Kink-like solitons in quantum droplet

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Abstract
Solitonic excitations of the one-dimensional quantum droplets are obtained, which smoothly connect vacuum with the flat-top droplet, akin to compactons in classical liquids. These solitons are of the kink type, necessarily residing on a constant pedestal, determined by the mean-field (MF) repulsion and beyond mean field quantum correction and having exactly one-third of the uniform condensate amplitude. Akin to the kinks, the propagating modes occur in pairs and are phase-locked with the background. The lowest chemical potential and condensate amplitude at the flat-top boundary matches with the self-trapped quantum droplet. More general excitations of analogous kind are obtained through the Möbius transform, which connect the required solutions to elliptic functions in general.

Keywords: kink-like soliton, quantum droplet, mean field and beyond mean field coupling

(Some figures may appear in colour only in the online journal)

1. Introduction
Solitons are ubiquitous in one-dimensional non-linear dynamical systems [1] and have manifested in diverse physical systems [2–4], ranging from polyacetylene [5] to optical fibres [6, 7] and Bose–Einstein condensates (BECs) in cigar shaped BEC [8, 9]. Characteristically, they take the form of dark [10, 11], grey [12] and bright [13–15] solitons, respectively, in the repulsive and attractive interaction regimes of BEC. Their existence and stability depend on the balancing effect of dispersion with non-linearity. On the other hand, the kink and anti-kink are solitonic solutions of the $\lambda \phi^4$-theory in one dimension [16], owing their stability to their topological charges [17, 18]. They interpolate between the two degenerate vacuum in the broken parity phase [19], while passing through the normal phase, where the order parameter vanishes. The analogous solutions for the complex order parameter are the previously mentioned, dark and grey solitons of the non-linear Schrödinger equation (NLSE), describing the MF dynamics of the one dimensional BEC. Akin to the kink and anti-kink, these extended objects asymptotically connect points on the degenerate vacuum manifold of the broken global U(1) symmetry phase of BEC [8, 20]. The grey soliton is the well-known Lieb mode, first obtained in the second-quantized model [21], and subsequently found as an exact solution of the NLSE [22–24], which connects points in the vacuum manifold without passing through the normal phase. It has been observed in BEC [12, 25, 26], and later as excitations in water body [27, 28] and other physical systems [29, 30]. Generically, these excitations are composed of the hyperbolic tangent function, which asymptotically takes positive and negative values. The well-known bright soliton has a characteristic profile in the form of the hyperbolic secant function, vanishing at spatial infinity [31].

The BEC is well described by the MF Gross–Pitaevskii (GP) equation, which in the cigar shaped geometry and for the weak coupling case, takes the form of NLSE [9]. Various solitonic excitations in BEC have been well studied [11, 13]. These solitons owe their stability to the balancing of dispersion with non-linearity. Recently, a new form of quantum matter, the quantum droplet [32, 33], has been identified in BEC, after taking into consideration beyond mean-field (BMF) Lee–Huang–Yang quantum correction [34]. The effect of BMF correction has been shown to lead to a MF equation, with cubic-quartic non-linearities [35]. In one dimension, an exact solution of a quantum droplet emerges due to the balancing of MF and
BMF effects. The size of the droplet depends on the particle number and for large number of particles, droplet has a flat-top profile, where number density remains nearly uniform [36]. The droplets are self-bound and exist in free space, recently observed experimentally in a number of systems like Bose–Bose (BB) mixture [37–39] and dipolar BEC [40]. Experimentally, variation of the coupling from weak to strong attractive domains yields a transition from expanding to localized state, which agrees with theoretical BMF predictions. The equilibrium properties of quantum droplets, e.g., size, critical number density and binding energy, have been found in agreement with the theoretical predictions [41, 42]. The dimension crossovers from 1D → 3D [43] and 2D → 3D in binary BECs have also been analysed [44]. Droplet with constituents having electric and magnetic moments have been explored [45], where it is observed that the droplets experience a crossover from the cigar to pancake shapes in terms of relative dipole orientations. A new type of droplet in binary magnetic gases has been recently found, where single component self-bound droplet can couple with other magnetic components, not in the droplet phase [46]. The study of collective excitations in the form of Goldstone modes, corresponding to the spontaneously broken internal and translational symmetries, is the subject of many recent investigations [47–50]. It is worth pointing that, in the presence of cubic-quintic interactions condensate solitons [51, 52], dipole soliton [53] as well as vortex soliton [54] have been studied, where the droplet like structure, dipole and vortex soliton mutually transforms into each other [52].

It is well-established that the BMF effect emerges from the zero-point energy summation of the Bogoliubov modes and depends on the dimensions of the system [32, 33, 39]. The BMF correction, in three-dimensions is attractive and scales as $n^{5/2}$ (n being the number density), whereas in one-dimension, scaling is proportional to $n^{3/2}$ and repulsive in nature [39, 44]. The corresponding mean field equation is the amended NLSE, having cubic and quadratic non-linearities. In one-dimension, exact self-bound droplet solution has been obtained by Petrov and Astrakharchik [33], revealing its characteristic flat-top nature. The dynamical properties of one-dimensional perturbed droplet has been studied by using the variational approach where period of small oscillation of the soliton is obtained [55]. As is well-known, modulation instability (MI) dictates the stability of the propagating modes in non-linear systems. Recently, the growth rate of MI for one-dimensional quantum droplet and BB mixture have been investigated [56, 57]. The possible MI of this system has been studied, with and without spin–orbit coupling, indicating the parameter domain where the instability of propagating plane waves may lead to solitonic excitation. The parameter domain beyond MI is conducive for generation of soliton and soliton trains [58].

Here, we explicitly demonstrate kink-like solitons in the droplet regime, having functional form similar to dark and grey solitons, but also showing significant differences. The dark/grey solitons of cigar shaped BEC, $\psi(x) = \sqrt{\sigma_0} (i \sin \theta + \cos \theta \tan(x \cos \theta / \xi)), \text{ asymptotically take the value of uniform vacuum density } \sigma_0$ [23, 31]. The two limiting cases, $\theta = 0$ and $\cos^2 \theta < 1$, represent the dark and grey soliton with lowest condensate density zero and finite, respectively, at the origin. On the other hand, kink-like solitons necessarily require the presence of a constant background, which is exactly one-third of the uniform condensate amplitude: $(\sqrt{\sigma_0} / 3 \sin \theta) g^{3/2} / \delta g$. They smoothly connect the normal zero-condensate vacuum with the droplet configuration and occur in pairs, similar to kink/anti-kinks in the $\lambda \phi^4$-theory [16]. It is worth mentioning that $\lambda \phi^4$-theory has two vacuum configurations with zero energy and kink solution of the form $\psi(x) = A \tanh x$, connecting these two vacuum with boundary conditions $\lim_{x \to -\infty} \phi(x) = -A$ and $\lim_{x \to \infty} \phi(x) = A$. However, in the present case, kink-like excitations asymptotically approach vanishing condensate density only at one end, unlike kink/anti-kinks and dark/grey solitons, which connect the degenerate vacua at both the asymptotic domain. They are similar to compactons [59], manifesting in the liquid-normal material boundary, having strictly compact support with vanishing derivatives. However, the quantum liquid solitons smoothly interpolate between the normal phase matching with the droplet phase located at the origin, with a suitably translated kink-like soliton. The fact that the droplet has a broad flat density profile makes it plausible that these solitonic excitations may appear at the boundary of the droplets, wherein the condensate phase smoothly joins with the normal phase.

The paper is organized as follows. In section 2, theory of quantum droplets is briefly described leading to the amended GP equation in the form of NLSE with quadratic coupling, governing its dynamics in one-dimension. Section 3 is devoted to obtain the consistency relations for the kink-like solution with background and obtain its stability criteria. It is to be mentioned that the consistency conditions lead to a bounded below chemical potential, which is identical to the lowest chemical potential of droplets. In section 4, we consider a more general exact solution of a kink-like behavior and find the interconnection between the background, amplitude and healing length of the solitonic profile. In section 5, using the suitable background subtraction, we obtain the ground state energy and momentum for kink-like soliton. Finally, we conclude with the summary of results and future directions for investigation.

2. Theory of quantum droplets in one dimension

Quantum droplets have been observed by exploiting the BB mixture of ultracold atoms. The formation of quantum droplets are the result of balance between the MF and BMF interactions such that $0 < \delta g \ll g$. where $g = g_{\uparrow \uparrow} = g_{\downarrow \downarrow}$, is the intra-particle interaction with each component having equal number of atoms, $n = n_{\uparrow} = n_{\downarrow}$ and $\delta g = g_{\uparrow \downarrow} + g_{\downarrow \uparrow}$, being the inter-particle interaction. In stability regime $0 < \delta g \ll g$, the droplet energy is [33, 60]

$$E_{1D} = \frac{(g_{\uparrow \uparrow}^{1/2} n_{\uparrow} - g_{\downarrow \downarrow}^{1/2} n_{\downarrow})^2}{2} + g \delta g (g_{\uparrow \downarrow}^{1/2} n_{\uparrow} + g_{\downarrow \uparrow}^{1/2} n_{\downarrow})^2 (g_{\uparrow \uparrow} + g_{\downarrow \downarrow})^2 \frac{2}{3 \pi} (g_{\uparrow \downarrow} n_{\uparrow} + g_{\downarrow \uparrow} n_{\downarrow})^{3/2}. \quad (1)$$
It has been illustrated earlier that, in three-dimensions, MF and BMF interaction energies scale as: $E_{\text{MF}} \propto n^2$ and $E_{\text{BMF}} \propto n^{3/2}$ [32, 39], whereas, in one-dimension, energy of the effective one component BEC from the above equation leads to [33],

$$E_{1D} = \delta gn^2 - \frac{4\sqrt{2}}{3\pi} (gn)^{3/2}. \quad (2)$$

The minimum energy per particle, $\mathcal{E} = E_{1D}/n$, yields the equilibrium density $n_0 = 8g^2/(\pi^2\delta g^2)$, with the corresponding chemical potential $\mu_0 = -\delta gn_0/2$. Thus, at certain density, MF and BMF effects compensate each other leading to the emergence of stable droplets. The negative chemical potential is essential as it prevents self-evaporation of droplets.

In one-dimension, BMF correction in energy of the BEC modifies the mean field dynamical equation in the form of amended GP equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \delta g |\Psi|^2 \Psi - \frac{\sqrt{2m}}{\pi \hbar} g^{3/2} |\Psi|^4 \Psi, \quad (3)$$

characterized by cubic and quadratic non-linearities. This MF expression arises from an effective two-components BEC, with attractive inter-particle and intra-particle interaction in the region, $0 < \delta g < g$.

We consider the transformation $\psi(x,t) = \Phi(x - vt)\exp\left[i(kx - \frac{\gamma}{2}t)\right]$ and substitute into the amended NLSE equation. Comparing the imaginary and real parts lead to two coupled equations. First one is the continuity equation as given below:

$$\frac{\partial \Phi}{\partial t} + \frac{\hbar k}{m} \frac{\partial \Phi}{\partial x} = 0, \quad (4)$$

with $X = x - vt$ and $v = \hbar k/m$. The second equation originating from the real part reads as,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial X^2} + \delta g |\Phi|^2 \Phi - \frac{\sqrt{2m}}{\pi \hbar} g^{3/2} |\Phi|^4 \Phi - \frac{\mu}{m} \Phi = 0. \quad (5)$$

The term $\hbar^2 k^2/2m$, coming from the kinetic part of amended GP equation, has been absorbed into $\mu = \mu - \hbar^2 k^2/2m$. In the next section, we take the soliton solution having the kink-like behavior and illustrate its properties.

3. Kink-like soliton

It is to be noted that the amended NLSE has non-linearity, similar to the weak and strong coupling domains of BEC [61]. Although the kink-type solutions exist in NLSE, the same has not been existed for the strong coupling. Therefore we consider a general ansatz having a propagating kink-like excitation, phase-locked with a non-zero background:

$$\Phi(X) = A + B \tanh\left(\frac{X}{\xi}\right), \quad (6)$$

where $A, B$ are constants with $\xi$ being the healing length. Substituting the above in equation (5), one gets the following relationships:

$$1 = \left(\frac{\delta gm}{\hbar^2}\right) B^2, \quad A = \left(\frac{\sqrt{2m}}{3\pi \hbar}\right) \frac{g^{3/2}}{\delta g},$$

$$B = \pm \left(\frac{\sqrt{2m}}{3\pi \hbar}\right) \frac{g^{3/2}}{\delta g}, \quad \frac{\hbar^2 k^2}{2m} = \mu + \left(\frac{4m}{9\pi^2 \hbar^2}\right) \frac{g^3}{\delta g}. \quad (7)$$

which illustrate that healing length is inversely proportional to soliton amplitude and controlled by the MF coupling. From the above equation, it is evident that the solitons necessarily reside on a constant condensate background, which is exactly one-third of the uniform condensate amplitude. They occur in pairs, $B = \pm A$, vanishing asymptotically at one end.

It is to be noted, soliton density for the kink-like case vanishes at asymptotic end $x = -\infty$ and remains finite: $(\frac{4m}{9\pi^2 \hbar^2}) \frac{g^3}{\delta g}$ at $x = \infty$, having characteristically different behavior from the kink solitons of $\lambda \phi^4$-theory. Similar is the case with anti-kink like soliton but at the opposite asymptotic ends. Kink and anti-kink like solitons connect the normal vacuum with the quantum droplets located at the origin, with an appropriate translation of soliton profile. This result is in agreement with the Petrov’s flat bulk region for droplet [33]. Moreover, the positive MF coupling leads to the minimum chemical potential based on dispersion relation: $\mu_{\text{min}} = 0$ and $\mu_0 = -(\frac{\sqrt{2m}}{\pi \hbar}) g^{3/2} / \delta g < 0$, which establishes that the chemical potential is bounded below and is identical to the self trapped droplet condition. The soliton amplitude lies between zero and twice the constant background, taking the form,

$$\psi_{\pm}(x,t) = \sqrt{\frac{2m}{3\pi \hbar}} \left(\frac{g^{3/2}}{\delta g}\right) \left[1 \pm \text{tanh}(X/\xi)\right] \times \exp\left[i(kx - \mu t / \hbar)\right], \quad (8)$$

where $\xi = \frac{2\pi \hbar^2}{m}(\delta g / g^4)^{1/2}$. In figure 1, variations of the soliton profile and its density are depicted as a function of position for different values of BMF repulsion ($g$). The solid and dashed curves are for the two solitons travelling with equal velocity in opposite directions, `ψ/µ = $A \left[1 \pm \text{tanh} \left(\frac{kx}{\sqrt{2}}\right)\right]$`. They can appear at the two boundaries of the droplet in a static configuration. The amplitude of the solitons increases by increase of the repulsive interspecies interaction. Interestingly, $B = 0$ and $B \to 0$ limits in the solution are not same. In case of $B = 0$, one obtains the constant background whereas $B \to 0$, profile has the constant background $\frac{\sqrt{2m}}{\pi \hbar} (g^{3/2}/\delta g)$ (figure 2).

We now investigate the stability of these solitons through the Vakhitov–Kolokolov (VK) stability analysis [65], which determines the ‘cost’ of increasing the particle number density incrementally. For this purpose, we determine the number density,

$$n = N/L = \frac{1}{L} \int_{-L/2}^{L/2} |\psi|^2 \, dx = \left(\frac{4m}{9\pi^2 \hbar^2}\right) \frac{g^3}{\delta g}. \quad (9)$$

the variation of number density with respect to the chemical potential determines the VK stability. For $L \to \infty$, VK stability
The solution of the amended GP equation in the form of kink-type soliton is:

\[ \Phi(x) = A + \frac{B f(x)}{1 + D f(x)} \]

where \( f(x) \) satisfy the elliptic function equation

\[ f''(x) + a f^2(x) + c f(x) = 0. \]

In the present case, we consider the kink-type scenario with \( f(x) = \tanh(X/\xi) \). The emerging inter-connecting relationships, from the amended GP equation, between \( A, B \) and \( D \) are given as following:

\[
\begin{align*}
\hbar^2 m \xi^2 (B - AD) - g_1 B^3 + g_2 B^2 D + \mu BD^2 &= 0, \\
\hbar^2 m \xi^2 (B - AD) D - 3g_1 A B^2 + g_2 B^2 + 2g_2 ABD + \mu AD^2 + 2\mu BD &= 0, \\
g_1 B^3 + 3g_1 A^2 B - 2g_2 A B^2 - g_2 A^2 D - g_2 B^2 D - 2\mu AD - B - \mu BD &= 0, \\
g_1 A^3 - g_2 A^2 B + 3g_1 A B^2 - 2g_2 ABD - g_2 B^2 - \mu A - \mu AD^2 + 2\mu BD &= 0.
\end{align*}
\]

A straightforward but lengthy calculation leads to,

\[ (A - B)^2 = \frac{\mu}{g_1} (1 - D)^2, \]

\[ g_1 (A^2 + B^2) - g_2 (B + AD) - \mu (1 + D^2) = 0, \]

explicitly, yielding the background \( A \) and soliton amplitude \( B \), in terms of \( D \)

\[ A = B \pm \left( \frac{\mu}{g_1} \right)^{1/2} [1 - D] \]

\[ B = \frac{1}{2g_2} \left[ \left( \pm 2(\mu g_1)^{1/2} (D - 1) + g_2 (D + 1) \right) \right. \]

\[ + \left. \left( \pm 2(\mu g_1)^{1/2} (D - 1) + g_2 (D + 1) \right)^2 - 4g_1 \left( \mp g_2 (\mu g_1)^{1/2} D (1 - D) - 4\mu D \right) \right]^{1/2}, \]

and the healing length is given by

\[ \frac{1}{\xi^2} = \left( \frac{m}{\hbar^2} \right)^2 \frac{1}{(3 - D^2)} \left[ 3g_1 B^2 - 2g_2 BD - \mu D^2 \right]. \]

The constant \( D \) can be obtained by exploiting the relationships given in equations (15) and (16), together with the first two equations in equation (13). It is worth mentioning that, solutions in equation (11) are of more general type, where \( 1 + D \tanh(X/\xi) \) behaves as a space dependent weight factor with \(-1 < D < 1\), to have non-singular solutions. The weight factor affects the soliton density as well as the healing length.

In our next section, we derive the energy and momentum for kink-like soliton. We also explain the BMF effect in energy density and show the procedure to counter the divergent background energy part.

5. Energy and momentum of the kink-like soliton

The ground state energy can be computed following the standard approach [23], with the Hamiltonian density,

\[ \mathcal{H} = \frac{\hbar^2}{2m} \left| \frac{\partial \psi}{\partial x} \right|^2 + \frac{1}{2} g_1 |\psi|^4 - \frac{2}{3} \frac{\sqrt{2m}}{\pi \hbar} g_2 |\psi|^3. \]
We consider $\Psi(x, t) = \sqrt{\sigma} \exp[i(kx - \frac{\mu}{2}t)]$, with $\sqrt{\sigma} = A + B \tanh(X/\xi)$ and use in the amended NLSE. As it will be seen later, $\sigma$ is positive semi-definite for our ansatz solution. From the imaginary and real parts, one gets,

$$\frac{\partial \sigma}{\partial t} = -v \frac{\partial \sqrt{\sigma}}{\partial X} \quad \text{with} \quad v = \frac{\hbar k}{m},$$

$$\frac{\hbar^2}{2m} \frac{\partial}{\partial X} \left( \frac{\partial \sqrt{\sigma}}{\partial X} \right)^2 = -\left( \frac{\mu}{3} - \frac{h^2 k^2 m}{m^2} \right) \frac{\partial \sigma}{\partial X} + \frac{\delta g}{\delta \sigma} \frac{\partial \sigma}{\partial X} - \frac{2}{\pi \hbar} \frac{\sqrt{2m}}{g^{3/2}} \frac{\sigma^{3/2}}{\partial X}. \quad (18)$$

As mentioned earlier, the density of grey soliton is constant at asymptotic ends. In order to obtain the converging energy, constant background density has to be subtracted from the Hamiltonian (See, e.g. [23]). However, for the kink-like case, inconsistent density distributions at the boundaries, lead to a non-trivial background subtraction. It is to be noted that the difference in densities, between two asymptotic ends, of the kink-like soliton can be represented as:

$$\Psi(x, t) = \sqrt{\sigma} \exp[i(kx - \frac{\mu}{2}t)],$$

where $\sigma$ is the anti-kink.

Substituting equation (19) in equation (20), energy for the kink-like soliton can be represented as:

$$E = \int_{-\infty}^{\infty} dx \mathcal{H} = \int_{-\infty}^{\infty} dx \left[ g_{1} \sigma^2 - \frac{2 \sqrt{2m}}{3 \pi \hbar} g^{3/2} \sigma^{3/2} - 2 \mu \sigma + \frac{32 m^2}{81 \pi^3 \hbar^4} \left( \frac{g^6}{\delta g^2} \right) + \frac{8m}{9 \pi^2 \hbar^2} \mu \left( \frac{g^3}{\delta g^2} \right) \right]. \quad (21)$$

It is important to point out that the last two constant terms are the background energy which compensate the divergent terms in the energy density. One obtains the final expression of total energy, by exploiting the relationships achieved in equation (7), as

$$E = \frac{8 \sqrt{2}}{27 \pi^2 \hbar^2} \left( \frac{m g^2}{\delta g^2} \right)^{5/2}. \quad (22)$$

It is evident from above that the total energy is controlled by interaction parameters $\delta g$ and $g$. The momentum can be obtained by considering the kink-like soliton in ann a box of finite length $L$ from $-L/2$ to $L/2$. For solitonic profile $\Psi(x)$, one gets

$$P = \frac{i \hbar}{2} \int_{-L/2}^{L/2} dx \left[ \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right] = \hbar k A^2 \left[ L - \xi \tanh(L/2) \right]. \quad (23)$$

As expected, the first term comes from the background $A \exp[i(kx - \mu t)/\hbar]$, whereas the second term arises from the soliton contribution. Interestingly, the two contributions have opposite sign which physically signifies that the background and solitonic profiles are moving in opposite directions. The soliton contribution in $L \to \infty$ limit yields

$$P = \frac{2 \sqrt{2}}{3 \pi \mu v} \left( \frac{g^3}{\delta g^2} \right)^{3/2} \equiv Mv, \quad (24)$$

We consider $\Psi(x, t) = \sqrt{\sigma} \exp[i(kx - \frac{\mu}{2}t)]$, with $\sqrt{\sigma} = A + B \tanh(X/\xi)$ and use in the amended NLSE. As it will be seen later, $\sigma$ is positive semi-definite for our ansatz solution. From the imaginary and real parts, one gets,

$$\frac{\partial \sigma}{\partial t} = -v \frac{\partial \sqrt{\sigma}}{\partial X} \quad \text{with} \quad v = \frac{\hbar k}{m},$$

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$$P = \frac{2 \sqrt{2}}{3 \pi \mu v} \left( \frac{g^3}{\delta g^2} \right)^{3/2} \equiv Mv, \quad (24)$$
where we used $v = \frac{h k}{m}$ and equation (7) and $M = -\frac{2v^2}{\hbar^2}$ is the effective mass akin to the earlier observed negative mass for the grey soliton in a trap [66–68]. The energy of kink-like solitons, in terms of effective mass, can be represented in the form

$$\mathcal{E} = -\frac{4M}{9\pi^2\hbar^2} \left( \frac{g^4}{\delta g} \right).$$

(25)

Remarkably, on comparing the above equation with the lowest value of chemical potential for self-bound droplet $\mu_0 = -\frac{4\pi^2}{3\hbar^2}(g^4/\delta g)$, one obtains $\mathcal{E} = (M/m)\mu_0$.

6. Conclusion

In conclusion, we have obtained the kink-like quantum soliton solutions in the quantum liquid, which smoothly interpolate between the normal phase and the flat-top droplet. They necessarily require a constant background of quantum nature. Remarkably, the background amplitude is exactly one-third of the uniform condensate amplitude. Evidently, these kink-like solitons are different from kink/anti-kink, dark, and grey solitons, as at one end, asymptotically they connect to the normal state. These dark and grey solitons represent localized defects. The chemical potential has a value $\mu_0$, which is identical to the condition for the self-trapped flat-top solution. The possibility and nature of extended objects like, Ma and Akhmediev breathers [69, 70], as well as rogue waves is worth studying for the droplets [71]. The collision of solitons and their role in the evaporation of the droplets is under investigation.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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