CALCULATING EIGENVECTORS IN MAX-ALGEBRA BY MUTATION METHOD

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Abstract. In this article we introduce a new method, called a mutation method, for calculating max-eigenvectors of a nonnegative irreducible $n \times n$ matrix $A$. The advantage of using a mutation method lies in its simplicity. Our method reduces to solving max-eigenproblems for simple mutation matrices that have exactly one positive entry in each row. Some instructive examples are presented.

Key words and phrases: mutation method; max algebra; circuit geometric mean; max-eigenvectors; critical circuits.

1. Introduction

The max-algebra system is one of the analogues of linear algebra that has recently attracted the attention of many researchers. The max-algebra system consists of nonnegative real numbers $\mathbb{R}_+$ equipped with the operations of multiplication $a \otimes b = ab$, and maximization $a \oplus b = \max\{a, b\}$. Furthermore, the max-algebra is isomorphic to the max-plus algebra, which consists of the set $\mathbb{R} \cup \{-\infty\}$ with operations of addition and maximization $[5, 8]$ and it is also isomorphic to the min-plus algebra consisting of the set $\mathbb{R} \cup \{\infty\}$ with operations of addition and minimization $[6, 19]$. This algebra system and its isomorphic versions raise the possibility of changing the non-linear phenomena in different areas such as parallel computation, transportation networks, timetabled programs, IT, dynamic systems, combinatorial optimization, computational biology, graph theory, and mathematical physics to linear algebra. To find out more details, we refer to references $[2, 6, 9, 10, 11, 26, 15, 17, 19, 12, 13]$. Furthermore, this algebra system has been used directly in areas such as algorithm, vetrbi, DNA analysis, and in AHP for ranking matrices. In this algebraic system, there is no obvious deduction but many of problems that appear in linear algebra like equation systems, eigenvalues, projections, subspaces, singular value decomposition, and duality theory have developed and have reached other areas such as functional analysis and combinatorial optimization $[4, 6, 19, 20, 1, 23, 24, 25, 27, 21, 18]$.

The article is organized in the following way. In Section 2 we present some preliminaries needed in our proofs. In Section 3 we present our main results by introducing a new mutation method for calculating a basis of a principal max-algebraic eigencone of an irreducible matrix $A$. The method reduces the calculation to solving max-eigenproblems for simple mutation matrices that have exactly one positive entry in each row. We illustrate the method by several examples.

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2. Preliminaries

For a $n \times n$ matrix $A$, the conventional eigenequation for eigenvalue $\lambda$ and corresponding eigenvector $x$ is $Ax = \lambda x$, $x \neq 0$. In the max-algebra system the eigenequation for a nonnegative matrix $A = [a_{ij}]$ is $A \otimes x = \lambda x$, where $(A \otimes x)_i = \max(a_{ij}x_j)$, $x = (x_1, x_2, \ldots, x_n)^T$, $x \neq 0$ and $\lambda$ is (geometric) max-eigenvalue corresponding to max-eigenvector $x$. For $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{R}_+)$ we denote $A \geq B$ if $a_{ij} \geq b_{ij}$ for all $i, j = 1, \ldots, n$. Similarly, for $x, y \in \mathbb{R}_+^n$ we write $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, n$.

We call a matrix $A \in M_n(\mathbb{R}_+)$ a reducible if either $n = 1$ and $A = 0$ or if $n \geq 2$, there exist a permutation matrix $P \in M_n$ and an integer $r$ with $1 \leq r \leq n - 1$ such that

$$P^TAP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where $B \in M_r(\mathbb{R}_+)$, $D \in M_{n-r}(\mathbb{R}_+)$, $C \in M_{r \times (n-r)}(\mathbb{R}_+)$, and $0 \in M_{(n-r) \times r}(\mathbb{R})$ is a zero matrix. Matrix $A \in M_n(\mathbb{R}_+)$ is said to be irreducible if it is not reducible.

Let $A = [a_{ij}] \in M_n(\mathbb{R}_+)$. The weighted directed graph associated with $A$ is denoted by $D(A) = (V, E)$, where $A$ has vertex set $V = \{1, 2, \ldots, n\}$ and edge $(i, j)$ from $i$ to $j$ with weight $a_{ij}$ if and only if $a_{ij} > 0$. A circuit of length $k$ is a sequence of $k$ edges $(i_1, i_2), \ldots, (i_k, i_1)$, where $i_1, i_2, \ldots, i_k$ are distinct (a circuit $(i, i)$ of length one which is called a loop [16]). The $k^{th}$ positive root of circuit product $a_{i_1, i_2} \cdots a_{i_{k-1}, i_k}$ is called a circuit geometric mean of matrix $A$. The maximum circuit geometric mean in $D(A)$ is denoted by $\mu(A)$. It is known that $\mu(A)$ is the largest max-eigenvalue of $A$. A circuit with circuit geometric mean equal to $\mu(A)$ is called a critical circuit, and vertices on critical circuits are called critical vertices. Assuming that simultaneous row and column permutations have been performed on $A$ so that the critical vertices are in the leading rows and columns, the critical matrix of $A$, denoted by $A^c = [a^c_{ij}]$, is formed from the principal submatrix of $A$ on the rows and columns corresponding to critical vertices by setting $a^c_{ij}$, where

$$a^c_{ij} = \begin{cases} 
    a_{ij} & \text{if } (i,j) \text{ is in a critical circuit}, \\
    0 & \text{otherwise}.
\end{cases}$$

Thus the critical graph $C(A) = D(A^c)$ has vertex set $N_c(A) = V_c$ of all critical vertices [9].

A matrix $A = [a_{ij}]$ is said to have a strongly connected (SC) property, if for every pair of distinct $p, q$ with $1 \leq p, q \leq n$ there is a sequence of distinct integers $k_1 = p, k_2, \ldots, k_m = q$, $1 \leq m \leq n$, such that all of the matrix entries $a_{k_1k_2}, a_{k_2k_3}, \ldots, a_{k_{m-1}k_m}$ are nonzero. A directed graph $D(A)$ is strongly connected if between every pair of distinct nodes $p_i, p_j$ in $D(A)$ there is a directed path of finite length that begins at $p_i$ and ends $p_j$ (see e.g. [16, 17]). A maximal strongly connected subgraph of $C(A)$ is called a strongly connected component of $C(A)$ ([7]).

**Theorem 2.1.** [16] Let $A \in M_n, n \geq 2$. The following are equivalent:

1. $A$ is irreducible.
2. $D(A)$ is strongly connected.
3. $A$ has SC property.

An important task in max-algebra is to calculate $\mu(A)$ and several methods may be used to calculate it [6, 9, 10]. The spectrum of nonnegative $n \times n$ matrix $A$ in
the max-algebra is denoted by $\sigma_\otimes(A)$ and is defined by

$$\sigma_\otimes(A) = \{ \lambda \geq 0 : \text{there exists } 0 \neq x \in \mathbb{R}^n_+ \text{ with } A \otimes x = \lambda x \}.$$ 

For $x \in \mathbb{R}^n_+$ the local spectral radius of $A$ at $x$ in max algebra is defined by

$$r_x(A) = \limsup_{k \to \infty} \|A^k \otimes x\|^{1/k}$$

The definition of $r_x(A)$ is independent of the choice of a norm $\| \cdot \|$. From now on for $x \in \mathbb{C}^n$, let $\|x\| = \max_{i=1,\ldots,n} |x_i|$. Let $e_i$ denote a standard basis vector for $i = 1, \ldots, n$. Let $A \in \mathbb{R}^{n \times n}$ and $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+, x \neq 0$. Then it is known (see [22, Theorem 2.4]) that

1. $\lim_{k \to \infty} \|A^k \otimes x\|^{1/k}$ exists;
2. $r_x(A) = \max\{r_{e_i}(A) : 1 \leq j \leq n, x_j \neq 0\}$.

It is known (see e.g. [22, Theorem 2.7]) that

$$\sigma_\otimes(A) = \{ \lambda \geq 0 : \text{there exists } i \in \{1, \ldots, n\} \text{ such that } \lambda = r_{e_i}(A) \}.$$ 

Recall the following lemma (see e.g. [22, Lemma 2.1]).

**Lemma 2.2.** Let $A \in \mathbb{R}^{n \times n}$, $j \in \{1, \ldots, n\}$. Then $r_{e_j}(A)$ is maximum of all $t \geq 0$ with the following property (*):

there exist $a \geq 0$, $b \geq 1$ and mutually distinct indices $i_0 := j, i_1, \ldots, i_a, i_{a+1}, \ldots, i_{a+b-1} \in \{1, \ldots, n\}$ such that

$$\prod_{s=0}^{a-1} A_{i_{s+1}, i_s} \neq 0 \text{ and } \prod_{s=a}^{a+b-1} A_{i_{s+1}, i_s} = t^b$$

where we set $i_{a+b} = i_a$.

If $A$ is irreducible, then the max spectrum of $A$ has a unique element $\mu(A)$ and all corresponding max eigenvectors are strictly positive ([4], [5], [6]). More details are found in the following result.

**Theorem 2.3.** ([4], [5], [6]) Let $A \in M_n$ be a nonnegative irreducible matrix. Then $\mu(A)$ is positive, it is the unique max-eigenvalue and there exists a positive max-eigenvector $x$, such that $A \otimes x = \mu(A)x$. This max-eigenvector is unique (up to scalar multiples) if and only if a graph of $A^c$ (a critical graph of $A$) is strongly connected.

Recall that for an irreducible matrix $A$ it may happen that the basis (in the sense of [6, Section 3.3]) of an max-eigencone corresponding to $\mu(A)$ may be of cardinality larger than one and this cardinality corresponds to a number of nonequivalent critical nodes (e.g., to the number of components of connectivity of a critical graph; see e.g. Theorem 4.3.5 of [6]).

The most efficient known algorithm for calculating $\mu(A)$ is Karp’s algorithm (see e.g. [6, 15]). In [9] a power method algorithm is given to compute the max eigenvalue $\mu(A)$ and max eigenvector $x$ of an irreducible nonnegative $n \times n$ matrix $A$. There are some limitations in the power method. How to select the primary vector $x(0)$ is one of these limitations since it affects the results. Howard’s algorithm is another in practice very efficient way of calculating max-eigenvectors (see e.g [15]). This algorithm utilizes the use of sunflower matrices (see e.g. [15, Sections 3.4 and 6.1]), which are related to our mutation matrices that we define in Section 4.
We now present some elements of the theory of max-linear cones, independency and bases in max-algebra, following [6]. Let $S \subseteq \mathbb{R}_+^n$. The set $S$ is called a max-algebraic cone if
\[ \alpha u \oplus \beta v \in S \]
for every $u, v \in S$ and $\alpha, \beta \in \mathbb{R}_+$. A vector $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n$ is called a max-combination of $S$ if
\[ v = \bigoplus_{x \in S} \alpha_x x, \quad \alpha_x \in \mathbb{R}_+, \]
where only a finite number of $\alpha_x$ are non-zero. The set of all max-combinations of $S$ is denoted by $\text{span}(S)$. We set $\text{span}(\emptyset) = \{0\}$. It is clear that $\text{span}(S)$ is a max-algebraic cone. If $\text{span}(S) = T$, then $S$ is called the set of generators for $T$. A vector $v \in S$ is called an extremal in $S$ if $v = \alpha u \oplus w$ for $u, w \in S$ implies $v = u$ or $v = w$. Clearly, if $v \in S$ is an extremal in $S$, then $\alpha v$ is also extremal in $S$.

Let $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}_+^n, v \neq 0$. Then $v$ is called scaled if $\|v\| = \max(v_1, v_2, \ldots, v_n) = 1$. The set $S$ is called scaled if all its elements are scaled.

The set $S$ is called dependent if $v$ is a max-combination of $S - \{v\}$ for some $v \in S$. Otherwise $S$ is independent. Let $S, T \subseteq \mathbb{R}_+^n$. The set $S$ is called a basis of $T$ if it is an independent set of generators for $T$. We recall two results from [6] ([6, Lemma 3.3.1, Theorem 3.3.9 and Corollary 3.3.11]).

**Lemma 2.4.** Let $S$ be a set of scaled generators of a max-algebraic cone $T \subseteq \mathbb{R}_+^n$ and let $v$ be a scaled extremal in $T$. Then $v \in S$.

**Theorem 2.5.** The set of scaled extremals of a max-algebraic cone $T$ is a basis of $T$ and it is a minimal set of generators for $T$. If $T$ is finitely generated, then the set of scaled extremals is non-empty and it is a unique scaled basis for $T$.

The Kleene star matrix for a nonnegative matrix $A$ is equal to
\[ \Delta(A) = I \oplus A \oplus A^2 \oplus \cdots \oplus A^{(n-1)}, \]
when $\mu(A) \leq 1$ ([6]). Let $V_\otimes(A, \mu(A))$ be a principal max eigencone of $A$ (see e.g. [6, Chapters 3 and 4]), i.e., $V_\otimes(A, \mu(A)) = \{ x \in \mathbb{R}_+^n : A \otimes x = \mu(A)x \}$. Clearly, $V_\otimes(A, \mu(A))$ is a subset of $\mathbb{R}_+^n$ closed under max-addition and nonnegative scalar multiplication, that is, it is a max-cone of $\mathbb{R}_+^n$. We rewrite [7, Theorem 6.2] (see also [6, Theorem 4.3.3]) with our notation in the following theorem.

**Theorem 2.6.** Let $A \in \mathbb{R}_+^{n \times n}$ be an irreducible matrix with $\mu(A) = 1$ and let $r$ denote the number of strongly connected components of $C(A)$. Then:

1. Each strongly connected component $C_s(1 \leq s \leq r)$ of $C(A)$ corresponds to an eigenvector defined as one of columns $\Delta(A)_i$ with $i \in N_{C_s}$, all such columns with $i \in N_{C_s}$ being multiple of each other.
2. $V_\otimes(A, \mu(A))$ is generated by columns of $\Delta(A)$ that are max-eigenvector of $A$.
3. Vectors in the set \{\(\Delta(A)_{i_1}, \ldots, \Delta(A)_{i_r}\)\}, where we take exactly one column of $\Delta(A)$ for each strongly connected component $C_s(1 \leq s \leq r)$ are extremals in $V_\otimes(A, \mu(A))$ and form a basis of $V_\otimes(A, \mu(A))$. 


Let $A$ be an irreducible matrix and let $r$ be the number of pairwise disjoint critical circuits (two circuits are called pairwise disjoint when they have no common vertices), i.e., $r$ equals the number of components of connectivity of a critical graph $C(A)$ of $A$. So there are exactly $r$ maximal positive max eigenvectors corresponding to $\mu(A)$ which form a basis of the principal max eigenvector $V_{\oplus}(A, \mu(A))$ of $A$ (see Theorem 2.6). Observe that since $A$ is irreducible every non-zero $x \in V_{\oplus}(A, \mu(A))$ is (strictly) positive by [6, Theorem 4.4.8].

We describe our mutation method for calculating a basis of $V_{\oplus}(A, \mu(A))$. We construct mutation matrices $A_1^*, \ldots, A_{i_0}^*$, which have one non-zero element in each row, where $i_0 \geq r$. Therefore it is easy to calculate $\mu(A_i^*)$ for all $i = 1, \ldots, i_0$ and the corresponding max eigenvectors. Moreover, by construction $\mu(A) = \mu(A_i^*)$ for all $j = 1, \ldots, i_0$ and it turns out that there exists a basis $\{x_1, \ldots, x_r\}$ of positive vectors for $V_{\oplus}(A, \mu(A))$, where each $x_i \in V_{\oplus}(A_i^*, \mu(A))$ for some $j_i \in \{1, \ldots, i_0\}$ (see Theorem 3.2 below).

Given a nonnegative irreducible $n \times n$ matrix $A$, the matrices $A_1^*, \ldots, A_{i_0}^*$ are defined by the following 4 steps algorithm.

**Algorithm 1:**

1. **Step (1):** Calculate $M_1 = \mu(A)$ and choose a critical circuit $C_1 = C$. Arrange all other circuits in decreasing order with respect to the maximum $k_i$th positive root $M_i$ of a product of nonzero elements in each circuit (where $k_i$ is the number of non-zero elements in the $i$th circuit). So

   $$M_2 \geq M_3 \geq \cdots \geq M_m.$$

   Suppose that $k = k_1$ is the number of elements in $M_1$.

2. **Step (2):** If $k = n$, then put the elements of $C$ as the corresponding entries in $A^*$ and other entries in $A^*$ set to be equal to zero, otherwise go to Step (3).

3. **Step (3):** First put elements of $C$ as the corresponding entries in $A^*$. The other entries in the rows of $A^*$ that contain these entries are set to be equal to zero. For other rows: choose the minimal $p \geq 2$, where a circuit $C_p$ corresponding to $M_p$ contains a node which lies in $C$, and then put the non-zero elements from $C_p$ that lie in not yet determined rows as the corresponding elements in $A^*$ and the other entries in these rows are set to zero. If $A^*$ has $n$ non-zero entries, then the algorithm ends. Otherwise replace $C$ with $C_p$ and repeat this step.

4. **Step (4):** Calculate all possible $A^*$ corresponding to $r$ pairwise disjoint critical circuits $C$. Choose all mutation matrices $A^*$ such that the product of elements from all $C_p$’s that are used in $A^*$ is maximal possible.

**Remark 3.1.** (i) Algorithm 1 completes the matrix $A^*$ (determines all $n$ rows of $A^*$), since $A$ is irreducible.

(ii) Observe that there are at least $r$ possible choices for $C$ (and we consider only $r$ pairwise disjoint critical circuits and corresponding mutation matrices), and there might be more than one possible choice also for $C_p$ corresponding to each critical circuit (if the elements from some of $C_p$’s that are used in $A^*$ have equal maximal product). Therefore the mutation matrix $A^*$ is not uniquely defined and we...
use Algorithm 1 to calculate all the possibilities. We denote them by $A^*_1, \ldots, A^*_s$. Clearly $r \leq \ell_0$.

(iii) Mutation matrices $A^*_1, \ldots, A^*_s$ that are related to critical pairwise disjoint circuits are called principal mutation matrices (observe that there might be several collections of such matrices, since each component of connectivity of a critical graph might contain several non-disjoint critical circuits). Each (positive) vector from $V_{\mathbb{R}}(A, \mu(A))$ that is also a max-eigenvector of some principal mutation matrix is called a principal max-eigenvector of $A$.

(iv) For each $i$ which is not in $C$ and each mutation matrix $A^*$ from Step (4) of Algorithm 1 it holds that $A^*$ contains a path with maximal possible product from $i$ to $C$ (since the matrices with nonmaximal products of paths are omitted in Step (4)).

**Theorem 3.2.** Let $A \in \mathbb{R}^{n \times n}_+$ be an irreducible matrix with $\mu(A) = 1$. Then each column of $\Delta(A)$ that is a max-eigenvector of $A$ is a multiple of a principal max-eigenvector of $A$.

Moreover, principal mutation matrices that are associated to critical circuits from the same component of connectivity of $C(A)$ have a common scaled max-eigenvector.

**Proof.** Let $g_i$ be a column of $\Delta(A)$, where $i \in N_C(A)$ and let $r$ be the number of components of connectivity of $C(A)$. Then there is $1 \leq s \leq r$ such that $i$ is a node in a critical circuit $C_s$. We show that $g_i$ is a max-eigenvector (and also a classical eigenvector) of a principal mutation matrix $A^*_s$ corresponding to $C_s$ (since there may be several principal mutation matrices related to $C_s$, $A^*_s$ could be any of them). Since $A^*_s \leq A$, we have $A^*_s \otimes g_i \leq A \otimes g_i = g_i$ and therefore we have the following system:

$$a_{1j_1}g_{j_1i} \leq g_{ii}$$
$$a_{2j_2}g_{j_2i} \leq g_{2i}$$
$$\vdots$$
$$a_{nj_n}g_{j_ni} \leq g_{ni},$$

where $a_{1j_1}, \ldots, a_{nj_n}$ are positive entries in $A^*_s$. Let us denote the nodes in $C_s$ by $j_1, \ldots, j_k$, $i_{k+1} = j_1$, where $k \leq n$.

It is clear that for every $t \leq p \leq k$ and $1 \leq q \leq n$, $a_{qj_q}g_{j_qi} = g_{qi}$ (if not, then by multiplying both sides of the associated inequalities in the above system one obtains $\mu(A) < 1$, which is a contradiction).

Now, let $a_{tj_t}g_{j_ti} < g_{ti}$ for some $l$ which is not a node of $C_s$ such that a $a_{tj_t}$ is positive entry in $A^*_s$. Then $a_{tj_t}$ lies in a closed path with a node from $C_s$ (since each nonzero element of $A^*_s$ lies in a critical circuit or in a circuit that contains a node from $C_s$, with maximum $k$th root of a product of $k$ nonzero elements in that circuit). Therefore by repeated use of the above inequalities related to this closed path we obtain

$$a_{tj_t}a_{r_1j_{r_1}} \cdots a_{r_kj_{r_k}}g_{ti} < g_{ti}$$

and consequently, $a_{tj_t}a_{r_1j_{r_1}} \cdots a_{r_kj_{r_k}} < g_{ti}$. This is contradiction, because both sides of the inequality are heaviest weights of paths from node $l$ to node $i$ by Remark 3.1(iv). Thus, $g_i$ is a max-eigenvector of $A^*_s$. Since $A^*_s$ has only one nonzero entry in each row, it has only one independent max-eigenvector. Hence $g_i$ is a principal max-eigenvector related to $A^*_s$. 
It also follows from the proof above and from Theorem 2.6 that principal mutation matrices that are associated to critical circuits from the same component of connectivity of \( C(A) \) have a common scaled max-eigenvector. □

Applying the above result to the matrix \( \frac{1}{\mu(A)} A \) we obtain the following result.

**Corollary 3.3.** Let \( A \in \mathbb{R}^{n \times n}_+ \) be an irreducible matrix. Then a basis of \( V_{\oplus}(A, \mu(A)) \) is formed by principal max-eigenvectors of \( A \).

**Corollary 3.4.** Let \( A \in \mathbb{R}^{n \times n}_+ \) be irreducible and let \( r \) be the number of pairwise disjoint critical circuits of \( A \) and let \( A^*_1, \ldots, A^*_{\ell_0} \) be the associated principal mutation matrices. Then

(i) \( \ell_0 \geq r \),

(ii) \( \mu(A) = \mu(A^*_j) \) for all \( j = 1, \ldots, \ell_0 \).

(iii) Let \( Y \) be the set of all \( x \in \mathbb{R}^n_+ \), \( x \neq 0 \), such that \( A^*_j \otimes x = \mu(A) x \) for some \( j = 1, \ldots, n \). Then there exists a subset \( X \subset Y \) of positive vectors such that \( |X| = r \) and that \( X \) is the basis of the principal max-eigencone \( V_{\oplus}(A, \mu(A)) \) of \( A \).

We can use the following algorithm to calculate the basis \( X \) of the principal max-eigencone of \( A \).

**Algorithm 2:** Input: let \( \ell_0 \) be the number of principal mutation matrices of \( A \) and let \( r \) be the number of strongly connected components of \( C(A) \).

\[ X_0 := \emptyset. \]

For \( i = 1, \ldots, l_0 \):

Define \( V_{i-1} := X_{i-1} \).

Calculate the basis \( \{v_1, \ldots, v_{j_i}\} \) of the principal max-eigencone of \( A^*_i \) (obtained by Algorithm 1)

For \( k = 1, \ldots, j_i \)

If \( V_{i-1} \cup \{v_k\} \) is a max independent set, then \( V_i := V_{i-1} \cup \{v_k\} \)

Return \( X_i = V_i \).

If \( |X_i| = r \), stop and return \( X = X_i \).

**Remark 3.5.** The set \( X \) calculated by Algorithm 2 is a basis of the principal max-eigencone of \( A \) by Corollary 3.4.

We illustrate our method with the following examples.

**Example 3.6.** Let

\[ A = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \]

Then \( r = 1 \) and

\[ \mu(A) = M_1 = a_{11} = 1 > M_2 = \sqrt{a_{12}a_{21}} = \frac{1}{2}. \]

Thus

\[ A^*_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}. \]

The principal max eigenvector of \( A \) is

\[ x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \]
which is also max-eigenvector (and a classical eigenvector) of $A_1^*$. Observe that

$$\Delta(A) = I \oplus A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

and only first column of $\Delta(A)$ is a multiple of $x_1$ (a principal max-eigenvector of $A$).

**Example 3.7.** Let

$$A = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}.$$  

Then $r = 1$ and

$$\mu(A) = M_1 = a_{11} = M_2 = a_{22} = 1 = M_3 = \sqrt{a_{12}a_{21}} = 1.$$

Thus

$$A_1^* = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}, \quad A_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad A_3^* = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

The principal max eigenvector of $A$ is

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

which is also max-eigenvector (and a classical eigenvector) of $A_1^*$, $A_2^*$ and $A_3^*$. Here $\{A_1^*, A_2^*\}$ and $\{A_3^*\}$ are sets of principal mutation matrices.

Observe that

$$\Delta(A) = I \oplus A = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}$$

and that each column of $\Delta(A)$ is a multiple of $x_1$ (a principal max-eigenvector of $A$).

**Example 3.8.** Let

$$A = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}.$$  

Then $r = 2$ and

$$\mu(A) = M_1 = a_{11} = M_2 = a_{22} = 1 > M_3 = \sqrt{a_{12}a_{21}} = \frac{1}{2}.$$

Thus

$$A_1^* = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad A_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The principal max eigenvectors of $A_1^*$ and $A_2^*$ are $x_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$ and $x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. These two vectors form the basis of the principal max-eigencone of $A$.

Observe that

$$\Delta(A) = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} = A$$

and that each column of $\Delta(A)$ is a principal max-eigenvector of $A$. 

Example 3.9. Let
\[ A = \begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & e \\ e & e & e^2 \end{pmatrix}. \]

Then \( r = 1 \) and we have
\[ \mu(A) = M_1 = \sqrt{a_{12}a_{21}} = \sqrt{e^6} > M_2 = a_{33} = e^2 > M_3 = \sqrt{a_{12}a_{23}a_{31}} = \sqrt{e^5} > M_4 = \sqrt{a_{23}a_{32}} = e. \]

Therefore by Algorithm 1 we have
\[ A^*_1 = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & 0 \\ e & 0 & 0 \end{pmatrix}. \]

The principal max-eigenvector of \( A^*_1 \) and of \( A \) is
\[ x_1 = \begin{pmatrix} e^2 \\ e^{-1} \end{pmatrix}. \]

If \( A' = e^{-\frac{5}{2}}A \), then \( \mu(A') = 1 \),
\[ \Delta(A') = \begin{pmatrix} 1 & e^\frac{5}{2} & e^{-1} \\ e^{-\frac{5}{2}} & 1 & e^\frac{-5}{2} \\ e^{-\frac{5}{2}} & e^{-1} & 1 \end{pmatrix} \]
and columns one and two of \( \Delta(A') \) are multiples of \( x_1 \).

Example 3.10. Let
\[ A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \]

Then \( r = 2 \) and we have
\[ \mu(A) = M_1 = \sqrt{a_{12}a_{21}} = M_2 = a_{22} = M_3 = a_{33} = 2 > M_4 = \sqrt{a_{12}a_{23}a_{31}} = M_5 = \sqrt{a_{21}a_{13}a_{32}} = \sqrt{2} > M_6 = \sqrt{a_{13}a_{31}} = M_7 = \sqrt{a_{23}a_{32}} = 1. \]

Therefore by the Algorithm 1 we have \( l_0 = 5 \) and the principal mutation matrices corresponding to critical circuits are \( \{(1, 2), (2, 1)\} \) and \( \{(2, 2)\} \) are
\[ A_1^* = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2^* = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]
\[ A_3^* = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_4^* = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

The principal max-eigenvector of \( A_1^*, A_2^*, A_3^*, A_4^* \) and of \( A \) is
\[ x_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}. \]
Also, by considering $\mu(A) = a_{33} = 2$ we have
\[ A_5^* = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}. \]
The principal max-eigenvector of $A_5^*$ and of $A$ is
\[ x_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \]
Thus, $X = \{x_1, x_2\}$. For $A' = \frac{1}{2}A$ we have
\[ \Delta(A') = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \end{pmatrix}. \]
The first two columns of $\Delta(A')$ are multiple of $x_1$ and the third column is multiple of $x_2$.

**Remark 3.11.** In the above example $x = (1, 1, 1)^T$ is an max-eigenvector of $A$ corresponding to $\mu(A) = 2$. It holds $x = \frac{1}{3}(x_1 \oplus x_2)$, where $x_1$ and $x_2$ are the above principal max-eigenvectors of $A$.

**Example 3.12.** Let
\[ A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{pmatrix}. \]
Then $r = 3$ and we have
\[ \mu(A) = M_1 = a_{11} = M_2 = a_{22} = M_3 = a_{33} = 3 > M_4 = \sqrt{a_{13}a_{32}a_{21}} = 2 \]
\[ > M_5 = \sqrt[3]{a_{12}a_{21}} = \sqrt{2} > M_6 = \sqrt{a_{23}a_{32}} = 1/7 > M_7 = \sqrt{a_{13}a_{23}} = 1. \]
Therefore by Algorithm 1 we have $l_0 = 3$ and so if we choose $M_1 = a_{11} > M_4 = \sqrt{a_{13}a_{32}a_{21}}$, $A_1^* = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, if we choose $M_1 = a_{22} > M_4 = \sqrt{a_{13}a_{32}a_{21}}$, $A_2^* = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and if choose $M_1 = a_{33} > M_4 = \sqrt{a_{13}a_{32}a_{21}}$, $A_3^* = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. 
The principal max-eigenvector of $A_1^*, A_2^*$ and $A_3^*$ are respectively
\[
x_1 = \begin{pmatrix} 9 \\ 6 \\ 4 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}, \quad \text{and} \quad x_3 = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}.
\]
Thus, $X = \{x_1, x_2, x_3\}$. If $A' = \frac{1}{2} A$, then
\[
\Delta(A') = \begin{pmatrix} 1 & \frac{4}{9} & \frac{2}{3} \\ \frac{2}{3} & 1 & \frac{4}{9} \\ \frac{4}{9} & \frac{2}{3} & 1 \end{pmatrix}
\]
and the columns of $\Delta(A')$ are multiples of $x_1, x_2$ and $x_3$, respectively.

**Example 3.13.** Let
\[
A = \begin{pmatrix} 0 & e^3 & e^2 \\ e^2 & 0 & e \\ e & e & e \end{pmatrix}.
\]
Then $r = 1$ and we have
\[
\mu(A) = M_1 = \sqrt{a_{12}a_{21}} = e^{\frac{5}{2}} \geq M_2 = \sqrt{a_{12}a_{23}a_{31}} = e^{\frac{5}{4}}
\]
\[
= M_3 = \sqrt{a_{13}a_{21}a_{32}}.
\]
After using the first three steps in Algorithm 1 we have
\[
A_1^* = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{\frac{3}{2}} & 0 \\ e^2 & 0 & 0 \\ e & 0 & 0 \end{pmatrix}
\quad \text{and} \quad A_2^* = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & 0 \\ 0 & e & 0 \end{pmatrix},
\]
but since $a_{12}a_{31} = e^4 > a_{21}a_{32} = e^3$, we omit $A_2^*$ by Step (4) in Algorithm 1. Therefore the principal max-eigenvector of $A_1^*$ and of $A$ is
\[
x_1 = \begin{pmatrix} e^{\frac{5}{4}} \\ 1 \\ e^{-1} \end{pmatrix}.
\]
If $A' = e^{-\frac{5}{4}} A$, then
\[
\Delta(A') = \begin{pmatrix} 1 & e^{\frac{5}{2}} & e^{-\frac{5}{4}} \\ e^{-\frac{5}{4}} & 1 & e^{-1} \\ e^{-\frac{5}{2}} & e^{-1} & 1 \end{pmatrix}
\]
and only columns one and two of $\Delta(A')$ are multiples of $x_1$.

**Example 3.14.** Let
\[
A = \begin{pmatrix} 1 & e^3 & 1 \\ e^2 & 1 & e^3 \\ 1 & e^2 & 1 \end{pmatrix}.
\]
Then $A$ is irreducible and $\mu(A) = e^{\frac{5}{2}}$, and we have $r = 1$.
\[
M_1 = \sqrt{a_{12}a_{21}} = M_2 = \sqrt{a_{23}a_{32}} = e^{\frac{5}{4}}
\]
\[
\geq M_3 = \sqrt{a_{12}a_{23}a_{31}} = e^2 \geq M_4 = \sqrt{a_{13}a_{21}a_{32}} = e^{\frac{5}{4}}.
\]
Therefore by Algorithm 1, we have two possibility for $M_1$. If we choose $M_1 = \sqrt{a_{12}a_{21}}$, then by using the first three steps in Algorithm 1 we have

$$A_1^* = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$A_2^* = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^3 & 0 \\ e^2 & 0 & 0 \\ 0 & e^2 & 0 \end{pmatrix},$$

but we omit $A_1^*$ since $a_{12}a_{31} = e^3 < a_{21}a_{32} = e^4$ by Step (4) in Algorithm 1.

The principal max-eigenvector of $A_2^*$ and of $A$ is

$$x_1 = \begin{pmatrix} e^{\frac{1}{2}} \\ 1 \\ e^{-\frac{1}{2}} \end{pmatrix}.$$

Similarly, if $M_1 = \sqrt{a_{23}a_{32}}$, then we have

$$A_3^* = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^3 & 0 \\ 0 & 0 & e^3 \\ 0 & e^2 & 0 \end{pmatrix}$$

and

$$A_4^* = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & e^3 \\ 0 & e^2 & 0 \end{pmatrix},$$

but we omit $A_3^*$ since $a_{13}a_{32} = e^2 < a_{21}a_{23} = e^6$ by Step (4) in Algorithm 1.

The principal max-eigenvector of $A_4^*$ and of $A$ is

$$x_1 = \begin{pmatrix} e^{\frac{1}{2}} \\ 1 \\ e^{-\frac{1}{2}} \end{pmatrix}.$$

If $A^* = e^{-\frac{1}{2}}A$, then

$$\Delta(A^*) = \begin{pmatrix} 1 & e^{\frac{1}{2}} & e \\ e^{-\frac{1}{2}} & 1 & e^{\frac{1}{2}} \\ e^{-1} & e^{-\frac{1}{2}} & 1 \end{pmatrix}$$

and all columns of $\Delta(A^*)$ are multiple of $x_1$.

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