Relativistic Quantum Tunnelling is Subluminal

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We prove that the classical Dirac equation in the presence of an external (non-dynamical) electromagnetic field is a relativistically causal theory. As a corollary, we show that it is impossible to use quantum tunnelling to transmit particles or information faster than light. When an electron tunnels through a barrier, it is bound to remain within its future lightcone. In conclusion, the relativistic quantum tunnelling (if modelled using the Dirac equation) is an entirely subluminal process.

I. INTRODUCTION

There is some debate over whether the “speed of tunnelling” could be faster than the speed of light [1–9]. Some authors claim that, yes, quantum tunnelling allows for superluminal signalling [1]. Other authors argue that, no, there is no superluminal propagation of particles or signals going on [2]. More recently, some authors [8] have proposed an intermediate interpretation: when a particle tunnels through a barrier, it may indeed emerge on the other side faster than light, but the probability for this to happen is so low that in practice photons arrive first, preventing an actual superluminal signalling.

What has made this subject so prone to interpretation is that, if one thinks just in terms of wavepackets and dispersion relations, then it is hard to define unambiguously terms like “signal”, or “tunnelling time”. On the other hand, the mathematical theory of partial differential equations provides us with all the tools that are needed for us to settle this matter once and for all. This is what we aim to do here.

Throughout the article, we adopt the spacetime signature (−, +, +, +), and work in natural units: \( c = \hbar = 1 \). We use standard rectangular coordinates \( \{x^\alpha\}_{\alpha=0}^3 \) in Minkowski space \( \mathbb{R}^{1+3} \), with \( t := x^0 \) denoting a time coordinate. Greek indices vary from 0 to 3, Latin indices from 1 to 3, and the sum convention is adopted.

II. A SIMPLE THEOREM

We follow the approach of Dumont et al. [8], and we consider electrons with quantum dynamics governed by the Dirac equation (we adopt the sign conventions of Weinberg [10]):

\[
(\gamma^\mu \partial_\mu + ie\gamma^\mu A_\mu + m)\Psi = 0.
\]

(1)

Here, \( \gamma^\mu \) are Dirac’s gamma matrices, \( \Psi = \Psi(x) \), \( x \in \mathbb{R}^{1+3} \), is a Dirac spinor (representing the electron), \( e \) and \( m \) are the electron’s charge and mass. The field \( A_\mu = A_\mu(x) \), \( x \in \mathbb{R}^{1+3} \), is the electromagnetic four-potential, and it is treated as a fixed, assigned, smooth function of the coordinates (it is not a dynamical degree of freedom). For tunnelling models, one should take

\[
e A_\mu = (V, 0, 0, 0),
\]

(2)

where \( V(x) \) is the potential energy barrier. However, here we may also keep the potential \( A_\mu \) completely general.

Our first task is to compute the characteristics of the system [11]. As a system of first-order partial differential equations, the Dirac equation (1) is naturally written in the standard matrix form (recall that \( \Psi \) is has four components):

\[
\mathcal{M}^\mu \partial_\mu \Psi + \mathcal{N} \Psi = 0,
\]

(3)

where \( \mathcal{M}^\mu = \gamma^\mu \) and \( \mathcal{N} = ie\gamma^\mu A_\mu + m \) are \( 4 \times 4 \) complex matrices. Working in the Weyl basis, we can write \( \mathcal{M}^\mu \) explicitly [12]:

\[
\mathcal{M}^0 = \gamma^0 = -i \begin{bmatrix} 0_{2 \times 2} & \mathbb{I}_{2 \times 2} \\ \mathbb{I}_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}, \quad \mathcal{M}^j = \gamma^j = -i \begin{bmatrix} 0_{2 \times 2} & \sigma_j \\ -\sigma_j & 0_{2 \times 2} \end{bmatrix},
\]

(4)

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\[\text{1 One should be careful about the sign: the potential energy of a particle with charge } q \text{ in an electrostatic potential } \phi \text{ is } V = q \phi. \text{ For the electron, } q = -e. \text{ Furthermore, given that our metric signature is } (-, +, +, +), \text{ we have that } \phi = A^0 = -A_0. \text{ Thus, } V = q \phi = eA_0.\]
where \( \sigma_j \) are the Pauli matrices:

\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]  

(5)

The characteristic surfaces are defined as the surfaces \( \Phi = \text{const} \) of any scalar field \( \Phi \) such that \( \det[M^\mu \xi_\mu] = 0 \), with \( \xi_\mu = \partial_\mu \Phi \). Using equation (4), we obtain

\[
0 = \det[M^\mu \xi_\mu] = (-i)^4 \det \begin{bmatrix}
0 & 0 & \xi_0 + \xi_3 & \xi_1 - i\xi_2 \\
0 & 0 & \xi_1 + i\xi_2 & \xi_0 - \xi_3 \\
-\xi_1 - i\xi_2 & -\xi_0 + \xi_3 & 0 & 0 \\
\end{bmatrix} = (\xi_\mu \xi_\mu)^2.
\]

(6)

Hence, \( \xi_\mu = \partial_\mu \Phi \) must be lightlike, meaning that the characteristic surfaces are null surfaces. This immediately allows us to derive the following

**Theorem 1** (Causality of the Dirac equation). Assume that \( A_\mu \) is continuously differentiable and let \( \Psi \) be a continuously differentiable solution to (1). Let \( \Sigma \subset \mathbb{R}^{1+3} \) be a Cauchy surface. Then, for any point \( x \) in the future of \( \Sigma \), the value of \( \Psi \) at \( x \), i.e., \( \Psi(x) \), depends only on the values of \( \Psi \) in the region \( J^-(x) \cap \Sigma \), and on the value of \( A_\mu \) in the region \( J^-(x) \cap J^+(\Sigma) \). Here, \( J^-(x) \) is the causal past of \( x \), and \( J^+(\Sigma) \) is the causal future of \( \Sigma \).

**Remark 1.** In practice, one usually takes \( \Sigma \) to be a surface where initial data for the system (1) is given (e.g., \( \Sigma = \{ t = 0 \} \)). In this case, the conclusion of the theorem can be rephrased in a more intuitive form as saying that the value of \( \Psi \) at \( x \), i.e., \( \Psi(x) \), depends only on the initial data in the region \( J^-(x) \cap \Sigma \), and on the value of \( A_\mu \) in the region \( J^- \cap J^+ \).

**Proof.** Fix \( x \) in the future of \( \Sigma \) and let \( \Psi_1 \) and \( \Psi_2 \) be two continuously differentiable solutions of the Dirac equation corresponding to two different choices of external potential. Then we have \( M^\mu \partial_\mu \Psi_1 + N_1 \Psi_1 = 0 \) and \( M^\mu \partial_\mu \Psi_2 + N_2 \Psi_2 = 0 \). Now assume that the external potential is the same on the spacetime region \( J^-(x) \cap J^+ \). Then, if we restrict our attention to such region, we have \( N_1 = N_2 \), and the field \( \Psi \) is a solution of \( M^\mu \partial_\mu \Psi + N \Psi = 0 \) on \( J^- \cap J^+ \). Finally, assume that \( \Psi_1 \) and \( \Psi_2 \) agree on \( J^-(x) \cap \Sigma \). Then \( \Psi \) is a solution of \( M^\mu \partial_\mu \Psi + N \Psi = 0 \) on \( J^-(x) \cap J^+ \). At this point, we can just apply John’s Global Holmgren Theorem (see Rauch [13], Section 1.8), considering that the characteristics of the Dirac equation are the same as those of the wave equation, and we find that \( \Psi \) is unique on \( J^- \cap J^+ \). This implies \( \Psi_1(x) = \Psi_2(x) \).

We observe that the conclusion of Theorem 1 is coordinate independent, even if we employed standard coordinates in the computation of the characteristics. This follows from the invariance of the characteristics (see, e.g., [14]) and of \( J^\pm \) as well as from standard theory of hyperbolic differential equations [15] [16]. We also remark that Theorem 1 is not new. The Dirac equation is known to be a hyperbolic partial differential equation (see, e.g., [17]) and thus Theorem 1 follows from textbook theory (above we quoted Rauch [13] in order to provide the reader with a precise reference, but there are plenty of sources explaining the properties we used for the proof, e.g., [15] [24], see the appendix of [25] for a summary). Nevertheless, we felt the need to state Theorem 1 and provide its proof because, as the literature review presented in the introduction demonstrates, there seems to be some confusion in the literature regarding the causal properties of the Dirac equation. In particular, properties that follow from standard hyperbolic theory seem to be neglected in these discussions.

Theorem 1 coincides with the principle of relativistic causality that we meet in all textbooks of General Relativity [14] [26] [27], and in the literature of relativistic hydrodynamics [28] [31]. It is the mathematical condition that people have in mind when they say: “no signal can exit the lightcone” [32] (see figure 1).

Let us make an interesting remark. If we set \( A_\mu = 0 \), then we recover the free Dirac equation. It is well known that, in this case, \( \Psi \) is also a solution of the free Klein-Gordon equation. Therefore, it is quite trivial to see that the free Dirac equation is a relativistically causal equation [39]. However, when \( A_\mu \neq 0 \) (e.g. inside a potential barrier), this becomes less intuitive. The key insight, here, is that the propagation of information is entirely determined by the characteristics of the system, which depend only on the principal part of the Dirac equation (the part with highest derivatives: \( \gamma^\mu \partial_\mu \Psi \)) and are completely unaffected by the presence of the external potential \( A_\mu \). In a nutshell, the presence of a potential barrier cannot increase (or shorten) the “speed of information”.

### III. IMPLICATIONS

Let us apply Theorem 1 to the relativistic quantum tunnelling, and let’s see what we can argue on a purely mathematical basis.
FIG. 1. Relativistic principle of causality. Let $\Sigma$ (blue plane) be the initial-data hypersurface, e.g. the hyperplane \( \{ t = 0 \} \). Pick an event $x$ in the future of $\Sigma$. Such event can be influenced only by that portion of $\Sigma$ that can be reached by a non-spacelike worldline emitted from $x$ (yellow line). In other words, the value of $\Psi(x)$ depends only on the initial data prescribed inside the past light-cone of $x$ (in red). Changes of initial data outside the past lightcone of $x$ cannot affect the value of $\Psi(x)$, when we solve the Dirac equation. Furthermore, we cannot change the value of $\Psi(x)$ even by altering the external potential $A_\mu$ outside the past lightcone of $x$.

(i) One direct implication of Theorem 1 is that, if $\Psi = 0$ in a region of space $\mathcal{R} \subset \Sigma$, then $\Psi = 0$ also on $\mathcal{D}^+(\mathcal{R})$, the Cauchy development of $\mathcal{R}$ \cite{26}. This is a no-go theorem for superluminal motion: the electron cannot move faster than light, because the support of the wavefunction cannot propagate outside the lightcone. For example, consider the situation illustrated in figure 2, upper panel. A potential barrier extends over the region $0 \leq z \leq L$. At $t = 0$, the electron is on the left of such barrier with probability 1, so that $\Psi = 0$ for $z > 0$. Then, for arbitrary $t > 0$, $\Psi$ must vanish in the region of space $z > t$. As a consequence, the probability for the electron to tunnel out of the barrier at a time $t < L$ is exactly zero.

(ii) Theorem 1 enables us also to answer the most important question: can we use tunnelling electrons to send signals faster than light? In the literature, the word “signal” often generates some debate. However, when we have a partial differential equation like (1), there is an unambiguous mathematical criterion to decide whether a superluminal signal can actually be sent or not. Consider the situation shown in figure 2, middle panel. Alice and Bob are on the opposite sides of a potential barrier, and they are spacelike-separated. The electron wavefunction $\Psi$ fills the space between them. Can a decision of Alice affect a measurement of Bob? No! Alice can use a device to modify the value of the potential $A_\mu$ at her spacetime location. This indeed generates a perturbation in $\Psi$. However, from Theorem 1, we know that changes in the value of $A_\mu$ outside the past lightcone of Bob cannot affect the value of $\Psi$ at Bob’s spacetime location. Thus, Alice has no way to influence Bob’s measurement.

(iii) When people say that “the tunnelling effect is superluminal”, they typically have in mind the following scenario. A wavepacket meets a potential barrier; most of it is reflected, but a small portion tunnels through, and it appears on the other side earlier than a hypothetical light-beam emitted by the initial wavepacket (see figure 2, lower panel). Recent findings already seem to question this picture \cite{9}, but let us say (for the sake of argument) that the idea is somehow correct. What does Theorem 1 have to say about that? Suppose that $\Psi(t = 0)$ were zero for $z > Q$ (figure 2, lower panel). Then, by Theorem 1 $\Psi(t)$ should vanish within the region $z > Q + t$, and there would be no tunnelled wavepacket. Therefore, the only way for us to observe a tunnelled wavepacket is to assume that $\Psi$ was already non-zero on the right of $Q$ at $t = 0$. In other words, to avoid a mathematical contradiction, we must assume that the incoming wavepacket had a long tail, which extended largely inside the barrier, and that the tunnelled wavepacket is just the (subluminal) evolution of such long tail.

The conclusion of our point (iii) is very similar to that of Büttiker and Washburn \cite{2}: the tunnelled wavepacket is the causal evolution of the right tail of the incoming wavepacket, which enters the barrier much earlier than the peak.

\footnote{Another thing that Alice may do is to make a measurement herself. However, here Quantum Field Theory comes to our aid, reassuring us that spacelike-separated observables always commute, meaning that their measurements cannot influence each other \cite{37–39}.}
FIG. 2. Visual representations of our arguments (i), (ii), and (iii), respectively, which are direct consequences of Theorem 1. The shades of red represent the electron density $\Psi^\dagger\Psi$ (red large, white small). Upper panel: the field $\Psi$ is bound to propagate within the lightcone. Hence, electrons cannot travel faster than light. Quantum tunnelling through a barrier is no exception. Middle panel: Alice can generate a disturbance in $\Psi$ by altering the value of $A_\mu$ at her location. However, such disturbance travels slower than light, and it cannot reach Bob, who is causally disconnected from Alice (no superluminal signalling). Lower panel: the only way for a tunnelled wavepacket to exit at a time $t < L$ is that $\Psi \neq 0$ on the right of $Q$ already at $t = 0$. 
so that, if we only focus on the peaks, we get the illusion of a superluminal motion. Dumont et al. [8] have criticised this interpretation, arguing that in quantum mechanics one should never say that one “piece” of the wavefunction originates from a corresponding “piece” in the past. Instead, the wavefunction should always be treated as whole. As a consequence, according to them, we cannot say that the tunnelled wavepacket “originates” from the right tail of the incoming wavepacket.

We do not wish to enter philosophical debates over the ontology of the wavefunction. On the other hand, we would like to point out that, when we say that the tunnelled wavepacket “originates” from the right tail, we are just making two rigorous mathematical statements (which follow from Theorem [1]). First, that if you change your initial data by removing the tail, i.e. by replacing \( \Psi(t = 0) \) with \( \Psi(t = 0)\Theta(Q - z) \), where \( \Theta \) is the Heaviside step function, the tunnelled wavepacket disappears. Second, if you instead replace \( \Psi(t = 0) \) with \( \Psi(t = 0)\Theta(z - Q) \), leaving only the tail and cutting all the rest, the tunnelled wavepacket still remains, and it is completely unaffected. These facts may not establish an “ontological relationship” between the tunnelled wavepacket and tail of the incoming wavepacket, but they tell us that the existence of the tunnelled wavepacket is a direct consequence of the existence of such tail in its causal past. And this is enough to rule out any claim of superluminal behaviour.

IV. SUBLUMINALITY IN AN INEQUALITY

There is a simple inequality which, we believe, will convince even the most ardent “superluminalist” that quantum tunnelling is an entirely subluminal process. Let us consider the probability current \( j^\mu = i\bar{\Psi}\gamma^\mu\Sigma \), where \( \Psi = i\bar{\Psi}^\dagger\Sigma^0 \). It can be easily shown that it has two properties [10–44]. First, it is conserved \( (\partial_\mu j^\mu = 0) \), also in the presence of an external potential \( A_\mu \). Second, it is non-spacelike, future-directed \( (j^\mu A_\mu \leq 0, j^0 \geq 0) \). In appendix A we verify explicitly that these properties hold also in the tunnelling model of Dumont et al. [8]. Then, considering that \( \Psi \) must decay to zero at spacelike infinity, we can apply the Gauss theorem over the (infinitely long) trapezoidal region shown in figure 3 and we obtain (we adopt the orientation conventions of Misner et al. [15], section 5)

\[
- \int_{\Sigma_t} j^\mu d\Sigma_\mu + \int_{\Sigma_L} j^\mu d\Sigma_\mu + \int_{\Sigma_F} j^\mu d\Sigma_\mu = 0. \tag{7}
\]

Since \( j^\mu \) is non-spacelike future-directed, and \( d\Sigma_\mu \), as a one-form, has positive sense towards the future (“standard orientation” [15]), then the integral over \( \Sigma_L \) is non-negative, so that

\[
\int_{\Sigma_F} j^\mu d\Sigma_\mu \leq \int_{\Sigma_t} j^\mu d\Sigma_\mu. \tag{8}
\]

On the other hand, on both \( \Sigma_t \) and \( \Sigma_F \) the integrand is just \( j^0 d^3x \). But \( j^0(x) = \bar{\Psi}^\dagger(x)\Sigma(x) \) is the probability density of observing the electron at \( x \). Therefore, the inequality (8) reduces the following constraint:

\[
\mathcal{P}_t(z > t) \leq \mathcal{P}_0(z > 0). \tag{9}
\]

In words: the probability of observing the electron on the right of \( z = t \) at time \( t \) will never exceed the initial probability of observing that same electron on the right of \( z = 0 \) at time 0. Again, this means that the probability cannot “flow” outside the lightcone, and an electron can never “overtake” a photon that is moving in the same direction (even during a tunnelling process). In fact, if an electron reaches us before a photon, then it must have had an advantage over the photon, and the probability of such advantage cannot be less than the probability of the electron arriving before the photon.

Let us apply this result to the setting discussed in point (iii) of section III. It is immediate to see that we can transport our Gauss-theorem argument of figure 3 into figure 2 (lower panel), locating the lower-left corner of the trapezium at \( Q \) with the lightlike path shown in the figure. Then, the inequality (9) becomes

\[
\mathcal{P}(\text{“tunnelled packet”}) \leq \mathcal{P}(\text{“tail on the right of } Q\text{”}). \tag{10}
\]

Again, this is showing us that the emerging wavepacket is just the subluminal evolution of the right tail of the incoming wavepacket. But now we know also something more: the probability associated to the tunnelled wavepacket cannot exceed that of this initial tail. This means that the tail cannot be used as “springboard”, or a means to “push” the electron through the barrier faster than light. No way. The probability stored in the tail is the maximum probability that the tunnelled wavepacket can carry.

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3 Readers might object that, by introducing the Heaviside function, we are no longer dealing with continuously differentiable functions, and thus Theorem 1 no longer applies. But since the Dirac equation is a linear equation, Theorem 1 remains true for distributional solutions (which will be the case for data involving the Heaviside function), see [19], Section 12.5. We assumed continuous differentiability only in order to avoid technicalities and keep the proof short.

4 This inequality is analogous to equation (10.1.11) of Wald [28], which is used to prove well-posedness and causality of the Klein-Gordon equation. There, instead of \( j^\mu \), Wald [28] uses an energy current, which is also conserved and future-directed non-spacelike. A similar theorem can also be found in [35].
FIG. 3. Visualization of the Gauss-theorem argument discussed in section IV. The yellow region represents the spectime volume where we integrate the divergence of the Dirac current (which vanishes). The hypersurfaces \( \Sigma_I \) and \( \Sigma_F \) have constant time, and thus they are spacelike. The hypersurface \( \Sigma_L \) is lightlike. Since the integral of \( \Psi^\dagger \Psi \) across all space is normalised to 1, we know that \( \Psi \) decays to zero at spacelike infinity, and thus we can extend the integration region up to \( z = +\infty \).

V. CONCLUSIONS

We believe that this article has finally settled a 20-year-old debate. The Dirac equation is a perfectly causal field equation, also when we turn on an extremely high potential barrier. That is because the electromagnetic four-potential \( A_\mu(x) \) does not enter the equation of the characteristics. Indeed, recent numerical tests [9] confirm our main message: tunnelling electrons cannot be faster than photons in vacuum. Interestingly, our inequality (9) can be equivalently rewritten in the form

\[
\mathcal{P}_t(\text{“Arrival electron”}) \leq \mathcal{P}_t(\text{“Arrival photon”}),
\]

which is precisely what has been observed in all tests performed by Dumont and Rivlin [9].

We hope that our work will also foster the interdisciplinary communication between the quantum physics community and the mathematical relativity community.

ACKNOWLEDGEMENTS

M.M.D. is partially supported by a Sloan Research Fellowship provided by the Alfred P. Sloan foundation, NSF grant DMS-2107701, and a Vanderbilt’s Seeding Success Grant. L.G. is partially supported by a Vanderbilt’s Seeding Success Grant.

Appendix A: Probability current for tunnelling models

The tunnelling model of Dumont et al. [8] is (1+1)-dimensional, and it evolves only two components of \( \Psi \) in the Dirac basis (as the other two components are fully decoupled). We call such components “\( f \)” and “\( h \)”. The Dirac equation then reads

\[
\begin{cases}
    i \partial_t f &= i \partial_z h + (V + m) f, \\
    i \partial_t h &= i \partial_z f + (V - m) h.
\end{cases}
\]

(A1)

Taking the complex conjugate, we get

\[
\begin{cases}
    i \partial_t f^* &= i \partial_z h^* - (V + m) f^*, \\
    i \partial_t h^* &= i \partial_z f^* - (V - m) h^*.
\end{cases}
\]

(A2)

5 To show this, just consider one electron and one photon with same initial probability distributions, and use the fact that, for photons, the inequality [9] is saturated.
Thus, it is immediate to verify that

\[
\begin{align*}
\partial_t (f^* f) &= f^* \partial_t h + f \partial_z h^*, \\
\partial_t (h^* h) &= h^* \partial_t f + h \partial_z f^*.
\end{align*}
\] (A3)

As we can see, all the terms with \( V \) cancel out. Taking the sum of these two equations, and bringing every term on the left-hand side, we obtain an equation of the form \( \partial_t j^\mu = 0 \), with

\[
\begin{align*}
j^\mu &= \left( f^* f + h^* h - f^* h - h^* f \right).
\end{align*}
\] (A4)

As we can see, \( j^0 = f^* f + h^* h \) is non-negative definite, and it has the usual form of a probability density. To prove that \( j^\mu \) is non-spacelike future-directed, we only need to show that \( j^0 \geq |j^2| \), namely \( f^* f + h^* h \geq |f^* h + h^* f| \). But this follows immediately from the chain of identities below:

\[
0 \leq |f \pm h|^2 = (f^* \pm h^*)(f \pm h) = f^* f + h^* h \pm (f^* h + h^* f). \] (A5)
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