Bound states of the Duffin-Kemmer-Petiau equation with a mixed minimal-nonminimal vector cusp potential

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Abstract

The problem of spin-0 and spin-1 bosons subject to a general mixing of minimal and nonminimal vector cusp potentials is explored in a unified way in the context of the Duffin-Kemmer-Petiau theory. Effects on the bound-state solutions due to a short-range interaction are discussed in some detail.

Keywords: DKP equation, Klein’s paradox, pair production, nonminimal coupling

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1 Introduction

The first-order Duffin-Kemmer-Petiau (DKP) formalism \cite{1-4} describes spin-0 and spin-1 particles and has been used to analyze relativistic interactions of spin-0 and spin-1 hadrons with nuclei as an alternative to their conventional second-order Klein-Gordon and Proca counterparts. The onus of equivalence between the formalisms represented an objection to the DKP theory for a long time and only recently it was shown that they yield the same results in the case of minimally coupled vector interactions, on the condition that one correctly interprets the components of the DKP spinor \cite{5-6}. However, the equivalence between the DKP and the Proca formalisms has already a precedent \cite{7}. The equivalence does not maintain if one considers partially conserved currents \cite{8}. Furthermore, the DKP formalism enjoys a richness of couplings not capable of being expressed in the Klein-Gordon and Proca theories.

Nonminimal vector potentials, added by other kinds of Lorentz structures, have already been used successfully in a phenomenological context for describing the scattering of mesons by nuclei \cite{9-16}. Nonminimal vector coupling with a quadratic potential \cite{17}, with a linear potential \cite{18}, and mixed space and time components with a step potential \cite{19-20} and a linear potential \cite{21} have been explored in the literature. See also Ref. \cite{21} for a comprehensive list of references on other sorts of couplings and functional forms for the potential functions. In Ref. \cite{21} it was shown that when the space component of the coupling is stronger than its time component the linear potential, a sort of vector DKP oscillator, can be used as a model for confining bosons.

The cusp potential in the form $e^{-|x|/\lambda}$ (screened Coulomb potential in a two-dimensional space-time world) has been analyzed and its analytical solutions have been found for the Dirac equation with vector \cite{22-24}, scalar \cite{25} and pseudoscalar \cite{26} couplings, and for the Klein-Gordon equation with vector \cite{27-29} and scalar \cite{30} couplings, and a mixed scalar-vector coupling \cite{31}. As has been emphasized in Refs. \cite{27} and \cite{30}, the solutions of relativistic equations with this sort of potential may find applications in the study of pionic atoms, doped Mott insulators, doped semiconductors, interaction between ions, quantum dots surrounded by a dielectric or a conducting medium, protein structures, etc. The bound states for the Klein-Gordon equation with a minimal vector coupling differ radically from those ones for the Dirac equation. For an enough deep and narrow potential the Klein-Gordon equation
exhibits the phenomenon called Schiff-Snyder-Weinberg effect (SSWE) [32]. Such an effect manifests by additional antiparticle bound states in a potential attractive only for particles. For critical depths, the particle and antiparticle energy levels coalesce and there unveils a new channel for the pair production. Popov [33] argued that the SSWE is inherent to short-range interactions and should not be expected for large-range potentials. Nevertheless, Klein and Rafelski [34] used a purported SSWE in a Coulomb potential for speculating about the Bose condensation and the stability of extremely high atomic number nuclei and, right away, were severely criticized [35]. As a matter of fact, the investigation of the bound-state solutions of the Klein-Gordon equation with different functional forms for the potential validates Popov’s conjecture [28]-[29], [36]-[37].

The main purpose of the present article is to report on the properties of the DKP theory with time components of minimal and nonminimal vector cusp potentials for spin-0 and spin-1 bosons in a unified way. This sort of mixing, beyond its potential physical applications, shows to be a powerful tool to obtain a deeper insight about the nature of the DKP equation and its solutions as far as it explores the differences between minimal and nonminimal couplings. The problem is mapped into an exactly solvable Sturm-Liouville problem of a Schrödinger-like equation with an effective symmetric Morse-like potential, or an effective cusp potential in the particular circumstance of a pure minimal coupling.

2 Vector couplings in the DKP equation

The DKP equation for a free boson is given by [4] (with units in which $\hbar = c = 1$)

$$(i\beta^\mu \partial_\mu - m) \psi = 0$$

(1)

where the matrices $\beta^\mu$ satisfy the algebra $\beta^\mu \beta^{\nu} \beta^\lambda + \beta^\lambda \beta^{\nu} \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu$ and the metric tensor is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. That algebra generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-0 particles and a ten-dimensional representation associated to spin-1 particles. The second-order Klein-Gordon and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP theory. A well-known conserved four-current is given by $J^\mu = \bar{\psi} \beta^\mu \psi / 2$ where the adjoint spinor $\bar{\psi}$ is given by $\bar{\psi} = \psi^\dagger \eta^0$ with $\eta^0 = 2 \beta^0 \beta^0 - 1$, provided $\beta^\mu = \eta^0 (\bar{\beta}^\mu)^\dagger \eta^0$. The time
component of this current is not positive definite but it may be interpreted as a charge density. Then the normalization condition \( \int d\tau J^0 = \pm 1 \) can be expressed as
\[
\int d\tau \bar{\psi} \beta^0 \psi = \pm 2
\]
(2)
where the plus (minus) sign must be used for a positive (negative) charge.

With the introduction of interactions, the DKP equation can be written as
\[
(\beta^\mu p_\mu - m - V) \psi = 0
\]
(3)
and \( J^\mu \) satisfies the equation [21]
\[
\partial_\mu J^\mu + \frac{i}{2} \bar{\psi} \left( V - \eta^0 V^\dagger \eta^0 \right) \psi = 0
\]
(4)
Thus, if \( V \) is Hermitian with respect to \( \eta^0 \) then the four-current will be conserved. The more general potential matrix \( V \) is written in terms of 25 (100) linearly independent matrices pertinent to the five(ten)-dimensional irreducible representation associated to the scalar (vector) sector and can be written in terms of well-defined Lorentz structures. For the spin-0 sector there are two scalar, two vector and two tensor terms [35], whereas for the spin-1 sector there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms [39]. Considering only the vector terms, \( V \) is in the form [38], [40]
\[
( i \beta^\mu \partial_\mu - m - \beta^\mu A^{(1)}_\mu - i [P, \beta^\mu] A^{(2)}_\mu ) \psi = 0
\]
(5)
where \( P \) and \( [P, \beta^\mu] \) are independent elements of the DKP subalgebra obtained by simple products of the \( \beta^\mu \) matrices plus the unit matrix in such a way that \( \bar{\psi} [P, \beta^\mu] \psi \) behaves like a vector under a Lorentz transformation as does \( \bar{\psi} \beta^\mu \psi \). Once again \( \partial_\mu J^\mu = 0 \), provided \( P = \eta^0 (P)^\dagger \eta^0 \) and \( i [P, \beta^\mu] = \eta^0 (i [P, \beta^\mu])^\dagger \eta^0 \). Notice that the vector potential \( A^{(1)}_\mu \) is minimally coupled but not \( A^{(2)}_\mu \). If the terms in the potentials \( A^{(1)}_\mu \) and \( A^{(2)}_\mu \) are time-independent one can write \( \psi(\vec{r}, t) = \phi(\vec{r}) \exp(-iEt) \), where \( E \) is the energy of the boson, in such a way that the time-independent DKP equation becomes
\[
\left[ \beta^0 \left( E - A^{(1)}_0 \right) + i \beta^i \left( \partial_i + i A^{(1)}_i \right) - (m + i [P, \beta^\mu] A^{(2)}_\mu) \right] \phi = 0
\]
(6)
In this case \( J^\mu = \bar{\phi} \beta^\mu \phi / 2 \) does not depend on time, so that the spinor \( \phi \) describes a stationary state. Note that the time-independent DKP equation is
invariant under a simultaneous shift of $E$ and $A^{(1)}_0$, such as in the Schrödinger equation, but the invariance does not maintain regarding $E$ and $A^{(2)}_0$. It can be shown (see Ref. [21]) that any two stationary states with distinct energies and subject to the boundary conditions

$$\int d\tau \partial_i (\bar{\phi}_k \beta^i \phi_{k'}) = 0$$

are orthogonal in the sense that $\int d\tau \bar{\phi}_k \beta^0 \phi_{k'} = 0$, for $E_k \neq E_{k'}$. In addition, in view of (2) the spinors $\phi_k$ and $\phi_{k'}$ are said to be orthonormal if

$$\int d\tau \bar{\phi}_k \beta^0 \phi_{k'} = \pm 2\delta_{E_k E_{k'}}$$

The charge-conjugation operation changes the sign of the minimal interaction potential, i.e. changes the sign of $A^{(1)}_\mu$. This can be accomplished by the transformation $\psi \rightarrow \psi_c = C\psi = CK\psi$, where $K$ denotes the complex conjugation and $C$ is a unitary matrix such that $C\beta^\mu = -\beta^\mu C$. The matrix that satisfies this relation is $C = \exp (i\delta_C) \eta^0 \eta^1$. The phase factor $\exp (i\delta_C)$ is equal to $\pm 1$, thus $E \rightarrow -E$. Note also that $J^\mu \rightarrow -J^\mu$, as should be expected for a charge current. Meanwhile $C$ anticommutes with $[P, \beta^\mu]$ and the charge-conjugation operation entails no change on $A^{(2)}_\mu$. The invariance of the nonminimal vector potential under charge conjugation means that it does not couple to the charge of the boson. In other words, $A^{(2)}_\mu$ does not distinguish particles from antiparticles. Hence, whether one considers spin-0 or spin-1 bosons, this sort of interaction can not exhibit Klein’s paradox.

For the case of spin 0, we use the representation for the $\beta^\mu$ matrices given by [11]

$$\beta^0 = \left( \begin{array}{cc} \theta & 0 \\ 0 & 0 \end{array} \right), \quad \beta^i = \left( \begin{array}{cc} 0 & \rho_i \\ -\rho_i^T & 0 \end{array} \right), \quad i = 1, 2, 3$$

where

$$\theta = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \rho_1 = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\rho_2 = \left( \begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \rho_3 = \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right)$$

$0, \tilde{0}$ and $0$ are $2\times3$, $2\times2$ and $3\times3$ zero matrices, respectively, while the superscript T designates matrix transposition. Here the matrix $P$ appearing
in (5) can be written as \[ P = (\beta^\mu \beta_\mu - 1)/3 = \text{diag}(1, 0, 0, 0, 0). \] In this case \( P \) picks out the first component of the DKP spinor. The five-component spinor can be written as \( \psi^T = (\psi_1, ..., \psi_5) \) in such a way that the time-independent DKP equation for a boson constrained to move along the \( X \)-axis, restricting ourselves to time-like components of four-dimensional vector potentials \( \vec{A}^{(1)} = \vec{A}^{(2)} = 0 \) depending only on \( x \), decomposes into

\[
\left( \frac{d^2}{dx^2} + k^2 \right) \phi_1 = 0
\]

\[
\phi_2 = \frac{1}{m} \left( E - A_0^{(1)} + iA_0^{(2)} \right) \phi_1
\]

\[
\phi_3 = \frac{i}{m} \frac{d}{dx} \phi_1, \quad \phi_4 = \phi_5 = 0
\]

where

\[
k^2 = \left( E - A_0^{(1)} \right)^2 - m^2 + \left( A_0^{(2)} \right)^2
\]

Meanwhile,

\[
J^0 = \frac{E - A_0^{(1)}}{m} |\phi_1|^2, \quad J^1 = \frac{1}{m} \text{Im} \left( \phi_1^* \frac{d\phi_1}{dx} \right)
\]

It is worthwhile to note that \( J^0 \) becomes negative in regions of space where \( E < A_0^{(1)} \) (a circumstance associated to Klein’s paradox) and that \( A_0^{(2)} \) does not intervene explicitly in the current. The orthonormalization formula \[8\] becomes

\[
\int_{-\infty}^{+\infty} dx \frac{E_k + E_{k'}}{2m} \frac{A_0^{(1)}}{m} \phi_{1\kappa}^* \phi_{1\kappa'} = \pm \delta_{E_k E_{k'}}
\]

regardless \( A_0^{(2)} \). Eq. (14) is in agreement with the orthonormalization formula for the Klein-Gordon theory in the presence of a minimally coupled potential \[43\]. This is not surprising, because, after all, both DKP equation and Klein-Gordon equation are equivalent under minimal coupling.

For the case of spin 1, the \( \beta^\mu \) matrices are \[42\]

\[
\beta^0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
\beta^i = \begin{pmatrix}
0 & \bar{0} & e_i \\
\bar{0}^T & 0 & 0 & -i s_i \\
-e_i^T & 0 & 0 & 0 \\
\bar{0}^T & -i s_i & 0 & 0
\end{pmatrix}
\]  \quad (15)

where \(s_i\) are the 3\times3 spin-1 matrices \((s_i)_{jk} = -i \varepsilon_{ijk}\), \(e_i\) are the 1\times3 matrices \((e_i)_{1j} = \delta_{ij}\) and \(\bar{0} = (0 \ 0 \ 0)\), while \(I\) and \(0\) designate the 3\times3 unit and zero matrices, respectively. In this representation \(P = \beta^\mu \beta^\nu - 2 = \text{diag} (1,1,1,1,0,0,0,0,0,0)\), i.e. \(P\) projects out the four upper components of the DKP spinor. With the spinor written as \(\psi^T = (\psi_1, ..., \psi_{10})\), and partitioned as

\[
\psi_{I}^{(\pm)} = \begin{pmatrix}
\psi_3 \\
\psi_4
\end{pmatrix}, \quad \psi_{I}^{(-)} = \psi_5
\]

\[
\psi_{II}^{(\pm)} = \begin{pmatrix}
\psi_6 \\
\psi_7
\end{pmatrix}, \quad \psi_{II}^{(-)} = \psi_2
\]

\[
\psi_{III}^{(\pm)} = \begin{pmatrix}
\psi_{10} \\
-\psi_9
\end{pmatrix}, \quad \psi_{III}^{(-)} = \psi_1
\]

the one-dimensional time-independent DKP equation with time-like components of vector potentials can be expressed as

\[
\left(\frac{d^2}{dx^2} + k^2\right) \phi_{I}^{(\pm)} = 0
\]

\[
\phi_{II}^{(\pm)} = \frac{1}{m} \left(E - A_0^{(1)} \pm i A_0^{(2)}\right) \phi_{I}^{(\pm)}
\]

\[
\phi_{III}^{(\pm)} = \frac{i}{m} \frac{d}{dx} \phi_{I}^{(\pm)}, \quad \phi_8 = 0
\]

where \(k\) is again given by (12). Now the components of the four-current are

\[
J^0 = \frac{E - A_0^{(1)}}{m} \left(|\phi_{I}^{(\pm)}|^2 + |\phi_{I}^{(-)}|^2\right)
\]

\[
J^1 = \frac{1}{m} \text{Im} \left(\phi_{I}^{(\pm)\dagger} \frac{d\phi_{I}^{(\pm)}}{dx} + \phi_{I}^{(-)\dagger} \frac{d\phi_{I}^{(-)}}{dx}\right)
\]

and the orthonormalization expression (8) takes the form

\[
\int_{-\infty}^{+\infty} dx \frac{E_{k} + E_{k'}}{2} \frac{A_0^{(1)}}{m} \left(\phi_{Ik}^{(\pm)\dagger} \phi_{Ik'}^{(\pm)} + \phi_{Ik}^{(-)\dagger} \phi_{Ik'}^{(-)}\right) = \pm \delta_{E_k E_{k'}}
\]  \quad (19)
Just as for scalar bosons, $J^0 < 0$ for $E < A_0^{(1)}$ and $A_\mu^{(2)}$ does not appear in the current. Similarly, $A_\mu^{(2)}$ do not manifest explicitly in the orthonormalization formula.

Comparison between the two sets of formulas for the spin-0 and spin-1 sectors of the DKP theory evidences that vector bosons and scalar bosons behave in a similar way.

3 The cusp potential

Now we are in a position to use the DKP equation with specific forms for vector interactions. Let us focus our attention on time components of minimal and nonminimal vector potentials in the form of a cusp potential

$$A = -V_0 \exp \left( -\frac{|x|}{\lambda} \right), \quad V_0 > 0$$

(20)

with

$$A_0^{(1)} = g_1 A, \quad A_0^{(2)} = g_2 A$$

(21)

where the coupling constants, $g_1$ and $g_2$, are dimensionless real parameters and $\lambda$, related to the range of the interaction, is a positive parameter. In this case the first equations of (11) and (17) transmutes into

$$-\frac{1}{2m} \frac{d^2 \Phi}{dx^2} + V_{\text{eff}} \Phi = E_{\text{eff}} \Phi$$

(22)

where $\Phi$ is equal to $\phi_1$ for the scalar sector, and to $\phi_I^{(\pm)}$ for the vector sector, with

$$V_{\text{eff}} = V_1 \exp \left( -\frac{|x|}{\lambda} \right) + V_2 \exp \left( -2\frac{|x|}{\lambda} \right)$$

$$E_{\text{eff}} = \frac{E^2 - m^2}{2m}$$

(23)

and

$$V_1 = -\frac{Eg_1 V_0}{m}, \quad V_2 = -\frac{g_1^2 + g_2^2}{2m} V_0^2$$

(24)

Therefore, one has to search for bounded solutions in an effective symmetric Morse-like potential for $g_1 \neq 0$, or in a cusp potential for the case of a pure
nonminimal vector potential \((g_1 = 0)\). Note carefully that the DKP equation can furnish a discrete spectrum only if

\[ V_1 < |V_2| \]  

(25)

This is so because the effective potential approaches zero as \(|x| \to \infty\) and has a minimum value equal to \(V_1 - |V_2|\). Only in this circumstance the effective potential presents a potential-well structure permitting bounded solutions in the range \(|E| < m (E_{\text{eff}} < 0)\). The energies in the range \(|E| > m\) correspond to the continuum. Also note that the spectrum changes sign under the transformation \(g_1 \to -g_1\), but it remains invariant under \(g_2 \to -g_2\). The DKP energies are obtained by inserting the effective eigenvalues into (23). When \(g_1 \neq 0\) the effective potential depends on the energy and the single-well potential is deeper and larger for one sign of energy than for the other one. When \(g_1 > 0\), for instance, the single-well potential is deeper and larger for positive energies than that one for negative energies. Thus, the capacity to hold bound states depends on the sign of the energy and one might expect that the number of positive-energy levels is greater than the number of negative-energy levels. By the way, the positive (negative) energy solutions are not to be promptly identified with the solutions for particles (antiparticles). Rather, whether it is positive or negative, an energy can be unambiguously identified with a bounded solution for a particle (antiparticle) only by observing if the energy level emerges from the upper (lower) continuum. When \(g_1 = 0\), though, the effective cusp potential allows energy levels symmetric about \(E = 0\). The easiness with which the nonminimal vector potential well has bound states for both signs of \(E\) is due to the fact that this kind of potential couples equally to particles and antiparticles, as has already been anticipated by the charge-conjugation properties. Condition (25) can also be expressed as

\[ g_1 E > -\frac{g_1^2 + g_2^2}{2} V_0 \]  

(26)

This inequality can be used to achieve the constraint on the potential parameters as well as the signs of \(E\). For \(g_1 > 0\) all positive values of \(E\) are allowed but negative values are allowed only if \(|E| < \frac{g_1^2 + g_2^2}{2g_1} V_0\). From this last formula, one can see that only if \(V_0 > 2m \frac{g_1}{g_1 + g_2}\) energy levels with \(E \simeq -m\) will be allowed. When \(g_1 = 0\), though, the spectrum can acquiesce both signs for the energies, regardless the values of \(E, g_2\) and \(V_0\). In this qualitative context, there is no hint or indication whether Popov’s conjecture
is valid. Nevertheless, since the energy levels for antiparticles must emerge from the lower continuum one can conclude that the SSWE is a strong-field phenomenon inasmuch as $E \simeq -m$ requires $V_0 > 2m$ in the case of a pure minimal coupling attractive for particles in a nonrelativistic scheme.

Now we move to consider a quantitative treatment of bound-state solutions. Since the effective potential is even under $x \to -x$, $\Phi$ can be expressed as a function of definite parity. Thus, we can concentrate our attention on the positive half-line and impose boundary conditions on $\Phi$ and $d\Phi/dx$ at $x = 0$ and $x = +\infty$. In addition to $\Phi(\infty) = 0$ in order to ensure normalizability of the DKP spinor, the boundary conditions can be met in two distinct ways: the even function obeys the homogeneous Neumann condition at the origin $(d\Phi/dx)|_{x=0} = 0$ whereas the odd function obeys the homogeneous Dirichlet condition $(\Phi(0) = 0)$.

Defining the dimensionless quantities

$$z = z_0 \exp\left(-\frac{|x|}{\lambda}\right), \quad z_0 = i2\lambda V_0 \sqrt{g_1^2 + g_2^2}$$

$$\kappa = \frac{2E\lambda^2 g_1 V_0}{z_0}, \quad \nu = \lambda m \sqrt{1 - \frac{E^2}{m^2}}$$

one obtains

$$zd^2\Phi/dz^2 + d\Phi/dz + \left(-\frac{z}{4} - \frac{\nu^2}{z} + \kappa\right)\Phi = 0$$

(28)

Let us define $w = z^{1/2} \Phi$ so that $w$ obeys the Whittaker equation [44]:

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \nu^2}{z^2}\right)w = 0$$

(29)

whose solution regular at $z = 0$ ($x = \infty$), with

$$a = \frac{1}{2} + \nu - \kappa, \quad b = 1 + 2\nu$$

(30)

is written as $w = N z^{1/2 + \nu} e^{-z/2} M(a, b, z)$. Here $N$ is a normalization constant and $M(a, b, z)$ is the regular confluent hypergeometric function (Kummer’s function)

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(b + n)} \frac{z^n}{n!}$$

(31)
where $\Gamma (z)$ is the gamma function. Thus,

$$\Phi = Nz^\nu e^{-z/2}M(a, b, z)$$  \hspace{1cm} (32)

From Eq. (32) one can see now that the boundary conditions at $x = 0$ ($z = z_0$) imply into

$$\frac{M(a+1, b+1, z_0)}{M(a, b, z_0)} = \frac{z_0+1-b}{2z_0^2}, \quad \text{for even states}$$  \hspace{1cm} (33)

$$M(a, b, z_0) = 0, \quad \text{for odd states}$$

Since the DKP energies are dependent on $a$ and $b$ via $\kappa$ and $\nu$, it follows that Eq. (33) is a quantization condition. The allowed values for the parameters $a$ and $b$, and $E$ as an immediate consequence, are determined by solving Eq. (33). Due to the presence of $\kappa$ in the definition of $a$ one can expect an asymmetry in the spectrum so that the presence of each sign of energy depends, of course, on the relative strength between $g_1$ and $g_2$. For the case of a pure nonminimal vector potential ($g_1 = 0$), though, one has $\kappa = 0$ and the negative- and positive-energy levels are disposed symmetrically about $E = 0$, as commented before, and there are as many positive-energy levels as negative ones. This particular case allows a simpler mathematical treatment as can be seen in Appendix A. Although the quantization condition has no closed form expressions in terms of simpler functions, the exact computation of the allowed eigenenergies can be done easily with a root-finding procedure of a symbolic algebra program. Proceeding in this way, the whole bound-state spectrum is found. The DKP energies are plotted in Figure 1, 2 and 3 for the lowest bound-state solutions as a function of $V_0$ for two different values of $\lambda$ ($\lambda_c = m^{-1}$ is the Compton wavelength). The energies for $g_1 < 0$ and $g_2 < 0$ can be obtained by using the charge-symmetries mentioned before, viz. the spectrum is invariant under $g_2 \to -g_2$, and $E \to -E$ when $g_1 \to -g_1$. For large $\lambda$ and small $V_0$, the spectrum consists of a finite set of energy levels of alternate parities and the energy level corresponding to the ground-state solution ($\Phi$ even) of particles (for $g_1 > 0$) always makes its appearance, as it happens in a nonrelativistic framework. Surprisingly, as $V_0$ increases the number of bound-state solutions can decrease and one might find an odd-parity ground-state solution, or no solution at all. Indeed, the situation is more complicated and the existing correlation between the number of bound-state solutions and the depth of the effective potential well in the nonrelativistic theory does not verify for a strong potential. In the case
of a pure minimal coupling, as illustrated in Figure 1, particle levels appear in the spectrum and those levels tend to sink at the continuum of negative energies for large $\lambda$. As $\lambda$ decreases, though, a new branch of solutions corresponding to antiparticle levels begin to emerge from the continuum of negative energies and coalesce with the particle levels. This is the signature of the SSWE. In Figure 2 one can see the tendency of the spectrum to be symmetrical about $E = 0$. In Figure 3, for the case of a pure nonminimal coupling, that symmetry shows itself perfect.

4 Conclusions

In summary, we have succeed in the proposal of searching the solution for a cusp potential with the DKP equation. An opportunity was given to analyze some aspects of the DKP equation which would not be feasibly only with the cases already approached in the literature. Thus, the use of the mixing of minimal and nonminimal vector Lorentz structures for other kinds of potentials, primarily because nonminimal vector couplings have no counterpart in the Klein-Gordon and Proca theories, may lead to a better understanding of the DKP equation and its solutions. In the case of a pure minimal coupling ($g_2 = 0$) the DKP equation reduces to the Klein-Gordon and Proca equations and our results are in accordance with those found in the literature for the Klein-Gordon case (see e.g. [28]). In particular, the Popov conjecture about the SSWE is supported as a short-range phenomenon for spin-0 particles as well as for spin-1 particles.

Finally, one very important point to note is that the matrix potential $i[P, \beta]A^{(2)}_\mu$ in (5) leads to a conserved four-current but the same does not happen if instead of the matrix $i[P, \beta]$ one uses either $P\beta^\mu$ or $\beta^\mu P$, as in [9]-[16], even though the linear forms constructed from those matrices behave as true Lorentz vectors.

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Appendix A

In the particular case of a pure nonminimal vector potential one can take advantage of the relation expressing the Bessel function of the first kind and order \( \nu \) in terms of the hypergeometric function \[44\]

\[ J_\nu(z) = \frac{(\frac{1}{2}z)^\nu e^{-iz}}{\Gamma(\nu + 1)} M \left( \nu + \frac{1}{2}, 2\nu + 1, 2iz \right) \]  

(34)
to write

\[ \Phi = N_\nu J_\nu(y) \]  

(35)
where \( N_\nu \) is a normalization constant and \( y = -iz/2 \). The boundary conditions at \( x = 0 \) \( (y = y_0) \) imply that

\[ \frac{dJ_\nu(y)}{dy} \bigg|_{y=y_0} = 0, \quad \text{for even states} \]  

\[ J_\nu(y_0) = 0, \quad \text{for odd states} \]  

(36)
The oscillatory character of the Bessel function and the finite range for \( y \) \( (0 < y \leq y_0) \) imply that there is a finite number of discrete DKP energies. The roots of \( J_\nu(y) \) and \( J'_\nu(y) \) are listed in tables of Bessel functions only for a few special values of \( \nu \). A bit of time and effort can be saved in the numerical calculation of the roots of \( J'_\nu(y) \) if one uses the recurrence relation

\[ J_{\nu-1} - J_{\nu+1} = 2J'_\nu, \]  
in such a manner that the quantization condition for even states translates into

\[ J_{\nu+1}(y_0) = J_{\nu-1}(y_0). \]
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Figure 1: Energies for the first bound states as a function of $V_0$ for $g_1 = 1$ and $g_2 = 0$. 
Figure 2: Energies for the first bound states as a function of $V_0$ for $g_1 = g_2 = 1/2$.

Figure 3: Energies for the first bound states as a function of $V_0$ for $g_1 = 0$ and $g_2 = 1$. 