Quantum liquids resulted from the models with four-fermion interaction

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A (nearly) perfect liquid discovered in the experiments with ultrarelativistic heavy ion collisions is investigated by studying the quark ensembles with four-fermion interaction as a fundamental theoretical approach. The comparative analysis of several quantum liquid models is performed and it results in the conclusion that the presence of gas—liquid phase transition is their characteristic feature. The problem of instability of the quark droplets of small quark number is discussed and argued to be rooted in the chiral soliton formation. An existence of mixed phase of the vacuum and baryon matter is proposed as a possible reason of the latter stability.

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Huge amount of data on relativistic heavy ion collisions obtained recently (perceptibly before the LHC began operating) in various experiments (first of all, at RHIC), were well understood and described in terms of concepts based on the equations of relativistic hydrodynamics. In particular, nearly ideal hydrodynamics, supplemented as needed by a variety of hadronic cascade models so as to correctly take into account a hadronic stage of the collision, quite successfully predicted an appearance of the radial and elliptic flows, their dependence on the collision, and confirmed predictions obtained by applying the past hydrodynamics.

At this point, it should be mentioned that the exploitation of such hydrodynamic notions dates back to the early fifties of the last century when L. D. Landau had developed the model of multiple particle production in collisions of hadrons and nuclei guided by the hydrodynamics in describing the evolution of nuclear matter that occurs right upon squeezing the latter at the collision point. Conceptually, this breakthrough idea had, at that time, not been particularly successful in applications because the nuclear matter had turned out to be a not very "suitable" liquid (as it was considered at the time), as the mean free path of nucleons in a produced system turned out comparable with the size of the latter.

A new generation of experiments carried out at much higher energies (reached at the LHC) quite remarkably confirmed predictions obtained by applying the past hydrodynamic ideas, rendering some of the latter, for instance, an observation of higher harmonics of flow induced by the fluctuations of original geometry, or a jet quenching effect initiated by heavy and light quarks, to be not only reliable experimental data, but also observations that bear a profound heuristic meaning.

The physics of ultrarelativistic heavy ion collisions needs to be described, at least, at the initial stage, in the language of quantum chromodynamics (QCD) for a strongly interacting system that is in a state far from equilibrium. At the same time, the data obtained by all three LHC collaborations, while being successfully described in terms of hydrodynamics, suggest a very fast thermalization, i.e., sufficient degree of local equilibrium or, rather, isotropization, since the equations of hydrodynamics do not include the temperature of produced matter with explicit collective properties, whose theoretical explanation at the macroscopic level is still far from clear. In recent years, there have appeared several scenarios of what could be the dynamics of a system transiting from the initial collision state to that when it becomes (almost) equilibrium. However, this problem is not discussed in the present paper. Instead, we focus on another aspect of the problem, namely, the smallness of the ratio \( \eta/s \) which corresponds to the presence of the strong interaction in a produced system or, in other words, small mean free path of its constituents, and try to understand the very nature of such interactions in a system, whose dynamics is governed by the coupling constant, which is likely not too large (at the LHC energies the running coupling constant in QCD as \( \alpha_s \sim 0.3–0.4 \) avoiding AdS/CFT duality (holographic QCD) arguments very popular at the moment).

Recently, such multiparticle (fermionic) systems are being intensively studied, particularly after they have successfully been experimentally realized as ultracold gas of fermionic atoms. This (unitary) Fermi-gas is a dilute system with short-range interaction, in which the s-wave scattering among fermions saturates a unitary limit for the cross section. Such a system is naturally characterized by the absence of any internal scale (conformality) and does not depend on the details of interaction. On
the other hand, the interaction in such a system needs to be described nonperturbatively, since no small parameter exists in the problem. An ideal liquid observed in heavy ion experiments is exactly another remarkable example of existence of such a strongly correlated fermionic system. An assumption of the existence of the lower bound for the ratio \( \eta/s \) of such fermionic systems formulated in \( [8] \), triggered even greater interest in their study after it was shown that \( \eta/s \) for the systems produced in heavy ion collisions and ultracold atomic gases turns out to be very small and close to each other in magnitude. It is interesting that the same value of that ratio is also predicted for low-energy electrons in graphene monolayers \( [9] \). The nature of these phenomena is, however, unclear which is seen from the behavior of, say, volume viscosity that for the quark–gluon systems turns out to be nonzero and can under certain circumstances (nearby phase transitions) be a significant source of dissipation, whereas for the unitary Fermi-gases it vanishes, just as a consequence of the scale invariance. The four-fermion (QCD-like) field theories still remain a most reliable source of quantitative information in the studies of the transport properties of strongly correlated systems and their thermodynamics, in particular, a chiral phase transition between massive hadrons and almost massless quarks. It is a thermodynamics that provides us with some general framework which lets one to understand how the properties of macroscopic matter and, in particular, its collective behavior, emerge from the laws that govern microscopic dynamics. The results of this work allows us to suppose with a sufficient, in our view, level of argumentation that the picture based on a complex collective behavior of quarks (antiquarks, gluons), which is expressed in the presence of vacuum condensates even under normal conditions, can be set by the nontrivial thermodynamic properties of vacuum, which eventually determine the observable properties of strongly interacting matter. In our opinion, this possibility was not sufficiently widely discussed and, even more so, used already at the initial stage of studies of the quark–gluon matter due to purely accidental circumstances. (As recent discussions of one of us (G.M.Z.) with E. Shuryak that have taken place during “Quark Matter 2012” have shown, similar thoughts are fully shared by him and, presumably, have occurred to him a bit earlier (see, for example, \( [10] \)).)

I. THERMODYNAMICS OF THE ENSEMBLE

In the present work some aspects of thermodynamical description of the quark ensemble with four-fermion interaction (generated as it is believed by strong stochastic gluon fields), with the Hamiltonian density

\[
\mathcal{H} = -\bar{q} \left( i \gamma \nabla + m \right) q - j^a_\mu \int d\mathbf{y} \left\langle A^a_\mu A^{b\nu}_\nu \right\rangle j^b_\nu, \tag{1}
\]

where \( j^a_\mu = \bar{q} \gamma^\mu \gamma^5 q \) is the quark current, with corresponding quark operators \( q, \bar{q} \), taken in spatial point \( \mathbf{x} \) (the variables with prime corresponds to the \( \mathbf{y} \) point), \( m \) is the current quark mass, \( t^a = \lambda^a/2 \) is the color gauge group \( SU(N_c) \) generators, \( \mu, \nu = 0, 1, 2, 3 \). We take the gluon field correlator \( \left\langle A^a_\mu A^{b\nu}_\nu \right\rangle \) in simple singlet in color form, with contact in time interaction (without retardation) \( [29] \)

\[
\left\langle A^a_\mu A^{b\nu}_\nu \right\rangle = G \delta^{ab} \delta_{\mu\nu} F(x - y), \tag{2}
\]

we do not include corresponding delta-function on time in this formula. This simple correlation function is a fragment of corresponding ordered exponent and besides the four-fermion interaction accompanied infinite number of multifermion vertexes arises. But for our purposes here it would be quite enough restrict ourself with this simple form. The mentioned above effective interactions appear in natural way by the coarse-grained description of the system with exploiting the corresponding averaging procedure, and having in mind that vacuum gluon field changed stochastically far enough (for example, in the form of instanton liquid, see \( [11] \)). But this elaboration of effective Hamiltonian resulting from the first principles of quantum chromodynamics (QCD) will be unessential for us, as it will be demonstrated below. The choice of correlation function in the simplest form with instantaneous interaction does not cause any problem at transiting from the Minkowski space to the Euclidean one and the formfactor \( F(x) \) is interpreted in a simple way as an interaction ‘potential’ of point-like particles. The correlation function itself looks, formally, as a gauge noninvariant object. \( [30] \). Nevertheless, there exists an effective way to significantly compensate for this shortcoming, if all similar ‘potentials’ are looked through, in some sense, (to be elucidated below). For example, this set will be wide enough, if it will turn out to be possible to confront two limits opposite in physics, starting from the form factor with a delta-like function in the coordinate space (the Nambu–Jona-Lasinio (NJL) model \( [12] \), the correlation length in this case is finite given the regularization is done) and ending up with a delta-like function.

FIG. 1: The most stable equilibrium angles \( \theta \) (in degrees) as function of momentum \( p \) in MeV. The solid line for NJL model, dashed one corresponds to the KKB model.

\[
\int \frac{d^2 p}{(2\pi)^2} \frac{\delta^{(2)}(\mathbf{x} - \mathbf{y})}{(p^2 + m^2)^2} = \frac{G}{4\pi}. \]

(3)
in the momentum space (it is clear that in this case the correlation length tends to infinity) analogous to that is well known in condensed matter physics, and named as the Keldysh model (KKB) [13].

It is worth to remark here that we will need only one of its properties, although [31], exceptionally important, which is related to fact that, due to the shape of the form factor, all the momentum integrations in the problem get factorized and effectively the problem becomes one-dimensional (only integration over energy are then in play). From this point of view, other models with an arbitrary form factor, including the NJL model, can then be represented as a superposition of elementary blocks obtained by using the KKB model. The utmost distributions mentioned above can be considered as a limiting case for the corresponding Gaussian correlators in the coordinate and momentum spaces, which, of course, look more realistic. The coupling constant scale $G$, that will turn out to be interesting for applications can be tuned by using corresponding PDG meson observables. Comparing the results obtained (by continuity arguments) one can make some conclusions about behavior of the system with practically any interaction potential.

We consider necessary to comment briefly on a case with a linear potential, which was always giving hope to discover an unusual feature in quark behavior thereby shedding some light on the nature of confinement. Meanwhile, at present it however appears that such a singular "potential" is even superfluous for our purposes, since the properties we are interested in are already revealed in the KKB model, which, in a sense, is like half way from the NJL model to that with a linear potential. Secondly, the quasiparticles in the model with a linearly growing potential can not basically be distinguished from those in, for example, the NJL model, provided an integrable infrared singularity in the former is eliminated (see, however, [16]). As a result the same massive objects appear without an anomalies in the energy spectrum. Additionally, the analysis shows that the multifermion contributions present in the problem in a general case can be reduced to the four-fermion interaction in an acceptable way by inserting the respective vacuum expectation values. In other words, even the Hamiltonian of the form (1) seems to capture the essential features of quark interactions.

It is believed that at sufficiently large interaction the ground state of the system transforms from a trivial vacuum $|0\rangle$ (the vacuum of free Hamiltonian) to the mixed state (with quark-anti-quark pairs with the opposite momenta and vacuum quantum numbers), which is presented as the Bogolyubov trial function (in that way some separate reference frame is introduced and chiral phase becomes fixed)

$$|\sigma\rangle = T|0\rangle, \quad T = \prod_{p,s} \exp[\varphi_p (a_{p,s}^+ b_{p,s}^+ + a_{p,s} b_{p,s})].$$

Here $a^+, a, b, b^+$ are the quarks creation and annihilation operators, $a|0\rangle = 0, b|0\rangle = 0$. The dressing transformation $T$ transmutes the quark operators to the creation and annihilation operators of quasiparticles $A = T a T^\dagger, B^+ = T b^+ T^\dagger$.

The termodynamic properties of a quark ensemble are known to be determined by solving the following problem. It is required to find such a statistical operator

$$\xi = \frac{e^{-\beta H_{\text{app}}}}{Z_0}, \quad Z_0 = \text{Tr} \{e^{-\beta H_{\text{app}}}\},$$

that at fixed mean charge

$$\overline{Q}_0 = \text{Tr}[\xi Q_0] = V \gamma \int d\vec{p} (n - \bar{n})$$

($Q_0 = \bar{q}^0 q$), and fixed mean entropy

$$\overline{S} = -\text{Tr}[\xi S] =$$

$$- V \gamma \int d\vec{p} \left[n \ln n + (1 - n) \ln (1 - n) + \bar{n} \ln \bar{n} + (1 - \bar{n}) \ln (1 - \bar{n})\right],$$

\[\text{FIG. 2: Three branches of solutions for dynamical quark mass in MeV for the KKB model as a function of momentum (MeV). The imaginary parts of the solutions are shown by dots.}\]

\[\text{FIG. 3: The group velocity of quasiparticles $v_F$ on the Fermi surface. The solid line describes the NJL model, the dashed one corresponds to the KKB model, the dots show the data for the KKB model tuned to the $\pi$-meson energy Figs.4–8.}\]
\( S = -\ln \xi \), the mean energy of the quark ensemble
\[ E = \text{Tr}\{\xi H\} \]

\( (H = \int dx \, \mathcal{H}) \) would be minimal. In other words we are interested in the minimum of the following functional
\[ \Omega = E - \mu \overline{Q}_0 - T \overline{S} \tag{6} \]

where \( \mu \) and \( T \) denote the Lagrangian multiplier for the chemical potential of the quark/baryon charge (which is in phenomenological considerations usually taken to be three times larger than the baryon one) and the temperature \((\beta = T^{-1})\), respectively. \( V \) is the volume the system is enclosed in \( d\mathbf{p} = dp/(2\pi)^3 \). \( \gamma = 2N_c \) (in the case of several quark flavors \( \gamma = 2N_cN_f \), where \( N_f \) is the flavor number), \( n = \text{Tr}\{\xi A^+A\}, \tilde{n} = \text{Tr}\{\xi B^+B\} \) are the components of the corresponding density matrix.

We restrict ourselves by considering the Bogolyubov–Hartree–Fock approximation in which the statistical operator is constructed on the basis of approximating effective Hamiltonian \( H_{\text{app}} \), quadratic in creation and annihilation operators for quasiparticles acting in the corresponding Fock space with a vacuum state \( |\sigma\rangle \). The average specific energy per quark \( w = E/(V\gamma) \) results in \cite{14}

\[ w = \int d\mathbf{p} \, p_0 - \int d\mathbf{p} \, (1 - n - \tilde{n}) \, p_0 \, \cos\theta - \frac{1}{2} \int d\mathbf{p} \, (1 - n - \tilde{n}) \sin(\theta - \theta_m) \, M(p) \tag{7} \]

where
\[ M(p) = 2G \int dq \, (1 - n' - \tilde{n}') \sin(\theta' - \theta'_m) \, F(p + q) \],
\[
\theta = 2\varphi, \quad p_0 = (p^2 + m^2)^{1/2}, \text{the primed variables, hereinafter correspond to the integration over momentum } q.\]

The auxiliary angle \( \theta_m \) is determined from the relation \( \sin\theta_m = m/p_0 \). The first term in Eq. \((7)\) is introduced in view of normalizing such that the energy of ground state is zero when an interaction is switched off. This constant is unessential for the following consideration and may be omitted, however it should be kept in mind that it will appear as a regularizer in singular expressions further down the text.

The most stable extremals of the functional \((7)\) are presented for comparison in Fig. \[11\] with the solid line for the NJL model and dashed one for the KKB model under normal conditions \((T = 0, \mu = 0)\). For delta-like potential in coordinate space (the NJL model) the expression \((7)\) diverges and to obtain the reasonable results the upper limit cutoff in the momentum integration \( \Lambda \) is introduced being one of the tuning model parameters along with the coupling constant \( G \) and current quark mass \( m \).

Below we use one of the standard sets of the parameters for the NJL model \cite{15}: \( \Lambda = 631 \text{ MeV}, \, G\Lambda^2/(2\pi^2) \approx 1.3, \, m = 5.5 \text{ MeV} \), whereas the KKB model parameters are chosen in such a way that for the same quark current masses the dynamical quark ones in both NJL and KKB models coincide at vanishing quark momentum. The moment \( p_0 \) (parameter) corresponds to the maximal attraction between quark and anti-quark. The value of this parameter inversely determines a characteristic size of quasiparticle. For the models considered it is of order of \( p_0 \sim (mM_q)^{1/2} \), where \( M_q \) is a characteristic quark dynamical mass, i.e. the quasiparticle size is comparable with the size of \( \pi \)-meson (Goldstone particle). It is a remarkable fact that the quasiparticle, as it is seen from Fig. \[11\] does not depend noticeably on the formfactor profile or, in other words, on the scale, but rather depends on the coupling constant. Using the properties of extremals the functional expression \((7)\) can be transformed to the form (see \cite{14})

\[ w = \int d\mathbf{p} \, p_0 - \int d\mathbf{p} \, (1 - n - \tilde{n}) \, p_0 + \frac{1}{4G} \int d\mathbf{p} d\mathbf{q} \, F(p + q) \, \tilde{M}(p)M(q) \tag{8} \]

where \( p_0 = [p^2 + M^2_q(p)]^{1/2} \) is the energy of quark quasiparticle with the dynamical quark mass
\[ M_q(p) = m + M(p) = m + \int d\mathbf{q} \, F(p + q) \, \tilde{M}(q) \tag{9} \]

\[ \begin{align*}
\text{FIG. 4: The compression module } K \text{ in MeV.} \\
\text{FIG. 5: The first sound velocity } C_1.
\end{align*} \]
Below we omit often the arguments of corresponding functions for the mass and quasiparticle energy. Varying the functional $\mathcal{F}$ with respect to the density of induced quasiparticle mass $M$ (in such a form it is convenient to take variational derivatives) we obtain the equation for dynamical quark mass as

$$M_q(p) = m + 2G \int d\tilde{q} (1 - n' - \tilde{n}') \frac{M_q}{P_0} F(p + q), \quad (10)$$

which corresponds exactly to the mean field approximation. In particular, under normal conditions ($T = 0$, $\mu = 0$) the dynamical quark mass in the NJL model is $M_q \sim 340$ MeV, whereas the dynamical quark mass of the KKB model is determined by the following equation

$$M(p) = 2G \frac{M_q(p)}{P_0}. \quad (11)$$

In practice, it is convenient to use an inverse function $p(M_q)$. In particular, in the chiral limit $M_q = (4G^2 - p^2)^{1/2}$, at $|p| < 2G$, and $M_0 = 0$ when $|p| > 2G$. In this case the quark states with momenta $|p| < 2G$ are degenerate in energy $P_0 = 2G$. Fig. 2 demonstrates three branches of the equation (11) solutions for dynamical quark mass. The dots show the imaginary part of solutions which are generated at the point where two real solution branches are getting merged.

## II. MEAN ENERGY AS A FUNCTIONAL OF QUANTUM LIQUID THEORY

The goal that we pursued while passing from the expression for specific energy $\mathcal{E}$ to the form $\mathcal{F}$ was to derive such a form that would easily be recognized as an energy functional of the Landau Fermi-liquid theory. Some aspects of this theory turn out to be interesting and useful to apply so as to be able to compare the results obtained in the NJL and KKB models. We will also discuss the first order phase transition, which is apparently typical for interacting fermions (relativistic Fermi-liquid).

Thus, the second term in (8) describes the contributions coming from quark and antiquark quasiparticles with occupation numbers $n$ and $\tilde{n}$ respectively. The unity in the expression $1 - n' - \tilde{n}'$ corresponds to the vacuum fluctuations of quarks and antiquarks (separately, each of their modes gives 1/2). The last term in (8) is due to the interaction of quasiparticles. The presence of contributions coming from antiparticles and relativistic form of the dynamics are those features which distinguish quark ensembles we study from the Fermi-liquids considered in condensed matter physics. The first variation of the functional $\mathcal{F}$ with respect to the particle (antiparticle) density leads as it should be to the energy of a quasiparticle:

$$\frac{\delta w}{\delta n} = P_0. \quad (12)$$

Consider first the case of zero temperature and discuss some aspects of filling up the Fermi sphere by quarks. Let us assume that the momentum distribution of quarks (antiquarks) is determined by the following expressions taken at the limit $\beta \to 0$

$$n = \left[e^{\beta(P_0 - \mu)} + 1\right]^{-1}, \quad \tilde{n} = \left[e^{\beta(P_0 + \mu)} + 1\right]^{-1}, \quad (13)$$

that is by the Fermi step function: $n = 1$, at $P_0 \leq \mu$ and $n = 0$ when $P_0 > \mu$. It is clear that for antiquarks $\tilde{n} = 0$. The quark density is determined by using the Fermi momentum:

$$\rho = \frac{2P_0^3}{6\pi^2}, \quad \rho = \frac{Q_0}{\sqrt{\nu}}, \quad (14)$$

with the quark chemical potential coinciding with the quasiparticle energy on the Fermi surface, as it follows from the relation (12)

$$\mu = \left[P_0^2 + M_q^2(P_0)\right]^{1/2}. \quad (15)$$

The group velocity of quasiparticles on the Fermi surface $v_f = \partial P_0/\partial p\big|_{P_0 = P_F}$, is shown in Fig. 3 as a function
of baryon (quark) density (by definition, the baryon density is three times smaller than the quark one \( \rho_B = \rho / 3 \)). A solid line describes the NJL model, while a dashed one corresponds to the KKB model. For comparison, there are points that show a version for the KKB model, when the parameters are tuned in such a way that the \( \pi \)-meson masses coincide in the NJL and KKB model (similar notation is used below in Figs. 4–5). Tending the group velocity to unity in the region of normal nuclear densities corresponds to the chiral symmetry restoration, when an induced quark mass tends to zero. In the KKB model, the group velocity turns to zero for quarks with momenta \( |p| < 2G \) in the chiral limit. The negative group velocities in the NJL model correspond to the regions of instability (see below). The points, which the group velocity vanishes in, give rise to the peaks in the density of states on the Fermi surface \( N_f \),

\[
N_f = \gamma \int d\mathbf{p} \delta (P_0 - \mu) = \frac{\gamma}{2\pi^2} P_\nu P_\nu^0 (1 + F_0)^{-1},
\]

\[
F_0 = \frac{M_q}{P_\nu} \frac{dM_q}{dP_\nu},
\]

where \( P_\nu^0 = P_0 \delta (|p| - P_\nu) \), \( N_f = d\rho / d\mu \). For more detail on how to determine the parameter \( F_0 \), see below. In an ideal gas, the interaction term in the functional (7) vanishes causing the derivative of the quark dynamical mass in the Fermi momentum to turn to zero: \( dM_q / dP_\nu = 0 \). Let us define the density of states of an ideal gas as

\[
\tilde{N}_f = \frac{\gamma}{(2\pi^2)^2} P_\nu P_\nu^0,
\]

then, the relation (15) can be brought to the form:

\[
N_f = \tilde{N}_f (1 + F_0)^{-1}.
\]

Another important characteristic is a compression coefficient

\[
K = 9\rho \frac{d\mu}{d\rho} = 3 \frac{P_\nu^2}{\mu} \left( 1 + F_0 \right).
\]

Fig. 4 demonstrates the data for the NJL and KKB models. They are consistent with specific values obtained for nuclear medium. One can also conclude that, in principle, these models admit a wide variety of state equations including sufficiently restrictive ones. The negative values of the compression coefficient are not allowed and correspond to the region of instability. The first sound velocity which is determined from the relation

\[
C_1^2 = K / \mu = \frac{v_\nu^2}{3} \left( 1 + F_0 \right),
\]

is shown in Fig. 5. When baryon densities are somewhat higher than that of normal nuclear matter, the sound velocity tends to its asymptotic value \( C_1 = 1/\sqrt{3} \), which is a natural manifestation of the chiral invariance restoration.

If the sound velocity of an ideal Fermi-gas \( C_1^2 = v_\nu^2 / 3 \) is introduced in a way similar to the \( \tilde{N}_f \) definition, then the expression (15) can be endowed with the form whose physical meaning is an equality of flow coming through the Fermi sphere of quasiparticles of (imaginary) ideal Fermi-gas and interacting Fermi-liquid (that is, there basically is a relativistic analogue of the Luttinger theorem [17])

\[
N_f \ C_1^2 = \tilde{N}_f \ C_1^2.
\]

The thermal conductivity at constant volume and low temperatures is given by the expression

\[
C_V = \frac{1}{3} \pi^2 N_f \ T.
\]

Fig. 6 shows the slope (the factor \( \frac{1}{3} \pi^2 N_f \) in Eq. (20), \( N_f = d\rho / d\mu \)), as a function of the baryon/quark density that demonstrates how informative it could be to measure the slope of a curve corresponding to the thermal conductivity. Yet another important characteristic of a Fermi-liquid is defined by a second variational derivative, which in the case of the functional (5) has only a scalar component:

\[
f_0 = \frac{\delta^2 \omega}{\delta n^2} = \frac{M_q}{P_0} \frac{dM_q}{dn}.
\]

For a Fermi-liquid at zero temperature, in particular, we have

\[
f_0 = \frac{2\pi^2}{\gamma P_\nu P_\nu^0} \frac{M_q}{P_\nu^2} \frac{dM_q}{dP_\nu}.
\]

For example, in the NJL model

\[
\frac{M_q}{P_\nu} \frac{dM_q}{dP_\nu} = -\frac{P_\nu}{P_\nu^0} \frac{1}{P_\nu^0 I + \pi^2 m/(GM_q^2),
\]

\[
I = \ln \frac{\Lambda + P_\nu^0}{P_\nu^0 + P_\nu^0} \frac{\Lambda}{P_\nu^0} + \frac{P_\nu^0}{P_\nu^0}.
\]
where \( P_\lambda^0 = P_0|_{|p|=\lambda} \). In the KKB model

\[
\frac{M_q}{P_q} \frac{dM_q}{dP_q} = -\frac{M M_q^2}{M^3 + m P_q^2}.
\]

In particular, in the chiral limit, when \( m = 0 \), we have \( (M_q/P_q)(dM_q/dP_q) = -1 \). The collective oscillation modes of a Fermi-liquid, the so-called zero sound (the collisionless mode) are found by using the parameter

\[
F_0 = \frac{\lambda}{\rho} f_0 = \frac{M_q}{P_q} \frac{dM_q}{dP_q},
\]

which is shown in Fig. 7. In particular, in the KKB model

\[
F_0 = -\frac{M M_q^2}{M M_q^2 + (P_0^0)^2 m} \geq -1.
\]

The zero sound oscillations are known to be determined by the solutions to the dispersion equation with a frequency parameter \( s \) (for details concerning this notation see the section devoted to the polarization operator) of the form:

\[
F_0 = \frac{s}{2} \log \frac{s + 1}{s - 1} - 1.
\]

When there is a repulsion in a system and the factor is positive \( F_0 > 0 \), the solutions to the dispersion equation \( s = \lambda + i \eta \) describe continuous oscillations (\( \eta = 0 \)). In the case of weak attraction, when \(-1 < F_0 < 0\), the damped oscillations of zero sound are possible with a purely imaginary frequency \( \lambda = 0 \), which is given by

\[
F_0 + 1 = s \arctan(1/\eta).
\]

In the case of strong attraction, when \( F_0 < -1 \), the solutions reside on a second sheet of the complex plane \( s \) and describe the damped oscillations, which are found from the solution to the equation

\[
F_0 + 1 = \eta [-\pi + \arctan(1/\eta)].
\]

It should, however, be recalled that these states of a Fermi-liquid are unstable (it will be discussed below).

In the chiral limit, all the states with momentum \( |p| < 2G \) are degenerate with respect to the chemical potential. \( M_q = (4G^2 - P^2)^{1/2}, P_0 = 2G \), when \( P_q < |p| < 2G, M_q = 0, P_0 = |p| \) if \( |p| < P_q \), and \( |p| > 2G \). Such a behavior of the chemical potential is a consequence of a rapid decrease of the dynamical quark mass with increasing Fermi momentum (see also (15)). From Eq. (3) it follows that the Fermi sphere is being filled as though from within. Those quarks with momenta smaller than the Fermi one \( |p| < P_q \) do not take part in forming a condensate. As a result, the quark dynamical mass can only decrease with the Fermi momentum increasing. In the NJL model, this dynamical mass is independent of the quark momentum because of the approximation assumed. In a more realistic case, as an analysis of the KKB model shows, this dependence must be taken into account.

It comes about that the pressure of some occupied states degenerate in the chemical potential almost coincides with that of vacuum (the pressure of a dilute Fermi-liquid).
gas) \((T=0)\)

\[ P = -\frac{dE}{dV} = -\mathcal{E} + \mu \rho , \]

where \(\mathcal{E} = E/V\) is the specific energy. Below we analyze respective data in a more detail, including the case with nonzero temperature. In the KKB model, the energy (and, hence, the pressure) of ensemble is a discontinuous functional of the quark current mass (see [11]). The integrands in (9) are in this case estimated as follows:

\[ p_0 - P_0 + \frac{1}{4G} M^2 \sim \frac{G m^2}{p^2} , \]

and, then, for the specific energy of ensemble we find a linearly diverging integral

\[ w \sim - \int \frac{dp}{2\pi^2} \frac{p^2}{G m^2} , \]

despite the fact that delta-like form factor in the momentum space is the strongest regularizer. It is paradoxical that any small value of the current mass \(m\) leads to the negative infinite energy of ensemble, while in the chiral limit the expression \(w|_{m=0}\) is well-defined. Even more so, a similar divergence occurs in the case of a delta-like form factor in the coordinate space. In the NJL model, this fact is concealed by introducing the cutoff momentum \(\Lambda\). By virtue of the singular pressure of ensemble in the KKB model mentioned above, it appears sensible to consider the relative pressure of quark ensemble in comparison with a (formally infinite) vacuum value. The pressure derivative in the ensemble density has the form:

\[ \frac{dP}{d\rho} = \rho \frac{d\mu}{d\rho} . \]

Therefore, by using an estimate given in (24), one can conclude that in the chiral limit the occupied states with momenta \(|p| < 2G\), are observed to also degenerate in the KKB model with respect to the pressure \((\mathcal{E} = 2G\rho, \mu = 2G)\). Beyond the chiral limit, the deviations are proportional to the quark current mass.

Now, determine some thermodynamic properties of a system and consider first the pressure of quark ensemble in a more detail:

\[ P = -\frac{dE}{dV} . \]

By definition, the volume derivative should be taken at the constant mean entropy, \(dS/dV = 0\). Treating this condition, one can, for example, extract the volume derivative of the chemical potential \(d\mu/dV\). However, this approach cannot be considered as acceptable because the mean charge \(Q_0\) might change. There is only one possibility to satisfy both conditions, namely, to introduce two independent chemical potentials for quarks and antiquarks. For the quark chemical potential we use a symbol \(\mu\) introduced earlier, whereas the antiquark chemical potential is taken opposite in charge and denoted as \(\bar{\mu}\). Then, for the quark and antiquark densities we have

\[ n = \frac{1}{e^{\beta (P_0 - \mu)} + 1} , \quad \bar{n} = \frac{1}{e^{\beta (P_0 + \bar{\mu})} + 1} . \]

In a general case, some nonequilibrium states of quark ensemble can also be described this way (formally, with loss of covariance, just like the case in the electrodynamics for an electron—positron gas). However, here we are only interested in a special situation when \(\bar{\mu} = \mu\). The partial derivative of the specific energy \(dw/dV\) can be represented in the following form:

\[ \frac{dw}{dV} = \int \frac{d\bar{p}}{2\pi^2} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) \left[ p_0 \cos \theta - 2G \times \sin (\theta - \theta_m) \left( n' + \bar{n}' - 1 \right) F \right] . \]

Dealing with the definition of an induced quark mass \([11]\) and expressing the trigonometric factors via the quark dynamical mass, the pressure of ensemble is found to be:

\[ P = -\frac{E}{V} - V 2N_c \int \frac{d\bar{p}}{2\pi^2} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) P_0 . \]

The condition for the mean charge conservation

\[ \frac{dQ_0}{dV} = \frac{\bar{Q}_0}{V} + V 2N_c \int \frac{d\bar{p}}{2\pi^2} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} - \frac{\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) = 0 , \]

gives the first equation that relates derivatives \(d\mu/dV\), \(d\bar{\mu}/dV\). Here, a regularized expression for the mean charge of quarks and antiquarks is assumed modulo respective vacuum contribution.

Manipulating the condition of constant mean entropy \(dS/dV = 0\) in a similar fashion, one can obtain the second equation of a system for the derivatives of chemical potentials:

\[ \int \frac{d\bar{p}}{2\pi^2} \ln \frac{n}{1-n} \frac{d\mu}{d\rho} - \int \frac{d\bar{p}}{2\pi^2} \ln \frac{\bar{n}}{1-\bar{n}} \frac{d\bar{\mu}}{d\rho} = \frac{S}{2N_c V^2} . \]

Substituting the expressions \(T \ln \frac{n}{1-n} = \mu - P_0\),
At low temperatures an antiquark contribution is small and thermodynamic description can be approximately constructed by using only the chemical potential $\mu$. If an antiquark contribution becomes significant, a thermodynamic description should be more sophisticated and should consider an influence of the chemical potential $\bar{\mu}$ with additional condition $\bar{\mu} = \mu$ and by using Eq. (25). Finally, for the pressure we have

$$P = -\frac{E}{V} + \frac{\bar{S} T}{V} + \frac{\bar{Q}_0 \mu}{V}.$$  \hspace{1cm} (27)

Here the thermodynamic potential $\Omega$ must satisfy the thermodynamic identity

$$\Omega = -PV = E - \mu \bar{Q}_0 - T \bar{S},$$  \hspace{1cm} (28)

and it does. At low temperatures an antiquark contribution is small and thermodynamic description can be approximately constructed by using only the chemical potential $\mu$. If an antiquark contribution becomes significant, a thermodynamic description should be more sophisticated and should consider an influence of the chemical potential $\bar{\mu}$ with additional condition $\bar{\mu} = \mu$. Fig. 11 shows the ensemble pressure $P$ in MeV/fm$^3$ as a function of the charge density $Q_0/3V$ for various temperatures. A lower curve is obtained at zero temperature. Next curves following upwards correspond to temperatures $T = 10$ MeV, $T = 50$ MeV (an upper curve) with a step $T = 10$ MeV. Let us also remember that, for the NJL model, the pressure of vacuum was estimated in [11] to be 40 to 50 MeV/fm$^3$, consistent with those obtained in the bag model. It was also demonstrated that there is a region of instability within a certain interval of the Fermi momenta generated by the anomalous behavior of the pressure $dP/dn < 0$ (see also [18]). Fig. 11 displays fragments of isotherms shown in Fig. 10 but now in different coordinates, in the form of chemical potential as a function of the pressure of ensemble. An upper curve is obtained at zero temperature. The isotherms following below are shown in steps of 10 MeV. A lowest curve is obtained at temperature 50 MeV. From the figure it is clearly seen that there are states on the isotherms which are in thermodynamic equilibrium. The pressure and chemical potential are the same for these states (see a characteristic Van der Waals triangle with intersecting curves). The equilibrium points obtained are shown in Fig. 11 by a dashed curve. The points at which a dashed curve intersects with isotherm give a boundary for a gas—liquid phase transition. The respective line $P = \text{const}$ cuts off nonequilibrium and unstable fragments of isotherm and describes a mixed phase. For the tuning parameters, the critical temperature turns out to be equal to $T_c \approx 46$ MeV with a critical charge density $Q_0 \approx 0.12$ charge/fm$^3$. Fig. 11 shows the isobars. Next to each curve the pressure in MeV/fm$^3$ is given. The vacuum pressure corresponds to approximately $\sim 50$ MeV/fm$^3$. It is possible to extrapolate isobars into the region of small charge densities, however, it is not really necessary. The figure clearly demonstrates presence of a dilute (a gas) and dense (a liquid) phases in the vicinity of the vacuum isobar.

FIG. 11: Isobars of NJL model. Pressure in MeV/fm$^3$ is indicated next to each curve. Vacuum pressure corresponds to approximately 50 MeV/fm$^3$.

FIG. 12: The dynamical quark mass $|M_q|$ (MeV) as a function of chemical potential $\mu$ (MeV) at the temperatures $T = 0$ MeV, ... , $T = 100$ MeV with spacing $T = 10$ MeV. The most right curve corresponds to zero temperature.
Its inverse value is given by the characteristic effective size of a quasiparticle $r_{\theta} = p_{\theta}^{-1}$. From the behavior of the quark dynamical mass as a function of temperature at small charge densities (see Fig. 13) one can conclude that a quasiparticle size grows with an energy increasing.

In [14] it was shown that if the quark chemical potential is defined as energy necessary to add (remove) one quasiparticle, $\mu = dE/dN$, then the chemical potential in vacuum coincides with the quark dynamical mass (see also [12], [15]). Therefore, it seems to be reasonable to consider a QCD phase diagram by starting from this value of the chemical potential, though formally it can be taken to be smaller than the quark dynamical mass, the way it was done in the pioneer paper [19]. In particular, by taking the chemical potential to be zero, we exactly reproduce a standard picture [15], [19]. The results obtained allowed to conjecture that the phase transition of (partial) restoration of the chiral invariance can already be realized in nature as a mixed phase of physical vacuum and baryonic matter. An indirect confirmation of this hypothesis can be seen in degenerate excited states of some baryons (see, for instance, [20]). It is, however, clear that the data presented, in particular, on the temperature and density of a critical point position should be understood as just an estimate. The critical temperature of a gas—liquid transition for nuclear matter extracted from experiment is estimated to be about 20 MeV. In addition, here (at $T = 0$) a gas component possesses the nonzero density of order of 0.01 of the normal nuclear density, whereas an observed value should correspond to physical vacuum, i.e., to zero baryon density. Although it should be noted that such an uncertainty is inherent in the other predictions of chiral symmetry restoration phase transition widely discussed in many papers, there are somewhere around two to six normal nuclear matter densities.

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\begin{align*}
\Pi^\Gamma(p, q) &= \int \frac{dk}{(2\pi)^3} \left( \hat{p} \Gamma + \hat{q} \right),
\end{align*}

where

\begin{align*}
\hat{p} \Gamma + \hat{q} &= \text{Tr} \left( S(k + p) \Gamma S(k - q) \Gamma \right),
\end{align*}

is a respective density of the polarization operator in channels $\Gamma = 1, i\gamma_5, \gamma\mu, \gamma_5\gamma\mu$, with the Green function of a quark with the dynamical mass $M_q$

\begin{align*}
S(k) &= \frac{1}{k + \tilde{\mu} - M_q(k)},
\end{align*}

$\tilde{\mu} = \mu \gamma^0$, where $p, q$ are the incoming and outgoing external momenta of quark quasiparticles. For our purposes, it will be enough to consider quasiparticles in the center of mass frame with momenta $p = q = Q/2$.

We analyze pseudoscalar and scalar channels, for which one can deduce

\begin{align*}
\Pi^{\pi, \sigma} &= N_c \int d^3k F(k) \left[ \frac{a + b \varepsilon}{(\varepsilon - (E_p + E_q))^2 + \varepsilon - (E_p + E_q)} + \frac{c}{\varepsilon - (E_p + E_q)} \right],
\end{align*}

\begin{align*}
a &= \left( E_p + E_q \right) \left[ 2 - n_p - n_q \right] \left[ \frac{Q^2/4 - k^2 \mp M_p M_q}{E_p E_q} - 1 \right],
\end{align*}

\begin{align*}
b &= \left[ n_q - n_p \right] \left[ \frac{Q^2/4 - k^2 \mp M_p M_q}{E_p E_q} - 1 \right],
\end{align*}

\begin{align*}
c &= \left[ n_p - n_q \right] \left[ \frac{Q^2/4 - k^2 \mp M_p M_q}{E_p E_q} + 1 \right],
\end{align*}

\begin{align*}
\end{align*}
where \( \varepsilon = p_0 - q_0 \) is the transferred energy, \( M_p = M_q(k + Q/2) \), \( E_p = (p^2 + M_q^2(p))^{1/2} \), and for the kinematics chosen \( p = k + Q/2 \). \( n_p \) is an occupation number for a quasiparticle with momentum \( k + Q/2 \). In particular, for zero temperature we have the Fermi step: \( n_p = n(E_p - \mu) \).

Similar notation is also used for a quasiparticle with momentum \( q = k - Q/2 \). Let us recall that the quantity \( F(k) \) is a form factor. The first term in Eq. (32) corresponds to the quark and anti-quark contributions, while the second one, to that coming from a quark and a hole. It is easy to see that in, the KKB model, \( F \) ermi sphere is filled from the inside, and quarks with momenta smaller than \( \epsilon \) ermi sphere turns out to be insignificant. For instance, for the quark and hole excitations in the Fermi sphere, it should be noted that for a quark ensemble we consider the medium properties are for the most part governed by the term \( A^{\pi,\sigma} \) which is responsible for a quark—anti-quark condensate, contrary to what we have in the condensed matter physics, where a dominant factor is known to be given by \( B^{\pi,\sigma} \). Therefore, the results obtained exclusively by using an analogy with the condensed matter physics should be taken with a grain of salt. In particular, in the present paper we have analyzed in some detail a situation with zero sound description taken as an example illustrating this point. With no antiquarks influence, zero sound would represent in itself highly damped oscillations described by the only scalar parameter \( F_0 \). A more careful analysis shows that, for example, in the case of KKB model there is, in addition to a paired quark—antiquark state, a stable branch of quark and hole excitations in the Fermi sphere. In the NJL model, by performing numeric integration one can observe a regular mass convergence for \( \pi \)- and \( \sigma \)-mesons when baryon density increases, which is related to the restoration of chiral invariance. An influence of a bound quark—hole state in the Fermi sphere turns out to be insignificant. For instance, for the densities of order of normal nuclear matter the dispersion law changes by several MeV, when the quark and hole momentum differs more than 200 MeV, but, as before in the KKB model, there are no damped oscillations.

One of the drawbacks of the models studied so far is considered to be the lack of quark confinement, which is understood as an impossibility to observe single-particle states with a regular (real) dispersion law. We see that if one argues formally, then one quasiparticle can freely propagate, indeed. But adding just another quasiparticle can dramatically change the picture due to existence of a bound channel. For example, in the KKB model the bound states in scalar, pseudoscalar, vector, and axial-

As it follows from Eq. (32), the polarization operator in the NJL model is defined by integrating over the running quark momentum \( k \) and represented as a superposition of branches of the KKB model, which has already been mentioned in the introduction. For the kinematics we chose, the most significant contributions turned out to be those coming from the terms denoted as \( a \) and \( c \) in Eq. (32). Integrating the angle out (it is more convenient to express the final formula by going to a nonsymmetric integration point, since in that case the corrections come about to be negligibly small, as analysis of rather bulky formulae, derived while integrating at a symmetric point,
vector channels appear at any quasiparticles momenta (details can be found in [21]). In particular, the bound state energy, obtained by using the dispersion equation

\[ E(p + q) = 2G_\Pi = 0, \]

in \( \pi \) and \( \sigma \) channels has the form

\[ \varepsilon^2_{\pi,\sigma} = \left( \frac{E_p + E_q}{E_p E_q} \right)^2 - 2G \left( \frac{E_p E_q \pm M p M q - p q}{E_p E_q} \right), \]

(upper sign corresponds to the pseudoscalar channel). The first term in this expression is the energy of free particles motion. The second one is strictly positive at any momenta \( p \) and \( q \) and plays a role of binding energy in \( \pi \) and \( \sigma \) channels (only in the configuration of \( q = p \) the binding energy vanishes for a scalar channel). Similarly, one can show that a quark and antiquark are always coupled in vector and axial—vector channels, i.e., the scattering matrix is always singular, except for a tensor channel, where it is trivial because of the initial interaction Hamiltonian taken as a product of two color currents. Similar bound states exist in a diquark channel. As a consequence, the states with any number of quark quasiparticles turn out to be bound in channels we have just mentioned. The same behavior is observed in the NJL model, where the bound states appear for the quarks with momenta somewhat lower than the cutoff momentum; that is, within this momentum interval the scattering matrix is also singular, as in the KKB model. It appears that the bound states are due to rather the fermion correlations than physical impact of the field, which is familiar in the quantum electrodynamics. Then, in order to understand what may be happening beyond the cutoff momentum one has to study respective nonlocal models.

IV. TRANSITION LAYER BETWEEN GAS AND LIQUID

The concept of a mixed phase of physical vacuum and baryonic matter would obtain the substantial confirmation if we are able to demonstrate an existence of the boundary (transition) layer where a transformation of the quark ensemble from one aggregate state to another takes place. As it was argued above the indicative characteristic to explore a homogeneous phase (at finite temperature) is the mean charge (density) of ensemble. All the other characteristics, for example, a chiral condensate, dynamical quark mass, etc. can be reconstructed if one knows the ensemble mean charge. So, here we analyze a specific case of the surface (transition) layer at zero temperature.

We assume that the quark ensemble parameters in a gaseous phase are approximately the same as those at zero charge \( \rho_q = 0 \), i.e. as in vacuum (minor differences in pressure, chemical potential and quark condensate are neglected). The dynamical quark mass develops here the maximal value, and for the parameter choice standard for the NJL model it is \( M = 335 \, \text{MeV} \). Then as the Van der Waals diagram shows a second phase (a liquid), being in equilibrium with a gas phase is characterized the density \( \rho_l = 3 \times 0.185 \, \text{charge/fm}^3 \) (by some reason which becomes clear below we correct it to take the value \( \rho_l = 3 \times 0.157 \, \text{charge/fm}^3 \)). The detached factor 3 here links again the magnitudes of quark and baryon matter densities. The quark mass is found to be approximately \( M \approx 70 \, \text{MeV} \) in this phase. Hereafter we focus on describing of two adjoining semi-infinite layers, i.e. assuming a plane symmetry of the corresponding one-dimensional problem).

The precursor experience teaches that an adequate description of heterogeneous states can be constructed on the basis of the mean field approximation [22]. Specifically, in our particular case it means making use the corresponding effective quark-meson Lagrangian [23] (the functional of Ginzburg-Landau type)

\[
\mathcal{L} = -\bar{q} (\hat{\partial} + M) q - \frac{1}{2} (\partial_\mu \sigma)^2 - U(\sigma) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m_v^2}{2} V_\mu V^\mu - g_\sigma \bar{q} q \sigma + ig_v \bar{q} \gamma_\mu q V_\mu, \tag{33}
\]

where

\[
F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu, \quad U(\sigma) = \frac{m_\sigma^2}{2} \sigma^2 + \frac{b}{3} \sigma^3 + \frac{c}{4} \sigma^4, \]

and \( \sigma \) is the scalar field, \( V_\mu \) is the field of vector mesons, \( m_\sigma, m_v \) are the masses of scalar and vector mesons and \( g_\sigma, g_v \) are the coupling constants of quark-meson interaction. The \( U(\sigma) \) potential includes the nonlinear \( \sigma \) field interaction terms up to the fourth order, for example. For simplicity we have not included the contributions coming from the pseudoscalar and axial-vector mesons.

The meson part of such a Lagrangian should be self-consistently treated by considering the corresponding
quark loops. Here we do not see any reason to go beyond the well elaborated and reliable one loop approximation \[33\], although recently the considerable progress has been reached (as we mentioned at the beginning of this paper) in scrutinizing the non-homogeneous quark condensates by application of the powerful methods of exact integration \[24\]. Here we believe it is more practical to adjust phenomenologically the effective Lagrangian parameters basing on the transparent physical picture. It is easy to see that handling \[33\] one loop approximation we come, in actual fact, to the Walecka model \[23\] but adopted for the quarks. In what follows we are working with the designations of that model and do hope it does not lead to the misunderstandings.

In the context of our paper we propose to interpret Eq. (33) in the following way. Each phase might be considered, in a sense, with regard to another phase as an excited state which requires the additional (apart from a charge density) set of parameters (for example, the meson fields) for its complete description, and those are characterizing the measure of deviation from the equilibrium state. Then the crucial question becomes whether it is possible to adjust the parameters of effective Lagrangian \[33\] to obtain the solutions in which the quark field interacts with the meson parameters \[24\] to the transparent physical picture. For all that the density of the filled-up state ensemble should asymptotically approach the equilibrium value of \(n_T\) and should turn to the zero value in the gas phase (vacuum).

The scale inherent in this problem may be assigned with one of the mass referred enlisted in the Lagrangian \[33\]. In particular, we bear in mind the dynamical quark mass in the vacuum \(M\). Besides, there are another four independent parameters in the problem and in order to compare them with the results of studying a nuclear matter we employ the form characteristic for the (nuclear) Walecka model

\[
\begin{align*}
C_s &= g_\sigma \frac{M}{m_\sigma}, \quad C_v = g_v \frac{M}{m_v}, \quad \bar{b} = \frac{b}{g_\sigma^2 M}, \quad \bar{c} = \frac{c}{g_\sigma^2}.
\end{align*}
\]

Parameterizing the potential \(U(\sigma)\) as \(b_\sigma = 1.5 m_\sigma^2 (g_\sigma/M), c_\sigma = 0.5 m_\sigma^2 (g_\sigma/M)^2\) we come to the sigma model whereas the choice \(b = 0, c = 0\) results in the Walecka model. As to standard nuclear matter application the parameters \(b\) and \(c\) demonstrate vital model dependent character and are quite different from the parameter values of sigma model. Truly, in that case their values are also regulated by additional requirement of an accurate description of the saturation property. On the other hand, for the quark Lagrangian \[33\] we could intuitively anticipate some resemblance with the sigma model and, hence, could introduce two dimensionless parameters \(\eta\) and \(\zeta\) in the form of \(b = \eta \, b_\sigma, c = \zeta^2 \, c_\sigma\) which characterize some fluctuations of the effective potential. Then the scalar field potential is represented as follows

\[
U(\sigma) = \frac{m_\sigma^2}{8} \frac{g_\sigma^2}{M^2} \left(4 \frac{M^2}{g_\sigma^2} + 4 \frac{M}{g_\sigma} \eta \, \sigma + \zeta^2 \sigma^2\right) \sigma^2.
\]

The meson and quark fields are determined by solving the following system of the stationary equations

\[
\Delta \sigma - m_\sigma^2 \sigma = b \sigma^2 + c \sigma^3 + g_\sigma \, \rho_\pi, \quad \Delta V - m_v^2 V = -g_v \rho, \quad (\vec{\nabla} + \hat{M}) q = (E - g_v \, V) q,
\]

where \(\hat{M} = M + g_\sigma \sigma\) is the running value of dynamical quark mass, \(E\) stands for the quark energy and \(V = -i v_\pi\). The density matrix describing the quark ensemble at \(T = 0\) has the form

\[
\xi(x) = \frac{P_F}{(2\pi)^3} \frac{dp}{2\pi} q_p(x) \bar{q}_p(x),
\]

in which \(p\) is the quasiparticle momentum and the Fermi momentum \(P_F\) is defined by the corresponding chemical potential. The densities \(\rho_\pi\) and \(\rho\) at the right hand sides of Eq. (33) are by definition

\[
\rho_\pi(x) = Tr \{\xi(x), 1\}, \quad \rho(x) = Tr \{\xi(x), \gamma_3\}.
\]

Here we confine ourselves to the Thomas–Fermi approximation while describing the quark ensemble. Then the densities which we are interested in are given with some local Fermi momentum \(P_F(x)\) as

\[
\rho(x) = \gamma \int \frac{dp}{(2\pi)^3} \frac{\gamma}{\delta^2} \frac{P_F^3}{P_F}, \quad \rho_\pi(x) = \gamma \int \frac{dp}{(2\pi)^3} \frac{\gamma}{\delta^2} \frac{P_F^3}{P_F} \left(1 + \lambda^2 \right)^{1/2} - \frac{\lambda^2}{2} \ln \left[\left(1 + \lambda^2 \right)^{1/2} + 1 \right] - \lambda M \left(1 + \lambda^2 \right)^{1/2} - 1 \right] \right\},
\]

where \(\gamma\) is a quark gamma-factor which for one flavour is \(\gamma = 2 N_c\) \((N_c\) is the number of colours), \(E = (p^2 + M^2)^{1/2}\) and \(\lambda = \hat{M} / P_F\). Under assumption adapted the ensemble chemical potential is constant and therefore, a local value of the Fermi momentum is defined by the running value of dynamical quark mass and vector field as

\[
\mu = M = g_v V + \left(P_F^2 + M^2\right)^{1/2}.
\]

Now we should tune the Lagrangian parameters in Eq. (33). For asymptotically large distances (in a homogeneous phase) we may neglect the gradients of scalar and vector fields and the equation for scalar field of the system \[34\] leads to the first equation that relates the parameters \(C_s, C_v, b, c\) as

\[
\frac{M^2 \bar{M} \bar{M}}{C_s^2} + b \bar{M} \bar{M} + c \bar{M} \bar{M} = -\rho_\pi.\]

\[
\frac{M^2 \bar{M} \bar{M}}{C_s^2} + b \bar{M} \bar{M} + c \bar{M} \bar{M} = -\rho_\pi.\]
The dynamical quark mass should coincide (or exceed) the liquid density (as it was argued above) to decrease the dynamical quark mass to the value which retains the identity [28] for the chemical potential looks like

\[
M = \frac{C_v^2 \rho}{M^2} + \left( P_f^2 + M^2 \right)^{1/2}.
\]  

(38)

If we know the liquid density we obtain the Fermi momentum \( P_f = 346 \) MeV from [39] and applying the identities [37], [38] we have for the particular case \( b = 0, c = 0 \) that \( C_v^2 = 25.3, C_s^2 = -0.471 \), i.e. the vector component \( C_v^2 \) is small (compared to \( C_s^2 \)) and has negative value that is unacceptable. Apparently, it looks necessary to abandon the contribution coming from the vector field or to reduce the dynamical quark mass \( M \) up to the value that retains the identity [28] with positive \( C_v^2 \) or even zero value. In the gaseous phase the dynamical quark mass can also be corrected to the value larger than the vacuum value. It is clear that in the situation of the liquid with the density \( \rho_i = 3 \times 0.185 \) ch/fm\(^3\) the dynamical quark mass should coincide (or exceed) \( M = 346 \) MeV in the gaseous phase. However, here we correct the liquid density (as it was argued above) to decrease its value up to \( \rho_i = 3 \times 0.157 \) ch/fm\(^3\) which is quite acceptable in the nucleation capacity. In fact, this possibility can be simply justified by another choice of the NJL model parameters. Thus, we obtain at \( M = 70 \) MeV and \( b = 0, c = 0 \) that \( C_s^2 = 28.4, C_v^2 = 0.015 \), i.e. we have a small but positive value for the vector field coefficient. At the same time, being targeted here to estimate the surface tension effects only we do not strive for the precise fit of parameters. In the Walecka model these coefficients are \( C_s^2 = 266.9, C_v^2 = 145.7 \), \( b = 0, c = 0 \). Moreover, there is another parameter set with \( C_s^2 = 64, C_v^2 \approx 0 \) [29] but it is rooted in an essential nonlinearity of the sigma-field due to the nontrivial values of the coefficients \( b \) and \( c \). The option (formally unstable) with negative \( c \) (b) has also been discussed.

The coupling constant of scalar field is fixed by the standard (for the NJL model) relation between the quark mass and the \( \pi \)-meson decay constant \( g_\pi = M/f_\pi \) (we put \( f_\pi = 100 \) MeV) although there is no any objection to treat this coupling constant as an independent parameter. As a result of all agreements done we have for the \( \sigma \)-meson mass \( m_\sigma = g_\sigma M/C_s \). In principle, we could even fix the \( \sigma \)-meson mass and coupling constant \( g_\sigma \) but all relations above mentioned lead eventually to quite suitable values of the \( \sigma \)-meson mass as will be demonstrated below. The vector field plays, as we see, a secondary role because of the small magnitude of constant \( C_v \). Then taking the vector meson mass as \( m_v \approx 740 \) MeV (slightly smaller value than the mass of \( \omega \)-meson because of simple technical reason only) we calculate the coupling constant of vector field from the relation similar to the scalar field \( m_v = g_v M/C_v \). Amazingly, its value comes about steadily small being compared to the value characteristic for the NJL model \( g_v = \sqrt{6} g_\sigma \). However, at the values of constant \( C_v \) which we are interested in it is very difficult to maintain the reasonable balance and to be specific in this paper we prefer to choose the massive vector field. Actually, it is unessential because we need this parameter (as we remember) to estimate the vector field strength only.

The key point of our interest here is the surface tension coefficient [20] which can be defined as

\[
\sigma_s = 4\pi \rho^2_0 \int_{-\infty}^{\infty} dx \left[ \mathcal{E}(x) - \frac{\xi_\sigma}{\rho_i} \rho(x) \right].
\]

(39)

The parameter \( \rho_0 \) will be discussed in the next section at considering the features of quark liquid droplet, and for the present we would like to notice only that for the parameters considered its magnitude for \( N_f = 1 \) is around \( \rho_0 = 0.79 \) fm. Recalling the factor \( 3^{1/3} \) which connects the baryon and quark numbers, we find the magnitude \( \rho_0 = 3^{1/3} \cdot 0.79 \approx 1.14 \) fm in full agreement with the magnitude standard for the nuclear matter calculations (in the Walecka model) \( \rho_0 = 1.1 - 1.3 \) fm.

In order to proceed we calculate \( \mathcal{E}(x) \) in the Thomas–Fermi approximation as

\[
\mathcal{E}(x) = \gamma \int_{\mathcal{P}_F(x)} \frac{d\mathcal{P}}{(2\pi)^3} \left[ p^2 + M^2(x) \right]^{1/2} + \frac{1}{2} g_\sigma \rho(x) V(x) - \frac{1}{2} g_\sigma \rho_\sigma(x) \sigma(x) .
\]

And to give some idea for the 'setup' prepared we present here the characteristic parameter values for some fixed \( b \) and \( c \) with \( \rho_i = 3 \times 0.157 \) ch/fm\(^3\). In the liquid phase they are \( M = 70 \) MeV \( (P_F = 327) \) MeV and \( \epsilon_i = 310.5 \) MeV (index \( l \) stands for a liquid phase and \( \epsilon(x) = \mathcal{E}(x) / (\rho(x)) \) defines the density of specific energy). Both relations [37] and [38] are obeyed by this state. There exist the solution with larger value of quark mass \( M = 306 \) MeV.
(\(P_F = 135\text{ MeV}\)) (we have faced the similar situation in the first section dealing with the gas of quark quasiparticles) and \(e = 338\text{ MeV} \sim e_g\) (\(e_g\) is the specific energy in the gas phase) that satisfies both equations as well. The specific energy of this solution occurs to be larger than specific energy of the previous solution. It is also worthwhile to mention the existence of an intermediate state corresponding to the saturation point with the mass \(M = 95\text{ MeV}\), \(P_F = 291\text{ MeV}\) and \(e = 306\text{ MeV}\). Obviously, it is the most favorable state with the smallest value of specific energy (and with the zero pressure of quark ensemble), and the system can fall into this state only in the presence of a significant vector field. This state (already discussed in the first section) corresponds to the minimal value of chemical potential \((T = 0)\) and can be reached at the densities typical for the normal nuclear matter. However, Eq. (38) is not valid for this state.

Two another parameters \(\eta, \zeta\) are fixed by looking through all the configurations in which the solution of equation system (34) with stable kink of the scalar field does exist and describes the transition of quarks from the gaseous phase to the liquid one. First, it is reasonable to scan the \(\eta, c\) \((\zeta = c\eta)\)-plane, in order to identify the domain in which the increase of specific energy \(\mathcal{E} - \mathcal{E}_t \rho/\rho_t \leq 0\) is revealed at running through all possible states which provide the necessary transition (without taking into account the field gradients). In practice one need to follow a simple heuristic rule. The state with \(P_F \sim 1\text{ MeV}\) (i.e. \(e\) and the corresponding \(\rho\)) and the state of characteristic liquid energy \(\mathcal{E}_t\) (together with \(\rho_t\)) should be compared while scanning the Lagrangian parameters \(\eta\) and \(c\). Just the domain in which they are commensurable could provide us with the solutions which we are interested in and Fig. 5 shows its boundary. The curve could be continued beyond the value \(\eta = 2.5\) but the values of corresponding parameter \(\eta\) are unrealistic and not shown in the plot.

We calculate the solution of equation system (34) numerically by the Runge–Kutta method with the initial conditions \(\sigma(L) \approx 0, \sigma'(L) \approx 0\) imposed at the large distance \(L \gg t\) where \(t\) is a characteristic thickness of transition layer (about 2 fm). Such a simple algorithm occurs quite suitable if the vector field contribution is considered as a small correction (what just takes place in the situation under consideration) and is presented as

\[
V(x) = \frac{1}{2m_v} \int_{-L}^{L} dz \ e^{-m_v|x-z|} g_v \rho(z),
\]

where the charge (density) \(\rho\) is directly defined by the scalar field. We considered the solutions including the contribution of the vector field as well and the corresponding results confirm the estimates obtained.

Rather simple analysis shows the interesting solutions are located along the boundary of discussed domain. Some of those are depicted in Fig. 16 as the dots. Fig. 17 shows the stable kinks of \(\sigma\)-field with the parameter \(c = 1.1\) for two existing solutions with \(\eta \approx 0.977\) \((m_\pi \approx 468\text{ MeV})\) (solid line) and \(\eta \approx 1.813\) \((m_\pi \approx 690\text{ MeV})\) (dashed line). For the sake of clarity we consider the gas (vacuum) phase is on the right. Then the asymptotic value of \(\sigma\)-field on the left hand side \((\sigma \approx 80\text{ MeV})\) corresponds to \(M = 70\text{ MeV}\). The thickness of transition layer for the solution with \(\eta \approx 0.977\) is \(t \approx 2\text{ fm}\) whereas for the second solution \(t \approx 1\text{ fm}\).

Characterizing the whole spectrum of the solutions obtained we should mention that there exist another more rigid (chiral) kinks which correspond to the transition into the state with the dynamical quark mass changing its sign, i.e. \(M \rightarrow -M\). In particular, the kink with the canonical parameter values \(\eta = 1, c = 1\) is clearly seen (marked by the star in Fig. 16) and its surface tension coefficient is about \(2m_\pi\) \((m_\pi\) is the \(\pi\)-meson mass). The most populated class of solutions consists of those having the meta-stable character. The system comes back to the starting point (after an evolution) pretty rapidly,
and usually the $\sigma$-field does not evolve in such an extent to reach the asymptotic value (which corresponds to the dynamical quark mass in the liquid phase $M = 70$ MeV). Switching on the vector field changes the solutions insignificantly (for our situation with small $C_v$ it does not exceed 2 MeV in the maximum).

The surface tension coefficient $u_s$ in MeV for the curve of stable kinks with parameter $\eta \leq 1.2$ as the function of parameter $c (\zeta = c\eta)$ is depicted in Fig. 18. The $\sigma$-meson mass at $c \approx 0$ is $m_\sigma \approx 420$ MeV and changes smoothly up to the value $m_\sigma \approx 500$ MeV at $c \approx 1.16$ (the maximal value of the coefficient $c$ beyond which the stable kink solutions are not observed). In particular, $m_\sigma \approx 450$ MeV at $c = 1$. Two kink solutions with $c = 1.1$ for $\eta \approx 0.977$ and for $\eta \approx 1.813$ (shown in Fig. 17) and the second one is not shown in Fig. 18 have the tension coefficient values $u_s \approx 35$ MeV and $u_s \approx 65$ MeV, correspondingly. The maximal value of tension coefficient for the normal nuclear matter does not exceed $u_s = 50$ MeV. The nuclear Walecka model claims the value $u_s \approx 19$ MeV [24] as acceptable and calculable. The reason to have this higher value of surface tension coefficient for quarks is rooted in the different values of the mass deficit. Indeed, for nuclear matter it does not exceed $M \approx 0.5M$ albeit more realistic values are considered around $M \approx 0.7M$ and for the quark ensemble the mass deficit amounts to $\tilde{M} \approx 0.3M$. We are also able to estimate the compression coefficient of quark matter $K$ which occurs significantly larger than the nuclear one. Actually, we see quite smooth analogy between the results obtained and the results of bag soliton model [27].

The thermodynamic treatment developed in the present paper allows us to formulate the adequate boundary conditions for the bag in physical vacuum and to diminish considerably the uncertainties in searching the true soliton Lagrangian. We believe, it was also shown here, that to single out one soliton solution among others (including even those obtained by the exact integration method [24]), which describes the transitional layer between two media, is not easy problem if the boundary conditions above formulated are not properly imposed.

V. DROPLET OF QUARK LIQUID

The results of previous sections have led us to put the challenging question about the creation and properties of finite quark systems or the droplets of quark liquid which are in equilibrium with the vacuum state. Thus, as a droplet we imply the spherically symmetric solution of the equation system (34) for $\sigma(r)$ and $V(r)$ with the obvious boundary conditions $\sigma'(0) = 0$ and $V'(0) = 0$ in the origin (the primed variables denote the first derivatives in $r$) and rapidly decreasing at the large distances $\sigma \to 0$, $V \to 0$, when $r \to \infty$.

A quantitative analysis of similar nuclear physics models which includes the detailed tuning of parameters is usually based on the comprehensive fitting of available experimental data. This way is obviously irrelevant in studying the quark liquid droplets. This global difficulty dictates a specific tactics of analyzing. We propose to start, first of all, with selecting the parameters which could be worthwhile to play a role of physical observables. Naturally, the total baryon number which phenomenologically (via factor 3) related to the number of valence quark in an ensemble is a reasonable candidate for this role. Besides, the density of quark ensemble $\rho(r)$, the mean size of droplet $R_0$ and the thickness of surface layer $t$ look like suitable for such an analysis.

It is argued above that the vector field contribution is negligible because of the small value of coefficient $C_v$ compared to the $C_s$ magnitude, and we follow this conclusion (or assumption) albeit understand it is scarcely justified in the context of finite quark system. Thus, we put down $g_c = 0$, $V = 0$ in what follows and it simplifies all the calculations enormously.

Fig. 19 shows the number of solutions ($\sigma$-field in MeV) to the system (34) at $N_f = 1$ and Fig. 20 presents the corresponding distributions of ensemble density $\rho$. 

---

**FIG. 19**: $\sigma$-field (MeV) as a function of the distance $r$ (fm) for several solutions of the equation system (34) which are characterized by the net quark number $N_q$ written down to the left of each curve.

**FIG. 20**: Distribution of the quark density $\rho$ (ch/fm$^3$) for the corresponding solutions presented in Fig. 20.
The parameters $C_s$, $C_v$, $b$ and $c$ are derived by the same algorithm as in the previous section, i.e. the chemical potential of quark ensemble $\lambda = 335$ MeV (and $\sigma \to 0$) is fixed at the spatial infinity. The filled-up states (of a liquid) are characterized by the parameters $\bar{M} = 70$ MeV, $\rho_0 = \rho_t = 3 \times 10^{15} \text{ch/fm}^3$. The $\sigma$-meson mass and the coupling constant $g_\sigma$ are derived at fixed coefficients $\eta$ and $\zeta$, and they just define the behaviour of solutions $\sigma(r)$, $\rho(r)$, etc. The magnitudes of functions $\sigma(r)$ and $\rho(r)$ at origin are not strongly correlated with the values characteristic for the filled-up states and are practically determined by solving the boundary value problem for system (34). In particular, the solutions presented in Fig. 19 have been derived with the running coefficient $\eta$ at $\zeta = \eta$. The most relevant parameter (instead of $\eta)$ from the physical view point is the total number of quarks in the droplet $N_q$ (as discussed above) and it is depicted to the left of each curve. (The variation of $M$, $\rho_0$ and $f_\pi$ could be considered as well instead of two mentioned parameters $\eta$ and $\zeta$.)

Analyzing the full spectrum of solutions obtained by scanning one can reveal a recurrent picture (at a certain scale) of kink-droplets which are easily parameterized by the total number of quarks $N_q$ in a droplet and by the density $\rho_0$. These characteristics are obviously fixed at completing the calculations. The sign which allows us to single out these solutions is related to the value of droplet specific energy (see below).

Table I exhibits the results of fitting the density $\rho(r)$ with the Fermi distribution

$$\rho_F(r) = \frac{\bar{\rho}_0}{1 + e^{(R_0-\rho)/b}} \ ,$$

where $\bar{\rho}_0$ is the density in origin, $R_0$ is the mean size of droplet and the parameter $b$ defines the thickness of surface layer $t = 4 \ln(3)b$. Besides, the coefficient $r_0$ which is absorbed in the surface tension coefficient $\sigma = r_0 N_q^{1/3}$, the $\sigma$-meson mass, $R_0 = r_0 N_q^{1/3}$ and the coefficient $\eta$ at which all other values have been obtained are also presented in the Table II.

The curves plotted in the Fig. 19 and results of Table II justify to conclude that the density distributions at $N_q \geq 50$ are in full agreement with the corresponding data typical for the nuclear matter. The thicknesses of transition layers in both cases are also similar and the coefficient $r_0$ with the factor $3^{1/3}$ included is in full correspondence with $\bar{\rho}_0$. The values of $\sigma$-meson mass in Table II look quite reasonable as well. However, the corresponding quantities are strongly different at small quark numbers in the droplet. We know from the experiments that in the nuclear matter some increase of the nuclear density is observed. It becomes quite noticeable for the Helium and is much larger than the standard nuclear density for the Hydrogen.

Obviously, we understand the Thomas–Fermi approximation which is used for estimating becomes hardly justified at small number of quarks, and we should deal with the solutions of complete equation system (33). However, one very encouraging hint comes from the chiral soliton model of nucleon [28], where it has been demonstrated that solving this system (33) the good description of nucleon and $\Delta$ can be obtained. Then our original remark could be that the soliton solutions obtained in [28] permit an interpretation as a ‘confluence’ of two kinks. Each of those kinks ‘works’ on the restoration of chiral symmetry since the scalar field approaches its zero value at the distance of $\sim 0.5$ fm from the kink center. Indeed, one branch of our solution corresponds to the positive value of dynamical quark mass, and another branch presents the solution with negative dynamical quark mass (in three-dimensional picture the pseudo-scalar fields appear just as a phase of chiral rotation from positive to negative value of quark mass). Such solutions develop the surface tension coefficient which is larger in factor two than the corresponding coefficient of single kink and as we believe signal some instability of a single kink solution.

The similar results are obtained for two flavours $N_f = 2 \ (\gamma = 2N_f N_c = 12)$ assuming all dynamical quark masses of SU(2) flavour multiplet are equal. The solutions for the $\sigma$-field and density distributions generally repeat the corresponding results presented in Fig. 19 and Fig. 20. The other data of fitting the solutions are shown in the following Table II. As it is seen the characteristic ensemble density is approximately in factor two larger than the density of normal nuclear matter (remember again the factor 3). The characteristic values of $\sigma$-meson mass are slightly larger than for $N_f = 1$ and, consequently, the thickness of transition layer is smaller almost in factor 1.4. The coefficient interrelating the mean size of droplet and the baryon (quark) number $\bar{\rho}_0 \sim 0.8$ is getting smaller. In principle, one can correct (increase) the

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$N_q$ & $\bar{\rho}_0$ & $R_0$ & $b$ & $t$ & $r_0$ & $\rho_0$ & $\eta$
\hline
15 & 0.34 & 1.84 & 0.51 & 2.24 & 0.74 & 351 & 0.65
\hline
43 & 0.43 & 2.19 & 0.52 & 2.28 & 0.75 & 384 & 0.73
\hline
159 & 0.46 & 4.19 & 0.52 & 2.29 & 0.77 & 409 & 0.78
\hline
303 & 0.47 & 5.23 & 0.52 & 2.29 & 0.78 & 417 & 0.795
\hline
529 & 0.47 & 6.37 & 0.52 & 2.27 & 0.79 & 423 & 0.805
\hline
742 & 0.47 & 7.15 & 0.52 & 2.27 & 0.79 & 426 & 0.81
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$N_q$ & $\bar{\rho}_0$ & $R_0$ & $b$ & $t$ & $r_0$ & $\rho_0$ & $\eta$
\hline
18 & 0.81 & 1.56 & 0.37 & 1.63 & 0.57 & 524 & 0.7
\hline
46 & 0.9 & 2.14 & 0.37 & 1.63 & 0.6 & 557 & 0.75
\hline
169 & 0.93 & 3.43 & 0.36 & 1.6 & 0.62 & 586 & 0.79
\hline
278 & 0.94 & 4.08 & 0.36 & 1.6 & 0.62 & 594 & 0.8
\hline
525 & 0.94 & 5.04 & 0.36 & 1.6 & 0.62 & 603 & 0.81
\hline
776 & 0.94 & 5.76 & 0.36 & 1.6 & 0.63 & 607 & 0.815
\hline
\end{tabular}
surface layer thickness and the parameter $\bar{r}_0$ by decreasing the $\sigma$-meson mass but the ensemble density remains higher than the normal nuclear one.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig21.png}
\caption{The specific binding energy at $N_f = 1$ and $N_f = 2$ in MeV as a function of quark number $N_q$.}
\end{figure}

Fig. 21 displays the specific binding energy of ensemble. It is defined by the expression similar to Eq. (39) in that the integration over the quark droplet volume is performed. The specific energy is normalized (compared) to the ensemble energy at the spatial infinity, i.e. in vacuum. Actually, Fig. 21 shows several curves in the upper part of plot which correspond the calculations with $N_f = 1$. The solid line is obtained by scanning over parameter $\eta$ and corresponds to the data presented in Table I. The dashed curve is calculated at fixed $\eta = 0.4$ but by scanning over parameter $M$. It is clearly seen if the specific energy data are presented as a function of quark number $N_q$, then the solutions, which we are interested in, rally in the local vicinity of the curve where the maximal binding energy $-|E^*_b|$ is reached. The similar solution scanning can be performed over the central density parameter $\rho_0$ in origin. The corresponding data are dotted for a certain fixed $M$ and $\rho_0$. It is interesting to notice that at scanning over any variable discussed a saturation property is observed and it looks like the minimum in $\varepsilon_b$ at $N_q \sim 200-250$. The results for the specific binding energy as a function of particle number are in the qualitative agreement with the corresponding experimental data. And one may say even about the quantitative agreement if the factor 3 (the energy necessary to remove one baryon) is taken into account. Another interesting fact to be mentioned is that there exist the solutions of system (34) with positive specific energy. For example, for $N_f = 2$ such meta-stable solutions appear at sufficiently large $\eta$ and with the density parameter in origin equal $\rho_0 \sim \rho_1 = 0.157$ ch/fm$^3$. In fact, the equation system (34) represents an equation of balance for the current quarks circulating between liquid and gas phases.

Summarizing we would like to emphasize that in the present paper we have demonstrated how a phase transition of liquid–gas kind (with the reasonable values of parameters) emerges in the NJL-type models. The constructed quark ensemble displays some interesting features for the nuclear ground state (for example, an existence of the state degenerate with the vacuum one), and the results of our study are suggestive to speculate that the quark droplets could coexist in equilibrium with vacuum under the normal conditions. These droplets manifest themselves as bearing a strong resemblance to the nuclear matter. Elaborating this idea in detail is a great challenge which will take a lot of special efforts and we do hope to undertake them in near future.

VI. CONCLUSION

In the present paper we described quantum liquids (Landau Fermi-liquids) resulting from the quark models with four-fermion interaction. This consideration is based on the identity of results obtained in [14] by using a dressing Bogolyubov transformation and mean field approximation. We demonstrated that the mean energy of ensemble serves as an energy functional of the Landau theory. It was shown that in a wide range of potentials interesting for applications one can expect the quantum liquids to behave in the essentially same way. For some of their properties a band of estimates was obtained. A comparison of NJL and KKB models, substantially different in many aspects, demonstrates that the properties of quantum liquids do not actually depend on a shape of the formfactor (a natural interaction length); rather, they are mainly determined by the coupling constant of interaction. It was shown that a common distinctive feature of ensembles is a presence of occupied states degenerate with respect to vacuum in chemical potential and pressure. Taking this observation the inhomogeneous states, which allowed at describing a transition layer, estimating a surface tension, as well as studying some properties of quark liquid droplets, were considered. It is noted that in the case of a small number of quarks in a droplet instability associated with lowering of the energy barrier, separating chiral phases, apparently manifests itself. This instability is seen in two kinks merging into one chiral soliton. An idea of dynamical equilibrium of a mixed phase consisting of baryon matter and vacuum was discussed as a possible scenario for explaining stability of nuclear matter.

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[29] Generally speaking, in such a correlation function the terms spanned on the vector of relative distance are allowed, but for simplicity we ignore them.

[30] It is evident that we are telling about an approximate calculation of corresponding generating functional for some specific conditions with the restricted area of applicability that does not imply the calculation of functional derivatives of arbitrary order.

[31] In the KKB model the fermion behavior is considered in the stochastic random field with infinite correlation length (the NJL model corresponds to the 'white noise' with zero correlation length). In this case one is lucky enough to be able to 'sum up' an entire diverging series and, therefore, to demonstrate that fermions are in general not on mass shell.

[32] If one decides to take the dynamical quark mass $M_q$, as a base variable, then it is seen from Eq. that it is
difficult to formulate an inverse transformation from $M_\eta$
to $\tilde{M}$ suitable for handling.