On \((\varepsilon)-para\) Sasakian 3-manifolds

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Abstract. In this paper we study the 3-dimensional \((\varepsilon)-para\) Sasakian manifolds. We obtain an necessary and sufficient condition for an \((\varepsilon)-para\) Sasakian 3-manifold to be an indefinite space form. We show that a Ricci-semi-symmetric \((\varepsilon)-para\) Sasakian 3-manifold is an indefinite space form. We investigate the necessary and sufficient condition for an \((\varepsilon)-para\) Sasakian 3-manifold to be locally \(\varphi\)-symmetric. It is proved that in an \((\varepsilon)-para\) Sasakian 3-manifold with \(\eta\)-parallel Ricci tensor the scalar curvature is constant. It is also shown that every \((\varepsilon)-para\) Sasakian 3-manifolds is pseudosymmetric in the sense of R. Deszcz.

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1 Introduction

In 1976, Säto [25] introduced a structure \((\varphi, \xi, \eta)\) satisfying \(\varphi^2 = I - \eta \otimes \xi\) and \(\eta(\xi) = 1\) on a differentiable manifold, which is now well known as an almost paracontact structure. The structure is an analogue of the almost contact structure [24, 10] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [27] introduced almost contact manifolds equipped with associated pseudo-Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as \((\varepsilon)-almost\) contact metric manifolds and \((\varepsilon)-Sasakian\) manifolds respectively [2, 14, 15]. Also, in 1989, K. Matsumoto [18] replaced the structure vector field \(\xi\) by \(-\xi\) in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field \(\xi\) is always timelike. These circumstances motivated the authors in [32] to associate a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite almost paracontact metric structure an \((\varepsilon)-almost\) paracontact structure, where the structure vector field \(\xi\) is spacelike or timelike according as \(\varepsilon = 1\) or \(\varepsilon = -1\).

In [32] the authors studied \((\varepsilon)-almost\) paracontact manifolds, and in particular, \((\varepsilon)-para\) Sasakian manifolds. They gave basic definitions, some examples of \((\varepsilon)-para\) paracontact manifolds and introduced the notion of an \((\varepsilon)-para\) Sasakian structure. The basic properties, some typical identities for curvature tensor and Ricci tensor of the \((\varepsilon)-para\)
Sasakian manifolds were also studied in [32]. The authors in [32] proved that if a semi-Riemannian manifold is one of flat, proper recurrent or proper Ricci-recurrent, then it can not admit an ($\varepsilon$)-para Sasakian structure. Also they showed that, for an ($\varepsilon$)-para Sasakian manifold, the conditions of being symmetric, semi-symmetric or of constant sectional curvature are all identical.

In this paper we study 3-dimensional ($\varepsilon$)-para Sasakian manifolds. The paper organized as follows. Section 2 is devoted to the some basic definitions and curvature properties of ($\varepsilon$)-para Sasakian manifolds. In section 2, we also prove that an ($\varepsilon$)-para Sasakian manifold is an indefinite space form if and only if the scalar curvature $r$ of the manifold is equal to $-6\varepsilon$. In section 3, we show that a Ricci-semi-symmetric ($\varepsilon$)-para Sasakian 3-manifold is an indefinite space form. In section 4, a necessary and sufficient condition for an ($\varepsilon$)-para Sasakian 3-manifold to be locally $\varphi$-symmetric is obtained. Section 5 contains some results on ($\varepsilon$)-para Sasakian 3-manifolds with $\eta$-parallel Ricci tensor. In last section 6, it is shown that every ($\varepsilon$)-para Sasakian 3-manifolds is pseudosymmetric in the sense of R. Deszcz.

2 Preliminaries

Let $M$ be an $n$-dimensional almost paracontact manifold [25] equipped with an almost paracontact structure ($\varphi, \xi, \eta$) consisting of a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

\begin{align}
\varphi^2 &= I - \eta \otimes \xi, \\
\eta(\xi) &= 1, \\
\varphi \xi &= 0, \\
\eta \circ \varphi &= 0.
\end{align}

Throughout this paper we assume that $X, Y, Z, U, V, W \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields in $M$, unless specifically stated otherwise. By a semi-Riemannian metric [23] on a manifold $M$, we understand a non-degenerate symmetric tensor field $g$ of type $(0,2)$. In particular, if its index is 1, it becomes a Lorentzian metric [1]. Let $g$ be a semi-Riemannian metric with $\text{index}(g) = \nu$ in an $n$-dimensional almost paracontact manifold $M$ such that

\begin{equation}
g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y),
\end{equation}

where $\varepsilon = \pm 1$. Then $M$ is called an ($\varepsilon$)-almost paracontact metric manifold equipped with an ($\varepsilon$)-almost paracontact metric structure ($\varphi, \xi, \eta, g, \varepsilon$) [32]. In particular, if $\text{index}(g) = 1$, then an ($\varepsilon$)-almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. In particular, if the metric $g$ is positive definite, then an ($\varepsilon$)-almost paracontact metric manifold is the usual almost paracontact metric manifold [25].

The equation (2.5) is equivalent to

\begin{equation}
g(X, \varphi Y) = g(\varphi X, Y)
\end{equation}
along with
\[ g(X, \xi) = \varepsilon \eta(X). \tag{2.7} \]
From (2.7) it follows that
\[ g(\xi, \xi) = \varepsilon, \tag{2.8} \]
that is, the structure vector field \( \xi \) is never lightlike. Defining
\[ \Phi(X, Y) \equiv g(X, \varphi Y), \tag{2.9} \]
we note that
\[ \Phi(X, \xi) = 0. \tag{2.10} \]

Let \((M, \varphi, \xi, \eta, g, \varepsilon)\) be an \((\varepsilon)-\)almost paracontact metric manifold (resp. a Lorentzian almost paracontact manifold). If \(\varepsilon = 1\), then \(M\) will be said to be a spacelike \((\varepsilon)-\)almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold). Similarly, if \(\varepsilon = -1\), then \(M\) will be said to be a timelike \((\varepsilon)-\)almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) [32]. Note that a timelike Lorentzian almost paracontact structure is a Lorentzian almost paracontact structure in the sense of Mihai and Rosca [20, 19], which differs in the sign of the structure vector field of the Lorentzian almost paracontact structure given by Matsumoto [18].

An \((\varepsilon)-\)almost paracontact metric structure is called an \((\varepsilon)-\)para Sasakian structure if
\[ \nabla_X \varphi Y = -g(\varphi X, \varphi Y)\xi - \varepsilon \eta(Y) \varphi^2 X, \tag{2.11} \]
where \(\nabla\) is the Levi-Civita connection with respect to \(g\). A manifold endowed with an \((\varepsilon)-\)para Sasakian structure is called an \((\varepsilon)-\)para Sasakian manifold [32]. In an \((\varepsilon)-\)para Sasakian manifold we have [32]
\[ \nabla \xi = \varepsilon \varphi, \tag{2.12} \]
\[ \Phi(X, Y) = g(\varphi X, Y) = \varepsilon g(\nabla_X \xi, Y) = (\nabla_X \eta) Y. \tag{2.13} \]

An \((\varepsilon)-\)almost paracontact metric manifold is called \(\eta\)-Einstein if its Ricci tensor \(S\) satisfies the condition
\[ S(X, Y) = ag(X, Y) + b \eta(X) \eta(Y). \tag{2.14} \]
The \(k\)-nullity distribution \(N(k)\) of a semi-Riemannian manifold \(M\) is defined by
\[ N(k) : p \to N_p(k) = \{ Z \in T_p M : R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y), \tag{2.15} \]
for all \(X, Y \in \mathfrak{X}(M)\), where \(k\) is some smooth function (see [29]). If \(M\) is an \(\eta\)-Einstein \((\varepsilon)-\)para Sasakian manifold and the structure vector field \(\xi\) belongs to the \(k\)-nullity distribution \(N(k)\) for some smooth function \(k\), then we say that \(M\) is an \(N(k)\)-\(\eta\)-Einstein \((\varepsilon)-\)para Sasakian manifold (see [31]).

In an \((\varepsilon)-\)para Sasakian manifold, the Riemann curvature tensor \(R\) and the Ricci tensor \(S\) satisfy the following equations [32]:
\[ R(X, Y) \xi = \eta(Y) X - \eta(X) Y, \tag{2.16} \]
\[ R(X, Y, Z, \xi) = -\eta(X)g(Y, Z) + \eta(Y)g(X, Z), \quad (2.17) \]
\[ \eta(R(X, Y) Z) = -\varepsilon \eta(X)g(Y, Z) + \varepsilon \eta(Y)g(X, Z), \quad (2.18) \]
\[ R(\xi, X) Y = -\varepsilon g(X, Y) \xi + \eta(Y) X, \quad (2.19) \]
\[ S(X, \xi) = -(n - 1)\eta(X). \quad (2.20) \]

It is known that in a semi-Riemannian 3-manifold
\[ R(X, Y) Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \]
\[ -\frac{r}{2} (g(Y, Z)X - g(X, Z)Y), \quad (2.21) \]
where \( Q \) is the Ricci operator and \( r \) is the scalar curvature of the manifold. If we substitute \( Z \) by \( \xi \) in (2.21) and use (2.16), we get
\[ \varepsilon(\eta(Y)QX - \eta(X)QY) = \left(1 + \frac{\varepsilon r}{2}\right)(\eta(Y)X - \eta(X)Y). \quad (2.22) \]
By putting \( Y = \xi \) in (2.22) and using (2.2) and (2.20) for \( n = 3 \), we obtain
\[ QX = \frac{1}{2}\{(r + 2\varepsilon)X - (r + 6\varepsilon)\eta(X)\xi\}, \]
that is,
\[ S(X, Y) = \frac{1}{2}\{(r + 2\varepsilon)g(X, Y) - \varepsilon(r + 6\varepsilon)\eta(X)\eta(Y)\}. \quad (2.23) \]
By using (2.23) in (2.21), we obtain
\[ R(X, Y) Z = \left(\frac{r}{2} + 2\varepsilon\right)\{g(Y, Z)X - g(X, Z)Y\} \]
\[ -\left(\frac{r}{2} + 3\varepsilon\right)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\} \]
\[ +\varepsilon\eta(Y)\eta(Z)X - \varepsilon\eta(X)\eta(Z)Y\}. \quad (2.24) \]

If an \((\varepsilon)\)-para Sasakian manifold is a space of constant curvature then it is an indefinite space form.

**Lemma 2.1** An \((\varepsilon)\)-para Sasakian 3-manifold is an indefinite space form if and only if the scalar curvature \( r = -6\varepsilon \).

**Proof.** Let a 3-dimensional \((\varepsilon)\)-para Sasakian manifold be an indefinite space form. Then
\[ R(X, Y) Z = c\{g(Y, Z)X - g(X, Z)Y\}, \quad X, Y, Z \in \mathfrak{X}(M), \quad (2.25) \]
where \( c \) is the constant curvature of the manifold. By using the definition of Ricci curvature and (2.25) we have
\[ S(X, Y) = 2c g(X, Y). \quad (2.26) \]
If we use (2.26) in the definition of the scalar curvature we get
\[ r = 6 c. \quad (2.27) \]
From (2.26) and (2.27) one can easily see that
\[ S(X, Y) = \frac{r}{3} g(X, Y). \]  
(2.28)

By putting \( X = Y = \xi \) in (2.23) and using (2.28) we obtain
\[ r = -6 \varepsilon. \]

Conversely, if \( r = -6 \varepsilon \) then from the equation (2.24) we can easily see that the manifold is an indefinite space form. This completes the proof. □

**Theorem 2.2** Every \((\varepsilon)-\text{para Sasakian} \) 3-manifold is an \(N(-\varepsilon)-\eta\)-Einstein manifold.

**Proof.** The proof follows from (2.23) and (2.16). □

3 **Ricci-semi-symmetric \((\varepsilon)-\text{para Sasakian} \) 3-Manifolds**

A semi-Riemannian manifold \( M \) is said to be Ricci-semi-symmetric [21] if its Ricci tensor \( S \) satisfies the condition
\[ R(X, Y) \cdot S = 0, \quad X, Y \in \mathfrak{X}(M), \]  
(3.1)

where \( R(X, Y) \) acts as a derivation on \( S \).

Let \( M \) be a Ricci-semi-symmetric \((\varepsilon)-\text{para Sasakian} \) 3-manifold. From (3.1) we have
\[ S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \]  
(3.2)

If we put \( Y = \xi \) and use (2.19), then we get
\[ \varepsilon g(X, U)S(\xi, V) - \eta(U)S(X, V) + \varepsilon g(X, V)S(U, \xi) - \eta(V)S(U, X) = 0. \]  
(3.3)

By using (2.20) in (3.3) we obtain
\[ 2\varepsilon g(X, U)\eta(V) + S(X, V)\eta(U) + 2\varepsilon g(X, V)\eta(U) + S(X, U)\eta(V) = 0. \]  
(3.4)

Consider that \( \{e_1, e_2, e_3\} \) be an orthonormal basis of the \( T_p M, \quad p \in M \). Then by putting \( X = U = e_i \) in (3.4) and taking the summation for \( 1 \leq i \leq 3 \), we have
\[ S(\xi, V) + 8\varepsilon \eta(V) + r\eta(V) = 0. \]  
(3.5)

Again by using (2.20) in (3.5), we get
\[ (r + 6\varepsilon)\eta(V) = 0, \]
which gives \( r = -6\varepsilon \). This implies, in view of Lemma 2.1, that the manifold is an indefinite space form.

Therefore, we can state the following

**Theorem 3.1** A Ricci-semi-symmetric \((\varepsilon)-\text{para Sasakian} \) 3-manifold is an indefinite space form.
4 Locally $\varphi$-Symmetric ($\varepsilon$)-para Sasakian 3-Manifolds

Analogous to the notion introduced by Takahashi [28] for Sasakian manifolds, we give the following definition.

**Definition 4.1** An ($\varepsilon$)-para Sasakian manifold is said to be locally $\varphi$-symmetric if

$$\varphi^2(\nabla_WR)(X, Y, Z) = 0,$$

for all vector fields $X$, $Y$, $Z$ orthogonal to $\xi$.

Now by taking covariant derivative of (2.24) with respect to $W$ and using (2.9) and (2.10) we have

$$\nabla_WR(X, Y, Z) = \frac{1}{2} (\nabla_W r) \left\{ g(Y, Z)X - g(X, Z)Y 
- g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi 
- \varepsilon \eta(Y)\eta(Z)X + \varepsilon \eta(X)\eta(Z)Y \right\}$$

$$+ \left( \frac{r}{2} + 3\varepsilon \right) \left\{ -g(Y, Z) (\Phi(X, W)\xi + \varepsilon \eta(X)\varphi W) 
+ g(X, Z) (\Phi(Y, W)\xi + \varepsilon \eta(Y)\varphi W) 
- \varepsilon (\Phi(Y, W)\eta(Z) + \Phi(Z, W)\eta(Y))X 
+ \varepsilon (\Phi(X, W)\eta(Z) + \Phi(Z, W)\eta(X))Y \right\}. \tag{4.1}$$

Then by taking $X$, $Y$, $Z$ orthogonal to $\xi$ and using (2.1), (2.3), (2.4) and (2.7), from (4.1) we obtain

$$\varphi^2(\nabla_WR)(X, Y, Z) = \frac{1}{2} (\nabla_W r) (g(Y, Z)X - g(X, Z)Y). \tag{4.2}$$

Hence from (4.2) we can state the following theorem:

**Theorem 4.2** A 3-dimensional ($\varepsilon$)-para Sasakian manifold is locally $\varphi$-symmetric if and only if the scalar curvature $r$ is constant.

If a 3-dimensional ($\varepsilon$)-para Sasakian manifold is Ricci-semi-symmetric then we have showed that $r = -6\varepsilon$ that is $r$ is constant. Therefore from (4.2), we have

**Theorem 4.3** A 3-dimensional Ricci-semi-symmetric ($\varepsilon$)-para Sasakian manifold is locally $\varphi$-symmetric.

In particular, by taking $Z = \xi$ in (4.1) we have

$$\nabla_WR(X, Y, \xi) = \left( \frac{\varepsilon r}{2} + 3 \right) \left\{ -\eta(Y)\Phi(X, W)\xi + \eta(X)\Phi(Y, W)\xi 
- \Phi(Y, W)X + \Phi(X, W)Y \right\}. \tag{4.3}$$

Applying $\varphi^2$ to the both sides of (4.3) we get

$$\varphi^2(\nabla_WR)(X, Y, \xi) = \left( \frac{\varepsilon r}{2} + 3 \right) \left\{ -\Phi(Y, W)\varphi^2 X + \Phi(X, W)\varphi^2 Y \right\}. \tag{4.4}$$
If we take $X, Y, W$ orthogonal to $\xi$ in (4.3) and (4.4) we have
\[
\varphi^2(\nabla_W R)(X, Y, \xi) = (\nabla_W R)(X, Y, \xi).
\]
Now we can state the following:

**Theorem 4.4** Let $M$ be an $(\varepsilon)$-para Sasakian 3-manifold such that
\[
\varphi^2(\nabla_W R)(X, Y, \xi) = 0
\]
for all $X, Y, W \in \mathfrak{X}(M)$, orthogonal to $\xi$. Then $M$ is an indefinite space form.

## 5 $\eta$-Parallel $(\varepsilon)$-para Sasakian 3-Manifolds

Motivated by the definitions of Ricci $\eta$-parallelity for Sasakian manifolds and $LP$-Sasakian manifolds were given by Kon [16] and Shaikh and De [26], respectively, we give the following

**Definition 5.1** Let $M$ be an $(\varepsilon)$-para Sasakian manifold. If the Ricci tensor $S$ satisfies
\[
(\nabla_X S)(\varphi Y, \varphi Z) = 0, \quad X, Y, W \in \mathfrak{X}(M),
\]
then the manifold $M$ is said to be $\eta$-parallel.

**Proposition 5.2** Let $M$ be an $(\varepsilon)$-para Sasakian 3-manifold with $\eta$-parallel Ricci tensor. Then the scalar curvature $r$ is constant.

**Proof.** From (2.23) by using (2.5) and (2.4)
\[
S(\varphi X, \varphi Y) = \left(\frac{r}{2} + \varepsilon\right)(g(X, Y) - \varepsilon\eta(X)\eta(Y)).
\]
If we take the covariant derivative of (5.2) with respect to $Z$ and (2.13), we get
\[
(\nabla_Z S)(\varphi X, \varphi Y) = \frac{1}{2} \{(\nabla_Z r)(g(X, Y) - \varepsilon\eta(X)\eta(Y))
- \varepsilon(r + 2\varepsilon)(\Phi(X, Z)\eta(Y) + \Phi(Y, Z)\eta(X))\}. \tag{5.3}
\]
Since $M$ is an $(\varepsilon)$-para Sasakian 3-manifold with $\eta$-parallel Ricci tensor, then from (5.1) we have
\[
(\nabla_Z r)\{g(X, Y) - \varepsilon\eta(X)\eta(Y)\} - \varepsilon(r + 2\varepsilon)\{\Phi(X, Z)\eta(Y) + \Phi(Y, Z)\eta(X)\} = 0. \tag{5.4}
\]
Consider that $\{e_1, e_2, e_3\}$ be an orthonormal basis of the $T_pM, \ p \in M$. Then by substituting both $X$ and $Y$ by $e_i$, $1 \leq i \leq 3$, in (5.4) and then taking summation over $i$ and using (2.10) we obtain
\[
\nabla_Z r = 0, \quad Z \in \mathfrak{X}(M).
\]
This completes the proof. ■

In view of Theorem 4.2 and Proposition 5.2 we have the following:
Theorem 5.3 An \((\varepsilon)\)-para Sasakian 3-manifold with \(\eta\)-parallel Ricci tensor is locally \(\varphi\)-symmetric.

Remark 5.4 An \((\varepsilon)\)-para Sasakian manifold is called Lorentzian para Sasakian manifold if \(\varepsilon = -1\) and \(\text{index}(g) = 1\). Therefore, some results we obtained in the previous three sections can be considered as a generalization of some results obtained by the authors in [26].

Remark 5.5 In an \((\varepsilon)\)-almost para contact 3-manifold, we observe that \(\text{trace}\varphi = 0\). Therefore, the assumption \(\text{trace}\varphi \neq 0\) in [26] may not help in proving several results and some proofs in these papers must be changed if the results are true anymore.

6 Pseudosymmetric \((\varepsilon)\)-para Sasakian 3-Manifolds

Now, we consider a well known generalization of the concept of an \(\eta\)-Einstein almost paracontact metric manifold in the following

Definition 6.1 [7] A non-flat \(n\)-dimensional Riemannian manifold \((M, g)\) is said to be a quasi Einstein manifold if its Ricci tensor \(S\) satisfies

\[
S = ag + b\eta \otimes \eta
\]

or equivalently, its Ricci operator \(Q\) satisfies

\[
Q = aI + b\eta \otimes \xi
\]

for some smooth functions \(a\) and \(b\), where \(\eta\) is a nonzero 1-form such that

\[
g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1
\]

for the associated vector field \(\xi\). The 1-form \(\eta\) is called the associated 1-form and the unit vector field \(\xi\) is called the generator of the quasi Einstein manifold.

B. Y. Chen and K. Yano [8] defined a Riemannian manifold \((M, g)\) to be of quasi-constant curvature if it is conformally flat manifold and its Riemann-Christoffel curvature tensor \(R\) of type \((0, 4)\) satisfies the condition

\[
R(X, Y, Z, W) = a\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}
+b\{g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W)
+ g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)\}
\]

for all \(X, Y, Z, W \in \mathfrak{X}(M)\), where \(a, b\) are some smooth functions and \(T\) is a non-zero 1-form defined by

\[
g(X, \rho) = T(X), \quad X \in \mathfrak{X}(M)
\]

for a unit vector field \(\rho\). On the other hand, Gh. Vrâncican [33] defined a Riemannian manifold \((M, g)\) to be of almost constant curvature if \(M\) satisfies (6.4). Later on, it was pointed out by A. L. Mocanu [22] that the manifold introduced by Chen and Yano and the manifold introduced by Vrâncican were identical, as it can be verified that if the curvature tensor \(R\) is of the form (6.4), then the manifold is conformally flat. Thus, a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor \(R\) satisfies (6.4). If \(b = 0\), then the manifold reduces to a manifold of constant curvature.
Example 6.2 A manifold of quasi-constant curvature is a quasi Einstein manifold [11, Example 1]. Conversely, a conformally flat quasi Einstein manifold of dimension \( n \) \(( n > 3)\) is a manifold of quasi-constant curvature [12, Theorem 4].

Let \((M, g)\) be a semi-Riemannian manifold with its Levi-Civita connection \( \nabla \). A tensor field \( F \) of type \((1, 3)\) is known to be curvature-like provided that \( F \) satisfies the symmetric properties of the curvature tensor \( R \). For example, the tensor \( R_g \) given by

\[
R_g(X, Y) Z \equiv (X \wedge g Y) Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y \in \mathfrak{X}(M),
\]

is a trivial example of a curvature like tensor. Sometimes, the symbol \( R_g \) seems to be much more convenient than the symbol \((X \wedge g Y) Z\). For example, a semi-Riemannian manifold \((M, g)\) is of constant curvature \( c \) if and only if \( R = cR_g \).

It is well known that every curvature-like tensor field \( F \) acts on the algebra \( \mathfrak{T}_s^1(M) \) of all tensor fields on \( M \) of type \((1, s)\) as a derivation [23, p. 44]:

\[
(F \cdot P)(X_1, \ldots, X_s; Y, X) = F(X, Y) \{P(X_1, \ldots, X_s)\} - \sum_{j=1}^s P(X_1, \ldots, F(X, Y) X_j, \ldots, X_s)
\]

for all \( X_1, \ldots, X_s \in \mathfrak{X}(M) \), \( P \in \mathfrak{T}_s^1(M) \). The derivative \( F \cdot P \) of \( P \) by \( F \) is a tensor field of type \((1, s + 2)\). A semi-Riemannian manifold \((M, g)\) is said to be semi-symmetric if \( R \cdot R = 0 \). Obviously, locally symmetric spaces \((\nabla R = 0)\) are semi-symmetric. More generally, a semi-Riemannian manifold \((M, g)\) is said to be pseudo-symmetric (in the sense of R. Deszcz) [13] if \( R \cdot R \) and \( R_g \cdot R \) in \( M \) are linearly dependent, that is, if there exists a real valued smooth function \( L : M \to \mathbb{R} \) such that

\[
R \cdot R = LR_g \cdot R
\]

is true on the set

\[
U = \left\{ x \in M : R \neq \frac{r}{n(n-1)} R_g \text{ at } x \right\}.
\]

A pseudo-symmetric space is said to be proper if it is not semi-symmetric. For details we refer to [5, 3].

In the literature, there is also another notion of pseudo-symmetry. A semi-Riemannian manifold \((M, g)\) is said to be pseudo-symmetric in the sense of Chaki [6] if

\[
(\nabla R)(X_1, X_2, X_3, X_4; X) = 2\omega(X)R(X_1, X_2, X_3, X_4) + \omega(X_1)R((X, X_2, X_3, X_4)
+ \omega(X_2)R((X_1, X, X_3, X_4) + \omega(X_3)R((X_1, X_2, X, X_4)
+ \omega(X_4)R((X_1, X_2, X_3, X)
\]

for all \( X_1, X_2, X_3, X_4; X \in \mathfrak{X}(M) \), where \( \omega \) is a 1-form on \((M, g)\). Of course, both the definitions of pseudo-symmetry for a semi-Riemannian manifold are not equivalent. For example, in contact geometry, every Sasakian space form is pseudo-symmetric in the sense of Deszcz [4, Theorem 2.3], but a Sasakian manifold cannot be pseudo-symmetric in the sense of Chaki [30, Theorem 1]. We assume the pseudo-symmetry always in the sense of Deszcz, unless specifically stated otherwise.

For Riemannian 3-manifolds, the following characterization of pseudosymmetry is known (cf. [17, 9]).
Proposition 6.3 A 3-dimensional Riemannian manifold \((M, g)\) is pseudo-symmetric if and only if it is quasi-Einstein, that is, if and only if there exists a 1-form \(\eta\) such that the Ricci tensor field \(S\) satisfies \(S = ag + b\eta \otimes \eta\) for some smooth functions \(a\) and \(b\).

In view of the above Proposition, we can state the following:

Theorem 6.4 Every 3-dimensional \(\eta\)-Einstein \((\varepsilon)\)-almost paracontact metric manifold is always pseudo-symmetric. In particular, each 3-dimensional \((\varepsilon)\)-para Sasakian manifold is pseudo-symmetric.

Problem 6.5 It would be interesting to know whether an \((\varepsilon)\)-almost para Sasakian manifold is pseudo-symmetric in the sense of Chaki or not.

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