ON THE GENERALIZED CHEEGER PROBLEM
AND AN APPLICATION TO 2D RECTANGLES

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Abstract. In this paper we consider the generalization of the Cheeger problem which comes by considering the ratio between the perimeter and a certain power of the volume. This generalization has been already sometimes treated, but some of the main properties were still not studied, and our main aim is to fill this gap. We will show that most of the first important properties of the classical Cheeger problem are still valid, but others fail; more precisely, long and thin rectangles will give a counterexample to the property of Cheeger sets of being the union of all the balls of a certain radius, as well as to the uniqueness. The shape of Cheeger set for rectangles is then studied as well as their Cheeger constant.

Introduction

The celebrated Cheeger problem consists in searching for sets \( E \subseteq \mathbb{R}^n \) minimizing the ratio

\[
\inf_{E \subseteq \Omega} \frac{P(E)}{|E|},
\]

where \( \Omega \subseteq \mathbb{R}^n \) is a given set of finite volume. Throughout the paper, we will write \( |E| \) to denote the volume (i.e., the Lebesgue \( L^n \) measure) of any Borel set \( E \), while \( P(E) \) will be its perimeter, that is, the \( \mathcal{H}^{n-1} \) Hausdorff measure of its reduced boundary \( \partial^* E \) (for definitions and properties of sets of finite perimeter, the reader can look in [2]). The main interesting questions in the Cheeger problem regard existence, uniqueness and main features of the Cheeger sets, that are the sets \( E \) realizing the infimum above. Notice that any set \( E \) which is Cheeger set for some \( \Omega \supseteq E \) is also Cheeger set for itself. Since the literature on this problem is huge and well known, we do not try to give here a list.

In this paper, we are interested in the following generalization of the Cheeger problem,

\[
h_\alpha(\Omega) := \inf_{E \subseteq \Omega} \frac{P(E)}{|E|^\frac{1}{\alpha}},
\]

for \( \alpha > 1 \). We will call this problem the “\( \alpha \)-Cheeger problem”, the above ratio is called “\( \alpha \)-Cheeger ratio of \( E \)”, and any set realizing the infimum will be called \( \alpha \)-Cheeger set. Actually, the problem is interesting only if \( 1 < \alpha < 1^* = (n-1)/n \), as we will see later. This generalized Cheeger problem has been already often considered in the literature, see for instance [4, 3] and the references therein, but up to our knowledge some of the main properties were not studied or not proved: the main aim of the present paper is to fill this gap. In particular, we will first concentrate, in Section 1 on the very basic a-priori known properties of the Cheeger problem, and we check how they generalize to the \( \alpha \)-Cheeger problem. Then, in Section 2 we show that the existence of an \( \alpha \)-Cheeger set \( E \) for any set \( \Omega \) is always true. The last Section 3 is devoted to study more extensively the case when \( \Omega \) is a rectangle, and checking how their Cheeger sets behave; indeed, as we will see, long and thing rectangles are a counterexample to the known properties of the classical Cheeger problem that, if \( \Omega \) is convex, then its Cheeger set is unique, and it is the union of all the balls of a certain radius contained in \( \Omega \).
1. A-priori properties

In this first section, we give a list of well known properties of the classical Cheeger problem, and see how to generalize them for the $\alpha$-Cheeger problem; most of them will be straightforward generalizations.

**Theorem 1.1.** The following properties hold:

1. The Cheeger problem is scale invariant while the Cheeger constant is not;
2. The constrained boundary of any Cheeger set, i.e. $\partial C_1 \cap \partial \Omega$, contains at least two points;
3. The free boundary of any Cheeger set, i.e. $\partial C_1 \cap \Omega$, is analytic possibly except for a closed singular set whose Hausdorff dimension does not exceed $n - 8$;
4. The mean curvature of the free boundary is constant at every regular point and, for $n = 2$, it equals $h_1(\Omega)$;
5. The free boundary of a Cheeger set meets $\partial \Omega$ only at regular points of $\partial \Omega$, and tangentially;
6. A Cheeger set of $\Omega \subseteq \mathbb{R}^2$ can not have corners with an angle smaller than $\pi$;
7. If a Cheeger set exists, then any of its connected components is also a Cheeger set;
8. If $\Omega \subseteq \mathbb{R}^2$ is convex, then its Cheeger set is unique and convex; in particular, it is the union of all the balls of radius $1/h(\Omega)$ which are contained in $\Omega$;
9. If $\Omega_1 \subseteq \Omega_2$ then $h_1(\Omega_1) \geq h_1(\Omega_2)$ but the strict inclusion does not imply the strict inequality.

All the previous properties are well known, a discussion can be found for instance in [8]. The only result of this section is the following, we we show that all the above properties generalize, with minor changes, to the case of the $\alpha$-Cheeger problem; in particular, a stronger version of 7 holds. We also add a couple of “new” properties, where we compare the Cheeger constants relative to different powers $\alpha_1$ and $\alpha_2$, which of course make no sense in the classical case. For the sake of shortness, we will write $C_\alpha$ to denote an $\alpha$-Cheeger set in $\Omega$.

**Theorem 1.2.** Let $\Omega, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open connected sets, let $\alpha, \alpha_1, \alpha_2$ be in $(1, 1^*)$ and let $B_1$ be the volume-unitary $n$-dimensional ball. Then the following is true:

1. The $\alpha$-Cheeger problem is scale invariant while the $\alpha$-Cheeger constant is not (in particular, it scales by a factor $t^{n-1-\frac{\alpha}{\alpha}}$);
2. The constrained boundary of an $\alpha$-Cheeger set, i.e. $\partial C_\alpha \cap \partial \Omega$, contains at least two points;
3. The free boundary of an $\alpha$-Cheeger set, i.e. $\partial C_\alpha \cap \Omega$, is analytic possibly except for a closed singular set whose Hausdorff dimension does not exceed $n - 8$;
4. The mean curvature of the free boundary is constant at every regular point and, for $n = 2$, it equals

$$\frac{h_\alpha(\Omega)}{\alpha} |C_\alpha|^{-\frac{1}{\alpha}}. \tag{3}$$

5. The free boundary of an $\alpha$-Cheeger set meets $\partial \Omega$ only at regular points of $\partial \Omega$, and tangentially;
6. An $\alpha$-Cheeger set of $\Omega \subseteq \mathbb{R}^2$ can not have corners with an angle smaller than $\pi$;
7. Any $\alpha$-Cheeger set is connected;
8. If $\Omega \subseteq \mathbb{R}^2$ is convex then any Cheeger set is convex;
9. If $\Omega_1 \subseteq \Omega_2$ then $h_\alpha(\Omega_1) \geq h_\alpha(\Omega_2)$ but the strict inclusion does not imply the strict inequality;
10. If $B_1 \subseteq \Omega$ and $\alpha_2 > \alpha_1$ then $h_{\alpha_2}(\Omega) \geq h_{\alpha_1}(\Omega)$;
11. If $|\Omega| \leq 1$ and $\alpha_2 > \alpha_1$ then $h_{\alpha_2}(\Omega) \leq h_{\alpha_1}(\Omega)$.
Proof. 1. Given any set $\Omega \subseteq \mathbb{R}^n$, let us consider its rescaling $t\Omega$ for some $t > 0$: there is a natural bijection between the subsets of $\Omega$ and the subsets of $t\Omega$ given by $E \mapsto tE$. Thus we have

$$h_\alpha(t\Omega) = \inf_{F \subseteq t\Omega} \frac{P(F)}{|F|^{1/\alpha}} = \inf_{E \subseteq \Omega} \frac{P(tE)}{|tE|^{1/\alpha}} = t^{n-1-\frac{\alpha}{n}} \inf_{E \subseteq \Omega} \frac{P(E)}{|E|^{1/\alpha}} = t^{n-1-\frac{\alpha}{n}} h_\alpha(\Omega),$$

which directly tells us that $E$ is a $\alpha$-Cheeger set for $\Omega$ if and only if $tE$ is a $\alpha$-Cheeger set for $t\Omega$. Note that we get an estimate not depending on $t$ if $\alpha = 1^*$, the reason for that will appear clear in Section 2.

2. First of all, assume the existence of a Cheeger set $C^\alpha_\Omega$ such that $C^\alpha_\Omega \subset \subset \Omega$. Then, the rescaled set $tC^\alpha_\Omega$ would still be contained in $\Omega$ for some $t > 1$, and this is again the minimality of $C^\alpha_\Omega$ since for any $\alpha < (n-1)/n$ it is

$$\frac{P(tC^\alpha_\Omega)}{|C^\alpha_\Omega|^{1/\alpha}} = t^{n-1} \frac{P(C^\alpha_\Omega)}{|C^\alpha_\Omega|^{1/\alpha}} = t^{n-1-\frac{\alpha}{n}} \frac{P(C^\alpha_\Omega)}{|C^\alpha_\Omega|^{1/\alpha}} < \frac{P(C^\alpha_\Omega)}{|C^\alpha_\Omega|^{1/\alpha}} = h_\alpha(\Omega).$$

Suppose now that the boundary of some $\alpha$-Cheeger set $C^\alpha_\Omega$ touches $\partial \Omega$ in a single point, say $\bar{x}$. We can deduce that $\bar{x}$ is a regular point of $\partial \Omega$ (see also point 5.), and then a slight translation of $C^\alpha_\Omega$ in the direction of the inward normal of $\partial \Omega$ at $\bar{x}$ is compactly contained in $\Omega$ and remains of course a Cheeger set; then, we found a contradiction as above.

3., 4., 5. Any $\alpha$-Cheeger set is in particular a minimizer of the perimeter among all the subsets of $\Omega$ having its same volume. Therefore, the regularity properties of the free boundary follow by well-known results, see for instance [4] [8], and it only remains to prove the formula for the curvature in the two-dimensional case.

To do so we observe that, in the two-dimensional case, a curve with constant curvature is nothing else than an arc of circle, hence the free boundary of a Cheeger set $C^\alpha_\Omega$ is a union of arcs of circle. Let us focus on a single arc, and let us in particular consider a subarc of an angle $\theta$ which is a strictly positive distance apart from $\partial \Omega$. Let us now modify this arc by fixing its two extremes and moving the center of a small distance $\varepsilon$ (hence, also its radius $r$ slightly changes). A trivial calculation ensures us that, if we call $A_\varepsilon$ the modified set, then

$$|A_\varepsilon| = |C^\alpha_\Omega| + 2 \frac{\varepsilon}{n} \frac{\pi}{2} r(\sin \theta - \theta \cos \theta) + o(\varepsilon), \quad P(A_\varepsilon) = P(C^\alpha_\Omega) + 2\varepsilon(\sin \theta - \theta \cos \theta) + o(\varepsilon).$$

Hence, we find that the $\alpha$-Cheeger ratio of $A_\varepsilon$ is strictly less than that of $C^\alpha_\Omega$ for a small (positive or negative) $\varepsilon$ unless

$$r = \frac{\alpha}{h_\alpha} |C^\alpha_\Omega|^{1/\alpha},$$

and since $C^\alpha_\Omega$ is an $\alpha$-Cheeger set we deduce that the above equality holds true, so the formula for the curvature follows.

6. Suppose that a Cheeger set $C^\alpha_\Omega$ contains an angle strictly smaller than $\pi$; by “cutting away” the corner at a distance $\varepsilon > 0$, we can lower the perimeter of a quantity which goes as $\varepsilon$, changing the volume only of a quantity proportional to $\varepsilon^2$. If $\varepsilon \ll 1$, the new set has a strictly better $\alpha$-Cheeger ratio than $C^\alpha_\Omega$, and since this is impossible we conclude also this point.

7. Let $C^\alpha_\Omega$ be an $\alpha$-Cheeger set with more than one connected component, and let $C_1$ and $C_2$ be two connected components of $C^\alpha_\Omega$. Since of course $P(C_1 \cup C_2) = P(C_1) + P(C_2)$, while for any $\alpha > 1$ it is $|C_1 \cup C_2|^{1/\alpha} = (|C_1| + |C_2|)^{1/\alpha} < |C_1|^{1/\alpha} + |C_2|^{1/\alpha}$, one has

$$\min \left\{ \frac{P(C_1)}{|C_1|^{1/\alpha}}, \frac{P(C_2)}{|C_2|^{1/\alpha}} \right\} \leq \frac{P(C_1) + P(C_2)}{|C_1|^{1/\alpha} + |C_2|^{1/\alpha}} < \frac{P(C_1 \cup C_2)}{|C_1 \cup C_2|^{1/\alpha}} \leq \frac{P(C^\alpha_\Omega)}{|C^\alpha_\Omega|^{1/\alpha}},$$

where the last inequality is emptily true if $C^\alpha_\Omega = C_1 \cup C_2$, while it can be achieved by choosing the “best” two connected components if $C^\alpha_\Omega$ has more than two connected components. Since the above inequality
says that there is some set which has $\alpha$-Cheeger ratio strictly better than $C_{\Omega}^{\alpha}$, the contradiction shows this point.

8. In the two-dimensional case, the convex hull of any set $E$ has bigger volume and smaller perimeter than $E$, with strict inequalities if $E$ is not already convex. The convexity of any Cheeger set corresponding to a convex set $\Omega$ is then obvious. Actually, even in the higher dimensional case the convexity of the Cheeger sets of convex sets is classically known, see [9, 1, 6].

9. This is obvious, since if $\Omega_1 \subseteq \Omega_2$ then any subset of $\Omega_1$ is also a subset of $\Omega_2$. The fact that the strict inclusion does not imply the strict inequality follows by trivial counterexamples.

10. Let us take any set $E \subseteq \Omega$ with $|E| \leq 1$, and let $B_E \subseteq B_1 \subseteq \Omega$ be a ball with the same volume as $E$. Then,

$$\frac{P(E)}{|E|^{1/\alpha}} \geq \frac{P(B_E)}{|B_E|^{1/\alpha}} \geq \frac{P(B_1)}{|B_1|^{1/\alpha}},$$

where the last inequality holds true whenever $\alpha < 1^\ast$. As a consequence, we derive that

$$h_{\alpha}(\Omega) = \inf_{E \subseteq \Omega, |E| \geq 1} \frac{P(E)}{|E|^{1/\alpha}}. \quad (4)$$

Observe now that, for any $E \subseteq \Omega$ with $|E| \geq 1$, it is

$$\frac{P(E)}{|E|^{1/\alpha_1}} \leq \frac{P(E)}{|E|^{1/\alpha_2}},$$

which by taking the infimum over the sets $E$ and recalling (4) concludes the claim.

11. The proof of this last claim is almost identical to the previous one: since $|\Omega| \leq 1$, then of course $|E| \leq 1$ for every set $E \subseteq \Omega$, thus

$$\frac{P(E)}{|E|^{1/\alpha_1}} \geq \frac{P(E)}{|E|^{1/\alpha_2}}$$

and the claim again follows by taking the infimum over sets $E$. \qed

Remark 1.3. If an $\alpha_2$-Cheeger set with volume strictly greater than 1 exists, then point 10 of the previous theorem holds with a strict inequality. If an $\alpha_1$-Cheeger set with volume strictly smaller than 1 exists, then point 11 of the previous theorem holds with a strict inequality.

Remark 1.4. As a straight consequence to the previous proof, we have obtained that to compute the $\alpha$-Cheeger constant of any set $\Omega$, one can reduce oneself to minimize the $\alpha$-Cheeger ratio among the sets with volume bigger than the bigger ball contained in $\Omega$.

2. Existence

This short section is devoted to show the existence of $\alpha$-Cheeger sets in any given set $\Omega$: the proof is identical to the one for the classical case, and we report it only for the sake of completeness.

Before starting, let us briefly compute the $\alpha$-Cheeger ratio of a ball $B_r$ of radius $r > 0$, which is

$$\frac{P(B_r)}{|B_r|^{1/\alpha}} = \frac{n \omega_n r^{n-1}}{(\omega_n r^n)^{1/\alpha}} = n \omega_n^{1-\frac{1}{\alpha}} r^{n-1-\frac{1}{\alpha}}.$$

Notice that the exponent $n-1-\frac{n}{\alpha}$ is negative if and only if $\alpha < 1^\ast$, and it is null when $\alpha = 1^\ast$. As a consequence, for $\alpha > 1^\ast$ the $\alpha$-Cheeger ratio of smaller and smaller balls converge to 0, while for $\alpha = 1^\ast$ all the balls have the same $\alpha$-Cheeger ratio, regardless of their radius. As a consequence, we can observe what follows.

Remark 2.1. If $\alpha > 1^\ast$, then the $\alpha$-Cheeger constant of any set $\Omega$ is 0, and there are no Cheeger sets. If $\alpha = 1^\ast$, every set $\Omega$ have the same $\alpha$-Cheeger constant, and the Cheeger sets are all and only the balls.
In other words, the $\alpha$-Cheeger problem would be trivial for $\alpha \geq 1^*$, and this is why one always chooses $1 < \alpha < 1^*$ for the generalized Cheeger problem.

We can now pass to the existence result.

**Theorem 2.2.** Let $\Omega \subseteq \mathbb{R}^n$ be any set with finite volume. Then, for every $1 < \alpha < 1^*$ there exists and $\alpha$-Cheeger set in $\Omega$; in other words, the infimum in (2) is a minimum.

**Proof.** Let $\{E_k\}$ be a minimizing sequence for (2). Then, $\chi_{E_k}$ is a bounded sequence in $BV(\mathbb{R}^n)$ since

$$\|\chi_{E_k}\|_{BV(\mathbb{R}^n)} = \|\chi_{E_k}\|_{L^1(\mathbb{R}^n)} + \|D\chi_{E_k}\|_{M(\mathbb{R}^n)} = |E_k| + P(E_k),$$

and $|E_k|$ is bounded by $|\Omega|$, while $P(E_k)$ is definitely bounded because $\{E_k\}$ is a minimizing sequence for (2). Thanks to the classical compactness results for $BV(\mathbb{R}^n)$ (see for instance [2]), and recalling that $\Omega$ has finite volume, we obtain that, up to a subsequence, $\chi_{E_k}$ weakly converges in $BV$ to some function $\varphi$. In particular, the convergence is strong in $L^1$, and this implies that $\varphi$ is also the characteristic function of some set $E \subseteq \Omega$; moreover, the lower semi-continuity implies that

$$P(E) = \|D\chi_{E}\|_{M(\mathbb{R}^n)} \leq \liminf_{k \to \infty} \|D\chi_{E_k}\|_{M(\mathbb{R}^n)} = \liminf_{k \to \infty} P(E_k).$$

Summarizing, we have proved the existence of a set $E \subseteq \Omega$ such that

$$|E| = \lim_{k \to \infty} |E_k|, \quad P(E) \leq \liminf_{k \to \infty} P(E_k). \quad (5)$$

The proof will be concluded once we show that $E$ is a Cheeger set in $\Omega$, which in turn is obvious from (5) as soon as we observe that $|E| > 0$. In fact, let us assume that $|E| = 0$: this implies that $|E_k| \to 0$, hence for $k \gg 1$ we can find a ball $B_k$ with radius $r_k$ having the same volume as $E_k$; thus,

$$\frac{P(E_k)}{|E_k|^{1/\alpha}} \geq \frac{P(B_k)}{|B_k|^{1/\alpha}} = n\omega_n^{1-\frac{1}{\alpha}}r_k^{-1-\frac{\alpha}{\alpha}} \to \infty,$$

where the last convergence holds since $1 < \alpha < 1^*$. Since this last estimate is in contradiction with the fact that $\{E_k\}$ is a minimizing sequence for (2), we have shown that it must be $|E| > 0$ and, as noticed above, this concludes the proof. $\square$

### 3. $\alpha$-Cheeger sets for rectangles

In this last section, we study in detail the $\alpha$-Cheeger problem for the case of the rectangles. This is not just an example; indeed, the case of rectangles is important already in the standard Cheeger problem as a basis to study the so-called “strips” (see for instance the papers [7, 8]). For the generalized Cheeger problem, we will see that rectangles give also counterexamples to standard facts which are true for the standard Cheeger problem. The plain of this section is the following: first we give a structure result for the $\alpha$-Cheeger sets in rectangles, then we show how this gives the abovementioned counterexamples, and finally, for the sake of completeness, we explicitly compute the $\alpha$-Cheeger constants of rectangles.

#### 3.1. A structure theorem for $\alpha$-Cheeger sets of rectangles

Through all this section, we denote by $R_L = (-L/2, L/2) \times (-1, 1)$ the rectangle of length $L$ and width $2$, and by $R_\infty = \mathbb{R} \times (-1, 1)$ the unbounded rectangle. Of course, by trivial rescaling one can treat also rectangles on any width. Let us recall the results which are known for the rectangles in the standard case.

**Theorem 3.1.** The infinite rectangle $R_\infty$ admits no Cheeger sets, but the rectangles $R_L \subseteq R_\infty$ are a minimizing sequence for (1) when $L \to \infty$, and $h(R_\infty) = 1$. Any rectangle $R_L$ admits a unique Cheeger set, which is the union of all the balls of radius $1/h(R_L) < 1$, according to point 8 of Theorem [1, 2]. in particular, this set is the whole rectangle with the four corners “cut away” by four arcs of circle.
We can immediately notice that the situation is quite different when \( \alpha > 1 \): we observe that the diameter of any minimising sequence for (2) is bounded, regardless how big \( L \) is.

**Lemma 3.2.** For every \( \alpha \in (1, 1^*) \), there exists \( d = d(\alpha) \) such that the diameter of any Cheeger set \( C_\alpha^L \) of \( R_L \) is less than \( d \).

**Proof.** For any \( L \), we already know by Theorem 2.2 that there exists an \( \alpha \)-Cheeger set \( C_\alpha^L \), and this set is connected by Theorem 1.2. Then, calling \( d \) the diameter of this set, we know that \( P(C_\alpha^L) \geq 2d \), and on the other hand \( |C_\alpha^L| \leq 2d \) because the width of the rectangle is 2. Hence, we can estimate the \( \alpha \)-Cheeger constant of \( C_\alpha^L \) as

\[
h_\alpha(R_L) = \frac{P(C_\alpha^L)}{|C_\alpha^L|^{1/\alpha}} \geq (2d)^{1-\frac{1}{\alpha}}.
\]

Since the latter quantity diverges for \( d \to \infty \), and on the other hand the \( \alpha \)-Cheeger constants of the rectangles \( R_L \) are uniformly bounded for \( L \) large thanks to point 9 of Theorem 1.2, we obtain the thesis. \( \square \)

An easy consequence of the above result is that the curvature of any \( \alpha \)-Cheeger set in \( R_L \) is 1 whenever \( L \) is big enough. Recall that the curvature of an \( \alpha \)-Cheeger set depends by the \( \alpha \)-Cheeger constant as well as by the volume of the set, thanks to (3); instead, in the standard Cheeger problem, the curvature depends only on the Cheeger constant, as follows also by (3) by substituting \( \alpha = 1 \).

**Lemma 3.3.** If \( L \) is big enough, then the curvature of the free boundary of any Cheeger set in \( R_L \) is 1.

**Proof.** We already know that an \( \alpha \)-Cheeger set \( C_\alpha^L \) for \( R_L \) exists, and that the curvature of its free boundary is constant; in particular, any connected component of the free boundary must be an arc of circle connecting two different sides of the rectangle. Moreover, if \( L \) is bigger than the constant \( d \) of Lemma 3.2, then \( C_\alpha^L \) cannot touch both the left and right side of \( R_L \).

As a consequence, there must be an arc of the free boundary connecting the two long sides of \( R_L \); since this arc must be tangent to both these sides, its radius must be 1, hence the curvature of the free boundary is 1. \( \square \)

**Corollary 3.4.** For \( L \) as in the Lemma 3.3, any Cheeger set in \( R_L \) is given by a rectangle \( R_M \subseteq R_L \) topped by two half-disks of radius 1, where

\[
M(\alpha) = \frac{\pi}{2} \cdot \frac{2-\alpha}{\alpha-1}.
\]

**Proof.** First of all, let us consider a Cheeger set \( C_\alpha^L \) inside a rectangle \( R_L \) as in Lemma 3.3: we already know that its boundary contains an half-circle connecting the two long sides of the rectangle; just to fix the ideas, we may assume that this half-circle is the left part of the boundary of \( C_\alpha^L \). Since this set cannot contain corners of the rectangle, by point 6 of Theorem 1.2, then also the two right corners of the rectangle must be cutted away by other arcs of the free boundary. There are then two possibilities: either both the corners are ruled out by a single arc of circle again connecting the two long sides –and then there cannot be other pieces of free boundary, thus \( C_\alpha^L \) is a rectangle \( R_M \subseteq R_L \) topped by two half-disks as claimed– or there are two distinct arcs ruling out the two corners. But also in this second case, since the radius of the arcs is 1, then both arcs must be centered at the same point \((L/2-1, 0)\), and then actually they form together a single arc, namely, an half-circle. So this second case in fact reduces to the first one, thus the claim about the shape of \( C_\alpha^L \) is proved and we only have to check the value of \( M \).
To do so, simply rewriting (5) we can express the radius of the arcs of the free boundary in terms of the $\alpha$-Cheeger constant and the area as

$$r = \frac{\alpha}{h_\alpha(R_L)} |C^\alpha_R|^\frac{1}{\alpha}.$$  \hfill (7)

Since $r = 1$, we derive

$$\frac{\alpha |C^\alpha_R|^\frac{1}{\alpha}}{h_\alpha(R_L)} = \frac{P(C^\alpha_R)}{|C^\alpha_R|^{\frac{1}{\alpha}}},$$

which substituting the values of area and perimeter of $C^\alpha_R$ gives (6). \hfill $\square$

We want now to understand how big must $L$ be in order for Lemma 3.3 and Corollary 3.4 to be true: of course, $L$ must be at least $M(\alpha) + 2$, because otherwise a rectangle of length $M$ topped with two half-circles cannot fit into $R_L$. Actually, we will see that this condition is also sufficient.

**Theorem 3.5** (Structure theorem of $\alpha$-Cheeger sets for rectangles). For any $\alpha \in (1, 2)$ the following holds:

(i) if $L < M(\alpha) + 2$, then $R_L$ has a unique $\alpha$-Cheeger set, obtained from the whole $R_L$ by cutting away the corners with four arcs of radius $r$, being

$$r = \frac{L + 1 - \sqrt{(L + 1)^2 - 2(4 - \pi)(2 - \alpha)La}}{(4 - \pi)(2 - \alpha)} < 1;$$  \hfill (8)

(ii) if $L = M(\alpha) + 2$, then $R_L$ has a unique $\alpha$-Cheeger set, obtained from the whole $R_L$ by cutting away the corners with four arcs of radius $r = 1$;

(iii) if $L > M(\alpha) + 2$, then $R_L$ has not a unique $\alpha$-Cheeger set; more precisely, its Cheeger sets are all rectangles of sides $M(\alpha)$ and 2 topped by two half disks of radius 1 which fit into $R_L$.

Proof. From the proofs of Lemma 3.3 and Corollary 3.4 we already know that a Cheeger set $C^\alpha_R$ can only have two shapes: if $r < 1$, then it coincides with the whole rectangle without the four corners, which are cut away by four arcs of circle with radius $r$; on the other hand, if $r = 1$, then it is some rectangle topped with two half-circles of radius 1. Moreover, in the second case, we have already calculated in Corollary 3.4 the exact length $M = M(\alpha)$ of the horizontal side of the rectangle contained in $C^\alpha_R$.

As a consequence, if $L < M(\alpha) + 2$ then for sure it cannot be $r = 1$, because the shape of the $\alpha$-Cheeger set would not fit into $R_L$. On the other hand, we want to show that, as soon as $L \geq M(\alpha) + 2$, then $r = 1$. Suppose that it is not so: this means that for some $\tilde{L} > M(\alpha) + 2$ there is an $\alpha$-Cheeger set $\tilde{C}$ corresponding to some radius $r < 1$; as a consequence, $\tilde{C}$ must reach both the short sides of $R_L$. For $L$ big enough, however, we already know that an $\alpha$-Cheeger set $C^\alpha_R$ is a rectangle of length $M$ topped with two half-circles; up to a translation, such $C^\alpha_R$ is entirely contained in $\tilde{C}$. Summarizing, both $\tilde{C}$ and $C^\alpha_R$ are contained both in $R_L$ and in $R_L$: but the $\alpha$-Cheeger ratio of $\tilde{C}$ is strictly bigger than that of $C^\alpha_R$, because Lemma 3.3 tells us that a set with $r < 1$ cannot be $\alpha$-Cheeger set for $R_L$ when $L \gg 1$, and this is a contradiction with the fact that $\tilde{C}$ is a Cheeger set in $R_L$.

Since the translations do not change neither perimeter nor area, then we have already proved the claim of this theorem, except formula (8). To obtain it, for any $L < M(\alpha) + 2$ and any $0 < t < 1$, let us call $R_t$ the rectangle $R_L$ with the four corners cut away with quarters of circle of radius $t$: we know already that the unique $\alpha$-Cheeger set of $R_L$ is $R_t$, and then $r$ will simply be the value of $t$ which minimizes the $\alpha$-Cheeger ratio of $R_t$. Since of course

$$P(R_t) = 2L + 2 - (8 - 2\pi)t, \quad |R_t| = 2L - (4 - \pi)t^2,$$

then formula (8) just comes by a straightforward minimization. \hfill $\square$
Remark 3.6. By sending $\alpha \to 1$ or $\alpha \to 2$ in the above result one derives, of course, the already known results for the standard case (when $\alpha = 1$) and the trivial results of the case $\alpha = 2^* = 2$, already discussed at the beginning of Section 2. In particular, our equation (8) extends the corresponding equation (11) of [6].

3.2. Counterexamples given by long and thin rectangles. Here we briefly discuss how the situation for the rectangles has completely changed from the standard case to the generalised one. In particular, all the claims of Theorem 3.1 fail. More in general, we are going to see that sufficiently long rectangles give a counterexample for all the properties of Theorem 1.1 whose analogue is not stated in Theorem 1.2.

First of all, we can observe that the infinite rectangle $R_\infty$ does admit $\alpha$-Cheeger sets, namely, any rectangle of length $M(\alpha)$ topped with two half-disks. Indeed, such a set is an $\alpha$-Cheeger set in $R_L$ for every $L \gg 1$, thus by continuity it is an $\alpha$-Cheeger set also for $R_\infty$. Moreover, by the same arguments of last section we can easily observe that these sets (which are all the same up to an horizontal traslation) are the unique Cheeger sets. In particular, the rectangles $R_L$ are not a minimising sequence for (2), since their $\alpha$-Cheeger constants explode when $L \to \infty$.

As soon as $L > M(\alpha) + 2$, $R_L$ does not admit a unique Cheeger set, and point 8 of Theorem 1.1 almost completely fails for $\alpha > 1$ (notice the difference with point 8 of Theorem 1.2): it is true that any $\alpha$-Cheeger set is convex, but it is not unique, and in particular it is not the union of all the balls with radius $r$.

Another property which is easily observed for the rectangles in the standard case is the following: if $L_1 < L_2$, then the Cheeger set corresponding to $R_{L_1}$ is contained in the Cheeger set corresponding to $R_{L_2}$. For the $\alpha$-Cheeger problem, this is only partially true: more precisely, any $\alpha$-Cheeger set in $R_{L_1}$ is contained in some $\alpha$-Cheeger set in $R_{L_2}$, but there are $\alpha$-Cheeger sets in $R_{L_2}$ which do not contain any $\alpha$-Cheeger set in $R_{L_1}$. Concerning different values of $\alpha$, the same happens: again by the non-uniqueness and the possible translations, in $R_L$ there are $\alpha_1$-Cheeger sets and $\alpha_2$-Cheeger sets which are not contained one into the other.

3.3. Computation of $h_\alpha$ for a given rectangle. In this last section, for the sake of completeness, we briefly compute the $\alpha$-Cheeger constant of some given rectangle $R_L$ depending on the power $\alpha \in (1, 2)$; we can assume without loss of generality that $L \geq 2$ (otherwise, it is enough to rotate and rescale the rectangle). Of course, first of all we need to determine when the situation is the one of case (i), or (ii), or (iii) of the structure Theorem 3.5. Actually, we already know that everything depends on the fact that $L$ is bigger or smaller than $M(\alpha) + 2$, and recalling (6) this is equivalent to $\alpha$ being bigger or smaller than

$$\bar{\alpha}(L) = \frac{2(\pi + L - 2)}{\pi + 2L - 4}.$$ 

It is immediate to observe that $L \mapsto \bar{\alpha}(L)$ is a strictly decreasing function, and that $\bar{\alpha}(2) = 2$ and $\bar{\alpha}(+\infty) = 1$.

**Proposition 3.7.** The $\alpha$-Cheeger constant of $R_L$ is given by

$$h_\alpha(R_L) = \begin{cases} 
\alpha \left( \frac{\pi}{\alpha - 1} \right)^{1 - \frac{1}{\alpha}} & \text{if } \alpha \geq \bar{\alpha}(L), \\
\frac{2L + 2 - (8 - 2\pi)r}{(2L - (4 - \pi)r^2)^{1/\alpha}} & \text{otherwise},
\end{cases}$$  

where $r$ is given by (8).
Proof. If $\alpha \geq \bar{\alpha}(L)$, then $L \geq M(\alpha) + 2$ and the Cheeger set is a rectangle of length $M(\alpha)$ plus two half-circles. As a consequence, by (7) together with the fact that $r = 1$ and using (6), we get
\[ h_{\alpha}(R_L) = \alpha|C_{\alpha}|^{1/\alpha} = \alpha\left(2M(\alpha) + \pi\right)^{1-1/\alpha} = \alpha\left(\frac{\pi}{\alpha - 1}\right)^{1-1/\alpha}, \]
according with (9).

If $\alpha < \bar{\alpha}(L)$, instead, we know precisely the shape of the $\alpha$-Cheeger set of $R_L$, which is the whole rectangle with the four corners cut away with arcs of circle of radius $r$, where $r$ is given by (8). Then, formula (9) follows straightforwardly. \(\square\)

Remark 3.8. By sending $L \to \infty$, the first line of formula (9) gives also the value of the $\alpha$-Cheeger constant $h_{\alpha}$ for the unbounded rectangle $R_{\infty}$. If we then send $\alpha \searrow 1$, we obtain that $h_{\alpha}(R_{\infty})$ converges to 1, which agrees with the fact that $h_1(R_{\infty}) = 1$, as well-known.

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