Ground state and far-infrared absorption of two-electron rings in a magnetic field

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Motivated by recent experiments [A. Lorke et al., Phys. Rev. Lett. 84, 2223 (2000)] an analysis of the ground state and far-infrared absorption of two electrons confined in a quantum ring is presented. The height of the repulsive central barrier in the confining potential is shown to influence in an important way the ring properties. The experiments are best explained assuming the presence of both high- and low-barrier quantum rings in the sample. The formation of a two-electron Wigner molecule in certain cases is discussed.

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I. INTRODUCTION

The possibility of confining a controlled number of electrons in artificial nanostructures at the interface of two semiconductors, achieved in recent years [1], has attracted much interest. An obvious reason is the potential application of these systems to sub-micron electronics technology but, from the theoretical point of view, much importance is attributed to their character of novel quantum systems in which new physics, as well as analogies with atomic clusters, atoms or nuclei, may be found. The most widely studied systems have been the so-called 'quantum dots', in which a certain number of electrons form a two-dimensional compact island, modelled by a confining potential of parabolic type (for small dots) plus the mutual electron-electron interactions. The electronic islands have also been formed in the shape of rings. In fact, mesoscopic rings were produced by Dahl et al. and their magnetoplasmon resonances measured for different ring widths [2]. Subsequent theoretical models, using semiclassical methods [3] and density functional theory [4] provided a good interpretation of the measured ring spectra. There also exist approaches based on exact diagonalization methods for very small numbers of electrons which have addressed the so-called 'persistent current' [5] and the optical excitations for very narrow rings (approaching the 1d limit) as well as for dots with a repulsive scatterer center [6]. Besides, a description of the optical properties in two electron rings based on an analytical treatment of the ground state was given by Wendler et al. [7].

Very recently, Lorke and co-workers reported the fabrication, using self-assembly techniques, of electron rings in the true quantum limit, not affected by random scatterers, containing two electrons [8]. The far-infrared (FIR) spectra as a function of the magnetic field has shown the appearance of electronic excitations peculiar of a ring geometry as well as a magnetic-field induced transition in the ground state angular momentum. The two electron system is the smallest one with electron-electron interaction and its reduced size places it within the reach of essentially exact methods, such as the Hamiltonian diagonalization in a basis of functions mentioned above. In this paper we have followed an alternative although also exact method, based on the solution of the Schrödinger equation by discretizing the coordinate space in a uniform grid of points. The method is therefore free from basis selection and truncation although its limitation lies, obviously, in the number of spatial points. This scheme can also be used to obtain the dynamical properties by solving the time-dependent Schrödinger equation and monitoring the evolution of relevant physical quantities with time. In this work we have addressed the linear response regime, for small amplitude oscillations. However, one of the potentialities of the real-time approach is the study of non-linear (large amplitude) dynamical processes. By finding the exact solution one can easily quantify the effect of the electron-electron interaction as well as check the validity of approximate descriptions, such as density-functional theory. In fact, an analysis of the experiment based on the local spin-density approximation (LSDA) in a symmetry-restricted approach has already been presented in Ref. [10].

An important input to the theory is the external confining potential acting on the electrons. Since we aim at a direct comparison with experiment, the strategy should be to introduce a reasonable model potential, depending only on a few parameters and explore the exact solution as a function of this parameters. Our approach has been to take the parabola parameter $\omega_0$ and ring radius $R_0$ as guessed in Ref. [8] and explore the effect of the central repulsive barrier by solving for a low-, an intermediate- and a high-barrier ring. With these barriers the system changes from a compact dot-like structure to a well developed ring one. We show that the barrier height modifies the ring properties and that the experiments are best explained assuming a mixed sample containing both low- and high-barrier rings.

In recent papers Yannouleas and Landman [11], as well as Koskinen et al. [12] have discussed the electronic localization, in the relative frame, known as the formation of Wigner molecules in two electron quantum dots and in rings with 6 electrons, respectively. Both groups have addressed the problem by calculating the low-lying
A. Hamiltonian and resolution method

The Hamiltonian of the two-electron system, in a magnetic field, using effective units \( \equiv \frac{1}{\hbar} \), is

\[
H = \sum_{i=1,2} \left[ \frac{1}{2} (-i\nabla_i + \gamma \mathbf{A}(\mathbf{r}_i))^2 + V(\mathbf{r}_i) \right] + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} + g^* m^* \frac{\gamma}{2} \mathbf{B} \cdot \mathbf{S},
\]

where \( \mathbf{A} \) is the vector potential, \( \gamma = e/c \) (we assume Gaussian magnetic fields) and \( V(\mathbf{r}) \) is the confining external potential. The electronic positions are restricted to the \( xy \) plane, i.e., \( \mathbf{r} \equiv (x, y) \), and we consider a constant and perpendicular magnetic field \( \mathbf{B} = Be_z \), described in the symmetric gauge with a vector potential \( \mathbf{A}(\mathbf{r}) \equiv B/2(-y, x) \). The last piece is the Zeeman term, depending on the total spin \( \mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2 \) the effective gyromagnetic factor \( g^* \) and electronic effective mass \( m^* \). \( \mathbf{S} \) is given by the well known singlet and triplet combination of spinors. Our method to obtain the spatial wave function is based on the only assumption that its gradient and Laplacian can be obtained by using finite difference formulas in a spatial grid of uniformly spaced points. Associating a macroindex \( I \equiv (x_1, y_1, x_2, y_2) \) to each point and discretizing the derivative operators (we use 5 point formulas), the Schrödinger eigenvalue equation transforms into the linear problem

\[
H_{1J}\Psi_J = E\Psi_J.
\]

It is worth to mention that the interaction and potential terms are diagonal (local) and only the kinetic contribution gives non diagonal elements. Besides, since the derivatives involve only a finite number of points (we have typically used 5-point formulas) the Hamiltonian matrix is very sparse. We have solved Eq. (3) for the ground state using the iterative imaginary time-step method \([6]\). In principle, one can also obtain excited states by introducing the constraint of orthogonalization to the lower eigenvectors, although the computational effort increases very rapidly. A very useful optimization is introduced by considering only symmetric (singlet) or antisymmetric (triplet) wave functions with respect to the exchange of the two particle positions, since this allows to reduce by a factor two the wave function dimension.

The above method has proved to be quite stable and the convergence with the number of points very fast. Typically, we have used from \( N_1 \approx 20 \) up to \( \approx 50 \) points for the discretization of a single dimension. Notice that the wave function dimension \( (N_1^4) \) would normally imply that the Hamiltonian matrix \( (N_1^4 \times N_1^4) \) exceed the storage capability. This is avoided by fully exploiting the sparseness of the Hamiltonian and not really dimensioning such a big matrix. Analogously, the imaginary-time step method allows us to determine the lower eigenstate of this matrix which would be unfeasible by direct diagonalization methods. We have also checked the convergence with the number of points used to discretize the Laplacian operator, \( e.g. \), when going from 5 to 9 point formulas a relative difference less than 1/10^3 in the ground state energy is found.

B. Results

In this subsection we present the results for the ground state of rings having \( R_0 = 1.4 \) \( a_0^* \) and \( \omega_0 = 1.1 \) \( H^* \). The corresponding physical values are, approximately, 14 nm and 13 meV and are taken as reasonable values from the experiment. Three different barrier heights are considered \( V_b = 0.75, 3 \) and 12 \( H^* \). As shown in Fig. 1 these three values correspond, going from low to high barrier, to decreasing central densities. In what follows the three rings will be referred to as V1, V2 and V3 for increasing
barrier height, respectively. While ring V1 has a compact density because of its rather low barrier, V3 has a real ring shape, with a hollow central region.

Figure 2 displays the expectation value of the angular momentum $\langle L_z \rangle$ as well as the energy of the lowest level for both singlet and triplet states as a function of the magnetic field. At $B = 0$ the ground state is a singlet and with increasing $B$ there is an alternation of triplet and singlet ground states in the three rings. The first singlet-triplet transition occurs approximately at 3, 2 and 1 T for V1, V2 and V3, respectively. The triplet-singlet energy differences remain quite small after the first crossing which hints at possible transitions induced by small non-spin conserving perturbations like spin-orbit interactions or magnetic impurities. We also notice how the ground state increases in steps its angular momentum (in absolute value) as a function of $B$. This reflects the exact property that angular momentum is a good quantum number since the Hamiltonian commutes with the $L_z$ operator. The angular momentum gain with magnetic field is faster for the higher barrier rings. In fact, for V3 the evolution is markedly linear, already indicating a rotor-like evolution for this ring. This is further discussed in Sec. IV.

### C. Non-interacting solution

When neglecting the Coulomb interaction the Hamiltonian (1) separates for each particle and, therefore, a solution built from single-particle (sp) orbitals with radial $n$ and angular $\ell$ quantum numbers is adequate (1). Figure 3 shows the sp eigenvalues for all, low and high barrier, rings. In fact, from the lowest maximum of the capacitance-voltage spectrum Lorke et al. obtained the single-electron ground-state shift with magnetic field. This manifests a change in slope at $B \sim 8T$ that was explained as the transition from $\ell = 0$ to $\ell = -1$ in a model potential. Of course, this is in agreement with the sp levels for V1, since the low-barrier ring essentially coincides with the model potential used in Ref. (1).

The first crossing for V2 is located around 4T, while as shown in Fig. 3 the high barrier case (V3) has again a change in slope at $B \sim 8T$, attributed to the $\ell = -1 \rightarrow \ell = -2$ transition, which may indicate that the experimental sample could contain both low and high barrier rings. This point will be further discussed when presenting the FIR spectra.

The non-interacting ground state with two electrons is obtained by filling the lowest orbitals of the sp scheme, with the constraint of having a singlet or triplet spin. The corresponding plots for the evolution of $\langle L_z \rangle$ and the ground state energy as a function of magnetic field are displayed in Fig. 4. By comparing it with Fig. 2 we notice the effects of the Coulomb interaction between the two electrons. Obviously, the non-interacting case gives lower ground state energies since the Coulomb repulsion is neglected. Due to the increasing effect of the confining potential the difference is higher for V1 than for V3. Differences in angular momentum are also higher for the low barrier ring. This is indicating that in well developed rings the interaction effects are less important than in more compact structures, in agreement with the findings of Ref. (1) and with the obvious expectation of a larger $V_b$ keeping the electrons more apart from each other than a lower barrier.

We also realize that the triplet solution is unfavoured in the non-interacting case. In V1 the gs has always $S = 0$, except for $B = 8T$ for which singlet and triplet are essentially degenerate. A similar thing happens in the other rings, with the difference that the singlet-triplet degeneracy points appear at lower B’s. These differences with respect to the exact results, in which clearer singlet-triplet oscillations are seen, indicate the importance of the Coulomb interaction for a proper ground state description.

### III. TIME INTEGRATION

The time evolution of the two-electron wave function is given by the time-dependent Schrödinger equation

$$\frac{i}{\hbar} \frac{\partial}{\partial t} \Psi(r_1, r_2; t) = H \Psi(r_1, r_2; t) , \quad (4)$$

where $H$ is the Hamiltonian operator (1). We have solved Eq. (1), in the same spirit as for the ground state, by discretizing the coordinate space in a uniform grid of points. Time is also discretized and the evolution from time $(n)$ to $(n+1)$ is obtained through the unitary Crank-Nicholson algorithm

$$\Psi^{(n+1)}_J = \Psi^{(n)}_J - \frac{i \Delta t}{2} \sum_J H_{IJ} \left( \Psi^{(n)}_J + \Psi^{(n+1)}_J \right) , \quad (5)$$

This is an implicit scheme, in which the new $\Psi^{(n+1)}_J$ may be obtained by iteration. The above algorithm allows to perform the dynamical simulation of Eq. (1) by using a small time step $\Delta t$. Of course, the feasibility of this approach to the solution of Eq. (1) relies on the very small number of constituents (two electrons) of the system.

In this paper we discuss the FIR absorption of two-electron rings and therefore our interest will focus on the dipole states corresponding to small amplitude charge oscillations. The ground state wave function is perturbed by an initial (small) rigid displacement in a certain direction $\hat{e}$, representing the electric field direction, and the dipole moment evolution is then followed in time. The absorption cross section is given by the frequency transform of the dipole signal

$$\sigma(\omega) \approx \frac{\omega}{2\pi} \int dt |\Psi(t)| D(|\Psi(t)|) \exp(i\omega t) \right| , \quad (6)$$

3
where the dipole operator is \( D = \hat{\mathbf{e}} \cdot (\mathbf{r}_1 + \mathbf{r}_2) \). As a test of the method, developed from previous calculations within mean-field approaches, we have checked that the oscillation frequencies when assuming a pure parabolic potential are given, as a consequence of the generalized Kohn theorem, by the values \( \omega_\pm \):

\[
\omega_\pm = \sqrt{\omega_0^2 + \omega_r^2/4} \pm \omega_r/2 ,
\]

where \( \omega_r = eB/c \) is the cyclotron frequency.

A. FIR spectra

Figure 5 shows the absorption spectra for singlet and triplet states corresponding to different magnetic fields. The intersection of the baseline for each spectrum with the horizontal axis indicates the magnetic field in each case. Notice that the horizontal scale for each spectrum is arbitrary, and therefore, it is not given by the horizontal axis. The spectra are drawn in linear scale and arbitrary units while the vertical scale gives the energies in meV. Not surprisingly the V1 spectrum is very similar to that of compact dots, with two branches of opposite frequencies, and secondly we do not think that the experimental features but misses others. A plausible explanation is that the sample contains a mixture of high and low barrier rings. In Fig. 6 we display a superposition of the V1 and V3 spectra in comparison with the available experimental points, assuming that the system takes the theoretical ground state at each magnetic field, although as already noticed in many cases the triplet-singlet energy difference is quite small. With this interpretation, the circles are attributed to high barrier rings while the triangles and rombuses correspond to low-barrier ones. Although the positions of the experimental peaks are not perfectly reproduced (this would involve a further optimization with respect to \( \omega_0 \) and \( R_0 \)) the qualitative agreement is good. A richer mixture including intermediate-barrier rings would induce a blurring of the different branches with the introduction of additional (intermediate) peaks, but would also yield the above discussion and comparison with experiment more involved. The crosses in Fig. 6 are not reproduced by any of our ring calculations and, as hinted by the experimental group, they could be due to pure quantum dots in the sample. We also mention that in the experiment an insufficient signal to noise ratio at low energies does not allow the detection of the lower branch at \( \omega < 10 \text{ meV} \).

B. Non-interacting spectra

Figure 7 shows the singlet and triplet spectra when the Coulomb interaction is neglected as in subsection II.C. Therefore, they correspond to pure particle-hole transitions in the single-particle level scheme. On a first look they seem quite similar to the spectra of Fig. 5. However, after a second inspection important differences appear. A dramatic one is the softening, i.e., a decrease in energy, of the lowest branch in all three cases. In fact for \( B \approx 8 \text{ T} \) the lowest state has an almost vanishing energy. This fact is explained as the formation of quasi Landau bands, even for such a small system, with the property that single-particle states of the same band are quasi-degenerate and therefore, particle-hole transitions are nearly gapless.

There are also sizeable shifts of peak positions and, in general, a higher fragmentation is present in the single particle picture. This is easy to understand since the collectivity asociated to the Coulomb interaction is absent when the dipole operator is neglected.
in the second case, and therefore the description of collective magnetoplasmon states is quite rough. Altogether, the correct qualitative result must be understood as an indication that interaction effects are not overwhelming for the spectroscopic features of a two-electron system.

IV. MEAN FIELD THEORY AND BROKEN-SYMMETRY SOLUTION

Usually, the exact solution of the many-body Schrödinger equation is not possible and one must content with approximate solutions. In this sense, the mean field approximation allows in many cases the description of relevant physics. Although in principle the mean field provides an accurate picture for a big enough number of particles, it is interesting to compare it with the exact solution in the present case for two particles. This will provide a strong test of the mean field result, that will allow us to quantify its validity. We remark that for systems confined in an external potential such as atomic or quantum dot electrons the restriction on having a sufficient number of particles in order to develop the mean field is not as strict as in self-bound systems, such as liquid drops or atomic nuclei. This is easy to understand since in confined systems the fixed external potential is already an important contribution to the mean field and, therefore it is not solely based on particle-particle interactions.

Imposing circular symmetry the mean field theory has indeed been applied to medium-large quantum dots \((N \approx 20 - 200)\), in the Hartree \([18]\), Hartree-Fock \([19]\) and LSDA schemes \([20,21]\). Nevertheless when using symmetry-unrestricted approaches, even at the mean field level, it is normally necessary to restrict to smaller sizes \((N \sim 10)\) because of the computational requirements \([22,23]\).

A. LSDA ground state

We have applied the LSDA symmetry-unrestricted approach developed by us in Ref. \([2]\) to the present two-electron rings. Figure 8 summarizes the ground state properties, for both singlet and triplet for V1, V2 and V3. The comparison with the energies for the exact solution (Fig. 2) prove that the the mean field is providing a quantitatively good description of the ground state energy, with relative energy differences around 2%. We notice that the MF description of the triplet is slightly better than that of the singlet. In fact, after the first singlet-triplet crossing the MF fails to reproduce a singlet state more bound than the triplet one. This can be attributed to a more important effect of correlations due to the interaction in the singlet than in the triplet which are only approximately taken into account within LSDA.

The overall evolution of the total angular momentum is also well reproduced in MF, especially for the triplet. Quite strikingly, however, the angular momentum of the singlet does not show the step-like evolution peculiar of the exact solution. In fact it is taking in most cases clear non-integer values. This must be attributed to a spontaneous breaking of the rotational symmetry in the frame of the mean field.

The symmetry of the MF solution is more easily appreciated in Fig. 9, that shows the MF ground state density \(\rho = \rho_\uparrow + \rho_\downarrow\) and magnetization \(m = \rho_\uparrow - \rho_\downarrow\), where \(\rho_\uparrow, \rho_\downarrow\) are the two spin densities, for V1 and V3 at \(B = 0\). Both cases correspond to singlet solutions but, while V1 has a circularly symmetric density and vanishing magnetization, V3 is showing a separation of spin up and down densities, leading to an oscillation of the magnetization along the ring perimeter (a spin density wave) as well as of the total density (a charge density wave). This behaviour is in fact common to all singlet states at \(B > 0\) for which \(|L_z|\) takes non-integer values. This peculiar states were predicted by Reimann \(et \ al.\) \([26]\) in narrow rings with a bigger number of electrons, and have been also studied by us in Ref. \([27]\).

The formation of broken-symmetry solutions within MF approaches is a well known phenomenon in nuclear physics \([28]\). In this field, it has been shown that the MF is in fact providing the system structure in an \(intrinsinc\) reference frame, given by proper \(intrinsic\) coordinates, while there is degeneracy with respect to the remaining \(collective\) coordinates. In a simplifying picture this corresponds to assume a certain structure of the system relative to its mean field, that in turn may be moving thus leading to a restoration of the exact symmetry. In fact, as shown in Fig. 1, when performing an angular average of the MF density for V3 an excellent agreement with the exact result is obtained. As we discuss further in the next subsection, the MF symmetry breaking can be interpreted as an electronic crystallization, i.e., the formation of a Wigner molecule.

B. The Wigner molecule

The symmetry breaking of the MF solution for V3 is pointing an incipient electron localization in the \(intrinsinc\) frame of this system. The formation of these localized structures has been recently discussed using \(exact\) solutions by Yannouleas and Landman for two electron parabolic dots \([11]\) and, for rings with 6 electrons by Koskinen \(et \ al.\) \([12]\). These authors have computed the exact excitation spectrum and shown that it adjusts to rotor bands, thus revealing the existence of the electron molecule.

Even when the exact wave function is known it is not a trivial task to notice the existence of a Wigner molecule.
in the intrinsic frame. It can be hinted by means of intrinsic distribution functions, like the conditional probability density \( \rho(r_2, \theta_2 | r_1 = \xi) \), which gives the probability of finding an electron at \((r_2, \theta_2)\) when the other one is fixed at \(r_1 = \xi\). Therefore, it provides valuable insight on the intrinsic structure of the two-electron system. In Ref. \[29\] Berry reviews the use of the conditional probability density to describe the formation of electron molecules in the doubly excited states of the helium atom. In what follows we show that a similar analysis can be used for the two-electron rings.

Figure 10 shows the conditional probability density when the first electron is placed at its most probable value for the radial coordinate in both V1 and V3 confining potentials. Since this position is taken as the origin for the angle \(\theta_2\) this implies that the first electron is placed in those graphs at \((\xi, 0)\), with \(\xi = 1.5, 1.8 a_0^*\) for V1 and V3, respectively. Notice that the two distributions yield a peak for the second electron that lies opposite to the first one, i.e., at \((-\xi, 0)\). On a first look both functions look rather similar although a closer inspection reveals that the inner hole is more pronounced in the case of V3. This is in agreement with a stronger localization of the second electron in this case. In order to manifest more clearly this result it is helpful to construct the density in the intrinsic frame and compare it with the MF one. To this end we assume that once the most probable relative positions are determined, \((x, y) = (\xi, 0)\) and \((-\xi, 0)\), the intrinsic frame density can approximately be obtained by centering in each position the conditional probability density. This procedure leads to the two lower plots of Fig. 10. Quite interestingly, the V3 result is clearly breaking the circular symmetry and both are in excellent agreement with the mean field densities of Fig. 9.

Although qualitative the preceding discussion supports the interpretation that the MF describes the intrinsic structure of the ring. Consequently, the existence of a high barrier in V3 produces an incipient Wigner molecule. However, it remains as an interesting task for the future to obtain the full set of excited states and discuss its rotor-like character, following Refs. \[11, 12\] as well as the analytical expressions of Ref. \[8\].

### C. Unrestricted time-dependent LSDA (TDLSDA)

Based on the mean-field picture, the TDLSDA allows to describe the collective dipole oscillations. These scheme has been used to describe both spin and density excitations in quantum dots and rings in a symmetry-restricted approach \[8, 10\]. In Ref. \[23\] we described the application of TDLSDA relaxing the constraint of circular symmetry and using a real-time approach similar to that of Sec. III. Within TDLSDA this implies the solution of the time-dependent Kohn-Sham equations, which are the analog of Eq. \[4\] within the LSDA. We have applied this technique to the present two-electron rings and show the corresponding singlet and triplet spectra for V1, V2 and V3 in Fig. 11.

The TDLSDA provides a rather good description of the rings’ dipole spectra, as compared to the exact ones. We notice that the overall evolution with \(B\) and the location of the strength is quite well reproduced for each branch. However, finer details of the spectra, such as the fragmentation structures are not reproduced in many cases. Although based on a single-particle picture the unrestricted TDLSDA corrects the lower branch instability found in Fig. 7. This is a manifestation of the interaction effects taken into account by the TDLSDA. From the present results we conclude that, despite the present rings only contain two electrons, the TDLSDA provides a rather sensible picture of the dipole oscillations.

### V. Conclusions

In this work we have theoretically analyzed two-electron rings motivated by recent measurements of their FIR absorption. We have found that in order to explain the experimental spectra it is necessary to assume that the sample contained a mixture of rings with different barrier height. As expected, high barrier rings are characterized by a clear violation of Kohn’s theorem, showing a sizeable amount of strength at energies \(\omega > 15\) meV for \(B < 6\) T. On the other hand, low barrier rings approximately fulfill Kohn’s theorem, with an effective \(\omega_p\), and have lower energy excitation branches more similar to the measured ones. A mixture of these two types of rings has been used to explain the main features of the experiment. A more detailed analysis would require including intermediate barrier rings (ideally knowing the actual barrier-height distribution in the sample), for which the spectra lie in between the preceding two extreme situations. Despite the very small number of constituents, these systems possess a rich FIR spectrum with an intricate \(B\)-dependence.

The evolution of angular momentum and total spin with barrier height and magnetic field has been discussed and contrasted with independent particle predictions. In agreement with previous studies we find that interaction effects become less important for increasingly narrow rings, obviously reflecting a bigger electronic separation than in a compact structure and a corresponding increase in angular momentum.

The comparison with the symmetry-unrestricted LSDA has allowed us to discuss the formation of an incipient Wigner molecule for the V3 case and, in general, has shown the good quality of this approximation. We have proposed a method to obtain the density in the intrinsic reference frame, based on the conditional probability density, that quantitatively agrees with the LSDA result.
The FIR absorption in TDLSDA provides a good overall strength distribution and dispersion with magnetic field, although it misses some finer details of the fragmentation patterns.

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[14] For bulk GaAs the effective gyromagnetic factor is $g^* = -0.44$.
[15] For a barrier height $V_b$ the condition of continuity up to the second derivative of the potential at $R_0$ gives $c = (2 - 4e_b)/R_0$ and $d = (1 - 3e_b)/R_0^2$, where $e_b = 2V_b/\omega_0^2 R_0^2$.
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FIG. 2. Evolution with magnetic field of the lowest singlet and triplet states for the three rings. Left panels show the state energy (in $\text{H}^*$) and right ones its corresponding total angular momentum in the perpendicular direction $\langle L_z \rangle$, in units of $\hbar$.

FIG. 3. Dependence of the single particle energy levels with $B$ for the three potentials. The different branches, taken in vertical order at $B = 1\text{T}$, correspond for all three cases to $(n \ell) = (1, 0), (1, -1), (1, 1), (1, -2), (1, -3)$, where $n$ (radial) and $\ell$ ($z$-component of angular momentum) are the usual quantum numbers. Each branch presents a small spin splitting for increasing $B$, with the spin up states placed below the spin down ones.

FIG. 4. Same as Fig. 2 for the non-interacting case.

FIG. 5. Evolution of the FIR spectrum for singlet and triplet ground states in the three rings. The results for V1, V2 and V3 are displayed in the three rows from top to bottom while left and right columns correspond to singlet and triplet states, respectively. For comparison, the dashed lines show the analytical prediction from Kohn’s theorem (7), see Sec. III.A.

FIG. 6. Superposition of V1 (solid lines) and V3 (dashed) spectra at different magnetic fields. The experimental data are taken from Lorke et al. [9]. See text (Sec. III.A) for details.

FIG. 7. Same as Fig. 5 for the non-interacting case.
FIG. 8. Same as Fig. 2 within the LSDA.

FIG. 9. Results obtained within the LSDA for $B = 0$ T: Lower plots display the density $\rho$ and magnetization $m$ for V3 while the upper one shows the density for V1. In the latter case the magnetization vanishes everywhere.

FIG. 10. Upper plots: conditional probability densities (CPD) $\rho(r_2, \theta_{21}|r_1 = \xi)$ obtained from the exact wave function at $B = 0$ T. The first electron has been fixed at $(\xi, 0)$, indicated by a solid dot, where $\xi = 1.5, 1.8$ $a_0^*$ is the most probable radius for a single electron, in V1 and V3 respectively. Lower plots: Intrinsic-frame density built from the upper conditional probability densities as explained in the text (Sec. IV.B).

FIG. 11. Same as Fig. 5 within the LSDA.
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