Isotopic liftings of Clifford algebras and applications in elementary particle mass matrices

R. da Rocha
Centro de Matemática, Computação e Cognição
Universidade Federal do ABC,
09210-170 Santo André, SP, Brazil
and
Instituto de Física "Gleb Wataghin",
Universidade Estadual de Campinas,
Unicamp, 13083-970 Campinas, SP, Brazil

J. Vaz, Jr.
Departamento de Matemática Aplicada, IMECC,
Unicamp, CP 6065, 13083-859, Campinas, SP, Brazil.

Isotopic liftings of algebraic structures are investigated in the context of Clifford algebras, where it is defined a new product involving an arbitrary, but fixed, element of the Clifford algebra. This element acts as the unit with respect to the introduced product, and is called *isounit*. We construct isotopies in both associative and non-associative arbitrary algebras, and examples of these constructions are exhibited using Clifford algebras, which although associative, can generate the octonionic, non-associative, algebra. The whole formalism is developed in a Clifford algebraic arena, giving also the necessary pre-requisites to introduce isotopies of the exterior algebra. The flavor hadronic symmetry of the six $u, d, s, c, b, t$ quarks is shown to be *exact*, when the generators of the isotopic Lie algebra $\mathfrak{su}(6)$ are constructed, and the unit of the isotopic Clifford algebra is shown to be a function of the six quark masses. The limits constraining the parameters, that are entries of the representation of the isounit in the isotopic group $\text{SU}(6)$, are based on the most recent limits imposed on quark masses.

PACS numbers: 02.10.De, 11.15.-q, 14.65.-q

I. INTRODUCTION

Some limitations concerning the description of physical theories, owning non-canonical, non-unitary and non-lagrangian character, have motivated investigations about a wider class of formalisms used to describe such theories, the so-called isotopies of mathematical structures. The isotopic lifting of such structures allows the physical theories to be described in a straightforward canonical, unitary and Lagrangian formalism [2, 3, 4, 5, 6, 7, 8, 9, 10, 11], by maps from Lagrangian, linear and local theories to more general ones, enrolling a non-linear, non-local and non-Lagrangian character. These later are led to the former when formulated in an isospace, endowed with a new product in the context of the Clifford algebras, with respect to which the unit is now a fixed, but arbitrary, element $\zeta$ of the Clifford algebra. The inverse of such element is called *isotopic element*, and shall be used to define the new product that endows the Clifford isotopic algebra, to be precisely defined in this article. These isotopic concepts are entirely related to the $q$-deformations of algebraic structures, to which have a one-to-one correspondence to the isotopic liftings of algebras [10]. Although in e.g. [4] isotopies of symplectic and other geometries are included, the present paper presents for the first time the isotopies of Clifford algebras with significant applications.

In what follows we define isotopic Clifford algebras, and subsequently the formalism developed is applied in some aspects of Quantum Field Theory, e.g., the description of the flavor $\text{SU}(6)$ symmetry as an *exact* symmetry among the six quarks, if they are to be viewed as components of an element of the carrier representation space of the isotopic group $\text{SU}(6)_\zeta$ associated with the group $\text{SU}(6)$, in the context of the isotopic Clifford algebra $\mathcal{C}_\ell_{12,0}$. As a consequence, all six quarks must have the same mass in *isospace*, which brings an immediate constraint among the elements that constitute the matrix representing the Santilli’s isounit, here emulated in a Clifford algebraic context. The isounit is shown to be a function of quark masses, whose original values are retrieved when an eigenvalue *isodequation*, or
equivalently, the expected value defined in isospace, is used. The isotopic Lie algebra \( \mathfrak{su}(6)_\zeta \), associated with the Lie algebra \( \mathfrak{su}(6) \), is constructed in the context of the isotopic lifting of the Clifford algebra \( \mathcal{C}_\ell_{12,0} \). More generally, the isotopic lifting of \( \mathfrak{su}(n) \) is described in the context of the isotopic lifting of the Clifford algebra \( \mathcal{C}_\ell_{2n,0} \), emulating a similar construction in \( [28] \).

We illustrate the general method to be used, by firstly describing the flavor symmetry among the \( u, d \) and \( s \) quarks as an exact symmetry of the isotopic SU(3) group, constructed via the isotopic lifting of the Dirac-Clifford algebra \( \mathcal{C}_\ell_{1,3}(\mathbb{C}) = \mathbb{C} \otimes \mathcal{C}_\ell_{1,3} \). In this context the isotopic group SU(3)\( \zeta \times SU(2) \) is obtained using solely the isotopic lifting of \( \mathcal{C}_\ell_{1,3}(\mathbb{C}) \). Here \( SU(3)_\ell \) denotes the flavor group SU(3) and has nothing to do with the SU(3) gauge group associated with strong interactions. Hereon we omit the index \( \ell \) and denote \( SU(3)\ell \) solely by \( SU(3) \). We emphasize that the isotope of SU(3) and the proof of their local isomorphism to the conventional SU(3) symmetry were first proved in \([11]\) and papers quoted therein. After introducing the iso-Gell-Mann matrices, as particular cases of the most general representation in the isotopic \( \mathfrak{su}(3) \) Lie algebra, analogously to \([16, 17, 18]\), the behavior of some quantum operators acting on the carrier fundamental representation space of the isotopic SU(3) is investigated.

In terms of its applications, the main aim of this paper is to obtain an exact flavor symmetry encompassing all the six quarks via the isotopic lifting of the generators of the group SU(6). The parameters that define the isotype are shown to be functions of the quark masses, and are delimited by the most recent limits of quark masses.

This article is organized as follows: in Sec. II a brief review on Clifford algebras is presented, and after discussing associative isotopies in Sec. III, in Sec. IV the isotopic liftings of non-associative algebras is presented. In Sec. V \( \zeta \)-fields are presented, and in Sec. VI we investigate the so-called Clifford admissible products in the context of the \( \zeta \)-applications. In Sec. VII the isotopic lifting of exterior algebras is introduced via Clifford isotropic algebras and in Sec. VIII and Sec. IX a complete formulation concerning the isotopic lifting of spacetime algebra is presented in order to introduce the heterodimensional isotopic lifting of the group SU(3). In Sec. X the more general case describing the isotopic generators of the Lie group SU(3) is constructed, in the light of the corresponding standard construction \([28]\). Finally in Sec. XI, applications to QFT are presented and we show how to suitably construct an isotype in such a way that in isospace the six quarks have equal masses, and consequently the \( SU(6) \) flavor symmetry becomes an exact symmetry in isospace. In the Appendix the isotopic lifting of SU(6) is presented via the isotopic lifting of the Clifford algebra \( \mathcal{C}_\ell_{12,0} \).

II. PRELIMINARIES

Let \( V \) be a finite \( n \)-dimensional real vector space and \( V^* \) denotes its dual. We consider the tensor algebra \( \bigoplus_{i=0}^{\infty} T^i(V) \) from which we restrict our attention to the space \( \Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V) \) of multivectors over \( V \). \( \Lambda^k(V) \) denotes the space of the antisymmetric \( k \)-tensors, isomorphic to the \( k \)-forms vector space. Given \( \psi \in \Lambda(V) \), \( \psi \) denotes the reversion, an algebra antiautomorphism given by \( \hat{\psi} = (-1)^{|k|/2} \psi \) (\( |k| \) denotes the integer part of \( k \)). \( \hat{\psi} \) denotes the main automorphism or graded involution, given by \( \hat{\psi} = (-1)^k \psi \). The conjugation is defined as the reversion followed by the main automorphism. If \( V \) is endowed with a non-degenerate, symmetric, bilinear map \( g : V^* \times V^* \to \mathbb{R} \), it is possible to extend \( g \) to \( \Lambda(V) \). Given \( \psi = u^1 \wedge \cdots \wedge u^k \) and \( \phi = v^1 \wedge \cdots \wedge v^l \), for \( u^i, v^j \in V^* \), one defines \( g(\psi, \phi) = \det(g(u^i, v^j)) \) if \( k = l \) and \( g(\psi, \phi) = 0 \) if \( k \neq l \). The projection of a multivector \( \psi = \psi_0 + \psi_1 + \cdots + \psi_n \), \( \psi_k \in \Lambda^k(V) \), on its \( p \)-vector part is given by \( \langle \psi \rangle_p = \psi_p \). Given \( \psi, \phi, \zeta \in \Lambda(V) \), the left contraction is defined implicitly by \( g(\psi \cdot_\phi \zeta) = g(\phi, \psi \wedge \zeta) \). For \( a \in \mathbb{R} \), it follows that \( \psi \cdot a = a \cdot \psi \). Given \( \psi \in V \), the Leibniz rule \( \psi \cdot (\psi \wedge \phi) = (\psi \cdot \psi) \wedge \phi + \psi \wedge (\psi \cdot \phi) \) holds. The right contraction is analogously defined \( g(\psi \cdot_\phi \zeta) = g(\phi, \psi \wedge \zeta) \) and its associated Leibniz rule \( (\psi \wedge \phi)_\cdot v = \psi \wedge (\phi \cdot v) + (\psi \cdot v) \wedge \phi \). Both contractions are related by \( \psi \cdot_\phi \psi = -\psi \wedge_\phi \psi \). The Clifford product between \( \psi \in V \) and \( \phi \in \Lambda(V) \) is given by \( \psi \phi = w \wedge_\phi w + w \cdot_\phi w \). The Grassmann algebra \( \Lambda(V), g \) endowed with the Clifford product is denoted by \( \mathcal{C}(V, g) \) or \( \mathcal{C}_{p,q} \), the Clifford algebra associated with \( V \simeq \mathbb{R}^p-q \), \( p+q = n \). In what follows \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \) denote respectively the real, complex and quaternionic fields.

III. ASSOCIATIVE ISOTOPIC ALGEBRAS

Consider a \( \mathbb{C} \)-associative algebra \( \mathcal{A} \) endowed with a product \( AB \) denoted by juxtaposition, where \( A, B \in \mathcal{A} \), and let \( \zeta \in \mathcal{A} \) be a fixed, but arbitrary element of \( \mathcal{A} \). The product \( \circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) is given by

\[
A \circ B := A \zeta^{-1} B = (A \zeta^{-1}) B = A (\zeta^{-1} B) \tag{1}
\]

Clearly \( \zeta \) is the unit of \( \mathcal{A} \) with respect to the \( \circ \)-product, since \( A \circ \zeta = \zeta \circ A = A \), for all \( A \in \mathcal{A} \). Since \( \zeta \) is assumed to be always invertible, the product \( \circ \) is not automorphic to the product of the original algebras \([20]\).
Given $A \in \mathcal{A}$, $\zeta$-applications are defined as

$$\diamond A := \zeta^{-1}A, \quad \bar{A} := A\zeta$$

(2)

where the juxtaposition denotes the product in $\mathcal{A}$. All the formalism to be developed here is motivated by definitions in Eqs. (2).

The isotope-$\zeta$ of the algebra $\mathcal{A}$, denoted by $\mathcal{A}_\zeta$, is defined as being the underlying vector space of the algebra $\mathcal{A}$, with multiplication given by Eq. (1). The action of the isotopic algebra $\mathcal{A}_\zeta$ on physical states, generally described by elements of a Hilbert space $\mathcal{H}$ — which is an ideal on which operators in $\mathcal{A}_\zeta$ acts on — comes from the definition of the isotope-$\zeta$ of an $\mathcal{A}$-module. Consider $V$ a left unital $\mathcal{A}$-module, with respect to the composition $A\mathbf{v}$, where $A \in \mathcal{A}$, $\mathbf{v} \in V$. Here $V$ must be a left ideal of $\mathcal{A}$. From the map

$$\mathcal{A}_\zeta \times V \rightarrow V$$

$$(A, \mathbf{v}) \mapsto A \diamond \mathbf{v} = A\zeta^{-1}\mathbf{v},$$

(3)

the $\mathcal{A}$-module $V$ becomes a left unital $\mathcal{A}_\zeta$-module $V_\zeta$, since $\zeta \diamond \mathbf{v} = \zeta\zeta^{-1}\mathbf{v} = \mathbf{v}$, for all $\mathbf{v} \in V$.

The product

$$\diamond : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$$

$$(A, B) \mapsto A \diamond B$$

(4)

can also be extended in order to encompass elements $\zeta, \bar{B} \in \mathcal{A}_\zeta$. Indeed, given $\bar{A}, \bar{B} \in \mathcal{A}_\zeta$, it is immediate that

$$\bar{A} \diamond \bar{B} = A\zeta\zeta^{-1}B\zeta = AB\zeta \in \mathcal{A}_\zeta,$$

(5)

i.e., with respect to the product $\diamond$, the elements $\bar{A}, \bar{B}$ inherit the structure of the product $AB$ in $\mathcal{A}$. This concept shall be useful in order to define exterior algebras isotopic liftings in Sec. IV.

### IV. NON-ASSOCIATIVE ISOTOPY

In this case the algebra $\mathcal{A}$ is a non-associative $\mathbb{C}$-algebra, and therefore the last equality in Eq. (1) does not hold anymore. Given $\zeta \in \mathcal{A}$ fixed, but arbitrary, the non-associative isotope-$\zeta$ of $\mathcal{A}$, denoted by $\mathcal{A}_{(\zeta)}$, is defined by the multiplication

$$A \circ B := A(\zeta^{-1}B)$$

(6)

while the $\zeta$-isotope of $\mathcal{A}$, denoted by $\mathcal{A}_{(\zeta)}\mathcal{A}$, is defined by the relation

$$A\zeta \circ B := (A\zeta^{-1})B$$

(7)

We verify that, while $\zeta$ is the right unit of the algebra $\mathcal{A}_{(\zeta)}$ with respect to the product given by Eq. (6), $\zeta$ is also the left unit of $\mathcal{A}_{(\zeta)}\mathcal{A}$ with respect to the product given by Eq. (7). The product $\circ$ defines uniquely the isotope-$\zeta$ $\mathcal{A}_{(\zeta)}$ of $\mathcal{A}$, while in a similar way the product $\circ$ defines the $\zeta$-isotope $\mathcal{A}_{(\zeta)}\mathcal{A}$ of $\mathcal{A}$. Naturally the product $\circ$ can be extended to elements $\bar{A}, \bar{B}$ in the isotope-$\zeta$ $\mathcal{A}_{(\zeta)}$ of $\mathcal{A}$, in such way that for this non-associative case it follows that

$$\bar{A} \circ \bar{B} := \bar{A}(\zeta^{-1}\bar{B})$$

$$= \bar{A}(\zeta^{-1}\bar{B})$$

(8)

In this way it is then possible to define the product $A \circ B := (A\zeta)(\zeta^{-1}B)$ from Eq. (8), which extends the $X$-product introduced in the octonionic algebra $\mathbb{O}$ context, to any non-associative algebra $\mathcal{A}$. The $X$-product was originally introduced in order to correctly define the transformation rules for bosonic (vector) and fermionic (spinor) fields on the tangent bundle over the 7-sphere $S^7$. This product is also closely related to the parallel transport of sections of the tangent bundle, at $X \in S^7$, i.e., $X \in \mathbb{O}$ such that $\hat{X}X = XX = 1$. The $X$-product is also shown to be twice the parallelizing torsion $\bar{T}^2$, given by the torsion tensor, and in particular, it is used to investigate the $S^7$ Kač-Moody algebra $\mathcal{K}_2$ and to obtain triality maps and $G_2$ actions on $\mathcal{K}_2$. Also, it leads naturally to remarkable
geometric and topological properties, for instance the Hopf fibrations $S^3 \rightarrow S^1$ and $S^8 \rightarrow S^7 \rightarrow S^7$ [21, 22]. Generalizations of these topics are developed in [20]. We also extend the product $\zeta$ to the $\zeta$-isotope $(\zeta)A$ of $A$, in such a way that for this case we have

$$A_\zeta \circ B := (A_\zeta^{-1})B = (A_\zeta \zeta^{-1})(B_\zeta) = A(B_\zeta).$$

(9)

The definitions of the left unital $A_{(\zeta)}$-module and the $(\zeta)A$-module for the cases given by Eqs.(6, 7) follow naturally from their respective definitions.

**Example 1**: The octonion algebra $\mathbb{O}$ can be generated by a basis $\{e_0 = 1, e_a\}_{a=1}^7$ in the underlying paravector space $\mathbb{R} \oplus \mathbb{R}^{0,7} \rightarrow \mathcal{C}_{0,7}$, endowed with the standard octonionic product $\circ : \mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$, which can be constructed using the Clifford algebras $\mathcal{C}_{0,7}$ as

$$A \circ B = \langle AB(1 - \psi) \rangle_0 \oplus 1, \quad A, B \in \mathbb{R} \oplus \mathbb{R}^{0,7},$$

(10)

where $\psi = e_1 e_2 e_3 + e_2 e_3 e_4 + e_3 e_4 e_5 + e_4 e_5 e_6 + e_5 e_6 e_7 + e_6 e_7 e_1 + e_7 e_1 e_2 + e_1 e_2 e_3 \in \Lambda^3(\mathbb{R}^{0,7}) \leq \mathcal{C}_{0,7}$ and the juxtaposition denotes the Clifford product [13]. The idea of introducing the octonionic product from the Clifford product in this context is to present hereon in this example our formalism using a Clifford algebraic arena. It is now immediate to verify the usual rules between basis elements under the octonionic product:

$$e_a \circ e_b = e_a e_b = -\delta_{ab} \quad (a, b, c = 1, \ldots, 7),$$

(11)

where we denote $e_a^b = 1$ for the cyclic permutations $(abc) = (124),(235),(346),(457),(561),(672)$ and (713). All the relations above can be expressed as $e_a \circ e_{a+1} = e_a e_{a+1} \equiv 1 \text{ mod } 7$.

Now, defining $\zeta = e_1$, the isotope-$\zeta$ $\mathbb{O}_{(\zeta)}$ related to the octonionic algebra $\mathbb{O}$, is given by the multiplication

$$A \circ \zeta = A \circ (e_1^{-1} \circ B),$$

(12)

and the $\zeta$-isotope $(\zeta)\mathbb{O}$ of $\mathbb{O}$ is defined by

$$A_\zeta \circ B = (A \circ e_1^{-1}) \circ B.$$  

(13)

For the particular cases where $A = e_2$ and $B = e_4$ it follows that

$$e_2 \circ \zeta e_5 = e_2 \circ (e_1^{-1} \circ e_5) = e_2 \circ (-e_6) = -e_7$$

(14)

while

$$e_2 \zeta \circ e_5 = (e_2 \circ e_1^{-1}) \circ e_5 = e_4 \circ e_5 = e_7.$$  

(15)

### V. $\zeta$-FIELDS AND ISOCOMPLEX FIELDS

An isotopy of the unit $1 \in \mathcal{A}$ is defined to be the map $1 \mapsto \zeta = \zeta(x)$. For consistency of the formalism, the associative products between operators are led to their corresponding isotopic (associative) partners:

$$AB \mapsto A \circ B = A\zeta^{-1}B, \quad \zeta \text{ fixed.}$$

(16)

As we have just seen, the element $\zeta$ is the unit with respect to the product $\circ$, also denominated isounit. On the other hand $\zeta^{-1}$ is called isotopic element.

The field $\mathbb{C} = \mathbb{C}(a, +, \times)$ with elements $a \in \mathbb{C}$, ordinary sum $a_1 + a_2$ and multiplication $a_1 \times a_2 = a_1 a_2$ is isotopically lifted to the isofield $\tilde{\mathbb{C}}(a, +, \times)$, where the isocomplex numbers (heretofore denoted by gothic characters) are given by $a := a\zeta$, the sum is expressed as $a_1 + a_2 := (a_1 + a_2)\zeta$ and the isomultiplication by $a_1 \circ a_2 = a_1\zeta^{-1}a_2 = (a_1 a_2)\zeta$. The fields $\mathbb{C}$ and $\tilde{\mathbb{C}}$ are shown to be isomorphic [2]. Note that given an operator $A \in \mathcal{A}$, the isoproduct between isoscalars and such operator is given by $a \circ A = a\zeta^{-1}A = aA$.

We should mention the effect that the lack of use of Santilli’s isofield activates the theorems of catastrophic inconsistences [5] in the case when non-canonical or non-unitary theory is not formulated on Santilli’s isofields.
VI. CLIFFORD ISOTOPIES VIA ASSOCIATIVE $\zeta$-PRODUCT

From this section on the algebra $\mathcal{A}$ is taken to be the Clifford algebra $\mathcal{C}_{p,q}$ over the quadratic space $\mathbb{R}^{p,q}$. It is well known that the Lie algebra $\mathfrak{so}(p,q)$ is isomorphic to the algebra $(\Lambda^2(\mathbb{R}^{p,q}), [\ , \ ])$ where $[\ , \ ]$ denotes the commutator — when $\text{Spin}(n,0) \simeq \text{Spin}(0,n)$ and $\text{Spin}_+ (n - 1,1) \simeq \text{Spin}_+ (1,n - 1)$, $n > 4$. Then besides the product given by Eq. [1], there is defined the isocommutator $(\mathbb{R}^{p,q}) \ [\ , \ ]$ in the isotope.

When $\xi, \psi \in \mathcal{C}_{p,q}$, the gensocommutator can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta$, can be used to investigate irreversible systems. Here $(\cdot , \cdot)_\zeta$, $(\cdot \cdot)_\zeta$, $(\cdot \cdot \cdot)_\zeta
and
\[ v \wedge \psi = \frac{1}{2} (v \psi - \hat{\psi} v) \] (23)
for all \( v \in \mathbb{R}^{p,q}, \psi \in \mathcal{C}_{p,q} \). The most natural manner to define the exterior product isotopic lifting \( v \wedge \psi \mapsto v \hat{\wedge} \psi \) is from Clifford algebras, via
\[ v \hat{\wedge} \psi := \frac{1}{2} (v \circ \psi + \hat{\psi} \circ v) = \frac{1}{2} (v \psi^{-1} \psi + \hat{\psi} \zeta^{-1} v) \] (24)
In a similar style, the isotopic (left) contraction is defined as being
\[ v \hat{\cdot} \psi = \frac{1}{2} (v \psi^{-1} \psi - \hat{\psi} \zeta^{-1} v) \] (25)
with an immediate analogue to the right contraction. Although these definitions above are correct from the formal viewpoint, we see from Eq. (18) that the product \( \hat{\circ} \) (endowing \( \mathcal{C}_{p,q} \)) inherit the structure of the original Clifford product \( \psi \phi \) only whether computed between elements \( \hat{\psi}, \hat{\phi} \in \mathcal{C}_{p,q} \). Therefore with respect to the physical applications concerning the formalism developed, the following extensions are very useful:
\[ \hat{\psi} \hat{\wedge} \psi = \frac{1}{2} (\hat{\psi} \circ \psi + \hat{\psi} \circ \hat{\psi}) = \frac{1}{2} (v \psi + \hat{\psi} v) \zeta \] (26)
and
\[ \hat{\psi} \hat{\cdot} \psi = \frac{1}{2} (\hat{\psi} \circ \psi - \hat{\psi} \circ \hat{\psi}) = \frac{1}{2} (v \psi - \hat{\psi} v) \zeta \] (27)
\( v \in \mathbb{R}^{p,q}, \psi \in \mathcal{C}_{p,q} \). From Eq. (20) it follows that
\[ \hat{u} \hat{\wedge} \psi = \frac{1}{2} (u \zeta \psi - \psi \zeta u) \zeta \]
\[ = (u \wedge v) \zeta, \quad u, v \in V. \] (28)
In this way we see that the isotopic exterior product \( u \hat{\wedge} v \) indeed induces the exterior product \( u \wedge v \) at the isospace.

VIII. THE SPACETIME ALGEBRA \( \mathcal{C}_{1,3} \)

Consider an orthonormal basis \{e_0, e_1, e_2, e_3\} in Minkowski spacetime \( \mathbb{R}^{1,3} \), where \( e_\mu e_\nu + e_\nu e_\mu = 2 \eta_{\mu\nu} = 2 \text{diag} (1, -1, -1, -1) \). An arbitrary element \( \Upsilon \in \mathcal{C}_{1,3} \) is written as \( \Upsilon = c + e^\mu e_\mu + e_\mu e_\nu + e_\mu e_\nu e_\sigma + e e_{0123} \), where \( c, e^\mu, e_\mu, e \in \mathbb{R} \). We use the notation \( e_\mu e_\nu = e_\mu e_\nu, e_\mu e_\nu e_\rho = e_\mu e_\nu e_\rho \) for \( \mu \neq \nu \neq \rho \).

The 4-vector \( e_{0123} \) is denoted by \( e_5 \) and satisfies \( (e_5)^2 = -1 \), besides anticommutating with vectors: \( e_\mu e_5 = -e_5 e_\mu \).

As \( \mathcal{C}_{1,3} \subset \mathcal{M}(2, \mathbb{H}) \), in order to obtain a representation of \( \mathcal{C}_{1,3} \) in terms of matrices with quaternionic entries, the primitive idempotent \( f = \frac{1}{2} (1 + e_0) \) is used. A left minimal ideal of \( \mathcal{C}_{1,3} \) is written as \( I_{1,3} = \mathcal{C}_{1,3} f \), which elements are written as
\[ \Xi = (a^1 + a^2 e_{23} + a^3 e_{31} + a^4 e_{12}) f + (a^5 + a^6 e_{23} + a^7 e_{31} + a^8 e_{12}) e_5 f, \] (29)
where
\[ a^1 = c + e^0, \quad a^2 = e^{23} + e^{023}, \quad a^3 = -e^{13} - e^{013}, \quad a^4 = e^{12} + e^{012}, \]
\[ a^5 = -e^{23} + e^{023}, \quad a^6 = e^1 - e^0, \quad a^7 = e^2 - e^0, \quad a^8 = e^3 - e^0. \] (30)
Since the equality \( e_\mu = f e_\mu f + f e_\mu e_5 f - f e_5 e_\mu f - f e_5 e_\mu e_5 f \) clearly holds, from the representation \( e_\mu \in \mathcal{C}_{1,3} \) in \( \mathcal{M}(2, \mathbb{H}) \) given by
\[ e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \] (31)
where \( i, j, k \) denote quaternionic units, the representations of the ideal generators

\[
 f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_5 f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (32)
\]

are obtained, and finally it is possible to write \( \mathcal{Y} \in \mathcal{C}\ell_{1,3} \) in \( \mathcal{M}(2, \mathbb{H}) \) as

\[
 \mathcal{Y} = \begin{pmatrix}
 c + e^0 + (c^{23} + e^{023})i & (c^{123} - e^{0123}) + (c^1 + e^{01})i + (c^2 + e^2)j + (c^3 + e^3)k \\
 +(-c^{13} - e^{013}) + (c^{12} + e^{012})i & -(c^{123} - e^{0123}) + (c^1 + e^{01})i + (c^2 + e^2)j + (c^3 + e^3)k \\
\end{pmatrix} (33)
\]

The \( \text{Spin}_+(1,3) \) group associated with \( \mathcal{C}\ell_{1,3} \) is given by

\[
 \text{Spin}_+(1,3) = \{ R \in \mathcal{C}\ell_{1,3}^+ \mid R \tilde{R} = 1 \} (34)
\]

Now taking a basis \( \{ e_i \} \) of Euclidean space \( \mathbb{R}^3 \), the Clifford algebra \( \mathcal{C}\ell_{3,0} \) over \( \mathbb{R}^3 \) is well known to be isomorphic to \( \mathcal{M}(2, \mathbb{C}) \) and that the quaternionic units can be written as \( i = e_3 e_2, j = e_3 e_1, k = e_1 e_2 \). If the isomorphism \( \mathcal{C}\ell_{3,0} \simeq \mathcal{M}(2, \mathbb{C}) \) given by \( e_i \mapsto \sigma_i \) is considered, where \( \sigma_i \) denotes the Pauli matrices given by

\[
 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

it follows that the matrix in Eq. (33) can be written as

\[
 \begin{pmatrix}
 a_1 & b_1 & d_1 & f_1 \\
-b_1 & a_1 & -f_1 & d_1 \\
 a_2 & b_2 & d_2 & f_2 \\
-b_2 & a_2 & -d_2 & f_2 \\
\end{pmatrix} (35)
\]

where \( a_1 = c + e^0 + i(c^{12} + e^{012}), b_1 = -c^{13} - e^{013} + i(c^{23} + e^{023}), d_1 = -c^{123} + e^{0123}, f_1 = c^2 - e^{02} + i(c^{1} - e^{01}), a_2 = -c^{123} - e^{0123} + i(c^{3} + e^{03}), b_2 = -c^{123} - e^{0123} - i(c^{3} + e^{03}), d_2 = -c^{23} + e^{023} + i(-c^{13} + e^{013}), f_2 = c^{23} - e^{023} + i(-c^{13} + e^{013}) \).

**IX. ISOTOPY \( \mathcal{C}\ell_{1,3} \) OF \( \mathcal{C}\ell_{1,3} \)**

In this case the basis \( \{ e_\mu \} \) of \( \mathbb{R}^{1,3} \) satisfies \( \zeta \circ e_\nu \circ e_\mu + e_\nu \circ \zeta \circ e_\mu = 2\eta_{\mu\nu}\zeta \), and an arbitrary element of \( \mathcal{Y}_\zeta \in \mathcal{C}\ell_{1,3}^\zeta \) can be now written as

\[
 \mathcal{Y}_\zeta = \zeta + e^\mu \circ e^-\zeta + e^\nu \circ e^-\zeta + e^\mu \circ e^-\zeta + e^\nu \circ e^-\zeta + e^\mu \circ e^-\zeta + \zeta \circ e_0 \circ e_1 \circ e_2 \circ e_3,
\]

where \( \zeta, e^\mu, e^\nu, h \in \mathbb{C} \).

The isomultivector \( \zeta_0 \circ \zeta_1 \circ \zeta_2 \circ \zeta_3 \) is denoted by \( e^\zeta_0 \) and satisfies \( e^\zeta_0 \circ e^\zeta_0 = -\zeta \). Now choosing a primitive idempotent, denoted by \( f_\zeta = \frac{1}{2}(\zeta + \zeta_0) \), a left minimal ideal \( I_{1,3}^\zeta \) associated with the isotopic algebra \( \mathcal{C}\ell_{1,3}^\zeta \) is written as \( \mathcal{C}\ell_{1,3}^\zeta \circ f_\zeta \), which elements are of the form

\[
 \Xi_\zeta = (a^1 \circ e^\zeta_2 \circ e^\zeta_3 + a^3 \circ e^\zeta_1 \circ e^\zeta_1 + a^4 \circ e^\zeta_1 \circ e^\zeta_2) \circ f_\zeta \\
+(a^5 \circ e^\zeta_2 \circ e^\zeta_3 + a^7 \circ e^\zeta_1 \circ e^\zeta_1 + a^8 \circ e^\zeta_1 \circ e^\zeta_2) \circ f_\zeta = a^m \circ e^\zeta_2 \\
\]

where

\[
 a^1 = \zeta + e^0, \quad a^2 = e^{23} + e^{023}, \quad a^3 = -e^{13} - e^{013}, \quad a^4 = e^{12} + e^{012}, \\
 a^5 = -e^{123} - e^{0123}, \quad a^6 = e^1 - e^0, \quad a^7 = e^2 - e^{02}, \quad a^8 = e^3 - e^{03}.
\]

(37)

Since \( \zeta_\mu = f_\zeta \circ e_\mu \circ f_\zeta + f_\zeta \circ e_\mu \circ e_\zeta \circ f_\zeta - f_\zeta \circ e_\zeta \circ e_\mu \circ f_\zeta - f_\zeta \circ e_\zeta \circ e_\zeta \circ e_\zeta \circ f_\zeta \) and from the representation of \( \zeta_\mu = e_\mu \zeta \in \mathcal{C}\ell_{1,3}^\zeta \) in \( \mathcal{M}(2, \mathbb{H})_\zeta \) it follows that

\[
 \zeta_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

(38)
and then it follows that

\[ f_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3^\zeta \circ f_\zeta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \] (39)

It follows that \( \Upsilon \in C\ell_{1,3}^\zeta \) is written in \( \mathcal{M}(2, \mathbb{H})_\zeta \) as

\[ \Upsilon_\zeta = \begin{pmatrix} a_1 & b_1 & d_1 & f_1 \\ -b_1 & a_1 & -f_1 & d_1 \\ a_2 & b_2 & d_2 & f_2 \\ -b_2 & a_2 & -f_2 & d_2 \end{pmatrix} \begin{pmatrix} 0 & \zeta \zeta_1 & \zeta \zeta_2 & \zeta \zeta_3 \\ -\zeta_1 & 0 & -\zeta \zeta_3 & \zeta \zeta_2 \\ \zeta \zeta_4 & \zeta \zeta_5 & 0 & \zeta \zeta_7 \\ -\zeta_7 & \zeta \zeta_4 & -\zeta \zeta_6 & \zeta \zeta_6 \end{pmatrix} \] (40)

where the right hand side matrix is the complexification of \( \zeta \in \mathcal{M}(2, \mathbb{H}) \). The isotopic \( \text{Spin}^\zeta_{1,3} \) group associated with \( C\ell_{1,3}^\zeta \) is now defined by

\[ \text{Spin}^\zeta_{1,3} = \{ R \in C\ell_{1,3}^\zeta \mid R \circ \bar{R} = \zeta \}. \] (41)

### A. Heterodimensional isotopic lifting of SU(3)

The isotopies of SU(3) and the proof of their local isomorphism to the conventional SU(3) symmetry were first proved in [11] and papers quoted therein.

From the examples above that show how to include the Lie algebra \( \text{su}(3) \), associated with the Lie group SU(3), in \( \mathbb{C} \otimes C\ell_{1,3} = C\ell_{1,3}(\mathbb{C}) \), it is possible to construct the isotopic lifting \( \text{su}(3)_\zeta \hookrightarrow C\ell_{1,3}^\zeta \) as:

\[ \text{SU}(3)_\zeta: \text{ case 1} \]

It is well known that considering an orthonormal basis \( \{ e^a \} \) of \( \mathbb{R}^{p,q} \), the relation \( e^a e^b = e^a \wedge e^b \) holds between the Clifford and the exterior product. Denoting \( e^\zeta \) by \( e_\zeta^\mu \), we define the isotopic lifting \( \text{su}(3)_\zeta \) of \( \text{su}(3) \), that generates the isotopic group \( \text{SU}(3)_\zeta \), as the isotopic lifting given by

\[
\begin{align*}
\lambda_1^\zeta &= \frac{1}{2}(e_\zeta^0 \wedge e_\zeta^1 + ie_\zeta^2 \wedge e_\zeta^3), \\
\lambda_2^\zeta &= \frac{1}{2}(e_\zeta^0 \wedge e_\zeta^2 - ie_\zeta^1 \wedge e_\zeta^3), \\
\lambda_3^\zeta &= \frac{1}{2}(e_\zeta^0 \wedge e_\zeta^3 + ie_\zeta^1 \wedge e_\zeta^2), \\
\lambda_4^\zeta &= \frac{1}{2} (e_\zeta^0 + ie_\zeta^1 \wedge e_\zeta^2 \wedge e_\zeta^3), \\
\lambda_5^\zeta &= \frac{1}{2} (i e_\zeta^3 - e_\zeta^1 \wedge e_\zeta^2 \wedge e_\zeta^3), \\
\lambda_6^\zeta &= \frac{1}{2} (e_\zeta^0 \wedge e_\zeta^2 \wedge e_\zeta^3 + ie_\zeta^1), \\
\lambda_7^\zeta &= \frac{i}{2} (e_\zeta^1 + e_\zeta^0 \wedge e_\zeta^2 \wedge e_\zeta^3), \\
\lambda_8^\zeta &= \frac{i}{2} (e_\zeta^1 \wedge e_\zeta^2 \wedge e_\zeta^3 + \frac{1}{2 \sqrt{3}} e_\zeta^0 \wedge e_\zeta^2 - \frac{i}{2 \sqrt{3}} e_\zeta^1 \wedge e_\zeta^3),
\end{align*}
\] (42)

where \( e_\zeta^5 := e_\zeta^0 \wedge e_\zeta^1 \wedge e_\zeta^2 \wedge e_\zeta^3 \).

**Example 2:** For instance, let us calculate, e.g., the isocommutator \( [\lambda_1^\zeta, \lambda_2^\zeta]_\zeta \). Using definitions in Eq. (42) and making use of the definitions [24] and (26) yields

\[
[\lambda_1^\zeta, \lambda_2^\zeta]_\zeta = \left[ \frac{1}{2}(e_\zeta^0 \wedge e_\zeta^1 + ie_\zeta^2 \wedge e_\zeta^3), \frac{1}{2} (e_\zeta^0 \wedge e_\zeta^2 - ie_\zeta^1 \wedge e_\zeta^3) \right]_\zeta \\
= \frac{1}{8} [e_\zeta^0 \circ e_\zeta^1 - e_\zeta^1 \circ e_\zeta^0 + ie_\zeta^2 \circ e_\zeta^3 - ie_\zeta^3 \circ e_\zeta^2, -ie_\zeta^1 \circ e_\zeta^0 + ie_\zeta^0 \circ e_\zeta^1 + e_\zeta^0 \circ e_\zeta^2 - e_\zeta^2 \circ e_\zeta^0]_\zeta
\] (43)
Using Eq. (20) it follows that

\[
[\lambda_1^\xi, \lambda_2^\zeta]\zeta = \frac{1}{8}[e^0 e^1 - e^1 e^0 + i e^2 e^3 - i e^3 e^2, -i e^1 e^3 + i e^3 e^1 + e^0 e^2 - e^2 e^0]\zeta
= \frac{1}{4}[e^0 e^1 - e^1 e^0 + i e^2 e^3 - i e^3 e^2](-i e^1 e^3 + i e^3 e^1 + e^0 e^2 - e^2 e^0)\zeta
= \frac{1}{4}(2i e^0 \wedge e^3 - 2 e^1 \wedge e^2)\zeta
= i\left(\frac{1}{2}(e^0 \wedge e^3 + i e^1 \wedge e^2)\right)\zeta
= i\left(\frac{1}{2}(e^0 \wedge e^3 + i e^1 \wedge e^2)\right), \quad \text{from Eq. (28)}
= i\lambda_2^\zeta.
\]

More generally the generators \(\lambda_i^\zeta\) satisfy the properties \(\lambda_i^\zeta, \lambda_j^\zeta\zeta = i f_{abc}\delta^{ij}_k \lambda_k^\zeta\), where the \(su(3)\) structure constants \(f_{abc}\) are given by \(f_{123} = 2 f_{147} = -2 f_{156} = 2 f_{246} = 2 f_{257} = -2 f_{345} = 2 f_{358}/\sqrt{3} = 2 f_{678}/\sqrt{3} = 1\).

It is also immediate to note that the elements \(\{\lambda_1^\xi, \lambda_2^\zeta, \lambda_3^\zeta\}\) and \(\lambda^\xi\) generates the isotopic subalgebra \(su(2)\zeta \times u(1)\zeta\).

\[SU(3)\zeta: \text{case 2}\]

By isotopically lifting the Lie algebra \(su(3)\) presented in \([28, 29]\) above it follows that

\[
\xi_1^\zeta = -\frac{i}{2}(e^0 \wedge e^3 + i e^1 \wedge e^2 + e^0 \wedge e^3 + e^1 \wedge e^2),
\xi_2^\zeta = \frac{i}{2}(e^1 \wedge e^3 + e^0 \wedge e^3 + e^1 \wedge e^3 + e^0 \wedge e^3),
\xi_3^\zeta = \frac{i}{2}(e^0 \wedge e^3 - i e^1 \wedge e^2 - e^0 \wedge e^3 - i e^1 \wedge e^2),
\xi_4^\zeta = \frac{i}{2}(e^1 \wedge e^3 + i e^0 \wedge e^3 + e^1 \wedge e^3 + i e^0 \wedge e^3),
\xi_5^\zeta = \frac{i}{2}(e^0 \wedge e^3 - i e^1 \wedge e^2 - e^0 \wedge e^3 - i e^1 \wedge e^2),
\xi_6^\zeta = \frac{i}{2}(e^1 \wedge e^3 + i e^0 \wedge e^3 + e^1 \wedge e^3 + i e^0 \wedge e^3).
\]

From the generators \(\{\xi^\zeta, \lambda_i^\zeta\}\) of the isotopic Lie algebra \(su(3)\zeta\), the isotopic Lie group \(SU(3)\zeta\) is constructed via the isosexpension defined by

\[\exp(\bar{\theta}_a \circ \bar{U}^\zeta), \quad \bar{\theta}_a \in \bar{\mathbb{C}}.\]

**X. ISOTOPIC LIFTING SU(\(n\)) OF SU(n)**

It has been stated in \([22]\) that all the elements of the compact spin groups \(Spin(n,0) \simeq Spin(0,n)\) are exponentials of bivectors when \(n > 1\). Also, the same holds for the other spin groups only for \(Spin_+(n-1,1) \simeq Spin_+(1,n-1)\), \(n > 4\). It is well-known that at least the Lie algebras of type \(spin(p,q) \simeq so(p,q)\) associated with the spin groups above can be described as elements of a Clifford algebra endowed with the commutator \((Cl_{p,q}, [\ , \ ]\) for \(p + q = n\) big enough \([1]\). Such statement for the Lie algebra \(so(p,q) \simeq (\Lambda^2(\mathbb{R}^{p,q}), [\ , \ ]\) associated with the group \(SO(p,q)\) can be immediately proved. The group \(SU(n)\) can be constructed in the context of the Clifford algebra \((Cl_{2(p+q)}, [\ , \ ]\), by taking a basis \(\{e_a\}_{a=1}^{2n}\) of \(\mathbb{R}^{p,q}\), where \(e_a^2 = 1\) and \(p + q = n\), and defining the 2-forms \([28]\)

\[
F^{pq} = e^p \wedge e^q + e^{p+n} \wedge e^{q+n},
F^{pq} = e^p \wedge e^{q+n} - e^{p+n} \wedge e^q,
H^q = e^r \wedge e^{r+n} - e^{r+n+1} \wedge e^{r+n+1}.
\]

for \(p, q = 1, \ldots, n, p \neq q\) and \(k = 1, \ldots, n-1\). It follows the expressions

\[
[F^{pq}, F^{rs}] = 2 F^{st}, \quad [F^{pq}, F^{pq}] = -2 H^q, \quad [H^{pq}, F^{st}] = 0, \quad [H^q, H^p] = 0.
\]
and also

\[ [F^{pq}, F^{ps}] = 2E^{qs}, \quad [H^{p}, E^{pq}] = -2F^{pq}, \quad [F^{pq}, F^{st}] = 0, \quad [H^{p}, E^{qs}] = 2F^{qs}, \]  

(49)

where the last commutator is non-trivial only when \( q = p + 1 \). Relations given by Eqs. (48, 49) completely define the Lie algebra \( su(n) \) associated with the group \( SU(n) \).

Now the generators of the isotopic group \( SU(\zeta(n)) \) are written using the isotopic exterior product defined by Eq. (24), denoting \( e_{\zeta}^m = e^m \), as

\[
E_{\zeta}^{pq} = e_{\zeta}^p \wedge e_{\zeta}^q + e_{\zeta}^{p+n} \wedge e_{\zeta}^{q+n},
\]

\[
F_{\zeta}^{pq} = e_{\zeta}^p \wedge e_{\zeta}^{q+n} - e_{\zeta}^{p+n} \wedge e_{\zeta}^q,
\]

\[
H_{\zeta}^r = e_{\zeta}^r \wedge e_{\zeta}^{r+n} - e_{\zeta}^{r+n+1} \wedge e_{\zeta}^{r+n+1},
\]

(50)

where \( p, q = 1, \ldots, n \), \( p \neq q \) and \( r = 1, \ldots, n - 1 \). It follows that the expressions

\[
[E_{\zeta}^{pq}, E_{\zeta}^{st}]_{\zeta} = 2E_{\zeta}^{qt}, \quad [E_{\zeta}^{pq}, F_{\zeta}^{pq}]_{\zeta} = -2H_{\zeta}^q, \quad [E_{\zeta}^{pq}, E_{\zeta}^{st}]_{\zeta} = 0, \quad [H_{\zeta}^q, H_{\zeta}^p]_{\zeta} = 0,
\]

(51)

and the relations

\[
[F_{\zeta}^{pq}, F_{\zeta}^{ps}]_{\zeta} = 2F_{\zeta}^{qs}, \quad [H_{\zeta}^p, E_{\zeta}^{pq}]_{\zeta} = -2F_{\zeta}^{pq}, \quad [F_{\zeta}^{pq}, F_{\zeta}^{st}]_{\zeta} = 0, \quad [H_{\zeta}^p, E_{\zeta}^{qs}]_{\zeta} = 2F_{\zeta}^{qs},
\]

(52)

completely define \( SU(\zeta(n)) \).

XI. APPLICATIONS IN FLAVOR \( SU(n) \) GROUP SYMMETRY

Consider a Hilbert space \( \mathcal{H} \) — and ideal with respect to the algebra defined by the operators acting on it — with elements \( \{\psi_i, \ldots\} \), where \( \langle \psi_i | \psi_j \rangle \in \mathbb{C} \), and the normalized states are given by \( \langle \psi_i | \psi_j \rangle = \delta_{ij} \). In order to formulate the isotopic quantum mechanics, denominated relativistic hadronic mechanics, (RHM), consider now a Hilbert isospace \( \mathcal{H}_\zeta \), which has operators acting on its elements satisfying the the rule given by Eq. (16). Elements \( \zeta | \psi \rangle \in \mathcal{H}_\zeta \), and elements of the dual space \( \langle \zeta | \phi \rangle \in \mathcal{H}_\zeta^* \) satisfy

\[
\langle \zeta | \phi \rangle := \langle \zeta | \zeta^{-1} \phi \rangle \zeta \in \zeta \in \zeta.
\]

(53)

In this case the normalized states are given by \( \langle \zeta | \zeta \phi \rangle = \zeta \in \zeta \). With these definitions, Santilli shows that Hermitean (observable) operators in the quantum mechanical formalism correspond to isohermitean states in RHM.

Hereon \( \zeta \text{-kets } \langle \cdot | \cdot \rangle \) are introduced

\[
| \psi \rangle := \zeta^{-1} | \psi \rangle
\]

(54)

together with the eigenvalue isoequation given by

\[
H \circ | \psi \rangle = H \zeta^{-1} | \psi \rangle = \zeta E \circ | \psi \rangle = E | \psi \rangle,
\]

(55)

where \( | \psi \rangle \) is an element of \( \mathcal{H}_\zeta \) and \( H \) denotes an arbitrary operator acting on \( \mathcal{H}_\zeta \).

A. Exact flavor \( SU(3) \) symmetry, isomesons and isobarions

Hereon the formalism used is implicitly the Clifford algebra \( \mathcal{C}l_{1,3}(\mathbb{C}) \), since \( SU(3) \) is described in terms of \( \mathcal{C}l_{1,3}(\mathbb{C}) \), as in Subsec. (XIA). In Subsecs. (XI A) and (XIB) we use the most recent limits of quark masses given by (12), page 36:

\[
1.5 \text{ MeV} \leq m_u \leq 3.0 \text{ MeV}, \quad 3 \text{ MeV} \leq m_d \leq 7 \text{ MeV},
\]

\[
70 \text{ MeV} \leq m_s \leq 110 \text{ MeV}, \quad 1.16 \text{ GeV} \leq m_c \leq 1.34 \text{ GeV},
\]

\[
4.13 \text{ GeV} \leq m_t \leq 4.27 \text{ GeV}, \quad 170.9 \text{ GeV} \leq m_{\psi} \leq 177.5 \text{ GeV},
\]

(56)
in order to determine the isotopic element $\zeta$, which is shown to be function of these masses. In this sense quark masses are responsible for the deformation of the algebraic structures involved, together with the induced deformation concerning the geometric structure associated with the formalism presented here.

In this Section we briefly recall the lifting of SU(3) [16, 17, 18], in the context introduced in this paper, which main aim is to extend the method to an exact symmetry in the isotopic lifting of SU(6). We must emphasize that since masses are responsible for the deformation of the algebraic structures involved, together with the induced deformation extend Santilli’s idea [17] describing the flavor SU(3) symmetry among quarks $0 = (0, 0)$ in order to determine the isotopic element $\zeta$

Using this choice the isonormalized isostates are given by

$$\lambda^\zeta_1 = \delta^{-1/2} \begin{pmatrix} 0 & g_{11} & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^\zeta_2 = \delta^{-1/2} \begin{pmatrix} 0 & -i g_{11} & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^\zeta_3 = \delta^{-1/2} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & -g_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^\zeta_4 = \delta^{-1/2} \begin{pmatrix} 0 & 0 & g_{11} \\ 0 & 0 & 0 \\ g_{33} & 0 & 0 \end{pmatrix},$$

$$\lambda^\zeta_5 = \delta^{-1/2} \begin{pmatrix} 0 & 0 & -i g_{11} \\ 0 & 0 & 0 \\ ig_{33} & 0 & 0 \end{pmatrix}, \quad \lambda^\zeta_6 = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g_{22} \\ 0 & g_{33} & 0 \end{pmatrix},$$

$$\lambda^\zeta_7 = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i g_{22} \\ 0 & ig_{33} & 0 \end{pmatrix}, \quad \lambda^\zeta_8 = \delta^{-1/2} \frac{1}{\sqrt{3}} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & -2g_{33} \end{pmatrix} \quad (57)$$

and satisfy the properties $|\lambda^\zeta_5, \lambda^\zeta_8\rangle = 2i f_{abc} \delta^{-1/2} \lambda^e_{c}$, where the $su(3)$ structure constants $f_{abc}$ are given by $f_{123} = 2 f_{147} = -2 f_{156} = 2 f_{246} = 2 f_{257} = -2 f_{345} = 2 f_{458} = 2 f_{567} = 2 f_{678} = 1$ [12].

With the condition $\det \zeta = 1$, we endow the Gell-Mann isomatrices with a standard adjoint representation character [15, 17]. Such condition implies that $\zeta$ can be written as

$$\zeta = \text{diag}(\alpha^{-1}, \beta^{-1}, \alpha\beta, 1), \quad \alpha, \beta \in \mathbb{R}. \quad (58)$$

Using this choice the isonormalized isostates are given by

$$|\tilde{\psi}_u\rangle = \left( \begin{array}{c} \alpha^{-1/2} \\ 0 \\ 0 \end{array} \right), \quad |\tilde{\psi}_d\rangle = \left( \begin{array}{c} 0 \\ \beta^{-1/2} \\ 0 \end{array} \right), \quad |\tilde{\psi}_s\rangle = \left( \begin{array}{c} 0 \\ 0 \\ (\alpha\beta)^{1/2} \end{array} \right) \quad (59)$$

and satisfy the relations $\langle \tilde{\psi}_i | \tilde{\psi}_j\rangle = \delta_{ij} \zeta$.

Since the mass operator in $su(3)$ is given by

$$M = \frac{1}{3}(m_u + m_d + m_s)I + \frac{1}{2}(m_u - m_d)\lambda^3 + \frac{\sqrt{3}}{6}(m_u + m_d - 2m_s)\lambda^8 = \text{diag}(m_u, m_d, m_s), \quad (60)$$

the isotopic lifting $SU(3)\zeta$ of SU(3) has the mass operator given by

$$\tilde{M} = \left( \begin{array}{c} \frac{1}{3}(m_u + m_d + m_s)\zeta + \frac{1}{2}(m_u - m_d)\lambda^3 + \frac{\sqrt{3}}{6}(m_u + m_d - 2m_s)\lambda^8 \end{array} \right) \zeta = \text{diag} \left( \alpha^{-1}m_u, \beta^{-1}m_d, \alpha\beta m_s \right). \quad (61)$$
In the simultaneous limits $\alpha \to 1$ and $\beta \to 1$, it can be verified that $\hat{M} \to M$. We now constrain the parameters $\alpha, \beta$ that compose the isounit $\zeta$, imposing that in isospace quarks $u, d$ and $s$ have the same mass $\hat{m} = \alpha^{-1}m_\zeta = \beta^{-1}m_d = \alpha\beta m_s$. Then, and $\alpha, \beta$ are shown to be functions of quarks $u, d, s$ masses, given explicitly by

$$\alpha = \left(\frac{m_u^2}{m_u m_d}\right)^{1/3}, \quad \beta = \left(\frac{m_d^2}{m_s m_u}\right)^{1/3}. \quad (62)$$

Taking the masses values in Eq. (56), the most recent limits of $\alpha$ and $\beta$ are given by

$$0.2204 \leq \alpha \leq 0.2638 \quad 0.2768 \leq \beta \leq 0.3057 \quad (63)$$

The exactness, or a better approximation for the values of $\alpha$ and $\beta$ relies on the precision in the determination of the masses $m_u, m_d$ and $m_s$.

Although the masses $m_u, m_d$ and $m_s$ are imposed to be equal in isospace, in physical space the conventional values of quarks $u, d, s$ masses are given by the eigenvalue isoequation

$$\hat{M} \cdot |\psi\rangle = M\zeta|\psi\rangle = M\zeta|\psi\rangle = \text{diag}(m_u, m_d, m_s)|\psi\rangle \quad (64)$$

or, equivalently, via expected values:

$$\langle \psi_u | \hat{M} | \psi_u \rangle = m_u, \quad \langle \psi_d | \hat{M} | \psi_d \rangle = m_d, \quad \langle \psi_s | \hat{M} | \psi_s \rangle = m_s. \quad (65)$$

The hypercharge operator $Y$ is naturally extended to isospace as

$$\hat{Y} = \frac{1}{2\sqrt{3}} \zeta_8 = \frac{1}{2\sqrt{3}} \text{diag}(\alpha^{-1}, \beta^{-1}, -2(\alpha\beta)), \quad (66)$$

while the $z$ isospin component $I_3$ is given by

$$\frac{1}{2} \zeta_3 = \text{diag}(\alpha, -\beta, 0). \quad (67)$$

Indeed, the expected eigenvalues for the operators above are

$$Y(u) = \langle \psi_u | Y | \psi_u \rangle = \frac{1}{6}, \quad Y(d) = \langle \psi_d | Y | \psi_d \rangle = \frac{1}{6}, \quad (68)$$

and

$$I_3(u) = \langle \psi_u | I_3 | \psi_u \rangle = \frac{1}{2}, \quad I_3(d) = \langle \psi_d | I_3 | \psi_d \rangle = -\frac{1}{2}, \quad (69)$$

The isotopic electric charge operator is obviously given by $Q = Y + I_3$.

Now denoting $|\psi\rangle$ a state describing any of the quarks $|\psi_u, \psi_d, \psi_s\rangle$, an isotopic lifting induces mesons, described by $|\psi\rangle \otimes |\bar{\psi}\rangle$, and barions, described by $|\psi\rangle \otimes |\psi\rangle \otimes |\bar{\psi}\rangle$, to have the corresponding states in isospace

$$|\zeta\rangle \otimes \zeta |\bar{\psi}\rangle \quad (70)$$

for the mesons

$$|\zeta\rangle \otimes \zeta |\bar{\psi}\rangle \otimes \zeta |\bar{\psi}\rangle \quad (71)$$

for the barions. The symbol $\otimes \zeta$ denotes the isotensorial product between spinor fields in $\mathbb{C} \otimes \mathbb{C} \ell_{1,3}$ by

$$(\cdot) \otimes \zeta (\cdot) := \zeta^{-1}(\cdot) \otimes (\cdot)(\zeta^{-1}e_3)^*$ \quad (72)$$

The isomeson can be expressed as

$$\text{isomeson} = \zeta^{-1} \text{ meson} (\zeta^{-1}e_3)^* \quad (73)$$
B. Exact flavor SU(6) isotopic symmetry

Up to now, there is no description of the generators of the group SU(6) in terms of elements of any of the minimal Clifford algebras $\mathcal{C} \ell_{1,6} \simeq \mathcal{C} \ell_{4,4} \simeq \mathcal{C} \ell_{5,2} \simeq \mathcal{M}(8, \mathbb{C})$. Up to our knowledge, there is not any explicit construction like in Section (IX), where in SU(3) is constructed inside the Dirac algebra $\mathcal{C} \ell_{1,3}$, and here we have extended it to the isotopic case. Although there is not such an explicit construction, it is still possible to consider the isotopic lifting of the generators of the representation of SU(6) — seen as a subgroup of $\mathcal{M}(6, \mathbb{C}) \hookrightarrow \mathcal{M}(8, \mathbb{C}) \simeq \mathcal{C} \ell_{1,6} \simeq \mathcal{C} \ell_{3,4} \simeq \mathcal{C} \ell_{5,2}$. Also, the characterization of SU(6) in $\mathcal{M}(8, \mathbb{C})$ can be accomplished if the trivial ‘block’ immersion of elements of SU(6)→ $\mathcal{M}(6, \mathbb{C})$ in $\mathcal{M}(8, \mathbb{C})$, as

$$B \mapsto \begin{pmatrix} B & \delta \\ \delta^T & 1 \end{pmatrix} \in \mathcal{M}(8, \mathbb{C}),$$

where $B \in \text{SU}(6) \hookrightarrow \mathcal{M}(6, \mathbb{C})$, $\delta = (0, 0, 0, 0, 0, 0)^T$, $0 \in \mathbb{C}$, and $1 \in \mathbb{R}$.

A particular case of Eqs. (52), relating exterior algebra elements in $\mathbb{R}^{12,0}$ and the generators of $\mathfrak{su}(6)$ is considered in details in the Appendix.

Now a basis $\{ | \psi_u \rangle, | \psi_d \rangle, | \psi_s \rangle, | \psi_c \rangle, | \psi_t \rangle, | \psi_b \rangle \}$ of the carrier representation space of SU(6) in a Hilbert space $\mathcal{H}$ is considered, an Santilli’s idea [10, 17, 18] is extended to describe the flavor SU(6) symmetry among quarks $u, d, s, c, b,$ and $t$, in such a way that they consequently have the same mass, in isospace.

If we choose the representation of the isonit in SU(6)→ $\mathcal{M}(6, \mathbb{C})$ → $\mathcal{M}(8, \mathbb{C}) \simeq \mathcal{C} \ell_{1,3}(\mathbb{C})$, as being $\zeta = \text{diag}(g_{11}, g_{22}, g_{33}, g_{44}, g_{55}, 1)$, we can describe SU(6) generators by

$$\lambda_1^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & g_{11} & 0 & 0 & 0 & 0 \\ g_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & -ig_{11} & 0 & 0 & 0 & 0 \\ ig_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3^\zeta = \delta^{-1/2} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & -g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & g_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_5^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & -ig_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{22} & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_7^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ig_{22} & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8^\zeta = \delta^{-1/2} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & -2g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(74)
\[
\lambda_9^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & g_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{10}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & -i g_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i g_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_{11}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{12}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i g_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i g_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_{13}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{14}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i g_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_{15}^\zeta = \frac{\delta^{-1/2}}{\sqrt{6}} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_{16}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{17}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & -i g_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i g_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_{18}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{19}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i g_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i g_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_{20}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{21}^\zeta = \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i g_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i g_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

(76)
\[
\begin{align*}
\lambda_{22}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 & 0 \\ 0 & 0 & g_{55} & 0 & 0 \\ 0 & g_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{23}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i g_{44} & 0 \\ 0 & 0 & i g_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{24}^\zeta &= \frac{\delta^{-1/2}}{2\sqrt{6}} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 g_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{25}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & g_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{66} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{26}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & -i g_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i g_{66} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{27}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{28}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i g_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{29}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{66} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{30}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & -i g_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{31}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{32}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i g_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{33}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ g_{55} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_{34}^\zeta &= \delta^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i g_{55} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{35}^\zeta &= \frac{\delta^{-1/2}}{2\sqrt{6}} \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 g_{66} \end{pmatrix}.
\end{align*}
\] (77)

With the condition \( \det \zeta = 1 \), we endow the Gell-Mann isomatrices with a standard adjoint representation character \([15, 17]\). Such condition implies that \( \zeta \) can be written as

\[
\zeta = \text{diag}(\alpha^{-1}, \beta^{-1}, \omega^{-1}, \kappa^{-1}, \tau^{-1}, \alpha \beta \omega \kappa \tau, 1, 1), \quad \alpha, \beta, \omega, \kappa, \tau \in \mathbb{R}.
\] (78)
Using this choice the isonormalized isostates are given by

\[
|\psi_u\rangle = \begin{pmatrix} \alpha^{-1/2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_d\rangle = \begin{pmatrix} 0 \\ \beta^{-1/2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_s\rangle = \begin{pmatrix} 0 \\ 0 \\ \omega^{-1/2} \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

(79)

and satisfy the relations \( \langle \psi_u | \psi_b \rangle = \delta_{ab} \zeta \), where here \( \delta_{ab} \) denotes the Kronecker delta.

Since the mass operator in SU(6) is given by

\[
M = \frac{1}{3}(m_u + m_d + m_s + m_c + m_b + m_t)I + \frac{107}{144}(m_u - m_d)\lambda_3
\]

\[- \frac{55\sqrt{3}}{144}(m_u + m_d - 2m_s)\lambda_8 - \frac{55\sqrt{6}}{144}(m_u + m_d + m_s - 3m_c)\lambda_{15}
\]

\[- \frac{11\sqrt{6}}{24}(m_u + m_d + m_s + m_c - 4m_b)\lambda_{24} - \frac{\sqrt{30}}{6}(m_u + m_d + m_s + m_c + m_b - 5m_t)\lambda_{35}
\]

\[= \text{diag}(m_u, m_d, m_s, m_c, m_b, m_t), \quad (80)
\]

the isotopic lifting SU(6)_c of SU(6) has the mass operator given by

\[
\tilde{M} = \frac{1}{3}(m_u + m_d + m_s + m_c + m_b + m_t)\zeta + \frac{107}{144}(m_u - m_d)\lambda_3\zeta
\]

\[- \frac{55\sqrt{3}}{144}(m_u + m_d - 2m_s)\lambda_8\zeta - \frac{55\sqrt{6}}{144}(m_u + m_d + m_s - 3m_c)\lambda_{15}\zeta
\]

\[- \frac{11\sqrt{6}}{24}(m_u + m_d + m_s + m_c - 4m_b)\lambda_{24}\zeta - \frac{\sqrt{30}}{6}(m_u + m_d + m_s + m_c + m_b - 5m_t)\lambda_{35}\zeta
\]

\[= \text{diag}(\alpha^{-1}m_u, \beta^{-1}m_d, \omega^{-1}m_s, \kappa^{-1}m_c, \tau^{-1}m_b, \alpha\beta\omega\kappa\tau m_t). \quad (81)
\]

By extending the process of the previous subsection to the SU(6) isotopic lifting describing an exact flavor symmetry, the mass operator in isospace, considering six quarks, is presented as

\[
\tilde{M} = M\zeta = \text{diag}(\alpha^{-1}m_u, \beta^{-1}m_d, \omega^{-1}m_s, \kappa^{-1}m_c, \tau^{-1}m_b, \alpha\beta\omega\kappa\tau m_t)
\]

(82)

\[= \text{diag}(\tilde{m}, \tilde{m}, \tilde{m}, \tilde{m}, \tilde{m}, \tilde{m})
\]

(83)

where in this case the isounit is given by \( \zeta = \text{diag}(\alpha^{-1}, \beta^{-1}, \omega^{-1}, \kappa^{-1}, \tau^{-1}, \alpha\beta\omega\kappa\tau) \). Imposing Eq. (82), we see that each term of the matrix in Eq. (82) must equal each other, i.e.,

\[
\alpha^{-1}m_u = \beta^{-1}m_d = \omega^{-1}m_s = \kappa^{-1}m_c = \tau^{-1}m_b = \alpha\beta\omega\kappa\tau m_t
\]

(84)

and in particular, let us isolate all the variables in terms of the variable \( \alpha \):

\[
\beta = \frac{m_d}{m_u}, \quad \omega = \frac{m_s}{m_u}, \quad \kappa = \frac{m_c}{m_u}, \quad \tau = \frac{m_b}{m_u}.
\]

(85)

Substituting Eqs. (82) in the last of Eqs. (84) \( \alpha^{-1}m_u = \alpha\beta\omega\kappa\tau m_t \) yields

\[
\alpha^{-1}m_u = \alpha^5 \left( \frac{m_d}{m_u} \right) m_u m_s m_c m_b m_t
\]

(86)
implying that

$$\alpha = \left( \frac{m_u^5}{m_d m_s m_c m_b m_t} \right)^{1/6}$$  \hspace{1cm} (87)

In the same way it can be shown that

$$\beta = \left( \frac{m_d^5}{m_u m_s m_c m_b m_t} \right)^{1/6}, \quad \omega = \left( \frac{m_s^5}{m_d m_u m_c m_b m_t} \right)^{1/6}$$  \hspace{1cm} (88)

$$\kappa = \left( \frac{m_c^5}{m_d m_u m_s m_b m_t} \right)^{1/6} \tau = \left( \frac{m_b^5}{m_d m_u m_s m_c m_t} \right)^{1/6}$$  \hspace{1cm} (89)

Substituting the values of quarks masses [12] in (56) yields

$$5.945 \times 10^{-3} \leq \alpha \leq 8.212 \times 10^{-3}$$
$$1.189 \times 10^{-2} \leq \beta \leq 1.920 \times 10^{-2}$$
$$2.774 \times 10^{-1} \leq \omega \leq 3.018 \times 10^{-1}$$
$$3.676 \leq \kappa \leq 4.598$$
$$486.938 \leq \tau \leq 677.379$$

XII. CONCLUDING REMARKS

This paper presents for the first time the isotopies of Clifford algebras with relevant applications in flavor symmetry of quarks. We have formulated the isotopic liftings in the context of Clifford algebras and highlighted the formal description concerning isotopies for non-associative general algebras. The structure of the isoalgebra identifies the six quarks is shown to be exact if the isotopic group SU(6) is regarded. We have shown that the unit of the isotopic Clifford algebra is a function of the six quark masses. It illustrates how phenomenological data concerning quark masses can constrain the geometry of spacetime, where the limits constraining the parameters, that are entries of the representation of the isounit in the isotopic group SU(6), are based on the most recent limits imposed on quark masses. We assert that the formulation of other theories in isospace can bring a new class of solutions of open questions in theoretical physics.

XIII. APPENDIX: CLIFFORD ALGEBRA GENERATORS OF \( su(6) \)

In order to completely define \( su(6) \) in terms of the exterior algebra of \( \mathcal{C}l_{12,0} \), in the light of Eqs. 52,

$$E_{\zeta}^{pq} = e_\zeta \wedge e_\zeta^q + e_\zeta^{p+n} \wedge e_\zeta^{q+n}$$

$$F_{\zeta}^{pq} = e_\zeta \wedge e_\zeta^{p+n} - e_\zeta^{p+n} \wedge e_\zeta^{q+n}$$

$$H_{\zeta}^r = e_\zeta^r \wedge e_\zeta^{r+n} - e_\zeta^{r+n+1} \wedge e_\zeta^{r+n+1}.$$  \hspace{1cm} (90)

where \( p, q = 1, ..., 6, \ p \neq q \) and \( r = 1, ..., 5 \), let us identify \{\( \lambda^\zeta_1, ..., \lambda^\zeta_{15} \)\} = \{\( E_{\zeta}^{pq} \)\}, \{\( \lambda^\zeta_{16}, ..., \lambda^\zeta_{30} \)\} = \{\( F_{\zeta}^{pq} \)\}, and \( \{\lambda^\zeta_{31}, ..., \lambda^\zeta_{35} \} = \{H^1_{\zeta}, H^2_{\zeta}, H^3_{\zeta}, H^4_{\zeta}, H^5_{\zeta} \} \). By this identification it can be verified that the set \{\( \lambda^\zeta_1, ..., \lambda^\zeta_{35} \)\}, explicitly constructed in Eqs. 76, 77 of generators of \( su(6) \) satisfy

$$[E_{\zeta}^{pq}, E_{\zeta}^{rql}]_{\zeta} = 2E_{\zeta}^{pq}, \quad [F_{\zeta}^{pq}, F_{\zeta}^{rql}]_{\zeta} = -2H^q_{\zeta}, \quad [E_{\zeta}^{pq}, E_{\zeta}^{rql}]_{\zeta} = 0, \quad [H^q_{\zeta}, H^p_{\zeta}]_{\zeta} = 0,$$  \hspace{1cm} (91)
and the relations

\[ [F^p_{\zeta}, F^q_{\zeta}]_\zeta = 2E^q_{\zeta}, \quad [H^p_{\zeta}, E^q_{\zeta}]_\zeta = -2F^p_{\zeta}, \quad [F^p_{\zeta}, F^q_{\zeta}]_\zeta = 0, \quad [H^p_{\zeta}, E^q_{\zeta}]_\zeta = 2F^q_{\zeta}, \quad (92) \]

for \( p, q = 1, \ldots, 6, p \neq q \) and \( r = 1, \ldots, 5 \).

It is well known that \( C_{12,0} \cong M(32, \mathbb{H}) \), and we included \( SU(6) \hookrightarrow M(6, \mathbb{C}) \hookrightarrow M(8, \mathbb{C}) \hookrightarrow M(32, \mathbb{H}) \), via Eq. (74) and the inclusion \( A \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), where \( A \in M(8, \mathbb{C}) \), and \( 0_8 \equiv 0_{8 \times 8} \in M(8, \mathbb{C}) \). Up to our knowledge, there is not any criterion or method to explicitly include \( su(n) \) in any Clifford algebra \( C_j \), where \( j < 2n \).

Acknowledgments

R. da Rocha thanks to Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) for financial support.

[1] Hestenes D and Sobczik G, Clifford Algebra to Geometric Calculus, D. Reidel, Dordrecht 1984.
[2] Kadeisvili J V, An Introduction to the Lie-Santilli Isotopic theory, Mathematical Methods in Applied Sciences 19, 1349-1395 (1996).
[3] Santilli R M, Isonumber and genonumbers of dimension 1, 2, 4, 8, their isoduals and pseudoduals, and hidden numbers of dimension 3, 5, 6, 7, Algebras, Groups and Geometries 10, 273-321 (1993).
[4] Santilli R M, Nonlocal integral isotoneries of differential calculus, geometries and mechanics, Rendiconti Circolo Matematico Palermo, Suppl. 42, 7-82 (1996).
[5] Santilli R M, Elements of Hadronic Mechanics, Vol. I: Mathematical Foundations (1993); Vol. II: Theoretical Foundations (1993), Vol. III: Recent Advances, Experimental Verifications and Industrial Applications in press, preliminary version available in pdf format at http://www.i-b-r.org/Hadronic-Mechanics.htm, Ukrainian Academy of Sciences, Kiev.
[6] Kadeisvili J V, Santilli’s Isotopies of Contemporary Algebras, Geometries and Relativities, 2nd Edition, Ukraine Academy of Sciences, Kiev 1997.
[7] Santilli R M, Lie-admissible invariant representation of irreversibility for matter and antimatter at the classical and operator levels, Nuovo Cimento B 121, 443-498 (2006).
[8] Santilli R M, Isotopic breaking of gauge theories, Phys. Rev. D20, 555-570 (1979).
[9] Kadeisvili J V, Kamiya N and Santilli R M, A characterization of isofields and their isoduals, Hadronic J. 16, 169-187 (1993).
[10] Klimyk S U e Santilli R M, Standard isorepresentations of isotopic Q-operator deformations of Lie algebras, Algebras, Groups and Geometries 10, 323-332 (1993).
[11] Santilli R M, Isotopic lifting of quark theories with exact confinement and convergent perturbative expansions, Comm. Theor. Phys. 4, 1-23 (1995).
[12] Alvarez-Gaumé L et al. (eds.), Review of Particle Physics, Phys. Letters B592, 1-1109 (2006).
[13] Loumesto P, Octonions and triality, Advances in Applied Clifford Algebras 11, (2) 191-213 (2001).
[14] Baylis W, The paravector model of spacetime, em Clifford (Geometric) Algebras with Applications in Physics, Mathematics and Engineering, Birkhäuser, Berlin 1995.
[15] Santilli R M, Isotopic lifting of SU(2) symmetry with application to nuclear physics, JINR (Joint Institute for Nuclear Research) Rapid Comm. 6, 24-38 (1993).
[16] Santilli R M, Isotopic lifting of quark theories, Int. J. Phys. 1, 1-26 (1995).
[17] Santilli R M, Isorepresentations of the Lie-isotopic SU(2) algebra with applications to nuclear physics and to local realism, Acta Applicandæ Mathematicæ 50, 177-190 (1998).
[18] Animalu A O E, A gauge-invariant relativistic theory of the Rutherford-Santilli neutron, Hadronic J. 27, 599-624 (2004).
[19] Cederwall M and Preitschopf C R, \( S^7 \) and its Kać-Moody Algebra, Commun. Math. Phys. 167, 373 (1995) [hep-th/9309030].
[20] da Rocha R and Vaz Jr. J, Clifford algebra-parametrized octonions and generalizations, Journal of Algebra 301, 459-473 (2006) [math-ph/0603053].
[21] Cederwall M, Introduction to Division Algebras, Sphere Algebras and Twisters, Talk presented at the Theoretical Physics Network Meeting at NORDITA, Copenhagen, sept. 1993 [hep-th/9310115].
[22] da Rocha R and Vaz Jr. J, Twisters, Generalizations and Exceptional Structures, PoS(WC2004) (2004) 022 [math-ph/0412037].
[23] Stfanumoor M and Misra K C (eds.), Kać-Moody Lie algebras and related topics: Ramanujan International Symposium on Kać-Moody Algebras and Applications, AMS Colloquium Publications, Providence 2004.
[24] Bengtsson I and Cederwall M, Particles, twisters and division algebras, Nucl. Phys. B302 (1988) 81-103.
[25] Lounesto P, Counterexamples in Clifford algebras with CLICAL, pp. 3-30 in Abianowicz R et al. (eds.): “Clifford Algebras with Numeric and Symbolic Computations”, Birkhäuser, Boston 1996; Counterexamples in Clifford algebras, Adv. Appl. Cliff. Algebras 6 (1996) 69-104.

[26] Dixon G M, Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics Kluwer, Dordrecht, 1994.

[27] Dixon G M, Octonion XY-product [hep-th/9503053]; Octonion X-product orbits [hep-th 9410202]; Octonion X-product and octonion lattices [hep-th 9411083].

[28] Keller J and Rodríguez-Romo S, Multivectorial representation of Lie groups, Int. J. Theor. Phys. 30 (2) 185-196 (1991).

[29] Chisholm J and Farewell R, Unified spin gauge theories of the four fundamental forces, in Micali A et al.(eds.) Clifford Algebras and Their Applications in Math. Physics, Kluwer Acad. Publishers, Dordrecht 1989.