CONTINUOUS HOMOTOPY FIXED POINTS FOR LUBIN-TATE SPECTRA

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Abstract. We construct a stable model structure on profinite spectra with a continuous action of an arbitrary profinite group. This provides a natural framework for a new definition of continuous homotopy fixed point spectra and of continuous homotopy fixed point spectral sequences for the action of the extended Morava stabilizer group on Lubin-Tate spectra. These continuous homotopy fixed point spectra and spectral sequences are equivalent to the ad hoc-constructions of Devinatz and Hopkins but have a drastically simplified construction.

1. Introduction

For the action of a discrete group $G$ on a spectrum $F$ there are well-known constructions for the homotopy fixed point spectrum $F^{hG}$ and for the homotopy fixed point spectral sequence. For fibrant $F$, the spectrum $F^{hG}$ is given by the $G$-fixed points of the function spectrum $\text{hom}(EG_+, F)$, where $EG$ is a contractible free $G$-space. For each spectrum $Z$, the spectral sequence

\[ H^*(G; F^* Z) \Rightarrow [Z, F^{hG}]^* \]

is induced by the usual filtration of the bar construction for $EG$. But in some cases of interest, the group $G$ and the spectrum $F$ carry additional structures that one would like to take care of. For example, this is the case for the most important group action in the chromatic approach to stable homotopy theory, namely, the action of the extended Morava stabilizer group $G_n$ on the $p$-local Landweber exact spectrum $E_n$. Let us shortly describe this famous example.

Let $p$ be a fixed prime, $n \geq 1$ an integer and $\mathbb{F}_{p^n}$ the field with $p^n$ elements. Let $S_n$ be the $n$th Morava stabilizer group, i.e. the automorphism group of the height $n$ Honda formal group law $\Gamma_n$ over $\mathbb{F}_{p^n}$. We denote by $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ and let $G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the semi-direct product. By the work of Lubin and Tate [22], there is a universal ring of deformations $E(\mathbb{F}_{p^n}, \Gamma_n) = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$ of $(\mathbb{F}_{p^n}, \Gamma_n)$, where $W(\mathbb{F}_{p^n})$ denotes the ring of Witt vectors of $\mathbb{F}_{p^n}$. The $MU_*$-module $E(\mathbb{F}_{p^n}, \Gamma_n)[u, u^{-1}]$ induces via the Landweber exact functor theorem a homology theory and hence a spectrum, denoted by $E_n$ and called Lubin-Tate spectrum, with $E_{n,*} = E(\mathbb{F}_{p^n}, \Gamma_n)[u, u^{-1}]$, $|u| = -2$. The profinite group $G_n$ acts on the ring $E_{n,*}$, cf. [9]. By Brown representability, this induces an action of $G_n$ by maps of rings in the stable homotopy category. Furthermore, Goerss, Hopkins and Miller have shown the crucial fact that there is even a $G_n$-action on the spectrum-level on $E_n$ that induces the action in the stable category, see [15] and [29].
Now $S_n$, $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ and hence also $G_n$ are profinite groups. Moreover each homotopy group $\pi_*E_n$ has the structure of a continuous profinite $G_n$-module. The continuity of the action of $G_n$ on each $\pi_*E_n$ is a very important property for stable homotopy theory. In fact, by Morava’s change of rings theorem, the $K(n)_*$-local $E_n$-Adams spectral sequence for the sphere spectrum $S^0$ has the form

$$H^*_c(T; E_n) \Rightarrow \pi_*L_{K(n)}S^0$$

where the $E_2$-term equals continuous cohomology of $G_n$ with profinite coefficients $E_{n_*}$. Here $K(n)$ denotes the $n$th Morava $K$-theory and $L_{K(n)}$ denotes $K(n)_*$-localization, cf. [23] and [7]. Hence $L_{K(n)}S^0$ looks like a continuous $G_n$-homotopy fixed point spectrum of $E_n$ and one would like to interpret the above spectral sequence as a continuous homotopy fixed point spectral sequence of the $G_n$-action. But the classical construction of homotopy fixed points and its spectral sequence [1] does not reflect the topology on $G_n$. The function spectrum $\text{hom}_{G}(EG_+, E_n)$ should consist of continuous maps in some sense and the $E_2$-term of the spectral sequence [1] should be continuous cohomology of $G$. Hence it is a fundamental problem in stable homotopy theory to understand in which way $E_n$ can be viewed as an object with a continuous action under $G_n$, see [10].

Devinatz and Hopkins [10] have circumvented this problem and given a complicated ad hoc argument for the construction of continuous homotopy fixed points of $E_n$. They proceeded in two steps by first constructing homotopy fixed points, here denoted by $E^{dhU}_n$ by adopting the notation in [5], for a proper open subgroup $U$ of $G_n$ using that $G_n/U$ is finite. In a second step they defined $E^{dhG}_n$ for a closed subgroup $G$. Since $G_n$ is a $p$-adic analytic group, it is possible to find a sequence of subgroups $G_n = U_0 \supset U_1 \supset \ldots$. Then $E^{dhG}_n$ is defined as an appropriate homotopy colimit of the $E^{dhU_i/G}_n$’s. Moreover, for every closed subgroup $G$ of $G_n$, they provided a construction of a descent spectral sequence for the homotopy fixed point spectrum $E^{dhG}_n$ with the correct continuous $E_2$-term.

But since the argument of [10] did not explain in which sense $G_n$ acts continuously on $E_n$, the question remained how to view $E_n$ as an actual continuous spectrum and to find a natural framework for the continuous homotopy fixed point spectral sequence. The purpose of this paper is to give a new and complete answer to this question.

A previous approach has been developed by Davis in [5] and by Behrens and Davis in [2] by studying discrete $G$-spectra. Davis used the idea of Devinatz and Hopkins to start with the homotopy fixed point spectrum $E^{dhU}_n$ of [10] for an open subgroup $U \subset G_n$ and defined a new spectrum $F_n := \text{colim}_i E^{dhU_i}_n$ where the $U_i$ run through a fixed sequence of open subgroups of $G_n$. The $K(n)_*$-localization of $F_n$ is equivalent to $E_n$. One can regard $F_n$ as a continuous discrete $G_n$-spectrum. Furthermore, Davis developed a stable homotopy theory for continuous discrete $G$-spectra, for an arbitrary profinite group $G$, based on the work of Goerss [13] and Jardine [21]. Then he defined systematically the homotopy fixed points for closed subgroups of $G_n$ and constructed a continuous homotopy fixed point spectral sequence.

A different method has been used by Fausk. In [12], Fausk constructed a model structure for pro-$G$-spectra, where $G$ denotes a compact Hausdorff topological group, e.g. a profinite group. He also obtained results on homotopy fixed points, descent spectral sequences and iterated homotopy fixed points. These results are equivalent to those of [5] if $G$ has finite virtual cohomological dimension.

But the crucial point is that if one wants to use the methods of Davis or Fausk...
for Lubin-Tate spectra $E_n$, one first has to apply the construction of \cite{10} for open subgroups and has to rewrite $E_n$ as a colimit of the $E_n^{dghU}$'s as in \cite{10}. Thereby one forces $E_n$ to look like a discrete object although its homotopy groups are not discrete $G_n$-modules. Hence the above question still remained open how to view $E_n$ as a continuous spectrum without using \cite{10} for open subgroups of $G_n$ and to give a unified construction for all closed subgroups of $G_n$ without \cite{10}. The approach of the present paper solves this problem. We give a new unified natural construction of continuous homotopy fixed points for any closed subgroup independent of \cite{10} and hence, in particular, also a new construction for open subgroups of $G_n$. The idea is straightforward. Since the homotopy groups $\pi_t E_n$ are not discrete but profinite $G_n$-modules, a natural guess would be to look for a profinite structure on $E_n$. And, in fact, there is one in the following sense. The spectrum $E_n$ can be built out of a sequence of simplicial profinite sets that carry a continuous $G_n$-action. Consequently, the natural setting to study the action of $G_n$ on $E_n$ is a suitable category of continuous profinite $G_n$-spectra.

Let us give a quick outline of the strategy and provide precise statements of the main results of this paper. We will study continuous actions on profinite spectra of an arbitrary profinite group $G$ in general. A profinite $G$-space is a simplicial object in the category of profinite sets with the limit topology and a continuous $G$-action. These profinite $G$-spaces form a category $\hat{S}_G$ with level-wise continuous $G$-equivariant maps as morphisms. A continuous profinite $G$-spectrum $F$ is then a sequence of profinite $G$-spaces $F_n$ with maps $S_G^1 \wedge F_n \to F_{n+1}$ for all $n$, where $S^1_G$ is a cofibrant replacement of the simplicial finite set $S^1$ with a continuous free $G$-action.

**Aside.** There is a completion functor from pro-$G$-spectra of \cite{12} to the category of profinite $G$-spectra, but this is not an equivalence even if one starts with the pro-category of what one would usually call finite spectra with a $G$-action. So in some sense the notion of profinite spectra might be misleading at first glance as it is not a pro-object of finite spectra in the usual sense. □

In the next section we will discuss the profinite completion functor from spaces to profinite spaces which will play a crucial role for the main argument and explain the connection to the work of Artin-Mazur, Morel and Sullivan. Then we show that the category $\text{Sp}(\hat{S}_G)$ of continuous profinite $G$-spectra is equipped with a natural stable model structure. The fibrant replacement functor $R$ of profinite spectra will allow us to give a natural definition for continuous homotopy fixed point spectra. In fact, the continuous homotopy fixed point spectrum $F^{hG}$ of a profinite $G$-spectrum $F$ is defined as the $G$-fixed points of the continuous function spectrum $\text{hom}(EG_n, RF)$ of level-wise continuous maps in $\text{Sp}(\hat{S}_G)$. The homotopy fixed point spectral sequence is then obtained from the usual bar construction for $EG$. The striking advantage of studying profinite actions in the category of profinite spectra is that $G$ and its classifying space $EG$ yield natural objects in $\hat{S}_G$. (But one should note that although $EG$ and $F$ are profinite, the function spectrum $\text{hom}_G(EG_n, F)$ does not in general inherit a profinite structure, since roughly speaking, the limit of $EG$ is turned into a colimit. Hence the homotopy groups of $F^{hG}$ are not profinite anymore in general.)

In order to be able to apply these techniques to the action of $G_n$ on the Lubin-Tate spectrum $E_n$, we have to prove that we may consider $E_n$ as a continuous profinite $G_n$-spectrum. The starting observation is that $E_n$ has a decomposition as a limit of
spectra \( \lim_I E_n \wedge M_I \), where the \( M_I \) denote generalized Moore spectra corresponding to an inverse system of ideals \( I \) in \( BP_\ast \), cf. \[18\]. These spectra have the important property that, for each such ideal \( I \) and for every \( t \), the homotopy groups \( \pi_t(E_n \wedge M_I) \) are finite (being trivial if \( t \) is odd). Since the profinite completion of spectra with finite homotopy groups preserves these homotopy groups, profinite completion yields a functorial model of each \( E_n \wedge M_I \) which is built out of profinite spaces. Since \( G_n \) is a finitely generated pro-\( p \)-group, every subgroup of finite index in \( G_n \) is open. This implies that \( G_n \) acts continuously on every finite discrete set and hence on every profinite set. Thus our functorial model for \( E_n \wedge M_I \) is built of profinite \( G_n \)-spaces in the above sense. This provides a \( G_n \)-equivariant model for \( E_n \) which is a continuous profinite \( G_n \)-spectrum being defined as the limit of continuous profinite \( G_n \)-spectra. This is the outline of the proof of the following statement that will be rigorously proven in the last section.

**Theorem 1.1.** \( E_n \) has a canonical model in the category of continuous profinite \( G_n \)-spectra, i.e. there is a \( G_n \)-equivariant map of spectra \( f : E_n \to E_n' \) such that \( E_n' \) is a continuous profinite \( G_n \)-spectrum and \( f \) is a stable equivalence.

This allows us to apply the techniques described above to \( E_n \) which yields the following theorem whose proof will be given in the last section.

**Theorem 1.2.** Let \( G \) be any closed subgroup of \( G_n \).

(i) There is a continuous homotopy fixed point spectrum \( E^{hG}_n \) of \( E_n \) which is natural in \( G \) and equivalent to the fixed point spectrum \( E^{dhG}_n \) of \[10\]. In particular, we obtain an equivalence \( E^{hG}_n \simeq E^{dhG}_n \simeq L_{K(n)^0} S^0 \).

(ii) For any spectrum \( Z \), there is a natural strongly convergent continuous homotopy fixed point spectral sequence starting from continuous cohomology

\[
H^\ast_{cts}(G; E_n^\ast Z) \Rightarrow (E^{hG}_n)^\ast Z
\]

which is canonically isomorphic to the spectral sequence obtained from mapping \( Z \) into a \( K(n)^0 \)-local \( E_n \)-Adams resolution of \( E^{dhG}_n \).

(iii) Let \( K \) be a normal subgroup of \( G \) such that \( G/K \) is finite. Then \( E^{hG}_n \) is naturally equivalent to \((E^{hK}_n)^{hG/K}\). For any spectrum \( Z \), there is a strongly convergent spectral sequence for iterated homotopy fixed points

\[
H^\ast(G/K; (E^{hK}_n)^\ast Z) \Rightarrow (E^{hG}_n)^\ast Z.
\]

This theorem restates the main results of \[10\]. But the point is that it has a drastically simplified and much more conceptual proof. While in \[10\] the special properties of \( G_n \) and \( E_n \) have been taken advantage of to define homotopy fixed points, the methods we develop to prove Theorem 1.2 are general and work for any continuous action of a profinite group on any profinite spectrum. (The only place where we use that \( G_n \) is a finitely generated pro-\( p \)-group is when we show that \( E_n \) has a canonical model in the category of continuous profinite \( G_n \)-spectra.) In particular, the definition of \( E^{hG}_n \) is the same for all closed (or open) subgroups. Moreover also the proof of the comparison statements in Theorem 1.2 is straightforward once we have recalled the constructions of \[10\]. There are canonical comparison maps \( E^{dhG}_n \to E^{hG}_n \) and canonical isomorphisms between the spectral sequences.

The proof that the natural map \( E^{hG}_n \to (E^{hK}_n)^{hG/K} \) is an equivalence and of the existence of the spectral sequence part (iii) of Theorem 1.2 is also much simpler
than in [10] and is independent of the methods in [10]. Nevertheless we also have to
assume that $G/K$ is finite, since after taking homotopy fixed points, the resulting
spectrum does not have to be profinite anymore. Hence it would not make sense
to speak of a continuous action in the previous sense if $G/K$ was not finite. We
remark that Devinatz [7] and Davis [6] proved a generalization of (iii) based on
[10].

Finally, the methods of this paper might also be of interest for studying Galois
extensions of ring spectra in the sense of [31]. In loc. cit. Rognes develops a Galois
theory for commutative $S^0$-algebras in analogy to the algebraic Galois theory. The
step from finite to infinite extensions involves the action of a profinite group on
such spectra. These infinite Galois extensions have been studied by Behrens and
Davis in [2]. The case of continuous profinite $G$-spectra might shed some light on
the picture from a different point of view.

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2. Homotopy theory of continuous profinite spaces

2.1. Profinite spaces. We start with basic definitions and important invariants
for profinite spaces that will be necessary to construct a homotopy category for
profinite spectra.

For a category $C$ with small limits, the pro-category of $C$, denoted pro-$C$, has as
objects all cofiltering diagrams $X : I \to C$. Its sets of morphisms are defined as

$$\text{Hom}_{\text{pro-}C}(X, Y) := \lim_{j \in J} \colim_{i \in I} \text{Hom}_C(X_i, Y_j).$$

A constant pro-object is one indexed by the category with one object and one
identity map. The functor sending an object $X$ of $C$ to the constant pro-object with
value $X$ makes $C$ a full subcategory of pro-$C$. The right adjoint of this embedding
is the limit functor $\text{lim} : \text{pro-}C \to C$, which sends a pro-object $X$ to the limit in $C$
of the diagram corresponding to $X$.

Let $E$ denote the category of sets and let $F$ be the full subcategory of finite sets.
Let $\hat{E}$ be the category of compact Hausdorff and totally disconnected topological
spaces. We may identify $F$ with a full subcategory of $\hat{E}$ in the obvious way. The
limit functor $\text{lim} : \text{pro-}E \to \hat{E}$ is an equivalence of categories. Moreover the forgetful
functor $\hat{E} \to E$ admits a left adjoint $(\cdot) : E \to \hat{E}$ which is called profinite completion.

We denote by $\hat{S}$ (resp. $S$) the category of simplicial profinite sets (resp. simplicial
sets). The objects of $\hat{S}$ (resp. $S$) will be called profinite spaces (resp. spaces). The
completion of sets induces a functor $(\cdot) : S \to \hat{S}$, which is also called profinite completion.

For a space $Z$, its profinite completion can be described as follows.
Let $\mathcal{R}(Z)$ be the set of simplicial equivalence relations on $Z$ such that the quotient
$Z/R$ is a simplicial finite set. Then $\mathcal{R}(Z)$ is again ordered by inclusion. The
profinite completion is defined as the limit of the $Z/R$ for all $R \in \mathcal{R}(Z)$, i.e.

$$\hat{Z} := \lim_{R \in \mathcal{R}(Z)} Z/R.$$ 

Profinite completion of spaces is again left adjoint to the
forgetful functor $| \cdot | : \hat{S} \to S$ which sends a profinite space to its underlying simplicial set.

Let $X$ be a profinite space and let $\pi$ be a topological abelian group. The continuous cohomology $H^*(X; \pi)$ of $X$ with coefficients in $\pi$ is defined as the cohomology of the complex $C^*(X; \pi)$ of continuous cochains of $X$ with values in $\pi$, i.e. $C^n(X; \pi)$ denotes the set $\text{Hom}_{\hat{S}S}(X_n, \pi)$ of continuous maps $\alpha : X_n \to \pi$ and the differentials $\delta^n : C^n(X; \pi) \to C^{n+1}(X; \pi)$ are the morphisms associating to $\alpha$ the map $\sum_{i=0}^{n+1} \alpha \circ d_i$, where $d_i$ denotes the $i$th face map of $X$. If $\pi$ is a finite abelian group and $Z$ a simplicial set, then the cohomologies $H^*(Z; \pi)$ and $H^*(\hat{Z}; \pi)$ are canonically isomorphic.

**Convention 2.1.** Here and in the rest of the paper we will not use a special notation for continuous cohomology. For a profinite space and a topological coefficient group, cohomology will always mean continuous cohomology.

If $G$ is an arbitrary profinite group, we may still define the first cohomology of $X$ with coefficients in $G$ as done by Morel in [24] p. 355. The functor $X \mapsto \text{Hom}_{\hat{S}}(X, G)$ is represented in $\hat{S}$ by a profinite space $EG$. We define the 1-cocycles $Z^1(X; G)$ to be the set of continuous maps $f : X_1 \to G$ such that $f(d_0x)f(d_2x) = f(d_1x)$ for every $x \in X_1$. The functor $X \mapsto Z^1(X; G)$ is represented by a profinite space $BG$. Explicit constructions of $EG$ and $BG$ may be given in the standard way. Furthermore, there is a map $\delta : \text{Hom}_{\hat{S}}(X, EG) \to Z^1(X; G) \cong \text{Hom}_{\hat{S}}(X, BG)$ which sends $f : X_0 \to G$ to the 1-cocycle $x \mapsto \delta f(x) = f(d_0x)f(d_1x)^{-1}$. We denote by $B^1(X; G)$ the image of $\delta$ in $Z^1(X; G)$ and we define the pointed set $H^1(X; G)$ to be the quotient $Z^1(X; G)/B^1(X; G)$. Finally, if $X$ is a profinite space, we define $\pi_0X$ to be the coequalizer in $\hat{E}$ of the diagram $d_0, d_1 : X_1 \rightrightarrows X_0$.

The profinite fundamental group of $X$ is defined via covering spaces in the spirit of Grothendieck, see [26]. There is a universal profinite covering space $(\hat{X}, \hat{x})$ of $X$ at a vertex $x \in X_0$. Then $\pi_1(\hat{X}, \hat{x})$ is defined to be the group of automorphisms of $(\hat{X}, \hat{x})$ over $(X, x)$. It has a natural structure of a profinite group as the limit of the finite automorphism groups of the finite Galois coverings of $(X, x)$. For details we refer the reader to [26].

**Definition 2.2.** A morphism $f : X \to Y$ in $\hat{S}$ is called,

1. a weak equivalence if the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of profinite sets, $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of profinite groups for every vertex $x \in X_0$ and $f^* : H^q(Y; \mathcal{M}) \to H^q(X; f^* \mathcal{M})$ is an isomorphism for every local coefficient system $\mathcal{M}$ of finite abelian groups on $Y$ for every $q \geq 0$;
2. a cofibration if $f$ is a level-wise monomorphism;
3. a fibration if it has the right lifting property with respect to every cofibration that is also a weak equivalence.

The following theorem has been proven in [26] (with the slight correction that in [26] as the generating sets of fibrations and cofibrations had been chosen too small; the revised proof is given in [27]).

**Theorem 2.3.** The above defined classes of weak equivalences, cofibrations and fibrations provide $\hat{S}$ with the structure of a fibrantly generated left proper simplicial model category. We denote the homotopy category by $\hat{H}$. 
We consider the category $\mathcal{S}$ of simplicial sets with the usual model structure of \cite{28}. We denote its homotopy category by $\mathcal{H}$. Then the next result follows immediately.

**Proposition 2.4.** 1. The level-wise completion functor $\hat{\cdot} : \mathcal{S} \to \hat{\mathcal{S}}$ preserves weak equivalences and cofibrations.

2. The forgetful functor $| \cdot | : \hat{\mathcal{S}} \to \mathcal{S}$ preserves fibrations and weak equivalences between fibrant objects.

3. The induced completion functor $\hat{\cdot} : \mathcal{H} \to \hat{\mathcal{H}}$ and the right derived functor $R| \cdot | : \hat{\mathcal{H}} \to \mathcal{H}$ form a pair of adjoint functors.

**Definition 2.5.** Let $X$ be a pointed profinite space and let $RX$ be a fibrant replacement of $X$ in the above model structure on $\hat{\mathcal{S}}$. Then we define the $n$th profinite homotopy group of $X$ for $n \geq 2$ to be the profinite group

$$\pi_n(X) := \pi_0(\Omega^n(RX)).$$

So in order to get the homotopy groups of a profinite space $X$ we take a fibrant replacement of $X$ in $\hat{\mathcal{S}}$, and then take the usual homotopy groups of the fibrant simplicial set $RX$. These groups carry a natural profinite structure.

### 2.2. Comparison with the work of Artin-Mazur, Morel and Sullivan.

Let $\mathcal{H}_{\text{fin}}$ denote the subcategory of $\mathcal{H}$ of finite spaces, i.e. of simplicial finite sets whose homotopy groups are all finite and only a finite number of them are nontrivial. The morphisms are the homotopy classes of maps. Artin and Mazur showed that, for every space $X$, the functor $\mathcal{H}_{\text{fin}} \to \mathcal{E}$, $F \mapsto [X,F]$ is pro-representable. The representing pro-object $\hat{X}^{AM} \in \text{pro}-\mathcal{H}_{\text{fin}}$ is called the (Artin-Mazur) profinite completion of $X$. Then Sullivan \cite{35} showed that the underlying diagram in $\mathcal{H}_{\text{fin}}$ has a limit $\hat{X}^{Su}$ in $\mathcal{H}$, which is called the profinite completion of Sullivan, see also \cite{24}. Moreover, there are analogues of these functors for various subclasses $\mathcal{C}$ of the class of finite groups, for which one replaces $\mathcal{H}_{\text{fin}}$ by its subcategory of finite spaces whose homotopy groups are all in $\mathcal{C}$. For example, one could consider the class of finite $p$-groups for a fixed prime number $p$.

Morel proved in \cite{24} that there is a model structure on $\hat{\mathcal{S}}$ for each prime number $p$ in which the weak equivalences are maps that induce isomorphisms on $\mathbb{Z}/p$-cohomology. The homotopy groups for this structure are pro-$p$-groups being defined as above using $R^p$. The generating fibrations and trivial fibrations are given by the canonical maps $L(\mathbb{Z}/p,n) \to K(\mathbb{Z}/p,n + 1)$, $K(\mathbb{Z}/p,n) \to \ast$, respectively by the maps $L(\mathbb{Z}/p,n) \to \ast$ for every $n \geq 0$, see also \cite{14} and \cite{26}. As we will explain below in the more general case, the fibrant replacement functor $R^p$ of \cite{24} yields a rigid version of the pro-$p$-finite-completion of Artin-Mazur and Sullivan.

One of the purposes of \cite{26} was to provide a rigid version of the full profinite completion functor of Artin-Mazur and Sullivan in the spirit of Morel’s approach in \cite{24}. In fact, the methods of \cite{20} and \cite{27} provide a rigid model of the Artin-Mazur and Sullivan pro-$\mathcal{C}$-completion for any subclass $\mathcal{C}$ of the class of finite groups. This rigidification is obtained as follows. Given a profinite space $Z$ we can decompose it into an inverse system of fibrant profinite spaces with finite homotopy groups. Together with taking finite Postnikov sections, this yields a decomposition of $Z$ into an inverse system of finite spaces in
By applying this to the completion \( \hat{X} \) of a space \( X \), we get a functor
\[
F : \mathcal{S} \to \hat{\mathcal{S}} \to \text{pro-} \mathcal{S}_{\text{fin}}
\]
where \( \mathcal{S}_{\text{fin}} \) is the subcategory of \( \mathcal{S} \) of finite spaces.

Now we can consider this functor on the homotopy level via Proposition 2.4
\[
\mathcal{H} \to \hat{\mathcal{H}} \to \text{pro-} \mathcal{H}_{\text{fin}}.
\]
It follows immediately from the previous results on profinite spaces and [1], Theorem 4.3, that \( F \) is isomorphic to the Artin-Mazur completion functor. Moreover, this implies that the fibrant replacement of \( \hat{X} \) in \( \hat{\mathcal{S}} \) is a rigid model for the Sullivan completion of \( X \), i.e. that \( |RX| \) is canonically isomorphic to \( \hat{X}^{Su} \) in \( \mathcal{H} \). Hence \( R\hat{X} \cong F(X) \) provides a rigid model for profinite completion.

### 2.3. Completion of spaces versus completion of groups.

Beside the completion functor of spaces \( \hat{\cdot} : \mathcal{S} \to \hat{\mathcal{S}} \) there is also a well-known profinite completion functor of discrete groups. By lack of better notation we will also denote the profinite completion of a group \( G \) by \( \hat{G} \). It is equipped with a natural map \( G \to \hat{G} \) which is universal among continuous maps from \( G \) to profinite groups.

Given a pointed space \( X \in \mathcal{S}_{*} \), the homotopy groups of Definition 2.5 of its profinite completion \( \hat{X} \in \hat{\mathcal{S}}_{*} \) are profinite groups. Hence the induced map \( \pi_{t}X \to \pi_{t}\hat{X} \) factors through the group completion of \( \pi_{t}X \), i.e. there is a commutative diagram
\[
\pi_{t}X \quad \rightarrow \quad \pi_{t}\hat{X} \\
\downarrow \quad \downarrow \\
\pi_{t}\hat{X}.
\]

It is a fundamental question how the completions of spaces and groups interact. Unfortunately, \( \varphi_{t} \) is not an isomorphism in general. A similar phenomenon is well known for group completion and cohomology. In [32], this led Serre to call a group \( G \) good if the induced map \( \psi : H^{*}(G; M) \to H^{*}(\hat{G}; M) \) from discrete to continuous group cohomology is an isomorphism for every finite \( G \)-module \( M \). It turns out that after modifying slightly the notion of good groups by considering the action of the fundamental group on the higher homotopy groups, we get a sufficient condition such that the completion of spaces commutes with the one of groups.

Following [35], for a pointed space \( X \), we call \( \pi_{1} := \pi_{1}X \) a good fundamental group, if it is a good group in the sense of Serre above with finite cohomology groups, i.e. if
\[
H^{i}(\pi_{1}; M) \cong H^{i}(\hat{\pi}_{1}; M)
\]
and if these groups are finite for all finite \( \pi_{1} \)-modules \( M \) and all \( i \geq 0 \).

Let \( \pi_{n} := \pi_{n}X \), \( n \geq 2 \), be a higher homotopy group of \( X \). It carries a canonical action of \( \pi_{1} \). Let \( \mathcal{P} \) be the filtered set of finite \( \pi_{1} \)-quotients of \( \pi_{n} \). We denote by \( \hat{\pi}_{n} := \lim_{Q \in \mathcal{P}} \pi_{n}/Q \) the \( \pi_{1} \)-completion of \( \pi_{n} \). This is, in particular, a profinite group on which \( \pi_{1} \) acts continuously. The \( \pi_{1} \)-module \( \pi_{n} \) is called a good higher homotopy group if
\[
H^{i}(\pi_{n}; A) \cong H^{i}(\hat{\pi}_{n}; A)
\]
and if these groups are finite for all finite coefficient groups \( A \) and all \( i \geq 0 \). With these definitions we obtain the following result.
Theorem 2.6. Let $X$ be a connected pointed space.
(a) For $t = 1$, the canonical map $\varphi_1 : \pi_1 \hat{X} \to \pi_1 \hat{X}$ is an isomorphism of profinite groups, i.e. $\pi_1 X \to \pi_1 \hat{X}$ equals group completion.
(b) If $X$ has a good fundamental group and good higher homotopy groups, then $\varphi_t : \pi_t \hat{X} \to \pi_t \hat{X}$ is an isomorphism of profinite groups for every $t$.

Proof. The first assertion follows immediately from the definition of the profinite fundamental group of a profinite space via profinite covering spaces with finite fibres and the analogue description of the usual fundamental group of $X$, see [26]. The second assertion follows from Theorem 3.1 in [35] together with the fact discussed above that $R\hat{X}$ is a rigid model for the profinite completion of Sullivan, i.e. that $R\hat{X}$ is canonically isomorphic to $\hat{X}_{Su}$ in $\mathcal{H}$. \hfill \Box

Furthermore, Sullivan shows that groups which are commensurable with solvable groups in which every subgroup is finitely generated are good fundamental groups and that finitely generated abelian groups are good higher homotopy groups. In particular, we have the following immediate consequence of the previous theorem.

Corollary 2.7. Let $X$ be a space whose homotopy groups are all finite. Then profinite completion induces an isomorphism $\pi_t X = \pi_t \hat{X}$ for every $t \geq 0$.

2.4. Profinite $G$-spaces. Let $G$ be a fixed profinite group. Let $S$ be a profinite set on which $G$ acts continuously, i.e. $G$ is acting on $S$ via a continuous map $\mu : G \times S \to S$. In this situation we say that $S$ is a profinite $G$-set. If $X$ is a profinite space and $G$ acts continuously on each $X_n$ such that the action is compatible with the structure maps, then we call $X$ a profinite $G$-space. We denote by $\hat{S}_C$ the category of profinite $G$-spaces with $G$-equivariant maps of profinite spaces as morphisms. If $X$ is a pointed profinite space with a continuous $G$-action that fixes the basepoint, then we call $X$ a pointed profinite $G$-space. We denote the corresponding category by $\hat{S}_C^*$. While a discrete $G$-space $Y$ is characterized as the colimit over the fixed point spaces $Y^U$ over all open subgroups, a profinite $G$-space $X$ is the limit over its orbit spaces $X^U$. More explicitly, for an open and hence closed normal subgroup $U$ of $G$, let $X_U$ be the quotient space under the action by $U$, i.e. the quotient $X/ \sim$ with $x \sim y$ in $X$ if both are in the same orbit under $U$. It is easy to show that the canonical map $\phi : X \to \lim_U X_U$ is a homeomorphism, where $U$ runs through the open subgroups of $G$.

In order to get a model structure on $\hat{S}_C$ one can find explicit sets of generating fibrations and trivial fibrations. They arise naturally by considering $G$-actions on the corresponding generating sets for the model structure on $S$. The following result has been proven in [27].

Theorem 2.8. There is a fibrantly generated left proper simplicial model structure on the category of profinite $G$-spaces such that a map $f$ is a weak equivalence in $\hat{S}_C$ if and only if its underlying map is a weak equivalence in $\hat{S}$. A map $f : X \to Y$ is a cofibration in $\hat{S}_C$ if and only if $f$ is a level-wise injection and the action of $G$ on $Y_n = f(X_n)$ is free for each $n \geq 0$. We denote its homotopy category by $\hat{H}_C$. 
3. Profinite $G$-spectra

3.1. Profinite spectra. Now we want to stabilize the model structure of profinite spaces. Since the simplicial circle $S^1 = \Delta^1/\partial\Delta^1$ is a simplicial finite set and hence an object in $\mathcal{S}_*$, we may stabilize $\mathcal{S}_*$ by considering sequences of pointed profinite spaces together with connecting maps for the suspension.

Definition 3.1. A profinite spectrum $X$ consists of a sequence of pointed profinite spaces $X_n \in \mathcal{S}_*$ and maps $\sigma_n : S^1 \wedge X_n \rightarrow X_{n+1}$ in $\mathcal{S}_*$ for $n \geq 0$. A morphism $f : X \rightarrow Y$ of spectra consists of maps $f_n : X_n \rightarrow Y_n$ in $\mathcal{S}_*$ for $n \geq 0$ such that $\sigma_n(1 \wedge f_n) = f_{n+1}\sigma_n$. We denote by $\text{Sp}(\mathcal{S}_*)$ the corresponding category of profinite spectra.

As for profinite spaces the word "profinite" in the notion of a profinite spectrum does not mean that we look at inverse systems of finite spectra in the usual sense. Instead we look at spectra that are built by simplicial profinite sets.

A profinite spectrum $E \in \text{Sp}(\mathcal{S}_*)$ is called an $\Omega$-spectrum if each $E_n$ is fibrant in $\text{Sp}(\mathcal{S}_*)$ and the adjoint structure maps $E_n \rightarrow \Omega E_{n+1} = \text{hom}_S(S^1,E_{n+1}) \in \mathcal{S}_*$ are weak equivalences for all $n \geq 0$. A map $f : X \rightarrow Y$ of profinite spectra is called a (stable) equivalence if it induces a weak equivalence of mapping spaces $\text{Map}(Y,E) \rightarrow \text{Map}(X,E)$ for all $\Omega$-spectra $E$: $f$ is called a (stable) cofibration if $X_0 \rightarrow Y_0$ and the induced maps $X_n \Pi_{S^1 \wedge X_{n-1}} S^1 \wedge Y_{n-1} \rightarrow Y_n$ are monomorphisms for all $n$; finally $f$ is called a (stable) fibration if it has the right lifting property with respect to all maps that are stable equivalences and stable cofibrations. The following result has been shown in [26], Theorem 2.36.

Theorem 3.2. These classes of morphisms define a stable simplicial model structure on profinite spectra. We denote the corresponding homotopy category by $\mathcal{S}H$. The levelwise profinite completion functor is a left Quillen functor from profinite spectra to Bousfield-Friedlander spectra $\text{Sp}(\mathcal{S}_*)$ of [3].

Via this model structure we can define profinite stable homotopy groups of a profinite spectrum. For a profinite spectrum $X$, we can profinite stable homotopy groups. Let $RX$ denote a functorial fibrant replacement of $X$ in the stable model structure on $\text{Sp}(\mathcal{S}_*)$ and let $S^k$ be the $k$th suspension of the sphere spectrum. We define $\pi_kX = \pi_kRX$ to be the abelian group $\text{Hom}_{\mathcal{S}H}(S^k, RX)$. Since $S^k$ is built out of simplicial finite sets, $\pi_kX$ inherits the structure of a profinite abelian group from $RX$. There is a lot more to explore about the relationship between the completion functor $(\cdot) : \text{Sp}(\mathcal{S}_*) \rightarrow \text{Sp}(\mathcal{S}_*)$ and profinite completion of groups. For the purposes of this paper, the following consequence of Theorem 2.6 and Corollary 2.7 will be sufficient.

Theorem 3.3. Let $X \in \text{Sp}(\mathcal{S}_*)$ be a spectrum whose homotopy groups are all finite groups. Then profinite completion induces an isomorphism $\pi_\ast X = \pi_\ast \hat{X}$.

Remark 3.4. There is also a more sophisticated category of profinite spectra with better properties based on the work of Hovey, Shipley and Smith and their category of symmetric spectra $\text{Sp}^S(\mathcal{S}_*)$. One can show that there is a stable model structure on the category $\text{Sp}^S(\mathcal{S}_*)$ of profinite symmetric spectra. A stable equivalence is a map $f$ such that $\text{Map}_{\text{Sp}^S(\mathcal{S}_*)}(f, E)$ is a weak equivalence of simplicial sets for every
injective profinite $\Omega$-spectrum $E$, where the injective symmetric spectra are defined as in [20]. Then the profinite completion functor $\hat{\cdot}$ is a monoidal left Quillen functor from symmetric spectra to profinite symmetric spectra. The proof follows from the dual machinery and ideas of [19] and [20] using the localization of [25], Theorem 6.

3.2. Continuous profinite $G$-spectra. Now let $G$ be a profinite group. Again we want to stabilize the homotopy category of $\hat{\mathcal{S}}_G$. This leads to the following notion of (naive) continuous $G$-spectra.

**Definition 3.5.** Let $S^1_G$ be a fixed cofibrant replacement of $S^1$ in $\hat{\mathcal{S}}_G$. A (continuous) profinite $G$-spectrum $X$ is a sequence of pointed profinite $G$-spaces $\{X_n\}$ together with pointed $G$-equivariant maps $S^1_G \wedge X_n \to X_{n+1}$ for each $n \geq 0$. A map of profinite $G$-spectra $X \to Y$ is a collection of maps $X_n \to Y_n$ in $\hat{\mathcal{S}}_G$ compatible with the structure maps of $X$ and $Y$. We denote the category of profinite $G$-spectra by $\text{Sp}(\hat{\mathcal{S}}_G)$.

Since $S^1_G$ is cofibrant in $\hat{\mathcal{S}}_G$, the functor $X \mapsto S^1_G \wedge X$ from $\hat{\mathcal{S}}_G$ to itself preserves weak equivalences and cofibrations and hence is a left Quillen endofunctor. The dual of the general procedure to stabilize a model structure with respect to a left Quillen endofunctor of [19] applied to Theorem 2.8 implies the following result, cf. also [26].

**Theorem 3.6.** There is a stable left proper simplicial model structure on $\text{Sp}(\hat{\mathcal{S}}_G)$ in which the equivalences are the stable equivalences of underlying profinite spectra in $\text{Sp}(\mathcal{S})$. We denote its homotopy category by $\hat{\text{SH}}_G$.

**Remark 3.7.** There is a naive version of symmetric profinite $G$-spectra using [19] just as in Remark 3.4. But, as for finite groups, there should be a more sophisticated category of profinite $G$-spectra that takes into account all continuous representations of $G$. Nevertheless, for the purposes of this paper, the previous definition is sufficient.

**Remark 3.8.** Let $j$ be a map in $\text{Sp}(\hat{\mathcal{S}}_G)$. If $j$ is a stable cofibration in $\text{Sp}(\hat{\mathcal{S}}_G)$, then its underlying map is a stable cofibration in $\text{Sp}(\mathcal{S})$. Moreover, by chasing through the mutual determinations of weak equivalences, cofibrations and fibrations, one sees that if the underlying map of a map $f$ in $\text{Sp}(\hat{\mathcal{S}}_G)$ is a stable fibration in $\text{Sp}(\mathcal{S})$, then $f$ is also a stable fibration in $\text{Sp}(\hat{\mathcal{S}}_G)$. Hence, if $R$ is a fibrant replacement functor in $\text{Sp}(\mathcal{S})$ and $X$ is a profinite $G$-spectrum, then $RX$ is by functoriality an object in $\text{Sp}(\hat{\mathcal{S}}_G)$ and moreover it is fibrant in $\text{Sp}(\hat{\mathcal{S}}_G)$ as well. Hence we may use $R$ as a functorial fibrant replacement in $\text{Sp}(\hat{\mathcal{S}}_G)$.

We have seen above that stable profinite homotopy groups of a profinite spectrum $X$ are defined as the stable homotopy groups of a fibrant replacement $RX$ in $\text{Sp}(\mathcal{S})$. If $X$ is equipped with the structure of a $G$-action, then there is an induced action of $G$ on each stable homotopy group $\pi_k X$. This action is easily seen to be compatible with the profinite structure.

**Proposition 3.9.** If $X$ is a profinite $G$-spectrum, then each stable profinite homotopy group $\pi_k X$ is a continuous profinite $G$-module.
4. Homotopy fixed point spectra

4.1. Homotopy fixed point spectra. Let $G$ be a profinite group. For a pointed space $X$ and a spectrum $E$, we denote by $\text{hom}(X, E) := \text{hom}_{\text{Sp}(\hat{S}_*)}(X, E)$ the function spectrum whose $n$th space is the mapping space $\text{hom}_{\hat{S}_*}(X, E_n) \in S$ of continuous maps in $\hat{S}_*$. Moreover, if $X$ is a pointed profinite $G$-space and $E$ is a profinite $G$-spectrum, then we denote by $\text{hom}_G(X, E) := \text{hom}_{\text{Sp}(\hat{S}_* G)}(X, E)$ the function spectrum of $G$-equivariant maps in $\text{hom}(X, E)$ or, equivalently, the $G$-fixed points of the spectrum $\text{hom}(X, E)$. One should note that $\text{hom}(X, E)$, and hence also $\text{hom}_G(X, E)$, do not carry a profinite structure in general. So $\text{hom}(X, E)$ is merely an object in $\text{Sp}(\hat{S}_*)$. This is due to the well-known fact that the set of continuous maps between two profinite sets is not profinite itself in general. But if $X$ is a simplicial finite set, then the profinite structure of $E$ induces a profinite structure on $\text{hom}(X, E)$.

Let $EG$ be a contractible free profinite $G$-space and let $R_G$ be a fixed functorial fibrant replacement in $\text{Sp}(\hat{S}_*)$. The following notion of a homotopy fixed point spectrum for a profinite $G$-spectrum $E$ is analogous to the definition but reflects the profinite structures of $G$ and $E$.

**Definition 4.1.** Let $E \in \text{Sp}(\hat{S}_*)$ be a profinite $G$-spectrum. We define the homotopy fixed point spectrum $E^h_G$ of $E$ to be the $G$-equivariant function spectrum $E^h_G := \text{hom}_G(EG, R_G E)$.

Since $EG$ is cofibrant and since $\text{Sp}(\hat{S}_*)$ is a simplicial model category we deduce the following lemma.

**Lemma 4.2.** If $E \rightarrow F$ is a stable equivalence in $\text{Sp}(\hat{S}_*)$, then $E^h_G \rightarrow F^h_G$ is a stable equivalence in $\text{Sp}(\hat{S}_*)$.

Let $E$ be a profinite $G$-spectrum. Its stable homotopy groups are continuous profinite $G$-modules. Let $\{I_k\}$ be a system of ideals in $\pi_* E$ such that

$$\pi_* E = \lim_k \pi_* E/I_k \pi_* E$$

is a compatible decomposition of $\pi_* E$ as an inverse limit of finite $G$-modules. Now if $Z$ is a spectrum, let $\{Z_\alpha\}$ be the directed system of its finite subspectra. Then we can equip the $E$-cohomology groups of $Z$ with the structure of a profinite $G$-module by observing

$$[Z, E]^t = E^t Z = \lim_{k, \alpha} E^t Z/I_k E^t Z_\alpha$$

where $[Z, E]^t$ denotes maps in the stable homotopy category $SH$ that lower dimension by $t$.

**Theorem 4.3.** Let $G$ be a profinite group, let $E$ be a profinite $G$-spectrum and let $Z$ be a spectrum. There is a continuous homotopy fixed point spectral sequence whose $E_2^{s,t}$-term is the $s$th continuous cohomology of $G$ with coefficients the profinite $G$-module $E^t Z$:

$$E_2^{s,t} = H^s(G; E^t Z) \Rightarrow (E^h_G)^{s+t} Z.$$  

This spectral sequence is strongly convergent if $\lim_{r} E_r^{s,t} = 0$ for all $s$ and $t$. 

Recall that, given a cosimplicial spectrum \( Y \). Let \( cSp(S_\ast) \) be the category of cosimplicial spectra equipped with the model structure of \([4\] X, §4\). There is a cosimplicial continuous replacement functor \( \Pi^\ast_{cts} E \in cSp(S_\ast) \) for a diagram of profinite spectra defined in codimension \( n \) by \( \Pi^\ast_{cts} E := \text{hom}_{Sp(S_\ast)}(G^{n+1}, E) \), i.e. the function spectrum of continuous maps from \( G^{n+1} \) to \( E \) in \( Sp(\hat{S}_\ast) \). Since \( E \) is fibrant in \( Sp(\hat{S}_\ast) \), its cosimplicial replacement is a fibrant object in \( cSp(S_\ast) \). Let \( \Delta^s \) be the cosimplicial space which in dimension \( n \) is the standard \( n \)-simplex with the usual coface and codegeneracy maps, and let \( \text{sk}_n \Delta^s \) be the cosimplicial space which in dimension \( n \) is the \( s \)-skeleton of \( \Delta^s \). Recall that, given a cosimplicial spectrum \( Y \), the total spectrum of \( Y \) is defined to be the limit

\[ \text{Tot} Y := \lim_s \text{Tot}_s Y \]

where \( \text{Tot}_s Y := \text{hom}(\text{sk}_s \Delta^s, Y) \) is the profinite function spectrum of continuous cosimplicial maps from \( \Delta^s \) to \( Y \). By mapping a spectrum \( Z \) into the tower of fibrations \( \{\text{Tot}_s Y\} \), we obtain a spectral sequence as in \([4\] X, §6\):

\[ E_2^{s,t} = \lim_s \pi^s([Z,Y]^t) \Rightarrow [Z, \text{Tot} Y]^{t+s} \]

where \( \lim^s \) is the \( s \)-th derived functor of the inverse limit. Now we apply this to \( Y = \Pi^\ast_{cts} E \). We have to check that the \( E_2 \)-term is continuous cohomology of \( G \). By \([4\] X, 7.2\), there are natural isomorphisms \( E_2^{s,t} = \pi^s[Z, \Pi^\ast_{cts} E]^t \) for \( t \geq s \geq 0 \), where \( \pi^s \) denotes the cohomotopy of a cosimplicial group. Since \( \Pi^\ast E \) is a fibrant cosimplicial object, there are natural isomorphisms \( [Z, \Pi^\ast_{cts} E]^t \cong \Pi^\ast_{cts} [Z, E]^t \) given in degree \( n \) by \( [Z, \text{hom}(G^{n+1}, E)]^t \cong \text{Hom}(G^{n+1}, [Z, E]^t) \), where the right hand \( \text{Hom} \)-term denotes the group of continuous group homomorphisms from \( G^{n+1} \) to \( [Z, E]^t \). This implies that the above cohomotopy groups are cohomology groups of the complex \( C^G([Z, E]^t) \) given in degree \( s \) by the set of continuous maps from \( G^s \to [Z, E]^t \), where \( [Z, E]^t = E^t Z \) is equipped with the above profinite structure. Hence we have identified the \( E_2 \)-term with the continuous cohomology groups of the statement.

It follows immediately from the definition of \( \Pi^\ast_{cts} E \) that the total spectrum of this cosimplicial object is equal to \( \text{hom}_{G}(EG_+, E) \in Sp(\hat{S}_\ast) \), i.e. the abutment of the spectral sequence is \( (E^h G)^{t+s} Z \). Finally, the strong convergence follows as in \([4\], IX §5\). \( \Box \)

**Corollary 4.4.** Let \( G \) be a profinite group and let \( E \) be a profinite \( G \)-spectrum. There is a descent spectral sequence, whose \( E_2^{s,t} \)-term is the \( s \)-th continuous cohomology of \( G \) with coefficients the profinite \( G \)-module \( \pi_s E \):

\[ H^s(G; \pi_s E) \Rightarrow \pi_s(E^h G). \]

This spectral sequence is strongly convergent if \( \lim_{r \to \infty} E_2^{s,t} = 0 \) for all \( s \) and \( t \).

4.2. Iterated homotopy fixed point spectra. Let \( H \) be a closed subgroup of \( G \). We will show now how the homotopy fixed points under the action of \( G \) and \( H \), if \( H \) is a normal subgroup, \( G/H \) are related to each other.

**Lemma 4.5.** Let \( X \) be a profinite \( G \)-spectrum and let \( H \) be a closed subgroup of \( G \). Then the fibrant profinite \( G \)-spectrum \( R_G X \) is fibrant as a profinite \( H \)-spectrum.
In particular, we can calculate the homotopy fixed point spectrum of $X$ under $H$ as $X^{hH} \cong \hom_H(\text{EH}_+, R_G X)$.

Proof. The following argument is essentially the same as in [2], Proposition 3.3.1(2). But in our profinite setting, it works for arbitrary closed and not only open subgroups. For a profinite $H$-spectrum $Y$, the Borel construction

$$\text{Ind}_H^G Y = G_+ \wedge_H Y$$

defines an induction functor $\text{Ind}_H^G : \text{Sp}(\hat{S}_H) \to \text{Sp}(\hat{S}_G)$. It is the left adjoint to the restriction $\text{Res}_H^G$ along the inclusion $H \to G$. Since there is a non-equivariant isomorphism $\text{Ind}_H^G Y = G/H_+ \wedge Y$, $\text{Ind}_H^G$ preserves cofibrations and stable equivalences. Hence $\text{Res}_H^G$ preserves fibrant objects.

Since $X \to R_G X$ is an $H$-equivariant trivial cofibration, it is a trivial cofibration in $\text{Sp}(\hat{S}_H)$. Hence we may compute $X^{hH}$ as $\hom_H(\text{EH}_+, R_G X).$ \hfill $\square$

Now we assume in addition that $H$ is a normal open subgroup of $G$, which implies that $F := G/H$ is a finite group. The homotopy fixed point spectrum $X^{hH}$ is then equipped with an $F$-action. Starting from the model structure of simplicial $F$-sets in [14, V §2], we can define the category of (naive) $F$-spectra $\text{Sp}(\hat{S}_F)$ as in Definition 3.3.3 (but with trivial action of $F$ on $S^1$ instead of $S^1_F$). Then the method of Hovey [19] yields again a stable model structure on $\text{Sp}(\hat{S}_F)$. Just as in Remark 3.8 we see that if for a map $f$ in $\text{Sp}(\hat{S}_F)$, its underlying map in $\text{Sp}(\hat{S}_*)$ is a fibration, then $f$ is a fibration in $\text{Sp}(\hat{S}_F)$. Since $X^{hH}$ is fibrant in $\text{Sp}(\hat{S}_*)$, we get the following lemma.

**Lemma 4.6.** If $H$ is a normal subgroup of $G$ with finite quotient $F := G/H$, then the fixed point spectrum $X^{hH}$ is fibrant in $\text{Sp}(\hat{S}_F)$.

For $H$ open and normal in $G$ with quotient $F$, the canonical map $X^{hG} \to X^{hH}$ factors through $(X^{hH})^F$. By composing with the canonical map from fixed points $(X^{hH})^F = \hom_F(\ast, X^{hH})$ to homotopy fixed points $(X^{hH})^{hF} = \hom_F(\text{EF}_+, X^{hH})$, we get the map

$$X^{hG} \to (X^{hH})^F \to (X^{hH})^{hF}.$$

It follows almost immediately from the definitions that this map is an equivalence. We summarize this in the following theorem.

**Theorem 4.7.** Let $X$ be a profinite $G$-spectrum and let $H$ be an open normal subgroup of $G$ with quotient $F = G/H$. Then the map $X^{hG} \cong (X^{hH})^{hF}$ is a stable equivalence in $\text{Sp}(\hat{S}_*)$. Moreover, for any spectrum $Z$, there is a spectral sequence for iterated homotopy fixed points

$$H^*(F; (X^{hH})^* Z) \Rightarrow (X^{hG})^* Z.$$

Proof. We can assume that $X$ is a fibrant profinite $G$-spectrum. By Lemma 4.5 and since $H$ acts freely on $EG$, we see that $X^{hH}$ is equivalent to the fixed point spectrum of $\hom_{\text{Sp}(\hat{S}_*)}(\text{EG}_+, X)$ under $H$. Moreover, $EG \times EF$ has a free $G$-action and hence $X^{hG}$ is equivalent to the $G$-fixed point spectrum of $\hom_{\text{Sp}(\hat{S}_* )}(\text{EG}_+ \times EF_+, X)$. 

Using adjunctions and the fact that $EF$ is a simplicial finite set with trivial $H$-action we get the following sequence of equivalences:

$$X^{hG} \simeq \text{hom}_{\text{Sp}(\mathcal{S}_*')}((EG_+ \times EF_+), X)^G$$
$$\cong \text{hom}_{\text{Sp}(\mathcal{S}_*')}((EF_+, \text{hom}_{\text{Sp}(\mathcal{S}_*')}((EG_+, X))^G$$
$$\cong \text{hom}_{\text{Sp}(\mathcal{S}_*')}((EF_+, (EG_+, X))^H)^F$$
$$\cong \text{hom}_{\text{Sp}(\mathcal{S}_*')}((EF_+, (EG_+, X)^H)^F$$
$$\simeq (X^{hH})^F.$$

This proves the first assertion. The second assertion now follows immediately from the first and Theorem 4.3 together with its older brother for finite groups. \hfill \Box

5. Morava stabilizer groups and Lubin-Tate spectra

5.1. $E_n$ is a continuous profinite $G_n$-spectrum. We return to our main example of the introduction. For a fixed prime number $p$ and an integer $n \geq 1$, $G_n$ denotes the extended Morava stabilizer group associated to the height $n$ Honda formal group law $\Gamma_n$ over $\mathbb{F}_p$. It is a profinite group and can also be described as the group of automorphisms of the Lubin-Tate spectrum $E_n$ in the stable homotopy category. Hopkins and Miller have shown that $G_n$ is in fact the automorphism group of $E_n$ by $A_\infty$-maps, cf. [22]. Later on, Goerss and Hopkins extended this result to the $E_\infty$-setting. Hence $G_n$ acts on the spectrum-level on $E_n$ by $E_\infty$-ring maps, cf. [15]. We will show now that this action is in fact a continuous action of $G_n$ on the profinite spectrum $E_n$. Let $BP$ be the Brown-Peterson spectrum for the fixed prime $p$. Its coefficient ring is $BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots]$, where $v_n$ has degree $2(p^n - 1)$. There is a canonical map $r : BP_* \to E_{n*} = W(\mathbb{F}_p)[[u_1, \ldots, u_{n-1}]](u, u^{-1})$ defined by $r(v_i) = u_i u_i^{-1} - p^i$ for $i < n$, $r(v_n) = u_n^{p^n} - p^n$ and $r(v_i) = 0$ for $i > n$. Let $I$ be an ideal in $BP_*$ of the form $(p^0, v_1^t, \ldots, v_{n-1}^t)$. Such ideals form a cofiltered system. Its images in $E_{n*}$ under $r : BP_* \to E_{n*}$ provide each $\pi_t E_n$ with the structure of a continuous profinite $G_n$-module as

$$\pi_t E_n \cong \lim_i \pi_i E_n / I \pi_i E_n.$$  

In fact, for $t$ odd these groups vanish, for $t$ even each quotient $\pi_t E_n / I \pi_t E_n$ is a finite discrete $G_n$-module and the decomposition as an inverse limit of these finite discrete $G_n$-modules is $G_n$-compatible. For a cofinal subsystem of such ideals $I$, there are finite generalized Moore spectra $M_I$ with trivial $G_n$-action whose Brown-Peterson homology is $BP_*(M_I) = BP_* / I$ and who have the property $\pi_t (E_n \wedge M_I) \cong \pi_t E_n / I \pi_t E_n$ for all $t$. Moreover there is a canonical equivalence, cf. [18],

$$E_n \simeq \lim_I E_n \wedge M_I.$$  

This observation is the starting point for all attempts to view $E_n$ as a continuous $G_n$-spectrum. Although each $\pi_t (E_n \wedge M_I)$ is a finite and hence discrete $G_n$-module, $G_n$ does not act discretely on the spectra $E_n \wedge M_I$, in the sense that there is no open subgroup $U$ of $G_n$ such that the $G_n$-action factors through $G_n / U$. But a slightly less demanding statement holds. Each $E_n \wedge M_I$ and hence $E_n$ can be viewed as an object in $\text{Sp}(\mathcal{S}_* G_n)$, i.e. as a continuous profinite $G_n$-spectrum.
Theorem 5.1. $E_n$ has a model in the category of continuous profinite $G_n$-spectra, i.e. there is a $G_n$-equivariant map of spectra $f : E_n \to E'_n$ such that $E'_n$ is a continuous profinite $G_n$-spectrum and $f$ is a stable equivalence.

Proof. To simplify the notation, we will set $E_{n,I} := E_n \land M_I$. Let $\hat{E}_{n,I} \in \text{Sp}(\hat{\mathcal{S}}_s)$ be the profinite completion of $E_{n,I}$ and let $R\hat{E}_{n,I}$ be a fixed functorial fibrant replacement in $\text{Sp}(\hat{\mathcal{S}}_s)$. Recall that we denote by $|\cdot| : \text{Sp}(\hat{\mathcal{S}}_s) \to \text{Sp}(\mathcal{S}_s)$ the forgetful functor that associates to a profinite spectrum its underlying spectrum. Since the homotopy groups of $E_{n,I}$ are finite, Theorem 5.3 implies that the canonical map

$$E_{n,I} \to |R\hat{E}_{n,I}|$$

obtained by functoriality induces isomorphisms

$$\pi_t(E_{n,I}) = \pi_t(|R\hat{E}_{n,I}|)$$

for every $t$ and hence it is a stable equivalence of spectra. Since the constructions we have applied are functorial, the action of $G_n$ on $E_{n,I}$ induces an action of $G_n$ on $R\hat{E}_{n,I}$. Furthermore, $G_n$ is a finitely generated pro-$p$-group and hence all its subgroup of finite index are open, or, in the terminology of [30] 4.2, $G_n$ is strongly complete. This implies that the action of $G_n$ on any finite and hence also on any profinite set is continuous. Thus, since the spectrum $R\hat{E}_{n,I}$ consists of simplicial profinite sets with a $G_n$-action, it is a continuous profinite $G_n$-spectrum. Finally, we define the continuous profinite $G_n$-spectrum $E'_n$ to be the limit of the $R\hat{E}_{n,I}$ over the cofinal subsystem of ideals $I$ for which $M_I$ exists. The continuous profinite $G_n$-spectrum $E'_n$ is equipped with a $G_n$-equivariant map of spectra

$$f : E_n \to \lim_I E_{n,I} \to \lim_I R\hat{E}_{n,I} =: E'_n$$

which is a stable equivalence as a map of underlying spectra by the previous arguments. This proves the assertion. \qed

Remark 5.2. Another example of a profinite spectrum is given by Morava $K$-theory. Let $K(n)$ be the $n$th $p$-primary Morava $K$-theory. Its coefficient ring is given by $K(n)_\ast = F_p[v_n, v_n^{-1}]$ where $v_n$ has degree $2(p^n - 1)$. Hence each homotopy group is a finite group. So by Theorem 5.3 the natural map $K(n) \to R\hat{K}(n)$ from $K(n)$ to the fibrant replacement of its profinite completion in $\text{Sp}(\hat{\mathcal{S}}_s)$ is an equivalence. Thus $K(n)$ has a model in the category of profinite spectra.

By restricting the action, the above theorem implies of course that $E_n$ is a continuous profinite $G$-spectrum for every closed subgroup $G$ of $G_n$. The homotopy fixed point gadget of the previous section then allows to define the continuous homotopy fixed spectrum of $E_n$ under the action of any closed or open subgroup of $G_n$ in the same breath. Since $E'_n$ is fibrant in $\text{Sp}(\hat{\mathcal{S}}_s)$ by construction, it is also fibrant in $\text{Sp}(\hat{\mathcal{S}}_{s,G})$ by Remark 5.3.

Definition 5.3. Let $G$ be any closed subgroup of $G_n$. The continuous homotopy fixed point spectrum $E_n^{hG}$ of the Lubin-Tate spectrum is the homotopy fixed point spectrum of Definition 4.4 of $E'_n$ considered via restriction as a profinite $G$-spectrum, i.e.

$$E_n^{hG} := \text{hom}_G(EG_+, E'_n).$$
Remark 5.4. Since $E_n$ is $K(n)_*$-local and since the operations used to build $E^{hG}_n$ preserve $K(n)_*$-local objects, we deduce that $E^{hG}_n$ is itself a $K(n)_*$-local spectrum.

5.2. Proof of Theorem 1.2. We have just defined the homotopy fixed point spectrum $E^{hG}_n$ for an arbitrary closed subgroup $G$ of $G_n$ in Definition 5.3. This proves the first assertion of (i) in Theorem 1.2. If $Z$ is a spectrum, let $\{Z_n\}$ be the directed system of its finite subspectra and let $I_n$ be the ideal $r(p, v_1, \ldots, v_{n-1}) \subseteq E_n$. The powers of $I_n$ already suffice to see that

$$E^t_n Z = \lim_{n,k} E^t_n Z/1^k E^t_n Z$$

is a profinite $G_n$-module for every $t$, see [10], Remark 1.3. The Bousfield-Kan spectral sequence of Theorem 1.3 then yields the continuous homotopy fixed point spectral sequence

$$H^*(G; E^*_n Z) \Rightarrow (E^{hG})^*_Z$$

for any closed subgroup $G$ of $G_n$, natural in $G$ and $Z$. Since $G_n$ is a $p$-adic analytic group, so is $G$ and its continuous cohomology groups with profinite coefficients are also profinite groups, cf. [34]. Hence the $\lim^1$ of the $E_r$-terms all vanish and the spectral sequence above is strongly convergent. This proves the first assertion of part (ii) of Theorem 1.2. Part (iii) on iterated homotopy fixed point spectra is an immediate consequence of Theorem 4.7. It remains to show that $E^{hG}_n$ and its associated spectral sequence are equivalent to the construction of Devinatz and Hopkins.

Let us first recall the construction of [10]. Let $\hat{L} = L_{K(n)}$ denote $K(n)_*$-localization. As in the introduction we will adopt the notation of [2] and [5] to denote the Devinatz-Hopkins fixed point spectra of $E_n$ by $E^{dhG}_n$. Let $R^+_G$ be the category of continuous finite left $G_n$-sets together with the left $G_n$-set $G_n$ to the category of spectra. For a finite $G_n$-set $S$, let $C^{*}_{\hat{S}}$ be the $E_\infty$-lift of [10], Theorem 3.2, of the cosimplicial diagram of spectra $C^{*}_{G_n/U}$ given in degree $j$ by

$$C^{j}_{\hat{S}} := \hat{L}(I \otimes E_n \wedge E^{(j)})$$

where $I \otimes E_n$ denotes the finite product of copies of $E_n$, one for each element of $S$, and the coface and codegeneracy maps are defined as in [10], 4.11. Let $\Pi^* C^{*}_{\hat{S}}$ be the cosimplicial replacement of $C^{*}_{\hat{S}}$ of Bousfield-Kan [4], XI §5. Since these operations are functorial in $S$, this yields a contravariant functor $F$ from $R^+_G$ to spectra which for finite $G_n$-set $S$ is defined by $F(S) := \text{Tot}(\Pi^* C^{*}_{\hat{S}})$, cf. [10], Definition 4.12. For the set $G_n$ with its natural left $G_n$-action, $C^{j}_{G_n}$ is defined as the cosimplicial object given in degree $j$ by

$$C^{j}_{G_n} := \hat{L}(E_n \wedge E^{(j+1)})$$

The spectrum $F(G_n)$ is defined to be $F(G_n) := \text{Tot}(\Pi^* C^{*}_{G_n})$. Finally, for a proper open subgroup $U$ of $G_n$, Devinatz and Hopkins set, cf. [10], Theorem 1:

$$E^{dhU}_n := F(G_n/U) = \text{Tot}(\Pi^* C^{*}_{G_n/U}).$$

In a second step the homotopy fixed points for a closed subgroup of $G_n$ are defined in [10]. Since $G_n$ is a $p$-adic analytic profinite group, it is possible to find a sequence of open normal subgroups of $G_n$

$$G_n = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_i \supseteq \cdots$$
with $\bigcap_i U_i = \{e\}$. After choosing such a sequence Devinatz and Hopkins define for a closed subgroup $G$ of $G_n$

$$E_{n}^{dhG} := \hat{L}(\text{holim}_i E_{n}^{dh(U_i,G)})$$

where holim denotes the homotopy colimit in the category $E$ of commutative $S^0$-algebras in the category of $S^0$-modules of [11].

In order to compare the different constructions of homotopy fixed points we use the functoriality of $F$. The canonical quotient map $G_n \to G_n/U$ defines a natural map of spectra $E_{n}^{dhU} = F(G_n/U) \to F(G_n)$. Since $F$ is a functor, this map is $G_n$-equivariant. Composition with the equivalence $F(G_n) \to E_n$ of [10], Theorem 1, provides a $G_n$-equivariant map $E_{n}^{dhU} \to E_n$. Moreover, as remarked in [10], p. 5, the maps in the colimit used to define $E_{n}^{dhG}$ are compatible with the $G_n$-action and induce a canonical map $E_{n}^{dhG} \to E_n$. Since it is $G_n$-equivariant and since $G$ acts trivially on $E_{n}^{dhG}$, this map factors through $E_{n}^{dhG} \to E_n$, i.e. through the $G$-fixed points of $E_n$. Furthermore there is a canonical map from the fixed points $E_{n}^{G} = \text{hom}_C(\ast, E_n)$ to the continuous homotopy fixed points $E_{n}^{hG}$ of Definition 1.3. Composition with the map $E_{n}^{dhG} \to E_{n}^{G}$ yields a map from $E_{n}^{dhG}$ to $E_{n}^{hG}$. Hence for any closed (or open) subgroup $G$ of $G_n$ we have constructed a canonical comparison map between the continuous homotopy fixed points fitting into a commutative diagram

$$
\begin{array}{ccc}
E_{n}^{dhG} & \rightarrow & E_{n}^{hG} \\
\downarrow & & \downarrow \\
E_n & & \\
\end{array}
$$

In order to show that the map $E_{n}^{dhG} \to E_{n}^{hG}$ is a weak equivalence for any closed $G$, we will prove that the commutative diagram 6 comes equipped with a natural isomorphism of spectral sequences between the homotopy fixed point spectral sequences of [10] converging to $(E_{n}^{dhG})^*Z$ and the continuous homotopy fixed point spectral sequence of Theorem 1.2 converging to $(E_{n}^{hG})^*Z$ for every spectrum $Z$. This will imply that the upper map in 5 is an equivalence and will complete the proof of Theorem 1.2.

Let $G$ be an arbitrary closed subgroup of $G_n$ and let $Z$ be a spectrum. Let $C_{G_n/g}^*$ be the cosimplicial diagram defined in degree $j$ by

$$C_{G_n/g}^j := \hat{L}(E_{n}^{dhG} \land E_{n}^{(j+1)}).$$

(Note that if $G = U$ is a proper open subgroup, this definition is weakly equivalent to the previous definition of $C_{G_n/U}^*$ in [3] by [10], Corollary 5.5.) Let $\hat{C}_{G_n/g}^*$ be the $E_\infty$-lift of $C_{G_n/g}^*$. By [10], Remark A.9, this cosimplicial spectrum is a canonical choice of a $K(n)_*$-local $E_n$-Adams resolution of $E_{n}^{dhG}$. The homotopy fixed point spectral sequence of [10] converging to $(E_{n}^{dhG})^*Z$ agrees with the spectral sequence obtained by mapping a spectrum $Z$ into this $K(n)_*$-local $E_n$-Adams resolution $\hat{C}_{G_n/g}^*$ of $E_{n}^{dhG}$. In [10], Proposition A.5, it is shown that this spectral sequence is canonically isomorphic to the Bousfield-Kan spectral sequence obtained by mapping $Z$ into the tower of fibrations of the total spectrum of the cosimplicial replacement $\Pi^*(\hat{C}_{G_n/g}^*)$ of $\hat{C}_{G_n/g}^*$. Hence we are reduced to compare the homotopy fixed point spectral sequence of Theorem 1.2(ii) to the spectral sequence obtained by mapping
$Z$ into the tower $\{\Tot_\ast \Pi^* (\bar{C}_{G_n/G}^\ast)\}$. We will compare them by proving that they are both canonically isomorphic to a third spectral sequence. The map $E_n^{dihG} \to E_n$ induces a map $\bar{C}_{G_n/G}^\ast \to \bar{C}_{G_n}^\ast$, where $\bar{C}_{G_n}^\ast$ is defined as in [4]. Moreover, there is a canonical map $\tau : \bar{C}_{G_n}^\ast \to \Pi_G^* E_n'$ to the continuous cosimplicial diagram defined in degree $j$ by

$$\Pi_G^* E_n' := \text{hom}_{\text{Sp}(\mathbb{S})}(G_{j+1}^+, E_n').$$

The cosimplicial map $\tau$ is the $E_\infty$-lift of the map that is defined in degree $j$ by

$$\tau_j(g_1, \ldots, g_{j+1}) : \tilde{L}(E_n^{(j+2)}) \to \tilde{L}(E_n^{(j+2)}) \to E_n \to E_n',$$

where $\mu$ denotes the ring spectrum multiplication map of $E_n$, see [10], (2.6) on p. 10. Furthermore the inclusion $G \subset G_n$ induces a map $\Pi_G^* E_n' \to \Pi_G^* E_n'$, where the right hand cosimplicial spectrum is defined in degree $j$ by

$$\Pi_G^* E_n' := \text{hom}_{\text{Sp}(\mathbb{S})}(G_{j+1}^+, E_n').$$

Hence we obtain a sequence of maps of cosimplicial diagrams

$$(6) \quad \bar{C}_{G_n/G}^\ast \to \bar{C}_{G_n}^\ast \to \Pi_G^* E_n' \to \Pi_G^* E_n'.$$

We remark that, since all these maps are $G_n$-equivariant, the images of the maps on the right lie in fact in the $G$-fixed points. Now we apply $\Tot \Pi^*$ to the cosimplicial diagrams and consider the map of spectral sequences that we obtain by mapping $Z$ into the towers of fibrations of total spectra induced by (6). We have seen that the spectral sequence obtained by mapping $Z$ into the tower $\{\Tot_\ast \Pi^* (\bar{C}_{G_n/G}^\ast)\}$ agrees with the homotopy fixed point spectral sequence of [10] converging to $(E_n^{dihG})^* Z$.

We claim that the the spectral sequence we get from mapping $Z$ into the tower $\{\Tot_\ast \Pi^* (\Pi_G^* E_n')\}$ is canonically isomorphic to the one of Theorem 5.8, (ii) defined by mapping $Z$ into the tower $\{\Tot_\ast \Pi_G^* E_n'\}$.

For, there is a canonical map of cosimplicial diagrams, see [4], XI §5.8,

$$\gamma : \Pi_G^* E_n' \to \Pi^* \Pi_G^* E_n'.$$

By [4], XI §7.5, or [10]. Proposition 4.16, it induces from the $E_2$-terms on an isomorphism of spectral sequences

$$\gamma^* : E_{\infty}^* (\Pi_G^* E_n') \to E_{\infty}^* (\Pi^* \Pi_G^* E_n').$$

This proves the claim and reduces our task to show that (6) induces an isomorphism of spectral sequences.

By Proposition 4.16 of [10], in order to show that (6) induces an isomorphism of spectral sequences after applying $\Tot \Pi^*$, it suffices to show that the map of cochain complexes of abelian groups induced by (6)

$$[Z, C_{G_n/G}^\ast]_f \to ([Z, C_{G_n}^\ast]_f)_{G} \to \text{Hom}_{\text{cts}}(G_n^+ + E_n^t Z)_{G} \to \text{Hom}_{\text{cts}}(G^+ + E_n^t Z)_{G}$$

is an isomorphism. Here the Hom-terms on the right denote continuous homomorphisms. By [10], 4.20, 4.21, 4.22 for open subgroups and Proposition 6.6, and by Theorem [5.1], this map is an isomorphism of cochain complexes. In particular, this map induces an isomorphism after taking cohomology of these complexes. Hence the composite of the maps of cosimplicial diagrams in (6) induces an isomorphism of spectral sequences. Summarizing the chain of arguments, by composition with
the inverse of $\gamma_*$ above, we have constructed a canonical and natural isomorphism of strongly convergent spectral sequences from $E_2$-terms on

$$H^*(G; E_n^*Z) \cong (E_{dhG}^n)^*Z$$

$$\cong H^*(G; E^n_*Z) \cong (E^{hG}_n)^*Z.$$

between the $K(n)_*$-local $E_n$-Adams spectral sequence converging to $(E_{dhG}^n)^*Z$ and the continuous homotopy fixed point spectral sequence converging to $(E^{hG}_n)^*Z$ of Theorem 1.2 (ii). In particular, since the map of spectra $E_{dhG}^n \to E^{hG}_n$ of (5) corresponds to the map of cosimplicial diagrams (6) by construction, we have proven that $E_{dhG}^n \to E^{hG}_n$ induces an isomorphism

$$(E_{dhG}^n)^*Z \cong (E^{hG}_n)^*Z$$

for every spectrum $Z$ and every closed subgroup $G$ of $G_n$. Now it suffices to apply the isomorphism of spectral sequences for $Z = S^0$ to conclude that the canonical map $E_{dhG}^n \to E_n^*$ is a stable equivalence of spectra for any closed subgroup $G$ of $G_n$.

Finally, the equivalences $E_{dhG_n}^n \simeq L_{K(n)}S^0$ and $E^{hG}_n \simeq E^{hG}_n$ give the canonical equivalence

$$E^{hG}_n \simeq L_{K(n)}S^0.$$ This completes the proof of Theorem 1.2.

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