G-Sasaki Manifolds and K-Energy

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Abstract
In this paper, we introduce a class of Sasaki manifolds, called $G$-Sasaki manifolds with a reductive $G$-group action on their Kähler cones. By proving the properness of K-energy on such manifolds, we obtain a sufficient and necessary condition for the existence of $G$-Sasaki–Einstein metrics. A similar result is also obtained for $G$-Sasaki–Ricci solitons. As an application, we construct many new examples of $G$-Sasaki–Ricci solitons by an established openness theorem.

Keywords K-energy · Lie group · Sasaki–Einstein metrics

Mathematics Subject Classification Primary: 53C25 · Secondary: 32Q20 · 53C55

1 Introduction

In this paper, we introduce a class of Sasaki manifolds $M$, called $G$-Sasaki manifolds with a Lie group $G$ action on their Kähler cones $C(M)$ (cf. Definition 3.1). One of our motivations is from the recent progress in Kähler geometry on the compactification of Lie Group $G$ (for simplicity, called the $G$-manifold if it is smooth and Kählerian). We refer the readers to see [3,18–20,34,35], etc.

In Sasaki geometry, a transverse Kähler metric is very closely related to a Kähler metric on a complex manifold (cf. [8,26]). In particular, if a Sasaki manifold $M$ is regular or quasi-regular, then $M$ is just an $S^1$-bundle over a Kähler manifold or an orbifold. Another relationship is that a transverse Sasaki–Einstein metric corresponds to
a Kähler–Ricci flat cone. Recently, Collins and Székelyhidi established a link between transverse Sasaki–Einstein metrics and stable Kähler cones in terms of the Yau–Tian–Donaldson conjecture [16]. The question of existence of Sasaki–Einstein metrics has received increasing attention in the physics community through their connection to the AdS/CFT correspondence (cf. [15,36,38]). We refer the reader to see many interesting examples of such metrics in a monumental work of Boyer and Galicki [8].

The purpose of this paper is to extend the K-energy method in the study of canonical metrics on $G$-manifolds [35] to $G$-Sasaki manifolds. In particular, we obtain a sufficient and necessary condition for the existence of transverse Sasaki–Einstein metrics on $G$-Sasaki manifolds (cf. Theorem 1.1). Our result also generalizes a beautiful theorem of Futaki, Ono, and Wang for the existence of transverse Sasaki–Ricci solitons on toric Sasaki manifolds [26].

To state our main results, let us recall some notations for the Lie group. As above, $G$ is a complex, connected, reductive group of complex dimension $(n + 1)$, which is the complexification of a compact Lie group $K$. Let $T$ be a maximal compact torus of $K$ and $T^c$ its complexification. We denote Lie algebras of $G$ and $T$ by $g$, $t$, respectively. Set $a = J_G t$, where $J_G$ is the complex structure of $G$. Then there is a system $R^+_G$ of positive roots of $(G, T^c)$, which defines a positive Weyl chamber $a^+ \subset a$.

By definition (cf. Definition 3.1), the Kähler cone $C(M)$ of $G$-Sasaki manifold $M$ contains a toric cone $Z$ generated by the torus $T^c$ (cf. Sect. 3). Then there are a moment polytope cone $\mathcal{C}$ associated to $Z$ and a restricted moment polytope $\mathcal{P} \subset \mathcal{C}$ associated to $Z \cap M$, respectively (cf. Sects. 3, 4). Let $\mathcal{P}_+ = \mathcal{P} \cap a^*_+$.

We define the barycenter of $\mathcal{P}_+$ by

$$
\text{bar}(\mathcal{P}_+) = \frac{\int_{\mathcal{P}_+} y \pi \, d\sigma_c}{\int_{\mathcal{P}_+} \pi \, d\sigma_c},
$$

where $d\sigma_c$ is the Lebesgue measure on $\mathcal{P}$, and $\pi(y) = \prod_{\alpha \in R^+_G} (y, \alpha)^2$ is a function on $a^*_+$ related to positive roots in $R^+_G$.

Let $\Xi$ be the relative interior of the cone generated by $R^+_G$ and set

$$
\sigma = \frac{1}{2} \sum_{\alpha \in R^+_G} \alpha.
$$

Then we can state our result for the existence of transverse Sasaki–Einstein metrics on $G$-Sasaki manifolds as follows.

**Theorem 1.1** Let $(M, g)$ be a $(2n + 1)$-dimensional $G$-Sasaki manifold with its transverse Kähler form $\omega^T_g \in \frac{n}{n+1} c^B_1(M) > 0$, where $c^B_1(M)$ denotes the first basic Chern class of $M$. Then $M$ admits a transverse Sasaki–Einstein metric if and only if the barycenter $\text{bar}(\mathcal{P}_+)$ satisfies

$$
\text{bar}(\mathcal{P}_+) - \frac{2}{n+1} \sigma + \frac{1}{n+1} \gamma_0 \in \Xi,
$$

(1.1)
where $\gamma_0$ is a rational vector in $a^*_z$ (the dual of $a_z$) uniquely determined in Proposition 6.1.

Theorem 1.1 can be regarded as a version of Delcroix’s theorem for the existence of Kähler–Einstein metrics on $G$-manifolds in case of $G$-Sasaki manifolds (cf. [18,35]). (1.1) is an obstruction to the existence of $G$-Sasaki–Einstein metrics. In fact, we will use an argument in [49] and [50] to derive an analytic obstruction to the existence of $G$-Sasaki metrics with constant transverse scalar curvature in terms of convex Weyl-invariant piecewise linear functions (cf. Proposition 7.4). Then by a construction of piecewise linear function in [35], the analytic obstruction implies (1.1).

To prove the sufficient part of Theorem 1.1, we show the properness of K-energy on the space of $K \times K$-invariant potentials on $\frac{n}{n+1}c^B_1(M)$ through computing the reduced K-energy (cf. Theorems 5.6 and 6.3). The reduced K-energy was first introduced by Donaldson to study the lower bound of Mabuchi K-energy on toric manifolds [23]. Subsequently, Zhou and Zhu extended Donaldson’s method to prove the properness of K-energy [49]. This method was recently generalized by Li, Zhou, and Zhu to $G$-manifolds [35], and by Delcroix to horosymmetric spaces [20], respectively. All those results for the properness of K-energy will imply the existence of constant scalar curvature metrics by a significant work of Chen and Cheng [13].

At present, the reduced K-energy $\mu(\cdot)$ is defined on a class of convex functions on $\iota^*(P_+)$, where $\iota^*$ is an isomorphism from $P_+$ to a polytope $P$ in a subspace of codimension one in $a^*$ (cf. Sect. 4). In general, $P$ does not satisfy the Delzant condition (cf. Sect. 6.1). Thus, we shall extend the method in the proof of main theorem [35, Theorem 1.2]. In fact, our method works for any $G$-Sasaki manifold to give a criterion for the properness of K-energy (cf. Theorem 5.6). It is interesting to mention that the form of $\mu(\cdot)$ may depend on the choice of $\iota^*$ if its transverse Kähler class of $G$-Sasaki manifold does not belong to a multiple of $c^B_1(M)$ (cf. Remark 5.7).

Another ingredient in the proof of Theorem 1.1 is that we prove a general existence result for transverse Sasaki–Einstein metrics on Sasaki manifolds under the properness of K-energy modulo in the transverse holomorphic automorphism group (cf. Proposition 7.3). The result also generalizes Zhang’s Theorem (cf. Theorem 7.1) to Sasaki manifolds which may be admitting transverse holomorphic vector fields.

An analogy of Theorem 1.1 will also be established for $G$-Sasaki Ricci solitons (cf. Theorem 8.1). Moreover, by deformation of Reeb vector fields as in [37,38], we prove the following openness theorem for transverse Sasaki–Ricci solitons.

**Theorem 1.2** Let $(M, \xi_0)$ be a $G$-Sasaki manifold which admitting a transverse Sasaki–Ricci soliton. Let $\mathcal{C}^\vee$ be the interior of the dual cone of $\mathcal{C}$ and $\Sigma = \mathcal{C}^\vee \cap a_z$. Set
\[ \Sigma_0 = \Sigma \cap \{ \xi | \gamma_0(\xi) = -(n+1) \}. \] 

---

1 According to a recent work of He, Theorem 5.6 gives a criterion for the existence of transverse constant scalar curvature metrics on $G$-Sasaki manifolds [29].
Then for any \( \xi \in \Sigma O \) sufficiently close to \( \xi_0 \), \( (M, \xi) \) admits a transverse Sasaki–Ricci soliton with respect to the Reeb vector field \( \xi \).

It is clear that \( \xi \in \Sigma O \) in (1.2) may not be rational. Thus, by Theorem 1.2, one can construct many irregular \( G \)-Sasaki–Ricci solitons from a \( G \)-Sasaki–Einstein metric in \( \frac{\pi}{n+1} c^B_1(M) \); see examples in Sect. 9, for details. We shall mention that there is a general openness theorem for transverse Sasaki–Ricci solitons established by He–Sun in [30, Theorem 4.1].\(^2\) However, in Theorem 1.2, we also show that the deformation of \( G \)-Sasaki–Ricci soliton \( (M, \xi_0) \) keeps \( G \)-Sasaki structure via Theorem 8.1 (cf. Sect. 8.1). By the way, our proof is different to the one of [30, Theorem 4.1]. In fact, we need to deform the soliton vector field through deformation of Reeb vector fields, so our proof here seems more directly by using the idea in [43].

The organization of paper is as follows. In Sect. 2, we recall some basic knowledge in Sasaki geometry, and in Sect. 3, we introduce the notion of \( G \)-Sasaki manifolds \( M \). In Sect. 4, we begin to study \( K \times K \)-invariant metrics and discuss the moment map restricted on a torus orbit in \( M \). The reduced \( K \)-energy \( \mu(\cdot) \) will be computed in Sect. 5 and then a criterion for the properness of \( \mu(\cdot) \) will be established (cf. Theorem 5.6). Section 6 is the main technical part, where we prove Theorem 5.6 in case of \( \omega^T_g \in \frac{\pi}{n+1} c^B_1(M) \). Theorems 1.1 and 1.2 will be proved in Section 7 and Section 8, respectively. In Sect. 9, we give several examples of \( G \)-Sasaki–Einstein metrics and \( G \)-Sasaki–Ricci solitons.

### 2 Sasaki Geometry

By definition, a \((2n + 1)\)-dimensional Riemannian manifold \( (M, g) \) is called a \textit{Sasaki manifold} if and only if its cone manifold \( (C(M), \bar{g}) \) is a Kähler manifold, where

\[
C(M) = M \times \mathbb{R}^+, \quad \bar{g} = d\rho^2 + \rho^2 g,
\]

and \( \rho \in \mathbb{R}^+ \). Following [26], we denote

\[
\xi = J \rho \frac{\partial}{\partial \rho}, \quad \eta(\cdot) = \frac{1}{\rho^2} \bar{g}(\cdot, \xi),
\]

where \( J \) is the complex structure of \( C(M) \). Identify \( M \) with \( \{ \rho = 1 \} \). The restriction of \( \xi \) on \( M \) is called the \textit{Reeb vector field} of \( M \). Let \( \nabla^g \) be the Levi-Civita connection of \( g \). Then \( \Phi(X) = \nabla^\bar{g} \xi \) defines an \((1, 1)\)-tensor \( \Phi \) on \( TM \). We call \( (g, \xi, \eta, \Phi) \) the \textit{Sasaki structure} of \( (M, g) \).

By [8], the relationship between \( J \) and \( \Phi \) on \( M \) is given by

\(^2\) The authors would like to thank the referee for telling them the reference [30].
\[ J(X) = \begin{cases} \Phi(X) - \eta(X)\rho \frac{\partial}{\partial \rho}, & X \in TM, \\ \xi, & X = \rho \frac{\partial}{\partial \rho}. \end{cases} \tag{2.2} \]

Thus, there are local coordinates \((x^0_\alpha, z^1_\alpha, \ldots, z^n_\alpha)\) on each chart \(U_\alpha \subset M\), where \(z^i_\alpha = x^i_\alpha + \sqrt{-1}x^{i+n}_\alpha \in \mathbb{C}, i = 1, \ldots, n\), such that
\[ \xi = \frac{\partial}{\partial x^0_\alpha}, \]
and
\[ \frac{\partial z^i_\beta}{\partial x^0_\alpha} = 0, \quad \frac{\partial z^i_\beta}{\partial \bar{z}^j_\alpha} = 0, \quad \forall 1 \leq i, j \leq n, \]
whenever \(U_\alpha \cap U_\beta \neq \emptyset\) [8, Chapter 6]. These local coordinates form a transverse holomorphic structure on \(M\). Then the corresponding complex structure \(\Phi^T_\alpha\) is given by
\[ \Phi^T_\alpha \left( \frac{\partial}{\partial x^i_\alpha} \right) = \frac{\partial}{\partial x^{i+n}_\alpha}, \tag{2.3} \]
which forms a global transverse complex structure \(\Phi^T\) on \(M\).

Denote the transverse holomorphic group of \(M\) by \(\text{Aut}^T(M)\) and the holomorphic transformations group of \(C(M)\) by \(\text{Aut}^\xi(C(M))\), which commutes with the holomorphic flow generated by \(\xi - \sqrt{-1}J\xi\). Then
\[ \text{Aut}^T(M) \cong \text{Aut}^\xi(C(M)). \]

To see this isomorphism, we note that for any \(f \in \text{Aut}^\xi(C(M))\), \(f\) commutes with \(\pi_r\), where \(\pi_r\) is the projection from \(C(M)\) to its level set \(M \cong \{\rho = 1\}\). Then one can define a map \(\hat{f}\) by
\[ \hat{f} = \pi_r \circ f : M \to M. \tag{2.4} \]
It is easy to see that \(f_*\xi = \xi\) and \(\pi_r*\xi = \xi\). This implies that \(\hat{f}\) preserves \(\xi\). On the other hand, the complex structure \(J\) on \(C(M)\) is preserved by \(f\). Thus, by (2.3), the transverse holomorphic structure \(\Phi^T\) is preserved by \(\hat{f}\).

### 2.1 Basic Forms and Transverse Kähler Structure

An \(m\)-form \(\Omega\) on \(M\) is called basic if
\[ L_\xi \Omega = 0, \quad i_\xi \Omega = 0. \tag{2.5} \]
This means that
\[
\Omega(\alpha)_{0i_2...i_m} = 0, \quad \frac{\partial}{\partial x^0(\alpha)} \Omega(\alpha)_{i_1...i_m} = 0, \quad \forall i_1, ..., i_m,
\]
if we let \( \Omega = \Omega(\alpha)_{i_1...i_m} dx^{i_1}_{(\alpha)} \wedge \cdots \wedge dx^{i_m}_{(\alpha)} \) under local transverse holomorphic coordinates \( x^0(\alpha), x^1(\alpha) = z^1(\alpha), \ldots, x^n(\alpha) = z^n(\alpha) \). Thus, \( \partial_B, \bar{\partial}_B \) operators are well defined for any basic \( m \)-form \( \Omega \), locally defined as
\[
\partial_B \Omega = \frac{\partial}{\partial z^j(\alpha)} \Omega(\alpha)_{Ij} dz^j(\alpha) \wedge d\bar{z}^I(\alpha)
\]
and
\[
\bar{\partial}_B \Omega = \frac{\partial}{\partial \bar{z}^j(\alpha)} \Omega(\alpha)_{Ij} dz^j(\alpha) \wedge d\bar{z}^I(\alpha)
\]
for multiple indexes \( I, J \subset \{1, ..., n\} \). Similar to the Hodge–Laplace operator, we introduce
\[
\Delta_B = \sqrt{-1} (\bar{\partial}_B \partial_B + \bar{\partial}_B \partial_B).
\]
In particular, for a basic function \( f \), we have
\[
\Delta_B f = \sqrt{-1} \text{tr}_g (\partial_B \bar{\partial}_B f).
\]
We are interested in basic \((1, 1)\)-forms. Since \( \bar{g} \) is a cone metric, we have
\[
\omega_{\bar{g}} = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \rho^2. \tag{2.6}
\]
It follows
\[
d\eta = 2 \sqrt{-1} \partial \bar{\partial} \log \rho. \tag{2.7}
\]
This means that \( \frac{1}{2} d\eta|_M = \omega^T_{\bar{g}} \) is a positive basic \((1, 1)\)-form. Usually, \( \omega^T_{\bar{g}} \) is called the transverse Kähler form. The following lemma shows that \( \hat{f} \in \text{Aut}^T(M) \) preserves transverse Kähler class.

**Lemma 2.1** Let \((M, g)\) be a compact \((2n + 1)\)-dimensional Sasaki manifold. Then \( f^* d\eta|_M \) is a basic form for any \( f \in \text{Aut}^\hat{\xi}(C(M)) \). Consequently, \( \hat{f}^* \omega^T_{\bar{g}} \in [\omega^T_{\bar{g}}]_B \).

**Proof** Note that
\[
f^* (d\eta) = f^* (\sqrt{-1} \partial \bar{\partial} \log \rho) = \sqrt{-1} \partial \bar{\partial} \log f^* \rho.
\]
Then to prove the lemma, it suffices to show that $\psi = \log f^* \rho - \log \rho$ satisfies

$$
\xi(\psi)(x) = 0 \text{ and } \frac{\partial}{\partial \rho} (\psi)(x) = 0, \ \forall \ x \in M.
$$

In fact,

$$
\xi(\psi)(x) = \frac{1}{f^* \rho(x)} f_\ast \xi(\rho)(f(x)) - \frac{1}{\rho} \xi(\rho)(x)
= \frac{1}{f^* \rho(x)} \xi(\rho)(f(x)) - \frac{1}{\rho} \xi(\rho)(x).
$$

Then $\xi(\psi)(x) = 0$, since $\xi(\rho) \equiv 0$. On the other hand, by $\xi = J \rho \frac{\partial}{\partial \rho}$, we have

$$
f_\ast \left( \rho \frac{\partial}{\partial \rho} \right) = \left( \rho \frac{\partial}{\partial \rho} \right).
$$

It follows

$$
\frac{\partial}{\partial \rho} (\psi)(x) = \frac{1}{f^* \rho(x)} \left( f_\ast \left( \frac{\partial}{\partial \rho} \right) \right)(f(x)) - \frac{1}{\rho(x)}
= \frac{1}{f^* \rho(x)} \frac{f^* \rho(x)}{\rho(x)} \left( \frac{\partial}{\partial \rho} \right) \rho(f(x)) - \frac{1}{\rho(x)}
= 0.
$$

$\square$

For a transverse Kähler form $\omega^T_g$, its transverse Ricci form is defined by

$$
\text{Ric}^T(g) = -\sqrt{-1} \partial_B \bar{\partial}_B \log \det(g^T_{i\bar{j}}), \text{ on } U_{(\alpha)},
$$

where $\omega^T_g = \sqrt{-1} g^T_{i\bar{j}} dz^i(\alpha) \wedge d\bar{z}^j(\alpha)$ on $U_{(\alpha)}$. Clearly, $\text{Ric}^T(g)$ is also a basic $(1, 1)$-form. Similar to the Kähler form, $\text{Ric}^T(g)$ is $d_B$-closed and the basic cohomology class $[\text{Ric}^T(g)]_B$ is independent with the choice of $\omega^T_g$ in $[\omega^T_g]_B$. We call $c^B_1(M) = \frac{1}{2\pi} [\text{Ric}^T(g)]_B$ the basic first Chern class. It was proved in [26, Proposition 4.3] that

**Proposition 2.2** The first basic Chern class is represented by $c d\eta$ for some constant $c$ if and only if $c_1(D) = 0$, where $D = \ker(\eta)$.

A Sasaki metric $g$ is called a Sasaki–Einstein metric on $M$ if it satisfies

$$
\text{Ric}(g) = 2ng.
$$
In case of $c_1(D) = 0$, the above equation is equivalent to the following transverse Sasaki–Einstein equation (cf. [26]),

$$\text{Ric}^T(g) = 2(n + 1)\omega_g^T.$$  (2.8)

In particular, $c^B_1(M) > 0$.

To solve (2.8), it turns to find a basic $C^\infty$-function $\psi$ in the following class (the space of transverse Kähler potentials),

$$\mathcal{H}\left(\frac{1}{2}d\eta\right) = \{\psi \in C^\infty(M) \text{ is a basic function} | \omega^T_{g\psi} = \frac{1}{2}d\eta + \sqrt{-1}\partial_B\bar{\partial}_B\psi > 0\}$$

such that $\omega^T_{g\psi} = \frac{1}{2}d\eta_{\psi} = \omega^T_g + \sqrt{-1}\partial_B\bar{\partial}_B\psi$ satisfies (2.8). Then (2.8) is reduced to a complex Monge-Ampère equation on each $U(\alpha)$ with transverse holomorphic coordinates $(z^1(\alpha), \ldots, z^n(\alpha))$,

$$\det(g^T(\alpha)_{i\bar{j}} + \psi(\alpha),_{i\bar{j}}) = \exp(-2(n + 1)\psi + h) \det(g^T(\alpha)_{i\bar{j}}),$$  (2.9)

where $h$ is a basic Ricci potential determined by

$$\text{Ric}^T(g) = 2(n + 1)\omega_g^T + \sqrt{-1}\partial_B\bar{\partial}_B h.$$  (2.10)

We will discuss (2.9) in Sect. 7 for details.

### 2.2 Futaki Invariant

In general, there is no solution of (2.9) since there are some obstructions to the existence of transverse Sasaki–Einstein metrics, such as Futaki invariant (cf. [10,26]). As in Kähler geometry, the Futaki invariant is defined for Hamiltonian vector fields. We call a complex vector field $X$ a Hamiltonian holomorphic vector field on a Sasaki manifold if $X$ satisfies (cf. [26, Definition 4.5]):

1. On each $U(\alpha)$, $\pi(\alpha)_*(X)$ is a (local) holomorphic vector field on $\mathbb{C}^n$, where $\pi(\alpha)(\cdot)$ is the projection given by

   $$\pi(\alpha)(x_{\alpha}^0, z^1(\alpha), \ldots, z^n(\alpha)) = (z^1(\alpha), \ldots, z^n(\alpha));$$

2. The complex-valued function $U_X = \sqrt{-1}\eta(X)$ satisfies

   $$\partial_B\bar{\partial}_B U_X = -\frac{\sqrt{-1}}{2}i_X d\eta.$$
Denote by \( \mathfrak{ham}(M) \) the Lie algebra of the Hamiltonian holomorphic vector fields. The Futaki invariant \( \text{Fut}(X) \) is defined by

\[
\text{Fut}(X) = -\int_M X(h) \left( \frac{1}{2} d\eta \right) \wedge \eta, \quad \forall \ X \in \mathfrak{ham}(M).
\]  

(2.11)

Clearly, \( \text{Fut}(X) = 0 \) for any \( X \in \mathfrak{ham}(M) \) if \( M \) admits a transverse Sasaki–Einstein metric. In case \( c_B^1(M) > 0 \), it has been shown that \( \text{aut}^T(M) \cong \mathfrak{ham}(M) \), where \( \text{aut}^T(M) \) is the space of transverse holomorphic vector fields, which can be identified with the Lie algebra of \( \text{Aut}^T(M) \) (cf. [14, Proposition 2.2]).

There is also a definition of Futaki invariant for general Sasaki metrics without assumption of \( \omega_T^g \in \pi n + 1 c_B^1(M) \). We refer the reader to [10, Sect. 5].

3 Sasaki Manifolds with Group Structure

In this section, we introduce \( G \)-Sasaki manifolds. Let \( G \) be a complex, connected, reductive group of complex dimension \( (n + 1) \), which is the complexification of a maximal compact subgroup \( K \). Assume that the center \( z(k) \) of Lie algebra of \( K \) is nontrivial.

**Definition 3.1** A \( G \)-Sasaki manifold \( (M, g, \xi) \) is a \((2n + 1)\)-dimensional Sasaki manifold with a holomorphic \( G \times G \)-action on \( C(M) \) such that the following properties are satisfied:

1. There is an open and dense orbit \( O \) in \( C(M) \) which is isomorphic to \( G \) as a \( G \times G \)-homogeneous space (we will identify it with \( G \));
2. The \( K \times K \)-action preserves \( \rho \) invariant;
3. \( \xi \in z(\mathfrak{k}) \).

By (2.6), the conditions (2) and (3) imply that the group \( K \times K \) acts on \( M \) and preserves its Sasaki structure \((g, \xi, \eta, \Phi)\) invariant. Clearly, if we take \( G \) an \((n + 1)\)-dimensional complex torus \( T^c \), then \( M \) is a \((2n + 1)\)-dimensional toric Sasaki manifold discussed in [26]. We will discuss more examples of \( G \)-Sasaki manifolds in Sect. 9 in the end of this paper.

Let \( Z \) be the closure of \( T^c \) in \( C(M) \). By [1, 2], \( Z \) is a toric manifold. Since \( \xi \in z(\mathfrak{k}) \subset \mathfrak{t} \), we have \( \rho \frac{\partial}{\partial \rho} = -J \xi \in \mathfrak{a} \), and so \( Z \) is a Kähler cone over \( Z \cap M \). This implies that \( Z \cap M \) is a toric Sasaki manifold with Sasaki structure \((g|_{Z \cap M}, \xi|_{Z \cap M}, \eta|_{Z \cap M}, \Phi|_{Z \cap M}) \) [8, 26, 37]. As in [1–3], the structure of \( G \)-Kähler manifold (a polarized \( G \)-group compactification) is determined by its toric submanifold, the structure of \( G \)-Sasaki manifold \((M, g)\) will be determined by its toric Sasaki submanifold \( Z \cap M \). In fact, we have

**Proposition 3.2** Let \((\hat{M}, \omega)\) be a \( K \times K \)-invariant Kähler manifold with holomorphic \( G \times G \)-action which satisfies (1) in Definition 3.1. Let \( Z \) be the closure of \( T^c \) in \( \hat{M} \).
Suppose that \((Z, \omega|_Z)\) is the Kähler cone over some toric Sasaki manifold \(M_Z\) such that the Reeb vector field \(\xi\) satisfies (3). Then \(\hat{M}\) is a Kähler cone over some \(G\)-Sasaki manifold \(M\).

The proof of Proposition 3.2 depends on the \(KAK\)-decomposition of the reductive group \(G\) [31, Sect. 7.3]. Let us choose a basis of right-invariant vector fields \(\{E_1, ..., E_{r+1}\}\) on \(G \subset C(M)\) such that \(\{E_1, ..., E_{r+1}\}\) spans \(\mathfrak{t}^c\), where \((r + 1)\) is the dimension of \(T^c\) (cf. [18, Sect. 1]). Denote the set of positive roots by \(R^+_G\) with roots \(\{\alpha_i\}_{i=1,..,\frac{n-r}{2}}\). For each \(\alpha = \alpha_i\), we set \(M_\alpha\)

\[
M_\alpha(x) = \frac{1}{2} \langle \alpha, \nabla \psi(x) \rangle \begin{pmatrix} \coth \alpha(x) & \sqrt{-1} \\ -\sqrt{-1} & \coth \alpha(x) \end{pmatrix}, \quad x \in \mathfrak{a}_+,
\]

where \(\mathfrak{a}_+ = \{x \in \mathfrak{a} | \alpha(x) > 0, \forall \alpha \in R^+_G\}\) is the positive Weyl chamber of \(\mathfrak{a}\).

Let \(W\) be the Weyl group of \((G, T^c)\). The following lemma gives a formula of complex Hessian for \(K \times K\)-invariant functions on \(G\) due to [18].

**Lemma 3.3** Any \(K \times K\)-invariant function \(\psi\) on \(G\) can descend to a \(W\)-invariant function (still denoted by \(\psi\)) on \(\mathfrak{a}\). Moreover, there are local holomorphic coordinates on \(G\) such that for \(x \in \mathfrak{a}_+\), the complex Hessian matrix of \(\psi\) is diagonal by blocks as follows,

\[
\text{Hess}_C(\psi)(\exp(x)) = \begin{pmatrix} \frac{1}{4} \text{Hess}_\mathbb{R}(\psi)(x) & 0 & 0 \\ 0 & M_{\alpha_1}(x) & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & M_{\alpha_p}(x) \end{pmatrix}. \quad (3.1)
\]

We apply Lemma 3.3 to prove Proposition 3.2.

**Proof of Proposition 3.2** Since \((Z, \omega|_Z)\) is a Kähler cone manifold, there is a smooth function \(\rho\) on \(Z\) such that

\[
\omega|_Z = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \rho^2
\]

as in (2.6). Then \(\rho\) can be extended to a smooth \(K \times K\)-invariant function on \(\hat{M}\) (still denoted by \(\rho\)). Note that \(\omega\) is a \(K \times K\)-invariant Kähler metric by the assumption. Thus, \(\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \rho^2\) on \(\hat{M}\) (cf. [3, Proposition 3.2]). Let \(M\) be the level set \(\{\rho = 1\}\) in \(\hat{M}\). Then \(M_Z = M \cap Z\). We need to prove that \(\omega\) is a cone metric over \(M\). By [38, Sect. 2.1], it is equivalent to show that

\[
\omega(\xi, X) = 0, \quad \forall X \in TM. \quad (3.2)
\]
It suffices to check (3.2) on $M \cap \mathcal{O}$. Note that (3.2) is true for any $X \in TZ$.

By the $KAK$-decomposition, for any $p \in M \cap \mathcal{O}$, there exists $k_1, k_2 \in K$ such that $p' = (k_1, k_2) p \in Z$. Moreover, $p' \in M$ since $\rho(p') = \rho(p) = 1$. Then by the $K \times K$-invariance of $\omega$, it holds

$$\omega(\xi, X)|_p = \omega(\xi, (k_1^{-1}, k_2^{-1})_* X)|_{p'}, \ \forall X \in T_pM.$$  

We need to check (3.2) in the following two cases:

Case 1, $(k_1^{-1}, k_2^{-1})_* X \in \mathcal{S} \{E_{r+2}, \ldots, E_{n+1}\}$. Applying $\rho^2$ to $\psi$ in Lemma 3.3, we see that (3.1) implies (3.2) since $\xi \in \mathfrak{g}(\mathfrak{k}) \subset \mathfrak{t}$.

Case 2, $(k_1^{-1}, k_2^{-1})_* X \in T_{p'}Z$. Then $(k_1^{-1}, k_2^{-1})_* X$ must lie in $T_{p'}M_Z$ since $M$ is $K \times K$-invariant. Thus,

$$\omega(\xi, (k_1^{-1}, k_2^{-1})_* X)|_{p'} = \omega(\xi, T\mathcal{Z})|_{p'} = 0.$$

(3.2) is also true. \qed

Let $\{e^{t\xi}\}_{t \in \mathbb{R}}$ be the one-parameter group generated by $\xi$. We call a Sasaki manifold quasi-regular if any orbit generated by $e^{t\xi}$ is closed. Otherwise, it is called irregular.

If the action $e^{t\xi}$ is in addition free, a quasi-regular Sasaki manifold is further called regular (cf. [8, 26]). We note that the regularity property of $M$ is also determined by the toric Sasaki submanifold $Z \cap M$. In fact, this follows from a result of Alexeev and Brion [1, Theorem 4.8]: For any $p \in \hat{M}$, there exists $g_1, g_2 \in G$ such that $p'' = (g_1, g_2) p \in Z$. Then, for any $p \in M$, there is a $\rho_p \in \mathbb{R}$ such that $p' = e^{\rho_p J\xi} p'' \in Z \cap M$. Since both $e^{t\xi}$ and $e^{tJ\xi}$ commute with the action of $(g_1, g_2)$ by (3) of Definition 3.1,

$$e^{t\xi} p = (g_1^{-1}, g_2^{-1}) e^{-\rho_p J\xi} e^{t\xi} p', \ \forall t \in \mathbb{R}.$$  

This means that the orbits of $p$ and $p'$ generated by $e^{t\xi}$ are isomorphic. Hence, $Z \cap M$ is regular (or quasi-regular, irregular) implies that $M$ is regular (or quasi-regular, irregular).

In the remaining of this section, we discuss the moment map $\mu_Z$ of $(Z, \bar{g}|Z)$. It is known that the image of $\mu$ is a cone minus the origin in $\mathbb{R}^{r+1} \cong a^*$ (cf. [26, 33]). Denote this cone by

$$\mathcal{C} = \bigcap_{A=1}^{d} \{y \in a^* | l_A(y) = u_A^i y_i \geq 0\}. \quad (3.3)$$

Without loss of generality, we may assume that this set of $\{u_A\}$ is minimal, which means that $\mathcal{C}$ will be changed if removing any $u_A$ in (3.3). Since $Z \cap M$ is smooth, the cone $\mathcal{C}$ is good in sense of [33] (cf. [37, Sect. 2]). Namely, $\mathcal{C}$ satisfies:

(C1) Each $u_A = (u^1_A, \ldots, u^{r+1}_A)$ is a prime vector in the lattice of one-parameter groups $\mathfrak{g}$.
(C2) Each codimension $N$ face $\mathcal{F} \subset \mathcal{C}$ can be realized uniquely as the intersection of some facets $\mathcal{F}_A = \{ y | l_A(y) = 0 \}$, where $A$ runs over a subset of cardinal $N$ of $\{1, \ldots, d\}$ and

$$\text{Span}_\mathbb{R}\{u_1, \ldots, u_N\} \cap \mathfrak{M} = \text{Span}_\mathbb{Z}\{u_1, \ldots, u_N\}.$$ 

Let

$$l_\infty(y) = \sum_A l_A(y).$$  \hfill (3.4)

Set

$$U_0^\xi(y) = \frac{1}{2} \sum_A l_A(y) \log l_A(y) + \frac{1}{2} l_\xi(y) \log l_\xi(y) - \frac{1}{2} l_\infty(y) \log l_\infty(y).$$  \hfill (3.5)

$U_0^\xi(y)$ is usually called Guillemin’s function on $\mathcal{C} \setminus \{O\}$ [27]. Then the Legendre function $\hat{F}_0$ of $U_0^\xi$ defined by

$$\left\{
\begin{array}{l}
\hat{F}_0(x) = y_i \frac{\partial U_0^\xi}{\partial y^i} - U_0^\xi \\
x^i = \frac{\partial U_0^\xi}{\partial y^i}
\end{array}\right.$$  \hfill (3.6)

is a Kähler potential on $Z$ [27], where

$$\frac{\partial U_0^\xi}{\partial y^i} = x^i : (y^1, \ldots, y^{r+1}) \to (x^1, \ldots, x^{r+1})$$

is a diffeomorphism from $\mathcal{C} \setminus \{O\}$ to $\mathbb{R}^{r+1}$. Conversely, for any toric cone metric with Kähler potential $F$ on $Z = C(Z \cap M)$, one can define a symplectic potential $U$ of $Z$ on $\mathcal{C} \setminus \{O\}$ by the Legendre transformation,

$$U(y) = x_i \frac{\partial F}{\partial x^i} - F.$$ 

As a version of Abreu’s result for toric cone metrics, the following proposition was proved in [37].

**Proposition 3.4** Any symplectic potential $U$ on $Z$ associated to a Kähler cone metric with the Reeb vector $\xi$ can be written as

$$U = U_0^\xi + U',$$  \hfill (3.7)
where $U'$ is a smooth homogenous function of degree 1 on $C \setminus \{O\}$ such that $U$ is strictly convex.

Since the cone metric $\bar{g}$ is $K \times K$-invariant in our case, $C$ is $W$-invariant [2]. We will further assume that all $U'$ are $W$-invariant.

### 4 $K \times K$-Invariant Metrics in a Transversely Holomorphic Orbit

In this section, we reduce a $K \times K$-invariant Sasaki metric $g$ in a transversely holomorphic $n$-dimensional Lie group orbit, using the idea of [26]. Let $\gamma \in \mathfrak{a}_\mathbb{C}$ be a rational element such that $J\gamma(\xi) \neq 0$. Set

$$\mathfrak{t}' = \ker\{J\gamma : \mathfrak{t} \to \mathbb{R}\} = \{\zeta \in \mathfrak{t} | J\gamma(\zeta) = 0\}.$$  

Then $\mathfrak{t}'$ is a rational Lie subalgebra of $\mathfrak{t}$. It follows that the subgroup $K'$ generated by $\exp(\mathfrak{t}')$ is a closed codimension 1 subgroup of $K$ [6]. Hence, its complexification $H = (K')^c$ is a closed (complex) codimension 1 reductive subgroup of $G$. Since $\xi \in \mathfrak{z}(\mathfrak{t})$, we see that $H \times H \subset \text{Aut} \xi(C(M)).$

Take a generic point $p \in M \cap O$. Then its $H \times H$-orbit $\text{Orb}(C(M)(p))$ is a complex submanifold of $C(M)$ and it is isomorphic to $H$ as a $H \times H$-homogenous space. By the isomorphism (2.4), $H \times H$ can be identified with a subgroup of $\text{Aut} T(M)$, and $\text{Orb}_M(p) = \pi(\text{Orb}(C(M)(p)))$ is its orbit of $p$ in $M$. Since $\xi \notin \mathfrak{t}'$, $\xi \notin T\text{Orb}_M(p)$. Thus, we can equip $\text{Orb}_M(p)$ with the transverse complex structure $\Phi_T$ so that $\omega_T g$ is a Kähler form on it. It can be shown that $\pi$ is a bi-holomorphic between $\text{Orb}(C(M)(p))$ and $\text{Orb}_M(p)$ (cf. [14,26]).

We claim that $\pi$ is an isometry between $(\text{Orb}(C(M)(p)), \frac{1}{2} d\eta|_{\text{Orb}(C(M)(p))})$ and $(\text{Orb}_M(p), \omega_T g)$. This is because

$$\pi_*(X) = X - \frac{1}{\rho^2} \bar{g}\left(X, \frac{\partial}{\partial \rho}\right) \frac{\partial}{\partial \rho}, \forall X \in TC(M),$$

and by (2.7),

$$i_{\rho \frac{\partial}{\partial \rho}} d\eta = -2L_{\rho \frac{\partial}{\partial \rho}} (Jd \log \rho) = 0.$$  

Thus, for any $X, Y \in TC(M)$, we get

$$\frac{1}{2} d\eta(X, Y) = \frac{1}{2} d\eta(\pi_{r*}(X), \pi_{r*}(Y)) = \omega_\xi(\pi_{r*}(X), \pi_{r*}(Y)) = \omega_T(\pi_{r*}(X), \pi_{r*}(Y)).$$

This verifies the claim. Hence, to study $\omega_T g$ on $M$, it suffices to compute $\frac{1}{2} d\eta$ on $\text{Orb}(C(M)(p))$.
Let $T' = \exp(t')$. As in Sect. 4, we consider the closure $Z'$ of $(T')^c$-orbit $\text{Orb}_M(p)$. Since $T'$ is a maximal compact torus of $K'$, $Z'$ is just the torus orbit corresponding to $(T')^c$ in $Z$. By (2.7), we see that

$$\omega_g^T(p) = \sqrt{-1} \partial \bar{\partial} \log \rho(p).$$

Then by the above claim, we get

$$\omega_g^{T'}|_{(T')^c \subset Z'} = \sqrt{-1} \partial \bar{\partial} \log \rho((T')^c(p))|_{(T')^c \subset Z'}. \quad (4.1)$$

It implies that $\varphi((T')^c) = \log \rho|_{(T')^c(p)}$ is a Kähler potential of the restriction of $\omega_g^T$ on the orbit $(T')^c \subset Z'$. Thus, $\log \rho$ can be regarded as a convex function in $\mathbb{R}^r$ since $\omega_g^T|_{Z'}$ is $K'$-invariant. We shall compute the polytope of moment $\mu'$ associated to $\omega_g^T$ with the action $(T')^c$ below.

### 4.1 Moment Polytope of $(Z', \omega_g^T|_{Z'})$

Let $\mu_Z$ be the moment map of $(Z, \bar{g}|_Z)$ as in Sect. 3. Since $\rho = 1$ on $M$, by a direct computation, the image of $Z \cap M$ under $\mu_Z$ is an intersection of $\mathcal{C}$ with the characteristic hyperplane \{y | $l_\xi(y) = \xi^i y_i = 1$\}, which is a polytope $\mathcal{P}$ in $a^*$,

$$\mathcal{P} = \{y | l_\xi(y) = 1, u_A^i y_i \geq 0, \forall A\}. \quad (4.2)$$

Thus, $\mathcal{C}$ is a cone over it. Since $M$ is compact, $\mathcal{P}$ must be bounded. Hence, $\xi$ lies in the interior of the dual cone of $\mathcal{C}$.

Let $a' = J t'$. Let $\iota : a' \to a$ be the inclusion and $\iota^* : a^* \to (a')^*$ its dual map. Then $\gamma(a') = 0$. It follows that

$$\iota^*(y) = y - \langle y, \gamma \rangle \gamma, \quad \forall y \in a^*. \quad (4.3)$$

Thus, we can identify $(a')^*$ with the image of $\iota^*$ in $a^*$, which is a codimension 1 subspace orthogonal to $\gamma$. Let $P$ be the image of $\mu'$ in $(a')^*$. The following proposition shows that $P$ is equal to $\iota^*(\mathcal{P})$, and consequently $P$ depends only on the choice of $\gamma$.

**Proposition 4.1** $P$ is equal to $\iota^*(\mathcal{P})$, which is a bounded, convex and $W$-invariant polytope. More precisely,

$$P = \{v \in (a')^* | l'_A(v) \geq 0, \ A = 1, \ldots, d\}, \quad (4.4)$$

where

$$l'_A(v) = \left( u_A^i - \frac{\gamma(u_A)}{\gamma(\xi)} \xi^i \right) v_i + \frac{\gamma(u_A)}{\gamma(\xi)}. \quad (4.5)$$
Furthermore, each codimension $N$ face of $P$ is exactly intersections of $N$ facets. In particular, each vertex of $P$ is exactly the intersection of $r$ facets.

**Proof** Note that the inclusion $\iota : (t')^c \to t^c$ of Lie algebras induces a holomorphic embedding (still denoted by $\iota$) of toric manifold $Z'$ into $Z$. Then for any holomorphic vector field $X$ on $Z'$, by (4.1), we have

$$i_X \left( \iota^* \left( \frac{1}{2} d\eta \right) \right) = \iota^* \left( i_{\iota_* X} \left( \frac{1}{2} d\eta \right) \right) = \sqrt{-1} \iota^* (\overline{\partial}(\iota_* X(\log \rho))).$$

Thus, by the definition of moment map, it follows that

$$\mu' = \iota^* (\nabla \log \rho).$$

On the other hand, by Proposition 3.4, $U'$ is homogenous of degree 1. Then

$$2y_k \frac{\partial x^i}{\partial y_k} = 2y_k \frac{\partial^2 U}{\partial y_k \partial y_i} = 2y_k \frac{\partial^2 U}{\partial y_k \partial y_i} = \xi^i.$$

Thus,

$$l_\xi(y) = \xi^i y_i = \rho^2,$$

and so

$$\frac{1}{2} d\eta = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log l_\xi(y).$$

As a consequence,

$$\mu' = \frac{\iota^*(y)}{l_\xi(y)}.$$

This means that

$$P = \{ v = \iota^*(y) | y \in C, l_\xi(y) = 1 \},$$

which is equivalent to $\iota^*(P)$. In particular, $P$ is bounded, convex and $W$-invariant.
By (4.2) and (4.3), it is easy to see that the inverse of $\iota$ is given by

$$\iota^{-1}(v) = v + \frac{1 - v(\xi)}{\gamma(\xi)} \gamma, \quad \forall v \in \mathfrak{a}^*.$$  

Thus, by (4.7), we obtain (4.4) immediately.

The second part in the proposition follows from the property of $\mathcal{P}$. In fact, according to (4.7), any codimension $N$ face of $\mathcal{P}$ is an intersection of a codimension $N$ face of $\mathcal{C}$ with the characteristic hyperplane $\{l_\xi = 1\}$. Then by the property (C2) of $\mathcal{C}$ in Sect. 3, each codimension $N$ face of $\mathcal{P}$ is exactly intersections of $N$ facets. □

4.2 Space of Legendre Functions

In this subsection, we determine the space of Legendre functions on $P$ associated to $K \times K$-invariant transverse Kähler potentials of $\omega_T^\psi \in [\omega_g^T]_B$. For convenience, we set the class of $K \times K$-invariant Kähler potentials of $(M, \frac{1}{2}d\eta)$ by $H_{K \times K}(\frac{1}{2}d\eta)$. Namely,

$$H_{K \times K}(\frac{1}{2}d\eta) = \{ \psi \mid \psi \in H(\frac{1}{2}d\eta) \mid \psi \text{ is } K \times K \text{-invariant} \}.$$  

Let $\hat{F}_0$ be the Kähler potential associated to the symplectic potential $U_{0}^\xi$ on $Z$ in (3.6). Then by Proposition 3.2, $\hat{F}_0$ extends to a function $\frac{1}{2}\rho_0^2$ on $C(M)$ which induces a $G$-Sasaki manifold $(M, \frac{1}{2}d\eta_0)$ with

$$\frac{1}{2}d\eta_0 = \sqrt{-1}d\bar{\partial}\log \rho_0.$$  

By (4.1), $\frac{1}{2} \log \hat{F}_0$ is a Kähler potential on $Z'$. Let $u_0$ be its Legendre function. Then

$$u_0(x) = \frac{1}{2} \left( (\log \hat{F}_0)_i x^i - \log \hat{F}_0 \right)$$

$$= \frac{1}{2} \left( \frac{\hat{F}_0_i x^i}{\hat{F}_0} - \log \hat{F}_0 \right)$$

$$= \frac{1}{2} \left( \frac{U_{0}^\xi (\nabla \hat{F}_0)}{\hat{F}_0} - \log \hat{F}_0 + 1 \right).$$  

(4.9)

On the other hand, by (4.6), we have

$$2 \hat{F}_0 = l_\xi(y),$$
where $y = \nabla \hat{F}_0$. Then by (3.5), we get

$$
\frac{U_0^\xi (\nabla \hat{F}_0)}{\hat{F}_0} - \log \hat{F}_0
= \sum_A \frac{l_A(y)}{l^\xi(y)} \log l_A(y) + \log l^\xi(y) - \frac{l^\infty(y)}{l^\xi(y)} \log l^\infty(y)
- \log l^\xi(y) + \log 2
= \sum_A \frac{l_A(y)}{l^\xi(y)} \log \frac{l_A(y)}{l^\xi(y)} + \sum_A \frac{l_A(y)}{l^\xi(y)} \log l^\xi(y)
- \frac{l^\infty(y)}{l^\xi(y)} \log \frac{l^\infty(y)}{l^\xi(y)} - \frac{l^\infty(y)}{l^\xi(y)} \log l^\xi(y) + \log 2.
$$

Hence, by (3.4), it follows that

$$
\frac{U_0^\xi (\nabla \hat{F}_0)}{\hat{F}_0} - \log \hat{F}_0
= \sum_A \frac{l_A(y)}{l^\xi(y)} \log \frac{l_A(y)}{l^\xi(y)} - \frac{l^\infty(y)}{l^\xi(y)} \log \frac{l^\infty(y)}{l^\xi(y)} + \log 2. \quad (4.10)
$$

Let

$$
v = \frac{\iota^* (\nabla \hat{F}_0)}{2\hat{F}_0} = \frac{\iota^* (y)}{\hat{\xi}^\xi y_i}.
$$

Then by (4.8),

$$
\frac{y}{\hat{\xi}^\xi y_i} = v + \frac{1 - v(\xi)}{\gamma(\xi)} \gamma.
$$

Thus, by (4.5), we see that

$$
\frac{l_A(y)}{l^\xi(y)} = l'_A(v),
$$

and

$$
\frac{l^\infty(y)}{l^\xi(y)} = \sum_A l'_A(v) = l'_\infty(v).
$$

Plugging (4.10) and the above two equalities into (4.9), we derive

$$
u_0(v) = \frac{1}{2} \sum_A l'_A(v) \log l'_A(v) - \frac{1}{2} l'_\infty(v) \log l'_\infty(v) + \log 2 + \frac{1}{2}. \quad (4.11)$$
Note that $l'_\infty(v)$ has strictly positive lower bound on $\overline{P}$. Then
\[ l'_\infty(v) \log l'_\infty(v) \in C^\infty(\overline{P}). \]

Set
\[ u_G = \frac{1}{2} \sum_A l'_A(v) \log l'_A(v). \quad (4.12) \]

We see that $u_0 - u_G \in C^\infty(\overline{P})$.

Set
\[ C_{P,W} = \{ u | u - u_G \in C^\infty(\overline{P}), u \text{ is strictly convex and } W \text{ - invariant} \}. \]

We prove

**Lemma 4.2** Let $\psi \in \mathcal{H}_{K \times K}(\frac{1}{2}d\eta)$ and $\varphi_\psi$ be the Kähler potential of $\omega_\psi^T \in [\omega_\xi^T]_B$. Then the Legendre function $u_\psi$ of $\varphi_\psi$ belongs to $C_{P,W}$.

**Proof** Without loss of generality, we may assume (cf. [26, Proposition 4.2]),
\[ F_\psi = e^{2\psi} \hat{F}_0. \]

Then $\varphi_\psi = \frac{1}{2} \log \hat{F}_0 + \psi$ is a toric Kähler potential of $\omega_\psi^T$. Since $\psi$ is basic, $\xi(\psi) = 0$. Thus, we get
\[ \xi^i (e^{2\psi} \hat{F}_0)_i = e^{2\psi} \xi^i \hat{F}_0,i = 2F_\psi. \quad (4.13) \]

Hence, if we set $y = \nabla F_\psi$, then
\[ \xi^i y_i = 2F_\psi. \]

On the other hand, by (3.7), the Legendre function $U_\psi$ of $F_\psi$ can be written as
\[ U_\psi = U_0^\xi + U' \]
for some smooth, homogenous degree 1 function $U'$ on $\mathcal{C}$. Then analogous to (4.9), the Legendre function of $\varphi_\psi = \frac{1}{2} \log F_\psi|_{a'}$, is given by
\[ u_\psi(x) = \frac{1}{2} \left( \frac{U_\psi(\nabla F_\psi)}{F_\psi} - \log F_\psi + 1 \right) \]
\[ = \frac{1}{2} \left( \frac{U_0^\xi(y)}{l^\xi(y)} - \log l^\xi(y) + \log 2 + 1 + U' \left( \frac{y}{l^\xi(y)} \right) \right). \]
Similarly as in the proof of (4.11), for \( v = t^\nu \left( \frac{y}{t_k(y)} \right) \), we get

\[
u \psi(v) = u_0(v) + \frac{1}{2} U'(v).
\]

The lemma then follows from the fact that \( u_0 - u_G \in C^\infty(P) \).

The convexity and \( W \)-invariance of \( \varphi_\psi \) follows exactly as in the Kähler case (cf. [18,35]). \( \square \)

5 The Reduced K-Energy \( \mu(\cdot) \)

In this section, we keep the notations in Sect. 4 to compute the K-energy \( \mathcal{K}(\cdot) \) on \( \mathcal{H}_{K \times K} \left( \frac{1}{2} d\eta \right) \) on a Sasaki manifold \((M, \frac{1}{2} d\eta)\) in terms of Legendre functions on \( P \) by the method in [35]. Recall that the average \( \bar{S}^T \) of transverse scalar curvature \( S^T \) of \( \frac{1}{2} d\eta \) is given by

\[
\bar{S}^T = \frac{1}{V} \int_M S^T \left( \frac{1}{2} d\eta \right)^n \wedge \eta,
\]

where

\[
V = \int_M \left( \frac{1}{2} d\eta \right)^n \wedge \eta
\]

is the volume of \( \frac{1}{2} d\eta = \omega_T^\psi \). As same as \( V \), \( \bar{S}^T \) is independent of the choice of \( \frac{1}{2} d\eta_\psi \) with \( \psi \in \mathcal{H} \left( \frac{1}{2} d\eta \right) \). The K-energy on \((M, \frac{1}{2} d\eta)\) is introduced by Futaki–Ono–Wang as follows [26],

\[
\mathcal{K}(\psi) = -\frac{1}{V} \int_0^1 \int_M \bar{\psi}_t (S^T_t - \bar{S}^T) \left( \frac{1}{2} d\eta_\psi \right)^n \wedge \eta_\psi \wedge dt,
\]

for any \( \psi \in \mathcal{H} \left( \frac{1}{2} d\eta \right) \), where \{\( \psi_t \)\}_{t \in [0,1]}\) is any smooth path in \( \mathcal{H} \left( \frac{1}{2} d\eta \right) \) joining 0 and \( \psi \), and \( S^T_t \) is the transverse scalar curvature of \( \frac{1}{2} d\eta_\psi \).

Let \( dh \) be a Haar measure of \( H \). Write the complex Monge–Ampère operator measure on \( \text{Orb}_M(p) \), induced by \( \frac{1}{2} d\eta_\psi \) as

\[
(\sqrt{-1} \bar{\partial}_c \varphi_\psi)^n = \text{MA}_C(\varphi_\psi) dh.
\]

Note that \( H \) and \( G \) have the same roots system. Then by Lemma 3.3, we have

\[
\text{MA}_C(\varphi_\psi)(\exp(x)) = \frac{1}{4r+p} \text{MA}_R(\varphi_\psi)(x) \frac{1}{J(x)} \prod_{\alpha \in R_+^G} (\alpha, \nabla \varphi_\psi(x))^2, \ \forall x \in \mathfrak{a}_+^*.
\]

(5.2)
where \( J(x) = \prod_{\alpha \in R^+_G} \sinh^2 \alpha(x) \).

The following lemma gives a version of \( KA K \)-integration formula on a \( G \)-Sasaki manifold.

**Lemma 5.1** Let \((M, \frac{1}{2} d\eta)\) be a \( G \)-Sasaki manifold with Reeb vector field \( \xi \). Then there is a constant \( C_0 \) which depends only on \( \xi \) and \( H \) such that for any \( K \times K \)-invariant function \( f \),

\[
\int_M f(\eta)^n \wedge \eta = C_0 \int_{a_+} f M A_{\mathbb{R}}(\varphi_0) \prod_{\alpha \in R^+_G} \langle \alpha, \nabla \varphi_0(x) \rangle^2 dx,
\]

(5.3)

where \( \varphi_0 \) is a transverse \( \mathbb{K} \-\)potential of \( \frac{1}{2} d\eta = \sqrt{-1} \partial \bar{\partial} \varphi_0 \).

**Proof** It suffices to do the integration on the open dense orbit \( M \cap O \). We claim that for any \( q \in M \cap O \), the flow line generated by \( \xi \) through \( q \) intersects \( \text{Orb}_M(p) \). In fact, by using \( KAK \)-decomposition, we may assume \( q \in Z \cap O \) without loss of generality. The claim then follows from [26, Proposition 7.2].

Note that all \( e^{t\xi} \)-orbits in \( M \cap O \) are isomorphic to each other. Then we have two cases.

**Case 1.** The \( e^{t\xi} \)-orbits in \( M \cap O \) are all compact, so they can be parameterized by \( S^1 \). In this case, the integration can be taken first along each \( e^{t\xi} \)-orbit and then over \( \text{Orb}_M(p) \). On the other hand, in the coordinates chosen in Sect. 2,

\[
\left( \frac{1}{2} d\eta \right)^n \wedge \eta = (\omega^T_g)^n \wedge dx^0_{(\alpha)}.
\]

Since \( f \) is \( e^{t\xi} \)-invariant, we have \( f = f(z_{(\alpha)}) \), which is a constant along each \( e^{t\xi} \)-orbit. Thus,

\[
\int_M f \left( \frac{1}{2} d\eta \right)^n \wedge \eta = C_0 \int_{\text{Orb}_M(p)} f(\omega^T_g|_{\text{Orb}_M(p)})^n,
\]

where \( C_0 \) is a constant independent of \( f \). By (5.2), we get (5.3).

**Case 2.** The \( e^{t\xi} \)-orbits in \( M \cap O \) are non-compact. In this case, let \( T_\xi \) be the closure of \( e^{t\xi} \). It is a compact torus in \( Z(K) \) dimension of which is at least 2. Thus, \( t_\xi \cap t' \neq \emptyset \). Take an \( \varsigma' \in t_\xi \cap t' \) such that \( \xi' = \xi + \varsigma' \) generates a compact group and let \( \theta' \) be the dual of \( \xi' \). Then

\[
\left( \frac{1}{2} d\eta \right)^n \wedge \eta = (\omega^T_g)^n \wedge \theta'.
\]

Since \( f \) is also \( e^{t\xi'} \)-invariant, (5.3) follows from the proof in Case 1. \( \square \)
For any $u \in C_{P,W}$, we denote

$$u_i = \frac{\partial u}{\partial v_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial v_i \partial v_j}$$

and $(u^{ij})$ the inverse matrix of $(u_{ij})$. By Proposition 4.1, near any point $p \in \partial P$, there exists local adapt coordinates introduced by [22]. That is, for any $p \in \partial P$, we can choose affine coordinates \( \{v_i\}_{i=1,...,r} \) on $\mathbb{R}^r$ such that a neighborhood of $p$ in $P$ is given by

$$v_1, ..., v_N \geq 0$$

for some $1 \leq N \leq r$. Thus, by (4.4) and [22, Proposition 2] we have for any $u \in C_{P,W},$

$$u^{ij}v_{Ai} \to 0 \quad \text{and} \quad u_{ij}v_{Ai} \to -\frac{2}{\lambda_A} (v, v_A), \quad (5.4)$$
as $v$ goes to a facet $\tilde{\delta}'_A = \{v \in (a')^* | l'_A(v) = 0\}$ of $P_+ = P \cap a^*_+$. Here

$$\lambda_A = \frac{\gamma(u_A)}{\gamma(\xi)}$$

and $v_A$ denotes the unit outer normal vector of $\tilde{\delta}'_A$.

Let

$$\Lambda_A = \frac{2}{\lambda_A} (1 - 2\sigma(u_A)), \quad \forall \tilde{\delta}'_A \cap a^*_+ \neq \emptyset. \quad (5.5)$$

For any $u \in C_{P,W}$, we define a functional $\mu(\cdot)$ by

$$\mu(u) = \frac{1}{V_P} \sum_{A | \tilde{\delta}'_A \cap a^*_+ \neq \emptyset} \Lambda_A \int_{\tilde{\delta}'_A \cap P_+} u(v, v_A) \pi d\sigma_0$$

$$- \frac{\tilde{S}}{V_P} \int_{P_+} u_\pi dv - \frac{4}{V_P} \int_{P_+} \sigma(\nabla u) \pi dv$$

$$- \frac{1}{V_P} \int_{P_+} \log \det (u_{ij}) \pi dv + \frac{1}{V_P} \int_{P_+} [\chi(\nabla u) + 4\sigma(\nabla u)] \pi dv, \quad (5.6)$$

where $\pi(v) = \prod_{\alpha \in R^+_G} (\alpha, v)^2$, $\chi(x) = -\log J(x)$ and $V_P = \int_{P_+} \pi dv$. The following proposition shows that K-energy is same to the functional $\mu(\cdot)$. 

\[ \text{Springer} \]
Proposition 5.2
\[ K(\psi) = \mu(u_\psi) + \text{const.}, \forall \psi \in H_{K \times K} \left( \frac{1}{2} \, d\eta \right). \] (5.7)

Proof By Lemma 5.1, we have
\[ K(\psi) = -\frac{C_0}{V} \int_0^1 \int_{(a')_+} \hat{\psi}_t (S^T - \bar{S}^T)(\omega_t^T)^n \wedge dr. \] (5.8)

On the other hand, analogous to [35, Lemma 2.4], we see that
\[ S^T_t = -u_{t,ij}^i - 2u_{t,j}^i \pi_t - u_t^i \pi_{t,ij}^j \]
\[ - u_t^k \frac{\partial^2 \chi}{\partial x^i \partial x^k} \bigg|_{x=\nabla u_t} - \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_t} \frac{\pi_{t,i}}{\pi}. \] (5.9)

Consequently,
\[ \bar{S}^T = \frac{1}{VP} \sum_{[A] \mid \bar{A} \cap a^*_+ \neq \emptyset} \Lambda_A \int_{\bar{A} \cap P_+} \langle v, v_A \rangle \pi \, d\sigma_0. \] (5.10)

Then substituting (5.9) and (5.10) into (5.8), and taking integration by parts together with (5.4), we get
\[ V_P \cdot K(\psi) = \sum_{[A] \mid \bar{A} \cap a^*_+ \neq \emptyset} \Lambda_A \int_{\bar{A} \cap P_+} u_\psi \langle v, v_A \rangle \pi \, d\sigma_0 - \bar{S} \cdot \int_{P_+} u_\psi \pi \, dv \]
\[ - \int_{P_+} \log \det (u_\psi, ij) \pi \, dv + \int_{P_+} \left[ \chi (\nabla u_\psi) + 4 \sigma (\nabla u_\psi) \right] \pi \, dv + \text{const}. \]

Note that
\[ V = \int_M (d\eta)^n \wedge \eta = C_0 \cdot V_P. \]

Thus, (5.7) is true. A detailed proof can be found in [35, Proposition 3.1]. \( \square \)

We call \( \mu(\cdot) \) the reduced K-energy of \( K(\cdot) \) as in [23,35,49]. By Proposition 5.2, \( \mu(\cdot) \) is well defined on \( C_{P,W} \). Note that the nonlinear part
\[ -\frac{1}{V_P} \int_{P_+} \log \det (u_{ij}) \pi \, dv + \frac{1}{V_P} \int_{P_+} [\chi (\nabla u) + 4 \sigma (\nabla u)] \pi \, dv \]
is invariant by adding a linear function which depends only on $a'_z = a' \cap \mathfrak{h}(\mathfrak{h})$. We will use the Futaki invariant to normalize $u$ in $C_{P,W}$. By Proposition 5.2, we observe

**Lemma 5.3** Let

$$\mathcal{L}(u) = \frac{1}{V_P} \sum_{\{A|S(A) \cap a'_z \neq \emptyset\}} \Lambda_A \int_{S(A) \cap P_+} u(v, v_A) \pi \ d\sigma_0$$

$$- \frac{\tilde{S}}{V_P} \int_{P_+} u \pi \ dv - \frac{4}{V_P} \int_{P_+} \sigma(\nabla u) \pi \ dv. \quad (5.11)$$

Then $M$ has vanishing Futaki invariant if and only if

$$\mathcal{L}(a^i v_i) = 0 \quad (5.12)$$

for any $a = (a^i)$ in $a'_z$.

**Proof** Let $\sigma_X(t)$ be a one parameter subgroup of $\text{Aut}^T(M)$ generated by some $X = \sum_{1 \leq i \leq r+1} a^i E_i \in a'_z$. Then by Lemma 2.1, we have

$$[\sigma_X(t)]^* \omega_g^T = \omega_g^T + \sqrt{-1} \partial \bar{\partial} \phi_t$$

for some basic function $\phi_t$. By [10, Proposition 5.2], it follows

$$\frac{d}{dt} \mathcal{K}(\psi_t) = - \frac{1}{V} \text{Fut}(X).$$

On the other hand, by Proposition 5.2, as in [35], we see that

$$\frac{d}{dt} \mathcal{K}(\psi_t) = \mathcal{L}(a^i v_i + c) = \mathcal{L}(a^i v_i),$$

where $c$ is some constant. Combining the above two relations, we prove the lemma. $\square$

Without loss of generality, we may choose $\gamma$ such that $O \in P$. When the Futaki invariant vanishes, $C_{P,W}$ can be normalized by a set

$$\hat{C}_{P,W} = \{ u \in C_{P,W} | u \geq u(O) = 0 \}.$$

In fact, we have

**Lemma 5.4** Assume that $\text{Fut}(\cdot) = 0$. Then for any $\psi \in \mathcal{H}_{K \times K}(\frac{1}{2}d\eta)$, there is $\sigma \in Z(H)$ such that the Legendre function $\hat{u}$ of $\varphi_\sigma$ belongs to $\hat{C}_{P,W}$.
Proof Let $u$ be the Legendre function of $\varphi_\psi$. Then $u \in C_{P, W}$ by the $W$-invariance, $a = \nabla u(O) \in a'$. Let $\sigma^a$ be the one parameter subgroup of $H^C$ generated by $-a$. By Lemma 2.1, there is a $\psi_\sigma \in \mathcal{H}_{K \times K} \left( \frac{1}{2}d\eta \right)$ with $(\varphi_0 + \psi_\sigma)(O) = 0$ such that

$$(\sigma^a)^* \frac{1}{2}d\eta_\psi = \frac{1}{2}d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \psi_\sigma,$$

where $\sigma = \sigma^a$. Then one can check that the Legendre function $\hat{u}$ of $\varphi_0 + \psi_\sigma$ is given by

$$\hat{u} = u - a' v_i - u(O).$$

Thus, $\hat{u} \in \hat{C}_{P, W}$. \hfill \Box

5.1 A Criterion for the Properness of K-Energy

Recall $I$-functional,

$$I(\psi) = I(\omega^T_g, \psi) = \frac{1}{V} \int_M \psi \left[ \left( \frac{1}{2}d\eta \right)^n \wedge \eta - \left( \frac{1}{2}d\eta_\psi \right)^n \wedge \eta_\psi \right],$$

where $\psi \in \mathcal{H} \left( \frac{1}{2}d\eta \right)$. We call $K(\cdot)$ proper on $\mathcal{H} \left( \frac{1}{2}d\eta \right)$ if there is an increasing function $f(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which satisfies $\lim_{t \to +\infty} f(t) = +\infty$ such that

$$K(\psi) \geq f(I(\psi)), \quad \forall \psi \in \mathcal{H} \left( \frac{1}{2}d\eta \right).$$

In view of Lemma 2.1, the action of $\text{Aut}^T(M)$ on $M$ preserves $[\omega^T_g]_B$. We introduce

Definition 5.5 Let $K$ be a maximal compact subgroup of $\text{Aut}^T(M)$ and $\mathcal{H}_K \left( \frac{1}{2}d\eta \right)$ the subset of $K$-invariant Sasaki metrics in $\mathcal{H} \left( \frac{1}{2}d\eta \right)$. Let $G_0$ be a reductive subgroup of $\text{Aut}^T(M)$. $K(\cdot)$ is called proper on $\mathcal{H}_K \left( \frac{1}{2}d\eta \right)$ modulo $G_0$ if there is a $f$ as in (8.6) such that

$$K(\psi) \geq \inf_{\psi_\sigma \in G_0} f(I(\psi_\sigma)), \quad \forall \psi \in \mathcal{H}_K \left( \frac{1}{2}d\eta \right),$$

where $\psi_\sigma$ is defined by $\frac{1}{2}d\eta_{\psi_\sigma} = \frac{1}{2}\sigma^*(d\eta_\psi) = \frac{1}{2}d\eta + \sqrt{-1} \partial_B \bar{\partial}_B \psi_\sigma$.

Let $\text{bar}(P_+)$ and $\widetilde{\text{bar}}(\partial P_+)$ be the weighted barycenters of $P_+$ and $\partial P_+$, respectively, which are defined by

\[ \text{Springer} \]
Let $\text{bar}_{ss}(P_+) \text{ and } \widetilde{\text{bar}}_{ss}(\partial P_+)$ are projections of $\text{bar}(P_+)$ and $\widetilde{\text{bar}}(\partial P_+)$ to the semi-simple part $a^*_{ss}$ in $a^*$, respectively. Then following the argument in the proof of main theorem in [35, Theorem 1.2], we have

**Theorem 5.6** Let $(M, g)$ be a compact $G$-Sasaki manifold with vanishing Futaki invariant. Suppose that there is a $\gamma \in a^*_c$ such that $\gamma(\xi) \neq 0$ and

\[
\begin{align*}
(\min_A \Lambda_A \cdot \text{bar}_{ss}(\partial P_+)) - 4\sigma & \in \Xi, \quad \forall \mathfrak{F}_A \cap a^*_{ss} \neq \emptyset, \quad (5.15) \\
\left(\text{bar}_{ss}(\partial P_+) - \text{bar}_{ss}(P_+)\right) & \in \widetilde{\Xi}, \quad (5.16)
\end{align*}
\]

\[
(n + 1) \cdot \min_A \Lambda_A - \tilde{S} > 0, \quad \forall \mathfrak{F}_A \cap a^*_{ss} \neq \emptyset, \quad (5.17)
\]

where $\Lambda_A$ are given by (5.5). Then the K-energy is proper on $H_{K \times K}(\frac{1}{2}d\eta)$ modulo $Z(H)$.

We will give a proof of Theorem 5.6 in case of $\omega^T_g \in \frac{\pi}{n+1}c^B_1(M)$ in next section. In this case, $\Lambda_A = 2(n + 1)$ for all $A$ and $\tilde{S} = 2n(n + 1)$. Thus, (5.16) and (5.17) are both automatically satisfied. Since $P$ does not satisfy the Delzant condition in general [27], we need to modify the argument in the proof of [35, Theorem 1.2]. For a general transverse Kähler class $[\omega^T_g]_B$, we leave the proof to the reader.

**Remark 5.7** Since the polytope $P$ in Theorem 5.6 depends on the choice of $\gamma$ in Proposition 4.1, we do not know whether the conditions (5.15)–(5.17) depend on $\gamma$ or not. But in case of $\omega^T_g \in \frac{\pi}{n+1}c^B_1(M)$, the conditions are independent of $\gamma$ (cf. Sect. 6).

### 6 Properness of $\mu(\cdot)$

In this section, we prove Theorem 5.6 in case of $\omega^T_g \in \frac{\pi}{n+1}c^B_1(M)$. In this case, we can chose a suitable $\gamma$ in Proposition 4.1 so that the quantities $\Lambda_A$ in (5.6) are all same. First we use Lemma 3.3 to give a criterion to verify $\omega^T_g \in \frac{\pi}{n+1}c^B_1(M)$ in terms of the moment cone $\mathfrak{C}$ given by (3.3). We need to introduce some notations below.

Set $\mathfrak{C}_+ = \mathfrak{C} \cap a^*_+$. We call a facet $\mathfrak{F}_A$ satisfies $\mathfrak{F}_A \cap a^*_+ \neq \emptyset$ an outer facet of $\mathfrak{C}_+$. Note that for any Weyl chamber $a^*_+$, there exists a unique $w' \in W$ such that $w'(a^*_+) = a^*_+$. Thus, for any $\mathfrak{F}_A'$ which intersects $a^*_+$, $w'^{-1}(\mathfrak{F}_A')$ is an outer facet by $W$-invariance of $\mathfrak{C}$, and it has prime normal vector $w'^{-1}(u_A')$. We associate to $\mathfrak{F}_A'$ a
vector $\sigma_{A'} := w'(\sigma)$. Obviously

$$
\sigma(w^{-1}u_{A'}) = \sigma_{A'}(u_{A'}).
$$

(6.1)

**Proposition 6.1**

$$
\omega_g^T \in \frac{\pi}{n+1} c^B_1 (M)
$$

(6.2)

holds if and only if there is a $\gamma_0 \in a^*_z$ such that

$$
\gamma_0(u_A) = -1 + 2\sigma_A(u_A), \ \forall \ A,
$$

(6.3)

and

$$
\gamma_0(\xi) = -(n + 1).
$$

(6.4)

**Proof** Suppose that (6.2) is true. Then by the relations (2.10), [26, (10)] and

$$
\text{Ric}(\bar{g})(X, Y) = \text{Ric}(g)(X, Y) - 2ng(X, Y), \ \forall X, Y \in TM,
$$

we have

$$
\text{Ric}(\bar{g}) = \sqrt{-1} \partial \bar{\partial} h.
$$

Using the Kähler potential $F = \frac{1}{2} \rho^2$, we get

$$
\partial \bar{\partial}(- \log \det(\partial \bar{\partial} F) - h) = 0, \ \text{in} \ Z,
$$

where the operators $\partial, \bar{\partial}$ are both defined in the affine coordinates on $Z$. On the other hand, by Proposition 3.4, the growth behavior of $F$ on the torus cone $Z$ is same as $\hat{F}$ in (3.6). Then one can check that $\log \det(\partial \bar{\partial} F)$ has at most the linear growth. Thus, there is $\gamma_0 \in a^*$ such that

$$
- \log \det(\partial \bar{\partial} F) = h + 2\gamma_0(x).
$$

By (5.2), it follows that

$$
- \log \text{MA}_{R}(F) - \log \prod_{a \in R^+_{G}} \langle a, \nabla F \rangle^2 - \chi(x) = 2\gamma_0(x) + h + C, \ \forall x \in a_+,
$$

(6.5)

where $\chi(x) = - \log J(x)$. Note that the function $\gamma_0(x)$ is $W$-invariant. It follows that $\gamma_0 \in a_z^* \subset J^*(\mathfrak{g})$.

3 Proposition 6.1 will be also used in the proof of Theorem 1.2 in Sect. 8.
Taking the Legendre transformation of $F$ in (6.5), we have

$$\log \det(U,ij) - 2 \sum_{\alpha \in R_G^+} \log \langle \alpha, y \rangle - \chi(\nabla U) = 2 \gamma_0 U, i + h + C, \quad (6.6)$$

where $U$ is the Legendre function of $F$ and

$$U, i = \frac{\partial U}{\partial y_i}, \quad U, ij = \frac{\partial^2 U}{\partial y_i \partial y_j}.$$ 

Since $\gamma_0$ is $W$-invariant, it suffices to prove (6.3) when $F_A$ is an outer facet. Let $y_0$ a point on a facet $F_A$ of $C$ in $\mathfrak{a}_+^*$, which is away from other facets and all Weyl walls. Then by (3.7), it is easy to see that the sum of singular terms at the left-hand side of (6.6) goes to

$$- \log I_A(y) + 2 \sigma(u_A) \log I_A(y)$$

as $y \to y_0$. Similar to the right-hand side of (6.6), we have

$$\gamma_0(u_A) \log I_A(y), \; \text{as} \; y \to y_0.$$ 

Thus, combining the above two relations, we derive (6.3). Furthermore, one can verify that $\gamma_0$ is uniquely determined by (6.3) and (C2)-condition for the good cone $C$ in Sect. 3. By (C1)-condition, $\gamma_0$ is also rational.

Next we determine the quantity $\gamma_0(\xi)$. We note that $\alpha(\xi) = 0$ for any $\alpha \in R_G$, since $\xi \in \mathfrak{z}(t)$. It follows

$$\xi^i \frac{\partial J}{\partial x^i} = 2J(x) \sum_{\alpha \in R_G^+} \alpha(\xi) \cdot \coth \alpha(x) = 0.$$ 

Thus, combining with (4.6), we get

$$y_i \frac{\partial}{\partial y_i} \det(U,ij)$$

$$= y_i \frac{\partial}{\partial y_i} \left( e^{2\gamma_0 U, i + h + C} \frac{\prod_{\alpha \in R_G^+} \langle \alpha, y \rangle^2}{J(\nabla U)} \right)$$

$$= \left( \gamma_0 \xi^i + n - r \right) \det(U,ij) \left( e^{2\gamma_0 U, i + h + C} \frac{\prod_{\alpha \in R_G^+} \langle \alpha, y \rangle^2}{J(\nabla U)} \right)$$

$$= \left( \gamma_0 \xi^i + n - r \right) \det(U,ij). \quad (6.7)$$

On the other hand, $\det(U,ij)$ is homogenous of degree $-(r + 1)$. Hence, by the Euler’s equation, we obtain (6.4) from (6.7) immediately.
To prove the sufficient part of proposition, it suffices to show that \(-lK_{C(M)}\) is trivial for some \(l \in \mathbb{N}_+\) as in the proof of [14, Theorem 1.2]. We reduce the problem to show that \(-lK_{C(M)}|_Z\) is trivial for some \(l\). Then we extend the property to \(C(M)\) by the \(K \times K\)-invariance through constructing a non-trivial meromorphic function on \(Z\).

By the work of Brion [11] (see also [39, Sect. 1.8]), we have

\[ -K_{C(M)}|_Z = \sum_A (1 - 2\sigma_A(u_A))D_A, \]

where \(D_A\) is the boundary prime divisor of \(Z\) associated to \(\mathfrak{F}_A\). By (6.3), it follows

\[ -K_{C(M)}|_Z = -\sum_A \gamma_0(u_A)D_A. \]

Recall that \(\gamma_0\) is rational. This means that there is an \(l \in \mathbb{N}_+\) such that \(l\gamma_0\) is a lattice point in \(\mathfrak{H}^*\). Thus, there is a global meromorphic function which defines the divisor \(-\sum_A l\gamma_0(u_A)D_A\) (cf. [25, Chapter 3]). Hence, \(-lK_{C(M)}|_Z\) is trivial, and so \(-lK_{C(M)}\) is. The proof is completed. \(\square\)

By (2.8), it is easy to see \(\bar{S} = 2n(n + 1)\). To simplify the reduced energy \(\mu(\cdot)\) in Proposition 5.2, which depends on the choice of \(H\)-orbit, we take a translation

\[ v' = v + \frac{1}{n + 1}t^*(\gamma_0), \]

where \(\gamma_0\) is given by Proposition 6.1. Then we get a translated polytope \(P' = P + \frac{1}{n + 1}t^*(\gamma_0)\) from (4.5), which is defined by

\[ l'_A(v') = \left(u^i_A - \frac{\gamma(u_A)}{\gamma(\xi)}\tilde{x}^i\right) v'_i + \frac{1 - 2\sigma_A(u_A)}{n + 1} > 0, \forall A. \]

It is also easy to see that the pull back of any function \(u \in C_{P,W}\) lies in \(C_{P',W}\).

The advantage of choice of \(P'\) is that \(\Lambda_A = 2(n + 1)\) for all \(A\). Then (5.11) becomes

\[ \mathcal{L}(u) = \frac{2(n + 1)}{V_P} \int_{P'} \left( v' - \frac{2}{n + 1}\sigma, \nabla u \right) \pi dv', \]

and

\[ \mu(u) = \mathcal{L}(u) - \frac{1}{V_P} \int_{P'_+} \log \det (u_{ij}) \pi dv' + \frac{1}{V_P} \int_{P'_+} [\chi (\nabla u) + 4\sigma (\nabla u)] \pi dv'. \quad (6.8) \]

One can check that (6.8) is just the reduced K-energy associated to the \(H_0\)-orbit determined by choosing \(\gamma = \gamma_0\). In the latter, we always assume that \(H = H_0\).
By the fact that \( \bar{\mathcal{S}} = 2n(n + 1) \) and \( \Lambda_A = 2(n + 1) \), we see that (5.15) is equivalent to
\[
\text{bars}_s(P_+) \in \frac{2}{n + 1} \sigma + \Xi,
\] (6.9)
which is also equivalent to the condition (1.1) in Theorem 1.1. Moreover, (5.12) is equivalent to
\[
\text{bar}(P_+) = \text{bars}_s(P_+) \in \rho_{ss, +},
\] (6.10)
where \( \rho_{ss, +} \) is the semi-simple part of \( \rho_+ \). Hence, (6.9) implies that \( M \) has vanishing Futaki invariant by Lemma 5.3. By Lemma 5.4, Theorem 5.6 turns to prove the following proposition in case of (6.2).

**Proposition 6.2** Assume that (6.9) is satisfied. Then \( \mu(\cdot) \) is proper on \( \hat{\mathcal{C}}_{P, W} \). More precisely, there are \( \delta, C_\delta > 0 \) such that
\[
\mu(u) \geq \delta \int_{P_+} u \pi(v) \, dv - C_\delta, \quad \forall \, u \in \hat{\mathcal{C}}_{P, W}.
\]

### 6.1 A Criterion for the Properness of General Functionals

In this subsection, we will establish a criterion to verify the properness of general functionals \( \mu(\cdot) \) for convex functions on a bounded polytope \( P \). Let us introduce a setting for such a \( P \) and related functionals as follows.

Let \( H = (K')^c \) be a reductive Lie group of dimension \( n \) with \( T' \) its maximal compact torus, and assume that the rank of \( H \) is \( r \). Let \( R \subset J(t')^* = (a')^* \) be the root system and \( R^+ \) a chosen set of positive roots. Set \( 2\sigma = \sum_{\alpha \in R^+} \alpha \) and denote the corresponding Weyl group by \( W \). We assume that a bounded polytope \( P \subset (a')^* \), which can be described as
\[
P = \bigcap_{\{A=1, \ldots, d\}} \{ l_A'(v) = \lambda_A - u_A' v_i > 0 \}
\]
with each \( \lambda_A > 0 \), which satisfies:

- (P1) \( P \) is convex and \( W \)-invariant, which contains the origin \( O \);
- (P2) Each codimension \( N \) face of \( P \) is exactly intersections of \( N \) facets. In particular, each vertex of \( P \) is exactly the intersection of \( r \) facets;
- (P3) Each \( u_A \) satisfies
\[
\alpha(u_A) \in \mathbb{Z}, \quad \forall \, \alpha \in R.
\] (6.11)
We note that $\mathcal{P}$ does not satisfy the Delzant condition [21] and $u_A$ need not to be a lattice vector in the lattice of one parameter groups. Also we remark that the moment polytope $P$ given in Sect. 4.1 satisfies (P1)-(P3). As before, we set $P_+ = P \cap (a')^*_+$, where $(a')^*_+$ is the positive Weyl chamber defined by $R^+$.

Define the Guillemin function of $P$ by

$$u_P(v) = \frac{1}{2} \sum_A l_A(v) \log l_A(v).$$

Then it has properties:

(F1) $u_P \in C^\infty(P) \cap C^0(\overline{P})$;
(F2) $u_P$ is $W$-invariant and strictly convex;
(F3) the derivatives of $u_P$ satisfies

$$u_P^{ij} \in C^\infty(\overline{P}),$$

where $u_P, ij = \frac{\partial^2}{\partial v_i \partial v_j} u_P$ and $(u_P^{ij}) = (u_P, ij)^{-1}$.

Let $a = (a') \in a'_z = a' \cap \mathfrak{g}(h)$, the central part of $a'$. Assume that

$$a_1 \leq a'v_i \leq a_2, \ \forall \ v \in \overline{P}$$

for some $a_1, a_2$. Let $f(t) : [a_1, a_2] \rightarrow \mathbb{R}$ be a smooth function which satisfies:

(W1) there are constants $m_f, M_f$ such that

$$0 < m_f \leq f(t) \leq M_f, \ \forall t \in [a_1, a_2];$$

(W2) there is constants $C_f$ such that

$$||f(t)||_{C^2} \leq C_f.$$

For simplicity, we denote $f_a(v) = f(a'v_i)$.

Set a space of normalized $W$-invariant strictly convex functions by

$$\hat{\mathcal{C}}_{P, W} = \{ u \in C^\infty(P) \cap C^0(\overline{P}) \mid u \text{ is strictly convex and } W \text{ -- invariant on } P, u \geq u(O) = 0 \}.$$ 

Let $\pi, \chi$ be functions as before. Given $f_a$ and a constant $\Lambda_L > 0$, we define a weighted functional $\mu(\cdot)$ associated to $f_a$ for any $u \in \hat{\mathcal{C}}_{P, W}$ by

$$\mu(u) = \frac{1}{\Lambda_L} \mathcal{L}(u) + \mathcal{N}(u),$$
where

\[ \mathcal{L}(u) = \int_{P_+} \langle v - 4\Lambda L\sigma, \nabla u \rangle f_a(v)\pi(v)dv, \quad (6.12) \]

and

\[ \mathcal{N}(u) = - \int_{P_+} \log \det (u,ij) f_a(v)\pi(v)dv \\
+ \int_{P_+} [\chi (\nabla u) + 4\sigma (\nabla u)] f_a(v)\pi(v)dv. \]

Clearly, \( \mathcal{L}(\cdot) \) is well defined on \( \hat{C}'_{P,W} \). We will show that \( \mathcal{N}(\cdot) \) is also well defined, so is \( \mu(\cdot) \) below. The following is the main result in this section.

**Theorem 6.3** Let \( \Xi \) is the relative interior of the cone generated by \( R^+ \). Suppose that \( P_+ \) satisfies

\[ \text{bar}_a(P_+) = \frac{\int_{P_+} \langle v, \nu \rangle f_a(v)\pi(v)dv}{\int_{P_+} f_a(v)\pi(v)dv} \in 4\Lambda L\sigma + \Xi. \quad (6.13) \]

Then there is a \( \delta > 0 \) and a constant \( C_\delta \) such that

\[ \mu(u) \geq \delta \int_{P_+} uf_a(v)\pi(v)dv - C_\delta, \quad \forall u \in \hat{C}'_{P,W}. \]

Clearly, Proposition 6.2 follows from Theorem 6.3 by taking \( f \equiv 1, \Lambda L = (2(n + 1))^{-1} \) and \( P = \nu^*(P) \). In the following, we will use the arguments in [35] to prove the theorem.

### 6.2 The Linear Part \( \mathcal{L}(\cdot) \).

Let \( d\sigma_0 \) be the Lebesgue measure of \( \partial P_+ \) and \( \nu \) the corresponding unit normal vector. By (W1)-condition for \( f_a \) and convexity of \( u \), there is a constant \( \Lambda \) such that for any \( W \)-invariant convex function \( u \) which is normalized at \( O \),

\[ \int_{P_+} uf_a(v)\pi(v)dv \leq \Lambda \int_{\partial P_+} u(v,\nu) f_a(v)\pi(v)d\sigma_0. \quad (6.14) \]

Taking integration by parts in (6.12), and using the fact that

\[ \nu_i\pi_i(v) = (n - r)\pi(v), \]
we have
\[
4\Lambda L \int_{P_+} u \sigma (\nabla \pi) f_a(v) dv \\
= \mathcal{L}(u) - \int_{\partial P_+} u \langle v - 4\Lambda L \sigma, v \rangle f_a \pi d\sigma_0 \\
+ n \int_{P_+} u f_a(v) \pi(v) dv + \int_{P_+} u \langle v - 4\Lambda L \sigma, a \rangle f' \pi dv.
\]
Then by using (W1), (W2) and (6.14), we get a constant \( C > 0 \) such that
\[
\int_{P_+} u \sigma (\nabla \pi) f_a(v) dv \\
\leq \frac{1}{4\Lambda L} \mathcal{L}(u) + C \int_{\partial P_+} u \langle v, v \rangle f_a(v) \pi(v) d\sigma_0, \quad \forall u \in \hat{C}_P', \quad (6.15)
\]
Combining (6.14), (6.15) and following the argument in the proof of [35, Proposition 4.3], we can prove

**Lemma 6.4** Under the assumption (6.13), there is a constant \( \lambda > 0 \), such that
\[
\mathcal{L}(u) \geq \lambda \int_{\partial P_+} u \langle v, v \rangle f_a(v) \pi(v) d\sigma_0, \quad \forall u \in \hat{C}_P', \quad (6.16)
\]

### 6.3 The Nonlinear Part \( \mathcal{N}(\cdot) \).

In this subsection, we estimate \( \mathcal{N}(\cdot) \). In particular, we show that \( \mathcal{N}(\cdot) \) is well defined on \( \hat{C}_P' \). We will use a method in [23] (also see [35,51]). In fact, it suffices to show that for any \( u \in \hat{C}_P' \),
\[
\mathcal{N}^+(u) = -\int_{P_+} \log \det (u_{,ij}) - \chi (\nabla u) - 4\sigma (\nabla u)]^+ f_a(v) \pi(v) dv \\
> -\infty. \quad (6.16)
\]
As in the proof of [35, Lemma 6.3], for any \( u \in \hat{C}_P' \), we define a \( W \)-invariant function \( \hat{u} \) such that
\[
\hat{u}|_{P_+} = u + \frac{1}{2} c|v|^2 + \sigma_i v_i.
\]
Then \( \hat{u} \) lies in \( C^\infty(P_+) \cap C^0(\overline{P}_+) \) and satisfies
\[
\mathcal{N}^+(\hat{u}) < \mathcal{N}^+(u).
\]
Thus, by replacing \( u \) with \( \hat{u} \), we may assume

\[
\log \det (u_{ij}) - \chi(\nabla u) - 4\sigma(\nabla u) > 0,
\]

and consequently \( \mathcal{N}^+(u) = \mathcal{N}(u) \).

By the convexity of \( \chi(\cdot) \) and \( -\log \det(\cdot) \), we have

\[
-\log \det(u_{ij}) + \chi(\nabla u) 
\geq -\log \det(u_{P,ij}) + \chi(\nabla u_P) - u_{ij}^P(u_{ij} - u_{P,ij}) + \frac{\partial \chi}{\partial x^i} \bigg|_{x = \nabla u_P} (u_{i} - u_{P,i}).
\]

(6.17)

On the other hand, by the condition (P2), we have (cf. [22]),

\[
u_{ij}^P v_{Ai} \to 0 \quad \text{and} \quad u_{ij}^P, j v_{Ai} \to -2\lambda_A \langle v, v_A \rangle,
\]

(6.18)
as \( v \) goes to a facet \( \tilde{F}_A = \{ v | l_A(v) = 0 \} \) of \( P \). Here \( v_A \) is the unit outer normal vector of \( \tilde{F}_A \). Thus, integrating both sides of (6.17) on \( P_+ \) and taking integration by parts for the terms \( u_{ij}^P u_{ij} \) and \( \frac{\partial \chi}{\partial x^i} \bigg|_{x = \nabla u_P} u_{i} \), we get

\[
\int_{P_+} \left[ -\log \det(u_{ij}) + \chi(\nabla u) \right] f_a \pi dv 
\geq -\int_{\partial P_+} u_{ij}^P u_{ij} v_{j} f_a \pi d\sigma_0 + \int_{P_+} u_{ij}^P u_{ij} f_a \pi dv + \int_{P_+} u_{ij}^P u_{ij} a_{ij} f_a' \pi dv
\]

\[
+ \int_{P_+} u_{ij}^P v_{ij} \pi f_a \pi dv + \int_{\partial P_+} u \frac{\partial \chi}{\partial x^i} \bigg|_{x = \nabla u_P} v^i f_a \pi d\sigma_0
\]

\[
- \int_{P_+} u \frac{\partial^2 \chi}{\partial x^i \partial x^j} \bigg|_{x = \nabla u_P} f_a \pi dv
\]

\[
- \int_{P_+} u \frac{\partial \chi}{\partial x^i} \bigg|_{x = \nabla u_P} a^i f_a' \pi dv - \int_{P_+} u \frac{\partial \chi}{\partial x^i} \bigg|_{x = \nabla u_P} f_a \pi f_{i} dv - C_0.
\]

(6.19)

We need to deal with each term in (6.19) in the following.

Note that \( u \) is convex and continuous on \( \overline{P} \). Then by (6.18), we have (cf. [23, Lemma 3.3.5]),

\[
-\int_{\partial P_+} u_{ij}^P u_{ij} v_{j} f_a \pi d\sigma_0 = 0
\]

(6.20)

and

\[
-\int_{\partial P_+} u_{ij}^P u_{ij} v_{j} f_a \pi d\sigma_0 = \sum_A \frac{2}{\lambda_A} \int_{\tilde{F}_A(\alpha')} \pi f_a d\sigma_0.
\]

(6.21)
Note that \( \pi \) vanishes quadratically on Weyl walls and
\[
\frac{\partial \chi}{\partial x^i}(x) \to -4\sigma_i \tag{6.22}
\]
as \( x \to \infty \) and away from Weyl walls. We see that
\[
\left| \frac{\partial \chi}{\partial x^i} \right|_{x=\nabla u \sigma} \to -4\sigma_i \tag{6.23}
\]
Moreover, by the fact that \( \alpha(a) = 0 \) for any \( \alpha \), we have
\[
\frac{\partial \chi}{\partial x^i} a^i = -2 \sum_{\alpha \in \mathbb{R}^+} (\alpha_i a^i) \coth \alpha(x) = 0. \tag{6.24}
\]
On the other hand, taking integration by parts with help of (6.18), and then by (F3), (W3), we get the following estimates,
\[
\left| \int_{P_+} u_{\partial P_+}^{ij} u_{\partial P_+}^{ij} f_a \pi dv \right| = \sum_{\lambda} 2 \int_{\partial \lambda \cap (a')^+} u f_a \pi d\sigma_0
\]
\[
- \int_{P_+} \left[ u_{\partial P_+}^{ij} f_a \pi + u_{\partial P_+}^{ij} a^i + u_{\partial P_+}^{ij} f_a \pi \right] dv \leq C \left( \int_{\partial \lambda \cap (a')^+} u \langle v, \nu \rangle f_a(v) \pi(v) dv \right), \tag{6.25}
\]
\[
\int_{P_+} u_{\partial P_+}^{ij} a^i u_{\partial P_+}^{ij} f_a \pi dv = - \int_{P_+} u \left[ i_{\partial P_+}^{ij} a^j + i_{\partial P_+}^{ij} f_a \pi \right] dv \leq C \left( \int_{P_+} u \langle v, \nu \rangle f_a(v) \pi(v) dv \right), \tag{6.26}
\]
\[
\int_{P_+} u_{\partial P_+}^{ij} u_{\partial P_+}^{ij} f_a \pi \pi dv = - \int_{P_+} u \left[ i_{\partial P_+}^{ij} f_a \pi + i_{\partial P_+}^{ij} f_a \pi \right] dv \leq C \left( \int_{P_+} u \langle v, \nu \rangle f_a(v) \pi(v) dv \right). \tag{6.27}
\]
Thus, substituting (6.20)–(6.27) into (6.19), we finally obtain

\[
\mathcal{N}(u) \geq -C_1 \int_{\partial P_+} u \langle v, v \rangle f_a(v)\pi(v) d\sigma_0 + C_2 \int_{P_+} u (1 + \sigma(\nabla \pi)) f_a(v) dv + \int_{P_+} u Q f_a(v)\pi(v) dv - C_3,
\]

where

\[
Q = -\frac{\partial \chi}{\partial x^i} \bigg|_{x = \nabla u_P} \frac{\pi, i}{\pi} - \frac{\partial^2 \chi}{\partial x^i \partial x^k} \bigg|_{x = \nabla u_P} \frac{\partial^2 u_P}{\partial v_i \partial v_k} - u_P^{ij} \frac{\pi, ij}{\pi}.
\]

(6.28)

Hence, by (6.14) and (6.15), we see that there are uniform constants $C_1, C_2, C_3 > 0$ such that for any $u \in \hat{C}'_{P,W}$,

\[
\mathcal{N}^+(u) \geq -C_1 \int_{\partial P_+} u \langle v, v \rangle f_a(v)\pi(v) d\sigma_0 - C_2 L(u) + \int_{P_+} u Q f_a(v)\pi(v) dv + C_3.
\]

(6.29)

In particular, (6.16) is true.

### 6.3.1 Estimate of $Q$

As in [35], we have to control the growth of $Q$ near Weyl walls. The goal is to show that

**Lemma 6.5** There is a uniform constant $C_Q$ such that

\[
|Q|\pi \leq C_Q \sigma(\nabla \pi), \forall v \in P_+.
\]

**Proof** From (6.28), a direct computation shows

\[
Q = \sum_{\alpha \in R^+} \left[ 4 \frac{|\alpha|^2 \coth \alpha(\nabla u_P)}{\langle \alpha, v \rangle} - 2 \frac{u_P^{ij} \alpha_i \alpha_j}{\sinh^2 \alpha(\nabla u_P)} - 2 \frac{u_P^{ij}}{\langle \alpha, v \rangle^2} \right] + 2 \sum_{\alpha \neq \beta \in R^+} \left[ \coth \alpha(\nabla u_P) \cdot \frac{\langle \alpha, \beta \rangle}{\langle \beta, v \rangle} + \coth \beta(\nabla u_P) \cdot \frac{\langle \alpha, \beta \rangle}{\langle \alpha, v \rangle} - 2 u_P^{ij} \frac{\alpha_i \beta_j}{\langle \alpha, v \rangle^2} \right].
\]

For simplicity, we denote each term in these two sums by $I_\alpha(v)$ and $I_{\alpha,\beta}(v)$, respectively.

To estimate $I_\alpha(v)$, it suffices to control it near the Weyl wall $W_\alpha = \{ v | \langle \alpha, v \rangle = 0 \}$. By the $W$-invariance of $P$, we can divide outer faces of $P$ exactly into three classes as in [35]. Fix a point $v_0 \in W_\alpha$, let $v \to v_0$. Following the arguments of [35],
Lemma 4.9, Lemma 4.11], we see that there is a neighborhood $U_{v_0}$ and a constant $C_{v_0}$ such that

$$|I_\alpha(v)| \leq \frac{C_{v_0}}{\langle \alpha, v \rangle}, \quad \forall v \in U_{v_0} \cap P_+.$$  

We should remark that by our assumption (6.11) it holds

$$\alpha(u_A) \in \mathbb{Z}_{>0},$$

for any outer facet $F_A$ which is not orthogonal to $W_\alpha$, although $u_A$ may not be a lattice vector. Thus, the arguments in Case (iii) of [35, Lemma 4.11] are still available. Following [35, Lemma 4.11], $I_{\alpha,\beta}$ can be estimated in a similar way. Since $\partial P_+ \cap W_\alpha$ is compact, there are uniform constants $C_\alpha, C_{\alpha,\beta}$ such that

$$|I_\alpha(v)| \leq C_\alpha \langle \alpha, v \rangle,$$

$$|I_{\alpha,\beta}(v)| \leq C_{\alpha,\beta} \left( \frac{1}{\langle \alpha, v \rangle} + \frac{1}{\langle \beta, v \rangle} \right),$$

for any $v \in P_+$. Recall that $\langle \sigma, \alpha \rangle > 0$ for any $\alpha \in R^+$. Then

$$\frac{\sigma(\nabla \pi(v))}{2\pi(v)} = \sum_{\alpha \in R^+} \frac{\langle \sigma, \alpha \rangle}{\langle \alpha, v \rangle} \geq \frac{C}{\langle \alpha, v \rangle}, \quad \forall \alpha \in R^+.$$  

Thus, Lemma 6.5 follows from (6.30) and the above inequality. \hfill \Box

Combining (6.14), (6.15), (6.29) and Lemma 6.5, we prove

**Proposition 6.6** There are uniform constants $C_0, C_L > 0$ such that for any $u \in \hat{\mathcal{C}}_{P,W}'$,

$$\mathcal{N}^+(u) \geq -C_L \mathcal{L}(u) - C_0.$$  

(6.31)

(6.31) implies (6.16). Thus, $\mathcal{N}(\cdot)$ is well defined on $\hat{\mathcal{C}}_{P,W}'$.

### 6.4 Proof of Theorem 6.3

**Proof of Theorem 6.3** Let $\epsilon \in (0, 1)$ be a small positive number. Note that

$$\mathcal{N}(\epsilon u) > \mathcal{N}^+(\epsilon u).$$
Then by Proposition 6.6, it is easy to see that (cf. [35, Proposition 4.1]),

\[ N'(u) \geq N'(\epsilon u) + n \log \epsilon \]
\[ \geq -C_0 + n \log \epsilon - \epsilon C_L \mathcal{L}(u). \]

Take \( \epsilon \) sufficiently small such that

\[ 1 - \epsilon \cdot \Lambda_L C_L = \delta' \cdot \Lambda_L > 0. \]

Thus, we get

\[ \mu(u) = \frac{1}{\Lambda_L} \mathcal{L}(u) + N(u) \]
\[ \geq \delta' \mathcal{L}(u) - C_0 + n \log \epsilon. \]

Combining (6.14) and Lemma 6.4, we derive

\[ \mu(u) \geq \frac{\delta' \lambda}{\Lambda} \int_M u f_a \pi d\nu - C_0 + n \log \epsilon. \]

The theorem is proved.

\[ \square \]

7 Existence of G-Sasaki–Einstein Metrics

As in the study of Kähler–Einstein metrics [41,46], we usually solve (2.9) via the following continuity method (cf. [24,26]),

\[ \det(g^T_{ij} + \psi_{,ij}) = \exp(-2t(n + 1)\psi + h) \det(g^T_{ij}), \quad t \in [0, 1], \quad (7.1) \]

where \( g^T \) is a transverse Kähler metric with its Kähler form \( \omega^T_g \in \frac{\pi}{n+1} c_B^B(M) \). It is known that (7.1) is solvable for sufficiently small \( t > 0 \) and \( \omega^T_g + \sqrt{-1} \partial \bar{\partial} \psi \) satisfies the Sasaki–Einstein metric equation (2.8) if \( \psi \) is a solution of (7.1) at \( t = 1 \). Thus, solving (2.9) turns to do a priori estimate for solutions \( \psi_t \) with \( t \in [t_0, 1] \) for some \( t_0 > 0 \). As shown in [24,26], we need to do the \( C^0 \)-estimate for solutions \( \psi_t \).

As a version of Tian’s theorem in case of Sasaki manifolds [41], Zhang proved the following analytic criterion for the existence of Sasaki–Einstein metrics [48].

**Theorem 7.1** Let \((M, \frac{1}{2} d\eta)\) be a \((2n + 1)\)-dimensional compact Sasaki manifold with
\[ \frac{1}{2}[d\eta]_B = \frac{\pi}{n+1} c_B^B(M). \] Suppose that there is no non-trivial transverse holomorphic vector field on \( M \). Then \((M, \frac{1}{2} d\eta)\) has a Sasaki–Einstein metric if and only if K-energy \( K(\cdot) \) is proper on \( \mathcal{H}(\frac{1}{2} d\eta) \).
7.1 A generalization of Zhang’s Theorem

In general, \((M, \frac{1}{2} \, d\eta)\) may admit Hamiltonian holomorphic vector fields. Note that \(K(\cdot)\) is invariant under \(\text{Aut}^T(M)\) if the Futaki-invariant vanishes. Thus, one shall modify Theorem 7.1 for the properness property of \(K(\cdot)\) in sense of Definition 5.5.

Similar to \(I\)-functional, one can define Aubin’s \(J\)-functional on \(\mathcal{H}(\frac{1}{2} \, d\eta)\) by

\[
J(\psi) = \int_0^1 \frac{1}{s} I(s \psi) \, ds.
\]

It can be checked that (cf. [48])

\[
0 \leq \frac{1}{n+1} I(\psi) \leq I(\psi) - J(\psi) \leq \frac{n}{n+1} I(\psi).
\]

**Lemma 7.2** Let \(\text{Aut}^T_0(M)\) be the connected component of \(\text{Aut}^T(M)\) which contains the identity. Then for any \(\psi \in \mathcal{H}(\frac{1}{2} \, d\eta)\), there exists a \(\sigma_0 \in \text{Aut}^T_0(M)\) such that

\[
(I - J)(\psi_{\sigma_0}) = \min_{\sigma \in \text{Aut}^T_0(M)} \{ (I - J)(\psi_{\sigma}) \},
\]

where \(\psi_{\sigma}\) is an induced potential defined by

\[
\frac{1}{2} \sigma^* d\eta_{\psi} = \omega^T_g + \sqrt{-1} \partial \bar{\partial} \psi_{\sigma},
\]

if and only if

\[
\int_M \text{real}(X)(\psi_{\sigma_0})(d\eta_{\psi_{\sigma_0}})^n \wedge \eta_{\psi_{\sigma_0}} = 0, \; \forall \; X \in \mathfrak{ham}(M).
\]

**Proof** Let \(\sigma_s\) be the one parameter subgroup in \(\text{Aut}^T_0(M)\) generated by \(\text{real}(X)\). Then by a direct computation, we have

\[
\frac{d}{ds} [I(\psi_{\sigma_s}) - J(\psi_{\sigma_s})]_{s=0}
\]

\[
= -\frac{n}{2^{n-1} V} \int_M \psi_{\sigma_0} d\psi_{\sigma_s} \wedge \eta_{\psi_{\sigma_0}}^{n-1} \wedge \eta_{\psi_{\sigma_0}}
\]

\[
= \frac{\sqrt{-1} n}{2^{n-1} V} \int_M \partial B \psi_{\sigma_0} \wedge \bar{\partial} B \psi_{\sigma_s} \wedge d\eta_{\psi_{\sigma_0}}^{n-1} \wedge \eta_{\psi_{\sigma_0}}
\]

\[
= \frac{1}{2^n V} \int_M \text{real}(X)(\psi_{\sigma_0}) d\eta_{\psi_{\sigma_0}}^{n} \wedge \eta_{\psi_{\sigma_0}}.
\]

Thus, if \(\sigma_0\) is a minimizer of \(F(\sigma) = I(\psi_{\sigma}) - J(\psi_{\sigma})\), then (7.3) holds. Conversely, we need to show that a critical point of \(F(\sigma)\) is also a minimizer. This follows from...
the convexity of $F(\sigma)$ along any one parameter subgroup $\sigma_s$. Namely, we have

$$\frac{d^2}{ds^2}[I(\psi_{\sigma_s}) - J(\psi_{\sigma_s})] \geq 0, \forall s \geq 0. \quad (7.5)$$

Rewrite the second identity in (7.4) as

$$\frac{d}{ds}[I(\psi_{\sigma_s}) - J(\psi_{\sigma_s})] = -\frac{1}{2^n V} \int_M \dot{\psi}_{\sigma_s} \triangle_B \psi_{\sigma_s} d\eta^{n}_{\psi_{\sigma_s}} \wedge \eta_{\psi_{\sigma_s}}. \quad (7.6)$$

Then

$$\frac{d^2}{ds^2}[I(\psi_{\sigma_s}) - J(\psi_{\sigma_s})]$$

$$= -\frac{1}{2^n V} \int_M \ddot{\psi}_{\sigma_s} \triangle_B \psi_{\sigma_s} d\eta^{n}_{\psi_{\sigma_s}} \wedge \eta_{\psi_{\sigma_s}} - \frac{1}{2^n V} \int_M \dot{\psi}_{\sigma_s} \triangle_B \dot{\psi}_{\sigma_s} d\eta^{n}_{\psi_{\sigma_s}} \wedge \eta_{\psi_{\sigma_s}}$$

$$- \frac{1}{2^n V} \int_M \dot{\psi}_{\sigma_s} \triangle_B \dot{\psi}_{\sigma_s} \triangle_B \psi_{\sigma_s} d\eta^{n}_{\psi_{\sigma_s}} \wedge \eta_{\psi_{\sigma_s}}$$

$$+ \frac{1}{2^n V} \int_M \dot{\psi}_{\sigma_s} \langle \partial_B \bar{\partial}_B \psi_{\sigma_s}, \partial_B \bar{\partial}_B \dot{\psi}_{\sigma_s} \rangle d\eta^{n}_{\psi_{\sigma_s}} \wedge \eta_{\psi_{\sigma_s}}. \quad (7.6)$$

Note

$$\ddot{\psi}_{\sigma_s} = |\partial_B \dot{\psi}_{\sigma_s}|^2 d\eta_{\psi_{\sigma_s}}.$$  

Taking integration by parts in (7.6), we get

$$\frac{d^2}{ds^2}[I(\psi_{\sigma_s}) - J(\psi_{\sigma_s})] = \frac{1}{2^n V} \int_M |X|^2_{d\eta^{n}_{\psi_{\sigma_s}}} d\eta^{n}_{\psi_{\sigma_s}} \wedge \eta_{\psi_{\sigma_s}} \geq 0, \forall s \geq 0.$$  

This verifies (7.5).

The following is an improvement of Theorem 7.1 in the sufficient part.

**Proposition 7.3** Let $(M, \frac{1}{2} d\eta)$ be a $(2n + 1)$-dimensional compact Sasaki manifold with $\frac{1}{2}[d\eta]_B = \frac{n}{n+1} c_1^B (M)$. Let $K$ and $G_0$ be two subgroups of $\text{Aut}^T (M)$ as in Definition 5.5. Then $(M, \frac{1}{2} d\eta)$ admits a transverse Sasaki–Einstein metric if $K(\cdot)$ is proper on $\mathcal{H}_K (\frac{1}{2} d\eta)$ modulo $G_0$.

**Proof** The proof is a slight modification of Tian’s argument for Kähler–Einstein metrics in [42, Theorem 2.6] (also see [41,48]). Without loss of generality, we may assume that $d\eta$ is $K$-invariant. Thus, all $\psi_t$ of (7.1) are $K$-invariant. It suffices to get a uniform bound of $I(\psi_t)$. We note that $\text{Fut}(\cdot) \equiv 0$ on $\text{Ham}(M)$ since $\mathcal{K}(\cdot)$ is proper on $\mathcal{H}_K (\frac{1}{2} d\eta)$ modulo $G_0$.  

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From the computation for solutions $\psi_t$ in (7.4), we have
\[
\frac{d}{ds} [I(\psi_{\sigma s}) - J(\psi_{\sigma s})]|_{s=0} = -\frac{1}{V} \int_M \text{real}(X)(\psi_t) d\eta^{\psi_t}_t \wedge \eta_t,
\]
Note that
\[
h_t + 2(n + 1)(1 - t)\psi_t = c_t,
\]
where $h_t$ is the basic Ricci potential of $\frac{1}{2} d\eta_{\psi_t}$ and $c_t$ is a constant. Thus,
\[
\frac{d}{ds} [I(\psi_{\sigma s}) - J(\psi_{\sigma s})]|_{s=0} = \frac{1}{(1 - t)V} \text{real}(\text{Fut}(X))
= 0, \forall X \in \mathfrak{ham}(M).
\]
This means that $\psi_t$ is a minimizer of $I(\psi_{\sigma}) - J(\psi_{\sigma})$ for $\psi_t$ by Lemma 7.2. Since $K(\psi_t)$ is uniformly bounded above for any $t \in [t_0, 1]$ (cf. [48]), $I(\psi_t) - J(\psi_t)$ and so $I(\psi_t)$ is uniformly bounded by the properness of $\mathcal{H}_K \left(\frac{1}{2} d\eta\right)$ modulo $G_0$. 

### 7.2 Proof of Theorem 1.1

First, we prove the necessary part. Here we will use an argument for extremal Kähler metrics from [49] and [50]. In fact, we have the following proposition.

**Proposition 7.4** Suppose that $M$ admits a $G$-Sasaki metric with constant transverse scalar curvature. Then for any convex $W$-invariant piecewise linear function $f$ on $P$, we have
\[
\mathcal{L}(f) \geq 0.
\]
Moreover, the equality holds if and only if
\[
f(v) = a^i v_i
\]
for some $a = (a^i) \in a'$. 

**Proof** As before, we assume that $\gamma$ is chosen such that $P$ contains $O$. A convex $W$-invariant piecewise linear function $f$ on $P$ can be written as
\[
f = \max_{1 \leq N \leq N_0} \{f_N\},
\]
where $f_N$ is $W$-invariant such that
\[
f_N|_{P_+}(v) = a^i_N v_i + c_N
\]
for some constant vector \( a_N = (a_N^i) \). It is showed that \( a_N \in \overline{a_T} \) (cf. [35, Proposition 3.4]). Then we can divide \( P_\tau \) into \( \tau_0 \) sub-polytopes \( P_1, \ldots, P_{\tau_0} \) such that for each \( \tau = 1, \ldots, \tau_0 \), there is an \( N(\tau) \in \{1, \ldots, N_0\} \) with

\[
f|_{P_\tau} = f_{N(\tau)}.
\]

For simplicity, we write \( f_\tau \) as \( f_{N(\tau)} \).

On the other hand, we may write a \( G \)-Sasaki metric with constant transverse scalar curvature as

\[
\omega^T g = \sqrt{-1} \bar{\partial} \bar{\partial} \varphi_0, \tag{5.9}
\]

where \( \varphi_0 \) is a \( K \times K \)-invariant function [10]. By (5.9), we have

\[
S^T(u_0) = -\frac{1}{\pi} \left( (u_0^i)^{ij} \pi_{ij} + \frac{\partial}{\partial v_i} \left( \pi \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} \right) \right)
\]

\[
= \tilde{S}.
\]

Then, on each \( P_\tau \),

\[
-\tilde{S} \int_{P_\tau} f \pi \, dv
\]

\[
= \int_{P_\tau} \left( (u_0^i)^{ij} \pi_{ij} + \frac{\partial}{\partial v_i} \left( \pi \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} \right) \right) f \, dv. \tag{7.7}
\]

Note that \( f_{,ij} = 0 \) on each \( P_\tau \). Taking integration by parts, we get

\[
\int_{P_\tau} (u_0^i)^{ij} \pi_{ij} f \pi \, dv = \int_{\partial P_\tau} \left( (u_0^i)^{ij} \pi_{ij} v_i + u_0^i \pi_{,i} v_j \right) f \, d\sigma_0
\]

\[
- \int_{\partial P_\tau} u_0^i v_i f \pi \, d\sigma_0
\]

and

\[
\int_{P_\tau} \frac{\partial}{\partial v_i} \left( \pi \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} \right) f \, d\sigma_0 = \int_{\partial P_\tau} v_i \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} f \pi \, d\sigma_0
\]

\[
- \int_{P_\tau} \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} f_{,i} \pi \, dv.
\]

Plugging the above relations into (7.7), it follows

\[
-\tilde{S} \int_{P_\tau} f \pi \, dv = \int_{\partial P_\tau} \left( (u_0^i)^{ij} \pi_{ij} v_i + u_0^i \pi_{,i} v_j + v_i \pi \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} \right) f \, d\sigma_0
\]

\[
- \int_{\partial P_\tau} u_0^i v_i f \pi \, d\sigma_0 - \int_{P_\tau} \frac{\partial \chi}{\partial x^i} \bigg|_{x=\nabla u_0} f_{,i} \pi \, dv.
\]
Thus, summing over \( \tau \), using (5.4) and the argument of [50, Proposition 2.2], we obtain

\[
- \bar{S} \int_{P_+} f \pi \, dv = \sum_{\tau_1 < \tau_2} \int_{\partial P_{\tau_1} \cap \partial P_{\tau_2}} \frac{u^{ij}_0 (a^{i}_{\tau_1} - a^{i}_{\tau_2})(a^{j}_{\tau_1} - a^{j}_{\tau_2})}{|a^{\tau_1}_r - a^{\tau_2}_r|} \pi \, d\sigma_0 \\
- \sum_A \Lambda_A \int_{\partial \tilde{P}_A \cap \partial P_+} f(v, v_A) \pi \, d\sigma_0 - \sum_\tau \int_{P_\tau} \frac{\partial \chi}{\partial x^i} \big|_{x = \nabla u_0} a^i_{\tau} \pi \, dv.
\]

(7.8)

Recall (6.12). We see that

\[
V_P \cdot L(f) = \sum_A \Lambda_A \int_{\partial \tilde{P}_A \cap \partial P_+} f(v, v_A) \pi \, d\sigma_0 - \bar{S} \int_{P_+} f \pi \, dv \\
- 4 \sum_\tau \int_{P_\tau} \sigma(a_\tau) \pi \, dv.
\]

(7.9)

Note that for any \( a_\tau = (a^i_\tau) \in \overline{a_\tau}' \),

\[-a^i_{\tau} \frac{\partial \chi}{\partial x^i} - 4 \sigma_i a^i_{\tau} = 2 \sum_{\alpha_\tau \in \Phi_+} (\coth \alpha(x) - 1) \alpha(a_\tau) \geq 0, \quad \forall \, x \in a_+.
\]

Hence, plugging (7.8) into (7.9), we derive

\[
V_P \cdot L(f) = \sum_{\tau_1 < \tau_2} \int_{\partial P_{\tau_1} \cap \partial P_{\tau_2}} \frac{u^{ij}_0 (a^{i}_{\tau_1} - a^{i}_{\tau_2})(a^{j}_{\tau_1} - a^{j}_{\tau_2})}{|a^{\tau_1}_r - a^{\tau_2}_r|} \pi \, d\sigma_0 \\
+ 2 \sum_\tau \sum_{\alpha_\tau \in \Phi_+} \int_{P_\tau} (\coth \alpha(x) - 1) \alpha(a_\tau) \pi \, dv \geq 0.
\]

(7.10)

It is easy to see that the equality in (7.10) holds if and only there is an \( a = (a^i) \in \overline{a_\tau}' \) such that

\[a_\tau = a, \quad \forall \, \tau\]

and

\[\alpha(a) = 0, \quad \forall \, \alpha \in \Phi_.\]

The second relation means that \( a \in \overline{a_\tau}' \). The proposition is proved. \( \square \)

**Proof of necessary part of Theorem 1.1** Suppose that (1.1) does not hold. Choosing \( \gamma = \gamma_0 \). Then

\[
\bar{b}ar(P_+) = \frac{2}{n + 1} \sigma \notin \Xi.
\]
We will follow a way in [35, Lemma 3.4] to construct a piecewise linear function. By (6.10), we may assume

$$\bar{\text{bar}}(P_+) - \frac{2}{n + 1}\sigma \in (a')^\ast_{ss},$$

otherwise the Futaki invariant does not vanishes. Let \(\{\alpha_{(1)}, ..., \alpha_{(r')}\}\) be the simple roots in \(\Phi^+_+.\) Without loss of generality, we can write

$$\bar{\text{bar}}(P_+) - \frac{2}{n + 1}\sigma = \lambda_1 \alpha_{(1)} + ... + \lambda_r' \alpha_{(r')} ,$$

where \(\lambda_1 \leq 0.\) Let \(\{\varpi_i\}\) be the fundamental weights for \(\{\alpha_{(1)}, ..., \alpha_{(r')}\}\) such that \(\frac{\langle \varpi_i, \alpha_{(j)} \rangle}{|\alpha_{(j)}|^2} = \delta_{ij}.\) Define a \(W\)-invariant rational piecewise linear function \(f\) on \(P\) by

$$f(v) = \max_{w \in W} \{\langle w \cdot \varpi_1, v \rangle\}.$$  

Then

$$f|_{P_+} = \langle \varpi_1, v \rangle.$$  

Note that \(\varpi_1 \in (a')^\ast_{ss}.\) However,

$$\mathcal{L}(f) = n(n + 1)|\alpha_{(1)}|^2 \lambda_1 \leq 0.$$  

This contradicts to Proposition 7.4. Hence, (1.1) is true.

To prove the sufficient part of Theorem 1.1, we need the following lemma.

**Lemma 7.5** For any \(\psi \in \mathcal{H}_{K \times K} \left(\frac{1}{2} d\eta\right)\) with \(u_\psi \in \hat{\mathcal{C}}_{P, W},\) there exists a uniform constant \(C\) such that

$$\left| J(\psi) - \frac{1}{V_P} \int_{P_+} u_\psi \, dv \right| \leq C.$$  

**Proof** First by Lemma 5.1, we have

$$J(\psi) = \frac{1}{V} \int_M \psi \, (d\eta)^n \wedge \eta + \frac{1}{V_P} \int_{P_+} (u_\psi - u_0) \pi \, dv.$$  

Then the lemma is reduced to prove

$$\left| \int_M \psi \, (d\eta)^n \wedge \eta \right| \leq C, \quad \forall u_\psi \in \hat{\mathcal{C}}_W.$$  

(7.11)
By the normalized condition, it follows
\[ \nabla u_\psi(O) = O, \ u_\psi(O) = 0. \]
Thus,
\[ \psi(O) = -\varphi_0(O). \]
On the other hand, since \( \psi \) is a basic function,
\[ \triangle_B \psi = \triangle_g \psi. \]
This means that the basic Laplace operator coincides with the Laplace operator of \( g \) on \( \psi \). Thus, by using the above two estimates and following the Green function argument in [49, Lemma 2.2], we can obtain a uniform \( C_0 \) such that
\[
\frac{1}{V_P} \int_M \psi(d\eta)^n \wedge \eta \geq \sup_M \psi - C_0 \geq -\psi_0(O) - C_0.
\]
(7.12)
On the other hand, by \( \xi(\psi) = 0 \), we have
\[ |\nabla \psi| = |\nabla \psi|_{Orb M(p)}. \]
It follows that
\[ |\nabla \psi| \leq |\nabla \varphi_0| + |\nabla \varphi| \leq 2\text{diam}(P). \]
Then by an argument in [49, Lemma 2.2] and (7.12), we get
\[
\sup_M \psi \leq C'
\]
for some large constant \( C' \). Hence, combining (7.12) and (7.13), we obtain (7.11). \( \square \)

**Proof of Sufficient Part of Theorem 1.1** First, we note that (6.9) is equivalent to (1.1) by the relation (4.8). On the other hand, by Lemma 5.4, we see that there is a \( \sigma \in Z(K') \) such that \( \hat{u} \in \hat{\Delta}_{P,W} \) for any \( \psi \in \mathcal{H}_{K \times K} \left( \frac{1}{2} d\eta \right) \), where \( \hat{u} \) is the Legendre function of \( \varphi_{\psi\sigma} \). Then by Proposition 6.2 and Lemma 7.5, we get
\[
\mathcal{K}(\psi) = \mu(\hat{u}) \geq \delta \int_{P_+} \hat{u}\pi(y) \ dy - C_\delta \\
\geq \delta J(\psi_\sigma) - C'_\delta \\
\geq \delta \inf_{\tau \in Z(H)} J(\psi_\tau) - C'_\delta.
\]
(7.14)
(7.14) means that \( \mathcal{K}(\cdot) \) is proper on \( \mathcal{H}_{K' \times K'} \left( \frac{1}{2} d\eta \right) \) modulo \( \mathcal{Z}(H) \). Hence, by Proposition 7.3, we prove the existence of \( G \)-Sasaki–Einstein metrics.

7.3 Strong Properness of \( K \)-Energy Modula \( \text{Aut}^{T}(M) \)

From the proof in Theorem 1.1, we actually prove the following strong properness of \( K \)-energy \( \mathcal{K}(\cdot) \) for a \( G \)-Sasaki–Einstein manifold.

**Corollary 7.6** Let \((M, g)\) be a \((2n + 1)\)-dimensional \( G \)-Sasaki manifold with \( \omega_{g}^{T} = \frac{n}{n + 1} c_{1}^{\mathcal{B}}(M) \). Suppose that \( M \) admits a transverse Sasaki–Einstein metric. Then there are \( \delta, C_{\delta} > 0 \) such that for any \( K \times K \)-invariant transverse Kähler potential \( \psi \) of \( \omega_{g}^{T} \) it holds

\[
\mathcal{K}(\psi) \geq \delta \inf_{\tau \in Z'(T^{c})} I(\psi_{\tau}) - C_{\delta},
\]

(7.15)

where \( Z'(T^{c}) \subset T^{c} \cap \text{Aut}^{T}(M) \) is a subgroup of the center \( Z(G) \) with codimension 1.

**Proof of Corollary 7.6** By the necessary part of Theorem 1.1, (6.9) holds. Then as in the proof for the sufficient part of Theorem 1.1 above, for any \( \psi \in \mathcal{H}_{K \times K} \left( \frac{1}{2} d\eta \right) \), (7.15) holds with \( Z'(T^{c}) \) chosen as \( Z(H) \). The corollary is proved.

For a general Sasaki manifold which admits a transverse Sasaki–Einstein metric, we propose the following conjecture.

**Conjecture 7.7** Let \((M, g)\) be a \((2n + 1)\)-dimensional Sasaki manifold with \( \omega_{g}^{T} = \frac{n}{n + 1} c_{1}^{\mathcal{B}}(M) \). Suppose that \( M \) admits a transverse Sasaki–Einstein metric. Then there are \( \delta, C_{\delta} > 0 \) such that for any \( K \)-invariant transverse Kähler potential \( \psi \) of \( \omega_{g}^{T} \) it holds

\[
\mathcal{K}(\psi) \geq \delta \inf_{\tau \in Z(\text{Aut}^{T}(M))} I(\psi_{\tau}) - C_{\delta},
\]

(7.16)

where \( K \) and \( Z(\text{Aut}^{T}(M)) \) are a maximal compact subgroup and the center of \( \text{Aut}^{T}(M) \), respectively.

Conjecture 7.7 can be regarded as a version of Tian’s conjecture for \( K \)-invariant Kähler potentials in case of transverse Sasaki–Einstein manifolds [41]. Recently, Darvas and Rubinstein proved Tian’s conjecture when \( Z(\text{Aut}(M)) \) is replaced by \( \text{Aut}(M) \) in case of Kähler–Einstein manifolds [17].

8 \( G \)-Sasaki–Ricci Solitons

In this section, we give a version of Theorem 1.1 for the existence of transverse Sasaki–Ricci solitons. As a generalization of transverse Sasaki–Einstein metrics, a
Sasaki metric \((M, g, \xi, \eta)\) is called a \textit{transverse Sasaki–Ricci soliton} if there is an \(X \in \mathfrak{ham}(M)\) such that (cf. [8,26,37,38], etc.)

\[
\text{Ric}^T(g) - 2(n + 1)\omega^T_g = L_X\omega^T_g,
\]

where \(X\) is called a soliton vector field \(X\) on \((M, \xi)\). Clearly, \(\frac{1}{2}[d\eta]_B = \frac{\pi}{n+1}c_1^B(M)\) by the definition. It has been proved that on a compact Sasaki manifold the soliton vector field \(X\) is determined by vanishing of the modified Futaki invariant (cf. [26, Proposition 5.3]),

\[
\text{Fut}_X(Y) = -\int_M u_Y e^{u_X} \left(\frac{1}{2} d\eta\right)^n \wedge \eta, \quad \forall v \in \mathfrak{ham}(M).
\] (8.1)

In case of the \(G\)-Sasaki manifold, by restricting the metric to the \(H_0\)-orbit as in Section 6, one can further show that the vanishing of (8.1) is equivalent to

\[
\int_{P^+} y^i v_i e^{X^i \gamma_k} \pi \, dv = 0, \quad \forall \, Y = (Y^i) \in \mathfrak{z}(h_0).
\] (8.2)

Clearly, \(X = (X^k)\) can be uniquely determined by (8.2) as in a toric manifold [44]. In particular, \(X \in \mathfrak{z}(h_0)\).

Define a weighted barycenter with respect to the above \(X\) by

\[
\text{bar}_X(P^+) = \frac{\int_{P^+} y e^{X^i v_i} \pi \, d\sigma_c}{\int_{P^+} e^{X^i v_i} \pi \, d\sigma_c}.
\]

We get a soliton version of Theorem 1.1 as follows.

**Theorem 8.1** Let \((M, g, \xi)\) be a \((2n + 1)\)-dimensional compact \(G\)-Sasaki manifold with \(\omega^T_g \in \frac{\pi}{n+1}c_1^B(M) > 0\). Then \(M\) admits a transverse Sasaki–Ricci soliton if and only if \(\text{bar}_X(P^+)\) satisfies

\[
\text{bar}_X(P^+) - \frac{2}{n + 1} \sigma + \frac{1}{n + 1} \gamma_0 \in \mathfrak{E}.
\] (8.3)

We note that an analogy of Theorem 8.1 for Kähler–Ricci solitons on \(G\)-manifolds has been recently established in [19] and [35], respectively. Similar to Kähler geometry, one can introduce a modified K-energy on \(\mathcal{H} \left(\frac{1}{2} d\eta\right)\) as in [12,35,43,45], etc. By following the argument in [35], one can extend the proof of Theorem 1.1 to Theorem 8.1 by taking \(f_a(v) = f_X(v) = e^{X^i v_i}\) in Theorem 6.3. We leave the details to the reader.
8.1 Deformation of Transverse Sasaki–Ricci Solitons

In [37,38], Martelli, Sparks and Yau introduced the deformation theory of Reeb vector fields $\xi$ on a compact Sasaki manifold. They showed that the volume of $M$ in fact depends only on $\xi$. Moreover, they proved that under the restriction of (6.2) the Sasaki structure $(M, g, \xi, \eta)$ has the vanishing Futaki invariant if $\xi$ is a critical point of $\text{Vol}(M, g)$. In particular, by applying their theory together with the Futaki–Ono–Wang’s result for the existence of transverse Sasaki–Ricci solitons on toric Sasaki manifolds [26], one will obtain a deformation theorem for transverse toric Sasaki–Ricci solitons. We want to extend the above deformation theorem to $G$-Sasaki manifolds. However, unlike the toric Sasaki manifolds, we need to overcome the obstruction condition (8.3).

Analogous to [37], we deform $\xi$ in $z(k)$ and see that $\xi$ must be in an open convex cone

$$\Sigma = C^\vee \cap a_z,$$

where $C^\vee$ is the interior of the dual cone of $C$. Fix a $\xi' \in \Sigma$, by Proposition 3.4, there is a function $\rho_{\xi'}$ defined on $Z$ such that

1. $F_{\xi'} = \frac{1}{2} \rho_{\xi'}^2$ is the Legendre function of $U_{\xi'}$;
2. $\omega' = \sqrt{-1} \partial \bar{\partial} F_{\xi'}$ is a Kähler cone metric on $Z$. Thus, $\{\rho_{\xi'} = 1\} \cap Z$ is a toric Sasaki manifold.

Note that the complex structure of $C(M)$ does not change. By Proposition 3.2 we see that there is a $G$-Sasaki manifold $M'$, which is diffeomorphic to $M$ and Kähler cone of which is $(C(M), \omega')$. Hence, we get a Sasaki structure $(M, g', \xi', \eta')$ [9, Sect. 3].

By Proposition 6.1, we see that $\omega'^T g' \in \pi_{n+1} c^B_1(M) > 0$ on the Sasaki manifold $(M, g', \xi', \eta')$ if and only if $\xi' \in \Sigma_O$, where $\Sigma_O$ is defined by (1.2). Now we prove Theorem 1.2.

**Proof of Theorem 1.2** By a change of variables $v = t^*(y)$, (8.2) is equivalent to

$$\left( \text{bar}_{\chi}(P_+) + \frac{1}{n+1} \gamma_0 \right)(Y) = 0, \forall Y \in a_z. \quad (8.4)$$

Choose coordinates $y_1, \ldots, y_{r+1}$ on $a_z$ such that $y_1, \ldots, y_r$ are the coordinates on $\ker(\gamma_0)$. Then by (8.4), it follows that

$$\Psi_i(X)_{\xi_0} = \int_{P_+} y_i e^{X_k Y_k} \pi(y) \, d\sigma_c = 0, \; i = 1, \ldots, r. \quad (8.5)$$
since \((M, \xi_0)\) admits a transverse Sasaki–Ricci soliton with respect to \(X = (X^1, \ldots, X^r)\). Define a function on the linear space \(\text{aut}^T(M, \xi_0) \cap a_z\) by

\[
F(\hat{X})_{\xi_0} = \int_{P^+} e^{\hat{X}^k y_k} \pi(y) d\sigma_c.
\]

By a result in [43], one sees that \(F(\hat{X})_{\xi_0}\) is strictly convex for \(\hat{X}\). Thus, \(X\) is in fact a critical point of \(F(\hat{X})_{\xi_0}\).

Let \(P(\xi')\) be a polytope defined as in (4.2) associated to \(\xi' \in \Sigma_O\). Note that \(\text{aut}^T(M, \xi') \cap a_z\) is isomorphic to \(\text{aut}^T(M, \xi) \cap a_z\). Then as in (8.5), we can also define a map

\[
\Psi(\hat{X})_{\xi'} = (\Psi_1(\hat{X})_{\xi'}, \ldots, \Psi_r(\hat{X})_{\xi'}) : \mathbb{R}^r \to \mathbb{R}^r
\]

by

\[
\psi_i(\hat{X})_{\xi'} = \int_{P^+(\xi')} y_i e^{\hat{X}^k y_k} \pi(y) d\sigma_c, \ i = 1, \ldots, r.
\]

Taking derivatives of the above \(\Psi(\hat{X})_{\xi'}\) with respect to \(\hat{X}^1, \ldots, \hat{X}^r\) at the pair \((X, \xi_0)\), we get

\[
\frac{\partial \psi_i}{\partial \hat{X}^j}(X)_{\xi_0} = \int_{P^+} y_i y_j e^{\hat{X}^k y_k} \pi(y) d\sigma_c.
\]

It is clear that \(\frac{\partial \psi_i}{\partial \hat{X}^j}(X)_{\xi_0}\) is a strictly positively definite \((r \times r)\)-matrix on \(\mathbb{R}^r\).

By the implicit function theorem, there exists a unique solution \(X'\) near \(X\) for any \(\xi'\) sufficiently closed to \(\xi_0\) such that

\[
\Psi(X')_{\xi'} = 0.
\]

Namely, \(X'\) is a soliton vector field on \((M, g', \xi')\). Since \(\Xi\) is open in \(a_{+ss}\), the condition (8.3) will be preserved on \((M, g', \xi')\) as long as \(\xi'\) sufficiently closed to \(\xi_0\).

Hence, we prove Theorem 1.2 by Theorem 8.1 immediately. \(\square\)

### 9 Examples

In this section, we give several examples of \(G\)-Sasaki manifolds and verify the existence of \(G\)-Sasaki Einstein metrics or \(G\)-Sasaki Ricci solitons on them.

**Example 9.1** Let \((M', \omega')\) be a Fano manifold and \(M\) the Kobayashi regular principle \(S^1\)-bundle over \(M'\). Then \(M\) is a regular Sasaki manifold.
The Kobayashi regular principle $S^1$-bundle over a Kähler manifold was constructed in [32]. Boyer-Galicki [8, Theorem 7.5.2] showed that $M$ is a regular Sasaki manifold with Reeb field of which is induced by the corresponding $S^1$-action. Furthermore, the contact form $\eta$ satisfies $\frac{1}{2} d\eta = \pi^* \omega'$ (cf. [5, Sect. 6.7.2], [28]), where $\pi$ is the projection to $M'$. Thus, $M$ admits a Sasaki–Einstein metric if $(M', \omega')$ admits a Kähler–Einstein metric (cf. [7, Corollary 2.1]).

If $M'$ is a Fano compactification of a connected reductive group $H$ and the $H \times H$-action can be lifted to $C(M)$ as a bundle isomorphism, $H \times H$ is a subgroup of $\text{Aut}^\xi (C(M))$, where $\xi$ is the Reeb field on $M$. Since $\xi$ generates a subgroup $C^\times$ of $\text{Aut}^\xi (C(M))$, $M$ is a $G$-Sasaki manifold with

$$G = (H \times C^\times)/\text{diag}(H \cap C^\times),$$

where $H \cap C^\times$ is embedded in $H$ by natural inclusion.

**Example 9.2** Let $(M_i^{2n_i+1}, g_i, \xi_i), i = 1, 2$ be two compact Sasaki manifolds and $(C(M_i^{2n_i+1}), \bar{g}_i)$ be their Kähler cones, respectively. Let $\omega_{\bar{g}_i} = \sqrt{-1} \partial \bar{\partial} \rho_i^2$ be their corresponding Kähler cone metrics. Take $\rho = \sqrt{\rho_1^2 + \rho_2^2}$ on the product $C(M_1) \times C(M_2)$ and let $M = \{ \rho = 1 \}$ be the corresponding level set. Then $\bar{g}$ is a Kähler metric associated to $\omega = \sqrt{-1} \partial \bar{\partial} \rho^2$ and $(M, g = \bar{g}_M)$ is a Sasaki manifold.

It can be verified that $\xi = \xi_1 + \xi_2$ is the Reeb field of $(M, g)$. If we further assume that each $M_i$ is a $G_i$-Sasaki manifold, then it is obvious that $M$ is a $G_1 \times G_2$-Sasaki manifold. Furthermore, the moment cone of $(M, g)$ is given by

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2,$$

where $\mathcal{C}_i$ is the moment cone of $(M_i, g_i)$. The normal vectors of facets of $\mathcal{C}$ are all given by $u_{A(i)}$, where $u_{A(i)}$’s are normals of facets of $\mathcal{C}_i$, considered as vectors in the product space. Thus, if $\omega_{\bar{g}_i}^T \in \frac{\pi}{n_i+1} c_i^B (M_i)$, then $\omega_{\bar{g}}^T \in \frac{\pi}{n_1+n_2+2} c_i^B (M)$. Moreover, $\gamma_0 = \gamma_{01} + \gamma_{02}$, where $\gamma_0 \in (a_{1z}^* + a_{2z}^*)$, $\gamma_{0i} \in a_{iz}^*$ are determined in Proposition 6.1 with respect to $M$, $M_i$, respectively.

The characteristic polytope of $(M, g)$ is given by

$$\mathcal{P} = \{ y = (y_1, y_2) | \xi(y) = 1 \} = \cup_{t \in [0,1]} [tP_1 + (1-t)P_2],$$

where $P_i$ is the characteristic polytope of $(M_i, g_i)$ embedded in the product cone $\mathcal{C}$. Then we have
Thus, $M$ admits a transverse $G$-Sasaki Einstein metric if and only if both $M_i$ do. Hence, by Theorem 1.2, we may deform to a family of non-product transverse $G$-Sasaki Ricci solitons from a product transverse $G$-Sasaki Einstein metric $(M, \xi)$.

**Example 9.3** Let $K = U(2)$ and $G = GL_2(\mathbb{C})$. Identify $\mathbb{C}^4 \setminus \{O\}$ with the set of non-zero $2 \times 2$ complex matrices $M_{2 \times 2}(\mathbb{C}) \setminus \{O\}$. For any $A \in \mathbb{C}^4 \setminus \{O\}$, define

$$\rho^2(A) = \text{tr}(AA^T).$$

Then we get a $GL_2(\mathbb{C})$-Sasaki manifold

$$S^7(1) = \{A \in \mathbb{C}^4 \setminus \{O\} | \rho(A) = 1\},$$

which is the standard Euclidean sphere.

It is easy to see that $\sqrt{-1} \partial \bar{\partial} \rho^2$ is the standard Euclidean metric on $\mathbb{C}^4$, thus $S^7(1)$ is the standard unit sphere. In the following, we verify that $S^7(1)$ is a $G$-Sasaki Einstein metric. We consider the $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ action on $\mathbb{C}^4 \setminus \{O\}$ given by

$$(GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \times (\mathbb{C}^4 \setminus \{O\}) \to \mathbb{C}^4 \setminus \{O\}$$

$$((X_1, X_2), Z) \to X_1ZX_2^{-1}.$$

Then $\mathbb{C}^4 \setminus \{O\}$ satisfies Definition 3.1 (1). Obviously, $\rho$ is $K \times K$-invariant. By a direct computation, we have
\[ g = \text{Span}_\mathbb{C} \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \right\} \]

and

\[ t^c = \text{Span}_\mathbb{C} \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right\}. \]

The Reeb vector field \( \xi \) is given by

\[ \xi = \left( \begin{array}{c} \sqrt{-1} \\ 0 \\ 0 \end{array} \right), \]

which satisfies Definition 3.1 (3).

We choose a maximal torus

\[ T^c = \left\{ \left( \begin{array}{cc} e^z & 0 \\ 0 & e^{-z} \end{array} \right) \mid z \in \mathbb{C}^* \right\}. \]

Then the restriction of \( \frac{1}{2} \rho^2 \) on it is given by

\[ \frac{1}{2} \rho^2(z) = \frac{1}{2} (|e^z|^2 + |e^{-z}|^2). \]

Choose \( E_1, E_2 \) as the generators of \( \mathfrak{a} \) and \( E_1^*, E_2^* \) be their dual in \( \mathfrak{a}_h \), we see that the lattice of characters of \( G \) is generated by \( E_1^* \) and \( E_2^* \) (See Fig. 1).

A direct computation shows that

\[ \mathcal{C} = \{ y_1 E_1^* + y_2 E_2^* \in \mathfrak{a}^* \mid y_1, y_2 \geq 0 \}. \]

Also, we have

\[ 2\sigma = E_1^* - E_2^*, \quad \xi = E_1 + E_2, \]

\[ u_1 = E_1, \quad u_2 = E_2, \]

\[ \gamma_0 = -2(E_1^* + E_2^*). \]

Then one can check that (1.1) holds. In fact, \( S^7(1) \) can be regarded as a Hopf \( S^1 \)-fiberation, which is a \( S^1 \)-bundle over \( \mathbb{C}P^3 \), and \( \mathbb{C}P^3 \) is a Fano compactification of \( PGL_2(\mathbb{C}) \) (cf. [3, Example 2.2]). However, the \( PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C}) \)-action cannot be lifted to \( \mathbb{C}^4 \), so it is not of the kind given in Example 9.1.

**Example 9.4** Let \( K = SU(2) \times S^1 \) and \( G = SL_2(\mathbb{C}) \times C^* \). Identify \( \mathbb{C}^5 \backslash \{ O \} \) with \( (M_{2 \times 2}(\mathbb{C}) \oplus \mathbb{C}) \backslash \{ O \} \). Consider the hypersurface

\[ \mathcal{H} := \{ (A, t) \in \mathbb{C}^5 \backslash \{ O \} \mid \det(A) = t^2 \}. \]

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For any \((A, t) \in \mathbb{C}^5 \setminus \{O\}\), define

\[
\rho^2(A, t) := \frac{1}{6} \left( \text{tr}(A\bar{A}^T) + |t|^2 \right).
\]

Then \(M = \mathcal{H} \cap \{ \rho = 1 \}\) is an \((SL_2(\mathbb{C}) \times \mathbb{C}^*)\)-Sasaki manifold of dimension 7, Kähler cone of which is \(\mathcal{H}\).

As in Example 9.3, \(\sqrt{-1} \frac{\partial}{\partial \rho} \rho^2\) is the standard Euclidean metric on \(\mathbb{C}^5\) and \(M\) is the intersection of \(\mathcal{H}\) and the unit sphere. Consider the \(G \times G\)-action on \(\mathcal{H}\) given by

\[
(G \times G) \times \mathcal{H} \rightarrow \mathcal{H}
\]

\[
((X_1, t_1), (X_2, t_2), (A, t)) \rightarrow (t_1X_1A^{-1}X_2^{-1}, t_1tt^{-1}).
\]

one can check directly that (1)–(2) in Definition 3.1 are satisfied.

By a direct computation, we have

\[
g = \text{Span}_\mathbb{C} \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), 0 \right\}, \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), 0 \right\}, \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), 0 \right\}, \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right\},
\]

and

\[
t^c = \text{Span}_\mathbb{C} \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), 0 \right\}, \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right\}.
\]

Let \(\xi = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \sqrt{-1} \right)\). Then \(\xi \in \mathfrak{z}(\mathfrak{t})\) and \(M\) is a \(G\)-Sasaki manifold with the Reeb vector field \(\xi\).
Let us determine the moment cone of this Sasaki manifold. Choose a maximal torus

\[ T^c = \left\{ \begin{pmatrix} e^z \\ 0 \\ e^{-z} \end{pmatrix} t, t \in \mathbb{C}^* \right\}. \]

Then the restriction of \( \frac{1}{2} \rho^2 \) on it is

\[ \frac{1}{2} \rho^2(z, t) = \frac{1}{6} (|e^z|^2 + |e^{-z}|^{-2} + 1)|t|^2. \]

Choose \( E_1, E_2 \) as the generators of \( a \) and \( E_1^*, E_2^* \) be their dual in \( a^* \). We see that the lattice of characters of \( G \) is generated by \( e_1^* = \frac{1}{2}(E_1^* + E_2^*) \) and \( e_2^* = \frac{1}{2}(E_1^* - E_2^*) \) (See Fig. 2).

A direct computation shows that

\[ C = \{ y_1 e_1^* + y_2 e_2^* \in a^* | -y_1 + y_2 \geq 0, y_1 + y_2 \geq 0 \}, \]

and the positive part is of

\[ C_+ = \{ y_1 e_1^* + y_2 e_2^* \in a^* | -y_1 + y_2 \geq 0, y_1 \geq 0 \}. \]

Also, we have

\[ 2\sigma = 2e_2^*, \quad u_1 = e_1 + e_2, \quad u_2 = e_1 - e_2, \quad \gamma_0 = -3e_1^*. \]

On the other hand, \( \xi = E_1 + E_2 \), and so \( \gamma_0(\xi) = -3 \). Thus, \( \xi \) does not define a Sasaki structure such that the corresponding transverse Kähler form lies in \( \pi_{n+1} \mathfrak{c}_1^B(M) \).

But by replacing \( \xi \) by \( \xi' = \frac{4}{3} \xi \), we get a Sasaki structure on \( M \) transverse Kähler form of which lies in \( \pi_{n+1} \mathfrak{c}_1^B(M) \). In fact, this new Sasaki structure can be derived from the original one by applying a \( D \)-homothetic deformation defined by Tanno [40] (see also [9]). It can be checked that (1.1) holds in this case. Thus, the Sasaki manifold \( M \) with its Reeb vector field \( \xi' \), admits a Sasaki–Einstein metric. In fact, in this case, \( M \) is an \( S^1 \)-bundle over \( M/e^t\xi \), which is the wonderful compactification of \( SL_2(\mathbb{C}) \).

It is known that the wonderful compactification of \( SL_2(\mathbb{C}) \) admits a Kähler–Einstein metric.

\[ \text{Example 9.5} \quad \text{Let} \ n = 4, \ G = PSL_2(\mathbb{C}) \times \mathbb{C}^* \ \text{and} \ \hat{G} = G \times \mathbb{C}^*. \ \text{Choose} \ 2\sigma = (0, 0, 0) \ \text{to be a positive root in} \ \hat{\mathfrak{t}} \cong \mathbb{R}^3. \ \text{Let} \ C \ \text{be the cone in} \ \hat{\mathfrak{g}} \ \text{given by} \]

\[ C = \{ y_3 - y_2 \geq 0, \ y_3 + y_2 \geq 0, \ 2y_3 - y_2 - y_1 \geq 0, \ 2y_3 - y_2 + y_1 \geq 0 \}. \]

Then there is a \( \hat{G} \)-Sasaki manifold of dimension 9 such that \( C \) is its moment cone.
Clearly, $\mathcal{C}$ is a good cone. Moreover, its facets intersect with Weyl wall orthogonally. Thus, by [3, Proposition 2.5], there is a smooth Kähler manifold $\hat{M}$ with an open dense $\hat{G} \times \hat{G}$-orbit isomorphic to $\hat{G}$. Furthermore, if we equip the toric orbit $Z$ in $\hat{M}$ with a toric cone metric, then it extends to a Kähler cone metric on $\hat{M}$ by Proposition 3.2.

Next, we choose a possible $\xi$ such that $\hat{M}$ is the Kähler cone of a Sasaki manifold $M$ with $c_1^B(M) > 0$. By Proposition 6.1, we see

$$\gamma_0 = (0, 0, -1),$$
$$\xi = (0, \xi_2, 5), \quad -5 < \xi_2 < 5.$$ 

Then the polytope $P_+$ in $(\alpha_1')^*$ is the convex hull of the following four points (See Fig. 3)

$$q_1 = \left(0, \frac{1}{\xi_2 + 5}\right), q_2 = \left(0, \frac{-1}{-\xi_2 + 5}\right),$$
$$q_3 = \left(\frac{1}{\xi_2 + 5}, \frac{1}{\xi_2 + 5}\right), q_4 = \left(\frac{3}{-\xi_2 + 5}, \frac{-1}{-\xi_2 + 5}\right).$$

The soliton vector field on $a'$ is of form,

$$X = (0, \lambda)$$

for some $\lambda \in \mathbb{R}$ since it lies in the center. Also $\frac{2\sigma}{n+1} = (\frac{1}{5}, 0)$. Let $\xi_2 \to 5$. Then $\lambda \to +\infty$. In this case, the barycenter of $P_+$,

$$\text{bar}_X(P_+) \to \left(\frac{3}{40}, 0\right).$$

Thus, $M$ admits no Sasaki–Ricci soliton when $\xi_2$ is chosen sufficiently close to 5. But if $\xi_2 \to -5$, we have $\lambda \to -\infty$. In this case.
and $M$ admits Sasaki–Ricci soliton metric when $\xi_2$ is chosen sufficiently close to $-5$. In particular, $\bar{X}(P_+) = \left(\frac{9}{40}, 0\right)$ when $\xi_2 = -\frac{5}{2}$. Hence, we proved the existence of Sasaki–Ricci solitons on $\hat{M}$.

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