Energy momentum tensor of scalar field with higher order derivatives

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Abstract. We considered a formulation of the energy momentum tensor of scalar field containing higher-order derivatives. Starting from the Lagrangian density, we can formulate the Hamiltonian density as well as the three-component momentum density by applying the total variation on the field. We claimed that this formulation can be used to derive some nonlinear wave equation, such as the KdV equation and the Burgers equation.

1. Introduction
Generally, physics contains two fundamental aspects, i.e., the particle and wave phenomena. In the early days, those two phenomena were considered as two separated aspects by classical physics. A thorough concern on the particle aspect was fully considered by Isaac Newton based on his famous Newton’s laws while the wave aspect was fully described by J. C. Maxwell through his famous Maxwell equations.

At the beginning of 20th century, some experiment results cannot be explained by considering the classical physics. One way to understand the phenomena was to unify the particle and wave aspects. This means that there is a dualism between a particle and a wave but never been observed simultaneously. To study the dualism, a field theory was then introduced in the modern physics and it is still used until now.

In the field theory, we always dealt with wave equation, by which the energy and other physical quantities can be obtained. Two mathematicians named J. L. Lagrange and W. R. Hamilton constructed the Lagrangian and Hamiltonian equations, both of them represent the motion equation. Through those two equations, one can predict the evolution of system and its characteristics due to the external potentials.

This paper discussed the energy momentum tensor containing the higher order derivatives. This form is the extension of the ordinary energy momentum tensor, which only contains second order derivative. One of the interesting discussion is that this formula can be used to consider the dynamic equation of nonlinear wave equation. Some literatures that discussed the solitary wave/soliton solution or the symmetry of nonlinear wave can be found in Refs. [1-8]. In addition, with this formulation, we are able to calculate the energy of the soliton solution from the nonlinear wave equations, if any, by using the Hamiltonian density.
2. Methods
In this section, we derive the ordinary energy momentum tensor which contains second order derivatives. By adopting this method, we can derive the higher order case, as explained in the next section. Consider the classical action of the field [9].

\[ S = \int_R \mathcal{L}(\varphi, \partial_\mu \varphi; x^\mu) \, d^4x, \]  

where \( \mu = 0, 1, 2, 3 \) is the flat space-time index. Besides, we also use the convention of the derivation operator

\[ \partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right), \]  

and its inverse

\[ \partial^\mu = \eta^{\mu\nu} \partial_\nu = \left( \frac{\partial}{\partial t}, -\nabla \right), \]  

with the definition of Minkowski tensor \( \eta_{\mu\nu} = (+, -, -, -) \). Through Eq. (1), the action variation can be written as Ryder [9].

\[ \delta S = \int_R \left( \delta \mathcal{L} + \mathcal{L} \delta x^\mu \right) d^4x, \]  

with the variation of Lagrangian density

\[ \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu, \]  

and the related transformation of space-time coordinates as

\[ x'^\mu = x^\mu + \delta x^\mu, \]  
\[ \varphi'(x^\mu) = \varphi(x^\mu) + \delta \varphi(x^\mu). \]  

Using an algebra and the Gauss theorem, we find two independent equations on the right-side

\[ \delta S = \int_R \left( \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right] \right) \delta \varphi d^4x + \int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi + \mathcal{L} \delta x^\mu \right) ds_\mu. \]  

On the right-side in Eq. (8), the first and second equations describe the motion equation and surface integral, respectively. To obtain the motion equation, we apply the boundary condition that on the \( R \) boundary, the variation for \( \varphi \) dan \( x^\mu \) should vanish

\[ \delta \varphi = 0, \ \delta x^\mu = 0. \]  

Applying the above condition and the minimum action principle \( \delta S = 0 \), we obtain the Euler-Lagrange equation from Eq. (8).

\[ \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right] = 0. \]  

To obtain the energy momentum tensor, we employ the total variation on the surface integral

\[ \varphi'(x'^\mu) = \varphi(x^\mu) + \Delta \varphi(x^\mu), \]  

with

\[ \Delta \varphi(x^\mu) = \delta \varphi(x^\mu) + (\partial_\mu \varphi) \delta x^\mu, \]  

which leads to

\[ \int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi + \mathcal{L} \delta x^\mu \right) d\sigma_\mu = \int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial \varphi} \frac{\delta \varphi + (\partial_\mu \varphi) \delta x^\alpha}{\Delta \varphi} \right) d\sigma_\mu \]  
\[ + \int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \partial_\gamma \varphi - \delta^\nu_\mu \mathcal{L} \right) \delta x^\nu d\sigma_\mu. \]  

Through Eq. (13), the energy momentum tensor can be formulated as
\[ T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \]  

(14)

With the above definition, we can conclude that the components of the energy-momentum tensor can be written as

- Hamiltonian density/energy density
  \[ T^0_0 = \frac{\partial \mathcal{L}}{\partial (\partial_\phi \phi)} \partial_\phi \phi - \mathcal{L}. \]  

(15)

- Momentum density
  \[ T^i_0 = \frac{\partial \mathcal{L}}{\partial (\partial_\phi \phi)} \partial_i \phi. \]  

(16)

with \( i = 1, 2, 3 \).

The above tensor in Eq. (14) is the ordinary energy-momentum tensor, which is always discussed in the textbook. In addition, the Euler-Lagrange in Eq. (8) can be used to derive one of the nonlinear wave equations, namely, the sine Gordon equation. The discussion that considers the Lagrangian of the nonlinear equation can be found in the textbooks [10,11]

3. Results and discussions

In this section, we adopt the previous method to derive two kinds of energy-momentum tensor, which can be used to derive the other nonlinear wave equation. For the intention, we divide into two forms of the Lagrangian density.

3.1 The Lagrangian density \( \mathcal{L}(\phi, \partial_\mu \phi, \partial^\mu \partial_\mu \phi; x^\mu) \)

The action of the Lagrangian density can be written as

\[ \delta S = \int_R \left( \delta \mathcal{L} + \mathcal{L} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \phi \right) d^4 x, \]  

(19)

with the variation of the lagrangian density

\[ \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial (\partial^\alpha \partial_\alpha \phi)} \delta (\partial^\alpha \partial_\alpha \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu. \]  

(20)

Then, by using the Gauss theorem, the surface integral can be found by decomposing Eq. (19)

\[ \delta S = \int_R \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}\right] + \partial^\mu \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial^\alpha \partial_\alpha \phi)}\right] \right) \delta \phi d^4 x + \int_{\partial R} \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi + \mathcal{L} \delta x^\mu + \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial^\alpha \partial_\alpha \phi)} \right] \delta \phi \right\} d x_\mu. \]  

(21)

Applying the boundary condition in Eq. (9) and employing the minimum action, we obtain the Euler-Lagrange

\[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}\right] + \partial^\mu \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial^\alpha \partial_\alpha \phi)}\right] = 0. \]  

(22)

By the same analogy, i.e., applying the total variation on field as in Eq. (11), we obtain the energy-momentum tensor

\[ T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - 2 \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial^\alpha \partial_\alpha \phi)}\right] \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \]  

(23)

This Lagrangian density can be used to derive the KdV equation.

3.2 The Lagrangian density \( \mathcal{L}(\phi, \partial_\mu \phi, \partial^\mu \partial_\mu \phi, \partial^\alpha \partial^\alpha \partial_\mu \phi; x^\mu) \)

The action of the Lagrangian density is given by

\[ \delta S = \int_R \left( \delta \mathcal{L} + \mathcal{L} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \phi \right) d^4 x, \]  

(24)
while the total variation reads
\[
\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \delta_\mu \phi)} \delta (\partial_\alpha \delta_\mu \phi) + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \delta (\partial_\alpha \partial_\beta \phi) + \frac{\partial \mathcal{L}}{\partial \epsilon} \delta \epsilon \mu . \tag{25}
\]

By using the same analogy as in the previous subsection, we find the integral
\[
S = \int_R \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \delta_\mu \phi)} \delta (\partial_\alpha \delta_\mu \phi) \right) \delta \phi d^4 \chi
+ \int_R \left( -2 \partial^\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \delta \phi \right) - \partial^\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \delta \phi \right) + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \delta \phi \right) d\sigma_\mu .
\]

Therefore, with the same procedure, we obtain the Euler–Lagrange equation
\[
\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \partial_\alpha \partial_\beta \phi = 0 , \tag{26}
\]

and energy momentum tensor
\[
T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi)} \delta_\alpha \delta_\nu - 2 \partial^\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \delta_\alpha \phi \right) \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \partial_\beta \phi)} \delta_\alpha \phi \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} . \tag{28}
\]

This Lagrangian density can be used to derive the Burgers equation.

The other nonlinear equations, which derived from the Lagrangian density can also be found in Refs. [12-17]. Since the general relativity is fully nonlinear differential equation, the solitary wave solution/soliton may exist [18-20] and its energy can be calculated by means of the Hamiltonian density.

4. Conclusions

We have formulated the energy momentum tensor, which can be used to derive the linear and nonlinear wave equation, by introducing the ansatz Lagrangian density. This derivation is based on quantum field theory, which represents the dualism between particle and wave. This method can be extended for the vector field, such as Maxwell equation.

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