Local Reasoning for Global Graph Properties

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Abstract. Separation logics are widely used for verifying programs that manipulate complex heap-based data structures. These logics build on so-called separation algebras, which allow expressing properties of heap regions such that modifications to a region do not invalidate properties stated about the remainder of the heap. This concept is key to enabling modular reasoning and also extends to concurrency. While heaps are naturally related to mathematical graphs, many ubiquitous graph properties are non-local in character, such as reachability between nodes, path lengths, acyclicity and other structural invariants, as well as data invariants which combine with these notions. Reasoning modularly about such graph properties remains notoriously difficult, since a local modification can have side-effects on a global property that cannot be easily confined to a small region.

In this paper, we address the question: What separation algebra can be used to avoid proof arguments reverting back to tedious global reasoning in such cases? To this end, we consider a general class of global graph properties expressed as fixpoints of algebraic equations over graphs. We present mathematical foundations for reasoning about this class of properties, imposing minimal requirements on the underlying theory that allow us to define a suitable separation algebra. Building on this theory we develop a general proof technique for modular reasoning about global graph properties over program heaps, in a way which can be integrated with existing separation logics. To demonstrate our approach, we present local proofs for two challenging examples: a priority inheritance protocol and the non-blocking concurrent Harris list.

1 Introduction

Separation logic (SL) [31,38] provides the basis of many successful verification tools that can verify programs manipulating complex heap-based data structures [15,20,28]. This success is due to the logic’s support for reasoning modularly about modifications to heap-based data. For simple inductive data structures such as lists and trees, much of this reasoning can be automated [2,12,22,33]. However, these techniques often fail when data structures are less regular (e.g. multiple overlaid data structures) or provide multiple traversal patterns (e.g. threaded trees). Such idioms are prevalent in real-world implementations such as the fine-grained concurrent data structures found in operating systems and databases. Solutions to these problems have been proposed [16] but remain difficult to automate. For proofs of general graph algorithms, the situation is even more dire. Despite substantial improvements in the verification methodology for such algorithms [36,39], significant parts of the proof argument still typically need to be carried out using non-local reasoning [8,9,14,26]. This paper presents a general technique for local reasoning...
As a motivating example, we consider an idealized priority inheritance protocol (PIP), which is a technique used in process scheduling [40]. The purpose of the protocol is to avoid (unbounded) priority inversion, i.e., a situation where a low-priority process blocks a high-priority process from making progress. The protocol maintains a bipartite graph with nodes representing processes and resources. An example graph is shown in Fig. 1. An edge from a process to a resource indicates that the process is waiting for the resource to become available whereas an edge in the other direction means that the resource is currently held by the process. Every node has an associated default priority as well as a current priority, both of which are natural numbers. The current priority affects scheduling decisions. When a process attempts to acquire a resource currently held by another process, the graph is updated to avoid priority inversion. For example, when process $p_1$ with current priority 3 attempts to acquire the resource $r_1$ that is held by process $p_2$ of priority 2, then $p_1$’s higher priority is propagated to $p_2$ and, transitively, to any other process that $p_2$ is waiting for ($p_3$ in this case). As a result, all nodes on the created cycle will be updated to current priority 3. The protocol thus maintains the following invariant: the current priority of each node is the maximum of its default priority and the current priorities of all its predecessors. Priority propagation is implemented by the method update shown in Fig. 1. The implementation represents graph edges by next pointers and handles both kinds of modifications to the graph: adding an edge (acquire) and removing an edge (release - code omitted). To recalculate the current priority of a node (line 12), each node maintains its default priority def_prio and a multiset prios which contains the priorities of all its immediate predecessors.

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3 The algorithm can then detect the cycle to prevent a deadlock, but this is not the concern of this data structure.
Verifying that the PIP maintains its invariant using established separation logic (SL) techniques is challenging. In general, SL assertions describe resources and express the fact that the program has permission to access and manipulate these resources. In what follows, we stick to the standard model of SL where resources are memory regions represented as partial heaps. We sometimes view partial heaps more abstractly as partial graphs (hereafter, simply graphs). Assertions describing larger regions are built from smaller ones using separating conjunction, $\phi_1 \ast \phi_2$. Semantically, the $\ast$ operator is tied to a notion of resource composition defined by an underlying separation algebra [6][7]. In the standard model, composition enforces that $\phi_1$ and $\phi_2$ must describe disjoint regions. The logic and algebra are set up so that changes to the region $\phi_1$ do not affect $\phi_2$ (and vice versa). That is, if $\phi_1 \ast \phi_2$ holds before the modification and $\phi_1$ is changed to $\phi_1'$, then $\phi_1' \ast \phi_2$ holds afterwards. This so-called frame rule enables modular reasoning about modifications to the heap and extends well to the concurrent setting when threads operate on disjoint portions of memory [4][10][11][37]. However, the mere fact that $\phi_2$ is preserved by modifications to $\phi_1$ does not guarantee that if a global property such as the PIP invariant holds for $\phi_1 \ast \phi_2$, it also still holds for $\phi_1' \ast \phi_2$.

For example, consider the PIP scenario depicted in Fig. 1. If $\phi_1$ describes the subgraph containing only node $p_1$, $\phi_2$ the remainder of the graph, and $\phi_1'$ the graph obtained from $\phi_1$ by adding the edge from $p_1$ to $r_1$, then the PIP invariant will no longer hold for the new composed graph described by $\phi_1' \ast \phi_2$. On the other hand, if $\phi_1$ captures $p_1$ and the nodes reachable from $r_1$ (i.e., the set of nodes modified by update), $\phi_2$ the remainder of the graph, and we reestablish the PIP invariant locally in $\phi_1$ obtaining $\phi_1'$ (i.e., run update to completion), then $\phi_1' \ast \phi_2$ will also globally satisfy the PIP invariant. The separating conjunction $\ast$ is not sufficient to differentiate these two cases; both describe valid partitions of a possible program heap. As a consequence, prior techniques have to revert back to non-local reasoning to prove that the invariant is maintained.

A first helpful idea towards a solution of this problem is that of iterated separating conjunction [29][45], which describes a graph $G$ consisting of a set of nodes $X$ by a formula $\Psi = \bigotimes_{x \in X} N(x)$ where $N(x)$ is some predicate that holds locally for every node $x \in X$. Using such node-local conditions one can naturally express non-inductive properties of graphs (e.g. “$G$ has no outgoing edges” or “$G$ is bipartite”). The advantage of this style of specification is two-fold. First, one can arbitrarily decompose and re-compose $\Psi$ by splitting $X$ into disjoint subsets. For example, if $X$ is partitioned into $X_1$ and $X_2$, then $\Psi$ is equivalent to $\bigotimes_{x \in X_1} N(x) \ast \bigotimes_{x \in X_2} N(x)$. Moreover, it is very easy to prove that $\Psi$ is preserved under modifications of subgraphs. For instance, if a program modifies the subgraph induced by $X_1$ such that $\bigotimes_{x \in X_1} N(x)$ is preserved locally, then the frame rule guarantees that $\Psi$ will be preserved in the new larger graph. Iterated separating conjunction thus yields a simple proof technique for local reasoning about graph properties that can be described in terms of node-local conditions. However, this idea alone does not actually solve our problem because general global graph properties such as “$G$ is a direct acyclic graph”, “$G$ is an overlay of multiple trees”, or “$G$ satisfies the PIP invariant” cannot be directly described this way.

Solution. The key ingredient of our approach is the concept of a flow of a graph: a function flow from the nodes of the graph to flow values. For the PIP, the flow maps each node to the multiset of its incoming priorities. In general, a flow is a fixpoint of
a set of algebraic equations induced by the graph. These equations are defined over a flow domain, which determines how flow values are propagated along the edges of the graph and how they are aggregated at each node. In the PIP example, an edge between nodes \((n, n')\) propagates the multiset containing \(\max(\text{flow}(n), n.\text{def}_prio)\) from \(n\) to \(n'\). The multisets arriving at \(n'\) are aggregated with multiset union to obtain \(\text{flow}(n')\). Flows enable capturing global graph properties in terms of node-local conditions. For example, the PIP invariant can be expressed by the following node-local condition: \(n.\text{curr}_prio = \max(\text{flow}(n), n.\text{def}_prio)\). To enable compositional reasoning about such properties we need an appropriate separation algebra allowing us to prove locally that modifications to a subgraph do not affect the flow of the remainder of the graph.

To this end, we make the useful observation that a separation algebra induces a notion of an interface of a resource: we say that two resources \(a\) and \(a'\) are equivalent if they compose with the same resources. The interface of a resource \(a\) is then given by \(a\)'s equivalence class. In the standard model of SL where resources are graphs and composition is disjoint graph union, the interface of a graph \(G\) is the set of all graphs \(G'\) that have the same domain as \(G\).

The interfaces of resources described by assertions capture the information that is implicitly communicated when these assertions are conjoined by separating conjunction. As we discussed earlier, in the standard model of SL, this information is too weak to enable local reasoning about global properties of the composed graphs because some additional information about the subgraphs' structure other than which nodes they contain must be communicated. For instance, if the goal is to verify the PIP invariant, the interfaces must capture information about the multisets of priorities propagated between the subgraphs. We define a separation algebra achieving exactly this: the induced flow interface of a graph \(G\) in this separation algebra captures how values of the flow domain must enter and leave \(G\) such that, when composed with a compatible graph \(G'\), the imposed local conditions on the flow of each node are satisfied in the composite graph.

This is the key to enabling SL-style framing for global graph properties. Using iterated separating conjunctions over the new separation algebra, we obtain a compositional proof technique that yields succinct proofs of programs such as the PIP, whose proofs with existing techniques would involve non-trivial global reasoning steps.

Contributions. In §2 we present mathematical foundations for flow domains, imposing minimal requirements on the underlying algebra that allow us to capture a broad range of data structure invariants and graph properties, and reason locally about them in a suitable separation algebra. Building on this theory we develop a general proof technique for modular reasoning about global graph properties that can be integrated with existing separation logics provided they support iterated separating conjunction based on the standard heap separation algebra (§3). We further identify general mathematical conditions that guarantee unique flows and provide local proof arguments to check the preservation of these conditions (§4). We demonstrate the versatility of our approach by presenting local proofs for two challenging examples: the PIP and the concurrent non-blocking list due to Harris[13].

Flows Redesigned. Our work is inspired by the recent flow framework explored by some of the authors [24]. We revisit the core algebra behind flows reasoning, and derive
a different algebraic foundation by analysing the minimal requirements for general local reasoning; we call our newly-designed reasoning framework the foundational flow framework. Our new mathematical foundation makes several significant improvements over [24] and eliminates its most stark limitations. First, we present a simplified and generalized meta theory of flows that makes the approach much more broadly applicable. For example, the original framework cannot reason locally about certain graph updates such as removing an edge that breaks a cycle (which can happen in the PIP). Our new framework provides an elegant solution to this problem by requiring that the aggregation operation on flow values is cancellative (see §2.2). This requirement is fundamentally incompatible with the algebraic foundation of the original framework, thus necessitating our new development. We show that requiring cancellativity does not limit expressivity. Moreover, the new framework is much more convenient to use because, unlike the original framework, it imposes no restrictions on how flow values are propagated along edges in the graph. Next, the proofs of programs shown in [24] depend on a bespoke program logic. This logic requires new reasoning primitives that are not supported by the logics implemented in existing SL-based verification tools. Our general proof technique eliminates the need for a dedicated program logic and can be implemented on top of standard separation logics and existing SL-based tools. Finally, the underlying separation algebra of the original framework makes it hard to use equational reasoning, which is a critical prerequisite for enabling proof automation. We provide a more detailed technical comparison to [24] and other related work in §5.

2 The Foundational Flow Framework

In this section, we introduce the foundational flow framework, explaining the motivation for its design with respect to local reasoning principles. We aim for a general technique for modularly proving the preservation of recursively-defined invariants over (partial) graphs, with well-defined decomposition and composition operations.

2.1 Preliminaries and Notation

The term \( b ? t_1 : t_2 \) denotes \( t_1 \) if condition \( b \) holds and \( t_2 \) otherwise. We write \( f : A \to B \) for a function from \( A \) to \( B \), and \( f : A \to B \) for a partial function from \( A \) to \( B \). For a partial function \( f \), we write \( f(x) = \bot \) if \( f \) is undefined at \( x \). We use lambda notation \((\lambda x. E)\) to denote a function that maps \( x \) to the expression \( E \) (typically containing \( x \)). If \( f \) is a function from \( A \) to \( B \), we write \( f[x \mapsto y] \) to denote the function from \( A \cup \{x\} \) defined by \( f[x \mapsto y](z) := (z = x ? y : f(z)) \). We use \( \{x_1 \mapsto y_1, \ldots, x_n \mapsto y_n\} \) for pairwise different \( x_i \) to denote the function \( \epsilon[x_1 \mapsto y_1] \cdots [x_n \mapsto y_n] \), where \( \epsilon \) is the function on an empty domain. Given functions \( f_1 : A_1 \to B \) and \( f_2 : A_2 \to B \) we write \( f_1 \bowtie f_2 \) for the function \( f : A_1 \bowtie A_2 \to B \) that maps \( x \in A_1 \) to \( f_1(x) \) and \( x \in A_2 \) to \( f_2(x) \) (if \( A_1 \) and \( A_2 \) are not disjoint sets, \( f_1 \bowtie f_2 \) is undefined).

We write \( \delta_{n=n'} : M \to M \) for the function defined by \( \delta_{n=n'}(m) := m \) if \( n = n' \) else 0. We also write \( \lambda_0 := (\lambda m. 0) \) for the identically zero function, \( \lambda_{id} := (\lambda m. m) \) for the identity function, and use \( e \equiv e' \) to denote function equality. For \( e : M \to M \) and \( m \in M \) we write \( m \diamond e \) to denote the function application \( e(m) \). We write \( e \circ e' \) to denote
function composition, i.e. \((e \circ e')(m) = e(e'(m))\) for \(m \in M\), and use superscript notation \(e^p\) to denote the function composition of \(e\) with itself \(p\) times.

For multisets \(S\), we use standard set notation when clear from the context. We write \(S(x)\) to denote the number of occurrences of \(x\) in \(S\). We write \(\{x_1 \mapsto i_1, \ldots, x_n \mapsto i_n\}\) for the multiset containing \(i_1\) occurrences of \(x_1\), \(i_2\) occurrences of \(x_2\), etc.

A partial monoid is a set \(M\), along with a partial binary operation \(+ : M \times M \mapsto M\), and a special zero element \(0 \in M\), such that (1) \(+\) is associative, i.e., \((m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)\); and (2) \(0\) is an identity, i.e., \(m + 0 = 0 + m = m\). Here, \(=\) means either both sides are defined and equal, or both are undefined. We identify a partial monoid with its support set \(M\). If \(+\) is a total function, then we call \(M\) a monoid. Let \(m_1, m_2, m_3 \in M\) be arbitrary elements of the (partial) monoid in the following. We call a (partial) monoid \(M\) commutative if \(+\) is commutative, i.e., \(m_1 + m_2 = m_2 + m_1\). Similarly, a commutative monoid \(M\) is cancellative if \(+\) is cancellative, i.e., if \(m_1 + m_2 = m_1 + m_3\) is defined, then \(m_2 = m_3\).

A separation algebra \([6]\) is a cancellative, partial commutative monoid.

2.2 Flows

Recursive properties of graphs naturally depend on non-local information; e.g. we cannot express that a graph is acyclic directly as a conjunction of per-node invariants. Our foundational flow framework captures non-local graph properties by defining flow values at each node; its entire theory is parametric with the choice of a flow domain, whose components will be explained and motivated in this section. We use two running examples of graph properties to illustrate our explanations in this section:

1. Firstly, we consider path-counting, defining a flow domain whose flow values at each node represent the number of paths to this node from a distinguished node \(n\). Path-counting provides enough information to express locally per node that e.g. (a) all nodes are reachable from \(n\) (the path count is non-zero), or (b) that the graph forms a tree rooted at \(n\) (all path counts are exactly 1).
2. Secondly, we use the PIP (Figure 1), defining flows with which we can locally capture the appropriate current node priorities as the graph is modified.

Definition 1 (Flow Domain). A flow domain \((M, +, 0, E)\) consists of a commutative cancellative (total) monoid \((M, +, 0)\) and a set of edge functions \(E \subseteq M \mapsto M\).

Example 1. The flow domain used for the path-counting flow is \((\mathbb{N}, +, 0, \{\lambda_{\text{id}}, \lambda_0\})\), consisting of the monoid on natural numbers under addition and the set of edge functions containing only the identity function and the zero function.

Example 2. We use \((\mathbb{N}^N, \cup, \emptyset, \{\lambda_0\} \cup \{\lambda m, \max(m \cup \{p\})\} | p \in \mathbb{N})\) as flow domain for the PIP example. This consists of the monoid of multisets of natural numbers under multiset union and two kinds of edge functions: \(\lambda_0\) and functions mapping a multiset \(m\) to the singleton multiset containing the maximum value between \(m\) and a fixed value \(p\) (representing a node’s default priority).
As explained below, edge functions are used to determine which flow values are propagated from node to node around the graph. For further definitions in this section we will assume a fixed flow domain \((M, +, 0, E)\) and a (potentially infinite) set of nodes \(\mathcal{N}\). For this section, we abstract heaps using directed partial graphs; integration of our graph reasoning with direct proofs over program heaps is explained in §3.

**Definition 2 (Graph).** A (partial) graph \(G = (N, e)\) consists of a finite set of nodes \(N \subseteq \mathcal{N}\) and a mapping from pairs of nodes to edge functions \(e : N \times N \rightarrow E\).

**Flow Values and Flows** Flow values (taken from \(M\); the first element of a flow domain) are used to capture sufficient information to express desired non-local properties of a graph. In Example 1, flow values are non-negative integers; for the PIP (Example 2) we instead use multisets of integers, representing relevant non-local information: the priorities of nodes currently referencing a given node in the graph. Given such flow values, a node’s correct priority can be defined locally per node in the graph. This definition requires only the maximum value of these multisets, but as we will see shortly these multisets enable local recomputation of a correct priority when the graph is changed.

For a graph \(G = (N, e)\) we express properties of \(G\) in terms of node-local conditions that may depend on the nodes’ flow. A flow is a function \(\text{flow} : N \rightarrow M\) assigning every node a flow value and must be some fixpoint of the following flow equation:

\[
\forall n \in N. \text{flow}(n) = \text{in}(n) + \sum_{n' \in N} \text{flow}(n') \triangleright e(n', n) \tag{FlowEqn}
\]

Intuitively, one can think of the flow as being obtained by a fold computation over the graph: the inflow \(\text{in} : N \rightarrow M\) defines an initial flow at each node. This initial flow is then updated recursively for each node \(n\): the current flow value at its predecessor nodes \(n'\) is transferred to \(n\) via edge functions \(e(n', n) : M \rightarrow M\). These flow values are aggregated using the summation operation \(+\) of the flow domain to obtain an updated flow of \(n\); a flow for the graph is some fixpoint satisfying this equation at all nodes.

**Example 3.** Consider the graph in Figure 1; if the flow domain is as in Example 2, the inflow function \(\text{in}\) assigns the empty multiset to every node \(n\) and we let \(\text{flow}(n)\) be the multiset labelling every node in the figure, then FlowEqn(\(\text{in}, e, \text{flow}\)) holds.

**Definition 3 (Flow Graph).** A flow graph \(H = (N, e, \text{flow})\) is a graph \((N, e)\) and function \(\text{flow} : N \rightarrow M\) such that there exists an inflow \(\text{in} : N \rightarrow M\) satisfying FlowEqn(\(\text{in}, e, \text{flow}\)).

We let \(\text{dom}(H) = N\), and sometimes identify \(H\) and \(\text{dom}(H)\) to ease notational burden. For \(n \in H\) we write \(H_n\) for the singleton flow subgraph of \(H\) induced by \(n\).

**Edge Functions** In any flow graph, the flow value assigned to a node \(n\) by a flow is propagated to its neighbours \(n'\) (and transitively) according to the edge function \(e(n, n')\) labelling the edge \((n, n')\). The edge function maps the flow value at the source node \(n\) to

\[\text{flow}(n) \triangleright e(n, n')\]

\[\text{flow}(n') = \text{flow}(n) \triangleright e(n, n')\]

We note that flows are not generally defined in this manner as we consider any fixpoint of the flow equation to be a flow. Nonetheless, the analogy helps to build an initial intuition.
one propagated on this edge to the target node \( n' \). Note that we require such a labelling for all pairs consisting of a source node \( n \) inside the graph and a target node \( n' \in \Omega \) (i.e., possibly outside the graph). The 0 flow value (the second element of our flow domains) is used to represent no flow; the corresponding (constant) zero function \( \lambda_0 = (\lambda m, 0) \) is used as edge function to model the absence of an edge in the graph. \(^{5}\) A set of edge functions \( E \) from which this labelling is chosen can, other than the requirement \( \lambda_0 \in E \), be chosen as desired. As we will see in \( \S 4.4 \), restrictions to particular sets of edge functions \( E \) can be exploited to further strengthen our overall technique.

For our PIP example, we choose the edge functions to be \( \lambda_0 \) where no edge exists in the PIP structure, and otherwise \((\lambda X, \{\max(X \cup \{m\})\})\) where \( m \) is the default priority of the source of the edge. For example, in Figure 1, \( e(r_3, p_2) = \lambda_0 \) and \( e(r_3, p_1) = (\lambda X, \{\max(X \cup \{0\})\}) \). Since the flow value at \( r_3 \) is \( \{1, 2, 2\} \), the edge \( (r_3, p_1) \) propagates the value \( \{2\} \) to \( p_1 \), correctly representing the current priority of \( r_3 \).

Edge functions can depend on the local state of the source node (e.g. default priorities \( e \), fault priority of the source of the edge. For example, in Figure 1, \( e \)

### Flow Aggregation and Inflows

The flow value at a node is defined by those propagated from other nodes \( \in \text{flow} \) to that node by edge functions, along with an additional inflow value, explained here. Since multiple non-zero flow values can be propagated to a single node, we require an aggregation of these values, for which a binary \( + \) operator on flow values must be defined: the third element of our flow domains. We require \( + \) to be commutative and associative, making this aggregation order-independent. The 0 flow value (representing no flow) must act as a unit for \( + \). For example, in the path-counting flow domain \( + \) means addition on natural numbers, while for the multisets employed for the PIP it means multiset union.

Each node in a flow graph has an inflow, modelling contributions to its flow value which do not come from inside the graph. Inflows play two important roles: first, since our graphs are partial, they model contributions from nodes outside of the graph. Second, inflow can be artificially added as a means of specialising the computation of flow values to characterise specific graph properties. For example, in the path-counting domain, we give an inflow of 1 to the node from which we are counting paths, and 0 to all others.

The flow equation \( \text{FlowEqn} \) defines the flow of a node \( n \) to be the aggregation of flow values coming from other nodes \( n' \) inside the graph (as given by the respective edge function \( e(n', n) \)) as well as the inflow \( \text{in}(n) \). Preserving solutions to this equation across updates to the graph structure is a fundamental goal of our technique. The following lemma (which relies on the fact that \( + \) is required to be cancellative) states that any correct flow values uniquely determine appropriate inflow values:

**Lemma 1.** Given a flow graph \((N, e, \text{flow}) \in \text{FG}\), there exists a unique inflow \( \text{in} : N \rightarrow M \) such that \( \text{FlowEqn}(\text{in}, e, \text{flow}) \).

**Proof.** Suppose \( \text{in} \) and \( \text{in}' \) are two solutions to \( \text{FlowEqn}(-, e, \text{flow}) \). Then, for any \( n \),

\[
\text{flow}(n) = \text{in}(n) + \sum_{n' \in \text{dom}(\text{in})} \text{flow}(n') \cdot e(n', n) = \text{in}'(n) + \sum_{n' \in \text{dom}(\text{in}')} \text{flow}(n') \cdot e(n', n)
\]

\(^{5}\) We will sometimes informally refer to paths in a graph as meaning sequences of nodes for which no edge function labelling a consecutive pair in the sequence is the zero function \( \lambda_0 \).
which, by cancellativity of the flow domain, implies that $\text{in}(n) = \text{in}'(n)$.

We now turn to how solutions of the flow equation can be preserved or appropriately updated under changes to the underlying graph.

**Graph Updates and Cancellativity** Given a flow graph with known flow and inflow values, suppose we remove an edge from $n_1$ to $n_2$ (replacing the edge function with $\lambda_0$). For the same inflow, such an update will potentially affect the flow at $n_2$ and nodes to which $n_2$ (transitively) propagates flow. Starting from the simple case that $n_2$ has no outgoing edges, we need to recompute a suitable flow at $n_2$. Knowing the old flow value (say, $m$) and the contribution $m' = \text{flow}(n_1) \triangleright e(n_1, n_2)$ previously provided along the removed edge, we know that the correct new flow value is some $m''$ such that $m' + m'' = m$. This constraint has a unique solution (and thus, we can unambiguously recompute a new flow value) exactly when the aggregation $\triangleright$ is cancellative; we therefore made cancellativity a requirement on the $+$ of any flow domain.

Cancellativity intuitively enforces that the flow domain carries enough information to enable adaptation to local updates (in particular, removal of edges$^6$). Returning to the PIP example, cancellativity requires us to carry multisets as flow values rather than only the maximum priority value: $+$ cannot be a maximum operation, as this would not be cancellative. The resulting multisets (similarly to the $\text{prio}$ fields in the actual code) provide the information necessary to recompute corrected priority values locally. For example, in the PIP graph shown in Figure 1, removing the edge from $p_6$ to $r_4$ would not affect the current priority of $r_4$ whereas if $p_7$ had current priority 1 instead of 2, then the current priority of $r_4$ would have to decrease. In either case, recomputing the flow value for $r_4$ is simply a matter of subtraction (removing $\{2\}$ from the multiset at $r_4$); cancellativity guarantees that our flow domains will always provide the information needed for this recomputation. Without this property, the recomputation of a flow value for the target node $n_2$ would, in general, entail recomputing the incoming flow values from all remaining edges from scratch. Cancellativity is also crucial for Lemma 1 above, forcing uniqueness of inflows, given known flow values in a flow graph. This allows us to define natural but powerful notions of flow graph decomposition and recomposition.

### 2.3 Flow Graph Composition and Abstraction

Building towards the core of our reasoning technique, we now turn to the question of decomposition and recomposition of flow graphs. Two flow graphs with disjoint domains always compose to a graph, but this will only be a flow graph if their flows are chosen consistently to admit a solution to the resulting flow equation (i.e. the flow graph composition operator defined below is partial).

$^6$ As we will show in §2.3, an analogous problem for composition of flow graphs is also directly solved by this choice to force aggregation to be cancellative.
Definition 4 (Flow Graph Algebra). The flow graph algebra \((\mathcal{F}G, \odot, H_\emptyset)\) for the flow domain \((M, +, 0, E)\) is defined by

\[
\mathcal{F}G := \{(N, e, \text{flow}) \mid (N, e, \text{flow}) \text{ is a flow graph}\}
\]

\[
(N_1, e_1, \text{flow}_1) \odot (N_2, e_2, \text{flow}_2) :=
\begin{cases}
H & H = (N_1 \sqcup N_2, e_1 \sqcup e_2, \text{flow}_1 \sqcup \text{flow}_2) \in \mathcal{F}G \\
\bot & \text{otherwise}
\end{cases}
\]

\[
H_\emptyset := (\emptyset, e_\emptyset, \text{flow}_\emptyset)
\]

where \(e_\emptyset\) and \(\text{flow}_\emptyset\) are the edge functions and flow on the empty set of nodes \(N = \emptyset\). We use \(H\) to range over \(\mathcal{F}G\).

Intuitively, two flow graphs compose to a flow graph if their contributions to each others’ flow (along edges from one to the other) are reflected in the corresponding inflow of the other graph. For example, consider the subgraph from Figure 1 consisting of the single node \(p_7\) (with 0 inflow). This will compose with the remainder of the graph depicted only if this remainder subgraph has an inflow which, at node \(r_4\) includes at least the multiset \(\{2\}\), reflecting the propagated value from \(p_7\).

We use this intuition to extract an abstraction of flow graphs which we call flow interfaces. Given a flow (sub)graph, its flow interface consists of the node-wise inflow and outflow (being the flow contributions its nodes make to all nodes outside of the graph). It is thus an abstraction that hides the flow values and edges wholly inside the flow graph. Flow graphs that have the same flow interface “look the same” to the external graph, as the same values are propagated inwards and outwards.

Our abstraction of flow graphs consists of two complementary notions. Recall that Lemma 1 implies that any flow graph has a unique inflow. Thus we can define an inflow function that maps each flow graph \(H = (N, e, \text{flow})\) to the unique inflow \(\text{inf}(H) : H \to M\) such that \(\text{FlowEqn}(\text{inf}(H), e, \text{flow})\). Dually, we define the outflow of \(H\) as the function \(\text{outf}(H) : \mathcal{N} \setminus N \to M\) defined by

\[
\text{outf}(H)(n) := \sum_{n' \in N} \text{flow}(n') \triangleright e(n', n).
\]

Definition 5 (Flow Interface). A flow interface is a tuple \(I = (\text{in}, \text{out})\) where \(\text{in} : N \to M\) and \(\text{out} : \mathcal{N} \setminus N \to M\) for some \(N \subseteq \mathcal{N}\).

Given a flow graph \(H \in \mathcal{F}G\), its flow interface \(\text{int}(H)\) is the tuple \((\text{inf}(H), \text{outf}(H))\) consisting of its inflow and its outflow. Returning to the previous example, if \(H\) is the singleton subgraph consisting of node \(p_7\) from Figure 1 with flow and edges as depicted, then \(\text{int}(H) = (\lambda n. \emptyset, \lambda n. (n=r_4 \Rightarrow \{2\}) : \emptyset)\).

We write \(I.\text{in}, I.\text{out}\) for the two components of the interface \(I = (\text{in}, \text{out})\). We again identify \(I\) and \(\text{dom}(I.\text{in})\) to ease notational burden.

This abstraction, while simple, turns out to be powerful enough to build a separation algebra over our flow graphs, allowing them to be decomposed, locally modified and recomposed in ways yielding all the local reasoning benefits of separation logics. In particular, for graph operations within a subgraph with a certain interface, we need to prove: (a) that the modified subgraph is still a flow graph (by checking that the flow
We next show the key result for this abstraction: the ability for two flow graphs to compose depends only on their interfaces; flow interfaces implicitly define a congruence relation on flow graphs.

**Definition 6 (Flow Interface Algebra).**

\[
\begin{align*}
I_1 := & \{ I \mid I \text{ is a flow interface} \} \\
I_0 := & \text{int}(H_0) \\
I_1 \oplus I_2 := & \begin{cases} \\
I & I_1 \cap I_2 = \emptyset \\
& \land \forall i \neq j \in \{1, 2\}, n \in I_i, I_i.in(n) = I.i.out(n) + I.j.out(n) \\
& \land \forall n \notin I. I.out(n) = I_1.out(n) + I_2.out(n) \\
& \top \quad \text{otherwise.}
\end{cases}
\end{align*}
\]

Flow interface composition is well defined because of cancellativity of the underlying flow domain (it is also, exactly as flow graph composition, partial). The interfaces of a singleton flow graph containing \(n\) capture the flow and the outflow values propagated by \(n\)’s edges:

**Lemma 2.** For any flow graph \(H = (N, e, \text{flow})\) and \(n, n' \in N\), if \(e(n, n) = \lambda_0\) then \(\text{int}(H_n).in(n) = \text{flow}(n)\) and \(\text{int}(H_n).out(n') = e(n, n')\).

**Proof.** Follows directly from \([\text{FlowEqn}]\) and the definition of outflow.

We next show the key result for this abstraction: the ability for two flow graphs to compose depends only on their interfaces; flow interfaces implicitly define a congruence relation on flow graphs.

**Lemma 3.** \(\text{int}(H_1) = I_1 \land \text{int}(H_2) = I_2 \Rightarrow \text{int}(H_1 \odot H_2) = I_1 \oplus I_2\).

**Proof.** If \(H_1 \odot H_2\) is defined and has interface \(I\), then we show that \(I_1 \oplus I_2\) is defined and equal to \(I\). Let \(H_1 = (N_1, e_i, \text{flow}_i), I_1 = (\text{in}, \text{out}), I_1 = (\text{in}_1, \text{out}_1),\) and \(I_2 = (\text{in}_2, \text{out}_2)\). Since \(H = H_1 \odot H_2 \in \text{FG}\) and \(\text{inf}(H) = I.\text{in} = \text{in}\), we know by definition that \(\forall i \neq j \in \{1, 2\}, n \in H_i,\) \(\text{flow}(n) = \text{in}(n) + \sum_{n' \in H_i} \text{flow}(n') \triangleright e(n', n)\)

\[\iff \text{flow}_i(n) = \text{in}(n) + \sum_{n' \in H_i} \text{flow}(n') \triangleright e(n', n)\]

\[\iff \text{in}_i(n) + \sum_{n' \in H_i} e_i(n', n, \text{flow}_i(n')) = \text{in}(n) + \sum_{n' \in H_i} e_i(n', n, \text{flow}_i(n')) + \sum_{n' \in H_j} e_j(n', n, \text{flow}_j(n'))\]

\[\iff \text{in}_i(n) = \text{in}(n) + \sum_{n' \in H_j} e_j(n', n, \text{flow}_j(n')) \text{ (By cancellativity)}\]

\[\iff \text{in}_i(n) = \text{in}(n) + \text{out}_j(n).\]
Secondly, let \( H = H_1 \circ H_2 \) and note that
\[
out(n) := \sum_{n' \in H} flow(n') \triangleright e(n', n)
\]
\[
= \sum_{n' \in H_1} flow_1(n') \triangleright e_1(n', n) + \sum_{n' \in H_2} flow_2(n') \triangleright e_2(n', n)
\]
\[
= out_1(n) + out_2(n).
\]
As \( H = H_1 \circ H_2 \) implies \( \text{dom}(H_1) \cap \text{dom}(H_2) = \emptyset \), this proves that \( I_1 \oplus I_2 = I \).

Conversely, if \( I_1 \oplus I_2 \) is defined and equal to \( I \) then we show that \( H_1 \circ H_2 \) is defined and has interface \( I \). First, \( I_1 \cap I_2 = \emptyset \), so we know that the graphs are disjoint. Note that the proof above works in both directions, so
\[
\forall i \neq j \in \{1, 2\}, n \in I_i, I_i.in(n) = I.in(n) + I_j.out(n)
\]
\( \Rightarrow flow(n) = in(n) + \sum_{n' \in H} flow(n') \triangleright e(n', n). \)

This tells us that \( H = H_1 \circ H_2 \in \text{FG} \) and \( \text{inf}(H) = \text{in} \). From above, we also know that \( out(n) = out_1(n) + out_2(n) \), so the interface composition condition \( I.out(n) = I_1.out(n) + I_2.out(n) \) gives us \( \text{outf}(H) = out \).

Flow Footprints Consider again the simple modification of changing the edge function labelling a single edge \((n_1, n_2)\); recall that we previously considered the simplified case above that \( n_2 \) has no outgoing edges. Cancellativity of \( + \) avoids a recomputation over arbitrary incoming edges, but once we remove this assumption, we also need to account for the propagation of the change transitively throughout the graph. For example, by adding the edge \((p_1, r_1)\) in Figure 1 and hence, 3 to the flow of \( r_1 \), we in turn add 3 to the flow of all other nodes reachable from \( r_1 \). On the other hand, adding an edge from \( r_4 \) to \( p_5 \) affects only the flow value of \( p_5 \). To capture the relative locality of the side-effects of such updates, we introduce flow footprints. A modification’s flow footprint is the smallest subset of the graph containing those nodes which are sources of modified edges, plus all those whose flow values need to be changed in order to obtain a new flow graph with an unchanged inflow. For example, the flow footprint for the addition of the edge \((p_1, r_1)\) in Figure 1 is \( p_1 \) and all nodes reachable from \( r_1 \) (including \( r_1 \) itself). On the other hand, the flow footprint for removing the edge \((p_2, r_2)\) is just these two nodes; the flow to and from the rest of the graph remains unchanged. We will exploit this idea to define when a subgraph can be replaced with another without disturbing its surroundings.

We next make this notion of flow footprint formally precise.

**Definition 7 (Flow Footprint).** Let \( H \) and \( H' \) be flow graphs such that \( \text{int}(H) = \text{int}(H') \), then the flow footprint of \( H \) and \( H' \), denoted \( \text{ffp}(H, H') \), is the smallest flow graph \( H'_1 \) such that there exists \( H_1, H_2 \) with \( H = H_1 \circ H_2 \), \( H' = H'_1 \circ H_2 \) and \( \text{int}(H_1) = \text{int}(H'_1) \).

The following lemma states that the flow footprint captures exactly those nodes in the graph that are affected by a modification (i.e. either their flow or their outgoing edges change).
Lemma 4. Let $H$ and $H'$ be flow graphs such that $\text{int}(H) = \text{int}(H')$, then for all $n \in H$, $n \in \text{ffp}(H, H')$ iff $H_n \neq H'_n$.

Crucially, the following result shows that we can use flow interfaces as an abstraction compatible with separation-logic-style framing.

Theorem 1. The flow interface algebra $(\mathcal{F}, \oplus, I_0)$ is a separation algebra.

Proof. We prove commutativity first, as it is used in the proof of associativity:

- $\oplus$ is commutative:
  This follows from the symmetry in the definition of $\oplus$ and the commutativity of the flow domain operator $+$.

- $\oplus$ is associative, i.e. $I_1 \oplus (I_2 \oplus I_3) = (I_1 \oplus I_2) \oplus I_3$:
  Note that if any two of the interfaces $I_1$, $I_2$, and $I_3$ are not disjoint, then both sides of the equation are equal to $\bot$. We now show that if the LHS is defined, then the RHS is also defined and equal to it. Let $I_{23} = I_2 \oplus I_3$, and $I = I_1 \oplus I_{23}$. Define $I_{12} = ((\lambda n. I.in(n) + I_3.out(n)), (\lambda n'. I_1.out(n') + I_2.out(n)))$. We first show that $I_{12} = I_1 \oplus I_2$. We know that the interfaces are disjoint, so let us check that the inflows are compatible. For $n \in I_1$,

$$I_{12}.in(n) + I_2.out(n) = I.in(n) + I_3.out(n) + I_2.out(n)$$

$$= I.in(n) + (I_2 + I_3).out(n)$$

$$= I_1.in(n).$$

(As $I = I_1 \oplus I_{23}$)

On the other hand, for $n \in I_2$,

$$I_{12}.in(n) + I_1.out(n) = I.in(n) + I_3.out(n) + I_1.out(n)$$

$$= I_2 + I_3).in(n) + I_3.out(n)$$

$$= I_2.in(n).$$

(As $I = I_{23} = I_2 \oplus I_3$)

Finally, the condition on the outflow follows by definition of $I_{12}$.

We now show that $I = I_{12} \oplus I_3$. Again, the disjointness condition is satisfied by assumption, so let us check the inflows. If $n \in I_{12}$, $I.in(n) + I_3.out(n) = I_{12}.in(n)$ by definition of $I_{12}$. And if $n \in I_3$,

$$I.in(n) + I_{12}.out(n) = I.in(n) + I_1.out(n) + I_2.out(n)$$

$$= I_2 + I_3).in(n) + I_2.out(n)$$

$$= I_3.in(n).$$

(As $I = I_{23} = I_2 \oplus I_3$)

The condition on outflows is true because

$$I.out(n) = I_1.out(n) + I_{23}.out(n)$$

$$= I_1.out(n) + I_2.out(n) + I_3.out(n)$$

$$= I_{12}.out(n) + I_3.out(n).$$

Thus, if the LHS is defined and equal to $I$, then the RHS is defined and equal to $I$.

By symmetry, and commutativity, the other direction is true as well.
- \( I_0 \) is an identity with respect to \( \oplus \):
  This follows directly from the definitions.
- \( \oplus \) is cancellative, i.e. \( I_1 \oplus I_2 = I_1 \oplus I_3 \Rightarrow I_2 = I_3 \):
  Let \( I = I_1 \oplus I_2 = I_1 \oplus I_3 \). Since the domains of \( I_2 \) and \( I_3 \) must be disjoint from \( I_1 \) and yet sum to the domain of \( I \), they must be equal. Now for \( n \in I_2 \), by the definition of \( \oplus \),
  \[
  I_2.\text{in}(n) = I.\text{in}(n) + I_1.\text{out}(n) = I_3.\text{in}(n),
  \]
  so the inflows are equal. As for the outflows, if \( n \not\in I \) then
  \[
  I.\text{out}(n) = I_1.\text{out}(n) + I_2.\text{out}(n) = I_1.\text{out}(n) + I_3.\text{out}(n),
  \]
  which, by cancellativity of the flow domain, implies that \( I_2.\text{out}(n) = I_3.\text{out}(n) \).
  On the other hand, if \( n \in I_3 \), then
  \[
  I_1.\text{in}(n) = I.\text{in}(n) + I_2.\text{out}(n) = I.\text{in}(n) + I_3.\text{out}(n),
  \]
  which again, by cancellativity, implies that \( I_2.\text{out}(n) = I_3.\text{out}(n) \).

This result forms the core of our reasoning technique; it enables us to make modifications within a chosen subgraph and, by proving preservation of its interface, know that the resulting subgraph composes with any context exactly as the original did. Flow interfaces capture precisely the information relevant about a flow graph, from the point of view of its context. In §B we provide additional examples of flow domains that demonstrate the range of data structures and graph properties that can be expressed using flows, including a notion of universal flow that in a sense provides a completeness result for the expressivity of the framework. We now turn to constructing proofs atop these new reasoning principles.

3 Proof Technique

This section shows how to integrate flow reasoning into a standard separation logic, using the priority inheritance protocol (PIP) algorithm to illustrate our proof techniques.

Since flow graphs and flow interfaces form separation algebras, it is possible in principle to define a separation logic (SL) using these notions as a custom semantic model (indeed, this is the proof approach taken in [24]). By contrast, we integrate flow interfaces with a standard separation logic without modifying its semantics. This has the important technical advantage that our proof technique can be naturally integrated with existing separation logics and verification tools supporting SL-style reasoning. We consider a standard sequential SL in this section, but our technique can also be directly integrated with a concurrent SL such as RGSep (as we show in §4.5) or frameworks such as Iris [21] supporting (ghost) resources ranging over user-defined separation algebras.

3.1 Encoding Flow-based Proofs in SL

Proofs using our flow framework can employ a combination of specifications enforced at the node-level and in terms of the flow graphs and interfaces corresponding to larger
heap regions such as entire data structures (henceforth, composite graphs and composite interfaces). At the node level, we write invariants that every node is intended to satisfy, typically relating the node’s flow value to its local state (fields). For example, in the PIP we use node-local invariants to express that a node’s current priority is the maximum of the node’s default priority and those in its current flow value. We typically express such specifications in terms of singleton (flow) graphs, and their singleton interfaces.

Specification in terms of composite interfaces has several important purposes. One is to define custom inflows: e.g. in the path-counting flow domain, specifying that the inflow of a composite interface is 1 at some designated node r and 0 elsewhere enforces in any underlying flow graph that each node n’s flow value will be the number of paths from r to n. Composite interfaces can also be used to express that, in two states of execution, a portion of the heap “looks the same” with respect to composition (it has the same interface, and so can be composed with the same flow graphs), or to capture by how much there is an observable difference in inflow or outflow; we employ this idea in the PIP proof below.

We now define an assertion syntax convenient for capturing both node-level and composite-level constraints, defined within an SL-style proof system. We assume a standard syntax and semantics of the underlying SL: see Appendix A for more details.

Node Predicates The basic building block of our flow-based specifications is a node predicate N(x, H), representing ownership of the fields of a single node x, as well as capturing its corresponding singleton flow graph H:

\[
N(x, H) := \exists fs, fl. x \mapsto fs \ast H = (\{x\}, (\lambda y. \text{edge}(x, fs, y)), fl) \ast \gamma(x, fs, fl(x))
\]

N is implicitly parameterised by fs, edge and γ; these are explained next and are typically fixed across any given flow-based proof. The N predicate expresses that we have a heap cell at location x containing fields fs (a list of field-name/value mappings). It also says that H is a singleton flow graph with domain \(\{x\}\) with some flow fl, whose edge functions are defined by a user-defined abstraction function \(\text{edge}(x, fs, y)\); this function allows us to define edges in terms of x’s field values. Finally, the node, its fields, and its flow in this flow graph satisfy the custom predicate \(\gamma\), used to encode node-local properties such as constraints in terms of the flow values of nodes.

Graph Predicates The analogous predicate for composite graphs is Gr. It carries ownership to the nodes making up potentially-unbounded graphs, using iterated separating conjunction over a set of nodes X as mentioned in §1:

\[
Gr(X, H) := \exists H. \bigotimes_{x \in X} N(x, H(x)) \ast \left(\bigcirc_{x \in X} H(x)\right) = H
\]

Note that the analogous property cannot be captured at the node-level; when considering singleton interfaces per node in a tree rooted at r, every singleton interface has an inflow of 1. For simplicity, we assume that all fields of a flow graph node are to be handled by our flow-based technique, and that their ownership (via \(\mapsto\) points-to predicates) is always carried around together; lifting these restrictions would be straightforward.
Gr\((X_1 \uplus X_2, H)\) \models \exists H_1, H_2 \cdot Gr(X_1, H_1) \ast Gr(X_2, H_2) * H_1 \odot H_2 = H \quad \text{(DECOMP)}

Gr(X_1, H_1) \ast Gr(X_2, H_2) * H_1 \odot H_2 \neq \perp \models Gr(X_1 \uplus X_2, H_1 \odot H_2) \quad \text{(COMP)}

N(x, H) \equiv Gr\{x\}, H \quad \text{(SING)}

\text{emp} \models Gr(\emptyset, H_\emptyset) \quad \text{(GREMP)}

Gr(X_1', H_1') \ast Gr(X_2, H_2) \wedge H = H_1 \odot H_2 \models Gr(X_1' \uplus X_2, H_1' \odot H_2) \quad \text{(REPL)}

\wedge \text{int}(H_1) = \text{int}(H_1') \wedge \text{int}(H) = \text{int}(H_1' \odot H_2)

Fig. 2: Some useful lemmas for proving entailments between flow-based specifications.

Gr is also implicitly parameterised by \(f_s\), edge and \(\gamma\). The existentially-quantified \(H\) is a logical variable representing a function from nodes in \(X\) to corresponding singleton flow graphs. \(Gr(X, H)\) describes a set of nodes \(X\), such that each \(x \in X\) is a \(N\) (in particular, satisfying \(\gamma\)), whose singleton flow graphs compose back to \(H\). As well as carrying ownership of the underlying heap locations, \(Gr\)'s definition allows us to connect a node-level view of the region \(X\) (each \(H(x)\)) with a composite-level view defined by \(H\), on which we can impose appropriate graph-level properties such as constraints on the region’s inflow.

**Lifting to Interfaces** Flow based proofs can often be expressed more elegantly and abstractly using predicates in terms of node and composite-level interfaces rather than flow graphs. To this end, we overload both our node and graph predicates with analogues whose second parameter is a flow interface, defined as follows:

\[
N(x, I) := \exists H. N(x, H) \wedge \text{int}(H) = I
\]

\[
Gr(X, I) := \exists H. Gr(x, H) \wedge \text{int}(H) = I
\]

We will use these versions in the PIP proof below; interfaces capture all relevant properties for decomposition and composition of these flow graphs.

**Flow Lemmas** We first illustrate our \(N\) and \(Gr\) assertions (capturing SL ownership of heap regions and abstracting these with flow interfaces) by identifying a number of lemmas which are generically useful in flow-based proofs. Reasoning at the level of flow interfaces is entirely in the pure world (mathematics independent of heap-ownership and resources) with respect to the underlying SL reasoning; these lemmas are consequences of our defined assertions and the foundational flow framework definitions themselves.

Examples of these lemmas are shown in Figure[2] (DECOMP) shows that we can always decompose a valid flow graph into subgraphs which are themselves flow graphs. Reconstruction (COMP) is possible only if the interfaces of the subgraphs compose (cf. Definition[3]). This rule, as well as (SING), and (GREMP) then all follow directly from the definition of \(Gr\) and standard SL properties of iterated separating conjunction. The final

---

9 In specifications, we implicitly quantify at the top level over free variables such as \(H\).
rule (\texttt{REPL}) is a direct consequence of rules (\texttt{COMP}), (\texttt{DECOMP}) and the congruence relation on flow graphs induced by their interfaces (cf. Lemma 3). Conceptually, it expresses that after decomposing any flow graph into two parts $H_1$ and $H_2$, we can replace $H_1$ with a new flow graph $H'_1$ with the same interface; when recomposing, the overall graph will be a flow graph with the same overall interface.

Note the connection between rules (\texttt{COMP}, \texttt{DECOMP}) and the algebraic laws of standard inductive predicates such as $\texttt{seg}$. For instance by combining the definition of $\texttt{Gr}$ with these rules and (\texttt{SING}) we can prove the following rule to fold or unfold the graph predicate:

$$
\text{Gr}(X \cup \{y\}, H) \equiv \exists H_y, H'. N(y, H_y) \ast \text{Gr}(X, H') \ast H = H_y \circ H' \quad (\text{UNFOLD})
$$

However, crucially (and unlike when using general inductive predicates \cite{32}), this rule is symmetrical for any node $x$ in $X$; it works analogously for any desired order of decomposition of the graph, and for any data structure specified using flows.

When working with our overloaded $N$ and $\text{Gr}$ predicates, similar steps to those described by the above lemmas are useful. Given the these overloaded predicates, we simply apply the lemmas above to the existentially-quantified flow-graphs in their definitions and then lift the consequence of the lemma back to the interface level using the congruence between our flow graph and interface composition notions (Lemma 3).

### 3.2 Proof of the PIP

We now have all the tools necessary to verify the priority inheritance protocol (PIP). Figure 3 gives the full algorithm with flow-based specifications; we also include some intermediate assertions to illustrate the reasoning steps for the \texttt{acquire} method, which we explain in more detail below.

We instantiate our framework in order to capture the PIP invariants as follows:

$$
\begin{align*}
\text{fs} & := \{\text{next}: y, \text{curr\_prio}: q, \text{def\_prio}: q^0, \text{prios}: Q\} \\
\text{edge}(x, \text{fs}, z) & := \\
& \begin{cases}
(\lambda M. \max(M \cup \{q^0\})) & \text{if } z = y \neq \text{null} \\
\lambda_0 & \text{otherwise}
\end{cases} \\
\gamma(x, \text{fs}, M) & := q^0 \geq 0 \land (\forall q' \in Q. q' \geq 0) \\
& \land M = Q \land q = \{\max(Q \cup \{q^0\})\} \\
\phi(I) & := I = (\{\_ \mapsto 0\}, \{\_ \mapsto 0\})
\end{align*}
$$

Each node has the four fields listed in $\text{fs}$. We abstract the heap into a flow graph by letting each node have an edge to its \textit{next} successor labelled by a function that passes to it the maximum incoming priority or the node’s default priority: whichever is larger. With this definition, one can see that the flow of every node will be the multiset containing exactly the priorities of its predecessors. The node-local invariant $\gamma$ says that all priorities are non-negative, the flow $M$ of each node is stored in the \texttt{prios} field, and its current priority is the maximum of its default and incoming priorities. Finally, the constraint $\phi$ on the global interface expresses that the graph is closed – it has no inflow or outflow.
// Let $\delta(M, q_1, q_2) := M \setminus \{q_1 \geq 0 \? \{q_1\} : \emptyset\} \cup \{q_2 \geq 0 \? \{q_2\} : \emptyset\}$

```java
method update(n: Ref, from: Int, to: Int)
  requires $N(n, I_n) \land Gr(X \setminus \{n\}, I') \land I = I'_n \oplus I' \land \varphi(I)$
  requires $I'_n = (n \to \delta(I_n.in(n), from, to)) \land n.out \land from \neq to$
  ensures $Gr(X, I)$
{
  n.prios := n.prios \ {from}
  if (to >= 0) {
    n.prios := n.prios \ {to}
  }
  from := n[curr_prio]
  n[curr_prio] := max(max(n.prios), n.def_prio)
  to := n[curr_prio]
  if (from != to && n.next != null) {
    update(n.next, from, to)
  }
}
```

```java
method acquire(p: Ref, r: Ref)
  requires $Gr(X, I) \land \varphi(I) \land p \in X \land r \in X \land p \neq r$
  ensures $Gr(X, I)$
{
  $\exists I_r, I_p, I_1, N(r, I_r) \land N(p, I_p) \land Gr(X \setminus \{r,p\}, I_1) \land I = I_r \oplus I_p \oplus I_1 \land \varphi(I)$
  if (r.next == null) {
    r.next := p;
    // Let $q_r = r[curr_prio]
    N(r, I_r') \land q_r \neq r.prios
    \exists I_r, I'_r, I_p, I_2, N(r, I_r') \land N(p, I_p) \land Gr(X \setminus \{r,p\}, I_2) \land I = I_r' \oplus I_p \oplus I_2 \land \varphi(I)$
    update(p, -1, r[curr_prio])
  }
  else {
    p.next := r; update(r, -1, p[curr_prio])
  }
}
```

```java
method release(p: Ref, r: Ref)
  requires $Gr(X, I) \land \varphi(I) \land p \in X \land r \in X \land p \neq r$
  ensures $Gr(X, I)$
{
  r.next := null; update(p, r[curr_prio], -1)
}
```

Fig. 3: Full PIP code and specifications, with proof sketch for `acquire`. The comments and coloured annotations (lines 29 to 32) are used to highlight steps in the proof, and are explained in detail the text.
Specifications and Proof Outline. Our end-to-end specifications of `acquire` and `release` guarantee that if we start with a valid flow graph (which is closed, according to $\phi$), we are guaranteed to return a valid flow graph with the same interface (i.e. the graph remains closed). For clarity of the exposition, we focus here on how we prove that being a flow graph (with the same composite interface) is preserved; extending this specification to one which proves e.g. that `acquire` adds the expected edge is straightforward in terms of standard separation logic, and we include such a specification in Appendix C.

The specification for `update` is more subtle. Remember that we call this function in states in which the current interfaces abstracting over the state of the whole graph’s nodes cannot compose to a flow graph; the propagation of priority information is still ongoing, and only once it completes will the nodes all satisfy their invariants and make up a flow graph. Instead, our precondition for `update` uses a “fake” interface $I'_n$ for the node $n$, while $n$’s current state actually matches interface $I_n$. The fake interface $I'_n$ is used to express that if $n$ could adjust its inflow according to the propagated priority change without changing its outflow, then it would compose back with the rest of the graph, and restore the graph’s overall interface. The shorthand $\delta$ defines the required change to $n$’s inflow. In general (except when $n$’s next field is null, or $n$’s flow value is unchanged), it is not possible for $n$ to satisfy $I'_n$; by updating $n$’s inflow, we will necessarily update its outflow. However, we can then construct a corresponding “fake” interface for the next node in the graph, reflecting the update yet to be accounted for.

To illustrate this idea more clearly, let us consider the first if-branch in the proof of `acquire`. Our intermediate proof steps are shown as purple annotations surrounded by braces. The first step, as shown in the first line inside the method body, is to apply $(\texttt{UNFOLD})$ twice (on the flow graphs represented by these predicates) and peel off $N$ predicates for each of $r$ and $p$. The update to $r$’s next field (line 27) causes the correct singleton interface of $r$ to change to $I'_r$: its outflow (previously none, since the next field was null) now propagates flow to $p$. We summarise this state in the assertion on line 29 (we omit e.g. repetition of properties from the function’s precondition, focusing on the flow-related steps of the argument). We now rewrite this state; using the definition of interface composition (Definition 6) we deduce that although $I'_r$ and $I_p$ do not compose (since the former has outflow that the latter does not account for as inflow), the alternative “fake” interface $I'_p$ for $p$ (which artificially accounts for the missing inflow) would do so (cf. line 30). Essentially, we show $I_r \oplus I_p = I'_r \oplus I'_p$ that the interface of $\{r, p\}$ would be unchanged if $p$ could somehow have interface $I'_p$. Now by setting $I_2 = I'_r \oplus I_1$ and using algebraic properties of interfaces, we assemble the precondition expected by `update`. After the call, `update`’s postcondition gives us the desired postcondition.

We focused here on the details of `acquire`’s proof, but very similar manipulations are required for reasoning about the recursive call in `update`’s implementation. The main difference there is that if the if-condition wrapping the recursive call is false then either the last-modified node has no successor (and so there is no outstanding inflow change needed), or we have `from = to` which implies that the “fake” interface is actually the same as the currently correct one.

Despite the property proved for the PIP example being a rather delicate recursive invariant over the (potentially cyclic) graph, the power of our framework enables extremely succinct specifications for the example, and proofs which require the application of
Fig. 4: A potential state of the Harris list with explicit memory management. $f_{\text{next}}$ pointers are shown with dashed edges, marked nodes are shaded gray, and null pointers are omitted for clarity.

relatively few generic lemmas. The integration with standard separation logic reasoning, and the parallel separation algebras provided by flow interfaces allow decomposition and recomposition to be simple proof steps. For this proof, we integrated with standard sequential separation logic, but in the next section we will show that compatibility with concurrent SL techniques is similarly straightforward.

4 Advanced Flow Reasoning & the Harris List

This section introduces some advanced foundational flow framework theory and demonstrates its use in the proof of the Harris list. We note that [24] presented a proof of this data structure in the original flow framework. The proof given here shows that the new framework eliminates the need for the customized concurrent separation logic defined in [24]. We start with a recap of Harris’ algorithm adapted from [24].

4.1 The Harris List Algorithm

The power of flow-based reasoning is exhibited in the proof of overlaid data structures such as the Harris’ list, a concurrent non-blocking linked list algorithm [13]. This algorithm implements a set data structure as a sorted list, and uses atomic compare-and-swap (CAS) operations to allow a high degree of parallelism. As with the sequential linked list, Harris’ algorithm inserts a new key $k$ into the list by finding nodes $k_1, k_2$ such that $k_1 < k < k_2$, setting $k$ to point to $k_2$, and using a CAS to change $k_1$ to point to $k$ only if it was still pointing to $k_2$. However, a similar approach fails for the delete operation. If we had consecutive nodes $k_1, k_2, k_3$ and we wanted to delete $k_2$ from the list (say by setting $k_1$ to point to $k_3$), there is no way to ensure with one CAS that $k_2$ and $k_3$ are still adjacent (another thread could have inserted/deleted in between them).

Harris’ solution is a two step deletion: first atomically mark $k_2$ as deleted (by setting a mark bit on its successor field) and then later remove it from the list using a single CAS. After a node is marked, no thread can insert or delete to its right, hence a thread that wanted to insert $k'$ to the right of $k_2$ would first remove $k_2$ from the list and then insert $k'$ as the successor of $k_1$.

In a non-garbage-collected environment, unlinked nodes cannot be immediately freed as there may be suspended threads continuing to hold a reference to them. A common
solution is to maintain a second “free list” to which marked nodes are added before they are unlinked from the main list (this is the so-called drain technique). These nodes are then labeled with a timestamp, which is used by a maintenance thread to free them when it is safe to do so. This leads to the kind of data structure shown in Figure 4, where each node has two pointer fields: a next field for the main list and an fnext field for the free list (shown as dashed edges). Threads that have been suspended while holding a reference to a node that was added to the free list can simply continue traversing the next pointers to find their way back to the unmarked nodes of the main list.

Even for seemingly simple properties such as that the Harris list is memory safe and not leaking memory, the proof will rely on the following non-trivial invariants:

(a) The data structure consists of two (potentially overlapping) lists: a list on next edges beginning at mh and one on fnext edges beginning at fh.
(b) The two lists are null terminated and next edges from nodes in the free list point to nodes in the free list or main list.
(c) All nodes in the free list are marked.
(d) ft is an element in the free list.

Challenges To prove that Harris’ algorithm maintains the invariants listed above we must tackle a number of challenges. First, we must construct flow domains that allow us to describe overlaid data structures, such as the overlapping main and free lists (§4.2). Second, the flow-based proofs we have seen so far center on showing that the interface of some modified region is unchanged. However, if we consider a program that allocates and inserts a new node into a data structure (like the insert method of Harris), then the interface cannot be the same since the domain has changed (it has increased by the newly allocated node). We must thus have a means to reason about preservation of flows.
by modifications that allocate new nodes (§ 4.3). The third issue is that in some flow
domains, there exist graphs $G$ and inflows $in$ for which no solutions to the flow equation
exist. For instance, consider the path-counting flow domain and the graph in
Figure 5a. Since we would need to use the path-counting flow in the proof of the Harris
list to encode its structural invariants, this presents a challenge (§ 4.4).

We will next see how to overcome these three challenges in turn, and then apply
those solution to the proof of the Harris list in § 4.5.

4.2 Product Flows for Reasoning about Overlays

An important fact about flows is that any flow of a graph over a product of two flow
domains is the product of the flows on each flow domain component.

\textbf{Lemma 5.} Given two flow domains $(M_1, +, 0_1, E_1)$ and $(M_2, +, 0_2, E_2)$, the product
domain $(M_1 \times M_2, +, (0_1, 0_2), E_1 \times E_2)$ where $(m_1, m_2) + (m'_1, m'_2) := (m_1 +_1 m'_1, m_2 +_2 m'_2)$ is a flow domain.

This lemma greatly simplifies reasoning about overlaid graph structures; we will use
the product of two path-counting flows to describe a structure consisting of two overlaid
lists that make up the Harris list.

4.3 Contextual Extensions and The Replacement Theorem

In general, when modifying a flow graph $H$ to another flow graph $H'$, requiring that $H'$
satisfies precisely the same interface $\text{int}(H)$ can be too strong a condition as it does not
permit allocating new nodes. Instead, we want to allow $\text{int}(H')$ to differ from $\text{int}(H)$ in
that the new interface could have a larger domain, as long as the new nodes are fresh and
edges from the new nodes do not change the outflow of the modified region.

\textbf{Definition 8.} An interface $I = (in, out)$ is contextually extended by $I' = (in', out')$, written $I \preceq I'$, if and only if (1) $\text{dom}(in) \subseteq \text{dom}(in')$, (2) $\forall n \in \text{dom}(in)$. $\text{in}(n) = \text{in}'(n)$, and (3) $\forall n' \notin \text{dom}(in)$. $\text{out}(n') = \text{out}'(n')$.

The following theorem states that contextual extension preserves composability and
is itself preserved under interface composition.

\textbf{Theorem 2 (Replacement Theorem).} If $I = I_1 \oplus I_2$, and $I_1 \preceq I'_1$ are all valid
interfaces such that $I'_1 \cap I_2 = \emptyset$ and $\forall n \in I'_1 \setminus I_1$, $I_2.\text{out}(n) = 0$, then there exists a
valid $I' = I'_1 \oplus I_2$ such that $I \preceq I'$.

In terms of our flow predicates, this theorem gives rise to the following adaptation of
the (REPL) rule:

$$\text{Gr}(X'_1, H'_1) \ast \text{Gr}(X_2, H_2) \land H = H_1 \circ H_2 \land \text{int}(H_1) \preceq \text{int}(H'_1)$$

\begin{center}
$\models \exists H'. \text{Gr}(X'_1 \cup X_2, H') \land H' = H'_1 \circ H_2 \land \text{int}(H) \preceq \text{int}(H')$ (REPL+)
\end{center}

The rule (REPL+) is derived from the Replacement Theorem by letting $I = \text{int}(H)$, $I_1 = \text{int}(H_1)$, $I_2 = \text{int}(H_2)$ and $I'_1 = \text{int}(H'_1)$. We know $I_1 \preceq I'_1$, $H = H_1 \circ H_2$ tells us (by
Lemma \([3]\) that \(I = I_1 \oplus I_2\), and \(\text{Gr}(X'_1, H'_1) \ast \text{Gr}(X_2, H_2)\) gives us \(I'_1 \cap I_2 = \emptyset\). The final condition of the Replacement Theorem is to prove that there is no outflow from \(X_2\) to any newly allocated node in \(X'_1\). While we can use additional ghost state to prove such constraints in our proofs, if we assume that the memory allocator only allocates fresh addresses and restrict the abstraction function edge to only propagate flow along an edge \((n, n')\) if \(n\) has a (non-ghost) field with a reference to \(n'\) then this condition is always true. For simplicity, and to keep the focus of this paper on the flow reasoning, we make this assumption in all subsequent proofs.

4.4 Existence and Uniqueness of Flows

We typically express global properties of a graph \(G = (N, e)\) by fixing a global inflow \(in: N \to M\) and then constraining the flow of each node in \(N\) using node-local conditions. However, as we discussed at the beginning of this section, there is no general guarantee that a flow exists or is unique for a given \(in\) and \(G\). The remainder of this section presents two complementary conditions under which we can prove that our flow fixpoint equation always has a unique solution. To this end, we say that a flow domain \((M, +, 0, E)\) has unique flows if for every graph \((N, e)\) over this flow domain and inflow \(in: N \to M\), there exists a unique flow that satisfies the flow equation \(\text{FlowEqn}(in, e, \text{flow})\). But first, we briefly recall some more monoid theory.

Positive Monoids and Endomorphisms We say \(M\) is positive if \(m_1 + m_2 = 0\) implies that \(m_1 = m_2 = 0\). For a positive monoid \(M\), we can define a partial order \(\leq\) on its elements as \(m_1 \leq m_2\) if and only if \(\exists m_3. m_1 + m_3 = m_2\). Positivity also implies that every \(m \in M\) satisfies \(0 \leq m\).

For \(e, e': M \to M\), we write \(e + e'\) for the function that maps \(m \in M\) to \(e(m) + e'(m)\). We lift this construction to a set of functions \(E\) and write it as \(\sum_{e \in E} e\).

Definition 9. A function \(e: M \to M\) is called an endomorphism on \(M\) if for every \(m_1, m_2 \in M\), \(e(m_1 + m_2) = e(m_1) + e(m_2)\). We denote the set of all endomorphisms on \(M\) by \(\text{End}(M)\).

Note that for cancellative \(M\), for every endomorphism \(e \in \text{End}(M)\), \(e(0) = 0\) by cancellativity. Note further that \(e + e' \in \text{End}(M)\) for any \(e, e' \in \text{End}(M)\). Similarly, for \(E \subseteq \text{End}(M)\), \(\sum_{e \in E} e \in \text{End}(M)\). We say that a set of endomorphisms \(E \subseteq \text{End}(M)\) is closed if for every \(e, e' \in E\), \(e \circ e' \in E\) and \(e + e' \in E\).

Nilpotent Cycles Let \((M, +, 0, E)\) be a flow domain where every edge function \(e \in E\) is an endomorphism on \(M\). In this case, we can show that the flow of a node \(n\) is the sum of the flow as computed along each path in the graph that ends at \(n\). Suppose we additionally know that the edge functions are defined such that their composition along any cycle in the graph eventually becomes the identically zero function. In this case, we need only consider finitely many paths to compute the flow of a node, which means the flow equation has a unique solution.

Formally, such edge functions are called nilpotent endomorphisms:
Definition 10. A closed set of endomorphisms $E \subseteq \text{End}(M)$ is called nilpotent if there exists $p > 1$ such that $e^p \equiv 0$ for every $e \in E$.

Example 4. The edge functions of the inverse reachability domain of §B are nilpotent endomorphisms (taking $p = 2$).

Before we prove that nilpotent endomorphisms lead to unique flows, we present some useful notions and lemmas when dealing with flow domains that are endomorphisms.

Lemma 6. If $(M, +, 0, E)$ is a flow domain such that $E$ is a closed set of endomorphisms, $G = (N, e)$ is a graph, $\text{in} : N \to M$ is an inflow such that $\text{FlowEqn}(\text{in}, e, \text{flow})$, and $L \geq 1$,

$\text{flow}(n) = \text{in}(n) + \sum_{n_1, \ldots, n_k \in N} \text{in}(n_1) \triangleright e(n_1, n_2) \cdots e(n_{k-1}, n_k) \triangleright e(n_k, n) + \sum_{n_1, \ldots, n_L \in N} \text{flow}(n_1) \triangleright e(n_1, n_2) \cdots e(n_{L-1}, n_L) \triangleright e(n_L, n)$.

We can now show that if all edges of a flow graph are labelled with edges from a nilpotent set of endomorphisms, then the flow equation has a unique solution:

Lemma 7. If $(M, +, 0, E)$ is a flow domain such that $M$ is a positive monoid and $E$ is a nilpotent set of endomorphisms, then this flow domain has unique flows.

Effectively Acyclic Flow Graphs There are some flow domains that compute flows useful in practice, but which do not guarantee either existence or uniqueness of fixpoints a priori for all graphs. For example, the path-counting flow from Example 1 is one where for certain graphs, there exist no solutions to the flow equation (see Figure 5a), and for others, there can exist more than one (in Figure 5b, the nodes marked with $x$ can have any path count, as long as they both have the same value).

In such cases, we explore how to restrict the class of graphs we use in our flow-based proofs such that each graph has a unique fixpoint; the difficulty is that this restriction must be respected for composition of our graphs. Here, we study the class of flow domains $(M, +, 0, E)$ such that $M$ is a positive monoid and $E$ is a set of reduced endomorphisms (defined below); in such domains we can decompose the flow computations into the various paths in the graph, and achieve unique fixpoints by restricting the kinds of cycles graphs can have.

Definition 11. A flow graph $H = (N, e, \text{flow})$ is effectively acyclic (EA) if for every $1 \leq k$ and $n_1, \ldots, n_k \in N$,

$\text{flow}(n_1) \triangleright e(n_1, n_2) \cdots e(n_{k-1}, n_k) \triangleright e(n_k, n_1) = 0$.

The simplest example of an effectively acyclic graph is one where the edges with non-zero edge functions form an acyclic graph. However, our semantic condition is weaker: for example, when reasoning about two overlaid acyclic lists whose union happens to form a cycle, a product of two path-counting domains will satisfy effective acyclicity because the composition of different types of edges results in the zero function.
Lemma 8. Let \((M, +, 0, E)\) be a flow domain such that \(M\) is a positive monoid and \(E\) is a closed set of endomorphisms. Given a graph \((N, e)\) over this flow domain and inflow \(\text{in}: N \to M\), if there exists a flow graph \(H = (N, e, \text{flow})\) that is effectively acyclic, then flow is unique.

While the restriction to effectively acyclic flow graphs guarantees us that the flow is the unique fixpoint of the flow equation, it is not easy to show that modifications to the graph preserve EA while reasoning locally. Even modifying a subgraph to another with the same flow interface (which we know guarantees that it will compose with any context) can inadvertently create a cycle in the larger composite graph. For instance, consider Figure 5c that shows a modification to nodes \(\{n_3, n_4\}\) (the boxed blue region). The interface of this region is \(\{n_3 \to 1, n_4 \to 1\}, \{n_5 \to 1, n_2 \to 1\}\), and so swapping the edges of \(n_3\) and \(n_4\) preserves this interface. However, the resulting graph, despite composing with the context to form a valid flow graph, is not EA (in this case, it has multiple solutions to the flow equation). This shows that flow interfaces are not powerful enough to preserve effective acyclicity. For a special class of endomorphisms, we show that a local property of the modified subgraph can be checked, which implies that the modified composite graph continues to be EA.

Definition 12. A closed set of endomorphisms \(E \subseteq \text{End}(M)\) is called reduced if \(e \circ e \equiv \lambda_0\) implies \(e \equiv \lambda_0\) for every \(e \in E\).

Note that if \(E\) is reduced, then no \(e \in E\) can be nilpotent. In that sense, this class of instantiations is complementary to those in §4.4.

Example 5. Examples of flow domains that fall into this class include positive semirings of reduced rings (with the additive monoid of the semiring being the aggregation monoid of the flow domain and \(E\) being any set of functions that multiply their argument with a constant flow value). Note that any direct product of integral rings is a reduced ring. Hence, products of the path counting flow domain are a special case.

For reduced endomorphisms, it is sufficient to check that a modification preserves the flow routed between every pair of source and sink node. This pairwise check ensures that we do not create any new cycles in any larger graph. Before we can define an analogous relation to contextual extension, we first define a useful notion:

Definition 13. The capacity of a flow graph \(G = (N, e)\) is \(\text{cap}(G) : N \times N \to (M \to M)\) defined inductively as \(\text{cap}(G) := \text{cap}^0(G)\), where \(\text{cap}^0(G)(n, n') := \delta_{n=n'}\) and

\[
\text{cap}^{i+1}(G)(n, n') := \delta_{n=n'} + \sum_{n'' \in G} \text{cap}^i(G)(n, n'') \circ e(n'', n').
\] (1)

For a flow graph \(H = (N, e, \text{flow})\), we write \(\text{cap}(H)(n, n') = \text{cap}((N, e))(n, n')\) for the capacity of the underlying graph. Intuitively, \(\text{cap}(G)(n, n')\) is the function that summarizes how flow is routed from any source node \(n\) in \(G\) to any other node \(n'\), including those outside of \(G\).
Lemma 9. The capacity is equal to the following sum-of-paths expression:
\[
\cap^i(G)(n, n') = \delta_{n=n'} + \sum_{n_1, \ldots, n_k \in G, 0 \leq k < i} e(n, n_1) \cdots e(n_k, n').
\]

We now define a relation between flow graphs that constrains us to modifications that preserve EA while allowing us to allocate new nodes\[^{10}\]

Definition 14. A flow graph \(H'\) is a subflow-preserving extension of \(H\), written \(H \preceq_s H'\), if \(\text{int}(H) \preceq \text{int}(H')\),
\[
\forall n \in H, n' \not\in H', m. m \leq \inf(H)(n) \Rightarrow m \triangleright \cap(H)(n, n') = m \triangleright \cap(H')(n, n'), \text{ and}
\forall n \in H' \setminus H, n' \not\in H', m. m \leq \inf(H')(n) \Rightarrow m \triangleright \cap(H')(n, n') = 0.
\]

We now show that it is sufficient to check our local condition on a modified subgraph to guarantee composition back to an effectively-acyclic composite graph:

Theorem 3. Let \((M, +, 0, E)\) be a flow domain such that \(M\) is a positive monoid and \(E\) is a reduced set of endomorphisms. If \(H = H_1 \circ H_2\) and \(H_1 \preceq_s H_1'\) are all effectively acyclic flow graphs such that \(H_1' \cap H_2 = \emptyset\) and \(\forall n \in H_1' \setminus H_1\), \(\text{outf}(H_2)(n) = 0\), then there exists an effectively acyclic flow graph \(H' = H_1' \circ H_2\) such that \(H \preceq_s H'\).

We define effectively acyclic versions of our flow graph predicates, \(\text{Gr}_a(X, H)\) and \(\text{Gr}_a(X_1, H_1')\), that take the same arguments but additionally constrain \(H\) to be effectively acyclic. The above theorem then implies the following entailment:
\[
\text{Gr}_a(X_1', H_1') + \text{Gr}_a(X_2, H_2) \land H = H_1 \circ H_2 \land H_1 \preceq_s H_1' \Rightarrow \exists H'. \text{Gr}_a(X_1', H_1') \land H' = H_1' \circ H_2 \land H \preceq_s H' \quad \text{(REPLEA)}
\]

4.5 Proof of the Harris List

We use the techniques seen in this section in the proof of Harris’ list. As the data structure consists of two potentially overlapping lists, we use Lemma 5 to construct a product flow domain of two path-counting flows: one tracks the path count from the head of the main list, and one from the head of the free list. We also work under the effectively acyclic restriction (i.e. we use the \(\text{Ns}\) and \(\text{Gr}_a\) predicates), both in order to obtain the desired interpretation of the flow as well as to ensure existence of flows in this flow domain.

We instantiate the framework using the following definitions of parameters:
\[
fs := \{\text{key}: k, \text{next}: y, \text{fnext}: z\}
\]
\[
\text{edge}(x, fs, v) := (v = \text{null} ? \lambda_0 : (v = y \land y \neq z ? \lambda_{(1, 0)} : (v 
\land (x = ft \Rightarrow I.\text{in}(x) = (\_1, 1)) \land (\neg M(y) \Rightarrow z = \text{null})
\]
\[
\varphi(I) := I.\text{in} = \{m_{h} \mapsto (1, 0), f_{h} \mapsto (0, 1), _{h} \mapsto (0, 0)\} \land I.\text{out} = \{\_ \mapsto (0, 0)\}
\]

\[^{10}\] The monoid ordering used in the following definition exists because we are working with a positive monoid.
Here, edge encodes the edge functions needed to compute the product of two path counting flows, the first component tracks path-counts from \( mh \) on \( \text{next} \) edges and the second tracks path-counts from \( fh \) on \( \text{fnext} \) edges (\( \lambda_{(1,0)} := (\lambda(m_1, m_2), (m_1, 0)) \) and \( \lambda_{(0,1)} := (\lambda(m_1, m_2), (0, m_2)) \)). The node-local invariant \( \gamma \) says: the flow is one of \( \{(1, 0), (0, 1), (1, 1)\} \) (meaning that the node is on one of the two lists, invariant (a)); if the flow is not \( (1, 0) \) (the node is not only on the main list, i.e. it is on the free list) then the node is marked (indicated by \( M(y) \), invariant (c)); and if the node is \( ft \) then it must be on the free list (invariant (d)). The constraint on the global interface, \( \varphi \), says that the inflow picks out \( mh \) and \( fh \) as the roots of the lists, and there is no outgoing flow (thus, all non-null edges from any node in the graph must stay within the graph, invariant (b)).

Since the Harris list is a concurrent algorithm, we perform the proof in rely-guarantee separation logic (RGSep) [42]. Like in §3, we do not need to modify the semantics of RGSep in any way; our flow-based predicates can be defined and reasoning using our lemmas can be performed in the logic out-of-the-box. For space reasons, the full proof can be found in Appendix D.

5 Related Work

An abundance of SL variants provide complementary mechanisms for modular reasoning about programs (e.g. [21, 37, 39]). Most are parameterized by the underlying separation algebra; our flow-based reasoning technique easily integrates with these existing logics.

Recursive data structures are classically handled in SL using recursive predicates [31, 38]. There is a rich line of work in automating such reasoning within decidable fragments (e.g. [2, 12, 18, 22, 33, 35]). However, recursive definitions are problematic for handling e.g. graphs with cycles, sharing and unbounded indegree, overlaid structures and unconstrained traversals.

The most common approach to reason about irregular graph structures in SL is to use iterated separating conjunction [29, 45] and describe the graph as a set of nodes each of which satisfies some local invariant. This approach has the advantage of being able to naturally describe general graphs. However, it is hard to express non-local properties that involve some form of fixpoint computation over the graph structure. One approach is to abstract the program state as a mathematical graph using iterated separating conjunction and then express non-local invariants in terms of the abstract graph rather than the underlying program state [16, 36, 39]. However, a proof that a modification to the state maintains a global invariant of the abstract graph must then often revert back to non-local and manual reasoning, involving complex inductive arguments about paths, transitive closure, and so on. Our technique also exploit iterated separating conjunction for the underlying heap ownership, with the key benefit that flow interfaces exactly capture the necessary conditions on a modified subgraph in order to compose with any context and preserve desired non-local invariants.

In recent work, Wang et al. present a Coq-mechanised proof of graph algorithms in C, based on a substantial library of graph-related lemmas, both for mathematical and heap-based graphs [43]. They prove rich functional properties, integrated with the VST tool. In contrast to our work, a substantial suite of lemmas and background properties are necessary, since these specialise to particular properties such as reachability. We believe
that our foundational flow framework could be used to simplify framing lemmas in a way which remains parameteric with the property in question.

Proofs of a number of graph algorithms have been mechanized in various verification tools and proof assistants, including Trajan’s SCC algorithm [9], union-find [8], Kruskal’s minimum spanning tree algorithm [14], and network flow algorithms [26]. These proofs generally involve non-local reasoning arguments about mathematical graphs.

The most closely related work is [24], for which we already provided a high-level comparison in §1. In addition to the technical innovations made here (general proof technique that integrates with existing SLs), the most striking difference is in the underlying meta theory. The prior flow framework required flow domains to form a semiring; the analogue of edge functions are restricted to multiplication with a constant, which must come from the same flow value set. Our foundational flow framework decouples the algebraic structure defining how flow is aggregated from the algebraic structure of the edge functions. In this way, we obtain a more general framework that applies to many more examples, and with simpler flow domains. Strictly speaking, the prior and our framework are incomparable as the prior did not require that flow aggregation is cancellative. As we argue in §2 cancellativity is, in general, necessary for local reasoning, and is critical for ensuring that the inflow of a composed graph is uniquely determined. Due to this issue, [24] requires proofs to reason about flow interface equivalence classes. This prevents the general modification of graphs with cyclic structures.

An alternative approach to using SL-style reasoning is to commit to global reasoning but remain within decidable logics to enable automation [19][23][25][27][44]. However, such logics are restricted to certain classes of graphs and certain types of properties. For instance, reasoning about reachability in unbounded graphs with two successors per node is undecidable [17]. Recent work by Ter-Gabrielyan et al. [41] shows how to deal with modular framing of pairwise reachability specifications in an imperative setting. Their framing notion has parallels to our notion of interface composition, but allows subgraphs to change the paths visible to their context. The work is specific to a reachability relation, and cannot express the rich variety of custom graph properties available in our technique.

6 Conclusions and Future Work

We have presented the foundational flow framework, enabling local modular reasoning about recursively-defined properties over general graphs. The core reasoning technique has been designed to make minimal mathematical requirements, providing great flexibility in terms of potential instantiations and applications. We identified key classes of these instantiations for which we can provide existence and uniqueness guarantees for the fixpoint properties our technique addresses and demonstrate our proof technique on several challenging examples. As future work, we plan to automate flow-based proofs in our new framework using existing tools that support SL-style reasoning such as Viper [28] and GRASShopper [34].
References

1. Appel, A.W.: Verified software toolchain. In: Goodloe, A., Person, S. (eds.) NASA Formal Methods - 4th International Symposium, NFM 2012, Norfolk, VA, USA, April 3-5, 2012. Proceedings. Lecture Notes in Computer Science, vol. 7226, p. 2. Springer (2012). [https://doi.org/10.1007/978-3-642-28891-3_2]

2. Berdine, J., Calcagno, C., O’Hearn, P.W.: A decidable fragment of separation logic. In: Lodaya, K., Mahajan, M. (eds.) FSTTCS 2004: Foundations of Software Technology and Theoretical Computer Science, 24th International Conference, Chennai, India, December 16-18, 2004. Proceedings. Lecture Notes in Computer Science, vol. 3328, pp. 97–109. Springer (2004). [https://doi.org/10.1007/978-3-540-30538-5_9]

3. Bornat, R., Calcagno, C., Yang, H.: Variables as resource in separation logic. Electron. Notes Theor. Comput. Sci. 155, 247–276 (May 2006). [https://doi.org/10.1016/j.entcs.2005.11.059]

4. Brookes, S., O’Hearn, P.W.: Concurrent separation logic. SIGLOG News 3(3), 47–65 (2016). [https://dl.acm.org/citation.cfm?id=2984457]

5. Calcagno, C., Distefano, D., Dubreil, J., Gabi, D., Hooimeijer, P., Luca, M., O’Hearn, P.W., Papakonstantinou, I., Purbrick, J., Rodriguez, D.: Moving fast with software verification. In: Havelund, K., Holzmann, G.J., Joshi, R. (eds.) NASA Formal Methods - 7th International Symposium, NFM 2015, Pasadena, CA, USA, April 27-29, 2015. Proceedings. Lecture Notes in Computer Science, vol. 9058, pp. 3–11. Springer (2015). [https://doi.org/10.1007/978-3-319-17524-9_1]

6. Calcagno, C., O’Hearn, P.W., Yang, H.: Local action and abstract separation logic. In: 22nd IEEE Symposium on Logic in Computer Science (LICS 2007), 10-12 July 2007, Wroclaw, Poland, Proceedings, pp. 366–378. IEEE Computer Society (2007). [https://doi.org/10.1109/LICS.2007.30]

7. Cao, Q., Cuellar, S., Appel, A.W.: Bringing order to the separation logic jungle. In: Chang, B.E. (ed.) Programming Languages and Systems - 15th Asian Symposium, APLAS 2017, Suzhou, China, November 27-29, 2017, Proceedings. Lecture Notes in Computer Science, vol. 10695, pp. 190–211. Springer (2017). [https://doi.org/10.1007/978-3-319-71237-6_10]

8. Charguéraud, A., Pottier, F.: Verifying the correctness and amortized complexity of a union-find implementation in separation logic with time credits. J. Autom. Reasoning 62(3), 331–365 (2019). [https://doi.org/10.1007/s10817-017-9431-7]

9. Chen, R., Cohen, C., Lévy, J., Merz, S., Théry, L.: Formal proofs of tarjan’s strongly connected components algorithm in why3, coq and isabelle. In: Harrison, J., O’Leary, J., Tolmach, A. (eds.) 10th International Conference on Interactive Theorem Proving, ITP 2019, September 9-12, 2019, Portland, OR, USA. LIPIcs, vol. 141, pp. 13:1–13:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2019). [https://doi.org/10.4230/LIPIcs.ITP.2019.13]

10. Dockins, R., Hobor, A., Appel, A.W.: A fresh look at separation algebras and share accounting. In: Hu, Z. (ed.) Programming Languages and Systems, 7th Asian Symposium, APLAS 2009, Seoul, Korea, December 14-16, 2009. Proceedings. Lecture Notes in Computer Science, vol. 5904, pp. 161–177. Springer (2009). [https://doi.org/10.1007/978-3-642-10672-9_13]

11. Dodds, M., Jagannathan, S., Parkinson, M.J., Svendsen, K., Birkedal, L.: Verifying custom synchronization constructs using higher-order separation logic. ACM Trans. Program. Lang. Syst. 38(2), 4:1–4:72 (Jun 2016). [https://doi.org/10.1145/2818638]
12. Enea, C., Lengál, O., Sighireanu, M., Vojnar, T.: SPEN: A solver for separation logic. In: Barrett, C.W., Davies, M., Kahsai, T. (eds.) NASA Formal Methods - 9th International Symposium, NFM 2017, Moffett Field, CA, USA, May 16-18, 2017, Proceedings. Lecture Notes in Computer Science, vol. 10227, pp. 302–309 (2017). https://doi.org/10.1007/978-3-319-57288-8_22

13. Harris, T.L.: A pragmatic implementation of non-blocking linked-lists. In: Welch, J.L. (ed.) Distributed Computing, 15th International Conference, DISC 2001, Lisbon, Portugal, October 3-5, 2001. Proceedings. Lecture Notes in Computer Science, vol. 2180, pp. 300–314. Springer (2001). https://doi.org/10.1007/3-540-45414-4_21

14. Haslbeck, M.P.L., Lammich, P., Biendarra, J.: Kruskal’s algorithm for minimum spanning forest. Archive of Formal Proofs 2019 (2019), https://www.isa-afp.org/entries/Kruskal.html

15. Hoare, C.A.R.: An axiomatic basis for computer programming. Commun. ACM 12(10), 576–580 (1969). https://doi.org/10.1145/363235.363259

16. Hobor, A., Villard, J.: The ramifications of sharing in data structures. In: Giacobazzi, R., Cousot, R. (eds.) The 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL ’13, Rome, Italy - January 23 - 25, 2013, pp. 523–536. ACM (2013). https://doi.org/10.1145/2429069.2429131

17. Immerman, N., Rabinovich, A.M., Reps, T.W., Sagiv, S., Yorsh, G.: The boundary between decidability and undecidability for transitive-closure logics. In: Marcinkowski, J., Tarlecki, A. (eds.) Computer Science Logic, 18th International Workshop, CSL 2004, 13th Annual Conference of the EACSL, Karpacz, Poland, September 20-24, 2004. Proceedings. Lecture Notes in Computer Science, vol. 3210, pp. 160–174. Springer (2004). https://doi.org/10.1007/978-3-540-30124-0_15

18. Iosif, R., Rogalewicz, A., Vojnar, T.: Deciding entailments in inductive separation logic with tree automata. In: Cassez, F., Raskin, J. (eds.) Automated Technology for Verification and Analysis - 12th International Symposium, ATVA 2014, Sydney, NSW, Australia, November 3-7, 2014, Proceedings. Lecture Notes in Computer Science, vol. 8837, pp. 201–218. Springer (2014). https://doi.org/10.1007/978-3-319-11936-6_15

19. Itzhaky, S., Banerjee, A., Immerman, N., Nanevski, A., Sagiv, M.: Effectively-propositional reasoning about reachability in linked data structures. In: Sharygina, N., Veith, H. (eds.) Computer Aided Verification - 25th International Conference, CAV 2013, Saint Petersburg, Russia, July 13-19, 2013. Proceedings. Lecture Notes in Computer Science, vol. 8044, pp. 756–772. Springer (2013). https://doi.org/10.1007/978-3-642-39799-8_53

20. Jacobs, B., Smans, J., Philippaerts, P., Vogels, F., Penninckx, W., Piessens, F.: Verifast: A powerful, sound, predictable, fast verifier for C and java. In: Bobaru, M.G., Havelund, K., Holzmann, G.J., Joshi, R. (eds.) NASA Formal Methods - Third International Symposium, NFM 2011, Pasadena, CA, USA, April 18-20, 2011. Proceedings. Lecture Notes in Computer Science, vol. 6617, pp. 41–55. Springer (2011). https://doi.org/10.1007/978-3-642-20398-5_4

21. Jung, R., Krebbers, R., Jourdan, J.H., Bizjak, A., Birkedal, L., Dreyer, D.: Iris from the ground up: A modular foundation for higher-order concurrent separation logic. Submitted for publication (2017)

22. Katelaan, J., Mathjea, C., Zuleger, F.: Effective entailment checking for separation logic with inductive definitions. In: Vojnar, T., Zhang, L. (eds.) Tools and Algorithms for the Construction and Analysis of Systems - 25th International Conference, TACAS 2019, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2019, Prague, Czech Republic, April 6-11, 2019, Proceedings, Part II. Lecture Notes in Computer Science,
23. Klarlund, N., Schwartzbach, M.I.: Graph types. In: Deusen, M.S.V., Lang, B. (eds.) Conference Record of the Twentieth Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, Charleston, South Carolina, USA, January 1993. pp. 196–205. ACM Press (1993). [https://doi.org/10.1145/158511.158628]

24. Krishna, S., Shasha, D.E., Wies, T.: Go with the flow: compositional abstractions for concurrent data structures. PACMPL 2 (POPL), 37:1–37:31 (2018). [https://doi.org/10.1145/3158125]

25. Lahiri, S.K., Qadeer, S.: Back to the future: revisiting precise program verification using SMT solvers. In: Necula, G.C., Wadler, P. (eds.) Proceedings of the 35th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2008, San Francisco, California, USA, January 7-12, 2008. pp. 171–182. ACM (2008). [https://doi.org/10.1145/1328438.1328461]

26. Lammich, P., Sefidgar, S.R.: Formalizing network flow algorithms: A refinement approach in isabelle/hol. J. Autom. Reasoning 62 (2), 261–280 (2019). [https://doi.org/10.1007/s10817-017-9442-4]

27. Madhusudan, P., Qiu, X., Stefanescu, A.: Recursive proofs for inductive tree data-structures. In: Field, J., Hicks, M. (eds.) Proceedings of the 39th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2012, Philadelphia, Pennsylvania, USA, January 22-28, 2012. pp. 123–136. ACM (2012). [https://doi.org/10.1145/2103656.2103673]

28. Müller, P., Schwerhoff, M., Summers, A.J.: Viper: A verification infrastructure for permission-based reasoning. In: Jobstmann, B., Leino, K.R.M. (eds.) Verification, Model Checking, and Abstract Interpretation (VMCAI). LNCS, vol. 9583, pp. 41–62. Springer-Verlag (2016)

29. Mueller, P., Schwerhoff, M., Summers, A.J.: Automatic verification of iterated separating conjunctions using symbolic execution. In: Chaudhuri, S., Farzan, A. (eds.) Computer Aided Verification - 28th International Conference, CAV 2016, Toronto, ON, Canada, July 17-23, 2016. Proceedings, Part I. Lecture Notes in Computer Science, vol. 9799, pp. 405–425. Springer (2016). [https://doi.org/10.1007/978-3-319-41528-4_22]

30. O'Hearn, P.W., Pym, D.J.: The logic of bunched implications. Bulletin of Symbolic Logic 5 (2), 215–244 (1999). [https://doi.org/10.2307/421090]

31. O'Hearn, P.W., Reynolds, J.C., Yang, H.: Local reasoning about programs that alter data structures. In: Fribourg, L. (ed.) Computer Science Logic, 15th International Workshop, CSL 2001. 10th Annual Conference of the EACSL, Paris, France, September 10-13, 2001. Proceedings. Lecture Notes in Computer Science, vol. 2142, pp. 1–19. Springer (2001). [https://doi.org/10.1007/3-540-44802-0_1]

32. Parkinson, M.J., Bierman, G.M.: Separation logic and abstraction. In: Falsberg, J., Abadi, M. (eds.) Principles of Programming Languages (POPL). pp. 247–258. ACM (2005)

33. Piskac, R., Wies, T., Zufferey, D.: Automating separation logic using SMT. In: Sharygina, N., Veith, H. (eds.) Computer Aided Verification - 25th International Conference, CAV 2013, Saint Petersburg, Russia, July 13-19, 2013. Proceedings. Lecture Notes in Computer Science, vol. 8044, pp. 773–789. Springer (2013). [https://doi.org/10.1007/978-3-642-39799-8_54]

34. Piskac, R., Wies, T., Zufferey, D.: Grasshopper - complete heap verification with mixed specifications. In: Abrahám, E., Havelund, K. (eds.) Tools and Algorithms for the Construction and Analysis of Systems - 20th International Conference, TACAS 2014. Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2014, Grenoble, France, April 5-13, 2014. Proceedings. Lecture Notes in Computer Science, vol. 8413, pp.
35. Qiu, X., Wang, Y.: A decidable logic for tree data-structures with measurements. In: Enea, C., Piskac, R. (eds.) Verification, Model Checking, and Abstract Interpretation - 20th International Conference, VMCAI 2019, Cascais, Portugal, January 13-15, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11388, pp. 318–341. Springer (2019). https://doi.org/10.1007/978-3-642-54862-8_9

36. Raad, A., Hobor, A., Villard, J., Gardner, P.: Verifying concurrent graph algorithms. In: Igarashi, A. (ed.) Programming Languages and Systems - 14th Asian Symposium, APLAS 2016, Hanoi, Vietnam, November 21-23, 2016, Proceedings. Lecture Notes in Computer Science, vol. 10017, pp. 314–334 (2016). https://doi.org/10.1007/978-3-319-47958-3_17

37. Raad, A., Villard, J., Gardner, P.: Colosl: Concurrent local subjective logic. In: Vitek, J. (ed.) Programming Languages and Systems - 24th European Symposium on Programming, ESOP 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015. Proceedings. Lecture Notes in Computer Science, vol. 9032, pp. 710–735. Springer (2015). https://doi.org/10.1007/978-3-662-46669-8_29

38. Reynolds, J.C.: Separation logic: A logic for shared mutable data structures. In: 17th IEEE Symposium on Logic in Computer Science (LICS 2002), 22-25 July 2002, Copenhagen, Denmark, Proceedings. pp. 55–74. IEEE Computer Society (2002). https://doi.org/10.1109/LICS.2002.1029817

39. Sergey, I., Nanevski, A., Banerjee, A.: Mechanized verification of fine-grained concurrent programs. In: Grove, D., Blackburn, S. (eds.) Proceedings of the 36th ACM SIGPLAN Conference on Programming Language Design and Implementation, Portland, OR, USA, June 15-17, 2015. pp. 77–87. ACM (2015). https://doi.org/10.1145/2737924.2737964

40. Sha, L., Rajkumar, R., Lehoczky, J.P.: Priority inheritance protocols: An approach to real-time synchronization. IEEE Trans. Computers 39(9), 1175–1185 (1990). https://doi.org/10.1109/12.57058

41. Ter-Gabrielyan, A., Summers, A.J., Müller, P.: Modular verification of heap reachability properties in separation logic. PACMPL 3(OOPSLA), 121:1–121:28 (2019). https://doi.org/10.1145/3360547

42. Vafeiadis, V.: Modular fine-grained concurrency verification. Ph.D. thesis, University of Cambridge, UK (2008), http://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.612221

43. Wang, S., Cao, Q., Mohan, A., Hobor, A.: Certifying graph-manipulating c programs via localizations within data structures. Proc. ACM Program. Lang. 3(OOPSLA), 171:1–171:30 (Oct 2019). https://doi.org/10.1145/3360597

44. Wies, T., Muñiz, M., Kuncak, V.: An efficient decision procedure for imperative tree data structures. In: Björner, N., Sofronie-Stokkermans, V. (eds.) Automated Deduction - CADE-23 - 23rd International Conference on Automated Deduction, Wrocław, Poland, July 31 - August 5, 2011. Proceedings. Lecture Notes in Computer Science, vol. 6803, pp. 476–491. Springer (2011). https://doi.org/10.1007/978-3-642-22438-6_36

45. Yang, H.: An example of local reasoning in BI pointer logic: the Schorr-Waite graph marking algorithm. In: Proceedings of the SPACE Workshop (2001)

46. Yang, H.: Local reasoning for stateful programs. University of Illinois at Urbana-Champaign (2001)
A Separation Logic

Separation logic (SL), is an extension of Hoare logic [15] that is tailored to perform modular reasoning about programs that manipulate mutable resources. The primary application of SL has been to verify heap-based data structures, but the core of SL is an abstract separation logic (based on the logic of bunched implications (BI) [30]) that can be instantiated to obtain various existing forms of SL by choosing an appropriate resource: any separation algebra (see §2.1).

Heaps Our separation logic uses standard partial heaps as its semantic model. Let us assume we have the following fixed countably infinite sets: Val, consisting of program values; Loc, consisting of memory addresses; and Field, consisting of field names. Partial heaps are partial maps from addresses to partial maps from field-names to values:

\[
\text{Heap} := \{ h \mid h : \text{Loc} \to (\text{Field} \to \text{Val}) \}
\]

It is easy to see that, under the disjoint union operator \( \uplus \), and using the empty heap \( h_\emptyset \), \((\text{Heap}, \uplus, h_\emptyset)\) forms a separation algebra.

Programming Language We consider the following simple imperative programming language:

\[
C \in \text{Com} ::= \text{skip} \quad \text{Empty command} \\
| c \quad \text{Basic command} \\
| C_1; C_2 \quad \text{Sequential composition} \\
| C_1 + C_2 \quad \text{Non-deterministic choice} \\
| C^* \quad \text{Looping}
\]

\[
c ::= \text{assume}(B) \quad \text{Assume condition} \\
| x := e \quad \text{Variable assignment} \\
| x := e.f \quad \text{Heap dereference} \\
| e_1.f := e_2 \quad \text{Heap write} \\
| x := \text{alloc}() \quad \text{Allocate heap cell}
\]

Here, \( C \) stands for commands, \( c \) for basic commands, \( x \) for program variables, \( e \) for heap-independent expressions, \( f \in \text{Field} \) for field names, and \( B \) for boolean expressions. Since we are only concerned with partial correctness in this dissertation, we can define the more familiar program constructs as the following syntactic shorthands:

\[
\text{if}(B) \quad C_1 \quad \text{else} \quad C_2 := (\text{assume}(B); C_1) + (\text{assume}(\neg B); C_2)
\]

\[
\text{while}(B) \quad C := (\text{assume}(B); C)^*; \text{assume}(\neg B)
\]

Assertions We assume that we start from a standard first-order logic over a signature that includes a countably infinite number of uninterpreted functions and predicates. The only requirement on the underlying logic is that it supports additional uninterpreted sorts, functions and predicates, which can be axiomatised in the pure part of the logic\(^{11}\).

\(^{11}\) We will use this power to express all the values associated with flows and flow interfaces.
Let $\text{Var}$ be an infinite set of variables (we omit sorts and type-checking from the presentation, for simplicity). The syntax of assertions $\phi$ is given by the following:

$$\phi ::= \text{P} \mid \text{true} \mid \phi \land \phi \mid \phi \Rightarrow \phi \mid \exists x. \phi \mid e \mapsto \{f_1: e_1, \ldots\} \mid \phi \ast \phi \mid \bigast_{x \in X} \phi$$

Here, the first line consists of first order assertions $\text{P}$ (called pure assertions in the SL world), the always valid assertion $\text{true}$, standard boolean connectives, and existential quantification. We can define the remaining boolean connectives and universal quantification as shorthands for the appropriate combination of these. The second line contains the new predicates and connectives introduced by SL (so-called spatial assertions). The points-to assertion $e \mapsto \{f_1: e_1, \ldots\}$ is a primitive assertion that denotes a heap cell at address $e$ containing fields $f_1$ with value $e_1$, etc. The key feature of SL is the new connective $\ast$, or separating conjunction, that is used to conjoin two disjoint parts of the heap. We use the $\bigast_{x \in X} \phi$ syntax to represent iterated separating conjunction (the bound variable $x$ ranges over a set $X$).

The semantics of the separation logic assertions are defined with respect to an interpretation of (logical and program) variables $i: \text{Var} \to \text{Val}$. We write $\rho e \kappa i$ for the denotation of expression $e$ under interpretation $i$. In particular, we have:

$$h, i \models e \mapsto \{f_1: e_1, \ldots, f_k: e_k\} \iff h([e]_i) = \{f_1 \mapsto e_1, \ldots, f_k \mapsto e_k\}$$

$$h, i \models \phi_1 \ast \phi_2 \iff \exists h_1, h_2. (h = h_1 \sqcup h_2) \land (h_1, i \models \phi_1) \land (h_2, i \models \phi_2)$$

Note that the logic presented here is garbage-collected [7] (also known as intuitionistic). Thus, the semantics of the points-to assertion $x \mapsto \{f_1: e_1, \ldots, f_k: e_k\}$ does not restrict the heap $h$ to only contain the address $x$, it only requires $x$ to be included in its domain. This restriction is not essential but simplifies presentation.

**Operational Semantics** We give a small-step operational semantics for our programming language. Configurations are either fault or a pair $(C, \sigma)$ of a command $C$ and a state $\sigma$ (i.e. a heap-interpretation pair). The following rules define a reduction relation $\rightarrow$ between configurations:

| Rule | Description |
|------|-------------|
| **SEQ1** | $(\text{skip}; C_2), \sigma \rightarrow C_2, \sigma$ |
| **SEQ2** | $C_1, \sigma \rightarrow C'_1, \sigma'; (C_1; C_2), \sigma \rightarrow (C'_1; C_2), \sigma'$ |
| **CHO1** | $(C_1 + C_2), \sigma \rightarrow C_1, \sigma$ |
| **CHO2** | $(C_1 + C_2), \sigma \rightarrow C_2, \sigma$ |
| **ASS** | $\text{assume}(B), \sigma \rightarrow \text{skip}, \sigma$ |
| **LOOP** | $C^*, \sigma \rightarrow (\text{skip} + (C; C^*)), \sigma$ |

Most presentations of SL also include the separating implication connective $\Rightarrow$. However, logics including $\Rightarrow$ are harder to automate and usually undecidable. By omitting $\Rightarrow$ we emphasize that we do not require it to perform flow-based reasoning.
While we can also give similar small-step semantics to basic commands (for instance, see [38]), it is easier to understand their axiomatic semantics, presented in the next paragraph.

Soundness of separation logic, especially the frame rule below, relies on the following locality property of the semantics of the programming language. By defining our basic commands via an axiomatic semantics, they automatically satisfy this property, and by construction all composite commands will have the locality property.

**Definition 15 (Locality).**

(1) If \((C, \sigma_1 \oplus \sigma) \rightarrow^* \text{fault}\), then \((C, \sigma_1) \rightarrow^* \text{fault}\).

(2) If \((C, \sigma_1 \oplus \sigma) \rightarrow^* (\text{skip}, \sigma_2)\), then either there exists \(\sigma'_2\) such that \((C, \sigma_1) \rightarrow^* \text{(skip), } \sigma'_2\) and \(\sigma_2 = \sigma \oplus \sigma'_2\), or \((C, \sigma_1) \rightarrow^* \text{fault}\).

**Proof Rules**

As with Hoare logic, programs are specified in separation logic by Hoare triples.

**Definition 16 (Hoare Triple).** We say \(\{ \phi \} \ C \{ \psi \}\) if for every state \(\sigma\) such that \(\sigma \models \phi\) we have (1) \((C, \sigma) \not\rightarrow^* \text{fault}\), and (2) for every state \(\sigma'\) such that \((C, \sigma) \rightarrow^* \text{(skip), } \sigma'\), \(\sigma' \models \psi\).

In the above definition, \(\rightarrow^*\) is the reflexive transitive closure of the reduction relation \(\rightarrow\). Intuitively, the judgment \(\{ \phi \} \ C \{ \psi \}\) means that if a command \(C\) is executed on a state satisfying the precondition \(\phi\), then it executes without faults. Moreover, if \(C\) terminates, then the resulting state satisfies the postcondition \(\psi\) (thus, this is a partial correctness criterion).

Separation logic inherits the standard Floyd-Hoare structural proof rules, and the rule of consequence:

\[
\begin{align*}
\text{SL-Skip} & \quad \text{SL-SEQ} \\
\vdash \{ \phi \} \text{skip} \{ \phi \} & \quad \vdash \{ \phi \} C_1 \{ \psi \} \quad \vdash \{ \psi \} C_2 \{ \rho \} \\
\vdash \{ \phi \} C_1; C_2 \{ \rho \} \\
\text{SL-Choice} & \quad \text{SL-LOOP} \\
\vdash \{ \phi \} C_1 \{ \psi \} \quad \vdash \{ \phi \} C_2 \{ \psi \} & \quad \vdash \{ \phi \} C \{ \phi \} \quad \vdash \{ \phi \} C^* \{ \phi \} \\
\vdash \{ \phi \} C_1 + C_2 \{ \rho \} \\
\text{SL-CONSEQ} & \\
P' \Rightarrow \phi & \quad \vdash \{ \phi \} C \{ \psi \} \quad \psi \Rightarrow Q' \\
\vdash \{ P' \} C \{ Q' \} \\
\text{SL-FRAME} & \\
\vdash \{ \phi \} C \{ \psi \} & \quad \vdash \{ \phi * \rho \} C \{ \psi * \rho \}
\end{align*}
\]

The scalability of SL-based reasoning arises due to the following frame rule:

\[
\begin{align*}
\text{SL-FRAME} & \\
\vdash \{ \phi \} C \{ \psi \} & \quad \vdash \{ \phi * \rho \} C \{ \psi * \rho \}
\end{align*}
\]

The frame rule allows one to lift a proof that a command \(C\) executes safely on a state satisfying \(\phi\), producing a state satisfying \(\psi\) if it terminates, to the setting where an
additional resource $\rho$ (the frame) is present. Since $C$ was safe when given only $\phi$, it does not access any resources outside $\phi$; hence, $\rho$ is untouched in the postcondition. The soundness of the frame rule relies on the disjointness of resources enforced by the separating conjunction operator $\ast$.

For the basic commands of the programming language, one can give small axioms, proof rules that specify the minimum resource they need in order to execute safely. The effect of basic commands on more complex states can be derived from these and the frame rule. Here are some of the small axioms:

$$\text{SL-ASSIGN} \quad \vdash \{ \psi[x \mapsto e] \} x := e \{ \psi \}$$

$$\text{SL-WRITE} \quad \vdash \{ e_1 \mapsto \{ f : _, \ldots \} \} e_1.f := e_2 \{ e_1 \mapsto \{ f : e_2, \ldots \} \}$$

$$\text{SL-READ} \quad \vdash \{ e \mapsto \{ f : z, \ldots \} \ast e = y \} x := e.f \{ y \mapsto \{ f : z, \ldots \} \ast x = z \}$$

Note that we write $\psi[x \mapsto e]$ for the assertion $\psi$ where all occurrences of $x$ are replaced with $e$, and _ for an anonymous existential variable (to denote expressions we do not care about).

Together with standard axioms of first-order logic, the proof rules presented above are known to be complete [46]. In other words, all valid Hoare triples can be derived by an appropriate combinations of these axioms.

### B Expressivity of Flows

We now give a few examples of flows to demonstrate the range of data structures whose properties can be expressed as local constraints on each node’s flow.

We start by describing a few interesting examples of flows that capture common graph properties. The path-counting flow defined in §2.2 is a very useful flow for describing the shape of common structures, e.g. lists (singly and doubly linked, cyclic), trees, and (by using product flow constructions) nested and overlaid combinations of these. By considering products with flows for data properties, we can also describe structures such as sorted lists, binary heaps, and search trees.

The next flow is similar to the PIP flow defined in §2.2 and can be used to specify the correctness of algorithms such as Dijkstra’s shortest paths algorithm.

**Definition 17 (Shortest Path Flow).** The shortest path flow uses the flow domain $(\mathbb{N}^C, \cup, \emptyset, \{ \lambda_n \mid n \in C \})$ of multisets over costs $C = \mathbb{N} \cup \{ \infty \}$ where $\lambda_n(S) = \{ n + \min(S) \}$.

The frame rule relies on a side condition that the program variables modified by $C$ do not overlap with the free variables in $\rho$, but this condition can be omitted using the “variables as resource” technique [3].

Note that Yang’s completeness result depends crucially on the separating implication $\Rightarrow$ being included in the assertion language.
Given a flow graph $H = (N, e, \text{flow})$ over this domain and a cost labeling function $c : N \times N \to C$ for edges, if $e(n, n')$ is $\lambda_{n,n'}$ and $\text{inf}(H) = (\lambda_n. (n = s ? \{0\} : 0))$ for some source node $s$, then $\text{flow}(n)$ is the multiset of costs of all shortest paths from $s$ to $n$ via $n$’s predecessors. That is, the cost of the shortest path from $s$ to $n$ is $\text{min}(\text{flow}(n))$.

The next flow can be used to reason about reachability properties in graphs.

**Definition 18 (Inverse Reachability Flow).** Consider the flow domain $(\mathbb{N}^{2^n}, \cup, \emptyset, E)$, consisting of the monoid of multisets of sets of nodes under multiset union and edge functions $E$ containing $\lambda_0$ and for every $n \in \mathbb{N}$ the function

$$
\lambda_n(S) := \{ P \to (n \in P ? S(P \setminus \{n\}) : 0) \}.
$$

Given a flow graph $H = (N, e, \text{flow})$, if $e(n, n') = \lambda_n$ and $\text{inf}(H) = (\lambda_n. (n = r ? \{0\} : 0))$, then $\text{flow}(n)$ is a multiset containing, for each simple path in $H$ from $r$ to $n$, the set of all nodes occurring on that path.

Finally, we demonstrate the full generality of the flow equation in terms of its ability to capture global graph properties. To this end, we define a universal flow that computes, at each node, sufficient information to reconstruct the entire graph. This shows that flows are powerful enough to capture any graph property of interest.

**Definition 19 (Universal Flow).** Say we are given a set of nodes $N \subseteq \mathbb{N}$ and a function $c : N \times \mathbb{N} \to A$ labelling each pair of nodes from some set $A$ (for instance, to encode an unlabelled graph, $A = \{0, 1\}$ and $c(n, n')$ is 1 iff an edge is present in the graph). Consider the flow domain $(\mathbb{N}^{2^n \times \mathbb{N} \times A}, \cup, \emptyset, E)$, consisting of the monoid of multisets of sets of tuples $(n, n', a)$ of edges $(n, n')$ and labels $a \in A$ under multiset union and edge functions $E$ containing $\lambda_0$ and for every $n, n' \in \mathbb{N}, a \in A$ the function

$$
\lambda_{n,n',a}(S) := \{ P \to ((n, n', a) \in P ? S(P \setminus \{(n, n', a)\}) : 0) \}.
$$

Given a flow graph $H = (N, e, \text{flow})$, if $e(n, n') = \lambda_{n,n',c(n,n')}$ and $\text{inf}(H) = (\lambda_n. \{(0\})$, then $\text{flow}(n)$ is a multiset containing, for each simple path in $H$ ending at $n$, a set of all edge-label tuples of edges occurring on that path.

To see why the universal flow computes the entire graph at each node, let us look at the edge functions in more detail. The way to think of $e(n, n') = \lambda_{n,n',c(n,n')}$ is that it looks at each path $P'$ in the input multiset $S$ and if $P'$ does not contain the tuple $(n, n', c(n, n'))$ then it adds the tuple to $P'$ and adds the resulting path $P$ to the output multiset. In order to convert this procedure into a multiset comprehension style definition, the formal definition above starts from each path $P$ in the output multiset and works backward (i.e. $P' = P \setminus \{(n, n', c(n, n'))\}$).

To understand the flow computation, let us start with the inflow to a node $n$, the singleton multiset containing the empty set $\{\emptyset\}$, and track its progress through a path. For every $n'$, the edge function $e(n, n')$ acts on $\{\emptyset\}$ and propagates the singleton multiset $\{(n, n', c(n, n'))\}$. In this way, if we consider a sequence of (distinct) edges $(n_1, n_2), \ldots, (n_{k-1}, n_k)$, then this value becomes

$$
\{(n_1, n_2, c(n_1, n_2)), \ldots, (n_{k-1}, n_k, c(n_{k-1}, n_k))\}.
$$
However, the minute we follow an edge \((n_i, n_{i+1})\) that has occurred on the path before, the edge function \(e(n_i, n_{i+1})\) will send this value to the empty multiset \(\emptyset\). Thus, the flow at each node \(n\) turns out to be the multiset containing sets of edge-label tuples for each simple path in the graph. Note that we label all pairs of nodes in the graph by edges of the form \(\lambda_{n, n'} e(n, n')\). This means that flow\((n)\) will contain one set for every sequence of pairs of nodes in the graph, even those corresponding to edges that do not “exist” in the original graph \(\epsilon\). From this information, one can easily reconstruct all of \(\epsilon\) and hence any graph property of the global graph.

The power of the universal flow to capture any graph property comes with a cost: the flow footprint of any modification is the entire global graph. This means that we lose all powers of local reasoning, and revert to expensive global reasoning about the program. This is to be expected, however, because the universal flow captures all details of the graph, even ones that are possibly irrelevant to the correctness of the program at hand. The art of using flows is to carefully define a flow that captures exactly the necessary global information needed to locally prove correctness of a given program.

For example, the inverse reachability flow is a simplified version of the universal flow in that for each edge it only keeps track of the source node. By capturing less information about the global graph, however, this flow permits more modifications: for instance, one can swap the order of two nodes in a simple path and only update the flows of the two nodes modified. This is an example of carefully tuning the flow domain to match the modifications performed by the program.

### C The PIP

In order to expose field values in the top level specification (e.g. to say that acquire results in the appropriate edge) we extend the signatures of our core predicates to allow extra custom parameters: \(\gamma(x, fs, m, \ldots)\) and \(N(x, H, \ldots)\). For the PIP, we instantiate the framework as follows, where \(\eta\) is a function from nodes to nodes storing the values of the next fields (as enforced by the last line of \(\gamma\)):

\[
fs := \{\text{next}: y, \text{curr_prio}: q, \text{def_prio}: q^0, \text{prios}: Q\}
\]

\[
\text{edge}(x, fs, z) := \begin{cases} 
\lambda M. \max(\max(M), q^0) & \text{if } z = y \neq \text{null} \\
\lambda_0 & \text{otherwise}
\end{cases}
\]

\[
\gamma(x, fs, M, \eta) := q^0 \geq 0 \land (\forall q' \in Q. q' \geq 0) \\
\land M = Q \land q = \{\max(\max(Q), q^0)\} \\
\land \eta(x) = y \land \eta(x) \neq x
\]

\[
\varphi(I) := I = (\{\_ \rightarrow \emptyset\}, \{\_ \rightarrow \emptyset\})
\]

---

\(^{15}\) This flow domain has the property that any graph has a unique solution to the flow equation (see §4.3).
Thus, \( N(x, H, \eta) \) describes a node \( x \) abstracted by flow graph \( H \) and whose next field is \( \eta(x) \).

```plaintext
1 // Let \( \delta(M, q_1, q_2) := M \setminus \{q_1 \geq 0 \? \{q_1\} : \emptyset\} \cup \{q_2 \geq 0 \? \{q_2\} : \emptyset\} \)
2 // Let \( \Delta(I, n, q_1, q_2) := \{(n \mapsto \delta(I.\text{in}(n), q_1, q_2)), I.\text{out}\} \)
3 method update(n: Ref, from: Int, to: Int)
4     requires \( N(n, I_n, \eta) \ast \text{Gr}(X \setminus \{n\}, I', \eta) \land I = I_n \cup I' \land \phi(I) \)
5     requires \( I'_n = \Delta(I_n, n, \text{from}, \text{to}) \land \text{from} \neq \text{to} \)
6     ensures \( \text{Gr}(X, I, \eta) \)
7     
8     n.prios := n.prios \setminus \{\text{from}\}
9     if (to >= 0) { \}
10     n.prios := n.prios \cup \{to\}
11     }
12     from := n.\text{curr}\_\text{prio}
13     n.\text{curr}\_\text{prio} := \max(\max(n.\text{prios}), n.\text{def}\_\text{prio})
14     to := n.\text{curr}\_\text{prio}
15     if (from != to \&\& n.next != null) { // Let \( n' := n.\text{next} \)
16         \exists I'_{n'}, I''_n, I_1. N(n', I'_{n'}, \eta) \ast N(n', I''_n, \eta) \ast \text{Gr}(X \setminus \{n', I'_{n'}, \eta\}, I_1, \eta) \land I = I''_n \cup I'_{n'} \cup I_1 \}
17             \land I''_n = (I_n.\text{in}, \{n' \mapsto \{\text{to}\}\}) \land I'_{n'} = \Delta(I_n, n', \text{from}, \text{to}) \land \text{from} \neq \text{to}
18         = \exists I'_{n'}, I'. N(n', I'_{n'}, \eta) \ast \text{Gr}(X \setminus \{n'\}, I', \eta) \land I' \cup I'_{n'} = I \}
19             \land I'_{n'} = \Delta(I_n, n', \text{from}, \text{to}) \land \text{from} \neq \text{to}
20         update(n.\text{next}, from, to)
21     }
22
23 method acquire(p: Ref, r: Ref)
24     requires \( \text{Gr}(X, I, \eta) \land \phi(I) \)
25     requires \( p \in X \land r \in X \land p \neq r \land \eta(p) = \emptyset \)
26     ensures \( \text{Gr}(X, I, \eta') \)
27     ensures \( \eta(r) = \emptyset ? \eta' = \eta[r \mapsto p] : \eta' = \eta[p \mapsto r] \)
28     
29     \{ \exists I_r, I_p, I_1. N(r, I_r, \eta) \ast N(p, I_p, \eta) \ast \text{Gr}(X \setminus \{r, p\}, I_1, \eta) \land I = I_r \cup I_p \cup I_1 \land \phi(I) \}
30     if (r.\text{next} == \emptyset) { \}
31         \exists I_r, I'_p, I_p, I_1. N(r, I'_p, \eta) \ast N(p, I_p, \eta) \ast \text{Gr}(X \setminus \{r, p\}, I_1, \eta) \land I = I_r \cup I_p \cup I_1 \land \phi(I) \}
32             \land q_r \geq 0 \land I_r.\text{out} = \lambda_0 \land \ldots
33     r.\text{next} := p
34     // Let \( \eta' = \eta[r \mapsto p] \)
35     \exists I_r, I'_p, I_p, I_1. N(r, I'_p, \eta') \ast N(p, I_p, \eta') \ast \text{Gr}(X \setminus \{r, p\}, I_1, \eta) \land I = I_r \cup I_p \cup I_1 \land \phi(I) \}
36             \land I'_p = (I_p.\text{in}, \{p \mapsto \{q_r\}\}) \land q_r \geq 0 \land I_p.\text{out} = \lambda_0 \land \ldots
37     r.\text{next} := p
38     
39     \exists I_p, I'_p, I_2. N(p, I_p, \eta) \ast \text{Gr}(X \setminus \{p\}, I_2, \eta) \land I = I'_p \cup I_2 \}
40             \land I'_p = (\{p \mapsto \delta(I_p.\text{in}(p), -1, q_r)\}, I_p.\text{out}) \land \ldots
```

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40 update\(p, -1, r.\text{curr\_prio}\) 
41 \{ \text{Gr}(X, I, \eta') \} 
42 \text{else} \{ 
43 \text{p.next := r} 
44 \text{update}(r, -1, p.\text{curr\_prio}) 
45 \} 
46 \}

D The Harris List

We perform the proof of the Harris list in rely-guarantee separation logic (RGSep) \cite{42}. RGSep is parametrised by the program states (any separation algebra), the language of assertions (a variant of separation logic), and the programming language (as long as the basic commands are local, see \S \ref{sec:background}). Unlike \cite{24}, where RGSep was instantiated with a bespoke separation algebra, assertion language, and programming language with custom constructs for flows, in this paper we instantiate RGSep with the standard partial heap separation algebra and the standard separation logic assertion language from \S \ref{sec:background}. We do, however, need to switch to a concurrent programming language, so we adopt the simple imperative language used in by the original RGSep presentation \cite{42}. As we did in \S \ref{sec:3}, we only need to define flow predicates \(N\) and \(Gr\) within the logic and assume the flow lemmas from Figure \ref{flowlemmas} in order to perform flow-based proofs in RGSep.

For the Harris, we instantiate the framework as follows, where we extend the \(\gamma\) and \(N\) predicates to also keep track of the mark status (the \(M\) predicate encodes whether a
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reference is marked), next and fnext values of each node:

\[ fs := \{ \text{key: } k, \text{next: } y, \text{fnext: } z \} \]

\[ \text{edge}(x, fs, v) := (v = \text{null ? } \lambda_0 \text{ : (v = y ∧ y \neq z ? } \lambda_{(1,0)} \text{ : (v \neq y ∧ y = z ? } \lambda_{(0,1)} \text{ : (v = y ∧ y = z ? } \lambda_\text{id : } \lambda_0))))) \]

\[ \gamma(x, fs, I, m, x_n, x_f) := (I.\text{in}(x) \in \{(1,0), (0,1), (1,1)\}) \]
\[ \land (I.\text{in}(x) \neq (1,0) \Rightarrow M(y)) \]
\[ \land (x = ft \Rightarrow I.\text{in}(x) = (_-, 1)) \]
\[ \land (\neg M(y) \Rightarrow z = \text{null}) \]
\[ \land (M(y) \Rightarrow m = \boldsymbol{\lambda} \land (\neg M(y) \Rightarrow m = \Diamond)) \]
\[ \land x_n = y \land x_f = z \]

\[ \varphi(I) := I.\text{in} = \{ mh \mapsto (1,0), fh \mapsto (0,1), _- \mapsto (0,0) \} \]
\[ \land I.\text{out} = \{ _- \mapsto (0,0) \} \]

\[ N(x, I, m, x_n, x_f) := x \mapsto fs * \gamma(x, fs, I, m, x_n, x_f) \]
\[ * \text{dom}(I) = \{ x \} * \forall y. I.\text{out}(y) = \text{edge}(x, fs, y)(I.\text{in}(x)) \]

Note that we extend \( \gamma \) with extra parameters \( m, x_n, \) and \( x_f \) that keep track of the mark status (the \( M \) predicate encodes whether a reference is marked), next and fnext values of \( x \). We also expose these parameters in the \( N \) predicate. In our proof, we ignore these additional parameters to \( N \) when we do not care about them (i.e. \( N(x, I) = N(x, I, _, _, _) \), etc.).

D.1 Actions

RGSep consists of two kinds of assertions: boxed assertions \( \boxed{\phi} \) talk about the shared state among threads, and unboxed assertions \( \phi \) talk about thread-local state. An RGSep proof requires an intermediate assertion in between every two atomic modifications to the shared state, along with a stability proof that this intermediate assertion is preserved by interference of other threads. Interference is formally specified via actions, two-state relations that describe modifications performed by threads to the shared state.
D.2 Proof

Since we work under the effectively acyclic restriction, we use the predicates $N_a$ and $Gr_a$ in our proof. As we did with $N$ and $Gr$, we overload $N_a$ and $Gr_a$ to talk about flow interfaces instead of flow graphs:

$$
N_a(x,I) := \exists H. N_a(x,H) \land \text{int}(H) = I
$$

$$
Gr_a(X,I) := \exists H. Gr_a(x,H) \land \text{int}(H) = I
$$

The only place where effective acyclicity reasoning comes in is when we reason about modifications to the shared state: we then have to show that the modified flow graph is a subflow-preserving extension (so that we can use the rule (R EPLEA)). We show some examples of this below with purple annotations in braces. Finally, the proof also requires showing stability of all intermediate assertions in blue. This is straightforward and follows the proof method used by [24, 42], so we omit these details here.

```plaintext
procedure search(k: Key) returns (l: Ref, r: Ref)
requires \(\exists X,I. Gr_a(X,I) \land \varphi(I)\)
ensures \(\exists X,I. Gr_a(X,I) \land \varphi(I) \land \{l,r\} \subseteq X\)
{
  \{ \exists X,I_{mh}, I_1. N_a(mh,I_{mh}) \land Gr_a(X \setminus \{mh\},I_1) \land \varphi(I_{mh} \oplus I_1) \land \{l,r\} \subseteq X \}
  l := mh; r := mh
  var n := head.next
  while (isMarkedRef(n) || r.key < k)
  invariant \(\exists X,I. Gr_a(X,I) \land \varphi(I) \land \{l,r\} \subseteq X \land (\neg M(n) \Rightarrow n \in X)\)
  {
    if (isMarkedRef(n)) { // r marked
      \{ \exists X,I, I_1. N_a(l,I_1) \land Gr_a(X \setminus \{l\},I_1) \land \varphi(I_1 \oplus I_1) \}
      if (CAS(l.next, r, unmarked(n))) { // try to unlink r
        // Success: now l.next == n, l still unmarked
        r := unmarked(n)
    } else {
      \{ \exists X,I, I_1. N_a(l,I_1) \land Gr_a(X \setminus \{l\},I_1) \land \varphi(I_1 \oplus I_1) \}
    }
  }
  \}
```

(Insert)

(Insert)

(Mark)

(Insert)

(Insert)

(Insert)

(Insert)

(Link)

(Link)

(Unlink)

(Unlink)
procedure insert(k: Key)
  requires $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I)$
  ensures $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I)$
  {
    Node n := new Node(k, null, null)
    { $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I) \land N_a(n, \_, \_, \null, \null)$ }
    while (true)
      invariant $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I) \land N_a(n, \_, \_, \null, \null)$
      var l, r := search(k)
      { $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I) \land \{l, r\} \subseteq X \land N_a(n, \_, \_, \null, \null)$ }
      if (r.key == k) {
        free(n)
        return false
      }
      n.next := r
      { $\exists X, I, I_1. N_a(l, I)$ $\land \text{Gr}_a(X \setminus \{l\}, I_1) \land \varphi(I_1 \oplus I_1) \land n \in X$ }
      if (CAS(l.next, r, n)) {
        { $N_a(l, H'_1) \land N_a(n, H'_n) \land \text{Gr}_a(X \setminus \{l\}, H_1) \land H_1 \leq_s (H'_1 \oplus H'_n) \land \varphi(\text{int}(H_1 \oplus H_1))$ }
        return true
      }
      }
  }
procedure delete(k: Key)
  requires $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I)$
  ensures $\exists X, I. \text{Gr}_a(X, I) \land \varphi(I)$
  { var l, r, n
while (true)
  invariant $\exists X, I. \text{Gr}_2(X, I) \land \varphi(I)$
  
  $l, r := \text{search}(k)$
  
  $\exists X, I. \text{Gr}_2(X, I) \land \varphi(I) \land \{l, r\} \subseteq X$
  
  $n := r.next$
  
  if (r.key != k) {
    return false
  }
  
  if (!isMarkedRef(n)) { // r unmarked
    CAS(r.next, n, marked(n)) { // mark r
      $\exists X, I. \text{Na}(r, I, \text{null}) \land \text{Gr}_2(X \setminus \{r\}, I_1) \land \varphi(I_r \oplus I_1) \land l \in X$ \land \neg M(n)$
      break
    }
  }
  
  if r already marked, we should have returned false, so retry
  
  $\exists X, I. \text{Na}(r, I, \text{null}) \land \text{Gr}_2(X \setminus \{r\}, I_1) \land \varphi(I_r \oplus I_1) \land l \in X \land r \neq ft$
  
  // Try to unlink r from main list
  CAS(l.next, r, n) // If this fails, next search will unlink
  
  $\exists X, I. \text{Na}(r, I, \text{null}) \land \text{Gr}_2(X \setminus \{r\}, I_1) \land \varphi(I_r \oplus I_1) \land l \in X \land r \neq ft$
  
  // Link r to free list
  while (!CAS(ft.next, null, r)) {
    $\exists X, I. \text{Na}(r, I, \text{null}) \land \text{Gr}_2(X \setminus \{r\}, I_1) \land \varphi(I_r \oplus I_1) \land l \in X \land r \neq ft \land I_r.in(r) = (\_1)$
  }
  
  ft := r
  
  return true
  
}