Holographic incoherent transport in Einstein-Maxwell-dilaton Gravity

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Recent progress in the holographic approach makes it more transparent that each conductivity can be decomposed into the coherent contribution due to momentum relaxation and the incoherent contribution due to intrinsic current relaxation. In this paper we investigate this decomposition in the framework of Einstein-Maxwell-dilaton theory. We derive the perturbation equations, which are decoupled for a large class of background solutions, and then obtain the analytic results of conductivity with the slow momentum relaxation in low frequency approximation, which is consistent with the known results from memory matrix techniques.
I. INTRODUCTION

Many strongly interacting many-body systems do not have quasiparticle excitations, for instance, the strange metal phase of cuprate superconductors \[1-3\]. It is still challenging for condensed matter physicists to understand the transport properties of strongly coupled systems without quasiparticles at finite temperature. In recent years, holographic duality provides a new approach for exploring these properties in the strongly correlated system\[4-6\]. The landmark achievement includes the universal bounds on diffusion rates of momentum, charge, and energy proposed in \[7, 8\].

To handle the transport properties in a strongly interacting system without quasiparticles, a traditional tool is the memory matrix framework \[12-20\], in which a perturbative expansion with small momentum relaxation rate $\Gamma$ can be performed. In particular, at leading order of expansion it can give rise to the universal Drude behavior of conductivity. Recently, the research on incoherent transport from the perspective of holography has been developed \[8, 20-28\]. These results can be extended by incorporating the laws of relativistic conformal hydrodynamics at the phenomenological level (see \[18\] or Eq. (1.3) in \[24\]), in which the conductivity can be decomposed into coherent contribution due to the momentum relaxation and incoherent contribution due to intrinsic current relaxation. Since the incoherent conductivity is decoupled from the momentum \[8\], it is supposed to capture the intrinsic behavior of the electric transport, and then to be a completely intrinsic characteristic of the system at low energy physics. Therefore, it is very crucial to understand its universal features in more generic circumstances.

For a system with no momentum dissipation, one can define an incoherent current $J_{inc}$, a particular combination of the charge and heat currents, which has no overlap with the momentum. Its corresponding conductivity $\sigma_{J_{inc}J_{inc}}$ (or $\sigma_{inc}$ in shorthand) is finite and may capture some universal transport properties. Its direct current (dc) conductivity turns out to be universal and can be obtained by the hydrodynamic method for a class of holographic

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1 Some extended studies have also been explored in \[3, 11\].
\[ \sigma_{inc} \equiv \sigma_Q = Z_+ \left( \frac{sT}{\epsilon + P} \right)^2. \] (1)

However, for a system with momentum dissipation, it usually becomes hard to obtain such incoherent current. Nevertheless, it is shown in a recent paper \cite{24} that it may be extracted out in some specific holographic models, for instance, in the Einstein-Maxwell (EM) theory with axions. The key observation in this framework is that the linearized perturbation equations can be decoupled by redefining perturbation variables such that we can construct a new orthogonal basis \( J_\pm \) for the currents instead of the usual charge current \( J \) and heat current \( J^Q \). As a result the corresponding response matrix is diagonal, namely, \( \chi_{J^+, J^-} = 0 \). More importantly, it can be found that these independent currents go back to \( J^{inc} \) and momentum, respectively, when the momentum dissipation vanishes. At slow momentum dissipation, they are responsible for the incoherent and coherent contribution of the low frequency conductivity, respectively.

In this paper, we intend to demonstrate that the incoherent part of conductivity can be extracted out for a large class of Einstein-Maxwell-dilaton (EMD) gravity theories with momentum dissipation by decoupling the linearized perturbation equations. Our work further confirms the universality of incoherent transport in the holographic approach. In addition, we systematically derive the low frequency behavior of conductivity at slow momentum dissipation for a large class of holographic models, including the Gubser-Rocha EMD model with vanishing ground state entropy \cite{29}.

Our paper is organized as follows. In Sec. \( \text{II} \) we derive the decoupling equations for a class of EMD-axion theories and define the independent currents. In Sec. \( \text{III} \) we simplify the analysis of the conductivity by relating it with the diagonal response matrix of the independent currents. And then the analytic expression of the low frequency conductivities in the presence of slow momentum dissipation is obtained in Sec. \( \text{IV} \). The conclusions and discussions are present in Sec. \( \text{V} \).

\section{EMD-Axion Model}

EMD theories have been widely studied in the context of holographic gravity (for instance, see \cite{30, 31} and the references therein). Here we are interested in a class of EMD gravity theories as \cite{18, 25}.
theories whose action can be written as a general form

\[
S = \int d^4x \sqrt{-g} \left( R - \frac{Z(\varphi)}{4} F^2 - \beta(\varphi) \sum_{I=x,y} (\partial \phi_I)^2 - \frac{1}{2} (\partial \varphi)^2 + V(\varphi) \right),
\]

(2)

where the coupling terms \(Z(\varphi), \beta(\varphi)\) and the potential \(V(\varphi)\) are general functions of dilaton field \(\varphi\), which allows us to study a large class of EMD models. The momentum dissipation is introduced by setting the axions \(\phi_x = mx, \phi_y = my\), as originally proposed in [32]. \(m\) is an arbitrary real number, characterizing the strength of momentum dissipation.

We consider the following ansatz for the background, which is supposed to be a black brane with asymptotically antiCde Sitter structure:

\[
ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 g(r) (dx^2 + dy^2),
\]

\[
A = A_t(r) dt, \quad \varphi = \varphi(r).
\]

(3)

(4)

If the position of the horizon is set at \(r_0\) by \(f(r_0) = 0\), then the bulk geometry described by the above metric is dual to a CFT with charge density \(q\), entropy density \(s\), and temperature \(T\) that are separately given by

\[
q = Z r^2 g A_t'(r), \quad s = 4\pi r_0^2 g(r_0), \quad T = r_0^2 f'(r_0)/(4\pi).
\]

(5)

Now we intend to investigate the transport properties of conductivity in the dual field theory. To do so we turn on the following perturbations around the background solution in (3) and (4):

\[
A_x = a_x(r) e^{-i\omega t}, \quad \delta \phi_x := \chi_x(r) e^{-i\omega t}
\]

\[
\delta g_{tx} := r^2 g(r) h_{tx}(r) e^{-i\omega t}, \quad \delta g_{rx} := r^2 g(r) h_{rx}(r) e^{-i\omega t}.
\]

(6)

Then, we can obtain three independent linearized perturbation equations, which read as

\[
2m^2 \beta h_{tx} - \frac{f}{g} (g r^4 \psi)' + i\omega 2m \beta \chi_x = \frac{q f}{g} a_x',
\]

\[
2m^2 \beta h_{rx} + \frac{i\omega g \psi}{f} - 2m \beta \chi_x = -\frac{i\omega q}{r^4 f g} a_x,
\]

\[
(Z r^2 f a_x')' + \frac{\omega^2 Z}{r^2 f} a_x + q \psi = 0.
\]

(7)

(8)

(9)

where \(\psi \equiv h_{tx}' + i\omega h_{rx}\) and the prime denotes a derivative with respect to \(r\). The coupling functions \(Z, \beta\) depend on \(r\) through the scalar field \(Z(r) = Z(\varphi(r)), \beta(r) = \beta(\varphi(r))\). The
axion perturbation equation of $\chi_x$ can be deduced by Eqs. (7) and (8) and we do not write it here.

After defining $y = g^2 r^4 \psi + qa_x$ and removing the variable $\chi_x$, these perturbation equations can be further simplified as

$$\left( \frac{f}{\beta g} y' \right)' + \left( \frac{\omega^2}{r^4} - \frac{2m^2}{g^2 r^4} \right) y + \frac{2m^2}{g^2 r^4} qa_x = 0, \tag{10}$$

$$\left( Zr^2 f a'_x \right)' + \left( \frac{\omega^2 Z}{r^2 f} - \frac{q^2}{g^2 r^4} \right) a_x + \frac{q}{g^2 r^4} y = 0, \tag{11}$$

which are two coupled differential equations of $y, a_x$. They take the same form as the differential equations (A1) and (A2) in Appendix A, with $\tilde{C} = 2m^2, \sqrt{\alpha} = (Z \beta gr^2)^{-1/2}$, and

$$K = \frac{1}{q} (-g^2 r^4 ((\sqrt{\alpha})' Zr^2 f)' + q^2 \sqrt{\alpha} - \frac{2m^2}{\sqrt{\alpha}}). \tag{12}$$

In general, it is not easy to figure out whether Eqs. (10) and (11) can be decoupled. However, if the quantity $K$ defined above is a constant, then we find it is completely possible to decouple them, as discussed in Appendix A. Here, if the background solution of the electric field satisfies

$$A_t(r) = \mu - q \sqrt{\alpha}, \text{ or } \beta = \left( \frac{1}{\sqrt{\alpha}} \right)', \tag{13}$$

then $K$ becomes

$$K = \frac{1}{q} [r^4 g^2 (\frac{f}{g})' - q(A_t(r) - \mu) - \frac{2m^2}{\sqrt{\alpha}}]. \tag{14}$$

With the use of the equations of motion, one can prove that $K$ is a constant indeed, which means the perturbation equations can be decoupled if the condition presented in (13) is satisfied. First of all, we point out that the following two typical solutions in holographic models satisfy this condition.

- The first one is simply the standard black brane solution in the EM model with momentum dissipation, which can be obtained by setting $\varphi = 0, Z = 1, \beta = 1/2$, and $V = 6 \ [24, 32]$.

- The second one is based on the Gubser-Rocha solution in the EMD model in [29]. The momentum dissipation effect can be introduced by explicitly incorporating axions into the action

$$S = \int d^4x \sqrt{-g} \left( R - \frac{e^\varphi}{4} F^2 - \sum_{I=x,y} (\partial \phi_I)^2 - \frac{3}{2} (\partial \varphi)^2 + 6 \cosh \varphi \right). \tag{15}$$
Setting $\phi_x = mx, \phi_y = my$, and under the ansatz (3) and (4), this model admits the following solution,

$$\varphi(r) = \frac{1}{3} \log g(r), \quad f(r) = h(r)g(r),$$

$$g(r) = (1 + Q/r)^{3/2},$$

$$h(r) = 1 - \frac{\mu^2(Q + r_0)^2}{3Q} \frac{1}{(Q + r)^3} - \frac{m^2}{(Q + r)^2},$$

$$A_t(r) = \mu(1 - \frac{Q + r_0}{Q + r}),$$

where $Q$ is a parameter, and $r_0, \mu$ represent the position of the horizon and the chemical potential, respectively. It is easy to check that this model satisfies the condition (13) and thus its linearized perturbation equations can be decoupled.

Secondly, we should emphasize that the assumption (13) is just a sufficient condition to ensure the decoupling. Whether it can be relaxed is still unclear and deserves further investigation.

Before proceeding, we point out that $K$ being a conserved quantity implies that it associates with a scale symmetry. We now elaborate this point as below. It is more convenient to discuss this problem by using the following ansatz:

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + c(r)(dx^2 + dy^2),$$

$$A = A_t(r)dt, \quad \phi_x = \phi_x(x), \quad \phi_y = \phi_y(x).$$

Substituting them into the action (2), one has

$$S = \int d^4x \frac{a^2b(c')^2 + 2abca'c + abc^2Z(A_t')^2}{2(ab)^{3/2}c} - (ab)^{1/2}\beta((\partial_x \phi_x)^2 + (\partial_y \phi_y)^2),$$

where we have ignored the total derivative terms. The above action is invariant under the scale transformation

$$c \to \lambda c, \quad a \to \lambda^{-2}a, \quad A_t \to \lambda^{-1}A_t, \quad \phi_x \to \lambda^{1/2}\phi_x, \quad \phi_y \to \lambda^{1/2}\phi_y.$$

Then, similar to the Noether conservation law, we can obtain the following equation:

$$\partial_c \left( \frac{1}{\sqrt{ab}} \left( c^2 \frac{a}{c} + cZ A'_t A_t \right) \right) - \sqrt{ab}\beta(\partial_x (\phi_x \partial_x \phi_x) + \partial_y (\phi_y \partial_y \phi_y)) = 0.$$
For our EMD-axion model, we have $a = 1/b = r^2 f$, $c = r^2 g$, $\phi_x = mx$, $\phi_y = my$, and then the above equation becomes

$$\partial_r (r^4 g^2 \left( \frac{f}{g} \right)' - q A_t) - 2m^2 \beta = 0,$$

where $r^2 g Z A'_t = q$ has been used. Since we have also introduced the decoupling condition (13), the above result becomes a conservation equation,

$$\partial_r (r^4 g^2 \left( \frac{f}{g} \right)' - A_t - \frac{2m^2}{\sqrt{\alpha}}) = 0.$$  

It means that the fact $K$ is a constant is a reflection of the scale symmetry of the EMD-axion model.

Since $K$ is a constant under the condition presented in (13), we may evaluate it at the horizon $r_0$, which can be rewritten as

$$K = \frac{1}{q} [sT + q \mu - \frac{2m^2 q}{\mu}] = \frac{1}{q} [\epsilon + P - \frac{2m^2 q}{\mu}]$$

(27)

where Eq.(5) and the relation $\epsilon + P = sT + q \mu$ have been used. Next, following the discussion at the end of Appendix A we may decouple Eqs. (10) and (11) by redefining new variables as [see Eq. (A7)]

$$v_\pm = \frac{1}{\sqrt{Z} \beta r^2 g} (r^4 g^2 (h'_t + i \omega q_{tx}) + q a_x) + \eta_\pm a_x,$$

(28)

where the constants $\eta_\pm$ are given by

$$\eta_\pm = \frac{-K \pm \sqrt{K^2 + 8m^2}}{2}.$$  

(29)

Consequently, the decoupled equations of new variables $v_\pm$ read as

$$(Z r^2 f v')' + M(\omega) v = 0,$$

(30)

with

$$M(\omega) = \frac{1}{r^4 g^2 \sqrt{\alpha}} \left( sT - \frac{2m^2 q}{\mu} + \eta q + q A_t(r) \right) + \frac{\omega^2 Z}{r^2 f},$$

(31)

where $v_\pm$ and $\eta_\pm$ have been simply denoted as $v$ and $\eta$, respectively.

Next, we intend to derive the relation between the currents sourced by the boundary value of the decoupled variables and the original momentum $T^{tx}$ and electric current $J^x$, which
are sourced by $h_{tx}^{(0)}$ and $a_{x}^{(0)}$, respectively. Before proceeding, it is convenient to introduce new decoupled variables $s_{\pm}$ as

$$s_{+} = -\frac{1}{\eta_{+}}v_{+}, \quad s_{-} = \frac{1}{\eta_{-}}v_{-},$$

which obey the same differential equation (30) as $v_{\pm}$. In the subsequent section, we alternately use $v_{\pm}$ and $s_{\pm}$, depending on the convenience of discussion. We also assume that near the boundary, a generic bulk field perturbation $\Phi(r)$ can be expanded like

$$\Phi(r) = \sum_{n} \frac{\Phi^{(n)}}{r^{n}}.$$  

Then, with the use of equations of motion (7) and (8) as well as expression (28), the pair of new sources $s_{\pm}^{(0)}$ can be expressed in terms of the original sources as

$$v_{\pm}^{(0)} = 2m^{2}h_{tx}^{(0)} + \eta_{\pm}a_{x}^{(0)},$$

$$s_{+}^{(0)} = -\frac{1}{\eta_{+}}v_{+}^{(0)}, \quad s_{-}^{(0)} = \frac{1}{\eta_{-}}v_{-}^{(0)}.$$  

Suppose the corresponding currents sourced by $s_{\pm}^{(0)}$ are $J_{s}^{\pm}$. Since the transformation of the sources and currents should preserve the form of the perturbed action, namely,

$$T_{tx}^{tx}h_{tx}^{(0)} + J_{x}^{x}a_{x}^{(0)} = J_{s}^{+}s_{+}^{(0)} + J_{s}^{-}s_{-}^{(0)},$$

from the relation of sources in (35) one can derive the relation of currents given by

$$J_{s}^{\pm} = \frac{T_{tx}^{tx} + \eta_{\pm}J_{x}^{x}}{\eta_{-} - \eta_{+}}.$$  

Now, we have obtained a new pair of sources [Eqs. (34) and (35)] and their corresponding currents [Eq.(37)], which are the major results from the decoupling process. So far, they are just a consequence of the special mathematical structure in the considered EMD-axion models. Next, we present a brief discussion on their property and find that this decoupling implies a decomposition of the incoherent and coherent transport.

(1) Because the current $J_{s}^{+}$ ($J_{s}^{-}$) depends only on $s_{+}^{(0)}$ ($s_{-}^{(0)}$), there is no overlap between these two currents $J_{s}^{\pm}$, i.e., the response function $\chi_{J_{s}^{+}J_{s}^{\pm}} = 0$, which indicates that $J_{s}^{+}$ and $J_{s}^{-}$ are independent with each other. We thus call them “independent currents”. As we see in the next sections, this property leads to a decomposition of the conductivity into two

\[^{3}\text{Here, we assume } Z(\varphi)|_{r \to \infty} = 1, \quad \beta(\varphi)|_{r \to \infty} = 1.\]
individual parts and makes it possible to obtain an analytic low frequency conductivity at small $m$.

(2) For the case of translational invariance, i.e., $m \to 0$, from Eqs. 29 and 37, we can see that $J^+_s$ describes the momentum. Then $J^-_s$ becomes the incoherent current since now it has no overlap with the momentum. This coincides with the result in [25] and $J^- = J^{inc}$ as

$$J^{inc} = J^- = \frac{sT J^x - q J^Q}{\epsilon + P},$$

where all the thermal quantities should take value at $m = 0$ and the heat current expression $J^Q = T^{tx} - \mu J^x$ has been used. It is then clear that the decoupling represents a decomposition of the incoherent and coherent transport at $m = 0$.

In the next section, we begin to study the conductivity and we find the decoupling also represents a decomposition of the incoherent and coherent transport for small momentum dissipation.

## III. CONDUCTIVITIES

In this section, we first derive the conductivities in terms of the response functions of the independent currents $J^\pm_s$. Since the modes of sources $s_\pm$ are decoupled, the matrix of the response function $\chi_s \equiv G^R_s(\omega) - G^R_s(0)$ of currents $J^\pm_s$ is diagonal and we denote $\chi_s = diag(\chi_{s+}, \chi_{s-})$.

Under the source transformation (34) and (35) and the current transformation (37), the Kubo formula [4],

$$\begin{pmatrix} J \\ J^Q \end{pmatrix} = \begin{pmatrix} \sigma_{JJ}, & \sigma_{JJQ} \\ \sigma_{JJQ}, & \sigma_{JJQJQ} \end{pmatrix} \begin{pmatrix} E \\ -\nabla T / T \end{pmatrix},$$

becomes

$$\begin{pmatrix} J^+_s \\ J^-_s \end{pmatrix} = \begin{pmatrix} \chi_{s+}, & 0 \\ 0, & \chi_{s-} \end{pmatrix} \begin{pmatrix} s_+^{(0)} \\ s_-^{(0)} \end{pmatrix}.$$
Then, it is straightforward to express the conductivities as

\[ \sigma_{JJ} = \frac{1}{i\omega}(\chi_{s+} + \chi_{s-}), \]

(41)

\[ \sigma_{JQ} = -\frac{1}{i\omega}((\eta_- + \mu)\chi_{s+} + (\eta_+ + \mu)\chi_{s-}), \]

(42)

\[ \sigma_{JQJQ} = \frac{1}{i\omega}((\eta_- + \mu)^2\chi_{s+} + (\eta_+ + \mu)^2\chi_{s-}). \]

(43)

As a consequence, the conductivities in Eq. (39) can be calculated through the above equations once all the response functions \( \chi_s \) are known. Next we explicitly derive their forms in terms of the perturbation fields following the standard procedure in the context of the linear response theory in the holographic approach.

Usually, we can expand the action with respect to some perturbation fields \( \Phi_I(k, r) \) as (up to the second order)

\[ S^{(2)}[\phi] = \int_{rB} d^d r \frac{d^d k}{(2\pi)^d} \Phi'(-k, r) A \Phi'(k, r) + \Phi'(-k, r) B \Phi(k, r) + \Phi(-k, r) C \Phi(k, r) \]

(44)

where \( kx = -\omega t + \mathbf{k} \cdot \mathbf{x} \). \( \Phi \) represents all \( \Phi_I \) and the matrices \( A(r), B(r, k), C(r, k) \) are independent of these perturbation fields. Then, the perturbed equation of motion of \( \Phi(r, k) \) read as\(^4\)

\[ (C(r, k) + C^T(r, -k)) \Phi = ((A + A^T) \Phi')' - B^T(r, -k) \Phi' + (B(r, k) \Phi)', \]

(45)

and the retarded Green function as pointed out in [34] is

\[ G^{R}_{IJ}(k) = -2(r^{-2}A^{(0)}_{IK})\kappa_{KJ}(k) + B^{(0)}_{IJ}(k), \]

(46)

where the index 0 again means taking value at the boundary and \( \kappa_{IJ}(k) \) denotes the linear relation \( \Phi_I^{(1)}(k) = \kappa_{IJ}(k)\Phi_J^{(0)}(k) \).

Now for \( \Phi_I = s = (s_+, s_-) \) with \( \mathbf{k} = 0 \), the equations in [35] should be decoupled and become

\[ (A s')' + \frac{1}{2}(B(r, \omega) - B(r, -\omega)) s' + \frac{1}{2}(B'(r, \omega) - C(r, \omega) - C(r, -\omega)) s = 0. \]

(47)

Comparing the above equation with Eq. [30] (which is also the differential equation of \( s \)), one has \( B(r, \omega) = 0 \) and \( A \) is diagonal, whose elements, denoted as \( \text{diag}(A_{++}, A_{--}) \), are proportional to \( Z r^2 f \). We can then choose

\[ (r^{-2}A_{++})^{(0)} = (r^{-2}A_{--})^{(0)} = -\frac{\Lambda_\pm}{2}, \]

(48)

\[^4\] Taking \( A \) as a symmetric matrix has no effect on the results.
where $\Lambda_\pm$ are arbitrary constants. Therefore, the response function $\chi_s$ can be derived through Eq.(46),

$$\chi_{s\pm}(\omega) = \Lambda_\pm \left( \frac{s_{\pm}^{(1)}(\omega)}{s_{\pm}^{(0)}(\omega)} - \frac{s_{\pm}^{(1)}(0)}{s_{\pm}^{(0)}(0)} \right).$$  

(49)

Finally, we can obtain the conductivities in terms of the perturbation fields through Eqs.(41)-(43) as

$$\sigma_{JJ} = \frac{1}{i\omega} (\Lambda_+ \Theta_+ + \Lambda_- \Theta_-)$$  

(50)

$$\sigma_{JQ} = -\frac{1}{i\omega} ((\eta_- + \mu) \Lambda_+ \Theta_+ + (\eta_+ + \mu) \Lambda_- \Theta_-),$$  

(51)

$$\sigma_{QJ} = \frac{1}{i\omega} ((\eta_- + \mu)^2 \Lambda_+ \Theta_+ + (\eta_+ + \mu)^2 \Lambda_- \Theta_-).$$  

(52)

where

$$\Theta_\pm = \frac{s_{\pm}^{(1)}(\omega)}{s_{\pm}^{(0)}(\omega)} - \frac{s_{\pm}^{(1)}(0)}{s_{\pm}^{(0)}(0)}.$$  

(53)

Note that the expression of $\Theta_\pm$ is invariant under the change of $s \rightarrow v$. Up to the current stage, we have found that the frequency dependence of conductivities can be determined once the quantities $\Theta_\pm$ and $\Lambda_\pm$ are known. Next we demonstrate that $\Theta_\pm$ can be obtained by solving the decoupled equation (30), while $\Lambda_\pm$ can be determined by the DC conductivity and the incoherent conductivity.

**IV. THE LOW FREQUENCY BEHAVIORS OF CONDUCTIVITIES**

In this section we study the property of conductivities in the low frequency limit. The decoupling results in Sec.II are a key step to obtain the analytic low frequency conductivities, since they decompose the conductivity into two individual contributions coming from $J^\pm$, respectively. Furthermore, we find that one part of the conductivity exhibits a Drude behavior corresponding to a coherent transport while the other part is dominant by the incoherent transport. Then, we not only give an analytic expression of the low frequency conductivities, but also achieve a decomposition of the incoherent and coherent transport.

To derive the low frequency conductivities, according to the results in Sec.III, we derive the expression of $\Theta_\pm$ by approximately solving decoupled equations (30) with low frequency expansion. For this purpose, it is convenient to write $v$ as

$$v = f^{-i\omega/(4\pi T)}v_0(r)F(r),$$  

(54)
where \( v_0 \) is a zero frequency solution to Eq.(30) with \( \omega = 0 \), which can be chosen as

\[
v_0(r) \equiv 1 + \frac{q}{\eta} \sqrt{\alpha},
\]

and \( F(r) \) is regular at the horizon which can be expanded as

\[
F(r) = F_0 + F_1(r)\omega + \mathcal{O}(\omega^2).
\]

For simplicity, here we still use one character to denote the two solutions of Eq.(30). Since the two equations are only different with parameter \( \eta \pm \), we can write, for instance, \( F = (F_+(\eta_+), F_-(\eta_-)) \) and so on. Without loss of generality, we can choose \( F_0 = 1 \). Then \( \Theta = (\Theta_+, \Theta_-) \) can be expressed as

\[
\Theta = \frac{v^{(1)}(\omega)}{v^{(0)}(\omega)} - \frac{v^{(1)}(0)}{v^{(0)}(0)} = \frac{F^{(1)}_1 \omega + \mathcal{O}(\omega^2)}{1 + F^{(0)}_1 \omega + \mathcal{O}(\omega^2)}. \tag{57}
\]

By substituting the expression in (54) into Eqs.(30), we can derive the following equation of \( F_1 \)

\[
F_1'(r) = i \frac{Z r^2 v_0^2 f' - 4\pi T (Z v_0^2) |_{r_+}}{4\pi T Z r^2 f v_0^2}, \tag{58}
\]

whose solution provides results to \( F_1^{(0)}, F_1^{(1)} \) and thus determines the low frequency conductivity. \( F_1^{(1)} \) can be obtained directly from Eq.(58) as

\[
F_1^{(1)} = i (Z v_0^2) |_{r_+} = i Z (r_+) \left( 1 + \frac{\mu}{\eta} \right)^2, \tag{59}
\]

which determines the DC conductivity. \( F_1^{(0)} \), which is usually hard to solve, can be obtained at small \( m \).

We first discuss the DC conductivity, which reads as [with the use of Eq.(50)]

\[
\sigma_{DC} = \Lambda_+ Z(r_+) \left( 1 + \frac{\mu}{\eta_+} \right)^2 + \Lambda_- Z(r_+) \left( 1 + \frac{\mu}{\eta_-} \right)^2. \tag{60}
\]

Obviously, the above expression is decomposed into two contributions from the independent currents \( J^\pm \). When \( m = 0 \), the first part from the momentum contribution is divergent due to the translational invariance, while the second part provides the incoherent conductivity which should be the same as Eq.(11),

\[
\sigma_{inc} = Z(r_+) \left( \frac{s T}{\epsilon + P} \right)^2 \bigg|_{m=0} \equiv \sigma_Q. \tag{61}
\]

\(^5\) We have assumed \( r^2 f' = 0 \) at \( r = \infty \).
Then the coefficient $\Lambda_-$ can be determined as an expansion form of slow momentum $\Lambda_-=1+\mathcal{O}(m^2)$. Now we can further determine $\Lambda_+$ by comparing the expression in (60) with the DC conductivity result calculated via the horizon data [35, 36]:

$$\sigma_{DC} = Z(r_+) \left(1 + \frac{\mu^2}{2m^2}\right).$$

(62)

In the case of slow momentum dissipation, we can simply rewrite Eq.(60) as

$$\Lambda_+ Z(r_+) \left(1 + \frac{\mu}{\eta_+}\right)^2 = \sigma_{DC} - \sigma_Q + \mathcal{O}(m^2).$$

(63)

Next, we study the low-frequency behavior of conductivities with slow momentum dissipation (small $m$). Explicitly, we treat $\omega$ and $m^2$ as the same order. Usually it is hard to obtain an analytical solution of $F_1^{(0)}(r)$ from Eq.(58). Nevertheless, what we are mainly concerned with is whether there exists the electric current dissipation, which can be signaled by the appearance of a pole in the expression of conductivity. Observing the expression in Eq.(57), we find only when $F_1^{(0)}$ becomes divergent as $m \to 0$, the conductivity may have a pole structure.

For $\eta = \eta_-$, since $F_1^{(0)}(r)$ is regular as $m \to 0$, and so does $F_1^{(0)}$, there is no pole structure in $\Theta_-$ and then we can write

$$\Theta_- = \sigma_Q + \mathcal{O}(m^2, \omega),$$

(64)

without losing the interesting properties. This implies that $J^-$ can be an incoherent current at small $m$, since its contribution to the conductivity is dominant by the incoherent conductivity.

For $\eta = \eta_+$, $F_1^{(0)}$ is divergent as $m \to 0$. Thus the pole structure totally comes from the contribution of $J^+$, which implies that $J^+$ is a coherent current, describing the dissipation process. To solve $F_1^{(0)}$ for $\eta = \eta_+$, it is convenient to introduce a new variable $\psi(r)$

$$\psi(r) \equiv F_1(r) - i \frac{1}{4\pi T} \ln f(r),$$

(65)

whose asymptotic behavior is like that of $F_1$

$$\psi^{(0)} = F_1^{(0)}, \quad \psi^{(1)} = F_1^{(1)}.$$

(66)

6 We can also refer to [37–42] for the analytical calculation on DC conductivity of holographic systems with momentum dissipation.
Then, to obtain $F_1^{(0)}$, we need only deal with the equation of $\psi(r)$, which is given from Eq. (58) as

$$\psi'(r) = -i \left( \frac{Z v_0^2}{r_+^2 f v_0^2} \right).$$

Integrating the above equation from the horizon to the boundary, we have

$$\psi(0) = -i \left( \frac{Z + (\eta + \mu)^2}{q \eta} \right) \frac{1}{\int_{r_+}^{\infty} \frac{dr}{Z r^2 f(\eta + q \sqrt{\alpha})^2}},$$

$$= \frac{-i(Z_+^2 \eta_+ + \mu)^2}{q \eta} + O(1),$$

$$\equiv -i \Gamma^{-1} + O(1).$$

Note that the divergence at $r_+$ is not a genuine singular and would not affect our discussion since the in-falling condition guarantees the regularity of $F_1$. As a result, we have

$$\Theta_+ = \frac{F_1^{(1)} \omega + O(\omega^2)}{1 - i \omega \Gamma^{-1} + O(1) \omega + O(\omega^2)},$$

with

$$\Gamma = \frac{q \eta_+}{Z_+^2 \eta_+ + \mu)^2} \approx \frac{2m^2 s \beta}{4\pi (\epsilon + P)} + O(m^4).$$

Therefore, $\Theta_+$ exhibits a Drude behavior at low frequency regime with the dissipation rate $\Gamma \sim m^2$ and thus is the coherent contribution to the conductivities. Note that since we have neglected the subleading term $O(1)$ in (68), which is the same order as $O(m^4)$ in (70), the above expression is only viable at $O(m^2)$.

Finally, combining the incoherent and coherent contribution and using Eq. (63) and the formulas in Eqs. (50)-(52) we obtain the low frequency conductivity at small momentum dissipation

$$\sigma_{JJ} = \frac{\sigma_{DC} - \sigma_Q + O(1, \omega)}{1 - i \omega / \Gamma} + \sigma_Q + O(1, \omega),$$

$$\mu \sigma_Q + O(1, \omega),$$

$$\sigma_{JJ,QQ} = \frac{\sigma_{DC}^2 - \sigma_Q^2 + O(1, \omega)}{1 - i \omega / \Gamma} + \mu^2 \sigma_Q + O(1, \omega).$$
The above results can also be rewritten as

\[ \sigma_{JJ} = \frac{q^2}{\epsilon + P} \frac{\mathcal{O}(\Gamma, \omega, \omega^2)}{\Gamma - i\omega} + \sigma_Q + \mathcal{O}(\Gamma, \omega) \]  

(74)

\[ \sigma_{JQ} = \frac{q^2 T}{\epsilon + P} \frac{\mathcal{O}(\Gamma, \omega, \omega^2)}{\Gamma - i\omega} - \mu \sigma_Q + \mathcal{O}(\Gamma, \omega), \]  

(75)

\[ \sigma_{QJ} = \frac{q^2 T^2}{\epsilon + P} \frac{\mathcal{O}(\Gamma, \omega, \omega^2)}{\Gamma - i\omega} + \mu^2 \sigma_Q + \mathcal{O}(\Gamma, \omega). \]  

(76)

Now, we see clearly that the decoupling process in Sec. II leads to a coherent and incoherent decomposition of the conductivities at small momentum dissipation. While the current \( J^+ \) provides a coherent Drude contribution, \( J^- \) responds to the incoherent contribution. This means that at small \( m \), the transport comes from two individual contributions, the dissipation and diffusion process, captured by the decoupling currents \( J^\pm \), respectively.

We also give the analytic low frequency conductivities at small momentum dissipation for a large class of EMD-axion theories that coincide with the results derived from the hydrodynamic memory matrix technique up to the leading order. It is the first time to obtain the analytic low frequency conductivities, which is valid for so many theories in holography. It also confirms that a hydrodynamical description of dual field in holography is usually suitable.

V. CONCLUSIONS AND DISCUSSIONS

In this paper we have investigated the incoherence of conductivity in the holographic framework explicitly in terms of EMD-axion theory. The key ingredient in this analysis is the decoupling of the linearized perturbation equations. Based on the decoupled equations, we have introduced a new pair of independent currents \( J_s^\pm \). When momentum is conserved, they go back to the momentum and the incoherent current \( J_{inc}^\pm \) respectively, which coincides with the discussion about incoherence in [8]. For slow momentum dissipation, we have derived the analytic expression for conductivity in the low frequency regime, which can be viewed as a generalization of the results presented in [24]. We have demonstrated that it can also be divided into two parts in the EMD-axion model as that in the Einstein-Maxwell model [24], including the coherent contribution from \( J^+ \) and the incoherent one from \( J^- \). Our holographic argument here also confirms the general results derived in the context of hydrodynamics up to the leading order of low frequency expansion.
Our work has made progress on the understanding of the incoherent part of conductivity in a more general holographic circumstance. It would be quite possible to push our investigation forward along the following directions in the future.

- In the fluid/gravity approach, the same results of low frequency conductivity as in [24] have been obtained for the EM-axion theory in [43]. It is very worth extending this to EMD-axion models and then making a comparison with our results in the current paper, which might be helpful for us to understand the physical implication of the decoupling condition as proposed in Eq.(13).

- Recently, from the memory matrix theory [44] as well as the fluid/gravity approach [45], the low frequency behavior of the conductivity in magnetotransport has been obtained. It is also worth exploring this subject from the incoherent and coherent point of view.

- As we know, the assumption (13) is applicable to the EM-axion model and Gubser-Rocha EMD-axion model. It would be interesting to figure out if we could find more background solutions subject to this condition in a wide range of holographic models such as hyperscaling-violating models.

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Appendix A: Useful tool for decoupling

In this appendix, we discuss the decoupling condition of two coupled differential equations

with the following general form,

\[ (\alpha Ay)' + \tilde{B}y + \tilde{C}a_x = 0, \quad (A1) \]
\[ (Aa_x)' + Ba_x + Cy = 0, \quad (A2) \]

where equation variables \( y, a_x \) are coupled to each other, and \( A, \alpha, B, C, \tilde{B}, \tilde{C} \) are all
functions of \( r \). Introducing a new variable \( u = y/k(r) \) with \( k = \frac{1}{\sqrt{\alpha}} \), the above equations become

\[ (Au')' + \left( \frac{\tilde{B}}{\alpha} - \frac{1}{\sqrt{\alpha}}(\alpha') \right)u + \frac{\tilde{C}}{\sqrt{\alpha}}a_x = 0, \quad (A3) \]
\[ (Aa_x')' + Ba_x + \frac{C}{\sqrt{\alpha}}u = 0. \quad (A4) \]

It can be easily found that if there is a constant \( \eta \) satisfying

\[ \frac{\tilde{B}}{\alpha} - \frac{1}{\sqrt{\alpha}}(\alpha')u + \frac{\tilde{C}}{\sqrt{\alpha}} = B + \frac{C}{\sqrt{\alpha}}\eta, \quad (A5) \]

then, multiplying Eq. \( (A4) \) with the constant \( \eta \) and combining it with Eq. \( (A3) \), we can obtain the following decoupled equations,

\[ (Av')' + \left( \frac{\tilde{B}}{\alpha} - \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha}')A' + \eta \frac{C}{\sqrt{\alpha}} \right)v = 0, \quad (A6) \]

with the decoupled variables defined as

\[ v = u + \eta a_x = \sqrt{\alpha}y + \eta a_x. \quad (A7) \]

The constant \( \eta \) is determined by the following equation:

\[ \eta^2 + K\eta - \frac{\tilde{C}}{C} = 0, \quad (A8) \]
\[ K \equiv \frac{\tilde{B}}{C\sqrt{\alpha}} - \frac{1}{C}(\sqrt{\alpha}')A' - \frac{B\sqrt{\alpha}}{C}. \quad (A9) \]

That is to say, if above Eq. \( (A8) \) has a constant solution, the perturbation equations \( (A1) \) and \( (A2) \) can be decoupled into Eq. \( (A6) \). A simple example of Eq. \( (A8) \) with a constant \( \eta \) solution is that the coefficients \( \frac{\tilde{C}}{C} \) and \( K \) are both constants.
Note that the decoupling conditions of the coupled equations (A1) and (A2) we discussed here are only the sufficient conditions, not necessary conditions.

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