On the metric dimension of line graphs

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Abstract

Let \( G \) be a (di)graph. A set \( W \) of vertices in \( G \) is a resolving set of \( G \) if every vertex \( u \) of \( G \) is uniquely determined by its vector of distances to all the vertices in \( W \). The metric dimension \( \mu(G) \) of \( G \) is the minimum cardinality of all the resolving sets of \( G \). Cáceres et al. \[3\] computed the metric dimension of the line graphs of complete bipartite graphs. Recently, Bailey and Cameron \[1\] computed the metric dimension of the line graphs of complete graphs. In this paper we study the metric dimension of the line graph \( L(G) \) of \( G \). In particular, we show that \( \mu(L(G)) = |E(G)| - |V(G)| \) for a strongly connected digraph \( G \) except for directed cycles, where \( V(G) \) is the vertex set and \( E(G) \) is the edge set of \( G \). As a corollary, the metric dimension of de Bruijn digraphs and Kautz digraphs is given. Moreover, we prove that \( \lceil \log_2 \Delta(G) \rceil \leq \mu(L(G)) \leq |V(G)| - 2 \) for a simple connected graph \( G \) with at least five vertices, where \( \Delta(G) \) is the maximum degree of \( G \). Finally, we obtain the metric dimension of the line graph of a tree in terms of its parameters.

Key words: Metric dimension; resolving set; line graph; de Bruijn digraph; Kautz digraph.

1 Introduction

Let \( G \) be a (di)graph. We often write \( V(G) \) for the vertex set of \( G \) and \( E(G) \) for the edge set of \( G \). A (di)graph \( G \) is (strongly) connected if for any two distinct vertices \( u \) and \( v \) of \( G \), there exists a path from \( u \) to \( v \). In this paper we only consider finite strongly connected digraphs, or undirected simple connected graphs. For two vertices \( u \) and \( v \) of \( G \), we denote the distance from \( u \) to \( v \) by \( d_G(u, v) \). A resolving set of \( G \) is a set of vertices \( W = \{w_1, \ldots, w_m\} \) such that for each \( u \in V(G) \), the vector \( D(u|W) = (d_G(u, w_1), \ldots, d_G(u, w_m)) \) uniquely determines \( u \). The metric dimension of \( G \), denoted by \( \mu(G) \), is the minimum cardinality of all the resolving sets of \( G \).

Metric dimension of graphs was introduced in the 1970s, independently by Harary and Melter \[10\] and by Slater \[13\]. Metric dimension of digraphs was first studied by Chartrand et al. in \[5\] and further in \[6\]. Fehr et al. \[8\] investigated the metric dimension of Cayley digraphs. In graph theory, metric dimension is a parameter that has appeared in various applications, as diverse as network discovery and verification \[2\], strategies for the Mastermind game \[7\], combinatorial optimization \[12\].

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and so on. It was noted in [9] p. 204 and [11] that determining the metric dimension of a graph is an NP-complete problem.

Let $L(G)$ denote the line graph of a (di)graph $G$. For the complete bipartite graph $K_{m,n}$, Cáceres et al. [3] proved that

$$\mu(L(K_{m,n})) = \begin{cases} \lceil \frac{2(m+n-1)}{3} \rceil, & m \leq n \leq 2m - 1, n \geq 2, \\ n - 1, & n \geq 2m. \end{cases}$$

For the complete graph $K_n$ when $n \geq 6$, Bailey and Cameron [1] proved that $\mu(L(K_n)) = \lceil \frac{2n}{3} \rceil$.

Motivated by these results, in this paper we study the metric dimension of the line graph of a (di)graph. In Section 2, we show that $\mu(L(G)) = |E(G)| - |V(G)|$ for a strongly connected digraph $G$ except for directed cycles. As a corollary, the metric dimension of de Bruijn digraphs and Kautz digraphs, which are two families of famous networks, is given. In Section 3, we prove that $\lceil \log_2 \Delta(G) \rceil \leq \mu(L(G)) \leq |V(G)| - 2$ for a connected graph $G$ with at least five vertices, where $\Delta(G)$ is the maximum degree of $G$. Finally, we obtain the metric dimension of the line graph of a tree in terms of its parameters.

## 2 Line graph of a digraph

Let $G$ be a digraph. For a directed edge $a = (x, y)$ of $G$, we say that $x$ is the head of $a$ and $y$ is the tail of $a$; we also say that $a$ is the out-going edge of $x$ and the in-coming edge of $y$. For $x \in V(G)$, we denote the set of all out-going edges of $x$ by $E_G^+(x)$ and the set of all in-coming edges of $x$ by $E_G^-(x)$. The line graph $G$ of the digraph $L(G)$ with the edges of $G$ as its vertices, and where $(a, b)$ is a directed edge in $L(G)$ if and only if the tail of $a$ is the head of $b$ in $G$. For two distinct vertices $a = (x_1, x_2), b = (y_1, y_2)$ of $L(G)$, we have

$$d_{L(G)}(a, b) = d_G(x_2, y_1) + 1. \quad (1)$$

Note that $\mu(L(G)) = 1$ if $G$ is a directed cycle.

**Theorem 2.1** If $G$ is a strongly connected digraph except for directed cycles, then

$$\mu(L(G)) = |E(G)| - |V(G)|.$$

**Proof.** Let $R$ be a resolving set of $L(G)$ with the minimum cardinality. For each vertex $x$ of $G$, since $G$ is strongly connected, $E_G^-(x) \neq \emptyset$. If $|E_G^-(x)| \geq 2$, pick two distinct edges $a, b \in E_G^-(x)$. For any $c \in V(L(G)) \setminus \{a, b\}$, since $d_{L(G)}(a, c) = d_{L(G)}(b, c), a \in R$ or $b \in R$. It follows that $|E_G^-(x) \cap R| \geq |E_G^-(x)| - 1$. If $|E_G^-(x)| = 1$, the above inequality is directed. By $R = \bigcup_{x \in V(G)}(E_G^-(x) \cap R)$, we obtain

$$\mu(L(G)) = |R| \geq \sum_{x \in V(G)}(|E_G^-(x)| - 1) = |E(G)| - |V(G)|. \quad (2)$$

Let $W$ be a set obtained from $E(G)$ by deleting one in-coming edge of each vertex of $G$. Since $G$ is not a directed cycle, $W \neq \emptyset$. We shall prove that $W$ is a resolving
set of $L(G)$. It suffices to show that, for any two distinct edges $a = (x_1, x_2)$ and $b = (y_1, y_2)$ in $E(G)\setminus W$, there exists an edge $c \in W$ such that

$$d_{L(G)}(a, c) \neq d_{L(G)}(b, c).$$

Let $A$ denote the set of all the heads of each edge of $W$. Pick $z_0 \in A$ satisfying $d_G(x_2, z_0) < d_G(x_2, z)$ for any $z \in A$.

**Case 1.** $d_G(x_2, z_0) \neq d_G(y_2, z_0)$. Pick $c \in E_G^+(z_0) \cap W$. By (1), (3) holds.

**Case 2.** $d_G(x_2, z_0) = d_G(y_2, z_0)$. Owing to $a, b \not\in W$, $x_2 \neq y_2$, which implies $z_0 \neq x_2$. Let $P_{x_2, z_0} = (v_0 = x_2, v_1, \ldots, v_k = z_0)$ be a shortest path from $x_2$ to $z_0$ and $P_{y_2, z_0} = (u_0 = y_2, u_1, \ldots, u_k = z_0)$ be a shortest path from $y_2$ to $z_0$. Suppose $i$ denotes the minimum index such that $v_i = u_i$. Since $d_G(x_2, v_{i-1}) < d_G(x_2, v_i) \leq d_G(x_2, z_0)$, we have $v_{i-1} \not\in A$, which implies $(v_{i-1}, v_i) \not\in W$. Hence $(u_{i-1}, u_i) \in W$ and $u_{i-1} \in A$. Pick $c = (u_{i-1}, u_i)$. By (1), we have

$$d_{L(G)}(a, c) = d_G(x_2, u_{i-1}) + 1$$

$$\geq d_G(x_2, z_0) + 1 = d_G(y_2, z_0) + 1$$

$$\geq d_G(y_2, u_i) + 1 = d_{L(G)}(b, c) + 1$$

$$> d_{L(G)}(b, c),$$

so (3) holds.

Therefore, $W$ is a resolving set of $L(G)$ with size $|E(G)| - |V(G)|$, which implies that $\mu(L(G)) \leq |E(G)| - |V(G)|$. By (2), the desired result follows.

Let $K_d$ be the complete digraph with $d$ vertices. A flowered complete digraph of order $d$, denoted by $K_d^+$, is a digraph obtained from $K_d$ by appending a self-loop at each vertex. Let

$$B(d, 1) = K_d^+, \quad B(d, n) = L(B(d, n - 1));$$

$$K(d, 1) = K_{d+1}, \quad K(d, n) = L(K(d, n - 1)).$$

Then $B(d, n)$ is the de Bruijn digraph and $K(d, n)$ is the Kautz digraph. By [14] Chapter 3, $B(d, n)$ and $K(d, n)$ are strongly connected and

$$|V(B(d, n))| = d^n, \quad |E(B(d, n))| = d^{n+1},$$

$$|V(K(d, n))| = d^n, \quad |E(K(d, n))| = d^{n+1} + d^n.$$

As a corollary of Theorem 2.1, we get the metric dimension of de Bruijn digraphs and Kautz digraphs, respectively.

**Corollary 2.2** Let integers $d \geq 2$ and $n \geq 1$. Then

(i) $\mu(B(d, n)) = d^{n-1}(d - 1);$

(ii) $\mu(K(d, n)) = \begin{cases} 
  d^n, & \text{if } n = 1, \\
  d^{n-2}(d^2 - 1), & \text{if } n \geq 2.
\end{cases}$
3 Line graph of a graph

Let $G$ be a graph with at least two vertices. The line graph of $G$ is the graph $L(G)$ with the edges of $G$ as its vertices, and where two edges of $G$ are adjacent in $L(G)$ if and only if they are adjacent in $G$.

If $G$ has at most four vertices, it is routine to compute the metric dimension of $L(G)$. Next we shall consider the case $|V(G)| \geq 5$.

**Theorem 3.1** If $G$ is a connected graph with at least five vertices, then

$$[\log_2 \Delta(G)] \leq \mu(L(G)) \leq |V(G)| - 2,$$

where $\Delta(G)$ is the maximum degree of $G$.

**Proof.** Let $v$ be a vertex of degree $\Delta(G)$, and let $\{f_1, \ldots, f_{\Delta(G)}\}$ be the set of all the edges incident to $v$. Suppose $W = \{e_1, \ldots, e_{\mu(L(G))}\}$ is a resolving set of $L(G)$ with the minimum cardinality. For each $j \in \{1, \ldots, \mu(L(G))\}$, let $d_j = \min\{d_G(v, w) \mid w \text{ is incident to } e_j\}$. Then $d_{L(G)}(f_i, e_j) = d_j$ or $d_j + 1$. Therefore, the size of $D = \{D(f_i\vert W)\mid i = 1, \ldots, \Delta(G)\}$ is at most $2^{\mu(L(G))}$. Since $D(f_i\vert W) \neq D(f_k\vert W)$ for $i \neq k$, $\Delta(G) \leq 2^{\mu(L(G))}$, which implies the lower bound.

Suppose $|V(G)| = 5$. If $G$ is isomorphic to the path $P_5$ or the cycle $C_5$, since $\mu(L(P_5)) = 1$ and $\mu(L(C_5)) = 2$, the upper bound is directed. If $G$ is not isomorphic to $P_5$ or $C_5$, then $G$ has a subgraph $S$ isomorphic to $K_{1,3}$. Since $E(S)$ is a resolving set of $L(G)$, $\mu(L(G)) \leq 3$, which implies the upper bound.

Now suppose $|V(G)| \geq 6$. Let $T$ be a spanning tree of $G$, and let $v$ be a vertex of degree 1 in $T$. Suppose $T_1$ is the subgraph of $T$ induced on $V(T) \setminus \{v\}$. We shall prove that $E(T_1)$ is a resolving set of $L(G)$. It suffices to show that, for any two distinct edges $a, b \in E(G) \setminus E(T_1)$, there exists an edge $e \in E(T_1)$ such that

$$d_{L(G)}(a, e) \neq d_{L(G)}(b, e). \tag{4}$$

**Case 1.** $a$ or $b$ is not incident to $v$. Without loss of generality, suppose $a$ is not incident to $v$. Let $a = uv$. Then there exists a unique path $P_{u,v'} = (u_0 = u, u_1, \ldots, u_k = v')$ between $u$ and $u'$ in $T$ where $k \geq 2$. If $b$ is not adjacent to $u_0u_1$, then (4) holds for $e = u_0u_1 \in E(T_1)$; if $b$ is not adjacent to $u_{k-1}u_k$, then (4) holds for $e = u_{k-1}u_k \in E(T_1)$. Now we assume that $b$ is adjacent to both $u_0u_1$ and $u_{k-1}u_k$.

**Case 1.1.** $k = 2$. Then $b$ is incident to $u_1$. Suppose $b = ux$, where $x \in V(G) \setminus \{u_0, u_1, u_2\}$. Let $S = \{u_0, u_1, u_2, x\}$ and $\overline{S} = V(T_1) \setminus S$. Since $|V(T_1)| = |V(G)| - 1 \geq 5$, there exists an edge $e \in [S, \overline{S}]_{T_1}$, where $[S, \overline{S}]_{T_1}$ is the set of edges between $S$ and $\overline{S}$ in $T_1$. If $e$ is incident to $u_0$ or $u_2$, then $d_{L(G)}(a, e) = 1$ and $d_{L(G)}(b, e) = 2$; if $e$ is incident to $u_1$ or $x$, then $d_{L(G)}(a, e) = 2$ and $d_{L(G)}(b, e) = 1$. So (4) holds.

**Case 1.2.** $k \geq 3$. Note that $b$ is incident to $u_1$ or $u_{k-1}$. Without loss of generality, assume that $b$ is incident to $u_1$. Let $e = u_1u_2 \in E(T_1)$. Then $d_{L(G)}(a, e) = 2 \neq 1 = d_{L(G)}(b, e)$, (4) holds.

**Case 2.** Both $a$ and $b$ are incident to $v$. Let $a = vx$, $b = vy$, $S = \{x, y\}$ and $\overline{S} = V(T_1) \setminus S$. Pick $e \in [S, \overline{S}]_{T_1}$. Note that $e$ is not incident to $v$. Similar to Case 1.1, $e$ satisfies (4).
Therefore, $E(T_1)$ is a resolving set of $L(G)$ with size $|V(G)| - 2$, and the upper bound is valid. \hfill \Box

The lower bound in Theorem 3.1 can be attained if $G$ is a path. The fact that $\mu(L(K_{1,n})) = n - 1$ implies that the upper bound in Theorem 3.1 is tight. It seems to be difficult to improve the bound for general graphs. However, for a tree $T$, we can obtain the metric dimension of $L(T)$ in terms of some parameters of $T$.

Let $T$ be a tree. A vertex of degree 1 in $T$ is called an end-vertex. A vertex of degree at least 3 in $T$ is called a major vertex. An end-vertex $u$ of $T$ is said to be a terminal vertex of a major vertex $v$ of $T$ if $d_T(u, v) < d_T(u, w)$ for every other major vertex $w$ of $T$. A major vertex $v$ of $T$ is an exterior major vertex of $T$ if there exists a terminal vertex of $v$ in $T$. We denote the set of all the exterior major vertices in $T$ by $\text{EX}(T)$; For $v \in \text{EX}(T)$, we denote the set of all the terminal vertices of $v$ by $\text{TER}(v)$. Let $\sigma(T) = \sum_{v \in \text{EX}(T)} |\text{TER}(v)|$ and $\text{ex}(T) = |\text{EX}(T)|$. Chartrand et al. [4] computed the metric dimension of a tree in terms of $\sigma(T)$ and $\text{ex}(T)$.

**Proposition 3.2** (4) If $T$ is a tree that is not a path, then $\mu(T) = \sigma(T) - \text{ex}(T)$.

Finally, we shall compute the metric dimension of the line graph of a tree. If $P$ is a path, then $\mu(L(P)) = 1$.

**Proposition 3.3** If $T$ is a tree that is not a path, then $\mu(L(T)) = \sigma(T) - \text{ex}(T)$.

**Proof.** Let $R$ be a resolving set of $L(T)$ with the minimum cardinality. For a given vertex $v \in \text{EX}(T)$, we claim that

$$\sum_{u \in \text{TER}(v)} |R \cap E(P_{u,v})| \geq |\text{TER}(v)| - 1, \quad (5)$$

where $P_{u,v}$ is the unique path between $u$ and $v$ in $T$. To the contrary, suppose that there exist two different terminal vertices $u_1, u_2$ of $v$ such that $R \cap E(P_{u_1,v}) = R \cap E(P_{u_2,v}) = \emptyset$. Let $e_1$ and $e_2$ be the edges incident to $v$ in $P_{u_1,v}$ and $P_{u_2,v}$, respectively. For each $e \in R$, we have $d_{L(T)}(e_1, e) = d_{L(T)}(e_2, e)$, contradicting the fact that $R$ is a resolving set of $L(T)$. Hence our claim is valid. Since $|R| \geq \sum_{v \in \text{EX}(T)} \sum_{u \in \text{TER}(v)} |R \cap E(P_{u,v})|$, by (5) we have

$$\mu(L(T)) = |R| \geq \sum_{v \in \text{EX}(T)} (|\text{TER}(v)| - 1) = \sigma(T) - \text{ex}(T). \quad (6)$$

Let $W$ be a set obtained from the end-vertex set of $T$ by deleting one terminal vertex of each exterior major vertex of $T$. In [4] Theorem 5, Chartrand et al. proved that $W$ is a resolving set of $T$ with size $\sigma(T) - \text{ex}(T)$. Let $W_L$ be the set of all the edges each of which is incident to one vertex of $W$. Then $|W_L| = |W|$. We will show that $W_L$ is a resolving set of $L(T)$.

For any two distinct edges $a$ and $b$ of $T$, there exists a unique path

$$(w_0, w_1, \ldots, w_k)$$
such that $a = w_0w_1$ and $b = w_{k-1}w_k$. Since $w_0 \neq w_k$, there exists a vertex $w \in W$ such that $d_T(w_0, w) \neq d_T(w_k, w)$. Without loss of generality, assume that $d_T(w_0, w) < d_T(w_k, w)$. Let $e$ be the edge incident to $w$. Then $e \in W_L$.

Case 1. $w_1 \in V(P_{w_0, w})$. Then

$$d_{L(T)}(a, e) = d_T(w_0, w) - 1 < d_T(w_k, w) - 1 \leq d_{L(T)}(b, e).$$

Case 2. $w_1 \not\in V(P_{w_0, w})$. Then $(w_k, w_{k-1}, \ldots, w_1, P_{w_0, w})$ is the unique path between $w_k$ and $w$. It follows that

$$d_{L(T)}(a, e) = d_T(w_0, w) < d_T(w_{k-1}, w) = d_{L(T)}(b, e).$$

Therefore, $W_L$ is a resolving set of $L(T)$, which implies that $\mu(L(T)) \leq \sigma(T) - \text{ex}(T)$. By \[\text{Proposition 3.3}\] and Proposition \[\text{Proposition 3.3}\] $\mu(T) = \mu(L(T))$ for a tree $T$. It seems to be interesting to characterize a graph $G$ satisfying $\mu(G) = \mu(L(G))$.

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