Multiblock ADMM for nonsmooth nonconvex optimization with nonlinear coupling constraints

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This paper proposes a multiblock alternating direction method of multipliers for solving a class of multiblock nonsmooth nonconvex optimization problem with nonlinear coupling constraints. We employ a majorization minimization procedure in the update of each block of the primal variables. Subsequential and global convergence of the generated sequence to a critical point of the augmented Lagrangian are proved. We also establish iteration complexity and provide preliminary numerical results for the proposed algorithm.

Key words: ADMM, nonlinear coupling constraints, majorization minimization, composite optimization

1. Introduction

We consider the following multiblock optimization problem with nonlinear coupling constraints

$$\min_{x,y} F(x_1, \ldots, x_m) + h(y) \quad \text{s.t} \quad \phi(x) + B y = 0,$$

where $x$ can be decomposed into $m$ blocks $x = (x_1, \ldots, x_m)$ with $x_i \in \mathbb{R}^{n_i}$, $n = \sum_{i=1}^{m} n_i$, $y \in \mathbb{R}^q$, $\phi$ is a nonlinear mapping from $\mathbb{R}^n$ to $\mathbb{R}^s$ defined by $\phi(x) = (\phi_1(x), \ldots, \phi_s(x))$, $B$ is a linear map, $F(x) = f(x) + g(x)$, $g(x) = \sum_{i=1}^{m} g_i(x_i)$, where $f: \mathbb{R}^n \to \mathbb{R}$ is a continuous (but possibly nonsmooth) function, $g_i: \mathbb{R}^{n_i} \to \mathbb{R} \cup \{+\infty\}$ are proper lower semi-continuous (lsc) functions for $i = 1, 2, \ldots, m$, and $h: \mathbb{R}^q \to \mathbb{R}$ is a differentiable function. Note that $F$ and $h$ can be nonconvex. Throughout the paper, we assume the following assumptions.

Assumption 1.  (A1) $F$ satisfies $\partial F(x) = \partial x_1 F(x) \times \ldots \times \partial x_m F(x)$, where $\partial F$ denote the limiting subdifferential of $F$ (see [26, Definition 8.3] for its definition).

(A2) $\nabla h$ is $L_h$-Lipschitz continuous.

(A3) $\sigma_B := \lambda_{\min}(BB^*) > 0$ ( $\sigma_B$ is the smallest eigenvalue of $BB^*$).

(A4) $F(x) + h(y)$ is lower bounded.

(Note that Assumption (A1) is satisfied when $f$ is a sum of a continuously differentiable function and a block separable function; see [1, Proposition 2.1].) While the Alternating Direction Method of Multipliers (ADMM) has gained significant attention and proven highly effective in addressing...
multiblock composite optimization problems with linear constraints, see e.g., [12, 28], the exploration of ADMM’s applicability to nonconvex nonsmooth problems with *nonlinear coupling constraints* remains relatively limited. The authors in [5] considers Problem (1) with $m = 1$ and a general nonlinear coupling constraint. They propose a universal framework to study global convergence analysis of Lagrangian sequences, introduce the notion of information zone, and propose an adaptive regime to detect this zone in finitely many steps and force the iterates to remain in the zone. Note that the essential assumption for ensuring the information zone’s role and the global convergence of the Lagrangian sequence in the framework of [5] is the boundedness of the multiplier sequence, as depicted in [5, Lemma 1]. Moreover, the choice of parameters within the context of [5] is closely tied to the upper bound constant of the multiplier sequence, as discussed in [5, Remark 7]. In practice, determining this upper bound for the multiplier can be an exceedingly challenging, if not impossible, task. The authors in [7] consider Problem (1) with $m = 1$. They design a proximal linearized ADMM in which the proximal parameter is generated dynamically by a backtracking procedure. Although the boundedness assumption of the multiplier sequence is not required in [7], the backtracking procedure used in [7], on one hand, requires to repeatedly evaluate the values of $f$ and $\phi$ to guarantee a descent property in updating $x$, and on the other hand, relies on the boundedness assumption of the generated sequence to guarantee the boundedness of the proximal parameters produced by the backtracking procedure. However, [7] does not provide a sufficient condition to guarantee the important boundedness assumption.

In this paper, we propose a multiblock alternating direction method of multipliers (mADMM) for solving Problem (1). Different from the general framework of [5], we specify the update of the primal block variables by embedding majorization minimization (MM) procedure. MM step can recover proximal point, proximal gradient and mirror descent step by using suitable surrogate functions depending on the structure of the objective function, see [13, 24]. Hence, employing MM and conducting the convergence analysis based on the surrogate functions allow us to not only better explore the structure of the problem but also unify the convergence analysis of special cases. We prove the subsequential convergence standard assumptions, and prove the global convergence with some additional assumptions. We also establish the iteration complexity and provide numerical results for mADMM (we note that iteration complexity was not established and numerical results were not presented in [5] and [7]).

The paper is organized as follows. In the next section, we provide some preliminary knowledge to support the forthcoming analysis. In Section 3, we describe mADMM and conduct the convergence analysis for it. We also provide a sufficient condition such that the generated sequence by mADMM is bounded. We report numerical results in Section 4 and conclude the paper in Section 5.

**Notation.** We denote $[m] = \{1, \ldots, m\}$. For a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^s$, we denote $\nabla \phi(x) = [\nabla \phi_1(x) \ldots \nabla \phi_s(x)]$ ($\nabla \phi(x)^\top$ would be the Jacobian), and $\nabla_{x_i} \phi(x) = [\nabla_{x_i} \phi_1(x) \ldots \nabla_{x_i} \phi_s(x)]$. 

2. Preliminaries

2.1. Augmented Lagrangian and ε-stationary point. The augmented Lagrangian function of Problem (1) is

\[ L_\beta(x, y, \omega) = \varphi_\beta(x, y, \omega) + \sum_{i=1}^m g_i(x_i), \]

where \( \varphi_\beta(x, y, \omega) = f(x) + h(y) + \langle \omega, \phi(x) + B_y \rangle + \frac{\beta}{2} \| \phi(x) + B_y \|^2. \) (2)

and \( \beta > 0 \) is a penalty parameter. Let \((x, y, \omega)\) be a critical point of \( L_\beta \), that is

\[ 0 \in \partial x_i F + \nabla x_i \phi(x) \omega, \quad \nabla h(y) + \nabla B_y (\omega + \beta (\phi(x) + B_y)) = 0, \quad \phi(x) + B_y = 0. \] (3)

The conditions in (3) imply that \( 0 \in \partial x_i F + \nabla x_i \phi(x) \omega, \quad \nabla h(y) + \nabla B_y (\omega + \beta (\phi(x) + B_y)) = 0, \quad \phi(x) + B_y = 0. \) Hence \((x, y)\) is a stationary point of Problem (1). Conversely, if \((x, y)\) is a stationary point of Problem (1) then there exists \( \omega \) such that \((x, y, \omega)\) satisfies (3), which means that \((x, y, \omega)\) is a critical point of \( L_\beta \). Therefore, finding a stationary point of (1) is equivalent to finding a critical point of the augmented Lagrangian \( L_\beta \).

**Definition 1.** We call \((x, y)\) an \( \varepsilon \)-stationary point of (1) if there exists \( \omega \) and \( \chi_i \in \partial x_i F(x) \) such that \( R_i = \| \chi_i + \nabla x_i \phi(x) \omega \| \leq \varepsilon, \) \( R_y = \| \nabla h(y) + B^* \omega \| \leq \varepsilon, \) and \( R_c = \| \phi(x) + B_y \| \leq \varepsilon. \)

2.2. Block surrogate function. The notations used in this subsection are independent of the notations of other sections. Let us first remind the proximal gradient method (PG), see e.g., [23], for solving the following composite optimization problem

\[ \min_{x \in \mathbb{R}^n} f(x) + g(x), \] (4)

where \( f \) is a continuous differentiable function, \( \nabla f(x) \) is assumed to be \( L \)-Lipschitz continuous and \( g \) is a lower semicontinuous function. The PG step is

\[ x^{k+1} \in \text{Prox}_{\frac{1}{L}g}(x^k - \frac{1}{L} \nabla f(x^k)), \] (5)

where the proximal mapping is defined as

\[ \text{Prox}_g(x) := \arg \min_{y} g(y) + \frac{1}{2} \| y - x \|^2. \]

As \( \nabla f \) is \( L \)-Lipschitz continuous, the descent lemma (see [21]) gives us

\[ f(x) \leq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + \frac{L}{2} \| x - x^k \|^2. \] (6)

Denote the right hand side of (6) by \( u(x, x^k) \). We notice that

\[ u(x, x) = f(x) \quad \text{and} \quad u(x, y) \geq f(x), \forall x, y. \] (7)
Each function $u(\cdot, \cdot)$ that satisfies (7) is called a surrogate function of $f$. The PG step in (5) can be rewritten in an equivalent form

$$x^{k+1} = \arg \min_x u(x, x^k) + g(x),$$

where $u$ is the Lipschitz gradient surrogate defined by the right hand side of (6). By choosing suitable surrogate functions, many first order methods such as proximal methods, mirror descent methods, etc., can also be rewritten in the majorization minimization (MM) form. Convergence analysis of MM algorithm (which can be roughly described in (8)) would unify the convergence analysis of the algorithms that correspond to different choices of the surrogates.

When $x$ has multiple blocks $x = (x_1, \ldots, x_m)$, $x_i \in \mathbb{R}^{n_i}$, and we assume that $g(x) = \sum g_i(x_i)$, block coordinate descent (BCD) method is a well-known approach to solve (4). BCD updates one block at a time while fixing the values of the other blocks. The condition in (7) is then extended to deal with functions that have multiblock variables as follows. We adopt the definition of block surrogate functions from [24].

**Definition 2 (Block surrogate function).** Let $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$, $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_m \subseteq \mathbb{R}^n$. A continuous function $u_i : \mathcal{X}_i \times \mathcal{X} \to \mathbb{R}$ is called a block surrogate function of $f$ on $\mathcal{X}$ with respect to block $x_i$ if

$$u_i(z_i, z) = f(z) \quad \text{and} \quad u_i(x_i, z) \geq f(x_i, z_{\neq i}) \quad \text{for all} \quad x_i \in \mathcal{X}_i, z \in \mathcal{X},$$

where $(x_i, z_{\neq i})$ denotes $(z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_m)$. The block approximation error is defined as

$$e_i(x_i, z) := u_i(x_i, z) - f(x_i, z_{\neq i}).$$

When writing the MM step for the update of block $i$, cf. (10), we can perceive that the current point is $z$ and the value $z_i$ of block $i$ will be updated by $\arg \min_{x_i} u_i(x_i, z_{\neq i}) + g_i(x_i)$.

**Example 1.** (i) A proximal surrogate (see e.g., [3, 2]) is

$$u_i(x_i, z) = f(x_i, z_{\neq i}) + \kappa_i D_{h_i}(x_i, z_i),$$

where $\kappa_i$ is a scalar that can depend on $z$, and $D_{h_i}(x_i, z_i)$ is a Bregman divergence defined by

$$D_{h_i}(x_i, z_i) = h_i(x_i) - h_i(z_i) - \langle \nabla h_i(z_i), x_i - z_i \rangle,$$

where $h_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is a strongly convex function and can be adaptively chosen in the course of the update of $x_i$. If $h_i(x_i) = \frac{1}{2} \langle x_i, Q_i x_i \rangle$, where $Q_i$ is a positive definite matrix that could also depend on $z$, then the proximal surrogate becomes the typical extended proximal surrogate

$$u_i(x_i, z) = f(x_i, z_{\neq i}) + \frac{1}{2} \kappa_i \| x_i - z_i \|^2_{Q_i}.$$
(ii) A quadratic surrogate (see e.g., [6, 22]) is defined as
\[ u_i(x_i, z) = f(z) + \langle \nabla z_i f(z), x_i - z_i \rangle + \frac{\kappa_i}{2} \|x_i - z_i\|_Q^2, \]
where \( \kappa_i \geq 1 \), \( f \) is assumed to be twice differentiable, and \( Q_i \) (which can depend on \( z \)) is a positive definite matrix such that \( Q_i - \nabla^2 z_i f(x_i, z_{\neq i}) \) is also a positive definite matrix. The approximation error function is
\[ e_i(x_i, z) = f(z) - f(x_i, z_{\neq i}) + \langle \nabla z_i f(z), x_i - z_i \rangle + \frac{\kappa_i}{2} \|x_i - z_i\|_Q^2. \]

If \( Q_i = L_i I \), where \( I \) is a identity matrix and \( L_i \) is a positive number, then the quadratic surrogate reduces to a Lipschitz gradient surrogate in the following.

(iii) A Lipschitz gradient surrogate (see e.g., [29]) is defined as
\[ u_i(x_i, z) = f(z) + \langle \nabla z_i f(z), x_i - z_i \rangle + \frac{\kappa_i L_i}{2} \|x_i - z_i\|^2, \]
where \( \kappa_i \geq 1 \) and we assume \( x_i \mapsto \nabla z_i f(x_i, z_{\neq i}) \) is \( L_i \)-Lipschitz continuous. Note that \( L_i \) can depend on \( z \).

(iv) A Bregman surrogate (see e.g., [11, 14, 20]) is defined as
\[ u_i(x_i, z) = f(z) + \langle \nabla z_i f(z), x_i - z_i \rangle + \kappa_i L_i D_{h_i}(x_i, z_i), \]
where \( \kappa_i \geq 1 \), \( h_i : \mathbb{R}^{m_i} \to \mathbb{R} \) is a strongly convex function, \( D_{h_i}(x_i, z_i) \) is a Bregman divergence defined in (9), the function \( x_i \mapsto f(x_i, z_{\neq i}) \) is assumed to be \( L_i \)-relative smooth (\( L_i \) can depend on \( z \)) to \( h_i \), that is, the function \( x_i \mapsto L_i h_i(x_i) - f(x_i, z_{\neq i}) \) is convex (see [17]). If \( h_i(x_i) = \frac{1}{2} \|x_i\|^2 \) then the block Bregman surrogate reduces to the Lipschitz gradient surrogate.

3. Multiblock ADMM for solving Problem (1) We describe mADMM for solving Problem (1) in Algorithm 1. We note that \( x_i, i \in [m], y \) and \( \omega \) are the blocks of variables of \( \varphi_\beta \) defined in (2).

Update of block \( x_i \). We choose block surrogate functions of \( \varphi_\beta \) with respect to \( x_i, i \in [m] \), such that they satisfy Condition 1 and one of the two conditions - Condition 2 or Condition 3.

Condition 1. For \( i \in [m] \), there exists an “upper bound” error \( \bar{e}_i(x_i, z) \) such that the approximation error \( e_i(x_i, z) = u_i(x_i, z) - \varphi_\beta(x_i, z_{\neq i}) \) satisfies
- \( e_i(x_i, z) \leq \bar{e}_i(x_i, z) \) for all \( (x_i, z) \), i.e., \( e_i \) is upper bounded by \( \bar{e}_i \), and
- we have \( \bar{e}_i(z_i, z) = 0 \) and \( \nabla z \bar{e}_i(z_i, z) = 0 \) for all \( z \).

Condition 2. The error \( e_i(x_i, z) \) satisfies \( e_i(x_i, z) \geq \eta_\beta(z, \beta) D_i(x_i, z_i) \) for all \( x_i, z \), where \( \eta_\beta(z, \beta) \) is a scalar that can depend on the values of \( z \) and \( \beta \). Here \( D_i(\cdot, \cdot) \) satisfies
\[ D_i(x_i^{k+1}, x_i^k) \geq 0, \text{ and if } D_i(x_i^{k+1}, x_i^k) \to 0 \text{ when } k \to \infty \text{ then } \|x_i^{k+1} - x_i^k\| \to 0. \quad (12) \]

A simple example of \( D_i(\cdot, \cdot) \) is \( D_i(x_i^{k+1}, x_i^k) = \frac{1}{2} \|x_i^{k+1} - x_i^k\|^2 \).
Algorithm 1 mADMM for solving Problem (1)

Notations. For a given $k$ (the outer iteration index) and $i$ (the cyclic inner iteration index) we denote $x^{k,i} = (x_1^{k,1}, \ldots, x_i^{k+1}, x_{i+1}^{k+1}, \ldots, x^k)$ and let $x^{k+1} = x^{k,n}$.

Choose $x^0$, $y^0$, $\omega^0$ and $\beta$ satisfies (28). Set $k = 0$.

repeat
  for $i = 1, \ldots, m$ do
    Update
    \[ x_i^{k+1} \in \arg\min_{x_i} \{ u_i(x_i, x^{k,i-1}, y^k, \omega^k) + g_i(x_i) \}. \tag{10} \]
  end for

Update $y$ as in (14).

Update $\omega$ as follows
\[ \omega^{k+1} = \omega^k + \beta(\phi(x^{k+1}) + B y^{k+1}). \tag{11} \]

Set $k \leftarrow k + 1$.

until some stopping criteria is satisfied.

**Condition 3.** The function $t(x_i, z) = u_i(x_i, z) + g_i(x_i)$ satisfies
\[ t(x_i, z) \geq t(x'_i, z) + \langle \xi, x_i - x'_i \rangle + \eta_i(z, \beta) D_i(x'_i, x_i) \]
for all $x_i, x'_i, z$ and any $\xi \in \partial_{x_i} t(x'_i, z)$. Here $D_i(\cdot, \cdot)$ satisfies the conditions in (12). For the case $D_i(x'_i, x_i) = 1/2\|x'_i - x_i\|^2$, Condition 3 reduces to the strong convexity of $x_i \mapsto t(x_i, z)$ with constant $\eta_i(z, \beta)$.

**Remark 1.** Note that all surrogates in Example 1 satisfy Condition 1 with $\bar{e}_i(x_i, z) = e_i(x_i, z)$.

The surrogate in Example 1(i) satisfies Condition 2 with $\eta_i = \kappa_i$ and $D_i = D_{h_i}$. Consider the Bregman surrogate in Example 1(iv). Since $x_i \mapsto L_i h_i(x_i) - f(x_i, z_{\neq i})$ is convex, we have
\[ L_i h_i(x_i) - f(x_i, z_{\neq i}) - (L_i h_i(z_i) - f(z_i, z_{\neq i})) - \langle L_i \nabla h_i(z_i), x_i - z_i \rangle \geq 0, \]
which implies $L_i D_{h_i}(x_i, z_i) + f(z) + \langle \nabla z_i f(z), x_i - z_i \rangle - f(x_i, z_{\neq i}) \geq 0$. Therefore, if we take $\kappa_i > 1$ then the approximation error of the Bregman surrogate satisfies Condition 2 with $\eta_i = (\kappa_i - 1)L_i$ and $D_i = D_{h_i}$. Specifically,
\[ e_i(x_i, z) = f(z) + \langle \nabla z_i f(z), x_i - z_i \rangle + \kappa_i L_i D_{h_i}(x_i, z_i) - f(x_i, z_{\neq i}) \geq (\kappa_i - 1)L_i D_{h_i}(x_i, z_i). \]
Similarly, we can show that the quadratic surrogate and the Lipschitz gradient surrogate in Example 1 also satisfy Condition 2.

Consider the Lipschitz gradient surrogate in Example 1(iii). If $g_i$ is convex then we see that $x_i \mapsto u_i(x_i, z) + g_i(x_i)$ is $\kappa_i L_i$-strongly convex. Hence, if $g_i$ is convex then Condition 3 is satisfied
with \( \eta_i(z, \beta) = L_i(z, \beta) \) and \( D_i(x'_i, x_i) = \frac{1}{2}\|x'_i - x_i\|^2 \). Similarly, we can prove that if \( g_i \) is convex, then the quadratic surrogate and the Bregman surrogate in Example 1 also satisfy Condition 3.

**Update of block \( y \).** As \( h \) is assumed to be \( L_h \)-smooth, we use the following surrogate

\[
u_{m+1}(y, \tilde{x}, \tilde{y}, \omega) = h(\tilde{y}) + \langle \nabla h(\tilde{y}), y - \tilde{y} \rangle + \frac{L_h}{2}\|y - \tilde{y}\|^2 + \tilde{\varphi}_\beta(\tilde{x}, y, \omega),
\]

(13)

where \( \tilde{\varphi}_\beta(x, y, \omega) := f(x) + \langle \omega, \phi(x) + By \rangle + \frac{\beta}{2}\|\phi(x) + By\|^2 \). The update of \( y \) is

\[
y^{k+1} = \arg\min_y \nu_{m+1}(y, x^{k+1}, y^k, \omega^k)
\]

\[
= \arg\min_y \{ \langle \nabla h(y^k) + B^*\omega^k, y \rangle + \frac{\beta}{2}\|\phi(x^{k+1}) + By\|^2 + \frac{L_h}{2}\|y - y^k\|^2 \}
\]

\[
= (\beta B^*B + L_hI)^{-1}(L_hy^k - \nabla h(y^k) - B^*(\omega^k + \beta(\phi(x^{k+1}))))
\]

(14)

3.1. Subsequential convergence. We establish subsequential convergence for \( \text{mADMM} \) in this section. Let us first prove some sufficient decreasing properties for the update of \( x_i \) and \( y \).

**Proposition 1.** (i) The update in (10) guarantees a sufficient decreasing:

\[\mathcal{L}_\beta(x^{k,i}, y^k, \omega^k) + \eta^k_iD_i(x^{k+1,i}, x^k_i) \leq \mathcal{L}_\beta(x^{k,i-1}, y^k, \omega^k),\]

(15)

where \( \eta^k_i = \eta_i(x^{k,i}, y^k, \omega^k, \beta) \), which is the scalar in Condition 2 or Condition 3.

(ii) The update in (14) guarantees a sufficient decreasing:

\[\mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^k) + \frac{\delta}{2}\|y^{k+1} - y^k\|^2 \leq \mathcal{L}_\beta(x^{k+1}, y^k, \omega^k),\]

(16)

where \( \delta = L_h + \beta\lambda_{\min}(B^*B) \).

**Proof.** We remark that, if we denote \( x_{m+1} = y, g_{m+1}(y) = 0 \) and \( D_{m+1}(y, y') = \frac{1}{2}\|y - y'\|^2 \), then we can write the update of block \( x_i, i \in [m] \), and the update of block \( y \) in (14) in the following unified form, supposed the current iterate is \( z = (\tilde{x}_1, \ldots, \tilde{x}_{m+1}, \omega) \),

\[x^+_i = \arg\min_{x_i} u_i(x_i, z) + g_i(x_i),\]

(17)

where \( u_i \) is a surrogate function of \( \varphi_\beta \) with respect to block \( x_i, i \in [m+1] \).

(i) In the following, we use \( \eta_i \) for \( \eta_i(z, \beta) \). Suppose Condition 2 holds. Then we have

\[u_i(x^+_i, z) - \varphi_\beta(x^+_i, z, \beta_i) \geq \eta_iD_i(x^+_i, z_i).\]

(18)

On the other hand, from (17) we have \( u_i(x_i, z) + g_i(x_i) \geq u_i(x^+_i, z) + g_i(x^+_i) \) for all \( x_i \). Choose \( x_i = z_i \), then combine with (18). We obtain

\[u_i(z_i, z) + g_i(z_i) \geq u_i(x^+_i, z) + g_i(x^+_i) \geq \varphi_\beta(x^+_i, z, \beta_i) + \eta_iD_i(x^+_i, z_i) + g_i(x^+_i).\]

(19)

Furthermore, \( u_i(z_i, z) = \varphi_\beta(z) \). The result follows from (19).

Suppose Condition 3 holds. Choosing \( x'_i = x^+_i \) and \( x_i = z_i \) in Condition 3, we have

\[u_i(z_i, z) + g_i(z_i) \geq t(x^+_i, z) + \langle \xi, z_i - x^+_i \rangle + \eta_iD_i(x^+_i, z_i), \quad \forall \xi \in \partial z_i t(x^+_i, z).\]

(20)
Moreover, we have $u_i(x_i^+, z) \geq \varphi(x_i^+, z_i)$. Hence the result follows.

(ii) Since $y \mapsto u_{m+1}(y, \tilde{x}, \tilde{y}, \omega)$ defined in (13) is $(L_h + \beta \lambda_{\min}(\mathcal{B}^* \mathcal{B}))$-strongly convex, similarly to the proof of Part (i) for the case that Condition 3 is satisfied, we can prove that (16) is satisfied. □

Denote $\Delta x_i^k = x_i^k - x_i^{k-1}$, $\Delta y_k = y_k - y_{k-1}$, $\Delta \omega_k = \omega_k - \omega_{k-1}$.

PROPOSITION 2. The values of $\eta_i^k$ and $\delta$ are given in Proposition 1. Let $\hat{\delta} = 2L_h$. We have

$$
\mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^{k+1}) + \sum_{i=1}^m \eta_i^k D_i(x_i^{k+1}, x_i^k) + \frac{1}{\beta} \|B^* \Delta \omega^{k+1}\|^2 
\leq \mathcal{L}_\beta(x^k, y^k, \omega^k) + \frac{1}{\beta G} \|B^* \Delta \omega^{k+1}\|^2, 
$$

(21)

and

$$
\|B^* \Delta \omega^{k+1}\|^2 \leq 3 \left( (L_h^2 + \hat{\delta}^2) \|\Delta y^{k+1}\|^2 + \hat{\delta}^2 \|\Delta y^k\|^2 \right). 
$$

(22)

Proof. Summing (15) from $i = 1$ to $m$ we obtain

$$
\mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^k) + \sum_{i=1}^m \eta_i^k D_i(x_i^{k+1}, x_i^k) \leq \mathcal{L}_\beta(x^k, y^k, \omega^k). 
$$

(23)

On the other hand,

$$
\mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^{k+1}) = \mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^k) + \langle \omega^{k+1} - \omega^k, \phi(x^{k+1}) + B y^{k+1} \rangle 
= \mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^k) + \frac{1}{\beta} \|\Delta \omega^{k+1}\|^2 
\leq \mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^k) + \frac{1}{\beta G} \|B^* \Delta \omega^{k+1}\|^2. 
$$

(24)

Therefore, Inequality (21) follows from (23), (16), and (24).

By definition of the approximation error, we have

$$
\nabla_y e_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k) = \nabla_y u_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k) - \nabla_y \varphi(x^{k+1}, y^{k+1}, \omega^k), 
$$

where $u_{m+1}$ is defined in (13). Furthermore, from (14), $\nabla_y u_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k) = 0$. Hence,

$$
-\nabla_y e_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k) = \nabla_y \varphi(x^{k+1}, y^{k+1}, \omega^k), 
$$

which implies that

$$
-\nabla_y e_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k) = \nabla h(y^{k+1}) + B^* \left( \omega^k + \beta (\phi(x^{k+1}) + B y^{k+1}) \right) 
= \nabla h(y^{k+1}) + B^* \omega^{k+1}. 
$$

(25)

Hence, we have

$$
\|B^* \Delta \omega^{k+1}\| = \| - \nabla_y e_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k) - \nabla h(y^{k+1}) + \nabla h(y^{k+1}) + \nabla h(y^k) \|. 
$$

(26)

On the other hand, $\nabla_y e_{m+1}(y, \tilde{x}, \tilde{y}, \omega) = \nabla h(\tilde{y}) - \nabla h(y) + L_h(y - \tilde{y})$. Hence,

$$
\|\nabla_y e_{m+1}(y^{k+1}, x^{k+1}, y^k, \omega^k)\| \leq \hat{\delta} \|y^{k+1} - y^k\|. 
$$

(27)

So from (26) we obtain Inequality (22). □
PROPOSITION 3. Let $\delta$ and $\eta^k_i$ be defined in Proposition 1 and $\hat{\delta} = 2L_h$. Suppose
\[ \beta (L_h + \beta \lambda_{\min}(B^*B)) \geq \frac{6L_h^2(5 + 4\hat{\delta})}{\sigma_B} \quad \text{and} \quad \eta^k_i \geq \eta, \] for some constants $\eta, \eta_i > 0$ and $\hat{\delta} > 1$. Denote $L^k = L_\beta(x^k, y^k, \omega^k)$.

(A) For $k > 0$, we have
\begin{align*}
L^{k+1} + \sum_{i=1}^m \eta^k_i D_i(x^k_{i+1}, x^k_i) + \frac{3\delta\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^{k+1}\|^2 & \leq L^k + \frac{3\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^k\|^2. 
\end{align*}

(B) For $K > 0$, we have
\begin{align*}
L^{K+1} + \frac{3\delta\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^{K+1}\|^2 + \sum_{i=1}^K \sum_{j=1}^m \eta^k_i D_i(x^k_{i+1}, x^k_i) + \frac{3\delta\hat{\delta}^2}{\beta \sigma_B} \sum_{i=1}^{K-1} \|\Delta y^{k+1}\|^2 & \leq L^1 + \frac{3\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^1\|^2. 
\end{align*}

(C) The sequences $\{\Delta x^k\}$, $\{\Delta y^k\}$ and $\{\Delta \omega^k\}$ converge to 0.

Proof. (A) Combining (21) with (22) gives us
\begin{align*}
L^{k+1} + \sum_{i=1}^m \eta^k_i D_i(x^k_{i+1}, x^k_i) + \frac{3\delta}{\beta \sigma_B} \|\Delta y^{k+1}\|^2 & \leq L^k + \frac{3\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^k\|^2.
\end{align*}

This implies the following recursive inequality
\begin{align*}
L^{k+1} + \sum_{i=1}^m \eta^k_i D_i(x^k_{i+1}, x^k_i) + \left( \frac{3\delta}{\beta \sigma_B} (L_h^2 + \hat{\delta}^2) \right) \|\Delta y^{k+1}\|^2 & \leq L^k + \frac{3\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^k\|^2.
\end{align*}

The condition in (28) implies that $\frac{\hat{\delta}^2}{3} - \frac{3}{\beta \sigma_B} (L_h^2 + \hat{\delta}^2) \geq \frac{3\delta\hat{\delta}^2}{\beta \sigma_B}$. Hence Inequality (31) implies (29).

(B) Summing (29) from $k = 1$ to $K$ and using the conditions $\eta^k_i \geq \eta$, we get (30).

(C) We now use the technique in [20, Proposition 2.9] to prove that
\[ \hat{L}^k := L^k + \frac{3\delta\hat{\delta}^2}{\beta \sigma_B} \|\Delta y^k\|^2 \geq \nu, \quad \text{for all } k \geq 1, \] where $\nu$ is a lower bound of $F(x^k) + h(y^k)$. From (29), $\{\hat{L}^k\}_{k \geq 1}$ is non-increasing. Suppose there exists $k_1 \geq 1$ s.t $\hat{L}^{k_1} < \nu$. Hence, $\hat{L}^k < \nu$ for all $k \geq k_1$. We have
\[ \sum_{k=1}^K (\hat{L}^k - \nu) \leq \sum_{k=1}^{k_1} (\hat{L}^k - \nu) + (K - k_1)(\hat{L}^{k_1} - \nu). \]

Hence, $\lim_{K \to \infty} \sum_{k=1}^K (\hat{L}^k - \nu) = -\infty$. On the other hand, we have
\[ \hat{L}^k \geq L^k \geq F(x^k) + h(y^k) + \frac{1}{\beta}(\omega^k - \omega_{k-1}^k) \geq \nu + \frac{1}{2\beta}(\|\omega^k\|^2 - \|\omega_{k-1}^k\|^2). \]

Thus, $\sum_{k=1}^K (\hat{L}^k - \nu) \geq \sum_{k=1}^K \frac{1}{2\beta}(\|\omega^k\|^2 - \|\omega_{k-1}^k\|^2) = \frac{1}{2\beta}(\|\omega^K\|^2 - \|\omega^0\|^2) \geq -\frac{1}{2\beta}\|\omega^0\|^2$. This gives a contradiction. Therefore, we get (32).

Inequality (30) together with the lower boundedness of $\{\hat{L}\}_{k \geq 0}$ we have $\sum_{k=0}^\infty D_i(x^k_{i+1}, x^k_i) < +\infty$,
\[ \sum_{k=1}^\infty \|\Delta y^{k+1}\| < +\infty. \] This implies that $\Delta x^k$ and $\Delta y^k$ converge to 0. On the other hand, it follows from (22) that $\sum_{k=0}^\infty \|B^*\Delta \omega^{k+1}\|^2 < +\infty$, leading to the convergence of $\{\Delta \omega^k\}$ to 0. \qed
Theorem 1 (Subsequential convergence). Suppose the parameters are chosen such that the conditions of Proposition 3 are satisfied. If there exists a subsequence \((x^{k_n}, y^{k_n}, \omega^{k_n})\) converging to \((x^*, y^*, \omega^*)\) then \((x^*, y^*, \omega^*)\) is a critical point of \(\mathcal{L}_\beta(x, y, \omega)\).

Proof. See Appendix A.1.

In the following proposition, by extending [28, Lemma 6], we provide a sufficient condition such that the generated sequence of mADMM is bounded.

Proposition 4. If \(\text{ran } \phi(x) \subseteq \text{Im}(\mathcal{B}), \lambda_{\min}(\mathcal{B}^*\mathcal{B}) > 0\) and \(F(x) + h(y)\) is coercive over the feasible set \(\{(x, y) : \phi(x) + \mathcal{B}y = 0\}\) then \(\{(x^k, y^k, \omega^k)\}_{k \geq 0}\) generated by Algorithm 1 is bounded.

Proof. See Appendix A.2.

3.2. Iteration complexity. We now establish the iteration complexity to obtain an \(\varepsilon\)-stationary point, see Definition 1. To this end, we need the following additional assumption.

Assumption 2. • For any \(z\) and \(x_i \in \text{dom}(g_i)\), we have
\[
\partial_{x_i}(u_i(x_i(z) + g_i(x_i))) = \partial_{x_i}u_i(x_i(z) + \partial g_i(x_i), \partial_{x_i}(f(x) + g_i(x_i)) = \partial_{x_i}f(x) + \partial g_i(x_i). 
\]

(33)

• For any \(S^k_i \in \partial_{x_i}u_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k)\) there exists \(\tilde{S}^k_i \in \partial_{x_i}\varphi(x_i^{k+1}, y^k, \omega^k)\) that
\[
\|S^k_i - \tilde{S}^k_i\| \leq \bar{L}_i\|x^{k+1} - x^{k,i-1}\|, 
\]

(34)

where \(\bar{L}_i\) is some nonnegative constant (we remark that the constant \(\bar{L}_i\) does not involve in how to choose the parameters in our framework; its existence is for the convergence proof).

Condition (33) says that \(x_i \mapsto u_i(x_i(z)\), \(x_i \mapsto f(x)\) and \(g_i\) follow the sum rule for the limiting subgradients. See [26, Corollary 10.9] for a sufficient condition. Note that if \(x_i \mapsto u_i(x_i(z)\) and \(x_i \mapsto f(x)\) are continuously differentiable then (33) is satisfied. Regarding (34), if we assume \((x^k, y^k, \omega^k)\) is bounded then (34) is satisfied when \(\nabla x_i u_i(x_i, x, y, \omega) = \nabla x_i \varphi(x, y, \omega)\) and \(u_i\) is twice continuously differentiable. Indeed, consider the bounded set containing \((x^k, y^k, \omega^k)\), then
\[
\|S^k_i - \tilde{S}^k_i\| = \|\nabla x_i u_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k) - \nabla x_i \varphi(x_i^{k+1}, y^k, \omega^k)\| \\
= \|\nabla x_i u_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k) - \nabla x_i u_i(x_i^{k+1}, x_i^{k+1}, y^k, \omega^k)\| \\
\leq \bar{L}_i\|x^{k+1} - x^{k,i-1}\|.
\]

The surrogate functions given in Example 1 satisfy (34) when \(f\) is twice continuously differentiable. We give an example when \(f\) is nonsmooth and (34) is still satisfied in Appendix B.

Proposition 5. Denote
\[
R^k_y = \|\nabla h(y^k) + \mathcal{B}^*\omega^k\|, R^k_i = \|\chi^{k} + \nabla x_i \phi(x^k)\omega^k\|, R^k_c = \|\phi(x^k) + \mathcal{B}y^k\|,
\]

where \(\chi^{k}\) is some nonnegative constant (we remark that the constant \(\chi^{k}\) does not involve in how to choose the parameters in our framework; its existence is for the convergence proof).
where $\chi^k_i \in \partial x_i F(x^k)$. Assume Assumption 2 hold, the conditions of Proposition 3 are satisfied, and $D_i(x^k_{i+1}, x^k_i) = \frac{1}{2} \| x^k_{i+1} - x^k_i \|^2$ in (15). Suppose $\{(x^k, y^k)\}_{k \geq 0}$ is bounded. For all $k > 0$, there exist $\chi^k_i \in \partial x_i F(x^k)$ such that

$$R_y^k = O(||\Delta y^k||), R^k = O(||\Delta x^k|| + ||\Delta y^k||), R^k_c = O(||\Delta y^k|| + ||\Delta y^{k-1}||)$$

(35)

and

$$ \min_{1 \leq j \leq k-1} \left\{ \frac{1}{2} \eta \| \Delta x^{j+1} \|^2 + \frac{3(\hat{\delta} - 1) \hat{\delta}^2}{2 \beta \sigma_B} (\| \Delta y^{j+1} \|^2 + \| \Delta y^j \|^2) \right\} = O\left(\frac{1}{k}\right),$$

(36)

where $\hat{\eta} = \min_{i \in [m]} \eta_i$.

Proof. From (25) and (27) we have

$$R_y^k = \| \nabla_y e_{m+1}(y^k, x^k, y^{k-1}, \omega^{k-1}) \| \leq \hat{\delta} \| \Delta y^k \|.$$  

(37)

Writing the optimality condition for (10) we get

$$0 \in \partial x_i \left( u_i(x^k_{i+1}, x^{k,i-1}, y^k, \omega^k) + g_i(x^k_{i+1}) \right) = \partial x_i u_i(x^k_{i+1}, x^{k,i-1}, y^k, \omega^k) + \partial x_i g_i(x^k_{i+1}).$$

In other words, there exist $S^k_i \in \partial x_i u_i(x^k_{i+1}, x^{k,i-1}, y^k, \omega^k)$ and $T^k_i \in \partial x_i g_i(x^k_{i+1})$ such that $S^k_i + T^k_i = 0$. Furthermore, it follows from (34) that there exists $\tilde{S}^k_i \in \partial x_i \varphi_\beta(x^k_{i+1}, y^k, \omega^k)$ such that $\| S^k_i - \tilde{S}^k_i \| \leq \tilde{L}_i \| x^{k,i-1} - x^k \|$. Hence, we have

$$\| T^k_i + \tilde{S}^k_i \| = \| S^k_i - \tilde{S}^k_i \| \leq \tilde{L}_i \| x^{k,i-1} - x^k \| \leq \tilde{L}_i \| \Delta x^k \|.$$  

(38)

Note that

$$\partial x_i \varphi_\beta(x^k_{i+1}, y^k, \omega^k) = \partial x_i f(x^k_{i+1}) + \nabla \varphi(x^{k+1}) (\omega^k + \beta (\varphi(x^{k+1}) + B y^k))$$

$$= \partial x_i f(x^{k+1}) + \nabla \varphi(x^{k+1}) \omega^{k+1} + \beta \nabla \varphi(x^{k+1}) (B y^k - B y^{k+1}).$$

Therefore, $\exists \xi_i^{k+1} \in \partial x_i f(x^{k+1})$ s.t $\xi_i^{k+1} + \nabla \varphi(x^{k+1}) \omega^{k+1} + \beta \nabla \varphi(x^{k+1}) (B y^k - B y^{k+1}) = \tilde{S}^k_i$. Moreover, it follows from (33) that $\chi^{k+1}_i = \xi^{k+1}_i + T^{k+1}_i \in \partial x_i F(x^{k+1})$. We then obtain

$$R^{k+1}_i = \| \xi^{k+1}_i + \nabla \varphi(x^{k+1}) \omega^{k+1} + T^{k+1}_i \|$$

$$= \| S^k_i - \beta \nabla \varphi(x^{k+1}) (B y^k - y^{k+1}) + T^{k+1}_i \|$$

$$\leq \tilde{L}_i \| \Delta x^k \| + \beta \| B \| \| \Delta y^{k+1} \|,$$

(39)

where in the last inequality we used: (38), the fact that $\| \nabla \varphi(x^{k+1}) \|$ $\leq M_\phi$ for some constant $M_\phi$ (since $x^k$ is bounded), $\phi$ is continuously differentiable.

On the other hand, we have

$$R^{k+1}_c = \| \phi(x^{k+1}) + B y^{k+1} \| = \frac{1}{\beta} \| \Delta \omega^{k+1} \| \leq \frac{1}{\beta \sigma_B} \| B^* \Delta \omega^{k+1} \|.$$

Together with (22), (37), and (39), we obtain (35).
From (30) and (32) we have
\[
\nu + \sum_{k=1}^{K} \sum_{i=1}^{m} \eta_i D_i(x_i^{k+1}, x_i^k) + \frac{3(\hat{\delta} - 1)\hat{\delta}^2}{2\beta}\sum_{k=1}^{K-1} \|\Delta y^{k+1}\|^2 \leq \mathcal{L}_1 + \frac{3\hat{\delta}^2}{\beta}\|\Delta y^1\|^2.
\] (40)

Note that \(2\sum_{k=1}^{K-1} \|\Delta y^{k+1}\|^2 \geq \sum_{k=1}^{K-1} \left( \|\Delta y^{k+1}\|^2 + \|\Delta y^k\|^2 \right) - \|\Delta y^1\|^2 \). Therefore from (40), we have
\[
\min_{1 \leq k \leq K-1} \left\{ \frac{1}{2}\|\Delta x^k\|^2 + \frac{3(\hat{\delta} - 1)\hat{\delta}^2}{2\beta}\left( \|\Delta y^{k+1}\|^2 + \|\Delta y^k\|^2 \right) \right\} \leq \frac{1}{K-1} C,
\]
where \(C = (\mathcal{L}_1 + \frac{3\hat{\delta}^2}{\beta}\|\Delta y^1\|^2 - \nu + \frac{3(\hat{\delta} - 1)\hat{\delta}^2}{2\beta}\|\Delta y^1\|^2 \). The result in (36) follows then. \(\square\)

Proposition (5) shows that \(\min_{0 \leq j \leq k-1} R_y^j + \sum_{i=1}^{m} R_i^j + R_c^j = O(\frac{1}{\sqrt{\epsilon}})\). Hence, after at most \(O(\frac{1}{\epsilon})\) iterations we obtain an \(\epsilon\)-stationary point of (1).

### 3.3. Global convergence under KL property and convergence rate.

The following theorem presents the global convergence of mADMM under the KL property (see [4] for its definition).

**Theorem 2.** Assume Assumption 2 hold, the conditions of Proposition 3 are satisfied, and
\(D_i(x_i^{k+1}, x_i^k) = \frac{1}{2}\|x_i^{k+1} - x_i^k\|^2 \) in (15). Furthermore, we assume that the generated sequence is bounded and the following Lyapunov function
\[
\hat{\mathcal{L}}_\beta(x, y, \omega, \bar{y}) = \mathcal{L}_\beta(x, y, \omega) + \frac{3\hat{\delta}^2}{\beta}\|y - \bar{y}\|^2
\] (41)
has the KL property with constant \(\sigma_L\), then \(\{z^k\}_{k \geq 0}\), where \(z^k = (x^k, y^k, \omega^k)\), converges to a critical point of \(\mathcal{L}_\beta\). Moreover, if \(\sigma_L = 0\) then mADMM converges after a finite number of steps; if \(\sigma_L \in (0, 1/2)\) then there exists \(k_1 \geq 1, w_1 > 0\) and \(w_2 \in [0, 1)\) such that \(\|z^k - z^*\| \leq w_1 w_2^k\) for all \(k \geq k_1\); and if \(\sigma_L \in (1/2, 1]\) then there exists \(k_1 \geq 1\) and \(w_1 > 0\) such that \(\|z^k - z^*\| \leq w_1 k^{-(1 - \sigma_L)/(\sigma_L - 1)}\) for all \(k \geq k_1\).

**Proof.** See Appendix A.3 \(\square\)

**Remark 2.** The assumption \(\sigma_B = \lambda_{\min}(B^*B) > 0\) can be relaxed to \(\text{ran} \phi(x) \subseteq \text{Im}(B)\). Then the constant \(\sigma_B\) in (28) would be replaced by the smallest positive eigenvalue of \(B^*B\), which is denoted by \(\lambda_{\min}^+(B^*B)\). It is because when \(\text{ran} \phi(x) \subseteq \text{Im}(B)\), we can derive from (11) that \(\Delta \omega^{k+1} \in \text{Im}(B)\). This implies \(\|\Delta \omega^{k+1}\|^2 \leq \lambda_{\min}^+(B^*B)\|B^*\Delta \omega^{k+1}\|^2\), which is the cornerstone to prove Proposition 2 (see Inequality (24)) and to derive \(\Delta \omega^k \to 0\) from \(B^* \Delta \omega^k \to 0\) in Proposition 3 (C).

### 4. Preliminary numerical results

We consider the following composite optimization problem
\[
\min_{x \in \mathbb{R}^n} g(x) + h(\phi(x)),
\] (42)
where \(g\) is a lower semicontinuous function, \(h\) is a differentiable function with \(L_h\)-Lipschitz continuous gradient, and \(\phi(x) = (\phi_1(x), \ldots, \phi_q(x))\) is a nonlinear mapping. Problem (42), which includes
regularized nonlinear least square problem as a special case (that is, when \(h(\phi(x)) = \frac{1}{2}\|\phi(x)\|^2\)), has appeared frequently in machine learning and statistics. For examples, Problem (42) covers the nonlinear regression problem \([10]\) \((h\) plays the role of a loss function, \(\phi\) represents the model to train and \(g\) is a regularizer), the risk parity portfolio selection problem \([19]\), the robust phase retrieval problem \([9]\), and the PDE-constrained inverse problem \([27]\). Problem (42) is rewritten in the form of (1) as follows.

\[
\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^q} g(x) + h(y) \quad \text{such that} \quad \phi(x) - y = 0. \tag{43}
\]

To illustrate the effect of using block surrogate functions in mADMM, let us consider the following specific example of (42)

\[
\min_{x=(x_1, x_2, x_3)} \frac{1}{q} \sum_{i=1}^{q} \log(1 + e^{-b_i(a_i, x_1)^2 + (a_i, x_2 + x_3)}) + \lambda \sum_{i=1}^{2} \|x_i\|_1, \tag{44}
\]

that is, \(h(y) = \frac{1}{q} \sum_{i=1}^{q} \log(1 + e^{-b_i y_i})\), \(g(x) = \lambda_1 \|x_1\|_1 + \lambda_2 \|x_2\|_1\) and \(\phi_i(x_1, x_2, x_3) = \langle a_i, x_1 \rangle^2 + \langle a_i, x_2 \rangle + x_3\), where \(\lambda_1\) and \(\lambda_2\) are regularizer parameters, \(a_i \in \mathbb{R}^d\) and \(b_i \in \{-1, 1\}\) are input data. Problem (44) has a form of a support vector machine problem \([15]\) that minimizes a logistic loss function with a nonlinear classifier and an \(l_1\) regularization.

4.1. Applying mADMM to solve Problem (44). The augmented Lagrangian for (43) is

\[
\mathcal{L}_\beta(x, y, \omega) = g(x) + h(y) + \sum_{i=1}^{q} \omega_i(\phi_i(x) - y_i) + \frac{\beta}{2} \sum_{i=1}^{q} (\phi_i(x) - y_i)^2.
\]

**Update of block \(x_1\).** Fix \(\omega\) and \(y\), and denote \(f_i(x) = \omega_i(\phi_i(x) - y_i) + \frac{\beta}{2} (\phi_i(x) - y_i)^2\). We have

\[
\nabla_{x_1} f_i(x) = 2\omega_i(a_i, x_1) a_i + 2\beta(\phi_i(x) - y_i) a_i + 4\beta(\langle a_i, x_1 \rangle)^2 a_i a_i^\top.
\]

This implies that

\[
\|\nabla_{x_1} f_i(x)\| \leq \|2(\omega_i - \beta y_i) a_i a_i^\top\| + 2\beta\|a_i a_i^\top\| \|\phi_i(x) + 2(\langle a_i, x_1 \rangle)^2\|
\]

\[
\leq 2\|\omega_i - \beta y_i\| \|a_i\|^2 + 2\beta\|a_i\|^2(\|a_i, x_2\| + x_3) + 3\|a_i\|^2 \|x_1\|^2
\]

\[
= 2\|\omega_i - \beta y_i\| + 2\|a_i, x_2\| + x_3) \|a_i\|^2 + 6\beta\|a_i\|^2 \|x_1\|^2
\]

\[
\leq 2\|a_i\|^2 \max \left\{\|\omega_i - \beta y_i\| + 2\|a_i, x_2\| + x_3), 3\beta\|a_i\|^2\right\} (1 + \|x_1\|^2).
\]

It follows from \([17, \text{Proposition 2.1}]\) that \(x_1 \mapsto f_i(x)\) is \(l_i\)-relative smooth to \(\mathcal{h}(x_1) = \frac{1}{4}\|x_1\|^2 + \frac{1}{2}\|x_1\|^2\) \(\tag{45}\)

with \(l_i = 2\|a_i\|^2 \max \left\{\|\omega_i - \beta y_i\| + 2\|a_i, x_2\| + x_3), 3\beta\|a_i\|^2\right\}\). Hence \(x_1 \mapsto \varphi_\beta(x_1, y, \omega)\) is \(\sum_{i=1}^{q} l_i\)-relative smooth to \(\mathcal{h}\) \((\varphi_\beta\) is defined in (2)). Denote \(l_1 = \sum_{i=1}^{q} l_i\). We use the Bregman surrogate:

\[
x_1^{k+1} \in \arg\min_{x_1} g_1(x_1) + \langle \nabla_{x_1} \varphi_\beta(x_1, x_2^k, x_3^k, y^k, \omega^k), x_1 - x_1^k \rangle + l_i D_\mathcal{h}(x_1, x_1^k)
\]

\[
= \arg\min_{x_1} \lambda_1 \|x_1\|^1 + \langle \nabla_{x_1} \varphi_\beta(x_1, x_2^k, x_3^k, y^k, \omega^k), x_1 - x_1^k \rangle + l_1 \mathcal{h}(x_1)
\]

\[
= \arg\min_{x_1} \lambda_1 \|x_1\|^1 + \langle \hat{c}, x_1 \rangle + l_1 \mathcal{h}(x_1),
\]

where \(\hat{c} = \nabla_{x_1} \varphi_\beta(x_1, x_2^k, x_3^k, y^k, \omega^k) - l_1 \nabla \mathcal{h}(x_1^k)\).
Lemma 1. Let $h$ be defined in (45). If $c \neq 0$, the update in (46) becomes

$$x_1^{k+1} = (s_1 + s_2)T(c)/\|T(c)\|_2,$$

where $s_1 = \sqrt{\frac{c}{2c_1} + \sqrt{\frac{1}{27} + \frac{c^2}{414}}}$, $s_2 = \sqrt{\frac{c}{2c_1} - \sqrt{\frac{1}{27} + \frac{c^2}{414}}}$, $c = \sqrt{\sum_{i=1}^{d}(|\tilde{c}_i| - \lambda_1)^2}$, and $T(c) = -((|c| - \lambda_1) + \text{sgn}(c))$. If $c = 0$, the solution of (46) is $x_1^{k+1} = (s_1 + s_2)x_1$, where $s_1$ and $s_2$ are determined as above with $c = -\lambda_1$, and $x_1$ is any vector with one component being 1 and the remaining components being 0.

Proof. See Appendix A.4. □

Update of block $x_2$. Fix $x_1$, $y$ and $\omega$. Note that $x_2 \mapsto \nabla_{x_2} f_i(x) = \omega_i a_i + \beta(\phi_i(x) - y_i) a_i$ is Lipschitz continuous with constant $\beta\|a_i\|^2$. Hence, $x_2 \mapsto \varphi_\beta(x,y,\omega)$ is $l_2$-smooth, with $l_2 = \beta\sum_{i=1}^q\|a_i\|^2$. Thus we use the Lipschitz gradient surrogate for updating $x_2$:

$$x_2^{k+1} = \arg\min_{x_2} \lambda_2\|x_2\|_1 + 1/2 \|\nabla_{x_2} \varphi_\beta(x_1^{k+1}, x_2^{k+1}, x_3^{k}, y^{k}, \omega^{k})\|_2^2 + l_2^2\|x_2 - x_2^k\|_2^2$$

$$= \text{Prox}_{\lambda_2\|x_1\|_1}\left(x_2^{k} - \frac{l_2}{2} \nabla_{x_2} \omega_i\varphi_\beta(x_1^{k+1}, x_2^{k}, x_3^{k}, y^{k}, \omega^{k})\right).$$

Update of block $x_3$. Similarly to the update of $x_2$, we use the Lipschitz gradient surrogate:

$$x_3^{k+1} = x_3^{k} - \frac{l_3}{2} \nabla_{x_3} \varphi_\beta(x_1^{k+1}, x_2^{k+1}, x_3^{k}, y^{k}, \omega^{k}),$$

where $l_3 = \beta q$. As noted in Remark 1, Condition 1 and Condition 2 are satisfied.

Update of $y$. We have

$$\nabla_{y} h(y) = -\frac{1}{q} \frac{b_i e^{-b_i y}}{1 + e^{-b_i y}} |_{i \in [q]}, \quad \nabla^2 h(y) = \frac{1}{q} \text{Diag}\left(\frac{b_i^2 e^{-b_i y}}{(1 + e^{-b_i y})^2} |_{i \in [q]}\right) \approx \frac{1}{4q} I.$$}

Hence $L_h = \frac{1}{4q}$. The update of $y$ is as in (14) (note that $B = -I$ and $b = 0$).

Choose $\beta$. We choose $\beta \geq 10L_h$ to satisfy (28).

4.2. Applying the method in [8]. Problem (44) can also be solved by the prox-linear method proposed in [8]

$$x^{k+1} \in \arg\min_{x} g(x) + \tilde{h}(\phi(x^k) + \nabla \tilde{\phi}(x^k)^T(x - x^k)) + \frac{1}{2\tau} \|x - x^k\|^2,$$

where $\tilde{h}(y) = 1/q \sum_{i=1}^q \log(1 + e^{-y_i}) + \tilde{\phi_i}(x_1, x_2, x_3) = b_i(\langle a_i, x_1 \rangle^2 + \langle a_i, x_2 \rangle + x_3)$ and $\tau = (L_h L_{\tilde{\phi}})^{-1}$.

The subproblem (47) does not have a closed-form solution, we thus use the accelerated proximal gradient method [21] to solve it approximately.

4.3. Numerical results for synthetic data sets. We take $\lambda_1 = 0.001$ and $\lambda_2 = 0.1$. We use the Matlab command rand to generate a random matrix $A \in \mathbb{R}^{d \times q}$ (the columns of $A$ are $a_i$, $i \in [q]$) and randsample to generate a random vector $b \in \mathbb{R}^q$, $b_i \in \{-1, 1\}$. The columns of $A$ are then normalized. For mADMM, we take $\beta = 2.5/q$. We run the algorithms with the same initial point generated by rand and with the same running time 15 seconds, 100 seconds and 300 seconds
Figure 1. Evolution of the fitting error with respect to the running time for 3 synthetic data sets with \((d,q) = (1000,100)\) (left), \((d,q) = (5000,1000)\) (middle) and \((d,q) = (10000,2000)\) (right).

Table 1. The final fitting error in solving Problem \((44)\) with synthetic data sets.

| \((d,q)\)          | mADMM     | prox-linear |
|--------------------|-----------|-------------|
| \((1000,100)\)     | 0.450111  | 0.668748    |
| \((5000,1000)\)    | 0.693019  | 0.693042    |
| \((10000,5000)\)   | 0.692809  | 0.694131    |

Table 2. The final fitting error of mADMM and prox-linear in solving \((44)\) with real data sets.

| Data set           | \((d,q)\)          | mADMM     | prox-linear |
|--------------------|--------------------|-----------|-------------|
| duke breast-cancer | \((7129,44)\)      | 0.440088  | 0.559264    |
| leukemia           | \((7129,38)\)      | 0.358154  | 0.455985    |
| colon-cancer       | \((2000,62)\)      | 0.33082   | 0.404217    |

for the size \((d,q) = (1000,100),(5000,1000)\) and \((10000,5000)\), respectively. We have tried different random initial points and observe that the results are similar. We report the evolution of the fitting error (i.e., the objective function of \((44)\)) with respect to time in Figure 1 and the final fitting error in Table 1. We observe from Figure 1 and Table 1 that mADMM converges faster than prox-linear and mADMM obtains better final fitting error than prox-linear.

4.4. Numerical results for real data sets. In this experiment, we test the algorithms on three data sets leukemia, duke breast-cancer and colon-cancer. The data sets are available at https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/. For each data set, we run each algorithm 30 seconds with the same random initial point (note that we have tried many random initial points and the results are similar). We set \(\lambda_1 = \lambda_2 = 0.001\) and normalize the column of the instance matrix \(A\) before running the algorithms. We report the evolution of the fitting error with respect to time in Figure 2 and the final fitting error in Table 2. The results observed from Table 2 and Figure 2 are consistent with the results for synthetic data sets: mADMM outperforms prox-linear, and, especially, mADMM obtained better final fitting errors than prox-linear.

5. Conclusion We have proposed mADMM, a multiblock alternating direction method of multipliers, for solving a class of multiblock nonconvex optimization problems with nonlinear...
coupling constraints. Subsequential convergence, iteration complexity and global convergence are studied for the proposed method. One flexible feature of mADMM is that it allows the usage of block surrogate functions in updating the primal variables. The flexibility and the advantage of this feature are illustrated through the application of mADMM to solve an $l_1$-regularized logistic regression problem with a nonlinear classifier. By choosing suitable block surrogate functions, subproblems for updating the block variables have closed-form solutions. The numerical results have shown certain efficacy of mADMM. We make an ending remark by proposing a potential future research direction, that is developing randomized/stochastic version of mADMM for solving Problem (43).

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Appendix

A. Technical proofs

A.1. Proof of Theory 1. The proof is similar to [12, Theorem 1]. It follows from Proposition 3 that if \( \{(x^k, y^k, \omega^k)\} \) converges to \( (x^*, y^*, \omega^*) \) then \( \{(x^{k+1}, y^{k+1}, \omega^{k+1})\} \) and \( \{(x^{k-1}, y^{k-1}, \omega^{k-1})\} \) also converge to \( (x^*, y^*, \omega^*) \). On the other hand, from (10), we have

\[
\tag{48}
 u_i(x_i^{k+1}, x_i^{k-1}, y_i^k, \omega^k) + g_i(x_i^{k+1}) \leq u_i(x_i, x_i^{k-1}, y_i^k, \omega^k) + g_i(x_i), \quad \forall x_i.
\]

Choose \( x_i = x_i^* \) in (48) and note that \( u_i(x_i, z) \) is continuous, we have \( \limsup_{n \to \infty} g_i(x_i^{k_n}) \leq g_i(x_i^*) \).

Furthermore, \( g_i(x_i) \) is l.s.c. Hence, \( g_i(x_i^{k_n}) \to g_i(x_i^*) \). Let \( k = k_n \to \infty \) in (48), for all \( x_i \) we have

\[
\varphi_\beta(x_i^*, y_i^*, \omega^*) + g_i(x_i^*) \leq u_i(x_i, x_i^*, y_i^*, \omega^*) + g_i(x_i) \leq \varphi_\beta((x_i, x_i^*), y_i^*, \omega^*) + \epsilon_i(x_i, x_i^*, y_i^*, \omega^*) + g_i(x_i).
\]

Hence, we have \( x_i^* \in \arg\min_{x_i} \varphi_\beta((x_i, x_i^*), y_i^*, \omega^*) + \epsilon_i(x_i, x_i^*, y_i^*, \omega^*) + g_i(x_i) \) since \( \epsilon_i(x_i^*, x_i^*, y_i^*, \omega^*) = 0 \). Thus, \( 0 \in \partial_{x_i} \left( \varphi_\beta(x_i^*, y_i^*, \omega^*) + \epsilon_i(x_i^*, x_i^*, y_i^*, \omega^*) + g_i(x_i^*) \right) \). Furthermore, \( \nabla_{x_i} \epsilon_i(x_i^*, x_i^*, y_i^*, \omega^*) = 0 \).

Hence, we have \( 0 \in \partial_{x_i} \mathcal{L}_\beta(x_i^*, y_i^*, \omega^*) \). Similarly, we can prove that \( 0 \in \partial_y \mathcal{L}(x^*, y^*, \omega^*) \). Moreover, we have \( \Delta \omega^k = \omega^k - \omega^{k-1} = \beta(\phi(x^k) + \mathcal{B}y^k) \to 0 \). Hence, \( \partial_{\omega} \mathcal{L}(x^*, y^*, \omega^*) = \phi(x^*) + \mathcal{B}y^* = 0 \). Finally, since we assume \( \partial F(x) = \partial_{x_1} F(x) \times \ldots \times \partial_{x_m} F(x) \), we have

\[
\partial \mathcal{L}_\beta(x, y, \omega) = \partial F(x) + \nabla \left( h(y) + \langle \omega, \phi(x) + \mathcal{B}y - b \rangle + \frac{\beta}{2} \| \phi(x) + \mathcal{B}y \|^2 \right)
= \partial_{x_1} \mathcal{L}_\beta(x, y, \omega) \times \ldots \times \partial_{x_m} \mathcal{L}(x, y, \omega) \times \partial_{y} \mathcal{L}(x, y, \omega) \times \partial_{\omega} \mathcal{L}(x, y, \omega).
\]

We conclude that \( 0 \in \partial \mathcal{L}_\beta(x^*, y^*, \omega^*) \). \( \square \)
A.2. Proof of Proposition 4. Since $\sigma_B > 0$ we have $B$ is surjective. On the other hand, since $\text{ran} \phi(x) \subseteq \text{Im}(B)$ we have there exist $\bar{y}^k$ such that $\phi(x^k) + B\bar{y}^k = 0$. We consider $k \geq 1$. Now we have

$$L^k = F(x^k) + h(y^k) + \frac{\omega}{2} \|\phi(x^k) + By^k\|^2 + \langle \omega, \phi(x^k) + By^k \rangle = F(x^k) + h(y^k) + \frac{\omega}{2} \|\phi(x^k) + By^k\|^2 + \langle B^* \omega, y^k - \bar{y}^k \rangle.$$  \hfill (50)

On the other hand, from (14) we have

$$\nabla h(u^k) + B^*(\omega + \beta(\phi(x^{k+1}) + By^{k+1})) + L_h(y^{k+1} - y^k) = 0.$$  \hfill (51)

Hence,

$$\langle B^* \omega, y^k - \bar{y}^k \rangle = \langle \nabla h(y^k) + L_h \Delta y^{k+1} + B^* \Delta w^{k+1}, \bar{y}^k - y^k \rangle \geq \langle \nabla h(y^k), \bar{y}^k - y^k \rangle - (L_h \| \Delta y^{k+1} \| + \| B^* \Delta w^{k+1} \|) \| \bar{y}^k - y^k \|.$$  \hfill (52)

Together with (50) and $L_h$-smooth property of $h$ we imply that

Moreover, we have

$$\|\bar{y}^k - y^k\|^2 \leq \frac{1}{\lambda_{\text{min}}(B^* B)} \| B(\bar{y}^k - y^k) \|^2 = \frac{1}{\lambda_{\text{min}}(B^* B)} \| \phi(x^k) + By^k \|^2 = \frac{1}{\lambda_{\text{min}}(B^* B)} \| \frac{1}{\beta} \Delta \omega^k \|^2.$$  \hfill (53)

On the other hand, Proposition 3 shows that $\| \Delta \omega^k \|$, $\| \Delta x^k \|$ and $\| \Delta y^k \|$ converge to 0, and from (29) we have $L^k$ is upper bounded. Therefore, (52) and (53) imply that $F(x^k) + h(\bar{y}^k)$ is upper bounded. So $\{x^k\}$ is bounded. Consequently, $\phi(x^k)$ is bounded. On the other hand, we have

$$\|y^k\|^2 \leq \frac{1}{\lambda_{\text{min}}(B^* B)} \| By^k \|^2 = \frac{1}{\lambda_{\text{min}}(B^* B)} \| \frac{1}{\beta} \Delta \omega^k - \phi(x^k) \|^2.$$  \hfill (54)

Hence, $\{y^k\}$ is bounded, which implies $\| \nabla h(y^k) \|$ is bounded. Finally, from (51), we have $\{\omega^k\}$ is bounded. \hfill \Box

A.3. Proof of Theorem 2. We do the analysis in the bounded set containing the generated sequence of mADMM. The Lyapunov sequence $\tilde{L}_\beta(z^k, y^{k-1})$ has the following properties.

(i) Sufficient decreasing property. We derive from (29) that

$$L^{k+1} + \frac{1}{2} \bar{\eta} \| \Delta x^{k+1} \|^2 + \frac{3\tilde{\delta}^2}{\beta \sigma_B} \| \Delta y^{k+1} \|^2 \leq L^k + \frac{3\tilde{\delta}^2}{\beta \sigma_B} \| \Delta y^k \|^2,$$

where $\bar{\eta} = \min_{i \in [m]} \eta_i$. Hence

$$\tilde{L}_\beta(z^{k+1}, y^k) + \frac{1}{2} \bar{\eta} \| \Delta x^{k+1} \|^2 + \frac{(\tilde{\delta} - 1)\tilde{\delta}^2}{\beta \sigma_B} \| \Delta y^{k+1} \|^2 \leq \tilde{L}_\beta(z^k, y^{k-1}).$$

(ii) Boundedness of subgradient. We have
\[ \partial_{x_i} \tilde{L}_\beta(z^{k+1}, y^k) = \partial_{x_i} F(x^{k+1}) + \nabla_{x_i} \phi(x^{k+1})(\omega^{k+1} + \beta(\phi(x^{k+1}) + B y^{k+1})) \]
\[ \nabla_\omega \tilde{L}_\beta(z^{k+1}, y^k) = \phi(x^{k+1}) + B y^{k+1}, \]
\[ \nabla_y \tilde{L}_\beta(z^{k+1}, y^k) = \nabla h(y^{k+1}) + B^* (\omega^{k+1} + \beta(\phi(x^{k+1}) + B y^{k+1}))) + \frac{\delta^2}{\beta \sigma_B} (y^{k+1} - y^k), \]
\[ \nabla_{y} \tilde{L}_\beta(z^{k+1}, y^k) = \frac{\delta^2}{\beta \sigma_B} (y^k - y^{k+1}). \]

On the other hand, Proposition 5 showed that there exists \( \lambda_i^k \in \partial_{x_i} F(x^k) \) such that (35) holds. Therefore, it is not difficult to prove that there exist \( \bar{\lambda}^{k+1} \in \partial \tilde{L}_\beta(x^{k+1}, y^{k+1}, \omega^{k+1}, y^k) \) such that
\[ \| \bar{\lambda}^{k+1} \| \leq a_1 \| \Delta x^{k+1} \| + a_2 \| \Delta y^{k+1} \| + a_3 \| \Delta y^k \| \]
for some positive constants \( a_1, a_2, a_3. \)

(iii) KL property. We assume that \( \hat{L} \) has the KL property with constant \( \sigma_{\hat{L}}. \)

(iv) A continuity property. Suppose a subsequence \( (z^k) \rightarrow (x^*, y^*, \omega^*). \) Proposition 3 showed that \( y^{k+1} \rightarrow y^*. \) Moreover, in the proof of Theorem 1 we proved that \( g_i(x_i^{k+1}) \rightarrow g_i(x_i^*) \). Therefore, \( \mathcal{L}_\beta(x^{k+1}, y^{k+1}, \omega^{k+1}) \rightarrow \mathcal{L}_\beta(x^*, y^*, \omega^*). \) Consequently, \( \tilde{L}_\beta(z^{k+1}, y^{k+1}) \rightarrow \tilde{L}_\beta(x^*, y^*, \omega^*, y^*). \)

We can prove \( \sum_{k=1}^{\infty} \| \Delta x^{k+1} \| + \| \Delta y^{k+1} \| + \| \Delta y^k \| < \infty \) by using the above properties and the same techniques of [4, Theorem 1] (as this is typical technique, see e.g., [12, 30], we omit the details). Then \( x^k \rightarrow x^* \) and \( y^k \rightarrow y^*. \) Moreover, from (35), we have \( R_s^k = \frac{1}{\beta} \| \Delta \omega^k \| = O(\| \Delta y^k \| + \| \Delta y^{k-1} \|). \)

Hence \( \sum_{k=1}^{\infty} \| \Delta \omega^k \| < \infty \), leading to \( \omega^k \rightarrow \omega^*. \) Finally, we use the same techniques of [3, Theorem 2] to obtain the convergence rate for \( \{z^k\}_{k \geq 1}. \)

A.4. Proof of Lemma 1. Suppose \( \hat{c} \neq 0. \) We have
\[ \min_{x_1} \{ \lambda \| x_1 \| + (\tilde{c}, x_1) : ||x_1||_2 = 1 \} \]
\[ \overset{(a)}{=} \min_{x_1} \{ \lambda \| x_1 \| + (\tilde{c}, x_1) : ||x_1||_2 \leq 1 \} \]
\[ \overset{(b)}{=} -\sqrt{\sum_{i=1}^{d} (\tilde{c}_i - \lambda_1)_+^2} \]
and is solved by \( x_1^* = T(\tilde{c})/||T(\tilde{c})||_2. \) Here we used [18, Proposition 4.6] for (b) and the fact \( ||x_1^*||_2 = 1 \) for (a). On the other hand, we have
\[ \min_{x_1 \in \mathbb{R}^d} \{ \lambda_1 || x_1 ||_1 + (\tilde{c}, x_1) + \frac{\lambda}{2} || x_1 ||_2^2 + \frac{\lambda}{2} || x_1 ||_2^2 \} \]
\[ \overset{(a)}{=} \min_{x_1 \in \mathbb{R}^d, t \in \mathbb{R}_+} \{ \lambda_1 || x_1 ||_1 + (\tilde{c}, x_1) + \frac{\lambda}{4} t^4 + \frac{\lambda}{2} t^2 : t^2 = ||x_1||_2^2 \} \]
\[ \overset{(b)}{=} \min_{t \in \mathbb{R}_+} \left\{ \frac{\lambda}{4} t^4 + \frac{\lambda}{2} t^2 + \min_{x_1} \{ \lambda_1 || x_1 ||_1 + (\tilde{c}, x_1) : ||x_1||_2^2 = t^2 \} \right\} \]
\[ \overset{(c)}{=} \min_{t \in \mathbb{R}_+} \left\{ \frac{\lambda}{4} t^4 + \frac{\lambda}{2} t^2 - t c \right\}, \]
where we have used (54) for (c). Note that \( t^* = s_1 + s_2 \) is the solution of the last minimization problem in (55)(which is the nonnegative real solution of the cubic equation \( t_1(t^*)^3 + t_1 t^* - c = 0 \)).

When \( \tilde{c} = 0 \), note that \( \min_{x_1} \{ \lambda_1 || x_1 ||_1 : ||x_1||_2^2 = 1 \} = \lambda_1 \) and the optimal value is obtained at any point \( x_1^* \) that has only one component being 1 and the remaining components being 0. \( \square \)
B. An example Suppose $f(x) = \sum_{i=1}^{m} \hat{f}_i(\|x_i\|_2)$, where $\hat{f}_i$ is a continuously differentiable concave function with Lipschitz gradient on any given bounded set. This covers many nonconvex regularizers of low rank representation problems, see e.g., [16]. Since $\hat{f}_i$ is concave, we have

$$\hat{f}_i(\|x_i\|_2) \leq \hat{f}_i(\|\bar{x}_i\|_2) + \nabla \hat{f}_i(\|\bar{x}_i\|_2)(\|x_i\|_2 - \|\bar{x}_i\|_2).$$

Hence $u_i$ defined in the following is a block surrogate function of $\varphi_\beta$

$$u_i(x_i, \bar{x}, y, \omega) = f(\bar{x}_i, x_i) + \nabla \hat{f}_i(\|\bar{x}_i\|_2)(\|x_i\|_2 - \|\bar{x}_i\|_2) + \hat{u}_i(x_i, \bar{x}, y, \omega),$$

where $\hat{u}_i(x_i, \bar{x}, y, \omega)$ is a surrogate of $\varphi_\beta(x, y, \omega) = \langle \omega, \phi(x) + B y \rangle + \frac{\beta}{2} \|y\|^2$ with respect to block $x_i$. Assume $\hat{u}_i$ is twice continuously differentiable and $\nabla_{x_i} \hat{u}_i(x_i, x, y, \omega) = \nabla_{x_i} \hat{\varphi}_\beta(x, y, \omega)$ for all $x, y, \omega$. Note that

$$\partial_{x_i} u_i(x_i, \bar{x}, y, \omega) = \nabla \hat{f}_i(\|\bar{x}_i\|_2)\partial(\|x_i\|_2) + \nabla_{x_i} \hat{u}_i(x_i, \bar{x}, y, \omega).$$

Hence, any subgradient in $\partial_{x_i} u_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k)$ has the form

$$S^k_i = \nabla \hat{f}_i(\|x_i^k\|_2)\xi^k_i + \nabla_{x_i} \hat{u}_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k),$$

where $\xi^k_i \in \partial(\| \cdot \|_2)(x_i^{k+1})$. Moreover, it follows from [25, Corollary 5Q] that $\partial \hat{f}_i(x_i) = \nabla f_i(\|x_i\|_2)\partial(\|x_i\|_2)$. Thus we take $\bar{S}^k_i = \nabla f_i(\|x_i^{k+1}\|_2)\xi^k_i + \nabla_{x_i} \hat{\varphi}_\beta(x^{k+1}, y^k, \omega^k) \in \partial_{x_i} \varphi(x^{k+1}, y^k, \omega^k)$. Assuming $(x^k, y^k, \omega^k)$ is bounded, we have

$$\|S^k_i - \bar{S}^k_i\| = \|(\nabla \hat{f}_i(\|x_i^k\|_2) - \nabla f_i(\|x_i^{k+1}\|_2))\xi^k_i + \nabla_{x_i} \hat{u}_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k) - \nabla_{x_i} \hat{u}_i(x_i^{k+1}, x_i^{k,i-1}, y^k, \omega^k)\| \leq L_{\hat{f}_i}(\|x_i^{k+1} - x_i^k\|_2)\|\xi^k_i\| + \bar{L}_{\hat{f}_i}\|x_i^{k+1} - x_i^k\|.$$