On the Cahn-Hilliard-Darcy system with mass source and strongly separating potential

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Abstract

We study an evolutionary system of Cahn-Hilliard-Darcy type including mass source and transport effects. The system may arise in a number of physical situations related to phase separation phenomena with convection, with the main and most specific application being related to tumoral processes, where the variations of the mass may correspond to growth, or shrinking, of the tumor. We prove existence of weak solutions in the case when the configuration potential for the order parameter $\varphi$ is designed in such a way to keep $\varphi$ in between the reference interval $(-1,1)$ despite the occurrence of mass source effects. Moreover, in the two-dimensional case, we obtain existence and uniqueness of strong (i.e., more regular) solutions.

Key words: Cahn-Hilliard-Darcy model, singular potential, tumor growth, weak solutions.

AMS (MOS) subject classification: 35D30, 35K61, 35Q35, 76D27, 92C30.

Dedicated to Maurizio Grasselli on the occasion of his 60th birthday, with friendship and admiration

1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$, with smooth boundary $\Gamma = \partial \Omega$, we consider the following PDE system:

\[
\begin{align*}
\varphi_t + u \cdot \nabla \varphi &= \Delta \mu + (1 - \varphi)S, \tag{1.1}
\mu &= -\Delta \varphi + f(\varphi) - \lambda \varphi, \tag{1.2}
\text{div} \, u &= S, \tag{1.3}
u &= -\nabla p + \mu \nabla \varphi, \tag{1.4}
S &= -(1 + \varphi)\gamma(x,t,\varphi). \tag{1.5}
\end{align*}
\]

Equations (1.1)-(1.2) are a variant of the well-known Cahn-Hilliard system \[4\] for phase separation, with $\varphi$ representing the order parameter, so normalized that the values $\pm 1$ correspond to the pure states, and $\mu$ is an auxiliary variable corresponding to the chemical potential of the process. In our setting the phase separation is influenced by transports effect driven by the movement of the particles of the substance; these effects are described by means of the macroscopic velocity $u$. In particular, the left hand side of (1.1) represents the material derivative of $\varphi$.

The velocity $u$ is assumed to satisfy a particular form of the Darcy law, stated by equations (1.3) and (1.4), where $p$ is the pressure and the coupling term $\mu \nabla \varphi$ represents a Korteweg force. It is worth noting that other expressions for the pressure and the Korteweg term could be considered,
leading to somehow different models (see the discussion in Subsec. 4.3 below). One of the main points in our analysis stands in the fact that we assume that there is no conservation of mass; in other words, the spatial mean value of \( \varphi \), corresponding to the proportion between the two components, or phases, may vary in time. The volumic mass source depends on the function \( S \) on the right hand side of (1.1), whose expression is specified by the last equation (1.5). In the sequel we will extensively comment about this choice; for now we just observe that the function \( \gamma \) is assumed to be measurable, globally bounded, and Lipschitz continuous with respect to \( \varphi \). We admit an explicit dependence on \((x,t)\) because, in applications, the above system may arise as a subsystem of a more complex model also involving other quantities (further details are given below). Finally, the terms \( f(\varphi) - \lambda \varphi \) in (2.2) represent the derivative of what, in the Cahn-Hilliard terminology, is generally noted as the “configuration potential” of the phase separation. In most cases, such a potential has a double-well structure, with the minima attained in proximity of the pure states \( \varphi = \pm 1 \). In our notation, \( F \) is the convex part of the potential, so that \( F = F' \), while the remainder (concave) part is given by \(-\lambda \varphi^2/2\), where \( \lambda \geq 0 \). Our specific choice for \( F \) will be thoroughly discussed in the sequel.

System (1.1) may have an independent mathematical interest in itself since it provides a prototypical coupling between the Cahn-Hilliard equation with mass source and the Darcy law. On the other hand, the choice of parameters and data (and, in particular, the expression (1.5) for the mass source term) is motivated by the specific situation of diffuse interface models for tumor growth. Indeed, the recent literature dealing with mathematical models for cancer evolution is very vast and rapidly expanding (we quote, with no claim of completeness, [6, 9, 10, 11, 12, 13, 16, 17, 20] for an overview). In tumor models based on the Cahn-Hilliard equation, the order parameter \( \varphi \) may represent the local concentration of one class of cells (e.g., “healthy”, “cancerous”, or “dead”), and the presence of a volumic mass source describes the fact that the tumor may grow, or shrink, depending on the effects of other factors (like nutrients, drugs, or blood vessels), here represented by the explicit dependence on \((x,t)\) of the term \( S \). Here, we will consider a scalar order parameter \( \varphi \), which means that we assume that there are only two types of cells; on the other hand, extending our results to the multi-phase case, i.e., to a vector-valued variable \( \varphi \), should be mathematically straightforward (see, e.g., [7, 10, 23] for related work).

Mathematically speaking, the main difficulty of system (1.1)-(1.6) arises because the mass source conflicts with the fact that \( \varphi \) represents a normalized order parameter and, for this reason, should take values into a bounded interval (in our setting \([-1,1]\)), whose extrema correspond to the pure states. More precisely, in tumor applications, \( \varphi(x,t) = 1 \) may tell us that, at some point \((x,t)\), only healthy cells are present; respectively, \( \varphi(x,t) = -1 \) means that, at \((x,t)\), only cancerous cells occur. Since the Cahn-Hilliard equation is of the fourth order in space and, consequently, its solutions do not obey a maximum principle, the property \( \varphi \in [-1,1] \) is usually enforced by choosing the convex part \( F \) of the potential to be “singular”, i.e. by assuming that \( F : \mathbb{R} \to [0, +\infty) \) takes finite values in the interval \([-1,1]\) only, with the most common example being given by the so-called “logarithmic potential”, whose convex part has the expression

\[
F(\varphi) = (1 + \varphi) \ln(1 + \varphi) + (1 - \varphi) \ln(1 - \varphi),
\]

so that \( f(\varphi) = F'(\varphi) = \ln(1 + \varphi) - \ln(1 - \varphi) \). (1.6)

In order to explain how the interaction with the mass source may give troubles, we consider the following “toy situation”, where no transport occurs, \( \lambda = 0 \), and the mass source is set equal to the constant value 1:

\[
\varphi_t - \Delta \mu = 1, \quad \mu = -\Delta \varphi + f(\varphi).
\] (1.7)

Then, integrating the first equation in space and assuming the (standardly used for the Cahn-Hilliard equation) no-flux (i.e., homogeneous Neumann) boundary conditions, one gets \( \varphi_\Omega(t) = 1 \) (here and below, \( \varphi_\Omega \) denotes the spatial mean of a function, or functional, \( \varphi \)), so that, independently of the inital value, at some time \( t \) one necessarily has \( \varphi_\Omega(t) > 1 \), which is inconsistent with the fact that, in the second equation, \( f \) is defined only as its argument ranges between \(-1\) and 1.

To fix this “inconsistency” issue, various strategies have been proposed. Let us now briefly discuss them.

(A) Replacing (1.6) with a “smooth” potential defined over the whole real line. For instance one may take \( F(\varphi) = \varphi^2 \) so that \( F(\varphi) - \lambda \varphi^2/2 \) is the standard quartic double well potential (cf. [9, 11, 12] for some examples in the tumor-related literature). This choice, of course, fixes the issue, but
the interpretation of $\varphi$ as an order parameter is lost since the values of $|\varphi|$ may evolve to become arbitrarily large at some points, even when the initial datum $\varphi_0$ is everywhere strictly between $-1$ and 1.

**(B)** Modifying the right hand side of (1.1) in such a way to include an explicit dependence on $\mu$ (cf. [5, 9, 21] for examples in the tumor-related literature). Referring to a “toy-case” similar to (1.7), we obtain something like

$$
\varphi_t - \Delta \mu + \mu = 1, \quad \mu = -\Delta \varphi + f(\varphi).
$$

(1.8)

Then, replacing the expression of $\mu$ given by the second relation into the first one, one may see that the function $f$ somehow influences the balance of mass, so contributing to keep $\varphi$ in between $-1$ and 1; on the other hand, the above choice may be questionable from the modeling viewpoint because it also adds an “artificial” dissipation term in the energy balance law.

**(C)** Designing the mass source term in such a way to constrain the mean value of $\varphi$ in between $-1$ and 1 (cf. [10, 15] for examples in the tumor-related literature). Namely, one may take

$$
\varphi_t - \Delta \mu = -K\varphi + s(x, t, \varphi),
$$

(1.9)

where $K > 0$ is large enough to dominate the nonlinear part $s$. In this case, integrating (1.9) in space leads to

$$
(\varphi_\Omega)_t + K \varphi_\Omega = \frac{1}{|\Omega|} \int_\Omega s(x, t, \varphi).
$$

(1.10)

At least if the right hand side is “small” in the $L^\infty$-norm (in a way that may depend on $K$ and on the initial datum), using elementary ODE techniques one can easily prove that $\varphi_\Omega$ is constrained between $-1$ and 1, which is the key step in order to avoid the inconsistency issue mentioned above and obtain existence of a weak solution.

We focus here on the approach (C), which was used in the recent paper [10] dealing with a model very similar to ours. Actually, when the macroscopic velocity is involved, a further complication due to the choice of the boundary conditions occurs. To realize this, we integrate (1.1) in space: applying the Gauss-Green formula we then obtain

$$
\frac{d}{dt} \int_\Omega \varphi = \int_\Omega S + \int_\Gamma \partial_n \mu - \int_\Gamma \varphi u \cdot n,
$$

(1.11)

where $n$ is the outer normal unit vector to $\Gamma$. Hence, if we take homogeneous Neumann boundary conditions for $\mu$, the last integral does not disappear and explicitly influences the balance of mass. Moreover, due to the poor regularity expected for $u$, it seems to be hard to estimate it. Hence, obtaining a manageable ODE like (1.10) seems to be out of reach in the no-flux case for $\mu$. To overcome this issue, in [10], in place of the Neumann condition for $\mu$, was assumed a “coupled” boundary condition of the form

$$
\partial_n \mu - \varphi u \cdot n \equiv 0 \quad \text{on } \Gamma.
$$

(1.12)

With such a choice, the two boundary integrals in (1.11) cancel with each other so that a constraint on the spatial mean of $\varphi$ can be reached provided that the mass source term is suitably designed. It is however worth discussing the “physical” meaning of condition (1.12), which corresponds to prescribing that the mass source effect is purely “volumic” (due, e.g., to the growth of existing cells and to the effects of nutrients or drugs), while there is no mass inflow or outflow due to transport.

In the present paper, we would like to address the case when the (probably more natural) no-flux condition for $\mu$ is taken. In such a situation only the first boundary integral in (1.11) disappears, which means that there may be some mass inflow or outflow through the boundary due to transport. For instance, if $u$ is oriented as $n$ (outflow) and $\varphi \sim 1$ (tumor-phase) on $\Gamma$, then the boundary integral tells us that the (prevailing) cancerous cells are being transported away, which is something expected. On the other hand, since the boundary integral in general cannot be estimated, in the no-flux case in principle one cannot exclude that the mean value of $\varphi_\Omega$ might exceed 1 (or $-1$), possibly leading to “inconsistency” of the model. Indeed, the only certainly available a-priori information comes from
the energy estimate (cf. (3.1) below), which, however, may be not sufficient to guarantee that \( \phi \) stays between \(-1\) and 1. In particular, inconsistency may occur in the case of the “logarithmic” potential (1.6) because the function \( F \) is bounded over the closed interval \([-1, 1]\) and, for this reason, it does not provide a sufficiently strong constraint on \( \phi \) in presence of a source of mass combined with a transport effect.

In order to avoid this problem, we will consider here what we will term as a “strongly separating” potential (cf. (2.7) below for the precise expression), i.e., we will consider a function \( F \) which is unbounded near \( \pm1 \) so to provide a stronger “separation effect” on \( \phi \) as a mere consequence of the energy estimate. Note that, from the qualitative point of view, the expression of \( f = F' \) is rather similar in the two cases (2.7) and (1.6); hence, this modification of the energy does not seem to affect the qualitative aspects of the model. On the other hand, we may show that, with (2.7), global existence of a weak solution can be obtained under very natural assumptions on the other parameters (and, in particular, on the volumic mass source \( S \)) and without adding any “artificial” term.

It is worth noting that, even with the help of (2.7), the procedure we use in order to obtain a suitable set of a-priori estimates is nontrivial; indeed, we still need to explicitly exclude the “inconsistency” phenomenon, i.e., the fact that \(|\phi_1|\) may, at some time, exceed 1. This is a real issue because the evolution of the mass does not depend, at least directly, on equation (1.2) (and, in turn, on the expression of \( f \)). Nevertheless, taking advantage of a proper approximation of the system, we are able to rigorously show that an estimate for the spatial mean value of \( \phi \) arises just as a direct consequence of the energy bound.

In the proof of existence of weak solutions, we will consider the two- and three-dimensional cases together since no variations are necessary at this level. Of course, one may wonder whether, at least in the two-dimensional case, further properties might be proved under additional assumptions of coefficients and data. This was, indeed, the spirit of the result proved in the recent paper [15]. Actually, we will show that the arguments of [15] can also be adapted to our situation so to provide existence and uniqueness of “strong” solutions in 2D; note that this adaptation is not completely straightforward because the different boundary conditions assumed here force us to estimate several terms in a different way compared to [15].

The remainder of the paper is organized as follows: in the next section, we introduce our notation, state our assumptions on coefficients and data, and present the precise statement of our results. The proof of existence mainly relies on a refined version of the energy estimate, which is first presented in a formal way in Section 3. Then, in the subsequent Section 4 we detail a suitable approximation of the system and the rigorous version of the a-priori estimates. Moreover, the limit with respect to the regularization parameter is achieved by weak compactness argument and a number of additional comments are given. Finally, in Section 5 we present the proof of the regularity and uniqueness result holding in two space dimensions.

2 Assumptions and main results

We assume \( \Omega \subset \mathbb{R}^d \), \( d \in \{2, 3\} \), be a smooth and bounded domain with boundary \( \Gamma = \partial \Omega \). In the existence proof, embeddings, interpolation inequalities, and related exponents, will be used by referring to the case \( d = 3 \); of course in the two-dimensional case the results remain valid and, in addition, the regularity properties of solutions may be improved.

We set \( H = L^2(\Omega) \), \( V := H^1(\Omega) \) and \( V_0 := H^1_0(\Omega) \). For notational simplicity we will use the same letters \( H, V \) and \( V_0 \) to denote vector-valued spaces (for instance, \( H \) may also refer to \( L^2(\Omega)^d \)). Then, identifying \( H \) with \( H' \) by means of the standard \( L^2 \)-scalar product, we obtain the Hilbert triplets \( V \subset H \subset V' \) and \( V_0 \subset H \subset V_0' \). We will denote by \( \| \cdot \| \) the norm in \( H \) and by \( \| \cdot \|_X \) the norm in a generic Banach space \( X \). We also indicate by \( A : V \to V' \) and \( B : V_0 \to V_0' \) the weak version of the Neumann and of the Dirichlet Laplacian, respectively.
We consider the system

\[
\begin{align*}
\varphi_t &= \Delta \mu + (1 - \varphi) S - u \cdot \nabla \varphi, \\
\mu &= -\Delta \varphi + f(\varphi) - \lambda \varphi, \\
\text{div} u &= S, \\
u &= -\nabla p + \mu \nabla \varphi, \\
S &= -(1 + \varphi) \gamma(x,t,\varphi),
\end{align*}
\]

(2.1)

\[
\begin{align*}
\mu &= -\Delta \varphi + f(\varphi) - \lambda \varphi, \\
\text{div} u &= S, \\
u &= -\nabla p + \mu \nabla \varphi, \\
S &= -(1 + \varphi) \gamma(x,t,\varphi),
\end{align*}
\]

(2.2)

\[
\begin{align*}
\text{div} u &= S, \\
u &= -\nabla p + \mu \nabla \varphi, \\
S &= -(1 + \varphi) \gamma(x,t,\varphi),
\end{align*}
\]

(2.3)

\[
\begin{align*}
(1 - \varphi) S &= -(1 - \varphi^2) \gamma(x,t,\varphi); \\
\text{div} u &= S, \\
u &= -\nabla p + \mu \nabla \varphi, \\
S &= -(1 + \varphi) \gamma(x,t,\varphi),
\end{align*}
\]

(2.4)

\[
\begin{align*}
\varphi|_{t=0} &= \varphi_0, \\
\text{div} u &= S, \\
u &= -\nabla p + \mu \nabla \varphi, \\
S &= -(1 + \varphi) \gamma(x,t,\varphi),
\end{align*}
\]

(2.5)

coupled with no-flux conditions for \(\varphi\) and \(\mu\) and with homogeneous Dirichlet conditions for \(p\). Namely, we assume

\[
\partial_n \mu = \partial_n \varphi = p = 0 \quad \text{on} \quad \Gamma.
\]

(2.6)

As said, \(\lambda \geq 0\) is a given constant and we will assume that the convex part \(F\) of the configuration potential takes the following “strongly separating” expression:

\[
F(\varphi) = -\ln(1 - \varphi^2), \quad \text{so that} \quad f(\varphi) = F'(\varphi) = \frac{2\varphi}{1 - \varphi^2}.
\]

(2.7)

Note that the above is just a prototypical choice; more general forms of \(F\) could indeed be taken. The key point stands in the fact that, differently from the case (1.6) of the “standard” Cahn-Hilliard logarithmic potential, here not only the derivative \(f\) is singular at \(\pm 1\), but the same is true also for \(F\). As noted in the introduction, this property is crucial in order to avoid the “inconsistency” phenomenon due to the mass source effect.

The function \(\gamma\) characterizing the mass source term is assumed to satisfy

\[
\gamma \in L^\infty(\Omega \times (0,T) \times \mathbb{R}), \quad \gamma(x,t,\cdot) \in \text{Lip}(\mathbb{R}) \quad \text{for a.e.} \quad (x,t) \in \Omega \times (0,T),
\]

(2.8)

\[
\gamma(x,t,r) \equiv 0 \quad \text{for all} \quad |r| \geq 2 \quad \text{and a.e.} \quad (x,t) \in \Omega \times (0,T).
\]

(2.9)

It is worth commenting a bit the above assumptions. First of all, since \(\gamma\) represents a source of mass, assuming global boundedness of it is a natural condition; moreover hypothesis (2.9) serves just as a normalization property for the sake of building a sound approximation of the system; indeed, in the limit, \(\varphi\) will take its values in the interval \((-1,1)\); hence the behavior of \(\gamma\) for \(\varphi\) outside \((-1,1)\) is factually irrelevant. The Lipschitz continuity with respect to \(\varphi\) stated by the second (2.8) is a natural requirement as we aim to apply a local existence result to a Faedo-Galerkin regularization.

Looking at the right hand side of (2.1), it is worth observing that

\[
(1 - \varphi) S = -(1 - \varphi^2) \gamma(x,t,\varphi);
\]

(2.10)

In a sense this prescribes that, as far as \(\varphi\) is close to 1, or to \(-1\), the volumic mass source tends to approach 0 (recall that \(\gamma\) is bounded). This ansatz is rather standard at least in tumor applications, meaning for instance that, when the tumor cells are strongly prevailing (compared to the healthy ones), very few new ones can be created (see also Remark 3.1 below for further comments).

Moreover, we complement the system with the initial condition

\[
\varphi|_{t=0} = \varphi_0,
\]

(2.11)

where the initial datum \(\varphi_0\) is assumed to satisfy the following regularity properties, which basically correspond to the finiteness of the physical energy at the initial time:

\[
\varphi_0 \in V, \quad F(\varphi_0) \in L^1(\Omega),
\]

(2.12)

Note that then, by Jensen’s inequality, there follows

\[
F\left(\int_\Omega \varphi_0(x) \frac{dx}{|\Omega|}\right) \leq \int_\Omega F(\varphi_0(x)) \frac{dx}{|\Omega|} < +\infty;
\]

(2.13)

Consequently, due to the choice (2.7), and more specifically to the fact that \(\lim_{|r| \to 1} F(r) = +\infty\), one has

\[
(\varphi_0)|_{\Omega} \in (-1,1).
\]

(2.14)
Theorem 2.1. Let assumptions (2.7), (2.8)-(2.10), and (2.12) hold. Then, there exists at least one quadruplet \((\varphi, \mu, p, u)\) of functions such that
\[
\varphi \in H^1(0, T; V' + L^1(\Omega)) \cap L^\infty(0, T; V) \cap L^4(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2.6}(\Omega)),
\]
\[
f(\varphi) \in L^2(0, T; L^6(\Omega)),
\]
\[
\mu \in L^2(0, T; V),
\]
\[
u \in L^2(0, T; H),
\]
\[
p \in L^2(0, T; W^{1.3/2}(\Omega)),
\]
satisfying system (2.1)-(2.5) with the boundary conditions (2.6) in the following weak form:
\[
\varphi_t + A\mu = (1 - \varphi)S - u \cdot \nabla \varphi \quad \text{in } L^2(0, T; V' + L^1(\Omega)),
\]
\[
\mu = -\Delta \varphi + f(\varphi) - \lambda \varphi, \quad \text{a.e. in } \Omega \times (0, T),
\]
\[
\partial_n \varphi = 0, \quad \text{a.e. on } \Gamma \times (0, T),
\]
\[
\int_{\Omega} u(t) \cdot \nabla \eta = -\int_{\Omega} S(:, t, \varphi) \eta \quad \text{for every } \eta \in V_0 \text{ and a.e. } t \in (0, T),
\]
\[
u = -\nabla p + \mu \nabla \varphi, \quad \text{a.e. in } \Omega \times (0, T),
\]
\[
S = -(1 + \varphi)\gamma(x, t, \varphi), \quad \text{a.e. in } \Omega \times (0, T),
\]
and complying with the initial condition (2.11).

We conclude this section by detailing our second result, which is devoted to existence and uniqueness of “strong” (i.e. more regular) solutions in two dimensions of space, so extending to the present setting [15] Theorem 4.1].

Theorem 2.2. Let assumptions (2.7), (2.8)-(2.10), and (2.12) hold and let \(\Omega \subset \mathbb{R}^2\). Moreover, let us set
\[
\mu_0 := -\Delta \varphi_0 + f(\varphi_0) - \lambda \varphi_0
\]
and, correspondingly, let us assume
\[
\varphi_0 \in H^2(\Omega), \quad f(\varphi_0) \in H, \quad \partial_n \varphi_0 = 0 \quad \text{on } \Gamma,
\]
\[
\mu_0 \in V.
\]
Then, there exists one and only one solution \((\varphi, \mu, p, u)\) to system (2.1)-(2.5) with the boundary conditions (2.6) in the following regularity class:
\[
\varphi \in W^{1,\infty}(0, T; V') \cap H^1(0, T; V) \cap L^\infty(0, T; W^{2.r}(\Omega)) \quad \text{for all } r \in [1, \infty),
\]
\[
f(\varphi) \in L^\infty(0, T; L^r(\Omega)) \quad \text{for all } r \in [1, \infty),
\]
\[
\mu \in L^\infty(0, T; V) \cap L^4(0, T; H^2(\Omega)),
\]
\[
u \in L^\infty(0, T; H^1(\Omega)),
\]
\[
p \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)).
\]
Note that the exponent 6 occurring in (2.15)-(2.16) has been replaced by \(r \in [1, \infty)\) in (2.29)-(2.30). This is a consequence of better 2D embeddings (and the same could be done in Theorem 2.1 when restricted to the two-dimensional setting).

The proof of the above results will occupy the remainder of the paper. In particular, in the proof of Theorem 2.2 presented in Section 5 below, we will mainly focus on the regularity part of the statement because uniqueness works very similarly with [15] and, for this reason, will be only sketched.

This fact is crucial in order to get a control of the spatial mean of \(\varphi\) over the interval \((0, T)\). We notice that, in the case of the logarithmic potential [1.9], (2.11) generally needs to be taken as an additional assumption (since \(F\) is finite over the closed interval \([-1, 1]\) in that case), while in our case is just a direct consequence of the expression (2.7).

We can now state the main result of this paper:
3 Formal energy estimate

In order to fix the main points of our procedure and understand the role of the assumptions, we derive, for the reader’s convenience, a formal version of the energy estimate. This means that we will directly work on the system (2.1)-(2.3) without referring to any approximation. The procedure will be made rigorous in the next section, where a regularization of the system will be proposed. We decided to present both versions of the estimate because the rigorous estimate a bit more technical.

That said, we test (2.1) by $\mu$ and (2.2) by $\varphi_t$ to get
\[
\frac{d}{dt} \mathcal{E} + \|\nabla \mu\|^2 = \int_\Omega (1 - \varphi) S \mu - \int_\Omega u \cdot \nabla \varphi \mu,
\]
where $\mathcal{E}$ denotes the standard Cahn-Hilliard energy, i.e.,
\[
\mathcal{E} = \mathcal{E}(\varphi) = \frac{1}{2} \|\nabla \varphi\|^2 + \int_\Omega \left( F(\varphi) - \frac{\lambda}{2} \varphi^2 \right).
\]

Now, using the expressions (2.2) for $\mu$ and (2.5) for $S$, it is easy to see that
\[
\int_\Omega (1 - \varphi) S \mu = - \int_\Omega (1 - \varphi^2) \gamma(x,t,\varphi)(-\Delta \varphi + f(\varphi) - \lambda \varphi).
\]
Then, using (2.4) and the fact that $\varphi$ takes values in $(-1,1)$, we have
\[
- \int_\Omega (1 - \varphi^2) \gamma(x,t,\varphi)(-\Delta \varphi + \lambda \varphi) \leq c + \frac{1}{4} \|\Delta \varphi\|^2.
\]
On the other hand, using the second (2.7) and the fact $\varphi \in (-1,1)$, we deduce
\[
- \int_\Omega (1 - \varphi^2) \gamma(x,t,\varphi)f(\varphi) = -2 \int_\Omega \gamma(x,t,\varphi) \varphi \leq c.
\]

Remark 3.1. In (3.5) we used the degenerate behavior of $(1 - \varphi^2)$ at $\pm 1$ in order to compensate the singular character of $f(\varphi)$. Note that the same argument would work also in the case of the standard “logarithmic” potential (1.4). On the other hand, for different expressions of the mass source term (for instance if one forgets the factor $(1 + \varphi)$ in the expression (2.5) for $S$), the argument may fail. However, an estimate could still be obtained if additional sign conditions on $\gamma$ are assumed. For instance, assuming for simplicity $\gamma$ to depend only on $\varphi$, if $\gamma(\varphi)$ has the same sign as $\varphi$ at least for $|\varphi|$ close to 1, then the integral in (3.5) may be moved to the left hand side and gives a positive contribution (so it does not need to be controlled). On the other hand, we preferred to avoid sign conditions on $\gamma$ because in applications its expression (and in particular its sign) may be determined by the effects of other quantities (cf., for instance, the discussion in [18, Sec. 3.4]). We may also observe that the expression (2.5) for $S$ appears to be realistic at least in tumor-related applications (see, e.g., [13]).

Next, testing (2.4) by $u$, integrating by parts, and using (2.3) with the boundary conditions (2.6), we obtain
\[
\|u\|^2 = \int_\Omega pS + \int_\Omega u \cdot \nabla \varphi \mu.
\]

Hence, summing (3.6) to (3.1) and using (3.4)-(3.5) we infer
\[
\frac{d}{dt} \mathcal{E} + \|\nabla \mu\|^2 + \|u\|^2 \leq \int_\Omega pS + c + \frac{1}{4} \|\Delta \varphi\|^2.
\]

To estimate the integral term on the right hand side, we define $\zeta$ as the solution of the following time-dependent family of elliptic problems:
\[
B \zeta = S.
\]
Taking the divergence of (2.4), and recalling the boundary conditions, we can write
\[
Bp = S - \text{div}(\mu \nabla \varphi).
\] (3.9)

Testing (3.9) by \( \zeta = B^{-1}S \) and using the boundary conditions, we infer
\[
\int_{\Omega} pS = \|S\|_{H^{-1}(\Omega)}^2 - \int_{\Omega} \text{div}(\mu \nabla \varphi) \zeta
= \|S\|_{H^{-1}(\Omega)}^2 + \int_{\Omega} \mu \nabla \varphi \cdot \nabla \zeta
= \|S\|_{H^{-1}(\Omega)}^2 + \int_{\Omega} (-\Delta \varphi + f(\varphi) - \lambda \varphi) \nabla \varphi \cdot \nabla \zeta
= \|S\|_{H^{-1}(\Omega)}^2 - \int_{\Omega} \Delta \varphi \nabla \varphi \cdot \nabla \zeta + \int_{\Omega} f(\varphi) \nabla \varphi \cdot \nabla \zeta - \int_{\Omega} \lambda \varphi \nabla \varphi \cdot \nabla \zeta
=: \|S\|_{H^{-1}(\Omega)}^2 + I_1 + I_2 + I_3,
\] (3.10)

the last inequality following from the fact that \( S \) is bounded in the \( L^\infty \)-norm.

We now provide a control of the integral terms on the right hand side. Firstly, by elementary use of interpolation and Sobolev’s embeddings, we have
\[
I_1 = -\int_{\Omega} \Delta \varphi \nabla \varphi \cdot \nabla \zeta \leq \|
\Delta \varphi \|_{L^\infty(\Omega)} \|
\nabla \varphi \|_{L^2(\Omega)} \|
\nabla \zeta \|_{L^2(\Omega)}
\leq \|
\Delta \varphi \|^3/2 \|
\nabla \varphi \|^{1/2} \|
\nabla \zeta \| \leq \frac{1}{4} \|
\Delta \varphi \|^2 + c \|
\nabla \varphi \|^2,
\] (3.11)

where, to deduce the last inequality, we also used (3.8) and the uniform boundedness of \( S \).

Second, we observe that
\[
I_2 = \int_{\Omega} f(\varphi) \nabla \varphi \cdot \nabla \zeta \leq \|f(\varphi)\|_{L^1(\Omega)} \|
\nabla \varphi \|_{L^2(\Omega)} \|
\nabla \zeta \|_{L^\infty(\Omega)}
\leq c \|f(\varphi)\|_{L^1(\Omega)} \|S\|_{L^1(\Omega)} \leq c \|f(\varphi)\|_{L^1(\Omega)} \|S\|_{L^\infty(\Omega)} \leq c \|f(\varphi)\|_{L^1(\Omega)}.
\] (3.12)

Finally, using again that \( \varphi \in (-1, 1) \), we have
\[
I_3 = -\int_{\Omega} \lambda \varphi \nabla \varphi \cdot \nabla \zeta \leq c \|
\nabla \varphi \| \leq c(1 + \|
\nabla \varphi \|^2).
\] (3.13)

To control the right hand sides of (3.11) and (3.12), the strongly separating behavior of \( F \) is crucial. To exploit it, we actually need to perform a further calculation by testing \( 2b \) by \(-\Delta \varphi \) to obtain
\[
\|
\Delta \varphi \|^2 + \int_{\Omega} f'(\varphi)|\nabla \varphi|^2 \leq \frac{1}{2} \|
\nabla \mu \|^2 + \left( \frac{1}{2} + \lambda \right) \|
\nabla \varphi \|^2.
\] (3.14)

Summing the above to (3.7) and using (3.11) and (3.12), we then infer
\[
\frac{d}{dt} \mathcal{E} + \frac{1}{2} \|
\nabla \mu \|^2 + \|
\mu \|^2 + \frac{1}{2} \|
\Delta \varphi \|^2 + \int_{\Omega} f'(\varphi)|\nabla \varphi|^2
\leq c \|f(\varphi)\|_{L^1(\Omega)} + c(1 + \|
\varphi \|^2).
\] (3.15)

Now, from (2.7) we may compute
\[
f'(\varphi) = \frac{2(1 + \varphi^2)}{(1 - \varphi^2)^2}.
\] (3.16)

Hence, it is easy to see that
\[
c \|f(\varphi)\|_{L^1(\Omega)} = \int_{\Omega} \frac{2|\varphi|}{1 - \varphi^2} |\nabla \varphi| \leq \frac{1}{2} \int_{\Omega} f'(\varphi)|\nabla \varphi|^2 + c.
\] (3.17)
Substituting (3.17) into (3.15), we finally arrive at the relation
\[
\frac{d}{dt} E + \frac{1}{2} \|\nabla \mu\|^2 + \|u\|^2 + \frac{1}{2} \|\Delta \phi\|^2 + \frac{1}{2} \int_{\Omega} f'(\phi) |\nabla \phi|^2 \leq c(1 + \|\phi\|^2).
\] (3.18)

Using the Gronwall lemma and noting that, by assumptions (2.12), the energy \( E \) is finite at the initial time, it is possible to deduce a global estimate. In the next section we will see that, in fact, such a procedure may be adapted to work in the framework of a rigorous approximation.

4 Approximation

In this part we introduce an approximation of system (2.11)-(2.5) and sketch a way to prove existence to the obtained regularized system.

Given \( \varepsilon \in (0, 1/4) \) which will be let go to 0 in the limit, the main point stands in providing a suitable regularization \( f_\varepsilon \) of the function \( f \) compatible with the a-priori estimates. To this aim, we take \( f_\varepsilon \) equal to \( f \) in the interval \([-1 + \varepsilon, 1 - \varepsilon]\) and \( f_\varepsilon \) given by the first order Taylor expansion of \( f \) centered in \( 1 - \varepsilon \) in \((1 - \varepsilon, +\infty)\) and, respectively, by the first order Taylor expansion of \( f \) centered in \(-1 + \varepsilon \) in \((-\infty, -1 + \varepsilon)\). Then, noting that
\[
\begin{align*}
F(1 - \varepsilon) & = F(-1 + \varepsilon) = -\ln(2 - \varepsilon) - \ln(\varepsilon) \sim -\ln \varepsilon, \\
f(1 - \varepsilon) & = \frac{2 - 2\varepsilon}{2\varepsilon - \varepsilon^2} \sim \frac{1}{\varepsilon}, \\
f(-1 + \varepsilon) & = -\frac{2 + 2\varepsilon}{2\varepsilon - \varepsilon^2} \sim -\frac{1}{\varepsilon}, \\
f'(1 - \varepsilon) = f'(-1 + \varepsilon) & = \frac{2(1 + (1 - \varepsilon)^2)}{(1 - (1 - \varepsilon)^2)^2} \sim \frac{1}{\varepsilon^2},
\end{align*}
\] (4.1)

and restricting our attention to the case \( \phi > 1 - \varepsilon \) (the case \( \phi < -1 + \varepsilon \) being analogous), we have
\[
\begin{align*}
f_\varepsilon(\phi) & = f(1 - \varepsilon) + f'(1 - \varepsilon)(\phi - 1 + \varepsilon) \sim \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}(\phi - 1 + \varepsilon), \\
f'_\varepsilon(\phi) & = f'(1 - \varepsilon) \sim \frac{1}{\varepsilon^2},
\end{align*}
\] (4.4)

so that, still for \( \phi > 1 - \varepsilon \), we also obtain
\[
F_\varepsilon(\phi) = F(1 - \varepsilon) + \int_{1-\varepsilon}^\phi f_\varepsilon(r) \, dr \sim -\ln \varepsilon + \frac{1}{\varepsilon}(\phi - 1 + \varepsilon) + \frac{1}{2\varepsilon^2}(\phi - 1 + \varepsilon)^2.
\] (4.6)

We then replace \( f \) with \( f_\varepsilon \) in equation (2.2). Since \( f_\varepsilon \) is globally Lipschitz, at the approximate level the order parameter will no longer take its values into \((-1, 1)\). For this reason, we also need to truncate the terms \( (1 - \varphi) \) in (2.1) and \( 1 + \varphi \) in (2.5) by replacing them with their positive parts, noted by \((\cdot)^+ \) in the sequel. Then, in order to prepare the ground for an existence result, we rewrite the system by eliminating \( u \) and using instead the pressure which is a somehow more natural variable because it satisfies an elliptic equation with proper (namely, Dirichlet) boundary conditions.

By the above considerations, noting as \((\varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)\) a candidate solution, we are led to the system
\[
\begin{align*}
\varphi_{\varepsilon,t} + A\varphi_\varepsilon + Bp_\varepsilon & = -(1 - \varphi_\varepsilon) S_\varepsilon + \nabla p_\varepsilon \cdot \nabla \varphi_\varepsilon - \mu_\varepsilon |\nabla \varphi_\varepsilon|^2, \\
\mu_\varepsilon & = A\varphi_\varepsilon + f_\varepsilon(\varphi_\varepsilon) - \lambda \varphi_\varepsilon, \\
Bp_\varepsilon & = S_\varepsilon - \text{div}(\mu_\varepsilon \nabla \varphi_\varepsilon), \\
S_\varepsilon & = -(1 + \varphi_\varepsilon)^+ \gamma(x, t, \varphi_\varepsilon),
\end{align*}
\] (4.7)

where the boundary conditions are incorporated into the operators \( A \) and \( B \).

The above system is naturally complemented with the initial conditions (2.11), where the initial data do not need to be regularized. Then, in order to prove a local existence result via a discretization scheme, it is also convenient to eliminate the auxiliary variable \( \mu_\varepsilon \). Actually, equations (4.7) and (4.8) can be combined as
\[
\varphi_{\varepsilon,t} + A^2 \varphi_\varepsilon + A f_\varepsilon(\varphi_\varepsilon) - \lambda A \varphi_\varepsilon = (1 - \varphi_\varepsilon) S_\varepsilon + \nabla p_\varepsilon \cdot \nabla \varphi_\varepsilon - (A \varphi_\varepsilon + f_\varepsilon(\varphi_\varepsilon) - \lambda \varphi_\varepsilon) |\nabla \varphi_\varepsilon|^2.
\] (4.11)
Analogously, one may insert (4.8) into (4.9) in order to eliminate \(\mu_\varepsilon\). This yields

\[
B p_\varepsilon = S_\varepsilon - \text{div} \left( (A \varphi_\varepsilon + f_\varepsilon(\varphi_\varepsilon) - \lambda \varphi_\varepsilon) \nabla \varphi_\varepsilon \right).
\]

(4.12)

The resulting system (4.11)-(4.12), with \(S_\varepsilon\) specified by (4.10), can then be solved, at least locally in time, by using the Faedo-Galerkin scheme and possibly implementing a fixed point argument. This procedure works similarly with other Cahn-Hilliard-based systems and is therefore omitted. Note that one may need to use two families of eigenfunctions as Faedo-Galerkin bases because \(\varphi_\varepsilon\) and \(p_\varepsilon\) satisfy different types of boundary conditions.

This procedure yields a local in time solution \((\varphi_\varepsilon,p_\varepsilon)\). Of course, once such a solution is obtained, one can go back to the formulation (4.7)-(4.10) by defining \(\mu_\varepsilon\) in the natural way. In addition to that, we note that, in view of the fact that the a-priori estimates derived below have a global in time character, by standard extension arguments one may prove that, in fact, the limit solution will be globally defined. For the sake of simplicity, we shall directly work on \((0,T)\) also at the approximated level leaving the details of the extension argument to the reader.

### 4.1 Rigorous a priori estimates

Here, we would like to show that, once one tries to adapt the a-priori estimates of Section 3 to the regularized system (written in the “extended” form (4.7)-(4.10)), the procedure still remains valid. To see this, we start with repeating the energy estimate (3.1) in the present setting. This corresponds, of course, to testing (4.7) by \(\mu_\varepsilon\).

Setting back (4.7) by \(\mu_\varepsilon\), (4.8) by \(\varphi_\varepsilon, t\), and (4.9) by \(p_\varepsilon\), so to obtain

\[
\frac{d}{dt} \mathcal{E}_\varepsilon + \|
abla \mu_\varepsilon\|^2 + \|
abla p_\varepsilon\|^2 + \int_\Omega \mu_\varepsilon^2 |
abla \varphi_\varepsilon|^2 - 2 \int_\Omega \mu_\varepsilon \nabla \varphi_\varepsilon \cdot \nabla p_\varepsilon = \int_\Omega (1 - \varphi_\varepsilon)^+ S_\varepsilon \mu_\varepsilon + \int_\Omega S_\varepsilon p_\varepsilon,
\]

(4.13)

where \(\mathcal{E}_\varepsilon\) denotes the regularized energy, i.e.,

\[
\mathcal{E}_\varepsilon = \mathcal{E}_\varepsilon(\varphi_\varepsilon) = \frac{1}{2} \|
abla \varphi_\varepsilon\|^2 + \int_\Omega \left( F_\varepsilon(\varphi_\varepsilon) - \frac{\lambda}{2} \varphi_\varepsilon^2 \right).
\]

(4.14)

Setting back

\[
u_\varepsilon := -\nabla p_\varepsilon + \mu_\varepsilon \nabla \varphi_\varepsilon,
\]

we then observe that

\[
\|\nu_\varepsilon\|^2 = \|
abla p_\varepsilon\|^2 + \int_\Omega \mu_\varepsilon^2 |
abla \varphi_\varepsilon|^2 - 2 \int_\Omega \mu_\varepsilon \nabla \varphi_\varepsilon \cdot \nabla p_\varepsilon.
\]

(4.15)

As before, we can also add to (4.13) the contribution of (4.8) tested by \(A \varphi_\varepsilon\), i.e., the analogue of (3.14). Performing standard manipulations, it is then not difficult to arrive at

\[
\frac{d}{dt} \mathcal{E}_\varepsilon + \frac{1}{2} \|
abla \mu_\varepsilon\|^2 + \|A \varphi_\varepsilon\|^2 + \|
u_\varepsilon\|^2 + \int_\Omega f_\varepsilon(\varphi_\varepsilon) |
abla \varphi_\varepsilon|^2 \leq \int_\Omega (1 - \varphi_\varepsilon)^+ S_\varepsilon \mu_\varepsilon + \int_\Omega S_\varepsilon p_\varepsilon + c \|
abla \varphi_\varepsilon\|^2,
\]

(4.17)

and we need to manage the right hand side. First of all, we can treat the integral of \(S_\varepsilon p_\varepsilon\) similarly as before. Namely, we obtain the analogue of the contributions \(I_j\), \(j = 1,2,3\), of (3.10), which we need to control. The estimate (3.11) of \(I_1\) can be repeated without any variation. Concerning \(I_3\), since we do not know the a-priori boundedness of \(\varphi_\varepsilon\) at this level, we can modify (3.13) as follows:

\[
I_3 = - \int_\Omega \lambda \varphi_\varepsilon \nabla \varphi_\varepsilon \cdot \nabla \zeta_\varepsilon \leq c \|
abla \varphi_\varepsilon\|_{L^2(\Omega)} \|
abla \varphi_\varepsilon\| \|
abla \zeta_\varepsilon\|_{L^\infty(\Omega)} \leq c \|
abla \varphi_\varepsilon\|^2,
\]

(4.18)

where standard embeddings have been used.
Actually, the main difference regards the control of the term $I_2$ in (3.10). Repeating (3.12) without any notational variations, we get the $L^1$-norm of $f_\varepsilon(\phi_\varepsilon)\nabla \phi_\varepsilon$ on the right hand side, and we would like to adapt (3.17) in order to estimate it. To this aim, for a.e. $t \in (0,T)$, we may split $\Omega = \Omega_+ \cup \Omega_0 \cup \Omega_-$, where

\[ \Omega_0 = \Omega_0(t) = \{ x \in \Omega : |\phi_\varepsilon(x,t)| \leq 1 - \varepsilon \}, \quad \Omega_+ = \Omega_+(t) = \{ x \in \Omega : \phi_\varepsilon(x,t) > 1 - \varepsilon \}, \]  

(4.19)

and $\Omega_-$ defined similarly (the dependence on $\varepsilon$ of the subdomains is not stressed in the notation). Then, the integral on the left hand side of (3.17) is decomposed into its components on the three subdomains. Clearly, the part on $\Omega_0$ may be treated as in the previous section, while we need to focus on the components on $\Omega_-$ and $\Omega_+$, and, for brevity, we will only consider the latter. Actually, using (4.4) and (twice) Young’s inequality, we obtain

\[
\int_{\Omega_+} f_\varepsilon(\phi_\varepsilon)|\nabla \phi_\varepsilon| \sim \frac{1}{\varepsilon} \int_{\Omega_+} |\nabla \phi_\varepsilon| + \frac{1}{\varepsilon^2} \int_{\Omega_+} (\phi_\varepsilon - 1 + \varepsilon)|\nabla \phi_\varepsilon| \\
\leq c + \frac{1}{4\varepsilon^2} \int_{\Omega_+} |\nabla \phi_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_{\Omega_+} (\phi_\varepsilon - 1 + \varepsilon)^2 + \frac{1}{2\varepsilon^2} \int_{\Omega_+} |\nabla \phi_\varepsilon|^2, \tag{4.20}
\]

where we notice that the constant $c$ may be taken independent of $\varepsilon$.

In order to control the above right hand side uniformly with respect to $\varepsilon$, we take advantage of the last term on the left hand side of (4.17), which, owing to (4.5), can be rewritten as follows:

\[
\int_{\Omega} f'_\varepsilon(\phi_\varepsilon)|\nabla \phi_\varepsilon|^2 \sim \int_{\Omega_0} f'_\varepsilon(\phi_\varepsilon)|\nabla \phi_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_{\Omega_+} |\nabla \phi_\varepsilon|^2 + \frac{1}{2\varepsilon^2} \int_{\Omega_-} |\nabla \phi_\varepsilon|^2. \tag{4.21}
\]

We also notice that there exists a constant $k > 0$, depending on $\lambda$ but independent of $\varepsilon$, such that

\[
\Phi_\varepsilon(r) := \frac{1}{2} F_\varepsilon(r) - \frac{\lambda}{2} r^2 + k \geq 0 \tag{4.22}
\]

for every $\varepsilon \in (0,1/4)$ and $r \in \mathbb{R}$. Using the above relations, (4.17) gives

\[
\frac{d}{dt} \left[ \frac{1}{2} |\nabla \phi_\varepsilon|^2 + \int_{\Omega} \Phi_\varepsilon(\phi_\varepsilon) + \frac{1}{2} \int_{\Omega} F_\varepsilon(\phi_\varepsilon) \right] + \frac{1}{2} \| A \phi_\varepsilon \|^2 + \frac{3}{4} \| u_\varepsilon \|^2 + \frac{1}{2\varepsilon^2} \int_{\Omega_+} |\nabla \phi_\varepsilon|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega_+} |\nabla \phi_\varepsilon|^2 \lesssim \int_{\Omega} (1 - \phi_\varepsilon)^+ S_\varepsilon \mu_\varepsilon + c \tag{4.23}
\]

Now, let us observe that, thanks to (4.6), there holds

\[
\frac{1}{2\varepsilon^2} \int_{\Omega_+} (\phi_\varepsilon - 1 + \varepsilon)^2 + \frac{1}{2\varepsilon^2} \int_{\Omega_-} (\phi_\varepsilon + 1 - \varepsilon)^2 \lesssim \int_{\Omega} F_\varepsilon(\phi_\varepsilon). \tag{4.24}
\]

Hence, (4.23) can be rewritten in the simpler form

\[
\frac{d}{dt} \left[ \frac{1}{2} |\nabla \phi_\varepsilon|^2 + \int_{\Omega} \Phi_\varepsilon(\phi_\varepsilon) + \frac{1}{2} \int_{\Omega} F_\varepsilon(\phi_\varepsilon) \right] + \frac{1}{2} \| A \phi_\varepsilon \|^2 + \frac{3}{4} \| u_\varepsilon \|^2 \lesssim \int_{\Omega} (1 - \phi_\varepsilon)^+ S_\varepsilon \mu_\varepsilon + c + \int_{\Omega} F_\varepsilon(\phi_\varepsilon) + c \| \phi_\varepsilon \|_{V}. \tag{4.25}
\]

It now remains to provide a control of the first summand on the right hand side. Using (4.10), we obtain

\[
\int_{\Omega} (1 - \phi_\varepsilon)^+ S_\varepsilon \mu_\varepsilon = - \int_{\Omega} (1 - \phi_\varepsilon^2)^+ \gamma(\cdot, \cdot) (A \phi_\varepsilon + f_\varepsilon(\phi_\varepsilon) - \lambda \phi_\varepsilon). \tag{4.26}
\]
Now, it is easy to see that, as before,
\[- \int_{\Omega} (1 - \varphi_{\varepsilon}^2) + \gamma(\cdot, \cdot, \varphi_{\varepsilon})(A\varphi_{\varepsilon} - \lambda \varphi_{\varepsilon}) \leq \frac{1}{4} \|A\varphi_{\varepsilon}\|^2 + c.\]  
(4.27)
Hence, it remains to control the quantity
\[- \int_{\Omega} (1 - \varphi_{\varepsilon}^2) + \gamma(x, t, \varphi_{\varepsilon})f_{\varepsilon}(\varphi_{\varepsilon}),\]  
(4.28)
which, however, is readily estimated simply by noting that \(|f_{\varepsilon}(r)| \leq |f(r)|\) for \(|r| \leq 1\) while \((1 - r^2)^+\) is zero for \(|r| \geq 1\). Hence, the argument in (3.55) basically remains valid up to notational variations.

Thanks to the above considerations, we finally obtain
\[
\frac{d}{dt} \left[ \frac{1}{2} \|\nabla \varphi_{\varepsilon}\|^2 + \int_{\Omega} \Phi_{\varepsilon}(\varphi_{\varepsilon}) + \frac{1}{2} \int_{\Omega} F_{\varepsilon}(\varphi_{\varepsilon}) \right] + \frac{1}{2} \|\nabla u_{\varepsilon}\|^2 + \frac{1}{2} \|A\varphi_{\varepsilon}\|^2 + \|u_{\varepsilon}\|^2
\]
\[+ \frac{1}{2} \int_{\Omega} f'(\varphi_{\varepsilon}) |\nabla \varphi_{\varepsilon}|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega} |\nabla \varphi_{\varepsilon}|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega} |\nabla \varphi_{\varepsilon}|^2 \]
\[\leq c + \int_{\Omega} F_{\varepsilon}(\varphi_{\varepsilon}) + c\|\varphi_{\varepsilon}\|_V^2 \lesssim c + c \left[ \frac{1}{2} \|\nabla \varphi_{\varepsilon}\|^2 + \int_{\Omega} \Phi_{\varepsilon}(\varphi_{\varepsilon}) + \frac{1}{2} \int_{\Omega} F_{\varepsilon}(\varphi_{\varepsilon}) \right],\]  
(4.29)
the last inequality following from the nonnegativity of \(\Phi_{\varepsilon}\) and the uniform (in \(\varepsilon\)) coercivity of \(F_{\varepsilon}\).

Hence the Grönwall lemma can be applied to deduce a number of a-priori estimates independent of the approximation parameter \(\varepsilon\).

Let us note in particular that \(F_{\varepsilon} \leq F\) everywhere; for this reason, the approximate energy (i.e., the quantity in square brackets) is controlled at the initial time by means of (2.12). Hence, we obtain
\[
\|\varphi_{\varepsilon}\|_{L^\infty(0,T;V)} + \|\varphi_{\varepsilon}\|_{L^2(0,T;H^2(\Omega))} \leq c, \tag{4.30}
\]
\[
\|F_{\varepsilon}(\varphi_{\varepsilon})\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{4.31}
\]
\[
\|u_{\varepsilon}\|_{L^2(0,T;H)} \leq c, \tag{4.32}
\]
\[
\|\nabla u_{\varepsilon}\|_{L^2(0,T;H)} \leq c, \tag{4.33}
\]
\[
\int_0^T \int_{\Omega_{\varepsilon}(t)} |\nabla \varphi_{\varepsilon}(t)|^2 \,dt + \int_0^T \int_{\Omega_{\varepsilon}(t)} |\nabla \varphi_{\varepsilon}(t)|^2 \,dt \leq c\varepsilon^2. \tag{4.34}
\]

We now provide a control of the mean value of \(\varphi_{\varepsilon}\). Namely, we prove that, at least for \(\varepsilon > 0\) small enough, \((\varphi_{\varepsilon})_t\) takes values into the reference interval \((-1, 1)\). As noted in the introduction, the validity of this property is tied to the choice of the “strongly separating” potential (2.7).

Using (4.31) and applying Jensen’s inequality, we observe that there exists a constant \(C_T\), possibly depending on \(T\) but independent of \(\varepsilon\), such that, for almost every \(t \in (0, T)\), there holds
\[
C_T \geq \int_{\Omega} F_{\varepsilon}(\varphi_{\varepsilon}(t, x)) \frac{dx}{|\Omega|} \geq F_{\varepsilon} \left( \int_{\Omega} \varphi_{\varepsilon}(t, x) \frac{dx}{|\Omega|} \right). \tag{4.35}
\]
Then, since \(F_{\varepsilon}\) is even and coincides with \(F\) on the interval \([-1 + \varepsilon, 1 - \varepsilon]\), we may combine (1.35) with the corresponding property at the initial time (2.13). Hence, using in particular (2.14), it is not difficult to check that there exists a number \(\delta \in (0, 1/4),\) which depends on the initial datum and on the constant \(C_T\), but is otherwise independent of \(\varepsilon\), such that
\[-1 + \delta \leq (\varphi_{\varepsilon}(t))_t \leq -1 + \delta \quad \text{for every } \varepsilon \in (0, \delta) \text{ and a.e. } t \in (0, T). \tag{4.36}\]
The above property permits us to estimate the mean value of \(f_{\varepsilon}(\varphi_{\varepsilon})\) by using a procedure which is rather standard for the Cahn-Hilliard system. Namely, we test (1.8) by \(\varphi_{\varepsilon} - (\varphi_{\varepsilon})_n\). Then, using also the Poincaré-Wirtinger inequality it is not difficult to obtain
\[
\int_{\Omega} f_{\varepsilon}(\varphi_{\varepsilon}) (\varphi_{\varepsilon} - (\varphi_{\varepsilon})_n) + \|\nabla \varphi_{\varepsilon}\|^2 = \int_{\Omega} \mu_{\varepsilon} (\varphi_{\varepsilon} - (\varphi_{\varepsilon})_n) + \lambda \int_{\Omega} \varphi_{\varepsilon} (\varphi_{\varepsilon} - (\varphi_{\varepsilon})_n)
\]
\[= \int_{\Omega} (\mu_{\varepsilon} - \mu_{\varepsilon}) (\varphi_{\varepsilon} - (\varphi_{\varepsilon})_n) + \lambda \int_{\Omega} (\varphi_{\varepsilon} - (\varphi_{\varepsilon})_n)^2 \]
\[\leq c \|\nabla \varphi_{\varepsilon}\| (\|\nabla \mu_{\varepsilon}\| + \|\nabla \varphi_{\varepsilon}\|) \leq c(1 + \|\nabla \mu_{\varepsilon}\|), \tag{4.37}
\]
where estimate (4.30) has been used in order to deduce the last inequality.

Now, reasoning as in [19, Prop. A.1], we deduce
\[
\int_{\Omega} f_{\varepsilon}(\varphi_{\varepsilon})(\varphi_{\varepsilon} - (\varphi_{\varepsilon})_{\Omega}) \geq \kappa \| f_{\varepsilon}(\varphi_{\varepsilon}) \|_{L^1(\Omega)} - c,
\]
where the constants \( \kappa > 0 \) (possibly small) and \( c > 0 \) (possibly large) may depend on \( \delta \) but are independent of \( \varepsilon \). Hence, replacing (4.35) into (4.37), squaring, and integrating in time, we infer
\[
\| f_{\varepsilon}(\varphi_{\varepsilon}) \|_{L^2(0,T;L^1(\Omega))}^2 \leq c \left( 1 + \| \nabla \mu_{\varepsilon} \|_{L^2(0,T;H)}^2 \right) \leq c,
\]
the last inequality following from (4.33).

Hence, integrating (4.38) in space, using (4.39) and performing standard manipulations, it is not difficult to deduce
\[
\| f_{\varepsilon}(\varphi_{\varepsilon}) \|_{L^2(0,T;L^1(\Omega))} \leq c,
\]
which, compared with (4.33), yields
\[
\| \mu_{\varepsilon} \|_{L^2(0,T;V)} \leq c.
\]

Let us now observe that, from (4.41) and (4.38), using standard interpolation and three-dimensional embedding inequalities, there follows
\[
\| \mu_{\varepsilon} \nabla \varphi_{\varepsilon} \|_{L^2(0,T;L^{3/2}(\Omega))} \leq c,
\]
whence, from (4.32), (4.15) and Poincaré’s inequality, we also get
\[
\| p_{\varepsilon} \|_{L^2(0,T;W^{1,3/2}(\Omega))} \leq c.
\]

4.2 Passage to the limit

We now prove that that it is possible take the limit \( \varepsilon \to 0 \) in the approximation scheme detailed above, so to obtain existence of a solution to the original system (2.1)-(2.5).

To this aim, we first observe that, thanks to the procedure carried out in the previous section, estimates (4.30)-(4.34), (4.41) and (4.43)-(4.45) hold uniformly with respect to the approximating parameter \( \varepsilon \). These estimates, as well as the ones that will follow, imply, by means of standard weak or weak star compactness results, appropriate convergence properties, which will be implicitly (i.e., not stressing it in the notation) intended to hold up to the extraction of subsequences. First of all, we have

\[
\varphi_{\varepsilon} \to \varphi \quad \text{weakly star in } L^{\infty}(0,T;V) \cap L^4(0,T;H^2(\Omega)) \cap L^2(0,T;W^{2,6}(\Omega)),
\]
\[
u_{\varepsilon} \to \nu \quad \text{weakly in } L^2(0,T;H),
\]
\[
p_{\varepsilon} \to p \quad \text{weakly in } L^2(0,T;W^{1,3/2}(\Omega)),
\]
\[
\mu_{\varepsilon} \to \mu \quad \text{weakly in } L^2(0,T;V).
\]

In order to detail an estimate for \( \varphi_{\varepsilon,t} \), we first replace the expression (4.15) for \( \nu_{\varepsilon} \) in the system so to rewrite (4.39) in the more natural form
\[
\varphi_{\varepsilon,t} + A \mu_{\varepsilon} = (1 - \varphi_{\varepsilon})^+ S_{\varepsilon} - \nu_{\varepsilon} \cdot \nabla \varphi_{\varepsilon}.
\]

Next, combining the first (4.46) with (4.47), we deduce
\[
\| \nu_{\varepsilon} \cdot \nabla \varphi_{\varepsilon} \|_{L^2(0,T;L^1(\Omega))} \leq c.
\]
Then, using (4.40) with the properties of $A$ as a bounded linear operator from $V$ to $V'$, comparing terms in (4.51) we readily deduce

$$\|\varphi_{e,t}\|_{L^2(0,T;V'+L^1(\Omega))} \leq c. \quad (4.52)$$

Combining the above with (4.40) and applying the Aubin-Lions lemma (see, e.g., [22]), we then obtain

$$\varphi_e \to \varphi \quad \text{strongly in } C^0([0,T]; H^{1-\sigma}(\Omega)) \cap L^2(0,T; W^{2-\sigma,6}(\Omega)),$$

for every $\sigma > 0$.

Next, we observe that, thanks to (4.47), there exists a function $\varphi$ such that

$$f_e(\varphi_e) \to \xi \quad \text{weakly in } L^2(0,T;L^6(\Omega)). \quad (4.54)$$

Hence, a fortiori, convergence (4.53) holds strongly in $L^2(0,T; H)$, while (4.54) holds weakly in $L^2(0,T; H)$. Thus, in view of the fact that $f_e$ converges to $f$ in the sense of graphs (or “G-convergence”, cf. [1, Def. 3.58]) in $\mathbb{R}$, and consequently the maximal monotone operators generated by $f_e$ on $L^2(0,T; H)$ converge, in the sense of graphs in $L^2(0,T; H)$, to the maximal monotone operator generated by $f$ on $L^2(0,T; H)$, we can apply a standard monotonicity argument in the Hilbert space $L^2(0,T; H)$ (see, e.g., [3] or [2]) to deduce that

$$\xi(x,t) = f(\varphi(x,t)) \quad \text{for a.e. } (x,t) \in \Omega \times (0,T). \quad (4.55)$$

We also notice that, thanks to (4.47), the second (4.53) and Sobolev’s embeddings, there holds at least

$$u_e \cdot \nabla \varphi_e \to u \cdot \nabla \varphi \quad \text{weakly in } L^1(0,T; H). \quad (4.56)$$

Moreover, using (4.40), (4.49) and (4.53) (recall also (4.42)), we infer

$$\mu_e \nabla \varphi_e \to \mu \nabla \varphi \quad \text{weakly in } L^2(0,T; L^{3/2}(\Omega)). \quad (4.57)$$

Next, note that, by (4.53), it follows that $\varphi(x,t) \in (-1,1)$ for almost every $(x,t) \in \Omega \times (0,T)$. Combining this with the Lipschitz continuity of the positive part function, we then obtain

$$(1 - \varphi_e^+) \to (1 - \varphi)^+ = (1 - \varphi) \quad \text{strongly, say, in } L^2(0,T; L^2(\Omega)), \quad (4.58)$$

$$(1 + \varphi_e^+) \to (1 + \varphi)^+ = (1 + \varphi) \quad \text{strongly, say, in } L^2(0,T; L^2(\Omega)). \quad (4.59)$$

Analogously, the boundedness of $\gamma$ and its Lipschitz continuity with respect to $\varphi$ (cf. (2.8)-(2.9)), thanks to the dominated convergence theorem, imply

$$\gamma(x,t,\varphi_e) \to \gamma(x,t,\varphi) \quad \text{strongly in } L^r((0,T) \times \Omega). \quad (4.60)$$

for every $r \in [1,\infty)$. This implies, in turn, that

$$S_e = -(1 + \varphi_e^+) \gamma(x,t,\varphi_e) \to -(1 + \varphi) \gamma(x,t,\varphi) =: S \quad \text{strongly in } L^r((0,T) \times \Omega), \quad (4.61)$$

still for every $r \in [1,\infty)$. Collecting the above relations, it is readily seen that we can take the limit as $e \searrow 0$ of the system given by (4.50) with (4.8)-(4.10), and get (2.1)-(2.5) in the limit. In particular, the limit of all nonlinear terms is correctly identified. Finally, we may notice that, in view of (4.53), $\varphi_0 = \varphi_e\big|_{t=0} \to \varphi\big|_{t=0}$ strongly in $H$; hence, the initial condition (2.11) is satisfied. This concludes the proof of Theorem (2.4).

### 4.3 Further remarks

With the existence proof at hand, we would like to give here some more comments regarding our assumptions, with particular reference to the choice of the boundary conditions.

First of all, it is worth observing that the strategy used to derive the energy estimate, and in particular the key step (namely, the control (3.17)) would work also in the case of the “standard”
logarithmic potential (1.6) up to purely technical variations. Indeed, (3.5) still holds since \( f \) is “less singular” in the logarithmic case; hence the degenerate character of \((1 - \varphi)^2\) balances it with no need of taking sign conditions on \( \gamma \). On the other hand, with the logarithmic potential (1.6), the outcome of the energy estimate is not sufficient to pass to the limit. Indeed, from the bound corresponding to (4.31) one would no longer be able to deduce (4.36) because \( F \) takes finite values into the closed interval \([-1, 1]\). When no mass source is present, that would not be a problem because the value \( \varphi_\Omega \) is conserved; On the other hand, in our situation, the only other possible strategy to get a control of \( \varphi_\Omega \) would be that of integrating (2.1) with respect to space. Nevertheless, as noted in the introduction, with the current choice of the boundary conditions, it seems not possible to control the last term in (1.11), whence the mentioned “inconsistency” phenomenon may occur, leading to ill-posedness of the system. As noted before, that issue was avoided in \([15]\) by assuming the boundary condition (1.12) and suitably designing the mass source term. As we consider, instead, the “strongly singular” potential (2.7) with the boundary conditions (2.6), the spatial mean \( \varphi_\Omega \) is automatically controlled from the energy bound, with no need of integrating (2.1) in space.

It is also worth observing that there are other significant choices of boundary conditions for which existence of a solution remains, up to our knowledge, an open issue. To explain this fact, we go back to the paper \([15]\), and observe that, there, equation (2.4) was in fact replaced by

\[
\mathbf{u} = -\nabla q - \varphi \nabla \mu,
\]

(4.62)

This means that, if \( p \) is “our” pressure and \( q \) is the pressure in \([15]\), there holds

\[
p = q + \varphi \mu,
\]

(4.63)

as one can readily see by comparing (2.4) with (4.62). “Incorporating” the term \( \varphi \mu \) into the pressure so to obtain a different expression of the Korteweg force is a standard procedure, and, indeed, our system (2.6)-(2.7) is perfectly equivalent to system (1.1) of \([15]\) as far as one looks at the equations on \( \Omega \). There is, however, an impact on the boundary conditions; indeed, in \([15]\) the Dirichlet condition \( q = 0 \) was considered, while here we are assuming \( p = 0 \), i.e., \( q + \varphi \mu = 0 \). Hence, it is a natural question to establish whether our arguments may be extended to the case

\[
\partial_n \mu = \partial_n \varphi = q = 0 \quad \text{on } \Gamma.
\]

(4.64)

Note that the above is different both from our (2.6) and from the choice of \([15]\), which in our notation reads as

\[
\partial_n \mu - \varphi \mathbf{u} \cdot \mathbf{n} = \partial_n \varphi = q = 0 \quad \text{on } \Gamma.
\]

(4.65)

However, it seems that conditions (4.64) are not straightforward to deal with, with the main issue arising already at the level of the energy estimate. Actually, repeating the computations at the beginning of Section 3, i.e. combining (3.1) with the analogue of (3.6), in the case (4.64) one would obtain

\[
\frac{d}{dt} \mathcal{E} + \|
abla \mu \|^2 + ||\mathbf{u}||^2 = \int_\Omega S_\mu + \int_\Omega Sq - \int_\Gamma \varphi \mu \mathbf{u} \cdot \mathbf{n},
\]

(4.66)

where the boundary term has no sign properties and seems very difficult to control by using trace theorems and embeddings because only the \( H \)-norm of \( \mathbf{u} \) appears on the left hand side. Finally, we observe that the case

\[
\partial_n \mu - \varphi \mathbf{u} \cdot \mathbf{n} = \partial_n \varphi = p = 0 \quad \text{on } \Gamma,
\]

(4.67)

seems to present mathematical difficulties similar to those occurring for (4.64); namely, a boundary integral, difficult to be controlled, remains on the right hand side of the energy inequality.

5 Additional regularity and uniqueness in dimension two

In this part we sketch the proof of Theorem 2.2. Actually, most of the procedure will follow the lines of the argument in \([15\) Sec. 4]; for this reason we will only outline the parts where the main differences arise. In fact, all the necessary variations depend only on the different choice of boundary conditions.
and to the corresponding definition of the pressure (as explained in the previous section), while at this level the "strongly separating" potential plays no role at all.

Of course, the main tool in the proof consists in the derivation of additional a-priori estimates, which, to avoid technicalities, will be carried out in a formal way, i.e. by working directly on the "original" system (2.1)-(2.5) without referring to the approximation. In particular, we will take advantage of all the information coming from the previous estimates, including the uniform bound $|\varphi| \leq 1$. Actually, to make the procedure completely rigorous one could proceed along the lines of [15, Sec. 4] where this issue is discussed in detail. Note in particular that the approximation scheme should also be refined, possibly operating some additional regularization of the initial data.

That said, we first observe that formulas [15, (4.39)-(4.41)] remain valid also in the current setting. In our notation, these properties, which are a direct consequence of the energy estimate, read

$$\|\mu\|_V \leq c(1 + \|\nabla\mu\|), \quad (5.1)$$

$$\|\varphi\|_{H^2(\Omega)} \leq c(1 + \|\nabla\mu\|), \quad (5.2)$$

$$\|\varphi\|_{W^{2,r}(\Omega)} \leq c_r(1 + \|\nabla\mu\|), \quad \text{for every } r \in [1, \infty), \quad (5.3)$$

where the constants $c$ (or $c_r$ in the last case, exploding as $r \nearrow \infty$), only depend on quantities that have already been estimated uniformly with respect both to the approximation parameter and to the time variable.

Then, the key tool we use in order to obtain higher order 2D estimates is the following Brezis-Gallouet-Wainger inequality (see, e.g., [8]):

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_V \log^{1/2} (e + \|f\|_{W^{1,r}(\Omega)}) + C, \quad (5.4)$$

valid for every $r > 2$ and $f \in W^{1,r}(\Omega)$, with $C > 0$ also depending on $r$. Of course, thanks to Sobolev’s embeddings, the $W^{1,r}$-norm may be replaced by the $H^2$- one when $f \in H^2(\Omega)$.

That said, we start detailing the regularity estimates. Basically, we need to test (2.4) by $\mu$; moreover, we take the time derivative of (2.2) and test it by $\varphi_t$. Then, comparing the resulting relations, we readily obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla\mu\|^2 + \|\varphi_t\|^2 + \int \phi'(\varphi)\varphi_t^2 - \int (1-\varphi)S\mu_t - \lambda\|\varphi_t\|^2 - \int \mu_t u \cdot \nabla \varphi. \quad (5.5)$$

Taking the time derivative of (2.1) and testing it by $u$, we also infer

$$\frac{1}{2} \frac{d}{dt} \|\mu\|^2 + \int p_t S = \int \mu_t u \cdot \nabla \varphi + \int \mu u \cdot \nabla \varphi_t. \quad (5.6)$$

Next, we integrate by parts in time the last terms on the left hand sides of (5.5) and (5.6). Then, we combine the two obtained relations and notice that two terms cancel out. Using also the monotonicity of $f$ (which permits us to neglect a further positive quantity), we then deduce

$$\frac{d}{dt} \left[ \frac{1}{2} \|\nabla\mu\|^2 + \frac{1}{2} \|\mu\|^2 - \int (1-\varphi)S\mu - \int pS \right] + \|\varphi_t\|^2 = \lambda \|\varphi_t\|^2 - \int \mu_t \frac{d}{dt} \left[ (1-\varphi)S \right] - \int pS_t + \int \mu u \cdot \nabla \varphi_t. \quad (5.7)$$

This relation basically corresponds to formula (4.56) of [15]. The main difference is provided by the last term on the right hand side, which in the setting of [15] has a different expression, namely

$$- \int \varphi_t u \cdot \nabla \mu, \quad (5.8)$$

due to the different choice of the pressure and of the Korteweg term taken there.

On the other hand, our situation is in several aspects simpler compared to [15]. Actually, here $\mu$ satisfies a no-flux boundary condition, while in [15] the boundary conditions for $\mu$ and $u$ were
coupled and consequently more complicated to deal with. Indeed, applying elliptic regularity results to (2.1) and recalling (5.1), it is easy to deduce
\[
\|\mu\|_{H^2(\Omega)} \leq c\left(\|\mu\| + \|\Delta \mu\|\right) \leq c\left(1 + \|\nabla \mu\| + \|\varphi_t\| + \|(1 - S)\varphi\| + \|u \cdot \nabla \varphi\|\right)
\leq c\left(1 + \|\nabla \mu\| + \|\varphi_t\| + \|u\|\|\nabla \varphi\|_{L^\infty(\Omega)}\right).
\]
(5.9)
This relation, whose analogue in [15] has a much longer proof, will play a key role in the sequel.

To proceed, we would like to provide a control of the right hand side of (5.7). To this aim, we need some preparatory computations. First of all, in order to get a tractable differential inequality, as in [15] we denote by \(H\) the quantity under time derivative in (5.7). Using the uniform boundedness of \(S\) and \(\varphi\), (5.1), the first (4.30), a suitable version of Poincaré’s inequality, and Young’s inequality, we may then notice that
\[
H = \frac{1}{2}\|\nabla \mu\|^2 + \frac{1}{2}\|u\|^2 - \int_{\Omega} (1 - \varphi) S \mu - \int_{\Omega} pS
\leq \frac{1}{2}\|\nabla \mu\|^2 + \frac{1}{2}\|u\|^2 + c\|\mu\|_{L^1(\Omega)}
\leq c\|\nabla \mu\|^2 + c + \frac{1}{2}\|u\|^2 + c\|\nabla p\|_{L^1(\Omega)}
\leq c\|\nabla \mu\|^2 + c + \frac{1}{2}\|u\|^2 + c\|u\|_{L^1(\Omega)} + c\|\mu\||\nabla \varphi||
\leq c\left(1 + \|\nabla \mu\|^2 + \|u\|^2\right).
\]
(5.10)
A similar procedure, which we do not detail for the sake of brevity, gives also the reverse inequality, i.e.,
\[
H \geq \frac{1}{4}\left(\|\nabla \mu\|^2 + \|u\|^2\right) - C,
\]
(5.11)
where the “large” constant \(C > 0\) only depends on quantities that have already been controlled uniformly in time in the previous estimates.

Let us now observe that, by (2.1) with the Poincaré-Wirtinger inequality, there follows
\[
\|
\varphi_t\| \leq \|
\varphi_t - (\varphi_t)_\Omega\| + c|((\varphi_t)_\Omega)| \leq c\|\nabla \varphi_t\| + c|((\varphi_t)_\Omega)|
\leq c\|\nabla \varphi_t\| + c\|(1 - \varphi) S\|_{L^1(\Omega)} + c\|u \cdot \nabla \varphi\|_{L^1(\Omega)}
\leq c\|\nabla \varphi_t\| + c + c\|u\|\|\nabla \varphi_t\| \leq c\left(\|\nabla \varphi_t\| + \|u\|\right).
\]
(5.12)
Then, on account of the above considerations, and using in particular (5.1)- (5.3), (5.11), (5.10), (5.11), and (5.12), the last term in (5.7) can be estimated as follows:
\[
\int_{\Omega} \mu u \cdot \nabla \varphi_t \leq \|\mu\|_{L^\infty(\Omega)} \|u\|\|\nabla \varphi_t\|
\]
\[
\leq c\left(1 + \|\nabla \mu\|\right) \log^{1/2} (e + \|\mu\|_{H^2(\Omega)}) \|u\|\|\nabla \varphi_t\|
\leq c\left(1 + H\right) \log^{1/2} (e + \|\mu\|_{H^2(\Omega)}) \|\nabla \varphi_t\|
\leq c\left(1 + H\right) \log^{1/2} (e + \|\nabla \mu\| + \|\varphi_t\| + \|u\|\|\nabla \varphi\|_{L^\infty(\Omega)}) \|\nabla \varphi_t\|
\leq c\left(1 + H\right) \log^{1/2} (e + H^{1/2} + \|\nabla \varphi_t\| + H^{1/2}\|u\|_{W^{2,1}(\Omega)}) \|\nabla \varphi_t\|
\leq c\left(1 + H\right) \log^{1/2} (e + \|\nabla \varphi_t\| + H) \|\nabla \varphi_t\|
\leq c\left(1 + H\right) \log^{1/2} (e + \|\nabla \varphi_t\| + H) \|\nabla \varphi_t\| + c(1 + H) \log^{1/2} (e + \|\nabla \varphi_t\|) \|\nabla \varphi_t\|
=: J_1 + J_2.
\]
(5.13)
Let us now go back to (5.7) and observe first that, by interpolation,
\[
\|\varphi_t\| \leq \|\varphi_t - (\varphi_t)_\Omega\| + |((\varphi_t)_\Omega)| \leq \|\varphi_t - (\varphi_t)_\Omega\|_{V'} \|\varphi_t - (\varphi_t)_\Omega\|_{V'}^{1/2} + |((\varphi_t)_\Omega)|
\leq c\|\varphi_t\|_{V'}^{1/2} \|\nabla \varphi_t\|^{1/2} + c\|\varphi_t\|_{V'} + c\|\varphi_t||V'|
\leq c\|\varphi_t\|_{V'}^{1/2} \|\nabla \varphi_t\|^{1/2} + c\|\varphi_t\|_{V'} + c\|\varphi_t||V',
\]
(5.14)
for every “small” $\epsilon > 0$ and correspondingly “large” $c_\epsilon > 0$.

Then, using the above relation (with $\epsilon$ taken small enough, which generates the constant $7/8$ below) it is very easy to provide a control of the first three terms on the right hand side of (5.7), all of which basically depend on the $L^2$-norm of $\varphi_t$ (see 1.5 for details). Managing the last summand in (5.7) by means of (5.13), it is then not difficult to obtain

$$\frac{d}{dt} \mathcal{H} + \frac{7}{8} \|\nabla \varphi_t\|^2 \leq c(1 + \mathcal{H})^2 + J_1 + J_2. \quad (5.15)$$

Moreover, using Young’s inequality it is apparent that

$$J_1 = c(1 + \mathcal{H}) \log^{1/2}(e + \mathcal{H}) \|\nabla \varphi_t\| \leq \frac{1}{8} \|\nabla \varphi_t\|^2 + c(1 + \mathcal{H})^2 \log(e + \mathcal{H}). \quad (5.16)$$

The control of $J_2$ is a bit more delicate. To achieve it, we consider the convex function $\psi : [0, \infty) \to \mathbb{R}$, $\psi(r) = r^2 \log(e + r)$ (5.17)

and we would like to provide some estimate near infinity for its convex conjugate

$$\psi^*(s) = \sup_{r \geq 0} \{rs - \psi(r)\}. \quad (5.18)$$

Actually, a simple computation permits us to check that, when $s$ is assigned large enough, the supremum is assumed at $r = r_0$ such that

$$r_0 = (\psi')^{-1}(s), \quad \text{i.e.,} \, s = \psi'(r_0) = 2r_0 \log(e + r_0) + \frac{r_0^2}{e + r_0} \sim 2r_0 \log(e + r_0), \quad (5.19)$$

where the last approximate equality holds as $s$, hence $r_0$, is large enough. In particular, using the relation $\log(e + r_0) \leq r_0/2$, also holding for $r_0$ large enough, we may estimate

$$\log(e + s) \lesssim \log(e + 2r_0 \log(e + r_0)) \leq \log(e + r_0^2) \leq \log(e + r_0)^2 = 2 \log(e + r_0). \quad (5.20)$$

Hence, replacing the above value of $s$ in (5.18), we deduce

$$\psi^*(s) \sim 2r_0^2 \log(e + r_0) - \psi(r_0) = r_0^2 \log(e + r_0) \sim \frac{s^2}{4 \log^2(e + r_0)} \log(e + r_0) = \frac{s^2}{4 \log(e + r_0)} \leq \frac{s^2}{2 \log(e + s)}. \quad (5.21)$$

Then, we use the Fenchel inequality

$$sr \leq \psi(r) + \psi^*(s) \quad (5.22)$$

with the choices

$$r = c(1 + \mathcal{H}), \quad s = \|\nabla \varphi_t\| \log^{1/2}(e + \|\nabla \varphi_t\|), \quad (5.23)$$

where the constant $c$ is the same as in the definition of $J_2$ in (5.13). It is then easy to verify that

$$\psi(r) \leq c(1 + \mathcal{H})^2 \log(e + \mathcal{H}). \quad (5.24)$$

On the other hand, by (5.21), we deduce

$$\psi^*(s) \lesssim \frac{s^2}{2 \log(e + s)} \sim \frac{\|\nabla \varphi_t\|^2 \log(e + \|\nabla \varphi_t\|)}{2 \log(e + \|\nabla \varphi_t\|) \cdot \log^{1/2}(e + \|\nabla \varphi_t\|)} \lesssim \frac{\|\nabla \varphi_t\|^2 \log(e + \|\nabla \varphi_t\|)}{2 \log(e + \|\nabla \varphi_t\|)} \lesssim \frac{1}{2} \|\nabla \varphi_t\|^2. \quad (5.25)$$

Consequently, by (5.10), (5.22) and (5.25), (5.13) reduces to

$$\frac{d}{dt}(e + \mathcal{H}) + \frac{1}{4} \|\nabla \varphi_t\|^2 \lesssim c(e + \mathcal{H})^2 \log(e + \mathcal{H}). \quad (5.26)$$
Hence, in view of the fact that, as a consequence of the energy estimate, \( H \in L^1(0,T) \) (cf. in particular (4.9) and (4.11)), we may apply a generalized version of Grönwall’s lemma to the above relation. This yields the regularity properties

\[
\nabla \mu \in L^\infty(0,T; H), \quad u \in L^\infty(0,T; H), \quad (5.27)
\]

\[
\varphi_t \in L^2(0,T; V), \quad (5.28)
\]

provided that the value of \( H \) is finite at the initial time \( t = 0 \), which, at least formally, corresponds to the conditions

\[
\mu_0 := \mu|_{t=0} \in V, \quad u_0 := u|_{t=0} \in H. \quad (5.29)
\]

Actually, due to the quasi-stationary character of the model, the above properties have to be deduced from the regularity assumed on \( \varphi_0 \), which is the sole initial datum associated with the system. Hence, we now verify that hypotheses (2.27)-(2.28) imply (5.29). Of course, this is just a formal check: indeed, (2.26), corresponds exactly to the first (5.29). Hence we just need to show that

\[
\text{Hence the initial value of } \varphi \text{, which in turn implies } \| \mu_0 \| \| \varphi_0 \|_{L^2(\Omega)} < + \infty, \quad (5.31)
\]

the last inequality following from (2.27)-(2.28). Hence, \( \text{div}(\mu_0 \nabla \varphi_0) \in H^{-1}(\Omega) \) and, by elliptic regularity results, \( p_0 \in H^1_0(\Omega) \), which in turn implies \( \nabla p_0 \in H \) and, by (5.31) again, \( u_0 \in H \), as desired. Hence the initial value of \( H \) is finite, which implies (5.27)-(5.28).

To conclude the proof, we need to show the regularity properties (2.27)-(2.28). Actually, a part of them follows from (5.27)-(5.28) and (5.32). To prove the missing ones, we proceed along the lines of [15] with some small variations deriving from the different expression of the Korteweg term.

First of all, we combine again (1.37) and (4.35). Then, by means of the improved time regularity of \( \nabla \mu \) in (5.27), it is not difficult to improve (1.39) and (4.11) as follows:

\[
f(\varphi) \in L^\infty(0,T; L^1(\Omega)), \quad \mu \in L^\infty(0,T; V). \quad (5.32)
\]

With this at hand, noting that in 2D one has \( V \subset L'(\Omega) \) for every \( r \in [1, \infty) \), and applying elliptic regularity results to the analogue of (1.5), we deduce (2.30) and the last (2.23).

Next, we observe that \( p \) solves the time dependent family of elliptic problems

\[
Bp = S - \nabla \mu \cdot \nabla \varphi - \mu \Delta \varphi, \quad (5.33)
\]

where it is easy to check that the right hand side lies in \( L^\infty(0,T; H) \) thanks to (5.32) and the last (2.23) (which implies, in particular, that \( \nabla \varphi \) is bounded in the uniform norm). From standard elliptic regularity results we then deduce (2.30). Now, reasoning on (2.24) and using the boundedness of \( A \) as a linear operator from \( V \) to \( V' \), it is not difficult to get the first (2.24). From (2.24) and interpolation, we then also obtain \( \varphi_t \in L^4(0,T; H) \). In turn, seeing (2.24) as an elliptic problem for \( \mu \), this yields the second (2.31). Finally, a direct check shows that \( \mu \nabla \varphi \in L^\infty(0,T; V) \). Combining this with (2.33), we then deduce (2.32), which concludes the proof of the regularity part.

We now sketch the proof of uniqueness, which, as noted above, may be carried out basically following the lines of [15]. To this aim, we consider two strong solutions \((\varphi_1, u_1, \mu_1, p_1)\) and \((\varphi_2, u_2, \mu_2, p_2)\) originating from the same initial condition \( \varphi_0 \). We then set

\[
\varphi := \varphi_1 - \varphi_2, \quad u := u_1 - u_2, \quad p := p_1 - p_2, \quad \mu := \mu_1 - \mu_2, \quad (5.34)
\]
so that the quadruplet \((\varphi, u, p, \mu)\) turns out to solve the system

\[
\begin{align*}
\varphi_t + u_1 \cdot \nabla \varphi + u \cdot \nabla \varphi_2 &= \Delta \mu + (1 - \varphi_1)S_1 - (1 - \varphi_2)S_2, \\
\mu &= -\Delta \varphi + f(\varphi_1) - f(\varphi_2) - \lambda \varphi, \\
\text{div } u &= S_1 - S_2, \\
u &= -\nabla p + \mu_1 \nabla \varphi + \mu \nabla \varphi_2,
\end{align*}
\]

(5.35)

(5.36)

(5.37)

(5.38)

where we have also set \(S_i = -(1 + \varphi_i)\gamma(x, t, \varphi_i)\), for \(i = 1, 2\).

We then test (5.36) by \(u\), (5.35) by \(\mu\), and (5.30) by \(\varphi_t\). Using standard tools, it is then not difficult to arrive at

\[
\frac{d}{dt} \left[ \frac{1}{2} \|\nabla \varphi\|^2 + \frac{1}{2} \int_{\Omega} \ell(\varphi_1, \varphi_2) |\varphi|^2 \right] + \|u\|^2 + \|\nabla \mu\|^2 \\
= \lambda \int_{\Omega} \varphi \varphi_t - \int_{\Omega} \mu u_1 \cdot \nabla \varphi + \int_{\Omega} \mu_1 u \cdot \nabla \varphi + \frac{1}{2} \int_{\Omega} |\varphi|^2 \frac{d}{dt} \ell(\varphi_1, \varphi_2) \\
+ \int_{\Omega} p(S_1 - S_2) + \int_{\Omega} ((1 - \varphi_1)S_1 - (1 - \varphi_2)S_2) \mu,
\]

(5.39)

where we have set

\[
\ell(\varphi_1, \varphi_2) := \int_0^1 f'(s \varphi_1 + (1 - s) \varphi_2) \, ds.
\]

(5.40)

Moreover, we also need to consider the elliptic problem associated with (5.38), namely

\[
Bp = S_1 - S_2 - \text{div}(\mu_1 \nabla \varphi + \mu \nabla \varphi_2).
\]

(5.41)

Testing (5.41) by \(B^{-1}p\), and using standard embedding inequalities, it is then easy to deduce

\[
\|p\|^2 = \int_{\Omega} (S_1 - S_2)B^{-1}p + \int_{\Omega} (\mu_1 \nabla \varphi + \mu \nabla \varphi_2) \cdot \nabla B^{-1}p \\
\leq \frac{1}{4} \|p\|^2 + c_\varphi \|\varphi\|^2 + \|\mu_1 \nabla \varphi + \mu \nabla \varphi_2\| \|\nabla B^{-1}p\| \\
\leq \frac{1}{2} \|p\|^2 + c(1 + \|\mu_1\|_{L^2(\Omega)}) \|\varphi\|_{L^2}^2 + c_\varphi \|\varphi\|^2,
\]

(5.42)

where we also used that \(\nabla \varphi_2\) is bounded in the uniform norm as a consequence of the last (2.22).

Then, we test (5.30) by \(\mu\) to control the last term (note that this will be used also to treat the other terms depending on \(\mu\) on the right hand side of (5.39)). Standard embedding inequalities permit us to arrive at

\[
\|\mu\|^2 \leq \epsilon \|\nabla \mu\|^2 + c_\epsilon (1 + \|\ell(\varphi_1, \varphi_2)\|_{L^2(\Omega)}) \|\varphi\|_{L^2(\Omega)}^2,
\]

(5.43)

for small \(\epsilon > 0\) and correspondingly large \(c_\epsilon > 0\) to be chosen later on.

Now, noticing that \(\ell \geq 2\) almost everywhere thanks to (3.16), it is easy to verify that

\[
\mathcal{K} := \left[ \frac{1}{2} \|\nabla \varphi\|^2 + \frac{1}{2} \int_{\Omega} \ell(\varphi_1, \varphi_2) |\varphi|^2 \right] \geq \frac{1}{2} \|\varphi\|_{L^2}^2.
\]

(5.44)

Comparing terms in (5.39), we then see that the first term on the right hand side of (5.39) can be treated as follows:

\[
\lambda \int_{\Omega} \varphi \varphi_t \leq \epsilon \|\varphi_t\|_{L^2}^2 + c_\epsilon \|\varphi\|_{L^2}^2 \\
\leq \epsilon \|u_1 \cdot \nabla \varphi + u \cdot \nabla \varphi_2 - \Delta \mu - (1 - \varphi_1)S_1 + (1 - \varphi_2)S_2\|_{L^2}^2 + c_\epsilon \|\varphi\|_{L^2}^2 \\
\leq c \|u_1 \cdot \nabla \varphi\|_{L^2(\Omega)}^2 + c \|\Delta \mu\|_{L^2(\Omega)}^2 + c \|u \cdot \nabla \varphi_2 - (1 - \varphi_1)S_1 + (1 - \varphi_2)S_2\|_{L^2(\Omega)}^2 + c_\epsilon \|\varphi\|_{L^2}^2 \\
\leq c \|u_1\|_{L^2(\Omega)}^2 \|\nabla \varphi\|^2 + c \|\nabla \mu\|^2 + c_\epsilon \|u\|_{L^2(\Omega)}^2 \|\nabla \varphi_2\|_{L^2(\Omega)}^2 + c_\epsilon \|\varphi\|_{L^2}^2 \\
\leq \frac{1}{8} (\|u_t\|^2 + \|\nabla \mu\|^2) + c(1 + \|u_1\|_{L^2(\Omega)}^2) \|\varphi\|_{L^2}^2,
\]

(5.45)
where in the last passage we have taken $\epsilon > 0$ small enough depending on the value of the $L^\infty$-norm of $\nabla \varphi_2$, which is a known quantity.

Then, it is readily seen that the remaining terms on the right hand side of (5.39) can be treated without further difficulties (the details are very similar to [15, Sec. 5]), provided that one can estimate the quantity $\ell$ defined in (5.40). This is in a sense the most delicate part because it involves the singular function $f$ and its derivatives. Namely, going back to (5.39) and (5.43), we need to provide a bound for the sum

$$\frac{1}{2} \int_{\Omega} |\varphi|^2 \frac{d}{dt} \ell(\varphi_1, \varphi_2) + c_\epsilon \|\ell(\varphi_1, \varphi_2)\|_{L^4(\Omega)}^2 \|\varphi\|^2_V.$$  \hfill (5.46)

On the other hand, with the strongly separating potential (2.7) this task is in fact simpler compared to the standard logarithmic potential (1.6) considered in [15]. Indeed, using (2.30) and performing direct computations, one can easily verify that

$$f^{(j)}(\varphi_i) \in L^\infty(0, T; L^r(\Omega))$$

for all $j \geq 0$, $i = 1, 2$, and $r \in [1, \infty)$. \hfill (5.47)

Then, we just sketch the estimation of the first summand in (5.46), the latter one being in fact simpler. Actually, by (5.40),

$$\frac{d}{dt} \ell(\varphi_1, \varphi_2) := \int_0^1 f''(s\varphi_1 + (1-s)\varphi_2)(s\varphi_{1,t} + (1-s)\varphi_{2,t}) \, ds,$$  \hfill (5.48)

whence, by standard embeddings,

$$\frac{1}{2} \int_{\Omega} |\varphi|^2 \frac{d}{dt} \ell(\varphi_1, \varphi_2) \leq c(1 + \|f''(\varphi_1)\|_{L^1(\Omega)} + \|f''(\varphi_2)\|_{L^1(\Omega)}) \left(\|\varphi_{1,t}\|_V + \|\varphi_{2,t}\|_V\right) \|\varphi\|_V^2. \hfill (5.49)$$

Then, going back to (5.39) and noting that the remaining terms on the right hand side can be controlled in a simple way (see [15] for details), it is readily seen that the Grönwall lemma can be applied to the functional $K$ defined by (5.44), so to obtain uniqueness of $\varphi$ and (using the latter summand on the left hand side of (5.39) together with (5.43) of $\mu$). Then, the uniqueness of $p$ is obtained from the elliptic problem (5.41) and, finally, that of $u$ from (5.38). This concludes the proof of Theorem 2.2.

**Remark 5.1.** Refining a bit the procedure (and in particular specifying the control of the remaining quantities on the right hand side of (5.39)), one may also obtain a continuous dependence estimate, so yielding well-posedness of the model in the class of “strong” solutions in the 2D setting.

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