta_n\) is an epimorphism so is \(\ker(\alpha_{n+1}) \to \ker(\alpha_n)\). Since every epimorphism of simplicial groups is a fibration, the left column exact sequence induces the following long exact sequence of homotopy groups

\[
\cdots \to \pi_1(\ker(\beta_n)) \to \pi_1(\ker(\alpha_{n+1})) \to \pi_1(\ker(\alpha_n)) \\
\to \pi_0(\ker(\beta_n)) \to \pi_0(\ker(\alpha_{n+1})) \to \pi_0(\ker(\alpha_n)) \to 1.
\]

Using some isomorphisms we can rewrite the above sequence as the following long exact sequence:

\[
\cdots \to \pi_1(\ker(\beta_n)) \to M^{(n)}(G, N) \to M^{(n-1)}(G, N) \\
\to \pi_0(\ker(\beta_n)) \to N_{\gamma_{n+1}(G, N)} \to N_{\gamma_n(G, N)} \to 1.
\]

(3.2)

Now for \(n = 2\), we discuss on the long exact sequence (3.2). First in the following lemma, we concentrate on \(\ker(\beta_2)\) which gives a relation between exterior product of a group and its quotient.

**Lemma 3.8.** Let \((G, N)\) be a pair of group, then we have the following exact sequence of groups

\[
1 \to \frac{N}{[N, G]} \wedge \frac{N}{[N, G]} \oplus \frac{N}{[N, G]} \otimes \left(\frac{G}{N}\right)^{ab} \to G^{ab} \wedge G^{ab} \to \left(\frac{G}{N}\right)^{ab} \wedge \left(\frac{G}{N}\right)^{ab} \to 1.
\]
Proof. It is known that if $F$ is a free group, then $\gamma_2(F)/\gamma_3(F)$ is a free abelian group which is isomorphic to $F^{ab} \wedge F^{ab}$. Hence computing the ranks of the terms of the first row exact sequence of commutative diagram (3.1) implies that

$$\ker(\beta_2) \cong (\ker(\alpha_2) \wedge \ker(\alpha_2)) \oplus (\ker(\alpha_2) \otimes L^{ab}).$$

This row exact sequence yields the following long exact sequence of homotopy groups as follows:

$$\cdots \to \pi_1(\ker(\beta_2)) \to \pi_1(K^{ab} \wedge K^{ab}) \to \pi_1(L^{ab} \wedge L^{ab}) \to \pi_0(\ker(\beta_2)) \to \pi_0(K^{ab} \wedge K^{ab}) \to \pi_0(L^{ab} \wedge L^{ab}) \to 1.$$ 

Since $\gamma_2(L)/\gamma_3(L)$ is a free abelian group so every right term of above long exact sequence splits. Burns and Ellis, (1997) showed that

$$\pi_0(K^{ab} \wedge K^{ab}) \cong G^{ab} \wedge G^{ab}.$$ 

Also we have

$$\pi_0(K^{ab} \wedge K^{ab}) \cong H_0(N(K^{ab} \wedge K^{ab})) \quad \text{(by Theorem 2.3[3])}$$

$$\cong H_0(N(K^{ab}) \otimes N(K^{ab})) \quad \text{(by Theorem 2.3[1])}$$

$$\cong H_0(N(K^{ab})) \otimes H_0(N(K^{ab})) \quad \text{(by Kunneth formula)}$$

$$\cong \pi_0(K^{ab}) \otimes \pi_0(K^{ab}) \quad \text{(by Theorem 2.3[3])}$$

$$\cong G^{ab} \otimes G^{ab}.$$ 

Therefore we have

$$\pi_0(\ker(\beta_2)) \cong \left(\pi_0(\ker(\alpha_2)) \wedge \pi_0(\ker(\alpha_2))\right) \oplus \pi_0(\ker(\alpha_2) \otimes L^{ab})$$

$$\cong \left(\frac{N}{[N,G]} \wedge \frac{N}{[N,G]}\right) \oplus \left(\frac{N}{[N,G]} \otimes \frac{G^{ab}}{N}\right).$$

\[\Box\]

Theorem 3.9. For each group $G$ and a normal subgroup $N$, there exists a functorial group $V(G, N)$ which fits into the following natural exact sequence

$$V(G, N) \oplus \text{Tor}(\frac{N}{[N,G]}, \frac{G}{N}) \oplus (M^{(1)}(G, N) \otimes (\frac{G}{N})^{ab}) \oplus (\frac{N}{[N,G]} \otimes M^{(1)}(\frac{G}{N}))$$

$$\to M^{(2)}(G, N) \to M^{(1)}(G, N) \to \frac{N}{[N,G]} \wedge \frac{N}{[N,G]} \oplus \frac{N}{[N,G]} \otimes (\frac{G}{N})^{ab}$$

$$\to \frac{N}{\gamma_3(G, N)} \to \frac{N}{[G,N]} \to 1.$$
Proof. It is sufficient to compute $\pi_1(\ker(\beta_2))$ and replace it in the long exact sequence (3.2). Similar to the proof of the above lemma the Künneth formula implies that

\[
\pi_1(\ker(\beta_2)) \cong \pi_1(\ker(\alpha_2) \land \ker(\alpha_2)) \oplus H_1\left(\frac{\langle N \rangle}{\langle N, \alpha_1, \alpha_2 \rangle} \otimes \frac{\langle G \rangle}{\langle N \rangle} \otimes \Gamma(\frac{\langle G \rangle}{\langle N \rangle}) \right)
\]

\[
\cong \pi_1(\ker(\alpha_2) \land \ker(\alpha_2)) \oplus M(G, N) \otimes (\frac{\langle G \rangle}{\langle N \rangle})^{ab}
\]

\[
\oplus \frac{\langle N \rangle}{\langle N, \alpha_1, \alpha_2 \rangle} \otimes M(\frac{\langle G \rangle}{\langle N \rangle}) \oplus \text{Tor}(\frac{\langle N \rangle}{\langle N, \alpha_1, \alpha_2 \rangle}, \frac{\langle G \rangle}{\langle N \rangle}).
\]

The group $V(G, N)$ is defined as an abelian group $\pi_1(\ker(\alpha_2) \land \ker(\alpha_2))$. □

Note that if we put $N = G$ in the above theorem, then we have the natural exact sequence which is proved in Burns and Ellis (1997).

4 The Baer invariant of a pair of the free product of groups

In this section we study the behavior of the Baer invariant of a pair of the free product of groups. For $i = 1, 2$, assume that $\alpha_i : K_i \to L_i$ are epimorphisms, where $K_i$ and $L_i$ are free simplicial resolutions corresponding to $G_i$ and $G_i/N_i$, respectively. Van-Kampen theorem for simplicial groups implies that $K_1 \ast K_2$ and $L_1 \ast L_2$ are the free simplicial groups corresponding to $G_1 \ast G_2$ and $(G_1/N_1 \ast G_2/N_2) \cong \frac{\langle G_1 \ast G_2 \rangle}{\langle N_1 \ast N_2 \rangle}$, respectively. So we can consider $\beta = \alpha_1 \ast \alpha_2$ as the corresponding epimorphism from $K_1 \ast K_2$ onto $L_1 \ast L_2$ (see Burns and Ellis, 1997).

Consider the following exact sequences of simplicial groups

\[
1 \to \ker(\alpha_1/V(\alpha_1)) \to \frac{K_1}{V(K_1)} \xrightarrow{\alpha_1/V(\alpha_1)} \frac{L_1}{V(L_1)} \to 1,
\]

\[
1 \to \ker(\alpha_2/V(\alpha_2)) \to \frac{K_2}{V(K_2)} \xrightarrow{\alpha_2/V(\alpha_2)} \frac{L_2}{V(L_2)} \to 1,
\]
Therefore we have the following commutative diagram

\[
\begin{array}{cccc}
1 & \rightarrow & \ker(\beta/V(\beta)) & \rightarrow \frac{K_1 \ast K_2}{V(K_1 \ast K_2)} \beta/V(\beta) & \rightarrow & \frac{L_1 \ast L_2}{V(L_1 \ast L_2)} & \rightarrow & 1. \\
1 & \rightarrow & \ker(\theta) & \rightarrow & \ker(\varphi_V) & \rightarrow & \frac{\beta/V(\beta)}{\beta/V(\beta)} & \rightarrow & 1 \\
1 & \rightarrow & \ker(\frac{\alpha_1}{V(\alpha_1)} \times \ker(\frac{\alpha_2}{V(\alpha_2)}) \varphi_V) & \rightarrow & \frac{K_1 \ast K_2}{V(K_1 \ast K_2)} & \rightarrow & \frac{\beta/V(\beta)}{\beta/V(\beta)} & \rightarrow & 1 \\
1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 1 \\
\end{array}
\]

Definition of \( \varphi_V \) implies that \( \varphi_V \mid \) is an epimorphism, and the epimorphism \( \varphi_V \mid \) yields the epimorphism \( \theta \). Since every epimorphism of simplicial groups is a fibration, the left column exact sequence of (4.2) induces the following long exact sequence of abelian groups

\[
\cdots \rightarrow \pi_n((\ker(\theta)) \rightarrow \pi_n(\ker \frac{\beta}{V(\beta)}) \rightarrow \pi_n(\ker \frac{\alpha_1}{V(\alpha_1)}) \oplus \pi_n(\ker \frac{\alpha_2}{V(\alpha_2)}) \rightarrow \cdots
\]

Consider the natural homomorphisms \( i_i : K_i/V(K_i) \rightarrow K_i \ast K_2/V(K_1 \ast K_2) \) for \( i = 1, 2 \). Since \( \pi_n(K_1/V(K_1)) \oplus \pi_n(K_2/V(K_2)) \) is a coproduct in the category of abelian groups for all \( n > 0 \), there exists

\[
\delta_{V_n} : \pi_n(K_1/V(K_1)) \oplus \pi_n(K_2/V(K_2)) \rightarrow \pi_n(K_1 \ast K_2/V(K_1 \ast K_2))
\]

such that \( \varphi_{V_n} \circ \pi_n(\delta_V) = id \). Similarly we have

\[
\tau_i : L_i/V(L_i) \rightarrow L_1 \ast L_2/V(L_1 \ast L_2),
\]

for \( i = 1, 2 \), therefore there exists

\[
\sigma_{V_n} : \pi_n(L_1/V(L_1)) \oplus \pi_n(L_2/V(L_2)) \rightarrow \pi_n(L_1 \ast L_2/V(L_1 \ast L_2))
\]

such that \( \psi_{V_n} \circ \pi_n(\sigma_V) = id \). 

13
The following commutative diagram

\[
\begin{array}{ccc}
\frac{K_1 \ast K_2}{V(K_1 \ast K_2)} & \xrightarrow{\beta/V(\beta)} & \frac{L_1 \ast L_2}{V(L_1 \ast L_2)} \\
\uparrow & & \uparrow \\
\frac{K_1}{V(K_1)} & \xrightarrow{\alpha} & \frac{L_1}{V(L_1)}
\end{array}
\]

gives rise \(\iota_i : \ker (\alpha_i/V(\alpha_i)) \to \ker (\beta/V(\beta))\). Similarly there exists

\[\eta : \pi_n\left(\ker (\alpha_1/V(\alpha_1))\right) \oplus \pi_n\left(\ker (\alpha_2/V(\alpha_2))\right) \to \pi_n\left(\ker (\beta/V(\beta))\right)\]

such that \(\varphi_V|_n \circ \pi_n(\delta_V) = id\).

Consequently, for all \(n > 0\), the exact sequence of \((4.3)\) splits, therefore

\[\pi_n\left(\ker (\theta)\right) \oplus \pi_n\left(\ker (\alpha_1/V(\alpha_1))\right) \oplus \pi_n\left(\ker (\alpha_2/V(\alpha_2))\right) \cong \pi_n\left(\ker (\beta/V(\beta))\right). \quad (4.4)\]

For \(n = 1\), using some isomorphisms, we can rewrite \((4.4)\) as follows:

\[
\mathcal{V}M(G_1 * G_2, \langle N_1 * N_2 \rangle_{G_1 \ast G_2}) \cong \mathcal{V}M(G_1, N_1) \oplus \mathcal{V}M(G_2, N_2) \oplus D,
\]

where \(D\) is defined as an abelian group \(\pi_1(\ker(\theta))\).

Now by the above notations we are in a position to state and prove the following theorem.

**Theorem 4.1.** Let \((G_i, N_i)\) be pairs of groups for \(i = 1, 2\), then

(i) \(M(G_1 * G_2, \langle N_1 * N_2 \rangle_{G_1 \ast G_2}) \cong M(G_1, N_1) \oplus M(G_2, N_2)\).

(ii) \(M^{(2)}(G_1 * G_2, \langle N_1 * N_2 \rangle_{G_1 \ast G_2}) \cong M^{(2)}(G_1, N_1) \oplus M^{(2)}(G_2, N_2)\)

\[
\oplus M(G_1, N_1) \otimes \frac{N_2}{[N_2, G_2]} \\
\oplus M(G_2, N_2) \otimes \frac{N_1}{[N_1, G_1]} \\
\oplus M(G_1, \frac{G_2}{N_2}) \otimes \frac{N_2}{[N_2, G_2]} \\
\oplus M(G_2, \frac{G_1}{N_1}) \otimes \frac{N_2}{[N_2, G_2]} \\
\oplus (\frac{G_1}{N_1})^{ab} \otimes M(G_2, N_2) \\
\oplus (\frac{G_2}{N_2})^{ab} \otimes M(G_1, N_1) \\
\oplus Tor(\frac{N_1}{[N_1, G_1]} \otimes \frac{N_2}{[N_2, G_2]}) \\
\oplus Tor(\frac{G_1}{N_1})^{ab} \otimes \frac{N_2}{[N_2, G_2]} \\
\oplus Tor(\frac{G_2}{N_2})^{ab} \otimes \frac{N_1}{[N_1, G_1]})
\]
Proof. (i) Let $\mathcal{V}$ be the variety of abelian groups. We have $(K_1 \ast K_2)^{ab} \cong K_1^{ab} \oplus K_2^{ab}$ since $K_1$ and $K_2$ are free groups. Hence in we have $\ker(\beta/\gamma_2(\beta)) \cong \ker(\alpha_1/\gamma_2(\alpha_1)) \oplus \ker(\alpha_2/\gamma_2(\alpha_2))$. Therefore the exact sequence (4.4) implies that

$$M(G_1 \ast G_2, \langle N_1 \ast N_2 \rangle^{G_1 \ast G_2}) \cong \pi_1\left(\ker(\beta/\gamma_2(\beta))\right) \cong \pi_1\left(\ker(\alpha_1/\gamma_2(\alpha_1))\right) \oplus \pi_1\left(\ker(\alpha_2/\gamma_2(\alpha_2))\right).$$

(ii) Let $\mathcal{V}$ be the variety of nilpotent groups of class at most 2. Burns and Ellis (1997) proved the isomorphisms $\ker(\varphi) \cong (K_1)^{ab} \otimes (K_2)^{ab}$ and $\ker(\psi) \cong (L_1)^{ab} \otimes (L_2)^{ab}$. Hence we have $\ker(\theta/\gamma_3(\theta)) \cong \ker(\alpha_1/\gamma_2(\alpha_1) \otimes \alpha_2/\gamma_2(\alpha_2))$.

Since $(K_1)^{ab} \otimes (K_2)^{ab}$ and $(L_1)^{ab} \otimes (L_2)^{ab}$ are free abelian simplicial groups, by computing the ranks of free abelian groups in an exact sequence, we can obtain $\ker(\theta/\gamma_3(\theta))$, as follows:

$$\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)} \otimes \frac{\alpha_2}{\gamma_2(\alpha_2)}\right) \cong \ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)}\right) \otimes \ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right) \oplus \frac{L_1}{\gamma_2(L_1)} \otimes \ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right) \oplus \ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)}\right) \oplus \frac{L_2}{\gamma_2(L_2)}.$$

Now we compute the fundamental group of $\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)} \otimes \frac{\alpha_2}{\gamma_2(\alpha_2)}\right)$. First we obtain $\pi_1\left(\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)} \otimes \ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right)\right)\right)$ in more details. By Theorem 2.3 (3) we must obtain $H_1\left(N(\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)} \otimes \ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right)\right))\right)$. Theorem 2.3 (4) and Künneth formula imply that

$$\pi_1\left(\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)} \otimes \ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right)\right)\right) \cong \pi_1\left(\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)}\right) \otimes \pi_0\left(\ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right)\right)\right) \oplus \pi_0\left(\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)}\right)\right) \otimes \pi_1\left(\ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right)\right) \oplus Tor\left(\pi_0\left(\ker\left(\frac{\alpha_1}{\gamma_2(\alpha_1)}\right)\right), \pi_0\left(\ker\left(\frac{\alpha_2}{\gamma_2(\alpha_2)}\right)\right)\right) \cong M(G_1, N_1) \otimes N_2 / N_2 G_2 \oplus N_1 / [N_1, G_1] \otimes M(G_2, N_2) \oplus Tor\left(\frac{N_1}{[N_1, G_1]}, \frac{N_2}{[N_2, G_2]}\right).$$
Similarly we have
\[
\pi_1 \left( \frac{L_1}{\gamma_2(L_1)} \otimes \ker \left( \frac{\alpha_1}{\gamma_2(\alpha_2)} \right) \right) \cong M \left( \frac{G_1}{N_1} \right) \otimes \frac{N_2}{[N_2,G_2]}
\oplus \left( \frac{G_1}{N_1} \right)^{ab} \otimes M(G_2, N_2)
\oplus \text{Tor} \left( \left( \frac{G_1}{N_1} \right)^{ab}, \frac{N_2}{[N_2,G_2]} \right).
\]

Also
\[
\pi_1 \left( \frac{L_2}{\gamma_2(L_2)} \otimes \ker \left( \frac{\alpha_1}{\gamma_2(\alpha_2)} \right) \right) \cong M \left( \frac{G_2}{N_2} \right) \otimes \frac{N_1}{[N_1,G_1]}
\oplus \left( \frac{G_2}{N_2} \right)^{ab} \otimes M(G_1, N_1)
\oplus \text{Tor} \left( \left( \frac{G_2}{N_2} \right)^{ab}, \frac{N_1}{[N_1,G_1]} \right).
\]

Now by replacing the above isomorphisms in (4.4), we conclude the following isomorphism:
\[
M^{(2)}(G_1 \ast G_2, [N_1 \ast N_2])^{G_1 \ast G_2} \cong M^{(2)}(G_1, N_1) \oplus M^{(2)}(G_2, N_2)
\oplus M(G_1, N_1) \otimes \frac{N_2}{[N_2,G_2]}
\oplus M(G_2, N_2) \otimes \frac{N_1}{[N_1,G_2]}
\oplus M \left( \frac{G_2}{N_2} \right) \otimes \frac{N_1}{[N_1,G_1]}
\oplus \left( \frac{G_1}{N_1} \right)^{ab} \otimes M(G_2, N_2)
\oplus \left( \frac{G_2}{N_2} \right)^{ab} \otimes M(G_1, N_1)
\oplus \text{Tor} \left( \left( \frac{G_1}{N_1} \right)^{ab}, \frac{N_2}{[N_2,G_2]} \right)
\oplus \text{Tor} \left( \left( \frac{G_2}{N_2} \right)^{ab}, \frac{N_1}{[N_1,G_1]} \right).
\]

\[\square\]

**Remark 4.2.** Theorems 4.1 and 3.2 imply that \(M(G_1 \ast G_2) \cong M(G_1) \oplus M(G_2)\) and \(M^{(2)}(G \ast H) \cong M^2(G) \oplus M^2(H) \oplus M(G) \otimes H^{ab} \oplus G^{ab} \otimes M(H) \oplus \text{Tor}(G^{ab}, H^{ab})\) which are proved by Miller (1952), and by Burns and Ellis (1997), respectively. Also note that the part (i) of the above theorem is proved by Mirebrahimi and Mashayekhy.
Now we intend to compute $M^{(c)}(G_1 \ast G_2, \langle N_1 \ast N_2 \rangle^{G_1 \ast G_2})$, for all $c \geq 1$, with some conditions.

**Theorem 4.3.** Let $(G_i, N_i)$ be pairs of groups for $i = 1, 2$ such that $G_1/N_1$ and $G_2/N_2$ satisfy in the following conditions:

\[
\begin{align*}
\frac{G_1}{N_1}^{ab} \otimes \frac{G_2}{N_2}^{ab} &= M^{(1)}(\frac{G_1}{N_1}) \otimes M^{(1)}(\frac{G_2}{N_2}) = \text{Tor}(\frac{G_1^{ab}}{N_1}, \frac{G_2^{ab}}{N_2}) \\
\frac{G_1}{N_1}^{ab} \otimes H_3(\frac{G_2}{N_2}) &= M^{(1)}(\frac{G_1}{N_1}) \otimes \frac{G_2^{ab}}{N_2} = \text{Tor}(\frac{G_1^{ab}}{N_1}, M^{(1)}(\frac{G_2}{N_2})) \\
\frac{G_1}{N_1}^{ab} \otimes H_3(\frac{G_2}{N_2}) &= M^{(1)}(\frac{G_2}{N_2}) \otimes \frac{G_1^{ab}}{N_1} = \text{Tor}(\frac{G_2^{ab}}{N_2}, M^{(1)}(\frac{G_1}{N_1})) = 0.
\end{align*}
\]

Also, let for $G_1$ and $G_2$ the following conditions hold:

\[
G_1^{ab} \otimes G_2^{ab} = M^{(1)}(G_1) \otimes G_2^{ab} = M^{(1)}(G_2) \otimes G_1^{ab} = \text{Tor}(G_1^{ab}, G_2^{ab}) = 0.
\]

Then for all $c \geq 1$, we have the following isomorphism:

\[
M^{(c)}(G_1 \ast G_2, \langle N_1 \ast N_2 \rangle^{G_1 \ast G_2}) \cong M^{(c)}(G_1, N_1) \oplus M^{(c)}(G_2, N_2).
\]

**Proof.** Consider the assumption, like the beginning of the section and let $V$ be variety of nilpotent groups of class at most $c$, also we note $\varphi_c$ by $\varphi_c$ and $\psi_c$ by $\psi_c$ in briefly.

The commutative diagram (4.2) implies the following commutative diagram:

\[
\begin{array}{ccc}
\pi_2(\ker \psi_c) & \rightarrow & \pi_1(\ker \varphi_c) \\
\downarrow \pi_2(\frac{L_1 \ast L_2}{N_1(N_1 \ast N_2)}) & \downarrow \pi_1(\ker(\frac{\varphi_c}{\varphi_c})) & \downarrow \pi_1(\frac{K_1 \ast K_2}{N_1(N_1 \ast N_2)}) \\
\pi_2(\frac{L_1}{N_1(\varphi_c)}) \oplus \pi_2(\frac{L_2}{N_2(\varphi_c)}) & \rightarrow & \pi_1(\ker(\frac{\alpha_1}{\alpha_1})) \oplus \pi_1(\ker(\frac{\alpha_2}{\alpha_2})) \\
\end{array}
\]

The assumption

\[
G_1^{ab} \otimes G_2^{ab} = M^{(1)}(G_1) \otimes G_2^{ab} = M^{(1)}(G_2) \otimes G_1^{ab} = \text{Tor}(G_1^{ab}, G_2^{ab}) = 0
\]

implies that $\pi_1(\ker \varphi_c)$ is a trivial group (See Vasagh, Mirebrahimi and Mashayekhy). Now like Vasagh, Mirebrahimi and Mashayekhy, by induction
on $c$ we prove that the other assumptions yield that $\pi_2(\ker(\psi_c))$ is trivial. Note that $\ker \psi_c$ satisfies in the following exact sequence

$$1 \rightarrow \frac{[L_1, L_2; c-2 F]^F}{[L_1, L_2; c-1 F]^F} \rightarrow \ker \psi_c \rightarrow \ker \psi_{c-1} \rightarrow 1,$$

where $F_c = L_1 \ast L_2$. Moreover

$$\frac{[L_1, L_2; c-2 F]^F}{[L_1, L_2; c-1 F]^F} \cong \bigoplus \sum_{i + j = c} L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab}.$$

For $c = 2$, we show that $\ker \psi_2 \cong L_1^{ab} \otimes L_2^{ab}$. Theorem 2.3 (3), (4) and Kunneth formula imply that

$$\pi_0(\ker \psi_2) \cong \pi_0(L_1^{ab} \otimes L_2^{ab}) \cong (G_1/N_1)^{ab} \otimes (G_2/N_2)^{ab} = 0.$$

Similarly

$$\pi_1(\ker \psi_2) \cong (G_1/N_1)^{ab} \otimes M(1)(G_2/N_2) \oplus M(1)(G_1/N_1) \otimes (G_2/N_2)^{ab} \oplus \text{Tor}((G_1/N_1)^{ab}, (G_2/N_2)^{ab}) = 0.$$

Also

$$\pi_2(\ker \psi_2) \cong \pi_2(L_1^{ab} \otimes L_2^{ab}) \cong (G_1/N_1)^{ab} \otimes H_3(G_2/N_2) \oplus (G_2/N_2)^{ab} \otimes H_3(G_1/N_1) \oplus M(1)(G_1/N_1) \otimes (G_2/N_2)^{ab} \oplus M(1)(G_2/N_2) \otimes (G_1/N_1)^{ab} \oplus \text{Tor}((G_1/N_1)^{ab}, M(1)(G_2/N_2)) \oplus \text{Tor}((G_2/N_2)^{ab}, M(1)(G_1/N_1)) = 0.$$
For $c > 2$, we have

$$\begin{align*}
\pi_2(L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab})^{i\text{--times}} \\
&\cong \pi_2(L_1^{ab} \otimes L_2^{ab}) \otimes \pi_0(L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab})^{(i-1)\text{--times}} \\
&\oplus \pi_1(L_1^{ab} \otimes L_2^{ab}) \otimes \pi_1(L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab})^{(j-1)\text{--times}} \\
&\oplus \pi_0(L_1^{ab} \otimes L_2^{ab}) \otimes \pi_2(L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab})^{(i-1)\text{--times}} \\
&\oplus \text{Tor}(\pi_0(L_1^{ab} \otimes L_2^{ab}), \pi_1(L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab}))^{(i-1)\text{--times}} \\
&\oplus \text{Tor}(\pi_1(L_1^{ab} \otimes L_2^{ab}), \pi_0(L_1^{ab} \otimes \ldots \otimes L_1^{ab} \otimes L_2^{ab} \otimes \ldots \otimes L_2^{ab}))^{(j-1)\text{--times}} \\
&\cong 0.
\end{align*}$$

Thus $\pi_2(\ker \psi_c) = 0$ and by (4.5) we have $\pi_1(\ker(\theta)) = 0$. Hence $M^{(c)}(G_1 \ast G_2, \langle N_1 \ast N_2 \rangle^{G_1 \ast G_2}) \cong M^{(c)}(G_1, N_1) \oplus M^{(c)}(G_2, N_2)$, for all $c \geq 1$. \hfill \qed

**Corollary 4.4.**

(i) Let $G_1$ and $G_2$ be two finite groups with $|G^{ab}|, |H^{ab}| = 1$, then for all $c \geq 1$

$$M^{(c)}(G_1 \ast G_2, \langle N_1 \ast N_2 \rangle^{G_1 \ast G_2}) \cong M^{(c)}(G_1, N_1) \oplus M^{(c)}(G_2, N_2).$$

(ii) Let $G_1$ and $G_2$ be two perfect groups such that $M^{(1)}(G_1, N_1) \otimes M^{(1)}(G_2, N_2)$ is trivial, then for all $c \geq 1$

$$M^{(c)}(G_1 \ast G_2, \langle N_1 \ast N_2 \rangle^{G_1 \ast G_2}) \cong M^{(c)}(G_1, N_1) \oplus M^{(c)}(G_2, N_2).$$
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