SUPERRADIATION IN SCATTERING OF A DIRAC PARTICLE OFF A POINT-LIKE NUCLEUS WITH $Z > 137$

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The main concepts of the recently developed approach to singular problems of quantum mechanics are extended to the Dirac particle in the Coulomb field of a point-like nucleus with its charge $Z > 137$. The reflection and transmission coefficients, which describe, respectively, the reflection of electron by the singularity and its falling onto it, are analytically calculated, using the exact solutions of the Dirac equation. The superradiation phenomenon is found. It suppresses the widely discussed effect of spontaneous electron-positron pair creation.
1 Introduction

In our previous publications [1, 2, 3] we developed a new approach - called black-hole approach - to the Schrödinger equation with attractive inverse-radius-squared singularity in its potential, possessing the fall-down onto the center behavior. Within this approach the solutions, oscillating near the point of singularity, are treated as free particles, emitted or absorbed by the singularity exactly in the same sense, as they are emitted or absorbed by the infinitely remote point - we mean, income or outcome from/to infinity.

In the present work we apply this approach to the relativistic problem of an electron placed in a Coulomb field of a point-like nucleus with its charge $Z$ greater than 137. The corresponding set of radial Dirac equations is known to possess in this case the singularity of the same mathematical character as in the Schrödinger problem above. Namely, the Coulomb potential is squared in the Dirac equation and produces singular attraction $(Z\alpha/r)^2$. Basing on the same concept and using the known explicit solution of the Dirac equation, we are able to calculate the coefficients of reflection and transmission for electrons that income from infinity and are then either reflected back or transmitted to the singularity point. In the negative energy continuum $\varepsilon < -m$, where $m$ is electron mass, these coefficients are responsible for the spontaneous electron-positron pair creation by the external field of the nucleus. This effect has been studied in a more conventional context (the basic references are [4], [5]), using an introduction of the nucleus finite-size core, acting as a cutoff. In the present approach the reflection and transmission coefficients are found analytically in a simple form, since they are not affected by a cutoff, whose presence is not meant at all. The result shows the superradiation phenomenon, characteristic of the Kerr black holes, when the reflected wave is more intense than the incident one. The reflection coefficient is greater, than unity, the transmission coefficient being, correspondingly, negative. This fact forbids the spontaneous pair creation owing to the Pauli principle applied within the Dirac sea picture.

The paper is organized as follows. In Section 2 we shortly outline the concepts of the black-hole approach, addressing the Dirac equation in the supercritical Coulomb field, with the reference to the original work [3] for details. In Subsection 2.1 we formulate the Kamke generalized eigenvalue problem with respect to $Z$ in what is called sector III of confined states $-m < \varepsilon < m$, $Z > 137$; find the corresponding integration measure in the orthogonality relations, that allows treating the solutions, oscillating near the
singularity, as free particles; perform the transformation (when $0 < \varepsilon < m$) that reduces the Kamke eigenvalue problem to the standard Liouville form and find the corresponding effective potential, confining the particle near the singularity. In Subsection 2.2 we analogously formulate the effective barrier problem in what is called sector IV, $\varepsilon^2 > m^2$, $Z > 137$, where particles emitted by the singularity or by the infinitely remote point are reflected back or transmitted to, resp., the infinitely remote point or the singularity. In Section 3 the reflection/transmission coefficients are calculated. The recipe for this calculation is determined by the concept, but is independent of any of the constructions in the previous Section 2.

2 Outline of the black-hole approach

The radial Dirac equation in the Coulomb field is two-component \[6, 7\]

\[
\frac{dG(r)}{dr} + \frac{\kappa}{r}G(r) - (\varepsilon + m + \frac{Z\alpha}{r})F(r) = 0,
\]

\[
\frac{dF(r)}{dr} + (\varepsilon - m + \frac{Z\alpha}{r})G(r) - \frac{\kappa}{r}F(r) = 0.
\]

(1)

Here the spinor components of the radial wave function $G(r)$ and $F(r)$ correspond to $gr$ and $fr$ in notations of Ref.\[6\], resp., $\varepsilon$ and $m$ are the electron energy and mass, $\alpha = 1/137$ is the fine structure constant, $Z$ is the charge of the nucleus, and $\kappa$ is the electron orbital momentum

\[
\kappa = -(l + 1) \quad \text{for} \quad j = l + \frac{1}{2},
\]

\[
\kappa = l \quad \text{for} \quad j = l - \frac{1}{2}.
\]

(2)

In what follows we confine ourselves to the lowest orbital state $\kappa = -1$. Solutions to equation (1) are known in terms of confluent hypergeometric functions. To illustrate the connection of the problem (1) with the Schrödinger problem, which possesses the fall-down onto the center property, let us for example consider the combination

\[
W(r) = \sqrt{r} \left( \frac{G(r)}{\sqrt{1 + \frac{\varepsilon^2}{m}}} - \frac{F(r)}{\sqrt{1 - \frac{\varepsilon^2}{m}}} \right).\]

(3)

It satisfies the second-order Schrödinger-like differential equation

\[
- \frac{d^2W}{dr^2} - \left( \frac{\sqrt{m^2 - \varepsilon^2}}{r} + \frac{1}{4r^2} + \varepsilon^2 - m^2 \right)W = \left( \frac{Z^2\alpha^2}{r^2} + \frac{2Z\alpha\varepsilon}{r} \right)W.
\]

(4)
We collected all terms containing the coupling $Z\alpha$ in the right-hand side. This fixes the operator, whose eigenfunctions will be studied as solutions to the generalized Kamke eigenvalue problem \cite{8} with respect to $Z\alpha$. The term $Z\alpha/r$ in \cite{4} corresponds to Coulomb attraction of an electron by a positively charged nucleus, when $Z\alpha$ and $\varepsilon$ are positive, while the term $Z^2\alpha^2/r^2$ produces singularity, which is attractive irrespective of the sign of $Z\alpha$. This property of the relativistic problem is well-known. The fall-down onto the center \cite{9} occurs, provided that

\begin{equation}
Z^2\alpha^2 - 1 > 0. \tag{5}
\end{equation}

Two fundamental solutions to equation \cite{4} behave near the origin $r \to 0$ like

\begin{equation}
r^{\pm i\gamma + \frac{1}{2}}, \tag{6}
\end{equation}

where $\gamma = \sqrt{Z^2\alpha^2 - 1}$.

Two fundamental solutions to equation \cite{4} behave near the infinity $r \to \infty$ like

\begin{equation}
e^{\pm r\sqrt{m^2 - \varepsilon^2}} \frac{\pm iZ\alpha}{r \sqrt{\varepsilon^2 - m^2} + \frac{1}{2}}. \tag{7}
\end{equation}

The usual Coulomb bound states of an electron lie within the strip

\begin{equation}
-m < \varepsilon < m, \quad \alpha Z < 1, \tag{8}
\end{equation}

corresponding to what was called sector I in \cite{11, 12, 13}. In this sector we choose the solution, which decreases exponentially at infinity (the upper sign in \cite{7}), it is a linear combination of two solutions, out of which one decreases near $r = 0$ ($\gamma$ is imaginary in sector I), whereas the other increases in accordance with \cite{6} and should be ruled out by the requirement of square-integrability. The nullification of the corresponding coefficient is possible for discrete values of $\varepsilon$, considered as functions of $\alpha Z$: $\varepsilon_{n_r}(\alpha Z), \ n_r = 0, 1, 2...$

On the contrary, in the strip

\begin{equation}
-m < \varepsilon < m, \quad \alpha Z > 1, \tag{9}
\end{equation}

corresponding to what was called sector III in \cite{11, 12, 13}, the both behaviors \cite{6} oscillate, and, in accord with the black-hole approach of Refs. \cite{11, 12, 13} we treat them as particles, free near the origin. These particles are emitted or absorbed by the singular center, but do not escape to infinity due to \cite{11}, with
the upper sign again. The corresponding states make a continuum, because the both oscillating solutions are acceptable. We call them confined states in order to distinguish from the bound states that also do not extend to infinity, but belong to discrete spectrum. The treatment of the solutions, oscillating near \( r = 0 \), as responsible for free particles, is possible, since the measure in the corresponding scalar product becomes infinite in this point. Within the present context this will be seen later.

In sector II

\[
\varepsilon < -m \quad \text{or} \quad \varepsilon > m, \quad \alpha Z < 1
\]  

(10)
electron and positron are elastically scattered from the Coulomb potential.

Finally, in sector IV

\[
\varepsilon < -m \quad \text{or} \quad \varepsilon > m, \quad \alpha Z > 1
\]  

(11)
the wave function oscillates both near the origin and at infinity, the particles being free in the both regions. This corresponds to any of the two inelastic scattering processes: An electron (or a positron) incoming from infinity, is partially scattered back with the probability, determined by the reflection coefficient, and partially penetrates to the origin, becoming a free particle there. Symmetrically, the particle, emitted by the center, escapes to infinity with a certain probability, determined by the transmission coefficient, and partially is reflected back to the center. We say that deconfinement processes go in sector IV.

2.1 Sector III of confined states \(-m < \varepsilon < m, \quad Z\alpha > 1\)

Following [1, 2, 3], equation (11) or (12) should be in sector III considered as a generalized eigenvalue problem of Kamke [8] for the parameter \( Z \), while its solutions should be classified as eigen-functions of the operator in the left-hand side of (11). Following the standard procedure, we may multiply eq. (11) by \( W_1^* \), the solution to the equation, complex conjugate to (11), written for a different eigen-value \( Z_1 \). After subtracting the complex conjugate of the same product with \( Z_1 \) and \( Z \) interchanged, and integrating by parts, we obtain

\[
- \left( W_1^* \frac{dW}{dr} - W \frac{dW^*_1}{dr} \right) \bigg|^{\infty}_{r_L} = \alpha(Z - Z_1) \int_{r_L}^{\infty} W_1^* W \, d\mu(r),
\]  

(12)
where

\[
d\mu(r) = \left( \frac{2\varepsilon}{r} + \frac{\alpha(Z + Z_1)}{r^2} \right) \, dr
\]  

(13)
is the integration measure with which two eigenfunctions with different eigenvalues \( Z_1 \neq Z \) and the same \( \varepsilon \) are orthogonal, provided that the boundary term on the left-hand side in (12) disappears. This is guaranteed if the system is placed in a box

\[
  r_L < r < \infty, \quad r_L \to 0,
\]

and the appropriate boundary condition

\[
  W(r_L) = 0
\]

is imposed at the wall of the box.

Let us make the transformation of the variable (\( r_0 \) is a free dimensional parameter)

\[
  \xi(r) = \int_{r_0}^r \left( \frac{1}{r_2 - \frac{2Z\alpha\varepsilon}{\gamma^2r}} \right) \frac{1}{2} dr
\]

and of the wave function

\[
  \tilde{W}(\xi) = \Phi(r)W(r), \quad \text{where} \quad \Phi(r) = \left( \frac{\gamma^2}{r^2} + \frac{2Z\alpha\varepsilon}{r} \right)^{\frac{1}{2}}.
\]

This transformation is nonsingular when the energy \( \varepsilon > 0 \) is positive, and reduces equation (4) to the standard Liouville form

\[
  -\frac{d^2\tilde{W}(\xi)}{d\xi^2} + U(\xi)\tilde{W}(\xi) = (\alpha^2Z^2 - 1)\tilde{W}(\xi)
\]

with the effective potential

\[
  U(\xi) = \frac{3 + \frac{2Z\alpha\varepsilon}{\gamma^2r}}{2(1 + \frac{2Z\alpha\varepsilon}{\gamma^2r})^2} - \frac{r\sqrt{\varepsilon^2 - \varepsilon^2} + r^2(\varepsilon^2 - m^2) + \frac{3}{2}}{1 + \frac{2Z\alpha\varepsilon}{\gamma^2r}}.
\]

It is understood that \( r \) is here the function of \( \xi \), inverse to (16). The free parameter \( r_0 \) enters the potential only through this function. Unlike the non-relativistic case of pure inverse square potential \( (V = 0 \text{ in Refs. [1, 2, 3])} \), where there is no dimensional parameter originally present in the theory, so that it is to be additionally introduced to define the problem, in the present case under consideration we have at our disposal the Compton length \( m^{-1} \).

For negative \( \varepsilon \) the potential (19) becomes singular as a result of the singularity of the transformation (16), (17).
In the vicinity of the singular point \( r = 0 \) the transformation (16), (17) behaves as

\[
\xi = \ln \frac{r}{r_0},
\]

\[
\left(\frac{\gamma}{r}\right)^\frac{1}{2} W(r) = \tilde{W}(\xi)
\]

The origin \( r = 0 \) is mapped onto \( \xi = -\infty \), and the infinitely remote point \( r = \infty \) to the point \( \xi = \infty \) \( (\xi \approx (2\sqrt{2}Z\alpha\varepsilon/\gamma)\sqrt{r}) \). The asymptotic behavior (6) in the point \( r = 0 \) becomes the oscillating asymptotic behavior at \( \xi \to -\infty \) (up to an unessential constant factor)

\[
\tilde{W}(\xi) \approx \exp(\pm i\xi\gamma)
\]

with \( \gamma = \sqrt{\alpha^2Z^2 - 1} \) playing the role of a "momentum". The potential (19) grows at \( \xi \to \infty \) as \((m^2 - \varepsilon^2)\gamma^2/2\alpha Z\varepsilon)\xi^2\) (see Fig.1) and prevents the electron from escaping to infinity.

![Figure 1: The confining potential (19) in sector III plotted for \( Z\alpha = 2 \). The lower curve corresponds to the positive energy value \( \varepsilon = 0.5m \), and the upper one to \( \varepsilon = 0.1m \). The value \( r_0 = m^{-1} \) is taken for \( r_0 \).](image)

The box (14) is mapped to the box , restricted by the limits

\[
-\xi_L < \xi < \infty,
\]

(23)
\[ r_L = r_0 \exp \left( -\xi_L \right), \quad \xi_L \to \infty, \quad \text{when} \quad r_L \to 0. \]  

(24)

The boundary condition (15) becomes

\[ \tilde{W}(-\xi_L) = 0 \]  

(25)

in full analogy with the customary way of quantization of particles, free at infinity. Eq. (25) makes the Kamke problem self-adjoint and provides the orthogonality and completeness of the wave functions (see [3] for detail.)

When necessary, the finite quantity \( r_L \) may be considered as a cutoff, alternative to the nucleus core introduced in [4].

As long as the box size \( r_L \) is finite, the spectrum of \( Z\alpha \) is discrete in the domain \( Z\alpha > 1 \) and is presented by an equidistant series of vertical lines, numbered by \( n = 0, 1, 2, \ldots \) from left to right in the plane with \( y = \varepsilon \) for the ordinate, and \( x = Z\alpha > 0 \) for the abscissa, which fill the domain, restricted by the borders of sector III, i.e. \(-m < y < m, \quad 1 < x < \infty\). As the spectral lines cannot intersect, these vertical lines must be adjusted to the Coulomb bound states levels, which are described in the same plane by a family of trajectories, numbered by the radial quantum number \( n_r \leq N \). \( N \) is finite, provided that \( r_L \) is finite. These trajectories go in sector I, as \( Z\alpha \) grows, from the point \( y = m, \quad x = 0 \) to the border \( x = 1 \), separating the two sectors. The lowest and leftmost trajectory \( n_r = 0 \) joins the leftmost vertical line \( n = 0 \) near the point \( y = 0, \quad x = 1 \) and is continued by it down to \( y = -m \). Other trajectories with \( n_r = 1, 2, \ldots N \) join with the vertical lines \( n = 1, 2, \ldots N \) above that point also to be continued down to \( \varepsilon = -m \). The other spectral lines with \( n > N \) remain unaffected and go almost vertically from \( y = m \) to \( y = -m \). When \( r_L \to 0 \), all the spectral lines to the right of the border \( x = 1 \) condense to make the continuum of confined states. In this limit the bound state trajectories approach that continuum at the border of sector III at \( x = Z\alpha = 1 \) and at discrete positions on the vertical axis, corresponding to the discrete energy eigenvalues, numbered by \( n_r \).

When \( \xi_L \to \infty \), the norm of solutions of eq. (18)

\[ \int_{-\xi_L}^\xi |\tilde{W}(\xi)|^2 d\xi = \gamma \int_{r_L}^\infty |W(r)|^2 \frac{dr}{r^2} \]  

(26)

diverges linearly with \( \xi_L \). This makes the \( \delta \)-normalization of the wave functions possible. Note the singularity of the volume element \( dr/r^2 \) in the initial \( r \)-space: the particle has enough volume near the origin to behave itself freely near it.

Such is the situation with sector III.
2.2 Sector IV of inelastic scattering $\alpha^2 Z^2 > 1, \varepsilon^2 > m^2$.

Effective barrier

Excluding $F(r)$ from equation (1), one comes to the equation for the upper component of the Dirac spinor $G(r)$

$$
- \frac{d^2 G}{dr^2} - \frac{Z \alpha}{r(Z \alpha + r(m + \varepsilon))} \frac{dG}{dr} - \left( \frac{2Z\alpha\varepsilon}{r} + \frac{-\kappa(\varepsilon + m)}{r(Z\alpha + r(m + \varepsilon))} \right) G = \left( \frac{Z^2 \alpha^2 - \kappa^2}{r^2} + \varepsilon^2 - m^2 \right) G. \quad (27)
$$

We placed the most singular term in the r.-h. side, as well the "kinetic energy" term $\varepsilon^2 - m^2$.

The corresponding equation for the lower component $F(r)$ is obtained from this by the substitution $G \rightarrow F, Z \rightarrow -Z, \kappa \rightarrow -\kappa, \varepsilon \rightarrow -\varepsilon$.

Set $\kappa = -1$ and perform the transformation of the wave function $G$, prescribed by the general theory of differential equations [8],

$$
\tilde{G}(\xi) = \Phi(r)G(r),
$$

$$
\Phi(r) = \left( \frac{\alpha^2 Z^2 - 1}{r^2} + \varepsilon^2 - m^2 \right)^{\frac{1}{4}} \left( \frac{r(\alpha Z + r_0(m + \varepsilon))}{r_0(\alpha Z + r(m + \varepsilon))} \right)^{\frac{1}{2}}, \quad (28)
$$

accompanied by the transformation of the variable

$$
\xi(r) = (Z^2 \alpha^2 - 1)^{-\frac{1}{2}} \int_{r_0}^{r} \frac{dr}{r} \sqrt{\alpha^2 Z^2 - 1 + r^2(\varepsilon^2 - m^2)}. \quad (29)
$$

When $r \rightarrow 0$ the new variable tends to $-\infty$ as $\ln r$. When $r \rightarrow \infty$ the new variable tends to $\infty$ as $\xi \rightarrow (\sqrt{\varepsilon^2 - m^2}/\gamma)r$. After the transformation (28), (29) equation (27) acquires the standard Liouville form

$$
- \frac{d^2 \tilde{G}(\xi)}{d\xi^2} + U(\xi)\tilde{G}(\xi) = (Z^2 \alpha^2 - 1)\tilde{G}(\xi) \quad (30)
$$

with the effective potential

$$
U(\xi) = -\frac{1}{(1 + y^2)} \left( 2\alpha Z \varepsilon r + \frac{(m + \varepsilon)r}{\alpha Z + r(m + \varepsilon)} + \frac{\alpha Z}{4} \frac{\alpha Z + 4r(m + \varepsilon)}{(\alpha Z + r(m + \varepsilon))^2} \right) - \frac{6y^2 + 1}{4(1 + y^2)^2}, \quad y^2 = r^2 \frac{\varepsilon^2 - m^2}{\alpha^2 Z^2 - 1}. \quad (31)
$$
If the transformation of the coordinate is not used - only the wave function is transformed - equation (30), (31) is reduced to eqs. (2,11), (2,12), (2,13) of Ref.[4].

Equation (30) should be supplemented by, e.g., periodic or antiperiodic boundary conditions in a box $-\xi_L < \xi < \xi_U$, $\xi_L \to \infty$, $\xi_U \to \infty$, where $\xi_L$ is defined as (24) (cf. [3]). The nonrelativistic case, considered in [3], teaches us that, in sector IV, the complete and orthogonal sets of solutions of the generalized Kamke eigenvalue problem make families, for each of which a certain relation between the two parameters $Z\alpha$ and $\varepsilon$ is fixed, while the eigenfunctions within each family are labelled by the remaining free parameter. This relation for the present relativistic case remains unknown to us.

![Figure 2: The barrier potential (31) in the upper continuum. The curve is drawn for the positive energy $\varepsilon = 2m$ and the charge twice the critical value $\alpha Z = 2$. The value $r_0 = m^{-1}$ is taken for $r_0$.](image)

In the upper continuum (a part of sector IV) $\varepsilon > m$ the potential $U(\xi)$ is plotted against $\xi$ in Fig.2. The most interesting situation appears in the lower continuum $\varepsilon < -m$. The potential $U(\xi)$ is plotted for this case in Fig.3. It has the barrier character and a singularity in the point $\xi_{\text{sing}} = \xi(r_{\text{sing}})$. 
where
\[ r_{\text{sing}} = \frac{-\alpha Z}{m + \varepsilon} > 0. \] (32)

This singularity first appeared in equation (27), resulted from the exclusion of

![Figure 3: The barrier potential (31) in the lower continuum. The curve is drawn for the negative energy \( \varepsilon = -2m \) and the charge twice the critical value \( \alpha Z = 2 \). The value \( r_0 = m^{-1} \) is taken for \( r_0 \).](image)

lower spinor component \( F \) from equation (1), due to the zero in the coefficient in front of \( F \) in (1). Vice versa, one might exclude the upper spinor component \( G \) from (1). Then one would encounter the second-order differential equation like (27) with the coefficients singular in another point \( r'_{\text{sing}} = (Z\alpha/(m - \varepsilon)) \), provided that \( \varepsilon < m \).

The singularity in the point (32) also appears in the part of the potential \( U_2 \), "responsible for the spin interaction", eq.(2,13) of Ref. [1], although is not discussed there. We admit, that the superradiation phenomenon discussed in the next Section may be attributed to this singularity.
3 Superradiation and the pair creation by the vacuum

The contents of this Subsection are not directly dependent on the procedures developed above, since here we deal with known solutions, not with equations.

In sector IV all solutions are meaningful, neither should be ruled out. Let us consider the two-component solution to equations (1), whose behavior near the origin contains one oscillating exponent

\[ G(r) \sim (2pr)^{i\sqrt{\alpha^2 Z^2 - 1}} \left[ i \left( \frac{\alpha Z \varepsilon}{p} + \sqrt{\alpha^2 Z^2 - 1} \right) \right] \left( 1 - \frac{\varepsilon}{m} \right)^{\frac{1}{2}}, \]

\[ F(r) \sim (2pr)^{i\sqrt{\alpha^2 Z^2 - 1}} \left[ -\frac{\alpha Z \varepsilon}{p} + \sqrt{\alpha^2 Z^2 - 1} \right] \left( \frac{\varepsilon}{m} - 1 \right)^{\frac{1}{2}}, \]

where \( p = \sqrt{\varepsilon^2 - m^2} \). The same solution behaves at infinity \( r \to \infty \) as

\[ G(r) \sim \mathcal{E} e^{-i\pi r/2pr} \left( \frac{\varepsilon}{m} + 1 \right)^{\frac{1}{2}} + \mathcal{G} e^{i\pi r/2pr} \left( \frac{\varepsilon}{m} + 1 \right)^{\frac{1}{2}}, \]

\[ F(r) \sim -i\mathcal{E} e^{-i\pi r/2pr} \left( \frac{\varepsilon}{m} - 1 \right)^{\frac{1}{2}} + i\mathcal{G} e^{i\pi r/2pr} \left( \frac{\varepsilon}{m} - 1 \right)^{\frac{1}{2}} \]  

(34)

The other independent solution differs from this one by the change of sign in the exponent in (33). (Correspondingly the constant coefficients \( \mathcal{E} \) and \( \mathcal{G} \) in (34) become different.) The solution, described by the asymptotic equations (33), (34) is a wave, falling from the infinity and reflected back to infinity, with a transmitted wave, absorbed by the center. The coefficient \( \mathcal{G} \) is responsible for transmission/absorption, and \( \mathcal{E} \) for reflection. As a whole, in sector IV, the inelastic process is described by two scattering phases and one mixing angle. These are arranged in a unitary \( 2 \times 2 \) S-matrix, the reflection and transmission coefficients being expressed in terms of these angles \[1, 2, 3\].

To find the absorption and reflection coefficients, note that the conservation \( \text{div} \mathbf{j} = 0 \) of the current \( \mathbf{j} = \bar{\psi} \gamma \psi \) in the Dirac equation implies that the flux

\[ i \int (\mathbf{j} \mathbf{n}) r^2 d\Omega = FG^* - F^*G, \]

(35)

where \( \mathbf{n} = \mathbf{r}/r \) and \( \Omega \) is the stereoscopic angle, be independent of \( r \). Eq.(35) is the Wronsky determinant for the set (11). By equalizing the two expressions for the flux (35) obtained near \( r = 0 \) and \( r = \infty \), we obtain the current
conservation property in the form of the identity
\[
1 = \frac{|E|^2}{|G|^2} + \frac{2\sqrt{\alpha^2 Z^2 - 1}}{|G|^2 \left( \frac{\alpha Z \sqrt{\varepsilon^2 - m^2}}{\sqrt{\varepsilon^2 - m^2}} - \sqrt{\alpha^2 Z^2 - 1} \right)}.
\] (36)

Here the second term in the r.-h. side is the transmission coefficient, and the first term the reflection coefficient. Using the known explicit solution, whose asymptotic behavior is (33), we find for the latter coefficient:
\[
R = \frac{|E|^2}{|G|^2} = e^{-2\pi\gamma} \frac{\sinh \pi(\zeta - \gamma)}{\sinh \pi(\zeta + \gamma)},
\] (37)

where
\[
\zeta = \frac{\alpha Z \varepsilon}{\sqrt{\varepsilon^2 - m^2}}, \quad \gamma = \sqrt{\alpha^2 Z^2 - 1}.
\] (38)

Figure 4: The reflection coefficient in the upper continuum (for positive energies $\varepsilon > m$). The curve is drawn for the charge twice the critical value $\alpha Z = 2$.

The reflection coefficient (37) is plotted against the energy in Figs.4,5. In the positive kinetic energy domain $\varepsilon > m$ the barrier, as shown in Fig.2, is negative (attractive). Correspondingly, the reflection coefficient (37) is very small.
Figure 5: The reflection coefficient in the lower continuum (for negative energies $\varepsilon < -m$). The curve is drawn for the charge twice the critical value $\alpha Z = 2$.

(see Fig.4). Its maximum achieved at the border $\varepsilon = m$ makes $\exp(-2\pi\gamma)$. The Coulomb center acts as a strong absorber: it reflects next to nothing and absorbs almost everything. On the contrary, in the negative energy range $\varepsilon < -m$ there is a repulsing barrier singular in the point $\varepsilon = m$, as shown in Fig.3. Correspondingly, the reflection coefficient is greater than unity, as it follows from (37) and is seen in Fig.5. The phenomenon of $\mathbb{R}$ being greater than unity is called superradiation. It is known for scattering off the Kerr (rotating) black holes [10] and is associated with the fact that there appears singularity within the definition domain of the differential equation. It is referred to as manifesting the possibility to pump out energy from a black hole.

In our present context, the superradiation forbids the electron-positron creation from the vacuum in the field of a point-like nucleus with $Z\alpha > 1$.

To explain this statement, let us first remind [4, 5] that this process may be traditionally understood within the second-nonquantized theory as following. The nucleus is supplied with a core, which acts as a cutoff. When, for sufficiently large charge $Z$, the electron level sinks into the negative continuum $\varepsilon < -m$, an electron taken from the filled Dirac sea can perform the under-barrier transition to become a tightly bound state near the nucleus. At the
same time, the hole in the sea left by that electron, becomes a positron. In this way, the pair, comprised of a free positron and a deeply bound electron is produced from the vacuum by the strong cut-off Coulomb field. The probability of this process is determined by the probability of the penetration through the barrier.

Our consideration of the nonregularized, point-like nucleus modifies this idea in only one respect. For us, the electron, after it passes through the barrier, becomes a confined state, free near the singularity. The process is described by the transition coefficient $T = 1 - R$. Now, once the superradiation takes place, the number of electrons reflected back into the Dirac sea, exceeds the number of those taken from the sea. This is forbidden by the Pauli principle, since all vacancies in the sea are occupied. In this way the superradiation supports the vacuum stability.

4 Conclusion

The mathematical basis for application of the present black-hole approach to the relativistic problem of an electron in the overcritical Coulomb field is not as firmly established as it was in the case of the Schrödinger equation [3]. Lacking is a description of families of sets of complete orthogonal solutions to the Kamke eigenvalue problem in sector IV, where inelastic processes occur. Nonetheless, the adjustment of the black-hole approach into relativistic context undertaken in Section 2 demonstrates its conceptual relevance, sufficient for calculating, in Section 3, the reflection and transmission coefficients, responsible for spontaneous pair creation. The establishing of superradiation in electron scattering off the Coulomb center with $Z > 137$, which suppresses the pair creation is independent of the way the Kamke problem is posed and solved.

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