Spin squeezing in a generalized one-axis twisting model

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Abstract. We investigate the dependence of spin squeezing on the polar angle of the initial coherent spin state $|\theta_0, \phi_0\rangle$ in a generalized one-axis twisting model, where the detuning $\delta$ is taken into account. We show explicitly that regardless of $\delta$ and $\phi_0$, previous results of the ideal one-axis twisting are recovered as long as $\theta_0 = \pi/2$. For a small departure of $\theta_0$ from $\pi/2$, however, the achievable variance $(V_-)_{\text{min}} \sim N^{1/3}$, which is larger than the ideal case $N^{1/3}$. We also find that the maximal squeezing time $t_{\text{min}}$ scales as $N^{-5/6}$. Analytic expressions of $(V_-)_{\text{min}}$ and $t_{\text{min}}$ are presented and they agree with numerical simulations.
1. Introduction

Spin squeezing arising from quantum correlation of collective spin systems [1] has potential applications in high-precision measurement [2, 3] and quantum information processes [4]–[9]. Kitagawa and Ueda [1] have studied the squeezing generated by a nonlinear Hamiltonian $\chi J_z^2$ due to one-axis twisting (OAT). Starting from a coherent spin state (CSS) $|\theta_0 = \pi/2, \phi_0 = 0\rangle$, the system evolves into a spin squeezed state (SSS), which shows reduced variance $(V^-)$ below the standard quantum limit (SQL): $N/4$, where $N$ is the total number of particles. The smallest variance $(V^-)_{\text{min}} \sim N^{1/3}$ is obtainable at the time that scaled as $\chi t_{\text{min}} \sim N^{-2/3}$.

The possible realization of OAT-induced squeezing in two-mode Bose–Einstein condensates (BECs) has been proposed [4], where the self-interaction parameter $\chi \sim (a_{aa} + a_{bb} - 2a_{ab})/2$ is inherently brought about from atomic intra- and inter-species collisions. Atomic collisions lead to both squeezing and phase diffusion [11, 12]. The dephasing process destroys phase coherence of the two-component BEC and thus sets a limit to the application of the condensates in high-precision measurement and quantum information processing. A straightforward way to suppress the diffusion is by the preparation of a number-squeezed state, a special case of the SSS with reduced variance along the $J_z$ component. Such a kind of squeezed state has been previously investigated both experimentally [13]–[19] and theoretically [20]–[24].

Besides the above schemes that rely on nonlinear interactions of the ultracold atoms, spin squeezing can be generated via light–matter interactions [2, 3], [25]–[27] and quantum nondemolition measurement [28]–[34]. Recently, the OAT-induced squeezing has been demonstrated in an ensemble of cesium atoms [31, 32] and ytterbium atoms [33, 34]. In their experiments, the CSS with $\theta_0 = \pi/2$ was adopted as the input state, which is the optimal initial state to obtain the strongest squeezing. Via optical pumping, it was shown that 98% of the atoms are in the CSS [32].

In this paper, we investigate the degree of OAT-induced squeezing for $\theta_0$ slightly departing from $\pi/2$. A generalized OAT model: $H = \delta J_z + \chi J_z^2$ is considered, which is the most important prototype in studying spin squeezing [1, 4] and quantum metrology [35, 36]. We prove explicitly that without particle losses, the detuning $\delta$ and the azimuth angle $\phi_0$ give vanishing contribution...
to the squeezing parameter, and the ideal OAT-induced spin squeezing can be reproduced as long as \( \theta_0 = \pi/2 \). As the main result of our paper, we investigate the dependence of the variance \( (V_-)_\text{min} \) and the timescale \( t_\text{min} \) on the particle number \( N \) and the polar angle \( \theta_0 \). Our results show that even for a small departure of \( \theta_0 \) from \( \pi/2 \), the power rule of the smallest variance \( (V_-)_\text{min} \) changes from \( N^{1/3} \) to \( N^{2/3} \) with the increase of particle number \( N \). The maximal squeezing is achievable at the time that scaled as \( \chi t_\text{min} \sim N^{-5/6} \).

Our paper is organized as follows. In section 2, we present general formulae of spin squeezing for the arbitrary spin-\( 1/2 \) system. In section 3, we study the quantum dynamics of the OAT model, which is exactly solvable for any initial CSS. In section 4, we present short-time solutions of the first- and second-order moments of the spin operators. An approximated expression of the reduced variance \( V_- \) is presented to obtain power rules of maximal squeezing and its timescale \( t_\text{min} \). Finally, a summary of our paper is presented.

### 2. Some formulae of the spin squeezing

Assume that an ensemble of \( N \) two-level atoms (i.e. spin-\( 1/2 \) particles) with ground state \( |a\rangle \) and excited state \( |b\rangle \) can be described by a collective spin operator \( \mathbf{J} = \sum_{k=1}^{N} \frac{1}{2} \sigma^{(k)} \), where \( \sigma^{(k)} \) is the Pauli operator of the \( k \)th atom. Spin components of \( \mathbf{J} \) obey SU(2) algebra, \([J_{n_1}, J_{n_2}] = iJ_{n_3}\) for any three orthogonal vectors \( n_1, n_2 \) and \( n_3 \). The associated uncertainty relation reads \( (\Delta J_{n_1})^2(\Delta J_{n_2})^2 \geq \frac{1}{4}|\langle J_{n_3}\rangle|^2 \), where the variance is defined as usual, \( (\Delta \hat{A})^2 = \langle \Psi | \hat{A}^2 | \Psi \rangle - \langle \Psi | \hat{A} | \Psi \rangle^2 \) for any spin state \( |\Psi\rangle \) and operator \( \hat{A} \). Considering the mean spin \( \langle \mathbf{J} \rangle = (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle) \), we choose the orthogonal vectors as

\[
\begin{align*}
n_1 &= (- \sin \phi, \cos \phi, 0), \\
n_2 &= (- \cos \theta \cos \phi, - \cos \theta \sin \phi, \sin \theta), \\
n_3 &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\end{align*}
\]

where the azimuth angle \( \phi = \tan^{-1}[\langle J_y \rangle/\langle J_x \rangle] \) and the polar angle \( \theta = \tan^{-1}[r/\langle J_z \rangle] \) with \( r = |\langle J_z \rangle|^2 = (\langle J_x \rangle^2 + \langle J_y \rangle^2)^{1/2} \) (see figure 1(a)). For the arbitrary spin state \( |\Psi\rangle \), it is easy to prove that the mean spin \( \langle \mathbf{J} \rangle \) is along the \( n_3 \)-direction, with the length of the mean spin \( R = |\langle \mathbf{J} \rangle| = |\langle J_y \rangle| \) (see the appendix). Now, let us consider the CSS [10]:

\[
|\theta, \phi\rangle = e^{-i J_{n_1} j} |j, j\rangle = e^{i \phi (J_z \sin \phi - J_y \cos \phi)} |j, j\rangle,
\]

which is an eigenstate of \( J_{n_3} \) with eigenvalue \( j = N/2 \) (where \( N \) is the total particle number), and thus \( \langle J_{n_3}\rangle = |\langle \mathbf{J} \rangle| = j \). In the single-particle picture, the CSS can be rewritten as a direct product, \( |\theta, \phi\rangle = \prod_{k=1}^{N} |\cos(\theta/2)|b\rangle_k + e^{i\phi} \sin(\theta/2) |a\rangle_k \), where \( |a\rangle_k \) and \( |b\rangle_k \) are the ground and excited states of the \( k \)th atom. Such a quantum uncorrelated state obeys the minimal uncertainty relationship: \( (\Delta J_{n_1})^2 (\Delta J_{n_2})^2 = \frac{1}{4}|\langle J_{n_3}\rangle|^2 = j/2 \), where the value \( j/2 \) is termed as the SQL.

Since the mean spin is parallel to \( n_3 \), one can introduce any spin component normal to the mean spin as

\[
J_\psi = \mathbf{J} \cdot \mathbf{n}_\psi = J_{n_1} \cos \psi + J_{n_2} \sin \psi,
\]

where the unit vector \( \mathbf{n}_\psi = n_1 \cos \psi + n_2 \sin \psi \), with \( \psi \) being an arbitrary angle between \( n_1 \) and \( n_\psi \). For any spin state \( |\Psi\rangle \), we have \( \langle J_\psi \rangle = 0 \) and therefore the variance of \( J_\psi \) reads

\[
(\Delta J_\psi)^2 = \frac{1}{2} \left[ C + A \cos(2\psi) + B \sin(2\psi) \right],
\]

where

\[
\begin{align*}
C &= \frac{1}{2} \sum_{k=1}^{N} \left| \langle J_k \rangle \right|^2, \\
A &= \frac{1}{2} \sum_{k=1}^{N} \left( \langle J_k \rangle \cos(\theta) - \sin(\theta) \right), \\
B &= \frac{1}{2} \sum_{k=1}^{N} \left( \langle J_k \rangle \sin(\theta) - \cos(\theta) \right).
\end{align*}
\]

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According to Wineland et al. [2], the squeezing parameter $\xi^2$ is defined as

$$
\xi^2 = \frac{2j}{|\langle J \rangle|^2} V_+ = \frac{j^2}{|\langle J \rangle|^2} \delta^2,
$$

where

$$
\delta^2 = \frac{2}{j} \left[ C + \sqrt{A^2 + B^2} \right],
$$

for the CSS, $\Delta J^2 = j/2$ and $\xi^2 < 1$. It should be mentioned that the coefficients $A$, $B$ and $C$ depend on only five quantities (see the appendix): $\langle J_z \rangle$, $\langle J_z^2 \rangle$, $\langle J_y^2 \rangle$ and $\langle J_x (2J_z + 1) \rangle$, from which one can solve the mean spin $\langle J \rangle$ and the squeezing parameter $\xi^2$. In addition, there are several definitions of the squeezing parameter. According to Wineland et al [2], the squeezing parameter is defined as

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where $\delta^2$ is the timescale to attain the strongest squeezing. For large $j$, it is given by equation (28).

Figure 1. The Husimi $Q$ function: $Q(\theta, \phi; t) = |\langle \theta, \phi | \Psi(t) \rangle|^2$ on the Bloch sphere for $j = 30$. (a) The unit vectors $n_1$ (green), $n_2$ (blue) and $n_3$ (red), as defined in equation (1). (b) The initial CSS $|\theta_0 = \pi/3, \phi_0 = \pi/3 \rangle$. (c) The SSS generated by the OAT Hamiltonian $H = \chi J_z^2$ at time $t_{\min} = 0.043 \chi^{-1}$, where $\chi t_{\min}$ is the timescale to attain the strongest squeezing. For large $j$, it is given by equation (28).

where the coefficients $A = \langle J_{n_1}^2 - J_{n_2}^2 \rangle$, $B = \langle J_{n_1} J_{n_2} + J_{n_2} J_{n_1} \rangle$ and $C = \langle J_{n_1}^2 + J_{n_2}^2 \rangle = j(j+1) - \langle J_{n_2}^2 \rangle$. Another orthogonal spin component with respect to $J_\psi$ and its variance can be obtained by replacing $\psi$ with $\psi + \pi/2$. For the CSS $|\theta, \phi \rangle$, it is easy to verify that the coefficients $A = B = 0$ and $C = j$, which gives the variance $\Delta J_\psi^2 = j/2$, indicating isotropically distributed variances of the CSS [1], as shown in figure 1(b).

An SSS is defined if the variance of a spin component normal to the mean spin is smaller than the SQL [1], i.e. $\Delta J_\psi^2 < j/2$. The SSS has anisotropic variances distribution in a plane normal to the mean spin (see figure 1(c)). Optimally squeezed angle $\psi_{op}$ is obtained via minimizing $(\Delta J_\psi^2)$ with respect to $\psi$, yielding $\tan(2\psi_{op}) = B/A$, so $\cos(2\psi_{op}) = \pm A/\sqrt{A^2 + B^2}$ and $\sin(2\psi_{op}) = \pm B/\sqrt{A^2 + B^2}$. Substituting these results into equation (4), we obtain the reduced and the increased variances [7, 22, 23]

$$
V_\pm = \frac{1}{2} \left[ C \pm \sqrt{A^2 + B^2} \right],
$$

where the reduced variance $V_- = \Delta J_\psi^2$ corresponds to the squeezing along $n_\psi$ with $\psi = \psi_{op} = [\pi + \tan^{-1}(B/A)]/2$, while the increased variance $V_+$ gives the so-called anti-squeezing for the angle $\psi = \psi_{op} + \pi/2$. The degree of spin squeezing can be quantified by the normalized variance

$$
\bar{\xi}^2 = \frac{2V_-}{j} = \frac{C - \sqrt{A^2 + B^2}}{j}.
$$

For the CSS, the variances $V_- = V_+ = j/2$ and $\bar{\xi}^2 = 1$, while for the SSS, $\bar{\xi}^2 < 1$. It should be mentioned that the coefficients $A$, $B$ and $C$ depend on only five quantities (see the appendix): $\langle J_z \rangle$, $\langle J_z^2 \rangle$, $\langle J_y^2 \rangle$ and $\langle J_x (2J_z + 1) \rangle$, from which one can solve the mean spin $\langle J \rangle$ and the squeezing parameter $\xi^2$. In addition, there are several definitions of the squeezing parameter. According to Wineland et al [2], the squeezing parameter is defined as

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\xi^2 = \frac{2j}{|\langle J \rangle|^2} V_+ = \frac{j^2}{|\langle J \rangle|^2} \delta^2,
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where $\delta^2$ is the timescale to attain the strongest squeezing. For large $j$, it is given by equation (28).
which is closely related both to frequency resolution in spectroscopy [2] and to many-body quantum entanglement [4].

3. The generalized OAT model and its exact solutions

The above formulae are valid for any spin-$\frac{1}{2}$ system with SU(2) symmetry. As an example, we consider two-component BECs [37, 38] confined in a deep three-dimensional (3D) harmonic potential. The total system can be described by the two-mode Hamiltonian ($\hbar = 1$) ([39] and references therein):

$$H = \omega_a \hat{N}_a + \omega_b \hat{N}_b + U_{ab} \hat{N}_a \hat{N}_b + \frac{U_{aa}}{2}(\hat{a}^\dagger \hat{a})^2 + \frac{U_{bb}}{2}(\hat{b}^\dagger \hat{b})^2,$$

where $\hat{a}$, $\hat{b}$ and $\hat{N}_i$ ($i = a, b$) are the annihilation and number operators for the two internal states $|a\rangle$ and $|b\rangle$; $\omega_i$ are single-particle kinetic energies; $U_{ij} = (4\pi a_{ij}/M) \int \! \! d^3r \! |\Phi_0(r)|^4$ are atom–atom interaction strengths. For a conserved total particle number $N = \hat{N}_a + \hat{N}_b$, the two-mode model can be rewritten as $H = \delta J_+ + \chi J_2^2$, where the detuning $\delta = \omega_b - \omega_a + (U_{bb} - U_{aa})(N - 1)/2$ and $\chi = (U_{aa} + U_{bb} - 2U_{ab})/2$. Angular momentum operators $J_+ = (J_-)^\dagger = \hat{b}^\dagger \hat{a}$ and $J_2 = (\hat{N}_b - \hat{N}_a)/2$ satisfying SU(2) algebra.

Assume that the two-mode system evolves from the CSS, $|\Psi(0)\rangle = |\theta_0, \phi_0\rangle = \sum_m c_m(0) |j, m\rangle$, with the probability amplitudes [10]

$$c_m = \sqrt{\frac{(2j)!}{(j + m)!(j - m)!}} \cos^{j+m} \left( \frac{\theta_0}{2} \right) \sin^j \left( \frac{\theta_0}{2} \right) e^{i(j-m)\phi_0},$$

where the polar angles $\theta_0$ and $\phi_0$ determine population imbalance and the relative phase between the two internal states [40, 41]. The state vector at any time $t$ reads

$$|\Psi(t)\rangle = \sum_{m=-j}^j c_m e^{-i(\delta m + \chi m^2)t} |j, m\rangle,$$

where the self-interaction $\chi$ scrambles the phase of each number state $|j, m\rangle$ and leads to spin squeezing [1, 4] and phase diffusion [11] of the two-mode BEC. In theory, the diffusion is quantified by correlation function $\langle \hat{b}^\dagger \hat{a} \rangle$ (i.e. $\langle J_+^2 \rangle$), which decays exponentially with the timescale $\chi t_a = j^{-1/2}$ for $\theta_0 = \pi/2$. Such a kind of dephasing process has been observed in experiment by extracting the visibility of the Ramsey fringe [12].

As an ideal case, spin squeezing induced by the OAT Hamiltonian $\chi J_2^2$ has been investigated for the initial CSS $|\theta_0 = \pi/2, \phi_0 = 0\rangle$ [1]. For this special CSS, it was shown that the smallest variance $(V_\min)^2 \sim (2j)^{1/5}$ is obtainable at the time $t_{\min} \sim (2j)^{-2/3}$. Based upon this, Sørensen et al [4] studied the possible realization of squeezing in $^{23}$Na atom BECs. More important, they proposed that the squeezing parameter can be used as a probe of many-body entanglement. In this paper, we investigate dynamical generation of the SSS in the generalized OAT model from arbitrary CSS. We find that the power rules change significantly even for $\theta_0 \sim \pi/2$.

At first, we determine the mean spin $\langle \mathbf{J} \rangle = (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$, where $\langle J_x \rangle = j \cos (\theta_0)$, $\langle J_y \rangle = \text{Re}(J_\theta)$ and $\langle J_z \rangle = \text{Im}(J_\theta)$, with

$$\langle J_\theta \rangle = j \sin (\theta_0) \exp [i(\phi_0 + \delta t)] [\cos (\chi t) + i \cos (\theta_0) \sin (\chi t)]^{(2j)^{-1}}.$$

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It is convenient to rewrite equation (11) as \( \langle J_x \rangle = r \exp (i \phi) \), which yields \( \langle J_x \rangle = r \cos \phi \) and \( \langle J_y \rangle = r \sin \phi \), as defined in equation (1). Therefore, we obtain

\[
\begin{align*}
    r &= j \sin(\theta_0)[1 - \sin^2(\theta_0) \sin^2(\chi t)]^{j-1/2}, \\
    \phi &= \phi_0 + \delta t + (2j - 1)\varphi(t),
\end{align*}
\]

where \( \varphi(t) = \tan^{-1}[\cos(\theta_0) \tan(\chi t)] \) is a dynamical phase. Note that in real calculations of the squeezing parameters, only \( \cos(\phi) \) and \( \sin(\phi) \) are needed and are given by equations (A.2) and (A.3). The explicit form of the phase \( \phi \) or \( \varphi \) is introduced to find out the roles of \( \delta \) and \( \phi_0 \) in the squeezing. Obviously, \( r \), \( \varphi \) and also \( R = (r^2 + \langle J_z \rangle^2)^{1/2} \) do not depend on them.

To proceed, we calculate the expectation values \( \langle J_z^2 \rangle \), \( \langle J_x^2 \rangle \) and \( \langle J_x(2J_z + 1) \rangle \), which are relevant to the coefficients \( A \), \( B \) and \( C \). The mean value \( \langle J_z^2 \rangle \) reads

\[
\langle J_z^2 \rangle = \frac{j}{2} \sin^2(\theta_0) + j^2 \cos^2(\theta_0) = \frac{j}{2} + j \left( j - \frac{1}{2} \right) \cos^2(\theta_0),
\]

which, together with \( \langle J_z \rangle = j \cos(\theta_0) \), gives atom number variance \( \Delta \hat{N}_a^2 = (\Delta \hat{N}_b)^2 \equiv (\Delta J_z)^2 = (j/2) \sin^2 \theta_0 \). The variance \( \langle \Delta J_z \rangle^2 \) for \( \theta_0 \neq \pi/2 \) becomes narrower than that for \( \theta_0 = \pi/2 \), which leads to relatively slow phase diffusion [42, 43]. After some tedious calculations, we further obtain

\[
\begin{align*}
    \langle J_z^2 \rangle &= j \left( j - \frac{1}{2} \right) \sin^2(\theta_0) \exp[2i(\phi_0 + \delta t)] \cos(2\chi t) + i \cos(\theta_0) \sin(2\chi t)]^{j-1/2}, \\
    \langle J_x(2J_z + 1) \rangle &= 2j \left( j - 1/2 \right) \sin(\theta_0) \exp[i(\phi_0 + \delta t)] \times [\cos(\chi t) + i \cos(\theta_0) \sin(\chi t)]^{2j-2} [\cos(\theta_0) \cos(\chi t) + i \sin(\chi t)].
\end{align*}
\]

From equations (A.4)–(A.6), one can find that the coefficients are fully determined by five quantities: \( \sin \theta \) (= \( R/R \), \( \cos \theta \) (= \( \langle J_z \rangle/R \), \( \langle J_z^2 \rangle \), \( \langle J_x^2 \rangle \) and \( \langle J_x(2J_z + 1) \rangle \) are independent of \( \delta \) and \( \phi_0 \), which remains true for the last two terms due to \( \exp[ik(\phi_0 + \delta t)]e^{-ik\phi} = \exp[-ik(2j - 1)\varphi(t)] \) (with \( k = 1, 2 \)), where \( \varphi(t) \) does not depend on \( \delta \) and \( \phi_0 \). As a result, we come to the conclusion that the detuning \( \delta \) and the azimuth angle \( \phi_0 \) change the mean spin direction (see also figures 1(b) and (c)), but do not present any contribution to the squeezing.

The squeezing parameters \( \xi \) and \( \zeta \) depend sensitively on the polar angle \( \theta_0 \) of the initial CSS, as shown in figure 2. The strongest squeezing can be obtained for \( \theta_0 = \pi/2 \), which corresponds to the initial CSS with equal atom population between the two internal states, i.e. \( \langle J_z \rangle = 0 \). From equation (11), we have \( \langle J_x \rangle = r \exp(i\phi) \) with \( R = j \cos^{2j-1}(\chi t) \) and \( \phi = \phi_0 + \delta t \). From equations (14)–(16), we further obtain \( \langle J_z^2 \rangle = j/2, \langle J_x^2 \rangle \) or \( \langle J_x(2J_z + 1) \rangle \) and \( \langle J_x^2(2J_z + 1) \rangle e^{-i\phi} = ij(2j - 1) \cos^{2j-2}(\chi t) \sin(\chi t) \). Substituting these results into equations (A.4)–(A.6), we obtain the coefficients

\[
\begin{align*}
    A &= \frac{j}{2} \left( j - \frac{1}{2} \right) \left[ 1 - \cos^{2j-2}(2\chi t) \right], \\
    B &= 2j \left( j - \frac{1}{2} \right) \cos^{2j-2}(\chi t) \sin(\chi t)
\end{align*}
\]
Figure 2. Time evolution of the squeezing parameters ($\zeta^2$, $\xi^2$) for various $\theta_0$ values of the initial CSS. From top to bottom: $\theta_0 = \pi/3$ (squares, blue lines), $0.98 \times \pi/2$ (crosses, red lines) and $\pi/2$ (empty circles, black lines). The arrows indicate the positions of the maximal squeezing time $t_{\text{min}}$ for different $\theta_0$s. Other parameters: $j = 30$ (a), $j = 2 \times 10^4$ (b) and $\delta = \phi_0 = 0$.

and $C = A + j$. From equation (5), we obtain the increased and the reduced variances

$$V_\pm = \frac{j}{2} \left[ 1 + \frac{j - \frac{1}{2}}{2} \left( \tilde{A}_\pm \sqrt{\tilde{A}^2 + \tilde{B}^2} \right) \right],$$

where the intermediate coefficients $\tilde{A} = 1 - \cos^{2j-2}(2\chi t)$ and $\tilde{B} = 2 \cos^{2j-2}(\chi t) \sin(\chi t)$. One can find that the variances are exactly the same as those of the ideal OAT case [1], even for nonzero $\delta$ and $\phi_0$.

Solid curves in figure 2 indicate the evolution of the normalized variance $\xi^2$. The minimal value of the squeezing parameter, $\xi^2_{\text{min}} = 2j^{-1}(V_-)_{\text{min}}$, appears at the time $t_{\text{min}}$ indicated by the arrows for different values of $\theta_0$. The smallest value of $\xi^2_{\text{min}}$ is obtained for the optimal initial state $\theta_0 = \pi/2$. For $\theta_0 \neq \pi/2$ and large $j$ values ($\gg 1$), the squeezing becomes worse than the optimal case. A closer look at the evolution of $\xi^2 = (|\langle J \rangle|^2)^{1/2}$ indicates that it is minimized before $t_{\text{min}}$ (see empty circles of figure 2(a)). This is because of different evolution rates of the variance $V_-$ and the mean spin $\langle J \rangle$. In addition, the minimal value $\zeta^2_{\text{min}}$ is slightly larger than $\xi^2_{\text{min}}$ due to the decreased mean spin $|\langle J \rangle| \leq j$. In the case of large $j$ values, however, the two squeezing parameters almost merge with each other in the short-time regime; see figure 2(b). As a result, one can assume that $\zeta^2_{\text{min}}$ obeys the same power rule as $\xi^2_{\text{min}}$ [34] and is determined by that of $(V_-)_{\text{min}}$.

4. Power rules of the strongest squeezing and its timescale

As shown by the red lines in figure 2(b), both $\xi^2_{\text{min}}$ and $t_{\text{min}}$ change significantly in comparison with the ideal case (i.e. $\theta_0 = \pi/2$). As a result, it is necessary to determine power rules of the variance $(V_-)_{\text{min}}$ and the time $t_{\text{min}}$ for $\theta_0 \neq \pi/2$. In this section, we calculate analytically the power rules by using standard treatments of [1]. We will focus on a small departure of $\theta_0$ from $\pi/2$ due to the fact that a relatively small population imbalance between two internal states favors the OAT effect.
4.1. The ideal OAT case with $\theta_0 = \pi/2$

In the short-time limit ($\chi t \ll 1$) and large particle number ($j \gg 1$), the increased and reduced variances equation (19) can be approximated as [1]:

$$V_+ \simeq \frac{j}{2} (4\alpha_0^2), \quad V_- \simeq \frac{j}{2} \left( \frac{1}{4\alpha_0^2} + \frac{2}{3} \beta_0^2 \right),$$

(20)

where $\alpha_0 = j\chi t > 1$ and $\beta_0 = j(\chi t)^2 \ll 1$. Equation (20) is the key point to obtain the strongest squeezing $\xi_{\text{min}}$ and its timescale $t_{\text{min}}$. Previously, the time $t_{\text{min}}$ was obtained by comparing the second term of $V_-$ with that of the first one [1]. Here, we solve $t_{\text{min}}$ via minimizing $V_-$ with respect to $t$, i.e.

$$\frac{d}{dt} (V_-)\bigg|_{t_{\text{min}}} = 0,$$

(21)

which yields the power rule of the maximal squeezing time:

$$\chi t_{\text{min}} \simeq 3^{1/6} (2j)^{-2/3}.$$  

(22)

Inserting $\chi t_{\text{min}}$ into equation (20), we further obtain the reduced variance as

$$(V_-)_{\text{min}} \simeq \frac{3}{8} \left( \frac{2j}{3} \right)^{1/3}$$

(23)

and also the smallest squeezing parameter $\xi_{\text{min}}^2 = 2j^{-1}(V_-)_{\text{min}} \simeq \frac{1}{2} \left( \frac{2j}{3} \right)^{-2/3}$. Power exponents of equations (22) and (23) are consistent with [1], but different in the coefficients. As shown by the black solid lines of figure 3, the revised results fit very well with their numerical results (empty circles).

4.2. The small departure case with $\theta_0 \sim \pi/2$

The power rules, equations (22) and (23), are valid only for $\theta_0 = \pi/2$. Now, we generalize them for the $\theta_0 \neq \pi/2$ case. To obtain the approximated expressions of the variances as equation (20), we calculate short-time solutions of $\langle J_z \rangle$, $\langle J_z^2 \rangle$ and $\langle J_+ (2J_z + 1) \rangle$.
In the short-time limit \((\chi t \ll 1)\), the dynamical phase \(\varphi(t) = \tan^{-1}[\cos(\theta_0) \tan(\chi t)] \simeq \chi t \cos(\theta_0)\) and equation (11) can be approximated as
\[
\langle J_z \rangle \simeq j \sin(\theta_0) e^{i\beta} e^{-\theta},
\]
(24)
where \(\beta = \beta_0 \sin^2(\theta_0) = j(\chi t)^2\sin^2(\theta_0)\) and \(\phi \simeq \phi_0 + \delta t + 2j \chi t \cos(\theta_0)\). We have assumed that the number of particles is large enough, so \(2j - 1 \simeq 2j\). The length of the correlation reads \(r = |\langle J_z \rangle| \simeq j \sin(\theta_0) e^{-\theta}\), which indicates that phase coherence of the two-mode BEC decays exponentially (i.e. phase diffusion [11]) with the coherence time scaled as \(\chi t_a = \sin^{-1}(\theta_0) j^{-1/2}\) [42, 43]. Similarly, short-time solutions of equations (15) and (16) can be written approximately as
\[
\langle J_z^2 \rangle \simeq j(j - 1/2) \sin^2(\theta_0) e^{2i\phi} e^{-4\theta}
\]
(25)
and
\[
\langle J_z(2J_z + 1) \rangle \simeq j(2j - 1) \sin(\theta_0)(\cos(\theta_0) + i\chi t) e^{i\beta} e^{-\theta},
\]
(26)
where the factor \(\cos(\theta_0)\) cannot be neglected since it is comparable with \(\chi t\). In fact, it is the presence of \(\cos(\theta_0)\) that leads to a significant change of \(t_{\min}\) and \((V_{-})_{\min}\) even for \(\theta_0 \sim \pi/2\).

To simplify the calculations, we make further approximations to the angles of \((a)\) decreases faster than the ideal OAT case; see also figure 3 and equation (32). By minimizing \(V_{-}\) with respect to \(t\), we obtain the power rule of the time as
\[
\chi t_{\min} \simeq \frac{3^{1/6} (2j \sin^2 \theta_0)^{-2/3}}{(1 + 9j \sin^2 \theta_0 \cos^2 \theta_0)^{1/6}}
\]
(28)
and that of the decreased variance as
\[
(V_{-})_{\min} \simeq \frac{3}{8} \left[ \frac{2j}{3 \sin^3 \theta_0} (1 + 9j \sin^2 \theta_0 \cos^2 \theta_0) \right]^{1/3}.
\]
(29)
For \(\theta_0 = \pi/2\), our results reduce to the ideal OAT case, i.e. equations (22) and (23), while for \(\theta_0 \neq \pi/2\) and large \(j\), equations (28) and (29) predict that the power rules change to
\[
\chi t_{\min} \sim (2j)^{-5/6}, \quad (V_{-})_{\min} \sim (2j)^{2/3},
\]
(30)
which are confirmed by numerical simulations. To see this more clearly, let us focus on red lines of figure 3. For \(\theta_0 \sim \pi/2\) and small \(j\), both the time \(\chi t_{\min}\) and the variance \((V_{-})_{\min}\) follow the same rule as the \(\theta_0 = \pi/2\) case. With the increase of \(j\), however, the red line (the crosses) of figure 3(a) decreases faster than the ideal OAT case; see also figure 2(b). The change of the power rule is shown more clearly in figure 3(b).

In figure 4, we show the dependence of \(t_{\min}\) and \((V_{-})_{\min}\) on \(\theta_0\) for a fixed value \(j\). It was shown that both \(t_{\min}\) and \((V_{-})_{\min}\) are symmetrical with respect to \(\theta_0 = \pi/2\). The strongest
squeezing (i.e. the smallest value of $\xi^2_{\text{min}}$) occurs for the optimal initial state $\theta_0 = \pi/2$. Our analytic results, equations (28) and (29), agree quite well with numerical simulations except $\theta_0 = 0$ or $\pi$. In this case, the state vector $|\Psi(t)\rangle = \exp[-i(\chi^2 J_z t)\hat{\sigma}_z] |\pm\rangle$, which is the CSS with the variances $\langle V_+ \rangle = \langle V_- \rangle = j/2$ and $\xi^2 = 1$. However, equation (29) diverges as $\theta_0 \to 0$ or $\pi$, inconsistent with the real situation. Equation (28) gives a relatively good estimate of the maximal squeezing time. As shown in figure 4(a), $t_{\text{min}}$ decreases monotonically in the small departure regime $|\theta_0 - \pi/2| < 0.27\pi/2$, which implies that the maximal squeezing occurs earlier and earlier; see also figure 2(b). Outside the regime, $t_{\text{min}}$ increases with the departure of $\theta_0$, and goes to infinity as $\theta_0 \to 0$ or $\pi$.

4.3. Dissipation effect due to atomic decay

So far, we have neglected the effects of dissipation on spin squeezing, such as particle losses and spatial dynamics of the atoms in the BECs [4, 40, 41]. For the squeezing generated in an atomic ensemble, the dominant dissipation source is the atomic decay due to spontaneous emission [32], which can be described by the master equation [26]:

$$\frac{\partial \rho}{\partial t} = i[\rho, H] + \frac{\gamma}{2} (2J_- \rho J_+ - J_+ J_- \rho - \rho J_+ J_-),$$

(31)

where $\rho$ is the density operator and $\gamma$ is the decay rate of the atoms. In the basis of $|j, m\rangle$, the elements $\rho_{m,n} = \langle j, m | \rho | j, n \rangle$ could be solved numerically by using the Runge–Kutta routine [26]. In real calculations of the squeezing parameters, only 6$j$ elements like $\rho_{m,m}$, $\rho_{m,m+1}$ and $\rho_{m,m+2}$ are needed.

In figure 5, we plot the time evolution of $\xi^2$ for small decay rates, e.g. $\gamma/\chi = 0.01$ and 0.1. Such a small dissipation can be realized by increasing $\chi$, which in turn leads to the preparation of the SSS within the lifetime of the atoms $\gamma^{-1}$ [32]. For the relatively small decay rate $\gamma/\chi = 0.01$ (red curves), both the maximal squeezing and its timescale change slightly in comparison with the $\gamma = 0$ case. The initial state with $\theta_0 = \pi/2$ looks more sensitive to atomic decay than the $\theta_0 = 0.8 \times \pi/2$ case. From the blue dotted lines of figure 5(b), we find that even for $\gamma/\chi = 0.1$, a considerable squeezing with $\xi^2$ (and also $\xi^2$) $\sim 0.22$ could be reached in an ensemble of 200 atoms, which occurs at a timescale given by equation (28).
5. Conclusion

In summary, we have presented general formulae to study spin squeezing in the spin-$\frac{1}{2}$ system. Instead of six fluctuation parameters as in [40, 41], only five parameters, i.e. $\langle J_z \rangle$, $\langle J_+ \rangle$, $\langle J_+^2 \rangle$ and $\langle J_+ (2J_z + 1) \rangle$, are needed to determine the mean spin and the squeezing parameters.

Spin squeezing of a generalized OAT model is investigated for the arbitrary CSS $|\psi_0, \phi_0 \rangle$. We show explicitly that $\theta_0 = \pi/2$ is the optimal initial state to obtain the minimum value of the variance $(V_-)_{\text{min}} \simeq \frac{1}{8} (2j/3)^{1/3}$, which takes place at the time scaled as $\chi t_{\text{min}} \simeq 3^{1/6} (2j)^{-2/3}$. The detuning $\delta$ and the azimuth angle $\phi_0$ alert the mean spin’s direction, but give vanishing contribution to the squeezing parameters. As the main result of our paper, we calculate analytically the dependence of the variance $(V_-)_{\text{min}}$ and the time $t_{\text{min}}$ on the polar angle $\theta_0$ as equations (28) and (29), respectively. What may be a little surprising is that even for a small departure of $\theta_0$ from $\pi/2$, the power rules become $(V_-)_{\text{min}} \sim (2j)^{2/3}$ and $\chi t_{\text{min}} \sim (2j)^{-5/6}$, deviating from the ideal case. The power rule, equation (28), is robust against atomic decay for $\gamma < 0.1 \chi$. Our results show that SSS in the OAT model depends sensitively on the initial state and the interaction time. A straightforward way to overcome these stringent requirements is still an open question.

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Appendix A. The coefficients $A$, $B$ and $C$

In equation (1), we have defined three orthogonal unit vectors $n_i$ ($i = 1, 2, 3$), which are valid for any spin state $|\Psi\rangle$. The angles $\theta$ and $\phi$ are determined by the mean spin $\langle \mathbf{J} \rangle = (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$,
with
\[
\sin \theta = \frac{r}{R}, \quad \cos \theta = \frac{\langle J_z \rangle}{R}, \quad (A.1)
\]
\[
\cos \phi = \frac{\langle J_x \rangle}{r} = \frac{\text{Re}\langle J_x \rangle}{r}, \quad (A.2)
\]
\[
\sin \phi = \frac{\langle J_y \rangle}{r} = \frac{\text{Im}\langle J_y \rangle}{r}, \quad (A.3)
\]
where the length of the mean spin \( R = |\langle J \rangle| = (\langle J_z \rangle^2 + \langle J_y \rangle^2 + \langle J_z \rangle^2)^{1/2} \) and \( r = |\langle J_x \rangle| = (\langle J_x \rangle^2 + \langle J_y \rangle^2)^{1/2} = R \sin \theta \). From equations (A.1)–(A.3), it is easy to verify that the unit vector \( \mathbf{n}_3 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = \mathbf{R}^{-1} \langle J \rangle \), i.e. the mean spin \( \langle J \rangle \) is parallel to the unit vector \( \mathbf{n}_3 \). Moreover, one can prove the expectation value \( \langle J_{n_1} \rangle = -\langle J_z \rangle \sin \phi + \langle J_y \rangle \cos \phi = -\langle J_z \rangle / r + \langle J_y \rangle / r = 0, \langle J_{n_2} \rangle = 0 \) and \( \langle J_{n_3} \rangle = |\langle J \rangle| \).

By using the above results, one can solve explicit expressions of the coefficients: \( \mathbf{A} = \langle J_{n_1}^2 - J_{n_2}^2 \rangle, \mathbf{B} = \langle J_{n_1} J_{n_2} + J_{n_2} J_{n_1} \rangle \) and \( \mathbf{C} = \langle J_{n_1}^2 + J_{n_2}^2 \rangle \), yielding
\[
2\mathbf{A} = \sin^2 \theta \left[ j(j+1) - 3 \langle J_z^2 \rangle \right] - (1 + \cos^2 \theta) \text{Re} \left[ [J_x^2] e^{-2i\phi} \right] + \sin(2\theta) \text{Re} \left[ \langle J_x (2J_z + 1) \rangle e^{-i\phi} \right], \quad (A.4)
\]
\[
\mathbf{B} = -\cos(\theta) \text{Im} \left[ [J_x^2] e^{-2i\phi} \right] + \sin(\theta) \text{Im} \left[ \langle J_x (2J_z + 1) \rangle e^{-i\phi} \right], \quad (A.5)
\]
\[
\mathbf{C} + \mathbf{A} = j(j+1) - \langle J_z^2 \rangle - \text{Re} \left[ [J_x^2] e^{-2i\phi} \right], \quad (A.6)
\]
where we have used the relations: \( \langle J_{n_1}^2 + J_{n_2}^2 \rangle = j(j+1) - \langle J_z^2 \rangle, \langle J_{n_1}^2 - J_{n_2}^2 \rangle = \text{Re} \langle J_x^2 \rangle, \langle J_{n_1} J_{n_2} + J_{n_2} J_{n_1} \rangle = \text{Re} \langle J_x (2J_z + 1) \rangle \) and \( \langle J_x J_z + J_z J_x \rangle = \text{Im} \langle J_x (2J_z + 1) \rangle \). Substituting the coefficients into equation (5), we obtain the variances \( V_x \) and the squeezing parameters \( \xi^2 \) and \( \zeta^2 \).

Now let us calculate the coefficients for any CSS \( |\theta, \phi \rangle \). The mean values \( \langle J_z \rangle = j \cos(\theta) \) and \( \langle J_x \rangle = j \sin(\theta) e^{i\phi} \), which yield \( \langle J_z \rangle = j \sin(\theta) \cos \phi \) and \( \langle J_x \rangle = j \sin(\theta) \sin \phi \). For the CSS, the length of the mean spin \( R = j \). These results can be directly obtained from equation (11) by taking time \( t = 0 \). Similarly, from equations (14)–(16), we further obtain \( \langle J_x^2 \rangle e^{-2i\phi} = j(j-1/2) \sin^2(\theta) \) and \( \langle J_x (2J_z + 1) \rangle e^{-i\phi} = j(2j - 1) \sin(\theta) \cos(\theta) \), from which we immediately obtain the coefficient \( \mathbf{B} = 0 \). Substituting these results into equations (A.4)–(A.6), we also get \( \mathbf{A} = 0 \) and \( \mathbf{C} = j \).

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