Study on Fractal Model for Predicting Permeability and Effective Thermal Conductivity to Fibres of Cut-Tobacco

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Abstract. The permeability and the effective thermal conductivity of cut-tobacco were studied using two-dimensional fractal theory. The porous structure of cut-tobacco samples on their cross sections was analysed and their box-counting dimensions were calculated. The permeability process of gas through cut-tobacco has been analysed and fractal model of the process has been established. The heat conduction process of a single hole was studied and the thermal conductivity of cut-tobacco was obtained by the thermal resistance method. The results show that the fractal dimension of porous structure of cut-tobacco approximately is equal to 1.3. With the increasing of the fractal dimension, high porosity cut-tobacco is in favor of permeability process. Oppositely, with the decreasing of the fractal dimension, the porosity of cut-tobacco decreases corresponding to the component of cut-tobacco increasing which lead to the strength of the thermal conductivity coefficient. Comparison between theoretical and experimental results, the results show that the fractal model can effectively predict the permeability and effective thermal conductivity to fibres of cut-tobacco.

1. Introduction
Cigarettes are tobacco products made by wrapping rolling paper around cut tobacco. They constitute a substantial market in the world. During smoking, mainstream smoke is emitted from the end of a lit cigarette. Numerous chemical changes take place in the anoxic hydrogen-rich environment as the burning tobacco produces carbon monoxide, carbon dioxide, hydrogen, and volatile hydrocarbons at different temperatures. To facilitate smoking cessation and reduce the harm of smoking, the market has launched various novel tobacco products that mainly consist of three types—smokeless tobacco, heat-not-burn tobacco, and electronic cigarettes [1]. While looking like traditional cigarettes, heat-not-burn products heat the tobacco at under 500 °C, just enough to release its flavour without burning it. Related studies show that traditional cigarettes produce an enormous amount of harmful substances at high temperatures of 350 °C to 600 °C. Meanwhile, low-temperature e-cigarettes discharge much fewer of these substances because they are heated under 300 °C. In addition, heat-not-burn products result in only 5% of the substances and 14% of the cytotoxicity produced by traditional cigarettes. Compared with other novel cigarettes, their flavour can better satisfy consumers’ needs and provide them with an authentic tobacco taste [2].

Inside heat-not-burn cigarettes is a porous material made of cut tobacco. Its structure, composition, and crafting affect its porosity, density, and, consequently, the air permeability and heat transfer during inhalation. Studies on the permeability and heat transfer coefficient are useful for developing a pyrolysis model of cut tobacco, which has a complex structure of randomness and irregularity. A substantial number of studies have indicated that the microstructure, pore size, and pore distribution of fibrous porous media made of cut tobacco carry fractal properties and that the transport and heat transfer in such media can be analysed with the fractal theory and method [3].
The fractal theory was devised by Mandelbrot during his study on the irregular and self-similar geometry of coastlines. In recent years, the theory and its applications have been developing so rapidly that it has become one of the most important topics in nonlinear science. Katz and Thompson [4] were the first to provide experimental data to justify that the pore spaces of sandstone samples follow a self-similar and fractal scaling law. The fractal theory has become an effective approach for studying the structure and transport of porous media. Adler [5] computed the thermal conductivity of the Sierpinski carpet, a fractal porous medium, with certainty. Based on the fractal geometric theory, Yu et al. [6, 7] proposed a scaling law followed by the cumulative pore-size distribution in porous media; based on the exact self-similar Sierpinski carpets, Yu and Li [6] obtained the correlation between the porosity and fractal dimensions of porous media. Liu [8] used a fractal model to describe the hydrate saturation and pore-scale behavior dependent permeability of hydrate-bearing sediments. Yu et al. [9] adopted the fractal theory to study the porous structure of wood sections perpendicular to fibres and obtained the effective thermal conductivity perpendicular to the grain. Yu et al. [6] employed the fractal theory to study the cut-tobacco size distribution and found its fractal properties and statistically significant self-similarity.

This study investigated the porous microstructure of cut tobacco using the fractal theory. First, it used the photomicrograph of a section of cut tobacco to analyse the fractal properties of the fine-cut tobacco in a cigarette and its porous structure. It also used the box-counting dimension to compute the fractal dimension of the porous tobacco. The effective permeability of the fractal porous media was then calculated with both the Hagen–Poiseuille equation and Darcy’s law. Then, the energy equation was employed to reveal the correlation between the thermal diffusivity and porosity of the cigarette. Eventually, this study compared the results of the fractal model and the experiment to prove the effectiveness of the fractal theory in predicting the permeability and heat transfer of cut tobacco.

2. Internal Microstructure of Cigarettes
A cigarette is a typical fibrous medium with a complicated inside porous structure. The experiment in this study selected a novel heat-not-burn cigarette available in the market as the target sample, the central part of which was observed with the HD4i VED tobacco test system manufactured by Rinc and Asia PTE Ltd. at 10X magnification. The photomicrograph is shown in Figure 1.

![Figure 1. Sectional view of cut tobacco](image-url)
From the photomicrograph, it is evident that the pores between the tobacco fibres display intricate and messy boundaries, random distribution, and great discrepancies in diameter. Irregularity and self-similarity are manifest in the porous structure.

Analysis of the porous structure reveals the difficulty in using conventional Euclidean geometry to describe its characteristics: no shapes with regular dimensions suffice to characterize the complexity of the porous structure. With reference to the properties of fractal figures, the porous medium exhibits a random coherent structure and a certain level of self-similarity. Hence, the porous structure of cut-tobacco fibres can be categorized as a fractal structure [10] and thus examined with the fractal theory.

3. Computation of Fractal Dimension
Fractal dimension is an important parameter for describing fractals and representing their basic properties. It is a tool for comparing fractals objectively and distinguishing one from another based on their irregularity. Its most distinctive characteristic is that it accepts fractional dimensions.

The fractal dimension is expressed as follows [11]:

$$D = \frac{\ln(N)}{\ln(s)}$$  \hspace{1cm} (1)

In this equation, $D$ denotes the fractal dimension and $N$ is the number of parts in the scaled down version of set $A$ by ratio $s$ that does not cover set $A$.

The commonly used fractal dimensions include the Hausdorff dimension, the self-similarity dimension, and the box-counting dimension. Among them, the box-counting dimension can perform approximation through experiments and has wide applications [12]. Chaudhury and Sarkar [13] suggested the calculation of fractal dimensions by differential box-counting. The computational thinking is: consider $A$ as an image with $M \times M$ pixels and $S$ a grid with $s \times s$ pixels; $A$ is partitioned by $S$, and $N(s)$ denotes the smallest number of boxes on $S$ needed to cover $A$. The box-counting dimension $D(A)$ is expressed as follows:

$$D(A) = \lim_{s \to 0} \frac{\ln(N(s))}{-\ln(s)}$$  \hspace{1cm} (2)

During the experiment, the first step of using the box-counting dimension to calculate the fractal dimension of a two-dimensional object was to pre-process and convert the digital image into a black-and-white binary image. The next step was to cover the fractal image with a grid consisting of square boxes measuring on each side and then count the number of boxes that contain image pixels, $N(s)$, which represent the black parts of the pore spaces. The same process was repeated by decreasing the box size of grids until the smallest square covering the pixels is found. Then, a linear regression line was fit to the log-log plot ($\ln(N(s)) - \ln(1/s)$) of the obtained data. The slope of the linear fit is the fractal dimension of the image. From equation (2), we can get:
\[ N(s) \sim (s)^{D(A)} \] 

(a) Sample 1  
(b) Sample 2

**Figure 3.** Curve fit of the box-counting dimension

An image of the porous structure is taken from the photomicrograph produced by the tobacco test system and then converted into a black-and-white binary image as shown in Figure 2. Analysis of the fractals in Figure 2 returns Figure 3, the curve fit of the box-counting dimension of the porous structure. Results show that the box-counting dimension demonstrated good linearity. The slope \( D \) of the linear fit represents the required box-counting dimension. The fractal dimension is smaller than 2 because the pores do not fill the entire plane. It is also relatively small because of the low porosity of the section. Hence, the value is between 1 and 1.5.

### 4. Fractal Model of Cigarette Permeability

The volume of airflow \( q \) passing through the capillaries in the porous structure satisfies the Hangen–Poiseulle equation [14]:

\[ q = \frac{\pi \Delta p \lambda^4}{128 L(\lambda) \mu} \]  

(4)

In this equation, \( L(\lambda) \) is the length of capillary with pore size as \( \lambda \), \( \mu \) is the viscosity of air, and \( \Delta p \) is the pressure difference between the two ends of a capillary.

Based on the fractal model of the capillaries in the cut tobacco, assuming the air passage resembles a straight line, the total volume of airflow \( Q \) can be obtained from the integrals in equation (4), \( l \) is the length of the capillary cell.

\[ Q = \lambda_{\text{max}} \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} q(\lambda) dN(\lambda) = \frac{\pi \Delta p}{128 \mu l} \frac{D}{4-D} \lambda_{\text{max}}^4 \]  

(5)

Integrating Darcy’s law, we obtain the effective permeability of the fractal porous medium:

\[ \zeta = \frac{\mu l Q}{\Delta p A} = \frac{\pi}{128 \frac{A}{A}} \frac{D}{4-D} \lambda_{\text{max}}^4 \]  

(6)
Based on the structural parameters of the cigarette, namely the sectional area $A$, length $l$, the maximum pore size $\lambda_{max}$, and the fractal dimension $D$, the permeability was computed as shown in Figure 4. The experimental results were in line with the theoretical results of the fractal model. The permeability increased with the fractal dimension because a higher fractal dimension leads to higher porosity, which facilitates the effusion of air. When the fractal dimension equal to 2, maximum permeability was achieved.

5. Fractal Model of Effective Thermal Conductivity

During inhalation, the heat-not-burn cigarette transfers heat through the thermal conductivity of the cut tobacco. It is essential to build a two-dimensional model of how the effective thermal conductivity changes with the fractal dimension. The unit structure of the cut tobacco can be simplified as in Figure 5. The convection and radiation in the pore spaces are ignored, and heat transfer is assumed a pure process of thermal conduction. Assume a uniform thickness of cut tobacco and a transfer of heat flux $\phi$ in the $x$-direction. The thermal resistance network of the porous structure is illustrated in Figure 6: the thermal resistance in each part can be expressed as follows:

$$r_1 = r_3 = l/(hk_{\text{w}})$$  \hspace{1cm} (7)

$$r_2 = r_5 = b/(hk_{\text{w}})$$  \hspace{1cm} (8)

$$r_3 = (l-2b)/(hk_{\text{w}})$$  \hspace{1cm} (9)

The total thermal resistance is:
Thus, the effective thermal conductivity is:

\[ k_i = \frac{(4b^2 + hl)k_wk_d + 2b(l - 2b)k_d^2}{2b(h + 2b)k_d + (h + 2b)(l - 2b)k_w} \]  

(11)

Substitute porosity \( \varepsilon \) into the equation. The above equation is transformed to:

\[ k_i = \frac{0.5(1 + \varepsilon)^2k_d k_w + \varepsilon (1 - \varepsilon)k_w^2}{\varepsilon (1 - \varepsilon)k_d + 2\varepsilon k_w} \]  

(12)

where

\[ \varepsilon = \left( \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}} \right)^{2-D} \]  

(13)

Through substitution of the above equation, we obtain the effective thermal conductivity of the cigarette.

Figure 6. Illustration of thermal resistance network.

Figure 7. Correlation between fractal dimension and effective thermal conductivity

The thermal conductivity \( k_w \) of solid tobacco is 0.086 W/mK. Assuming the pores are filled with air, we obtain the thermal conductivity of \( k_a = 0.024 \) W/mK. The computational results are shown in Figure 7. To test the reliability of the fractal model, the thermal conductivity of different pores was measured with an apparatus for measuring the thermophysical properties based on the transient plane source method manufactured by Hot Disk AB. During measurement, the temperature distribution in the sample constitutes an unsteady temperature field that is constantly changing over time. The rate of temperature change is used to measure the heat diffusivity of the sample and, in turn, the thermal conductivity of the material. This method is suitable for measuring a wide range of thermal
conductivity and has a short measurement time and high measurement accuracy. Figure 7 shows that the results of experimental and computational are consistent which means the computational results of the fractal model are reliable. The effective thermal conductivity declined as the fractal dimension increased. The lower the fractal dimension, the smaller the porosity, which means a higher concentration of tobacco. Since the cut tobacco has a higher thermal conductivity than air, the porous medium has greater thermal conductivity. However, the closer the fractal dimension is to 2, the greater the porosity, which means a higher concentration of air and, consequently, lower thermal conductivity.

6. Conclusion
This study used the fractal theory to build two-dimensional computational models of the fractal dimension, fractal permeability, and effective thermal conductivity of the porous medium of cut tobacco. According to the comparison of the model outcomes and the experimental results, the fractal model was effective in predicting the permeability and heat transfer of the cut tobacco. The fractal dimension of porous structure of cut-tobacco approximately is equal to 1.3 The permeability increases and the effective thermal conductivity declines as the fractal dimension increasing. This provides a significant theoretical foundation and research method for follow-up studies.

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