Nonsmoothness of the boundary and the relevant heat kernel coefficients

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Abstract. The contributions to the heat kernel coefficients generated by the corners of the boundary are studied. For this purpose the internal and external sectors of a wedge and a cone are considered. These sectors are obtained by introducing, inside the wedge, a cylindrical boundary. Transition to a cone is accomplished by identification of the wedge sides. The basic result of the paper is the calculation of the individual contributions to the heat kernel coefficients generated by the boundary singularities. In the course of this analysis certain patterns, that are followed by these contributions, are revealed. The implications of the obtained results in calculations of the vacuum energy for regions with nonsmooth boundary are discussed. The rules for obtaining all the heat kernel coefficients for the minus Laplace operator defined on a polygon or in its cylindrical generalization are formulated.

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1. Introduction

The asymptotic expansion of the heat kernel proves to be important in a series of physical applications. Its coefficients specify the divergences and conformal anomalies taking place in a concrete field theory model [1], the high temperature behaviour of the thermodynamic functions [2, 4] and so on [4].

For a flat manifold with a smooth boundary the heat kernel coefficients are determined by local characteristics of the boundary [5, 6, 7, 8]. If the boundary has discontinuities (for example, it is piecewise smooth), then the latter give additional contributions to the heat kernel coefficients [9]. Usually in physical applications one assumes that the boundary is smooth. However there is a series of problems where such assumption is certainly not acceptable. A typical example here is supplied by fields defined between two plates which cross at a given angle, i.e., inside a dihedral angle. Such a configuration is considered when calculating the Casimir effect for a conducting wedge [10]. If the fields inside a dihedral are subjected to periodicity condition with respect to the angular variable then one is concerned in fact with the fields on a cone, and the point where the boundary has discontinuity becomes an internal point of the cone surface. However the origin of this singularity is the same as in the case of fields inside the wedge. The conical singularity proved to be very important in many areas of mathematical physics [11, 12] and lately it is investigated in connection with studies of quantum fields on the background of black holes [13] and cosmic strings [14].

The general consideration of the boundary nonsmoothness in terms of the heat kernel expansion lacks till now [15]. Such contributions to the heat kernel coefficients $C_{3/2}$ and $C_2$ have been investigated in papers [9]. For a plane domain the contribution to $C_1$ generated by an edge of the boundary is known [16, 17] and by its limiting configuration, by cusp. It is interesting to note that a cusp pointing outward, with respect to the domain under study, leads to change of the power of the asymptotic variable (time) in the heat kernel expansion [5] in comparison with the standard case. In references [4, 10, 8, 13] the asymptotic expansion of the heat kernel with allowance for the boundary nonsmoothness has been built by calculating the relevant Green function of the heat conduction equation. The present paper seeks to show the effectiveness of applying the spectral zeta functions for the calculation of the contributions to the heat kernel coefficients caused by such boundary discontinuities as the corners. For this goal we shall use the technique for constructing the zeta functions developed in [20] and extended in [21]. This method proves to be very effective for calculating the heat kernel coefficients for different boundary conditions given on a sphere and cylinder [3, 20, 22]. A close approach has been used in [23, 24].

We shall consider internal and external parts of a plane sector formed by two infinite radial rays emerging from the center of a circle of radius $R$ at angle $\alpha$ to one another (see figure 1, where I is the internal circular sector and II is the external circular sector). The choice of such domains with nonsmooth boundary is caused by the possibility of constructing for them the global zeta functions. The latter cannot be done, for example,
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for an open angle (figure 1 without circular arc inside the angle). This point will be discussed in detail in Section II. In both the sectors the Laplace operator is defined acting on scalar real functions subjected to the Dirichlet or Neumann boundary conditions. If we substitute in figure 1 the radial rays by crossed planes and the circular arc 1 – 2 by an appropriate part of a cylinder surface we arrive at the boundary value problems which have the same heat kernel coefficients as in the plane case.

Identifying the points of the boundary with the polar coordinates \((r, \theta = 0)\) and \((r, \theta = \alpha)\), i.e., imposing the periodicity condition with respect to the angular variable \(\theta\) with a period \(\alpha\), we arrive obviously at the spectral problem for the Laplace operator on two parts of lateral surface of a cone \(C_\alpha\): internal part \((r \leq R)\) and external part \((r \geq R)\). At \(r = R\) we can, as before, impose the Dirichlet or Neumann conditions.

In the present paper, six coefficients of the asymptotic expansion of the heat kernel for the boundary value problems specified above will be calculated by making use of the relevant zeta functions. It will be shown that each of these coefficients, starting from the third one, is the sum of contributions generated by the corners of the boundary and by the curvature of the arc 1–2. Analysis of the obtained results enables one to reveal some regularities obeyed by the contributions of the boundary nonsmoothness to the heat kernel coefficients, namely: i) the corner contribution substantially depends on whether the corner is made up by crossing two straight lines or two lines with nonzero curvature; ii) contributions of adjacent angles to the heat kernel coefficients \(B_{3/2}\) and \(B_{5/2}\) have opposite signs in the same way as the contributions of a circular arc to the coefficients \(B_1\) and \(B_2\) for the internal and external regions. In the case of a polygon or its cylindrical generalization the rules are formulated for obtaining all the heat kernel coefficients for the minus Laplace operator. The implications of the obtained results in the Casimir energy calculations with employment of the zeta function technique are also considered. In particular, it is shown that nonsmoothness of the boundary does not make always worse the situation with calculation of the vacuum energy in the framework of the zeta regularization method. For example, the corner contributions can simply be mutually cancelled in the same way as the contributions of the curved boundary are cancelled when taking into account the internal and external regions. Besides, the analysis conducted enables one to infer that it is very unlikely to get a finite and unique result for the vacuum energy by smoothing the boundary singularities, e.g., by taking into account the atomic structure of the boundary or quantum fluctuations of the boundary.

The paper is organized as follows. Section 2 is devoted to the detailed discussion of the choice of the domains with piece-wise smooth boundary (internal and external circular sectors), for which the complete spectral zeta functions can be constructed. In section 3 the heat kernel coefficients are calculated for internal circular sector by making use of the counter integral representation for the corresponding zeta functions \([20, 4]\). In section 4 the contributions to the heat kernel coefficients of the individual boundary discontinuities (corners of different angles) are identified. Further (in section 5) the heat kernel coefficients for the union of both the sectors are calculated and the coefficients
for the external sector are obtained as the regarding differences. Technically it turns out to be simpler in comparison with calculation of the heat kernel coefficients for the external sector alone. The asymptotic expansion of the heat kernel for the union of internal and external sectors is constructed by differentiation of the logarithm of the function specifying the frequency equation [23, 24]. In section 6 the identification of the individual contributions of boundary nonsmoothness to heat kernel coefficients for external sector is carried out. Some general rules for these contributions are revealed here for both sectors. In section 7 we conclude with a few summarizing remarks.

2. Choice of domains with piecewise smooth boundaries

When choosing the domain for investigating the contribution to the heat kernel coefficients of the boundary discontinuities we pursue two goals: the boundary of a domain should have a sufficient number of discontinuities and at the same time for this domain one can construct the spectral zeta function. In order to investigate the corner singularities of the boundary one should at first sight take the most simple configuration, namely, the angle on plane formed by two radial unrestricted rays or dihedral in space. However, for these domains the global zeta functions cannot be constructed. Let us explain this point in detail. We consider the Laplace operator $\Delta$ acting on the scalar functions defined inside the dihedral of opening angle $\alpha$ (wedge of an angle $\alpha$, $W_\alpha \times \mathbb{R}^1$) and subjected to the Dirichlet conditions on the wedge sides. In cylindrical coordinate system $(r, \theta, z)$ the eigenfunctions in this problem are

$$u_{\lambda nk}(r, \theta, z) = \frac{e^{ikz}}{\sqrt{\pi \alpha}} J_{np}(\lambda r) \sin(n \theta),$$

$$0 \leq \theta \leq \alpha, \quad p = \pi/\alpha, \quad n = 1, 2, \ldots, \quad 0 \leq \lambda < \infty,$$  

(2.1)

where $J_{\nu}(z)$ is the Bessel function. The operator $-\Delta$ has the following eigenvalues

$$\omega^2(k, \lambda) = k^2 + \lambda^2, \quad -\infty < k < \infty, \quad 0 \leq \lambda < \infty.$$  

(2.2)

These eigenvalues do not depend on the quantum number $n$, i.e., there is a degeneracy with respect to this number, the multiplicity of this degeneracy being infinite\(^\S\)

$$N = \sum_{n=1}^{\infty} 1 = \infty.$$  

(2.3)

The global spectral zeta function of the operator $L$, $\zeta(s)$, is determined as the trace of the operator $L^{-s}$

$$\zeta(s) = \text{Tr} \ L^{-2s} = \sum_j N_j \lambda_j^{-2s},$$  

(2.4)

\(^\S\) In addition to infinite multiplicity, the every point of the spectrum with fixed values of $k$ and $\lambda$ is a nonisolated point. In fact, the spectrum is continuous according to the terminology of spectral theory of operators in Hilbert space [24, 26, 27] and $\omega^2(k, \lambda)$ are not eigenvalues as the eigenfunctions are not square integrable.
where $\lambda_j^2$ is the $j$th eigenvalue of the operator $L$, and $N_j$ is the degeneracy of this eigenvalue. Obviously, this definition does not work when $N_j$ is infinite.

In order to remove this degeneracy we put inside a dihedral a cylindrical boundary as it is shown in figure 1 (the arc 1–2). On the internal and external sides of this boundary the scalar field will obey the Dirichlet or Neumann conditions. Thus, we are considering the internal (I) and external (II) sectors. Certainly, the union of these two sectors is not identical to unrestricted dihedral (to a wedge), because now the values of the field on the arc 1–2 are not arbitrary but they are determined by the corresponding boundary conditions. In the new configuration there appear additional discontinuities of the boundary at the points 1 and 2. However at these points the angle, at which the involved boundary surfaces intersect, is fixed (it is equal to $\pi/2$). It will be shown below that the contribution of such boundary singularities to the heat kernel coefficients can be easily separated from the contribution of the corner at the origin. The latter is proportional to the difference $\pi - \alpha$, because at $\alpha = \pi$ the singularity at the origin disappears.

We shall concern with the standard asymptotic expansion of heat kernel

$$K(t) = \sum_j e^{-\lambda_j^2t} = (4\pi t)^{-d/2} \sum_{n=0}^{\infty} t^{n/2} B_{n/2} + ES,$$

where $d$ is the dimension of the manifold under study and ES stands for the exponentially small corrections as $t \to 0$. This definition leads to the same heat kernel coefficients $B_{n/2}$ for a dihedral and for corresponding plane problem obtained by crossing of the dihedral by a transverse plane. In fact, the eigenvalues of the operator $(-\Delta)$ in these two problems obey the relation

$$\lambda_j^2(d = 3) = k^2 + \lambda_j^2(d = 2).$$

Hence

$$K_{d=3}(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-k^2t} K_{d=2}(t) = \frac{1}{4\pi t} K_{d=2}(t).$$

Taking into account the definition (2.5) one easily deduce from (2.7) that the heat kernel coefficients $B_{n/2}$ in two eigenvalue boundary problems, mentioned above, are equal. Here one should bear in mind that the heat kernel $K_{d=3}(t)$ and its coefficients are referred to a unite length along the OZ axes. In view of this, when calculating the heat kernel coefficients we shall consider either the spectral problem on a plane ($d = 2$) or in the space ($d = 3$), pursuing only simplicity of calculation.

3. Heat kernel coefficients for internal sector

3.1. Dirichlet boundary conditions

At first we consider only internal sector (the region I in figure 1) for $d = 3$, i.e., for internal sector of a dihedral or a wedge\[1\]. We employ here the technique developed\[1\] in the framework of the quantum billiard studies this problem has been investigated in \[12\].
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in [20] and extended to the generalized bounded cone in [21] (see also book [4]). In this approach the spectral zeta function \( \zeta(s) \) should be constructed in the beginning, and then the relevant heat kernel coefficients are calculated through the relation [3]

\[
\frac{B_n}{(4\pi)^{d/2}} = \lim_{s \to d/2 - n} (s + n - d/2) \zeta(s) \Gamma(s), \quad n = 0, 1/2, 1, \ldots.
\] (3.1)

For the Dirichlet boundary conditions the eigenfunctions of the operator \((-\Delta)\) in the region \( I \) are defined by equation (2.1), with \( \lambda \) being the roots of the equations

\[
J_{np}(\lambda_{nm}R) = 0, \quad p = \pi/\alpha, \quad n = 1, 2, \ldots.
\] (3.2)

Here the subscript \( m = 1, 2, \ldots \) numbers the nonzero roots of these equations for fixed \( n \). The relevant eigenvalues are defined by equation (2.2) with \( \lambda = \lambda_{nm} \)

\[
\omega^2 = k^2 + \lambda_{nm}^2, \quad -\infty < k < \infty.
\] (3.3)

In view of the behavior of the Bessel function near zero

\[
J_\nu(z) \sim z^{\nu}/(2^{\nu} \Gamma(\nu+1))
\]

it follows that the frequency equation (3.2) has the zero root of ‘multiplicity’ \( \nu \). Such roots should be removed from the definition of the spectral zeta function (2.4). Therefore instead of equation (3.2) we shall use the following frequency equation

\[
(\lambda R)^{-\nu} J_\nu(\lambda R) = 0, \quad \nu = np, \quad n = 1, 2, \ldots.
\] (3.4)

According to the general definition (2.4) the zeta function in the problem under study is given by

\[
\zeta_D(s) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{n=1}^{\infty} \sum_{m} (k^2 + \lambda_{nm}^2)^{-s}.
\] (3.5)

As usual we substitute the sum over \( m \) in this formula by the contour integral in the plane of a complex variable \( \lambda \)

\[
\zeta_D(s) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint_C d\lambda (\lambda^2 + k^2)^{-s} \frac{d}{d\lambda} \ln[(\lambda R)^{-\nu} J_\nu(\lambda R)],
\] (3.6)

where the contour \( C \) encloses counter clockwise the positive roots of equations (3.4). On deforming the contour \( C \) in an appropriate way and integrating with respect to \( k \) by means of the formula

\[
\int_{-\infty}^{\infty} \frac{dk}{(k^2 + \lambda^2)^s} = \frac{\sqrt{\pi} \Gamma(s - 1/2)}{\Gamma(s)}, \quad \text{Re } s > 1/2,
\] (3.7)

we arrive at the result

\[
\zeta_D(s) = \frac{(R/p)^{2s-1}}{2\sqrt{\pi} \Gamma(s) \Gamma(\frac{3}{2} - s)} \sum_{n=1}^{\infty} n^{1-2s} \int_{0}^{\infty} dy y^{1-2s} \frac{d}{dy} \ln[(\nu y)^{-\nu} I_\nu(\nu y)],
\] (3.8)

where \( I_\nu(z) \) is the modified Bessel function.

The analytical continuation of (3.8) into the left half-plane of the complex variable \( s \) is accomplished by making use of the uniform asymptotic expansion of the function \( I_\nu(\nu y) \)

\[
I_\nu(\nu y) \simeq \frac{1}{\sqrt{2\pi \nu}} \frac{e^{\nu y}}{(1 + y^2)^{1/4}} \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right),
\] (3.9)
where
\[ \eta = \sqrt{1 + y^2} + \ln \frac{y}{1 + \sqrt{1 + y^2}}, \quad \frac{d\eta}{dy} = \frac{\sqrt{1 + y^2}}{y}, \] (3.10)
and \( u_k(t) \) are the known polynomials in \( t = 1/\sqrt{1 + y^2} \). Their explicit form and the corresponding recurrent relations can be found in references [28, 29]. Keeping in this expansion all the terms proportional to \( \nu^{-k} \) with \( k \leq 4 \) one can write
\[ \ln \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right) \simeq \sum_{k=1}^{4} \frac{F^D_k(t)}{\nu^k}, \] (3.11)
where
\[ F^D_1(t) = \frac{1}{8} t - \frac{5}{24} t^3, \]
\[ F^D_2(t) = \frac{1}{16} t^2 - \frac{3}{8} t^4 + \frac{5}{16} t^6, \]
\[ F^D_3(t) = \frac{25}{384} t^3 - \frac{531}{640} t^5 + \frac{221}{128} t^7 - \frac{1105}{1152} t^9, \]
\[ F^D_4(t) = \frac{13}{128} t^4 - \frac{71}{32} t^6 + \frac{531}{64} t^8 - \frac{339}{32} t^{10} + \frac{565}{128} t^{12}. \] (3.12)
In accordance with this expansion we represent the zeta function in equation (3.9) as the following sum
\[ \zeta^D(s) = \sum_{j=-1}^{4} Z^D_j(s), \] (3.13)
where
\[ Z^D_{-1}(s) = C(s)p^{2-2s}\zeta_R(2s - 2) \int_0^\infty dy \frac{y^{1-2s}}{1 + \sqrt{1 + y^2}} (\eta - \ln y), \] (3.14)
\[ Z^D_0(s) = -\frac{1}{4} C(s)p^{1-2s}\zeta_R(2s - 1) \int_0^\infty dy \frac{y^{1-2s}}{1 + \sqrt{1 + y^2}} \ln(1 + y^2), \] (3.15)
\[ Z^D_j(s) = C(s)p^{j-2s-1}\zeta_R(2s + j - 1) \int_0^\infty dy y^{1-2s} \frac{d}{dy} F^D_j(t), \quad j = 1, 2, 3, 4. \] (3.16)
In these formulas \( \zeta_R(s) \) is the Riemann zeta function and
\[ C(s) = \frac{R^{2s-1}}{2\sqrt{\pi} \Gamma(s) \Gamma(3/2 - s)}. \] (3.17)
Substituting (3.10) into equation (3.14) and taking into account the value of the integral
\[ \int_0^\infty dy \frac{y^{2-2s}}{1 + \sqrt{1 + y^2}} = \frac{1}{4} \frac{\Gamma(s-1) \Gamma(3/2 - s)}{s-1/2}, \] (3.18)
we obtain for the function \( Z^D_{-1}(s) \) the final expression
\[ Z^D_{-1}(s) = \frac{R}{8\sqrt{\pi}} \left( \frac{R}{p} \right)^{2s-2} \frac{\Gamma(s-1)}{(s-1/2) \Gamma(s)} \zeta_R(2s - 2). \] (3.19)
The product \( \Gamma(s) Z_{D}^{1}(s) \) has simple poles at the points \( s = 3/2, \ 1, \ \text{and} \ 1/2 \) with the respective residua

\[
\frac{\alpha R^{2}}{16\pi^{3/2}}, \quad -\frac{R}{8\pi}, \quad \frac{\sqrt{\pi}}{48\alpha}.
\]  

(3.20)

Multiplying these residua by \((4\pi)^{3/2}\) we obtain, according to equation \(3.4\), the contributions of the function \( Z_{D}^{1}(s) \) to the heat kernel coefficients \( B_{0}, B_{1/2}, \text{and} \ B_{1} \), respectively.

By making use of the table integral \[^{30}\]

\[
2 \int_{0}^{\infty} \frac{dy}{y} \frac{y^{2s}}{1+y^{2}} = \Gamma \left( \frac{3}{2} - s \right) \Gamma \left( s - \frac{1}{2} \right),
\]

(3.21)

we recast equation \(3.13\) to the form

\[
Z_{D}^{0}(s) = -\frac{1}{8\sqrt{\pi}} \left( \frac{R}{p} \right)^{2s-1} \zeta_{R}(2s-1) \frac{\Gamma(s-1/2)}{\Gamma(s)}.
\]

(3.22)

The expression \( \Gamma(s) Z_{D}^{0}(s) \) has simple poles at the points \( s = 1/2 \) and \( s = 1 \) with the residua

\[
-\frac{R}{16\pi}, \quad \frac{1}{16\sqrt{\pi}},
\]

(3.23)

which give the respective contributions to the heat kernel coefficients \( B_{1/2} \) and \( B_{1} \) (see table 1).

Calculation of the functions \( Z_{j}(s), \ j = 1, \ldots, 4, \) defined in equation \(3.16\), can be carried out with hardly any trouble. Their contribution to the heat kernel coefficients are given in table 1. In order to get the complete values of these coefficients one should sum all the elements of respective rows.

3.2. Neumann boundary conditions

In this case the operator \(-\Delta\) has the following eigenfunctions in internal circular sector I (see figure 1)

\[
v_{\lambda_{nk}}(r, \theta, z) = \eta_{n0} \frac{e^{ikz}}{\sqrt{\pi\alpha}} J_{np}(\lambda_{nm}r) \cos(np\theta), \quad n = 0, 1, 2, \ldots,
\]

\[
0 \leq \theta \leq \alpha, \quad p = \pi\alpha, \quad \eta_{n0} = \begin{cases} 
\frac{1}{\sqrt{2}}, & n = 0, \\
1, & n = 1, 2, \ldots.
\end{cases}
\]

(3.24)

Here \( \lambda_{nm} \) are the roots of the equations

\[
J_{np}'(\lambda_{nm}R) = 0, \quad n = 0, 1, 2, \ldots.
\]

(3.25)

Taking into account the behavior of the derivative of the Bessel function at the origin \( J_{\nu}'(z) \sim z^{\nu-1}/(2^{\nu}\Gamma(\nu)) \), we multiply equation \(3.25\) by \((\lambda_{nm}R)^{1-np}\) in order to exclude the zero multiple roots

\[
(\lambda R)^{1-\nu} J_{\nu}'(\lambda R) = 0, \quad \nu = np, \quad n = 0, 1, \ldots.
\]

(3.26)
By making use of the frequency equations (3.26) we can write immediately the
integral representations for the zeta function in the problem at hand analogous to
equation (3.8)
\[ \zeta_N(s) = C(s) \left[ \int_0^\infty dy \, y^{1-2s} \frac{d}{dy} \ln(y I_0'(y)) \right. \]
\[ \left. + \sum_{n=1}^\infty n^{1-2s} \int_0^\infty dy \, y^{1-2s} \frac{d}{dy} \ln(y^{1-\nu} I'_\nu(\nu y)) \right] , \]
where the coefficient \( C(s) \) is has been defined in equation (3.17). An important
distinction of this equation is a new term with \( n = 0 \), that was absent in the equation
(3.8) for the Dirichlet boundary condition.

We again use the uniform asymptotic expansion [29, 28] but now for the derivative
of the Bessel function \( I'_\nu(\nu y) \)
\[ I'_\nu(\nu y) \simeq \frac{1}{\sqrt{2\pi\nu}} \frac{(1 + y^2)^{1/4}}{y} e^{\nu\eta} \left( 1 + \sum_{k=1}^\infty \frac{v_k(t)}{\nu^k} \right) , \]
where \( \eta \) is defined in equation (3.10) and the functions \( v_k(t) \) are the known polynomials
[28] in \( t(y) \). Again we use the approximation
\[ \ln \left( 1 + \sum_{k=1}^\infty \frac{v_k(t)}{\nu^k} \right) \simeq \sum_{k=1}^4 \frac{F_k^N(t)}{\nu^k} , \]
where
\[ F_1^N(t) = -\frac{3}{8} t + \frac{7}{24} t^3 , \]
\[ F_2^N(t) = -\frac{3}{16} t^2 + \frac{5}{8} t^4 - \frac{7}{16} t^6 , \]
\[ F_3^N(t) = -\frac{21}{128} t^3 + \frac{869}{640} t^5 - \frac{315}{128} t^7 + \frac{1463}{1152} t^9 , \]
\[ F_4^N(t) = -\frac{27}{128} t^4 + \frac{109}{32} t^6 - \frac{733}{64} t^8 + \frac{441}{32} t^{10} - \frac{707}{128} t^{12} . \]

Instead of equation (3.13) we have now
\[ \zeta_N(s) = \tilde{Z}^N(s) + \sum_{j=1}^4 Z_j^N(s) , \]
where
\[ \tilde{Z}^N(s) = C(s) \int_0^\infty dy \, y^{1-2s} \frac{d}{dy} \ln[y I_0'(y)] , \]
\[ Z_{-1}^N(s) = Z_{-1}^D(s) , \quad Z_0^N(s) = Z_0^D(s) , \]
\[ Z_j^N(s) = C(s) p^{-2s-1+j} \zeta_R(2s + j - 1) \int_0^\infty dy \, y^{1-2s} \frac{d}{dy} F_j^N(t) , \quad j = 1, 2, 3, 4. \]

Finding the functions \( Z_j^N(s) , \quad j = 1, \ldots, 4 \) in the expansion (3.31) presents no
difficulty while the function \( \tilde{Z}^N(s) \) needs more thorough treatment. By making use of
the asymptotics
\[
\frac{d}{dy} \ln [y I_0'(y)] \simeq \frac{2}{y} + \frac{1}{4} y + O(y^3), \quad y \to 0
\]
\[
\frac{d}{dy} \ln [y I_0'(y)] \simeq 1 + \frac{1}{2y} + \frac{3}{8y^2} + \frac{3}{8y^3} + \frac{63}{128y^4} + O(y^{-5}), \quad y \to \infty
\]
it is easy to make sure that the integral in the definition of the function \( \tilde{Z}_N(s) \) does not converge at any values of \( s \). In order to overcome this drawback first the zeta function \( \zeta_N(s) \) for a scalar field with nonzero mass \( m \) should be constructed and on calculating the residua according to equation (3.1) with this zeta function the mass \( m \) should be put equal to zero. Following this line we consider the function
\[
\tilde{Z}_N^m(s) = C(s) \int_m^\infty dy (y^2 - m^2)^{-s+1/2} \frac{d}{dy} \ln [y I_0'(y)].
\]
(3.37)

It is defined in the region \( 1 < \text{Re } s < 3/2 \), the lower (upper) limit in these inequalities being determined by the convergence of the integral when \( y \to \infty \) (\( y \to m \)).

In order to find, by equation (3.1), the contribution of the function \( \tilde{Z}_N^m(s) \) to the coefficient \( B_0 \) the analytical continuation of \( \tilde{Z}_N^m(s) \) to the region \( \text{Re } s \geq 3/2 \) is required. In the most simple way it can be done by adding and subtracting from the integrand its asymptotics when \( y \to m \). For our goals it is sufficient to take only the first term in this asymptotics
\[
\frac{\tilde{Z}_N^m(s)}{C(s)} = \int_m^\infty dy (y^2 - m^2)^{1/2-s} \left\{ \frac{d}{dy} \ln [y I_0'(y)] - f_m \right\} + f_m \int_m^\infty dy (y^2 - m^2)^{1/2-s},
\]
(3.38)

where
\[
f_m = \left. \frac{d}{dy} \ln [y I_0'(y)] \right|_{y=m}.
\]

The first term in equation (3.38) is regular at the point \( s = 3/2 \), but the integral in the second term gives a simple pole at this point
\[
\int_m^\infty dy (y^2 - m^2)^{1/2-s} = \frac{m^{1-2s} \Gamma(s-1) \Gamma(3/2-s)}{\Gamma(1/2)}.
\]

However the gamma function \( \Gamma(3/2-s) \) responsible for this pole is canceled by the same multiplier in the denominator of the coefficient \( C(s) \) (see equation (3.17)). As a result the function \( \tilde{Z}_N^m(s) \) does not give the contribution to the heat kernel coefficient \( B_0 \).

For calculating the contribution of the function \( \tilde{Z}_N^m(s) \) to the rest heat kernel coefficients by equation (3.1), this function should be analytically continued to the region \( \text{Re } s \leq 1 \). It can again be done by adding and subtracting the asymptotics of the integrand when \( y \to \infty \) now. Further the residua are found according to equation (3.1) and only after that the mass \( m \) is put equal to zero. The corresponding results are presented in table 2. Finding the contributions of the rest functions \( Z_j^N(s) = j = -1, \ldots, 4 \) to the heat kernel coefficients presents no trouble (see table 2).

\footnote{Another way to treat \( n = 0 \) case for the Neumann boundary conditions has been explained in detail in \cite{31}.}
3.3. Spectral problems on a cone

Imposing on the eigenfunctions the periodicity condition with respect to the angular variable $\theta$ with a period $\alpha$ we pass from a wedge to a cone because in this case the respective points on the radial rays $O1$ and $O2$ are identified (see figure 1), the circular arc 1–2 being converted into a circumference. Apparently, the boundary discontinuities at the points 1 and 2 disappear. This circumference separates two parts (internal and external) of the cone surface which for simplicity will be refereed to as the internal and external sectors of the cone. On the circumference separating them we impose on the eigenfunctions, as before, the Dirichlet or Neumann conditions. The corresponding (unnormalized) eigenfunctions for internal cone sector are

$$u_{nm}(r, \theta) = J_{np}(\lambda_{nm}r) \begin{pmatrix} \sin np \theta \\ \cos np \theta \end{pmatrix}, \quad p = \frac{2\pi}{\alpha}, \quad n = 0, 1, 2, \ldots, \quad (3.39)$$

where $\lambda_{nm}$ are the roots of the frequency equations for the Dirichlet boundary conditions at $r = R$

$$J_{np}(\lambda_{nm}R) = 0, \quad n = 0, 1, 2, \ldots \quad (3.40)$$

or of those for the Neumann conditions

$$J'_{np}(\lambda_{nm}R) = 0, \quad n = 0, 1, 2, \ldots \quad (3.41)$$

For the external sector the Bessel functions in equations (3.39)–(3.41) are replaced by the Hankel functions $H_{np}^{(1)}(\lambda r)$.

From (3.39) it follows that on the cone all the states with $n \neq 0$ are double degenerate

$$N_0 = 1, \quad N_n = 2, \quad n = 1, 2, \ldots. \quad (3.42)$$

Thus, in order to find the heat kernel coefficients for the internal cone sector one should put in previous calculations

$$p = 2\pi/\alpha, \quad (3.43)$$

take into account the degeneracy of states (3.43), and sum up over $n$ starting with $n = 0$.

Let us consider the internal sector of a cone with the Dirichlet conditions on the circumference 1–2. For the corresponding zeta function $\xi_D(s)$ the representation (3.38) holds with $p$ defined in (3.43) and with the summation replaced by $2\sum_{n=0}^{\infty}$, where the prime on the summation sign means that the $n = 0$ term is counted with half weight. For $\xi_D(s)$ in the sum (3.13) there arises an additional term with $n = 0$

$$\xi_D(s) = \tilde{Z}^D(s) + 2 \sum_{j=-1}^{4} Z_j^D(s), \quad (3.44)$$

where

$$\tilde{Z}^D(s) = C(s) \int_0^\infty dy y^{1-2s} \frac{d}{dy} \ln I_0(y), \quad (3.45)$$

and the functions $Z_j^D(s), \ j = -1, 0, \ldots, 4$ are defined by equations (3.14)–(3.16) with $p = 2\pi/\alpha$. 

The asymptotics
\[
\frac{d}{dy} \ln I_0(y) \simeq \frac{y}{2} - \frac{y^3}{16} + \mathcal{O}(y^5), \quad y \to 0, \tag{3.46}
\]
\[
\frac{d}{dy} \ln I_0(y) \simeq 1 - \frac{1}{2y} - \frac{1}{8y^2} - \frac{1}{8y^3} - \frac{25}{128y^4} + \mathcal{O}(y^{-5}), \quad y \to \infty \tag{3.47}
\]
imply that the function \( \tilde{Z}^D(s) \) in equation (3.45) is defined in the region
\[
1 < \text{Re } s < 3/2. \tag{3.48}
\]
To single out in the integral (3.45) the pole contribution at the point \( s = 3/2 \) we rewrite the function \( \tilde{Z}^D(s) \) as follows
\[
\tilde{Z}^D(s) = C(s) \int_0^1 dy \, y^{1-2s} \left[ \frac{d}{dy} \ln I_0(y) - \frac{y}{2} \right]
+ \frac{C(s)}{2} \int_0^1 dy \, y^{2-2s} + C(s) \int_1^{\infty} dy \, y^{1-2s} \frac{d}{dy} I_0(y). \tag{3.49}
\]
The integrals in the first and third terms in equation (3.49) are regular at the point \( s = 3/2 \). Substituting the second term from equation (3.49) into definition (3.1) we get
\[
\text{Res}_{s \to 3/2 - 0} \left( \tilde{Z}^D(s) \Gamma(s) \right) = \text{Res}_{s \to 3/2 - 0} \left( \frac{R^{2s-1}}{2\sqrt{\pi}(3/2 - s)} \frac{1}{3 - 2s} \right) = 0.
\]
Thus the function \( \tilde{Z}^D(s) \) does not give any contribution to the coefficient \( B_0 \).

So as to find the contribution of the function \( \tilde{Z}^D(s) \) to the heat kernel coefficients \( B_n, \ n = 1/2, 1, \ldots \) we again split the domain of integration in equation (3.45) into two intervals \((0, 1)\) and \((1, \infty)\). When integrating over the second interval we add and subtract under the integral sign the asymptotics (3.47). When calculating the residua at the points \( s = 1, \ s = 1/2, \ 0, \ -1/2, \ -1 \) we shall take the right-hand limits in equation (3.1). It gives the following contributions to the heat kernel coefficients \( B_{1/2}, \ B_1, \ B_{3/2}, \ B_2, \) and \( B_{5/2}, \) respectively
\[
2 R \sqrt{\pi}, \ -\pi, \ -\frac{\sqrt{\pi}}{2 R}, \ -\frac{\pi}{4R^2}, \ -\frac{25 \sqrt{\pi}}{48 R^3}. \tag{3.50}
\]
In order to evaluate the contributions of the functions \( Z^D_j(s), \ j = -1, 0, \ldots, 4 \) to the heat kernel coefficients, in addition to (3.50), one should substitute in table 1 \( \alpha \) by \( \alpha/2 \) and multiply all the elements of this table by 2. Summing the contributions (3.50) and the data from table 1 we obtain the heat kernel coefficients for the internal sector I on the cone with the Dirichlet conditions on the circle 1–2
\[
B_0 = \frac{\alpha}{2} R^2, \quad B_{1/2} = -\frac{1}{2} \alpha R \sqrt{\pi}, \quad B_1 = \frac{2}{3} \frac{\pi^2}{\alpha} + \frac{\alpha}{6},
B_{3/2} = \frac{\alpha \sqrt{\pi}}{64 R}, \quad B_2 = \frac{4 \alpha}{315 R^2}, \quad B_{5/2} = \frac{37 \alpha \sqrt{\pi}}{213}. \tag{3.51}
\]
For Neumann boundary conditions the spectral zeta function for internal sector on a cone is given by
\[
\zeta_N(s) = \tilde{Z}^N(s) + 2 \sum_{j=-1}^{4} Z^N_j(s), \tag{3.52}
\]
where the functions $\tilde{Z}^N$ and $Z_j^N$ are determined in equations (3.32) – (3.34) with $\alpha$ replaced by $\alpha/2$. The corresponding coefficients of the heat kernel expansion are

$$B_0 = \frac{\alpha R^2}{2}, \quad B_{1/2} = \frac{1}{2} \alpha R \sqrt{\pi}, \quad B_1 = \frac{2 \pi^2}{3 \alpha} + \frac{\alpha}{6},$$

$$B_{3/2} = \frac{5 \alpha \sqrt{\pi}}{64 R}, \quad B_2 = \frac{4 \alpha}{45 R^2}, \quad B_{5/2} = \frac{269 \alpha \sqrt{\pi}}{213}.$$  

(3.53)

4. Identification of the individual contributions to the heat kernel coefficients for internal sector

Let us envisage the internal sector I (see figure 1). Its boundary possesses the following peculiarities which give contribution to the heat kernel coefficients: nonzero curvature of the arc 1–2; right-angled corners at the points 1 and 2; corner of angle $\alpha$ at the origin. The arc contribution is proportional to its length, i.e., to $\alpha$, the contributions of the right-angled corners does not depend on $\alpha$, contribution of the corner at the origin vanishes when $\alpha = \pi$. It is sufficient to separate in each of the heat kernel coefficients the contribution due to each boundary singularity enumerated above. We demonstrate this considering the heat kernel coefficients for internal circular sector I with Dirichlet conditions on its boundary (see table 1).

The first coefficient

$$B_0 = \alpha \frac{R^2}{2} = |\Omega|$$

(4.1)

is the area $|\Omega|$ of the circular sector I. The second heat kernel coefficient

$$B_{1/2} = -\sqrt{\pi} \frac{2 R + \alpha R}{2} = -\sqrt{\pi} \frac{L}{2},$$

(4.2)

where $L$ is the length of the sector boundary (its perimeter). As concerns the coefficient

$$B_1 = \frac{1}{6} \left( \frac{\pi^2}{\alpha} + \alpha \right) + \frac{\pi}{2},$$

(4.3)

the situation is more complicated. It is clear that $\pi/2$ is the contribution of two right-angled corners at the points 1 and 2. The term $\alpha/6$ contains the contribution of the curvature of the arc 1–2 (denote it by $k_{\text{arc}} \alpha$) and a part of the contribution of the corner at the origin $O$ (the latter is equal to $(1/6 - k_{\text{arc}}) \alpha$). In terms of these notations the complete contribution of the corner at the origin is

$$c(\alpha) = \frac{\pi^2}{6 \alpha} + \left( \frac{1}{6} - k_{\text{arc}} \right) \alpha.$$  

(4.4)

From the condition

$$c(\pi) = \pi \left( \frac{1}{3} - k_{\text{arc}} \right) = 0$$

(4.5)

it follows that

$$k_{\text{arc}} = 1/3.$$  

(4.6)
Nonsmoothness of the boundary

Further we find
\[ c(\alpha) = \frac{\pi^2 - \alpha^2}{6\alpha}. \tag{4.7} \]

Thus the corner of an angle \( \alpha \) on the boundary gives the contribution to \( B_1 \) defined by equation (4.7). At first time this contribution has been calculated in \[16\]. Another method to derive it is described in \[17\]. In both the cases the Green’s function of the equation of heat conductivity was considered. In our approach it is found by making use of the spectral zeta function technique.

Now we are in position to check the consistency of our reasoning, namely, we can calculate the contribution to the coefficient \( B_1 \) due to the right-angled corners at the points 1 and 2 by making use of equation (4.7). It gives
\[ 2c(\pi/2) = \pi/2. \tag{4.8} \]

It is this value that has been attributed to this contribution above (see equation (4.3)).

Identification of individual contributions to the rest of heat kernel coefficients can be done in a direct way. The terms independent of the angle \( \alpha \) are attributed to the right-angled corners at the points 1 and 2, while the linear in \( \alpha \) terms are due to the curvature of the arc 1–2 (see table 3). Merely such a separation of individual peculiarity contributions leads to a correct value of the arc curvature contribution which is known from the heat kernel expansion for a smooth boundary (see below). It is worth noting that the angle \( \alpha \) at the origin does not contribute to the heat kernel coefficients \( B_n \) with \( n > 1 \) even when \( \alpha = \pi/2 \). It may be explained only taking into account that the higher derivatives of the radius vector of the boundary curve behave in a different way at the origin and at the points 1 and 2.

Let \( \Omega \) be a simply connected region of a plane bounded by a smooth curve \( \Gamma \). For the heat kernel of the minus Laplace operator with the Dirichlet conditions on \( \Gamma \) the following asymptotic expansion holds when \( t \to 0 \) (see, for example, references \[5, 16, 17\])
\[
K(t) \simeq \frac{|\Omega|}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{12\pi} \int_\Gamma k(s) \, ds + \frac{\sqrt{\pi t}}{256\pi} \int_\Gamma k^2(s) \, ds + \frac{t}{315\pi} \int_\Gamma k^3(s) \, ds \\
+ \sqrt{\pi t^3} \left[ \frac{37}{215\pi} \int_\Gamma k^4(s) \, ds - \frac{11}{211\pi} \int_\Gamma (k'(s))^2 \, ds \right] + O(t^2), \tag{4.9}
\]
where \(|\Omega|\) is the area of \( \Omega \), \( L \) is the length of \( \Gamma \), \( k(s) \) is the curvature of the curve \( \Gamma \) at the point \( s \), \( k^2(s) = (d^2r/ds^2)^2 \), where \( r(s) \) is a parametric representation of the curve \( \Gamma \); \( s \) is the natural parameter along \( \Gamma \): \( ds^2 = (dr)^2 \); \( k'(s) = dk(s)/ds \). For convex portions of \( \Gamma \) \( k(s) \) is considered to be positive, and for concave parts of \( \Gamma \) \( k(s) \) is assumed to be negative.

In the expansion (4.9) the numerical coefficients of \( k \), \( k^2 \), \( k^3 \), and \( k^4 \) are derived from the contributions, proportional to \( \alpha \), to the heat kernel coefficients \( B_1 \), \( B_{3/2} \), \( B_2 \), and \( B_{3/2} \), respectively (see table 3). Here it should be taken into account that in the problem under study \( k(s) = 1/R \), \( ds = R \, d\alpha \) and the coefficients \( B_n \) enter the heat kernel expansion (2.3) with the multiplier \( 1/(4\pi) \).
If we go from a wedge to a cone by identifying the radial rays $O1$ and $O2$ the corners at the points 1 and 2 disappear. The sole singular point remains the origin $O$ which becomes an internal point of the cone surface. However the contribution of this singularity to the coefficient $B_1$ has the same nature as in the case of wedge.

The heat kernel coefficients for internal sector I on the cone with Dirichlet condition on the boundary 1–2 are listen in equation (3.51). The coefficients $B_0$ and $B_{1/2}$ obey general equations (4.1) and (4.2) with corresponding values for $|\Omega|$ and $L$. Separation of the contributions to $B_1$ generated by singularity at the origin and by the curvature of the circle 1–2 can be conducted in the same way as for a wedge. Let $k_{arc} \alpha$ be the contribution of the boundary 1–2 and

$$d(\alpha) = \frac{2}{3} \frac{\pi^2}{\alpha} + \frac{\alpha}{6} - k_{arc} \alpha$$

be the contribution due to the singularity at the origin. When $\alpha = 2\pi$ the surface of a cone becomes a plane and the singularity at the origin $O$ disappear. Therefore, $d(2\pi) = 0$. It gives

$$k_{arc} = 1/3,$$

i.e., for this quantity we have the same value as in the case of a wedge (see (4.6)). With (4.11) allowed for, one deduces from the equation (4.10)

$$d(\alpha) = \frac{2}{3} \frac{\pi^2}{\alpha} - \frac{\alpha}{6} = 2c(\alpha/2),$$

where $c(\alpha)$ is defined in equation (4.7).

Identification of the individual contributions to the heat kernel coefficients for the Neumann boundary conditions is conducted in the same way (see table 3). The first coefficient $B_0$ is, as before, the area $|\Omega|$ of the sector I. For the second coefficient $B_{1/2}$ we have equation (4.2) with opposite sign in the right-hand side

$$B_{1/2} = \sqrt{\pi} \frac{L}{2}, \quad L = 2R + \alpha R.$$  

The contribution to the coefficient $B_1$ generated by the singularity at the origin proves to be the same as for the Dirichlet conditions, $c(\alpha)$.

For Neumann boundary conditions we failed to find in the literature the formula analogous to equation (4.9), i.e., the asymptotic expansion of the heat kernel for the operator $-\Delta$ defined in the region of a plane with smooth boundary curve. However, as was noted earlier, the spectral problem on the plane, envisaged by us, and its cylindrical generalization have the same coefficients $B_n$. Therefore, for verification of our results, concerning the contributions to $B_n$ due to the smooth parts of the boundary, we used the expansion of the heat kernel for Robin conditions with smooth boundary that has the dimension greater than 1 (see references [7, 32]).
5. Heat kernel coefficients for external sector

5.1. Basic formulas

Here we shall use the technique applied in [23]. It is close to the method developed in [4, 20, 21] and employed in preceding subsections.

Let us consider the spectral zeta function depending on a parameter $x^2$

$$\zeta(s, x^2) = \sum_n (\lambda_n^2 + x^2)^{-s}. \quad (5.1)$$

It may be regarded as an extension to the general spectral problem of the Epstein-Hurwitz zeta function

$$\zeta_{EH}(s, a^2) = \sum_{n=1}^{\infty} (n^2 + a^2)^{-s}. \quad (5.1)$$

It turns out that the heat kernel coefficients $B_n$ can be found from the expansion of the function $\zeta(s, x^2)$ in terms of inverse powers of $x$ developed for a certain value of $s$. It is convenient to chose this value to be equal to $d/2$. In fact, from the definition of the gamma function it follows that

$$\Gamma(s) (\lambda_n^2 + x^2)^{-s} = \int_0^\infty rm(t) t^{-s} e^{-(\lambda_n^2 + x^2)t}. \quad (5.2)$$

For $s = 1 + d/2$ equation (5.2) gives

$$\Gamma \left(1 + \frac{d}{2}\right) \sum_n (\lambda_n^2 + x^2)^{-1-d/2} = \int_0^\infty dt t^{d/2} e^{-x^2t} \sum_n e^{-\lambda_n^2t} = \int_0^\infty dt t^{d/2} e^{-x^2t} K(t). \quad (5.3)$$

On substituting the asymptotic expansion (2.5) in equation (5.3) we obtain

$$\Gamma(1 + d/2) \zeta(1 + d/2, x^2) \simeq \sum_{n=6}^{\infty} \frac{B_n/2}{(4\pi)^{d/2}} \Gamma \left(1 + \frac{n}{2}\right) x^{-n-2} \quad (5.4)$$

$$\times \left[ \frac{B_0}{x^2} + \frac{B_{1/2} \Gamma(3/2)}{x^3} + \frac{B_1 \Gamma(2)}{x^4} + \frac{B_{3/2} \Gamma(5/2)}{x^5} + \frac{B_2 \Gamma(3)}{x^6} + \frac{B_{5/2} \Gamma(7/2)}{x^7} + \frac{B_3 \Gamma(4)}{x^8} + O(x^{-9}) \right].$$

Let $F(z) = 0$ be the frequency equation which determines the spectrum $\lambda_n$ in the problem under consideration. We also suppose that the function $F(z)$ allows one to rewrite this equation in the form

$$\prod_n (\lambda_n^2 - z^2) = 0. \quad (5.5)$$

Taking into account the equality

$$\frac{1}{(\lambda_n^2 + x^2)^m} = \frac{(-1)^m}{m!} \left( \frac{d}{2x \, dx} \right)^m \ln(\lambda_n^2 + x^2), \quad z = ix, \quad (5.6)$$

we recast the left-hand side of equation (5.4) to the form

$$\Gamma \left(1 + \frac{d}{2}\right) \zeta \left(1 + \frac{d}{2}, x^2\right) = -\left( \frac{1}{2x \, dx} \right)^{1+d/2} \ln F(ix) . \quad (5.7)$$
Obviously formula (5.7) is applicable only to the manifolds of even dimension.

Rather than to calculate the heat kernel coefficients $B_{n}^{II}$ for the external sector alone it is simpler to find first the coefficients $B_{n}^{I+II}$ for the union of the sectors I and II. Then the coefficients $B_{n}^{II}$ are obtained as the corresponding differences

$$B_{n}^{II} = B_{n}^{I+II} - B_{n}^{I}, \quad n = 0, 1/2, 1, \ldots.$$ 

When calculating the heat kernel coefficients for the union of the internal (I) and external (II) sectors on should take into account the following. The technique used by us [4, 20, 21, 23, 24] gives the difference between the zeta function for the region $I+II$ and the zeta function for the corresponding part of the Euclidean space. The last contribution is usually referred to as the Minkowski space-time contribution [33]. In the case under consideration the zeta function for an open angle $\alpha$ (without circular arc 1-2) is subtracted from the zeta function sought for. The heat kernel coefficients $B_{n}$, corresponding to the Minkowski space-time contribution, can be calculated in the following way: in the respective heat kernel coefficients for the sector I one should put $R = R_1 \to \infty$ and discard the curvature contribution of the arc 1-2 and of two rightangled internal corners at the points 1 and 2.

5.2. Internal and external circular sectors of a wedge with Dirichlet condition on separating arc

Now we proceed to practical using the general formulas (5.4) and (5.7) for calculation of the heat kernel coefficients. We consider the scalar Laplace operator on the union of internal and external circular sectors on a plane (see figure 1) with Dirichlet conditions on the arc 1–2 separating these sectors. As it was explained in Section 2 the heat kernel coefficients for the corresponding boundary value problem in space ($d = 3$) The frequency equations are will be the same.

$$J_{np}(Rz) = 0, \quad \text{(internal sector I)}, \quad (5.8)$$

$$H_{np}^{(1)}(Rz) = 0, \quad \text{(external sector II)}, \quad n = 1, 2, \ldots, \quad p = \pi/\alpha. \quad (5.9)$$

Further we shall concern with the product $J_{\nu}(ix)H_{\nu}^{(1)}(ix)$ and use for it the uniform asymptotic expansion [28] which depends only on $x^2$. In this case the condition (5.5) is apparently satisfied. This point can also be explained in the following way. In view of the formula [30, 34]

$$J_{\nu}(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right), \quad \nu \neq -1, -2, \ldots \quad (5.10)$$

for the function $z^{-\nu}J_{\nu}(z)$ the representation of type (5.7) holds. Here $z_{\nu,m}$ ($m = 1, 2, 3, \ldots$) are the nonzero roots of the function $J_{\nu}(z)$. The multiplier $(z/2)^\nu$ in equation (5.10) is canceled in the product $J_{\nu}(ix)H_{\nu}^{(1)}(ix) \sim I_{\nu}(x)K_{\nu}(x)$ with the multiplier $(x/2)^{-\nu}$ following from the small $x$ asymptotics of the function $K_{\nu}(x)$: $K_{\nu}(x) \sim (1/2)\Gamma(\nu)(x/2)^{\nu}, \quad \nu > 0$. Hence the requirement (5.5) is satisfied.
Nonsmoothness of the boundary

Setting in (5.7) \( d = 2 \) and substituting in it the left-hand sides of the frequency equations (5.8) and (5.9) we obtain

\[
\zeta_D(2, x^2) = - \left( \frac{1}{2x} \frac{d}{dx} \right)^2 \sum_{n=1}^{\infty} \ln I_\nu(Rx)K_\nu(Rx) .
\]

(5.11)

Now we use the uniform asymptotic expansion for the product of the modified Bessel functions [28]

\[
\ln I_\nu(Rx)K_\nu(Rx) = - \ln 2 \nu + \ln t + \sum_{j=1}^{\infty} \frac{G_D^{2j}(t)}{\nu^{2j/2}} ,
\]

(5.12)

where

\[
t = \frac{1}{\sqrt{1 + z^2}} , \quad z = \frac{Rx}{\nu} , \quad \nu = np = n \frac{\pi}{\alpha} ,
\]

(5.13)

and \( G_D^{2j}(t) \) are the polynomials in \( t \) expressed in terms of the known functions \( u_k(t) \). In order to calculate the first six coefficients in the expansion (5.4) it is sufficient to keep two terms of the sum in (5.12). The relevant coefficients \( G_D^{2j}(t) \) and \( G_D^{4j}(t) \) are

\[
G_D^{2}(t) = \frac{1}{8} t^2 - \frac{3}{4} t^4 + \frac{5}{8} t^6 ,
\]

(5.14)

\[
G_D^{4}(t) = \frac{13}{64} t^4 - \frac{71}{16} t^6 + \frac{531}{32} t^8 - \frac{339}{16} t^{10} + \frac{565}{64} t^{12} .
\]

(5.15)

Substituting in (5.11) differentiation with respect to \( x \) we get

\[
\zeta_D(2, x^2) = - \sum_{n=1}^{\infty} \left( \frac{R}{\nu} \right)^4 \left( \frac{t^3}{2} \frac{d}{dt} \right)^2 \left[ \ln t + \sum_{j=1}^{2} \frac{G_D^{2j}(t)}{\nu^{2j/2}} \right]
\]

\[
= - R^4 \sum_{n=1}^{\infty} \left[ \frac{t^4}{2\nu^4} + \frac{t^6}{4\nu^6} (1 - 18t^2 + 30t^4)
\right.
\]

\[
 + \frac{3t^8}{32\nu^8} (13 - 568t^2 + 3540t^4 - 6780t^6 + 3955t^8) \right] .
\]

(5.16)

It will be recalled that \( t \) depends on \( x \) and \( n \) through (5.13).

All the sums in the equation (5.16) are finite. Hence the problem of analytic continuation does not emerge here. In order to do the summation exactly we use the formula [30, 35]

\[
\sum_{n=1}^{\infty} \frac{1}{y^2 + n^2} = \frac{\pi}{2y} \left( \coth \frac{\pi y}{2} - \frac{1}{\pi y} \right) \approx \frac{\pi}{2y} - \frac{1}{2y^2} \equiv S_1(y) ,
\]

(5.17)

When \( y \to \infty \), the function \( S_1(y) \) affords the value of the sum on the left-hand side of equation (5.17) up to exponentially small corrections. Step-by-step differentiation with respect of \( y \) of the left and right hand sides of the equation (5.17) gives the values of all the sums entering (5.16)

\[
\sum_{n=1}^{\infty} \frac{1}{(y^2 + n^2)^2} \approx - \frac{1}{2y} \frac{d}{dy} S_1(y) = \frac{\pi}{4y^3} - \frac{1}{2y^4} \equiv S_2(y) ,
\]

(5.18)
Nonsmoothness of the boundary

\[ \sum_{n=1}^{\infty} \frac{1}{(y^n + n^2)^3} \simeq - \frac{1}{2} \frac{1}{2y} \frac{d}{dy} S_2(y) = \frac{3\pi}{16y^5} - \frac{1}{2y^6} \equiv S_3(y), \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ \sum_{n=1}^{\infty} \frac{1}{(y^n + n^2)^{k+1}} \simeq - \frac{1}{k} \frac{1}{2y} \frac{d}{dy} S_k(y) \equiv S_{k+1}(y). \] (5.18)

In order to express the zeta function \( \zeta_D(2, x^2) \) in terms of \( S_k(x) \), \( k = 1, 2, \ldots, 8 \) explicitly the following substitutions should be accomplished in equation (5.16)

\[ t^2 = \nu^2 = \frac{1}{p^2(n^2 + \bar{x}^2)}, \] (5.19)

\[ t^2 = 1 - \frac{\bar{x}^2}{n^2 + \bar{x}^2}, \quad \bar{x} = \frac{Rx}{p}, \] (5.20)

the change (5.20) being done only in the round brackets in this equation. As a result the zeta function \( \zeta_D(2, x^2) \) acquires the form

\[ \zeta_D(2, x^2) = - \frac{R^4}{2p^2} S_2(\bar{x}) - \frac{R^4}{4p^6} \left[ 13 S_3(\bar{x}) - 42 \bar{x}^2 S_4(\bar{x}) + 30 \bar{x}^4 S_5(\bar{x}) \right] \\
- \frac{3R^4}{32p^8} \left[ 160 S_4(\bar{x}) - 1992 \bar{x}^2 S_5(\bar{x}) + 6930 \bar{x}^4 S_6(\bar{x}) \right. \\
\left. - 9040 \bar{x}^6 S_7(\bar{x}) + 3955 \bar{x}^8 S_8(\bar{x}) \right]. \] (5.21)

Substitution of the explicit expressions for the functions \( S_k(\bar{x}) \) from (5.18) to (5.21) gives

\[ \zeta_D(2, x^2) = - \frac{R^4}{8x^2} + \frac{3\alpha}{4x^4} + \frac{1}{2} \frac{1}{4R^2} \left[ \frac{512R^5}{8x^6} + \frac{555\alpha}{217R^3x^7} + \frac{39}{26R^4x^8} + \mathcal{O}(x^9) \right]. \] (5.22)

Comparing the expansions (5.4) and (5.22) we obtain the values of the first seven heat kernel coefficients in the problem under consideration

\[ \bar{B}_0^{I+II} = 0, \quad \bar{B}_{1/2}^{I+II} = -\alpha R\sqrt{\pi}, \quad \bar{B}_1^{I+II} = \pi, \]

\[ \bar{B}_{3/2}^{I+II} = \frac{\alpha}{32R}, \quad \bar{B}_2^{I+II} = \frac{\alpha}{4R^2}, \quad \bar{B}_{5/2}^{I+II} = \frac{37}{212R^{3}}, \quad \bar{B}_3^{I+II} = \frac{13\pi}{32R^4}, \] (5.23)

where \( \bar{B}_n^{I+II} = \bar{B}_n^{I+II} - \bar{B}_n \), and \( \bar{B}_n \) are the heat kernel coefficients corresponding to the open angle \( \alpha \) (without the arc 1-2)

\[ \bar{B}_0 = \frac{1}{2} \alpha R_1^2, \quad \bar{B}_{1/2} = -\sqrt{\pi} R_1 - \frac{\alpha}{2} \alpha R_1, \quad \bar{B}_1 = \frac{\pi^2 - \alpha^2}{6\alpha} = c(\alpha), \quad R_1 \to \infty. \] (5.24)

All the rest of coefficients \( \bar{B}_j \) vanish \( \bar{B}_j = 0, \quad j \geq 3/2 \).

Using equations (5.23), (5.24) and the results for the internal sector I, presented in table 1 we deduce the heat kernel coefficients for the external sector II with Dirichlet boundary condition

\[ B_0^{II} = \frac{\alpha}{2} (R_1^2 - R^2), \quad B_1^{II} = -\sqrt{\pi} (R_1 - R) - \frac{\alpha}{2} \alpha \frac{1}{2} (R_1 + R), \quad B_1^{II} = \frac{\pi}{2} - \frac{\alpha}{3}; \]

\[ B_{3/2}^{II} = -\frac{\sqrt{\pi}}{4R} + \frac{\alpha \sqrt{\pi}}{64R^2}, \quad B_2^{II} = \frac{\frac{\alpha}{8R^2} - \frac{4}{315} \frac{\alpha}{R^2}, \quad B_3^{II} = -\frac{25}{96} \frac{\sqrt{\pi}}{R^3} + \frac{37}{213R^{3}} \frac{\alpha}{R^3}. \] (5.25)
5.3. Internal and external circular sectors of a wedge with Neumann condition on separating arc

In the case of Neumann boundary conditions on a separating arc the heat kernel coefficients for the union of internal and external sectors are calculated completely in a similar line. Now the frequency equations read

\[
\frac{d}{dr} J_{np}(rz) \bigg|_{r=R} = 0, \quad \text{(internal sector I)},
\]

\[
\frac{d}{dr} H_{np}^{(1)}(rz) \bigg|_{r=R} = 0, \quad \text{external sector II}, \quad n = 0, 1, 2, \ldots, \quad p = \pi/\alpha.
\]

(5.26) (5.27)

The angular part of the eigenfunctions is proportional to \(\cos(np\theta)\), therefore the index \(n\) takes integer values starting with zero.

For the zeta function in this eigenvalue problem the representation analogous to (5.11) holds

\[
\zeta_N(2, x^2) = -\left(\frac{1}{2x} \frac{d}{dx}\right)^2 \sum_{n=0}^{\infty} \ln \left[-I_{np}'(Rx)K_{np}'(Rx)\right].
\]

(5.28)

Again we use the uniform asymptotic expansion \[28\]

\[
\ln \left[-I_{np}'(Rx)K_{np}'(Rx)\right] \simeq - \ln 2\nu - \ln t + \sum_{j=1}^{2} \frac{G_N^{2j}(t)}{\nu^{2j}},
\]

(5.29)

where

\[
G_2^{N}(t) = -\frac{3}{8} t^2 + \frac{5}{4} t^4 - \frac{7}{8} t^6,
\]

\[
G_4^{N}(t) = -\frac{27}{64} t^4 + \frac{109}{16} t^6 - \frac{733}{32} t^8 + \frac{441}{16} t^{10} - \frac{707}{64} t^{12}.
\]

(5.30)

On substituting (5.29) and (5.30) and differentiating in equation (5.28), the zeta function under consideration assumes the form

\[
\zeta_N(2, x^2) = - R^4 \sum_{n=0}^{\infty} \left[-\frac{t^4}{2\nu^4} - \frac{3t^6}{4\nu^6} (1 - 10 t^2 + 14 t^2)
\right.

\[
-\frac{t^8}{32\nu^8} (81 - 2616 t^2 + 14660 t^4 - 26460 t^6 + 14847 t^8)\right].
\]

(5.31)

In order to take the sum over \(n\) in (5.31) we apply the formula that follows from (5.17)

\[
\sum_{n=0}^{\infty} \frac{1}{y^2 + n^2} = \frac{\pi}{2y} \left(\coth \pi y + \frac{1}{\pi y}\right) \simeq \frac{\pi}{2y} + \frac{1}{2y^2} \equiv \tilde{S}_1(y).
\]

(5.32)

Sequential differentiation of equation (5.32) gives

\[
\sum_{n=0}^{\infty} \frac{1}{(y^2 + n^2)^{k+1}} \simeq - \frac{1}{k} \frac{1}{2y} \frac{d}{dy} \tilde{S}_k(y) \equiv \tilde{S}_{k+1}(y).
\]

(5.33)
On making use of the change of variables (5.19) and (5.20) in equation (5.31) the zeta function $\zeta_N(2, x^2)$ assumes the form
\[
\zeta_N(2, x^2) = \frac{R^4}{2p^8} \bar{S}_2(x) + \frac{3R^4}{4p^6} \left[ 5\bar{S}_3(x) - 18x^2 \bar{S}_4(x) + 14x^4 \bar{S}_5(x) \right]
+ \frac{R^4}{32p^8} \left[ 512 \bar{S}_4(x) - 6712x^2 \bar{S}_5(x) + 24362x^4 \bar{S}_6(x) \right]
- 32928x^6 \bar{S}_7(x) + 14847x^8 \bar{S}_8(x).
\]
(5.34)
Substitution of the function $\bar{S}_k(x)$ with $k = 2, 3, \ldots, 8$ from (5.32) and (5.33) gives
\[
\zeta_N(2, x^2) = \frac{\alpha R}{8x^3} + \frac{1}{4x^4} + \frac{15}{512} \frac{\alpha}{R^2 x^5} + \frac{3}{8} \frac{1}{R^2 x^6} + \frac{4035}{131072} \frac{\alpha}{R^3 x^7} + \frac{81}{64} \frac{1}{R^4 x^8} + O(x^{-9}).
\]
(5.35)
Comparison of the expansions (5.35) and (5.4) gives the following values for the heat kernel coefficients $\bar{B}_n^{l+II}$ for Neumann boundary conditions
\[
\bar{B}_0^{l+II} = 0, \quad \bar{B}_1^{l+II} = \alpha R\sqrt{\pi}, \quad \bar{B}_2^{l+II} = \pi, \quad \bar{B}_3^{l+II} = \frac{5\alpha \sqrt{\pi}}{32R}, \quad \bar{B}_4^{l+II} = \frac{3\pi}{4R^2}, \quad \bar{B}_5^{l+II} = \frac{269}{4096} \frac{\alpha \sqrt{\pi}}{R^3}, \quad \bar{B}_6^{l+II} = \frac{27\pi}{32R^4}.
\]
(5.36)
The heat kernel coefficients corresponding to the Minkowski space-time contribution in the case under consideration are derived from the respective coefficients for internal sector I by putting there $R = R_1 \to \infty$ and omitting the contribution due to the curvature of the arc 1-2 and contributions of the rightangled corners at the points 1 and 2 (see Tables 2 and 3)
\[
\bar{B}_0 = \frac{\alpha}{2} R^2, \quad \bar{B}_{1/2} = \sqrt{\pi} R_1 + \frac{\alpha}{2} \sqrt{\pi} R_1, \quad \bar{B}_1 = c(\alpha), \quad \bar{B}_j = 0, \quad j = 3/2, 2, \ldots.
\]
(5.37)
By making use of the equations (5.36), (5.37) and table 2 we derive the heat kernel coefficients for the external sector II with Neumann condition
\[
\bar{B}_0^{II} = \frac{\alpha}{2} (R_1^2 - R^2), \quad \bar{B}_1^{II} = \sqrt{\pi} (R_1 - R) + \frac{\alpha}{2} \sqrt{\pi} (R_1 + R), \quad \bar{B}_1^{II} = \frac{\pi}{2} - \frac{\alpha}{3}, \quad \bar{B}_2^{II} = \frac{3\pi}{8R^2} - \frac{\alpha}{45} R^2, \quad \bar{B}_3^{II} = \frac{-21}{32} \sqrt{\pi} R^3 + \frac{269}{213} \frac{\alpha \sqrt{\pi}}{R^4}.
\]
(5.38)
5.4. Internal and external sectors on a cone

In the case of a cone the eigenfunctions are defined in (3.33) with $\lambda_{nm}$ being the roots of the frequency equations (5.8), (5.9) and (5.26), (5.27) where $p = 2\pi/\alpha$. For both Dirichlet and Neumann conditions on the circle 1–2 the index $n$ ranges from $n = 0$. The state degeneracy in both the cases is determined by equation (3.42).

All this implies that in order to proceed to a cone one should put in the relevant formulas for a wedge $p = 2\pi/\alpha$ and use for summation over $n$ the following relations (instead of equations (5.18), (5.32), and (5.33))
\[
\frac{1}{y^2} + 2 \sum_{n=1}^{\infty} \frac{1}{y^2 + n^2} = \frac{\pi}{y} \coth \pi y \cong \frac{\pi}{y} \equiv S_k^c(y),
\frac{1}{y^{2(k+1)}} + 2 \sum_{n=1}^{\infty} \frac{1}{(y^2 + n^2)^{k+1}} \cong -\frac{1}{k} \frac{1}{2y} \frac{d}{dy} S_k^c(y) \equiv S_{k+1}^c(y), \quad k = 1, 2, \ldots.
\]
(5.39)
It is essential that these summation formulas should be employed for both Dirichlet and Neumann boundary conditions given on the circle 1–2 and separating the internal and external sectors of a cone.

First we consider Dirichlet conditions. Carrying out in (5.21) the change of variables (3.43) and using for summation the functions $S_{\kappa}(\bar{x})$ from equation (5.39) we obtain the expansion for the zeta function in the spectral problem at hand

$$\zeta_{\kappa}(2, x^2) = -\frac{\alpha R}{8 x^3} + \frac{3 \alpha}{512 R x^5} + \frac{555}{217} \frac{\alpha}{R^3 x^7} + \mathcal{O}(x^{-9}).$$  \hspace{1em} (5.40)

This expansion can formally be derived from the relevant zeta function for a wedge, equation (5.22), by omitting the terms with even powers of $x$. Comparison of the series (5.40) with equation (5.4) affords the values of the first seven heat kernel coefficients $\bar{B}^{I+II}_n$

$$B^{I+II}_0 = B^{I+II}_1 = B^{I+II}_2 = B^{I+II}_3 = 0, \quad B^{I+II}_{1/2} = -\alpha \sqrt{\pi} R,$$

$$\bar{B}^{I+II}_{3/2} = \frac{\alpha \sqrt{\pi}}{32 R}, \quad \bar{B}^{I+II}_{5/2} = \frac{37 \alpha \sqrt{\pi}}{2^{12} R^3}. \hspace{1em} (5.41)$$

The heat kernel coefficients for Minkowski spacetime contribution are

$$\bar{B}_0 = \frac{\alpha}{2} R_1, \quad \bar{B}_{1/2} = -\frac{\alpha}{2} \sqrt{\pi} R_1, \quad \bar{B}_1 = 2 c(\alpha/2), \quad \bar{B}_j = 0, \quad j \geq 3/2. \hspace{1em} (5.42)$$

We deduce from equations (5.41), (5.42), and (3.51) the heat kernel coefficients for external sector of a cone with Dirichlet boundary conditions on the circle 1–2

$$B^{II}_0 = \frac{\alpha}{2} (R^1 - R^2), \quad B^{II}_{1/2} = -\frac{\alpha}{2} \sqrt{\pi} (R^1 - R), \quad B^{II}_1 = -\frac{\alpha}{3},$$

$$B^{II}_{3/2} = \frac{\alpha \sqrt{\pi}}{64 R}, \quad B^{II}_2 = -\frac{4 \alpha}{315 R^2}, \quad B^{II}_{5/2} = \frac{37 \alpha \sqrt{\pi}}{2^{13} R^3}. \hspace{1em} (5.43)$$

By analogy with this we obtain from equation (5.34) for Neumann boundary conditions

$$\zeta_{\kappa}(2, x^2) = \frac{\alpha R}{8 x^3} + \frac{15 \alpha}{512 R x^5} + \frac{4035 \alpha}{217} \frac{1}{R^3 x^7} + \mathcal{O}(x^{-9}).$$  \hspace{1em} (5.44)

This expansion can formally be derived from the relevant zeta function for a wedge equation (5.35) by omitting the terms with even powers of $x$. Comparing the equations (5.44) and (5.4) we obtain the values of the first seven heat kernel coefficients $\bar{B}^{I+II}_n$

$$\bar{B}^{I+II}_0 = \bar{B}^{I+II}_1 = \bar{B}^{I+II}_2 = \bar{B}^{I+II}_3 = 0, \quad \bar{B}^{I+II}_{1/2} = \alpha \sqrt{\pi} R,$$

$$\bar{B}^{I+II}_{3/2} = \frac{5 \alpha \sqrt{\pi}}{32 R}, \quad \bar{B}^{I+II}_{5/2} = \frac{269 \alpha \sqrt{\pi}}{2^{12} R^3}. \hspace{1em} (5.45)$$

The heat kernel coefficients for Minkowski spacetime contribution in this case are

$$\bar{B}_0 = \frac{\alpha}{2} R_1, \quad \bar{B}_{1/2} = \frac{\alpha}{2} \sqrt{\pi} R_1, \quad \bar{B}_1 = 2 c(\alpha/2), \quad \bar{B}_j = 0, \quad j \geq 3/2. \hspace{1em} (5.46)$$

From equations (5.43), (5.46), and (3.53) we derive the heat kernel coefficients for the external sector II of the cone with Neumann boundary conditions on the circle 1–2

$$B^{II}_0 = \frac{\alpha}{2} (R^1 - R^2), \quad B^{II}_{1/2} = \frac{\alpha}{2} \sqrt{\pi} (R^1 - R), \quad B^{II}_1 = -\frac{\alpha}{3},$$

$$B^{II}_{3/2} = \frac{5 \alpha \sqrt{\pi}}{64 R}, \quad B^{II}_2 = \frac{4 \alpha}{45 R^2}, \quad B^{II}_{5/2} = \frac{269 \alpha \sqrt{\pi}}{2^{13} R^3}. \hspace{1em} (5.47)$$
6. Specification of individual contributions to heat kernel coefficients for external sector. Some general rules

The heat kernel coefficients for external sector are given in equations (5.25), (5.38), (5.43), and (5.47). Identification of the individual contributions to these coefficients due to the boundary nonsmoothness does not differ basically from the analogous procedure for internal sector carried out in section 4. The sole point that should be noted here is the following. The area $|\Omega|$ and the perimeter $L$ are in this case infinite. Therefore they should be treated as the limit, when $R_1$ tends to infinity, of the expressions

$$|\Omega| = \frac{\alpha}{2} (R_1^2 - R^2), \quad L = 2 (R_1 - R) + \alpha (R_1 + R). \quad (6.1)$$

The coefficients $B_0$ and $B_1/2$ are defined, as before, by equations (4.1), (4.2), and (4.13) with $|\Omega|$ and $L$ defined in (6.1). Contributions to the rest of the heat kernel coefficients, which are proportional to the angle $\alpha$ are due to the curvature of the arc 1–2 and the contributions which are independent of $\alpha$ are due to the right-angled corners at the points 1 and 2. The results of this analysis are represented in table 3 together with the heat kernel coefficients for the internal sector I. Analysis of this table enables one to reveal some general rules for corner contributions.

It is interesting to note that the right-angled corners at the points 1 and 2 give the contributions to the coefficients $B_{3/2}$ and $B_{5/2}$ which have opposite signs for internal and external sectors, it being valid for all configurations and boundary conditions considered. Such a behaviour seems to be related with convexity (internal sector) or concavity (external sector) of the arc 1–2. It is analogous to the contributions of the smooth segments of the boundary to the heat kernel coefficients $B_1$ and $B_2$.

The boundary discontinuities due to the corner at the origin $O$ and to the corners at the points 1 and 2 contribute to the heat kernel coefficients basically in different ways. Corner at the origin gives contribution only to the coefficient $B_1$ (even when $\alpha = \pi/2$). The corners at the points 1 and 2 contribute to all the heat kernel coefficients starting with $n = 1$. Obviously the reason of this distinction is a geometrical one, the corner at the origin is formed by crossing two straight lines, while the corners at the points 1 and 2 are the result of intersection of lines one of which has a nonzero curvature.

This general assertion concerning the corner contribution to the heat kernel coefficients can be illustrated by a known expression for the heat kernel expansion for a rectangle with sides $a$ and $b$

$$K(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp \left[ - \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 t \right] = \left( \frac{a}{\sqrt{4\pi t}} - \frac{1}{2} \right) \left( \frac{b}{\sqrt{4\pi t}} - \frac{1}{2} \right) + ES$$

$$= \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + ES, \quad (6.2)$$

where ES denotes the exponentially small corrections as $t \to 0$ (see, for example, [19]). Here the scalar operator $-\Delta$ with Dirichlet boundary conditions is considered. For a rectangle the coefficient $B_1$ is obviously equal to the contributions of four right-angled
corners. Indeed, the third term in the expansion (6.2) can be represented in the form
\[ \frac{1}{4} = \frac{1}{4\pi} B_1 = \frac{1}{4\pi} 4c(\alpha = \pi/2), \]
where \( c(\alpha) \) is given in equation (4.7). Besides these corners the boundary of a rectangle has no other singularities, therefore the heat kernel coefficients \( B_n \) with \( n > 1 \) vanish in this problem.

These rules for obtaining the heat kernel coefficients are directly generalized to an arbitrary polygon with the angles \( \alpha_i \). The first two coefficients \( B_0 \) and \( B_{1/2} \) are defined by equations (4.1) and (4.2), respectively, where \( |\Omega| \) is the area of the polygon and \( L \) is its perimeter. The third coefficient \( B_1 \) is equal to the sum of the contributions due to the angles \( \alpha_i \)
\[ B_1 = \sum_i c(\alpha_i). \quad (6.3) \]
The rest of the coefficients \( B_n, \ n > 1 \) vanish. In particular, it implies that the zeta function technique should provide a finite value of the Casimir energy for a polygon on a plane (\( B_{3/2} = 0 \)) and for a cylindrical generalization of the polygon spectral problem (\( B_2 = 0 \)). These subjects have been discussed earlier in papers [36].

In reference [37] the vacuum energy of massless fields including electromagnetic field was calculated for the boundary configuration shown in figure 1 with \( \alpha = \pi \). Both versions of this boundary value problem were considered, three-dimensional one (a semicircular infinite cylinder) and two-dimensional spectral problem on the plane. In both the cases the zeta regularization didn’t give a finite value of the Casimir energy. As known [4], the reason of this is nonzero heat kernel coefficients \( B_2 \) for \( d = 3 \) and \( B_{3/2} \) for \( d = 2 \). Using the results of the present paper we can elucidate the geometrical origin of this fact. Let us consider electromagnetic filed in internal and external sectors of the wedge with \( \alpha = \pi \). For the boundaries with cylindrical symmetry electromagnetic field reduces to two massless fields subjected to Dirichlet and Neumann boundary conditions [38, 39]. From table 3 it follows that
\[ B_{e-m}^2 = 2 \frac{\pi}{8 R^2} + 2 \frac{3}{8} \frac{\pi}{R^2}, \quad B_{e-m}^{3/2} = 2 \frac{\pi \sqrt{\pi}}{64 R} + 2 \frac{5}{64} \frac{\pi \sqrt{\pi}}{R} = \frac{3}{16} \frac{\pi \sqrt{\pi}}{R}. \quad (6.4) \]
The nonzero value of the coefficient \( B_2 \) is due to the contribution of four right-angled corners at the points 1 and 2. It has been noted at first time in reference [15]. In the case of the coefficient \( B_{3/2} \) the contributions of the corners from internal and external sectors are mutually canceled, while the contributions of the curvature of the arc 1–2 from internal and external sectors are added.

Different geometrical origins of the zeta function failure to provide a finite value of the vacuum energy in the two- and three-dimensional versions of the boundary value problem in question probably imply the impossibility of obtaining a finite and unique value of this quantity by taking advantage of the atomic structure of the boundary [40] or its quantum fluctuations [41]. It is clear because any physical reason of the Casimir energy divergences should be valid simultaneously in the two- and three-dimensional versions of the boundary configuration under consideration.
7. Conclusion

The basic result of the paper is the calculation of the individual contributions to the heat kernel coefficients generated by such particularities of the boundary as the corners. In the course of this analysis certain patterns, that are followed by these contributions, have been revealed. As a by product, the rules for obtaining all the heat kernel coefficients for the minus Laplace operator defined on a polygon or in its cylindrical generalization are formulated, these rules being valid both for Dirichlet and Neumann boundary conditions. Implications of the obtained results in calculations of the vacuum energy for regions with nonsmooth boundary are discussed. Our calculations comport the conventional point of view according to which the heat kernel coefficients are determined, in the case under consideration, by the local properties of the boundary.

In any case, the heat kernel coefficients obtained in the present paper can be used for verification of the general methods of calculating the contributions of boundary discontinuities to the heat kernel coefficients which may be developed in the future. These general methods must in particular allow one to calculate the contribution of an arbitrary corner to the coefficients $B_n$ with $n > 1$ in terms of the jumps of the derivatives of the boundary curve (or its geometrical invariants) at this corner, i.e., the formulas analogous to equation [L.7] should be found for $B_n$ with $n > 1$.

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Nonsmoothness of the boundary

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Figure 1. The cross section of a dihedral angle with circular boundary of radius $R$ inside.

Table 1. The contributions of the functions $Z_j^{D}$, $j = -1, 0, \ldots, 4$ to the heat kernel coefficients (internal circular sector I with Dirichlet boundary conditions).

| $Z_{-1}^{D}(s)$ | $Z_0^{D}(s)$ | $Z_1^{D}(s)$ | $Z_2^{D}(s)$ | $Z_3^{D}(s)$ | $Z_4^{D}(s)$ |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| $B_0 \ (s \to -3/2)$ | $\frac{1}{2} \alpha R^2$ |               |             |             |             |
| $B_{1/2} \ (s \to -1)$ | $-\sqrt{\pi} R$ | $-\frac{1}{2} \alpha R \sqrt{\pi}$ |             |             |             |
| $B_1 \ (s \to 1/2)$ | $\frac{\pi^2}{6 \alpha}$ | $\frac{\pi}{2}$ | $\frac{\alpha}{6}$ |             |             |
| $B_{3/2} \ (s \to 0)$ |               |             | $\frac{\sqrt{\pi}}{14 \pi}$ | $\frac{\alpha \sqrt{\pi}}{64 \pi}$ |             |
| $B_2 \ (s \to -1/2)$ |               |             |             | $\frac{4}{315} \frac{\alpha}{R^2}$ |             |
| $B_5/2 \ (s \to -1)$ |               |             |             |             | $\frac{25 \sqrt{\pi}}{90 R^4}$ | $\frac{37 \sqrt{\pi}}{8192 R^5} \alpha$ |
Table 2. The contributions of the functions $\tilde{Z}_N$, $Z_N^N$, $j = -1, 0\ldots, 4$, to the heat kernel coefficients (internal circular sector I with Neumann boundary conditions).

|                | $\tilde{Z}_N (s)$ | $Z_{-1}^N (s)$ | $Z_0^N (s)$ | $Z_1^N (s)$ | $Z_2^N (s)$ | $Z_3^N (s)$ | $Z_4^N (s)$ |
|----------------|-------------------|----------------|-------------|-------------|-------------|-------------|-------------|
| $B_0$          |                   |                |             |             |             |             |             |
| $(s \to 3/2)$  |                   |                |             |             |             |             |             |
|                | $\frac{1}{2} \alpha R^2$ |                |             |             |             |             |             |
| $B_{1/2}$      |                   |                |             |             |             |             |             |
| $(s \to 1)$    |                   | $2 \sqrt{\pi} R$ | $\sqrt{\pi} R$ | $\frac{\sqrt{\pi} R}{2} \alpha R$ |             |             |             |
| $B_1$          |                   | $\pi$          | $\frac{\pi^2}{6 \alpha}$ | $-\frac{\pi}{2}$ | $\frac{\alpha}{6}$ |             |             |
| $(s \to 1/2)$  |                   |                |             |             |             |             |             |
| $B_{3/2}$      |                   | $\frac{3 \sqrt{\pi}}{2 R}$ | $-\frac{3 \sqrt{\pi}}{4 R}$ | $\frac{5 \alpha \sqrt{\pi}}{64 R}$ |             |             |             |
| $(s \to 0)$    |                   |                |             |             |             |             |             |
| $B_2$          |                   | $\frac{3 \pi}{4 R^2}$ | $-\frac{3 \pi}{8 R^2}$ | $\frac{4 \alpha}{45 R^2}$ |             |             |             |
| $(s \to -1/2)$ |                   |                |             |             |             |             |             |
| $B_{5/2}$      |                   | $\frac{21 \sqrt{\pi}}{16 R^3}$ | $-\frac{21 \sqrt{\pi}}{32 R^3}$ | $\frac{269 \sqrt{\pi} \alpha}{8192 R^3}$ |             |             |             |
| $(s \to -1)$   |                   |                |             |             |             |             |             |
Table 3. The contributions of different parts of the boundary to heat kernel coefficients; \( D \) and \( N \) stand for the Dirichlet and Neumann boundary conditions for a wedge, \( D_C \) and \( N_C \) denote these conditions for a cone; the upper (lower) sign is referred to the internal I (external II) sector.

|          | Curvature of the arc 1–2 | Right-angled corners at the points 1 and 2 | Corner of angle \( \alpha \) at the origin |
|----------|--------------------------|-------------------------------------------|------------------------------------------|
| \( B_1 \) | \( \pm \frac{\alpha}{3} \) | \( \frac{\pi}{2} \)                                    | \( c(\alpha) \)                                    |
| \( D \)  | \( \pm \frac{\alpha}{3} \) | \( \frac{\pi}{2} \)                                    | \( c(\alpha) \)                                    |
| \( N \)  | \( \pm \frac{\alpha}{3} \) | \( \frac{\pi}{2} \)                                    | \( c(\alpha) \)                                    |
| \( D_C \) | \( \pm \frac{\alpha}{3} \) | \( 2c(\alpha/2) \)                | \( 0 \)                                    |
| \( N_C \) | \( \pm \frac{\alpha}{3} \) | \( 2c(\alpha/2) \)                | \( 0 \)                                    |
| \( B_{3/2} \) | \( \pm \frac{\sqrt{\pi} \alpha}{64R} \) | \( \pm \frac{\sqrt{\pi}}{4R} \) | \( \pm 3 \frac{\sqrt{\pi}}{4R} \) | \( \pm \frac{\sqrt{\pi}}{4R} \) |
| \( D \)  | \( \pm \frac{5\sqrt{\pi} \alpha}{64R} \) | \( \pm \frac{3\sqrt{\pi}}{4R} \)                      | \( \pm \frac{\sqrt{\pi}}{4R} \)                      |
| \( N \)  | \( \pm \frac{\sqrt{\pi} \alpha}{64R} \) | \( \pm \frac{\sqrt{\pi}}{4R} \)                      | \( \pm \frac{\sqrt{\pi}}{4R} \)                      |
| \( D_C \) | \( \pm \frac{\sqrt{\pi} \alpha}{64R} \) | \( \pm \frac{\sqrt{\pi}}{4R} \)                      | \( \pm \frac{\sqrt{\pi}}{4R} \)                      |
| \( N_C \) | \( \pm \frac{5\sqrt{\pi} \alpha}{64R} \) | \( \pm \frac{3\sqrt{\pi}}{4R} \)                      | \( \pm \frac{\sqrt{\pi}}{4R} \)                      |
| \( B_2 \) | \( \pm \frac{4 \alpha}{315R^2} \) | \( \frac{1 \pi}{8R^2} \)                              | \( \frac{1 \pi}{8R^2} \)                              |
| \( D \)  | \( \pm \frac{4 \alpha}{45R^2} \) | \( \frac{3 \pi}{8R^2} \)                              | \( \frac{3 \pi}{8R^2} \)                              |
| \( N \)  | \( \pm \frac{4 \alpha}{315R^2} \) | \( \frac{1 \pi}{8R^2} \)                              | \( \frac{1 \pi}{8R^2} \)                              |
| \( D_C \) | \( \pm \frac{4 \alpha}{315R^2} \) | \( \frac{1 \pi}{8R^2} \)                              | \( \frac{1 \pi}{8R^2} \)                              |
| \( N_C \) | \( \pm \frac{4 \alpha}{45R^2} \) | \( \frac{3 \pi}{8R^2} \)                              | \( \frac{3 \pi}{8R^2} \)                              |
| \( B_{5/2} \) | \( \pm \frac{37\sqrt{\pi} \alpha}{8192R^3} \) | \( \frac{25\sqrt{\pi}}{96R^3} \) | \( \frac{25\sqrt{\pi}}{96R^3} \) | \( \frac{25\sqrt{\pi}}{96R^3} \) |
| \( D \)  | \( \pm \frac{37\sqrt{\pi} \alpha}{8192R^3} \) | \( \pm \frac{21\sqrt{\pi}}{32R^3} \)     | \( \pm \frac{21\sqrt{\pi}}{32R^3} \)     |
| \( N \)  | \( \frac{269\sqrt{\pi} \alpha}{8192R^3} \) | \( \pm \frac{21\sqrt{\pi}}{32R^3} \)     | \( \pm \frac{21\sqrt{\pi}}{32R^3} \)     |
| \( D_C \) | \( \frac{37\sqrt{\pi} \alpha}{8192R^3} \) | \( \frac{21\sqrt{\pi}}{32R^3} \)     | \( \frac{21\sqrt{\pi}}{32R^3} \)     |
| \( N_C \) | \( \frac{269\sqrt{\pi} \alpha}{8192R^3} \) | \( \frac{21\sqrt{\pi}}{32R^3} \)     | \( \frac{21\sqrt{\pi}}{32R^3} \)     |