Thermodynamic Properties and Phase Transitions in a Mean-Field Ising Spin Glass on Lattice Gas: the Random Blume-Emery-Griffiths-Capel Model

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The study of the mean-field static solution of the Random Blume-Emery-Griffiths-Capel model, an Ising-spin lattice gas with quenched random magnetic interaction, is performed. The model exhibits a paramagnetic phase, described by a stable Replica Symmetric solution. When the temperature is decreased or the density increases, the system undergoes a phase transition to a Full Replica Symmetry Breaking spin-glass phase. The nature of the transition can either be of the second order (like in the Sherrington-Kirkpatrick model) or, at temperature below a given critical value, of the first order in the Ehrenfest sense, with a discontinuous jump of the order parameter and accompanied by a latent heat. In this last case coexistence of phases takes place. The thermodynamics is worked out in the Full Replica Symmetry Breaking scheme, and the relative Parisi equations are solved using a pseudo-spectral method down to zero temperature.

Since its discovery, the spin glass (SG) phase has played and still plays a fundamental role in the investigation and understanding of many basic properties of disordered and complex systems. The analysis of the mean-field approximation of theoretical models displaying such a phase has revealed different possible scenarios, including different kinds of transition from the paramagnetic phase to the SG phase, as well as different kinds of SG phases. Most of the work, however, has been concentrated on just two scenarios.

In order of appearance in literature the first scenario is described by a Full Replica Symmetry Breaking (FRSB) solution characterized by a continuous order parameter function which continuously grows from zero by crossing the transition. The prototype model is the Sherrington-Kirkpatrick (SK) model a fully connected Ising-spin model with quenched random magnetic interactions.

The second scenario, initially introduced by Derrida by means of the Random Energy Model (REM) provides a transition with a jump in the order parameter to a stable low temperature phase in which the replica symmetry is spontaneously broken only once. The order parameter is a step function taking two values and the so-called Edwards-Anderson order parameter , (else said self-overlap), with . In the paramagnetic phase they are both equal to zero. At the transition, grows to a value larger than . No discontinuity appear, however, in the thermodynamic functions. Actually, at the transition to the one step Replica Symmetry Breaking (1RSB) SG phase, the Edwards-Anderson order parameter can either grow continuously from zero or jump discontinuously to a finite value. The first case of this second scenario includes Potts-glasses with three or four states, the spherical p-spin spin-glass model in strong magnetic fields and some inhomogeneous spherical p-spin model with a mixture of and interactions. The latter case includes, instead, Potts-glasses with more than four states, quadrupolar glass models p-spin interaction spin-glass models with and the spherical p-spin glass model in weak magnetic field. Because of the discontinuity of the overlap parameter across the transition the models belonging to this second case, often referred to as “discontinuous spin glasses” (see, for instance, Ref. [13]), have been widely investigated in the last years because of their relevance for the structural glass transition.

In all scenarios discussed above no latent heat occurs, i.e. the phase transition is continuous in the Ehrenfest sense. In this paper, instead, we consider a spin glass model undergoing (below a given critical point) a true thermodynamic first order phase transition between a paramagnetic (PM) and a Full Replica Symmetry Breaking (FRSB) SG phase, presenting coexistence of phases and latent heat, completing the work presented in Ref. [15].

Such a model is a generalization of the Blume-Emery-Griﬃths-Capel (BEGC) model for the transition and the phase separation in the mixtures of He³-He⁴ in a crystal ﬁeld, in which a quenched disordered magnetic interaction is introduced. In that case the phase diagram consisted of a ﬂuid phase, a super-ﬂuid one and a mixed phase (see Fig. 1 for a pictorial representation of the diagram).

The ﬁrst study of a spin-glass model undergoing a genuine ﬁrst order thermodynamic transition is the Ghatak-Sherrington model (GS), a simplified version of the model under current investigation. In this spin-1 model in crystal ﬁeld, no biquadratic coupling is considered: the Replica Symmetric (RS) solution and its stability have been carried out by Lage and Almeida Mottishaw and Sherrington and da Costa et al., albeit not always with compatible results. There the evidence for a ﬁrst order phase transition was found, in the neighborhood of the tricritical point.

In the last few years some work has been devoted to the comprehension of the Random generalization of the BEGC model (RBEGC), mostly at the level of the RS...
There exist two completely equivalent versions of this disordered model (at least as far as the statics is concerned). One version is a direct generalization of the BEGC model, with spin 1 variables \( \sigma_i = 1, 0, -1 \) on site \( i \),\(^{22,23} \) the other one is a lattice gas of Ising spins (spin \( S_i = 1, -1 \), with occupation numbers \( n_i = 0, 1 \) on site \( i \))\(^{24,23} \).

In this paper we will use the second formulation described, in the mean-field approximation, by the Hamiltonian\(^{23} \)

\[
\mathcal{H} = -\sum_{i<j} J_{ij} S_i S_j n_i n_j - \frac{K}{N} \sum_{i<j} n_i n_j - \mu \sum_i n_i - h \sum_i S_i n_i,
\]

where the Ising spin glass lattice gas in an external magnetic field is coupled to a spin reservoir by the chemical potential \( \mu \). \( K \) is the particle-particle coupling constant and \( h \) the external magnetic field. The magnetic interaction is described by quenched Gaussian random variables \( J_{ij} \), \( J_{ij} \) symmetric in \( i \leftrightarrow j \), with mean \( J_{ij} = J_0/N \) and variance \( \mathcal{J}_{ij}^2 = \mathcal{J}_1^2 = \mathcal{J}^2/N \). Here and in the following the overline denotes average with respect to disorder.

Just for completeness we also report the Hamiltonian in the original formulation:

\[
\mathcal{H} = -\sum_{i<j} J_{ij} \sigma_i \sigma_j - \frac{K}{N} \sum_i (\sigma_i \sigma_j)^2 + D \sum_i \sigma_i^2 - h \sum_i \sigma_i,
\]

The transformation \( \sigma_i = S_i n_i \), between the spin lattice gas dynamic variables and the spin-1 variables \( \sigma_i \), and the transformation \( D = -\mu - T \log 2 \), between \( \mu \) and the crystal field \( D \) of the spin-1 system, allow for a perfect equivalence of two versions of the model.

Some limiting cases of the model are the SK model\(^{2} \) (for \( \mu/J \to \infty \)) and the site frustrated percolation model\(^{15} \) (for \( K = -1 \) and \( J/\mu \to \infty \)).

In most cases studied up to now,\(^{18,19,20,21,22,23,24,25,26,27} \) the analysis was mainly limited to the RS solution. The general picture which emerged was an instability of the RS solution below some transition line in the region of low temperature and large density of particles. The nature of the SG phase (1RSB of FRSB) and the type of transition (SK-like, \( p \)-spin-like, or different\(^{22} \) were, however, not clear, also due to some "anomalous" properties at the RS level such as, for instance, complex stability eigenvalues. Despite this fact, the possibility of a \( p \)-spin-like transition has put new interest into this model as a possible, more realistic, model for the structural glasses, and its finite dimensions version has been numerically investigated in a search for evidence of a structural glass transition scenario\(^{32,33} \). Performing the quenched averages in the most general Replica scheme of computation it has been possible to show that the stable solution in the mean-field case, is FRSB everywhere in the SG phase, where spins are frozen.\(^{15} \)
II. THE REPLICA TRICK FOR THE THERMODYNAMICS OF DISORDERED SYSTEMS

For any fixed (quenched) coupling realization \( J \), the partition function of a system of \( N \) dynamic (annealed) variables \( \sigma \) is given by

\[
Z_N[J] = \text{Tr}_\sigma \exp(-\beta \mathcal{H}[J; \sigma])
\]

and the quenched free energy per spin is

\[
f_N = -\frac{1}{N\beta} \log Z_N = -\frac{1}{N\beta} \int d[J] P[J] \log Z_N[J]
\]

where \( \langle \cdots \rangle \) indicates the average over the couplings realizations. The thermodynamic limit of the free energy, \(-\lim_{N \to \infty} \frac{1}{N\beta} \log Z_N[J] / N\beta \) is well defined and equal to the quenched free energy \( f \) for almost any coupling realization \( J \) (self-average property).

The analytic computation of the quenched free energy, i.e., of the average of the logarithm of the partition function, is quite a difficult problem, even in simple cases as nearest neighbor one dimensional models. However, since the integer moments of the partition function are easier to compute, the standard method involves the so called “replica trick” by considering the annealed free energy \( f(n) \) of \( n \) non-interacting ‘replicas’ of the system

\[
f(n) = -\lim_{N \to \infty} \frac{1}{N\beta n} \log \left( \frac{Z^n_N[J]}{N^n} \right).
\]

The quenched free energy of the original system is then recovered as the continuation of \( f(n) \) down to the unphysical limit \( n = 0 \).

\[
f = -\lim_{N \to \infty} \frac{1}{N\beta n} (Z_N[J])^n - 1 = \lim_{n \to 0} f(n).
\]

In the last equality we assumed that the replica limit and the thermodynamic limit can be exchanged. This procedure replaces the original interactions in the real space with couplings among different replicas. The interested reader can find a complete and detailed presentation of the replica method for disordered statistical mechanical systems in Refs. [34] and [35].

Applying the replica trick to the computation of the thermodynamic potential of the random BEGC model we find, in the saddle point approximation for large \( N \),

\[
\bar{Z}_N[J] = \exp \left\{ -n N \beta f \left( \{ \rho \}, \{ m \}, \{ q \} \right) \right\}
\]

\[
\beta f = \frac{\beta \bar{K}}{2} n \sum_{a=1}^{\infty} \rho_a^2 + \frac{\beta J_0}{2} n \sum_{a=1}^{\infty} m_a^2
\]

\[
+ \frac{(\beta J)^2}{4} n \sum_{a \neq b} q_{ab} - \frac{1}{n} \log Z'
\]

\[
Z' \equiv \sum_{\{ S \}, \{ n \}} \exp \left\{ -\beta H' \left[ \{ d \}, \{ m \}, \{ q \} \right] \right\}
\]

where the sum in the one-site partition \( Z' \) is taken over all the possible values of the \( n \) spins \( S_a \) and the \( n \) occupation numbers \( n_a \). Here and everywhere else in this paper we use the abbreviation

\[
\tilde{K} \equiv K + \frac{\beta J^2}{2}
\]

The saddle point equations, coming from the extremization of Eq. [5], give the self-consistent relations

\[
q_{ab} = \langle S_a S_b \rangle
\]

\[
p_a = \langle n_a \rangle
\]

\[
m_a = \langle S_a n_a \rangle
\]

where \( \langle \cdots \rangle \) is the average computed over the measure \( \exp (-\beta H') \).

The parameter \( p_a \) represents the density of the replica \( a \). In the thermodynamic limit this is equal to \( 1/N \sum_i n_i = \bar{n} \) which, in the Hamiltonian [11], is only coupled to the chemical potential \( \mu \), that is a replica independent quantity. The density is therefore equal for each replica: \( \rho_a = \rho \), \( \forall a = 1, \ldots, n \).

The same holds for \( m_a \) which is coupled to the external field \( h \): \( m_a = m \), \( \forall a = 1, \ldots, n \). It generally holds that one index quantities are replica invariant [34].

A. Replica Symmetric Solution and Stability of the Paramagnetic Phase

The replica symmetric free energy is obtained by evaluating Eq. [5] at \( q_{ab} = q_0 \), for \( a \neq b \), \( p_a = \rho \) and \( m_a = m \) for every \( a, b \) and it reads

\[
\beta f = \frac{\beta \bar{K}}{2} \rho^2 + \frac{\beta J_0}{2} m^2 - \frac{(\beta J)^2}{4} q_0^2 - \beta J \int_{-\infty}^{\infty} dy P_0(y) \phi_0(y)
\]

with

\[
\Theta_0 \equiv \left( \frac{(\beta J)^2}{2} (\rho - q_0) + \beta K \rho + \beta \mu \right)
\]

\[
\left( \frac{\beta \bar{K}}{2} + \beta \mu - \frac{(\beta J)^2}{2} q_0 \right) P_0(y) + \frac{1}{\sqrt{2\pi q_0^2}} \exp \left[ -\frac{(y - m J_0 / J - h / J)^2}{2 q_0^2} \right]
\]

\[
\phi_0(y) \equiv \frac{1}{\beta J} \log \left( 2 + 2 e^{\Theta_0} \cosh \beta J y \right)
\]
derivation):

\[
\rho = \int_{-\infty}^{\infty} dy \ P_0(y) \ \hat{\rho}(y) \quad (19)
\]

\[
q_0 = \int_{-\infty}^{\infty} dy \ P_0(y) \ \hat{m}^2(y) \quad (20)
\]

\[
m = \int_{-\infty}^{\infty} dy \ P_0(y) \ \hat{m}(y) \quad (21)
\]

with the following definitions

\[
\hat{\rho}(y) = \frac{\cosh \beta J y}{e^{-\Theta_0} + \cosh \beta J y} \quad (22)
\]

\[
\hat{m}(y) = \frac{\sinh \beta J y}{e^{-\Theta_0} + \cosh \beta J y} \quad (23)
\]

The eigenvalues of the Hessian of Eq. 8 computed in the RS approximation are derived in appendix VI, where the stability analysis is carried out.

In the case \( J_0 = h = 0 \) only the paramagnetic solution \( (q_0 = 0) \) is stable, so that Eqs. (19)-(21) reduce to

\[
q_0 = 0 ; \quad \rho = \frac{1}{1 + e^{-\beta (K \rho - \beta \mu)} ; \quad m = 0 . \quad (24)
\]

and the eigenvalues of independent modes are

\[
\Lambda_0 = (\beta J)^2 \ [1 - (\beta J)^2 \rho^2] \quad (25)
\]

\[
\Lambda_1 = \beta \mid \hat{K} \mid \ [1 - \beta J \rho (1 - \rho)] \quad (26)
\]

where \( \Lambda_0 \) is connected to \( \delta q \ \delta q \) fluctuations, whereas \( \Lambda_1 \) is the stability eigenvalue of the density-density fluctuations. In the above formulas we have considered both the case in which \( \hat{K} > 0 \) and \( \hat{K} < 0 \). The last one occurring only if the particle-particle interaction \( K \) is negative and when \( T \) is bigger than \( -1/(2\hat{K}) \).

The lines \( \Lambda_0 = 0 \) and \( \Lambda_1 = 0 \) delimit the stability region for the RS solution on the phase diagram. In the \( T - \rho \) phase diagram the stable region is for

\[
T > J \rho
\]

\[
T > K \rho (1 - \rho) \left[ 1 + \sqrt{1 + \frac{J^2 \rho (1 - \rho)}{K^2}} \right] \quad (28)
\]

corresponding, respectively, to \( \Lambda_0 > 0 \) and \( \Lambda_1 > 0 \).

For any value of \( K \), there is one intersection between \( \Lambda_0 = 0 \) and \( \Lambda_1 = 0 \), namely the tricritical point:

\[
\frac{T_c}{J} = \frac{\rho_c}{2K} \left[ -\frac{3}{2} + \frac{2}{J} + \sqrt{\frac{J^2}{K^2} - \frac{K}{J} + \frac{9}{4}} \right] \quad (29)
\]

\[
\frac{\mu_c}{J} = -\frac{1}{2} - \rho_c \left[ \frac{K}{J} + \log \left( \frac{\rho_c}{J} - 1 \right) \right] \quad (30)
\]

where \( \mu_c \) is obtained from the paramagnetic expression \( 19 \) for \( \rho \). In table I we list some values of interest of the tricritical values and in Fig. 2 we plot their behavior as function of the particle-particle coupling constant.

As \( K \) decreases, the critical temperature goes to zero. As we will see in the following, in Sec. V where we study the phase diagrams, this implies that the phase diagram region of phase coexistence progressively reduces itself. The critical value of the chemical potential grows to zero as \( K \) tends to \( -\infty \): in order to contrast large particle repulsion a larger chemical potential (in the limit non-negative) is needed.

1. Thermodynamic Observables in the Replica Symmetric Solution

The internal energy and the entropy take the form:

\[
u = \frac{K + \beta J^2}{2} \rho^2 - J_0 \rho m^2 - h \rho m - \mu + \frac{\beta J^2}{4} \rho^2 \quad (31)
\]

\[
s = \frac{-(\beta J)^2}{4} (\rho - q_0)^2 - \rho \Theta_0
\]

\[
+ \beta J \int_{-\infty}^{\infty} dy \ P_0(y) \ [\phi_0(y) - y \ \hat{m}(y)]
\]

Integrating by part we can derive:

\[
\int_{-\infty}^{\infty} dy \ P_0(y) \ y \ \hat{m}(y) = \beta J (\rho - q_0) \quad (33)
\]

| \( K/J \) | \( \infty \) | 1 | 0 | -1 | \( -\infty \) |
|---|---|---|---|---|---|
| \( T_c/J \) | \( \infty \) | 1/2 | 1/3 | 0.21922 | 0 |
| \( \mu_c/J \) | \( -\infty \) | -1 | -0.73105 | -0.55923 | 0 |
The entropy, then, becomes

\[ s = - \frac{(\beta J)^2}{4} (\rho - q_0)^2 - \rho \Theta_0 \]

\[ - (\beta J)^2 q_0 (\rho - q_0) + \beta J \int \int \limits_{-\infty}^{\infty} dy \ P_0(y) \ \phi_0(y) \]

\[ (34) \]

\section*{B. Low Temperature Behavior of the RS Solution}

It can be of some help for later purposes and also to make a comparison with existing results, to probe how the observables behave in the very low temperature limit, even in this case for which the SG solution is unstable. We simplify the discussion to the case for which the SG solution is unstable. We put the observables in the very low temperature limit,

\[ \text{We define} \]

\[ a \equiv \frac{\Theta_0}{\beta J} \]

\[ C \equiv \beta J (\rho - q_0) \]

\[ \Delta I \equiv \int \int \limits_{-\infty}^{\infty} dy \ e^{-z^2/2} \{ \phi_0(z) - \sqrt{q_0} \ z \ \tilde{m}(z) \} \]

\[ \text{We consider two cases separately in the zero temperature limit.} \]

\subsection*{1. Full system:} \( a \geq 0 \)

\[ \lim_{\beta \to \infty} \rho = 1 \]

\[ \lim_{\beta \to \infty} C = \sqrt{\frac{2}{\pi}} \]

\[ \lim_{\beta \to \infty} a = \frac{1}{\sqrt{2\pi}} + \frac{K + \mu}{J} \]

\[ \Delta I = \Theta_0 + O(T) \]

\[ s = - \frac{1}{2\pi} + O(T) \]

\subsection*{2. Partially diluted system:} \( a < 0 \)

\[ \lim_{\beta \to \infty} \rho = \bar{\rho} < 1 \]

\[ \lim_{\beta \to \infty} C = \bar{C} \]

\[ \lim_{\beta \to \infty} a = \frac{\bar{C}}{2} + \frac{K\bar{\rho} + \mu}{J} \]

\[ \text{with } \bar{\rho} \text{ and } \bar{C} \text{ given by the zero temperature self-consistency equations:} \]

\[ \bar{\rho} = \text{erfc} \left( \frac{\pi}{2 \bar{\rho}} \right) \]

\[ \bar{C} = \sqrt{\frac{2}{\pi \bar{\rho}}} \exp \left( - \frac{\bar{\rho}}{\sqrt{2 \bar{\rho}}} \right) \]

\[ (37) \]

Notice that as \( a \to 0 \) one obtains from the above equations: \( \bar{\rho} = 1 \) and \( \bar{C} = \sqrt{2/\pi} \).

Finally one gets

\[ \Delta I = \bar{\rho} \Theta_0 + (1 - \bar{\rho}) \log 2 - \bar{C} \bar{\rho} \log 2 + O(T) \]

\[ s = - \frac{\bar{C}^2}{4} + (1 - \bar{\rho}) \log 2 - \bar{C} \bar{\rho} \log 2 + O(T) \]

For fixed \( K \) and \( J \) there is, thus, a limiting (negative) value of the chemical potential below which the lattice will not be full at zero temperature. For

\[ \frac{\mu^*}{J} < - \frac{1}{\sqrt{2\pi}} - \frac{K}{J} \]

\[ \frac{\mu^*}{J} = \frac{1}{\sqrt{2\pi}} - \frac{K}{J} \]

\[ \frac{\mu^*}{J} = - \frac{1}{\sqrt{2\pi}} - \frac{K}{J} \]

\subsection*{III. A VARIATIONAL METHOD FOR THE FULL REPLICA SYMMETRY BREAKING SOLUTION IN DISORDERED SYSTEMS}

To clarify which kind of transition takes place we improve the study of the static properties of the SG phase of the mean-field RBEGC, making use of the FRSB Parisi Ansatz. The first thing to notice is that the stable SG phase is always of FRSB type. The transition between the PM phase and the SG phase can either be of the SK-type or, below a given \( T_c \), discontinuous with a jump in the entropy and hence a latent heat. Moreover for a certain range of parameters, the two phases, PM and SG, coexist. For any parameter choice we find no evidence for a \( p \)-spin-like transition with discontinuous order parameter.

The aim is to evaluate the \( n \to 0 \) limit in Eq. \[ \mathbf{8} \] with the Ansatz that the structure of the matrix \( Q \) follows a FRSB scheme. In order to be as general as possible, we shall use the RSB scheme introduced for the SK model by de Dominicis, Gabay and Orland,\[ 36,37 \] which besides the Edwards-Anderson order parameter also involves the anomaly to the linear response function, otherwise called Sompolinsky’s anomaly.\[ 28 \] The more usual Parisi’s RSB scheme is, if necessary, eventually recovered by a proper gauge fixing, once that the limit of infinite number of replica symmetry breakings has been taken [see below Eq. \[ \mathbf{12} \] ].

By applying the RSB scheme infinite times and introducing the two functions \( q(x) \) (overlap function) and
\( \Delta(x) \) (anomaly function), \( 0 \leq x \leq 1 \), the free energy functional, Eq. (53), becomes:
\[
\beta f = \frac{\beta^2 K}{2} + \frac{\mu^2}{2} J_0 - \frac{(\beta J)^2}{4} q(1)^2 
- \frac{\beta J}{2} \int_0^1 dx \langle q(x) \rangle \frac{\Delta(x)}{\nu} - \beta J \int_0^\infty dy \ P_0(y) \phi(0, y) ,
\]

where \( P_0(y) \) is defined as
\[
P_0(y) = \frac{1}{\sqrt{2\pi q(0)}} \exp \left\{ -\frac{(y - (h + J_0 m)/J)^2}{2q(0)} \right\} \tag{54}
\]

and \( \phi(0, y) \) is the solution, evaluated at \( x = 0 \), of the Parisi parabolic equation
\[
\dot{\phi}(x, y) = -\frac{\dot{q}(x)}{2} \phi''(x, y) + \frac{\Delta(x)}{2} \phi'(x, y)^2 , \tag{55}
\]

with the boundary condition at \( x = 1 \)
\[
\phi(1, y) = \phi_1(y) = (\beta J)^{-1} \log \left( 2 + 2e^{\Theta_1} \cosh \beta J y \right) , \tag{56}
\]

and
\[
\Theta_1 \equiv \frac{(\beta J)^2}{2} [\rho - q(1)] + \beta (\mu + K \rho) \tag{57}
= \frac{\beta K \rho + \mu - q(1)}{2} \left( \frac{\beta J^2}{2} \right) .
\]

The overlap \( q(x) \), the density of occupied sites \( \rho \) and the anomaly \( \Delta(x) \) are the order parameters. We have used the standard notation and denoted derivatives with respect to \( x \) by a dot and derivatives with respect to \( y \) by a prime. In this notation Sompolinsky’s \( \Delta' \) in Ref. [38] becomes our \( T \Delta \).

The Parisi equation [55] can be included into a free energy variational functional via the Lagrange multiplier \( P(x, y) \) and the initial condition at \( x = 1 \), Eq. (56), via \( P(1, y) \). The free energy functional is then
\[
\beta f_v = \beta f 
+ \beta J \int_0^\infty dy P(1, y) \left[ \phi(1, y) - \phi_1(y) \right] 
- \beta J \int_0^1 dx \int_0^\infty dy P(x, y) \left[ \dot{\phi}(x, y) 
+ \frac{\dot{q}(x)}{2} \phi''(x, y) - \frac{\Delta(x)}{2} \phi'(x, y)^2 \right] ,
\]

with \( P_0(y) \) and \( \phi_1(y) \) defined in Eqs. (54) and (56).

By such a construction \( f_v \) is stationary with respect to variations of \( P(x, y) \), \( P(1, y) \), \( \phi(x, y) \), \( \phi(0, y) \), \( q(x) \), \( \Delta_q(x) \) and deriving with respect to \( \rho \). Variations of \( P(x, y) \) and \( P(1, y) \) simply give back Eqs. (55) and (56). Stationarity with respect to variations of \( \phi(x, y) \) and \( \phi(0, y) \) leads to a partial differential equation for \( P(x, y) \):
\[
\dot{P}(x, y) = \frac{\dot{q}(x)}{2} P''(x, y) + \frac{\Delta(x)}{2} [P(x, y) \phi'(x, y)]' , \tag{59}
\]

and to the boundary condition at \( x = 0 \)
\[
P(0, y) = P_0(y) . \tag{60}
\]

Eventually, variations of \( f_v \) with respect to \( q(x) \), \( \Delta(x) \) and the derivative with respect to \( \rho \) lead to
\[
\Delta(x) = -\beta J [\rho - q(1)] \int_0^\infty dy P(x, y) \phi''(x, y) \tag{61}
\]
\[
q(x) = \int_0^\infty dy P(x, y) \phi'(x, y)^2 \tag{62}
\]
\[
\rho = \int_0^\infty dy P(1, y) e^{-\Theta_1 + \cosh \beta J y} \tag{63}
\]

with \( \Delta(1) = 0 \), the anomaly at the shortest time-scale, corresponding to \( x = 1 \), being zero by construction: the Fluctuation-Dissipation Theorem (FDT) holds at short time-scales.

The Lagrange multiplier \( P(x, y) \) represents the distribution of local fields. One may indeed associate a given overlap \( q(x) \) with a time scale \( \tau_q \) such that for times of order \( \tau_q \) states with an overlap equal to \( q(x) \) or greater can be reached by the system (these time-scales completely decouple in the thermodynamic limit). In this picture the \( P(x, y) \) becomes the probability distribution of frozen local fields \( y \) at the time scale labeled by \( x \).

For the numerical treatment of the FRB equations and also to allow for a clearer physical interpretation of the functions that we are analyzing, we define the local magnetization \( m(x, y) \equiv \phi'(x, y) \), whose differential antiparabolic equation we derive from Eqs. (55), (56) as
\[
m(x, y) = -\frac{\dot{q}}{2} m''(x, y) + \Delta(x) m(x, y) m'(x, y) , \tag{64}
\]
\[
m(1, y) = m_1(y) = \frac{\sinh(\beta y)}{e^{-\Theta_1} + \cosh(\beta y)} . \tag{65}
\]

The average equilibrium magnetization \( m \) can, then, be computed in terms of the local magnetization at \( x = 0 \):
\[
m = -\frac{\partial f}{\partial h} = \int_0^\infty dy P(0, y) m(0, y) . \tag{66}
\]

Another useful relation for the numerical evaluation of the order parameters and the thermodynamic functions built on them is got from the computation of the term \( \int_0^1 dx \int_0^\infty dy P(x, y) \phi(x, y) \) in two different ways: once using Eq. (59) and, in the other case, applying Eq. (66) and integrating by part. From the comparison of the two results the following equation is obtained:
\[
\int_0^\infty dy P(1, y) \phi(1, y) - \int_0^\infty dy P(0, y) \phi(0, y) \tag{67}
= \frac{1}{2} \int_0^1 dx \ q(x) \ \Delta(x) 
\]

Deriving Eq. (61) with respect to \( x \) yields
\[
\int_0^\infty dy P(x, y) m'(x, y)^2 = 1 \tag{68}
\]
valid for every \(x\) and guaranteeing the marginal stability of the FRSB Ansatz.

The coupled equations \(59\), \(60\), with border conditions \(65\), \(66\) are the FRSB equations. In the following we are going to show how they can be numerically solved and what are the gauges we will use in order to get the most general SG phase and the order parameters characterizing the transition to it.

Differentiating once more Eq. \(65\) one finds

\[
-\frac{\Delta}{q} = \frac{\int_{-\infty}^{\infty} dy \ P(x, y) \ m''(x, y)x^3}{\int_{-\infty}^{\infty} dy \ P(x, y) \ m''(x, y)x^2}
\]

that will become useful is the following when we will discuss the choice of the gauge to perform the numerical computation (see Sec. 11.13).

A. Numerical Integration of the FRSB Equations: the Pseudo-Spectral method

In order to study the low temperature regime of the RBEGC in the limit of a large number of clauses we have numerically integrated the FRSB equations \(59\), \(60\) to determine \(q(x)\), \(P(x, y)\) and \(m(x, y)\). We followed the iterative scheme of Refs. \(14\), \(15\), but with an improved numerical method which allows for very accurate results for all temperatures (see Refs. \(12\), \(13\)).

We start from an initial guess for \(q(x)\), \(\Delta(x)\) and \(\rho\) then \(m(x, y)\), \(P(x, y)\) and the associated \(q(x)\) are computed in the order as:

1. Compute \(m(x, y)\) integrating from \(x = 1\) to \(x = 0\) Eqs. \(59\) with initial condition \(66\).

2. Compute \(P(x, y)\) integrating from \(x = 0\) to \(x = 1\) Eqs. \(65\) with initial condition \(65\).

3. Compute \(q(x)\), \(\Delta(x)\) and \(\rho\) using Eqs. \(62\), \(63\).

The steps \(1 \rightarrow 2 \rightarrow 3\) are repeated until a reasonable convergence is reached, typically we require a mean square error on \(q\), \(P\) and \(m\) of the order \(O(10^{-6})\) and we checked that the identities \(64\), \(59\) and \(1 - T = 7\) were satisfied to this precision as well. The number of iterations necessary are a few hundreds. The core of the integration scheme is the integration of the partial differential equations \(59\) and \(60\). In previous works this was carried out through direct integration in the real space which requires a large grid mesh to obtain precise results. To overcome this problem we move to the Fourier space where we can apply a pseudo-spectral method of integration.

Indicating by \(\text{FT}[o(x, k)]\) the Fourier transform of function \(o(x, y)\),

\[
\text{FT}[o(x, k)] = \frac{1}{N_y} \int_{-N_y/2}^{N_y/2} dy \ e^{-iky} \ o(x, y),
\]

the FRSB Eq. \(59\), written in terms of the wave number \(k\), becomes

\[
m(x, k) = k^2 \frac{q(x)}{2} m(x, k) + ik \frac{\Delta(x)}{2} \text{FT}[m^2](x, k)
\]

and the FRSB Eq. \(60\) takes the form

\[
P(x, k) = -k^2 \frac{q(x)}{2} P(x, k) + ik \frac{\Delta(x)}{2} \text{FT}[P \ m](x, k)
\]

For each value of \(k\) these are ordinary differential equations which can be integrated using standard methods. To avoid the time consuming calculation of the convolution in the nonlinear terms we use the pseudo-spectral code on a grid mesh of \(N_x \times N_y\) points, covering the \(x\) interval \([0,1]\) and the \(y\) interval \([−y_{\text{max}}, y_{\text{max}}]\). The truncation of the wave number may bring to anisotropic effects for large \(k\). De-aliasing is, thus, performed by a \(N_y/2\) truncation, which ensures a better isotropy of the numerical treatment. Eventually, the \(x\) integration is carried out by means of a third order Adam-Bashford scheme which reduces the number of fast Fourier transforms (FFT) calls.

Typical values used are \(N_x = 500 \div 1000\), \(N_y = 1024 \div 4096\) and \(y_{\text{max}} = 24 \div 48\). The difference between the values for \(N_x\) and the values for \(N_y\) comes from the fact that, if the solution in \(y\) is smooth enough, only a few low wave numbers \(k\) are exited. The value of the parameter \(y_{\text{max}}\) fixes the \(y\) range where the solution is assumed different from zero. Indeed, in the numerical algorithm is assumed that \(P(x, y) = m(x, y) = 0\) for \(|y| > y_{\text{max}}\). This explains the rather large value of \(y_{\text{max}}\) used.

B. Choice of Gauge

The solution to the spin-glass mean-field models obtained using the scheme of Sommers is overconstrained and the functional expression of \(\Delta(x)\) can be, thus, chosen in different ways, selecting, in this way, a gauge for the order parameters. The usually studied gauges are

\[
\begin{align*}
\Delta(x) &= -\beta J x \ q(x) \quad \text{Parisi gauge}\quad (73) \\
\Delta(x) &= -\Delta(0) \quad \text{Sommers gauge}\quad (74)
\end{align*}
\]

where the anomaly is given by the stationarity Eq. \(74\) at \(x = 0\). We rewrite it here in terms of the local magnetization

\[
\Delta(0) = -\beta J [\rho - q(1)] + \int_{-\infty}^{\infty} dy \ P(0, y) \ m'(0, y)
\]

The anomaly at the largest time-scale is gauge invariant since it depends on the Edwards-Anderson parameter and on the derivative of the equilibrium local magnetization. \(\Delta(x)\) measures the violation of the linear response on the time-scale labeled by \(x\). As such, it’s a decreasing function of \(x\), being zero at the shortest time-scale.
We introduce the cumulative function of a cumulative distribution, we can use Eq. (69) and to build a general definition of 

$$\hat{\Delta}(x) = -\gamma \Delta(0)(1-x)^{\gamma^{-1}}, \quad \gamma = 1, 2, \ldots \quad (76)$$

that, for $\gamma = 1$, is the Sommers gauge.

In practice, the most common choice is the Parisi gauge $\Delta(x) = -\beta J x \dot{q}(x)$, useful from a numerical point of view, since for $x$ larger than a certain critical value $x_c$, at which the overlap function displays a cusp and reaches the plateau value $q_{E_A}$, $\dot{q}(x)$ tends to zero and the integration domain can, thus, be reduced. For $T \to 0$, however, the overlap function in this gauge tends to a step function, yielding, for some $x$, a diverging factor $\dot{q}(x)$ in both terms of the rhs of equations (71) and (72).

For this reason the alternative Sommers gauge can be adopted $\Delta(x) = -\Delta(0)$, so that the non linear terms of the rhs of Eqs. (71) and (72) is automatically kept finite. The function $q(x)$ comes out to be a smoother function of $x$ in this gauge, not only continuous and monotonous, but also without any cusp at any $x$. When the external magnetic field is zero $\dot{q}$ is still divergent, but only exactly at $x = 0$, thus yielding no sharp change in convexity, whereas for $h \neq 0$ even this divergence disappears and the FRSB equations do not display any integration problem because of $q(x)$ behavior.

1. **Gauge dependent parameters and physical observables**

The Parisi gauge is, thus, advantageous down to temperatures where it starts bringing numerical instabilities. For lower temperatures the Sommers gauge (or some similar, e.g. Eq. (70)) stabilizes the results. To choose a gauge means to select the $x$ behavior of the order parameter functions will be different. This will not change, anyway, the physical quantities, such as density, Edwards-Anderson parameter ($q_{E_A} = q(1)$), energy, entropy, etc., that we are going to show in the next section.

Also the overlap probability distribution $P(q)$ is gauge invariant. However, one has to consider the fact that the definition $P(q) = dq(x)/dq$ only holds in the Parisi gauge. To build a general definition of $P(q)$ as the derivative of a cumulative distribution we can use Eq. (70) and introduce the cumulative function $\xi$

$$\xi(x) \equiv -T \frac{\Delta(x)}{q(x)} \quad (77)$$

where $x$ here is the gauge-dependent RSB parameter and $\Delta$ and $q$ are the gauge-dependent order parameter. In the Parisi gauge is $\xi(x) = x$, and we get back the usual definition. In the Sommers gauge is $\xi(x) = T \Delta(0)/\dot{q}(x)$ and $\xi(q)$ can be obtained, e.g. parametrically, using Eq. (72). Thus, the expression of the overlap distribution has to be derived as

$$P(q) = \frac{d\xi(q)}{dq} \quad (78)$$

2. **Parisi gauge as fluctuation-dissipation ratio**

Following a dynamical interpretation of the order parameters $q(x)$ and $\Delta(x)$, respectively as spin-spin correlation function and anomaly in the susceptibility at time-scale $x$, the Parisi gauge is actually a rewriting of the Fluctuation-Dissipation Relation. That is the generalization of the FDT for aging systems dynamically stuck out of equilibrium, such as spin-glasses, where the coefficient between correlation and response functions is not $T$ but some function of the time-scale on which the system is relaxing (often referred to as effective temperature). We saw in Eq. (59) that $\Delta(x) = \chi(x) - \chi(1)$ is the anomaly of the susceptibility with respect to the linear response value (i.e. the ZFC susceptibility). Thus, $\Delta(x)$ is the anomalous response function, whereas $\dot{q}$ is the derivative of the correlation function and $T_{\text{eff}} = 1/(\beta x)$ the effective temperature. The time dependence of $T_{\text{eff}}$ is expressed through the time-scale index $x$.

**IV. THERMODYNAMIC OBSERVABLES**

All thermodynamic quantities can be written in terms of the order parameter functions derived by solving Eqs. (50), (57), namely overlap, density, anomaly, local magnetization and distribution of local fields.

3. **Internal energy**

The internal energy $u = \partial \beta f/\partial \beta$ can be computed either taking the derivative of Eq. (5) (and then breaking the replica symmetry infinite times) or directly deriving Eq. (55).

In the first case the energy comes out to be

$$u = \lim_{N_B \to \infty} \lim_{n \to 0} \left[ -\frac{K + \beta J^2}{2} \frac{1}{n} \sum_a \rho_a^2 - J_0 \frac{1}{n} \sum_a m_a^2 - h \frac{1}{n} \sum_a m_a - \mu \frac{1}{n} \sum_a \rho_a - \frac{\beta J^2}{2} \frac{1}{n} \sum_{a \neq b} q_{ab}^2 \right]$$

$$= -\frac{K + \beta J^2}{2} \rho^2 - \frac{J_0}{2} m^2 - h m - \mu \rho + \frac{\beta J^2}{2} q(1)^2 + J \int_0^1 dx q(x) \hat{\Delta}(x) \quad (79)$$

where $N_B$ is the number of replica symmetry breakings. Otherwise, deriving Eq. (55) and using Eqs. (50, 61), one
finds

\[ u = - \frac{K + \beta J^2}{2} \rho^2 - \mu \rho + \frac{J_0}{2} m^2 + \beta J^2 q(1) \rho - \frac{\beta J^2}{2} q(1)^2 - J \int_0^1 dx \, q(x) \, \Delta(x) \]

\[ - J \int_{-\infty}^{\infty} dy \, P(1, y) \, m(1, y) \]

The comparison between Eq. (79) and Eq. (80) yields the relation

\[ 2 \int_0^1 dx \, q(x) \, \Delta(x) = \beta J \, q(1) \, [\rho - q(1)] \]

or, exploiting Eq. (81), as

\[ s = \frac{\beta J^2}{4} [\rho - q(1)]^2 - \rho \Theta_1 - (\beta J)^2 q(1) \, [\rho - q(1)] \]

\[ - \beta J_0 m^2 - \beta h m + 2 \beta J \int_0^1 dx \, q(x) \, \Delta(x) \]

\[ + \beta J \int_{-\infty}^{\infty} dy \, P(1, y) \, \phi(1, y) \]

4. Entropy density

The entropy density \( s = \beta^2 \partial f / \partial \beta \) can be expressed either as

\[ s = -\rho \Theta_1 - \frac{(\beta J)^2}{4} [\rho - q(1)]^2 \]

\[ + \beta J \int_{-\infty}^{\infty} dy \, P(1, y) \, [\phi(1, y) - y \, m(1, y)] \]

or, exploiting Eq. (81), as

\[ s = \frac{(\beta J)^2}{4} [\rho - q(1)]^2 - \rho \Theta_1 - (\beta J)^2 q(1) \, [\rho - q(1)] \]

\[ - \beta J_0 m^2 - \beta h m + 2 \beta J \int_{-\infty}^{1} dx \, q(x) \, \Delta(x) \]

\[ + \beta J \int_{-\infty}^{\infty} dy \, P(1, y) \, \phi(1, y) \]

5. Compressibility

The compressibility in terms of the density of occupied sites and its conjugated field, the chemical potential, can be expressed in the FRSB formulation as:

\[ \kappa = \frac{1}{\rho^2} \frac{\partial \rho}{\partial \mu} = \frac{\beta}{\rho^2} \left( \rho - \int_{-\infty}^{\infty} dy \, P(1, y) \, \rho_1^2(y) \right) \]

where we define

\[ \rho_1(y) = \frac{\cosh(\beta J y)}{e^{-\beta_1} + \cosh(\beta J y)} \]

In Fig. 4 we show the compressibility behavior versus temperature of the system with particle-particle interaction constant \( K = J \) for three values of the chemical potential. Respectively above, at and below the critical value \( \mu_c \) below which the transition happens to be first order in the Ehrenfest sense. For \( \mu < \mu_c \) a cusp shows up and, decreasing the temperature, \( \kappa \) shrinks to a much lower value as the transition point is crossed.
6. Susceptibility

The magnetic susceptibility of the RBEGC model at equilibrium can be written either as

$$\chi_0 = \beta J \left[ \rho - \int_0^1 dx \, q(x) \right]$$  \hspace{1cm} (86)

or, with the help of Eq. (66), as

$$\chi_0 = \frac{\partial m}{\partial h} = \int_{-\infty}^{\infty} dy \, P(0, y) \, m'(0, y) ,$$  \hspace{1cm} (87)

whereas the susceptibility obtained when the system is constrained to stay in a single minimum of the free energy function comes out to be

$$\chi_1 = \beta J \left[ \rho - q(1) \right] .$$  \hspace{1cm} (88)

The equilibrium susceptibility is a function of the average $\overline{q} = \int_0^1 dx \, q(x)$ of the overlap over all possible values it can take at all time scales ($0 \leq x \leq 1$). It corresponds to the Field-Cooled susceptibility. The second susceptibility, instead, only depends on the Edwards-Anderson parameter $q_{EA} = q(1)$. It physically expresses the self-overlap of configurations belonging to the same state. Indeed, at time scale $\tau_1$, i.e., the shortest time scale, the system has not yet visited but one metastable state, thus the response to a field perturbation only depends on the self-overlap. The experimental analogue of $\chi_1$ is the Zero Field Cooled susceptibility. As a matter of fact, also in that case the system remains in one single state during the cooling down to the SG phase, since it is not driven by any external field.

From a dynamical point of view, the equilibrium susceptibility $\chi_0$ can otherwise be expressed in terms of the function $\Delta(x)$ defined by Sompolinsky to encode the anomalous response to a field perturbation at large time scales ($x < 1$, $\tau_x > \tau_1$ in the parametric representation used so far). The anomaly function is a direct way to measure ergodicity breaking occurring in spin glasses (see Refs. 31,32), even at infinite time ($x = 0$). Defining the susceptibility function at the time-scale labeled by $x$ as

$$\chi(x) = \beta J[\rho - q(1)] + \Delta(x) ,$$  \hspace{1cm} (89)

where $\Delta(x)$ is given by the stationarity Eq. (61), the equilibrium susceptibility, Eq. (86), can be rewritten as

$$\chi_0 = \chi(0) = \beta J[\rho - q(1)] + \Delta(0) .$$  \hspace{1cm} (90)

The anomaly $\Delta(0)$ can, thus, be interpreted as the difference between the theoretical descriptions of the zero-field-cooled and the field-cooled susceptibility.

$$\Delta(0) = \chi(1) - \chi(0) = \beta \left[ q(1) - \int_0^1 dx \, q(x) \right]$$  \hspace{1cm} (91)

$$= \beta (q_{EA} - \overline{q}) .$$

7. Free Energy Integral $\int_{-\infty}^{\infty} dy \, P(0, y) \, \phi(0, y)$

The often occurring integral $\int_{-\infty}^{\infty} dy \, P(0, y) \, \phi(0, y)$ [see Eqs. (65) and (67)] can be expressed as a function of the local magnetization $m(0, y)$, numerically found as solution of coupled Eqs. (69), (71). $x = 0$. From the magnetization definition we write the identity

$$\phi(0, y) = \phi(0, 0) + \int_0^y dy' \, m(0, y') ,$$  \hspace{1cm} (92)

that is valid for every $y$. Combining this with the identity $\phi(1, y) - \phi(0, y) = \int_0^1 dx \, \delta(x, y)$ and taking the limit
FIG. 7: $\varphi - \rho$ as a function of temperature for $K = J$.
For $\mu < \mu_c = -J \varphi < 1$ and there is a discontinuity at the transition temperature.

$|y| \gg 1$, we get for the constant $\phi(0,0)$ the value

$$\phi(0, 0) = \lim_{|y| \to \infty} \left\{ T \Theta_1 + |y| - \frac{1}{2} \int_0^1 dx \Delta(x) \right\} \equiv \Theta_1$$

$$= T \Theta_1 - \frac{1}{2} \int_0^1 dx \Delta(x) + \int_0^\infty dy \left[ 1 - m(0, y) \right]$$

This leads to the more convenient expression, for a numerical computation,

$$\int_{-\infty}^{\infty} dy \ P(0, y) \phi(0, 0)$$

$$= T \Theta_1 - \frac{1}{2} \int_0^1 dx \Delta(x) + \int_0^\infty dy \left[ 1 - m(0, y) \right]$$

$$+ \int_{-\infty}^\infty dy \ P(0, y) \int_0^y dy' m(0, y').$$

1. Full lattice at zero temperature: $a > 0$

For $T \to 0$ the last term of the entropy in Eq. (S2) goes as

$$\beta J \int_{-\infty}^\infty dy \ P(1, y) \left[ \phi(1, y) - y \ m(1, y) \right] = \Theta_1(0) + O(e^{-1/T})$$

where the boundary functions $\phi(1, y)$ and $m(1, y)$ are given by Eqs. (S5), (S9).

1. Partially empty lattice: $a < 0$

The parameter $\Theta_1$ tends to $-\infty$ at zero temperature and $q(1)$ and $\rho$ are less than one, namely

$$\lim_{\beta \to \infty} q(1) = \lim_{\beta \to \infty} \rho \equiv \varphi = 2 \int_{-\infty}^\infty dy \ P(1, y) \int_{-\infty}^{\infty} dy P(1, y - J K / J - \mu / J)$$

If $\Theta_1 < 0$, in general, $e^{\Theta_1} \cosh \beta J y$ will not be much larger than one for any value of $y$ in the argument of $\phi(1, y)$ and in $y \ m(1, y)$. The zero $T$ limit depends on the sign of $\alpha_{|y|} \equiv |y| + a / J$:

$$\lim_{\beta \to \infty} \phi(1, y) = \begin{cases} \alpha_{|y|} & \text{if } \alpha_{|y|} > 0 \\ \frac{\beta}{2} \log 2 & \text{if } \alpha_{|y|} < 0 \end{cases}$$

The analogue of Eq. (JE) is, in this case,

$$\beta J \int_{-\infty}^\infty dy \ P(1, y) \left[ \phi(1, y) - y \ m(1, y) \right]$$

$$= (1 - \varphi) \log 2 + \varphi \Theta_1(0) + O(e^{-1/T}).$$
Since the zero temperature density is less than one, there will be a fraction \( 1 - \rho \) of spins whose orientation is irrelevant for measuring any observable. This brings to a degeneracy in the ground state, of \( 2^{N(1-\rho)} \) equivalent configurations. Then, for the entropy density, we get

\[
s = \lim_{\beta \to \infty} \left\{ \frac{(-\beta J)^2}{4} \left[ \rho - q(1) \right]^2 + (\rho - \rho_0) \Theta_1 + (1 - \rho) \log 2 \right\} = (1 - \rho) \log 2.
\]

Also in this case we can see from numerical data at low temperature that the behavior of \( \rho - q(1) \) and \( s \) is \( T^2 \), whereas \( \rho - \rho \sim T^3 \) is still consistent with our numerical data.

At zero temperature the free and internal energy are:

\[
u = f = -\frac{K}{2} \rho - \mu \rho + \int_0^1 dx q(x) \tilde{\Delta}(x)
\]  

where \( \rho \) is either 1 or \( \rho \) depending on the sign of \( \Theta_1 \).

V. PHASE DIAGRAMS

Analyzing the stability of the RS solution [see Eqs. (24)-(26)] one gets the critical lines

\[1 - (\beta J \rho)^2 = 0, \quad 1 - \beta K (1 - \rho) \rho = 0,\]

above which the only solution is the PM solution \( q(x) \equiv 0 \) for \( x \in [0, 1] \), \( \rho = 1/[1 + e^{-\Theta_1}] \) [see Eq. (24)], stable for any value of \( K \). In the \( T - \rho \) plane, these are, respectively, the straight line and the left branch of the spinodal line shown in Fig. 8 for \( K = J \). The two lines meet at the tricritical point \( (\mu_c, \rho_c) \). Between them, under the two hard line curves branching out from the tricritical point, there is a region of coexistence of phases (as indicated in the plot of Fig. 8). The broken line curves are the first order transition lines. When \( \mu < \mu_c \), in cooling at fixed chemical potential, the density of the system jumps discontinuously from the PM to the higher SG phase. As an example the line at \( \mu = -1.05 \) is plotted.

By crossing the critical line Eq. (102) above the tricritical point \( (\rho > \rho_c, T > T_c, \mu > \mu_c) \) the system undergoes a continuous phase transition of the SK-type to a FRSB SG phase, with a non-trivial continuous order parameter function \( q(x) \) which smoothly grows from zero.

Below the tricritical point the scenario is completely different, displaying a discontinuous transition from the PM phase to a FRSB SG phase with \( q(x) \) which discontinuously jumps from zero to a non-trivial (continuous) function. At the critical temperature the entropy is discontinuous, see Fig. 9 and hence a latent heat is involved in the transformation from the PM to the SG phase, implying that the transition is of the first order in the thermodynamic sense. The transition line is determined by the free energy balance between the PM and the SG phase, and is shown as a broken line in the phase diagrams. The line yielded by Eq. (103) where the PM solution becomes unstable, and the equivalent line from the SG side are the spinodal lines. Also interesting is the \( \mu - \rho \) phase diagram represented in Figs. 8, 9. We indeed see that the isothermal lines cross the instability lines with vanishing derivative (Fig. 10) and hence a diverging compressibility \( \kappa \) occurs crossing these lines.

It can be shown that the first order transition line can be determined in the \( \mu - \rho \) phase diagram from the isothermal and spinodal lines by using a Maxwell con-

FIG. 8: \( T - \rho \) phase diagram of the RBEGC for \( K/J = 1 \). The dot marks the tricritical point \( \mu_c/J = -1, T_c = J/2, \rho_c = 1/2 \). In the upper-left region we have the paramagnetic (PM) phase. In the upper-right region the FRSB spin glass (SG) phase. Following the iso-potential line at \( \mu/J = -1.05 \) one sees the jump in density at the first order phase transition.

FIG. 9: \( \mu - \rho \) phase diagram of the RBEGC for \( K/J = 1 \). Above the tricritical point (star) the transition is second order. Two isothermal lines are shown for temperature \( T = 0.5, 0.6 \), i.e. at and above.
Below the spinodal lines (in the $T - \rho$ plane) no pure phase can exist and the system is in a mixture of PM and SG phase (phase coexistence).

Eventually, the phase diagram in the $T - \mu$ plane, for $K = 1$, is shown in Fig. 11. Since our code, even in 'most suitable' gauge (see Sec. III B), is unstable in region of both low temperature and low chemical potential we were not able to reliably compute the points belonging to the SG spinodal line and the first order phase transition line for $T < 0.1 = T_c / 5$. The prolongation of the lines down to zero temperature are, hence, computed by fit. Therefore the zero-$T$ transition are estimates rougher than the others. The first order line goes to $\mu^{\text{st}}_0 = -1.256 \pm 0.009$ and the spinodal one reaches zero at $\rho^{\text{SG}}_0 = -1.465 \pm 0.008$. Nevertheless, it seems that we can rule out a reentrance to the PM phase as $T \to 0$.

By varying $K$ the scenario remains qualitatively unchanged [look at Ref. 15 for the phase diagrams of the Ghatak-Sherrington model (K = 0) and the frustrated Ising lattice gas model (K = -J)]. The only effect of a strong repulsive particle-particle interaction is to increase the phase diagram zone where the empty system ($\rho = 0$) is the stable solution. In order to find further phases, e.g. an antiquadrupolar phase, a generalization of the present analysis to a two component magnetic model including quenched disorder, has to be carried out.

If we now look at the qualitative reproduction of the phase diagram of the original BEG model, with ferromagnetic interaction, in Fig. 11, one can imagine a correspondence between the PM phase of the disordered magnetic material and the fluid phase of the $\text{He}^3-\text{He}^4$ mixture, and between the SG phase and the superfluid phase. The density $\rho$ corresponds to the $\text{He}^4$ density and the density $x$ of $\text{He}^3$ particles is $1 - \rho$. The second order transition line corresponds to the $\lambda$ transition points from fluid to superfluid.

### A. Transition Lines

The starting point to analytically determine the transition lines around the tricritical point is the expansion of Eq. 5 for small $q$ and densities next to $\rho_c$ that we report in appendix [Eq. 165]. There the various coefficients are expressed in terms of the function $\rho_0 = (1 + \exp(-\beta \mu - \beta KT/J))^{-1}$ coinciding with the paramagnetic density of the system evaluated along the second order transition line $A_0 = 0$ (along which is $\rho = \rho_0 = T/J$). The expansion parameters are the elements of the overlap matrix $q_{ab}$ and $r \equiv \rho - \rho_0$ (replica independent).

One can obtain the approximated analytical expression of the spinodal line at which the FRSB solution disappears and the first order transition line, at which the paramagnetic free energy overcomes the approximated spin glass free energy, around the tricritical point.

In figure 12 we show both the behavior of the spinodal

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**FIG. 10:** $\mu - \rho$ phase diagram of the RBEGC for $K = 1$. From the tricritical point (star) two solid lines come out: the spinodal lines at which the paramagnetic (upper-left curve) and the spin glass (lower-right curve) phases cease to exist, even as metastable. The first order transition lines, also branching out of the tricritical point, are plotted as dashed curves. An isothermal line is shown, for a temperature $T = 0.3$ below $T_c = 1/2$. Along such curve a first order phase transition occurs. In the plot, also the metastable branches are shown, both in the RS PM phase and in the FRSB SG phase (broken curves continuing the full curves). They reach the spinodal lines with zero derivative. In this plane of conjugated thermodynamic variables a Maxwell construction can be explicitly performed.

**FIG. 11:** $T - \mu$ phase diagram of the RBEGC for $K = 1$. For low $\mu$ (lower than -1.2 in this specific case) the code at $T = 0$ - in the Sommers gauge - is very unstable. Therefore the zero-$T$ transition points are obtained by fit.
line of the spin glass phase and the behavior of the first order transition line for the case $K = J$, in the neighborhood of the tricritical point in the $T$-$\mu$ plane. The lines in the plot are obtained by numerically solving the FRSB differential equations. Notice that, at the order of approximation used, the first order line displays a reentrance that does not occur, instead, performing the exact computation.

Notice that, at the present degree of approximation, already at such small distance from $(\mu_c, T_c)$ the first order line displays a reentrance, whereas the exact computation shows that this does not take place.

In the following we consider $\mu/J \rightarrow \mu$ and $T/J \rightarrow T$ and we express everything in terms of the small quantities $\delta \mu \equiv \mu - \mu_c$, $\delta T \equiv T - T_c$ and

$$\epsilon \equiv \delta \mu - \frac{\partial \mu_0(T)}{\partial T} \delta T , \quad (104)$$

This last auxiliary variable represents the distance from the second order phase transition line. If $\epsilon < 0$, in the $T$-$\mu$ plane, we are above the second order line (PM phase), otherwise below (SG phase). The function $\mu_0(T)$ is defined as:

$$\mu_0(T) = -\frac{1}{2} - T K - T \log \left( \frac{1}{T} - 1 \right) \quad (105)$$

The expressions of $r = \rho - \rho_0$ and $q_1$, the highest value of $q(x)$, are

$$q_1 = -\frac{3}{2} \delta T + \frac{1}{2} \sqrt{\epsilon - \delta T^2} \quad (106)$$

$$r = -\frac{\delta T}{2} + \frac{1}{2} \sqrt{\epsilon - \delta T^2} \quad (107)$$

Since $\delta T$ is always negative the above parameters are always positive, as far as they exist ($\epsilon \geq \delta T^2$). The solution breaks down at the spinodal line, therefore given by $\epsilon = \delta T^2$, i.e.

$$\mu = \mu_c - T_c + T + (T - T_c)^2 \quad (108)$$

From the comparison of the free energy values for the PM and the SG phases the approximated first order transition line turns out to be

$$\mu = \mu_c - T_c + T + 11.19(T - T_c)^2 \quad (109)$$

VI. CONCLUSIONS

In the present paper we have shown, in some detail, the properties of the Random Blume-Emery-Griffiths-Capel model. Such a model can be seen both as a 1-spin model or a Ising-spin model on a lattice gas. The two formulations are completely equivalent, at least from a static point of view. In the second representation, that we adopted, the interactions involved are a quenched random magnetic coupling $J_{ij}$ between spins and an attractive/repulsive coupling between (full) sites $(K)$. The system is embedded in a reservoir and the exchange of particles is controlled by the chemical potential $\mu$. The quenched random interaction is the source of a spin-glass phase at low temperature, provided that the chemical potential is large enough. If, on the contrary, $\mu$ is lower than a certain value, the system happen to be always in the paramagnetic phase, becoming progressively empty as $T \rightarrow 0$. We analytically studied the system in the mean field approximation. As already shown by the authors in Ref. the qualitative features of the system do not depend of the value of $K$, nor on the coupling being repulsive, attractive or zero.

The external parameters are the temperature and the chemical potential. In a certain region of the phase diagram (see figure 11 in Sec. V) for the $T - \mu$ diagram at particle-particle interaction $K = J$, or, else, figures 8, 9 and 10 the system undergoes a second order transition varying $T$ or $\mu$, down to a given 'tricritical' point at which the second order line ends. Indeed, the main feature of this model is that for low temperature, or high chemical potential, a continuous transition (in the Ehrenfest sense) from a pure PM phase to a pure SG phase does not occur anymore. In its place a first order transition takes place, with consumption/production of latent heat (see Fig. 14 in Sec. IV) and the appearance of a region of the parameter space where the two phases (PM and SG) do coexist. The SG phase comes out to be stable exclusively in the FRSB scheme of computation. We can, thus, rule
out the existence of a glass-like phase, by this meaning a
phase with one step RS in the static, corresponding to
a dynamic decoupling of the characteristic time scales of
the processes involved in two time sectors. Only one kind
of spin glass phase exist in the frozen phase and this is the
one typical of mean-field models for amorphous systems
(i.e. spin-glasses in the proper sense). In order to recover
a system with a stable 1RSB phase and displaying a first
order transition a lattice gas of spins interacting through
multiple spins interaction (i.e. a p-spin glass on lattice
gas) has to be considered (see e.g. Ref. 15). It is
interesting to notice that a discontinuous transition be-
tween liquid and glass, with coexistence of phases, has
been recently found in lattice heteropolymers with ran-
dom interactions 16,17. We also mention that generalizing
the present model to a two component magnetic model 18,
and including quenched disorder, further phases can be
found, e.g. an antiquadrupolar phase.

Computing the state of the system down to very low
temperature (including zero for not extremely low values)
it has been possible to see that the reentrance dis-
played in the $T$-μ phase diagram in the Replica Symmet-
ric approximation is just an artifact (see e.g. Ref. 16,
but also the ‘small q’ expansion in Appendix B). There is
no μ-range for which lowering the temperature the spin
glass can transform itself in a paramagnet.

We have discussed in detail the numerical method that
we use to solve the antiparabolic differential Parisi equa-
tions for the present model, allowing us to compute the
overlap order parameter and all the thermodynamic ob-
servables in the SG phase. In particular we face the prob-
lem of making the code converge at very low temperature,
including zero. Section III is dedicated to the presenta-
tion and explanation of the variational approach to the
problem of computing the FRSB anti-parabolic differen-
tial equations and the pseudo-spectral method employed
to solve them.

Besides the numerical resolution of the FRSB equa-
tions we have also studied the phase diagrams around
the tricritical point making an expansion for small over-
lap values and densities next to the density at the tricriti-
cal point (see Sec. V A and appendix VI). The expansion
formalism is valid all along, and around, the second order
transition line. The only interesting point around which
is worth probing the parameter space is, however, the
tricritical point. From there one can build analytical ap-
proximated expressions for the spinodal line of the spin
glass phase and for the first order transition line as well.
Above the tricritical point the transition is always second
order and qualitatively identical to the phase transition
taking place in the Sherrington-Kirkpatrick model, that
is easily recovered as limit of the present model (density
one, zero chemical potential, full lattice).

APPENDIX A: Stability Analysis in the Replica
Formalism

Rescaling $βJ → β$, $J_0/J → J$, $K/J → K$, $h/J → h$,$\mu/J → μ$ and, consequently $\tilde{K} → K/J$ we write the
replica thermodynamic potential 18 as

$$ G(\{\rho\}, \{m\}, \{q\}) \equiv nβf(\{\rho\}, \{m\}, \{q\}) $$

$$ = \frac{\beta K}{2} \sum_{a=1}^{n} \rho_a^2 + \frac{\beta J_0}{2} \sum_{a=1}^{n} m_a^2 + \frac{\beta^2}{4} \sum_{a \neq b} q_{ab}^2 - \log Z' $$

with $Z'$ and $H'$ given respectively in Eqs. (9), (10) and
repeated here for clarity:

$$ Z' = \sum_{\{\delta\}, \{\nu\}} \exp \{-βH'[\{\delta\}, \{\nu\}]\} $$

$$ -βH'[\{\rho\}, \{m\}, \{q\}] = \sum_{\nu} n_{\nu} (\beta K \rho_{\nu} + \beta \mu) $$

$$ + \sum_{\nu} \sum_{a=1}^{n} S_{\nu} m_a (\beta J_0 m_a + \beta h) + \frac{\beta^2}{2} \sum_{a \neq b} q_{ab} S_{\nu} n_a S_{\nu} n_b $$

The variation of $G$ with respect to the replica param-
eters is

$$ δG(\{\rho\}, \{m\}, \{q\}) = \beta \tilde{K} \sum_{a=1}^{n} \rho_a \ δ\rho_a $$

$$ + \beta J_0 \sum_{a=1}^{n} \sum_{b \neq a} m_a \ δm_a + \frac{\beta^2}{2} \sum_{a \neq b} \sum_{b \neq a} q_{ab} \ δq_{ab} $$

$$ - \sum_{a=1}^{n} \beta \tilde{K} \ δ (n_{a}) \ δ\rho_a + \sum_{a=1}^{n} \beta J_0 \ δ (S_{a} n_{a}) \ δm_a $$

$$ + \frac{\beta^2}{2} \sum_{a \neq b} \delta \left( (S_{a} n_{a} S_{b} n_{b}) \right) \ δq_{ab} $$

where the average $\langle ... \rangle$ is performed with the measure
given by Eq. (112). From $δG$ the saddle point equations
are immediately derived:

$$ q_{ab} = \langle S_{a} n_{a} S_{b} n_{b} \rangle $$

$$ \rho_a = \langle n_a \rangle $$

$$ m_a = \langle S_{a} n_{a} \rangle $$

Eventually the fluctuations functional is equal to

$$ δ^2 G(\{\rho\}, \{m\}, \{q\}) = \beta \tilde{K} \sum_{a=1}^{n} (δ\rho_a)^2 $$

$$ + \beta J_0 \sum_{a=1}^{n} (δm_a)^2 + \frac{\beta^2}{2} \sum_{a \neq b} (δq_{ab})^2 $$

$$ - \sum_{a=1}^{n} \beta \tilde{K} \ δ (n_{a}) \ δ\rho_a + \sum_{a=1}^{n} \beta J_0 \ δ (S_{a} n_{a}) \ δm_a $$

$$ + \frac{\beta^2}{2} \sum_{a \neq b} \delta \left( (S_{a} n_{a} S_{b} n_{b}) \right) \ δq_{ab} $$

without the existence of a glass-like phase, by this meaning a
phase with one step RS in the static, corresponding to
da dynamic decoupling of the characteristic time scales of
the processes involved in two time sectors. Only one kind
of spin glass phase exist in the frozen phase and this is the
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tions we have also studied the phase diagrams around
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lap values and densities next to the density at the tricriti-
cal point (see Sec. V A and appendix VI). The expansion
formalism is valid all along, and around, the second order
transition line. The only interesting point around which
is worth probing the parameter space is, however, the
tricritical point. From there one can build analytical ap-
proximated expressions for the spinodal line of the spin
glass phase and for the first order transition line as well.
Above the tricritical point the transition is always second
order and qualitatively identical to the phase transition
taking place in the Sherrington-Kirkpatrick model, that
is easily recovered as limit of the present model (density
one, zero chemical potential, full lattice).
where
\[
\delta \langle o_1 \ldots o_k \rangle = \delta \frac{\sum \{S,n\} o_a \ldots \delta o_k e^{-\beta H'}}{Z'}
\]
= \langle o_1 \ldots o_k \delta (-\beta H') \rangle - \langle o_1 \ldots o_k \rangle \langle \delta (-\beta H') \rangle
\]
and
\[
\delta (-\beta H') = \beta K \sum_{a=1}^{n} n_a \delta \rho_a
\]
+ \beta J_0 \sum_{a=1}^{n} \left( S_a n_a \delta m_a + \frac{\beta^2}{2} \sum_{b \neq a} S_a n_a S_b n_b \delta \rho_{ab} \right).
\]

In (117) we have, thus, six kinds of terms: the quadratic ones and the mixed ones whose coefficients are given in Table II.

The functional \( \delta^2 G \), that has to be definite positive in order for the solution around which the expansion is performed to be stable, takes the form
\[
\delta^2 G = \beta^2 \sum_{ab} (\delta q_{ab})^2 + \beta K \sum_a (\delta \rho_a)^2
\]
+ \beta J_0 \sum_a (\delta m_a)^2 + \sum_{(ab),(cd)} D_{(ab)(cd)} \delta q_{cd}
\]
+ \sum_{(ab),c} \delta q_{ab} E_{(ab)c} \delta m_c
\]
and the eigenvalues equations, then, come out to be
\[
\beta^2 \delta q_{ab} + \sum_{(ab),(cd)} A_{(ab)(cd)} \delta q_{cd}
\]
+ \sum_{(ab),c} D_{(ab)c} \delta \rho_c + \sum_{(ab),c} E_{(ab)c} \delta m_c = \Lambda \delta q_{ab},
\]
\[
\beta K \delta \rho_a + \sum_{c,d} D_{a(cd)} \delta q_{cd}
\]
+ \sum_{c,d} B_{ac} \delta \rho_c + \sum_{c} F_{ac} \delta m_c = \Lambda \delta \rho_a,
\]
\[
\beta J_0 \delta m_a + \sum_{c,d} E_{a(cd)} \delta q_{cd}
\]
+ \sum_{c} F_{ac} \delta \rho_c + \sum_{c} C_{ac} \delta m_c = \Lambda \delta m_a.
\]

| fluctuation term | coefficient |
|------------------|-------------|
| \( \delta q_{ab} \delta q_{cd} \) | \( A_{(ab)(cd)} = -\beta^4 \left( \langle n_a S_a n_b S_b S_c n_d S_d \rangle - \langle n_a S_a n_b S_b \rangle \langle n_c S_c n_d S_d \rangle \right) \) |
| \( \delta \rho_a \delta \rho_c \) | \( B_{ac} = -(\beta K)^2 \left( \langle n_a n_c \rangle - \langle n_a \rangle \langle n_c \rangle \right) \) |
| \( \delta m_a \delta m_c \) | \( C_{ac} = -(\beta J_0)^2 \left( \langle n_a S_a n_c \rangle - \langle n_a S_a \rangle \langle n_c \rangle \right) \) |
| \( \delta q_{ab} \delta \rho_c \) | \( D_{(ab)c} = -\beta^3 K \left( \langle n_a S_a n_b S_b n_c \rangle - \langle n_a S_a n_b \rangle \langle n_c \rangle \right) \) |
| \( \delta q_{ab} \delta m_c \) | \( E_{(ab)c} = -\beta^3 J_0 \left( \langle n_a S_a S_b n_c \rangle - \langle n_a S_a n_b \rangle \langle n_c \rangle \right) \) |
| \( \delta \rho_a \delta m_c \) | \( C_{ac} = -\beta^2 K J_0 \left( \langle n_a n_c \rangle - \langle n_a \rangle \langle n_c \rangle \right) \) |

TABLE II: Different contributions to the fluctuation functional of the Random BEGC model in the replica formalism. All the averages have to be evaluated at the saddle point given by Eqs. (122) - (124). The subscript \((ab)\) means distinct pairs of indexes.

A.1 The Stability of the RS Solution

Inserting in the above expressions the Ansatz that all replicas are equivalent (same density, same magnetization and same overlap between all of them) we can study the stability of the RS solution. The RS Ansatz is implemented by the substitutions:
\[
q_{ab} = (1 - \delta_{ab}) q_0 \quad ; \quad \rho_a = \rho \quad ; \quad m_a = m.
\]
Replica Symmetric Expressions for the Coefficients in $\delta^2 G$

| $A_{(ab)(ab)}$ | $-\beta^4 \left[ (n_a n_b) - (n_a S_a n_b S_b) \right]^2 = -\beta^4 \left[ \int D\rho \tilde{\rho}^2 (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $A_2$ |
| $A_{(ab)(cd)}$ | $-\beta^4 \left[ (n_a n_b n_c n_d) - (n_a S_a n_b S_b n_c n_d S_d) - (n_a S_a n_b S_b n_c) + (n_a S_a n_b n_c S_d) \right] = -\beta^4 \left[ \int D\rho \tilde{m}^4 (\rho) - \left( \int D\rho \tilde{m} (\rho) \right)^2 \right]$ | $A_0$ |
| $B_{aa}$ | $-\beta^3 \tilde{K}^2 \left[ (n_a) - (n_a) \right] = -\beta^2 \tilde{K}^2 \left[ \int D\rho \tilde{\rho} (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $B_1$ |
| $B_{ac}$ | $-\beta^3 \tilde{K}^2 \left[ (n_a n_c) - (n_a) (n_c) \right] = -\beta^2 \tilde{K}^2 \left[ \int D\rho \tilde{\rho}^2 (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $B_0$ |
| $C_{aa}$ | $-\beta^2 J_0^2 \left[ (n_a) - (n_a S_a) \right]^2 = -\beta^2 J_0^2 \left[ \int D\rho \tilde{\rho} (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $C_1$ |
| $C_{ac}$ | $-\beta^2 J_0^2 \left[ (n_a n_c) - (n_a S_a) (n_c) \right] = -\beta^2 J_0^2 \left[ \int D\rho \tilde{\rho}^2 (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $C_0$ |
| $D_{(ab)a}$ | $-\beta^2 \tilde{K} \left[ (n_a n_b S_b) - (n_a S_a n_b S_b) \right] = -\beta^2 \tilde{K} \left[ \int D\rho \tilde{\rho} (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $D_1$ |
| $D_{(ab)c}$ | $-\beta^2 \tilde{K} \left[ (n_a n_c n_d) - (n_a n_c S_d) \right] = -\beta^2 \tilde{K} \left[ \int D\rho \tilde{\rho}^2 (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $D_0$ |
| $E_{(ab)a}$ | $-\beta^3 J_0 \left[ (n_a n_b) - (n_a S_a n_b S_b) \right] = -\beta^3 J_0 \left[ \int D\rho \tilde{\rho} (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $E_1$ |
| $E_{(ab)c}$ | $-\beta^3 J_0 \left[ (n_a n_c n_d) - (n_a n_c S_d) \right] = -\beta^3 J_0 \left[ \int D\rho \tilde{\rho}^2 (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $E_0$ |
| $F_{aa}$ | $-\beta^2 J_0 \tilde{K} \left[ (n_a) - (n_a S_a) \right] = -\beta^2 J_0 \tilde{K} \left[ \int D\rho \tilde{\rho} (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $F_1$ |
| $F_{ac}$ | $-\beta^2 J_0 \tilde{K} \left[ (n_a n_c) - (n_a S_a) \right] = -\beta^2 J_0 \tilde{K} \left[ \int D\rho \tilde{\rho}^2 (\rho) - \left( \int D\rho \tilde{\rho} (\rho) \right)^2 \right]$ | $F_0$ |

TABLE III: Here we report the coefficients of the second order term in the expansion of the free energy functional of the Random BEGC model around the RS solution. The left hand side of equalities shows the generic expression of the coefficients, whereas the right hand side gives the coefficients evaluated with the Ansatz of replica symmetry. On the far right column the abbreviation for the RS coefficients are listed.

Using the definition

$$\Theta_0 = \beta \tilde{K} \rho + \beta \mu - \frac{\beta q_0}{2},$$

the replica one site Hamiltonian becomes:

$$-\beta H'(\rho, m, q_0) = \Theta_0 \sum_a n_a + \beta (J_0 m + h) \sum_a n_a s_a + \frac{\beta^2 q_0}{2} \left( \sum_a n_a s_a \right)^2 .$$

With this measure we have to compute the averages $\langle o_{a_1} \ldots o_{a_k} \rangle$ occurring in the saddle point equations and in the coefficients of the eigenvalues equations (see Tab. II).

"Opening", with the Hubbard-Stratonovic method, the squared sum in Eq. 125 into the Gaussian integral of an exponential with linear exponent we get

$$e^{-\beta H'} = \int_{-\infty}^{\infty} D\rho \exp \left[ -\sum_a H_a (\rho) \right] ,$$

with the one-index Hamiltonian

$$H_a (\rho) = \Theta_0 n_a + \tilde{h}(y) n_a S_a, \tag{128}$$

and

$$\tilde{h}(y) \equiv y \frac{\beta \sqrt{q_0}}{2} + J_0 m + h . \tag{129}$$

We also define the one-index partition sum as

$$Z_a (y) \equiv \sum_{\rho, m_n} e^{-\beta H_a (y)} = 2 + e^{\Theta_0} \cosh \tilde{h}(y) , \tag{130}$$

and the one-index average

$$\langle o_{a} \rangle \equiv \frac{\sum_{\rho, m_n} o_a e^{-\beta H_a (y)}}{Z_c (y)} , \tag{131}$$

that is a function of $y$.

With the help of Eqs. 129, 130, 125 we are now able to compute the numerator of the average $\langle o_{a_1} \ldots o_{a_k} \rangle$:

$$\sum_{\{S\}, \{n\}} o_{a_1} \ldots o_{a_k} e^{-\beta H'} = \int_{-\infty}^{\infty} D\rho \prod_{i=1}^{k} \sum_{S_{a_i}, m_{a_i}} o_{a_i} e^{-\beta H_{a_i} (y)} \prod_{i=k+1}^{n} \sum_{S_{a_i}, m_{a_i}} e^{-\beta H_{a_i} (y)} = \int_{-\infty}^{\infty} D\rho \left[ Z_1 (y) \right] \sum_{i=1}^{n} \langle o_{a_i} \rangle_{a_i} . \tag{132}$$
so that the complete expression of the average is eventually given by:

\[
\lim_{n \to 0} \langle o_{a_1} \ldots o_{a_k} \rangle = \lim_{n \to 0} \frac{1}{Z'} \int_{-\infty}^{\infty} \mathcal{D}y \left[ Z_1(y) \right]^n \prod_{i=1}^k \langle o_{a_i} \rangle_{a_i} = \int_{-\infty}^{\infty} \mathcal{D}y \prod_{i=1}^k \langle o_{a_i} \rangle_{a_i},
\]

since \( Z' = \int_{-\infty}^{\infty} \mathcal{D}y \left[ Z_1(y) \right]^n \to 1 \) in the zero replicas limit.

In our case \( o_{a} \) can be any combination of \( n_a \) and \( n_a S_a \) occurring in Eqs. \[143, 146\] and in the coefficients of the second order expansion term of the thermodynamic potential \[\text{Eq. 117}\] reported in Tab. III. In Eq. \[128\] \( n_a \) is coupled to \( \Theta_0 \), whereas \( n_a S_a \) is coupled to \( h \). Their one-index averages can, then, be obtained as:

\[
\langle n_a \rangle_{a} = \frac{\partial \log Z_{a}(y)}{\partial \Theta_0} = \frac{\cosh \hat{h}(y)}{e^{-\theta_0} + \cosh \hat{h}(y)} \equiv \tilde{\rho}(y),
\]

\[
\langle n_a S_a \rangle_{a} = \frac{\partial \log Z_{a}(y)}{\partial h} = \frac{\sinh \hat{h}(y)}{e^{-\theta_0} + \cosh \hat{h}(y)} \equiv \tilde{m}(y).
\]

Using these results we get Eqs. \[139 - 142\] and the expression of the coefficients of \( \delta^2 G \) for the RS solution. We report the complete list in Table III. Filling in those terms, the sum occurring in the eigenvalues Eqs. \[124 - 129\] can, thus, be written in the RS scheme as:

\[
\sum_{(cd)} A_{(ab)(cd)} \delta q_{cd} = (A_2 - 2A_1 + A_0) \delta q_{ab} + \left( A_1 - A_0 \right) \sum_{c} \delta q_{ac} + \frac{A_0}{2} \sum_{cd} \delta q_{cd}
\]

\[
+ \left( D_1 - D_0 \right) \sum_{c} \delta \rho_c = (D_1 - D_0)(\delta \rho_a + \delta \rho_b) + D_0 \sum_{c} \delta \rho_c
\]

\[
\sum_{c} E_{(ab)c} \delta m_c = (E_1 - E_0)(\delta m_a + \delta m_b) + E_0 \sum_{c} \delta m_c
\]

\[
\sum_{(cd)} D_{(ab)(cd)} \delta q_{cd} = (D_1 - D_0) \sum_{c} \delta q_{ac} + \frac{D_0}{2} \sum_{cd} \delta q_{cd}
\]

\[
\sum_{c} B_{ac} \delta \rho_c = (B_1 - B_0) \delta \rho_a + B_0 \sum_{c} \delta \rho_c
\]

\[
\sum_{c} F_{ac} \delta m_c = (F_1 - F_0) \delta m_a + F_0 \sum_{c} \delta m_c
\]

\[
\sum_{(cd)} E_{(ab)(cd)} \delta q_{cd} = (E_1 - E_0) \sum_{c} \delta q_{ac} + \frac{E_0}{2} \sum_{cd} \delta q_{cd}
\]

\[
\sum_{c} F_{ac} \delta \rho_c = (F_1 - F_0) \delta \rho_a + F_0 \sum_{c} \delta \rho_c
\]

\[
\sum_{c} C_{ac} \delta m_c = (C_1 - C_0) \delta m_a + C_0 \sum_{c} \delta m_c
\]

In the case \( J_0 = 0 \) (no ferromagnetic phase) \( C_{ab} = E_{(ab)c} = F_{ab} = 0, \forall a, b, c \) and we are left with a system of equations for \( n(n - 1)/2 \) variables \( \delta q_{ab} \) and \( n \) variables \( \delta \rho_a \) (\( \delta m_a \) are not involved, since always coupled with \( J_0 \)). In total, the dimension of the space of solutions is \( d_{tot} = n(n + 1)/2 \).

- \( \Lambda_0 \)
  To obtain the first eigenvalue \( \Lambda_0 \) we analyze the equations \[121 - 122\] in the subspace

\[
\sum_{a} \delta q_{ab} = 0, \forall b; \quad \sum_{a} \delta \rho_a = 0, \forall b.
\]

These are \( 2n \) equations and the dimension of this subspace is \( d_{\Lambda_0} = d_{tot} - 2n = n(n - 3)/2 \), corresponding to the degeneracy of \( \Lambda_0 \). Eq. \[146\] also implies that \( \sum_{ab} \delta q_{ab} = \sum_{a} \delta \rho_a = 0 \).

Using the conditions \[146\], the equations \[121, 122\] are easily reduced, in this case, to the single equation

\[
(\beta^2 + A_2 - 2A_1 + A_0) \delta q_{ab} = \Lambda_0 \delta q_{ab},
\]

yielding

\[
\Lambda_0 = \beta^2 + A_2 - 2A_1 + A_0.
\]

- \( \Lambda_1 \)
  We look at the second eigenvalue in the subspace

\[
\sum_{ab} \delta q_{ab} = 0; \quad \sum_{a} \delta \rho_a = 0,
\]

where, as opposed to the previous case, \( \sum_{a} \delta q_{ab} \neq 0, \forall b \) and \( \sum_{a} \delta \rho_a \neq 0, \forall a \). The \( \Lambda_0 \) and the \( \Lambda_1 \) subspaces are orthogonal.

Subtracting two [the equalities given in Eq. \[149\]] and the degeneracy of the \( \Lambda_0 \) to the total dimension we get the degeneracy of \( \Lambda_1 \): \( d_{\Lambda_1} = d_{tot} - d_{\Lambda_0} = 2(n - 1) \).

Eqs. \[121 - 122\] reduce in this subspace to

\[
[\beta^2 + A_2 + (n - 4)A_1 - (n - 3)A_0] \sum_{b} \delta q_{ab} + (n - 2)(D_1 - D_0) \sum_{c} \delta \rho_c = \Lambda_1 \sum_{b} \delta q_{ab}
\]

\[
(D_1 - D_0) \sum_{b} \delta q_{ab} + (\beta K + B_1 - B_0) \sum_{a} \delta \rho_a = \Lambda_1 \sum_{b} \delta q_{ab}
\]

for which the eigenvalue comes out to be
\[ \Lambda_1 = \frac{1}{2} \left[ \beta^2 + A_2 + (n - 4)A_1 - (n - 3)A_0 + \beta K + B_1 - B_0 \right] \]
\[ \pm \frac{1}{2} \sqrt{\left[ \beta^2 + A_2 + (n - 4)A_1 - (n - 3)A_0 - (\beta K + B_1 - B_0) \right]^2 + 4(n - 2)(D_1 - D_0)^2} \]

\[ \Lambda_2 = \frac{1}{2} \left[ \beta^2 + A_2 + 2(n - 2)A_1 + \frac{(n - 2)(n - 3)}{2}A_0 + D_1 + \frac{n - 2}{2}D_0 \right] \]
\[ \pm \frac{1}{2} \sqrt{\left[ \beta^2 + A_2 + (n - 2) \left( 2A_1 + \frac{n - 3}{2}A_0 \right) - \left( D_1 + \frac{n - 2}{2}D_0 \right) \right]^2 + 2(n - 1)[2D_1 + (n - 2)D_0]^2} \]

In the limit for \( n \to 0 \) the eigenvalues \( \Lambda_1 \) and \( \Lambda_2 \) are degenerate. They reduce both to
\[ \Lambda_1 = \Lambda_2 = \frac{1}{2} \left( \beta^2 + A_2 - 4A_1 + 3A_0 + \beta K + B_1 - B_0 \right) \]
\[ \pm \frac{1}{2} \sqrt{\left( \beta^2 + A_2 - 4A_1 + 3A_0 - \beta K - B_1 + B_0 \right)^2 - 8(D_1 - D_0)^2} \]

If we also put \( \hbar = 0 \) we simplify things much, since in this case the stable solution is \( q_0 = 0 \). This brings to
\[ \tilde{\rho}(y) = \frac{1}{e^{-\theta_0} + 1} \equiv \rho_0, \]
\[ \tilde{m}(y) = 0, \]
and, therefore, no integral in the coefficients of table III has to be carried out anymore. This leads to \( A_1 = A_0 = D_1 = D_0 = B_0 = 0 \) and
\[ A_2 = -\beta^4 \rho_0^2, \]
\[ B_1 = -\beta^2 K^2 \rho_0 (1 - \rho_0). \]
Substituting these coefficients into Eqs. (148)-(157) we obtain
\[ \Lambda_0 = \beta^2 (1 - \beta^2 \rho_0^2), \]
\[ \Lambda_1 = \beta \tilde{K} \left[ 1 - \beta \tilde{K} \rho_0 (1 - \rho_0) \right]. \]

The above analysis is valid for \( \tilde{K} > 0 \). This is always the case if the biquadratic coupling is \( K \geq 0 \). When a negative \( \tilde{K} \) occurs, however, at high enough temperature \( (T > 1/(2|K|)) \) \( \tilde{K} \) becomes negative. Taking this into account and repeating the whole scheme of computation the final result for \( \tilde{K} < 0 \) is, in the case of out interest,
APPENDIX B: Small q Expansion

The expansion to the fourth order in \( r = \rho - \rho_0 \) and \( q_{ab} \) is:

\[
\beta f(q_{ab}, r) = \beta f_0 - \beta \tilde{K} \left( \rho_0 - \frac{T}{J} \right) r
+ \frac{\Lambda_1}{2} r^2 - \frac{(\beta \tilde{K})^3}{6} \rho_0 (1 - \rho_0) (1 - 2 \rho_0) r^3
- \frac{(\beta \tilde{K})^4}{24} \rho_0 (1 - 7 \rho_0 + 12 \rho_0^2 - 6 \rho_0^3) r^4
+ \frac{\Lambda_0}{4} n \text{Tr} q^2 - \frac{(\beta J)^4}{2} \beta \tilde{K} \rho_0^2 (1 - \rho_0)
\times \left[ r + \frac{\beta \tilde{K}}{2} (2 - 3 \rho_0) r^2 \right] \frac{1}{n} \text{Tr} q^2
- \frac{(\beta J)^6}{6} \rho^3 \left[ 1 + 3 \beta \tilde{K} (1 - \rho_0) r \right] \frac{1}{n} \text{Tr} q^3
- \frac{(\beta J)^8}{48} \rho_0^2 \left[ \sum_{(ab)} q_{ab}^4 + 6 \rho_0 \sum_{(abc)} q_{ab} q_{ac}^2 
+ \frac{3}{2} \rho_0^2 \sum_{(abcd)} q_{ab} q_{bc} q_{cd} q_{da} \right]
\]

where \( \sum_{(\ldots)} \) is a sum over distinct indexes. The paramagnetic contribution is

\[
\beta f_0 = \frac{\beta \tilde{K}}{2} \rho_0^2 + \log(1 - \rho_0)
\]

and

\[
\rho_0 = \frac{1}{1 + \exp \left[ -\beta \mu - \beta \tilde{K} T/J \right]}
\]

is the value of the paramagnetic density evaluated along the second order phase transition line.

The above expansion is valid in the neighborhood of the second order transition line, \( \Lambda_0 = 0 \), down to, and including the tricritical point (\( \Lambda_0 = 0 \cup \Lambda_1 = 0 \)).

The various terms appearing in the expansion of the free energy functional, Eq. (165), are expressed in the following. In order to compute it in a generic \( N_B \)-RSB scheme we just need the following expression for the form of the overlap matrix \( q_{ab} \):

\[
q_{ab} = -q_{N_B} \delta_{ab} + \sum_{i=1}^{N_B} (q_i - q_{i-1}) \epsilon_{ab}^{(i)} + q_0
\]

with \( q_{N_B+1} = q = 1 \), and \( n = n, \epsilon_{ab} = \delta_{ab} \).

The matrix \( \epsilon_{ab}^{(i)} \) has \( m_i/m_{i+1} \) blocks along the diagonal and its summation rules are:

\[
\sum_b \epsilon_{ab}^{(i)} \delta_{bc} = \epsilon_{ac}^{(i)}
\]

where \( \epsilon_{ab}^{(i)} \) is the value of the paramagnetic density evaluated along the second order transition line. In the full replica symmetry breaking limit and in the zero replicas limit the breaking parameters become continuous between 0 and 1:

\[
m_i \to i, \; m_k - m_{i+1} \to -dx.
\]

Moreover the same structure of Eq. (166) holds for any \( \sum_{ab} q_{ab}^p \), thus

\[
\lim_{n \to 0} \sum_{ab} q_{ab}^p = \lim_{n \to 0} \sum_{i=0}^{N_B} q_{ab}^p (m_i - m_{i+1}) = \int_0^1 dx \; q_0^p (x)
\]
Thus the above expressions for the terms contributing to the sum reduce to

$$\sum_{i<j<k} q_i q_j^2 (m_k - m_{k+1})(m_i - m_{i+1})$$

where we have neglected terms of order $\mathcal{O}$. In the full replica symmetry breaking limit (and for number of replicas $n \to 0$):

$$m_k \to x \quad m_k - m_{k+1} \to -dx$$
$$m_i \to y \quad m_i - m_{i+1} \to -dy$$

Thus the above expressions for the terms contributing to the sum reduce to

$$B \to \int_0^1 dx \ q(x) \int_0^x dy \ q(y)^2$$
$$D \to \int_0^1 dx \ x \ q(x)^3$$

where we have neglected terms of order $(dx)^2$. Summing up, the trace comes out to be:

$$\lim_{n \to 0} \frac{1}{n} \text{Tr} \ q^3 = 3B + D$$

$$= 3 \int_0^1 dx \ q(x) \int_0^x dy \ q(y)^2 + \int_0^1 dx \ x \ q(x)^3$$

**Term** $\text{Tr} q^3 = \sum_{abc} q_{ab} \ q_{bc} \ q_{ca}$

Exploiting the formulation of Eq. (170) once again we can obtain

$$\sum_{bcd} \ q_{bcd} \ q_{cd} \ q_{da} = \sum_{i,j,k} q_i q_j q_k \ \sum_{bcd} \left( \epsilon_{ab} \epsilon_{bc} \epsilon_{cd} \epsilon_{da} - \epsilon_{ab} \epsilon_{bc} \epsilon_{cd} \epsilon_{da} - \epsilon_{ab} \epsilon_{bc} \epsilon_{cd} \epsilon_{da} - \epsilon_{ab} \epsilon_{bc} \epsilon_{cd} \epsilon_{da} \right)$$

Contribution and multiplicity are reported in table IV

| type of sum | contribution | multiplicity |
|-------------|--------------|--------------|
| A: $i < j < k$ | 0 | 6 |
| B: $i = j < k$ | $q_i q_j^2 (m_k - m_{k+1})(m_i - m_{i+1})$ | 3 |
| C: $i < j = k$ | 0 | 3 |
| D: $i = j = k$ | $q_k [ (m_k - m_{k+1})^2 - m_{k+1}(m_k - m_{k+1})]$ | 1 |

TABLE IV: Contributions to the trace of $q^4$ for a generic number of breakings of the replica symmetry: only two kind of terms are different from zero.

For $N_B \to \infty$ and $n \to 0$ the terms are

$$B \to - \int_0^1 dx \ q(x) \int_0^x dy \ q(y) \int_0^y dq^2(z)$$

(184)
TABLE V: Contributions and multiplicities of the terms in the sum of Eq. (184). Only for kinds of terms are non-zero. Of them, term $E$ turns to be of order $(dx)^2$ once the zero replica limit has been performed.

\[ E \to 0 \]  
\[ F \to -\int_0^1 dx \int_0^x dy \int_0^y dz \]  
\[ H \to -\int_0^1 dx \int_0^x dy \int_0^y dz \]

Eventually this last object reads:

\[ \sum_{abc} q_{ab} q_{ac}^2 = \sum_b q_{ab} q_{ac}^2 = \sum_c q_{ac} q_{ac}^2 \]
\[ = \sum_i q_i^2 (m_i - m_{i+1}) \sum_j q_j^2 (m_j - m_{j+1}) \]

the FRSB limit (for $n \to 0$) being

\[ \lim_{n \to 0} \frac{1}{n} \sum_{abc} q_{ab} q_{ac}^2 = 2 \int_0^1 dx \int_0^x dy \int_0^y dz \]

1. Full RSB free energy for disordered BEG around $T_c$

Computing the expansion for small $q$ and $\rho \approx \rho_0$, Eq. (165), in the FRSB scheme, yields

\[ \beta f = \beta f_0 - \beta K (\rho_0 - \frac{T}{J}) - \frac{\Delta_0}{4} \int_0^1 dx \int_0^x dy \int_0^y dz \]

\[ + \frac{1}{2} \left( \frac{\beta K}{6} \rho_0 (1 - \rho_0) \right)^2 \]

\[ - \frac{1}{24} \left( \beta K \right)^4 \rho_0 (1 - \rho_0) \]

\[ + \frac{1}{2} \beta K \rho_0 (1 - \rho_0) \left[ \frac{3}{2} + \beta K (1 - \rho_0) \right] \]

\[ \beta f = \beta f_0 - \frac{\beta K}{4} (1 - \rho_0)^2 \]

\[ - \frac{\beta K}{6} \rho_0 (1 - \rho_0) \left( 1 + \beta K (1 - \rho_0) \right) \]

\[ + \frac{\beta K}{24} \left( 1 + 3 \beta K (1 - \rho_0) \right) \int_0^1 dx \int_0^x dy \int_0^y dz \]

The expansion is valid along the critical line $\rho = T$ down to the tricritical point $(\rho_c, T_c)$ given by Eqs. (219-220).

The last term is introduced for the case $K = 0$, i.e. for the Ghatak-Sherrington model, in which $1 - 3 \rho_0$ goes to zero at the tricritical point $\rho_c = 1/3$ (e.g., decreasing temperature or density along the second order line). In this case the relevant quartic term $\int_0^1 dx q(x)^4$, responsible for the replica symmetry breaking, vanishes. This means that, in this case, the symmetry breaking is weaker than, for instance, in the SK case.

Putting $\rho \to 1$ and $r = 0$ in the above expression we obtain the same result of Ref. [51], where the same kind of expansion for SK model is performed, up to the forth order. To allow a straightforward check we rewrite the above formula in the way of Ref. [51].
\[-2\rho_0^3 (1 - 3 \rho_0) q(x)^2 \int_0^x dy \, q(y)^2
\]
\[+12 \rho_0^4 q(x) \int_0^x dy \, y \, q(y)^3
\]
\[+36 \rho_0^4 q(x) \int_0^x dy \, y \int_0^y dz \, q(z)^2 \}
\[+O(q^6) + O(r)
\]

Variation with respect to \(q(x)\) and further differentiation with respect to \(x\) lead to an integral equation that can be reduced to
\[
\frac{\partial_x q(x, \rho)}{q(x, \rho)} = \frac{1}{x - (1 - 3 \rho)^2} + \frac{6 x \rho^2}{(1 - 3 \rho)^2 + 6 x \rho^2} \tag{192}
\]
or
\[
\partial_x q(x, \rho) = 0 \tag{193}
\]

Below the critical temperature the solution is:
\[
q(x, \rho) = \frac{C_1}{(1 - 3 \rho)^2 \sqrt{(1 - 3 \rho)^2 + 6 \rho^2 x^2}} \tag{194}
\]
as far as \(x \leq x_M(\rho)\), otherwise \(q(x, \rho) = q_1\) for \(x > x_M(\rho)\). The value \(x_M\) is given by equating \(q_1 = q(x_M, \rho)\):
\[
x_M(\rho) = \frac{q_1 (1 - 3 \rho)^3}{\sqrt{C_1^2 - 6 \rho^2 (1 - \rho)^4 q_1^2}} \approx \frac{q_1 (1 - 3 \rho)^3}{C_1} + O(q^3) \tag{195}
\]
Notice that in Eqs. (192), (195) \(\rho = \rho_0 + r\) and the formulas have yet to be expanded in \(r\). The above computation can be simplified neglecting the quartic terms in Eq. (195) which are irrelevant with respect to the RSB, i.e. all but the one involving \(q_{ab}^4\). In this case \(q(x)\) is simply
\[
q(x) = A(r) \, x \tag{196}
\]
with
\[
A(r) = \frac{2 \rho_0}{(1 - 3 \rho_0)^2} + \beta \tilde{\kappa} (\beta J)^6 \rho_0^4 \frac{(1 - \rho_0)}{(1 - 3 \rho_0)^2} \, r \tag{197}
\]

for \(x \leq x_M \sim q_1/A(r)\) and \(q(x) = q_1\) for \(x > x_M\). If \(r \to 0\) and \(\rho_0 \to 1\) \(A(0)\) reduces to 1/2 (SK model) \(\Box\)

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Indeed, the important aspect of mean field models is that this calculation can actually be carried out, whereas for short range disordered models the free energy is beyond the reach of analytical computation.

In the dynamical approach of Sompolinsky\textsuperscript{38,39} to the spin glass problem $\Delta$ represents, in the Fluctuation-Dissipation Relation (FDR), the anomaly in the linear response with respect to the equilibrium value.