ON THE THEORY OF SPACES OF CONSTANT CURVATURE

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Abstract

Some examples of three-dimensional metrics of constant curvature defined by the solutions of non-linear integrable differential equations and their generalizations are constructed. The properties of Riemann extensions of the metrics of constant curvature are studied. The connection with the theory of normal Riemann spaces are discussed.

1 Introduction

The metrics of constant curvature are defined by a following condition on a curvature tensor

\[ R_{ijkl} - \lambda (g_{ik}g_{jl} - g_{il}g_{jk}) = 0. \]  

(1)

In the case of diagonal metric

\[ ds^2 = A(x, y, z)^2 dx^2 + B(x, y, z)^2 dy^2 + C(x, y, z)^2 dz^2 \]

the conditions (1) are equivalent the system of differential equations on the functions \( A, B, C \). They looks as

\[
\frac{\partial^2}{\partial y \partial z} A(x, y, z) = \frac{\left( \frac{\partial}{\partial y} B(x, y, z) \right) \frac{\partial}{\partial z} A(x, y, z)}{B(x, y, z)} + \frac{\left( \frac{\partial}{\partial y} C(x, y, z) \right) \frac{\partial}{\partial z} A(x, y, z)}{C(x, y, z)},
\]

(2)

\[
\frac{\partial^2}{\partial x \partial z} B(x, y, z) = \frac{\left( \frac{\partial}{\partial x} A(x, y, z) \right) \frac{\partial}{\partial z} B(x, y, z)}{A(x, y, z)} + \frac{\left( \frac{\partial}{\partial x} C(x, y, z) \right) \frac{\partial}{\partial z} B(x, y, z)}{C(x, y, z)},
\]

(3)

\[
\frac{\partial^2}{\partial x \partial y} C(x, y, z) = \frac{\left( \frac{\partial}{\partial y} A(x, y, z) \right) \frac{\partial}{\partial x} C(x, y, z)}{A(x, y, z)} + \frac{\left( \frac{\partial}{\partial y} B(x, y, z) \right) \frac{\partial}{\partial x} C(x, y, z)}{B(x, y, z)},
\]

(4)

\[
\lambda C(x, y, z) B(x, y, z) + \frac{\left( \frac{\partial}{\partial y} C(x, y, z) \right) \frac{\partial}{\partial z} B(x, y, z)}{(A(x, y, z))^2} + \frac{\frac{\partial^2}{\partial x \partial y} B(x, y, z)}{C(x, y, z)} - \frac{\left( \frac{\partial}{\partial y} B(x, y, z) \right) \frac{\partial}{\partial z} C(x, y, z)}{(B(x, y, z))^2} = 0,
\]

(5)

\[
\lambda A(x, y, z) C(x, y, z) - \frac{\left( \frac{\partial}{\partial x} A(x, y, z) \right) \frac{\partial}{\partial y} C(x, y, z)}{(C(x, y, z))^2} + \frac{\frac{\partial^2}{\partial x \partial y} A(x, y, z)}{C(x, y, z)} +
\]
2 \textbf{THE SPACE OF ZERO CURVATURE } \lambda = 0

The properties of the spaces of curvature \( \lambda = 0 \) \( for \) diagonal metrics have been studied in ([4]).

Here we consider the metric in non diagonal form ([5])

\[
\begin{align*}
\frac{ds^2}{\lambda} &= y^2 dx^2 + 2 \left(l(x, z) y^2 + m(x, z)\right) dx dz + 2 dy dz + \\
+ \left((l(x, z))^2 y^2 - 2 \left(\frac{\partial}{\partial x} l(x, z)\right) y + 2 l(x, z) m(x, z) + 2 l(x, z)\right) dz^2
\end{align*}
\]

(8)

with some functions \( l(x, z) \) and \( m(x, z) \).

The condition on the curvature tensor

\[
R_{ijkl} = 0
\]

for the metric (8) lead to the relations

\[
R_{1313} = \left(\frac{\partial^3}{\partial x^3} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) + \frac{\partial}{\partial z} l(x, z)\right) y + \\
+ \left(-l(x, z) \frac{\partial^2}{\partial x^2} m(x, z) + \frac{\partial^2}{\partial x \partial z} m(x, z) - 3 \left(\frac{\partial}{\partial x} l(x, z)\right) \frac{\partial}{\partial x} m(x, z) - 2 m(x, z) \frac{\partial^2}{\partial x^2} l(x, z) - \frac{\partial^2}{\partial x^2} l(x, z)\right) - \\
\left(-m(x, z) \frac{\partial}{\partial z} m(x, z) + m(x, z) l(x, z) \frac{\partial}{\partial x} m(x, z) + \left(\frac{\partial}{\partial x} l(x, z)\right) m(x, z) + 2 \left(\frac{\partial}{\partial x} l(x, z)\right) (m(x, z))^2\right) / y = 0,
\]

and

\[
R_{1323} = \left(\frac{\partial}{\partial x} l(x, z) - \frac{\partial}{\partial z} m(x, z) + 2 \left(\frac{\partial}{\partial x} l(x, z)\right) m(x, z) + l(x, z) \frac{\partial}{\partial x} m(x, z)\right) / y = 0
\]

which are equivalent the system of equations for the functions \( l(x, z) \) and \( m(x, z) \)

\[
\begin{align*}
\frac{\partial^3}{\partial x^3} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) + \frac{\partial}{\partial z} l(x, z) &= 0, \\
\frac{\partial}{\partial x} l(x, z) - \frac{\partial}{\partial z} m(x, z) + 2 \left(\frac{\partial}{\partial x} l(x, z)\right) m(x, z) + l(x, z) \frac{\partial}{\partial x} m(x, z) &= 0.
\end{align*}
\]

(9) \hspace{1cm} (10)

So we can formulate the following
Theorem 1 There exists a class of 3-dimensional flat metrics defined by the solutions of the system of equations (9-10).

Remark that the first equation of the system is the famous KdV-equation and this fact may be used for studying of the properties of orthogonal metrics.

Let us consider some examples.

1. \[ l(x, z) = -\frac{x}{3z}, \quad m(x, z) = -1/2 + F_1\left(\frac{z}{x^3}\right)x^{-2}. \]

2. \[ l(x, z) = -4 (\cosh(x - 4z))^{-2}, \quad m(x, z) = -\frac{1}{2} \]

Remark 1 In the simplest case three-dimensional metrics of zero curvature look as

\[ ds^2 = y^2 dx^2 + 2 \left(l(x, z)y^2 - 1/2\right) dx dz + 2 dy dz + \left(l(x, z)^2 y^2 - 2 \left(\frac{\partial}{\partial x}l(x, z)\right) y + l(x, z)\right) dz^2 \]

where the function \( l(x, z) \) is solution of classical KdV-equation

\[ \frac{\partial}{\partial z}l(x, z) - 3 l(x, z) \frac{\partial}{\partial x}l(x, z) + \frac{\partial^3}{\partial x^3}l(x, z) = 0. \]

In particular the functions

\[ l(x, z) = -4 (\cosh(x - 4z))^{-2}, \]

and

\[ l(x, z) = -24 \frac{4 \cosh(2x - 8z) + \cosh(4x - 64z) + 3}{(3 \cosh(x - 28z) + \cosh(3x - 36z))^2} \]

give us the examples of such type of metrics.

In spite of the fact that the determinant of the metric do not depends from the function \( l(x, z) \) it is possible to distinguish the properties of metrics with the help of eigenvalue equation for the Laplace operator defined on the 1-forms

\[ A(x, y, z) = A_i(x, y, z)dx^i. \]

It has the form

\[ g^{ij} \nabla_i \nabla_j A_k - R^i_{jk} A_i = -\lambda A_k. \]

In particular case \( l(x, z) = 0 \) and \( A_i(x, y, z) = [h(y), q(y), f(y)] \) these equations take the form

\[ -1/4 \frac{d}{dy}h(y) - 4 \left(\frac{d}{dy}f(y)\right) y^2 - \left(\frac{d^2}{dy^2}h(y)\right) y + 4 \lambda h(y)y^3 = 0, \]

\[ -1/4 \frac{d}{dy}h(y) - 3 q(y) + 3 \left(\frac{d}{dy}q(y)\right) y + 4 \left(\frac{d}{dy}h(y)\right) y - \left(\frac{d^2}{dy^2}q(y)\right) y^2 + 4 f(y)y^2 + 4 \lambda q(y)y^4 = 0, \]

\[ -1/4 \frac{d}{dy}f(y) - \left(\frac{d^2}{dy^2}f(y)\right) y + 4 \lambda f(y)y^3 = 0. \]

The solutions of this system are dependent from the eigenvalues \( \lambda \) and characterize the properties of corresponding flat metrics.
Remark 2 In theory of varieties the Chern-Simons characteristic class is constructed from a matrix gauge connection $A^i_{jk}$ as

$$W(A) = \frac{1}{4\pi^2} \int d^3x e^{ijk} \text{tr} \left( \frac{1}{2} A_i \partial_j A_k + \frac{1}{3} A_i A_j A_k \right).$$

This term can be translated into three-dimensional geometric quantity by replacing the matrix connection $A^i_{jk}$ with the Christoffel connection $\Gamma^i_{jk}$.

For the density of Chern-Simons invariant can be obtained the expression ([6])

$$CS(\Gamma) = \epsilon^{ijk} (\Gamma^p_{iq} \Gamma^q_{jp}) \Gamma^r_{kp}.$$

For the metric (8) we find the quantity

$$CS(\Gamma) = -\frac{5 l(x,z) \frac{\partial}{\partial z} l(x,z) - 5 \frac{\partial}{\partial x} l(x,z)}{\sqrt{y^2}} - \frac{3 l(x,z) \frac{\partial}{\partial x} m(x,z) - 4 \left( \frac{\partial}{\partial z} l(x,z) \right) m(x,z) - 3 \frac{\partial}{\partial z} m(x,z) - 2 \frac{\partial}{\partial x} l(x,z)}{y^2 \sqrt{y^2}}.$$

Using this formulae for the first example one get

$$CS(\Gamma) = \frac{10}{9} \frac{x \text{csgn}(y)}{z^2 y} - 10/3 \int F_1 \left( \frac{z}{x^2} \right) \text{csgn}(y) z^{-2} x^{-2} y^{-3}.$$

For the second example this quantity is

$$CS(\Gamma) = 160 \frac{\text{cosh}(x - 4 z) \left( \text{sinh}(x - 4 z) \right)^3 \text{csgn}(y)}{y}.$$

3 The metrics of nonzero curvature $\lambda \neq 0$

The metric of the space of positive curvature $\lambda = 1$ is defined by

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{(1 + (x^2 + y^2 + z^2)/4)^2}. \quad (12)$$

The metric of the space of negative curvature $\lambda = -1$ is defined by

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}. \quad (13)$$

Starting from these expressions it is possible to get more general examples of the metrics of constant curvature.

For example, the substitution

$$A(x,y,z) = \frac{1}{z}, \quad B(x,y,z) = \frac{1}{z} + v(z), \quad C(x,y,z) = \frac{1}{z} + u(z)$$

into the system (2)-(7) and integration of corresponding equations for the functions $u(z)$ and $v(z)$ give us the metric

$$ds^2 = \frac{dx^2}{z^2} + (z^{-2} - 1) dy^2 + \frac{dz^2}{z^2 (1 - z^2)}$$

of negative curvature $\lambda = -1$. 
3 THE METRICS OF NONZERO CURVATURE $\lambda \neq 0$

The substitution

$$A(x, y, z) = (1 + 1/4 x^2 + 1/4 y^2 + 1/4 z^2)^{-1} + U(x, y, z)$$

into the system (2)-(7) and integration of corresponding equations lead to the metrics

$$ds^2 = A(x, y, z)^2 dx^2 + \frac{dy^2 + dz^2}{(1 + (x^2 + y^2 + z^2/4))},$$

with the function $A(x, y, z)$ in form

$$A(x, y, z) = 4 \sqrt{x^2 (x^2 - 4 \lambda + 4) (4 + x^2 + y^2 + z^2)}.$$

The Chern-Simons invariant of orthogonal 3-dimensional metrics with conditions (2)-(4) on components is given by the expression

$$CS(\Gamma) = 10 \left( \frac{\partial}{\partial y} A(x, y, z) \right) \left( \frac{\partial}{\partial z} B(x, y, z) \right) \left( \frac{\partial}{\partial x} C(x, y, z) \right) - \left( \frac{\partial}{\partial y} A(x, y, z) \right) \left( \frac{\partial}{\partial z} B(x, y, z) \right) \left( \frac{\partial}{\partial x} C(x, y, z) \right) = 0.$$  

After substitution of the components of metric (13) into this expression one obtain

$$CS(\Gamma) = 0.$$  

As it is follows from (14) the class of 3-dimensional orthogonal metrics with vanishing Chern-Simons invariant is defined by the condition

$$\left( \frac{\partial}{\partial z} A(x, y, z) \right) \left( \frac{\partial}{\partial x} B(x, y, z) \right) \left( \frac{\partial}{\partial y} C(x, y, z) \right) - \left( \frac{\partial}{\partial y} A(x, y, z) \right) \left( \frac{\partial}{\partial z} B(x, y, z) \right) \left( \frac{\partial}{\partial x} C(x, y, z) \right) = 0.$$  

The metrics of the form

$$ds^2 = 2E(x, y, z) dx dy + dz^2$$

are the metrics of constant curvature if the function $E(x, y, z)$ is defined by

$$E(x, y, z) = 1/4 \left( \frac{F(x, y) \sin(\sqrt{\lambda} z) - F(x, y) \cos(\sqrt{\lambda} z)}{\lambda} \right)^2,$$

where

$$-4 \left( \frac{\partial}{\partial y} F(x, y) \right) \left( \frac{\partial}{\partial x} F(x, y) \right) + 4 \frac{\partial^2}{\partial x \partial y} F(x, y) + (F(x, y))^4 = 0,$$

or

$$4 \frac{\partial^2}{\partial x \partial y} U(x, y) + e^{2U(x, y)} = 0$$

after the substitution

$$F(x, y) = \exp(U(x, y)).$$

Last equation is the famous Liouville equation with general solution

$$U(x, y) = 1/2 \ln \left( -4 \left( \frac{d}{dx} a(x) \right) \left( \frac{d}{dy} b(y) \right) \right).$$

The Chern-Simons invariant for this example of metrics is

$$CS(\Gamma) = 0.$$
Let us consider the metrics of the form
\[ ds^2 = dx^2 + 2 \cos(u(x, y)) dy^2 + A(x, y)^2 dz^2. \]

The conditions (1) lead to the linear system of equations
\[
\frac{\partial^2}{\partial x \partial y} A(x, y) + \lambda A(x, y) \cos(u(x, y)) = 0,
\]
\[
\frac{\partial^2}{\partial x^2} A(x, y) - \frac{\cos(u(x, y))}{\sin(u(x, y))} \left( \frac{\partial}{\partial x} A(x, y) \right) + \lambda A(x, y) + \frac{\left( \frac{\partial}{\partial x} A(x, y) \right) \frac{\partial}{\partial y} u(x, y)}{\sin(u(x, y))} = 0,
\]
\[
\frac{\partial^2}{\partial y^2} A(x, y) - \frac{\cos(u(x, y))}{\sin(u(x, y))} \left( \frac{\partial}{\partial y} A(x, y) \right) + \lambda A(x, y) + \frac{\left( \frac{\partial}{\partial y} A(x, y) \right) \frac{\partial}{\partial x} u(x, y)}{\sin(u(x, y))} = 0,
\]
which is compatible on the solutions of the "sin–Gordon" equation \( \lambda = -\mu \)
\[
\frac{\partial^2}{\partial x \partial y} u(x, y) + \lambda \sin(u(x, y)) = 0.
\]

So for any solution of the "sin–Gordon" equation one possible to find the function \( A(x, y) \) with the help of solution of the corresponding linear system of equations.

To take one example.
The simplest solution of the equation
\[
\frac{\partial^2}{\partial x \partial y} u(x, y) - \sin(u(x, y)) = 0
\]
is given by
\[ u(x, y) = 4 \arctan(\exp(x + y)). \]

By this condition the linear system looks as
\[
\frac{\partial^2}{\partial x \partial y} A(x, y) + \frac{6 e^{2x+2y} - 1 - e^{4x+4y}}{(1 + e^{2x+2y})^2} A(x, y) = 0,
\]
\[
(1 - e^{4x+4y}) A(x, y) + (-2 e^{2x+2y} - e^{4x+4y} - 1) \frac{\partial}{\partial y} A(x, y) + (6 e^{2x+2y} + 1 + e^{4x+4y}) \frac{\partial}{\partial x} A(x, y) +
\]
\[
+ (-1 + e^{4x+4y}) \frac{\partial^2}{\partial x^2} A(x, y) = 0,
\]
\[
(1 - e^{4x+4y}) A(x, y) + (-6 e^{2x+2y} + 1 + e^{4x+4y}) \frac{\partial}{\partial y} A(x, y) + (2 e^{2x+2y} - e^{4x+4y} - 1) \frac{\partial}{\partial x} A(x, y) +
\]
\[
+ (-1 + e^{4x+4y}) \frac{\partial^2}{\partial y^2} A(x, y) = 0.
\]

Its solution is
\[ A(x, y) = \frac{e^{x+y}}{1 + e^{2x+2y}}. \]
4 An examples of essentially three-dimensional metrics

We consider a family of three-dimensional metrics in form
\[ ds^2 = dx^2 + 2 \cos(B(x, y, z)) dx dy + dy^2 + \left( \frac{\partial}{\partial z} B(x, y, z) \right)^2 dz^2 \]

The conditions (1) for a such metrics lead to a following system of equations
\[
\frac{\partial^2}{\partial x \partial y} B(x, y, z) + 1/4 \sin(B(x, y, z)) (-1 + 4 \lambda) = 0,
\]
\[
\frac{\partial^3}{\partial y^2 \partial z} B(x, y, z) - \frac{\cos(B(x, y, z))}{\sin(B(x, y, z))} \left( \frac{\partial^2}{\partial y \partial z^2} B(x, y, z) \right) + \left( \frac{\partial^2}{\partial x \partial z} B(x, y, z) \right) \frac{\partial}{\partial y} B(x, y, z) - \frac{\partial^3}{\partial x \partial y \partial z} B(x, y, z) \sin(B(x, y, z)) = 0,
\]
\[
\frac{\partial^3}{\partial x^2 \partial z} B(x, y, z) - \frac{\cos(B(x, y, z))}{\sin(B(x, y, z))} \left( \frac{\partial^2}{\partial x \partial y} B(x, y, z) \right) + \left( \frac{\partial^2}{\partial x \partial z} B(x, y, z) \right) \frac{\partial}{\partial x} B(x, y, z) - \frac{\partial^3}{\partial y \partial z} B(x, y, z) \sin(B(x, y, z)) = 0.
\]

This system is compatible and its solutions gives us the examples of three-dimensional metrics of constant curvature.

The Chern-Simons term in this case is defined by
\[ CS(\Gamma) = -\frac{5}{2} \left( \frac{\partial^2}{\partial x \partial z} B(x, y, z) \right) \frac{\partial}{\partial y} B(x, y, z) + \frac{5}{2} \left( \frac{\partial}{\partial x} B(x, y, z) \right) \frac{\partial^2}{\partial y \partial z} B(x, y, z). \]

Let us consider an example.

For the sake of simplicity we present the metric in equivalent form
\[ ds^2 = dx^2 + 2 u(x, y, z) dx dy + dy^2 + \left( \frac{\partial}{\partial z} u(x, y, z) \right)^2 dz^2, \]
where the function \( u(x, y, z) \) is determined from the condition
\[ u(x, y, z) = \arccos(B(x, y, z)). \]

A system of equations for the function \( B(x, y, z) \) grade at that into more simple system on the function \( u(x, y, z) \).

Its integration at the condition \( \lambda = 1/4 \) lead to the metric
\[ ds^2 = dx^2 + 1/2 \left( \frac{z^4 f(x)^2 (h(y))^2 + 1}{z^2 f(x) h(y)} \right) dy^2 + 4 \frac{dz^2}{z^2}, \]
where \( f(x), h(y) \) are arbitrary functions.

The Ricci tensor of this case has a form
\[ R_{ab} = \begin{bmatrix}
1/2 & 1/4 z^4 f(x)^2 (h(y))^2 + 1 & 0 \\
1/4 z^4 f(x)^2 (h(y))^2 + 1 & 1/2 & 0 \\
0 & 0 & -2 z^{-2}
\end{bmatrix}. \]

The Chern-Simons invariant of this metric equal to zero.

Remark 3 The absolute value of curvature \( \lambda = 1/4 \) is special in theory of Riemann manifolds and is connected with the theory of manifolds with pinched curvature.
5 The Rieman extensions of the spaces of constant curvature $\lambda = \pm 1, 0$

The space with diagonal metrics has a following coefficients of connections

\[ \Gamma^1_{11} = \frac{\partial}{\partial x} A(x, y, z), \quad \Gamma^2_{11} = -\frac{A(x, y, z) \partial y}{(B(x, y, z))^2}, \quad \Gamma^3_{11} = -\frac{A(x, y, z) \partial z}{(C(x, y, z))^2} \]

\[ \Gamma^2_{12} = \frac{\partial}{\partial y} A(x, y, z), \quad \Gamma^2_{12} = \frac{\partial B(x, y, z)}{B(x, y, z)}, \quad \Gamma^2_{12} = 0, \quad \Gamma^1_{13} = \frac{\partial}{\partial z} A(x, y, z), \]

\[ \Gamma^2_{23} = -\frac{B(x, y, z) \partial B(x, y, z)}{(C(x, y, z))^2}, \quad \Gamma^2_{23} = 0, \quad \Gamma^2_{23} = \frac{\partial B(x, y, z)}{B(x, y, z)}, \quad \Gamma^2_{33} = \frac{\partial C(x, y, z)}{C(x, y, z)}, \]

\[ \Gamma^3_{33} = \frac{C(x, y, z) \partial C(x, y, z)}{A(x, y, z)}, \quad \Gamma^3_{33} = -\frac{C(x, y, z) \partial C(x, y, z)}{(B(x, y, z))^2}, \quad \Gamma^3_{33} = \frac{\partial C(x, y, z)}{C(x, y, z)}. \]  

Using these expressions we introduce a six-dimensional metric

\[ 6ds^2 = -2\Gamma^i_{jk} dx^j dx^k \psi_i - 2dx^i d\psi_i, \]  

where $\psi_1, \psi_2, \psi_3$ are an additional coordinates ([7])-([10]).

The Ricci tensor of the metric (17) is vanished at the conditions (2-7), $\lambda = 0$, and the Riemann tensor $^6R_{ijkl} = 0$ in this case also is vanished.

So after the Riemian extension of the flat 3-dimensional diagonal metric $\lambda = 0$ with help of the coefficients (16) we get a six-dimensional flat $^6R_{ijkl} = 0$ space having the signature $[++]$. 

The Ricci tensor of the metric (1) at the conditions (2-7) has the components

\[ R_{11} = 2\lambda A(x, y, z)^2, \quad R_{22} = 2\lambda B(x, y, z)^2, \quad R_{33} = 2\lambda C(x, y, z)^2, \]

\[ R_{12} = 0, \quad R_{13} = 0, \quad R_{23} = 0 \]

and corresponding space in the case $\lambda = \pm 1$ is symmetric space with conditions on the curvature tensor

\[ ^3R_{ijkl;m} = 0. \]

It is remarkable fact that after the Riemann extension of the space of constant curvature one get symmetric space.

Theorem 2 The Riemann extension of 3-dimensional space of constant curvature is a six-dimensional symmetric space

\[ ^6R_{ijkl;m} = 0. \]

The proof of this statement can be checked by direct calculations.

Remark 4 In the case $\Gamma^1_{23} \neq 0, \quad \Gamma^3_{12} \neq 0, \quad \Gamma^2_{13} \neq 0$ it is possible to get a six-dimensional curved metric

\[ 6ds^2 = -2\Gamma^i_{jk} dx^j dx^k \psi_i - 2dx^i d\psi_i, \]  

(18)
with conditions

\[
\Gamma^2_{13} = \frac{C(x, y, z)A(x, y, z)}{B(x, y, z)}, \quad \Gamma^3_{12} = \frac{A(x, y, z)B(x, y, z)}{C(x, y, z)}, \quad \Gamma^1_{23} = \frac{B(x, y, z)C(x, y, z)}{A(x, y, z)}
\]

for the spaces with \( \lambda = 1 \) and

\[
\Gamma^2_{13} = \frac{iC(x, y, z)A(x, y, z)}{B(x, y, z)}, \quad \Gamma^3_{12} = \frac{iA(x, y, z)B(x, y, z)}{C(x, y, z)}, \quad \Gamma^1_{23} = \frac{iB(x, y, z)C(x, y, z)}{A(x, y, z)}
\]

for the spaces with \( \lambda = -1 \).

For such form of connection coefficients we get the generalization of the Darboux system of equations

\[
\begin{aligned}
\frac{\partial^2}{\partial y \partial z} A(x, y, z) &= \left(\frac{\partial}{\partial y} B(x, y, z)\right) \frac{\partial}{\partial z} A(x, y, z) + \left(\frac{\partial}{\partial y} C(x, y, z)\right) \frac{\partial}{\partial z} A(x, y, z) + \\
&+ 2 \left(\frac{\partial}{\partial x} (B(x, y, z)C(x, y, z))\right), \\
\frac{\partial^2}{\partial x \partial z} B(x, y, z) &= \left(\frac{\partial}{\partial z} A(x, y, z)\right) \frac{\partial}{\partial x} A(x, y, z) + \left(\frac{\partial}{\partial z} C(x, y, z)\right) \frac{\partial}{\partial x} B(x, y, z) + \\
&+ 2 \left(\frac{\partial}{\partial y} (A(x, y, z)C(x, y, z))\right), \\
\frac{\partial^2}{\partial x \partial y} C(x, y, z) &= \left(\frac{\partial}{\partial y} A(x, y, z)\right) \frac{\partial}{\partial x} C(x, y, z) + \left(\frac{\partial}{\partial y} B(x, y, z)\right) \frac{\partial}{\partial x} C(x, y, z) + \\
&+ 2 \left(\frac{\partial}{\partial z} (A(x, y, z)B(x, y, z))\right),
\end{aligned}
\]

(19)

which can be useful in applications.

For example the simplest solution of this system is

\[
A(x, y, z) = \exp\left(\frac{z}{4} - x\right), \quad B(x, y, z) = \exp\left(y - \frac{z}{4}\right), \quad C(x, y, z) = \exp\left(\frac{z}{4}\right)
\]

which corresponds the metrics (18) of a six-dimensional manifold with a corresponding conditions on the curvature tensor.

6 The normal Riemann spaces

The notion of normal Riemann space was introduced first by Eisenhart. Their properties have been studied in ([11]).

Definition 1 An \( n \)-dimensional Riemannian space \( N(x^k) \) with the metrics

\[
ds^2 = g_{ij} dx^i dx^j
\]

is normal if a following conditions on their main curvatures \( K_i \) hold

\[
\frac{\partial K_i}{\partial x^j} = 3 \lambda_i + 3 \mu_i K_i, \quad i = l,
\]

and

\[
\frac{\partial K_i}{\partial x^j} = \lambda_i + \mu_i K_i, \quad i \neq l,
\]

\[
\frac{\partial \ln(g_{ij})}{\partial x^l} = \frac{2}{K_i} \frac{\partial K_i}{\partial x^l} \quad i \neq l,
\]

where \( \lambda \) and \( \mu \) are the functions of coordinates \( x^k \).
Remark 5 The values $K_i$ are the roots of algebraic equation

$$|b_{ij} - \kappa g_{ij}| = 0$$

relatively of a second range tensor $b_{ij}$ defined in a given Riemann space.

In 3-dimensional case principal curvatures $K_i$ satisfy the system of equations

$$(K_2 - K_3) \frac{\partial K_1}{\partial x} + 3(K_3 - K_1) \frac{\partial K_2}{\partial x} + 3(K_1 - K_2) \frac{\partial K_3}{\partial x} = 0,$$

$$3(K_2 - K_3) \frac{\partial K_1}{\partial y} + (K_3 - K_1) \frac{\partial K_2}{\partial y} + 3(K_1 - K_2) \frac{\partial K_3}{\partial y} = 0,$$

$$3(K_2 - K_3) \frac{\partial K_1}{\partial z} + 3(K_3 - K_1) \frac{\partial K_2}{\partial z} + 3(K_1 - K_2) \frac{\partial K_3}{\partial z} = 0. \tag{20}$$

The relations between the systems of Darboux equations (2)-(4) and the system (20) were established first in ([12]).

Theorem 3 The system of equations (20) for principal curvatures of three-dimensional normal spaces transforms into the system of equations

$$\frac{\partial^2}{\partial x \partial y} K_1(x, y, z) + \left( \frac{\partial}{\partial y} A(x, y, z) + \frac{\partial}{\partial x} B(x, y, z) \right) \frac{\partial}{\partial z} K_1(x, y, z) = 0,$$

$$\frac{\partial^2}{\partial x \partial z} K_1(x, y, z) + \left( \frac{\partial}{\partial z} A(x, y, z) + \frac{\partial}{\partial x} B(x, y, z) \right) \frac{\partial}{\partial y} K_1(x, y, z) = 0,$$

$$\frac{\partial^2}{\partial y \partial z} K_1(x, y, z) + \left( \frac{\partial}{\partial z} A(x, y, z) + \frac{\partial}{\partial y} B(x, y, z) \right) \frac{\partial}{\partial x} K_1(x, y, z) = 0, \tag{21}$$

if the following relations are hold

$$\frac{\partial}{\partial y} K_1(x, y, z) - \left( \frac{\partial}{\partial y} A(x, y, z) \right) (K_2(x, y, z) - K_1(x, y, z)) = 0,$$

$$\frac{\partial}{\partial z} K_1(x, y, z) + \left( \frac{\partial}{\partial z} A(x, y, z) \right) (-K_3(x, y, z) + K_1(x, y, z)) = 0,$$

$$\frac{\partial}{\partial x} K_2(x, y, z) + \left( \frac{\partial}{\partial x} B(x, y, z) \right) (K_3(x, y, z) - K_1(x, y, z)) = 0,$$

$$\frac{\partial}{\partial z} K_2(x, y, z) + \left( \frac{\partial}{\partial z} B(x, y, z) \right) (K_3(x, y, z) - K_2(x, y, z)) = 0.$$
\[
\frac{\partial}{\partial x} K_3(x, y, z) - \left( \frac{\partial}{\partial z} C(x, y, z) \right) \left(-K_3(x, y, z) + K_1(x, y, z) \right) \frac{1}{C(x, y, z)} = 0
\]
\[
\frac{\partial}{\partial y} K_3(x, y, z) - \left( \frac{\partial}{\partial z} C(x, y, z) \right) \left(K_3(x, y, z) - K_3(x, y, z) \right) \frac{1}{C(x, y, z)} = 0.
\]

**Remark 6** Similar equations and for the components \( K_2 \) and \( K_3 \) can be written. As example for the value \( K_2(x, y, z) \) one get

\[
\frac{\partial^2}{\partial x \partial y} K_2(x, y, z) + \frac{\partial}{\partial x} B(x, y, z) \frac{\partial}{\partial y} K_2(x, y, z) +
\]
\[
+ \left( \frac{\partial}{\partial y} B(x, y, z) \right) \left( \frac{\partial^2}{\partial x \partial z} B(x, y, z) \right) \left( \frac{\partial}{\partial y} B(x, y, z) \right) \left( \frac{\partial}{\partial z} B(x, y, z) \right) \frac{\partial}{\partial x} K_2(x, y, z) = 0,
\]
\[
\frac{\partial^2}{\partial x \partial z} K_2(x, y, z) + \left( \frac{\partial}{\partial y} B(x, y, z) \right) \left( \frac{\partial^2}{\partial x \partial z} B(x, y, z) \right) \left( \frac{\partial}{\partial z} B(x, y, z) \right) \frac{\partial}{\partial y} K_2(x, y, z) = 0,
\]
\[
+ \left( \frac{\partial}{\partial z} B(x, y, z) \right) \left( \frac{\partial^2}{\partial x \partial z} B(x, y, z) \right) \left( \frac{\partial}{\partial z} B(x, y, z) \right) \frac{\partial}{\partial x} K_2(x, y, z) = 0,
\]

(22)

It is important to note that the equations (21) and (22) are present the linear systems of equations with conditions of compatibility in form of the Darboux system (2)-(4).

This property will be used for the studying of solutions of the complete Lame system of equations (2)-(7).

Now we present some solutions of the system (20).

Remark that from the system (20) follows the relations

\[ K_i(x, y, z)(\phi_j - \phi_i) + K_j(x, y, z)(\phi_i - \phi_j) + K_1(x, y, z)(\phi_i - \phi_j) = 0, \quad i \neq j \neq l \]

where \( \phi_i = \phi_i(x) \) are arbitrary functions depending from the variable \( x \).

This restrictive clause together with equations (20) lead to the system of equations on the functions \( K_i(x, y, z) \).

As example for the value \( K_1(x, y, z) \) we get

\[
\frac{\partial^2}{\partial x \partial y} K_1(x, y, z) - \frac{1}{2} \frac{\partial}{\partial x} b(y) + \frac{1}{2} \frac{\partial}{\partial y} a(x) \frac{\partial}{\partial y} K_1(x, y, z) = 0,
\]
\[
\frac{\partial^2}{\partial x \partial z} K_1(x, y, z) - \frac{3}{2} \frac{\partial}{\partial z} a(x) \frac{\partial}{\partial z} K_1(x, y, z) - \frac{1}{2} \frac{\partial}{\partial z} c(z) - \frac{1}{2} \frac{\partial}{\partial z} K_1(x, y, z) = 0,
\]
\[
\frac{\partial^2}{\partial y \partial z} K_1(x, y, z) - \frac{1}{2} \frac{\partial}{\partial y} c(z) + \frac{3}{2} \frac{\partial}{\partial y} b(y) - \frac{1}{2} \frac{\partial}{\partial y} K_1(x, y, z) = 0
\]

where \( a(x), \ b(y), \ c(z) \) are arbitrary functions.
With the help of equations (21) it is possible to show that the components of metrics of the Darboux space with a given conditions on principal curvatures are

\[
A(x,y,z) = \frac{U(x)}{\sqrt{a(x)-b(y)}\sqrt{a(x)-c(z)}}, \]

\[
B(x,y,z) = \frac{V(y)}{(a(x)-b(y))^{3/2}\sqrt{b(y)-c(z)}}, \]

\[
C(x,y,z) = \frac{W(z)}{(a(x)-c(z))^{3/2}\sqrt{b(y)-c(z)}}. \]

References

[1] V.S.Dryuma, Projective properties of a family operators, IX- Geometric Conference, Kishinev, 20-22 September, 1988, Theses of communications, pp.104-105 (in Russian).

[2] V.S.Dryuma, Three dimensional exactly integrable system of nonlinear equations and its applications, Matematicheskie issledovaniya, Kishinev, Stiintsa, 1992, v.124, pp.56-68 (in Russian).

[3] V.S.Dryuma, Geometrical properties of the multidimensional nonlinear differential equations, Teoreticheskaya i Matematicheskaya Fizika, v.99, No.2, 1993, pp.241-249.

[4] Zakharov V., Description of the n-orthogonal curvalinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type: 1. Integration of the Lame Equations, Duke Math. Journal, 1998, v.94 No.1, p.103–139.

[5] Wolf T., About vacuum solutions of Einstein’s field equation with flat three-dimensional hypersurfaces, Journal of Math. Phys., v.27(9), 1986, 2354-2359.

[6] Jackiw R., A Pure Cotton Kink in a Funny Place, ArXiv: math-ph/0403044, v.2, 21 July 2004.

[7] Paterson E.M. and Walker A.G., Riemann extensions, Quart. J. Math. Oxford, 1952, V.3,19–28.

[8] Dryuma V., The Riemann Extensions in theory of differential equations and their applications, Matematicheskaya fizika, analiz, geometriya, 2003, v.10, No.3,1–19.

[9] Dryuma V., The Riemann and Einstein-Weyl geometries in the theory of ODE’s, their applications and all that, New Trends in Integrability and Partial Solvability, 115-156, (eds.A.B.Shabat et al.) 2004, Kluwer Academic Publishers, ArXiv: nlin: SI/0303023, 11 March, 2003, 1–37.

[10] Dryuma V., Applications of Riemannian and Einstein-Weyl Geometry in the theory of second order ordinary differential equations, Theoretical and Mathematical Physics, 2001, V.128, N 1, 845–855.

[11] Sobchuk V., About one classe of normal Riemann spaces, Seminar on vector and tensor analyse, v.15, 1968, 65-76 (in Russian).

[12] Dryuma V., On the law of transformation of affine connection and its integration. Part 1. Generalization of the Lame equations, Buletinul Academiei de Stiinte a Republicii Moldova, matematika, 1998, V.1(26), 55–68.