Holography of charges in gauge theories

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In this short review we compare the rigid Noether charges to topological gauge charges. One important extension is that one should consider each boundary component of spacetime independently. The argument that relates bulk charges to surface terms can be adapted to the perfect fluid situation where one can recognise the helicity and enstrophies as Noether charges. More generally a forcing procedure that increases for instance any Noether charge is demonstrated. In the gauge theory situation, the key idea can be summarized by one sentence: “go to infinity and stay there”. A new variational formulation of Einstein’s gravity is given that allows for local GL(D,R) invariance. The a priori indeterminacy of the Noether charges is emphasized and a covariant ansatz due to S. Silva for the surface charges of gauge theories is analysed, it replaces the (non-covariant) Regge-Teitelboim procedure.

1. Introduction

Between 1918 and 1961 there was no clearly defined total charge in gauge theories and in particular in General Relativity. The few physicists who have read Noether’s paper will recall what we might call Hilbert’s disaster; namely the property of any global current in a gauge theory to be a total divergence. In those days the asymptotic structure of spacetime had not been formalised not to speak of the topological charges. We shall explain this statement and recall that in the Maxwell case it is so obvious that we do not pay attention, what happens is that the Noether current is on-shell equal to a topological current (by Gauss’ law).

The mathematical reasoning can be adapted to the situation of time independent gauge invariance as it is encountered in perfect fluids. There the dynamical object is a map from lagrangian coordinate volume to the laboratory vessel, even for a compressible fluid the adiabaticity and isentropy conditions ensure that all Lagrangian volume elements are indistinguishable and hence guarantee time-independent diffeomorphism invariance, actually volume preserving ones. We are in a situation where gauge invariance is realised without gauge fields, no metric is available (general relativistic fluids would be presented with a metric in laboratory coordinates). For Gedanken vessels without boundary there still can be helicity or enstrophies, the conserved charges in 3 (resp.2) space dimensions. The helicity has been analysed by H. K. Moffatt see \cite{18} and the reference to J. J. Moreau. What may be surprising for a general relativist is that we do not need to use a boundary component nor special asymptotics, what replaces that is a field dependent one-parameter subgroup of the time independent volume preserving diffeomorphisms, its existence is in turn a consequence of the gauge symmetry namely the Helmholtz-Kelvin theorem \cite{17}. The topological looking character of helicity, namely its interpretation as an average linking number of vortex lines, clashes clearly with our Noetherian interpretation especially in view of a variant of the following result.

In passing we mention a general lemma obtained in \cite{3} about the forcing of Noether charges under rather general conditions, the assumption of velocity independent transformation can be re-
laxed for fluids. The idea is that contrary to the use of a boost in relativistic or galilean physics which requires 3 symmetries, for instance time and space (x) translations and boosts mixing x and t coordinates and their charges, we actually need only one symmetry to be able to generate a solution with nonzero-momentum from a static solution. The price to be paid is that one learns only how to modify initial conditions, yet although the resolution of the dynamical problem is left open, it is not needed and one knows that the charge has increased after the appropriate kick.

The transmutation of bulk charges into boundary charges in gauge theories carries difficulties and advantages. The main misconception of most textbooks is that the so-called improvement terms of the currents is under control, it actually requires serious work. On the other hand the possibility to work at infinity (more precisely at one component of infinity) allows for singularities inside and the discussion should be entirely focussed away from those (but for the variational principle which strictly speaking requires handling of all boundary components!).

Special cases like global (bulk) Killing vectors permit a standard Noetherian treatment when they exist. Subtle global questions arise as well for locally asymptotically flat solutions for instance; there one must distinguish the local measurement of the asymptotic field and the total charge value assuming spherical symmetry, both may obey a different relationship than in the classical case. Finally p-form gauge invariances follow a similar pattern [14].

2. Surface terms and arbitrary “improvements”

It is easy to prove that when a rigid symmetry is actually a subgroup of a gauge symmetry its Noether current is a total divergence. The proof [15] see the works of Bergmann’s school relies on locality and an expansion in derivatives of the symmetry parameter at a point: if they are all independent the result follows. For instance in the case of second order equations of motion one obtains

\[ J_\xi := \xi \cdot J + d\xi \cdot \Lambda U \]  

which is valid for all \( \xi \) displacements. Henceforth \( dJ_\xi = 0 \) implies \( J = dU \). \( U \) is traditionally called a superpotential.

This had been also discussed (actually independently) in [14]. There gauge invariance was slightly enlarged by allowing gauge parameters that are closed but not exact, then the generalised Noether theorem for closed one-forms is precisely this total divergence formula. The extension to arbitrary closed \( p \)-forms is straightforward, and the closed 0-forms are precisely the constant parameters of the original Noether theorem.

Here comes an obvious corollary namely that the currents which are well known to be defined up to a total divergence when one modifies the action by a surface term that changes the boundary variation, are now totally arbitrary in a gauge context unless restricted by some extra requirement [3].

Let us assume that the variation of the fields depend on the gauge parameter and its derivative, then \( U \) depends up to a surface contribution only on the derivative part of the variation. When gauge invariance is implemented without gauge field it happens that \( U \) vanishes and hence the current as well (2d conformal theories, kappa symmetry...). In the fluid case the invariance is only under volume preserving transformations the current is given by the sole Lagrange multiplier for this constraint and the “conservation” of vorticity obtains after taking a curl that kills the latter.

3. Fluid flow invariants

The action of isentropic fluids is given by [17]

\[ L = \frac{1}{2} \left( \frac{\partial x^i}{\partial \tau} \right)^2 - c \left( \det \frac{\partial x^i}{\partial a^a}, s(a^a) \right) - \Phi(x^i) \]  

The basic fields of our theory are the fluid-particle Eulerian coordinates \( x^i(a^a, \tau) \). The cells of fluid are labelled by the \( a^a \) at a given time \( \tau \). The \( i, j, k... \) indices will be used for the laboratory space (called \( x \)-space) whereas the \( a, b, c... \) will be for the internal label space (called \( a \)-space) both
with the same dimension \( D \). The labelling follows the fluid particles along the dynamics. The labels are the Lagrangian coordinates. \( \Phi(x^i) \) is the potential of some external force. \( e \) is just the specific internal energy, a given thermodynamic function of \( \det \frac{\partial x}{\partial a} \) and of the specific entropy \( s \). The important hypothesis here is that the entropy \( s \) depends neither on the labels nor on the time \( \tau \), these are respectively homentropy and adiabaticity or isentropy conditions.

In a gauge theory there are infinitely many one parameter subgroups of the gauge group, each one of them has its conserved Noether charge, they are in general useless as they are gauge dependent. What saves our day is that for a very specific field dependent transformation the current is in fact physical. We can choose diffeomeorphisms that preserve the volume and are time independent by taking them as constant multiples of the vorticity. The resulting charge in 3 dimensions is nothing but the celebrated helicity that has important applications in turbulence and in particular in magnetohydrodynamics. The Moreau Moffatt helicity reads in Eulerian coordinates as the Hopf index for the velocity (viewed as a potential one form):

\[
H_M = \int v \wedge dv. \tag{3}
\]

Let us now present our forcing lemma which might be useful at least for numerical simulations as it will involve in the present situation the Lagrangian coordinates. The latter are notoriously important in turbulence and regrettably hard to access experimentally. So let us assume that a Noether rigid invariance does not involve the time derivatives of the coordinates of a Lagrangian system. The fluid case requires more general hypotheses and we refer to the paper for details.

Lemma: given a global or rigid Noether symmetry and the associated charge given, with \( x \) representing the coordinates or field variables, by \( \delta x = \xi(x) \) and \( Q = \int J_\xi \), then the change of \( Q \) under the impulsive forcing at some time \( t \) given with \( u \) the corresponding velocities by

\[
\delta x = 0, \delta u = \xi(x) \tag{4}
\]
is precisely equal to

\[
\delta Q = (\frac{\partial^2 L}{\partial u \partial u}) . \xi . \xi
\]

This is a positive quantity in view of the positivity of the acceptable kinetic terms.

Clearly a random change in the initial conditions will change the value of the charge as it is not in general a symmetry transformation, the interesting fact is that our specific kick is in a sense optimised to increase the charge. Let us mention also that in all odd space dimensions one finds one conserved charge (at least?) and in the even case an infinite number of them.

4. Local GL(d,R) formulation of General Relativity

Let us now return to our main discussion of relativistic gauge invariant theories. It turns out to be much easier to use first order formulations to analyse the conserved currents. Surprisingly a new formulation of Einstein’s gravity remained to be discovered. It lies above both the Palatini formalism in which metric and torsion free connection are taken as the independent variables and the Cartan-Weyl tetrad formulation where local Lorentz invariance is implemented through the introduction of the moving frames and the compensating non-propagating Lorentz connection. In this new variational principle the metric, the moving frames \( \theta^a \) and the linear connections \( \omega^a_b \) are all independent. One obtains the previous formulations by gauge fixing and partial extremisation. Let us mention one interesting feature namely a projective invariance under modifications of the linear connection by an arbitrary Weyl type diagonal part

\[
\delta_\kappa \omega^a_b = \kappa \delta^a_b \tag{5}
\]

\[
\delta_\kappa \theta^a = \delta_\kappa g^{ab} = 0 \tag{6}
\]

where \( \kappa \) is an arbitrary one-form.

The charge corresponding to a careful treatment of boundary terms turns out to be the KBL energy in the case of asymptotically flat boundary conditions.

A general consequence of symmetries is the existence of relations between equations of motion.
In the rigid Noetherian case the conservation of the current is such a differential relation. In the gauge case the Noether identities are algebraic relations expressing the fact that some variations vanish identically. The interesting situation arises here \[3\], see references therein and in particular \[7\], that all the equations of motion are in fact consequences of the symmetry in the sense that they follow from the conservation laws, this is typical of unique geometric Lagrangians (at a given order).

5. Holography

In the rigid symmetry situation where the Noether procedure does provide a simple and general construction of currents up to topological terms, the conservation of charges follows from an extra hypothesis namely that there is no leak of charge at infinity, the corresponding flux should vanish in the use of Stokes theorem between two equal time surfaces.

In the gauge case where the charge is given by a flux at infinity at a given time, again one must assume that there is no leak but the proof uses Stokes theorem at infinity only. It may be useful to emphasize that in that context not only the current is ill defined, pseudo-energy momentum tensors proliferate out of control, but the total bulk charge is also ill-defined. At least this is the case whenever there is a singularity even when it is hidden behind an horizon. It is only recently and in special cases that a mass at the horizon has been constructed. Clearly one needs the analog of the asymptotic flatness condition there or some well posed boundary condition, this has been realised only for the so-called isolated case \[13\] up to now. The bulk charge only exists in fine as the sum of all the boundary contributions.

Let us now recall the Regge Teitelboim hypothesis \[3\]. First of all a variational principle requires a well defined choice of boundary conditions for each component of the boundary. The symmetries to be discussed are those that preserve these asymptotic constraints. For example in the asymptotically flat situation the metric tends in a well defined way to the flat metric and the remaining symmetry at infinity is the rigid Poincaré group. In the case of the Einstein-Hilbert scalar action it is well known that one must modify the action by a surface term involving the second fundamental form, alternatively there is a non scalar action that only involves first derivatives of the metric through its quadratic dependence on the connection.

The gauge constraints due to reparametrisation invariance can be written as

\[
G(ξ, t) = \int ξ^A G_A dx \tag{7}
\]

where the gauge parameter $ξ$ tends to zero at infinity for proper gauge transformations. Under an arbitrary variation one may following \[9\] require that the variation of $G$ be a bulk integral of an expression proportional to the local variations of the fields, physically it means that the degrees of freedom are in the bulk. Now if one applies this principle to the conserved charges ie the same expressions $G$ for parameters that are nonzero at infinity one finds that they must be modified by surface terms so as to restore the bulk feature of the variation, in fact the evaluation of charges would have been zero otherwise as the constraints vanish on-shell. It is a little bit paradoxical to restore bulkiness through a surface term, this is holography at its best. What has been shown in \[10\] is that the resulting variational equation for the superpotential $U$ can be formulated in a covariant fashion. The integrability condition has been recently discussed in \[11\] see also the many references to R. Wald’s earlier work in there. A different approach can be found in \[12\]. In \[8\] we show that the construction of the symplectic form actually resembles the current discussion and compare various approaches.

Numerous applications confirm the above hypothesis \[16,5\]. Typically naive approaches are prone to mistakes by factors of two as there is no systematics at all.

In conclusion we believe there are important open questions left open. The canonical treatment of null infinity is not systematic, horizons are still problematic in general, extended objects bring many new questions to our mind, all of them are rather urgent.
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