We present an improved calculation of all $B \to \pi, K, \eta$ decay formfactors from light-cone sum rules, including one-loop radiative corrections to twist-2 and twist-3 contributions, and leading order twist-4 corrections. The total theoretical uncertainty of our results at zero momentum transfer is 10 to 13%. The dependence of the formfactors on the momentum transfer $q^2$ is parametrized in a simple way that is consistent with their analytical properties and is valid for all physical $q^2$. The uncertainty of the extrapolation in $q^2$ on the semileptonic decay rate $\Gamma(B \to \pi e \nu)$ is estimated to be 5%.

1. Introduction and Definitions

In a recent paper\(^1\) we have reported a new calculation of $B \to \pi, K, \eta$ decay formfactors from QCD sum rules on the light-cone (LCSRs). The paper improves upon our previous publications\(^2,3\) by:

- including radiative corrections to twist-3 contributions to one-loop accuracy, for all formfactors;
- a precisely defined method for fixing sum rule specific parameters;
- using updated values for input parameters;
- a careful analysis of the uncertainties of the formfactors at zero momentum transfer;
- a new parametrization of the dependence of the formfactors on
momentum transfer, which is consistent with the constraints from
analyticity and heavy-quark expansion;
• a detailed breakdown of the dependence of formfactors on non-
perturbative hadronic parameters describing the \( \pi, K, \eta \) mesons,
  which facilitates the incorporation of future updates of their num-
erical values and also allows a consistent treatment of their effect
on nonleptonic decays.

The key idea of LCSRs is to consider a correlation function of the weak cur-
rent and a current with the quantum-numbers of the \( B \) meson, sandwiched
between the vacuum and, in the present context, the pseudoscalar meson \( P \),
i.e. \( \pi, K \) and \( \eta \). For large (negative) virtualities of these currents, the corre-
lation function is, in coordinate-space, dominated by distances close to the
light-cone and can be discussed in the framework of light-cone expansion.
In contrast to the short-distance expansion employed by conven-
tional QCD sum rules à la SVZ\(^4\), where nonperturbative effects are encoded in vacuum
expectation values of local operators with vacuum quantum num-
bers, the condensates, LCSRs rely on the factorisation of the underlying corre-
lation function into genuinely nonperturbative and universal hadron distrib-
ution amplitudes (DAs) \( \phi \) that are convoluted with process-dependent amplitudes
\( T \), which are the analogues to the Wilson-coefficients in the short-distance
expansion and can be calculated in perturbation theory. Schematically, one
has

\[
\text{correlation function} \sim \sum_n T^{(n)} \otimes \phi^{(n)}. \tag{1}
\]

The sum runs over contributions with increasing twist, labelled by \( n \), which
are suppressed by increasing powers of, roughly speaking, the virtualities
of the involved currents. The same correlation function can, on the other
hand, be written as a dispersion-relation, in the virtuality of the current
coupling to the \( B \) meson. Equating dispersion-representation and the light-
cone expansion, and separating the \( B \) meson contribution from that of
higher one- and multi-particle states, one obtains a relation (QCD sum
rule) for the formfactor describing the \( B \to P \) transition.

The particular strength of LCSRs lies in the fact that they allow the
inclusion not only of hard-gluon exchange contributions, which have been
identified, in the seminal papers that opened the study of hard exclusive
processes in the framework of perturbative QCD (pQCD)\(^5\), as being dom-
inant in light-meson form factors, but that they also capture the so-called
Feynman-mechanism, where the quark created at the weak vertex carries
nearly all momentum of the meson in the final state, while all other quarks are soft. This mechanism is suppressed by two powers of momentum-transfer in processes with light mesons, but there is no suppression in heavy-to-light transitions, and hence any reasonable application of pQCD to $B$ meson decays should include this mechanism. It is precisely LCSRs that accomplish this task and have been applied to a variety of problems in heavy-meson physics. A more detailed discussion of the rationale of LCSRs and of the more technical aspects of the method can be found e.g. in Ref. 8.

The formfactors in question can be defined as ($q = p_B - p$)

$$\langle P(p)|\bar{q}\gamma_{\mu}b|B(p_B)\rangle = f^P_+(q^2)\left\{ (p_B + p)_{\mu} - \frac{m^2_B - m^2_P}{q^2} q_{\mu} \right\} + \frac{m^2_B - m^2_P}{q^2} f^P_0(q^2) q_{\mu},$$  \hspace{1cm} (2)

$$\langle P(p)|\bar{q}\sigma_{\mu\nu}q^{\nu}b|B(p_B)\rangle = i \left\{ (p_B + p)_{\mu}q^2 - q_{\mu}(m^2_B - m^2_P) \right\} \frac{f^P_T(q^2)}{m_B + m_P}.$$  \hspace{1cm} (3)

The starting point for the calculation of e.g. $f^\pi_+$ is the correlation function

$$i \int d^4y e^{iqy} \langle \pi(p)|T[\bar{q}\gamma_{\mu}b](y)[m_b\bar{b}i\gamma_5q](0)|0 \rangle = \Pi_+, 2p_{\mu} + \ldots,$$  \hspace{1cm} (4)

where the dots stand for other Lorentz structures. For a certain configuration of virtualities, namely $m^2_b - p^2_B \geq O(\Lambda_{QCD}m_b)$ and $m^2_b - q^2 \geq O(\Lambda_{QCD}m_b)$, the integral is dominated by light-like distances and can be expanded around the light-cone:

$$\Pi_+(q^2, p^2_B) = \sum_n \int_0^1 du \phi^{(n)}(u; \mu_F) T^{(n)}(u; q^2, p^2_B; \mu_F).$$  \hspace{1cm} (5)

As in Eq. (1), $n$ labels the twist of operators and $\mu_F$ denotes the factorisation scale. The restriction on $q^2$, $m^2_b - q^2 \geq O(\Lambda_{QCD}m_b)$, implies that $f^\pi_+$ is not accessible at all momentum-transfers; to be specific, we restrict ourselves to $0 \leq q^2 \leq 14$ GeV$^2$. As $\Pi_+$ is independent of $\mu_F$, the above formula implies that the scale-dependence of $T^{(n)}$ must be canceled by that of the DAs $\phi^{(n)}$.

In Eq. (5) it is assumed that $\Pi_+$ can be described by collinear factorisation, i.e. that the only relevant degrees of freedom are the longitudinal momentum fractions $u$ carried by the partons in the $\pi$, and that transverse momenta can be integrated over. Hard infrared (collinear) divergences occurring in $T^{(n)}$ should be absorbable into the DAs. Collinear factorisation is trivial at tree-level, where the $b$ quark mass acts effectively as regulator, but
can, in principle, be violated by radiative corrections, by so-called “soft” divergent terms, which yield divergences upon integration over \( u \). Actually, however, it turns out that for all formfactors calculated in Ref.\(^1\) the \( T \) are nonsingular at the endpoints \( u = 0, 1 \), so there are no soft divergences, independent of the end-point behavior of the distribution amplitudes. In Ref.\(^1\) Eq. (5) has been demonstrated to be valid to \( O(\alpha_s) \) accuracy for twist-2 and twist-3 contributions for all correlation functions \( \Pi_{+,0,T} \) from which to determine the formfactors \( f_{+,0,T} \).

As for the distribution amplitudes (DAs), they have been discussed intensively in the literature\(^9\). For pseudoscalar mesons, there is only one DA of leading-twist, i.e. twist-2, which is defined by the following light-cone matrix element \((x^2 = 0)\):

\[
\langle 0 | \bar{u}(x)\gamma_\mu\gamma_5 d(-x)|\pi(p)\rangle = i f_\pi p_\mu \int_0^1 du e^{i p u} \phi_\pi(u),
\]

where \( \zeta = 2u - 1 \) and we have suppressed the Wilson-line \([x, -x]\) needed to ensure gauge-invariance. The higher-twist DAs are of type

\[
\langle 0 | \bar{u}(x)\Gamma d(-x)|\pi(p)\rangle \quad \text{or} \quad \langle 0 | \bar{u}(x)\Gamma G_{\mu\nu}(vx)\lambda^a/2d(-x)|\pi(p)\rangle,
\]

where \( v \) is a number between 0 and 1 and \( \Gamma \) a combination of Dirac matrices. The sum rule calculations performed in Refs.\(^1,2,3\) include all contributions from DAs up to twist-4. The DAs are parametrized by their partial wave expansion in conformal spin, which to NLO provides a controlled and economic expansion in terms of only a few hadronic parameters\(^9\).

The LCSR for \( f_\pi^T \) is derived in the following way: the correlation function \( \Pi_+ \), calculated for unphysical \( p_B^2 \), can be written as dispersion relation over its physical cut. Singling out the contribution of the \( B \) meson, one has

\[
\Pi_+ = f_+^T(q^2) \frac{m_B^2 f_B}{m_B^2 - q^2} + \text{higher poles and cuts},
\]

where \( f_B \) is the leptonic decay constant of the \( B \) meson, \( f_B m_B^2 = m_b \langle B | \bar{b}i\gamma_5 d |0\rangle \). In the framework of LCSRs one does not use (7) as it stands, but performs a Borel transformation, \( 1/(t - p_B^2) \to \hat{B} 1/(t - p_B^2) = 1/M^2 \exp(-t/M^2) \), with the Borel parameter \( M^2 \); this transformation enhances the ground-state \( B \) meson contribution to the dispersion representation of \( \Pi_+ \) and suppresses contributions of higher twist to the light-cone expansion of \( \Pi_+ \). The next step is to invoke quark-hadron duality to approximate the contributions of hadrons other than the ground-state \( B \) me-
son by the imaginary part of the light-cone expansion of $\Pi_+$, so that

$$\hat{\Pi}_+^{LCE} = \frac{1}{M^2} m_B^2 f_B \, f^\pi_+(q^2) \, e^{-m_B^2/M^2} + \frac{1}{M^2} \frac{1}{\pi} \int_{s_0}^{\infty} dt \, \text{Im} \Pi_+^{LCE}(t) \, \exp(-t/M^2).$$

(8)

Subtracting the 2nd term on the right-hand side from both sides, one obtains

$$\frac{1}{M^2} \frac{1}{\pi} \int_{m_b^2}^{s_0} dt \, \text{Im} \Pi_+^{LCE}(t) \, \exp(-t/M^2) = \frac{1}{M^2} m_B^2 f_B \, f^\pi_+(q^2) \, e^{-m_B^2/M^2}.$$  (9)

Eq. (9) is the LCSR for $f^\pi_+$. $s_0$ is the so-called continuum threshold, which separates the ground-state from the continuum contribution. At tree-level, the continuum-subtraction in (9) introduces a lower limit of integration in $u$, the momentum fraction of the quark in the $\pi$: $u \geq (m_b^2 - q^2)/(s_0 - q^2)$, in (5), which behaves as $1 - \Lambda_{QCD}/m_b$ for large $m_b$ and thus corresponds to the dynamical configuration of the Feynman-mechanism, as it cuts off low momenta of the $u$ quark created at the weak vertex. At $O(\alpha_s)$, there are also contributions with no cut in the integration over $u$, which correspond to hard-gluon exchange contributions. The task now is to find sets of parameters $M^2$ (the Borel parameter) and $s_0$ (the continuum threshold) such that the resulting formfactor does not depend too much on the precise values of these parameters.

2. Results

For a detailed discussion of the procedure used to determine the hadronic and sum rule specific input parameters we refer to Ref.1. One main feature is that $f_B$, the decay constant of the $B$ meson entering Eq. (9) is calculated from a sum rule itself10, which reduces the dependence of the resulting formfactors on the input parameters, in particular $m_b$, which is the one-loop pole mass and taken to be $(4.80 \pm 0.05)$ GeV. This procedure does not, however, reduce the formfactors’ dependence on the parameters describing the twist-2 DAs, which turns out to be rather crucial. Despite much effort spent on both their calculation from first principles and their extraction from experimental data, these so-called Gegenbauer moments, $a_1$ (only for $K$), $a_2$ and $a_4$ (for all $P$) are not known very precisely. Figure 1 shows the dependence of $f^\pi_+(0)$ on $a_2$ and $a_4$; the dots represent different determinations of these parameters and illustrate the resulting spread in values of the formfactor. The situation is even more disadvantageous for
Figure 1. Dependence of $f_\pi^+(0)$ on $a_2$ and $a_4$, for central values of input parameters. The lines are lines of constant $f_\pi^+(0)$. The dot labeled BZ denotes our preferred values of $a_2$, BMS the values from the nonlocal condensate model and BF from sum rule calculations.

Figure 2. (a) Dependence of $f_K^+(0)$ on the Gegenbauer moment $a_1$. (b) $f_K^+(q^2)$ as function of $q^2$ for different values of $a_1$: solid line: $a_1^K = 0.17$, short dashes: $a_1^K = 0$, long dashes: $a_1^K = -0.18$.

the $K$, whose formfactors depend on the SU(3) breaking parameter $a_1^K$, whose size and even sign are under discussion: at present, values as different as $-0.18$ and $+0.17$ (at $\mu = 1$ GeV) are being quoted. Figure 2 shows the dependencies of (a) $f_K^+(0)$ and (b) $f_K^+(q^2)$ on this parameter; evidently it is very important to determine its value more precisely.

Summarizing the detailed analysis of the uncertainties induced by both external input and LCSR parameters, the final results for the formfactors at zero momentum transfer obtained in Ref. are:

\[
\begin{align*}
f_\pi^+(0) & = 0.258 \pm 0.031, & f_T^+(0) & = 0.253 \pm 0.028, \\
f_K^+(0) & = 0.331 \pm 0.041 + 0.25\delta_{a_1}, & f_T^K(0) & = 0.358 \pm 0.037 + 0.31\delta_{a_1}, \\
f_\eta^+(0) & = 0.275 \pm 0.036, & f_T^\eta(0) & = 0.285 \pm 0.029.
\end{align*}
\]

$\delta_{a_1}$ is defined as $a_1^K(1 \text{ GeV}) - 0.17$, i.e. the deviation of $a_1^K$ from the central value used in Ref. For $f_{\pi,\eta}$ the total theoretical uncertainty ranges between 10% to 13%, for $f_K^K$ it is 12%, plus the uncertainty in $a_1$, which hopefully will be clarified through an independent calculation in the not
too far future. The intrinsic, irreducible uncertainty of the sum rule calculation is related to the dependence of the result on the sum rule specific parameters $M^2$ and $s_0$ and estimated to be $\sim 7\%$.

Turning to the $q^2$-dependence of formfactors, it has to be recalled that LCSRs are only valid if the energy $E_P$ of the final state meson, measured in the rest frame of the decaying $B$, is large, i.e. if $q^2 = m_B^2 - 2m_BE_P$ is not too large; specifically, we choose $E_P > 1.3$ GeV, i.e. $q^2 \leq 14$ GeV$^2$. The resulting formfactors are plotted in Fig. 3, using central values for the input parameters. In order to allow a simple implementation of these results in actual applications, and also in order to provide predictions for the full physical regime $0 \leq q^2 \leq (m_B - m_P)^2 \approx 25$ GeV$^2$, it is necessary to find parametrizations of $f(q^2)$ that

- reproduce the data below 14 GeV$^2$ with good accuracy;
- provide an extrapolation to $q^2 > 14$ GeV$^2$ that is consistent with the expected analytical properties of the formfactors and reproduces the lowest-lying resonance (pole) with $J^P = 1^-$ for $f_+$ and $f_T$.

\footnote{For $f_0$, the lowest pole with quantum numbers $0^+$ lies above the two-particle threshold}
Table 1. Fit parameters for $f(q^2)$. $m_1$ is the vector meson mass in the corresponding channel: $m_1^{\pi,\eta} = m_{B^*} = 5.32$ GeV and $m_1^K = m_{B^*} = 5.41$ GeV. The scale of $f_T$ is $\mu = 4.8$ GeV.

|       | $r_1$ | $r_2$ | $(m_1^2)^2$ | $m_{fit}^2$ |
|-------|-------|-------|-------------|-------------|
| $f_+^\pi$ | 0.744 | -0.486 | 40.73       |             |
| $f_0^\pi$ | 0     | 0.258 | -33.81      |             |
| $f_T^\pi$ | 1.387 | -1.134 | 32.22       |             |
| $f_+^K$ | 0.162 | 0.173 | -           | -           |
| $f_0^K$ | 0     | 0.330 | -37.46      |             |
| $f_T^K$ | 0.161 | 0.198 | -           | -           |
| $f_+^\eta$ | 0.122 | 0.155 | -           | -           |
| $f_0^\eta$ | 0     | 0.273 | 31.03       |             |
| $f_T^\eta$ | 0.111 | 0.175 | -           | -           |

As shown in Ref.1, the following parametrizations are appropriate:

- for $f_+^\pi$:
  $$f(q^2) = \frac{r_1}{1 - q^2/m_1^2} + \frac{r_2}{1 - q^2/m_{fit}^2},$$
  \hspace{1cm} (10)
  where $m_1$ is the mass of $B^*(1^-)$, $m_1 = 5.32$ GeV; the fit parameters are $r_1$, $r_2$ and $m_{fit}$;

- for $f_+^K$:
  $$f(q^2) = \frac{r_1}{1 - q^2/m_1^2} + \frac{r_2}{(1 - q^2/m_1^2)^2},$$
  \hspace{1cm} (11)
  where $m_1$ is the mass of the $1^-$ meson in the corresponding channel, i.e. 5.32 GeV for $\eta$ and 5.41 GeV for $K$; the fit parameters are $r_1$ and $r_2$;

- for $f_0$:
  $$f_0(q^2) = \frac{r_2}{1 - q^2/m_{fit}^2},$$
  \hspace{1cm} (12)
  the fit parameters are $r_2$ and $m_{fit}$.

The central results for the fit parameters are collected in Tab. 1. The quality of all fits is very good and the maximum deviation between LCSR and fitted result is 2% or better. The impact of the extrapolation of the fit formulas to $q^2 > 14$ GeV$^2$ is of phenomenological relevance mainly for $B \to \pi e\nu$, relevant for the determination of $|V_{ub}|$ from experiment. We have starting at $(m_B + m_P)^2$ and hence is not expected to feature prominently.
estimated the effect of the extrapolation on the decay rate by implementing
different parametrisations for $f_{\pi}^+$, which all fit the LCSR result very well for
$q^2 < 14 \text{ GeV}^2$, but differ for larger $q^2$, the main distinguishing feature being
the positions of the poles. We find that for reasonable parametrisations,
that is such that do not exhibit too strong a singularity at $q^2 = m^2_1$, the
total rates differ by not more than 5%, the difference becoming smaller if an
cut-off on the maximum invariant mass of the lepton pair is implemented,
which implies that the extrapolation is well under control.

3. Summary & Conclusions
LCSRs provide accurate results for weak decay formfactors of the $B$ meson
into light mesons, in particular $\pi$, $K$ and $\eta$. The results depend on sum
rule specific input parameters which generate an irreducible “systematic”
uncertainty of the approach estimated to be $\sim 7\%$. Additional uncertainties
are induced by imprecisely known hadronic input parameters, in particular
the Gegenbauer moments $a_{1,2,4}$ describing the leading-twist light-meson
distribution amplitudes. An improved determination of these parameters
would be very welcome. The present total uncertainty of the formfactors at
zero momentum transfer varies between 10 and 13\%, but becomes smaller
at larger $q^2$. LCSR calculations require the energy of the final state meson
to be large in the rest-frame of the decaying $B$ and hence are valid only for
not too large momentum transfer $q^2$; the maximum eligible $q^2$ is chosen to
be $14 \text{ GeV}^2$. The $q^2$-dependence of the formfactors can be cast into simple
parametrisations in terms of two or three parameters, which also capture
the main features of the analytical structure and are expected to be valid
in the full kinematical regime $0 \leq q^2 \leq (m_B - m_P)^2$. The total uncertainty
introduced by the extrapolation of the formfactors to $q^2$ larger than the
sum rule cut-off $14 \text{ GeV}^2$ is estimated to be $\sim 5\%$ for the semileptonic rate
$\Gamma(B \rightarrow \pi e \nu)$.

Ref.\(^1\) also contains a detailed breakdown of the dependence of the form-
factors on the Gegenbauer moments, which not only allows one to recalcu-
late the formfactors once these parameters are determined more precisely,
but also makes it possible to consistently assess their impact on nonleptonic
decay amplitudes (e.g. $B \rightarrow \pi \pi$) treated in QCD factorisation.

The LCSR approach is complementary to standard lattice calculations,
in the sense that it works best for large energies of the final state meson
(i.e. small $q^2$), whereas lattice calculations work best for small energies – a
situation that may change in the future with the implementation of moving
NRQCD. Previously, the LCSR results for $f_{+0}^\pi$ at small and moderate $q^2$ were found to nicely match the lattice results obtained for large $q^2$. The situation will have to be reassessed in view of our new results and it will be very interesting to see if and how it will develop with further progress in both lattice and LCSR calculations.

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