Controllability of ensembles of linear dynamical systems

Michael Schönlein and Uwe Helmke∗

Institute for Mathematics
University of Würzburg
Emil-Fischer Straße 40
97074 Würzburg
Germany

April 24, 2015

Abstract

We investigate the task of controlling ensembles of initial and terminal state vectors of parameter-dependent linear systems by applying parameter-independent open loop controls. Necessary, as well as sufficient, conditions for ensemble controllability are established, using tools from complex approximation theory. For real analytic families of linear systems it is shown that ensemble controllability holds only for systems with at most two independent parameters. We apply the results to networks of linear systems and address the question of open-loop robust synchronization.

Keywords: Polynomial approximations, ensemble control, parameter-dependent linear systems.

1 Introduction

Driven by recent engineering applications the task of controlling ensembles of system by open loop controls has gained considerable attraction. The motivation for this study originates, for instance, from quantum control [13], the control of spatially invariant systems such as platoons [1] [4], the control of partial differential equations [3], and flocks of systems [2].

From a functional analytic point of view, the problem of ensemble controllability is equivalent to approximate controllability of infinite-dimensional systems defined on Banach or Hilbert spaces [9]. Standard characterizations of approximate controllability in Hilbert spaces can be found in [5] [7]. However, the results in [5] [11] are, except for very special cases, not easily applicable for ensemble control as they depend on the existence of a Riesz basis of eigenvectors. A new function theory approach to uniform ensemble controllability has been developed in [9], using classical approximation theoretic results, such as the Stone-Weierstrass Theorem [6] and Mergelyan’s Theorem [17].

In this paper, recent results on ensemble control by [9] [15] [16] are extended in several directions. The necessary and sufficient conditions for ensemble control in [9] are extended to finite unions of disjoint parameter intervals. Somewhat surprisingly, we show that ensemble controllability of real analytic families of systems holds only if the parameter space is at most two-dimensional. New explicit characterizations of uniform ensemble controllability are derived for a special class of one-parameter families of systems. Finally, we discuss an application to robust synchronization for circular interconnected homogeneous networks.

∗(helmke,schoenlein)@mathematik.uni-wuerzburg.de
2 Polynomial characterizations of uniform ensemble controllability

We consider parameter-dependent linear time-invariant systems, described in continuous-time by
\[
\Sigma(\theta) : \begin{cases}
\frac{\partial}{\partial t}x(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t) \\
x(0, \theta) = 0,
\end{cases}
\] (1)
and in discrete-time, by
\[
\Sigma(\theta) : \begin{cases}
x_{t+1}(\theta) = A(\theta)x_t(\theta) + B(\theta)u_t \\
x_0(\theta) = 0.
\end{cases}
\] (2)

Since we want explore reachability properties of these systems, the initial state is taken to be zero. Note that in the continuous-time case there is no difference between reachability and controllability. In the sequel we will only use the term controllability, with the caveat that for discrete-time systems this has to be interpreted as reachability. Recall that a system (1), or (2), is reachable for a fixed parameter \(\theta\) if and only if the Kalman matrix has full rank, i.e.
\[
\text{rank}(B(\theta)A(\theta)B(\theta) \cdots A(\theta)^{n-1}B(\theta)) = n.
\]

Throughout the paper we assume that the parameter \(\theta\) varies in a compact subset \(P \subset \mathbb{R}^d\) and the system matrices \(A(\theta) \in \mathbb{R}^{n \times n}, B(\theta) \in \mathbb{R}^{n \times m}\) are continuous functions in \(\theta\), i.e. \(A \in C(P, \mathbb{R}^{n \times n})\) and \(B \in C(P, \mathbb{R}^{m \times n})\). We refer to such parameter-dependent families of systems and initial/final states as an ensemble. Throughout this paper we assume that \(P\) is a nonempty compact subset of \(\mathbb{R}^d\) with \(\text{int} P = P\). This implies that the dimension of \(P\) is well defined and satisfies \(\text{dim} P = d\). It excludes the well-understood case of parallel interconnected linear systems, where \(P\) is a finite set.

In this paper we address the following open loop control task for parametric families (1) and (2). Let
\[
\varphi(T, \theta, u) = \int_0^T e^{A(\theta)(T-s)}B(\theta)u(s) \, ds
\]
and
\[
\varphi(T, \theta, u) = \sum_{k=0}^{T-1} A(\theta)^kB(\theta)u_{T-1-k},
\]
denote the solutions of (1) and (2), respectively. An ensemble \(\Sigma(\theta)\) is called uniformly ensemble controllable, if for any \(x^* \in C(P, \mathbb{R}^n)\) and any \(\varepsilon > 0\) there exists a \(T > 0\) and an input function \(u \in L^1([0, T], \mathbb{R}^m)\) (or, in discrete-time, a finite input sequence \(u_0, ..., u_{T-1}\)) such that
\[
\sup_{\theta \in P} \|\varphi(T, \theta, u) - x^*(\theta)\| < \varepsilon.
\] (3)

They key point to note is that the input function (or input sequence) is assumed to be independent of the parameter values \(\theta\). Thus the open loop input function \(u\) achieves (3) universally for all \(\theta\). For the subsequent analysis the following necessary conditions are important. For the proof we refer to [9, Lemma 1]. Let \(\sigma(A) \subset \mathbb{C}\) denote the spectrum of an \(n \times n\)-matrix \(A\), i.e., the set of real and complex eigenvalues of \(A\).
Proposition 1. Suppose that the ensemble (1) (or (2)) is uniform ensemble controllable. Then

(E1) For every \( \theta \in \mathbf{P} \) the linear system \((A(\theta), B(\theta))\) is reachable.

(E2) For each number \( s \geq m + 1 \) of distinct parameters \( \theta_1, \ldots, \theta_s \in \mathbf{P} \), the spectra of \( A(\theta) \) satisfy
\[
\sigma(A(\theta_1)) \cap \cdots \cap \sigma(A(\theta_s)) = \emptyset.
\]

The preceding two conditions are useful in order to rule out families that are not ensemble controllable. In particular, condition (E2) implies that \( A(\theta) \) cannot have a \( \theta \)-independent eigenvalue. We next show that necessary and sufficient conditions for uniform ensemble controllability can be stated in terms of a polynomial approximation property.

For discrete-time ensembles with a scalar input sequence \( u_0, \ldots, u_{T-1} \), the solution at \( T > 0 \) is
\[
\varphi(T, \theta, u) = (u_{T-1}I + u_{T-2}A(\theta) + \cdots + u_0A(\theta)^{T-1})b(\theta).
\]
This implies that a family of single-input, discrete-time systems \((A(\theta), b(\theta))\) is uniformly ensemble controllable if and only if for all \( \varepsilon > 0 \) and all \( x^* \in C(\mathbf{P}, \mathbb{R}^n) \), there is a real scalar polynomial \( p \in \mathbb{R}[z] \) such that
\[
\sup_{\theta \in \mathbf{P}} \| p(A(\theta))b(\theta) - x^*(\theta) \| < \varepsilon.
\]

For a multivariable discrete-time system, the ensemble control condition (3) is similarly seen as being equivalent to
\[
\sup_{\theta \in \mathbf{P}} \left\| \sum_{j=1}^m p_j(A(\theta))b_j(\theta) - x^*(\theta) \right\| < \varepsilon.
\]

More generally, we obtain a polynomial characterization of ensemble controllability for both continuous-time and discrete-time systems. First we need a lemma.

Lemma 1. Let \( \mathbf{P} \) be compact and assume that the connected components of
\[
K = \bigcup_{\theta \in \mathbf{P}} \sigma(A(\theta)) \subset \mathbb{C}
\]
are simply connected. Assume further that the eigenvalues of \( A(\theta) \) are simple for all \( \theta \in \mathbf{P} \). Let \( \tau > 0 \) be sufficiently small and define
\[
F(\theta) = e^{\tau A(\theta)}, \quad G(\theta) = \left( \int_0^\tau e^{sA(\theta)} \, ds \right) B(\theta).
\]
Let \( p_1, \ldots, p_m \in \mathbb{R}[z] \) be real polynomials and \( \varepsilon > 0 \). Then there are real polynomials \( \psi_1, \ldots, \psi_m \in \mathbb{R}[z] \) such that
\[
\left\| \sum_{j=1}^m p_j(A(\theta))b_j(\theta) - \sum_{j=1}^m \psi_j(F(\theta))g_j(\theta) \right\| < \varepsilon.
\]

Proof. The set \( K \) is compact. Thus there exists \( \tau > 0 \) small enough such that \( \tau K \subset \mathbb{R} \times (-\pi, \pi) \). Then the complex exponential function maps \( \tau K \) homeomorphically onto the compact subset \( L = \{ e^{\tau z} \mid z \in K \} \subset \mathbb{C} \setminus (-\infty, 0] \). Since \( K \) is simply connected, so is \( L \) and therefore the complement \( \mathbb{C} \setminus L \) is connected. By the theorem of Mergelyan [17] there exists a real polynomial \( q(z) \in \mathbb{R}[z] \) such that
\[
|q(w) - \frac{1}{\tau} \log w| < \delta \quad \text{for all } w \in L.
\]
This is equivalent to
\[ |q(e^{\tau z}) - z| < \delta \text{ for all } z \in K. \]

Let \( \varepsilon > 0 \). Applying the mean value theorem and by choosing \( \delta \) sufficiently small we obtain
\[ |(p_j \circ q)(e^{\tau z}) - p_j(z)| < \varepsilon \text{ for all } z \in K, j = 1, \ldots, m. \]

For each \( n \times n \)-matrix \( X \) one has the convergent power series expansion
\[
\frac{e^X - I}{X} := \sum_{k=0}^{\infty} \frac{1}{(k+1)!} X^k.
\]

Since \( A(\theta) \) has simple eigenvalues we can apply standard estimates for norms to obtain
\[
\|p_j \circ q(e^{\tau A(\theta)}) \frac{e^{\tau A(\theta)}}{\tau A(\theta)} - p_j(A(\theta))\| < C\varepsilon, \quad j = 1, \ldots, m.
\]

Here \( C \) is a suitable positive constant. Define the polynomials \( \psi_j(z) := \frac{1}{\tau} p_j(q(z)) \). Then
\[
\left\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - \sum_{j=1}^{m} \psi_j(F(\theta))g_j(\theta) \right\| = \left\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - \sum_{j=1}^{m} \psi_j(e^{\tau A(\theta)}) \frac{e^{\tau A(\theta)}}{\tau A(\theta)} b_j(\theta) \right\|
\leq \sum_{j=1}^{m} \left\| p_j(A(\theta)) b_j(\theta) - \psi_j(e^{\tau A(\theta)}) \frac{e^{\tau A(\theta)}}{A(\theta)} b_j(\theta) \right\| \cdot \|b_j(\theta)\|
< mC\varepsilon \sum_{j=1}^{m} \|b_j(\theta)\|.
\]

This completes the proof.

\( \square \)

We note that the condition on \( K \) of being simply connected is implied by (but does not imply) the following two assumptions:

(K1) \( A(\theta) \) has simple spectra for all \( \theta \).

(K2) For all \( \theta \neq \theta' \) the spectra of \( A(\theta) \) and \( A(\theta') \) are disjoint.

The next result specifies conditions that uniform ensemble controllability for continuous-time and discrete-time systems \( \text{(1)} \) and \( \text{(2)} \) are equivalent.

**Theorem 1.** Let \( A \in C(P, \mathbb{R}^{n \times n}) \) and \( B \in C(P, \mathbb{R}^{n \times n}) \). Let \( b_1(\theta), \ldots, b_m(\theta) \) denote the columns of \( B(\theta) \). The following assertions are equivalent.

(a) The discrete-time ensemble \( \text{(2)} \) is uniformly ensemble controllable.

(b) For each \( \varepsilon > 0 \) and each \( x^* \in C(P, \mathbb{R}^n) \) there exist real scalar polynomials \( p_1, \ldots, p_m \in \mathbb{R}[z] \) such that
\[
\sup_{\theta \in P} \left\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - x^*(\theta) \right\| < \varepsilon.
\]

(c) The set
\[
\text{span}\{A(\theta)^k b_j(\theta) \mid k \in \mathbb{N}_0, j = 1, \ldots, m\}
\]

of continuous functions in \( \theta \) is dense in \( C(P, \mathbb{R}^n) \) with respect to the sup-norm.
(d) Suppose that $A(\theta)$ has simple eigenvalues for all $\theta$ and that the connected components of $K = \bigcup_{\theta \in P} \sigma(A(\theta))$ are simply connected. Then each of the preceding conditions is equivalent to uniform ensemble controllability of the continuous-time ensemble $\Sigma$.

**Proof.** Let $\varepsilon > 0$ and $x^* \in C(P, \mathbb{R}^n)$ be fixed. The preceding remarks prove the equivalence of (a) and (b). The equivalence of (b) and (c) is obvious. We next prove that (d) implies (b). If the continuous-time system $\Sigma(\theta)$ is uniformly ensemble controllable there is a $T > 0$ and in input $u \in L^1([0, T], \mathbb{R}^m)$ so that

$$
\sup_{\theta \in P} \| \varphi(T, \theta, u) - x^*(\theta) \| < \frac{\varepsilon}{2}.
$$

Moreover, as $t \mapsto \sum_{k=0}^{\infty} \frac{A(\theta)^k t^k}{k!}$ is convergent uniformly on compact sets we have

$$
\varphi(T, \theta, u) = \int_0^T e^{A(\theta)(T-s)} B(\theta) u(s) \, ds \\
= \sum_{k=0}^{\infty} \frac{1}{k!} A(\theta)^k B(\theta) \int_0^T (T-s)^k u(s) \, ds.
$$

Furthermore, there exists $N \in \mathbb{N}$ such that

$$
\sup_{\theta \in P} \left\| \int_0^T e^{A(\theta)(T-s)} B(\theta) u(s) \, ds - \sum_{k=0}^{N} \frac{1}{k!} A(\theta)^k B(\theta) \int_0^T (T-s)^k u(s) \, ds \right\| < \frac{\varepsilon}{2}.
$$

Define a vector of real polynomials

$$
(p_1(z), \ldots, p_m(z))^\top := \sum_{k=0}^{N} c_k z^k, \quad c_k := \frac{1}{k!} \int_0^T (T-s)^k u(s) \, ds.
$$

Then

$$
\sup_{\theta \in P} \left\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - x^*(\theta) \right\| \leq \sup_{\theta \in P} \| \varphi(T, \theta, u) - x^*(\theta) \|
$$

$$
+ \sup_{\theta \in P} \left\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - \int_0^T e^{A(\theta)(T-s)} B(\theta) u(s) \, ds \right\| < \varepsilon.
$$

This completes the proof that (d) implies (b).

To prove that (b) implies (d) we impose the assumptions on $A(\theta)$, i.e., $A(\theta)$ has simple spectra and the connected components of $K = \bigcup_{\theta \in P} \sigma(A(\theta))$ are simply connected. By (b), for each $\varepsilon > 0$ and $x^* \in C(P, \mathbb{R}^n)$ there exists real polynomials $p_1, \ldots, p_m$ such that

$$
\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - x^*(\theta) \| < \varepsilon.
$$

Applying Lemma 1 we conclude that for every that there exists real polynomials $\psi_1, \ldots, \psi_m$ with

$$
\| \sum_{j=1}^{m} p_j(A(\theta)) b_j(\theta) - \sum_{j=1}^{m} \psi_j(F(\theta)) g_j(\theta) \| < \varepsilon.
$$
Therefore,
\[ \left\| \sum_{j=1}^{m} \psi_j(F(\theta))g_j(\theta) - x^*(\theta) \right\| < 2\varepsilon. \]

Applying the equivalence of (a) and (b) for the discrete-time system
\[ x_{t+1}(\theta) = F(\theta)x_t(\theta) + G(\theta)u_t \]
shows that \((F(\theta), G(\theta))\) is uniformly ensemble controllable. But \((F(\theta), G(\theta))\) is obtained from the continuous-time system
\[ \dot{x} = A(\theta)x + B(\theta)v \] (5)
by sampling with a constant sampling rate \(\tau > 0\). Take an equidistant partition of \([0, T]\) by intervals of lengths \(\tau\) and let \(v^\tau : [0, T] \to \mathbb{R}^m\) denote the piecewise constant input function, whose constant values coincide with the finite inputs \(u_k\) for uniform ensemble control of \((F(\theta), G(\theta))\). Then \(v^\tau\) achieves the uniform ensemble control task for (5). This completes the proof.

Remark 1. Lemma 1 and Theorem 1 remain true, if the assumption that \(A(\theta)\) has simple eigenvalues for all \(\theta\) is replaced by the assumption that \(A(\theta) = T(\theta)\Lambda(\theta)T(\theta)^{-1}\) is diagonalizable by a similarity transformation \(T(\theta) \in \text{GL}_n(\mathbb{C})\) with bounded condition number \(\kappa_T = \sup_{\theta \in \mathbb{P}} \|T(\theta)\| \|T(\theta)^{-1}\| < \infty\). This guarantees the inequality for the spectral norm
\[ \|f(A(\theta))\|_2 \leq \kappa_T \sup_{z \in K} |f(z)| \]
for all polynomials \(f(z)\). This is instrumental in order to pass from bounds on polynomials to bounds on matrix norms. In particular, if \(A(\theta)\) is normal for every \(\theta \in \mathbb{P}\) one has
\[ \|f(A(\theta))\|_2 = \sup_{z \in \mathbb{K}} |f(z)|. \]

3 Refined characterizations of uniform ensemble controllability

The preceding results show that the problem of uniform ensemble controllability is intimately connected to the classical problem of approximating a continuous function by a polynomial. In the sequel, unless stated otherwise, all results hold for uniform ensemble controllability of both continuous-time and discrete-time systems. In this section we will derive more explicit necessary and sufficient conditions. In particular, we will show that uniform controllability can only hold for systems depending on at most two parameters.

We begin with a technical result. Given an array of linear parameter dependent systems \((A_{ij}, B_{ij}) \in C(\mathbb{P}, \mathbb{R}^{n_i \times n_j}) \times C(\mathbb{P}, \mathbb{R}^{n_i \times m_j})\) with \(1 \leq i \leq j \leq N\). Let \(\underline{m} = \sum_{j=1}^{N} n_j\) and \(\overline{m} = \sum_{j=1}^{N} m_j\). Define the associated upper triangular ensemble of systems by
\begin{equation}
A = \begin{pmatrix}
A_{11} & \cdots & A_{1N} \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{NN}
\end{pmatrix} \in C(\mathbb{P}, \mathbb{R}^{\underline{m} \times \overline{m}}), \quad B = \begin{pmatrix}
B_{11} & \cdots & B_{1N} \\
\vdots & \ddots & \vdots \\
0 & \cdots & B_{NN}
\end{pmatrix} \in C(\mathbb{P}, \mathbb{R}^{\underline{m} \times \overline{m}}). \tag{6}
\end{equation}

**Proposition 2.** The upper triangular family of systems \((A(\theta), B(\theta))\) is uniform ensemble controllable if and only if the families \((A_{ii}(\theta), B_{ii}(\theta))\) are uniform ensemble controllable for \(i = 1, \ldots, N\).
Proof. The necessity part is obvious. Thus assume that \((A_i, B_i)\) are uniform ensemble controllable. We consider continuous-time systems; the arguments for discrete-time systems go mutatis mutandis (and are even easier). For simplicity we focus on \(N = 2\), i.e., on

\[
\begin{align*}
\dot{x}_1(t) &= A_{11}(\theta)x_1(t) + A_{12}(\theta)x_2(t) + B_{11}(\theta)u_1(t) + B_{12}(\theta)u_2(t) \\
\dot{x}_2(t) &= A_{22}(\theta)x_2(t) + B_{22}(\theta)u_2(t).
\end{align*}
\]

The general case is treated, proceeding by induction. Let \(x_i(t)\) denote the solution of the \(i\)-th component. Given \(x^* = \text{col}(x_1^* \cdots x_N^*) \in C(\mathbb{P}, \mathbb{R}^T)\) and let \(\varepsilon > 0\). By uniform ensemble controllability of \((A_{22}, B_{22})\) there exists an input \(u_2 \in L^1([0, T], \mathbb{R}^{m_2})\) such that

\[
\sup_{\theta \in \mathbb{P}} \|x_2(T) - x_2^*(\theta)\| < \varepsilon.
\]

Let \(u = \text{col}(u_1 \cdots u_N) \in L^1([0, T], \mathbb{R}^{m})\) with \(u_i \in L^1([0, T], \mathbb{R}^{m_i})\). Applying the variations of constant formula we have

\[
x_2(t) = \int_0^t e^{(t-s)A_{22}}B_{22}u_2(s) \, ds
\]

and thus

\[
x_1(T) = z_1(T) + \int_0^T e^{(T-s)A_{11}}B_{11}u_1(s) \, ds,
\]

where

\[
z_1(T) = \int_0^T e^{(T-s)A_{11}} \left( A_{12} \int_0^s e^{(s-\tau)A_{22}}B_{22}u_2(\tau) \, d\tau + B_{12}u_2(s) \right) \, ds.
\]

By uniform ensemble controllability of \((A_{11}, B_{11})\) there exists an input \(u_1 \in L^1([0, T], \mathbb{R}^{m_1})\) with

\[
\sup_{\theta \in \mathbb{P}} \| \int_0^T e^{(T-s)A_{11}}B_{11}u_1(s) \, ds - x_1^*(\theta) + z_1(T) \| < \varepsilon.
\]

But this implies

\[
\sup_{\theta \in \mathbb{P}} \|x_1(T) - x_1^*(\theta)\| < \varepsilon
\]

and we are done. \(\square\)

Using an appropriate similarity transformation, every system can be transformed into Hermite canonical form \([12][10]\), which has the upper triangular form \([9]\). Given a matrix pair \((A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\), where \(b_i\) is the \(i\)-th column of \(B\). Select from left to right in the permuted Kalman matrix

\[
\begin{pmatrix}
b_1 & Ab_1 & \cdots & A^{n-2}b_1 & \cdots & b_m & Ab_m & \cdots & A^{n-1}b_m
\end{pmatrix},
\]

the first linear independent columns. Then one obtains a list of basis vectors of the reachability subspace as

\[
b_1, \ldots, A^{K_1-1}b_1, \ldots, b_m, \ldots, A^{K_m-1}b_m.
\]

The integers \(K_1, \ldots, K_m\) are called the Hermite indices, where \(K_i := 0\) if the column \(b_i\) has not been selected. One has \(K_1 + \cdots + K_m = n\) if and only if \((A, B)\) is reachable. The next result has been proven in \([9]\) for the special case that \(\mathbb{P}\) is a single compact interval.
Theorem 2. Let $P \subset \mathbb{R}$ be a finite union of disjoint compact intervals. The ensemble of linear systems $\Sigma = \{(A(\theta), B(\theta)) \mid \theta \in P\}$ is uniformly ensemble controllable if the following conditions are satisfied:

(i) $(A(\theta), B(\theta))$ is reachable for all $\theta \in P$.

(ii) The input Hermite indices $K_1(\theta), \ldots, K_m(\theta)$ of $(A(\theta), B(\theta))$ are independent of $\theta \in P$.

(iii) For any pair of distinct parameters $\theta, \theta' \in P$, $\theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint:

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.$$ 

(iv) For each $\theta \in P$, the eigenvalues of $A(\theta)$ have algebraic multiplicity one.

Proof. We show the claim for $P = P_1 \cup P_2$, where $P_1, P_2$ are disjoint compact intervals in $\mathbb{R}$. Define $K_1 := \bigcup_{\theta \in P_1} \sigma(A(\theta))$. The union of $N$ disjoint compact intervals can be concluded by induction. Let $x^* \in C(P, \mathbb{R}^n)$ and $\varepsilon > 0$ be fixed. According to the proof of Theorem 1 in \cite{6} it is sufficient to prove the assertion for discrete-time single-input ensembles. Then, by \cite{9} Theorem 1 the ensembles $\{(A(\theta), B(\theta)) \mid \theta \in P_1\}$ and $\{(A(\theta), B(\theta)) \mid \theta \in P_2\}$ are uniformly ensemble controllable. Then, as is shown above there are polynomials $p_1 \in \mathbb{R}[z]$ and $p_2 \in \mathbb{R}[z]$ such that

$$\sup_{\theta \in P_1} \|p_1(A(\theta))b(\theta) - x^*(\theta)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{\theta \in P_2} \|p_2(A(\theta))b(\theta) - x^*(\theta)\| < \frac{\varepsilon}{2}.$$ 

Then, by the Stone-Weierstrass Theorem \cite{4} Theorem 6.6.3 there is a polynomial $q \in \mathbb{R}[z]$ satisfying

$$\sup_{z \in K_1} \|p_1(z) - q(z)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{z \in K_2} \|p_2(z) - q(z)\| < \frac{\varepsilon}{2}.$$ 

Then, for any $\theta \in P$, w.l.o.g. $\theta \in P_1$, we have

$$\|q(A(\theta))b(\theta) - x^*(\theta)\| \leq \|q(A(\theta))b(\theta) - p_1(A(\theta))b(\theta)\| + \|p_1(A(\theta))b(\theta) - x^*(\theta)\| < \varepsilon.$$ 

This shows the assertion. \hfill $\square$

In the continuous-time case, conditions (i)-(iv) imply that uniform ensemble controllability of $\Sigma$ can be achieved in arbitrary time $T > 0$. As pointed out earlier, uniform ensemble controllability is related to approximation theory. The proof of Theorem 2 is based on Mergelyan’s Theorem from complex approximation.

Remark 2. Condition (iv) in Theorem 2 can be replaced by the following two conditions that are sometimes easier to check:

1. $A(\theta)$ is diagonalizable by a similarity transformation with uniformly bounded condition number.
2. $\mathbb{C} \setminus K$ is connected for $K = \bigcup_{\theta \in P} \sigma(A(\theta))$.

As we next show, there do not exist ensembles of linear systems with more than three parameters that are uniformly ensemble controllable. We need the following lemma from \cite{10}.

Lemma 2. Let $P \subset \mathbb{R}^d$ with $\text{int} \, P = P$. Let $\Pi: \mathbb{R}^d \to \mathbb{R}^{n^2 + nm}$, $\theta \mapsto (A(\theta), B(\theta))$ be real analytic. Then the Hermite indices of the ensemble $\Sigma = \{(A(\theta), B(\theta)) \mid \theta \in P\}$ are generically constant, i.e. the Hermite indices are constant on an open and dense subset of $P$. 

8
Proof. Let \( K_{n,m} \) denote the set of all \( K = (K_1, \ldots, K_m) \in \mathbb{N}_0^m \) with \( K_1 + \cdots + K_m \leq n \). For \( K \in K_{n,m} \) let \( \text{Her}(K) := \{(A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \mid \text{Hermite indices are } K\} \) denote the subset in \( \mathbb{R}^{n^2 + nm} \) of all systems with Hermite indices \( K \). The set \( \text{Her}(K) \) is a constructible algebraic subset of \( \mathbb{R}^{n^2 + nm} \), which induces a disjoint partition \( \bigcup_{K \in K_{n,m}} \text{Her}(K) = \mathbb{R}^{n^2 + nm} \). Consider the real analytic map \( \Pi: \mathbb{R}^d \rightarrow \mathbb{R}^{n^2 + nm} \), \( \theta \mapsto (A(\theta), B(\theta)) \). Then, the preimage \( \Pi^{-1}(\text{Her}(K)) \) is a constructible analytic subset of \( \mathbb{R}^d \) (or the empty set) and there exists a \( K_1 \in K_{n,m} \) such that \( \dim \Pi^{-1}(\text{Her}(K_1)) \) has an interior point. But any constructible analytic subset \( S \subset \mathbb{R}^d \) of dimension \( d \) contains an open and dense subset of \( \mathbb{R}^d \). Thus the set of interior points of \( \Pi^{-1}(\text{Her}(K_1)) \) is an open and dense subset of \( \mathbb{R}^d \). In particular, the intersection \( \Pi^{-1}(\text{Her}(K_1)) \cap \mathbb{P} \) contains an open and dense subset of \( \mathbb{P} \). It remains to show that \( K_1 \) is unique. Suppose there exist two different \( K_1, K_2 \in K_{n,m} \) such that their preimage \( \Pi^{-1}(\text{Her}(K_1)) \), \( i = 1, 2 \) contains open and dense subsets. Then the intersection \( \Pi^{-1}(\text{Her}(K_1)) \cap \Pi^{-1}(\text{Her}(K_2)) \neq \emptyset \), implying \( K_1 = K_2 \). This shows the assertion. 

Theorem 3. Let \( \mathbb{P} \subset \mathbb{R}^d \) with \( \overline{\text{int } \mathbb{P}} = \mathbb{P} \). Let \( \Pi: \mathbb{R}^d \rightarrow \mathbb{R}^{n^2 + nm} \), \( \theta \mapsto (A(\theta), B(\theta)) \) be real analytic. Suppose the ensemble \( \Sigma(\theta) = \{(A(\theta), B(\theta)) \mid \theta \in \mathbb{P} \} \) is uniformly ensemble controllable.

(a) Then \( \dim \mathbb{P} \leq 2 \).

(b) Let \( \lambda_1(\theta), \ldots, \lambda_n(\theta) \) denote the eigenvalues of \( A(\theta) \). Assume that at least one branch \( \{\lambda(\theta) \mid \theta \in \mathbb{P}\} \) of the eigenvalues is contained in a one-dimensional real subspace \( S \) of \( \mathbb{C} \). Then \( \dim \mathbb{P} = 1 \).

(c) If there is at least one eigenvalue such that \( S := \{s(\theta) \mid \theta \in \mathbb{P}\} \subset \mathbb{R} \) then \( \dim \mathbb{P} = 1 \).

Proof. As the Hermite indices are generically constant, i.e. independent of the parameter, we may assume that \( (A(\theta), B(\theta)) \) satisfies the assumptions (i) and (ii) of Theorem 2. As the Hermite indices of the family \( (A(\theta), B(\theta)) \) are independent of parameter \( \theta \in \mathbb{P} \), there exists a continuous family of invertible coordinate transformations \( S(\theta) \) such that \( (S(\theta)A(\theta)S(\theta)^{-1}, S(\theta)B(\theta)) \) is in Hermite canonical form. Thus, w.l.o.g. we can assume that \( (A(\theta), B(\theta)) \) is in Hermite canonical form

\[
\begin{pmatrix}
A_{11}(\theta) & \cdots & A_{1m}(\theta) \\
0 & \ddots & \vdots \\
0 & \cdots & A_{mm}(\theta)
\end{pmatrix},
\begin{pmatrix}
b_1 & 0 \\
0 & \cdots & b_m
\end{pmatrix},
\]

where the \( m \) single-input subsystems \( (A_{kk}(\theta), b_k) \in \mathbb{R}^{n_k \times n_k} \times \mathbb{R}^{n_k} \) are reachable and in control canonical form. Note that \( b_k \) denotes the first standard basis vector and thus is independent of \( \theta \). Partition the desired state vector as \( x^*(\theta) = \text{col}(x_1^*(\theta) \cdots x_N^*(\theta)) \) with \( x_k^*(\theta) \in \mathbb{R}^{n_k} \). By Proposition 2 the single-input systems defined by \( (A_{kk}(\theta), b_k) \) are uniform ensemble controllable. Applying (E2) for the single input case we conclude that the spectra of \( A_{kk}(\theta) \) and \( A_{kk}(\theta') \) are disjoint for all distinct parameters \( \theta \neq \theta' \) in \( \mathbb{P} \). Let \( \lambda: \mathbb{P} \rightarrow \mathbb{C} \) be a locally defined branch of the eigenvalues of \( A(\theta) \). Let \( S \) denote real subspace of \( \mathbb{C} \) of smallest dimension such that \( \lambda(\mathbb{P}) \subset S \). Since the eigenvalues of a matrix depends continuously on the parameters we see that locally \( \lambda \) is continuous. By injectivity of \( \lambda \) we get \( \dim \mathbb{P} \leq \dim \mathbb{R} S \). This completes the proof. 

4 Special classes of ensembles

Consider an ensemble of harmonic oscillators defined by

\[
\theta A = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
where the parameter is contained in the union $P$ of finitely many compact intervals with $0 \in P$. Note that for $\theta = 0$ the matrix $A(0)$ has a double eigenvalue 0. The Hermite indices are

$$K_1(\theta) = \begin{cases} 2 & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0 \end{cases} \quad \text{and} \quad K_2(\theta) = \begin{cases} 0 & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0 \end{cases},$$

i.e. they are not constant. Thus the ensemble does not satisfy the condition (ii) in Theorem 2. Nevertheless we will show that this family of systems is uniformly ensemble controllable. The next result applies Theorem 2 to prove an extension of [14, Theorem 1].

For fixed matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ we consider continuous-time ensembles defined by

$$\frac{\partial}{\partial t} x(t, \theta) = \theta Ax(t, \theta) + Bu(t), \quad (7)$$

and discrete-time ensembles defined by

$$x_{t+1}(\theta) = \theta Ax_t(\theta) + Bu_t. \quad (8)$$

For this special class of system families, a simple characterization of uniform ensemble controllability holds.

**Theorem 4.** Let $P \subset \mathbb{R}$ be the union of compact intervals. Assume that $0 \in P$. The family $\Sigma = \{(\theta A, B) \mid \theta \in P\}$ is uniformly ensemble controllable if and only if $\text{rank } A = n$ and $\text{rank } B = n$.

**Proof.** We focus on the continuous-time case; the discrete-time case goes mutatis mutandis. Suppose that $\Sigma = \{(\theta A, B) \mid \theta \in P\}$ is uniformly ensemble controllable. Then, since $0 \in P$, the necessary condition (E1) implies $\text{rank } B = n$. In particular, we have $m \geq n$. To show the second claim, suppose that $\text{rank } A < n$. Then zero is an eigenvalue of $A$ and for distinct parameter values $\{\theta_1, ..., \theta_{n+1}\} \in P$ we have

$$0 \in \sigma(\theta_1 A) \cap \cdots \cap \sigma(\theta_{n+1} A)$$

contradicting the necessary condition (E2).

Conversely, assume that $\text{rank } A = n$ and $\text{rank } B = n$. Without loss of generality we can assume that $B = I_n$. The reachability condition (E1) is implied by the rank condition on $B$. Let $\Lambda$ denote the Jordan canonical form. It is sufficient to consider the ensemble

$$\frac{\partial}{\partial t} x(t, \theta) = \theta \Lambda x(t, \theta) + I u(t). \quad (9)$$

Using Proposition 2 it remains to prove the assertion for one Jordan block. For simplicity we focus on the case of a two dimensional Jordan block; the higher dimensional case follows by an induction argument. Let

$$\frac{\partial}{\partial t} z(t, \theta) = \begin{pmatrix} \theta \lambda & \theta \\ 0 & \theta \lambda \end{pmatrix} z(t, \theta) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(t). \quad (10)$$

The solution to (10) is given by

$$\varphi(T, \theta, u) = \int_0^T \begin{pmatrix} e^{\theta \lambda (T-s)} u_1(s) + \theta (T-s) e^{\theta \lambda (T-s)} u_2(s) \\ e^{\theta \lambda (T-s)} u_2(s) \end{pmatrix} ds.$$ 

Given $z^* = \text{col}(z_1^*, z_2^*) \in C(P, \mathbb{R}^2)$ and $\varepsilon > 0$. By applying Theorem 2 to the ensemble

$$\frac{\partial}{\partial t} z_2(t, \theta) = \theta \lambda z_2(t, \theta) + u(t),$$

and
there is an input function \( u_2 : [0, T] \to \mathbb{R} \) so that
\[
|z_2^*(\theta) - \varphi_2(T, \theta, u_2)| < \varepsilon \text{ for all } \theta \in \mathbf{P}.
\]
Let
\[
w^*(\theta) := z_1^*\theta - \int_0^T \theta (T - s) e^{\theta \lambda(T-s)} u_2(s) \, ds \in C(\mathbf{P}, \mathbb{R}).
\]
Following the same reasoning there is an input \( u_1 : [0, T] \to \mathbb{R} \) so that
\[
|w^*(\theta) - \int_0^T e^{\theta \lambda(T-s)} u_1(s) \, ds| < \varepsilon.
\]
Consequently, we have
\[
\sup_{\theta \in \mathbf{P}} \|z^*(\theta) - \varphi(T, \theta, u)\| < \varepsilon
\]
and we are done. \( \square \)

Theorem 4 dealt with the situation \( m \geq n \). In the subsequent result we do not make this assumption. We use the notation \( \lambda \mathbf{P} := \{ \lambda \theta \mid \theta \in \mathbf{P} \} \).

**Theorem 5.** Let \( \mathbf{P} \) be the union of compact real intervals with \( 0 \notin \mathbf{P} \).

(a) If the family \( \Sigma = \{(\theta A, B) \mid \theta \in \mathbf{P} \} \) is uniformly ensemble controllable then \((A, B)\) is controllable and \( A \) is invertible.

(b) Suppose that \( A \) is diagonalizable. Then the family \( \mathbf{7} \) is uniformly ensemble controllable if and only if the family \( \mathbf{8} \) is uniformly ensemble controllable.

(c) Let \((A, B)\) be controllable and let \( A \) be invertible and diagonalizable such that \( \lambda_{k \mathbf{P}} \cap \lambda_{l \mathbf{P}} = \emptyset \) for all \( k \neq l \in \{1, ..., r\} \). Then the family \( \Sigma = \{(\theta A, B) \mid \theta \in \mathbf{P} \} \) is uniformly ensemble controllable.

**Proof.** (a) Let the family \( \Sigma = \{(\theta A, B) \mid \theta \in \mathbf{P} \} \) be uniformly ensemble controllable. Then, by (E1) the pair \((\theta A, B)\) is controllable for every \( \theta \in \mathbf{P} \). Using the Kalman matrix we have \((A, B)\) is controllable. To show the second claim, suppose that \( \text{rank } A < n \). Then zero is an eigenvalue of \( A \) and for distinct parameter values \( \{\theta_1, ..., \theta_{m+1}\} \in \mathbf{P} \) we have
\[
0 \in \sigma(\theta_1 A) \cap \cdots \cap \sigma(\theta_{m+1} A)
\]
contradicting the necessary condition (E2).

(b) We show that the assumptions (d) in Theorem 1 are satisfied. The set
\[
K = \bigcup_{\theta \in \mathbf{P}} \sigma(\theta A) = \bigcup_{k=1}^r \{ \lambda_k \theta \mid \theta \in \mathbf{P} \}
\]
is compact and the finitely many connected components of \( K \) are simply connected. Let \( S \in \mathbb{C}^{n \times n} \) be invertible such that \( SAS^{-1} = \Lambda := \text{diag}(\lambda_1, ..., \lambda_n) \). For all polynomials \( f \in \mathbb{R}[z] \) one has
\[
\|f(\theta A)\|_2 \leq \|S\| \|S^{-1}\| \sup_{z \in K} |f(z)|.
\]
The assertion follows by applying Remark 2.

(c) To show the claim, we verify the sufficient conditions of Theorem 2. The reachability of the pair \((A, B)\) and the fact that \( 0 \neq \mathbf{P} \) implies that \((\theta A, B)\) is reachable for every \( \theta \in \mathbf{P} \). Due to the fact that \( 0 \notin \mathbf{P} \) the Hermite indices of \((\theta A, B)\) are independent of \( \theta \). Moreover, as \( \lambda_{k \mathbf{P}} \cap \lambda_{l \mathbf{P}} = \emptyset \) for all \( k \neq l \in \{1, ..., r\} \) we have \( \sigma(\theta A) \cap \sigma(\theta' A) = \emptyset \) for all \( \theta \neq \theta' \in \mathbf{P} \). The assertion then follows from Remark 2. \( \square \)
5 Ensembles of networks of linear dynamical systems

We consider ensembles of networks of $N$ identical single-input-single-output systems $\Sigma = (A,b,c)$, whose dynamics are given by

$$
\dot{x}_i(t) = A x_i(t) + b v_i(t) \\
y_i(t) = c x_i(t).
$$

Here $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1 \times n}$. We assume that $(A,b,c)$ is controllable and observable. The $N$ identical systems $\Sigma$ are coupled via directed links and the interconnection structure is described by a directed graph $G = (V,E)$. Here $V = \{1,\ldots,N\}$ denotes the set of vertices, i.e. the $N$ SISO systems, and $E$ describes the set of edges, i.e. the couplings. We examine the situation where the coupling strength is uncertain and is assumed to vary over a compact interval $P \subset (0,\infty)$. The weighted graph adjacency matrix is given by

$$K(\theta) = \begin{cases} k_{ij}(\theta) & \text{if } (i,j) \in E \\ 0 & \text{else,} \end{cases}$$

where $k_{ij} : P \to (0,\infty)$ are known continuous functions, $(i,j) \in E$. Thus, $K_P = \{K(\theta) | \theta \in P\}$ denotes a family of adjacency matrices. We assume that there is an external input $u$ which is broadcasted to the systems $\Sigma$ via the input-to-state interconnection vector $B \in \mathbb{R}^N$. The setting is illustrated in Figure 1.

![Figure 1: Block diagram of an ensemble of homogenous networks with parameter-dependent couplings.](image)

Thus, $\Sigma = \{(\Sigma, K(\theta), B), \theta \in P\}$ defines an ensemble of networks. Using the state vector $x = \text{col}(x_1 \cdots x_N) \in \mathbb{R}^{nN}$, the network is described by the following control system

$$\frac{\partial}{\partial t} x(t,\theta) = (I \otimes A + K(\theta) \otimes bc)x(t,\theta) + (B \otimes b)u(t). \quad (11)$$

Let $1 = (1 \cdots 1)^\top \in \mathbb{R}^N$ and $x_0 \in \mathbb{R}^n$ denote the initial state of the $i$th node system and $x_0 = \text{col}(x_1^0 \cdots x_N^0) \in \mathbb{R}^{nN}$. The solution to (11) at time $T > 0$ starting in $x_0$ under the interconnection $K(\theta)$ and the input $u$ is denoted by $\varphi(T,K(\theta),x_0,u)$.

We emphasize that the input $u$ is broadcasted to the systems $\Sigma$ within the network according to the input-to-state vector $B$. This phenomenon may be interpreted as that $u$ serves as an universal input for a whole ensemble of networks that steers the initial state $x_0$ to a desired terminal state $x^* \in C(P,\mathbb{R}^n)$ in finite time $T$ uniformly for all interconnection matrices $K(\theta)$, $\theta \in P$.

The ensemble $\{(\Sigma, K(\theta), B) | \theta \in P\}$ is called robustly synchronizable to $x^* \in C(P,\mathbb{R}^n)$ if for every $\varepsilon > 0$ there is a $T > 0$ and an input-function $u \in L^1([0,T],\mathbb{R})$ such that

$$\sup_{\theta \in P} \|\varphi(T,K(\theta),x_0,u) - (1 \otimes x^*(\theta))\| < \varepsilon.$$
The reason for this terminology is that the desired states to which we want to control have identical components, i.e.,

\[ 1 \otimes x^*(\theta) = \begin{pmatrix} x^*(\theta) \\ \vdots \\ x^*(\theta) \end{pmatrix} \in \mathbb{R}^{nN}. \]

Thus the single input function achieves synchronization in finite time \( T \), starting from non-synchronous initial condition \( x_0 = \text{col}(x_1(0), \ldots, x_N(0)) \in \mathbb{R}^{nN} \). We emphasize that the inputs may depend both on the initial condition \( x_0 \) and on the family of terminal state \( x^*(\theta) \).

As an illustration we consider \( N \) identical single-input-single-output (SISO) harmonic oscillators

\[
A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c := \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{12}
\]

The oscillators are coupled in a circular manner. The network topology is described by a directed graph \( G \) with \( N \) nodes. The weighted adjacency matrix \( K(\theta) \) is given by the circulant matrix

\[
K_{\text{ring}}(\theta) := \begin{bmatrix} 0 & \theta & \cdots & \cdots & \theta \\ \theta & 0 & \cdots & \cdots & \theta \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ \theta & \cdots & \cdots & 0 & \theta \end{bmatrix}, \quad \theta \in \mathbb{P} := [\theta^-, \theta^+]. \tag{13}
\]

The network is depicted in Figure 2.

![Figure 2: Ensemble of rings of 5 identical harmonic oscillators.](image)

Without loss of generality, assume that the harmonic oscillators are numbered such that the external input is applied to the first oscillator. Thus, the input-to-state interconnection vector is \( B = e_1 = (1, 0, \ldots, 0)^\top \). The dynamics of the overall network of systems is of the form

\[
\frac{\partial}{\partial t} x(t, \theta) = (I \otimes A + K_{\text{ring}}(\theta) \otimes bc)x(t) + (e_1 \otimes b)u(t) \\
x(0, \theta) = x_0 \in \mathbb{R}^{2nN}. \tag{14}
\]

Let \( x^* \in C(\mathbb{P}, \mathbb{R}^2) \) denote the desired terminal states of the harmonic oscillators.

**Proposition 3.** Let \( P \) denote a compact interval in \((0, 1) \cup (1, \infty)\). The circular network ensemble of harmonic oscillators (14) is robustly synchronizable to \( 1 \otimes x^* \).
Proof. Let \( \omega := e^{2\pi i/N} \) denote the primitive \( N \)-th root of unity. The adjacency matrix is a circulant matrix with spectrum

\[
\sigma(K(\theta)) = \left\{ \theta e^{2\pi i l/N} \mid l = 0, \ldots, N - 1 \right\}.
\]

The family \( \{K(\theta) \mid \theta \in \mathbb{P}\} \) of circulant matrices is simultaneously diagonalizable using the unitary Vandermonde matrix

\[
S = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & \omega & \cdots & \omega^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2}
\end{bmatrix}, \tag{15}
\]

Applying the change of coordinates \( S \otimes I \) to the network dynamics yields the state space equivalent system

\[
\frac{\partial}{\partial t} z(t, \theta) = \begin{pmatrix} A + \lambda_1(\theta)bc \end{pmatrix} x(t) + (1 \otimes b) u(t) \\
z(0, \theta) = (S \otimes I)x_0. \tag{16}
\]

Since \( S1 = Ne_1 \), the corresponding desired terminal states are \( Ne_1 \otimes x^*(\theta) \in C(\mathbb{P}, \mathbb{R}^{2N}) \). We show that Theorem 2 can be applied to (16). A simple calculation shows that

\[
\bigcup_{l=1}^{N} \sigma(A + \lambda_l(\theta)bc) = \bigcup_{l=1}^{N} \left\{ w \in \mathbb{R} \mid w^2 - (\theta e^{2\pi i l/N} - 1) = 0 \right\}. \tag{17}
\]

Thus

\[
\sigma(A + \lambda_l(\theta)bc) \cap \sigma(A + \lambda_k(\theta)bc) = \emptyset
\]

holds for all \( \theta \in \mathbb{P} \) and \( l \neq k \in \{1, \ldots, N\} \). Moreover, since \( (A + \lambda_l(\theta)bc, b) \) is controllable for all \( \theta \in \mathbb{P} \) and \( l \in \{1, \ldots, n\} \), we conclude that the parallel connected system (16) is controllable. Furthermore, by inspection from (17), conditions (iii) and (iv) in Theorem 2 are satisfied. This completes the proof. \( \square \)

Proposition corrects an error in [16], where the corresponding result was claimed for undirected circular graphs, with the symmetric circulant adjacency matrix

\[
\begin{bmatrix}
0 & \theta & \cdots & \theta \\
\theta & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \theta \\
\theta & \cdots & \theta & 0
\end{bmatrix}. \tag{18}
\]

This matrix has real eigenvalues \( \theta \cos \left(\frac{2\pi l}{N}\right) \), \( l = 0, \ldots, N-1 \). Thus, for \( N = 4 \), the necessary condition (E2) is not satisfied as \( \cos(\frac{\pi}{2}) = \cos(\frac{3\pi}{2}) = 0 \). Therefore, the ring of oscillators with symmetric coupling matrix (18) is not robust synchronizable.

Next, we discuss a scenario where the node systems depend on a single parameter and are interconnected by a fixed graph adjacency matrix \( K \). That is, let \( \Sigma(\theta) = (A(\theta), b(\theta), c(\theta)) \) be an ensemble of SISO systems, where \( A \in C(\mathbb{P}, \mathbb{R}^{n \times n}) \), \( b \in C(\mathbb{P}, \mathbb{R}^{n}) \) and \( c \in C(\mathbb{P}, \mathbb{R}^{1 \times n}) \). We assume that the
system \( (A(\theta), b(\theta), c(\theta)) \) is controllable and observable for every \( \theta \in \mathbf{P} \). Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) be a directed graph with a fixed weighted adjacency matrix \( K \in \mathbb{R}^{N \times N} \).

Apply a single, parameter-independent external input \( u \) to the systems \( \Sigma(\theta) \), using an input-to-state interconnection vector \( B \in \mathbb{R}^{N} \). Let \( x = \text{col}(x_1 \cdots x_N) \in \mathbb{R}^{nN} \) denote the state vector of the network. The ensemble of networks we are interested in is described by the control system

\[
\frac{\partial}{\partial t} x(t, \theta) = (I \otimes A(\theta) + K \otimes b(\theta) c(\theta)) x(t, \theta) + (B \otimes b(\theta)) u(t). \tag{19}
\]

We note that (19) is a special case of the more general setting described in [8]. In particular, the controllability properties of the network are easily established. In fact, (19) is controllable if and only if \( (K, B) \) and \( (A(\theta), b(\theta), c(\theta)) \) is controllable for all \( \theta \); cf. [8]. In order to simplify the analysis, we assume that the adjacency matrix \( K \) has distinct eigenvalues \( \lambda_1, \ldots, \lambda_N \), which implies that there is an invertible matrix \( S \) with \( SKS^{-1} = D = \text{diag}(\lambda_1, \ldots, \lambda_N) \). Using the change of coordinates \( z = (S \otimes I) x \), system (19) is state-space equivalent to

\[
\frac{\partial}{\partial t} z(t, \theta) = (I \otimes A(\theta) + D \otimes b(\theta) c(\theta)) z(t, \theta) + (S B \otimes b(\theta)) u(t). \tag{20}
\]

Note that \( \lambda_k \neq \lambda_l \) for all \( k \neq l \in \{1, \ldots, N\} \). System (20) is equivalent to

\[
\frac{\partial}{\partial t} z(t, \theta) = \begin{pmatrix} A(\theta) + \lambda_1 b(\theta) c(\theta) \\ \vdots \\ A(\theta) + \lambda_N b(\theta) c(\theta) \end{pmatrix} z(t, \theta) + (S B \otimes b(\theta)) u(t).
\]

Thus the spectrum of \( I \otimes A(\theta) + K \otimes b(\theta) c(\theta) \) is

\[
\sigma \left( A(\theta) + \lambda_1 b(\theta) c(\theta) \right) \cup \cdots \cup \sigma \left( A(\theta) + \lambda_N b(\theta) c(\theta) \right).
\]

Since \( (A(\theta), b(\theta), c(\theta)) \) is controllable and observable we obtain

\[
\det \left( zI - (A(\theta) - \lambda_k b(\theta) c(\theta)) \right) = q_\theta(z) + \lambda_k p_\theta(z)
\]

with coprime polynomials \( p_\theta(z), q_\theta(z) \). This shows, for fixed \( \theta \in \mathbf{P} \) and \( \lambda_k \neq \lambda_l \), that there is no \( z \in \mathbb{C} \) such that

\[
q_\theta(z) + \lambda_k p_\theta(z) = q_\theta(z) + \lambda_l p_\theta(z).
\]

Therefore, one can apply Theorem 2, which proves the next result.

---

Figure 3: Block diagram of the ensemble of homogenous networks with parameter-dependent node systems.
Proposition 4. Let $P \subset \mathbb{R}$ be compact. Suppose the system $\Sigma(\theta) = (A(\theta), b(\theta), c(\theta))$ is controllable and observable for every $\theta \in P$. Assume that the pair $(K, B) \in \mathbb{R}^{n \times (n+1)}$ is controllable and $K$ is diagonalizable. Then the ensemble of networks \(19\) is robustly synchronizable if the eigenvalues of $A(\theta) + \lambda b(\theta)c(\theta)$ have algebraic multiplicity one for each $\theta \in P$ and $\lambda \in \sigma(K)$ and for each $\theta \neq \theta' \in P$ and $\lambda_l \neq \lambda_k \in \sigma(K)$ it holds
\[
\sigma (A(\theta) + \lambda_l b(\theta)c(\theta)) \cap \sigma (A(\theta') + \lambda_k b(\theta')c(\theta')) = \emptyset.
\]

6 Conclusions

An approximation theoretical approach to controlling state ensembles of linear systems has been proposed. Up to a technical condition the ensemble control problem for continuous-time and discrete-time systems are shown to be equivalent. For analytic parameter-dependent linear systems it is shown that uniform ensemble controllability is not possible for more than three parameters. A complete characterization of uniform ensemble controllability is presented for a special class of ensembles. An application to robust synchronization using broadcasted open-loop controls is given. Our approach is based on information of the spectrum of the system matrices (see assumption (iv) in Theorem 2). An open problem is to find relaxed necessary and sufficient conditions using the concept of pseudospectra.

Acknowledgements

This research is partially supported by the grant HE 1858/13-1 from the German Research Foundation. Parts of this work has been presented at the 8th Workshop Structural Dynamical Systems: Computational Aspects (SDS2014). We thank the organizers for this beautiful workshop.

References

[1] B. Bamieh, F. Paganini, and M. A. Dahleh. Distributed control of spatially invariant systems. IEEE Transactions on Automatic Control, 47(7):1091–1107, 2002.
[2] R. W. Brockett. On the control of a flock by a leader. Proceedings of the Steklov Institute of Mathematics, 268(1):49–57, 2010.
[3] J.-M. Coron. Control and nonlinearity. Providence, RI: American Mathematical Society (AMS), 2007.
[4] R. Curtain, O. V. Iftime, and H. Zwart. System theoretic properties of a class of spatially invariant systems. Automatica, 45(7):1619–1627, 2009.
[5] R. F. Curtain and H. Zwart. An introduction to infinite-dimensional linear systems theory. New York, NY: Springer-Verlag, 1995.
[6] P. Davis. Interpolation and approximation. A Blaisdell Book in the Pure and Applied Sciences. New York-Toronto-London: Blaisdell Publishing Company., 1963.
[7] P. A. Fuhrmann. On weak and strong reachability and controllability of infinite-dimensional linear systems. Journal of Optimization Theory and Applications, 9(2):77–89, 1972.
[8] P. A. Fuhrmann and U. Helmke. Reachability, observability and strict equivalence of networks of linear systems. Mathematics of Control, Signals, and Systems, 25:437–471, 2013.
[9] U. Helmke and M. Schönlein. Uniform ensemble controllability for one-parameter families of time-invariant linear systems. *Systems and Control Letters*, 71:69–77, 2014.

[10] D. Hinrichsen and A. Linnemann. Normalformen vom Hermite-Typ und die Berechnung dominanter Hermite-Indizes strukturierter Systeme / Hermite-type canonical forms and the computation of dominant Hermite-indices for structured systems. *At - Automatisierungstechnik*, 32:124–130, 1984.

[11] B. Jacob and J. R. Partington. On controllability of diagonal systems with one-dimensional input space. *Systems & Control Letters*, 55(4):321–328, 2006.

[12] T. Kailath. *Linear systems*. Prentice-Hall, Inc., Englewood Cliffs Publ., N.J., 1980.

[13] J.-S. Li and N. Khaneja. Control of inhomogeneous quantum ensembles. *Physical review A*, 73(3):030302, 2006.

[14] J. Qi and J.-S. Li. Ensemble controllability of time-invariant linear systems. In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pages 2709–2714, Dec 2013.

[15] B. Scherlein, M. Schönlein, and U. Helmke. Open-loop control of parameter-dependent discrete-time systems. *PAMM*, 14(1):939–940, 2014.

[16] M. Schönlein and U. Helmke. Robust synchronization by open-loop control. In *Proc. of the 21st International Symposium on Mathematical Theory of Networks and Systems*, pages 500–505, 2014.

[17] J. Walsh. Interpolation and approximation by rational functions in the complex domain. 4th ed. American Mathematical Society (AMS). Colloquium Publications. 20. Providence, R.I.: American Mathematical Society (AMS), 1965.