A non-local quasi-linear ground state representation and criticality theory

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1 Introduction

The quasi-linear $p$-Laplacian on Riemannian manifolds, and especially in the Euclidean space, is one of the best studied non-linear local elliptic operators. There are many beautiful monographs related to this operator, see e.g. [2, 8, 18, 26, 39], or in general metric spaces, see [9].

First results for $p$-Laplacians on finite graphs are given in [5, 13, 27, 28, 44, 45]. On locally finite graphs, $p$-Laplacians were studied in [31, 55, 61, 62], and on almost locally finite graphs in [42, 57].

Here, we study an extended class of operators including the $p$-Laplacian as a special case, i.e., $p$-Schrödinger operators. Moreover, we study this class on locally summable weighted graphs.

Recently, the potential theory of local $p$-Schrödinger operators with not necessarily non-negative potential term was studied more closely, see e.g. [2, 15, 29, 50–53]. In this theory, the ground state representation is of fundamental importance. This representation is an equivalence between functionals. It states that the $p$-energy functional associated with the $p$-Schrödinger operator is equivalent to a simplified energy functional consisting of non-negative terms only.
For non-local $p$-Schrödinger operators in the Euclidean space, which includes graphs as a special case, a one-sided inequality for $p \geq 2$ is given in [23].

Here, we show a ground state representation for non-local $p$-Schrödinger operators for all $p > 1$ in terms of an equivalence between the corresponding $p$-energy functional and the simplified energy. We show this statement on graphs in Theorem 3.1 and Corollary 3.2.

The ground state representation is an essential tool in criticality theory (which is sometimes also called parabolic theory). A $p$-energy functional is called critical if it is non-negative and the $p$-Hardy inequality does not hold. In the continuum, many characterisations of criticality are known, see e.g. [29, 50].

Criticality theory on almost locally finite graphs for the standard $p$-Laplacian was studied in [57], see [55, 61] for locally finite graphs. They show a connection between positive $p$-harmonic functions and the variational $p$-capacity.

In this paper, we establish a characterisation of criticality for $p$-Schrödinger operators in terms of null-sequences, the variational $p$-capacity, as well as positive $p$-harmonic functions, see Theorem 4.1. This will be achieved mainly by the aid of the ground state representation. These characterisations are the discrete counterpart to results in [29, 50].

Moreover, we show a Liouville comparison principle which is a discrete analogue to results in [51, 53]. It is another application of the ground state representation and gives the criticality of an energy functional if a $p$-subharmonic function can be estimated properly by the ground state of another energy functional.

In this paper, we focus on graphs but since the fundamental methods are based on pointwise estimates, also corresponding results in other non-local settings are valid. To be more specific, our results can also be extended to non-local $p$-Schrödinger operators on the Euclidean space in the spirit of [23], confer also with [6, 7, 11, 12, 16, 37]. We chose graphs because they do not have local regularity issues and therefore the presentation is less technical than, e.g., for non-local $p$-Schrödinger operators on $\mathbb{R}^d$.

There is an abundance of literature for the linear ($p = 2$)-case: For linear Schrödinger operators on locally summable graphs see [25, 32–34], for random walks see [60], for Schrödinger forms see [24, 41, 54, 58], for Jacobi matrices see [22], and references therein. In probabilistic settings, the ground state representation is also known as Doob’s $h$-transform.

In the local and linear case, ground state representations are classical and have shown their powerfulness in many applications, see [23, Sect. 1] for a list of applications with references and more details.

This paper is organised as follows: In Sect. 2, we briefly introduce the basic notation and show connection between the $p$-Schrödinger operators and $p$-energy functionals via a Green’s formula. Then, we turn in Sect. 3 to the main results of this paper, Theorem 3.1 and Corollary 3.2. After we stated the results, we discuss them in detail in Sect. 3.2. This includes a comparison with the local case in the continuum. The proofs of Theorem 3.1 and Corollary 3.2 are then divided into two parts: The proof of an elementary equivalence in Sect. 3.3, and then the application of this equivalence in Sect. 3.4. Thereafter, we show in Sect. 4 how this ground state representation can be used to prove some characterisations of criticality. Here, also one part of an Agmon-Allegretto-Piepenbrink-type theorem is needed, as well as a local $p$-Harnack inequality. This is part of Sect. 4.1. We end this paper with two simple applications of Theorems 3.1 and 4.1, one of them is a Liouville comparison principle. In the appendix, we show the proof of Lemma 3.8.
2 Setting the scene

In this section, we start by introducing graphs. Thereafter, we define quasi-linear Schrödinger operators on graphs. We end this section by introducing $p$-energy functionals and showing a connection to $p$-Schrödinger operators via Green’s formula.

2.1 Graphs and Schrödinger operators

Let an infinite set $X$ equipped with the discrete topology and a symmetric function $b: X \times X \to [0, \infty)$ with zero diagonal be given such that $b$ is locally summable, i.e., the vertex degree satisfies

$$\deg(x) = \sum_{y \in X} b(x, y) < \infty, \quad x \in X.$$  

We refer to $b$ as a graph over $X$ and elements of $X$ are called vertices. Two vertices $x, y$ are called connected with respect to the graph $b$ if $b(x, y) > 0$, in terms $x \sim y$. A subset $V \subseteq X$ is called connected with respect to $b$, if for every two vertices $x, y \in V$ there are vertices $x_0, \ldots, x_n \in V$, such that $x = x_0, y = x_n$ and $x_{i-1} \sim x_i$ for all $i \in \{1, \ldots, n - 1\}$. For $V \subseteq X$ let $\partial V = \{ y \in X \setminus V : y \sim z \in V \}$. Throughout this paper we will always assume that $X$ is connected with respect to the graph $b$.

We now turn to functions: Let $S$ be some arbitrary set. A function $f: S \to \mathbb{R}$ is called non-negative, positive, or strictly positive on $I \subseteq S$, if $f \geq 0$, $f \geq 0$, $f > 0$ on $I$, respectively. If for two non-negative functions $f_1, f_2: S \to \mathbb{R}$ there exists a constant $C > 0$ such that $C^{-1} f_1 \leq f_2 \leq Cf_1$ on $I \subseteq S$, we write

$$f_1 \asymp f_2 \quad \text{on } I,$$

and call them equivalent on $I$.

The space of real valued functions on $V \subseteq X$ is denoted by $C(V)$ and is a subspace of $C(X)$ by extending the functions of $C(V)$ by zero on $X \setminus V$. The space of functions with compact support in $V$ is denoted by $C_c(V)$.

A strictly positive function $m \in C(X)$ extends to a measure with full support via $m(V) = \sum_{x \in V} m(x)$ for $V \subseteq X$.

The next fundamental definition is the one of the $p$-Laplacian. But first, we have to introduce some notation. For showing the connection to the counterpart in the continuum, we introduce the difference operator $\nabla$ on $C(X)$ via

$$\nabla_{x,y} f = f(x) - f(y), \quad x, y \in X.$$  

Let $p \in [1, \infty)$. For $V \subseteq X$, let the formal space $F(V) = F_{b,p}(V)$ be given by

$$F(V) = \left\{ f \in C(X) : \sum_{y \in X} b(x, y) |\nabla_{x,y} f|^{p-1} < \infty \text{ for all } x \in V \right\}.$$  

If $V = X$ we write $F = F(X)$.

For $1 < p < 2$ we make the convention that $|t|^{p-2} t = 0$ if $t = 0$, i.e., $0 \cdot \infty = 0$. Then, we can write for all $p \geq 1$,

$$(t)^{(p-1)} := |t|^{p-1} \text{sgn}(t) = |t|^{p-2} t, \quad t \in \mathbb{R}.$$
In particular, \( C \) is the sign function, that is \( \text{sgn}(t) = 1 \) for all \( t > 0 \), \( \text{sgn}(t) = -1 \) for all \( t < 0 \), and \( \text{sgn}(0) = 0 \). We remark that \( F(V) = C(X) \) if \( p = 1 \), by the local summability assumption on the graph.

Next, we show a basic lemma, which states an alternative representation for the formal Laplacian: Let \( m \) be a measure on \( X \). The first sum on the right-hand side is finite by the local summability property of the graph.

**Proof** The case \( p = 1 \) is trivial. Let \( p > 1 \), and denote the set on the right-hand side by \( \hat{F}(V) \). We obviously have that \( C_c(V) \subseteq \ell^\infty(V) \subseteq \hat{F}(V) \). Furthermore, let \( f \in \hat{F}(V) \). Then, using the elementary inequality (2.1), we get for any \( x \in V \) that

\[
\sum_{y \in X} b(x, y) |\nabla_x y f|^{p-1} \leq 2^{p-1} \left( |f(x)|^{p-1} \sum_{y \in X} b(x, y) + \sum_{y \in X} b(x, y) |f(y)|^{p-1} \right).
\]

The first sum on the right-hand side is finite by the local summability property of the graph \( b \). The second sum is finite since \( f \in \hat{F}(V) \). This shows \( f \in F(V) \).

Moreover, if \( f \in F(V) \) we obtain \( f \in \hat{F}(V) \) since for all \( x \in V \)

\[
\sum_{y \in X} b(x, y) |f(y)|^{p-1} \leq 2^{p-1} \left( |f(x)|^{p-1} \sum_{y \in X} b(x, y) + \sum_{y \in X} b(x, y) |\nabla_x y f|^{p-1} \right) < \infty.
\]

Now, we are in a position to define the Laplacian: Let \( m \) be a measure on \( X \). Then, the **\( p \)-Laplace operator** \( L = L_{b,m,V,p} : F(V) \to C(V) \) is defined via

\[
Lf(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y) \left( \nabla_x y f \right)^{(p-1)}, \quad x \in V.
\]

Let \( p \geq 1 \). If we have additionally \( m = 1, b(X \times X) \subseteq [0, 1] \), then \( L \) is called **standard \( p \)-Laplacian**.

**Remark 2.2** Following [42, 55, 59], there is the following analogy to \( p \)-Laplacians in the continuum: A vector field \( v \) is a function in \( C(X \times X) \) such that \( v(x, y) = -v(y, x) \), \( x, y \in X \). Moreover, define \( \text{div} \) on the space of absolutely summable vector fields in the second entry via

\[
(\text{div} v)(x) = \frac{1}{m(x)} \sum_{y \in X} v(x, y).
\]

Then, for all \( f \in F \), and \( p \geq 1 \),

\[
Lf(x) = \text{div}(b |\nabla f|^{p-2} \nabla f)(x), \quad x \in X.
\]

This shows that our Laplacian is a discrete analogue to weighted Laplace–Beltrami-type operators on manifolds.
Remark 2.3 The definition of the action of the $p$-Laplacian to vanish outside of $V \subseteq X$ is an arbitrary choice. It evolves from the main result, where we are taking test functions only from $C_c(V)$. This choice implies that the functions $u$ in Theorem 3.1 can be taken from the largest possible set, namely $F(V)$, and that the action of the $p$-Laplacian outside of $V$ does not matter. If we enlarge the set of test functions then we need to be more careful with the definition of the action of the $p$-Laplacian outside of $V$. Then, we might also reconsider to define the $p$-Laplacian only weakly. A larger set of test functions would on the other hand result in a smaller set of functions $u$ to which we can apply the main result, confer Theorem 3.1. Since the main result is proven by summing up pointwise estimates, similar results hold mutatis mutandis when considering other sets of test functions and related $p$-Laplacians, see [30, 42] for definitions of Dirichlet and Neumann $p$-Laplacians.

Finally, we can define Schrödinger operators as follows: Let $c \in C(X)$. Then the $p$-Schrödinger operator $H = H_{b,c,m,V,p} : F(V) \to C(V)$ is given by

$$Hf(x) = Lf(x) + \frac{c(x)}{m(x)} (f(x))^{(p-1)}, \quad x \in V.$$ 

The function $c$ is then usually called the potential of $H$. If $c$ is non-negative, then $H$ is called $p$-Laplace-type operator.

A function $u \in F(V)$ is said to be $p$-harmonic, ($p$-superharmonic, $p$-subharmonic) on $V \subseteq X$ with respect to $H$ if

$$Hu = 0 \quad (Hu \geq 0, \; Hu \leq 0) \quad \text{on } V.$$ 

If $V = X$ we only speak of $p$-super-/sub-/harmonic functions.

### 2.2 Energy functionals associated with graphs

Let $D = D_{b,c,p}$ be given by

$$D = \left\{ f \in C(X) : \sum_{x,y \in X} b(x, y) \left| \nabla_{x,y} f \right|^p + \sum_{x \in X} c(x) \left| f(x) \right|^p < \infty \right\}.$$ 

Then, the $p$-energy functional $h = h_{b,c,p} : D \to \mathbb{R}$ is defined via

$$h(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y) \left| \nabla_{x,y} f \right|^p + \sum_{x \in X} c(x) \left| f(x) \right|^p.$$ 

If $p = 2$, then the corresponding energy functional is a quadratic form, and called Schrödinger form.

As in the continuum or the linear case on graphs, there exists a so-called Green’s formula which shows a connection between $H$ and $h$ on $C_c(X)$. The Green’s formula seems to be folklore in both worlds. However, for the convenience of the reader we include a proof here. A similar proof of the Green’s formula for the normalised $p$-Laplacian, that is $m = \deg$ and $c = 0$, is given in [59].
Lemma 2.4 (Green’s formula) Let \( p \geq 1, V \subseteq X, f \in F(V) \) and \( \varphi \in C_c(X) \). Then, all of the following sums converge absolutely and

\[
\sum_{x \in V} Hf(x)\varphi(x)m(x) = \frac{1}{2} \sum_{x, y \in V} b(x, y) (\nabla_{x,y} f)^{(p-1)} (\nabla_{x,y} \varphi) + \sum_{x \in V} c(x) (f(x))^{(p-1)} \varphi(x) \\
+ \sum_{x \in V, y \in \partial V} b(x, y) (\nabla_{x,y} f)^{(p-1)} \varphi(x).
\]

In particular, the formula can be applied to \( f \in C_c(X) \), or \( f \in D \), and

\[
h(\varphi) = \sum_{x \in V} H\varphi f(x)\varphi(x)m(x), \quad \varphi \in C_c(V).
\]

Proof Since \( \varphi \in C_c(X) \), the absolute convergence follows from

\[
\sum_{x \in V} |Lf(x)\varphi(x)| m(x) \leq \sum_{x \in V} |\varphi(x)| \sum_{y \in X} b(x, y) |\nabla_{x,y} f|^{p-1} < \infty,
\]

for any \( f \in F(V) \).

Applying Fubini’s theorem, using the absolute convergence of the sums and the symmetry of \( b \), we get

\[
\sum_{x \in V} Lf(x)\varphi(x)m(x) \\
= \sum_{x \in V, y \in X} b(x, y) (\nabla_{x,y} f)^{(p-1)} \varphi(x) \\
= \frac{1}{2} \sum_{x, y \in V} b(x, y) (\nabla_{x,y} f)^{(p-1)} \varphi(x) - \frac{1}{2} \sum_{\hat{x}, \hat{y} \in V} b(\hat{x}, \hat{y}) (\nabla_{\hat{x},\hat{y}} f)^{(p-1)} \varphi(\hat{y}) \\
+ \sum_{x \in V, y \in \partial V} b(x, y) (\nabla_{x,y} f)^{(p-1)} \varphi(x) \\
= \frac{1}{2} \sum_{x, y \in V} b(x, y) (\nabla_{x,y} f)^{(p-1)} \nabla_{x,y} \varphi + \sum_{x \in V, y \in \partial V} b(x, y) (\nabla_{x,y} f)^{(p-1)} \varphi(x).
\]

The assertions for the Schrödinger operator \( H \) follow now easily.

By Lemma 2.1, \( C_c(X) \subseteq F(V) \). Note that \( F(X) \subseteq F(V) \). It remains to show that \( D \subseteq F(X) \). This follows from Hölder’s inequality for all \( x \in X \) by

\[
\sum_{y \in X} b(x, y) |\nabla_{x,y} f|^{p-1} \leq \left( \sum_{y \in X} b(x, y) \right)^{1/p} \left( \sum_{y \in X} b(x, y) |\nabla_{x,y} f|^p \right)^{(p-1)/p} < \infty.
\]

This ends the proof.

The Green’s formula is sometimes taken to define a \( p \)-Schrödinger operator weakly. By taking the test function \( \varphi = 1_z \) for \( z \in V \subseteq X \), we see that the action of a weakly defined operator agrees with the action of our (strongly) defined \( p \)-Schrödinger operator on \( V \).
3 The ground state representation on graphs

In the classical linear case, ground state representations are transformations which use a superharmonic function to turn a quadratic energy form associated with a linear Schrödinger operator into a quadratic energy form associated with a linear Laplace operator, see e.g. [34, Proposition 4.8] for such a statement on graphs, and e.g. [14, p. 109] for a counterpart in the continuum.

In the non-linear (\( p \neq 2 \))-case, we do not have an equality via a transformation between functionals anymore. But instead, we achieve an equivalence between functionals, providing that a positive \( p \)-superharmonic function exists. The equivalent functional has the property that it consists of non-negative terms only.

Our representations in Theorem 3.1 and Corollary 3.2 can be seen as the non-local analogues to the local and non-linear representations in [51, 53], where \( p \)-Schrödinger operators on domains in \( \mathbb{R}^d \) are discussed.

First applications of our representations are given in Sect. 4. Moreover, other applications can be found in the follow-up papers [20, 21].

3.1 The statement

Let \( p > 1 \), and \( 0 \leq u \in F(V) \) for some \( V \subseteq X \). The simplified energy (functional) \( h_u \) of \( h \) with respect to \( u \) on \( C_c(V) \) be given by

\[
h_u(\varphi) := \sum_{x,y \in X} b(x,y)u(x)u(y)(\nabla_{x,y} \varphi)^2 \cdot \left( (u(x)u(y))^{1/2} |\nabla_{x,y} \varphi| + \frac{|\varphi(x)| + |\varphi(y)|}{2} |\nabla_{x,y} u| \right)^{p-2},
\]

where we set \( 0 \cdot \infty = 0 \) if \( 1 < p < 2 \).

Moreover, we define a weighted bracket \( \langle \cdot, \cdot \rangle \) on \( C(X) \times C_c(X) \) via

\[
\langle f, \varphi \rangle := \sum_{x \in X} f(x) \varphi(x)m(x), \quad f \in C(X), \varphi \in C_c(X).
\]

We state now the main result of this paper.

**Theorem 3.1** (Ground state representation) Let \( p > 1 \) and \( 0 \leq u \in F(V) \) for some \( V \subseteq X \). Then, we have

\[
h(u \varphi) - \langle H u, u |\varphi|^p \rangle \approx h_u(\varphi), \quad \varphi \in C_c(V).
\]

Furthermore, the equivalence becomes an equality if \( p = 2 \).

In many applications the function \( u \) is assumed to be \( p \)-harmonic in \( V \subseteq X \). In this case the representation in (3.1) reduces to

\[
h(u \varphi) \approx h_u(\varphi), \quad \varphi \in C_c(V).
\]

A further consequence of (3.1) is, that the corresponding left-hand side is non-negative, i.e.,

\[
h(u \varphi) \geq \langle H u, u |\varphi|^p \rangle, \quad \varphi \in C_c(V).
\]
This inequality is known as Picone’s inequality, see [1, 4, 5, 11, 21, 23, 45, 46, 53] for applications of this inequality in various contexts.

From the inequalities in Theorem 3.1, we get as consequences estimates between the energy associated with the Schrödinger operator and other functionals, which are usually also referred to as simplified energies (see e.g. [15, 53]). They all are called simplified, because they consist of non-negative terms only, and the difference operator \( \nabla \) also referred to as energy functional to the Schrödinger operator \( \Delta_1 \).

The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.2** Let \( p > 1 \). If \( 1 < p \leq 2 \), then there is a positive constant \( c_p \) such that for all \( 0 \leq u \in F(V) \)

\[
h(u\varphi) - \langle Hu, u \varphi \rangle \leq c_p h_{u,1}(\varphi), \quad \varphi \in C_c(V),
\]

and for \( p \geq 2 \), we define on \( C_c(V) \)

\[
h_{u,2}(\varphi) := \sum_{x,y \in X} b(x, y) u(x)u(y) \left| \nabla_{x,y} \varphi \right|^p,
\]

The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.2** Let \( p > 1 \). If \( 1 < p \leq 2 \), then there is a positive constant \( c_p \) such that for all \( 0 \leq u \in F(V) \)

\[
h(u\varphi) - \langle Hu, u \varphi \rangle \leq c_p h_{u,1}(\varphi), \quad \varphi \in C_c(V),
\]

and if \( p \geq 2 \) the reversed inequality in (3.2) holds true, i.e.,

\[
h(u\varphi) - \langle Hu, u \varphi \rangle \geq c_p h_{u,1}(\varphi), \quad \varphi \in C_c(V).
\]

Furthermore, both inequalities become equalities if \( p = 2 \).

Moreover, if \( p \geq 2 \), we have for all \( 0 \leq u \in F(V) \),

\[
h(u\varphi) - \langle Hu, u \varphi \rangle \leq h_{u,1}(\varphi) + h_{u,2}(\varphi), \quad \varphi \in C_c(V).
\]

The statements in Theorem 3.1 and Corollary 3.2 will follow mainly by pointwise inequalities without summation. Then, we will sum over \( X \times X \) and use Green’s formula to obtain the results. The elementary inequalities are basically given in the upcoming Lemma 3.8.

The proof does not include the case \( p = 1 \). This is because we use a quantification of the strict convexity of the mapping \( x \mapsto \left| x \right|^p, \ p \geq 1 \).

### 3.2 Some remarks on the main result

**Remark 3.3** (Comparison with the local non-linear analogue) We compare our ground state representation with results in [53]. Similar results associated with weighted \( p \)-Schrödinger operators can be found in [51].

Fix \( p \in (1, \infty) \) and a domain \( \Omega \subseteq \mathbb{R}^d \). Let \( u \in W_{\text{loc}}^{1,p}(\Omega) \) and \( \Delta(u) := -\text{div}(\left| \nabla u \right|^{p-2} \nabla u) \) be the \( p \)-Laplacian on \( \Omega \). Furthermore, let \( V \in L_{\text{loc}}^{\infty}(\Omega) \). The corresponding energy functional to the Schrödinger operator \( \Delta + V \) is given by

\[
Q(\varphi) := \int_{\Omega} \left| \nabla \varphi \right|^p + V \left| \varphi \right|^p \, dx, \quad \varphi \in C_c^{\infty}(\Omega).
\]

Then, by [53, Lemma 2.2], we have the following: If \( u \) is a positive \( \ p \)-harmonic function of \( \Delta + V \) in the weak sense, i.e.,

\[
\int_{\Omega} \left| \nabla u \right|^{p-2} \nabla u \cdot \nabla \varphi + V \left| u \right|^{p-2} u \varphi \, dx = 0 \text{ for all } \varphi \in C_c^{\infty}(\Omega),
\]

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In particular, for $p$ we can take any subset of the graph.

Comparison of the terms in the right-hand side (RHS) of (3.5) with $h_u(\varphi)$

| RHS of (3.5) | $h_u(\varphi)$ |
|--------------|-----------------|
| $u^2 |\nabla \varphi|^2$ | $(u(x))u(y)|\nabla_{x,y}\varphi|^2$ |
| $u |\nabla \varphi| + \varphi |\nabla u|$ | $(u(x))u(y)(1/2)|\nabla_{x,y}\varphi| + (1/2)(|\varphi(x)| + |\varphi(y)|)|\nabla_{x,y}\varphi|$ |

Comparison of the terms in the right-hand side (RHS) of (3.6) with $h_{u,1}(\varphi) + h_{u,2}(\varphi)$

| RHS of (3.6) | $h_{u,1}(\varphi) + h_{u,2}(\varphi)$ |
|--------------|---------------------------------|
| $u^p |\nabla \varphi|^p$ | $(u(x))u(y)|\nabla_{x,y}\varphi|^p$ |
| $u^2 |\nabla u|^p - |\nabla \varphi|^p \varphi^2 |\nabla \varphi|^2$ | $(u(x))u(y)|\nabla_{x,y}\varphi|^p - (1/2)(|\varphi(x)| + |\varphi(y)|)|\nabla_{x,y}\varphi|^2$ |

then

$$Q(u\varphi) \leq \int_{\Omega} u^2 |\nabla \varphi|^2 (u |\nabla \varphi| + \varphi |\nabla u|)^{p-2} \, dx, \quad 0 \leq \varphi \in C^1_c(\Omega).$$

In particular, for $p > 2$, we have

$$Q(u\varphi) \leq \int_{\Omega} u^p |\nabla \varphi|^p + u^2 |\nabla u|^p - 2 \varphi^2 |\nabla \varphi|^2 \, dx, \quad 0 \leq \varphi \in C^1_c(\Omega).$$

In the case of $1 < p < 2$, we have by [53, Remark 1.12] that

$$\int_{\Omega} u^2 |\nabla \varphi|^2 (u |\nabla \varphi| + \varphi |\nabla u|)^{p-2} \, dx \leq \int_{\Omega} u^p |\nabla \varphi|^p. \quad (3.7)$$

Now, we do the comparison: In the continuum, domains of $\mathbb{R}^d$ are considered. On graphs, we can take any subset of the graph.

Recall that $u$ is $p$-harmonic. It is very easy to compare $h_u(\varphi)$ with the right-hand side in (3.5), see Table 1.

This motivates to call the simplified energy $h_u$ the analogue to the simplified energy in the local non-linear case. Note that in the continuum, we only consider non-negative compactly supported functions $\varphi$, whereas on graphs, we allow $\varphi$ to take negative values. Thus, the version in the continuum contains hidden moduli of $\varphi$.

Furthermore, we see that the equivalence (3.6) has the same structure as the equivalence (3.4). For a comparison of $h_{u,1}(\varphi) + h_{u,2}(\varphi)$ with the right-hand side in (3.6) see Table 2.

Furthermore, we see that the estimate in (3.7) together with (3.5) has the same structure as the upper bound (3.2).

It should be mentioned that the strategy to prove the ground state representation in [53] and here are similar. There, an elementary equivalence is the key ingredient and then a Picone identity is used. Here, we use different elementary equivalences and the Green’s formula. However, the proof of the elementary equivalences in the discrete is technically much harder than the proof of the corresponding one in the continuum. Thus, the differences above might come from the fact that in the continuum we have a Picone identity (see [53, Sect. 2]) which is established via the chain rule. Whereas in the discrete, we only have a one-sided Picone inequality. A general version of this one-sided Picone inequality is discussed in a follow-up paper by the author [21], see also [11].

Moreover, in [53, Proposition 5.1] it was shown that for $p > 2$ both summands in the integral in (3.6) are needed in general for an upper bound. We expect that the same holds true.
on graphs, i.e., we expect that both \( h_{u,1} \) and \( h_{u,2} \) are needed in general for an upper bound of \( h \).

**Remark 3.4** (Discussion of the constants) By comparing Theorem 3.1 with [23, Proposition 2.3] and Lemma 3.8 (the lemma below) with [23, Lemma 2.6], we see that \( c_p \) in (3.3) can be stated explicitly as a minimiser, i.e., for \( p \geq 2 \)

\[
c_p = \frac{1}{2} \min_{t \in (0,1/2)} \left( (1-t)^p - t^p + pt^{p-1} \right) \in (0,1/2].
\]

Note that \( c_2 = 1/2 \). Moreover, we expect that the best constants in Theorem 3.1 are between 0 and 1.

**Remark 3.5** (Hardy inequality) In [15] the non-linear ground state representation of [53] was used to prove optimality of certain \( p \)-Hardy weights associated with \( p \)-Schrödinger operators on domains in \( \mathbb{R}^d \). The discrete counterpart does hold as well using the here presented discrete ground state representation and are topic of a follow-up paper by the author, [20]. This generalises the results of the linear case in [33] to \( p \neq 2 \). A consequence of the ground state representation and some results in this yet unpublished paper is that the improved \( p \)-Hardy inequality on \( \mathbb{N} \) in [19] is indeed optimal.

**Example 3.6** (Standard \( p \)-Laplacian on \( \mathbb{N}_0 \)) Here, we calculate the representation for one of the simplest cases: for the graph \( b \) on \( \mathbb{N} \) with \( b(n,m) = 1 \) if \( |n-m| = 1 \) and \( b(n,m) = 0 \) elsewhere for all \( n,m \in \mathbb{N} \).

The standard (or combinatorial) \( p \)-Laplacian \( \Delta \) for real valued functions on \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) is given by

\[
\Delta f(n) = \sum_{m=n\pm 1} \text{sgn} (f(n) - f(m)) |f(n) - f(m)|^{p-1}
\]

for all functions \( f \in C(\mathbb{N}) \) and \( n \geq 1 \). The corresponding energy functional reads then as

\[
h(\varphi) = \frac{1}{2} \sum_{n=1}^{\infty} |\varphi(n) - \varphi(m)|^p = \sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^p,
\]

for all \( \varphi \in C_c(\mathbb{N}) \). From [19, Proposition 4] it follows that \( u \in F \) defined via \( u(n) = n^{(p-1)/p} \) is a positive \( p \)-superharmonic function such that \( \Delta u = wu^{p-1} \), where \( w \) is the improved \( p \)-Hardy weight in [19, Theorem 1]. Let \( q := p/(p-1) \) and \( \alpha(n) := (1-1/n)^{1/q}, n \in \mathbb{N} \). Then, the equivalence (3.1) reads as follows: for all \( \varphi \in C_c(\mathbb{N}) \), we have

\[
\sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^p - w(n) |\varphi(n)|^p
\]

\[
\leq \sum_{n=2}^{\infty} \frac{1}{\alpha^{p-1}(n)} \left( \alpha(n)\varphi(n) - \varphi(n-1) \right)^2
\]

\[
\cdot \left( \alpha^{1/2}(n) |\varphi(n) - \varphi(n-1)| + \frac{\alpha(n) |\varphi(n)| + |\varphi(n-1)|}{2} (1 - \alpha(n)) \right)^{p-2}.
\]

If \( p = 2 \), then the equivalence is an equality and gives exactly the result of [36, Theorem 1].

Moreover, the inequality (3.3) \( (p \geq 2) \) in Corollary 3.2 is here

\[
\sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^p - w(n) |\varphi(n)|^p \geq c_p \sum_{n=2}^{\infty} \frac{1}{\alpha^{p/2}(n)} |\alpha(n)\varphi(n) - \varphi(n-1)|^p
\]
for all $\varphi \in C_c(\mathbb{N})$. By (3.2), the reversed inequality holds for $1 < p \leq 2$.

**Remark 3.7** (Simplified Energy Functionals) In the linear case, an application of the ground state representation formula is to get from Schrödinger forms $h$ with arbitrary potential part via an equality to a new Schrödinger form $h_u + \langle uHu, \cdot \rangle^2$ with non-negative potential part. This new form corresponds then to the graph $b_u$ given by $b_u(x, y) := b(x, y)u(x)u(y)$, $x, y \in X$, where $u$ is a positive superharmonic function with respect to the Schrödinger operator associated with $h$, and has the potential $c_u := uHu$, see [33, 35]. An advantage of Schrödinger forms with non-negative potential part is that they are Markovian.

For $p \neq 2$, the simplified energy $h_u$ is not a $p$-energy functional anymore and the described method from the linear situation cannot be applied directly. However, there is a partial workaround via $h_{u,1}$. By applying Corollary 3.2 for some fixed positive superharmonic $u$ in $X$, we see that there is a positive constant $c_p$ such that

$$h(u\varphi) \leq c_ph_{u,1}(\varphi) + \langle uHu, |\varphi|^p \rangle \quad \varphi \in C_c(X), p \in (1, 2].$$

The right-hand side is a $p$-energy functional associated with the graph $b_u$ given by $b_u(x, y) := c_pb(x, y)(u(x)u(y))^{p/2}$, $x, y \in X$, and has the non-negative potential $c_u := uHu$.

In the case of $p > 2$, the situation is more complicated because the right-hand side gets additionally the functional $h_{u,2}$, which is not a $p$-energy functional. Nevertheless, the following observation has proven to be useful (see [15, 20]): applying Hölder’s inequality to $h_{u,2}$, we get the existence of a positive constant $C_p$ such that

$$h(u\varphi) \leq C_p(h_{u,1}(\varphi) + \left(\frac{h_{u,1}(\varphi)}{h_{u,3}(\varphi)}\right)^{2/p} h_{u,3}(\varphi)) + \langle uHu, |\varphi|^p \rangle, \quad p > 2,$$

where

$$h_{u,3}(\varphi) := \sum_{x,y \in X} b(x, y) \left(\frac{|\varphi(x)| + |\varphi(y)|}{2}\right)^p \left|\nabla_{x,y} u\right|^p,$$

which can be interpreted as a $p$-energy functional on $F$ for some fixed $\varphi \in C_c(X)$.

### 3.3 Elementary inequalities and equivalences

The next lemma is the most important tool in order to derive the ground state representations, Theorem 3.1 and Corollary 3.2. It provides us with pointwise estimates which result in the desired estimates of the energy functionals by summing over all vertices.

**Lemma 3.8** (Fundamental inequalities and equivalences) Let $a \in \mathbb{R}$, $0 \leq t \leq 1$, and $p > 1$. Then we have

$$|a - t|^p - (1 - t)^{p - 1}(|a|^p - t|a - 1|^2) \asymp t|a - t|^p + (1 - t)^{p - 2}, \quad \text{ (3.8)}$$

where the right-hand side is understood to be zero if $1 < p < 2$ and $a = t = 1$. Moreover, we have

$$|a - t| + 1 - t \asymp t^{1/2} |a - 1| + (1 - t)\frac{|a| + 1}{2}, \quad \text{ (3.9)}$$

where the right-hand side is an upper bound with optimal constant $c = 2$, and it is a lower bound with optimal constant $c = 1/2$. 

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Furthermore, if $1 < p \leq 2$, then
\[ t |a - 1|^2 \leq t^{p/2} |a - 1|^p (|a - t| + 1 - t)^{2-p}, \]  
and for $p \geq 2$, the reserved inequality holds, i.e.,
\[ t |a - 1|^2 (|a - t| + 1 - t)^{p-2} \geq t^{p/2} |a - 1|^p. \]

Moreover, we have the following refinement of the elementary inequality (2.1): for all $p \geq 0$, we have
\[ \alpha^p + \beta^p \approx (\alpha + \beta)^p, \quad \alpha, \beta \geq 0, \]  
where the right-hand side is an upper bound with optimal constant $c_p = 2^{1-p}$ if $0 \leq p \leq 1$ and $c_p = 1$ if $p \geq 1$, and it is a lower bound with optimal constant $c_p = 1$ for $0 \leq p \leq 1$ and $c_p = 2^{1-p}$ for $p \geq 1$.

A proof is given in Appendix 1.

We do not claim that the constants we get in (3.8) are optimal. We expect that they can be improved and that the best constants should be either on the boundary of $[0, 1] \times \mathbb{R}$, or at $(t, 0)$, $(t, t)$, $(t, 1)$, $t \in [0, 1]$. Moreover, we expect that the optimal constants are between 0 and 2.

Also note that the inequalities (3.8) and (3.11) show that we improved an elementary one-sided result in [23] for $p > 2$.

Moreover, in the case of $1 < p < 2$, the "$\geq$"-inequality in (3.8) was proven in [1, Lemma 3.3]. However, the basic strategy to prove the remaining inequalities in (3.8) up to a certain point will be the similar, i.e., we start the proof with the same substitution and then use the same Taylor-Maclaurin formula (confer this also with the proof of [40, Lemma 4.2]).

Furthermore, note that (3.8) is false for $p = 1$ as the left-hand side vanishes for $a > 1 \geq t > 0$ but the right-hand side does not. A similar argument can also be made for (3.10).

### 3.4 Proof of the ground state representation theorem

Now we prove our main results, Theorem 3.1 and Corollary 3.2. The basic strategy is to use the pointwise equivalences and estimates from Lemma 3.8 and then sum over all vertices.

**Proof of Theorem 3.1** Let $\varphi \in C_c(V)$ and $0 \leq u \in F(V)$ for some $V \subseteq X$. If either $u(x) = 0$ or $u(y) = 0$ for some $x, y \in V \cup \partial V$, then
\[ |\nabla_{x,y}(u\varphi)|^p - (\nabla_{x,y}u)^{(p-1)} \nabla_{x,y}(u |\varphi|^p) = 0. \]

Moreover, $u(x)u(y)(\nabla_{x,y}\varphi)^2 = 0$. Thus, it remains in the following to consider the case $u(x), u(y) > 0$.

Firstly, let $u(x) \geq u(y) > 0$ for some fixed $x, y \in V \cup \partial V$. Moreover, assume that $\varphi(y) \neq 0$. Then, setting $t = u(y)/u(x)$ and $a = \varphi(x)/\varphi(y)$ in (3.8) combined with (3.9) results in
\[ |\nabla_{x,y}(u\varphi)|^p - (\nabla_{x,y}u)^{p-1} \nabla_{x,y}(u |\varphi|^p) \]
\[ \approx u(x)u(y)(\nabla_{x,y}\varphi)^2 ((u(x)u(y))^{1/2} |\nabla_{x,y}\varphi| + \frac{|\varphi(x)| + |\varphi(y)|}{2} \nabla_{x,y}u)^{p-2}. \]

If $\varphi(y) = 0$, then we get the equivalence above if we can show that
\[ 1 - (1 - t)^{p-1} \approx t(t^{1/2} + (1 - t)/2)^{p-2}, \quad t = u(y)/u(x) \in (0, 1]. \]
Using \((3.9)\) with \(a = 0\), we see that \(t^{1/2} + (1 - t)/2 \simeq 1\). Moreover, the left-hand side lies between \(t \mapsto (p - 1)t\) and the identity on \((0, 1]\). This shows the claim.

By a symmetry argument, we also get for all \(x, y \in V \cup \partial V\) such that \(u(y) \geq u(x) > 0\),
\[
\frac{\partial_x \partial_y u^p}{\partial_y^2 u} - \frac{(\partial_y u)^{p-1}}{\partial_x u} \simeq u(y) (\partial_y \partial_x u)^2 \left( (\partial_x u)^{1/2} \right) + \frac{|\partial_x u| + |\partial_y u|}{2} \partial_x \partial_y u^{p-2}.
\]
Note that by Green’s formula, Lemma 2.4 for the \(p\)-Laplacian \(L\),
\[
\sum_{x, y \in V \cup \partial V} b(x, y) (\partial_x \partial_y u)^{p-1} \partial_x \partial_y u |\varphi|^p = 2 \sum_{x \in V} u(x) L u(x) |\varphi(x)|^p m(x).
\]
Summing over all \(x, y \in X\) with respect to \(b\) and using the calculation above yields then \((3.1)\).

Now, we can directly continue with proving Corollary 3.2.

**Proof of Corollary 3.2** The proof of \((3.2)\) and \((3.3)\) can simply be read off \((3.1)\).

The inequalities in \((3.4)\) follow easily from \((3.1)\) and \((3.12)\).

Alternatively, one can also deduce \((3.2)\) and \((3.3)\) from \((3.11), (3.10)\) and \((3.8)\). The proof can then be mimicked from the proof of Theorem 3.1.

### 4 Characterisations of criticality

In this section, we will discuss the notion of criticality. For the history of this notion see [49, Remark 2.7] or [34, Sect. 5]. There it is stated that in the continuum the notion goes back to [56] and was then generalised in [43, 48]. On locally summable weighted graphs, [34] is the first paper discussing criticality in the context of linear Schrödinger operators. See also [32, Chapter 6] (and references therein) for corresponding results for linear Laplace-type operators on graphs.

Non-negative energy functionals associated with Schrödinger operators seem to divide naturally into two categories: the ones which are strictly positive, i.e., for which a Hardy inequality holds true, and the ones which are not strictly positive, i.e., for which the Hardy inequality does not hold. In the linear \((p = 2)\)-case, there are surprisingly many equivalent formulations to the statement that the Hardy inequality does (not) hold, for graphs confer [34]. For \(p = 2, c = 0\) and \(m = \text{deg}\), this is exactly the division of graphs into transient and recurrent graphs.

Using our recently developed ground state representation, we will see that many of the characterisations in [34] remain characterisations also if \(p \neq 2\).

Let \(h\) be a functional which is non-negative on \(C_c(V), \quad V \subseteq X\). Then, \(h\) is called **subcritical** in \(V\) if the **Hardy inequality** holds true in \(V\), that is, there exists a positive function \(w \in C(V)\) such that
\[
h(\varphi) \geq \langle w, |\varphi|^p \rangle, \quad \varphi \in C_c(V).
\]
If such a positive \(w\) does not exist, then \(h\) is called **critical** in \(V\). Moreover, \(h\) is called **supercritical** in \(V\) if \(h\) is not non-negative on \(C_c(V)\).

Other names for a subcritical functional are sometimes strictly positive, coercive, or hyperbolic.
Before we can state the main result of this section, we need the following definition: A sequence \((e_n)\) in \(C_c(V)\), \(V \subseteq X\), of non-negative functions is called null-sequence in \(V\) if there exists \(o \in V\) and \(\alpha > 0\) such that \(e_n(o) = \alpha\) and \(h(e_n) \to 0\).

Moreover, we define the variational capacity of \(h\) in \(V \subseteq X\) at \(x \in V\) via
\[
\text{cap}_h(x, V) := \inf_{\varphi \in C_c(V), \varphi(x) = 1} h(\varphi).
\]

**Theorem 4.1** (Characterisations of criticality) Let \(p > 1\). Furthermore, assume that there exists a positive \(p\)-superharmonic function in \(X\). Then the following statements are equivalent:

(i) \(h\) is critical in \(X\).

(ii) For any \(o \in X\) and \(\alpha > 0\) there is a null-sequence \((e_n)\) in \(X\) such that \(e_n(o) = \alpha\), \(n \in \mathbb{N}\).

(iii) \(\text{cap}_h(x, X) = 0\) for all \(x \in X\).

(iv) There exists a strictly positive \(p\)-harmonic function \(u\) in \(X\) such that \(h_u\) is critical in \(X\).

(v) For all positive \(p\)-harmonic functions \(u\) in \(X\), the ground state representation \(h_u\) is critical in \(X\).

(vi) For any positive \(p\)-superharmonic function \(u \in F(X)\) in \(X\) and any null-sequence \((e_n)\) in \(X\) there exists a positive constant \(\tilde{c}\) such that \(e_n(x) \to \tilde{c} u(x)\) for all \(x \in X\) as \(n \to \infty\).

(vii) There exists a strictly positive \(p\)-harmonic function \(u \in F(X)\) in \(X\) and a null-sequence \((e_n)\) in \(X\) such that \(e_n(x) \to u(x)\) for all \(x \in X\) as \(n \to \infty\). If \(p \geq 2\), the sequence can be chosen such that \(0 \leq e_n \leq u\) for all \(n \in \mathbb{N}\).

In particular, if one of the equivalent statements above is fulfilled, then there exists a unique positive \(p\)-superharmonic function in \(X\) (up to linear dependence) and this function is strictly positive and \(p\)-harmonic in \(X\).

We remark that in the continuum, the corresponding characterisation holds true on any subdomain of \(\mathbb{R}^d\), confer [50, Theorem 4.15]. In Proposition 4.6, we show that \(h\) cannot be critical in \(V\) whenever \(V \subsetneq X\), assumed that a positive function exists which is \(p\)-superharmonic on \(V\), and which is strictly positive somewhere at the boundary.

We divide the proof of this main theorem into two subsections. In the first subsection, we show some more general auxiliary lemmata, and in the second subsection, we show the equivalences.

Some other characterisations of criticality for \(p\)-Schrödinger operators, involving the non-existence of positive minimal Green’s functions, can be found in [21].

### 4.1 Preliminaries

We start with a direct consequence of the ground state representation.

**Lemma 4.2** Let \(p > 1\) and \(V \subseteq X\). Assume that there exists a function \(u \in F(V)\) which is strictly positive in \(V\) and \(p\)-superharmonic on \(V\). Then, we have
\[
h(\varphi) \geq \langle Hu, u^{1-p} |\varphi|^p \rangle, \quad \varphi \in C_c(V).
\]
In particular, \(h\) is non-negative on \(C_c(V)\).

**Proof** By the ground state representation, Theorem 3.1, the statement follows easily since for some \(c_p > 0\),
\[
h(\varphi) - \langle Hu, u^{1-p} |\varphi|^p \rangle \geq c_p h_u(\varphi/u) \geq 0, \quad \varphi \in C_c(V).
\]
\[\square\]
Note that the reversed statement in Lemma 4.2 is also true which is proven in a follow-up paper by the author [21]. The corresponding statement is known as an Agmon-Allegretto-Piepenbrink-type theorem (see [3, 10, 47] for a linear version in the continuum, [17, 34] for a linear version in the discrete setting, [50] for a recent non-linear version in the continuum, and [38] for a corresponding result on strongly local Dirichlet forms).

An immediate consequence of the definition of criticality is the following statement.

**Lemma 4.3** Let $p > 1$ and $V \subseteq X$. Let $h$ be critical in $V$. Then any function $u \in F(V)$ which is strictly positive in $V$ and $p$-superharmonic in $V$ is $p$-harmonic on $V$.

**Proof** Let $u$ be such a strictly positive $p$-superharmonic function. By Lemma 4.2 we have $h(\varphi) \geq \langle u^{1-p} Hu, |\varphi|^p \rangle$ for all $\varphi \in C_c(V)$. Because $h$ is critical we get that $u^{1-p} Hu = 0$ on $V$, and thus, $u$ is $p$-harmonic on $V$. \hfill $\Box$ 

Next we show that locally, i.e., on finite and connected sets, our graphs fulfill a so-called Harnack inequality for non-negative supersolutions. This inequality implies that non-negative supersolutions are either strictly positive or the zero function, and they do not tend to infinity in the interior of the graph.

There is a long list of proofs of various Harnack-type inequalities for the $p$-Laplacian, see for instance for metric spaces [9, Theorem 8.12] where a $p$-Poincaré inequality is assumed. The corresponding analogue for linear Schrödinger operators on locally summable graphs can be found in [34]. The basic idea of the following proof of the local Harnack inequality can also be found in [55], where the standard $p$-Laplacian on locally finite graphs without potential (i.e., $c = 0$) is considered, and [27], where the standard $p$-Laplacian on finite graphs without potential, is considered.

**Lemma 4.4** (Local Harnack inequality) Let $p > 1$, $V \subseteq X$ be connected and $f \in C(X)$. Let $u \in F(V)$ be non-negative on $V \cup \partial V$ such that $Hu \geq fu^{p-1}$ on $V$. Then, for any $x \in V$ and $y \sim x$ we have

$$u(y) \leq \left( \left( \frac{\deg(x) + c(x) - f(x)m(x)}{b(x, y)} \right)^{\frac{1}{p-1}} + 1 \right) u(x).$$

In particular, we have $\deg \geq f m$ on $V$. Furthermore, if $u(x) = 0$ for some $x \in V$, then $u(x) = 0$ for all $x \in V \cup \partial V$. In other words, any function which is positive on $V \cup \partial V$ and $p$-superharmonic on $V$ is strictly positive on $V$.

If $V$ is also finite, then there exists a positive constant $C_{V,H,f}$ depending only on $V$, $H$ and $f$, such that

$$\max_V u \leq C_{V,H,f} \min_V u.$$ 

The constant $C_{V,H,f}$ can be chosen to be monotonous in the sense that if $f \leq g \in C(X)$ then $C_{V,H,f} \geq C_{V,H,g}$.

**Proof** Let $V \subseteq X$ be connected and let $u \in F(V)$ be such that $u \geq 0$ on $V \cup \partial V$ and $Hu \geq fu^{p-1}$ on $V$ for some $f \in C(X)$.

If $u(x_0) = 0$ for some $x_0 \in V$, then we have

$$0 = f(x_0)u^{p-1}(x_0)m(x_0) \leq Hu(x_0)m(x_0) = -\sum_{y \in X} b(x_0, y)u(y)^{p-1} \leq 0.$$ 

Thus, for all $x \sim x_0$, we have $u(x) = 0$ and since $V$ is connected we infer by induction that $u(x) = 0$ for all $x \in V \cup \partial V$. 

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Hence, we can assume that \( u > 0 \) on \( V \). Because of \( Hu \geq fu^{p-1} \) on \( V \), we have
\[
\sum_{x \in X, \forall x, u \leq 0} b(x, y) \left( \frac{u(y)}{u(x)} - 1 \right)^{p-1} \\
\leq \sum_{x \in X, \forall x, u > 0} b(x, y) \left( 1 - \frac{u(y)}{u(x)} \right)^{p-1} + c(x) - f(x)m(x)
\]
for any \( x \in V \). The right-hand side can be estimated as follows:
\[
\ldots \leq \sum_{x \in X, \forall x, u > 0} b(x, y) + c(x) - f(x)m(x) \leq \deg(x) + c(x) - f(x)m(x).
\]

Let \( d_f := \deg + c - fm \). The previous calculations imply that \( d_f \geq 0 \) on \( V \). Now assume that there is a vertex \( y_0 \sim x \) such that \( u(x) \leq u(y_0) \). Then the previous calculations also imply
\[
b(x, y_0) \left( \frac{u(y_0)}{u(x)} - 1 \right)^{p-1} \leq d_f(x), \text{ i.e., } u(y_0) \leq \left( \left( \frac{d_f(x)}{b(x, y_0)} \right)^{\frac{1}{p-1}} + 1 \right)u(x).
\]
Hence, for all \( y \sim x \) we have
\[
u(y) \leq \left( \left( \frac{d_f(x)}{b(x, y)} \right)^{\frac{1}{p-1}} + 1 \right)u(x).
\]

Let \( x \in V \) and \( y \in V \cup \partial V \). Since \( V \) is connected, there is a path \( x_1 \sim \ldots \sim x_n \) in \( V \) such that \( x_1 \sim x_0 := y \) and \( x_n = x \). Then we derive
\[
u(y) \leq \prod_{i=0}^{n-1} \left( \left( \frac{d_f(x_i)}{b(x_i, x_{i+1})} \right)^{\frac{1}{p-1}} + 1 \right)u(x).
\]

Note that the obtained product does not only depend on \( V, H \) and \( f \) but also on \( x \) and \( y \), and the chosen path. The dependence on the path can be overcome by considering all possible paths in \( V \) such that the starting vertex is connected to \( y \). Taking then the infimum of all resulting products, we get a function \( C_{V, H, f} : V \times V \cup \partial V \to [1, \infty) \) which is still dependent on the vertices but independent of a specific path. As we would like to get a unifying constant for all vertices, one possibility is to simply take the supremum of \( C_{V, H, f}(x, y) \) over all \( x \in V, y \in V \cup \partial V \). However, this supremum might not be finite. A way to ensure finiteness is to assume additionally that \( V \) is finite and \( x, y \in V \). Then, \( C_{V, H, f} := \max_{x, y \in V} C_{V, H, f}(x, y) < \infty \) which yields the desired statement. Moreover, if \( f \leq g \in C(X) \) then \( d_f \geq d_g \) and hence \( C_{K, H, f} \geq C_{K, H, g} \).

The following lemma is the discrete analogue of [50, Proposition 4.11].

**Lemma 4.5** Let \( p > 1 \) and assume that there exists a positive \( p \)-superharmonic function \( u \in F \). Furthermore, assume that \( (e_n) \) is a null-sequence in \( X \) such that \( e_n(\alpha) = \alpha \) for some \( o \in X \) and \( \alpha > 0 \). Then, \( e_n \to (\alpha / u(\alpha))u \) pointwise on \( X \) as \( n \to \infty \). In particular, for all \( (x, y) \in X \times X \) we have \( \nabla_{x,y}(e_n/u) \to 0 \) as \( n \to \infty \).

**Proof** By the Harnack inequality, Lemma 4.4, any positive \( p \)-superharmonic function \( u \) in \( X \) is strictly positive in \( X \).

Let \( o \in X \) and \( \alpha > 0 \) be arbitrary. Set \( \varphi_n := e_n/u \). Then, by the ground state representation, Theorem 3.1,
\[
0 \leq h_u(\varphi_n) \leq h(e_n) \to 0, \quad n \to \infty.
\] (4.1)
Firstly, let \( p \geq 2 \). Then, (4.1) implies \( \lim_{n \to \infty} \nabla_{x,y} \varphi_n = 0 \) for all \( x, y \in X, x \sim y \). Since \( X \) is connected, we have for any \( x \in X \) a \( k \in \mathbb{N} \) such that \( x = x_1 \sim \ldots \sim x_k = o \). Thus, we obtain

\[
\lim_{n \to \infty} \varphi_n(x) = \lim_{n \to \infty} \left( \sum_{i=1}^{k-1} \nabla_{x_i,x_{i+1}} \varphi_n + \varphi_n(o) \right) = \alpha/u(o).
\]

Rearranging, yields \( e_n \to (\alpha/u(o))u \) pointwise on \( X \) as \( n \to \infty \).

Secondly, let \( 1 < p < 2 \). Then Eq. (4.1) implies either

1. \( \left| \nabla_{x,y} \varphi_n \right| \to 0 \), or
2. \( \left| \nabla_{x,y} \varphi_n \right| \to \infty \), or
3. \( (\varphi_n(x) + \varphi_n(y)) \left| \nabla_{x,y} u \right| \to \infty \)

for each \((x, y) \in X \times X, x \sim y \). We show that (2) and (3) cannot apply: Using the triangle inequality, it is easy to see that (2) and (3) are equivalent for the pair \((x, o)\) with \( x \sim o \). They are also equivalent to \( e_n(x) \to \infty \) for \( x \sim o \). Set

\[
\Phi_n(x, o) := (u(x)u(o))^{1/2} \left| \nabla_{x,o} \varphi_n \right| + 1/2(|\varphi_n(x)| + |\varphi_n(o)|) \left| \nabla_{x,o} u \right|.
\]

Then using Hölder’s inequality with \( \bar{p} = 2/p > 1 \), and \( \bar{q} = 2/(2 - p) \), we calculate

\[
b(x, o)(u(x)u(o))^{p/2} \left| \nabla_{x,o} \varphi_n \right|^p
\]

\[
\leq \left( b(x, o)(u(x)u(o)) \right)^{p/2} \left| \nabla_{x,o} \varphi_n \right|^p \Phi_n^{p/2}(x, o) \cdot (b(x, o)\Phi_n^{p}(x, o))^{(2-p)/2}
\]

\[
\leq c_1(p) \cdot h_u^{p/2}(\varphi_n)
\]

\[
\cdot \left( (u(x)u(o))^{p/2} \left| \nabla_{x,o} \varphi_n \right|^p + c_2(p)(|\varphi_n(x)|^p + (\alpha/u(o))) \left| \nabla_{x,o} u \right|^p \right)^{(2-p)/2}
\]

\[
\leq c_1(p) \cdot h_u^{p/2}(\varphi_n) \cdot (b(x, o)((u(x)u(o))^{p/2} + c_3(p)) \left| \nabla_{x,o} \varphi_n \right|^p + c_4(p) + 1),
\]

where \( c_i(p), i \leq 4 \), are positive constants depending only on \( p \) (and not on \( n \)). Since \( b(x, o) \), \( u(x) \), \( u(o) \) are also independent of \( n \) and strictly positive, we can rewrite the inequality above as

\[
\left| \nabla_{x,o} \varphi_n \right|^p \leq C_1(p) \cdot h_u^{p/2}(\varphi_n) \cdot \left( \left| \nabla_{x,o} \varphi_n \right|^p + C_2(p) \right),
\]

for some positive constants \( C_i(p), i = 1, 2 \). Since \( h_u(\varphi_n) \to 0 \) as \( n \to \infty \), we conclude that \( \left| \nabla_{x,o} \varphi_n \right| \to 0 \), and \( e_n(x) \) does not converge to \( \infty \) for all \( x \sim o \). Hence, (2) and (3) cannot apply for all \( x \sim o \), and only (1) holds true for all \( x \sim o \). Thus, we can continue as in the case \( p \geq 2 \) to get that \( e_n(x) \to (\alpha/u(o))u(x) \) for all \( x \sim o \).

Arguing similarly, we have for all \( y \sim x \sim o \) that

\[
\left| \nabla_{y,x} \varphi_n \right|^p \leq C_1(p) \cdot h_u^{p/2}(\varphi_n) \cdot \left( \left| \nabla_{y,x} \varphi_n \right|^p + C_2(p) |\varphi_n(x)|^p + C_3(p) \right),
\]

for some positive constants \( C_i(p), i \leq 3 \). Thus, as before, (2) and (3) cannot apply for all \( y \sim x \sim o \) which results in \( e_n(y) \to (\alpha/u(o))u(y) \). Since \( X \) is connected, we get by induction that \( e_n(y) \to (\alpha/u(o))u(y) \) for all \( y \in X \). This proofs the statement for \( 1 < p < 2 \).

\( \square \)
4.2 Proof of the characterisations of criticality

Here, we proof Theorem 4.1. We show the equivalences in the following order: (i) \(\iff\) (ii) \(\iff\) (iii), and (i) \(\iff\) (iv) \(\iff\) (v), and under the assumption \(V = X\), (ii) \(\iff\) (vi), and ((i) & (ii) & (vi)) \(\iff\) (vii) \(\implies\) (i). From (vii), we deduce the last assertion of the theorem.

**Proof of Theorem 4.1**

Ad (i) \(\implies\) (ii): Let \(w_n = 1_o/n\) for \(o \in X\) and \(n \in \mathbb{N}\). Then by the criticality of \(h\) in \(X\) we have the existence of a function \(e_n \in C_c(X)\) such that \(\langle w_n, |e_n|^p \rangle < 0\). By the reverse triangle inequality, we have \(h(|e_n|) \leq h(e_n)\) and thus, we can assume that \(e_n \geq 0\). By Lemma 4.2 we have that \(h\) is non-negative in \(C_c(X)\), and therefore we get

\[
0 \leq h(e_n) < \langle w_n, |e_n|^p \rangle = e_n^p(o) m(o)/n.
\]

Hence, we can normalise \(e_n\) such that \(e_n(o) = \alpha\) for any \(\alpha > 0\). Altogether, \(h(e_n) < \alpha^p m(o)/n\) and \((e_n)\) is a null sequence in \(X\).

Ad (ii) \(\implies\) (i): Let \((e_n)\) be a null-sequence in \(X\) with \(e_n(o) = \alpha > 0\) for some \(o \in X\). Let \(w \geq 0\) on \(X\) such that \(h(\varphi) \geq \langle w, |\varphi|^p \rangle\) for \(\varphi \in C_c(X)\). Then,

\[
0 = \lim_{n \to \infty} h(e_n) \geq \lim_{n \to \infty} \langle w, |e_n|^p \rangle \geq \lim_{n \to \infty} w(o) e_n^p(o) m(o) = w(o) \alpha^p m(o).
\]

Since \(o \in X\) is arbitrary, \(m(o) > 0\) and \(\alpha > 0\), we get \(w = 0\) on \(X\).

Ad (ii) \(\iff\) (iii): This follows immediately from the definitions.

Ad (i) \(\iff\) (iv) \(\iff\) (v): This follows from the ground state representation, Theorem 3.1. Note that the existence of such a strictly positive \(p\)-harmonic function is ensured by Lemma 4.3.

Ad (ii) \(\implies\) (vi): This is Lemma 4.5.

Ad ((i) and (ii) and (vi)) \(\implies\) (vii): The preamble ensures the existence of a positive \(p\)-superharmonic function \(u\). By the the Harnack inequality, Lemma 4.4, the function \(u\) is a strictly positive \(p\)-superharmonic function in \(X\). By Lemma 4.3, the criticality of \(h\) in \(X\) implies that any strictly positive \(p\)-superharmonic in \(X\) is a strictly positive \(p\)-harmonic function in \(X\).

By (vi), any null-sequence converges to a constant multiple of \(u\). The existence of a null-sequence is ensured by (ii). This shows the first part.

Let \(p \geq 2\), and let \((e_n)\) be a null sequence such that \(e_n(o) = u(o)\) for some \(o \in X\), and \(e_n \to u\). Consider the sequence \((e_n \wedge u)\), where \(\wedge\) denotes the minimum. We show that it is a null-sequence. Indeed, since for all \(\alpha, \beta, \gamma \in \mathbb{R}\), we have

\[
|\alpha \wedge \gamma - \beta \wedge \gamma| \leq |\alpha - \beta|,
\]

we conclude,

\[
0 \leq h(e_n \wedge u) \leq h_u(u^{-1}(e_n \wedge u)) = h_u(u^{-1}e_n \wedge 1) \leq h_u(u^{-1}e_n) \approx h(e_n).
\]

Thus, \(h(e_n \wedge u) \to 0\), and \(e_n(o) = u(o) > 0\), i.e., \((e_n \wedge u)\) is a null sequence. Since \(e_n \to u\), we conclude that \((e_n \wedge u) \to u\).

Ad (vii) \(\implies\) (i): By Lemma 4.2, \(0 \leq h(e_n) \to 0\). Hence, \(h\) is critical.

Thus, we have completed the proof of the equivalences. The last statement follows immediately from (vi). This finishes the proof.

\(\square\)

Now, we show that \(h\) cannot be critical on any proper subset of \(X\).
**Proposition 4.6** Let $p > 1$, $V \subseteq X$, and $h$ be non-negative on $C_c(V)$. Assume that there is a function $u \in F(V)$ which is positive in $V \cup \partial V$, $p$-superharmonic in $V$ and there is a vertex $o \in \partial V$ such that $u(o) > 0$. Then, $h$ is subcritical in $V$.

Furthermore, there does not exists a null-sequence $(e_n)$ in $V$ with $e_n(x_o) = \alpha$ for $o \sim x_o \in V$ and some $\alpha > 0$, and we have $\text{cap}(x_o, V) > 0$.

**Proof** Without loss of generality, we can assume that $V \neq \emptyset$, and thus there is at least one non-empty connected component of $V$. By the Harnack inequality, Lemma 4.4, we have that $u$ is strictly positive in every connected component of $V$ which contains a vertex that is connected to $o$.

Assume that $h$ is critical in $V$. Note that then $u$ is $p$-harmonic in $V$: On the connected components where it is strictly positive it follows from Lemma 4.3, and on the other components $\bar{U}$ it has to vanish on $\bar{U} \cup \partial \bar{U}$ by Lemma 4.4. Furthermore, let us fix a connected component $U$ of $V$ with $o \in \partial U$. Take $x_o \in U$ with $x_o \sim o$, and set $w_n = 1_{x_o}/n$ for $n \in \mathbb{N}$. Then, by the definition of criticality in $V$ we have the existence of a sequence $(\varphi_n)$ in $C_c(V)$ such that

$$0 \leq h(\varphi_n) < |\varphi_n(x_o)|^p m(x_o)/n, \quad \varphi \in C_c(V).$$

(4.2)

By the reversed triangle inequality, we can assume without loss of generality that $\varphi_n \geq 0$ on $V$ for all $n \in \mathbb{N}$. Furthermore, we can normalise $\varphi_n$ such that $\varphi_n(x_o) = 1$ for all $n \in \mathbb{N}$. Then, $(\varphi_n)$ is a null-sequence in $V$.

Using the ground state representation, Theorem 3.1, we get that $h(u(\psi)) \approx h_u(\psi)$ for all $\psi \in C_c(V)$. Let us define $\psi_n \in C_c(V)$ for all $n \in \mathbb{N}$ via $\psi_n = \varphi_n//u$ on $V$ wherever $u$ is strictly positive and $\psi_n = 0$ otherwise. Then, $(\psi_n)$ is a null-sequence of $h_u$ in $V$.

Firstly, let $p \geq 2$. Since $(\psi_n)$ is a null-sequence of $h_u$ in $V$, we have

$$u(x)u(y) \langle \nabla_{x,y} \psi_n \rangle^2 \to 0, \quad x, y \in V \cup \partial V, x \sim y.$$  

In particular, since $u > 0$ on $U \cup \{o\}$, we have

$$\psi_n(y) = \langle \nabla_{o,y} \psi_n \rangle \to 0, \quad y \in U, y \sim o.$$  

But this is a contradiction, because $\psi_n(x_o) = 1//u(x_o) > 0$ for all $n \in \mathbb{N}$.

Secondly, let $1 < p < 2$. Since $(\psi_n)$ is a null-sequence of $h_u$ in $V$, we have for each $(x, y) \in (U \cup \{o\})^2$, $x \sim y$ either

1. $\langle \nabla_{x,y} \psi_n \rangle \to 0$, or
2. $\langle \nabla_{x,y} \psi_n \rangle \to \infty$, or
3. $(\psi_n(x) + \psi_n(y)) \langle \nabla_{x,y} u \rangle \to \infty$.

Since $\langle \nabla_{o,x_o} \psi_n \rangle = \psi_n(x_o) = 1//u(x_o) > 0$ for all $n \in \mathbb{N}$ and $o \sim x_o \in U$,

we see that neither (1) nor (2) can apply for the pair $(o, x_o)$. Because of

$$(\psi_n(o) + \psi_n(x_o)) \langle \nabla_{o,x_o} u \rangle = \frac{1}{u(x_o)} \langle \nabla_{o,x_o} u \rangle \in [0, \infty),$$  

also (3) cannot apply. Hence, we also have a contradiction in the case of $1 < p < 2$.

Thus, $(\varphi_n)$ cannot be a null-sequence of $h$ in $V$, and therefore the strict inequality in (4.2) does not hold, i.e., $h$ cannot be critical in $V$. \hfill \Box

**Remark 4.7** Proposition 4.6 has the following interpretation for $p = 2, m = \deg,$ and $c = 0$: Given any connected graph, the induced graph on any proper subset is then a graph with boundary and thus transient.
We end this subsection by giving a connection between the energy functional associated with the graph $b$, and the energy functional associated with the graph $b_u$, where $b_u(x, y) = b(x, y)u(x)u(y)^{p/2}$ for $0 \leq u \in F$.

If $h$ is critical in $X$ then the unique $p$-harmonic function $\psi$ such that $\psi(o) = 1$ is called (Agmon) ground state of $h$ normalised at $o$.

**Corollary 4.8** Let $p > 1$, and $0 \leq u \in F$.

(a) If $p > 2$ and $u$ is a ground state of $h$, then $1$ is a ground state of $h_{u,1}$.

(b) If $1 < p < 2$ and $h_{u,1}$ is critical in $X$, then $h(-) - \langle u^{1-p}H, |·|^{p} \rangle$ is critical in $X$.

**Proof** Ad (a): Recall that a ground state is $p$-harmonic, i.e., $uHu = 0$. Moreover, $h$ is critical. Then, by the ground state representation, Corollary 3.2, we have that $h_{u,1}$ is critical. Since $1$ is a $p$-harmonic function with respect to the Laplace operator associated with $h_{u,1}$, it is a ground state.

Ad (b): This is a direct consequence of the ground state representation. \( \Box \)

### 4.3 Liouville comparison principle

Here, we show a consequence of the characterisations of criticality and the ground state representation which is usually referred to as a Liouville comparison principle, confer [49, Sect. 11] and references therein for the linear case. For the counterpart in the continuum see [51, Theorem 8.1], or [53, Theorem 1.9].

**Proposition 4.9** (Liouville comparison principle) Let $p > 1$. Let $b$ and $\tilde{b}$ be two graphs on $X$, and $c, \tilde{c} \in C(X)$ be two potentials. Let denote $h$ and $\tilde{h}$ the energy functionals with corresponding Schrödinger operators $H := H_{b,c,m,x,p}$ and $\tilde{H} := H_{\tilde{b},\tilde{c},m,x,p}$, respectively. Assume that the following assumptions hold true:

(a) The energy functional $h$ is critical in $X$ with ground state $u$.

(b) There exists a positive $p$-superharmonic function with respect to $\tilde{H}$, and also a positive $p$-subharmonic function $\tilde{u}$ with respect to $\tilde{H}$.

(c) There exists a constant $\alpha > 0$, such that for all $x, y \in X$ we have

$$b^{2/p}(x, y)u(x)u(y) \geq \alpha \tilde{b}^{2/p}(x, y)\tilde{u}(x)\tilde{u}(y).$$

(d) There exists a constant $\beta > 0$ such that for all $x, y \in X$ we have for $p \geq 2$,

$$b^{1/p}(x, y) |\nabla_{x,y}u| \geq \beta \tilde{b}^{1/p}(x, y) |\nabla_{x,y}\tilde{u}|,$$

and the reversed inequality holds for $1 < p < 2$.

Then the energy functional $\tilde{h}$ is critical in $X$ with ground state $\tilde{u}$.

**Proof** By Theorem 4.1, there exists a null-sequence $(e_n)$ with respect to $h$ such that $e_n \to u$ pointwise as $n \to \infty$. Denote $\varphi_n = e_n/u$, $n \in \mathbb{N}$. From the ground state representation, Theorem 3.1, we get $h(e_n) \asymp h_u(\varphi_n)$ for all $n \in \mathbb{N}$. Hence, using (c) and (d), we infer

$$h_u(\varphi_n) \geq \gamma_1 h_{\tilde{u}}(\varphi_n)$$

for some constant $\gamma_1 > 0$. Now, (b) implies by Lemma 4.2 that $\tilde{h}$ is non-negative in $C_c(X)$. Using this fact, the calculation before, and the ground state representation, we get for some constants $\gamma_2, \gamma_3 > 0$ that

$$0 \leq \tilde{h}(\tilde{u}\varphi_n) \leq \gamma_2 \tilde{h}_{\tilde{u}}(\varphi_n) \leq \gamma_3 h(e_n) \to 0, \quad n \to \infty.$$ 

Thus, $(\tilde{u}\varphi_n)$ is a null-sequence for $\tilde{h}$ and by Theorem 4.1, $\tilde{h}$ is critical in $X$. Since $\varphi_n \to 1$, we get $\tilde{u}\varphi_n \to \tilde{u}$, and by Theorem 4.1, $\tilde{u}$ is the ground state. \( \Box \)
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Appendix A: Proof of Lemma 3.8

Before we can proof Lemma 3.8, we need the following quantification of the strict convexity of the mapping $x \mapsto |x|^p$, $p > 1$. In the following lemma, $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$.

Lemma A.1 (Lindqvist’s lemma, Lemma 4.2 in [40]) Let $a, b \in \mathbb{R}^n$. Then, for all $p \geq 2$ we have

$$|a|^p - |b|^p \geq p |b|^{p-2} \langle b, a - b \rangle_{\mathbb{R}^n} + c_p |a - b|^p,$$

where $c_p = 1/(2^{p-1} - 1) > 0$. If $1 < p < 2$, then

$$|a|^p - |b|^p \geq p |b|^{p-2} \langle b, a - b \rangle_{\mathbb{R}^n} + c_p \frac{|a - b|^2}{(|a| + |b|)^{2-p}},$$

where $c_p = 3p(p-1)/16 > 0$, and the fraction is interpreted to be zero if $a = b = 0$.

In the previous lemma, the constant $c_p$ does not seem to be optimal. However, this is not important for our further investigations.

Proof of Lemma 3.8 Ad (3.8): Recall that we have to show that for $p > 1$,

$$|a - t|^p - (1 - t)^{p-1}(|a|^p - t) \leq (a - t)^2 (|a - t| + 1 - t)^{p-2}, \quad a \in \mathbb{R}, 0 \leq t \leq 1.$$

The strategy of the proof is as follows: We start with some simple special cases for which the equivalence can be shown very easily. Thereafter, we do a substitution to bring the equivalence in a simpler form for the remaining cases. Then, we divide $\mathbb{R}$ into the three intervals $[1, +\infty)$, $(t, 1)$, and $(-\infty, t]$ for some $t \in [0, 1]$. In the two intervals $[1, +\infty)$ and $(t, 1)$, we then distinguish between proving lower bounds and upper bounds, as well as having $p > 2$ or $1 < p < 2$. In the remaining interval $(t, 1)$, we show that we can deduce the equivalence from the validity of the equivalence in $[1, +\infty)$.
1. The three cases \( t \in \{0, 1\}, a = t, \text{ and } p = 2 \): If \( p = 2 \), then it is obvious that we have equality for all \( a \in \mathbb{R} \) and \( t \in [0, 1] \).

An easy computation shows that we have indeed equality for \( t \in \{0, 1\} \).

If \( a = t \), we have to show that for all \( p > 1 \)

\[-(1-t)^{p-1}(t^p - t) \asymp t(1-t)^p.\]

Thus, let us consider the function

\[ q(t) := \frac{(1-t)^{p-1}(t - t^p)}{t(1-t)^p} = \frac{1-t^{p-1}}{1-t}. \]

If \( 1 < p < 2 \), then \( t^{p-1} \geq t \) for \( t \in (0, 1) \), and thus, \( q \) is decreasing. If \( p > 2 \), we have \( t^{-1} \leq t \) for \( t \in (0, 1) \), and \( q \) is increasing. Moreover, by L'Hôpital's rule \( q(1) = p - 1 \).

Hence, for \( p > 2 \) we have \( 1 = q(0) \leq q(t) \leq q(1) = p - 1 \) and for \( 1 < p < 2 \), we have \( p - 1 = q(1) \leq q(t) \leq q(0) = 1 \).

2. The remaining cases \( t \in (0, 1), a \neq t, \text{ and } p \neq 2 \): We do the following substitution:

Set \( \alpha := (a - t)/(1 - t) \), then we have to show that

\[ |\alpha|^p - \frac{|\alpha(1-t) + t|^p - t}{1-t} \asymp \frac{t(\alpha - 1)^2}{(|\alpha| + 1)^{2-p}}. \quad \text{(A.1)} \]

We will do this by considering the following three cases separately

- \( \alpha \geq 1 \),
- \( 1 > \alpha > 0 \), and
- \( \alpha < 0 \).

Furthermore, let

\[ f_\alpha(t) := \frac{|\alpha(1-t) + t|^p - t}{1-t} = \frac{|\alpha + t(1-\alpha)|^p - t}{1-t}. \]

Note that \( f_\alpha(0) = |\alpha|^p \).

2.1 The case \( \alpha \geq 1 \). The basic strategy is to use the Taylor-Maclaurin formula. Thus, let us calculate the first and the second derivatives with respect to \( t \). Note that for \( \alpha \geq 1 \), we have \( |\alpha + t(1-\alpha)| = \alpha + t(1-\alpha) \). Hence, we calculate

\[ f'_\alpha(t) = \frac{p(1-\alpha)(\alpha + t(1-\alpha))^{p-1} - 1}{1-t} + \frac{f_\alpha(t)}{1-t}, \]
and using \( \alpha + t(1 - \alpha) = \beta + 1 \), where \( \beta := (\alpha - 1)(1 - t) \geq 0 \), we get
\[
f''_\alpha(t) = \frac{p(p - 1)(1 - \alpha)^2(\alpha + t(1 - \alpha))^{p-2}}{1 - t} + \frac{p(1 - \alpha)(\alpha + t(1 - \alpha))^{p-1} - 1}{(1 - t)^2} \\
\quad + \frac{f'_\alpha(t)}{1 - t} + \frac{f_\alpha(t)}{(1 - t)^2} \\
= \frac{(\alpha + t(1 - \alpha))^{p-2}}{(1 - t)^3} \left( p(p - 1)(\alpha - 1)^2(1 - t)^2 \\
- 2p(\alpha - 1)(1 - t)(\alpha + t(1 - \alpha)) + 2(\alpha + t(1 - \alpha))^2 \right) - \frac{2}{(1 - t)^3} \\
= \frac{(\beta + 1)^{p-2}}{(1 - t)^3} \left( p(p - 1)\beta^2 - 2p\beta(\beta + 1) + 2(\beta + 1)^2 \right) - \frac{2}{(1 - t)^3} \\
= \frac{g(\beta) - 2}{(1 - t)^3},
\]
(A.2)
where
\[
g(\beta) := \left( (p - 1)(p - 2)\beta^2 + 2(2 - p)\beta + 2 \right)(\beta + 1)^{p-2}, \quad \beta \geq 0.
\]
Let us analyse \( g(\beta) \) for \( \beta \geq 0 \). Then, \( g'(\beta) = p(p - 1)(p - 2)(\beta + 1)^{p-3} \beta^2 \), which is positive for \( p > 2 \) and negative for \( 1 < p < 2 \). Hence, \( g(0) = 2 \) is a minimum for \( p > 2 \) and a maximum for \( 1 < p < 2 \). This implies that for all \( t \in (0, 1) \)
\[
f''_\alpha(t) \begin{cases} 
< 0 & \text{if } 1 < p < 2, \\
\geq 0 & \text{if } p > 2.
\end{cases}
\]
Now, we apply the Taylor-Maclaurin formula
\[
f_\alpha(t) = f_\alpha(0) + tf'_\alpha(0) + \int_0^t (t-s)f''_\alpha(s)\,ds.
\]
Since \( f_\alpha(0) = \alpha^p \), we have
\[
|\alpha|^p - \frac{|\alpha(1-t) + t|^p - t}{1 - t} = f_\alpha(0) - f_\alpha(t) \\
= -tf'_\alpha(0) - \int_0^t (t-s)f''_\alpha(s)\,ds \\
= t((p-1)\alpha^p - p\alpha^{p-1} + 1) - \int_0^t (t-s)f''_\alpha(s)\,ds.
\]
(A.3)
This term will be analysed in the following for upper and lower bounds and different values of \( p \).

2.1.1 Lower bound for \( 1 < p < 2 \) and \( \alpha \geq 1 \). Then \( f''_\alpha \leq 0 \) on \((0, 1)\). Thus we conclude from (A.3),
\[
|\alpha|^p - \frac{|\alpha(1-t) + t|^p - t}{1 - t} \geq t((p-1)\alpha^p - p\alpha^{p-1} + 1).
\]
Using Lindqvist’s lemma, Lemma A.1, with \( b = \alpha \) and \( a = 1 \), we see
\[
t((p-1)\alpha^p - p\alpha^{p-1} + 1) = t(1 - \alpha^p - p\alpha^{p-2} \alpha(1 - \alpha)) \geq C_p \frac{t(\alpha - 1)^2}{(\alpha + 1)^{2-p}}.
\]
This is the desired lower bound in (A.1) for \( 1 < p < 2 \) and \( \alpha \geq 1 \).

### 2.1.2 Upper bound for \( p > 2 \) and \( \alpha \geq 1 \)

Then \( f''_\alpha \geq 0 \) on \((0, 1)\). Thus we conclude from (A.3),
\[
|\alpha|^p - \frac{|\alpha(1-t) + t|^p - t}{1-t} \leq t((p-1)\alpha^p - p\alpha^{p-1} + 1).
\]
Hence, it remains to show that there exists \( C_p > 0 \) such that
\[
((p-1)\alpha^p - p\alpha^{p-1} + 1) \leq C_p(\alpha - 1)^2(\alpha + 1)^{p-2}.
\]
For any positive constant \( C_p \) we have using \((1 + \alpha^{-1})^{p-2} \geq 1\),
\[
j(\alpha) := \alpha^{p-2} \left( (p-1)\alpha^2 + p\alpha + \alpha^{2-p} \right) - C_p(\alpha - 1)^2(1 + \alpha^{-1})^{p-2}
\]
\[
\leq \alpha^{p-2} \left( (p-1)\alpha^2 - p\alpha + \alpha^{2-p} \right) - C_p(\alpha - 1)^2
\]
\[
= \alpha^{p-2} \left( (p-1 - C_p)\alpha^2 + (2C_p - p)\alpha + \alpha^{2-p} - C_p \right).
\]
Let \( g(\alpha) := (p-1 - C_p)\alpha^2 + (2C_p - p)\alpha + \alpha^{2-p} - C_p \) for \( \alpha > 0 \), then
\[
g'(\alpha) = 2(p-1 - C_p)\alpha + (2C_p - p) - (p-2)\alpha^{1-p}
\]
has a root at \( \alpha = 1 \). If we can show that \( g \) is concave on \([1, \infty)\), then \( g(1) = 0 \) is a maximum. Since
\[
g''(\alpha) = 2(p-1 - C_p) + (p-2)(p-1)\alpha^{-p} \leq 2(p-1 - C_p) + (p-2)(p-1),
\]
g is concave on \([1, \infty)\) for all \( C_p \geq p(p-1)/2 \), we found a possible constant such that \( j(\alpha) \leq 0 \). In other words, we have the desired upper bound for \( p > 2 \). However, it is obvious that the constant can be improved.

### 2.1.3 Upper bound for \( 1 < p < 2 \) and \( \alpha \geq 1 \)

For \( 1 \leq p \leq 2 \), the function \(|\cdot|^{p-1}\) is concave on \((0, \infty)\), thus
\[
|\alpha(1-t) + t|^{p-1} \geq (1-t)\alpha^{p-1} + t.
\]
Using this estimate in the left-hand side of (A.1), we get
\[
|\alpha|^p - \frac{|\alpha(1-t) + t|^p - t}{1-t} \leq t(\alpha^{p-1} - 1)(\alpha - 1) \tag{A.4}
\]
Define for \( \alpha \geq 1 \),
\[
g(\alpha) := (\alpha + 1)^{2-p}(\alpha^{p-1} - 1) - \alpha + 1,
\]
then
\[
g(\alpha) = (\alpha^{p-1} - 1)\alpha^{2-p}(1 + \alpha^{-1})^{2-p} - \alpha + 1 = (\alpha - \alpha^{2-p}) \sum_{k=0}^{\infty} \binom{2-p}{k} \alpha^{-k} - \alpha + 1.
\]
Since for all $k \in 2\mathbb{N}$, $1 \leq p \leq 2$ and $\alpha \geq 1$, we have
\[
\left(\frac{2-p}{k}\right)^{\alpha-k} + \left(\frac{2-p}{k+1}\right)^{\alpha-k-1} \leq 0,
\]
we get for all $1 \leq p \leq 2$ and $\alpha \geq 1$,
\[
g(\alpha) \leq (\alpha - \alpha^2 - p)(1 + \frac{2-p}{\alpha}) - \alpha + 1 = (2-p) + 1 - \alpha^2 - p - (2-p)\alpha^{1-p} =: l(\alpha).
\]
Since $l'(\alpha) = (2-p)((p-1) - \alpha)\alpha^{-p} \leq 0$ for $\alpha \geq 1$ and $1 \leq p \leq 2$, we get
\[
g(\alpha) \leq l(\alpha) \leq l(1) = 0.
\]
Thus, using that $g \leq 0$ on $[1, \infty]$ results in (A.4) in
\[
(\alpha^{p-1} - 1)(\alpha - 1) \leq \frac{(\alpha - 1)^2}{(\alpha + 1)^{2-p}}.
\]
This results in the right-hand side of (A.1) with constant 1.

2.1.4 Lower bound for $p > 2$ and $\alpha \geq 1$. For $p \geq 2$, the function $|\cdot|^{p-1}$ is convex on $(0, \infty)$, thus
\[
|\alpha(1-t) + t|^{p-1} \leq (1-t)\alpha^{p-1} + t.
\]
Using this estimate in the left-hand side of (A.1), we get
\[
|\alpha|^p - \frac{|\alpha(1-t) + t|^{p-1} - t}{1-t} \geq t(\alpha^{p-1} - 1)(\alpha - 1) \tag{A.5}
\]
Define for $\alpha \geq 1$, and some constant $C_p > 0$,
\[
g(\alpha) := \alpha^{p-1} - 1 - C_p(\alpha - 1)(\alpha + 1)^{p-2},
\]
then
\[
g'(\alpha) = \alpha^{p-2}\left(p - 1 - C_p\left(\left(1 + \frac{1}{\alpha}\right)^{p-2} + (p-2)\left(1 + \frac{1}{\alpha}\right)^{p-3}\left(1 - \frac{1}{\alpha}\right)\right)\right).
\]
If $p \geq 3$, then
\[
\ldots \geq \alpha^{p-2}(p-1-C_p(2^{p-2}+(p-2)2^{p-3})).
\]
Choosing $C_p = 2^{3-p}(p-1)/p$, we get $g' \geq 0$ on $[1, \infty)$. In particular,
\[
g(\alpha) \geq g(1) = 0.
\]
Thus, for $p \geq 3$,
\[
(\alpha^{p-1} - 1)(\alpha - 1) \geq C_p(\alpha - 1)^2(\alpha + 1)^{p-2}. \tag{A.6}
\]
If $2 \leq p \leq 3$, then
\[
g'(\alpha) \geq \alpha^{p-2}(p - 1 - C_p(2^{p-2} + p - 2)).
\]
Choosing $C_p = (p-1)/(2^{p-2} + p - 2)$, we get $g' \geq 0$ on $[1, \infty)$. Thus, for $2 \leq p \leq 3$,
\[
(\alpha^{p-1} - 1)(\alpha - 1) \geq C_p(\alpha - 1)^2(\alpha + 1)^{p-2}. \tag{A.7}
\]
Applying (A.6) and (A.7) to (A.5), results in the right-hand side of (A.1).
Moreover, this was the last puzzle stone to show (A.1) for $\alpha \geq 1$ and all $1 < p < \infty$.

2.2 The case $0 < \alpha < 1$. We have shown that (A.1) holds for all $\alpha > 1$ and $t \in (0, 1)$. Then it holds in particular for $s = 1 - t$, i.e.,

$$|\alpha|^p - \frac{|\alpha s + 1 - s|^p - (1 - s)}{s} \leq \frac{(1 - s)(\alpha - 1)^2}{(|\alpha| + 1)^{2-p}}.$$ 

Now, for any $\alpha > 1$ let $\beta := 1/\alpha \in (0, 1)$. Then, we get by multiplying both sides with $\beta^p s/(1 - s)$,

$$|\beta|^p - \frac{\beta(1 - s) + s|^p - s}{1 - s} \leq \frac{s(\beta - 1)^2}{(|\beta| + 1)^{2-p}},$$

which is the desired equivalence.

2.3 The case $\alpha < 0$. Set $\beta := -\alpha$. Then, substituting into (A.1), we have to show that for all $\beta > 0$ and $t \in (0, 1)$,

$$|\beta|^p - \frac{|\beta(1 - t) - t|^p - t}{1 - t} \leq \frac{t(\beta + 1)^2}{(|\beta| + 1)^{2-p}} = t(\beta + 1)^p.$$  

We have

$$|\beta|^p - \frac{|\beta(1 - t) - t|^p - t}{1 - t} = |\beta|^p - \frac{|\beta(1 - t) + t|^p - t}{1 - t} + g_t(\beta),$$

where

$$g_t(\beta) := \frac{1}{1 - t}((\beta(1 - t) + t)^p - |\beta(1 - t) - t|^p), \quad \beta > 0, \ t \in (0, 1).$$

Before we continue with the estimates, let us note that

$$g_t \geq 0 \quad \text{and} \quad g_t' \geq 0.$$ 

The first inequality can be seen as follows: let $\gamma > 0$. Firstly assume that $\gamma > t$. Then,

$$(\gamma + t)^p - (\gamma - t)^p = 2\gamma^p \sum_{k \in 2N - 1} \binom{p}{k} \left(\frac{t}{\gamma}\right)^k > 0.$$ 

Secondly, if $\gamma \leq t$, then a similar calculation can be done to get the desired inequality (factor $t$ out of the sum and use the binomial theorem).

Note that for all $p \geq 1$,

$$g_t'(\beta) = p(|\beta(1 - t) + t|^{p-1} - |\beta(1 - t) - t|^{p-1} \text{sgn}(\beta(1 - t) - t)) \geq 0.$$ 

Now we continue with showing (A.8): By the first parts of the proof, i.e., the proof of (A.1), we have that for all $\beta > 0$,

$$|\beta|^p - \frac{|\beta(1 - t) + t|^p - t}{1 - t} \leq \frac{t(\beta - 1)^2}{(|\beta| + 1)^{2-p}}.$$  

(A.9)

The strategy for the upper bound will be as follows: Clearly, $(\beta - 1)^2 \leq (\beta + 1)^2$ for all $\beta > 0$. If we apply this estimate to (A.9), we are left to show that also

$$g_t(\beta) \leq C_p t(\beta + 1)^p,$$

for some positive constant $C_p$ in order to show the upper bound in (A.8).
Let us turn to the strategy for the lower bound: It is obvious, that there does not exists a positive constant $C_p$ such that $(β - 1)^2 ≥ C_p(β + 1)^2$ since the left-hand side has a root at $β = 1$. However, fix $0 < ε < 1$, then we clearly have for all $β ∈ (0, ∞) \setminus (1 - ε, 1 + ε)$ that $(β - 1)^2 ≥ C_p ε(β + 1)^2$ for some constant $C_p ε > 0$. Since $g ≥ 0$, we have the desired lower bound of (A.8) using (A.9) in $(0, ∞) \setminus (1 - ε, 1 + ε)$.

For the lower bound, we are left to discuss the compact interval $[1 - ε, 1 + ε]$. On this interval, we clearly have $(β + 1)^p ≍ 1$. The equivalence (A.9) shows in particular that the corresponding left-hand side is positive. Thus, we are left to show that there exists $C_p,ε > 0$ such that

$$g_t ≥ C_p,ε t$$ on $[1 - ε, 1 + ε]$.

2.3.1 Lower bound for $1 < p < 2$ and $β = −α ≥ 0$. By the discussion before we only have to show that $g_t ≥ C_p,ε t$ on $[1 - ε, 1 + ε]$. Since $g'_t ≥ 0$, we have for all $β ∈ [1 - ε, 1 + ε],$

$$g_t(β) ≥ g_t(1 - ε) = \frac{1}{1 - t} \left( (1 - ε)(1 - t) t - |(1 - ε)(1 - t) - t|^p \right).$$

Using Lindqvist’s lemma, Lemma A.1, we get with $a = (1 - ε)(1 - t) + t$ and $b = |(1 - ε)(1 - t) - t|$ that

$$|a|^p - |b|^p ≥ p |(1 - ε)(1 - t) - t|^{p-1} \left( (1 - ε)(1 - t) t - |(1 - ε)(1 - t) - t| \right) + C_p \frac{ \left( (1 - ε)(1 - t) t - |(1 - ε)(1 - t) - t| \right)^2 }{ \left( (1 - ε)(1 - t) + t + |(1 - ε)(1 - t) - t| \right)^{2-p} }.$$ (A.10)

If $(1 - ε)(1 - t) - t ≥ 0$, i.e., $t ∈ (0, (1 - ε)/(2 - ε))$, the latter reduces to

$$\ldots = p \left( (1 - ε)(1 - t) - t \right)^{p-1} (2t) + C_p \frac{(2t)^2}{(2(1 - ε)(1 - t))^{2-p}} = t \left( 2p \left( (1 - ε)(1 - t) - t \right)^{p-1} + 4C_p \frac{t}{(2(1 - ε)(1 - t))^{2-p}} \right).$$

Using this, we get

$$g_t(β) ≥ g_t(1 - ε) ≥ t \left( 2p \frac{ \left( (1 - ε)(1 - t) - t \right)^{p-1} }{1 - t} + \frac{4C_p}{(2(1 - ε))^{2-p}} \cdot \frac{t}{(1 - t)^{3-p}} \right).$$

Since $t ↦ \left( (1 - ε)(1 - t) - t \right)^{p-1}/(1 - t)$ is continuous on $[0, (1 - ε)/(2 - ε)]$, strictly positive on $[0, (1 - ε)/(2 - ε)]$ and has a root at $t = (1 - ε)/(2 - ε)$, and $t ↦ t/(1 - t)^{3-p}$ is continuous and strictly positive on $(0, 1)$, has a root at $t = 0$, we conclude that there is a positive constant which bounds the sum from below on $[0, (1 - ε)/(2 - ε)] ⊂ [0, 1]$.

If $(1 - ε)(1 - t) - t < 0$, i.e., $t ∈ ((1 - ε)/(2 - ε), 1)$, then (A.10) reduces instead to

$$\ldots = p \left( -(1 - ε)(1 - t) + t \right)^{p-1} (2(1 - ε)(1 - t)) + C_p \frac{(2(1 - ε)(1 - t))^2}{(2t)^{2-p}}.$$ \hfill (A.10)

Using this, we get

$$g_t(β) ≥ g_t(1 - ε) ≥ t \left( 2p(1 - ε) \frac{-(1 - ε)(1 - t) + t}{t}^{p-1} + 2pC_p(1 - ε)^2 \frac{1 - t}{t^{3-p}} \right).$$
Since $t \mapsto (-(1-\varepsilon)(1-t)+t)^{p-1}/t$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon), 1]$ and vanishes at $t = (1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto (1-t)/t^{3-p}$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon), 1)$ (and is only zero at $t = 1$), we conclude that there is a positive constant which bounds the sum from below.

This shows the desired lower bound for $1 < p < 2$ and $\beta \geq 0$.

### 2.3.2 Lower bound for $p > 2$ and $\beta = -\alpha \geq 0$.

As in the case for $p < 2$, it suffices to show that $g_t \geq t \cdot C_{p, e} > 0$ on $[1-\varepsilon, 1+\varepsilon]$. Since $g_t' \geq 0$, we have for all $\beta \in [1-\varepsilon, 1+\varepsilon]$

$$g_t(\beta) \geq g_t(1-\varepsilon) = \frac{1}{1-\varepsilon} \left( ((1-\varepsilon)(1-t)+t)^{p-1} - |(1-\varepsilon)(1-t)-t|^{p} \right).$$

Using Lindqvist’s lemma, Lemma A.1, we get with $a = (1-\varepsilon)(1-t)+t$ and $b = |(1-\varepsilon)(1-t)-t|$ that

$$|a|^{p} - |b|^{p} \geq p \left( |(1-\varepsilon)(1-t)-t|^{p-1} \cdot (1-\varepsilon)(1-t)+t - |(1-\varepsilon)(1-t)-t| \right) + C_{p} \left( |(1-\varepsilon)(1-t)-t| - (1-\varepsilon)(1-t) \right)^{p}. \tag{A.11}$$

If $(1-\varepsilon)(1-t)-t \geq 0$, i.e., $t \in (0, (1-\varepsilon)/(2-\varepsilon))$, the latter reduces to

$$\ldots = 2t p ((1-\varepsilon)(1-t)-t)^{p-1} + C_{p}(2t)^{p}.$$ 

Using this, we get

$$g_t(\beta) \geq g_t(1-\varepsilon) \geq t \left( 2p \frac{(1-\varepsilon)(1-t)-t}{1-t}^{p-1} + 2p C_{p} \frac{t^{p-1}}{1-t} \right).$$

Since $t \mapsto ((1-\varepsilon)(1-t)-t)^{p-1}/(1-t)$ is continuous on $[0, (1-\varepsilon)/(2-\varepsilon)]$, strictly positive on $[0, (1-\varepsilon)/(2-\varepsilon)]$ and vanishes at $t = (1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto t^{p-1}/(1-t)$ is continuous and strictly positive on $(0, 1)$, and vanishes at $t = 0$, we conclude that there is a positive constant which bounds the sum from below on $[0, (1-\varepsilon)/(2-\varepsilon)]$.

If $(1-\varepsilon)(1-t)-t < 0$, i.e., $t \in ((1-\varepsilon)/(2-\varepsilon), 1)$, then (A.11) reduces instead to

$$\ldots = p ((1-\varepsilon)(1-t)+t)^{p-1} (2(1-\varepsilon)(1-t)) + 2p C_{p} (1-\varepsilon)^{p} (1-t)^{p}.$$ 

Using this, we get

$$g_t(\beta) \geq g_t(1-\varepsilon) \geq t \left( 2p (1-\varepsilon) \frac{-(1-\varepsilon)(1-t)+t}{t}^{p-1} + 2p C_{p} (1-\varepsilon)^{p} \frac{(1-t)^{p-1}}{t} \right).$$

Since $t \mapsto (-(1-\varepsilon)(1-t)+t)^{p-1}/t$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon), 1]$ and vanishes only at $t = (1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto (1-t)^{p-1}/t$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon), 1)$ and vanishes only at $t = 1$, we conclude that there is a positive constant which bounds the sum from below.

This shows the desired lower bound for $p > 2$ and $\beta \geq 0$, and we are left to show the upper bounds.

### 2.3.3 Upper bound for $1 < p < 2$ and $p > 2$, and $\beta = -\alpha \geq 0$.

It remains to show that $g_t(\beta) \leq C_{p,t}(\beta + 1)^{p}$ for all $\beta \geq 0$.

Recall that by the convexity of $|\cdot|^{p}$, we have

$$|a|^{p} - |b|^{p} \leq p |a|^{p-2} a(a-b), \quad a, b \in \mathbb{R}.$$
Let $a = \beta(1 - t) + t$ and $b = |\beta(1 - t) - t|$, then we get by the convexity that

$$g_t(\beta) \leq p(\beta(1 - t) + t)^{P-1}(\beta(1 - t) + t - |\beta(1 - t) - t|).$$

Since $\beta(1 - t) + t \leq \beta + 1$ and $(\beta + 1)^{P-1} \leq (\beta + 1)^p$ for all $\beta \geq 0$, $1 < p < \infty$ and $t \in [0, 1]$, we get

$$\ldots \leq p(\beta + 1)^p(\beta(1 - t) + t - |\beta(1 - t) - t|).$$

If $\beta(1 - t) \geq t$, then $\beta(1 - t) + t - |\beta(1 - t) - t| = 2t$. If $\beta(1 - t) \leq t$, then $\beta(1 - t) + t - |\beta(1 - t) - t| = 2\beta(1 - t) \leq 2t$. Thus, we get altogether,

$$g_t(\beta) \leq 2pt(\beta + 1)^p.$$

This finishes the proof of (A.8) and moreover, it also finishes the proof of (3.8).

Ad (3.9): The assertion follows by a simple case analysis. Here are the details: Let

$$f_{t,C}(a) := C \left( t^{1/2} |a - 1| + (1 - t) \left( C \frac{|a| + 1}{2} - 1 \right) \right) - |a - t|, \quad a \in \mathbb{R}.$$

We have to show that $f_{t,C} \geq 0$ for all $t \in [0, 1]$ and $C \geq 2$, $f_{t,C} \leq 0$ for all $t \in [0, 1]$ and $0 \leq C \leq 1/2$, and for every $C \in (1/2, 2)$, the function $f_{t,C}(\cdot)$ changes sign.

1. The cases $t \in \{0, 1\}$ and $a = t$: For $t = 0$, we have

$$f_{0,C}(a) = \frac{C - 2}{2} (|a| + 1),$$

which is non-negative for $C \geq 2$ and strictly negative for $C < 2$. If $t = 1$, then

$$f_{1,C}(a) = (C - 1) |a - 1|,$$

which is non-negative for $C \geq 1$ and strictly negative for $C < 1$. If $a = t$, then

$$f_{t,C}(t) = (1 - t) \left( C \frac{2t^{1/2} + t + 1}{2} - 1 \right),$$

which is non-negative for $C \geq 2$, non-positive for $C \leq 1/2$ and changes sign from negative to positive as $t$ increases in $1/2 < C < 2$. Hence, it is easy to see that $f_{t,C}(\cdot)$ changes sign for any $1/2 < C < 2$ and an appropriate choice of $t$ by evaluating $f_{t,C}$ at $0$, $t$ and $1$.

2. The remaining cases $t \in (0, 1)$, $a \neq t$: Note that for $a \notin \{0, t, 1\}$, we can calculate the derivative, i.e.,

$$f'_{t,C}(a) = C t^{1/2} \text{sgn}(a - 1) + \frac{C}{2} (1 - t) \text{sgn}(a) - \text{sgn}(a - t).$$

We have for all $t \in (0, 1)$ and $C \geq 2$,

$$f'_{t,C}(a) = \begin{cases} 
- C t^{1/2} + \frac{C}{2} t + \frac{2-C}{2} & \leq 0, \\
- C t^{1/2} - \frac{C}{2} t + \frac{2+C}{2} & \geq 0, \\
- C t^{1/2} - \frac{C}{2} t + \frac{C-2}{2} & \leq 0, \\
C t^{1/2} - \frac{C}{2} t + \frac{C-2}{2} & \geq 0.
\end{cases}$$

If $0 \leq C \leq 1/2$, then $f'_{t,C}$ has opposite sign on every subinterval.

Hence, $f_{t,C}$ has two extrema, one at $a = 0$ and one at $a = t$. If $C \geq 2$, the extrema are minima, and if $0 \leq C \leq 1/2$, the extrema are maxima. By the computations in the first case and since

$$f_{t,C}(0) = C t^{1/2} - \frac{C}{2} t + \frac{C - 2}{2},$$
which is non-negative for $C \geq 2$ and non-positive for $0 \leq C \leq 1/2$, it follows that $f_{r,C}$ is non-negative if $C \geq 2$ and non-positive if $0 \leq C \leq 1/2$ for all $t \in [0, 1]$ and we have shown that the right-hand side in (3.9) is an upper bound for every $C \geq 2$, and lower bound if $0 \leq C \leq 1/2$.

Ad (3.10) and (3.11) We will show these inequalities similarly as we showed (3.8). Recall that we have to show that

$$t |a - 1|^2 \leq t^{p/2} |a - 1|^p (|a - t| + 1 - t)^{2-p}, \quad 1 < p \leq 2,$$

and

$$t |a - 1|^2 (|a - t| + 1 - t)^{p-2} \geq t^{p/2} |a - 1|^p, \quad p \geq 2.$$  

Note that the inequalities basically come from the fact that for $t \in [0, 1]$, we have $t^{p/2} \geq t$ for $1 < p \leq 2$, whereas $t^{p/2} \leq t$ for $p \geq 2$. Here are the details:

1. The three cases $t \in \{0, 1\}, a = t$, and $p = 2$: If $p = 2$, then it is obvious that we have equality for all $a \in \mathbb{R}$ and $t \in [0, 1]$.

An easy computation shows that we indeed have equality for $t \in \{0, 1\}$.

If $a = t$, then note that $t \in [0, 1]$ implies $t^{p/2} \geq t$ for $1 < p \leq 2$, and $t^{p/2} \leq t$ for $p \geq 2$. This immediately yields the desired inequalities.

2. The remaining cases $t \in (0, 1), a \neq t$, and $p \neq 2$: We consider the cases $a > t$ and $a < t$ separately.

2.1. The case $a > t$: Here, we have to show that

$$t |a - 1|^2 \leq t^{p/2} |a - 1|^p (a + 1 - 2t)^{2-p}, \quad 1 < p \leq 2$$  

as well as

$$t |a - 1|^2 (a + 1 - 2t)^{p-2} \geq t^{p/2} |a - 1|^p, \quad p \geq 2.$$  

Firstly, consider the case $1 < p < 2$. We clearly have $a + 1 - 2t \geq a - 1$. Thus, $(a + 1 - 2t)^{2-p} \geq (a - 1)^{-p}$. Moreover, $t \leq t^{p/2}$ for $t \in (0, 1)$. This shows the inequality (A.12).

Secondly, consider the case $p > 2$. Because $(a + 1 - 2t)^{p-2} \geq (a - 1)^{p-2}$ as well as $t \geq t^{p/2}$ for $t \in (0, 1)$, we get the desired inequality (A.13).

2.2. The case $a < t$: Note that $a < t < 1$. Thus, we have to show that

$$t(1-a)^2 \leq t^{p/2}(1-a)^p(1-a)^{2-p}, \quad 1 < p \leq 2$$

as well as

$$t(1-a)^2(1-a)^{p-2} \geq t^{p/2}(1-a)^p, \quad p \geq 2.$$  

Since $t \leq t^{p/2}$ for $1 < p < 2$ and $t \geq t^{p/2}$ for $p > 2$, we get the desired result.

Ad (3.12): Recall that there we assume that $p \geq 0$. The desired inequality is clearly fulfilled if $\alpha = \beta = 0$. Thus, assume that both do not vanish at the same time. Setting $t = \alpha/(\alpha + \beta)$, then (3.12) is equivalent to

$$f(t) := t^p + (1-t)^p \leq 1, \quad t \in [0, 1].$$

If $0 \leq p < 1$, then $f$ has a minimum at 0 and 1, and a maximum at 1/2. If $p \geq 1$, then $f$ has a maximum at 0 and 1, and a minimum at 1/2. Since $f(0) = f(1) = 1$ and $f(1/2) = 2^{1-p}$, we finished the proof. \hfill $\square$
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