Cumulants and the moment algebra: tools for analysing weak measurements

Johan Åberg\textsuperscript{1,2} and Graeme Mitchison\textsuperscript{3,†}

\textsuperscript{1}Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland
\textsuperscript{2}Communication Technology Laboratory, ETH Zurich, 8092 Zurich, Switzerland
\textsuperscript{3}Centre for Quantum Computation, DAMTP, University of Cambridge, Cambridge CB3 0WA, UK

Recently it has been shown that cumulants significantly simplify the analysis of multipartite weak measurements. Here we consider the mathematical structure that underlies this, and find that it can be formulated in terms of what we call the moment algebra. Apart from resulting in simpler proofs, the flexibility of this structure allows generalizations of the original results to a number of weak measurement scenarios, including one where the weakly interacting pointers reach thermal equilibrium with the probed system.

I. INTRODUCTION

For many readers, the word “cumulant”, if it means anything, probably evokes a slight feeling of discomfort: a recollection, perhaps, of a baffling definition and a paper left half-read. Yet, as we hope to show here, cumulants should have pleasurable associations. They arise as part of an algebraic structure, the moment algebra, that can be defined very simply yet has striking properties. It has the familiar operations of complex analysis – a multiplication and inverse, functions like log and exp, a derivative operation, power series expansions, etc. – but all transposed into a very different setting, with functions defined on a lattice of finite subsets instead of the continuum of a complex space, and with curious-looking new definitions for the functions. In the moment algebra, the cumulant is just the log function, though many textbooks do an impeccable job of concealing this fact.

Cumulants have a long history, with roots in statistics. They were probably first considered by Thorwald N. Thiele\textsuperscript{1,2} as “half-invariants”; they then went through a protean sequence of name changes\textsuperscript{3, 4, 5} until the current name finally stuck. Since they are tools of statistics and probability theory\textsuperscript{6}, it is perhaps unsurprising that they have been applied in statistical mechanics\textsuperscript{5, 8, 9, 10}, notably for the calculation of virial coefficients and perturbation expansions of the free energy, as well as in solid state physics, quantum chemistry, and quantum field theory (see e.g.,\textsuperscript{11, 12, 13, 14}). Other studies have focused more directly on the cumulants as, in some sense, genuine multipartite correlation measures, both in a classical setting\textsuperscript{15}, and a quantum setting\textsuperscript{16}.

The second component of this paper is weak measurement\textsuperscript{17, 18}. This is a way of obtaining information about a system while perturbing it only a little, by coupling the system weakly to a pointer. The imprecision of the measurement outcome is compensated for by running many repeats of the protocol, each time with a freshly prepared system. The measurement results obtained this way are often surprising\textsuperscript{19}, and give insight into the underlying physics, as in the analysis\textsuperscript{20} of Hardy’s paradox\textsuperscript{21}. Other examples include phenomena in fiber optics\textsuperscript{22} and photonic crystals\textsuperscript{23}.

Here we consider multipartite weak measurements, by which we mean measurements involving more than one pointer\textsuperscript{24, 25, 26, 27}. The moments of pointer observables (i.e. the expectations of products of those observables) turn out to depend in an extremely complicated way on weak values. Despite this, it has recently been shown that the cumulants of pointer moments are very simply related to cumulants of weak values\textsuperscript{28}. This suggests that the role cumulants play in simplifying perturbation expansions in statistical mechanics may have an analogue in weak measurement. Note that, given the weak value cumulants, one may if one wishes obtain the weak values themselves by the inverse operation to the cumulant (the exponential in the moment algebra); this gives an operational procedure for computing weak values.

With the help of the moment algebra, we uncover some of the mathematical structure that underlies the favourable interplay between cumulants and weak measurements. We also show that these results can be generalized by relaxing some of the assumptions behind weak measurement. For instance, we consider a situation where weak measurements are performed over an extended period during which the system undergoes continuous evolution. This is related, by what can be broadly described as an imaginary time transformation, to another scenario where the pointers and the
system correlate via thermalization; we call this “thermal weak measurement”. Thus in general we get a broader view of weak measurement, together with some tools for making its analysis more tractable.

II. THE MOMENT ALGEBRA AND CUMULANTS

Here we introduce what we call the moment algebra. This can be regarded as a natural setting for discussions about cumulants, and provides a handy formalism for the proofs in the rest of this paper.

Let Ωn be the set of integers 1, 2, ..., n, for some n > 0. Let Mn denote the set of functions that assign a complex number f(a) ∈ ℂ to every subset a of Ωn, including the empty set ∅. We will refer to such an f as an M-map.

If f and g are two M-maps in Mn, we can define their product fg simply as the new M-map [fg](a) = f(a)g(a), for any subset a of Ωn. However, there is another product, the convolution product f * g, which has particularly interesting properties. If a = {a1, ..., ak}, let ∂a f denote the formal derivative (∂/∂ξa1 ... ∂/∂ξak)f, where the ξi are notional variables that we never deal with explicitly. We now define

\[(f * g)(a) = ∂a(fg),\]  (1)

where the right hand side is interpreted as follows: In the standard expression for the derivative of the product fg, we make the replacements ∂b f → f(b) and ∂b g → g(b) for any subset b ⊂ a; i.e. we replace the derivative ∂b f by the value f(b) of the M-map f on b, and similarly with ∂b g. We also replace plain f by f(∅). As an example, suppose a = {1, 2}. Then we have

\[∂1,2(fg) = (∂1f)(g) + (∂2f)(∂1g) + (∂2g)(∂1f) + f(∂1,2g),\]  (2)

and after replacing the derivatives we obtain the convolution product evaluated at a = {1, 2},

\[(f * g)(1,2) = f(1,2)g(∅) + f(1)g(2) + f(2)g(1) + f(∅)g(1,2).\]  (3)

In general, (1) gives the explicit rule

\[(f * g)(a) = ∑_{a1∪a2=a} f(a1)g(a2)\]  (4)

for any subset a ≠ ∅, where the sum runs over all ordered bipartitions (a1, a2) of a, including (∅, a) and (a, ∅) (treated as distinct). In the case a = ∅ we find (f * g)(∅) = f(∅)g(∅). The vector space of M-maps, Mn, together with the convolution product becomes an algebra, which we also denote by Mn and call the moment algebra.

The terms “moment” and “convolution” are inspired by the following construction. Let φ(x1, ..., xn) be a complex valued integrable function of n complex variables. Define an M-map by

\[f_φ(a) = ∫ x_{a_k} ... x_{a_1} φ(x_1, ..., x_n)dx_1 ... dx_n,\]  (5)

for each subset a = {ak, ..., a1}. Hence, the M-map f_φ assigns to each subset a the moment of the function φ corresponding to a. If we now choose another function ψ, then f_φ * f_ψ gives moments of the usual convolution φ∗ψ(y1, ..., yn) = ∫ φ(x1, ..., xn)ψ(y1 - x1, ..., yn - xn)dx1 ... dxn. We can write this succinctly as f_φ * f_ψ = f_φ∗ψ, where the subscript ’φ∗ψ’ is the usual convolution product. In fact, any M-map can be represented by an f_φ in this way, for some (non-unique) φ, but we do not make use of this, and proceed entirely within our abstract algebra framework.

There is an identity, 1∗ in Mn, defined by

\[1∗(∅) = 1, \quad 1∗(a) = 0 \text{ for } a ≠ ∅.\]  (6)

We have

\[f * 1∗ = 1∗ * f = f.\]  (7)

Given a sufficiently differentiable mapping F : ℂ → ℂ we can define a mapping F∗ : Mn → Mn via

\[(F∗(a)) = ∂a F(f),\]  (8)
where we assume that the formal derivative operates according to the standard chain rule, giving $\partial_a F(f)$ as a function of the formal derivatives $\partial_b f$ of $f$. Note that $F^* f(\emptyset) = F(f(\emptyset))$. Equation (3) defines an operation “*” taking us from a function $F: \mathbb{C} \to \mathbb{C}$ to a function $F^*: \mathcal{M}_n \to \mathcal{M}_n$. Furthermore “*” is a homomorphism under composition; viz.,

$$ (FG)^* = F^*G^*, $$(9)

for any two functions $F, G: \mathbb{C} \to \mathbb{C}$.

We can readily extend the definition (3) of the “*” operation to operate on functions $F$ with several complex variables, e.g., if $F: \mathbb{C}^2 \to \mathbb{C}$, and we have two $\mathcal{M}$-maps $f$ and $g$, then

$$ F^*(f, g)(a) = \partial_a F(f, g). $$(10)

If we now choose $G(f, g) = fg$, we can use the extended definition to find $G^*(f, g) = f * g$. We can furthermore apply (9) to obtain

$$ F^*(f * g) = (F(fg))^*. $$ (11)

Thus “*” enables us to carry over maps of complex numbers to $\mathcal{M}_n$, preserving all their properties. For instance, if $F(f) = f^{-1}$, we obtain an inverse in $\mathcal{M}_n$, defined whenever $f(\emptyset) \neq 0$ by

$$ (f^{-1})^*(a) = \partial_a (f^{-1}), $$(12)

where $f^{-1}$ is the multiplicative inverse $1/f(a)$. Thus $f * f^{-1} = 1^*$. Two other operations are

$$ (\log^* f)(a) = \partial_a \log(f), $$ (13)

$$ (\exp^* f)(a) = \partial_a \exp(f), $$ (14)

where “log” here and elsewhere means $\log_e$. From (9) we deduce that $\exp^*(\log^*) = 1^*$, and from (11) that

$$ \log^*(f^* g) = \log^* f + \log^* g, $$ (15)

and $\log^*(f^{-1}^*) = -\log^* f$.

The prescription for the inverse, (12), can easily be turned into an explicit rule by formal differentiation. One finds

$$ f^{-1^*}(\emptyset) = 1/f(\emptyset), $$ (16)

$$ f^{-1^*}(1) = -f(1)/f(\emptyset)^2, $$ (17)

$$ f^{-1^*}(1, 2) = -f(1, 2)/f(\emptyset)^2 + 2f(1)f(2)/f(\emptyset)^3, $$ (18)

and in general

$$ f^{-1^*}(a) = \sum_{p \in \Lambda(a)} \frac{|p|!(−1)^{|p|}|p|!}{f(\emptyset)^{|p|+1}} \Pi_{c \in p} f(c), $$ (19)

the sum being taken over all partitions $\lambda(a)$ of $a$, with $|p|$ denoting the number of parts (subsets) of the partition $p$.

This follows at once from the combinatorial version of Faà di Bruno’s rule (29) (and the moment algebra acquires an added grace by this use of the theorem of a beatified mathematician). It is remarkable that the simple definition in (3) leads to such a rich structure.

Similarly, Faà di Bruno’s rule gives

$$ \log^* f(a) = \sum_{p \in \Lambda(a)} |p|! (-1)^{|p|} \Pi_{c \in p} f(c) f(\emptyset)^{|p|}, \quad a \neq \emptyset, $$ (20)

which is well-defined for $\mathcal{M}$-maps $f$ such that $f(\emptyset) \neq 0$. From (13) and (20) we find

$$ \log^* f(\emptyset) = \log f(\emptyset), $$ (21)

$$ \log^* f(1) = f(1)/f(\emptyset), $$ (22)

$$ \log^* f(1, 2) = f(1, 2)/f(\emptyset) - f(1)f(2)/f(\emptyset)^2. $$ (23)
We can make a similar calculation for \( \exp^* f \) and find
\[
\exp^* f(a) = e^{f(\emptyset)} \sum_{p \in \mathcal{P}(a)} \Pi_{c \in p} f(c).
\] (24)

We shall refer to \( \log^* f \) as the cumulant of the \( \mathcal{M} \)-map \( f \), and \( \exp^* f \) as its anticumulant. Cumulants were originally introduced in the context of statistics and probability theory. To formulate these “classical” cumulants within this framework we let \( X_1, \ldots, X_n \) be a collection of random variables, and define an \( \mathcal{M} \)-map \( f \) by \( f(a) = \langle \Pi_{j \in a} X_j \rangle \), for each subset \( a \subseteq \{1, \ldots, n\} \) where \( \langle \cdots \rangle \) is the expectation value of the product of the random variables. (We also take \( f(\emptyset) = \langle 1 \rangle = 1 \).) Then
\[
\log^* f(a) \equiv \log^*(\langle \Pi_{j \in a} X_j \rangle)
\] (25)
is precisely the classical joint cumulant \(^{10}\) of the random variables \( \{X_j\}_{j \in a} \), and \( \exp^* f(a) \) is their classical anticumulant. This justifies us in using the same terminology for \( \log^* f \) and \( \exp^* f \) in the more general situation where \( f \) is an arbitrary \( \mathcal{M} \)-map.

We now use the moment algebra to rederive some properties of cumulants in this more general setting. Suppose that an \( \mathcal{M} \)-map \( f \) satisfies
\[
f(c) = f(c \cap A)f(c \cap B), \quad \forall c \subseteq \Omega_n,
\] (26)
where \( \{A, B\} \) is a bipartition of \( \Omega_n \), i.e., \( A \cap B = \emptyset \) and \( A \cup B = \Omega_n \). We say that \( f \) factorizes with respect to \( \{A, B\} \) on \( \Omega_n \). Then \( \log^* f(c) = 0 \) for any nonempty \( c \subseteq A \Omega_n \) such that \( c \not\subseteq A \) or \( c \not\subseteq B \), and in particular \( \log^* f(\Omega_n) = 0 \). In other words the cumulant of factorizing \( \mathcal{M} \)-maps vanishes. To see this, we note that we can write \( f = f_A \ast f_B \), where \( f_A(c) = f(c) \) if \( c \subseteq A \) and \( f_A(c) = 0 \) otherwise, and \( f_B(c) = f(c) \) if \( c \subseteq B \) and \( f_B(c) = 0 \) otherwise. Then
\[
\log^* f(c) = \log^* f_A(c) + \log^* f_B(c),
\] (27)
and this is zero for any \( c \) that is not either entirely in \( A \) or entirely in \( B \), since then there is some element \( a \) of \( c \) with \( a \notin A \), and \(^{20}\) shows that \( \log^* f_A \) must vanish; but this is also true of \( \log^* f_B \). It can be shown that this property characterizes cumulants \(^{30, 31}\).

We mention here a property of cumulants that we will frequently make use of. Given \( \alpha \in \mathbb{C} \), define the scalar \( \alpha^* \) by \( \alpha^*(\emptyset) = \alpha \), \( \alpha^*(c) = 0 \) if \( c \neq \emptyset \). Thus in the special case \( \alpha = 1 \), \( \alpha^* \) is what we have previously called \( 1^* \), so our usage is consistent. Now \( f \ast \alpha^*(c) = (af)(c) \). Thus the convolution product with \( \alpha^* \) is equivalent to scaling the value of \( f \) for all subsets of \( \Omega_n \) by the factor \( \alpha \). Note also that \( \log^* \alpha^*(c) = 0 \), for all \( c \neq \emptyset \). Thus, by \(^{[15]}\),
\[
\log^* (f \ast \alpha^*) (c) = \log^* \alpha^*(c) + \log^* f(c) = \log^* f(c),
\] (28)
for \( c \neq \emptyset \), so scaling \( f \) by a constant factor leaves the cumulant unchanged on all non-empty subsets of \( \Omega_n \).

The comparison between \( \mathcal{M} \) and complex analysis is further strengthened by the existence of the power series
\[
\log^*(1^* + f) = f - \frac{f \ast f}{2} + \frac{f \ast f \ast f}{3} - \cdots,
\] (29)
which converges whenever \( |f(\emptyset)| < 1 \). When applied to the empty set this becomes
\[
\log^*(1^* + f)(\emptyset) = f(\emptyset) - \frac{f(\emptyset)^2}{2} + \frac{f(\emptyset)^3}{3} - \cdots,
\] (30)
which is the familiar power series for the complex function \( \log(1 + f(\emptyset)) \), as it should be according to \(^{[21]}\). When \(^{[29]}\) is applied to sets other than \( \emptyset \), checking its validity is a pleasant exercise. For instance, using the product rule \(^{[4]}\) to evaluate the repeated convolutions gives
\[
\log^*(1^* + f)(1, 2) = f(1, 2) - [f(1, 2)f(\emptyset) + f(1)f(2)] + [f(1, 2)f(\emptyset)^2 + 2f(1)f(2)f(\emptyset)] + \cdots
\]
\[
= \frac{f(1, 2)}{1 + f(\emptyset)} - \frac{f(1)f(2)}{(1 + f(\emptyset))^2},
\]
which one sees is the correct expression if one compares it with \(^{[23]}\) and bears in mind the definition of \( 1^* \) by \(^{[6]}\).

As a final ingredient, we define
\[
(\partial_i^* f)(a) = \partial_a(\partial_i f), \quad \text{if } i \notin a.
\] (31)
(The annoying restriction \( i \notin a \) can be removed by means of the multiset formalism in Section \(^{[VIII]}\).) This operation behaves like a partial derivative; for instance, we have \( \partial_i \log^* f(a) = (\partial_i^* f)(\ast f^{-1*}(a)) \). (In the case of the \( \mathcal{M} \)-map defined by \(^{[3]}\), \( \partial_i \) corresponds to the operation \( \phi \to \partial \phi/\partial x_i \).) It can also be interpreted as a sort of “raising” operator, taking \( f(a) \) to \( f(a \cup i) \). This lets us immediately derive \( \log^*(1^* + f)(a) \) for any \( a \) from \( \log^*(1^* + f)(\emptyset) \), which, as we have just seen, is the complex function expansion \(^{[50]}\). More generally, we can derive a moment algebra equivalent from any complex function power series.
III. WEAK MEASUREMENTS

Weak measurement, developed by Aharonov and his colleagues, is a strategy for extracting information from a quantum system $S$, and has a number of distinct ideas behind it:

(i) A measuring system $P$ is weakly coupled to the original system $S$, and $P$ is then uncoupled and measured. This allows some limited information about $S$ to be gained with little disturbance to $S$.

(ii) By repeating the entire procedure many times, with the system identically prepared on each occasion, the noisy information about $S$ obtained by the weak coupling of $P$ can be averaged to give a definite answer.

(iii) The system $S$ can be both preselected and postselected, the latter meaning that the procedure concludes with a measurement on $S$, and only if a particular outcome is obtained are the data included in the averaging process.

These ideas combine constructively and enable one to probe a system in new ways. In particular, one can express a weak measurement result in terms of a quantity called the weak value, akin to the standard expectation value. This depends upon both the pre- and post-conditioning states, and can take very curious-seeming values, that can nevertheless be shown to have a natural physical meaning.

In the standard weak measurement setup the pointer system $P$ has a continuous degree of freedom, like the position of a single particle. Another common assumption is that this pointer particle initially is in a pure state $\phi$ for which the expectation value of both the position and momentum observables are zero. Hence, the probability density $P$ is a gaussian centered at zero. Furthermore it is assumed that the coupling between the pointer and system is given by the impulsive Hamiltonian $H = \gamma p \otimes A\delta(t)$, where $p$ is the momentum operator on $P$ and $A$ is any Hermitian operator on $S$, and where $\delta(t)$ is the delta distribution centered at time $t = 0$. Suppose the system is initially in state $|\psi_i\rangle$ and is postselected in state $|\psi_f\rangle$. The state of the pointer after the interaction and postselection is proportional to $\langle \psi_f | \exp(-i \gamma p \otimes A)|\phi\rangle |\psi_i\rangle$. After the interaction, the pointer position $q$ is measured, and by averaging over many repeats one obtains the expectation value to any desired accuracy.

To model the weakness of the interaction we expand the resulting expectation value up to the first order in the interaction parameter $\gamma$, resulting in

$$\langle q \rangle = \gamma ReA_w,$$

where $A_w$ is the weak value of the observable $A$ defined by

$$A_w = \frac{\langle \psi_f | A |\psi_i\rangle}{\langle \psi_f | \psi_i \rangle}.$$

This basic type of weak measurement can readily be generalised. The pointer can be an arbitrary quantum system. We do not necessarily have to have a continuous degree of freedom. The Hilbert space could be finite-dimensional; e.g., the pointer could be the spin degree of freedom of a particle. The coupling can be $H = \gamma \delta(t)s \otimes A$, where $s$ is now any Hermitian operator, and likewise one can measure any Hermitian operator $r$ on $P$. We further allow the initial pointer state $|\phi\rangle$ to be arbitrary. Then (32) becomes

$$\langle r \rangle = \langle r \rangle_\phi + \gamma Re(\xi A_w),$$

where

$$\xi = -2i (\langle rs \rangle_\phi - \langle r \rangle_\phi \langle s \rangle_\phi)$$

and $\langle r \rangle_\phi = \langle \phi | r | \phi \rangle$. We can conclude that, by measuring the expectation value of the observable $r$ on the pointer, we can obtain the weak value $\langle A \rangle_w$ on the system.

A different direction of generalisation is to weakly measure several operators $A_k$ either simultaneously or sequentially. In the latter case, one couples pointers at successive times $t_k$ via the Hamiltonians $H_k = \gamma_k \delta(t-t_k)s_k \otimes A_k$, and assumes that the system evolves by unitaries $U_k$ between these times. One can then calculate the moment $\langle r_1 \cdots r_n \rangle$, i.e. the expectation of the product of pointer measurements. It turns out that this can be expressed in terms of sequential weak values given by

$$\langle A_n, \ldots, A_1 \rangle_w = \frac{\langle \psi_f | U_{n+1}A_nU_n \ldots A_1U_1 |\psi_i\rangle}{\langle \psi_f | U_{n+1}U_n \ldots U_1 |\psi_i\rangle}.$$

The expression for $\langle r_1 \cdots r_n \rangle$ in terms of sequential weak values is horrendously complicated. It undergoes a striking simplification, however, if one looks at cumulants. We note that $f(a) = \langle \Pi_{j \in a} r_j \rangle$ is an $\mathcal{M}$-map, since it is
well defined for every subset \( a \), if we add the assumption that \( f(0) = 1 \). We will denote the cumulant \( \log^* f(a) \) by \( \log^* (\Pi_{j \in a} \gamma_j) \).

We can also define another \( M \)-map using sequential weak values on subsets as

\[
A_w(a) = \frac{\langle \psi_f | U_{n+1} F_n U_n \cdots F_2 U_2 F_1 U_1 | \psi_i \rangle}{\langle \psi_f | U_{n+1} U_n \cdots U_1 | \psi_i \rangle}, \quad F_j = \begin{cases} A_j & \text{if } j \in a \\ 1 & \text{if } j \notin a. \end{cases}
\]  

(37)

If we compare this with (36) we see that we only insert the operator \( A_j \) if \( j \in a \). As an explicit example, consider the case of four pointers and the sequential weak value \( A_w(2, 4) \), which would be

\[
A_w(2, 4) = \frac{\langle \psi_f | U_5 A_4 U_4 A_3 A_2 U_2 U_1 | \psi_i \rangle}{\langle \psi_f | U_5 U_4 U_3 U_2 U_1 | \psi_i \rangle}.
\]  

(38)

We will refer to \( \log^* A_w \) as the sequential weak value cumulant.

The following theorem was proved in [28].

**Theorem III.1** (Cumulant theorem). To the lowest joint order in the variables \( \gamma \),

\[
\log^* (\Pi_{j \in \Omega} \gamma_j) = (\Pi_{j \in \Omega} \gamma_j) \Re \{ \xi \log^* A_w(\Omega) \},
\]  

(39)

where \( \xi \) is given by

\[
\xi = 2(-i)^{\vert \Omega \vert} (\Pi_{j \in \Omega} \langle r_j, s_j \rangle_{\phi_j} - \Pi_{j \in \Omega} \langle r_j \rangle_{\phi_j} \langle s_j \rangle_{\phi_j}).
\]  

(40)

As explained above, the parameters \( \gamma_j \) signify the interaction strength between the system and the pointers, and we obtain the weak measurement scenario by expanding the cumulant \( \log^* (\Pi_{j \in \Omega} \gamma_j) \) in the strength parameters \( \gamma \), keeping only the lowest order coefficients. This is also the sense in which the equality in equation (39) has to be interpreted: as an equality up to the lowest orders. Note furthermore that the “joint order” of \( \gamma_2 \gamma_1 \) is 2, for \( \gamma_5 \gamma_3 \gamma_2 \) it is 3, etc. Hence, in the above theorem the lowest (nonzero) joint order in the expansion is \( |\Omega| = n \). By this theorem we can conclude that the weak measurement setup can be used to measure the sequential weak value cumulant on the system, since in the weak limit the joint cumulant of the pointer observables is simply related to the sequential weak value cumulant on the system. This can thus be regarded as a natural generalization of the weak measurement scenario in the single pointer case, where the expectation value of the single pointer observable corresponds to the weak value on the system.

In the case of simultaneous weak measurement [24, 23, 26], one couples \( n \) pointers at a time \( t_0 \) via the Hamiltonian \( H_k = \sum_k \delta(t - t_0) \gamma_k s_k \otimes A_k \), and the simultaneous weak values are given by

\[
(A_1, \ldots, A_n)_{ws} = \frac{1}{n!} \sum_{\pi} \frac{\langle \psi_f | A_{\pi(n)} \cdots A_{\pi(1)} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle},
\]  

(41)

where \( \pi \) runs over all permutations of \( 1, \ldots, n \). Again, one can define an \( M \)-map and a cumulant denoted by \( \log^* A_{ws} \). There is an analogue to Theorem III.1 for the simultaneous case [28]:

**Theorem III.2** (Cumulant theorem for simultaneous measurement). If \( (\Pi_{j \in \Omega} \gamma_j) \) is the expected value of the product of \( n \) pointers in a simultaneous weak measurement, then, to the lowest joint order in the variables \( \gamma \),

\[
\log^* (\Pi_{j \in \Omega} \gamma_j) = (\Pi_{j \in \Omega} \gamma_j) \Re \{ \xi \log^* A_{ws}(\Omega) \},
\]  

(42)

where \( \xi \) is as in Theorem III.1.

**IV. A NEW CUMULANT THEOREM**

The original proof of Theorem III.1 was by no means transparent. We will show how the moment algebra setting allows a better proof. However, we begin by considering a different way of gathering information from the pointers, where the corresponding theorem can be proved more simply (and in section VII we prove the original result, which requires an extra flourish of cumulant technology).

The assumption behind Theorem III.1 is that the moment \( (\Pi_{j \in a} \gamma_j) \) for a subset \( a \) of the total collection of pointers \( \Omega \) is obtained by coupling just that set of pointers to the system: in other words, to obtain \( (\Pi_{j \in a} \gamma_j) \), one does an experiment in which just the pointers in \( a \) are coupled, so a separate experiment is needed for each subset. The
alternative approach that we now adopt is to suppose that a single experiment is carried out in which all the pointers are coupled and measured, but a subset of the measurement results is used for calculating each moment.

Another way to put this is to say that in the present case we have the total time dependent Hamiltonian $H(t) = 1_p \otimes H_S(t) + \sum_{k=1}^{n} \gamma_k \delta(t - t_k) s_k \otimes A_k$, where $H_S(t)$ is the Hamiltonian that generates the unitary evolution between the coupling with the pointers. We then define the solution of the theorems in Sec. III, and is more in line with the characterization in Sec. II of cumulants as mappings from $M$-maps to $M$-maps. Just as the $M$-map of its own, generating a unitary evolution between the couplings to the pointers. We let $U(t)$.

We shall see that the joint coupling of all pointers leads to a result that differs from Theorem III.1. This comes about because of the perturbation of the system by those pointers whose readings are not being used to calculate the cumulants of the pointers. Hence, $H_S(t)$ generates the sequence of unitary operators $U_k$. In Theorem III.1, however, we had a separate experiment with a separate Hamiltonian $H_{a}(t) = 1_p \otimes H_S(t) + \sum_{i \in a} \gamma_i \delta(t - t_i) s_i \otimes A_i$ for each subset $a$.

To state the theorem, let $\eta$ be the state of all the pointers after coupling and the postselection on the system. (We define $\eta$ more precisely in (47) below.) The moment of the pointer observables in the subset $a$ is $\langle \eta | j a \rangle \pi_j$, where the subscript $\pi_j$ indicates that all the pointers are coupled.

**Theorem IV.1 (Cumulant theorem with all pointers coupled).** Suppose all $n$ pointers are coupled sequentially. Let $a$ be a subset of the finite collection of pointers $\gamma$. To the lowest joint order in the variables $\gamma$,

$$\log^* (\Omega_{j \in a} r_j)_{\eta} = (\Omega_{j \in a} \gamma_j) \Re \{ \xi \log^* A_w(a) \},$$

where $\xi$ is given by

$$\xi = 2(-i)^{|a|} \prod_{j \in a} (\langle r_j s_j | \phi_j - \langle r_j | \phi_j | s_j \rangle | \phi_j) = 2(-i)^{|a|} \prod_{j \in a} \log^* (r_j | s_j \rangle | \phi_j).$$

This theorem allows us to consider the cumulants of subsets of the collection of pointers. This is a slight generalization of the theorems in Sec. III and is more in line with the characterization in Sec. II of cumulants as mappings from $\mathcal{M}$-maps to $\mathcal{M}$-maps. Just as the $\mathcal{M}$-map of $\Omega_{j \in a} r_j$ is defined for every subset of pointers $a$, so is the corresponding cumulant $\mathcal{M}$-map $\log^* (\Omega_{j \in a} r_j)_{\eta}$.

**Proof.** First, we establish some notation. Let $| \phi_k \rangle$ be the initial state of pointer $k$ in state space $\mathcal{H}_k$, and denote the total state space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ of pointers by $\mathcal{P}$, where we assume $|\Omega| = n$. Let $|\psi_i \rangle$ be the initial state of the system $\mathcal{S}$, and $|\psi_f \rangle$ the postselected final state. Define projectors on $\mathcal{S} \otimes \mathcal{P}$ by

$$P_i = |\psi_i \rangle \langle \psi_i | \otimes (|\phi_1 \rangle \langle \phi_1 | \cdots |\phi_n \rangle \langle \phi_n |),$$

$$P_f = |\psi_f \rangle \langle \psi_f | \otimes I \mathcal{P}.$$

Let $r_k$ be the pointer variable measured on pointer $k$ and $H_k = s_k \otimes A_k$ be the coupling between pointer $k$ and the system $\mathcal{S}$. For any state $\rho$ on $\mathcal{S} \otimes \mathcal{P}$, define

$$L_k \rho = -i[H_k, \rho],$$

so

$$e^{\gamma_k L_k} \rho = e^{-i\gamma_k H_k} \rho \ e^{i\gamma_k H_k}.$$

Apart from the coupling to the pointer we also assume that the system $\mathcal{S}$ has a (possibly time dependent) Hamiltonian of its own, generating a unitary evolution between the couplings to the pointers. We let $U_k$ be the unitary operator on $\mathcal{S}$ that gives the evolution between the coupling with pointer $k - 1$ and the coupling with pointer $k$, and let $U_0$ be the unitary that maps the initial state to the application of the first pointer. We then define

$$U_k \rho = (1_p \otimes U_k) \rho (1_p \otimes U_k)^\dagger$$

(46)

to be the corresponding unitary channel. The state of the pointers after the interaction and postselection on the system can thus be written as

$$\eta = \frac{\text{tr}_S (P_f U_{n+1} e^{\gamma_n L_n} U_n \cdots e^{\gamma_1 L_1} U_1 P_i)}{\text{tr} (P_f U_{n+1} e^{\gamma_n L_n} U_n \cdots e^{\gamma_1 L_1} U_1 P_i)}$$

(47)
Let us define the \( \mathcal{M} \)-map \( \langle \Pi_{j \in a} r_j \rangle_{\eta} = \text{tr} [ \langle \Pi_{j \in a} r_j \rangle_{\eta} \rangle ] \). Suppose now that we replaced one of the observables \( r_k \) in \( \langle \Pi_{j \in a} r_j \rangle_{\eta} \) with the identity operator \( 1_k \). We would then obtain a new \( \mathcal{M} \)-map \( u_k(a) = \langle \Pi_{j \in a \setminus \{ k \}} r_j \rangle_{\eta} \). This \( \mathcal{M} \)-map factorizes on \( a \) with respect to the partition \( \{ \{ k \}, a \setminus \{ k \} \} \), i.e., for all \( b \subset a \) we have \( u_k(b) = u_k(b \cap (a \setminus \{ k \})) u_k(b \cap \{ k \}) \).

Thus we can conclude that \( \log^* u_k(a) = 0 \). If we combine this with the multilinearity of the cumulant in the pointer observables we find that

\[
\log^* \langle \Pi_{j \in a} (r_j - \langle r_j \rangle_{\rho_j}) \rangle_{\eta} = \log^* \langle \Pi_{j \in a} r_j \rangle_{\eta}.
\]

Define

\[
v(a) = \frac{\text{tr} \left[ P_f \langle \Pi_{j \in a} (r_j - \langle r_j \rangle_{\rho_j}) \rangle_{\eta} \{ U_{a+1} e^{\gamma_1 L_1} U_n \cdots e^{\gamma_n L_n} U_1 P_i \} \right]}{|\langle \psi_f | U_{n+1} \cdots U_1 | \psi_i \rangle|^2}.
\]

The denominators of (49) and (47) do not depend on \( a \), and are thus scalars in \( \mathcal{M}_n \). The scale-invariance of the cumulant, (28), thus yields

\[
\log^* \langle \Pi_{j \in a} (r_j - \langle r_j \rangle_{\rho_j}) \rangle_{\eta} = \log^* v.
\]

We shall now expand the above cumulant to the lowest joint order in the parameters \( \gamma_j \). Let us write \( \partial^\gamma \) for the derivative with respect to the variables \( \gamma_j \) with labels in the set \( b \), i.e.,

\[
\partial^\gamma = \frac{\partial}{\partial \gamma_b(1)} \cdots \frac{\partial}{\partial \gamma_b(n)}.
\]

(These “proper” derivatives \( \partial^\gamma \) should not be confused with the formal derivatives \( \partial a \) introduced in section III.) We next prove the following

\[
\partial^\gamma \log^* v(a) |_{\gamma = 0} = \left\{ \begin{array}{ll} 0 & \text{if } a \setminus b \neq \emptyset \\ \log^* w(a) & \text{if } b = a, \end{array} \right.
\]

where

\[
w(a) = \partial^\gamma v(a) |_{\gamma = 0} = \frac{\text{tr} \left[ P_f U_{n+1} W_n U_n \cdots W_1 U_1 P_i \right]}{|\langle \psi_f | U_{n+1} \cdots U_1 | \psi_i \rangle|^2}, \quad W_j = \left\{ \begin{array}{ll} (r_j - \langle r_j \rangle_{\rho_j})_j & \text{if } a \setminus j \neq \emptyset, \\ 1 & \text{if } j \notin a, \end{array} \right.
\]

and \( \gamma = 0 \) means \( \gamma_j = 0 \), for \( 1 \leq j \leq n \). Equation (52) tells us that the first potentially nonzero expansion coefficient of the cumulant \( \log^* v(a) \) can itself be regarded as a cumulant, but of the new \( \mathcal{M} \)-map \( w \). This method of regarding the expansion coefficient of a cumulant as a cumulant in its own right is a technique that we will use repeatedly.

To prove (52), let us first suppose that \( a \setminus b \neq \emptyset \). Hence, there must be some element \( j \in a \) such that \( j \notin b \). Recall the expression for the cumulant in terms of partitions, (20), and suppose \( p \) is a partition of \( a \). Then \( \partial^\gamma_\Pi_{\ell \in p} v(c) |_{\gamma = 0} = 0 \), since for that \( c \in p \) that contains \( j \) the derivative \( \partial j \partial \gamma j \) is not applied to \( v(c) \), and consequently this term contains \( r_j - \langle r_j \rangle_{\rho_j} \) but not \( L_j \) and must therefore vanish. This gives the first part of (52). For the case \( b = a \), we again use the fact that, for any \( j \in a \), a term containing \( r_j - \langle r_j \rangle_{\rho_j} \) but not \( L_j \) must vanish, to find that \( \partial^\gamma_\Pi_{\ell \in p} v(c) |_{\gamma = 0} = \Pi_{\ell \in p} \partial \gamma \partial \rho \partial v(c) |_{\gamma = 0} \), for any partition \( p \) of \( a \). The statement in (52) follows.

It remains to evaluate the cumulant \( \log^* w(a) \). Let us rephrase the definition of \( L_j \) in (15) as

\[
L_j = L_j^{\text{left}} + L_j^{\text{right}}, \quad L_j^{\text{left}}(\rho) = (-i s_j \otimes A_j) \rho, \quad L_j^{\text{right}}(\rho) = \rho (i s_j \otimes A_j),
\]

so the subscript ‘left’ (‘right’) indicates which side the operator is applied to. We can write

\[
w(a) = \sum_{c_1, c_2} w_{c_1, c_2}(a),
\]

where the sum is over all ordered bipartitions \( (c_1, c_2) \) of \( a \), and where \( w_{c_1, c_2}(a) \) is defined as in (53) but with \( L_j \) replaced by \( L_j^{\text{left}} \) when \( j \in c_1 \), and by \( L_j^{\text{right}} \) when \( j \in c_2 \). By using the fact that \( L_j^{\text{left}} \) and \( L_j^{\text{right}} \) act independently, and that \( P_j \) is a projector onto pure product states, one can show that \( w_{c_1, c_2} \) factorizes on \( a \) with respect to the partition \( \{ c_1, c_2 \} \). Thus \( \log^* w_{c_1, c_2}(a) = 0 \) except when either \( c_1 = \emptyset \) or \( c_2 = \emptyset \); so \( \log^* w(a) = \log^* w_{\emptyset, \emptyset}(a) + \log^* w_{\emptyset, a}(a) \). Direct calculation shows that \( \log^* w_{a, \emptyset} = (-i)^{|a|} \Pi_{k \in a} \langle (r_k s_k) - \langle r_k \rangle_{\rho_k} \rangle A_w(a) \), and \( \log^* w_{\emptyset, a} \) gives the complex conjugate. The theorem follows.
V. SIMULTANEOUS WEAK MEASUREMENT WITH SYSTEM EVOLUTION

The last section focused on sequential weak measurement. In the case of simultaneous weak measurement, it is assumed \( \sum_{k=1}^{n} g_k s_k \otimes A_k \) that the Hamiltonian has the form \( H_k = \sum_{k=1}^{n} \delta(t-t_0) g_k s_k \otimes A_k \), and the evolution of the system, which occurred between coupling of pointers in sequential weak measurement, can be ignored here since the coupling occurs impulsively and simultaneously for all pointers, and any evolution before or after the coupling can be incorporated into the initial and final states by writing \( \langle \psi_f | = U_1 | \psi_i \rangle \), \( \langle \psi_f | = (\psi_f | U_2 \rangle \).

But suppose the coupling occurs over a finite time. Then \( \langle \psi_f | = \psi_i \rangle \), \( \langle \psi_f | = (\psi_f | U_2 \rangle \).

To prove this, we begin with the following

\( D(a) = \frac{1}{\tau k} \sum_{\pi} \int_{\tau_{k+1}-\tau_k-\cdots-\tau_1 \geq 0} (A_{\sigma(k)}, \ldots, A_{\sigma(1)}) w[\tau_{k+1}, \ldots, \tau_1] \delta(\tau - \sum_{j=1}^{k+1} \tau_j) d\tau_1 \cdots d\tau_k d\tau_{k+1} \).

\( D(a) = \frac{1}{\tau k} \sum_{\pi} \int_{\tau_{k+1}-\tau_k-\cdots-\tau_1 \geq 0} (A_{\sigma(k)}, \ldots, A_{\sigma(1)}) w[\tau_{k+1}, \ldots, \tau_1] \delta(\tau - \sum_{j=1}^{k+1} \tau_j) d\tau_1 \cdots d\tau_k d\tau_{k+1} \).

\( D(a) = \frac{1}{\tau k} \sum_{\pi} \int_{\tau_{k+1}-\tau_k-\cdots-\tau_1 \geq 0} (A_{\sigma(k)}, \ldots, A_{\sigma(1)}) w[\tau_{k+1}, \ldots, \tau_1] \delta(\tau - \sum_{j=1}^{k+1} \tau_j) d\tau_1 \cdots d\tau_k d\tau_{k+1} \).

\( D(a) = \frac{1}{\tau k} \sum_{\pi} \int_{\tau_{k+1}-\tau_k-\cdots-\tau_1 \geq 0} (A_{\sigma(k)}, \ldots, A_{\sigma(1)}) w[\tau_{k+1}, \ldots, \tau_1] \delta(\tau - \sum_{j=1}^{k+1} \tau_j) d\tau_1 \cdots d\tau_k d\tau_{k+1} \).

To prove this, we begin with the following

**Theorem V.1** (Weak simultaneous measurement with system evolution). Let \( a \) be a subset of the finite collection of pointers \( \Omega \). To the lowest joint order in the variables \( \gamma \),

\[ \log^+ \langle \Pi_{j \in a} \tau_j \rangle_\sigma = \langle \Pi_{j \in a} \gamma_j \rangle_\sigma \langle \Pi_{j \in a} \gamma_j \rangle_{\sigma} = \langle \Pi_{j \in a} \gamma_j \rangle_{\sigma} \langle \Pi_{j \in a} \gamma_j \rangle_{\sigma} \]

\[ \text{where } \xi \text{ is as in Theorem IV.7} \]

To prove this, we begin with the following

**Lemma V.2.** Let \( a = \{a_1, \ldots, a_k\} \). Then

\[ \frac{\partial^\gamma e^{Y+\sum_{j=1}^{\gamma} a_j X_j}}{\partial \gamma_{a_j}} \bigr|_{\gamma=0} = \sum_{\pi} \int_{\tau_{k+1}, \tau_k, \ldots, \tau_1 \geq 0} e^{\tau_{k+1} Y} \cdots e^{\tau_k Y} \cdots e^{\tau_2 Y} \cdots e^{\tau_{k+1} Y} \delta(\tau - \sum_{j=1}^{k+1} \tau_j) d\tau_1 \cdots d\tau_k d\tau_{k+1}, \]

where the sum is over all permutations \( \pi \) of the set \( \{1, \ldots, k\} \).

**Proof.** We use the Dyson series (proof: differentiate both sides with respect to \( \tau \))

\[ e^{Y+V} = e^{Y} + \int_{0}^{\tau} e^{Y} \cdot V e^{Y} \cdot dt_1 + \int_{0}^{\tau} \int_{0}^{t_2} e^{(\tau-t_2)Y} V e^{(t_2-t_1)Y} V e^{Y} \cdot dt_2 \cdots + \]

\[ e^{Y+V} = e^{Y} + \int_{0}^{\tau} e^{Y} \cdot V e^{Y} \cdot dt_1 + \int_{0}^{\tau} \int_{0}^{t_2} e^{(\tau-t_2)Y} V e^{(t_2-t_1)Y} V e^{Y} \cdot dt_2 \cdots + \]

Putting \( V = \sum_{j=1}^{n} \gamma_j X_j \), the only term that survives the combined operations of differentiation by \( \partial^\gamma_\eta \) and setting \( \gamma = 0 \) is the \( k \)-times repeated integral

\[
\partial^\gamma_\eta e^{\tau(Y+V)}|_{\gamma=0} = \sum_{\pi} \int_0^\tau \int_0^{t_k} \cdots \int_0^{t_2} e^{(\tau-t_k)Y} X_{\pi(1)} e^{(t_k-t_{k-1})Y} X_{\pi(k-1)} \cdots e^{(t_1-t_1)Y} X_{\pi(1)} e^{t_1 Y} dt_1 \cdots dt_k.
\] (63)

To obtain \( \Pi \), make the change of variables \( \tau_1 = t_1, \tau_2 = t_2 - t_1, \ldots, \tau_k = t_k - t_{k-1}, \tau_{k+1} = \tau - t_k \).

**Proof of Theorem.** Consider the \( \mathcal{M} \)-map \( \log^* (\Pi_{j=0} r_j) \). Following the proof of Theorem \( \Pi \), we can replace all the observables \( r_j \) with \( r_j - \langle r_j \rangle_1 \), without changing the cumulant. Still following Theorem \( \Pi \), we find that the lowest order term in the expansion has joint degree \( |a| \) and corresponding expansion coefficient \( \partial^\gamma_\eta \log^* (\Pi_{j=0} (r_j - \langle r_j \rangle_1))_{\gamma=0} = \log^* d(a) \) with the new \( \mathcal{M} \)-map

\[
d(a) = \frac{\partial^\gamma_\eta \text{tr} \left\{ P_f (\Pi_{j=0} (r_j - \langle r_j \rangle_1)) e^{-irH} P_1 e^{irH} \right\} |_{\gamma=0}}{|\langle \psi_f | e^{-irHS} | \psi_1 \rangle|^2} \sum_{(c_1, c_2)} \text{tr} \left\{ P_f (\Pi_{j=0} (r_j - \langle r_j \rangle_1)) \left[ \partial^\gamma_\eta e^{-irH} P_1 \partial^\gamma_\eta e^{irH} \right] \right\} |_{\gamma=0},
\] (64)

where the sum is over all ordered bipartitions \( (c_1, c_2) \) of \( a \). Next, we apply Lemma \( \Pi \) to both \( \left[ \partial^\gamma_\eta e^{-irH} \right] |_{\gamma=0} \) and \( \left[ \partial^\gamma_\eta e^{irH} \right] |_{\gamma=0} \), with \( Y = -i \hat{1}_F \otimes H_S, X_k = -\frac{i}{\beta} s_k \otimes A_k \), and \( Y = i \hat{1}_F \otimes H_S, X_k = \frac{i}{\beta} s_k \otimes A_k \), respectively. We find that \( d = D' \) where

\[
D(a) = (-i)^{|a|} (\Pi_{j=0} (r_j s_j - \langle r_j \rangle \langle s_j \rangle)) D(a),
\] (65)

and where \( D' \) is the complex conjugate of \( D \). The statement of Theorem \( \Pi \) follows from \( d = D + D' \) together with the fact that \( \log^* d = \log^* D + \log^* D' \).

The interpretation of this theorem is quite intuitive. If there were no evolution, i.e. \( H_S = 0 \), then \( D(a) \) would be the simultaneous weak value, \( 24, 25, 26 \) and \( \Pi \), given by symmetrizing over all orders of applying operators, viz.

\[
D(a) = \frac{1}{n!} \sum_{\pi} (A_{\pi(1)}, \ldots, A_{\pi(n)}) w,
\] (66)

where the factor \( 1/n! \) comes from integrating \( \tau_1, \ldots, \tau_n, \tau_{n+1} \geq 0 \) with the constraint \( \sum_{j=1}^{n+1} \tau_j = \tau \). When \( H_S \) is nonzero, we must in addition average over episodes of evolution under \( e^{-iH_S t} \) between application of the \( A_k \) with the lengths of all episodes summing to \( \tau \).

The theorem implies that simultaneous weak measurement can be simulated by collections of sequential weak measurements, by sampling over permutations of the ordering of the applications of the pointers, as well as over the time steps between the applications of the pointers.

**VI. THERMAL WEAK MEASUREMENT**

In the previous section we stretched the concept of weak measurement a little by allowing the system to evolve while the pointers are coupled. Here we stretch it further by abandoning the notion of preselection and postselection (key ingredients of the original weak measurement philosophy \( 32, 33 \)), and instead considering a system in thermal equilibrium. As we will see, this thermal weak measurement concept is, formally speaking, closely related to the simultaneous weak measurement with system evolution considered in the previous section. The correspondence between these two scenarios is analogous to that between path integrals and equilibrium systems under the imaginary time transformation \( t \leftrightarrow it \) \( 14 \).

For thermal equilibrium systems with Hamiltonian \( H \), the Helmholtz free energy \( 34 \) can be written

\[
F = -\frac{1}{\beta} \log Z(\beta) = -\frac{1}{\beta} \log \text{tr} e^{-\beta H},
\]

where \( \beta = 1/kT \), with \( T \) being the temperature. When external parameters, e.g. fields, are changed infinitely slowly, the difference between the final and initial free energy is equal to the work performed on the system, under the assumption that the system is kept in contact with a heat bath at constant temperature \( T \) \( 34 \). The Taylor expansion
of the free energy with respect to the external fields thus characterizes the system’s response to small changes. This picture can be extended to several fields \( g_j \) coupling to the system via observables \( A_j \), for instance with linear coupling

\[
H_C = H_S + \sum_j g_j A_j,
\]

leading to a free energy

\[
F = -\frac{1}{\beta} \log \text{tr} e^{-\beta H_C}.
\]

Comparing (74) and (75) with (59) and (58), respectively, the picture can be extended to several fields \( g_j \) coupling to the system via observables \( A_j \), for instance with linear coupling

\[
H_C = H_S + \sum_j g_j A_j,
\]

leading to a free energy

\[
F = -\frac{1}{\beta} \log \text{tr} e^{-\beta H_C}.
\]

We follow [30] and refer to the expansion coefficients of \( F \) with respect to \( \gamma_k \), i.e. \( \partial^k_a \frac{\partial}{\partial \gamma} F |_{\gamma = 0} \) for a subset \( a \) of the \( \gamma \)'s, as generalized susceptibilities. We shall now construct a weak measurement scenario where the correlation of the pointers, as measured by the joint cumulant of the pointer observables, turns out to be directly proportional to these generalized susceptibilities.

Instead of letting a collection of pointers weakly interact with the system for specific times, here we let the pointers and system equilibrate under the assumption of weak interactions. When this combined system has equilibrated we separate the pointers and, as before, measure a collection of observables on them. We therefore consider a total Hamiltonian

\[
H = \hat{1}_p \otimes H_S + \sum_{j=1}^n \gamma_j \frac{s_j}{\beta} \otimes A_j,
\]

where \( \gamma_j = \beta g_k \), and we assume that the system and the pointers reach the thermal equilibrium state under this Hamiltonian, yielding

\[
\rho = \frac{e^{-\beta H}}{\text{tr} e^{-\beta H}}.
\]

The expectation for pointer measurements is the \( \mathcal{M} \)-map \( (\Pi_{j \in a} r_j)_{\rho} = \text{tr}(\rho \Pi_{j \in a} r_j) \). We also define the \( \mathcal{M} \)-map

\[
\mathcal{E}(a) = \frac{\partial^k_a \text{tr} \{ e^{-\beta H_S - \sum_j \gamma_j A_j} \} |_{\gamma = 0}}{\beta^{|a|} \text{tr} e^{-\beta H_S}},
\]

which gives, up to a normalising factor, the Taylor coefficients in the expansion of the partition function with respect to \( \gamma \).

**Theorem VI.1** (Weak measurement of a system in equilibrium). Let \( a \) be a subset of the finite collection of pointers \( \Omega \). To the lowest joint order in \( \gamma \) we find

\[
\log^* \langle \Pi_{j \in a} r_j \rangle_{\rho} = \langle \Pi_{j \in a} \gamma_j \rangle \xi \log^* \mathcal{E}(a) = -\beta \langle \Pi_{j \in a} \gamma_j \rangle \xi \partial^2_a \frac{\partial}{\partial \gamma} F |_{\gamma = 0},
\]

and

\[
\xi = \Pi_{j \in a} \text{tr}(r_j s_j) - \text{tr}(r_j) \text{tr}(s_j).
\]

This assumes that the various traces \( \text{tr}(r_j) \), \( \text{tr}(s_j) \), and \( \text{tr}(r_j s_j) \) are well defined, which in the case of infinite dimensional Hilbert spaces requires them to be trace class [33]. The theorem tells us that the joint cumulant of the pointers is directly proportional to the generalized susceptibility of the system.

As an application of these ideas, consider a collection of pointers in equilibrium with a heat bath. Correlations between the pointers will be generated by the heat bath, and these can be characterised by local observables. Our theorem says that, if the coupling of the pointers to the heat bath is weak, the cumulants of these local observables will be proportional to the generalized susceptibilities of the heat bath.

As mentioned above, there is an analogy between thermal weak measurement and simultaneous weak measurement. To see this, we can use Lemma [V.2] to expand the \( \mathcal{M} \)-map \( \mathcal{E} \) in weak values as

\[
\mathcal{E}(a) = \frac{1}{\beta^{|a|}} \sum_{\sigma} \int_{\tau_k+1, \ldots, \tau_1 \geq 0} (A_{\sigma(1)}, \ldots, A_{\sigma(k+1)}) e^{-\beta \sum_{j=1}^{k+1} \tau_j} \delta(\beta - \sum_{j=1}^{k+1} \tau_j) d\tau_1 \cdots d\tau_{k+1},
\]

where

\[
(A_k, \ldots, A_1) e^{-\beta \sum_{j=1}^{k+1} \tau_j} = \frac{\text{tr} [ e^{-\tau_k+1 H_S} A_k e^{-\tau_k H_S} \cdots A_1 e^{-\tau_1 H_S}]}{\text{tr} [ e^{-\beta H_S}]}.
\]

Comparing (74) and (75) with (59) and (58), respectively, the \( t \leftrightarrow it \) correspondence is clear; this allows us to carry over a large part of the proof of Theorem [V.1] to the present theorem.
Proof of Theorem. Consider the $\mathcal{M}$-map $(\Pi_{j\in a} r_j)_\rho = \text{tr}(\rho \Pi_{j\in a} r_j)$. In the analogue of (19), instead of replacing $r_j$ by $(r_j - r_j 1_j)$, we replace it by $(r_j - \text{tr}(r_j 1_j))$, the trace playing the role previously taken by the expectation. As before, this modification of the pointer observables does not change the cumulants, $\log^* (\Pi_{j\in a} r_j)_\rho = \log^* (\Pi_{j\in a} (r_j - \text{tr}(r_j 1_j))_\rho$.

Defining

$$X_\rho = \frac{\Pi_{j\in a} (r_j - \text{tr}(r_j 1_j))}{\text{tr} e^{-\beta H}}$$

we use scale-invariance, (28), to show

$$\partial_\alpha^\gamma \langle \Pi_{j\in a} (r_j - \text{tr}(r_j 1_j)) \rangle_\rho |_{\gamma = 0} = \partial_\alpha^\gamma \log^* \text{tr} (e^{-\beta H} X_\rho) |_{\gamma = 0}$$

$$= \log^* \partial_\alpha^\gamma \text{tr} (e^{-\beta H} X_\rho) |_{\gamma = 0}$$

$$= \log^* \text{tr} (\Pi_{j\in a} s_j X_\rho) \partial_\alpha^\gamma \text{tr} (e^{-\beta H c}) |_{\gamma = 0}$$

$$= \xi \log^* \mathcal{E}(a),$$

where (78) follows from (77) by the same argument that derived (52) from (53). Furthermore,

$$\xi \log^* \mathcal{E}(a) = \xi \log^* \partial_\alpha^\gamma (e^{-\beta H c}) |_{\gamma = 0}$$

$$= \frac{\xi \partial_\alpha^\gamma \log \text{tr} (e^{-\beta H c}) |_{\gamma = 0}}{\gamma = 0}$$

$$= -\beta \xi \partial_\alpha^\gamma F |_{\gamma = 0},$$

where (79) follows from scale-invariance, (28), and (80) follows directly from the definition of $\log^*$ using (13); see the Appendix for details.

Note that in the proof of Theorem [11] $e^{-i\tau H}$ operates on the left of $P_i$ and its conjugate operates on the right, which leads to the real part, $\text{Re} \{\xi \log^* \mathcal{E}(a)\}$, appearing in (60). In the above theorem $e^{-\beta H}$ appears without its conjugate, so we get the whole of $\xi \log^* \mathcal{E}(a)$ in (72).

VII. NEW PROOF OF THE ORIGINAL THEOREM

Finally, we shall show how our moment algebra methods can be used to prove the original theorem in (28). Note that we here prove a slight generalization of Theorem [11] in the sense that we allow the cumulants to be taken over arbitrary subsets $a$ of the total collection of pointers $\Omega$.

Proof of Theorem [11].

$$x(a) = \frac{\text{tr} (P_i U_{n+1} X_n U_n \cdots X_1 U_1 P_i)}{|\langle \psi_j | U_{n+1} \cdots U_1 | \psi_i \rangle|^2}, \quad X_j = \begin{cases} r_j e^{\gamma_j L_j} & \text{if } j \in a \\ 1 & \text{if } j \notin a \end{cases}$$

$$y(a) = \frac{\text{tr} (P_i U_{n+1} Y_n U_n \cdots Y_1 U_1 P_i)}{|\langle \psi_j | U_{n+1} \cdots U_1 | \psi_i \rangle|^2}, \quad Y_j = \begin{cases} e^{\gamma_j L_j} & \text{if } j \in a \\ 1 & \text{if } j \notin a \end{cases}$$

$$z(a) = \langle \Pi_{j\in a} r_j \rangle = \frac{x(a)}{y(a)},$$

so the $\mathcal{M}$-map $z(a)$ is the expectation for pointer measurements in the subset $a$. Note that the normalizations of the $\mathcal{M}$-maps $x$ and $y$ have been chosen so that $x(\emptyset) = 1$ and $y(\emptyset) = 1$. For $y$ another convenient property is that $y(a) |_{\gamma = 0} = 1$ for all $a$.

The idea of the proof is as follows: if the ratio in (52) were defined in terms of convolution operations, so we had $z = x * y^{-1}$ instead of $z = xy^{-1}$, then this would imply $\log^* z = \log^* x - \log^* y$, leaving us with the much simpler task of calculating $\log^* x$ and $\log^* y$. In fact, it turns out that, by expanding the $\mathcal{M}$-map $z$ in powers of the $\gamma$'s, we can achieve this switch from multiplicative to convolution operations: see (80).

We first prove the equivalent of (52) for $z$:

$$\partial_\alpha^\gamma \log^* z(a) |_{\gamma = 0} = \begin{cases} \log^* \hat{z}(a) & \text{if } b = a, \\ 0 & \text{if } b \subset a, b \neq a \end{cases}$$

(84)
where \( \tilde{z}(a) = \partial^\gamma_0 z(a) |_{\gamma=0} \). To this end, for any \( b \subseteq a \) define the \( M \)-map \( z_b \) by \( z_b(c) = \partial^\gamma_{b^c c} z(c) |_{\gamma=0} \) for all \( c \subseteq a \). By \( 20 \),

\[
\partial^\gamma_b \log^* z(a) |_{\gamma=0} = \sum_{p \in \pi(a)} (|p| - 1)! (|p| - 1)^{-1} \partial^\gamma_p \prod_{c \in p} z(c) |_{\gamma=0}
\]

\[
= \sum_{p \in \pi(a)} (|p| - 1)! (|p| - 1)^{-1} \prod_{c \in p} \partial^\gamma_{b^c c} z(c) |_{\gamma=0} = \log^* z_b(a),
\]

(85) using the fact that \( z(c) \) depends only on the \( \gamma \)'s in \( c \). Since \( z_a = \tilde{z} \), \( 31 \) follows for the case \( b = a \). If \( b \neq a \), there is some \( j \in a \) with \( j \notin b \). When we put \( \gamma_j = 0 \), since \( 33 \) shows the term \( e^{\gamma_j L_j} \) is not differentiated, the operator \( L_j \) does not appear in \( z_b(a) \). Therefore \( r_j \) is not coupled to \( S \) and \( z_b \) factorises on \( a \). We conclude that \( \log^* z_b(a) = 0 \).

Equation \( 34 \) tells us that \( \log^* \tilde{z}(a) \) is the first non-vanishing term in the expansion of the cumulant \( \log^* (\prod_{j \in \gamma} r_j) \) in the \( \gamma \)'s. In other words, the relevant expansion coefficient of the latter cumulant can be regarded as the cumulant of the new \( M \)-map \( \tilde{z} \). To calculate this new cumulant we use the usual law for differentiation of a product, together with the observation that if \( (b_1, b_2) \) is bipartition of \( b \) then

\[
[\partial^\gamma_{b_1} x(b)] |_{\gamma=0} = (\prod_{j \in b_2} \langle r_j, \phi_j \rangle) [\partial^\gamma_{b_1} x(b_1)] |_{\gamma=0}, \quad [\partial^\gamma_{b_2} \frac{1}{y(b_2)}] |_{\gamma=0} = [\partial^\gamma_{b_2} \frac{1}{y(b_2)}] |_{\gamma=0},
\]

which yields

\[
\tilde{z}(b) = \partial^\gamma_b x(b) |_{\gamma=0} = \sum_{b_1 \cup b_2 = b} [\partial^\gamma_{b_1} x(b_1)] [\partial^\gamma_{b_2} \frac{1}{y(b_2)}] |_{\gamma=0}
\]

\[
= \sum_{b_1 \cup b_2 = b} (\prod_{j \in b_2} \langle r_j, \phi_j \rangle) [\partial^\gamma_{b_1} x(b_1)] [\partial^\gamma_{b_2} \frac{1}{y(b_2)}] |_{\gamma=0}
\]

\[
= \sum_{b_1 \cup b_2 = b} \tilde{x}(b_1) \tilde{y}^{-1}(b_2)
\]

\[
= \tilde{x} \ast \tilde{y}^{-1}(b),
\]

(86) where we have the new \( M \)-maps

\[
\tilde{x}(b) = \partial^\gamma_b x(b) |_{\gamma=0},
\]

\[
\tilde{y}(b) = (\prod_{j \in b} \langle r_j, \phi_j \rangle) \partial^\gamma_b y(b) |_{\gamma=0}.
\]

(87) \( 88 \) With the aid of \( 15 \) we thus find that the cumulant of the \( M \)-map \( \tilde{z} \) can be decomposed as

\[
\log^* \tilde{z}(a) = \log^* \tilde{x}(a) - \log^* \tilde{y}(a).
\]

(89) We next turn to the evaluation of \( \log^* \tilde{x}(a) \). Equations \( 81 \) and \( 87 \) imply

\[
\tilde{x}(a) = \frac{1}{\langle \psi_f | U_{n+1} U_n \cdots U_1 | \psi_i \rangle^2} \bigg| \langle \psi_f | U_{n+1} U_n \cdots U_1 | \psi_i \rangle \bigg|^2
\]

\[
= \sum_{r_j L_j} \left\{ \begin{array}{ll}
\rho_j L_j & \text{if } j \in a \\
1 & \text{if } j \notin a
\end{array} \right\}
\]

(90) This should be compared with \( 33 \) in the proof of Theorem \( IV.1 \). Just as in that proof, where we defined \( w_{c_1, c_2} \), here we define \( \tilde{x}_{c_1, c_2} \) where \( L_j \) is replaced by \( L_j^{\text{left}} \) if \( j \in c_1 \) and by \( L_j^{\text{right}} \) if \( j \in c_2 \). By the same argument, we find

\[
\log^* \tilde{x}(a) = \log^* \tilde{x}_{a, 0}(a) + \log^* \tilde{x}_{\emptyset, a}(a),
\]

(91) and \( \tilde{x}_{\emptyset, a}(a) \) gives the complex conjugate. Thus

\[
\log^* \tilde{x}(a) = \text{Re} \{ 2(-i)^{|a|} (\prod_{j \in a} \langle r_j, s_j \rangle \phi_j) \log^* A_w(a) \}
\]

(92) and this gives \( 39 \) with the first part of \( \xi \) (see \( 40 \)). The evaluation of the cumulant of \( \log^* \tilde{y}(a) \) is analogous to the above, and results in the second half of \( \xi \).
VIII. MULTISETS AND MULTISET CUMULANTS

Our definition of cumulants in Section III was somewhat unconventional. A more standard definition for the classical cumulant is

$$\log^* (\Pi_{j \in a} X_j) = \partial^\gamma \log (e^{\sum_{k=1}^{|a|} \gamma_k X_k})|_{\gamma = 0},$$

(93)

which is readily seen (Appendix) to be equivalent to ours given by [25]. Thus cumulants can be thought of as coefficients in a formal power series expansion in the $\gamma$'s. To a statistician, the expression being expanded is the logarithm of the moment generating function. To a physicist, a comparison with the Helmholtz free energy, [25], is compelling, and indeed there is a strong connection with thermodynamics [10]. A combinatorialist can also stake a claim [30, 37].

The formal expansion for two variables begins

$$\log \langle e^{\gamma_1 X_1 + \gamma_2 X_2} \rangle = \gamma_1 \langle X_1 \rangle + \gamma_2 \langle X_2 \rangle - \frac{\gamma_1^2}{2} \langle X_1^2 \rangle - \langle X_1 \rangle^2 - \frac{\gamma_2^2}{2} \langle X_2^2 \rangle - \langle X_2 \rangle^2 - \gamma_1 \gamma_2 \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle + \ldots,$$

(94)

and the coefficient of $\gamma_1 \gamma_2$ is familiar as the classical cumulant $\log^* \langle X_1 X_2 \rangle$. However, this expansion also forces on one's attention terms in higher powers of the $\gamma$s, such as $\gamma_1^2 (\langle X_1^2 \rangle - \langle X_1 \rangle^2)/2$. The moment algebra, as we introduced it in Sec. III, does not include such terms. However, with a slight modification of the construction of the algebra, all of these higher order terms can be incorporated. The natural setting for it is not subsets of a set, but multisets [38]. In a multiset, an element may occur any finite number of times. For instance, if the underlying set is $\Omega_3 = \{1, 2, 3\}$, then $\{1,1\}$, and $\{1,1,1,2,3,3\}$ are two examples of multisets in $\Omega_3$ (Note that, as with ordinary sets, the ordering does not matter, i.e., $\{1,1,2,3,3\} = \{3,1,2,3,1,1\}$.)

To extend the moment algebra we define $\mathcal{M}$-maps on multisets rather than subsets, so an $\mathcal{M}$-map $f$ assigns a complex number $f(a)$ to every multiset $a$ of $\Omega$. The whole machinery of formal derivatives and their action on composite functions, as presented in Sec. III goes through essentially unaltered. In particular, given an $\mathcal{M}$-map $f$ defined on multisets, we can define higher order cumulants $\log^* f(a)$ for any multiset $a$.

As an example, consider the multiset that consists of $\{1\}$, $\{1,1\}$, $\{1,1,1\}$, etc.. Suppose $f$ is a function on this multiset. Then we can apply (8) to $a = \{1,1,1\}$, for instance, so that from

$$\partial_{\{1,1,1\}} \log f = \frac{f'''}{f} - \frac{3 f'' f}{f^2} + 2 \frac{(f')^3}{f^3},$$

(with primes denoting differentiation by variable 1), we deduce

$$\log^* f(\{1,1,1\}) = \frac{f(\{1,1,1\})}{f(\emptyset)} - 3 \frac{f(\{1,1\})f(\{1\})}{f(\emptyset)^2} + 2 \frac{f(\{1\})^3}{f(\emptyset)^3}.$$

As a special case, we define $f(\{1,1,\ldots,1\}) = \langle X^k \rangle$, for some random variable $X$. Then we can rewrite the above expression as

$$\log^* f(\{1,1,1\}) = \langle X^3 \rangle - 3 \langle X^2 \rangle \langle X \rangle + 2 \langle X \rangle^3 \equiv \kappa_3(X).$$

In this way we obtain the classical cumulant $\kappa_3(X)$, some others in the series being $\kappa_1(X) = \langle X \rangle$ and $\kappa_2(X) = \langle X^2 \rangle - \langle X \rangle^2$. These cumulants are widely used in statistics. They have nice properties; for instance, $\kappa_j(X) = 0$ for $j \geq 3$ is a necessary and sufficient condition for $X$ to be Gaussian [35].

It is easy to see that any multiset cumulant can be obtained from our original cumulant with distinct variables simply by setting certain subsets of its variables equal. For instance, $\log^* f(\{1,1,1\})$ can be derived from $\log^* f(\{1,2,3\})$ by setting every ‘2’ and ‘3’ to a ‘1’ (and leaving ‘1’s unchanged). Similarly $\log^* f(\{1,1,2\})$ can be obtained from $\log^* f(\{1,2,3\})$ by sending ‘3’ to ‘1’.

We can mimic this procedure in the case of multipartite weak measurements by treating subsets of pointers identically: i.e., we can couple all the pointers in each subset via the same system observable and afterwards measure the same pointer observable on each of them. As a simple case, suppose we have just two pointers and we couple both to the system through the interaction Hamiltonian $H = s \otimes A$ and finally carry out the same measurement $r$ on them; note though we have to give labels to $s$ and $r$ to indicate which pointer they belong to; so we have $s_1$, $s_2$ and $r_1$, $r_2$, etc.
We first note that respectively, for pointers 1 and 2. To avoid the complication of ordering of the couplings of pointers, let us consider simultaneous weak measurement. Then Theorem III.2 gives

$$\langle r_1 r_2 \rangle - \langle r_1 \rangle \langle r_2 \rangle = \gamma^2 \text{Re} \{ \kappa_2(A)_w \}$$

where we have suggestively written $\kappa_2(A)_w$ for the simultaneous weak value $(A^2)_w - (A_w)^2$ given by III.

This gives us a procedure for gaining information about the multiset cumulants of weak values. With thermal weak measurement we can use this procedure to measure higher order susceptibilities. If we couple $m_1$ independent pointers to observable $A_1$, $m_2$ pointers to $A_2$, etc, Theorem VII.I gives a direct relationship between the correlation of the pointers and the relevant susceptibility:

$$\log^* \langle \Pi_{j=1}^m \Pi_{l=1}^n r_j^{(l_j)} \rangle = -\beta \langle \Pi_{j=1}^m \Pi_{l=1}^n \gamma_{j,l} \rangle \xi \frac{\partial^{m_1}}{\partial \gamma_1^{m_1}} \cdots \frac{\partial^{m_n}}{\partial \gamma_n^{m_n}} F |_{\gamma=0}.$$  

(96)

**IX. CONCLUSIONS**

In physics we often study the effects of weak coupling between systems. By focussing on the effect of one system upon the other, weak measurement gives a way of understanding the nature of such interactions. If we weakly couple two systems, $S$ and $P$, say, and then carry out a strong measurement on $P$, the effects of the coupling can be expressed in terms of weak values [17, 18]. When $P$ consists of a very simple system, e.g., a “pointer” or particle in a given initial state, the measurement results depend on weak values in a very simple way [23]. As $P$ becomes more complicated, the dependency becomes rapidly more complicated, and in fact already assumes a highly baroque form when $P$ consists of two pointers applied at different times (see the Appendix of [28]). However, this complication vanishes if one takes cumulants: the cumulant of the measured variables and the cumulant of the weak values are once more simply related. The aim of this paper has been to give proofs of this fact that illuminate why this phenomenon occurs.

The proofs of our various theorems repeatedly use two properties of cumulants: first, that they are logarithms in the moment algebra $\mathcal{M}_n$, and turn a convolution product into a sum; second, that they vanish on maps that factorise, i.e., that can be written as a product of two maps that are defined on disjoint sets of variables. All the maps that we construct are elements of $\mathcal{M}_n$, which is therefore the natural setting for the proofs. This algebra is in itself an interesting object. Although some of its features have been thoroughly described in the literature, we are not aware of any explicit formulation of the algebra as an object in its own right. We feel that it deserves this recognition, because of the simplicity of the definitions of its operations, and the surprising richness of the structure that this gives rise to.

Cumulants have long played a role in statistical mechanics [5, 8, 9, 10], where they are used to simplify perturbation expansions. This suggests alternative weak measurement scenarios. In this spirit we consider a collection of pointers that reach thermal equilibrium with a probed system, and call this a “thermal weak measurement”. By its very nature, this excludes pre- and postselection, which are standard components of weak measurement. We lose thereby some of the strengths of weak measurement: many of the more intriguing phenomena in the standard setting arise from postselection. However, we retain the advantages of minimal perturbation of a system, and the possibility of applying several probes simultaneously or sequentially opens up some new territory for exploration.

**X. APPENDIX: DEFINING CUMULANTS VIA GENERATING FUNCTIONS**

Here we show that the standard definition of the classical cumulant via the generating function (98) is equivalent to our definition (13) of $\log^* f$ for the $\mathcal{M}$-map $f(a) = \langle \Pi_{j \in a} X_j \rangle$. In other words, we wish to show that, for any multiset $a$,

$$\log^* f(a) = \partial^a \log h(\gamma)|_{\gamma=0},$$

(97)

where

$$h(\gamma) = \langle e^{\sum_k \gamma_k X_k} \rangle.$$

(98)

We first note that

$$\partial^a \log h = \Lambda(h, \partial^1 h, \partial^2 h, \partial^2_{1,2}, \ldots),$$

(99)
where $\Lambda$ is a function of all the relevant partial derivatives of $h$, obtained via the chain rule when we apply $\partial^2_{a} h$ to the logarithm. The same function $\Lambda$ appears when we express the cumulant log to the formal derivatives

$$\log^\ast f(a) = \partial_a \log f(a) = \Lambda(f, \partial_1 f, \partial_2 f, \partial_1 2f, \ldots)$$

(100)

$$= \Lambda(f(\emptyset), f(1), f(2), f(1, 2), \ldots).$$

To obtain (97) from (99) and (100) we only need observe that $\partial^2_{a} h|_{\gamma=0} = \langle \Pi_{j \in C} X_j \rangle = f(c)$. This equivalence of definitions can be extended to other functions $h$ if we take $f(a) = \partial^2_{a} h|_{\gamma=0}$. For example, let $h(\gamma) = \text{tr}(e^{-\beta H_c}) = \text{tr}(e^{-\beta H_S - \sum_i \gamma_i A_i})$. Then the above arguments prove (80), which occurs in the proof of Theorem VI.1. Note that the arguments extend without difficulty to non-commutative observables.

[1] T. N. Thiele, *The Theory of Observations* (C & E Layton, London, 1903).
[2] T. N. Thiele, Ann. Math. Stat. 2, 165 (1931).
[3] R. A. Fisher, Proc. London Math. Soc. Ser. 2, 30, 199 (1929).
[4] C. C. Craig, Ann. Math. Stat. 2, 154 (1931).
[5] G. S. Sylvester, Commun. Math. Phys. 42, 209 (1975).
[6] R. A. Fisher and J. Wishart, Proc. London Math. Soc. Ser. 2, 33, 195 (1931).
[7] M. Kendall and A. Stuart, *The advanced theory of statistics, Volume 1* (Charles Griffin, London and High Wycombe, 1977).
[8] R. Kubo, J. Phys. Soc. Japan, 17, 1100 (1962).
[9] B. Kahn and G. E. Uhlenbeck, Physica 5, 399 (1938).
[10] A. Royer, J. Math. Phys. 24, 897 (1983).
[11] P. Fulde *Electron Correlations in Molecules and Solids*, 3rd Ed. (Springer Verlag, New York, 1995).
[12] K. Klaidko and P. Fulde, Int. J. Quant. Chem 66, 377 (1998).
[13] G. Archontis and M. Karplus, J. Chem. Phys. 105, 11246 (1996).
[14] X.-G. Wen. Quantum field theory of many-body systems (Oxford University Press, Oxford, 2004).
[15] P. Carruthers, Phys. Rev. A 43, 2632 (1991).
[16] D. L. Zhou, B. Zeng, Z. Xu, and L. You, Phys. Rev. A 74, 052110 (2006).
[17] Y. Aharonov and D. Rohlich, Quantum Paradoxes (Wiley-VCH, Weinheim, Germany, 2005).
[18] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).
[19] Y. Aharonov, D. Z. Albert, A. Casher, and L. Vaidman, Phys. Lett. A 124, 199 (1987).
[20] Y. Aharonov, A. Botero, S. Popescu, B. Reznik, and J. Tollaksen, Phys. Lett. A 301, 130 (2002).
[21] L. Hardy, Phys. Rev. Lett. 68, 2981 (1992).
[22] N. Brunner, A. Acín, D. Collins, N. Gisin, and V. Scarani, Phys. Rev. Lett. 91, 180402 (2003).
[23] D. R. Solli, C. F. McCormick, and R. Y. Chiao, Phys. Rev. Lett. 92, 043601 (2004).
[24] K. J. Resch and A. M. Steinberg, Phys. Rev. Lett. 92, 130402 (2004).
[25] K. J. Resch, J. Opt. B 6, 482 (2004).
[26] J. S. Lundeen and K. J. Resch, Phys. Lett A 344, 337 (2005).
[27] G. Mitchison, R. Jozsa, and S. Popescu, Phys. Rev. A 76, 062105 (2007).
[28] G. Mitchison, Phys. Rev. A 77, 052102 (2008).
[29] W. B. Johnson, Am. Math. Monthly 109, 217 (2002).
[30] J. K. Percus, Commun. Math. Phys. 40, 283 (1975).
[31] B. Simon Functional Integration and Quantum Physics (Academic Press, New York, San Francisco, London, 1979).
[32] Y. Aharonov, P. G. Bergmann, J. L. Lebowitz, Phys. Rev. 134, B1410 (1964).
[33] Y. Aharonov, L. Vaidman, J. Phys. A: Math. Gen. 24, 2315 (1991).
[34] L. D. Landau and E. M. Lifshitz, Statistical Physics 3rd Ed. (Pergamon Press, Oxford, 1990).
[35] M. Reed and B. Simon Methods of Modern Mathematical Physics I: Functional Analysis (Academic Press, San Diego, London, Toronto, 1980).
[36] M. Aigner A Course in Enumeration (Springer, Berlin, Heidelberg, 2007).
[37] H. S. Wilf generatingfunctionology (Academic Press, Boston, San Diego, New York, 1990).
[38] W. D. Blizard, Notre Dame Journal of Formal Logic, 30, 36 (1989).
[39] J. Marcinkiewicz, Math. Z. 44, 612 (1939).