OPTIMAL THRESHOLD STRATEGIES WITH CAPITAL INJECTIONS IN A SPECTRALLY NEGATIVE LÉVY RISK MODEL

MANMAN LI*
College of Economics and Business Administration, Chongqing University
Chongqing 400030, China

GEORGE YIN
Department of Mathematics, Wayne State University
MI, USA, 48202

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ABSTRACT. This paper focuses on optimal threshold strategies for a spectrally negative Lévy (SNL) risk process with capital injections and proportional transaction costs. Restricted to solvency constraint, our model requires the shareholders of dividends prevent ruin by injecting capitals. Value function of the firm is assumed to be an expected discounted total dividends less discounted capital injection. Under such a setup, we derive certain key identities in connection with value function of the firm of a maximum dividend rate. Under restricted dividend rates and capital injection, we give analytical description of the maximum value function of the firm and the optimal threshold strategy explicitly.

1. Introduction. Recently, much attention has been devoted to exploring the interactions of classical risk models and properties of Lévy processes. The main effort is to gain insights for a wider range of applications; see for example, the treatments of the barrier dividend problems Avram et al. [5], Loeffen [22, 23], Kyprianou et al. [15], and the threshold dividend policies Kyprianou and Loeffen [14]). In reference to the existing literature, this paper further reveals the analytical properties of scale functions for spectrally negative Lévy (SNL) processes to solve threshold dividend optimization problems.

Concerning the Sparre-Andersen models, a host of researchers have concentrated on the threshold strategies with a bounded dividend rate only (e.g., Jeamblanc and Shiryaev [12], Asmussen and Taksar [3], Asmussen [2], Gerber and Shiu [9, 10], Lin and Pavlova [21], Ng [26], Yin and Yuen [33]). Using such a strategy, dividends are paid at a constant rate whenever the modified risk process is above the threshold level $b > 0$, and no dividends are paid whenever the modified risk process is below $0$.

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* Corresponding author: Manman Li.
b. Thus, the risk process undergoes the so-called stochastic refraction; see Gerber and Shiu [9, 10], both a threshold dividend strategy and a linear barrier dividend strategy in Alberedeer et al [1].

One of the drawbacks of the bang-bang dividend strategies is that the risk process still has a positive probability of ruin. To overcome the difficulties, many researchers suggested to use capital injections to prevent ruin over unrestricted dividend rates recently. Two types of capital injection strategies have been considered in the literature, namely, forced bail outs (e.g., Harrison and Taylor [11], Avram et al. [5], Kulenko and Schmidt [13]), and optional bail outs (e.g., Løkka and Zervos [25], Avanzi et al. [4], Dai et al. [8]). For forced bail outs, the reserve process stays positive such that the insurance company never go bankrupt. While one is not forced to avoid the event of bankruptcy for the case of optional bail out. Indeed, financial institutions and insurance companies are restricted by regulations to maintain non-negative or even positive reserves at all times in order to operate. Capital injections provide a strategy that can be used by such institutions in meeting their regulatory requirements. Considering capital injections, there are a growing number of papers investigating the optimal dividend strategy under unrestricted/restricted dividend rates such as Zhou et al. [34], Zhu and Yang [35] (forced bail outs), Yao et al. [32] (optional bail outs) among others.

In this paper, we consider the optimal threshold strategies for SNL processes with “forced bail outs”. From a practical consideration, we assume that a proportional cost is incurred each time a dividend payment or a capital injection is made. The main difficulties that we face are analyzing the occupation times of intervals for the SNL processes under threshold strategies, as well as complicated calculations of the explicit key identities based on scale functions. With the maximum dividend rate, we derive certain key identities, which are generalization of [12, Lemma 3.1]. Using these identities, [5, Theorem 1] can be obtained as a special case of our results. We also show that the optimal threshold strategies exist for the restricted dividend strategies with capital injections. Moreover, value function of the firm can be expressed explicitly by use of key identities and scale functions. This paper was originally written in 2014 [20]. Thanks to one of the reviewers for letting us know the independent work of [29].

The rest of the paper is arranged as follows. Section 2 presents the problem setup and reviews of some existing results of SNL Processes. Certain key identities are obtained in Section 3, which are expressed in terms of scale functions. In Section 4, we derive the expected total discounted dividends and the total discounted injected capital in the threshold strategies, in terms of scale functions of Lévy processes. Then we apply these results in Section 5 to obtain optimal strategies for the restricted dividend problems with capital injections. The main effort in this section is to show the existence and to obtain explicit expressions of the value function and the optimal capital injections and threshold strategies.

2. Problem setup and notation.

2.1. Lévy insurance risk processes. Let $X = \{X_t, t \geq 0\}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a filtered probability space that satisfies the usual conditions. Denote the probability with $X_0 = x$ and the corresponding expectation by $\mathbb{P}_x$ and $\mathbb{E}_x$, respectively. The reader is referred to Sato [30] for more details. It is well known that the Laplace exponent is $\psi(\theta) = \log \mathbb{E}_0(e^{\theta X_1})$ for $\theta \geq 0$, and $\psi(\theta)$ is strictly convex on $(0, \infty)$ with $\lim_{\theta \to \infty} \psi(\theta) = +\infty$. Moreover, if we denote
Lemma 2.1. (Kyprianou [18].) These identities are to be used throughout the rest of the paper.

the right-inverse function of \( \psi \) by \( \Phi : [0, \infty) \to [\Phi(0), \infty) \). Denote the extended generator of the process \( X \) by \( \Gamma \), which acts on \( C^2 \) functions with compact support. Then

\[
\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + r f'(x) + \int_{-\infty}^{+\infty} [f(x + y) - f(x) - f'(x) y I_{|y|<1}] \nu(dy),
\]

(1)

where \( \nu \) is a measure on \( \mathbb{R} - \{0\} \) such that \( \int (1 + x^2) \nu(dx) < \infty \). Throughout the paper, we assume that \( X \) is spectrally negative if the measure \( \nu \) is carried by \( (-\infty, 0) \). That is, \( \nu(0, \infty) = 0 \). We exclude the trivial case of monotone paths.

Recall from [18] that for \( q \geq 0 \), there exits a function \( W^{(q)} : \mathbb{R} \to [0, \infty) \), called the \( q \)-scale function such that \( W^{(q)}(x) = 0 \) for all \( x < 0 \) that is continuous and increasing on \( (0, \infty) \) with Laplace transform

\[
\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q).
\]

(2)

For convenience, we write \( W \) instead of \( W^{(0)} \). Associated to the functions \( W^{(q)} \) are the functions \( Z^{(q)} : \mathbb{R} \to [1, \infty) \) defined by

\[
Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy
\]

for \( q \geq 0 \). The functions \( W^{(q)} \) and \( Z^{(q)} \) are collectively known as scale functions.

Let us recall some properties of \( W^{(q)}(x) \) known in the literature. If \( X \) has bounded variation, it takes the form \( X_t = ct - S_t \) for a subordinator \( S_t \) and a constant \( c = r - \int_0^1 y \nu(dy) \), and then it is straightforward to deduce that \( W^{(q)}(0) = 1/c \). Otherwise \( W^{(q)}(0) = 0 \) for the case of unbounded variation. In all cases, if \( X \) drifts to \( \infty \) then \( W(\infty) = 1/E_x(X_1) \). When \( X \) has unbounded variation or \( \nu \) has no atoms, for any \( q \geq 0 \), \( W^{(q)} \in C^2(0, \infty) \) (e.g., Kyprianou et al. [15], Lambert [19]). It was shown in Chan and Kyprianou [7] that when \( X \) has a Gaussian component \((\sigma > 0)\), then \( W^{(q)} \in C^2(2, \infty) \) with \( \frac{dW^{(q)}(x)}{dx} |_{x=0} = 2/\sigma^2 \).

The theorem below is a collection of known fluctuation identities; see [18, Chapter 8]. These identities are to be used throughout the rest of the paper.

**Lemma 2.1.** (Kyprianou [18].) Let \( X \) be a SNL Process and \( \tau_a^+ = \inf \{ t > 0 : X_t > a \} \), and \( \tau_0^- = \inf \{ t > 0 : X_t < 0 \} \).

(i) (Theorem 8.1) For each \( q \geq 0 \) and \( x \in \mathbb{R} \)

\[
E_x(e^{-q \tau_0^-} I_{\{\tau_0^- < \infty\}}) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x).
\]

(3)

(ii) (Theorem 8.1) For \( q \geq 0 \) and \( x \leq a \)

\[
E_x(e^{-q \tau_0^-} I_{\{\tau_0^- > \tau_a^+\}}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}.
\]

(4)

(iii) (Exercises 10.6) Let \( a > 0 \), \( x \in [0, a] \), \( q \geq 0 \) and \( f, g \) be positive, bounded, and measurable functions. Suppose that \( X \) is of bounded variation or \( f(0)g(0) = 0 \).

Then

\[
E_x(e^{-q \tau_0^-} f(X_{\tau_0^-}) g(X_{\tau_0^-} - I_{\{\tau_0^- < \tau_a^+\}}))
\]

\[
= \int_0^a \int_0^\infty f(y - \theta) g(y) \left\{ \frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} \nu(d\theta) dy.
\]

(5)
2.2. Under-reflected Lévy processes. We denote the Lévy process $X$ reflected at its past infimum $I$ by $\hat{Y} = X - I$, where $I_t = \inf_{0 \leq s \leq t} (X_s \wedge 0)$. By duality and the Wiener-Hopf factorization of $X$ (e.g., [6]), it follows that

$$\hat{Y}_e(q) \sim \exp(\Phi(q)),$$

where $e(q)$ is an independent random variable with parameter $q$. In view of (6), $W(q)(x)/W(q)'(x)$ is an increasing function on $(0, \infty)$ with $\lim_{x \to \infty} \frac{W(q)(x)}{W(q)'(x)} = \frac{1}{\Phi(q)}$. Furthermore, it was shown in Pistorius [28] that the Laplace transform of the entrance time $\hat{\tau}_b$ of the reflected process $\hat{Y}$ into $(b, \infty)$ can be expressed in terms of the functions $Z(q)$ and $W(q)$ as

$$\mathbb{E}_y[e^{-q\hat{\tau}_b}] = \frac{Z(q)(y)}{Z(q)(b)}, \quad 0 \leq y \leq b,$$

where $\hat{Y}_0 = y$ under $\mathbb{P}_y$.

2.3. Over-refracted Lévy processes. The Lévy process $X$ models the uncontrolled risk process of an insurance company. Let $\pi_b$ be a threshold strategy at level $b > 0$, which corresponds to subtracting a constant rate $f_b(t)$ from the increments of the Lévy process whenever it exceeds the level $b$. We confine ourselves to $0 < f_b(t) \leq \delta < E_x(X_t)$. Here $\delta$ is a maximum dividend rate. Denote by $L_t^{\pi_b} = \int_0^t I_{\pi_b} ds$ the total dividend paid out by the company up to time $t$ with $L_0^{\pi_b} = 0$, which is a non-decreasing left-continuous $\mathbb{F}$-adapted process. The controlled risk process under the strategy $\pi_b = \{L_t^{\pi_b}, t \geq 0\}$ is then given by

$$U_t^{\pi_b} = X_t - L_t^{\pi_b}, \quad t \geq 0,$$

with $X_0 = x$ and the corresponding ruin time is denoted by $\tau_b^- = \inf\{t > 0 : U_t^{\pi_b} < 0\}$. We also refer to $U_t^{\pi_b}$ as an over-refracted Lévy process.

Let $U^b = U^{\pi_b}$ and $L^b = L^{\pi_b}$ be the modified risk process and the total dividend process, respectively. Consider

$$\tau_a^+ := \inf\{t > 0 : U_t^a > a\}, \quad \tau_a^- := \inf\{t > 0 : U_t^a < a\}.$$  

As in [14], let $Y = \{Y_t := X_t - \delta t, t \geq 0\}$. Let $l^{\pi_b} = \delta$, then for $b \leq x \leq a$, we also have

$$\tau_a^+ := \inf\{t > 0 : Y_t > a\}, \quad \tau_b^- = \inf\{t > 0 : Y_t < b\}.$$  

For each $q \geq 0$, $W^{(q)}$ and $Z^{(q)}$ are the $q$-scale functions associated with $X$ and $\Psi^{(q)}$ and $\Phi^{(q)}$ are the $q$-scale functions associated with $Y$. We also extend $\Psi^{(q)}$ to the whole real line by setting $\Psi^{(q)}(y) = 0$ for $y < 0$. Denote the probabilities of $Y$ and the associated expectation by $\{\mathbb{P}_x : x \in R\}$ and $\{\mathbb{E}_x : x \in R\}$, respectively. For any $x, a \geq 0$, under either the law $\mathbb{P}_x$ or $\mathbb{P}_x$, $\tau_a^+$ and $\tau_a^-$ are the first passage into $(a, \infty)$ and $(-\infty, a)$, respectively. Moreover $\varphi$ is defined as the right inverse of the Laplace exponent of $Y$ so that

$$\varphi(q) = \sup\{\theta \geq 0 : \psi(\theta) - \delta \theta = q\}.$$  

2.4. Over-refracted and under-reflected Lévy processes. Here we consider the situation where the shareholders of the dividends are required to inject capital into the insurance company to prevent bankruptcy. Let $\pi_{0,b}$ be a bail-out threshold strategy, which consists of imposing over-refracted level at $b$ and under-reflected barrier at $0$, respectively. Denote by $R^{\pi_{0,b}} = \{R_t^{\pi_{0,b}}, t \geq 0\}$ a right-continuous
The controlled risk process
The total dividends
The total injected capital
Threshold strategy level

Figure 1. The modified Lévy risk process

process describing the total injected capital. Under policy \( \pi_{0,b} = \{ L^{0,b}, R^{0,b} \} \), the controlled risk process with initial reserve \( x \geq 0 \) satisfies

\[
V_t^{\pi_{0,b}} = X_t - L_t^{0,b} + R_t^{0,b}, \quad t \geq 0,
\]

where \( dR_t^{\pi_{0,b}} \) is a random measure supported by \( \{ V_t^{\pi_{0,b}} \} \) such that

\[
R_t^{\pi_{0,b}} = - \inf_{0 \leq s \leq t} \{ U_t^{\pi_{0,b}} \wedge 0 \}.
\]

A dividend policy \( \pi_{0,b} \) is admissible, \( \pi_{0,b} \in \Pi \), if it is a pair of non-decreasing \( \mathcal{F} \)-adapted processes with \( R_{0}^{\pi_{0,b}} = L_{0}^{\pi_{0,b}} = 0 \), absolutely continuous dividend rate in \( [0, \delta] \), \( V_t^{\pi_{0,b}} \) is non-negative for \( t > 0 \) and

\[
\int_0^\infty e^{-qt} dR_t^{\pi_{0,b}} < \infty, \quad \mathbb{P}_x - \text{a.s.} \tag{12}
\]

Define the first passage times by

\[
k^+_a := \inf \{ t > 0 : V_t^b > a \}, \quad k^-_a := \inf \{ t > 0 : V_t^b < a \}.
\]

For the optimization problem of absolutely continuous dividend and capital injection, the value function associated with the strategy \( \pi_{0,b} \in \Pi \) is then given by

\[
\varpi_{\pi_{0,b}} = \mathbb{E}_x [ \beta \int_0^\infty e^{-qt} dL_t^{\pi_{0,b}} - \eta \int_0^\infty e^{-qt} dR_t^{\pi_{0,b}} ], \tag{13}
\]

where \( x \geq 0 \) is the initial reserve, \( 0 < \beta \leq 1 \) is the net dividend factor, and \( \eta \geq 1 \) is the cost per unit of injected capital.

Remark 1. For \( \beta = 1 \) or \( \eta = 1 \), there doesn’t exist any proportional transaction cost, when shareholders receive dividends or inject capital. For \( \beta = \eta = 1 \), this can lead to a situation where a linear drift at rate \( \delta > 0 \) is subtracted from the increments of a Lévy process immediately. This situation is a direct result of the later Lemma 5.2.

The associated objective function is given by

\[
\varpi_*(x) = \sup_{\pi_{0,b} \in \Pi} \varpi_{\pi_{0,b}}(x). \tag{14}
\]
To ensure that the value function is finite and to avoid degeneracy, we assume that $E_x[X_1] > -\infty$, $q > 0$. We need the following hypotheses that will be used throughout the rest of the paper.

\begin{itemize}
\item[(H1)] $0 < l_{\alpha,b} \leq \delta < E(X_1)$, $\psi'(0+) > -\infty$;
\item[(H2)] $X$ has bounded variation, and $\nu$ is absolutely continuous w.r.t. the Lebesgue measure.
\end{itemize}

\textbf{Remark 2.} According to [14], $l_{\alpha,b} \leq \delta < E(X_1)$ ensures that there exists a unique strong solution (in the a.s. sense) to (8) and the path of $U^{\pi_0}$ is not monotone.

\textbf{Remark 3.} (H2) assures that $W^{(q)}(x)$ belongs to $C^1(0,\infty)$ at least, then $\mathcal{W}^{(q)}_\nu \in C^1(0,\infty)$ is obviously. If $\nu$ is absolutely continuous w.r.t. the Lebesgue measure, then absolutely continuity of the resolvent measure $u^\nu_0(x,y)$ is satisfied.

We present some properties of the value function.

\textbf{Lemma 2.2.} The function $\tau_\nu(x)$ is bounded by $\frac{\beta \delta}{q}$, increasing, and concave.

\textbf{Proof.} Note that $\tau_\nu(x)$ being increasing and $\tau_\nu(x) \leq \int_0^\infty \beta \delta e^{-q}dt = \frac{\beta \delta}{q}$ is clear. In view of Kulenko and Schmidli [13], since $X_t$ is a risk model with independent increments, the function $\tau_\nu(x)$ is concave. \hfill $\square$

\textbf{3. Key identities.} To proceed, we focus on the role played by scale functions in connection with the value function of over-refracted and under-reflected SNL processes.

It follows from (11) that $L^b$ and $R^0$ are some increasing and adapted processes. For $t \geq 0$, $V^b_t$ can then be expressed as

$$V^b_t = X_t - L^b_t + R^0_t,$$

where the supports of the Stieltjes measures $dL^b_t$ and $dR^0_t$ are included in the closures of the sets $\{t : V^b_t \geq b\}$ and $\{t : V^b_t = 0\}$, respectively. We can conclude the strong Markovian property of $V^b$ similar to $U^b$ in [18] from the following theorem.

\textbf{Theorem 3.1.} For the bounded variation case, there exists a unique strong solution to (15) for $l_{\alpha,b} = \delta$, and the solution is a strong Markov process.

\textbf{Proof.} Similarly to Avram et al. [5], we first give the path construction of $U^b$, $L^b$, $R^0$ and $V^b$. It is divided into two cases. Set $k = 1$.

\begin{itemize}
\item \textbf{Case (i)} $X_0 \in [0,b]$:
\item \textbf{Step 1.} Set $T = \tau^-_0 \wedge \tau^+_b$, $U^b_T = X_T$. For $0 \leq t < T$, set $L^b_t = R^0_t = 0$ and $V^b_t = X_t$. If $X_T \leq 0$, set $S_1 = T$, $T_1 = 0$, $\xi := X_{S_1}$, $k = k + 1$, and go to Step 3; else set $T_1 = T$, $S_1 = 0$, $L^b_{T_1} = 0$ and $V^b_{T_1} = b$, $k = k + 1$ and go to Step 2.
\item \textbf{Step 2.} Set $Q_t = X_t - X_{T_{k-1}}$. For $t \geq T_{k-1}$, put

$$U^b_t = b + Q_t - \delta(t - T_{k-1}).$$

For $T_{k-1} \leq t < S_k := \inf\{t \geq T_{k-1} : U^b_t < b\}$, set $L^b_t = L^b_{T_{k-1}} + \delta(t - T_{k-1})$, $R^0_t = R^0_{T_{k-1}}$, $V^b_t = U^b_t = b + Q_t - (L^b_t - L^b_{T_{k-1}})$. If $0 \leq U_{S_k} < b$, $k = k + 1$, and go to Step 4. Else set $\xi = U^b_{S_k}$, $k = k + 1$, and go to Step 3.
\item \textbf{Step 3.} Set $Q_t = X_t - X_{S_{k-1}}$. For $t \geq S_{k-1}$, put

$$U^b_t = Q_t.$$
\end{itemize}
For $S_{k-1} \leq t < T_k := \inf\{t \geq S_{k-1} : U_t^b - \inf_{s \leq t} \{U_s^b \wedge 0\} \geq b\}$, set

$$R_t^0 = R_{S_{k-1}}^0 - \xi - \inf_{s \leq t} \{Q_s \wedge 0\}, V_t^b = U_t^b - \inf_{s \leq t} \{U_s^b \wedge 0\} = Q_t + R_t^0 - R_{S_{k-1}}^0,$$

and let $L_t^b = L_{S_{k-1}}^b$. Set $k = k + 1$ and go to Step 2.

**Step 4.** Since $U_{S_{k-1}}^b = b + X_{S_{k-1}} - X_{T_{k-2}} - \delta(S_{k-1} - T_{k-2})$, set

$$\tau_{tk}^b = \inf\{t > S_{k-1} : U_{S_{k-1}}^b + X_t - X_{S_{k-1}} < 0\},$$

$$\tau_{tk}^b = \inf\{t > S_{k-1} : U_{S_{k-1}}^b + X_t - X_{S_{k-1}} < 0\},$$

$A_k = \tau_{tk}^- \wedge \tau_{tk}^+.$

For $S_{k-1} \leq t < A_k$, $V_{S_{k-1}}^b = U_{S_{k-1}}^b + X_t - X_{S_{k-1}} = b + X_t - X_{T_{k-2}} - \delta(S_{k-1} - T_{k-2})$, $L_t^b = L_{S_{k-1}}^b$, $R_t^0 = R_{S_{k-1}}^0$.

If $b + X_{A_k} - X_{T_{k-2}} - \delta(S_{k-1} - T_{k-2}) < 0$, set $S_k = A_k$, $L_t^b = L_{S_{k-1}}^b$, $V_{A_k}^b = U_{A_k}^b = b + X_{A_k} - X_{T_{k-2}} - \delta(S_{k-1} - T_{k-2})$, $k = k + 1$, and go to Step 3. Else set $T_k = A_k$, $L_{T_k}^b = L_{S_{k-1}}^b$, $R_{T_k}^0 = R_{S_{k-1}}^0$, and $V_{T_k}^b = b$, $k = k + 1$, and go to Step 2.

**Case (ii) $X_0 > b$:** The above construction can be modified as: in Step 1 set $V_0^b = x$ and $T_1 = 0$, in Step 2 set $U_t^b = x + Q_t - \delta(t - T_1)$ for $T_1 \leq t < S_2$, and repeat the rest of the construction.

We have defined the process $U_t^b$ by (16) in Step 2 $T_k \leq t < S_k$, (17) in Step 3 for $S_k \leq t < T_{k+1}$ respectively, $k = 1, 2, 3, \ldots$. By [27], [5] and [14], the SDE (16), (17) has a unique strong solution respectively. Thus the capital injection process $R_t^0 = -\inf_{0 \leq s \leq t} (U_s^b \wedge 0)$ can be uniquely determined by $U_t^b$, which ensure the uniqueness of $V_t^b$.

We can verify that the process $V_t^b$ constructed in this way satisfies $V_t^b \in [0, \infty)$ and $L_t^b$ and $R_t^0$ are processes with the required properties such that (15) holds. Standard arguments lead to the existence of a unique strong solution to (15) for $X_0 = x$, resulting in that $V_t^b$ is a strong Markov process. We suppose that $T$ is a stopping time w.r.t. the natural filtration generated by $U_t^b$, and define $\tilde{V}_t^b$ whose dynamics are those of $\{V_t^b : t \leq T\}$ starting from $x$. Starting from $V_t^b$, the unique solution $\tilde{V}_t^b$ on $[T, \infty)$ is driven by the process $\tilde{U}_t^b = \{U_{T+s}^b - U_T^b, s \geq 0\}$. By construction, on $\{T < \infty\}$, the dependency of $\{\tilde{V}_t^b : t \geq T\}$ on $\{\tilde{V}_t^b : t \leq T\}$ occurs only when $\tilde{V}_t^b = V_t^b$. Then we have for $t > 0$

$$\tilde{V}_{T+t}^b = \tilde{V}_T^b + \tilde{V}_T^{b} - \inf_{0 \leq s \leq t} (\tilde{U}_s^b \wedge 0) = U_T^b - \inf_{0 \leq s \leq T} (U_s^b \wedge 0) + U_{T+t}^b - U_T^b - \inf_{0 \leq s \leq t} ((U_{T+s}^b - U_T^b) \wedge 0) = U_{T+t}^b - \inf_{0 \leq s \leq T+t} (U_s^b \wedge 0),$$

which show that $\tilde{V}_t^b$ solves (15) starting from $x$. Following from the uniqueness of $\tilde{V}_t^b$ to (15) a.s., we obtain the strong Markov property of $\tilde{V}_t^b$. □

**Definition 3.2.** For $q, b \geq 0$, we define $H_b^q$ to be a function space including positive and measurable functions $h^{(q)}(x)$ that satisfy

$$E_x[e^{-qT} h^{(q)}(X_{T_x}^b \wedge 1_{\{T_x^b < \tau_x^+\}})] = h^{(q)}(x) - \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)} h^{(q)}(a)$$ (18)

The next result is obtained for over-refracted and under-reflected SNL processes $V$ defined by (11), which is a generalization of [24, Lemma 2.1].
Lemma 3.3. Let \( q, b \geq 0 \) and \( h^{(q)} \in H^{(q)}_b \cap C^1(0, \infty) \) on \( \mathbb{R} \). Give a SNL process \( X \) with the following assumptions:

\[
\int_0^{\infty} e^{-\lambda x} h^{(q)}(x) dx < \infty \text{ for } \lambda > \varphi(q).
\]  

(19)

and \( I^{\varphi(q)} = \delta \) hold, then for all \( b \leq x \leq a \),

\[
E_x[e^{-q_{k_x} h^{(q)}(V_{k_x}^-)} I_{\{k_x^- < k_x^+\}}] = h^{(q)}(x) + \frac{W^{(q)}(q-\omega)_x}{W^{(q)}(x)} h^{(q)}(a) + \frac{1}{W^{(q)}(a-b)} (h^{(q)}(a) + \delta) I_{\{k_x^- < k_x^+\}}(y)dy
\]  

(20)

Proof. Since \( \{Y_i, t < \tau_b^-\} \) under \( P_x \) and \( \{V_i, t < k_b^-\} \) under \( P_x \) have the same law when \( x \geq b \), we have

\[
E_x[e^{-q_{k_x} h^{(q)}(V_{k_x}^-)} I_{\{k_x^- < k_x^+\}}] = E_x[e^{-q_{k_x} h^{(q)}(Y_{\tau_b}^-)} I_{\{k_b^- < k_b^+\}}].
\]

Similar to the proof of [14, Theorem 16], we take Laplace transforms on both sides of the equations. Now setting \( x = b \) and using (5), we have

\[
E_b[e^{-q_{k_b} h^{(q)}(X_{\tau_b}^-)} I_{\{\tau_b^- < \tau_b^+\}}] = h^{(q)}(b) - \frac{W^{(q)}(0)}{W^{(q)}(a-b)} h^{(q)}(a)
\]  

(21)

Then it follows from \( W^{(q)}(0) = \frac{1}{c} \) that

\[
I_1 = \int_0^{a-b} \int_y^{\infty} h^{(q)}(y-x)W^{(q)}(b-y-x) \nu(dx)dy = cW^{(q)}(a-b) h^{(q)}(b) - h^{(q)}(a).
\]  

(22)

We also set \( a = x \) in the above identity and take Laplace transforms from \( b \) to \( \infty \) of both sides of (22). By Fubini’s theorem one can show that

\[
\int_0^{\infty} \int_y^{\infty} e^{-\lambda y} h^{(q)}(y-x) \nu(dx)dy = ch^{(q)}(b) - (\psi(\lambda) - q)e^{\lambda b} \int_b^{\infty} e^{-\lambda x} h^{(q)}(x)dx
\]  

(23)

for \( \lambda > \Phi(q) \). In view of [14, Theorem 16], we need to show that for \( q \geq 0 \) and \( x \in [b, a] \),

\[
I_2(x) = \int_0^{\infty} \int_y^{\infty} h^{(q)}(y-x)W^{(q)}(x+y-x) \nu(dx)dy
\]  

\[
= -h^{(q)}(x) + (c - \delta) h^{(q)}(b)W^{(q)}(b-x) - \delta \int_b^{\infty} W^{(q)}(x-y)h^{(q)}(y)dy
\]  

(24)

Let \( \lambda > \varphi(q) \). Indeed (24) follows by taking Laplace transforms on both sides in \( x \), that is, by virtue of Fubini’s theorem,

\[
\int_b^{\infty} e^{-\lambda x} I_2(x)dx
\]  

\[
= \frac{e^{-\lambda b}}{\psi(\lambda) - \delta \lambda - q} \left(ch^{(q)}(b) - (\psi(\lambda) - q)e^{\lambda b} \int_b^{\infty} e^{-\lambda x} h^{(q)}(x)dx\right)
\]  

(25)
Hence (24) holds for almost every $x \geq b$. By the continuity of (24) in $x$, we have that (24) holds for all $x \geq b$, and that (20) can be obtained by use of (5) and (24). Then Lemma 3.3 holds for all $b \leq x \leq a$. \hfill \Box

**Lemma 3.4.** For any $q > 0$ and $b \leq x \leq a$, the conclusion of Lemma 3.3 holds for (i) $h^q_1(x) = W^q(x)$, (ii) $h^q_2(x) = Z^q(x)$, (iii) $h^q_3(x) = \overline{Z^q}(x) + \frac{\psi'(0+)}{q}$, where $\overline{Z^q}(x) = \int_0^x Z^q(y)dy$.

**Proof.** In accordance with [4], $e^{-q(t \wedge \tau^a_b)}h^q(X_{t \wedge \tau^a_b})$ is a martingale, where

$$h^q_1(x) = W^q(x), \quad h^q_2(x) = Z^q(x), \quad h^q_3(x) = \frac{\psi'(0+)}{q}.$$ 

Then using the strong Markov property, (4) and the lack of upward jumps, for $b \leq x \leq a$, $t \to e^{-q(t \wedge \tau^a_b \wedge \tau^b_a)}h^q(X_{t \wedge \tau^a_b \wedge \tau^b_a})$ is also a $P_x$-martingale. By taking expectations and the limit as $t \to \infty$, (20) holds. We omit the remaining details, which are similar to the proof of [24, Lemma 2.2]. \hfill \Box

**Corollary 1.** For $q > 0$, $x \geq b > 0$, and $\psi'(0+) > -\infty$, denote

$$h_1(x) = E_x[e^{-\psi(t \wedge \tau^a_b)}W^q(Y_{t \wedge \tau^a_b})I_{\{\tau^a_b < \infty\}}],$$ 

$$h_2(x) = E_x[e^{-\psi(t \wedge \tau^a_b)}(Z^q(Y_{t \wedge \tau^a_b}) + \frac{\psi'(0+)}{q})I_{\{\tau^a_b < \infty\}}].$$  

If $W^q \in C^1(0, \infty)$, we have

$$h_1(x) = Z^q(x) + \frac{\psi'(0+)}{q} \int_b^x W^q(x-y)W^q(y)dy + q \int_0^x \psi'(y)W^q(y)dy,$$

$$h_2(x) = Z^q(x) + \frac{\psi'(0+)}{q} \int_b^x W^q(x-y)Z^q(y)dy - q \int_0^x \psi'(y)W^q(y)dy - \int_0^x \psi'(x-y)W^q(y)dy.$$

**Proof.** We prove the corollary by taking $a \to \infty$ in Lemma 3.4. Let us use the following identity (cf [14], Section 8) for $q, a > 0$

$$\delta \int_0^a W^q(a-y)W^q(y)dy = \int_0^a W^q(y)dy - \int_0^a W^q(y)dy,$$

and (see [18], Chapter 8)

$$\lim_{a \to \infty} \frac{W^q(a)}{W^q(a)^t} = \frac{1}{\psi(q)}, \quad W^q(a) = e^{\psi(a)q}W^q(q),$$

where $W^q(a)$ is the 0-scale function of SNL processes with Laplace exponent $\psi(q)(\theta) = \psi(q) + \delta(q) + q$. Further $\lim_{a \to \infty} W^q(a) < \infty$ except if simultaneously $q = 0$ and $\psi'(0) = \delta$. (i). Recalling (28) and by dominated convergence theorem,

$$\lim_{a \to \infty} \frac{Z^q(a) + \delta \int_b^a W^q(a-y)W^q(y)dy}{W^q(a-b)} = \lim_{a \to \infty} \left[ \frac{Z^q(a)}{W^q(a-b)} - \delta q e^\psi(a) \int_0^b e^{-\psi(y)}W^q(y)dy \right]$$

$$= \lim_{a \to \infty} \left[ \int_b^a \frac{Z^q(x)}{W^q(x-b)}dx - \delta q e^\psi(a) \int_0^b e^{-\psi(y)}W^q(y)dy \right].$$
The last line follows from l’Hôpital’s rule, (29), and the Laplace transform of $W^{(q)}$,

\[
\lim_{a \to \infty} \frac{Z^{(q)}(a)}{W^{(q)}(a-b)} = \lim_{a \to \infty} \frac{qW^{(q)}(a)}{qW^{(q)}(a-b) + qW^{(q)}(a-b)W^{(q)}(a-b)} = \frac{q \varphi(q) \delta}{\varphi(q)},
\]

which proves (26) in Corollary 1. (ii). Similarly, we obtain

\[
\lim_{a \to \infty} \frac{Z^{(q)}(a) + \frac{\varphi'(0+)}{2} + \delta \int_{0}^{a} W^{(q)}(a-y)Z^{(q)}(y)dy - \delta \int_{0}^{b} W^{(q)}(a-y)Z^{(q)}(y)dy - \delta \int_{0}^{b} W^{(q)}(b-y)Z^{(q)}(y)dy}{qW^{(q)}(a-b)} = \frac{q \varphi(q) \delta}{\varphi(q)}.
\]

In the last line, we used l’Hôpital’s rule, (29), and (30),

\[
\lim_{a \to \infty} \frac{Z^{(q)}(a)}{W^{(q)}(a-b)} = \lim_{a \to \infty} \frac{qW^{(q)}(a-b)}{qW^{(q)}(a-b) + qW^{(q)}(a-b)W^{(q)}(a-b)} = \frac{q \varphi(q) \delta}{\varphi(q)}.
\]

This proves (27) in Corollary 1. 

\[\Box\]

4. SNL processes with threshold strategies and capital injections. We are in a position to compute the value function corresponding to the strategy $\pi_{0,b}$. The strategy $\pi_{0,b} = \{L^b_t, R^b_t, t \geq 0\}$ consists of imposing over-refracted level $L^b$ and under-reflecting barrier $R^b$ at $b$ and $0$ respectively, which, in words, is the strategy where dividends are paid out at the maximum rate $\delta$ when the controlled process is above $b$ and at the minimum rate $0$ when below $b$. With the initial reserve $X_0 = x \in [0, b]$ holds, the risk process $V^b_t := V^{x, \pi_{0,b}}_t$ (defined in (15) with hypotheses (H1) and (H2)) moves as a Lévy process whilst it is inside $[0, b]$ but each time it attempts to down-cross 0 or up-cross $b$ with positive probability. We consider the discounted expectations of $L^b$ and $R^b$ with an over-refracted and under-reflected process, in which shareholders are divided by a threshold strategy with maximum dividend rate $\delta$.

**Theorem 4.1.** For the bounded variation risk process $V^b_t$, let $b > 0$, $\pi^{x,\delta} = \delta$, and $q > 0$, we define $f(x) = E_x[\int_{0}^{\infty} e^{-\vartheta t} dL^b_t]$, $g(x) = E_x[\int_{0}^{\infty} e^{-\vartheta t} dR^b_t]$. Then it holds that

\[
f(x) = \begin{cases} 
\frac{Z^{(q)}(x)}{q \varphi(q)} \int_{0}^{h_1(x,b)} e^{-\vartheta t} W^{(q)}(y+b)dy, & 0 \leq x \leq b, \\
\frac{Z^{(q)}(b)}{q \varphi(q)} \int_{0}^{h_1(x,b)} e^{-\vartheta t} W^{(q)}(y+b)dy + \frac{q}{\varphi(q)} \psi_0(x-b), & x > b.
\end{cases}
\]

\[
g(x) = \begin{cases} 
\frac{Z^{(q)}(x)}{q \varphi(q)} \int_{0}^{h_2(x,b)} e^{-\vartheta t} W^{(q)}(y+b)dy + q - (Z^{(q)}(x) + \frac{\psi_0(0+)}{q}), & 0 \leq x \leq b, \\
\frac{Z^{(q)}(b)}{q \varphi(q)} \int_{0}^{h_2(x,b)} e^{-\vartheta t} W^{(q)}(y+b)dy + q - h_2(x,b), & x > b,
\end{cases}
\]

where $h_1(x,b)$ and $h_2(x,b)$ is defined by (26), (27), respectively.
Proof. **Step 1.** We first prove Eq. (31). Denote $\tau'_b := \inf \{ t \geq 0 : V^b_t = a \}$ for the first hitting time of \{a\}. Applying the strong Markov property of $V^b$ at $\tau'_b$, we then have

$$f(x) = \mathbb{E}_x [\int_0^{\tau'_b} e^{-qt} dL^b_t] + \mathbb{E}_x [e^{-q\tau'_b}] f(0).$$

(33)

Since $L^b$ does not increase until $V^b$ reaches the level $b$, we find that $f(0) = \mathbb{E}_0 [e^{-q\tau'_b}] f(b) = \frac{f(b)}{Z^{(q)}(b)}$ by use of (7). Then for $0 \leq x \leq b$, we have

$$f(x) = \frac{Z^{(q)}(x)}{Z^{(q)}(b)} f(b),$$

(34)

in view of (7) and the fact that \{V^b_t, t \leq \tau'_b, V^b_0 = x \} has the same law as \{\hat{Y}, t \leq \hat{\tau}_b, \hat{Y}_0 = x \}. Inserting formulas (ii) of Theorem 5, (10.25) in [14] into (33) results in

$$f(x) = \frac{W^{(q)}(x)}{\varphi(q) \int_0^\infty e^{-\varphi(q) y} W^{(q)}(y+b) dy} + \frac{f(b)}{Z^{(q)}(b)} \left[ Z^{(q)}(x) - W^{(q)}(x) \right] \frac{q}{\varphi(q)} \int_0^\infty e^{-\varphi(q) y} W^{(q)}(y+b) dy.$$

(35)

As (35) remains valid for $x = b$, we have $f(b) = \frac{Z^{(q)}(b)}{\varphi(q) \int_0^\infty e^{-\varphi(q) y} W^{(q)}(y+b) dy}$. Inserting $f(b)$ into (35) yields Eq. (31) when $0 \leq x \leq b$. For $x > b$, by using the strong Markov property of $V^b$, (3), and (34), we derive that

$$f(x) = \mathbb{E}_x [\delta \int_0^{\tau'_{b_+}} e^{-qt} dt + \delta \int_{\tau'_{b_+}}^\infty e^{-qt} I_{\{V^b_t > b\}} dt]$$

$$= \frac{\delta}{q} \left( 1 - Z^{(q)}(x-b) + \frac{q}{\varphi(q)} \mathbb{W}^{(q)}(x-b) \right) + \mathbb{E}_x (e^{-q\tau'_b} f(Y^{\tau'_b})).$$

(36)

Here we also used the fact that 0 is irregular for $(-\infty, 0)$ for $\hat{Y}$ in (36). By the boundedness of $Y^{\tau'_b} \leq b$ and the increasing property of $Z^{(q)}$,

$$\mathbb{E}_x (e^{-q\tau'_b} f(Y^{\tau'_b})) = \frac{f(b)}{Z^{(q)}(b)} \mathbb{E}_x (e^{-q\tau'_b} Z^{(q)}(Y^{\tau'_b}))$$

$$= \frac{f(b)}{Z^{(q)}(b)} \mathbb{E}_x (e^{-q\tau'_b} Z^{(q)}(Y^{\tau'_b}) I_{\{\tau'_b < \infty\}}).$$

Then inserting (26) into (36) leads to Eq. (31).

**Step 2.** Now we prove the remaining Eq. (32). Applying the strong Markov property of $V^b$ at $\tau'_b$ yields that for $0 \leq x \leq b$

$$g(x) = \mathbb{E}_x [\int_0^{\tau'_b} e^{-qt} dR^0_t] + \mathbb{E}_x [e^{-q\tau'_b}] g(b).$$

(37)

Using the argument of [5, Theorem 1], we get the identity

$$-\mathbb{E}_x [\int_0^{\tau'_b} e^{-qt} dR^0_t] = Z^{(q)}(x) + \frac{\psi'(0+)}{q} - \frac{Z^{(q)}(x)}{Z^{(q)}(b)} (Z^{(q)}(b) + \frac{\psi'(0+)}{q}).$$

(38)

Inserting (7) and (38) into (37) results in the equation

$$g(x) = -Z^{(q)}(x) - \frac{\psi'(0+)}{q} + \frac{Z^{(q)}(x)}{Z^{(q)}(b)} (Z^{(q)}(b) + \frac{\psi'(0+)}{q}) + \frac{Z^{(q)}(x)}{Z^{(q)}(b)} g(b).$$

(39)
Next, let \( x > b \), and by using the strong Markov property, (39), we have

\[
g(x) = \mathbb{E}_x[e^{-\eta x} g(Y_{\tau_b^-})]
\]

\[
= \frac{Z^{(q)}(x) + \psi'(0+)}{Z^{\varphi(q)}(b)} + g(b) \mathbb{E}_x[e^{-\eta x} Z^{(q)}(Y_{\tau_b^-}) I_{\{\tau_b^- < \infty\}}]
\]

\[
- \mathbb{E}_x[e^{-\eta x} Z^{(q)}(Y_{\tau_b^-}) + \frac{\psi'(0+)}{q} I_{\{\tau_b^- < \infty\}}],
\]

where we also used the fact that \( \overline{Z}^{(q)}(Y_{\tau_b^-}) \leq \overline{Z}^{(q)}(b) \) and \( Z^{(q)}(Y_{\tau_b^-}) \leq Z^{(q)}(b) \). In particular, we suppose that \( X_t \) has bounded variation. Letting \( x = b \) and solving for \( g(b) \), recalling that \( \mathbb{V}(q)(0) = \frac{1}{c^2} \), (26) and (27), we get

\[
g(b) = \frac{Z^{(q)}(x) Z^{(q)}(b) + \psi'(0+)}{q \varphi(q)} + \frac{Z^{(q)}(x)}{\varphi(q)} - \frac{Z^{(q)}(b)}{\varphi(q)} - \frac{\psi'(0+)}{q}.
\]

Taking account of (41), we may now write for \( x \leq b \)

\[
g(x) = \frac{Z^{(q)}(x) Z^{(q)}(b) + \psi'(0+)}{q \varphi(q)} + \frac{Z^{(q)}(x)}{\varphi(q)} - \frac{Z^{(q)}(b)}{\varphi(q)} - \frac{\psi'(0+)}{q}.
\]

Substituting (41) back into (39) results in (32) for \( x > b \), which finishes the proof. \( \Box \)

### 4.1 Asymptotic analysis

In this subsection, we consider the case of unrestricted dividend rate with \( \delta \to \infty \). Then the dividend strategy becomes a barrier strategy, which has been discussed in [5]. Below is a restatement of [5, Theorem 1], which becomes a direct consequence of our Theorem 4.1.

**Corollary 2.** (Avram et al. [5]). Let \( \delta \to \infty \), the dividend problem (13) becomes unrestricted dividend case. For \( q > 0 \) it holds that

\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\eta x} dL^q_b \right] = \frac{Z^{(q)}(x)}{q W^{(q)}(b)}, \quad 0 \leq x \leq b,
\]

\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\eta x} dR^q_b \right] = -\left( \overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q} \right) + \frac{Z^{(q)}(b)}{q W^{(q)}(b)} Z^{(q)}(x), \quad 0 \leq x \leq b.
\]

**Proof.** The maximum dividend rate \( \delta \to \infty \) results in a barrier strategy, and the SNL process \( V^b \) always stays in \([0, b]\). Then we assume \( x \in [0, b] \). By comparing Theorem 4.1 with [4, Theorem 1], we need only prove

\[
\lim_{\delta \to \infty} \varphi(q) \int_0^\infty e^{-\varphi(q)v} W^{(q)}(y + b) dy = W^{(q)}(b), \quad \lim_{\delta \to \infty} q/\varphi(q) = 0.
\]

Recalling the definition of \( \varphi(q) \) in (10), \( \psi(\varphi(q)) \to \infty \) for \( \delta \to \infty \), then \( \varphi(q) \to \infty \) from the convexity of \( \psi(x) \). The second limit is obvious in (43). To obtain the first limit in (43), we make use of the fact that \( W^{(q)}(x) / W^{(q)}(x) \) is increasing for \( q, x \geq 0 \). Then by use of (2), (10), and partial integration, we have

\[
\varphi(q) \int_0^\infty e^{-\varphi(q)y} W^{(q)}(y + b) dy = \int_0^\infty e^{-\varphi(q)y} W^{(q)}(y + b) dy + W^{(q)}(b) \leq \frac{W^{(q)}(b)}{\varphi(q)} \int_0^\infty e^{-\varphi(q)y} W^{(q)}(y + b) dy + W^{(q)}(b).
\]

Taking \( \delta \to \infty \) in the above equation, the first limit in (43) holds. Then the assertion follows for \( x \in [0, b] \). \( \Box \)

### 5. Optimal threshold strategies

In this section, for the bounded variation case, we investigate the existence and explicit analytical expression of the optimal problem (14) by optimal absolutely continuous dividend with capital injection. Our method draws on classical optimal control literature, such as Jeanblanc and Shiryaev [12] and Asmussen and Taksar [3], both of which deal with the dividend problem of the classical threshold type in a Brownian motion setting.
5.1. Threshold dividend strategy with capital injection. From Theorem 4.1, the value function corresponding to the strategy \( \pi_{0,b} \) of putting reflecting barrier at the level 0 and refracting level at \( b \geq 0 \) with \( \pi_{0,b} = \delta \) is given by \( \pi_{\pi_{0,b}} = \pi_b \), where

\[
\begin{aligned}
\pi_b(x) &= \begin{cases} 
\frac{Z^{(q)}(x)(\beta - \eta Z^{(q)}(b))}{q \varphi(q)} \int_0^\infty e^{-\varphi(q)Z^{(q)}(y+b)dy} + \eta \big( (Z^{(q)}(x) + \psi'(0^+)) \big), & 0 \leq x \leq b, \\
\frac{h_1(x,b)}{\varphi(q)} \left( \int_0^\infty e^{-\varphi(q)W(q)(y+b)dy} - \eta q \right) + \eta h_2(x,b) + \frac{\beta \delta}{q} \left( 1 - Z^{(q)}(x-b) - \frac{Z^{(q)}(x-b)}{\varphi(q)} \right), & x > b,
\end{cases}
\end{aligned}
\]

and \( h_1(x,b), h_2(x,b) \) is defined by (26), (27) for \( x > b \) respectively. Particularly, for initial value \( x = 0 \), the value function keeping the risk process refracted at zero is written by

\[
\pi_0(0) = \frac{\delta}{q} \left( \beta - \eta \right) + \eta \left( \frac{\psi'(0^+)}{q} - \frac{1}{\varphi(q)} \right),
\]

and \( \pi_0(0) < 0 \) holds if \( \psi'(0^+) < \frac{\pi}{\varphi(q)} \). By use of (46) and detailed calculations, for initial value \( x > 0 \), the value function refracted at zero is

\[
\pi_0(x) = \pi_0(0) + \eta \left( Z^{(q)}(x) + \int_0^\infty \frac{\varphi(q)}{\varphi(q)} dy \right) - \eta \left( \frac{q}{\varphi(q)} \right) \int_0^\infty \varphi(q)(y)dy.
\]

We specify the threshold level as

\[
d^* = \inf \{ b > 0 : G(b) \leq 0 \},
\]

where

\[
G(b) = (\eta Z^{(q)}(b) - \beta) \int_0^\infty e^{-\varphi(q)W^{(q)}(y+b)dy} - \eta q W^{(q)}(b) \int_0^\infty e^{-\varphi(q)W^{(q)}(y+b)dy}.
\]

In Lemma 5.2, we derive properties of \( d^* \). Then we can conclude that the constructed solution \( \pi_{d^*} \) is the value function of the optimal threshold problem (14), which will be proved later.

**Theorem 5.1.** Let \( q > 0 \) and suppose that \( \psi'(0^+) < \infty \). Then for the dividend problem (14) with hypotheses (H1), (H2), and absolutely continuous dividend,

1. \( d^* < \infty \),
2. The value function and the optimal strategy is given by \( \pi_*(x) = \pi_{d^*}(x) \) and \( \pi_* = \pi_{0,d^*} \), respectively.

**Remark 4.** By use of Equations (43) and (44), we can easily have

\[
\lim_{\delta \to \infty} G(b) \varphi(q) = (\eta Z^{(q)}(b) - \beta W^{(q)}(b) - \eta q W^{(q)}(b))^2,
\]

which corresponds to the function \( G(a) \) determining the optimal barrier level in Avram et al. (2007). That is, the threshold strategy will be changed to the barrier strategy as \( \delta \to \infty \).
5.2. Optimal threshold strategies with capital injection. Before showing the optimality of $\pi_{0,d^*}$, we give some properties of $d^*$.

**Lemma 5.2.** Suppose that $q > 0$. If $\eta(1 - \frac{q}{\varphi(q)(c - \sigma)}) \leq \beta$ and $\sigma = 0$, then $d^* = 0$, or else $d^* > 0$.

**Proof.** With $G$ defined in (47), $G(b) = 0$ can be rewritten as $F(b) = 0$, where

$$F(b) := \int_0^\infty e^{-\varphi(q)y}W(q')(y + b)dy = \eta E_b[e^{-\varphi(q)}I_{(\tau_0 < \infty)}] - \beta. \quad (49)$$

Note that (49) is derived from one-sided exit problem in [14] with $x = b$. Since $b \mapsto \tau_0^-$ is monotonically increasing with $\lim_{b \to \infty} \tau_0^- = \infty$ a.s., it follows that $E_0[e^{-\varphi(q)}I_{(\tau_0 < \infty)}]$ monotonically decreases to zero as $b \to \infty$, and $\lim_{b \to \infty} F(b) = -\beta$. Therefore we have $F(b) \leq 0$ for all $b > 0$ if $F(0+) \leq 0$. Further, as $F$ is continuous, it follows from the intermediate value theorem that the equation $F(b) = 0$ has a root in $(0, \infty)$ if $F(0+) \in (0, \infty]$. And if $X$ is a compound Poisson process, we have $F(0+) \leq 0$ if and only if both $\sigma = 0$ and $\eta(1 - \frac{q}{\varphi(q)(c - \sigma)}) \leq \beta$ are satisfied. The assertions follow. \hfill \Box

As a consequence of Lemma 5.2, we have the following proposition.

**Proposition 1.** Let $q > 0$, and suppose that $\psi'(0+) < \infty$. It holds that $d^* < \infty$ and $\pi_{0,d^*}$ is the optimal threshold strategy with capital injections for any $x \geq 0$. That is,

$$\pi_{0}(x) \leq \pi_{d^*}(x), \quad b \geq 0. \quad (50)$$

To prove Proposition 1, we need another Lemma.

**Lemma 5.3.** Let $q > 0$, for any $x > 0$

$$\frac{\partial \pi_{0}(x)}{\partial b} = \begin{cases} F(b) \frac{Z(q)(x)}{q\varphi(q)} \int_0^\infty e^{-\varphi(q)y}W(q')(y + b)dy, & 0 < x \leq b, \\ F(b) \frac{h_1(x,b)}{q\varphi(q)} \int_0^\infty e^{-\varphi(q)y}W(q')(y + b)dy, & x > b, \end{cases} \quad (51)$$

where $h_1(x,b)$ and $F(x)$ is defined in (26), (49) respectively.

**Proof.** It follows from (45) and (49) that for $0 < x \leq b$

$$\frac{\partial \pi_{0}(x)}{\partial b} = F(b) \frac{Z(q)(x)}{q\varphi(q)} \int_0^\infty e^{-\varphi(q)y}W(q')(y + b)dy.$$

For $x > b$, we also have

$$\frac{\partial \pi_{0}(x)}{\partial b} = F(b) \frac{h_1(x,b)}{q\varphi(q)} \int_0^\infty e^{-\varphi(q)y}W(q')(y + b)dy + \frac{\partial h_1(x,b)}{q\varphi(q)} \int_0^\infty e^{-\varphi(q)y}W(q)(y + b)dy \beta - \eta Z(q)(b)$$

$$- \eta \frac{\partial h_1(x,b)}{\varphi(q)} + \eta \frac{\partial h_2(x,b)}{\varphi(q)} - \beta q W(q)(\pi(x) - b) - \varphi(q) W(q)(x - b)). \quad (52)$$

From the definition of $h_1(x,b), h_2(x,b)$, after some algebra we can get that

$$\frac{\partial h_1(x,b)}{\partial b} = \delta q(W(q)'(x - b) - \varphi(q)W(q)(x - b)) \int_0^\infty e^{-\varphi(q)y}W(q)(y + b)dy,$$

$$\frac{\partial h_2(x,b)}{\partial b} = \frac{\delta}{\varphi(q)}(W(q)'(x - b) - \varphi(q)W(q)(x - b))$$

$$\times [Z(q)(b) + q \int_0^\infty e^{-\varphi(q)y}W(q)(y + b)dy], \quad (53)$$
Now we compute results of the last four terms on the r.h.s. of (52). Based on (53),

\[
I_4 = \frac{\partial h_{x,b}}{\partial \psi(x)} \int_0^\beta - \eta Z(q)(b) \frac{\phi}{\phi(q)} e^{-\phi(q)y} W(q)(y + b) dy
\]

\[
= \frac{\partial}{\partial \psi(q)} \left( \beta - \eta Z(q)(b) \left( \mathcal{W}(q)'(x - b) - \phi(q) \mathcal{W}(q)(x - b) \right) \right)
\]

\[
I_5 = \eta \frac{\partial h_{x,b}}{\partial b} - \eta \frac{\partial h_{x,b}}{\partial \phi(q)} = \eta \delta \left( \mathcal{W}(q)'(x - b) - \phi(q) \mathcal{W}(q)(x - b) \right)
\]

then we have

\[
I_4 + I_5 = \frac{\beta \delta}{\partial \psi(q)} \left( \mathcal{W}(q)'(x - b) - \phi(q) \mathcal{W}(q)(x - b) \right)
\]

Substituting (54) back into (52) results in (51) for \( x > b \), which finishes the proof.

\[\square\]

Proof of Proposition 1. The proof is similar to [5, Proposition 2 (ii)]. In view of (51), we need only verify that, for any \( x > 0 \), the derivative of \( b \mapsto \pi_b(x) \) in \( b > x \) is equal to \( F(b) \times \{ Z(q)(x) \} \) and in \( 0 < b < x \) is equal to \( F(b) \times \{ h_t(x,b) \} \) respectively, where \( F(b) \) is given in (49). It is easily seen that \( b \mapsto \pi_b(x) \) attains its maximum over \( b \in (0, \infty) \) in \( d^* \), with the help of the arguments in Lemma 5.2 and the definition of \( d^* \).

Next, we obtain the following properties of \( \pi_{d^*} \). It is stated as a lemma for later use.

**Lemma 5.4.** Let \( q, x, b > 0 \). The following are true:

1. \( b \mapsto \pi_b(x) \) is non-increasing for \( b > d^* \).
2. \( \beta \leq \pi_{d^*}(x) \leq \eta \) for \( 0 < x < d^* \). Further, if \( d^* > 0 \), \( \pi_{d^*}(d^*) = \beta \) and \( \pi_{d^*}(0+) < \eta \) if \( X \) has bounded variation.
3. The function \( \pi_{d^*} : (0, \infty) \to R : x \mapsto \pi_{d^*}(x) \) is concave.

**Proof.** (1) The assertion follows from the proof of Proposition 1.

(2) From Lemma 5.2 and the argument in Proposition 1, it follows that if \( d^* > 0 \) and \( 0 < x < d^* \),

\[
\pi_{d^*}(x) = \eta Z(q)(x) - \eta q W(q)(x) \int_0^\beta - \eta Z(q)(d^*) \frac{\phi}{\phi(q)} e^{-\phi(q)y} W(q)(y + d^*) dy
\]

\[
= \eta Z(q)(x) - \eta q W(q)(x) \int_0^\beta - \eta Z(q)(d^*) \frac{\phi}{\phi(q)} e^{-\phi(q)y} W(q)(y + d^*) dy = \eta E_x [e^{-q_x} \mathcal{I}(x < \infty)],
\]

when the threshold level \( b = d^* \). From the argument in Lemma 5.2,

\[
\eta E_x [e^{-q_x} \mathcal{I}(x < \infty)]
\]

decreases both in \( b \) and \( x \),

\[
\pi_{d^*}(x) \geq \eta Z(q)(d^*) - \eta q W(q)(d^*) \int_0^\beta - \eta Z(q)(d^*) \frac{\phi}{\phi(q)} e^{-\phi(q)y} W(q)(y + d^*) dy = \beta
\]

If \( d^* > 0 \) and \( 0 < x < d^* \), it also holds that

\[
\pi_{d^*}(x) \leq \eta Z(q)(0+) - \eta q W(q)(0+) \int_0^\beta - \eta Z(q)(d^*) \frac{\phi}{\phi(q)} e^{-\phi(q)y} W(q)(y) dy \leq \eta,
\]

If \( x > d^* > 0 \),

\[
\pi_{d^*}(x) = \eta Z(q)(x) + \eta \delta \int_0^x W(q)(x - y) W(q)(y) dy
\]

\[
- \eta q \int_0^\beta - \eta Z(q)(d^*) \frac{\phi}{\phi(q)} e^{-\phi(q)y} W(q)(y + d^*) dy \leq \eta E_x [e^{-q_x} \mathcal{I}(x < \infty)] < \beta,
\]
where we used $G(d^*) \leq 0$. The other assertions of (2) follow easily from the definitions of $\bar{V}_{d^*}(x)$ and $Z^{(q)}$, the form of $W^{(q)}$.

(3) Suppose that $d^* > 0$. As in the above discussion, for $x > 0$, $v^*_u(x) = \eta E_x[e^{-q\tau\bar{\xi}} I_{(\tau\bar{\xi} < \infty)}]$ decreases both in $b$ and $x$. It is obvious that the function $\bar{V}_{d^*}(x)$ is concave, then the assertion follows. \hfill \Box

5.3. Verification Theorem. Suppose (H1) and (H2) hold. The value function $\bar{V}_s$ in (14) should satisfy the following variational inequality

$$\max_{0 \leq t \leq T} \{ V_{s}(x) - q\bar{V}_s(x) - q\bar{V}_s(x) + \beta \bar{V}_s(x) \} = 0, \quad x > 0, \quad \bar{V}_s(x) \leq \eta, \quad x > 0, \quad \bar{V}_s(x) = \eta, \quad x < 0,$$

where the extended generator $\Gamma$ of $\bar{V}_s$ is indeed an optimal bail-out threshold strategy. We extend $\bar{V}_s$ to the negative half-axis by setting $\bar{V}_d(x) = \bar{V}_d(0) + \eta x, \bar{V}_s(x) = \bar{V}_s(0) + \eta x$ for $x < 0$. Recalling that $W^{(q)}(x) = 0$, $Z^{(q)}(x) = 1$ and $Z^{(q)}(x) = x$ for $x < 0$, we see that these are natural extensions of (45).

Proof of Theorem 5.1. First, we show the Itô expansion of $e^{-qt}\bar{V}_s(V_t)$. Let $\pi \in \Pi$ be any admissible policy and denote by $L = L^\pi$, $R = R^\pi$ the corresponding pair of cumulative dividend and cumulative injected capital processes respectively and by $V = V^\pi$ the corresponding risk process. Applying Itô’s Lemma to $e^{-qt}\bar{V}_s(V_t)$, we can verify that

$$e^{-qt}\bar{V}_s(V_t) - \bar{V}_s(V_0)$$

$$= -q \int_0^t e^{-qs}\bar{V}_s(V_s - L_s + R_s)ds + \int_0^t e^{-qs}\bar{V}_s(V_s - L_s + R_s)d(X_s - L_s + R_s)$$

$$+ \sum_{0 < s \leq t} e^{-qs}\bar{V}_s(V_s - L_s + R_s)\Delta X_s - \bar{V}_s(V_s - L_s + R_s)\Delta I_s(V_s + \Delta X_s \geq 0)$$

$$+ \sum_{0 < s \leq t} e^{-qs}\bar{V}_s(0) - \bar{V}_s(0) - \bar{V}_s(V_s - L_s + R_s)\Delta I_s(V_s + \Delta X_s < 0),$$

where we used the following notation:

$$\Delta R_s = R_s - R_s - \pi,$$

$$\bar{V}_s(0)\{V_s + \Delta X_s < 0\} = (\bar{V}_s(V_s + \Delta X_s) + \eta \Delta R_s)\Delta I_s(V_s + \Delta X_s < 0),$$

$$(-\bar{V}_s)\{V_s + \Delta X_s < 0\} = (\Delta X_s + \Delta R_s)\Delta I_s(V_s + \Delta X_s < 0),$$

respectively, and rewriting Eq. (5) leads to

$$e^{-qt}\bar{V}_s(V_t) - \bar{V}_s(V_0)$$

$$= \int_0^t e^{-qs}(\Gamma - q)\bar{V}_s(V_s - L_s + R_s)ds + \int_0^t e^{-qs}\bar{V}_s(V_s - L_s + R_s)dL_s$$

$$+ \eta \sum_{0 < s \leq t} e^{-qs}\Delta R_s\Delta I_s(V_s + \Delta X_s < 0)$$

$$+ \int_0^t e^{-qs}\bar{V}_s(V_s - L_s + R_s)dR_s - \sum_{0 < s \leq t} e^{-qs}\bar{V}_s(V_s - L_s + R_s)\Delta I_s(V_s + \Delta X_s < 0)$$

$$+ \left\{ \left( \int_0^t e^{-qs}\bar{V}_s(V_s - L_s + R_s)dL_s \left( e - \int_0^t x\nu(dx) \right) - \sum_{0 < s \leq t} \Delta X_s \Delta I_s(\Delta X_s > 1) \right) \right\}$$

$$+ \left\{ \sum_{0 < s \leq t} e^{-qs}(\bar{V}_s(V_s + \Delta X_s) - \bar{V}_s(V_s - \Delta X_s)) - \bar{V}_s(0)\Delta X_s\Delta I_s(\Delta X_s \leq 1) \right\}$$

$$- \int_0^t \Delta X_s e^{-qs}\bar{V}_s(0)\{V_s - \pi(V_s) + \bar{V}_s(V_s - \pi(V_s))\}dI_s(\Delta X_s \leq 1)$$

Note that $R$ jumps at time $s$ if and only if $X$ jumps at time $s$ and $\Delta X_s$ is larger than $V_s$, and the measure $d(R^s_0)$ has support inside $\{ s : V_s = 0 \}$. Thus

$$\int_0^t e^{-qs}\bar{V}_s(V_s - L_s + R_s)dR_s - \sum_{0 < s \leq t} e^{-qs}\bar{V}_s(V_s - L_s + R_s)\Delta I_s(V_s + \Delta X_s < 0) = \int_0^t e^{-qs}\bar{V}_s(0)\{V_s - \pi(V_s) + \bar{V}_s(V_s - \pi(V_s))\}dR_s.$$
Furthermore, \((R^x_s)^c = 0\) (if \(X\) has bounded variation). By the Lévy-Itô decomposition, the expressions between the first and the second pairs of curly brackets is zero-mean martingale, respectively. Hence we derive that

\[
\nu_s(V_0) = \beta \int_0^t e^{-qs} dL_s^* - \eta \int_0^t e^{-qs} dR_s^0 - M_t
- \int_0^t e^{-qs}(\Gamma - q)\nu_s(V_s^-) - l^x(s)\pi_s^a(V_s^-) + \beta l^x(s))ds + e^{-qt}\nu_s(V_t),
\]

(58)

where \(\{M_t : t \geq 0\}\) is a zero-mean martingale. Next we show that the function \(\nu_{d_s}\) satisfies the variational inequality (55). Similar to [5, Lemma 5], let \(d^* > 0\). In view of [28, Proposition 3], [4, Proposition 2], \(e^{-q(t\wedge \tau_0^a \wedge \tau_0^d)}\nu_{d_s}(X_{t\wedge \tau_0^a \wedge \tau_0^d})\) is a martingale. By use of Itô’s lemma, \((\Gamma \nu_{d_s} - q\nu_{d_s})(x) = 0\) if \(x \in (0, d^*)\). Recalling Lemma 5.4 (2), we can derive that \(\nu_{d_s}\) satisfies (55) for \(x \in (0, d^*)\).

For \(a > d^*\) and \(x \in (0, a)\), let \(V^a\) the process over-refracted and under-reflecting at \(a\) and \(0\) respectively. According to [5, Lemma 5], applying Itô’s lemma to \(e^{-q\nu_{d_s}}(V^a_t)\) as in (58), we have

\[
e^{-qt}\nu_{d_s}(V^a_t) - \nu_{d_s}(x)
= \int_0^t e^{-qs}((\Gamma - q)\nu_{d_s}(V^a_s)ds - \int_0^t e^{-qs}\nu_{d_s}(V^a_s)ds
+ \eta \int_0^t e^{-qs} dR_s^0 + M_t.
\]

Then taking expectations and letting \(t \to \infty\) in (59), it holds that

\[
\nu_a(x) - \nu_{d_s}(x) = E_x[\int_0^\infty e^{-qs}((\Gamma - q)\nu_{d_s}(V^a_s)
+ (\beta \nu_{d_s} - \nu_{d_s}(V^a_s))\eta_{d_s}(V^a_s > a))ds
\]

(60)

where \(u^a_2(x, y)\) is the resolvent measure of \(V^a_s\) and \(\eta_{d_s}\) is the controlled dividend rate w.r.t. \(V^a_s\).

Referring to [14, Theorem 6] and (H2), \(u^a_2(x, y)\) is absolutely continuous on \((0, a)\) with positive density for \(a > 0\). Then for \(y \in (d^*, a)\),

\[(\Gamma - q)\nu_{d_s}(y) < 0\]

(61)

holds from Proposition 1. For \(x > d^*\), we have

\[
e^{-q(t\wedge \tau_0^a)}(\nu_{d_s}(X_{t\wedge \tau_0^a}) - \beta l^x(t\wedge \tau_0^a)) - \nu_{d_s}(x) - \beta l^x(t\wedge \tau_0^a))ds.
\]

According to the proof of [17, Lemma 6], \(e^{-q(t\wedge \tau_0^a)}\nu_{d_s}(X_{t\wedge \tau_0^a}) - \beta l^x(t\wedge \tau_0^a)\nu_{d_s}(V^a_{t\wedge \tau_0^a})\) is martingale for \(x > d^*\). We shall show that for \(y \in (d^*, \infty)\), \((\Gamma - q)\nu_{d_s}(y) - \beta l^x\nu_{d_s}(V^a_t)\) is obvious. Then

\[
(\Gamma - q)\nu_{d_s}(y) - l^x\nu_{d_s}(y) + \beta l^x \leq 0,
\]

holds from \(l^x \in [0, \delta]\), \(\nu_{d_s}(y) \leq \beta\), and (61). That is, \(\nu_{d_s}\) satisfies the inequality (55).

Now we prove the verification result \(\nu_{d_s} \geq \nu_s\), which implies that \(\nu_s = \nu_{d_s}\) and the strategy \(\pi_{0, d_s}\) is optimal. Using \(\nu_{d_s} \geq 0\) and (58), (55) leads to

\[
\nu_{d_s}(x) \geq \beta \int_0^t e^{-qs} dL_s^* - \eta \int_0^t e^{-qs} dR_s^0 + M_t + e^{-qt}\nu_{d_s}(V_t)
\]

(62)
Taking expectations and letting that $t \to \infty$ in (62), in view of the fact that $q > 0$ and the condition (12) in conjunction with the monotone convergence theorem it follows that $\bar{v}_e(x) \leq \bar{v}_{d^*}(x)$. Since $\bar{v}$ is arbitrary we can conclude that $\bar{v}_{d^*} \geq \bar{v}_e$. 

**Remark 5.** Since $v_b(x)$ is increasing, concave, and bounded, there exists a $d^* := \inf \{b > 0 : G(b) \leq 0\}$, where $G(b)$ is defined as in (48). For $t > 0$, the optimal dividend rate is

$$
l^\pi(t) = \begin{cases} 
0, & x < d^* \implies \bar{v}'_{d^*}(V^\pi_t) > \beta, \\
\delta, & x > d^* \implies \bar{v}'_{d^*}(V^\pi_t) < \beta.
\end{cases}
$$

(63)

If $v'_{d^*}(V^\pi_t) = \beta$, the dividend rate $l^\pi$ can be any value between 0 and $\delta$.

**Remark 6.** For the case of unrestricted dividend rate, the optimal dividend problem has been investigated in [5] and [13]. It was shown that a barrier strategy is optimal.

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**REFERENCES**

[1] H. Albrecher, J. Hartinger and S. Thonhauser, On exact solutions for dividend strategies of threshold and linear barrier type in a Sparre Andersen model, *ASTIN Bull.*, **37** (2007), 203–233.

[2] S. Asmussen, *Applied Probability and Queues*, World Scientific, 2003.

[3] S. Asmussen and M. Taksar, Controlled diffusion models for optimal dividend pay-out, *Insurance Math. Econom.*, **20** (1997), 1–15.

[4] B. Avanzi, J. Shen and B. Wong, Optimal dividends and capital injections in the dual model with diffusion, *ASTIN Bull.*, **41** (2011), 611–644.

[5] F. Avram, Z. Palmowski and M. R. Pistorius, On the optimal dividend problem for a SNLP, *Ann. Appl. Prob.*, **17** (2007), 156–180.

[6] J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.

[7] T. Chan, A. E. Kyprianou and M. Savov, Smoothness of scale functions for spectrally negative Lévy processes, *Probab. Theory Relat. Fields*, **150** (2011), 691–708.

[8] H. S. Dai, Z. M. Liu and N. Luan, Optimal dividend strategies in a dual model with capital injections, *Math. Meth. Oper. Res.*, **72** (2010), 129–143.

[9] H. Gerber and E. Shiu, On optimal dividend strategies in the compound poisson model, *North American Actuarial J.*, **10** (2006), 76–93.

[10] H. Gerber and E. Shiu, On optimal dividends: from reflection to refraction, *J. Comput. Appl. Math.*, **186** (2006), 4–22.

[11] J. M. Harrison and A. J. Taylor, Optimal control of a Brownian storage system, *Stoch. Process. Appl.*, **6** (1978), 179–194.

[12] M. Jeanblanc-Picqué and A. N. Shiryaev, Optimization of the flow of dividends, *Russian Math. Surveys*, **50** (1995), 257–277.

[13] N. Kulenko and H. Schmidli, Optimal dividend strategies in a Cramer-Lundberg model with capital injections, *Insurance Math. Econom.*, **43** (2008), 270–278.

[14] A. E. Kyprianou and R. L. Loeffen, Refracted Lévy processes, *Annales de l’Instut Henri Poincaré*, **46** (2010), 24–44.

[15] A. E. Kyprianou, V. Rivero and R. Song, Convexity and smoothness of scale functions and de Finetti’s control problem, *J. Th. Probab.*, **23** (2010), 547–564.

[16] A. E. Kyprianou and F. Hubalek, Old, new examples of scale functions for spectrally negative Lévy processes, *Seminar on Stochastic Analysis, Random Fields and Applications VI Progress in Probability*, **63** (2011), 119–145.

[17] A. E. Kyprianou, R. Loeffen and J. Perez, Optimal control with absolutely continuous strategies for spectrally negative Lévy processes, *J. Appl. Probab.*, **49** (2012), 150–166.

[18] A. E. Kyprianou, *Fluctuations of Lévy Processes with Applications*, 2nd ed. Springer, 2014.
[19] A. Lambert, Completely asymmetric Lévy processes confined in a finite interval, *Ann. Inst. H. Poincaré Probab. Statist.*, 36 (2000), 251–274.
[20] M. Li and G. Yin, Optimal threshold strategies with capital injections in a spectrally negative Lévy risk model, preprint, 2014.
[21] X. S. Lin and K. P. Pavlova, The compound Poisson risk model with a threshold dividend strategy, *Insurance Math. Econom.*, 38 (2006), 57–80.
[22] R. L. Loeffen, On optimality of the barrier strategy in de Finetti’s dividend problem for spectrally negative Lévy processes, *Ann. Appl. Probab.*, 18 (2008), 1669–1680.
[23] R. L. Loeffen, An optimal dividend problem with a terminal value for spectrally negative Lévy processes with a completely monotone jump density, *J. Appl. Probab.*, 46 (2009), 85–98.
[24] R. L. Loeffen, J. F. Renaud and X. W. Zhou, Occupation times of intervals until first passage times for spectrally negative Lévy processes, *Stoch. Proc. Appl.*, 124 (2014), 1408–1435.
[25] A. Løkka and M. Zervos, Optimal dividend and issuance of equity policies in the presence of proportional costs, *Insurance Math. Econom.*, 42 (2008), 954–961.
[26] A. C. Y. Ng, On a dual model with a dividend threshold, *Insurance Math. Econom.*, 44 (2009), 315–324.
[27] M. R. Pistorius, On doubly reflected completely asymmetric Lévy processes, *Stoch. Proc. Appl.*, 107 (2003), 131–143.
[28] M. R. Pistorius, On exit and ergodicity of the completely asymmetric Lévy process reflected at its infimum, *J. Th. Probab.*, 17 (2004), 183–220.
[29] J. L. Pérez and K. Yamazakic, On the refracted-reflected spectrally negative Lévy processes, *Stochastic Process. Appl.*, 128 (2018), 306–331.
[30] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.
[31] S. E. Shreve, J. P. Lehoczky and D. P. Gaver, Optimal consumption for general diffusions with absorbing and reflecting barriers, *SIAM J. Control Optim.*, 22 (1984), 55–75.
[32] D. J. Yao, H. L. Yang and R. M. Wang, Optimal risk and dividend control problem with fixed costs and salvage value: Variance premium principle, *Economic Modelling*, 37 (2014), 53–64.
[33] C.C. Yin and K.C. Yuen, Optimality of the threshold dividend strategy for the compound Poisson model, *Statistics and Probability Letters*, 81 (2011), 1841–1846.
[34] M. Zhou and K. C. Yuen, Optimal reinsurance and dividend for a diffusion model with capital injection: Variance premium principle, *Economic Modelling*, 29 (2012), 198–207.
[35] J. X. Zhu and H. L. Yang, Optimal financing and dividend distribution in a general diffusion model with regime switching, *Advances in Applied Probability*, 48 (2016), 406–422.

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E-mail address: lmm@cqu.edu.cn
E-mail address: gyin@math.wayne.edu