Trading Permutation Invariance for Communication in Multi-Party Non-Locality Distillation

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Abstract—Quantum theory puts forward phenomena unexplainable by classical physics—or information, for that matter. A prominent example is non-locality. Non-local correlations cannot be explained, in classical terms, by shared information but only by communication. On the other hand, the phenomenon does not allow for (potentially faster-than-light) message transmission. The fact that some non-local and non-signaling correlations are predicted by quantum theory, whereas others fail to be, asks for a criterion, as simple as possible, that characterizes which joint input-output behaviors are “quantum” and which are not. In the context of the derivation of such criteria, it is of central importance to understand when non-local correlations can be amplified by a non-interactive protocol, i.e., whether some types of weak non-locality can be distilled into stronger by local operations. Since it has been recognized that the searched-for criteria must inherently be multi-partite, the question of distillation, extensively studied and understood two-party scenarios, should be addressed in the multi-user setting, where much less is known. Considering the space of intrinsically n-partite correlations, we show the possibility of distilling weak non-local boxes to the algebraically maximal ones without any communication. Our protocols improve on previously known methods which still required partial communication. The price we have to pay for dropping the need for communication entirely is the assumption of permutation invariance: Any correlation that can be realized between some set of players is possible between any such set. This assumption is very natural since the laws of physics are invariant under spacial translation.

I. MOTIVATION AND OUTLINE

Einstein, Podolsky, and Rosen [1] raised the question “Can quantum-mechanical description of physical reality be considered complete?” In direct response to that question, Bell [2] showed that quantum mechanics is incompatible with a local hidden variable theory: The theory predicts correlations that are, in classical terms, not explainable by shared information but only by communication. It is important to note, however, that, on the other hand, the arising correlations do not allow for message transmission.

With the goal of a systematic and generalized understanding of non-local correlations, quantum and beyond, Popescu and Rohrlich described an input-output behavior that maximally violates the Bell-inequality, but that still fulfills the non-signaling condition [3]. Their bipartite input-output behavior or box can classically be realized with 75% only, by quantum states with 85% [4], whereas even the perfect approximation would still be compatible with the non-signaling principle. This means that, strangely enough, quantum physics is not maximally non-local and cannot be singled out by the non-signaling principle. It is a fascinating and conceptually important question whether there is an (information-theoretical) principle that is able to describe exactly the quantum correlations. One such attempt has been to generalize the non-signaling principle to parties that are allowed to use limited communication, to the so-called information-causality principle [5]. Other authors have looked for principles characterizing quantum correlations as the ones that do not improve the efficiency of non-local computation [6] or do not collapse communication complexity [7]. Two more physically motivated principles are macroscopic locality [8] and local orthogonality [9]. In each case, it has been shown that quantum physics respects the corresponding principle, whereas some “super-quantum” correlations violate it. For none of the principles, however, it was possible to show that every non-quantum behavior is in violation.

In the search of a principle exactly singling out quantum theory, the possibility of making (weak) non-local correlations stronger by local wirings is paramount since it offers the possibility of generating systems violating some principle from correlations which respect it. Therefore, a systematic understanding of the power and limitations of distillation of non-locality potentially leads to deep insights into the mysterious nature of quantum theory.

The question of non-locality distillation was mainly studied in the bipartite scenario: There exist weak non-local correlations that can be distilled to an almost perfect Popescu-Rohrlich box by an adaptive protocol [10], [11], [12]. On the other hand, isotropic correlations seem to be undistillable [13].

In the context both of information principles able to single out quantum theory [14] as well as for information-processing tasks such as randomness generation [15], it has turned out that multi-party, as opposed to only bipartite correlations, play a crucial role. Nevertheless, much less is known for that case. One effort was to generalize the XOR protocol [16], but it fails to distill maximal non-local boxes. It was shown in [17], [18] that the large class of full-correlation boxes can be distilled by a multipartite version of Brunner-Skrzypczyk’s protocol [11] under the (strong) assumption that partial communication is allowed. This latter assumption, unfortunately, puts into question the relevance of the protocol in the context described above, namely of finding information-based criteria singling...
out quantum theory.

We introduce a new kind of multi-party distillation protocols by showing that the need for communication can be dropped entirely. The price for this is the need for the assumption that any correlation which can be realized between some set of players is also possible between any other set. We believe this assumption to be quite natural since we imagine the correlations to arise from the interaction with a concrete physical system. In this sense, the assumption is true in every world in which the laws of physics are invariant under spacial translation.

This is an outline of the present article. We first characterize full-correlation boxes and give for them a criterion for being maximally non-local (Section III). Second, we present a new distillation protocol that does not use partial communication, but by shared randomness only. This kind of boxes has the property that it displays correlation only with respect to the full set of parties. An n-partite full-correlation box takes as inputs \( x = (x_1, x_2, \ldots, x_n) \) and as outputs \( a = (a_1, a_2, \ldots, a_n) \), where all \( x_i, a_i \in \{0, 1\} \). The input-output behavior is characterized by the following conditional distribution:

\[
P(a|x) = \begin{cases} \frac{1}{2^n} \sum_{i} a_i \equiv f(x) \pmod{2} & \text{otherwise}, \\
0 & \text{otherwise},
\end{cases}
\]

where \( f(x) \) is a Boolean function of the inputs.

A special case of the full-correlation boxes is the n-party generalization of the Popescu-Rohrlich box \(n\)-PR box that is defined by the conditional distribution:

\[
P^{PR}_{n}(a|x) = \begin{cases} \frac{1}{2^n} \bigoplus_{i=1} n a_i = \prod_{i=1}^k x_i & \text{otherwise}, \\
0 & \text{otherwise},
\end{cases}
\]

Note that an n-PR box and an \((n,n)\)-PR box are identical.

### III. Full-Correlation Boxes

#### A. Definition and Related Boxes

We focus our attention to one of the most general type of boxes: the \(n\)-partite full-correlation boxes that were introduced by Barrett and Pironio \[19\]. This kind of boxes has the property that it displays correlation only with respect to the full set of parties. An \(n\)-partite full-correlation box takes as inputs \( x = (x_1, x_2, \ldots, x_n) \) and as outputs \( a = (a_1, a_2, \ldots, a_n) \), where all \( x_i, a_i \in \{0, 1\} \). The input-output behavior is characterized by the following conditional distribution:

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#### B. Construction of n-Partite PR Boxes

In \[20\] is shown how a 3-PR box can be constructed from three PR boxes. This construction can be used to construct an \(n\)-PR recursively: Assume that the first \(n-1\) – 1 parties share a \((n-1)\)-PR box and every of this parties input their input \( x_i \) to this box. The outputs (say \( a'_i \)) of the box fulfills

\[
i = 1^{n-1} a'_i = \bigwedge_{i=1}^{n-1} x_i.
\]

Every of these \(n-1\) parties inputs the output bit in the PR box shared with the \(n\)th party and the \(n\)th party inputs in every PR box his input bit. The output bit of the \(n\)th party is the XOR of all his outputs from the PR boxes (see Fig. 1). In the end, the outputs fulfill

\[
\bigoplus_{i=1}^n a_i = \bigoplus_{i=1}^{n-1} a_i \oplus a_n = \bigoplus_{i=1}^{n-1} (a_i \oplus b_i) = \bigoplus_{i=1}^{n-1} x_i \land a'_i = \bigwedge_{i=1}^n x_i.
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#### B. Definition

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\]
simulated by local operations and shared randomness.

C. Construction of Full-Correlation Boxes

In the same way as in [17, 18], we look how full-correlation boxes can be constructed by generalized PR-boxes and how they can be characterized.

Lemma 1 If \( f \) is a Boolean function of the input elements \( x_1, x_2, ..., x_n \), then it can be written as

\[
f(x_1, ..., x_n) = \bigoplus_{I \in \mathcal{I}} (a_I \cdot \bigwedge_{i \in I} x_i),
\]

where \( \mathcal{I} = \mathcal{P}(\{1, 2, ..., n\}) \) and \( a_I \in \{0, 1\} \) for all \( I \in \mathcal{I} \).

Hence, it is obvious that the full-correlation box associated to the Boolean function \( f \) can be constructed by \( \sum_{I \in \mathcal{I}} a_I \) n-PR boxes. For an example, see Fig. 2. Note that the \( n \)-PR boxes belonging to an \( a_I \) where \( |I| \leq 1 \) are local and can be simulated by local operations and shared randomness.

\[
\begin{align*}
x & \quad y & \quad z \\
1 & \oplus & x \cdot y & \oplus & x \cdot z \\
a & \quad b & \quad c
\end{align*}
\]

Figure 2. Construction of the \( 1 \oplus xy \oplus xz \)-Box

We define the set of all non-local \( n \)-PR boxes that are needed to simulate the full-correlation box: Let

\[
\mathcal{J} := \{ I \in \mathcal{I} | a_I = 1 \text{ and } |I| \geq 2 \}.
\]

This set can be partitioned into pairwisely disjoint subsets \( \{J_1, J_2, ..., J_{|\mathcal{J}|}\} \) such that all \( A \in J_i \) and \( B \in J_j \) fulfill \( A \cap B = \emptyset \) for all \( i \neq j \). We define the maximal number of such subsets as \( n_{\mathcal{J}} \) and denote this partition as the empty-overlap partition of \( \mathcal{J} \).

D. Non-Locality of Full-Correlation Boxes

We write the Boolean function that characterizes a full-correlation box as in Lemma 1 so it is easy to determine if this box is local or not.

It is obvious that a full-correlation box is local if the associated Boolean function can be written as the XOR of single inputs of the box and a constant. Assume that the function consists of at least one AND-term, then this box can be reduced to a PR box by distributing all input-and output-interfaces only to two parties such that both of them get at least one input that belongs to the AND-term. Therefore, a full-correlation box is local if and only if the Boolean function can be written as the XOR of a constant and single inputs of the box

\[
f(x_1, ..., x_n) = \bigoplus_{I \in \mathcal{L}} (a_I \cdot \bigwedge_{i \in I} x_i),
\]

where \( \mathcal{L} = \{\emptyset, \{x_1\}, \{x_2\}, ..., \{x_n\}\} \) and \( a_I \in \{0, 1\} \) for all \( I \in \mathcal{L} \).

We show that for every non-local full-correlation box, there exists a closest local box (measured in the \( L^1 \)-norm) that is also a full-correlation box. Let \( P \) be the joint probability distribution of a non-local full-correlation box. Then the joint probability distribution of the closest local full-correlation box \( P^* \) is defined by

\[
\parallel P - P^* \parallel_1 = \min_{P' \text{ loc. full-corr. box}} (\parallel P - P' \parallel_1).
\]

Lemma 2 Let \( P \) be the joint probability distribution of an \( n \)-partite full-correlation box. Then the closest local full-correlation box is one of the closest local boxes. That means

\[
\parallel P - P^* \parallel_1 = \min_{P' \in \mathcal{P}_{loc. full-corr. box}} (\parallel P - P' \parallel_1).
\]

for some random variable \( \mathcal{R} \).

Proof: It is obvious that every deterministic local strategy \( r \in \mathcal{R} \) can achieve at most the same number of input-output behaviors (XOR of the outputs equal to a Boolean function of the inputs) as the closest local full-correlation box. So every local box (that is a convex combination of these deterministic local strategies) has at least the same distance from the given full-correlation box as the closest local full-correlation box.

E. Extremal Boxes of the Non-Signaling Polytope

It is a well-known fact that all full-correlation boxes are non-signaling, since they can be simulated by PR boxes [19]. In [21], all tripartite extremal boxes of the non-signaling polytope have been characterized, but for more parties it is not known which of the (full-correlation) boxes are extremal.

Theorem 1 (Extremal Full-Correlation Boxes) Let \( P \) be an \( n \)-partite full-correlation box associated to the Boolean function \( f \) that depends on \( k \) input variables. Then \( P \) is
an extremal box of the non-signaling polytope if and only if \( n \neq 1 \) and \( k = n \) hold.

Theorem 1 follows from Lemmas 3, 4, 5, and 6.

**Lemma 3** Let \( P \) be an \( n \)-partite full-correlation box with associated function \( f \) that depends on \( k \) input variables. If \( k \neq n \), then \( P \) is not an extremal box.

**Proof:** \( P \) can be written as a convex combination of the following two non-signaling boxes:

\[
P^1(a|x) = \begin{cases} \frac{1}{2^m} \sum_{i=1}^{m} a_i = f(x_1, x_2, \ldots, x_m) \\ \text{and} \sum_{i=m+1}^{n} a_i = 0 \end{cases}
\]

(14)

and

\[
P^2(a|x) = \begin{cases} \frac{1}{2^{m+1}} \sum_{i=1}^{m} a_i = 1 \oplus f(x_1, x_2, \ldots, x_m) \\ \text{and} \sum_{i=m+1}^{n} a_i = 1 \end{cases}
\]

(15)

So \( P = \frac{1}{2}P^1 + \frac{1}{2}P^2 \). Therefore, \( P \) is not an extremal box of the non-signaling polytope.

**Lemma 4** Let \( P \) be an \( n \)-partite full-correlation box with associated function \( f \) that depends on \( k \) input variables. Let \( k = n \). If \( n \neq 2 \), then \( P \) is not extremal.

**Proof:** Since \( n \neq 2 \), we are able to split the Boolean function \( f \) in two other Boolean functions, \( f_1 \) and \( f_2 \), such that they do not depend on the same input variables. Without loss of generality, we assume that \( f_1 \) depends on the input variables \( x_1, x_2, \ldots, x_m \) and \( f_2 \) depends on \( x_{m+1}, \ldots, x_n \). So the box \( P \) can be written as a convex combination of the following two boxes:

\[
P^1(a|x) = \begin{cases} \frac{1}{2^m} \sum_{i=1}^{m} a_i = f_1(x_1, x_2, \ldots, x_m) \\ \text{and} \sum_{i=m+1}^{n} a_i = f_2(x_{m+1}, \ldots, x_n) \end{cases}
\]

(16)

and

\[
P^2(a|x) = \begin{cases} \frac{1}{2^{m+1}} \sum_{i=1}^{m} a_i = \neg f_1(x_1, x_2, \ldots, x_m) \\ \text{and} \sum_{i=m+1}^{n} a_i = \neg f_2(x_{m+1}, \ldots, x_n) \end{cases}
\]

(17)

So \( P = \frac{1}{2}P^1 + \frac{1}{2}P^2 \). Therefore, \( P \) is not an extremal box of the non-signaling polytope.

**Lemma 5 (Existence)** Every \( n \)-PR box is extremal.

**Proof:** The proof is based on the same argument as in [22] for showing that any non-locality implies some secrecy.

Assume that the \( n \)-PR box \( P \) can be written as a convex combination of two other non-signaling boxes \( P^1 \) and \( P^2 \),

\[
P = \epsilon P^1 + (1 - \epsilon)P^2,
\]

where \( 0 < \epsilon < 1 \). It is obvious that both of the boxes must fulfill that the XOR of their output elements is equal to the AND of their input elements, i.e.,

\[
\text{Prob} \left[ \bigoplus_{i=1}^{n} A_i = \prod_{i=1}^{n} X_i \mid X_i = x_i \forall 1 \leq i \leq n \right] = 1
\]

(19)

for all input elements \( x_i \in \{0, 1\} \). We will show that all possible biases, \( p_i := \text{Prob} [A_i = 0] = X_k = 0 \) for all \( k \) such that \( 1 \leq i \leq n \) such that the box is non-signaling, must be \( p_i = 1/2 \). Therefore, \( P \) cannot be written as a convex combination of other non-signaling boxes.

Assume without loss of generality that all \( p_i \geq 1/2 \) for all \( 1 \leq i \leq n - 1 \). Because of Equation (19), the bias \( p_n \) can be computed from the biases \( p_i \) for \( i \in \{1, 2, \ldots, n - 1\} \).

Since our box is non-signaling, all biases are independent of the other parties’ inputs. We determine step by step the biases \( p_i' := \text{Prob} [A_i = 0] = x_k = 1 \) for all \( n \) and get that \( p_i' = p_i \). If not all biases are \( 1/2 \), then this is a contradiction to Equation (19) for the input \( (1, 1, \ldots, 1) \).

**Lemma 6** Let \( P^1 \) and \( P^2 \) be extremal \( m \) and \( k \)-partite full-correlation boxes with associated functions \( f_1 \) and \( f_2 \), where \( f_1 \) depends on the input variables \( x_1, x_2, \ldots, x_m \) and \( f_2 \) depends on \( x_{m+1}, \ldots, x_{m+k-1} \) (\( l \leq m \)). Then the box \( P \) with associated function

\[
f(x_1, \ldots, x_{l+k-1}) = f_1(x_1, \ldots, x_m) \oplus f_2(x_m, \ldots, x_{l+k-1})
\]

(20)

is also extremal.

**Proof:** We assume that the box with associated function \( f \) can be written as a convex combination of two other non-signaling boxes \( P_1 \) and \( P_2 \),

\[
P = \epsilon P_1 + (1 - \epsilon)P_2,
\]

where \( 0 < \epsilon < 1 \). As before, it is obvious that both of the boxes must fulfill that the XOR of their output elements is equal to the XOR of the Boolean functions \( f_1 \) and \( f_2 \). Therefore, we define \( f(X_1, \ldots, X_{l+k+1}) = f_1(X_1, \ldots, X_m) \oplus f_2(X_m, \ldots, X_{l+k+1}) \). We have

\[
\text{Prob} \left[ \bigoplus_{i=1}^{n} A_i = f(X_1, \ldots, X_{l+k+1}) \mid X_i = x_i \forall i \right] = 1
\]

(22)

for all input elements \( x_i \in \{0, 1\} \).

Assume that all parties \( i \in \{x_m, \ldots, x_{l+k-1}\} \) input 0 to the box. Therefore, the box acts like the box \( P_1 \) (assume that the parties \( m \) to \( k+l-1 \) are the same or are able to communicate to each other), and we have found a convex combination of this box. This is in contradiction to the assumption that \( P_1 \) is extremal. Therefore, the new box is also extremal.

Note that the \( n \)-partite full-correlation box associated to the function \( f(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i \oplus x_1 \) is also an extremal box since it can be constructed with an \( n \)-PR and an \((n-1)\)-PR box by flipping the input bit \( x_1 \).
IV. DISTILLATION OF FULL-CORRELATION BOXES

We introduce a new noncommunicative protocol for distillation which requires the parties to arbitrarily distribute the input-and output-interfaces of the weak boxes between the parties. Therefore, the parties have no longer a fixed access to the box, it is even possible that one party has no access to a box, but another one has multiple ones.

A. Distilling n-Partite PR Boxes

Using the generalization of the Brunner-Skrzypczyk protocol [11] that were presented in [17], [18] we are able to distill imperfect n-partite PR boxes $P_{n, \varepsilon}^{PR}$, where

$$P_{n, \varepsilon}^{PR} = \varepsilon P_{n}^{PR} + (1 - \varepsilon) P_{n}^{C},$$

and

$$P_{n}(a|x) = \begin{cases} \frac{1}{n} \sum_{i=1}^{n} a_i = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Protocol 1 (Gen. BS Protocol for n-PR Boxes [17], [18])

All $n$ parties share two boxes, where we denote by $x_i$ the value that the $i$th party inputs to the first box and by $y_i$ the value that the $i$th party inputs to the second box. The output bit of the first box for the $i$th party is $a_i$, and the output bit of the second box is $b_i$. The $n$ parties proceed as follows: $y_i = x_i \overline{a_i}$ and they output, finally, $c_i = a_i \oplus b_i$ (see also Fig. 3).

Figure 3. Generalized BS Protocol for n-PR boxes

Theorem 2 The generalized BS protocol takes two copies of an arbitrary box $P_{n, \varepsilon}^{PR}$ with $0 < \varepsilon < 1$ to an $n$-partite correlated non-local box $P_{n, \varepsilon}^{PR}$ with $\varepsilon' > \varepsilon$, i.e., is distilling non-locality. In the asymptotic case of many copies, any $P_{n, \varepsilon}^{PR}$ with $0 < \varepsilon$ is distilled arbitrarily closely to the n-PR box.

B. Equivalence Between n-and (n, k)-PR Boxes

We show that $k$-PR boxes and $(n, k)$-PR boxes are equivalent in the sense that using one of the boxes and shared randomness, the other box can be simulated and vice versa. This property is transitive.

Assume $k$ parties share a $k$-PR box, where each of the parties has an input and an output. All of these $k$ parties share a random variable with $n - k$ additional parties. This random variable helps to create a local distribution between the parties that has the property that the XOR of all $n$ outputs is zero. Combining these two distributions with an XOR, the new distribution correspond the distribution of the $(n, k)$-PR box.

To see the opposite implication, let $k$ parties share a $(n, k)$-PR box, where every party has an input interface that has influence on the distribution in the end, and the corresponding output. The left inputs (and the corresponding outputs) can be distributed to arbitrary parties. In the end, these parties have to take the XOR of their output to get their final output. Therefore, the $k$ parties simulate a $k$-PR box.

C. Distillation of (n, k)-PR Boxes

Using the generalized BS Protocol we are able to distill imperfect $(n, k)$-PR boxes $P_{(n, k), \varepsilon}^{PR}$, where

$$P_{(n, k), \varepsilon}^{PR} = \varepsilon P_{(n, k)}^{PR} + (1 - \varepsilon) P_{n}^{C}.$$  

With the same construction as we showed the equivalence between $n$-and $(n, k)$-PR boxes, we are able to show equivalence between $P_{(n, k), \varepsilon}^{PR}$ and $P_{(n, k)}^{PR}$. Because of Theorem 2 we know that $P_{(n, k), \varepsilon}^{PR}$ can be distilled arbitrarily closely to $P_{(n, k)}^{PR}$. Again, $P_{(n, k), \varepsilon}^{PR}$ is equivalent to $P_{(n, k), \varepsilon}^{PR}$. Therefore, $P_{(n, k), \varepsilon}^{PR}$ can be distilled arbitrarily closely to $P_{(n, k)}^{PR}$.

D. Distillation of Full-Correlation Boxes

As shown in Lemma 1 every (imperfect) $n$-partite full-correlation box can be simulated by (imperfect) $n$-PR boxes, where some parties input the constant one. These boxes are exactly the $(n, k)$-PR boxes. If we could isolate each of these (imperfect) $(n, k)$-PR boxes, then we could distill each of them to almost perfect, and so, the whole full-correlation box can be distilled to almost perfect.

Let us assume that $P$ is the full-correlation box, $P^*$ the closest local full-correlation box, and $\varepsilon$ a parameter between 0 and 1. We show that every convex combination

$$P_{\varepsilon} = \varepsilon P + (1 - \varepsilon) P^*$$

can be distilled arbitrarily closely to the full-correlation box $P$. Without loss of generality, assume that $P$ has no local part, since we change only the local part of the box that can be reached by a local strategy. That implies that $P^* = P_{n, \varepsilon}^{PR}$.

An $(n, k)$-PR box belonging to $a_I$, $I \subseteq J$, can be isolated from the full-correlation box, if there exist no $J \subseteq J$ such that $J \subseteq I$. Then, the box can be isolated when every party $i \notin J$ inputs 0 to the full-correlation box. Otherwise, the box cannot be isolated in this way, but there exist two other possibilities: Assume there is an $(n, k)$-PR box belonging to $a_I$ and an $(n, l)$-PR box $(k > l)$ belonging to $a_J$, $J \subset I$.

1) If there exists an $(n, m)$-PR box with $m \geq k$ that can be isolated, then we replace our box with it.
Therefore, every imperfect full-correlation box can be distilled in this way and we are also in the multipartite case able to distill boxes that are close to the local bound to maximal non-local ones.

V. EXAMPLE

In this example we distill the following full-correlation box:

\[
P^1(a|x) = \begin{cases} 
\frac{4}{n} \sum_{i=1}^{4} a_i = x_1 x_2 x_3 x_4 \oplus x_1 x_2 x_3 \oplus x_3 x_4 & \text{if } x_1, x_2, x_3 \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]  

The closest local box is \( P^1 \). Let us assume that we have imperfect boxes that are close to the local bound

\[
P^* = \epsilon P^1 + (1-\epsilon) P^n.
\]

First, we isolate the tripartite PR box between the first three parties. Therefore, the first two parties take the first two inputs and the corresponding outputs of the box, and the third party takes the third and fourth inputs and outputs of the box. Into the fourth input, the third party inputs 0 (see Fig. 4). We apply the same method to isolate the bipartite PR box. As soon as the boxes are isolated, they can be distilled to almost perfect.

VI. CONCLUSION

We have studied the problem of non-locality distillation in the multipartite setting, where the parties are not allowed to communicate with each other. First, we characterized maximally non-local full-correlation boxes and showed that for every full-correlation box, there exists a closest local box which is also a full-correlation box. Second, based on the generalized Brunner/Skrzypczyk protocol, we showed that every full-correlation box can be distilled to almost perfect without communication if the interfaces of the box can be arbitrarily distributed by the parties. This implies that it is possible in the multipartite case to distill boxes that are arbitrarily close to the local bound to boxes that are maximally non-local. It remains an open question to classify and find distillation protocols for multipartite non-local boxes that are not full-correlation boxes.

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REFERENCES

[1] A. Einstein, B. Podolsky, and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete?” Physical Review, vol. 47, pp. 777–780, May 1935.

[2] J. S. Bell, “On the Einstein-Podolsky-Rosen paradox,” Physics, vol. 1, 1964.

[3] S. Popescu and D. Rohrlich, “Quantum nonlocality as an axiom,” Foundations of Physics, vol. 24, Mar 1994.

[4] B. S. Cirel’son, “Quantum generalizations of Bell’s inequality,” Letters in Mathematical Physics, vol. 4, Mar 1980.

[5] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, “Information causality as a physical principle,” Nature, vol. 461, no. 7267, pp. 1101–1104, 2009.

[6] N. Linden, S. Popescu, A. J. Short, and A. Winter, “Quantum nonlocality and beyond: limits from nonlocal computation,” Physical review letters, vol. 99, no. 18, pp. 180502–180502, 2007.

[7] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, “Limit on nonlocality in any world in which communication complexity is not trivial,” Physical Review Letters, vol. 96, no. 25, p. 250401, 2006.

[8] M. Navascués and H. Wunderlich, “A glance beyond the quantum model,” Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, vol. 466, no. 2115, pp. 881–890, 2010.

[9] T. Fritz, A. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acín, “Local orthogonality: a multipartite principle for correlations,” arXiv preprint arXiv:1210.3018, 2012.

[10] M. Forster and P. Skrzypczyk, “Nonlocality distillation and postquantum theories with trivial communication complexity,” Physical Review Letters, vol. 102, p. 120401, Mar 2009.

[11] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, “Limit on nonlocality in any world in which communication complexity is not trivial,” Physical Review Letters, vol. 96, no. 25, p. 250401, 2006.

[12] P. Høyer and J. Rashid, “Optimal protocols for nonlocality distillation,” Physical Review A, vol. 82, no. 4, p. 042118, 2010.

[13] D. D. Dukaric and S. Wolf, “A limit on nonlocality-distillation,” arXiv preprint arXiv:0808.3317, 2008.

[14] R. Gallego, L. Massanes, G. De La Torre, C. Dícura, L. Aolita, and A. Acín, “Full randomness from arbitrarily deterministic events,” arXiv preprint arXiv:1210.6514, 2012.

[15] L.-Y. Hsu and K.-S. Wu, “Multipartite nonlocality distillation,” Physical Review A, vol. 82, no. 5, p. 052102, 2010.

[16] H. Ebbe and S. Wolf, “Distillation of multi-party non-locality with and without partial communication,” Information Theory Proceedings (ISIT), pp. 739–743, 2013.

[17] ——, “Multi-user non-locality amplification,” arXiv preprint arXiv:1307.7927, 2013.

[18] J. Barrett, S. Popescu and D. Roberts, “Popescu-Rohrlich correlations as a unit of nonlocality,” Physical Review Letters, vol. 95, p. 140401, Sep 2005.

[19] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, “Nonlocal correlations as an information-theoretic resource,” Physical Review A, vol. 71, p. 022101, Feb 2005.

[20] S. Pironio, J.-D. Bancal, and V. Scarani, “Extremal correlations of the tripartite no-signaling polytope,” Journal of Physics A: Mathematical and Theoretical, vol. 44, p. 065303, Jan 2011.
[22] E. Hänggi, R. Renner, and S. Wolf, “Quantum cryptography based solely on Bell’s theorem,” EUROCRYPT 2010, Jan 2010.