Geometry of the extreme Kerr black hole

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Abstract

Geometrical properties of the extreme Kerr black holes in the topological sectors of nonextreme and extreme configurations are studied. We find that the Euler characteristic plays an essential role to distinguish these two kinds of extreme black holes. The relationship between the geometrical properties and the intrinsic thermodynamics are investigated.

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I. Introduction

The entropy of the extreme black hole (EBH) has been an intriguing subject of investigations, recently. Based upon the basic difference between the topology of the EBH and non-extreme black hole (NEBH), Hawking et al.[1,2] claimed that the EBH is a different object from its non-extreme counterpart and the Bekenstein-Hawking formula of the entropy fails to describe the entropy of EBH. The EBH must have zero entropy, despite the non-zero area of the event horizon, and can be in thermal equilibrium at arbitrary temperature.

On the other hand, starting from grand canonical ensemble, Zaslavskii [3] argued that a black hole can approach the extreme state as closely as one likes in the topological sector of non-extreme configurations. In so doing, the thermodynamic equilibrium can be fulfilled at every stage of the limiting process and the Bekenstein-Hawking formula of entropy is still valid for the final EBH. To study the geometry of non-extreme Reissner-Nordstrom (RN) black hole near the extreme state, Zaslavskii [4] found that the limiting geometry of the RN black hole depends only on one scale factor and the whole Euclidean manifold is described by the Bertotti-Robinson (BR) spacetime.

The above contradiction seems to imply that the geometric properties, in particular, the spacetime topology, play an essential role in the explanation of intrinsic thermodynamics of the extreme black holes. To exhibit the relationship between the topology and the thermodynamical features of gravitational instantons, many authors [5,6] introduced the Euler characteristic in the new formulation of entropy and found that the Euler characteristic determines the entropy of NEBH directly. They also found that if the EBH satisfies the topological requirement of Hawking et al[1], the Euler characteristic is zero [5]. But for the limiting metric of NEBH near the extreme state suggested by Zaslavskii, the Euler characteristic has not been calculated and the relationship between the entropy and Euler characteristic for BR metric has not been addressed.

In this paper we hope to extend the results of RN black hole to rotating Kerr black hole. We will focus our attentions on the extreme Kerr black hole obtained in the grand canonical ensemble and compare its Euler characteristic, thermodynamical behaviors with that of the original Kerr EBH. We will show the geometrical properties as well as the thermodynamical properties of the Kerr EBH in the topological sectors of non-extreme and extreme configurations are quite different. Since the extreme conditions are both satisfied
for these two configurations, we speculate that perhaps there are two kinds of EBH in the nature.

The organization of this paper is as follows: in Sec.II and Sec.III, we present the geometric properties of the Kerr EBH in the topological sectors of non-extreme and extreme configurations respectively. Sec.IV is devoted to the calculation of the thermodynamical quantities. The discussions and conclusions will be presented in the last section.
II. Extreme black hole with non-extreme topology

The metric of the Kerr black hole reads[7]:

\[
ds^2 = -\frac{\Delta}{\Sigma} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2) d\phi - adt]^2 \\
+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2
\]  

(1)

where

\[
\Delta = r^2 - 2Mr + a^2, a = J/M \tag{2}
\]

\[
\Sigma = r^2 + a^2 \cos^2 \theta \tag{3}
\]

\( J \) and \( M \) are respectively the angular momentum and the mass of the Kerr black hole. It displays an event horizon and Cauchy horizon provided that

\[
a^2 < M^2
\]

and locate at

\[
r_+ = M + \sqrt{M^2 - a^2} \quad \text{and} \quad r_- = M - \sqrt{M^2 - a^2}
\]

respectively. For the extreme case \( M = a \), these two horizons degenerate and only event horizon \( r_+ = M \) exists.

As we will be interested in the metric near the horizon \( r_+ \), it is convenient to redefine the angular variable [4] according to

\[
d\phi = d\phi + \frac{a - a\sqrt{f}}{r^2 + a^2} dt \tag{4}
\]

where

\[
f = \frac{\Delta}{r^2 + a^2} = \frac{(r - r_+)(r - r_-)}{r^2 + a^2} \tag{5}
\]

The metric (1) can be rewritten as

\[
ds^2 = -\frac{r^2 + a^2}{\Sigma} \left[ \sqrt{f} dt - a \sin^2 \theta \sqrt{f} d\phi - \sqrt{\frac{a^2 \sin^2 \theta}{r^2 + a^2}} (1 - \sqrt{f}) dt \right]^2 \\
+ \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2) d\phi - a\sqrt{f} dt]^2 + \frac{\Sigma}{r^2 + a^2} dt^2 + \Sigma d\theta^2
\]

(6)

where \( l \) is the proper distance between \( r_+ \) and \( r \).

Following the general approach for finite-size thermodynamics [8], we consider the grand-canonical ensemble and put the hole into a cavity. The boundary of the cavity is \( r_B \). For the spacetime[3], the equilibrium condition reads

\[
\beta = \beta_0 [f(r_B)]^{1/2}, T_0 = T_H = \frac{f'(r_+)}{4\pi} \tag{7}
\]
As in ref[4], we normalize the time by the condition $t_1 = 2\pi T_0 t$ and choose the coordinate according to

$$r - r_+ = 4\pi T_0 b^{-1} \sinh^2 \frac{x}{2}, b = f''(r_+)/2$$  \hspace{1cm} (8)

In the limit $r_+ \to r_B$, where the hole tends to occupy the entire cavity, the region $r_+ \leq r \leq r_B$ shrinks and we can expend $f(r)$ in a power series $f(r) = 4\pi T_0(r-r_+)+b(r-r_+)^2+\cdots$ near $r = r_+$. After substituting Eqs(7,8) into (6) and taking the extremal limit $r_+ = r_0 = r_B$, we obtain

$$ds^2 = \Sigma_B[-\sinh^2 x dt_1^2+dx^2+d\theta^2]+ \frac{\sin^2 \theta}{\Sigma_B}[(r_B^2+a^2)d\phi-a\sinh x \sqrt{r_B^2+a^2} dt_1]^2$$  \hspace{1cm} (9)

where

$$\Sigma_B = r_B^2 + a^2 \cos^2 \theta, \, dx^2 = \frac{dl^2}{r_B^2 + a^2}$$  \hspace{1cm} (10)

This is an extension of the Bertotti-Robinson (BR) spacetime [9] metric to the case of the limiting form of rotating four-dimensional Kerr black hole. It can easily be seen that this metric has the properties of BR spacetime, namely, nonsingular and static [10]. This is the asymptotic form of the metric and fields near the extremal Kerr black hole horizon. Extending this spacetime to Kerr-Newman black hole is straightforward.

Now we are in a position to discuss the properties of Eq(9). The horizon of the black hole is determined by

$$\Delta = f(r_B^2+a^2) = 0$$  \hspace{1cm} (11)

In the extreme case $T_0 = T_H = \frac{f'(r_+)}{4\pi} = 0$, therefore Eq(11) can be written as

$$\Delta = (r_B^2 + a^2)\frac{f'^2(r_+)}{4} (b^{-1} \sinh^2 x) = (r_B^2 + a^2)^2\frac{f'^2(r_+)}{4} \sinh^2 x = 0$$  \hspace{1cm} (12)

So the horizon can locate at finite $x$, say $x = 0$. The proper distance between the horizon and any other point is finite.

It is of interest to study the topology of this extreme Kerr black hole. Since this EBH is obtained by first taking the boundary condition $r_+ \to r_B$,
and then adopting the extremal limit, the formula of the Euler characteristic[6] can be used directly. We obtain

\[\chi = \frac{Mr_+ (r_+ - M)}{4\pi^2} \int_0^{\beta_0} dt \int_0^{2\pi} d\phi \int_0^\pi \frac{(r_+^2 - 3a^4 \cos^4 \theta)}{(r_+^2 + a^2 \cos^2 \theta)^3} \sin^2 \theta d\theta \mid_{\text{extr}}\]

\[= \frac{2}{\pi} \beta_0 (r_+ - M) \frac{Mr_+}{(r_+^2 + a^2)^2} \mid_{\text{extr}} \]

Taking account of

\[\beta_0 = \frac{1}{T_0} = \frac{4\pi Mr_+}{\sqrt{M^2 - a^2}} \]

We find that even in the extreme case \((M = a)\), \(\chi = 2\). This result is the same as that of the non-extreme Kerr black hole. Therefore if we first take the boundary condition and then let the hole become extreme, we obtain the final extreme Kerr black hole is still in the topological sector of nonextreme configuration.

III. Extreme black hole with extreme topology

Now we turn to concentrate our attention on the original extreme Kerr black hole. This black hole satisfy \(M = a\) from the very beginning. We put it in a cavity with boundary \(r_B\). The metric has the form

\[ds^2 = -\frac{(r - r_+)^2}{\Sigma} \left[dt - a \sin^2 \theta d\phi\right]^2 + \frac{\sin^2 \theta}{\Sigma} \left[(r^2 + a^2) d\phi - adt\right]^2 + \frac{\Sigma}{(r - r_+)^2} dr^2 + \Sigma d\theta^2 \]

(15)

where \(f = \frac{(r - r_+)^2}{r^2 + a^2}\) now. Expanding the metric coefficients near \(r = r_+\) and introducing \(r - r_+ = r_B \rho^{-1}[4]\), one obtains,

\[ds^2 = \Sigma_B \rho^{-2} \left\{-\frac{r_B^2}{\Sigma_B} \left[dt - a \sin^2 \theta d\phi\right]^2 + \frac{\rho^2 \sin^2 \theta}{\Sigma_B} \left[(r_B^2 + a^2) d\phi - adt\right]^2 \right\}
\]

\[+ d\rho^2 + \rho^2 d\theta^2 \]

(16)

in the limit \(r_+ \to r_B\).

By using

\[\Delta = (r_B^2 + a^2)f = (r_B^2 + a^2)\frac{r_B^2 \rho^{-2}}{r_B^2 + a^2} = r_B^2 \rho^{-2} = 0 \]

(17)
to determine the horizon, we find that the horizon locates at infinity $\rho = \infty$. So the distance between the horizon and any other $\rho < \infty$ is infinite. It is this property that gives rise to the qualitatively different topological feature of this black hole from that of Sec.II and plays an important role to determine its Euler characteristic and entropy. The metric of the hole with infinite proper distance does not show any conical structure near its event horizon, so no conical singularity removal is required. It means that $\beta_0$ can not be fixed. Applying the argument in [5,2], this feature will lead unambiguously to $\chi = 0$ for the original extreme Kerr black hole. The topology of the original extreme Kerr black hole differs greatly from the extreme Kerr black hole obtained from its nonextremal counterpart in the grand canonical ensemble.

IV. Thermodynamical properties

By means of the relation between the Euler characteristic and the entropy derived in [6]

$$S = \frac{A}{8} \chi$$

and the different Euler characteristic obtained in Secs.II and III, naturally one can conclude that the extreme Kerr black hole with non-extreme topology has the entropy of $A/4$, while for the black hole with extreme topology zero entropy emerges. These results can also be deduced from the direct thermodynamic study discussed below.

We focus our attention on the extreme Kerr black hole developed from its nonextreme counterpart discussed in Sec.II first.

The temperature on the boundary of the cavity is $T = 1/\beta$. In the grand canonical ensemble, only this temperature has physical meaning. The condition of thermal equilibrium has the form of Eq(7). By setting $r_+ \rightarrow r_B$ at first and imposing the extreme condition afterwards

$$\beta = 2\pi \sinh x_B \sqrt{r_B^2 + a^2}$$

The finite $\beta$ here is similar to that in the RN case[3]. Therefore there exists a well defined, in thermodynamical sense, extreme Kerr black hole state of its non-extreme counterpart in a grand canonical ensemble.

The action for Kerr black hole derived in [8] has the form

$$I = -\frac{1}{4} A_H + \oint_B d^2x \sqrt{\sigma} \left[ \beta \frac{dE}{dA} - (\beta \omega) \frac{dJ}{dA} \right]$$

7
where $E = \frac{1}{8\pi} \oint_B d^2 x \sqrt{\sigma} (k - k^0)$. $k$ is the extrinsic curvature of the boundary embedded into two-dimensional space. $k^0$ is a constant and can be chosen to zero to normalize $E = 0$ in a flat spacetime. Choosing the boundary $B$ as an isothermal surface, then the energy term in $I$ becomes $(\beta E)_B = \text{constant}$ [8].

The free energy

$$F = \frac{I}{\beta} = -\frac{1}{4} A_T T + \oint_B d^2 x \sqrt{\sigma} \left[ \frac{dE_B}{dA} - \omega \frac{dJ}{dA} \right]$$

(21)

For the extreme Kerr black hole developed from the non-extremal Kerr black hole in the grand canonical ensemble, $T \neq 0$. Using the formula $S = -(\frac{\partial F}{\partial T})_D$, where $D$ indicates the thermal quantity, we have

$$S = \frac{A}{4}$$

(22)

But for the original extreme Kerr black hole, even if one let $r_+ \to r_B$ in the end,

$$\beta = \frac{4\pi r_B}{f'(r_+) \sqrt{r_B^2 + a^2 \rho_B}}$$

(23)

On the cavity $\rho_B$ is finite, therefore $\beta$ still diverges because of $1/f'(r_+)$. The temperature detected on the cavity boundary for the original extreme Kerr black hole is zero. Directly using the approach of [8] and Eq(21), we have

$$F = \oint_B d^2 x \sqrt{\sigma} \left[ \frac{dE_B}{dA} - \omega \frac{dJ}{dA} \right]$$

(24)

We note that the free energy is dependent on only the thermodynamic quantity, namely $J$, therefore

$$S = 0$$

(25)

These results are in consistent with the different topological properties of these extreme Kerr black holes.

V. Conclusions and discussions

In this paper we have studied the geometrical properties of the extreme Kerr black hole developed from the non-extreme one and the extreme Kerr black hole at the very beginning respectively. We have shown that there
exists the extreme state of non-extreme Kerr black hole which has the universal form of the limiting metric. From this limiting metric and the Euler characteristic, we found that these two kinds of extreme Kerr black holes are in the different topological sectors, say nonextreme and extreme configurations, respectively. And due to the differences in the spacetime topology of these two kinds of extreme Kerr black holes, the intrinsic thermodynamical properties are quite different. The result obtained here is an extension of that of the spherically symmetrical RN black holes.

Combining the spherical results given in [1-4] and the nonspherical rotating results got above, we have an impression that there are two kinds of extreme black holes which have different topologies ($\chi = 2$ or zero) and different thermodynamical properties, ($S = A/4$ or zero) in the nature.

This conclusion affects not only the understanding of the geometry and thermodynamics of EBH, but also the phase transition of the black hole. It has been shown by many authors [11-14] that a phase transition exists for the Kerr black holes at the extreme limit. The transitional point is a critical point and the critical behavior can be described by various critical components satisfying the scaling law[13]. As was argued by Hawking et al [1], one should regard NEBH and EBH with extreme topological configuration ($\chi = 0$) as qualitatively different objects and a NEBH cannot be turned to this kind of EBH. But as shown in Secs.II and IV, a NEBH can be transformed to an EBH with nonextreme topological configuration ($\chi = 2$). At the extreme limit, a phase transition happens. Since the entropy changes continuously from NEBH to EBH with $\chi = 2$, we come to a conclusion that this is a second order phase transition. This result is in consistent with previous studies [11,13].

It is widely believed that black holes retain only very limited information about the matter that collapsed to form them. This information is reflected in the number of parameters characterizing the black hole. For Kerr black hole, such parameters are the mass $M$ and angular momentum $J$, and the properties of the hole are completely determined by these parameters. This is known as the “no hair theorem”. While the spherical results in[1-4] and the nonspherical results obtained in our paper suggested that the no hair theorem is violated. A topological hair should be introduced, at least in the extreme cases, to describe two kinds of extreme black hole with profoundly different properties. This is another challenge to the “no hair theorem” besides those proposed in [15,16].
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