A new type of complementarity between quantum and classical information

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Physical systems contain information which can be divided between classical and quantum information. Classical information is locally accessible and allows one to perform tasks such as physical work, while quantum information allows one to perform tasks such as teleportation. It is shown that these two kinds of information are complementarity in the sense that two parties can either gain access to the quantum information, or to the classical information but not both. This complementarity has a form very similar to the complementarities usually encountered in quantum mechanics. For pure states, the entanglement plays the role of Planck’s constant. We also find another class of complementarity relations which applies to operators, and is induced when two parties can only perform local operations and communicate classical. In order to formalize this notion we define the restricted commutator. Observables such as the parity and phase of two qubits commute, but their restricted commutator is non-zero. It is also found that complementarity is pure in the sense that can be "decoupled” from the uncertainty principle.

I. INTRODUCTION

The idea of complementarity appeared together with the birth of quantum mechanics in 1926 [1] [2, 3]. In quantum mechanics, observables such as the position and momentum of a particle are complementary. An accurate measurement of momentum will make a subsequent measurement of position yield random results - the position information is destroyed during the momentum measurement. In this work, we find new complementarities which arise in the context of quantum information theory.

In quantum information theory, one is often interested in measurements that can be made by two parties in distant labs who can only communicate classically and perform local operations (LOCC). We find that this restriction on the class of allowable operations induces new complementarity principles. In this paper, we will explore two different types of complementarity.

The first, is a complementarity relationship between classical and quantum information. This is a generalization of an idea we briefly noted in [4]. In Section II we will first demonstrate how to divide information into classical and quantum parts in an operational way. The essentially idea is that classical information is locally accessible and can be used to perform tasks such as physical work, while quantum information can be used to perform tasks such as teleportation and double dense coding. Classical information can be measured by how many pure separable bits can be obtained from a state, while one can obtain quantum information by distilling singlets from the state.

Then in Section III we show that the two types of information are complimentary. One finds that one can exploit the quantum information to perform teleportations, but then the ability to use the classical information is completely destroyed. Or, one can obtain classical information (locally accessible), but then the ability to perform teleportations is completely destroyed. We will show that this idea can be expressed as an information theoretic bound which has the same form as an uncertainty relation. For pure states, the bound has the feature that the entanglement plays the role of Planck’s constant $\hbar$. Pure states can also be though of as the counterpart of coherent states (i.e. minimum uncertainty wave packets).

In Section IV we introduce a complementarity principle involving individual measurements. For example, if one has two qubits, one can globally measure the parity and phase. However, when two parties each hold one of the qubits in distant labs, then they find that all parity and phase measurements will be complimentary. They can measure the parity of the state, or the phase of the state, but not both. To quantify this, we introduce the idea of the LOCC commutator. We find that two observables can be complimentary without being uncertain, demonstrating that the two concepts can be decoupled. We conclude in Section V and mention some open questions.

II. CLASSICAL AND QUANTUM INFORMATION

Before showing how information can be divided into classical and quantum contributions, a few preliminaries are necessary. Consider a bipartite state $\rho_{AB}$ composed of $n$ qubits which could be shared between two parties Alice and Bob [5]. Let $\rho_A$ and $\rho_B$ be the reduced density matrix for each party, and let $n_A$ and $n_B$ be the number of qubits
that each party holds.

The total information in the state is given by

$$I = n - S(\rho_{AB})$$ (1)

where the symbol $S(\rho)$ is the von-Neumann entropy of a state $\rho$. The more we know about the state of a system, the lower it’s entropy and the greater the informational content of the state. This information can be divided into local information contents $I_A = n_A - S(\rho_A)$, $I_B = n_B - S(\rho_B)$ and the mutual information $I_M = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ so that

$$I = I_A + I_B + I_M$$ (2)

Typically, the mutual information $I_M$ is used as a measure of the total correlations between $\rho_A$ and $\rho_B$. It tells us how much information the two systems have in common.

In classical information theory [6], a classical system $\rho_{cl}$ has a mutual information which is always smaller than the total Shannon entropy $H$ of the state.

$$I_M(\rho_{cl}) \leq H(\rho_{cl})$$ (3)

This means that the correlations are always accompanied by a lack of information about the total system: only mixed states can have nonzero correlations. Also, for two $d$-level classical systems, the correlations cannot exceed $\log d$.

For quantum system there is no restriction like (3). Therefore pure states can contain correlations, and the mutual information can be twice as much as in the classical case. In general we have

$$I_M(\rho_{qu}) \leq 2 \log d$$ (4)

Thus two qubits can share two bits of mutual information as in the case of a maximally entangled state such as the singlet

$$\psi^- = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) .$$ (5)

There is a basic question: For the singlet, what is the meaning of the fact that the amount of mutual information is two? One possible answer comes from superdense coding [8]. By using a singlet, one can communicate 2 bits of information through 1 qubit [9]. It has also been argued that the additional correlations are related to negative conditional entropies [10]. We will propose a different answer to this fundamental question. We will argue that 2 is not equal to 1 + 1 but rather it is equal to either 1 or 1. In other words, the two bits of mutual information can be divided into one bit of quantum information, and one bit of classical information, but that these two types of information are complimentary - one can retrieve the one bit of classical information, or the one bit of quantum information, but not both. In general, we shall find that the correlations of a quantum state consist of two complementary parts - one classical, the other quantum.

We now show how to divide the total information into quantum and classical parts. The first attempt at quantifying quantum contents of correlations other than entanglement is due to Zurek [11]. Quantifying classical correlations of a quantum state and the division into classical and quantum correlations was proposed in [12]. An operational proposal of quantifying different types of correlations was first proposed in [4]. Another method, using the entanglement of purification was given in [13]. Here we will follow [4] where the division emerges from thermodynamical considerations.

The idea is to define information operationally - classical information is defined to be information that can be manipulated into a locally accessible form. In other words, it is the information that can be localized by two parties, Alice and Bob, and used to perform classical tasks. For example, as discussed in [4], it can be used to extract real physical work from a local heat bath, using a Szilard engine [14] or (for quantum states) a Von Neumann engine [15]. Information that is locally accessible is equal to the maximum amount of work which can be drawn from a local heat bath by Alice and Bob under LOCC in units of $kT$ where $T$ is the temperature of the bath and $k$ is Boltzmann’s constant. We will henceforth set $kT = 1$ so that the amount of work drawn is measured in bits. One can think of the classical information as the maximum amount of pure separable states which can be extracted from a state. We will therefore talk of extracting classical information from a state, with the understanding that it could refer to extracting physical work from the information encoded in the state, or extracting a number of pure separable states. It should also be mentioned that this definition of classical information is independent of the interpretation of quantum mechanics one uses (Copenhagen, Many Worlds, Bohmian, etc.).

On the other hand quantum information is defined to be the information which can be used to perform tasks which have no classical counterpart, such as teleportation and double-dense coding. One bit of quantum information, can
be used to teleport one qubit. One can think of teleportation (sending qubits) as analogous to a form of quantum logical work \[23\].

Let us first look at that case where Alice and Bob are only allowed to perform local operations (LO). In this case, the amount of information \( I_{LO} \) they can obtain is

\[
I_{LO} = n_A - S(\rho_A) + n_B - S(\rho_B) = I_A + I_B
\]

(6)

On the other hand, if we allow Alice and Bob to perform any local operations (LO) and send qubits to each other through a classical channel (CC) \[17\], then they will be able to obtain more information from the state by exploiting correlations. Alice and Bob can then transform the state \( \rho_{AB} \) into a new state \( \rho'_{AB} \) such that the amount of local information is maximized. The amount of obtainable local information is

\[
I_c = I_A(\rho'_A) + I_B(\rho'_B) = n - S(\rho'_A) - S(\rho'_B)
\]

(7)

then the difference

\[
\Delta_c \equiv I_c - I_{LO}
\]

(8)
tells us the additional information that can be obtained, if the two parties are also given access to a classical communication channel (CC). Since the channel is classical, we will refer to \( \Delta_c \) as the classical deficit.

On the other hand, if Alice and Bob had access to a quantum channel (QC) rather than to the classical channel, they will be able to localize all the information (and draw all the work from the state). The information they can obtain under these operations (LOQC) is just

\[
I_{LOQC} = I = n - S(\rho_{AB})
\]

(9)
since Alice can just send her part of the state to Bob, who can then perform local operations on it to draw all the information. The quantity

\[
\Delta_q \equiv I_{LOQC} - I_c
\]

(10)
then tells us how much more information can be obtained when the channel is changed from a classical channel to a quantum channel. It is the quantum deficit.

It is easy to verify that the classical deficit \( \Delta_c \) plus the quantum deficit \( \Delta_q \) are equal to the total amount of correlations contained in the state. I.e.

\[
I_M = \Delta_q + \Delta_c
\]

(11)

Remarkably, for pure states, it was found that \( \Delta_q = E_D \) where \( E_D \) is the amount of distillable entanglement contained in the state (i.e. the number of singlets per state \( \rho_{AB} \) that can be drawn under LOCC from a large number of copies of \( \rho_{AB} \)) \[4\]. This was also conjectured to be true for sets of states such as the ”maximally correlated” state or \[20\]. In general, the quantum deficit \( \Delta_q \) can be due to entanglement, as in the case of pure states, but it also appears that separable states can have a non-zero \( \Delta_q \) as in the case of mixtures of states which are separable, but indistinguishable \[21\]. States such as Werner states \[22\] are believed to have \( \Delta_q > E_D \). However, it is not yet clear whether \( \Delta_q > E_D \) in the case of many copies.

Under LOCC, \( I_c \) is the amount of classical information (pure separable states) that can be extracted using the state \( \rho \), and used to perform physical work. \( E_D \) has the interpretation of the maximal amount of useful quantum logical work which can be extracted from the state \( \rho_{AB} \) (since each singlet can be then used for such tasks as the teleportation of one qubit, or double dense coding). Here, we will take quantum work to mean teleportation of qubits, but it is certainly not excluded that there are other forms of quantum work \[24\].

Rewriting Equation (11) therefore divides the total informational content into the classical information \( I_c \) which can be used to perform classical tasks such as physical work, and the quantum informational part \( \Delta_q \) which for pure states is equal to the amount of entanglement which can be used to perform teleportations.

\[
I = I_c + \Delta_q
\]

(12)

However, in the next section, we will see that the amount of extractable information is not \( I_c + \Delta_q \) but rather \( I_c \) or \( \Delta_q \).
III. COMPLEMENTARITY BETWEEN CLASSICAL AND QUANTUM INFORMATION

We will now show that classical information and quantum information are complimentary – one can use the classical information, or the quantum information, but not both. Before discussing the general case, it may be useful to first show how this complementarity principle plays out with a simple state such as the singlet of Equation (5). This will be done in Subsection III A, and we will also show how this can be extended to other states. In Subsection III B we will discuss this complementarity in more generality, and show two different and useful ways that it can be expressed. Furthermore, for pure states, one can express the relationship in a particularly simple form, where the entanglement plays the role of Planck’s constant $\hbar$. In Subsection III C we show that these complementarities are of the same form as the more familiar ones encountered in quantum mechanics between conjugate observables such as position and momentum. Finally, in Subsection III D we give an example of information extraction which illustrates our complementarity principle, and shows that pure states can be thought of as the counterpart to coherent states (i.e. minimum uncertainty wavepackets).

A. An example of complementarity between quantum and classical information

Although the mutual information of the singlet is two bits, initially, neither Alice nor Bob can obtain any information, since their local density matrices are maximally mixed. However, in [25] we showed that one bit of information can be obtained by the two parties. This can be done using the following process which was proven to be optimal.

(a) Alice uses a measuring device represented by a qubit prepared in the standard state $|0\rangle$ [26]. She performs a cnot [26] using her original state as the control qubit, and the measuring qubit as the target. (b) The measurement qubit is now in the same state as her original bit and can be dephased (i.e. decohered) in the $|0\rangle$, $|1\rangle$ basis so that the information is purely ”classical” (dephasing simply brings the off-diagonal elements of the density matrix to zero, destroying all quantum coherence). It is worth noting that, during the dephasing process, one bit of information is irreversibly transferred into the environment and is no longer available. (c) The measuring qubit can now be sent to Bob who (d) performs a cnot using the measuring qubit as the control. His original qubit is now in the standard state $|0\rangle$. (e) Bob sends the measuring qubit back to Alice who (f) resets the measuring device by performing a cnot using her original bit as the control. Alice’s state is now maximally mixed, while Bob’s state is known. They have obtained one bit of information. This information can be used to extract one bit of physical work from a heat bath using a Szilard heat engine.

This process, though optimal, only extracts one bit of classical information, even though two bits of information could be extracted by someone who is not constrained by LOCC. However, the singlet also has one bit of quantum information, which can be used to teleport a single qubit. In this case, the ability to obtain classical information will be lost. If Alice wishes to teleport the state $\psi_A$ using a singlet, the total initial state is

$$\psi_A \otimes \psi_{AB}$$

The final state (after Alice resets her measuring device) is

$$\frac{1}{4} I_{A'A} \otimes \psi_B$$

Thus the state (excluding the teleported state $\psi_A$) is now maximally mixed, containing no classical information. One might think that there could be some other, more sophisticated protocol that allows one to teleport a qubit in such a way that the final state will not be maximally mixed. However this is not the case. All perfect fidelity teleportation schemes were considered by Werner [22] who showed that essentially the standard teleportation protocol is unique.

We therefore see that for the singlet, there appears to be a complementarity between teleportation, and classical information - one must choose which one to obtain, and one bit of information gets destroyed. The example of the singlet leads to the following general procedure and result.

For a given state, one can use a particular process to distill singlets (quantum information) and perform quantum logical work. For this process, the amount of singlets need not be optimal (i.e. less than $E_D$). Similarly, the classical correlations can be exploited to obtain classical information under a process which need not be optimal (i.e. less than $I_c$). After distilling singlets from a state, one can then use the rest of the state to gain classical information and visa versa. One can also consider more general processes $\mathcal{P}$ where both classical and quantum information is extracted. We will denote by $E_D(\mathcal{P})$ and $I_c(\mathcal{P})$ the amount of quantum and classical information that can be extracted under this process.

We will now show that for any LOCC process $\mathcal{P}$

$$E_D(\mathcal{P}) + I_c(\mathcal{P}) \leq I_c$$

(15)
I.e. the amount of classical information plus quantum information which can be drawn from a state under any process cannot exceed the optimal amount of classical information that can be drawn under LOCC. To see this, we will demonstrate that if there is a process $P$ such that the bound of Equation (13) is violated, then there must exist a process $P'$ which would enable us to draw a greater amount of classical work than the optimal amount $I_c$. The process $P'$ is as follows: We first apply the process $P$ to draw an amount $I_c(P)$ of quantum information (pure separable states), and $E_D(P)$ of singlets from the state $\rho_{AB}$ (we don’t perform teleportations yet).

Alice and Bob can then obtain more classical information by converting the $E_D(P)$ singlets into $E_D(P)$ pure separable states using the optimal procedure described above to convert each singlet into one local pure state. Using this process, Alice and Bob, can draw $I_c(P) + E_D(P)$ bits of classical information from the state $\rho_{AB}$. Since $I_c$ is the optimal amount of information, the bound given by Equation (15) follows.

Eq. (15) shows that there is a trade-off between two different processes: if we define the goal (i) as having one bit of classical information on either site and goal (ii) as sending one bit of quantum information from one site to another, then only one of the goals can be reached. This represents the trade-off. However, there is more going on here than a trade-off. Namely reaching (i) we irreversibly destroy the possibility of access to (ii) and vice versa. This is what corresponds to complementarity. All this can be seen easily in the scenario before the teleportation process: Alice and Bob share a singlet and Alice has an unknown qubit. The latter does not change the balance because, as an additional resource, it must be counted in both the input and the output. Then to achieve (i) we can only spend the singlet which can finally lead to one classical bit according to the result of Ref. [4]. This destroys all the quantum correlations and consequently the possibility to reach goal (ii). If, on the other hand Alice and Bob decide to teleport, then goal (ii) is reached but finally Alice’s state is completely mixed (Bob’s qubit is in an unknown pure qubit that does not enter the balance), so classical information has been irreversibly destroyed to enable us to obtain (i).

It is worthwhile to compare Equation (15) with Equation (12) since in a number of cases [4], $\Delta_c = E_D$. In this case, Equation (12) gives $E_D + I_c \leq I$. One can see that the optimal amount of distillable quantum information plus the amount of distillable classical information is in general much greater than the amount that is actually extracted because of the complementarity between the two. There is an irreversible process which destroys our ability to obtain one kind of information, if the other kind is obtained.

### B. Complementarity between quantum and classical information expressed in terms of entropies

Although Equation (15) essentially expresses the complementarity between classical and quantum information, however, it is not in a form usually associated with uncertainty relations. We will therefore re-express our bound in two different ways. First, we will rewrite it as an informational bound, where the right hand side is a constant, as opposed to something which depends on the state. We will also re-express it as a bound on entropies. In this case, the right hand side is related to the entanglement of a state. Both these bounds have a form like those associated with complimentary observables.

Let us first re-express (15) so that the right hand side is independent of the particular state chosen. To do this, we write it in terms of the classical information which is extractable from correlations (as opposed to the local informational content). Defining the information which can be extracted from correlations as $I_{cor}(P, \rho_{AB}) \equiv I_c - I_{LO}$, we can subtract $I_{LO}$ from both sides of (15) and use Equation (8) to give

$$E_D(P, \rho_{AB}) + I_{cor}(P, \rho_{AB}) \leq \Delta_c.$$  \hfill (16)

Under some assumptions in [4] we have proved that $I_c \leq n - S_X$, $X = A, B$. We believe, that this is true in general. Since $\Delta_c = n - S_A - S_B - I_c$, we than would obtain $\Delta_c \leq \min(S_A, S_B) \leq \log d$, so that (16) would take the form

$$E_D(P, \rho_{AB}) + I_{cor}(P, \rho_{AB}) \leq \log d$$  \hfill (17)

This bound is the tightest bound one can have which is state independent, as it is saturated by maximally entangled states, since for two qubit states we have $E_D(P, \rho_{AB}) + I_{cor}(P, \rho_{AB}) \leq 1$.

We can also re-express the complementarity relation in terms of entropies, which will also be useful in relating our complementarity to the ones usually encountered in quantum mechanics. We therefore rewrite Eq. (15) in the form

$$H_{LOCC(P)} + H_B(P) \geq n + E_f - I_c$$  \hfill (18)

where $H_{LOCC(P)}$ is defined, in analogy with Eq. (1) through

$$I_{LOCC}(P) \equiv n - H_{LOCC(P)}.$$  \hfill (19)
The quantity $H_{\text{LOCC}}(P)$ can be thought of as the Shannon entropy as Alice and Bob would perceive it during the classical information localizing procedure. In fact, it has been advocated [28] that entropy should always be defined with respect to ones measuring apparatus and how they can be used. For example, the coarse-grained entropy is defined with respect to detectors which can only probe with a finite resolution. Here, the measuring devices of Alice and Bob, are restricted to LOCC operations.

Also in analogy with Eq. (3), $H_B(P)$ is defined through

$$E_D(P) \equiv E_f - H_B(P) \quad (20)$$

Instead of $n$ which is the number of qubits needed to create the state $\rho_{AB}$, one ought to define $H_B(P)$ with respect to $E_f$ - the number of singlets needed to create the state under LOCC (called the entanglement of formation). The definition of $H_B(P)$ simply reflects the fact that not all the entanglement can be distilled to perform teleportations - there is “bound entanglement” [27]. Here, since the process is not necessarily optimal, $H_B(P)$ can be less than the bound entanglement. The relationship between bound entanglement and entropy (or heat) was discussed in [29].

Our definitions help elucidate the strong parallels between entanglement and classical information. $n$ separable pure states enable one to perform $n$ bits of physical work, while $E_f$ singlets allow one to perform $E_f$ bits of quantum work such as teleportation. To create a state $\rho_{AB}$ Alice and Bob will also need to use $n$ pure separable states, but they will also need $E_f$ singlets. The entropy $H_{\text{LOCC}}(P)$ prevents Alice and Bob from extracting the full $n$ bits of classical information, while the bound entanglement $H_B(P)$ prevents them from extracting the full $E_f$ bits of quantum information.

The information-theoretic version of our complementarity relation takes a particularly simple form for pure states. For pure states, it was shown in [3] that $I_c = n - E_D = n - E_f$. We therefore have

$$H_{\text{LOCC}}(P)(\psi) + H_B(P)(\psi) \geq 2E_D(\psi) \quad . \quad (21)$$

C. complementarity under LOCC compared with ordinary complementarity

Although the relations given above, may seem unfamiliar, they actually have a logical structure similar to the usual complementarity principle between non-commuting observables such as $x$ and $p$.

The reason that Eqs. (15), (18) and (21) don’t immediately strike one as being like the usual complementarity relationship, is because we are used to seeing them written like a Heisenberg uncertainty principle such as

$$\Delta x \Delta p \geq \hbar \quad (22)$$

or for general operators $M$, $N$, the Robertson inequality [30]

$$\Delta M \Delta N \geq |\langle \psi |[M,N]|\psi \rangle| \quad . \quad (23)$$

However, it is now recognized that these inequalities can be better expressed as relationships between entropies. This method of writing the uncertainty principle was begun by Bialynicki-Birula and Mycielski [31], and later advocated by Deutsch [32] who was dissatisfied with the fact that the bound on the right hand side of Eq. (24) is not a constant but instead depends on the state. His bound was improved by Partovi [33], Kraus [34], and Maassen and Uffink [35].

The latter bound can be written as

$$H_M(\psi) + H_N(\psi) \geq -2 \ln (\sup |\langle m|n \rangle|) \quad (24)$$

where $m$ and $n$ are the eigenstates of two operators $\hat{M}$, $\hat{N}$ and the entropies $H_M$ and $H_N$ of the state $\psi$ are the usual Shannon entropies defined for example by

$$H_M(\psi) = -\sum_m |\langle m|\psi \rangle|^2 \ln |\langle m|\psi \rangle|^2 \quad . \quad (25)$$

That an uncertainty principle can be written in such a form makes intuitive sense, because having a larger Shannon entropy in a certain basis corresponds to a larger uncertainty in measurements which correspond to that basis. We therefore see that our complementarity principle is closely related to the more familiar one encountered in quantum mechanics.

For position and momentum, the Partovi bound takes the form

$$H_x(\psi) + H_p(\psi) \leq 2 \ln[2/(1 + \delta x \delta p/2\pi \hbar)] \quad (26)$$
for small values of $\delta x \delta p / 2\pi \hbar$, where $\delta x$ and $\delta p$ are the resolution of the detector (i.e. phase space is divided into cells).

Comparing this to Equation (21) we see that for pure states, the entanglement plays a role analogous to Planck’s constant $\hbar$. The difference of course is that $\hbar$ is a constant which is independent of the state. In our case, fixing the amount of entanglement in the allowable states is equivalent to fixing $\hbar$ and the detector resolution. The right hand side only depends on the amount of entanglement of the state, and not on any other properties. It is the addition of entanglement into the system which acts like $\hbar$ and creates this complementarity.

Our informational complementarity principle, expressed by Equation (17) does have the appealing feature that the right hand side is completely independent of the state. It has the form of the informational bound derived by Hall [30] for complimentary observables, which is given by

$$I_M + I_N \leq \log N$$

where $I_M$ and $I_N$ gives the amount of information obtainable from a measurement of complimentary observables $\hat{M}$ and $\hat{N}$, and $N$ is the dimension of the Hilbert space. The similarity between this equation, and Equation (17) is striking.

D. Example: drawing quantum and classical information from pure states

We will now consider a protocol $\mathcal{P}$ on pure state where both quantum and classical information is extracted optimally. It will be used to show the balance between classical and quantum information. We will also see that pure states can be thought of as being analogous to coherent states.

Essentially, the procedure is that Alice will perform a measurement which determines how much entanglement is available. Depending on the result of the measurement, the parties can choose whether they want to extract quantum information, or classical information. For example, they may choose to extract quantum information when they find a lot of entanglement (i.e. more than the average optimal amount $E_D$) and extract classical information when there is a small amount of entanglement (since in this case, they can extract more classical information than the optimal amount $I_c$).

The scenario is similar to the concentration of entanglement scheme of Ref. [18]. Alice and Bob share $n$ pairs of a pure state $\psi_{AB} = a|00\rangle + b|11\rangle$. Alice performs a measurement with $n+1$ outcomes. As a result Alice and Bob share a maximally entangled state with Schmidt rank $d_k = \binom{n}{k}$ with probability $p_k = \binom{n}{k} a^{2k} b^{2(n-k)}, \ k = 0,...,n$. The singlet is "diluted" into all $2n$ qubits. However, it is not a maximally entangled state of those $2n$ qubits, so that it can be swapped into a smaller number log $d_k$ of qubit pairs. Then the remaining pairs will be in product states.

Each process $\rho \rightarrow \{p_k, \rho_k\}$ after which information $I_k$ is extracted from $\rho_k$ with probability $p_k$, provides $I_c = \sum_k p_k I_k - H(\{p\})$ of information. The Shannon entropy $H(\{p\})$ of distribution $\{p_k\}$ equals the cost of erasure of information which allows Alice and Bob to work with an ensemble of $\rho_k$’s [19]. Thus in our example Alice and Bob have to put $I_{er} = H(\{p\})$ of erasure to pay for the next part of the scheme, in which they draw the $\sum_k p_k I_k$ amount of information.

In our protocol, Alice and Bob will decide whether to extract entanglement or classical information based on the result of Alice’s measurement (i.e. what value of $k$ she measures). They divide the outcomes of $k$ into two sets, $K_q$ and $K_c$.

If they obtain outcome $k \epsilon K_q$ they i) concentrate the “diluted” singlet, obtaining on average

$$E_D(\mathcal{P}) = \sum_{k \epsilon K_q} p_k \log d_k$$

singlets

ii) draw classical information from the rest of qubits, obtaining on average

$$I_{c1}(\mathcal{P}) = \sum_{k \epsilon K_q} p_k(2n - 2 \log d_k)$$

If instead they obtained the outcomes with $k \epsilon K_c$ they

iii) draw classical information directly from the state with the average result

$$I_{c2}(\mathcal{P}) = \sum_{k \epsilon K_c} p_k(2n - \log d_k) .$$
Summing up all information drawn from the system, we have
\[ I(\mathcal{P}) = E_D(\mathcal{P}) + I_{c_1}(\mathcal{P}) + I_{c_2}(\mathcal{P}) - I_{er}, \]  
(31)
which gives
\[ I(\mathcal{P}) = 2n - \sum_{k=0}^{n} p_k \log d_k - H(p). \]  
(32)

Passing to intensive quantities, \( \tilde{i} = \frac{I(\mathcal{P})}{n} \) is asymptotically equal to the maximum possible amount of classical information per pair which can be obtained starting with \( |\psi_{AB}\rangle \langle \psi_{AB}| \otimes^n \), namely \( \tilde{i} \approx 2 - S_A \), where \( S_A \) is the entropy of reduction of \( \psi_{AB} \). Indeed the last term in (32) is of order of \( \log n \), so that its contribution vanishes in the asymptotic limit. Thus the inequality (13) of complementarity is saturated. It follows that for pure states it is possible to obtain partially quantum and partially classical information, without any loss. However for mixed states it is rather unlikely that an optimal protocol which saturates the inequality would exist.

Since the bound is saturated, and Alice and Bob may obtain any amount of either \( I_c(\mathcal{P}) \) or \( E_D(\mathcal{P}) \) (up to their maximal value), we can therefore think of pure states as being analogous to coherent states i.e. minimally uncertain states. Maximally entangled states are also the only ones which also saturate the constant bound of (17).

IV. LOCC COMMUTATOR

In the previous sections, we introduced a complementarity principle between quantum and classical information. Physically, it referred to general processes, rather than any particular implementation. It would therefore be useful to see if there is a complementarity principle which just refers to general measurements or operations. Indeed, we will find that when the implementation of a measurement is restricted to LOCC, it induces a new set of complementarities. One can generalize this further and consider complementarities when one is restricted to other classes of operations.

We shall start with two examples. In Subsection IV A we will demonstrate that the parity and phase operator, which normally commutes in quantum mechanics, no longer commutes under LOCC. In subsection IV B we will look at measurements which distinguish between the orthogonal states of reference [21]. Finally, in Subsection IV C we will introduce the notion of the restricted commutator that is induced between two observables, if our means to measure them are somehow restricted. Our definition will hold for any class of operations, but for clarity we will talk about LOCC operations.

A. The parity and phase observable

Consider two observables \( \Sigma_x = \sigma_x^{(A)} \otimes \sigma_x^{(B)} \) and \( \Sigma_z = \sigma_z^{(A)} \otimes \sigma_z^{(B)} \), where \( \sigma_x \) and \( \sigma_z \) are the usual Pauli spin matrices and the superscript refers to which subsystem it acts upon. They commute,
\[ [\Sigma_x, \Sigma_z] = 0 \]  
(33)
and their eigenbasis is the Bell basis
\[ \psi^\pm = |01\rangle \pm |10\rangle \]
\[ \Phi^\pm = |00\rangle \pm |11\rangle \]  
(34)
\( \Sigma_z \) measures the parity bit, and will therefore distinguish between the \( \psi \) and \( \Phi \) eigenstates, while \( \Sigma_x \) measures the phase bit, and will distinguish between the eigenstates which have a + as the relative phase, or a −. E.g, if one finds \( \Sigma_z = 0 \) and \( \Sigma_x = 0 \), then we have a singlet.

However if Alice and Bob are restricted to LOCC operations and want to measure such observables on their shared system, it is impossible. Indeed, to measure \( \Sigma_x \), Alice and Bob must separately measure \( \sigma_x \), while to measure \( \Sigma_z \) they have to measure separately \( \sigma_z \). Clearly, since \( \sigma_x \) does not commute with \( \sigma_z \), they cannot measure both the parity and the phase. One might suspect, that there could exist some complicated LOCC protocol that measures them jointly, somehow avoiding measuring directly local non-commuting observables. Later we will show by a simple argument that this is impossible in general by any LOCC operation. However, here we would like to grasp the rough idea of the difference between the global and LOCC measurement.

To this end, note, that in the distant labs case, Alice and Bob measure too much. Indeed, measuring \( \Sigma_x \) globally, gives one bit of information (phase bit), because \( \Sigma_x \) has only two eigenvalues. In contrast, measuring the phase locally
(by having Alice and Bob measure $\sigma_x$ on their subsystem), the two parties will acquire 2 bits of information. Thus in local measurements the measurement is nondegenerate, while $\Sigma_x$ and $\Sigma_z$ are degenerate. In fact, a local determination of parity and phase must acquire two bits of information.

We can simulate a global measurement of parity or phase by using the local operators

$$\Sigma^{LOCC}_z = \sigma_z^{(A)} + \alpha_z \sigma_z^{(B)}$$

$$\Sigma^{LOCC}_x = \sigma_x^{(A)} + \alpha_x \sigma_x^{(B)}$$

where the parameters $\alpha$ act to break the degeneracy in the operators $\Sigma_x$ and $\Sigma_z$. Then these local measurements of parity and phase no longer commute

$$[\Sigma^{LOCC}_x, \Sigma^{LOCC}_z] = -i(\sigma_y^{(A)} + \alpha \sigma_y^{(B)})$$



Here, we have only given one possible local realization of the parity and phase measurement. This therefore may be possible that one can find a clever procedure, perhaps involving positive operator valued measured (POVM’s), such that the parity and phase measurements commute. This, however, cannot be the case.

Let us imagine that there exists local implementations of $\Sigma_z$ and $\Sigma_x$ which are jointly measurable (i.e. commute). Then we would be able to use these locally implementable operators to distinguish between the four Bell states. This is however impossible, as shown in [37]. Naively, this is because if one could distinguish between the Bell states, then one can produce entanglement from the identity state (which is separable). This would contradict the fact that entanglement cannot be created under LOCC. In fact the problem is more subtle. One could, in principle be able to distinguish the Bell states but in so doing, the entanglement could be destroyed. Indeed, in the case of two entangled states, one can distinguish them by LOCC [38] For this case, the entanglement is necessarily destroyed during the measurement. However, distinguishing between the four Bell states would lead to entanglement creation under LOCC [37]. We therefore see that that the parity and phase cannot be jointly measurable. In fact, parity and phase must be completely complimentary, since if one was able to measure the parity, and get even partial knowledge of the phase, then one would be able to create entanglement.

It is also interesting to ask how much entanglement is needed in order to implement the parity and phase operators in such a way that they commute. The answer, is one bit of entanglement. To see this, we note that 1 bit of entanglement is clearly sufficient, since Alice can use a singlet to teleport her qubit to Bob, who can then measure the parity and phase. 1 bit of entanglement must also be necessary, since measuring parity and phase under LOCC allows one to create 1 bit of entanglement [37]. Since one cannot create entanglement under LOCC, one must use up at least 1 bit of entanglement to make the measurement. It is therefore rather interesting that if we act the commutator of Eq. (37) on a separable state then we can get an entangled state, which is maximally entangled for $\alpha = 1$.

Finally, it is worth asking whether one can find other examples for 2X2 systems. In other words, are there other observables which commute globally, but do not commute under LOCC. It appears that the number of examples is very limited. For pairs of product observables of the form $A \otimes B$ there is (up to local unitary transformations) only one other pair of operators which commute globally, namely $\sigma_z^{(A)} \otimes \sigma_y^{(B)}$ and $\Sigma^{LOCC}_z$.

The proof of this result is contained in the appendix to this paper. For now, we simply state the result:

**Proposition** - If for some products of two qubit observables $[A \otimes B, C \otimes D] = 0$ then up to unitary product transformation $U_1 \otimes U_2$ and constant factor, one has

$$A = B = \hat{\sigma}_z,$$

$$C = D = \hat{\sigma}_x.$$
example, we will construct two such operators which although they commute globally, and are implementable locally, do not commute under LOCC.

The nine sausage states are (using a different numbering scheme from [21] for convenience)

\[
\begin{align*}
A & \quad B \\
\psi_1 &= |0 + 1\rangle \langle 2| \\
\psi_2 &= |0 - 1\rangle \langle 2| \\
\psi_3 &= |0\rangle \langle 0 + 1| \\
\psi_4 &= |0\rangle \langle 0 - 1| \\
\psi_5 &= |1 + 2\rangle \langle 0| \\
\psi_6 &= |1 - 2\rangle \langle 0| \\
\psi_7 &= |1\rangle \langle 1| \\
\psi_8 &= |2\rangle \langle 1 + 2| \\
\psi_9 &= |2\rangle \langle 1 - 2| 
\end{align*}
\]

(40)

Now we can construct an operator \( O_1 \) which has it’s eigenstates, the first seven states with seven different, non-zero eigenvalues and remaining two eigenstates with zero eigenvalues. We can also construct an operator \( O_2 \) which has \( \psi_8 \) and \( \psi_9 \) as eigenstates with two different, non-zero eigenvalues and remaining seven eigenstates with zero eigenvalues. These operators clearly commute globally, since all the \( \psi_i \) are orthogonal. However, under LOCC they clearly cannot commute. If they did, one could measure \( O_1 \) and \( O_2 \) simultaneously, and therefore, distinguish between all nine sausage states, in violation of the indistinguishability proof given in [21].

Now, \( O_2 \) can easily be implemented under LOCC. Bob can simply use the projectors \( |1 + 2\rangle \) and \( |1 - 2\rangle \) to implement \( O_2 \). These projectors distinguish between \( \psi_8 \) and \( \psi_9 \), \( O_1 \) can also be implemented under LOCC, although some effort is needed. Consider for example an implementation \( O_1' \) which instead has the following orthogonal eigenbasis

\[
\begin{align*}
A & \quad B \\
\psi_1 &= |0 + 1\rangle \langle 2| \\
\psi_2 &= |0 - 1\rangle \langle 2| \\
\psi_3 &= |0\rangle \langle 0 + 1| \\
\psi_4 &= |0\rangle \langle 0 - 1| \\
\psi_5 &= |1 + 2\rangle \langle 0| \\
\psi_6 &= |1 - 2\rangle \langle 0| \\
\psi_7 &= |1\rangle \langle 1| \\
\psi_{10} &= |2\rangle \langle 2| \\
\psi_{11} &= |2\rangle \langle 1|
\end{align*}
\]

(41)

The first seven eigenstates are identical to the eigenstates of \( O_1 \), and so \( O_1' \) is an implementation of \( O_1 \). Furthermore, \( O_1' \) can be implemented under LOCC using a sequence of von Neumann projection measurements which was given in [21]. The detailed procedure is contained in Appendix B.

The difficulty, is that while the eigenvectors \( \psi_1 - \psi_7 \) commute with \( O_2' \), the projectors onto \( \psi_{10} \) and \( \psi_{11} \) do not. If we write \( O_1' \) and \( O_2' \) as

\[
O_1' = O_1 + |2\rangle \langle 2| \otimes \sigma_z^{(B)}, \quad O_2' = |2\rangle \langle 2| \otimes \sigma_x^{(B)}
\]

(42)

where we once again use the Pauli matrices, this time written in the \( |1\rangle, |2\rangle \) basis, then we find that while \( [O_1, O_2] = 0 \), we have

\[
[O_1', O_2'] = -i|2\rangle \langle 2| \otimes \sigma_y^{(B)}
\]

(43)

Unlike the case of the parity-phase commutator, this commutator, operating on a separable state, cannot create entanglement. This may be related to the fact that for parity and phase, the eigenstates are entangled, while for \( O_1 \) and \( O_2 \), the eigenstates are separable.

C. The LOCC commutator

In general, we may define the LOCC commutator as follows. Consider two operators \( M \) and \( N \) which are implementable under LOCC. For all LOCC implementations \( M_{LOCC} \) and \( N_{LOCC} \) we can define the LOCC commutator as

\[
[M, N]_{LOCC} = \min [M_{LOCC}, N_{LOCC}]
\]

(44)
where the minimum is taken over all LOCC implementations of $M$ and $N$. It is not clear what the physical significance is of the right hand side of this equation. In the parity-phase it was an operator which could create one bit of entanglement (which was also the amount of entanglement needed to implement both measurements). This is perhaps intriguing in light of Equation (21) where entanglement played the role of $\hbar$.

The LOCC commutator can also be generalized to any class of operations. If we consider the set of all allowable operations $A$, and a restricted subset of these $R \subseteq A$, then we can define a restricted commutator much in the way we have done here.

There is however, one key difference between the interpretation of this commutator, and the usual commutator we are familiar with. The fact that two operators $M$ and $N$ do not LOCC commute can imply that they are complementarity observables. It will also imply that one cannot prepare a state under LOCC which has a definite value of the observable $M$ and $N$. But it does not imply uncertainty of measurements.

To see this, recall that a singlet (for example) has definite parity and phase. If Alice and Bob are given an ensemble of singlets, and measure the phase on half the ensemble they will always get the same result ($-$). If they measure the parity on the other half of the ensemble, then they will also always get a definite result (0). There is nothing uncertain about what the outcome of a measurement of parity or phase will be. However, if Alice and Bob are given an unknown state (perhaps a singlet) then their measurement of phase will completely destroy their ability to determine what the parity is, and visa versa. For a single state, they cannot determine both the parity and the phase. Likewise, there is no way for them to prepare a system with definite parity and phase (which would amount to creating entanglement).

This decoupling of complementarity and uncertainty shows that they are independent concepts. The essential reason for this decoupling is that the measurement does not prepare the system in an eigenstate of the observable we are trying to measure. The measurement irreversibly alters the state. Usually, the von Neumann postulate holds – after a measurement, the state is in an eigenstate of the observable. Therefore it was hard to distinguish between complementarity and uncertainty, and discussions concerning this difference could seem speculative and philosophical. However, as we have seen, the situation changes in the LOCC paradigm.

V. CONCLUSION

We have found that when one is restricted to making only local measurements and communicating classically, then new types of complementarities are induced. One type of complementarity was between classical and quantum information, and was given by Equation (15). If one attempts to maximize the amount of classical information (pure separable states) then the ability to extract quantum information and perform quantum operations such as teleportations is lost. Likewise, extracting quantum information destroys the ability to obtain classical information.

In Equations (17) and (18), we wrote this complementarity in a form which was of the same kind as those ordinarily encountered in quantum mechanics. For pure states, we found that the entanglement plays the role of $\hbar$.

We also found a complementarity that gets induced between operators when implemented under LOCC. For example, the parity and the phase of a 2X2 state is no longer jointly measurable. It is remarkable that in this case, the uncertainty principle and complementarity are decoupled, and one can have complementarity without uncertainty. We therefore see that they are indeed separate concepts.

We then introduced the notion of the LOCC commutator, to quantify this complementarity. How to interpret this quantity is an interesting open question. In this regard, it is perhaps interesting that for parity and phase, the LOCC commutator can create one bit of entanglement. Likewise, being able to measure parity and phase simultaneously also results in the creation of one bit of entanglement.

It therefore might be interesting to ask how much entanglement would be needed such that one can jointly measure two observables. Quantifying this "entanglement assisted commutator" might help answer some of the questions raised here.

It also would be interesting to relate the complementarity principle between operators, and between classical and quantum information. The latter involves comparisons between two types of restricted operations (LO and LOCC), while the complementarity principle between operators only involves LOCC. However, both seem to involve the notion of entanglement.

In this paper, the part of quantum information which was discussed was entanglement. However, the quantum deficit $\Delta q$ is also non-zero for unentangled states (at least for a finite number of copies). It would therefore be of interest to also consider the case of date hiding, where Alice and Bob are essentially unable to obtain the classical information encoded in a state.

Finally, the above results support the view that quantum states carry two complementary kinds of information, the classical information which is locally accessible and quantum information which can be used for such tasks as teleportation (see in this context). This complementarity lies at the foundations of quantum mechanic more
deeply than it might seem. We believe that complementarity in general is a fundamental and intrinsic feature of information carried by physical systems which cannot be derived from any probabilistic models.

**APPENDIX A: PROOF OF PROPOSITION 1**

Let us provide a simple lemma first:

**Lemma** - If $X \otimes Y = R \otimes S$ for some operators $X, Y, R, S$ then $X = \alpha R, Y = \alpha^{-1}S$ for some nonzero number $\alpha$.

Proof of the above lemma is immediate. Without loss of generality we can consider $X, Y$ to be of full rank and utilize their inverses (otherwise they are pseudoinverses) getting $I \otimes I = X^{-1}R \otimes Y^{-1}S$. Comparing the eigenvectors of both sides of the latter formula gives $Y^{-1}S = \alpha I, X^{-1}R = \alpha^{-1}I$, concluding the proof of the lemma.

Now we shall provide the simple proof of the following

**Proposition** - If for some products of two qubit observables $[A \otimes B, C \otimes D] = 0$ then up to unitary product transformations $U_1 \otimes U_2$ and constant factor, one has

$$A = B = \hat{\sigma}_z,$$

$$C = D = \hat{\sigma}_x.$$  \hspace{1cm} (A1)

Proof - By adding and subtracting the term $CA \otimes BD$, it is immediate that vanishing of the commutator from the Proposition is equivalent to

$$[A, C] \otimes BD = CA \otimes [D, B]$$  \hspace{1cm} (A2)

Applying the lemma to the above we get

$$[A, C] = \alpha CA, \quad [D, B] = \alpha BD$$  \hspace{1cm} (A3)

Now for two qubits we put $A = aI + \tilde{a}\hat{\sigma}, B = bI + \tilde{b}\hat{\sigma}, C = cI + \tilde{c}\hat{\sigma}, D = dI + \tilde{d}\hat{\sigma}$. We then perform a simple calculation taking into account the fact that (because of linear independence of $I, \sigma_x, \sigma_y, \sigma_z$) absence of $I$ on one side implies the same for the other side. This gives two equations:

$$\alpha[\tilde{a}\hat{\sigma}, \tilde{b}\hat{\sigma}] = (a\tilde{a} + c\tilde{c})\hat{\sigma} + i\tilde{a} \times \tilde{c}\hat{\sigma}$$

$$\alpha[\tilde{b}\hat{\sigma}, \tilde{d}\hat{\sigma}] = (b\tilde{b} + d\tilde{d})\hat{\sigma} + i\tilde{b} \times \tilde{d}\hat{\sigma}$$  \hspace{1cm} (A5)

Calculating the LHS for both sides and using the linear independence of Pauli matrices we get finally:

$$a\tilde{a} + c\tilde{c} + i(1 - 2\alpha)\tilde{a} \times \tilde{c} = 0$$

$$b\tilde{b} + d\tilde{d} + i(1 - 2\alpha)\tilde{b} \times \tilde{d} = 0$$  \hspace{1cm} (A6)

Now let us observe that (i) $\tilde{a} \times \tilde{c} \neq 0$ and hence also (ii) $\tilde{a}, \tilde{c} \neq 0$. Indeed if $\tilde{a} \times \tilde{c} = 0$ then $[A, C] = 0$ and consequently (see (A3)) either $AC = 0$ (which leads to the trivial solution because some operators must completely vanish) or $AC \neq 0$ but then (again because of (A3)) also $[B, D] = 0$ which would be trivial again.

Because of (i) and (ii) the LHS of the first line of (A6) is a linear combination of three nonzero and linearly independent vectors $\tilde{a}, \tilde{c}, \tilde{a} \times \tilde{c}$ so all the coefficients in the combination must vanish giving in particular $a = c = 0$.

In a similar way we get $b = d = 0$. This simplifies our observables: $A = \tilde{a}\hat{\sigma}, B = \tilde{b}\hat{\sigma}, C = \tilde{c}\hat{\sigma}, D = \tilde{d}\hat{\sigma}$. Putting them again into (A4) we get immediately

$$(\tilde{a} \times \tilde{c}) \otimes (\tilde{b}\hat{\sigma} I + \tilde{b} \times \tilde{d}\hat{\sigma}) = (\tilde{a}\tilde{c}\hat{\sigma} I + i\tilde{a} \times \tilde{c}\hat{\sigma}) \otimes (\tilde{b} \times \tilde{d}\hat{\sigma})$$  \hspace{1cm} (A7)

which for nonzero $\tilde{a} \times \tilde{c}$ and $\tilde{b} \times \tilde{d}$ is satisfied iff $\tilde{a}\tilde{c} = \tilde{b}\tilde{d} = 0$. We can put $\tilde{a} = \tilde{b} = \tilde{z}$ since we can always choose such a local basis for Alice and Bob. We then have $\tilde{c} = \tilde{d} = \tilde{x}$ (again using our choice of label for the direction orthogonal to $\tilde{z}$). This concludes the proof of the Proposition.
Here we show how to implement $O'_1$ using a ping-pong process between Alice and Bob. Essentially, the procedure is:

- **b1)** Bob first does a projection on $|2\rangle$ and communicates his result to Alice.
  - **a1)** If his result is positive, then Alice can project onto the three states $|0+1\rangle$, $|0-1\rangle$ which will distinguish between $\psi_1$ and $\psi_2$. However, if she finds neither $\psi_1$ or $\psi_2$ she will know that the state is $\psi_{10}$ which is in some sense superfluous information which she would not get if she was measuring $O_1$ globally.
  - **a1')** If Bob’s first projection yielded a negative result, then Alice instead projects onto the $|0\rangle$ state and communicates her result to Bob.
- **b2)** If her projection found the state $|0\rangle$ then Bob projects onto $|0+1\rangle$ and $|0-1\rangle$ which distinguishes between $\psi_4$ and $\psi_5$.
  - **b2')** If her result was negative, Bob projects onto $|0\rangle$ and $|1\rangle$ and communicates the result to Alice.
- **a2)** Alice can then make the final orthogonal projection, either onto $|1+2\rangle$ and $|1-2\rangle$, or onto $|1\rangle$ and $2\rangle$ depending on Bob’s result. This distinguishes between $\psi_5$, $\psi_6$ and $\psi_7$, as desired, but it also singles out $\psi_{11}$ which is again, surplus information which is not required to implement $O_1$.

Acknowledgments: This work is supported by EU grant EQUIP, Contract No. IST-1999-11053. M. H., P. H. and R.H. acknowledge support of KBN grant No. 2 P03B 103 16. M.H. would like to thank Viacheslav Belavkin and Berthold-Georg Englert for discussion on uncertainty principles. J.O. would like to thank Jacob Bekenstein for interesting discussions and the Vergelle Institute for their hospitality while part of this work was completed. He acknowledges the support of the Lady Davis Fellowship Trust, and grant No. 129/00-1 of the Israel Science Foundation.

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