ON THE DIMENSION OF SYZYGIES

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ABSTRACT. In this note we compute length, support and dimension of syzygy modules of certain modules. This partially answers questions asked by Huneke et al.

1. INTRODUCTION

In this note $(R, m, k)$ is a commutative noetherian local ring of dimension $d > 0$ and $0 \neq M$ is a finitely generated $R$-module. Let $i \in \mathbb{N}_0$. The notation $\text{p. dim}(\cdot)$ stands for the projective dimension and $\lambda(\cdot)$ is the length function. The $i$th betti number of $M$ is given by $\beta_i(M) := \dim_k(\text{Tor}_i^R(k, M))$. If there is no danger of confusion we will use $\beta_i$ instead of $\beta_i(M)$. A minimal free resolution of $M$ is of the form

$$
\cdots \longrightarrow R^{\beta_{i+1}} \xrightarrow{f_{i+1}} R^{\beta_i} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0. 
$$

The $i$th syzygy module of $M$ is $\text{Syz}_i(M) := \ker(f_{i+1}) = \text{coker}(f_{i-1})$ for all $i > 0$. Computing numerical invariants of the syzygy modules is of some interest for a variety of reasons. Our first aim is to compute the length of syzygy modules:

**Question 1.1.** (See [5, Question 1.2]) Let $M$ be such that $\text{p. dim}_R(M) = \infty$ and $\lambda(M) < \infty$. Is $\lambda(\text{Syz}_i(M)) = \infty$ for all $i > d + 1$?

The assumption $d > 0$ is needed. Indeed, let $R$ be a zero-dimensional local ring which is not a field. Then $\text{p. dim}(R/m) = \infty$ and each of its syzygy modules are nonzero. Since $R$ is zero-dimensional, any finitely generated module is of finite length. So, $d$ should be positive. There are few progress concerning Question 1.1. Let us recall an achievement from literature. Recently, Huneke and his coauthors showed that Question 1.1 is true over 1-dimensional Buchsbaum rings. Also, they showed the requirement of $i > d + 1$ is necessary (over 1-dimensional rings):

**Example 1.2.** Let $R := k[[x, y]]/(x^2, xy)$ and $M := R/(y)$. Then $\text{p. dim}_R(M) = \infty$, $\lambda(M) < \infty$, and $\lambda(\text{Syz}_2(M)) < \infty$. We should remark that the module $N = R/(x)$ does not do the same job.

In support of Question 1.1 we present four observations. The first one drops the dimension restriction from the Buchsbaum rings:

**Observation A.** Let $(R, m)$ be a Buchsbaum ring of dimension $d > 1$, $\text{p. dim}_R(M) = \infty$ and $\lambda(M) < \infty$. Then $\text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R)$ for all $i > 0$. In particular, $\lambda(\text{Syz}_i(M)) = \infty$.

We show a little more:

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Corollary 1.3. Let \((R, m, k)\) be a \(d\)-dimensional ring for which \(H^0_m(R)\) is a \(k\)-vector space (e.g. \(R\) is Buchsbaum) and \(d > 0\). Let \(M\) be locally free on the punctured spectrum such that \(\text{p. dim}_R(M) = \infty\). Then \(\lambda(\text{Syz}_i(M)) = \infty\) for all \(i > 2\). Suppose in addition that \(R\) is equidimensional. Then \(\dim(\text{Syz}_i(M)) = \dim R\) for all \(i > 2\).

Here, \(H^0_m(R) := \bigcup_{n \in \mathbb{N}} (0 :_R m^n)\) is the \(0\)th local cohomology module of \(R\) with respect to \(m\). We observed in Example 1.2 that the second syzygy module \(\text{Syz}_2(M)\) of a finite length module \(M\) may be of finite length. If we focus on the simple module the story will changes:

Observation B. We reprove a result of Okiyama by a short argument:

i) If \(R\) is regular, then \(\text{Syz}_i(R/m) = 0\) for all \(i > d\) and \(\dim(\text{Syz}_i(R/m)) = d\) for all \(i \leq d\).

ii) If \(R\) is not regular, then \(\text{Supp}(\text{Syz}_i(R/m)) = \text{Spec}(R)\) for all \(i > 0\).

We avoid the Tate (and Gulliksen-Levin) approach of homology of local rings. Our second aim is to investigate the following question:

Question 1.4. (See [4] and [11]) Is \(\dim(\text{Syz}_i(M))\) constant for all \(i \gg 0\)?

To find a connection between Question 1.1 and Question 1.4 let us revisit Example 1.2, where we observed that \(\lambda(\text{Syz}_1(M)) = \infty\). In fact, \(\text{Syz}_1(M) = yR\). Also, \(\text{Ann}(yR) = xR\). Thus, \(\text{Supp}(\text{Syz}_1(M)) = V(xR) = \text{Spec}(R)\). This is a sample of a general fact, which extends and corrects some known results:

Observation C. Let \(d > 0\) and \(0 \neq M\) be a finite length module of infinite projective dimension. Then, for all \(r \geq 0\) the following conditions are equivalent:

i) \(\text{Supp}(\text{Syz}_{r+1}(M)) = \text{Spec}(R)\),

ii) \(\dim(\text{Syz}_{r+1}(M)) = \dim R\),

iii) \(\lambda(\text{Syz}_{r+1}(M)) = \infty\).

By \(\text{Assh}(R)\) we mean the set of all prime ideals \(p\) such that \(\dim(R) = \dim(R/p)\). Here is an immediate application: Let \(R\) be a ring such that \(\text{Ass}(R) = \text{Assh}(R)\) (e.g. \(R\) is Cohen-Macaulay), \(\lambda(M) < \infty\) and \(\text{p. dim}_R(M) = \infty\). Then \(\text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R)\) for all \(i > 0\). In particular, Question 1.1 and Question 1.3 are true over integral domains. To see more applications of Observation C, please see Corollary 2.3 and Corollary 2.5. Of course the last four results only work for finite length modules, please see Example 3.3. Section 2 is devoted to the proof of Observation C. In Section 3 we show the following:

Observation D. Let \(R\) be a reduced local ring and \(M\) a finite length module of infinite projective dimension. Then \(\text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R)\) for all \(i > 0\).

We show a little more, see Lemma 3.4. Section 4 is devoted to the proof of Observation A and Observation B. Section 5 runs Question 1.1 over some rings. Neither of these are Buchsbaum nor reduced. Complete reduced rings are quotients of regular local rings by a radical ideal. Let us define the following related class of rings:
Definition 1.5. A ring is called weakly reduced if it is quotient of a local ring by a nonzero and proper integrally closed ideal.

Recall that \( I \subseteq T \subseteq \text{rad}(I) \). Thus, any complete reduced ring is weakly reduced. Also, we show:

Corollary 1.6. Let \( R \) be a weakly reduced ring of dimension \( d > 1 \) and \( M \) a finite length module of infinite projective dimension. Then \( \text{Supp} \text{Syz}_i(M) = \text{Spec}(R) \) for all \( i > 0 \).

Question 1.4 asks whenever \( \dim \text{Syz}_i(M) \) asymptotically is constant. Our partial results compute \( \dim \text{Syz}_i(\cdot) \) for all \( i > 0 \).

2. From Question 1.1 to Question 1.4

A finitely generated module \( M \) is called locally free on the punctured spectrum if \( M_q \) is free over \( R_q \) for all \( q \in \text{Spec}(R) \setminus \{m\} \). For example, any finite length module is locally free.

Lemma 2.1. Let \( (R, m) \) be equidimensional and \( M \) be locally free on the punctured spectrum. Then either \( \dim \text{Syz}_i(M) = \dim R \) or \( \lambda(\text{Syz}_i(M)) < \infty \).

In the next argument, we are not in a position to drop the equidimensional assumption.

Proof. If \( d = \dim R \leq 1 \) there is nothing to prove. In particular, we may assume that \( d > 0 \) and that \( \text{Syz}_i(M) \neq 0 \). If \( \min(\text{Supp}(\text{Syz}_i(M))) \subset \min(R) \) were the case, then we should have \( \dim(\text{Syz}_i(M)) = \dim R \), because \( R \) is equidimensional. Thus, we can assume \( \min(\text{Supp}(\text{Syz}_i(M))) \not\subseteq \min(R) \). Let \( p \in \min(\text{Supp}(\text{Syz}_i(M))) \setminus \min(R) \). We claim that \( p \) is the maximal ideal. Suppose on the contrary that \( p \neq m \). To search a contradiction, we look at \( F \rightarrow M \rightarrow 0 \) the minimal free resolution of \( M \). Since \( M \) is locally free, \( M_p \) is free. Consequently, \( F_p \rightarrow M_p \rightarrow 0 \) splits. It turns out that \( \text{Syz}_i(M)_p \) is free. Therefore, \( \dim(\text{Syz}_i(M)_p) = \dim(R_p) > 0 \). Since \( p \in \min(\text{Supp}(\text{Syz}_i(M))) \) we get to a contradiction.

However, if \( M \) is of finite length we are able to drop the equidimensional assumption:

Lemma 2.2. Let \( d > 0 \) and \( 0 \neq M \) be finite length module of infinite projective dimension. Then, for all \( r \geq 0 \) the following conditions are equivalent.

\[ \begin{align*}
\text{i) } & \lambda(\text{Syz}_{r+1}(M)) = \infty, \\
\text{ii) } & \sum_{i=0}^{r} (-1)^{r-i} \beta_i(M) > 0, \\
\text{iii) } & \text{Supp}(\text{Syz}_{r+1}(M)) = \text{Spec}(R), \\
\text{iv) } & \dim(\text{Syz}_{r+1}(M)) = \dim R.
\end{align*} \]

Proof. First we recall a routine fact. Let \( p \neq m \) be a prime ideal. Such a thing exists, because \( d > 0 \). Keep in mind that \( M \) is of finite length. Since \( M_p = 0 \), we have the following split exact sequence:

\[ 0 \rightarrow \text{Syz}_{r+1}(M)_p \rightarrow R_p^\beta_r \rightarrow \cdots \rightarrow R_p^\beta_0 \rightarrow 0. \]

Since the sequence splits, \( \text{Syz}_{r+1}(M)_p \) is free. So,

\[ \sum_{i=0}^{r} (-1)^{r-i} \beta_i(M) = \text{rank}(\text{Syz}_{r+1}(M)_p) \geq 0 \quad (*) \]
Now, we prove the lemma:

\( i \Rightarrow ii \): Let \( p \neq m \) be a prime ideal in \( \text{Supp} (\dim \text{Syz}_{r+1}(M)) \). By the assumption such a \( p \) exists. Thus \( \text{Syz}_{r+1}(M)_p \) is a nonzero free module. Therefore, \( \text{rank}(\text{Syz}_{r+1}(M)_p) > 0 \). From (*) we get that \( \sum_{i=0}^{r} (-1)^{r-i} \beta_i(M) > 0 \), as claimed.

\( ii \Rightarrow iii \): Let \( p \in \text{Spec}(R) \setminus \{m\} \). By the assumption, \( \sum_{i=0}^{r} (-1)^{r-i} \beta_i(M) > 0 \). In view of (*) we have \( \text{rank}(\text{Syz}_{r+1}(M)_p) > 0 \). Therefore, \( p \in \text{Supp}(\text{Syz}_{r+1}(M)) \). Thus, \( \text{Spec}(R) \setminus \{m\} \subset \text{Supp}(\text{Syz}_{r+1}(M)) \).

One has \( \text{Syz}_{r+1}(M) \neq 0 \). Hence \( m \in \text{Supp}(\text{Syz}_{r+1}(M)) \). Consequently,

\[ \text{Spec}(R) \subset \text{Supp}(\text{Syz}_{r+1}(M)). \]

The reverse inclusion always hold. So, \( \text{Spec}(R) = \text{Supp}(\text{Syz}_{r+1}(M)) \) as claimed.

\( iii \Rightarrow iv \): Recall that modules with the same support have a same dimension. This implies that \( \dim \text{Syz}_{r+1}(M) = \dim R \), as claimed.

\( iv \Rightarrow i \): Since \( d > 0 \), a finitely generated module of dimension \( d \) is of infinite length. Thus, \( \lambda(\text{Syz}_{r+1}(M)) = \infty \). This finishes the proof. \( \square \)

**Corollary 2.3.** Let \( M \) be a finite length module and of infinite projective dimension over a 1-dimensional local ring \( R \). Then \( \text{Supp}(\text{Syz}_1(M)) = \text{Supp}(\text{Syz}_3(M)) = \text{Spec}(R) \).

**Proof.** In view of [5, Corollary 5.10], we know that \( \lambda(\text{Syz}_1(M)) = \lambda(\text{Syz}_3(M)) = \infty \). Lemma 2.2 shows that \( \text{Supp}(\text{Syz}_1(M)) = \text{Supp}(\text{Syz}_3(M)) = \text{Spec}(R) \), as claimed. \( \square \)

**Lemma 2.4.** (See [5, Proposition 5.5]) Let \( R \) be a local ring of positive dimension. Suppose there is an \( R \)-module \( M \) of infinite projective dimension and finite length such that \( \lambda(\text{Syz}_{i+1}(M)) < \infty \) for some fixed \( i > 0 \). If \( \beta_i(M) \geq \beta_{i-1}(M) \), then \( \lambda(\text{Syz}_{i-1}(M)) < \infty \).

**Corollary 2.5.** Let \( 0 \neq M \) be a finite length module such that \( \beta_i(M) \leq \beta_{i+1}(M) \) for all \( i > 0 \). Then \( \text{Supp}(\text{Syz}_{2i+1}(M)) = \text{Spec}(R) \) for all \( i \geq 0 \).

**Proof.** We may assume that \( d > 0 \). First note that \( p, \dim(M) \) is infinite. Clearly, \( \lambda(\text{Syz}_1(M)) = \infty \). This follows by looking at the following short exact sequence

\[ 0 \rightarrow \text{Syz}_1(M) \rightarrow R^\beta_1 \rightarrow M \rightarrow 0, \]

and the fact that \( d > 0 \). This implies that \( \text{Supp}(\text{Syz}_1(M)) = \text{Spec}(R) \), see Lemma 2.2. Suppose on the contrary that \( \text{Supp}(\text{Syz}_{2i+1}(M)) \neq \text{Spec}(R) \) for some \( i > 0 \). By revisiting Lemma 2.2, we see that \( \lambda(\text{Syz}_{2i+1}(M)) < \infty \). We apply this along with Lemma 2.4 to observe that \( \lambda(\text{Syz}_{2i-1}(M)) < \infty \). If \( 2i - 1 \neq 1 \) we can repeat the argument to observe that \( \lambda(\text{Syz}_1(M)) < \infty \), a contradiction. \( \square \)

Let \( d(M) \) be the smallest integer \( l \) such that \( \dim(\text{Syz}_l(M)) \) is constant for all \( i > l \).

**Question 2.6.** Let \( C \) be a class of finitely generated modules. Suppose \( d(M) \) is finite for all \( M \in C \). Is \( \sup \{ d(M) : M \in C \} < \infty \)?

The classes that we are interested on it are the class of finitely generated modules, the class of finite length modules and the class of modules with fixed certain numerical invariants.
3. Dealing with Reduced Rings

In the Cohen-Macaulay case and for all \( i > \dim R \) the following fact is in \cite{I1}.

**Fact 3.1.** (Okuyama) Let \( R \) be a ring such that \( \text{Ass}(R) = \text{Assh}(R) \) (e.g. \( R \) is Cohen-Macaulay or \( R \) is a domain) and \( p, \dim_R(M) = \infty \). Then \( \dim(\text{Syz}_i(M)) = \dim R \) for all \( i > 0 \).

**Proof.** First note that the property \( \text{Ass}(R) = \text{Assh}(R) \) is one of the basic properties Cohen-Macaulay rings but does not characterize Cohen-Macaulay rings. By looking at the following exact sequence

\[
0 \to \text{Syz}_i(M) \to R^{\delta_i-1} \to \text{Syz}_{i-1}(M) \to 0,
\]

we observe that \( \text{Ass}(\text{Syz}_i(M)) \subset \text{Ass}(R) \). Since \( \text{Syz}_i(M) \neq 0 \), \( \text{Ass}(\text{Syz}_i(M)) \neq \emptyset \). Let \( p \in \text{Ass}(\text{Syz}_i(M)) \). Then \( p \in \text{Ass}(R) = \text{Assh}(R) \). Thus, \( \dim R/p = \dim R \). So, \( \dim(\text{Syz}_i(M)) = \dim R \) as claimed.

**Corollary 3.2.** Let \( R \) be a ring such that \( \text{Ass}(R) = \text{Assh}(R) \), \( \lambda(M) < \infty \) and \( p. \dim_R(M) = \infty \). Then \( \text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R) \) for all \( i > 0 \).

**Proof.** By Fact 3.1, \( \dim(\text{Syz}_i(M)) = \dim R \) for all \( i > 0 \). It is enough to apply Lemma 3.2.

The finite length assumption in Corollary 2.3, Corollary 2.3, lemma 2.2 and Corollary 3.2 is important:

**Example 3.3.** We look at the Cohen-Macaulay ring \( R := k[[X, Y]]/(XY) \) and the infinite length module \( M := R/xR \). Thus, \( \text{Ass}(R) = \text{Assh}(R) \). Then \( \text{Syz}_i(M) = R/xR \) and \( \text{Syz}_{2i-1}(M) = R/yR \) for all \( i > 0 \). So, \( \text{Supp}(\text{Syz}_i(M)) \neq \text{Spec}(R) \) for all \( i > 0 \). But \( \dim(\text{Syz}_i(M)) = \dim R \) for all \( i > 0 \).

**Lemma 3.4.** Let \( R \) be a ring of positive depth. Then Question 1.1 is true. In fact, if \( M \) is such that \( p. \dim_R(M) = \infty \) and \( \lambda(M) < \infty \), then \( \text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R) \) for all \( i > 0 \).

**Proof.** Recall that depth of a finitely generated module \( L \) is defined by \( \inf\{ j \geq 0 : \text{Ext}_R^j(R/m, L) \neq 0 \} \). Let \( i > 0 \) and look at the exact sequence

\[
0 \to \text{Syz}_i(M) \to R^{\delta_i-1} \to \text{Syz}_{i-1}(M) \to 0.
\]

Note that depth\( (R^{\delta_i-1}) > 0 \). Apply the long exact sequence of Ext-modules \( \text{Ext}_R^i(R/m, -) \) to deduce that

\[
\text{depth}(\text{Syz}_i(M)) \geq \inf\{ \text{depth}(R^{\delta_i-1}), \text{depth}(\text{Syz}_{i-1}(M)) + 1 \} > 0.
\]

In particular, \( \text{depth}(\text{Syz}_i(M)) > 0 \) for all \( i > 0 \). Since depth of any nonzero finite length module is zero, we get that \( \lambda(M) = \infty \). Also, Lemma 2.2 shows that \( \text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R) \) for all \( i > 0 \).

**Corollary 3.5.** Let \( R \) be equidimensional and of positive depth. If \( M \) is locally free, then \( \dim(\text{Syz}_i(M)) \) is constant for all \( i > 0 \).

**Proof.** We may assume that \( p. \dim(M) = \infty \). By Lemma 3.4, \( \lambda(\text{Syz}_i(M)) = \infty \). In view of Lemma 2.1, \( \dim(\text{Syz}_i(M)) = \dim R \) for all \( i > 0 \).

The following is our main result in this section:

**Corollary 3.6.** Let \( R \) be a reduced local ring and \( M \) a finite length module of infinite projective dimension. Then \( \text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R) \) for all \( i > 0 \).
Proof. We may assume that \( \dim R > 0 \). Reduced rings satisfy in the Serre’s property \((S_1)\). One may read as follows: depth\((R_p) \geq \min\{1, \text{ht}(p)\}\) for all \( p \in \text{Spec}(R) \). We apply this for the maximal ideal to observe that depth\((R) > 0\). Now Lemma 3.4 yields the claim. \(\square\)

4. LOOKING THROUGH BUCHSBAUM GLASSES

We start with:

**Lemma 4.1.** (Vanishing result) Let \( M \) be locally free over punctured spectrum that \( \lambda(\text{Syz}_{i+1}(M)) < \infty \) for some fixed \( i > 0 \). Then \( \text{Tor}^R_i(M, R/ H^0_m(R)) = 0 \).

**Proof.** The proof in the case \( M \) is of finite length is in [5, Lemma 5.2]. Again, such a proof works for locally free modules. \(\square\)

**Lemma 4.2.** Let \( M \) be of finite length. Then \( \lambda(\text{Syz}_i(M)) = \infty \) for all \( 1 \leq i \leq d \) provided \( \text{Syz}_i(M) \neq 0 \). In fact \( \text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R) \).

**Proof.** Suppose on the contradiction that \( \lambda(\text{Syz}_i(M)) < \infty \) for some \( 1 \leq i \leq d \). We look at the following complex of free modules with finite length homologies:

\[
0 \longrightarrow R^{\beta_i-1} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow 0
\]

The new intersection theorem [9] implies that \( i - 1 \geq \dim R \) which excluded by the assumption. Now, the proof of Lemma 2.2 shows that \( \text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R) \). \(\square\)

We need to recall the following result: Let \( R \) be a noetherian ring and \( 0 \neq I \) an ideal with a finite free resolution. Then \( I \) contains an \( R \)-regular element, see [2, Corollary 1.4.7].

**Corollary 4.3.** Let \((R, m, k)\) be a \( d \)-dimensional ring with \( d > 0 \). Then \( \text{Syz}_i(R/m) \) is of infinite length provided \( \text{Syz}_i(R/m) \) is nonzero for all \( i > 0 \). In fact, \( \text{Supp}(\text{Syz}_i(R/m)) = \text{Spec}(R) \) for all \( i > 0 \).

**Proof.** In view of \( 0 \rightarrow \text{Syz}_1(R/m) \rightarrow R^{\beta_0} \rightarrow R/m \rightarrow 0 \) we observe that \( \text{Syz}_1(R/m) \) is of infinite length. Then we may assume that \( i > 1 \). Suppose on the contradiction that \( \text{Syz}_{i+1}(R/m) \) is of finite length for some \( i > 0 \). Then by the vanishing result we have \( \text{Tor}^R_i(k, R/ H^0_m(R)) = 0 \). Keep in mind that

\[
\text{p. dim}(R/ H^0_m(R)) = \sup\{j \geq 0 : \text{Tor}^R_j(k, R/ H^0_m(R)) \neq 0\}.
\]

Consequently, \( H^0_m(R) \) has a finite free resolution. In view of Lemma 3.4 we may and do assume that \( H^0_m(R) \neq 0 \). Also, \( H^0_m(R) \neq R \), because \( d > 0 \). We can apply [2, Corollary 1.4.7] to conclude that \( H^0_m(R) \) contains an \( R \)-regular element. Since each element of \( H^0_m(R) \) is annihilated by some power of \( m \) we get to a contradiction. In the light of Lemma 2.2 we observe that \( \text{Supp}(\text{Syz}_i(R/m)) = \text{Spec}(R) \). \(\square\)

Let \((R, m)\) be a local ring. Recall that a sequence \( x_1, \ldots, x_r \subseteq m \) is called a weak sequence if \( m((x_1, \ldots, x_{i-1}) : x_i) \subseteq (x_1, \ldots, x_{i-1}) \) for all \( i \). The ring \( R \) is called Buchsbaum if every system of parameters is a weak sequence. Now, let \( R \) be Buchsbaum. Recall from [10, Lemma 2.4] that \( mH_m^i(R) = 0 \) for all \( i \neq \dim R \). The converse of this is not true, see [10, Page 75].
Proposition 4.4. Let $(R, m, k)$ be a $d$-dimensional ring for which $m H^0_m(R) = 0$ (e.g. $R$ is Buchsbaum) and that $d > 1$. Let $M$ be finite length such that $p. \dim_R(M) = \infty$. Then $\text{Supp(Syz}_i(M)) = \text{Spec}(R)$ for all $i > 0$. In particular, $\lambda(\text{Syz}_i(M)) = \infty$.

Proof. In view of Lemma 3.4 we can assume that $\text{depth}(R) = 0$. In particular, $H^0_m(R) \neq 0$. Thus, $H^0_m(R)$ is a nonzero $k$-vector space. Suppose first that $i > 2$. Suppose on the contradiction that $\text{Syz}_i(M)$ is of finite length. We can apply Lemma 4.1. This states that $\text{Tor}_i^R(M, R/H^0_m(R)) = 0$, because $i - 1 > 0$. Since $i - 2 > 0$ we have $\text{Tor}_{i-2}^R(M, R) = \text{Tor}_{i-1}^R(M, R) = 0$. Look at $0 \to H^0_m(R) \to R \to R/H^0_m(R) \to 0$. This implies that

$$0 = \text{Tor}_{i-1}^R(M, R/H^0_m(R)) \simeq \text{Tor}_{i-2}^R(M, H^0_m(R)) \simeq \bigoplus \text{Tor}_{i-2}^R(k, M).$$

Recall that

$$p. \dim(M) = \sup \{ j \geq 0 : \text{Tor}_j^R(k, M) \neq 0 \}.$$ 

Consequently, $p. \dim(M) < \infty$ which is impossible. This contradiction yields that $\text{Syz}_i(M)$ is of infinite length. Clearly, $\lambda(\text{Syz}_i(M)) = \infty$. This follows by looking at the following short exact sequence

$$0 \to \text{Syz}_1(M) \to R^b_0 \to M \to 0.$$ 

Thus, $\text{Syz}_2(M)$ is the only possible case for which the length may be finite. This excluded from Lemma 4.2. Here is a place that we use the assumption $d > 1$. In sum, $\lambda(\text{Syz}_i(M)) = \infty$ for all $i > 0$. Finally, we deduce from Lemma 2.2 that $\text{Supp(Syz}_i(M)) = \text{Spec}(R)$ for all $i > 0$. □

In a similar vein we have:

Corollary 4.5. Let $(R, m, k)$ be a $d$-dimensional ring for which $H^0_m(R)$ is a $k$-vector space (e.g. $R$ is Buchsbaum) and $d > 0$. Let $M$ be locally free such that $p. \dim_R(M) = \infty$. Then $\lambda(\text{Syz}_i(M)) = \infty$ for all $i > 2$. Suppose in addition that $R$ is equidimensional. Then $\dim(\text{Syz}_i(M)) = \dim R$ for all $i > 2$.

Proof. The claim in the case $d = 1$ follows from [5] Proposition 5.3] under the assumption $\lambda(M) < \infty$. The same argument works for locally free modules. Then we may assume that $d > 1$. Now, the first desired claim is in Proposition 4.4. If $R$ is equidimensional, we deduce from Lemma 2.2 that $\dim(\text{Syz}_i(M)) = \dim R$ for all $i > 2$. □

The new intersection theorem answers the question provided $\dim R \neq 1$. Recall that the new intersection theorem gives a lower bound on the size of perfect complexes with finite length homologies. Reversely, we ask:

Question 4.6. Let $F : 0 \to F_n \to \ldots \to F_0 \to 0$ be a complex of free modules and with finite length homologies. Under what conditions there is an integer $\ell$ independent of $n$ such that $n < \ell \dim R$?

In Question 1.1 we focus on perfect complexes of the form $F : 0 \to F_n \to \ldots \to F_0 \to 0$ with the additional assumption $H_i(F) = 0$ for all $0 < i < n$. 
5. SOME NON-REDUCED AND NON-BUCHSBAUM EXAMPLES

We start by proving Corollary 1.6. First, we recall the main point for dealing with weakly reduced rings:

**Fact 5.1.** (See [7, Corollary 1.2]) Let $S$ be a local ring with an integrally closed ideal $I$. Let $M$ be a finitely generated $R$-module. Suppose $R := S/I$ is of zero depth. Then $\{\beta_i(M)\}$ is not decreasing.

Now, we extend Observation D in the following sense:

**Corollary 5.2.** Let $R$ be a weakly reduced local ring of dimension $d > 1$ and $M$ a finite length module of infinite projective dimension. Then $\text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R)$ for all $i > 0$.

**Proof.** In view of Lemma 4.1 we may assume that $\text{depth}(R) = 0$. Thus, we are in a situation to apply Fact 5.1 i.e., $\{\beta_i(M)\}$ is not decreasing. We apply Corollary 2.5 to observe that $\text{Supp}(\text{Syz}_{2i+1}(M)) = \text{Spec}(R)$ for all $i > 0$. In view of Lemma 2.2, $\lambda(\text{Syz}_2(M)) = \infty$. Suppose for some $i > 1$ we have $\text{Supp}(\text{Syz}_{2i}(M)) \neq \text{Spec}(R)$. Due to Lemma 2.2, $\lambda(\text{Syz}_{2i}(M)) < \infty$. We apply this along with Lemma 2.4 to observe that $\lambda(\text{Syz}_{2i-2}(M)) < \infty$. If $2i - 2 \neq 2$ we can repeat the argument to observe that $\lambda(\text{Syz}_2(M)) < \infty$ a contradiction. Thus, $\text{Supp}(\text{Syz}_{2i}(M)) = \text{Spec}(R)$. This completes the proof.

We need to recall the following result:

**Lemma 5.3.** (See [7, Proposition 2.1]) Let $I$ be a non-nilpotent ideal in a local ring $(S, n)$. Set $R := \frac{I}{m}$. Let $M$ be a finitely generated $R$-module. Then $M$ has increasing betti numbers.

**Example 5.4.** Look at the ring $R := k[[X, Y]]/X(X, Y)^n$ for some $n > 0$. Then $\lambda(\text{Syz}_i(R/m^n)) = \infty$ for all $i > 0$. In fact, $\text{Supp}(\text{Syz}_i(R/m^n)) = \text{Spec}(R)$.

**Proof.** Clearly, $\lambda(\text{Syz}_i(R/m^n)) = \lambda(m^n) = \infty$. Set $I := X(X, Y)^{n-1} \lhd k[[X, Y]]$. Let $i > 0$. In view of Lemma 5.3, $\beta_i(M) \geq \beta_{i-1}(M)$. Thus, Lemma 2.4 implies that $\lambda(\text{Syz}_{2i+1}(R/m^n)) = \infty$. Similarly, $\lambda(\text{Syz}_2(R/m^n)) = \infty$ provided $\lambda(\text{Syz}_2(R/m^n)) = \infty$. Hence, things reduce to show $\lambda(\text{Syz}_2(R/m^n)) = \infty$. One has $H^0_m(R) = xR$. This annihilated by $m^n$. Suppose on the contrary that $\lambda(\text{Syz}_2(R/m^n)) < \infty$. In the light of the vanishing result (see Lemma 2.1) we deduce that $\text{Tor}^R_1(R/m^n, R/H^0_m(R)) = 0$. But $\text{Tor}^R_1(R/m^n, R/H^0_m(R)) = \frac{m^n \cap H^0_m(R)}{m^n H^0_m(R)} = \frac{m^n \cap H^0_m(R)}{m^n H^0_m(R)}$. To get a contradiction it is enough to note that $0 \neq x^n \in m^n \cap H^0_m(R) = m^n \cap (x)$. Therefore, $\lambda(\text{Syz}_2(R/m^n)) = \lambda(m^n) = \infty$. We conclude from Lemma 2.2 that $\text{Supp}(\text{Syz}_i(M)) = \text{Spec}(R)$ for all $i > 0$.

**Example 5.5.** Let $0 \neq f$ be a nonunit power series in $k[[X_1, \ldots, X_n]]$ with $n > 2$ and let $R := k[[X_1, \ldots, X_n]]/fm$. Let $M$ be locally free and of infinite projective dimension. Then $\dim(\text{Syz}_n(M)) = \dim R$ for all $i > 2$. If $\lambda(M) < \infty$, then the same claim holds for $i = 1$ and also for $i = 2$.

In the above example we have $\text{Ass}(R) \neq \text{Assh}(R)$, because $m \in \text{Ass}(R) \setminus \text{Assh}(R)$. Also, $R$ is not reduced, e.g. $f^2 = 0$ and $f \neq 0$. 


Proof. We have $H^0_m(R) = fR$. Thus, $H^0_m(R) \neq 0$ and that $m H^0_m(R) = 0$. Consequently, $H^0_m(R)$ is a nonzero $k$-vector space. In view of Proposition 4.4, $\lambda(Syz_i(M)) = \infty$ for all $i > 0$. Note that

$$\min(R) = \{(f_i) : f_i \text{ is an irreducible component of } f\}.$$ 

Thus $R$ is equidimensional. We deduce from the above corollary that $\dim(Syz_i(M)) = \dim R$ for all $i > 2$. If $\lambda(M) < \infty$, then we use Proposition 4.4 to observe that $\dim(Syz_i(M)) = \dim R$ for all $i > 0$. 

Now, we deal with $R := \frac{k[[X_1, \ldots, X_n]]}{(f_1, \ldots, f_m)t^m}$ where $m > 1$. Recall from [5] Theorem 3.2 the following fact:

**Fact 5.6.** Let $I$ be an ideal in a normal local ring $(S, n)$ which is not contained in any height one prime. Set $R := S/I_n$. Let $M$ be finitely generated and non-free. Then $M$ has strictly increasing betti numbers.

The following is taken from [4] and plays a role in the sequel.

**Fact 5.7.** i) If $\beta_i(M) > \beta_{i-1}(M)$, then $\text{Supp}(Syz_{i+1}(M)) = \text{Spec}(R)$. In particular, $\dim(Syz_{i+1}(M)) = \dim R$. Suppose on the contradiction that there is a $p \in \text{Spec}(R) \setminus \text{Supp}(Syz_{i+1}(M))$. We may assume that $p \in \min(R)$. Thus $\ker(f_i)_p = Syz_{i+1}(M)_p = 0$. So $(f_i)_p : R^\beta_p \rightarrow R^\beta_{p-1}$ is injective. This contradicts $\beta_i(M) > \beta_{i-1}(M)$. Similarly:

ii) If $\beta_i(M) < \beta_{i-1}(M)$, then $\text{Supp}(Syz_{i-1}(M)) = \text{Spec}(R)$. In particular, $\dim(Syz_{i-1}(M)) = \dim R$.

**Remark 5.8.** Let us stress that the results are in the realm of commutative rings. We just present a funny point: a ring $A$ is said to have invariant basis number property if $A^n \simeq A^m$ implies that $n = m$ for all $n$ and $m$. There are rings without invariant basis number property. It may be worth to give a ring $A$ and an $A$-module $M$ such that $M^n \simeq M^m$ for all $m$ and $n$.

Now, we present the following example:

**Example 5.9.** Let $p$ be a height two prime ideal in $k[[X_1, \ldots, X_n]]$ and let $R := \frac{k[[X_1, \ldots, X_n]]}{pm^t}$ for some $t \geq 0$. Let $M$ be finitely generated and non-free. Then $\text{Supp}(Syz_i(M)) = \text{Spec}(R)$ for all $i > 1$. If $\lambda(M) < \infty$, then the same claim holds for $i = 1$.

**Proof.** If $t = 0$, then $R = k[[X_1, \ldots, X_n]]/p$ is an integral domain. In view of Fact 3.1, we get the claim. Thus, we may assume that $t > 0$. Set $I := pm^{t-1}$. Then $I$ is not contained in any height one prime ideal. In the light of Fact 5.6, $\beta_{i+1}(M) > \beta_i(M)$. Due to Fact 5.7, $\text{Spec}(Syz_i(M)) = \text{Spec}(R)$ for all $i > 1$. Without loss of generality we assume that $\dim R > 0$. Now if $\lambda(M) < \infty$, in view of $0 \rightarrow Syz_1(M) \rightarrow R^{\beta_1} \rightarrow M \rightarrow 0$, we get that $\dim(Syz_1(M)) = \dim R$. By Lemma 2.2, we have $\text{Supp}(Syz_i(M)) = \text{Spec}(R)$. 

We close the paper by the following specialized question:

**Question 5.10.** When is $\lambda(Syz_2(M)) < \infty$?

If such a thing happens, then the ring is 1-dimensional with a nonzero nilpotent.
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