Analytical solution of the Navier-Stokes equations reduced to the third-order equation for the problem of fluid motion in a round pipe

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Abstract. The paper defines a rheological law that takes into account the molecular and molar transfer of fluid particles between the layers of the flow; the equation of fluid motion, taking into account two mechanisms of molecular and molar exchange of momentum in the flow; the form of the obtained new equations in the form of an equation of the boundary layer, neglecting the terms whose order is much lower than the order held in the equations; statement of the problem of stationary fluid flow in cylindrical coordinates with the corresponding boundary conditions using the transition to new dimensionless variables; a technique for solving the Navier-Stokes equation reduced to a third-order differential equation for studying the motion of a fluid in a round pipe; analytical solution of the formulated problem; the role of the newly introduced molar transfer coefficient in describing the flow pattern. An analytical solution of the problem of fluid motion in a cylindrical pipe is obtained, taking into account these two transfer mechanisms where third-order terms are formed in the Navier-Stokes equations. For small Reynolds numbers, the influence of the newly introduced term on the flow pattern is a shortening of the length of the initial segment of motion. A decrease in the value of the new number is associated with an increase in this region.

1. Introduction

The fluid motion is mainly studied taking into account the momentum exchange of molecules between the layers of the flow. The Navier-Stokes equation used in this case is based on the rheological law on the proportionality of stress to fluid velocity. The solution to this equation is considered in [1-14]. This equation is a second-order differential equation and is obtained on the basis of Newton’s rheological law, which takes into account the molecular exchange between the layers of a flow. The problem of fluid flow in a semi-infinite pipe according to the rheological law, where both the molecular and molar momentum exchange in the flow are taken into account, is considered below. It is known that the momentum exchange at the molecular level in Newton’s law is taken into account by the ratio of the direct proportionality of the stress to the derivative of the velocity normal. At the molar level, for the problem under consideration, the stress is taken directly proportional to the derivative of normal acceleration. In this case, third-order terms are formed in the Navier-Stokes equations.

Object:
• derivation of basic equations of fluid motion according to the rheological law of the joint consideration of the molecular (single) and molar (group) transfer of particles between the layers in a flow,
• formulation of the problem of fluid flow in a round pipe for a given rheological law,
• obtaining an analytical solution to the formulated problem,
• finding out the role of the newly introduced molar transfer coefficient in the dynamics of the flow process.

2. Research methods
The research methods are based on Newton’s rheological law, where during molecular transfer of particles between the layers in the flow, the stress is taken directly proportional to the derivative of the velocity normal. In contrast, the concept of molar transfer of particles in a flow is used in the work, where the stress is directly proportional to the derivative of normal acceleration of the fluid. The process of fluid flow is considered when the molecular and molar mechanisms of particle transfer in a flow are taken into account.

3. Research results
The results of computational experiments on the obtained analytical solution are presented in the form of graphs.

Figure 1. Velocity profiles at Re=500.

Figure 2. Change in pressure along the length of the pipeline.
3.1. Mathematical description of the problem solution

The Navier-Stokes equation is generalized, taking into account two mechanisms of molecular and molar momentum exchange in a flow. This equation takes the form of a third order differential equation. An analytical solution of this equation is obtained for the problem of the stationary flow of a compressible viscous fluid in a semi-infinite round pipe.

Rheological equations relate the components of the tensors of stresses, strains and their derivatives. These equations can be the same for various motions of the medium under consideration [15] or depend on the nature of its various possible motions, in particular, on the design of apparatuses [16] in which motions occur from the flow history effects or under the action of certain action fields [17].

Such problems are solved by numerical iterative and variational finite element methods [13], Taylor methods without a grid [12], or a family of convergent numerical schemes [14], the Fourier analytic method [18], or involving the positions of operational calculi [19].

Under the above conditions, the liquid particles, along with the main movement in a certain preferred direction, move from layer to layer. Conditions are created for the accumulation of inhomogeneities, the alignment of which does not occur during the individual movement of individual molecules, but entire groups of molecules [18, 19]. In the case of group transfer of the momentum of molecules in a fluid flow, both the laws of momentum transfer and their mechanism, as well as the velocity distribution, differ from the case of viscous fluid flow, and the stress in this case is considered proportional to the derivative of acceleration.

When considering the exchange of momentum between the layers of the flow of transfer mechanisms of individual molecules and their groups, the stress is taken in the form

$$\tau = \mu \frac{\partial u}{\partial n} + m_i \frac{\partial w}{\partial n},$$

or in component form:

$$\tau = \begin{cases} \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + m_i \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) & \text{when } j \neq i, \\
-p + 2\mu \frac{\partial v_i}{\partial x_i} + m_i \frac{\partial w_i}{\partial x_i} & \text{when } j = i \quad (i, j = 1, 2, 3) \end{cases}$$

(1)

Here $u$ – flow velocity; $\mu$ – dynamic viscosity coefficient of the fluid; $m_i$ – molar transfer coefficient; $w$ – acceleration of the flow, which has the following form in the two-dimensional case:
To compose the equations of motion of an incompressible fluid, the general equation of dynamics in stresses is used [19].

\[
\begin{align*}
\rho \frac{dv_1}{dt} &= \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}, \\
\rho \frac{dv_2}{dt} &= \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3}, \\
\rho \frac{dv_3}{dt} &= \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3},
\end{align*}
\]

Substituting (1) in (2), we neglect the terms whose order is much lower than the order of the terms kept in the equations.

Let an incompressible fluid flow in a round pipe of radius \( R \) unlimited in one direction, and the fluid has a constant velocity \( u_0 \) in the inlet section. Let’s combine the beginning of the axes of the cylindrical coordinates with the center of the inlet section and direct the axis \( Ox_1 \) along the axis of the pipe towards the flow.

According to the estimates, the system of equations of motion of an incompressible fluid for this model, taking into account the continuity equation in cylindrical coordinate systems, has the form:

\[
\begin{align*}
\rho v_1 \frac{\partial v_1}{\partial x_1} + \rho v_2 \frac{\partial v_2}{\partial x_2} &= -\frac{dp}{dx_1} + \mu \frac{\partial^2 v_1}{\partial x_2^2} + m_l \frac{1}{x_2} \left( \frac{\partial v_1}{\partial x_2^2} \frac{\partial v_1}{\partial x_1} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial x_1^2} \right) + \\
&+ v_1 \left( \frac{\partial v_1}{\partial x_2} + \frac{1}{x_2} \frac{\partial v_1}{\partial x_2^2} \right) + \frac{\partial v_1}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_2} \frac{\partial^2 v_1}{\partial x_2^2}, \\
\frac{\partial v_1}{\partial x_1} + \frac{1}{x_2} \frac{\partial v_1}{\partial x_2} &= 0.
\end{align*}
\]

Turning to the new variables:

\[
v_x = \frac{v_1 - u_0}{u_0}, \quad v_y = \frac{v_2}{u_0}, \quad P = \frac{p - p_0}{\rho u_0^2}, \quad x = \frac{x_1}{h}, \quad y = \frac{x_2}{h},
\]

we get the system of equations:

\[
\begin{align*}
\frac{\partial^2 v_x}{\partial y^2} + \frac{1}{y} \frac{\partial v_x}{\partial y} - \text{Re} \left( \frac{\partial v_x}{\partial x} + \frac{dp}{dx} \right) + a \left( \frac{\partial^3 v_x}{\partial y^2 \partial x} + \frac{1}{y} \frac{\partial^2 v_x}{\partial y \partial x} \right) \\
= \frac{\partial v_x}{\partial x} + \frac{1}{y} \frac{\partial (yv_y)}{\partial y} = 0.
\end{align*}
\]

Here \( \text{Re} = \frac{hu_0}{v} \) – Reynolds number, \( a = \frac{m_l \text{Re}}{\rho h^2} \) – dimensionless molar transfer number.

This system of equations is solved under the following boundary conditions:
\[ v_x = 0, P = 0 \quad \text{where} \quad x = 0, \]
\[ \frac{\partial v_x}{\partial y} = 0, v_y = 0, |v_x| < \infty \quad \text{where} \quad y = 0, \]
\[ v_x = -1, v_y = 0 \quad \text{where} \quad x > 0 \quad \text{and} \quad y = \pm 1. \]

In order to solve the problem, let’s apply the Laplace integral transform with respect to the variable \( x \) to the system of equations (5) and take into account the conditions at \( x=0 \). Then we get:

\[
\begin{align*}
\frac{\partial^2 v_x}{\partial y^2} + \frac{1}{y} \frac{\partial v_x}{\partial y} - \alpha^2(s) v_x &= \alpha^2(s) P, \\
-s v_x + \frac{1}{y} \frac{\partial (y v_y)}{\partial y} &= 0.
\end{align*}
\]

Accordingly, the boundary conditions will take the form

\[
\begin{align*}
\mid v_x = -\frac{1}{s}, v_x &= 0 \quad \text{where} \quad y = 1, \\
v_x &= 0, |v_x| < \infty \quad \text{where} \quad y = 0.
\end{align*}
\]

Since \( P \) is independent of \( y \), and \( v_x \) is bounded for \( y=0 \), then:

\[
\bar{v}_x = c_1(s) I_0(wy) - \bar{P}
\]

Using the boundary conditions for \( y=1 \) and determining \( c_1 \), we obtain the following solution in the image field

\[
\bar{v} = (\bar{P} - \frac{1}{s}) \frac{I_0(\alpha y)}{I(\alpha)} - \bar{P}.
\]

To determine \( \bar{P} \), we multiply both sides of the second equation (7) by \( ydy \) and integrate over \( y \) from 0 to 1. Then we find

\[
\bar{P} = -\frac{2I_1(\alpha)}{s \alpha I_2(\alpha)}.
\]

Using the relation given by Targ in [21]

\[
\bar{F}(s) = \frac{2I_1(\sqrt{s})}{s \sqrt{s} I_0(\sqrt{s})} = 8x + \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{\beta_k^3} e^{-\beta_k^3 x} = f(x),
\]

where \( \beta_k \) – positive roots of the equation \( J_2(\beta)=0 \), we define the original expression

\[
\Phi(s) = \frac{2I_1(\alpha)}{3\alpha I_2(\alpha)}.
\]

Let’s use the Efros theorem [18]:

If the image \( F(s) = f(x) \) is known, and analytical functions \( G(s) \) and \( q(s) \) are so that

\[ G(s) e^{-\tau q(s)} = g(x, \tau), \]

\[
\Phi(s) = \frac{2I_1(\alpha)}{3\alpha I_2(\alpha)}.
\]
then
\[ F[q(s)]G(s) = \int_{0}^{\infty} f(\sigma)g(x, \sigma) \, d\sigma. \]

Let’s integrate the relation given in [18]
\[ \frac{1}{s} - e^{i\eta} = \frac{1}{s} + 1\, e^{i\eta} - 1 = 1 + \int_{0}^{\infty} \frac{-\eta x}{\sqrt{x/\pi \Delta^* x}} I_0(2\sqrt{\pi \Delta^* x}) \, dx \]
and get the relation
\[ G(s) e^{-\eta q(s)} = e^{-\eta \text{Re} e^{i\eta} g(x, \tau)} = g(x, \tau) = e^{-\eta \tau} I_0(2\sqrt{\pi \Delta^* x}) + \eta \int_{0}^{\infty} e^{-\eta x} I_0(2\sqrt{\pi \Delta^* x}) \, dx \]
(13)

Now, taking into account relations (12) and (14) and Efros theorem, we find the formula
\[ \Phi(x) = 8e^{-\eta x} \int_{0}^{\infty} e^{-\eta \sigma \text{Re} I_0(2\sqrt{\pi \Delta^* x})} \, d\sigma + \]
\[ 8\eta \int_{0}^{\infty} e^{-\eta \sigma} \int_{0}^{\infty} e^{-\eta x} I_0(2\sqrt{\pi \Delta^* x}) \, dx \, d\sigma + \]
\[ 4e^{-\eta x} \sum_{k=1}^{\infty} e^{-(\beta^2 + \eta \text{Re})^{\sigma} I_0(2\sqrt{\pi \Delta^* x})} \, d\sigma + \]
\[ 4\eta \int_{0}^{\infty} e^{-\eta x}(2\sqrt{\pi \Delta^* x}) \, dx \sum_{k=1}^{\infty} e^{-(\beta^2 + \eta \text{Re})^{\sigma} I_0(2\sqrt{\pi \Delta^* x})} \, d\sigma. \]
(14)

Having performed calculations in relation (14), taking into account [18].
\[ \int_{0}^{\infty} e^{-ax} I_{2\nu}(2\sqrt{\beta a x}) \, dx = \frac{e^{2a}}{\sqrt{a \beta}} \frac{M_{\mu, \nu}^{\frac{a}{2}}}{\Gamma(\nu + 1)} \frac{M_{-\frac{a}{2}, \nu}^{\frac{a}{2}}}{\Gamma(2\nu + 1)}. \]
(15)

\[ M_{\lambda, \mu}(z) = z^{\frac{1}{2}}e^{-z^{2}} \Phi(\mu - \lambda + \frac{1}{2}, 2\mu + 1; z). \]
(16)

\[ \Phi(\alpha, \gamma; z) = F_1(\alpha, \gamma; z) = 1 + \frac{az}{\gamma!} + \frac{a(\alpha + 1) z^2}{\gamma(\gamma + 1)} + \ldots, \]
(17)

\[ M_{-\frac{a}{2}, \nu}^{\frac{a}{2}}(z) = \sqrt{\pi} e^{-z^{2}} \Phi(1, 1; z) = \sqrt{\pi} e^{\frac{z^2}{2}}, \]
(18)

\[ \int_{0}^{\infty} e^{-ax} I_0(2\sqrt{\beta a x}) \, dx = \frac{e^{\beta a}}{a}. \]
(19)
where \( \Gamma (\cdots) \) – Euler gamma function, (15) can be rewritten in the form

\[
\Phi(x) = \frac{8}{\text{Re}} (x + a_{1}) + 4 \sum_{k=1}^{\infty} \frac{a_{1} \beta_{k}^{2} + \text{Re}[1 - \exp(-\frac{x \beta_{k}^{2}}{a \beta_{k}^{2} + \text{Re}})]}{\beta_{k}^{2} (a \beta_{k}^{2} + \text{Re})}.
\] (20)

Thus, we determine that the original \( P \) has the following form

\[
P(x) = -\frac{2 I_1(\sqrt{\text{Re}/a})}{\sqrt{\text{Re}/a} I_2(\sqrt{\text{Re}/a})} - \frac{4 (x - a)}{\text{Re}} - 2 \sum_{k=1}^{\infty} \frac{a \beta_{k}^{2} + \text{Re}[1 - \exp(-\frac{x \beta_{k}^{2}}{a \beta_{k}^{2} + \text{Re}})]}{\beta_{k}^{2} (a \beta_{k}^{2} + \text{Re})}.
\]

Hence, for \( x = 0 \) (i.e. in the inlet section), we obtain

for \( a = 0 \)

\[
P = \frac{8a}{\text{Re}} - 4a \sum_{k=1}^{\infty} \frac{1}{a \beta_{k}^{2} + \text{Re}} = 0,
\]

for \( a \neq 0 \)

\[
- \frac{8a}{\text{Re}} - 4 \sum_{k=1}^{\infty} \frac{1}{\beta_{k}^{2} + \text{Re}/a} = 0
\]

or

\[
\sum_{k=1}^{\infty} \frac{1}{\beta_{k}^{2} + \text{Re}/a} = -\frac{2a}{\text{Re}}
\]

We turn to the original in formula (10), for which we represent it in the form:

\[
\bar{v}_{i} = -2 \frac{I_1(\omega) I_0(\omega y)}{s \omega I_2(\omega)} - \bar{P} - \frac{1}{3} \frac{I_0(\omega y)}{I_0(\omega)}.
\] (21)

Let’s consider separately the expression:

\[
- \bar{\varphi} = 2 \frac{I_0(s \sqrt{s}) I_0(\sqrt{s} y)}{s \sqrt{s} I_2(s \sqrt{s}) I_0(\sqrt{s})}.
\]

Expanding \( I_0(\sqrt{s}), I_1(\sqrt{s}), I_2(\sqrt{s}) \) into series [14], we have:

\[
I_0(\sqrt{s}) = 1 + \frac{s}{2} + \frac{s^2}{24} + ...
\]

\[
I_1(\sqrt{s}) = \frac{\sqrt{s}}{2} + \frac{s \sqrt{s}}{4} + ...
\]
When \( s \to 0 \) for \( \varphi \), we obtain:

\[
\varphi_{s \to 0} = 2 \left( \frac{\sqrt{s}}{2} \right)^2 \left( \frac{1}{2^3} + \frac{s}{12} + \frac{s^2}{2^3} + \frac{s^3}{12} + \frac{s^4}{2^3} + \cdots \right) \frac{1}{\sqrt{s}} = \frac{1 + s/2^3}{1 + s/12 + s/2^3 + \cdots}.
\]

Using the property of the Bessel function, we have:

\[
\text{res}\varphi(0) = 8 \lim_{s \to 0} \frac{d}{ds} e^{3s} \left[ \frac{1 + s/8}{1 + 7s/12 + \cdots} \right] = 8 \frac{x + \left( \frac{y}{2} + \frac{1}{8} \right) - \frac{7}{12}}{1} = 8x + 4y - \frac{11}{3},
\]

\[
I_2(\sqrt{s}) = 0; s = -\beta_k^2; I_2(i\beta_k) = 0 \to -iJ_2(\beta_k) = 0;
\]

\[
I_0(i\mu_k) = 0 \to J_0(\mu_k) = 0.
\]

Finally, we have:

\[
V_x = 4 \sum_{k=1}^{\infty} \left\{ \frac{a\beta_k^2 + \text{Re} \left[ 1 - \exp \left( -\frac{x\beta_k^2}{a\beta_k^2 + \text{Re}} \right) \right]}{\beta_k^2 (a\beta_k^2 + \text{Re})} \left( 1 - \frac{J_0(\beta_k y)}{J_0(\beta_k)} \right) \right\} + \sum_{k=1}^{\infty} \frac{a\mu_k^2 + \text{Re} \left[ 1 - \exp \left( -\frac{x\mu_k^2}{a\mu_k^2 + \text{Re}} \right) \right]}{\mu_k^2 (a\mu_k^2 + \text{Re})} \left( \frac{2J_1(\mu_k) - \mu_k J_2(\mu_k)}{J_1(\mu_k) J_2(\mu_k)} \right) J_0(\mu_k y).
\]

(22)

where \( \mu_k \) – positive roots of the equation \( J_0(\mu) = 0 \).

Passing to dimensional quantities, we obtain:

\[
\frac{p - p_0}{\rho u_0^2} = -\frac{8\nu x_l}{u_0 R^2} - \frac{8 m_l}{\rho R^2} - 4 \sum_{k=1}^{\infty} \frac{\beta_k^2 m_l + \rho R^2 \left[ 1 - \exp \left( -\frac{\rho v_{x_l} \beta_k^2}{u_0 (m_l + \rho R)} \right) \right]}{\beta_k^2 (m_l \beta_k^2 + \rho R^2)}.
\]

(23)

\[
\frac{v_l - u_0}{u_0} = 4 \sum_{k=1}^{\infty} \frac{\beta_k^2 m_l + \rho R^2 \left[ 1 - \exp \left( -\frac{\rho v_{x_l} \beta_k^2}{u_0 (m_l + \rho R)} \right) \right]}{\beta_k^2 (m_l \beta_k^2 + \rho R^2)} \left( \frac{J_0(\frac{x_l}{R} \beta_k)}{J_0(\beta_k)} \right) + \sum_{k=1}^{\infty} \frac{\mu_k^2 m_l + \rho R^2 \left[ 1 - \exp \left( -\frac{\rho v_{x_l} \mu_k^2}{u_0 (m_l + \rho R)} \right) \right]}{\mu_k^2 (m_l \mu_k^2 + \rho R^2)} \left( \frac{2J_1(\mu_k) - \mu_k J_2(\mu_k)}{J_1(\mu_k) J_2(\mu_k)} \right) J_0(\mu_k \frac{x_l}{R}).
\]

(24)
Thus, the system of equations of fluid motion according to the rheological law of molecular and molar transfer in the narrow channel approximation (5) with boundary conditions (7) has a solution in the form of (23) and (24).

4. Discussion
1. Formulas (23) and (24) at \( m_l = 0 \) are an analytical solution to the problem of the motion of a viscous incompressible fluid in a round pipe, taking into account molecular transfer in the flow, the determination of which is not yet complete [21]. When \( m_l \) differs from zero, (23) and (24) are an analytical solution to this problem, taking into account both molecular and molar transfer in a flow.

2. The calculation results showed that in the initial section, the velocity distribution is uniform over the entire cross section. Moving away from the initial section, the shape of the velocity profile changes. In laminar flow, the reduction in the size of the flow core and the establishment of the regime occurs uniformly. Such uniformity is not observed in the calculation according to (24). If a layered inhibition of the liquid occurs under the influence of molecular viscosity, then under the influence of the molar viscosity, the profile will be rebuilt with a large maximum speed and a small size of the flow core. At a certain distance from the inlet, the location of the places of maximum speeds acquires the opposite character, i.e. the maximum flow rate at \( a \neq 0 \) becomes greater than at \( a = 0 \). The above corresponds to a velocity distribution at a distance of 70r. Further, the profile and velocity value are constantly approaching each other, and a flow regime that does not depend on the initial data at the inlet is established. This distance corresponds to 208r.

3. Comparison of the nature of the flow at \( Re = 500 \), \( a = 0.5 \); 2.5; 5; 25 showed that at a distance of 10r from the inlet, the velocity profile in non-zero values of the number \( a \) in the middle zone of the pipe is greater than at \( a = 0 \). Starting from \( x = 0.4r \), the values of the velocities at \( a = 0 \) become larger than at \( a \neq 0 \). At a distance of \( x = 40r \) from the inlet in the middle zone, large velocity values are achieved at \( a = 0 \). In the peripheral region, starting from \( x = 0.7r \), the flow rate at \( a = 0 \) becomes greater. At a certain distance from the inlet, more precisely at \( x = 160...200r \) and further, the flow velocity profile for all values of \( a \) is the same parabolic, and the values of the corresponding transverse coordinates, too. To demonstrate the above, figure 1 shows the velocity profiles at \( Re = 500 \), \( a = 0 \) and \( a = 5 \) at distances of 10r, 40r from the inlet.

4. Qualitatively and quantitatively, the energy loss due to friction leads to a change in the longitudinal pressure drop, which is shown in figure 2. But note that the pressure changes more smoothly than the shear stress on the wall.

5. The velocity profiles at a distance of 160r from the inlet at \( Re = 25000 \), \( a = 100 \) and \( Re = 125000 \), \( a = 500 \) are shown in figure 3. It can also be noted here that a flow in which the same velocities are reached at many points of the cross section corresponds more to the number \( Re \). This theoretical result makes it possible to explain the formation of the flow core at high velocities and its deformation at low velocities.

6. From the above it follows that there is an interconnecting regularity between the dimensionless parameters \( a = m_l Re/(pr^2) \) and the number \( Re \). It was revealed that in the range of parameter \( a \), for which the calculations were performed, the additional term does not generate additional maxima and minima in the velocity diagram.

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