Conformal symmetry of the phase space formulation for topological string actions

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It is proved the conformal invariance of the phase space formulation for topological string actions associated with the number of handles and the number of self-intersections of the world surface. Differences and similarities with the phase space formulation of an Abelian gauge theory are discussed.

I. Introduction

As it is well known, the conformal symmetry plays a crucial role in the quantum consistency of string theory, and constitutes a way in which the celebrated critical dimension anomaly occurs. Specifically, the anomalous scaling behavior at a quantum level for bosonic string theory disappears only if the background dimension is 26, and the classical conformal symmetry is thus preserved. These results have been established considering basically a string action that is proportional to the area swept out by the worldsheet.

On the other hand, the topological invariants in Lagrangians for string theory have been completely neglected because they do not give dynamics to the string, and additionally such topological actions only contribute as global factors in the path integral formulation of the theory. Thus, one may think that the topological terms may have not relevant quantum effects, and particularly will have not effects on the quantum consistency of the theory.

However, this is only illusory, since recently [1, 2] it has been shown that the topological terms have a dramatic effect on the phase space formulation for the theory, and even have, by themselves, a nontrivial phase space formulation, which mimics in fact the symplectic structure of an Abelian gauge...
theory [3]. Consequently the topological invariants will have by themselves a nontrivial quantum field theory, despite having trivial classical dynamics.

With these preliminaries, it is natural to ask on the role that the conformal invariance will play in the quantum aspects of topological string actions. In this work we want to show, as a first step in that direction, that the phase space formulation for these topological actions preserves the conformal symmetry. Considering the analogy established with an Abelian gauge theory, we make additionally a comparison of the roles that the conformal invariance plays in both cases. For this purpose, this work is organized as follows.

In the next section we set up the basic ideas on a conformal transformation of the background metric and related formulas in the imbedding supported background tensor approach for the differential geometry of an imbedded surface developed by Carter in [4]. In Section III we outline the conformal properties of an Abelian gauge theory, with the purpose of making a comparison with the case of a topological string action developed in Section IV. We finish in Section V with some remarks and prospects for the future.

II. Conformal transformation

The general conformal transformation of the background metric

\[ g_{\mu\nu} \rightarrow e^{-2\sigma} g_{\mu\nu}, \]

where \( \sigma \) is a local arbitrary function, induces a shift on the corresponding covariant derivative acting on a vector field \( V_\nu \),

\[ \nabla_\mu V_\nu \rightarrow \nabla_\mu V_\nu - L^\lambda_{\mu\nu} V_\lambda, \]

where

\[ L^\lambda_{\mu\nu} = -2\delta^\lambda_{(\mu} \nabla_{\nu)} \sigma + g_{\mu\nu} \nabla^\lambda \sigma, \]

which is symmetric in \( (\mu\nu) \). Additionally (1) induces a change on the fields with support confined to the world surface [4]

\[ n^{\mu\nu} \rightarrow e^{2\sigma} n^{\mu\nu}, \quad \perp^{\mu\nu} \rightarrow e^{2\sigma} \perp^{\mu\nu}, \]
\[ i_A^\mu \to e^\sigma i_A^\mu, \quad \mathcal{E}^{\mu\nu} \to e^{2\sigma} \mathcal{E}^{\mu\nu}, \quad (4) \]

where \( n^{\mu\nu} \) is the (first) fundamental tensor of the world-surface, and \( \perp^{\mu\nu} \) its complementary orthogonal projection. Additionally the frame vectors \( \{ i_A^\mu \} \) are tangential to the world-surface and \( \mathcal{E}^{\mu\nu} = \mathcal{E}^{[\mu\nu]} = \mathcal{E}^{AB} i_A^\mu i_B^\nu \), where \( \mathcal{E}^{AB} \) is the constant components of the standard two-dimensional flat space alternating tensor.

Obviously \( n^{\mu\nu} \to n^{\mu\nu}, \perp^{\mu\nu} \to \perp^{\mu\nu}, \) and \( \mathcal{E}^{\mu\nu} \to \mathcal{E}^{\mu\nu} \). Considering the definition of the internal gauge connection \[4\]
\[ \rho_{\lambda}^{\mu\nu} = n^{\mu\alpha} i_\nu^A \nabla_\lambda i_{\alpha A}, \quad \nabla_\lambda = n_\alpha^\lambda \nabla_\alpha, \]
and using Eqs. (2)-(4), it is very easy to find the shift of \( \rho_{\lambda}^{\mu\nu} \) under the transformation (1):
\[ \rho_{\lambda}^{\mu\nu} \to \rho_{\lambda}^{\mu\nu} + 2 n_{\lambda[\nu} \nabla^{\mu]} \sigma, \quad (5) \]
and thus, considering that \( \rho_{\lambda} = \rho_{\lambda}^{\mu\nu} \mathcal{E}^{\nu}_{\mu} \), we have
\[ \rho_{\lambda} \to \rho_{\lambda} - 2 \mathcal{E}^{\nu}_{\lambda} \nabla_{\nu} \sigma. \quad (6) \]

Similarly, considering the expression \( R = \nabla_\mu (\mathcal{E}^{\mu\nu} \rho_\nu) = \mathcal{E}^{\mu\nu} \nabla_\mu \rho_\nu \) for the internal scalar curvature of the two-dimensional world surface \[4\], we can find that
\[ R \to e^{2\sigma} [R + 2 \nabla_\mu \nabla^\mu \sigma], \quad (7) \]
which implies that the Euler characteristic of the embedding two-surface \[4\]
\[ \chi = \int \sqrt{-\gamma} \ R \ d\Sigma, \quad (8) \]
related geometrically with the number of handles of the world-surface, undergoes the transformation
\[ \chi = \chi + 2 \int \sqrt{-\gamma} \ \nabla_\mu \nabla^\mu \sigma \ d\Sigma, \quad (9) \]
and then \( \chi \) is, modulo a total divergence, conformally invariant, as it is well known in the literature. In Eq. (9) we have considered that \( \sqrt{-\gamma} \to e^{-2\sigma} \sqrt{-\gamma} \).
III. Conformal symmetry in an Abelian gauge theory

As it is well known in the literature, the Abelian gauge theory described by the equations

\begin{align}
\nabla_\mu F_{\mu \nu} &= 0, \\
\nabla_{(\mu} F_{\alpha \beta)} &= 0,
\end{align}

is conformally invariant only if the background spacetime dimension and the conformal weight of $F_{\mu \nu}$ are fixed to be 4 and 0 respectively. Additionally, if we set up the conformal weight of $A_\nu$ to be 0, the covariant and gauge-covariant symplectic structure for the theory given by [5]

\[ \tilde{\omega} = \int_\Sigma \delta F_{\mu \nu} \delta A_\nu \, d\Sigma, \]

where $\Sigma$ is a spacelike hypersurface, is also conformally invariant. Under these conditions, the left-hand side of Eq. (10) transforms homogeneously under (1),

\[ (\nabla_\mu F_{\mu \nu})' = e^{2\sigma} \nabla_\mu F_{\mu \nu}, \]

which implies that if (and only if) the Eq. (10) is satisfied, the conformally transformed version also is.

Similarly, the left-hand side of Eq. (11) transforms as

\[ (\nabla_{(\mu} F_{\alpha \beta)})' = \nabla_{(\mu} F_{\alpha \beta)}. \]

IV. Conformal symmetry of the phase space formulation for $\chi$

As shown in [3], the phase space formulation for the topological string action $\chi$ is given in terms of the equations

\begin{align}
\nabla_\mu E_{\mu \nu} &= 0, \\
\n\nabla \{ \beta n^\rho \nabla_\gamma (R E_{\sigma \rho}) &= 0, \quad (16)
\end{align}
and the covariant and gauge-invariant symplectic structure given by
\[
\omega = \int \delta(\sqrt{-\gamma} \, E^{\mu\nu}) \, \delta \rho \, d\Sigma, \quad (17)
\]
which mimic entirely the mathematical structure of that of the Abelian gauge theory given in terms of Eqs. (10), (11), (12), in the indicated order [3].

As we shall see in this section, despite the close analogy between the phase space formulation of both theories, there exist important differences in the conditions required for ensuring the conformal invariance of those formulations.

We suppose first that we do not know the conformal weight of \( E^{\mu\nu} \) (-2, according to (Eq. (4)), and to consider that in general
\[
E^{\mu\nu} \rightarrow e^{s\sigma} E^{\mu\nu},
\]
and let us prove that the conformal invariance of Eq. (15) requires precisely that conformal weight induced on \( E^{\mu\nu} \) by the transformation (1).

Considering the left-hand side of Eq. (15) conformally transformed, and the Eqs. (2)-(4), we have
\[
(n^{\mu\alpha})' \nabla'_{\alpha} E'_{\mu\nu} = e^{2\sigma} n^{\mu\alpha}[\nabla_{\alpha}(e^{s\sigma} E^{\mu\nu}) - L^{\lambda}_{\alpha\mu}(e^{s\sigma} E^{\lambda\nu}) - L^{\lambda}_{\alpha\nu}(e^{s\sigma} E^{\mu\lambda})],
\]
\[
(\nabla' E_{\mu\nu})' = e^{(s+2)\sigma} \nabla' E_{\mu\nu} - (s + 2)e^{(s+2)\sigma} E_{\mu\nu} \nabla' \sigma, \quad (18)
\]
where we have considered the symmetry of \( n^{\mu\nu} \), the antisymmetry of \( E^{\mu\nu} \), and that \( n^{\mu\nu} n_{\mu\nu} = 2 \) for a two-dimensional world surface [4].

Therefore, Eq. (18) shows in a manifest way that if \( s = -2 \), the conformal weight naturally induced on \( E_{\mu\nu} \) by (1), \( \nabla' E_{\mu\nu} \) is strictly a conformal invariant without any restriction on the background dimension. In the case of the Abelian gauge theory, \( \nabla' F_{\mu\nu} \) is not strictly a conformal invariant (see Eq. (13)), but it transforms homogeneously under (1). In this manner, if (and only if) Eq. (15) holds, its conformally transformed version is also satisfied.

On the other hand, Eq. (16) corresponds to the closure of the two-form \( RE_{\mu\nu} \), which ensures that locally it can be written as the exterior derivative (\( \bar{\partial} \)) of the one-form \( \rho_{\mu} \) [4],
\[
\bar{\partial}(RE) = 0 \iff RE = \bar{\partial}\rho. \quad (19)
\]
In passing, we point out a mistake in the equivalence relations (A.10), and (A.14) in Ref. [4], whose left-hand sides must be not \( \partial\bar{\partial}F \), and \( \bar{\partial}\partial F \), but only \( \partial F \) (or \( \partial\partial A \)), and \( \bar{\partial}F \) (or \( \bar{\partial}\partial A \)) respectively.
In components, the right-hand side of the equivalence relation (19) is expressed as

\[ R E_{\mu \nu} = (\partial_{\rho})_{\mu \nu} = 2 n_{[\nu}^{\sigma} \nabla_{\rho]} \rho_{\sigma}, \quad (20) \]

and Eq. (16) corresponds, in components, to the left-hand side of (19).

Let us prove that this closure-exactness property is preserved under (1). Considering Eqs. (2), (6), and (20), \( R E \) undergoes the transformation

\[ (R E_{\mu \nu})' = R E_{\mu \nu} + 2 n^{\alpha}_{[\nu} \nabla_{\mu]} B_{\alpha}, \quad (21) \]

where \( B_{\alpha} = 2 \nabla_{\nu} (E_{\alpha}^{\nu}) \) is the conformal shift of \( \rho_{\mu} \) (see Eq. (6)). Equation (21) implies that \( R E \) changes by the exterior derivative of \( B \);

\[ (\partial B)_{\mu \nu} = 2 n^{\alpha}_{[\mu} \nabla_{\nu]} B_{\alpha}. \quad (22) \]

Therefore, considering Eqs. (2), and (21) we have that the left-hand side of (19) conformally transformed is given by

\[ [n^{\sigma}_{[\beta} n^{\rho}_{\gamma} \nabla_{\alpha]} (R E_{\sigma \rho})]' = n^{\sigma}_{[\beta} n^{\rho}_{\gamma} n^{\mu}_{\alpha} \nabla_{\mu} (R E_{\sigma \rho})' = n^{\sigma}_{[\beta} n^{\rho}_{\gamma} \nabla_{\alpha]} (R E_{\sigma \rho}) + n^{\sigma}_{[\beta} n^{\rho}_{\gamma} \nabla_{\alpha]} [2 n^{\lambda}_{\sigma} \nabla_{\rho] B_{\lambda}], \quad (23) \]

where the terms proportional to \( L_{\mu \nu}^{\lambda} \) vanish as a consequence of the symmetry of \( L_{\mu \nu}^{\lambda} \), and the antisymmetry of \( E_{\mu \nu} \) and of the exterior derivative of \( B \) (22). Equation (23) corresponds, in components, to the identity

\[ \partial (R E)' = \partial (R E) + \partial \partial B. \quad (24) \]

The last term in (24) vanishes identically because of the nilpotency of the exterior derivative \( \partial^2 = 0 \) [4] , and then \( \partial (R E) \) is a strict conformal invariant (such as the case of an Abelian gauge theory, Eq. (14)), and its vanishing is guaranteed under a conformal transformation.

We analyze now the conformal symmetry of the symplectic structure (17). Considering that \( E_{\mu \nu} \rightarrow e^{2 \sigma} E_{\mu \nu} \), and \( \sqrt{-\gamma} \rightarrow e^{-2 \sigma} \sqrt{-\gamma} \), then \( \sqrt{-\gamma} E_{\mu \nu} \) turns out to be a conformal invariant, and consequently its variation \( \delta(\sqrt{-\gamma} E_{\mu \nu}) \) is also a strict conformal invariant. On the other hand, the conformal shift on the connection \( \rho_{\lambda} \) in Eq. (6), induces a corresponding shift on its variation given by

\[ \delta \rho_{\lambda} \rightarrow \delta \rho_{\lambda} - 2 E'_{\nu \lambda} \nabla_{\nu} \rho_{\sigma}, \quad (25) \]
and then the conformal shift on the symplectic structure $\omega$ is given by

$$
\omega' = \omega + \int E^\alpha_\nu \delta(\sqrt{-\gamma} \, \mathcal{E}^{\mu\nu}) \nabla_\alpha \sigma \, d\Sigma_\mu. \tag{26}
$$

Let us prove now that the second term on the right-hand side is a total divergence. Considering the symmetry $E^\alpha_\nu \delta(\sqrt{-\gamma} \, \mathcal{E}^{\mu\nu}) = E^\mu_\nu \delta(\sqrt{-\gamma} \, \mathcal{E}^{\alpha\nu})$, and that $\nabla_\mu[\delta(\sqrt{-\gamma} \, \mathcal{E}^{\mu\nu})] = 0$ [3], such a term can be rewritten as

$$
\int E^\alpha_\nu \delta(\sqrt{-\gamma} \, \mathcal{E}^{\mu\nu}) \nabla_\alpha \sigma \, d\Sigma_\mu = \int E^\mu_\nu \nabla_\alpha[\delta(\sqrt{-\gamma} \, \mathcal{E}^{\alpha\nu}) \, \sigma] \, d\Sigma_\mu \tag{27}
$$

$$
= \int_{\Sigma} \nabla_\alpha[\delta(\sqrt{-\gamma} \, \mathcal{E}^{\alpha\nu}) \, \sigma] \, d\Sigma_\mu - \int_{\Sigma} \sigma \delta(\sqrt{-\gamma} \, \mathcal{E}^{\alpha\nu}) (\nabla_\alpha \mathcal{E}^\mu_\nu) \, d\Sigma_\mu,
$$

where the integrand of the last term vanishes, because $\nabla_\alpha \mathcal{E}^\mu_\nu = K^\alpha_{\tau \nu} [\mathcal{E}^\mu_\nu]$ and then $K^\alpha_{\tau \mu} d\Sigma_\mu = 0$, and $\delta(\sqrt{-\gamma} \, \mathcal{E}^{\mu\nu}) K^\alpha_{\tau \nu} = 0$ ($K^\alpha_{\tau \mu}$ is orthogonal in its last indice, and $d\Sigma_\mu$ and $\delta(\sqrt{-\gamma} \, \mathcal{E}^{\mu\nu})$ are tangential in their indices). Therefore, the symplectic structure changes by a total derivative, such as the action $\chi$ itself (see Eq. (9)), under a conformal transformation, and imposing the appropriate compactness properties on the field variations at the boundary $\partial \Sigma$, we have finally that $\omega$ is a conformal invariant, without any restriction on the background dimension.

V. Remarks and prospects

In this manner, considering the basic idea of a phase space formulation of preserving the relevant symmetries of the theory[5], the conformal symmetry of the action $\chi$ is preserved in its corresponding canonical formalism given by Eqs. (15), (16), and (17), without any restriction.

As it is well known in (bosonic) string theory, the classical symmetry is preserved in the quantum domain only if the background dimension is fixed to be 26. One can ask if $\chi$ has a relevant effect on this particular question, since the quantization of $\chi$ can be achieved, in principle, using the phase space formulation studied here. Works along these lines are in progress, and will be subject of future communications.

There exists another topological invariant associated with the two-dimensional world surface, and related geometrically with the number of self-intersections of such a surface. The corresponding
phase space formulation proves to have also the form of an Abelian gauge theory [3], and preserves
the conformal symmetry of the original action in an entirely similar way to the case developed here.
However, this case is from the beginning dimensionally restricted, since the topological invariance
of the action requires a 4-dimensional background. This dimensional restriction makes its analogy
with an Abelian gauge theory closer than that of $\chi$, in accordance with the results presented in Sec.
III.

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References

[1] R. Cartas-Fuentevilla, J. Math. Phys., 45, 602 (2004), math-ph/0404004.
[2] R. Cartas-Fuentevilla, and A. Escalante, Topological terms and the global symplectic geometry of
the phase space in string theory, Trends in Math. Phys., Nova publishing, to be published (2004),
math-ph/0404001.
[3] R. Cartas-Fuentevilla, Fluctuating topological invariants in string theory as an Abelian gauge
theory, submitted to Phys. Lett. B., (2004), math-ph/0404011.
[4] B. Carter, J. Geom. Phys., 8, 53 (1992).
[5] C. Crncović and E. Witten, in Three Hundred Years of Gravitation, edited by S. W. Hawking
and W. Israel (Cambridge University Press. Cambridge, 1987).