Fokker-Planck equations for nonlinear dynamical systems driven by non-Gaussian Lévy processes *

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Abstract

The Fokker-Planck equations describe time evolution of probability densities of stochastic dynamical systems and are thus widely used to quantify random phenomena such as uncertainty propagation. For dynamical systems driven by non-Gaussian Lévy processes, however, it is difficult to obtain explicit forms of Fokker-Planck equations because the adjoint operators of the associated infinitesimal generators usually do not have exact formulation. In the present paper, Fokker-Planck equations are derived in terms of infinite series for nonlinear stochastic differential equations with non-Gaussian Lévy processes. A few examples are presented to illustrate the method.

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1 Introduction and statement of the problem

The Fokker-Planck equations are one of the deterministic tools to quantify how randomness propagates or evolves in nonlinear dynamical systems. For stochastic differential equations (SDEs) with Gaussian processes such as Brownian motion, the Fokker-Planck equations are well established [6, 3]. However, for SDEs with non-Gaussian processes such as Lévy processes, explicit forms of the Fokker-Planck equations are not easily available, except in some special cases [1, 9]. The difficulty is to obtain the expressions for the adjoint operators of the infinitesimal generators associated with these SDEs.

Lévy processes are a class of stochastic processes having independent and stationary increments, as well as stochastically continuous sample paths [1, 8]. Given a sample space $\Omega$, together with a probability measure $\mathbb{P}$ and the corresponding mathematical expectation $E$. A Lévy process $L_t$, taking values in $\mathbb{R}^d$, is characterized by a drift parameter $b \in \mathbb{R}^d$, a $d \times d$ positive-
definite covariance matrix $A$ and a measure $\nu$ defined on $\mathbb{R}^d$ and concentrated on $\mathbb{R}^d \setminus \{0\}$. In fact, this measure $\nu$ satisfies the following condition \[ \int_{\mathbb{R}^d \setminus \{0\}} (\|y\|^2 \wedge 1) \nu(dy) < \infty, \] (1) or equivalently \[ \int_{\mathbb{R}^d \setminus \{0\}} \frac{\|y\|^2}{1 + \|y\|^2} \nu(dy) < \infty. \] (2)

Here $\| \cdot \|$ is the usual Euclidean norm or length in $\mathbb{R}^d$. This measure $\nu$ is called a Lévy jump measure for the Lévy process $L_t$. A Lévy process with the generating triplet $(b, A, \nu)$ has the Lévy-Itô decomposition

$$L_t = bt + B_t + \int_{\|y\| < 1} y \tilde{N}(t, dy) + \int_{\|y\| \geq 1} y N(t, dy),$$

(3)

where $N(dt, dx)$ is the Poisson random measure, $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx) dt$ is the compensated Poisson random measure, and $B_t$ is an independent $d$-dimensional Brownian motion (i.e., Wiener process) with covariance matrix $A$. Equation (3) can be formally rewritten in a differential form as

$$dL_t = b dt + dB_t + \int_{\|y\| < 1} y \tilde{N}(dt, dy) + \int_{\|y\| \geq 1} y N(dt, dy).$$

(4)

We shall consider stochastic dynamical systems described by the following SDE in the Itô form

$$dX_t = f(X_t, t) dt + \sigma(X_{t-}, t) dL_t, \quad X_0 = x,$$

(5)

or in the Marcus form

$$dX_t = f(X_t, t) dt + \sigma(X_{t-}, t) \circ dL_t, \quad X_0 = x,$$

(6)

where $L_t$ is a Lévy process with the generating triplet $(b, A, \nu)$. Equation (5) is meaningful in the sense of

$$X_t = X_0 + \int_0^t f(X_s, s) ds + \int_0^t \sigma(X_{s-}, s) dL_s,$$

(7)
where the last term is an Itô integral. Note that Lévy processes are semi-
martingales, and Itô integrals with respect to semimartingales are well de-

fined [7]. The equation (6) is interpreted as

\[ X_t = X_0 + \int_0^t f(X_s, s)ds + \int_0^t \sigma(X_{s-}, s) \diamond dL_s, \]  

where “\diamond” indicates Marcus integral [4, 5, 1] defined by

\[ \int_0^t \sigma(X_{s-}, s) \diamond dL_s = \int_0^t \sigma(X_{s-}, s)dL_s + \frac{1}{2} \int_0^t \sigma(X_{s-}, s)\sigma'(X_{s-}, s)d[L_s, L_s] + \sum_{0 \leq s \leq t} [\xi(\Delta L(s), \sigma(X_{s-}, s), X_{s-}) - X_{s-} - \sigma(X_{s-}, s)\Delta L_s], \]

with \( \xi(r, g(x), x) \) being the value at \( z = 1 \) of the solution of the following ordinary differential equation:

\[ \frac{dy(z)}{dz} = rg(y(z)), \quad y(0) = x. \]  

Assumption \((H_1)\):

We assume that the appropriate Lipschitz and growth conditions on the

drift \( f \) and noise intensity \( \sigma \) are satisfied so that both equations (5) and (6)

have unique solutions.

The main objective of this paper is to derive an expression of the Fokker-

Planck equations for nonlinear dynamical systems described by SDEs (5)

and (6), in \( \S 2 \) and \( \S 3 \) respectively. For simplicity, we only consider one-
dimensional case, i.e., \( X_t \) and \( L_t \) in (5) and (6) are all scalar processes. The

conclusion can be generalized into higher dimensional cases which, however,

will not be considered in this paper. A few examples will be presented in

\( \S 4 \).

2 Fokker-Planck equations for SDEs with Itô integrals

In this section, we derive the Fokker-Planck equation for Itô SDE (5), in

which \( L_t \) being a scalar Lévy process with the triplet \( (b, A, \nu) \). In this case,
A is a non-negative scalar.

Substituting (11) into (5), we get

$$dX_t = f(X_t, t)dt + b\sigma(X_{t-}, t)dt + \sigma(X_{t-}, t)dB_t + \int_{|y|<1} \sigma(X_{t-}, t)y \tilde{N}(dt, dy)$$

$$+ \int_{|y|\geq 1} \sigma(X_{t-}, t)y N(dt, dy).$$

(11)

By Itô formula [1], it follows from (11) that for any smooth function $\phi(x)$,

$$\phi(X_{t+\Delta t}) - \phi(X_t) = \int_t^{t+\Delta t} f(X_s, s) \frac{\partial}{\partial x} \phi(X_s) ds + \int_t^{t+\Delta t} b\sigma(X_{s-}, s) \frac{\partial}{\partial x} \phi(X_s) ds$$

$$+ \int_t^{t+\Delta t} \sigma(X_{s-}, s) \frac{\partial}{\partial x} \phi(X_s) dB_s + \frac{A}{2} \int_t^{t+\Delta t} \sigma^2(X_{s-}, s) \frac{\partial^2}{\partial x^2} \phi(X_s) ds$$

$$+ \int_t^{t+\Delta t} \int_{|y|\geq 1} \left[ \phi(X_{s-} + y\sigma(X_{s-}, s)) - \phi(X_{s-}) \right] N(ds, dy)$$

$$+ \int_t^{t+\Delta t} \int_{|y|<1} \left[ \phi(X_{s-} + y\sigma(X_{s-}, s)) - \phi(X_{s-}) - y\sigma(X_{s-}, s) \frac{\partial}{\partial x} \phi(X_{s-}) \right] \nu(dy) ds.$$

(12)

Let $Q$ be the infinitesimal generator associated with the Markovian solution process $X_t$. Then for any function $\phi$ in $C_0^\infty(\mathbb{R})$, consisting of smooth functions with compact support on $\mathbb{R}$, we obtain

$$Q\phi(x) = \lim_{\Delta t \to 0} \frac{\mathbb{E}\{\phi(X_{t+\Delta t})|X_t = x\} - \phi(x)}{\Delta t}$$

$$= f(x, t) \frac{\partial}{\partial x} \phi(x) + b\sigma(x, t) \frac{\partial}{\partial x} \phi(x) + \frac{A}{2} \sigma^2(x, t) \frac{\partial^2}{\partial x^2} \phi(x)$$

$$+ \int_{|y|<1} \left[ \phi(x + y\sigma(x, t)) - \phi(x) - y\sigma(x, t) \frac{\partial}{\partial x} \phi(x) \right] \nu(dy)$$

$$+ \int_{|y|\geq 1} \left[ \phi(x + y\sigma(x, t)) - \phi(x) \right] \nu(dy),$$

(13)

or equivalently,

$$Q\phi(x) = f(x, t) \frac{\partial}{\partial x} \phi(x) + b\sigma(x, t) \frac{\partial}{\partial x} \phi(x) + \frac{A}{2} \sigma^2(x, t) \frac{\partial^2}{\partial x^2} \phi(x)$$

$$+ \int_{\mathbb{R}\setminus\{0\}} \left[ \phi(x + y\sigma(x, t)) - \phi(x) - I_{(-1, 1)}(y) y\sigma(x, t) \frac{\partial}{\partial x} \phi(x) \right] \nu(dy),$$

(14)
where $I_{(-1,1)}$ the indicator function of the set $(-1,1)$. To get the second identity in (13), we have used the fact that

$$
E \left\{ \int_t^{t+\Delta} \sigma(X_{s-},s) \frac{\partial}{\partial x} \phi(X_s) \, dB_s \bigg| X_t = x \right\} = 0,
$$

(15)

$$
E \left\{ \int_t^{t+\Delta} \int_{|y|<1} \left[ \phi(X_{s-} + y \sigma(X_{s-})) - \phi(X_{s-}, s) \right] \tilde{N}(ds, dy) \bigg| X_t = x \right\} = 0,
$$

(16)

and

$$
E \left\{ \int_t^{t+\Delta} \int_{|y|\geq1} \left[ \phi(X_{s-} + y \sigma(X_{s-})) - \phi(X_{s-}, s) \right] N(ds, dy) \bigg| X_t = x \right\}
= \int_t^{t+\Delta} \int_{|y|\geq1} \left[ \phi(x + y \sigma(x, s)) - \phi(x) \right] \nu(dy) ds.
$$

(17)

Denoting $u(x,t) = E(\phi(X_t) | X_0 = x)$, it can then be shown that

$$
\frac{\partial}{\partial t} u(x,t) = Qu(x,t),
$$

(18)

which is the well known backward Kolmogorov equation. Let $p(y,t)$ be the probability density function for $X_t$ associated with the SDE (5). Then (18) becomes

$$
\int_{\mathbb{R}} \frac{\partial}{\partial t} \left( \phi(y)p(y,t) \right) \, dy = \int_{\mathbb{R}} Q\phi(y) \, p(y,t) \, dy.
$$

(19)

Let $Q^*$ be the adjoint operator of $Q$. Therefore,

$$
\int_{\mathbb{R}} Q\phi(y) \, p(y,t) \, dy = \int_{\mathbb{R}} \phi(y) Q^* p(y,t) \, dy.
$$

(20)

With relation (20) in mind, equation (19) becomes

$$
\int_{\mathbb{R}} \phi(y) \left( \frac{\partial}{\partial t} p(y,t) - Q^* p(y,t) \right) \, dy = 0.
$$

(21)

Since (21) is true for any $\phi \in C^\infty_0(\mathbb{R})$, we get

$$
\frac{\partial}{\partial t} p(y,t) = Q^* p(y,t),
$$

(22)
which is the Fokker-Planck equation, the governing equation for the transition probability density $p$ for the stochastic dynamical system (5). 

Now we try to derive an expression for the adjoint operator $Q^*$. It follows from (14) that the operator $Q$ can be written as 

$$ Q = A_1 + A_2 $$

where $A_1$ and $A_2$ are defined as 

$$ A_1 \phi(x) = f(x,t) \frac{\partial}{\partial x} \phi(x) + b \sigma(x,t) \frac{\partial}{\partial x} \phi(x) + \frac{A}{2} \sigma^2(x,t) \frac{\partial^2}{\partial x^2} \phi(x), $$

and 

$$ A_2 \phi(x) = \int_{\mathbb{R}\setminus\{0\}} \left[ \phi(x + y \sigma(x,t)) - \phi(x) - I_{(-1,1)}(y) y \sigma(x,t) \frac{\partial}{\partial x} \phi(x) \right] \nu(dy), $$

respectively. Note that $A_1^*$ is expressed as 

$$ A_1^* p(x,t) = -\frac{\partial}{\partial x} \left[ p(x,t) \left( f(x,t) + b \sigma(x,t) \right) \right] + \frac{A}{2} \frac{\partial^2}{\partial x^2} \left( \sigma^2(x,t) p(x,t) \right). $$

We now find an expression for $A_2^*$. Once $A_2^*$ is obtained, the adjoint operator $Q^*$ can be expressed as 

$$ Q^* = A_1^* + A_2^* $$

and then the Fokker-Planck equation $p_t = Q^* p$ will be obtained. 

By Taylor expansion, $\phi(x + y \sigma(x))$ can be written as 

$$ \phi(x + y \sigma(x,t)) = \phi(x) + \sum_{k=1}^{\infty} \frac{y^k}{k!} \sigma^k(x,t) \frac{\partial^k}{\partial x^k} \phi(x). $$

 Substitute (28) into (14) and (25), respectively, we get 

$$ Q \phi(x) = f(x,t) \frac{\partial}{\partial x} \phi(x) + b \sigma(x,t) \frac{\partial}{\partial x} \phi(x) + \frac{A}{2} \sigma^2(x,t) \frac{\partial^2}{\partial x^2} \phi(x) 
+ \int_{\mathbb{R}\setminus\{0\}} \left[ \sum_{k=1}^{\infty} \frac{y^k}{k!} \sigma^k(x,t) \frac{\partial^k}{\partial x^k} \phi(x) - I_{(-1,1)}(y) y \sigma(x,t) \frac{\partial}{\partial x} \phi(x) \right] \nu(dy), $$

(29)
and

\[ A_2 \phi(x) = \int_{\mathbb{R} \setminus \{0\}} \left[ \sum_{k=1}^{\infty} \frac{y^k}{k!} \sigma^k(x, t) \frac{\partial^k}{\partial x^k} \phi(x) - I_{(-1,1)}(y) y \sigma(x, t) \phi(x) \right] \nu(dy). \]

(30)

It follows from (30) that

\[ \int_{\infty}^{\infty} A_2 \phi(x) p(x, t) \, dx \]

\[ = \int_{\mathbb{R} \setminus \{0\}} \left( \int_{\mathbb{R}} \left[ \sum_{k=1}^{\infty} \frac{y^k}{k!} \sigma^k(x, t) \frac{\partial^k}{\partial x^k} \phi(x) - I_{(-1,1)}(y) y \sigma(x, t) \frac{\partial}{\partial x} \phi(x) \right] \nu(dy) \right) p(x, t) \, dx \]

\[ = \int_{\mathbb{R} \setminus \{0\}} \left( \int_{\mathbb{R}} \left[ \sum_{k=1}^{\infty} \frac{y^k}{k!} \sigma^k(x, t) \frac{\partial^k}{\partial x^k} \phi(x) - I_{(-1,1)}(y) y \sigma(x, t) \frac{\partial}{\partial x} \phi(x) \right] p(x, t) \, dx \right) \nu(dy) \]

\[ = \int_{\mathbb{R} \setminus \{0\}} \left( \int_{\mathbb{R}} \left[ \sum_{k=1}^{\infty} \frac{(-y)^k}{k!} \frac{\partial^k}{\partial x^k} \sigma^k(x, t) p(x, t) + I_{(-1,1)}(y) y \frac{\partial}{\partial x} (\sigma(x, t) p(x, t)) \right] \phi(x) \, dx \right) \nu(dy) \]

\[ = \int_{\mathbb{R}} \left( \int_{\mathbb{R} \setminus \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-y)^k}{k!} \frac{\partial^k}{\partial x^k} \sigma^k(x, t) p(x, t) + I_{(-1,1)}(y) y \frac{\partial}{\partial x} (\sigma(x, t) p(x, t)) \right] \nu(dy) \right) \phi(x) \, dx \]

\[ = \int_{\mathbb{R}} \phi(x, t) A_2^* p(x, t) \, dx, \quad (31) \]

where \( A_2^* \), the adjoint of \( A_2 \), is as follows

\[ A_2^* p(x, t) = \int_{\mathbb{R} \setminus \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-y)^k}{k!} \frac{\partial^k}{\partial x^k} \sigma^k(x, t) p(x, t) + I_{(-1,1)}(y) y \frac{\partial}{\partial x} (\sigma(x, t) p(x, t)) \right] \nu(dy). \]

(32)

Note that in obtaining the third identity in (31), we have made use of

\[ \int_{\mathbb{R}} p(x, t) \sigma^k(x, t) \frac{\partial^k}{\partial x^k} \phi(x) \, dx = (-1)^k \int_{\mathbb{R}} \phi(x) \frac{\partial^k}{\partial x^k} (\sigma^k(x, t) p(x, t)) \, dx, \quad \forall k \in \mathbb{N}. \]

(33)

Here \( \mathbb{N} \) is the set of natural numbers. Given (32), (26) and (27), it follows from (22) to get the desired Fokker-Planck equation. We summarize this result in the following theorem.
Theorem 1. Under the Assumption \((H_1)\), the Fokker-Planck equation for the \(\text{Itô SDE (5)}\) is
\[
\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} (f(x,t)p(x,t)) - b \frac{\partial}{\partial x} (\sigma(x,t)p(x,t)) + \frac{1}{2} A \frac{\partial^2}{\partial x^2} (\sigma^2(x,t)p(x,t)) \\
+ \int_{\mathbb{R}\setminus\{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-y)^k}{k!} \frac{\partial^k}{\partial x^k} (\sigma^k(x,t)p(x,t)) + I_{(-1,1)}(y) y \frac{\partial}{\partial x} (\sigma(x,t)p(x,t)) \right] \nu(dy).
\]
\hspace{10cm} (34)

Remark 1. Two special cases are noted here.

(i) When \(\sigma(x,t) = 1\), the Fokker-Planck equation \((34)\) reduces to
\[
\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} (f(x,t)p(x,t)) - b \frac{\partial}{\partial x} p(x,t) + A \frac{\partial^2}{\partial x^2} p(x,t) \\
+ \int_{\mathbb{R}\setminus\{0\}} \left( p(x-y,t) - p(x,t) + I_{(-1,1)}(y) y \frac{\partial}{\partial x} p(x,t) \right) \nu(dy).
\]
\hspace{10cm} (35)

This is the Fokker-Planck equation for an \(\text{Itô SDE with an additive Lévy process.}\)

(ii) When the Lévy jump measure \(\nu\) satisfies \(\int_{|x| \geq 1} |x|^k \nu(dx) \leq \infty\) for \(k \in \mathbb{N}\), which is true if and only if \(\mathcal{E}(|X_t|^k) \leq \infty\) (see [1]), the Fokker-Planck equation \((34)\) can be rewritten as
\[
\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} (f(x,t)p(x,t)) + \sum_{k=1}^{\infty} c_k \frac{\partial^k}{\partial x^k} (\sigma^k(x,t)p(x,t)),
\]
where
\[
c_1 = -b - \int_{|y| \geq 1} y \nu(dy),
\]
\hspace{10cm} (37)

\[
c_2 = \frac{1}{2} A + \frac{1}{2} \int_{\mathbb{R}\setminus\{0\}} y^2 \nu(dy),
\]
\hspace{10cm} (38)

and
\[
c_k = \int_{\mathbb{R}\setminus\{0\}} \frac{(-y)^k}{k!} \nu(dy), \quad k \geq 3.
\]
\hspace{10cm} (39)
3 Fokker-Planck equations for SDEs with Marcus integrals

In this section, we derive the Fokker-Planck equation for Marcus SDE (6), in which \( L_t \) being a scalar Lévy process with the triplet as \((b, 1, \nu)\). For simplicity, we make the following assumption.

**Assumption \((H_2)\)**

Assume that the noise intensity \( \sigma(x, t) = \sigma(x) \neq 0 \).

Then (6) becomes

\[
dX_t = f(X_t, t)dt + \sigma(X_t) \diamond dL_t. \tag{40}
\]

Define

\[
Y_t = H(X_t) = \int_0^{X_t} \frac{1}{\sigma(u)} du. \tag{41}
\]

The transform \( H \) in (41) is the Lamperti transform [2].

By the chain rule of the Marcus integral [1], we have

\[
dY_t = H'(X_t) \diamond dX_t \\
= H'(X_t)f(X_t, t)dt + (H'(X_t)\sigma(X_t)) \diamond dL_t \\
= \frac{f(X_t, t)}{\sigma(X_t)} dt + dL_t \\
= \frac{f(H^{-1}(Y_t), t)}{\sigma(H^{-1}(Y_t))} dt + dL_t. \tag{42}
\]

Introducing

\[
\tilde{f}(Y_t, t) = \frac{f(H^{-1}(Y_t))}{\sigma(H^{-1}(Y_t))}, \tag{43}
\]

equation (42) becomes

\[
dY_t = \tilde{f}(Y_t, t)dt + dL_t. \tag{44}
\]

By Theorem [2] in the last section, the Fokker-Planck equation for \( Y_t \) is

\[
\frac{\partial}{\partial t} p(y, t) = -\frac{\partial}{\partial y} \left( \tilde{f}(y, t)p(y, t) \right) - b \frac{\partial}{\partial y} p(y, t) + A \frac{\partial^2}{\partial y^2} p(y, t) \\
+ \int_{\mathbb{R} \setminus \{0\}} \left( p(y - r) - p(y) + I_{(-1,1)}(r)r \frac{\partial}{\partial y} p(y, t) \right) \nu(dr). \tag{45}
\]
Let \( q(x, t) \) represents the probability density function of \( X_t \), from the transform (41) and the Fokker-Planck equation (45), we get the following result for the desired Fokker-Planck equation for SDE (40) defined by Marcus integrals.

**Theorem 2.** Under the Assumptions \((H_1)\) and \((H_2)\), the Fokker-Planck equation for the Marcus SDE (40) is

\[
\sigma(x) \frac{\partial}{\partial t} q(x, t) = -\sigma(x) \frac{\partial}{\partial x} (f(x, t)q(x, t)) - b \sigma(x) \frac{\partial}{\partial x} (\sigma(x)q(x, t)) + \frac{A}{2} \sigma(x) \frac{\partial}{\partial x} \left( \sigma(x) \frac{\partial}{\partial x} (\sigma(x)q(x, t)) \right) \\
+ \int_{R \setminus \{0\}} \left( \sigma(H^{-1}(H(x) - r))q(H^{-1}(H(x) - r), t) - \sigma(x)q(x, t) + I_{(-1,1)}(r) \sigma(x) \frac{\partial}{\partial x} (\sigma(x)q(x, t)) \right) \nu(dr).
\]

**Remark 2.** Consider a special case. When \( \sigma(x) = 1 \), (46) reduces to

\[
\frac{\partial q(x, t)}{\partial t} = -\frac{\partial}{\partial x} (f(x, t)q(x, t)) - b \frac{\partial}{\partial x} q(x, t) + \frac{A}{2} \frac{\partial^2}{\partial x^2} q(x, t) \\
+ \int_{R \setminus \{0\}} \left( q(x - y, t) - q(x, t) + I_{(-1,1)}(y) \frac{\partial}{\partial x} q(x, t) \right) \nu(dy),
\]

which is equivalent to (55), the Fokker-Planck equation for a SDE with an additive Lévy process. This is the consequence of the fact that for additive Lévy noise, the Marcus integral and Itô integral are the same.

### 4 Examples

In this section, we present the Fokker-Planck equations for some SDEs with special Lévy processes, such as a Brownian motion together with a Poisson process, a compound Poisson process, and finally a \( \alpha \)-stable Lévy motion, respectively.

For simplicity, in all these examples, we take \( f(x, t) = x \), and \( \sigma(x, t) = x \). Then the Itô SDE (5) and the Marcus SDE (4) become

\[
dX_t = X_t dt + X_t dL_t,
\]
and
\[
\text{d}X_t = X_t \text{d}t + X_t \circ \text{d}L_t,
\]
respectively.

**Example 1.** When \( L_t \) is a standard Brownian motion together with a Poisson process with parameter \( \lambda \), the triplet \((b, A, \nu)\) of \( L_t \) is \((0, 1, \lambda \delta_1)\), where \( \delta_1 \) is the Dirac measure defined as
\[
\delta_1(D) = \begin{cases} 
1, & \text{if } 1 \in D, \\
0, & \text{if } 1 \notin D.
\end{cases}
\]

By using (34), we get the Fokker-Planck equation for (48)
\[
\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x} (xp(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 p(x, t)) + \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} (x^k p(x, t)).
\]

By using (46), we get the Fokker-Planck equation for (49)
\[
x \frac{\partial q(x, t)}{\partial t} = -x \frac{\partial}{\partial x} (xq(x, t)) + \frac{1}{2} x \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x} (xp(x, t)) \right) + \lambda e^{-1} x |q(e^{-1} x, t)| - \lambda x q(x, t).
\]

**Example 2.** When \( L_t \) is a compound Poisson process defined as
\[
L_t = \sum_{i=1}^{N_t} \xi_i,
\]

where \( N_t \) is a Poisson process with parameter \( \lambda \) and \( \xi_i \) \((i = 1, 2, \cdots)\) are i.i.d. random numbers with standard normal distribution \( \mathcal{N}(0, 1) \), the triplet \((b, A, \nu)\) of \( L_t \) is \(b = 0, A = 0, \nu(dx) = \frac{\lambda}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{d}x\). By using (34), we get the Fokker-Planck equation for (48)
\[
\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x} (f(x, t)p(x, t)) + \lambda \sum_{k=1}^{\infty} c_k \frac{\partial^k}{\partial x^k} (x^k p(x, t)),
\]

where
\[
c_k = \begin{cases} 
0, & \text{if } k \text{ is odd}, \\
(k - 1)!!, & \text{if } k \text{ is even}.
\end{cases}
\]
and !! is the double factorial defined by
\[
 n!! = \begin{cases} 
 n \cdot (n - 2) \cdots 1, & \text{if } n \text{ is odd,} \\
 n \cdot (n - 2) \cdots 2, & \text{if } n \text{ is even.} 
\end{cases}
\] (56)

By using (46), we get the Fokker-Planck equation for (49)
\[
x \frac{\partial q(x, t)}{\partial t} = -x \frac{\partial}{\partial x} (xq(x, t)) + \lambda \int_{\mathbb{R} \setminus \{0\}} \left( e^{-r|x|} q(e^{-r|x|}, t) - xq(x, t) \right) dr.
\] (57)

**Example 3.** When \( L_t \) is a symmetric \( \alpha \)-stable Lévy motion with the triplet as \( b = 1, A = 0 \) and \( \nu(dx) = \frac{C_\alpha dx}{|x|^{1+\alpha}} \), where \( C_\alpha \) is a constant (depending on \( \alpha \)). Then by using (34), we get the Fokker-Planck equation for (48)
\[
\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x} (f(x, t)p(x, t)) + C_\alpha \int_{\mathbb{R} \setminus \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-y)^k}{k!} \frac{\partial^k}{\partial x^k} \left( \sigma^k(x, t)p(x, t) \right) + I_{(-1, 1)}(y) y \frac{\partial}{\partial x} (\sigma(x, t)p(x, t)) \right] \frac{dy}{|y|^{1+\alpha}}.
\] (58)

By using (46), we get the Fokker-Planck equation for (19)
\[
x \frac{\partial q(x, t)}{\partial t} = -x \frac{\partial}{\partial x} (xq(x, t)) + \lambda C_\alpha \int_{\mathbb{R} \setminus \{0\}} \left[ e^{-r|x|} q(e^{-r|x|}, t) - xq(x, t) I_{(-1, 1)}(r) r \frac{\partial}{\partial x} (\sigma(x, t)p(x, t)) \right] \frac{dr}{|r|^{1+\alpha}}.
\] (59)
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