IN INVARIANCE THEOREMS FOR NEVANLINNA FAMILIES

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Abstract. A complex function \( f(z) \) is called a Herglotz-Nevanlinna function if it is holomorphic in the upper half-plane \( \mathbb{C}_+ \) and maps \( \mathbb{C}_+ \) into itself. By a maximum principle a Herglotz-Nevanlinna function which takes a real value \( a \) in a single point \( z_0 \in \mathbb{C}_+ \) should be identically equal to \( a \). In the present note we prove similar invariance results both for the point and the continuous spectra of an operator-valued Herglotz-Nevanlinna function with values in the set of bounded or unbounded linear operators (or relations) in a Hilbert space. The proof of this invariance result for continuous spectrum is based on Harnack’s inequality. This inequality is systematically used to characterize operator-valued Herglotz-Nevanlinna functions with form-domain invariance property for their imaginary parts or Herglotz-Nevanlinna functions with values in the Schatten-von Neumann classes.

1. Introduction

The class of Herglotz-Nevanlinna functions plays an important role in function theory, probability theory, mathematical physics, etc. In particular, the m-function of a Sturm-Liouville operator on a half-line belongs to this class; \([33], [8]\). Similarly, the M-function of an elliptic operators is an operator-valued Herglotz-Nevanlinna function; see \([1]\). Also the Krein’s formula for (generalized) resolvents involves an another source for various important applications of this class. In particular, Krein’s formula allows to parametrize sets of solutions of various classical interpolation and moment problems with a parameter ranging over the class of Herglotz-Nevanlinna families; cf. \([24], [26]\). For basic properties of Herglotz-Nevanlinna functions see e.g. the survies in \([22], [7], [17]\).

The class \( R[\mathcal{H}] \) of Herglotz-Nevanlinna functions with values in the set \( \mathcal{B}(\mathcal{H}) \) of bounded linear operators in a separable Hilbert space \( \mathcal{H} \) is defined as follows.

Definition 1.1. An operator-valued function \( F(z) \) holomorphic on \( \mathbb{C} \setminus \mathbb{R} \), with values in \( \mathcal{B}(\mathcal{H}) \) is said to belong to the class \( R[\mathcal{H}] \), if:

(i) for every \( z \in \mathbb{C}_+ \setminus \mathbb{R} \) the operator \( F(z) \) is dissipative (resp. accumulative);
(ii) \( F(z)^* = F(\bar{z}) \), \( z \in \mathbb{C} \setminus \mathbb{R} \);

In what follows an operator \( T \in \mathcal{B}(H) \) is called dissipative (resp. accumulative), if its imaginary part

\[
\text{Im} (T) = \frac{1}{2i} (T - T^*)
\]

is a nonnegative (resp. nonpositive) operator in \( \mathcal{H} \); cf. \([23], [30]\).

Each operator-valued function \( F \in R[\mathcal{H}] \) admits the following integral representation

\[
F(z) = B_0 + B_z + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\Sigma(t),
\]

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where $B_0 = B_0^* \in \mathcal{B}(\mathcal{H})$, $0 \leq B_1 = B_1^* \in \mathcal{B}(\mathcal{H})$, and $\Sigma(\cdot)$ is a $\mathcal{B}(\mathcal{H})$-valued operator measure, such that
\[
K_{\Sigma} := \int_{\mathbb{R}} \frac{d\Sigma(t)}{t^2 + 1} \in \mathcal{B}(\mathcal{H}).
\]
Here integral in (1.2) is uniformly convergent in $\mathcal{B}(\mathcal{H})$; cf. [7], [22], [19].

The next result summarizes some invariance results on the spectra properties of operator-valued functions $F \in R[\mathcal{H}]$ (cf. [16, Proposition 1.2]).

**Theorem 1.2.** Let $F \in R[\mathcal{H}]$, $z_0 \in \mathbb{C}_+$ and $a = \bar{a}$. Then the following equivalences hold:

1. $0 \in \sigma_\rho(\text{Im} \,(F(z_0))) \iff 0 \in \sigma_\rho(\text{Im} \,(F(z)))$ for all $z \in \mathbb{C}_+$;
2. $0 \in \sigma_c(\text{Im} \,(F(z_0))) \iff 0 \in \sigma_c(\text{Im} \,(F(z)))$ for all $z \in \mathbb{C}_+$;
3. $0 \in \rho(\text{Im} \,(F(z_0))) \iff 0 \in \rho(\text{Im} \,(F(z)))$ for all $z \in \mathbb{C}_+$;
4. $a \in \sigma_p(F(z_0)) \iff a \in \sigma_p(F(z))$ for all $z \in \mathbb{C}_+$;
5. $a \in \sigma_s(F(z_0)) \iff a \in \sigma_s(F(z))$ for all $z \in \mathbb{C}_+$;
6. $a \in \rho(F(z_0)) \iff a \in \rho(F(z))$ for all $z \in \mathbb{C}_+$.

The following two subclasses of the class $R[\mathcal{H}]$ appear in the theory of $Q$-functions of symmetric operators, [27], and in the boundary triplet approach to the extension theory of symmetric operators, [11], [15].

\[
R^s[\mathcal{H}] = \{ F(\cdot) \in R[\mathcal{H}] : \ker \text{Im} \, F(z) = \{0\} \text{ for all } z \in \mathbb{C} \setminus \mathbb{R} \};
\]
\[
R^n[\mathcal{H}] = \{ F(\cdot) \in R^s[\mathcal{H}] : 0 \in \rho(\text{Im} \, F(z)) \text{ for all } z \in \mathbb{C} \setminus \mathbb{R} \}.
\]

It follows from Theorem 1.2 that each of the subclasses $R^s[\mathcal{H}]$ and $R^n[\mathcal{H}]$ can be single out by a single condition:

\[
R^s[\mathcal{H}] = \{ F \in R[\mathcal{H}] : \ker \text{Im} \, F(i) = \{0\} \};
\]
\[
R^n[\mathcal{H}] = \{ F \in R^s[\mathcal{H}] : 0 \in \rho(\text{Im} \, F(i)) \}.
\]

The classes $R^n[\mathcal{H}]$, $R^s[\mathcal{H}]$, and $R[\mathcal{H}]$ are ordered by inclusion

(1.5) $R^n[\mathcal{H}] \subset R^s[\mathcal{H}] \subset R[\mathcal{H}]$.

It follows from (1.3) and Theorem 1.2 that the operator function $F(z)$ with the integral representation (1.1) belongs to the class $R^s[\mathcal{H}]$ (or $R^n[\mathcal{H}]$), if $0 \not\in \sigma_\rho(\Sigma_0)$ ($0 \in \rho(\Sigma_0)$, respectively).

Recall that every Nevanlinna family can be realized (uniquely up to unitary equivalence) as a Weyl function (or Weyl family) of a (minimal unitary) boundary relation; see [11], [12, Theorem 3.9]. Various other subclasses of the class $\bar{\mathcal{R}}(\mathcal{H})$ appearing above and below can be characterized in boundary triplet and boundary relation context. Each property of $R$-function in Theorem 1.2 when treated as a Weyl function of a (generalized) boundary triplet has its geometrical counterpart. For instance, the class of $R^n[\mathcal{H}]$-functions is known to characterize the class of Weyl functions corresponding to ordinary boundary triplets of $A^*$, where $A$ is a not necessarily densely defined symmetric operator in $\mathcal{H}$ (see [27], [14, 15]). The class $R^s[\mathcal{H}]$ gives a precise characterization of Weyl functions of symmetric operators, corresponding to the so-called $B$-generalized boundary triplets (see [15, 11]).

The paper is organized as follows. For later purposes a proof of items (1) – (4) in Theorem 1.2 will be presented in Section 2, while items (5) and (6) will be treated in Section 3, where all of these invariance results are extended to the class $\bar{\mathcal{R}}(\mathcal{H})$ of Herglotz-Nevanlinna functions with values in the set $C(\mathcal{H})$ of closed linear operators and to the class $\bar{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families. The proof of the first half of these invariance results is based on the maximum principle for Herglotz-Nevanlinna functions or alternatively for contractive holomorphic operator functions.
The rest is proven then with the help of the Harnack’s inequality for harmonic functions. It is emphasized that no realization results via operator or functional models, or boundary triplets methods, for functions from these classes of operator functions are involved in the given arguments.

Harnack’s inequality is systematically used in Section 4 to characterize invariance properties of operator-valued harmonic functions as well as for Herglotz-Nevanlinna functions whose imaginary parts have a so-called form-domain invariance property and for Herglotz-Nevanlinna functions with values in the Schatten-von Neumann classes. For instance, by applying such analytic arguments it is shown (see Proposition 4.9) that under certain additional assumptions a Herglotz-Nevanlinna function $F(\cdot)$ whose imaginary part is bounded at one point admits a representation

$$(1.6) \quad F(z) = G(z) + T, \quad z \in \mathbb{C}_+,$$

where $G(\cdot) \in R[H]$ and $T = T^* \in C(H)$ if and only if $F(z_0) \in B(H)$ for some $z_0 \in \mathbb{C}_+$. Functions of the form (1.6) with $G(\cdot) \in R[\mathcal{H}]$ characterize Weyl functions of generalized boundary triplets with a selfadjoint operator $A_0 = A^*[\ker \Gamma_0]$; see [11, Section 4], [12, Theorem 7.39]. Similar functions appear also in a connection with the so-called quasi-boundary triplets introduced and investigated in [3]. In fact, an arbitrary function of the form,

$$F(z) = T + F_0(z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where $T$ is a symmetric densely defined operator (not necessarily self-adjoint or even closed) on $\mathcal{H}$ and $F_0(\cdot)$ belongs to the class $R[\mathcal{H}]$. Such $F(\cdot)$ appears as a Weyl function of a so-called almost $B$-generalized boundary triplet (possibly multi-valued); a concept that is originated in a forthcoming paper [13] by the authors. Such functions play a central role in the study of form-domain invariant Nevanlinna families (see Definition 4.5 below). Characteristic properties of such functions are investigated in [13] within boundary triplet (and boundary relation) setting: in particular, it is shown therein that a Herglotz-Nevanlinna function having a form-domain invariant imaginary part can be realized as the Weyl function of a unitary boundary triplet (boundary relation) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with an essentially selfadjoint kernel $A_0 = \ker \Gamma_0$. Such functions appear in applications, e.g. in the study of local point interactions, [25, 29], in PDE setting as M-functions, Dirichlet-to-Neumann maps, and their analogs; see e.g. [32, Section 7.7] for a treatment of the Zaremba problem.

Finally, in Section 5 we present several examples which reflect various invariance properties of associated Herglotz-Nevanlinna functions and stability properties of quadratic forms generated by the imaginary parts of such functions.

2. Preliminaries

2.1. Linear relations in Hilbert spaces. Let $\mathcal{H}$ be a separable Hilbert space. The set of bounded (closed) linear operators in $\mathcal{H}$ is denoted by $B(\mathcal{H})$, $(C(\mathcal{H}))$, respectively).

Recall that a linear relation $T$ in $\mathcal{H}$ is a linear subspace of $\mathcal{H} \times \mathcal{H}$. Systematically a linear operator $T$ will be identified with its graph. The set of closed linear relations in $\mathcal{H}$ is denoted by $\tilde{C}(\mathcal{H})$. It is convenient to interpret the linear relation $T$ as a multi-valued linear mapping from $\mathcal{H}$ into $\mathcal{H}$. For a linear relation $T \in \tilde{C}(\mathcal{H})$ the symbols dom $T$, ker $T$, ran $T$, and mul $T$ stand for the domain, kernel, range, and the multi-valued part, respectively. The inverse $T^{-1}$ is a linear relation in $\mathcal{H}$ defined by \{ $\{ f', f \} : \{ f, f' \} \in T$ \}. The adjoint $T^*$ is the closed linear relation from $\mathcal{H}$ defined by (see [2], [6, 8])

$$T^* = \{ \{ h, k \} \in \mathcal{H} \oplus \mathcal{H} : (k, f)_{\mathcal{H}} = (h, g)_{\mathcal{H}}, \{ f, g \} \in T \}. $$
The sum $T_1 + T_2$ and the componentwise sum $T_1 \hat{\oplus} T_2$ of two linear relations $T_1$ and $T_2$ are defined by

$$T_1 + T_2 = \{(f, g + h) : (f, g) \in T_1, (f, h) \in T_2\},$$
$$T_1 \hat{\oplus} T_2 = \{(f + h, g + k) : (f, g) \in T_1, (h, k) \in T_2\}.$$

If the componentwise sum is orthogonal it will be denoted by $T_1 \oplus T_2$. Moreover, $\rho(T)$ ($\hat{\rho}(T)$) stands for the set of regular (regular type) points of $T$. The closure of a linear relation $T$ will be denoted by $\text{clo} T$.

Recall that a linear relation $T$ in $\mathcal{H}$ is called symmetric (dissipative or accumulative) if $\text{Im} (h', h) = 0$ ($\geq 0$) or $\leq 0$, respectively) for all $\{h, h'\} \in T$. These properties remain invariant under closures. By polarization it follows that a linear relation $T$ in $\mathcal{H}$ is symmetric if and only if $T \subset T^*$. A linear relation $T$ in $\mathcal{H}$ is called selfadjoint if $T = T^*$, and it is called essentially selfadjoint if $\text{clo} T = T^*$. A dissipative (accumulative) linear relation $T$ in $\mathcal{H}$ is called maximal dissipative (maximal accumulative) if it has no proper dissipative (accumulative) extensions.

Assume that $T$ is closed. If $T$ is dissipative or accumulative, then $\text{mul} T \subset \text{mul} T^*$. In this case the orthogonal decomposition $\mathcal{H} = (\text{mul} T)^\perp \oplus \text{mul} T$ induces an orthogonal decomposition of $T$ as

$$T = T_s \oplus T_\infty, \quad T_\infty = \{0\} \times \text{mul} T, \quad T_s = \{(f, g) \in T : g \perp \text{mul} T\},$$

where $T_\infty$ is a selfadjoint relation in $\text{mul} T$ and $T_s$ is an operator in $\mathcal{H} \ominus \text{mul} T$ with $\text{dom} T_s = \text{dom} T = (\text{mul} T^*)^{-1}$, which is dissipative or accumulative. Moreover, if the relation $T$ is maximal dissipative or accumulative, then $\text{mul} T = \text{mul} T^*$. In this case the orthogonal decomposition $(\text{dom} T)^\perp = \text{mul} T^*$ shows that $T_s$ is a densely defined dissipative or accumulative operator in $(\text{mul} T)^\perp$, which is maximal (as an operator); see e.g. [21, Sec. 3, Cor. 4.16]. In particular, if $T$ is a selfadjoint relation, then there is such a decomposition where $T_s$ is a selfadjoint operator (densely defined in $(\text{mul} T)^\perp$).

2.2. Nevanlinna families.

**Definition 2.1.** A family of linear relations $\mathcal{F}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space $\mathcal{H}$ is called a \textit{Nevanlinna family} if:

1. \textit{(NF1)} for every $z \in \mathbb{C}_+(\mathbb{C}_-)$ the relation $\mathcal{F}(z)$ is maximal dissipative (resp. accumulative);
2. \textit{(NF2)} $\mathcal{F}(z)^* = \mathcal{F}(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$;
3. \textit{(NF3)} for some (and hence for all) $w \in \mathbb{C}_+(\mathbb{C}_-)$ the operator family $(\mathcal{F}(z) + w)^{-1}(\in [\mathcal{H}])$ is holomorphic in $z \in \mathbb{C}_+(\mathbb{C}_-)$.

The condition $(\mathcal{F}(z) + w)^{-1}(\in [\mathcal{H}])$ implies that $\mathcal{F}(z)$ is maximal dissipative or accumulative relation $\mathcal{F}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, and thus, in particular, closed. The class of all Nevanlinna families in a Hilbert space is denoted by $\tilde{R}(\mathcal{H})$. If the multi-valued part $\text{mul} \mathcal{F}(z)$ of $\mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H})$ is nontrivial, then it is independent of $z \in \mathbb{C} \setminus \mathbb{R}$, so that

$$\mathcal{F}(z) = \mathcal{F}_s(z) \oplus \mathcal{F}_\infty, \quad \mathcal{F}_\infty = \{0\} \times \text{mul} \mathcal{F}(z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where $\mathcal{F}_s(z)$ is a Nevanlinna family of single-valued linear relations in $\mathcal{H} \ominus \text{mul} F(z)$. [20].

Clearly, if $\mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H})$, then $\mathcal{F}_\infty \subset \mathcal{F}(z) \cap \mathcal{F}(z)^*$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

2.3. Class $R(\mathcal{H})$. Class $R(\mathcal{H})$ is defined below as a class of all single-valued Nevanlinna families.

**Definition 2.2.** An operator-valued function $F(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, with values in $\mathcal{C}(\mathcal{H})$ is said to belong to the class $R(\mathcal{H})$, if:

1. \textit{(NF1)} for every $z \in \mathbb{C}_+(\mathbb{C}_-)$ the operator $F(z)$ is maximal dissipative (resp. accumulative);
(NF2) $F(z)^* = F(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$;

(NF3) for some (and hence for all) $w \in \mathbb{C}_+(\mathbb{C}_-)$ the operator function $(F(z) + w)^{-1}(\in [\mathcal{H}])$ is holomorphic in $z \in \mathbb{C}_+(\mathbb{C}_-)$. 

The following subclass

$$R[\mathcal{H}] = \{ F(\cdot) \in R(\mathcal{H}) : \text{dom } F(z) = \mathcal{H} \text{ for all } z \in \mathbb{C} \setminus \mathbb{R} \}$$

of the class $R(\mathcal{H})$ consist of operator-valued functions $F(z)$ with values in $B(\mathcal{H})$.

Analogous to (1.3), the following subclasses of the class $R(\mathcal{H})$ can be determined

$$R^s(\mathcal{H}) = \{ F(\cdot) \in R(\mathcal{H}) : \ker (\text{Im } F(z)) = \{0\} \text{ for all } z \in \mathbb{C} \setminus \mathbb{R} \} ;$$

$$R^u(\mathcal{H}) = \{ F(\cdot) \in R(\mathcal{H}) : F(z) \uplus F(z)^* = \mathcal{H}^2 \text{ for all } z \in \mathbb{C} \setminus \mathbb{R} \} ,$$

where $F(z)$ is the graph of the linear operator $F(z)$. As will be shown in Proposition 3.12 (see also [11], Proposition 2.18) the classes $R^s(\mathcal{H})$ and $R^u[\mathcal{H}]$ coincide. The Nevanlinna functions in $R^s(\mathcal{H})$ and $R^u[\mathcal{H}]$ will be called strict and uniformly strict, respectively.

These definitions give rise to the following chain of inclusions:

$$(2.2) \quad R^u(\mathcal{H}) \subset R^s(\mathcal{H}) \subset R(\mathcal{H}) \subset \tilde{R}(\mathcal{H}).$$

In the infinite-dimensional situation each of the inclusions in (2.2) is strict.

It follows from the integral representation (1.1) for every $F \in R[\mathcal{H}]$ the kernel

$$(2.3) \quad N_F(z, w) := \begin{cases} \frac{F(z) - F(w)^*}{z - \bar{w}}, & \text{if } z \neq \bar{w}; \\ F'(z), & \text{if } z = \bar{w}. \end{cases}$$

is nonnegative in $\mathbb{C}_+ \cup \mathbb{C}_-$. This observation leads to a couple of the invariance results in the class $R[\mathcal{H}]$, which were stated in Theorem [12] and can be considered to be well known. For the convenience of the reader, a short proof for items (1)-(4) is given; the rest will follow from a more general Theorem 5.9 given below.

**Proof of Theorem [12] (1)-(4).** (1) Assume that $0 \in \sigma_p(\text{Im } (F(z_0)))$ and $\text{Im } (F(z_0))h_0 = 0$ for some $h_0 \in \mathcal{H}$, $h_0 \neq 0$. The matrix

$$(2.4) \quad \begin{pmatrix} (N_F(z_0, z_0)h_0, h_0) & (N_F(z_0, z)h_0, h_0) \\ (N_F(z_0, z)h_0, h) & (N_F(z, z)h_0, h) \end{pmatrix}$$

is nonnegative for all $z \in \mathbb{C}_+ \cup \mathbb{C}_-, z \neq \bar{z}_0$. Since the left-upper corner of this matrix vanishes, then also $(N_F(z_0, z_0)h_0, h_0) = 0$ and hence

$$(2.5) \quad (F(z)h_0, h_0) = (F(z_0)^*h_0, h_0) = (h_0, F(z_0)h_0) \quad \text{for all } z \in \mathbb{C}_+ \cup \mathbb{C}_- (z \neq \bar{z}_0).$$

Hence, $(\text{Im } (F(z))h_0, h_0) = 0$ and since $\text{Im } (F(z)) \geq 0$ (or $\leq 0$) this gives $\text{Im } (F(z))h_0 = 0$.

(2) Assume that $0 \in \sigma(\text{Im } (F(z_0)))$ and $\text{Im } (F(z_0))h_n \to 0$ as $n \to \infty$ for some sequence $h_n \in \mathcal{H}$, such that $\|h_n\| = 1$. Then for all $z \in \mathbb{C}_+ \cup \mathbb{C}_-, z \neq \bar{z}_0$, 

$$|(N_F(z, z_0)h_n, h_n)|^2 \leq (N_F(z_0, z_0)h_n, h_n)(N_F(z, z)h_n, h_n) \to 0 \quad (n \to \infty).$$

This implies that $\text{Im } (F(z))h_n \to 0$. Hence, $\|\text{Im } (F(z))^{1/2}h_n\|^2 \to 0$ (if e.g. $\text{Im } z > 0$) and therefore also $\text{Im } (F(z))h_n \to 0$ as $n \to \infty$.

(3) This statement is implied by (1) and (2).

(4) Assume that $F(z_0)h_0 = ah_0$ for some $h_0 \in \mathcal{H}$, $h_0 \neq 0$. Then the left-upper corner of the matrix in (2.4) equals to 0 and therefore (2.5) holds. Hence one obtains $F(z)h_0 = ah_0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

The proof of (5) and (6) is postponed until Theorem 3.9 (v), (vi).
3. Nevanlinna pairs

In abstract eigenvalue depending boundary value problems Nevanlinna family is often represented via its counterpart – a Nevanlinna pair, see e.g. [15], [9], [10]. In this section connections between Nevanlinna families and Nevanlinna pairs are investigated in the general Hilbert space setting.

Every closed linear relation \( T \) in a separable Hilbert space \( \mathcal{H} \) can be represented as
\[
T = \{ (\Phi h, \Psi h) : h \in \mathcal{L} \},
\]
where \( \mathcal{L} \) is a parameter Hilbert space and the operators \( \Phi, \Psi \) belong to \([\mathcal{L}, \mathcal{H}]\). To show this it is enough to take \( T \) as \( \mathcal{L} \) and the projections \( \pi_1, \pi_2 \) onto the first and the second components of \( T \subset \mathcal{H} \times \mathcal{H} \) as \( \Phi \) and \( \Psi \). Clearly, each pair \( \{ \Phi, \Psi \} \) of operators in \([\mathcal{L}, \mathcal{H}]\) gives rise to a linear relation \( T \) in \( \mathcal{H} \) via (3.1). In the infinite-dimensional case (\( \dim \mathcal{H} = \infty \)) the parameter Hilbert space \( \mathcal{L} \) can be taken to be equal to \( \mathcal{H} \). Note that when \( \rho(T) \) is not empty and \( z_0 \in \rho(T) \) then
\[
T = \{ (T - z_0)^{-1} h, (I + z_0 (T - z_0)^{-1}) h : h \in \mathcal{H} \},
\]
so that \( \mathcal{L} = \mathcal{H} \) and there is a natural choice for the pair \( \{ \Phi, \Psi \} \) in \( \mathcal{B}(\mathcal{H}) \).

For linear relations given by the equation (3.1) its properties can be characterized in terms of the pair \( \{ \Phi, \Psi \} \).

**Proposition 3.1** ([10]). Let \( T \) be a linear relation \( T \) in \( \mathcal{H} \), defined by (3.1). Then:

(i) the adjoint \( T^* \) is a linear relation given by
\[
T^* = \{ (h, h') \in \mathcal{H}^2 : \Psi^* h - \Phi^* h' = 0 \}.
\]

(ii) \( T \) is a dissipative (accumulative) relation if and only if
\[
-i(\Phi^* \Psi - \Psi^* \Phi) \geq 0, \quad (\leq 0);
\]

(iii) \( T \) is symmetric if and only if
\[
\Phi^* \Psi - \Psi^* \Phi = 0;
\]

If, additionally, \( \ker \Phi \cap \ker \Psi = \{0\} \), then

(iv) \( z \in \rho(T) \) if and only if the operator \( \Psi - z \Phi \) has a bounded inverse;

(v) \( T \) is maximal dissipative (accumulative) if and only if (3.3) holds and the operator \( \Psi + i \Phi \) (\( \Psi - i \Phi \)) has a bounded inverse;

(vi) \( T \) is selfadjoint if and only if (3.4) holds and the operators \( \Psi \pm i \Phi \) have bounded inverses.

**Proof.** (i) For \( \{ g, g' \} \in T^* \) and arbitrary \( h \in \mathcal{H} \) one has the equality
\[
0 = (g', \Phi h) - (g, \Psi h) = (\Phi^* g' - \Phi^* g, h),
\]
which implies \( \Psi^* g - \Phi^* g' = 0 \).

(ii), (iii) If \( T \) is symmetric then for \( \{ \Phi h, \Psi h \} \in T \) one obtains
\[
0 = -i[(\Psi h, \Phi h) - (\Phi h, \Psi h)]] = -i((\Phi^* \Psi - \Psi^* \Phi) h, h), \quad h \in \mathcal{H},
\]
and conversely. Similarly, one obtains the conditions (3.3) for dissipative and accumulative linear relations.

(iv) It follows from (3.1) that
\[
T - z = \{ (\Phi h, (\Psi - z \Phi) h) : h \in \mathcal{L} \},
\]
Assume that \( z \in \rho(T) \) and \( (\Psi - z \Phi) h = 0 \). Then \( \Psi h = \Phi h = 0 \) and, hence, \( h = 0 \) (by the assumption \( \ker \Phi \cap \ker \Psi = \{0\} \)). Since \( \text{ran} (\Psi - z \Phi) = \text{ran} (T - z) = \mathcal{H} \), it follows
$0 \in \rho(\Psi - z\Phi)$. Similarly, if $0 \in \rho(\Psi - z\Phi)$ one obtains from \((3.5)\) that $z \in \rho(T)$ and $(T^{-1} - z)^{-1} = \Phi(\Psi - z\Phi)^{-1}$.

(v), (vi) are immediate from (ii), (iii) and (iv).

Now let a family of linear relations $F(\cdot)$ be represented in the form
\begin{equation}
F(z) = \{\Phi(z), \Psi(z)\} := \{\Phi(z)h, \Psi(z)h\} : h \in H\}
\end{equation}
where $\Phi(\cdot), \Psi(\cdot)$ is a pair of holomorphic operator functions on $C_+ \cup C_-$. In the case when $F(\cdot)$ is a Nevanlinna family the corresponding pair $\{\Phi(\cdot), \Psi(\cdot)\}$ in the representation \((3.6)\) is called the Nevanlinna pair.

**Definition 3.2.** A pair $\{\Phi, \Psi\}$ of $B(H)$-valued functions $\Phi(\cdot), \Psi(\cdot)$ holomorphic on $C \setminus \mathbb{R}$ is said to be a Nevanlinna pair if:

\begin{align*}
(NP1) \quad & \text{Im } \Phi(z)\Psi(z)/\text{Im } z \geq 0, \quad z \in C_+ \cup C_-; \\
(NP2) \quad & \Psi(z)\Phi(z) - \Phi(z)\Psi(z) = 0, \quad z \in C \setminus \mathbb{R}; \\
(NP3) \quad & 0 \in \rho(\Psi(z) \pm i\Phi(z)), \quad z \in C_+. 
\end{align*}

Two Nevanlinna pairs $\{\Phi(\cdot), \Psi(\cdot)\}$ and $\{\Phi'_1(\cdot), \Psi'_1(\cdot)\}$ are said to be *equivalent*, if they generate the same graph $F(z)$ in \((3.7)\) for every $z \in C \setminus \mathbb{R}$. In fact, the formula \((3.6)\) establishes a one-to-one correspondence $\{\Phi, \Psi\} \mapsto F$ between the equivalence classes of Nevanlinna pairs and Nevanlinna families $F(\cdot) \in \tilde{R}(H)$; cf. [10, Proposition 2.4].

**Proposition 3.3 ([10]).** Let $\{\Phi, \Psi\}$ be a Nevanlinna pair of $B(H)$-valued functions on $C_+ \cup C_-$, and let $F(\cdot)$ be defined by \((3.6)\). Then $F(\cdot)$ is a Nevanlinna family.

Conversely, if $F(\cdot) \in \tilde{R}(H)$ then there exists a Nevanlinna pair $\{\Phi, \Psi\}$ of $B(H)$-valued functions on $C_+ \cup C_-$ such that \((3.6)\) holds.

**Proof.** Let $\{\Phi, \Psi\}$ be a Nevanlinna pair. Then it follows from (NP1), (NP3) and Proposition \(3.1\) that the linear relation $F(z)$ is maximal dissipative (maximal accumulative) for all $z \in C_+$ ($z \in C_-$). The assumption (NP2) concerning $\{\Phi, \Psi\}$ means that $F(z) \subset F(z)^*$. According to \(3.2\)
\begin{equation}
F(z)^* = \{\{h, h'\} : \Psi(z)^*h - \Phi(z)^*h' = 0\},
\end{equation}
and, hence
\begin{equation}
F(z)^* \pm i = \{\{h, g\} : (\Psi(z)^* \pm i\Phi(z)^*)h = \Phi(z)^*g\}.
\end{equation}
Using (NP3) one obtains $\ker (F(z)^* \pm i) = \{0\}$ and $\text{ran } (F(z)^* \pm i) = H$ for $z \in C_z$. Similarly $\ker (F(z) \pm i) = \{0\}$ and $\text{ran } (F(z) \pm i) = H$, $z \in C_z$. Hence, $(F(z)^* \pm i)^{-1}$ and $(F(z) \pm i)^{-1}$, $z \in C_z$, both are everywhere defined operators and, thus, the inclusion $F(z) \subset F(z)^*$ must hold as an equality $F(z) = F(z)^*$, $z \in C \setminus \mathbb{R}$. This proves (NP2) and (NP3) with $w = \pm i$.

Conversely, assume that $F(\cdot) \in \tilde{R}(H)$. Define $\Phi(\cdot)$ and $\Psi(\cdot)$ by
\begin{equation}
\Phi(z) = (F(z) \pm i)^{-1}, \quad \Psi(z) = I \mp i(F(z) \pm i)^{-1}, \quad z \in C_+.
\end{equation}
Then $F(\cdot)$ has the representation \((3.6)\). The property (NP3) implies that $\Phi(\cdot), \Psi(\cdot)$ are holomorphic on $C_+ \cup C_-$ with the values in $B(H)$. Clearly, $\Psi(z) \pm i\Phi(z) = I$ and hence (NP3) holds. Moreover, the symmetry condition (NP2) is obvious and the positivity condition (NP1) follows from (NF1) in view of
\begin{equation}
\frac{(\Phi(z)^*\Psi(z) - \Psi(z)^*\Phi(z))h, h}{\text{Im } z} = \frac{\text{Im } (\Psi(z)h, \Phi(z)h)}{\text{Im } z} \geq 0.
\end{equation}
This completes the proof. \(\square\)
Let \( \{\Phi, \Psi\} \) be a Nevanlinna pair and let \( \mathcal{F}(\cdot) \) be a family of linear relations associated with the Nevanlinna pair \( \{\Phi, \Psi\} \). The Cayley transform \( \mathcal{C}(z) \) of \( \mathcal{F}(z) \) is given by

\[
\mathcal{C}(z) = (\Psi(z) - i\Phi(z))(\Psi(z) + i\Phi(z))^{-1} \quad (z \in \mathbb{C}_+).
\]

The operator-valued function \( \mathcal{C}(z) \) belongs to the Schur class \( \mathcal{S}(\mathcal{H}) \), i.e. \( \mathcal{C}(z) \) is holomorphic on \( \mathbb{C}_+ \) and takes values in the set of contractive operators on \( \mathcal{H} \) for all \( z \in \mathbb{C}_+ \). For every operator-valued function \( \mathcal{C} \in \mathcal{S}(\mathcal{H}) \) the kernel

\[
K(z, w) = \frac{I - \mathcal{C}(w)^*\mathcal{C}(z)}{-i(z - \bar{w})}, \quad z, w \in \mathbb{C}_+.
\]

is nonnegative on \( \mathbb{C}_+ \) in a sense that for every choice of \( z_j \in \mathbb{C}_+ \) and \( h_j \in \mathcal{H} \) \((j = 1, \ldots, n)\) the quadratic form

\[
\sum_{j=1}^{n} (K(z_j, z_j) h_j, h_j) \xi_j \xi_j
\]

is nonnegative.

**Proposition 3.4.** Let \( \{\Phi, \Psi\} \) be a Nevanlinna pair. Then the following kernel is nonnegative on \( \mathbb{C}_+ \):

\[
N_{\Phi, \Psi}(z, w) = \frac{\Phi(w)^*\Psi(z) - \Psi(w)^*\Phi(z)}{z - \bar{w}}, \quad z, w \in \mathbb{C}_+.
\]

**Proof.** Let \( \mathcal{F}(\cdot) \) be a family of linear relations associated with the Nevanlinna pair \( \{\Phi, \Psi\} \) and let \( \mathcal{C}(z) \) be the Cayley transform of \( \mathcal{F}(z) \) given by (3.8). It follows from the equality

\[
K(z, w) = 2(\Psi(w) + i\Phi(w))^*N_{\Phi, \Psi}(z, w)(\Psi(z) + i\Phi(z))^{-1}
\]

that the kernel \( N_{\Phi, \Psi}(z, w) \) is nonnegative on \( \mathbb{C}_+ \); cf. [31]. \( \square \)

Nonnegativity of the kernel \( N_{\Phi, \Psi} \) implies the following properties for Nevanlinna families and reflect maximum principle in the class \( \tilde{R}(\mathcal{H}) \).

**Proposition 3.5.** Let \( \{\Phi(\cdot), \Psi(\cdot)\} \) be a Nevanlinna pair and let \( \mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H}) \) be the corresponding Nevanlinna family. Let \( z_0 \in \mathbb{C} \setminus \mathbb{R} \) be fixed and let \( S \) be a symmetric relation in \( \mathcal{H} \). Then

\[
S \subset \mathcal{F}(z_0) \Rightarrow S \subset \mathcal{F}(z) \quad \text{for every} \ z \in \mathbb{C} \setminus \mathbb{R}.
\]

In particular,

\[
\mathcal{F}(z) \cap \mathcal{F}(\bar{z}) \equiv \mathcal{F}(z_0) \cap \mathcal{F}(\bar{z_0}) \quad (z \in \mathbb{C} \setminus \mathbb{R})
\]

is a maximal symmetric subspace \( S \) satisfying the inclusion \( S \subset \mathcal{F}(z) \) for some (equivalently for every) \( z \in \mathbb{C} \setminus \mathbb{R} \). Moreover,

\[
\mathcal{F}(z) \cap \mathcal{F}(\bar{z}) = \{ \{\Phi(z) u, \Psi(z) u\} : u \in \ker(N_{\Phi, \Psi}(z, z)) \} \quad (z \in \mathbb{C}_+).
\]

**Proof.** Assume that \( S \subset \mathcal{F}(z_0) \) and let \( \{\Phi(z_0) h_0, \Psi(z_0) h_0\} \subset S \) and \( \{\Phi(z) h, \Psi(z) h\} \subset \mathcal{F}(z) \) with \( z \neq z_0 \). By Proposition 3.4 the matrix

\[
\begin{pmatrix}
(N_{\Phi, \Psi}(z_0, z_0) h_0, h_0) & (N_{\Phi, \Psi}(z_0, z) h_0, h) \\
(N_{\Phi, \Psi}(z, z_0) h, h_0) & (N_{\Phi, \Psi}(z, z) h, h)
\end{pmatrix}
\]

is nonnegative, and since \( S \) is symmetric the left-upper corner equals to 0. This implies that \( (N_{\Phi, \Psi}(z_0, z) h_0, h) = 0 \) for all \( h \in \mathcal{H} \) or, equivalently,

\[
(\Psi(z_0) h_0, \Phi(z) h)_H = (\Phi(z_0) h_0, \Psi(z) h)_H
\]
for all \( h \in \mathcal{H} \). Therefore, \( \{ \Phi(z_0)h_0, \Psi(z_0)h_0 \} \in \mathcal{F}(z)^* = \mathcal{F}(\bar{z}) \) for all \( z \neq z_0 \). This proves \((3.11)\).

Since \( \mathcal{F}(z) \cap \mathcal{F}(\bar{z}) = \mathcal{F}(z) \cap \mathcal{F}(z)^* \), this subspace is symmetric and hence its maximality as a symmetric subset of \( \mathcal{F}(z) \) follows from \((3.11)\). Moreover, the invariance property \((3.12)\) is also immediate from \((3.11)\).

Finally, the equality \((3.13)\) follows from the general formula \( F \cap F^* = F \cap \ker(\text{Im } F) \) for a linear relation \( F \) with \( \mu(F) = \mu(F^*) \); see [21, Section 5.1]. In fact, here in view of \((3.2)\) \( \{ \Phi(z)u, \Psi(z)u \} \in \mathcal{F}(z)^* \) precisely, when

\[
\Psi(z)^* \Phi(z)u - \Phi(z)^* \Psi(z)u = 0,
\]
or, equivalently, \( N_{\Phi,\Psi}(z, z)u = 0. \) \( \square \)

**Corollary 3.6.** Let \( \{ \Phi(\cdot), \Psi(\cdot) \} \) and \( \mathcal{F}(\cdot) \in \tilde{\mathcal{R}}(\mathcal{H}) \) be as in Proposition 3.5 and let \( z_0 \in \mathbb{C} \setminus \mathbb{R} \) and \( a \in \mathbb{R} \). Then the following statements hold for all \( z \in \mathbb{C} \setminus \mathbb{R} \):

(i) \( \ker(\mathcal{F}(z) - a) = \ker(\mathcal{F}(z_0) - a) \);

(ii) \( \ker(\mathcal{F}(z) - \mathcal{F}(\bar{z})) = \{ f \in \mathcal{H} : \{ f, g \} \in \mathcal{F}(z_0) \cap \mathcal{F}(\bar{z}_0) \} \);

(iii) \( \ker(\mathcal{F}(z)^{-1} - \mathcal{F}(\bar{z})^{-1}) = \{ g \in \mathcal{H} : \{ f, g \} \in \mathcal{F}(z_0) \cap \mathcal{F}(\bar{z}_0) \} \);

(iv) \( \mu(\mathcal{F}(z)) = \mu(\mathcal{F}(z_0)) \).

**Proof.** The statements (i) and (iv) follow from Proposition 3.5 since \( \ker(\mathcal{F}(z) - a) \) with \( a \in \mathbb{R} \) and \( \mu(\mathcal{F}(z)) \) are symmetric subspaces of \( \mathcal{F}(z) \).

On the other hand, the formulas (ii) and (iii) clearly hold when \( z = z_0 \). The independence from \( z \in \mathbb{C} \setminus \mathbb{R} \) follows from \((3.12)\). \( \square \)

The statements of the following lemma are based on the maximum principle for the class \( \mathcal{S}(\mathcal{H}) \) and, apparently, are well-known.

**Lemma 3.7.** Let \( \mathcal{C}(\cdot) \in \mathcal{S}(\mathcal{H}), z_0 \in \mathbb{C}_+, \alpha \in \mathbb{C} \) and \( |\alpha| = 1 \). Then the following statements hold:

1. If \( 0 \in \sigma_p(\mathcal{K}(z_0, z_0)) \), then \( 0 \in \sigma_p(\mathcal{K}(z, z)) \) for all \( z \in \mathbb{C}_+ \) and in this case
   \[
   \ker(\mathcal{K}(z_0, z_0)) = \ker(\mathcal{K}(z, z))\]
2. If \( 0 \in \rho(\mathcal{K}(z_0, z_0)) \), then \( 0 \in \rho(\mathcal{K}(z, z)) \) for all \( z \in \mathbb{C}_+ \).
3. If \( 0 \in \sigma_c(\mathcal{K}(z_0, z_0)) \), then \( 0 \in \sigma_c(\mathcal{K}(z, z)) \) for all \( z \in \mathbb{C}_+ \).
4. If \( \alpha \in \sigma_p(\mathcal{C}(z_0)) \), then \( \alpha \in \sigma_p(\mathcal{C}(z)) \) for all \( z \in \mathbb{C}_+ \) and in this case
   \[
   \ker(\mathcal{C}(z_0) - \alpha) = \ker(\mathcal{C}(z) - \alpha) ;
   \]
5. If \( \alpha \in \rho(\mathcal{C}(z_0)) \), then \( \alpha \in \rho(\mathcal{C}(z)) \) for all \( z \in \mathbb{C}_+ \);
6. If \( \alpha \in \sigma_c(\mathcal{C}(z_0)) \), then \( \alpha \in \sigma_c(\mathcal{C}(z)) \) for all \( z \in \mathbb{C}_+ \).

**Proof.** (1) Let \( 0 \in \sigma_p(\mathcal{K}(z_0, z_0)) \) and let \( \mathcal{K}(z_0, z_0)h_0 = 0 \) for some \( z_0 \in \mathbb{C}_+ \) and \( h_0 \neq 0 \). Then the following matrix

\[
\begin{pmatrix}
(\mathcal{K}(z_0, z_0)h_0, h_0) & (\mathcal{K}(z_0, z_0)h_0, h) \\
(\mathcal{K}(z, z)h_0, h_0) & (\mathcal{K}(z, z)h, h)
\end{pmatrix},
\]
is nonnegative for all \( z \in \mathbb{C}_+ \), \( h \in \mathcal{H} \) and hence \( (\mathcal{K}(z_0, z)h_0, h) = 0 \) for all \( h \in \mathcal{H} \). This implies that the contraction \( T = \mathcal{C}(z)^* \mathcal{C}(z_0) \) has an eigenvector \( h_0 \) corresponding to the eigenvalue 1: \( Th_0 = h_0 \). Therefore, \( h_0 \) is also an eigenvector for the operator \( T^* = \mathcal{C}(z_0)^* \mathcal{C}(z) \) \( (T^* h_0 = h_0) \) and, hence,

\[
\|h_0\| = \|\mathcal{C}(z_0)^* \mathcal{C}(z)h_0\| \leq \|\mathcal{C}(z)h_0\|.
\]
This implies the inequality

\[
(0 \leq) (\mathcal{K}(z, z)h_0) = (h_0, h_0) - (\mathcal{C}(z)h_0, \mathcal{C}(z)h_0) \leq 0,
\]
which means that $K(z, z)h_0 = 0$. This proves the statement (1).

(2) Let $0 \in \rho(K(z_0, z_0))$ for some $z_0 \in C_+$. Then the operator $C(z_0)$ is a strict contraction and the space $H^2$ admits the decomposition

$$
H^2 = \text{ran} \left( \begin{pmatrix} I & C(z_0) \end{pmatrix} \right) + \text{ran} \left( \begin{pmatrix} C(z_0)^* \end{pmatrix} \right).
$$

Assume that $0 \in \sigma_r(K(z, z))$. Then there exists a sequence $h_n \in H$, such that $\|h_n\| = 1$ and $K(z, z)h_n \to 0$ as $n \to \infty$. Using the decomposition (3.15), one obtains

$$
(3.16) \quad \begin{pmatrix} h_n \\ C(z)h_n \end{pmatrix} = \begin{pmatrix} h_n' \\ C(z_0)h_n' \end{pmatrix} + \text{ran} \left( \begin{pmatrix} C(z_0)^*h_n'' \\ h_n'' \end{pmatrix} \right),
$$

where $h_n', h_n'' \in H$. Since the matrix

$$
\begin{pmatrix} (K(z_0, z_0)h_n', h_n') & (K(z_0, z)h_n', h_n) \\ (K(z, z)h_n, h_n') & (K(z, z)h_n, h_n) \end{pmatrix},
$$

is nonnegative, then

$$
(K(z_0, z)h_n', h_n) \to 0
$$

as $n \to \infty$. Using (3.16) one obtains

$$
((I - C(z_0)^*C(z_0))h_n', h_n') \to 0.
$$

By the assumption $0 \in \rho(K(z_0, z_0))$ this implies $h_n' \to 0$. Next, the equality

$$
((I - C(z)^*C(z))h_n, h_n) = ((I - C(z_0)^*C(z_0))h_n', h_n') - ((I - C(z_0)C(z_0)^*)h_n'', h_n'')
$$

yields

$$
((I - C(z_0)C(z_0)^*)h_n'', h_n'') \to 0,
$$

which, in view of the condition $0 \in \rho(I - C(z_0)^*C(z_0))$ implies that $h_n'' \to 0$. Therefore, $h_n \to 0$ as $n \to \infty$ and this contradicts the equalities $\|h_n\| = 1$.

(3) If $0 \in \sigma_r(K(z_0, z_0))$, then by (1) and (2) $0 \not\in \sigma_p(K(z, z)) \cup \rho(K(z, z))$ and hence $0 \in \sigma_c(K(z, z))$.

(4) Let $\alpha \in \sigma_p(C(z_0)$ $(|\alpha| = 1)$ and $C(z_0)h_0 = \alpha h_0$ for some vector $h_0 \neq 0$. Then $h_0$ is an eigenvector for the contraction $T = C(z)^*$, corresponding to the eigenvalue $\alpha^{-1}$ and for the contraction $T^* = C(z)$, corresponding to the eigenvalue $\alpha^{-*} = \alpha$. This proves the equality (3.14).

(5) Let $\alpha \in \rho(C(z_0))$ $(|\alpha| = 1)$. Then for every $z \in C_+$ and $u \in H$ the harmonic function $h_u(z) := \text{Re} \{e^{-i\arg(\alpha)}((\alpha - C(z))u, u)\} \geq 0$ is nonnegative and satisfies the inequality $h_u(z_0) \geq q\|u\|^2 > 0$ for some $q \in (0, 1)$. By Harnack’s inequality (cf. Section 4 below) for every $z \in C_+$ there are positive constants $c_1(z)$ and $c_2(z)$, such that

$$
c_1(z)h_u(z_0) \leq h_u(z) \leq c_2(z)h_u(z_0).
$$

It is emphasized that constants $c_1(z)$ and $c_2(z)$ do not depend on $u \in H$. Therefore,

$$
h_u(z) \geq q c_1(z)\|u\|^2 > 0 \quad \text{for all} \quad u \in H
$$

and hence $\alpha \in \rho(C(z))$.

(6) Let $\alpha \in \sigma_c(C(z_0))$ $(|\alpha| = 1)$. Then by (4) and (5) $\alpha \not\in \sigma_p(C(z)) \cup \rho(C(z))$. Moreover, $\alpha \not\in \sigma_c(C(z))$, since otherwise we would have $\tilde{\alpha} \in \sigma_p(C(z)^*)$ and hence $\tilde{\alpha} \in \sigma_p(C(z_0)^*)$ which contradicts the assumption $\alpha \in \sigma_c(C(z_0))$. This completes the proof of (6). \hfill $\square$
In order to adapt the above statements to the class \( \tilde{R}(\mathcal{H}) \) we will need the following lemma connecting the spectral properties of a Nevanlinna family \( \mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H}) \) with the spectral properties of its Cayley transform \( \mathcal{C}(z) \).

**Lemma 3.8.** Let \( \{\Phi(\cdot), \Psi(\cdot)\} \) be a Nevanlinna pair, let \( \mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H}) \) be the corresponding Nevanlinna family and let the operator function \( C(z) \) and the kernel \( K(z, w) \) be defined by (3.8) and (3.9). Let \( a \in \mathbb{R}, \alpha = (a - i)(a + i)^{-1} \) and \( z \in \mathbb{C}_+ \). Then the following equivalences hold:

(i) \( 0 \in \sigma_p(N_{\Phi, \Psi}(z, \cdot)) \iff 0 \in \sigma_p(K(z, \cdot)) \);
(ii) \( 0 \in \sigma_c(N_{\Phi, \Psi}(z, \cdot)) \iff 0 \in \sigma_c(K(z, \cdot)) \);
(iii) \( 0 \in \rho(N_{\Phi, \Psi}(z, \cdot)) \iff 0 \in \rho(K(z, \cdot)) \);
(iv) \( a \in \sigma_p(\mathcal{F}(z)) \iff \alpha \in \sigma_p(\mathcal{C}(z)) \);
(v) \( a \in \sigma_c(\mathcal{F}(z)) \iff \alpha \in \sigma_c(\mathcal{C}(z)) \);
(vi) \( a \in \rho(\mathcal{F}(z)) \iff \alpha \in \rho(\mathcal{C}(z)) \);
(vii) \( \mathcal{F}(z) \in \mathcal{B}(\mathcal{H}) \iff 1 \in \rho(\mathcal{C}(z)) \).

**Proof.** The equivalences (i)-(iii) are implied by the identity (3.10).

Notice, that \( a \in \sigma_p(\mathcal{F}(z)) \) if and only if

\[
(3.17) \quad (\Psi(z) - a\Phi(z))u = 0 \quad \text{for some} \quad u \in \mathcal{H} \setminus \{0\}.
\]

If (3.17) holds then by (NP3) \( h := (\Psi(z) + i\Phi(z))u = (a + i)\Phi(z)u \neq 0 \) and in view of (3.8)

\[
C(z)h = (a - i)\Phi(z)u = \frac{a - i}{a + i}h = \alpha h.
\]

Therefore, \( \alpha \in \sigma_p(\mathcal{C}(z)) \). Conversely, if \( \alpha \in \sigma_p(\mathcal{C}(z)) \), and \( C(z)h = \alpha h \) for some \( h \in \mathcal{H} \setminus \{0\} \), then (3.17) holds for \( u = (\Phi(z) + i\Phi(z))^{-1}h(\neq 0) \) and hence \( a \in \sigma_p(\mathcal{F}(z)) \). This proves (iv).

The equivalences (v)-(vi) follows from the equality (3.8) and the equivalences

\[
a \in \sigma_c(\mathcal{F}(z)) \iff \ker(\Psi(z) - a\Phi(z)) = \{0\}, \quad \text{and} \quad \ker(\Psi(z) - a\Phi(z)) \quad \text{is dense in} \quad \mathcal{H};
\]

\[
a \in \rho(\mathcal{F}(z)) \iff 0 \in \rho(\Psi(z) - a\Phi(z)).
\]

Similarly, (vii) follows from the equality (3.8) and the equivalence

\[
\mathcal{F}(z) \in \mathcal{B}(\mathcal{H}) \iff 0 \in \rho(\mathcal{F}(z)) \).
\]

**Theorem 3.9.** Let \( \{\Phi(\cdot), \Psi(\cdot)\} \) be a Nevanlinna pair and let \( \mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H}) \) be the corresponding Nevanlinna family. Let \( \zeta_0 \in \mathbb{C}_+ \) and let \( a \in \mathbb{R} \). Then the following statements hold:

(i) if \( 0 \in \sigma_p(N_{\Phi, \Psi}(\cdot, \zeta_0)) \), then \( 0 \in \sigma_p(N_{\Phi, \Psi}(z, \cdot)) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \);
(ii) if \( 0 \in \sigma_c(N_{\Phi, \Psi}(\cdot, \zeta_0)) \), then \( 0 \in \sigma_c(N_{\Phi, \Psi}(z, \cdot)) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \);
(iii) if \( 0 \in \rho(N_{\Phi, \Psi}(\cdot, \zeta_0)) \), then \( 0 \in \rho(N_{\Phi, \Psi}(z, \cdot)) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \);
(iv) if \( a \in \sigma_p(\mathcal{F}(\zeta_0)) \), then \( a \in \sigma_p(\mathcal{F}(z)) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \); and in this case

\[
\ker(\mathcal{F}(z) - a) = \ker(\mathcal{F}(\zeta_0) - a);
\]

(v) if \( a \in \sigma_c(\mathcal{F}(\zeta_0)) \), then \( a \in \sigma_c(\mathcal{F}(z)) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \);
(vi) if \( a \in \rho(\mathcal{F}(\zeta_0)) \), then \( a \in \rho(\mathcal{F}(z)) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \);
(vii) if \( \mathcal{F}(\zeta_0) \in \mathcal{B}(\mathcal{H}) \), then \( \mathcal{F}(z) \in \mathcal{B}(\mathcal{H}) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \);
(viii) \( \mu(\mathcal{F}(z)) \) does not depend on \( z \in \mathbb{C} \setminus \mathbb{R} \).

**Proof.** Statements (i), (iv) and (viii) have been derived already in Corollary 3.6.

Statements (ii) and (iii) follow from Lemma 3.7 (2)-(3) and Lemma 3.8 (ii)-(iii). Statements (iv) – (vii), follow from Lemma 3.7 (4)-(6) and Lemma 3.8 (iv) – (vii).
Remark 3.10. The invariance results in Theorem 3.9 can be obtained from the realization of a Nevanlinna family as a Weyl family of a boundary relation. Such an approach for proving these facts was used in [11]; see, in particular, [11, Lemma 4.1, Prop. 4.18]. Also other models giving realizations for Nevanlinna families can be used in establishing such invariance results; we mention, in particular, the functional models which can be found from [4, 5].

Proposition 3.11. Let \( \{\Phi, \Psi\} \) be a Nevanlinna pair and let \( \mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H}) \) be the corresponding Nevanlinna family. Let \( z \in \mathbb{C}_+ \). Then:

(i) \( \mathcal{F}(\cdot) \in R^a(\mathcal{H}) \) if and only if \( 0 \not\in \sigma_p(N_{\Phi, \Psi}(z, z)) \);

(ii) \( \mathcal{F}(\cdot) \in R^u(\mathcal{H}) \) if and only if \( 0 \in \rho(N_{\Phi, \Psi}(z, z)) \).

Proof. (i) Let \( h \in \ker N_{\Phi, \Psi}(z, z) \), that is \( (\Phi(z)^*\Psi(z) - \Psi(z)^*\Phi(z))h = 0 \). Then it follows from (3.17) that

\[
\{\Phi(z)h, \Psi(z)h\} \in \mathcal{F}(z) \cap \mathcal{F}(z)^* \quad \text{and, therefore,} \quad h = 0 \text{ if and only if } \mathcal{F}(z) \cap \mathcal{F}(z)^* = \{0\}.
\]

(ii) Let \( f, f' \in \mathcal{H} \) and let \( h, g \) satisfy the equations

\[
(3.18) \quad \Phi(z)h + \Phi(\bar{z})g = f, \quad \Psi(z)h + \Psi(\bar{z})g = f'.
\]

Then it follows from (NP2) that

\[
(3.19) \quad N_{\Phi, \Psi}(z, z)h = \Psi(z)^*f - \Phi(z)^*f', \quad N_{\Phi, \Psi}(z, \bar{z})g = \Psi(z)^*f - \Phi(z)^*f'.
\]

Assume that \( 0 \in \rho(N_{\Phi, \Psi}(z, z)) \). Then it follows from (3.19) and Theorem 3.9 that the system (3.18) has a unique solution for all \( f, f' \in \mathcal{H} \) and, therefore, \( \mathcal{F}(z) \oplus \mathcal{F}(z)^* = \mathcal{H}^2 \).

Conversely, let \( \mathcal{F}(\cdot) \in R^u(\mathcal{H}) \) and thus the system (3.18) has a unique solution for all \( f, f' \in \mathcal{H} \). Then it follows from the first equation in (3.19) and the hypothesis (NP3) that \( \operatorname{ran} N_{\Phi, \Psi}(z, z) = \mathcal{H} \). This implies that \( 0 \in \rho(N_{\Phi, \Psi}(z, z)) \). \( \square \)

Proposition 3.12. Let \( \mathcal{F}(\cdot) \in R^u(\mathcal{H}) \). Then \( \mathcal{F}(z) \in \mathcal{B}(\mathcal{H}) \) and \( \mathcal{F}(z)^{-1} \in \mathcal{B}(\mathcal{H}) \) for every \( z \in \mathbb{C} \setminus \mathbb{R} \). In particular, the following equality holds \( R^u(\mathcal{H}) = R^u[\mathcal{H}] \).

Proof. Let \( \{\Phi, \Psi\} \) be a Nevanlinna pair associated to \( M \). It is enough to prove that \( \Phi(z) \) and \( \Psi(z) \) are invertible. Now assume, for instance, that \( \Phi(z)h_n \to 0 \) for some sequence \( h_n \in \mathcal{H} \), \( \|h_n\| = 1 \). This together with \( 0 \in \rho(N_{\Phi, \Psi}(z, z)) \) shows that for some \( \alpha > 0 \) one has

\[
\alpha \leq (N_{\Phi, \Psi}(z, z)h_n, h_n)_{\mathcal{H}} \to 0,
\]

a contradiction. Since \( \operatorname{ran} \Phi(z) = \operatorname{dom} \mathcal{F}(z) \) (ran \( \Psi(z) = \operatorname{ran} \mathcal{F}(z) \)) is dense in \( \mathcal{H} \), one concludes that \( \Phi(z) \) must be invertible. A similar argument shows that \( \Psi(z) \) is invertible. \( \square \)

We finish this section with some further properties of Nevanlinna pairs. The next statement can be found e.g. from [11, Proposition 2.4].

Lemma 3.13. Two Nevanlinna pairs \( \{\Phi, \Psi\} \) and \( \{\Phi_1, \Psi_1\} \) are equivalent if and only if \( \Phi_1(z) = \Phi(z)\chi(z) \) and \( \Psi_1(z) = \Psi(z)\chi(z) \) for some operator function \( \chi(\cdot) \in \mathcal{B}(\mathcal{H}) \) which is holomorphic and invertible on \( \mathbb{C}_+ \cup \mathbb{C}_- \).

Proof. By definition the Nevanlinna pairs \( \{\Phi, \Psi\} \) and \( \{\Phi_1, \Psi_1\} \) are equivalent if and only if the ranges of the block operators

\[
T(z) = \left( \frac{\Phi(z)}{\Psi(z)} \right), \quad T_1(z) = \left( \frac{\Phi_1(z)}{\Psi_1(z)} \right)
\]

coincide with the graph of the corresponding Nevanlinna family \( \mathcal{F}(z), z \in \mathbb{C} \setminus \mathbb{R} \). It is well known (from Douglas’ lemma) that the equality \( \operatorname{ran} T(z) = \operatorname{ran} T_1(z) \) implies the existence of
a bounded operator $\chi(z) \in \mathcal{B}(\mathcal{H})$ such that $T(z) = T_1(z)\chi(z)$. Thus, $\Phi(z) = \Phi_1(z)\chi(z)$ and $\Psi(z) = \Psi_1(z)\chi(z)$, and hence

$$\Phi(z) \pm i\Psi(z) = (\Phi_1(z) \pm i\Psi_1(z))\chi(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$ 

In view of (NP3) this implies that $\chi(z)$ is bounded with bounded inverse and holomorphic in $z \in \mathbb{C}_\pm$. □

As follows from Proposition 3.14, the conditions (NP3) and (NF3) can be replaced, for instance, by

$$0 \in \rho(\Psi(z) + w\Phi(z)) \quad \text{and} \quad 0 \in \rho(\Phi(z) + wI),$$

respectively, for some (equivalently for every) $w \in \mathbb{C}_\pm$ and for all $z$ in the same halfplane as $w$. Moreover, the following more general statement holds.

**Proposition 3.14.** Let $\{\Phi(\cdot), \Psi(\cdot)\}$ be a Nevanlinna pair, let $W$ be a unitary operator in the Kreǐn space $(\mathcal{H}_2, J_H)$, and let

$$
\begin{pmatrix}
\Phi(z) \\
\Psi(z)
\end{pmatrix}
= W
\begin{pmatrix}
\tilde{\Phi}(z) \\
\tilde{\Psi}(z)
\end{pmatrix}.
$$

Then $\{\tilde{\Phi}(\cdot), \tilde{\Psi}(\cdot)\}$ is also a Nevanlinna pair. In particular, if $X = X^* \in \mathcal{B}(\mathcal{H})$, $Y$ is an invertible operator in $\mathcal{B}(\mathcal{H})$ and $M(\cdot) \in \mathbb{R}^m[\mathcal{H}]$, each of the following pairs is also a Nevanlinna pair:

$$(3.20) \quad \{\Phi(\cdot), \Psi(\cdot) + X\Phi(\cdot)\}, \quad \{Y^{-1}\Phi(\cdot), Y^*\Psi(\cdot)\}, \quad \{-\Psi(\cdot), \Phi(\cdot)\}, \quad \{\Phi(\cdot), \Psi(\cdot) + M(\cdot)\Phi(\cdot)\}.$$

**Proof.** Consider $\mathcal{F}(z)$ and $\tilde{\mathcal{F}}(z)$ as the ranges of the block operators

$$T(z) = \begin{pmatrix}
\Phi(z) \\
\Psi(z)
\end{pmatrix}, \quad \tilde{T}(z) = \begin{pmatrix}
\tilde{\Phi}(z) \\
\tilde{\Psi}(z)
\end{pmatrix}. $$

Then the kernel $N_{\Phi,\Psi}(z, w)$ can be represented as follows:

$$(3.21) \quad N_{\Phi,\Psi}(z, w) = \frac{T(w)^*J_HT(z)}{-i(z - \bar{w})}.$$ 

The properties (NP1), (NP2) for $\{\tilde{\Phi}(\cdot), \tilde{\Psi}(\cdot)\}$ are implied by the equalities

$$(3.22) \quad N_{\tilde{\Phi},\tilde{\Psi}}(z, w) = \frac{\tilde{T}(w)^*J_H\tilde{T}(z)}{-i(z - \bar{w})} = \frac{T(w)^*J_HT(z)}{-i(z - \bar{w})} = N_{\Phi,\Psi}(z, w).$$

The graph $\mathcal{F}(z)$ can be treated as a maximal nonnegative subspace of the Kreǐn space $(\mathcal{H}_2, J_H)$ for $z \in \mathbb{C}_+\setminus \mathbb{R}$; see [11, Section 2]. Since $\tilde{\mathcal{F}}(z)$ is the range of $\tilde{T}(z)$ it has the same property and, therefore, $\tilde{\mathcal{F}}(\cdot) \in \tilde{R}_H$. By Proposition 3.3 the pair $\{\tilde{\Phi}, \tilde{\Psi}\}$ is a Nevanlinna pair.

Applying this statement to the pair $\{\Phi, \Psi\}$ and the matrices

$$W = \begin{pmatrix}
I & 0 \\
X & I
\end{pmatrix}, \quad W = \begin{pmatrix}
Y^{-1} & 0 \\
0 & Y^*
\end{pmatrix}, \quad W = \begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}$$

one shows that the first three pair in (3.20) are Nevanlinna pairs. The properties (NP1), (NP2) for the pair $\{\tilde{\Phi}(\cdot), \tilde{\Psi}(\cdot)\} = \{\Phi(\cdot), \Psi(\cdot) + M(\cdot)\Phi(\cdot)\}$ are implied by the identity

$$N_{\tilde{\Phi},\tilde{\Psi}}(z, w) = N_{\Phi,\Psi}(z, w) \Phi(w) \frac{M(z) - M(w)^*}{z - \bar{w}} \Phi(z).$$

To show that the operator $\tilde{\Psi}(z) + i\tilde{\Phi}(z)$ is invertible for some $z \in \mathbb{C}_+$, set $X = \text{Re } M(z)$, $Y = \text{Im } M(z)$ and apply the previous statement to the pairs:

$$\{\Phi_1(z), \Psi_1(z)\} = \{(Y + I)^{1/2}\Phi(z), (Y + I)^{-1/2}\Psi(z)\},$$

$\Phi_1(z)$ and $\Psi_1(z)$ being Nevanlinna pairs by Proposition 3.14.
\{\Phi_2(z), \Psi_2(z)\} = \{\Phi_1(z), \Psi_1(z) + (Y + I)^{-1/2}X(Y + I)^{-1/2}\Phi_1(z)\}.

Since these pairs are maximal dissipative it follows that the operator

\[ \widetilde{\Psi}(z) + i\widetilde{\Phi}(z) = (Y + I)^{1/2}(\Psi_2(z) + i\Phi_2(z)) \]

is also invertible. \qed

**Remark 3.15.** The connection between Nevanlinna pairs \(\{\Phi(\cdot), \Psi(\cdot)\}\) and Nevalinna families \(\mathcal{F}(\cdot) \in \overline{R(\mathcal{H})}\) in Proposition 3.3 implies some invariance properties for the pair \(\{\Phi(\cdot), \Psi(\cdot)\}\) via Theorem 3.9. We indicate here a couple of the underlying connections.

(i) If \(\{\Phi(\cdot), \Psi(\cdot)\}\) corresponds to \(\mathcal{F}(\cdot)\) then the transformed pair \(\{\Psi(\cdot), -\Phi(\cdot)\}\) corresponds to the inverse \(-\mathcal{F}(z)^{-1}\) and, moreover,

\[ N_{\Phi, -\Phi}(z, w) = N_{\Phi, \Psi}(z, w), \quad z, w \in \mathbb{C} \setminus \mathbb{R}. \]

(ii) The kernels of \(\Phi(z)\) and \(\Psi(z)\) do not depend on \(z \in \mathbb{C} \setminus \mathbb{R}\); namely,

\[ \text{mul} \mathcal{F}(z) = \text{mul} (\mathcal{F}(z) \pm iI) = \text{mul} \{\Phi(z), \Psi(z) \pm i\Phi(z)\} = \ker \Phi(z), \]

and, similarly, \(\ker \Psi(z) = \ker \mathcal{F}(z)\).

(iii) It is not difficult to see that the condition \(0 \in \rho(N_{\Phi, \Psi}(z, z))\) implies that \(\ker \Phi(z) = \ker \Psi(z) = 0\) and that \(\text{ran} \Phi(z)\) and \(\text{ran} \Psi(z)\) are closed. Moreover,

\[ (\text{ran} \Phi(z))^{-1} = (\text{dom} \mathcal{F}(z))^{-1} = \text{mul} \mathcal{F}(z) = \ker \mathcal{F}(z) = \{0\}. \]

Consequently, \(\text{ran} \Phi(z_0) = \mathcal{H}\) and hence \(0 \in \rho(\Phi(z))\). Then in view of (i) we have also \(0 \in \rho(\Psi(z))\). The Nevanlinna kernel for \(\mathcal{F}(\cdot)\) as defined in (2.3) is given by

\[ N_\mathcal{F}(z, w) = (\Phi(w))^{-*}(N_{\Phi, \Psi}(z, w))(\Phi(z))^{-1}, \quad z \neq \bar{w}, z, w \in \mathbb{C} \setminus \mathbb{R}, \]

and there is a similar formula for \(-\mathcal{F}(\cdot)^{-1}\). Therefore,

\[ 0 \in \rho(N_{\Phi, \Psi}(z, z)) \iff \mathcal{F}(\cdot) \in R^u[\mathcal{H}] \iff -\mathcal{F}(\cdot)^{-1} \in R^u[\mathcal{H}]. \]

4. **Invariance Theorems for Harmonic Operator-Valued Functions and Quadratic Forms**

Let us recall from [23, Section 7.1] the definition of a boundedly holomorphic function \(T(\cdot)\) with values in the set \(\mathcal{C}(\mathcal{H})\) of closed (not necessarily bounded) operators acting in \(\mathcal{H}\).

**Definition 4.1.** Let \(T(\kappa)\) be a family of operators with values in \(\mathcal{C}(\mathcal{H})\) and defined in a neighborhood of \(\kappa_0 \in \mathbb{C}\) and let \(\zeta \in \rho(T(\kappa_0))\). The family \(T(\cdot)\) is called holomorphic at \(\kappa_0 \in \mathbb{C}\) if \(\zeta \in \rho(T(\kappa))\) and the resolvent \(R(\zeta, \kappa) = (T(\kappa) - \zeta)^{-1}\) is boundedly holomorphic in \(\kappa\) for \(|\kappa - \kappa_0|\) small enough.

It is shown in [23, Theorem 7.1.3] that in this case the resolvent \(R(\zeta, \kappa)\) of the family \(T(\kappa)\) is holomorphic in both variables \((\zeta, \kappa)\) in an appropriate domain in \(\mathbb{C}^2\).

The following definition of holomorphic \(R\)-function is crucial in the sequel.

**Definition 4.2.** A function \(F \in R(\mathcal{H})\) will be called a *strongly holomorphic function* if the following two conditions are satisfied:

(i) the set

\[ D(F) := \cap_{z \in \mathbb{C}_+} \text{dom} F(z) \] is dense in \(\mathcal{H}\).

(ii) vector function \(F(z)u\) is holomorphic in a domain \(\Omega \subset \mathbb{C}_+ \cup \mathbb{R}\) for each \(u \in D(F)\).
Remark 4.3. Note that in general, the domain of holomorphy $\Omega$ in (ii) might be broader than the corresponding domain in Definition 4.1. Namely, in general conditions (i) and (ii) do not imply the local boundedness of the resolvent $(F(\cdot) + i)^{-1}$ at real points. For instance, consider the function
\[ F(z) = Az, \quad 0 \leq A = A^* \in \mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H}), \quad \text{dom } F(z) = \text{dom } A \neq \mathcal{H}, \quad z \in \mathbb{C} \setminus \{0\}, \]
and $\text{dom } (F(0)) = \mathcal{H}$. It is easily seen that $\text{Im } F(z) = Ay \geq 0$ for $z = x + iy \in \mathbb{C}_+$ and conditions (i), (ii) are satisfied with $\Omega = \mathbb{C}_+ \cup \mathbb{R}$ and $\mathcal{D}(F) = \text{dom } A$.

However, $F(\cdot)$ is not holomorphic at zero in the sense of Definition 4.1. Indeed, $F(z)^{-1} = z^{-1}A^{-1}$ is not boundedly holomorphic at zero even if $A^{-1} \in \mathcal{B}(\mathcal{H})$.

This example shows that the domain of holomorphy $\Omega$ in Definition 4.2(ii) might be broader than the corresponding domain in Definition 4.1.

An unbounded Nevanlinna function is not in general strongly holomorphic. In fact, in the next example an extreme situation of a Nevanlinna function is constructed, such that the domains of $F(\lambda)$ and $F(\mu)$ have a zero intersection:
\[ \text{dom } F(\lambda) \cap \text{dom } F(\mu) = \{0\} \text{ for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. \]

Example 4.4. Let $A$ and $B \geq 0$ be two bounded selfadjoint operators on a Hilbert space $\mathcal{H}$ with $\ker A = \ker B = \{0\}$ and such that
\[ \text{ran } A \cap \text{ran } B = \{0\}. \]

Then $M(z) = A - \frac{1}{z}B, z \neq 0$, is a Nevanlinna function from the class $R[\mathcal{H}]$. The transform $F(z) := -M(z)^{-1}$ is an operator-valued Nevanlinna function in the class $R(\mathcal{H})$:
\[ F(z) := -\left(A - \frac{1}{z}B\right)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \]

To consider the domain of $F(z)$ at two points $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ and assume that there is a nonzero vector $k \in \text{dom } F(\lambda) \cap \text{dom } F(\mu)$. This means that $k = M(\lambda)f = M(\mu)g$ for some $f, g \in \mathcal{H}$, i.e.,
\[ (A - \frac{1}{\lambda}B)f = (A - \frac{1}{\mu}B)g \]
or, equivalently,
\[ A(f - g) = B \left(\frac{1}{\lambda}f - \frac{1}{\mu}g\right). \]

Since $\text{ran } A \cap \text{ran } B = \{0\}$ and $\ker A = \ker B = \{0\}$, we get $f = g$ and $\frac{1}{\lambda}f = \frac{1}{\mu}g$ which leads to $\lambda = \mu$. Therefore,
\[ \text{dom } F(\lambda) \cap \text{dom } F(\mu) = \{0\}, \quad \lambda \neq \mu, \quad \lambda, \mu \neq 0. \]

Recall that there exist bounded nonnegative selfadjoint operators on a Hilbert space $\mathcal{H}$ with $\ker A = \ker B = \{0\}$ and such that
\[ \text{ran } A \cap \text{ran } B = \{0\}, \quad \text{ran } A^{1/2} = \text{ran } B^{1/2}; \]
see e.g. [15] Example, p.278] for an example of such operators. Then it follows from $\text{ran } A^{1/2} = \text{ran } B^{1/2}$ that there exists a bounded and boundedly invertible positive operator $C$ such that
\[ A = B^{1/2}CB^{1/2}. \]

Hence, this choice of $A$ and $B$ implies that $M(z) = B^{1/2}(C - \frac{1}{z})B^{1/2}$ and
\[ F(z) = -B^{-1/2}\left(C - \frac{1}{z}\right)^{-1}B^{-1/2} = B^{-1/2}F(z)B^{-1/2}, \]
where $\tilde{F}(z) = -(C - \frac{1}{z^2})^{-1}$ satisfies, $\tilde{F}(\cdot), -\tilde{F}^{-1}(\cdot) \in R^v[\mathcal{H}]$; cf. Remark 3.15 (iii). Consequently, for every $z \in \mathbb{C} \setminus \mathbb{R}$ the form

\[ t_{F(z)[u,v]} := \frac{1}{z - \bar{z}} [(F(z)u,v) - (u,F(z)v)] = (N_F(z)B^{-1/2}u,B^{-1/2}v), \quad u,v \in \text{dom}\ F(z), \]

is closable and its closure has the same formula which is defined on a constant domain $\text{ran} \ B^{1/2}$. Nevanlinna functions with this property are studied systematically in a forthcoming paper [13] by the authors and they are called form-domain invariant Nevanlinna functions.

The general definition of the class of form-domain invariant Nevanlinna families $\mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H})$ reads as follows; cf. [13] for a treatment of such functions as Weyl functions of boundary triplets.

**Definition 4.5.** A Nevanlinna family $\mathcal{F}(\cdot) \in \tilde{R}(\mathcal{H})$ is said to be form-domain invariant if its operator part $\mathcal{F}_s(\cdot) \in \tilde{R}(\mathcal{H}_s)$ is form-domain invariant, which means that the quadratic form $t_{\mathcal{F}_s(\lambda)}$ in $\mathcal{H}_s$ generated by the imaginary part of $\mathcal{F}_s(\lambda)$ via

\[ t_{\mathcal{F}_s(\lambda)}[u,v] = \frac{1}{\lambda - \bar{\lambda}} [(\mathcal{F}_s(\lambda)u,v) - (u,\mathcal{F}_s(\lambda)v)], \]

is closable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the closure of the form $t_{\mathcal{F}_s(\lambda)}$ has a constant domain.

In what follows the set of nonnegative harmonic functions in $\mathbb{C}_+$ is denoted by $Har_+(\mathbb{C}_+)$. In this section we will systematically make use of the classical Harnack’s inequality: Given a pair of points $z_1, z_2 \in \mathbb{C}_+$, there exists positive constants $c_j = c_j(z_1, z_2), \ j \in \{1, 2\}$, such that

\[ c_1 h(z_1) \leq h(z_2) \leq c_2 h(z_1), \quad h(\cdot) \in Har_+(\mathbb{C}_+). \]

It is emphasized that the constants $c_1$ and $c_2$ do not depend on $h(\cdot) \in Har_+(\mathbb{C}_+)$.  

With any strongly holomorphic $R$-function $F(\cdot) \in R(\mathcal{H})$ one associates a family of the quadratic forms $t(z)[:,]$ given by

\[ t(z)[u] = \text{Im} (F(z)[u]) := \text{Im} (F(z)u,u) \geq 0, \quad u \in \text{dom} (F(z)), \quad z \in \mathbb{C}_+. \]

Equipping $\text{dom} F(z)$ with the inner product

\[ (u, v+) = (u, v)_{\mathcal{H}} + t(z)[u,v], \quad u,v \in \text{dom} F(z), \]

we obtain a pre-Hilbert space $\mathcal{H}_+(z)$. The corresponding energy space is denoted by $\mathcal{H}_+(z)$, i.e., it is the completion of $\mathcal{H}_+(z)$ with respect to the norm $\| \cdot \|_{+,z}$. 

Recall that the form $t(z)$ is called closable if the norms $\| \cdot \|_{\mathcal{H}}$ and $\| \cdot \|_{+,z}$ are compatible. The latter means that the completion of $\mathcal{H}_+(z)$ holds within $\mathcal{H}$, i.e. $\mathcal{H}_+(z) \subset \mathcal{H}$. The form $t(z)$ is called closed if it is closable and $\mathcal{H}_+(z) = \mathcal{H}_+(z)$, i.e. $\mathcal{H}_+(z)$ is a Hilbert space.

In the following proposition we investigate certain stability properties of the family $[4.3]$.

**Proposition 4.6.** Let $F(\cdot) \in R(\mathcal{H})$ be strongly holomorphic. Assume in addition that for some $z_0 \in \mathbb{C}_+$ the form $t(z_0)$ is closable and $D(F)$ is a core for its closure $\overline{t}(z_0)$. Then:

(i) The form $t(z)$ is closable for any $z \in \mathbb{C}_+$;

(ii) $D(F)$ is a core for the closure $\overline{t}(z)$ of the form $t(z)$ for each $z \in \mathbb{C}_+$. Moreover, the corresponding energy spaces $\mathcal{H}_+(z)$, $z \in \mathbb{C}_+$, coincide algebraically and topologically,

\[ \mathcal{H}_+(z) = \mathcal{H}_+(z_0), \quad z \in \mathbb{C}_+. \]

In particular,

\[ D[F] := \cap_{z \in \mathbb{C}_+} \mathcal{H}_+(z) = \mathcal{H}_+(z_0). \]

(iii) For any pair $u, v \in D[F]$ the function $\overline{t}(\cdot)[u,v]$ is a harmonic (hence real analytic) function in $\mathbb{C}_+$. 

Then by the Harnack’s inequality there exist positive constants $c_1 = c_1(z_0, z)$, $c_2 = c_2(z_0, z)$, depending only on $z_0, z$, and such that

$$
0 \leq c_1t(z_0)[u_n - u_m] \leq t(z)[u_n - u_m] \leq c_2t(z_0)[u_n - u_m], \quad u_n \in \mathcal{D}(F).
$$

Combining (4.6) with the first inequality in (4.7) we get $t(z_0)[u_n - u_m] \to 0$ as $n, m \to \infty$. Since in addition $\lim_{n \to \infty} \|u_n\| = 0$ and the form $t(z_0)$ is closable, one has $t(z_0)[u_n] \to 0$ as $n \to \infty$. In turn, the second inequality in (4.7) with $u_m = 0$ yields

$$
\lim_{n \to \infty} t(z)[u_n] = 0, \quad z \in \mathbb{C}_+.
$$

Thus, (4.6) implies (1.8). This means the closability of $t_z$, hence the identical embedding $\mathcal{H}'_+(z) \hookrightarrow \mathcal{H}$ is extended to a (continuous) embedding of the energy space $\mathcal{H}_+(z)$ into $\mathcal{H}$. Moreover, it follows from (4.7) that the norms in $\mathcal{H}'_+(z)$ and $\mathcal{H}'_+(z_0)$ are equivalent. Hence completing the spaces $\mathcal{H}'_+(z)$ and $\mathcal{H}'_+(z_0)$ we conclude that $\mathcal{H}_+(z)$ and $\mathcal{H}_+(z_0)$ coincide algebraically and topologically.

(iii) Since $\mathcal{D}(F)$ is a core for $\mathfrak{T}(z_0)$, it follows form the definition of the closure that for any $u \in \mathcal{D}[F] = \mathcal{H}_+(z_0)$ there exists $u_n \in \mathcal{D}(F)$ such that

$$
\mathfrak{T}(z_0)[u] = \lim_{n \to \infty} t(z_0)[u_n], \quad u_n \in \mathcal{D}(F).
$$

It follows from (4.7) (with $u_m = 0$) that $\mathfrak{T}(z)[u] = \lim_{n \to \infty} t(z)[u_n]$ uniformly on compact subsets of $\mathbb{C}_+$. So, by the first Harnack theorem, $\mathfrak{T}(-)[u]$ is a nonnegative harmonic function in $\mathbb{C}_+$. Using the polarization identity one proves that $\mathfrak{T}(-)[u, v]$ is also harmonic for any pair $u, v \in \mathcal{D}[F]$.

Proposition 4.6 makes it possible to introduce the imaginary part of the function $F(\cdot)$.

**Definition 4.7.** Let $F(\cdot) \in R(\mathcal{H})$ satisfy the conditions of Proposition 4.6. Denote by $F_I(z), \ z \in \mathbb{C}_+$, the nonnegative self-adjoint operator associated with the closed form $\mathfrak{T}(z)$ in accordance with the first representation theorem (see [23, Theorem 6.2.1]).

Note that the operator $F_I(z) = F_I(z)^*$ is a self-adjoint extension of the operator

$$
F_I'(z) := (F(z) - F(z)^*)/2i, \quad \text{dom} (F_I'(z)) = \text{dom} (F(z)) \cap \text{dom} (F(z)^*),
$$

which is only nonnegative symmetric not necessarily essentially self-adjoint.

**Remark 4.8.** (i) Note that in accordance with the second representation theorem (see [23, Theorem 6.2.23]) and Definition 4.7, equalities (4.4)-(4.5) can be rewritten as

$$
\mathcal{H}_+(z) = \text{dom} (F_I(z)^{1/2}) = \text{dom} (F_I(z_0)^{1/2}) = \mathcal{H}_+(z_0) = \mathcal{D}[F] \quad \text{for each} \quad z \in \mathbb{C}_+.
$$

Here the spaces $\mathcal{H}_+(z)$ and $\text{dom} (F_I(z)^{1/2})$ (equipped with the graph norm) coincide algebraically and topologically.

(ii) Proposition 4.6 shows that the family $F(\cdot)$ is a holomorphic family in $\mathbb{C}_+$ of the type $(B)$ in the sense of T. Kato [23, Section 7.4.2]

**Proposition 4.9.** Let $F(\cdot) \in R(\mathcal{H})$ and let the conditions of Proposition 4.6 be satisfied. Assume also that $F_I(z_0) \in \mathcal{B}(\mathcal{H})$ for $z_0 \in \mathbb{C}_+$. Then:

(i) $F_I(\cdot)$ takes values in $\mathcal{B}(\mathcal{H})$;
(ii) The function $F(z)$ admits a representation
\begin{equation}
F(z) = G(z) + T, \quad z \in \mathbb{C}_+,
\end{equation}
where $G(\cdot) \in R[\mathcal{H}]$ and $T = T^* \in \mathcal{C}(\mathcal{H})$.

**Proof.** (i) Since $F \in R(\mathcal{H})$ is a strongly holomorphic function, the family
\[\mathcal{F} = \{\text{Im} \ (F(\cdot)u, u) : u \in \mathcal{D}(F)\}\]
is well defined and constitutes the family of nonnegative harmonic functions, $\mathcal{F} \subset H_{\text{ar}}(\mathbb{C}_+)$. Fix $z \in \mathbb{C}_+$. Then, by Proposition 4.6(ii) the form $t(z)[\cdot]$ is closable and by the Harnack’s inequality (4.12),
\[
0 \leq t(z)[u] := \text{Im} \ (F(z)u, u) \leq c_2 \text{Im} \ (F(z_0)u, u) \leq c_2 \|F_I(z_0)\| \cdot \|u\|^2, \quad u \in \mathcal{D}(F).
\]
It follows that the form $t(z)$ is bounded on $\mathcal{D}(F)$. Since $\mathcal{D}(F)$ is a core for $t(\cdot)$, the form $t(z)$ admits a bounded continuation on $\mathcal{H}$ and by the Riesz representation theorem,
\begin{equation}
t(z)[u, v] = (T(z)u, v)_\mathcal{H}, \quad 0 \leq T(z) = T^*(z) \in \mathcal{B}(\mathcal{H}), \quad u, v \in \mathcal{B}(\mathcal{H}).
\end{equation}
Using the polarization identity we obtain from (4.1) that
\[
t(z)[u, v] = (2i)^{-1} \left((F(z)u, v) - (u, F(z)v)\right), \quad u, v \in \text{dom } F(z).
\]
Combining this identity with (4.1.1) we derive
\[
((F(z) - iT(z))u, v) = (u, (F(z) - iT(z))v), \quad u, v \in \text{dom } F(z).
\]
Since $\text{dom } F(z)$ is dense in $\mathcal{H}$, it follows that $v \in \text{dom } (F(z)^* + iT(z))$, i.e.
\[
\text{dom } F(z) \subset \text{dom } (F(z)^* + iT(z)).
\]
On the other hand, since $T(z) = T^*(z)$ is bounded, then
\[
\text{dom } (F(z)^* + iT(z)) = \text{dom } F(z)^* \quad \text{and} \quad \text{dom } F(z) \subset \text{dom } F(z)^*.
\]
By symmetry,
\[
\text{dom } F(z)^* = \text{dom } F(\overline{z}) \subset \text{dom } F^*(\overline{z}) = \text{dom } F(z)
\]
Thus, $\text{dom } F(z)^* = \text{dom } F(z)$ and the imaginary part $F_I(\cdot) := (2i)^{-1} \left((F(\cdot) - F^*(\cdot))\right)$ of $F(\cdot)$ is well defined and
\begin{equation}
F_I(z)u = T(z)u, \quad u \in \text{dom } F(z) \subset \mathcal{D}(F).
\end{equation}
Hence $F_I(z)$ is bounded and its closure coincides with $T(z)$.

(ii) Being a nonnegative harmonic $\mathcal{B}(\mathcal{H})$-valued function in $\mathbb{C}_+$, $T(\cdot)$ admits a representation
\begin{equation}
T(z) = B_0 + B_1z + \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} d\Sigma(t),
\end{equation}
where $B_j = B_j^* \in \mathcal{B}(\mathcal{H})$, $j \in \{0, 1\}$, $B_1 \geq 0$, and $\Sigma(\cdot)$ is the $\mathcal{B}(\mathcal{H})$-valued operator measure satisfying
\begin{equation}
K_{\Sigma} := \int_{\mathbb{R}} (1 + t^2)^{-1} d\Sigma(t) \in \mathcal{B}(\mathcal{H}).
\end{equation}
Define $R[\mathcal{H}]$-function $G(\cdot)$ by setting
\begin{equation}
G(z) = B_0 + B_1z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{1}{1 + t^2}\right) d\Sigma(t).
\end{equation}
Further we let
\begin{equation}
G_1(z) := F(z) - G(z)
\end{equation}
and note that $G_{1}(\cdot)$ is holomorphic in $\mathbb{C}_{+}$. Moreover, it follows from (4.12), (4) and (4.15) that

$$\text{Im} \left( G_{1}(z)u, u \right) = 0, \quad z \in \mathbb{C}_{+}, \quad u \in \mathcal{D}(F).$$

Hence the operator $G_{1}(z)$ is symmetric for any $z \in \mathbb{C}_{+}$. Let us show that $G_{1}(z)$ is self-adjoint. Since $F(z)$ is $m$-dissipative for $z \in \mathbb{C}_{+}$ and $G(\cdot)$ takes values in $\mathcal{B}(\mathcal{H})$, one has $\rho(G_{1}(z)) \cap \mathbb{C}_{-} \neq \emptyset$.

Further, since $F(z)^{*}$ is $m$-accumulative for $z \in \mathbb{C}_{+}$ and $G(z)^{*} \in \mathcal{B}(\mathcal{H})$, we get

$$G_{1}(z)^{*} = F(z)^{*} - G(z)^{*} \quad \text{and} \quad \rho(G_{1}(z)^{*}) \cap \mathbb{C}_{+} \neq \emptyset.$$

Thus, $G_{1}(z) = G_{1}(z)^{*}$ for any $z \in \mathbb{C}_{+}$. Being holomorphic in $\mathbb{C}_{+}$, the operator-valued function $G_{1}(\cdot)$ is constant, $G_{1}(z) = T = T^{*}$, $z \in \mathbb{C}_{+}$. Combining this with (4.16) leads to (4.10). □

Next we investigate the invariance property of real continuous spectrum.

Recall that $\lambda_{0} \in \sigma_{c}(T)$ if $\lambda_{0} \not\in \sigma_{p}(T)$ and there exists a non-compact (quasi-eigen) sequence $f_{n} \in \text{dom}(T) \subset \mathcal{H}$ such that

$$\lim_{n \to \infty} \| (T - \lambda_{0})f_{n} \| = 0.$$

**Proposition 4.10.** Let $F \in R(\mathcal{H})$ and satisfy the conditions of Proposition 4.6. Let also $F_{1}(\cdot)$ be its imaginary part in the sense of Definition 4.7. Then the following holds:

(i) If $a = \overline{a} \in \sigma_{c}(F_{1}(z_{0}))$ for some $z_{0} \in \mathbb{C}_{+}$, then

$$a \in \sigma_{c}(F_{1}(z)) \quad \text{for} \quad z \in \mathbb{C}_{+};$$

(ii) If $a = \overline{a} \in \sigma_{p}(F_{1}(z_{0}))$ for some $z_{0} \in \mathbb{C}_{+}$, then

$$a \in \sigma_{p}(F_{1}(z)) \quad \text{and} \quad \ker(F_{1}(z) - a) = \ker(F_{1}(z_{0}) - a) \quad \text{for} \quad z \in \mathbb{C}_{+}.$$

**Proof.** (i) Without loss of generality we can assume that $a = 0$. Since $0 \in \sigma_{c}(F_{1}(z_{0}))$, there exists an non-compact (quasi-eigen) sequence $\{v_{k}\}_{k \in \mathbb{N}} \in \text{dom}(F_{1}(z_{0}))$ such that

$$\|v_{k}\| = 1 \quad \text{and} \quad \lim_{k \to \infty} \|F_{1}(z_{0})v_{k}\| = 0.$$

By Proposition 4.6 (ii), $\{v_{k}\}_{k \in \mathbb{N}} \in \text{dom}(F_{1}(z_{0})) \subset \mathcal{D}[F]$. Using Definition 4.7 and relation (4.9) one rewrites the right-hand side of inequality (4.7) as

$$0 \leq \|F_{1}(z)^{1/2}v_{k}\|^{2} = t(z)|v_{k}|^{2} \leq c_{2}t(z_{0})|v_{k}|^{2} = \|F_{1}(z_{0})^{1/2}v_{k}\|^{2}, \quad v_{k} \in \mathcal{D}[F] = \mathcal{H}_{+}(z).$$

Combining (4.17) with (4.18) and noting that the sequence $\{v_{k}\}$ is not compact, one gets that

$$0 \in \sigma_{c}(F_{1}(z)^{1/2}). \quad \text{Hence} \quad 0 \in \sigma_{c}(F_{1}(z)).$$

(ii) Let $a = 0 \in \sigma_{p}(F_{1}(z_{0}))$ and $u \in \ker(F_{1}(z_{0}))$. Hence $u \in \ker(F_{1}(z_{0})^{1/2})$. By Proposition 4.6 (ii), $u \in \text{dom}(F_{1}(z)) \subset \mathcal{D}[F] = \text{dom}(F_{1}(z_{0})^{1/2})$ for each $z \in \mathbb{C}_{+}$. It follows from (4.18) with $u$ in place of $v_{k}$ that

$$F_{1}(z)^{1/2}u = 0. \quad \text{Hence} \quad F_{1}(z)u = 0 \quad \text{for each} \quad z \in \mathbb{C}_{+}. \quad \Box$$

Next we slightly improved Proposition 4.10 (i).

**Proposition 4.11.** Let $F \in R(\mathcal{H})$ and satisfy the conditions of Proposition 4.6. Let also $F_{1}(\cdot)$ be its imaginary part in the sense of Definition 4.7. If $a = \overline{a} \in \sigma_{c}(F_{1}(z_{0}))$ for some $z_{0} \in \mathbb{C}_{+}$, then

$$a \in \sigma_{c}(F_{1}(z)) \quad \text{for} \quad z \in \mathbb{C}_{+}.$$

Moreover, a quasi-eigen sequence can be chosen to be common for all $F_{1}(z)$, $z \in \mathbb{C}_{+}$. 
Proof. (i) First we reduce the proof to the case of $R$-function with bounded imaginary part.

In fact, this sequence $\{v_k\}_{k \in \mathbb{N}} \in \text{dom}(F_I(z_0))$ in (4.17) can be chosen to be orthonormal. Passing if necessary to a subsequence $\{v_m\}_{k \in \mathbb{N}}$, we can assume that

$$\sum_{k=1}^{\infty} \| F_I(z_0)v_k \|^2 =: \alpha_F(z_0) < \infty.$$ 

Since $\mathcal{D}(F)$ is a core for the form $t(z_0)$, it is dense in $\text{dom}(F_I(z_0))(\subset \mathcal{H})$ equipped with the graph’s norm. So, there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(F)$ such that

$$(4.19) \sum_{k=1}^{\infty} \| u_k - v_k \|^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \| F_I(z_0)(u_k - v_k) \|^2 =: \beta_F(z_0) < \infty.$$ 

Let $\mathcal{H}_0$ be a subspace spanned by the sequence $\{u_k\}_{k \in \mathbb{N}}$, $\mathcal{H}_0 := \text{span}\{u_k\}_{k \in \mathbb{N}}$. It is known (see [20 Theorem 6.2.3]) that the system $\{u_k\}_{k \in \mathbb{N}}$ forms (after possible replacement of a finite number of vectors by another system of linearly independent vectors) a Riesz basis in $\mathcal{H}_0$. Assume for convenience that such replacement is not needed, i.e. the system $\{u_k\}_{k \in \mathbb{N}}$ itself constitutes the Riez basis in $\mathcal{H}_0$. Denote by $P_0$ the orthoprojector in $\mathcal{H}$ onto $\mathcal{H}_0$ and put

$$F_0(\cdot) := P_0 F(\cdot)|_{\mathcal{H}_0} \quad \text{and} \quad F_{I,0}(z) := P_0 F_{I,1}(z)|_{\mathcal{H}_0}, \quad z \in \mathbb{C}_+.$$ 

First we show that $F_{I,0}(\cdot) \in R[\mathcal{H}_0]$. Indeed, since the system $\{u_k\}_{k \in \mathbb{N}}$ forms the Riesz basis in $\mathcal{H}_0$, any $u \in \mathcal{H}_0$ admits a decomposition $u = \sum_k c_k u_k$ with $c := \{c_k\}_{k \in \mathbb{N}} \in l^2(\mathbb{N})$. Clearly,

$$\|F_{I,0}(z_0)\sum_{k=1}^{n} c_k u_k\|^2 \leq \left( \sum_{k=1}^{n} |c_k| \cdot \|F_{I,0}(z_0)u_k\| \right)^2 \leq \left( \sum_{k=1}^{n} |c_k|^2 \right) \left( \sum_{k=1}^{n} \|P_0 F_I(z_0)u_k\|^2 \right) \leq 2(\alpha_F(z_0) + \beta_F(z_0)) \cdot \|c\|^2_2, \quad n \in \mathbb{N}.$$ 

Hence $F_{I,0}(z_0) \in [\mathcal{H}_0]$. Moreover, it is easily seen that $F_0(\cdot)$ satisfies the conditions of Proposition 4.3 together with $F(\cdot)$. Thus, by Proposition 4.3, $F_{I,0}(\cdot)$ takes values in $\mathcal{B}(\mathcal{H}_0)$.

(ii) It follows from (4.17) and (4.19) that the sequence $\{u_k\}_{k \in \mathbb{N}}$ is a quasi-eigen sequence for the operator $F_{I,0}(z_0)$ corresponding to the point $a = 0$, i.e. it is bounded, non-compact and

$$\lim_{k \to \infty} \| F_{I,0}(z_0)u_k \| = 0.$$ 

Define a family of scalar nonnegative harmonic functions $h_k(\cdot) := (F_{I,0}(\cdot)u_k, u_k)$ in $\mathbb{C}_+$. Since the sequence $\{u_k\}_{k \in \mathbb{N}}$ is bounded, relation (4.20) yields

$$\lim_{k \to \infty} h_k(z_0) = \lim_{k \to \infty} \langle F_{I,0}(z_0)u_k, u_k \rangle = 0.$$ 

By the Harnack inequality (4.22), this relation implies similar relation for any $z \in \mathbb{C}_+$ (cf. (4.17)),

$$\lim_{k \to \infty} h_k(z) = \lim_{k \to \infty} \langle F_{I,0}(z)u_k, u_k \rangle = \lim_{k \to \infty} \| F_{I,0}^{1/2}(z)u_k \|^2 = 0, \quad z \in \mathbb{C}_+.$$ 

Since $F_{I,0}(\cdot)$ takes values in $\mathcal{B}(\mathcal{H}_0)$, the latter implies $\lim_{k \to \infty} \| F_{I,0}(z)u_k \| = 0$ which proves the result. \hfill \Box

**Corollary 4.12.** Let $F(\cdot) \in R(\mathcal{H})$ and $F(z_0) \in \mathcal{B}(\mathcal{H})$ for some $z_0 \in \mathbb{C}_+$. Then $F(\cdot) \in R[\mathcal{H}]$.

**Proof.** By Proposition 4.3, $F(\cdot) = F_1(\cdot) + T$ where $F_1(\cdot) \in R[\mathcal{H}]$ and $T = T^*$. Since

$$F(z_0) = F_1(z_0) + T \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad F_1(z_0) \in \mathcal{B}(\mathcal{H}),$$

the operator $T$ is bounded and $F \in R[\mathcal{H}]$. \hfill \Box
Next we present another proof of statement (iv) in Theorem 3.9.

**Proposition 4.13.** Let \( F(\cdot) \in R(\mathcal{H}) \), \( a = \sigma \in \sigma_p(F(z_0)) \) for \( z_0 \in \mathbb{C}_+ \). Then
\[
a \in \sigma_p(F(z)) \quad \text{and} \quad \ker (F(z) - a) = \ker (F(z_0) - a) \quad z \in \mathbb{C}_+.
\]

**Proof.** Since \( F(\cdot) - a \in R(\mathcal{H}) \) for any \( a \in \mathbb{R} \), we can assume without loss of generality that \( a = 0 \). Let us put \( G(\cdot) := -(F(\cdot) + i)^{-1} \). Since \( F(z) \) is \( m \)-dissipative for \( z \in \mathbb{C}_+ \), \( G(\cdot) \in R[\mathcal{H}] \). Moreover, due to the classical estimate
\[
\|G(z)\| = \|F(z) + i\|^{-1} \leq 1, \quad z \in \mathbb{C}_+,
\]
\( G(\cdot) \) is a contractive holomorphic operator-valued function in \( \mathbb{C}_+ \). Hence its imaginary part \( G_1(\cdot) \) is also contractive, \( 0 \leq G_1(z) \leq 1, \ z \in \mathbb{C}_+ \). Further, let us assume that \( u_0 \in \ker (F(z_0)) \) and for definiteness \( \|u_0\| = 1 \). Then \( h(\cdot) := (G_1(\cdot)u_0, u_0) \) is a scalar nonnegative contractive harmonic function in \( \mathbb{C}_+ \). Moreover, since \( (F(z_0) + i)u_0 = iu_0 \) we get
\[
G_1(z_0)u_0 = u_0 \quad \text{and} \quad h(z_0) = (G_1(z_0)u_0, u_0) = \|u_0\|^2 = 1.
\]
According to the Maximum Principle applied to the contractive harmonic function \( h(\cdot) \), one gets \( h(z) = h(z_0) = 1, \ z \in \mathbb{C}_+ \). Rewriting this identity in the form
\[
((I - G_1(z))u_0, u_0) = 0, \quad z \in \mathbb{C}_+,
\]
and noting that \( I - G_1(z) \geq 0 \), we derive \( G_1(z)u_0 = u_0, \ z \in \mathbb{C}_+ \). Since \( G(\cdot) \) is contractive, the previous identity yields \( G(z)u_0 = iu_0 \), i.e. \( F(z)u_0 = 0 \) for \( z \in \mathbb{C}_+ \). \( \square \)

**Corollary 4.14.** Assume the conditions of Proposition 4.13 and let \( \mathcal{H}_0 := \ker (F(i) - a) \). Then \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) and \( F(\cdot) \) admits the following orthogonal decomposition
\[
F(z) = aI_{\mathcal{H}_0} \oplus F_\alpha(z),
\]
where \( F_\alpha(\cdot) \in R(\mathcal{H}_1) \) and \( \ker (F_\alpha(z) - a) = \{0\}, \ z \in \mathbb{C}_+ \).

**Proposition 4.15.** Let \( F \in R(\mathcal{H}) \) satisfy the conditions of Proposition 4.10, and let \( a = \sigma \in \sigma_c(F(z_0)) \) for some \( z_0 \in \mathbb{C}_+ \). Then
\[
a \in \sigma_c(F(z)) \quad \text{for} \quad z \in \mathbb{C}_+.
\]
Moreover, a quasi-eigen sequence can be chosen to be common for all \( F(z), \ z \in \mathbb{C}_+ \).

**Proof.** Repeating the procedure applied in the proof of Proposition 4.11 one reduces the proof to the case of \( F = F_0 \) with values in \( \mathcal{B}(\mathcal{H}) \). Therefore \( F_0(\cdot) \) admits the integral representation (4.15) with \( B_0 \geq 0, \ B_1 = B_1^* \in \mathcal{H}_0 \) and \( \Sigma(\cdot) \) being the \( \mathcal{B}(\mathcal{H}_0) \)-valued operator measure satisfying condition (4.14). Clearly, \( K_\Sigma \geq 0 \) and \( K_\Sigma \in \mathcal{B}(\mathcal{H}_0) \).

Repeating the reasoning of Proposition 4.11 one shows that there exists a quasi-eigen sequence for \( F_0(z_0) \) and such that \( \{u_k\} \subset \mathcal{D}(F) \), i.e.
\[
\lim_{k \to \infty} \|F_0(z_0)u_k\| = 0.
\]
Setting
\[
H(z) := \text{Im} F_0(z) := (2\pi)^{-1} (F_0(z) - F_0(z)^*)
\]
one defines a family of scalar nonnegative harmonic functions \( h_n(\cdot) := (H(\cdot)u_n, u_n) \) in \( \mathbb{C}_+ \). It follows from (4.21) that \( \lim_{n \to \infty} h_n(z_0) = \lim_{n \to \infty} (H(z_0)u_n, u_n) = 0 \). By Proposition 4.11 similar conclusion holds for any \( z \in \mathbb{C}_+ \), i.e.
\[
\lim_{n \to \infty} h_n(z) = \lim_{n \to \infty} (H(z)u_n, u_n) = 0, \quad z \in \mathbb{C}_+.
\]
On the other hand, it follows from (4.13) with account of (4.14) that
\begin{equation}
H(i) = -i(F_0(i) - B_0) = B_1 + K_\Sigma.
\end{equation}
Combining (4.22) with (4.23) and noting that $B_1 \geq 0$ and $K_\Sigma > 0$, one gets
\begin{equation}
\lim_{n \to \infty} \|B_1 u_n\| = \lim_{n \to \infty} \|K_\Sigma u_n\| = 0.
\end{equation}

Further, for any fixed $z \in \mathbb{C}_+$ we set
\begin{equation}
c_2(z) := \max_{t \in \mathbb{R}} \left| \frac{1 + zt}{t - z} \right|.
\end{equation}
Combining (4.13) with (4.14), applying the Cauchy-Bunyakovskii inequality for integrals, and taking the notation (4.24) into account we derive (cf. [28] Section 7)
\begin{align*}
\left| ((F_0(z) - B_1 z - B_0) u, v) \right|^2 & \leq \left| \int_{\mathbb{R}} \frac{1 + zt}{(t - z)(1 + t^2)} d(\Sigma(t)u, v) \right|^2 \\
& \leq c_2(z)^2 \left| \int_{\mathbb{R}} \frac{1 + t^2}{1 + t^2} d(\Sigma(t)u, u) \int_{\mathbb{R}} \frac{1}{1 + t^2} d(\Sigma(t)v, v) \right| \\
& \leq c_2(z)^2 \|K^{1/2}_\Sigma u, u\| \|K^{1/2}_\Sigma v, v\| \\
& = c_2(z)^2 \|K^{1/2}_\Sigma u\|^2 \cdot \|K^{1/2}_\Sigma v\|^2.
\end{align*}
This ”weak” estimate is equivalent to the following ”strong” one
\begin{equation}
\left| ((F_0(z) - B_1 z - B_0) u) \right| \leq c_2(z) \|K^{1/2}_\Sigma u\| \cdot \|K^{1/2}_\Sigma v\|, \quad z \in \mathbb{C}_+.
\end{equation}
Inserting in this inequality $u = u_n$ and taking into account (4.24) yields
\begin{equation}
\lim_{n \to \infty} \|(F_0(z) - B_0) u_n\| = 0, \quad z \in \mathbb{C}_+.
\end{equation}
Setting here $z = z_0$ and using the assumption $\lim_{n \to \infty} \|F_0(z_0) u_n\| = 0$, we get $\lim_{n \to \infty} B_0 u_n = 0$. Finally, combining this relation with (4.24) implies
\begin{equation}
\lim_{n \to \infty} F_0(z) u_n = 0, \quad z \in \mathbb{C}_+.
\end{equation}
Since the sequence $\{u_n\}_{n \in \mathbb{N}}$ is non-compact, the latter means that $0 \in \sigma_c(F_0(z))$. Hence $0 \in \sigma_c(F(z))$ and the result is proved.
\end{proof}

\begin{proposition}
Let $F(\cdot) \in R[\mathcal{H}]$ and $F(z_0) \in \mathcal{G}_p(\mathcal{H})$ for some $z_0 \in \mathbb{C}_+$ and $p \in (0, \infty)$. Then $F(\cdot)$ takes values in $\mathcal{G}_p(\mathcal{H})$,
\begin{equation*}
F(\cdot) : \mathbb{C}_+ \to \mathcal{G}_p(\mathcal{H}).
\end{equation*}
\end{proposition}

\begin{proof}
Since $F(\cdot) \in R[\mathcal{H}]$, it admits integral representation (4.15). According to (4.26) the following estimate holds
\begin{equation}
\left| ((F(z) - B_1 z - B_0) u, v) \right| \leq c_2(z) \|K^{1/2}_\Sigma u\| \cdot \|K^{1/2}_\Sigma v\|, \quad u, v \in \mathcal{H}, \quad z \in \mathbb{C}_+,
\end{equation}
where $K_\Sigma \geq 0$ is a nonnegative bounded operator in $\mathcal{H}$ given by (4.14). This estimate is equivalent to the following representation
\begin{equation}
F(z) - B_1 z - B_0 = K^{1/2}_\Sigma T(z) K^{1/2}_\Sigma, \quad z \in \mathbb{C}_+,
\end{equation}
where $T(z)$ is an operator-valued function with values in $\mathcal{B}(\mathcal{H})$ and $\|T(z)\| \leq c_2(z)$.

On the other hand, setting as above $\dot{H}(z) := \text{Im} F(z) := F_1(z)$, applying the Harnack’s inequality (1.2), and taking (1.23) into account, we obtain
\begin{align*}
C_1(H(z_0) u, u) & \leq \dot{H}(i) u, u) = (B_1 u, u) + (K_\Sigma u, u) \\
& \leq C_2(H(z_0) u, u), \quad u \in \mathcal{H}.
\end{align*}
It follows that there exists an operator $T_0 \in \mathcal{B}(\mathcal{H})$ with bounded inverse $T_0^{-1} \in \mathcal{B}(\mathcal{H})$ and such that

$$B_1 + K_\Sigma = T_0 H(z_0)T_0^* = T_0 F_I(z_0)T_0^* \in \mathfrak{S}_p(\mathcal{H}).$$

Since both operators $B_1$ and $K_\Sigma$ are nonnegative, one gets

$$s_j(B_1) = \lambda_j(B_1) \leq \lambda_j(B_1 + K_\Sigma) = s_j(B_1 + K_\Sigma), \quad j \in \mathbb{N}.$$ 

Hence $B_1 \in \mathfrak{S}_p(\mathcal{H})$ and $K_\Sigma \in \mathfrak{S}_p(\mathcal{H})$. Combining these inclusion with (4.28) we get

$$F(z) - B_0 \in \mathfrak{S}_p(\mathcal{H}).$$

Setting here $z = z_0$, yields $B_0 \in \mathfrak{S}_p(\mathcal{H})$. Thus, $F(z) \in \mathfrak{S}_p(\mathcal{H})$ for any $z \in \mathbb{C}_+$. \hfill $\square$

**Corollary 4.17.** Let $F(\cdot) \in R[\mathcal{H}]$ and $F_I(z_0) \in \mathfrak{S}_p(\mathcal{H})$ for some $z_0 \in \mathbb{C}_+$ and $p \in (0, \infty]$. Then $F_I(\cdot)$ takes values in $\mathfrak{S}_p(\mathcal{H})$,

$$F(\cdot) : \mathbb{C}_+ \rightarrow \mathfrak{S}_p(\mathcal{H}).$$

**Remark 4.18.** The result is valid for any two-sided ideal $\mathfrak{S}(\mathcal{H})$ instead of $\mathfrak{S}_p(\mathcal{H})$. In particular, for any $F(\cdot) \in R[\mathcal{H}]$ the following implication holds

$$s_j(F(z_0)) = O(j^{-1/p}) \implies s_j(F(z)) = O(j^{-1/p}), \quad z \in \mathbb{C}_+.$$ 

5. **Examples**

**Example 5.1.** Let $\varphi(\cdot)$ be a scalar $R$-function and $\mathcal{H} = L^2(0, \infty)$. Consider an operator-valued function $F_\varphi(\cdot)$ given by

$$F_\varphi(z)u = -\frac{d^2 u}{dx^2}, \quad \text{dom}(F_\varphi(z)) = \{u \in W^2_2(\mathbb{R}_+) : u'(0) = \varphi(z)u(0)\}, \quad z \in \mathbb{C}_+.$$ 

Clearly, $F(\cdot) \in R(\mathcal{H})$ and

$$\mathcal{D}(F) := \cap_{z \in \mathbb{C}_+} \text{dom } F(z) = W^2_{2,0}(\mathbb{R}_+) := \{u \in W^2_2(\mathbb{R}_+) : u(0) = u'(0) = 0\}$$

is dense in $\mathcal{H}$. The corresponding family of closed quadratic forms reads as follows

$$F_\varphi(z)[u] = \int_{\mathbb{R}_+} |u'(x)|^2 dx + \varphi(z)|u(0)|^2, \quad u \in \text{dom}(F_\varphi(z)) = W^1_2(\mathbb{R}_+), \quad z \in \mathbb{C}_+.$$ 

However, the imaginary parts of these forms constitute a family of non-closable (singular) forms

$$t_\varphi(z)[u] := \text{Im } F_\varphi(z)[u] = \text{Im } \varphi(z) \cdot |u(0)|^2, \quad u \in W^1_2(\mathbb{R}_+), \quad z \in \mathbb{C}_+.$$ 

In accordance with Proposition 4.6 they are non-closable for all $z \in \mathbb{C}_+$ simultaneously.

On the other hand, taking the real part of the form (5.1) one gets

$$t_\varphi(z)[u] := \text{Re } F_\varphi(z)[u] = \int_{\mathbb{R}_+} |u'(x)|^2 dx + \varphi(z) \cdot |u(0)|^2, \quad u \in W^1_2(\mathbb{R}_+), \quad z \in \mathbb{C}_l.$$ 

If $\varphi(\cdot) \in S_+$, then this form is nonnegative for each $z \in \mathbb{C}_l$, hence $F_\varphi(\cdot) \in S_+(\mathcal{H})$. This example demonstrates Proposition 4.6 applied to the real part of $F_\varphi(\cdot)$ in place of its imaginary part: the form $t_\varphi(z)$ is closed for each $z \in \mathbb{C}_l$,

$$\mathcal{H}_{+,r}(z) = \mathcal{H}_{+,r}(z_0) = W^1_2(\mathbb{R}_+), \quad z \in \mathbb{C}_l, \quad \text{and } \mathcal{D}_r[F] = W^1_2(\mathbb{R}_+).$$

Moreover, the operator associated with the form (5.2) is given by

$$F_{\varphi(z),r}(z)u = -\frac{d^2 u}{dx^2}, \quad \text{dom}(F_{\varphi(z),r}(z)) = \{u \in W^2_2(\mathbb{R}_+) : u'(0) = (\text{Re } \varphi(z))u(0)\}, \quad z \in \mathbb{C}_l.$$ 

In accordance with Definition 4.7 (in fact, with its real counterpart) $F_{\varphi(z),r}(\cdot)$ is the real part of the function $F_{\varphi(z)}(\cdot) \in S_+(\mathcal{H})$. 

**NEVANLINNA FAMILIES**

23
Example 5.2. Let \( \varphi(\cdot) \) and \( \mathcal{H} \) be as above. Define an operator-valued function \( G_{\varphi}(\cdot) \) by

\[
G_{\varphi}(z) f = -\frac{i d^2 u}{dx^2}, \quad \text{dom}(G_{\varphi}(z)) = \{ u \in W^2_2(\mathbb{R}_+) : u'(0) = \varphi(z) u(0) \}, \quad z \in \mathbb{C}_+.
\]

Clearly, \( \rho(G_{\varphi}(z)) \neq \emptyset \) for each \( z \in \mathbb{C}_+ \). Furthermore, the corresponding family of quadratic forms have the description

\[
(5.3) \quad G_{\varphi(z)}[u] = i \int_{\mathbb{R}_+} |u'(x)|^2 dx + \varphi(z) |u(0)|^2, \quad u \in \text{dom} G_{\varphi(z)} = \text{dom} (G_{\varphi}(z)).
\]

It follows that the form \( G_{\varphi(z)}[\cdot] \) is dissipative for each \( z \in \mathbb{C}_+ \), hence \( G(\cdot) \in \mathcal{R}(\mathcal{H}) \) and \( \mathcal{D}(G) = W^2_2(\mathbb{R}_+) \) is dense in \( \mathcal{H} \). Taking imaginary part in (5.3) we get

\[
t_{\varphi(z)}[u] := \text{Im} G_{\varphi(z)}[u] = \int_{\mathbb{R}_+} |u'(x)|^2 dx + \text{Im} \varphi(z) |u(0)|^2, \quad \text{dom} t_{\varphi(z)} = \text{dom} (G_{\varphi}(z)).
\]

This form is closable and its closure is given by

\[
\tilde{t}_{\varphi(z)}[u] := \text{Im} G_{\varphi(z)}[u] = \int_{\mathbb{R}_+} |u'(x)|^2 dx + \text{Im} \varphi(z) |u(0)|^2, \quad \text{dom} \tilde{t}_{\varphi(z)} = W^1_2(\mathbb{R}_+), \quad z \in \mathbb{C}_+.
\]

The latter is in accordance with Proposition 4.6

\[
\mathcal{H}_+(z) = W^2_2(\mathbb{R}_+), \quad z \in \mathbb{C}_+, \quad \text{and} \quad \mathcal{D}[G_{\varphi}] = W^1_2(\mathbb{R}_+).
\]

The operator associated with the form \( \tilde{t}_{\varphi(z)} \) (the imaginary part of the operator \( G_{\varphi}(z) \) in the sense of Definition 4.7) is given by

\[
G_{\varphi(z),1} u = -\frac{i d^2 u}{dx^2}, \quad \text{dom} (G_{\varphi(z),1}) = \{ u \in W^2_2(\mathbb{R}_+) : u'(0) = (\text{Im} \varphi(z)) u(0) \}, \quad z \in \mathbb{C}_+.
\]

According to Proposition 4.10 \( a \in \sigma_c(G_{\varphi(z),1}) \) for each \( z \in \mathbb{C}_+ \) and \( a \geq 0 \). Moreover, by Proposition 4.13 \( 0 \in \sigma_c(G_{\varphi(z)}) \) for each \( z \in \mathbb{C}_+ \).

Example 5.3. Let \( \varphi(\cdot) \) be a scalar \( \mathcal{R} \)-function and \( \mathcal{H} = L^2(0,1) \). Consider an operator-valued function \( G_{\varphi}(\cdot) \) given by

\[
G_{\varphi}(z) u = -\frac{i d^2 u}{dx^2}, \quad \text{dom}(G_{\varphi}(z)) = \{ u \in W^2_2(0,1) : u'(0) = \varphi(z) u(0), \quad u(1) = 0 \}, \quad z \in \mathbb{C}_+.
\]

It is easily seen that \( \rho(G_{\varphi(z)}) \neq \emptyset \) for each \( z \in \mathbb{C}_+ \) and the operator \( G_{\varphi(z)} \) has discrete spectrum. Moreover, the corresponding quadratic form is

\[
(5.4) \quad G_{\varphi(z)}[u] = \int_0^1 |u'(x)|^2 dx + \varphi(z) |u(0)|^2, \quad u \in \text{dom} G_{\varphi(z)} = \text{dom} (G_{\varphi}(z)).
\]

Clearly, the form is dissipative, hence \( G(\cdot) \in \mathcal{R}(\mathcal{H}) \) and

\[
\mathcal{D}(G) := \cap_{z \in \mathbb{C}_+} \text{dom} G(z) = \{ u \in W^2_2(0,1) : u(0) = u'(0) = u(1) = 0 \}
\]

is dense in \( \mathcal{H} \). Taking imaginary part in (5.4) one obtains a nonnegative closable form \( t_{\varphi(z)}[\cdot] \) defined on \( \text{dom} (G_{\varphi(z)}) \). Its closure is given by

\[
\tilde{t}_{\varphi(z)}[u] := \text{Im} G_{\varphi(z)}[u] = \int_0^1 |u'(x)|^2 dx + \text{Im} \varphi(z) |u(0)|^2, \quad \text{dom} (\tilde{t}_{\varphi(z)}) = \widetilde{W}^2_2(0,1), \quad z \in \mathbb{C}_+,
\]

where \( \widetilde{W}^2_2(0,1) := \{ u \in W^2_2(0,1) : u(1) = 0 \} \).
The latter is in accordance with Proposition 4.6
\[ \mathcal{H}_+(z) = \widetilde{W}_{2,0}^2(0,1), \quad z \in \mathbb{C}_+, \quad \text{and} \quad \mathcal{D}[G_\varphi] = \widetilde{W}_{2,0}^2(0,1). \]

The operator associated with the form \( t_\varphi(z) \) (the imaginary part of \( G_\varphi(z) \)) is given by
\[ G_{\varphi, I}(z)u = -\frac{d^2u}{dz^2}, \quad \text{dom} (G_{\varphi, I})(z) = \{ u \in W_{2}^2(\mathbb{R}_+) : u'(0) - (\text{Im} \varphi(z))u(0) = u(1) = 0 \}. \]

Since the spectrum of \( G_\varphi(z) \) is discrete, \( \sigma_c(G_\varphi(z)) = \sigma_c(G_{\varphi, I}(z)) = \emptyset \) for each \( z \in \mathbb{C}_+ \). This fact is in accordance with Propositions 4.15 and 4.10. Moreover, the estimate \( s_j((G_{\varphi(z)})^{-1}) = O(j^{-2}), \quad j \in \mathbb{N}, \) holds for each \( z \in \mathbb{C}_+ \). This fact correlates with Remark 4.18.

References

[1] W.O. Amrein, D.B. Pearson, *M-operator: a generalisation of Weyl-Titchmarsh theory*, J. Comp. Appl. Math., 171 (2004), 1–26.
[2] T.Ya. Azizov and I.S. Iokhvidov, *Linear operators in spaces with indefinite metric*, John Wiley and Sons, New York, 1989.
[3] J. Behrndt, M. Langer, *Boundary value problems for elliptic partial differential operators on bounded domains*, J. Funct. Anal., 243 (2007), 536–565.
[4] J. Behrndt, V. Derkach, S. Hassi, and H.S.V. de Snoo, *A realization theorem for generalized Nevanlinna families*, Operators and Matrices, 5, No. 4 (2011), 679–706.
[5] J. Behrndt, S. Hassi, and H.S.V. de Snoo, *Boundary relations, unitary colligations, and functional models*, Complex Analysis and Operator Theory, 3 (2009), 57–98.
[6] C. Benewitz, *Symmetric relations on a Hilbert space*, Lect. Notes Math., 280 (1972), 212–218.
[7] M.S. Brodskii, *Triangular and Jordan representations of linear operators*, Nauka, Moscow, 1968.
[8] E.A. Coddington, *Extension theory of formally normal and symmetric subspaces*, Mem. Amer. Math. Soc., 134 (1973), 1–80.
[9] B. Ćurgus, A. Dijksma, and T.T. Read, *The linearization of boundary eigenvalue problems and reproducing kernel Hilbert spaces*, Linear Algebra and Appl., 329 (2001), 97–136.
[10] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Methods of Functional Analysis and Topology, 6, no.3 (2000), 24–55.
[11] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, *Boundary relations and Weyl families*, Trans. Amer. Math. Soc., 358 (2006), 5351–5400.
[12] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, *Boundary triplets and Weyl functions. Recent developments*, London Mathematical Society Lecture Notes, 404, (2012), 161–220.
[13] V.A. Derkach, S. Hassi, M.M. Malamud, *On a class of generalized boundary triplets and form domain invariant Nevanlinna functions*, (in preparation).
[14] V.A. Derkach and M.M. Malamud, *Generalized resolvents and the boundary value problems for hermitian operators with gaps*, J. Funct. Anal., 95 (1991), 1–95.
[15] V.A. Derkach and M.M. Malamud, *The extension theory of hermitian operators and the moment problem*, J. Math. Sciences, 73 (1995), 141–242.
[16] V.A. Derkach and M.M. Malamud, *On some classes of Holomorphic Operator Functions with Non-negative Imaginary Part*, 16th OT Conference Proceedings, Operator theory, operator algebras and related topics (Timisoara, 1996), 113-138, Theta Found., Bucharest, 1997.
[17] W.F. Donoghue, *Monotone matrix functions and analytic continuation*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
[18] P.A. Fillmore and J.P. Williams, *On operator ranges*, Adv. Math., 7 (1971), 254–281.
[19] F. Gesztesy, E. Tsekanovskii, *On matrix-valued Herglotz functions*, Math. Nachr. 218 (2000), 61-138.
[20] I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear non-selfadjoint operators in Hilbert space*, Nauka, Moscow, 1965, 448 pp.
[21] S. Hassi, H.S.V. de Snoo, and F.H. Szafraniec, *Componentwise and Cartesian decompositions of linear relations*, Dissertationes Mathematicae 465, Polish Academy of Sciences, Warszawa, 2009, 59 pp.
[22] I.S. Kac and M.G. Krein, *R-functions – analytic functions mapping the upper halfplane into itself*, Supplement to the Russian edition of F.V. Atkinson, *Discrete and continuous boundary problems*,
[23] T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin – Heidelberg – New York, 1966.

[24] M.G. Kreǐn, *On the resolvents of an Hermitian operator with the deficiency-index $(m,m)$*. (Doklady) Acad. Sci. URSS (N.S.) 52, (1946), 651654.

[25] A.S. Kostenko, M.M. Malamud, *1–D Schrœdinger operators with local point interactions on a discrete set*, J. Differential Equations 249 (2010), no. 2, 253–304.

[26] M.G. Kreǐn and H. Langer, *On defect subspaces and generalized resolvents of Hermitian operator in Pontryagin space*, Funkts. Anal. i Prilozhen. 5, no.2 (1971), 59–71; ibid. 5 no. 3 (1971), 54–69 (Russian) (English translation: Funct. Anal. Appl., 5 (1971), 136–146; ibid. 5 (1971), 217–228).

[27] M.G. Kreǐn and H. Langer, *Über die $Q$-function eines π-hermiteschen Operators in Raume $\Pi_\kappa$*, Acta. Sci. Math. (Szeged), 34 (1973), 191–230. Siberian Math. J., 18 (1977), 728–746.

[28] M.M. Malamud and S.M. Malamud, *Spectral theory of operator measures in Hilbert space*, St. Petersburg Math. Journal, 15, no. 3 (2003), 1-77.

[29] M.M. Malamud and H. Neidhardt, *Sturm-Liouville boundary value problems with operator potentials and unitary equivalence*, J. Differential Equations, 252 (2012), 5875–5922.

[30] R.S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc., 90 (1959), 192–254.

[31] B. Sz.-Nagy and C. Foías, *Harmonic analysis of operators in Hilbert space*, Budapest, 1967.

[32] O. Post, *Boundary pairs associated with quadratic forms*, arXiv:1210.4707

[33] E.C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations. Part I. Second Edition Clarendon Press*, Oxford 1962 vi+203 pp. 34.30