Influence of the coupling mechanism on the stochastic resonance response in extended systems

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Abstract. We analyze the stochastic resonance response in an extended system, considering different transport/coupling mechanisms: diffusion, KPZ, and also include the possibility of a non-local interaction. Our aim, since these mechanisms correspond to different forms of coupling of resonant units leading to an extended system, is to obtain information about the way to optimize the system’s response to weak signals. To reach such a goal, we exploit the knowledge of the so called “non-equilibrium potential” for the above indicated situations.

Stochastic resonance (SR) has become a paradigm of the constructive effects of fluctuations on nonlinear systems [1, 2]. In a simplified picture, the phenomenon occurs whenever the Kramers’ rate for the transition between attractors matches the typical frequency of a signal which is incapable by itself to trigger that transition (i.e. it is subthreshold).

In a recent series of papers we have studied the SR phenomenon for the transitions between two different patterns in extended systems, exploiting the concept of nonequilibrium potential (NEP). Some recent review-like papers [3, 4, 5], summarize the indicated results including several references to our work as well. Such NEP is a special Lyapunov’s function of the associated deterministic system which, for non-equilibrium systems, plays a role similar to that of a thermodynamic potential in equilibrium thermodynamics [6, 7, 8, 9, 10]. It is closely related to the stationary solution of the system’s Fokker–Planck equation, characterizing the global properties of the dynamics: attractors, linear and relative stability of these attractors, height of the barriers separating attraction basins. Moreover, it allows to evaluate the transition rates among the different attractors [6, 7, 8, 9, 10]. Regarding the problem of SR in extended systems, it was shown that the knowledge of the NEP allows to obtain a rather complete picture of the behavior of the output signal-to-noise ratio (SNR), and eventually other response measures. The novelty with non-equilibrium extended systems is that even point-like attractors in the medium’s infinite-dimensional phase-space can be nontrivial field configurations (real-space patterns).

In this contribution we exploit previous results, and new ones as well, in order to discuss the role of the coupling mechanism on the SR response of a bistable extended system. We will consider three different situations: a simple reaction-diffusion mechanism [3, 4, 5, 11], a KPZ-like contribution [12, 13, 14], and finally the existence of a non-local coupling [15].

We start briefly sketching some relevant points related to the NEP concept. We refer to [3, 4] for details. If we have a kinetic equation defined by

$$\partial_t \phi(x, t) = \mathcal{L} \phi(x, t) + f(\phi(x, t)) + \xi(x, t),$$

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where \( L \) is the coupling or transport operator; \( \xi(x,t) \) is a Gaussian white noise [of zero mean, correlation \( \langle \xi(x,t)\xi(x',t') \rangle = 2\gamma\delta(x-x')\delta(t-t') \), and intensity \( \gamma \)]; and \( f(\phi(x)) = -\partial V(\phi(x))/\partial \phi \), [here we adopt \( V(\phi(x)) = \frac{2}{\beta} \phi(x)^4 - \frac{\alpha}{2} \phi(x)^2 \)]. We denote by \( \Phi[\phi] \) the NEP of such a system, whose first variation gives \( \frac{d}{dt}\phi(x,t) = -\frac{\delta \Phi[\phi]}{\delta \phi(x,t)} + \xi(x,t) \), and that in the deterministic case fulfills the Lyapunov condition \( \frac{d}{dt}\Phi[\phi] = -\left(\frac{\delta \Phi[\phi]}{\delta \phi(x,t)}\right)^2 \leq 0 \).

As indicated in [3, 4], for a scalar bistable system and within the “two-state approximation” [1] (that implies an adiabatic assumption), the SNR is given by \( \text{SNR} = \frac{3\pi}{2\gamma} \frac{\mu_l\mu_r}{\mu_l + \mu_r} \), where \( \mu_i \sim \exp\{\Delta \Phi[\phi_i]/\gamma\} \). Here \( \Delta \Phi[\phi_i] = \Phi[\phi_o] - \Phi[\phi_i] \), with \( \phi_i(x) \) (\( i = l, r \)) one of the two stable stationary states, and \( \phi_o(x) \) corresponds to the (unstable) pattern at the saddle separating both stable attractors. We will restrict the system to the interval \( 0 \leq x \leq \pi \) with Dirichlet b.c.

The first case we discuss here corresponds to a reaction-diffusion system described by

\[
\partial_t \varphi(x,t) = \nu \partial_x^2 \varphi(x,t) + f(\varphi(x,t)) + \xi(x,t).
\]

The associated NEP reads [3, 4, 11]

\[
\Phi[\varphi] = \int dx \left[ \frac{\nu}{2} (\partial_x \varphi)^2 + V(\varphi(x)) \right].
\]

As is known [3, 4, 11], for the above indicated form of \( V(\varphi(x)) \) and Dirichlet b.c., this system has two stable and one unstable solutions. We can readily use the expression for the SNR indicated above and obtain the result indicated by the continuous line on Fig. 1.

![Figure 1. SNR vs noise intensity (\( \gamma \)). For the reaction-diffusion system in Eq. (1) we have the solid line on the left panel. Also, for the KPZ case we have \( \lambda = 0 \) (solid line), \( \lambda = 0.25 \) (dotted line) and \( \lambda = 0.5 \) (dashed line). The rest of parameters are \( \nu = 2 \), \( \alpha = \beta = 30 \).](image)

![Figure 2. Nonequilibrium potential \( \Phi(\eta) \) vs \( \eta \), for the KPZ case, where \( \eta \) indicates a projection along the line joining the attractors through the saddle. We have \( \lambda = 0 \) (solid line), \( \lambda = 0.25 \) (dotted line) and \( \lambda = 0.5 \) (dashed line). The rest of parameters are \( \nu = 2 \), \( \alpha = \beta = 30 \).](image)

Now we analyze the case of KPZ equation. For the 1-d case it reads

\[
\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda}{2} (\partial_x h(x,t))^2 + f(h(x,t)) + \xi(x,t).
\]

This nonlinear stochastic partial differential equation describes the height of a fluctuating growing interface with surface tension \( \nu \), \( \lambda \) being proportional to the average growth velocity
and arises because in this growth process the surface slope is transported in parallel. In Ref. [14] it was shown that the functional

$$\Phi[h] = \int dx \left[ \frac{\nu}{2} (\partial_x h)^2 - \frac{\lambda}{2} \int_{h_{ref}}^{h(x)} d\psi (\partial_x \psi)^2 + V(h(x)) \right],$$

is the NEP for KPZ. The system is bounded by the potential $V$ indicated before. In this case $h_i(x)$ (i = l, r) are the two stable (nonhomogeneous) stationary states, while $h_o(x)$ corresponds to the flat (unstable) pattern at the saddle separating both stable attractors. We will work with the NEP’s approximated form $\Phi[h_i(x)] \approx \int dx \left[ \left( \frac{\nu}{2} - \frac{\lambda}{2} h_i(x) \right) (\partial_x h_i(x))^2 + V(h(x)) \right]$.

Figure 1 plots the SNR as a function of the noise intensity, for different values of $\lambda$. The trend is that for increasing $\lambda$, the system’s response diminishes. This can be understood within the NEP picture, since the KPZ term resembles an effective ($\lambda$ dependent) diffusivity [16, 17], indicating a variation of the effective diffusivity with the spatial variable. This originates a breaking of the NEP’s symmetry as $\lambda$ is varied, yielding a SNR reduction [5]. Figure 2 depicts a projection of that NEP (along the line joining the attractors and crossing the saddle), where the rupture of symmetry due to the effect of $\lambda$ is apparent.

The last example corresponds to the inclusion of a non-local coupling [1, 5]. In this case the equation has the form

$$\partial_t \phi(x,t) = \mathcal{L}\phi(x,t) + f(\phi) - \vartheta \int_0^\pi \mathcal{K}(x,x') \phi(x',t) dx' + \xi(x,t),$$

where $\mathcal{K}(x,x')$ is the non-local coupling [3, 4] that, in order to simplify the analysis, we choose as $\mathcal{K}(x,x') = \frac{1}{2l}$ if $|x-x'| \leq l$, and $= 0$ if $|x-x'| > l$ [15]. Hence, we can do the analysis just varying the interaction range $2l$. Such a contribution could arise from the adiabatic elimination of an inhibitor-like field in an activator-inhibitor system [3, 4]. The associated NEP has the form

$$\mathcal{F}[\phi] = \int_0^\pi dx \left\{ \frac{1}{2} (\partial_x \phi)^2 + V(\phi(x)) + \frac{\vartheta}{2} \int_0^\pi dx' \phi(x') \mathcal{K}(x,x') \phi(x) \right\}.$$  

This NEP contains both, local and nonlocal couplings corresponding to the diffusive and the nonlocal contribution, respectively. The last one contains the nonlocal kernel given above, with a variable range $2l$, corresponding to the field interaction at points $x'$ belonging to the region $[x-l, x+l]$. However, such points will contribute if and only if are inside the domain $[0, \pi]$. With this form of NEP we can directly use the expression for the SNR indicated above and obtain the result indicated on Fig. 3.
When comparing the results for the three cases there are some apparent trends. Firstly, for the 2nd case we see a reduction of the SNR when including a KPZ-like contribution [14]. Secondly, when considering nonlocal terms, at least for moderate values of $\vartheta$ and the range $2l$, there is an increase of the system’s response. However, a deeper analysis of this problem requires to make a detailed comparison with Monte Carlo simulations, that will be the subject of a future work.

When searching for enhancement mechanisms, an interesting possibility involves analyzing the extension of the previous studies to systems with selective coupling as studied in [16, 17]. Other possibility could be to exploit the coupling with an auxiliary field as in [18], that seems to present differences respect to the activator-inhibitor-like systems studied in [3, 4, 5]. Also, to study the effect of a coupling based on a Swift-Hohenberg mechanism [19]. Such studies will be the subject of future work.

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