A shadow of algebraic topology and variational method - *Prandtl Batchelor* problem

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Abstract

In this paper we study the existence of nontrivial weak solution to a *Prandtl-Batchelor* type free boundary value elliptic problem involving a $p$-Laplacian operator and a power nonlinearity. Topics from algebraic topology will be used to establish the existence of a solution to the approximating problem, whereas, the variational technique will be used to fix the claim of existence of a solution to the main problem. In the process, a couple of classical results were also improved to suit the purpose of establishing the existence of a nontrivial solution.

Keywords: Dirichlet free boundary value problem, Sobolev space, Morse relation, cohomology group.

AMS Classification: 35J35, 35J60.

1. Introduction

We will investigate the existence of solution to the following free boundary value problem.

\[-\Delta_p u = \lambda \chi_{\{u > 1\}} (u - 1)^{\frac{p}{p-1}}, \quad \text{in } \Omega \setminus H(u),\]

\[|\nabla u^+|^p - |\nabla u^-|^p = \frac{p}{p-1}, \quad \text{in } H(u)\]

\[u = 0, \quad \text{on } \partial \Omega.\]

Here, $\lambda > 0$ is a parameter, $(u - 1)_+ = \max\{u - 1, 0\}$ and

\[H(u) = \partial\{u > 1\}.\]

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Also $\nabla u^\pm$ are the limits of $\nabla u$ from the sets $\{u > 1\}$ and $\{u \leq 1\}$ respectively. The domain $\Omega \subset \mathbb{R}^N (N \geq 2)$ is bounded with a sufficiently smooth boundary $\partial \Omega$. The relation between the exponents are assumed in the order $1 < p \leq q - 1$, with $q < p^* = \frac{Np}{N - p}$. The solution(s) satisfy the free boundary condition in the following sense: for all $\phi \in C^1_0(\mathbb{R}^N)$ such that $u \neq 1$ a.e. on the support of $\phi$,

$$\lim_{\epsilon^+ \to 0} \int_{u=1+\epsilon^+} \left( \frac{p}{p-1} - |\nabla u|^p \right) \phi \cdot \hat{n} dS - \lim_{\epsilon^- \to 0} \int_{u=1-\epsilon^-} |\nabla u|^p \phi \cdot \hat{n} dS = 0,$$

(1.2)

where $\hat{n}$ is the outward drawn normal to $\{1 - \epsilon^- < u < 1 + \epsilon^+\}$. Note that the sets $\{u = 1 \pm \epsilon^\pm\}$ are smooth hypersurfaces for almost all $\epsilon^\pm > 0$ by the Sard’s theorem. The limit above in (1.2) is taken by running such $\epsilon^\pm > 0$ towards zero.

A rich literature survey has been done in the book due to Perera et al. [10] where the author has discussed problems of several variety involving the $p$-Laplacian operators which could be studied using the Morse theory. The motivation for the current work has been drawn from the work due to Perera [13]. The treatment used to address the existence of atleast one (or two) solution(s) to the approximating problem may be classical (section 3, Theorems 3.3 and 3.5) but the result concerning the reguarity of the free boundary is very new and the question of existence of solution to the problem (1.1) has not been answered till now (section 4, Lemma 4.1), to the best of my knowledge. Two more results due to Alt-Caffarelli [1] (section 4, Lemma 4.2) and Caffarelli et al. [6] (Appendix, Lemma 4.3) were improved to the best possible extent to suit the purpose of the problem in this paper.

### 1.1 A physical motivation

Consider the problem

$$-\Delta u = \lambda \chi_{\{u>1\}}(x), \text{ in } \Omega \setminus H(u),$$

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2, \text{ in } H(u)$$

$$u = 0, \text{ on } \partial \Omega.$$  \hspace{1cm} (1.3)

This is the well known Prandtl-Batchelor free boundary value problem, where the phase $\{u > 1\}$ is a representation of the vortex patch bounded by the vortex line $u = 1$ in a steady fluid flow for $N = 2$ (refer Batchelor [2, 3]). Thus the current problem is a more generalized version of (1.3). For a more physical application to this problem we direct the reader’s attention to the work due to Caflisch [4], Elcrat and Miller [7].

Another instance of occurrence of such a phenomena is in the non-equilibrium system of melting of ice. In a given block of ice, the heat equation can be solved with a given
set of appropriate initial/boundary conditions in order to determine the temperature. However, if there exists a region of ice in which the temperature is greater than the melting point of ice, this subdomain will be filled with water. The boundary thus formed due to the ice-water interface is controlled by the solution of the heat equation. Thus encountering a free boundary in the nature is not unnatural. The problem in this paper is a large enough generalization to this physical phenomena which besides being a new addition to the literature can also serve as a note to bridge the problems in elliptic PDEs with algebraic topology.

2. Preliminaries

We begin by giving the relevant definitions and results besides defining the function space which will be used very frequently in the article. Let $X$ be a topological space and $A \subset X$ be a topological subspace. Roughly, a homology group is an algebraic group constructed from a topological object or a space. Following is the fundamental tool that will be used to work with, namely the homology theory [12].

**Definition 2.1.** A homology group on a family of pairs of spaces $(X, A)$ consists of:

1. A sequence $\{H_k(X, A)\}_{k \in \mathbb{N}_0}$ of abelian groups is known as homology group for the pair $(X, A)$ (note that for the pair $(X, \phi)$, we write $H_k(X), k \in \mathbb{N}_0$). Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. To every map of pairs $\varphi : (X, A) \to (Y, B)$ is associated a homomorphism $\varphi^* : H_k(X, A) \to H_k(Y, B)$ for all $k \in \mathbb{N}_0$.

3. To every $k \in \mathbb{N}_0$ and every pair $(X, A)$ is associated a homomorphism $\partial : H_k(X, A) \to H_{k-1}(A)$ for all $k \in \mathbb{N}_0$.

These items satisfy the following axioms.

(A1) If $\varphi = id_X$, then $\varphi_* = id|_{H_k(X, A)}$.

(A2) If $\varphi : (X, A) \to (Y, B)$ and $\psi : (Y, B) \to (Z, C)$ are maps of pairs, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. 

(A3) If $\varphi : (X, A) \to (Y, B)$ is a map of pairs, then $\partial \circ \varphi_* = (\varphi|_A)_* \circ \partial$.

(A4) If $i : A \to X$ and $j : (X, \phi) \to (X, A)$ are inclusion maps, then the following sequence is exact

$$\ldots \xrightarrow{\partial} H_k(A) \xrightarrow{j_*} H_k(X) \xrightarrow{i_*} H_k(X, A) \xrightarrow{\partial} H_{k-1}(A) \xrightarrow{\partial} H_{k-2}(A) \xrightarrow{\partial} \ldots$$
Recall that a chain \( \cdots \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}} C_{k-2}(X) \xrightarrow{\partial_{k-2}} \cdots \) is said to be exact if \( \text{im}(\partial_{k+1}) = \ker(\partial_k) \) for each \( k \in \mathbb{N}_0 \).

(A\(_5\)) If \( \varphi, \psi : (X, A) \to (Y, B) \) are homotopic maps of pairs, then \( \varphi_* = \psi_* \).

(A\(_6\)) (Excision): If \( U \subseteq X \) is an open set with \( \bar{U} \subseteq \text{int}(A) \) and \( i : (X \setminus U, A \setminus U) \to (X, A) \) is the inclusion map, then \( i_* : H_k(X \setminus U, A \setminus U) \to H_k(X, A) \) is an isomorphism.

(A\(_7\)) If \( X = \{\ast\} \), then \( H_k(\ast) = 0 \) for all \( k \in \mathbb{N} \).

**Definition 2.2.** A continuous map \( F : X \times [0, 1] \to X \) is a deformation retraction of a space \( X \) onto a subspace \( A \) if, for every \( x \in X \) and \( a \in A \), \( F(x, 0) = x, F(x, 1) \in A \), and \( F(a, 1) = a \).

A crucial notion in analysis is the idea of compactness and the **Palais-Smale** condition is a special type of compactness which is given as follows.

**Definition 2.3.** (S. Kesavan \([11]\)) Let \( V \) be a Banach space and \( J : V \to \mathbb{R} \) a \( C^1 \) functional. It is said to satisfy the Palais-Smale condition (PS) if the following holds: whenever \( (u_n) \) is a sequence in \( V \) such that \( (J(u_n)) \) is bounded and \( J'(u_n) \to 0 \) in \( V' \) (the dual space of \( V \)), then \( (u_n) \) has a strongly convergent subsequence.

The following is a deformation lemma which will be quintessential in computing the homology groups.

**Lemma 2.4.** (S. Kesavan \([11]\)) Let \( J : V \to \mathbb{R} \) be a \( C^1 \) functional satisfying the Palais-Smale condition. Let \( c, a \) be real numbers. Define \( K_{J,c} = \{ v \in X : J(v) = c, J'(v) = 0 \} \), \( K^a = \{ v \in X : J(v) \leq a \} \) (likewise we define \( K_a = \{ v \in X : J(v) \geq a \} \)). Let \( K_{J,c} = \emptyset \). Then there exists \( \epsilon' > 0 \) and a continuous homotopy \( \eta : [0, 1] \times V \to V \) such that \( \forall \ 0 < \epsilon \leq \epsilon' \)

1. \( \eta(0, v) = v \) for all \( v \in X \).
2. \( \eta(t, v) = v \) for all \( t \in [0, 1], v \neq J^{-1}([c - \epsilon, c + \epsilon]) \).
3. \( \eta(1, K^{c+\epsilon}) \subset K^{c-\epsilon} \).

**Definition 2.5.** Morse index of a functional \( J : V \to \mathbb{R} \) is defined to be the maximum subspace of \( V \) such that \( J'' \), the second Fréchet derivative, is negative definite on it.
2.1 Space description

We begin by defining the standard Lebesgue space $L^p(\Omega)$ for $1 \leq p < \infty$ as

$$L^p(\Omega) = \left\{ u: \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^p dx < \infty \right\}$$

endowed with the norm $\|u\|_p = \left( \int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}$. We will define the Sobolev space as

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N \}$$

with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. We further define

$$W^{1,p}_0(\Omega) = \{ u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega \}.$$

The associated norm is $\|u\|_p = \|\nabla u\|_p$.

With these norms, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ are separable, reflexive Banach spaces. We now state the Hölder’s inequality and embedding results in the following propositions.

**Proposition 2.6.** For any $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, where $L^{p'}(\Omega)$ is the conjugate space of $L^p(\Omega)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$,

$$|\int_\Omega uv \, dx| \leq \|u\|_p \|v\|_{p'}.$$

**Proposition 2.7.** If $p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [p,p^*)$ and compact for $r \in [p,p^*)$. If $p = N$, then $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous and compact for $r \in [p,\infty)$. Further, if $p > N$, then $W^{1,p}(\Omega) \hookrightarrow C^{1-\frac{N}{p}}(\bar{\Omega})$.

3. The way to tackle the problem using Morse theory

We at first define an energy functional associated to the problem in (1.1) which is as follows.

$$I(u) = \int_\Omega \frac{|
abla u|^p}{p} \, dx + \int_\Omega \chi_{\{u>1\}}(x)dx - \lambda \int_\Omega (u - 1)^q \, dx.$$

This functional is not even differentiable and hence poses serious issues as far as the application of variational theorems are concerned. Thus we approximate $I$ using the following functionals that varies with respect to a parameter $\alpha > 0$. This method is adapted from the work of Jerison-Perera [9]. We define a smooth function $g : \mathbb{R} \to [0,2]$ as follows:

$$g(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
\text{a positive function}, & \text{if } 0 < t < 1 \\
0, & \text{if } t \geq 1 
\end{cases}$$

and $\int_0^1 g(t)dt = 1$. We further let $G(t) = \int_0^t g(t)dt$. Clearly, $G$ is smooth and nondecreasing function such that

$$G(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \text{a positive function < 1,} & \text{if } 0 < t < 1 \\ 1, & \text{if } t \geq 0. \end{cases}$$

We thus define

$$I_\alpha(u) = \int_\Omega \left| \nabla u \right|^p dx + \int_\Omega G \left( \frac{u - 1}{\alpha} \right) dx - \lambda \int_\Omega \frac{(u - 1)^q}{q} dx.$$

This functional $I_\alpha$, is of at least $C^2$ class and hence

$$\langle I''_\alpha(u), v \rangle = \int_\Omega \left[ \left| \nabla u \right|^{p-2} \nabla u \cdot \nabla v + (p - 2)\left| \nabla u \right|^{p-4} (\nabla u \cdot \nabla v)(\nabla u \cdot \nabla w) \right] dx$$

$$+ \int_\Omega \frac{1}{\alpha^2} g' \left( \frac{u - 1}{\alpha} \right) v w dx - \lambda \int_\Omega (u - 1)^{q-2} v w dx.$$

Following is an important result in Morse theory which explains the effect of the associated Homology groups on the set $K_{J,(\infty, a]}$. 

**Theorem 3.1.** Let $J \in C^2(V)$ satisfy the Palais-Smale condition and let ‘a’ be a regular value of $J$. Then if, $H_*(V, J^a) \neq 0$, implies that $K_{J,(\infty, a]} \neq \emptyset$.

**Remark 3.2.** Before we apply the Morse lemma we recall that for a Morse function the following holds

1. $H_*(J^c, f^c \setminus \text{Crit}(J, c)) = \bigoplus_j H_*(J^c \cap N_j, J^c \cap N_j \setminus \{x_j\})$, where $\text{Crit}(J, c) = \{x \in V : J(x) = c, J'(x) = 0\}$, $N_j$ is a neighbourhood of $x_j$.

2. $H_k(J^c \cap N, J^c \cap N \setminus \{x\}) = \begin{cases} \mathbb{R}, & k = m(x) \\ 0, & \text{otherwise} \end{cases}$

where $m(x)$ is a Morse index of $x$, a critical point of $J$.

3. Further

$$H_k(J^a, J^b) = \bigoplus_{i : m(x_i) = k} \mathbb{R} = \mathbb{R}^{m_k(a,b)}$$

where $m_k(a, b) = n(\{i : m(x_i) = k, x_i \in K_{J,(a,b)}\})$. Here $n(S)$ denotes the number of elements present in the set $S$. 
4. Morse relation

\[ \sum_{u \in K_{J,(a,b)}} \sum_{k \geq 0} \dim(C_k(J,u)) t^k = \sum_{k \geq 0} \dim(H_k(J^a,J^b)) t^k + (1 + t) \Omega_t \]

for all \( t \in \mathbb{R} \). Here \( Q_t \) is a nonnegative polynomial in \( \mathbb{N}_0[t] \).

**Theorem 3.3.** The functional \( I_\alpha \) has at least one nontrivial critical point when \( 0 < \lambda \leq \lambda_1 \), \( \lambda_1 \) being the first eigen value of \( (-\Delta)_p \).

**Proof.** We observe that \( I_\alpha(tu) \to -\infty \) as \( t \to \infty \). A key observation here is that there exists \( R \) sufficiently small such that \( I_\alpha(u) \geq \alpha > 0 \) whenever \( \|u\| = R \). We choose \( \epsilon > 0 \) such that \( c = \epsilon \) is a regular value of \( I_\alpha \). Thus, \( I^c_\alpha \) is not path connected since it has at least two path connected components namely in the form of a neighbourhood of 0 and a set \( \{ u : \|u\| \geq R \} \) for \( R \) sufficiently large. From the theory of homology groups we get that \( \dim(H_0(I^c_\alpha)) \geq 2 \), ‘dim’ denoting the dimension of the Homology group. From the Definition 2.1 let us consider the following exact sequence

\[ ... \to H_1(W_0^{1,p(x)}(\Omega), I^c_\alpha) \xrightarrow{\partial_1} H_0(I^c_\alpha, \emptyset) \xrightarrow{i_0} H_0(W_0^{1,p(x)}(\Omega), \emptyset) \to ... \]

Obviously \( \dim(H_0(W_0^{1,p(x)}(\Omega), \emptyset)) = 1 \) and \( \dim(H_0(I^c_\alpha)) \geq 2 \). Due to the exactness of the sequence we conclude that \( \dim H_1(W_0^{1,p(x)}(\Omega), I^c_\alpha) \geq 1 \). Thus by the Remark we have \( K_{I_\alpha,(-\infty,\epsilon]} \neq \emptyset \).

Suppose that the only critical point to (1.1) is \( u = 0 \) at which the energy of the functional \( I_\alpha \) is also 0. Thus from the discussion above and the Remark (3.2)-(4) we have from the Morse relation we have the following identity over \( \mathbb{R} \)

\[ 1 = t + P(t) + (1 + t) \Omega_t, \]

\( g \) being a power series in \( t \), \( \Omega_t \geq 0 \). This is a contradiction. Thus there exists at least one \( u \neq 0 \) which is a critical point to \( I_\alpha \) whenever \( \lambda \leq \lambda_1 \).

**Definition 3.4 (Krasnoselskii genus).** Let \( V \) be a Banach space and \( S \subset V \). A set \( S \) is said to be symmetric if \( u \in S \) implies \( -u \in S \). Let \( S \) be a close, symmetric subset of \( V \) such that \( 0 \notin S \). We define a genus \( \gamma(S) \) of \( S \) by the smallest integer \( k \) such that there exists an odd continuous mapping from \( S \) to \( \mathbb{R}^k \setminus \{0\} \). We define \( \gamma(S) = \infty \), if no such \( k \) exists.
To each closed and symmetric subsets \( M \) of \( W^{1,p}_0(\Omega) \) with the Krasnoselskii genus \( \gamma(M) \geq k \), define
\[
\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} I_\alpha(u).
\]
Here \( \mathcal{F}_k = \{ M \subset W^{1,p}_0(\Omega), \text{closed and symmetric : } \gamma(M) \geq k \} \). A natural question at this point will be to ask if the same conclusion can be drawn when \( \lambda_k < \lambda \leq \lambda_{k+1} \). We will define \( \lambda_0 = 0 \). The next theorem answers this question.

**Theorem 3.5.** The problem in (1.1) has at least one nontrivial solution when \( \lambda_k < \lambda \leq \lambda_{k+1} \), \( \lambda_k \) being as defined above.

**Proof.** We at first show that \( H_k(W^{1,p}_0(\Omega), I^{-a}_\alpha) = 0 \) for all \( k \geq 0 \). Pick a \( u \in \{ v : \|v\| = 1 \} = \partial B^\infty \), where \( B^\infty = \{ v : \|v\| \leq 1 \} \). Then \( I_\alpha(tu) = \int_\Omega \frac{|\nabla(tu)^p|}{p} dx + \int_\Omega G \left( \frac{tu-1}{\alpha} \right) dx - \lambda \int_\Omega \frac{(tu-1)^q}{q} dx < -a < 0 \) for all \( t \geq t_0 \). It can be easily seen that for a fixed \( u \), we have \( I'(tu) > 0 \). Further, for any \( t \geq t_0 \) we have \( I_\alpha(tu) < -a < 0 \). Thus, there exists \( t(u) \) such that \( I'_\alpha(tu) = 0 \) by the continuity of \( I'_\alpha \). We can thus say that there exists a \( C^1 \)-function \( T : W^{1,p}_0(\Omega) \setminus \{0\} \to \mathbb{R}^+ \). We now define a standard deformation retract \( \eta \) of \( W^{1,p}_0(\Omega) \setminus B^R(0) \) into \( I^{-a}_\alpha \) as follows (refer Definition 2.2).

\[
\eta(s, u) = \begin{cases} 
(1 - s)u + sT \left( \frac{u}{\|u\|} \right) \frac{u}{\|u\|}, & \|u\| \geq R, I_\alpha(u) \geq -a \\
u, & I_\alpha(u) \leq -a.
\end{cases}
\]

It is not difficult to see that \( \eta \) is a \( C^1 \) function over \([0, 1] \times W^{1,p}_0(\Omega) \setminus B^R(0) \). On using the map \( \delta(s, u) = \frac{u}{\|u\|} \), for \( u \in W^{1,p}_0(\Omega) \setminus B^R(0) \) we claim that \( H_k(W^{1,p}_0(\Omega), W^{1,p}_0(\Omega) \setminus B^R(0)) = H_k(B^\infty, S^\infty) \) for all \( k \geq 0 \). This is because, \( H_k(B^\infty, S^\infty) \cong H_k(\ast, 0) \). From elementary computation of homology groups with two 0-dimensional simplices it is easy to see that \( H_k(\ast, 0) = \{0\} \) for each \( k \geq 0 \). A result in [10] says that
\[
C_m(I, u) = \begin{cases} 
\mathbb{R}, & \text{if } m(u) = m \\
0, & \text{otherwise}
\end{cases}
\]

Therefore, from the Morse relation in the Remark 3.2-4 and the result above, we have for \( b > 0 \)
\[
\sum_{u \in \mathcal{F}_{1,\ast,-a}} \sum_{k \geq 0} \dim(C_k(I, u)) t^k = t^{m(u)} + \mathcal{P}(t)
\]  \hspace{1cm} (3.1)

where \( m(u) \) is the Morse index of \( u \) and \( \mathcal{P}(t) \) contains the rest of the powers of \( t \) corresponding to the other critical points, if any. The Morse index is finite because of the following reason. From the argument which helped in establishing a ‘maxima’, say \( u_0 \), using the mountain pass geometry around 0, we had to assume \( \lambda < C^{-q} \|u\|^{p-q} \).
Owing to \( u_0 \) being a maxima, we have \( I''_\alpha(u_0) < 0 \) which necessarily requires \( \lambda > C^{-q/p-1} \|u\|^{p-q} \). Thus we have

\[
C^{-q/p-1} \|u\|^{p-q} < \lambda < C^{-q/p} \|u\|^{p-q}.
\]

This implies that \( \lambda_i < \lambda < \lambda_j \) for some \( i,j \in \mathbb{N}_0 \). On further using the Morse relation we obtain

\[
t^m(u) + \mathcal{P}(t) = (1 + t)\mathcal{Q}_t.
\]

This is because the \( H_{ks} \)s are all trivial groups. Hence, \( \mathcal{Q}_t \) either contains \( t^m(u) \) or \( t^m(u)^{-1} \) or both. Thus there exists at least one nontrivial \( u \in \mathcal{K}_{I\alpha,(-\infty,\infty)} \) with \( m(u) \leq n+1 \).

**Remark 3.6.** If \( 0 < \lambda \leq \lambda_{k+1} \), then there exists at least \( k \) solutions to the equation (1.1).

### 4. Existence of solution to the main problem (1.1) and smoothness of the boundary \( \partial\{u > 1\} \)

**Lemma 4.1.** Let \( \alpha_j \to 0 \ (\alpha_j > 0) \) as \( j \to \infty \) and \( u_j \) be a critical point of \( I_{\alpha_j} \). If \( (u_j) \) is bounded in \( W^1_p(\Omega) \cap L^\infty(\Omega) \), then there exists \( u \), a Lipschitz continuous function, on \( \overline{\Omega} \) such that \( u \in W^1_p(\Omega) \cap C^2(\overline{\Omega} \setminus H(u)) \) and a subsequence (still denoted by \( (u_j) \)) such that

1. \( u_j \to u \) uniformly over \( \overline{\Omega} \),
2. \( u_j \to u \) locally in \( C^1(\overline{\Omega} \setminus \{u = 1\}) \),
3. \( u_j \to u \) strongly in \( W^1_p(\Omega) \),
4. \( I(u) \leq \liminf I_{\alpha_j}(u_j) \leq \limsup I_{\alpha_j}(u_j) \leq I(u) + |\{u = 1\}| \), i.e. \( u \) is a nontrivial function if \( \liminf I_{\alpha_j}(u_j) < 0 \) or \( \limsup I_{\alpha_j}(u_j) > 0 \).

Furthermore, \( u \) satisfies

\[
-\Delta_p u = \lambda \chi_{\{u > 1\}}(x)(u - 1)^{q-1}_+
\]

classically in \( \Omega \setminus H(u) \), the free boundary condition is satisfies in the generalized sense and vanishes continuously on \( \partial\Omega \). In the case of \( u \) being nontrivial, then \( u > 0 \) in \( \Omega \), the set \( \{u < 1\} \) is connected and the set \( \{u > 1\} \) is nonempty.

An important result that will be used to pass the limit in the proof of the Lemma 4.1 is the following theorem which is in line to the theorem due to CAFFARELLI ET AL. in [6, Theorem 5.1].
Lemma 4.2. Let $u$ be a Lipschitz continuous function on the unit ball $B_1(0) \subset \mathbb{R}^N$ satisfying the distributional inequalities

$$
\pm \Delta_p u \leq A \left( \frac{1}{\alpha} \chi_{\{|u-1|<\alpha\}}(x) + 1 \right)
$$

for constants $A > 0$ and $0 < \alpha \leq 1$. Then there exists a constant $C > 0$ depending on $N, A$ and $\int_{B_1(0)} u^p dx$, but not on $\alpha$, such that

$$
\text{esssup}_{x \in B_1(0)} \{ |\nabla u(x)| \} \leq C.
$$

Proof. Given that $u$ is a Lipschitz continuous function on the unit ball $B_1(0) \subset \mathbb{R}^N$, so $u$ is also bounded in the unit ball say by a constant $M_0$. Not just that, $u$ is also differentiable a.e. in $B_1(0)$. We will prove the result stated in the lemma for $u_+$, as the proof for $u_-$ will follow suit. Denote $v(x) = \frac{15}{\alpha} u_+(ax/15)$ and

$$
v_1 = v + \max_{B_{1/2}(0)} \{ v^- \}.
$$

Therefore, $0 \leq v_1 \leq M_1$. Let us choose a test function $\eta \in C_0^\infty(B_{1/4})$ which is such that $0 \leq \eta \leq 1$ in $B_{3/4}$ and $\eta = 1$ in $B_{1/2}$. Thus

$$
\int_\Omega \eta^p |\nabla v_1|^p = - \int_\Omega (pv_1 \eta^{p-1} |\nabla v_1|^{p-2} (\nabla v_1 \cdot \nabla \eta) + \eta^p v_1 \Delta_p v_1) dx \\leq \int_\Omega \eta^p |\nabla v_1|^p + \int_\Omega \eta^p |\nabla \eta|^p dx + \frac{1}{p} \int_\Omega \eta^p |\nabla v_1|^p dx + pM_1 \int_\Omega |\nabla \eta|^p dx
$$

(4.1)

$$
\leq \int_\Omega \eta^p |\nabla v_1|^p + pM_1 \int_\Omega |\nabla \eta|^p dx +\frac{1}{p} \int_\Omega \eta^p \left( \chi_{\{|u-1|<\alpha\}}(x) + 1 \right) dx
$$

$$
\leq 1 + AM_1 \int_\Omega \eta^p \left( \chi_{\{|u-1|<\alpha\}}(x) + 1 \right) dx.
$$

(4.2)

It is now established that

$$
\int_{B_{1/2}(0)} |\nabla v_1|^p dx \leq M_2.
$$

However, $u$ being Lipschitz continuous, the gradient $\nabla u$ is bounded a.e. in $B_1(0)$ and hence in $B_{1/2}(0)$. Thus $\text{esssup}_{B_{1/2}(0)} \{ |\nabla u| \} \leq C$, for some $C > 0$. $\square$

Proof of Lemma 4.1. Let $0 < \alpha_j < 1$. Consider the problem sequence $(P_j)$

$$
-\Delta_p u_j = -\frac{1}{\alpha_j} g \left( \frac{(u_j - 1)_+}{\alpha_j} \right) + \lambda(u - 1)_{+}^{q-1} \text{ in } \Omega
$$

$$
u_j > 0 \text{ in } \Omega
$$

$$
u_j = 0 \text{ on } \partial \Omega.
$$

(4.3)
The nature of the problem being a sublinear one allows us to conclude by an iterative technique that the sequence \((u_j)\) is bounded in \(L^\infty(\Omega)\). Therefore, there exists \(C_0\) such that \(0 \leq g \left( \frac{(u_j - 1)_+}{\alpha_j} \right) (u - 1)^{q-1}_+ \leq C_0\). Let \(\varphi_0\) be a solution of
\[
-\Delta_p \varphi_0 = \lambda C_0 \text{ in } \Omega \\
\varphi_0 = 0 \text{ on } \partial \Omega. 
\]
(4.4)

Now since \(g \geq 0\), we have that \(-\Delta_p u_j \leq \lambda C_0 = -\Delta \varphi_0 \text{ in } \Omega\). Therefore by the maximum principle,
\[
0 \leq u_j(x) \leq \varphi_0(x) \forall x \in \Omega. 
\]
(4.5)

Since \(\{u_j \geq 1\} \subset \{\varphi_0 \geq 1\}\), hence \(\varphi_0\) gives a uniform lower bound, say \(d_0\), on the distance from the set \(\{u_j \geq 1\}\) to \(\partial \Omega\). Thus \((u_j)\) is bounded with respect to the \(C^2\) norm. Therefore, it has a convergent subsequence in the \(C^2\)-norm in a \(\frac{d_0}{2}\) neighbourhood of the boundary \(\partial \Omega\). Obviously \(0 \leq g \leq 2\chi_{(-1,1)}\) and hence
\[
\pm \Delta u_j = \pm \frac{1}{\alpha_j} g \left( \frac{(u_j - 1)_+}{\alpha_j} \right) \mp \lambda (u_j - 1)^{q-1}_+ \\
\leq \frac{2}{\alpha_j} \chi_{\{|u_j-1|<\alpha_j\}}(x) + \lambda C_0. 
\]
(4.6)

Since \((u_j)\) is bounded in \(L^2(\Omega)\) and by Lemma \(4.2\) it follows that there exists \(A > 0\) such that
\[
\text{esssup}_{x \in B_r(x_0)} \{|\nabla u_j(x)|\} \leq \frac{A}{r} \quad (4.7)
\]
for a suitable \(r > 0\) such that \(B_r(0) \subset \Omega\). However, since \((u_j)\) is a sequence of Lipschitz continuous functions that are also \(C^1\), therefore
\[
\sup_{x \in B_r(x_0)} \{|\nabla u_j(x)|\} \leq \frac{A}{r}. 
\]
(4.8)

Thus \((u_j)\) is uniformly Lipschitz continuous on the compact subsets of \(\Omega\) such that its distance from the boundary \(\partial \Omega\) is at least \(\frac{d_0}{2}\) units.

Thus by the Ascoli-Arzel\(\grave{a}\) theorem applied to \((u_j)\) we have a subsequence, still named the same, such that it converges uniformly to a Lipschitz continuous function \(u\) in \(\Omega\) with zero boundary values and with strong convergence in \(C^2\) on a \(\frac{d_0}{2}\)-neighbourhood of \(\partial \Omega\). By the Eberlein-Šmulian theorem we conclude that \(u_j \rightharpoonup u \text{ in } W^{1}_0, p(\Omega)\).

We now prove that \(u\) satisfies
\[
-\Delta_p u = \alpha \chi_{\{u>1\}}(x)(u - 1)^{q-1}_+ 
\]
(4.9)
in the set \( \{ u \neq 1 \} \). Let \( \varphi \in C_0^\infty(\{ u > 1 \}) \) and therefore \( u \geq 1 + 2\delta \) on the support of \( \varphi \) for some \( \delta > 0 \). On using the convergence of \( u_j \) to \( u \) uniformly on \( \Omega \) we have \( |u_j - u| < \delta \) for any sufficiently large \( j, \delta_j < \delta \). So \( u_j \geq 1 + \delta_j \) on the support of \( \varphi \). On testing (4.9) with \( \varphi \) yields

\[
\int_{\Omega} |\nabla u_j|^{p-2}\nabla u_j \cdot \nabla \varphi dx = \lambda \int_{\Omega} (u_j - 1)_+^{q-1} \varphi dx. \tag{4.10}
\]

On passing the limit \( j \to \infty \) to (4.9), we get

\[
\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx = \lambda \int_{\Omega} (u - 1)_+^{q-1} \varphi dx. \tag{4.11}
\]

To arrive at (4.11) we have used the weak convergence of \( u_j \) to \( u \) in \( W_0^{1,p}(\Omega) \) and the uniform convergence of the same in \( \Omega \). Hence \( u \) is a weak solution of \( -\Delta_p u = \lambda(u-1)_+^{q-1} \) in \( \{ u > 1 \} \). Since \( u \) is a Lipschitz continuous function, hence by the Schauder estimates we conclude that it is also a classical solution of \( -\Delta_p u = \lambda(u-1)_+^{q-1} \) in \( \{ u > 1 \} \). Similarly on choosing \( \varphi \in C_0^\infty(\{ u < 1 \}) \) one can find a \( \delta > 0 \) such that \( u \leq 1 - 2\delta \). Therefore, \( u_j < 1 - \delta \).

Let us now analyze the nature of \( u \) in the set \( \{ u \leq 1 \}^c \). On testing (4.9) with any nonnegative function and passing the limit \( j \to \infty \) and using the fact that \( g \geq 0 \), \( G \leq 1 \) we can show that \( u \) satisfies

\[
-\Delta_p u \leq \lambda(u-1)_+^{q-1} \text{ in } \Omega \tag{4.12}
\]

in the distributional sense. Furthermore, \( \mu = \Delta_p u \) is a positive Radon measure supported on \( \Omega \cap \partial \{ u < 1 \} \) (refer Lemma 4.3 in Appendix). From (4.12), the positivity of the Radon measure \( \mu \) and the usage of Section 9.4 in GILBARG-TRUDINGER [8] we conclude that \( u \in W_0^{2,p}(\{ u \leq 1 \}^c) \), \( 1 < p < \infty \). Thus \( \mu \) is supported on \( \Omega \cap \partial \{ u < 1 \} \cap \partial \{ u > 1 \} \) and \( u \) satisfies \( -\Delta_p u = 0 \) in the set \( \{ u \leq 1 \}^c \).

In order to prove (ii), we will show that \( u_j \to u \) locally in \( C^1(\Omega \setminus \{ u = 1 \}) \). Note that we have already proved that \( u_j \to u \) in the \( C^2 \) norm in a neighbourhood of \( \partial \Omega \). Suppose \( M \subset \subset \{ u > 1 \} \). In this set \( M \) we have \( u \geq 1 + 2\delta \) for some \( \delta > 0 \). Thus for sufficiently large \( j \), with \( \delta_j < \delta \), we have \( |u_j - u| < \delta \) in \( \Omega \) and hence \( u_j \geq 1 + \delta_j \) in \( M \).

From (4.9) we have

\[
-\Delta_p u_j = \lambda(u_j - 1)_+^{q-1} \text{ in } M.
\]

Clearly, \( (u_j - 1)_+^{q-1} \to (u - 1)_+^{q-1} \) in \( L^p(\Omega) \) for \( 1 < p < \infty \) and \( u_j \to u \) uniformly in \( \Omega \). This analysis says something more stronger - since \( (-\Delta_p)u_j = \lambda(u_j - 1)_+^{q-1} \) in \( M \), we have that \( u_j \to u \) in \( W^{2,p}(M) \). By the embedding \( W^{2,p}(M) \hookrightarrow C^1(M) \) for \( p > 2 \), we have \( u_j \to u \) in \( C^1(M) \). This shows that \( u_j \to u \) in \( C^1(\{ u > 1 \}) \). Working on similar lines we can also show that \( u_j \to u \) in \( C^1(\{ u < 1 \}) \).
We will now prove $(iii)$. Since $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, we have that by the weak lower semicontinuity of the norm $\| \cdot \|$ that
\[
\|u\| \leq \lim \inf \|u_j\|.
\]

It is sufficient to prove that $\lim \sup \|u_j\| \leq \|u\|$. To achieve this, we multiply \eqref{4.3} with $(u_j - 1)$ and then integrate by parts. We will also use the fact that $tg \left( \frac{t}{\delta_j} \right) \geq 0$ for any $t \in \mathbb{R}$. This gives,
\[
\int_\Omega |\nabla u_j|^p \, dx \leq \lambda \int_\Omega f(u_j - 1)^q \, dx - \int_{\partial \Omega} \frac{\partial u_j}{\partial n} \, dS \rightarrow \lambda \int_\Omega (u - 1)^q \, dx - \int_{\partial \Omega} \frac{\partial u}{\partial \hat{n}} \, dS \quad \text{(4.13)}
\]
as $j \rightarrow \infty$. Here $\hat{n}$ is the outward drawn normal to $\partial \Omega$.

We choose $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$ such that $u \neq 1$ a.e. on the support of $\varphi$. On multiplying $\nabla u_n \cdot \varphi$ to the weak formulation of \eqref{4.3} and integrating over the set $\{1 - \epsilon^- < u_n < 1 + \epsilon^+\}$ gives
\[
\int_{\{1 - \epsilon^- < u_n < 1 + \epsilon^+\}} \left[ -\Delta_p u_n + \frac{1}{\alpha_n} g \left( \frac{u_n - 1}{\alpha_n} \right) \right] \nabla u_n \cdot \varphi \, dx = \int_{\{1 - \epsilon^- < u_n < 1 + \epsilon^+\}} (u_n - 1)^{q-1} \nabla u_n \cdot \varphi \, dx. \tag{4.14}
\]
The term on the left hand side of \eqref{4.14} can be expressed as follows.
\[
\nabla \cdot \left( \frac{1}{p} |\nabla u_n|^p \varphi - (\nabla u_n \cdot \varphi) |\nabla u_n|^{p-2} \nabla u_n \right) + (\nabla \varphi \cdot \nabla u_n) \cdot \nabla u_n |\nabla u_n|^{p-2} - \frac{1}{p} |\nabla u_n|^p \nabla \cdot \varphi + \nabla G \left( \frac{u_n - 1}{\alpha_n} \right) \cdot \varphi. \tag{4.15}
\]

Using \eqref{4.15} and on integrating by parts we obtain
\[
\int_{\{u_n = 1 + \epsilon^+\} \cup \{u_n = 1 - \epsilon^-\}} \left[ \frac{1}{p} |\nabla u_n|^p \varphi - (\nabla u_n \cdot \varphi) |\nabla u_n|^{p-2} \nabla u_n + G \left( \frac{u_n - 1}{\alpha_j} \right) \varphi \right] \cdot \hat{n} \, dS
\]
\[
= \int_{\{1 - \epsilon^- < u_n < 1 + \epsilon^+\}} \left( \frac{1}{p} |\nabla u_n|^p \nabla \cdot \varphi - (\nabla \varphi \cdot \nabla u_n) |\nabla u_n|^{p-2} \nabla u_n \right) \, dx
\]
\[
+ \int_{\{1 - \epsilon^- < u_n < 1 + \epsilon^+\}} \left[ G \left( \frac{u_n - 1}{\alpha_n} \right) \nabla \cdot \varphi + \lambda (u_n - 1)^{q-1} (\nabla u_n \cdot \varphi) \right] \, dx. \tag{4.16}
\]
The integral on the left of equation (4.16) converges to
\[ \int_{\{u_n=1+\epsilon\} \cup \{u_n=1-\epsilon\}} \left( \frac{1}{p} |\nabla u|^p \bar{\varphi} - (\nabla u_n \cdot \bar{\varphi}) |\nabla u_n|^{p-2} \nabla u_n \right) \cdot \hat{n}dS + \int_{\{u_n=1+\epsilon\}} \bar{\varphi} \cdot \hat{n}dS \]
(4.17)

\[ = \int_{\{u_n=1+\epsilon\}} \left[ 1 - \left( \frac{p-1}{p} \right) |\nabla u_n|^p \right] \bar{\varphi} \cdot \hat{n}dS - \int_{\{u_n=1-\epsilon\}} \left( \frac{p-1}{p} \right) |\nabla u_n|^p \bar{\varphi} \cdot \hat{n}dS. \]
(4.18)

Thus the equation (4.17) under the limit \( \epsilon \to 0 \) becomes
\[ 0 = \lim_{\epsilon \to 0} \int_{\{u=1+\epsilon\}} \left[ \left( \frac{p}{p-1} \right) - |\nabla u|^p \right] \bar{\varphi} \cdot \hat{n}dS - \lim_{\epsilon \to 0} \int_{\{u=1-\epsilon\}} |\nabla u|^p \bar{\varphi} \cdot \hat{n}dS \]
(4.19)

This is because \( \hat{n} = \pm \frac{\nabla u}{|\nabla u|} \) on the set \( \{u = 1 + \epsilon^+\} \cup \{u = 1 - \epsilon^-\} \). This proves that \( u \) satisfies the free boundary condition. The solution cannot be trivial as it satisfies the free boundary condition. Thus a solution to (1.1) exists that obeys the free boundary condition besides the Dirichlet boundary condition.

**Appendix**

**Lemma 4.3.** \( u \) is in \( W^{1,p}_{\text{loc}}(\Omega) \) and the Radon measure \( \mu = \Delta_p u \) is nonnegative and supported on \( \Omega \cap \{ u < 1 \} \).

**Proof.** We follow the proof due to Alt-Caffarelli [1]. Choose \( \delta > 0 \) and a test function \( \varphi^p \chi_{\{u < 1-\delta\}} \) where \( \varphi \in C_0^\infty(\Omega) \). Therefore,

\[ 0 = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (\varphi^p \min\{u - 1 + \delta, 0\}) dx \]

\[ = \int_{\Omega \cap \{u < 1-\delta\}} |\nabla u|^{p-2} \nabla u \cdot \nabla (\varphi^p \min\{u - 1 + \delta, 0\}) dx \]
(4.20)

\[ = \int_{\Omega \cap \{u < 1-\delta\}} |\nabla u|^p \varphi^p dx + p \int_{\Omega \cap \{u < 1-\delta\}} \varphi^{p-1}(u - 1 + \delta) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx, \]

and so by Caccioppoli like estimate we have

\[ \int_{\Omega \cap \{u < 1-\delta\}} |\nabla u|^p \varphi^p dx = -p \int_{\Omega \cap \{u < 1-\delta\}} \varphi^{p-1}(u - 1 + \delta) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \]
\[ \leq c \int_\Omega u^p |\nabla \varphi|^p dx. \]
(4.21)
Since $\int_\Omega |u|^p \, dx < \infty$, therefore on passing the limit $\delta \to 0$ we conclude that $u \in W^{1,p}_{\text{loc}}(\Omega)$. Furthermore, for a nonnegative $\zeta \in C_0^\infty(\Omega)$ we have

$$-\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta \, dx = \left( \int_{\Omega \cap \{0 < u < 1 - 2\delta\}} + \int_{\Omega \cap \{1 - 2\delta < u < 1 - \epsilon\}} + \int_{\Omega \cap \{1 - \delta < u < 1\}} \right)$$

$$+ \int_{\Omega \cap \{u > 1\}} \left[ |\nabla u|^{p-2} \nabla u \cdot \nabla \left( \zeta \max \left\{ \min \left\{ 2 - \frac{1 - u}{\delta}, 1 \right\}, 0 \right\} \right) \right] \, dx$$

$$\geq \int_{\Omega \cap \{1 - 2\delta < u < 1 - \delta\}} \left[ |\nabla u|^{p-2} \nabla u \cdot \left( 2 - \frac{1 - u}{\delta} \right) \nabla \zeta + \frac{\zeta}{\delta} |\nabla u|^p \right] \, dx \geq 0.$$

(4.22)

On passing the limit $\delta \to 0$ we obtain $\Delta_p (u - 1)_- \geq 0$ in the distributional sense and hence there exists a Radon measure $\mu$ (say) such that $\mu = \Delta (u - 1)_- \geq 0$. \hfill \Box

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