Abstract. We develop a general theory of bisets: sets with two commuting group actions. They naturally encode topological correspondences.

Just as van Kampen’s theorem decomposes into a graph of groups the fundamental group of a space given with a cover, we prove analogously that the biset of a correspondence decomposes into a graph of bisets: a graph with bisets at its vertices, given with some natural maps. The fundamental biset of the graph of bisets recovers the original biset.

We apply these results to decompose the biset of a Thurston map (a branched self-covering of the sphere whose critical points have finite orbits) into a graph of bisets. This graph closely parallels the theory of Hubbard trees.

This is the first part of a series of five articles, whose main goal is to prove algorithmic decidability of combinatorial equivalence of Thurston maps.

1. Introduction

This is the first of a series of five articles, and develops the theory of decompositions of bisets. For an overview of the series, see [1].

1.1. Bisets. Bisets are algebraic objects used to describe continuous maps up to isotopy. Classically, a map $f: (Y, \uparrow) \to (X, \ast)$ between pointed spaces induces a homomorphism between the fundamental groups $f_*: \pi_1(Y, \uparrow) \to \pi_1(X, \ast)$. If, however, $f$ does not preserve any natural base points, it is much more convenient to consider a weaker kind of relation between $\pi_1(Y, \uparrow)$ and $\pi_1(X, \ast)$: this is precisely a biset, namely a set $B(f)$ with two commuting group actions, of $\pi_1(Y, \uparrow)$ on the left and of $\pi_1(X, \ast)$ on the right. Bisets may naturally be multiplied, and the product of bisets corresponds to composition of the maps.

Another advantage of bisets is that only bijective maps may be inverted, while every biset $B$ has a contragredient $B^\vee$, obtained by dualizing the actions. Thus it is almost as easy to handle correspondences $Y \xleftarrow{\iota} Z \xrightarrow{f} X$ as genuine maps $Y \to X$, see §4: the biset of the correspondence is $B(i)^\vee \otimes B(f)$.

Van Kampen’s theorem is a cornerstone in algebraic topology: given a space $X$ covered by (say) open subsets $(U_\alpha)$, the theorem expresses the fundamental group of a space in terms of the fundamental groups of the pieces $U_\alpha$ and of their intersections. This is best expressed in terms of a graph of groups, namely a simplicial graph with groups attached to its vertices and edges. The fundamental group of a graph of groups is algebraically constructed in terms of the graph data, and recovers the fundamental group of $X$. 

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The main construction in this article is a graph of bisets expressing the decomposition of a continuous map, or more generally a topological correspondence, between spaces given with compatible covers, see Definition 3.8. We single out the subclass of fibrant graphs of bisets. Graphs of bisets can be multiplied, and the product of fibrant graphs of bisets is again fibrant.

We construct in Definition 3.13 the fundamental biset of a graph of bisets, and show in Corollary 3.20 how they can be conveniently computed for fibrant graphs of bisets.

Our central result, Theorem 4.8, is an analogue of van Kampen’s theorem in the language of bisets. It expresses the biset of a correspondence as the fundamental biset of a graph of bisets constructed from restrictions of the correspondence to elements of the cover.

1.2. Applications. We then apply these results, in §5, to complex dynamics. Following Thurston, we consider branched coverings of the sphere, namely self-maps $f: S^2 \to S^2$ that are locally modeled on $z \mapsto z^d$ in complex charts. The subset of $S^2$ at which the local model is $z^d$ with $d > 1$ is called the critical set of $f$, and the post-critical set $P(f)$ is the strict forward orbit of the critical set. A Thurston map is a branched covering of the sphere for which $P(f)$ is finite. Examples are rational maps such as $f_p(z) = z^2 - 1$.

The simplest example of all is the map $f(z) = z^d$ itself, with $P(f) = \{0, \infty\}$. The map $f$ is a covering on $\mathbb{C} \setminus \{0, \infty\}$, and $\pi_1(\mathbb{C} \setminus \{0, \infty\}, 1) = \mathbb{Z}$. In this case, we can give the biset of $f$ quite explicitly: it is $B(f) = \mathbb{Z}$, with right and left actions given by $m \cdot b \cdot n = dm + b + n$. We call such bisets regular cyclic bisets, and we define a cyclic biset as a transitive biset over cyclic groups.

The next simplest maps are complex polynomials $f(z)$ with finite post-critical set. They have been intensively studied, in particular via their Hubbard tree, see [6, 7, 14]. This is a dynamically-defined $f$-invariant tree containing $P(f)$ and embedded in $\mathbb{C}$. We apply the van Kampen Theorem 4.8 to the Hubbard tree and obtain in this manner a decomposition of the biset of $f$ as a graph of cyclic bisets, see Theorem 5.5.

One of the advantages in working with branched coverings rather than rational maps is that surgery operations are possible. For example, given a Thurston map $f$ with a fixed point $z$ mapping locally to itself by degree $d > 1$, and given a topological polynomial $g$ of degree $d$, a small neighbourhood of $f$ may be removed to be replaced by a sphere, punctured at $\infty$, on which $g$ acts. One calls the resulting map the tuning of $f$ by $g$.

This operation has a transparent interpretation in terms of graphs of bisets: it amounts to replacing a cyclic biset, at a vertex of the graph of bisets of $f$, with the biset $B(g)$; see Theorem 5.8.

In case $f$ itself also has degree $d$, one calls the resulting map $h$ the mating of $f$ and $g$. The sphere on which $h$ acts is naturally covered by the spheres on which $f$, respectively $g$ act, and the “equator” on which they overlap. Thus $h$ is naturally expressed by a graph of bisets with two vertices corresponding to $f$ and $g$ respectively, and an edge corresponding to the equator; see Theorem 5.9.

Laminations allow Julia sets of polynomials to be obtained out of the Julia set of $z^d$, namely a circle, by pinching. Similarly, van Kampen’s Theorem 4.8 applied to the lamination of $f$ decomposes the biset $B(f)$ into a graph of bisets made of trivial bisets and one regular cyclic biset (the biset of $z^d$). We compute it explicitly
for the map \( f(z) = z^2 - 1 \), and describe in this manner, in \S 5.6, the mating of \( z^2 - 1 \) with an arbitrary quadratic polynomial.

1.3. Notation. We introduce some convenient notations, which we will follow throughout the series of articles. For a set \( S \), we write \( Si \) for its permutation group. This has the mnemonic advantage that \( \#(Si) = (\#S)! \). We always write \( \mathbb{1} \) for the identity map from a set to itself. The restriction of a map \( f: Y \to X \) to a subset \( Z \subset Y \) is written \( f|_Z \). Self-maps are written \( f: X \circlearrowleft \) in preference to \( f: X \to X \).

Paths are continuous maps \( \gamma: [0, 1] \to X \); the path starts at \( \gamma(0) \) and ends at \( \gamma(1) \). We write \( \gamma \# \delta \) for concatenation of paths, following first \( \gamma \) and then \( \delta \); this is defined only when \( \gamma(1) = \delta(0) \). We write \( \gamma^{-1} \) for the inverse of a path.

We write \( \approx \) for isotopy of paths, maps etc, \( \sim \) for conjugacy or combinatorial equivalence, and \( \cong \) for isomorphism of algebraic objects.

A graphs of groups is a graph \( X \) with groups associated with \( X \)'s vertices and edges. We always write \( G_x \) for the group associated with \( x \in X \); thus if \( Y \) is another graph of groups, we will write \( G_y \) for the group associated with \( y \in Y \), and no relationship should be assumed between \( G_x \) and \( G_y \). Similarly, in a graph of bisets \( \mathcal{B} \) there are bisets \( B_z \) associated with \( z \in \mathcal{B} \), and different graphs of bisets will all have their bisets written in this manner.

2. Bisets

We show, in this section, how topological data can be conveniently converted to group theory. We shall extend, along the way, the classical dictionary between topology and group theory.

Consider a continuous map \( f: Y \to X \) between path connected topological spaces, and basepoints \( \star \in Y \) and \( \ast \in X \). Denote by \( H = \pi_1(Y, \star) \) and \( G = \pi_1(X, \ast) \) their fundamental groups. If \( f(\star) = \ast \), then \( f \) induces a homomorphism \( f_*: H \to G \); however, no such natural map exists if \( f \) does not preserve the basepoints.

A solution would be to express \( f \) in the fundamental groupoid \( \pi_1(X) \), whose objects are \( X \) and whose morphisms from \( x \) to \( y \) consist of all paths from \( x \) to \( y \) in \( X \) up to homotopy rel their endpoints. However, for computational purposes, a much more practical solution exists: one chooses basepoints in \( X \) and \( Y \), and expresses \( f \) as an \( H \)-\( G \)-biset.

**Definition 2.1** (Bisets). Let \( H, G \) be two groups. An \( H \)-\( G \)-biset is a set \( B \) equipped with a left \( H \)-action and a right \( G \)-action that commute; namely, a set \( B \) and maps \( H \times B \to B \) and \( B \times G \to B \), both written \( \cdot \), such that

\[
h \cdot h' \cdot (b \cdot g') = hh' \cdot (b \cdot gg') = (hh' \cdot b) \cdot gg' = (hh' \cdot b) \cdot g \cdot g',
\]

so that no parentheses are needed to write any product of \( h \)'s, \( b \), and \( g \)'s. We will also omit the \( \cdot \), and write the actions as multiplication.

A \( G \)-biset is a \( G \)-biset.

An \( H \)-\( G \)-set \( B \) is left-free if, qua left \( H \)-set, it is isomorphic to \( H \times S \) for a set \( S \) (where, implicitly, the left action of \( H \) is by multiplication on the first coordinate); it is right-free if, qua right \( G \)-set, \( B \) is isomorphic to \( T \times G \). It is left-principal, respectively right-principal, if furthermore \( S \), respectively \( T \) may be chosen a singleton, so that the respective action is simply transitive; more generally, it is left, respectively right free of rank \( r \) if \( \#S = r \), respectively \( \#T = r \). \( \triangleleft \)
Bisets should be thought of as generalizations of homomorphisms. Indeed, if \( \phi : H \to G \) is a group homomorphism, one associates with it the \( H \)-\( G \)-set \( B_\phi \), which, qua right \( G \)-set, is plainly \( G \); the left \( H \)-action is by

\[
h \cdot b = h^{\phi(b)}.
\]

All bisets of the form \( B_\phi \) are right-principal; and they are left-free if and only if \( \phi \) is injective. Note that we write the evaluation of maps as ‘\( f(x) \)’ unless the map is a group homomorphism, in which case we write ‘\( x \phi \)’.

Let us return to our continuous map \( f : Y \to X \), but drop the assumption \( f(\uparrow) = * \). The \( H \)-\( G \)-biset of \( f \) is defined as homotopy classes of paths rel their endpoints:

\[
B(f) = B(\uparrow, *, *) = \{ \gamma : [0, 1] \to X \mid \gamma(0) = f(\uparrow), \gamma(1) = * \} / \sim.
\]

For paths \( \gamma, \delta : [0, 1] \to X \) with \( \gamma(1) = \delta(0) \), we denote by \( \gamma \# \delta \) their concatenation, defined by

\[
(\gamma \# \delta)(t) = \begin{cases} 
\gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\delta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1;
\end{cases}
\]

and, for a path \( \gamma : [0, 1] \to X \), its reverse \( \gamma^{-1} \) is defined by \( \gamma^{-1}(t) = \gamma(1 - t) \).

The left action of \( H \) on \( B(f) \) is, for a loop \( \lambda \) in \( Y \) based at \( \uparrow \),

\[
[\lambda] \cdot [\gamma] = [f \circ \lambda \# \gamma],
\]

and the right action of \( G \) is, for a loop \( \mu \) in \( X \) based at \(*\),

\[
[\gamma] \cdot [\mu] = [\gamma \# \mu].
\]

It is then clear that \( B(f) \) is a right-principal biset. If \( f(\uparrow) = * \), then \( B(f) \) is naturally isomorphic to \( B_{f*} \).

As we will see in §4, bisets actually encode topological correspondences, as generalizations of continuous maps.

2.1. Morphisms. We consider three different kinds of maps between bisets.

**Definition 2.2** (Biset morphisms). Let \( H B_G \) and \( H B'_G \) be two \( H \)-\( G \)-bisets. A *morphism* between them is a map \( \beta : B \to B' \) such that

\[
h \beta^g b = (h \beta g)^\beta \quad \text{for all } h \in H, b \in B, g \in G.
\]

The definitions of endomorphism, isomorphism, automorphism, monomorphism, and epimorphism are standard.

Let now \( H B_G \) and \( H B'_G \) be two bisets. A *congruence* is a triple \((\psi, \beta, \phi)\) of maps, with \( \psi : H \to H' \) and \( \phi : G \to G' \) group homomorphisms, and \( \beta : B \to B' \) a map, such that

\[
h \psi b \beta^g \phi = (h \beta g)^\beta \quad \text{for all } h \in H, b \in B, g \in G.
\]

The map \( \beta \) itself is called a \((\psi, \phi)\)-congruence; therefore a morphism is the same thing as a \((1, 1)\)-congruence. A congruence in injective if all \( \psi, \beta, \phi \) are injective, and similarly for surjective etc.
Consider finally a $G$-biset $GB_G$ and a $G'$-biset $GB'_G$. Apart from morphisms and congruences, a third (intermediate) notion relates them: a semiconjugacy is a pair $(\phi, \beta)$ of maps, with $\phi: G \to G'$ and $\beta: B \to B'$, such that

$$h^\phi b^\beta g^\phi = (hb)^\beta \quad \text{for all } g, h \in G, b \in B.$$ 

In other words, $\beta$ is a $(\phi, \phi)$-congruence. Note that we do not require, as is sometimes customary, that $\beta$ be surjective. A conjugacy is an invertible semiconjugacy, i.e. a semiconjugacy in which both $\phi$ and $\beta$ are invertible.

Note now the following important, if easy, fact: the isomorphism class of a biset $B_{\phi}$ remembers precisely the homomorphism $\phi$ up to inner automorphisms. More precisely,

**Lemma 2.3.** Let $\phi, \psi: H \to G$ be two homomorphisms. Then the bisets $B_{\phi}$ and $B_{\psi}$ are congruent if and only if there exists an automorphism $\eta$ of $G$ such that $\psi = \phi \eta$; and they are isomorphic if and only if $\eta$ is inner.

**Proof.** We only prove the second assertion ("isomorphic bisets if and only if $\eta$ is inner").

Assume first $\psi = \phi \eta$, and let $\eta$ be conjugation by $g$. Then an isomorphism between the bisets $B_{\phi}$ and $B_{\psi}$ is given by $\beta: b \mapsto g^{-1}b$. Indeed,

$$(h \cdot b)^\beta = (h^\phi b)^\beta = g^{-1}h^\phi b = h^\phi \eta g^{-1}b = h^\psi b^\beta = h \cdot b^\beta.$$ 

Conversely, if $B_{\phi}$ and $B_{\psi}$ are isomorphic, let $\beta$ be such an isomorphism, and set $g = (1^\beta)^{-1}$. Because $\beta$ commutes with the right $G$-action, which is principal, the map $\beta$ must have the form $b^\beta = 1^\beta b = g^{-1}b$, and the same computation as above shows that $\psi = \phi \eta$, with $\eta$ conjugation by $g$. \hfill $\square$

2.2. **Products.** The product of the $H$-$G$-biset $B$ with the $G$-$F$-biset $C$ is the $H$-$F$-biset $B \otimes_G C$, defined as

$$B \otimes_G C = B \times C / \{(b, g \cdot c) = (b \cdot g, c) \text{ for all } b \in B, g \in G, c \in C\};$$ 

it is naturally an $H$-$F$-biset for the left $H$-action on $B$ and the right $F$-action on $C$. We have the easy

**Lemma 2.4.** Let $\phi: H \to G$ and $\psi: G \to F$ be group homomorphisms. Then

$$B_{\phi \psi} \cong B_{\phi} \otimes_G B_{\psi}. \quad \square$$ 

Note that we stick to the usual topological convention that $g \circ f$ is first $f$, then $g$; but we use the algebraic order on composition of homomorphisms and bisets, so that $fg$ means ‘first $f$, then $g’ for composable algebraic objects $f$ and $g$. In other words, we may write $g \circ f = fg$.

The contragredient of the $H$-$G$-biset $B$ is the $G$-$H$-biset $B^\vee$, which is $B$ as a set (but with elements written $b$), and actions

$$g \cdot b \cdot h = (h^{-1} \cdot b \cdot g^{-1}).$$ 

If $\phi: H \to G$ is invertible, then we have $(B_{\phi})^\vee = B_{\phi^{-1}}$. In all cases, we have a canonical isomorphism $(B \otimes C)^\vee = C^\vee \otimes B^\vee$.

**Remark 2.5.** Answering a question of Adam Epstein, we may define $\text{Hom}(B, C) = B^\vee \otimes C$, and then note that ‘$\otimes$’ is the adjoint to this internal $\text{Hom}$ functor, namely $\text{Hom}(C \otimes B, D) = \text{Hom}(B, \text{Hom}(C, D))$ is natural.
Products allow us to write congruences in terms of morphisms:

**Lemma 2.6.** Let $\psi: H \to H'$ and $\phi: G \to G'$ be homomorphisms, let $B$ be an $H$-$G$-biset, and let $B'$ be an $H'$-$G'$-biset. Then there is an equivalence between congruences $(\psi, \beta, \phi): B \to B'$ and morphisms $\gamma: B_\psi \otimes B \otimes B_\phi \to B'$, in the following sense: there is a natural map $\theta: B \to B_\psi \otimes B \otimes B_\phi$ defined by $b \mapsto 1^\gamma \otimes b \otimes 1$, and the congruence $(\psi, \beta, \phi)$ factors as $(\psi, \beta, \phi) = \theta \gamma$.

In particular, the bisets $G_B G$ and $G_B' G'$ are conjugate if and only if there exists an isomorphism $\phi: G \to G'$ with $B' \cong B_\psi \otimes B \otimes B_\phi = B_{\phi^{-1}} \otimes B \otimes B_\phi$.

### 2.3. Transitive bisets.

We call an $H$-$G$-biset transitive if it consists of a single $H \times G$-orbit. If desired, transitive bisets may be viewed as quotients of right-principal bisets (namely, bisets coming from homomorphisms), as follows:

**Lemma 2.7.** Let $B$ be a transitive $H$-$G$-biset. Then there exist a group $K$ and homomorphisms $\phi: K \to G$, $\psi: K \to H$ such that $B = B_\psi \otimes_K B_\phi$. Furthermore, there exists a unique minimal such $K$, in the sense that if $(K', \phi', \psi')$ also satisfy $B = B_\psi \otimes_{K'} B_{\phi'}$ then there exists a homomorphism $\tau: K' \to K$ with $\tau \phi = \phi'$ and $\tau \psi = \psi'$.

**Proof.** Choose a basepoint $b \in B$, and define

$$K = \{(h, g) \in H \times G \mid bg = hb\}.$$ 

The homomorphisms $\psi, \phi$ are given by projection on the first, respectively second coordinate. To construct an isomorphism $\beta: B_\psi \otimes B_\phi \to B$, set $(h^\psi \otimes g)^\phi := h^{-1}bg$; note that this is well-defined because $h^\psi \otimes g = (h')^\psi \otimes g'$ if and only if there exists $(h'', g'') \in K$ with $g' = g''g$ and $h' = h''h$; and then $h^{-1}bg = (h')^{-1}bg'$.

To prove the unicity of $K$, consider a group $K'$ with homomorphisms $\phi': K' \to G$ and $\psi': K' \to H$, and an isomorphism $\beta: B_{\psi'} \otimes_{K'} B_{\phi'} \to B$. Define then the map $\tau: K' \to K$ as follows. Write $\beta^{-1}(b) = (b_1, b_2)$. For $k' \in K'$, define $h \in H, g \in G$ by

$$\beta(b_1, k'b_2) = hb = bg$$

and then $\tau(k') = (h, g)$.

Note that the product of two bisets, when expressed as groups $K, L$ with homomorphisms as in the lemma, is nothing but the fibre product of the corresponding groups $K, L$.

If furthermore $B$ is left-free, then the construction can be made even more explicit: the biset $B_\psi$ is a subbiset of $B$, and $B_\phi$ is a left- and right-free. We summarize this in the

**Proposition 2.8.** Let $B$ be a left-free transitive $H$-$G$-biset. Choose $b \in B$. Define

- $G_b := \{g \in G \mid bg \in Hb\}$, the right stabilizer of $b$;
\[ D := GB_G, \text{ the natural } Gb-G \text{ biset}; \]
\[ C^\triangledown := Hb, \text{ an } H-Gb \text{ subtree of } HBG_b. \]

Then \( C \) is a right-principal biset and \( B \cong C^\triangledown \otimes_{G_b} D. \)

\[ \square \]

2.4. Combinatorial equivalence of bisets. Let \( GB_G \) be a left-free biset. For every \( n \geq 0 \) we have a right \( G \)-action on \( 1 \otimes_G B_G^\otimes n \), which gives a well defined action on the set

\[ T := \bigcup_{n \geq 0} 1 \otimes_G B_G^\otimes n. \]

The \( G \)-set \( T \) naturally has the structure of a rooted tree, by putting an edge from

\[ 1 \otimes b_1 \otimes \cdots \otimes b_n \rightarrow 1 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1} \]

for all \( b_i \in B \). The action of \( G \) on \( T \) is self-similar in an appropriate labeling of \( T \), see [12, Chapter 2].

Set \( G' := G/\ker(\text{action}); \) thus \( G' \) is the quotient of \( G \) so that the induced action on \( T \) is faithful. As in [13] we say that \( GB_G \) is combinatorially equivalent to \( GB'_G \),

\[ GB'_G := G' \otimes_G B \otimes_G G'. \]

(The motivation is dynamical: if \( GB_G \) is contracting, then so is \( GB'_G \) and, moreover, the limit dynamical systems associated with \( GB_G \) and \( GB'_G \) are topologically conjugate, see e.g. [12, Corollary 3.6.7].)

We say that two left-free \( GB_G \) and \( HC_B \) are combinatorially equivalent if \( GB'_G \) and \( HC_H \) are conjugate.

It is easy to check that any surjective semi-conjugacy \( (\phi, \beta): GB_G \rightarrow HC_B \) between left-free bisets descents into a semi-conjugacy \( (\phi', \beta'): GB'_G \rightarrow HC'_B \). We say that a surjective semi-conjugacy \( (\phi, \beta): GB_G \rightarrow HC_B \) between left-free bisets respects combinatorics if \( (\phi', \beta'): GB'_G \rightarrow HC'_B \) is a conjugacy (so that \( GB_G \) and \( HC_B \) are combinatorially equivalent).

2.5. Biset presentations. Let \( B \) be a left-free \( H-G \)-biset. Choose a basis of \( B \), namely a subset \( S \) of \( B \) such that \( B \), qua left \( H \)-set, is isomorphic to \( H \times S \). In other words, \( S \) contains precisely one element from each \( H \)-orbit of \( B \).

The structure of the biset is then determined by the right action in that description. For \( g \in G \) and \( s \in S \), there are unique \( (h, t) \in H \times S \) such that \( sg = ht \). The choice of basis therefore leads to a map \( S \times G \rightarrow H \times S \), or, which is the same, a map \( \Phi: G \rightarrow (H \times S)^S \).

Associativity of the biset operations yields, as is easy to see, that \( \Phi \) is a group homomorphism \( G \rightarrow H \wr S \). This last group, the wreath product of \( H \) with the symmetric group \( S \), is by definition the semidirect product of \( H \) with \( S \), in which \( S \) acts by permutations of the co-ordinates in \( H \). We call \( \Phi \) a wreath map of the biset \( B \).

Wreath products are best thought of in terms of decorated permutations: one writes permutations of \( S \) in the standard arrow diagram notation, but adds a label belonging to \( H \) on each arrow in the permutation. Permutations are multiplied by concatenating arrow diagrams and multiplying the labels.

A presentation of \( B \) is a choice of generating sets \( \Gamma, \Delta \) for \( G, H \) respectively, and for each \( g \in \Gamma \) an expression of the form

\[ g = \langle h_1, \ldots, h_d \rangle(i_1, \ldots), \]

describing \( \Phi(g) \): the \( h_i \) are words in \( \Delta \), and \( (i_1, \ldots) \) is a permutation of \( S \cong \{1, \ldots, d\} \) in disjoint cycle format.
Let $H$ be a closed surface and consider the group $\pi_1(X,*)$ of loops at $1$. Define a wreath map $\Phi: G \to H \wr \pi_1(X,*)$ by
$$\Phi(g) = \langle g_1, \ldots, g_d \rangle \sigma, \quad \Psi(g) = \langle h_1, \ldots, h_d \rangle \tau.$$ 
We therefore have $s_i g = g_i s_i \sigma$ and $t_i g = h_i t_i \tau$ for all $i$. Now
$$h_i t_i \tau = t_i s_i g = w_i^{-1} s_i g = w_i^{-1} g_i s_i \sigma = w_i^{-1} g_i w_i \sigma t_i \tau,$$
so $\Phi(g)^w = \Psi(g)$. \hfill $\square$

It may help to introduce an example here. Consider the map $f(z) = z^d$ from the cylinder $X = \mathbb{C} \setminus \{0\}$ to itself. Choose $s = \uparrow = 1$ as basepoints, and identify $\pi_1(X,*)$ with $\mathbb{Z}$ by choosing as generator the loop $\tau(t) = \exp(2\pi it)$.

The biset $B(f)$ is the set of homotopy classes of loops at $1$, and so may be naturally identified with $\mathbb{Z}$. The biset structure is given by $n \cdot b \cdot m = dn + b + m$ for $n, m \in \mathbb{Z} \cong \pi_1(X,*)$ and $b \in \mathbb{Z} \cong B(f)$. Thus $B(f)$ is left-free of rank $d$, and a basis is a complete set of congruence representatives modulo $d$, for example $\{0, 1, \ldots, d-1\}$. In that basis, the wreath map reads
$$\Phi(\tau) = \langle 1, \ldots, 1, \tau \rangle \langle 1, 2, \ldots, d \rangle,$$
or in diagram notation

\[
\begin{array}{ccc}
\tau & \sim & \tau \\
\downarrow & \sim & \downarrow \\
\uparrow & \sim & \uparrow \\
\end{array}
\]

### 2.6. Conjugacy classes in groups.

We will, in later articles, consider conjugacy classes so as to represent unbased loops in topological spaces. There are various applications; for example, a fundamental construction by Goldman [8] defines a Lie bracket structure on the vector space spanned by conjugacy classes in $\pi_1(X,*)$ for a closed surface $X$. We consider, here, the general case of groups and bisets.

Let $G^G$ denote the set of conjugacy classes of the group $G$, and consider a left-free $H$-biset $B$. Choose a basis $S$ of $B$, whence a wreath map $\Phi: G \to H \wr S$. Consider $g^G \in G^G$, write $\Phi(g) = \langle h_1, \ldots, h_d \rangle \pi$, and let $S_j$ be the orbits of $\pi$ on $S$. For each $j = 1, \ldots, \ell$, let $j_k$ be the product of the $h_i$’s along the orbit $S_j$; namely, if $S_j = \{s_1, \ldots, s_d\}$ with $s_i \equiv s_{i+1}$, indices being computed modulo $d$, then $j_k = h_{s_1} h_{s_2} \cdots h_{s_d}$. The multiset $\{(d_j, k^H_j) \mid i = j, \ldots, \ell\}$ consisting of degrees and conjugacy classes in $H$ is called the lift of $g^G$.

**Lemma 2.10.** The lift of $g^G$ is independent of all the choices made: of $g$ in its conjugacy class, of the basis $S$, and of the cyclic ordering of the orbit $S_j$.

**Proof.** Different choices of bases give conjugate wreath maps, by Lemma 2.9. \hfill $\square$

Let $QG^G$ denote the vector space spanned by the conjugacy classes $G^G$, and consider again a left-free $H$-biset $B$. Then the lift operation gives rise to a linear
map $B^*: \mathbb{Q}G^G \to \mathbb{Q}H^H$, defined on the basis by

\begin{equation}
B^*(g^G) := \sum_{(d_i, h^H_i) \text{lift of } g^G} \frac{k^H_i}{d_i},
\end{equation}

called the **Thurston endomorphism** of $B$.

### 3. Graphs of bisets

We define here the fundamental notion in this article: viewing groups as fundamental groups of spaces, bisets describe continuous maps between spaces. Graphs of groups describe decompositions of spaces, and graphs of bisets describe continuous maps that are compatible with the decompositions.

#### 3.1. Graphs of groups

We begin by precising the notion of graph we will use; it is close to Serre’s [15], but allows more maps between graphs. In particular, we allow graph morphisms that send edges to vertices.

**Definition 3.1 (Graphs).** A graph $\mathfrak{X}$ is a set $\mathfrak{X} = V \sqcup E$, decomposed in two subsets called vertices and edges respectively, equipped with two self-maps $x \mapsto x^-$ and $x \mapsto \overline{x}$, and subject to axioms

\begin{equation}
\forall x \in \mathfrak{X}: \quad \overline{x} = x, \quad x^- \in V, \quad \text{and} \quad x = x^- \iff x = \overline{x} \iff x \in V.
\end{equation}

The object $\overline{x}$ is called the reverse of $x$. Setting $x^+ = (\overline{x})^-$, the vertices $x^-, x^+$ are respectively the origin and terminus of $x$. \hfill $\triangle$

We write $V(\mathfrak{X})$ and $E(\mathfrak{X})$ for the sets of vertices and edges respectively in a graph $\mathfrak{X}$. A **path** is a sequence $(e_1, \ldots, e_n)$ of edges with $e_i^\pm = e_{i+1}^\mp$ for all $i = 1, \ldots, n-1$. A graph is **connected** if there exists a path joining any two objects. A **circuit** is a sequence $(e_1, \ldots, e_n)$ of edges with $e_i^\pm = e_{i+1}^\mp$ for all $i$, indices taken modulo $n$. A **tree** is a graph with no non-trivial circuits; that is in every circuit $(e_1, \ldots, e_n)$ one has $e_i^\pm = e_{i+1}^\mp$ for some $i$.

A **graph morphism** is a map $\theta: \mathfrak{Y} \to \mathfrak{X}$ satisfying $\theta(\overline{y}) = \overline{\theta(y)}$ and $\theta(y^-) = \theta(y)^-$ for all $y \in \mathfrak{Y}$. Note that $\theta$ maps the vertices of $\mathfrak{Y}$ to those of $\mathfrak{X}$. It is **simplicial** if furthermore $\theta$ maps the edges of $\mathfrak{Y}$ to those of $\mathfrak{X}$.

A **graph of groups** is a connected graph $\mathfrak{X} = V \sqcup E$, with a group $G_x$ associated with every $x \in \mathfrak{X}$, and homomorphisms $(\cdot)^-: G_x \to G_{x^-}$ and $(\cdot): G_x \to G_{\overline{x}}$ for each $x \in \mathfrak{X}$, satisfying the same axioms as (3), namely the composition $G_x \to G_{\overline{x}} \to G_{\overline{x}^-} = G_x$ is the identity for every $x \in \mathfrak{X}$, and if $x \in V$ then the homomorphisms $G_x \to G_{x^-}$ and $G_x \to G_{\overline{x}}$ are the identity. For $g \in G_e$ we write $g^+ = \overline{g^-}$. The graph of groups is still denoted $\mathfrak{X}$.

Let $\mathfrak{X} = V \sqcup E$ be a graph of groups. For $v, w \in V$, consider the set

\begin{equation}
\Pi_{v,w} = \{ (x_0, x_1, \ldots, x_n, g_n) \mid x_i \in \mathfrak{X}, x_i^- = v, x_i^+ = x_{i+1}^-, x_n^+ = w, g_i \in G_{x_i^+} = G_{x_{i+1}^-} \},
\end{equation}

of (group-decorated) paths from $v$ to $w$. As a special case, if $v = w$ then we allow $n = 0$ and $g_0 \in G_v = G_w$. Say that two paths are equivalent, written $\sim$, if they differ by a finite sequence of elementary local transformations of the form $(gh) \leftrightarrow (g, x, 1, \overline{1}, h)$ for some $x \in \mathfrak{X}$ and $g, h \in G_{x^-}$, or of the form $(gh^-, x, k) \leftrightarrow (g, x, h^+k)$ for some $x \in \mathfrak{X}$ and $g \in G_{x^-}, h \in G_{x^+}, k \in G_{x^+}$, or of the form $(g, x, h) \leftrightarrow (gh)$ for some $x \in V$ and $g, h \in G_x$. 

The product of two paths \((g_0, x_1, \ldots, g_m)\) and \((h_0, y_1, \ldots, h_n)\) is defined if \(x^+_n = y^+_1\), and equals \((g_0, x_1, \ldots, g_m h_0, y_1, \ldots, h_n)\); this product \(\Pi_{u,v} \times \Pi_{v,w} \to \Pi_{u,w}\) is compatible with the equivalence relation. The fundamental groupoid \(\pi_1(\mathcal{X})\) of \(\mathcal{X}\) is a groupoid with object set \(V\), and with morphisms between \(v\) and \(w\) the set \(\pi_1(\mathcal{X}, v, w) = \Pi_{v,w}/\sim\) of equivalence classes of paths from \(v\) to \(w\).

In particular, for \(v \in V\), the fundamental group \(\pi_1(\mathcal{X}, v)\) is \(\Pi_{v,v}/\sim\). If \(\mathcal{X}\) is connected, then \(\pi_1(\mathcal{X}, v)\) is up to isomorphism independent of the choice of \(v\) in \(V\).

In fact, the elements \(x_i\) in a path may be assumed to all belong to \(E\), and if \(x_{i+1} = \pi\) then the element \(g_{i+1}\) may be assumed not to belong to \((G_{x_{i+1}})^-\). Such paths are called reduced, and \(\Pi_{v,w}/\sim\) may be identified with the set of reduced paths from \(v\) to \(w\).

In a more algebraic language, the fundamental groupoid \(\pi_1(\mathcal{X})\) is the universal groupoid with object set \(V\), and whose morphism set is generated by \(\mathcal{X} \cup \bigcup_{v \in V} G_v\); the source and target of \(x \in \mathcal{X}\) are \(x^-\) and \(x^+\) respectively; the source and target of \(g \in G_v\) are \(v\); the relations are those of the \(G_v\) as well as \(v = 1 \in G_v\) for all \(v \in V\), and \(x^+ = 1 \in G_{x^-}\) and \(g^- x = x g^+\) for all \(x \in \mathcal{X}\) and \(g \in G_x\). The path \((g_0, x_1, \ldots, g_n)\) is identified with \(g_0 x_1 \ldots g_n\). The following property follows easily from the definitions:

**Lemma 3.2** (e.g. Serre [15, §5.2]). Let \(\mathcal{X}\) be a graph of groups. If all morphisms \(G_x \to G_{x^-}\) are injective, then all natural maps \(G_x \to \pi_1(\mathcal{X}, \ast)\) are injective. \(\square\)

3.2. Decompositions and van Kampen’s theorem. Graphs of groups, and their fundamental group, generalize decompositions of spaces and their fundamental group. Here is a useful example of graphs of groups:

**Example 3.3.** If \(X\) be a path connected surface with at least one puncture, consider a graph \(\mathcal{X}\) drawn on \(X\) which contains precisely one puncture in each face. Then \(X\) deformation retracts to \(\mathcal{X}\), and the groups \(\pi_1(X, \ast)\) and \(\pi_1(\mathcal{X}, \ast)\) are isomorphic. Here we consider \(\mathcal{X}\) as a graph of groups in which all groups \(G_x\) are trivial.

**Definition 3.4** (Finite 1-dimensional covers). Consider a path connected space \(X\), covered by a finite collection of path connected subspaces \((X_v)_{v \in V}\). It is a finite 1-dimensional cover of \(X\) if

- for every \(v, w \in V\) and for every path connected component \(X'\) of \(X_u \cap X_v\) there are an open neighborhood \(\tilde{X}' \supset X'\) and an \(X_w \subset X'\) such that \(X_u \to \tilde{X}'\) is a homotopy equivalence;
- if \(X_u \subset X_v \subset X_w\) then \(u = v\) or \(v = w\).

We order \(V\) by writing \(u < v\) if \(X_u \not\subset X_v\). \(\triangle\)

**Remark 3.5.** In a classical formulation of van Kampen’s theorem it is assumed that all \(X_v\) are open subsets of \(X\). Then every curve \(\gamma \subset X\) is a concatenation of finitely many curves \(\gamma_i\) where each \(\gamma_i\) lies entirely in some \(X_{v(i)}\).

For our dynamical purposes it is more convenient to allow the \(X_v\) to be closed subsets of \(X\). Thus we add the condition that \(X_w\) is homotopy equivalent to an open neighborhood of \(X_w\). Under this assumption every curve \(\gamma \subset X\) is, up to homotopy rel endpoints, a concatenation of finitely many curves \(\gamma_i\) which all lie entirely in some \(X_{v(i)}\).

**Definition 3.6** (Graphs of groups from covers). Consider a path connected space \(X\) with a 1-dimensional cover \((X_v)_{v \in V}\). It has an associated graph of groups \(\mathcal{X}\),
defined as follows. The vertex set of \( \mathfrak{X} \) is \( V \). For every pair \( u < v \) there are edges \( e \) and \( \bar{e} \) connecting \( u = e^- = \bar{e}^+ \) and \( v = e^+ = \bar{e}^- \), and we let \( E \) be the set of these edges. Set \( \mathfrak{X} = V \cup E \).

Choose basepoints \( *_v \in X_v \) for all \( v \in V \), and set \( G_v := \pi_1(X_v, *_v) \). For every edge \( e \) with \( e^- < e^+ \) choose a path \( \gamma_e : [0, 1] \to X_{e^+} \) from \( *_{e^-} \) to \( *_{e^+} \), set \( G_e := G_{e^-} \) and define \( G_e \to G_{e^-} := \mathbb{1} \) and \( G_e \to G_{e^+} \) by \( \gamma \mapsto \gamma_{e^-} \# \gamma \# \gamma_{e^+} \). Finally for edges \( e \) with \( e^- > e^+ \) set \( G_e := G_{e^-} \) and \( \gamma_e = \gamma_{e^-} \) and define morphisms \( G_e \to G_{e^-} \) and \( G_e \to G_{e^+} \) as \( G_e \to G_{e^-} \) and \( G_e \to G_{e^+} \) respectively. \( \square \)

The graph of groups \( \mathfrak{X} \) depends on the choice of basepoints \( *_v \) and paths \( \gamma_e \), but mildly: we will show in Lemma 4.9 that the congruence class of \( \mathfrak{X} \) is independent of the choice of \( *_v \) and \( \gamma_e \).

Here is a simple example: on the left, the subspaces \( X_v \) are the simple loops and the two triple intersection points; on the right, the corresponding graph of groups, with trivial or infinite cyclic groups.

\[ \begin{array}{c}
\includegraphics[width=0.4\textwidth]{example_graph.png}
\end{array} \]

**Theorem 3.7** (van Kampen’s Theorem). Let \( X \) be a path connected space with finite 1-dimensional cover \( (X_v) \) and a choice of basepoints \( *_v \in X_v \) as well as paths connecting \( *_u \) and \( *_v \) if \( X_u \cong X_v \). Let \( \mathfrak{X} \) be the associated graph of groups.

Then, for every \( v \), the groups \( \pi_1(X, *_v) \) and \( \pi_1(\mathfrak{X}, v) \) are isomorphic.

**Sketch of proof.** The isomorphism \( \theta : \pi_1(\mathfrak{X}, v) \to \pi_1(X, *_v) \) is defined by

\[
\theta(g_0, e_1, g_1, \ldots, e_n, g_n) = g_0 \# \gamma_{e_1} \# \cdots \# \gamma_{e_n} \# g_n.
\]

Every loop \( \gamma : [0, 1] \to X \) with \( \gamma(0) = \gamma(1) = *_v \) is homotopic to a concatenation of the above type, see Remark 3.5; thus \( \theta \) is surjective. By the classical van Kampen argument (see [11, Chapter IV]), the path \( \theta(g) \) is homotopic to \( \theta(g') \) in \( X \) if and only if \( g \) and \( g' \) are equivalent in \( \pi_1(\mathfrak{X}, v) \); thus \( \theta \) is injective. \( \square \)

In case \( \mathfrak{X} \) is a tree, \( \pi_1(\mathfrak{X}, *) \) is an iterated free product with amalgamation of the \( G_v \)'s, amalgamated over the \( G_e \)'s. It may be constructed explicitly as follows: consider the set of finite sequences over the alphabet \( \bigcup_{v \in V} G_v \); identify two sequences if they differ on subsequences respectively of the form \( \{(1, ())\} \) or \( \{(g, h), (gh)\} \) for \( g, h \) in the same \( G_v \); or \( \{(g), (g)\} \) where the first, respectively second, \( 'g' \) denote the image of \( g \in G_e \) in \( G_{e^-} \), respectively \( G_{e^+} \); and quotient by the equivalence relation generated by these identifications.

Note the following three operations on a graph of groups \( \mathfrak{X} \), which does not change its fundamental group:

1. **split an edge:** choose an edge \( e \in E \). Add a new vertex \( v \) to \( V \), and replace \( e, \bar{e} \) by new edges \( e_0, \bar{e}_0, e_1, \bar{e}_1 \) with \( e_0^+ = e_1^- = v \) and \( e_0^- = e_1^+ = e^- \) and \( e_0^+ = e_1^- = e^+ \). Define the new groups by \( G_v = G_{e_0} = G_{e_1} = G_e \), with the obvious maps between them;
(2) add an edge: choose a vertex \( v \in V \) and a subgroup \( H \subseteq G_v \). Add a new vertex \( w \) to \( V \) and add new edges \( e, \overline{e} \) with \( e^- = v \) and \( e^+ = w \). Define the new groups by \( G_e = G_w = H \), with the obvious maps between them;

(3) barycentric subdivision: construct a new graph \( \mathcal{X}' = V' \sqcup E' \) with \( V' = \mathcal{X}/\{x = \pi\} \) and \( E' = E \times \{+,-\} \); for \( e \in E \) and \( \varepsilon \in \{-,+,\} \), set \((e,\varepsilon)^\circ = e^\varepsilon\) and \((e,\varepsilon)^{-\circ} = [e]\) and \((e,\varepsilon) = (\varepsilon, -\varepsilon)\). Define finally a graph of groups structure \( (G'\}_{x}\) on \( \mathcal{X}' \) by setting \( G'\}_{x} \) to be \( G_x \) for all \( x \in V' \) (recalling that \( G_z \) and \( G_{\overline{z}} \) are isomorphic), and \( G'\}_{(e,\varepsilon)} = G_e \), with the obvious maps between these groups.

Recall that graph morphisms may send edges to vertices, but not vertices to (midpoints of) edges. If such a map between the topological realizations of the graphs is needed, it may be expressed as a map between their barycentric subdivisions.

If \( \mathcal{X} \) be a graph of groups, let \( \mathcal{X}' \) denote the underlying graph of \( \mathcal{X} \), but with trivial groups. Then, on the one hand, \( \pi_1(\mathcal{X}') \) is the usual fundamental groupoid of \( \mathcal{X}' \), and \( \pi_1(\mathcal{X}, v) \) is a free group; on the other hand, there exists a natural morphism of groupoids \( \pi_1(\mathcal{X}) \to \pi_1(\mathcal{X}') \).

If \( \mathcal{X}, \mathcal{Y} \) be graphs of groups, \( \theta : \mathcal{Y} \to \mathcal{X} \) be a graph morphism, and \( \theta_y \) be, for every \( y \in \mathcal{Y} \), a homomorphism \( G_y \to G_\theta(y) \) such that \( \theta_y(g)^\circ = \theta_y(g)^- \) and \( \theta_y(g) = \theta_\theta(y) \) for all \( g \in G_y \), then there is an induced morphism of groupoids \( \pi_1(\mathcal{Y}) \to \pi_1(\mathcal{X}) \). We do not give a name to these kinds of maps, because they are too restrictive; see the discussion in the introduction of [4]. We will give later, in §3.9, more examples of graphs of groups; we now describe in more detail the precise notion of morphisms between graphs of groups that we will use.

3.3. Graphs of bisets. Just as bisets generalize appropriately to our setting homomorphisms between groups, so do “graphs of bisets” also generalize the notion of homomorphisms between graphs of groups. In fact, our notion also extends the definition of morphisms between graphs of groups given in [4], in that a “right-principal graph of bisets” is the same thing as a morphism between graphs of groups in the sense of Bass.

Definition 3.8 (Graph of bisets). Let \( \mathcal{X}, \mathcal{Y} \) be two graphs of groups. A graph of bisets \( \mathcal{Y},{\mathcal{X}} \) between them is the following data:

- a graph \( \mathfrak{B} \), not necessarily connected;
- graph morphisms \( \lambda : \mathfrak{B} \to \mathcal{Y} \) and \( \rho : \mathfrak{B} \to \mathcal{X} \);
- for every \( z \in \mathfrak{B} \), a \( G_{\lambda(z)}-G_{\rho(z)} \)-biset \( B_z \), a congruence \( (\cdot)^- : B_z \to B_z^- \) with respect to the homomorphisms \( G_{\lambda(z)} \to G_{\lambda(z)}^- \) and \( G_{\rho(z)} \to G_{\rho(z)}^- \), and a congruence \( (\cdot)^- : B_z \to B_z^- \) with respect to the homomorphisms \( G_{\lambda(z)} \to G_{\lambda(z)}^- \) and \( G_{\rho(z)} \to G_{\rho(z)}^- \).

These homomorphisms must satisfy the same axioms as (3), namely: the composition \( B_z \to B_z^- \to B_z = B_z \) is the identity for every \( z \in \mathcal{X} \), and if \( z \in V \), then the homomorphisms \( B_z \to B_z^- \) and \( B_z \to B_z^- \) are the identity.

For \( b \in B_z \), we write \( b^+ = \overline{b}^- \).

We call \( \mathfrak{B} \) a \( \mathcal{Y}-\mathcal{X} \)-biset. \( \triangle \)

Example 3.9. A graph morphism \( \theta : \mathcal{Y} \to \mathcal{X} \), given with homomorphisms \( \theta_y : G_y \to G_{\theta(y)} \) for all \( y \in \mathcal{Y} \) such that \( (\cdot)^- \circ \theta_y = \theta_y \circ (\cdot)^- \), naturally gives rise to a \( \mathcal{Y}-\mathcal{X} \)-biset \( \mathfrak{B}_\theta \). Its underlying graph is \( \mathfrak{B} = \mathcal{Y} \); the maps are \( \lambda = 1 \) and \( \rho = \theta \). For every \( z \in \mathfrak{B} \),
the biset $B_z$ is $G_{\theta(z)}$ with its natural right $G_{\theta(z)}$-action, and its left $G_z$-action is $g \cdot b = \theta_z(g)b$.

**Definition 3.10** (Product of graphs of bisets). Let $\mathfrak{B}$ be a $\mathfrak{Y}$-$\mathfrak{X}$-biset, and let $\mathfrak{C}$ be a $\mathfrak{X}$-$\mathfrak{M}$-biset. The *product* of $\mathfrak{B}$ and $\mathfrak{C}$ is the following graph of bisets $\mathfrak{N} = \mathfrak{B} \otimes_\mathfrak{X} \mathfrak{C}$: its underlying graph is the fibre product

$$\mathfrak{D} = \{(b, c) \in \mathfrak{B} \times \mathfrak{C} \mid \rho(b) = \lambda(c)\},$$

with $(b, c)^- = (b^-, c^-)$ and $(b, c) = (b, \tau)$. The map $\lambda: \mathfrak{D} \rightarrow \mathfrak{N}$ is $\lambda(b, c) = \lambda(b)$, and the map $\rho: \mathfrak{D} \rightarrow \mathfrak{M}$ is $\rho(b, c) = \rho(c)$. The biset $B_{(b, c)}$ is $B_b \otimes_{\rho(b)} B_c$.

Properties of the product will be investigated in §3.7. For now we content ourselves with some illustrative examples.

**Lemma 3.11** (See Example 3.9). Let $\theta: \mathfrak{Y} \rightarrow \mathfrak{X}$ and $\kappa: \mathfrak{X} \rightarrow \mathfrak{M}$ be graph morphisms, given with homomorphisms $\theta_y: G_y \rightarrow G_{\theta(y)}$ and $\kappa_x: G_x \rightarrow G_{\kappa(x)}$. Then their bisets satisfy $\mathfrak{B}_y \otimes_\mathfrak{X} \mathfrak{B}_\kappa = \mathfrak{B}_{\theta\kappa}$.

Proof: We have natural identifications $G_y \otimes G_x G_x = G_y$ between the bisets of $\mathfrak{B}_y \otimes_\mathfrak{X} \mathfrak{B}_\kappa$ and those of $\mathfrak{B}_{\theta\kappa}$. □

The contragredient $\mathfrak{B}^\vee$ of the graph of bisets $\mathfrak{B}$ is defined by exchanging $\rho$ and $\lambda$, and taking the contragredients of all bisets $B_z$. If $\theta: \mathfrak{Y} \rightarrow \mathfrak{X}$ is bijective and all $\theta_y$ are isomorphisms, then the contragredient of $\mathfrak{B}_\theta$ is $\mathfrak{B}_{\theta^{-1}}$.

The identity biset for products is the following biset $\mathfrak{X} \otimes_\mathfrak{X} \mathfrak{X}$: its underlying graph is $\mathfrak{X}$, with maps $\lambda = \rho = 1$, and bisets $B_z = G_z$.

**Example 3.12** (See Lemma 2.7; a partial converse to Example 3.9). Let $\mathfrak{Y} \otimes_\mathfrak{X} \mathfrak{X}$ be a graph of bisets such that the graph $\mathfrak{B}$ is connected and $B_z$ is transitive for every $z \in \mathfrak{B}$. Suppose that for every $z \in \mathfrak{B}$ we are given an element $b_z \in B_z$ such that $b_z = b_z$ and $(b_z)^- = b_z^-$ for all $z \in \mathfrak{B}$. Then there exist a graph of groups $\mathfrak{Z}$ and morphisms of graphs of groups $\lambda: \mathfrak{Z} \rightarrow G_\mathfrak{Y}$ and $\rho: \mathfrak{Z} \rightarrow \mathfrak{X}$ as in Example 3.9 such that $\mathfrak{B}$ is isomorphic to $\mathfrak{Z} \otimes_\mathfrak{X} \mathfrak{X}$.

Note, however, that it is not always possible to make coherent choices of basepoints $b_z \in B_z$. For example, it may happen that $(B_e)^- \cap (B_f)^- = \emptyset$ for two edges $e, f$ with $e^- = f^-$. Here is a dynamical situation in which this happens. Let $f$ be a rational map whose Julia set is a Sierpiński carpet, and assume that its Fatou set has two invariant components $U, V$, perforate with disjoint closures, on which $f$ acts as $z \mapsto z^n, z^n$ respectively. Remove these components, and attach on each of them a sphere with a self-map that has an invariant Fatou component mapped to itself by the same degree. To these data correspond a graph of groups $\mathfrak{X}$ with three vertices (the main sphere and the two grafted ones), and a graph of bisets $\mathfrak{X} \otimes_\mathfrak{X} \mathfrak{X}$ describing the self-map. The basepoint on the main sphere cannot be chosen to coincide with the basepoints of the grafted spheres, because the Fatou components of the Sierpiński map have disjoint closures.

3.4. The fundamental biset.

**Definition 3.13** (Fundamental biset of graph of bisets). Let $\mathfrak{B}$ be a $\mathfrak{Y}$-$\mathfrak{X}$-biset; choose $* \in \mathfrak{X}$ and $\dagger \in \mathfrak{Y}$. Write $G = \pi_1(\mathfrak{X}, *)$ and $H = \pi_1(\mathfrak{Y}, \dagger)$. The *fundamental
biset of \( \mathfrak{B} \) is an \( H\)-\( G \)-biset \( B = \pi_1(\mathfrak{B}, \dagger, \ast) \), constructed as follows.

\[
B = \bigcup_{z \in \text{V}(\mathfrak{B})} \pi_1(\mathfrak{B}, \dagger, \lambda(z)) \otimes_{G_{\lambda(z)}} B_z \otimes_{G_{\rho(z)}} \pi_1(\mathfrak{X}, \rho(z), \ast)
\]

\( \forall \ q \in \pi_1(\mathfrak{B}, \dagger, \lambda(z)^{-1}), \ b \in B_z, \ p \in \pi_1(\mathfrak{X}, \rho(z)^{-1}, \ast), \ z \in E(\mathfrak{B}) \)

In other words, elements of \( B \) are sequences \( (h_0, y_1, h_1, \ldots, y_n, b, x_1, \ldots, g_{m-1}, x_n, g_n) \) subject to the equivalence relations used previously to define \( \pi_1(\mathfrak{X}) \), as well as \( (y_n, h, b^{-1}g, x_1) \leftrightarrow (y_n, h, \lambda(z), b^+, \rho(z), g, x_1) \) for all \( z \in \mathfrak{B} \), \( b \in B_z \), \( h \in G_{\lambda(z)^{-1}} \), \( g \in G_{\rho(z)^{-1}} \).

In particular, if \( \mathfrak{B} = \mathfrak{X} \) are graphs with one vertex and no edges and associated groups \( H, G \) respectively, the definition simplifies a little: then \( B \) is the quotient of \( \bigcup_{z \in \text{V}(\mathfrak{B})} B_z \) by the relation identifying \( b^- \) with \( b^+ \), for all edges \( z \in \mathfrak{B} \) and all \( b \in B_z \), with respective images \( b^\pm \) in \( B_z^\pm \).

If \( f : Y \to X \) be a continuous map and the spaces \( X, Y \) admit decompositions giving a splitting of their fundamental group as a graph of groups, as in Theorem 3.7, and if \( f \) is compatible with the decompositions of \( X \) and \( Y \), then \( B(f) \) will decompose into a graph of bisets, as we will see in Theorem 4.8.

Let \( \mathfrak{X} \) be a graph of groups. Choose \( \ast \in \mathfrak{X} \), and consider the graph \( \{ \ast \} \) with a single vertex and no edge. We treat it as a graph of groups, with group \( \pi_1(\mathfrak{X}, \ast) \) attached to \( \ast \), and denote it by \( \mathfrak{X}_0 \). Consider now the following biset \( \chi(\mathfrak{X}, \ast)_{\mathfrak{X}_0} : \) its underlying graph is \( \mathfrak{X} \); the maps are \( \lambda = \mathbb{1} \) and \( \rho(z) = \ast \) for all \( z \in \mathfrak{X} \); and the bisets are \( B_z = \pi_1(\mathfrak{X}, z^-, \ast) \); for vertices, this is just \( \pi_1(\mathfrak{X}, z, \ast) \), while for edges this is a left \( G_z \)-set via the embedding \( G_z \to G_z^- \). The embedding \( B_z \to B_z^- \) is the identity, while the map \( B_z \to B_\mathfrak{X} \) is \( b \to z b \).

**Lemma 3.14.** Write \( G = \pi_1(\mathfrak{X}, \ast) \). Then we have an isomorphism of \( G\)-\( G \)-bisets

\[
\pi_1((\mathfrak{X}, \ast), \ast, \ast) \cong G G G.
\]

The bisets \( (\mathfrak{X}, \ast) \) and \( (\mathfrak{B}, \dagger) \) enable us, using (5), to write \( \pi_1(\mathfrak{B}, \dagger, \ast) \) more simply. We will prove Lemma 3.15 in §3.7:

**Lemma 3.15.** Let \( \mathfrak{g} \mathfrak{B} \mathfrak{X} \) be a graph of bisets, and choose \( \ast \in \mathfrak{X}, \dagger \in \mathfrak{B} \). Then

\[
\pi_1((\mathfrak{X}, \ast), \dagger, \ast) = \pi_1((\mathfrak{B}, \dagger) \otimes \mathfrak{B} \otimes \mathfrak{X}, \ast, \dagger, \ast).
\]

Note, in particular, the expression \( \pi_1((\mathfrak{X}, \ast) \otimes_{\mathfrak{X}} (\mathfrak{X}, \ast)) = \pi_1(\mathfrak{X}, \dagger, \ast) \) of the set of paths from \( \dagger \) to \( \ast \) in \( \mathfrak{X} \) in terms of graphs of bisets.

3.5. **Fibrant and covering bisets.** We now define a class of graphs of bisets for which the fundamental biset admits a convenient description.

**Definition 3.16** (Fibrant and covering graph of bisets). Let \( \mathfrak{B} \) be a \( \mathfrak{B} \)-\( \mathfrak{X} \)-graph of bisets. Then \( \mathfrak{B} \) is a \( \ast \)-fibrant graph of bisets if \( \rho : \mathfrak{B} \to \mathfrak{X} \) is a simplicial graph-map, and for every vertex \( v \in \mathfrak{B} \) and every edge \( f \in \mathfrak{X} \) with \( f^- = \rho(v) \) the map

\[
\bigcup_{e \in \rho^{-1}(f)} G_{\lambda(v)} \otimes_{G_{\lambda(v)}} B_v \to B_v, \quad g \otimes b \mapsto gb^{-}
\]

is a \( G_{\lambda(v)} \)-\( G_f \) biset isomorphism, for the action of \( G_f \) on \( B_v \) given via \((\cdot)^- : G_f \to G_{\rho(v)} \).
If in addition \( B_z \) is a left-free biset for every object \( z \in \mathfrak{X} \), then \( \mathfrak{B} \) is a covering, or left-free, graph of bisets.

If furthermore \( B_z \) is left-principal for every \( z \in \mathfrak{B} \) and \( \rho: \mathfrak{B} \to \mathfrak{Y} \) is a graph isomorphism, then \( \mathfrak{B} \) is left-principal. Right-fibrant, -free, and -principal graphs of bisets are defined similarly. For example, the biset associated with a morphism of groups as in Example 3.9 is right-principal.

We think about (6) as a unique lifting property: we may always rewrite \( pbq \in \pi_1(\mathfrak{B}, \vartriangle, \ast) \) as \( p'b' \in \pi_1(\vartriangle, \ast, \ast) \) in an essentially unique way, as we will see in Corollary 3.20. We will show in Lemma 3.28 that for left-principal graphs of bisets (6) follows automatically.

For every object \( y \in \mathfrak{Y} \) let us denote by \( \hat{G}_y \) the image of \( G_y \) in the fundamental groupoid \( \pi_1(\mathfrak{Y}) \); we write \( g \to \hat{g} \) the associated natural quotient map \( G_y \to \hat{G}_y \).

We define \( \hat{G}_y \) and \( g \to \hat{g} \) similarly for \( x \in \mathfrak{X} \). Finally, for \( z \in \mathfrak{B} \) we set

\[
\hat{B}_z := \hat{G}_{\lambda(z)} \otimes G_{\rho(z)} B_z \otimes \hat{G}_{\rho(z)}
\]

and denote by \( b \mapsto \hat{b} \) the natural quotient map \( B_z \to \hat{B}_z \).

Let \( \hat{\mathfrak{B}} \) be the graph of bisets obtained from \( \mathfrak{B} \) by replacing all \( G_x, G_y, B_z \) by \( \hat{G}_x, \hat{G}_y, \hat{B}_z \) respectively. It is immediate that \( \pi_1(\hat{\mathfrak{B}}, \vartriangle, \ast) \) and \( \pi_1(\mathfrak{B}, \vartriangle, \ast) \) are naturally isomorphic via \( ybx \mapsto \hat{y}b\hat{x} \).

**Theorem 3.17.** Suppose that \( \mathfrak{B} \) is a left-fibrant biset. Then

1. \( \hat{\mathfrak{B}} \) is left-fibrant;
2. \( \hat{B}_z \cong \hat{G}_{\lambda(z)} \otimes G_{\rho(z)} B_z \) via the natural map \( 1 \otimes b \otimes 1 \mapsto 1 \otimes \hat{b} \);
3. \( (-)^{-} : \hat{B}_e \to \hat{B}_{e^{-}} \) are monomorphisms for all \( e \in \mathfrak{B} \); and
4. if \( \mathfrak{B} \) is left-free, respectively left-principal, then so is \( \hat{\mathfrak{B}} \).

**Proof.** For \( z \in \mathfrak{B} \) let us write \( B'_z := \hat{G}_{\lambda(z)} \otimes B_z \), viewed as a right \( G_{\rho(z)} \)-set. For \( x \in \mathfrak{X} \) let us write

\[
C_x := \bigsqcup_{z \in \rho^{-1}(x)} B'_z,
\]

viewed as a right \( G_{x} \)-set. In particular, if \( f \in \mathfrak{X} \) is an edge, then (6) gives a bijection

\[
\bigsqcup_{z \in \rho^{-1}(f)} \left( \hat{G}_{\lambda(e^{-})} / \hat{G}_{\lambda(e)} \right) \times B'_e \to C_{f^{-}}
\]

of right \( G_{f} \)-sets. Finally, for \( x \in \mathfrak{X} \) we denote by \( \text{Aut}(C_x) \) the set of pure automorphisms of \( C_x \):

\[
\text{Aut}(C_x) := \{ \phi = (\phi_z)_{z \in \rho^{-1}(x)} \mid \phi_z \in (B'_z)_{\ast} \}, \quad (gb)^{\phi_{z}} = gb^{\phi_{z}} \quad \forall b \in B'_z, g \in \hat{G}_{\lambda(z)} \}.
\]

The right action of \( G_{x} \) on \( C_{x} \) induces a homomorphism

\[
\theta_{x} : G_{x} \to \text{Aut}(C_{x}).
\]

**Lemma 3.18.** Suppose \( \mathfrak{B} \) is a left-fibrant biset. Then for every edge \( f \in \mathfrak{X} \) there is a natural embedding

\[
(\cdot)^{-} : \left\{ \begin{array}{c}
\text{Aut}(C_f) \hookrightarrow \text{Aut}(C_{f^{-}}) \\
(\phi_{e})_{e \in \rho^{-1}(f)} \mapsto \left[ (g,b) \in \left( \hat{G}_{\lambda(e^{-})} / \hat{G}_{\lambda(e)} \right) \times B'_{e} \to (g,b^{\phi_{e}}) \right]
\end{array} \right.
\]

under the identification of \( C_{f^{-}} \) given by (7). Moreover, \( (\cdot)^{-} \) embeds \( \theta_{f}(G_{f}) \) into \( \theta_{f^{-}}(G_{f^{-}}) \).
Proof. The contents of (7) that we use is that every $G_f$-set occurring in $C_f$ occurs as a summand of $C_f$, with multiplicity $\geq 1$. Therefore $\theta^-$ is injective. Since the map $\theta^+: G_f \to G_f^-$ is equivariant with respect to the actions on $C_f$ and $C_f^-$, we get $\theta^-/(G_f^-) \cong (\theta_f(G_f))^\sim \leq \theta^-/(G_f^-)$.

Let $X'$ be the graph of groups obtained from $X$ by replacing each $G_x$ with $\theta_x(G_x)$, with maps $\theta^-$ given by (8). By Lemma 3.2, every $\theta_x(G_x)$ embeds into the fundamental groupoid $\pi_1(X')$. We also have an epimorphism $\theta: \pi_1(X) \to \pi_1(X')$ induced by the epimorphisms $\theta_x: G_x \to \theta_x(G_x)$. Therefore, $\theta_x: G_x \to \theta_x(G_x)$ descends to a map $\widehat{\theta}_x: \hat{G}_x \to \theta_x(G_x)$. The action of $\hat{G}_\rho(z)$ on $\hat{B}_z$ therefore lifts to $B'_z$, and this completes the proof of Claim (2).

Multiplying (6) on the left by $\hat{G}_\lambda(v)$ gives then Claim (1) as well as Claim (3).

Finally, if $B_z$ is left-free then so is $\hat{G}_\lambda(z) \otimes \hat{G}_\lambda(x) B_z = B'_z \cong \hat{B}_z$, and similarly if $B_z$ is left-principal then so is $\hat{B}_z$; this finishes the proof of Theorem 3.17. □

3.6. Canonical form for fibrant bisets. We now derive a canonical expression for element in a left-fibrant biset. Let us set

$$S := \bigsqcup_{z \in V(\mathcal{B})} \pi_1(\mathcal{Q}, \dag, \lambda(z)) \otimes \hat{G}_\lambda(x) \hat{B}_z \otimes \hat{G}_\rho(z) \pi_1(\mathcal{X}, \rho(z), *)$$

$$\cong \bigsqcup_{z \in V(\mathcal{B})} \pi_1(\mathcal{Q}, \dag, \lambda(z)) \otimes \hat{G}_\lambda(x) B_z \otimes \hat{G}_\rho(z) \pi_1(\mathcal{X}, \rho(z), *).$$

Then $\pi_1(\mathcal{B}, \dag, *) \cong S/\sim$, where $\sim$ is the equivalence relation from (5). For any path $p \in \pi_1(\mathcal{X}, x, *)$ starting at some vertex $x \in \mathcal{X}$, we consider the following subset of $S$:

$$S(p) := \bigsqcup_{z \in \rho^{-1}(x)} \pi_1(\mathcal{Q}, \dag, \lambda(z)) \otimes \hat{G}_\lambda(x) \hat{B}_z \otimes \hat{G}_\rho(z) p$$

viewed as a left $\pi_1(\mathcal{Q}, \dag)$-set.

Proposition 3.19. Suppose $\mathcal{B}$ is a left-fibrant graph of bisets. Then for every $p_1, p_2 \in \pi_1(\mathcal{X}, -, *)$ the relation $\sim$ on $S(p_1) \times S(p_2)$ is, in fact, a $\pi_1(\mathcal{Q}, \dag)$-set isomorphism between $S(p_1)$ and $S(p_2)$.

Proof. If $p \in \pi_1(\mathcal{X}, x, *)$ then, clearly, $S(p) = S(gp)$ for every $g \in G_x$. We now show that for every edge $f \in \mathcal{X}$ with $f^+ = x$ the equivalence $q_f p = q\lambda(z)b^+\rho(z)p$ in (5) induces an isomorphism between $S(fp)$ and $S(p)$. Since $\hat{B}_x^+$ is isomorphic to $\hat{B}_x^-$.
by Theorem 3.17(3) we have the required isomorphism:

\[
S(fp) = \bigsqcup_{z \in p^{-1}(f^-)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(z)) \otimes \hat{B}_z \otimes fp
\]

\[
= \bigsqcup_{z \in p^{-1}(f^-)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(z)) \otimes \hat{G}_{\lambda(e^-)} \hat{B}_e^- \otimes fp
\]

\[
= \bigsqcup_{e \in p^{-1}(f)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(e^-)) \otimes \hat{B}_e^- \otimes fp
\]

\[
\cong \bigsqcup_{e \in p^{-1}(f)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(e^-)) \otimes (\lambda(e) \hat{B}_e^+ \rho(e)) \otimes fp
\]

\[
= \bigsqcup_{e \in p^{-1}(f)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(e^+)) \otimes \hat{B}_e^+ \otimes p = S(p).
\]

**Corollary 3.20** (Canonical form of \(\pi_1(\mathcal{B}, \uparrow, \ast)\)). Let \(\mathcal{B}_X\) be a left-fibrant graph of bisets. Then its fundamental graph of bisets has the following description

\[(9) \quad \pi_1(\mathcal{B}, \uparrow, \ast) = \bigsqcup_{z \in p^{-1}(\ast)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(z)) \otimes \hat{B}_z,
\]

with right action given by lifting of paths in \(\pi_1(\mathcal{X}, \ast)\).

**Proof.** Follows immediately from Proposition 3.19, since the right-hand side of (9) is \(S(1)\). \(\square\)

Since \(\pi_1(\mathcal{B}, \uparrow, \ast) \cong \pi_1(\hat{\mathcal{B}}, \uparrow, \ast)\), by Theorem 3.17 we may rewrite (9) as

\[(10) \quad \pi_1(\hat{\mathcal{B}}, \uparrow, \ast) \cong \bigsqcup_{z \in p^{-1}(\ast)} \pi_1(\hat{\mathcal{G}}, \uparrow, \lambda(z)) \otimes \hat{B}_z.
\]

**Corollary 3.21.** Let \(\mathcal{B}_X\) be a left-free graph of bisets. Then \(\pi_1(\mathcal{B}, \uparrow, \ast)\) is a left-free biset of degree equal to that of \(\bigsqcup_{z \in p^{-1}(\ast)} \hat{B}_z\).

In particular, the number of left orbits of \(\bigsqcup_{z \in p^{-1}(\ast)} \hat{B}_z\) is independent on \(\ast \in \mathcal{B}\).

**Proof.** Follows immediately from (10) and Theorem 3.17 Claim 4. \(\square\)

**Lemma 3.22.** Suppose \(\mathcal{B}\) is a left-fibrant graph of bisets. Then \(\pi_1(\mathcal{B}, \uparrow, \ast)\) is left-free if and only if \(\hat{\mathcal{B}}\) is a left-free graph of bisets.

**Proof.** It follows from (10) that \(\pi_1(\mathcal{B}, \uparrow, \ast)\) is left-free if and only if every \(\hat{B}_z\) is left-free. \(\square\)

The following corollary is an adaptation of Lemma 3.2 to the context of bisets:

**Corollary 3.23.** Let \(\mathcal{B}\) be a left-fibrant graph of bisets. If for \(y \in \mathcal{Y}\) the maps \((\cdot)^{-1}: G_y \rightarrow G_{y^{-}}\) are injective, then the natural maps

\[B_z \rightarrow \pi_1(\mathcal{B}, \lambda(z), \rho(z)) \quad b \mapsto 1 \otimes b \otimes 1
\]

are also injective for all vertices \(z \in \mathcal{B}\).

**Proof.** By Lemma 3.2 we have \(G_y \cong \hat{G}_y\) for all \(y \in \mathcal{Y}\). Therefore, by Theorem 3.17(2) we have \(B_z \cong \hat{B}_z\) for all \(z \in \mathcal{B}\). In particular, all \(B_z \rightarrow \pi_1(\mathcal{B}, \lambda(z), \rho(z))\) are injections by Corollary 3.20. \(\square\)
Consider a graph of bisets $\mathcal{B}_X$. As for graphs of groups, $\mathcal{B}_X$ has a barycentric subdivision $\mathcal{B}'_X$: all graphs in $\mathcal{B}'_X$ are the barycentric subdivisions of those in $\mathcal{B}_X$, the vertex groups and bisets $G_x, G_z, B_z$ with $y \in \mathcal{Y}, x \in \mathcal{X}, z \in \mathcal{B}$ are the groups and bisets of the associated objects in $\mathcal{B}_X$, the edge groups and bisets in $\mathcal{B}'_X$ represent the group morphisms and the biset congruences of $\mathcal{B}_X$.

**Lemma 3.24.** Let $\mathcal{B}'_X$ be the barycentric subdivision of a graph of bisets $\mathcal{B}_X$. Then for $\mathcal{Y}$ and $\mathcal{X}$ we have a natural isomorphism $\pi_1(\mathcal{B}, \mathcal{Y}, \mathcal{X}) \cong \pi_1(\mathcal{B}'_X, \mathcal{Y}, \mathcal{X})$.

**Proof.** Write $\mathcal{B}'_{*, \mathcal{Y}} := (\mathcal{Y}') \vee \mathcal{B}' \vee (\mathcal{X}', \mathcal{X})$ and $\mathcal{B}_{*, \mathcal{Y}} := (\mathcal{Y}, \mathcal{Y}) \vee \mathcal{B} \vee (\mathcal{X}, \mathcal{X})$. By Lemma 3.15, the fundamental bisets of $\mathcal{B}_{*, \mathcal{Y}}$ and $\mathcal{B}_{*, \mathcal{Y}}$ are isomorphic to the fundamental bisets of $\mathcal{B}'$ and $\mathcal{B}$ respectively. It is also easy to see that $\mathcal{B}'_{*, \mathcal{Y}}$ is (isomorphic to) the barycentric subdivision of $\mathcal{B}_{*, \mathcal{Y}}$ because $\pi_1(\mathcal{B}'_{*, \mathcal{Y}}, \mathcal{Y}) \cong \pi_1(\mathcal{B}_{*, \mathcal{Y}}, \mathcal{Y})$ and $\pi_1(\mathcal{X}', \mathcal{Y}) \cong \pi_1(\mathcal{X}, \mathcal{Y})$. This reduces the problem to the case when $\mathcal{Y}$ and $\mathcal{X}$ are graphs with one vertex. In this case we have a simple description of the fundamental biset $B$ of $\mathcal{B}$, and similarly of $\mathcal{B}'$, see §3.4: $B$ is the quotient of $\bigsqcup_{v \in V(\mathcal{B})} B_v$ by the relation identifying $b^-$ with $b^+$, for all edges $z \in \mathcal{B}$ and all $b \in B_z$, with respective images $b^\pm$ in $B_z$. The first claim of the lemma easily follows.

The second part of the lemma is straightforward: the unique lifting property (6) is respected by passing to the barycentric subdivision and (6) holds for the new vertices of $\mathcal{B}'$ which are the former edges of $\mathcal{B}$. □

### 3.7. Properties of products.

**Lemma 3.25.** For every graphs of bisets $\mathcal{B}_X$ and $\mathcal{C}_Y$ there exists a biset morphism

\begin{equation}
\pi_1(\mathcal{B} \otimes \mathcal{Y}, \mathcal{C}, \mathcal{Y}, \mathcal{C}) \to \pi_1(\mathcal{B}, \mathcal{Y}, \mathcal{C}) \otimes \pi_1(\mathcal{C}, \mathcal{Y}, \mathcal{C})
\end{equation}

defined as follows: consider $(v, w) \in V(\mathcal{B} \otimes \mathcal{C})$ and $b \otimes c \in B(v, w)$. Consider paths $q \in \pi_1(\mathcal{B}, \mathcal{Y}, \mathcal{C}), r \in \pi_1(\mathcal{C}, \mathcal{Y}, \mathcal{C})$. Then a typical element of $\pi_1(\mathcal{B} \otimes \mathcal{C}, \mathcal{Y}, \mathcal{C})$ is of the form $q(b \otimes c)r$, and its image under (11) is defined to be $(qbp) \otimes (p^{-1}cr)$ for any choice of path $p \in \pi_1(\mathcal{C}, \mathcal{Y}, \mathcal{C})$.

**Proof.** To show that (11) is well-defined, we must see that it is independent of the choice of $q(b \otimes c)r$ in its $\sim$-equivalence class and of the choice of $p$. Clearly the product over $\mathcal{X}$ is independent of the choice of $p$. Changing $b \otimes c$ into $bg \otimes g^{-1}$ is the same as changing $p$ into $gp$. For an edge $(e, f)$ in $\mathcal{B} \otimes \mathcal{C}$, changing $q(b \otimes c)r$ into $q(e)(b \otimes c)^{-1} r(f)r$ is the same as changing $p$ into $\lambda(e)p$. It is immediate that (11) is a biset morphism. □

In general, the morphism in Lemma 3.25 need not be an isomorphism. For instance, $\mathcal{B} \otimes \mathcal{C}$ is the empty graph of bisets if the images of the graphs $\mathcal{B}$ and $\mathcal{C}$ do not intersect in $\mathcal{X}$. However,

**Lemma 3.26.** Let $\mathcal{B}$ be a left-fibrant $\mathcal{Y}$-$\mathcal{X}$-biset, and let $\mathcal{C}$ be a $\mathcal{X}$-$\mathcal{Y}$-biset. Then (11) induces an isomorphism

\begin{equation}
\pi_1(\mathcal{B}, \mathcal{Y}, \mathcal{C}) \otimes \pi_1(\mathcal{C}, \mathcal{Y}, \mathcal{C}) \cong \pi_1(\mathcal{B} \otimes \mathcal{C}, \mathcal{Y}, \mathcal{C}).
\end{equation}

**Proof.** By definition, every element in $\pi_1(\mathcal{B}, \mathcal{Y}, \mathcal{C}) \otimes \pi_1(\mathcal{C}, \mathcal{Y}, \mathcal{C})$ is of the form $qbpgr$ for paths $q, p, r, g \in \mathcal{B}, \mathcal{Y}, \mathcal{C}$ respectively. By Corollary 3.20 the subexpression
$qbp$ can be rewritten in a unique way as $q'b'$. This defines a biset morphism

$$\pi_1(\mathfrak{B}, \star, \mathfrak{C}) \otimes \pi_1(\mathfrak{X}, \star, \mathfrak{C}) = \pi_1(\mathfrak{B} \otimes \mathfrak{X}, \star, \mathfrak{C})$$

by $qbp \mapsto q'(b \otimes c)r$, which is clearly the inverse of (11).

\textbf{Lemma 3.27.} Let $\mathfrak{B}_X$ and $\mathfrak{X}_W$ be left-fibrant (respectively left-free, left-principal) bisets. Then $\mathfrak{B} \otimes \mathfrak{X}$ is a left-fibrant (respectively left-free, left-principal) graph of bisets.

\textit{Proof.} Observe first that $\rho_1$ is a left-fibrant (respectively left-free, left-principal) graph of bisets.

Let us verify (6). Note first that if $p_q p_w$, then $p_{wq}$ because morphisms send vertices to vertices, so $v \in p^{-1}(\lambda(w))$ is a vertex of $\mathfrak{B}$ because $\rho_p \mathfrak{B} \rightarrow \mathfrak{X}$ is simplicial. Similarly, if $p(v, w)$ is an edge, then so is $w \in \mathfrak{B}$ because $\rho_p \mathfrak{B} \rightarrow \mathfrak{X}$ is simplicial; thus $(v, w)$ is an edge in $\mathfrak{B} \otimes \mathfrak{X}$.

Consider now a vertex $(v, w) \in \mathfrak{B} \otimes \mathfrak{X}$ and an edge $f \in \mathfrak{W}$ with $f^- = \rho(v, w)$. Let us verify (6). Note first that if $(\ell, e) \in p^{-1}(f)$ with $(\ell, e)^- = (v, w)$, then $\ell = v$ if and only if $\lambda(\ell) = \lambda(v)$, because $\rho_p \mathfrak{B} \rightarrow \mathfrak{X}$ is simplicial. We have

$$B_{(v, w)} = B_v \otimes B_w \cong B_v \otimes \bigcup_{e \in p^{-1}(f), e^-=w} G_{\lambda(w)} \otimes G_{\lambda(e)} B_e$$

using (6) for $\mathfrak{B}_w$

$$= \bigcup_{e \in p^{-1}(f), e^-=w} (B_v \otimes G_{\lambda(w)}) \otimes G_{\lambda(e)} B_e \cong \bigcup_{e \in p^{-1}(f), e^-=w} B_v \otimes G_{\lambda(e)} B_e$$

because $\rho_p = \lambda(w)$

$$= \bigcup_{\ell \in p^{-1}(\lambda(e)), \ell^- = v} G_{\lambda(\ell)} \otimes G_{\lambda(e)} B_{(\ell, e)}$$

This shows that $\mathfrak{B} \otimes \mathfrak{X}$ is a left-fibrant graph of bisets. If, moreover, $\mathfrak{B}_X$ and $\mathfrak{X}_W$ are left-free (respectively left-principal), then all $B_{(v, w)}$ are left-free (respectively left-principal); thus $\mathfrak{B} \otimes \mathfrak{X}$ is a left-free (respectively left-principal) graph of bisets.

We are now ready to prove Lemma 3.15:

\textit{Proof of Lemma 3.15.} Both $(\mathfrak{B}, \uparrow)$ and $(\mathfrak{X}, \ast)$ are left-principal graphs of bisets. The proof now follows from Lemmas 3.26 and 3.27.
3.8. Principal and biprincipal graphs of bisets. Let us now simplify the part of Definition 3.16 concerning principal graphs of bisets.

**Lemma 3.28.** A graph of bisets \( \mathcal{B} \) is left-principal if and only if \( \rho : \mathcal{B} \to \mathcal{X} \) is a graph isomorphism and \( B_z \) are left-principal for all \( z \in \mathcal{B} \).

**Proof.** We verify that the conditions stated in Lemma 3.28 imply (6).

Since \( \rho : \mathcal{B} \to \mathcal{X} \) is a graph isomorphism, (6) takes the following form: for every \( z \in \mathcal{B} \) the map

\[
G_{\lambda(e^-)} \otimes_{G_{\lambda(e)}} B_e \to B_{e^-}, \quad g \otimes b \mapsto gb^-
\]

is a \( G_{\lambda(e^-)} \otimes_{G_{\lambda(e)}} \) biset isomorphism (for the action of \( G_{\rho(e)} \) on \( B_{e^-} \) given via \( (-)^{-} : G_{\rho(e)} \to G_{\rho(e)^{-}} \)). Let us verify (12).

Consider an edge \( e \in \mathcal{B} \). We claim that \( (G_{\lambda(e)})^{-1} (B_e) \) is isomorphic to \( (G_{\lambda(e)} \otimes_{G_{\lambda(e)}} B_e) \). Indeed, \( (G_{\lambda(e)})^{-1} (B_e) \) is left-free as a subiset of a left-free biset \( B_{e^-} \). Furthermore, \( (G_{\lambda(e)})^{-1} (B_e) \) has a single orbit under the action of \( (G_{\lambda(e)})^{-1} \) because this property holds for \( B_e \). Therefore, \( (G_{\lambda(e)})^{-1} (B_e) = (G_{\lambda(e^-)} \otimes_{G_{\lambda(e)}} B_e) \). Since \( B_{e^-} \) is left-principal we obtain a natural isomorphism of \( G_{\lambda(e^-)} \otimes_{G_{\lambda(e^-)}} B_e \)

\[
B_{e^-} = G_{\lambda(e^-)} (B_e)^{-1} \cong G_{\lambda(e^-)} \otimes_{G_{\lambda(e^-)}} B_e
\]

which is (12).

An example of left-principal graph of bisets is \( \mathcal{X}(\mathcal{X}, \cdot) \), defined in §3.4.

A biset \( _HB_G \) is biprincipal if it is left- and right-principal. Clearly, if \( _HB_G \) is biprincipal, then the groups \( G \) and \( H \) are isomorphic. Similarly, a graph of bisets \( \mathcal{G} \mathcal{Y} \mathcal{X} \) is biprincipal if it is left- and right-principal. By Lemma 3.28 a graph of bisets \( \mathcal{G} \mathcal{Y} \mathcal{X} \) is biprincipal if and only if

1. \( \lambda : \mathcal{I} \to \mathcal{Y} \) and \( \rho : \mathcal{I} \to \mathcal{X} \) are graph isomorphisms; and
2. \( B_z \) are biprincipal for all objects \( z \in \mathcal{I} \).

**Definition 3.29.** (Congruence of graphs of groups and bisets) Two graphs of groups \( \mathcal{G} \mathcal{X} \) are called congruent if there is a biprincipal graph of bisets \( \mathcal{G} \mathcal{Y} \mathcal{X} \).

Isomorphism of graphs of bisets is meant in the strongest possible sense: isomorphism of the underlying graphs, and isomorphisms of the respective bisets.

Two graphs of bisets \( \mathcal{G} \mathcal{X} \) and \( \mathcal{G} \mathcal{Y} \mathcal{X} \) are congruent if there are biprincipal graph of bisets \( \mathcal{G} \mathcal{Y} \mathcal{X} \) and \( \mathcal{X} \mathcal{L} \mathcal{Y} \) such that \( \mathcal{G} \mathcal{X} \) is isomorphic to \( \mathcal{I} \otimes \mathcal{G} \mathcal{X} \mathcal{L} \mathcal{Y} \).

Finally, two graph of bisets \( \mathcal{G} \mathcal{X} \) and \( \mathcal{X} \mathcal{L} \) are conjugate if \( \mathcal{X} \mathcal{L} \mathcal{Y} \) is isomorphic to \( \mathcal{I} \otimes \mathcal{X} \mathcal{L} \mathcal{Y} \) for a biprincipal graph of bisets \( \mathcal{X} \mathcal{L} \mathcal{Y} \).

Let \( _HB_G \) be a biprincipal biset and let \( H' \) be a subgroup of \( H \). For every \( b \in B \) we consider the subgroup \((H')^b \) of \( G \) defined by

\[
H' = b(H')^b.
\]

It is easy to see that the conjugacy class of \((H')^b \) in \( G \) is independent of \( b \).

**Lemma 3.30.** Let \( \mathcal{G} \), \( \mathcal{X} \) be graphs of groups and let \( \mathcal{U} : \mathcal{G} \to \mathcal{X} \) be an isomorphism of the underlying graphs. Suppose also that \( G_y \to B_y \) is biprincipal biset \( B_y \) is given for every object \( y \in \mathcal{G} \). Set \( \mathcal{I} \) to be \( \mathcal{G} \) as a graph. Then the data \( \{ B_z \}_{z \in \mathcal{I}} \) extends (via appropriate embeddings of edge bisets into vertex bisets) to a biprincipal graph
of bisets \( \mathcal{Y} \mathcal{X} \) if and only if for every edge \( e \in \mathcal{I} \) the following holds. For any, or equivalently for every, \( b \in B_e \) the groups \( (G_e^-)^b \) and \( G_{h(e)}^- \) are conjugate as subgroups of \( G_{h(e)}^- \).

**Proof.** Follows immediately from the definition: if \( (G_e^-)^b \) and \( G_{h(e)}^- \) are conjugate, say \( ((G_e^-)^b)^g = G_{h(e)}^- \), then we may define \((-)^- : B_e \to B_e^- \) by mapping \( B_e \) into \( G_e^- (bg)G_{h(e)}^- \). The converse is obvious. \( \square \)

### 3.9. Examples

We will give more examples and applications in §5. Nevertheless, we give here an example that illustrates the main features of Definition 3.13. First let us introduce some conventions simplifying descriptions of the examples.

Let the graph \( \mathcal{X} = V \sqcup E \) be a graph as in Definition 3.1. By an undirected graph we mean \( \mathcal{X}_0 := \mathcal{X}/(\mathcal{Z} = \mathcal{Z}) \) with vertex set \( V_0 = V \) and geometric edge set \( E_0 := E/(\mathcal{E} = \mathcal{E}) \). The undirected graph \( \mathcal{X}_0 \) is endowed with an adjacency relation. A map \( \theta : \mathcal{X}_0 \to \mathcal{Y}_0 \) is a weak morphism if \( \theta \) sends adjacent objects into adjacent or equal objects. If \( \mathcal{X}' \) and \( \mathcal{Y}' \) are the barycentric subdivisions of \( \mathcal{X} \) and \( \mathcal{Y} \) as in §3.2, then \( \theta : \mathcal{X}_0 \to \mathcal{Y}_0 \) uniquely determines a graph morphism \( \theta : \mathcal{X}' \to \mathcal{Y}' \).

**Convention.** We will often describe a graph of bisets as \( \mathcal{Y}_0 \mathcal{B}_0 \mathcal{X}_0 \) with

- undirected graphs \( \mathcal{Y}_0, \mathcal{B}_0, \mathcal{X}_0 \);
- weak graph morphisms \( \lambda : \mathcal{B}_0 \to \mathcal{Y}_0 \) and \( \rho : \mathcal{B}_0 \to \mathcal{X}_0 \);
- groups \( G_y, G_x \) and \( G_{\lambda(z)} - G_{\rho(z)} \) bisets \( B_z \) for all \( y \in \mathcal{Y}_0, x \in \mathcal{X}_0 \) and \( z \in \mathcal{B}_0 \) respectively;
- congruences from \( B_e \) to \( B_e \) and \( B_w \) for all edges \( e \in \mathcal{B}_0 \) adjacent to vertices \( v, w \in \mathcal{B}_0 \) and, similarly, group morphisms from \( B_e \) to \( B_e \) and \( B_w \) for all edges \( e \in \mathcal{B}_0 \sqcup \mathcal{X}_0 \) adjacent to vertices \( v, w \).

By passing to a barycentric subdivision of \( \mathcal{Y}_0 \mathcal{B}_0 \mathcal{X}_0 \) we obtain a graph of bisets \( \mathcal{Y}' \mathcal{B}' \mathcal{X}' \) satisfying Definition 3.8. Following Lemma 3.24 we set \( \pi_1(\mathcal{Y}_0 \mathcal{B}_0 \mathcal{X}_0, \uparrow, *) := \pi_1(\mathcal{Y}' \mathcal{B}' \mathcal{X}', \uparrow, *) \).

**Example 3.31.** Consider the biset corresponding to the “Basilica map” \( f(z) = z^2 - 1 \), from \( \mathbb{C}, \{ -1, 0, 1 \} \) to \( \mathbb{C}, \{ 0, -1 \} \).

On the one hand, choose a basepoint \( x_0 \equiv -0.6 \) in \( \mathbb{C}\{0, -1\} \), and choose a graph \( \mathcal{X} \) with one vertex \( x_0 \) to which \( \mathbb{C}\{0, -1\} \) deformation retracts; let \( \mathcal{Y} \) be the full preimage of \( \mathcal{X} \) under \( f \\

\[ y_0' \] \[ y_1' \]
\[ y_0'' \] \[ y_1'' \]
\[ f \]
\[ x_0 \] \[ x_0 \]

There, the graphs of groups \( \mathcal{X} \) and \( \mathcal{Y} \) have trivial groups, and the biset \( \mathcal{B} \) corresponding to \( f \) is \( \mathcal{Y} \) as a graph, with \( \lambda = 1 \) and \( \rho = f \).

This biset may be encoded differently, by considering rather an orbispace structure on \( \mathbb{C} \) with non-trivial groups at 0, \( \pm 1 \); the complex plane deformation retracts to the intervals \([-1, 0]\) and \([-1, 1]\) respectively:

\[ y'_0 \] \[ y_1 \] \[ y'_0 \]
\[ z \] \[ z \] \[ z \]
\[ f \]
\[ x_1 \] \[ x_0 \]
\[ z \] \[ z \]
The graphs $\mathfrak{X}$ and $\mathfrak{Y}$ have the group $Z$ attached to each vertex, and the edge groups are all trivial. The biset $\mathfrak{B}$ has $\mathfrak{Y}$ as underlying graph; the map $\lambda$ is the retraction to the segment $[y_0, y_1] = [x_1, x_0]$ and $\rho$ folds $\mathfrak{Y}$ around its middle point. The bisets $B_{y_0}$ and $B_{y_1}$ are $\mathbb{Z}\mathbb{Z}$, while $B_{y_i} = 2\mathbb{Z}\mathbb{Z}$ is free of rank 2 on the left. The edge bisets embed in $B_{y_i}$, as two elements of opposite parity.

Note that the underlying graph of $\mathfrak{B}$ is the “Hubbard tree” of $f$; that will be explained in more details in §5.2.

**Proposition 3.32.** Let $\phi: H \to G$ be a group homomorphism, and let $\mathfrak{X}$ be a graph of groups with fundamental group $G$. Then there exists a graph of groups decomposition of $H$ and a graph of bisets with limit $f$. Furthermore, there exists a unique minimal such decomposition for $H$ in the sense that other decompositions are refinements of it.

**Proof.** Let $G$ act on the tree $\hat{\mathfrak{X}}$, the universal cover of $\mathfrak{X}$ (also known as the Bass-Serre tree of $G$, see [15]), with stabilizers the groups $G_x$ of $\mathfrak{X}$. Then $H$ acts on $\hat{\mathfrak{X}}$ via $\phi$. We define the graph of groups $\mathfrak{Y}$ as $\hat{\mathfrak{X}}/H$, namely as the quotient of $\hat{\mathfrak{X}}$ by the action of $H$, remembering the stabilizers in $H$ of vertices and edges in the quotient.

We next define a graph of bisets $\mathfrak{B}$ as follows. Its underlying graph is $\hat{\mathfrak{X}}/H$. The graph morphism $\lambda: \mathfrak{B} \to \mathfrak{Y}$ is the identity, and the graph morphism $\rho: \mathfrak{B} \to \mathfrak{X}$ is the natural quotient map $xH \to xG$. The biset at the vertex or edge $xH$ is $G_{\rho(xH)}$, on which $G_{\rho(xH)}$ acts naturally by left multiplication, while $H_{xH}$ acts on the right via $\phi$. □

**Remark 3.33.** In a manner similar to Lemma 2.7, every graph of bisets $\mathfrak{Y}\mathfrak{B}_\mathfrak{X}$ may be factored as $\mathfrak{B} = \mathfrak{A}\mathfrak{Y} \oplus_{\mathfrak{A}} \mathfrak{C}$ for some graph of groups $\mathfrak{A}$ and some right-principal bisets $\mathfrak{A}\mathfrak{Y}$ and $\mathfrak{A}\mathfrak{C}$. Furthermore, there is an essentially unique minimal such $\mathfrak{A}$.

**Lemma 3.34.** Let $\mathfrak{Y}\mathfrak{B}_\mathfrak{X}$ and $\mathfrak{Y}\mathfrak{C}_\mathfrak{X}$ be graphs of bisets over the same graphs, and let $\theta: \mathfrak{B} \to \mathfrak{C}$ be a “semiconjugacy of graphs of bisets”, namely a graph morphism $\theta: \mathfrak{B} \to \mathfrak{C}$ and compatible surjective biset morphisms $\theta_z: B_z \to C_{\theta(z)}$. Then $\theta$ may be factored as $\theta = \kappa \circ i$ through semiconjugacies $i: \mathfrak{B} \to \mathfrak{C}'$ and $\kappa: \mathfrak{C}' \to \mathfrak{C}$, in such a manner that $i$ induces an isomorphism between $\pi_1(\mathfrak{B})$ and $\pi_1(\mathfrak{C}')$, and the underlying graph of $\mathfrak{C}'$ is the same as that of $\mathfrak{C}$.

4. **Correspondences**

Bisets are well adapted to encode more general objects than continuous maps, topological correspondences.

**Definition 4.1 (Correspondences).** Let $X, Y$ be topological spaces. A topological correspondence from $Y$ to $X$ is a triple $(Z, f, i)$ consisting of a topological space $Z$ and continuous maps $f: Z \to X$ and $i: Z \to Y$. It is often abbreviated $(f, i)$, and written $Y \leftarrow Z \to X$.

In the special case that $f$ is a covering, $(Z, f, i)$ is called a covering correspondence. If $X = Y$, then $(Z, f, i)$ is called a self-correspondence. If $X = Y$ and $f$ is a covering, then $(Z, f, i)$ is called a covering pair, and is also written $f, i: Z \to X$. △

Ultimately, we will be interested in dynamical systems that are partially defined topological coverings $f: X \to X$, for path connected topological spaces $X$; the dashed arrow means that the map is defined on an open subset $Y \subseteq X$; these can naturally be viewed as correspondences $(Y, f, i)$ with $i$ the inclusion.
4.1. The biset of a correspondence. Let \( Y \leftarrow Z \to X \) be a correspondence, and assume that \( X \) and \( Y \) are path connected. Assume first that \( Z \) is path connected, and fix a basepoint \( \hat{z} \in Z \). Then we have (1) a \( \pi_1(Z, \hat{z})-\pi_1(X, \ast) \)-biset \( B(f) \) and a \( \pi_1(Z, \hat{z})-\pi_1(Y, \hat{z}) \)-biset \( B(i) \); we define the biset \( B(f, i) \) to be \( B(i) \overset{\gamma}{\ast} \pi_1(Z, \hat{z}) \) \( B(f) \); this amounts to first “lifting” from \( Y \) to \( Z \), and then projecting \( Z \) back to \( X \). If \( Z \) is not path connected, then we define \( B(f, i) \) to be the disjoint union of the bisets \( B(i) \overset{\gamma}{\ast} \pi_1(Z, \hat{z}) \) \( B(f) \) with one \( p \) per path connected component of \( Z \).

The following alternate construction of \( B(f, i) \) avoids an unnecessary reference to the basepoint and fundamental group of \( Z \):

**Lemma 4.2.** The biset \( B(f, i) \) may be constructed directly as follows, from the correspondence:

\[
(13) \quad B(f, i) = \left\{ \left( \begin{array}{c}
\delta: [0, 1] \to Y \\
p \in Z \\
\gamma: [0, 1] \to X
\end{array} \right) \mid \begin{array}{c}
\delta(0) = \uparrow, \\
\delta(1) = i(p) \\
\gamma(0) = f(p), \\
\gamma(1) = \ast
\end{array} \right\} / \sim
\]

in which the equivalence relation \( \sim \) allows \( p \) to move in \( Z \) as long as \( \delta, \gamma \) move smoothly with it\(^1\).

The left action is by pre-composition of \( \delta \) by loops in \( Y \) at \( \uparrow \), and the right action is by post-composition of \( \gamma \) by loops in \( X \) at \( \ast \):

\[
[\alpha] \cdot ([\delta], p, [\gamma]) \cdot [\varepsilon] = ([\alpha \# \delta], p, [\gamma \# \varepsilon]).
\]

**Proof.** By the definition of products of bisets, \( B(f, i) = \bigsqcup \{(\delta, \gamma)\} \) \( / \sim \), with \( \delta: [0, 1] \to Y \) ending at \( \uparrow \) and \( \gamma: [0, 1] \to X \) ending at \( \ast \) and \( \delta(0) = i(z) \) and \( \gamma(0) = f(z) \), and with one \( z \) per path connected component of \( Z \). By the homotopy relation, we may also allow \( z \) to move freely, arriving at the formulation of the lemma in which no condition on \( z \) is imposed. \( \square \)

Recall that a continuous map \( f: Y \to X \) is a fibration, or a fibrant map if it has the homotopy lifting property with respect to arbitrary spaces: for every space \( Z \), every homotopy \( g: Z \times [0, 1] \to X \) and every \( h: Z \times \{0\} \to Y \) with \( f \circ h = g \) on \( Z \times \{0\} \) there exists an extension of \( h \) to \( Z \times [0, 1] \) with \( f \circ h = g \). Clearly, the composition of fibrations is a fibration. If \( X \) is path connected, then in particular all fibres \( f^{-1}(x) \) are homotopy equivalent.

**Remark 4.3.** In fact, we may consider at no cost a larger class of maps: let us say that a map \( f \) is a \( \pi_1 \)-fibration if \( f \) has the lifting property with respect to \footnote{Note the similarity with Definition 3.13}
Lemma 4.4. Assume that \( f: Z \to X \) is a fibration. Then the biset of \( f \) has the following description:

\[
B(f) = \{ \beta: [0, 1] \to Z \mid \beta(0) = \ast, f(\beta(1)) = \ast \}/\sim,
\]

with \( \beta \sim \beta' \) if and only if there is a path \( \epsilon: [0, 1] \to f^{-1}(\ast) \) connecting \( \beta(1) \) to \( \beta'(1) \) such that \( \beta \# \epsilon \) is homotopic to \( \beta' \).

Assume that \( (f, i) \) is a correspondence with \( f \) fibrant. Then in (13) we may assume that \( \gamma \) is constant, and write

\[
B(f, i) = \{ \delta: [0, 1] \to Y, p \in Z \mid \delta(0) = \ast, f(p) = \ast \}/\sim,
\]

with \( (\delta, p) \sim (\delta', p') \) if and only if there exists a path \( \epsilon: [0, 1] \to f^{-1}(\ast) \subseteq Z \) connecting \( p \) to \( p' \), such that \( \delta \# \epsilon \) is homotopic to \( \delta' \).

Proof. Recall from (1) that \( B(f) \) is defined as \( \{ \gamma: [0, 1] \to X \mid \gamma(0) = f(\ast), \gamma(1) = \ast \}/\sim \). Since \( f \) is fibrant, every \( \gamma \in B(f) \) has a lift, say \( \beta \), with \( \beta(0) = \ast \) and \( f \circ \beta = \gamma \). We need to show that if \( \gamma_0 \) is homotopic to \( \gamma_1 \), say via \( \gamma_t \), then their lifts \( \beta_0 \) and \( \beta_1 \) have the property that \( \beta_0 \) is homotopic to \( \beta_1 \# \epsilon \) for some \( \epsilon \) in \( f^{-1}(\ast) \).

Consider \( \gamma_t \) as a map \( \gamma: [0, 1] \times [0, 1] \to X \), where \( t \) is the first variable and the second variable parameterizes \( \gamma_t \). By construction, \( \gamma(t, 0) = f(\ast) \) and \( \gamma(t, 1) = \ast \).

Since \( f \) is fibrant, there is a lift \( \epsilon: [0, 1] \times [0, 1] \to Z \) of \( \gamma \) such that \( \epsilon(t, 0) = \ast \), \( \epsilon(t, 1) = \ast \), \( \epsilon(0, \tau) = \beta_0(\tau) \), and \( \epsilon(1, \tau) = \beta_1(\tau) \). Define \( \epsilon(\tau) := \epsilon(1, \tau) \). Then \( \beta_0 \# \beta_1 \epsilon \) is homotopic to \( \beta_1 \# \epsilon \).

Let us now prove the second statement. Using the first part, the biset \( B(f, i) \) in the sense of (13) is identified with

\[
\{(\delta: [0, 1] \to Y, \beta: [0, 1] \to Z) \mid \delta(0) = \ast, \delta(1) = i(\beta(0)), f(\beta(1)) = \ast \}/\sim,
\]

where \( (\delta_0, \beta_0) \sim (\delta_1, \beta_1) \) if there is a path \( \epsilon: [0, 1] \to f^{-1}(\ast) \) such that \( (\delta_0, \beta_0) \) is homotopic to \( (\delta_1, \beta_1 \epsilon) \). Projecting \( \beta_1 \epsilon \) to \( Y \) yields the required description. □

Example 4.5. Here are some extreme examples of correspondences worth keeping in mind. If \( Z = \{ \ast \} \) is a point, then \( B(f, i) = \pi_1(Y, \ast) \times \pi_1(X, \ast) \). On the other hand, if \( Z = X = Y \) and \( f \) and \( i \) are the identity, then \( B(f, i) \cong \pi_1(Y, \ast) \cong \pi_1(X, \ast) \).

Finally, one may consider \( Z = Y \times X \) with \( i, f \) the first and second projections respectively; then \( B(f, i) \) consists of a single element.

If \( f \) is a covering, then the biset \( B(f, i) \) is left-free. Indeed, consider (15) as a description of \( B(f, i) \). Suppose \( f^{-1}(\ast) = \{ p_i \}_{i \in I} \). For every \( i \) choose \( \delta_i: [0, 1] \to Y \) with \( \delta_i(0) = \ast \) and \( \delta_i(1) = i(p_i) \). Then \( (\delta_i, p_i)_{i \in I} \) forms a basis of \( B(f, i) \).

4.2. Van Kampen’s theorem for correspondences. We describe in this section the decomposition of the biset of a correspondence into a graph of bisets, using a decomposition of the underlying spaces. We start by the following straightforward lemma.

Lemma 4.6. Let

- \( f: Y \to X \) be a continuous map;
- \( X' \) and \( Y' \) be path connected subsets of \( X \) and \( Y \) with \( f(Y') \subseteq X' \);
- \( \ast, \ast', \ast \), and \( \ast' \) be base points in \( Y, Y', X, \) and \( X' \); and
- \( \gamma_\ast \subset X \) and \( \gamma_{\ast'} \subset Y \) be curves connecting \( \ast \) to \( \ast' \) and \( \ast \) to \( \ast' \) respectively.
We allow $X' = X$; in this case we suppose that $\gamma_*$ is the constant path and $\ast' = \ast$.

Then the maps $\delta \to \gamma_{\delta}^{-1} \gamma_{\ast}', \delta \to \gamma_{\ast}^{-1} \gamma_{\delta}'$, and $b \to f(\gamma_1)b\gamma_1^{-1}$ define a biset congruence

$$\pi_1(Y', \gamma)' \to \pi_1(Y, \gamma) \pi_1(X, \ast').$$

Let $f : Z \to X$ be a continuous map between path connected spaces. Suppose that $(U_{\delta})$ and $(W_{\gamma})$ are finite 1-dimensional covers $X$ and $Z$ respectively, as in Definition 3.4. Choose base points $\ast_\delta \in U_{\delta}$ and $\ast_\gamma \in W_{\gamma}$ as well as connections between $*_{\alpha}$ and $*_{\gamma}$ if $U_{\alpha'} \not\subseteq U_{\alpha}$; and similarly for $(W_{\gamma})$. Consider the associated graphs of groups $X$ and $Z$ according to Definition 3.6. We have $\pi_1(X, \ast_\alpha) = \pi_1(X, \alpha)$ and $\pi_1(Z, \ast_\gamma) = \pi_1(Z, \gamma)$, by Theorem 3.7. Suppose furthermore that $(U_{\delta})$ and $(W_{\gamma})$ are compatible with $f$ in the following sense: for every $\gamma$ there is an $\alpha =: \rho(\gamma)$ with $f(W_{\gamma}) \subset U_{\alpha}$.

To the above data there is an associated graph of bisets $\mathfrak{B}(f)_X$ defined as follows. As a graph, $\mathfrak{B}$ is $F$, with $\lambda : \mathfrak{B} \to \mathfrak{X}$ the identity and $\rho : \mathfrak{B} \to \mathfrak{X}$ given above. For every vertex $\gamma \in \mathfrak{B}$, the biset $B_\gamma$ is $B(f : W_{\gamma} \to U_{\rho(\gamma)}, \lambda_{\rho(\gamma)}, *_{\rho(\gamma)})$. For every edge $e \in \mathfrak{B}$ representing the embedding $W_{\gamma'} \subseteq W_{\gamma}$ the biset $B_e$ is $B_{\gamma'}$. If $e$ is oriented so that $e^- = \gamma'$ and $e^+ = \gamma$, the congruences $\langle \rangle^\pm$ are $\langle \rangle^- = \lambda : B_e \to B_{\gamma'}$, and $\langle \rangle^+ : B_e \to B_{\gamma}$. We define the graph of bisets of the correspondence $(f, i)$ as

$$\mathfrak{B}(f, i)_X := \bigsqcup_k \mathfrak{B}(i|Z_k)_{S_k} \otimes \mathfrak{B}(f|Z_k)_X.$$  

The following equivalent description of $\mathfrak{B}(f, i)_X$ follows immediately from the definition; see Lemma 4.2

**Lemma 4.7.** Let $(Z, f, i)$ be a topological correspondence from $Y$ to $X$, for path connected spaces $X, Y$. Let $(U_{\delta})$, $(V_{\beta})$, and $(W_{\gamma})$ be finite 1-dimensional covers of $X, Y$, and $Z$ compatible with $f$ and $i$ as above. Then the graph of bisets $\mathfrak{B}(f, i)_X$ of $(f, i)$ with respect to the above data is as follows:

- the graphs of groups $\mathfrak{G} = \mathfrak{B}$ are constructed as in Definition 3.6 using the covers $(U_{\delta})$ and $(V_{\beta})$ of $X, Y$ respectively. Choices of paths $\ell_\gamma$, $m_\alpha$ were made for edges $e$ in $\mathfrak{G}$, $\mathfrak{F}$ respectively;
- the underlying graph of $\mathfrak{B}$ is similarly constructed using the cover $(W_{\gamma})$ of $Z$. Note that neither $Z$ nor the underlying graph of $\mathfrak{B}$ need be connected;
- for every vertex $z \in \mathfrak{B}$ the biset $B_{\gamma_{\lambda(\gamma)}} B_{\gamma_{\rho(\gamma)}}$ is $B(f|W_z, i|W_z)$;
- for every edge $e \in \mathfrak{B}$ representing the embedding $W_{\gamma'} \subseteq W_{\gamma}$ the biset $B_e$ is $B_{\gamma'}$, and if $e$ is oriented so that $e^- = z'$ then the congruences $\langle \rangle^\pm$ are the maps $\langle \rangle^- = \lambda : B_e \to B_{\gamma'}$, and $\langle \rangle^+ : B_e \to B_{\gamma}$ given by $\langle \gamma^{-1}, \delta \rangle \mapsto (m_\gamma^{-1}, \#\gamma^{-1}, \delta \#\ell_{\rho(\gamma)})$ in the description of $B_e$ as $B(f|W_{\gamma})^\gamma \otimes B(f|W_{\gamma})$, see Lemma 4.6.  

$\square$
Theorem 4.8 (Van Kampen’s theorem for correspondences). Let \((f, i)\) be a topological correspondence from a path connected space \(Y\) to a path connected space \(X\), and let \(g \mathcal{B}_X\) be the graph of bisets subject to compatible finite 1-dimensional covers of spaces in question.

Then for every \(v \in \mathcal{Y}\) and \(u \in \mathcal{X}\) we have

\[
B(f, i, \uparrow_v, *_u) \cong \pi_1(\mathcal{B}, v, u),
\]

where \(\uparrow_v\) and \(*_u\) are basepoints.

Proof. Recall from \(\S 3.2\) that for all vertices \(y \in \mathcal{Y}\) there are chosen basepoints \(\uparrow_y \in V_y\) identifying the groups \(G_y\) with \(\pi_1(V_y, \uparrow_y)\) and for all edges \(e \in \mathcal{Y}\) there are curves \(\gamma_e\) connecting \(\uparrow_{e^-}\) to \(\uparrow_{e^+}\) satisfying \(\gamma_y = \gamma_e^{-1}\) and describing \((-\cdot)\)-maps. Similarly, for objects in \(\mathcal{X}\) there are basepoints \(*_x\) \(U_x\) and curves \(\gamma_x \subset X\).

Then every biset element \(\overline{b} = (h_0, y_1, \ldots, b, \ldots, x_n, g_n) \in \pi_1(\mathcal{B}, \uparrow, *)\) defines a certain element \(\theta(\overline{b}) \in B((Z, f, i), \uparrow, *)\) that is the concatenation \(h_0 \# \gamma y_1 \# \ldots \# b \# \ldots \# \gamma x_n g_n\).

We get a biset morphism

\[
\theta: \pi_1(\mathcal{B}, \uparrow, *) \to B((Z, f, i), \uparrow, *).
\]

Clearly \(\theta\) is surjective, because every element in \(B((Z, f, i), \uparrow, *)\) can be presented as a concatenation \(h_0 \# \gamma y_1 \# \ldots \# b \# \ldots \# \gamma x_n g_n\), see Remark 3.5. To prove that \(\theta\) is injective, we show that if \(\theta(\overline{b}_1)\) is homotopic to \(\theta(\overline{b}_2)\), then \(\overline{b}_1 = \overline{b}_2\).

By the classical van Kampen argument (see [11, Chapter IV]) a homotopy between \(\theta(\overline{b}_1)\) and \(\theta(\overline{b}_2)\) can be expressed as a combination of the following operations: (1) homotopies within \(U_i\) and in \(V_j\); (2) replacement of \(g_k\) or \(h_k\) by \(\gamma_z g_k \gamma_z^{-1}\) or \(\gamma_z h_k \gamma_z^{-1}\) for some \(z\), and (3) replacement of \(b\) by \(\gamma_{\lambda(z)} \# b \# \gamma_{\rho(z)}\) for some \(z \in \mathcal{B}\).

All the above operations fix the corresponding elements in \(\pi_1(\mathcal{B}, \uparrow, *)\). \(\square\)

Lemma 4.9. Consider a path connected space \(X\) with a 1-dimensional cover \((X_v)_{v \in V}\) and the associated graph of groups \(\mathcal{X}\) as in Definition 3.6.

Then, up to congruence, \(\mathcal{X}\) and the graph of bisets associated with a topological correspondence are independent of the choices of basepoints \(*_v\) and connecting paths \(\gamma_v\).

Proof. Let \(\mathcal{X}, \mathcal{X}'\) be two graphs of groups associated with \(X\) and with a fixed 1-dimensional cover \((X_v)_{v \in V}\) but with different choices of basepoints and connecting paths. Consider a topological correspondence \((X, f, i)\) such that \(f: X \supset \emptyset\) and \(i: X \supset \emptyset\) are the identity maps. We may assume that \(\mathcal{X}', \text{resp } \mathcal{X},\) is the graph of groups associated with the range of \(f, \text{resp } i\). Let \(g \mathcal{B}_{X'}\) be the graph of bisets associated with \((X, f, i)\) and the cover \((X_v)_{v \in V}\). Then \(\mathcal{B}\) is a biprincipal graph of bisets because \(B(f|_{X_v}, i|_{X_v})\) is biprincipal for every \(v\).

The second claim follows from the first. \(\square\)

4.3. Products of correspondences. Correspondences, just as continuous maps, may be composed: the operation is given by fiber products.

Let \((Z_1, f_1, i_1)\) and \((Z_2, f_2, i_2)\) be two correspondences such that \(f_1: Z_1 \to X\) and \(i_2: Z_2 \to X\) have the same range \(X\). Their product is the correspondence \((Z, f, i)\) given by

\[
Z = \{(z_1, z_2) \in Z_1 \times Z_2 \mid f_1(z_1) = i_2(z_2)\}, \quad f(z_1, z_2) = f_2(z_2), \quad i(z_1, z_2) = i_1(z_1).
\]

We have natural maps \(\tilde{i}_2: Z \to Z_1\) and \(\tilde{f}_1: Z \to Z_2\) given respectively by

\[
\tilde{i}_2(z_1, z_2) = z_1, \quad \tilde{f}_1(z_1, z_2) = z_2.
\]
It is easy to check that $f$ is a fibration, respectively a covering, if both $f_1$ and $f_2$ are fibrations, respectively coverings.

The biset of a product of two correspondences is, in favourable cases, the product of the corresponding bisets:

**Lemma 4.10.** Let $(Z, f, i)$ be the product of two correspondences $(Z_1, f_1, i_1)$ and $(Z_2, f_2, i_2)$, with $f_1 : Z_1 \to X$ and $i_2 : Z_2 \to X$ path connected. Then there is a biset morphism

$$
\begin{align*}
B(f, i) & \to B(f_1, i_1) \otimes B(f_2, i_2) \\
(\delta, p, \gamma) & \mapsto (\delta, i_2(p), \varepsilon^{-1}) \otimes (\varepsilon, f_1(p), \gamma),
\end{align*}
$$

for any choice of path $\varepsilon$ from the basepoint of $X$ to $f_1(i_2(p)) = i_2(f_1(p)).$ \hfill $\square$

**Example 4.11.** In general, the map in the above lemma need not be an isomorphism; for instance, the ranges of $i_2$ and $f_1$ need not intersect, in which case $Z = \emptyset$ so $B(f, i) = \emptyset$, while $B(f_1, i_1)$ and $B(f_2, i_2)$ are non-empty so their product is not empty.

For a less artificial example, consider the correspondence shown on Figure 4.3. Denote by 1 the trivial group; we also view 1 as a set consisting of one element. Then $B(i_1, f_1) = 1_{Z_1}$, $B(i_2, f_2) = 1_{Z_2}$; thus $B(i_1, f_1) \otimes B(i_2, f_2) = 1_{Z_1 Z_2}$. On the other hand, $B(i_2 i_1, f_1 f_2) = 1_{Z_1 Z_2}$.

Sufficient conditions on the map (16) being an isomorphism are given by Lemma 4.12, in analogy with Lemma 3.26.

**Lemma 4.12.** If in Lemma 4.10 at least one of the maps $i_2, f_1$ is fibrant, then (16) is an isomorphism of bisets.

**Proof.** We construct an inverse to the map (16), keeping the notations of Lemma 4.10. Suppose without loss of generality that $i_2$ is fibrant. Consider $b = (\delta_1, p_1, \gamma_1) \otimes (\delta_2, p_2, \gamma_2) \in B(f_1, i_1) \otimes B(f_2, i_2)$. Changing the basepoint in $X$ to $\ast = f_1(p_1)$ and denoting by $\ast$ the constant path in $X$ at $\ast$, we rewrite $b$ as $(\delta_1, p_1, \ast) \otimes (\gamma_1 \# \delta_2, p_2, \gamma_2)$. Since $i_2$ is fibrant, the curve $\gamma_1 \# \delta_2$ admits a lift $\varepsilon$ to $Z_2$ with $\varepsilon(1) = p_2$. Write $p_2' = \varepsilon(0)$. We then have $b = (\delta_1, p_1, \ast) \otimes (\ast, p_2', (f_2 \circ \varepsilon) \# \gamma_2)$, see Lemma 4.4. Since $f_1(p_1) = \ast = i_2(p_2')$, there is a unique $p \in Z$ with $i_2(p) = p_1$ and $f_1(p) = p_2$. Then $(\delta_1, p, (f_2 \circ \varepsilon) \# \gamma_2) \in B(f, i)$ is a preimage of $b$ under (16). \hfill $\square$
4.4. **Fibrant maps and covers.** Suppose \( f: Z \to X \) is a fibrant map between path connected spaces and \((U_\alpha)\) is a cover of \( X \) consisting of path connected sets \( U_\alpha \). The pullback \( (W_\gamma) := f^*(U_\alpha) \) is the cover of \( Z \) consisting of all path connected components of \( f^{-1}(U_\alpha) \), for all \( U_\alpha \) in the cover. It is easy to see that if \( (U_\alpha) \) is a finite cover and \( f^{-1}(*) \) consists of finitely many connected components for some or equivalently any \( * \in X \), then \((V_\alpha)\) is a finite cover. If, furthermore, \((U_\alpha)\) is 1-dimensional, then so is \((W_\gamma)\). It is also immediate that all \( f \vert_W \) are fibrant maps.

**Proposition 4.13.** Suppose \( \mathfrak{B}_X \) is the graph of bisets of a topological correspondence \((Z, f, i)\) from \( Y \) and \( X \) subject to finite 1-dimensional covers \((U_\alpha), (V_\beta)\), and \((W_\gamma)\) of \( X, Y \), and \( Z \) respectively.

If \( f: Z \to X \) is fibrant, respectively a covering, and \((W_\gamma) = f^*(U_\alpha)\), then \( \mathfrak{B}_X \) is a left-fibrant, respectively a left-free, graph of bisets.

The proof of the above proposition is based on the following property:

**Lemma 4.14.** Let

- \( f: W \to U \) be a fibrant map between path connected spaces, with biset \( B(f, \dagger, *) \);
- \( U' \subset U \) be path connected with basepoint \( u' \);
- \( \{W'_\ell\}_{\ell \in I} \) be the set of connected components of \( f^{-1}(U') \);
- \( \mathcal{B}(f \big|_{W'_\ell}, u', *) \) be the biset of \( f: W'_\ell \to U' \);
- \( (\cdot)^-: \pi_1(W_\gamma, \dagger, \gamma')B(f \big|_{W_\gamma}, W'_\ell, \gamma') \to \pi_1(U, \dagger, \gamma)B(f, \dagger, \gamma) \) be congruences as in Lemma 4.6.

Then we have a \( \pi_1(W, \dagger) \to \pi_1(U, *) \) isomorphism

\[
(17) \quad \bigsqcup_{\ell \in I} \pi_1(W, \dagger) \otimes_{\pi_1(W'_\ell, \gamma')} B(f \big|_{W'_\ell}, u', \gamma') \to B(f \big|_{W}, u, u'), \quad g \otimes b \mapsto \gamma g^{-1}
\]

with right action of \( \pi_1(U, *) \) on \( B(f \big|_{W}, u, u') \) given via \((\cdot)^-: \pi_1(U, \gamma) \to \pi_1(U, \gamma')\).

**Proof.** The statement is clearly stable under motion of \( u' \), thus we may assume that \( \gamma = * \) and that \((\cdot)^-: \pi_1(U', \gamma) \to \pi_1(U, \gamma)\) is given by taking each \( \gamma \in \pi_1(U', \gamma) \) modulo homotopy in \( U \). By Lemma 4.4

\[
B(f, \dagger, *) = (\beta: [0, 1] \to W \mid \beta(0) = \dagger, f(\beta(1)) = *)/\sim,
\]

with \( \beta \sim \beta' \) if and only if there is a path \( \epsilon: [0, 1] \to f^{-1}(* \cdot )\) connecting \( \beta(1) \) to \( \beta'(1) \) such that \( \beta \# \epsilon \) is homotopic to \( \beta' \). Therefore, each \( \beta: [0, 1] \to W \) in \( B(f) \) ends at a unique \( W'_\ell \) independent on the choice representing \( \beta \). It is now easy to see that the \( \pi_1(W, \dagger) \to \pi_1(U, \gamma) \) subbiset of \( B(f, \dagger, \gamma) \) consisting of all \( b \in B(f, \dagger, \gamma) \) terminating at \( W'_\ell \) is exactly \( \pi_1(W, \dagger) \otimes_{\pi_1(W'_\ell, \gamma')} B(f \big|_{W'_\ell}, u', \gamma') \). This implies \( (17) \).

**Proof of Proposition 4.13.** Let \( (Z_k) \) be the path connected components of \( Z \). By definition, \( \mathfrak{B}_X = \bigsqcup_k \mathfrak{B}(i \vert_{Z_k})_{\gamma_k} \otimes_{3_k} 3_k \mathfrak{B}(f \big|_{Z_k})_X \). Lemma 4.14 implies that every \( 3_k \mathfrak{B}(f \big|_{Z_k})_X \) is left-fibrant: Equation \( (17) \) is exactly \( (6) \). Since the product of left-fibrant bisets is left-fibrant (Lemma 3.27), \( \mathfrak{B}_X \) is a left-fibrant biset. This proves the part of Proposition 4.13 concerning fibrant maps. The case of covering maps observe that all bisets in \( \mathfrak{B}(f \big|_{Z_k}) \) are left-free because \( f \big|_{Z_k} \) are coverings while all bisets in \( \mathfrak{B}(i \vert_{Z_k}) \) are left-principal; hence all bisets in \( \mathfrak{B}_X = \bigsqcup_k \mathfrak{B}(i \vert_{Z_k})_{\gamma_k} \otimes_{3_k} 3_k \mathfrak{B}(f \big|_{Z_k})_X \) are left-free.
4.5. Partial self-coverings. We now turn to a restricted class of covering correspondences \((Z,f: Z \rightarrow X, i: Z \rightarrow Y)\), in which the map \(i\) is an inclusion, namely a homeomorphism on its image. We then view \(Z\) as a subset of \(Y\), and write the correspondence as a partial covering \(f: Y \rightarrow X\). Here the dashed arrow means that the map is defined on the image of \(i\).

In this case, the definition of the biset of a correspondence (see §4.1) can be simplified as follows

**Lemma 4.15.** Let \(f: X \rightarrow X\) be a partially defined self-covering. The biset \(B(f)\) may then be constructed directly as follows:

\[
B(f) = \{ \gamma: [0,1] \rightarrow X \mid \gamma(0) = * = f(\gamma(1)) \}/\sim.
\]

The left action is by pre-composition by loops in \(X\) at *, and the right action is by lifting loops through \(f\):

\[
[\alpha] \cdot [\gamma] = [\alpha \# \gamma], \quad [\gamma] \cdot [\alpha] = [\gamma \# \tilde{\alpha}]
\]

for the unique \(f\)-lift \(\tilde{\alpha}\) of \(\alpha\) that starts at \(\gamma(1)\).

**Proof.** Simply remark that, in the notation of Lemma 4.2, the point \(p\) is determined as \(i^{-1}(\gamma(1))\), so may be removed from the definition. 

Let us consider now a self-correspondence \((Z,f: Z \rightarrow X, i: Z \rightarrow X)\). It may arise as follows: \(f: X \supseteq\) is a branched covering, \(Z\) is the subset of \(X\) on which \(f\) is a genuine covering, and \(i\) is the inclusion.

Such correspondences may be iterated; however, their iterates are not defined on the same \(Z\) anymore. We set \(X^{(1)} := Z\) and, more generally,

\[
X^{(n)} = \{(x_1, \ldots, x_n) \in Z^n \mid f(x_i) = i(x_{i+1})\}.
\]

Define the maps \(f^{(n)}(x_1, \ldots, x_n) = f(x_n)\) and \(i^{(n)}(x_1, \ldots, x_n) = i(x_1)\). Then the \(n\)-th power, in the sense of §4.3, of \((f, i)\) is \((f^{(n)}, i^{(n)})\).

If \(f\) is fibrant, then so are all \(f^{(n)}\), and naturally (Lemma 4.12), the biset of \((f^{(n)}, i^{(n)})\) is none other than \(B(f, i)^{\otimes n}\).

If \(i: X^{(1)} \rightarrow X\) is an inclusion, then so are all \(i^{(n)}: X^{(n)} \rightarrow X\). Then \((f, i)\) is identified with a partially defined map \(f: X \rightarrow X\); and all \((f^n, X^{(n)} \rightarrow X, i^n: X^{(n)} \rightarrow X)\) are identified with \(f^{(n)}: X \rightarrow X\). If \(f\) is fibrant, then the bisets of \((f^{(n)}, i^{(n)})\) and of \(f^{(n)}\) are naturally isomorphic.

Iteration may also be interpreted purely in the language of homomorphisms, using Lemma 2.7. Given a \(G\)-\(G\)-biset \(B\), write \(G_0 = G\), and find a group \(G_1\) such that \(B\) decomposes as \(B_{\psi_1} \otimes G_1 B_{\phi_1}\), for homomorphisms \(\phi_1, \psi_1: G_1 \rightarrow G_0\). Define then iteratively \(G_{n+1}\) as the fibre product of \(G_n\) with \(G_n\) over \(G_{n-1}\):

\[
\begin{array}{c}
G_{n+1} \\
\downarrow \phi_{n+1} \\
G_n \\
\downarrow \psi_n \\
G_n \phi_n \rightarrow G_{n-1}
\end{array}
\]

Then \(B^{\otimes n} = (B_{\psi_1} \cdots \psi_1)^{\otimes n} \otimes G_n B_{\phi_1} \cdots \phi_1\).
4.6. **Generic maps.** We can slightly relax the previous setting, in which \( i: Z \to Y \) is an inclusion, to “generic” maps in the following sense.

**Definition 4.16** (Generic maps). A continuous map \( f: Y \to X \) is *generic* if there exists a continuous map \( g: X \to Y \) such that \( f \circ g \) is isotopic to the identity on \( X \).

Here is a typical example: Consider a topological space \( X \) and an injective path \( \gamma: [0, 1] \to X \), such that a neighbourhood of the image of \( \gamma \) is contractible in \( X \). Then the inclusion \( f: X \setminus \{ \gamma(0), \gamma(1) \} \to X \setminus \{ \gamma(0) \} \) is a generic map. Indeed contractibility of the image of \( \gamma \) implies the existence of a homeomorphism \( g: X \setminus \{ \gamma(0) \} \to X \setminus \{ 0, 1 \} \).

**Lemma 4.17.** Let \( f: Y \to X \) be a generic map, and let \( \dagger \in Y \) be a basepoint. Write \( f(\dagger) = \ast \). Then \( f_\ast: \pi_1(Y, \dagger) \to \pi_1(X, \ast) \) is a split epimorphism.

Therefore, the biset of \( f \) is left-invertible and right-principal; namely, it is a \( \pi_1(Y, \dagger) \)-\( \pi_1(X, \ast) \)-biset \( B \) such that \( B \ast \otimes \pi_1(Y, \dagger) B \cong \pi_1(X, \ast) \otimes \pi_1(X, \ast) \).

**Proof.** Let \( g \) be a homotopy left inverse of \( f \). The first statement follows simply from \( g_\ast f_\ast = id_X \). The second one follows from the first. \( \Box \)

### 4.7. Conjugacy classes in bisets.

Let \( G \) be a biset. Its set of conjugacy classes is \( B^G := GB_G/\{ b = gbg^{-1} \mid b \in B, g \in G \} \).

Consider a self-correspondence \( f, i: Z \to X \). A homotopy pseudo-fixed point \([9]\) is the data \((p, \gamma)\) such that \( p \in Z \) and \( \gamma: [0, 1] \to X \) with \( \gamma(0) = f(p) \) and \( \gamma(1) = i(p) \). In other words, \( \gamma \) encodes a homotopy difference between \( f(p) \) and \( i(p) \). If \( \gamma \) is a constant path, then \( p \) is a fixed point of \((f, i)\). Two homotopy pseudo-fixed points \((p, \gamma)\) and \((q, \delta)\) are conjugate if there is a path \( \ell: [0, 1] \to Z \) with \( \ell(0) = p, \ell(1) = q \) such that \( f(\ell) \# \delta \# i(\ell^{-1}) \) is homotopic to \( \gamma \).

The set of fixed points conjugate to a given fixed point \( p \in Z \) is also known as the Nielsen class of \( p \). The Nielsen number \( N(f, i) \) is the set of fixed points of \((f, i)\) considered up to conjugacy.

Every \((\delta^{-1}, p, \gamma) \in B(f, i)\), in the notation of Lemma 4.2, naturally defines a homotopy pseudo-fixed point \((p, \gamma \# \delta)\). Conversely, if \( X \) is path connected, then for every homotopy pseudo-fixed point \((p, \gamma)\) we may choose a path \( \ell: [0, 1] \to X \) with \( \ell(0) = \ast \) and \( \ell(1) = f(p) \) and construct an element \((\ell^{-1}, p, \gamma \# \ell) \in B(f, i)\) encoding \((p, \gamma)\).

The following proposition is immediate.

**Proposition 4.18.** Two elements \((\delta^{-1}, p, \gamma), (\delta'^{-1}, p', \gamma') \in B(f, i)\) are conjugate as elements of the biset \( B(f, i) \) if and only if the homotopy fixed points \((p, \gamma \# \delta), (p', \gamma' \# \delta')\) are conjugate.

As a corollary, if \( X \) is path connected, then the Nielsen number \( N(f, i) \) is bounded by the cardinality of \( B^G \).

In [3] we will further investigate homotopy pseudo-periodic orbits of a Thurston map.

5. **Dynamical systems**

We turn now to applications of the previous sections. They will mainly be to the theory of iterations of branched self-coverings of surfaces. The main objective
is an algorithmic understanding of these maps up to isotopy, and will be developed
in later articles. Here are some more elementary byproducts.

Consider first a polynomial $p(z) \in \mathbb{C}[z]$. We recall some basic definitions and
properties; see [6, 7] for details. Let us denote by $\hat{P}(p) \subset \mathbb{C}$ the forward orb
of $p$’s critical points:

$$\hat{P}(p) := \{p^n(z) \mid p'(z) = 0, n \geq 0, z \in \mathbb{C}\}.$$ 

The post-critical set $P(p) := p(\hat{P}(p))$ is the forward orb of $p$’s critical values. The
polynomial $p$ is post-critically finite if $P(p)$ is finite. The Julia set $J(p)$ of $p$ is the
closure of the repelling periodic points of $p$. Equivalently, $J(p)$ is the boundary of
the filled-in Julia set $K(p)$:

$$K(p) := \{z \in \mathbb{C} \mid P(p) \text{is bounded}\}, \quad J(p) = \partial K(p).$$

The Fatou set $F(p)$ is the complement of $J(p)$.

Let us assume that $p$ is post-critically finite. This assumption is essential to obtain
simple combinatorial descriptions of $p$. Then each bounded connected component
of $F(p)$ is a disk, and $p$ acts on this set of disks. Furthermore, the boundary
of each disk component is a circle in $J(p)$.

Furthermore, every grand orb of $p$: $\hat{P}(p) \to \hat{P}(p)$ contains a unique periodic
cycle. If a periodic cycle $C \subset \hat{P}(p)$ contains a critical point of $p$, then $C$ lies entirely
in the Fatou set, and moreover each element $c \in C$ lies in a different disk of $F(p)$.
On the other hand, if a cycle $C$ contains no critical point of $p$, then $C$ lies in $J(p)$. If
no cycle of $P(p)$ contains a critical point of $p$, then $F(p)$ has no bounded component
and $J(p)$ is a dendrite.

Let $\mathcal{T}$ be the smallest tree in $K(p)$ containing $\hat{P}(p)$ and containing $P(p)$ in its
vertex set such that $\mathcal{T}$ intersects $F(p)$ along radial arcs. Since $\hat{P}(p)$ is forward
invariant we get a self-map $p$: $\mathcal{T} \subset \mathcal{T}$. For every pair of adjacent edges $e_1, e_2$ there
is the angle $\hat{x}(e_1, e_2) \in \mathbb{Q}/\mathbb{Z}$ uniquely specified by the following conditions, see [14]:

- $\hat{x}(e_1, e_2) = 0$ if and only if $e_1 = e_2$;
- if $v$ is a common vertex of $e_1$ and $e_2$ and $\deg_v(p)$ is the local degree of $p$ at
  $v$, then
  $$\hat{x}(p(e_1), p(e_2)) = \deg_v(p) \hat{x}(e_1, e_2);$$
- for all edges $e_1, e_2, e_3$ adjacent to a common vertex we have
  $$\hat{x}(e_1, e_3) = \hat{x}(e_1, e_2) + \hat{x}(e_2, e_3).$$

**Definition 5.1** (Hubbard trees). The (angled) Hubbard tree of a complex polynomial
$p$ is the data consisting of $p$: $\mathcal{T} \subset \mathcal{T}$, the values $\deg(v) \in \mathbb{N}$ measuring the local
degrees of $p$ at vertices $v \in p^{-1}(\mathcal{T})$, and the angle structure “$\hat{x}$” on $\mathcal{T}$ satisfying
the above axioms. △

In [14] post-critically finite polynomials are classified in terms of their Hubbard
trees.

**Remark 5.2.** If $p$ has degree 2, then the angled structure of $p$: $\mathcal{T} \subset \mathcal{T}$ is uniquely
determined by how $\mathcal{T}$ is topologically embedded into the plane. However, in general,
the planarity of $\mathcal{T}$ is insufficient to recover $p$; one needs to endow $p$: $\mathcal{T} \subset \mathcal{T}$ with extra
information sufficient to recover the embedding of $p^{-1}(\mathcal{T})$ into the plane.

It may be more convenient to consider a variant, the Hubbard complex, which is
a special case of topological automaton (see [13, §3]):
Definition 5.3 (Hubbard complexes). Let $p$ be a complex polynomial. Its **Hubbard complex** is the self-correspondence $p, i: H^1 \to H^0$ defined as follows:

- $H^0$ is the smallest 1-dimensional subcomplex of the plane containing $P(p) \cap J(p)$ and all $\partial D$ for $D \subseteq F(p)$ a Fatou component intersecting $P(p)$;
- $H^1$ is the smallest 1-dimensional subcomplex of the plane containing $P(p) \cap J(p)$ and all $\partial D$ for $D \subseteq F(p)$ a Fatou component intersecting $p^{-1}(P(p))$;
- $p: H^1 \to H^0$ is the restriction of $p$ to $H^1$ and is a covering map;
- $i: H^1 \to H^0$ retracts $H^1$ into $H^0$.

(As for the Hubbard tree, we assume that $H^1, H^0$ intersects $F(p)$ along radial lines. Points in $H^0 \cap P(p)$ and in $H^1 \cap \hat{P}(p)$ are treated as orbifold points.)

![Figure 2. The Julia sets of the maps $z^2 + i$, $z^2 - 1$ (the “Basilica”) and $z^2 + c$ for $(c^2 + c)^2 + c = 0$ (the “Rabbit”). The Hubbard complex is drawn in red, and (if it differs) the Hubbard tree is drawn in black.](image)

5.1. **Thurston maps.** We may consider the more general situation of a **branched self-covering** of the sphere $S^2$, namely a map $p: S^2 \to S^2$ that is locally modelled, in complex charts, by $z \mapsto z^n$ for some $n \in \mathbb{N}$. Those points $z \in S^2$ at which $n \geq 2$ are **critical points**, and $P(p)$ is the forward orbit of $p$’s critical values. If furthermore there is a point $\infty \in S^2$ with $p^{-1}(\infty) = \{\infty\}$, then $p$ is a topological polynomial. If $P(p)$ is finite, then $p$ is called a **Thurston map**.

Unless $p$ expands a metric on $S^2$, there is no well-defined notion of Julia set. There is, however, a convenient encoding of $p$ by a biset. One sets $X = Y = S^2 \setminus P(p)$ and $Z = S^2 \setminus p^{-1}(P(p))$, with maps $i: Z \to Y$ the inclusion and $p: Z \to X$ the restriction of $p$. Thus $p: S^2 \to S^2$ is given by a covering correspondence $p, i: Z \to X$.

Fix a basepoint $* \in X$, and write $G = \pi_1(X, *)$. Let $B(p)$ denote the biset of the above correspondence associated with $p$; it is a $G$-$G$-biset, left-free of degree $\deg(p)$. 
Let \( p_0, p_1 \) be Thurston maps. They are called \textit{combinatorially equivalent} if there is an isotopy \((p_t)_{t \in [0,1]}\) from \( p_0 \) to \( p_1 \) along which \( P(p) \) moves continuously (and, in particular, has constant cardinality). The biset of a Thurston map is a complete invariant for combinatorial equivalence:

\textbf{Theorem 5.4} (Kameyama \cite{10}, Nekrashevych \cite[Theorem 6.5.2]{12}). Let \( p_0, p_1 \) be Thurston maps. Then \( p_0, p_1 \) are Thurston equivalent if and only if the bisets \( B(p_0), B(p_1) \) are conjugate by an isomorphism \( \pi_1(S^2 \setminus P(p_0), \ast_0) \to \pi_1(S^2 \setminus P(p_1), \ast_1) \) induced by a surface homeomorphism.

By definition, \( p \) behaves locally as \( z \to z^{\deg_z(p)} \) at a point \( z \in S^2 \); set

\[ \text{ord}(v) = \text{l.c.m.}\{\deg_z(p^n) \mid n \geq 0, z \in p^{-n}(v)\}. \]

Clearly, \( \text{ord}(z) > 1 \) if and only if \( z \in P(p) \), and \( \text{ord}(v) = \infty \) if and only if \( v \) is in a periodic cycle containing a critical point. For example, the degrees at \( P(p) \) are all \( \infty \) for the Basilica and the Rabbit maps, and are \( 2 \) at \( -i, i, i-1 \) for \( z^2 + i \).

This order function \( \text{ord} \) defines an orbispace structure on \( S^2 \): in our simplified context, a topological space with the extra data of a non-trivial group attached at a discrete set of points. We will not go into details of orbispaces, but simply note that, if \( v \) is a point with group \( G_v \) attached to it, then \( v \) has canonical neighbourhoods with fundamental group isomorphic to \( G_v \). In our situation, the group attached to \( v \in P(p) \) is cyclic of order \( \text{ord}(v) \). If \( \text{ord}(v) = \infty \), then the point \( v \) may be treated as a punctured rather than as a point with \( Z \) attached to it.

For each \( z \in P(p) \), let \( \gamma_z \) denote a small loop around \( z \), and identify \( \gamma_z \) with a representative of a conjugacy class in \( \pi_1(S^2 \setminus P(p), \ast) \). It follows that the fundamental group of the orbispace defined by \( \text{ord} \) is given as follows:

\begin{equation}
G_p = \pi_1(S^2 \setminus P(p), \ast)/\langle \gamma_z^{\text{ord}(z)} : z \in P(p) \rangle.
\end{equation}

For every \( z \in S^2 \) define \( \text{ord}^1(z) := \text{ord}(p(z))/\deg_z(p) \). Then \( p: (S^2, \text{ord}^1) \to (S^2, \text{ord}) \) is a covering between orbispaces while \( (S^2, \text{ord}^1) \to (S^2, \text{ord}) \) is an orbispace inclusion. This defines the \( G_p \)-biset \( B(p) \). Note that Theorem 5.4 was stated for bisets over the group \( \pi_1(S^2 \setminus P(p)) \), not for bisets over the orbispace fundamental group \( G_p \). However, an analogue of Theorem 5.4 is also true in that context; this will be proven in \cite{2}.

Much structure in the space of Thurston maps can be obtained by comparing, or deriving, maps from the simplest example \( f(z) = z^d \). This map has \( J(f) = \{ |z| = 1 \} \) and \( P(f) = \{ 0, \infty \} \) so that \( \pi_1(\mathbb{C} \setminus P(f), \ast) = \langle t \rangle \cong \mathbb{Z} \). We call the corresponding biset the \textit{regular cyclic biset} of degree \( d \): as a left \( \langle t \rangle \)-set, it is \( \langle t \rangle \times \{ 1, \ldots, d \} \); and the right action on the basis \( \{ 1, \ldots, d \} \) is given by

\[ 1 \cdot t = 2, \ldots, (d - 1) \cdot t = d, \quad d \cdot t = t \cdot 1. \]

It may also be defined more economically as \( B(z^d) = \{ t^{j/d} : j \in \mathbb{Z} \} \), with left and right actions given by \( t^i \cdot t^{j/d}, t^{k} = t^{i+j/d+k/d} \).

Since topological polynomials of degree \( d \) behave as \( z^d \) in a neighbourhood of \( \infty \), their bisets contain a copy of the regular degree-\( d \) cyclic biset. More generally, if \( p \) has a fixed point in a neighbourhood of which it acts as \( z \to z^n \), then \( p \) contains a regular degree-\( n \) cyclic subbiset.

The graph of bisets decompositions that we shall consider essentially attempt to describe bisets of Thurston maps in terms of cyclic bisets.
5.2. (Graphs of) bisets from Hubbard trees. Let us consider the (angled) Hubbard tree \( p: T \rightharpoonup \) of a complex polynomial \( p \). We will now apply Van Kampen’s theorem to \( p: T \rightharpoonup \) to decompose \( B(p) \) as a graph of bisets \( \mathfrak{X}_X \) as in Example 3.31.

Let \( T^1 \) be the preimage of \( T \) under \( p: C \rightharpoonup \). We note that \( T^1 \) is easily reconstructable from the data \( p: T \rightharpoonup \) and \( \mathfrak{X} \) and \( \deg_e(p) \) for all \( v \in T \). The angled structure of \( T \) lifts via \( p: T^1 \to T \) to an angled structure on \( T^1 \). We denote by \( \lambda: T^1 \to T \) the natural retraction of \( T^1 \) into its subtree. We write \( \lambda(z) = v \) if an object \( z \in T^1 \) retracts into a vertex \( v \in T \) and we write \( \lambda(z) = e \) if an object \( z \in T^1 \) retracts into a subset of an edge \( e \in T \) but \( \lambda(z) \) is not a vertex of \( T \). This defines a graph morphism \( \lambda: T^1 \to T \) between undirected graphs, see§3.9 such that the images of adjacent objects are adjacent or equal objects.

Let \( X \) be the space obtained from \( T \) by blowing up each vertex \( v \in T \) into a closed unit disc \( D_v \) with \( \partial D_v \cong \mathbb{R}/\mathbb{Z} \) and by blowing down each edge \( e \), say adjacent to vertices \( v \) and \( w \), into a point \( D_e \) with \( \{D_e\} = D_v \cap D_w \) such that for every pair of edges \( e_1, e_2 \) adjacent to a common vertex \( v \) we have

\[
\mathfrak{X}(e_1, e_2) = D_{e_1} - D_{e_2} \in \mathbb{R}/\mathbb{Z} \cong \partial D_v.
\]

For every vertex \( v \in T \) put an orbifold point of order \( \ord(v) \) at the center of \( D_v \), so that the fundamental group of \( D_v \) is a cyclic group of order \( \ord(v) \).

Similarly, let \( X^1 \) be the space obtained from \( T^1 \) by blowing up each vertex \( v \in T^1 \) into a closed disc \( D_v \) with orbifold point of order \( \ord(v) \) at the center of \( D_v \) and by blowing down each edge \( e \) into a point \( D_e \) satisfying same properties as above.

Then \( p: T^1 \to T \) naturally induces a covering map \( p: X^1 \to X \) specified so that all maps between unit discs are of the form \( z \to z^d \). Furthermore, we have a natural retraction \( \iota: X^1 \to X \) satisfying \( \iota(D_z) \subset D_{\lambda(z)} \) for every object \( z \in T^1 \). Applying Van Kampen’s theorem Theorem 4.8 to the covering pair \((X^1, p, \iota)\) subject to covers \( \{D_z\}_{z \in T^1}, \{D_z\}_{z \in T}, p, \lambda: T^1 \to T \) we get the graph of bisets \( \mathfrak{X}_X \).

This graph of bisets can in fact also directly be described out of the Hubbard tree data. We present below an algorithm that computes \( \mathfrak{X}_X \).

We say that a vertex \( v \in T^1 \) is essential if it is the image of a vertex under the embedding \( T \to T^1 \). By definition, every vertex \( v \in T \) has a unique essential preimage under \( \lambda \). We say a vertex \( v \in T^1 \) is critical if \( \ord(v) > 1 \). Observe that if \( v \) is a critical but non-essential, then \( \lambda(v) \) is an edge.

On the level of graphs, \( \mathfrak{X} \xleftarrow{\lambda} \mathcal{T} \xrightarrow{\iota} \mathcal{P} \) is the barycentric subdivision of \( \lambda: T^1 \to T \). For convenience let us write \( \ord(e) = 1 \) for every edge \( e \in T \). For every object \( z \in T \) set

\[
G_z := \mathbb{Z}/\ord(z).
\]

In particular, \( G_z \) is a non-trivial group if and only if \( z \) is an essential vertex. This constructs the graph of groups \( \mathfrak{X} \). The fundamental group of \( \mathfrak{X} \) is isomorphic to a free product of \( G_v \) over all essential vertices \( v \in T \). For every \( z \in T^1 \) set

\[
B_z := \left( \frac{1}{\deg_z(p)} \mathbb{Z} \right)/\ord(\lambda(v))
\]
as a set with \( G_{\lambda(z)} \cdot G_{p(z)} \) actions given by

\[
m \cdot b \cdot n = \begin{cases} m + b + \frac{n}{\deg_z(p)} & \text{if } z \text{ is an essential or a critical vertex}, \\ m + b & \text{otherwise}. \end{cases}
\]

It remains to specify a congruence from \( B_e \) into \( B_v \) for every edge \( e \in T^1 \) adjacent to \( v \in T^1 \). Suppose first that \( v \) is a non-essential non-critical vertex. Then
\( \lambda(e) = \lambda(v) \) and we define \( B_e \to B_v \) to be the natural bijection coming from (19); this is a well defined congruence between bisets because the right actions are trivial.

Let \( e \in \mathcal{T}^1 \) be an edge adjacent to either an essential or a critical vertex \( v \in \mathcal{T}^1 \). Then the congruence \( B_e \to B_v \) is given by

\[
b \to b + [D_e]
\]

with \( D_e \) treated as an element of \( \mathbb{R}/\mathbb{Z} \cong \partial D_v \) and \( [D_e] \) means “round \( D_e \) down to an element in \( B_v \)”.

**Theorem 5.5.** The fundamental group of \( X \) is isomorphic to \( G_p \) (18). The fundamental biset of \( X \) constructed as above is isomorphic to \( B(p) \). All bisets \( B_z, z \in \Sigma \) are cyclic.

**Proof.** By construction, the covering pair \(((\Sigma, \text{ord}^1), \rho, \Sigma)\) is homotopic to the covering pair \((X^1, p, \iota)\) along a homotopy path of covering pairs. Therefore, by Theorem 3.7 the fundamental group of \( X \) is isomorphic to \( G_p \); and by Theorem 4.8 the fundamental biset of \( \pi_1(X) \) is isomorphic to \( B(p) \). Let us verify that the above algorithm computes \( x\Sigma x \).

Let us assume that \( 0 \in \mathbb{R}/\mathbb{Z} = \partial D_v \) is the basepoint of \( \pi_1(D_e) \) for every vertex \( v \in \mathcal{T} \). For every edge \( e \) adjacent to \( v \) set \( [0, D_v] \subset \mathbb{R}/\mathbb{Z} \cong \partial D_v \) to be the path connecting the basepoint of \( D_v \) to the basepoint of \( D_e \) (recall that \( D_e \) is a singleton).

For every vertex \( v \in \mathcal{T}^1 \) there is an \( a_v \in \mathbb{R}/\mathbb{Z} \) such that the map \( p: \partial D_v \to \partial D_{p(v)} \) is given by \( x \to \deg_{p(v)}(p)x + a_v \) in the \( \mathbb{R}/\mathbb{Z} \)-coordinates. Moreover, for every essential vertex \( v \in \mathcal{T}^1 \) we may choose coordinates \( \partial D_v \cong \mathbb{R}/\mathbb{Z} \) such that the map \( \iota: \partial D_v \to \partial D_{\lambda(v)} \) is the identity in \( \mathbb{R}/\mathbb{Z} \)-coordinates.

Clearly, \( G_z \) is isomorphic to \( \mathbb{Z}/\text{ord}(z) \) for every object \( z \in \mathcal{T} \).

For every essential vertex \( v \in \mathcal{T}^1 \) the biset \( B_v \) is computed by (15) as

\[
B_v = \{ b: [0, 1] \to \mathbb{R}/\mathbb{Z} \cong \partial D_{\lambda(v)} \mid b(0) = 0, \deg_{p(v)}(p)b(1) + a_v = 0 \}/\sim
\]

with \( G_{\lambda(v)} \) actions are given by pre-concatenation and by post-concatenation via lifting. Writing

\[
t = \deg_{p(v)}(p) := \left[ \frac{0}{\deg_{p(v)}(p)}, \frac{t - a_v}{\deg_{p(v)}(p)} \right] \subset \mathbb{R}/\mathbb{Z} \cong \partial D_{\lambda(v)}
\]

we see that \( B_v \) takes the form given in (19) and (20). The case of non-essential vertices is immediate because the right action is trivial.

If \( e \in \mathcal{T}^1 \) is an edge adjacent to a non-essential vertex \( v \in \mathcal{T}^1 \), then the congruence \( B_e \to B_v \) respects (19). Suppose \( e \in \mathcal{T}^1 \) is an edge adjacent to an essential vertex \( v \in \mathcal{T}^1 \). If \( \lambda(e) \) is an edge, then \( B_e \cong \{ D_e \} \) embeds into \( B_v \) as \( [0, D_e] \subset \mathbb{R}/\mathbb{Z} \cong \partial D_v \) followed by the lift of \( \deg_{p(v)}(p)D_e + a_v, 0 \subset \partial D_{p(v)} \) under \( p: \partial D_v \to \partial D_{p(v)} \) starting at \( D_e \in \partial D_v \). This gives an element \( [D_e] \in B_v \) with \( B_v \) viewed in the form (19).

If \( \lambda(e) \) is a vertex, then \( \lambda(e) = v \) and the 0-element of \( B_e \) is \( [0, D_e] \subset \mathbb{R}/\mathbb{Z} \cong D_v \).

Again, this element is mapped to \( [D_e] \) via the congruence \( B_e \to B_v \).

### 5.3. (Graphs of) bisets from Hubbard complexes

Consider now a Hubbard complex \( p, \iota: H^1 \rightrightarrows H^0 \). As in the case of Hubbard trees \( \S 5.2 \) we blow up each vertex \( v \in H^1 \cup H^0 \) into a closed disc \( D_v \) with an orbifold point of order \( \text{ord}(v) \) at the center of \( D_v \), and we blow down each edge \( e \in H^1 \cup H^0 \) into a point \( D_v \). Observe that all \( D_v \) have finite fundamental groups. Then \( p, \iota: H^1 \rightrightarrows H^0 \) naturally induces a correspondence \( p, \iota: Y^1 \rightrightarrows Y^0 \) normalized so that all maps between unit discs are of the form \( z \to z^d \). As in \( \S 5.2 \) we specify a graph morphism \( \lambda: H^1 \to H^0 \).
by \(i(D_z) \subset D_{\lambda(z)}\) for every object \(z \in H^1\). Applying Van Kampen’s Theorem 4.8 to the covering pair \((Y^1, p, i)\) subject to covers \(\{D_z\}_{z \in H^1}, \{D_z\}_{z \in H^0}\) with maps \(p, \lambda: H^1 \to H^0\) we get a graph of bisets \(\mathcal{Y} \mathcal{B}_{\mathcal{Y}}\). We remark that \(\mathcal{Y} \mathcal{B}_{\mathcal{Y}}\) could be explicitly computed out of \(p, i: H^1 \to H^0\) in the same way as \(x \Sigma \lambda\) was computed out of the Hubbard tree \(\S 5.2\).

**Theorem 5.6.** The fundamental group of \(\mathcal{Y}\) is isomorphic to \(G_p\) (18). The fundamental biset of \(\mathcal{Y} \mathcal{B}_{\mathcal{Y}}\) is isomorphic to \(B(p)\).

All groups \(G_y, y \in \mathcal{Y}\) and all bisets \(B_z, z \in \mathcal{Y}\) are finite.

**Proof.** The proof is the same as of Theorem 5.5. The claim about finiteness of \(G_y\) and \(B_z\) is straightforward. \(\square\)

5.4. (Graphs of) bisets from subdivision rules. Let us now generalize the setup of \(\S 5.2\). Suppose that \(p: S^2 \to \mathcal{Y}\) is a topological Thurston map. By a subdivision rule or a “puzzle partition” we mean graphs \(G^0 \subset G^1 \subset S^2\) such that

- \(G^1 = p^{-1}(G^0)\);
- there is a retraction \(\iota: G^1 \to G^0\), with \(\iota|_{G^1} = \mathbb{1}\) such that \(G^1\) is homotopic in \(S^2\) to \(\iota(G^1)\) rel the post-critical set;
- each connected component \(S^2 \setminus G^1\) contains at most one post-critical point.

The last condition guarantees that \(p, \iota: G^1 \Rightarrow G^0\) captures all combinatorial information of \(p: S^2 \to \mathcal{Y}\).

Assume, furthermore, that \(\iota\) can be chosen in such a way that it maps vertices and edges of \(G^1\) into vertices and edges of \(G^1\). We may construct a graph of bisets \(3\mathcal{B}_3\) associated with \(p, \iota: G^1 \Rightarrow G^0\). As in \(\S 5.2\) we blow up each vertex \(v \in G^1 \cup G^0\) into a closed disc \(D_v\) with orbifold point of order \(\text{ord}(v)\) at the center of \(D_v\) and we blow down each edge \(e \in G^1 \cup G^0\) into a point \(D_e\). Then \(p, \iota: G^1 \Rightarrow G^0\) naturally descents into a correspondence \(p, \iota: \mathcal{Z} \Rightarrow \mathcal{Y}\) normalized so that all maps between unit discs are of the form \(z \to z^d\). As in \(\S 5.2\) we specify a graph morphism \(\lambda: G^1 \to G^0\) by \(\iota(D_z) \subset D_{\lambda(z)}\) for every object \(z \in G^1\). Applying Van Kampen’s Theorem 4.8 to the covering pair \((\mathcal{Z}^1, p, i)\) subject to covers \(\{D_z\}_{z \in G^1}, \{D_z\}_{z \in G^0}\) with \(p, \lambda: G^1 \Rightarrow G^0\) we get the graph of bisets \(3\mathcal{B}_3\).

**Theorem 5.7.** There is a natural epimorphism \(\phi: \pi_1(3, \ast) \to G_p\) (18) and there is a natural surjective map \(\beta: \pi_1(3\mathcal{B}_3, \ast) \to B(p)\) such that

\[
(\phi, \beta): \pi_1(3\mathcal{B}_3) \to G_p B(p) G_p
\]

is a semi-conjugacy respecting combinatorics (see \(\S 2.4\)).

**Proof.** The space \(\mathcal{Z}^0\) has a natural embedding into \(S^2\), unique up to homotopy rel the post-critical set, such that \(\mathcal{Z}^0\) separates post-critical points. Therefore, there is a natural epimorphism \(\phi: \pi_1(\mathcal{Z}) \to G_p\).

The embedding of \(p, i: \mathcal{Z}^1 \Rightarrow \mathcal{Z}^0\) into \(S^2\) defines a semi-conjugacy \((\phi, \beta): B(\mathcal{Z}^1, p, i) \to B(p)\). Observe that if a loop \(\gamma \in \pi_1(\mathcal{Z}^0)\) is trivial in \(G_p\), then all lifts of \(\gamma\) via \(p: \mathcal{Z}^1 \to \mathcal{Z}^0\) are loops. Therefore, \(\gamma\) has trivial monodromy action, so \((\phi, \beta)\) respects combinatorics. \(\square\)

5.5. Tuning and mating. The tuning operation takes as input a topological polynomial \(p\), a periodic cycle \(z_0, z_1, \ldots, z_n = z_0\) for \(p\), and \(n\) topological polynomials \(q_0, \ldots, q_{n-1}\) with \(\deg(q_i) = \deg_{z_i}(p)\) for all \(i = 0, \ldots, n - 1\); and produces a topological polynomial of degree \(\deg(p)\).
For each $i$, we give ourselves a set $P_i$ containing the critical values of $q_{i-1}$ and such that $q_i(P_i) \subset P_{i+1}$. In particular, $P_i$ contains the post-critical set of $q_{i-1} \circ \cdots \circ q_1 \circ q_i$.

For simplicity, let us assume that the cycle $z_0, z_1, \ldots, z_n = z_0$ has no iterated critical preimages outside of the cycle. This condition could be lifted, but the construction would require slight modifications.

Up to isotopy, we may assume that around each $z_i$ there is a small closed topological disc $F_i$ such that $p$ restricts to a covering map $p: F_i \setminus \{z_i\} \to F_{i+1} \setminus \{z_{i+1}\}$ and such that there are homeomorphisms (known as “Böttcher coordinates”) $\psi_i: \text{int}(F_i) \to \mathbb{C}$ so that $\psi_{i+1} \circ p \circ \psi_i^{-1}$ is the map $z \mapsto z^{\deg(q_i)}$. Also, up to isotopy, we may assume that $q_i(z) = z^{\deg(q_i)} + o(z^{\deg(q_i)})$ so that the extension of $q_i$ to the circle at infinity coincides with the extension of $z \mapsto z^{\deg(q_i)}$.

We consider the following \emph{tuning} map $t(p, \{z_i\}, \{q_i\})$:

$$t(z) = \begin{cases} \psi_{i+1}^{-1}(q_i(\psi_i(z))) & \text{if } z \in \text{int}(F_i) \text{ for some } i, \\ p(z) & \text{otherwise.} \end{cases}$$

Note that if $q_i = z^{\deg(q_i)}$ for all $i$ then $t$ is isotopic to $p$.

**Theorem 5.8.** Suppose that $t(p, \{z_i\}, \{q_i\})$ is the tuning of a complex post-critically finite polynomial $p$ with polynomials $q_i: \mathbb{C} \setminus P_i \to \mathbb{C} \setminus P_{i+1}$ as above. Suppose also that $\text{ord}(z_i) = \infty$ for all $i$. Let $x(\mathbb{C} \setminus X)$ denote the graph of bisets constructed out of the Hubbard tree of $p$, see Theorem 5.5. Set $G_i = \pi_1(\mathbb{C} \setminus P_i, *_i)$ for a basepoint $*_i > 0$, and let $\phi_i: G_{z_i} \to G_i$ be the monomorphism defined by identifying $\text{int}(F_i)$ with $\mathbb{C}$ via $\psi_i$, so that the circle $\partial F_i$ corresponds to the circle at infinity in $\mathbb{C} \setminus P_i$. Let $\beta_i: B_{z_i} \to B(q_i)$ be the $\phi_i$-congruence defined by viewing $p: \partial F_i \to \partial F_{i+1}$ as a map at infinity of $q_i: \mathbb{C} \setminus P_i \to \mathbb{C} \setminus P_{i+1}$.

For all $i$, replace the cyclic group $G_{z_i}$ at $z_i \in \mathbb{C}$ by $G_i := \pi_1(\mathbb{C} \setminus P_i)$, and replace the regular cyclic biset $B_{z_i}$ at $z_i \in \mathbb{B}$ by the $G_i$-$G_{i+1}$-biset $B(q_i)$. Modify accordingly all non-essential bisets: replace all $B_z$ with $z \in \lambda^{-1}(z_i) \setminus \{z_i\}$ by $G_i \otimes G_z$, and declare for all $z \in p^{-1}(z_{i+1}) \setminus \{z_i\}$ the right $G_{i+1}$-action on $B_z$ to be trivial. Congruences of edge bisets into $B(q_i)$ are given through the congruence $\beta_i: B_{z_i} \to B(q_i)$. This defines a new graph of bisets $\mathbb{B}$ with same underlying graph as $\mathbb{X}$.

Then $B(t)$ and $\pi_1(\mathbb{B})$ are isomorphic.

**Proof.** Let $p, \iota: X^1 \to X$ be the correspondence from §5.2 associated with the Hubbard tree of $p$. Let $t, \iota: \tilde{X}^1 \to \tilde{X}$ be the correspondence obtained by replacing each $p: \text{int}(D_{z_i}) \to \text{int}(D_{z_{i+1}})$ with $q_i: \mathbb{C} \setminus P_i \to \mathbb{C} \setminus P_{i+1}$ appropriately glued with $\partial D_i, \partial D_{i+1}$. Since $t$ is isotopic to $\iota, \tilde{X}^1 \to \tilde{X}$, the biset of $t$ is isomorphic to the biset of a covering pair $(\tilde{X}^1, t, \iota)$ which by Van Kampen’s Theorem 4.8 is $\pi_1(\mathbb{B})$. \hfill $\square$

This operation may be performed with $p$ a Thurston map, not necessarily a topological polynomial. As a special case, a topological polynomial has a critical fixed point $\infty$. The \emph{formal mating} $p \sqcup q$ of two topological polynomials $p, q$ of same degree is the tuning $t(p, \{\infty\}, q)$. In other words, $p \sqcup q$ is a topological map on a two dimensional sphere where $p$ acts on the southern hemisphere while $q$ acts on the northern hemisphere.

Since the vertex $\infty$ does not belong to the Hubbard tree of $p$, Theorem 5.8 cannot apply. We do have a simple description of the mating in terms of graphs of bisets, though:
Theorem 5.9. Let \( p, q \) be two topological polynomials of same degree \( d \). Write \( G_p = \pi_1(\mathbb{C}\setminus \mathbb{P}(p), \ast) \) and \( G_q = \pi_1(\mathbb{C}\setminus \mathbb{P}(q), \dagger) \) with \( \ast, \dagger \) close to \( \infty \). Consider the graph of groups \( X \) with one edge \( e \) and two vertices \( p, q \). The groups at \( p, q \) are \( G_p, G_q \) respectively; the group at \( e \) is \( \mathbb{Z} \), included in \( G_p \) and \( G_q \) as the loop around infinity. Consider the graph of bisets \( \mathcal{B} \) with one edge and two vertices. The graph maps \( \lambda, \rho \) are the identity, the biset \( \mathcal{B}_e \) on the edge is regular cyclic of degree \( d \), and the bisets at the vertices are \( \mathcal{B}_p, \mathcal{B}_q \) respectively. The congruences \( \mathcal{B}_e \equiv \mathcal{B}_p \) and \( \mathcal{B}_e \equiv \mathcal{B}_q \) are defined by viewing \( z \equiv z^d : S^1 \) as a map on the circle at infinity of \( p, q \) and \( q \). Consider the graph of bisets \( \mathcal{B} \) with one edge and two vertices.

Then \( \pi_1(\mathcal{B}) \) is isomorphic to \( \mathcal{B}(p \sqcup q) \).

Proof. Applying Theorem 4.8 to \( p \sqcup q \) subject to the cover consisting of the closed southern hemisphere, the closed northern hemisphere and the equator, we obtain a graph of bisets isomorphic to the barycentric subdivision of \( \mathcal{B} \). □

5.6. Laminations. Consider a complex polynomial \( p \) of degree \( d \). The Fatou component around \( \infty \) admits a Böttcher parameterization \( \phi: F_\infty \rightarrow \{ |z| < 1 \} \) such that \( \phi(p(z)) = \phi(z)^d \). Since \( J(p) = \partial F_\infty \), every element of \( J(p) \) may be described as \( \lim_{r \to 1} \phi^{-1}(re^{i\theta}) \) for some (non-unique) angle \( \theta \); this gives a surjective map \( c_p: \{ |z| = 1 \} \rightarrow J(p) \) encoding the Julia set. In fact, the biset of \( p \) may be read from the base-\( d \) expansion of the kernel of \( c_p \), namely the equivalence relation \( \Xi_p = \{ (z_1, z_2) \mid c_p(z_1) = c_p(z_2) \} \).

The equivalence relation may be presented by a lamination of the disk \( \{ |z| \leq 1 \} \): the disjoint collection of subsets of the closed disk, called leaves, that are convex hulls (in the hyperbolic metric, say) of equivalence classes of \( \Xi_p \).

The Julia set \( J(p) \) is the quotient of the circle by the relation \( \Xi_p \), so the filled-in Julia set is the quotient of the disk obtained by contracting all leaves of the lamination to points. The dynamics on the circle is the “multiply the angle by \( d \)” map. The van Kampen theorem 3.7 may therefore be applied to the covering consisting of the boundary circle and the leaves. In fact, if \( p \) is post-critically finite it suffices to consider a finite collection of leaves. We consider, as a simple illustration, the Basilica map \( z^2 - 1 \) obtained from the circle by pinching the above lamination.
The space under consideration consists of a circle $C$ and an arc $A$ connecting the points $x = \exp(2i\pi/3)$ and $y = \exp(4i\pi/3)$, see Figure 5.6. We fix basepoints $* = 1$ on $C$ and $\dagger = x$ on $A$.

Let us denote by $S, T, U$ the bisets of $f, i : C \rightrightarrows C$, $f, i : A \rightrightarrows A$, and $f : A \rightarrow A$, $i : -A \rightarrow C$ respectively. We thus have the following graph of bisets $\mathfrak{B}$:

\[
\begin{array}{ccc}
T & & \pi_1(A, \dagger) \\
T & & \pi_1(A, \dagger) \\
S & & \pi_1(C, *) \\
S & & \pi_1(C, *) \\
U & & \pi_1(C, *) \\
U & & \pi_1(C, *) \\
\end{array}
\]

in which $S = \mathbb{Z} \times \mathbb{Z}$ is the regular cyclic biset encoding the doubling map on $C$ and the other bisets $T, U$ are trivial. The maps $\rho, \lambda$ are given by

\[
\rho(T) = \rho(U) = \lambda(T) = \pi_1(A, \dagger), \quad \rho(S) = \lambda(S) = \lambda(U) = \pi_1(C, *).
\]

The structure of $\mathfrak{B}$ will have been completely given when we fix bases of $S, T, U$ and describe the edge bisets $x_1, y_1, x_2, y_2$ as maps from $T$ or $U$ into $S$. Let us write $H = \pi_1(C, *) = \langle t \rangle$ and $1 = \pi_1(A, \dagger)$. Then

\[
\mu S = H \times \{1, 2\}, \quad T = 1 \times \{3\}, \quad U = H \times \{4\}.
\]

Recall that elements of a biset $B(f, i, *, *)$ are written, in their most general form (13), as $(\alpha, p, \beta)$ for paths $\alpha, \beta$ with $\alpha(0) = *\alpha, \beta(0) = f(p), \beta(1) = *\beta$. We fix paths $p, x, \ell, y$ starting at $* = 1$, turning counterclockwise on $C$, and ending respectively at $\exp(2i\pi/6), x, *2, y$. We also write $\epsilon$ for any constant path.

In this notation, the bases $1, 2, 3, 4$ are respectively

\[
1 = (\epsilon, *1, \epsilon), \quad 2 = (\ell, *2, \epsilon), \quad 3 = (\epsilon, \dagger1, \epsilon), \quad 4 = (p, \dagger2, \epsilon).
\]

We may now view $y_1, x_2$ as maps $T \to S$ and $x_1, y_2$ as maps $U \to S$. They are given by

\[
y_1(3) = (x, \dagger1, y^{-1}) = 1, \quad x_1(4) = (p, \dagger2, x^{-1}) = 1, \\
x_2(3) = (y, \dagger1, x^{-1}) = 2, \quad y_2(4) = (p, \dagger2, y^{-1}) = t^{-1}2.
\]
We are finally ready to compute the fundamental biset $B$ of $\mathfrak{B}$. It is a $G$-$G$-biset for the group $G = H \ast \langle yx^{-1} \rangle = \langle t, yx^{-1} \rangle$. We may keep the basis $\{1, 2\}$ of $S$, so that $G_B = G \times \{1, 2\}$, and we have, just as in $S$,$$1 \cdot t = 2, \quad 2 \cdot t = t \cdot 1.$$We now compute$$1 \cdot yx^{-1} = y_1(3)yx^{-1} = x3x^{-1} = xy^{-1}x_2(3) = xy^{-1} \cdot 2,$$and$$2 \cdot yx^{-1} = ty_2(4)yx^{-1} = tx_1(4) = t \cdot 1.$$In this manner, we formally recover the presentation of the Basilica biset from the graph of bisets $\mathfrak{B}$, using the relations $y_1(3)y = x^3$ etc. appearing in Definition 3.13.

There is another sort of mating, obtained directly from the Julia sets. The geometric mating of two polynomials $p, q$ of same degree is the map obtained from $z \mapsto z^2$: $(C \cup \{ x \}) \preceq$ through the quotient of $\{ |z| \leq 1 \}$ by the lamination of $p$ and through the quotient of $\{ |z| \geq 1 \}$ by the lamination of $q$ embedded into the outer disc via the map $z \to \frac{1}{z}$. In case both $p$ and $q$ are post-critically finite quadratic polynomials that are not in the conjugate limbs of the Mandelbrot set, the formal and geometric matings of $p$ and $q$ are isotopic. See [5] for a discussion on the various types of mating.

**Theorem 5.10.** Let $p$ be a post-critically finite quadratic polynomial that lies outside the Basilica limb. Write $H = \pi_1(\mathbb{C}\backslash P(p), \ast)$. Then the biset of $(z^2 - 1) \cup p$ is the fundamental biset of the following graph of bisets:

\[
\begin{array}{ccc}
T & B(p) & U \\
\begin{array}{cc}
\{y_1\} & \{x_1\} \\
\{x_2\} & \{y_2\}
\end{array} & \begin{array}{c}
x \\
y
\end{array} & H
\end{array}
\]

with the same maps $\rho, \lambda$ and inclusions as above.

**Proof.** The map $(z^2 - 1) \cup p$ is homotopic to the map shown on Figure 5.6 with $\{ |z| \geq 1 \}$ replaced by the filled-in Julia set of $p$. The statement becomes a corollary of Theorem 4.8. $\square$

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