On a characterisation theorem for probability distributions on discrete Abelian groups

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Let $X$ be a countable discrete Abelian group containing no elements of order 2, $\alpha$ be an automorphism of $X$, $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. The main result of the article is the following statement. The symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ implies that $\mu_j$ are shifts of the Haar distribution of a finite subgroup of $X$ if and only if the automorphism $\alpha$ satisfies the condition $\text{Ker}(I + \alpha) = \{0\}$. This theorem is an analogue for discrete Abelian groups the well-known Heyde theorem where Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another. We also prove some generalisations of this theorem.

Key words and phrases: conditional distribution, Haar distribution, discrete Abelian group

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1. Introduction

According to Heyde’s theorem the Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form of $n$ independent random variables given another ([14], see also [15 §13.4.1]). Some generalisations of Heyde’s theorem, where independent random variables $\xi_j$ take values in a locally compact Abelian group and coefficients of linear forms are topological automorphisms of the group, were studied in [2–4], [7–11], [16–18], see also [5, Chapter VI] and [6, Chapter VI].

Let $X$ be a second countable locally compact Abelian group. We will consider only such groups, without mentioning it specifically. In particular, if $X$ is a discrete group, it means that $X$ is countable. Denote by $\text{Aut}(X)$ the group of topological automorphisms of a locally compact Abelian group $X$, and by $I$ the identity automorphism of a group. The following theorem was proved in [4], see also [6, Corollary 17.31].

**Theorem A.** Let $X$ be a discrete Abelian group containing no elements of order 2. Let $\alpha$ be an automorphism of the group $X$ satisfying the conditions

$$I \pm \alpha \in \text{Aut}(X).$$

Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j$ are shifts of the Haar distribution of a finite subgroup $K$ of the group $X$. Moreover, $\alpha(K) = K$.

It is well known that Gaussian distributions on discrete Abelian groups are degenerated ([19 Chapter IV]). Shifts of the Haar distributions of finite subgroups on such groups play the role of Gaussian distributions. For this reason we can consider Theorem A, as an analogue of Heyde’s theorem for discrete Abelian groups.

The main result of the article, Theorem 1, is the following. We prove that in fact Theorem A is valid under a condition which is significantly weaker then (1), namely

$$\text{Ker}(I + \alpha) = \{0\}.$$
Moreover, condition (2) can not be relaxed. We prove Theorem 1 in §2, and some generalisations of Theorem 1 in §3.

Recall some definitions and agree on notation. For an arbitrary locally compact Abelian group $X$ denote by $Y$ its character group, and by $(x, y)$ the value of a character $y \in Y$ at an element $x \in X$. If $K$ is a closed subgroup of $X$, denote by $A(Y, K) = \{y \in Y : (x, y) = 1 \text{ for all } x \in K\}$ its annihilator. Denote by $b_K$ the subgroup consisting of all compact elements of the group $X$, and by $c_X$ the connected component of zero of the group $X$. If $\alpha$ is a continuous endomorphism of the group $X$, then the adjoint continuous endomorphism $\tilde{\alpha}$ of the group $Y$ is defined as follows $(x, \tilde{\alpha}y) = (\alpha x, y)$ for all $x \in X$, $y \in Y$. Note that $\alpha \in \text{Aut}(X)$ if and only if $\tilde{\alpha} \in \text{Aut}(Y)$. Let $K$ be a closed subgroup of the group $X$ and $\alpha \in \text{Aut}(X)$. If $\alpha(K) = K$, then the restriction of $\alpha$ to $K$ is a topological automorphism of the group $K$. We denote it by $\alpha_K$. Let $p$ be a prime number. The $p$-torsion subgroup of an Abelian group $X$ is the set of all elements of $X$ that have order a power of $p$. Denote by $X_p$ the $p$-torsion subgroup of $X$. Let $n$ be an integer. Denote by $f_n : X \mapsto X$ an endomorphism of the group $X$, defined by the formula $f_n(x) = nx$, $x \in X$. Put $f_n(X) = X^{(n)}$, $\text{Ker}f_n = X^{(n)}$. Denote by $\mathbb{R}$ the group of real numbers. Let $f(y)$ be a function on the group $Y$, and let $h \in Y$. Denote by $\Delta_h$ the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y), \quad y \in Y.$$ 

A function $f(y)$ on $Y$ is called a polynomial if

$$\Delta^n_h f(y) = 0$$

for some natural $n$ and all $y, h \in Y$.

Denote by $M^1(X)$ the convolution semigroup of probability distributions on the group $X$. Let $\mu \in M^1(X)$. Denote by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x), \quad y \in Y,$$

the characteristic function of the distribution $\mu$, and by $\sigma(\mu)$ the support of $\mu$. Define the distribution $\bar{\mu} \in M^1(X)$ by the formula $\bar{\mu}(B) = \mu(-B)$ for any Borel subset $B$ in $X$. Then $\hat{\mu}(y) = \bar{\mu}(y)$. Denote by $m_K$ the Haar distribution of a compact subgroup $K$ of the group $X$, and by $E_x$ the degenerate distribution concentrated at an element $x \in X$. We note that the characteristic function of a distribution $m_K$ is of the form

$$\hat{m}_K(y) = \begin{cases} 1, & \text{if } y \in A(Y, K), \\ 0, & \text{if } y \notin A(Y, K). \end{cases} \quad (3)$$

Denote by $\Gamma(\mathbb{R}^n)$ the set of Gaussian distributions on the group $\mathbb{R}^n$.

We will use in the article the standard results on the structure of locally compact Abelian groups and the duality theory (see [13]).

2. Proof of the main theorem

The main result of the article is the proof of the following statement.

**Theorem 1.** Let $X$ be a discrete Abelian group containing no elements of order 2. Let $\alpha$ be an automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = m_K * E_{x_j}$, where $K$ is a finite subgroup of the group $X$, $x_j \in X$, $j = 1, 2$. Moreover, $\alpha(K) = K$.

To prove Theorem 1 we need some lemmas.
Lemma 1 ([5 Lemma 16.1]). Let $X$ be a locally compact Abelian group, $Y$ be its character group. Let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. The conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$
\hat{\mu}_1(u + v)\hat{\mu}_2(u + \tilde{\alpha}v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - \tilde{\alpha}v), \quad u, v \in Y.
$$

(4)

It is convenient for us to formulate as a lemma the following well-known statement (see e.g. [5 Proposition 2.13]).

Lemma 2. Let $X$ be a locally compact Abelian group, $Y$ be its character group. Let $\mu \in M^1(X)$. Then the set $E = \{ y \in Y : \hat{\mu}(y) = 1 \}$ is a closed subgroup of the group $Y$ and $\sigma(\mu) \subset A(X,E)$.

Lemma 3 ([17]). Let $X$ be a locally compact Abelian group, $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then the linear forms $M_1 = (I + \alpha)\xi_1 + 2\alpha \xi_2$ and $M_2 = 2\xi_1 + (I + \alpha)\xi_2$ are independent.

The following lemma is standard and was proved under assumption that $\alpha_j, \beta_j$ are topological automorphisms. The proof holds true if $\alpha_j, \beta_j$ are continuous endomorphisms. We formulate the lemma for two independent random variables.

Lemma 4 ([5 Lemma 10.1]). Let $X$ be a locally compact Abelian group, $Y$ be its character group. Let $\alpha_j, \beta_j$ be continuous endomorphisms of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. The linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ are independent if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$
\hat{\mu}_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v)\hat{\mu}_2(\tilde{\alpha}_2 u + \tilde{\beta}_2 v) = \hat{\mu}_1(\tilde{\alpha}_1 u)\hat{\mu}_2(\tilde{\alpha}_2 u)\hat{\mu}_1(\tilde{\beta}_1 v)\hat{\mu}_2(\tilde{\beta}_2 v), \quad u, v \in Y.
$$

(5)

Lemma 5. Let $X$ be a discrete torsion Abelian group, containing no elements of order 2. Let $Y$ be its character group. Let $\alpha$ be an automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$ such that $\hat{\mu}_j(y) \geq 0$, $j = 1, 2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\hat{\mu}_1(y) = \hat{\mu}_2(y) = 1$, $y \in B$, where $B$ is an open subgroup of $Y$.

Proof. By Lemma 3, the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ implies that the linear forms $M_1 = (I + \alpha)\xi_1 + 2\alpha \xi_2$ and $M_2 = 2\xi_1 + (I + \alpha)\xi_2$ are independent. Then, by Lemma 4, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (5) which takes the form

$$
\hat{\mu}_1((I + \tilde{\alpha})u + 2v)\hat{\mu}_2(2\tilde{\alpha}u + (I + \tilde{\alpha})v) = \hat{\mu}_1((I + \tilde{\alpha})u)\hat{\mu}_2(2\tilde{\alpha}u)\hat{\mu}_1(2v)\hat{\mu}_2((I + \tilde{\alpha})v), \quad u, v \in Y.
$$

(6)

Since $X$ is a discrete torsion Abelian group, it follows that $Y$ is a compact totally disconnected Abelian group. Then any neighborhood of zero of the group $Y$ contains an open subgroup. Hence, we can chose an open subgroup $U$ in the group $Y$ such that $\hat{\mu}_j(y) > 0$, $y \in U$, $j = 1, 2$. Put $\psi_j(y) = -\log \hat{\mu}_j(y)$, $y \in U$, $j = 1, 2$. Since $\hat{\mu}_j(y) \leq 1$, it follows that

$$
\psi_j(y) \geq 0, \quad y \in U, \quad j = 1, 2.
$$

(7)

Put $V = U \cap \tilde{\alpha}^{-1}(U) \cap \tilde{\alpha}^{-2}(U)$. Then $V$ is an open subgroup in $Y$. It follows from (6) that the functions $\psi_j(y)$ satisfy the equation

$$
\psi_1((I + \tilde{\alpha})u + 2v) + \psi_2(2\tilde{\alpha}u + (I + \tilde{\alpha})v) = P(u) + Q(v), \quad u, v \in V,
$$

where

$$
P(y) = \psi_1((I + \tilde{\alpha})y), \quad Q(y) = \psi_1(2y) + \psi_2((I + \tilde{\alpha})y), \quad y \in V.
$$

(9)
To solve equation (8) we use the finite difference method. Let \( h_1 \) be an arbitrary element of the group \( V \). Substitute in (8) \( u + (I + \tilde{\alpha})h_1 \) for \( u \) and \( v - 2\tilde{\alpha}h_1 \) for \( v \). Subtracting equation (8) from the resulting equation, we get

\[
\Delta_{(I-\tilde{\alpha})2h_1} \psi_1((I + \tilde{\alpha})u + 2v) = \Delta_{(I+\tilde{\alpha})h_1} P(u) + \Delta_{-2\tilde{\alpha}h_1} Q(v), \quad u, v \in V. \tag{10}
\]

Let \( h_2 \) be an arbitrary element of the group \( V \). Substitute in (10) \( u + 2h_2 \) for \( u \) and \( v - (I + \tilde{\alpha})h_2 \) for \( v \). Subtracting equation (10) from the resulting equation, we obtain

\[
\Delta_{2h_2} \Delta_{(I+\tilde{\alpha})h_1} P(u) + \Delta_{-(I+\tilde{\alpha})h_2} \Delta_{-2\tilde{\alpha}h_1} Q(v) = 0, \quad u, v \in V. \tag{11}
\]

Let \( h \) be an arbitrary element of the group \( V \). Substitute in (11) \( u + h \) for \( u \). Subtracting equation (11) from the resulting equation, we have

\[
\Delta_h \Delta_{2h_2} \Delta_{(I+\tilde{\alpha})h_1} P(u) = 0, \quad u \in V. \tag{12}
\]

Put \( W_P = V \cap V^{(2)} \cap (I + \tilde{\alpha})(V) \). Since \( h, h_1, h_2 \) in (12) are arbitrary elements of the group \( V \), it follows from (12) that the function \( P(y) \) satisfies the equation

\[
\Delta_h^2 P(y) = 0, \quad y, h \in W_P. \tag{13}
\]

Put \( W_Q = V \cap (I + \tilde{\alpha})(V) \cap (\tilde{\alpha}(V))^{(2)} \). Reasoning similarly, we obtain from (11) that the function \( Q(y) \) satisfies the equation

\[
\Delta_h^2 Q(y) = 0, \quad y, h \in W_Q. \tag{14}
\]

Set \( W = W_P \cap W_Q \).

Since the automorphism \( \alpha \) satisfies condition (2), the group \((I + \tilde{\alpha})(Y)\) is dense in \( Y \), and taking into account that \( Y \) is a compact group, we have

\[
(I + \tilde{\alpha})(Y) = Y. \tag{15}
\]

It follows from this that the continuous endomorphism \( I + \tilde{\alpha} \) is open, and hence \((I + \tilde{\alpha})(V)\) is an open subgroup in \( Y \). Since \( X \) is a torsion group containing no elements of order 2, we have \( f_2 \in \text{Aut}(X) \).

It follows from this that \( f_2 \in \text{Aut}(Y) \), and hence \( V^{(2)} \) and \((\tilde{\alpha}(V))^{(2)}\) are open subgroups in \( Y \). This implies that \( W \) is an open subgroup in \( Y \). Hence, \( W \) is a closed subgroup in \( Y \), so that \( W \) is a compact subgroup. It follows from (13) and (14) that \( P(y) \) and \( Q(y) \) are continuous polynomials on the group \( W \). It is well known, see e.g. [5, Proposition 5.7], that a continuous polynomial on a compact Abelian group is a constant. Since \( P(0) = Q(0) = 0 \), we have

\[
P(y) = Q(y) = 0, \quad y \in W. \tag{16}
\]

Put \( B = (I + \tilde{\alpha})(W) \). It is obvious that \( B \subseteq U \). Since \((I + \tilde{\alpha})\) is a continuous open endomorphism, \( B \) is an open subgroup in \( Y \). It follows from (7), (9) and (16) that \( \psi_1(y) = \psi_2(y) = 0, \quad y \in B \). Hence, \( \hat{\mu}_1(y) = \hat{\mu}_1(y) = 1, \quad y \in B \). Lemma 5 is proved.

It should be noted that if an automorphism \( \alpha \) satisfies conditions (11), then Lemma 5 is a particular case of a general statement which was proved in [1, Lemma 6], see also [5, Lemma 17.24].

It is convenient for us to formulate as a lemma the following easily verified statement.

**Lemma 6.** Let \( X \) be a locally compact Abelian group, \( Y \) be its character group. Let \( G \) be a compact subgroup of a group \( X \) and \( \beta \) be a continuous endomorphism of the group \( X \). Then the following statements are equivalent:

(i) \( \beta(G) \supseteq G \);
If $\tilde{\beta}y \in A(Y,G)$, then $y \in A(Y,G)$.

**Lemma 7.** Let $X$ be a locally compact Abelian group containing no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in a group $X$ and distributions $\mu_1 = m_{K_1}$ and $\mu_2 = m_{K_2}$, where $K_1$ and $K_2$ are finite subgroups of $X$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $K_1 = K_2 = K$ and $\alpha(K) = K$.

**Proof.** Denote by $Y$ the character group of the group $X$. Put $f(y) = \hat{m}_{K_1}(y), g(y) = \hat{m}_{K_2}(y)$, $E_j = A(Y,K_j), j = 1,2$. It follows from [3] that

$$f(y) = \begin{cases} 1, & \text{if } y \in E_1, \\ 0, & \text{if } y \notin E_1, \end{cases} \quad g(y) = \begin{cases} 1, & \text{if } y \in E_2, \\ 0, & \text{if } y \notin E_2. \end{cases} \quad (17)$$

By Lemma 1, the characteristic functions $f(y)$ and $g(y)$ satisfy equation (1) which takes the form

$$f(u + v)g(u + \tilde{\alpha}v) = f(u - v)g(u - \tilde{\alpha}v), \quad u, v \in Y. \quad (18)$$

Put in (18) $u = v = y$. We get

$$f(2y)g((I + \tilde{\alpha})y) = g((I - \tilde{\alpha})y), \quad y \in Y. \quad (19)$$

Assume that

$$(I - \tilde{\alpha})y \in E_2. \quad (20)$$

Then, it follows from (17) and (19) that

$$2y \in E_1 \quad (21)$$

and

$$(I + \tilde{\alpha})y \in E_2. \quad (22)$$

Since group $X$ contains no elements of order 2 and $K_2$ is a finite group, we have

$$(K_2)^{(2)} = K_2. \quad (23)$$

It follows from (20) and (22) that

$$2y \in E_2. \quad (24)$$

Taking into account (23) and applying Lemma 6 to $\beta = f_2, G = K_2$, we get from (24) that $y \in E_2$. So, we proved that (20) implies that $v \in E_2$. Applying Lemma 6 again to $\beta = I - \alpha, G = K_2$, we obtain that $(I - \alpha)(K_2) \supset K_2$. Since $K_2$ is a finite group, it follows from this that

$$(I - \alpha)(K_2) = K_2, \quad (25)$$

and hence,

$$\alpha(K_2) = K_2. \quad (26)$$

Assume that $y \in E_2$. It is obvious that (25) implies the inclusion $(I - \tilde{\alpha})(E_2) \subset E_2$. Hence, (20) holds, and then (21) follows from (17) and (19). Since group $X$ contains no elements of order 2 and $K_1$ is a finite group, we have $(K_1)^{(2)} = K_1$. Taking into account (21) and applying Lemma 6 to $\beta = f_2, G = K_1$, it follows from $(K_1)^{(2)} = K_1$ that $y \in E_1$. So, we proved that

$$E_2 \subset E_1. \quad (27)$$
Put in \( (18) \ u = \alpha y, \ v = y \). We get

\[
f((I + \bar{\alpha})y)g(2\bar{\alpha}y) = f((I - \bar{\alpha})y), \quad y \in Y.
\]

Arguing similarly as above we get that \( (28) \) implies the inclusion

\[
(I - \bar{\alpha})(E_1) \subset E_1.
\]

Let \( y \in E_1 \). It follows from \( (29) \) that \( (I - \bar{\alpha})y \in E_1 \), and we get from \( (17) \) and \( (28) \) that

\[
2\bar{\alpha}y \in E_2.
\]

It is easy to see \( (23) \) and \( (26) \) imply \( 2\alpha(K_2) = K_2 \). Taking into account \( (30) \) and applying Lemma 6 to \( \beta = 2\alpha \), \( G = K_2 \), it follows from \( 2\alpha(K_2) = K_2 \) that \( y \in E_2 \).

So, we proved that \( E_1 \subset E_2 \). Considering \( (27) \), this implies that \( E_1 = E_2 \), and hence, \( K_1 = K_2 = K \). The equality \( \alpha(K) = K \) follows from \( (20) \). Lemma 7 is proved.

Note that if \( f_2 \in \text{Aut}(X) \) and a topological automorphism \( \alpha \) satisfies conditions \( (1) \), then Lemma 7 was proved in \( [3] \), see also \( [5] \) Lemma 17.25.

**Remark 1.** Generally speaking, Lemma 7 fails if a group \( X \) contains an element of order 2. Denote by \( Z(2) = \{0, 1\} \) the group of residues modulo 2. Let \( X = (Z(2))^2 \). Denote by \( x = (x_1, x_2) \), where \( x_j \in Z(2) \), elements of the group \( X \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \). It follows from Lemma 1 that for any automorphism \( \alpha \) the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric. Put \( K_1 = \{(x_1, 0)\}, K_2 = \{(0, x_2)\}, x_j \in Z(2) \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( m_{K_1} \) and \( m_{K_2} \). Then the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, whereas \( K_1 \neq K_2 \).

**Remark 2.** Generally speaking, Lemma 7 fails if \( K_1 \) and \( K_2 \) are compact but not finite subgroups of the group \( X \). Here is an example. (compare with \( [5] \) Remark 13.16). Let \( G \) be an arbitrary compact Abelian group. Consider the direct product

\[
X = \mathbf{P}_{j \in Z} G_j,
\]

where all \( G_j = G \). Put \( H = G^* \). Then the group \( Y \) is topologically isomorphic to a weak direct product of groups \( H_j \), where \( H_j = H \),

\[
Y \cong \mathbf{P}_{j \in Z} H_j.
\]

Let \( \alpha \in \text{Aut}(X) \) be an automorphism of the form

\[
\alpha(g_j) = (g_{j-2}), \quad (g_j) \in X.
\]

Then

\[
\tilde{\alpha}(h_j) = (h_{j+2}), \quad (h_j) \in Y.
\]

It is obvious that

\[
\text{Ker}(I - \alpha) = \{(g_j) \in X : g_{2k} = g_2, \ g_{2k-1} = g_1, \ g_1, g_2 \in G, \ k \in Z\}.
\]

Since \( (I - \tilde{\alpha})(Y) = A(Y, \text{Ker}(I - \alpha)) \), we have

\[
(I - \tilde{\alpha})(Y) = \{(h_j) \in Y : \sum_{k \in Z} h_{2k} = 0, \sum_{k \in Z} h_{2k-1} = 0, \ k \in Z\}.
\]
Consider the subgroups

\[ K_1 = \bigoplus_{j \in \mathbb{Z}, j \neq 1} G_j, \quad K_2 = \bigoplus_{j \in \mathbb{Z}, j \neq 2} G_j. \]

Obviously, \( K_1 \neq K_2 \). Put \( L_j = A(Y, K_j), \ j = 1, 2 \). Denote by \( L \) the subgroup of \( Y \), generated by \( L_1 \) and \( L_2 \). It is easy to see that \( L \cap (I - \alpha)(Y) = \{0\} \) and \( L_1 \cap L_2 = \{0\} \).

We will check that the characteristic functions \( f(y) = \hat{m}_{K_1}(y) \) and \( g(y) = \hat{m}_{K_2}(y) \) satisfy equation (18). Let the left-hand side of (18) be equal to 1. Then \( u + v \in L_1, u + \alpha v \in L_2 \). It follows from this that \((I - \alpha)v \in L \). Hence, \((I - \alpha)v = 0 \). Since, obviously, \( \text{Ker}(I - \alpha) = \{0\} \), we have \( v = 0 \). This implies that \( u \in L_1 \cap L_2 \), and hence, \( u = 0 \). So, the right-hand side of (18) is also equal to 1. Similarly we verify that if the right-hand side of (18) is also equal to 1, then the left-hand side of (18) is also equal to 1. It means that the characteristic functions \( f(y) \) and \( g(y) \) satisfy equation (18).

Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( m_{K_1} \) and \( m_{K_2} \). By Lemma 1, the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric.

**Lemma 8** ([2], see also [4, Corollary 17.2 and Remark 17.5]). Let \( X \) be a finite Abelian group containing no elements of order 2. Let \( \alpha \) be an automorphism of \( X \) satisfying condition (2). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = m_K * E_{x_j} \), where \( K \) is a subgroup of the group \( X \), \( x_j \in X, j = 1, 2 \). Moreover, \( \alpha(K) = K \).

**Lemma 9** ([11]). Let \( Y \) be a connected compact Abelian group, \( X \) be its character group. Let \( \alpha \) be a topological automorphism of the group \( Y \) satisfying condition (15). Let \( \mu_1 \) and \( \mu_2 \) be distributions on the group \( X \) such that their characteristic functions \( \hat{\mu}_j(y) \) satisfy equation (4). Then \( \hat{\mu}_j(y) = \hat{x}_j, y \), where \( x_j \in X, j = 1, 2 \).

**Lemma 10.** Let \( X \) be a discrete Abelian group, \( \alpha \) be an automorphism of the group \( X \) satisfying condition

\[ \text{Ker}(I + \alpha) \subset b_X. \] (31)

Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then there exist elements \( x_1', x_2' \in X \) such that the distributions \( \mu_j' \) of the random variables \( \xi_j' = \xi_j - x_j' \) are supported in the subgroup \( b_X \), and the conditional distribution of the linear form \( L_2' = \xi_1' + \alpha \xi_2' \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric.

**Proof.** Denote by \( Y \) the character group of the group \( X \). Since \( X \) is a discrete group, \( b_X \) is the subgroup consisting of all elements of finite order of the group \( X \). Consider the factor-group \( X/b_X \) and denote by \([x]\) its elements. Note that the character group of the group \( X/b_X \) is topologically isomorphic to the annihilator \( A(Y, b_X) \), and \( A(Y, b_X) = c_Y \). Note that \( c_Y \) is a compact group. It follows from

\[ \alpha(b_X) = b_X \] (32)

that \( \alpha \) induces an automorphism \( \hat{\alpha} \) of the factor-group \( X/b_X \) by the formula \( \hat{\alpha}[x] = [\alpha x] \). We note that \( \hat{\alpha}(c_Y) = c_Y \), and a topological automorphism of the group \( c_Y \) which is adjoint to \( \hat{\alpha} \) is a restriction of \( \hat{\alpha} \) to \( c_Y \). Verify that

\[ (I + \hat{\alpha}c_Y)(c_Y) = c_Y. \] (33)

Taking into account that for a discrete Abelian group \( X \) conditions (2) and (15) are equivalent, it suffices to verify that

\[ \text{Ker}(I + \hat{\alpha}) = \{0\}. \] (34)

Let \( x_0 \in X \) and \([x_0] \in \text{Ker}(I + \hat{\alpha}) \). Then \([I + \alpha]x_0 = 0 \), and this implies that \( (I + \alpha)x_0 \in b_X \), i.e. \( n(I + \alpha)x_0 = 0 \) for some integer \( n \). Hence, \( (I + \alpha)nx_0 = 0 \). Taking into account (33) this implies that \( nx_0 \in b_X \). Thus, \( x_0 \in b_X \). So, (34) holds, thus (33) is proved.
Consider the restriction of equation (1) to \( c_Y \). Taking into account that by Lemma 1, the characteristic functions \( \tilde{\mu}_j(y) \) satisfy equation (1), and applying Lemma 9 to the group \( c_Y \), we get that the restrictions of the characteristic functions \( \tilde{\mu}_j(y) \) to the subgroup \( c_Y \) are characters of the subgroup \( c_Y \). Extending these characters to characters of the group \( Y \), we obtain that there exist elements \( x_j \in X \), \( j = 1, 2 \), such that

\[
\tilde{\mu}_j(y) = (x_j, y), \quad y \in c_Y, \quad j = 1, 2 \tag{35}
\]

holds. Substituting (35) into (1) and taking into account that \( A(X, c_Y) = b_X \), we get

\[
2(x_1 + \alpha x_2) \in b_X. \tag{36}
\]

Since \( b_X \) consists of all elements of finite order of the group \( X \), it follows from (35) that

\[
x_1 + \alpha x_2 \in b_X. \tag{37}
\]

Put \( x_1' = -\alpha x_1, \ x_2' = x_2 \). It is easy to see that (35) and (37) imply that

\[
\tilde{\mu}_j(y) = (x_j', y), \quad y \in c_Y, \quad j = 1, 2. \tag{38}
\]

Since \( x_1' + \alpha x_2' = 0 \), the characteristic functions \( f_j(y) = (-x_j', y), \ j = 1, 2 \), on the group \( Y \) satisfy equation (1). Put \( \xi_j = \xi_j' - x_j' \), and denote by \( \mu_j' \) the distribution of the random variable \( \xi_j' \). It follows from

\[
\mu_j' = \mu_j \ast E_{-x_j}, \quad j = 1, 2, \tag{39}
\]

that the characteristic functions \( \mu_j' \) also satisfy equation (1). By Lemma 1, the conditional distribution of the linear form \( L_2 = \xi_1' + \alpha \xi_2' \) given \( L_1 = \xi_1' \) is symmetric. Moreover, (38) and (39) imply that \( \mu_j'(y) = 1 \), \( y \in c_Y \). Then, by Lemma 2, \( \sigma(\mu_j') \subset A(X, c_Y) = b_X \). Lemma 10 is proved.

Note that if an automorphism \( \alpha \) satisfies conditions (1), then Lemma 10 is a particular case of a general statement which was proved in [4, Corollary 1], see also [5, Lemma 17.22].

**Remark 3.** Generally speaking, Lemma 10 fails if we omit condition (31). Put \( G = \text{Ker}(I + \alpha) \). Let \( x_0 \in G, \ x_0 \notin b_X \) and let \( \mu \) be a distribution on \( X \) such that \( \sigma(\mu) = \{0, x_0\} \). Let \( \xi_1 \) and \( \xi_2 \) be independent identically distributed random variables with values in the group \( G \) and distribution \( \mu \). It is obvious that \( \alpha x = -x \) for all \( x \in G \). Hence, applying Lemma 1 to the group \( G \) we get that the conditional distribution of the linear form \( L_2 = \xi_1 - \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric. Hence, if we consider the random variables \( \xi_1 \) and \( \xi_2 \), as random variables with values in the group \( X \), then the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is also symmetric. Since \( \sigma(\mu \ast E_x) = \{x, x_0 + x\} \) for any \( x \in X \), it is obvious that does not exist an element \( x \in X \) such that the distribution \( \mu_j' \) of the random variables \( \xi_j' = \xi_j + x \) is supported in the subgroup \( b_X \).

**Proof of Theorem 1.** We will follow the scheme of the proof of Theorem 17.26 in [5]. Denote by \( Y \) the character group of the group \( X \). Since \( \alpha(b_X) = b_X \), we can apply Lemma 10 and assume that \( X \) is a torsion group. By Lemma 1, the symmetry of the conditional distribution of the linear form \( L_2 \) given \( L_1 \) implies that the characteristic functions \( \tilde{\mu}_j(y) \) satisfy equation (1). Put \( \nu_j = \mu_j \ast \tilde{\nu}_j \). Then equation (1) takes the form (18). We will prove that in this case \( f(y) = g(y) = \tilde{\mu}_j(y) \). Then equation (1) takes the form (18). We will prove that in this case \( f(y) = g(y) = \tilde{\mu}_j(y) \).
Lemma 5, there exists an open subgroup $B$ such that $B \subset E_f \cap E_g$. Put $S = A(X, B)$. Then $F$ and $G$ are subgroups in $S$. Since $B$ is an open subgroup, $S$ is a compact subgroup, and taking into account that $X$ is a discrete group, $S$ is a finite subgroup. Hence, $F$ and $G$ are also finite subgroups.

It follows from \[ (13) \] that
\[
 f^n(u + v) g^n(u + \tilde{a}v) = f^n(u - v) g^n(u - \tilde{a}v), \quad u, v \in Y, \tag{40}
\]
holds for any natural $n$. It is obvious that there exist limits
\[
 \tilde{f}(y) = \lim_{n \to \infty} f^n(y) = \begin{cases} 1, & \text{if } y \in E_f, \\ 0, & \text{if } y \not\in E_f, \end{cases} \quad \bar{g}(y) = \lim_{n \to \infty} g^n(y) = \begin{cases} 1, & \text{if } y \in E_g, \\ 0, & \text{if } y \not\in E_g. \end{cases} \tag{41}
\]
Since $E_f = A(Y, F)$, $E_g = A(Y, G)$, it follows from \[ (3) \] that
\[
 \hat{m}_F(y) = \begin{cases} 1, & \text{if } y \in E_f, \\ 0, & \text{if } y \not\in E_f, \end{cases} \quad \hat{m}_G(y) = \begin{cases} 1, & \text{if } y \in E_g, \\ 0, & \text{if } y \not\in E_g. \end{cases}
\]
Hence,
\[
 \hat{m}_F(y) = \tilde{f}(y), \quad \hat{m}_G(y) = \bar{g}(y).
\]
We now return to the random variables $\xi_1$ and $\xi_2$ and the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha \xi_2$. Since $\sigma(\mu_j) \subset F$, the random variables $\xi_j$ take values in the finite group $F$. Taking into account that $\alpha(F) = F$, we can apply Lemma 8 to the group $F$. Since $\mu_j(y) \geq 0$, by Lemma 8, $\mu_1 = \mu_2 = m_K$, where $K$ is a finite subgroup of the group $X$, and $\alpha(K) = K$. Theorem 1 is proved.

**Remark 4.** Let $X$ be a discrete Abelian group containing no elements of order 2, and $\alpha$ be an automorphism of the group $X$. Assume that $G = \text{Ker}(I + \alpha) \neq \{0\}$. Let $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in the group $G$ and distribution $\mu$. It is obvious that $\alpha x = -x$ for all $x \in G$. Hence, applying Lemma 1 to the group $G$ we get that the conditional distribution of the linear form $\tilde{L}_2 = \xi_1 - \xi_2$ given $\tilde{L}_1 = \xi_1 + \xi_2$ is symmetric. It follows from this that if we consider the random variables $\xi_1$ and $\xi_2$, as random variables with values in the group $X$, then the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is also symmetric. Taking into account that $\mu$ is an arbitrary distribution, we see that Theorem 1 fails if condition \[ (2) \] is not fulfilled.

We complement Theorem 1 by the following statement.

**Proposition 1.** Let $X$ be a locally compact Abelian group containing no elements of order 2, $K$ be a finite subgroup of $X$, $\alpha$ be a topological automorphism of $X$. Let $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in the group $X$ and distribution $m_K$. Then the following statements are equivalent:

(i) the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric;

(ii) $(I - \alpha)(K) = K$.

**Proof.** Denote by $Y$ the character group of the group $X$. Put $L = A(Y, K)$, $f(y) = \hat{m}_K(y)$. 
(i) $\Rightarrow$ (ii). By Lemma 7, it follows from (i) that $\alpha(K) = K$. It means that we can assume that $X = K$, and the characteristic function $f(y)$ is of the form

$$f(y) = \begin{cases} 1, & \text{if } y = 0, \\ 0, & \text{if } y \neq 0. \end{cases}$$

(42)

By Lemma 1, the characteristic function $f(y)$ satisfies equation

$$f(u + v)f(u + \tilde{\alpha}v) = f(u - v)f(u - \tilde{\alpha}v), \quad u, v \in Y.$$

(43)

Put in (43) $u = v = y$. We get

$$f(2y)f((I + \tilde{\alpha})y) = f((I - \tilde{\alpha})y), \quad y \in Y.$$

(44)

Let $y \in \text{Ker}(I - \tilde{\alpha})$. Then it follows from (44) that $2y = 0$. Since $X$ is a finite group and contains no elements of order 2, the group $Y$ also contains no elements of order 2, and hence $y = 0$, i.e. $\text{Ker}(I - \tilde{\alpha}) = \{0\}$. This implies (ii).

(ii) $\Rightarrow$ (i). The characteristic function $f(y)$ is of the form (3). We shall verify that $f(y)$ satisfies equation (13). Assume that for some $u, v \in Y$ the left hand-side of equation (13) is equal to 1. Then $u + v, u + \tilde{\alpha}v \in L$. This implies that $(I - \tilde{\alpha})v \in L$. Taking into account (ii), by Lemma 6, applying to $\beta = I - \alpha$, we get $v \in L$. Hence, $u, v \in L$, and then $\tilde{\alpha}v \in L$. It follows from this that $u - v, u - \tilde{\alpha}v \in L$. We get that the right hand-side of equation (13) is also equal to 1. Reasoning similarly we check that if the right hand-side of equation (13) is equal to 1, than the left hand-side of equation (13) is equal to 1. Thus the characteristic function $f(y)$ satisfies equation (13). By Lemma 1, (i) holds. Proposition 1 is proved.

Note that the proof of the statement (ii) $\Rightarrow$ (i) is based on Lemmas 1 and 6 only. Therefore, the statement (ii) $\Rightarrow$ (i) also holds in the case, when $K$ is a compact subgroup of $X$.

Remark 5. Generally speaking, Proposition 1 fails if a group $X$ contains elements of order 2. Let a group $X$ and subgroups $K_j$ be as in Remark 1. Put $\alpha(x_1, x_2) = (x_2, x_1 + x_2)$. It is obvious that $\alpha \in \text{Aut}(X)$ and $\alpha$ satisfies condition (2). We have $(I - \alpha)K_2 = K_1$. Let $\xi_1$ and $\xi_2$ be independent identically distributed random variables with values in the group $X$ and distribution $m_{K_2}$. Then condition (i) holds, whereas (ii) is not.

3. Generalizations of Theorem 1

First we prove the following generalization of Theorem 1.

Theorem 2. Let $X = \mathbb{R}^n \times G$, where $n \geq 0$, and $G$ is a discrete Abelian group containing no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$ satisfying condition (2). Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \gamma_j * m_K * E_{g_j}$, where $\gamma_j \in \Gamma(\mathbb{R}^n)$, $K$ is a finite subgroup of the group $G$, $g_j \in G$, $j = 1, 2$. Moreover, $\alpha(K) = K$.

To prove Theorem 2 we need the following lemma.

Lemma 11 ([11, see also [5, Theorem 13.3]]). Let $X = \mathbb{R}^n \times G$, where $n \geq 0$, and $G$ is a finite Abelian group. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. Let $\alpha_j, \beta_j, j = 1, 2$, be topological automorphisms of the group $X$. If the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ are independent, then $\mu_j = \gamma_j * m_K * E_{g_j}$, where $\gamma_j \in \Gamma(\mathbb{R}^n)$, $K_j$ is a finite subgroup of the group $G$, $g_j \in G$, $j = 1, 2$.

Proof of Theorem 2. First we reduce the proof of the theorem to the case when $G$ is a torsion group, and then to the case when $G$ is a finite group. Denote by $Y$ the character group of the group $X$.
The group $Y$ is topologically isomorphic to the group $\mathbb{R}^n \times H$, where $H$ is the character group of the group $G$. In order not to complicate the notation we assume that $Y = \mathbb{R}^n \times H$. Denote by $x = (t, g)$, where $t \in \mathbb{R}^n$, $g \in G$, elements of the group $X$, and by $y = (s, h)$, where $s \in \mathbb{R}^n$, $h \in H$, elements of the group $Y$.

1. Consider the factor-group $X/(\mathbb{R}^n \times b_G)$, and denote by $[(t, g)]$ its elements. Obviously, $X/(\mathbb{R}^n \times b_G)$ is a discrete group. The character group of the group $X/(\mathbb{R}^n \times b_G)$ is topologically isomorphic to the annihilator $A(Y, \mathbb{R}^n \times b_G)$. It is easy to see that $A(Y, \mathbb{R}^n \times b_G) = c_H$. Since $\mathbb{R}^n$ is a connected component of zero of the group $X$, we have $\alpha(\mathbb{R}^n) = \mathbb{R}^n$. It follows from $b_G = b_X$, that $\alpha(b_G) = b_G$. Hence, $\alpha(\mathbb{R}^n \times b_G) = \mathbb{R}^n \times b_G$ and $\alpha$ induces an automorphism $\hat{\alpha}$ on the factor-group $X/(\mathbb{R}^n \times b_G)$ by the formula $\hat{\alpha}[x] = [\alpha x]$. We note that $\hat{\alpha}(c_H) = c_H$ and a topological automorphism of the group $c_H$ which is adjoint to $\hat{\alpha}$, is a restriction of $\hat{\alpha}$ to $c_H$. Verify that

$$(I + \hat{\alpha}_{c_H})(c_H) = c_H.$$  

(45)

Taking into account that for a discrete Abelian group $X$ conditions (2) and (15) are equivalent, it suffices to verify that

$$\text{Ker}(I + \hat{\alpha}) = \{0\}.$$  

(46)

Take $(t_0, g_0) \in X$, and let $[(t_0, g_0)] \in \text{Ker}(I + \hat{\alpha})$. Then $[((I + \alpha)(t_0, g_0)) = 0$. Hence, $(I + \alpha)(t_0, g_0) \in \mathbb{R}^n \times b_G$. Since $b_G$ is a torsion group, we have $k(I + \alpha)(t_0, g_0) \in \mathbb{R}^n$ for some natural $k$. This implies that $(I + \alpha)(k(t_0, g_0)) = \mathbb{R}^n$, i.e.

$$(I + \alpha)(k(t_0, g_0)) = (t', 0).$$  

(47)

It is obvious that $(I + \alpha)(\mathbb{R}^n) \subset \mathbb{R}^n$. It follows from (2) that the restriction of the continuous endomorphism $I + \alpha$ of the group $X$ to the subgroup $\mathbb{R}^n$ is a topological automorphism of the group $\mathbb{R}^n$. Hence,

$$(t', 0) = (I + \alpha)(\tilde{t}, 0)$$  

(48)

for an element $\tilde{t} \in \mathbb{R}^n$. Taking into account (2), it follows from (47) and (48) that $k g_0 = 0$. Hence, $g_0 \in b_G$, and it means that $(t_0, g_0) \in \mathbb{R}^n \times b_G$. This implies that $|[t_0, g_0]| = 0$. So, (15) holds, and thereby (15) is proved.

By Lemma 1, the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ implies that the characteristic functions $\tilde{\mu}_j(y)$ satisfy equation (1). Since $\hat{\alpha}(c_H) = c_H$, consider the restriction of equation (1) to the subgroup $c_H$. Taking into account (15), apply Lemma 9 to the group $c_H$. We obtain that $\tilde{\mu}_j(y) = (y, j, y, j, 1, 2)$. By the extension theorem for characters from a closed subgroup to the group, we can assume that $x_j \in X, j = 1, 2$. Substituting these expressions for $\tilde{\mu}_j(y)$ to equation (1) and taking into account that $A(X, c_H) = \mathbb{R}^n \times b_G$, we get

$$2(x_1 + \alpha x_2) \in \mathbb{R}^n \times b_G.$$  

(49)

Since $b_G$ consists of all elements of finite order of the group $X$, it follows from (49) that

$$x_1 + \alpha x_2 \in \mathbb{R}^n \times b_G.$$  

(50)

Consider new independent random variables $\eta_1 = \xi_1 + \alpha x_2$ and $\eta_2 = \xi_2 - x_2$ with values in the group $X$. Denote by $\lambda_j$ the distribution of the random variable $\eta_j$. Obviously, $\lambda_1 = \mu_1 * E_{\alpha x_2}$, $\lambda_2 = \mu_2 * E_{x_2}$. It is easy to see that the characteristic functions $\lambda_j(y)$ also satisfy equation (1). Hence, by Lemma 1, the conditional distribution of the linear form $N_2 = \eta_1 + \alpha \eta_2$ given $N_1 = \eta_1 + \eta_2$ is symmetric. Obviously, $\lambda_2(y) = 1, y \in c_H$. It follows from (50) that $\lambda_1(y) = 1, y \in c_H$. By Lemma 2, this implies that $\sigma(\lambda_j) \subset A(X, c_H) = \mathbb{R}^n \times b_G, j = 1, 2$. Since $\alpha(\mathbb{R}^n \times b_G) = \mathbb{R}^n \times b_G$, we reduced the proof of the theorem to the case, when $G$ is a torsion group. So, we will assume that.
2. Since \( G \) is a torsion group, we have \( \alpha(G) = G \). Hence, \( \alpha(t, g) = (\alpha t, \alpha g) \) and \( \tilde{\alpha}(s, h) = (\tilde{\alpha} s, \tilde{\alpha} h) \).

Write equation (41) in the form
\[
\hat{\mu}_1(s + s', h + h') \hat{\mu}_2(s + \tilde{\alpha} s', h + \tilde{\alpha} h') = \hat{\mu}_1(s - s', h - h') \hat{\mu}_2(s - \tilde{\alpha} s', h - \tilde{\alpha} h'), \quad (s, h), (s', h') \in Y.
\] (51)

Substituting in (4) \( s = s' = 0 \), we get
\[
\hat{\mu}_1(0, h + h') \hat{\mu}_2(0, h + \tilde{\alpha} h') = \hat{\mu}_1(0, h - h') \hat{\mu}_2(0, h - \tilde{\alpha} h'), \quad h, h' \in H.
\] (52)

It follows from Theorem 1, Lemma 1 and (3) that solutions of equation (52) are of the form
\[
\hat{\mu}_1(0, h) = (g_1, h) \hat{\mu}_2(h), \quad \hat{\mu}_2(0, h) = (g_2, h) \hat{\mu}_2(h), \quad h \in H,
\] (53)

where \( K \) is a finite subgroup of the group \( G, g_1, g_2 \in G \) and \( \alpha(K) = K \). Put \( L = A(H, K) \). Substituting (53) in (52), taking into account (3), and considering the restriction of of the resulting equation to \( L \), we obtain that \( 2(g_1 + \alpha g_2) \in K \). Since the group \( G \) contains no elements of order 2, this implies that
\[
g_1 + \alpha g_2 \in K.
\] (54)

Consider new independent random variables \( \eta_1 = \xi_1 + \alpha g_2 \) and \( \eta_2 = \xi_2 - g_2 \) with values in the group \( X \).

Denote by \( \lambda_j \) the distribution of the random variable \( \eta_j \). Obviously, \( \lambda_1 = \mu_1 * E_{\alpha g_2}, \lambda_2 = \mu_2 * E_{-g_2} \).

It is easy to see that the characteristic functions \( \hat{\lambda}_j(y) \) also satisfy equation (41). Hence, by Lemma 1, the conditional distribution of the linear form \( P_2 = \eta_1 + \alpha \eta_2 \) given \( P_1 = \eta_1 + \eta_2 \) is symmetric. Obviously, \( \hat{\lambda}_2(y) = 1, y \in L \). It follows from (54) that \( \hat{\lambda}_1(y) = 1, y \in L \). By Lemma 2, this implies that \( \sigma(\lambda_j) \subset A(X, L) = \mathbb{R}^n \times K, j = 1, 2 \). Since \( \alpha(\mathbb{R}^n \times K) = \mathbb{R}^n \times K \), we reduced the proof of the theorem to the case, when \( G \) is a finite group, and we will assume that.

3. By Lemma 3, the linear forms \( M_1 = (I + \alpha) \xi_1 + 2 \alpha \xi_2 \) and \( M_2 = 2 \xi_1 + (I + \alpha) \xi_2 \) are independent. Since \( G \) is a finite group containing no elements of order 2, it follows from (2) that the coefficients of the linear forms \( M_1 \) and \( M_2 \) are topological automorphisms of the group \( X \). By Lemma 11, this implies that \( \mu_j = \gamma_j * m_{K \eta} * E_{g_j}, \) where \( \gamma_j \in \Gamma(\mathbb{R}^n), K_j \) is a finite subgroup of \( G, g_j \in G, j = 1, 2 \).

The equality \( K_1 = K_2 = K \) follows from (53). Theorem 2 is proved.

We note that Theorem 2 was proved in [10] for the groups \( X = \mathbb{R}^n \times G, \) where \( n \geq 0, \) and \( G \) is a discrete Abelian group containing no elements of order 2 and such that its torsion part is a finite group. We also note that if a topological automorphism \( \alpha \) satisfies conditions (1), then the statement of Theorem 2 was proved in [4, Theorem 2 and Remark 4], see also [5, Remark 17.27]. Reasoning as in Remark 4, we see that Theorem 2 fails if we omit condition (2).

We prove now the following generalization of Theorem 1.

**Theorem 3.** Let \( X \) be a discrete Abelian group, \( F \) be the 2-component of the group \( X, G \) be the subgroup generated by all elements of odd order of the group \( X \). Let \( \alpha \) be an automorphism of \( X \) satisfying condition (2). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = \rho_j * m_{K \xi_j} * E_{x_j}, \) where \( \sigma(\rho_j) \subset F, K \) is a finite subgroup of \( G, x_j \in X, j = 1, 2 \).

To prove Theorem 3 we need the following lemma.

**Lemma 12.** Let \( X \) be a discrete torsion Abelian group, \( \alpha \) be an automorphism of the group \( X \) satisfying condition (2). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j \) are supported in a subgroup generated by \( X_{(2)} \) and a finite subgroup, and hence, in a subgroup \( X_{(n)} \) for some \( n \).
Proof. First we prove the lemma supposing that \( \hat{\mu}_j(y) \geq 0 \), \( j = 1, 2 \). Reasoning as in the proof of Lemma 5 and retaining notation of Lemma 5 we obtain that

\[
\hat{\mu}_1(y) = \hat{\mu}_1(y) = 1, \quad y \in B,
\]

but in contrast to Lemma 5, if the group \( X \) contains elements of order 2, then the subgroup \( B \) does not need to be open.

We have \( W = V^{(2)} \cap (I + \tilde{\alpha})(V) \cap (\tilde{\alpha}(V))^{(2)} \). Put \( L = V \cap (I + \tilde{\alpha})(V) \cap \tilde{\alpha}(V), \ M = (I + \tilde{\alpha})(L) \). By (15), the continuous endomorphism \( I + \tilde{\alpha} \) is open. This implies that \( L \) and hence \( M \) are open subgroups in \( Y \). It is obvious that \( W \supset L^{(2)} \). Hence, \( B \supset M^{(2)} \). Put \( G = A(X, M^{(2)}) \). By Lemma 2, it follows from (55) that \( \sigma(\mu_j) \subset A(X, B) \subset G, \ j = 1, 2 \).

We have \( G = \{ x \in X : (x, 2y) = 1 \text{ for all } y \in M \} = \{ x \in X : (2x, y) = 1 \text{ for all } y \in M \} = \{ x \in X : 2x \in A(X, M) \} = f_2^{-1}(A(X, M)) \). Note that \( \text{Ker} f_2 = X^{(2)} \). Since \( M \) is an open subgroup in \( Y \), the annihilator \( A(X, M) \) is a compact subgroup in \( X \), and hence a finite subgroup in \( X \). This implies that the group \( G = f_2^{-1}(A(X, M)) \) is generated by \( X^{(2)} \) and a finite subgroup of \( X \).

We proved the lemma, assuming that \( \hat{\mu}_j(y) \geq 0 \), \( j = 1, 2 \). Get rid of this restriction. Put \( \nu_j = \mu_j * \tilde{\mu}_j \). Then \( \nu_j(y) = |\tilde{\mu}_j(y)|^2 \geq 0 \), \( y \in Y \). Obviously, the characteristic functions \( \nu_j(y) \) also satisfy equation (4). Denote by \( \xi_j \) the independent random variables with values in the group \( X \) and distributions \( \nu_j, \ j = 1, 2 \). By Lemma 1, the conditional distribution of the linear form \( \tilde{\xi}_2 = \xi_1 + \alpha \tilde{\xi}_2 \) given \( \tilde{L}_1 = \xi_1 + \tilde{\xi}_2 \) is symmetric. As has been proved above \( \nu_j \) are supported in a subgroup \( S \), generated by \( X^{(2)} \) and a finite subgroup. It follows from this that each distribution \( \mu_j \) is supported in a set \( x_j + S \) for some \( x_j \in X \). Since \( X \) is a torsion group, each of the elements \( x_j \) has a finite order. Hence, the subgroup generated by \( S \) and \( x_j \) also has the required form. Lemma 12 is proved.

Proof of Theorem 3. Since \( \alpha(b_X) = b_X \), we can apply Lemma 10 and assume that \( X \) is a torsion group. Since \( \alpha(X^{(n)}) = X^{(n)} \), by Lemma 12, we can assume that \( X = X^{(n)} \) for some \( n \). Since \( X \) is a torsion group, \( X \) is a weak direct product of its \( p \)-torsion subgroups (12 Theorem 8.4)). This implies that, \( X = F \times G \). Denote by \( Y \) the character group of the group \( X \). The group \( Y \) is topologically isomorphic to the group \( L \times H \), where \( L \) is the character group of \( F \), and \( H \) is the character group of \( G \). In order not to complicate the notation we assume that \( Y = L \times H \). Since \( \alpha(F) = F \) and \( \alpha(G) = G \), we have \( \tilde{\alpha}(L) = L, \tilde{\alpha}(H) = H \). Denote by \( y = (l, h) \), where \( l \in L, \ h \in H \), elements of the group \( Y \).

Reasoning as in the proof of item 2 of Theorem 2, and considering the group \( F \) instead of \( \mathbb{R}^n \), we reduce the proof of the theorem to the case, when \( G \) is a finite group, and

\[
\hat{\mu}_1(0, h) = \hat{\mu}_2(0, h) = \hat{m}_G(h), \quad h \in H.
\]

By Lemma 1, the symmetry of the conditional distribution of the linear form \( L_2 \) given \( L_1 \) implies that the characteristic functions \( \hat{\mu}_j(y) \) satisfy equation (4). Substitute in (4) \( u = v = (l, h) \). We get

\[
\hat{\mu}_1(2l, h))\hat{\mu}_2((I + \tilde{\alpha})(l, h)) = \hat{\mu}_2((I - \tilde{\alpha})(l, h)), \quad (l, h) \in Y.
\]

Substitute in (56) \( l = 0, \ h \neq 0 \). Since \( 2h \neq 0 \), it follows from (56) that the left-hand side of the resulting equation is equal to zero. Thus \( \mu_2(0, (I - \tilde{\alpha}_H)h) = 0 \). Hence, if \( h \neq 0 \), then \( (I - \tilde{\alpha}_H)h \neq 0 \), i.e. \( \text{Ker}(I - \tilde{\alpha}_H) = \{0\} \). Since \( H \) is a finite group, this implies that

\[
I - \tilde{\alpha}_H \in \text{Aut}(H).
\]

Obviously, the automorphism \( \alpha_F \) satisfies the condition

\[
\text{Ker}(I + \alpha_F) = \{0\}.
\]
It is easy to see that (59) is equivalent to the condition
\[ \operatorname{Ker}(I - \alpha_F) = \{0\}. \tag{60} \]
Indeed, assume that (59) is fulfilled and \( x \in \operatorname{Ker}(I - \alpha_F), x \neq 0 \). We can assume without loss of generality that \( 2x = 0 \). Then the equality \((I + \alpha_F)x = (I - \alpha_F)x + 2\alpha_Fx = 0\) implies the contradiction. Reasoning similarly, we get that (60) implies (59).

Consider the factor group \( X/X(2) \). Its character group is topologically isomorphic to the annihilator \( A(Y, X(2)) \). Obviously, \( A(Y, X(2)) = Y(2) \). It follows from \( (X(2)) = X(2) \) that \( \alpha \) induces an automorphism \( \hat{\alpha} \) of the factor-group \( X/X(2) \) by the formula \( \hat{\alpha}[x] = [\alpha x] \). In so doing, a topological automorphism of the group \( Y(2) \) which is adjoint to \( \hat{\alpha} \) is the restriction of \( \hat{\alpha} \) to \( Y(2) \).

Obviously, \( \hat{\alpha}(Y(2)) = Y(2) \). Verify that (54) is fulfilled. Take \( x_0 \in X \) such that \( [x_0] \in \operatorname{Ker}(I + \hat{\alpha}) \). Then \( [(I + \alpha)x_0] = 0 \), and hence \((I + \alpha)x_0 \in X(2)\), i.e. \( 2(I + \alpha)x_0 = 0 \). It follows from (2) that \( 2x_0 = 0 \), i.e. \([x_0] = 0 \). Thus (54) holds true.

Since we assume that \( X = X(n) \), the group \( X \) is bounded. Hence, the group \( F \) is also bounded. Denote by \( k_X \) the least nonnegative integer such that \( F_{2^{k_X}} = \{0\} \). If \( k_X = 0 \), then \( X = G \), and the statement of the theorem follows from Theorem 1. We prove by induction that if the theorem holds true for the groups \( X \) satisfying the condition \( k_X = m - 1 \), then it is valid for the groups \( X \) satisfying the condition \( k_X = m \).

So, let \( X \) be a group such that \( k_X = m \). Put \( \hat{\mu}_1(l, 0) = a_1(l) \), \( \hat{\mu}_2(l, 0) = a_2(l) \) and prove that
\[ \hat{\mu}_1(l, h) = \begin{cases} a_1(l), & \text{if } h = 0, \\ 0, & \text{if } h \neq 0, \end{cases} \quad \hat{\mu}_2(l, h) = \begin{cases} a_2(l), & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases} \tag{61} \]
The statement of the theorem easily follows from this.

The restrictions of the characteristic functions \( \hat{\mu}_j(l, h) \) to the subgroup \( Y(2) \) are the characteristic functions of some independent random variables \( \eta_1 \) and \( \eta_2 \) with values in the factor-group \( X/X(2) \). By Lemma 1, the conditional distribution of the linear form \( \eta_2 = \eta_1 + \hat{\alpha}\eta_2 \) given \( \eta_1 = \eta_1 + \eta_2 \) is symmetric. As has been noted above, (34) is fulfilled. Obviously, \( k_{X/X(2)} = m - 1 \). Then by induction hypothesis (61) holds true for \( y \in Y(2) \). Let \( (l, h) \in Y \), and \( h \neq 0 \). Since \((2(l, h)) \in Y(2) \) and \( 2h \neq 0 \), the left-hand side of equation (57) is equal to zero. Hence, \( \hat{\mu}_2((I - \hat{\alpha})(l, h)) = 0 \) for all \((l, h) \in Y \), \( h \neq 0 \). We have, \((I - \hat{\alpha})(l, h) = ((I - \hat{\alpha}_L)(I - \hat{\alpha}_H)h) \). It follows from (60) that \((I - \hat{\alpha}_L)(L) = L \). Taking into account (58), we get that representation (61) for the function \( \hat{\mu}_2(l, h) \) holds true for all \((l, h) \in Y \). Substituting in (44) \( u = (l, h), v = \hat{\alpha}^{-1}(l, h) \) and reasoning similarly, we obtain the required representation for the function \( \hat{\mu}_1(l, h) \). Theorem 3 is proved.

Note that if an automorphism \( \alpha \) satisfies conditions (11), then Theorem 3 is proved in [16], see also [6], Theorem 17.33.

Condition (2) is necessary and sufficient for Theorem 1 and 2. As to Theorem 3 the situation is more complicated. The reasoning similar to that used in Remark 4 shows that the condition
\[ \operatorname{Ker}(I + \alpha) \subset X_2 \tag{62} \]
is necessary if we want that Theorem 3 be valid. Below we consider the case, when condition (62) holds, but condition (2) fails, i.e.
\[ \operatorname{Ker}(I + \alpha) \neq \{0\}. \tag{63} \]
We complement Theorem 3 by the following statement, which shows that, generally speaking, Theorem 3 fails if we change condition (2) for (62).
Proposition 2. Let $X$ be a discrete Abelian group, $F$ be the 2-torsion subgroup of $X$, $G$ be the subgroup generated by all elements of odd order of $X$. Let $\alpha$ be an automorphism of the group $X$ satisfying condition \((62)\). Then the following statements hold.

1. Assume that the only finite subgroup $K$ of the group $G$, satisfying the condition $\langle I-\alpha \rangle (K) = K$, is $K = \{0\}$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \rho_j \ast E x_j$, where $\sigma(\rho_j) \subset F$, $x_j \in X$, $j = 1, 2$.

2. Assume that there exists a non-zero finite subgroup $K_0$ of the group $G$, satisfying the condition $\langle I-\alpha \rangle (K_0) = K_0$. Assume that the automorphism $\alpha$ satisfies the condition \((63)\). Then there exist independent identically distributed random variables $\xi_1$ and $\xi_2$ with values in the group $X$ and distribution $\mu$ such that $\mu$ can not be represented in the form $\mu = \rho \ast m_K \ast E x$, where $\sigma(\rho) \subset F$, $K$ is a finite subgroup of the group $G$, $x \in X$, whereas the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.

Proof. 1. By Lemma 10, we can assume that $X$ is a torsion group. Then we have $X = F \times G$. Denote by $Y$ the character group of the group $X$. The group $Y$ is topologically isomorphic to the group $L \times H$, where $L$ is the character group of $F$, and $H$ is the character group of $G$. In order not to complicate the notation we assume that $Y = L \times H$. Since $\alpha(F) = F$ and $\alpha(G) = G$, we have $\hat{\alpha}(L) = L$, $\hat{\alpha}(H) = H$. Denote by $y = (l, h)$, where $l \in L$, $h \in H$, elements of the group $Y$. By Lemma 1, the characteristic functions $\hat{\nu}_j(y)$ satisfy equation \((4)\). Put $\nu_j = \mu_j \ast \hat{\nu}_j$. Then $\hat{\nu}_j(y) = |\hat{\nu}_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation \((4)\), which takes the form

$$
\hat{\nu}_1(l + l', h + h')\hat{\nu}_2(l + \alpha l, h + \alpha h') = \hat{\nu}_1(l, h - h')\hat{\nu}_2(l - \alpha l, h - \alpha h'), \quad l, l' \in L, \quad h, h' \in H.
$$

Putting here $l = l' = 0$, we obtain

$$
\hat{\nu}_1(0, h + h')\hat{\nu}_2(0, h + \alpha h') = \hat{\nu}_1(0, h - h')\hat{\nu}_2(0, h - \alpha h'), \quad h, h' \in H.
$$

(64)

It follows from \((62)\) that $\operatorname{Ker}(I + \alpha G) = \{0\}$. This implies by Lemma 1 and Theorem 1, applying to the group $G$ that $\hat{\nu}_1(0, h) = \hat{\nu}_2(0, h) = \hat{m}_K(0, h)$, $h \in H$, where $K$ is a finite subgroup of $G$. We have by Lemma 1 and Proposition 1 that $\langle I-\alpha \rangle (K) = K$. It follows from the condition of Proposition 2 that $K = \{0\}$. Hence, $\hat{\nu}_1(0, h) = \hat{\nu}_2(0, h) = 1$, $h \in H$. By Lemma 2, this implies that $\sigma(\nu_j) \subset F$, and hence, $\nu_j = \rho_j \ast E x_j$, where $\sigma(\rho_j) \subset F$, $x_j \in X$, $j = 1, 2$. Thus, we proved statement 1.

2. It is easy to see that $\alpha(K_0) = K_0$. Consider the subgroup $T = F_2(2) \times K_0$ of the group $X$. Obviously, $\alpha(T) = T$, i.e. the restriction of $\alpha$ to $T$ is an automorphism of the group $T$. Denote by $\alpha_T$ this restriction. Then the group $T$ and the automorphism $\alpha_T$ satisfy the conditions of statement 2. Taking this into account we can prove statement 2, assuming that $X = F \times G$,

$$
F = F_2(2),
$$

(66)

and $G$ is a finite group such that $\langle I-\alpha G \rangle (G) = G$. Since $G$ is a finite group, it means that

$$
I - \alpha G \in \operatorname{Aut}(G).
$$

(67)

It follows from \((62)\) and \((63)\) that $\operatorname{Ker}(I + \alpha F) \neq \{0\}$, and hence, $\langle I + \alpha L \rangle (L) = A(L, \operatorname{Ker}(I + \alpha F)) \neq L$. Take $l_0 \notin (I + \alpha L)(L)$. Since $F$ is a torsion group, $L$ is a compact and totally disconnected group. For this reason there exists an open in $L$ subgroup $U$ such that $U \cap (l_0 + U) = \emptyset$ and $(l_0 + U) \cap (I + \alpha L)(L) = \emptyset$. Consider the factor-group $A = Y/U \cong L/U \times H$, and denote by $[y] = ([l], h)$, where $[l] \in L/U$, $h \in H$, $Y/U$. 

\[ Y/U \]
its elements. Denote by \( n \) the number of elements of the group \( H \). Consider on the group \( A \) the function
\[
f([y]) = f([l], h) = \begin{cases} 
1, & \text{if } [l] = 0, \; h = 0, \\
0, & \text{if } [l] = 0, \; h \neq 0, \\
\frac{1}{n}, & \text{if } [l] = [l_0], \; h \in H, \\
0, & \text{if } [l] \notin \{0, [l_0]\}, \; h \in H. 
\end{cases}
\]

Put \( B = A(F, U) \times G \). By the duality theorem the groups \( A \) and \( B \) are the character groups of each other. Put
\[
p(x) = 1 + \sum_{[y] \in A, \; [y] \neq 0} f([y])(x, [y]), \; x \in B.
\]
It is obvious that \( p(x) \geq 0 \) and \( \int_B p(x) \, dm_B(x) = 1 \). Let \( \mu \) be the distribution on the group \( B \) with density \( p(x) \) with resect to \( m_B \). We have \( \hat{\mu}([y]) = f([y]), \; [y] \in A \). If we consider \( \mu \) as a distribution on the group \( X \), then the characteristic function \( \hat{\mu}(y) \) is of the form
\[
\hat{\mu}(y) = \hat{\mu}(l, h) = \begin{cases} 
1, & \text{if } l \in U, \; h = 0, \\
0, & \text{if } l \in U, \; h \neq 0, \\
\frac{1}{n}, & \text{if } l \in l_0 + U, \; h \in H, \\
0, & \text{if } l \notin U \cup (l_0 + U), \; h \in H. 
\end{cases}
\]

Obviously, \( \mu \) can not be represented in the form \( \mu = \rho \ast m_K \ast E_x \), where \( \sigma(\rho) \subseteq F, \; K \) is a finite subgroup of the group \( G, \; x \in X \). Let \( \xi_1 \) and \( \xi_2 \) be independent identically distributed random variables with values in the group \( X \) and distribution \( \mu \). It follows from (66) that \( L_{(2)} = L_1 \). Hence \( l = -l, \; l \in L \). We shall verify that the characteristic function \( f(l, h) = \hat{\mu}(l, h) \) satisfies the equation
\[
f(l + l', h + h')f(l + \tilde{\alpha}_Ll', h + \tilde{\alpha}_Hh') = f(l + l', h - h')f(l + \tilde{\alpha}_Ll', h - \tilde{\alpha}_Hh'), \; l, l' \in L, \; h, h' \in H. \tag{69}
\]
Then, by Lemma 1, the conditional distribution of the linear form \( L_2 = \xi_1 + \alpha \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric. The following cases exhaust all possibilities for \( l \) and \( l' \).

(i) \( l + l' \in U \) and \( l + \tilde{\alpha}_Ll' \in U \). It follows from (68) that the left-hand side and the right-hand side of equation (69) can take values either 1 or 0. Assume that the left-hand side of equation (69) is equal to 1. This implies that \( h + h' = 0 \) and \( h + \tilde{\alpha}_Hh' = 0 \). Hence,
\[
(I - \tilde{\alpha}_H)h' = 0. \tag{70}
\]
It follows from (67) that \( I - \tilde{\alpha}_H \in \text{Aut}(H) \), and (70) implies that \( h' = 0 \), and hence \( h = 0 \). It means that the right-hand side of equation (69) is also equal to 1. Reasoning similarly, we verify that if right-hand side of equation (69) is equal to 1, then the left-hand side of equation (69) is equal to 1. Thus, in this case both sides of equation (69) are equal.

(ii) \( l + l' \in l_0 + U \) and \( l + \tilde{\alpha}_Ll' \in l_0 + U \). It follows from (68) that both sides of equation (69) are \( \frac{1}{n} = \frac{1}{n} \).

(iii) \( l + l' \in U \) and \( l + \tilde{\alpha}_Ll' \in l_0 + U \). This implies that \( (I + \tilde{\alpha}_L)l' \in l_0 + U \). But this contradicts the fact that \( (l_0 + U) \cap (I + \tilde{\alpha}_L)(L) = \emptyset \). Thus, this case is impossible.

(iv) \( l + l' \in l_0 + U \) and \( l + \tilde{\alpha}_Ll' \in U \). The reasoning is similar to the case (iii).

(v) \( l + l' \notin l_0 + U \) and \( l + l' \notin U \). Then in view of (68) we have \( f(l + l', h + h') = f(l + l', h - h') = 0 \), and both sides of equation (69) are zero.

(vi) \( l + \tilde{\alpha}_Ll' \notin l_0 + U \) and \( l + \tilde{\alpha}_Ll' \notin U \). Then in view of (68) we have \( f(l + \tilde{\alpha}_Ll', h + \tilde{\alpha}_Hh') = f(l + \tilde{\alpha}_Ll', h - \tilde{\alpha}_Hh') = 0 \), and both sides of equation (69) are zero.

Proposition 2 is proved.
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