JACK POLYNOMIALS AND THE MULTI-COMPONENT CALOGERO-SUTHERLAND MODEL

P.J. Forrester
Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

Using the ground state $\psi_0$ of a multicomponent generalization of the Calogero-Sutherland model as a weight function, orthogonal polynomials in the coordinates of one of the species are constructed. Using evidence from exact analytic and numerical calculations, it is conjectured that these polynomials are the Jack polynomials $J_{\kappa}^{(1+1/\lambda)}$, where $\lambda$ is the coupling constant. The value of the normalization integral for $\psi_0 J_{\kappa}^{(1+1/\lambda)}$ is conjectured, and some further related integrals are evaluated.

1. INTRODUCTION

The Schrödinger operator [1-3]

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{2\pi^2}{L^2} \sum_{1 \leq k < l \leq N} \frac{\lambda(\lambda - M_{jk})}{\sin^2(\pi(x_k - x_l)/L)};$$

(1.1)

where $M_{jk}$ is the operator which exchanges the internal ‘spin’ coordinates of particles $j$ and $k$, is a multi-component generalization of the periodic $1/r^2$ many body Schrödinger operator originally considered by Sutherland [4]. In fact if we seek eigenfunctions of $\mathcal{H}$ which are symmetric under exchange of spin coordinates, $M_{jk}$ can be replaced by unity and (1.1) is precisely the Sutherland Hamiltonian. Then it has been proved [5] that the complete set of eigenfunctions are

$$\psi_{q,\kappa}(z_1, \ldots, z_N) := \prod_{1 \leq j < k \leq N} |z_k - z_j|^\lambda \prod_{j=1}^{N} z_j^{-q} J_{\kappa_j}^{(1/\lambda)}(z_1, \ldots, z_N), \quad z_j := e^{2\pi i x_j/L}$$

(1.2)

where $\kappa = (\kappa_1, \ldots, \kappa_N)$, $(\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_N \geq 0)$ is a partition, $q \in Z_{\geq 0}$ and $J_{\kappa_j}^{(1/\lambda)}$ denotes the Jack polynomial.

The eigenfunctions (1.2) occur in two integration formulas [6-8] of fundamental importance in the calculation of correlations functions [9] (see also [10,11])

$$\left( \prod_{l=1}^{N} \int_{-L/2}^{L/2} dx_l z_l^{(a-b)/2} |1 + z_l|^{a+b} \right) \psi_{0,0}^*(z_1, \ldots, z_N) \psi_{0,\kappa}(z_1, \ldots, z_N)$$

$$\frac{L^N J_{\kappa}^{(1/\lambda)}(1^N)}{\prod_{j=1}^{N} \Gamma(1 + a + b + \lambda(j - 1)) \Gamma(1 + a + \lambda(N - j) + \kappa_j) \Gamma(1 + b + \lambda(j - 1) - \kappa_j) \Gamma(1 + \lambda)}$$

(1.3)

and

$$\left( \prod_{l=1}^{N} \int_{-L/2}^{L/2} dx_l \right) |\psi_{q,\kappa}(z_1, \ldots, z_N)|^2 = L^N \frac{\Gamma(\lambda N + 1)}{f_N(\kappa)} \frac{(J_{\kappa}^{(1/\lambda)}(1^N))^2}{f_N(\kappa)},$$

(1.4)

1email: matpjf@maths.mur.oz.au; supported by the ARC
where
\[ f_n^\lambda(\kappa) := \prod_{1 \leq i < j \leq n} \frac{(j - i)\lambda + \kappa_i - \kappa_j)}{(j - i)\lambda}, \quad \bar{f}_n^\lambda(\kappa) := \prod_{1 \leq i < j \leq n} \frac{(1 - \lambda + (j - i)\lambda + \kappa_i - \kappa_j)}{(1 - \lambda + (j - i)\lambda)}, \]
with
\[ (a)_\lambda := \frac{\Gamma(a + \lambda)}{\Gamma(a)}. \]

The * in (1.3) denotes complex conjugation. The explicit value of \( J^{(1/\lambda)}_\kappa(1^N) \) is given by [12]
\[ (1/\lambda)^{[\kappa]} \prod_{j=1}^{N} \frac{\Gamma(\lambda N - \lambda(j - 1) + \kappa_j)}{\Gamma(\lambda N - \lambda(j - 1))}. \]

Kato and Kuramoto [3] have proved that if \( M_{jk} \) in (1.1) acts as a position coordinate exchange operator, then the ground state wave function \( \psi_0 \) which is symmetric in the variables \( x_1, \ldots, x_N \) and antisymmetric in the variables \( x_j^{(\alpha)} := x_{N_0 + \ldots + N_{\alpha - 1} + j} \) \( (j = 1, \ldots, N\alpha) \) for each \( \alpha = 1, \ldots, p \) \( (N_0 + \ldots + N_p = N) \) is given by
\[ \psi_0(\{z_j^{(\alpha)}\}_{i=1}^{p}, \{w_j\}_{i=1}^{p}) \prod_{\alpha=1}^{p} \prod_{j=1}^{N_\alpha} z_j^{(\alpha)(N_\alpha - 1)/2} \]
\[ = \prod_{\alpha=1}^{p} \prod_{1 \leq j < k \leq N_\alpha} |z_j^{(\alpha)} - z_k^{(\alpha)}|^{\lambda} z_j^{(\alpha)} z_k^{(\alpha)} \prod_{1 \leq j < k \leq N_0} |w_{j'} - w_{j''}|^{\lambda} \]
\[ \times \prod_{\alpha=1}^{p} \prod_{j=1}^{N_\alpha} \prod_{k=1}^{N_\alpha} |z_j^{(\alpha)} - z_k^{(\beta)}|^{\lambda} \prod_{\alpha=1}^{p} \prod_{j=1}^{N_\alpha} \prod_{j'=1}^{N_\alpha} |z_j^{(\alpha)} - w_{j'}|^{\lambda}, \tag{1.5} \]
where \( w_j := e^{2\pi i x_j/L} \) and \( z_j^{(\alpha)} := e^{2\pi i x_j^{(\alpha)}/L} \).

In a previous work [13], the present author has considered the analogue of the integration formula (1.3) in the case \( \kappa = 0 \), with \( |\psi_{0,0}|^2 \) replaced by the modulus squared of (1.5):
\[ D_p(N_1, \ldots, N_p; N_0; a, b, \lambda) \]
\[ = \left( \prod_{i=1}^{N_0} \int_{-1/2}^{1/2} dx_i w_i^{(a-b)/2} |1 + w_i|^{a+b} \right) \left( \prod_{\alpha=1}^{p} \prod_{i=1}^{N_\alpha} \int_{-1/2}^{1/2} dx_i^{(\alpha)} z_i^{(\alpha)(a-b)/2} |1 + z_i^{(\alpha)}|^{a+b} \right) \]
\[ \times |\psi_0(\{z_j^{(\alpha)}\}_{i=1}^{p}, \{w_j\}_{i=1}^{p})|^2 \] (1.6)
(for convenience we have set \( L = 1 \)). Guided by data from exact numerical integration, some analytic results and the structure of (1.3) in the case \( \kappa = 0 \), we were led to conjecture the functional property
\[ D_p(N_1, \ldots, N_p; N_0; a, b, \lambda) = f_{p-1}(N_1, \ldots, N_{p-1}; N_0; a, b, \lambda) A_p(N_1, \ldots, N_p; N_0; a, b, \lambda) \] (1.7a)
where
\[ A_p(N_1, \ldots, N_p; N_0; a, b, \lambda) \]
\[ = \prod_{j=0}^{N_p - 1} \frac{(j + 1)! \Gamma(\lambda + 1) j + a + b + \lambda \sum_{j=0}^{p-1} N_j + 1) \Gamma((\lambda + 1)(j + 1) + \lambda \sum_{j=0}^{p-1} N_j)}{\Gamma(1 + \lambda) \Gamma((\lambda + 1) j + a + \lambda \sum_{j=0}^{p-1} N_j + 1) \Gamma((\lambda + 1) j + b + \lambda \sum_{j=0}^{p-1} N_j + 1)} \] (1.7b)
\( f_{p-1} \) can be specified by a recurrence and \( N_p \geq N_j - 1 \) \((j = 1, \ldots, p - 1)\) (a proof was given in the case \( p = 1, \lambda = 1 \)). This functional property can be used as a recurrence and solved ((1.3) with \( \kappa = 0 \) provides the initial condition) to give \( D_p \) in terms of products of gamma functions.

The objective of this paper is to continue the study of [13] by extending the ‘ground state integrand’ in (1.5) to a state analogous to (1.2), and to develop some of the integration properties of these states. These tasks are addressed in Sections 2 and 3 respectively. In Section 4 comments and conclusions are made, and in the Appendix a further integration formula is noted.

2. OCCURRENCE OF THE JACK POLYNOMIALS

2.1 The approach of Kato and Kuramoto

The most natural generalization of (1.2) to the multicomponent case would be to provide the complete set of eigenfunctions of (1.1), subject to some specification of the symmetry of the coordinates. However, since there are both spin and position coordinates, this does not appear tractable. Another approach is to follow Kato and Kuramoto [3] and to consider (1.1) as describing a spinless model with \( M_{jk} \) acting as the exchange operator for the position coordinates. Then it is shown in [3] that seeking eigenfunctions of (1.1) of the form

\[
\psi = \prod_{1 \leq j < k \leq N} |z_k - z_j|^\lambda \Phi(z_1, \ldots, z_N)
\]  

(2.1)

gives the eigenvalue equation for \( \Phi \)

\[
\mathcal{H}^\prime_N \Phi := \left[ \sum_{j=1}^{N} \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \lambda(N - 1) \sum_{j=1}^{N} \frac{\partial}{\partial z_j} + 2 \lambda \sum_{1 \leq j < k \leq N} \frac{z_j z_k}{z_j - z_k} \right] \Phi = \epsilon \Phi
\]

(2.2)

Note that for symmetric \( \Phi \), \( 1 - M_{jk} = 0 \). The analytic solutions of the resulting eigenfunction equation are then the Jack polynomials [12].

To compute eigenfunctions for the multicomponent system, let \( \sigma \) be a partition of length \( \leq N \) and \( P \) a permutation of \( \{1, \ldots, N\} \), and put

\[
\phi_{\sigma, P} = \prod_{j=1}^{N} z_{P(j)}^{\sigma_j}
\]

(2.3)

From [3] we know

\[
\mathcal{H}^\prime_N \phi_{\sigma, P} = \sum_{j=1}^{N} \left[ \sigma_j^2 + \lambda(N - 1) \sigma_j \right] \phi_{\sigma, P} + 2 \lambda \sum_{j < k} h_{P(j)P(k)} \phi_{\sigma, P}
\]

(2.4a)

where

\[
h_{P(j)P(k)} = -\sigma_k + \left\{ \begin{array}{ll} \sum_{l=1}^{\sigma_k-1} (\sigma_j - \sigma_k - l)(z_{P(j)}^{-1} z_{P(k)})^l, & \sigma_j \geq \sigma_k + 2 \\ 0, & \text{otherwise} \end{array} \right.
\]

(2.4b)
Suppose in particular we seek a solution of (2.2) of the form
\[ \Phi = A^{(\alpha)} \left( \prod_{\alpha=1}^{p} \prod_{i=1}^{N_\alpha} z_{i_{\alpha}}^{N_\alpha-1} \right) f(w_1, \ldots, w_{N_0}) \] (2.5a)
where the symbol \( A^{(\alpha)} \) denotes the antisymmetrization in each of the sets of variables \( \{z_{1^{(\alpha)}}, \ldots, z_{N_{\alpha}}^{(\alpha)}\} \alpha = 1, \ldots, p \) and
\[ f(w_1, \ldots, w_{N_0}) = \prod_{j=1}^{N_0} w_{j}^{\kappa_j} + \sum_{\rho<\kappa} a_{\rho} \prod_{j=1}^{N_0} w_{j}^{p_{\rho}} \] (2.5b)
with \( \kappa = (\kappa_1, \ldots, \kappa_{N_0}) \) a partition such that
\[ \kappa_1 \leq \min(N_1, \ldots, N_p) \] (2.5c)
\( (N_1, \ldots, N_p \) are assumed non-zero). Inspection of (2.4) shows that in the action of \( H'_N \) on \( \Phi \) all terms in the sum over pairs which couple different species vanish and thus
\[ H'_N \Phi = \sum_{\alpha=0}^{p} H'_{N_{\alpha}} \Phi \] (2.6)
where \( H'_{N_{\alpha}} \) is defined as in (2.2) except that the coordinates are \( w_1, \ldots, w_{N_0} \) for \( \alpha = 0 \) and \( z_{j}^{(\alpha)} \) for \( \alpha = 1, \ldots, p \). The condition (2.5c) is essential for the validity of (2.6).

As noted in [3], \( \Phi \) is an antisymmetric eigenfunction of \( H'_{N_{\alpha}} \) for each \( \alpha = 1, \ldots, p \) (this is true independent of the particular function \( f \)). But we also know that the symmetric eigenfunction of \( H'_{N_{\alpha}} \) of the form (2.5b) is \( J^{(1/\lambda)}_{\kappa}(w_1, \ldots, w_{N_0}) \). Thus choosing \( f = J^{(1/\lambda)}_{\kappa} \) in (2.5a) with \( \kappa_1 \) restricted as in (2.5c) gives a class of eigenfunctions of \( H'_N \). Substituting this result in (2.1) and performing the antisymmetrizations explicitly using the Vandermonde formula gives
\[ \psi_{\kappa}(\{z_{1^{(\alpha)}}\}_{j=1,\ldots,N_{\alpha}}, \{w_{j}\}_{j=1,\ldots,N_0}) = \psi_0(\{z_{j}^{(\alpha)}\}_{j=1,\ldots,N_{\alpha}}, \{w_{j}\}_{j=1,\ldots,N_0}) \prod_{\alpha=1}^{p} \prod_{j=1}^{N_{\alpha}} z_{j}^{(\alpha)(N_{\alpha}-1)/2} J^{(1/\lambda)}_{\kappa}(w_1, \ldots, w_{N_0}). \] (2.7)

To proceed further and study integration formulas, it becomes apparent that the states (2.7) are not orthogonal. This transpires because the operator (1.1), with \( M_{jk} \) regarded as the position coordinate exchange operator, is not Hermitian with respect to the inner product
\[ \langle f | g \rangle_I := \prod_{l=1}^{N_0} \int_{-1/2}^{1/2} dx_l \prod_{\alpha=1}^{p} \prod_{l=1}^{N_{\alpha}} \int_{-1/2}^{1/2} dx_{l}^{(\alpha)} f^* g. \] (2.8)

2.2 A Gram-Schmidt approach
In view of the above difficulty with the approach of Kato and Kuramoto, we abandoned all reference to the operator (1.1) and sought to replace the eigenstates (1.2) by the corresponding orthogonal polynomials constructed by the Gram-Schmidt procedure.

**Definition 2.1** Let \( \kappa \) denote a partition of length \( \leq N_0 \). Define a symmetric polynomial in the variables \( w_1, \ldots, w_{N_0} \), denoted \( p_{\kappa}(w_1, \ldots, w_{N_0}) \), by the following properties:
(i) \( p_\kappa(w_1, \ldots, w_{N_0}) = m_\kappa + \sum_{\mu < \kappa} a_\mu m_\mu \), where \( |\mu| = |\kappa| \), \( \mu < \kappa \) is with respect to reverse lexicographical ordering of the partitions, \( m_\mu \) refers to the monomial symmetric function with exponents \( \mu \) in the variables \( w_1, \ldots, w_{N_0} \) and \( a_\mu \) is the corresponding coefficient.

(ii) For all \( N_1, \ldots, N_p \geq \kappa_1 \), \( \langle \psi_0 p_\kappa | \psi_0 p_\sigma \rangle_I = 0 \) for \( \kappa \neq \sigma \).

We remark that the restriction \( N_1, \ldots, N_p \geq \kappa_1 \) is motivated by the condition (2.5c) necessary for (2.7) to be eigenfunctions of (1.1).

The existence of these polynomials is assured by the Gram-Schmidt procedure applied to \( \{ m_\kappa \} \), where for each value of \( |\kappa| \) the polynomials \( p_\kappa \) are constructed from \( m_\kappa \) in reverse lexicographical ordering. Note in particular that

\[
p_{1k}(w_1, \ldots, w_{N_0}) = m_{1k} := \sum_{1 \leq j_1 < \ldots < j_k \leq N_0} w_{j_1} w_{j_2} \ldots w_{j_k}, \tag{2.9}
\]

### 2.1.1 Exact specification of \( p_{21k} \) for \( p = \lambda = 1 \)

The practical application of the Gram-Schmidt procedure requires the computation of multidimensional integrals, which appear intractable in general. An exception is the case \( p = \lambda = 1 \), for which we have previously presented [14] an integration procedure based on determinants to compute the normalization of (1.5). Generalizing this procedure, we can compute, for example, the inner products

\[
\langle \psi_0 m_{1k+2} | \psi_0 m_{21k} \rangle_I \quad \text{and} \quad \langle \psi_0 m_{1k+2} | \psi_0 m_{1k+2} \rangle_I
\]

necessary to compute \( a_{1k+2} \) in the Definition 2.1 of \( p_{21k} \).

**Proposition 2.1** For \( p = \lambda = 1 \) we have

\[
p_{21k}(w_1, \ldots, w_{N_0}) = m_{21k} + \frac{(k + 1)(k + 2)}{(k + 3)} m_{1k+2}.
\]

**Proof** From (2.9) and property (i) of Definition 2.1

\[
p_{21k} = m_{21k} + a_{1k+2} p_{1k+2}.
\]

Forming the inner product with \( p_{1k+2} \) on both sides and using property (ii) of Definition 2.1 gives

\[
a_{1k+2} = -\frac{\langle \psi_0 m_{1k+2} | \psi_0 m_{21k} \rangle_I}{\langle \psi_0 p_{1k+2} | \psi_0 p_{1k+2} \rangle_I}, \tag{2.10}
\]

so our task is to compute these two inner products.

Let us consider the denominator of (2.10). For \( p = \lambda = 1 \)

\[
|\psi_0(\{ z_j \}_{j=1,\ldots,N_1}; \{ w_j \}_{j=1,\ldots,N_0})|^2 = (-1)^{N_0+N_1} \prod_{k=1}^{N_0} w_k^{N_1} \prod_{\alpha=1}^{N_1} z_\alpha^{N_0+2N_1-2} A_1 A_2 \tag{2.11a}
\]

where

\[
A_1 = \prod_{1 \leq j < k \leq N_0} (w_k - w_j) \tag{2.11b}
\]

\[
A_2 = \prod_{1 \leq j < k \leq N_0} (w_k^{-1} - w_j^{-1}) \prod_{j=1}^{N_0} \prod_{\alpha=1}^{N_1} (w_j^{-1} - z_\alpha^{-1})^2 \prod_{1 \leq \alpha < \beta \leq N_1} (z_\beta^{-1} - z_\alpha^{-1})^4, \tag{2.11c}
\]
and furthermore, from (2.9)

\[ p_{1k+2} = m_{1k+2} = s_{1k+2} = \sum_{j_1 < \ldots < j_{k+2}} \prod_{j=1}^{k+2} w_{j_n}, \quad (2.12) \]

where \( s_\kappa \) denotes the Schur polynomial. Thus we can write

\[ \langle \psi_0 p_{1k+2} | \psi_0 p_{1k+2} \rangle_I = (-1)^{N_0+N_1} \prod_{l=1}^{N_0} \int_0^1 dx_l w_l^{N_0} \prod_{\alpha=1}^{N_1} \int_0^1 dx_\alpha^{(1)} z_\alpha^{N_0+2N_1-2} \times s_{1k+2}^i A_1 A_2 s_{1k+2}. \quad (2.13) \]

We proceed as in [14] and use a confluent form of the Vandermonde identity:

\[ A_2 = \begin{bmatrix} w_j^{-(l-1)} & \ldots & w_j^{-(l-1)} \\ z_j^{-(l-1)} & \ldots & z_j^{-(l-1)} \\ (l-1)z_j^{-(l-2)} & \ldots & (l-1)z_j^{-(l-2)} \end{bmatrix}_{j=1,\ldots,N_0}, \quad (2.14) \]

as well as the determinant formula for the Schur polynomials:

\[ A_1 s_{1k+2} = \det[w_j^{l+kN_0-j+1-1}]_{j,l=1,\ldots,N_0} \quad (2.15) \]

where \( \kappa_j = 1 \) \((j = 1, \ldots, k+2)\), \( \kappa_j = 0 \) otherwise.

Since (2.14) and (2.15) are antisymmetric with respect to interchanges of \( w_1, \ldots, w_{N_0} \), in the integral (2.13) we can replace (2.15) by \( N_0! \) times its diagonal term. Multiply each factor \( w_j^{j+kN_0-j+1-1} \) of this term into the row of (2.14) dependent on \( w_j \). Now take the sum in the definition (2.12) of \( s_{1k+2}^i \) outside the integral and multiply the factors of the summand into appropriate rows of (2.14). Row-by-row integration of the determinant with respect to \( w_1, \ldots, w_{N_0} \) gives a non-zero contribution in row \( 2N_1 + j \) only in column

\[ l = N_1 + j + \kappa_{N_0-j+1} - \xi_j, \quad \xi := \begin{cases} 1 & \text{if } j = j_1, \ldots, j_n \\ 0 & \text{otherwise} \end{cases} \quad (2.16) \]

and this term is equal to unity.

For the determinant to be non-zero we require each of the columns (2.16) to be distinct. Thus we require

\[ \xi_1 = \ldots = \xi_\nu = 1, \quad \xi_{N_0-k-1} = \ldots = \xi_{N_0-\nu} = 1, \quad \xi_j = 0 \text{ otherwise} \quad (2.17) \]

for some \( \nu = 0, \ldots, k+2 \) or equivalently

\[ \{j_1, \ldots, j_\nu\} = \{1, \ldots, \nu, N_0 - k - 1, \ldots, N_0 - 2 - \nu\}. \quad (2.18) \]

Assuming (2.17) and expanding the integrated determinant by the columns (2.16) with non-zero entries, gives, after expanding the remaining terms and grouping in pairs

\[ \langle \psi_0 p_{1k+2} | \psi_0 p_{1k+2} \rangle_I \]

\[ = N_0! \sum_{P(2\alpha) > P(2\alpha-1)} \epsilon(P) \prod_{\alpha=1}^{N_1} (P(2\alpha) - P(2\alpha-1)) \int_{-1/2}^{1/2} dx_\alpha^{(1)} z_\alpha^{N_0+2N_1-1-P(2\alpha)-P(2\alpha-1)} \]

\[ \quad (2.19a) \]
where
\[ P(\alpha) \in \{1, \ldots, N_1-1\} \cup \{N_1+\nu\} \cup \{N_1+N_0-\nu+1\} \cup \{N_1+N_0+2, \ldots, N_0+2N_1\} \]  
(2.19b)

We see that it is possible to choose
\[ P(2\alpha-1) = N_0 + 2N_1 + 1 - P(2\alpha), \quad \alpha = 1, \ldots, N_1 \]  
(2.20a)
and thus have a non-zero contribution to (2.19a) if and only if
\[ P(2\alpha) \in \{N_1 + N_0 + 1 - \nu\} \cup \{N_1 + N_0 + 2, \ldots, 2N_1 + N_0\} \]  
(2.20b)
Each of the \( N_1! \) different choices (2.20b) give the same contribution to (2.19a) and so
\[
\langle \psi_0 p_{1^{k+2}} | \psi_0 p_{1^{k+2}} \rangle_I = N_0! N_1! \prod_{l=1}^{N_1-1} (N_0 + 2N_1 + 1 - 2l) \sum_{\nu=0}^{k+2} (N_0 + 1 - 2\nu)
= N_0! N_1! (k+3)(N_0 - 1 - k) \prod_{l=1}^{N_1-1} (N_0 + 2N_1 + 1 - 2l) \]  
(2.21)
A similar calculation gives
\[
\langle \psi_0 p_{21^k} | \psi_0 p_{1^{k+2}} \rangle_I = -N_0! N_1! (k+1)(k+2)(N_0 - 1 - k) \prod_{l=1}^{N_1-1} (N_0 + 2N_1 + 1 - 2l) \]  
(2.22)
The stated result follows by substituting (2.21) and (2.22) in (2.10).

For \( \lambda \in \mathbb{Z}_{\geq 0} \) we noted in [13] a simple method of exact numerical integration for integrals of the type in (2.10), which relies on the fact that the integrand can be written as a finite (multivariable) Fourier series. Applying this method to compute (2.10) for \( \lambda = 2, 3, N_1 = 1 \) and various (small) values of \( k \) and \( N_0 \) indicates that Proposition 2.1 has a simple generalization to the general \( \lambda \) case

**Conjecture 2.1** For \( p = 1 \) and \( \lambda \geq 0 \),
\[ p_{21^k}(w_1, \ldots, w_{N_0}) = m_{21^k} + \frac{\lambda(k+1)(k+2)}{(\lambda(k+2) + 1)} m_{1^{k+2}}. \]

A striking feature of Proposition 2.1 and more generally Conjecture 2.1 is that the coefficients are independent of both \( N_0 \) and \( N_1 \). Exact numerical evaluation of \( p_\kappa \) for other \( \kappa \) indicate that this is a general property, which is valid provided \( N_1 \geq \kappa_1 - 1 \).

**Conjecture 2.2** For general \( p \in \mathbb{Z}^+ \) and \( \lambda \geq 0 \) the coefficients \( a_\nu \) in the polynomials of Definition 2.1 are independent of \( N_0, N_1, \ldots, N_p \). Also, in (ii) of Definition 2.1 the condition \( N_1, \ldots, N_p \geq \kappa_1 \) can be replaced by \( N_1, \ldots, N_p \geq \kappa_1 - 1 \).

The independence of the coefficients on \( N_1, \ldots, N_p \) can be understood in terms of the following conjecture which generalizes the recurrence (1.7a).

**Conjecture 2.3** Let \( h(w_1, \ldots, w_{N_0}) \) be a Laurent polynomial of the form
\[ h = \sum_{\sigma \leq \rho} c_\sigma m_\sigma, \]
where \( \rho = (\rho_1, \ldots, \rho_{N_0}) \), \( |\rho_1| \geq \cdots \geq |\rho_{N_0}| \). Suppose \( N_1 \geq |\rho_1| \) and \( N_1 \leq \ldots \leq N_p \). Then

\[
\langle \psi_0 | \psi_0 \prod_{l=1}^{N_0} w_l^{(a-b)/2} | 1 + w_l |^{a+b} \prod_{\alpha=1}^{p} \prod_{l=1}^{N_0} z_{l}^{(a)(a-b)/2} | 1 + z_{l}^{(a)} |^{a+b} \rangle = f_{p-1}^p(N_1, \ldots, N_{p-1}; N_0; a, b, \lambda) A_p(N_1, \ldots, N_p; N_0; a, b, \lambda)
\]

where \( A_p \) is given by (1.7b) and \( f_{p-1}^p \) satisfies the recurrence

\[
f_{p-1}^p(N_1, \ldots, N_{k-1}; N_0; a, b, \lambda) = A_{k-1}(N_1, \ldots, N_{k-1}; N_0; a, b, \lambda) f_{k-2}^p(N_1, \ldots, N_{k-2}; N_0; a, b, \lambda)
\]

where

\[
A_{k-1}(N_1, \ldots, N_{k-1}; N_0; a, b, \lambda) := \prod_{j=1}^{N_{k-1}} \frac{A_k(N_1, \ldots, N_{k-2}, j - 1, j; N_0; a, b, \lambda)}{A_k(N_1, \ldots, N_{k-2}, j, j - 1; N_0; a, b, \lambda)}
\]

with \( f_0^p(N_0; a, b, \lambda) = \) equal to the inner product in the case \( p = 0 \).

The recurrence for \( f_{p-1}^p \) is analogous to the recurrence for \( f_{p-1}^p \) in (1.7b), which is given in ref. [13]. Taken in the order \( k = p, p - 1, \ldots, 2 \) these equations explicitly determine \( f_{p-1}^p \) and thus give the inner product in terms of the inner product in the case \( p = 0 \). Now, the ratio of inner products specifying each \( a_v \) in Definition 2.1 are of the same form as the inner product in Conjecture 2.3 with \( a = b = 0 \). By introducing the recurrence in Conjecture 2.3 we conclude that the dependence on \( N_1, \ldots, N_p \) factorizes from the dependence on \( \kappa \) and \( N_0 \) and thus cancels out of the ratios.

By definition (see e.g. [12]), the Jack polynomials satisfy property (i) of Definition 2.1 with the coefficients \( a_v \) independent of \( N_0 \). Let us compare \( p_{21k} \), as given by Conjecture 2.1, with \( J_{21k}^{(\alpha)} \), which is given by [12, Proposition 7.2 with the normalization chosen so that the coefficient of \( m_{21k} \) is unity]:

\[
J_{21k}^{(\alpha)}(w_1, \ldots, w_{N_0}) = m_{21k} + \frac{(k + 1)(k + 2)}{k + 1 + \alpha} m_{1k+2}.
\]

We see

\[
p_{21k} = J_{21k}^{(1+1/\lambda)}
\]

which together with our exact numerical data suggests that the polynomials \( p_{\kappa} \) are simply related to the Jack polynomials.

**Conjecture 2.4** For general \( \lambda \geq 0 \) and partitions \( \kappa \), the polynomials \( p_{\kappa} \) of Definition 2.1 are given in terms of the Jack polynomials by

\[
p_{\kappa}(w_1, \ldots, w_{N_0}) = J_{\kappa}^{(1+1/\lambda)}(w_1, \ldots, w_{N_0}).
\]

**3. NORMALIZATION INTEGRAL**

In this Section we will provide the (in general conjectured) exact evaluation of the normalization integral, which is given by the inner product

\[
\langle \psi_0 J_{\kappa}^{(1+1/\lambda)}(w_1, \ldots, w_{N_0}) | \psi_0 J_{\kappa}^{(1+1/\lambda)}(w_1, \ldots, w_{N_0}) \rangle = N_{\kappa}^p(N_1, \ldots, N_p; N_0; \lambda)
\]

(3.1)
some general properties that our expression must possess. First, from (3.2), we require numerical data, we were led to formulate the following conjecture.

\[ \mathcal{N}_1^\kappa(N_1; N_0; \lambda) = \mathcal{N}_1^\kappa(\kappa_1; N_0; \lambda) \prod_{j=\kappa_1}^{N_1-1} \frac{(j+1)}{\Gamma(1+\lambda)}((\lambda+1)j + \lambda N_0 + 1) \lambda \]  

(3.2)

3.1 Analytic result for \( \lambda = p = 1, \kappa = 1^k \)

In the case \( \lambda = p = 1, \kappa = 1^k \), analytic results can be obtained by reading off a result obtained in the proof of Proposition 2.1.

Proposition 3.1 Consider the case \( \lambda = p = 1, \kappa = 1^k \) of (3.1). For \( N_1 \geq 1 \) we have

\[ \mathcal{N}_1^\kappa(N_1; N_0; \lambda) = N_0! N_1!(k+1)(N_0 + 1 - k) \prod_{l=1}^{N_1-1} (N_0 + 2N_1 + 1 - 2l) \]

\[ = N_0!(k+1)(N_0 + 1 - k) \prod_{l=1}^{N_1-1} (l+1)(2l + N_0 + 1). \]

Proof The first line is precisely (2.21) with \( k \) replaced by \( k - 2 \). The second line, which is of the general form (3.2), follows after replacing \( l \) by \( N_1 - l \) in the product and noting \( N_1! = \prod_{l=1}^{N_1-1}(l+1) \).

3.2 The conjecture

Let us write the partition \( \kappa = (\kappa_1, \ldots, \kappa_{N_0}) \) as

\[ \prod_{j=1}^{\kappa_1} (\kappa_1 + 1 - j)^{f_{\kappa_1+1-j}} \]  

(3.3)

so that \( f_j \) gives the frequency of the integer \( j \) in the partition, and denote

\[ \mathcal{N}_1^\kappa(N_1; N_0; \lambda) = \mathcal{N}_1^{f_{\kappa_1} f_{\kappa_1-1} \cdots f_1}(N_1; N_0; \lambda) \]  

(3.4)

We seek to evaluate (3.4) with \( N_1 = \kappa_1 \) in terms of the variables \( f_{\kappa_1}, \ldots, f_1 \). There are some general properties that our expression must possess. First, from (3.2), we require

\[ \mathcal{N}_1^{(f_{\kappa_1}=0)f_{\kappa_1-1} \cdots f_1}(\kappa_1; N_0; \lambda) = \mathcal{N}_1^{f_{\kappa_1-1} \cdots f_1}(\kappa_1 - 1; N_0; \lambda) \frac{\kappa_1}{\Gamma(1+\lambda)}((\lambda+1)(\kappa_1 - 1) + \lambda N_0 + 1) \lambda \]  

(3.5)

Second, in the cases \( \kappa_{N_0} > 0 \), since [12]

\[ J_0^{(1+1/\lambda)}(w_1, \ldots, w_{N_0}) = \prod_{j=1}^{N_0} w_j^{\kappa_{N_0}} J_0^{(1+1/\lambda)}(w_1, \ldots, w_{N_0}), \]

where \( \kappa - \kappa_{N_0} = (\kappa_1 - \kappa_{N_0}, \ldots, \kappa_{N_0} - \kappa_{N_0}) \), we require

\[ \mathcal{N}_1^\kappa(N_1; N_0; \lambda) = \mathcal{N}_1^{\kappa - \kappa_{N_0}}(N_1; N_0; \lambda). \]  

(3.6)

Guided by Proposition 3.1, the general properties (3.5) and (3.6), and some exact numerical data, we were led to formulate the following conjecture.
Conjecture 3.1 Define $N_1^\kappa(N_1; N_0; \lambda)$ by (3.1) and introduce the notation (3.3) and (3.4). Then

$$N_1^f_{\kappa_1,f_{\kappa_1-1},...,f_1}(\kappa_1; N_0; \lambda) = \frac{\kappa_1! \Gamma(\lambda N_0 + 1)}{(\Gamma(1 + \lambda))^{N_0+\kappa_1}} \prod_{j=1}^{\kappa_1} (\lambda f_j + 1)_\lambda \times \prod_{j=1}^{\kappa_1} \left( (\lambda + 1) j + 1 + \lambda \sum_{k=1}^{\kappa_1+1-j} f_{j+k-1} \right)_\lambda \left( (\lambda + 1) (j - 1) + 1 + \lambda (N_0 - \sum_{k=1}^{\kappa_1+1-j} f_{j+k-1}) \right)_\lambda$$

3.2 Evaluation of some multidimensional integrals

Knowledge of the normalization allows some multidimensional integrals represented by inner products to be evaluated. Suppose $f(w_1, \ldots, w_{N_0})$ is a polynomial of degree $K$ and has a known expansion in terms of Jack polynomials:

$$f(w_1, \ldots, w_{N_0}) = \sum_{k=0}^{K} \sum_{|\sigma|=k} c_{\sigma} f^{(1+1/\lambda)}_\sigma(w_1, \ldots, w_{N_0}) \quad (3.7)$$

Multiplying both sides by $|\psi_0|^2 J^{(1+1/\lambda)}_\sigma(w_1^*, \ldots, w_{N_0}^*)$, where $\psi_0$ is given by (1.5) with $N_1 \geq K$, and integrating using the orthogonality property of Definition 2.1 gives

$$\langle \psi_0 J^{(1+1/\lambda)}_\sigma(w_1, \ldots, w_{N_0}) | \psi_0 f(w_1, \ldots, w_{N_0}) \rangle = c_{\sigma} N_{\sigma}^{\kappa}(N_1, \ldots, N_p; N_0; \lambda). \quad (3.8)$$

Assuming Conjecture 3.1, the multidimensional integral defining the inner product is thus evaluated.

Perhaps the most straightforward example of an application of (3.8) is given by the following result.

Proposition 3.2 Suppose $N_1 \geq 1$ and put $N = N_0 + N_1$. We have

$$\prod_{l=1}^{N} f^{1/2}_{l-1/2} dx_l \prod_{1 \leq j < k \leq N} \left| e^{2\pi i x_k} - e^{2\pi i x_j} \right|^{2\lambda} \times \prod_{N_0+1 \leq j < k \leq N} \left| e^{2\pi i x_k} - e^{2\pi i x_j} \right|^{2} \prod_{l=1}^{N_0} (1 + u e^{2\pi i x_l}) J^{(1+1/\lambda)}_{1_k}(e^{-2\pi i x_1}, \ldots, e^{-2\pi i x_{N_0}})$$

$$= u^k \frac{\Gamma(\lambda N_0 + 1)}{(\Gamma(1 + \lambda))^{N_0+1}} \prod_{l=1}^{N_0} (\lambda (N_0 - k))^{(\lambda + 1)l + \lambda N_0 + 1}_\lambda (\lambda + 1) l + \lambda N_0 + 1$$

Proof This follows immediately from the simple expansion formula

$$\prod_{l=1}^{N_0} (1 + uw_l) = \sum_{k=0}^{N_0} u^k m_{1_k} = \sum_{k=0}^{N_0} u^k J^{(1+1/\lambda)}_{1_k}(w_1, \ldots, w_{N_0}), \quad (3.9)$$

(3.8) and Conjecture 3.1 with $\kappa_1 = 1$ and $f_{\kappa_1} = k$.

In the Appendix we give an integration formula which replaces

$$\prod_{l=1}^{N_0} (1 + u e^{2\pi i x_l})$$

by

$$\prod_{l=1}^{N} e^{i \pi x_l(a-b)} |1 + e^{2\pi i x_l}|^{a+b}$$
in Proposition 3.2, and thus can be viewed as a generalization of (1.3) in the case $\kappa = 1^k$.

4. CONCLUSION

Our objective of extending the ground state integrand in (1.5) to a state analogous to (1.2) led us to define (Definition 2.1) a class of multivariable symmetric polynomials. As stated in Conjecture 2.4, it appears that these polynomials are precisely the Jack polynomials $J_{\lambda}^{1+1/\lambda}$. An immediate difficulty in proving this result is the lack of a differential operator defining the states as eigenfunctions. In particular, the operator (1.1) with $M_{jk}$ interpreted as acting on the particle position coordinates does not have these states as eigenfunctions, and in fact is not an Hermitian operator.

The states we defined appear to possess closed form, product-of-gamma-function-type expressions for their normalization (Conjecture 3.1). However they do not form a complete set as they are restricted to the coordinates $w_1, \ldots, w_{N_0}$. Even for functions of the coordinates $w_1, \ldots, w_{N_0}$ they do not form a complete set, due to the restriction $N_1 \geq \kappa_1$. In particular, this means that the results obtained so far are not sufficient to compute correlation functions in multicomponent Calogero-Sutherland-type systems.

ACKNOWLEDGEMENTS

I thank the referee for a careful reading.

APPENDIX

Here we will evaluate the integral in Proposition 3.2, modified so that

$$\prod_{l=1}^{N_0} (1 + ue^{2\pi i x_l}) \text{ is replaced by } \prod_{l=1}^{N} e^{\pi i x_l (a-b)} |1 + e^{2\pi i x_l |a+b}$$

in the integrand. Thus we are seeking to evaluate

$$D_1^{1k} (N_1; N_0; a, b, \lambda) := \left( \prod_{l=1}^{N} \int_{-1/2}^{1/2} dx_l e^{\pi i x_l (a-b)} |1 + e^{2\pi i x_l |a+b} \right) \times \prod_{1 \leq j < k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{2\lambda} \prod_{N_0+1 \leq j < k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{2j_{1+1/\lambda}^{1k} (e^{2\pi i x_1}, \ldots, e^{2\pi i x_{N_0}})}$$

(A.1)

According to Conjecture 2.3

$$D_1^{1k} (N_1; N_0; a, b, \lambda) = D_1^{1k} (1; N_0; a, b, \lambda) \frac{A_1 (N_1; N_0; a, b, \lambda)}{A_1 (1; N_0; a, b, \lambda)},$$

(A.2)

where $A_1$ is given by (1.7b). Since $j_{1+1/\lambda}^{1k} = m_1$, we see that

$$D_1^{1k} (1; N_0; a, b, \lambda) = \frac{N_0 + 1 - k}{N_0 + 1} D_0^{1k} (N_0 + 1; a, b, \lambda)$$

(A.3)

which is an explicit evaluation, as the value of $D_0^{1k}$ is given explicitly by (1.3) (in this formula the Jack polynomial is assumed to have the normalization of Stanley [12]; to
normalize $J_{(1+1/\lambda)}^{(1+1/\lambda)}$ as in Definition 2.1 it is necessary to divide by $k!$). Thus

$$D_1^k(N_1; N_0; a, b, \lambda) = \frac{(-N_0)_k(b/\lambda)_k}{k!(-N_0 - (a + 1)/\lambda)_k}D_0^0(N_0 + 1; a, b, \lambda) \times \prod_{j=1}^{N_1-1} \frac{(j + 1)(\lambda + 1)j + a + b + \lambda N_0 + 1}{\Gamma((\lambda + 1)j + a + \lambda N_0 + 1)\Gamma((\lambda + 1)j + b + \lambda N_0 + 1)}$$

(A.4)

$(D_0^0$ is given by (1.3) with $\kappa = 0$, $N = N_0 + 1$ and $L = 1$).

From (A.4) a generalization of the Selberg integral of the type first given by Aomoto [15] can be deduced. Thus using (3.9) we see from (3.1) that

$$\sum_{k=0}^{N_0} u^k D_1^k(N_1; N_0; a, b, \lambda) = \left( \prod_{l=1}^{N} \int_{-1/2}^{1/2} dx_l e^{\pi i x_l (a-b)} |1 + e^{2\pi i x_l}|^{a+b} \right) \times \prod_{1 \leq j < k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^{2\lambda} \prod_{N_0+1 \leq j < k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^2 \prod_{l=1}^{N_0} (1 + ue^{2\pi i x_l}),$$

(A.5)

where $N = N_0 + N_1$ and $N_1 \geq 1$. On the other hand, from (A.4) we have

$$\sum_{k=0}^{N_0} u^k D_1^k(N_1; N_0; a, b, \lambda) = 2F_1(-N_0, b/\lambda; -N_0 - (a + b)/\lambda; u)D_0^0(N_0 + 1; a, b, \lambda) \times \prod_{j=1}^{N_1-1} \frac{(j + 1)(\lambda + 1)j + a + b + \lambda N_0 + 1}{\Gamma((\lambda + 1)j + a + \lambda N_0 + 1)\Gamma((\lambda + 1)j + b + \lambda N_0 + 1)},$$

(A.6)

which says the integral in (A.5) is proportional to a certain Gauss hypergeometric function, with proportionality constant given in terms of products of gamma functions.

REFERENCES

1 A.P. Polychronakos, Phys. Rev. Lett. 69, (1992) 703.
2 Z.N.C. Ha and F.D.M. Haldane, Phys. Rev. Lett. B 46 (1992) 9359
3 Y. Kato and Y. Kuramoto, Phys. Rev. Lett. 74 (1995) 1222
4 B. Sutherland, Phys. Rev. A 4 (1971) 2019
5 P.J. Forrester, Nucl. Phys. B 416 (1994) 377
6 I.G. Macdonald, Hall polynomials and symmetric functions 2nd ed. (Oxford Univ. Press, Oxford, 1995)
7 K.W.J. Kadell, Compos. Math. 87 (1993) 5
8 J. Kaneko, SIAM J. of Math. Analysis 24 (1993) 1086
9 P.J. Forrester, Mod. Phys. Lett. 9 (1995) 359
10 Z.N.C. Ha, Nucl. Phys. B 435 (1995) 604
11 F. Lesage, V. Pasquier and D. Serban, Nucl. Phys. B 435 (1995) 585
12 R.P. Stanley, Adv. Math. 77 (1989) 1243
13 P.J. Forrester, Int. J. of Mod. Phys. B 9 (1995) 1243
14 P.J. Forrester and B. Jancovici, J. Physique Lett. 45 (1984) L583
15 K. Aomoto, SIAM J. of Math. Analysis 18 (1987) 545