Transverse Dynamics and Regions of Stability for Nonlinear Hybrid Limit Cycles

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Abstract: This paper presents an algorithm for computing inner estimates of the regions of attraction of limit cycles of a nonlinear hybrid system. The basic procedure is: (1) compute the dynamics of the system transverse to the limit cycle; (2) from the linearization of the transverse dynamics construct a quadratic candidate Lyapunov function; (3) search for a new Lyapunov function verifying maximal regions of orbital stability via iterated sum-of-squares programs. The construction of the transverse dynamics is novel, and valid for a broad class of nonlinear hybrid systems.

Keywords: Verification, Stability Analysis, Periodic Motion

1. INTRODUCTION

Nonlinear dynamical systems exhibiting oscillating solutions are found in an extraordinary variety of engineering and scientific problems. Stability analysis of such oscillations has a long history, with the modern theory going back to Poincaré. For planar systems substantial qualitative insight can be gained, however for higher-order systems almost all stability results are local. In this paper, we aim to characterize regions of stability to limit cycles of nonlinear systems. The problem of orbital stabilization via feedback control will also be addressed.

A major motivation for the work in this paper is control of underactuated “dynamic walking” robots (Collins et al. (2005)). These robots can exhibit efficient, naturalistic, and highly dynamic gaits. However, control design and stability analysis for such robots is a challenging task since their dynamics are intrinsically hybrid and highly nonlinear. Local stabilizing control design has been investigated via hybrid zero dynamics (Westervelt et al. (2007)) and transverse linearization (see, e.g., Shiriaev et al. (2008); Manchester et al. (2011a)). As well as being of interest in their own right, estimates of regions of stability will be an enabling technology for planning transitions among a library of stabilized walking gaits (Gregg et al. (2010)), and for constructive control design and motion-planning (Tedrake et al. (2010)).

The most well-known tool for analysis of limit cycles is the Poincaré map: orbital stability is characterized by stability of an associated “first-return map”, describing the repeated passes of the system through a single transversal hypersurface. However, since the system’s evolution is only analyzed on a single surface, regions of stability in the full state-space are difficult to evaluate via the Poincaré map.

A related technique known variably as “transverse coordinates” or “moving Poincaré sections” also has a long history but has not been much used in applications until recently due to difficulty in the relevant computations (see Hale (1980)). With this technique, a new coordinate system is defined on a family of transversal hypersurfaces which move about the orbit under study. In most cases, it is also used to study local stability, however as we will show it can be adapted to characterize regions of stability in the full state space.

Previous work by the author and colleagues has utilized a construction specifically for Lagrangian mechanical systems (see Shiriaev et al. (2008); Manchester et al. (2011a)). The new construction is also useful for design of stabilizing controllers for non-periodic trajectories of highly nonlinear systems without this special structure, e.g. in Shkolnik et al. (2011) a quadruped robot bounding over rough terrain was stabilized using a preliminary version of the construction in the present paper.

In this paper we present an algorithm to compute a conservative estimate of the region of stability to a known periodic solution of hybrid nonlinear system. The method we propose is to construct the transverse dynamics in regions of the orbit, and then utilize the well-known sum-of-squares (SoS) relaxation of polynomial positivity which is amenable to efficient computation via semidefinite programming (see, e.g., Parrilo (2000)). The SoS relaxation has been previously used to characterize regions of stability of equilibria of nonlinear systems (see, e.g., Topcu et al. (2008); Tan and Packard (2008)).

The method we propose has been tested on a number of example systems, presented in the companion paper Manchester et al. (2011b). Numerical details are provided in Tobenkin et al. (2011). There is comparatively little work on computing regions of stability of limit cycles. The proposed method has aspects in common with the surface Lyapunov functions proposed in Goncalves (2005), however that method was restricted to piecewise linear systems. The technique of cell-to-cell mapping has been used in analysis of walking robots (Schwab and Wisse
(2001), but the computational cost is exponential in the dimension of the system.

An extended version of this paper with proofs and control design for hybrid limit cycles can be found in Manchester (2010).

2. PRELIMINARIES AND PROBLEM STATEMENTS

2.1 Stability of Limit Cycles

Consider an autonomous dynamical system, which may be continuous or hybrid, with a state \( x \in \mathbb{R}^n \). The solution, or flow of the system is denoted by \( \Phi(x_0, t) \), i.e. \( x(t) = \Phi(x_0, t) \) is the solution at time \( t > 0 \) of the dynamical system in question from an initial state \( x(0) = x_0 \). If this solution exists and is unique then \( \Phi(x_0, t) \) is well-defined.

Suppose the system has a non-trivial \( T \)-periodic orbit, i.e. \( T > 0 \) is the minimal period such that \( x^*(t) = x^*(t + T) \) for all \( t \), and one would like to analyze the stability of this orbit. It is well-known that such a solution cannot be asymptotically stable in the standard sense, since perturbations in phase are persistent. The more appropriate notion is orbital stability. The definitions in this section are all standard (see, e.g., Hale (1980); Hauser and Chung (1994)).

Definition 1. Consider non-trivial \( T \)-periodic solution \( x^*(t) \) of a dynamical system with flow \( \Phi(t, \cdot) \), and let \( \Gamma^* = \{ x \in \mathbb{R}^n : \exists t \in [0, T) : x = x^*(t) \} \). The solution \( x^*(\cdot) \) is said to be asymptotically orbitally stable if there exists a \( b > 0 \) such that for any \( x_0 \) satisfying \( d(x, \Gamma^*) < b \) the solution exists, is unique, and \( d(\Phi(x_0, t), \Gamma^*) \to 0 \) as \( t \to \infty \).

The distance to a set is defined in the usual way: \( d(x, \Gamma^*) = \inf_{y \in \Gamma^*} |y - x| \) with \( | \cdot | \) the Euclidean norm in \( \mathbb{R}^n \).

A stronger statement is exponential orbital stability.

Definition 2. A \( T \)-periodic solution \( x^*(\cdot) \) is said to be exponentially orbitally stable if it is orbitally stable and furthermore there exists a \( b > 0, K > 0, c > 0 \) such that for any \( x_0 \) satisfying \( d(x_0, \Gamma^*) < b \) we have

\[
\text{dist}(\Phi(x_0, t), \Gamma^*) \leq K \text{dist}(x_0, \Gamma^*) e^{-ct}.
\]

The primary aim of this paper is to characterize regions of orbital stability:

Definition 3. A set \( R \subset \mathbb{R}^n \) with non-empty interior \( R^i \) and \( \Gamma^* \subset \mathbb{R}^i \) is said to be an inner estimate of the region of stability of \( x^*(\cdot) \) if for all \( x_0 \in R \) we have \( \text{dist}(\Phi(x_0, t), \Gamma^*) \to 0 \) as \( t \to \infty \).

2.2 Problem Statements

This paper will suggest algorithms for three analysis and control problems for limit cycles:

Problem 1. Given an autonomous smooth nonlinear system with state \( x \in \mathbb{R}^n \):

\[
\dot{x} = f(x) \tag{1}
\]

and a non-trivial \( T \)-periodic solution \( x^*(t) = x^*(t + T) \) of (1) such that \( f(x^*) \neq 0 \) for any \( t \in [0, T] \). Characterize the stability of \( x^*(\cdot) \) and if it is exponentially orbitally stable then compute an inner estimate of its region of stability. \( \Box \)

Problem 2. Consider an autonomous hybrid nonlinear system with state \( x \in \mathbb{R}^n \) and switching dynamics defined between hyperplanes:

\[
\dot{x} = f(x), \ x \notin S^- \tag{2}
\]

\[
x^+ = \Delta(x), \ x \in S^- \tag{3}
\]

Here \( f(\cdot) \) and \( \Delta(\cdot) \) are smooth functions with \( \Delta : S^- \to S^+ \) where

\[
S^- = \{ x : c_-^l x = d_-^l g(x) \geq 0 \}, \tag{4}
\]

\[
S^+ = \{ x : c_+^l x = d_+^l \}, \tag{5}
\]

\( c_-^l, c_+^l \in \mathbb{R}^n \) and \( d_-^l, d_+^l \in \mathbb{R} \). Suppose \( x^*(\cdot) \) is a non-trivial \( T \)-periodic solution that undergoes \( N \) impacts at times \( \{ t_1, t_2, ..., t_N \} + kT \) for integer \( k \). We will assume that the impacts are not “grazing”, i.e. \( c_-^l f(x^*(t_i)) \neq 0 \) and \( c_+^l f(x^*(t_i)) \neq 0 \) for all \( i \).

The problem statement is to characterize the stability of \( x^*(\cdot) \) and if it is exponentially orbitally stable compute an inner estimate of its region of stability. \( \Box \)

For simplicity of expression we will consider the problem with a single switching surface and a single set of continuous dynamics, however the extension to multiple switches and multiple continuous phases is trivial. It is also straightforward to have the dimension of the continuous system change between different phases.

Problem 3. Consider a controlled, possibly hybrid, system with a control input \( u \in \mathbb{R}^m \):

\[
\dot{x} = f(x, u), \ x \notin S^- \tag{6}
\]

\[
x^+ = \Delta(x), \ x \in S^- \tag{7}
\]

Suppose this system has a \( T \)-periodic solution \( x^*(\cdot) \) that undergoes \( N \) impacts per period at times \( \{ t_1, t_2, ..., t_N \} + kT \) for integer \( k \), and is generated by a piecewise continuous control signal, \( u^*(t) = u^*(t + T) \). If this solution is not exponentially orbitally stable then, if possible, construct a state-feedback exponentially orbitally stabilizing controller and compute an inner estimate of its region of orbital stability. \( \Box \)

3. REGIONS OF STABILITY FOR CONTINUOUS SYSTEMS

In this section we propose a solution to Problem 1.

The process we propose for finding regions of orbital stability is based on the construction of a smooth local change of coordinates \( x \to (x_\perp, \tau) \). At each point \( t \in [0, T] \) we define a hyperplane \( S(t) \), with \( S(0) = S(T) \), which is transversal to the solution \( \Gamma^* \), i.e. \( x^*(t) \notin S(t) \).

Given a point \( x \) nearby \( x^*(\cdot) \), the scalar \( \tau \in [0, T] \) represents which of these transversal surfaces \( S(\tau) \) the current state \( x \) inhabits; the vector \( x_\perp \in \mathbb{R}^{n-1} \) is the “transversal” state representing the location of \( x \) within the hyperplane \( S(\tau) \), with \( x_\perp = 0 \) implying that \( x = x^*(\tau) \). This is visualised in Figure 1.
3.1 Selection of a Set of Transversal Surfaces

Suppose we have a periodic orbit \( x^*(t) = x^*(t + T) \) with \( \dot{x}^*(t) \neq 0 \) \( \forall t \in [0, T] \). At each point \( x^*(\tau) \) of the target orbit we define a transversal surface \( S(\tau) \) in the following way:
\[
S(\tau) = \{ y \in \mathbb{R}^n : z(\tau)'(y - x^*(\tau)) = 0 \}
\]
where \( z(\tau) : [0, T) \to \mathbb{R}^n \) is a smooth periodic vector function to be chosen. We will enforce that \( z(\tau) \) has bounded derivative. In the literature on the use of transversal coordinates to prove local properties about periodic solutions, it is common to choose \( z(\tau) = f(x^*(\tau)) \) (Hale, 1980). That is, the transversal planes are orthogonal to the current motion of the system.

However, orthogonal transversal planes are often a bad choice when performing analysis on larger regions around the orbit, and allowing some freedom in \( z(\tau) \) is highly beneficial. The reason is, roughly speaking, that singularities will occur in the change of coordinates \( x \to (x_\perp, \tau) \) near sections of \( x^* \) with large curvature. This will be made more precise in Section 5.

The primary requirement is that the resulting \( S(\tau) \) are still transversal to the orbit, which is guaranteed if there is some \( \delta > 0 \) such that \( z(\tau)'f(x^*(\tau)) > \delta \) for all \( \tau \in [0, T] \). I.e., \( z(\tau) \) is never orthogonal to \( f(x^*(\tau)) \).

As a technical condition, we require that \( z(\tau) \) be Lipschitz on each continuous interval. For simplicity of derivations, and without loss of generality, we will further enforce that \( ||z(\tau)|| = 1 \) for all \( \tau \). If planes orthogonal to the system motion are desired, we can take \( z(\tau) = f(x^*(\tau))/||f(x^*(\tau))|| \).

3.2 Construction of a Moving Coordinate System

Having chosen a set of hyperplanes, defined by \( z(\tau) \), we now construct a smoothly \( \tau \)-varying coordinate system upon this subspace. For \( n = 2 \), construction of a basis is trivial: pick, e.g., \([-z_2, z_1]'\) as the basis vector for \( \Pi \). For \( n \geq 3 \), a constructive can be adapted from a method in Diliberto and Hufford (1956).

1. Choose a vector \( w \) such that \( w \) is not collinear with \( z(\tau) \) for any \( \tau \).
2. Choose a fixed orthonormal basis \( \eta_j, j = 1, 2, ..., n \) for \( \mathbb{R}^n \) with \( w \) as its first element.
3. For each \( \tau \in [0, T] \), define the plane containing both \( w \) and \( z(\tau) \), and the rotation matrix \( R(\tau) \) which takes \( w \) to be collinear with \( z(\tau) \) rotating in this plane.
4. Define \( \xi_j(\tau) = R(\tau)\eta_j \) then \( \xi_2(\tau), ..., \xi_n(\tau) \) form a basis for the transversal coordinates.

An explicit formula for \( \xi_j(\tau) \) in terms of \( \eta_j, \eta_z \) and \( z(\tau) \) is
\[
\xi_j(\tau) = \eta_j - \frac{\eta_j'z(\tau)'}{1 + \eta_z'z(\tau)'}(\eta_1 + z(\tau)), \quad j = 2, 3, ..., n. \quad (11)
\]

Note that since \( \eta_1 \) and \( z(\tau) \) are, by construction, unit vectors which are not collinear, \( \eta_1'z(\tau) < 1 \) for all \( \tau \) so (11) is well-defined.

Having this basis defined, we also construct the projection operator:
\[
\Pi(\tau) = \begin{bmatrix} 
\xi_2(\tau)' \\
\vdots \\
\xi_n(\tau)' 
\end{bmatrix}
\]
which defines the mapping \( x \to (x_\perp, \tau) \). That is, if \( x \in S(\tau) \) then 
\[ x_\perp = \Pi(\tau)(x - x^*(\tau)). \]
Note that a given \( x \in \mathbb{R}^n \) will in general be in more than one transversal plane, i.e. we can have \( x \in S(\tau_1) \) and \( x \in S(\tau_2) \) with \( \tau_1 \neq \tau_2 \), though this will not cause any problems for the proposed method.

3.3 Transverse Dynamics and Linearization

**Theorem 1.** The dynamics of the system in the new coordinates \((x_\perp, \tau)\) are given by:

\[
\begin{align*}
\dot{x}_\perp &= \dot{\tau} \left[ \frac{d}{d\tau} \Pi(\tau) \right] \Pi(\tau)' x_\perp + \Pi(\tau) f(x^*(\tau) + \Pi(\tau)' x_\perp) \\
-\Pi(\tau) f(x^*(\tau)) \dot{\tau}, \\
\dot{\tau} &= \frac{z(\tau)' f(x^*(\tau)) + \Pi(\tau)' x_\perp}{z(\tau)' f(x^*(\tau)) - \frac{dz(\tau)}{d\tau} \Pi(\tau)' x_\perp} \quad (12) \\
\end{align*}
\]

**Transverse Linearization** The transverse linearization is the following \((n-1)\)-dimensional \( T \)-periodic linear system:

\[
\hat{z} = A(t) z 
\]

where \( A(t) \) comes from differentiating (12) with respect to \( x_\perp \):

\[
A(t) = \left[ \frac{d}{d\tau} \Pi(t) \right] \Pi(\tau)' + \Pi(\tau) \frac{\partial f(x^*(\tau))}{\partial x} \Pi(\tau)' - \Pi(\tau) f(x^*(\tau)) \left. \frac{\partial \dot{\tau}}{\partial x_\perp} \right|_{x_\perp=0} \quad (15)
\]

where

\[
\frac{\partial \dot{\tau}}{\partial x_\perp} \bigg|_{x_\perp=0} = \frac{z(t)' \Pi(\tau)' + \frac{dz(\tau)}{d\tau} \Pi(\tau)' - \frac{\partial f(x^*(\tau))}{\partial x} \Pi(\tau)' - \Pi(\tau) f(x^*(\tau)) \left. \frac{\partial \dot{\tau}}{\partial x_\perp} \right|_{x_\perp=0}}{z(\tau)' f(x^*(\tau)) - \frac{dz(\tau)}{d\tau} \Pi(\tau)' x_\perp}. \quad (16)
\]

**Remark 1.** In two special cases some simplifications are possible: if the system is planar, i.e. \( n = 2 \), then \([\frac{d}{d\tau} \Pi(\tau)]\Pi(\tau)' = 0\) so (12) and (15) can be simplified by removing the first term from each. If transversal planes orthogonal to the system motion are chosen then \( \Pi(\tau)f(x^*(\tau)) = 0 \) so (12) and (15) can be simplified by removing the last term from each.

**Remark 2.** Note that in the selection of \( z(\tau) \) we have enforced that \( z(\tau)' f(x^*(\tau)) \geq \delta \) for all \( \tau \), so if \( x_\perp \) remains sufficiently small we have:

\[
z(\tau)' f(x^*(\tau)) - \frac{dz(\tau)}{d\tau} \Pi(\tau)' x_\perp \geq \alpha > 0
\]

for some \( \alpha \), so the dynamics of \( \tau \) are well-defined. This condition breaks if there is a continuum of \( \tau \) such that \( x^*(\tau) \) satisfies \( f(x^*(\tau))' y - x^*(\tau) = 0 \). E.g. for constant speed curves \( \langle |\dot{x}| = 0 \rangle \) means \( |y - x^*(\tau)| \) is exactly the radius of curvature of the the target orbit, i.e. \( 1/|\dot{x}| \). In Section 5 we will discuss optimizing \( z(\tau) \) so as to maximize the regions in which the dynamics of \( \tau \) are well-defined.

3.4 Verification of Orbital Stability

We now state conditions which guarantee regions of stability, giving a solution to Problem 1. It will then be shown how to optimize regions for polynomial systems via a SoS relaxation.

**Theorem 2.** Suppose there exists a function \( V : \mathbb{R}^{n-1} \to \mathbb{R} \) such that \( D := \{ (x_\perp, \tau) : V(x_\perp, \tau) \leq 1 \} \) is compact and for which following inequalities hold for all \((x_\perp, \tau) \in D\):

\[
V(x_\perp, \tau) > 0, \quad \frac{d}{dt} V(x_\perp, \tau) < 0, \quad x_\perp \neq 0 \quad (17)
\]

\[
V(0, \tau) = \frac{d}{dt} V(0, \tau) = 0, \quad (18)
\]

\[
z(\tau)' f(x^*(\tau)) - \frac{dz(\tau)}{d\tau} \Pi(\tau)' x_\perp > 0. \quad (19)
\]

then the \( D \) is an inner estimate for the region of orbital stability of \( x^*(\tau) \).

The optimization problem is then to search over Lyapunov functions \( V(x_\perp, \tau) \) so as to maximize the size of the regions \( D \) such that these conditions can be verified.

**Sums-of-Squares Relaxation**

We now show how the regions can be computed using a SoS relaxation. For the purposes of this section, we assume that the vector field \( f(x) \) is polynomial in \( x \). This being the case, let us fix a particular value of \( \tau \) and examine the formula for \( \dot{\tau} \):

\[
\dot{\tau} = \frac{z(\tau)' f(x^*(\tau)) + \Pi(\tau)' x_\perp}{z(\tau)' f(x^*(\tau)) - \frac{dz(\tau)}{d\tau} \Pi(\tau)' x_\perp} = \frac{n(x_\perp, \tau)}{d(x_\perp, \tau)}.
\]

It is clear that both the numerator and denominator, \( n(x_\perp, \tau) \) and \( d(x_\perp, \tau) \) respectively, are both polynomial in \( x_\perp \).

Furthermore, the well-posedness condition ensures that \( d(x_\perp, \tau) > 0 \), hence we can multiply both sides of condition \( \dot{V} \) by \( d(x_\perp, \tau) \) without changing its validity, resulting in the condition \( \dot{V}(x_\perp, \tau) \leq 0 \), where \( \dot{V}(x_\perp, \tau) \) is given by (20). It is straightforward to verify that this condition is also polynomial in \( x_\perp \).

We can verify these conditions regionally using Lagrange multipliers \( l(x_\perp) \) and \( m(x_\perp) \) that are polynomial in \( x_\perp \) with the following SoS constraints:

\[
-D V(x_\perp, \tau) - l(x_\perp)(1 - V(x_\perp, \tau)) = \text{SoS}, \quad (21)
\]

\[
d(\tau, x_\perp) - \delta l - m(x_\perp)(1 - V(x_\perp, \tau)) = \text{SoS}, \quad (22)
\]

\[
l(x_\perp) = \text{SoS}, \quad (23)
\]

\[
m(x_\perp) = \text{SoS}. \quad (24)
\]

In practice we sample a sufficiently fine finite sequence \( \tau_i, i = 1, 2, ..., N_t \) such that \( \tau_1 = 0, \tau_{N_t} = T. \) Then for each \( \tau_i \) and verify the conditions (21) – (24) for each fixed \( \tau_i \).

The objective is to maximize the regions satisfying (21) – (24). The decision parameters are the coefficients of \( V \) (within some restricted class) and the Lagrange multipliers \( l \) and \( m \) at each sample of \( \tau \). This optimization is bilinear in the decision parameters, however a reasonable approach which has proven successful is to iterate between optimizing over multipliers and optimizing over \( V \).

If \( V \) is a polynomial of higher order than quadratic, then what exactly should be optimized in the search for \( V \) is open to some choice, but a natural candidate is to maximize the size of some ball in \(|x_\perp| \) contained in the set.
\[ DV(x_{\perp}, \tau) := \frac{\partial}{\partial \tau} V(x_{\perp}, \tau) n(x_{\perp}, \tau) + \frac{\partial}{\partial x_{\perp}} V(x_{\perp}, \tau) \left[ n(x_{\perp}, \tau) \tau d \tau \Pi(\tau) \right] \]
\[ + d(x_{\perp}, \tau) \Pi(\tau) f(x^{\star}(\tau) + \Pi(\tau)^{\prime} x_{\perp}) \]

for some sequence of scalars \( \alpha_i \neq 0 \). That is, \( z(t) \) aligns with the normals to the switching surfaces, and always makes a “small angle” with \( f(x^{\star}) \). The non-grazing condition on impact surfaces ensures that many such vector functions exist.

With the coordinate system defined to line up with the switching surfaces, an entire region of the form \( V(x_{\perp}, \tau) \leq 1 \) for a time \( t_i \) will map into the image of \( S_i \) under \( \Delta_i \).

The switching update of the transversal coordinates is:
\[ x_{\perp}^{\tau} = \Pi(\tau_{i}^{+}) \Delta_i \left( x^{\star}(\tau_{i}^{-}) + \Pi(\tau_{i}^{-}) x_{\perp}^{-} \right) - x^{\star}(\tau_{i}^{+}) \]
and the transverse linearization of the impact map is:
\[ A_{\Pi} = \Pi(\tau_{i}^{+}) \frac{\partial}{\partial x} \Pi(\tau_{i}^{-}) \]
evaluated at \( x = x^{\star}(\tau_{i}^{-}) \).

4. HYBRID SYSTEMS

In this section, we extend the above results to the case of hybrid systems, and propose an algorithm for solving Problem 2.

Suppose we find a vector function \( z(t) \in \mathbb{R}^n \) for \( t \in [0, T) \) such that
\[ z(t) f(x^{\star}(t)) > 0, \forall \ t \in [0, T) \]
\[ z(t_i) = \alpha_i c_i, \ \forall \ i = 1, 2, \ldots, N, \]

4.1 Stability Conditions

For systems with impacts, conditions for the continuous phases will be (17), (18), (19) as well as
\[ \left( c^{\prime}_{-} (x^{\star}(\tau_{i}^{-}) + \Pi(\tau_{i}^{-}) x_{\perp}^{-}) - d_{-} \right) l_{s}(x_{\perp}, \tau) \]
\[ + g(x^{\star}(\tau_{i}^{-}) + \Pi(\tau_{i}^{-}) x_{\perp}^{-}) < 0. \]

Here \( l_{s}(x_{\perp}, \tau) \) is a Lagrange multiplier which is not constrained to be positive. This constraint ensures that the switching surface is not hit before it is expected. For some systems, it may be obvious that this will not happen, and the third condition could be dropped. At the switching times the conditions to be verified are:
\[ V \left( \Pi(\tau_{i}^{+}) \Delta_i \left( x^{\star}(\tau_{i}^{-}) + \Pi(\tau_{i}^{-}) x_{\perp}^{-} \right) \right) \]
\[ - x^{\star}(\tau_{i}^{+}) \right] \tau_{i}^{+} \right) - V(x_{\perp}, \tau_{i}) \leq 0, \]
\[ g(x^{\star}(\tau_{i}^{-}) + \Pi(\tau_{i}^{-}) x_{\perp}^{-}) > 0. \]

The first verifies the stability, the second that the state is within the region of definition of the impact on the switching hyperplane. All constraints evaluated over the regions \( V(x_{\perp}, \tau) \leq 1 \) via Lagrange multipliers.

4.2 Initial Candidate Lyapunov Function

A natural candidate Lyapunov function for the hybrid case is the unique SPPD solution of the jump-Lyapunov function:
\[ -\dot{P} = A'P + PA + Q, \ t \neq t_i \]
\[ P(\tau_i^-) = A_d(\tau_i)'P(\tau_i^+)A_d(\tau_i) + Q_i, \ t = t_i \]
A similar statement to Theorem 3 can be made applying the results of De Nicolao and Strada (1998) and Bainov and Simeonov (1989).

5. OPTIMIZATION OF TRANSVERSAL SURFACES

The method described above allows great flexibility in the choice of transversal surfaces, parameterized by their normal vectors \( z(\tau) \). In some cases, problem specific information might suggest a natural candidate (see examples). In others, the classical choice \( z(\tau) = f(x^*(\tau))/\|f(x^*(\tau))\| \) may be sufficient. However, it is likely that for many practical examples some optimization of \( z(\tau) \) is appropriate.

The size of the smallest \( x_\perp \) at which well-posedness can break is given by

\[ |x_\perp(\tau)| = \left| \frac{z(\tau)'f(x^*(\tau))}{\partial z(\tau)/\partial \tau} \right| \]

which should be maximized in some sense. However, notice that the denominator has \( \partial z(\tau)/\partial \tau \) as a factor. The optimum may have \( z(\tau) \) constant for some intervals of \( \tau \) which means \( |x_\perp(\tau)| \) would go to infinity, which may make an optimization difficult. It is therefore better-posed to minimize its inverse. E.g. one can minimize

\[ z(\tau)^* = \arg \min \left( \int_0^T \frac{\partial z(\tau)/\partial \tau |^p}{|z(\tau)'f(x^*(\tau))| |^p} d\tau \right)^{1/p} \]  
\[ \text{s.t. } z(\tau)'f(x^*(\tau)) > \delta \]
\[ |z(\tau)| = 1 \quad \forall \tau \in [0, T], \]

for some small positive \( \delta \). For systems with impacts, the integrand in the above optimization can be multiplied by a smooth shaping function \( \phi(\tau) \) with \( \phi(\tau_i) = 0 \), since the transversal surfaces must be aligned with the switching surfaces and cannot be optimized.

This is a nonconvex optimization, so it requires a good initial seed. A natural candidate optimization is \( z(\tau) = f(x^*(\tau))/\|f(x^*(\tau))\| \). This initial guess always satisfies the transversality conditions, but may not always satisfy the condition of alignment to switching surfaces.

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