A surface theoretic model of quantum gravity

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Abstract

A surface theoretic view of non-perturbative quantum gravity as “spin-foams” was discussed by Baez. A possibility of constructing such a model was studied some time ago based on (2+1) dimensional general relativity as a reformulation of the Ponzano-Regge model in Riemannian spacetime. In the present work, a model based on (3+1) dimensional general relativity in Riemannian spacetime is presented. The construction is explicit and calculable in details. For a physical application, a computation formula for spacetime volume density correlations is presented. Remarks for further investigations are made.

1 Introduction

A surface theoretic view of non-perturbative quantum gravity as “spin-foams” was discussed by Baez [1]. A possibility of constructing such a model was studied some time ago based on (2+1) dimensional general relativity as a reformulation of the Ponzano-Regge model in Riemannian spacetime [2]. In the present work, a model based on (3+1) dimensional general relativity in Riemannian spacetime is presented. The main difference is that (3+1) general relativity is not a topological field theory but has local degrees of freedom while (2+1) general relativity is identical to the non-degenerate sector of BF theory (a topological field theory) in (2+1) dimensions. The construction is explicit and calculable in details. This model allows one to investigate the difference between the quantum theories for general relativity and BF theory from a surface theoretic point of view. For a physical application, a computation formula for spacetime volume density correlations is presented. For a family of models allowing spin-foam interpretation of quantum gravity, see the references [1, 2, 3, 4, 5, 6]. In particular, surface theoretic views are discussed by Baez [1], Reisenberger [4], Reisenberger and Rovelli [4] and the author [2]. In the present work, the calculation technologies developed in the connection representation [7] and the loop representation [8] of non-perturbative canonical quantum gravity play important roles.

1The kind of surfaces playing important roles is called in terms of different names by different authors. The name “spin-foam” is due to Baez.
Among the models revealing aspects of the spin-foam interpretation, the philosophy of the present model is similar to that of the Reisenberger model [3] in the sense that both models are based on a path integral of general relativity. The other models are based on lattice topological field theories [2, 4], canonical quantization of general relativity [4] or others [6]. However, the technical difference of the present model from that of the Reisenberger model is that the present model uses multiple pairs of lattices. In each pair, one lattice is dual to the other. The pairs of lattices are used to regularize the exterior product of two forms. The exterior product of two 2-forms, which is present in the action functional of general relativity in 4-dimensions, is naturally defined on the lattice at the intersection of a face and its dual face. In addition, the degenerate and non-degenerate sectors of the theory are clearly characterized by the number of pairs of lattices.

In a limit, the partition function of the model is reduced to a product of copies of the partition function of lattice BF theory, each of which is independently defined on one of the lattices. This fact implies that the contribution from each lattice is decoupled in this limit and the use of the multiple lattices for topological field theory is redundant. In general, however, the contribution from each lattice cannot be decoupled and the partition function of the model is different from that of lattice BF theory. This difference is supposed to correspond to the fact that general relativity contains local degrees of freedom while BF theory does not. In this way, the model provides a way of clarifying local degrees of freedom of quantum gravity.

The present model is formulated such a way that the inclusion of local degrees of freedom of general relativity, which is absent in BF theory, is incorporated to coefficients in the partition function and the “trivialization” of the coefficients leads to (a product of copies of) lattice BF theory. The computation of the coefficients is done by the regularization in terms of the multiple pairs of lattices. Each pair consists of two lattices dual to each other. The “trivialization” corresponds to the limit mentioned above.

In the next section, the partition function of the model is presented at the beginning, and then the rest of the section is devoted to the determination of coefficients in the partition function, followed by the presentation of a computation formula for spacetime volume density correlations. Remarks for further investigations are made in the conclusion section.

2 The model

The partition function of the model is

\[ Z_{\beta,\lambda}[\xi] := \prod_{p=1}^{P} \prod_{e \in \Delta_p \Delta_p^*} \int dU(e) \times \prod_{f \in \Delta_p \Delta_p^*} \sum_{j} C^{(j)}_{\beta,\lambda}[\xi] \chi_j \left( \prod_{e \in f} U(e) \right), \]

(1)

2 The partition function of lattice BF theory in 4-dimensions defined on a single lattice was written by Ooguri [9].
\[ C_{\beta,\lambda}[\xi] := \sum_{\{l\}} W_{\beta,\lambda}(\{l\}, \xi) \Omega^{(j)}(\{l\}). \]  

Here, \( \beta \) is 1 or \( i \) (the imaginary unit number) and \( \lambda \) is a real or imaginary number. \( \xi \) is an external field coupled to spacetime volume density, \( f \, d^4x \sqrt{g} \xi(x) \). This field is a mathematical tool to calculate spacetime volume density correlations. \( P \) is the number of pairs of lattices and \( \Delta_p \) and \( \Delta_p^* \) are a pair of lattices dual to each other. The index \( p \) runs from 1 through \( P \). A family of spin-foams are defined in terms of each of the lattices. More details of the lattices and spin-foams are described below.

\( \chi_j(U) \) is the character of SU(2) element \( U \) in the spin-\( j \) representation. \( \Pi^{path} \) means the path ordered product. \( dU \) is the Haar measure for SU(2) element \( U \). \( e \in \Delta_p \) and \( f \in \Delta_p \) mean that they are an edge (1-cell) and a face (2-cell) of \( \Delta_p \) respectively. \( e \in f \) means that \( e \) is an edge shared by \( f \) and other faces; in other words, \( e \) is a boundary of \( f \). \( j \) and \( l \) are spins associated with the faces taking values 0, \( \frac{1}{2} \), 1, \( \frac{3}{2} \), \( \cdots \). \( W_{\beta,\lambda}(\{l\}, \xi) \) and \( \Omega^{(j)}(\{l\}) \) are coefficients to be determined below. A particular property of these coefficients are as follows. When \( \beta = 1 \), \( \lambda = 0 \) and \( \xi = 0 \), then

\[ C_{1,0}^{(j)}[0] = \prod_{p=1}^{P} \prod_{f \in \Delta_p} \chi_j(\prod_{e \in f} U(e)) \]  

\[ = \left[ \prod_{e \in \Delta} dU(e) \times \prod_{f \in \Delta} \sum_{j} (2j + 1) \chi_j(\prod_{e \in f} U(e)) \right]^{2P}. \]  

Notice that this is not just the partition function of lattice SU(2) BF theory in (3+1) dimensions, a topological invariant, but the multiple power of it. We have used the fact that the partition function of lattice BF theory does not depend on the choice of lattice and let \( \Delta \) represent one of the lattices. A line of studies of lattice BF theory is detailed in the reference [3].

The lattices \( \Delta_p \) and \( \Delta_p^* \) used in the partition function are defined as follows. They are 4-dimensional piecewise linear cell manifolds consisting of 0-cells (vertices), 1-cells (edges), 2-cells (polygons), 3-cells (polyhedrons) and 4-cells. They are dual to each other. That is, every k-cell of \( \Delta_p \) (and \( \Delta_p^* \)) intersects with a (4-k)-cell of \( \Delta_p^* \) (and \( \Delta_p \) respectively) at a point inside the cells. We call the intersection of a k-cell of \( \Delta_p \) and a (4-k)-cell of \( \Delta_p^* \) the “center” of each of the cells. Every cell of \( \Delta_p \) and \( \Delta_p^* \) has one and only one center. We define the two lattices such that the center of a k-cell is not on the boundary of but inside the cell. The exception is that the center of a 0-cell is identified to itself. A spin-foam is defined as a 2-dimensional sub-complex of \( \Delta_p \) or \( \Delta_p^* \) with 2-cells labeled by spins and 1-cells labeled by intertwiners. A 2-cell labeled by the 0-spin is understood as the absence of the cell in the 2-dimensional sub-complex under consideration.

In the model, we use the multiple of such pairs of lattices and denote the set of the lattices by \( \{\Delta_p, \Delta_p^*(p = 1, \cdots, P)\} \). In addition, we impose two
conditions to the lattices. One condition is that the position of the center of every face of $\Delta_p$ coincides with the position of the center of a face of $\Delta_q$ for all $p, q = 1, \cdots, P$. This condition is needed in order to characterize the degenerate and non-degenerate sectors of the theory. The other condition is that $\Delta_p$ and $\Delta_p^*$ are isomorphic to $\Delta_q$ and $\Delta_q^*$ respectively for all $p, q = 1, \cdots, P$ and hence the faces of them correspond one-to-one. In other words, $\Delta_1, \cdots, \Delta_P$ are copies of the same (abstract) lattice embedded in different locations of the spacetime manifold with a restriction due to the first condition and so are their respective dual lattices $\Delta_1^*, \cdots, \Delta_P^*$. The positions of the one-to-one corresponding faces of $\Delta_p$ and $\Delta_q$ under the isomorphism do not necessarily coincide with each other. The second condition regularizes the kind of lattices utilized for the model and guarantees that copies of lattice BF theory independently defined on single lattices produce an identical partition function. (A way of constructing the pairs of lattices is discussed in Appendix A.)

In order to determine the coefficients $W_{\beta, \lambda}(\{l\}, \xi)$ and $\Omega^{(j)}(\{l\})$ we compute a path integral of general relativity. The action for general relativity we consider is the Plebanski action [10]. It consists of two terms, both of which is spacetime diffeomorphism invariant separately in addition to their internal gauge invariance, and is

$$S[A, B, \phi] := \int d^4 x \delta_{ij} \epsilon^{abcd} B^i_{ab} F^j_{cd} - \frac{\lambda}{2} \int d^4 x \phi_{ij} \epsilon^{abcd} B^i_{ab} B^j_{cd}. \quad (4)$$

Here $a, b, c \cdots$ are spacetime indices $\{0, 1, 2, 3\}$ and $i, j \cdots$ are internal SU(2) adjoint indices $\{1, 2, 3\}$. $\epsilon^{abcd}$ is the alternating tensor with density weight 1. $F^i_{ab} := \partial_i A^i_b - \partial_b A^i_a + \epsilon^{ijk} A^j_d A^k_b$ are the curvature of an SU(2) gauge connection $A^i_a$, $B^i_{ab}$ is an SU(2) algebra valued 2-form field and $\phi_{ij}$ are scalar fields (traceless symmetric with respect to the SU(2) indices). The dimensions of $A^i_a$, $B^i_{ab}$ and $\phi_{ij}$ are of length inverse, length inverse squared and length inverse squared respectively. $\lambda$ is a parameter with the dimension of length squared and its presence can be understood if one rescales $A^i_a$ and $B^i_{ab}$ appropriately. The Euclidean sector is obtained by real valued fields and imaginary time coordinate. The five scalar fields $\phi_{ij}$ are Lagrange multipliers forcing the non-trace parts of $\tilde{\epsilon}^{abcd} B^i_{ab} B^j_{cd}$ (about the internal indices $i$ and $j$) vanish. This condition implies the existence of non-degenerate tetrad fields $e_a^i$ such that $B^i_{ab} = e_a^i (e_b^i + \frac{1}{2} \epsilon^{ijk} e_a^j e_b^k)$ if $\delta_{ij} \epsilon^{abcd} B^i_{ab} B^j_{cd} \neq 0$. In other words, in its non-degenerate sector the action has the value of the self dual Hilbert-Palatini action evaluated on the tetrad. The first term of (4) is known to be the action of the BF theory, a topological field theory. The second term amounts to introduce local degrees of freedom contained in general relativity.

We define a path integral in terms of an action projected to the pairs of lattices $\{\Delta \Delta^*\}$. The variables projected to the lattices are defined as follows. For an edge $e$, define $U(e) := p \exp[\int_{e} dx^a A^i_a \tau_i]$, a path ordered parallel transport along $e$. Here $\tau_i$ is the Pauli matrix divided by $2i$. For a face $f$, define $\eta^j(f) := \frac{1}{2} \int_f dx^a dx^b B^i_{ab}$. The scalar fields $\phi_{ij}$ are defined at the centers of faces. The integrals over edge and face are oriented such that they are compatible with the orientation of $\Delta_p$ and $\Delta_p^*$. 

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The action projected to the pairs of lattices \( \{ \Delta \Delta^* \} \) is defined by

\[
S_{\{\Delta \Delta^*\}}[U, \eta, \phi] := \sum_{p=1}^{P} \sum_{\Delta \in \Delta_p} \left( \eta^i(f) \frac{1}{2} \text{Tr}[\tau_i U(f^*) \prod_{e \in f^*} U(e) U^{-1}(f^*)] \right.
\]

\[
\left. - \frac{\lambda}{2} \phi_{ij}(f^*) [\eta^i(f) \eta^j(f^*) + \eta^j(f) \eta^i(f^*)] \right),
\]

where \( f^* \) is the face sharing its center with \( f \) on a pair of lattices \( \Delta_p \) and \( \Delta_p^* \), and \( f f^* \) means the center of \( f \) and \( f^* \). \( U(f^*) \) is a parallel transport defined not on an edge of the lattice but on a curve from the center of \( f^* \) to one of the vertices belonging to \( f^* \). The choice of the vertex is arbitrary and does not affect the definition of the partition function of the model. This parallel transport is necessary in order to make the first term gauge invariant since the two faces \( f \) and \( f^* \) intersect with each other only at their center and the untraced parallel transport must base on the center.

The exponential of the action can be considered as (an extension of) the graph-cylindrical function discussed in \( [7] \). The integral measure we use for the variable \( A^i_a \) is the Ashtekar-Lewandowski measure \( \Pi \). The measure for the scalar fields \( \phi_{ij} \) is the one discussed in \( [4] \). The measure for the variable \( B_{ab} \) is analogously defined. The path integral has the form of

\[
\int d\mu(A) d\mu(B) d\mu(\phi) \Psi_{\{\Delta \Delta^*\}, \psi}(A, B, \phi) :=
\]

\[
\int \prod_{e \in \{\Delta \Delta^*\}} dU(e) \prod_{f \in \{\Delta \Delta^*\}} d\eta(f) \prod_{f \in \{\Delta \}} d\phi(f f^*) \times
\]

\[
\psi(U(e_1, A), \cdots U(e_t, A); \eta(f_1, B), \cdots \eta(f_m, B); \phi(v_1) \cdots \phi(v_n)),
\]

where \( \Psi_{\{\Delta \Delta^*\}, \psi} \) is a cylindrical function defined on the pairs of the lattices \( \{ \Delta \Delta^* \} \) in terms of a complex valued integrable function \( \psi \) on \([SU(2)]^l \times [su(2)]^m \times R^n \). Note that \( e \in \{ \Delta \Delta^* \} \) does not mean that \( e \) is a lattice but means that \( e \) is an edge of one of the lattices. In the same way, \( f \in \{ \Delta \Delta^* \} \) means that \( f \) is a face of one of the lattices. We loosely use this kind of notation. This integral form is diffeomorphism invariant.

The path integral which will be identified to \( \Pi \) is

\[
Z'_{\beta, \lambda}[\xi] := \int dU \int_{-\infty}^\infty d\eta d\phi e^{i\beta \left[ S_{\{\Delta \Delta^*\}}[U, \eta, \phi] + \sum_{p=1}^{P} \sum_{f \in \Delta_p} \xi(f f^*) \eta_i(f) \eta^i(f^*) \right]}
\]

\[
= \prod_{e \in \{\Delta \Delta^*\}} \int dU(e) \prod_{f \in \{\Delta \Delta^*\}} \int_{-\infty}^\infty d\eta(f) \times
\]

\[
\prod_{f \in \{\Delta \}} \int_{-\infty}^\infty d\phi(f f^*) \delta(\phi_{11} + \phi_{22} + \phi_{33}) \times
\]

\[
\prod_{p=1}^{P} \prod_{f \in \Delta_p} \int dU(f^*) e^{i\beta \eta_i(f) \frac{1}{2} \text{Tr}[\tau_i U(f^*) \prod_{e \in f^*} U(e) U^{-1}(f^*)] \times
\]

\[
\prod_{p=1}^{P} \prod_{f \in \Delta_p} e^{-\frac{i}{2} \beta \lambda \phi_{ij}(ff^*) [\eta^i(f) \eta^j(f^*) + \eta^j(f) \eta^i(f^*)]} e^{i\beta \xi(f f^*) \eta_i(f) \eta^i(f^*)}.
\]
Here, the external field $\xi$ is defined at the centers of faces as $\phi_{ij}$ is defined. We note that if $P = 1$ then one can easily solve the constraints on $\eta^i$ imposed by $\phi_{ij}$ and finds only degenerate solutions corresponding to $\delta_{ij}e^{\alpha \beta 0} B_{ab} B_{cd} = 0$. In order to include the non-degenerate sector, one needs $P > 1$. (See Appendix B for more discussion on the non-degenerate sector.)

Let us expand the exponential containing the trace of an SU(2) element to its characters as follows.

$$\int dV e^{i\beta \eta^i \frac{1}{2} \text{Tr} [\tau^i V UV^{-1}]} = \int dV \sum_{l_1,l_2,l_3} \frac{J_{2l_1+1}(\beta \eta^1)}{\beta \eta^1} \frac{J_{2l_2+1}(\beta \eta^2)}{\beta \eta^2} \frac{J_{2l_3+1}(\beta \eta^3)}{\beta \eta^3} \times$$

$$2^3(-1)^{l_1+l_2+l_3} (2l_1+1)(2l_2+1)(2l_3+1) \times$$

$$\chi_{l_1}[\tau_1 V UV^{-1}] \chi_{l_2}[\tau_2 V UV^{-1}] \chi_{l_3}[\tau_3 V UV^{-1}]$$

$$= \sum_{l_1,l_2,l_3} \frac{J_{2l_1+1}(\beta \eta^1)}{\beta \eta^1} \frac{J_{2l_2+1}(\beta \eta^2)}{\beta \eta^2} \frac{J_{2l_3+1}(\beta \eta^3)}{\beta \eta^3} \sum_j \Omega^{(j)}(l_1,l_2,l_3) \chi_j(U). \quad (8)$$

Here, $J_m(x)$ is the Bessel function of the first kind and we have used the following formulae.

$$e^{ix \frac{1}{2} \text{Tr} U} = \sum_j 2^{2j+1} x \frac{J_{2j+1}(x)}{x} \chi_j(U), \quad (9)$$

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{ix \sin \theta - im \theta}. \quad (10)$$

The first formula can be easily proved and the second is a definition of the Bessel function. $\Omega^{(j)}(l_1,l_2,l_3)$ is defined as follows.

$$\Omega^{(j)}(l_1,l_2,l_3) := 2^3(-1)^{l_1+l_2+l_3} (2l_1+1)(2l_2+1)(2l_3+1) \times$$

$$\sum_{m_{l_1},m_{l_2},m_{l_3}} D^{(j)}_{m_{l_1}m_{l_2}m_{l_3}}(\tau_1) D^{(l_2)}_{m_{l_2}m_{l_3}}(\tau_2) D^{(l_3)}_{m_{l_3}m_{l_3}}(\tau_3) (-1)^{n_3-m_3} \sum_{l_{12},m_{12},n_{12}} (2l_{12}+1) \times$$

$$\left( \begin{array}{ccc} l_1 & l_2 & l_{12} \\ n_1 & n_2 & n_{12} \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & l_{12} \\ m_{1} & m_{2} & m_{12} \end{array} \right) \left( \begin{array}{ccc} l_{12} & l_3 & j \\ n_{12} & n_3 & n \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & l_{12} \\ m_1 & m_2 & m_{12} \end{array} \right). \quad (11)$$

Here, $D_{m_{l_1}m_{l_2}m_{l_3}}^{(j)}(U)$ is the spin-$j$ representation matrix of SU(2) element $U$ and $m$ and $n$ run from $-j$ through $j$ with the increment 1. $\langle j_1 m_1; j_2 m_2 | j, m \rangle$ is the so-called 3j-coefficient and defined by

$$\left( \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right) := \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}} \langle j_1 m_1; j_2 m_2 | j, -m \rangle, \quad (12)$$

with the Clebsch-Gordan coefficient $\langle j_1 m_1; j_2 m_2 | jm \rangle$ and we have used properties of $D_{m_{l_1}m_{l_2}m_{l_3}}^{(j)}(U)$ such as

$$(-1)^{n-m} D_{m_{l_1}m_{l_2}m_{l_3}}^{(j)}(U) = D_{n_{l_1}n_{l_2}n_{l_3}}^{(j)}(U), \quad (13)$$

$$D_{n_{l_1}m_{l_1}m_{l_2}}^{(j_1)}(U) D_{n_{l_2}m_{l_2}m_{l_3}}^{(j_2)}(U) = \sum_{j,n,m} (2j+1) \left( \begin{array}{ccc} j_1 & j_2 & j \\ n_1 & n_2 & n \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right) D_{m_{l_3}m_{l_3}}^{(j)}(U), \quad (14)$$

$$\int dU D_{m_{l_1}m_{l_2}m_{l_3}}^{(i)}(U) D_{m_{l_1}m_{l_2}m_{l_3}}^{(j^*)}(U) = \frac{1}{2j+1} \delta_{ij} \delta_{m_{l_1}m_{l_1}} \delta_{m_{l_2}m_{l_2}}, \quad (15)$$
where the asterisks mean the complex conjugate and the sum of \( j \) is taken over \(|j_1 - j_2|\) through \( j_1 + j_2 \) and the sums of \( n \) and \( m \) over \(-j\) through \( j \). Then the partition function can be written as follows.

\[
Z'_{\beta, \lambda}[\xi] = \prod_{f \in \{\Delta\}} \int dU(e) \times \prod_{j \in \{\Delta^*\}} \int_{-\infty}^{\infty} d\eta(f) \times \prod_{f \in \{\Delta\}} \int_{-\infty}^{\infty} d\phi(ff^*) \delta(\phi_{11} + \phi_{22} + \phi_{33}) \times \prod_{p=1}^{P} \prod_{f \in \Delta_p \{\Omega\}} \sum_{j(f)} \frac{J_{2f_1}(f^*)}{\beta \eta^1(f)} \frac{J_{2f_2}(f^*)}{\beta \eta^2(f)} \frac{J_{2f_3}(f^*)}{\beta \eta^3(f)} \times e^{-\frac{i}{2} \beta \lambda \phi_{01}(ff^*)} \eta^1(f) \eta^1(f^*) e^{i \beta \lambda \phi_{02}(ff^*)} \eta^2(f) \eta^2(f^*) \times \sum_{j(f^*)} \Omega^{j(f^*)}(l_1(f), l_2(f), l_3(f)) \chi_{j(f^*)} \left( \prod_{e \in f} U(e) \right) \times \sum_{j(f^*)} \Omega^{j(f^*)}(l_1(f^*), l_2(f^*), l_3(f^*)) \chi_{j(f^*)} \left( \prod_{e \in f^*} U(e) \right). \tag{16}
\]

Here, \( \{l\}_{ff^*} \) stands for \( l_1(f), l_2(f), l_3(f), l_1(f^*), l_2(f^*), l_3(f^*) \). Let us write the other exponentials explicitly and expand them as a power series as follows.

\[
e^{-\frac{i}{2} \beta \lambda \phi_{01}(ff^*)} \eta^1(f) \eta^1(f^*) e^{i \beta \lambda \phi_{02}(ff^*)} \eta^2(f) \eta^2(f^*)
\]

\[
= e^{-\frac{i}{2} \beta \lambda \phi_{01}(ff^*)} \eta^1(f) \eta^1(f^*) e^{-i \beta \lambda \phi_{02}(ff^*)} \eta^2(f) \eta^2(f^*) \times e^{-i \beta \lambda \phi_{03}(ff^*)} \eta^3(f) \eta^3(f^*) e^{-i \beta \lambda \phi_{04}(ff^*)} \eta^4(f) \eta^4(f^*) \times e^{-i \beta \lambda \phi_{05}(ff^*)} \eta^5(f) \eta^5(f^*) e^{-i \beta \lambda \phi_{06}(ff^*)} \eta^6(f) \eta^6(f^*) \times
\]

\[
= \sum_{N_{11}=0}^{\infty} \sum_{M_1=0}^{\infty} \frac{[\eta^1(f) \eta^1(f^*)]^{M_1}}{N_{11}! M_1!} \times \sum_{N_{22}=0}^{\infty} \sum_{M_2=0}^{\infty} \frac{[\eta^2(f) \eta^2(f^*)]^{M_2}}{N_{22}! M_2!} \times \sum_{N_{33}=0}^{\infty} \sum_{M_3=0}^{\infty} \frac{[\eta^3(f) \eta^3(f^*)]^{M_3}}{N_{33}! M_3!} \times \sum_{N_{12}=0}^{\infty} \sum_{N_{21}=0}^{\infty} \frac{[\eta^1(f) \eta^2(f^*)]^{M_2}}{N_{12}! N_{21}!} \times \sum_{N_{23}=0}^{\infty} \sum_{N_{32}=0}^{\infty} \frac{[\eta^2(f) \eta^3(f^*)]^{M_3}}{N_{23}! N_{32}!} \times \sum_{N_{31}=0}^{\infty} \sum_{N_{13}=0}^{\infty} \frac{[\eta^3(f) \eta^1(f^*)]^{M_1}}{N_{31}! N_{13}!} \times \sum_{N_{11}+N_{12}+N_{21}=0}^{\infty} \frac{[\eta^1(f) \eta^2(f^*)]^{N_{11}+N_{12}+N_{21}}}{N_{11}! N_{12}! N_{21}!} \times \sum_{N_{22}+N_{23}+N_{32}=0}^{\infty} \frac{[\eta^2(f) \eta^3(f^*)]^{N_{22}+N_{23}+N_{32}}}{N_{22}! N_{23}! N_{32}!} \times \sum_{N_{33}+N_{31}+N_{13}=0}^{\infty} \frac{[\eta^3(f) \eta^1(f^*)]^{N_{31}+N_{33}+N_{13}}}{N_{31}! N_{33}! N_{13}!}. \tag{17}
\]
Here, the non-negative integers $N_{ij}$ and $M_i$ with $i, j = 1, 2, 3$ are associated with the pair of faces $f$ and $f^*$. Then the partition function becomes

\[
Z'_{\beta, \lambda}[\xi] = \prod_{e \in \{\Delta \Delta^*\}} \int dU(e) \times \prod_{p=1}^{P} \prod_{f \in \Delta_p} \sum_{\text{path}} W_{\beta, \lambda}(\{l\}_{ff^*}, \xi) \times \\
\sum_{j(f)} \Omega^{j(f)}(l_1(f), l_2(f), l_3(f)) \chi_j(f) \left( \prod_{e \in f} U(e) \right) \times \\
\sum_{j(f^*)} \Omega^{j(f^*)}(l_1(f^*), l_2(f^*), l_3(f^*)) \chi_j(f^*) \left( \prod_{e \in f^*} U(e) \right),
\]

with

\[
W_{\beta, \lambda}(\{l\}_{ff^*}, \xi) := \int_{-\infty}^{\infty} d\eta(f) d\eta(f^*) \int_{-\infty}^{\infty} d\phi(f f^*) \delta(\phi_{11} + \phi_{22} + \phi_{33}) \times \\
\sum_{N, M_{ff^*}} \left[ -i\beta \phi_{11} (f f^*) \right]^{N_{11}} \left[ -i\beta \phi_{22} (f f^*) \right]^{N_{22}} \left[ -i\beta \phi_{33} (f f^*) \right]^{N_{33}} \times \left. [N_{12}! N_{21}! N_{23}! N_{32}! N_{31}! N_{13}!] \right]^{N_{M_{ff^*}}} \\
\times \left( \frac{1}{M_1! M_2! M_3!} \right) \times \\
\frac{[\eta^1(f)]^{M_1+N_{11}+N_{12}+N_{13}} [\eta^2(f)]^{M_2+N_{22}+N_{21}+N_{23}} [\eta^3(f)]^{M_3+N_{33}+N_{32}+N_{31}}}{} \\
\times \frac{[\eta^1(f^*)]^{M_1+N_{11}+N_{12}+N_{13}} [\eta^2(f^*)]^{M_2+N_{22}+N_{21}+N_{23}} [\eta^3(f^*)]^{M_3+N_{33}+N_{32}+N_{31}}}{}
\]

\[
J_{2l_1(f^*)+1}(\beta \eta^1(f)) J_{2l_2(f^*)+1}(\beta \eta^2(f)) J_{2l_3(f^*)+1}(\beta \eta^3(f)) \\
\beta \eta^1(f) \beta \eta^2(f) \beta \eta^3(f) \times \\
J_{2l_1(f)+1}(\beta \eta^1(f^*)) J_{2l_2(f)+1}(\beta \eta^2(f^*)) J_{2l_3(f)+1}(\beta \eta^3(f^*)) \beta \eta^1(f^*) \beta \eta^2(f^*) \beta \eta^3(f^*)
\]

Here, $\{N\}_{ff^*}$ stands for the non-negative integers $N_{ij}$ and $M_i$ associated with the pair of faces $f$ and $f^*$. The integrations over $\eta^i$ can be performed. Here, for concreteness, we compute the coefficient for the case that $\beta$ is positive real number. A useful formula is $J(m, n) := \int_{-\infty}^{\infty} x^{n-1} J_m(x) dx = \beta^n \int_{-\infty}^{\infty} x^{n-1} J_m(\beta x) dx$ and

\[
J(m, n \geq 1) = i^{n-1} \left[ \left( \frac{d}{d\theta} \frac{1}{\cos \theta} \right)^{n-1} e^{-im\theta} \bigg|_{\theta=0} + \left( \frac{d}{d\theta} \frac{1}{\cos \theta} \right)^{n-1} e^{-im\theta} \bigg|_{\theta=\pi} \right], \quad \text{for } m \geq 1
\]

\[
J(m, 0) = \frac{1}{m} \left[ 1 - (-1)^m \right]. \quad \text{for } m \geq 1
\]

Here $(\frac{d}{d\theta} \frac{1}{\cos \theta})^{n-1}$ means $\frac{d}{d\theta} \frac{1}{\cos \theta}$ acts $n - 1$ times. For given $m$ and $n$, $J(m, n)$ exists and can be explicitly computed. In terms of $J(m, n)$, the coefficient $W_{\beta, \lambda}(\{l\}_{ff^*}, \xi)$ can be written as follows.

\[
W_{\beta, \lambda}(\{l\}_{ff^*}, \xi) = \beta^{-6} \int_{-\infty}^{\infty} d\phi(f f^*) \delta(\phi_{11} + \phi_{22} + \phi_{33}) \times \\
\left( \frac{1}{M_1! M_2! M_3!} \right) \times \\
\left[ \frac{[\eta^1(f)]^{M_1+N_{11}+N_{12}+N_{13}} [\eta^2(f)]^{M_2+N_{22}+N_{21}+N_{23}} [\eta^3(f)]^{M_3+N_{33}+N_{32}+N_{31}}}{} \\
\times \frac{[\eta^1(f^*)]^{M_1+N_{11}+N_{12}+N_{13}} [\eta^2(f^*)]^{M_2+N_{22}+N_{21}+N_{23}} [\eta^3(f^*)]^{M_3+N_{33}+N_{32}+N_{31}}}{} \\
\times \left( \frac{1}{M_1! M_2! M_3!} \right) \times \\
\left( \frac{\beta \eta^1(f)}{\beta \eta^1(f^*)} \frac{\beta \eta^2(f)}{\beta \eta^2(f^*)} \frac{\beta \eta^3(f)}{\beta \eta^3(f^*)} \right) \left( \frac{\beta \eta^1(f)}{\beta \eta^1(f^*)} \frac{\beta \eta^2(f)}{\beta \eta^2(f^*)} \frac{\beta \eta^3(f)}{\beta \eta^3(f^*)} \right)
\]

\[
\left( \frac{\beta \eta^1(f)}{\beta \eta^1(f^*)} \frac{\beta \eta^2(f)}{\beta \eta^2(f^*)} \frac{\beta \eta^3(f)}{\beta \eta^3(f^*)} \right)
\]

\[
\left( \frac{\beta \eta^1(f)}{\beta \eta^1(f^*)} \frac{\beta \eta^2(f)}{\beta \eta^2(f^*)} \frac{\beta \eta^3(f)}{\beta \eta^3(f^*)} \right)
\]
and (24) with (22) are the final forms for the coefficients (2) in the partition function of the model.

\[ \lambda = 0 \] and \( \xi \)

Now, define the coefficients \( \Omega_{ij}^{(l)}(l) \) and \( W_{\beta,\lambda}(l, \xi) \) such that \( Z'_{\beta,\lambda}^\ddagger[\xi] \) is identified to \( Z_{\beta,\lambda}[\xi] \) as follows.

\[
\Omega_{ij}^{(l)}(l) := \prod_{p=1}^{P} \prod_{f \in \Delta_p} 2^{-9} \Omega_{ij}^{(l)}(l_1(f), l_2(f), l_3(f)),
\]

(23)

\[
W_{\beta,\lambda}(l, \xi) := \prod_{p=1}^{P} \prod_{f \in \Delta_p} \beta^6 W_{\beta,\lambda}(l)_{ff^*}, \xi).
\]

(24)

Here, \( 2^{-9} \) and \( \beta^6 \) have been inserted for convenient normalization. (23) with (21), and (24) with (22) are the final forms for the coefficients (9) in the partition function of the model.

For a completeness, let us compute the coefficient \( C_{i0}^{(j)}[\xi] \) for the case that \( \beta = 1, \lambda = 0 \) and \( \xi = 0 \). In that case,

\[
W_{1,0}(l, 0) = \prod_{p=1}^{P} \prod_{f \in \Delta_p} J(2l_1(f) + 1, 0) J(2l_2(f) + 1, 0) J(2l_3(f) + 1, 0)
\]

\[
= \prod_{p=1}^{P} \prod_{f \in \Delta_p} \left[ \frac{1 - (-1)^{2l_1(f)+1}}{2l_1(f)+1} \right] \left[ \frac{1 - (-1)^{2l_2(f)+1}}{2l_2(f)+1} \right] \left[ \frac{1 - (-1)^{2l_3(f)+1}}{2l_3(f)+1} \right],
\]

(25)

and

\[
C_{1,0}^{(j)}[0] = \sum_{\{l\}} W_{1,0}(l, 0) \Omega_{ij}^{(l)}(l)
\]

\[
= \prod_{p=1}^{P} \prod_{f \in \Delta_p} \sum_{l_1, l_2, l_3} \left[ 1 - (-1)^{2l_1+1} \right] \left[ 1 - (-1)^{2l_2+1} \right] \left[ 1 - (-1)^{2l_3+1} \right] 2^{-6} (-1)^{l_1+l_2+l_3} \times
\]

\[
\sum_{m_1, m_2, m_3} D_{m_1, m_2, m_3}^{(l_1)}(1) D_{m_2, m_3, l_3}^{(l_1)}(1) D_{m_3, l_3, l_2}^{(l_1)}(1) D_{l_2, l_3, l_2}^{(l_1)}(1) \left( \begin{array}{ccc}
1 & l_1 & l_2 \\
2 & l_2 & l_3 \\
3 & l_3 & l_2
\end{array} \right)
\]

\[
= \prod_{p=1}^{P} \prod_{f \in \Delta_p} \sum_{l_1, l_2, l_3} \chi_{l_1}(\tau_1) \chi_{l_2}(\tau_2) \chi_{l_3}(\tau_3) \chi_{l_12}(1) \times
\]

\[
\int dV dW \chi_{l_1}(\tau_1 V) \chi_{l_2}(\tau_2 V) \chi_{l_12}(V W^{-1}) \chi_{l_3}(\tau_3 W) \chi_j(W).
\]

(26)
Here, we have used the formulae that $\chi_l(\tau_i) = \frac{1}{2}(-1)^l$ if $l$ is an integer and $\chi_l(\tau_i) = 0$ if $l$ is a half-integer and that $\chi_l(1) = 2l + 1$ and that
\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  n_1 & n_2 & n_3
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
  m_1 & m_2 & m_3
\end{pmatrix} = \int dV D_{n_1m_1}^{(j_1)}(V) D_{n_2m_2}^{(j_2)}(V) D_{n_3m_3}^{(j_3)}(V). \tag{27}
\]
Let us work out the coefficient $C^{(j)}_{1,0}[0]$. From the last expression, $V$ and $W$ are forced to be $V = W = 1$ after the sums over $l_1, l_2, l_3$ and $l_{12}$ and the integrations. Then we get
\[
C^{(j)}_{1,0}[0] = \prod_{p=1}^{P} \prod_{f \in \Delta_p \Delta^*_p} \chi_j(1) = \prod_{p=1}^{P} \prod_{f \in \Delta_p \Delta^*_p} (2j + 1) \tag{28}
\]
if $j$ is an integer; otherwise, vanishes. Here, $j$ is associated with each $f$. This “trivial” coefficient makes the partition function of the model the multiple power of that of the lattice BF theory as claimed already.

Finally, for a physical application, we present a computation formula for space-time volume density correlations. From the partition function of the model, define $Z^{(x_1 \ldots x_n)}_{\beta, \lambda}[\xi]$ which is $Z_{\beta, \lambda}[\xi]$ with $W_{\beta, \lambda}(\{l\}, \xi)$ replaced by $\delta_{\xi(x_1)} \ldots \delta_{\xi(x_n)} W_{\beta, \lambda}(\{l\}, \xi)$. Then the space-time volume density correlation function is
\[
\langle \sqrt{g(x_1)} \cdots \sqrt{g(x_n)} \rangle := \frac{Z^{(x_1 \ldots x_n)}_{\beta, \lambda}[0]}{Z_{\beta, \lambda}[0]} \tag{29}
\]
This quantity is the average of a product of space-time volume densities defined at the centers of faces with respect to the partition function. The space-time volume density at $x$ is computed by
\[
\sqrt{g(x)} := \sum_{ff^* = x} \eta_i(f) \eta^i(f^*), \tag{30}
\]
where the sum is taken over faces $f$ and $f^*$ whose center $ff^*$ is located at $x$.

3 Conclusion

We have presented a model for quantum gravity, which could reveal an aspect of the spin-foam interpretation of spacetime. Its partition function is defined by (1) with coefficients (2). We have determined the coefficients based on a path integral of $(3+1)$ dimensional general relativity in Riemannian spacetime. To do so, we projected the Plebanski action (4) to multiple pairs of lattices, each pair of which consists of two lattices dual to each other, (5) and then computed its path integral (7). The final forms of the coefficients are given by (23) with (11), and (24) with (22). In terms of the partition function, we have presented a computation formula for space-time volume density correlations.

We conclude with remarks for further investigations:
(i) The variables $U$ and $\phi_{ij}$ are left unintegrated. The integration for $U$ is known to produce functions of spins and intertwiners labeling respectively the faces and edges of a spin-foam. This integration is already present in the case of the BF theory and the resulting functions consist of the so-called 3nj-symbols. A spin-foam can be decomposed to a family of surfaces and these functions can be understood as telling the number of “distinguishable” ways with respect to a given lattice (dual to the lattice the spin-form bases on) the family of surfaces are allocated as studied in [2]. The integrations for $\phi_{ij}$ are particular in quantization of general relativity and are supposed to induce local degrees of freedom. In the present model, physics of the local degrees of freedom may be translated from the information how single spin-foams “interact” with each other. The coefficient $W_{\beta,\lambda}^{\mu}(\{l\}, \xi)$ has roughly the form of $\int dx_1 \cdots dx_k \sum_{n_1 \cdots n_k} f_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$ with $x'$s corresponding to $\phi_{ij}$ and naive term-by-term integration leads to divergences. We have to understand the properties of $f_{n_1 \cdots n_k}$ before the integrations of $\phi_{ij}$.

(ii) To perform the path integral of general relativity, we have used diffeomorphism invariant measures. However, we have to check the resulting partition function is really diffeomorphism invariant. In a limiting case, we have shown that the partition function is reduced to the multiple power of the partition function of the lattice BF theory, which is a topological invariant and certainly diffeomorphism invariant.

(iii) A technical peculiarity in the construction of the present model is the use of multiple pairs of lattices. Each pair consists of two lattices dual to each other. Because of this, physics comes out not from single spin-foams but from “interactions” between them. We would like to understand whether the use of the multiple lattices is necessary or redundant.

(iv) In the standard lattice gauge theory, in which the number of degrees of freedom is finite and each variable is compactified, the use of lattice regularizes the divergences due to the integrals along the gauge orbits as long as computations of gauge invariant quantities are concerned. However, in the present model, the regularization is not yet under control because not all the variables are compactified. In particular, to perform computations of gauge dependent quantities such as propagators, the knowledge of how to fix a gauge is crucial.

(v) As physical applications, one wants to compute physical quantities. For example, a detailed study of the spacetime volume density correlation function we have presented is to be pursued.

(vi) In order to understand the physical meaning of spin-foams, a possible way of studying it is the following. Couple external fields to the variables $A^i_a$, $B^i_{ab}$ and $\phi_{ij}$ and then compute the partition function with the external field. In terms of the partition function, one may define Schwinger equations corresponding to the classical Einstein equations with those variables. Detailed study may allow one to understand the physical meaning of the spin-foams corresponding to classical spacetime geometries.

(vii) Ultimately one wants to formulate a quantum gravity model in Lorentzian spacetime. As an extension of the present model, Lorentzian spacetime gravity is defined in terms of the path integral of complex variables with certain reality condi-
tions. A possibility of including the reality conditions into the path integral ought to be studied.

## A Pairs of lattices

We discuss in this appendix a way of constructing the multiple pairs of lattices described in Sec.2. Because 4-dimensional lattices are difficult to visualize, we first utilize 2-dimensional lattices, the role of whose edges and dual edges mimics the role of faces and dual faces of 4-dimensional lattices, to construct pairs of lattices. Then we repeat the construction in 4-dimensional case. Note that these constructions are not oriented to show a mathematical proof of the existence of the pairs of lattices.

On a 2-dimensional manifold with the topology of $S^2$ for simplicity, draw a lattice and its dual lattice. They consist of 0-cells (vertices), 1-cells (edges), and 2-cells (polygons or faces). The edges are not necessarily straight lines but may be curved. Every edge of the lattice (and the dual lattice) intersects an edge of the dual lattice (and the lattice respectively) at their “center.” Let us plot a dot at every center of edges (and dual edges). Draw curves connecting the every two centers of edges sharing a vertex of the lattices. This drawing creates another lattice, whose vertices are located at the centers of the edges of the original lattices. We call the created lattice “child” of the original lattices. Erase the original lattices. We have now only the child on the 2-dimensional manifold with dots indicating the positions of the vertices of the child.

Move the child as follows while the dots are fixed. Every vertex is moved from one dot to another or is not moved so that every dot is occupied by only one vertex. The edges sharing a vertex are moved together with the vertex if the vertex is moved. Let us call the resulting lattice “cousin” of the child. They are isomorphic to each other.

Draw a pair of lattices (a lattice and its dual lattice) from the cousin such that their relationship is analogous to the relationship between the original pair of lattices and their child. Erase the cousin. Now, we have only the second pair of lattices with dots indicating the positions of the vertices of the child.

By repeating the same process one can generate a third pair, a fourth pair, and so forth.

Next, let us repeat the analogous construction for 4-dimensional case. In a 4-dimensional spacetime manifold with the topology of $S^4$ for simplicity, draw a lattice and its dual lattice. They consist of 0-cells (vertices), 1-cells (edges), 2-cells (polygons or faces), 3-cells (polyhedrons) and 4-cells. The edges and polygons are not necessarily straight lines or flat surfaces respectively but may be curved ones. Every face of the lattice (and the dual lattice) intersects a face of the dual lattice (and the lattice respectively) at their “center.” Let us plot a dot at every center of faces (and dual
faces). Draw curves connecting the every two centers of faces sharing a vertex of 
the lattices. The curves connecting the centers of faces sharing a vertex form a 
polygon. This drawing creates another lattice. Its vertices are located at the centers 
of the faces of the original lattices. We call the created lattice “child” of the original 
lattices. Erase the original lattices. We have now only the child in the spacetime 
manifold with dots indicating the positions of the vertices of the child.

Move the child as follows while the dots are fixed. Every vertex is moved from one 
dot to another or is not moved so that every dot is occupied by only one vertex. The 
edges sharing a vertex are moved together with the vertex if the vertex is moved. Let 
us call the resulting lattice “cousin” of the child. They are isomorphic to each other.

Draw a pair of lattices (a lattice and its dual lattice) from the cousin such that 
their relationship is analogous to the relationship between the original pair of lattices 
and their child. Erase the cousin. Now, we have only the second pair of lattices with 
dots in the spacetime manifold. The centers of faces of this pair of lattices are located 
at the dots and hence the position of the center of every face of this pair of lattices 
coincides with the position of the center of a face of the original pair of lattices. One 
of the original pair is isomorphic to one of the second pair while the other lattice of 
the original pair is isomorphic to the other lattice of the second pair. By repeating 
the same process one can generate a third pair, a fourth pair, and so forth.

\section{Non-degenerate sector}

Pairs of lattices in 4-dimensions are used in the model. Each pair consists of two 
lattices dual to each other. That is, every k-cell of one lattice intersects with a (4-k)-
cell of the other lattice at a point inside the cells. The intersection is called “center.” 
In particular, every face (2-cell) of one lattice shares a center with a face of the other 
lattice. These two faces are said to be dual to each other. Let $\Delta_p$ and $\Delta_p^\ast$ denote a 
pair of lattices with $p = 1, \cdots, P$. The $P$ is the number of pairs. If they form the 
multiple pairs described in Sec.\[3,\] then there exist a face $f_p$ in $\Delta_p$ and its dual face 
$f_p^\ast$ in $\Delta_p^\ast$ with their center denoted by $f_p f_p^\ast$ such that $f_1 f_1^\ast = \cdots = f_p f_p^\ast$. (Here the 
equality means that their positions coincide.)

Let $f f^\ast$ represent the position of the coincident centers. The Lagrange multipliers 
$\phi_{ij}$ are defined at $f f^\ast$. The constraints imposed by the multipliers at $f f^\ast$ are

$$
\sum_{p=1}^{P} [\eta^i(f_p)\eta^j(f_p^\ast) + \eta^j(f_p)\eta^i(f_p^\ast)] = C(f f^\ast)\delta^{ij}.
$$

(31)

Here $C(f f^\ast)$ is an arbitrary non-negative real number determined by $\eta^i(f_p)$ and $\eta^i(f_p^\ast)$ 
and proportional to spacetime volume density at $f f^\ast$. $C(f f^\ast) > 0$ and $C(f f^\ast) = 0$ 
mean that the solutions for $\eta^i$ are non-degenerate and degenerate respectively at $f f^\ast$.

It is easy to check that if $P = 1$, then solutions for $\eta^i$ at $f f^\ast$ do not exist unless 
$C(f f^\ast) = 0$. This fact means that the model needs more than a pair of lattices in 
order to contain the non-degenerate sector. Once the number of pairs, $P$, is fixed,
then the non-degenerate and degenerate sectors can be clearly characterized. Since
the constraints have the form of the inner product for three vectors \((i, j = 1, 2, 3)\)
with their norms proportional to spacetime volume density, the non-degenerate sector
of the model is analogous to the space spanned by non-degenerate orthogonal vectors
in a 3-dimensional linear space.

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