ABSTRACT. It is a well known result that the fixed point subalgebra of a finite dimensional complex simple Lie algebra under a finite order automorphism is a reductive Lie algebra so is a direct sum of finite dimensional simple Lie subalgebras and an abelian subalgebra. We consider this for the class of extended affine Lie algebras and are able to show that the fixed point subalgebra of an extended affine Lie algebra under a finite order automorphism (which satisfies certain natural properties) is a sum of extended affine Lie algebras (up to existence of some isolated root spaces), an abelian subalgebra and a subspace which is contained in the centralizer of the core.

0. Introduction

In 1955, A. Borel and G.D. Mostow [BM] proved that the fixed point subalgebra of a finite dimensional complex simple Lie algebra under a finite order automorphism is a reductive Lie algebra. A natural question which arises here is what can we say about the fixed points of a finite order automorphism of an extended affine Lie algebra (EALA for short). EALAs are natural generalizations of finite dimensional complex simple Lie algebras and affine Kac–Moody Lie algebras. They are axiomatically defined (see [AABGP], [HK-T]) and the axioms guarantee the existence of analogues of Cartan subalgebras, root systems, invariant forms, etc. A root of an EALA is called isotropic if it is orthogonal to itself, with respect to the form. The dimension of the real span of the isotropic roots is called the nullity of the Lie algebra. A finite dimensional simple Lie algebra is an EALA of nullity zero, and an EALA is an affine Lie algebra if and only if its nullity is 1 (see [ABGP] for details). Thus EALAs form a natural class of algebras in which to consider extensions of the result of Borel and Mostow.

Here we would like to explain a procedure which has been the most general theme of constructing affine Lie algebras and their generalizations, since the birth of Kac–Moody Lie algebras in 1968.

Let \((\mathcal{G}, (\cdot,\cdot), \mathcal{H})\) be an EALA with root system \(R\) (in particular \(\mathcal{G}\) can be a finite dimensional simple Lie algebra or an affine Lie algebra). Let \(\sigma\) be a finite order automorphism of \(\mathcal{G}\) which stabilizes \(\mathcal{H}\) and leaves the form...
invariant. Assume also that the fixed point subalgebra of $\mathcal{H}$ (with respect to $\sigma$) is a Cartan subalgebra of the fixed point subalgebra of $\mathcal{G}$. Consider the Lie algebra

$$\text{Aff}(\mathcal{G}) := (\mathcal{G} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $c$ is central, $d = t \frac{d}{dt}$ is the degree derivation so that $[d, x \otimes t^n] = nx \otimes t^n$, and multiplication is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{m+n, 0}c.$$

Extend the form $(\cdot, \cdot)$ to $\text{Aff}(\mathcal{G})$ so that $c$ and $d$ are naturally paired. Then the triple

$$(\text{Aff}(\mathcal{G}), (\cdot, \cdot), \mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d)$$

is again an EALA with root system $\tilde{R} = R + \mathbb{Z}\delta$ where $\delta$ is the linear functional on $\mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ defined by $\delta(d) = 1$ and $\delta(\mathcal{H} \oplus \mathbb{C}c) = 0$. Extend $\sigma$ to an automorphism of $\text{Aff}(\mathcal{G})$ by

$$\sigma(x \otimes t^i + rc + sd) = \zeta^{-i} \sigma(x) \otimes t^i + rc + sd,$$

where $\zeta = e^{2\pi \sqrt{-1}/m}$ and $\sigma^m = \text{id}$. Then the fixed point subalgebra of $\text{Aff}(\mathcal{G})$, under $\sigma$, is an EALA (see [ABP] and Example [A18]). As one can see in [K] and [H], when $\mathcal{G}$ is a finite dimensional simple Lie algebra, all affine Kac–Moody Lie algebras can be constructed this way. In fact all the examples of EALA’s which are presented in [W], [P] and [HK-T] can be obtained from the above procedure. It is also shown in [A2] that most of the examples of EALA’s constructed in [AABGP] can be put in the above context (see Section 3 and [A2]).

In this paper, we consider the theorem of [BM] for the class of EALA’s. Namely we investigate what is the fixed point subalgebra of an EALA under a finite order automorphism which satisfies certain conditions. In a very real sense this paper found it’s inspiration from the paper [ABP] which also studies Lie algebras constructed from finite order automorphisms of an EALA. The difference is this; in [ABP] the algebras studied are affinizations of the EALA one starts with while here we study fixed point subalgebras. But our basic techniques are similar to [ABP] and, as the reader will see, we will rely on some of the results from [ABP] in certain crucial places. Thus, although both studies are independent of each other there is certainly an interdependence.

In Section 1 of this paper, we state a modified version of the definition of an extended affine root system (EARS for short). The usual definition has indecomposability built into it. We need to consider root systems which are unions of such root systems so have broken down the usual definition. This should not cause the reader any difficulties. The definition given here is basically the same as [AABGP] except for a rearrangement of the axioms. Part of the reason this is done is to make more compatible the two different definitions of EARSs found in the literature. It is shown that an EARS, modulo some isotropic roots, is a union of a finite number of irreducible EARS’s which are orthogonal with respect to the form.
In Section 2, which forms the core of the paper, we show that the fixed point subalgebra $G^\sigma$ of an EALA $G$ under a finite order automorphism $\sigma$ which satisfies certain conditions is of the form $G^\sigma = \sum_{i=1}^k G_i^\sigma \oplus W \oplus I$, where for each $i$, $G_i^\sigma$ satisfies axioms EA1–EA5(a) of an EALA, $I$ is a subspace of $G^\sigma$ contained in the centralizer of the core of $G^\sigma$, and $W$ is an abelian subalgebra contained in the centralizer of $G_i^\sigma$ for each $i$ (see Theorem 2.63). This agrees with the result of [BM] when $G$ is a finite dimensional simple Lie algebra over $\mathbb{C}$, as in this case $I = \{0\}$ and for each $i$, $G_i^\sigma$ is a finite dimensional simple Lie algebra (see Corollary 2.64). In Lemma 2.54 certain relations between the core of $G$ and the core of $G^\sigma$ and some results regarding the tameness of $G^\sigma$ in terms of the tameness of $G$ are obtained.

In Section 3, the last section, a large number of examples are presented. Examples 3.68–3.72 illustrate how the terms $G_i^\sigma$’s, $W$ and $I$ (see Theorem 2.63) appear as the fixed points of automorphisms. In 3.75, 3.79 and 3.81 some examples from [A2] are restated in such a way that they fit in the setting presented in Section 2. In 3.83 an example regarding the results in [ABP] is provided. This example shows how the affinizations in [ABP] can be viewed as fixed point subalgebras of the type considered here.

The authors would like to thank Professor B. Allison and Professor A. Pianzola for some helpful discussions. It was at the Fields Institute Program on Infinite Dimensional Lie Theory and Its Applications where this work was originally conceived.

1. Terminology and prerequisites

There are two different definitions for the term extended affine root systems (EARS for short) in the literature. This term first was introduced by K. Saito [Sa1] in 1985. In 1997, another definition for EARS was introduced in [AABGP]. In [A1], to distinguish between these two definitions, the author used SEARS to refer to the definition introduced by K. Saito and used EARS for the one which is introduced in [AABGP]. The basic difference between SEARS and EARS is that in SEARS all roots are nonisotropic while in EARS there are isotropic roots as well. Also in [AABGP] it is assumed that an EARS is both reduced and indecomposable while this is not assumed in the case of a SEARS. In [A1], it is shown that there is a one to one correspondence between reduced indecomposable SEARS and EARS. In fact, if we drop the axioms which are related to an EARS being reduced and indecomposable (axioms R4 and R7 from [AABGP]) it follows from [A1] that there is a one to one correspondence between SEARS and EARS. As we will see in the sequel, such root systems will naturally arise as the root systems of the fixed point subalgebras (under certain finite order automorphisms) of some extended affine Lie algebras. Also, such root systems arise in [Y].

We start by modifying the definition of an EARS. Our new definition is basically the same as [AABGP], except, since we want to consider more
general types of root systems, we leave off the axioms concerning reducibility and indecomposability.

**Definition 1.1.** Let $\mathcal{V}$ be a finite dimensional real vector space with a nontrivial positive semidefinite symmetric bilinear form $(.,.)$ and let $R$ be a subset of $\mathcal{V}$. Let

$$R^\times = \{\alpha \in R : (\alpha, \alpha) \neq 0\} \quad \text{and} \quad R^0 = \{\alpha \in R : (\alpha, \alpha) = 0\}.$$  

Then $R = R^\times \uplus R^0$ where $\uplus$ means disjoint union. Then we will say $R$ is an extended affine root system (EARS) in $\mathcal{V}$ if $R$ satisfies the following 4 axioms:

(R1) $R = -R$,
(R2) $R$ spans $\mathcal{V}$,
(R3) $R$ is discrete in $\mathcal{V}$,
(R4) if $\alpha \in R^\times$ and $\beta \in R$, then there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\} \quad \text{and} \quad d - u = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}.$$

The EARS $R$ is called *tame* if it satisfies:

(R5) for any $\delta \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \delta \in R$. We say a root satisfying this condition is *nonisolated* and call isotropic roots which do not satisfy this *isolated*.

The EARS $R$ is called *indecomposable* if it satisfies:

(R6) $R^\times$ cannot be decomposed into a disjoint union of two nonempty subsets which are orthogonal with respect to the form.

A tame indecomposable EARS $R$ is called *irreducible*. Finally, the EARS $R$ is called *reduced* if it satisfies:

(R7) $\alpha \in R^\times \Rightarrow 2\alpha \notin R$.

Since the form is nontrivial, it follows from R2 that $R^\times \neq \emptyset$. This, together with R1 and R4, implies that $0 \in R$. Note that in an EARS as used here could have both isolated and nonisolated roots. Thus Axiom R5 indicates that in a tame EARS isotropic roots are nonisolated. By [A1], there is a one to one correspondence between irreducible (reduced) EARS and indecomposable (reduced) SEARS.

**Lemma 1.2.** Let $R$ be an EARS and $R_1$ be a subset of $R$ with $R_1^\times \neq \emptyset$. Suppose that

(a) $R_1 = -R_1$,
(b) $\{\delta \in R^0 : \alpha' + \delta \in R_1 \text{ for some } \alpha' \in R_1^\times\} \subseteq R_1$,
(c) $\alpha' \in R_1$, $\beta \in R$, $(\alpha', \beta) \neq 0 \Rightarrow \beta \in R_1$.

Then $R_1$ is an EARS in its real span. Moreover, if we set

$$R'_1 = R_1^\times \cup (\langle R_1 \rangle \cap R^0),$$

then $R'_1$ is also an EARS in the real span of $R_1$ $(\langle R_1 \rangle$ denotes the $\mathbb{Z}$-span of $R_1$).
Proof. Clearly (R1)–(R3) hold for $R_1$. We now check R4. Let $\alpha' \in R_1^x$ and $\beta' \in R_1$. Since $R4$ holds for $R$, it is enough to show that for $n \in \mathbb{Z}$,

$$\beta' + n\alpha' \in R \implies \beta' + n\alpha' \in R_1.$$ 

Since $\beta' \in R_1$, we may assume that $n \neq 0$, and by (a), we may also assume that $n > 0$. So let $\beta' + n\alpha' \in R$, $n > 0$. If $\beta' + n\alpha' \in R^x$, then $(\beta' + n\alpha', \beta') \neq 0$ or $(\beta' + n\alpha', \alpha') \neq 0$. In either cases, we get from (c) that $\beta' + n\alpha' \in R_1$. Next, let $\beta' + n\alpha' \in R^0$. Since $R4$ holds for $R$ and $n > 0$, we have $\beta' + (n-1)\alpha' \in R^x$. So repeating our previous argument we get $\beta' + (n-1)\alpha' \in R_1$. Since

$$\beta' + n\alpha' + (-\alpha') = \beta' + (n-1)\alpha' \in R_1^x$$

it follows from (a) and (b) that $\beta' + n\alpha' \in R_1$. This completes the proof of the first assertion.

Next let $R_1'$ be as in the statement. Clearly $R_1$ and $R_1'$ have the same real span. Since $R_1^x = (R_1')^x$, it is easy to check that $R_1^x$ satisfies conditions (a)-(c), and so is an EARS. □

Let $R$ be an EARS in $\mathcal{V}$. Let $\mathcal{V}^0$ be the radical of the form $(\cdot, \cdot)$ on $\mathcal{V}$. Set $\overline{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$, and let $\overline{v}$ be the image of an element $v$ of $\mathcal{V}$ under the projection map $\mathcal{V} \rightarrow \overline{\mathcal{V}}$. For $\alpha, \beta \in \mathcal{V}$ define $(\overline{\alpha}, \overline{\beta}) := (\alpha, \beta)$. Then $(\cdot, \cdot)$ is positive definite on $\overline{\mathcal{V}}$, and by [AABGP, Chapter II] $\overline{R}$ is a finite root system in $\overline{\mathcal{V}}$. Note that it is not assumed here that $R$ is irreducible, but the same argument as in [AABGP, Chapter II] shows that $\overline{R}$ is a finite root system (which is not necessarily irreducible).

For an EARS $R$, we set

$$R_{\text{iso}} = \{ \delta \in R^0 \mid \alpha + \delta \notin R \text{ for any } \alpha \in R^x \},$$

$$R_{\text{niso}} = \{ \delta \in R^0 \mid \alpha + \delta \in R \text{ for some } \alpha \in R^x \} = R^0 \setminus R_{\text{iso}}.$$

That is $R_{\text{iso}}$ ($R_{\text{niso}}$) is the set of isolated (nonisolated) isotropic roots of $R$, so $R$ is tame if and only if $R_{\text{iso}} = \emptyset$. We also set

$$R_t = R^x \cup R_{\text{niso}}.$$

Then $R_t^x = R^x$ and

$$R = R^x \uplus R_{\text{niso}} \uplus R_{\text{iso}} = R_t \uplus R_{\text{iso}}.$$

**Lemma 1.4.** Let $R$ be an EARS. Then

(i) $R_t$ is a tame EARS in its real span.

(ii) $R = (\bigcup_{i=1}^k R_i) \cup R_{\text{iso}}$, where each $R_i$ is an irreducible EARS, and for $i \neq j$, $R_i$ and $R_j$ are orthogonal with respect to the form. Furthermore, if we set

$$R_i' = R_i^x \uplus ((R_i) \cap R^0),$$

then $R_i'$ is an indecomposable EARS.

Proof. Clearly (R1)–(R3) hold for $R_1$. We now check R4. Let $\alpha' \in R_1^x$ and $\beta' \in R_1$. Since $R4$ holds for $R$, it is enough to show that for $n \in \mathbb{Z}$,
Proof. (i) It follows from definition of $R_t$ that conditions (a)-(c) of Lemma 1.2 hold for $R_t$. So $R_t$ is an EARS. It is clear from definition that $R_t$ is tame.

(ii) We have $R = R_t \uplus R_{iso}$, and by part (i), $R_t$ is a tame EARS. Since $\tilde{R}_t = \tilde{R}$ is a finite root system, we have $\tilde{R}_t \setminus \{0\} = \uplus_{i=1}^{k} \tilde{R}_i$, where for each $i$, $\tilde{R}_i \cup \{0\}$ is an irreducible finite root system, and $\tilde{R}_i$’s are orthogonal with respect to the form. Let $R_i^\times$ be the preimage of $\tilde{R}_i$ under the projection map $\tilde{\cdot}$. Then

$$R_t^\times = \uplus_{i=1}^{k} R_i^\times$$

and $(R_i^\times, R_j^\times) = \{0\}$ if $i \neq j$.

Set

$$R_i = R_i^\times \cup \{\delta \in R_t^0 | \delta + \alpha \in R_t \text{ for some } \alpha \in R_i^\times\}.$$

Since $R_t$ is tame, the isotropic roots of $R_t$ are nonisolated so

$$R_t = \bigcup_{i=1}^{k} R_i.$$

It is easy to see that conditions (a)-(c) of Lemma 1.2 hold for $R_i$. Thus $R_i$ is an EARS. By Lemma 1.2 the last assertion also holds. □

2. Fixed Point Subalgebras

In this section we study the structure of fixed point subalgebra of an extended affine Lie algebra under a finite order automorphism which satisfies certain conditions. For a systematic study of extended affine Lie algebras reader is referred to [AABGP].

An extended affine Lie algebra (EALA for short) is a triple $(G, (\cdot, \cdot), H)$ where $G$ and $H$ are two Lie algebras over $\mathbb{C}$ and $(\cdot, \cdot)$ is a complex valued bilinear form on $G$ satisfying the following five axioms:

EA1. The form $(\cdot, \cdot)$ is symmetric, nondegenerate and invariant on $G$.
EA2. $H$ is a nontrivial finite dimensional abelian subalgebra of $G$ which is self-centralizing and $ad(h)$ is diagonalizable for all $h \in H$.

Consider the root space decomposition $G = \bigoplus_{\alpha \in H^*} G_{\alpha}$. Then $R = \{\alpha \in H^* | G_{\alpha} \neq \{0\}\}$ is called the root system of $G$. Let $R^\times$ and $R^0$ be as in Definition 1.1. The next three axioms are as follows:

EA3. For any $\alpha \in H^*$ and $x \in G_{\alpha}$, $ad_G(x)$ is locally nilpotent on $G$.
EA4. $R$ is a discrete subset of $H^*$.
EA5a. $R^\times$ is indecomposable (in the sense of Definition 1.1).
EA5b. Isotropic roots of $R$ are nonisolated (in the sense of Definition 1.1).

Consider a fixed EALA $(G, (\cdot, \cdot), H)$ with root system $R$ and the corresponding root space decomposition $G = \bigoplus_{\alpha \in H^*} G_{\alpha}$. It is shown in [AABGP, Chapter I] that $R$ is a reduced irreducible EARS (in the sense of Definition 1.1). Consider a fix automorphism $\sigma$ of $G$ and set

$$G^\sigma = \{x \in G | \sigma(x) = x\} \quad \text{and} \quad H^\sigma = \{h \in H | \sigma(h) = h\}.$$
That is \( \mathcal{G}^\sigma \) (resp. \( \mathcal{H}^\sigma \)) is the fixed point subalgebra of \( \mathcal{G} \) (resp. \( \mathcal{H} \)) with respect to \( \sigma \).

Let \( m \geq 1 \) and suppose that

A1. \( \sigma^m = 1 \).
A2. \( \sigma(\mathcal{H}) = \mathcal{H} \).
A3. \( (\sigma(x), \sigma(y)) = (x, y) \) for all \( x, y \in \mathcal{G} \).
A4. \( C_{\mathcal{G}^\sigma}(\mathcal{H}^\sigma) = \mathcal{H}^\sigma \).

We start by recording some facts related to A1-A3, so only need to assume A1-A3 hold for now. Let \( \tilde{i} \) denote the image of \( i \in \mathbb{Z} \) in \( \mathbb{Z}/m\mathbb{Z} \). From A1 and A2, we have

\[
\mathcal{G} = \bigoplus_{i=0}^{m-1} \mathcal{G}_i \quad \text{and} \quad \mathcal{H} = \bigoplus_{i=0}^{m-1} \mathcal{H}_i,
\]

where \( \mathcal{G}_i \) (resp \( \mathcal{H}_i \)) is the eigenspace corresponding to the \( i \)-th power of the \( m \)-th root of unity \( \zeta = e^{2\sqrt{-1}\pi/m} \). Then

\[
\mathcal{G}_i = \{ x \in \mathcal{G} | \sigma(x) = \zeta^i x \}.
\]

Note that \( \mathcal{G}^\sigma = \mathcal{G}_0 \) and \( \mathcal{H}^\sigma = \mathcal{H}_0 \). It follows from A3 that

\[
(\mathcal{G}_i, \mathcal{G}_j) = \{0\} = (\mathcal{H}_i, \mathcal{H}_j) \quad \text{if} \quad i + j \neq 0.
\]

Set \( \mathcal{G}^c = \bigoplus_{i=1}^{m-1} \mathcal{G}_i \) and \( \mathcal{H}^c = \bigoplus_{i=1}^{m-1} \mathcal{H}_i \). Then \( \mathcal{G} = \mathcal{G}^\sigma \oplus \mathcal{G}^c \) and \( \mathcal{H} = \mathcal{H}^\sigma \oplus \mathcal{H}^c \). Moreover,

\[
(\mathcal{G}^\sigma, \mathcal{G}^c) = \{0\} = (\mathcal{H}^\sigma, \mathcal{H}^c).
\]

Note also that \( \sigma \) induces an automorphism \( \sigma \in \text{Aut}(\mathcal{H}^*) \) by

\[
(2.12) \quad \sigma(\alpha)(h) = \alpha(\sigma^{-1}(h)), \quad \text{for} \ \alpha \in \mathcal{H}^* \text{ and } h \in \mathcal{H}.
\]

Let \( \pi \) denote the projection map from \( \mathcal{G} \) (resp. \( \mathcal{H} \)) onto \( \mathcal{G}^\sigma \) (resp. \( \mathcal{H}^\sigma \)). Then for \( x \in \mathcal{G} \) we have \( x - \pi(x) \in \mathcal{G}^c \). Since \( \mathcal{G}^\sigma \) is stable under \( \sigma \) and \( \sigma \pi = \pi \), we obtain

\[
\sum_{i=0}^{m-1} \sigma^i(x - \pi(x)) \in \mathcal{G}^c \cap \mathcal{G}^\sigma = \{0\}.
\]

So

\[
(2.13) \quad \pi(x) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(x), \quad (x \in \mathcal{G}).
\]

Similarly,

\[
(2.14) \quad \pi(h) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(h), \quad (h \in \mathcal{H}).
\]

The map \( \pi \) induces a map, denoted again by \( \pi \), from the dual space \( \mathcal{H}^* \) of \( \mathcal{H} \) onto the dual space \( (\mathcal{H}^\sigma)^* \) of \( \mathcal{H}^\sigma \). Namely, for \( \alpha \in \mathcal{H}^* \) and \( h \in \mathcal{H}^\sigma \), define

\[
(2.15) \quad \pi(\alpha)(h) = \alpha(\pi(h)),
\]
that is $\pi(\alpha)$ for $\alpha \in \mathcal{H}^*$ is the restriction of $\alpha$ to $\mathcal{H}^\sigma$. Then from (2.12) we have, for $\alpha \in \mathcal{H}^*$ and $h \in \mathcal{H}^\sigma$,

$$\pi(\alpha)(h) = \alpha(\pi(h)) = \alpha\left(\frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(h)\right) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha)(h).$$

Thus for $\alpha \in \mathcal{H}^*$, we have

(2.16) $$\pi(\alpha) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha).$$

Since both $\mathcal{G}$ and $\mathcal{G}_i$ are $\mathcal{H}^\sigma$-modules, we have

(2.17) $$\mathcal{G} = \sum_{\tilde{\alpha} \in (\mathcal{H}^\sigma)^*} \mathcal{G}_{\tilde{\alpha}} = \sum_{\alpha \in \mathcal{H}^*} \mathcal{G}_{\pi(\alpha)} = \sum_{\alpha \in R} \mathcal{G}_{\pi(\alpha)}$$ and

$$\mathcal{G}_i = \sum_{\alpha \in R} \mathcal{G}_{i,\pi(\alpha)},$$

where

$$\mathcal{G}_{\tilde{\alpha}} = \{x \in \mathcal{G} \mid [h, x] = \tilde{\alpha}(h)x \text{ for all } h \in \mathcal{H}^\sigma\}$$

and

$$\mathcal{G}_{i,\pi(\alpha)} = \mathcal{G}_i \cap \mathcal{G}_{\pi(\alpha)}.$$

Note that $\mathcal{G}^\sigma$ as an $\mathcal{H}^\sigma$-submodule of $\mathcal{G}$ has a weight space decomposition

(2.18) $$\mathcal{G}^\sigma = \bigoplus_{\tilde{\alpha} \in (\mathcal{H}^\sigma)^*} \mathcal{G}_{\tilde{\alpha}},$$

where

$$\mathcal{G}_{\tilde{\alpha}}^\sigma = \mathcal{G}^\sigma \cap \mathcal{G}_{\tilde{\alpha}}.$$

Set

$$\mathcal{R}^\sigma = \{\tilde{\alpha} \in (\mathcal{H}^\sigma)^* | \mathcal{G}_{\tilde{\alpha}}^\sigma \neq \{0\}\}.$$

Then

$$\mathcal{R}^\sigma \subseteq \pi(R).$$

As we will see in the next section, in many examples $\mathcal{R}^\sigma$ is in fact a proper subset of $\pi(R)$. Denote the set of nonisotropic (isotropic) roots of $\mathcal{R}^\sigma$ by $(\mathcal{R}^\sigma)^\times$ ($(\mathcal{R}^\sigma)^0$), respectively. That is,

$$(\mathcal{R}^\sigma)^\times = \{\tilde{\alpha} \in \mathcal{R}^\sigma | (\tilde{\alpha}, \tilde{\alpha}) \neq 0\} \text{ and } (\mathcal{R}^\sigma)^0 = \{\tilde{\alpha} \in \mathcal{R}^\sigma | (\tilde{\alpha}, \tilde{\alpha}) = 0\}.$$

It is easy to see that

(2.19) $$\sigma(\mathcal{G}_{\alpha}) = \mathcal{G}_{\sigma(\alpha)} \quad (\alpha \in R),$$

and so

$$\sigma(R) = R.$$

Also we have

$$\pi(\mathcal{G}_{\alpha}) \subseteq \mathcal{G}_{\pi(\alpha)}^\sigma.$$

Thus

(2.20) $$\mathcal{G}_{\pi(\alpha)}^\sigma = \sum_{\beta \in R \atop \pi(\beta) = \pi(\alpha)} \pi(\mathcal{G}_{\beta}).$$
According to (2.24) the eigenvalues of $ad_{\sigma}$ (2.26), $h$ and $y$ have nondegenerate form (2.25). Identifying ($\star$) on $H$, it follows that $\text{ad}_{\sigma}$ acts locally nilpotently on $H$. So for some $\gamma \in H$ (2.23) \[ \text{G}^\sigma = \{0\} \quad \text{unless} \quad \alpha + \beta = 0. \]

In particular, we have

\begin{equation}
R^\sigma = -R^\sigma.
\end{equation}

Also for any $\alpha, \beta \in R$,

\begin{equation}
[\text{G}^\sigma_{\pi(\alpha)}, \text{G}^\sigma_{\pi(\beta)}] = 0 \quad \text{or} \quad [\text{G}^\sigma_{\pi(\alpha)}, \text{G}^\sigma_{\pi(\beta)}] \subseteq \text{G}^\sigma_{\pi(\gamma)},
\end{equation}

for some $\gamma \in R$ with $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$. Then from (2.20) and (2.18), we have

\begin{equation}
\text{G}^\sigma = \sum_{\alpha \in R} \text{G}^\sigma_{\pi(\alpha)} = \bigoplus_{\alpha \in R} \text{G}^\sigma_{\alpha} = \bigoplus_{\alpha \in R^\sigma} \text{G}^\sigma_{\alpha}.
\end{equation}

For $\alpha \in H^*$ let $t_\alpha$ be the unique element in $H$ which represents $\alpha$ via the nondegenerate form $(\cdot, \cdot)$. That is,

\[ \alpha(h) = (t_\alpha, h) \quad (h \in H). \]

Then for $\alpha \in H^*$,

\begin{equation}
\sigma(t_\alpha) = t_{\sigma(\alpha)} \quad \text{and} \quad \pi(t_\alpha) = t_{\pi(\alpha)}.
\end{equation}

So

\begin{equation}
H^\sigma = \{t_\alpha | \alpha \in (H^\sigma)^*\} \quad \text{and} \quad H^c = \{t_\alpha | \alpha \in (H^c)^*\}.
\end{equation}

Transfer the form $(\cdot, \cdot)$ to $H^*$ by

\[ (\alpha, \beta) = (t_\alpha, t_\beta), \quad (\alpha, \beta \in H^*). \]

Now similar to (2.11), we have

\begin{equation}
H^* = (H^*)^\sigma + (H^*)^c, \quad \text{and} \quad ((H^*)^\sigma, (H^*)^c) = \{0\}.
\end{equation}

Identifying $(H^*)^\sigma$ with $(H^*)^*$ and $(H^*)^c$ with $(H^c)^*$, we see that the map $\pi$ on $H^*$ is in fact the restriction to $(H^*)^*$. 

**Lemma 2.28.** Let $\alpha \in R$ and $(\pi(\alpha), \pi(\alpha)) \neq 0$. Then for $x \in \mathcal{G}^\sigma_{\pi(\alpha)}$, $ad_{\mathcal{G}^\sigma} x$ acts locally nilpotently on $\mathcal{G}^\sigma$.

**Proof.** Fix $0 \neq x \in \mathcal{G}^\sigma_{\pi(\alpha)}$. By (2.21), it is enough to show that if $\beta \in R$ and $y \in \mathcal{G}^\sigma_{\pi(\beta)}$, then $(ad x)^N(y) = 0$ for sufficiently large $N$. By (2.25) and (2.20), $h := t_{\pi(\alpha)} \in H^\sigma$ and so

\[ (h, (ad x)^k(y)) = (k(\pi(\alpha), \pi(\alpha)) + \pi(\beta)(h))(ad x)^k(y). \]

According to (2.24) the eigenvalues of $ad_{\mathcal{G}^\sigma} h$ are of the form $\pi(\gamma)(h), \gamma \in R$. Now

\[ \pi(\gamma)(h) = (\pi(\alpha), \pi(\gamma)) = \frac{1}{m} \sum_{i=0}^{m-1} (\alpha, \sigma^i(\gamma)). \]
Lemma 2.29. \( \pi(R) \) is a discrete subset of \((\mathcal{H}^*)^*\). In particular \( R^\times \) as a subset of \( \pi(R) \) is discrete.

Proof. This is clear as \( \pi(\alpha) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha) \in \frac{1}{m}(R) \), and \( \langle R \rangle \) is a discrete subset of \( \mathcal{H}^* \).

The parts (i) and (ii) of the following lemma are proved in [ABP]. For part (iii) we need to recall that the structure of the root system \( R \) of type \( X_\ell \) is of the form

\[
(S + S) \cup (R + S) \quad \text{if } X_\ell = A_\ell, \ D_\ell, \ E_\ell
\]
\[
(S + S) \cup (R_{sh} + S) \cup (R_{lg} + L) \quad \text{if } X_\ell = B_\ell, \ C_\ell, \ F_4, \ G_2
\]
\[
(S + S) \cup (R_{sh} + S) \cup (R_{ex} + E) \quad \text{if } X_\ell = BC_1
\]
\[
(S + S) \cup (R_{sh} + S) \cup (R_{lg} + L) \cup (R_{ex} + E) \quad \text{if } X_\ell = BC\ell
\]

where \( R_{sh}, \ R_{lg} \) and \( R_{ex} \) are the sets of short, long and extra long roots of a finite root system \( R \), and \( S, \ L, \ E \) are the so called semilattices or translated semilattices involved in the structure of \( R \) which interact in a prescribed way. Moreover, \( R_{sh} \cup R_{lg} \subseteq R \) (for details see [AABGP, Chapter II].)

Lemma 2.31. Suppose \( \pi(R)^\times \neq \emptyset \). Then

(i) For \( \beta \in R^\times \), there exists \( \alpha \in R \) such that \( (\pi(\alpha), \pi(\alpha)) \neq 0 \) and \( (\alpha, \beta) \neq 0 \).

(ii) \( \pi(R)^\times \) is indecomposable. That is, \( \pi(R)^\times \) cannot be written as a disjoint union of two nonempty subsets which are orthogonal with respect to the form.

(iii) Let \( x \in R \) and \( (\pi(x), \pi(x)) = 0 \). Then there exists \( \tilde{\beta} \in \pi(R)^\times \) such that \( \tilde{\beta} + \pi(x) \in \pi(R)^\times \).

Proof. For parts (i) and (ii) see [ABP Lemma 3.49]. To see (iii), we consider the two cases (a) \( x \in R^0 \) and \( x \in R^\times \), separately. Let \( \delta := \pi(x) \).

(a) \( x \in R^0 = S + S \). Then \( x = \delta_1 + \delta_2 \) for some \( \delta_1, \delta_2 \in S \subseteq R^0 \). By assumption, there exists \( \alpha \in R \) such that \( (\pi(\alpha), \pi(\alpha)) \neq 0 \). Since \( R_{sh} + S \) and \( R \) have the same real span, we may assume that \( \alpha \in R_{sh} + S \). Also \( \pi(V^0) \subseteq V^0 \), so we may assume that \( \alpha = \dot{\alpha} \in R_{sh} \). Set

\[
\beta := \dot{\alpha} + \delta_1 \in R_{sh} + S \subseteq R^\times.
\]

Then \( (\pi(\beta), \pi(\beta)) = (\pi(\dot{\alpha}), \pi(\dot{\alpha})) \neq 0 \). So \( \tilde{\beta} := \pi(\beta) \in \pi(R)^\times \). Also

\[
\beta + x = \dot{\alpha} + \delta_1 + \delta_1 + \delta_2 \in \dot{\alpha} + 2S + S \subseteq \dot{\alpha} + S \subseteq R^\times.
\]

Finally, \( \tilde{\beta} + \tilde{\delta} = \pi(\beta + x) \in \pi(R)^\times \).

(b) \( x \in R^\times \). By part (i), there exists \( \alpha \in R^\times \) such that \( (\alpha, x) \neq 0 \) and \( (\pi(\alpha), \pi(\alpha)) \neq 0 \). Now it follows from the root string property that either...
\( \alpha + x \in R \) or \( -\alpha + x \in R \). Take \( \tilde{\beta} = \pi(\alpha) \) in the first case and \( \tilde{\beta} = \pi(-\alpha) \) in the latter case. Then \( \tilde{\beta} \in \pi(R)^\times \) and \( \tilde{\beta} + \tilde{\delta} = \pi(\pm\alpha + x) \in \pi(R)^\times \). This finishes the proof of lemma. \( \square \)

Up to now we have only used properties A1–A3 of the automorphism \( \sigma \).

From now on we assume that \( \sigma \) satisfies A1–A4.

As we will see in the next proposition, the conditions A1–A4 imply that \( (G^\sigma, (\cdot, \cdot)_{H^\sigma}) \) satisfies axioms EA1–EA4 of an EALA. It is shown in [AABGP Chapter I] that most of the structural properties of an EALA rely on these axioms.

Since \( \sigma(R) = R \), \( \sigma \) restricted to \( V \) is an automorphism of \( V \). Then \( \pi(V) \) is the fixed point subspace of \( \sigma \) under \( \sigma \). Set

\[
V^\sigma := \text{span}_R R^\sigma \quad \text{and} \quad V^\pi := \text{span}_R \pi(R) = \pi(V).
\]

We will see in the next section that in general \( V^\sigma \) is a proper subspace of \( V^\pi \), so here the upper index \( \sigma \) does not stand for the fixed points of \( \sigma \) on \( V \).

**Proposition 2.33.** Let \( (G, (\cdot, \cdot), H) \) be an EALA and \( \sigma \) be an automorphism of \( G \) such that A1–A4 hold.

(i) If \( (R^\sigma)^\times \neq \emptyset \), then \( (G^\sigma, (\cdot, \cdot), H^\sigma) \) satisfies axioms EA1–EA4 of an EALA.

(ii) If \( (R^\sigma)^\times \neq \emptyset \), then \( R^\sigma \) is a reduced EARS in \( V^\sigma \) (in the sense of Definition 1.7).

(iii) If \( \pi(R)^\times \neq \emptyset \), then \( \pi(R)^\times \) is an indecomposable SEARS in \( V^\pi \).

**Proof.** (i) EA1 holds for \( G^\sigma \) by (2.11) and the fact that the form \( (\cdot, \cdot) \) is nondegenerate and invariant on \( G \). EA2 holds by A4 and (2.24). EA3 holds by Lemma 2.28. Finally EA4 holds by Lemma 2.29.

(ii) Since \( (R^\sigma)^\times \neq \emptyset \), the form is nontrivial and positive semidefinite on \( V^\sigma \). By (2.22), R1 holds for \( R^\sigma \). R2 holds by the way \( V^\sigma \) is defined. Since \( R^\sigma \subseteq \pi(R) \), R3 holds by Lemma 2.29. By part (i), \( G^\sigma \) satisfies the axioms EA1–EA4 of an EALA. Therefore by [AABGP, Theorem 1.29], the root system \( R^\sigma \) of \( G^\sigma \) satisfies R4 and R7.

(iii) Clearly the form is positive semidefinite on \( V^\pi \). By Lemma 2.31(iii), \( \pi(R) \) and \( \pi(R)^\times \) have the same real span. So it remains to show that if \( \tilde{\alpha}, \tilde{\beta} \in \pi(R)^\times \), then

\[
2 \left( \frac{\tilde{\beta}, \tilde{\alpha}}{\tilde{\alpha}, \tilde{\alpha}} \right) \in \mathbb{Z}, \quad \text{and} \quad \tilde{\beta} - 2 \left( \frac{\tilde{\beta}, \tilde{\alpha}}{\tilde{\alpha}, \tilde{\alpha}} \right) \tilde{\alpha} \in \pi(R).
\]

To see (2.34), we use [ABP, Theorem 3.63] (see Remark 2.35 below), where it is shown that there exists a reduced irreducible EARS \( \tilde{R} \) and an isotropic element \( \tilde{\delta} \) such that

\[
\text{span}_R \tilde{R} = V^\pi \oplus \mathbb{R} \tilde{\delta} \quad \text{and} \quad \pi(R) \subseteq \tilde{R} \pmod{\mathbb{Z} \tilde{\delta}}.
\]

Now we may consider \( \tilde{\alpha} \) and \( \tilde{\beta} \) as elements of \( \tilde{R} \pmod{\mathbb{Z} \tilde{\delta}} \) and so (2.34) holds for them as elements of \( \tilde{R} \pmod{\mathbb{Z} \tilde{\delta}} \). Thus (2.34) holds for \( \tilde{\alpha}, \tilde{\beta} \) as elements of \( \pi(R) \). \( \square \)
Remark 2.35. The extended affine root system $\tilde{R}$ which appeared in the proof of Proposition 2.32 is in fact the root system of an EALA $\tilde{G}$ which is a covering algebra of $G$ constructed by affinization of the loop algebra of $G$ with respect to $\sigma$. In the statement of [ABP, Theorem 3.63], the EALA $(G, (\cdot, \cdot), H)$ is assumed to be tame. However, what we really need in the proof of Proposition 2.32 is only the fact that $R_4$ holds for $\tilde{R}$. But this holds if axioms EA1–EA3 of an EALA hold for $\tilde{G}$ (see [AABGP, Chapter I]). Now checking [ABP], one can see that the notion of tameness is not used in the proof of the fact that EA1–EA3 hold for $\tilde{G}$.

As in (1.3), let $R_{\sigma}^\sigma (R_{\sigma}^\sigma_{\text{iso}})$ denotes the set of isolated (nonisolated) isotropic roots of $R^\sigma$.

Corollary 2.36. Let $(R^\sigma)^\times \neq \emptyset$. Then

$$R^\sigma = (\bigcup_{i=1}^k R_i^\sigma) \cup R_{\text{iso}}^\sigma,$$

where for each $i$, $R_i^\sigma$ is an irreducible reduced EARS in the real span $\mathcal{V}_i^\sigma$ of $R_i^\sigma$, with $(R_i^\sigma, R_j^\sigma) = \{0\}$ if $i \neq j$. Moreover if

$$R_i := R_i^\sigma \cup (R_i^\sigma \cap (R^\sigma)^0),$$

then $R_i$ is a reduced indecomposable EARS in $\mathcal{V}_i^\sigma$.

Proof. See Proposition 2.33(ii) and Lemma 1.4. \qed

Note that

$$R_i^\sigma \subseteq R_i \quad \text{and} \quad \text{Span}_R R_i = \text{Span}_R R_i^\sigma = \mathcal{V}_i^\sigma.$$

Since $R_i^\sigma$ is an EARS in $\mathcal{V}_i^\sigma$, we may use the same notation as in the second paragraph after Definition 1.3. So $R_i^\sigma$ is an irreducible finite root system in $\mathcal{V}_i^\sigma$. Fix a choice of a fundamental system $\Pi_i$ for $R_i^\sigma$. Also choose a fixed preimage $\hat{\Pi}_i$ in $R_i^\sigma$ of $\hat{\Pi}_i$ under $\hat{\cdot}$. Let $\hat{\mathcal{V}}_i^\sigma$ be the real span of $\hat{\Pi}_i$. Then

$$\mathcal{V}_i^\sigma = \hat{\mathcal{V}}_i^\sigma \oplus (\mathcal{V}_i^\sigma)^0 \quad \text{and} \quad (\mathcal{V}_i^\sigma, \mathcal{V}_j^\sigma) = \{0\} \text{ for } i \neq j.$$

Then $\hat{\cdot}$ restricts to an isometry of $\hat{\mathcal{V}}_i^\sigma$ onto $\hat{\mathcal{V}}_i^\sigma$. Let

$$\hat{R}_i^\sigma = \{\hat{\alpha} \in \hat{\mathcal{V}}_i^\sigma \mid \hat{\alpha} + \delta \in R_i^\sigma \text{ for some } \delta \in (\mathcal{V}_i^\sigma)^0\}.$$

Then $\hat{R}_i^\sigma$ is a finite root system in $\hat{\mathcal{V}}_i^\sigma$ which is isometrically isomorphic under $\hat{\cdot}$ to $R_i^\sigma$. Set

$$(R_i^\sigma)^0 = R_i^\sigma \cap (\mathcal{V}_i^\sigma)^0.$$

Let $\nu_i = \dim(\mathcal{V}_i^\sigma)^0$ and $l_i = \dim \hat{\mathcal{V}}_i^\sigma$. Then by [AABGP, Chapter II], $(R_i^\sigma)^0 = S_i + S_i$, where $S_i$ is a semilattice in $(\mathcal{V}_i^\sigma)^0$ of rank $\nu_i$ (see also 2.30). Also the $\mathbb{Z}$-span of $S_i$ is a free abelian group of finite rank with a basis $B_i \subseteq S_i$. Now

$$B_i \cup \hat{\Pi}_i \subseteq R_i^\sigma \subseteq (\mathcal{H}^\sigma)^* \quad \text{and} \quad R_i^\sigma \subseteq \langle B_i \cup \hat{\Pi}_i \rangle.$$
Corresponding to the subspaces $\dot{\mathcal{V}}_i^\sigma$ and $(\mathcal{V}_i^\sigma)^0$ of $(\mathcal{H}_i^\sigma)^*$ we consider two subspaces $(\dot{\mathcal{V}}_i^\sigma)_C$ and $(\mathcal{V}_i^\sigma)^0_C$ of $\mathcal{H}^\sigma$ as follows. Let

\begin{equation}
(\dot{\mathcal{V}}_i^\sigma)_C = \sum_{\dot{\alpha} \in \Pi_i} \mathbb{C}t_\dot{\alpha} \subseteq \mathcal{H}^\sigma \quad \text{and} \quad (\mathcal{V}_i^\sigma)^0_C = \sum_{\delta \in B_i} \mathbb{C}t_\delta \subseteq \mathcal{H}^\sigma.
\end{equation}

Set

\[\dot{\mathcal{V}}_C^\sigma = \sum_{i=1}^k (\dot{\mathcal{V}}_i^\sigma)_C \quad \text{and} \quad (\mathcal{V}_C^\sigma)^0 = \sum_{i=1}^k (\mathcal{V}_i^\sigma)^0_C.\]

**Lemma 2.43.** Suppose $(R^\sigma)_{iso} \neq \emptyset$. If $R^\sigma_{iso} = \emptyset$ then

\[\dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0 = \bigoplus_{\alpha \in (R^\sigma)^x} [\mathcal{H}_0^\sigma, \mathcal{H}_{-\alpha}^\sigma].\]

**Proof.** Since $\mathcal{G}^\sigma$ satisfies axioms EA1-EA4 of an EALA (see Proposition 2.33), we have from [AABGP, Chapter I] that if $\alpha \in (R^\sigma)^x$ then $[\mathcal{G}_0^\sigma, \mathcal{G}_{-\alpha}^\sigma] = \mathbb{C}t_\alpha$. So if $\dot{\alpha} \in \dot{\Pi}_i \subseteq (R^\sigma)^x$, then $t_{\dot{\alpha}} \in [\mathcal{G}_0^\sigma, \mathcal{G}_{-\dot{\alpha}}^\sigma]$. If $\delta \in S_i$ then from (2.40), we have $\dot{\alpha} + \delta \in (R^\sigma)^x \subseteq (R^\sigma)^x$ for some $\dot{\alpha} \in \dot{R}_i^\sigma$. Then $t_{\dot{\alpha} + \delta} \in [\mathcal{G}_{\dot{\alpha} + \delta}^\sigma, \mathcal{G}_{-\dot{\alpha} - \delta}^\sigma]$. Thus $t_\delta$ is contained in the $\mathbb{C}$-span of $[\mathcal{G}_0^\sigma, \mathcal{G}_{-\dot{\alpha}}^\sigma]$, $\alpha \in (R^\sigma)^x$, and so $\dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0$ is contained in the sum appearing in the statement. Conversely, let $\dot{\alpha} \in (R^\sigma)^x$. Since $R^\sigma_{iso} = \emptyset$, $\in \in R^\sigma_{iso}$ for some $1 \leq i \leq k$. So $\dot{\alpha}$ is in the real span of $\dot{\Pi}_i \cup S_i$. It follows that $[\mathcal{G}_0^\sigma, \mathcal{G}_{-\dot{\alpha}}^\sigma] = \mathbb{C}t_{\dot{\alpha}} \subset \dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0$. This completes the proof. \qed

Assume that $(R^\sigma)^x \neq \emptyset$. Since the form $(\cdot, \cdot)$ is real valued and positive definite on $\dot{\mathcal{V}}_C^\sigma$, it follows that the form on $\mathcal{H}^\sigma$ restricted to $(\dot{\mathcal{V}}_C^\sigma)_C$ is nondegenerate, and so by (2.40) the form on $\dot{\mathcal{V}}_C^\sigma$ is nondegenerate. Since the form $(\cdot, \cdot)$ is nondegenerate on $\mathcal{H}^\sigma$, and $(\dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0, (\mathcal{V}_C^\sigma)^0) = \{0\}$, it follows that there exists a subspace $\mathcal{D}$ of $\mathcal{H}^\sigma$ such that

\begin{equation}
\dim \mathcal{D} = \dim \sum_{i=1}^k (\mathcal{V}_i^\sigma)_C^0,
\end{equation}

\begin{equation}
(\mathcal{D}, \dot{\mathcal{V}}_C^\sigma) = \{0\},
\end{equation}

$(\cdot, \cdot)$ is nondegenerate on $\dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0 \oplus \mathcal{D}$. Next consider a complement $\mathcal{W}$ of $\dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0 \oplus \mathcal{D}$ in $\mathcal{H}^\sigma$ such that

\begin{equation}
(\dot{\mathcal{V}}_C^\sigma \oplus (\mathcal{V}_C^\sigma)^0 \oplus \mathcal{D}, \mathcal{W}) = \{0\},
\end{equation}

$(\cdot, \cdot)$ is nondegenerate on $\mathcal{W}$.

For $1 \leq j \leq k$, consider a subspace $\mathcal{D}_j$ of $\mathcal{D}$ such that

\begin{equation}
\dim (\mathcal{V}_j^\sigma)^0_C = \dim \mathcal{D}_j,
\end{equation}

$(\cdot, \cdot)$ is nondegenerate on $(\mathcal{V}_j^\sigma)_C \oplus (\mathcal{V}_j^\sigma)^0 \oplus \mathcal{D}_j$.

Set

\begin{equation}
\mathcal{H}_i^\sigma = (\dot{\mathcal{V}}_i^\sigma)_C \oplus (\mathcal{V}_i^\sigma)^0 \oplus \mathcal{D}_i.
\end{equation}
Then the form on $\mathcal{H}_i^\sigma$ is nondegenerate and

$$\mathcal{H}^\sigma = \sum_{i=1}^{k} \mathcal{H}_i^\sigma \oplus W.$$ 

Put

$$\mathcal{G}_i^\sigma = \mathcal{H}_i^\sigma \oplus \sum_{\tilde{\alpha} \in R_i \setminus \{0\}} \mathcal{G}_\tilde{\alpha}^\sigma.$$ 

**Proposition 2.48.** (i) $\mathcal{G}_i^\sigma$ is a subalgebra of $\mathcal{G}^\sigma$.  
(ii) $\mathcal{H}_i^\sigma$ is an abelian subalgebra of $\mathcal{G}_i^\sigma$. Moreover the form restricted to $\mathcal{H}_i^\sigma$ is nondegenerate.  
(iii) $C_{\mathcal{G}_i^\sigma}(\mathcal{H}_i^\sigma) = \mathcal{H}_i^\sigma$.

**Proof.** (i) Set

$$\mathcal{K}_i = \sum_{\tilde{\alpha} \in R_i \setminus \{0\}} \mathcal{G}_\tilde{\alpha}^\sigma \text{ and } T_i = \sum_{\tilde{\alpha} \in R_i} \mathcal{C}_t\tilde{\alpha} = \sum_{\tilde{\alpha} \in R_i^0} \mathcal{C}_t\tilde{\alpha} = \sum_{\tilde{\alpha} \in (R_i^0)^\times} \mathcal{C}_t\tilde{\alpha},$$

and let $\mathcal{M}_i$ be the subalgebra of $\mathcal{G}^\sigma$ generated by $\mathcal{K}_i$. By (2.41) and (2.42),

$$T_i = (\mathcal{V}_i^\sigma)_C \oplus (\mathcal{V}_i^\sigma)_0.$$

We first claim that $\mathcal{M}_i = \mathcal{K}_i \oplus T_i$. To see this note that by Proposition 2.33.

EA1–EA4 hold for $\mathcal{G}^\sigma$, therefore for any $\tilde{\alpha} \in R^\sigma$,

$$[\mathcal{G}_\tilde{\alpha}^\sigma, \mathcal{G}_{-\tilde{\alpha}}^\sigma] = \mathcal{C}_t\tilde{\alpha}$$

(See [AABGP, Chapter I]). Thus $T_i \subseteq \mathcal{M}_i$. Thus $T_i \oplus \mathcal{K}_i \subseteq \mathcal{M}_i$. To see the equality it is enough to show that $T_i \oplus \mathcal{K}_i$ is closed under $[\cdot, \cdot]$. But $T_i \subseteq \mathcal{H}^\sigma$, so $[T_i, T_i \oplus \mathcal{K}_i] \subseteq \mathcal{K}_i$. To see $[\mathcal{K}_i, \mathcal{K}_i] \subseteq T_i \oplus \mathcal{K}_i$, let $\tilde{\alpha}, \tilde{\beta} \in R_i \setminus \{0\}$, and $[\mathcal{G}_\tilde{\alpha}^\sigma, \mathcal{G}_\tilde{\beta}^\sigma] \neq \{0\}$. In particular $\tilde{\alpha} + \tilde{\beta} \in R^\sigma$. If $\tilde{\beta} = -\tilde{\alpha}$, then $[\mathcal{G}_\tilde{\alpha}^\sigma, \mathcal{G}_{-\tilde{\alpha}}^\sigma] = \mathcal{C}_t\tilde{\alpha} \subseteq T_i$. So we may assume that $\tilde{\beta} + \tilde{\alpha} \neq 0$. If $\tilde{\alpha} + \tilde{\beta}$ is nonisotropic, then either $(\tilde{\alpha} + \tilde{\beta}, \tilde{\alpha}) \neq 0$ or $(\tilde{\alpha} + \tilde{\beta}, \tilde{\beta}) \neq 0$. In either cases $\tilde{\alpha} + \tilde{\beta}$ is a nonisotropic root in $R^\sigma$ which is not orthogonal to $\tilde{\alpha} \in R_i^\sigma$ or $\tilde{\beta} \in R_i^\sigma$, so $\tilde{\alpha} + \tilde{\beta} \in (R_i^\sigma)^\times$. Thus

$$[\mathcal{G}_\tilde{\alpha}^\sigma, \mathcal{G}_\tilde{\beta}^\sigma] \subseteq \mathcal{G}_{\tilde{\alpha} + \tilde{\beta}}^\sigma \subseteq \mathcal{K}_i.$$ 

If $\tilde{\alpha} + \tilde{\beta}$ is isotropic, then $\tilde{\alpha} + \tilde{\beta} \in (R_i^\sigma) \cap (R^\sigma)^0 \subseteq R_i$. Thus $[\mathcal{G}_\tilde{\alpha}^\sigma, \mathcal{G}_\tilde{\beta}^\sigma] \subseteq \mathcal{G}_{\tilde{\alpha} + \tilde{\beta}}^\sigma \subseteq \mathcal{K}_i$, as $\tilde{\alpha} + \tilde{\beta} \in R_i \setminus \{0\}$. This completes the proof of our claim.

Next note that

$$\mathcal{G}_i^\sigma = \mathcal{H}_i^\sigma \oplus \sum_{\tilde{\alpha} \in R_i \setminus \{0\}} \mathcal{G}_\tilde{\alpha}^\sigma = T_i \oplus \mathcal{K}_i \oplus D_i = \mathcal{M}_i \oplus D_i.$$

Since $D_i \subseteq \mathcal{H}_i^\sigma \subseteq \mathcal{H}^\sigma$, $[D_i, \mathcal{D}_i \oplus \mathcal{K}_i] \subseteq \mathcal{K}_i$. But $\mathcal{M}_i$ is the subalgebra generated by $\mathcal{K}_i$, so $[D_i \oplus \mathcal{M}_i, D_i \oplus \mathcal{M}_i] \subseteq \mathcal{M}_i$.

(ii) The first assertion is clear. It follows from (2.46) that the form is nondegenerate.
(iii) Suppose to the contrary that \( C_{G^a}(\mathcal{H}_i^a) \not\subseteq \mathcal{H}_i^a \). Then there exists 
\( x = x_0 + x_{\tilde{\alpha}_1} + \cdots + x_{\tilde{\alpha}_t} \in C_{G^a}(\mathcal{H}_i^a) \) such that 
\( x_0 \in \mathcal{H}_i^a \), \( x_{\tilde{\alpha}_i} \)'s are nonzero 
and \( \tilde{\alpha}_j \)'s are distinct roots of \( R_i \setminus \{0\} \). Then for any \( h \in \mathcal{H}_i^a \),
\[
0 = [h, x] = \tilde{\alpha}_1(h)x_{\tilde{\alpha}_1} + \cdots + \tilde{\alpha}_t(h)x_{\tilde{\alpha}_t}.
\]
Thus \( \tilde{\alpha}_j(h) = 0 \) for \( 1 \leq j \leq t \), \( h \in \mathcal{H}_i^a \). That is \( (t_{\tilde{\alpha}_j}, \mathcal{H}_i^a) = \{0\} \). Since \( t_{\tilde{\alpha}_j} \in T_i \subseteq \mathcal{H}_i^a \), it follows from part (ii) that \( \tilde{\alpha}_j = 0 \) for all \( j \), which is a contradiction. \( \square \)

**Proposition 2.49.** Suppose that \( (R^a)^* \neq \emptyset \). Then \((G^a_i, (\cdot, \cdot), \mathcal{H}_i^a)\) satisfies axioms EA1-EA5(a) of an EALA.

**Proof.** By Proposition 2.33, EA1-EA2 hold for \( G^a \). So for any \( \tilde{\alpha} \in R^a \), the form restricted to \( G^a_\tilde{\alpha} \oplus G^a_{-\tilde{\alpha}} \) is nondegenerate. In particular, this holds for any \( \tilde{\alpha} \in R_i \). Thus \( (\cdot, \cdot) \) is nondegenerate on \( G^a_i \) and EA1 holds for \( G^a_i \). From (2.41) we have \( R_i \subset (\mathcal{H}^a)^* \). Now from the way \( G^a_i \) is defined it follows that \( \text{ad}_{G^a_i} h \) is diagonalizable for all \( h \in \mathcal{H}_i^a \). This together with Proposition 2.45(ii) imply that EA2 holds for \( G^a_i \). EA3 holds for \( G^a_i \) as it holds for \( G^a \). EA4 holds for the root system \( R_i^a \) of \( G^a \) since \( R_i^a \subseteq R^a \) and EA4 holds for \( R^a \). Finally EA5(a) holds by Corollary 2.36 \( \square \)

Set
\[
\mathcal{I} = \sum_{\tilde{\delta} \in R_i \setminus \{0\}} G^a_{\tilde{\delta}},
\]
where sum runs through
\[
\tilde{\delta} \in R_i^a \setminus (\cup_{i=1}^k R_i) = R_i^a \setminus \left( \cup_{i=1}^k (R_i)_{\text{iso}} \right). \tag{2.51}
\]
Then
\[
G^a = \mathcal{H}^a \oplus \sum_{\tilde{\delta} \in R^a \setminus \{0\}} G^a_{\tilde{\delta}} = \sum_{i=1}^k G^a_i \oplus \mathcal{W} \oplus \mathcal{I}. \tag{2.52}
\]

For our next proposition we need to state some results regarding the core \( G^c \) of \( G^a \). In particular, we would like to obtain some criteria for the tameness of \( G^a \). The notions of core and tameness are defined for an EALA but as we will see in next section \( G^a \) may not be in general an EALA. In the sequel, we also have some other Lie algebras which satisfy EA1-EA4 but are not in general an EALA. So let us define the core \( G_c \) of a triple \((G, (\cdot, \cdot), \mathcal{H})\) satisfying EA1-EA4 to be the subalgebra of \( G \) generated by root spaces \( G_{\alpha} \), \( \alpha \in R^a \). \( G \) is called **tame** if the centralizer of \( G_c \) in \( G \) is contained in \( G_c \). It follows that \( G_c \) is a perfect ideal of \( G \).

Since \( \sigma(G_{\alpha}) = G_{\sigma(\alpha)} \), we have \( \sigma(G_c) = G_c \). Thus
\[
G_c = \bigoplus_{i=0}^{m-1} (G_c)_i, \tag{2.53}
\]
where \((G_c)_i = G_c \cap G_i\).
Lemma 2.54. Suppose \((R^\sigma)^\times \neq \emptyset\). Then

(i) \(G \cap G^\sigma\) is the sum of spaces of the forms:

\[
G^\sigma_{\pi(\alpha)}, \quad \alpha \in R, \quad \pi(\alpha) \in \pi(R)^\times,
\]

and

\[
G^\sigma_{\pi(\alpha)}, \quad \alpha, \beta \in R, \quad \pi(\alpha), \pi(\beta) \in \pi(R)^\times, \quad i \in \mathbb{Z}.
\]

(ii) \(G^\sigma\) is the sum of spaces of the forms \((2.55)\) and their commutators.

(iii) \(G^\sigma \subseteq G^\sigma \cap G_c^\sigma\).

(iv) If \(G^\sigma_c \cap H^\sigma = G^\sigma \cap H^\sigma\), then \(C_{G^\sigma}(G^\sigma \cap G_c^\sigma) \subseteq G^\sigma \cap G_c^\sigma\). In particular if \(G\) is tame and \(G^\sigma \cap G_c^\sigma\), then \(G^\sigma\) is tame.

Proof. (i) This is an immediate consequence of \((2.55)\) and [ABP, Lemma 3.53].

(ii) Let \(S\) be the sum of the spaces in \((2.55)\) and their commutators. Then \(S \subseteq G^\sigma\). To show the reverse inclusion it is enough to show that \(S\) is closed under bracket, since \(S\) contains all the generators of \(G^\sigma\). So it is enough to show that for any three spaces \(G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}\) as in \((2.55)\),

\[
\{G^\sigma_{\pi(\alpha)}, [G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}]\} \subseteq S.
\]

Certainly, we may assume that the two brackets involved in \((2.55)\) are nonzero. Then by \((2.23)\), \(\pi(\beta) + \pi(\gamma) = \pi(\alpha')\) for some \(\alpha' \in R\) and \([G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}] \subseteq G^\sigma_{\pi(\alpha')}\). Now if \(\pi(\alpha') \notin \pi(R)^\times\), then

\[
\{G^\sigma_{\pi(\alpha)}, [G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}]\} \subseteq \{G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\alpha')}\} \subseteq S.
\]

If \(\pi(\alpha') \in \pi(R)^0\), then \(\pi(\alpha) + \pi(\alpha')\) is nonisotropic and by \((2.23)\), \(\pi(\alpha) + \pi(\alpha') = \pi(\beta')\) for some \(\beta' \in R\), and \([G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\beta')}] \subseteq G^\sigma_{\pi(\beta')}\). But \(G^\sigma_{\pi(\beta')} \subseteq S\), since \(\pi(\beta') \in \pi(R)^\times\). This completes the proof of part (ii). Now (iii) is an immediate consequence of (i) and (ii).

(iv) Let \(x \in C_{G^\sigma}(G^\sigma \cap G_c^\sigma)\). Then \(x = \sum_{i=0}^n x_{\tilde{\alpha}_i}\) where \(\tilde{\alpha}_i\) are distinct roots of \(R^\sigma\) with \(\tilde{\alpha}_0 = 0\), and \(x_{\tilde{\alpha}_i} \in G^\sigma_{\tilde{\alpha}_i}\) for \(i \neq 0\). Let \(\tilde{\alpha}, \tilde{\beta} \in (R^\sigma)^\times\). By part (i),

\[
\sum_{i=0}^n [x_{\tilde{\alpha}_i}, G^\sigma_{\tilde{\alpha}_i}] = [x, G^\sigma_{\tilde{\alpha}_i}] = \{0\} = \sum_{i=0}^n [x_{\tilde{\alpha}_i}, [G^\sigma_{\tilde{\alpha}_i}, G^\sigma_{-\tilde{\alpha}_i}]] = [x, [G^\sigma_{\tilde{\alpha}_i}, G^\sigma_{-\tilde{\alpha}_i}]],
\]

for \(j \in \mathbb{Z}\), where \(x_{\tilde{\alpha}_i} \subseteq G^\sigma_{\tilde{\alpha}_i+\tilde{\alpha}}\) and \([x_{\tilde{\alpha}_i}, [G^\sigma_{\tilde{\alpha}_i}, G^\sigma_{-\tilde{\alpha}_i}]] \subseteq G^\sigma_{\tilde{\alpha}_i+\tilde{\alpha}+\tilde{\beta}}\). It follows that for \(0 \leq i \leq n\),

\[
[x_{\tilde{\alpha}_i}, G^\sigma_{\tilde{\alpha}_i}] = \{0\} \quad \text{and} \quad [x_{\tilde{\alpha}_i}, [G^\sigma_{\tilde{\alpha}_i}, G^\sigma_{-\tilde{\alpha}_i}]] = \{0\},
\]

and so \(x_{\tilde{\alpha}_i} \in C_{G^\sigma}(G^\sigma \cap G_c^\sigma)\) for \(0 \leq i \leq n\). Therefore we must show that for each \(i\), \(x_{\tilde{\alpha}_i} \in G^\sigma_{\tilde{\alpha}_i+\tilde{\alpha}}\). By \((2.10)\), it is enough to show that for each \(i\), \(x_{\tilde{\alpha}_i}, G^\sigma \cap G_c^\sigma\) = \{0\}. Since the form is nondegenerate and invariant on \(G^\sigma\),
and since $G_0^\sigma = H^\sigma$, it follows that
\[
(x_0, G^\sigma \cap G_c) = (x_0, H^\sigma \cap G_c) = (x_0, H^\sigma \cap G_c^\sigma)
\]
\[
= (x_0, G_c^\sigma, G_c^\sigma) 
\]
\[
= ([x_0, G_c^\sigma], G_c^\sigma) \subseteq ([x_0, G^\sigma \cap G_c], G_c^\sigma) = \{0\}.
\]
Thus $x_0 \in G_c^\perp$. Next, we show that for $1 \leq i \leq n$, $\hat{\alpha}_i \in G_c^\perp$ (note that $\hat{\alpha}_i \neq 0$). Fix $1 \leq i \leq n$. Let $\hat{\alpha}, \hat{\beta} \in (R^\sigma)^\times$ and $\hat{\gamma} = \hat{\alpha} + \hat{\beta}$. Let $y_\hat{\gamma} \in G_{\hat{\gamma}}^\sigma$ or $y_\hat{\gamma} \in [G_{\hat{\beta}, \hat{\beta}}, G_{\hat{\beta}, \hat{\beta}}]$ for some $j \in \mathbb{Z}$. We must show that $(x_\hat{\alpha}, y_\hat{\gamma}) = 0$. If $\hat{\alpha} + \hat{\gamma} \neq 0$ then by (2.24), $(x_\hat{\alpha}, y_\hat{\gamma}) = 0$. So we may assume that $\hat{\gamma} = -\hat{\alpha}_i$. Since $y_\hat{\gamma} \in G^\sigma \cap G_c$, and since EA1-EA2 hold for $G^\sigma$, we see from [AABGP, I.(110)] that
\[
0 = [x_\hat{\alpha}, y_\hat{\gamma}] = (x_\hat{\alpha}, y_\hat{\gamma})t_{\hat{\alpha}_i}.
\]
Thus $(x_\hat{\alpha}, y_\hat{\gamma}) = 0$. This completes the proof of the first statement in (iv). Now if $G$ is tame then $G_c^\perp \subseteq G_c$. So the second statement of (iv) is an immediate consequence of the first statement.

\[\square\]

**Remark 2.59.** As we will see in the next section, in many examples $G_c^\sigma$ is a proper subspace of $G^\sigma \cap G_c$.

Note that from Proposition 2.54(ii), we have
\[
G_c^\sigma \subseteq \sum_{i=1}^k G_i^\sigma \subseteq G^\sigma.
\]

**Proposition 2.61.** (i) $W \subseteq C_{G^\sigma}(\sum_{i=1}^k G_i^\sigma)$.
(ii) $\sum_{\delta \in R^\sigma_{iso}} G_\delta^\sigma \subseteq C_{G^\sigma}(G_c^\sigma)$. In particular $W \oplus I \subseteq C_{G^\sigma}(G_c^\sigma)$.

**Proof.** (i) Since $W \subseteq H^\sigma$, we have $[W, H^\sigma] = \{0\}$. Therefore, it remains to show that $[W, G_\alpha^\sigma] = \{0\}$ for any $\alpha \in R_i \setminus \{0\}$, $1 \leq i \leq k$. Let $\alpha \in R_i \setminus \{0\}$. Then $\alpha \in (R_i^\sigma)^\times$. By Corollary 2.39, $R_i^\sigma$ is tame, so $(R_i^\sigma)^\times = ((R_i^\sigma)^\times)$. By (2.41), (2.42) and (2.45), we have $W \subseteq \ker \hat{\beta}$, for any $\hat{\beta} \in (R_i^\sigma)^\times$. Therefore $[W, G_\alpha^\sigma] = \{0\}$.

(ii) Let $\delta \in R^\sigma_{iso}$. Since $\delta + \alpha \notin R^\sigma$ for any $\alpha \in (R^\sigma)^\times$, $[G_\delta^\sigma, G_\delta^\sigma] = \{0\}$. But $G_c^\sigma$ is generated by root spaces $G_\alpha^\sigma$, $\alpha \in (R^\sigma)^\times$, so $[G_\delta^\sigma, G_\delta^\sigma] = \{0\}$. Now the proof is complete, using part (i) and (2.60). \[\square\]

**Corollary 2.62.** If $(R^\sigma)^\times \neq \emptyset$, then for each $i$, the triple
\[
(G_i^\sigma \oplus W, (\cdot, \cdot), H^\sigma_i \oplus W)
\]
satisfies EA1-EA5(a).

**Proof.** By Proposition 2.61, $G_i^\sigma \oplus W$ is a Lie algebra. By Proposition 2.49, the form $(\cdot, \cdot)$ is nondegenerate on $G_i^\sigma$, so by (2.43) and (2.48), the form $(\cdot, \cdot)$ is nondegenerate on the Lie algebra $G_i^\sigma \oplus W$. By Propositions 2.48(iii)
and Proposition 2.61(ii), $\mathcal{H}^\sigma_i \oplus \mathcal{W}$ is self-centralizing in $\mathcal{G}^\sigma_i \oplus \mathcal{W}$. By acting as zero on $\mathcal{W}$ we may identify elements of $(\mathcal{H}^\sigma_i)^*$ as elements of $(\mathcal{H}^\sigma_i \oplus \mathcal{W})^*$. Then it follows from Proposition 2.49 that EA2 holds for $\mathcal{G}^\sigma_i \oplus \mathcal{W}$ and that $\mathcal{G}^\sigma_i$ and $\mathcal{G}^\sigma_i \oplus \mathcal{W}$ have the same root system. Now it is clear that EA3-EA5(a) hold for $\mathcal{G}^\sigma_i \oplus \mathcal{W}$.

Recall that the EALA $G$ is called nondegenerate if the real space $V^0$ and its complex span in $\mathcal{H}^\sigma$ have the same dimension. We summarize our results in the following theorem.

**Theorem 2.63.** Let $(G, (\cdot, \cdot), \mathcal{H})$ be an EALA with corresponding root system $R$. Let $\sigma$ be an automorphism of $G$ satisfying A1–A4. Let $\mathcal{G}^\sigma$ $(\mathcal{H}^\sigma)$ be the fixed point subalgebra of $G$ $(\mathcal{H})$, under $\sigma$, and let $R^\sigma$ be the root system of $\mathcal{G}^\sigma$ with respect to $\mathcal{H}^\sigma$. Suppose $(R^\sigma)^\times \neq \emptyset$, then

(i) $(\mathcal{G}^\sigma, (\cdot, \cdot), \mathcal{H}^\sigma)$ satisfies axiom EA1–EA4 of an EALA and $R^\sigma$ is a reduced EARS.

(ii) $R^\sigma = (\bigcup_{i=1}^k R_i) \cup R^\sigma_{iso}$ where for each $i$, $R_i$ is an indecomposable reduced EARS with $(R_i^\times, R_j^\times) = \{0\}$ if $i \neq j$. (The union is not necessarily disjoint.)

(iii) $\mathcal{H}^\sigma = \sum_{i=1}^k \mathcal{H}^\sigma_i \oplus \mathcal{W}$ where $\mathcal{H}^\sigma_i$ and $\mathcal{W}$ are some subspaces of $\mathcal{H}^\sigma$ with $(\mathcal{H}^\sigma_i, \mathcal{W}) = \{0\}$ and $[\mathcal{W}, \mathcal{G}^\sigma_i] = \{0\}$, for each $i$.

(iv) $\mathcal{G}^\sigma = \sum_{i=1}^k \mathcal{G}^\sigma_i \oplus \mathcal{W} \oplus \mathcal{I}$, where for each $i$, $(\mathcal{G}^\sigma_i, (\cdot, \cdot), \mathcal{H}^\sigma_i)$ is a Lie algebra satisfying axioms EA1–EA5(a) of an EALA, and $\mathcal{I}$ is a subspace of $\mathcal{G}^\sigma$, satisfying $\mathcal{I} \subseteq C_{\mathcal{G}^\sigma}(\mathcal{G}^\sigma_c)$. Moreover, $\mathcal{I} = \{0\}$ if $R^\sigma_{iso} = \emptyset$.

(v) If $i \neq j$, then $[(\mathcal{G}^\sigma_i)_c, (\mathcal{G}^\sigma_j)_c] = \{0\}$. Moreover $\mathcal{G}^\sigma_c = \sum_{i=1}^k (\mathcal{G}^\sigma_i)_c$. ($(\mathcal{G}^\sigma)^c$ is the core of $\mathcal{G}^\sigma$.)

(vi) If $G$ is nondegenerate then for each $i$, dim $\mathcal{H}^\sigma_i = l_i + 2\nu_i$.

**Proof.** For (i) see Proposition 2.33. For (ii) see Corollary 2.36 and Proposition 2.61. For (iii) see Proposition 2.45 and Proposition 2.61. For (iv) see (2.50), Proposition 2.49 and Proposition 2.61.

For the first statement of (v), it is enough to show that if $\pi(\alpha) \in R^\times_i$ and $\pi(\beta) \in R^\times_j$, then $\pi(\alpha) + \pi(\beta)$ is not a root in $R^\sigma$. By part (ii), $(\pi(\alpha), \pi(\beta)) = 0$. Therefore $\pi(\alpha) + \pi(\beta)$ is not orthogonal to both $\pi(\alpha)$ and $\pi(\beta)$ and so cannot be a root (by part (ii)). This in particular shows that $\sum_{i=1}^k (\mathcal{G}^\sigma_i)_c$ is a Lie algebra which contains all generators of $\mathcal{G}^\sigma_c$. So the second statement of (v) holds. (vi) is clear from the construction of $\mathcal{H}^\sigma_i$.

As a corollary we can state a weak version of a result which is due to [BM] (see Remark 2.65).

**Corollary 2.64.** Let $(G, (\cdot, \cdot), \mathcal{H})$ be a finite dimensional complex simple Lie algebra, where $\mathcal{H}$ is a Cartan subalgebra and $(\cdot, \cdot)$ is the Killing form on $G$. Let $\sigma$ be an automorphism of $G$ satisfying A1–A2 and A4. Then $(\mathcal{G}^\sigma, (\cdot, \cdot), \mathcal{H}^\sigma)$ is a reductive Lie algebra.
Proof. It is easy to see that $A_3$ is a consequence of $A_1$. $G$ is an EALA of nullity zero and so there is no nonzero isotropic root. Therefore by (2.50) and (2.51), $I = \{0\}$. By Corollary 2.36 and part (ii) of Theorem 2.63, $R_i = R_i^\sigma$ is an irreducible finite roots system in $V_i^\sigma$. By Theorem 2.63(iv) and (2.47), each $G_i^\sigma$ is an EALA of nullity zero, where $H_i^\sigma$ is of dimension equal to the rank of $R_i^\sigma$. Thus $G_i^\sigma$ is a finite dimensional simple Lie algebra.

Finally $W$ is an abelian subalgebra of $G$. In fact it follows from Proposition 2.61(i) that $W = Z(G^\sigma)$. $\square$

Remark 2.65. (i) According to [BM], if $\sigma$ satisfies only $A_1$, then $G^\sigma$ is a reductive Lie algebra. A version of conjugacy for Cartan subalgebras is used in the proof and no such result is known for a general EALA.

(ii) According to Theorem 2.63 and Corollary 2.62, we may express the result of [BM] in another way. Namely, if $\sigma$ is a finite order automorphism of a finite dimensional complex simple Lie algebra $G^\sigma$, then $G^\sigma = J_1 \oplus J_2$ where $J_1$ is a semisimple Lie algebra and $J_2$ is an EALA such that $J_2/Z(J_2)$ is a finite dimensional simple Lie algebra.

(iii) Similar to part (ii), we may state Theorem 2.63(iv) in a different way. In fact by Corollary 2.62, for a fixed $i$, we can consider $G_i^\sigma \oplus W$ as a Lie algebra satisfying EA1–EA5(a). So part (iv) of the Theorem can be restated to say $G^\sigma$ is a direct sum of some EALAs and a subspace $I$ satisfying $I \subseteq C_{G^\sigma}(G_i^\sigma)$.

(iv) One may define a notion of nondegeneracy for a triple $(G, (\cdot, \cdot), H)$ satisfying EA1–EA5(a) exactly in the same way which one defines this for an EALA. Therefore by Theorem 2.63, for each $i$, $G_i^\sigma$ is nondegenerate.

Corollary 2.66. Let $(G, (\cdot, \cdot), H), \sigma$ and $R^\sigma$ be as in Theorem 2.63. If $R^\sigma$ is an irreducible EARS, then $(G^\sigma, (\cdot, \cdot), H^\sigma)$ is an EALA.

Proof. By Theorem 2.63, $G^\sigma$ satisfies EA1–EA5(a). Now EA5(b) also holds as $R^\sigma$ is irreducible. $\square$

Corollary 2.67. Let $(G, (\cdot, \cdot), H)$ be as in Theorem 2.63, and assume that $(R^\sigma)^\times \neq \emptyset$ and that $G$ is tame. If $G^\sigma \cap G_c = G_c^\sigma$, then $G^\sigma = \sum_{i=1}^k G_i^\sigma$, where each $G_i^\sigma$ is an EALA.

Proof. (i) By Lemma 2.51(iv), $C_{G^\sigma}(G_i^\sigma) \subseteq G_i^\sigma$. Therefore by Theorem 2.63, $(G^\sigma, (\cdot, \cdot), H^\sigma)$ is a tame Lie algebra satisfying EA1–EA4. Now it follows from [ABP, Lemma 3.62] that $G^\sigma$ also satisfies EA5(b) (that is $R^\sigma_{\text{iso}} = \emptyset$). In particular $I = \{0\}$ and each $R_i$ is an irreducible reduced EARS. Therefore, by part (iv) of Theorem 2.63, each $G_i^\sigma$ is an EALA. Since $G^\sigma$ is tame we have from Proposition 2.61 that $W \subseteq G^\sigma_c \cap H^\sigma$. By Lemma 2.63, $G^\sigma_c \cap H^\sigma = \sum_{\alpha \in (R^\sigma)^\times} [G^\sigma_c, G^\sigma_{-\alpha}] = V^\sigma_c \oplus (V^\sigma)_C^0$. But $V^\sigma_c \oplus (V^\sigma)_C^0$ and $W$ have zero intersection. Thus $W = \{0\}$. $\square$
3. EXAMPLES

In this section we present several examples which elaborate on the results obtained in Section 2. In 3.68-3.72 below a large class of examples is introduced which illustrate how the terms $\mathcal{G}_\sigma$'s, $W$ and $I$ (see Theorem 2.63) appear as the fixed points of automorphisms. In 3.75-3.79 and 3.81 we recall some examples from [A2], in which certain finite order automorphisms of EALA are given. We show that these automorphisms satisfy conditions A1-A4. Then, using Theorem 2.63, we are able to give a new proof of the results obtained in [A2], namely to prove that many examples of EALA (of types $D_{\ell}$, $A_{\ell}$, $B_{\ell}$, $C_{\ell}$, and $BC_{\ell}$) can be obtained as the fixed points of automorphisms of some other EALA (of types $A_{\ell}$, $D_{\ell}$ and $C_{\ell}$) which may have a simpler structure. Finally in 3.83, we present an example regarding the results in [ABP].

Example 3.68. Let $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ be an EALA of type $X_{\ell}$ and nullity $\nu$. Let $R$ be the corresponding irreducible extended affine root system and denote its root lattice by $Q$. Then

$$Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{\ell} \oplus \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_{\nu},$$

where $\{\alpha_1, \ldots, \alpha_{\ell}\}$ is a subset of $R$ which form a basis for a finite root system of type $X_{\ell}$ and $\{\delta_1, \ldots, \delta_{\nu}\}$ is a subset of $R$ which form a basis of the radical of the form restricted to the real span of $R$. Consider any group homomorphism $\phi: Q \rightarrow \mathbb{C} \setminus \{0\}$. $\phi$ is uniquely determined by specifying $\phi(\alpha_i)$ for $1 \leq i \leq \ell$ and $\phi(\delta_j)$ for $1 \leq j \leq \nu$. The homomorphism $\phi$ induces an automorphism $\sigma$ of $\mathcal{G}$ by letting

$$\sigma_{|_{\alpha}} = \phi(\alpha)\text{id}_{\mathcal{G}_\alpha} \quad \text{for } \alpha \in R.$$  

Since $\mathcal{G}_0 = \mathcal{H}$ and $\phi(0) = 1$, we have $\sigma(h) = h$ for all $h \in \mathcal{H}$. Note that $\sigma$ is of finite order if and only if $\phi(\alpha_i)$'s for $1 \leq i \leq \ell$ and $\phi(\delta_j)$'s for $1 \leq j \leq \nu$ are roots of unity. Assume that $\sigma^m = \text{id}$, for some positive integer $m$. Let $\mathcal{G}^\sigma$ be the set of fixed points of $\sigma$. Clearly $\sigma(\mathcal{H}) = \mathcal{H}$ and $\mathcal{H}^\sigma = \mathcal{H}$. Since

$$\mathcal{G}_\alpha \cap \mathcal{G}_\beta = \{0\} \quad \text{unless} \quad \alpha + \beta = 0,$$

it follows that

$$\langle \sigma(x), \sigma(y) \rangle = (x, y) \quad \text{for all } x, y \in \mathcal{G}.$$  

Since $\sigma(\alpha) = \alpha$ for all $\alpha \in (\mathcal{H}^\sigma)^*$, it follows from [ABP, Proposition 3.25] that $C_{\mathcal{G}^\sigma}(\mathcal{H}) = \mathcal{H}$. Therefore $\sigma$ satisfies conditions A1–A4. Thus by Theorem 2.63 $\mathcal{G}^\sigma$ satisfies axioms EA1–EA4. Therefore if $(R^\sigma)^\times \neq \emptyset$, then

$$(3.69) \quad \mathcal{G}^\sigma = \sum_{i=1}^{k} \mathcal{G}_i^\sigma \oplus W \oplus I,$$

where for each $i$, $(\mathcal{G}_i^\sigma, (\cdot, \cdot), \mathcal{H}_i^\sigma)$ satisfies EA1–EA5(a) and $W$ and $I$ are as in Theorem 2.63. Note that $\mathcal{G}^\sigma \cap \mathcal{G}_\alpha \neq \{0\}$ if and only if $\phi(\alpha) = 1$. In
\[ R^\sigma = \{ \alpha \in R \mid \mathcal{G}^\sigma \cap \mathcal{G}_\alpha \neq \{0\} \} = \{ \alpha \in R \mid \mathcal{G}_\alpha \subseteq \mathcal{G}^\sigma \} = \{ \alpha \in R \mid \phi(\alpha) = 1 \}. \]

**Example 3.70.** In Example 3.68 assume that \( X_\ell \) has one of the types \( A_\ell, \) \( D, \) \( E, \) \( B_\ell, \) \( C_\ell, \) \( F_4 \) or \( G_2. \) Let \( S \) and \( L \) be semilattices which appear in the structure of \( R \) (see \( [AABGP, \text{II.2.32}]) \). We want to impose some restrictions on the semilattices \( S \) and \( L \) as follows. For types \( A_\ell (\ell \geq 2), \) \( D \) and \( E \) we impose no restriction as we know from \( [AABGP, \text{II.2.32}] \) that for these types the semilattice \( S \) is always a lattice. For type \( A_1 \) assume that \( S \) is a lattice and for the remaining types assume that both \( S \) and \( L \) are lattices and that \( S = L. \) (The root systems of toroidal Lie algebras, with some derivations added, are of this form). We claim that for such types and under the above restrictions the axiom EA5(b) also holds.

Suppose that \( (R^\sigma)^\times \neq \emptyset. \) So there exists \( \alpha \in R^\times \) such that \( \phi(\alpha) = 1. \) Let \( \delta \in (R^\sigma)^0. \) That is \( \phi(\delta) = 1. \) Under the above assumptions, it follows from the structure of EARS of type \( X_\ell \) that \( \alpha + \delta \in R. \) Now \( \phi(\alpha + \delta) = 1 \) and so \( \alpha + \delta \in R^\sigma. \) This shows that EA5(b) holds. Thus by Example 3.68,

\[
(3.71) \quad \mathcal{G}^\sigma = \bigoplus_{i=1}^{k} \mathcal{G}_i^\sigma \oplus \mathcal{W},
\]

where each \( \mathcal{G}_i^\sigma \) is an EALA and \( \mathcal{W} \) is an abelian subalgebra of \( \mathcal{G}. \)

**Example 3.72.** Let \( \mathcal{G} \) and \( \sigma \) be as in Example 3.68 and let \( X_\ell = A_1. \) We have

\[ R = (S + S) \cup (\pm \hat{\alpha} + S), \]

where \( S \) is a semilattice in the real span \( V^0 \) of \( R^0. \) If \( S \) is a lattice, then by Example 3.70, axiom EA5(b) holds for \( \mathcal{G}^\sigma. \)

Next suppose that \( S \) is not a lattice. We show that in this case it might happen that the axiom EA5(b) does not hold. To see this let the nullity \( \nu \) of \( R \) be 3. Let

\[ S = \{ \sum_{i=1}^{3} m_i \delta_i \mid m_i \in \mathbb{Z} \text{ and } m_im_j \equiv 0 \text{ mod } 2, \text{ if } i \neq j \}. \]

Then \( S \) is a semilattice in \( V^0 \) and \( Q = \mathbb{Z} \hat{\alpha} \oplus \mathbb{Z} \delta_1 \oplus \mathbb{Z} \delta_2 \oplus \mathbb{Z} \delta_3. \) Define

\[ \phi(\hat{\alpha}) = -1, \quad \phi(\delta_1) = 1, \quad \phi(\delta_2) = 1, \quad \text{and} \quad \phi(\delta_3) = -1. \]

Now \( \phi(\delta_1 + \delta_2) = 1 \) and so \( \delta := \delta_1 + \delta_2 \in (R^\sigma)^0. \) We claim that \( \delta \) is isolated, that is \( \delta \in R^\sigma_{is}. \) Suppose to the contrary that there exists \( \alpha = \pm \hat{\alpha} + m_1\delta_1 + m_2\delta_2 + m_3\delta_3 \in R^\sigma \) such that \( \alpha + \delta \in R^\sigma. \) We have

\[ 1 = \phi(\alpha) = -\phi(\delta_3)^{m_3} = -(-1)^{m_3}, \]

and so \( m_3 = 2k_3 + 1 \) for some \( k_3 \in \mathbb{Z}. \) Since \( \alpha \in R^\sigma \subseteq R, \) we must have \( m_1 = 2k_1 \) and \( m_2 = 2k_2 \) for some \( k_1, k_2 \in \mathbb{Z}. \) Then

\[ \alpha + \delta = \pm \hat{\alpha} + (2k_1 + 1)\delta_1 + (2k_2 + 1)\delta_2 + (2k_3 + 1)\delta_3 \in R^\sigma \subseteq R. \]

But this contradicts the fact that \( R \) contains no such root.
For our next few examples we need the following setting (see [BGK], [AABGP] and [A2]). Let $\nu \geq 1$. Let $e = (e_1, \ldots, e_\nu)$ be a vector in $\mathbb{C}^\nu$ and let $q = (q_{ij})$ be a $\nu \times \nu$-matrix such that

$$e_i = \pm 1, \quad q_{ii} = 1 \text{ for } 1 \leq i \leq \nu, \quad \text{and} \quad q_{ij} = q_{ji} \text{ for } 1 \leq i \neq j \leq \nu.$$  

Let $A$ be the associative algebra over $\mathbb{C}$ with generators $x_i, x_i^{-1}$ subject to the relations $x_ix_j = q_{ij}x_jx_i$. Then

$$A = \bigoplus_{\delta \in \mathbb{Z}^\nu} \mathbb{C}x^\delta, \text{ where } x^\delta = x_1^{n_1} \cdots x_\nu^{n_\nu} \text{ for } \delta = (n_1, \ldots, n_\nu) \in \mathbb{Z}^\nu.$$ 

Let $-$ be the involution on $A$ such that $\bar{x}_i = e_i x_i$, for all $i$. The pair $(A, -)$ is called the quantum torus with involution determined by the vector $e$ and the matrix $q$. We have $A = A_+ \oplus A_-$ where $A_+ = \{ h \in A \mid \bar{h} = h \}$ and $A_- = \{ s \in A \mid \bar{s} = -s \}$. Set

$$I_e = \{ i \mid 1 \leq i \leq \nu, \ e_i = -1 \} \quad \text{and} \quad J_q = \{ (i, j) \mid 1 \leq i < j \leq \nu, \ q_{ij} = -1 \}.$$ 

Put

$$Z_{e, q} = \{ \delta \in \mathbb{Z}^\nu \mid \sum_{i \in I_e} n_i + \sum_{(i, j) \in J_q} n_i n_j = 0 \} \quad \text{and} \quad Z_{e, q}^c = \mathbb{Z}^\nu \setminus Z_{e, q}.$$ 

Then $Z_{e, q}$ is a semilattice in $\mathbb{R}^\nu$.

Fix $n \geq 1$ and let $M_n(A)$ be the Lie algebra of $n \times n$ matrices with entries from $A$. Next define a bilinear form on $M_n(A)$ as follows. Define $\epsilon$ in the dual space of $A$ by the linear extension of $\epsilon(1) = 1$ and $\epsilon(x^\delta) = 0$ for any nonzero $\delta \in \mathbb{Z}^\nu$. Then for $A, B \in M_n(A)$ define $(A, B) = \epsilon(\text{tr}(AB))$. This defines a symmetric nondegenerate invariant bilinear form on $M_n(A)$.

Let $K$ be the Lie subalgebra $sl_n(A)$ of $M_n(A)$ consisting of matrices $A$ such that $\text{tr}(A) \equiv 0, \mod [A, A]$. Let $\mathcal{H}$ be the abelian subalgebra of $sl_n(A)$ with basis $e_{ii} - e_{i+1, i+1}$, for $1 \leq i \leq n-1$. Define $\epsilon_i \in \mathcal{H}^*$ by $\epsilon_i (e_{jj} - e_{j+1, j+1}) = \delta_{ij}$. Then, with respect to $\mathcal{H}$, we have the root space decomposition

$$K = \bigoplus_{\delta \in \mathcal{R}} K_{\delta}, \text{ where } \mathcal{R} = \{ \pm (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n - 1 \}.$$ 

Suppose that $K$ has a $\mathbb{Z}^\nu$ grading, say $K = \sum_{\delta \in \mathbb{Z}^\nu} K^\delta$, such that

$$\delta + \tau \neq 0 \implies (K^\delta, K^\tau) = \{ 0 \}, \quad \text{and}$$

$$\{ \delta \in \mathbb{Z}^\nu \mid K^\delta \neq \{ 0 \} \} \text{ spans } \mathbb{C}^\nu.$$ 

Later we would like to consider the fixed point subalgebra of $K$ under some automorphism $\sigma$. According to our previous notation, we should write $K^\sigma$ for the corresponding fixed point subalgebra. But this might cause some confusion with the notation which we used for the grading on $K$. To prevent this, we devote the upper index $\sigma$ only to indicate the fixed points, and we use other symbols such as $\delta, \tau, \gamma, \ldots$ for the grading on $K$. 


Thus, by [ABP, Proposition 3.25], A4 holds. One can see that $\epsilon$ satisfies conditions A2 and A3. To see A4 holds, note that the root system of $G$ with respect to $H$ is

$$ R = 2\mathbb{Z}^\nu \cup \{\epsilon_i - \epsilon_j + 2\delta \mid \delta \in \mathbb{Z}^\nu, \ 1 \leq i \neq j \leq n-1\}, $$

where $\epsilon_i$ is the element of $H^*$ which acts on $H$ by $\epsilon_i(e_{jj} - e_{j+1,j+1}) = \delta_{ij}$. Also note that $H^\sigma = \{\sum_{i=1}^{\ell} a_i(e_{ii} - e_{i+i,i+i}) \mid a_i \in \mathbb{C}\}$. Then it is clear that

$$ 0 \neq \alpha \in R \Rightarrow \pi(\alpha) = \alpha|_H^\sigma \neq 0. $$

Thus, by [ABP, Proposition 3.25], A4 holds. One can see that

$$ G^\sigma = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B^t = -B, \ C^t = -C, \ A, B, C \in M_\ell(A) \right\}. $$

The root system of $G^\sigma$ with respect to $H^\sigma$ is

$$ R^\sigma = 2\mathbb{Z}^\nu \cup \{\pm(\epsilon_i \pm \epsilon_j) + 2\delta \mid \delta \in \mathbb{Z}^\nu, \ 1 \leq i \neq j \leq \ell\}, $$
which is an irreducible reduced EARS of type $D_\ell$. Thus by Corollary 2.66, $G^\sigma$ is an EALA of type $D_\ell$. It is not difficult to show that $G^\sigma = K^\sigma \oplus \mathcal{C}$. Thus $G^\sigma \cap G = G^\sigma$. Therefore by Lemma 2.54 $G^\sigma$ is a tame EALA.

**Example 3.76.** Let $(A, -)$ be the quantum torus determined by $e$ and $q$ as in (3.75). Suppose that $m \geq 1$ and let $\tau_1, \ldots, \tau_m$ represent distinct cosets of $2\mathbb{Z}'$ in $\mathbb{Z}'$ with $\tau_1 = 0$ and $\tau_i \in \bar{Z}_{e,q}$ for all $i$. Define $\deg(x^q e_{pq}) = 2\delta + \lambda_p - \lambda_q$ where $\lambda_1 = \cdots = \lambda_{2\ell} = 0$ and $\lambda_{2\ell+i} = \tau_i$ for $1 \leq i \leq m$. Set

$$F = \begin{pmatrix} x^{\tau_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^{\tau_m} \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & I_\ell & 0 \\ I_\ell & 0 & 0 \\ 0 & 0 & F \end{pmatrix},$$

where $I_\ell$ is the $\ell \times \ell$ identity matrix. Then $G$ is a $n \times n$ matrix with $n = 2\ell + m$. For $Y \in K = sl_n(A)$ define $\sigma(Y) = -G^{-1} Y^t G$. Then $\sigma$ defines a period two automorphism of $G$. It is straightforward to see that $\sigma$ satisfies $A2^\alpha - A3$. Elements of $K^\sigma$ are of the form

$$X = \begin{pmatrix} A & S & -\bar{D}^t F \\ T & -\bar{A}^t & -\bar{C}^t F \\ C & D & B \end{pmatrix},$$

where $\bar{S} = -S$, $\bar{T} = -T$, $F^{-1} \bar{B}^t F = -B$, and $\text{tr}(X) \equiv 0 \mod |A, A|$.

We next want to check $A_4$. Note that $\mathcal{H}^\sigma = \{ \sum_{i=1}^\ell a_i (e_{ii} - e_{\ell+i,\ell+i}) \mid a_i \in \mathbb{C} \}$. It is easy to see that

$$\mathcal{G} = \bigoplus_{\alpha \in \mathcal{H}^\sigma} \mathcal{G}_\alpha = \sum_{\delta \in \mathbb{Z}'} \sum_{\alpha \in \hat{R}} \mathcal{G}_{\alpha + \delta},$$

where $\hat{R} = \{ \pm (e_i - e_j) \mid 1 \leq i < j \leq n \}$,

$$\mathcal{G}_0 = \mathcal{H} \oplus \mathcal{C} \oplus \mathcal{D}, \quad \text{and} \quad \mathcal{G}_{\alpha + \delta} = K_\alpha \cap K^\delta \quad \text{if} \quad \alpha + \delta \neq 0.$$  

In particular, the root system $R$ of $\mathcal{G}$ is of the form

$$R = \{ \alpha + \delta \mid \alpha \in \hat{R}, \delta \in \mathbb{Z}' \text{ and } K_\alpha \cap K^\delta \neq \{0\} \}.$$  

To describe the root system of $\mathcal{G}$ note that if $1 \leq i < j \leq 2\ell$, then $\deg(e_{ij}) = \lambda_i - \lambda_j = 0$ and so $e_{ij} \in K_{\epsilon_i - \epsilon_j} \cap K^0$. Therefore $\epsilon_i - \epsilon_j \in R$. If $1 \leq i \leq 2\ell$ and $1 \leq j \leq m$, then $\deg(x^\delta e_{i,2\ell+j}) = 2\delta + \lambda_i - \lambda_j = 2\delta - \tau_j$. Therefore $x^\delta e_{i,2\ell+j} \in K_{\epsilon_i - \epsilon_{2\ell+j}} \cap K^{2\delta - \tau_j}$. So $\epsilon_i - \epsilon_{2\ell+j} + 2\delta - \tau_j \in R$. Finally if $1 \leq i \leq j \leq m$, then $\deg(x^\delta e_{2\ell+i,2\ell+j}) = 2\delta + \tau_i - \tau_j$. So

$$x^\delta e_{2\ell+i,2\ell+j} \in K_{\epsilon_{2\ell+i} - \epsilon_{2\ell+j}} \cap K^{2\delta + \tau_i - \tau_j}.$$  

Thus $\epsilon_{2\ell+i} - \epsilon_{2\ell+j} + 2\delta + \tau_i - \tau_j \in R$. This in particular shows that $\epsilon_{2\ell+i} - \epsilon_{2\ell+j} \notin R$, as $\tau_i - \tau_j \notin 2\mathbb{Z}'$ if $i \neq j$. Set

$$\epsilon'_i = \epsilon_i \quad \text{for} \ 1 \leq i \leq 2\ell \quad \text{and} \quad \epsilon'_{\ell+i} = \epsilon_{2\ell+i} + \tau_i \quad \text{for} \ 1 \leq i \leq m.$$  

Then

$$R = 2\mathbb{Z}' \cup \{ \pm (\epsilon'_i - \epsilon'_j) + 2\delta \mid \delta \in \mathbb{Z}', \ 1 \leq i < j \leq n \}.$$
It now follows that
\[ \alpha \in R \setminus \{0\} \Rightarrow \pi(\alpha) = \alpha|_{H^\sigma} \neq 0. \]
Thus by [ABP, Proposition 3.25], the condition A4 also holds. It is shown in [A2, Proposition 3.23], the root system \( R^\sigma \) of \( G^\sigma \) with respect to \( H^\sigma \) is an irreducible reduced EARS of type
\[
\begin{aligned}
A_1 & \quad \text{if } \ell = 1, \ e = 1_\nu \text{ and } q = 1_{\nu \times \nu} \\
B_\ell & \quad \text{if } \ell \geq 2, \ e = 1_\nu \text{ and } q = 1_{\nu \times \nu} \\
BC_\ell & \quad \text{if } e \neq 1_\nu \text{ or } q \neq 1_{\nu \times \nu}.
\end{aligned}
\]
Thus by Corollary 2.66, \( G^\sigma \) is an EALA of the above type. It is not difficult to show that \( G^\sigma \) is an EALA. It is not difficult to show that \( G^\sigma \) is a tame EALA.

Example 3.77. Suppose \( \ell \geq 2 \) and \( n = 2\ell \). Let \( K = sl_n(\mathbb{A}) \) and consider the EALA \((G, \langle \cdot, \cdot \rangle, H)\) as in the setting before Example 3.75. Define a gradation on \( K \) by \( \deg(a_{i,j}) = 2\delta \). This gradation satisfies (3.74). Let \( K = \left( \begin{array}{cc} 0 & I_\ell \\ -I_\ell & 0 \end{array} \right) \) and for \( Y \in K \) define \( \sigma(Y) = -K^{-1}Y^tK \). Extend \( \sigma \) to a period 2 automorphism of \( G \). Check that A2–A3 hold for \( \sigma \). Let \( X \in K \). It is easy to see that \( X \in K^\sigma \) if and only if
\[
X = \begin{pmatrix} A & S \\ T & -A^t \end{pmatrix}
\]
with \( S^t = S \) and \( T^t = T \).

We also have
\[
\hat{H}^\sigma = \left\{ \sum_{i=1}^\ell a_i(e_{ii} - e_{\ell+i,\ell+i}) \mid a_i \in \mathbb{C} \right\}.
\]
Next we check A4. Note that the root system \( R \) of \( G \) is of the form
\[
R = 2\mathbb{Z}^\nu \cup \{ \epsilon_i - \epsilon_j + 2\mathbb{Z}^\nu \mid 1 \leq i \neq j \leq n \},
\]
(see Example 3.75). It is now easy to see that
\[
\alpha \in R \setminus \{0\} \Rightarrow \pi(\alpha) = \alpha|_{H^\sigma} \neq 0.
\]
Thus by [ABP, Proposition 3.25], \( \sigma \) satisfies A4. The root system \( R^\sigma \) of \( G^\sigma \) with respect to \( H^\sigma \) is an irreducible reduced EARS of type \( C_\ell \) (see [AABGP, III. Theorem 4.7]). Thus by Corollary 2.66, \( G^\sigma \) is an EARS of type \( C_\ell \). As in the previous examples we can see that \( G^\sigma \) is a tame EALA.

Example 3.79. Let \( \nu \geq 1 \) and let \( A \) be the associative commutative Laurent polynomials in \( \nu \)-variables \( x_1, \ldots, x_\nu \). Let \( \ell \geq 1, m \geq 1, n = \ell + m \). Let
\[
K = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B^t = -B, \ C^t = -C, \ A, B, C \in M_n(\mathbb{A}) \right\}
\]
and
\[
\hat{H} = \left\{ \sum_{i=1}^n a_i(e_{i,i} - e_{n+i,n+i}) \mid a_i \in \mathbb{C} \right\}.
\]
Let \( G = \mathcal{K} \oplus \mathcal{C} \oplus \mathcal{D} \) and \( H = \check{\mathcal{H}} \oplus \mathcal{C} \oplus \mathcal{D} \). Then by Example 3.75 \((G, \langle \cdot , \cdot \rangle, H)\) is an EALA of type \( D_n \) \((n \geq 4)\).

Let \( \tau_1, \ldots, \tau_m \) represent distinct cosets of \( 2\mathbb{Z}' \) in \( \mathbb{Z}' \) with \( \tau_1 = 0 \). Define 
\[
\deg(x^\delta e_{pq}) = 2\delta + \lambda_p - \lambda_q \quad \text{where}
\]
\[
\lambda_i = \lambda_{n+i} = 0 \quad \text{for } 1 \leq i \leq \ell \quad \text{and} \quad \lambda_{\ell+i} = -\tau_i, \quad \lambda_{n+\ell+i} = \tau_i \quad \text{for } 1 \leq i \leq m.
\]
This defines a gradation on \( \mathcal{K} \) satisfying (3.74).

Put 
\[
F = \begin{pmatrix} x^{\tau_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x^{\tau_m} \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} I_\ell & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & I_\ell & 0 \\ 0 & F^{-1} & 0 & 0 \end{pmatrix}.
\]

Define a period 2 automorphism of \( \mathcal{K} \) by \( \sigma(Y) = KYK \) and extend it to an automorphism of \( \mathcal{G} \) as before. It is easy to check that \( \sigma \) satisfies \( A2-A3 \).

Next we check \( A4 \). It is not difficult to see that the root system \( R \) of \( \mathcal{G} \) is of the form
\[
R = 2\mathbb{Z}' \cup \{ \pm(\epsilon_i^\ell + \epsilon_j^\ell) + 2\mathbb{Z}' \mid 1 \leq i < j \leq n \},
\]
where 
\[
\epsilon_i^\ell = \epsilon_i \quad \text{for } 1 \leq i \leq \ell \quad \text{and} \quad \epsilon_i^\ell = \epsilon_{\ell+i} - \tau_i \quad \text{for } 1 \leq i \leq m.
\]
Also we have
\[
\mathcal{H}^\sigma = \left\{ \sum_{i=1}^\ell a_i(e_{ii} - e_{n+i,n+i}) \mid a_i \in \mathbb{C} \right\} \oplus \mathcal{C} \oplus \mathcal{D}.
\]

It follows that
\[
\alpha \in R \setminus \{0\} \Rightarrow \pi(\alpha) = \alpha|_{\mathcal{H}^\sigma} \neq 0.
\]
This proves that \( A4 \) holds for \( \sigma \). One can check that each element of \( \mathcal{K}^\sigma \) is of the form
\[
(3.80) \quad \begin{pmatrix} A & -D^t & S & -D^tF \\ FC & -B^t & FD & FPF \\ T & -C^t & -A^t & -C^tF \\ C & P & D & B \end{pmatrix} \quad \text{with} \quad S^t = -S, \quad T^t = -T, \quad P^t = -P
\]
where \( A, S, T \in M_\ell(\mathcal{A}), \quad C, D \in M_{m \times \ell}(\mathcal{A}) \) and \( B, P \in M_m(\mathcal{A}) \). It is shown in [A2, Proposition 3.24] that the root system \( R^\sigma \) of \( \mathcal{G}^\sigma \) with respect to \( \mathcal{H}^\sigma \) is an irreducible reduced EARS of type
\[
A_1 \quad \text{if } \ell = 1 \quad \text{and} \quad B_\ell \quad \text{if } \ell \geq 2.
\]
Thus by Corollary 2.68 \( \mathcal{G}^\sigma \) is an EALA of the above type.

Note that \( \mathcal{G} \) is a tame EALA and \( \mathcal{G}_c = \mathcal{K} \oplus \mathcal{C} \) (see [AABGP, Chapter III]). Let \( \mathcal{K}^\sigma_c \) be the subalgebra of \( \mathcal{K}^\sigma \) consisting of all matrices of the form (3.80) with \( P = BF^{-1} \). By [A2, Lemma 3.17], \( \mathcal{G}^\sigma_c = \mathcal{K}^\sigma_c \oplus \mathcal{C} \). It follows that \( \mathcal{G}_c^\sigma = \mathcal{G}^\sigma \cap \mathcal{G}_c \) if and only if \( m = 1 \). Now Lemma 2.54 implies that \( \mathcal{G}^\sigma \) is tame if \( m = 1 \). In fact one can see that \( \mathcal{G}^\sigma \) is tame if and only if \( m = 1 \) (see
This example in particular shows that $G^\sigma_c$ in general is a proper subalgebra of $G^\sigma \cap G_c$.

**Example 3.81.** Consider the quantum torus $(A, \cdot)$ determined by the vector $e$ and the matrix $q$. Assume that $e \neq 1_\nu$ or $q \neq 1_{\nu \times \nu}$. Then $Z_{e,q}^c \neq \emptyset$. Let $\ell \geq 1$ and $n \geq 1$. Consider the subalgebra $K$ of $sl_{2n}(A)$ consisting of matrices of the form (3.82). Let $H = \{\sum_{i=1}^n a_i(e_{ii} - e_{n+i,n+i}) | a_i \in \mathbb{C}\}$. Set $G = K \oplus C \oplus D$ and $H = H \oplus C \oplus D$. By Example 3.71, $(G, (\cdot, \cdot), \mathcal{H})$ is a tame EALA of type $C_\ell$. Let $\tau_1, \ldots, \tau_m$ represent distinct cosets of $2\mathbb{Z}^\nu$ in $\mathbb{Z}^\nu$ such that $\tau_i \in Z_{e,q}$. Define a gradation on $K$ by $\deg(e_{pq}) = 2\delta + \lambda_p - \lambda_q$ where $\lambda_i$'s are defined as in Example 3.79. Put

$$F = \begin{pmatrix} x_{\tau_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{\tau_m} \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 & I_{\ell} & 0 \\ 0 & -F^{-1} & 0 & 0 \\ -I_{\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & F \end{pmatrix}.$$ 

Define a period 2 automorphism of $K$ by $\sigma(Y) = -K^{-1}Y^t K$ and extend it to an automorphism of $G$ as before. We have seen in the previous examples that $\sigma$ satisfies $A_3$. It is easy to check that $\sigma$ satisfies $A_2$. The root system $R$ of $G$ is of the form

$$R = 2\mathbb{Z}^\nu \cup \{\pm(e_i + e_j) | 1 \leq i \neq j \leq n\} \cup \{2e_i | 1 \leq i \leq n\},$$

where

$$e'_i = e_i \quad \text{for } 1 \leq i \leq \ell \quad \text{and} \quad e'_{i+\ell} = e_{i+\ell} - \tau_i \quad \text{for } 1 \leq i \leq m.$$ 

Also note that $\mathcal{H}^\sigma = \{\sum_{i=1}^\ell a_i(e_{ii} - e_{n+i,n+i}) | a_i \in \mathbb{C}\}$. It follows that

$$\alpha \in R \setminus \{0\} \Rightarrow \pi(\alpha) = \alpha|_{H^\sigma} \neq 0.$$ 

Thus $\sigma$ satisfies $A_4$. It is shown in [A2, (3.27)] that $X \in K^\sigma$ if and only if (3.82)

$$X = \begin{pmatrix} A & -\tilde{D}^t & S & \tilde{D}^t \\ -FC & -B^t & -FD & FPFF \\ T & C^t & -A^t & -C^tF \\ C & P & D & B \end{pmatrix} \quad \text{with} \quad S^t = S, \quad T^t = T, \quad P^t = P, \quad F^{-1}B^tF = -B, \quad \text{and} \quad \text{tr}(X) \equiv 0 \mod[A, A],$$

where $A, S, T \in M_\ell(A)$, $C, D \in M_{m \times \ell}(A)$ and $B \in M_m(A)$. It is also shown in [A2, Proposition 3.50] that the root system of $G^\sigma$ with respect to $H^\sigma$ is an irreducible reduced EARS of type $BC_\ell$ ($\ell \geq 1$). Thus $(G^\sigma, (\cdot, \cdot), H^\sigma)$ is an EALA of type $BC_\ell$.

The Lie algebra $G^\sigma$ is not tame. To see this let $K^\sigma_c$ be the subalgebra of $K^\sigma$ consisting of matrices of the form (3.82) with $P = -BF^{-1}$. By [A2, Lemma 3.51] if the intersection of $A_-$ and the center of $A$ is nonzero, then $G^\sigma$ is not tame. Also if this intersection is zero, then the matrix $X$ of the form (3.82) with $A = S = T = 0$, $C = D = 0$, $P = I_m$ and $B = F$ is orthogonal to $G^\sigma_c$ but $X \notin G^\sigma_c$. Thus $G^\sigma$ is not tame. It follows from Lemma 2.51 that $G^\sigma_c$ is a proper subspace of $G^\sigma \cap G_c$. 

[A2, Corollary 3.22].
Example 3.83. (See [ABP]) Let \((G, (\cdot, \cdot), H)\) be an EALA with root system \(R\). Consider the so called affinization

\[
\text{Aff}(G) = (G \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

of \(G\) introduced in [ABP]. Then \(\text{Aff}(G)\) is a Lie algebra where \(c\) is central, \(d = \frac{d}{dt}\) is the degree derivation so that \([d, x \otimes t^n] = n x \otimes t^n\), and

\[
[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{m+n,0} c.
\]

Extend the form \((\cdot, \cdot)\) to \(\text{Aff}(G)\) such that \(c\) and \(d\) are dually paired. Then the triple

\[
(\text{Aff}(G), (\cdot, \cdot), H \oplus \mathbb{C}c \oplus \mathbb{C}d)
\]

is an EALA with root system \(\tilde{R} = R + \mathbb{Z}\delta\) where \(\delta \in \tilde{H}\) is defined by \(\delta(d) = 1\) and \(\delta(H + \mathbb{C}c) = 0\). Moreover, \(\text{Aff}(G)\) is tame if and only if \(G\) is tame. Next consider an automorphism \(\sigma\) of \(G\) satisfying A1-A4. Extend \(\sigma\) to an automorphism of \(\text{Aff}(G)\) by

\[
\sigma(x \otimes t^i + rc + sd) = \zeta^{-i}\sigma(x) \otimes t^i + rc + sd.
\]

It is easy to see that \(\sigma\) as an automorphism of \(\text{Aff}(G)\) satisfies A1-A4. By [ABP, Lemmas 3.49, and 3.62] the root system of \(\text{Aff}(G)\sigma\) is irreducible. Thus by Corollary 2.66 \(\text{Aff}(G)\sigma\) is an EALA. Therefore all Lie algebras \(\tilde{G}\) constructed in [ABP] can be considered as fixed point subalgebras of the loop algebra of some EALA.

References

[AABGP] B. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola, Extended affine Lie algebras and their root systems, Mem. Amer. Math. Soc. 603 (1997), 1–122.

[A1] S. Azam, Extended affine root systems, J. Lie Theory, Vol. 12:2 (2002).

[A2] S. Azam, Construction of extended affine Lie algebras by the twisting process, Comm. Alg. 28 (2000), 2753–2781.

[ABP] B. Allison, S. Berman, A. Pianzola, Covering Algebras I. Extended affine Lie algebras, J. Alg. 250, (2002), 458–516.

[BGK] S. Berman, Y. Gao, Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, J. Funct. Anal 135 (1996), 339–386.

[BGKN] S. Berman, Y, Gao, Y, Krylyuk, E. Neher, The alternative torus and the structure of elliptic quasi-simple Lie algebras of type A_2, Trans. Amer. Math. Soc 347 (1995) 4315–4363.

[BM] A. Borel, G.D. Mostow, On semisimple automorphisms of Lie algebras, Ann. of Math., 61 (1955), 389–504.

[H] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.

[H-KT] R. Høegh-Krohn and B. Torresani, Classification and construction of quasi-simple Lie algebras, J. Funct. Anal. 89 (1990), 106–136.

[K] V. Kac, Infinite dimensional Lie algebras, third edition, Cambridge University Press, 1990.

[P] A. Pianzola. On automorphisms of semisimple Lie algebras, Algebras, Groups and Geometries, 2 (1985) 95–116.

[Sa1] K. Saito, Extended affine root systems 1 (Coxeter transformations), RIMS., Kyoto Univ. 21 (1985) 75–179.
[W] M. Wakimoto, *Extended affine Lie algebras and a certain series of Hermitian representations*, preprint.

[Y] Y. Yoshii, *Root systems extended by an abelian group, their Lie algebras and the classification of Lie algebras of type $B$*, preprint.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ISFAHAN, ISFAHAN, IRAN, P.O. BOX 81745.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SASKATCHEWAN, SASKATOON, SASKATCHEWAN, CANADA, S7N 5E6.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ISFAHAN, ISFAHAN, IRAN, P.O. BOX 81745.