Statistical modelling of tropical cyclone tracks: modelling the autocorrelation in track shape

Tim Hall, GISS∗ and Stephen Jewson
December 19, 2021

Abstract

We describe results from the third stage of a project to build a statistical model for hurricane tracks. In the first stage we modelled the unconditional mean track. In the second stage we modelled the unconditional variance of fluctuations around the mean. Now we address the question of how to model the autocorrelations in the standardised fluctuations. We perform a thorough diagnostic analysis of these fluctuations, and fit a type of AR(1) model. We then assess the goodness of fit of this model in a number of ways, including an out-of-sample comparison with a simpler model, an in-sample residual analysis, and a comparison of simulated tracks from the model with the observed tracks. Broadly speaking, the model captures the behaviour of observed hurricane tracks. In detail, however, there are a number of systematic errors.

1 Introduction

There are six distinct regions in which tropical cyclones occur: the North Atlantic, the North East Pacific, the North West Pacific, the South West Pacific, the Northern Indian Ocean and the Southern Indian Ocean. Tropical cyclones in the North Atlantic (the stronger ones of which are often known as hurricanes) affect the Caribbean, Central America and the US. Those in the North East Pacific (also known as hurricanes) can affect Mexico. Storms in the North West Pacific (also known as typhoons) frequently impact Japan, Taiwan, Korea and China. South Pacific storms may hit the east coast of Australia or New Zealand. Northern Indian Ocean storms, although infrequent, can make landfall in India or Bangladesh, while Southern Indian Ocean storms can hit the West Coast of Australia, and Indian ocean islands such as Madagascar.

Tropical cyclones in all of these regions have the potential to cause significant damage and loss of life. As a result they affect many groups of people in each of these communities including home-owners, businesses, health-care providers and, in the wealthier countries, the insurance industry. Many of these groups might benefit from a more accurate assessment of the tropical cyclone risk they face, since a more accurate understanding of the risk could lead to better decisions about preventative measures. Making approximate risk assessments is rather easy, since information about historical tropical cyclones is widely available. However, many factors conspire to make more accurate risk assessments difficult. These include the fact that the most severe tropical cyclones are rare, that the geometry of islands and coastlines is often complicated, which can make the comparison of wind records from nearby locations difficult, and that long term fluctuations in the climate, including anthropogenic climate change, may be changing underlying activity rates (and these changes may be different in different regions).

One class of methods for estimating tropical cyclone risk that reduces these difficulties is the basin-wide track modelling approach, as followed by Fujii and Mitsuta (1973), Drayton (2000), Vickery et al. (2000) and Emanuel et al. (2005). In this approach statistical models are constructed that attempt to represent the behaviour of the tracks and intensities of tropical cyclones across a whole basin. The potential advantages of this approach are that it allows the use of all the available track information for a particular basin, and that the total impact of a storm can be calculated rather easily. The potential disadvantage is that the modelling may be rather difficult because of the complicated nature of tropical cyclone tracks. A poor model may then give less accurate estimates of risk than a much simpler approach that did not use basin-wide track modelling.

∗Correspondence address: Email: tmh1@columbia.edu
We are currently building a new basin-wide tropical cyclone model from scratch. Because of the complex nature of the problem we are taking a systematic approach that includes extensive validation and cross-validation of our models. Our initial focus is on Atlantic hurricanes, and so far we have completed two stages in the modelling process. The first stage was to model the mean tracks of Atlantic hurricanes (see Hall and Jewson (2005a)). We do this by using a weighted nearest neighbour approach, in which the definition of ‘near’ is optimised using a jack-knife out-of-sample fitting procedure. The second stage was to model the variance of fluctuations around these mean tracks (see Hall and Jewson (2005b)), and again we use a weighted nearest neighbour approach, with optimal definition of near. Putting together the first two stages, one can generate random artificial hurricane tracks, and we show such tracks in figure 6 of Hall and Jewson (2005b). These tracks are clearly not realistic, however (they are too jagged, or irregular, relative to real tracks), and it seems that the most important missing element is along-track memory. The third stage of our research, which we describe in this article, is therefore to attempt to model this memory (or ‘inertia’) in the standardised anomalies, in the hope that this will lead to more realistic track shapes.

The structure of this paper is as follows. First, in section 2 below, we describe the data set we use for this study. Secondly, in section 3 we present various diagnostics of this data. These diagnostics motivate the choice of a statistical model, which we describe in section 4. In sections 5 and 6 we then present results from this model, at microscopic and macroscopic scales, respectively. Finally in section 7 we summarise our results.

2 The data

The basic data set for this study is a 54 year subset (1950-2003, inclusive) of the Hurdat data (Jarvinen et al., 1984). This 54 year period has 524 hurricanes, corresponding to 15,607 six-hourly observations of storm position. In Hall and Jewson (2005a) we used a nearest neighbours method to estimate unconditional mean hurricane tracks from this data. In Hall and Jewson (2005b) we then used a similar method to estimate the unconditional variance of fluctuations around these mean tracks. In this study we again consider these fluctuations, but now standardized using the variance model. The 6-hour steps are projected onto the local optimized mean track. The mean track (parallel and perpendicular components) are then subtracted, and the results divided by the local optimized variance. Assuming the mean and variance models to be accurate, the result is 524 time series of varying length, each with zero mean and unit variance. We will generally refer to these standardized fluctuations as ‘standardized anomalies’ or just ‘anomalies’.

3 Properties of the standardised anomalies

We now investigate the statistical properties of the standardized hurricane track anomalies, beginning with the scatter plots in figure 1. Figure 1a shows u (i.e. along mean track) anomalies versus u anomalies one step (6 hours) prior, and figure 1b shows the same for v (i.e. across mean track) anomalies. Data are drawn from the full set of storm tracks, but only every 10th location is plotted for clarity. In figure 1c we plot u versus v at the same location, while in figure 1d u is plotted versus v at the previous step. We see that u and v are closely and linearly related to their values at the previous step. The presence of such a strong relationship in time is expected, since hurricane motion is driven by large-scale weather patterns, which themselves have memory. By contrast, there is no evidence for a relationship between u and v, either contemporaneously or with a one-step lag, suggesting that independent modelling of u and v would be appropriate.

To diagnose the nature of the u and v memory we compute autocorrelation functions (ACFs). Figure 2 shows an example from a randomly selected storm track. In figure 2a, we show the u and v anomaly time series for this track. That there is memory is apparent in the lack of step-to-step variability. Equally apparent is the lack of any obvious u-v correlation. The natural logs of the ACFs, shown in figure 2b, display approximately linear decline. A purely exponential ACF is expected for a process that only carries information forward from the previous time step (sometimes called a Markov process).

To test the robustness of these observations we also compute the ACFs from the anomalies of all storms. These ACFs, up to a three-day lag, are shown in figure 3. There is a large amount of data, and so these ACFs should be estimated well (although the longer leads are based on fewer values than shorter leads because some storms last less than 3 days). The u log ACF is very close to linear. The v log ACF is slightly convex, indicating that v anomalies have less memory than u anomalies.
In order to isolate the direct relationship between an anomaly and its value at various past lags we perform a regression analysis (ACFs are non-zero at lags greater than one even if the process is purely Markov because of the lag-one correlations between intermediate steps). Figure 3 shows the coefficients obtained by a multiple regression of an anomaly against the anomalies of the preceding 10 steps on the same track, computed over all tracks of at least 10 steps. At lag one the coefficients in $u$ and $v$ are about 0.8, but they are near zero at all subsequent lags. We conclude from this analysis that a lag-one process considered separately for $u$ and $v$ is a reasonable model for the anomalies.

A lag-one ACF has the form $f(t) = e^{-t/\tau}$, where the autocorrelation time, $\tau$, can be interpreted as the time scale for the track memory to decay. We estimate this time scale in days from the observational ACFs using the value $f(t)$ at $t$ of lag one; that is $t = 0.25$ days. We plot $\tau$ as a function of space in figure 5 using a 900km smoothing window (the justification for which is presented in section 4 below). There is spatial structure in $\tau$, with generally longer memory at higher latitudes. Note that the memory in $u$ is somewhat larger than $v$, consistent with figure 6.

Finally, we ask to what degree the anomalies have a probability distribution that is normal (in [Hall and Jewson (2005)] we assumed a normal distribution when we derived the form of the likelihood that we used to optimize the variance). Figure 4 shows QQ plots of the $u$ and $v$ anomalies versus a normal distribution. Were the observed anomalies purely normal, then the scatter would fall along a straight line. In fact, the scatter only falls along a straight line inside ±2 standard deviations. Outside this range, the distribution of observed anomalies displays ‘fat tails’, that is, large anomalies are more common than would be the case if the distribution were normal.

In summary, we have investigated the local statistical properties of the observed hurricane track anomalies. We find that the temporal dependence is simple: $u$ is linearly related to its value at the previous time step, and likewise for $v$, but there is no evidence for a relationship to values at earlier time steps, and no evidence for a relationship between $u$ and $v$ (note, however, that in [Hall and Jewson (2005)] we saw weak, but significant, $u$-$v$ correlations that varied in sign depending on geographic region). The probability distributions of $u$ and $v$ are more complex. They are normally distributed within ±2 standard deviations, but beyond that show significantly fatter tails than the normal distribution would.

## 4 A linear autoregressive model

Having investigated the local properties of the observed hurricane track anomalies, we now consider how they might be modelled. The simplest standard statistical models for the modelling of autocorrelated time-series are the autoregressive moving average (ARMA) models (see [Box and Jenkins (1970)] or almost any modern book on time-series analysis). In the case of two time series, ARMA can be generalised to vector ARMA, which models cross-correlations and cross-lag-correlations between the time-series. There are also other generalisations of ARMA to include, for instance, long memory. It would be convenient if we could use these standard models to capture the anomaly behaviour described above. So we ask ourselves: based on the evidence we have seen so far, are the ARMA models, or some generalisation of them, likely to be good for this purpose, and, if so, which of the various ARMA models?

The lack of correlation between $u$ and $v$ suggests we need only consider separate univariate ARMA models rather than a vector ARMA model that would model $u$ and $v$ as correlated. The fact that the autoregression functions drop to zero after one lag suggests that we can simplify even further and consider separate AR(1) models. The linearity of the lag one dependency structure also suggests that using linear models such as AR(1) is reasonable. The only catch seems to be that the QQ-plots show non-normality in the tails of the distribution, and the ARMA models are based on an assumption of pure normality. Given all of the above, we tentatively fit AR(1) models to the two time series, bearing in mind that one of the conditions for such a model to work (the distributional assumption) has not been completely satisfied. The AR(1) models for $u$ and $v$ are then given by:

$$ u_{n+1} = \phi_n u_n + s_n e_n^u \quad (1) $$

$$ v_{n+1} = \phi_n v_n + s_n e_n^v \quad (2) $$

There are two coefficients in each equation: the memory parameter $\phi_n$ and the standard deviation of the noise forcing $s_n$. Since our anomalies are already standardised, there is a simple constraint between these two parameters. This constraint, and other details of the model, are described in detail in appendix A.

The simplest implementation of this model would have the memory parameter and noise variance constant in space. However, as we have seen above in figure 5, this would not be entirely realistic since...
the autocorrelation time-scale of the observations clearly varies. We therefore allow both the memory parameter and the noise variance to vary in the model, and in fact the value of $\phi$ (and the value of $s$, which is derived from $\phi$) is derived directly from the data used to create figure 5. The spatial smoothing used to create $\phi$ was derived optimally using the same method as used to define the optimal smoothing in the modelling of the mean and the variance: each year of the 54 years of historical data was eliminated in turn, and for a range of values of the smoothing lengthscales the log-likelihood (given by equation 13 in appendix A) for the eliminated year was calculated using the AR(1) model fitted to the rest of the data. The smoothing window has a Gaussian shape, and is normalized so that the weights sum to one. The resulting dependence of log-likelihood on lengthscales, which gives an optimum lengthscales of 900 km, is shown in figure 7.

How well does the AR(1) model work? There are many tests that we can do to try and answer this question, and the rest of this article consists of results from a variety of such tests, and some discussion and speculation about what the results might be telling us.

5 Microscopic diagnostics

We will start our assessment of the performance of the AR(1) model using what we call ‘microscopic’ diagnostics. By this we mean an assessment of how well the model reproduces the statistics of individual steps of hurricane tracks. We will then continue our assessment (in section 6 below) by looking at ‘macroscopic’ diagnostics, by which we mean an assessment of how well simulated tracks from the model reproduce the large scale features of the observed hurricane track distribution.

In our earlier articles we discussed the importance of performing model selection using out-of-sample testing, and we implemented such testing using the Quenouille-Tukey jack-knife. Our first evaluation of the AR(1) model is to apply such testing and compare the AR(1) model to the simpler AR(0) model. The AR(0) model neglects the memory in $u$ and $v$ entirely and models them both as stationary ‘white noise’. Thus comparing AR(0) with AR(1) is a test for the importance of modelling the memory. The comparison between the AR(0) and AR(1) models is performed by missing out each year of the data in turn, fitting the model to the rest of the data, and evaluating the likelihood of the missed year for both models. The likelihoods for all missed years are then compared across the two models on a year by year basis. We like this test because it directly compares the ability of the models to extrapolate the distribution of possible hurricanes, and extrapolation of the hurricane distribution is exactly what is needed for good assessments of hurricane risk. The year by year comparison of out of sample likelihoods is shown in figure 8 (the left panel shows log-likelihood scores for the two models and the right panel shows the differences between them). We see a dramatic difference between the two models: the AR(1) model beats AR(0) in every year except one. If the models were in fact equally good this could only occur with extremely low probability, and so this seems to be conclusive proof that, by this measure, modelling the correlations in the track fluctuations gives an improvement over not modelling them. This is not entirely surprising, of course.

Our second set of microscopic diagnostics concerns the behaviour of the residuals from the model. If the model were completely correct then the residuals would be uncorrelated in time, normally distributed and not cross-correlated. On the other hand, if any of these properties do not hold, that immediately gives an indication that the model is not ideal. And since we already know that the original anomaly data didn’t perfectly satisfy the assumptions behind the AR(1) model, we might already anticipate some problems with the residuals too.

We start our residual analysis by showing a number of superimposed time series of residual anomalies from different storms, in figure 9. Various lag and cross-correlations derived from the residual anomalies of all storms are then shown in figure 10 (note that this figure looks very similar to figure 8 but is now for residuals rather than the anomalies themselves). The first panel shows the lag one dependence for $u$, the second panel shows the lag one dependence for $v$, the third panel shows the lag zero dependence between $u$ and $v$ and the fourth panel shows the lag one dependence between $u$ and $v$, with $u$ leading. In all four cases the results are very good and there is no discernible structure. This suggests that the model is doing a good job of capturing the behaviour of the anomalies in time, and this accords well with the analysis of the anomalies themselves, discussed in section 4 above, in which we saw that the regression coefficients dropped to zero after one lag and the relation between adjacent anomalies was close to linear (both of which suggest that AR(1) may work well).

The distributions of the $u$ and $v$ residuals are shown in figure 11. The horizontal axes are shown as quantiles while the vertical axis is in km. We see that the residuals are close to normal out to about 2 standard deviations, but then deviate sharply from normal, and show definite fat tails. This is very
similar to the behaviour of the anomalies themselves, and is a clear fault with the model, since the residuals should be normally distributed. It would seem likely that the fat tails of the residuals are a direct consequence of the fat tails in the original data. What might be causing these fat tails? Possibly that a small fraction of storms behave very differently to the majority. This could be storms in certain seasons, or certain regions, or in certain large-scale weather conditions. Or maybe it is wrong to look for a particular cause: why, after all, should the distributions of the anomalies and the residuals be normal in the first place?

Based on all of the microscopic diagnostics discussed above we conclude that our model captures some, but not all, of the observed behaviour of the motion of hurricanes at a microscopic level. The temporal dependence of the tracks seems to be captured very well. The main shortcoming would seem to be the inability of the model to capture the correct distribution of the anomalies, which is fatter than normal in the tails. At a statistical level there is no single obvious way to remedy this distribution problem. Two of the things one might consider trying would be:

- using non-normal residuals in the subsequent simulations, either modelled or resampled from the residuals generated during the fitting process.
- transforming the anomalies to normal before fitting the AR(1) model, and transforming them back to the original distribution.

However, rather than just blindly testing these, or other, statistical methods we will first consider the macroscopic diagnostics. This may lead to some more ‘physical’ insight into why the model is not capturing the correct distribution.

6 Macroscopic diagnostics

We now move on to consider macroscopic diagnostics from our fitted track model. These are diagnostics that assess whether the tracks that can be generated from the track model are realistic or not. If the microscopic diagnostics indicated that the model were perfect, then we would expect the tracks to be realistic. However, we have seen that the microscopic diagnostics are not perfect, and that there are systematic biases in the tails of the distribution of simulated track steps. Furthermore, there may be other biases that we haven’t detected (for instance, we haven’t considered the microscopic diagnostics on a region by region or month by month basis, and this could be obscuring problems in the model). It is very hard to know how important any microscopic biases are likely to be when it comes to simulating full tracks, and whether they will somehow cancel out, or whether they will lead to large systematic errors in the track shapes. All of the following diagnostics are therefore based on integrating the track model to generate simulated tracks, and comparing these tracks with the observed.

Our first comparison is based on the simulation of tracks from a single point with the AR(0) and AR(1) models (see figures 12 and 13). The red lines in both cases show the mean track from that point, the blue lines shows a real track and the black lines show a number of simulated tracks. The AR(0) model (figure 12) gives tracks that are rather similar to each other and very bunched together. It is clear that the real track is qualitatively different from the simulated tracks, and hence that the simulated tracks do not come from an appropriate model. The AR(1) model (figure 13) performs much better: the simulated tracks are now very different from each other, and spread out rapidly. It now seems quite plausible that the real track could have been generated as one of these simulations, and hence by this test the AR(1) model is a good one. One interesting point in this comparison is that the AR(0) and AR(1) models have the same variance of the anomalies: all that differs is the autocorrelation. We thus see that the rapid spreading of the AR(1) tracks is due entirely to this autocorrelation.

So far we have seen that the AR(1) model performs well in comparison with the simpler AR(0) model. But AR(0) is an easy target to beat, and we do not yet have any idea how the AR(1) model performs in an absolute sense. Figure 14 is an attempt to look at that question. It shows the 524 observed tracks for our historical data set (in panel (a)), and three realisations of the same number of simulated tracks (in the other three panels). The simulated tracks were generated from the same starting points as the observed tracks, and each track has the same number of steps as the corresponding real track. We shouldn’t expect the simulated tracks to look exactly like the observations. But if the AR(1) model were a perfect model, and if track origin and track shape are independent, then it should look as if the observations could have come from another realisation of the model. The first thing we note from making this comparison is that the simulations show a singularity of some sort in the eastern Atlantic. The cause for this becomes rather clear when we consider the forward speed from the mean track model (figure 15).
The forward speed in this region is very small compared to the forward speeds elsewhere in the basin: any simulated track that enters this region will likely take a long time to leave it. Also, comparison with the observed tracks shows that there are no tracks in this region at all. We conclude that the singularity is therefore just an artefact of the model, for two reasons. Firstly, we are running a track model but no simulation of the hurricane intensity. In reality, any hurricane entering this region of the Atlantic would die because the sea surface temperatures are too low to support the evaporation needed to drive the hurricane. Thus the singularity would likely disappear if we were to add an appropriate intensity model (which we intend to do in due course). Secondly, our mean track speeds are given by the average observed speeds of nearby hurricane tracks. In regions where there are no tracks this average takes values from far away and ceases to be physically meaningful. We could equally well set the mean track speed in this region to some arbitrary (but reasonable) fixed value, and this would eliminate the singularity.

Looking more generally at the shapes of the simulated tracks we see that overall the tracks from the model seem reasonably similar to the observed tracks, and certainly capture the broad behaviour of the observations. However, in detail, there are differences. Too many tracks penetrate deep into the America continent, and too many cross central America into the Pacific. These two faults would probably be solved by the inclusion of an intensity model that would presumably kill these tracks a little earlier. More seriously, perhaps, the density of tracks off the east coast of the US is higher in the observations than any of the sets of simulations. This is highlighted in figure 16 which shows a close-up of the US coastline, with the observed and simulated track sets as before.

Figure 17 considers in more detail the number of observed and simulated tracks that cross individual lines of latitude moving from South to North (left panel) and North to South (right panel). The various sets of lines correspond to different latitudes (which are, from bottom to top, 10N, 20N, 30N, 40N and 50N), the bold line shows the numbers of hurricanes in 54 years of observations and the dashed lines show numbers of hurricanes from 3 sets of simulations. If the model were perfectly correct than the observations should lie close to the sets of simulations. For the northward moving storms the model seems to perform well at 10N and 20N, but at 30N and 80W there is a greater number of observed storms than simulated. This is also the case at 40N and 70W. This corresponds exactly to the region off the US east coast in figure 16 where the simulations do not generate a great enough density of storm tracks.

Figure 18 counts observed and simulated storms moving from West to East (upper panel) and from East to West (lower panel) in a similar way. The longitudes used are 80W, 70W, 60W, 50W, 40W, 30W and 20W (from left to right). The model does reasonably well for the storms moving East to West in the subtropics, but less well for storms moving from West to East in the extratropics. In particular, there are too few storms moving from West to East at longitudes of 80W, 70W, 60W and 50W, and for latitudes between 30N and 45N.

Our next diagnostic is designed to test whether the speed of the simulated tracks is correct. We test this in two ways, both shown in figure 15. The left hand panel shows the distribution of the total length of the trajectories: model and observations agree very well. The right hand panel shows great circle distance from the start to end of each trajectory: again, model and observations show fairly good agreement, although there are more simulated tracks than observed tracks that reach distances greater than 7000km from their origins (again, this may be related to the lack of a physical death mechanism, since some storms reach unrealistically high latitudes).

Our final diagnostic is an attempt to create a measure of track density in space i.e. to count how many storms are passing through different parts of the basin. We calculate this measure as follows. First, we divide the whole basin into 100km-by-100km grid cells. Within each grid cell we then count the number of ‘storm points’, which are six-hourly locations of storms. Slow moving storms may be counted several times in a single box. We plot both the observations, and results from an ensemble of 34 simulations, each of which contains 54 years of simulated tracks starting from the observed genesis points as before. This allows us to compare the observed track density against the distribution of results generated by the track model. Panel (a) is the number of simulated storm points per 100km-by-100km box accumulated over 54 years and averaged over the ensemble. Panel (b) is the historical distribution of the same quantity, and panel (c) is the difference between the historical and simulated. We see immediately that the model is failing to create the very high density of observed storms off the US coast that is seen in the observations, exactly as discussed above. To be rigorous, however, one might question whether these differences are significant (they certainly look significant, but we should test to be sure). To this end, panel (d) is the standard deviation across the simulation ensemble members. Comparison of this with panel (c) makes it clear that the differences between the observed and simulated fields are 5 or 6 standard deviations of the distribution of the simulated field, which confirms that these are significant differences, and not just due to randomness.
7 Summary

We have described the third stage of our attempt to build a statistical model that gives faithful simulations of the tracks of Atlantic hurricanes. The first stage was to model the mean, the second stage was to model the variance, and the third stage, described above, has been to attempt to model the autocorrelations. We have shown that modelling the autocorrelations allows us, for the first time in this study, to create simulations of hurricane tracks that look somewhat realistic.

Detailed analysis of the residuals from the model shows that we are capturing the temporal dependency structure of hurricane track anomalies very well, but that the distribution of track anomalies is fat-tailed and hence that our model, which assumes Gaussian distributions, is deficient in this respect.

Comparison of simulations of tracks from the model with observed tracks show that the model is broadly correct, but has a number of systematic errors when analysed in detail. Some, but not all, of these systematic errors would be eliminated if we included an intensity model in our simulations. In particular, the observations show a very high density of tracks off the East Coast of the US, and this is not replicated by the model. This shows up in a number of diagnostics such as simulated track sets, counts of storms crossing longitude and latitude lines and storm point density plots.

The next stage in our research will be to try and better understand, and hopefully correct, this systematic bias. There are many candidates for conditioning the calculations of the mean, variance and autocorrelation, such as time of year, storm intensity, geographic region of storm origin, and climate indices such as ENSO and the North Atlantic Oscillation. Also, including some representation of the ‘fat tails’ of the anomaly probability distribution would certainly improve the microscopic performance of the simulations, and, possibly, the macroscopic performance.

Clearly, a physical death process is needed to stop the propagation of simulated storms in meteorologically unfavourable regions. We believe, however, that the infrastructure we have developed, including the out-of-sample validation and the micro- and macroscopic diagnostics, will allow us to assess readily if new features results in significant improvement.

A The AR(1) model

A.1 Likelihood for AR(1)

Consider an observed hurricane track with $m + 1$ points. To define the likelihood for AR(1) we need to use an $m$ dimensional normal distribution ($m$ not $m + 1$ because the first point is considered fixed, and only subsequent points are random).

We consider the $u$ component ($v$ is analogous).

The AR(1) model has the form:

$$ u_{n+1} = \phi_n u_n + s_n \epsilon_n $$

where $s_n$ is the standard deviation of the noise forcing (normally one would use $\sigma$ rather than $s$, but we’ve used $\sigma$ already in equation 1 in Hall and Jewson (2005b)).

Writing out the first few terms of the model:

$$ u_0 = 0 \quad \text{(starting point of the track)} $$
$$ u_1 = s_0 \epsilon_0 $$
$$ u_2 = \phi_1 u_1 + s_1 \epsilon_1 $$
$$ u_3 = \phi_2 u_2 + s_2 \epsilon_2 $$
$$ \quad \vdots $$
$$ u_m = \phi_{m-1} u_{m-1} + s_{m-1} \epsilon_{m-1} $$

It looks like there are two parameters ($\phi$ and $s$), but really there is only one. This is because if we square equation 3 take expectations, and use the fact that the variance of $u$ is 1 (because we have used the variance model to standardize the anomalies) then we find that

$$ (1 - \phi^2) = \sigma^2 $$

So all we have to do is to estimate $\phi_n$ and we can calculate the amplitude of the noise, $s_n$.

$\phi_n$ is the correlation between $u_{n+1}$ and $u_n$, so we can estimate it using nearest neighbours in the same way that we do for mean and variance.
To evaluate the performance of the model for different values of the smoothing parameter we need the likelihood. The density is just the multivariate normal:

\[ f = \frac{1}{(2\pi)^{\frac{p}{2}} D^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (z - \mu)^T \Sigma^{-1} (z - \mu) \right) \tag{11} \]

where \( z \) is the vector of \( m \) anomalies along the track. If we set \( \mu = 0 \) this becomes

\[ f = \frac{1}{(2\pi)^{\frac{p}{2}} D^{\frac{1}{2}}} \exp \left( -\frac{1}{2} z^T \Sigma^{-1} z \right) \tag{12} \]

and the log likelihood over many independent tracks is:

\[ \ln f = \sum -\ln[(2\pi)^{\frac{p}{2}} D^{\frac{1}{2}}] + \sum -\frac{1}{2} z^T \Sigma^{-1} z \tag{13} \]

What is \( \Sigma \)? Since the anomalies are normalised, the diagonal of \( \Sigma \) is all 1’s. The first off diagonal is then given by the various values of \( \phi_n \) along the track, the second off diagonal by products of pairs of these \( \phi \)’s, and so on. This gives:

\[
\begin{pmatrix}
1 & \phi_1 & \phi_1 \phi_2 & \phi_1 \phi_2 \phi_3 & \ldots \\
. & 1 & \phi_2 & \phi_2 \phi_3 & \ldots \\
. & . & 1 & \phi_3 & \ldots \\
. & . & . & 1 & \ldots \\
. & . & . & . & \ldots
\end{pmatrix}
\]

In the case where we assume that \( \phi \) is constant everywhere, this then simplifies to

\[
\begin{pmatrix}
1 & \phi & \phi^2 & \phi^3 & \ldots \\
. & 1 & \phi & \phi^2 & \ldots \\
. & . & 1 & \phi & \ldots \\
. & . & . & 1 & \ldots \\
. & . & . & . & \ldots
\end{pmatrix}
\]

So to calculate the likelihood we have to:

- loop over the observed tracks
- for each track...
- find all the \( \phi \)’s on that track using the nearest neighbour model for \( \phi \)
- calculate \( \Sigma \)
- calculate \( \Sigma^{-1} \) (numerically)
- calculate \( D \) (numerically)
- calculate \( \ln f \) for that track
- move on to the next track
- add up the \( \ln f \)’s for all the tracks
A.2 Simulation for AR(1)

To simulate, we have to do the following:

- generate the 1st point on the track from the genesis model (or fix it from observations)
- generate the 2nd point by using the mean track and the variance model
- generate the 3rd and subsequent points using the mean track, the variance model and the memory model

So for the 3rd point onward the simulation works as follows:

- Find the mean track from the mean track model, save that
- Find the variances from the variance model, save that
- Find the $\phi_u$ and $\phi_v$ from the autocorrelation model
- Derive the $s_u$ and $s_v$ from the $\phi$’s using $s^2 = (1 - \phi^2)$
- Apply $u_{n+1} = \phi_u u_n + s_u \epsilon$
- Apply $v_{n+1} = \phi_v v_n + s_v \epsilon$
- Multiply $u_{n+1}$ and $v_{n+1}$ by the appropriate variances from the variance model
- Add on the appropriate mean from the mean track model
- And we have a new point on the track

Note that if we did simplify the model so that $\phi$ is constant in space then this would be much easier since we could just simulate the whole track in one go. But in the general model we don’t know what the $\phi$ along the track will be in advance, so we don’t know the correlation matrix in advance.

B Legal statement

SJ was employed by RMS at the time that this article was written.

However, neither the research behind this article nor the writing of this article were in the course of his employment, (where ‘in the course of their employment’ is within the meaning of the Copyright, Designs and Patents Act 1988, Section 11), nor were they in the course of his normal duties, or in the course of duties falling outside his normal duties but specifically assigned to him (where ‘in the course of his normal duties’ and ‘in the course of duties falling outside his normal duties’ are within the meanings of the Patents Act 1977, Section 39). Furthermore the article does not contain any proprietary information or trade secrets of RMS. As a result, the authors are the owner of all the intellectual property rights (including, but not limited to, copyright, moral rights, design rights and rights to inventions) associated with and arising from this article. The authors reserve all these rights. No-one may reproduce, store or transmit, in any form or by any means, any part of this article without the authors’ prior written permission. The moral rights of the authors have been asserted.

The contents of this article reflect the authors’ personal opinions at the point in time at which this article was submitted for publication. However, by the very nature of ongoing research, they do not necessarily reflect the authors’ current opinions. In addition, they do not necessarily reflect the opinions of the authors’ employers.

References

G Box and G Jenkins. Time series analysis, forecasting and control. Holden-Day, 1970.

M Drayton. A stochastic basin-wide model of Atlantic hurricanes. 2000. 24th Conference on Hurricanes and Tropical Meteorology. http://ams.confex.com/ams/last2000/24Hurricanes/abstracts/12797.htm.

K Emanuel, Ravela S, Vivant E, and Risi C. A combined statistical-deterministic approach of hurricane risk assessment. Unpublished manuscript, 2005.
T Fujii and Y Mitsuta. Synthesis of a stochastic typhoon model and simulation of typhoon winds. Technical report, Kyoto University Disaster Prevention Research Institute, 1975.

T Hall and S Jewson. Statistical modelling of tropical cyclone tracks part 1: a semi-parametric model for the mean trajectory. arXiv:physics/0503231, 2005a.

T Hall and S Jewson. Statistical modelling of tropical cyclone tracks part 2: a comparison of models for the variance of trajectories. arXiv:physics/0505103, 2005b.

B Jarvinen, C Neumann, and M Davis. A tropical cyclone data tape for the North Atlantic Basin, 1886-1983: Contents, limitations, and uses. Technical report, NOAA Technical Memorandum NWS NHC 22, 1984.

P Vickery, P Skerlj, and L Twisdale. Simulation of hurricane risk in the US using an empirical track model. Journal of Structural Engineering, 126:1222–1237, 2000.
Figure 1: Scatter plots that show some of the dependencies between anomalies. Panel (a) shows the lag one dependency structure for along-track anomalies, panel (b) shows the lag one dependency structure for across-track anomalies, panel (c) shows the simultaneous dependency structure between along-track and across-track anomalies and panel (d) shows the lag one dependency structure between along-track and across-track anomalies, with across-track anomalies leading.
Figure 2: The top panel shows a randomly chosen hurricane track expressed in terms of the standardized fluctuations around the unconditional mean track. The solid line shows the along-track fluctuations and the dashed line shows the across track fluctuations. The lower panel shows log-ACFs calculated from these tracks.
Figure 3: Log-ACFs derived from the standard fluctuations from 524 storms over a 54 year period. The solid line corresponds to along-track fluctuations while the dashed line corresponds to across track fluctuations.
Figure 4: Regression coefficients derived by regressing the standardised anomalies onto previous values. The solid line corresponds to along-track fluctuations while the dashed line corresponds to across track fluctuations.
Figure 5: The lag one correlation for along-track and across-track anomalies, converted into a timescale with units of days.
Figure 6: QQ plots comparing the distributions of along-track and across-track anomalies with the normal distribution.
Figure 7: The out-of-sample log-likelihood versus the smoothing length-scale used to estimate the lag one autocorrelation, showing an optimal length-scale of 900 km.
Figure 8: The left panel shows out-of-sample log-likelihood scores for AR(0) (dashed) and AR(1) (solid) models. The right panel shows the difference between them.
Figure 9: Several examples of time series of residual anomalies in the along-track and across-track directions.
Figure 10: Scatter plots that show some of the dependencies between anomaly residuals. Panel (a) shows the lag one dependency structure for along-track residuals, panel (b) shows the lag one dependency structure for across-track residuals, panel (c) shows the simultaneous dependency structure between along-track and across-track residuals and panel (d) shows the lag one dependency structure between along-track and across-track residuals, with across-track residuals leading.
Figure 11: QQ plots comparing the distributions of the residual anomalies with the normal distribution.
Figure 12: For a fixed origin point, the red line shows the unconditional mean track, the blue line shows a real track and the black lines show simulated tracks from an AR(0) model.
Figure 13: As for the previous figure, but now for simulations with an AR(1) model.
Figure 14: Panel (a) shows historical hurricane tracks for a period of 54 years. The other panels show simulated hurricane tracks, starting at the same origin points as the historical tracks, generated using the AR(1) model.
Figure 15: The forward speed in the mean track model, in units of km per 6-hour time step.
Figure 16: Same as figure 14 but for a smaller region.
Figure 17: The number of tracks from observations and from simulations that cross certain lines of latitude (equally spaced from 10N to 50N, from bottom to top), in a northward direction (left panel) and in a southward direction (right panel).
Figure 18: The number of tracks from observations and from simulations that cross certain lines of longitude (equally spaced from 80W to 20W, from left to right), in an eastward direction (panel (a)) and in a westward direction (panel (b)).
Figure 19: Panel (a) shows the distributions of track lengths for model and observations. Panel (b) show the distributions of distances from the start of each track to the end of the track for model and observations.
Figure 20: Track densities for model and observations. Panel (a) shows the track density for the AR(1) model, averaged over 34 realisations of 524 storms. Panel (b) shows the track density for observations, for 524 storms. Panel (c) shows the difference of these densities, and panel (d) shows the standard deviation of the density from the model (across the 34 simulations of 524 storms).