Weak approximation of an invariant measure and a low boundary of the entropy, II

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Abstract

For a measurable map $T$ and a sequence of $T$-invariant probability measures $\mu_n$ that converges in some sense to a $T$-invariant probability measure $\mu$, an estimate from below for the Kolmogorov–Sinai entropy of $T$ with respect to $\mu$ is suggested in terms of the entropies of $T$ with respect to $\mu_1$, $\mu_2$, . . . . This result is obtained under the assumption that some generating partition has finite entropy. By an explicite example it is shown that, in general, this assumption cannot be removed.

Keywords: invariant measure, metric entropy, Markov shift

1 Introduction

In problems of ergodic theory and thermodynamic formalism it is sometimes necessary to estimate the entropy of a measure preserving map. If this map acts in a compact metric space and is expansive, one can use the fact that the entropy is semi-continuous from above on the space of invariant probability measures with weak topology. Both conditions — the compactness and expansiveness are essential in this context, and if at least one of them fails, one has to use other tools. Two such tools are suggested in this paper (see theorems 1 and 5). In both cases a countable generating partition appears. In Theorem 1 we assume that the entropy of this partition with respect to the measure of interest is finite and in Section 3 we show that this assumption cannot be dropped.

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Although we give up topological assumptions, primarily the compactness, the results of this type turn out to be useful for smooth dynamics (see, e.g., [2]).

We use standard notation, terminology and results of entropy theory (see, e.g., [1] – [4], [6]). Let $T$ be an automorphism of a measurable space $(X, F)$ and $\mu$ a $T$-invariant probability measure. For a countable partition $\eta$ of $X$ we write $B \in \eta$ and $B \subset \eta$ if the set $B$ is an atom of $\eta$ or a union of such atoms, respectively. We call $\eta$ a generating partition (or a generator) for $(X, F, T, \mu)$ (usually just for $(T, \mu)$) if the $\sigma$-algebra generated by the partitions $T^n \eta$, $i \in \mathbb{Z}$, coincides $\mu$-mod 0 with $F$. In what follows we usually omit $F$ from the notation.

Theorem 1. Assume that for a countable measurable partition $\xi$ of $X$ and a $T$-invariant probability measure $\mu$, the entropy $H_{\mu}(\xi)$ is finite and that there exist a sequence of $T$-invariant probability measures $\mu_n$ and sequences of numbers $r_n \in \mathbb{N}$, $\varepsilon_n > 0$ such that

$$\lim_{n \to \infty} r_n = \infty, \quad \lim_{n \to \infty} \varepsilon_n = 0,$$

$$|\mu(A) - \mu_n(A)| \leq \varepsilon_n \mu_n(A) \text{ for all } A \in \vee_{i=0}^{r_n} T^{-i} \xi, \ n \geq 1,$$

$$\xi \text{ is a generator for } (T, \mu) \text{ and } (T, \mu_n), n \geq 1.$$ (1.3)

Then

$$h_{\mu}(T) \geq \limsup_{n \to \infty} h_{\mu_n}(T).$$ (1.4)

This theorem will be proved in Section 2. In Section 3 we show that the assumption $H_{\mu}(\xi) < \infty$ cannot be in general omitted and in Section 4 establish another estimate for $h_{\mu}(T)$ without the requirement that $H_{\mu}(\xi) < \infty$.

## 2 Proof of theorem

We begin the proof of Theorem 1 with two simple lemmas.

**Lemma 2.** Let $p := (p_i)_{i \in \mathbb{N}}$, $q := (q_i)_{i \in \mathbb{N}}$, where $p_i, q_i \geq 0$ for all $i$ and $\sum_i p_i = \sum_i q_i = 1$. Let also $H(p) := -\sum_{i \in \mathbb{N}} p_i \ln p_i$ (with $0 \ln 0 = 0$) and $H(q)$ be defined similarly. Assume that for some $c \in (0, 1/3)$,

$$|p_i - q_i| \leq cq_i, \ i = 1, 2, \ldots.$$ (2.1)

Then

$$H(p) \leq (1 + c)H(q) + c \ln 3.$$
Proof. Denote \( \varphi(t) := -t \ln t, \ t \geq 0 \). It is clear that (a) \( \varphi(t) \) increases when \( 0 \leq t \leq e^{-1} \), (b) \( \varphi(t) \leq 0 \) when \( t \geq 1 \), (c) \( -1 \leq \varphi'(t) \leq \ln 3 \) when \( (3e)^{-1} \leq t \leq 1 \). Hence (see also (2.1))

\[
H(p) = \sum_{i \in \mathbb{N}} \varphi(p_i) = \sum_{i \geq 1/2e} \varphi(p_i) + \sum_{i > 1/2e} \varphi(p_i) \leq \sum_{i \geq 1/2e} \varphi((1+c)p_i)
+ \sum_{i > 1/2e} \left[ \varphi(p_i) + |p_i - q_i| \ln 3 \right] = \sum_{i \leq 1/2e} \left[ q_i \varphi(1+c) + (1+c)\varphi(q_i) \right]
+ \sum_{i > 1/2e} \left[ \varphi(q_i) + |p_i - q_i| \ln 3 \right] \leq (1+c) \sum_{i \in \mathbb{N}} \varphi(q_i) + c \ln 3
= (1+c)H(q) + c \ln 3.
\]

\[\square\]

Lemma 3. If \( \eta \) is a countable measurable partition of the space \( X \) and if, for probability measures \( \nu \) and \( \nu \) on \( (X,F) \), every \( A \in \eta \) and some \( \varepsilon > 0 \), we have \( |\mu(A) - \nu(A)| \leq \varepsilon \nu(A) \), then the same is true for every \( A \subset \eta \).

The proof is evident and is omitted.

We continue the proof of Theorem [1]. For all \( k,l \in \mathbb{Z}, k \leq l \), we denote \( \xi_k^l(T) := \bigvee_{i=k}^l T^i \xi \).

It is known \([6, 4]\) that if \( H_{\mu_0}(\xi) < \infty \) and (1.3) holds for \( n = 0 \), then

\[
h_{\mu_0}(T) = \lim_{n \to \infty} \frac{1}{n} H_{\mu_0}(\xi_{-n}^{-1}(T)),
\]

and the sequence on the right hand side is non-increasing.

It easy to verify that if \( \varepsilon \leq 1/2 \), then \( |a-b| \leq \varepsilon b, a,b \geq 0 \) imply that \( |a-b| \leq 2\varepsilon a \). Therefore by (1.2) for all \( n \) such that \( \varepsilon_n \leq 1/2 \) and all \( A \in \xi(-r_n,0) \), we have

\[
|\mu_0(A) - \mu_n(A)| \leq 2\varepsilon_n \mu_0(A).
\]

(2.2)

For such \( n \) we compare \( H_{\mu_0}(\xi_{-r_n}^{-1}(T)) \) and \( H_{\mu_n}(\xi_{-r_n}^{-1}(T)) \).

An arbitrary numbering of the atoms \( A \in \xi_{-r_n}^{-1}(T) \) yields a sequence \( A_1, A_2, \ldots \). Let \( p := \mu_n(A_i), q := \mu_0(A_i) \). By applying Lemma 2 with \( c = 2\varepsilon_n \) (see (2.2)) we obtain

\[
H_{\mu_n}(\xi_{-r_n}^{-1}(T)) \leq (1+2\varepsilon_n)H_{\mu_0}(\xi_{-r_n}^{-1}(T)) + 2\varepsilon_n \ln 3.
\]

For all sufficiently large \( n \), this implies that

\[
h_{\mu_n}(T, \xi) \leq \frac{1}{r_n} H_{\mu_n}(\xi_{-r_n}^{-1}(T))
\leq \frac{1}{r_n}(1+2\varepsilon_n)H_{\mu_0}(\xi_{-r_n}^{-1}(T)) + \frac{2}{r_n}\varepsilon_n \ln 3,
\]

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or

\[ h_{\mu_n}(T, \xi) \leq \frac{1}{r_n}(1 + 2\varepsilon_n)\nu(A_{-r_n}^{-1}(T)) + \frac{2}{r_n}\varepsilon_n \ln 3. \]

Therefore (see (1.1))

\[ h \leq \limsup_{n \to \infty} h_{\mu_n}(T, \xi) \leq h_{\mu_0}(T). \]

The proof is completed.

3 Pitskel’s example

Here we use an example constructed by B. Pitskel [5] for a different purpose. Let \( Y = \mathbb{N}^\mathbb{Z} \) be the sequence space equipped with the cylinder \( \sigma \)-algebra \( C \) and \( \tau \) the right one step shift on \( Y \) defined by \( \tau y = y' \), where

\[ y = (y_i, \ i \in \mathbb{Z}), \ y' = (y_i', \ i \in \mathbb{Z}), \ y'_i = y_i-1, \ i \in \mathbb{Z}. \]

We denote the partition of \( Y \) into the one-dimensional cylinders \( C_k = \{ y \in Y : y_0 = k \}, \ k \in \mathbb{N}, \ \text{by} \ \eta \) and introduce the \( \tau \)-invariant product measure \( \nu \) on \( Y \) by \( \nu(C_k) = 1/2^k \). Then \((Y, \nu, \tau)\) is a Bernoulli shift and \( h_\nu(\tau) = H_\nu(\eta) < \infty. \)

We now define a partition \( \zeta > \eta \) as follows: for every \( k \in \mathbb{N} \), each atom of \( \zeta \) lying in \( C_k \) is an atom of \( \eta_{\leq 2^k}(\tau) \).

One can easily check that \( H_\nu(\zeta|((\zeta^{-1})_n(\tau)) = H_\nu(\eta) = h_\nu(\tau) \). But the following is true.

Proposition 4. For every \( n \in \mathbb{N}, \)

\[ H_\nu(\zeta|((\zeta^{-1})_n(\tau)) = \infty. \quad (3.1) \]

Proof. By definition, for an arbitrary atom \( A \) of \( \zeta^{-1}_n(\tau) \), we have \( A = \tau^{-1}A_1 \cap \tau^{-2}A_2 \cap \cdots \cap \tau^{-n}A_n \), where \( A_i \) is an atom of \( \zeta \) and \( A_i \subset C_{k_i} \) for some \( k_i \in \mathbb{N} \). Moreover, for each \( i \),

\[ A_i = C_{m_i(0)} \cap \tau^{-1}C_{m_i(1)} \cap \cdots \cap \tau^{-2^{k_i}}C_{m_i(2^{k_i})}, \quad (3.2) \]

where \( m_i(0) = k_i \) and \( m_i(j) \in \mathbb{N}, \ 1 \leq j \leq 2^{k_i} \). Hence

\[ \tau^{-l}A_i = \tau^{-l}C_{m_i(0)} \cap \tau^{-i-l}C_{m_i(1)} \cap \cdots \cap \tau^{-i-2^{k_i}}C_{m_i(2^{k_i})}, \ 1 \leq i \leq n. \quad (3.3) \]

In order to evaluate \( H(\zeta|((\zeta^{-1})_n(\tau))_n) \), we will describe the partition induced by \( \zeta \) on \( A \). Let

\[ l(A) := \max_{1 \leq i \leq n}(i + 2^{k_i}). \]

Then \( A \) is a cylinder defined by fixing the coordinates \( y_r \) with \(-l(A) \leq r \leq -1. \) For every atom \( A' \) of \( \zeta \), we have \( A' \subset C_{k'}, \) where \( k' \in \mathbb{N} \). Then \( A' \) is a cylinder defined by fixing coordinates \( y_r \) with \(-2^{k'} \leq r \leq 0. \) Therefore, if \( 2^{k'} \leq l(A) \), then

\[ A \cap A' = \emptyset \text{ or } A \cap C_{k'}, \quad (3.4) \]
and if $2^{k'} > l(A)$, then

$$A \cap A' = 0 \text{ or } A \cap C_{m'}(0) \cap \tau^{-l(A)-1}C_{m'}(1) \cap \cdots \cap \tau^{-2^{k'}}C_{m'}(2^{k'}),$$

(3.5)

where $m'(0) = k'$, $m'(r) \in \mathbb{N}$ for $1 \leq r \leq 2^{k'}$ (see (3.2), (3.3)).

By definition

$$H_\nu \left( \zeta | \zeta_{-n}^{-1}(\tau) \right) = \sum_{A \in \zeta_{-n}^{-1}(\tau)} \nu(A) H_\nu(\zeta | A).$$

(3.6)

Using (3.5), the independence of the partitions $\tau^i \eta$ for different indices $i$, and standard properties of entropy, we obtain

$$H_\nu(\zeta | A) \geq \sum_{k \geq 1} \mu(C_k | A) H_\nu(\zeta | C_k \cap A) \geq \sum_{k : 2^k > l(A)} \nu(C_k | A) H_\nu(\zeta | C_k \cap A)$$

$$= \sum_{k : 2^k > l(A)} \frac{1}{2^k} H_\nu \left( \eta_{-2^k}(A)^{-1}(\tau) \right) = \sum_{k : 2^k > l(A)} \frac{1}{2^k} \left( 2^k - l(A) \right) H_\nu(\eta) = \infty.$$ (3.7)

Now (3.6) and (3.7) yield (3.1).

We now label the atoms of $\zeta$ by positive integers, introduce the set $X := \mathbb{N}^2$ and, for every point $y \in Y$, write $\varphi(y)_i = k$ if $\tau^{-1}y$ belongs to the atom of $\zeta$ labeled by $k$, $k \in \mathbb{N}$, $i \in \mathbb{Z}$. The mapping $\varphi : Y \to X$ so defined is clearly an embedding. Let $T$ be the shift on $X$ (similar to $\tau$) and $\mu := \varphi \circ \nu$. Then the dynamical systems $(Y, \nu, \tau)$ and $(X, mu, T)$ are isomorphic. Hence

$$h_\mu(T) = h_\nu(\tau) = H_\nu(\eta) < \infty.$$ (3.8)

Let $\xi$ be the partition of $X$ into the sets $\{x \in X : x_0 = k\}$, $k \in \mathbb{N}$ (the 1-dimensional cylinders in $X$ with support $\{0\}$). For $n = 1, 2, \ldots$ we define a measure $\mu_n$ on $X$ such that

(a) it is $T$-invariant,

(b) it satisfies $\mu_n(A) = \mu(A)$ for all $A \in \xi_0^{-m}(T)$,

(c) it is a Markov measure of order $n$.

Property (c) means that, for all $A \in \xi$, all $m \in \mathbb{N}$, and all $A' \in \xi_{-n-m}^{-1}(T)$ with $\mu(A') > 0$, we have $\mu_n(A|A') = \mu_n(A|A'')$, where $A''$ is the atom of $\xi_{-n}^{-1}(T)$ containing $A$. It is easy to check that there exists a unique measure with properties (a) – (c). We call it the $n$-Markov hull of $\mu$.

From (3.7) it follows that

$$H_{\mu_n}(\zeta | \zeta_{-\infty}^{-1}(T)) = H_{\mu_n}(\zeta | \zeta_{-1}^{-1}(T)) = H_\nu(\zeta | \zeta_{-n}^{-1}(\tau)) = \infty.$$ (3.9)

We claim that

$$H_{\mu_n}(\zeta | \zeta_{-\infty}^{-1}(\tau)) = h_{\mu_n}(T).$$ (3.10)

Indeed, if the entropy $H_{\mu_n}(\xi)$ of the generator $\xi$ for $(T, \mu_n)$ were finite, (3.10) would be a standard property of the entropy. Otherwise it does not generally hold.
However, Pitskel [5] observed that (3.10) does hold for every Markov measure of order 1, be the entropy of the generator finite or not. From this it follows that (3.10) holds for Markov measures of all orders, so that $h_{\mu_n}(T) = \infty$ for all $n$.

On the other hand, it is seen that if we put $r_n = n$, then conditions (1.1) – (1.3) will hold for any $\varepsilon_n \to 0$. However, by (3.8) – (3.10) the conclusion (1.4) fails.

4 Another estimate for $h_{\mu_0}(T)$

We number in an arbitrary way the atoms of $\xi$, pick an increasing sequence of positive integers $q(m), m = 1, 2, \ldots$, and replace all the atoms with labels $\geq q(m)$ by their union. The partition thus obtained will be denoted by $\xi_m$ and its atoms by $A_1, \ldots, A_{q(m)}$.

**Theorem 5.** If all the conditions of Theorem 2 with one possible exception, namely, $h_{\mu}(\xi) < \infty$, are fulfilled, then

$$h_{\mu}(T) \geq \limsup_{m \to \infty} \limsup_{n \to \infty} h_{\mu_n}(T, \xi_m). \quad (4.1)$$

**Proof.** We fix $m$ and introduce the sequence space $Y_m := \{1, \ldots, q(m)\}^\mathbb{Z}$ with a standard metric $\rho_m$, the Borel $\sigma$-algebra $B_m$, and the one step right shift $\sigma_m$ on $Y_m$. Then we define a mapping $\psi_m : X \to Y_m$ by $\psi_m x = y$, where $y_n = i$ provided that $T^{-n}x \in A_i$. It is easy to check that $\psi_m T x = \sigma_m \psi_m x$ for all $x \in X$.

For every $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ consider the probability measures $\mu^m := \psi_m \circ \mu$ and $\mu^m_n := \psi_m \circ \mu_n$ on $(Y_m, B_m)$. From (1.2) it follows that $\mu_n^m$ converges weakly to $\mu^m$ as $n \to \infty$. Since the space $(Y_m, \rho_m)$ is compact and $\sigma_m$ is expansive, we have

$$h_{\mu^m}(\sigma_m) \geq \limsup_{n \to \infty} h_{\mu^m_n}(\sigma^m), \quad m \in \mathbb{N}.$$ 

But it is clear that

$$h_{\mu^m}(\sigma_m) = h_{\mu}(T, \xi_m), \quad h_{\mu^m_n}(\sigma_m) = h_{\mu_n}(T, \xi_m), \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}^+.$$ 

Hence

$$h_{\mu}(T, \xi_m) \geq \limsup_{n \to \infty} h_{\mu_n}(T, \xi_m), \quad m \in \mathbb{N}.$$ 

From well-known properties of entropy it follows that $h_{\mu}(T, \xi_m)$ tends to $h_{\mu}(T)$ as $m \to \infty$. This implies (4.1). \hfill \Box

In conclusion we give a simple example where the assumptions of Theorem 5 are satisfied and this theorem yields the correct value of $h_{\mu}(T)$.

**Example 6.** Let $Y, C, \tau$, and $\eta$ be as in Section 3, and let $\nu$ be a $\tau$-invariant probability measure on $(Y, C)$. Consider a function $f : Y \to \mathbb{Z}_+$ with $\nu(f) < \infty$ such that $f$ is constant on each atom of $\eta$. The integral (suspension) automorphism
constructed by \( \tau \) and \( f \) acts on the space \( \hat{Y} := \{(y, u) : x \in Y, u \in \mathbb{Z}_+, u \leq f(x)\} \) by the formulas \( \hat{\tau}(y, u) = (y, u + 1) \) if \( u < f(y) \) and \( \hat{\tau}(y, u) = (\tau y, 0) \) if \( u = f(y) \). Clearly \( \hat{\tau} \) preserves the probability measure \( \hat{\nu} := \frac{1}{\nu(f)^i}(\nu \times \lambda)_{\hat{Y}} \), where \( \lambda \) is the counting measure on \( \mathbb{Z}_+ \).

Let \( \eta = \{C_1, C_2, \ldots \} \) and \( f(y) = f_i \) when \( y \in C_i \). By construction \( \sum_{n=1}^{\infty} f_i \nu(C_i) < \infty \). We may also assume that \( \sum_{n=1}^{\infty} f_i \nu(C_i) \log \nu(C_i) = \infty \), where \( \hat{\eta} \) is the partition of \( \hat{Y} \) whose atoms are \( C_{i,k} = \{(y, u) : y \in C_i, u = k\} \), \( i \in \mathbb{N}, k = 0, 1, \ldots, f_i \). Clearly \( \hat{\xi} \) is a countable generator for \((\hat{\nu}, \hat{\tau})\).

We denote the \( n \)-Markov hull of \( \mu \) (see Section 3 for a definition) by \( \hat{\mu}_n \) and define a measure \( \hat{\mu}_n \) on \( \hat{X} \) by

\[
\hat{\mu}_n|_{\hat{T}^i C_k} := \frac{1}{\mu(f) + 1} \hat{T}^i \circ (\mu|_{C_k}), \quad k \in \mathbb{N}, \quad 0 \leq i \leq f_k.
\]

Let \( \hat{\xi}_m \) be the partition of \( \hat{X} \) whose atoms are \( C_{i,k}, 1 \leq i \leq m, 0 \leq k \leq f_i \), and the union of \( C_{i,k} \) over all \( i > m \) and \( 0 \leq k \leq f_i \).

Taking \( \hat{X}, \hat{\mu}, \hat{T}, \hat{\xi} \), and \( \hat{r}_m := \sum_{i=1}^{m}(f_i+1) \) for \( X, \mu, T, \xi, \) and \( r_m \), respectively, in Theorem 5 one can easily check that the condition of this theorem are satisfied for any \( \varepsilon_n \to 0 \).

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