A SECOND LOOK AT
A GEOMETRIC PROOF OF THE SPECTRAL THEOREM FOR UNBOUNDED
SELF-ADJOINT OPERATORS

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Abstract. A new geometric proof of the spectral theorem for unbounded self-
adjoint operators \(A\) in a Hilbert space \(H\) is given based on a splitting of \(A\)
in positive and negative parts \(A_+ \geq 0\) and \(A_- \leq 0\). For both operators \(A_+\)
and \(A_-\) the spectral family can be defined immediately and then put together
to become the spectral family of \(A\). Of course crucial methods and results of
\[Lei79\] are used.

The underlying note is a second look at my article A Geometric Proof of the
Spectral Theorem for Unbounded Self-Adjoint Operators that appeared 1979 in the
Math. Ann. \[Lei79\]. A second look means that we concentrate in this note
on semi-bounded self-adjoint operators in a Hilbert space \(H\). For semi-bounded
self-adjoint operators \(A\), say \(A \geq 0\), we can define the spectral family \((E(\lambda))_{\lambda \in \mathbb{R}}\)
immediately by setting

\[E(\lambda) = P_{F(A,\lambda)} \quad (\lambda \in \mathbb{R}),\]

where \(P_{F(A,\lambda)}\) is the projection of \(H\) onto the subspace

\[F(A,\lambda) = \{ x \mid x \in D(A^n), \| A^n x \| \leq \lambda^n \| x \| \ (n \in \mathbb{N}) \} \]

The general situation of a totally unbounded (unbounded from above and from
below) self-adjoint operator \(A\) is handled by splitting the operator \(A\) in two semi-
bounded operators \(A_+ \geq 0\) and \(A_- \leq 0\) such that \(A = A_- \oplus A_+\). Here again a
(with respect to \(A\)) reducing subspace \(F(B + \beta, \beta)\) with \(B = A(1 + A^2)^{-1}\) and
\(B + \beta \geq 0\) is of importance.

Moreover we use the possibility to present certain improved and extended lemmas
of \[Lei79\], namely \[Lei79\, Lemma 1\] and \[Lei79\, Lemma 4\].

Concerning definitions, notations and basic results of Hilbert spaces and in partic-
ular of the calculus of projections we refer the reader to \[Wei80\].

We now recall those results of \[Lei79\] that are used decisively in this note. Notice
that an inspection of the proofs of these results in \[Lei79\] shows that the basic
assumption ‘\(A\) symmetric’ may be weakened to ‘\(A\) Hermitian’.

Lemma 1. Suppose \(A\) is a closed Hermitian operator in the Hilbert space \(H\) and
\(\epsilon, \delta, \lambda\) are non-negative real numbers. Then

i) \(F(A,\lambda)\) is a subspace of \(H\), which is left invariant by the operator \(A\).

ii) Every bounded linear operator \(B\) satisfying \(BA \subset AB\) maps \(F(A,\lambda)\) into
itself. Similarly we have \(B^* (F(A,\lambda))^\perp \subset F(A,\lambda)^\perp\).

iii) \(F(A + \delta, \epsilon) \subset F(A, \delta + \epsilon)\) and \(F(A^2, \epsilon^2) = F(A, \epsilon)\).

\[\text{1} \text{Notice that our scalar products are linear in the first argument.}\]
Proof: For the proof of i) and ii) we refer to [Lei79] Lemma 1]. To prove the first inclusion of iii) we know by property i) of Lemma [1] that \( F(A + \delta, \epsilon) \) is invariant under \( A + \delta \) and thus also under \( A \) itself. Hence for \( x \in F(A + \delta, \epsilon) \) we may conclude \( A^n x \in F(A + \delta, \epsilon) \) for all \( n \in \mathbb{N} \), i.e. \( x \in D^\infty(A) = \bigcap_{n=1}^\infty D(A^n) \) and

\[
\|Ax\| \leq \|(A + \delta)x\| + \delta \|x\| \leq (\epsilon + \delta) \|x\|.
\]

Suppose \( \|A^n x\| \leq (\epsilon + \delta)^n \|x\| \) is valid for given \( n \in \mathbb{N} \) and \( x \in F(A + \delta, \epsilon) \), then

\[
\|A^{n+1} x\| = \|AA^n x\| = \|(A + \delta)A^n x - \delta A^n x\|
\leq \epsilon \|A^n x\| + \delta \|A^n x\| = (\epsilon + \delta) \|A^n x\|
\leq (\epsilon + \delta)^n \|x\|.
\]

Thus by induction on \( n \in \mathbb{N} \) we see that \( x \in F(A, \delta + \epsilon) \).

To prove the second identity of iii) we first remark that \( A^2 \) is a closed operator. Clearly \( D^\infty(A^2) = D^\infty(A) \) and for \( x \in F(A^2, \epsilon^2) \) we have

\[
\|A^n x\|^2 = \langle A^n x, A^n x \rangle = \langle x, A^{2n} x \rangle
\leq \|x\| \|(A^2)^n x\| \leq \|x\| \|x\| \|(\epsilon^2)^n \|x\|\|
\]

which gives \( \|A^n x\| \leq \epsilon^n \|x\| \) and thus \( x \in F(A, \epsilon) \). This gives \( F(A^2, \epsilon^2) \subset F(A, \epsilon) \). The inverse inclusion \( F(A, \epsilon) \subset F(A^2, \epsilon^2) \) is obvious.

Lemma 2. Suppose \( A \) is a Hermitian operator in a finite dimensional Hilbert space \( H \). Then we have for all \( x \in F(A, \lambda)^1, x \neq 0 \), and all \( \lambda \geq 0 \):

i) \( \|Ax\| > \lambda \|x\| \)

ii) \( \langle Ax, x \rangle > \lambda \langle x, x \rangle \), provided \( A \geq 0 \)

Proof: For a proof see [Lei79] Lemma 2].

Proposition 1. Suppose \( A \) is a closed Hermitian operator in a Hilbert space \( H \) and \( \lambda, \mu \) are non-negative real numbers. Then for every \( x \in F(A, \mu) \cap F(A, \lambda)^1 \) there is a sequence \( \{A_n\} \) of Hermitian operators \( A_n : H_n \to H_n \) defined in finite dimensional subspaces \( H_n \subset H \) and a sequence \( \{x_n\} \) of Elements \( x_n \) belonging to \( H_n \) in such a way that

\[
x_n \in F(A_n, \lambda)^1 \quad \text{and} \quad \lim_{n \to \infty} \|x_n - x\| = 0 = \lim_{n \to \infty} \|Ax_n - Ax\|.
\]

If \( A \geq 0 \) then \( A_n \) can be chosen non-negative.

Proof: For a proof see [Lei79] Lemma 3].

Lemma 3. Let \( A \) be a closed Hermitian operator in \( H \), and \( \lambda, \mu \) nonnegative real numbers. Then for all \( x \in F(A, \mu) \cap F(A, \lambda)^1 \)

i) \( \lambda \|x\| \leq \|Ax\| \leq \mu \|x\| \)

ii) \( \lambda \langle x, x \rangle \leq \langle Ax, x \rangle \leq \mu \langle x, x \rangle \), provided \( A \geq 0 \).

Remark. Note that Lemma [3] can be written in the equivalent form

\[
\lambda^2 \langle x, x \rangle \leq \langle A^2 x, x \rangle \leq \mu^2 \langle x, x \rangle.
\]

Proof: For \( x \in F(A, \mu) \) the inequalities \( \|Ax\| \leq \mu \|x\| \) and \( \langle Ax, x \rangle \leq \mu \langle x, x \rangle \) follow directly from the definition of \( F(A, \mu) \) whereas for \( x \in F(A, \mu) \cap F(A, \lambda)^1 \) the inequalities \( \lambda \|x\| \leq \|Ax\| \) and \( \lambda \langle x, x \rangle \leq \langle Ax, x \rangle \) can be derived from Lemma [2] using the limiting process described in Proposition [1]
Lemma 4. Let \( A \) be a closed Hermitian operator in a Hilbert space \( H \). Then the following statements are equivalent:

i) \( \bigcup_{\epsilon > 0} F(A, \epsilon) \) is dense in \( H \).

ii) \( A \) is self-adjoint.

Proof: The proof of i) \( \Rightarrow \) ii) can be found in \cite{Lei79} Lemma 4 and remains unchanged. To see ii) \( \Rightarrow \) i) we present a new shortened and simplified proof.

Suppose \( A \) is self-adjoint and assume in addition that \( A \) is bounded from below by 1, i.e. \( A \geq 1 \). Then \( A^{-1} \) exists, is bounded and the following relation holds true:

\[
F(A^{-1}, \epsilon^{-1}) \subset F(A, \epsilon) \quad (\epsilon > 0)
\]

Assume for the moment that (1) is already proven then

\[
\left( \bigcup_{\epsilon > 0} F(A, \epsilon) \right) \perp = \bigcap_{\epsilon > 0} F(A, \epsilon \perp) \subset \bigcap_{\epsilon > 0} F(A^{-1}, \epsilon^{-1}) = N(A^{-1}) = \{0\}
\]

and \( \bigcup_{\epsilon > 0} F(A, \epsilon) \) is dense in \( H \).

For general \( A \) we consider the self-adjoint operator \( S = A^2 + 1 \geq 1 \) being self-adjoint in view of \cite{Kat76} Theorem 3.24. We use the inclusions iii) in Lemma 4 to get

\[
F(S, \epsilon) \subset F(A^2, \epsilon + 1) \subset F(A, \sqrt{\epsilon + 1}) \quad (\epsilon > 0)
\]

and the density of \( \bigcup_{\epsilon > 0} F(A, \epsilon) \) in \( H \) is proven.

To end the proof we have to justify inclusion (1). In order to show (1) we consider \( H_\epsilon = F(A^{-1}, \epsilon^{-1}) \perp \) and \( B_\epsilon = A^{-1}|H_\epsilon \). Applying assertion ii) of Lemma 1 we have \( B_\epsilon(H_\epsilon) \subset H_\epsilon \) and using assertion i) of Lemma 3 with \( A \) replaced by \( A^{-1} \), \( \lambda = \epsilon^{-1} \) and \( \mu = \|A^{-1}\| \) we conclude that \( B_\epsilon : H_\epsilon \rightarrow H_\epsilon \) is bijective and \( B_\epsilon^{-1} = A|H_\epsilon \) with \( \|A|H_\epsilon\| \leq \epsilon \). Hence \( \|(A|H_\epsilon)^n\| \leq \epsilon^n \) for \( n \in \mathbb{N} \) and consequently \( \|A^n x\| \leq \epsilon^n \|x\| \) for \( x \in H_\epsilon \). It follows \( x \in F(A, \epsilon) \) which implies inclusion (1). □

Remark. A closed Hermitian operator \( A \) is self-adjoint if and only if \( A^2 \) is self-adjoint.

Proof: This follows from Lemma 4 and the identity \( F(A^2, \epsilon^2) = F(A, \epsilon) \) being valid in view of Lemma 4 for all \( \epsilon \geq 0 \). Notice that for a closed Hermitian operator \( A \) the operator \( A^2 \) is always closed. □

Lemma 5. Suppose \( A \) is a closed Hermitian operator in a Hilbert space \( H \) and \((P_n)_{n \in \mathbb{N}}\) an increasing sequence of projections such that \( R(P_n) \subset D(A) \), \( AP_n = P_n AP_n \) and \( P_n \rightarrow I \) strongly. Then the following assertions hold:

i) \( D(A) = \{ x \in H \mid (AP_n x) \text{ converges} \} = \{ x \in H \mid (\|AP_n x\|) \text{ converges} \} \)

ii) \( Ax = \lim_{n \rightarrow \infty} AP_n x \) for all \( x \in D(A) \)

Proof: For a proof see \cite{Lei79} Lemma 5. □

Lemma 6. Suppose \( A \) is a self-adjoint operator in a Hilbert space \( H \). Then there exist subspaces \( H_\pm \subset H \), self-adjoint operators \( A_\pm = A_{D(A) \cap H_\pm} \) in \( H_\pm \) such that

\[
H = H_- \oplus H_+ , \quad A = A_- \oplus A_+ \quad \text{and} \quad A_- \leq 0 \leq A_+.
\]

Proof: We put \( B = A(1 + A^2)^{-1} \) and \( E = P_{F(B + \beta, \beta)} \) with \( \beta \geq 0 \) such that \( B + \beta \geq 1 \), where \( E \) is the projection of \( H \) onto the subspace \( F(B + \beta, \beta) \). Let us
note, that $B$ is a bounded self-adjoint operator. From Lemma 1 ii) with $A$ replaced by $B + \beta$ and $B$ replaced by $E$ we conclude $EB = BE$. Hence

$$EA(A^2 + 1)^{-1} = EB = BE = A(A^2 + 1)^{-1}E = AE(A^2 + 1)^{-1}$$

where we used $(A^2 + 1)^{-1}E = E(A^2 + 1)^{-1}$, being valid by Lemma 1 ii) since $B$ and $(A^2 + 1)^{-1}$ commute. If we drop the middle terms in (3) and apply both sides of the resulting equation to $y = (A^2 + 1)x$ with $x \in D(A^2)$ we get

$$EAx = AEx \quad (x \in D(A^2))$$

Let us extend (4) to elements $x \in D(A)$. Since $D(A^2)$ is a core of $D(A)$ (see [Kat76] Theorem 3.24) there is for each $x \in D(A)$ a sequence $(x_n) \subset D(A^2)$ such that $x_n \to x, Ax_n \to Ax$. Using (4) we conclude $Ex_n \to Ex$ as well as $AEx_n = EAx_n \to EAx$. Since $A$ is a closed operator $Ex \in D(A)$ and

$$AEx = EAx \quad (x \in D(A)),$$

which means $EA \subseteq AE$.

We put $H_- = R(E), H_+ = R(I - E)$ which gives $H = H_- \oplus H_+$ and we remind the reader that $E = P_{H_-}$ is the projection of $H$ onto the subspace $H_-$. Because of $EA \subseteq AE$ the subspaces $H_-$ and $H_+$ are reducing subspaces for the self-adjoint operator $A$ and the operators

$$A_- = A|_{D(A) \cap H_-} \quad \text{and} \quad A_+ = A|_{D(A) \cap H_+}$$

are self-adjoint operators in the Hilbert spaces $H_-$ and $H_+$ with $A = A_- \oplus A_+$. See [Wei80] Theorem 7.28 for details of this facts.

Now it remains to prove $A_- \leq 0$ and $A_+ \geq 0$. For all $x \in D(A)$ we have

$$Ax = (A^2 + 1)(A^2 + 1)^{-1}Ax = (A^2 + 1)A(A^2 + 1)^{-1}x = (A^2 + 1)Bx$$

which gives $\langle Ax, x \rangle = \langle Bx, x \rangle + (A^2 Bx, x)$ and thus

$$\langle Ax, x \rangle = \langle Bx, x \rangle + (B Ax, Ax)$$

where we have used $BA \subseteq AB$. Now for $x \in H_+ = F(B + \beta, \beta)$ we have because of Lemma 3 ii) with $A$ replaced by $B + \beta$ and $\lambda = 0, \mu = \beta$

$$\langle (B + \beta)x, x \rangle \leq \beta \langle x, x \rangle$$

or

$$\langle Bx, x \rangle \leq 0.$$  

Hence for $x \in D(A_-)$ we have $Ax \in H_-$ and thus in view of (7) the inequality

$$\langle B Ax, Ax \rangle \leq 0$$

holds true. Together with equation (8) we obtain $\langle Ax, x \rangle \leq 0$, i.e. $A_- \leq 0$.

In analogue way we conclude $\langle Ax, x \rangle \geq 0$ for all $x \in D(A_+)$, i.e. $A_+ \geq 0$. Indeed for $x \in H_+ = F(B + \beta, \beta)$ we have because of Lemma 3 ii) with $A$ replaced by $B + \beta$ and $\lambda = \beta, \mu = \|B + \beta\|$ the inequality

$$\beta \langle x, x \rangle \leq \langle (B + \beta)x, x \rangle$$

or

$$0 \leq \langle Bx, x \rangle.$$
For $x \in D(A_+)$ it’s clear that $A_+ x = Ax \in H_+$ and thus
\begin{equation}
0 \leq \langle B Ax, Ax \rangle
\end{equation}
which together with (9) and (6) gives $0 \leq \langle Ax, x \rangle$, i.e. $0 \leq A_+$. \hfill \Box

**Corollary 1.** Suppose the self-adjoint operators $A_-$ and $A_+$ in Lemma\ref{lem:spec_rep} admit spectral representations $A_{\pm} = \int_{\mathbb{R}} \lambda dE_{\pm}(\lambda)$, then with $E(\lambda) = E_-(\lambda) \oplus E_+(\lambda)$ the operator $A = A_- \oplus A_+$ admits a spectral representation
\begin{equation}
A = \int_{\mathbb{R}} \lambda dE(\lambda).
\end{equation}

**Proof:** Let us write
\begin{equation}
x = x_- + x_+, \quad x \in D(A), \quad x_{\pm} \in D(A_{\pm}), \quad Ax = A_-x_- + A_+x_+
\end{equation}
\begin{equation}
A_{\pm} = \int_{\mathbb{R}} \lambda dE_{\pm}(\lambda) \quad \text{with spectral families} \quad (E_{\pm}(\lambda))_{\lambda \in \mathbb{R}}.
\end{equation}
We have $x \in D(A) \Leftrightarrow x_{\pm} \in D(A_{\pm}) \Leftrightarrow \int_{\mathbb{R}} \lambda^2 dE_{\pm}(\lambda) < \infty$. If we define $E(\lambda) = E_-(\lambda) \oplus E_+(\lambda)$ then in view of
\begin{equation}
\langle E(\lambda)x, y \rangle = \langle E_-(\lambda)x_-, y_- \rangle + \langle E_+(\lambda)x_+, y_+ \rangle
\end{equation}
\begin{equation}
\langle Ax, y \rangle = \langle A_-x_-, y_- \rangle + \langle A_+x_+, y_+ \rangle
\end{equation}
we conclude
\begin{equation}
x \in D(A) \Leftrightarrow \int_{\mathbb{R}} \lambda^2 dE_{\pm}(\lambda) < \infty \Leftrightarrow \int_{\mathbb{R}} \lambda^2 dE(\lambda) < \infty.
\end{equation}
\begin{equation}
\langle Ax, y \rangle = \langle A_-x_-, y_- \rangle + \langle A_+x_+, y_+ \rangle = \int_{\mathbb{R}} \lambda d\langle E(\lambda)x, y \rangle.
\end{equation}
\hfill \Box

**Theorem 1.** Every self-adjoint operator $A$ in a Hilbert space $H$ admits one and only one spectral family $(E(\lambda))_{\lambda \in \mathbb{R}}$ such that
\begin{equation}
A = \int_{\mathbb{R}} \lambda dE(\lambda).
\end{equation}

**Proof:** **Uniqueness.** The proof of the uniqueness of the representing spectral family is given in [Lei79, Theorem 1].

**Existence.** We first prove the spectral theorem for positive self-adjoint operators $A > 0$. We define a family of projections by setting
\begin{equation}
E(\lambda) = P_{F(A, \lambda)} (\lambda \in \mathbb{R}).
\end{equation}
Notice that $E(\lambda) = 0$ if $\lambda \leq 0$. It is not difficult to check that $(E(\lambda))_{\lambda \in \mathbb{R}}$ is actually a spectral family. The only nontrivial point is the property $\lim_{\lambda \to \infty} E(\lambda) = I$. But this follows from Lemma\ref{lem:spec_rep}. So it remains to show the validity of the formula
\begin{equation}
A = \int_{\mathbb{R}} \lambda dE(\lambda).
\end{equation}
We take $n \in \mathbb{N}$, $x \in F(A, n)$ and fix these elements. Also for fixed $k \in \mathbb{N}$ we define $\lambda_i = \frac{i}{k}$ for all $i \in \{0, 1, 2, \ldots, nk\}$. With
\begin{equation}
x_i = E(\lambda_i)x - E(\lambda_{i-1})x \in F(A, \lambda_i) \cap F(A, \lambda_{i-1})^\perp \quad \text{for all} \quad 1 \leq i \leq nk.
\end{equation}
we have
\[ x = \sum_{i=1}^{nk} x_i \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{nk} \|x_i\|^2, \]

since \( \langle x_i, x_j \rangle = 0 \) if \( i \neq j \).

We use Lemma 3 to obtain the following inequalities.

(16) \[ \lambda_{i-1}\langle x_i, x_i \rangle \leq \langle Ax_i, x_i \rangle \leq \lambda_i\langle x_i, x_i \rangle \quad (1 \leq i \leq nk) \]

(17) \[ \lambda^2_{i-1}\langle x_i, x_i \rangle \leq \langle A^2x_i, x_i \rangle \leq \lambda^2_i\langle x_i, x_i \rangle \quad (1 \leq i \leq nk) \]

hence

(18) \[ |\langle (A - \lambda_i)x_i, x_i \rangle| \leq (\lambda_i - \lambda_{i-1}) \|x_i\|^2 \leq \frac{1}{k} \|x_i\|^2 \quad (1 \leq i \leq nk) \]

(19) \[ |\langle (A^2 - \lambda^2_i)x_i, x_i \rangle| \leq (\lambda^2_i - \lambda^2_{i-1}) \|x_i\|^2 \leq \frac{2n}{k} \|x_i\|^2 \quad (1 \leq i \leq nk) \]

The identity \( \langle Ax, x \rangle = \sum_{i=1}^{nk} \langle Ax_i, x_i \rangle \) and equation (18) gives

\[ \left| \langle Ax, x \rangle - \sum_{i=1}^{nk} \lambda_i\langle x_i, x_i \rangle \right| \leq \frac{1}{k} \sum_{i=1}^{nk} \|x_i\|^2 = \frac{1}{k} \|x\|^2 \]

A similar estimate (using (19)) holds for \( A^2 \) so that we have the following set of estimates

(20) \[ \left| \langle Ax, x \rangle - \sum_{i=1}^{nk} \lambda_i\langle x_i, x_i \rangle \right| \leq \frac{1}{k} \|x\|^2 \]

(21) \[ \left| \langle A^2x, x \rangle - \sum_{i=1}^{nk} \lambda^2_i\langle x_i, x_i \rangle \right| \leq \frac{2n}{k} \|x\|^2 \]

For fixed \( n \in \mathbb{N} \) we let tend \( k \to \infty \) and obtain for \( x \in F(A, n) \)

(22) \[ \langle Ax, x \rangle = \int_0^n \lambda \ d(E(\lambda)x, x) = \int_{[0,n]} \lambda \ d(E(\lambda)x, x) \]

(23) \[ \|Ax\|^2 = \langle A^2x, x \rangle = \int_0^n \lambda^2 \ d(E(\lambda)x, x) = \int_{[0,n]} \lambda^2 \ d(E(\lambda)x, x) \]

Notice that the first integrals in (22), (23) are Riemann-Stieltjes integrals and the second ones are Lebesgue-Stieltjes integrals.

Now we take an arbitrary \( x \in D(A) \), put \( P_n = E(n) \) then \( P_nx \in F(A, n) \) and thus using (22) and (23) we get

(24) \[ \langle AP_nx, P_nx \rangle = \int_{[0,n]} \lambda \ d(E(\lambda)P_nx, P_nx) = \int_{[0,n]} \lambda \ d(E(\lambda)x, x) \]

(25) \[ \|AP_nx\|^2 = \langle A^2x, x \rangle = \int_{[0,n]} \lambda^2 \ d(E(\lambda)x, x) \]
since $P_n E(\lambda) P_n = E(\lambda)$ for $0 \leq \lambda \leq n$. In a last step we apply Lemma 5 together with (24) and (25) to get
\[
A = \int_{\mathbb{R}} \lambda \, dE(\lambda),
\]
i.e. for exactly $x \in D(A)$ we have
\[
\|Ax\|^2 = \int_{\mathbb{R}} \lambda^2 \, d\langle E(\lambda)x, x \rangle < \infty, \quad (26)
\]
and
\[
\langle Ax, x \rangle = \int_{\mathbb{R}} \lambda \, d\langle E(\lambda)x, x \rangle. \quad (27)
\]
Notice that actually $E(\lambda) = 0$ for all $\lambda \leq 0$.

Now let us extend the validity of a spectral representation to arbitrary self-adjoint operators. First we remind the reader of the simple fact that if a self-adjoint operator $A$ has a spectral representation $A = \int_{\mathbb{R}} \lambda \, dE(\lambda)$ then the families
\[
F(\lambda) := E(\lambda - c) \quad \text{as well as} \quad G(\lambda) := I - E(-\lambda) \quad (\lambda \in \mathbb{R})
\]
are spectral resolutions for $A + c$ and $-A$ respectively. Notice that $G(\lambda)$ is actually left-continuous. So to be formally correct one has in fact to chose the right-continuous spectral family $(G(\lambda+))_{\lambda \in \mathbb{R}}$. We know now that every semi-bounded self-adjoint operator admits a spectral representation. In a last step we apply Lemma 6 and Corollary 1 to guarantee the existence of a spectral representation for all self-adjoint operators. \(\square\)

**Concluding Remarks.** 1. Let us mention that (27) can be extended (by using the polarization identity) to
\[
\langle Ax, y \rangle = \int_{\mathbb{R}} \lambda \, d\langle E(\lambda)x, y \rangle \quad (x, y \in D(A)). \quad (28)
\]
Even elements $y \in H$ are allowed in (28) (by a specific interpretation of the integral), but we will not prove this here.

2. As a consequence of the proof of Lemma 4 the spectral theorem is now proven directly in real as well as in complex Hilbert spaces. (The existence of a resolvent $R(A, \lambda)$ with non-real $\lambda$ is no more needed!)

**References**

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