Evolution method and HOMFLY polynomials for virtual knots

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Abstract

Following the suggestion of [1] to lift the knot polynomials for virtual knots and links from Jones to HOMFLY, we apply the evolution method to calculate them for an infinite series of twist-like virtual knots and antiparallel 2-strand links. Within this family one can check topological invariance and understand how differential hierarchy is modified in virtual case. This opens a way towards a definition of colored (not only cabled) knot polynomials, though problems still persist beyond the first symmetric representation.

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1 Introduction

Virtual knots were introduced in [2] as associated with the link diagrams, laying on non-trivial Riemann surfaces. Since then the theory of virtual knots is slowly developing [3]. From the point of view of knot polynomials the biggest achievement until recently seems to be evaluation of Jones polynomials for the simplest examples, summarized in [4]. The main problem is the apparent breakdown of group-theoretical approach of [5, 6, 7], because the structure of representation theory is violated by additional “sterile” vertex.

Recently in [1] it was suggested to overcome this problem by applying the alternative calculus of [8], originally devised to substitute the overcomplicated construction of Khovanov-Rozansky polynomials [9]. Though this formalism is of the same (low) complexity level as the Khovanov calculus for (super)Jones polynomials [10], for ordinary knots and links it immediately reproduces HOMFLY polynomials for generic algebras \( SL(N) \) – but this is more or less a triviality (non-trivial part of [8] is a possibility to describe KR polynomials). However, for virtual knots this formalism (which supposedly remains absolutely the same as in the ordinary case) is a real way out: at this moment it is the only one, providing HOMFLY polynomials for virtual knots and links.

Moreover, the very first attempt, made in [1], demonstrated that the answers exhibit a structure, which looks like just a minor deformation of the group-theoretical one – and thus provides a hope for building up a relevant modification of the approach of [7].

The goal of the present paper is to extend the first observations of [1] to a little wider families of virtual knots and links. This provides new evidence that the approach of [8, 1] indeed provides the topological invariant HOMFLY polynomials, and also extends the observation about the needed group theory deformation.

We do not repeat here the details of the construction from [8] and [1]: it involves the standard conversion of link diagram into a hypercube [10] and subsequent reading of HOMFLY polynomial from data at the hypercube vertices. Specifics of [8] is a modification of one of the two resolutions, which make vector spaces at hypercube vertices into factor-spaces, and the split of hypercube construction into two steps: that of the “primary” and then the ”main” hypercube, which, in turn is first built in the “classical” approximation and then “quantized”. Details can be found in [8, 1] or will hopefully get clear from consideration of the simple examples below.

The paper is organized as follows. In section 2 we describe the answers for the fundamental link polynomials in the warming-up 2-strand torus case, partly known, partly new. For their derivations see the next section 3.

Then in sections 4 and 5 we consider the family of twist knots, ordinary and virtual – the former are old results, the latter are new. The main new result here is eq. (4.14), it is a direct generalization of the well-known (4.12). Like in the torus case we observe striking similarity between the answers in ordinary and virtual cases – despite the violation of representation theory rules for the latter. Moreover, the answers naturally exhibit the peculiar structure, first observed in [11] for the figure-eight knot and further promoted in [12] and [13] to the universal “differential hierarchy” structure, which probably reflect and materialize the original observations in [14] and became a powerful tool for calculations of colored knot polynomials – see also [15].

In section 6 we demonstrate that, like in the case of ordinary knots/links, such universal differential structure (appropriately but universally modified) persists for fundamental HOMFLY polynomials of virtual knots beyond the family of twist knots. Moreover, it can still be used for generalization to colored polynomials – which can be introduced by the standard cabling method. This is, however, a tedious task, both technically and conceptually, because now there is no natural projection on particular representation in the cable. Using available data about cabled Jones polynomials for simple virtual knots, we demonstrate in section 7 some positive evidence for the
first symmetric representation [2] and some immediate problems for the next symmetric representation [3]. This is one of the obvious direction for further investigation.

## 2 Evolution method for the two-stand braids

### 2.1 Two parallel strands

Following [8], we use black and white vertices in oriented link diagrams to denote the two types of crossings:

\[
\mathcal{R} = 
\begin{array}{c}
\blacklozenge \\
\blacklozenge
\end{array}
\quad \mathcal{R}^{-1} = 
\begin{array}{c}
\blacklozenge \\
\blacklozenge
\end{array}
\]

where the quantum \( \mathcal{R} \)-matrix and its inverse are inserted in the RT formalism. The big white circle is used to denote the "sterile" crossing, where nothing is inserted:

\[
\mathcal{P} = 
\begin{array}{c}
\blacklozenge \\
\blacklozenge
\end{array}
\quad \mathcal{\tilde{P}} = 
\begin{array}{c}
\blacklozenge \\
\blacklozenge
\end{array}
\]

In [1] the fundamental HOMFLY polynomials were introduced for all the simplest (triple-intersection) virtual knots from [4]. A more far-going example in [1] concerns the family of two-strand knots of the shape

\[
\begin{array}{c}
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge
\end{array}
\quad \begin{array}{c}
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge \\
\blacklozenge
\end{array}
\]

In the left picture we have ordinary knots and links, in the right – the virtual ones appear. It is sufficient to put just one sterile vertex, because they commute with the \( \mathcal{R} \)-matrices and two adjacent sterile vertices cancel each other.

It is assumed that there are \( n \) black vertices in the left picture and \( n - 1 \) black vertices in the right one. For negative \( n \) the vertices are white rather than black. When \( n \) is odd, we have a knot, and it is independent of orientation. When \( n \) is even, we get a two-component link – and it depends on the mutual orientation of components: they can be either parallel or antiparallel. The two-strand knots (for odd \( n \)) are always parallel.

The main idea of the group-theoretic (RT) approach is that \( \mathcal{R} \)-matrices act as unit matrices in the irreducible representations, only numerical eigenvalues should be taken into account. If only fundamental representations \( [1] = \square \) are ascribed to the lines of the link diagram (reversion of arrow changes in to the anti-fundamental \( [1^{N-1}] = \blacksquare \)), then there are just two eigenvalues in the parallel channel and two in the anti-parallel:

\[
\square \otimes \square = [2] + [11], \quad \mathcal{R}_{[2]} = \frac{A}{q} \cdot \text{Id}, \quad \mathcal{R}_{[11]} = -qA \cdot \text{Id}.
\]

This implies that the HOMFLY polynomial in the parallel case is just

\[
H_{[2,n]} = \left( \frac{A}{q} \right)^n \cdot D_{[2]} + \left( -qA \right)^n \cdot D_{[11]} = \left( \frac{A}{q} \right)^n \cdot \frac{[N][N+1]}{2} + \left( -qA \right)^n \cdot \frac{[N][N-1]}{2},
\]

where \( D_{[2]} = \frac{[N][N+1]}{2} \) and \( D_{[11]} = \frac{[N][N-1]}{2} \) are quantum dimensions of the symmetric and antisymmetric representations respectively. Here \( A = q^k \) and the quantum number is defined as

\[
[k] = \frac{\{q^k\}}{\{q\}} = \frac{q^k - q^{-k}}{q - q^{-1}},
\]

\[
[D_{[2]}] = \frac{[N+1][N]}{2}, \quad [D_{[11]}] = \frac{[N-1][N]}{2},
\]

where \( [N] = q \cdot q^{-1} \) is the quantum number for the fundamental representation of the simple Lie algebra \( \mathfrak{g} \).
where \( \{ x \} = x - x^{-1} \).

Eq. (2.2) can be obtained by the “evolution method” [16, 12]: as an “average” of the \( R \)-matrix in “representation form”,

\[
H^{[2,n]} = \left\langle \left( \frac{A}{q} - qA \right)^n \cdot P_{2\text{-strands}} \right\rangle = \text{Tr}_{\bigotimes \triangle} \hat{R}^n ,
\]

(2.4)
corresponding to the splitting of the diagram into the “evolution braid” and “vacuum projector”:

\[
\bullet \quad \bullet
\]

The idea of the evolution method is that whenever there is a box of this type in a link diagram, with an \( n \)-point two-strand braid inside, the dependence of HOMFLY polynomial on \( n \) is fully described by the linear combination

\[
\alpha \cdot \left( \frac{A}{q} \right)^n + \beta \cdot \left( - qA \right)^n ,
\]

(2.5)
where \( \alpha \) and \( \beta \) depend on the ”outside” of the box, but does not depend on \( n \). These coefficients can be calculated if the answer is known for some two values of \( n \), e.g. \( n = 0 \) and \( n = \pm 1 \).

In the particular case of the 2-strand parallel braid

\[
H^{[2,0]} = [N]^2 \quad \text{and} \quad H^{[2,\pm 1]} = [N]
\]

(2.6)
(two and one unknot respectively), thus the coefficients \( \alpha \) and \( \beta \) are equal to \( \frac{[N][N+1]}{2} \), as stated in (2.2).

Similarly, one can find the coefficients \( \alpha \) and \( \beta \) for the right diagram, i.e. for virtual knots and links:

\[
V^{[2,1]} = \text{Tr}_{\bigotimes \triangle} \hat{R}^n P = \left( \frac{A}{q} \right)^{n-1} \cdot \left( [N][N-1] \left[ \frac{2}{2} \right] + [N] \right) - \left( -qA \right)^{n-1} \cdot \left( [N][N-1] \left[ \frac{2}{2} \right] \right)
\]

(2.7)
in accordance with [1]. We denote HOMFLY polynomials for virtual knots by \( V \), to emphasize that these are still new, under-investigated quantities with a more shaky status.

Operator \( P \) stands for the “sterile” vertex, and it breaks the group-representation structure. The most interesting fact is that this breaking is not as radical as it could: (2.7) differs from (2.2) by the substitution

\[
D^{[2]} = \frac{[N][N+1]}{2} \rightarrow \frac{[N][N-1]}{2} + [N], \quad D^{[11]} = \frac{[N][N-1]}{2} \rightarrow \frac{[N][N-1]}{2}
\]

(2.8)
These are the intriguing formulae which imply that HOMFLY polynomials for virtual knots and links can one day acquire a representation-theory interpretation.

To avoid possible confusion, we note that the middle part of eq. (2.7) is somewhat symbolic – no generally-applicable matrix representation is known for operators \( \hat{R} \) and \( P \) together, thus one can not use this type of formulae for practical calculations with virtual knots and links. The only possibility at this moment is to use an alternative approach of [8] – as suggested in [1]. This is how the r.h.s. of (2.7) is actually derived.
2.2 Two antiparallel strands

In the antiparallel case the two product of two representations is a combination of adjoint $\text{Adj} = [21^{N-2}]$ and a singlet $[0]$:

$$ \square \otimes \square = \text{Adj} + [0], \quad \mathcal{R}_{\text{Adj}} = A \cdot \text{Id}, \quad \mathcal{R}_{[0]} = 1 \cdot \text{Id}. \quad (2.9) $$

For the antiparallel link ($n = 2k$ is obligatory even in this case)

we get

$$ \gamma \cdot A^{2k} + \delta \quad (2.10) $$

with $n = 2k$-independent $\gamma$ and $\delta$.

Note additional significant difference between evolutions in parallel and antiparallel cases. In variance with

for parallel strands, the two diagrams

with even and odd number of $R$-matrices in the 2-strand braid inside the circle are essentially different. Therefore in the antiparallel case evolution works for insertion of even number of vertices, while the two series with different parities of vertex numbers can be quite different. In the case of 2-strand antiparallel links this situation is just extreme: the series with odd number of vertices simply does not exist.

Another word of caution is that the eigenvalues in (2.1) and (2.9) do not coincide for $N = 2$, though from representation-theory point of view in this case $[1] \cong [1]$. This is the feature of “topological framing”, which is adjusted to ensure topological invariance under the Reidemeister moves – it requires that $A/q$ in (2.1) coincides with 1 in (2.9) at $N = 1$ rather than $(A/q, -Aq)$ with $(A, 1)$ at $N = 2$.

For ordinary links we get (see eq.(116) in [12]):

$$ H_{\square \times \square}^{[2,2k]} = \text{Tr} \square \otimes \square \mathcal{R}^{2k} = 1 + A^{2k} [N - 1] [N + 1], \quad (2.11) $$

while for the virtual links –

$$ V_{\square \times \square}^{[2,2k]} = \text{Tr} \mathcal{R}^{2k} \tilde{\mathcal{P}} = 1 + A^{2k-1} [N - 1] (|N| + 1). \quad (2.12) $$

This follows from evaluation of two particular examples in the next section 3.

Again, the difference between ordinary and virtual cases – caused by insertion of the sterile-vertex operator $\tilde{\mathcal{P}}$ (tilde means that it acts in another – transposed – channel) – is reduced to the intriguing change

$$ D_{\text{Adj}} = [N + 1][N - 1] \rightarrow [N - 1] (|N| + 1) $$

$$ D_{[0]} = 1 \rightarrow 1 \quad (2.13) $$
3 Torus two-strand links and knots

In this section we derive the four formulae (2.2), (2.7), (2.11) and (2.12) for particular values of \( n \) and \( k \) (what is sufficient to determine the coefficients \( \alpha, \beta, \gamma \) and \( \delta \) in the evolution formulae) by the method of [8,1]. In the case (2.2) and (2.11) of ordinary knots and links this can be also done by the RT method [7], but for (2.7) and (2.12) there are still no alternative derivations and even alternative definition of what is the HOMFLY polynomial for virtual knots and links.

Later, in section 4 we consider another family – of twist knots, and there we encounter another, topologically equivalent, realization of the same virtual knots. HOMFLY polynomials will turn to be the same – thus providing an evidence, that our definition is indeed topologically invariant.

3.1 Resolution and the Seifert cycles

The main idea of Khovanov’s approach is to consider all colorings of the given link diagram at once and substitute the diagram by a hypercube of its resolutions.

Namely, consider the diagram with all vertices black and resolve each vertex in the following way:

\[
\begin{array}{c}
\black\black\black\\
\black\black\black
\end{array} \quad \rightarrow \quad \begin{array}{c}
\black\black\black\\
\black\black\black
\end{array}
\]

Then for the ordinary link or knot the diagram decomposes into a collection of non-intersecting (Seifert) cycles, while for virtual knots each sterile vertex remains an intersection. However, the resolutions of the parallel and antiparallel braids are significantly different:

\[
\begin{array}{c}
\black\black\black\\
\black\black\black
\end{array} \quad \rightarrow \quad \begin{array}{c}
\black\black\black\\
\black\black\black
\end{array}
\]

i.e. in the parallel case we get just two Seifert cycles, while in the antiparallel case there will be \( n = 2k \):

\[
\begin{array}{c}
\black\black\black\\
\black\black\black
\end{array} \quad \rightarrow \quad \begin{array}{c}
\black\black\black\\
\black\black\black
\end{array}
\]

For virtual knots/links the number of Seifert cycles will be again different: just one in the parallel case

\[
\begin{array}{c}
\black\black\black\\
\black\black\black
\end{array} \quad \rightarrow \quad \begin{array}{c}
\black\black\black\\
\black\black\black
\end{array}
\]

and \( n - 1 = 2k - 1 \) in the antiparallel case:
3.2 Primary hypercube

Above resolution of the link diagram stands at the main vertex of the hypercube. Other vertices correspond to alternative resolutions, while edges of the hypercube are associated with the flip of the resolution at a particular vertex of the link diagram. Different versions of Khovanov formalism differ by the choice of the second resolution. The choice of $[S]$ – generating knot polynomials for arbitrary $N$ – is rather complicated and is introduced in three steps. At the first step the flip to the second resolution is

\[ \begin{array}{c}
\text{primary}\end{array} \]

i.e. introduces intersections of the Seifert cycles.

Hypercube introduced with the help of this flip is called primary and in four above cases it looks rather different:

- parallel ordinary case:

\[
\begin{array}{c|c|c|c|c|c|c|c}
2 & n \times 1 & C_n^2 \times 2 & C_n^3 \times 1 & C_n^4 \times 2 & \ldots & 1 \text{ for odd } n & 2 \text{ for even } n \\
\end{array}
\]

- antiparallel ordinary case:

\[
\begin{array}{c|c|c|c|c|c|c|c}
\text{n = 2k} & n \times n - 1 & C_n^2 \times n - 2 & C_n^3 \times n - 3 & \ldots & n \times 1 & 2 \\
\end{array}
\]

- parallel virtual case:

\[
\begin{array}{c|c|c|c|c|c|c|c}
1 & (n - 1) \times 2 & C_{n-1}^2 \times 1 & C_{n-1}^3 \times 2 & C_{n-1}^4 \times 1 & \ldots & 1 \text{ for odd } n & 2 \text{ for even } n \\
\end{array}
\]

- antiparallel virtual case:

\[
\begin{array}{c|c|c|c|c|c|c|c}
\text{n - 1 = 2k - 1} & (n - 1) \times n - 2 & C_{n-1}^2 \times n - 3 & \ldots & (n - 1) \times 1 & 2 \\
\end{array}
\]

Shown are the multiplicities of vertices at a given distance (number of flips) from the main (Seifert) vertex with the given (underlined) number of cycles. The main vertex is put into a box. Binomial coefficients are the ordinary $C_n^i = \binom{n}{i}$. The number of cycles in the last vertex is equal to the number of components in the original link, i.e. in our case it is 1 for odd $n$ and 2 for even $n$. In the virtual case the number of vertices in original link diagram is $n - 1$ rather than $n$, thus the dimension of hypercube is also less by one.

3.3 The full classical hypercube

The second step in the construction of $[S]$ is modification of the flip:

\[ \begin{array}{c}
\text{full classical hypercube}\end{array} \]
At \( N = 2 \) the following procedure gives the same results as Kauffman’s orientation-violating flip \cite{17}

\[
\begin{array}{c}
\text{\LARGE \boldsymbol{N}} \\
\text{\LARGE \boldsymbol{N}} \\
\text{\LARGE \boldsymbol{N}} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{\LARGE \boldsymbol{N}} \\
\text{\LARGE \boldsymbol{N}} \\
\text{\LARGE \boldsymbol{N}} \\
\end{array}
\]

(there is no orientation-dependence at \( N = 2 \), i.e. for Jones polynomials), but for \( N \neq 2 \) there will be essential differences.

Since resolution is now a linear combination of two, a hypercube vertex, which is at the distance \( h \) from the main (Seifert) one will be associated with a formal linear combination of \( 2^h \) resolved diagrams, half of them with negative signs. If the diagram has \( \nu \) cycles, we substitute it by \( N^\nu \) and take a linear combination of these powers – in Khovanov theory this can be considered as a K-theory dimension (in the case of virtual knots/links it can even be negative) of a “factor-space”, associated to the hypercube vertex. Invariance with the widely familiar \( N = 2 \) case, this association is ”non-local” – depends on the \( h \)-dimensional sub-hypercube, connecting the given vertex with the main (Seifert) vertex.

This construction provides what we call classical hypercube – it can be easily restored once the primary one with marked (boxed) Seifert vertex is given. In our above examples we get:

**parallel ordinary case:**

\[
\begin{array}{c|c|c|c|c|c|c|}
\text{ordinary case:} & N^2 & n \times N(N-1) & C_n^2 \times 2N(N-1) & C_n^3 \times 2^2N(N-1) & C_n^4 \times 2^3N(N-1) & \ldots & 2^{n-1}N(N-1) \\
\end{array}
\]

**antiparallel ordinary case:**

\[
\begin{array}{c|c|c|c|c|c|c|}
\text{ordinary case:} & N^n & n \times N^{n-1}(N-1) & C_n^2 \times N^{n-2}(N-1)^2 & C_n^3 \times N^{n-3}(N-1)^3 & \ldots & n \times N(N-1)^{n-1} & N(N-1) Y_{n-1}^{cl} \\
\end{array}
\]

**parallel virtual case:**

\[
\begin{array}{c|c|c|c|c|c|c|}
\text{virtual case:} & N^2 & (n-1) \times -N(N-1) & C_n^2 \times -2N(N-1) & C_n^3 \times -2^2N(N-1) & \ldots & -2^{n-1}N(N-1) \\
\end{array}
\]

**antiparallel virtual case:**

\[
\begin{array}{c|c|c|c|c|c|c|}
\text{virtual case:} & N^{n-1} & (n-1) \times N^{n-2}(N-1) & C_n^2 \times N^{n-3}(N-1)^2 & \ldots & (n-1) \times N(N-1)^{n-2} & N(N-1) y_{n-2}^{cl} \\
\end{array}
\]

Here

\[
Y_{n-1}^{cl} = \frac{N^n - C_n^1 N^{n-1} + C_n^2 N^{n-2} - \ldots - C_n^{n-1} N + N^2}{N(N-1)} = \frac{(N-1)^n + N^2 - 1}{N(N-1)} = \frac{(N-1)^{n-1} + N + 1}{N} \tag{3.1}
\]

and

\[
y_{n-2}^{cl} = \frac{N^{n-1} - C_n^1 N^{n-2} + C_n^2 N^{n-3} - \ldots - C_n^{n-2} N - N^2}{N(N-1)} = \frac{(N-1)^{n-1} - N^2 + 1}{N(N-1)} = \frac{(N-1)^{n-2} - N - 1}{N} \tag{3.2}
\]

For even \( n = 2k \) the numerators at the r.h.s. are divisible by \( N \).

Note that in the parallel virtual case most numbers are negative, in the antiparallel virtual case this happens only to \( y_0^{cl} = -1 \).

### 3.4 The full quantum hypercube and HOMFLY polynomial

The last step – if we do not proceed further to Khovanov-Rozansky polynomials – is to introduce \( q \)-dependence. We call this procedure “quantization”, in terms of vector spaces this means conversion to \( q \)-graded spaces.

If quantum (graded) hypercube – the set of numbers \( \{N_v\} \) is known, one can construct HOMFLY polynomials, by taking alternated sums over the hypercube vertices, starting from initial vertex \( v_c \), defined by the coloring \( c \) of original link diagram \( \mathcal{L}_c \):

\[
H^{\mathcal{L}_c} = q^{n^0(N-1)} \sum_{v \in H(\mathcal{L})} (-q)^{h_v - h_{v_c}} N_v. \tag{3.3}
\]
Note that the only dependence on coloring is in the distance $|h_c - h_c|$ – the hypercube $\mathcal{H}(L)$ is the same for all colorings.

Usually, for a given knot/link its link diagram possesses alternative colorings, associated with simpler links – for which the answers are already known. This can be used to specify the quantization procedure for the new, yet unknown, items.

Most of quantities in above examples are quantized in a trivial way: when the number is factorized into items $N - i$, each factor is just substituted by a quantum number $[N - i]$. In particular, this is sufficient for a full description of parallel families.

In antiparallel case one still needs quantization of $Y$ and $y$. Applying the above-mentioned re-coloring trick, one can check that naively-quantized are the differences:

$$Y^e_{n+1} - Y^e_{n-1} = \frac{(N-1)^{n+1} - (N-1)^{n-1}}{N} = (N-1)^{n-1}(N-2) \rightarrow [N-1]^{n-1}[N-2] = Y_{n+1} - Y_{n-1}$$

and

$$y^e_n - y^e_{n-2} = \frac{(N-1)^n - (N-1)^{n-2}}{N} = (N-1)^{n-2}(N-2) \rightarrow [N-1]^{n-2}[N-2] = y_n - y_{n-2}.$$  

To this one should add initial conditions $Y_1 = [2]$ and $y_0 = -1$.

### 3.5 The four series

It remains to substitute all these prescriptions and deduce the answers for our four examples.

#### 3.5.1 Ordinary parallel case

For all black vertices we obtain

$$H^{[2,n]}_{\square} = q^{n(N-1)}\left([N]^2 - qC_n^1[N][N-1] + q^2C_n^2[2][N][N-1] + \ldots + (-q)^n[2]^{n-1}[N][N-1]\right) =$$

$$= q^{n(N-1)}\left([N]^2 - \frac{[N][N-1]}{2} + \frac{[N][N-1]}{2} \left(1 - q \cdot 2\right)\right)^n = q^{n(N-1)}\left(\frac{[N]}{2} + (-q^2)^n \cdot \frac{[N][N-1]}{2}\right).$$

This is exactly eq. (2.2) from [8] – a particular (2-strand) case of the celebrated Rosso-Jones formula for torus knots and links [18, 16].

#### 3.5.2 Ordinary antiparallel case

For all black vertices we obtain (for $n = 2k$)

$$H^{[2,2k]}_{\square \times \square} = q^{n(N-1)}\left([N]^n - qC_n^1[N]^{n-1}[N-1] + q^2C_n^2[2]^n[N-1]^2 - \ldots - (-q)^{n-1}C_n^{n-1}[N][N-1]^{n-1} + q^n[N][N-1][Y_{n-1}]\right) =$$

$$= q^{n(N-1)}\left\{([N] - q[N-1])^n + q^n[N][N-1]Y_{n-1} - q^n[N-1]^n\right\} =$$

$$= 1 + q^nN [N-1] (\frac{N}{2} Y_{n-1} - [N-1]^{n-1}),$$

where we used the identity $[N] - q[N-1] = q^{1-N}$.

In order to understand what stands in the remaining bracket it is sufficient to note that it is independent of $n$: the difference $[N-1]^{n+1} - [N-1]^{n-1} = [N][N-1]^{n-1}[N-2]$ is exactly the same as $[N](Y_{n+1} - Y_{n-1})$. Thus it remains to look at particular example of $n = 2$, when $[N]Y_1 - [N-1] = [2][N] - [N-1] = [N+1]$. Thus

$$H^{[2,2k]}_{\square \times \square} = 1 + q^{2kN}[N+1][N-1],$$

what is exactly (2.11) from [12]. Note that the only coincidence between (3.6) and (3.8) is at $n = 2$, the Hopf link is independent of orientation:

$$H^{[2,2]}_{\square \times \square} = q^{2N-2}[N][N+1] = 1 + q^{2N}[N-1][N+1] = H^{[2,2]}_{\square \times \square}. $$
In this example one needs to justify the quantization rule for \( Y_{n-1} \), thus it also deserves considering some other colorings. If one vertex is white, then we should get the link with \( n \rightarrow n - 2 \). On the other hand, this corresponds to taking as initial an vertex in the hypercube, which is adjacent to the Seifert vertex. Such checks were performed, at least partly, in [8].

### 3.5.3 Virtual parallel case

For all black vertices

\[
V^{[2,n]}_{\square \cdot \square} = q^{(n-1)(N-1)} \left( [N] + qC_{n-1}^1 [N][N-1] - q^2 C_{n-1}^2 [N][N-1] + q^3 C_{n-1}^3 [N][N-1] - \ldots \right) = \\
= q^{(n-1)(N-1)} \left\{ [N] + \frac{[N][N-1]}{2} - \frac{1}{2} (1 - q [2])^{n-1} [N][N-1] \right\} = \\
= q^{(N-1)(n-1)} \cdot \left( \frac{[N][N-1]}{2} + [N] \right) - q^{N+1} n-1 \cdot \frac{[N][N-1]}{2} ,
\]

what is exactly the answer (2.7) from [1]. Note the change of signs in most terms in the original sum, as compared to the case of ordinary (non-virtual) knots and links – it is caused by negativity of most ”dimensions” at hypercube vertices. Also note that the combination \( \frac{[N][N-1]}{2} + [N] \) can contain odd, not only even powers of \( q \) – while for ordinary knots and links the knot polynomials always depend on even powers of \( q \).

### 3.5.4 Virtual antiparallel case

This time we get for the link diagram with all black vertices (we remind that \( n - 1 = 2k - 1 \) is obligatory odd in this case):

\[
V^{[2,2k]}_{\square \cdot \square} = q^{(n-1)(N-1)} \left( [N]^{n-1} - qC_{n-1}^1 [N]^{n-2}[N-1] + q^2 C_{n-1}^2 [N]^{n-3}[N-1]^2 - \ldots \\
\ldots + q^{n-2} C_{n-1}^{n-2} [N][N-1]^{n-2} - q^{n-1} [N][N-1] y_{n-2} \right) = \\
= q^{(n-1)(N-1)} \left\{ \left([N] - q[N-1]\right)^{n-1} + q^{n-1} \left((N-1)^{n-1} - [N][N-1] y_{n-2}\right) \right\} = \\
= 1 + q^{N(n-1)} [N-1] \left( [N-1]^{n-2} - [N] y_{n-2} \right).
\]

Like in the ordinary antiparallel case the last bracket is independent of \( n \) and for \( n = 2 \) it is equal to \( 1 + [N] \), so that finally

\[
V^{[2,2k]}_{\square \cdot \square} = 1 + q^{(2k-1)N} [N-1] \left( [N] + 1 \right).
\]

This is the same as (2.12) – the first essentially new answer for a virtual family in the present paper.

Like in the ordinary antiparallel case one can validate (or derive) the quantization rule for \( y_{n-2} \) by considering another coloring, i.e. taking another vertex of the hypercube. Another confirmation of the quantization rule is that it leads to the answer, which is consistent with the evolution in \( n \), i.e. has the form (2.10) with the \( k \)-independent coefficients \( \gamma \) and \( \delta \).

### 4 Twist knot series

#### 4.1 The four-vertex twist knot

We begin from the first non-trivial example. From the knot diagram with all black vertices and its resolution

\[ \text{\includegraphics[width=0.2\textwidth]{twist_knot.png}} \]

we read the following primary and the main classical hypercubes:
To avoid overloading the picture we do not show the edges – but they are implicitly used in the construction of the main hypercube. The only non-trivial is the quantization of the underlined item – and it can be obtained from the requirement that alternative coloring,

\[
- q^{2N-3}[N] \left\{ [N][N-1] - q \left( [N]^2 + 2[N-1]^2 + [2][N][N-1] \right) + q^2 \left( 3[N][N-1] + 3[2][N-1]^2 \right) - q^3 \left( 2[N-1]^2 + [2][N][N-1] + [N-1] \left[ (N-1) + [2]|N-2| \right] \right) + q^4[2][N-1]^2 \right\} = [N].
\]  

(4.2)

To obtain \([N]\) we had to take the underlined item to be what it is, i.e. the relevant quantization rule in this case is \((N-1)(3N-5) \rightarrow [N-1] (|N-1| + [2]|N-2|)\) – as already known from [8].

Using this prescription, we can obtain the HOMFLY polynomial for original (all black) coloring, starting from the boxed main (Seifert) vertex of the hypercube:

\[
q^{4(N-1)}[N] \left\{ [N]^2 - 4q[N][N-1] + q^2 \left( 2[2][N][N-1] + 4[N-1]^2 \right) - 4q^3[2][N-1]^2 + q^4[N-1] \left( |N-1| + [2]|N-2| \right) \right\} =
\]

\[
= [N] \left\{ \left( q^2 + \frac{1}{q^2} \right) A^2 - A^4 \right\} = \left[ N \right] \left\{ 1 - A^2 \{ Aq \} \{ A/q \} \right\} = H_{G_1}^{3\lambda_1},
\]

where \(A = q^N\) and \(\{ Aq^{\pm 1} \} = (q - q^{-1})[N \pm 1] = q^{N\pm 1} - q^{-N\mp 1}\).

For another initial vertex,

\[
\]

marked by a triple box in (4.1), we get:

\[
q^{-2}[N] \left\{ (1 + q^4)[2][N][N-1] - (q + q^3) \left( 2[N][N-1] + 2[2][N-1]^2 \right) + q^2 \left( [N]^2 + 4[N-1]^2 + [N-1] \left( |N-1| + [2]|N-2| \right) \right) \right\} =
\]

\[
= [N] \left\{ 1 + \{ Aq \} \{ A/q \} \right\} = H_{G_1}^{4\lambda_1}.
\]  

(4.4)
4.2 The three-vertex virtual twist knot

For the similar virtual knot

![Virtual Knot Diagram]

the initial and full hypercubes are:

\[
\begin{array}{c|c|c|c}
& N^2 & -N^3 & 2(N^2 - N) \\
\hline
1 & N^2 - N & -N^3 + 2N^2 - N & -N^3 + 4N^2 - 3N \\
\end{array}
\]

Quantization of the underlined item is prescribed by the requirement that

\[
V^{s \bullet} = -q^{N-2}[N] \left\{ [N - 1] - q \left( [N] + [2][N - 1] - [N - 1]^2 \right) + q^2 \left( [N - 1] - [N][N - 1] - [N - 1]([N - 2] - 1) \right) + q^3[1 - N] \right\} = [N],
\]

i.e. the relevant quantization rule in this case is \((N - 1)(N - 3) \rightarrow [N - 1][[N - 2] - 1]\) – as already known from \[1\].

Likewise the triple-boxed initial vertex, describing the coloring

![Triple-Boxed Initial Vertex]

also provides the unknot:

\[
V^{s \bullet} = q^{-N-1}[N] \left\{ -[N - 1]^2 - q \left( [N - 1] - [N][N - 1] - [N - 1]([N - 2] - 1) \right) + q^2 \left( [N] + [2][N - 1] - [N - 1]^2 \right) - q^3[N - 1] \right\} = [N].
\]

Then the HOMFLY polynomial for original (all black) coloring (with the single-boxed initial vertex) is

\[
V^{s \bullet} = q^{3(N-1)}[N] \left\{ [N] - q \left( 2[N - 1] - [N][N - 1] \right) + q^2 (2[N - 1] - 2[N - 1]^2) + q^3[N - 1] \right\} = V^{2-1} = [N] \left( 1 - q^{N-1}(q^{N+1} - 1)(q^{N-1} - q^{1-N}) \right) = [N] \left( 1 - \frac{4}{q}(Aq - 1)\{A/q\} \right).
\]

For yet another coloring,
initial is the double boxed vertex in

\[
\begin{array}{c|c|c}
[N]^2 & [N][N-1] & -[N][N-1]^2 \\
-[N]^2(N-1) & -[2][N][N-1] & -[N][N-1](N-2-1) \\
\end{array}
\]

and

\[
V^{\bigbullet\bigbullet}_\square = -q^{N-2}[N] \left\{ -[N][N-1] - q \left( [N] - 2[N-1]^2 \right) + q^2 \left( 2[N-1] - [N-1](N-2-1) \right) \right\} = [N] \left( 1 + (q^{-N+1} - q^{-1-N}) \right) = [N] \left( 1 + (Aq - 1)\{A/q\} \right) = V^{\bigbullet\bigbullet}_\square \quad (4.8)
\]

while for the triple-boxed initial vertex,

we would get:

\[
V^{\bigcirc\bigcirc}_\square = q^{-N-1}[N] \left\{ [2][N-1] - q \left( 2[N-1] - [N-1](N-2-1) \right) \right\} = [N] \left( 1 + q^{-N-1}(q^{N+1} - q^{-1-N}) \right) = [N] \left( 1 + \frac{1}{Aq}(Aq - 1)\{A/q\} \right) = V^{\bigcirc\bigcirc}_\square (q^{-1},A^{-1}).
\]

Finally, for the last coloring

we can get the answer directly from (4.7):

\[
V^{\bigcirc\bigcirc}_\square (q,A) = V^{\bigbullet\bigbullet}_\square (q^{-1},A^{-1}) = [N] \left( 1 - \frac{1}{A^2}(Aq - 1)\{A/q\} \right).
\]

### 4.3 Evolution for ordinary and virtual twist knots

Consideration of the single knot diagram in the previous section is actually enough to obtain the answer for entire family of twist knots – such is the power of the evolution method. As mentioned in 2.2 for antiparallel strands evolution actually changes the number of vertices by two, however in the case of twist knots there is an additional topological identity

\[
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\end{array}
\end{array}
\end{array}
\]

which allows one to relate the two series – with even and odd number of vertices:

\[
V^{2k+1}_{\bigbullet\bigbullet} = V^{2k}_{\bigbullet\bigbullet},
\]

(4.11)
where \(2k + 1\) is the number of black vertices on the vertical (sterile crossing not counted). This is the analogue of the same relation holds for the ordinary twist knots \([12, 8]\):

\[
H_{\square}^{2k} = H_{\square}^{2k-1} = 1 - \frac{A^{k+1}\{A\}}{\{A\}} \{Aq\}\{A/q\}. 
\]

(4.12)

with the following knots being different members of the series:

\[
\begin{align*}
\ldots \\
5_2 & \quad k = 2 \quad \bullet\bullet = \bullet\bullet \\
3_1 & \quad k = 1 \quad \bullet\bullet = \bullet\bullet \\
\text{unknot} & \quad k = 0 \quad \bullet\bullet = \bullet\bullet \\
4_1 & \quad k = -1 \quad \bullet\bullet = \bullet\bullet \\
\ldots
\end{align*}
\]

(4.13)

Note only that in our notation \(V_{\square}^{2k+1}\) is similar to \(H_{\square}^{2k+2}\) and therefore relation (4.11) is somewhat different from that in (4.12).

From (4.7) and (4.8) with \(2k + 1 = 1\) and \(2k + 1 = -1\) respectively we can get the coefficients \(\gamma\) and \(\delta\) in the evolution formula (2.10):

\[
V_{\square}^{2k+1} = [N] \left( 1 + \frac{(Aq-1)\{A/q\}}{\{A\}} \left( 1 - \left( 1 + \frac{A}{q} \right) \frac{A^{2k+2-1}}{A^{2}-1} \right) \right) = \\
= [N] \left( 1 + \frac{A(Aq)\{A/q\}}{\{A\}} - \frac{(A+q)(Aq-1)\{A/q\}}{\{A\}} . \frac{A^{2k+1}}{q\{A\}} \right) = \\
= [N] \left( 1 + \frac{(Aq+1) - (A+q)A^{2k+1}}{q\{A\}}(Aq-1)\{A/q\} \right). 
\]

(4.14)

Similarly from (4.9) and (4.10) we deduce:

\[
V_{\square}^{2k+1} = [N] \left( -\frac{1}{A} + \frac{A^{2k+2-1}}{A^{2}-1} \left( \frac{1}{Aq} + \frac{1}{A} \right) \right) = \\
= [N] \left( 1 - \frac{(Aq+1) - (A+q)A^{2k+1}}{qA^{2}(A)}(Aq-1)\{A/q\} \right). 
\]

(4.15)

In fact, these two formulae are not independent:

\[
V_{\square}^{2k+1}(q, A) = V_{\square}^{-2k-1}(q^{-1}, A^{-1}). 
\]

(4.16)
Together with (4.11) eq.(4.14) provides the full answer for the family of twist virtual knots. In particular,

\[ G_{-6} = G_{-5} = 1 + \frac{1}{qA} + \frac{1}{A^2} + \frac{1}{A^3} + \frac{1}{A^4}, \]

\[ G_{-4} = G_{-3} = 1 + \frac{1}{qA} + \frac{1}{A^2}, \]

\[ G_{-2} = G_{-1} = G_{0} = 1, \]

\[ G_{2} = G_{3} = -\frac{A}{q}(1 + qA + A^2), \]

\[ G_{4} = G_{5} = -\frac{A}{q}(1 + qA + A^2 + qA^3 + A^4), \] (4.17)

\[ ... \]

From (4.16) it follows that \( qA \cdot G_{-2k+1} = G_{-2k-1} \) because of the change of the factor \((Aq - 1)\).

In the next section 5 we check the prediction (4.17) of evolution method by explicit calculation of a few additional examples. The four quantities that we already calculated – and used to make this prediction – are put in boxes.

5 More examples of (virtual) twist knots

5.1 The six-vertex twist knot \((2k + 1 = \pm 3)\)

Switching to more complicated examples we start from direct generalization of what we just studied:

We temporarily omitted the 3-point case, because resolved diagrams there would have a different structure – see 5.2 below.

The primary hypercube is now made from the following vertices (underlined are the cycle numbers, and the
numbers in front of them are their multiplicities):

| 2 \times 2 | 5 | 2 \times [N] | 2 \times [2][N][N-1] | 2 \times [N][N-1] |
|----------|-----------------|-----------------|-----------------|-----------------|
| 4 \times 4 | 8 \times 3 | 4 \times 4 | 4 \times [N][N-1] | 8 \times [N][N-1]^2 | 4 \times [2][N][N-1]^2 |
| 6 \times 2 | 12 \times 2 | 6 \times 2 | 6 \times [N][N-1]^2 | 12 \times [N][N-1]^3 | 6 \times [2][N][N-1]^3 |
| 4 \times 2 | 8 \times 1 | 4 \times 2 | 4 \times [N][N-1]^3 | 8 \times [N][N-1]^4 | 4 \times [2][N][N-1]^4 |
| \underline{3} | \underline{2} \times \underline{2} | 1 | \underline{\ell}[N][N-1] Y_3 | 2 \times [N][N-1]^2 Y_3 | [N][N-1] \left( [N-1]^3 + [N-2] Y_3 \right) |

where \( Y_3 = [N-1][N-2] + [2] \) is the quantization of \( N^2 - 3N + 4 \), obtained from the requirement that we get the unknot, if we take for initial the double boxed vertex, see s.5.7.4 of [5].

For the virtual knot with a similar knot diagram

we get the following primary and main hypercubes:

| 2 \times 2 | 4 | \underline{[N]^4} | 2 \times \underline{[N][N-1]} | \underline{[2][N][N-1]} |
|----------|-----------------|-----------------|-----------------|-----------------|
| 3 \times 2 | 6 \times 2 | 3 \times 2 | 3 \times [N][N-1] | 6 \times [N][N-1]^2 | 3 \times [2][N][N-1]^2 |
| 3 \times 2 | 6 \times 1 | 3 \times 2 | 3 \times [N][N-1]^2 | 6 \times [N][N-1]^3 | 3 \times [2][N][N-1]^3 |
| \underline{3} | \underline{2} \times \underline{2} | 1 | \underline{\ell}[N][N-1] (\underline{[N-2]-1}) | 2 \times [N][N-1]^2 (\underline{[N-2]-1}) | [N][N-1] y_2 |

Starting from the double boxed vertex, we should get the unknot and this defines the quantization rule for the underlined items in last row, in particular, \( 2N^2 - 7N + 7 \rightarrow y_2 = [2][N-1][N-2] - [N-2] + 1 \). Indeed

\[ -q^{2N-4}[N] \left\{ [N]^3 [N-1] - q \left( [N]^3 + 3[N][N-1]^2 + [2][N][N-1] \right) \right. \]
\[ -0^2 \left( 3[N]^2[N-1] + [N]^2[N-1] + 3[2][N][N-1]^2 + 3[N][N-1]^3 \right) - \]
\[ -q^3 \left( 3[N][N-1]^2 + [N-1]^3 ([N-2]-1) + 3[N][N-1]^2 + 3[2][N][N-1]^3 \right) - \]
\[ + q^4 \left( 3[N-1]^3 + [N][N-1]([N-2]-1) + [N-1]y_2 \right) - \]
\[ -q^5 ([N-1]^2 ([N-2]-1)) \} \right. ] = [N]. \]

Now we can find HOMFLY polynomial for the original diagram (all vertices black):

\[ q^{5N-5}[N] \left\{ [N]^3 - 5q[N]^2[N-1] + q^2 \left( [N][N-1]^2 + [2][N][N-1] \right) - \right. \]
\[ -q^3 \left( [N][N-1]([N-2]-1) + 6[N-1]^3 + 3[2][N][N-1]^2 \right) + \]
\[ + q^4 \left( 2[N-1]^2 ([N-2]-1) + 3[2][N][N-1]^3 \right) - \]
\[ -q^5 [N-1] \{ (2[N-1])[N-2] - [N-2] + 1 \} = [N] \{ 1 - q^{N-1}(q^{N+1} - 1)(A/q)(q^{2N} + q^{N+1} + 1) \}. \]
If the triple boxed vertex is taken for initial one we get:

\[
q^{N-3}[N] \left\{ [2][N][N - 1] - q \left( 2[N]^2[N - 1] + 3[2][N][N - 1]^2 \right) + 
+ q^2 \left( [N]^3 + 6[N][N - 1]^2 + 3[2][N - 1]^3 \right) - q^3 \left( 3[N]^2[N - 1] + 6[N - 1]^3 + [N - 1]y_2 \right) + 
+ q^4 \left( 3[N][N - 1]^2 + 2[N - 1]^2 ([N - 2] - 1) \right) - q^5[N][N - 1]((N - 2) - 1) \right\} = 
= [N] \left( 1 + q^{N-1} - 1 \right) (A/q)(q^{2N} + q^{N+1} + 1). 
\]  

(5.3)

5.2 The three-vertex twist knot \((2k = 0)\)

This example would be of course the first one. We did not begin from it, because it provides just one point on the evolution line and would not be sufficient to restore the evolution for virtual knots - thus we began from the four-vertex example with two virtual points \(2k + 1 = \pm 1\). Once we did that, the case \(2k + 1 = \pm 3\) was the natural next example, because the structure of Seifert cycles differs drastically depending on the parity of the points number. Still we would like to look at this second series also.

From the knot diagram with all black vertices and its resolutions

![Knot Diagram]

we read the following primary and the main classical hypercubes:

\[
\begin{array}{ccc}
1 & 2 & \quad [N][N - 1] \\
2 & 1 & 2 & 1 \\
1 & 2 & [N][N - 1] \\
\end{array} 
\] 

(5.4)

The HOMFLY polynomial for original (all black) coloring (with the boxed initial vertex) is

\[
q^{3(N-1)}[N] \left( [N] - 3q[N - 1] + 3q^2[2][N - 1] - q^3[2]^2[N - 1] \right) = 
= [N] \left( 1 - \frac{1}{q^2}(Aq - 1)(Aq + 1)(A - q)(A + q) \right) = 
= [N] \left( 1 - A^2 \{A/q\} \right). 
\] 

(5.5)

For another initial vertex

![Knot Diagram]

marked by a double box in (5.4), we naturally get the answer for the unknot:

\[
-q^{2N-1}[N] \left( [N - 1] - q ([N] + 2[2][N - 1]) + q^2(2[N - 1] + 2[2][N - 1]) - q^3[2][N - 1] \right) = [N]. 
\] 

(5.6)

5.3 The two-vertex virtual twist knot

For the similar virtual knot

![Knot Diagram]
primary and the main classical hypercubes are:

\[
\begin{array}{c|c|c}
2 & [N][1-N] & [2][N][1-N] \\
1 & [N][1-N] & \\
\end{array}
\] (5.7)

Fortunately, we can easily quantize all items in this example and obtain the HOMFLY polynomial for original (all black) coloring, starting from the boxed vertex of the hypercube:

\[
q^{2(N-1)}[N] \left(1 - 2q[1 - N] + q^2[2][1 - N] \right) = [N] \left(1 - \frac{A}{q} (Aq - 1) \{A/q\} \right). \quad (5.8)
\]

For another coloring of the knot with initial double-boxed vertex HOMFLY polynomial is

\[
-q^{-1}[N] \left( [1 - N] - q (1 + [2][1 - N]) + q^2[1 - N] \right) = [N]. \quad (5.9)
\]

### 5.4 The five-vertex twist knot \((2k = \pm 2)\)

The primary hypercube has the structure:

\[
\begin{array}{c|c|c|c|c|c|c}
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 2 & 1 & 2 & 3 & 3 & 2 \\
4 & 3 & 4 & 3 & 2 & 1 & 3 \\
4 & 3 & 4 & 3 & 2 & 1 & 3 \\
3 & 2 & 1 & 2 & 3 & 3 & 2 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

and the main classical hypercube is

\[
\begin{array}{c|c|c|c|c|c|c|c}
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
N^3(N-1) & N^2(N-1)^2 & N(N-1)^3 & N(N-1)^3 & N(N-1)^3 & N(N-1)(N^2 - 3N + 4) & 2N(N-1)^3 & 2N(N-1)(N^2 - 3N + 4) \\
\end{array}
\]
We can predict quantization of the underlined item by the requirement that the coloring with the double-boxed initial vertex is the trefoil, polynomial of which is already known from (5.5):

\[
q^{3N-4}[N]\left([N]^2[N - 1] - q([N]^3 + 4[N][N - 1]^2) + q^2(4[N]^2[N - 1] + 5[N - 1]^3 + 2[N][N - 1]^2) - q^3(5[N][N - 1]^2 + 2[N]^2[N - 1] + 2[N - 1]([N - 1][N - 2] + [2]) + 2[2][N - 3]^3) + q^4(2[N - 1]^3 + 2[2][N][N - 1]^2 + [2][N - 1]([N - 1][N - 2] + [2]) - q^5[2][N - 1]^3\right) = [N] (1 - A^2 \{Aq\} \{A/q\}),
\]

i.e. the relevant quantization rule in this case is \((N - 1)(N^2 - 3N + 4) \rightarrow [N - 1] ([N - 1][N - 2] + [2])\). Now we can get HOMFLY polynomial for original (all black) diagram:

\[
q^{3N-5}[N]\left([N]^3 - 5q[N]^2[N - 1] + q^2(9[N][N - 1]^2 + 2[N]^2[N - 1]) - q^3(3[2][N][N - 1]^2 + 7[N - 1]^3) + q^4(2[N - 1]([N - 1][N - 2] + [2]) + 3[2][N - 1]^3) - q^5[2][N - 1]([N - 1][N - 2] + [2]) \right) = [N] (1 - A^2(A^2 + 1)\{Aq\}\{A/q\}) .
\]

5.5 The four-vertex virtual twist knot

For the similar virtual knot the primary and main classical hypercubes are:

\[
\begin{array}{c|c|c|c}
1 & N^2(N - 1) & N(N - 1)^2 & N(N - 1)(N - 3) \\
2 1 2 & N(N - 1)^2 & 2N(N - 1)^2 & 2N(N - 1)^2 \\
2 1 2 & N^2(N - 1) & 2N^2(N - 1) & 2N^2(N - 1) \\
2 3 2 & N^2(N - 1) & 2N^2(N - 1) & 2N^2(N - 1) \\
2 1 2 & N^2(N - 1) & N(N - 1)^2 & N(N - 1)^2 \\
1 & N^2(N - 1) & N(N - 1)^2 & 2N(N - 1)^2 \\
\end{array}
\]

We can find out the quantization of the underlined item using the requirement that the coloring for the initial double-boxed initial vertex is the unknot:
\[-q^{2N-3}[N]\left([N][N-1] - q([N]^2 + 2[N-1]^2 + [2][N][N-1]) + \\
+q^2(3[N][N-1] + 2[2][N-1]^2 + [N-1]([N-2] - 1)) - \\
-q^3(3[N-1]^2 + [2][N-1]([N-2] - 1)) + q^4[N-1]([N-2] - 1)\right) = [N], \tag{5.13}\]

i.e. the relevant quantization rule in this case is $(N - 1)(N - 2) \to [N-1][N-2] - 1)$. Now we can find HOMFLY for original (all black) coloring:

\[q^k(N-1)[N]\left([N]^2 - 4q[N][N-1] + q^2(5[N-1]^2 + [2][N][N-1]) - \\
-q^3(2[2][N-1]^2 + 2[N-1]([N-2] - 1)) + q^4[2][N-1]([N-2] - 1)\right) = \\
[N] (1 - (A/q)(A^2 + Aq + 1)(Aq - 1)\{A/q\}). \tag{5.14}\]

6 On differential expansion for the fundamental HOMFLY

6.1 The claim

Let us now return to eq. (4.18), which we just checked in a number of examples. In \cite{11,12,13} a powerful idea of differential expansion was introduced for knot polynomials – and proved to be useful for constructing colored polynomials, i.e. those in non-fundamental (primarily, symmetric and antisymmetric) representations. The starting point in \cite{11} was exactly like (4.18): the statement that the deviation \(h_{\square} - 1\) of reduced (normalized) HOMFLY polynomial \(h_{\square} = [N]^{-1}H_{\square}\) from unity vanishes at \(A = q^{\pm 1}\). This should be true in the fundamental representation, in higher symmetric representations

\[h_{\square_1} \sim \{\sqrt{Aq}\}\{A/q\} \sim (q^{N+1} -(q^{1-N})(q^{N-1} - q^{1-N}) \sim (q^{N+1} - 1)(q^{N-1} - 1), \tag{6.2}\]

for virtual ones

\[v_{\square} \sim (Aq - 1)\{A/q\} \sim \sqrt{Aq}\{A/q\} \sim (q^{N+1} - 1)(q^{N-1} - q^{1-N}) \sim (q^{N+1} - 1)(q^{N-1} - 1). \tag{6.3}\]

In this section we provide certain evidence that this happens not only for the series of virtual twist knots, but in more general situation, presumably – always. Moreover, this property is true not only for reduced HOMFLY for knots, it seems also to hold for unreduced HOMFLY for links.
6.2 Small virtual knots

Fundamental HOMFLY polynomials, found in [1] for the first few virtual knots from the table [4], can be rewritten as:

\[ V_{\square}^{2.1} = -\frac{A^2}{q} + \frac{q^2}{q^2} + Aq = 1 - \frac{A}{q} \left( Aq - 1 \right) \{ A/q \}, \]
\[ V_{\square}^{3.1} = 1, \]
\[ V_{\square}^{3.2} = A^2 - \frac{A}{q} - q^2 + 1 + \frac{q}{q} = 1 + \left( Aq - 1 \right) \{ A/q \}, \]
\[ V_{\square}^{3.3} = V_{\square}^{2.1}, \]
\[ V_{\square}^{3.4} = V_{\square}^{3.2}, \]
\[ V_{\square}^{3.5} = -A^4 + (q^2 + q^{-2})A^2 = 1 - A^2 \left\{ Aq \right\} \{ A/q \} = 1 - \frac{A}{q} \left( Aq + 1 \right) \left( Aq - 1 \right) \{ A/q \}, \]
\[ V_{\square}^{3.6} = V_{\square}^{3.5} = H_{[2,3]}^{[2,3]}, \]
\[ V_{\square}^{3.7} = 1, \]
\[
\vdots
\]

All these knots are topologically distinct, however there are many coincidences between their fundamental HOMFLY polynomials.

In application to the fundamental HOMFLY the differential hierarchy structure is the appearance of the universal factor in a box – presumably, for arbitrary virtual knot. For ordinary knots this property is enhanced: the universal factor is promoted to that in the double box – in our small table this is represented by the example of 3.6, which is actually an ordinary trefoil 31, i.e. the ordinary 2-strand torus knot [2, 3].

6.3 2-strand torus knots

From (6.6) we get:

\[ h_{[2,3]}^{[2,3]} = 1 - q^{2N} \left\{ Aq \right\} \left\{ A/q \right\}, \]
\[ h_{[2,5]}^{[2,5]} = 1 - q^{2N} \left\{ Aq \right\} \left( 1 + (q^2 + q^{-2})q^{2N} \right), \]
\[ h_{[2,7]}^{[2,7]} = 1 - q^{2N} \left\{ Aq \right\} \left( 1 + (q^2 + q^{-2})q^{2N} + (q^4 + 1 + q^{-4})q^{4N} \right), \]
\[ h_{[2,9]}^{[2,9]} = 1 - q^{2N} \left\{ Aq \right\} \left( 1 + (q^2 + q^{-2})q^{2N} + (q^4 + 1 + q^{-4})q^{4N} + (q^6 + q^2 + q^{-2} + q^{-6})q^{6N} \right), \]
\[ \cdots \]

and from (6.10) —

\[ v_{[2,3]}^{[2,3]} = V_{\square}^{2.1} = 1 - q^{N-1} \left( q^{N+1} - 1 \right) \{ A/q \}, \]
\[ v_{[2,5]}^{[2,5]} = 1 - q^{N-1} \left( q^{N+1} - 1 \right) \{ A/q \} \left( 1 + q^{N+1} + (q^2 + q^{-2})q^{2N} \right), \]
\[ v_{[2,7]}^{[2,7]} = 1 - q^{N-1} \left( q^{N+1} - 1 \right) \{ A/q \} \left\{ 1 + q^{N+1} \left( 1 + (q^2 + q^{-2})q^{2N} \right) + (q^4 + 1 + q^{-4})q^{4N} \right\}, \]
\[ v_{[2,9]}^{[2,9]} = 1 - q^{N-1} \left( q^{N+1} - 1 \right) \{ A/q \} \left\{ 1 + q^{N+1} \left( 1 + (q^2 + q^{-2})q^{2N} + (q^4 + 1 + q^{-4})q^{4N} \right) \right. \right. \]
\[ \left. \left. + (q^6 + q^2 + q^{-2} + q^{-6})q^{6N} \right\} \right. \right. \]
\[ \cdots \]

This extends differential hierarchy from twist to the simplest torus family. However, the coefficients in front of the universal factors in the torus case are considerably more complicated than for twist knots – this gets even worse in the colored case [13].
6.4 Parallel 2-strand links

Moreover, the structure survives also for two-component torus links, only now we take unreduced HOMFLY. From the same (3.3) and (3.10) we deduce

\[ H_{n}^{[2,n]} = q^{-n} \left( [N]^2 - \frac{[N][N - 1]}{2} \right) + q^{n} \left( \frac{[N][N - 1]}{2} \right) \tag{6.7} \]

and

\[ V_{n+1}^{[2,n+1]} = q^{-n} \left( [N] + \frac{[N][N - 1]}{2} \right) + q^{n} \left( \frac{[N][N - 1]}{2} \right) \tag{6.8} \]

These give the following answers for simplest 2-strand links:

\[
\begin{align*}
H_{[2]}^{[2,2]} &= 1 + q^{2N} \left( \frac{[Aq][A/q]}{[q^2 - 1]^2} \right), \\
H_{[2]}^{[2,4]} &= 1 + q^{2N} \left( \frac{[Aq][A/q]}{[q^2 - 1]^2} \right), \\
H_{[2]}^{[2,6]} &= 1 + q^{2N} \left( \frac{[Aq][A/q]}{[q^2 - 1]^2} \right)
\end{align*}
\tag{6.9}
\]

and

\[
\begin{align*}
V_{[2]}^{[2,2]} &= 1 + q^{N-1} \left( \frac{[Aq - 1][A/q]}{[q^2 - 1]^2} \right), \\
V_{[2]}^{[2,4]} &= 1 + q^{N-1} \left( \frac{[Aq - 1][A/q]}{[q^2 - 1]^2} \right), \\
V_{[2]}^{[2,6]} &= 1 + q^{N-1} \left( \frac{[Aq - 1][A/q]}{[q^2 - 1]^2} \right)
\end{align*}
\tag{6.10}
\]

6.5 Antiparallel 2-strand links

Similarly, for antiparallel links we obtain from (3.8) and (3.12), again for unreduced HOMFLY:

\[
\begin{align*}
H_{[2] \times \square}^{2,2k} &= 1 + A^{2k}[N + 1][N - 1] = 1 + \frac{A^{2k} \{Aq\} \{A/q\}}{(q - q^{-1})^2}, \\
\end{align*}
\tag{6.11}
\]

and

\[
\begin{align*}
V_{[2] \times \square}^{2,2k} &= 1 + A^{2k-1}[N + 1][N - 1] = 1 + \frac{A^{2k-2} \{Aq - 1\} \{A/q\}}{(q - q^{-1})^2} \cdot (q^{N-1} + 1). \\
\end{align*}
\tag{6.12}
\]

It deserves noting, that at \( q = 1 \) un-reduced HOMFLY for \( l \)-component links are equal to \( N^l \). In this limit also \( A = 1 \), but the ratio \( \frac{A^{l-1}}{q} \rightarrow N \pm 1 \), thus both terms in the differential decomposition contribute in this limit.

6.6 Differential expansion for Jones polynomials

Above examples exhaust all the virtual knots with already known HOMFLY polynomials. For a wider class of examples the Jones polynomials (i.e. HOMFLY at \( N = 2 \)) can be found in [2].

For fundamental reduced Jones our differential hierarchy structure implies that the difference \( J_{\square} - 1 \) is divisible by \( (Aq - 1)\{A/q\} \rightarrow (q^3 - 1)(q^2 - 1) \) - but not by \( (q^6 - 1)(q^2 - 1) \) as it does for the ordinary knots. This is indeed the case for all the examples discussed in [1] and for several checked examples from [3]. The coefficient \( g^\square_c \) introduced using

\[ J^\square_c = 1 + g^\square_c(q) \frac{(q^3 - 1)(q - q^{-1})}{(q^2 - 1)} \tag{6.13} \]
is equal (after a substitution $q^{-1/2} \rightarrow q$ in (4)) to:

\[
\begin{align*}
g_4^{4.1} &= q^6 - q^4 - q^3 - q, \\
g_4^{4.2} &= -q^{-1} - q^{-2} = -q^{-2}(q + 1), \\
g_4^{4.3} &= g_4^{4.1}, \\
g_4^{4.4} &= -q, \\
&\vdots \\
g_4^{4.49} &= g_4^{4.4}, \\
g_4^{4.50} &= -q^2 - q = -q(q + 1), \\
&\vdots \\
g_4^{4.107} &= 0, \\
g_4^{4.108} &= \boxed{1+q^{-3}} = q^{-3}(q^3 + 1). 
\end{align*}
\]

This is the ordinary knot 4_1.

7 On colored HOMFLY for virtual knots

7.1 Conjectures

Since the very beginning in [11] differential expansion was used to promote fundamental HOMFLY in two directions: to higher representations (at least, (anti)symmetric) and to superpolynomials. It is a very interesting question, whether this can be also done for virtual knots and links. In this section we say just a few words about the first of these options.

As already mentioned in section 6.1, differential expansion for colored polynomials forms possesses a deep hierarchical structure [13], but it starts from (6.1) – a direct analogue of what we discussed in the previous section for the fundamental representation. However, even this first step can be a problem when we switch to virtual knots. Relation (6.1),

\[
h_{[r]} - 1 \sim \{Aq^r\} \{A/q\},
\]

(7.1)
could be hypothetically promoted to

\[
v_{[r]} - 1 \sim \{Aq^r - 1\} \{A/q\},
\]

(7.2)
but the problem is to define what $v_{[r]}$ could be – because representation-theory structure is broken by the sterile vertices. The question is – to what extent it is broken: whether just internal structure is deformed within irreducible representations or different representations are intermixed. This section describes simple observations, which can help thinking about this issue.

While definition of colored HOMFLY $V_R$ with arbitrary Young diagram $R$ is not at all straightforward for virtual knots and links, one can easily define the cabled HOMFLY $V_{\square \otimes r}$ – by substituting a line in the knot by an $r$-strand cable:

and taking HOMFLY of this virtual link. It is sometime suggested to consider $\widetilde{V}_{\square \otimes r}$ – a HOMFLY for polynomial for a link with additional twistings inside the cable, added to make the linking numbers of every (?) two strands in the cable vanishing. Similar twistings are used to make projections on particular representations $R$ in the decomposition

\[
H_{\square \otimes r} = \sum_{R, |R|=r} H_R
\]

(7.3)
of cabled into colored HOMFLY for ordinary knots and links. Immediate question is if a similar decomposition can exist in the virtual case:

$$V_{\square r} \equiv \sum_{R,|R|=r} V_R.$$  \hspace{1cm} (7.4)

A possible obstacle is that the sterile vertex can provide non-vanishing transitions between different representations:

![Diagram](image)

even if there is a single virtual vertex in the knot, so that $$R_3 = R_1$$ and $$R_4 = R_2$$ because of topology of the rest of the graph, $$R_2$$ can still be different from $$R_1$$ and in that case (7.4) should then be substituted by a far more complicated decomposition

$$V_{\square r} \equiv \sum_{R_1, R_2, |R_i|=r} V_{R_1, R_2}$$  \hspace{1cm} (7.5)

and even worse – when the number of sterile vertices increases.

To make the choice between (7.4) and (7.5) phenomenologically one needs to know examples of cabled HOMFLY. In the very simplest case of 2.1 and $$r = 2$$ the relevant link diagram has eight vertices – and still needs to be calculated (after all, the present paper is just the second after [1], where HOMFLY for virtual knots are considered). Known, however, are many cabled Jones [4].

Since Jones polynomials in these tables are reduced, while (7.4) involves unreduced ones, the relation should be rewritten as follows:

$$(q + q^{-1})J_{\square \times \square} \equiv (q^2 + 1 + q^{-2})J_{[2]} + 1,$$  \hspace{1cm} (7.6)

$$(q + q^{-1})J_{\square \times \square \times \square} \equiv (q^3 + q + q^{-1} + q^{-3})J_{[3]} + 2 (q + q^{-1})J_{\square},$$  \hspace{1cm} (7.7)

Thus (7.6) and (7.7) were true, one could use them to define $$J_{[2]}$$ and $$J_{[3]}$$ – and then, iteratively, next $$J_{[r]}$$ – from known cabled Jones. The minimal criterium of truth is that the resulting quantities are indeed Laurent polynomials in $$q$$ – i.e. the difference $$(q + q^{-1})J_{\square \times \square \times \square} - 1$$ is divisible by $$(q^2 + 1 + q^{-2}) = [3]$$ and so on.

### 7.2 Check for ordinary knots

This is of course true for ordinary knots, i.e. for 3.6 and 4.108 of [4], which are respectively the ordinary trefoil and figure eight knots:

$$J_{[2]}^{3.6} = \frac{[2] J_{\square \times \square}^{3.6} - 1}{[3]} = q^{22} - q^{20} - q^{18} + q^{16} - q^{14} + q^{10} + q^4,$$  \hspace{1cm} (7.8)

$$J_{[3]}^{3.6} = -q^{22} + q^{40} + q^{38} - q^{34} + q^{30} - q^{28} - q^{26} + q^{22} - q^{20} + q^{14} + q^6.$$
and

\[ J^{108}_{[2]} = \frac{[2]J^{108} - 1}{[3]} = q^{12} - q^{10} - q^8 + 2q^6 - q^4 - q^2 + 3 - q^{-2} - q^{-4} + 2q^{-6} - q^{-8} - q^{-10} + q^{-12}, \]

\[ J^{108}_{[3]} = q^{24} - q^{22} - q^{20} + 2q^{16} - 2q^{12} + 3q^8 - 3q^6 + 3 - 3q^{-4} + 3q^{-8} - 2q^{-12} + 2q^{-16} - q^{-20} - q^{-22} + q^{-24}. \]  

These are the well known correct expressions, moreover

\[ J_{[2]}^{3,6} - 1 \sim (q^4 - 1)(q^2 - 1), \]

\[ J_{[2]}^{108} - 1 \sim (q^5 - 1)(q^2 - 1), \]

\[ J_{[3]}^{3,6} - 1 \sim (q^5 - 1)(q^2 - 1), \]

\[ J_{[3]}^{108} - 1 \sim (q^5 - 1)(q^2 - 1), \]

\[ \ldots \]

as conjectured in (7.7). However, in these two cases this is just a corollary of (7.1).

7.3 Validation of (7.6) and support to (7.2) for \( r = 2 \)

A real surprise is that (7.6) works not only for ordinary knots but also for virtual ones from [3]. Moreover, so defined \( J_2 \) do indeed possess the property (7.2). \( J_{[2]} - 1 \sim (q^4 - 1)(q^2 - 1) \). Some examples are listed int the table 1. The coefficients are defined as

\[ J_{[2]} - 1 = g_{[2]}(q^4 - 1)(q - q^{-1}), \]

\[ J_{[2]} - 1 = g_{[2]}(q^4 - 1)(q - q^{-1}) = \left( [2] g_{[2]}^{(2)} \cdot (q^4 - 1)(q - q^{-1}) \right), \]

while colored Jones \( J_{[2]} \) is extracted from

\[ (q + q^{-1})J_{[2, \times]} + (q^2 + 1 + q^{-2})J_{[2]} + 1 = 0. \]  

7.4 Violation of (7.7)

However, the same trick does not work for (7.7). One cannot extract \( J_{[3]} \) for virtual knots from the cabled polynomials from [3]. For example the conjecture (7.7) gives for the knot 2:1

\[ J^{1,1}_{[3]} = -\frac{1}{q^2 + q^{-2}}(q^{29} - q^{27} + q^{25} - q^{23} - 3q^{22} - 2q^{21} - q^{20} + q^{19} + 3q^{18} + 2q^{17} + 4q^{16} + 4q^{15} - 2q^{13} - 3q^{12} - q^{11} - q^{10} - 2q^9 - q^8 + q^6 - q^4). \]  

Likewise for 3.1

\[ J^{3,1}_{[3]} = -\frac{1}{q^2 + q^{-2}}(q^{31} - q^{19} - 2q^{18} - 5g^{17} - 7q^{16} - 14q^{15} - 10q^{14} - 14q^{13} - 7q^{12} + 2q^{11} + 13q^{10} + 26q^9 + 33q^8 + 24q^7 + 19q^6 + 7q^5 - 7q^4 - 11q^3 - 21q^2 - 13q - 12 - 4q^{-1} + q^{-2} - 2q^{-3} + q^{-4} + 2q^{-5} - 2q^{-6} + q^{-8}). \]

As one can see these answers are not polynomials and thus can hardly be interpreted as the colored Jones polynomials.

It is an intriguing question why (7.9) works and (7.7) does not – as we explained, it would be natural to expect that neither one does. Perhaps, the success with the first symmetric representation [2] will be limited to Jones (\( N = 2 \)) case, when antisymmetric representation [11] trivializes to a singlet. Perhaps, the sterile vertex fails to mix a singlet with [2], but does so with [11] and [2] in general.

Note also that we did not really distinguish \( \hat{J} \) from \( J \) (i.e. Jones polynomials for cables with additional crossings corresponding to the vanishing linking numbers and Jones polynomial for plain cable), while, strictly speaking (7.6) and (7.7) are about \( J \) (for \( \hat{J} \) extra powers of \( q \) can appear for twisting insertions), while the data in [3] is presumably for \( \hat{J} \). It is unclear whether this kind of modification could cure (7.7) and provide a reasonable “definition” of \( J_{[3]} \).
| knot | $J_\square$ | $g_\square = \frac{J_\square^{-1}}{(q-1)(q-q^{-1})}$ | $J_\square \times \square$ | $J_\square$ from \cite{7, 6} | $g_{[2]} = \frac{J_\square^{-1}}{(q-1)(q-q^{-1})}$ |
|------|------------|----------------------------------|----------------|-----------------|----------------|
| 2.1  | $-q^5 + q^3 + q$ | $-q$ | $q^{15} - q^{13} - q^{11} - q^9 + q^7 + 2q^5 + q$ | $q^{14} - q^{12} - 2q^{10} + q^8 + q^6 + q^4$ | $q^9 - q^5 - q^3 - q$ |
| 3.1  | $1$ | $0$ | $-q^9 - q^7 + q^5 + 2q^3 + 2q - q^{-3}$ | $-q^8 - q^6 + 2q^4 + 2q^2 - q^{-2}$ | $-q^3 - 2q - q^{-1}$ |
| 3.2  | $q^4 - q^2 - q + 1 + q^{-1}$ | $1$ | $q^{13} - q^{11} - q^9 + 2q^5 + q^3 - q + q^{-5}$ | $q^{12} - q^{10} - 2q^8 + 2q^6 + 2q^4 - 2q^{-2} - 1 + q^{-4}$ | $q^7 - q^3 + q^{-1} + q^{-3}$ |
| 3.3  | $-q^5 + q^3 + q^2 - q$ | $-q$ | $-q^{17} + q^{11} + q^5 + q$ | $-q^{16} + q^{12} + q^4$ | $-q^{11} - q^9 - q^7 - q^5 - q^3 - q$ |
| 3.4  | $q^4 - q^2 - q + 1 + q^{-1}$ | $1$ | $q^{13} - 2q^{11} - 2q^9 + q^7 + 4q^5 + 2q^3 - 2q - q^{-1} + q^{-5}$ | $q^{12} - 2q^{10} - 3q^8 + 4q^6 + 4q^4 - 2q^2 - 2 + q^{-4}$ | $q^7 - q^5 - 3q^3 - q + q^{-1} + q^{-3}$ |
| 3.5  | $-q^6 + q^2 + q^2$ | $-q^4 - q$ | $q^{21} - q^{19} - q^7 - 2q^{15} - q^{13} + q^{11} + 3q^5 + q$ | $q^{22} - q^{20} + q^{16} - 4q^{14} + 4q^{12} - q^6 + q^4$ | $q^{17} + q^{13} + q^{11} - 2q^9 - 2q^7 - q^5 - q^3 - q$ |
| 3.6  | $-q^6 + q^2 + q^2$ | $-q^4 - q$ | $q^{23} - q^{21} - q^{17} + q^9 + q^5 + q$ | $q^{22} - q^{20} - q^{18} + q^{16} - q^{14} + q^{10} + q^4$ | $q^{17} - q^9 - q^7 - q^5 - q^3 - q$ |
| 3.7  | $1$ | $0$ | $q^{11} - 2q^7 - q^5 + q^3 + 3q + q^{-1} - q^{-3}$ | $q^{10} - 3q^6 + 3q^2 + 1 - q^{-2}$ | $q^5 + q^3 - q - q^{-1}$ |

Table 1: In this table the Jones, cabled Jones and colored Jones in representation [2] are listed. Also provided are coefficients $g_\square$ and $g_{[2]}$. 


Still another problem is that the deeper levels of differential hierarchy involve factorization of differences like \((G_{[r]} - [r]G_c)\) in \(h_{[r]} = 1 = G_{[r]}\{A/q\}\) – but such differences are considerably changed after switching from coefficients in front of \(\{Aq\}\{A/q\}\) to those in front of \(\{\sqrt{Aq}\}\{A/q\}\), which seem to be relevant in virtual case.

In any case, to handle all these problems there is a big need to calculate cabled HOMFLY – Jones is clearly insufficient. This is a straightforward task in the approach of [8], [1] and the present paper – the problem is just technical: to effectively computerize calculations for arbitrary \(N\), as it was once done for Jones [19]. It is only needed to include new resolutions, step-by-step construction of the full quantum hypercube – on one hand, and, to include link diagrams with sterile vertices – on another.

8 Conclusion

In this paper we provided a new significant evidence that the formalism of [8] allows to define topologically invariant HOMFLY polynomials for virtual knots and links – despite the usual RT approach is not directly applicable. We extended the 2-strand torus example from original paper [1] to the families of antiparallel 2-strand braids and twist knots. This is important not only for the check of topological invariance – there are non-trivial equivalencies within the twist family – but also they provide new evidence for partial survival of group-theory structures and their implication, like the differential-hierarchy structure (which is modified by a straightforward, still mysterious substitution \(\{Aq\}\{A/q\} \rightarrow \{\sqrt{Aq}\}\{A/q\}\)). Moreover, these observations open a way to the introduction of colored (not just cabled) HOMFLY polynomials: we provided some optimistic evidence about representation [2], and described immediate problems in the case of representation [3].

Our conclusion is that at least the fundamental HOMFLY polynomials clearly exist for virtual knots, and perhaps even the RT method can be modified to describe them and introduce colored HOMFLY. This however, requires further investigation. Another open way is to further develop the method of [8] – which now proved capable to provide essentially new results. From the perspective of virtual knots, an interesting question is if Khovanov-Rozansky and super-polynomials can also be constructed for them.

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