Gabriel localization in functor categories

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ABSTRACT

Gabriel showed that for a unital ring $R$, there exists a bijective correspondence between the set of Gabriel filters of $R$ and the set of Giraud subcategories of $\text{Mod}(R)$. In this paper we prove a result analogous to the one given by Gabriel: for a small preadditive category $C$, there exists a bijective correspondence between the left Gabriel filters of $C$ and Giraud subcategories of $\text{Mod}(C)$.

1. Introduction

The idea that preadditive categories are rings with several objects was convincingly developed by Barry Mitchell, who showed that a substantial part of the theory of noncommutative rings remains true in this generality; see [9]. Here we would like to emphasize that clarity in concepts, statements, and proofs is sometimes gained by dealing with additive categories, and that familiar theorems for rings arise from the natural development of category theory. For example, the notions of radical of a preadditive category, perfect and semisimple rings, global dimensions, among other topics, have been studied extensively in the context of rings with several objects.

In 1962, Gabriel introduced in [6] the concept of localization in the setting of abelian categories, and he proposed the now so named Gabriel filter on $R$, where $R$ denotes a unital ring and $\text{Mod}(R)$ the category of its left unital $R$-modules, to study localization in rings and modules; see [1] and [14, Chapter VI.5]. Moreover, P. Gabriel showed that there is a bijective correspondence between the set of Gabriel filters of $R$ and the set of the class of the isomorphisms of Giraud subcategories of $\text{Mod}(R)$; see [6]. Recall that a subcategory $\mathcal{X}$ of $\text{Mod}(R)$ is a Giraud subcategory if the inclusion functor has a left adjoint that is left exact.

Over the years, Gabriel filters have been studied by several authors in different contexts. For instance, in [7], L. Angeleri Hügel and S. Bazzoni studied Gabriel filters in Grothendieck...
categories with a generator. Notice that their definition is slightly different from the one we use in this paper.

In 2015, Díaz-Alvarado and Ortíz-Morales introduced in [10] the notion of Gabriel filter for a preadditive category \( C \). They proved that there is a bijective correspondence between Gabriel filters of \( C \) and hereditary torsion classes of \( \text{Mod}(C) \). Recently, in [11], Parra, Saorín and Virili have studied torsion pairs in categories of modules for a preadditive category, as well as abelian recollements in functor categories, and centrally splitting TTFs. In their research, these authors have provided similar definitions and results that are related to the work of Díaz-Alvarado and Ortíz-Morales.

Following Mitchel’s philosophy and the definition of a Gabriel filter given in [10], our goal in this paper is to study Gabriel’s analogous result for the context of rings with several objects. One of the main results in this work is explained below:

**Theorem 4.22.** Let \( C \) be a small preadditive category and let \( \text{Mod}(C) \) be the category of additive covariant functors from \( C \) to the category of abelian groups \( \text{Ab} \). Then, there is a bijective correspondence between the left Gabriel filters on \( C \), and the classes of isomorphisms of Giraud subcategories in \( \text{Mod}(C) \).

Our final result is related to the localization by Serre subcategories developed by P. Gabriel. The notion of quotient and localization of abelian categories by dense subcategories (i.e., Serre classes) was introduced by P. Gabriel in his famous doctoral thesis “Des catégories abéliennes” [6], and it plays an important role in ring theory. This notion achieves some goal as quotients in other areas of mathematics. In particular, in this paper we prove that there is an equivalence of categories \( \text{Mod}(C)/T \cong \text{Mod}(C,F) \), where \( \text{Mod}(C,F) \) is a certain Giraud subcategory associated with a left Gabriel filter \( F \) and \( T \) is a hereditary torsion class; see 4.24.

It is worth mentioning that techniques of localization in lattice-contexts have also been studied, for example, by Albu and Smith; see [2, 3], and [1], as well as by Harold Simmons in a series of unpublished papers.

This article is organized as follows. In section 2, we recall definitions needed in the paper, and then we collect for later use a variety of results from various backgrounds. In section 3, we define a prelocalization functor \( \mathbb{L} \), and we prove several technical results related to the functor \( \mathbb{L} \) that will be needed to define the Gabriel localization on \( \text{Mod}(C) \). In section 4, we define the Gabriel localization functor \( \mathbb{G} \), and we prove our main result, Theorem 4.22. Finally, in 4.25 we give an example of a Gabriel filter in the category of representations of an infinite quiver.

### 2. Preliminaries

We recall that a category \( C \) together with an abelian group structure on each of the sets of morphisms \( C(C_1,C_2) \) is called a preadditive category provided all the composition maps \( C(C_1,C_2) \times C(C_1,C') \to C(C_2,C') \) in \( C \) are bilinear maps of abelian groups. A covariant functor \( F : C_1 \to C_2 \) between preadditive categories \( C_1 \) and \( C_2 \) is said to be additive if for each pair of objects \( C \) and \( C' \) in \( C_1 \), the map \( F : C_1(C,C') \to C_2(F(C),F(C')) \) is a morphism of abelian groups. We will denote by \( \text{Ab} \) the category of abelian groups. Throughout this paper, \( C \) will be an arbitrary small preadditive category.

#### 2.1. The category \( \text{Mod}(C) \)

We will denote by \( \text{Mod}(C) \) the category of additive covariant functors from \( C \) to the category of abelian groups \( \text{Ab} \), called the category of \( C \)-modules. This category has as objects the additive covariant functors from \( C \) to \( \text{Ab} \), and a morphism \( f : M_1 \to M_2 \) of \( C \)-modules is a natural transformation, that is, the set of morphisms \( \text{Hom}_C(M_1,M_2) \) from \( M_1 \) to \( M_2 \) is given by \( \text{Nat}(M_1,M_2) \).
Sometimes, we will write for short, $\mathcal{C}(-, ?)$ instead of $\text{Hom}_\mathcal{C}(-, ?)$ and when it is clear from the context we will simply use $(-, ?)$.

We now recall some properties of the category $\text{Mod}(\mathcal{C})$. The category $\text{Mod}(\mathcal{C})$ is a Grothendieck abelian category with the following properties:

1. A sequence
   $$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$
   is exact in $\text{Mod}(\mathcal{C})$ if and only if
   $$M_1(C) \xrightarrow{f} M_2(C) \xrightarrow{g} M_3(C)$$
   is an exact sequence of abelian groups for each $C$ in $\mathcal{C}$.

2. Let $\{M_i\}_{i \in I}$ be a family of $\mathcal{C}$-modules indexed by the set $I$. The $\mathcal{C}$-module $\prod_{i \in I} M_i$ defined by $(\prod_{i \in I} M_i)(C) = \prod_{i \in I} M_i(C)$ for all $C$ in $\mathcal{C}$ is a direct sum for the family $\{M_i\}_{i \in I}$ in $\text{Mod}(\mathcal{C})$, where $\prod_{i \in I} M_i(C)$ is the direct sum in $\text{Ab}$ of the family of abelian groups $\{M_i(C)\}_{i \in I}$. The $\mathcal{C}$-module $\prod_{i \in I} M_i$ defined by $(\prod_{i \in I} M_i)(C) = \prod_{i \in I} M_i(C)$ for all $C$ in $\mathcal{C}$ is a product for the family $\{M_i\}_{i \in I}$ in $\text{Mod}(\mathcal{C})$, where $\prod_{i \in I} M_i(C)$ is the product in $\text{Ab}$.

3. (Yoneda lemma) The $\mathcal{C}$-module $(C, -)$ given by $(C, -)(X) = C(C, X)$ for each $X$ in $\mathcal{C}$ has the property that for $M \in \text{Mod}(\mathcal{C})$ the map $((C, -), M) \rightarrow M(C)$ given by $f \mapsto f_{C(1_C)}$ for each $\mathcal{C}$-morphism $f : (C, -) \rightarrow M$, is an isomorphism of abelian groups. We will often consider this isomorphism an identification. Thus:
   (a) The functor $P : \mathcal{C} \rightarrow \text{Mod}(\mathcal{C})$ given by $P(C) = (C, -)$ is fully faithful.
   (b) For each family $\{C_i\}_{i \in I}$ of objects in $\mathcal{C}$, the $\mathcal{C}$-module $\prod_{i \in I} P(C_i)$ is a projective $\mathcal{C}$-module.
   (c) Given a $\mathcal{C}$-module $M$, there is a family $\{C_i\}_{i \in I}$ of objects in $\mathcal{C}$ such that there is an epimorphism $\prod_{i \in I} P(C_i) \rightarrow M \rightarrow 0$.

The reader can see [8] and [4] for more details.

### 2.2. Linear filters of $\mathcal{C}$

In [10], the authors introduced the notion of a linear filter in preadditive categories and they gave the definitions of Gabriel filter, torsion theory and annihilator of ideals in the category $\text{Mod}(\mathcal{C})$. They also established a bijective correspondence between hereditary torsion theories and linear filters.

The following are some of the basic notions introduced in [10].

**Definition 2.1.** An additive subfunctor $I(C, -)$ of the functor $\text{Hom}_\mathcal{C}(C, -)$ is called a **left ideal** of $\text{Hom}_\mathcal{C}(C, -)$.

We will sometimes write $I$ instead of $I(C, -)$ when it is clear from the context that $I$ is a subfunctor of $\text{Hom}_\mathcal{C}(C, -)$. With the above definition, we recall that a left ideal in an additive category $\mathcal{C}$ is a collection of left ideals

$$\{I(C, -) \subseteq \text{Hom}_\mathcal{C}(C, -) \mid C \in \mathcal{C}\}.$$

Similarly, we can define a right ideal of $\mathcal{C}$. A two sided ideal of the category $\mathcal{C}$ is an additive subfunctor of the two variable functor $\text{Hom}_\mathcal{C}(-, ?) : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \text{Ab}$.

**Definition 2.2.**

(a) Let $N \in \text{Mod}(\mathcal{C})$ be and $K$ a submodule of $N$. By considering $C \in \mathcal{C}$ and $x \in N(C)$, we define the following $\mathcal{C}$-module

$$(K(-) : x) : \mathcal{C} \rightarrow \text{Ab},$$

as follows: for $C' \in \mathcal{C}$ we set

$$K(C') = K(C') \cap N(C')$$
(K(−) : x)(C′) := \{f ∈ \text{Hom}_C(C, C′) | N(f)(x) ∈ K(C′)\}.

(b) Let 0 be the zero module in Mod(\mathcal{C}). Since 0 is a submodule of N, we define the **annihilator of** \(x \in N(\mathcal{C})\) denoted by \(\text{Ann}(x, −)\) as follows:

\[
\text{Ann}(x, −) := (0(−) : x).
\]

That is, \(\text{Ann}(x, −)(C′) := \{f ∈ \text{Hom}_C(C, C′) | N(f)(x) = 0\} \) for \(C′ ∈ \mathcal{C}\).

It is easy to see that \((K(−) : x)\) is a left ideal of \(\text{Hom}_C(−, −)\) and that for each \(x ∈ N(\mathcal{C})\) we have the left ideal \(\text{Ann}(x, −)\) of \(\text{Hom}_C(−, −)\).

**Remark 2.3.**

(a) Taking \(N = \text{Hom}_C(−, −)\) for some \(C ∈ \mathcal{C}\) and \(K = I(−, −)\) a left ideal of \(N\), for \(h ∈ N(B) = \text{Hom}_C(B, C)\) we get the following \(\mathcal{C}\)-module

\[
(I(−, −) : h) : \mathcal{C} → \text{Ab},
\]

given as follows: for \(C′ ∈ \mathcal{C}\),

\[
(I(−, −) : h)(C′) := \{f ∈ \text{Hom}_C(B, C′) | f \circ h ∈ I(C, C′)\}.
\]

Thus, \((I(−, −) : h)\) is a left ideal of \(\text{Hom}_C(B, −)\).

(b) Notice that \((I(−, −) : h)\) is given by the following pullback in \(\text{Mod}(\mathcal{C})\)

\[
\begin{array}{ccc}
(I(−, −) : h) & \xrightarrow{\gamma_f^C} & I(−, −) \\
\downarrow & & \downarrow \delta_f \\
\text{Hom}_C(B, −) & \xrightarrow{\text{Hom}_C(h, −)} & \text{Hom}_C(C, −)
\end{array}
\]

where \(\gamma_f^C := \text{Hom}_C(h, −)|_{(I(−, −) : h)}\) and \(\delta_f\) is the inclusion of \(I(−, −)\) into \(\text{Hom}_C(C, −)\). In other words, if \(α := \text{Hom}_C(h, −)\) we have that

\[
(I(−, −) : h) = α^{-1}(I(−, −)).
\]

For future reference, let us denote by \(\delta_{α^{-1}(I)} : α^{-1}(I(−, −)) → \text{Hom}_C(B, −)\) the canonical inclusion.

Now, we will consider the following definitions.

**Definition 2.4.** [10, Definition 2.2] Let \(\mathcal{F}_C\) be a family of left ideals of \(\text{Hom}_C(−, −)\). It is said that \(\mathcal{F}_C\) is a **left filter** of \(\text{Hom}_C(−, −)\) if the following conditions below hold:

\(\mathcal{T}_1\) If \(I ∈ \mathcal{F}_C\) and \(I ⊆ J\) then \(J ∈ \mathcal{F}_C\),

\(\mathcal{T}_2\) If \(I, J ∈ \mathcal{F}_C\) then \(I ∩ J ∈ \mathcal{F}_C\).

A collection \(\mathcal{F} := \{\mathcal{F}_C\}_{C ∈ \mathcal{C}}\) is a **left linear filter for the category** \(\mathcal{C}\) if \(\mathcal{F}_C\) is a filter for \(\text{Hom}_C(−, −)\) for all \(C ∈ \mathcal{C}\) and

\(\mathcal{T}_3\) If \(I ∈ \mathcal{F}_C\) and \(h : C → B\) is a morphism in \(\mathcal{C}\), then \((I(−, −) : h) ∈ \mathcal{F}_B\) (see 2.3(a)).

A collection \(\mathcal{F} := \{\mathcal{F}_C\}_{C ∈ \mathcal{C}}\) is a **left Gabriel filter for the category** \(\mathcal{C}\) if \(\mathcal{F} = \{\mathcal{F}_C\}_{C ∈ \mathcal{C}}\) is a linear filter and the following holds:

\(\mathcal{T}_4\) Let \(J(−, −) ∈ \mathcal{F}_C\) be and \(I(−, −)\) an ideal satisfying that for each \(B ∈ \mathcal{C}\) the ideal \((I(−, −) : h)\) belongs to \(\mathcal{F}_B\) for all \(h ∈ J(−, −) ⊆ \text{Hom}_C(C, B)\), then \(I(−, −) ∈ \mathcal{F}_C\).

**Definition 2.5.** [10]

(i) Let \(\mathcal{F} := \{\mathcal{F}_C\}_{C ∈ \mathcal{C}}\) be a left linear filter for \(\mathcal{C}\). We define

\[
\mathcal{T}_\mathcal{F} := \{M ∈ \text{Mod}(\mathcal{C}) | \text{for each } C ∈ \mathcal{C}, \text{ Ann}(x, −) ∈ \mathcal{F}_C \text{ } ∀x ∈ M(C)\}.
\]
Let \( T \) be a hereditary pretorsion class in \( \text{Mod}(\mathcal{C}) \). We define \( \mathcal{F}_T := \{ \mathcal{F}_C \}_{C \in \mathcal{C}} \) where

\[
\mathcal{F}_C := \left\{ I \subseteq \text{Hom}_C(C, -) \mid \frac{\text{Hom}_C(C, -)}{I} \in T \right\}.
\]

**Definition 2.6.** Let \( \mathcal{F} = \{ \mathcal{F}_C \}_{C \in \mathcal{C}} \) be a left filter in \( \mathcal{C} \). We say that \( M \in \text{Mod}(\mathcal{C}) \) is an \( \mathcal{F} \)-torsion module if \( M \in \mathcal{T}_F \).

We recall that a class \( \mathcal{A} \subseteq \text{Mod}(\mathcal{C}) \) is a pretorsion class if it is closed under quotient objects and coproducts. A pretorsion class is called hereditary if it is closed under subobjects. A hereditary torsion class is a hereditary pretorsion class that is closed under extensions. Therefore, we can consider the following results.

**Theorem 2.7.** [10, Theorem 2.5] The maps \( \mathcal{F} \to \mathcal{T}_F, \mathcal{T} \to \mathcal{F}_T \) induce a bijection between hereditary pretorsion classes of \( \text{Mod}(\mathcal{C}) \) and left linear filters on \( \mathcal{C} \).

**Theorem 2.8.** [10, Theorem 2.6] The maps \( \mathcal{F} \to \mathcal{T}_F, \mathcal{T} \to \mathcal{F}_T \) induce a bijection

\[
\begin{array}{c}
\{ \text{Left Gabriel filters of } \mathcal{C} \} \\
\uparrow \psi^{-1} \\
\downarrow \psi \\
\{ \text{Hereditary torsion classes of } \text{Mod}(\mathcal{C}) \} \end{array}
\]

We also recall that a preradical \( t \) of \( \text{Mod}(\mathcal{C}) \) is simply a subfunctor of the identity functor \( 1_{\text{Mod}(\mathcal{C})} : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C}) \). A preradical \( t \) is called radical if \( t(M) = 0 \) for all \( M \in \text{Mod}(\mathcal{C}) \).

Let \( \mathcal{T} \) be a pretorsion class in \( \text{Mod}(\mathcal{C}) \). We can construct a preradical \( t_T \) associated with this pretorsion class as follows: For \( M \in \text{Mod}(\mathcal{C}) \)

\[
t_T(M) = \sum_{N \in T, N \subseteq M} N,
\]

and for \( f : M \to N \), we have \( t_T(f) := f_{| t_T(M)} \).

Conversely, let \( t \) be a preradical in \( \text{Mod}(\mathcal{C}) \). We construct the class \( \mathcal{T}_t := \{ M \in \text{Mod}(\mathcal{C}) \mid t(M) = M \} \). We have the following well-known results.

**Proposition 2.9.** The maps \( \mathcal{T} \to t_T \) and \( t \to \mathcal{T}_t \) give a bijective correspondence between left exact preradicals of \( \text{Mod}(\mathcal{C}) \) and hereditary pretorsion classes of \( \text{Mod}(\mathcal{C}) \).

**Proof.** See [14, Corollary 1.8] on page 138.

**Proposition 2.10.** The maps \( \mathcal{T} \to t_T \) and \( t \to \mathcal{T}_t \) give a bijective correspondence

\[
\begin{array}{c}
\{ \text{Hereditary torsion classes of } \text{Mod}(\mathcal{C}) \} \\
\uparrow \psi^{-1} \\
\downarrow \psi \\
\{ \text{Left exact radicals of } \text{Mod}(\mathcal{C}) \} \end{array}
\]

**Proof.** See [14, Proposition 3.1] on page 141.

Now, consider a left Gabriel filter \( \mathcal{F} := \{ \mathcal{F}_C \}_{C \in \mathcal{C}} \) in \( \mathcal{C} \). By 2.8, we obtain the hereditary torsion class
\[ T_F := \{ M \in \text{Mod}(C) \mid \text{for each } C \in C, \quad \text{Ann}(x, -) \in F_C \quad \forall x \in M(C) \}. \]

By 2.10, we get the corresponding left exact radical \( t \), (we use \( t \) instead of \( t_{T_F} \) for simplified notation) defined as:

\[ t(M) = \sum_{N \in T_F, N \subseteq M} N. \]

**Remark 2.11.** Let \( F := \{ F_C \}_{C \in C} \) be a left Gabriel filter in \( C \) and \( t \) the radical associated with the filter \( F \) via the bijections 2.8 and 2.10. Let us note that, \( M \) is an \( F \)-torsion module if and only if \( t(M) = M \).

### 3. Prelocalization functor

Let \( F := \{ F_C \}_{C \in C} \) be a left linear filter on \( C \) as defined in 2.4. For each \( C \in C \) the set of left ideals \( F_C \) is a directed set with the order defined as follows:

\[ J \leq I \iff I \subseteq J. \]

Recall that we will write \( I \) instead of \( I(C, -) \) when it is clear from the context that \( I \) is a subfunctor of \( \text{Hom}_C(C, -) \). Now, let \( I \subseteq J \) in \( F_C \), and let us denote by \( \mu_{I,J} : I \to J \) the canonical inclusion. For \( M \in \text{Mod}(C) \), we have a morphism of abelian groups

\[ \hat{\lambda}_{I,J} := \text{Hom}_\text{Mod}(C)(\mu_{I,J}, M) : \text{Hom}_\text{Mod}(C)(J, M) \to \text{Hom}_\text{Mod}(C)(I, M). \]

We then have a directed system of abelian groups

\[ \{ \hat{\lambda}_{I,J} : \text{Hom}_\text{Mod}(C)(I, M) \to \text{Hom}_\text{Mod}(C)(J, M) \}_{J \leq I}. \]

Thus, we can form the abelian group \( \varinjlim_{I \in F_C} \text{Hom}_\text{Mod}(C)(I, M) \). For each \( I \in F_C \), we will denote by \( \varphi_{I,M} : \text{Hom}_\text{Mod}(C)(I, M) \to \varinjlim_{I \in F_C} \text{Hom}_\text{Mod}(C)(I, M) \) the canonical morphisms into the direct limit. Now we define a functor

\[ \mathbb{L}(M) : C \to \text{Ab} \]

as follows:

(i) \( \mathbb{L}(M)(C) := \varinjlim_{I \in F_C} \text{Hom}_\text{Mod}(C)(I, M) \).

(ii) If \( h : C \to B \) is a morphism in \( C \), we need to construct

\[ \mathbb{L}(M)(h) : \mathbb{L}(M)(C) \to \mathbb{L}(M)(B). \]

Indeed, we have \( \alpha := \text{Hom}_C(h, -) : \text{Hom}_C(B, -) \to \text{Hom}_C(C, -) \). By 2.3(b) we get that \( (I(C, -) : h) := \alpha^{-1}(I(C, -)) \), and the following commutative diagram

\[ \begin{array}{ccc}
\alpha^{-1}(I(C, -)) & \xrightarrow{\gamma_{I,C}} & I(C, -) \\
\downarrow & & \downarrow \delta_I \\
\text{Hom}_C(B, -) & \xrightarrow{\text{Hom}_C(h, -)} & \text{Hom}_C(C, -)
\end{array} \]

where \( \gamma_{I,C} := \text{Hom}_C(h, -)|_{(I(C, -) : h)} \) and \( \delta_I : I(C, -) \to \text{Hom}_C(C, -) \) is the inclusion. Using the universal property of the pullback for \( I(C, -) \subseteq J(C, -) \in F_C \) with inclusion \( \mu_{I,J} : I(C, -) \to J(C, -) \), we obtain the diagram
where $\mu_{x^{-1}(I), x^{-1}(J)} : x^{-1}(I) \to x^{-1}(J)$ denotes the inclusion of $x^{-1}(I)$ into $x^{-1}(J)$. Applying $\text{Hom}_C(-, M)$ to the previous diagram, we get

\[
\begin{array}{ccc}
\text{Hom}_C(I, M) & \xrightarrow{\eta_{I, M}} & \text{Hom}_C(I, M) \\
\downarrow{\lambda_{I, J}} & & \downarrow{\lambda_{I, J}} \\
\text{Hom}_C(I, M) & \xrightarrow{\eta_{I, M}} & \text{Hom}_C(I, M)
\end{array}
\]

Now, let us consider the canonical morphisms into the direct limit

\[
\phi_{I, M}^C : \text{Hom}_C(I, M) \to \varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(I, M).
\]

Since $x^{-1}(I) \in \mathcal{F}_B$ ($\mathcal{F} = \{ \mathcal{F}_C \}_{C \in C}$ satisfies $T3$), we also have the following canonical morphisms

\[
\phi_{x^{-1}(I), M}^B : \text{Hom}_C(x^{-1}(I), M) \to \varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(x^{-1}(I), M)
\]

into the direct limit $\varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(I, M)$. So, there exists a unique morphism $\mathbb{L}(M)(h) : \mathbb{L}(M)(C) \to \mathbb{L}(M)(B)$ such that the following diagram commutes for all $I \in \mathcal{F}_C$

\[
\begin{array}{ccc}
\mathbb{L}(M)(C) & \xrightarrow{\phi_{I, M}^C} & \mathbb{L}(M)(B) \\
\downarrow{\phi_{I, M}^C} & & \downarrow{\phi_{I, M}^C} \\
\text{Hom}_C(I, M) & \xrightarrow{\eta_{I, M}^C} & \text{Hom}_C(I, M)
\end{array}
\]

(1)

**Definition 3.1.** We define the functor $\mathbb{L} : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C})$ as follows:

(a) For $M \in \text{Mod}(\mathcal{C})$ we set $\mathbb{L}(M) \in \text{Mod}(\mathcal{C})$ as the functor defined above.

(b) Let $\eta : M \to N$ be a natural transformation. For each $C \in \mathcal{C}$, there exists a unique morphism of abelian groups

\[
\tilde{\eta}_C : \varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(I, M) \to \varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(I, N)
\]

such that the following diagram commutes

\[
\begin{array}{ccc}
\varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(I, M) & \xrightarrow{\tilde{\eta}_C} & \varinjlim_{I \in \mathcal{F}_C} \text{Hom}_C(I, N) \\
\downarrow{\phi_{I, M}^C} & & \downarrow{\phi_{I, N}^C} \\
\text{Hom}_C(I, M) & \xrightarrow{\eta_{I, M}} & \text{Hom}_C(I, N)
\end{array}
\]

(2)

We define $\mathbb{L}(\eta := \tilde{\eta}$, where $\eta := \{ \tilde{\eta}_C \}_{C \in \mathcal{C}}$. 
Proposition 3.2. The functor \(L : \text{Mod}(C) \to \text{Mod}(C)\) is left exact.

Proof. This follows from the fact that \(\text{Hom}_{\text{Mod}(C)}(I, -)\) is left exact and \(\varprojlim\) is exact because \(\text{Ab}\) is a Grothendieck category. \(\square\)

Now, let us consider \(C \in C\) and \(I \subseteq J\) in \(\mathcal{F}_C\). We then have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{\delta^C_{I,M}} & \text{Hom}_{\text{Mod}(C)}(I, M) \\
\mbox{id} & & \mbox{id} \\
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{\delta^C_{I,M}} & \text{Hom}_{\text{Mod}(C)}(I, M)
\end{array}
\]

where \(\delta^C_{I,M} := \text{Hom}_{\text{Mod}(C)}(\delta_I, M)\) and \(\delta^C_{I,M} := \text{Hom}_{\text{Mod}(C)}(\delta_I, M)\).

So, it induces a morphism

\[
[\psi^C_M] : \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) \to \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M).
\]

such that the following diagram commutes for each \(I \in \mathcal{F}_C\):

\[
\begin{array}{ccc}
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{[\psi^C_M]} & \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M) \\
\mbox{id} & & \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M) \\
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{\delta^C_{I,M}} & \text{Hom}_{\text{Mod}(C)}(I, M)
\end{array}
\]

(3)

Now consider the Yoneda isomorphism \(Y_C : M(C) \to \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M)\), then we obtain the following commutative diagram for each \(I \in \mathcal{F}_C\):

\[
\begin{array}{ccc}
M(C) & \xrightarrow{Y_C} & \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) \\
\mbox{id} & & \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M) \\
M(C) & \xrightarrow{Y_C} & \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) \\
\mbox{id} & & \text{Hom}_{\text{Mod}(C)}(I, M)
\end{array}
\]

Now, let us consider \(C \in C\) and \(I \subseteq J\) in \(\mathcal{F}_C\). We then have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{\delta^C_{I,M}} & \text{Hom}_{\text{Mod}(C)}(I, M) \\
\mbox{id} & & \mbox{id} \\
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{\delta^C_{I,M}} & \text{Hom}_{\text{Mod}(C)}(I, M)
\end{array}
\]

where \(\delta^C_{I,M} := \text{Hom}_{\text{Mod}(C)}(\delta_I, M)\) and \(\delta^C_{I,M} := \text{Hom}_{\text{Mod}(C)}(\delta_I, M)\).

So, it induces a morphism

\[
[\psi^C_M] : \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) \to \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M).
\]

such that the following diagram commutes for each \(I \in \mathcal{F}_C\):

\[
\begin{array}{ccc}
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{[\psi^C_M]} & \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M) \\
\mbox{id} & & \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M) \\
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) & \xrightarrow{\delta^C_{I,M}} & \text{Hom}_{\text{Mod}(C)}(I, M)
\end{array}
\]

(3)

Now consider the Yoneda isomorphism \(Y_C : M(C) \to \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M)\), then we obtain the following commutative diagram for each \(I \in \mathcal{F}_C\):

\[
\begin{array}{ccc}
M(C) & \xrightarrow{Y_C} & \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) \\
\mbox{id} & & \varprojlim_{I \in \mathcal{F}_C} \text{Hom}_{\text{Mod}(C)}(I, M) \\
M(C) & \xrightarrow{Y_C} & \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), M) \\
\mbox{id} & & \text{Hom}_{\text{Mod}(C)}(I, M)
\end{array}
\]

(3)

Definition 3.3. Let \(M \in \text{Mod}(C)\). For each \(C \in C\), we define \([\varphi^C_M] : M(C) \to \mathbb{L}(M)(C)\) as \([\varphi^C_M] := [\psi^C_M] \circ Y_C\).

Thus, we have the following result.

Proposition 3.4. There exists a morphism \(\varphi^C_M : M \to \mathbb{L}(M)\) in \(\text{Mod}(C)\) defined as \([\varphi^C_M] := [\psi^C_M] \circ Y_C\) for each \(C \in C\).

Proof. Let \(h : C \to B\) be a morphism in \(C\). We must check that the following equality holds:

\[\mathbb{L}(M)(h) \circ [\varphi^C_M] = [\varphi^B_M] \circ M(h).\]

Let \(x \in M(C)\), then \(Y_C(x) := \eta_x : \text{Hom}_C(C, -) \to M\). By the commutativity of the diagram (3) above, we get that \([\varphi^C_M](x) = \varphi^C_{I,M}(\delta^C_{I,M}(\eta_x)) \in \mathbb{L}(M)(C)\). Since \(\delta^C_{I,M}(\eta_x) \in \text{Hom}_{\text{Mod}(C)}(I, M)\), by diagram (1) before Definition 3.1, we obtain
\[ \mathbb{L}(M)(h)([\varphi_M]_C(x)) = \varphi_{x^{-1}(I),M}^C(\text{Hom}_{\text{Mod}(C)}(\gamma^C_1,M)(\theta^C_{I,M}(\eta_x))) \]
\[ = \varphi_{x^{-1}(I),M}^C((\theta^C_{I,M}(\eta_x)) \circ \gamma^C_1) \]
\[ = \varphi_{x^{-1}(I),M}^C((\eta_x \circ \delta_I) \circ \gamma^C_1), \quad \text{[def. of } \theta^C_{I,M} \text{]} \]
\[ = \varphi_{x^{-1}(I),M}^C((\eta_x \circ \text{Hom}_C(h,-) \circ \delta_{x^{-1}(I)}), \quad \text{[Remark 2.3(b)]} \]

We have, however, that \( \eta_x \circ \text{Hom}_C(h,-) : \text{Hom}_C(B,-) \to M \) is such that \( [\eta_x \circ \text{Hom}_C(h,-)]_B([1_B]) = [\eta_x]_B(h) = M(h)(x) \) \[ \text{[def. of Yoneda iso].} \]

Thus, we get that \( \eta_x \circ \text{Hom}_C(h,-) = \eta_{M(h)(x)}. \) Therefore, we conclude that

\[ \mathbb{L}(M)(h)([\varphi_M]_C(x)) = \varphi_{x^{-1}(I),M}^C(\eta_{M(h)(x)} \circ \delta_{x^{-1}(I)}). \]

On the other hand, by diagram (3) above \( (B \text{ instead of } C \text{ and } x^{-1}(I) \text{ instead of } I) \) and since \( Y_B(M(h)(x)) = \eta_{M(h)(x)} : \text{Hom}_C(B,-) \to M, \) we get the equalities

\[ [\varphi_M]_B(M(h)(x)) = [\psi_M]_B(Y_B(M(h)(x))) \quad \text{[def. of } [\varphi_M]_B \text{]} \]
\[ = \varphi_{x^{-1}(I),M}^B(\theta^B_{\varphi^{-1}(I),M}(Y_B(M(h)(x)))) \]
\[ = \varphi_{x^{-1}(I),M}^B(\theta^B_{\varphi^{-1}(I),M}(\eta_{M(h)(x)})) \]
\[ = \varphi_{x^{-1}(I),M}^B(\eta_{M(h)(x)} \circ \delta_{\varphi^{-1}(I)}) \quad \text{[def. of } \theta^B_{\varphi^{-1}(I),M} \text{].} \]

This proves the equality required. Thus \( \varphi_M : M \to \mathbb{L}(M) \) is a morphism in \( \text{Mod}(C). \)

\[ \square \]

**Proposition 3.5.** Consider the functors \( 1_{\text{Mod}(C)}, \mathbb{L} : \text{Mod}(C) \to \text{Mod}(C). \) There is a natural transformation \( \varphi : 1_{\text{Mod}(C)} \to \mathbb{L}. \)

**Proof.** For \( C \in C, \) we must check that the following diagram commutes

\[ \begin{array}{ccc}
M(C) & \xrightarrow{[\varphi_M]_C} & \mathbb{L}(M)(C) \\
\downarrow \eta_C & & \downarrow \eta_C \\
N(C) & \xrightarrow{[\varphi_N]_C} & \mathbb{L}(N)(C).
\end{array} \]

On one hand, we have that

\[ \eta_C \circ [\varphi_M]_C = [\eta]_C \circ [\psi_M]_C \circ Y_C \]
\[ = [\eta]_C \circ \varphi_{I,M}^C \circ \theta_{I,M}^C \circ Y_C \quad \text{[diagram (2)]} \]
\[ = \varphi_{I,N}^C \circ \text{Hom}_{\text{Mod}(C)}(I,\eta) \circ \theta_{I,M}^C \circ Y_C \quad \text{[diagram (3)]} \]

For \( x \in M(C), \) we get \( Y_C(x) := \alpha_x : \text{Hom}_C(C,-) \to M. \) Thus, \( \theta_{I,M}^C(\alpha_x) = \alpha_x \delta_I \) with \( \delta_I : I \to \text{Hom}_C(C,-) \) the inclusion. Therefore, \( \eta_C \circ [\varphi_M]_C(x) = \varphi_{I,N}^C(\eta \circ \alpha_x \circ \delta_I). \)

On the other hand, we have that

\[ [\varphi_N]_C \circ \eta_C = \varphi_{I,N}^C \circ \theta_{I,N}^C \circ Y_C \circ \eta_C \quad \text{[diagram (3) with } N \text{ instead of } M]. \]

For \( x \in M(C) \) we get \( y := \eta_C(x) \in N(C), \) then \( Y_C(y) = \gamma_y : \text{Hom}_C(C,-) \to N. \) Hence, we obtain that \( ([\varphi_N]_C \circ \eta_C)(x) = \varphi_{I,N}^C(\gamma_y \circ \delta_I). \)

We assert that \( \gamma_y = \eta \circ \alpha_x. \) Indeed, for \( C' \in C, \) the morphism \( [\eta \circ \alpha_x]_{C'} : \text{Hom}_C(C',-) \to N(C') \) is defined as \( [\eta \circ \alpha_x]_{C'}(f) = [\eta]_{C'}([\alpha_x]_{C'}(f)) = \eta_C(M(f)(x)). \)
On the other hand, \([\gamma_\gamma]_C(f) := N(f)(y) = N(f)(\eta_C(x))\). Since \(\eta : M \to N\) is a natural transformation, we get that \(N(f) \circ \eta_C = \eta_C \circ M(f)\), and then \([\gamma_\gamma]_C(f) := N(f)(y) = N(f)(\eta_C(x)) = \eta_C(M(f)(x)) = [\eta \circ \alpha_x]_C(f)\). We conclude that \(\gamma_\gamma = \eta \circ \alpha_x\). Hence,

\[
(\tilde{\eta}_C \circ [\varphi_M]_C)(x) = \varphi_{I,N}^C(\eta \circ \alpha_x \circ \delta_I) = \varphi_{I,N}^C(\gamma_\gamma \circ \delta_I) = ([\varphi_N]_C \circ \eta_C)(x)
\]

Therefore, the required diagram commutes. \(\square\)

**Proposition 3.6.** Let \(\mathcal{F} := \{\mathcal{F}_C\}_{C \subseteq C}\) be a left Gabriel filter in \(\text{Mod}(C)\), and consider the morphism \(\varphi_M : M \to I(M)\) given in 3.4. The \(\text{Ker}(\varphi_M) = t(M)\) where \(t\) is the radical associated with the filter \(\mathcal{F}\).

**Proof.** Let \(K = \text{Ker}(\varphi_M)\) be and \(\psi : K \to M\) the canonical inclusion. Given the filter \(\mathcal{F} := \{\mathcal{F}_C\}_{C \subseteq C}\), the corresponding torsion class is

\[
\mathcal{T}_\mathcal{F} := \{M \in \text{Mod}(C) \mid \text{for each } C \in \mathcal{C}, \text{ Ann}(x, -) \in \mathcal{F}_C \forall x \in M(C)\},
\]

and the radical \(t\) is defined as

\[
t(M) = \sum_{N \in \mathcal{T}_\mathcal{F}, N \subseteq M} N.
\]

Therefore, \(t(M)(C) = \sum_{N \in \mathcal{T}_\mathcal{F}, N \subseteq M} N(C)\). Let \(x \in K(C)\) be, then we get that \(0 = [\varphi_M]_C(x)\).

Thus, by definition of \([\varphi_M]_C\) and by diagram (3), we conclude that \(\phi_{f,I,M}^C(\theta_{f,I}(Y_C(x))) = 0\) for some \(I \in \mathcal{F}_C\) where \(Y_C(x) := \eta_x : \text{Hom}_C(C, -) \to M\) is such that \([\eta_x]_C(1_C) = x\) (Yoneda isomorphism). Hence, \(\theta_{f,I}(Y_C(x)) = \text{Hom}_{\text{Mod}(C)}(\delta_I, M)(\eta_x) = \eta_x \circ \delta_I : I \to M\).

Since \(\mathcal{F}_C\) is a directed set, we have that \(\phi_{f,I,M}^C(\theta_{f,I}(Y_C(x))) = \phi_{f,I,M}^C(\eta_x \circ \delta_I) = 0\) implies that there exists \(I \geq J (I \subseteq J)\) in \(\mathcal{F}_C\) such that \(\lambda_{j,I}(\eta_x \circ \delta_I) = 0\) in \(\text{Hom}_{\text{Mod}(C)}(I, M)\), see, Lemma 5.30 (ii) in [13]. We recall that \(\lambda_{j,I} := \text{Hom}_{\text{Mod}(C)}(\mu_{I,J}, M)\), then we get that

\[
0 = \lambda_{j,I}(\eta_x \circ \delta_I) = \text{Hom}_{\text{Mod}(C)}(\mu_{I,J}, M)(\eta_x \circ \delta_I) = \eta_x \circ \delta_I \circ \mu_{I,J} = \eta_x \circ \delta_I,
\]

where \(\delta_I : I \to \text{Hom}_C(C, -)\) is the inclusion. By 2.7, it follows that

\[
\mathcal{F}_C := \left\{I(C, -) \subseteq \text{Hom}_C(C, -) \mid \frac{\text{Hom}_{\text{Mod}(C)}(C, -)}{I(C, -)} \in \mathcal{T}_\mathcal{F}\right\}.
\]

Hence, \(\frac{\text{Hom}_{\text{Mod}(C)}(C, -)}{I(C, -)} \in \mathcal{T}_\mathcal{F}\) since \(I \in \mathcal{F}_C\) (by the above equality). Since \(\eta_x \circ \delta_I = 0\), there exists \(\tilde{\eta}_x : \frac{\text{Hom}_{\text{Mod}(C)}(C, -)}{I(C, -)} \to M\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Hom}_C(C, -) & \xrightarrow{\eta_x} & M \\
\downarrow{\pi_C} & & \\
\text{Hom}_C(C, -)/I(C, -) & \xrightarrow{\tilde{\eta}_x} & M \\
\end{array}
\]

Therefore, \(\tilde{\eta}_x(1_C + I(C, C)) = x\). Now, let us consider the factorization of \(\tilde{\eta}_x\)

\[
\begin{array}{ccc}
\text{Hom}_C(C, -)/I(C, -) & \to & N \\
\downarrow{\tilde{\eta}_x} & & \downarrow{M} \\
M & \to & M
\end{array}
\]

through its image. Since \(\mathcal{T}_\mathcal{F}\) is closed under quotients, we conclude that \(N \in \mathcal{T}_\mathcal{F}\) and \(N \subseteq M\).

Next, \(\tilde{\eta}_x(1_C + I(C, C)) = x\) implies that \(x \in N(C)\). Therefore, \(x \in t(M)(C) = \sum_{N \in \mathcal{T}_\mathcal{F}, N \subseteq M} N(C)\), proving that \(K(C) \subseteq t(M)(C)\).

On the other hand, since \(t\) is the radical associated with \(\mathcal{T}_\mathcal{F}\), by 2.10, we get that \(t(M) \in \mathcal{T}_\mathcal{F}\).
Next, recall that \( t(M)(C) \subseteq M(C) \). If \( x \in t(M)(C) \), we get that \( I(C, -) := \text{Ann}(x, -) \in \mathcal{F}_C \), since \( t(M) \in \mathcal{T}_F \) and the description of \( \mathcal{T}_F \) in 2.5. We know that \( \text{Ann}(x, C') := \{ f \in \text{Hom}(C, C') \mid M(f)(x) = 0 \} \). Let \( \delta_1 : I(C, -) \to \text{Hom}_C(C, -) \) be the inclusion. Consider \( Y_C(x) := \eta_x : \text{Hom}_C(C, -) \to M \) such that \( [\eta_x]_C(1_C) = x \) (Yoneda isomorphism). Let us show that \( \eta_x \circ \delta_1 = 0 \).

For \( C' \in C \), we must to see that \( [\eta_x]_{C'} \circ [\delta_1]_{C'} = 0 \). By the construction of the Yoneda isomorphism, we obtain that \( [\eta_x]_{C'} : \text{Hom}_C(C', C') \to M(C') \) satisfies that \( [\eta_x]_{C'}(f) := M(f)(x) \) \( \forall \ f \in \text{Hom}_C(C, C') \). So, for \( f \in I(C, C') \) we have that \( [\eta_x]_{C'}([\delta_1]_{C'}(f)) = [\eta_x]_{C'}(f) = M(f)(x) = 0 \) since \( [\delta_1]_{C'} \) is the inclusion and \( f \in I(C, C') = \text{Ann}(x, C') \). We conclude that \( \eta_x \circ \delta_1 = 0 \).

Therefore, by diagram (3), we get that

\[
[\varphi_M]_C(x) = \varphi_{I,M}^C(\theta_{I,M}^C(Y_C(x))) = \varphi_{I,M}^C(\theta_{I,M}(\eta_x)) = \varphi_{I,M}^C(\eta_x \circ \delta_1) = 0.
\]

This implies by definition of \( K \) that \( x \in K(C) \). Therefore, \( t(M)(C) \subseteq K(C) \), and then we conclude that \( K = t(M) \).

**Proposition 3.7.** Let \( M \in \text{Mod}(C) \). Then \( M \) is an \( \mathcal{F} \)-torsion module if and only if \( L(M) = 0 \).

**Proof.** Let us suppose that \( M \) is an \( \mathcal{F} \)-torsion module. We known that an element \( w \in L(M)(C) \) is of the form \( w = \varphi_{I,M}^C(\beta) \) for some ideal \( I(C, -) \in \mathcal{F}_C \) and \( \beta : I(C, -) \to M \), see, Lemma 5.30 (i) in [13]. Let \( x : K(C, -) \to I(C, -) \) be the kernel of \( \beta \). We are going to prove that \( K(C) \in \mathcal{F}_C \).

Let \( C' \in C \) and \( f \in I(C, C') \), then \( \beta_{C'}(f) \in M(C') \). By hypothesis, \( M \) is an \( \mathcal{F} \)-torsion module, and then \( \text{Ann}(\beta_{C'}(f), -) \in \mathcal{F}_{C'} \), see 2.6. Let us recall that, \( \text{Ann}(\beta_{C'}(f), -)(X) = \{ h : C' \to X \mid M(h)(\beta_{C'}(f)) = 0 \} \) for all \( X \in C \).

Now, we have the Yoneda isomorphism \( \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C', -), I(C, -)) \cong I(C, C') \). Therefore, \( f \) induces a morphism \( \eta_f : \text{Hom}_C(C', -) \to I(C, -) \) in \( \text{Mod}(C) \) such that \( [\eta_f]_X : \text{Hom}_C(C', X) \to I(C, X) \) is defined as \( [\eta_f]_X(h) := I(C, h)(f) = hf \) \( \forall h \in \text{Hom}_C(C', X) \) and for all \( X \in C \).

By the Yoneda isomorphism, \( \beta \circ \eta_f : \text{Hom}_C(C', -) \to M \) corresponds to \( [\beta \circ \eta_f]_C(1_C) \in M(C') \), but \( [\beta \circ \eta_f]_C(1_C) = \beta_{C'}(f) \). Similarly, \( \beta_{C'}(f) \in M(C') \) corresponds to \( \eta : \text{Hom}_C(C', -) \to M \) such that \( \eta_X(h) = M(h)(\beta_{C'}(f)) \) for all \( X \in C \) and for all \( h \in \text{Hom}_C(C', X) \). Thus, we conclude that \( \eta = \beta \circ \eta_f \).

We assert that \( (\beta \circ \eta_f) \mid \text{Ann}(\beta_{C'}(f), -) \) is 0. Indeed, we must see that for all \( X \)

\[
[\beta \circ \eta_f]_X(h) = 0.
\]

Indeed, let \( h : C' \to X \) in \( \text{Ann}(\beta_{C'}(f), X) \). Hence \( [\beta \circ \eta_f]_X(h) = \eta_X(h) = M(h)(\beta_{C'}(f)) = 0 \). This proves that \( (\beta \circ \eta_f) \mid \text{Ann}(\beta_{C'}(f), -) \) is 0.

Let \( u_C : \text{Ann}(\beta_{C'}(f), -) \to I(C, -) \) be the inclusion. Since \( \nu = \text{Ker}(\beta) \), there exists a unique morphism \( \psi_{C,f}^C \) such that the following diagram commutes:

\[
(*) : \quad \begin{array}{ccc}
\text{Ann}(\beta_{C'}(f), -) & \xrightarrow{\nu} & I(C, -) \\
\psi_{C,f}^C \downarrow & & \downarrow \beta \\
K(C, -) & \xrightarrow{\eta_f \circ \nu} & M.
\end{array}
\]

The family of morphisms \( \{ \psi_{C,f}^C \}_{f \in I(C, C')} \) induces a a morphism

\[
\Psi : \bigoplus_{f \in I(C, C')} \text{Ann}(\beta_{C'}(f), -) \to K(C, -).
\]
We define \( J(C, -) := \text{Im}(\Psi) \). We assert that \( \text{Ann}(\beta_C(f), -) \subseteq (I(C, -) : f) \). Indeed, since \( f \in I(C, -)(C') = I(C, C') \) and \( I(C, -) \subseteq I(C, -) \), then by definition (see 2.2(a)):
\[
(J(C, -) : f)(X) = (I(C, X) : f) = \{ h \in \text{Hom}_C(C', X) \mid I(C, h)(f) = hf \in I(C, X) \}.
\]

Now, by definition of \( J(C, -) \), we have that \( J(C, X) := \sum_{f \in I(C, C')} \text{Im}([\psi_{C,f}^C]_X) \).

Since \( v \) and \( u_C \) are the inclusions, for \( h \in \text{Ann}(\beta_C(f), X) \), we get that \([\psi_{C,f}^C]_X(h) = v_X([\psi_{C,f}^C]_X(h)) = [\eta]_X([\psi_{C,f}^C]_X(h)) = [\eta]_X(h) = hf \) (because of the diagram (+) and the definition of \( \eta \)). We conclude that \( hf \in \text{Im}([\psi_{C,f}^C]_X) \subseteq J(C, X) \). This tells us that \( h \in (J(C, -) : f)(X) \).

Therefore, \( \text{Ann}(\beta_C(f), -) \subseteq (J(C, -) : f) \; \forall f \in I(C, C') \). Since \( \text{Ann}(\beta_C(f), -) \in \mathcal{F}_C \) \( (M \text{ is an } \mathcal{F}-\text{torsion module}), \) by \( T_1 \) in definition 2.4, we obtain that
\[
(J(C, -) : f) \in \mathcal{F}_C \; \forall f \in I(C, C').
\]

Now, \( I(C, -) \in \mathcal{F}_C \) implies that \( J(C, -) \in \mathcal{F}_C \) by \( T_4 \). Since \( J(C, -) \subseteq K(C, -) \), by \( T_1 \) we conclude that \( K(C, -) \in \mathcal{F}_C \).

Let us define \( \mu_{K,I} = v : K(C, -) \rightarrow I(C, -) \) as the canonical inclusion. Thus, it follows that \( \lambda_{K,I} \beta = \text{Hom}_{\text{Mod}(C)}(\mu_{K,I}, M)(\beta) = \beta \circ \mu_{K,I} = \beta \circ v = 0 \) since \( v : K(C, -) \rightarrow I(C, -) \) is the kernel of \( \beta \). By [13, Lemma 5.30 (ii)], we conclude that \( w = \phi_{I,M}^C(\beta) = 0 \) in \( \mathbb{L}(M)(C) \). Therefore, \( \mathbb{L}(M)(C) = 0 \), and thus \( \mathbb{L}(M) = 0 \).

Conversely, suppose that \( \mathbb{L}(M) = 0 \). Thus by 3.6, we have that \( M = \text{Ker}(\phi_M) = t(M) \), proving that \( M \) is an \( \mathcal{F}-\text{torsion module} \).

**Proposition 3.8.** Let \( w = \phi_{I,M}^C(\beta) \in \mathbb{L}(M)(C) \) for some ideal \( I(C, -) \in \mathcal{F}_C \) and \( \beta : I(C, -) \rightarrow M \). Then, the following diagram commutes
\[
\begin{array}{ccc}
I(C, -) & \xrightarrow{\delta_1} & \text{Hom}_C(C, -) \\
\downarrow{\beta} & & \downarrow{\psi} \\
M & \xrightarrow{\varphi_M} & \mathbb{L}(M)
\end{array}
\]
where \( \delta_1 \) is the canonical inclusion and \( \psi \) is the natural transformation corresponding to \( w = \phi_{I,M}^C(\beta) \in \mathbb{L}(M)(C) \) via the Yoneda isomorphism \( Y : \mathbb{L}(M)(C) \rightarrow \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), \mathbb{L}(M)) \).

**Proof.** First, we recall that \( \psi : \text{Hom}_C(C, -) \rightarrow \mathbb{L}(M) \) is such that given any \( C' \in \mathcal{C} \), \( \psi_C : \text{Hom}_C(C, C') \rightarrow \mathbb{L}(M)(C') \) is defined as \( \psi_C(f) := \mathbb{L}(M)(f)(w) \) for all \( f \in \text{Hom}_C(C, C') \).

Now, given any \( C' \in \mathcal{C} \) we will show that the following equality holds:
\[
\psi_C \circ [\delta_1]_{C'} = [\varphi_M]_{C'} \circ \beta_C.
\]

Let \( f \in I(C, C') \). Since \([\delta_1]_{C'} \) is the inclusion and by diagram (1) before definition 3.1 we have that
\[
\psi_C([\delta_1]_{C'}(f)) = \psi_C(f) = \mathbb{L}(M)(f)(\varphi_{I,M}^C(\beta)) = \varphi_{\pi^{-1}(I),M}^C(\text{Hom}(\gamma_{I,M}^C, M)(\beta)) = \varphi_{\pi^{-1}(I),M}^C(\beta \circ \gamma_{I})
\]
where \( \gamma_{I} : \pi^{-1}(I(C, -)) \rightarrow I(C, -) \) is such that the following diagram commutes
\[
\begin{array}{ccc}
\pi^{-1}(I(C, -)) & \xrightarrow{\gamma_{I}} & I(C, -) \\
\downarrow{\delta_{\pi^{-1}(I)}} & & \downarrow{\delta_1} \\
\text{Hom}_C(C', -) & \xrightarrow{a=\text{Hom}_C(f, -)} & \text{Hom}_C(C, -)
\end{array}
\]
with $\gamma_I^C := \text{Hom}_C(f, -)|_{(I(C, -))}$ and $\delta_I : I(C, -) \to \text{Hom}_C(C, -)$ is the inclusion; see 2.3(b).

On the other hand, consider $\gamma := \beta_C(f) \in M(C')$, via the Yoneda isomorphism $Y_C : M(C') \to \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C', -), M)$ it corresponds to the natural transformation $\eta_y : \text{Hom}_C(C', -) \to M$ such that

$$[\eta_y]_X(h) = M(h)(\beta_C(f)) \quad \forall X \in C, \quad \forall h \in \text{Hom}_C(C', X).$$

Since $\alpha^{-1}(I(C, -)) \in \mathcal{F}_C$ (by T3) and $\eta_y : \text{Hom}_C(C', -) \to M$, by definition of $\varphi_M$ we have that

$$[\varphi_M]_C(\beta_C(f)) = \varphi_{\alpha^{-1}(I(C, -)), M}(\theta_{\alpha^{-1}(I(C, -)), M}(Y_C(\beta_C(f)))) = \varphi_{\alpha^{-1}(I(C, -)), M}(\eta_y).$$

In order to show that $\psi_C([\delta_I]_C(f)) = [\varphi_M]_C(\beta_C(f))$, it is enough to show that $\beta \circ \gamma^C_I = \theta_{\alpha^{-1}(I(C, -)), M}(\eta_y)$. However, if we denote by $\delta'_{\alpha^{-1}(I)} : \alpha^{-1}(I) \to \text{Hom}_C(C', -)$ the canonical inclusion we have that $\theta_{\alpha^{-1}(I), M}(\eta_y) = \text{Hom}_{\text{Mod}(C)}(\delta'_{\alpha^{-1}(I)}, M)(\eta_y) = \eta_y \circ \delta'_{\alpha^{-1}(I)}$. Therefore, for $X \in C$ we must show that the following diagram commutes

$$\begin{array}{ccc}
\alpha^{-1}(I(C, -))(X) & \xrightarrow{[\gamma^C_I]_X} & I(C, X) \\
[\delta'_{\alpha^{-1}(I)}]_X & \downarrow & \beta_X \\
\text{Hom}_C(C', X) & \xrightarrow{[\eta_y]_X} & M(X).
\end{array}$$

Indeed, let $h : C' \to X$ in $\alpha^{-1}(I(C, -))(X)$. Since $[\delta'_{\alpha^{-1}(I)}]_X$ is the inclusion, we get that $[\eta_y]_X([\delta'_{\alpha^{-1}(I)}]_X(h)) = [\eta_y]_X(h) := M(h)(\beta_C(f))$ (see def. of $\eta_y$ above).

On the other hand, $\beta_X([\gamma^C_I]_X(h)) := \beta_X(hf)$ (since $\gamma_I^C := \text{Hom}_C(f, -)|_{(I(C, -))f}$).

Now, since $\beta : I(C, -) \to M$ is a natural transformation, by considering $h : C' \to X$ we obtain that $M(h) \circ \beta_C = \beta_X \circ I(C, h)$.

Hence, $M(h)(\beta_C(f)) = \beta_X(I(C, h)(f)) = \beta_X(hf)$. This shows that $\beta \circ \gamma^C_I = \eta_y \circ \delta'_{\alpha^{-1}(I)}$, proving that $\psi_C([\delta_I]_C(f)) = [\varphi_M]_C(\beta_C(f))$. Therefore the required diagram commutes. \hfill \Box

**Proposition 3.9.** For each $M \in \text{Mod}(C)$, we have that $	ext{Coker}(\varphi_M)$ is an $\mathcal{F}$-torsion module.

**Proof.** Let us consider $N := \text{Im}(\varphi_M)$ (that is, $N(C) := \text{Im}([\varphi_M]_C)$ for $C \in C$) and $u : N \to \mathbb{L}(M)$ the inclusion. Then, we have that $\pi : \mathbb{L}(M) \to \mathbb{L}(M)/N$ is the cokernel of $\varphi_M$, where $(\mathbb{L}(M)/N)(C) := \mathbb{L}(M)(C)/N(C)$ for $C \in C$. Let $w := w + N(C) \in \mathbb{L}(M)(C)/N(C)$ with $w \in \mathbb{L}(M)(C)$. We know that $w = \varphi_M(\beta)$ for some ideal $I(C, -) \in \mathcal{F}_C$ and $\beta : I(C, -) \to M$. By 3.8, for each $C' \in C$ we obtain the following commutative diagram

$$\begin{array}{ccc}
I(C, C') & \xrightarrow{[\delta_I]_C} & \text{Hom}_C(C, C') \\
\beta_C \downarrow & & \psi_C \\
M(C') & \xrightarrow{[\varphi_M]_C} & \mathbb{L}(M)(C') \to \mathbb{L}(M)(C')/N(C') \to 0.
\end{array}$$

For $f \in I(C, C')$, we get that $\psi_C(f) = \psi_C([\delta_I]_C(f)) = [\varphi_M]_C(\beta_C(f)) \in \text{Im}([\varphi_M]_C)$. By definition of $\psi_C$ we have that $\psi_C(f) = \mathbb{L}(M)(f)(\varphi_M(\beta)) = \mathbb{L}(M)(f)(w)$. Therefore, we conclude that

$$(*) : \quad (\mathbb{L}(M)(f))(w) = [\varphi_M]_C(\beta_C(f)) \in N(C') = \text{Im}([\varphi_M]_C) \quad \forall f \in I(C, C').$$
Since $\bar{w} \in (\mathbb{L}(M)/N)(C)$, we get that $\text{Ann}(\bar{w}, -)$ is a left ideal of $\text{Hom}_C(C, -)$ defined as
\[
\text{Ann}(\bar{w}, -)(C') := \{ f \in \text{Hom}_C(C, C') \mid ((\mathbb{L}(M)/N)(f))(\bar{w}) = 0 \} = \{ f \in \text{Hom}_C(C, C') \mid (\mathbb{L}(M)(f))(w) \in N(C') \}.
\]

By the assertion given in (9) above, it follows that $I(C, -) \subseteq \text{Ann}(\bar{w}, -)$. Since $I \in \mathcal{F}_C$ and $\mathcal{F}$ is a Gabriel filter, we conclude that $\text{Ann}(\bar{w}, -) \in \mathcal{F}_C$. This proves that $\mathbb{L}(M)/N$ is an $\mathcal{F}$-torsion module; see def. 2.6. □

4. Gabriel localization

We now recall the following construction: given a radical $t : \text{Mod}(C) \to \text{Mod}(C)$ we can construct a functor $q_t : \text{Mod}(C) \to \text{Mod}(C)$ defined as $q_t(M) := \frac{M}{t(M)}$ for all $M \in \text{Mod}(C)$.

**Definition 4.1.** Let $C$ be a preadditive category and $\mathcal{F} := \{ \mathcal{F}_C \}_{C \in C}$ be a left Gabriel filter in the category $C$. The *Gabriel localization functor* with respect to $\mathcal{F}$ is the functor
\[
\mathcal{G} := \mathbb{L} \circ q_t : \text{Mod}(C) \to \text{Mod}(C),
\]
where $t$ is the radical associated with the filter $\mathcal{F}$. The *Gabriel localization* of $M$ with respect to $\mathcal{F}$ is the $C$-module $\mathcal{G}(M) := \mathbb{L}(\frac{M}{t(M)})$.

For the results in this section, we consider the following definition.

**Definition 4.2.** We define $\Delta_M : M \to \mathcal{G}(M)$ as the composition
\[
M \xrightarrow{\pi_M} \frac{M}{t(M)} \xrightarrow{\theta_M} \mathbb{L}(\frac{M}{t(M)}).
\]

**Proposition 4.3.** There exists an isomorphism $\mathcal{G}(M) \simeq \mathbb{L}\mathbb{L}(M)$.

**Proof.** We can construct the following exact sequence
\[
0 \longrightarrow t(M) \longrightarrow M \xrightarrow{\theta_M} \mathbb{L}(M) \longrightarrow \text{Coker}(\theta_M) \longrightarrow 0.
\]

We know that $\text{Im}(\theta_M) \simeq \frac{M}{t(M)}$. Hence, we can consider the factorization of $\varphi_M$ through its image as the following composition $M \xrightarrow{\pi_M} \frac{M}{t(M)} \xrightarrow{\gamma_M} \mathbb{L}(M)$. Then we get the exact sequence $0 \longrightarrow M/t(M) \xrightarrow{\gamma_M} \mathbb{L}(M) \longrightarrow \text{Coker}(\varphi_M) \longrightarrow 0$. By applying $\mathbb{L}$, to the previous sequence we obtain the following diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & M/t(M) & \xrightarrow{\gamma_M} & \mathbb{L}(M) & \longrightarrow & \text{Coker}(\varphi_M) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{L}(M/t(M)) & \xrightarrow{\mathbb{L}(\gamma_M)} & \mathbb{L}(\mathbb{L}(M)) & \longrightarrow & \mathbb{L}(\text{Coker}(\varphi_M)) & & \\
& & & & & & & & & \\
\end{array}
\]

Since $\text{Coker}(\varphi_M)$ is an $\mathcal{F}$-torsion module, we have that $\mathbb{L}(\text{Coker}(\varphi_M)) = 0$, proving that $\mathbb{L}(\gamma_M)$ is an isomorphism. That is, $\mathcal{G}(M) \simeq \mathbb{L}(\mathbb{L}(M))$. □

**Remark 4.4.**

(a) By the proof of 4.3, we get the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & M/t(M) & \xrightarrow{\gamma_M} & \mathbb{L}(M) & \longrightarrow & \text{Coker}(\varphi_M) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{L}(M/t(M)) & \xrightarrow{\mathbb{L}(\gamma_M)} & \mathbb{L}(\mathbb{L}(M)) & \longrightarrow & \mathbb{L}(\text{Coker}(\varphi_M)) & & \\
& & & & & & & & & \\
\end{array}
\]
M \xrightarrow{\varphi_M} \mathbb{L}(M)
\downarrow \pi_M 
M/t(M) \xrightarrow{\pi} \mathbb{L}(M)
\downarrow \varphi_{t(M)}
\mathbb{L}(M/t(M)) \xrightarrow{L(\varphi_M)} \mathbb{L}(\mathbb{L}(M))

where \( \mathbb{L}(\varphi_M) \) is an isomorphism. Then, we could have defined the morphism \( \Delta_M \) as the composition \( \varphi_{\mathbb{L}(M)} \circ \varphi_M : M \to \mathbb{L}(M) \).

(b) The morphism \( \varphi_{\mathbb{L}(M)} : \frac{M}{\pi(M)} \to \mathbb{L}(\frac{M}{\pi(M)}) \) is a monomorphism. Thus, the composition

\( M \xrightarrow{\pi_M} \frac{M}{\pi(M)} \xrightarrow{\varphi_{\mathbb{L}(M)}} \mathbb{L}(\frac{M}{\pi(M)}) \)

is the factorization of \( \Delta_M \) through its image. Indeed, this follows from 3.6 and the fact that \( t(\frac{M}{\pi(M)}) = 0 \) since \( t \) is a radical.

**Corollary 4.5.** There is an exact sequence

\[
0 \rightarrow t(M) \rightarrow M \rightarrow \Delta_M \rightarrow \text{G}(M) \rightarrow \text{Coker}(\varphi_{\mathbb{L}(M)}) \rightarrow 0.
\]

In particular, \( \text{Ker}(\Delta_M) \) and \( \text{Coker}(\Delta_M) \) are \( \mathcal{F} \)-torsion modules.

**Proof.** This follows from 4.4(b), 3.9 and 3.6. \( \square \)

**Proposition 4.6.** For \( M \in \text{Mod}(C) \), there is an isomorphism \( \text{G}(M) \simeq \text{G}(\frac{M}{\pi(M)}) \).

**Proof.** Since \( t(M/t(M)) = 0 \), then \( q_t(M/t(M)) = M/t(M) \). Hence, \( \text{G}(M/t(M)) = \mathbb{L}(q_t(M/t(M))) = \mathbb{L}(M/t(M)) = \mathbb{L}q_t(M) = \text{G}(M) \). \( \square \)

**Definition 4.7.** A \( C \)-module \( M \) is \( \mathcal{F} \)-closed if

\[
\theta^C_{I,M} : \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C,-),M) \rightarrow \text{Hom}_{\text{Mod}(C)}(I,M)
\]

is an isomorphism for all \( C \in \mathcal{C} \) and \( I \in \mathcal{F}_C \) (recall that \( \theta^C_{I,M} := \text{Hom}_{\text{Mod}(C)}(\delta_I,M) \) where \( \delta_I : I \rightarrow \text{Hom}_C(C,-) \) is the inclusion).

**Proposition 4.8.** Let \( M \in \text{Mod}(C) \) be a \( \mathcal{F} \)-closed module. Let \( C \in \mathcal{C} \) be and \( x \in M(C) \) such that \( \text{Ann}(x,-) \in \mathcal{F}_C \), then \( x = 0 \). In particular \( t(M) = 0 \).

**Proof.** By Yoneda lemma we get a morphism \( \eta_x : \text{Hom}_C(C,-) \rightarrow M \) such that \( [\eta_x]_B(f) = M(f)(x) \) for all \( f \in \text{Hom}_C(C,B) \). Let \( I(C,-) := \text{Ann}(x,-) \) and \( \delta_I : I \rightarrow \text{Hom}_C(C,-) \) be the canonical inclusion. Since \( M \) is \( \mathcal{F} \)-closed we have an isomorphism

\[
\theta^C_{I,M} : \text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C,-),M) \rightarrow \text{Hom}_{\text{Mod}(C)}(I,M)
\]

Nevertheless, \( \theta^C_{I,M}(\eta_x) = \eta_x \circ \delta_I : I \rightarrow M \) satisfies that \( [\eta_x \circ \delta_I]_B(f) = [\eta_x]_B(f) = M(f)(x) = 0 \) for all \( B \in \mathcal{C} \) and for all \( f \in \text{Hom}_C(C,B) \), since \( f \in I(C,B) = \text{Ann}(x,-)(B) \). Thus, \( \eta_x \circ \delta_I = 0 \) and, since \( \theta^C_{I,M} \) is an isomorphism it follows that \( \eta_x = 0 \). Therefore, by Yoneda lemma, we conclude that \( x = 0 \).

Now, since \( t(M) = \sum_{N \in \mathcal{F}_S, N \leq M} N \), we obtain that \( t(M) = 0 \). \( \square \)
Corollary 4.9. If $M$ is an $\mathcal{F}$-closed $C$-module, then $\Delta_M : M \to \mathbb{G}(M)$ is an isomorphism.

Proof. By 4.8, we conclude that $\pi_M = 1, M = M/\mathfrak{p}(M)$ and $\varphi_M = \varphi_{\mathfrak{p}(M)}$. Now by diagram (3), and the definition of $\mathcal{F}$-closed, we have that $\varphi_M$ is an isomorphism. This proves that $\Delta_M$ is an isomorphism.

Proposition 4.10. Let $\mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ be a left Gabriel filter. Let $I, J \in \mathcal{F}_C$ with $\mu_{I,J} : I \to J$ the canonical inclusion and $M$ a torsion-free module (that is, $t(M) = 0$). Let $f, g : J \to M$ be morphisms such that $f \circ \mu_{I,J} = g \circ \mu_{I,J}$ then $f = g$.

Proof. Consider the exact sequence

$$0 \rightarrow I(C, -) \xrightarrow{\mu_{I,J}} J(C, -) \xrightarrow{\pi} \frac{I(C, -)}{I(C, -)} \rightarrow 0.$$ 

Since $(f - g) \circ \mu_{I,J} = 0$, there exists $\theta : \frac{I(C, -)}{I(C, -)} \rightarrow M$ such that $f - g = \theta \pi$.

Let $\bar{w} \in \frac{I(C, C')}{I(C, C')}$ be with $w \in I(C, C') \subseteq \text{Hom}_C(C, C')$, then we have that $\text{Ann}(\bar{w}, -)$ is a left ideal of $\text{Hom}_C(C', -)$. In addition, we easily get that

$$\text{Ann}(\bar{w}, -)(X) = \{f \in \text{Hom}_C(C', X) | f\bar{w} \in I(C, X)\}.$$ 

Hence, $\text{Ann}(\bar{w}, -) = (I(C, -) : w)$. By $T_3$, it follows that $\text{Ann}(\bar{w}, -) \in \mathcal{F}_C$. Thus, we get that $\frac{I(C, -)}{I(C, -)} \in \mathcal{F}$ (that is, $\frac{I(C, -)}{I(C, -)}$ is an $\mathcal{F}$-torsion module). Since $\frac{I(C, -)}{I(C, -)}$ is an $\mathcal{F}$-torsion module (equiv-
lently, $t(\frac{I(C, -)}{I(C, -)}) = 0$) and $M$ is a torsion free module, we conclude that $\theta : \frac{I(C, -)}{I(C, -)} \rightarrow M$ is the zero morphism. Thus, $f - g = \theta \pi = 0$, proving that $f = g$.

Proposition 4.11. Let $\mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ be a left Gabriel filter, $M \in \text{Mod}(\mathcal{C})$ and $t$ the radical associated with $\mathcal{F}$. If $t(M) = 0$, then $t(\mathbb{L}(M)) = 0$.

Proof. Consider the filter $\mathcal{F} := \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ in $\mathcal{C}$. By 2.8, we obtain the following torsion class

$$\mathcal{T}_\mathcal{F} := \{M \in \text{Mod}(\mathcal{C}) | \text{ for each } C \in \mathcal{C}, \ Ann(x, -) \in \mathcal{F}_C \ \forall x \in M(\mathcal{C})\}.$$ 

By 2.10, there exists a radical $t$ associated with $\mathcal{T}_\mathcal{F}$. Then, we get that

$$t(\mathbb{L}(M)) = \sum_{N \in \mathcal{T}_\mathcal{F}, \ N \subseteq \mathbb{L}(M)} N.$$ 

Let $w \in t(\mathbb{L}(M))(C) \subseteq \mathbb{L}(M)(C)$. Thus, it follows that $J(C, -) := \text{Ann}(w, -) \in \mathcal{F}_C$, since $t(\mathbb{L}(M)) \in \mathcal{T}_\mathcal{F}$ and the description of $\mathcal{T}_\mathcal{F}$ in 2.5. Recall that $\text{Ann}(w, C') := \{f \in \text{Hom}(C, C') | \mathbb{L}(M)(f)(w) = 0\}$.

We know that $w = \varphi^C_{M,M}(\beta) \in \mathbb{L}(M)(C)$ for some ideal $I(C, -) \in \mathcal{F}_C$ and $\beta : I(C, -) \rightarrow M$. Hence, by 3.8, we have that $\psi \circ \delta_I = \varphi_M \circ \beta$, where $\delta_I$ is the canonical inclusion and $\psi$ is the natural transformation corresponding to $w = \varphi^C_{M,M}(\beta) \in \mathbb{L}(M)(C)$ via the Yoneda $Y : \mathbb{L}(M)(C) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{C})}(\text{Hom}_C(C, -), \mathbb{L}(M))$. Then, we obtain the following commutative diagram

$$
\begin{array}{ccc}
J(C, -) \cap I(C, -) & \xrightarrow{\epsilon^j} & J(C, -) \\
\downarrow{\epsilon_I} & & \downarrow{\delta_I} \\
I(C, -) & \xrightarrow{\delta_I} & \text{Hom}_C(C, -) \\
\downarrow{\beta} & & \downarrow{\psi} \\
M & \xrightarrow{\varphi_M} & \mathbb{L}(M).
\end{array}
$$
Thus, for $C' \in C$ we have the commutative diagram

$$
\begin{array}{ccc}
J(C, C') \cap I(C, C') & \xrightarrow{[\varepsilon]_{C'}} & J(C, C') \\
\downarrow{[\varepsilon]_{C'}} & & \downarrow{[\delta]_{C'}} \\
I(C, C') & \xrightarrow{[\delta]_{C'}} & \text{Hom}_C(G, C') \\
\downarrow{\beta_{C'}} & & \downarrow{\psi_{C'}} \\
M(C') & \xrightarrow{[\varphi_{M}]_{C'}} & \mathbb{L}(M)(C')
\end{array}
$$

where $[\delta]_{C'}, [\delta]_{C'}, [\varepsilon]_{C'}$ and $[\varepsilon]_{C'}$ are the inclusions as sets. For $f \in J(C, C') \cap I(C, C')$ we get that $\psi_{C'}([\delta]_{C'}[\varepsilon]_{C'}(f)) = \psi_{C'}(f) = \mathbb{L}(M)(f)(w) = 0$ since $f \in \text{Ann}(w, C')$. Therefore, it follows that $\varphi_{M} \circ \beta \circ \varepsilon_{I} = \psi \circ \delta \circ \varepsilon_{I} = 0$. Since $M$ is torsion free ($t(M) = \ker(\varphi_{M}) = 0$), we have that $\varphi_{M}$ is a monomorphism and then we get that $\beta \circ \varepsilon_{I} = 0$. Since $I(C, -) \cap I(C, -) \in \mathcal{F}_{C}$, it follows that $I(C, -) \cap I(C, -) \in \mathcal{F}_{C}$ (by property $T_2$). Hence, by Lemma 5.30 (ii) in [13], we have that $w = \varphi_{C,M}(\beta) = 0$ in $\mathbb{L}(M)$, and we conclude that $t(\mathbb{L}(M)) = 0$.

**Proposition 4.12.** $G(M)$ is an $\mathcal{F}$-closed module for each $M \in \text{Mod}(C)$.

**Proof.** (1) First, let us see that

$$
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), \mathbb{L}(\frac{M}{t(M)})) \xrightarrow{\theta_{C \mathbb{L}(\frac{M}{t(M)})}} \text{Hom}_{\text{Mod}(C)}(J, \mathbb{L}(\frac{M}{t(M)}))
$$

is injective for all $J = J(C, -) \in \mathcal{F}_{C}$. By applying 4.11 to $M/t(M)$, we have that $0 = t(\mathbb{L}(\frac{M}{t(M)})) = \ker(\varphi_{\mathbb{L}(\frac{M}{t(M)})})$. Then, there is an exact sequence

$$
0 \longrightarrow \mathbb{L}(\frac{M}{t(M)}) \xrightarrow{\varphi_{\mathbb{L}(\frac{M}{t(M)})}} \mathbb{L}^2(\frac{M}{t(M)}) \longrightarrow \text{Coker}(\varphi_{\mathbb{L}(\frac{M}{t(M)})}) \longrightarrow 0.
$$

By definition of $\varphi_{\mathbb{L}(\frac{M}{t(M)})}$ for each $C \in C$, we get that $[\varphi_{\mathbb{L}(\frac{M}{t(M)})}]_{C} := [\psi_{\mathbb{L}(\frac{M}{t(M)})}]_{C} \circ Y_{C}$ where $Y_{C}$ is the Yoneda isomorphism (see 3.3), and $[\psi_{\mathbb{L}(\frac{M}{t(M)})}]_{C}$ is such that the following commutes for all $J(C, -) \in \mathcal{F}_{C}$:

$$
\begin{array}{ccc}
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), \mathbb{L}(\frac{M}{t(M)})) & \xrightarrow{[\psi_{\mathbb{L}(\frac{M}{t(M)})}]_{C}} & \lim_{J \in \mathcal{F}_{C}} \text{Hom}_{\text{Mod}(C)}(J, \mathbb{L}(\frac{M}{t(M)})) \\
\uparrow & & \uparrow \\
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), \mathbb{L}(\frac{M}{t(M)})) & \xrightarrow{\theta_{C \mathbb{L}(\frac{M}{t(M)})}} & \text{Hom}_{\text{Mod}(C)}(J, \mathbb{L}(\frac{M}{t(M)})).
\end{array}
$$

Since $[\varphi_{\mathbb{L}(\frac{M}{t(M)})}]_{C} := [\psi_{\mathbb{L}(\frac{M}{t(M)})}]_{C} \circ Y_{C}$ is a monomorphism and $Y_{C}$ is an isomorphism, we conclude that $[\psi_{\mathbb{L}(\frac{M}{t(M)})}]_{C}$ is a monomorphism. By the above diagram, we conclude that $\theta_{C \mathbb{L}(\frac{M}{t(M)})}$ is a monomorphism for all $J_{C} \in \mathcal{F}_{C}$.

In other words, we have that

$$
\text{Hom}_{\text{Mod}(C)}(\text{Hom}_C(C, -), \mathbb{L}(\frac{M}{t(M)})) \xrightarrow{\theta_{C \mathbb{L}(\frac{M}{t(M)})}} \text{Hom}_{\text{Mod}(C)}(J, \mathbb{L}(\frac{M}{t(M)})),
$$

is injective for all $J = J(C, -) \in \mathcal{F}_{C}$. 


We will show that there exists an isomorphism $\Phi_{M,N} : \text{Hom}_{\text{Mod}(C,F)}(G(M), N) \rightarrow \text{Hom}_{\text{Mod}(C)}(M, N)$ for every $N \in \text{Mod}(C,F)$ and $M \in \text{Mod}(C)$. We consider the morphism $\varphi_{M,N} : \frac{M}{t(M)} \rightarrow \mathbb{L}(\frac{M}{t(M)}) = G(M)$ and $\pi_M : M \rightarrow \frac{M}{t(M)}$ the projection. Given $\alpha : G(M) \rightarrow N$, we define $\Phi_{M,N}(\alpha) = \alpha \circ \varphi_{M,N} \circ \pi_M = \alpha \circ \Delta_M$.

Now, given $f : M \rightarrow N$ in $\text{Mod}(C)$, we get the following commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\Delta_M} & & \downarrow{\Delta_N} \\
G(M) & \xrightarrow{G(f)} & G(N)
\end{array}
$$
Since \( N \in \text{Mod}(C, \mathcal{F}) \), it follows that \( \Delta_N \) is an isomorphism (see 4.9). Therefore, we define \( \alpha := \Delta_N^{-1} \circ \mathcal{G} \) and we have that \( \Phi_{M, N}(\alpha) = \alpha \circ \Delta_M = \Delta_N^{-1} \circ \mathcal{G} \circ \Delta_M = f \). This proves that \( \Phi_{M, N} \) is surjective. Now, consider the exact sequence

\[
0 \longrightarrow \frac{M}{t(M)} \longrightarrow \mathcal{L}(\frac{M}{t(M)}) \longrightarrow \text{Coker}(\frac{M}{t(M)}) = Z \longrightarrow 0
\]

By applying \( \text{Hom}_{\text{Mod}(C)}(-, N) \), we have the exact sequence

\[
\text{Hom}_{\text{Mod}(C)}(Z, N) \longrightarrow \text{Hom}_{\text{Mod}(C)}(\mathcal{G}(M), N) \longrightarrow \text{Hom}_{\text{Mod}(C)}(\frac{M}{t(M)}, N).
\]

Since \( Z \) is torsion module and \( N \) is a torsion-free module (see 3.6 and 4.8), it follows that \( \text{Hom}_{\text{Mod}(C)}(Z, N) = 0 \). Hence, we conclude that

\[
\text{Hom}_{\text{Mod}(C)}(\frac{M}{t(M)}, N) \longrightarrow \text{Hom}_{\text{Mod}(C)}(\frac{M}{t(M)}, N)
\]

is injective. Now, we assert that

\[
\text{Hom}_{\text{Mod}(C)}(\pi_M, N) : \text{Hom}_{\text{Mod}(C)}\left(\frac{M}{t(M)}, N\right) \rightarrow \text{Hom}_{\text{Mod}(C)}(M, N)
\]

is an isomorphism. This follows from the following exact sequence

\[
0 \longrightarrow \left(\frac{M}{t(M)}, N\right) \longrightarrow (M, N) \longrightarrow (t(M), N) = 0,
\]

where \((t(M), N) = 0\) because \( t(M) \) is a torsion module and \( N \) is torsion-free module. This implies that

\[
\text{Hom}_{\text{Mod}(C, \mathcal{F})}(\mathcal{G}(M), N) \longrightarrow \text{Hom}_{\text{Mod}(C)}(\Delta_M, N) \rightarrow \text{Hom}_{\text{Mod}(C)}(M, N)
\]

is injective. But \( \Phi_{M, N} = \text{Hom}_{\text{Mod}(C)}(\Delta_M, N) \), proving that \( \Phi_{M, N} \) is bijective.

We recall the following notion. Let \( \mathcal{X} \subseteq \text{Mod}(C) \) be a class of objects and \( f : M \rightarrow X \) a morphism with \( X \in \mathcal{X} \), it is said that \( f \) is a left \( \mathcal{X} \)–approximation of \( M \) if for every morphism \( g : M \rightarrow X' \in \mathcal{X} \) there exists a morphism \( h : X \rightarrow X' \) such that \( g = h \circ f \). If every object in \( \text{Mod}(C) \) admits a left \( \mathcal{X} \)-approximation, we say that \( \mathcal{X} \) is covariantly finite in \( \text{Mod}(C) \).

**Corollary 4.15.** The morphism \( \Delta_M : M \rightarrow \mathcal{G}(M) \) is a left \( \text{Mod}(C, \mathcal{F}) \)-approximation of \( M \). In particular, \( \text{Mod}(C, \mathcal{F}) \) is a covariantly finite subcategory of \( \text{Mod}(C) \).

**Proof.** This follows by the previous propositions since \( \mathcal{G}(M) \) is closed.

**Corollary 4.16.** The functor \( \mathcal{G} \) is exact.

**Proof.** Since \( \mathcal{L} \) is left exact and by 4.4(b), we conclude that \( \mathcal{G} \) is left exact. Now, \( \mathcal{G} \) is right exact since \( \mathcal{G} \) is left adjoint to \( i \); see [15, Theorem 2.6.1].

**Proposition 4.17.** Let \( \mathcal{F} = \{\mathcal{F}_C\}_{C \in C} \) be a left Gabriel filter and \( M \in \text{Mod}(C) \). Then, \( \mathcal{G}(M) = 0 \) if and only if \( M \) is a \( \mathcal{F} \)-torsion module. In other words, \( \mathcal{G}(M) = 0 \) if and only if \( t(M) = M \), where \( t \) is the radical associated with \( \mathcal{F} \).

**Proof.** Let \( t : \text{Mod}(C) \rightarrow \text{Mod}(C) \) the radical associated with \( \mathcal{F} \). If \( M \) is an \( \mathcal{F} \)-torsion module, we get that \( t(M) = M \). Then, \( \mathcal{G}(M) = \mathcal{L}(\frac{M}{t(M)}) = 0 \).
If \( G(M) = \mathbb{L}(\frac{M}{\tau(M)}) = 0 \), by 3.7, it follows that \( \frac{M}{\tau(M)} \) is an \( \mathcal{F} \)-torsion module. That is \( t(\frac{M}{\tau(M)}) = \frac{M}{\tau(M)} \), but since \( t \) is radical we have that \( t(\frac{M}{\tau(M)}) = 0 \). Hence, \( t(M) = M \), proving that \( M \) is an \( \mathcal{F} \)-torsion module.

**Definition 4.18.** Let \( \mathcal{A} \) be a complete Grothendieck category, and let \( \mathcal{B} \) a full subcategory of \( \mathcal{A} \).

(a) \( \mathcal{B} \) is a reflective subcategory of \( \mathcal{A} \) if the inclusion functor \( i: \mathcal{B} \to \mathcal{A} \) has a left adjoint \( \alpha: \mathcal{A} \to \mathcal{B} \).
(b) \( \mathcal{B} \) is a Giraud subcategory of \( \mathcal{A} \) if \( \mathcal{B} \) is reflective such that the functor \( \alpha: \mathcal{A} \to \mathcal{B} \) preserves kernels.

It is well known that if \( \mathcal{B} \) is a Giraud subcategory of \( \mathcal{A} \), then the functor \( \alpha: \mathcal{A} \to \mathcal{B} \) is exact and \( \mathcal{B} \) is a Grothendieck category.

Given the adjoint pair \( (\alpha, i) \), we get the unit and counit morphisms

\[
\eta: 1_{\mathcal{A}} \to i \circ \alpha, \quad \epsilon: \alpha \circ i \to 1_{\mathcal{B}}.
\]

We have the following well-known result.

**Proposition 4.19.** [14, Chapter X, Proposition 1.5, page 215] Let \( \mathcal{A} \) be a complete Grothendieck category, \( \mathcal{B} \) a Giraud subcategory of \( \mathcal{A} \) and \( \alpha: \mathcal{A} \to \mathcal{B} \) the left adjoint to the inclusion functor. We set \( \mathcal{T}_{\mathcal{B}} := \{ A \in \mathcal{A} | \alpha(A) = 0 \} \) and \( \mathcal{F}_{\mathcal{B}} := \{ A \in \mathcal{A} | \eta_A: A \to i \circ \alpha(A) \) is a monomorphism \}. Then \( (\mathcal{T}_{\mathcal{B}}, \mathcal{F}_{\mathcal{B}}) \) is a hereditary torsion theory for \( \mathcal{A} \).

**Proposition 4.20.** The category \( \text{Mod}(\mathcal{C}, \mathcal{F}) \) is a Giraud subcategory of \( \text{Mod}(\mathcal{C}) \)

**Proof.** This follows from 4.16.

**Corollary 4.21.** \( \text{Mod}(\mathcal{C}, \mathcal{F}) \) is a Grothendieck abelian category.

Now, we will prove the main result of this work. As mentioned at the beginning of this article, it is a generalization of a classical result given by P. Gabriel in his doctoral thesis (see [6, Lemme 1]).

**Theorem 4.22.** Let \( \mathcal{C} \) be a small preadditive category and let \( \text{Mod}(\mathcal{C}) \) be the category of additive covariant functors from \( \mathcal{C} \) to the category of abelian groups \( \text{Ab} \). Then, there is a bijective correspondence between the left Gabriel filters on \( \mathcal{C} \), and the classes of isomorphisms of Giraud subcategories in \( \text{Mod}(\mathcal{C}) \).

**Proof.** By 4.19, there is a map

\[
\Upsilon: \{ \text{Giraud subcategories of } \text{Mod}(\mathcal{C}) \} \to \{ \text{hereditary torsion classes of } \text{Mod}(\mathcal{C}) \},
\]

given by \( \Upsilon(B) := \mathcal{T}_B = \{ M \in \text{Mod}(\mathcal{C}) | a'(M) = 0 \} \), where \( a': \text{Mod}(\mathcal{C}) \to \mathcal{B} \) is the left adjoint to the inclusion \( i': \mathcal{B} \to \text{Mod}(\mathcal{C}) \). By 2.8, there is a bijective correspondence

\[
\psi: \{ \text{left Gabriel filters of } \mathcal{C} \} \leftrightarrow \{ \text{hereditary torsion classes of } \text{Mod}(\mathcal{C}) \}.
\]

We define a correspondence

\[
\Phi: \{ \text{Giraud subcategories of } \text{Mod}(\mathcal{C}) \} \to \{ \text{left Gabriel filters of } \mathcal{C} \}
\]
as $\Phi := \Psi^{-1} \circ \mathbb{T}$. Hence, for $B$ a Giraud subcategory of $\text{Mod}(C)$, $\Phi(B) := \mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ where

$$
\mathcal{F}_C := \left\{ I \subseteq \text{Hom}_C(C, -) \mid \frac{\text{Hom}_C(C, -)}{I} \in \mathcal{T}_B \right\} = \left\{ I \subseteq \text{Hom}_C(C, -) \mid a'\left(\frac{\text{Hom}_C(C, -)}{I}\right) = 0 \right\}.
$$

We define $\Gamma : \{\text{left Gabriel filters of } C\} \to \{\text{Giraud subcategories of } \text{Mod}(C)\}$, as $\Gamma(\mathcal{F}) = \text{Mod}(C, \mathcal{F})$.

Let us see that $\Phi \circ \Gamma = 1$. Indeed, let $\mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ be a left Gabriel filter. By 2.8, $\mathcal{F}$ corresponds bijectively with $\mathcal{T}_\mathcal{F} := \left\{ M \in \text{Mod}(C) \mid \text{for each } C \in \mathcal{C}, \ \text{Ann}(x, -) \in \mathcal{F}_C \ \forall x \in M(C) \right\}$.

By 2.11, it follows that $\mathcal{T}_\mathcal{F} = \{M \in \text{Mod}(C) \mid t(M) = M\}$, where $t$ is the radical associated with $\mathcal{F}$.

On the other hand, by the construction of $\Phi$, the left Gabriel filter $\Phi(\Gamma(\mathcal{F})) = \Phi(\text{Mod}(C, \mathcal{F}))$ corresponds bijectively with the torsion class $\{M \in \text{Mod}(C) \mid G(M) = 0\}$. Also, by 4.17, $\mathcal{T}_\mathcal{F} = \{M \in \text{Mod}(C) \mid G(M) = 0\}$. Hence, $\Phi(\Gamma(\mathcal{F})) = \mathcal{F}$, proving that $\Phi \circ \Gamma = 1$.

Let $B$ be a Giraud subcategory of $\text{Mod}(C)$ and $a' : \text{Mod}(C) \to B$ the left adjoint to the inclusion $i' : B \to \text{Mod}(C)$. Consider the torsion class $\mathcal{T}(B) := \mathcal{T}_B = \{M \in \text{Mod}(C) \mid a'(M) = 0\}$. By the definition of $\Phi$ and the bijective correspondences 2.8 and 2.10, it follows that the torsion class $\mathcal{T}_B$ determines a unique left Gabriel filter $\Phi(\mathcal{T}_B) = \mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$, and a unique radical $t : \text{Mod}(C) \to \text{Mod}(C)$. In this case,

$$
\mathcal{F}_C = \left\{ I \subseteq \text{Hom}_C(C, -) \mid a'\left(\frac{\text{Hom}_C(C, -)}{I}\right) = 0 \right\}.
$$

That is, we get the assignation:

\[ (*) : \Phi(\mathcal{T}_B) = \mathcal{F} = \{\mathcal{F}_C\} \leftrightarrow \mathcal{T}_B \leftrightarrow t. \]

Now, we construct $\text{Mod}(C, \mathcal{F})$ and $G : \text{Mod}(C) \to \text{Mod}(C, \mathcal{F})$. By 4.17, we have that $G(M) = 0$ if and only if $t(M) = M$. However, by the correspondence 2.10, we know that $t(M) = M$ if and only if $M \in \mathcal{T}_B$ (the torsion class associated with $t$ is $\mathcal{T}_B$ by ($\ast$)). Thus,

\[ (*) : \mathcal{T}_B = \{M \in \text{Mod}(C) \mid a'(M) = 0\} = \{M \in \text{Mod}(C) \mid G(M) = 0\}. \]

Now, we will show that $B$ is equivalent to $\text{Mod}(C, \mathcal{F})$. Consider the following commutative diagram

$$
\begin{array}{ccc}
\text{Mod}(C) & \xrightarrow{a'} & \text{Mod}(C, \mathcal{F}) \\
\downarrow{G} & & \downarrow{t} \\
B & \xrightarrow{i} & \text{Mod}(C)
\end{array}
$$

We assert that $a' \circ i \circ G \simeq a'$. Indeed, by 4.5, we get the short exact sequence

$$
\gamma : 0 \to t(M) \to M \xrightarrow{\Delta_M} G(M) \to \text{Coker}(\varphi_{M \mid M}) \to 0.
$$

In particular, $\text{Ker}(\Delta_M)$ and $\text{Coker}(\Delta_M)$ are $\mathcal{F}$-torsion modules (that is, $\text{Ker}(\Delta_M), \text{Coker}(\Delta_M) \in \mathcal{T}_B$). Since $\mathcal{T}_B = \{M \in \text{Mod}(C) \mid a'(M) = 0\}$ (see equality ($\ast$)), by applying $a'$ to the exact sequence $\gamma$, we get an isomorphism
\[ a'(M) \xrightarrow{d'(I M)} a'(G(M)) \].

It is easy to show that this is a natural isomorphism, and therefore, \( a' \circ i \circ G \simeq a' \). Now, consider the unit \( \eta' : 1_{\text{Mod}(C)} \to i' \circ a' \). Since \( i' \) is full and faithful, by [5, Proposition 3.4.1], it follows that the counit \( \epsilon' : a' \circ i' \to 1_B \) is an isomorphism and we have that \( a' \ast \eta' \) is isomorphism. In other words, \( a'(\eta_M) \) is an isomorphism for all \( M \in \text{Mod}(C) \). Consider the exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \ker(\eta_M) & \longrightarrow & M & \longrightarrow & \iota'(M) & \longrightarrow & \text{Coker}(\eta_M) & \longrightarrow & 0.
\end{array}
\]

Since \( a' \) is exact and \( a'(\eta_M) \) is an isomorphism, it follows that \( a'(\ker(\eta_M)) = 0 \) and \( a'(\text{Coker}(\eta_M)) = 0 \). This implies that \( \ker(\eta_M), \text{Coker}(\eta_M) \in \mathcal{T}_B \). Hence, \( \ker(\eta_M), \text{Coker}(\eta_M) \) are \( \mathcal{F} \)-torsion modules. By applying \( G \), to the exact sequence \((\delta)\), we get the isomorphism \( G(\eta_M) : G(M) \to G \circ i' \circ a'(M) \). Therefore,

\[ G \simeq G \circ i' \circ a'. \]

For the same reason that \( \epsilon' \) is an isomorphism, we obtain that the counit \( \epsilon : G \circ i \to \text{Mod}(C, F) \) is an isomorphism. Hence, \((G \circ i') \circ (a' \circ i) = (G \circ i' \circ a') \circ i \simeq G \circ i \simeq 1\), where the last isomorphism is via \( \epsilon \). Similarly, \((a' \circ i) \circ (G \circ i') = (a' \circ i \circ G) \circ i' \simeq a' \circ i' \circ 1\). Therefore, \( B \) is equivalent to \( \text{Mod}(C, F) \). Hence, \( \Gamma \circ \Phi = 1 \).

The notion of quotient and localization of abelian categories by dense subcategories (i.e. Serre classes) was introduced by P. Gabriel in his famous doctoral thesis “Des catégories abéliennes” [6], and plays an important role in ring theory. This notion achieves some goal as quotients in other areas of mathematics.

Let \( A \) be an abelian category. Recall that a Serre subcategory \( B \) of \( A \) is a subcategory closed under forming subobjects, quotients and extensions. In this case, we can construct the quotient category \( A/B \) of \( A \) with respect to \( B \) and a functor \( Q : A \to A/B \), which is called the quotient functor. For basic properties of quotient categories we refer to [12] and [6]. We recall the following well-known result.

**Theorem 4.23** (Gabriel). Let \( B \) be a Serre subcategory of a locally small abelian category \( A \), and consider \( \Sigma = \{ f \in A \mid \ker(f), \text{coker}(f) \in B \} \). Then, there exists an abelian category \( A/B \) and an exact functor \( Q : A \to A/B \) such that \( Q(f) \) is an isomorphism for all \( f \in \Sigma \). Moreover, if \( F : A \to D \) is a functor satisfying that \( F(f) \) is an isomorphism for all \( f \in \Sigma \), then there exists a unique functor \( G : A/B \to D \) such that \( F = G \circ Q \).

**Proposition 4.24.** There exists an equivalence of categories

\[ \text{Mod}(C, F) \simeq \text{Mod}(C)/\mathcal{T}_F. \]

**Proof.** We have an exact functor \( G : \text{Mod}(C) \to \text{Mod}(C, F) \) with right adjoint \( i : \text{Mod}(C, F) \to \text{Mod}(C) \) that is full and faithful. By [12, Theorem 4.9] on page 180, we get that \( \ker(G) \) is a localizing subcategory and

\[ \text{Mod}(C, F) \simeq \text{Mod}(C)/\ker(G). \]

By 4.17, we conclude that \( \ker(G) = \mathcal{T}_F \).

Finally, we give an example of a left Gabriel filter using the path category of a quiver.

**Example 4.25.** Consider a field \( K \) and the quiver

\[ \begin{array}{c}
\text{Object} \\
\downarrow \\
\text{Arrow} \\
\end{array}
\]

with arrows going in both directions. Then, \( \mathcal{F} \) is a Gabriel filter if and only if the dimension of each component of \( \text{Hom}(\text{Ob}(C'), \text{Ob}(C')) \) is finite.
and $C = KQ/\langle \rho \rangle$ the path category with the relations given by $\alpha_i \circ \alpha_{i-1} = \beta_i \circ \beta_{i+1} = \alpha_i \circ \beta_i = 0$. Thus, the functor $(i,-) := \text{Hom}_{KQ/\langle \rho \rangle}(i,-)$ can be thought of as a representation in the category $\text{Rep}(Q, \rho)$ given by

$((i,j), u : i \to j)_{(i,j) \in Q_0, u \in Q_1}.$

Thus $(i,i) := \{f : i \to i\} = \langle 1, \beta_i \alpha_i \rangle \simeq K^2, (i,i+1) = \{f : i \to i+1\} = \langle \alpha_i \rangle \simeq K$ and $(i,i-1) = \{f : i \to i-1\} = \langle \beta_{i-1} \rangle \simeq K$. Then, the representation corresponding to $(i,-)$ is the following

Computing the ideals, we obtain that the functor $(i,-)$ has seven ideals: $0$, $(i,-)$ and the given by the following representations:

$[\beta_i \alpha_i] : \cdots \begin{array}{ccc} 0 & K & 0 \\ 0 & 0 & 0 \end{array} \cdots.$

$[\beta_{i-1}] : \cdots \begin{array}{ccc} K & 0 & 0 \\ 0 & 0 & 0 \end{array} \cdots.$

$[\alpha_i] : \cdots \begin{array}{ccc} 0 & K & K \\ 0 & 1 & 0 \end{array} \cdots.$

$[\beta_{i-1}] \oplus [\alpha_i \beta_i] : \cdots \begin{array}{ccc} K & K & 0 \\ 0 & 0 & 0 \end{array} \cdots.$

$[\beta_{i-1}] \oplus [\alpha_i] : \cdots \begin{array}{ccc} K & K & K \\ 0 & 1 & 0 \end{array} \cdots.$

We get the following lattice of ideals:
An easy calculation shows that \( \mathcal{F} = \{ [\beta_{i-1}], [z_i], [z_i] \oplus [\beta_{i-1}], (i, -) \} \) is a left Gabriel filter. It is easy to show in this case that \((i, -) \in \mathcal{T}_{\mathcal{F}}\), so we have that \( C((i, -)) = 0 \).

**Acknowledgements**

This research was initiated after a series of talks that the first and the third authors gave at the Universidad Autónoma Metropolitana campus Iztapalapa, invited by the second author.

The authors are grateful for the referee’s valuable comments and suggestions, which have improved the quality and readability of the article.

**Funding**

The authors thanks Project PRODEP PTC-2019 grant UAM-PTC-700 Num. 12613411 awarded by the Mexican Secretariat of Public Education (SEP).

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