Hilfer-Hadamard fractional differential equations; Existence and Attractivity

Fatima Si Bachir\textsuperscript{a}, Saïd Abbas\textsuperscript{b}, Maamar Benbachir\textsuperscript{c}, Mouffak Benchohra\textsuperscript{d}

\textsuperscript{a}Laboratory of Mathematics and Applied Sciences, University of Ghardaia, 47000, Algeria.
\textsuperscript{b}Department of Mathematics, University of Saida–Dr. Moulay Tahar, P.O. Box 138, EN-Nasr, 20000 Saida, Algeria.
\textsuperscript{c}Department of Mathematics, Saad Dahlab Blida1, University of Blida, Algeria.
\textsuperscript{d}Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbê, P.O. Box 89, Sidi Bel-Abbê 22000, Algeria.

Abstract

This work deals with a class of Hilfer-Hadamard differential equations. Existence and stability of solutions are presented. We use an appropriate fixed point theorem.

Keywords: Hilfer-Hadamard fractional derivative, Schauder fixed-point Theorem, uniformly locally attracting.

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1. Introduction

The beginning of the fractional calculus in 1695, the fractional differential equation has been used in fields like mathematics, engineering, bioengineering, physics, etc.[16, 30], to see interesting results in the theory of fractional calculus and fractional differential equations, the reader may consult the monographs by; Abbas \textit{et al.} \cite{8, 9}, Kilbas \textit{et al.} \cite{22}, Oldham \textit{et al.} \cite{26}, Podlubny \cite{27}, Samko \textit{et al.} \cite{28}, Zhou \textit{et al.} \cite{33}, and the papers by Abbas \textit{et al.} \cite{3, 5}, Benchohra \textit{et al.} \cite{12}, Lakshmikantham \textit{et al.} \cite{23, 24, 25}. Other recent results are provided in \cite{11, 13, 17, 18, 19, 20, 21, 29, 31, 32}. Attractivity results for various classes of fractional differential equations are considered in \cite{1, 2, 4, 6, 10}.

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In [7], Abbas et al. studied some existence and Ulam stability results of the following problem

\[
\begin{align*}
(H D_{1+}^{\tau,\theta} i)(t) &= \chi(t, i(t)); \quad t \in [1, T], \\
(H I_{1+}^{1-\theta} i)(1) &= d, \quad \varrho = \tau + \theta(1 - \tau).
\end{align*}
\]

This work is devoted to the existence and attractivity of solutions of the following problem

\[
\begin{align*}
(H D_{1+}^{\tau,\theta} i)(t) &= \chi(t, i(t)); \quad t \in [c, +\infty), \quad c > 0, \\
(H I_{c+}^{1-\theta} i)(c) &= d, \quad \varrho = \tau + \theta(1 - \tau),
\end{align*}
\]

where \(d \in \mathbb{R}, \chi : [c, +\infty) \times \mathbb{R} \to \mathbb{R}, H I_{c+}^{1-\theta}\) is the left-sided Hadamard fractional of order \(\tau > 0\) and \(H D_{c+}^{\tau,\theta}\) is the Hilfer-Hadamard derivative operator of order \(\tau (0 < \tau < 1)\) and type \(\theta (0 \leq \theta \leq 1)\).

\section{Preliminaries}

We will introduce some spaces. We denote by \(C_{\varrho,\log}[c, e], (0 < c < e < \infty)\), the space \(C_{\varrho,\log}[c, e] = \{\nu : (c, e) \to \mathbb{R} : (\log \frac{1}{c})^{1-\varrho} \nu(t) \in C[c, e]\}\), with the norm

\[
\|\nu\|_{C_{\varrho,\log}} = \sup_{t \in [c, e]} \left| \left( \log \frac{1}{c} \right)^{1-\varrho} \nu(t) \right|.
\]

\(BC^*: = BC([c, +\infty))\) denotes the space continuous and bounded functions \(\nu : [c, +\infty) \to \mathbb{R}\).

\(BC_\varrho = \{\nu : (c, +\infty) \to \mathbb{R} : (\log \frac{1}{c})^{1-\varrho} \nu(t) \in BC^*\}\), with the norm

\[
\|\nu\|_{BC_\varrho} := \sup_{t \in [c, +\infty]} \left| \left( \log \frac{1}{c} \right)^{1-\varrho} \nu(t) \right|.
\]

Denote \(\|\nu\|_{BC_\varrho}\) by \(\|\nu\|_{BC^*}\).

\begin{definition} \cite{22} \end{definition}
Let \((c, e) (0 \leq c < e \leq \infty)\) and \(\tau > 0\). The Hadamard left-sided fractional integral \(H I_{c+}^{\tau} j\) of order \(\tau > 0\) is defined by

\[
(H I_{c+}^{\tau} j)(x) := \frac{1}{\Gamma(\tau)} \int_{c}^{x} \left( \log \frac{x}{t} \right)^{\tau-1} j(t) dt, \quad c < x < e.
\]

When \(\tau = 0\), we set

\[
H I_{c+}^{0} j = j.
\]

\begin{definition} \cite{22} \end{definition}
Let \((c, e)(0 \leq c < e \leq \infty)\) be a finite or infinite interval of the half-axis \(\mathbb{R}_+\) and let \(\tau > 0\). The Hadamard right-sided fractional integral \(H I_{e-}^{\tau} j\) of order \(\tau > 0\) is defined by

\[
(H I_{e-}^{\tau} j)(x) := \frac{1}{\Gamma(\tau)} \int_{x}^{e} \left( \log \frac{t}{x} \right)^{\tau-1} j(t) dt, \quad c < x < e.
\]

When \(\tau = 0\), we set

\[
H I_{e-}^{0} j = j.
\]

\begin{example} \end{example}
For each \(\tau > 0\) and \(\lambda \in \mathbb{R}\), we have

\[
H I_{1}^{\lambda}(\log x)^{\lambda-1} := \frac{\Gamma(\lambda)}{\Gamma(\tau + \lambda)} (\log x)^{\tau+\lambda-1}, \quad x \geq 1.
\]
Definition 2.4. \cite{22} The left-sided Hadamard fractional derivative of order \(\tau(0 \leq \tau < 1)\) on \((c, e)\) is defined by
\[
(\mathcal{H}D^{\tau}_{c^+} j)(x) = \frac{1}{\Gamma(1-\tau)} \left( x \frac{d}{dx} \right) \int_{c}^{x} \left( \log \frac{x}{t} \right)^{-\tau} j(t) \frac{dt}{t}, \quad c < x < e.
\]
In particular, when \(\tau = 0\) we have
\[
\mathcal{H}D^{0}_{c^+} j = j.
\]

Definition 2.5. \cite{22} The right-sided Hadamard fractional derivative of order \(\tau(0 \leq \tau < 1)\) on \((c, e)\) is defined by
\[
(\mathcal{H}D^{\tau}_{e^-} j)(x) = -\left( x \frac{d}{dx} \right) \frac{1}{\Gamma(1-\tau)} \int_{x}^{e} \left( \log \frac{t}{x} \right)^{-\tau} j(t) \frac{dt}{t}.
\]
In particular, when \(\tau = 0\) we have
\[
\mathcal{H}D^{0}_{e^-} j = j.
\]

Definition 2.6. Let \((c, e)\) be a finite interval of the half-axis \(\mathbb{R}_+\). The fractional derivative \(\mathcal{H}cD^{\tau}_{c^+} j\) of order \(\tau(0 < \tau < 1)\) on \((c, e)\) defined by:
\[
\mathcal{H}cD^{\tau}_{c^+} j = \mathcal{H}I^{1-\tau}_{c^+} \delta j,
\]
where \(\delta = x(d/dx)\), is called the Hadamard-Caputo fractional derivative of order \(\tau\).

Lemma 2.7. \cite{22} Let \(\tau > 0, \theta > 0\) and \(0 \leq \mu < 1\). If \(0 < c < e < \infty\), then for \(j \in C_{\mu,\log}[c, e]\) the equality
\[
\mathcal{H}I^{\tau}_{c^+} \mathcal{H}I^{\theta}_{c^+} j = \mathcal{H}I^{\tau+\theta}_{c^+} j
\]
does not hold.

Theorem 2.8. \cite{22} Let \(0 < \tau < 1\) and \(0 < c < e < \infty\). If \(j \in C_{\mu,\log}[c, e](0 \leq \mu < 1)\) and \(\mathcal{H}I^{1-\tau}_{c^+} j \in C^{1}_{\delta,\mu}[c, e]\) then
\[
(\mathcal{H}I^{\tau}_{c^+} \mathcal{H}D^{\tau}_{c^+} j)(x) = j(x) - \frac{\mathcal{H}I^{1-\tau}_{c^+} j(c)}{\Gamma(\tau)} \left( \log \frac{x}{c} \right)^{1-\tau},
\]
holds at any point \(x \in (c, e)\). If \(j \in C[c, e]\) and \(\mathcal{H}I^{1-\tau}_{c^+} j \in \mathcal{C}^{1}_{\delta,\mu}[c, e]\), then the relation holds at any point \(x \in [c, e]\).

Definition 2.9. (Hilfer-Hadamard fractional derivative). The left sided fractional derivative of order \(\tau(0 < \tau < 1)\) and type \(0 \leq \theta \leq 1\) with respect to \(x\) is defined by
\[
(\mathcal{H}D^{\tau,\theta}_{c^+} j)(x) = (\mathcal{H}I^{\theta(1-\tau)}_{c^+} \mathcal{H}D^{\tau-\theta\tau}_{c^+} j)(x).
\]

Corollary 2.10. \cite{21} Let \(\sigma \in C_{\theta,\log}(I)\). Then the problem
\[
\begin{align*}
(\mathcal{H}D^{\tau,\theta}_{c^+} i)(t) &= \sigma(t), \quad t \in I := [c, e] \\
(\mathcal{H}I^{\tau-\theta\tau}_{c^+} i)(c) &= d,
\end{align*}
\]
admits the following unique solution
\[
i(t) = \frac{d}{\Gamma(\theta)} \left( \log \frac{t}{c} \right)^{\theta-1} + (\mathcal{H}I^{\tau}_{c^+} \sigma)(t).
\]  \quad (2)

Lemma 2.11. Let \(\chi : (c, e) \times \mathbb{R} \to \mathbb{R}\) be a function such that \(\chi(\cdot, i(\cdot)) \in BC_{\theta}\) for any \(i \in BC_{\theta}\). Then the problem (1) is equivalent to the integral equation
\[
i(t) = \frac{d}{\Gamma(\theta)} \left( \log \frac{t}{c} \right)^{\theta-1} + (\mathcal{H}I^{\tau}_{c^+} \chi(\cdot, i(\cdot)))(t).
\]  \quad (3)

Let \(\emptyset \neq H \subset BC^*\) and let \(T : H \to H\). Let the equation
\[
(Ti)(t) = i(t).
\]  \quad (4)
Definition 2.12. Solutions of equation (4) are locally attractive if there exists a ball $B(i_0, \delta)$ in the space $BC^*$ such that, for any solutions $w = w(t)$ and $\Theta = \Theta(t)$ of equations (4) that belong to $B(i_0, \delta) \cap H$, we can write
\[
\lim_{t \to \infty} (w(t) - \Theta(t)) = 0.
\] (5)

If limit (5) is uniform with respect to $B(i_0, \delta) \cap H$, then (4) is uniformly locally attractive.

Lemma 2.13. [14] Let $P \subset BC^*$. Then $P$ is relatively compact in $BC^*$ if the following conditions are satisfied:

(a) $P$ is uniformly bounded in $BC^*$;

(b) the functions belonging to $P$ are almost equicontinuous in $\mathbb{R}_+$, i.e., equicontinuous on every compact set in $\mathbb{R}_+$

(c) the functions from $P$ are equiconvergent, i.e., given $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that
\[
\left| i(t) - \lim_{t \to \infty} i(t) \right| < \varepsilon,
\]
for any $t \geq M(\varepsilon)$ and $i \in P$.

Theorem 2.14. (Schauder Fixed-Point Theorem [15]). Let $X$ be a Banach space, let $D$ be a nonempty bounded convex and closed subset of $X$, and let $L : D \to D$ be a compact and continuous map. Then $L$ has at least one fixed point in $D$.

3. Existence and Attractivity Results

Definition 3.1. A measurable function $i \in BC_\rho$ is a solution of (1) if it verifies $(H I^c i)(c) = d$, and the equation $(H D^c i)(t) = \chi(t, i(t))$ on $[c, +\infty)$.

We will give the following hypotheses:

$(H_1)$ The function $t \mapsto \chi(t, i)$ is measurable on $[c, +\infty)$ for each $i \in BC_\rho$, and $i \mapsto \chi(t, i)$ is continuous.

$(H_2)$ There exists a continuous function $l : [c, +\infty) \to [0, +\infty)$ such that
\[
|\chi(t, i)| \leq \frac{l(t)}{1 + |i|} \text{ for a.e. } t \in [c, +\infty) \text{ and each } i \in \mathbb{R},
\]

and
\[
\lim_{t \to \infty} \left( \log \frac{t}{c} \right)^{1-\varepsilon} (H I^c t)(t) = 0.
\]

Set
\[
l^* = \sup_{t \in [c, +\infty)} \left( \log \frac{t}{c} \right)^{1-\varepsilon} (H I^c t)(t).
\]

Theorem 3.2. If $(H_1)$ and $(H_2)$ hold, then (1) has at least one solution which is uniformly locally attractive.

Proof. Define the operator $L$ by
\[
(Li)(t) = \frac{d}{\Gamma(\delta)} \left( \log \frac{t}{c} \right)^{\delta-1} + \frac{1}{\Gamma(\tau)} \int_c^t \left( \log \frac{s}{t} \right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s}.
\]
We can prove that the operator $L$ maps $BC_\varrho$ into $BC_\varrho$. Indeed; the map $L(i)$ is continuous on $[c, +\infty)$, and for any $i \in BC_\varrho$ and, for each $t \in [c, +\infty)$, we have

$$\left| \left( \log \frac{t}{c} \right)^{1-\varrho} (Li)(t) \right| \leq \frac{|d|}{\Gamma(\varrho)} + \frac{(\log \frac{t}{c})^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left( \log \frac{t}{s} \right)^{\tau-1} \left| \chi(s, i(s)) \right| \frac{ds}{s}$$

$$\leq \frac{|d|}{\Gamma(\varrho)} + \frac{(\log \frac{t}{c})^{1-\varrho}}{\Gamma(\tau)} \int_c^t \left( \log \frac{t}{s} \right)^{\tau-1} l(s) \frac{ds}{s}$$

$$\leq \frac{|d|}{\Gamma(\varrho)} + t^* := R^*,$$

so

$$\|L(i)\|_{BC_\varrho} \leq R^*. \quad (6)$$

Therefore, $L(i) \in BC_\varrho$, which proves that the operator $L(BC_\varrho) \subset BC_\varrho$. Equation (6) implies that $L$ maps $B_{R^*} := B(0, R^*) = \{ v \in BC_\varrho : \|v\|_{BC_\varrho} \leq R^* \}$ into itself.

**Step 1.** $L$ is continuous.

Let $\{i_n\}_{n \in \mathbb{N}}$ be a sequence converging to $i$ in $B_{R^*}$. Then,

$$\left| \left( \log \frac{t}{c} \right)^{1-\varrho} (Li_n) (t) - \left( \log \frac{t}{c} \right)^{1-\varrho} (Li) (t) \right|$$

$$\leq \frac{1}{\Gamma(\tau)} \int_c^t \left( \log \frac{t}{s} \right)^{\tau-1} \left| \left( \log \frac{t}{c} \right)^{1-\varrho} \chi(s, i_n(s)) - \left( \log \frac{t}{c} \right)^{1-\varrho} \chi(s, i(s)) \right| \frac{ds}{s}. \quad (7)$$

Case 1. If $t \in [c, T], T > 0$, then, since $i_n \to i$ as $n \to \infty$ and from the continuity of $\chi$, we get

$$\|L(i_n) - L(i)\|_{BC_\varrho} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Case 2. If $t \in (T, \infty), T > 0$, then (7) implies that

$$\left| \left( \log \frac{t}{c} \right)^{1-\varrho} (Li_n) (t) - \left( \log \frac{t}{c} \right)^{1-\varrho} (Li) (t) \right| \leq \frac{2(\log \frac{t}{c})^{1-\varrho}}{\Gamma(\tau)}$$

$$\times \int_c^t \left( \log \frac{t}{s} \right)^{\tau-1} l(s) \frac{ds}{s}, \quad (8)$$

since $i_n \to i$ as $n \to \infty$ and $(\log \frac{t}{c})^{1-\varrho} (H_{c+1}^* x) (t) \to 0$ as $t \to \infty$, it follows from (8) that

$$\|L(i_n) - L(i)\|_{BC_\varrho} \to 0 \quad \text{as} \quad n \to \infty.$$ 

**Step 2.** $L(B_{R^*})$ is uniformly bounded and equicontinuous.
Since \( L(B_{R^+}) \subset B_{R^+} \) and \( B_{R^+} \) is bounded, then \( L(B_{R^+}) \) is uniformly bounded.

Next let \( t_1, t_2 \in [c, T], \ t_1 < t_2 \), and let \( i \in B_{R^+} \). This yields

\[
\begin{align*}
&\left| \left( \log \frac{t_2}{c} \right)^{1-\gamma} (Li)(t_2) - \left( \log \frac{t_1}{c} \right)^{1-\gamma} (Li)(t_1) \right| \\
\leq & \left( \log \frac{t_2}{c} \right)^{1-\gamma} \left[ \frac{d}{\Gamma(\theta)} \left( \log \frac{t_2}{c} \right)^{\theta-1} + \frac{1}{\Gamma(\tau)} \int_{c}^{t_2} \left( \log \frac{t_2}{s} \right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s} \right] \\
& - \left( \log \frac{t_1}{c} \right)^{1-\gamma} \left[ \frac{d}{\Gamma(\theta)} \left( \log \frac{t_1}{c} \right)^{\theta-1} + \frac{1}{\Gamma(\tau)} \int_{c}^{t_1} \left( \log \frac{t_1}{s} \right)^{\tau-1} \chi(s, i(s)) \frac{ds}{s} \right]
\end{align*}
\]

Then, we get

\[
\begin{align*}
&\left| \left( \log \frac{t_2}{c} \right)^{1-\gamma} (Li)(t_2) - \left( \log \frac{t_1}{c} \right)^{1-\gamma} (Li)(t_1) \right| \\
\leq & \frac{\left( \log \frac{t_2}{c} \right)^{1-\gamma}}{\Gamma(\tau)} \int_{c}^{t_1} \left( \log \frac{t_2}{s} \right)^{\tau-1} |\chi(s, i(s))| \frac{ds}{s} \\
& + \frac{1}{\Gamma(\tau)} \int_{c}^{t_1} \left( \log \frac{t_2}{c} \right)^{\tau-1} \left( \log \frac{t_1}{s} \right)^{\tau-1} \frac{l(s) ds}{s}
\end{align*}
\]

Thus, we obtain

\[
\begin{align*}
&\left| \left( \log \frac{t_2}{c} \right)^{1-\gamma} (Li)(t_2) - \left( \log \frac{t_1}{c} \right)^{1-\gamma} (Li)(t_1) \right| \\
\leq & \frac{l_s \left( \log \frac{t_2}{c} \right)^{1-\gamma}}{\Gamma(\tau)} \int_{c}^{t_1} \left( \log \frac{t_2}{s} \right)^{\tau-1} ds \\
& + \frac{l_s}{\Gamma(\tau)} \int_{c}^{t_1} \left( \log \frac{t_2}{c} \right)^{\tau-1} \left( \log \frac{t_1}{s} \right)^{\tau-1} \frac{l(s) ds}{s}
\end{align*}
\]

As \( t_1 \to t_2 \), the right-hand side of the inequality tends to zero.

**Step 3.** \( L(B_{R^+}) \) is equiconvergent.
Let $t \in [c, +\infty)$ and let $i \in B_{R^*}$. We have

$$\left| \left( \log \frac{t}{c} \right)^{1-\theta} (L_i)(t) \right| \leq \frac{|d|}{\Gamma(\theta)} + \frac{\left( \log \frac{t}{c} \right)^{1-\theta}}{\Gamma(\tau)} \int_c^t \left( \log \frac{s}{c} \right)^{\tau-1} |\chi(s, i(s))| \frac{ds}{s}$$

$$\leq \frac{|d|}{\Gamma(\theta)} + \frac{\left( \log \frac{t}{c} \right)^{1-\theta}}{\Gamma(\tau)} \int_c^t \left( \log \frac{s}{c} \right)^{\tau-1} l(s) \frac{ds}{s}$$

$$\leq \frac{|d|}{\Gamma(\theta)} + \frac{\left( \log \frac{t}{c} \right)^{1-\theta}}{\Gamma(\tau)} (Hf^+(t)).$$

Since

$$\left( \log \frac{t}{c} \right)^{1-\theta} (Hf^+(t)) \rightarrow 0 \text{ as } t \rightarrow +\infty,$$

we find

$$|(L_i)(t)| \leq \frac{|d|}{\left( \log \frac{t}{c} \right)^{1-\theta} \Gamma(\theta)} + \frac{\left( \log \frac{t}{c} \right)^{1-\theta}}{\left( \log \frac{t}{c} \right)^{1-\theta}} (Hf^+(t)) \rightarrow 0 \text{ as } t \rightarrow +\infty.$$ Hence

$$|(L_i(t)) - (L_i(+\infty))| \rightarrow 0 \text{ as } t \rightarrow +\infty.$$ As a consequence of Steps 1 - 3, we conclude that $L : B_{R^*} \rightarrow B_{R^*}$ is compact and continuous. Applying Schauder’s fixed point theorem, we get that $L$ has a fixed point $i$, which is a solution of problem (1) on $[c, +\infty)$.

**Step 4.** Assume that $i_0$ is solution of (1). Set $i \in B(i_0, 2l^*)$, we have

$$\left| \left( \log \frac{t}{c} \right)^{1-\theta} (L_i)(t) - \left( \log \frac{t}{c} \right)^{1-\theta} i_0(t) \right|$$

$$= \left| \left( \log \frac{t}{c} \right)^{1-\theta} (L_i)(t) - \left( \log \frac{t}{c} \right)^{1-\theta} (L_{i_0})(t) \right|$$

$$\leq \frac{\left( \log \frac{t}{c} \right)^{1-\theta}}{\Gamma(\tau)} \int_c^t \left( \log \frac{s}{c} \right)^{\tau-1} \left| \chi(s, i(s)) - \chi(s, i_0(s)) \right| \frac{ds}{s}$$

$$\leq 2 \frac{\left( \log \frac{t}{c} \right)^{1-\theta}}{\Gamma(\tau)} \int_c^t \left( \log \frac{s}{c} \right)^{\tau-1} l(s) \frac{ds}{s}$$

$$\leq 2l^*.$$ We get

$$\|L(i) - i_0\|_{BC^*_c} \leq 2l^*.$$ So, we conclude that $L$ is a continuous function such that

$$L(B(i_0, 2l^*)) \subset B(i_0, 2l^*).$$ Moreover, if $i$ is a solution of problem (1), then

$$|i(t) - i_0(t)| = |(L_i)(t) - (L_{i_0})(t)|$$

$$\leq \frac{1}{\Gamma(\tau)} \int_c^t \left( \log \frac{s}{c} \right)^{\tau-1} \left| \chi(s, i(s)) - \chi(s, i_0(s)) \right| \frac{ds}{s}$$

$$\leq 2 \left( Hf^+ \right)(t).$$
Therefore,
\[ |i(t) - i_0(t)| \leq \frac{2 (\log \frac{t}{c})^{1-\theta} (H I^\gamma_{c+} t) (t)}{(\log \frac{t}{c})^{1-\theta}}. \tag{9} \]
By (9) and
\[ \lim_{t \to \infty} \left( \log \frac{t}{c} \right)^{1-\theta} (H I^\gamma_{c+} t) (t) = 0, \]
we get
\[ \lim_{t \to \infty} |i(t) - i_0(t)| = 0. \]
Hence, solutions of (1) are uniformly locally attractive.

4. An Example

Consider the problem
\[
\begin{cases}
(H D^{\frac{1}{2} \frac{1}{2}}_{1+}) i)(t) = \chi(t, i(t)); & t \in [1, +\infty), \\
(H I^{\gamma}_{1+} i)(1) = 1,
\end{cases}
\tag{10}
\]
where
\[
\begin{cases}
\chi(t, i) = \frac{(t-1)^2 (\log t)^{-1} \cos t}{64 (t^2+1)(1+i(t))}, & t \in (1, \infty), \ i \in \mathbb{R}, \\
\chi(1, i) = 0, & i \in \mathbb{R}.
\end{cases}
\tag{11}
\]
Clearly, the function \( \chi \) is continuous, and \((H_2)\) is satisfied with
\[
\begin{cases}
l(t) = \frac{(t-1)^2 (\log t)^{-1} \cos t}{64 (t^2+1)}; & t \in (1, \infty), \\
l(1) = 0,
\end{cases}
\tag{12}
\]
and
\[
(\log t)^{\frac{1}{2}} H I^\gamma_{1-} l(t) = \frac{(\log t)^{1/4}}{\Gamma \left( \frac{1}{2} \right)} \int_1^t \left( \log \frac{t}{s} \right)^{-1/2} l(s) \frac{ds}{s},
\]
\[
\leq \frac{(\log t)^{1/4}}{\Gamma \left( \frac{1}{2} \right)} \int_1^t \left( \log \frac{t}{s} \right)^{-1/2} (\log s)^{-1} \frac{ds}{s},
\]
\[
\leq \frac{1}{\sqrt{\pi}} (\log t)^{-1/4} \to 0 \quad \text{as} \quad t \to \infty.
\]
Hence, problem (10) has at least one solution which is uniformly locally attractive.

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