QUANTITATIVE GEOMETRIC CONTROL IN LINEAR KINETIC THEORY

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Abstract. We consider general linear kinetic equations combining transport and a linear collision on the kinetic variable with a spatial weight that can vanish on part of the domain. The considered transport operators include external potential forces and boundary conditions, e.g. specular, diffusive and Maxwell conditions. The considered collision operators include the linear relaxation (scattering) and the Fokker-Planck operators and the boundary conditions include specular, diffusive and Maxwell conditions. We prove quantitative estimates of exponential stabilisation (spectral gap) under a geometric control condition. The argument is new and relies entirely on trajectories and weighted functional inequalities on the divergence operators. The latter functional inequalities are of independent interest and imply quantitatively weighted Stokes and Korn inequalities. We finally show that uniform control conditions are not always necessary for the existence of a spectral gap when the equation is hypoelliptic, and prove weaker control conditions in this case.

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1. Introduction

1.1. Summary. This manuscript is part of a novel recent development in kinetic theory, namely the control theory of hypocoercive structures (with or without hypoellipticity). We consider a general class of linear kinetic equations as prototypes of such structures. They combine a degenerate relaxation (in $v$) with a first-order transport dynamics, and with the addition of a further degeneracy of the relaxation along the spatial variable: this corresponds to a thermalisation degeneracy. A natural question in control theory, that of exponential stabilisation, is to determine conditions upon the thermalising region under which a spectral gap can be obtained.

Our main contribution is to provide a quantitative exponential stability estimate under a general control condition, and for a large class of operators, boundary conditions and potential confinements. Moreover, we obtain quantitative exponential stability for a hypoelliptic equation in a critical case where standard control conditions fail, by exploiting the hypoellipticity. Our results are new for most concrete equations considered, apart from the linear relaxation without boundary conditions. The general approach is also new.

We first present our ideas in an abstract result, see the assumptions in Section 1.5 and the abstract Theorem 1.15. Before the abstract discussion we present informally four key examples in
Section 1.4. The application to some concrete models is proved in Corollary 1.16. The key new feature of our method is to follow trajectories and to use the divergence inequality (i.e. constructing a non-unique inverse to the divergence in $H^1$ with quantitative estimate). Even without thermalisation degeneracy it leads to novel proofs of hypocoercivity and covers cases with boundaries and confining potentials that were not known before. The required divergence inequalities (see Theorem 1.19) are extensions (with weight and boundary conditions) of key results of Bogovskii, Bourgain and Brézis. They are proved in Section 2 and have interest per se.

1.2. The problem at hand. We consider the abstract kinetic equation

$$\begin{align*}
\frac{\partial}{\partial t} f + T f &= \sigma \mathcal{L} f \quad \text{in } \Omega \times V,
\mathcal{R} \gamma_+ f &= \int_{\mathbb{R}^d} \mathcal{R} \gamma_+ f (x,v) R(x,v,v_s) (n \cdot v_s) \, dv_s,
\end{align*}$$

for a time-dependent probability density $f = f(t,x,v) \geq 0$ on the phase space $(x,v) \in \Omega \times V$, where $\Omega$ is a smooth open set of $\mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ and $V \subset \mathbb{R}^d$. The transport operator $T := v \cdot \nabla - \nabla x \phi \cdot \nabla v$ includes an external potential $\phi : \mathbb{R}^d \to \mathbb{R}$; the linear dissipation operator $\mathcal{L}$ acts on $v$ only and models the thermalisation effect due to collisions with a background medium; and the thermalisation degeneracy function $\sigma = \sigma(x) \geq 0$ can vanish on part of $\Omega$.

If $\Omega$ has a boundary, $\partial \Omega$ is assumed to be smooth and we denote $n : \partial \Omega \to S^{d-1}$ the unit outgoing normal vector and $\Gamma_+ := \{ (x,v) \in \partial \Omega \times V : \pm (n \cdot v) \geq 0 \}$. We then denote $\gamma_+ \varphi$ the trace of $\varphi$ at $\pm (n \cdot v) \geq 0$, and $\gamma_- \varphi(v) := \varphi(v - 2(n \cdot v)n)$ on $\Gamma_+$. Then the boundary conditions in (1.1) are defined by the operator (following the spirit of the framework in [23])

$$\mathcal{R} \gamma_+ f (x,v) := \int_{\mathbb{R}^d} (\gamma_+ f) (x,v,v_s) R(x,v,v_s) (n \cdot v_s) \, dv_s,$$

for $(x,v) \in \Gamma_+$, with a measurable kernel $R = R(x,v,v_s) \geq 0$ preserving mass:

$$\forall (x,v_s) \in \partial \Omega \times V \text{ with } (n \cdot v_s) \geq 0, \quad \int_{(n \cdot v) \geq 0} R(x,v,v_s) (n \cdot v) \, dv \equiv 1.$$

The question addressed in this paper is the relaxation to equilibrium $f(t,x,v) \to f_{\infty}(x,v)$ for large time, and more specifically whether the linear evolution (1.1)-(1.2) has a spectral gap.

1.3. Previous results and contribution. When $\sigma$ vanish on part of $\Omega$, [10] proved exponential relaxation by non-constructive methods when $\mathcal{L}$ is the linear Boltzmann operator. $\Omega = \mathbb{T}^d$ is the flat torus, $\phi = 0$, $V$ is bounded, and under a geometric control condition on $\sigma$ inspired from wave equations [8]. The same authors obtained lower bound on the rate of convergence for similar models in [9] when the geometric control condition fails. Later [47] extended this non-constructive result to $\phi \neq 0$ and unbounded velocities $V = \mathbb{R}^d$. Finally the recent preprint [41] gave the first constructive proof, in the same setting, by a probabilistic approach based on Döblin’s theorem. When there is no thermalisation degeneracy ($\sigma = 1$) but $\Omega$ has a boundary, the paper [46] proved exponential relaxation in $L^2$ for the related linearised Boltzmann equation with specular boundary conditions by non-constructive compactness arguments, and the paper [17] proved it for the linearised Boltzmann equation with diffusive boundary conditions by constructive methods. The more recent preprint [11] finally proved it for specular, diffusive and Maxwell boundary conditions by extending the constructive method of [34] to initial-boundary-value problems, which includes solving specific difficulties.

Our method here recovers all these previous results (and others) in a quantitative manner. We propose a novel quantitative approach based on trajectories which, apart from unifying and simplifying previous works, gives the first quantitative estimates when a thermalisation degeneracy is combined with a diffusive operator in velocity, or when a thermalisation degeneracy is combined with boundary conditions. In the terminology of control theory, we prove quantitative and unconditional exponential stabilisation for linear kinetic equations and our estimates readily imply the unique continuation property for the set $\omega := \text{supp } \sigma$. When $\mathcal{L}$ is bounded, they also straightforwardly imply the quantitative observability of this set (see in particular (3.12)); the estimates
established here are however stronger than the latter property. We also initiate the study of how geometric control conditions and hypoellipticity interplay (see Example 1.5).

1.4. Setting and concrete examples. The main theorem handles the abstract equations (1.1)-(1.2) in a general form. Before stating the precise abstract conditions, we discuss the following key examples in kinetic theory to motivate the general theory and help the reader gain intuition.

Take for the velocity either \( v \in V := \mathbb{R}^d \) with the equilibrium measure \( M(v) = (2\pi)^{-d/2}e^{-|v|^2/2} \) or \( v \in \mathbb{S}^{d-1} \) with the equilibrium measure \( M(v) = |\mathbb{S}^{d-1}|^{-1} \). Consider then the collision operator to be the linear Boltzmann operator (also called scattering operator)

\[
\mathcal{L}f(v) := \int_{V} \left[ k(v, v_*) f(v_*) - f(v) k(v_*, v) \right] \, dv_*
\]

with \( 0 \leq k \in C^0(V^2) \) so that \( M \) is the only invariant measure and \( \mathcal{L} \) has a spectral gap in \( L^2(M^{-1}) \) (see Section 4.1 for a discussion of sufficient conditions) or the linear Fokker-Planck operator

\[
\mathcal{L}f(v) := \begin{cases} 
\Delta_{\text{LB}} f & \text{if } V = \mathbb{S}^{d-1} \\
\nabla_v \cdot (\nabla_v f + vf) & \text{if } V = \mathbb{R}^d 
\end{cases}
\]

where \( \Delta_{\text{LB}} \) denotes the Laplace-Beltrami operator on \( \mathbb{S}^{d-1} \).

We now discuss four paradigmatic examples of increasing complexity:

Example 1.1 (Periodic confinement). Assume that \( \Omega = \mathbb{T}^d \) and let

\[
(x_0, v_0) \mapsto (X_t(x_0, v_0), V_t(x_0, v_0))
\]

be the characteristic map starting from \( (x_0, v_0) \) associated to the transport \( \mathcal{T} \). Suppose there exists a good set \( \Sigma \) so that \( \sigma \geq 1_\Sigma \) and \( \chi \in C^\infty(\Omega) \), \( c, T > 0 \) with \( \text{supp} \, \chi \subset \Sigma \) so that

\[
\forall (x_0, v_0) \in \Omega \times V, \quad \int_0^T \chi(X_t(x_0, v_0)) \, dt \geq c.
\]

Then our method yields exponential convergence with quantitative estimate.

In other words, we prove exponential convergence if there exists a set \( \Sigma \) in which thermalisation occurs and which is exposed to all configuration by the transport flow, i.e. for any initial point the trajectory is thermalised a strictly positive amount in the set \( \Sigma \) over some time interval \([0, T]\) (heuristically we can think of \( \chi \) as \( 1_\Sigma \), omitting technical regularity requirements on \( \chi \)). For (nearly) massless particles like neutrons, the velocities can be modelled by \( V = \mathbb{S}^{d-1} \) and without external potential \( (\phi \equiv 0) \) the condition (1.6) reduces to whether straight lines hit the good set \( \Sigma \), see Fig. 1.

Example 1.2 (Boundaries and potential). In order to cover boundaries, we face two problems: (i) in the case of non-convex geometries the evolution may create discontinuities and (ii) with diffusive boundary conditions the transport flow \( \mathcal{S}_t^{\text{trans}} \) associated with the transport operator \( \mathcal{T} \) and the boundary conditions \( \mathcal{R} \) ceases to admit deterministic characteristic trajectories. To overcome these issues, we rewrite (1.6) as

\[
\forall (x_0, v_0) \in \Omega \times V, \quad \int_{0}^{T} \int_{\Omega \times V} (\mathcal{S}_t^{\text{trans}} \delta_{(x_0, v_0)})(x, v) \chi(x, v) \, dx \, dv \, dt \geq c > 0
\]

and our method applies, once characteristic flow are adequately treated more abstractly, as long as the operator \( \mathcal{L} \) is bounded and the transport flow propagates enough regularity.

To overcome the regularity issue when the latter is not satisfied, we introduce the full semigroup \( \mathcal{S}_t^{\text{full}} \) associated to (1.1). In the stochastic interpretation of the collision operator, we have a positive probability that the particle has undergone no collision (in the linear Boltzmann setting (1.4)) or the change of velocity has been arbitrary small (in the Fokker-Planck setting (1.5)). Hence, one expects in general that (1.7) implies the same condition with \( \mathcal{S}_t^{\text{trans}} \) replaced by \( \mathcal{S}_t^{\text{full}} \). Using \( \mathcal{S}_t^{\text{full}} \), we can replace the regularity condition by a \( \Gamma \) condition on the collision operator which we can justify for the Fokker-Planck operator and all linear Boltzmann operators satisfying reversibility.
Figure 1. Illustration of (1.6) for a periodic domain with $V = S^{d-1}$ (massless particles) and no external potential $\phi \equiv 0$. On the left, all lines hit and spend a controlled fraction of time in the thermalisation set $\Sigma = \text{supp} \sigma$ on a time interval $[0,T]$, yielding exponential convergence. On the right, there is a set of configurations with zero measure whose trajectories never hit $\Sigma$, and around this set the time to hit $\Sigma$ can be arbitrarily large: the control condition is not satisfied, and one typically expects polynomial rate of convergence.

As for the boundary conditions, we impose a compatibility condition which includes all standard cases. In particular, it holds for the specular, diffusive and Maxwell boundary conditions, whose kernel in (1.2) is

$$R(x,v,v_*) := (1 - \alpha(x)) \frac{\delta_{v=v_*}}{(n \cdot v_*)} + \alpha(x) \sqrt{2\pi} M(v)$$

with an accommodation coefficient $\alpha : \partial \Omega \to [0,1]$. Specular boundary conditions correspond to $\alpha = 0$, while diffusive boundary conditions correspond to $\alpha = 1$.

Then (1.1) converges exponentially if there exists a set $\Sigma \subset \Omega$ with $\sigma \gtrsim 1$ such that a weighted Poincaré inequality with potential $\phi$ holds on $\Sigma$ and the set $\Sigma$ is exposed to all initial data in the following sense: there exists $\chi \in L^\infty(\Omega)$ with $\text{supp} \chi \subset \Sigma$ such that

$$\forall (x_0,v_0) \in \Omega \times V, \quad \int_0^T \int_{\Omega \times V} (S_t^{\text{full}}(\delta_{(x_0,v_0)}))(x,v) \chi(x,v) \, dx \, dv \, dt \geq c.$$  

As an illustration, consider the spherical vessel $\Omega = B(0,1)$ in dimension 3 with diffusive boundary conditions and $\sigma = 1_{B(0,1/2)}$ in Fig. 2. In such case, it is proved in [3] (with optimal polynomial rate) that the transport flow with diffusive boundary conditions starting from initial data in $L^\infty(\Omega \times V; \, dx \, d\mu)$ relaxes polynomially towards equilibrium in $L^2(\Omega \times V; \, dx \, d\mu)$. Our method then yields exponential convergence in spite of the fact that the uniform geometric control condition would fail for the specular transport flow; in other words our theorem genuinely uses the dissipativity at the boundary.

Remark 1.3. Note that although we only require the Poincaré inequality on the potentially smaller set $\Sigma$, and not $\Omega$, the condition (1.9) requires that the whole domain $\Omega$ be connected in finite time to $\Sigma$.

Example 1.4 (Harmonic potential with thermalisation only around zero). Consider the harmonic potential $\phi(x) = x^2/2$ in one dimension $\Omega = \mathbb{R}$ and a thermal degeneracy given by $\sigma = 1_{|x| \leq 1}$. The transport control condition (1.6) then fails for $(x_0,v_0)$ with $r_0 = \sqrt{x_0^2 + v_0^2}$ large since the time the trajectories spend in $\{|x| \leq 1\}$ is proportional to $r_0^{-1}$. Since the speed with which such bad trajectories cross the good region $\text{supp} \sigma$ is high, one can however recover exponential convergence if the local coercivity estimate of the collision operator gains a weight $|v|$ at large velocities, see Fig. 3. For instance (1.1) converges exponentially when $L$ is given Fokker-Planck operator over $V = \mathbb{R}$. 


Example 1.5 (Hypoellipticity with control condition failing at a point). In the case of the linear Boltzmann equation with deterministic transport flow, the geometric control condition (1.7) is not only sufficient but necessary to the exponential relaxation, see [47]: this follows from considering initial data whose supports concentrate on trajectories where the uniform geometric control condition (1.7) fails. However, for regularising collision operators like the Fokker-Planck operator (1.5), it can be relaxed to some extent, as first noticed in [32]. The argument from [47] then fails because of hypoellipticity. Let us consider the following prototypical example, described in Fig. 4.
Consider $\Omega = V = \mathbb{R}$ in dimension $d = 1$, with $\mathcal{L}$ given by the Fokker-Planck operator (1.5), with the harmonic potential $\phi(x) = x^2/2$ and with the thermalisation degeneracy $\sigma(x) = \min(x^2, 1)$ around $x = 0$. Around $x = 0$, the spatial degeneracy $\sigma(x)$ behaves as $x^2$ and the characteristics describe circle with period $2\pi$ around $(x, v) = (0, v)$, so that the geometric control conditions (1.7) and (1.9) both fail around $(0, v)$. However, thermalisation is effective apart from the latter point, and the stationary probability measure $e^{-\phi(x)}M(v)$ is still unique. We then prove the existence of a spectral gap thanks to corrector $C$ in the control condition, and (1.1) relaxes exponentially.

1.5. Abstract assumptions.

**Hypothesis 1** (Geometric constraints). The domain $\Omega$ is either a smooth and open subset of $\mathbb{R}^d$, or $\Omega = \mathbb{T}^d$, and the velocity space $V \subset \mathbb{R}^d$ is even (−$V = V$) and spans $\mathbb{R}^d$. The thermalisation degeneracy function $\sigma$ is non-negative and bounded, and the potential $\phi \in C^2(\Omega)$ with $\int_{\Omega} e^{-\phi} = 1$.

**Remark 1.6.** There are no further assumptions on the velocity space $V$. In particular, a discrete velocity space $V$ is possible, as used in lattice Boltzmann methods and in simplified models. The assumption that $V$ is even is only necessary for handling boundary data. Otherwise, it suffices to have two bases of $\mathbb{R}^d$ in $V$ which differ by at least one element, in order to be able to construct the functions $\varphi_0, \varphi_1, \ldots, \varphi_d$ in Section 3.5. Note also that the boundedness of $\sigma$ on $\Omega$ can be relaxed to boundedness merely on the control set $\Sigma$ introduced below in (H5).

**Hypothesis 2** (Equilibria and semigroup). There is a positive $M \in L^1(V)$ with mass 1 so that $f_\infty(x, v) := e^{-\phi(x)}M(v)$ achieves equilibrium: $\mathcal{L}f_\infty = T f_\infty = 0$ and $R\gamma_+ f_\infty = \gamma_+ f_\infty$ on $\Gamma_+$. 

![Figure 4. The prototypical example of hypoellipticity overcoming some level of degeneracy in the control condition.](image-url)
There are no further assumptions on the profile $M$, in particular non-Gaussian equilibria are possible as long as the equilibrium condition is satisfied separately by $T$ and $L$.

**Hypothesis 3** (Microscopic coercivity). $L$ and $L^*$ are real closable operator with dense domains including $C^0_0(V)$ in $L^2_t(M^{-1})$, with Range($L$) $\perp 1$ (preservation of mass), and so that $L + L^*$ has a spectral gap in the velocity space with a weight $w \in L^1_t(M)$ satisfying $w \geq 1$, i.e. there is $\lambda_1 > 0$ such that for any real-valued $g \in \text{Domain}(L)$

$$
\int_V g(v) L g(v) \frac{dv}{M(v)} \leq -\lambda_1 \int_V \left[ g(v) - \left( \int_V g(v) \frac{dv}{M(v)} \right) M(v) \right]^2 w(v) \frac{dv}{M(v)}. \tag{1.10}
$$

**Remark 1.7.** Whether the previous spectral gap condition holds for the operator $L$ is a standard question discussed in Section 4.1. Note that $L$ and $w$ could depend on $x$, provided conditions and estimates on them are made uniform in $x$, i.e. degeneracy is entirely captured by $\sigma$. The theory could also cover cases with a degenerate spectral gap, i.e. $w$ goes to zero towards infinity, which is then compensated by a suitable growth in $\sigma$.

**Hypothesis 4** (Boundary compatibility). The operator $R$ is real and bounded in $L^2(\Gamma_+, d\nu)$ and is contractive. There is $C_r > 0$ such that for any $g \in L^2(\Gamma_+, d\nu)$ and $\varphi \in L^\infty(\Gamma_+)$ it holds that

$$
\int_{\Gamma_+} \varphi \left[ f_{\infty} R (f_{\infty}^{-1} g^2) - (R g)^2 \right] d\nu \leq C_r \|\varphi\|_{L^\infty(\Gamma_+)} \int_{\Gamma_+} \left[ g^2 - (R g)^2 \right] d\nu \tag{1.11}
$$

where we denote by $dS$ the surface measure of $\partial \Omega$ and $d\nu := (n \cdot v) f_{\infty}^{-1}(x,v) dS dv$ on $\Gamma_+$.

**Remark 1.8.** The condition (1.11) means that the boundary terms coming from estimates of the squared solution with $L^\infty$ test function can be controlled by the entropy production at the boundary. It is satisfied in all standard cases, as discussed in Section 4.2, and in particular $R$ correspond to Maxwell boundary conditions (1.8) with an arbitrary accommodation coefficient $\alpha : \partial \Omega \to [0,1]$. Note also that the left hand side of (1.11) rewrites $\int_{\Gamma_+} [R^T(\varphi) g^2 - \varphi R(g)^2] d\nu$ with $R^T \varphi := f_{\infty}^{-1} R^* (f_{\infty} \varphi)$ (the $L^2$ adjoint), and that with such reformulation, $R^T$ could be replaced, in this assumption and the next one, by another operator $R'$ provided it satisfies, for any $g \in L^2(\Gamma_+, d\nu)$ and $\varphi \in L^\infty(\Gamma_+)$,

$$
\begin{cases}
\int_{\Gamma_+} \left[ R'(\varphi) g^2 - \varphi (R g)^2 \right] d\nu \leq C_r \|\varphi\|_{L^\infty(\Gamma_+)} \int_{\Gamma_+} \left[ g^2 - (R g)^2 \right] d\nu \\
\int_{\Gamma_+} \left[ R'(\varphi) g f_{\infty} g - \varphi f_{\infty} R g \right] d\nu \leq C_r \|\varphi\|_{L^\infty(\Gamma_+)} \left( \int_{\Gamma_+} \left[ g^2 - (R g)^2 \right] d\nu \right)^{\frac{1}{2}} \tag{1.12}
\end{cases}
$$

**Hypothesis 5** (Transport control condition – abstract). Let $B$ be either zero or $\sigma L^T$ where $\sigma L^T \eta := \sigma f_{\infty}^{-1} L^*(f_{\infty} \eta)$. There is a connected domain $\Sigma \subset \Omega$ so that $\inf_{x \in \Sigma} \sigma(x) > 0$, and there is $0 \leq \chi \in L^\infty(\Omega \times V)$ with $\text{supp} \chi \subset \Sigma \times V$ and an operator $C$ in $\sigma$ so that the solution $\varphi$ to the evolution problem

$$
\begin{cases}
\partial_t \varphi - T \varphi - B \varphi = -C \varphi - \chi \varphi & \text{in } \Omega \times V, \\
\gamma_+ \varphi = R^T \gamma_- \varphi & \text{on } \Gamma_+ \\
\varphi|_{t=0} = \varphi_0 = 1 & \text{on } \{t = 0\} \tag{1.13}
\end{cases}
$$

is bounded and converges to zero in $L^\infty$, and so that the operators $B$ and $C$ satisfy (for smooth $f$)

$$
\begin{align}
\int_{\Omega \times V} [B^*(f^2) - 2 f \sigma L f] (1 - \varphi_t) d\mu & \lesssim \|f\|_{L^2(\Omega \times V, d\mu)} \int_{\Omega \times V} \sigma f L f d\mu \frac{1}{2} + \int_{\Omega \times V} \sigma f L f d\mu, \\
\int_{\Omega \times V} [B^* (f f_{\infty}) - \sigma L f] (1 - \varphi) d\mu & \lesssim \|f\|_{L^2(\Omega \times V, d\mu)} \int_{\Omega \times V} \sigma f L f d\mu \frac{1}{2}, \tag{1.14}
\end{align}
$$

where $\sigma \in C(\Sigma)$, $\sigma \geq \lambda_2 > 0$, and $\sigma = \mu_\sigma = 0$ on $\Gamma_+ \setminus \Sigma$.
Remark 1.11. The assumption that $\Sigma$ is connected could be relaxed: our method works provided the different connected components of $\Sigma$ are connected through the transport semigroup. For a simple setting, this is discussed in [33].

\[ \left\{ \begin{array}{l}
\int_{\Omega \times V} \mathcal{E}^*(f^2) \varphi_t \, d\mu \lesssim \|f\|_{L^2(\Omega \times V, d\mu)} \left( \int_{\Omega \times V} \sigma f L f \, d\mu \right)^{1/2} + \int_{\Omega \times V} \sigma f L f \, d\mu,
\end{array} \right. \tag{1.15} \]

where we denote $d\mu := f_\infty(x, v)^{-1} \, dx \, dv$.

Remark 1.9. In order to capture many cases, and in particular Example 1.5 in the hypoelliptic case, (H5) is formulated generally but we provide below easier conditions for the main practical cases apart from Example 1.5. In practice, note that the operator $\mathcal{E}$ is only used in order to overcome degeneracy for hypoelliptic equations and otherwise one can take $\mathcal{E} = 0$. Regarding the operator $\mathcal{B}$, one can take $\mathcal{B} = 0$ if regularity is propagated by the transport semigroup or if $\mathcal{L}$ is bounded. Otherwise we can drop the latter assumption by taking $\mathcal{B} = \sigma \mathcal{L}$, at the price of a $\Gamma$ condition (see below). Note that we could allow in (H5) a more general $\mathcal{B}$ provided that $\mathcal{T} + \mathcal{B}$ combined with the boundary conditions associated with $\mathcal{R}^T$ (or more generally $\mathcal{R}'$) generates a contractive semigroup in $L^\infty$ admitting 1 as equilibrium, and whose adjoint admits $f_\infty^2$ as equilibrium.

Hypothesis 5' (Transport control condition – practical). Assume the weight $w = 1$ in (H3). Then (H5) is satisfied in the following cases:

Case 1. Consider the transport semigroup $\mathcal{S}_t^{\text{trans}}$ created by $\mathcal{T}$ and $\mathcal{R}$ and assume that the collision operator $\mathcal{L}$ is bounded. Then assume a connected domain $\Sigma \subset \Omega$ so that $\inf_{x \in \Sigma} \sigma(x) > 0$ and there is a non-negative $\chi \in L^\infty(\Omega \times V)$ with $\text{supp} \chi \subset \Sigma \times V$, and a time $T > 0$ and constant $c > 0$ such that

\[ \forall (x_0, v_0) \in \Omega \times V \quad \int_0^T \int_{\Sigma \times V} (\mathcal{S}_t^{\text{trans}} \delta_{x_0, v_0}) \chi(x) \, dx \, dv \, dt \geq c. \]

Case 1'. Consider the transport semigroup $\mathcal{S}_t^{\text{trans}}$ created by $\mathcal{T}$ and $\mathcal{R}$ and assume $\mathcal{S}_t^{\text{trans}}$ propagates regularity in $W^{1,\infty}$ and that the collision operator $\mathcal{L}$ is the Fokker-Planck operator. Then assume a connected domain $\Sigma \subset \Omega$ so that $\inf_{x \in \Sigma} \sigma(x) > 0$ and there is a non-negative $\chi \in W^{1,\infty}(\Omega \times V)$ with $\text{supp} \chi \subset \Sigma \times V$, and a time $T > 0$ and constant $c > 0$ such that

\[ \forall (x_0, v_0) \in \Omega \times V \quad \int_0^T \int_{\Sigma \times V} (\mathcal{S}_t^{\text{trans}} \delta_{x_0, v_0}) \chi(x) \, dx \, dv \, dt \geq c. \]

Case 2. Assume (1.1) generates a bounded semigroup $\mathcal{S}_t^{\text{full}}$ in $L^1(\Omega \times V)$ and $\mathcal{L}$ satisfies the condition

\[ f_\infty \mathcal{L} \left( \frac{f^2}{f_\infty} \right) - 2f \mathcal{L} f \geq 0. \tag{1.16} \]

Then assume a connected domain $\Sigma \subset \Omega$ so that $\inf_{x \in \Sigma} \sigma(x) > 0$, and there is a non-negative $\chi \in L^\infty(\Omega \times V)$ with $\text{supp} \chi \subset \Sigma \times V$ and a time $T > 0$ and constant $c > 0$ such that

\[ \forall (x_0, v_0) \in \Omega \times V \quad \int_0^T \int_{\Sigma \times V} (\mathcal{S}_t^{\text{full}} \delta_{x_0, v_0}) \chi(x) \, dx \, dv \, dt \geq c. \tag{1.17} \]

The condition (1.16) corresponds to the positivity of the $\Gamma$ function in the Bakry-Émery calculus, that follows from reversibility (see Subsection 4.5):

Proposition 1.10 (\Gamma condition). Assume that $\mathcal{L}$ is either the Fokker-Planck operator or the linear Boltzmann operator with a reversible kernel, then $\mathcal{L}$ satisfies (1.16).

Remark 1.11. The assumption that $\Sigma$ is connected could be relaxed: our method works provided the different connected components of $\Sigma$ are connected through the transport semigroup. For a simple setting, this is discussed in [33].
Figure 5. Representation of a regular-domain-potential pair \((U, \Phi)\).

For the functional inequalities capturing the macroscopic coercivity on \(\Sigma\), we introduce a notion of regularity for domain-potential pairs, represented in Fig. 5:

**Definition 1.12.** Given \(\epsilon > 0\), a domain \(U \subset \mathbb{R}^n\), \(n \geq 1\), and a potential \(\Phi \in C^2(U)\), we say that this domain-potential pair \((U, \Phi)\) satisfies the \(\epsilon\)-regularity condition if \(|\nabla^2 \Phi| \leq 1 + |\nabla \Phi|\) and for every boundary point \(x \in \partial U\) there exists an isometry \(T : \mathbb{R}^n \to \mathbb{R}^n\) with \(T(x) = 0\) and \(f \in C^1(\mathbb{R}^{n-1})\) with \(\|\nabla f\| \leq 1/8\) such that, denoting \(B(x, r)\) the ball of radius \(r\) around \(x\) and \(|\nabla \Phi| := (1 + |\nabla \Phi|^2)^{1/2}\),
\[
T\left(B(x, \epsilon |\nabla \Phi|^{-1}) \cap U\right) = T\left(B(x, \epsilon |\nabla \Phi|^{-1})\right) \cap \{\xi' \in \mathbb{R}^{n-1} \times \mathbb{R} : \xi_n > f(\xi')\}.
\]

**Remark 1.13.** When the boundary \(\partial U\) is bounded and at least \(C^1\) this is automatically satisfied.

**Hypothesis 6** (Macroscopic coercivity). Given the set \(\Sigma \subset \Omega\) from (H5), the pair \((\Sigma, \phi)\) is \(\epsilon\)-regular for some \(\epsilon > 0\), and the potential \(\phi\) satisfies a Poincaré-Wirtinger inequality on \(\Sigma\), i.e. there is \(\lambda_2 > 0\) so that
\[
\forall \rho \in H^1(\Sigma; e^{\phi}), \quad \int_\Sigma |\nabla_x \rho + \rho \nabla_x \phi|^2 e^\phi \, dx \geq \lambda_2 \int_\Sigma \rho - \left(\int_\Sigma \rho\right) e^{-\phi} \|\nabla \phi\|^2 e^\phi \, dx. \tag{1.18}
\]

**Remark 1.14.** Such inequality, with or without the additional weight \(|\nabla \phi|^2\) on the right hand side of (1.18), is a classical concentration estimate. We discuss sufficient conditions in Section 2.1.

### 1.6. Main results.

For considering the evolution problem (1.1) we need a notion of solution that admits suitable traces. This is provided in Appendix B: given initial data \(f_{in} \in L^2(\Omega \times \mathcal{V}, \, d\mu)\), there is a unique solution \(f = f(t, x, v) \in C^0([0, +\infty); L^2(\Omega \times \mathcal{V}, \, d\mu))\) to (1.1) that admits traces \(\gamma f \in L^2([0, +\infty) \times \partial \Omega; \mathcal{H}(\mathcal{V}, (n \cdot v)^2 \, d\mu))\) for the Fokker-Plank operator. Under the abstract condition of the previous subsection, we establish a spectral gap in the following theorem; however note that when \(\mathcal{L}\) is given by the Fokker-Plank operator (and is unbounded) and one uses the full semigroup \(S_t^{full}\) for the control condition, then to fully justify the a priori estimates of the proof, it would be necessary to approximate the operator by bounded operators, so that the trilinear boundary terms are all well-defined (note that integrals involved are however perfectly controlled from above by the a priori estimates).

**Theorem 1.15** (Quantitative relaxation to equilibrium). Assume (H1)-(H2)-(H3)-(H4)-(H5)-(H6). There are \(C > 1\), \(\Lambda > 0\) such that given any \(f_{in} \in L^2(\Omega \times \mathcal{V}, \, d\mu)\), any \(f = f(t, x, v) \in C^0([0, +\infty); L^2(\Omega \times \mathcal{V}, \, d\mu))\) admitting traces \(\gamma f \in L^2([0, +\infty) \times \partial \Omega; H^{-1}(\mathcal{V}, (n \cdot v)^2 \, d\mu))\) and solution to (1.1) satisfies
\[
\left\|f_t - \left(\int_{\Omega \times \mathcal{V}} f_{in} \right) f_{in}\right\|_{L^2(\Omega \times \mathcal{V}, \, d\mu)} \leq C e^{-\Lambda t} \left\|f_{in} - \left(\int_{\Omega \times \mathcal{V}} f_{in} \right) f_{in}\right\|_{L^2(\Omega \times \mathcal{V}, \, d\mu)}. \tag{1.19}
\]

The constants \(C\) and \(\Lambda\) can be computed from the constants in the assumptions.

This covers the following common cases.

Curvature does not grow faster than \(|\nabla\Phi|\) as \(|x| \to \infty|\nabla\Phi|\).
Corollary 1.16 (Application to concrete equations). We list the possible settings:

Regarding the geometric constraints:
- The spatial domain is either $\Omega = \mathbb{T}^d$ or a smooth open subset $\Omega \subset \mathbb{R}^n$.
- The velocity space $V$ and potential $\phi$ are either $(\mathbb{S}^{d-1}, 0)$ or $(\mathbb{R}^d, \phi)$ with $\phi \in C^2(\Omega)$.
- There is a connected open subset $\Sigma \subset \Omega$ so that $(\Sigma, \phi)$ is $\epsilon$-regular for some $\epsilon > 0$, and $\sigma \geq 1$ and $|\nabla \phi| - \Delta \phi \sim_{\infty} 0$ on $\Sigma$ and $n \cdot \nabla \phi \geq 0$ on $\partial \Sigma$.

Regarding the collision process:
- The operator $\mathcal{L}$ is the linear Boltzmann operator (1.4) or the Fokker-Planck operator (1.5).
- There is unique equilibrium measure $M = M(v)$ for $\mathcal{L}$.
- The operator $\mathcal{L}$ is symmetric non-negative and satisfies a weighted spectral gap inequality (1.10) with weight $1 \leq w \in L^1(M)$.

Regarding the transport control conditions:
- When the transport flow is deterministic, e.g. when there is no boundaries or for specular or bounce-back boundary conditions in (1.2), there are characteristics $(X_t(x,v), V_t(x,v))$ of $\mathcal{S}_t^{\text{trans}}$. Assume that there is $\chi \in C_\infty(\Sigma)$ and $T > 0$ such that
  \[ \forall (x,v) \in \Omega \times V, \quad \int_0^T \chi(X_t(x,v)) w(V_t(x,v)) \, dt \geq 1. \tag{1.20} \]
  Additionally, assume that $\mathcal{L}$ is bounded or that $\mathcal{L}$ is the Fokker-Planck operator, $\mathcal{S}_t^{\text{trans}}$ propagates $W^{1,\infty}$ regularity and the weight $w$ is such that $|\nabla w| \lesssim w$ and
  \[ \int_0^T 1_{\text{supp} \chi}(X_t(x,v)) w(V_t(x,v)) \, dt \lesssim 1. \tag{1.21} \]
- Assume general Maxwell conditions (1.8) with $\alpha : \partial \Omega \to [0, 1]$. Assume that $\mathcal{L}$ is a scattering operator satisfying detailed-balance (including Fokker-Planck). Then assume that there is $\chi \in C_\infty(\Sigma)$, $c > 0$ and $T > 0$ such that
  \[ \forall (x_0,v_0) \in \Omega \times V, \quad \int_0^T \int_{\Sigma \times V} \left( \mathcal{S}_t^{\text{full}} \delta_{x_0,v_0} \right) \chi(x) \, dx \, dv \, dt \geq c. \tag{1.22} \]

Then Theorem 1.15 applies and implies exponential relaxation to equilibrium for solutions in $L^2_{x,v}(\mu)$ with quantitative estimate on the rate.

Remark 1.17. The condition (1.20) for a deterministic flow covers Example 1.4 with the weight $w(v) = \sqrt{1 + |v|^2}$. For this application, we only use the necessary gain from the weight so that we also have the upper bound (1.21).

The proof of the abstract Theorem 1.15 is given in Section 3 but makes use of quantitative divergence inequalities proved in Section 2. The proof of Corollary 1.16 is given in Section 4.

The trajectorial approach can also be used to show convergence without a rate for a non-uniform control condition, which is discussed in Appendix C.

1.7. Geometric control conditions and hypoellipticity. We now come back to Example 1.5 in which we relax the control condition when the collision operator $\mathcal{L}$ is regularising, i.e. (1.1) is hypoelliptic. To achieve such a relaxation, we use the additional corrector operator $\mathcal{C}$ (H5). In the case of the Fokker-Planck operator with the Hörmander-type structure $\sigma \mathcal{L} = -A^* A$, a natural choice is $\mathcal{C} = f_{\infty}^{-1} A^*(f_{\infty} a \cdot)$ for a well-chosen bounded function $a \in L^\infty(\Omega \times V)$ that satisfies (1.15) and ensures the decay of the solution to (1.13). In the simple but illuminating Example 1.5, we deduce the existence of a spectral gap with quantitative estimates, in spite of the fact that the transport control condition is not satisfied (and neither are more standard uniform geometric control conditions); moreover the corresponding stochastic process suggests that the quadratic degeneracy of $\sigma$ is critical for the existence of a spectral gap. Note that we also give an alternative proof using the commutator method developed in [63], see Appendix A.
Theorem 1.18. Consider $\Omega = \mathbb{V} = \mathbb{R}$, the potential $\phi(x) = x^2/2$, a thermalisation degeneracy function $\sigma \in L^\infty$ that satisfies $\sigma(x) \gtrsim \min(1, x^2)$, and the Fokker-Planck collision operator (1.5). Then the semigroup is relaxing exponentially to equilibrium.

1.8. Functional inequalities. In the good set $\Sigma$ of (H5), we perform a standard micro-macro decomposition and note that the dissipation gives some control on the macroscopic density. This was already implicitly understood in [50] and more recently formulated in the kinetic setting in [2]. This approach has also been used to show hypocoercivity in much more restricted cases [21, 18].

A key ingredient is the use of a suitable divergence inequality (also known as Bogovskiĭ operator to invert the divergence), that we extend to cases combining weight and boundaries. This is of independent interest, and it is proved in Section 2.

Theorem 1.19 (Divergence, Poincaré-Lions, Korn and Stokes inequalities). Consider $n \geq 1$ and $U \subset \mathbb{R}^n$ open and consider a $C^2$ potential $\Phi : U \to \mathbb{R}$ so that $(U, \Phi)$ is $\epsilon$-regular for some $\epsilon > 0$, and $e^{-\Phi(x)}\, dz$ satisfies the Poincaré-Wirtinger inequality on $U$:

$$
\forall \rho \in H^1(U; e^\Phi) \text{ with } \int_U \rho = 0, \quad \int_U |\nabla \rho + \rho \nabla \Phi|^2 e^\Phi \, dz \geq \int_U \rho^2 |\nabla \Phi|^2 e^\Phi \, dz.
$$

Then the following inequalities hold with quantitative estimates:

1. **Divergence inequality.** There is $C_D > 0$ and a linear map $D$ mapping any $g \in L^2(U; e^\Phi)$ with $\int_U g = 0$ to a $F : U \to \mathbb{R}^n$ in $H^1(U; e^\Phi)$ that satisfies

$$
\begin{cases}
\nabla \cdot F = g \text{ in } U, \\
F = 0 \text{ on } \partial U,
\end{cases}
$$

(1.23)

where $(H^1(U; e^\Phi))'$ is the standard dual space.

2. **Poincaré-Lions inequality.** There is $C_{PL} > 0$ so that for any $h \in L^2(U; e^\Phi)$ one has

$$
\left\| h - \left( \frac{1}{U} \int_U h \right) e^{-\Phi} \right\|_{L^2(U; e^\Phi)} \leq C_{PL} \left\| \nabla h + h \nabla \Phi \right\|_{(L^1(U; e^\Phi))^\prime},
$$

(1.24)

satisfies

$$
\left\| u - \left( \frac{1}{U} \int_U u \right) e^{-\Phi} \right\|_{L^2(U; \nabla \Phi^2; e^\Phi)} + \left\| (\nabla + \nabla \Phi) u \right\|_{L^2(U; e^\Phi)} + \left\| p \right\|_{L^2(U; e^\Phi)} \leq C_{KL} \left\| s \right\|_{L^2(U; e^\Phi)}.
$$

(1.25)

3. **Stokes inequality.** There is $C_S > 0$ so that for any $s : U \to \mathbb{R}^n$ in $L^2(U; e^\Phi)$ with $\int_U s = 0$, the unique solution $(u, p) \in H^1(U; e^\Phi) \times L^2(U; e^\Phi)$ with $\int_U p = 0$ to

$$
\begin{cases}
\nabla \cdot u = 0 \text{ in } U, \\
u = 0 \text{ on } \partial U,
\end{cases}
$$

(1.26)

satisfies

$$
\begin{cases}
\nabla \cdot (\nabla + \nabla \Phi) u + (\nabla + \nabla \Phi) p = s \text{ in } U, \\
u = 0 \text{ on } \partial U,
\end{cases}
$$

(1.27)

satisfies

$$
\left\| (\nabla + \nabla \Phi) u \right\|_{L^2(U; e^\Phi)} \leq C_{KL} \left\| (\nabla + \nabla \Phi) \text{sym} u \right\|_{L^2(U; e^\Phi)}.
$$

(1.28)

with

$$
\left\| (\nabla + \nabla \Phi) \text{sym} u \right\|_{L^2(U; e^\Phi)}^2 := \sum_{i,j=1,...,n} \left\| \left( \partial_{z_i} + \partial_{z_j} \Phi \right) u_j + \frac{1}{2} \left( \partial_{z_i} + \partial_{z_j} \Phi \right) u_i \right\|_{L^2(U; e^\Phi)}^2.
$$
Remark 1.20. Without the assumption (1.27), the estimate (1.28) holds if the boundary prohibits all rotations and drifts. Let us illustrate this with the case of non-penetration boundary condition \( \mathbf{n} \cdot \mathbf{u} = 0 \) on \( \partial U \). Let \( p = \int_\partial U z e^{-\Phi} \in \mathbb{R}^n \) be the weighted centre and for \( 1 \leq i < j \leq n \) let \( E^{ij} = e^i \otimes e^j - e^j \otimes e^i \), where \( e^1, \ldots, e^n \) is an orthonormal basis of \( \mathbb{R}^n \). Then suppose that there are scalar functions \( \chi^i, \chi^{ij} \in C_c^1(\overline{U}) \), \( 1 \leq i \neq j \leq n \), such that for \( 1 \leq k \neq l \leq n \),

\[
\begin{align*}
\int_{\partial U} \chi^i (\mathbf{n} \cdot e^k) &= \delta_{i=k}, \\
\int_{\partial U} \chi^{ij} (\mathbf{n} \cdot e^k) &= 0, \\
\int_{\partial U} \chi^{ij} (\mathbf{n} \cdot e^k) &= 0, \\
\int_{\partial U} \chi^{ij} (\mathbf{n} \cdot (E^{kl}(z - p))) &= \delta_{(i,j)=(k,l)}.
\end{align*}
\]

Then (1.28) holds. When (1.29) is not imposed, the symmetric gradient has a non-trivial kernel. This kernel can, e.g., be controlled by \( \|\mathbf{u} \cdot \nabla \Phi\| \), as in [22], if \( \Phi \) has no rotation symmetry, which can be quantified by the constant

\[
\sup_{J \in \mathcal{E}} \left| \int_0^1 J(z) \nabla e^{-\Phi} \right| \|J\|
\]

where the supremum is taken over the set of affine functions

\[
\mathcal{E} := \left\{ J(z) = \sum_{i,j} b_{ij} E^{ij}(z - p) + \sum_i b_i e^i, \quad b_i, b_{ij} \in \mathbb{R}, \quad \text{compatible with the boundary} \right\}.
\]

Remark 1.21. The above form of the divergence inequality seems more general than the existing literature, due to the addition of the potential force and also its combination with a boundary. We refer to [13, 14, 37, 43, 1, 28, 38, 16, 39, 15] among an important literature. What is called Poincaré-Lions inequality above is used in [40] and mentioned in [30] and we introduced the terminology in [22]. The Stokes equation is a classical equation of fluid dynamics, see [42, 48] for an overview and [44] for some extension to unbounded domains without potential. The Korn inequality was discovered in [51, 52, 53] in the case of bounded domain with Dirichlet conditions (see also [49] for a somehow recent review), and extended to non-penetration conditions in [30], and to the whole space with confining potential in [36] (non-constructive argument) and [22] (constructive argument). Our statement includes these previous works and extend them. We deduce the Korn inequality from the Poincaré-Lions inequality arguing as in [22], however the Poincaré-Lions inequality is proved in new cases and by a novel method.

2. Weighted divergence and related inequalities

We prove Theorem 1.19 in this section. Let us denote \( \nabla \Phi := \nabla + \nabla \Phi \).

2.1. The Poincaré inequality. We first extend the standard Poincaré-Wirtinger inequality. In the case \( U = \mathbb{R}^n \), standard arguments, see for instance [31, Proof of Theorem 6.2.21] and [63, Theorem A.1 in A.19], show that the Poincaré inequality follows from \( \frac{|
abla \Phi|^2}{2} - \Delta \Phi \rightarrow \infty \) as \( |z| \rightarrow \infty \). The additional weight \( |
abla \Phi|^2 \) in (1.18) is classically obtained under the assumption \( |
abla^2 \Phi| \lesssim 1 + |
abla \Phi| \), see for instance [63, Lemma A.24 in Section A.23]. In order to deal with boundaries, we will assume furthermore that \( \mathbf{n} \cdot \nabla \Phi \geq 0 \) on \( \partial U \), where \( \mathbf{n} \) is the unit outgoing normal on \( U \). (Note that the latter assumption could likely be replaced by simply assuming the \( \epsilon \)-regularity of \( (U, \Phi) \)).

Lemma 2.1. Consider \( n \geq 1 \), \( U \subset \mathbb{R}^n \) open, \( \Phi : U \rightarrow \mathbb{R} \) in \( C^2 \) so that \( \mathbf{n} \cdot \nabla \Phi \geq 0 \) on \( \partial U \) and \( \frac{|
abla \Phi|^2}{2} - \Delta \Phi \rightarrow \infty \) as \( |z| \rightarrow \infty \). Then it satisfies the weighted Poincaré-Wirtinger inequality

\[
\forall \rho \in H^1(U; e^\Phi) \text{ with } \int_U \rho = 0, \quad \int_U |
abla \Phi \rho|^2 e^\Phi \geq \int_U \rho^2 (|
abla \Phi|^2 + e^\Phi) \cdot (\nabla \Phi)^2. \tag{2.1}
\]
Proof. Assume first that $\rho = 0$ on $\partial U$. Then a standard calculation yields
\[
\int_U |\nabla \rho|^2 e^\Phi = \int_U |\nabla (\rho e^\Phi)|^2 e^{-\Phi}
\]
\[
= \int_U |\nabla (\rho e^{\Phi/2})|^2 + \int_U \rho^2 \left( \frac{|\nabla \Phi|^2}{4} - \frac{\Delta \Phi}{2} \right) e^\Phi + \frac{1}{2} \int_{\partial U} \rho^2 (n \cdot \nabla \Phi) e^\Phi.
\]  \hspace{1cm} (2.2)

Since we assume that $\frac{|\nabla \Phi|^2}{2} - \Delta \Phi \to \infty$ as $|z| \to \infty$, this controls the $L^2(U; e^\Phi)$ norm of $\rho$ for large $z$. As explained in [63, Thm A.1 in A.19] this can be combined with a standard Poincaré-type inequality on a ball to deduce
\[
\int_U |\nabla \rho|^2 e^\Phi \lesssim \int_U \rho^2 e^\Phi
\]  \hspace{1cm} (2.3)

and if moreover $|\nabla^2 \Phi| \lesssim 1 + |\nabla \Phi|$ one has $\frac{|\nabla \Phi|^2}{2} - \Delta \Phi \lesssim |\nabla \Phi|^2$ and thus the combination of (2.2) and (2.3) implies (2.1).

2.2. The divergence inequality $L^2 \to L^2$. The assumptions on the potential $\Phi$ imply the following Poincaré-type inequality on $\tilde{\Phi} := \Phi + 2 \ln |\nabla \Phi|$: for any $h \in H^1(U; e^\Phi)$ with $\int_U h = 0$,
\[
\int_U h^2 |\nabla \Phi|^2 e^\Phi \lesssim \int_U |\nabla h + h \nabla \Phi|^2 e^\Phi
\]  \hspace{1cm} (2.4)

with quantitative estimate on the constant (it follows from applying the Poincaré inequality on $e^{-\Phi}$ to $h$ and $h|\nabla \Phi|$ and combining linearly the two estimates by the assumption on $|\nabla^2 \Phi|$).

Given $g \in L^2(U; e^\Phi)$ with $\int_U g = 0$ we consider the problem
\[
\nabla \cdot F_0 = g \text{ in } U, \quad F_0 \cdot n = 0 \text{ on } \partial U
\]  \hspace{1cm} (2.5)

for a vector field $F_0 : U \to \mathbb{R}^n$ in $L^2(U; e^\Phi)$. To solve it, we consider the following elliptic problem
\[
\nabla \cdot (\nabla \psi + \psi \nabla \Phi) = 0 \text{ in } U, \quad (\nabla \psi + \psi \nabla \Phi) \cdot n = 0 \text{ on } \partial U
\]  \hspace{1cm} (2.6)

and then define $F_0 := \nabla \psi + \psi \nabla \Phi$. The existence of a unique solution to (2.6) in $L^2(e^\Phi)$ follows from (2.4) and the Lax-Milgram theorem. This solution then satisfies
\[
\|F_0\|_{L^2(U; |\nabla \Phi|^\frac{1}{2}e^\Phi)} = \|F_0\|_{L^2(U; e^\Phi)} \lesssim \|g\|_{L^2(U; |\nabla \Phi|^{-\frac{1}{2}}e^\Phi)} = \|g\|_{L^2(U; e^\Phi)}.
\]

(Note that the same argument also shows $\|F_0\|_{L^2(U; e^\Phi)} \lesssim \|g\|_{H^1_0(U; e^\Phi)}$).

2.3. The divergence inequality $L^2 \to H^1$. The idea is to use the $L^2 \to L^2$ divergence inequality from the previous subsection to reduce the problem to balls $B_k$ such that on $B_k$ the weight $e^\Phi$ is not changing significantly and after a change of variable to flatten the boundary we can then use the explicit representation formula of Bogovskii. To start, note that $|\nabla^2 \Phi| \lesssim 1 + |\nabla \Phi|$ implies by a direct Gronwall argument that there exists $\varepsilon > 0$ so that
\[
2 \left(1 + |\nabla \Phi(x)|\right) \geq 1 + |\nabla \Phi(y)| \geq \frac{1}{2} \left(1 + |\nabla \Phi(x)|\right) \quad \forall x, y \in U \text{ with } |x - y| \leq 2\varepsilon,
\]  \hspace{1cm} (2.7)

which directly implies
\[
4 |\nabla \Phi(x)| \geq |\nabla \Phi(y)| \geq \frac{1}{4} |\nabla \Phi(x)| \quad \forall x, y \in U \text{ with } |x - y| \leq 2\varepsilon.
\]  \hspace{1cm} (2.8)

By reducing $\varepsilon$ if necessary, we may assume without loss of generality that (2.7) and (2.8) hold with the same $\varepsilon$ as the $\varepsilon$-regularity of $(U, \Phi)$ in Definition 1.12. We then need the following covering lemma.

Lemma 2.2. Consider a domain $U \subset \mathbb{R}^n$ with $C^2$ potential $\Phi : U \to \mathbb{R}$ and $\varepsilon > 0$ so that the $\varepsilon$-regularity of $(U, \Phi)$ from Definition 1.12 and (2.8) holds. There is a cover $(B_k)_{k \in I}$ of $U$ and a subordinate partition of unity $(\theta_k)_{k \in I}$ such that
\begin{enumerate}
  \item $B_k = B(z_k, r_k)$ is either a ball with $z_k \in U$ and $r_k = (\varepsilon/40)|\nabla \Phi|^{-1}$ and $B_k \subset U$ or a ball around $z_k \in \partial U$ of radius $r_k = \varepsilon|\nabla \Phi|^{-1}$ in which case there exists $B'_k = B(y_k, r_k/4)$ and $|z_k - y_k| = r_k/2$ so that $B_k$ is star-shaped with respect to $B'_k$;
\end{enumerate}
(2) each point is covered at most $C_d$ times where $C_d \in \mathbb{N}$ only depends on the dimension $d$,
(3) one has $\|\nabla \theta_k\|_{L^\infty(B_k)} \lesssim r_k^{-1}$ and $e^{\Phi(z)} \lesssim e^{\Phi(z_\epsilon)}$ for $z \in B_k$.

Proof. Introduce the radius $r_\epsilon(z) = \epsilon/40|\nabla \Phi|^{-1}(z)$ for $z \in \partial U$ and $r_\epsilon(z) = (\epsilon/40)|\nabla \Phi|^{-1}(z)$ for $z \in U$. By (2.8) we can cover $U$ by the balls $B(z) = B(z, r_\epsilon(z)/10)$ for $z \in \partial U$ and the balls $B(z) = B(z, r_\epsilon(z)/10)$ for $z \in U$ satisfying $B(z, r_\epsilon(z)) \subset U$.

By Vitali’s covering lemma, there exists a disjoint subcollection $(\tilde{B}(z_k))_{k \in I}$ of balls such that $U \subset \bigcup_k \tilde{B}(z_k)$, and we consider the covering $(B_k)_{k \in I}$. Around any $z \in U$, the radii used are comparable by (2.8) so that the fact that the $(\tilde{B}(z_k))_{k \in I}$ are disjoint implies that every point is covered at most $C_d$ times with a dimensional constant $C_d$. Then (2.8) and the mean-value theorem imply $e^{\Phi(z)} \lesssim e^{\Phi(z_\epsilon)}$ on $z \in B_k$.

Let $\zeta \in C^\infty(\mathbb{R}^n)$ with $\zeta(x) = 1$ if $|x| \leq 1/2$ and $\zeta(x) = 0$ if $|x| \geq 1$, and let

$$ w(z) : = \sum_k \zeta \left( \frac{z - z_k}{r_k} \right). $$

Then $w(z) \geq 1$ on $U$ and $|\nabla w| \lesssim |\nabla \Phi|$ so that $\theta_k(z) := \frac{1}{w(z)} \zeta \left( \frac{z - z_k}{r_k} \right)$ is a partition of unity satisfying the claimed properties. Finally, for a ball $B_k$ centred around a boundary point, the claimed star-shaped ball $B_k'$ follows from the Lipschitz bound in Definition 1.12 of $\epsilon$-regularity. □

Using the partition of unity from the previous lemma, define $g_k := \nabla \cdot (\theta_k F_0)$ for $k \in I$, where $F_0$ was constructed in the previous subsection. Each $g_k$ has support included in $B_k$ and has zero average on $U \cap B_k$ due to the divergence structure and $F_0 \cdot n = 0$ on $\partial U$. Moreover $g_k = \nabla \theta_k \cdot F_0 + \theta_k g$ and $|\nabla \theta_k| \lesssim |\nabla \Phi|$ on $B_k$, therefore (using the bound on the overlaps of the covering)

$$ \sum_{k \in I} \|g_k\|^2_{L^2(U;dx^*)} \lesssim \sum_{k \in I} \int_{\partial U \cap B_k} g_k^2 e^{\Phi} \, dz \lesssim \sum_{k \in I} \int_{U \cap B_k} \left( |F_0|^2 |\nabla \theta_k|^2 + |g|^2 |\theta_k|^2 \right) e^{\Phi} \, dz $$

$$ \lesssim \int_U \left( |F_0|^2 |\nabla \Phi|^2 + |g|^2 \right) e^{\Phi} \, dz \lesssim \|g\|^2_{L^2(U;dx^*)}, $$

$$ \sum_{k \in I} \||\nabla \Phi| \theta_k F_0\|^2_{L^2(U;dx^*)} \lesssim \int_U |F_0|^2 |\nabla \Phi|^2 e^{\Phi} \, dz \lesssim \|g\|^2_{L^2(U;dx^*)}. $$

On $U \cap B_k$ we claim (following [14, 16]) that there exists a linear map mapping $g_k$ to a vector field $F_k$ on $U \cap B_k$ so that

$$ \begin{aligned}
\nabla \cdot F_k &= g_k = \nabla \cdot (\theta_k F_0) \quad \text{in } U \cap B_k, \\
F_k &= 0 \quad \text{on } \partial (U \cap B_k) \\
\|F_k\|_{L^2(U \cap B_k)} &\lesssim \|\theta_k F_0\|_{L^2(U \cap B_k)} \\
\|\nabla F_k\|_{L^2(U \cap B_k)} &\lesssim \|g_k\|_{L^2(U \cap B_k)}
\end{aligned} $$

(2.11)

where the constants are independent of $k \in I$. The vector field $F := \sum_{k \in I} F_k$ on $U$ then solves

$$ \begin{aligned}
\nabla \cdot F &= g \quad \text{in } U, \\
F &= 0 \quad \text{on } \partial U, \\
\|F\|_{L^2(U;dx^*)} + \|\nabla F\|_{L^2(U;dx^*)} &\lesssim \|g\|_{L^2(U;dx^*)}
\end{aligned} $$

(2.12)

by (2.9)–(2.10)–(2.11), which concludes the proof.

Let us now prove (2.11). We use the explicit construction of Bogovskii [13, 14] (inspired from an older idea of Sobolev [61]). As it is important that the estimates in (2.11) are independent of $k \in I$ (so in particular independent of the scale $r_k$), we explicitly scale the problem by the map $T_k : x \to (x - y_k)/r_k$: the inner ball $B_k'$ becomes a ball around the origin of radius $1/4$ and $T_k(B_k \cap U)$ is a star-shaped domain around $T_k B_k'$ and contained in the ball of radius $3/2$ around the origin, see Fig. 6. We then denote $\bar{g}_k(x) = g_k(T_k x)$, $\bar{F}_k(x) = r_k F_k(T_k x)$ and $\bar{G}_k(x) =$
where \( L \) is the first equation in (2.13), we argue by density again and consider \( \tilde{g}_k = \nabla \cdot \tilde{G}_k \) and (2.11) is equivalent to

\[
\begin{aligned}
\nabla \cdot \tilde{F}_k &= \tilde{g}_k \text{ in } T_k(U \cap B_k), \\
\tilde{F}_k &= 0 \text{ on } \partial T_k(U \cap B_k) \\
\|\tilde{F}_k\|_{L^2(T_k(U \cap B_k))} &\lesssim \|\tilde{G}_k\|_{L^2(T_k(U \cap B_k))} \\
\|\nabla \tilde{F}_k\|_{L^2(T_k(U \cap B_k))} &\lesssim \|\tilde{g}_k\|_{L^2(T_k(U \cap B_k))}
\end{aligned}
\tag{2.13}
\]

We then define \( \tilde{F}_k \) through the explicit integral representation

\[
\tilde{F}_k(x) := \int_{T_k(U \cap B_k)} \tilde{g}_k(y) K(x, y) \, dy \quad \text{with} \quad K(x, y) := \int_1^{+\infty} (x - y) \psi(y + \tau(x - y)) \tau^{n-1} \, d\tau
\]

where \( \psi : \mathbb{R}^n \to \mathbb{R}_+ \) is a smooth function with support in the ball of radius 1/4 around the origin (which is \( T_k(B'_k) \)) and \( \int \psi = 1 \).

The estimates in the third and fourth equations in (2.13) follow from Calderón-Zygmund theory [19] since all the components of \( V \mathbf{K} \) behave like \(|x - y|^{-n}\) (obtained by changing variable \( \tau = |x - y|^{-1} \tau' \)) and satisfy the Calderón-Zygmund conditions (for details see e.g. [42, Section III.3 (p 161)]). Alternatively, the bounds can be obtained by Fourier analysis [37]. To prove the first equation in (2.13), we argue by \( L^2 \)-density and assume \( \tilde{g}_k \in C^0_c(T_k(U \cap B_k)) \) and compute

\[
\nabla \cdot \tilde{F}_k(x) = \nabla \cdot \left[ \lim_{\ell \to +\infty} \int_{T_k(U \cap B_k)} \tilde{g}_k(y) \int_1^{\ell} (x - y) \psi(y + \tau(x - y)) \tau^{n-1} \, d\tau \, dy \right]
\]

\[
= \lim_{\ell \to +\infty} \int_{T_k(U \cap B_k)} \tilde{g}_k(y) \left[ \psi(y + \tau(x - y)) \tau^{n-1} \right]_{\tau=1}^{\tau=\ell} \, dy
\]

\[
= \lim_{\ell \to +\infty} \left[ \int_{T_k(U \cap B_k)} \tilde{g}_k(y) \psi(y + \ell(x - y)) \, dy \right] = \tilde{g}_k(x) \left( \int_{\mathbb{R}^n} \psi(y) \, dy \right) = \tilde{g}_k(x)
\]

where we have used \( \int \tilde{g}_k = 0 \) and \( \int \psi = 1 \). By density one deduces \( \nabla \cdot \tilde{F}_k = \tilde{g}_k \) in \( L^2 \). Finally to prove the second equation in (2.13), we argue by density again and consider \( \tilde{g}_k \in C^0_c(T_k(U \cap B_k)) \). Then the vector field \( \tilde{F}_k \) is continuous and if \( x \notin T_k(U \cap B_k) \) and \( y \in T_k(U \cap B_k) \) then the half-line \( y + \tau(x - y) \) for \( \tau \geq 1 \) never intersects \( T_k(B'_k) \), hence \( \tilde{F}_k \) vanishes on \( \partial T_k(U \cap B_k) \). By density, we thus deduce that \( \tilde{F}_k \in H^1_0(T_k(U \cap B_k)) \). This concludes the proof of (2.11) and thus concludes the proof of (1.23) in Theorem 1.19.
2.4. Consequences and related inequalities. The weighted Poincaré-Lions inequality (1.24) follows from integrating \( h \in L^2(U; e^\Phi) \) against \( \nabla \cdot F = g \) for any \( g \in L^2(U; e^\Phi) \) with zero mass and \( F \) satisfying (2.12). Regarding the Stokes equations (1.25), the a priori estimate (1.26) follows from integrating the first equation on \( U \) against \( u e^\Phi \) and using the Poincaré inequality, and then integrating the first equation against \( F e^\Phi \) where \( F \) solves (2.12) with any \( g \in L^2(U; e^\Phi) \) with zero mass. In fact the right hand side in (1.26) could be replaced by \( \|s\|_{L^2(U; |\nabla \Phi|^{-2} e^\Phi)} \) or \( \|s\|_{(L^2(U; e^\Phi))'} \).

The weighted Korn inequality is slightly more involved (see also [22]). Consider a vector field \( u : U \to \mathbb{R}^n \) in \( H^1(U; e^\Phi) \) satisfying the average conditions (1.27). To prove (1.28), we work on \( v = u e^\Phi \in L^2(U; e^{-\Phi}) \) and write

\[
\|\nabla e u\|_{L^2(U; e^\Phi)} = \|\nabla v\|_{L^2(U; e^{-\Phi})} = \|\nabla_{\text{sym}} v\|_{L^2(U; e^{-\Phi})} + \|\nabla_{\text{anti}} v\|_{L^2(U; e^{-\Phi})}.
\]

where we have used the previous Poincaré-Lions inequality since \( \int_0^1 (\nabla_{\text{anti}} v) e^{-\Phi} = 0 \) and we have used the fact that all second-order derivatives are linear combination of those in \( \nabla \nabla_{\text{sym}} \). This proves (1.28) under the average conditions (1.27).

In Remark 1.20 the average conditions (1.27) are replaced by boundary conditions. Here we write

\[
v = v_0 + \sum_i b_i e^i + \sum_{1 \leq i < j \leq n} b_{ij} e^{ij}(z - p)
\]

such that \( \int_U v_0 \, dx = 0 \) and \( \int_U (\nabla_{\text{anti}} v_0) e^{-\Phi} = 0 \). By the non-penetration boundary condition, we find for \( \chi^k \) from Remark 1.20 that \( \int_U \nabla \cdot (\chi^k v) = 0 \) for \( k = 1, \ldots, n \) and

\[
\int_U \nabla \cdot \left( \chi^k \sum_i b_i e^i + \chi^k \sum_{1 \leq i < j \leq n} b_{ij} e^{ij}(z - p) \right) = b_k.
\]

Regarding \( v_0 \) we have

\[
\left| \int_U \nabla \cdot (\chi^k v_0) \right| \leq \int_U |\nabla \chi^k \cdot v_0| + \int_U |\chi^k \nabla \cdot v_0| \\
\leq \|\nabla \chi^k\|_{L^2(U; e^\Phi)} \|v_0\|_{L^2(U; e^{-\Phi})} + \|\chi^k\|_{L^2(U; e^\Phi)} \|\nabla \cdot v_0\|_{L^2(U; e^{-\Phi})} \\
\lesssim \|\nabla_{\text{sym}} v_0\|_{L^2(U; e^{-\Phi})},
\]

where we used the Poincaré inequality and the control by the symmetric gradient for the zero average of the antisymmetric gradient from the first part. This shows \( |b_k| \lesssim \|\nabla_{\text{sym}} v\|_{L^2(U; e^{-\Phi})} \), and we likewise control \( b_{ij} \) by testing against \( \chi^{ij} \).

3. The quantitative trajectorial method

In this section we prove Theorem 1.15.

3.1. Transport mapping condition. In (H5), we only assume a lower bound on the transport condition. As a first step, we show that (H5) implies the following transport mapping condition which is then used in the following trajectorial method.

Consider the evolution (1.13), and introduce the semigroup \( G_t = e^{(T + B) t} \) with boundary condition \( R^T \). Remembering that \( B = 0 \) or \( B = \sigma L^T \) in (H5), observe that \( G_t \) is a semigroup in \( L^\infty \) and has stationary state 1 and \( G_t^2 \) has stationary state \( f_\infty^2 \).

Hypothesis 5M (Transport mapping condition). Let \( B \) be an operator such that the evolution

\[
\begin{cases}
\partial_t \varphi - T \varphi - B \varphi = 0 & \text{in } \Omega \times V, \\
\gamma_+ \varphi = R^T \tilde{\gamma}_- \varphi & \text{on } \Gamma_+
\end{cases}
\]  

(3.1)
defines a semigroup \( G_t \) in \( L^\infty \) with the constant 1 as stationary state and \( f_\infty^2 \) as stationary state of \( G_t^* \).

Assume a connected domain \( \Sigma \subset \Omega \) so that \( (\Sigma, \phi) \) is \( \epsilon \)-regular for some \( \epsilon > 0 \), \( \inf_{x \in \Sigma} \sigma(x) > 0 \) and a non-negative \( \chi \in L^\infty(\Omega \times V) \) with \( \text{supp} \chi \subset \Sigma \times V \). Assume that there exists a time \( T > 0 \) and functions \( \psi, \tilde{\psi} : [0, T] \times \Omega \times V \rightarrow \mathbb{R} \) such that

\[
1 - \int_0^T G_t(\psi_t + \tilde{\psi}_t) \, dt \leq \frac{1}{8}. \tag{3.2}
\]

Moreover, it satisfies the following compatibility conditions

- The sources \( \psi_t \) is controlled as
  \[
  \forall t \in [0, T], \quad |\psi_t| \lesssim \chi w \tag{3.3}
  \]
  and \( \tilde{\psi}_t \) satisfies for all \( t \in [0, T] \) the estimates
  \[
  \begin{align*}
  \left| \int_{\Omega \times V} f^2 \tilde{\psi}_t \, d\mu \right| & \lesssim \|f\|_{L^2(\Omega \times V, d\mu)} \left( \int_{\Omega \times V} \sigma f \mathcal{L} f \, d\mu \right)^{\frac{1}{2}} + \left( \int_{\Omega \times V} \sigma f \mathcal{L} f \, d\mu \right)^{\frac{1}{2}}, \\
  \left| \int_{\Omega \times V} ff_\infty \tilde{\psi}_t \, d\mu \right| & \lesssim \left( \int_{\Omega \times V} \sigma f \mathcal{L} f \, d\mu \right)^{\frac{1}{2}}. \tag{3.4}
  \end{align*}
  \]

- For all \( t \in [0, T] \) the function
  \[
  G_t = \int_{\tau=0}^{T-t} G_{\tau}(\psi_{t+\tau} + \tilde{\psi}_{t+\tau}) \, d\tau \tag{3.5}
  \]
  is uniformly bounded in \( L^\infty(\Omega \times V) \) and satisfies
  \[
  \int_{\Omega \times V} [B^*(f^2) - 2f \mathcal{L} f] G_t \, d\mu \lesssim \|f\|_{L^2(\Omega \times V, d\mu)} \left( \int_{\Omega \times V} \sigma f \mathcal{L} f \, d\mu \right)^{\frac{1}{2}} + \left( \int_{\Omega \times V} \sigma f \mathcal{L} f \, d\mu \right)^{\frac{1}{2}}. \tag{3.6}
  \]

and
  \[
  \left| \int_{\Omega \times V} [B^*(ff_\infty) - f_\infty \mathcal{L} f] G_t \, d\mu \right| \lesssim \left( \int_{\Omega \times V} \sigma f \mathcal{L} f \, d\mu \right)^{\frac{1}{2}}. \tag{3.7}
  \]

The relation to (H5) is captured by the following lemma.

**Lemma 3.1.** The hypothesis (H5) implies (H5M).

**Proof.** By the assumed decay of (1.13) in (H5), there exists a time \( T > 0 \) such that \( \|\varphi_T\|_\infty \leq 1/8 \). As the constant 1 is a stationary state, we find by Duhamel that

\[
1 - \int_0^T G_t(\mathcal{C}\varphi_{T-t} + \chi w \varphi_{T-t}) \, dt = \varphi_T.
\]

We therefore find functions \( \psi \) and \( \tilde{\psi} \) satisfying (3.2) by setting

\[
\psi_t = \chi w \varphi_{T-t} \quad \text{and} \quad \tilde{\psi}_t = \mathcal{C}\varphi_{T-t}.
\]

By (H5) \( \varphi_t \) is bounded in \( L^\infty \) so that the choice of \( \psi_t \) implies \( |\psi_t| \lesssim \chi w \). The bounds on \( \tilde{\psi} \) in (3.4) follows directly from the assumption (1.15) in (H5).

Then note again by Duhamel that

\[
G(t, x, v) = \int_{\tau=0}^{T-t} G_{\tau}(\psi_{t+\tau} + \tilde{\psi}_{t+\tau}) \, d\tau = 1 - \varphi_{T-t}.
\]

Hence by the boundedness of \( \varphi_t \), we find that \( G_t \) is uniformly bounded in \( L^\infty \). The last assumptions (3.6) and (3.7) then follow directly from (1.14). \( \square \)
3.2. Decay criterion. We now prove the decay claimed in Theorem 1.15, where we replace (H5) by (H5M) using Lemma 3.1 from the previous subsection. Note that \( \psi \) is only appearing in the case when \( \xi \neq 0 \) for capturing hypocoercivity effects. Hence at first reading, \( \psi \) could be ignored.

Assume (H1)-(H2)-(H3)-(H4)-(H5M)-(H6) and consider \( f \in L^\infty(\mathbb{R}_+; L^2(\Omega \times \mathcal{V}, d\mu)) \) a real-valued solution to (1.1) with zero global average \( \int_{\Omega \times \mathcal{V}} f_{in} = 0 \) (the complex case follows by linearity). Its associated equilibrium is zero. The dissipation is given by

\[
\mathcal{D}(f) := -\frac{d}{dt} \|f_t\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2 = -2 \int_{\Omega \times \mathcal{V}} \sigma f \mathcal{L} f \, d\mu - \int_{\Gamma_+} \left[ (\mathcal{R} \gamma + f)^2 - (\gamma + f)^2 \right] d\nu \geq 0. \tag{3.8}
\]

The following decay criterion is standard in semigroup theory and is used at least since [45] in kinetic theory. Assume there is \( \eta \in (0,1) \) so that

\[
\int_0^T \mathcal{D}(f_t) \, dt \geq \eta \|f_{in}\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2 \tag{3.9}
\]

for a solution \( f_t \in L^2(\mu) \), then

\[
\|f_t\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2 \leq C e^{-\Lambda t} \|f_{in}\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2 \]

with \( C := (1-\eta)^{-1} > 1 \) and \( \Lambda := -\frac{\ln(1-\eta)}{T} > 0 \). We now prove (3.9) for some \( \eta \in (0,1) \).

3.3. Following trajectories. The underlying idea for the first step is to use the transport to relate the initial condition \( \|f_{in}\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2 \) to the value of \( f^2 \) over \([0,T] \times \Omega \times \mathcal{V} \) where it can be controlled by the dissipation; in particular, to \( f^2 \) in \([0,T] \times \Omega \times \Sigma \). To achieve this idea, note that \( f^2 \) satisfies the transport equation up to error terms and that the semigroup \( \mathcal{G}_t \) is solving up to the effect of \( \mathcal{B} \) the dual transport equation.

To implement this idea, we use (3.2) to relate \( \|f_{in}\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2 \) to \( \int f^2(\psi + \tilde{\psi}) \, d\mu \). By the evolution equations (1.1) and (3.1) we find for \( s \in [0,t] \) and \( t \in [0,T] \) that

\[
\frac{d}{ds} \int_{\Omega \times \mathcal{V}} f_s^2 \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) \, d\mu = -\int_{\Omega \times \mathcal{V}} \mathcal{T} \left[ f_s^2 \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) \right] \, d\mu + \int_{\Omega \times \mathcal{V}} \left( 2f_s \sigma \mathcal{L} f_s - B^*(f_s^2) \right) \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) \, d\mu. \tag{3.10}
\]

Using \( \mathcal{T} f_s = \mathcal{T} f_s^{-1} = 0 \), the transport term can be integrated and then yields boundary terms. Further integrating \( s \) over \([0,t] \) and then \( t \) over \([0,T] \) yields

\[
\int_0^T \int_{\Omega \times \mathcal{V}} f_s^2 \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) \, d\mu \, dt = \int_0^T \int_{\Omega \times \mathcal{V}} f_{in}^2 \mathcal{G}_t(\psi_t + \tilde{\psi}_t) \, d\mu \, dt \\
+ \int_0^T \int_0^t \int_{\Omega \times \mathcal{V}} \left( 2f_s \sigma \mathcal{L} f_s - B^*(f_s^2) \right) \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) \, d\mu \, ds \\
+ \int_0^T \int_0^t \int_{\Gamma_+} \left[ (\mathcal{R} \gamma + f_s)^2 \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) - (\gamma + f_s)^2 \mathcal{R}^T \mathcal{G}_{t-s}(\psi_t + \tilde{\psi}_t) \right] \, d\nu \, ds \\
=: I_1 + I_2 + I_3.
\]

The first term \( I_1 \) gives a control on \( \|f_{in}\| \) by (3.2) in (H5M) as

\[
I_1 = \int_0^T \int_{\Omega \times \mathcal{V}} f_{in}^2 \mathcal{G}_t(\psi_t + \tilde{\psi}_t) \, d\mu \, dt \geq \frac{7}{8} \int_{\Omega \times \mathcal{V}} f_{in}(x,v)^2 \, d\mu \, dt = \frac{7}{8} \|f_{in}\|_{L^2(\Omega \times \mathcal{V}, d\mu)}^2.
\]

The second term \( I_2 \) can be written with \( \mathcal{G}_t \) of (3.5) and bounded by (3.6) as

\[
-I_2 = \int_0^T \int_{\Omega \times \mathcal{V}} \left( B^*(f_s^2) - 2f_s \sigma \mathcal{L} f_s \right) G_s \, d\mu \, ds \leq \int_0^T \|f_s\|_{L^2(\Omega \times \mathcal{V}, d\mu)} \sqrt{\mathcal{D}(f_s)} \, ds + \int_0^T \mathcal{D}(f_s) \, ds
\]
The third term $I_3$ is controlled by (1.11) in (H4) as

$$-I_3 = \int_0^T \int_{\Gamma_+} \left[ (\gamma + f_s)^2 R^T (\gamma - G_s) - (R \gamma + f_s)^2 \gamma - G_s \right] \, dv \, ds \leq C_T \|G_s\|_{L^\infty([0,T] \times \Omega \times V)} \int_0^T D(f_s) \, ds.$$

For the LHS we keep at this stage the contribution of $\int_0^T \int_{\Omega \times V} f_t^2 \psi_t \, d\mu \, dt$ and note for the other term by (3.4) that

$$\int_0^T \int_{\Omega \times V} f_t^2 \psi_t \, d\mu \, dt \leq \int_0^T \|f_t\|_{L^2(\Omega \times V, d\mu)} \sqrt{D(f_t)} \, dt + \int_0^T D(f_t) \, dt. \quad (3.11)$$

As the evolution of $f$ is dissipative, we have the trivial bound $\|f_t\|_{L^2(\Omega \times V, d\mu)} \leq \|f_{in}\|_{L^2(\Omega \times V, d\mu)}$ for $t \in [0, T]$ which we can use to further estimate the bound on $-I_2$ and (3.11). Combining the different parts, we therefore arrive at

$$\frac{3}{4} \|f_{in}\|_{L^2(\Omega \times V, d\mu)}^2 - \int_0^T \int_{\Omega \times V} f_t^2 \psi_t \, d\mu \, dt \leq \int_0^T D(f_t) \, dt. \quad (3.12)$$

### 3.4. Removing the local and global averages.

Using the spatial averages

$$\langle f_t \rangle(x) = \langle f \rangle(t, x) := \int_{\Omega} f(t, x, v) \, dv, \quad \langle \psi_t \rangle_M(x) := \int_{\Omega} \psi(t, x, v) M(v) \, dv,$$

split the term remaining to control by (1.10) of (H3) as

$$\int_0^T \int_{\Omega \times V} f_t(x, v)^2 \psi_t(x, v) \, d\mu \, dt \leq 2 \int_0^T \int_{\Omega \times V} \langle f_t \rangle^2 \psi_t \, d\mu \, dt + 2 \int_0^T \int_{\Omega \times V} [f_t - \langle f_t \rangle]^2 \psi_t \, d\mu \, dt$$

$$\leq 2 \int_0^T \int_{\Omega} \langle f_t \rangle^2 \psi_t M e^\theta \, dx \, dt + 2\lambda_1 \int_0^T \int_{\Omega} \|\psi_t\|_\infty^2 D(f_t) \, dt. \quad (3.13)$$

and by assumption (3.3) we know that $\psi_t/(\sigma w)$ is uniformly bounded. Using the global projection

$$\langle f \rangle_\psi := \frac{1}{m_\psi} \int_0^T \int_{\Omega} \langle f_t \rangle(x) \langle \psi_t \rangle_M(x) \, dx \, dt, \quad m_\psi := \int_0^T \int_{\Omega} \langle \psi_t \rangle_M(x) e^{-\phi(x)} \, dx \, dt,$$

we further split the integral over $\langle f_t \rangle^2$ as

$$\int_0^T \int_{\Omega} \langle f_t \rangle^2 \langle \psi_t \rangle_M e^\theta \, dx \, dt - m_\psi \langle f \rangle_\psi^2 \leq \int_0^T \int_{\Omega} \left[ \langle f_t \rangle - \langle f \rangle_\psi \right] e^{-\phi} \langle \psi_t \rangle_M e^\theta \, dx \, dt. \quad (3.14)$$

### 3.5. Control of the local average.

Let us prove

$$\int_{\Sigma_T} \left[ \langle f_t \rangle - \langle f \rangle_\psi e^{-\phi} \right]^2 \langle \psi_t \rangle_M e^\theta \, dz \leq \epsilon' \int_{\Sigma_T \times V} f_t(x, v)^2 \psi_t(x, v) \, d\mu \, dt + C(\epsilon') \int_0^T D(f_t) \, dt \quad (3.15)$$

for any $\epsilon' > 0$ and some corresponding constant $C(\epsilon') > 0$, and with the notation $z := (t, x)$ and $\Sigma_T := (0, T) \times \Sigma$.

For proving (3.15), consider

$$g := \langle f \rangle_\psi M - \langle f \rangle_\psi \langle \psi \rangle_M e^{-\phi} \quad (3.16)$$

and note that $g$ has zero mass and $g \in L^2(\Sigma_T; e^\phi)$ with the bound

$$\int_{\Sigma_T} |g|^2 e^\phi \, dz \leq \int_{\Sigma_T \times V} f_t^2 \psi_t \, d\mu \, dt + \int_0^T D(f_t) \, dt \quad (3.17)$$

which follows from

$$\langle f_t \rangle(x) \langle \psi_t \rangle_M(x) = \int_{v, v_\ast \in V} f_t(x, v) \psi_t(x, v_\ast) M(v_\ast) \, dv \, dv_\ast$$

$$= \int_{v_\ast \in V} f_t(x, v_\ast) \psi_t(x, v_\ast) \, dv_\ast + \int_{v_\ast \in V} \left[ f_t(x, v_\ast) - \langle f_t \rangle M(v_\ast) \right] \psi_t(x, v_\ast) \, dv_\ast.$$
For this $g$ we can therefore apply the divergence inequality of Theorem 1.19 which provides a solution $F = F(z) : \Sigma_T \to \mathbb{R}^{1+d}$ to (1.23). Using the vector field $\phi$, we then find
\[
\int_{\Sigma_T} \left(\langle f \rangle - \langle \langle f \rangle \rangle e^{-\phi}\right)^2 \langle \psi \rangle_M e^\phi \, dz = \int_{\Sigma_T} \left(\nabla_z \cdot F \right) \left(\langle f \rangle e^\phi - \langle \langle f \rangle \rangle e^\phi\right) \, dz = - \int_{\Sigma_T} F \cdot \nabla_z (\langle f \rangle e^\phi) \, dz
\]
where the integration by part has no boundary term since $F \in H_0^1(\Sigma_T; e^\phi)$. Let us denote $\partial_t = \partial_\tau$, and $\partial_i = \partial_{x_i}$ for $i = 1, \ldots, d$ and prove $\partial_i (\langle f \rangle e^\phi) = K_i + \sum_{j=0}^d \partial_j J_{ij}$, $i, j = 0, \ldots, d$, with
\[
\int_{\Sigma_T} \left(\nabla \phi \right)^2 K_i(t, x)^2 + |J_{ij}(t, x)|^2 e^{-\phi} \, dz \lesssim \int_0^T \mathcal{D}(f_i) \, dt.
\]  
(3.18)

Let us first accept (3.18) and conclude the proof of (3.15) (using (1.23), (3.17) and $|\nabla^2 \phi| \lesssim 1 + |\nabla \phi|$):
\[
\int_{\Sigma_T} \left(\langle f \rangle - \langle \langle f \rangle \rangle e^{-\phi}\right)^2 \langle \psi \rangle_M e^\phi \, dz = - \int_{\Sigma_T} F \cdot \nabla_z (\langle f \rangle e^\phi) \, dz
\]
\[
= - \sum_{i=0}^d \int_{\Sigma_T} F_i(z) K_i(z) \, dz + \sum_{i,j=0}^d \int_{\Sigma_T} \partial_i F_i(z) J_{ij}(z) \, dz
\]
\[
\lesssim \left( \int_{\Sigma_T} |F|^2 (\nabla \phi)^2 e^\phi \, dz + \int_{\Sigma_T} |\nabla F|^2 e^\phi \, dz \right)^{\frac{1}{2}} \left( \int_{\Sigma_T} (\nabla \phi)^2 K_i(z)^2 + |J_{ij}(z)|^2 e^{-\phi} \, dz \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \int_{\Sigma_T} f^2 \psi \, d\mu \, dt \right)^{\frac{1}{2}} \left[ \int_0^T \mathcal{D}(f_i) \, dt \right]^{\frac{1}{2}} + \int_0^T \mathcal{D}(f_i) \, dt.
\]

which proves (3.15) by splitting the product into squares adequately. Let us now prove (3.18). Define $\phi_i \in C_0^\infty(\mathbb{V})$, $i = 0, \ldots, d$ so that, denoting $v_0 = 1$, $\int_\mathbb{V} \phi_i(v) v_j M(v) \, dv = \delta_{ij}$. Then $T f_\infty = 0$ from (H2) implies
\[
\int_\mathbb{V} \left\{ (\partial_t + v \cdot \nabla - \nabla x \cdot \nabla) [\langle f \rangle M] \right\} \phi_i e^\phi \, dv = \partial_i (\langle f \rangle e^\phi)
\]
so that the evolution (1.1) implies
\[
\partial_t (\langle f \rangle e^\phi) = \int_\mathbb{V} \left\{ (\partial_t + v \cdot \nabla - \nabla x \cdot \nabla) [\langle \langle f \rangle M - f \rangle] \right\} \phi_i e^\phi \, dv + \int_\mathbb{V} (\sigma L f) \phi_i e^\phi \, dv.
\]
The terms on the RHS can be collected as
\[
\int_\mathbb{V} \left\{ (\partial_t + v \cdot \nabla - \nabla x \cdot \nabla) [\langle \langle f \rangle M - f \rangle] \right\} \phi_i e^\phi \, dv + \int_\mathbb{V} (\sigma L f) \phi_i e^\phi \, dv = K_i + \sum_{j=0}^d \partial_j J_{ij}
\]
where
\[
\begin{cases}
K_i(t, x) := \int_\mathbb{V} (\langle f \rangle M - f) \nabla x \cdot \nabla \phi_i e^\phi \, dv + \int_\mathbb{V} (\sigma L f) \phi_i e^\phi \, dv, \\
J_{i0}(t, x) := \int_\mathbb{V} (\langle f \rangle M - f) \phi_i e^\phi \, dv, \\
J_{ij}(t, x) := \int_\mathbb{V} \langle f \rangle M - f \, v_j \phi_i e^\phi \, dv.
\end{cases}
\]
On the one hand, (H3) implies as $\phi_i \in \text{Domain } L^*$ that
\[
\int_{\Sigma_T} \left| \int_\mathbb{V} (\sigma L f_i)(x, v) \phi_i(v) e^\phi \, dv \right|^2 e^{-\phi} \, dz \lesssim \int_0^T \mathcal{D}(f_i) \, dt
\]
and on the other hand \( \langle f \rangle M - f \) is controlled by assumption (1.10) as \( \sigma \geq 1 \) on \( \Sigma \) so that

\[
\forall i, j = 0, \ldots, d, \quad \int_{\Omega_T} |\nabla x \phi|^2 \frac{e^{-\phi}}{2} dz + \int_{\Omega_T} J_\Omega^2 e^{-\phi} dz
\]

\[
\lesssim \int_0^T \int_\Omega \sigma |\langle f \rangle M - f| \langle f \rangle |dv| e^\phi dz \lesssim \int_0^T D(f_t) dt.
\]

The combination of (3.13), (3.14) and (3.15) yields by choosing \( \epsilon' \) small enough:

\[
\int_0^T \int_{\Omega \times \mathbb{V}} f_t(x, v)^2 \psi(x, v) d\mu dt - 4\langle \langle f \rangle \rangle^2_\psi m_\psi \lesssim \int_0^T D(f_t) dt.
\]  

(3.19)

Together with (3.12) it implies

\[
\frac{3}{4} \| f_{in} \|_{L^2_{\psi, t}(\mu)}^2 - 4m_\psi \langle \langle f \rangle \rangle^2 \lesssim \int_0^T D(f_t) dt.
\]  

(3.20)

### 3.6. Control of the global average

We decompose the global average as

\[
\langle \langle f \rangle \rangle_\psi = \frac{1}{m_\psi} \int_0^T \int_{\Omega \times \mathbb{V}} \langle f_t \rangle(x) \psi_t(x, v) M(v) dx dv dt
\]

\[
= \frac{1}{m_\psi} \int_0^T \int_{\Omega \times \mathbb{V}} \left[ \langle f_t \rangle(x) M(v) - f_t(x, v) \right] \psi_t(x, v) dx dv dt
\]

\[
+ \frac{1}{m_\psi} \int_0^T \int_{\Omega \times \mathbb{V}} f_t(x, v) \psi_t(x, v) dx dv dt =: A_1 + A_2.
\]

The first term \( A_1 \) is controlled using (1.10) by

\[
|A_1| \lesssim \left( \int_0^T \int_{\Omega \times \mathbb{V}} \sigma |\langle f \rangle M - f| \langle f \rangle |dv| d\mu dt \right)^\frac{1}{2} \lesssim \left[ \int_0^T D(f_t) dt \right]^\frac{1}{2}.
\]

The second term \( A_2 \) is controlled similarly to the idea in Section 3.3, except that we can work directly with \( f \). Here we find for \( s \in [0, t] \) and \( t \in [0, T] \) that

\[
\frac{d}{ds} \int_{\Omega \times \mathbb{V}} f_s f_\infty G_{t-s}(\psi_t + \bar{\psi}_t) d\mu dt = - \int_{\Omega \times \mathbb{V}} \mathcal{T} \left[ f_s f_\infty G_{t-s}(\psi_t + \bar{\psi}_t) \right] d\mu dt
\]

\[
+ \int_{\Omega \times \mathbb{V}} \sigma \mathcal{L} f_s - \frac{1}{f_\infty} \mathcal{B}^*(f_s f_\infty) f_\infty G_{t-s}(\psi_t + \bar{\psi}_t) d\mu dt.
\]

(3.21)

Following the same calculation as in Section 3.3 we arrive at

\[
\int_0^T \int_{\Omega \times \mathbb{V}} f_t(\psi_t + \bar{\psi}_t) dx dv = \int_{\Omega \times \mathbb{V}} f_{in} \left( \int_0^T \mathcal{G}_t(\psi_t + \bar{\psi}_t) dt \right) dx dv
\]

\[
+ \int_0^T \int_{\Omega \times \mathbb{V}} \left( \sigma \mathcal{L} f_s - \frac{1}{f_\infty} \mathcal{B}^*(f_s f_\infty) \right) f_\infty G_s d\mu ds
\]

\[
+ \int_0^T \int_{\Omega \times \mathbb{V}} \left[ \mathcal{R}(\gamma_t f_s) f_\infty \tilde{\gamma}_s - \mathcal{R}(\gamma_t f_s) f_\infty \tilde{\gamma}_s \right] dv ds.
\]

(3.22)

The last term vanishes by the definition of \( \mathcal{R}^T \) as the appropriately weighted adjoint. As \( f_{in} \) has mass zero, we find by (3.2) that

\[
\left| \int_{\Omega \times \mathbb{V}} f_{in} \left( \int_0^T \mathcal{G}_t(\psi_t + \bar{\psi}_t) dt \right) dx dv \right| \leq \frac{1}{8} \| f_{in} \|_{L^2(\mu)}.
\]

Hence (3.22) shows by (3.4), (3.7) for \( A_2 \) that

\[
\int_0^T \int_{\Omega \times \mathbb{V}} f_t \psi_t dx dv - \frac{1}{8} \| f_{in} \|_{L^2(\mu)} \lesssim \int_0^T \sqrt{D(f_t)} dt.
\]
We therefore arrive at the bound
\[ 4m_{\psi} \langle f \rangle_{\psi}^2 - \frac{1}{8m_{\psi}} \| f_{\text{in}} \|^2_{L^2(\mu)} \lesssim \int_0^T D(f_t) \, dt. \]

For the final conclusion, we note that \( m_{\psi} \) is close to 1 due to (3.2). As \( \int_{\Omega \times V} \psi f_{\infty} \, dx \, dv = 0 \) due to (3.4) and \( f_{\infty}^2 \) is a stationary state of \( G_{\tau} \) we find that
\[
m_{\psi} = \int_0^T \int_{\Omega \times V} \psi_t f_{\infty} \, dx \, dv \, dt = \int_0^T \int_{\Omega \times V} (\psi_t + \psi_t) f_{\infty} \, dx \, dv \, dt = \int_0^T \int_{\Omega \times V} G_t(\psi_t + \psi_t) f_{\infty} \, dx \, dv \, dt
\]
so that \( 7/8 \leq m_{\psi} \leq 9/8 \). We therefore arrive at the bound
\[
4m_{\psi} \langle f \rangle_{\psi}^2 - \frac{1}{8} \| f_{\text{in}} \|^2_{L^2(\mu)} \lesssim \int_0^T D(f_t) \, dt. \tag{3.23}
\]

3.7. Conclusion. We combine (3.20) and (3.23) to get
\[
\frac{1}{2} \| f_{\text{in}} \|^2_{L^2(\mu)} \lesssim \int_0^T D(f_t) \, dt
\]
which implies the exponential convergence as discussed in Section 3.2. This concludes the proof of Theorem 1.15.

4. Application to concrete equations

4.1. Proof of local spectral gap (H3). In the described geometric settings, we can directly verify \( L M = 0 \) and \( R f_{\infty} = 0 \). For the Fokker-Planck operator (1.5), we have the equilibrium measure \( M(v) = (2\pi)^{-d/2}e^{-v^2/2} \) for \( V = \mathbb{R}^d \) or the uniform probability measure on \( S^{d-1} \) for \( V = S^{d-1} \). By the weight we find directly that
\[
\int_{V} g(v) L g(v) \frac{dv}{M(v)} = \begin{cases} - \int_{V} |\nabla g|^2 \frac{dv}{M(v)} & \text{if } V = S^{d-1}, \\ - \int_{V} |\nabla g + v g|^2 \frac{dv}{M(v)} & \text{if } V = \mathbb{R}^d, \end{cases}
\]
so that the spectral gap follows from the Poincaré inequality of the Gaussian measure [63, 35].

For the linear Boltzmann operator (1.4) and a given equilibrium measure \( M = M(v) \), the symmetry condition \( k(v, v_*) M(v_*) = k(v_*, v) M(v) \) corresponds to the detailed balance and implies that \( L \) is symmetric. For the dissipation, we find
\[
\int_{V} g(v) L g(v) \frac{dv}{M(v)} = -\frac{1}{2} \int_{v, v_\ast \in V} k(v, v_\ast) M(v_\ast) \left[ \frac{g(v)}{M(v)} - \frac{g(v_\ast)}{M(v_\ast)} \right]^2 dv dv_\ast.
\]
For the specific kernels of the linear Boltzmann operator, several conditions for the spectral condition are known [57, 12, 56, 20].

Another viewpoint for the spectral gap can be obtained from the Cheeger’s inequality which is another popular tool to establish spectral gaps [26, 27, 54, 62, 29].

Adapting to our case, denote the size of a set \( A \subset V \) measured by \( |A|_M = \int_A M(v) \, dv \).

Then the condition
\[
\Phi := \inf_{A \subset V} \int_{v, v_\ast \in V} q(v, v_\ast) M(v) M(v_\ast) dv dv_\ast \frac{dv}{\min(|A|_M, |A^c|_M)} > 0 \text{ with } q(v, v_\ast) = k(v, v_\ast) M(v_\ast)
\]
implies a spectral gap as a weighted Cheeger’s inequality, which intuitively says that we cannot split the velocity space \( V \) into two separate parts between which the mass is equiliberating slowly.
Lemma 4.1 (Cheeger’s inequality). For the linear Boltzmann operator, introduce the Rayleigh coefficients

\[ R(g) = \frac{-\langle g, \mathcal{L}g \rangle_{L^2(M^{-1})}}{\|g\|_{L^2(M^{-1})}^2}. \]

Then the spectral gap

\[ \lambda_1 = \inf_{g \perp M} R(g) \]

is bounded by

\[ 2\lambda_1 \geq \Phi^2. \]

Proof. Given \( g \perp M \), assume wlog that \( \|g\|_{L^2(M^{-1})} = 1 \). We first claim that there are two vectors \( g_- \) and \( g_+ \) with disjoint support such that \( \max(R(g_-), R(g_+)) \leq 2R(g) \). Indeed, for \( s \in \mathbb{R} \), define the vector \( g^s \) by \( g^s(v) = g(v) + sM(v) \) and set

\[ g^s = \min(g^s, 0) \text{ and } g^s_+ = \max(g^s, 0). \]

As \( \|g^s\|_{L^2(M^{-1})} \) varies continuously for changing \( s \) and \( \|g^s\|_{L^2(M^{-1})} \to \infty \) as \( s \to -\infty \) and \( \|g^s\|_{L^2(M^{-1})} \to 0 \) as \( s \to +\infty \), we can find some \( \bar{s} \) such that \( \|g^\bar{s}\|_{L^2(M^{-1})}^2 = 1/2 \). Moreover,

\[ \|g^\bar{s}_-\|^2_{L^2(M^{-1})} + \|g^\bar{s}_+\|^2_{L^2(M^{-1})} = \|g\|_{L^2(M^{-1})}^2 = 1 + \bar{s}^2 \geq 1, \]

where we used that \( g \perp M \). Hence we also have that \( \|g^\bar{s}_\pm\|^2_{L^2(M^{-1})} \geq 1/2 \). From the expression of the dissipation

\[ \langle g^\bar{s}_\pm, -\mathcal{L}g^\bar{s}_\pm \rangle \leq \langle g^\bar{s}_-, -\mathcal{L}g^\bar{s}_- \rangle = \langle g, -\mathcal{L}g \rangle, \]

where we used in the last step that \( \mathcal{L}M = 0 \). Hence we indeed find that \( g^\bar{s}_\pm \) have disjoint support and

\[ \max(R(g_-), R(g_+)) \leq 2R(g). \]

By taking the one of \( g^\bar{s}_\pm \) with smaller support, we can find \( \bar{g} \) such that \( R(\bar{g}) \leq 2R(g) \) and \( |\text{supp } \bar{g}|M \leq 1/2 \). By rescaling \( \bar{g} \), we may also assume that \( \sup_{v \in \text{V}} \bar{g}(v)M^{\pm}(v) = 1 \).

Take \( t \) as a uniform random variable on the interval \( [0, 1] \) and define the random set

\[ S_t := \{ v \in \text{supp } \bar{g} : |\bar{g}(v)|^2 \geq tM^2(v) \}. \]

Then we find directly that

\[ \mathbb{E}[|S_t|M] = \|\bar{g}\|_{L^2(M^{-1})}^2. \]

Also

\[
\begin{align*}
\mathbb{E} \left[ \int_{v \in S_t} \int_{v^* \in S_t^c} \sqrt{q(v, v^*)M(v)M(v^*)} \, dv_* \, dv \right] & = \frac{1}{2} \int_{v, v^* \in \text{V}} \left( \frac{\bar{g}(v)}{M(v)} \right)^2 - \left( \frac{\bar{g}(v^*)}{M(v^*)} \right)^2 \sqrt{q(v, v^*)M(v)M(v^*)} \, dv_* \, dv \\
& = \frac{1}{2} \int_{v, v^* \in \text{V}} \left( \frac{\bar{g}(v)}{M(v)} - \frac{\bar{g}(v^*)}{M(v^*)} \right) \left( \frac{\bar{g}(v)}{M(v)} + \frac{\bar{g}(v^*)}{M(v^*)} \right) \sqrt{q(v, v^*)M(v)M(v^*)} \, dv_* \, dv \\
& \leq \frac{1}{2} \left( \int_{v, v^* \in \text{V}} \left( \frac{\bar{g}(v)}{M(v)} - \frac{\bar{g}(v^*)}{M(v^*)} \right)^2 q(v, v^*) \, dv_* \, dv \right)^{1/2} \times \\
& \quad \times \left( \int_{v, v^* \in \text{V}} \left( \frac{\bar{g}(v)}{M(v)} + \frac{\bar{g}(v^*)}{M(v^*)} \right)^2 M(v)M(v^*) \, dv_* \, dv \right)^{1/2} \\
& \leq \sqrt{R(\bar{g})} \|\bar{g}\|_{L^2(M^{-1})}^2.
\end{align*}
\]

Hence there exists a \( t \in (0, 1) \) such that

\[ \int_{v \in S_t} \int_{v^* \in S_t^c} \sqrt{q(v, v^*)M(v)M(v^*)} \, dv_* \, dv \leq \sqrt{R(\bar{g})} |S_t|M. \]
As $|S_t|_M \leq 1/2$ by construction, this set $S_t$ implies
\[ \Phi \leq \sqrt{R(g)} \leq \sqrt{2R(g)}, \]
which yields the claimed bound. \qed

4.2. **Boundary compatibility.** As already noted in [23, Eq. (5)], it is natural to derive the boundary interaction like the Boltzmann equation from some reversible dynamics. Assuming that the boundary has the final temperature so that the system converges to equilibrium, this yields the detailed-balance condition
\[ r(x,v,v_*)f_\infty(x,v) = r(x,v_*,v)f_\infty(x,v). \] (4.1)

By the mass conservation (1.3), the operator $\mathcal{R}$ can be understood as step for a time-discrete Markov chain on $\Gamma_+$ and then the detailed-balance equation (4.1) means that it is a reversible Markov chain with stationary state $f_\infty$.

For a given $f \in L^2(\Gamma_+,(\mathbf{n} \cdot v) \, d\nu)$, the corresponding limiting state of the Markov chain is denoted by $\Pi f$ and takes the form
\[ (\Pi f)(x,v) = \lambda(x,v)f_\infty(x,v) \] (4.2)
where $\lambda(x,v) = \lambda(x,v_*)$ if $r(x,v,v_*) > 0$. For Maxwell boundary conditions (1.8) it is explicitly given by
\[ (\Pi f)(x,v) = \begin{cases} f(x,v) & \text{if } \alpha(x) = 0, \\ \sqrt{2\pi}M(v) \int_{\{\mathbf{n} \cdot v_*\}} f(x,v_*) (\mathbf{n} \cdot v_*) \, dv_* & \text{if } \alpha(x) \in (0,1]. \end{cases} \] (4.3)

**Remark 4.2.** In terms of Markov chains, (4.2) states that $\Pi f$ is proportional to $f_\infty$ where the proportionality factor can be different in different communicating classes. For many boundary conditions with a diffusive component, e.g. Maxwell boundary conditions with $\alpha > 0$, all velocities are related so that $(\Pi f)(x,v) = \lambda(x)f_\infty(x,v)$. A case with different communicating classes would be boundary conditions which only thermalise the normal velocity component but keep the tangential velocity component unchanged.

Indeed we can verify (H4) in this framework, if there exists either a uniform spectral gap or we have the special algebra (4.3) (in which case $\alpha$ can be arbitrary). Note that it handles further boundary conditions discussed in [23].

**Proposition 4.3** (Boundary compatibility). Let $r$ be a boundary condition kernel with a projection $\Pi$ satisfying (4.2). Assume either

- the uniform bound
  \[ \int_{\Gamma_+} [(f - \Pi f)^2 + (\mathcal{R}f - \Pi f)^2] \, d\nu \lesssim \int_{\Gamma_+} [f^2 - (\mathcal{R}f)^2] \, d\nu, \] (4.4)

- or that $r$ has the form of Maxwell boundary conditions (1.8) for any $\alpha : \partial\Omega \to [0,1]$. Then the boundary compatibility condition (H4) is satisfied.

**Proof.** By (4.2) it holds that
\[ \int_{\Gamma_+} \left[ \frac{R^*(f_\infty \varphi)}{f_\infty} f^2 - \varphi (\mathcal{R}f)^2 \right] \, d\nu \leq \|\phi\|_\infty + \int_{\Gamma_+} \left[ f^2 - (\mathcal{R}f)^2 \right] \, d\nu. \]
Furthermore, note that
\[
\int_{\Gamma_+} \left[ \left( \frac{\mathcal{R}^*(\mathcal{F}_\infty \phi) - \Pi(\mathcal{F}_\infty \phi)}{\mathcal{F}_\infty} \right)^2 - \frac{(\mathcal{F}_\infty \phi - \Pi(\mathcal{F}_\infty \phi))}{\mathcal{F}_\infty} (\mathcal{R} f)^2 \right] d\nu
= \int_{\Gamma_+} \left[ \left( \frac{\mathcal{R}^*(\mathcal{F}_\infty \phi) - \Pi(\mathcal{F}_\infty \phi)}{\mathcal{F}_\infty} \right)^2 - \frac{(\mathcal{F}_\infty \phi - \Pi(\mathcal{F}_\infty \phi))}{\mathcal{F}_\infty} (\mathcal{R} \mathcal{F} - \Pi \mathcal{F})^2 \right] d\nu
\]
because over regions that \( R \) is relating (\( \Pi \mathcal{F} \))/\( \mathcal{F}_\infty \) is constant by (4.2) so that by the definition of the adjoint
\[
\int_{\Gamma_+} \left[ \left( \frac{\mathcal{R}^*(\mathcal{F}_\infty \phi) - \Pi(\mathcal{F}_\infty \phi)}{\mathcal{F}_\infty} \right)^2 \frac{f}{\mathcal{F}_\infty} \Pi f - \frac{(\mathcal{F}_\infty \phi - \Pi(\mathcal{F}_\infty \phi))}{\mathcal{F}_\infty} \right] d\nu = 0
\]
and by the conservation of mass (1.3)
\[
\int_{\Gamma_+} \left[ \left( \frac{\mathcal{R}^*(\mathcal{F}_\infty \phi) - \Pi(\mathcal{F}_\infty \phi)}{\mathcal{F}_\infty} \right)^2 - \frac{(\mathcal{F}_\infty \phi - \Pi(\mathcal{F}_\infty \phi))}{\mathcal{F}_\infty} (\mathcal{R} \mathcal{F} - \Pi \mathcal{F})^2 \right] d\nu = 0.
\]

In the case of the uniform bound (4.4), the resulting form in (4.5) can be directly estimated by the boundary dissipation as required. In the case that \( R \) is a Maxwell boundary condition, we have that \( R \mathcal{F} = (1 - \alpha) \mathcal{F} + \alpha \Pi \mathcal{F} \) so that the final expression from (4.5) is
\[
\int_{\Gamma_+} \left[ \left( \frac{\mathcal{R}^*(\mathcal{F}_\infty \phi) - \Pi(\mathcal{F}_\infty \phi)}{\mathcal{F}_\infty} \right)^2 - \frac{(\mathcal{F}_\infty \phi - \Pi(\mathcal{F}_\infty \phi))}{\mathcal{F}_\infty} (\mathcal{R} \mathcal{F} - \Pi \mathcal{F})^2 \right] d\nu
= \int_{\Gamma_+} \left[ (1 - \alpha) - (1 - \alpha)^2 \right] \left[ \frac{(\mathcal{F}_\infty \phi - \Pi(\mathcal{F}_\infty \phi))}{\mathcal{F}_\infty} (\mathcal{F} - \Pi \mathcal{F})^2 \right] d\nu
\leq \| \phi \|_\infty \int_{\Gamma_+} (1 - \alpha) (\mathcal{F} - \Pi \mathcal{F})^2 d\nu.
\]

In this case the boundary dissipation is
\[
\int_{\Gamma_+} \left[ \mathcal{F}^2 - (\mathcal{R} \mathcal{F})^2 \right] d\nu = \int_{\Gamma_+} \alpha (2 - \alpha) (\mathcal{F} - \Pi \mathcal{F})^2 d\nu,
\]
which yields the uniform bound for any \( \alpha : \partial \Omega \rightarrow [0, 1] \).

4.3. Proof of the control condition for deterministic transport. In this subsection, we cover the case of deterministic transport (1.20) in Corollary 1.16. This conclusion also proves the decay in Example 1.4. Instead of using the transport control condition (H5), we can directly verify the transport mapping condition (H5M) with vanishing \( \tilde{\psi} = 0 \).

In this case, we take \( \mathcal{B} = 0 \) so that \( \mathcal{G}_t \) is the dual transport semigroup following the trajectories backward. We then set \( \tilde{\psi} \equiv 0 \) and
\[
\psi(t, x, v) := \frac{\chi(x)w(v)}{\int_0^t \chi(X_{t-s}(x, v))w(V_{t-s}(x, v)) \, ds}, \quad \forall (t, x, v) \in [0, T] \times \Omega \times \mathcal{V}.
\]

It is well-defined as the denominator is uniformly bounded from below. Moreover it is \( W^{1, \infty}([0, T] \times \Omega \times \mathcal{V}) \), non-negative, \( \supp \psi(t, \cdot, v) = \supp \chi \subset \mathcal{C} \) and importantly
\[
\forall (x, v) \in \Omega \times \mathcal{V}, \quad \int_0^T (\mathcal{G}_t \psi_t) (x, v) \, dt = 1.
\]

For the required bounds in (H5M) note that \( G \) from (3.5) becomes in this choice
\[
G(t, x, v) = \frac{\int_0^T \chi(X_{t-s}(x, v))w(V_{t-s}(x, v)) \, ds}{\int_0^T \chi(X_{t-s}(x, v))w(V_{t-s}(x, v)) \, ds}.
\]

Then by construction \( |G| \leq 1 \). In the case of a bounded collision operator, this implies the required bounds (3.6) and (3.7) by the spectral gap of \( \mathcal{L} \).
For the case of the Fokker-Planck operator, the assumed propagation of regularity along the transport shows with $|\nabla w| \lesssim w$ and (1.21) that $G \in W^{1,\infty}$. In the case $V = \mathbb{R}^d$, we find that

$$-\int_0^T \int_{\Omega \times V} G(t, x, v)(\sigma f \mathcal{L} f)(t, x, v) \, d\mu \, dt = \int_0^T \int_{\Omega \times V} \sigma G |(\nabla_v + v)f|^2 \, d\mu \, dt + \int_0^T \int_{\Omega \times V} \sigma f \nabla_v G \cdot (\nabla_v + v)f \, d\mu \, dt$$

$$\leq \int_0^T \int_{\Omega \times V} |G| \sigma |(\nabla_v + v)f|^2 \, d\mu \, dt$$

$$+ \int_0^T \int_{\Omega} \|\nabla_v G(t, x, \cdot)\|_{L^\infty} \sqrt{\sigma} \|f(t, x, \cdot)\|_{L^2(M^{-1})} \|\sqrt{\sigma}(\nabla_v + v)f(t, x, \cdot)\|_{L^2(M^{-1})} \, e^\Phi \, dx \, dt$$

and

$$\left| \int_V G(t, x, v)(\sigma \mathcal{L} f)(t, x, v) \, dv \right|^2 \leq \int_V \sigma \nabla_v G(t, x, v) \cdot (\nabla_v + v)f(t, x, v) \, dv$$

$$\leq \left( \int_V \sigma |\nabla_v G|^2 M(v) \, dv \right) \left( \int_V |(\nabla_v + v)f|^2 \frac{dv}{M(v)} \right).$$

and the desired bounds follow from the fact that $G_t$ and $\sqrt{\sigma}\nabla_v G_t$ are uniformly bounded by construction. The case of the Laplace-Baltrami operator is similar.

4.4. **Proof that** (H5') **implies** (H5) **(with vanishing corrector C = 0).** We first cover the case 2 of (H5'). We claim (1.17) implies that

$$\begin{cases}
\partial_t f + Tf - \sigma \mathcal{L} f = -\chi f & \text{in } \Omega \times V, \\
\gamma_- f(v) = (R\gamma_+) (v - 2(n \cdot v)n) & \text{in } \partial\Omega \times V \text{ with } n \cdot v \leq 0,
\end{cases} \quad (4.7)$$

decays in $L^1$. To prove the claimed decay, it suffices by linearity to take the positive part and note the decay as

$$\|f^+\|_{L^1} = \|f_\infty\|_{L^1} - \int_{s=0}^T |f_s \chi| \, dx \, dv \, ds.$$

By the assumption (1.17) it holds that

$$\int_{s=0}^T \mathcal{S}_s^{\text{full}} f \chi \, dx \, dv \, ds \geq c$$

and by Duhamel

$$\int_{s=0}^T \int |f_s - \mathcal{S}_s^{\text{full}} f| \, dx \, dv \, ds \leq \int_{s=0}^T \int |f_s \chi| \, dx \, dv \, ds.$$

As $\chi$ is bounded we thus find the claimed decay in $L^1$.

By duality, the dual evolution to (4.7) decays in $L^\infty$ which is the required decay of (1.13) in (H5) with $B = \sigma \mathcal{L}^T$. To verify the bounds of (1.14) note by the choice of $B$ that $B^*(ff_\infty - f_\infty \sigma \mathcal{L} f) = 0$ which proves the second part. For the first part note that

$$\int_{\Omega \times V} |B^*(f^2) - 2f \sigma \mathcal{L} f| \, d\mu \lesssim \|1 - \phi_t\|_{L^\infty} \int_{\Omega \times V} |B^*(f^2) - 2f \sigma \mathcal{L} f| \, d\mu$$

As $\phi_t$ is bounded in $L^\infty$, the factor $\|1 - \phi_t\|_{L^\infty}$ is bounded. By the assumed sign in (1.16), we can drop the absolute value and find

$$\int_{\Omega \times V} |B^*(f^2) - 2f \sigma \mathcal{L} f| \, d\mu = \int_{\Omega \times V} B^*(f^2) - 2f \sigma \mathcal{L} f \, d\mu = -2 \int_{\Omega \times V} f \sigma \mathcal{L} f \, d\mu$$

where we used that $\int_{\Omega \times V} B^*(f^2) \, d\mu = \int \sigma \mathcal{L}(f^2/f_\infty) f_\infty \, d\mu = 0$ by mass conservation. This shows the required first bound in (1.14).
For the case 1 and 1’ of (H5’), the given transport control assumptions imply by the same decay of (1.13) with \( B = 0 \). Then (1.14) can be verified as in the previous subsection for the deterministic case.

4.5. Proof of the \( \Gamma \) condition. Here we prove Proposition 1.10. It is classical for the Fokker-Planck operator so that we focus on the linear Boltzmann operator of the form (1.4) with a reversible kernel, i.e. for all \( v, v_* \in V \) it holds that \( k(v, v_*)M(v_*) = k(v_*, v)M(v) \).

**Proof of Proposition 1.10.** For the linear Boltzmann operator we find

\[
\left[ ML \left( \frac{f^2}{M} \right) - 2fL \right] (v) = \int k(v_*, v) dv_* \left[ f(v) - \frac{\int k(v, v_*)f(v_*) dv_*}{\int k(v_*, v) dv_*} \right]^2
\]

\[
- \left[ \int k(v, v_*)f(v_*) dv_* \right]^2 + M(v) \int k(v, v_*) f(v_*) dv_*
\]

Under the reversibility condition \( k(v, v_*)M(v_*) = k(v_*, v)M(v) \) we find

\[
M(v) \int k(v, v_*) \frac{f(v_*)^2}{M(v_*)} dv_* - \left[ \int k(v, v_*)f(v_*) dv_* \right]^2 \int k(v_*, v) f(v_*) dv_*
\]

\[
= M(v) \int \left( \frac{f(v_*)}{M(v_*)} - \frac{\int k(v, w)f(w) dw}{\int k(v_*, v)M(v_*) dw} \right)^2 k(v, v_*)M(v_*) dv_* \geq 0
\]

which implies the claimed sign. \( \square \)

4.6. Proof of (H5) in the hypoelliptic case. We now complete the proof of Theorem 1.18 by constructing the corrector operator \( \mathcal{C} \) in (H5) for the setting of Example 1.5. Take \( B = 0 \) in (H5).

The characteristics are given by

\[
\begin{pmatrix} X_t \\ V_t \end{pmatrix} = r \begin{pmatrix} \sin(\theta + t) \\ \cos(\theta + t) \end{pmatrix}
\]

for parameters \( r \) and \( \theta \) determined by \( X_0, V_0 \). Introduce a cutoff function \( \gamma : \mathbb{R} \to [0,1] \) with \( \text{supp} \gamma \in [-2,2] \) and \( \gamma(x) = 1 \) for \( |x| \leq 1 \) and introduce

\[
a(x, v) := -f^{-1}_\infty(x, v)\gamma(x)\gamma(v)x \max \left(-1, \min \left( \frac{v}{x'}, 1 \right) \right).
\]

Using the Hörmander notation for the Fokker-Planck operator \( \mathcal{L} = -\mathcal{A}^* \mathcal{A} \) with \( \mathcal{A} = (\nabla_v + v) \) and \( \mathcal{A}^* = -\nabla_v \), we then take

\[
\mathcal{C}\varphi = \epsilon f^{-1}_\infty \mathcal{A}^* (f_\infty a\varphi)
\]

for a small constant \( \epsilon > 0 \). Then (1.13) takes the form

\[
\partial_t \varphi - \mathcal{T} \varphi - \epsilon a \nabla_v \varphi = -\epsilon f^{-1}_\infty \nabla_v \left( \gamma(x)\gamma(v)x \max \left(-1, \min \left( \frac{v}{x'}, 1 \right) \right) \right) \varphi - \chi \varphi
\]

where we take \( \chi \) smooth and zero around \( x = 0 \) and with \( \chi \gtrsim 1 \) for \( |x| \geq 1/2 \). The LHS then defines a transport semigroup which is still essentially a circle for small enough \( \epsilon \). For \( ||(x, v)|| \leq 3/4 \) the new term on the right hand side will create some decay. Outside it creates some growth which can be absorbed by \( \chi \) when choosing \( \epsilon \) small enough. As for the compatibility condition (1.14) note that \( |\nabla \varphi| \lesssim r^{-1} \) so that we can absorb it with the factor \( \sqrt{\sigma} \).

**Appendix A. The commutator method for hypoelliptic control**

In this appendix, we provide another viewpoint on why the uniform transport control condition can be partially relaxed in the case of hypoelliptic operators, based on Villani’s commutator conditions for hypocoercivity [63], itself inspired by Hörmander [50]. It is an interesting example of the commutator method that requires three commutators. We consider a slight generalisation of the setting of Example 1.5 as in Theorem 1.18, with \( \Omega = V = \mathbb{R} \) and \( \phi(x) = x^2/2 \) (in fact the method can cover small variations of the harmonic potential), and \( \sigma = \kappa^2 \) with

- \( \kappa \) and \( \kappa' \) are bounded and \( \kappa' \geq -1/4 \), and \( |\kappa| \) is strictly bounded away from zero in the complement of every neighbourhood of \( x = 0 \).
• \( \kappa'(0) = 1 \) and \( \kappa'(x)|x| \) is uniformly bounded for \( x \in \mathbb{R} \),
• \( \kappa \) is smooth enough (as needed in the proof).

Following the setup of \([63]\) we equivalently prove the decay of \( h := f/f_\infty \) in \( L^2(f_\infty) \) where the equation write \( \partial_t h + \mathcal{B} h = -A^* A h \) with

\[
A = \kappa \partial_x, \quad \mathcal{B} = v \partial_x - \phi' \partial_x,
\]

where \( \mathcal{B}^* = -\mathcal{B} \). Then \([63, \text{Thm 24 and Remark 26}]\) shows exponential decay if there exist operators \( \mathcal{C}_0, \ldots, \mathcal{C}_{N_c+1}, \mathcal{R}_1, \ldots, \mathcal{R}_{N_c+1} \) and \( Z_1, \ldots, Z_{N_c+1} \) such that

\[
\mathcal{C}_0 = A,
\]

\[
[\mathcal{C}_j, \mathcal{B}] = Z_{j+1} \mathcal{C}_{j+1} + \mathcal{R}_{j+1}, \quad \text{for } 0 \leq j \leq N_c,
\]

\[
\mathcal{C}_{N_c+1} = 0
\]

and for \( k = 0, \ldots N_c \)

\[
\begin{cases}
[\mathcal{A}, \mathcal{C}_k] \text{ is bounded relative to } (\mathcal{C}_j)_{0 \leq j \leq k} \text{ and } (\mathcal{C}_j A)_{0 \leq j \leq k-1} \\
[\mathcal{C}_k, \mathcal{A}^*] \text{ is bounded relative to id and } (\mathcal{C}_j)_{0 \leq j \leq k} \\
\mathcal{R}_k \text{ is bounded relative to } (\mathcal{C}_j)_{0 \leq j \leq k-1} \text{ and } (\mathcal{C}_j A)_{0 \leq j \leq k-1} \\
\text{there are positive constants } \lambda_k \text{ and } \Lambda_k \text{ such that } \lambda_j \text{id} \leq Z_j \leq \Lambda_j \text{id}.
\end{cases}
\]

and

\[
\sum_{j=0}^{N_c} C_j^* C_j \text{ is coercive.}
\]

In order to fix the operators, take a cutoff \( \gamma \in C^\infty(\mathbb{R}) \) with \( \gamma : \mathbb{R} \to [0,1] \) and \( \gamma(x) = 1 \) for \( |x| \leq 1 \) and supp \( \gamma \subset [-2,2] \). As the rescaled version define \( \gamma_\delta(x) = \gamma(x/\delta) \). Then consider

\[
w := \gamma_\delta \kappa' + (1 - \gamma_\delta) \quad \text{and} \quad \tilde{\kappa} := \frac{\kappa}{w}
\]

for a sufficiently small \( \delta \). By choosing \( \delta \) sufficiently small we can ensure that

\[
\frac{1}{2} \leq w \leq \frac{3}{2} \quad \text{and} \quad \tilde{\kappa} \geq -1/2.
\]

We then build a sequence of commutators \( (\mathcal{C}_n)_n \) where we take away known parts using \( (Z_n)_n \) and \( (R_n)_n \) in order to avoid problematic additional commutator terms. Starting with \( \mathcal{C}_0 = A \) find

\[
[\mathcal{C}_0, \mathcal{B}] = \kappa \partial_x - v \kappa' \partial_x = w(\tilde{\kappa} \partial_x - v \partial_x) + (1 - \gamma_\delta)(1 - \kappa')v \partial_x
\]

so that we take

\[
Z_1 = w
\]

\[
\mathcal{C}_1 = \tilde{\kappa} \partial_x - v \partial_x
\]

\[
\mathcal{R}_1 = (1 - \gamma_\delta)(1 - \kappa')v \partial_x.
\]

In the next iteration we find

\[
[\mathcal{C}_1, \mathcal{B}] = -\tilde{\kappa} \phi'' \partial_x - v \partial_x - v \kappa' \partial_x - \phi' \partial_x = (1 + \tilde{\kappa}')(-v \partial_x - \phi' \partial_x) + (\tilde{\kappa}' \phi' - \tilde{\kappa} \phi'') \partial_x
\]

so that we take

\[
Z_2 = 1 + \tilde{\kappa}'
\]

\[
\mathcal{C}_2 = -v \partial_x - \phi' \partial_x
\]

\[
\mathcal{R}_2 = (\tilde{\kappa}' \phi' - \tilde{\kappa} \phi'') \partial_x.
\]

In the next iteration we find

\[
[\mathcal{C}_2, \mathcal{B}] = 2\phi'''(\tilde{\kappa} \partial_x - \mathcal{C}_1) - 2\phi' \partial_x = 2(\phi''' \tilde{\kappa} - \phi') \partial_x - 2\phi'' \mathcal{C}_1
\]
so that we take

$$Z_3 = 2$$
$$C_3 = (\phi'' \kappa - \phi') \partial_x$$
$$\mathcal{R}_3 = 2\phi'' \mathcal{C}_1.$$ 

In the last step we find

$$\mathcal{R}_4 := [C_3, B] = (-\phi'' \kappa - \phi') \phi'' \partial_v - \nu (\phi'' \kappa - \phi') \partial_x$$

where we have taken \(Z_4 = 1\) and \(C_4 = 0\).

The operators \(C_1, C_2, C_3\) give a good weighted control.

**Lemma A.1.** There exists a constant \(c\) such that

$$\|(x^2 + v^2)|\nabla h|^2\|_{L^2(f_{\infty})}^2 \leq c \left(\|C_1 h\|_{L^2(f_{\infty})}^2 + \|C_2 h\|_{L^2(f_{\infty})}^2 + \|C_3 h\|_{L^2(f_{\infty})}^2\right).$$

**Proof.** Split into the cases \(|x| \leq \delta\) and \(|x| \geq \delta\). For \(|x| \leq \delta\) find by elementary algebra

$$\int_{|x| \leq \delta} \int_{v \in \mathbb{R}} (x^2 + v^2)|\nabla h|^2 f_{\infty} \, dx \, dv = \int_{|x| \leq \delta} \int_{v \in \mathbb{R}} \left[C_2 h|^2 + \frac{x}{\kappa} |C_1 h|^2 + (x^2 - x \kappa) |\partial_x h|^2 + v^2 \left(1 - \frac{x^2}{\kappa}\right) |\partial_v h|^2\right] f_{\infty} \, dx \, dv.$$ 

The last two summands in the integral of the RHS can be absorbed into the LHS giving the claimed inequality.

For \(|x| \geq \delta\), first note that there exists a constant \(c\) with

$$\int_{|x| \geq \delta} \int_{v \in \mathbb{R}} |x^2| |\partial_x h|^2 f_{\infty} \, dx \, dv \leq c \int_{|x| \geq \delta} \int_{v \in \mathbb{R}} |C_3 h|^2 f_{\infty} \, dx \, dv.$$ 

Hence we can find a constant \(c\) such that

$$\int_{|x| \geq \delta} \int_{v \in \mathbb{R}} |v^2 |\partial_v h|^2 f_{\infty} \, dx \, dv \leq c \int_{|x| \geq \delta} \int_{v \in \mathbb{R}} \left[|C_1 h|^2 + |C_3 h|^2\right] f_{\infty} \, dx \, dv.$$ 

Finally with \(C_2\) we can control the remaining terms \(v^2 |\partial_v h|^2\) and \(x^2 |\partial_x h|^2\) giving the claimed result. \(\square\)

For the bounds we use a simple adaptation of the Poincaré inequality:

**Lemma A.2.** There exists a constant \(c\) such that

$$\|\kappa^2 v \partial_v h\|_{L^2(f_{\infty})}^2 \leq c \left[\|\mathcal{A} h\|_{L^2(f_{\infty})}^2 + \|\mathcal{A} \mathcal{C} h\|_{L^2(f_{\infty})}^2\right].$$

**Proof.** Use the refined Poincaré inequality

$$\int_{\mathbb{R}} v^2 |f|^2 e^{-v^2/2} \, dv \leq c \left[\int_{\mathbb{R}} |\partial_v f|^2 e^{-v^2/2} \, dv + \left(\int_{\mathbb{R}} f(v) e^{-v^2/2} \, dv\right)^2\right]$$

for a constant \(c\), which immediately implies

$$\int_{\mathbb{R}} v^2 |f|^2 e^{-v^2/2} \, dv \leq c \left[\int_{\mathbb{R}} |\partial_v f|^2 e^{-v^2/2} \, dv + \int_{\mathbb{R}} |f|^2 e^{-v^2/2} \, dv\right].$$

Applying this to \(\partial_v h\) yields the claimed result. \(\square\)

We now prove the relevant error bounds.

**Lemma A.3.** The operators \(C_0, \ldots, C_{N_c+1}\), \(\mathcal{R}_1, \ldots, \mathcal{R}_{N_c+1}\), \(Z_1, \ldots, Z_{N_c+1}\) satisfy the bounds (A.2).
Proof. For the commutator $[\mathcal{A}, \mathcal{C}_k]$ and $[\mathcal{C}_k, \mathcal{A}^*]$, note that $\mathcal{A}^* = \kappa v - \mathcal{A}$ and we find

$$
\begin{align*}
[\mathcal{A}, \mathcal{C}_0] &= 0 & [\mathcal{C}_0, \mathcal{A}^*] &= \kappa^2 \\
[\mathcal{A}, \mathcal{C}_1] &= -\kappa(1 + \kappa') \partial_v & [\mathcal{C}_1, \mathcal{A}^*] &= [\mathcal{A}, \mathcal{C}_1] + \kappa(\kappa' - 1)v \\
[\mathcal{A}, \mathcal{C}_2] &= -\kappa \partial_x - v \kappa' \partial_v & [\mathcal{C}_2, \mathcal{A}^*] &= [\mathcal{A}, \mathcal{C}_2] - v^2 \kappa - \kappa x \\
[\mathcal{A}, \mathcal{C}_3] &= (\kappa - x) \kappa' \partial_v & [\mathcal{C}_3, \mathcal{A}^*] &= [\mathcal{A}, \mathcal{C}_3] - (\kappa - x) \kappa' v
\end{align*}
$$

Here are $[\mathcal{A}, \mathcal{C}_1]$ and $[\mathcal{A}, \mathcal{C}_3]$ relatively bounded to $\mathcal{C}_0 = \mathcal{A}$. For $\mathcal{C}_2$ note that

$$
\|\kappa \partial_x h\|_{L^2(f_{\infty})}^2 + \|v \partial_v h\|_{L^2(f_{\infty})}^2 \leq \|\mathcal{C}_1 h\|_{L^2(f_{\infty})}^2 + \|\mathcal{C}_2 h\|_{L^2(f_{\infty})}^2.
$$

Therefore, $[\mathcal{A}, \mathcal{C}_2]$ is bounded relative to $\mathcal{C}_1$ and $\mathcal{C}_2$.

Now consider the additional terms in the commutator with $\mathcal{A}^*$. In $[\mathcal{C}_0, \mathcal{A}^*]$ the additional term is bounded relative to the identity. As in Lemma A.2, we find from the refined Poincaré inequality that the additional term in $[\mathcal{C}_1, \mathcal{A}^*]$ and $[\mathcal{C}_3, \mathcal{A}^*]$ is bounded relative to id and $\mathcal{C}_0 = \mathcal{A}$. For the additional term in $[\mathcal{C}_2, \mathcal{A}^*]$ use the refined Poincaré inequality with $|v \partial_v h|^2$ which as before is controlled. Hence the additional term is bounded relative to id, $\mathcal{C}_1$ and $\mathcal{C}_2$.

This proves the assumptions (1) and (2).

For the bound on the error terms note that $\mathcal{R}_1$ is bounded relative to $\mathcal{C}_0$ and $\mathcal{C}_0 \mathcal{A}$. The remainder $\mathcal{R}_2$ is bounded relative to $\mathcal{C}_0$ and $\mathcal{R}_3$ is bounded relative to $\mathcal{C}_1$. For $\mathcal{R}_4$ use Lemma A.1 to show that it is bounded relative to $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Finally, the explicit form of the factors $\mathcal{Z}_1, \ldots, \mathcal{Z}_4$ shows the required control (4). 

The last remaining part is a weighted Poincaré inequality.

Lemma A.4. There exists a constant $c$ such that

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} |h|^2 f_{\infty} \, dx \, dv \leq c \int_{\mathbb{R}} \int_{\mathbb{R}} (x^2 + v^2) |\nabla h|^2 f_{\infty} \, dx \, dv
$$

for all $h$ with

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} h(x) f_{\infty} \, dx \, dv = 0.
$$

The proof of the remaining Poincaré inequality can be done explicitly using the expansion into Hermite polynomials, see [32]. However, it can also be shown by adapting Theorem A.1 of [63]:

Proof. Define the measure $\nu$ by

$$
d\nu(x, v) = e^{-V(x, v)} \, dx \, dv
$$

with

$$
V(x, v) = \frac{x^2}{2} + \frac{v^2}{2} - \log(x^2 + v^2).
$$

By expanding $0 \leq \int \nabla (he^{-V}) \, dx \, dv$, we arrive at

$$
\int_{\mathbb{R} \times \mathbb{R}} |h|^2 (|\nabla V|^2 - 2\Delta V) \, dx \, dv \leq 4 \int_{\mathbb{R} \times \mathbb{R}} |\nabla h|^2 \, dx \, dv.
$$

In our case

$$
\int_{\mathbb{R} \times \mathbb{R}} |h|^2 (4 - 6(x^2 + v^2) + (x^2 + v^2)^2) e^{-x^2/2-v^2/2} \, dx \, dv \leq 4 \int_{\mathbb{R} \times \mathbb{R}} (x^2 + v^2) |\nabla h|^2 e^{-x^2/2-v^2/2} \, dx \, dv,
$$

which shows the required inequality for $(x, v) \to 0$ and $(x, v) \to \infty$.

Following the proof in [63], this shows the claimed inequality together with the standard Poincaré inequality on bounded domains. This inequality ensures that for every $r > 0$, there exists a constant
need enough integrability at infinity to make sure that During the trace we loose a weight (related to the return time and typically like |v|). It makes sense.

The key difficulty with traces in the proof of Section 3.3 is dealing with the term $\int_{\mathcal{L}} (A.5)$ to find the result.

As $\mu(h) = 0$ we find

$$\left[ \int_{r^{-1} \leq ||(x,v)|| \leq r} h \mathcal{Z}^{-1}(r) e^{-x^2/2-v^2/2} \, dx \, dv \right]^2 = \mathcal{Z}^{-2}(r) \left[ \int_{[r^{-1},r]} h e^{-x^2/2-v^2/2} \, dx \, dv \right]^2 \leq \epsilon(r) \int_{r^{-1} \leq ||(x,v)|| \leq r} |h|^2 e^{-x^2/2-v^2/2} \, dx \, dv$$

where

$$\epsilon(r) = \mathcal{Z}^{-2}(r) \int_{r^{-1} \leq ||(x,v)|| \leq r} e^{-x^2/2-v^2/2} \, dx \, dv.$$ 

As $r \to \infty$, we have that $\epsilon(r) \to 0$, so that for large enough $r$ we can combine this inequality with (A.5) to find the result.

Hence with this commutator setup we find exponential convergence to equilibrium.

**Appendix B. Traces for the considered solutions**

We adapt the techniques in [58] to conclude the following lemma (the result in [58] only applies to bounded collision operators in $L^1$):

**Proposition B.1** (Existence of the trace). Consider $f \in L^\infty([0, +\infty); L^2(\Omega \times \mathbb{V}, d\mu))$ so that $\partial t f + Tf \in L^\infty([0, +\infty); L^2(\partial \Omega; \mathcal{H}(V, \lambda^{-k} d\mu)))$ for some $k \in \mathbb{N}$, then the traces in time and at $\partial \Omega$ exist in $L^2(L^2(\partial \Omega; \mathcal{H}(V, (n \cdot v)^2 \lambda^{-k} d\mu)))$, where $\mathcal{H} = L^2$ for the bounded linear Boltzmann operator and $\mathcal{H} = H^{-1}$ for the Fokker-Planck operator, satisfy the Green formula, and can be renormalised by any $\beta \in W^{1,\infty}(\mathbb{R}_+)$.

For the transport semigroup in $L^1$ we can construct a semigroup which contracts the mass (in general the mass can escape to infinity), see [64, 55, 5, 60, 4]. These works also include trace theorems with Green’s formula. The oldest reference for the Green’s formula seems to be [24, 25] but they can also be found in [58, 59]. For completeness [7, 6] constructs the transport semigroup in $L^p$.

The key difficulty with traces in the proof of Section 3.3 is dealing with the term

$$\int_{\Omega \times \mathbb{V}} \mathcal{T} \left[ f^2 \mathcal{G}_{t-s}(\psi_t + \psi_t) \right] d\mu.$$ 

Assume that $\mathcal{L}$ is bounded. Then first note that then $f \in C(0, \infty, L^2)$ and $(\partial_t + T)f \in L^2$ and its trace exists in $L^2$; moreover if $g \in C(0, \infty, L^\infty)$ and $(\partial_t + T)g \in L^\infty$ its trace exists in $L^\infty$. Therefore we have traces for $f, f^2$ and $f^2 \mathcal{G}_{t-s}(\psi_t + \psi_t)$ by the product rule for the weak derivative in the interior, since $g := \mathcal{G}_{t-s}(\psi_t + \psi_t)$ satisfies the $L^\infty$ a priori bounds. By weak-strong convergence of approximations we also see that the trace commutes, i.e. $\gamma f^2 = (\gamma f)^2$ and likewise with $\psi$. During the trace we loose a weight (related to the return time and typically like $|v|^{-1}$) and we need enough integrability at infinity to make sure that $f$ preserves mass and the boundary operator makes sense.
APPENDIX C. Non-uniform control conditions

We can treat non-uniform geometric control conditions, with less quantitative estimates and additional non-concentration and tightness assumptions on the solution. For simplicity we formulate as Case 2 of \((H5')\) which we replace by the following condition.

**Hypothesis 5"** (Non-uniform Geometry Control Condition). Assume \(\mathcal{L}\) satisfies \((1.16)\) and that there is a connected domain \(\Sigma \subset \Omega\) with the following properties: (i) \((\Sigma, \phi)\) is \(\epsilon\)-regular for some \(\epsilon > 0\), (ii) \(\inf_{x \in \Sigma} \sigma(x) > 0\), and (iii) there is a non-negative \(\chi \in L^\infty(\Omega)\) with \(\text{supp} \chi \subseteq \Sigma\) so that
\[
\forall (x,v) \in \Omega \times V, \quad \exists T = T(x,v) > 0 \text{ such that } \int_0^{T(x,v)} \mathcal{G}_t \chi(x,v) \, dt > 0.
\]
Moreover we assume there is a Banach space \(\mathcal{B}\) dense in \(L^2(\Omega \times V, d\mu)\) so that \(\|f\|_{\mathcal{B}} \lesssim \|f_{in}\|_{\mathcal{B}}\) uniformly in time, and the norm of \(\mathcal{B}\) provides non-concentration and tightness on the solution: that there is \(a : \mathbb{R}_+^+ \to \mathbb{R}_+^+\) going to zero at zero and infinity so that for any \(B \subset \Omega \times V\) Borel set and \(M > 0\):
\[
\int_{\Omega \times V} \mathbf{1}_B f^2 \, d\mu \leq a(|B|) \|f\|_{\mathcal{H}}^2 \quad \text{and} \quad \int_{\Omega \times V} (\mathbf{1}_{|x| \geq M} + \mathbf{1}_{|v| \geq M}) f^2 \, d\mu \leq a(M) \|f\|_{\mathcal{H}}^2.
\]

**Theorem C.1** (Non-uniform control condition). Assume \((H1)-(H2)-(H3)-(H4)-(H6)-(H5'')\). Then given any \(f_{in} \in L^2(\Omega \times V, d\mu)\), any \(f = f(t,x,v) \in C^0([0,\infty); L^2(\Omega \times V, d\mu))\) admitting traces \(\gamma f \in L^2([0,\infty) \times \Omega; H^{-1}(V, (n \cdot v)^2 \, d\mu))\) and solution to \((1.1)\) satisfies
\[
\left\| f_t - \left( \int_{\Omega \times V} f_{in} \right) \right\|_{L^2(\Omega \times V, d\mu)} \xrightarrow{t \to \infty} 0.
\]
This implies the convergence to equilibrium without rate for the concrete equations treated in Corollary 1.16 when \((H5)\) is replaced by \((H5'')\).

A similar result was already obtained in [47] for the linear Boltzmann equation and we adapt here their idea in our setting. It is enough to prove the relaxation to equilibrium for initial data \(f_{in} \in L^2(\mu) \cap \mathcal{B}\) by density of \(\mathcal{B}\) in \(L^2(\mu)\). Assume by contradiction that there is \(f_{in} \in L^2(\mu) \cap \mathcal{B}\) with zero global average so that the solution does not converge to zero in \(L^2(\mu)\). There is therefore \(t_k \to \infty\) with \(t_{k+1} - t_k \to \infty\) as \(k \to \infty\) so that \(\|f(t_k)\|_{L^2(\mu)}\) is uniformly bounded below in \(k \geq 1\). The sequence \(g_k(t) := f(t_k + t)\|f(t_k)\|^{-1}_{L^2(\mu)}\) then satisfies \(g_k \in L^2(\mu) \cap \mathcal{B}\) with
\[
\forall k \geq 1, \quad \forall t \geq 0, \quad \|g_k(0)\|_{L^2(\mu)} = 1, \quad \|g_k(t)\|_{\mathcal{B}} \lesssim 1,
\]
and for any fixed \(T > 0\),
\[
\int_0^T \int_{\Omega \times V} \sigma_{g_k} \mathcal{L} g_k \, d\mu \, dt \xrightarrow{k \to \infty} 0.
\]
By weak compactness of the unit ball in \(L^2_{t,loc}(\Omega \times V, \mu)\) we can also assume that \((g_k)\) has a weak limit in this space. Such a weak limit \(\tilde{g}\) satisfies the transport equation without collision operator and with \(\tilde{g} = \langle \tilde{g} \rangle \mu\) on supp \(\psi\). Since \((C.1)\) implies that all points can be connected to supp \(\psi\) by a trajectory, \(\tilde{g}\) is at local equilibrium everywhere, and is thus zero since it solves the transport equation. We then argue as in \((3.12)-(3.13)-(3.14)-(3.15)\) on \(g_k\) with the weight \(\chi\) to get
\[
\int_0^T \int_{\Omega \times V} g_k(0,x,v)^2 \mathcal{G}_t \chi(x,v) \, d\mu \, dt \lesssim \langle g_k \rangle_{\chi} + \int_0^T \int_{\Omega \times V} \sigma_{g_k} \mathcal{L} g_k \, d\mu \, dt.
\]

We then use \((C.1)-(C.2)\) to find \(B \subset \Omega \times V\) small enough and \(T\) large enough so that
\[
\forall (x,v) \in (\Omega \times V) \setminus B, \quad \left( \int_0^T \mathcal{S}_t^* \psi_t(x,v) \, dt \right) \geq 1 \quad \text{and} \quad \int_B g_k(0,x,v)^2 \, d\mu \leq \frac{1}{2}.
\]
Then, given this choice of $T$, the RHS of (C.4) goes to zero as $k \to \infty$ since $g_k$ weakly converges to zero and the dissipation vanishes asymptotically on any time interval, and thus for $k$ large enough
\[
\int_{\Omega \times V} g_k(0, x, v)^2 \, d\mu < 1
\]
which contradicts the assumptions.

Acknowledgements

All authors acknowledge partial support from the ERC grant MATKIT grant. HD & CM acknowledge partial support from the ERC grant MAFRAN and would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme “Frontiers in kinetic theory”. This work was supported by EPSRC grant no EP/R014604/1 and a grant from the Simons Foundations. HD also acknowledges support from the UK CDT EPSRC grant EP/H023348/1 Cambridge Centre for Analysis, Université Sorbonne Paris Cité in the framework of the “Investissements d’Avenir” convention ANR-11-IDEX-0005, and the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n. PCOFUND-GA-2013-600102, through the PRESTIGE programme coordinated by Campus France. IH also acknowledges the support of the EPSRC programme grant Mathematical fundamentals of Metamaterials for multiscale Physics and Mechanics (EP/L024926/1).

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