The Gauged (2,1) Heterotic Sigma-Model

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**Abstract**

The geometry of (2,1) supersymmetric sigma-models with isometry symmetries is discussed. The gauging of such symmetries in superspace is then studied. We find that the coupling to the (2,1) Yang-Mills supermultiplet can be achieved provided certain geometric conditions are satisfied. We construct the general gauged action, using an auxiliary vector to generate the full non-polynomial structure.
1 Introduction

It has been proposed by Kutasov and Martinec \cite{1} that the heterotic strings with (2,1) world-sheet supersymmetry provide an appropriate framework for implementing and extending an earlier idea due to Green \cite{2}, who suggested that the world-sheet theories of various string theories are obtainable from the target space theories of two-dimensional strings. Furthermore, the authors of \cite{1} showed that different target vacua for (2,1) heterotic strings correspond to the type IIB string, to the membrane of M theory and to their compactifications.

The $N = 2$ strings, and especially the (2,1) heterotic strings, have a number of remarkable features \cite{3} whose consideration within the framework of string and M theory duality lead to the above picture. First, their spectrum is simple in that it contains only a finite number of massless modes (there is no tower of massive modes as in other critical string theories), and their interactions are likewise simple, as all $n$-point scattering amplitudes vanish for $n \geq 4$. Although their critical dimension is four, supersymmetry implies that the signature of the target manifold is either $4 + 0$ or $2 + 2$, the latter case being relevant to the heterotic theory. Furthermore, the $N = 2$ superconformal algebra contains a $U(1)$ current whose left-moving component must be gauged in the (2,1) heterotic version of the theory \cite{3}. This in turn necessitates the introduction of a new set of ghosts which raises the critical dimension in the left-moving sector by 2; hence the theory, when embedded in ten-dimensional spacetime, contains a left-moving internal $N = 1$ SCFT with $\hat{c} = 8$. In order to ensure the absence of BRST anomalies one must require the left-moving $U(1)$ current to be of the form $v \cdot \partial X$ where $v$ is a null Killing vector, $v^2 = 0$. If the component of this vector along the internal directions vanishes, the string theory lives effectively in $1 + 1$ dimensions, and one can recover the bosonic, type II and heterotic world-sheet theories in a physical gauge for different choices of the internal $\hat{c} = 8$ SCFT \cite{1}. On the other hand, if the vector $v$ lies partly in the internal sector, one obtains an effective $2 + 1$ theory that corresponds to the world-volume theory of the supermembrane of eleven-dimensional supergravity \cite{1}. It was further suggested that a similar construction could yield other $p$-branes and that the (2,1) heterotic string is the unifying structure that underlies all M theory vacua \cite{1}. Also, consideration of the scattering amplitudes of the (2,1) heterotic string has led to the construction of the exact classical action for the target field theory (see refs. \cite{1, 3}). This describes the dynamics of a set of self-dual Yang-Mills fields coupled to self-dual gravity \cite{3}.

In this context, it is important to study the geometry of the gauged two-dimensional (2,1) supersymmetric sigma-models, which describe the propagation of (2,1) heterotic strings on certain hermitian manifolds (with torsion) that admit isometry symmetries. The information obtained concerning the geometry and quantum dynamics of the sigma-model should be useful in further studies of the target field theory of the (2,1) heterotic string.

The general two-dimensional non-linear sigma-model with two right-moving and one left-moving supersymmetries was considered in \cite{10} (following ref. \cite{6}) and was formulated in a superspace with (1,1) supersymmetry manifest. The geometric
conditions the model must satisfy in order to possess a second right-moving supersymmetry and to be finite at one loop were found \[10\]. A formulation of the model in (2,1) extended superspace was given in \[11\], while an alternative extended superspace formulation was proposed by Howe and Papadopoulos \[16\]. In this ‘HP-formalism’, the action is an integral with the usual (1,1) superspace measure \(\int d^2\sigma d^2\theta\) of a Lagrangian constructed from (2,1) superfields. Checking that the action is independent of the extra \(\theta\) is necessary to show (2,1) supersymmetry.

The gauging of isometry symmetries of the (2,1) sigma-model in the HP formalism was achieved in ref. \[15\]. However, only (1,1) supersymmetry was manifest in this approach and for many purposes, such as the coupling to supergravity, an approach based on a conventional superspace formalism is more convenient. The purpose of this paper is to construct an alternative formulation of the gauged (2,1) sigma-model with torsion. The problem is that the action has a complicated non-polynomial dependence on the scalar gauge prepotentials which is hard to find directly. There is a similar non-polynomial structure in the \(N = 1\) sigma-model in four dimensions and in the (2,2) sigma-model in two dimensions. There the ungauged Lagrangian is given in terms of a Kähler potential and the isometry in general changes this by a Kähler gauge transformation, \(K(z, \bar{z}) \rightarrow K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})\). In ref. \[8\], an extra superfield coordinate was introduced to construct a higher-dimensional target space in which the isometry became a conventional symmetry of the Lagrangian that could be gauged by minimal coupling. Eliminating the extra superfield coordinate then generated the non-polynomial structure of the gauged action. This method was later used in ref. \[13\] to construct the gauged (2,2) sigma-model with torsion in two dimensions, and our purpose here is to develop it further to derive the full non-polynomial structure of the gauged (2,1) supersymmetric sigma-model. This generalises the results of our previous paper \[9\] in which the gauging was achieved for a special class of (2,1) target space geometries with isometries.

The paper is organised as follows. In section 2 we review the (1,1) supersymmetric non-linear sigma-model with Wess-Zumino term and the geometric conditions for the model to be invariant under an extra chiral supersymmetry. This introduces some notation and conventions which we use to formulate the (2,1) supersymmetric model in (2,1) superspace, following \[6, 10, 11\]. We then review the analysis of the conditions under which the (1,1) and the (2,1) supersymmetric models have isometry symmetries and introduce certain potentials which will play a central role in the gauging of these models in section 3. A more detailed discussion was recently given in \[9\], and we refer the reader to that paper for details of the construction. We begin our discussion of the gauging in section 4 by recalling some of the results of \[14\], where the (1,1) and (2,1) supersymmetric models were coupled to the (1,1) and (2,1) Yang-Mills supermultiplets using the approach of \[14\]. In this approach, only (1,1) supersymmetry is manifest. As explained above, this has a number of disadvantages, which leads us to seek a new form of the gauged action for the (2,1) heterotic model with manifest (2,1) supersymmetry. In section 5 we present the (2,1) Yang-Mills supermultiplet and discuss the transformation properties of the scalar and gauge multiplets under the gauge and the isometry groups. In sections 6, 7 and 8 we give an alternative formulation of the gauged (2,1) model, utilizing and
adapting the approach of [8]. The gauged superspace action is constructed first for a special class of models in section 6. The generic models are then considered in sections 7 and 8, and a new gauged superspace action is found. We conclude in section 9, where we briefly mention some possible applications of our work in the context of the recent developments referred to above.

2 The (2,1) Heterotic Sigma-Model

Consider first the general non-linear sigma-model with (1,1) supersymmetry with a Wess-Zumino term [6, 12]. The (1,1) superspace is parametrised by two Bose coordinates $\sigma^\dagger, \sigma^-$ and two Fermi coordinates $\theta_{1\pm}$ of opposite chirality. The (1,1) superspace action for this model is [7]

$$S_{(1,1)} = \frac{1}{4!} \int d^2\sigma d^2\theta [g_{ij}(\phi) + b_{ij}(\phi)] D_{1+} \phi^i D_{1-} \phi^j,$$

(1)

where the $\phi^i, i = 1 \ldots d$, can be viewed as coordinates on some $d$-dimensional manifold $M$ with metric $g_{ij}$ and torsion $H$ given by the curl of the antisymmetric tensor $b_{ij}$,

$$H_{ijk} = \frac{1}{2} (b_{ij,k} + b_{jk,i} + b_{ki,j});$$

(2)

here

$$D_{1+} = \frac{\partial}{\partial \theta_{1+}} + i \theta_{1+} \frac{\partial}{\partial \sigma^\dagger}, \quad D_{1-} = \frac{\partial}{\partial \theta_{1-}} + i \theta_{1-} \frac{\partial}{\partial \sigma^-}.$$  

(3)

The action (1) is manifestly invariant under (1,1) supersymmetry, target space general coordinate transformations and antisymmetric tensor gauge transformations $\delta b_{ij} = \partial_i \lambda_j$. Furthermore, it was shown in [3, 7, 8] that (1) will be invariant under an extra chiral supersymmetry

$$\delta \phi^i = J^i_j(\phi) \varepsilon_- D_{1+} \phi^j$$

(4)

and so have (2,1) supersymmetry provided that $d$ is even and that (i) $J^i_j$ is a complex structure satisfying

$$J^i_j J^j_k = -\delta^i_k$$

$$N^k_{ij} \equiv J^l_i J^k_{[j,l]} - J^l_j J^k_{[i,l]} = 0$$

(5)

(ii) $J^i_j$ is covariantly constant

$$\nabla_i J^j_k \equiv J^j_{k,i} + \Gamma^j_{il} J^l_k - \Gamma^j_{ik} J^l_j = 0$$

(6)

with respect to the connection

$$\Gamma^i_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + g^{jl} H_{jkl}$$

(7)
which differs from the usual Christoffel connection by the (gauge invariant) totally antisymmetric torsion \( T \) and (iii) the metric \( g_{ij} \) is hermitian with respect to the complex structure,
\[
g_{ij} J^j_k J^i_l = g_{kl}.
\]

In a complex coordinate system \( z^\alpha, \bar{z}^\beta = (z^\beta)^* \), \( \alpha, \beta = 1, \ldots, 2d \) in which the line element is \( ds^2 = 2g_{\alpha\beta} dz^\alpha d\bar{z}^\beta \) and the complex structure is constant and diagonal,
\[
J^i_j = i \begin{pmatrix} \delta^\beta_\alpha & 0 \\ 0 & -\delta^\beta_\bar{\alpha} \end{pmatrix},
\]
these conditions imply that the torsion is given by
\[
H_{\alpha\beta\gamma} = \frac{1}{2} (g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}), \quad H_{\alpha\beta\gamma} = 0
\]
and that the metric satisfies
\[
g_{\alpha[\beta,\gamma]} - g_{\beta[\gamma,\alpha]} = 0.
\]

Then the geometry is determined locally by some vector field \( k_\alpha(z, \bar{z}) \),
\[
\begin{align*}
g_{\alpha\beta} &= \partial_\alpha \bar{\kappa}_\beta + \partial_\beta \kappa_\alpha \\
b_{\alpha\beta} &= \partial_\alpha \bar{\kappa}_\beta - \partial_\beta \kappa_\alpha \\
H_{\alpha\beta\gamma} &= \frac{1}{2} \partial_\gamma (\partial_\alpha k_\beta - \partial_\beta k_\alpha).
\end{align*}
\]

If the torsion \( H = 0 \), the manifold \( M \) is Kähler with \( k_\alpha = \frac{\partial}{\partial z^\alpha} K(z, \bar{z}) \) where \( K(z, \bar{z}) \) is the Kähler potential, and the (2,1) supersymmetric model in fact has (2,2) supersymmetry, while for \( H \neq 0 \), \( M \) is a hermitian manifold with torsion of the type introduced in [6, 7].

We now wish to formulate the (2,1) model directly in the (2,1) superspace. The latter is parametrised by \( \sigma^+, \sigma^- \), two real Fermi coordinates of the same chirality \( \theta_{1+} \) and \( \theta_{2+} \), and a single real Fermi coordinate \( \theta_- \) of the opposite chirality. \( \theta_{1+} \) and \( \theta_{2+} \) can be combined into a complex coordinate \( \theta_+ = (\theta_{1+} + i\theta_{2+})/\sqrt{2} \) and its complex conjugate \( \bar{\theta}_+ = (\theta_{1+} - i\theta_{2+})/\sqrt{2} \), and it is natural to define the complex conjugate supercovariant derivatives
\[
D_+ = \frac{1}{\sqrt{2}} (D_{1+} + iD_{2+}), \quad \bar{D}_+ = \frac{1}{\sqrt{2}} (D_{1+} - iD_{2+})
\]
with \( D_{1+} \) and \( D_{2+} \) as in [3]. The supersymmetric sigma model can then be formulated in (2,1) superspace in terms of scalar superfields \( \emptyset^\alpha \) which are constrained to satisfy the chirality conditions
\[
\bar{D}_+ \emptyset^\alpha = 0, \quad D_+ \bar{\emptyset}^\alpha = 0.
\]

\(^1\)We will use the notation \( \emptyset \) for (2,1) scalar superfields to distinguish them from the (1,1) scalar superfields \( \phi \).
The lowest components of the superfields $\phi^\alpha|_{\theta=0} = z^\alpha$ are the bosonic complex coordinates of the space-time. The most general renormalizable and Lorentz invariant (2,1) superspace action is then

$$S = i \int d^2\sigma d\theta_+ d\overline{\theta}_- \left( k_\alpha D_- \varphi^\alpha - \overline{\kappa}_\bar{\alpha} D_- \varphi^{\bar{\alpha}} \right),$$

(15)

and will be gauged in sections 3, 6 and 7. At this stage, we simply note that the relations (12) which determine the geometry in terms of the vector field $k_\alpha$ can be recovered from (15) by performing the integration over $\theta^2_2$ in the usual way and identifying the resulting (1,1) superspace action with (1), where $g_{ij}$ and $b_{ij}$ are given by eq. (12). Also, notice that the geometry is left invariant by the following transformation

$$\delta k_\alpha = \rho_\alpha$$

(16)

provided $\rho_\alpha$ satisfies

$$\overline{\partial}_\bar{\beta} \rho_\alpha = i \partial_\alpha \overline{\partial}_\bar{\beta} \chi$$

(17)

for some arbitrary real $\chi$. This implies that $\rho$ is of the form

$$\rho_\alpha = i \partial_\alpha \chi + f_\alpha,$$

$$\overline{\partial}_\bar{\beta} f_\alpha = 0$$

(18)

for some holomorphic $f_\alpha$. The symmetry (16) turns out to be the analog of the generalised Kähler gauge transformation discussed in [7]. It leaves the metric and torsion invariant, but changes $b_{ij}$ by an antisymmetric tensor gauge transformation of the form $\delta b_{ij} = \partial_i \lambda_{\bar{j}}$.

3 Rigid Symmetries

We now consider the isometry symmetries of the target geometry. Let $G$ be a continuous subgroup of the diffeomorphism group of $M$. The action of $G$ on $M$ is generated by vector fields $\xi^i_a (a = 1, \ldots, \dim G)$ which satisfy the Lie bracket algebra

$$[\xi^i_a, \xi^j_b]^i \equiv \xi^i_a \partial_j \xi^j_b - \xi^j_b \partial_j \xi^i_a \equiv \mathcal{L}_a \xi^i_b = f^{ci}_{ab} \xi^c_i,$$

(19)

where $\mathcal{L}_a$ denotes the Lie derivative with respect to $\xi_a$ and $f^{ci}_{ab}$ are the structure constants of the group $G$. The infinitesimal transformations of the (1,1) sigma-model superfields

$$\delta \varphi^i = \lambda^a \xi^i_a (\phi)$$

(20)

with constant parameters $\lambda^a$ induce a change in the (1,1) supersymmetric action (1) which can be cancelled by the following compensating transformations of the metric and torsion:

$$\delta g_{ij} = \lambda^a (\mathcal{L}_a g)_{ij}, \quad \delta H_{ijk} = \lambda^a (\mathcal{L}_a H)_{ijk}.$$  

(21)

Although this is not a conventional Noether symmetry in general (as the infinite number of coupling constants encoded in $g_{ij}$ and $H_{ijk}$ transform under (20)), the
action of $G$ generates a group of proper symmetries of the sigma-model field equations if \((21)\) vanishes, i.e. if the Lie derivatives with respect to the vector fields $\xi^i_a$ of the metric and torsion vanish,
\[
(\mathcal{L}_a g)_{ij} = 0, \quad (\mathcal{L}_a H)_{ijk} = 0. \tag{22}
\]
This requires that the $\xi^i_a$ are Killing vectors of the metric $g$,
\[
\nabla_{(i} \xi_{j)a} = 0, \tag{23}
\]
so that $G$ is a group of isometries of $M$, and that $\xi^i_a H_{ijk}$ is curl-free, so that there is a locally defined one-form $u_a$ such that \([12]\)
\[
\xi^i_a H_{ijk} = \partial_{[j} u_{k]a}. \tag{24}
\]
For the transformations \((20)\) to define a symmetry of the sigma-model action, it is necessary in addition for $u$ to be globally defined. The one-forms $u_a$ are only defined up to the addition of an exact piece:
\[
u_{ia} \to u_{ia} + \partial_i \alpha_a. \tag{25}
\]
Taking the Lie derivative of \((24)\), we find that
\[
D_{(ba} = \mathcal{L}_b u_{ai} - f^{c}_{ba} u_{ic}. \tag{26}
\]
is a closed one-form. If it is exact, it is often possible to use the ambiguity \((25)\) in the definition of $u$ to choose it to be equivariant, i.e. to choose it so that it transforms covariantly,
\[
\mathcal{L}_a u_{bi} = f^{c}_{ab} u_{ic}. \tag{27}
\]
However, in general there can be obstructions to choosing an equivariant $u$ which have an interpretation in terms of equivariant cohomology \([12, 13]\). It will be useful to define \([12, 13]\)
\[
c_{ab} = \xi^i_a u_{bi}. \tag{28}
\]
While the conditions \((22)\) are sufficient for the isometry generated by the Killing vector $\xi_a$ to be a symmetry of the (1,1) supersymmetric model, a further condition is necessary in order for it to be compatible with (2,1) supersymmetry. Under the infinitesimal transformations of the (2,1) sigma model superfields
\[
\delta \varphi^i = \lambda^a \xi^i_a (\varphi), \tag{29}
\]
one finds that the complex structure undergoes the compensating transformation \([14]\)
\[
\delta J^i_{\ j} = \lambda^a (\mathcal{L}_a J)^i_{\ j}. \tag{30}
\]
It follows that the necessary conditions for the isometry to constitute a proper Noether symmetry is that the metric, torsion and complex structure are invariant under the diffeomorphisms generated by $\xi_a$, i.e. that \((22)\) and
\[
(\mathcal{L}_a J)^i_{\ j} = 0 \tag{31}
\]
are satisfied. Then the $\xi_a$ are Killing vectors which are holomorphic with respect to $J$, so that

$$\partial_\alpha \xi^\alpha_a = 0.$$  \hfill (32)

If the torsion vanishes, then $M$ is Kähler and for every holomorphic Killing vector $\xi^i_a$, the one-form with components $J_{ij} \xi^i_a$ is closed so that locally there are functions $X_a$ such that $J_{ij} \xi^i_a = \partial_\alpha X_a$; these are the Killing potentials which play a central role in the gauging of the supersymmetric sigma-models without torsion [17, 8]. In complex coordinates, this becomes $\xi_\alpha = -\partial_\alpha X_a$. When the torsion does not vanish, this generalises straightforwardly [15]: if $\xi^i_a$ is a holomorphic Killing vector field satisfying (24) and (31), then the one-form with components $\omega_\beta \equiv J_{ij} (\xi^i_a + u^j_a)$ satisfies $\partial_\alpha w_\beta = 0$, so that there are generalised complex Killing potentials $Z_a \equiv Y_a + iX_a$ such that

$$\xi_\alpha + u_{\alpha a} = \partial_\alpha Y_a + i\partial_\alpha X_a.$$  \hfill (33)

The $X_a$ and $Y_a$ are locally defined functions on $M$ and they are determined up to the addition of constants. Their role in the construction of gauge invariant actions for the (2,1) model will become apparent in the following sections. Note that the transformation $u_{\alpha a} \rightarrow u_{\alpha a} + \partial_\alpha \alpha_a$ leaves (33) invariant provided $Y_a$ also transforms as $Y_a \rightarrow Y_a + \alpha_a$. It will be useful to absorb $Y$ into $u$, defining

$$u'_{\alpha a} = u_{\alpha a} - \partial_\alpha Y_a,$$  \hfill (34)

so that $\xi_\alpha + u'_{\alpha a} = i\partial_\alpha X_a$, as in [15]; we henceforth drop the prime on $u$.

Under the rigid symmetries (29), the variation of the Lagrangian in (15) is

$$\delta L = i\lambda^a \left( \mathcal{L}_a k_\alpha D_- \varphi^a - \mathcal{L}_a \overline{\varphi} D_+ \varphi^a \right),$$  \hfill (35)

where the Lie derivative of $k_\alpha$ is

$$\mathcal{L}_a k_\alpha = \xi_\beta^a \partial_\beta k_\alpha + \overline{\xi}_\gamma^a \partial_\gamma k_\alpha + k_\beta \partial_\alpha \xi^\beta_a.$$  \hfill (36)

In general the symmetries (29) will not leave the action (15) invariant; they will leave it invariant only up to a gauge transformation of the form (16). This requires that

$$\mathcal{L}_a k_\alpha = i\partial_\alpha X_a + \vartheta_{\alpha a}$$  \hfill (37)

for some real functions $\chi_a$ and holomorphic one-forms $\vartheta_{\alpha a}$, $\partial_\gamma \vartheta_{\alpha a} = 0$.

In ref. [2], explicit forms for $\chi$ and $\vartheta$ were found using the geometric relations reviewed in the foregoing. Several other results concerning the relation of the isometry subgroup $G$ of $M$ to its geometry were also obtained. We shall now summarise these results, but the reader is referred to [1] for the derivations.

The explicit forms for $\chi$ and $\vartheta$ are found from (23), (24), (32), (33), and (12) to be

$$\chi_a = X_a + i \left( \overline{\xi}_\alpha^a k_\alpha - \xi_\alpha^a k_\beta \right)$$  \hfill (38)

and

$$\vartheta_{\alpha a} = 2\xi_\gamma^a \partial_\gamma k_{\alpha a} + \xi_{\alpha a} - i\partial_\alpha X_a.$$  \hfill (39)
Substituting (38) and (39) into (37), one finds the Lie derivative (36). The holo-
morphy of $\vartheta_{\alpha a}$ in (39) can be checked using (32) and (33). Note that the ambiguity $X_a \rightarrow X_a + C_a$ in the definition of $X_a$ (for some constant function $C_a$) does not affect $\vartheta$. On the other hand, $\vartheta$ should transform as $\delta \vartheta_{\alpha a} = L_a f_{\alpha}$ under the transformations (16), (18) and the form (39) indeed transforms in this way.

We also find the following expression for $u_{\alpha a}$:

$$u_{\alpha a} = 2\xi^\gamma_a \partial_\gamma k_a - \vartheta_{\alpha a}$$

and this satisfies (24).

The one-form $D_{ab}$ defined by (26) is closed, implying the local existence of a real potential $E_{ba}$ such that

$$D_{ba\alpha} = i\partial_\alpha E_{ba}$$

(note that (41) does not imply that $D_{ba}$ is exact). The potential $E_{ba}$ is only defined up to the addition of real constants, and is determined by the imaginary part of the generalised Killing potential. This is seen by taking the Lie derivative of eq. (33) and integrating, which yields

$$E_{ba} = L_b X_a - f_{ba}^c X_c + e_{ba}$$

where the $e_{ba}$ are real constants which we henceforth absorb into the definition of $E_{ba}$.

Using the relations (12), (32) and (33), the Lie derivative (36) of $k_\alpha$ can be written as in (37) where $\vartheta$ and $\chi$ are given by the forms (39) and (38). Further information into the relation of the isometry subgroup $G$ of $M$ to its geometry subgroup was obtained in ref. [9] by deriving the action of the Lie bracket algebra on $k_\alpha$,

$$[\mathcal{L}_a, \mathcal{L}_b] k_\alpha = f_{ab}^\gamma \mathcal{L}_\gamma k_\alpha .$$

First, the Lie derivatives of the potentials $\chi$ and $\vartheta$ satisfy

$$\mathcal{L}_b \chi_a - \mathcal{L}_a \chi_b - f_{ba}^c \chi_c = \mathcal{L}_b X_a - f_{ba}^c X_c + i \left( \xi_a^\gamma \vartheta_{\gamma b} - \xi_b^\gamma \vartheta_{\gamma a} \right)$$

and

$$\mathcal{L}_b \vartheta_{\alpha a} - \mathcal{L}_a \vartheta_{\alpha b} = f_{ba}^c \vartheta_{\alpha c} - \partial_\alpha (\xi_a^\gamma \vartheta_{\gamma b} + i E_{ba}) .$$

Moreover, eq. (37) implies the relation

$$[\mathcal{L}_b, \mathcal{L}_a] k_\alpha = f_{ba}^\gamma \mathcal{L}_\gamma k_\alpha + i \partial_\alpha (\mathcal{L}_b \chi_a - \mathcal{L}_a \chi_b - f_{ba}^c \chi_c) + (\mathcal{L}_b \vartheta_{\alpha a} - \mathcal{L}_a \vartheta_{\alpha b} - f_{ba}^c \vartheta_{\alpha c})$$

Then it was seen that the algebras (14) and (15), together with eqs. (11) and (12), imply that the sum of the last two terms on the right hand side of (14) explicitly cancels, so that (16) indeed reduces to (13).

Another important consequence of (15) follows from symmetrization with respect to group indices: this implies that the quantities

$$\hat{d}_{(ab)} \equiv \xi^\alpha_{(a} \vartheta_{\gamma b)} + i E_{(ba)}$$

(47)
are antiholomorphic functions $\hat{d}_{(ab)} = \hat{d}_{(ab)}(z)$. Then, defining $-c_{(ab)}$ as twice the real part of $\hat{d}_{(ab)}$, one finds

$$-c_{(ab)} \equiv \hat{d}_{(ab)} + \overline{\hat{d}_{(ab)}} = \xi^i(\alpha \theta_{ib}).$$

(48)

Contracting (39) with $\xi^a_\alpha$ (noting (34)) and using the relation (40), we find

$$\xi^a_\alpha \theta_{ab} = 2\xi^a_\alpha \beta \partial_{[\beta k_{a\alpha]}} - \xi^a_\alpha u_{ab}.$$  

(49)

Then, symmetrization with respect to group indices yields

$$\xi^a_\alpha \theta_{ab} = -\xi^a_\alpha u_{[a|b]}$$  

(50)

so that (48) can be rewritten as

$$c_{(ab)} = \xi^i(\alpha u_{ib}),$$

(51)

which is precisely the definition of the real constants $c_{(ab)}$ given in [12, 15], where it was shown that their vanishing is a necessary condition for the gauging of the sigma model to be possible.

The equivariance condition on the imaginary part of the generalised Killing potential,

$$\mathcal{L}_b X_a = f^c_{ba} X_c,$$

(52)

was found in [12, 15] to be another necessary condition for the gauging of the isometries generated by the $\xi^a_\alpha$ to be possible. If (52) holds, then it follows from (12) that the potential $E_{ba}$ defined in (41) is a constant and can be chosen to vanish,

$$E_{ba} = 0,$$

(53)

and that eqns. (26), (44) and (45) simplify. The equations (26), (41) then imply that $u$ is equivariant.

Summarizing, the action of a group $G$ generated by the vector fields $\xi^a_\alpha$ as in (20) is a symmetry provided the $\xi^a_\alpha$ are holomorphic Killing vectors, i.e. eqs. (32) and (23) hold, so that the metric and complex structure are invariant, and in addition the torsion is invariant, i.e. eqs. (24) and (31) hold. In general, the isometry symmetries will not leave the potential $k_{\alpha}$ invariant, but will change it by a gauge transformation of the form (16), so that the action (15) is unchanged. The geometry and Killing potentials then determine the quantity $\mathcal{L}_a k_{\alpha}$ appearing in the gauge transformation to take the form (37) with $\chi$, $\vartheta$ as in (38) and (39). The potentials $\chi$ and $\vartheta$ satisfy (44) and (45). Using the latter, it is found that the action of the Lie bracket algebra on the vector potential $k_{\alpha}$ reduces to (13), as it must. Also, the quantities $c_{(ab)}$ defined in (48) are real constants equal to those defined in ref. [15]. When the imaginary part of the generalised Killing potential is chosen to be equivariant, i.e. when (52) holds, it is found that the potential $E_{ba}$ defined in (41) vanishes. Then the one-forms $u_{a}$ defined in (34) are equivariant and the geometry simplifies. We

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2This definition corrects a sign mistake in eq. (54) of ref. [1].
note the result of ref. [13], where it was shown that the equivariance condition (52) on the imaginary part of the generalised Killing potential must hold in order for the gauging of the supersymmetric sigma model to be possible.

The discussion given here also applies to the geometry and isometries of the target space of (2,0) heterotic strings. The corresponding formulae can be obtained from those given in the foregoing by appropriate truncation of the (2,1) superfields.

4 Gauging the Isometries

For any supersymmetric sigma-model with a rigid isometry of the type discussed in the previous section, one can attempt to promote the rigid symmetry to a local one by coupling to the appropriate super Yang-Mills multiplet. As a warm-up, in this section we shall briefly review the gauging of the (1,1) model and that of the (2,1) model in the HP formalism as presented in [13]. In the following sections we consider the gauging of the isometries of the (2,1) model in conventional (2,1) superspace formalism (adapted from [8]) in which extended supersymmetry is manifest. The new gauged actions will be given in section 8.

Let us consider first the (1,1) sigma-model with superspace action (1) and no extra supersymmetry. The (1,1) Yang-Mills supermultiplet \(A_{(1,1)} = (A^a_+, A^a_-, A^a_+^+, A^a_-^-)\) can be used to define supercovariant derivatives \(\nabla\), \(\nabla_+\), \(\nabla_\pm\) and \(\nabla_-\):

\[
\nabla_\mu \phi^i = \partial_\mu \phi^i + A^a_\mu \xi_a^i, \quad \mu = (+, =)
\]

\[
\nabla_\pm \phi^i = D_\pm \phi^i + A^a_\pm \xi_a^i,
\]

and these are required to satisfy the super-commutation relations

\[
\begin{align*}
[\nabla_+, \nabla_+] &= 2i\nabla_+ \\
[\nabla_+, \nabla_-] &= W \\
[\nabla_-, \nabla_+] &= -i\nabla_+ W \\
[\nabla_+, \nabla_+] &= 2i\nabla_+ \\
[\nabla_-, \nabla_-] &= 2i\nabla_-
nabla_+, \nabla_-] &= F_{\pm}
\end{align*}
\]

(with all other super-commutators vanishing). The Bianchi identities imply that \(F_{\pm}\) can be written in terms of the unconstrained field strength \(W\).

A gauge-invariant kinetic term is obtained by minimal coupling, i.e. by replacing the superspace derivatives \(D_{\pm}\) by the gauge-covariant derivatives \(\nabla_{\pm}\). The Wess-Zumino term can be gauged if there is a globally defined \(u\) satisfying eq. (24), which is equivariant (eq. (27)) and for which

\[
c_{(ab)} = 0.
\]

If all these conditions are satisfied, then the action for the gauged (1,1) sigma-model is given in two-dimensional form by [12]

\[
S = \int d^2 x d\theta_1^- d\theta_1^+ \left( g_{ij} \nabla_+ \phi^i \nabla_- \phi^j + b_{ij} D_1+ \phi^i D_1- \phi^j - A^a_+ u_a D_1- \phi^i - A^a_- u_a D_1+ \phi^i + A^a_- A^b_- c_{[ab]} \right)
\]

3 We use a unified notation where the super-commutator is an ordinary commutator except when both quantities in it are anti-commuting, in which case it is the ordinary anti-commutator.
Now suppose the conditions under which the (1,1) model with action (1) in fact has (2,1) supersymmetry hold (cf. section 2). A (2,1) supersymmetric gauge-invariant action can be constructed as follows. Let $\varphi^i$ now be (2,1) superfields and impose the chirality constraint

$$D_{2+} \varphi^i = J^i_j D_{1+} \varphi^j,$$  \hspace{1cm} (58)

which is equivalent to (14). Consider the HP form of the action for the (2,1) supersymmetric model given by [16]

$$S = \int d^2 \sigma d\theta_1^+ d\theta_1^- g_{ij} D_{1+} \varphi^i D_{1-} \varphi^j + \int d^2 \sigma dt d\theta_1^+ d\theta_1^- H_{ijk} \partial_t \varphi^i D_{1+} \varphi^j D_{1-} \varphi^k \hspace{1cm} (59)$$

where the $\varphi^i(\sigma^\mu, \theta^+, \theta, t)$ are interpolating superfields satisfying

$$\varphi^i(\sigma^\mu, \theta^+, \theta^+, \theta, 0) = 0 \hspace{1cm} \varphi^i(\sigma^\mu, \theta^+, \theta^+, \theta, 1) = \varphi^i(\sigma^\mu, \theta^+, \theta^+, \text{t})$$

$$D_{2+} \varphi^i(\sigma^\mu, \theta^+, \theta^+, \theta, \text{t}) = -i J^i_j D_{1+} \varphi^j(\sigma^\mu, \theta^+, \theta^+, \text{t}).$$  \hspace{1cm} (60)

Using the chirality constraint (58) (or (14)), it is straightforward to show that the action (59) is independent of the extra supercoordinate $\theta^2$ (i.e. $\delta S/\delta \theta^2 = 0$ up to surface terms) provided eq. (6) holds, which implies that it is invariant under the non-manifest extra supersymmetry generated by the supercharge $Q_{2+}$.

To construct the gauged (2,1) supersymmetric action in this formalism, one replaces the chiral constraints (58) by the gauge covariant constraints

$$\nabla_{2+} \varphi^i = J^i_j \nabla_{1+} \varphi^j$$  \hspace{1cm} (61)

where the gauge covariant derivatives are defined using the super-connections of the (2,1) Yang-Mills supermultiplet, which will be described in the next section. Then, using the defining properties of the (2,1) Yang-Mills supermultiplet and super-curvatures $W$ and $\mathcal{W}$, it can be shown that the action [16]

$$S = \int d^2 \sigma d\theta_1^+ d\theta_{1-} \left[ g_{ij} \nabla_{1+} \phi^i \nabla_{1-} \phi^j - i \frac{1}{\sqrt{2}} X_a (W^a - \mathcal{W}^a) \right] + \int d^2 \sigma dt d\theta_1^+ d\theta_{1-} \left[ H_{ijk} \nabla_t \phi^i \nabla_{1+} \phi^j \nabla_{1-} \phi^k - \frac{1}{\sqrt{2}} u_{ia} \nabla_t \phi^i \left( W^a + \mathcal{W}^a \right) \right]$$  \hspace{1cm} (62)

is (2,1) supersymmetric provided that the conditions (8) and (33) are satisfied. This action is also gauge-invariant provided that $X$ and $u$ are equivariant, i.e. if (27) and (52) hold. Further, if the conditions (27) and (56) hold, then the field equations are two-dimensional and the action (62) can be put in a two-dimensional gauge-invariant form similar to (57).
5 The (2,1) Gauge Multiplet and Gauge Symmetries

We now wish to discuss the gauging of the isometries in a formalism which is manifestly (2,1) supersymmetric. The aim then is to promote the local isometry symmetries (29) to local ones in which the constant parameters $\lambda^a$ are replaced by (2,1) superfields $\Lambda^a$, 

$$\delta \phi^\alpha = \Lambda^a \xi_\alpha^a, \quad \delta \bar{\phi}^{\dot{\alpha}} = \overline{\Lambda^a} \overline{\xi}_{\dot{\alpha}}^a.$$  

(63)

In order to ensure that these transformations preserve the chirality constraints (14), one must require the $\Lambda^a$ to be chiral superfields, 

$$\overline{D}_- \Lambda^a = 0, \quad D_+ \overline{\Lambda}^a = 0.$$  

(64)

Under a finite transformation, 

$$\phi \to \phi' = e^{L \xi} \phi, \quad \bar{\phi} \to \bar{\phi}' = e^{L \Xi} \bar{\phi},$$  

(65)

where 

$$L \cdot \xi \equiv \Lambda^a \xi_\alpha^a \frac{\partial}{\partial \phi^\alpha}$$  

(66)

and $L_{\Lambda \xi} \phi^\alpha$ denotes the action of the infinitesimal diffeomorphism with parameter $L \cdot \xi$, 

$$L_{\Lambda \xi} \phi^\alpha \equiv \{ L \cdot \xi, \phi \}^\alpha,$$  

(67)

and acts on tensors as the Lie derivative with respect to $L \cdot \xi$.

The (2,1) super Yang-Mills multiplet is given in (2,1) superspace by a set of superconnections $A_{(2,1)} = (A^a_{1+}, A^a_{2+}, A^a_{-}, A^a_{\downarrow}, A^a_{\uparrow})$, and these can be used to define gauge covariant derivatives $\nabla_{1+}, \nabla_{2+}, \nabla_{-}, \nabla_{\downarrow}$ and $\nabla_{\uparrow}$ as in (54). It is convenient (cf. section 2) to combine $A^a_{1+}$ and $A^a_{2+}$ into a complex superconnection. We define $A^a_{\pm} = \frac{1}{\sqrt{2}} (A^a_{1+} + iA^a_{2+})$ and its complex conjugate $\overline{A}^a_{\pm}$. Then $A_{(2,1)} = (A^a_{\pm}, \overline{A}^a_{\downarrow}, A^a_{\downarrow}, A^a_{\uparrow}, A^a_{\uparrow})$ and the corresponding covariant derivatives $\nabla_{\pm}, \nabla_{\downarrow}, \nabla_{\downarrow}, \nabla_{\uparrow}$ and $\nabla_{\uparrow}$ satisfy the algebra

$$[\nabla_{\pm}, \nabla_{\pm}] = 0, \quad [\nabla_{\pm}, \nabla_{\mp}] = 0$$

$$[\nabla_{\pm}, \nabla_{\downarrow}] = 2i \nabla_{\downarrow}, \quad [\nabla_{\downarrow}, \nabla_{\downarrow}] = 2i \nabla_{\downarrow}$$

$$[\nabla_{\downarrow}, \nabla_{\uparrow}] = W, \quad [\nabla_{\uparrow}, \nabla_{\downarrow}] = \overline{W}$$

$$[\nabla_{\downarrow}, \nabla_{\uparrow}] = F_{\downarrow \uparrow}.$$  

(68)

The super-curvatures on the right-hand side of these super-commutators are not all independent, as they are constrained by the Bianchi identities. For example, the Bianchi identity for the covariant derivatives $\nabla_{\downarrow}, \nabla_{\downarrow}, \nabla_{\downarrow}$ is

$$\nabla_{\downarrow} W + \nabla_{\downarrow} \overline{W} = 0.$$  

(69)
The constraints (68) can be solved to give all connections in terms of a scalar pre-potential \( V^a \) and the spinorial connection \( A^a_\alpha \). In the chiral representation, the right-handed spinorial derivatives that appear in the algebra (68) are given by [18]

\[
\nabla_+ \equiv D_+ , \quad \nabla_+ = e^V D_+ e^{-V},
\]

while the left-handed covariant derivative is defined by

\[
\nabla_- \phi^\alpha = D_- \phi^\alpha - A^a_- \xi^\alpha_a.
\]

The gauge transformations of the Yang-Mills supermultiplet are as follows. The real superfield prepotential \( V^a \) transforms as

\[
e^V \rightarrow e^{\Lambda} e^V e^{-\Lambda}
\]

under a finite transformation. The superconnection \( A^a_\alpha \) has the infinitesimal gauge transformation

\[
\delta A^a_\alpha = D_- \Lambda^a + [A_- , \Lambda]^a.
\]

Because the parameters \( \Lambda^a \) are complex superfields, this implies that the connection \( A^a_\alpha \) is also complex, so that a reality condition should be imposed on it. The complex conjugate superconnection \( \overline{A}^a_\alpha \) transforms as

\[
\delta \overline{A}^a_\alpha = D_- \overline{\Lambda}^a + [\overline{A}_- , \overline{\Lambda}]^a
\]

and defines the complex conjugate covariant derivative

\[
\nabla_- \overline{\phi} \equiv D_- \overline{\phi} - \overline{A}^- \xi^a_\alpha.
\]

A natural choice for the non-vanishing supercommutators involving the covariant derivative \( \nabla_- \) is then

\[
\begin{align*}
[\nabla_+ , \nabla_-] &= \mathcal{W} , \\
[\nabla_+ , \nabla_-] &= \mathcal{W} \\
[\nabla_- , \nabla_-] &= 2i\nabla_+ , \\
[\nabla_- , \nabla_-] &= 0,
\end{align*}
\]

where the field strength \( \mathcal{W} \) is obtained from \( W \) by replacing \( A^a_\alpha \) with \( \overline{A}^a_\alpha \). However, \( e^V \nabla_- e^{-V} \) transforms in the same way as \( \nabla_- \), so that it is consistent to identify them; this generalised reality constraint reduces the number of degrees of freedom of the complex field \( A_- \) by a factor of two, to get the correct counting.

As explained at the end of the previous section, the scalar fields \( \phi , \overline{\phi} \) transform under the local isometry symmetries as in (63). Now let us define (following [8])

\[
\overline{\phi} = e^{L_V} \xi \overline{\phi},
\]

where

\[
L_V \xi = V^a \xi_a \frac{\partial}{\partial \overline{\phi}^a}.
\]
Then the fields $\varphi, \tilde{\varphi}$ satisfy the covariant chiral constraints
\[
\nabla_+ \varphi^\alpha = 0, \quad \nabla_+ \tilde{\varphi}^\alpha = 0,
\]
and transform under the isometry symmetries as
\[
\delta \varphi^\alpha = \Lambda^a \xi^\alpha_a, \quad \delta \tilde{\varphi}^\alpha = \Lambda^a \tilde{\xi}^\alpha_a(\tilde{\varphi}).
\]
Note that the transformation of $\tilde{\varphi}$ involved the parameter $\Lambda$ while that for $\varphi$ involved $\Lambda$. The left-handed covariant derivative of $\tilde{\varphi}$ is
\[
\nabla_- \tilde{\varphi}^\alpha = D_- \tilde{\varphi}^\alpha - A^a \tilde{\xi}^\alpha_a(\tilde{\varphi}).
\]

6 The Gauging in Superspace when $\vartheta = 0$

We now discuss the gauged (2,1) model based on the approach of ref. [8]. In this section we will set the stage and perform the gauging for the special class of model for which $\vartheta = 0$ (cf. eq. (37)), following [9], while the generic gauging will be given in the following sections.

We start by recalling that the general action (15) for the (2,1) model as well as the metric and torsion (12) are left invariant under the gauge transformation (16) with $\rho$ taking the form (18). The presence of this gauge invariance implies that the isometries (63) will not in general leave the action invariant, but will leave it invariant only up to gauge transformations of the form (16). This is analogous to the situation in the more familiar Kähler case where the model has in fact (2,2) supersymmetry and the Kähler potential is left invariant up to Kähler gauge transformations [7].

Now consider the variation of the Lagrangian
\[
L = i \left( k_\alpha D_- \varphi^\alpha - \bar{k}_\alpha D_- \bar{\varphi}^\alpha \right)
\]
under the infinitesimal rigid transformations (29). It is straightforward to check that
\[
\delta L = i \left( \mathcal{L}_a k_\alpha D_- \varphi^\alpha - \mathcal{L}_a \bar{k}_\alpha D_- \bar{\varphi}^\alpha \right),
\]
where the Lie derivative of $k_\alpha$ is given by the expression (36). The gauge invariance (18) then requires generically that
\[
\mathcal{L}_a k_\alpha = i \partial_\alpha \chi_a + \vartheta_{\alpha a}
\]
with $\chi$ a real function and $\vartheta_{\alpha a}$ a holomorphic one-form which were shown in ref. [9] to take the explicit forms (38) and (39).

Let us consider first the special class of models for which
\[
\mathcal{L}_a k_\alpha = 0,
\]
so that the Lagrangian (82) and hence the action (15) are invariant under the rigid transformations (29). Then the gauged sigma models belonging to this class are
obtained by minimal coupling. This coupling is achieved by replacing $\varphi$ with $\tilde{\varphi}$ and replacing the supercovariant derivative $D_-$ with the gauge covariant derivative $\nabla_-$ defined in (71) and (81). This gives the Lagrangian

$$L_0 = i \left( k_\alpha (\varphi, \tilde{\varphi}) \nabla_- \varphi^\alpha - \bar{k}_\alpha (\varphi, \tilde{\varphi}) \nabla_- \tilde{\varphi}^\alpha \right),$$

(86)

This is indeed invariant under the transformations (73), (72) and (80) provided (85) holds.

In the remaining part of the present section, we will gauge the more general class of models for which the holomorphic part of the Lie derivative (84) vanishes, i.e., the conditions

$$\vartheta_{aa} = 0 \quad (87)$$

and

$$L_a k_\alpha = i \partial_\alpha \chi_a \quad (88)$$

hold for these models, while the generic case $\vartheta \equiv 0$ will be treated in the following sections.

When (87) and (88) hold, the action based on (86) is no longer gauge invariant. Using (73), (84), (87), and the infinitesimal variation of the fields (80), we find

$$\delta L_0 = i \Lambda^a D_- \chi_a (\varphi, \tilde{\varphi}) + \Lambda^a A^b \left( \partial_\alpha \chi_a \xi^\alpha_b (\varphi) + \partial_\alpha \chi_a \xi^\alpha_b (\tilde{\varphi}) \right) (\varphi, \tilde{\varphi}).$$

(89)

This can be cancelled by adding the following term to $L_0$:

$$\hat{L}_0 = -A^a \chi_a (\varphi, \tilde{\varphi}).$$

(90)

The expression (38) of the potential $\chi$ implies that the terms multiplying the gauge connection $A_-$ combine to yield the generalised Killing potential $X$:

$$L_g^{(0)} = L_0 + \hat{L}_0 = i \left( k_\alpha D_- \varphi^\alpha - \bar{k}_\alpha D_- \tilde{\varphi}^\alpha \right) (\varphi, \tilde{\varphi}) - A^a X_a (\varphi, \tilde{\varphi}).$$

(91)

We claim that the action based on the Lagrangian $L_g^{(0)}$ in (91) is the full gauge-invariant action for the gauged (2,1) model in the special case where $\vartheta = 0$ provided that the generalised Killing potential $X$ transforms covariantly under the isometries (83), i.e.,

$$\delta X_a = f_{ab}^c \Lambda^b X_c.$$  

(92)

To see this, note that the variation of the first term in (91) is given by

$$\delta \left[ i \left( k_\alpha D_- \varphi^\alpha - \bar{k}_\alpha D_- \tilde{\varphi}^\alpha \right) (\varphi, \tilde{\varphi}) \right] = -\Lambda^a D_- \chi_a (\varphi, \tilde{\varphi}) - i D_- \Lambda^a \left( \xi^\alpha_a k_\alpha - \xi^\alpha_a k_\alpha \right) (\varphi, \tilde{\varphi}) = -D_- \Lambda^a X_a (\varphi, \tilde{\varphi}),$$

(93)

the manipulations being similar to those which lead to the expression (89); the last identity follows upon integrating by parts, discarding a surface term and using the
expression (38) (notice that the second term on the right-hand side of (38) has cancelled). On the other hand, the variation of the second term in (91) is

\[
\delta \left[ -A_a^a X_a(\varphi, \bar{\varphi}) \right] = -D_- A_a^a X_a(\varphi, \bar{\varphi}) - A_a^a \left( \delta X_a + f_{cb}^c A_b^b X_c \right)(\varphi, \bar{\varphi}),
\]

where we have used the variations (73) of the gauge connection. Adding (93) and (94), a cancellation occurs, and one is left with

\[
\delta L_g^{(0)} = -A_a^a \left( \delta X_a + f_{cb}^c A_b^b X_c \right)(\varphi, \bar{\varphi}).
\]

This cannot be cancelled by the variation of any further addition to the action or transformation rules, but vanishes if \( \delta X_a + f_{cb}^c A_b^b X_c = 0 \), so we find that the equivariance condition (92) is a necessary condition for the gauging to be possible. As seen in section 3, the condition (92) implies the equivariance of \( u \). Here the relation (13) together with the assumption that \( \vartheta = 0 \) implies \( c_{(ab)} = 0 \). Thus the conditions necessary for gauging to be possible found in [12, 15] (the equivariance of \( X \) and \( u \), and \( c_{(ab)} = 0 \)) are all satisfied.

Summarizing, we find that, in the special case where \( \vartheta = 0 \), the action (15) for the (2,1) model can be gauged provided the same geometric condition as that found in ref. [15] is satisfied, namely the equivariance of the generalized Killing potential \( X \). Moreover, if (92) holds, then the gauged (2,1) sigma-model action in this case is the superspace integral of the gauge invariant Lagrangian (71).

7 Noether Gauging in Superspace

Let us now turn to the discussion of the gauging in the generic situation where the holomorphic part of the Lie derivative (84) is arbitrary, i.e. \( \vartheta \) may not vanish. We shall first use the Noether method to obtain the gauging to lowest and first order in the gauge coupling constant \( q \) in order to gain some insight into the structure of the full all-orders gauge invariant action. In the next section, we will introduce a procedure which enables us to reduce the analysis to that of the special case \( \vartheta = 0 \) up to some subtleties which will be discussed in detail.

We now reinstate the gauge coupling constant \( q \), which has until now been set to 1, and note the following first order infinitesimal variation:

\[
q \delta V^a = \Lambda^a - \nabla^a + \frac{q}{2} f_{bc}^a V^b \left( \Lambda^c + \nabla^c \right) + O(q^2)
\]

which is the infinitesimal form of (72). Taylor expanding (77) gives

\[
\varphi \rightarrow \varphi + q V^a \xi^a + \frac{q^2}{2} V^a V^b \xi^b \xi^a + O(q^3).
\]

We shall also need the infinitesimal gauge transformations (73), (74) of the super-connections \( A_a^a \) and \( \nabla_a^a \). Note however that, when gauging an abelian isometry subgroup, it is sufficient to consider the lowest order variations of the gauge fields since the structure constants \( f_{bc}^a \) are zero in that case.
We start with the Lagrangian $L_g^{(0)}$ of the previous section, given in eq. (92), which was the full gauged Lagrangian when $\vartheta = 0$. In the general case, its variation depends on $\vartheta$. Under the variations (73) and (96) of the gauge superconnections and (80) of the superfields $\varphi$ and $\tilde{\varphi}$, we find

$$
\delta L_g^{(0)} = -i\Lambda^a\overline{\varphi}_{\alpha a} D_- \varphi^\alpha - iq\Lambda^a D_- V^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} - iq\Lambda^a V^b \mathcal{L}_b \overline{\varphi}_{\alpha a} D_- \varphi^\alpha \\
- qA_+ \left( \delta X_a + f_{ba} \Lambda^b X_c \right) + O(q^2)
$$

(98)

up to surface terms. The term independent of $q$ can be cancelled by adding

$$
\tilde{L}_1 = iqV^a \overline{\varphi}_{\alpha a} (\tilde{\varphi}) \nabla_- \tilde{\varphi}_{\alpha a}
$$

(99)

which can be expanded to second order in $q$ using the expansion of $\tilde{\varphi}$ in (77), yielding

$$
\tilde{L}_1^{(2)} = iqV^a \overline{\varphi}_{\alpha a} D_- \varphi^\alpha + iq^2 V^a D_- V^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} \\
+ iq^2 V^a \mathcal{L}_b (a \overline{\varphi}_{\alpha a}) D_- \varphi^\alpha - iq^2 V^a A_\alpha^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} + O(q^3)
$$

(100)

where we have used the definition (71) of the left-handed covariant derivative.

Then the gauge invariant Lagrangian to order $q^2$ is of the form

$$
L_g^{(2)} = L_g^{(0)} + L_1^{(2)}
$$

(101)

with $L_g^{(0)}$ the Lagrangian (71), which is gauge invariant to that order when $\vartheta = 0$ and the equivariance condition (52) holds, and $L_1^{(2)}$ a second-order Lagrangian which includes the term (100). The variation to order $q$ is now given by

$$
\delta \left( L_g^{(0)} + L_1^{(2)} \right) = -iq\Lambda^a D_- V^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} - iq\Lambda^a V^b \mathcal{L}_b \overline{\varphi}_{\alpha a} D_- \varphi^\alpha + \frac{q}{2} f_{ba} V^b (\Lambda^c + \overline{\Lambda}^c) \overline{\varphi}_{\alpha a} D_- \varphi^\alpha \\
+ iqV^a \overline{\varphi}_{\alpha a} \overline{\varphi}_{\alpha a} \nabla_- \tilde{\varphi}_{\alpha a} + iqV^a \overline{\varphi}_{\alpha a} D_- \tilde{\varphi}_{\alpha a} + iqV^a \mathcal{L}_b (a \overline{\varphi}_{\alpha a}) \nabla_- \tilde{\varphi}_{\alpha a} \\
+ 2iq(\Lambda^a - \overline{\Lambda}^a) V^b \mathcal{L}_b (a \overline{\varphi}_{\alpha a}) D_- \varphi^\alpha \\
- iq(\Lambda^a - \overline{\Lambda}^a) A_\alpha^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} - iqV^a D_- \Lambda_\alpha^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} \\
- qA_+ \left( \delta X_a + f_{ba} \Lambda^b X_c \right) + O(q^2)
$$

(102)

(up to surface terms) which must be cancelled by the variation of additional contributions to the lagrangian (modulo certain geometric conditions stated below). It turns out that three such contributions are needed, namely

$$
\hat{L}_1^{(2)} = -iq^2 V^a A_\alpha^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} - \frac{i}{2} q^2 V^a D_- V^b \mathcal{L}_b (a \overline{\varphi}_{\alpha a}) D_- \varphi^\alpha - \frac{i}{2} q^2 V^a D_- V^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} + O(q^3)
$$

(103)

Varying as in (73), (80) and (96), it can be checked that the choice

$$
L_1^{(2)} = \hat{L}_1^{(2)} + \tilde{L}_1^{(2)} \\
= iqV^a \overline{\varphi}_{\alpha a} D_- \varphi^\alpha + \frac{i}{2} q^2 V^a D_- V^b \xi_b^{\alpha} \overline{\varphi}_{\alpha a} + \frac{i}{2} q^2 V^a V^b \mathcal{L}_b (a \overline{\varphi}_{\alpha a}) D_- \varphi^\alpha \\
+ 2iq^2 V^a A_\alpha^b \overline{\varphi}_{\alpha a} + O(q^3)
$$

(104)
gives a Lagrangian $L_g^{(2)}$ in (104) that is gauge invariant to first order in $q$ (up to surface terms) when the equivariance condition (102) and the condition

$$\hat{d}_{(ab)} = \xi^\alpha (\vartheta_{ab}) = 0.$$  

(105)

hold. The latter condition is in fact equivalent to that of vanishing constants $c_{(ab)}$ (eq. (56)). To see this, recall the defining equation for the Killing potential $X_a$,

$$J_{ij} (\xi^j_a + u^j_a) = \partial_i X_a.$$  

(106)

Contracting (106) with $\xi^i_b$, we find

$$- J_{ij} \xi^i_a \xi^j_b + J_{ij} \xi^i_b u^j_a = \mathcal{L}_b X_a.$$  

(107)

If $X_a$ is equivariant, i.e. if the condition (52) holds, then the right-hand side of (104) equals $f^c_{ba} X_c$ and, symmetrizing with respect to group indices, we find the condition

$$J_{ij} \xi^i_b u^j_a = 0,$$  

(108)

which implies

$$\xi^\alpha_b u|\alpha|a = \xi^\alpha_b (\vartheta_{|\alpha|b} |a|).$$  

(109)

It follows that the constants $c_{(ab)}$ are given by

$$c_{(ab)} = \xi^i_{(a} u_{|a|b)} = 2\xi^\alpha_{(a} u|\alpha|b).$$  

(110)

Then, using eqs. (50) and (110), we find that the quantities $\hat{d}_{(ab)}$ are related to the constants $c_{ab}$ as follows

$$\hat{d}_{(ab)} = \xi^\alpha (\vartheta_{ab}) = -\xi^\alpha_{(a} u_{|a|b)} = -\frac{1}{2} c_{(ab)}.$$  

(111)

Thus the condition (103) of vanishing $\hat{d}_{(ab)}$ is equivalent to that of vanishing constants $c_{(ab)}$, eq. (50).

It is important to notice that, apart from the contribution involving the $\hat{d}_{(ab)}$ (which, as shown above, vanishes when the condition (56) holds), the net effect of adding the contribution $\hat{L}_1^{(2)}$ to $\hat{L}_1^{(2)}$ is to modify some of the numerical coefficients multiplying the individual terms in the latter; this yields the specific coefficients appearing in (104). We shall see shortly that this behaviour is generic and would be observed at each order in the perturbative expansion: the necessity of substracting terms such as those appearing in $\hat{L}_1^{(2)}$ from those in $\hat{L}_1$ in order to obtain a gauge invariant result to the order considered reflects the general structure of the all-orders gauged action, to the construction of which we now turn.
8 General Gauging in Superspace

From the analysis of section 6, we know that, given any local one-form \( w^\alpha \) such that
\[
\mathcal{L}_a w^\alpha = i \partial^\alpha W^a
\]
for some \( W^a \), then the Lagrangian
\[
L = i \left( w^\alpha \nabla_\alpha \varphi - \bar{w} \pi \nabla_\alpha \bar{\varphi} \right) - q A^a_\alpha W^a (\varphi, \bar{\varphi})
\]
is gauge invariant provided the ‘Killing potential’
\[
X^{(w)}_a \equiv W^a - i \left( \xi_a^\alpha \pi^{\alpha} - \xi_a^\alpha w^\alpha \right)
\]
is equivariant, i.e. it satisfies eq. (52). The lagrangian (113) can be rewritten as
\[
L = i \left( w^\alpha D_\alpha \varphi - \bar{w} \pi D_\alpha \bar{\varphi} \right) - q A^a_\alpha X^{(w)}_a (\varphi, \bar{\varphi}).
\]
The potential \( k^\alpha \) does not satisfy (112) as its Lie derivative is given by (37). In the spirit of [8], we seek a ‘correction’ to \( k^\alpha \) such that (112) is satisfied, but the correction does not modify the geometry. We therefore seek to define
\[
w^\alpha = k^\alpha - \kappa^\alpha
\]
which satisfies (112) for some \( \kappa^\alpha \), which will be locally defined in general. Then the Lie derivative of \( \kappa^\alpha \) is determined by (37), (112) and (114) to be of the form
\[
\mathcal{L}_a \kappa^\alpha = \vartheta^{\alpha a} + 2i \partial^\alpha \gamma_a
\]
with
\[
\gamma_a = \Im (\xi^a \cdot \kappa^\alpha) = \frac{1}{2i} \left( \xi^a \kappa^\alpha - \bar{\xi}^a \bar{\kappa}^\alpha \right).
\]

If in addition \( \kappa^\alpha \) is holomorphic, then the replacement \( k^\alpha \to w^\alpha \) leaves the ungauged action (13) unchanged, so that \( \kappa^\alpha \) does not change the sigma-model geometry. As we shall see below, this holomorphy also results in the elimination of \( \kappa^\alpha \) from the gauged action. We now show that although the auxiliary field \( \kappa^\alpha \) will not in general exist globally, its local existence will be guaranteed by standard arguments provided the geometric conditions (52) and (56) hold. We emphasize that our final results are independent of \( \kappa^\alpha \).

We shall seek a field \( \kappa^\alpha \) which satisfies the following two conditions: (i) \( \kappa^\alpha \) is holomorphic, i.e.
\[
\partial^{\beta} \kappa^\alpha = 0
\]
and (ii) the Lie derivative of \( \kappa^\alpha \) with respect to \( \xi^a \) is given by
\[
\mathcal{L}_a \kappa^\alpha = \vartheta^{\alpha a} + 2i \partial^\alpha \gamma_a
\]
for some real function \( \gamma_a \). The integrability conditions on \( \kappa^\alpha \) following from eqs. (119) and (120) will now be derived.
Taking the Lie derivative of (120) with respect to $\xi_b$ and antisymmetrizing with respect to group indices yields
\[ \mathcal{L}_b [\mathcal{L}_a \kappa_\alpha] = \mathcal{L}_b [\partial_{a\alpha}] + 2i \partial_\alpha \left( \mathcal{L}_b [\gamma_a] \right). \] (121)

The left-hand side of this equation can be rewritten using the Lie algebra of $G$ and eq. (120), while the first term on the right-hand side is given by
\[ \mathcal{L}_b [\partial_{a\alpha}] = \frac{1}{2} f^c_{ba} \partial_{ac} - \frac{1}{2} \partial_\alpha (\xi^a_\gamma \partial_{\gamma b}), \] (122)

where we have set the potential $E_{ba}$ to zero, as in eq. (53); this is possible provided the equivariance condition (52) on the imaginary part of the Killing potential holds. Substituting (122) into (121), we find upon integration of the resulting equation that the compatibility of the condition (120) with the equivariance condition (122) requires the function $\gamma_a$ to satisfy
\[ \mathcal{L}_a [\gamma_b] = \frac{1}{2} f^c_{ab} \gamma_c - \frac{i}{4} \left( \xi^a_\alpha \partial_{\gamma a} - \bar{\kappa}_{ab}(\tau) \right). \] (123)

for some antiholomorphic function $\bar{\kappa}$.

It turns out that there is a simple solution to the condition (123), namely
\[ \gamma_a = \Im (\xi_a \cdot \kappa) = \frac{1}{2i} \left( \xi^a_\alpha \kappa_\alpha - \bar{\kappa}_{a\beta} \bar{\kappa}_{a\alpha} \right). \] (124)

Taking the Lie derivative of (124) with respect to $\xi_b$, substituting eq. (120) and antisymmetrizing with respect to group indices, we find that $\gamma_a$ in (124) solves the condition (123) provided the antiholomorphic function $\bar{\kappa}$ takes the form
\[ \bar{\kappa}_{ab} = \bar{\kappa}_{a\beta} \bar{\kappa}_{a\alpha}. \] (125)

In what follows we shall suppose that the vector field $\kappa_\alpha$ satisfies (120) with the function $\gamma_a$ as in (124). Then the assumed holomorphy of $\kappa_\alpha$ implies the condition
\[ \mathcal{L}_a \kappa_\alpha = \partial_{a\alpha} + \partial_\alpha \left( \xi^\beta_{a\beta} \kappa_\beta \right). \] (126)

In fact, no loss of generality in the following arguments will be involved in replacing condition (120) with condition (124). This can be seen by checking the compatibility of eq. (124) with eq. (123) as follows. Taking the Lie derivative of (124) with respect to $\xi_b$ and antisymmetrizing with respect to group indices, we find
\[ [\mathcal{L}_b, \mathcal{L}_a] \kappa_\alpha = 2 \mathcal{L}_b [\partial_{a\alpha}] + 2 \partial_\alpha \left\{ f^c_{ba} \xi^c_\beta \kappa_\beta + \xi^\beta_{a\beta} \partial_{b\beta} + \xi^\beta_{a\beta} \partial_\beta \left( \xi^a_\beta \kappa_\gamma \right) \right\}. \] (127)

Using the Lie algebra of $G$, eq. (125) and the following relation
\[ \xi^\alpha_{[a} \mathcal{L}_b \kappa_{\alpha]} = \xi^\alpha_{[a} \partial_{ab]} + \xi^\alpha_{[a} \partial_\alpha \left( \xi^\beta_{b\beta} \kappa_\beta \right) \] (128)
(which follows from (126) upon contracting with $\xi^\alpha_b$ and antisymmetrizing with respect to group indices), this can be rewritten as

$$
\xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} = \xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} - \xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} + 2 \partial_\alpha \left( \xi^\beta_a \nabla_{[\beta} \xi^\gamma_{\gamma]} \right), \tag{129}
$$

where the second equality follows from the definition of the Lie derivative and the holomorphy of $\kappa_\alpha$. Moreover, we find from eq. (128) that

$$
\xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} - \xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} = 2 \xi^\alpha_a \partial^\beta \xi^\gamma_{\gamma}. \tag{130}
$$

Substituting in eq. (129) and using the antiholomorphy of $d_{(ab)} = \xi^\alpha_{(a} \nabla_{\beta)}$ (cf. eq. (46) and below) then yields the equivariance condition (122).

Now we show that the condition (56) of vanishing constants $c_{(ab)}$ necessarily holds if a vector field satisfying the conditions (119) and (126) exists. Contracting eq. (126) with $\xi^\alpha_a$ and symmetrizing with respect to group indices, we find the relation

$$
\xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} = \xi^\alpha_a \nabla_{[a} \xi^\beta_{b]} - \xi^\alpha_a \partial^\beta \xi^\gamma_{\gamma}. \tag{131}
$$

Then, taking the Lie derivative as in (36) and using the assumed holomorphy of $\kappa_\alpha$, one finds after some simple manipulations that the right-hand side of (131) vanishes identically. Hence two integrability conditions on the vector field $\kappa_\alpha$ satisfying the conditions (119) and (126) are the equivariance condition (52) and the condition (105) of vanishing $\hat{d}_{(ab)}$, which is equivalent to the condition (56) of vanishing constants $c_{(ab)}$ as shown above. Recall that (56) was found in [15] to be a necessary condition for the gauging of the (2,1) supersymmetric sigma model to be possible (cf. section 4), and that we required this same condition to hold in the perturbative analysis given in the end of the previous section.

Finally, we return to the issue of the holomorphy of $\kappa_\alpha$. The condition (124) with $\gamma$ given by (124) implies that $\kappa$ satisfies

$$
\xi^\alpha_a \partial_\beta \kappa_\alpha - \xi^\alpha_a \partial_\beta \kappa_\beta + \xi^\beta_a (\partial_\alpha \kappa_\alpha + \partial_\alpha \kappa_\beta) = \nabla_{[a}, \tag{132}
$$

which is an inhomogeneous first-order partial differential equation for the holomorphic vector field $\kappa_\alpha$. Choosing adapted coordinates for one of the Killing vectors in which $\xi^\alpha_a \partial_\alpha = \partial/\partial z$ for some particular value of $a$, the equation (132) becomes

$$
\partial_z \kappa_\alpha - \partial_\alpha \kappa_z + [\partial_z \kappa_\alpha + \partial_z \kappa_\beta] = \nabla_{[a}, \tag{133}
$$

As $\nabla_{[a}$ is holomorphic, this can clearly be integrated for holomorphic $\kappa$. Thus $\kappa$ can be chosen to be holomorphic with respect to the coordinates corresponding to any commuting set of Killing vectors. If the integrability conditions considered above, and in particular the condition (124), hold, then the equation (132) will have local solutions $\kappa$ which are holomorphic. This establishes the local existence of an auxiliary vector field $\kappa_\alpha$ with the desired properties.
Then, using the definitions (114) and (116) and the holomorphy of \( \kappa_\alpha \), we find that the Lagrangian (115), which formally is gauge invariant, can be rewritten as

\[
L = i \left( k_\alpha D_- \varphi^\alpha - \bar{k}_\alpha D_- \bar{\varphi}^\alpha \right) (\varphi, \bar{\varphi}) - qA^a_{\alpha} X_a (\varphi, \bar{\varphi})
+ i \bar{\kappa}_{\alpha}(\bar{\varphi}) D_- \bar{\varphi}^\alpha
\]  

(134)

where we have discarded a term \( \kappa_\alpha D_- \varphi^\alpha \), which is chiral as a result of the holomorphy of \( \kappa_\alpha \). Note that other terms have cancelled from (134).

The expression (134) of the gauged Lagrangian is not satisfactory as it involves the vector field \( \kappa_\alpha \), which is only defined implicitly; the action obtained using the Noether method involves no such vector. We must therefore endeavour to rewrite it in such a way that no explicit dependence on \( \kappa \) remains. To this end, we write the last term in (134) in the following way:

\[
\bar{\kappa}(\bar{\varphi}) D_- \bar{\varphi}^\alpha = e^L (\bar{\kappa}_{\alpha}(\bar{\varphi}) D_- \bar{\varphi}^\alpha)
= \bar{\kappa}_{\alpha}(\bar{\varphi}) D_- \bar{\varphi}^\alpha + \frac{e^L - 1}{L} L \left[ \bar{\kappa}_{\alpha}(\bar{\varphi}) D_- \bar{\varphi}^\alpha \right],
\]  

(135)

where we have defined

\[
L \equiv \mathcal{L}_{V}\bar{\tau}
\]  

(136)

and used the identity

\[
e^L = 1 + \frac{e^L - 1}{L}. \tag{137}
\]

The first term in the last line of (135) is antichiral by holomorphy of \( \kappa_\alpha \), and can be discarded. Moreover, a simple calculation utilizing the definition (136) of the operator \( L \) and the relations (120) and (123) yields

\[
L \left[ \bar{\kappa}_{\alpha}(\bar{\varphi}) D_- \bar{\varphi}^\alpha \right] = qV^a \bar{\tau}_{\alpha a} D_- \bar{\varphi}^\alpha + D_- \left( qV^a \bar{\tau}_{\alpha a} \bar{\kappa}_{\alpha} \right). \tag{138}
\]

Hence we find that the \( \kappa \)-dependent term in (113) can be rewritten as

\[
i \bar{\kappa}_{\alpha}(\bar{\varphi}) D_- \bar{\varphi}^\alpha = \frac{i}{L} qV^a \bar{\tau}_{\alpha a} D_- \bar{\varphi}^\alpha + \frac{i}{L} D_- \left( qV^a \bar{\tau}_{\alpha a} \bar{\kappa}_{\alpha} \right)
= \frac{i}{L} qV^a \bar{\tau}_{\alpha a} D_- \bar{\varphi}^\alpha + D_- \left[ \frac{e^L - 1}{L} qV^a \bar{\tau}_{\alpha a} \bar{\kappa}_{\alpha} \right] \tag{139}
\]

where the second inequality follows from the fact that the operator \( L \) in (136) is the generator of infinitesimal gauge transformations with parameter the prepotential \( V \), and hence must commute with the supercovariant derivative \( D_- \). As a result, all terms in the expansion of the second term in the first line of eq. (139) can be recast into a total derivative term, as indicated in the second line.

The gauge invariant action (to all orders) for the gauged (2,1) heterotic sigma model is the superspace integral of the Lagrangian (134). Hence, upon substitution of the expression (139) in (134), we find the action

\[
S_g = \int d^2\sigma d\theta d\bar{\theta} d\bar{\theta}_- \left\{ \left[ i \left( k_\alpha D_- \varphi^\alpha - \bar{k}_\alpha D_- \bar{\varphi}^\alpha \right) - qA^a_{\alpha} X_a \right] (\varphi, \bar{\varphi})
+ \frac{i}{L} qV^a \bar{\tau}_{\alpha a} D_- \bar{\varphi}^\alpha \right\}. \tag{140}
\]
Several comments on this result are in order. First, it is obvious that in the special case where \( \vartheta = 0 \), (140) reduces to the superspace integral of the gauged Lagrangian (91), as indeed it must. Moreover, the geometric conditions for gauge invariance of both (140) and (91) are the equivariance of the generalized Killing potential \( X_a \) and the vanishing of the constants \( c_{(ab)} \) defined in (48), i.e. eqs. (52) and (56) respectively.

Second, it can be checked using the definition (136) of the operator \( L \) that the expansion of the second term in (140) to second order in the gauge coupling constant \( q \) yields precisely those terms appearing in the second-order Lagrangian (101), which was shown above (by explicit variation) to be gauge invariant to first order in \( q \) when the geometric conditions (52) and (56) hold. Furthermore, we have also checked by a (somewhat lengthy) direct calculation that the expansion of the last term in (134) to second order in \( q \), which contains many terms involving the vector field \( \kappa_\alpha \) as well as its first and second order derivatives, can indeed be recast as the sum of the expected terms (104) appearing in the second-order gauged Lagrangian (101) and a total derivative term, as in (139):

\[
\tilde{\kappa}_\pi(\varphi)D_\varphi \varphi^\pi = qV^a \partial^{\alpha}D_\alpha \varphi^\pi + \frac{q}{2} V^a D_\pi \xi_b \varphi^\pi V_{a \pi} + \frac{q^2}{2} V^a V^b L_{(b \bar{\sigma})} D_\sigma \varphi^\pi \\
+ D_\pi \left[ qV^a \xi_{a \pi} \varphi^\pi + \frac{q^2}{2} V^a V^b \xi_{(a \bar{\sigma})} \varphi^\pi + \frac{q^2}{2} V^a V^b \xi_{(a \bar{\sigma})} \partial_\pi \left( \bar{\xi}_{b \bar{\sigma}} \right) \right] (141) \\
+ O(q^3). (142)
\]

Using the condition (105) and the definition (136), it is easily seen that this is precisely the expansion of eq. (139) to that order. Notice that this derivation relies only on the two defining conditions for the vector field \( \kappa_\alpha \), namely (119) and (120) (with \( \gamma_a \) as in (124)). This is a non-trivial test of our results, particularly of the structure given in (139).

Summarizing, we find that the (2,1) superspace action (15) can be gauged provided the geometric conditions (52) and (56) hold, in which case the gauged superspace action is given in (140). Although our construction utilized a vector field \( \kappa_\alpha \) satisfying certain requirements, we stress that our final action (140) is independent of such an object, and its gauge invariance can be checked directly.

9 Conclusion

The main results of this paper can be summarized as follows. If the (2,1) sigma-model with torsion is formulated in extended superspace, as in (1)-(8) and (15), then its isometries can be gauged by coupling the model to the usual (2,1) Yang-Mills supermultiplet provided the geometric condition (52) and (56) are satisfied. If (52) and (56) hold, then there are two alternative forms of the gauged action: the result (62) was found in ref. [15], while the manifestly (2,1) supersymmetric gauged action (140) is new.

In the special case where the geometry of the target space is Kähler or twisted Kähler, the (2,1) gauged action (140) should reduce to the (2,2) gauged actions.
found in \[8, 15\], although we have not presented a proof here. Also, a new action for the gauged (2,0) supersymmetric sigma model with torsion may be obtained from (140) by appropriate truncation of the (2,1) superfields. This gives

$$S_g = \int d^2\sigma d\theta_1 d\bar{\theta}_1 \left\{ i \left(k_\alpha \partial_\sigma \varphi^\alpha - k_{\bar{\sigma}} \partial_{\bar{\sigma}} \varphi^{\bar{\sigma}}\right) - q A^a X_a \right\} (\varphi, \bar{\varphi}).$$

(143)

where all fields are (2,0) superfields. Here the (2,0) superspace is parametrised by two bosonic null coordinates ($\sigma^+, \sigma^-$) and two Grassmann coordinates ($\theta^1, \theta^2$) of the same chirality; the supercovariant derivatives $D_{1,2+} = \partial/\partial \theta_{1,2+} + i \theta_{1,2+} \partial/\partial \sigma^+$ satisfy $D_{1,2+}^2 = i \partial^+ 1$ and $[D_{1+}, D_{2+}] = 0$.

As was mentioned in the introduction, our work is of relevance to the study of the geometry of the $N = (2,1)$ heterotic string theory because it shows how the $U(1)$ current of the internal left-moving sector of that theory can be gauged, as indeed it must be. Since the latter sector contains 8 chiral bosons, it is important to construct a covariant action with manifest (2,1) supersymmetry which describes the chiral bosons off-shell as well as on-shell. This can be achieved by coupling the (2,1) heterotic sigma model to supergravity. The methods of the present paper can then be applied to the gauging of the resulting action. This will be discussed in more detail elsewhere [19].

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