Abstract

In this note we present a method for constructing CMC-1 surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ in terms of holomorphic data first introduced in Bianchi’s Lezioni di Geometria Differenziale of 1927, therefore predating by many years the modern approaches due to Bryant, Small and others. Besides its obvious historical interest, this note aims to complement Bianchi’s analysis by deriving explicit formulae for CMC-1 surfaces and comparing the various approaches encountered in the literature.

1 Introduction

It is generally accepted that the theory of surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ with constant mean curvature equal to one (CMC-1 surfaces, for short) started with a seminal paper by R. Bryant ([B]), where he derives a representation for such surfaces in terms of holomorphic data. More precisely, Bryant’s recipe works as follows. Start with a holomorphic curve $S \subset \mathbb{P}SL(2, \mathbb{C})$ which is null with respect to the conformal structure inherited from the Killing-Cartan form and recall the natural projection $\pi : \mathbb{P}SL(2, \mathbb{C}) \to \mathbb{H}^3(-1)$, $\pi(\omega) = \omega \omega^t$, where $\mathbb{H}^3(-1)$ is realized via the hermitian model and $\mathbb{P}SL(2, \mathbb{C})$ is the group of orientation preserving isometries of $\mathbb{H}^3(-1)$. It turns out that, at least locally, any CMC-1 surface is of the form $\pi(S)$ for some such $S$. In this setting, the primary object is a pair of holomorphic functions defining another (holomorphic) curve on the Lie algebra of $\mathbb{P}SL(2, \mathbb{C})$ from which $\Sigma$ is obtained after solving a certain first order differential equation. Thus an integration procedure appears in the description and this partially accounted for the inherent difficulties in constructing examples at the very early stages of the theory. But notice however the striking analogy with the well-known Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ ([L]), as these are obtained as real slices of null curves in $\mathbb{C}^3$.

After the appearance of Bryant’s investigation, many other researchers contributed to the subject. For example, starting with the paper [UY], M. Umehara and K. Yamada refined
substantially Bryant’s approach and were able to construct a varied class of examples of CMC-
1 surfaces, besides developing many interesting global aspects in the theory. On the other hand,
inspired by Hitchin’s generalization \cite{H} of Penrose’s twistor theory, A. J. Small \cite{S} gave a
clean characterization of null curves in $\mathbb{P}SL(2, \mathbb{C})$. In fact, Small realizes $\mathbb{P}SL(2, \mathbb{C})$ as the
complement of the quadric $Q_2 = \{ad - bc = 0\} \subset \mathbb{P}^3$ and observes that twistors lead naturally
to the construction of a Gauss transform $\Gamma_S$ of a null curve $S \subset \mathbb{P}SL(2, \mathbb{C})$ as a curve lying in
$Q_2^*$, the quadric dual to $Q_2$. Moreover, he shows that, aside from the technical issue of missing
points lying in $Q_2$, one has

$$S = \Gamma_S^*,$$  \hspace{1cm} (1)

where $\Gamma_S^*$ is the dual to $\Gamma_S$. This gives a classical procedure for recovering $S$ from $\Gamma_S$ and in
this context one is naturally compelled to regard the Gauss transform as the primary object.
The final blow is the well-known fact that $Q_2^* = Q_2 = \mathbb{P} \times \mathbb{P}$, a product of two projective lines,
and we can locally represent

$$\Gamma_S = (f, g),$$  \hspace{1cm} (2)

for holomorphic functions $f$ and $g$ on the Riemann surface $M$ underlying $\Gamma_S$. Since dualization
is carried out by applying algebraic operations on the derivatives up to second order of $(f, g)$ one
ends up with an explicit formula for CMC-1 surfaces involving no integration whatsoever! We
stress that this approach has the obvious drawback of being local in nature but the elimination
of quadratures reminds us of an alternative (and much lesser known) way of describing the
classical Weierstrass representation, which was rediscovered in modern times by Hitchin \cite{H2}
and Small \cite{S2} again by using twistorial methods.

The above historical account reflects an attempt to interpret the prevalent view in the CMC-
1 surfaces community. Consensual as it may be, the main purpose of this note is to point out
that it is not entirely correct. In effect, in Bianchi’s Lezioni di Geometria Differenziale \cite{B},
edited in 1927, we may find a recipe for constructing CMC-1 surfaces out of holomorphic data.
However, the motivation for writing this note goes beyond this historical curiosity, for it seems
that there are at least two other reasons for exhibiting this old method to a wider audience.

The first one is that the method allows one to start with an arbitrary holomorphic map $f$ defined in a region $\Omega \subset \mathbb{C}$ and, elaborating upon Bianchi’s ideas, to end up with explicit
formulae for a CMC-1 surface (see Theorem 3.2). Moreover, this map $f$ has an immediate
geometric interpretation: it is simply the parametrized hyperbolic Gauss map, or in other
words, the expression for the hyperbolic Gauss map in terms of a local complex parameter on
$\Omega$. In addition, the resulting formulae involve algebraic expressions in terms of the derivatives
of $f$ up to second order and no integration at all. As such, they come close to the spirit of
Small’s approach as described above. In fact, and this is an important issue here, our formulae
coincide with Small’s if the pertinent transformation between models for $\mathbb{H}^3(-1)$ is carried out
(see Section 3).

The second reason is that in his way toward the construction of CMC-1 surfaces, Bianchi
translates to hyperbolic geometry the solution of a strictly Euclidean-geometric problem involving
the rolling of a pair of isometric surfaces, thereby establishing a surprising linking between

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these two geometries. As a bonus, his approach is entirely elementary in the sense that for its understanding it is only required familiarity with the fundamentals of three definitely classical disciplines: Euclidean geometry, hyperbolic geometry and complex function theory. This should be compared to the heavy algebraic-geometric machinery used by Small.

This article is organized as follows. In Section 2 we review some classical concepts in order to describe the rolling problem in Euclidean geometry. In particular we derive, following Bianchi, the so called Calò’s formulae. In Section 3 we relate the solution of the Euclidean problem of Section 2 to the construction of CMC-1 surfaces in $\mathbb{H}^3(-1)$. We obtain an explicit parametrization for a CMC-1 surface in terms of a conformal map. Finally in Section 4 we exhibit some simple examples constructed via the Bianchi-Calò method.

2 Classical concepts and the Bianchi-Calò method

2.1 Congruence of spheres and envelopes.

Here is the first classical concept we shall meet. A congruence of spheres is a smooth two-parameter family of spheres in $\mathbb{R}^3$, that we will suppose parametrized by coordinates $(u, v)$. To each such congruence we may associate a function $R = R(u, v)$, the radius function, describing the radii of the spheres in the congruence. We also assume that the vector function $X = X(u, v)$ describing the centers of the spheres defines a regular surface which we call the surface of centers. Such a congruence of spheres will be denoted by $[X, R]$.

Generically there are two surfaces, the so-called envelopes, associated to a given congruence $[X, R]$. In effect, a point $p \in \mathbb{R}^3$ belongs to an envelope $\xi$ if $p \in S$ for some sphere $S$ in $[X, R]$ and moreover $T_p \Sigma = T_p S$. The next proposition gives the expression for the envelopes in terms of the unit normal vector $N = N(u, v)$ and the metric of the surface of centers.

Proposition 2.1. In coordinates $(u, v)$,

$$\xi = X - R \left( \Delta(X, R) \pm \sqrt{1 - \Delta_1 R N} \right),$$  \hspace{1cm} (3)

where

$$\Delta(X, R) = (R_u A_{11} + R_v A_{12}) X_u + (R_u A_{21} + R_v A_{22}) X_v,$$  \hspace{1cm} (4)

and

$$\Delta_1 R = R_u^2 A_{11} + 2 R_u R_v A_{12} + R_v^2 A_{22}.$$  \hspace{1cm} (5)

Here, the matrix $A = [A_{ij}]$ is the inverse of the matrix defined by the metric in the given coordinates.

Proof. The conditions defining the envelopes are expressed as

$$\xi = X - R v,$$  \hspace{1cm} (6)
\[ \langle \nu, \xi_u \rangle = \langle \nu, \xi_v \rangle = 0, \]  

where \( \nu \) is the unit vector in the direction of \( \mathbf{X} - \xi \), so that taking derivatives of (3) and using the fact that \( |\nu| = 1 \) we obtain

\[ \langle \nu, \mathbf{X}_u \rangle = R_u, \quad \langle \nu, \mathbf{X}_v \rangle = R_v \]  

(8)

Now, write

\[ \nu = a \mathbf{X}_u + b \mathbf{X}_v + c \mathbf{N}, \]  

(9)

take inner products of this with \( \mathbf{X}_u \) and \( \mathbf{X}_v \) and solve the linear system

\[ g_{11}a + g_{12}b = R_u, \]
\[ g_{21}a + g_{22}b = R_v, \]

so as to obtain

\[ a = A_{11}R_u + A_{12}R_v, \]
\[ b = A_{21}R_u + A_{22}R_v. \]

Finally, \( c \) is further determined using that \( |\nu| = 1 \):

\[ c = \pm \sqrt{1 - \Delta_1 R}. \]  

(10)

Substitution yields the desired formula for the envelopes.

If a congruence of spheres has two distinct envelopes we then have a natural correspondence between their points, namely, points on distinct envelopes correspond if they are the contact points of the envelopes with a given sphere of the congruence. We will allow the degenerate case where one of the envelopes reduces to a point and notice that even in this situation the unit vector \( \nu \) appearing in the above proposition is still well defined, a fact we shall use in the following result due originally to Beltrami.

**Proposition 2.2.** Let \([\mathbf{X}, R]\) be a congruence of spheres with \( \xi \) being one of its envelopes (possibly degenerated to a point), obtained say by choosing the positive sign in (3), and consider the angles \( \omega_1, \omega_2 \) and \( \sigma \) formed between the unit vector \( \nu \) in the direction of \( \mathbf{X} - \xi \), \( \mathbf{X}_u \), \( \mathbf{X}_v \) and \( \mathbf{N} \), respectively. Suppose further that another surface \( \tilde{\mathbf{X}} \) isometric to \( \mathbf{X} \) is given and consider the congruence of spheres \([\tilde{\mathbf{X}}, \tilde{R}]\). Finally, let \( \tilde{\xi} \) be the envelope for \([\tilde{\mathbf{X}}, \tilde{R}]\) obtained by choosing the same sign as we did in order to get \( \xi \) starting from \([\mathbf{X}, R]\). Then the angles between the unit vector \( \tilde{\nu} \) in the direction of \( \tilde{\mathbf{X}} - \tilde{\xi} \), \( \tilde{\mathbf{X}}_u, \tilde{\mathbf{X}}_v \) and \( \tilde{\mathbf{N}} \), respectively, where \( \tilde{\mathbf{N}} \) is the unit normal to \( \tilde{\mathbf{X}} \), coincide with the corresponding angles for \([\mathbf{X}, R]\).

**Proof.** From (8), (9) and (10) one has

\[ \cos \omega_1 = \frac{\langle \nu, \mathbf{X}_u \rangle}{\sqrt{g_{11}}} = \frac{R_u}{\sqrt{g_{11}}}, \]
\[ \cos \omega_2 = \frac{\langle \nu, \mathbf{X}_v \rangle}{\sqrt{g_{22}}} = \frac{R_v}{\sqrt{g_{22}}}, \]
\[ \cos \sigma = \langle \nu, \mathbf{N} \rangle = \sqrt{1 - \Delta_1 R}, \]
and since the surfaces of centers of the congruences, which have the same radius function, are isometric to each other, the right hand sides above are the same when we consider the angles of both congruences.

\[ \sim \]

2.2 Rolling of isometric surfaces

We now describe the rolling of isometric surfaces and show how a congruence of spheres is associated to such a rolling. Consider a pair \((S, \tilde{S})\) of isometric surfaces in \(\mathbb{R}^3\), and let \(p \in S\) and \(\tilde{p} \in \tilde{S}\) be points corresponding under the isometry. Suppose \(S\) is fixed in space and consider the two-parameter family of positions of congruent copies of \(\tilde{S}\) such that to each \(p \in S\) we consider a rigid motion of \(\mathbb{R}^3\) (call it \(H_p\)) sending \(\tilde{p}\) to \(p\), \(T_p\tilde{S}\) to \(T_pS\), and further adjusted so that the differential of the isometry composed with \(H_p\) is the identity map. This two-parameter family of positions for copies of \(\tilde{S}\) is called the rolling of \(\tilde{S}\) over \(S\). The surface \(\tilde{S}\) is called the rolled surface and \(S\) the support surface.

Now fix a point \(O \in \mathbb{R}^3\) and consider its image under the two-parameter family of rigid motions associated to the rolling of \(\tilde{S}\) over \(S\). In the generic case, the motion of \(O\) defines a surface \(\Sigma\) called the rolling surface with respect to the satellite point \(O\).

The crucial point now is that the two concepts introduced so far, namely congruence of spheres and rolling of surfaces, share a close relationship. More precisely, we have

**Proposition 2.3.** Given a rolling of \(\tilde{S}\) over \(S\) as above, the rolling surface \(\Sigma\) can be viewed as an envelope of a congruence of spheres having \(S\) as its surface of centers and the sizes of the corresponding line segments joining points of \(\tilde{S}\) to \(O\) as radii.
Proof. Look at $O$ as a degenerate envelope corresponding to the congruence of spheres having $\tilde{S}$ as surface of centers and passing through $O$, see figure 2.1. By Proposition 2.2, an envelope of the congruence with $S$ as the surface of centers, and same radius function as the congruence just considered, has the property that the unit vector joining a point of it to the corresponding point of $S$ makes the same angles (with respect to the obvious fixed basis) than the corresponding unit vector joining a point $p \in \tilde{S}$ to $O$ makes with the corresponding basis. But this shows that the point of the envelope coincides with the point of the rolling surface.

2.3 The Calò’s formulae

We are now in a position to formulate the problem in Euclidean geometry that will lead us, according to Bianchi, to a method for constructing CMC-1 surfaces in $\mathbb{H}^3(-1)$:

**Find pairs** $(S, \tilde{S})$ **of isometric surfaces such that**, for a convenient satellite point $O$, the rolling surface $\Sigma$ is contained in a plane.

To solve this problem we work with Cartesian coordinates $(x, y, z)$ in $\mathbb{R}^3$, suppose that the plane in question is $\{z = 0\}$ and moreover that the satellite point $O$ is the origin of our coordinate system. Let $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{z})$ and $p = (x, y, z)$ denote corresponding points for $\tilde{S}$ and $S$, respectively. The necessary and sufficient conditions for a pair $(S, \tilde{S})$ to be a solution to our problem are

\begin{align}
\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 &= z^2, \\
\tilde{R}^2 = z^2 &\text{ and choosing } R = z, \text{ so that } S \text{ now lies in the upper half-space, we obtain, after using (11)},
\end{align}

On the other hand, (12) and (13) imply $R^2 = z^2$ and choosing $R = z$, so that $S$ now lies in the upper half-space, we obtain, after using (11),

\begin{align}
R^2(d\theta^2 + \sin^2 \theta d\phi^2) &= dx^2 + dy^2, \tag{14}
\end{align}

which we may interpret as follows: the central projection of $\tilde{S}$ over $S^2$ and the orthogonal projection of $S$ onto the plane $\{z = 0\}$ define a conformal map from $S^2$ to the plane $\{z = 0\}$. Thus we make our first contact with complex function theory.

Via stereographic projection we introduce on $S^2$ the complex variable

\[ \tau = \cot \frac{\theta}{2} e^{i\phi}, \tag{15} \]
so that in terms of this parameter one gets the coordinates of $\tilde{S}$:

$$\tilde{x} = R \frac{\tau + \bar{\tau}}{|\tau|^2 + 1}, \quad \tilde{y} = R \frac{\tau - \bar{\tau}}{i|\tau|^2 + 1}, \quad \tilde{z} = R \frac{|\tau|^2 - 1}{|\tau|^2 + 1}. $$

Much in the same vein, we can parametrize the corresponding piece in $S$ via a local inverse of the orthonormal projection by means of the complex parameter $\zeta = x + iy$. The discussion above allows us to consider $\zeta$ as a holomorphic function of $\tau$: $\zeta = f(\tau)$, see figure 2.2. In other words, $f$ describes the isometry between the surfaces in terms of the above chosen complex parameters.

![Diagram](image)

**Fig. 2.2**

The final step is to determine $R$ in terms of $f$. Using (14) and (15) we get

$$\frac{4R^2 |d\tau^2|}{(|\tau|^2 + 1)^2} = |d\zeta^2| = |f'(\tau)|^2 |d\tau^2|,$$

from which we conclude that

$$R = \frac{1 + |\tau|^2}{2} |f'(\tau)|,$$  

and we have finally met the solution to our problem, namely, the coordinates of $\tilde{S}$ and $S$ are respectively given in terms of the holomorphic data as

$$\tilde{x} = |f'(\tau)| \frac{\tau + \bar{\tau}}{2}, \quad \tilde{y} = |f'(\tau)| \frac{\tau - \bar{\tau}}{2i}, \quad \tilde{z} = |f'(\tau)| \frac{|\tau|^2 - 1}{2},$$  

and

$$x = \text{Re} f(\tau), \quad y = \text{Im} f(\tau), \quad z = |f'(\tau)| \frac{|\tau|^2 + 1}{2}. $$

The above expressions are called *Calò’s formulae* since they have been originally published by B. Calò in 1899 (14) in another context involving isometric surfaces.
3 The Bianchi-Calò method

In last section, starting with a holomorphic map \( f \), we have determined a pair of isometric surfaces such that one of the envelopes of the associated congruence of spheres was a plane. Notice that we also arranged things so that the surface of centers was placed in the upper half-space. Now, in principle we could also determine the second envelope of the congruence associated to the rolling, which is then also contained in the upper half space. It can be shown that the correspondence between the envelopes of the congruence associated to the Calò’s pair \( S, \tilde{S} \) considered in the last section is a conformal map. This is proved in Bianchi’s Lezioni when he considers Darboux congruencies and is one of the ingredients in the proof of the following central result, also due to Bianchi.

**Theorem 3.1.** To each pair \( (\tilde{S}, S) \) of isometric surfaces such that the rolling surface \( \Sigma \) of the rolling of \( \tilde{S} \) over \( S \) is a plane there corresponds a CMC-1 given by the second envelope of the associated congruence of spheres considered as a surface in the standard upper half space model of \( \mathbb{H}^3(-1) \).

**Proof.** Look at the spheres of our congruence as horospheres by using the upper half-space containing the surface of centers as a model for \( \mathbb{H}^3(-1) \). In this way, the correspondence between the envelopes becomes the hyperbolic Gauss map for the second envelope, which is known to be conformal, exception made for totally umbilical surfaces, exactly when the surface is a CMC-1 surface (see [Br] or [B]).

Although Bianchi indicates how one can find CMC-1 surfaces starting with an arbitrary horlomorphic map \( f \) via Theorem 3.1, he does not complete his analysis by deriving explicit formulae. In the following, we carry out the calculations and exhibit a CMC-1 surface in \( \mathbb{H}^3(-1) \) in terms of the map \( f \). The method is indeed very simple: we use Calò’s formulae (16) and (18) to compute the surface of centers \( S \) and the radius function \( R \), then we calculate the envelopes of this congruence of spheres, one of them being a piece of the plane \( \{z = 0\} \) and the other one being our CMC-1 surface. In the end of this section, we check that these formulae coincide with Small’s and this proves that they define a CMC-1 surface indeed and moreover that any CMC-1 arises locally in this way. We insist however that a proof of these assertions can be given in an elementary way and entirely avoiding Small’s results: one just has to check directly the validity of the above statements regarding conformality. We stick to the approach presented in the sequel just to stress the equivalence between the two methods.

**Theorem 3.2.** In the above situation, the parametrization of a CMC-1 surfaces in terms of \( f \)
is given by

\[
x = \text{Re } f - \frac{|f'|^2 \text{Re } (f' \tau) + \frac{1+|\tau|^2}{2} \text{Re } ((f')^2 \bar{f}'' \tau)}{|f'|^2 + \text{Re } (f' \bar{f}'' \tau) + \frac{|f''|^2}{4}(|\tau|^2+1)},
\]

\[
y = \text{Im } f - \frac{|f'|^2 \text{Im } (f' \tau) + \frac{1+|\tau|^2}{2} \text{Im } ((f')^2 \bar{f}'' \tau)}{|f'|^2 + \text{Re } (f' \bar{f}'' \tau) + \frac{|f''|^2}{4}(|\tau|^2+1)},
\]

\[
z = \frac{|f|^3}{|f|^2 + \text{Re } (f' \bar{f}'' \tau) + \frac{|f''|^2}{4}(|\tau|^2+1)}.
\]

(19)

Proof. In terms of \( \tau = u + iv, f = f_1 + if_2 \) and the radius function \( R \), the surface \( S \) is written as

\[X(\tau) = (f_1, f_2, R),\]

so that the coefficients of the metric become

\[g_{11} = |f'|^2 + R_u^2, \quad g_{12} = R_u R_v, \quad g_{22} = |f'|^2 + R_v^2,\]

where we used the Cauchy-Riemann equations for \( f \).

The corresponding determinant is

\[\chi = g_{11} g_{22} - g_{12}^2 = |f'|^2 \left(|f'|^2 + R_u^2 + R_v^2\right) = |f'|^2 \left(|f'|^2 + |\nabla R|^2\right),\]

and moreover

\[X_u \wedge X_v = \left(\alpha_1, \alpha_2, |f'|^2\right),\]

where

\[\alpha_1 = R_v f_{2,u} - R_u f_{2,v}, \quad \alpha_2 = R_u f_{1,v} - R_v f_{1,u},\]

so that the unit normal vector is

\[N = \frac{1}{\sqrt{\chi}} \left(\alpha_1, \alpha_2, |f'|^2\right).\]

On the other hand, the inverse of the matrix associated to the metric is

\[A_{11} = \frac{|f'|^2 + R_v^2}{\chi}, \quad A_{12} = \frac{-R_u R_v}{\chi}, \quad A_{22} = \frac{|f'|^2 + R_u^2}{\chi}.\]

From (4) and (5) we obtain

\[\Delta(X, R) = \frac{R_u |f'|^2}{\chi} (f_{1,u}, f_{2,u}, R_u) + \frac{R_v |f'|^2}{\chi} (f_{1,v}, f_{2,v}, R_v)\]

\[= \frac{|f'|^2}{\chi} (-\alpha_1, -\alpha_2, |\nabla R|^2),\]
and

\[ \Delta_1 R = R_u \left| f' \right|^2 + \frac{R_v^2}{\chi} + 2R_uR_v - R_uR_v + R_u^2 \frac{\left| f' \right|^2}{\chi} + R_v^2 \frac{\left| f' \right|^2}{\chi} \]

so that in particular,

\[ \sqrt{1 - \Delta_1 R} = \frac{|f'|^2}{\sqrt{\chi}}. \]

In order to calculate the envelopes we use (3) in the form

\[ \xi = X - R \left( \Delta(X, R) \pm \sqrt{1 - \Delta_1 R} N \right), \]

so that \( \xi_+ \), the envelope contained in \( \{ z = 0 \} \), is given by

\[ \xi_+ = (f_1, f_2, R) - R \left( \frac{|f'|^2}{\chi} (-\alpha_1, -\alpha_2, |\nabla R|^2) + \frac{|f'|^2}{\chi} \frac{1}{\sqrt{\chi}} \frac{1}{\sqrt{\chi}} (\alpha_1, \alpha_2, |f'|^2) \right) \]

\[ = (f_1, f_2, R) - R \left( \frac{|f'|^2}{\chi} (0, 0, |\nabla \phi|^2 + |f'|^2) \right) = (f_1, f_2, 0), \]

as expected, and \( \xi_- \), our CMC-1 surface, is

\[ \xi_- = (f_1, f_2, R) - R \left( \frac{|f'|^2}{\chi} (-\alpha_1, -\alpha_2, |\nabla R|^2) - \frac{|f'|^2}{\chi} \frac{1}{\sqrt{\chi}} \frac{1}{\sqrt{\chi}} (\alpha_1, \alpha_2, |f'|^2) \right) \]

\[ = \left( f_1 + \frac{2\alpha_1 R}{|f'|^2 + |\nabla R|^2}, f_2 + \frac{2\alpha_2 R}{|f'|^2 + |\nabla R|^2}, \frac{2R}{|f'|^2 + |\nabla R|^2} \right). \]

An explicit formula entirely in terms of \( f \) can be found if we use (16) to obtain

\[ R_u = u |f'| + \frac{1 + |\tau|^2}{2 |f'|} \text{Re} \left( f' \bar{f}'' \right), \]

\[ R_v = v |f'| + \frac{1 + |\tau|^2}{2 |f'|} \text{Im} \left( f' \bar{f}'' \right). \]

It then follows from (20) that

\[ \alpha_1 = \left( v |f'| + \frac{1 + |\tau|^2}{2 |f'|} \text{Im} \left( f' \bar{f}'' \right) \right) f_{2,u} - \left( u |f'| + \frac{1 + |\tau|^2}{2 |f'|} \text{Re} \left( f' \bar{f}'' \right) \right) f_{1,u} \]

\[ = - |f'| \text{Re} \left( f' \tau \right) - \frac{1 + |\tau|^2}{2 |f'|} \text{Re} \left( (f')^2 \bar{f}'' \right), \]
and similarly,

\[ \alpha_2 = -|f'| \Im(f'\tau) - \frac{1 + |\tau|^2}{2|f'|} \Im((f')^2 \bar{f}''). \]

On the other hand,

\[
|f'|^2 + |\nabla R|^2 = |f'|^2 + \left( u |f'| + \frac{1 + |\tau|^2}{2 |f'|} \Re(f' \bar{f}'')^2 + v |f'| + \frac{1 + |\tau|^2}{2 |f'|} \Im(f' \bar{f}'') \right)^2
\]

\[
= (|\tau|^2 + 1) \left( |f'|^2 + \Re(f' \bar{f}'') + \frac{|f''|^2 (|\tau|^2 + 1)}{4} \right).
\]

Assembling together all the pieces of our computation we can finally write down the coordinates for the sought CMC-1 surface as described in the theorem.

**Remark 3.3.** Bianchi’s construction is local in the sense that he assumes that the projections defining \( f \) are both bijective. But as the ruled example in Section 3 suggests, we may imagine a global construction by considering function elements on appropriate Riemann surfaces.

**Remark 3.4.** Calò also published formulae similar to (17)-(18) corresponding to a rolling problem where the plane is replaced by the sphere (see [B]). A variant of the argument above applied to this case then furnishes explicit formulae for CMC-1 surfaces in the Poincaré model.

**Remark 3.5.** Retracing through the above construction, it is not hard to check that \( f \) is the hyperbolic Gauss map \( G \) of the corresponding CMC-1 surface. More precisely, we have \( f = G \circ X \) where \( X = (x, y, z) \) is given by (19).

We finish this section by briefly indicating how Small’s result relates to the one presented here. Recall from (P) the local representation for the Gauss transform \( \Gamma_S \) of a null curve \( S \subset \mathbb{P}SL(2, \mathbb{C}) \) in terms of a pair \((f, g)\) of holomorphic functions. According to (I), one has to apply dualization to this in order to recover \( \Sigma \). After doing this, one obtains that \( S \) is given by the map \( \omega : M \to \mathbb{P}SL(2, \mathbb{C}) \),

\[
\omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} (f')^{1/2} - \frac{1}{2} f(f')^{-3/2} f'' & f \left( (f')^{-1/2} + \frac{1}{2} g(f')^{-3/2} f'' \right) - g(f')^{1/2} \\ -\frac{1}{2} (f')^{-3/2} f'' & (f')^{-1/2} + \frac{1}{2} g(f')^{-3/2} f'' \end{pmatrix},
\]

(21)

where \( f' = df/dg \) and \( f'' = d^2 f/dg^2 \). The expression for the CMC-1 surface in terms of the hermitian model is, according to Bryant, given by \( \omega \bar{\omega} \), but in order to compare this with (19) one has to perform the transformation to the upper half-space model. In terms of the entries of \( \omega \), this is given by

\[
x + iy = \frac{\alpha \bar{\gamma} + \beta \bar{\delta}}{|\gamma|^2 + |\delta|^2}, \quad z = \frac{1}{|\gamma|^2 + |\delta|^2}.
\]

We now take \( g = \tau \) and Small’s \( f \) to be our \( f \). A straightforward computation yields the equivalence between the methods.
4 Some examples and final remarks

We illustrate the method by retrieving two well-known examples. First, if we take \( f(\tau) = \tau^2 \), substitution in (19) after writing \( \tau = r e^{i\theta} \) yields

\[
\begin{align*}
x &= -r^2 (\cos 2\theta) \frac{5r^2 + 3}{7r^2 + 1}, \\
y &= -r^2 (\sin 2\theta) \frac{5r^2 + 3}{7r^2 + 1}, \\
z &= \frac{8r^3}{7r^2 + 1}.
\end{align*}
\]

This is a catenoid cousin. A rough picture is given in figure 4.1

![Fig. 4.1](image)

Now let \( f(\tau) = \ln \tau \). Again, substitution in (19) yields

\[
\xi = (\ln r - 2 \frac{(r - r^{-1})}{(r + r^{-1})}, \theta, \frac{4}{(r + r^{-1})}).
\]

Or, writing \( r = e^s \),

\[
\xi = (s - 2 \tanh s, \theta, \frac{2}{\cosh s}).
\]

This is a ruled example as figure 4.2 makes it clear.
Remark 4.1. Strictly speaking we used the Riemann surface associated to the log function in this last example, showing that the construction might work in more general cases.

One would like to add a few questions:

1. Can one understand the interplay between the Euclidean problem (rolling of isometric surfaces) and the construction of CMC-1 surfaces in terms of some underlying structure? Can the relationship be thought of as a natural one from some other point of view?

2. Is there a way to view directly the congruence of spheres in a twistorial perspective (or in any other complexified way)?

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