Nonperturbative partonic quasidistributions of the pion from chiral quark models

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Abstract

We evaluate nonperturbatively the quark quasidistribution amplitude and the valence quark quasidistribution function of the pion in the framework of chiral quark models, namely the Nambu–Jona-Lasinio model and the Spectral Quark Model. We arrive at simple analytic expressions, where the nonperturbative dependence on the longitudinal momentum of the pion can be explicitly assessed. The model results for the quark quasidistribution amplitude of the pion compare favorably to the data obtained from the Euclidean lattice simulations. The quark distribution amplitude, arising in the limit of infinite longitudinal momentum of the pion, agrees, after suitable QCD evolution, to the recent data extracted from Euclidean lattices, as well as to the old data from transverse lattice simulations.

Keywords: Partonic quasidistribution amplitude and function, Nambu–Jona-Lasinio model, Spectral Quark Model, nonperturbative pion structure.

1. Introduction

Partonic distributions provide direct access to the internal structure of hadrons in terms of the Bjorken variable, $x$, but their \textit{ab initio} determination directly from QCD has remained elusive during many years, with the only exception of isolated and courageous efforts on the transverse lattice formulated \textit{directly} on the light cone \cite{1,2} (for a review see, e.g., \cite{3}). There, both the parton distribution functions (PDF) and the parton distribution amplitudes (PDA) of the pion have been determined. The case of the pion, which is of our concern here, is particularly interesting, since not only it is the lightest hadron but also its properties are strongly constrained by the chiral symmetry and its spontaneous breakdown, a feature difficult to implement on the light-cone. In addition, the gauge invariance, relativity, and positivity are essential ingredients in the calculation; they guarantee correct normalization, proper support of the parton distributions, and a probabilistic interpretation. However, for the most popular implementation of QCD on Euclidean lattices only moments of distributions in $x$ can be accessed as matrix elements of local operators. The breaking of the Lorentz invariance of the lattice generates operator mixing with the result that up to now only a few lowest moments can be extracted with an acceptable signal to noise ratio.

Within this state of affairs, Ji has proposed a scheme where nonperturbative partonic physics could be accessed on a Euclidean lattice by boosting the space-like correlators to a finite momentum and, eventually, extrapolate the results to the infinite-momentum frame \cite{4}. The resulting objects, termed parton quasidistributions, may offer a unique missing link between nonperturbative physics and the light-cone dynamics; their peculiar properties have received a great deal of attention both from a theoretical as well as from a practical perspective. Following this proposal, in the non-singlet case, the one-loop matching for parton distributions \cite{5}, the renormalization of parton quasidistributions \cite{6}, as well as the one-loop matching for generalized parton distributions \cite{7} have been studied. Power divergences, ubiquitous in lattice QCD, can be safely removed by introducing an improved parton quasidistribution through the Wilson line renormalization \cite{8}. A quite appealing feature is the possibility of reformulating the whole problem as an effective large-momentum effective theory approach to parton physics, where the separation scale is the boosting momentum $P_z$ \cite{9,10,11,12}.

From a lattice perspective, the pion has always been a challenge, since realistic calculations require very small quark masses which need very large lattice volumes. The most impressive implication of the mentioned series of works is the recent determination of PDA of the pion on the Euclidean lattice \cite{13}, with very promising results for $m_q = 310$ MeV, and with errors dominated by the uncertainty in the Wilson-line self-energy. Nonetheless, the agreement with the largely forgotten transverse-lattice calculations is worth mentioning \cite{1,2}.

Along these lines Radyushkin has analyzed the nonperturbative $P_z$-evolution of parton quasidistributions \cite{14}.
and explored the connection between PDA, quasidistributions and the transverse-momentum distributions (TMD) [15]. He also recalled the Ioffe-time distributions, where the longitudinal distance rather than momentum display more clearly the physics of the light cone kinematics [16], and introduced the concept of pseudo-distributions [17], motivating an exploratory study on the lattice [18].

A key and subtle issue in the whole discussion regards the renormalization. The meaning of renormalization in the transverse lattice is quite transparent, as changing the renormalization scale \( \mu \) corresponds to varying the transverse lattice spacing \( a_\perp \sim \mu \), such that for wavelengths smaller than \( a_\perp \), all the dynamics gets frozen and evolution stops. The original transverse lattice calculation of both PDF and PDA of the pion [2] considered \( a_\perp \sim 0.4 \text{ fm} \), corresponding to a low scale \( \mu \sim 500 \text{ MeV} \), for which the partonic expansion should be expected to converge.

The nonperturbative renormalization of nonlocal quark bilinears, characteristic of parton quasidistributions, has been addressed by several groups and, inspired by an analogy and experience with heavy quark physics, it has been recently found how the use of an auxiliary spinless color field reduces the problem to the well understood renormalization of local and composite operators within an effective field theory viewpoint [10], as well as on the lattice [20]. Amazingly, the multiplicative renormalizability of QDFs in coordinate space has also been established to all orders in perturbation theory [21].

The purpose of this paper is to illustrate the above-discussed concepts in an explicit nonperturbative model calculation for the pion, where QDA and QDF for the valence quarks can be obtained analytically. In our study, we use the Nambu–Jona Lasinio (NJL) model [22, 23] with quarks in the Pauli-Villars (PV) regularization, as well as the Spectral Quark Model (SQM) [24–26]. In both models, all the theoretical constraints of chiral symmetry, relativity, gauge invariance, and positivity are fulfilled for PDFs [27, 28], PDAs [30] (for a review of chiral quark models, see e.g., Ref. [31] and references therein), and GPD [32, 33].

We obtain very simple analytic expressions for QDA, QDF, their transverse-momentum unintegrated analogues, as well as for the Ioffe-time distributions. We illustrate with our formulas the Radyushkin relation [14], linking via simple integration the QDA and QDF to the transverse-momentum distributions corresponding to PDA and PDF. Finally, we compare the QDA and PDA to the available lattice data and find very reasonable agreement.

A major issue in nonperturbative model calculations is the proper identification of the renormalization scale where the model is supposed to work. We assume that in the chiral quark model, where no other degrees of freedom are present, the valence quarks carry 100% of the pion momentum. This renormalization condition sets the initial scale for the QCD evolution, while phenomenological studies of parton distributions for the pion extracted from Drell-Yan and prompt photon experiments [31] yield 47(2)% at \( Q = 2 \text{ GeV} \) for the valence quarks (see also Ref. [35, 36]), which is used to determine the evolution ratio. The perturbative scheme using standard DGLAP to LO and NLO with \( \mu \sim Q \), despite providing an unusually low initial scale \( \mu_0 = Q_0 = 313^{+20}_{-10} \text{ MeV} \), gives small differences between LO and NLO [27, 28]. As explicitly shown in Ref. [32], the approach mimics quite reasonably the nonperturbative evolution observed in transverse lattice simulations [2]. It also predicts somewhat less gluon content than expected [33–35], a problem which has never been understood within the NJL model, as it requires some gluon content present at the low scale \( Q_0 \), with the radiative generation of gluons not sufficient.

For completeness, we mention lattice simulations for the nucleon [37], where a complete non-perturbative renormalization prescription QDFs has been carried out [38].

Other model calculations of QDFs have also been performed. A quark-diquark model for the nucleon was proposed in Ref. [39]. More recently, QDA in a nonlocal model for the pion was calculated [40]. However, the nonlocal model suffers from ambiguities related to the identification of the momentum fraction carried by the valence quarks, and hence to the scale.

2. Derivation

In the NJL model (see e.g. Ref. [31] for a review) sufficiently strong point-like 4-quark interactions lead to dynamical chiral symmetry breaking, attributing the quarks, via a gap mechanism, with a constituent mass \( M \sim 300 \text{ MeV} \). The model is suited for evaluation of soft matrix elements with pions (and photons), in particular those relevant for PDA and PDF.

The model needs to be regularized to get rid of the ultraviolet divergences, leaving the soft-momentum degrees of freedom in the dynamics. The simplest regularization scheme satisfying the necessary requirements is PV regularization [31], where the one-quark-loop quantity \( A(M^2) \) is replaced with

\[
A(M^2)|_{\text{reg}} \equiv A(M^2) - A(M^2 + \Lambda^2) + \Lambda^2 \frac{d}{d\Lambda^2} A(M^2 + \Lambda^2).
\]

The PV cutoff \( \Lambda \) is adjusted to reproduce the value of the pion decay constant \( f \). For \( M = 300 \text{ MeV} \) we use \( \Lambda = 731 \text{ MeV} \) in the chiral limit, yielding \( f = 86 \text{ MeV} \). An interesting alternative for regularization is offered by the Spectral Quark Model (SQM), where one-quark loop is regularized by introducing a generalized spectral density \( \rho(\nu) \) of the spectral mass \( \nu \), integrated over a suitably chosen complex contour [25]. The construction implements

\[\text{In a model where a finite cut-off must be introduced, the implementation of these constraints without violation of positivity is highly non-trivial (see, e.g., a recent discussion in Ref. [33]).}\]
The evaluation of these distributions in the momentum space yields the scalar two-point function (conventionally, all our momenta are Euclidean)

$$I(x, P \cdot n, P^2, n^2) = \frac{4N_c M^2}{f^2} \int \frac{d^4 k}{(2\pi)^4} \delta(x - k \cdot n) S_k S_{k - P},$$

where $S_k = 1/(k^2 + M^2)$, $M$ is the constituent quark mass, $P^2 = -m_n^2$, and $n$ is the four-vector yielding the kinematic constraint. For the PDA case it is a null vector, $n^2 = 0$, whereas for QDA $n = (0, 0, 0, 1/P_z)$, hence $n^2 = -1/P_z^2$. In both cases $P \cdot n = 1$. Using the Schwinger representation $S_k = \int_0^\infty \! da \exp[-(a^2 + M^2)]$ and proceeding along the lines of Appendix A of Ref. [32], we can easily find the generic formulas for PDA and QDA. The scalar two-point function is

$$I = \frac{N_c M^2}{4\pi^2 f^2} \int \frac{dk_i^2}{2\pi} \int_0^\infty \! da \int_0^\infty \! dx \frac{P_z}{\sqrt{M^2 + k_f^2 + P_z^2 y^2}} \times
$$

$$e^{-(\alpha + \beta)(k_i^2 + M^2 - \lambda^2 x^2/4 + m_n^2 \alpha \beta/(\alpha + \beta) + \alpha \lambda x - \beta(\alpha + \beta)x)} \times
$$

$$+ (y \leftrightarrow 1 - y),$$

where

$$f^2 = -\frac{N_c M^2}{4\pi^2} \ln(M^2)|_{\text{reg.}}.$$}

The corresponding QDA and PDF for the valence quarks, obtained from Eq. (8) via integration over $dk_f^2$, are

$$\tilde{\phi}(y, P_z) = \frac{N_c M^2}{4\pi^2 f^2} \times$$

$$\text{sgn}(y) \ln\left(\frac{P_z |y| + \sqrt{M^2 + P_z^2 y^2}}{M}\right)|_{\text{reg.}} + (y \leftrightarrow 1 - y),$$

In NJL, the analogous formulas read:

$$\Psi(y, k_f^2, P_z) = V(y, k_f^2, P_z) = \frac{m_n}{\pi^2} \times$$

$$\frac{3\pi}{(4k_f^2 + m_n^2)^{5/2}} - \frac{6 \arctg^{-1} \left( \frac{2P_z \sqrt{4k_f^2 + m_n^2}}{4k_f^2 + m_n^2 + (4y - 1)y P_z^2} \right)}{(4k_f^2 + m_n^2)^{5/2}} \times$$

$$+ \frac{4P_z}{(4k_f^2 + m_n^2)^2} \left[ y (20k_f^2 + 5m_n^2 + 12y^2 P_z^2) + (y \leftrightarrow 1 - y) \right]$$

with explicit formulas for the two regularization methods, following from the analytic integration over $\alpha$ and $\beta$, provided below. The other quantity of interest is the transverse-momentum distribution amplitude (TMD), obtained with Eq. (4) with $n^2 = 0$ by leaving the $k_T$ momentum unintegrated. We find

$$\mathcal{F}(x, k_T^2) = \frac{N_c M^2}{4\pi^2 f^2} \int_0^\infty \! dx \int_0^\infty \! dx \delta(\alpha - \alpha x) \times$$

$$e^{-(\alpha + \beta)(M^2 + k_f^2 + m_n^2 \alpha \beta/(\alpha + \beta))} \times$$

$$\frac{N_c M^2}{4\pi^2 f^2} \frac{\theta|x(1 - x)|}{(1 - x)}|_{\text{reg.}},$$

where $\theta$ is the Heaviside step function.
and

\[ \tilde{\phi}(y, P_z) = V(y, P_z) = \frac{1}{2} - \frac{1}{n} \arctg \left( \frac{2m_y P_z}{m_y^2 - 4(1-y) y P_z^2} \right) + \frac{2m_y P_z}{\pi} \left( \frac{m_y^2 + 4(1-y) y P_z^2}{m_y^2 + 4y^2 P_z^2} \right). \]

The above quantities satisfy the proper normalization

\[ \int_{-\infty}^{\infty} dy \tilde{\phi}(y, P_z) = \int_{-\infty}^{\infty} dy V(y, P_z) = 1, \]

and the limit

\[ \lim_{P_z \to \infty} \tilde{\phi}(y, P_z) = \lim_{P_z \to \infty} V(y, P_z) = \theta[y(1-y)]. \]

We also obtain, in the chiral limit, very simple expressions for the Ioffe-time distribution [16] [17],

\[ \mathcal{M}(\nu, z_3) = \int_{-\infty}^{\infty} dy e^{i(y-\frac{1}{2})\nu} \tilde{\phi}(y, P_z) \]

(we shift y by \( \frac{1}{2} \) to get real expressions), where \( z_3 = \nu / P_z \). The distribution \[ \mathcal{M}(0, 0) = 1 \], corresponding to Eq. (13). In the NJL model we get

\[ \mathcal{M}(\nu, z_3) = \frac{N_c M_F^2}{2\pi^3 f^2} \frac{\sin \left( \frac{\nu}{2} \phi \left( \frac{M_z}{z_3} \right) \right)}{\nu} K_0(M_z), \]

whereas in SQM the result reads

\[ \mathcal{M}(\nu, z_3) = \frac{\sin \left( \frac{\nu}{2} \right)}{\nu} e^{-\frac{m_y z_3}{\nu}} (m_y z_3 + 2). \]

We note the dependence on \( \nu \) and \( z_3 \) is factorized. The large-\( z_3 \) behavior of the two models is similar, as \( K_0(t) \sim e^{-t/\sqrt{t}} \), and \( m_y \approx 2M \).

3. Numerical Results

After listing in the previous section simple formulas obtained in the chiral quark model for QDA of the pion and the related quantities, we now show our results numerically and compare to the available lattice data [13]. Note that whereas QDA and QDF have originally been thought of as auxiliary quantities needed to obtain PDA and PDF on Euclidean lattices, they may well be used to verify model predictions obtained at particular finite values of \( P_z \), thus acquiring practical significance in their own right.

In Fig. 1 we present the quark quasidistribution amplitude (QDA) and the valence quark quasidistribution function (QDF) of the pion in the NJL model, obtained at the constituent quark scale \( \mu_0 \) and in the chiral limit of \( m_\pi = 0 \), evaluated according to Eq. (10). The results are given for various values of the longitudinal pion momentum \( P_z \) and are plotted as functions of the fraction \( P_z \), carried by the valence quark, \( y \). We note that as \( P_z \) increases to infinity, the curves tend to the PDA and PDF at the quark model scale \( \mu_0 \), which are equal to \( \phi(x) = \theta[x(1-x)] \) and \( 2\nu V(x) = 2x\theta[x(1-x)] \), respectively. Evolution in \( \mu \) is certainly necessary for PDA and PDF to compare these quantities to the data obtained at higher scales \( \mu \) (cf. Fig. 3 in the following and its discussion).

We compare our NJL model (a) and SQM (b) results for QDA of the pion to the lattice data in Fig. 2. The model curves are obtained at the constituent quark scale \( \mu_0 \), whereas the lattice data correspond to the scale \( \mu = 2 \) GeV, as inferred from the lattice spacing. The presented points were used in the extrapolation of the LaMET Collaboration data in Ref. 13 to obtain the pion PDA. We use \( P_z = 0.9 \) and 1.3 GeV and \( m_\pi = 310 \) MeV, exactly as used in Ref. 13.

While the agreement of a direct comparison is quite remarkable, one should note that there is room for improvement, particularly from the QCD evolution point of view. In a future work, the model QDA could be evolved in \( \mu \) with the equations outlined in Ref. 7 and modified for...
the lattice calculations [13] to account for a finite cut-off effect with a Wilson line self-energy correction [8].

For PDA or PDF, the evolution equations have been efficiently implemented (ERBL [42, 43] and DGLAP [44, 45]) and used to evolve the model results from $\mu_0$ to experimental or lattice scales [27]. The results of this evolution, from the quark model scale $\mu_0 = 313$ MeV, where PDA of the pion is constant in the chiral limit, $\phi(x) = \theta(x(1-x))$ [30], to the scales $\mu = 2$ GeV and $\mu = 0.5$ GeV are given in Fig. 3 where we also compare to the Euclidean lattice extraction [12] and to the transverse lattice data [2].

We recall that some low moments of the parton distributions of the pion on the Euclidean lattice have been evaluated. In the case of PDF, the second moment corresponding to the pion momentum fraction at $\mu = 2$ GeV in the MS-scheme, takes the value $\langle x \rangle^2 = 0.214(15)_{-0.09}^{+0.12}$ [46]. In the case of PDA, only the second moment has been extracted and the most accurate result, also at $\mu = 2$ GeV in the MS-scheme, is found to be $\langle x \rangle^2 = 0.2077(43)(32)$, or equivalently, $a_2 = 0.0762(127)$ [17] for the second Gegenbauer coefficient. In the chiral quark models evolved from $\mu_0 = 290$ MeV to $\mu = 2$ GeV, we get $\langle x \rangle^2 = 0.21$ and $a_2 = 0.10$, which is compatible with the above-quoted lattice values.

We remark that one of the features implied by our choice of the normalization scale $\mu \sim Q$ and the low energy scale $\mu_0$ is the large-$x$ behavior of the PDFs, where after the DGLAP evolution one has

$$V(x,Q)/V(x,Q_0) \sim (1-x)^{(C_F/2\beta_0)} \ln[Q(Q)/Q_0]$$

which for $V(x,Q_0) = 1$ yields $V(x,Q) \sim (1-x)^{1.1\pm0.1}$ for $Q = 2$ GeV [31], complying quite accurately with experimental extractions [18]. Actually, the large-$x$ dependence of PDF of the pion has been a subject of controversy [49, 50] (see, e.g., Fig. 8 of Ref. [32] for a comparison). From this viewpoint, a QDF-based lattice calculation would be most helpful to settle the issue.

### 4. Conclusions

Hadronic physics on the light cone has been a topic of phenomenological and experimental discussion ever since the parton model was proposed by Feynman, but the actual ab initio calculations of the corresponding parton distributions confront enormous theoretical and practical difficulties. In this paper, we have analyzed, within chiral quark models of the pion, the recently proposed quasi-distributions. These objects have been proposed as means to extract, from Euclidean lattices, the space-like correlators boosted to the infinite momentum frame.

Our model calculation describes the pion as a composite $q\bar{q}$ state and complies with all the a priori requirements imposed by the chiral symmetry the gauge invariance, relativity, and positivity; it provides a helpful and convenient playground to explore the peculiar features of the new way of extracting the Minkowskian parton distributions from the Euclidean formulation. We believe that our study is useful from two viewpoints. First, it provides nonperturbative predictions for the soft objects of interest from a dynamical model, with the results expressed by very simple and intuitive formulas. The predictions for the quasidistributions may be confronted with the lattice extractions, whereby these quantities acquire practical meaning. On the other hand, our formulas may be used to illustrate and verify the methods and various relations between the pertinent quantities, such as for instance the Radyushkin relation of the quasidistributions with the transverse-momentum distributions.

We have shown that our model (without the QCD evolution implementing radiative corrections) compares favorably to the recent lattice extractions of the valence quark PDA of the pion, and thus mimics quite accurately the space-like boosted dynamics observed on the lattice. We also find that finite pion mass effects are a few percent away from the chiral limit. However, the finite pion mass destroys the factorization between the transverse momentum and the Bjorken $x$ variable.

We also predict similar features for the QDF and PDF...
of the pion, suggesting that a lattice calculation based on quasi-distributions should be equally feasible and realistic, as it has been in the PDA case. Therefore, most interesting would be an analysis of the large-$x$ behavior and the extension to singlet distributions, as the chiral quark models fail to fully reproduce the phenomenologically needed gluon content of hadrons. We hope that our paper will stimulate such studies on the lattice, hence filling the gap in our understanding of the pion structure on the lightcone from a fundamental QCD viewpoint.

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![Figure 3: The distribution amplitude (PDA) of the pion, obtained from the chiral quark models (the same for NJL and SQM) with $m_\pi = 0$, evolved to the scale $\mu = 2$ GeV (a) and $\mu = 0.5$ GeV (b), and compared to the lattice data of the LaMET Collaboration [12] (a) and to the transverse-lattice results [2] (b).](image-url)
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