QUALITATIVE ANALYSIS OF A GENERALIZED NOSÉ-HOOVER OSCILLATOR

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Abstract. In this paper, we analyze the qualitative dynamics of a generalized Nosé-Hoover oscillator with two parameters varying in certain scope. We show that if a solution of this oscillator will not tend to the invariant manifold \(\{(x, y, z) \in \mathbb{R}^3 | x = 0, y = 0\}\), it must pass through the plane \(z = 0\) infinite times. Especially, every invariant set of this oscillator must have intersection with the plane \(z = 0\). In addition, we show that if a solution is quasiperiodic, it must pass through at least five quadrants of \(\mathbb{R}^3\).

1. Introduction. In 1986, by reformulating the Nosé equation which models a particle interacting with a heat-bath in [3],[4], Hoover in [2],[5] got the Nosé-Hoover oscillator as follows:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - yz, \\
\dot{z} &= \alpha(y^2 - 1).
\end{align*}
\]

where the variable \(x\) represents the oscillator, the variable \(y\) represents the momentum, the variable \(z\) represents the friction coefficient, and \(\alpha\) is a positive coupling parameter.

A lot of dynamic phenomena of the Nosé-Hoover oscillator have been discovered. The authors in [5] showed that this oscillator has both regular and chaotic dynamical trajectories with different initial values. Wang and Yang in [9],[8] obtained that the Nosé-Hoover oscillator has averagely conservative regions which are full of an infinite sequence of nested tori with various knot types numerically.

After the work in [9],[8], in [10] the authors found a modified Nosé-Hoover oscillator, which possesses horseshoe chaos and “fat fractal” structure. In particular, they also studied this modified oscillator in [11] mathematically and showed that

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every trajectory not belonging to the invariant manifolds (one is unit circle and the other is a straight line) must have at least one $\omega$-limiting point on the disk.

Based on the equation studied in [7], the author in [6] considered a generalized Nosé-Hoover oscillator with two parameters given by

$$
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - yz \\
\dot{z} &= \alpha(y^2 - 1 - \epsilon \tanh(x)).
\end{align*}
$$

(1)

By numerical simulation, the author characterized the dynamics of this model at each point of the two-dimensional parameter-space and demonstrated that there are dissipative quasiperiodic structures embedded in a chaotic region. Moreover, the same parameter-space also confirms the multistability phenomenon in this model.

Although it is very hard to study system (1) by analytic approach, we still try to study the behavior of this system by mathematical analysis. In this paper, we will investigate some general qualitative dynamics of this generalized Nosé-Hoover oscillator with $\epsilon \in (0, 1]$ and $\alpha \in (0, \frac{1}{1 + \epsilon})$ as the first step. Since for $\alpha \in \left[\frac{1}{1 + \epsilon}, +\infty\right)$, it is not easy to determine the sign of the inner product of the normal vector of the surface $x = -yz$ and the vector field of system (1) on this surface, which makes it difficult to give an analysis of the qualitative behavior of the solution starting from the surface $x = -yz$, thus we will consider this in the future. Then we will show that the solution of system (1) must pass through the plane $z = 0$ infinite times unless they tend to the invariant manifold $\{(x, y, z) \in \mathbb{R}^3 | x = 0, y = 0\}$. Especially, every invariant set of this oscillator must have intersection with the plane $z = 0$. Moreover, we will also prove that if an orbit of system (1) is quasiperiodic, it must pass through at least five regions of $\mathbb{R}^3$, which conforms with the numerical simulation results in [6].

2. Main results. Throughout this paper, we denote $\{\phi(t, p_0) = (x(t, p_0), y(t, p_0), z(t, p_0)) | t \in \mathbb{R}\}$ as the solution of system (1) with the initial condition $p_0 = (x_0, y_0, z_0)$ at $t = 0$.

For the discussions that follow, let us denote the eight quadrants of $\mathbb{R}^3$ as follows

\begin{align*}
A_1 &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y > 0, z > 0\}, \\
A_2 &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y > 0, z < 0\}, \\
B_1 &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y > 0, z > 0\}, \\
B_2 &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y > 0, z < 0\}, \\
C_1 &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y < 0, z > 0\}, \\
C_2 &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y < 0, z < 0\}, \\
D_1 &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y < 0, z > 0\}, \\
D_2 &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y < 0, z < 0\}.
\end{align*}

Meanwhile, let

\begin{align*}
\mathcal{L} &= \{(x, y, z) \in \mathbb{R}^3 | x = 0, y = 0\}, \\
\mathcal{S}_1 &= \{(x, y, z) \in \mathbb{R}^3 | y = \sqrt{1 + \epsilon \tanh(x)}\}, \\
\mathcal{S}_2 &= \{(x, y, z) \in \mathbb{R}^3 | y = -\sqrt{1 + \epsilon \tanh(x)}\}, \\
\mathcal{Z}^+ &= \{(x, y, z) \in \mathbb{R}^3 | z > 0\}, \\
\mathcal{Z}^- &= \{(x, y, z) \in \mathbb{R}^3 | z < 0\}.
\end{align*}

and

\begin{align*}
\mathcal{I} &= \{(x, y, z) \in \mathbb{R}^3 | -\sqrt{1 + \epsilon \tanh(x)} < y < \sqrt{1 + \epsilon \tanh(x)}\}, \\
\mathcal{E} &= \{(x, y, z) \in \mathbb{R}^3 | y > \sqrt{1 + \epsilon \tanh(x)} \text{ or } y < -\sqrt{1 + \epsilon \tanh(x)}\}.
\end{align*}

See Fig.1 and Fig.2.

From system (1), we can easily obtain the following facts:
1. For every $\epsilon \in (0, 1], \alpha \in (0, \frac{1}{1+\epsilon})$, $L$ is a one-dimensional invariant manifold and the solution obtained with the initial condition of $p_0 \in L$ can be formulated as

$$\phi(t, p_0) : x(t) = 0, y(t) = 0, z(t) = z_0 - \alpha t.$$ 

2. For each $(x(t), y(t), z(t)) \in I$, we have

$$\dot{z}(t) = \alpha(y^2(t) - 1 - \epsilon \tanh(x(t))) < 0.$$
Similarly, for each \((x(t), y(t), z(t)) \in E\), we have \(\dot{z}(t) > 0\). And \(\dot{z}(t) = 0\) for each \((x(t), y(t), z(t)) \in S_1\) or \((x(t), y(t), z(t)) \in S_2\).

The behavior of the solution of system (1) can be described in the following proposition.

**Proposition 1.** The qualitative behavior of the solution of system (1) is as follows:

1. The solution with initial condition in \(A_1\) will tend to \(L\) or pass \(A_2\), \(B_1\) or \(B_2\).
2. The solution with initial condition in \(A_2\) will tend to \(L\) or pass either \(A_1\) or \(D_1\).
3. The solution with initial condition in \(B_1\) will tend to \(L\) or pass \(A_1\), \(A_2\) or \(B_2\).
4. The solution with initial condition in \(B_2\) will tend to \(L\) or pass \(A_1\), \(A_2\) or \(B_1\).
5. The solution with initial condition in \(C_1\) will tend to \(L\) or pass \(B_1\), \(B_2\) or \(C_2\).
6. The solution with initial condition in \(C_2\) will tend to \(L\) or pass either \(B_2\) or \(C_1\).
7. The solution with initial condition in \(D_1\) will tend to \(L\) or pass \(C_1\), \(C_2\) or \(D_2\).
8. The solution with initial condition in \(D_2\) will tend to \(L\) or pass \(C_1\), \(C_2\) or \(D_1\).

For clarity, the above behavior except that tend to \(L\) can be given in the diagram showing in Fig.3:

**Figure 3.** \(A_1 \rightarrow A_2\) means there are solutions from \(A_1\) to \(A_2\), \(B_1 \rightarrow A_2\) means there are solutions from \(B_1\) to \(A_2\) and these solutions have intersection with \(X\)-axis or \(Y\)-axis.

From the above proposition, we give the following theorems.

**Theorem 2.1.** If a solution of system (1) will not tend to the invariant manifold \(L\), it must pass through the plane \(z = 0\) infinite times. Especially, every invariant set of system (1) must have intersection with the plane \(z = 0\).

**Theorem 2.2.** Let

\[ S = \{A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2\}. \]

Then if an orbit of system (1) is quasiperiodic, it must pass through at least five regions of \(S\) provided it has no intersection with \(X\)-axis or \(Y\)-axis; Otherwise, it must pass through at least four regions of \(S\).

The proof of Proposition 1 will be given in section 3, the proof of Theorem 2.1 will be given in section 4 and the proof of Theorem 2.2 will be given in section 5.
3. Proof of Proposition 1. Firstly, we consider the behavior of solutions with initial conditions on $S_1$ and $S_2$.

Let us denote the intersection of the surface $x = -yz$ with $S_1$ as $P_1 = (x_1, y_1, z)$, $z \in \mathbb{R}$, and the intersection of the surface $x = -yz$ with $S_2$ as $P_2 = (x_2, y_2, z)$, $z \in \mathbb{R}$.

Lemma 3.1. The qualitative behavior of the solution with initial condition on $S_1$ and $S_2$ is as follows:

1. For each $p_0 = (x_0, y_0, z_0) \in S_1$, $\phi(t, p_0)$ passes $E$ for $t > 0$ provided that $x_0 < x_1$; $\phi(t, p_0)$ passes $I$ for $t > 0$ provided that $x_0 \geq x_1$.

2. For each $p_0 = (x_0, y_0, z_0) \in S_2$, $\phi(t, p_0)$ passes $E$ for $t > 0$ provided that $x_0 > x_2$; $\phi(t, p_0)$ passes $I$ for $t > 0$ provided that $x_0 \leq x_2$.

Proof. 1. On $S_1$, system (1) becomes

$$
\begin{aligned}
\dot{x} &= \sqrt{1 + \epsilon \tanh(x)} \\
\dot{y} &= -x - \sqrt{1 + \epsilon \tanh(x)} \cdot z \\
\dot{z} &= 0.
\end{aligned}
$$

It is obvious that $\dot{x} > 0$ and

$$
\begin{aligned}
\dot{y} &> 0, \text{ for } x < -yz = -\sqrt{1 + \epsilon \tanh(x)} \cdot z, \\
\dot{y} &< 0, \text{ for } x > -yz = -\sqrt{1 + \epsilon \tanh(x)} \cdot z.
\end{aligned}
$$

Then $\phi(t, p_0)$ will pass $E$ for $t > 0$ if $x_0 < x_1$. And $\phi(t, p_0)$ will pass $I$ for $t > 0$ if $x_0 > x_1$. When $x_0 = x_1$, then $p_0 = P_1$, we have

$$
\dot{x} > 0 \text{ and } \dot{y} = 0.
$$

Thus $\phi(t, p_0)$ will pass $I$ since $y = \sqrt{1 + \epsilon \tanh(x)}$ is monotonically increasing with respect to $x$.

2. On $S_2$, system (1) becomes

$$
\begin{aligned}
\dot{x} &= -\sqrt{1 + \epsilon \tanh(x)} \\
\dot{y} &= -x + \sqrt{1 + \epsilon \tanh(x)} \cdot z \\
\dot{z} &= 0.
\end{aligned}
$$

The argument is analogous to that in (1) in this lemma. It is straightforward so is omitted.

Next we will show the behavior of the solution with initial condition on planes $y = 0$ and $x = 0$. For simplicity, let

$$
\begin{aligned}
a_1 &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y = 0, z > 0\}, \\
a_2 &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y = 0, z > 0\}, \\
a_3 &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y = 0, z < 0\}, \\
a_4 &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y = 0, z < 0\}, \\
l_{10} &= \{(x, y, z) \in \mathbb{R}^3 | x > 0, y = 0, z = 0\}, \\
l_{20} &= \{(x, y, z) \in \mathbb{R}^3 | x < 0, y = 0, z = 0\}, \\
b_1 &= \{(x, y, z) \in \mathbb{R}^3 | x = 0, y > 0, z > 0\}, \\
b_2 &= \{(x, y, z) \in \mathbb{R}^3 | x = 0, y < 0, z > 0\}, \\
b_3 &= \{(x, y, z) \in \mathbb{R}^3 | x = 0, y < 0, z < 0\}, \\
b_4 &= \{(x, y, z) \in \mathbb{R}^3 | x = 0, y > 0, z < 0\}.
\end{aligned}
$$
Especially, we have $P_1 = (0, 1, 0)$ and $P_2 = (0, -1, 0)$ when $z = 0$. Thus, let us denote

$$l_{01} = \{(x, y, z) \in \mathbb{R}^3|x = 0, y \geq 1, z = 0\},$$

$$l_{02} = \{(x, y, z) \in \mathbb{R}^3|x = 0, 0 < y < 1, z = 0\},$$

$$l_{03} = \{(x, y, z) \in \mathbb{R}^3|x = 0, -1 \leq y < 0, z = 0\},$$

$$l_{04} = \{(x, y, z) \in \mathbb{R}^3|x = 0, y < -1, z = 0\}.$$

As in Fig.4 and Fig.5.

**Figure 4.** From right to left are $l_{10}$ and $l_{20}$.

**Figure 5.** From right to left are $l_{01}, l_{02}, l_{03}$ and $l_{04}$.
Lemma 3.2. For each $p_0 = (x_0, y_0, z_0)$ on the plane $y = 0$, we can obtain the following statements:

1. $\phi(t, p_0)$ passes $D_1$ or $D_2$ for $t > 0$ provided that $p_0 \in a_1$ or $p_0 \in a_4$ respectively.
2. $\phi(t, p_0)$ passes $B_1$ or $B_2$ for $t > 0$ provided that $p_0 \in a_2$ or $p_0 \in a_3$ respectively.
3. $\phi(t, p_0)$ passes $D_2$ for $t > 0$ provided that $p_0 \in l_{10}$ and passes $B_2$ for $t > 0$ provided that $p_0 \in l_{20}$.

Proof. On the plane $y = 0$, system (1) becomes

$$
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= -x \\
\dot{z} &= \alpha(-1 - \epsilon \tanh(x)).
\end{align*}
$$

Thus $\dot{z} < 0$ and

$$
\begin{align*}
\dot{y} &> 0, \text{ for } x < 0, \\
\dot{y} &< 0, \text{ for } x > 0.
\end{align*}
$$

And then it is easy to get this lemma. \qed

Lemma 3.3. For each $p_0 = (x_0, y_0, z_0)$ on the plane $x = 0$, we can obtain the following statements:

1. $\phi(t, p_0)$ passes $A_1$ or $A_2$ for $t > 0$ provided that $p_0 \in b_1$ or $p_0 \in b_4$ respectively.
2. $\phi(t, p_0)$ passes $C_1$ or $C_2$ for $t > 0$ provided that $p_0 \in b_2$ or $p_0 \in b_3$ respectively.
3. $\phi(t, p_0)$ passes $A_1$ or $C_1$ for $t > 0$ provided that $p_0 \in l_{01}$ or $p_0 \in l_{04}$ respectively.
4. $\phi(t, p_0)$ passes $A_2$ or $C_2$ for $t > 0$ provided that $p_0 \in l_{02}$ or $p_0 \in l_{03}$ respectively.

Proof. On the plane $x = 0$, system (1) becomes

$$
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -yz \\
\dot{z} &= \alpha(y^2 - 1).
\end{align*}
$$

Thus we have

$$
\begin{align*}
\dot{x} &> 0, \text{ for } y > 0, \\
\dot{x} &< 0, \text{ for } y < 0,
\end{align*}
$$

and

$$
\begin{align*}
\dot{y} &> 0, \text{ for } y \cdot z < 0, \\
\dot{y} &< 0, \text{ for } y \cdot z > 0,
\end{align*}
$$

meanwhile,

$$
\begin{align*}
\dot{z} &> 0, \text{ for } \|y\| > 1, \\
\dot{z} &< 0, \text{ for } \|y\| < 1.
\end{align*}
$$

Then this lemma can be easily proved. \qed

Next we will show the qualitative behavior of the solution with initial condition in the eight quadrants $A_i$, $B_i$, $C_i$ and $D_i$, $i = 1, 2$, which are mentioned in section 2. Firstly, we consider $A_1$. 
Lemma 3.4. For each \( p_0 = (x_0, y_0, z_0) \in A_1 \), \( \phi(t, p_0) \) will tend to \( L \) or pass \( A_2, D_1 \) or \( D_2 \).

Proof. We have known the property of the solution with initial condition on \( A_1 \cap S_1 \) by Lemma 3.1. Then to prove Lemma 3.4, we need only show the qualitative behavior of solutions of system (1) in \( A_1 \cap E \) and \( A_1 \cap I \) in the following two claims.

Claim 1. For each \( p_0 = (x_0, y_0, z_0) \in A_1 \cap E \), \( \phi(t, p_0) \) will pass \( A_1 \cap I \).

Now we give the proof of this claim. In \( A_1 \cap E \), we have
\[
\dot{x} = y > 0,
\]
so
\[
x(t, p_0) > x_0 > 0.
\]
Thus
\[
\dot{y} = -x - yz < -x < -x_0 < 0,
\]
and
\[
y(t, p_0) < -x_0t + y_0,
\]
which means \( \phi(t, p_0) \) will reach \( A_1 \cap S_1 \) in a finite time. And then \( \phi(t, p_0) \) will pass \( A_1 \cap I \) according to Lemma 3.1(1).

Claim 2. For each \( p_0 = (x_0, y_0, z_0) \in A_1 \cap I \), if \( \phi(t, p_0) \) will not tend to \( L \), it will pass \( A_2 \cap I \) or \( D_1 \cap I \) or \( D_2 \cap I \).

To prove this claim, it is easy to see that if \( x_0, y_0 \) are small enough or \( z_0 \) is small enough, \( \phi(t, p_0) \) will tend to the one-dimensional invariant manifold \( L \) or cross the plane \( z = 0 \) and pass \( A_2 \cap I \) by the continuous dependence on initial values.

Suppose \( \phi(t, p_0) \) will neither tend to \( L \) nor pass \( A_2 \cap I \), then as the proof of claim 1 in this lemma, we have
\[
\dot{y} < -x_0 < 0,
\]
which means \( \phi(t, p_0) \) will pass \( D_1 \cap I \) due to Lemma 3.3.

Especially, if there is \( t_1^* > 0 \), such that
\[
z(t_1^*, p_0) = 0,
\]
and
\[
y(t_1^*, p_0) = 0,
\]
that is \( \phi(t_1^*, p_0) \in l_{10} \), which is mentioned in Lemma 3.2. Then \( \phi(t, p_0) \) passes \( D_2 \cap I \) for \( t > t_1^* \). The proof is completed.

For the discussions that follow, let
\[
H_1 = \{(x, y, z) \in \mathbb{R}^3 | x < -yz\},
\]
\[
H_2 = \{(x, y, z) \in \mathbb{R}^3 | x > -yz\}.
\]

Lemma 3.5. For each \( p_0 = (x_0, y_0, z_0) \in A_2 \), \( \phi(t, p_0) \) will tend to \( L \) or pass either \( A_1 \) or \( D_2 \).

Proof. Firstly, for each \( X = (x, y, z) \) on the surface \( x = -yz \), the vector field of system (1) at this point is
\[
H(X) = (y, -x - yz, \alpha(y^2 - 1 - \epsilon \tanh(x))) = (y, 0, \alpha(y^2 - 1 - \epsilon \tanh(x))).
\]

(2)
At the same time, the normal vector of the surface $x = -yz$ at $X$ is
\[
\mathbf{n} = (-1, -z, -y).
\] (3)

Then we have
\[
H(X) \cdot \mathbf{n} = -y[1 + \alpha(y^2 - 1 - \epsilon \tanh(x))].
\] (4)

For $\epsilon \in (0, 1]$ and $\alpha \in (0, \frac{1}{1+\epsilon})$, we have
\[
H(X) \cdot \mathbf{n} < 0,
\] (5)

which means that the solution with initial condition on the surface $x = -yz$ will pass $A_2 \cap H_2$.

On the other hand, according to Lemma 3.1, we have known the property of the solution with initial condition on $A_2 \cap S_1$. Thus it suffices to give the property of the solution with initial condition in $A_2 \cap E \cap H_2$ and $A_2 \cap I \cap H_2$, $i = 1, 2$.

**Claim 1.** For each $p_0 \in A_2 \cap E \cap H_2$, $\phi(t, p_0)$ will pass either $A_1 \cap E$ or $A_2 \cap I \cap H_2$.

In $A_2 \cap E \cap H_2$, we have
\[
\begin{cases}
\dot{x} = y > 0 \\
\dot{y} = -x - yz < 0 \\
\dot{z} = \alpha(y^2 - 1 - \epsilon \tanh(x)) > 0.
\end{cases}
\] (6)

By the continuous dependence on initial values, $\phi(t, p_0)$ will pass $A_1 \cap E$ provided that the absolute value of $z_0$ is sufficiently small.

If $\phi(t, p_0)$ will not pass $A_1 \cap E$, by (6), it is obvious that
\[
-x(t, p_0) < -x_0 < 0,
\]
\[
y_0 > y(t, p_0) > 0,
\]

and
\[
-z_0 > -z(t, p_0) > 0.
\]

Therefore
\[
\dot{y} = -x - yz < -x_0 - y_0z_0 < 0.
\]

Furthermore, according to Lemma 3.1, it follows that $\phi(t, p_0)$ will cross $A_2 \cap S_1$ and pass $A_2 \cap I \cap H_2$.

**Claim 2.** For each $p_0 \in A_2 \cap I \cap H_2$, $\phi(t, p_0)$ will either pass $D_2 \cap I$ or tend to $L$.

In $A_2 \cap I \cap H_2$, we have
\[
\begin{cases}
\dot{x} = y > 0 \\
\dot{y} = -x - yz < 0 \\
\dot{z} = \alpha(y^2 - 1 - \epsilon \tanh(x)) < 0.
\end{cases}
\]

It can easily be seen that there exists $p_0 \in A_2 \cap I \cap H_2$ such that $x_0$ and $y_0$ are sufficiently small. Then thanks to the continuous dependence on initial values, $\phi(t, p_0)$ will tend to the invariant manifold $L$.

Assume $\phi(t, p_0)$ will not tend to $L$, it will cross the plane $y = 0$ and pass $D_2 \cap I$.

Follow the idea in [1], to prove this, denote
\[
\theta = \arctan \frac{y}{x},
\] (7)

we have
\[
\dot{\theta} = \frac{-x(x + yz) - y^2}{x^2 + y^2},
\] (8)
by taking derivative to the both sides of the equality with respect to $t$. Firstly, we show that $\phi(t, p_0)$ will close the plane $y = 0$ as much as possible as $t$ increasing.

Proof by contradiction, suppose there is a $p_0 \in A_2 \cap I \cap H_2$, such that

$$\theta(t, p_0) \to \bar{\theta} > 0, \text{ as } t \to +\infty,$$

because of $\dot{\theta} < 0$ in this region. Then from (8), we obtain

$$\dot{\theta} < \frac{-y^2}{x^2 + y^2} = -\sin^2 \theta,$$

since in $A_2 \cap I \cap H_2$, $-x(x + yz) < 0$. By integration of the inequality (9) with respect to $t$, it follows that

$$T_1 < \cot \theta_1 - \cot \theta_0 + T_0,$$

where $T_1$ is the time for $\phi(t, p_0)$ using from $\theta_0$ to $\theta_1$ and $0 < \theta_1 < \theta_0 < \frac{\pi}{2}$. Then $\phi(t, p_0)$ will reach $\theta = \bar{\theta}$ in a finite time since $T_1$ is bounded, which is contrary to $\bar{\theta}$ is a limited angle and $\dot{\theta} < 0$.

Secondly, the solutions which are near the plane $y = 0$ enough will cross it by the continuous dependence on initial values and Lemma 3.2. We get this claim.

**Claim 3.** For each $p_0 \in A_2 \cap E \cap H_1$, $\phi(t, p_0)$ will pass either $A_1 \cap E$ or $A_2 \cap E \cap H_2$.

In region $A_2 \cap E \cap H_1$, due to the continuous dependence on initial values and the analysis of (5) in this lemma, $\phi(t, p_0)$ will pass $A_2 \cap E \cap H_2$ if $p_0$ close the surface $x = -yz$ enough.

If $\phi(t, p_0)$ will not pass $A_2 \cap E \cap H_2$, it will cross the plane $z = 0$ and pass $A_1 \cap E$.

In fact, suppose this is not true, then according to the analyses of (5) in Lemma 3.5, Lemma 3.1(1) and the vector field on the plane of $x = 0$ in Lemma 3.3, it follows that $\phi(t, p_0)$ will be in $A_2 \cap E \cap H_1$ forever. Consequently, we have

$$\dot{x} = y > y_0 > 0 \text{ and } \dot{z} > 0,$$

which imply that

$$x(t, p_0) \to +\infty, \text{ as } t \to +\infty,$$

and

$$\|z(t, p_0)\| < \|z_0\|, \text{ for } t > 0.$$

Moreover, since $\phi(t, p_0) \in H_1$ for $t \in \mathbb{R}$, we obtain

$$y(t, p_0) \to +\infty, \text{ as } t \to +\infty.$$

Then for a fixed $\epsilon_1 > 0$, there is $t_* > 0$, such that

$$y^2(t_*, p_0) > 1 + \epsilon + \epsilon_1.$$

Furthermore, in region $A_2 \cap E \cap H_1$, we have $\dot{y} > 0$, therefore for $t > t_*$,

$$\dot{z} = \alpha(y^2 - 1 - \epsilon \tanh(x)) > \alpha \epsilon_1 > 0,$$

$$z(t, p_0) > \alpha \epsilon_1 t + z_0,$$

which means $\phi(t, p_0)$ will cross the plane $z = 0$ in a finite time. This is contrary to our supposition.

**Claim 4.** For each $p_0 \in A_2 \cap I \cap H_1$, if $\phi(t, p_0)$ will not tend to $L$, it will pass either $A_2 \cap I \cap H_2$ or $A_2 \cap E \cap H_1$.

In $A_2 \cap I \cap H_1$, by the fact (1) in section 2 and (5) in Lemma 3.5, $\phi(t, p_0)$ will tend to $L$ or $A_2 \cap I \cap H_2$ provided that $x_0$ and $y_0$ are small enough or $p_0$ close the surface $x = -yz$ enough respectively because of the continuous dependence on initial values.
Assume \( \phi(t, p_0) \) will not pass \( A_2 \cap E \cap H_1 \). Since in region \( A_2 \cap I \cap H_1 \), we have \( \dot{y} > 0 \), and

\[
0 < y < \sqrt{1 + \epsilon \tanh(x)},
\]

there exists \( 0 < c < \sqrt{1 + \epsilon \tanh(x)} \), such that

\[
\lim_{t \to +\infty} y(t, p_0) = c.
\]

Then for any \( \mu > 0 \), there exists \( t_\mu > 0 \), such that for any \( t > t_\mu \), we obtain

\[
c - \mu < y(t, p_0) < c + \mu.
\]

Thus for \( t > t_\mu \),

\[
\dot{x} = y > c - \mu,
\]

\[
\dot{z} = \alpha(y^2 - 1 - \epsilon \tanh(x)) > \alpha[(c - \mu)^2 - 1 - \epsilon],
\]

which means

\[
x(t, p_0) > (c - \mu)t + x_0,
\]

and

\[
z(t, p_0) > \alpha[(c - \mu)^2 - 1 - \epsilon]t + z_0.
\]

Therefore

\[
-\frac{x(t, p_0)}{y(t, p_0)} < -\frac{(c - \mu)t + x_0}{c + \mu} = -\frac{c - \mu}{c + \mu}t - \frac{x_0}{c + \mu},
\]

and

\[
z(t, p_0) > -\alpha[1 + \epsilon - (c - \mu)^2]t + z_0.
\]

Then as \( \mu \to 0 \), we get

\[
\lim_{\mu \to 0^+} \frac{c - \mu}{c + \mu} = 1, \quad \lim_{\mu \to 0^+} \alpha[1 + \epsilon - (c - \mu)^2] = \alpha(1 + \epsilon - c^2).
\]

Note that in this case, we have

\[
0 < \alpha < \frac{1}{1 + \epsilon},
\]

thus since \( 0 < c < \sqrt{1 + \epsilon \tanh(x)} \),

\[
0 < \alpha(1 + \epsilon - c^2) < 1,
\]

which implies

\[
z(t, p_0) > -\frac{x(t, p_0)}{y(t, p_0)}.
\]

This is contrary to our assumption. We complete the proof of Lemma 3.5. \( \square \)

**Lemma 3.6.** For each \( p_0 \in D_1 \), \( \phi(t, p_0) \) will tend to \( L \) or pass \( C_1, C_2 \) or \( D_2 \).

**Proof.** For the parameter scope we study in this paper, in \( D_1 \), from (4), we have

\[
H(X) \cdot n > 0,
\]

which means that the solution with initial condition on the surface \( x = -yz \) will pass \( D_1 \cap H_1 \). Moreover, from Lemma 3.1, we have known the property of the solution with initial condition on \( D_1 \cap S_2 \). Thus to prove this Lemma, it is sufficient to show the property of the solution with initial condition in \( D_1 \cap E \cap H_i \) and \( D_1 \cap I \cap H_i, i = 1, 2 \).

**Claim 1.** For each \( p_0 \in D_1 \cap I \cap H_2 \), if \( \phi(t, p_0) \) will not tend to \( L \), it will pass \( D_2 \cap I, D_1 \cap I \cap H_1 \) or \( D_1 \cap E \cap H_2 \).
In $D_1 \cap I \cap H_2$, by the continuous dependence on initial values, $\phi(t, p_0)$ will pass $D_2 \cap I$ provided that $z_0$ is small enough since $\dot{z} < 0$. And $\phi(t, p_0)$ will also pass $D_1 \cap I \cap H_1$ or tend to $L$ provided that $p_0$ close the surface $x = -yz$ enough or $x_0$ and the absolute value of $y_0$ are sufficiently small since (10) in this lemma and the fact (1) in section 2 respectively.

Suppose $\phi(t, p_0)$ will not do any of the three dynamical behavior above, it will cross $D_1 \cap S_2$ and pass $D_1 \cap E \cap H_2$. Assume this statement would not hold, then in $D_1 \cap I \cap H_2$, since $-\sqrt{1 + \epsilon \tanh(x)} < y < 0$ and $\dot{y} < 0$, there exists $\epsilon_2 > 0$ such that

$$y(t, p_0) \to -\epsilon_2, \text{ as } t \to +\infty,$$

and

$$-\sqrt{1 + \epsilon \tanh(x)} < -\epsilon_2 < y_0.$$ Then

$$\dot{x} = y \to -\epsilon_2, \text{ as } t \to +\infty,$$

$$x(t, p_0) \to -\infty, \text{ as } t \to +\infty,$$

which means $\phi(t, p_0)$ will go to the surface $x = -yz$ in $D_1 \cap I$ and cross it. This contradicts our hypothesis.

**Claim 2.** For each $p_0 \in D_1 \cap E \cap H_2$, $\phi(t, p_0)$ will pass $D_1 \cap E \cap H_1$.

In $D_1 \cap E \cap H_2$, since $\dot{y} < 0$, we have

$$y(t, p_0) < y_0 < 0.$$ Then

$$\dot{x} = y < y_0 < 0,$$

which means

$$x(t, p_0) < y_0t + x_0,$$

thus there is a finite time $t_2^*$ such that $\phi(t, p_0)$ will cross the surface $x = -yz$ in $D_1 \cap E$ at $t_2^*$ and pass $D_1 \cap E \cap H_1$ by (10) in this lemma.

**Claim 3.** For each $p_0 \in D_1 \cap E \cap H_1$, $\phi(t, p_0)$ will pass either $C_1 \cap E$ or $D_1 \cap I \cap H_1$.

In $D_1 \cap E \cap H_1$, if $x_0$ is sufficiently small, $\phi(t, p_0)$ will cross the plane $x = 0$ and pass $C_1 \cap E$ by Lemma 3.3 and the continuous dependence on initial values.

To prove this claim, suppose $\phi(t, p_0)$ will pass neither $C_1 \cap E$ nor $D_1 \cap I \cap H_1$. Then since $\dot{y} > 0$, we have

$$y_0 < y(t, p_0) < -\sqrt{1 + \epsilon \tanh(x)}.$$ There exists $\epsilon_3 > 0$, such that

$$y(t, p_0) \to -\epsilon_3, \text{ as } t \to +\infty,$$

and

$$-\epsilon_3 < -\sqrt{1 + \epsilon \tanh(x)}.$$ Therefore

$$\dot{x} = y \to -\epsilon_3, \text{ as } t \to +\infty,$$

which means

$$x(t, p_0) \to -\infty, \text{ as } t \to +\infty.$$
Thus $\phi(t, p_0)$ will cross the plane $x = 0$, which is contrary to our supposition.

**Claim 4.** For each $p_0 \in D_1 \cap I \cap H_1$, if $\phi(t, p_0)$ will not pass $C_1 \cap I$, $C_2 \cap I$ or $D_2 \cap I$, it will tend to $L$.

In $D_1 \cap I \cap H_1$, by the continuous dependence on initial values, $\phi(t, p_0)$ will pass $D_2 \cap I$ or tend to $L$ provided that $z_0$ is small enough or $x_0$ and $|y_0|$ are sufficiently small because of $\dot{z} < 0$ or the fact that $L$ is one invariant manifold respectively.

If $\phi(t, p_0)$ will neither tend to $L$ or pass $D_2 \cap I$, to prove our claim, assume $\phi(t, p_0)$ will not pass the plane $x = 0$. Then give the similar proof with that in claim 3 of this lemma, we obtain

$$x(t, p_0) \to -\infty, \text{ as } t \to +\infty.$$  

Then $\phi(t, p_0)$ will cross the plane $x = 0$ and pass $C_1 \cap I$ by Lemma 3.3. This contradicts our assumption. Especially, if there exists a $t^*_3 > 0$, such that

$$x(t^*_3, p_0) = 0 \text{ and } z(t^*_3, p_0) = 0,$$

$\phi(t, p_0)$ will pass $C_2 \cap I$ according to Lemma 3.3.

This completes the proof of Lemma 3.6.

**Lemma 3.7.** For each $p_0 \in D_2$, $\phi(t, p_0)$ will tend to $L$ or pass $C_1$, $C_2$ or $D_1$.

**Proof.** We have known the property of the solution with initial condition on $D_2 \cap S^2$ by Lemma 3.1. Then to prove Lemma 3.7, it is enough to give the behavior of solutions of system (1) in region $D_2 \cap E$ and $D_2 \cap I$.

**Claim 1.** For each $p_0 \in D_2 \cap I$, if $\phi(t, p_0)$ will not tend to $L$, it will pass either $C_2 \cap I$ or $D_2 \cap E$.

In $D_2 \cap I$, we have

$$\begin{cases} 
\dot{x} < 0 \\
\dot{y} < 0 \\
\dot{z} < 0.
\end{cases} \quad (11)$$

Due to the continuous dependence on initial values, $\phi(t, p_0)$ will tend to $L$ or pass $C_2 \cap I$ if $x_0$ and $|y_0|$ are sufficiently small or $x_0$ is small enough according to the fact (1) in section 2 or Lemma 3.3 respectively.

Suppose $\phi(t, p_0)$ will not tend to $L$ or pass $C_2 \cap I$, from (11), we have

$$-x_0 < -x(t, p_0) < 0,$$

$$y(t, p_0) < y_0 < 0,$$

$$z(t, p_0) < z_0 < 0.$$

Then

$$\dot{y} = -x - yz < -yz < -y_0z_0 < 0,$$

in addition, according to Lemma 3.1, it follows that $\phi(t, p_0)$ will cross $D_2 \cap S^2$ and pass $D_2 \cap E$.

**Claim 2.** For each $p_0 \in D_2 \cap E$, $\phi(t, p_0)$ will pass $C_1 \cap E$, $C_2 \cap E$ or $D_1 \cap E$.

In $D_2 \cap E$, thanks to the continuous dependence on initial values, $\phi(t, p_0)$ will pass $D_1 \cap E$ if $|z_0|$ is sufficiently small because of $\dot{z} > 0$. Especially, if there is a $t^*_1 > 0$ such that

$$z(t^*_1, p_0) = 0 \text{ and } x(t^*_1, p_0) = 0,$$

$\phi(t, p_0)$ will pass $C_1 \cap E$ by Lemma 3.3.
If \( \phi(t, p_0) \) will pass neither \( C_1 \cap E \) nor \( D_1 \cap E \), to prove this claim, suppose that
\( \phi(t, p_0) \) will not pass \( C_2 \cap E \). Since \( \dot{y} < 0 \), we obtain
\[
y(t, p_0) < y_0 < 0.
\]
Then
\[
\dot{x} = y < y_0 < 0,
\]
moreover, according to Lemma 3.3, it follows that \( \phi(t, p_0) \) will cross the plane \( x = 0 \).
This contradicts our supposition.

This completes the proof of Lemma 3.7.

**Lemma 3.8.** For each \( p_0 \in C_2 \), \( \phi(t, p_0) \) will tend to \( L \) or pass either \( C_1 \) or \( B_2 \).

**Proof.** Firstly, in \( C_2 \), for \( \epsilon \in (0, 1] \) and \( \alpha \in (0, \frac{1}{1+\alpha}) \), from (5),
\[
H(X) \cdot n > 0,
\]
which means that the solution with initial condition on the surface \( x = -yz \) will pass \( C_2 \cap H_1 \). Moreover, according to Lemma 3.1, we have known the property of the solution with initial condition on \( C_2 \cap S_2 \). To prove Lemma 3.8, we need only to show the behavior of solutions of system (1) in \( C_2 \cap E \cap H_1 \) and \( C_2 \cap I \cap H_1 \), \( i = 1, 2 \).

**Claim 1.** For each \( p_0 \in C_2 \cap I \cap H_2 \), if \( \phi(t, p_0) \) will not tend to \( L \), it will pass either \( C_2 \cap I \cap H_3 \) or \( C_2 \cap E \cap H_2 \).

This claim can be proved in a similar way as shown in the proof of Claim 4 in Lemma 3.5, so we omit it.

**Claim 2.** For each \( p_0 \in C_2 \cap E \cap H_2 \), \( \phi(t, p_0) \) will pass either \( C_1 \cap E \) or \( C_2 \cap E \cap H_1 \).

In \( C_2 \cap E \cap H_2 \), we have
\[
\begin{cases}
    \dot{x} < 0 \\
    \dot{y} < 0 \\
    \dot{z} > 0.
\end{cases}
\]
Since \( \dot{z} > 0 \), \( \phi(t, p_0) \) will cross the plane of \( z = 0 \) and pass \( C_1 \cap E \) if \( |z_0| \) is small enough because of the continuous dependence on initial values.

To prove our claim, suppose that \( \phi(t, p_0) \) will pass neither \( C_1 \cap E \) nor \( C_2 \cap E \cap H_1 \), which means that \( \phi(t, p_0) \) will not cross the surface \( x = -yz \) in \( C_2 \cap E \). Then the reminder proof of this claim can be proved by the same method as employed in the proof of Claim 3 in Lemma 3.5, so we omit it.

**Claim 3.** For each \( p_0 \in C_2 \cap E \cap H_1 \), \( \phi(t, p_0) \) will pass either \( C_1 \cap E \) or \( C_2 \cap I \cap H_1 \).

The proof of this claim can be completed by the method analogous to that used in the proof of claim 1 in Lemma 3.5, so we omit it.

**Claim 4.** For each \( p_0 \in C_2 \cap I \cap H_1 \), \( \phi(t, p_0) \) will tend to \( L \) or pass \( B_2 \cap I \).

By similar argument as in the proof of Claim 2 in Lemma 3.5, we can easily carry out the proof of this claim, so we omit it.

This completes the proof of Lemma 3.8.

**Lemma 3.9.** For each \( p_0 \in C_1 \), \( \phi(t, p_0) \) will tend to \( L \) or pass \( B_1 \), \( B_2 \) or \( C_2 \).

**Proof.** Like the beginning of the proof of Lemma 3.7, here we need only to show the qualitative behavior of the solution with initial condition in \( C_1 \cap E \) and \( C_1 \cap I \).

**Claim 1.** For each \( p_0 \in C_1 \cap I \), if \( \phi(t, p_0) \) will not tend to \( L \), it will pass \( B_1 \cap I \), \( B_2 \cap I \) or \( C_2 \cap I \).
The proof of this claim can be completed by the method analogous to that used in the proof of Claim 2 in Lemma 3.4, so we omit it.

**Claim 2.** For each $p_0 \in C_1 \cap E$, $\phi(t, p_0)$ will pass $C_1 \cap I$.

In $C_1 \cap E$, we have

$$-xyz < 0.$$ 

And from (8) in Lemma 3.5, it follows that

$$\dot{\theta} = -1 - \frac{xyz}{x^2 + y^2} < -1,$$

which means $\phi(t, p_0)$ will rotate a angle of $\frac{\pi}{2}$ in a finite time. Thus $\phi(t, p_0)$ will pass $C_1 \cap I$ by the result of $S_2 \cap C_1$ in Lemma 3.1.

This completes the proof of Lemma 3.9. $\square$

**Lemma 3.10.** For each $p_0 \in B_1$, $\phi(t, p_0)$ will tend to $L$ or pass $A_1$, $A_2$ or $B_2$.

**Proof.** For $\epsilon \in (0, 1]$ and $\alpha \in (0, \frac{1}{1-r^2})$, in $B_1$, from (5), we obtain

$$H(X) \cdot n < 0,$$

which means that the solution with initial condition on the surface $x = -yz$ will pass $B_1 \cap H_2$. Moreover, according to Lemma 3.1, we have known the property of the solution with initial condition on $B_1 \cap S_1$. Thus to prove Lemma 3.10, it suffices to show the property of solutions of system (1) in $B_1 \cap E \cap H_1$ and $B_1 \cap I \cap H_1$, $i = 1, 2$.

**Claim 1.** For each $p_0 \in B_1 \cap I \cap H_1$, if $\phi(t, p_0)$ will not tend to $L$, it will pass $B_2 \cap I$, $B_1 \cap I \cap H_2$ or $B_1 \cap E \cap H_1$.

**Claim 2.** For each $p_0 \in B_1 \cap E \cap H_1$, $\phi(t, p_0)$ will pass $B_1 \cap E \cap H_2$.

**Claim 3.** For each $p_0 \in B_1 \cap E \cap H_2$, $\phi(t, p_0)$ will pass either $A_1 \cap E$ or $B_1 \cap I \cap H_2$.

**Claim 4.** For each $p_0 \in B_1 \cap I \cap H_2$, if $\phi(t, p_0)$ will not tend to $L$, it will pass $A_1 \cap I$, $B_2 \cap I$ or $A_2 \cap I$.

The Claim $i$ ($i = 1, 2, 3, 4$) in this lemma can be proved in a similar way as shown in the proof of Claim $i$ ($i = 1, 2, 3, 4$) in Lemma 3.6 respectively, so we omit it.

This completes the proof of Lemma 3.10. $\square$

**Lemma 3.11.** For each $p_0 \in B_2$, $\phi(t, p_0)$ will tend to $L$ or pass $A_1$, $A_2$ or $B_1$.

**Proof.** From Lemma 3.1, we have known the dynamical behavior of the solution with initial condition on $B_2 \cap S_1$. Thus it is enough to show the behavior of the solution with initial condition on $B_2 \cap E$ and $B_2 \cap I$.

**Claim 1.** For each $p_0 \in B_2 \cap I$, if $\phi(t, p_0)$ will not tend to $L$, it will pass either $A_2 \cap I$ or $B_2 \cap E$.

**Claim 2.** For each $p_0 \in B_2 \cap E$, $\phi(t, p_0)$ will pass $A_1 \cap E$, $B_1 \cap E$ or $A_2 \cap E$.

Using the similar argument as in the proof of claim $i$ ($i = 1, 2$) in Lemma 3.7, we can easily carry out the proof of claim $i$ ($i = 1, 2$) in this lemma respectively. So we omit it. And we get this lemma. $\square$

By the lemmas above, we complete the proof of Proposition 1.
4. **Proof of Theorem 2.1.** When $\alpha = 0$, system (1) becomes

\[
\begin{cases}
\dot{x} = y \\
\dot{y} = -x - yz \\
\dot{z} = 0.
\end{cases}
\]  

(13)

Firstly, consider system (13) on a fixed plane $z = -\rho < 0$, which is

\[
\begin{cases}
\dot{\bar{x}} = \bar{y} \\
\dot{\bar{y}} = -\bar{x} + \rho \bar{y},
\end{cases}
\]  

(14)

where $\rho > 0$. Let us denote

\[
X(t) = \begin{pmatrix} \bar{x}(t) \\ \bar{y}(t) \end{pmatrix}
\]

as the solution of system (14). Since the proof of Theorem 2.1 is quite similar to the proof of Theorem 2.1 in [1], here we just give the outline of the proof of Theorem 2.1 as follows.

**Lemma 4.1.** For system (14), we have the following statements:

1. When $\rho \geq 2$, the equilibrium point $(0, 0)$ is a source and the solution of system (14) with initial value $\bar{p}_0 = (\bar{x}_0, \bar{y}_0)$ satisfying $\|\bar{y}_0\| > \sqrt{1 + \epsilon \tanh(\bar{x}_0)}$ will stay in the region of $\|\bar{y}\| > \sqrt{1 + \epsilon \tanh(\bar{x})}$ forever after a finite time.

2. When $0 < \rho < 2$, the equilibrium point $(0, 0)$ is a spiral source. Let $T_\rho$ denote the time of the solution of system (14) rotating a angle of $2\pi$ for a fixed $\rho$, and $\Delta t$ denote the time of the solution of system (14) staying in the region of $\|\bar{y}\| < \sqrt{1 + \epsilon \tanh(\bar{x})}$ when it rotates a angle of $2\pi$ for a fixed $\rho$. Then $T_\rho$ decreases with the decrease of $\rho$. Moreover,

\[
\lim_{\rho \to 0^+} T_\rho = 2\pi,
\]

and $\Delta t \to 0$ as the initial value of the solution of system (14) is far away from $(0, 0)$.

For the discussion below, let us first define

\[V(x, y, z) = x^2 + y^2.\]

The derivative of $V$ along the trajectories of system (1) is

\[\dot{V} = 2x\dot{x} + 2y\dot{y} = -2y^2 z.\]

And denote $\tilde{L}$ as the set of solutions with initial conditions in $I \cap Z^-$ which tend to the invariant manifold $L$, then let

\[I_* = Z^- \setminus \tilde{L} \neq \emptyset.\]

We give the following lemmas.

**Lemma 4.2.** For each $p_0 \in I_*$, there exists a $\tau_1 > 0$ such that $\phi(\tau_1, p_0) \in Z^+$.

**Proof.** We divide our proof in three steps.

Firstly, we will show that $\phi(t, p_0)$ with $p_0 \in \{(x, y, z) \in \mathbb{R}^3 | z < -2\} \setminus I_*$ will go to the region $-2 < z < 0$.

The next thing to do in the proof is to certify that $\phi(t, p_0)$ with $p_0 \in \{(x, y, z) \in \mathbb{R}^3 | -2 < z < 0\}$ will not stay there forever.

Finally, we need to prove $\phi(t, p_0)$ with $p_0 \in I_*$ will not cross the plane $z = -2$ infinitely many times.
This completes the proof of Lemma 4.2.

**Lemma 4.3.** For each \( p_0 \in Z^+ \), there exists a \( \tau_2 > 0 \) such that \( \phi(\tau_2, p_0) \in Z^- \).

**Proof.** By contradiction, assume there is a solution \( \phi(t, p_0) \) staying in \( Z^+ \) forever. Due to

\[
\dot{V} = -2y^2z \leq 0,
\]

and note that

\[
\dot{V} = 0 \iff y = 0,
\]

\( \phi(t, p_0) \) will go into \( I \cap Z^+ \) because there are no invariant sets near the plane \( y = 0 \) in \( Z^+ \). Then according to the fact (2) in section 2, it follows that \( \phi(t, p_0) \) will pass \( Z^- \) in a finite time. This contradicts our assumption.

According to Lemma 4.2 and Lemma 4.3, it follows that if the solution of system (1) do not tend to the invariant manifold \( L \), they must pass through the plane \( z = 0 \). Thus it is easy to see that every invariant set of system (1) must have intersection with the plane \( z = 0 \). This completes the proof of Theorem 2.1.

5. **Proof of Theorem 2.2.** We divided our proof of Theorem 2.2 in three steps which are given in the following three lemmas. And note that \( S \) is defined in Theorem 2.2.

**Lemma 5.1.** There is no quasiperiodic orbit of system (1) staying in \( S_1 \cup S_2 \), where \( S_1 \in S \) and \( S_2 \in S \).

**Proof.** We have known that every invariant set of system (1) must cross the plane \( z = 0 \) by Theorem 2.1. Then according to the diagram in Proposition 1, if there is a quasiperiodic orbit staying in \( S_1 \cup S_2 \), \( S_1 \in S \) and \( S_2 \in S \), it follows that this quasiperiodic orbit can only stay in \( A_1 \cup A_2 \), \( B_1 \cup B_2 \), \( C_1 \cup C_2 \), or \( D_1 \cup D_2 \).

Proof by contradiction, suppose there is a quasiperiodic orbit \( \phi(t, p_0) \) staying in \( A_1 \cup A_2 \). Then there exists a \( \beta_1 > 0 \), such that for \( t > 0 \),

\[
y(t, p_0) > \beta_1 > 0.
\]

Then

\[
x = y > \beta_1 > 0,
\]

which implies

\[
x(t, p_0) \to +\infty, \text{ as } t \to +\infty.
\]

This contradicts the fact that \( \phi(t, p_0) \) is bounded. The proofs of other three cases are similar, so we omit it.

**Lemma 5.2.** There is no quasiperiodic orbit of system (1) staying in the union of three regions of \( S \).

**Proof.** From the diagram in Proposition 1, it is evident that this lemma holds.

**Lemma 5.3.** The quasiperiodic orbit of system (1) staying in the union of four regions of \( S \) must have intersection with \( X \)-axis or \( Y \)-axis.

**Proof.** According to Theorem 2.1, the quasiperiodic orbits of system (1) must cross the plane of \( z = 0 \). And then by the diagram in Proposition 1, it can be directly seen there is no quasiperiodic orbit staying in the union of four regions of \( S \) except that the behavior of this quasiperiodic orbit contains the dashed lines showing in the diagram of Proposition 1, which implies that the quasiperiodic orbit intersects with \( X \)-axis or \( Y \)-axis.
Consequently, if there is a quasiperiodic orbit in system (1), then it must pass through at least five regions of $S$ provided it does not intersect with $X$-axis or $Y$-axis, otherwise it must pass through four regions of $S$. This completes the proof of Theorem 2.2.

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