RECOGNIZING $\text{PSL}(2, p)$ IN THE NON-FRATTINI CHIEF FACTORS OF FINITE GROUPS

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Abstract. Given a finite group $G$, let $P_G(s)$ be the probability that $s$ randomly chosen elements generate $G$, and let $H$ be a finite group with $P_G(s) = P_H(s)$. In this paper, we prove that if the nonabelian composition factors of $G$ and $H$ are $\text{PSL}(2, p)$ then $G$ and $H$ have the same non-Frattini chief factors.

1. Introduction

Let $G$ be a finite group. The probability $P_G(s)$ that $s$ randomly chosen elements generate $G$ is calculated as follows ([11]):

$$P_G(s) = \sum_{n \geq 1} \frac{a_n(G)}{n^s}, \quad \text{where} \ a_n(G) = \sum_{[G:H]=n} \mu_G(H).$$

Here $\mu_G$ is the Möbius function on the subgroup lattice of $G$ defined recursively by $\mu_G(G) = 1$ and $\mu_G(H) = -\sum_{H < K \leq G} \mu_G(K)$ if $H < G$. Considering (1) as a formal Dirichlet series associated to a group $G$, if $G = \mathbb{Z}$ then

$$P_\mathbb{Z}(s) = \sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

where $\mu$ is the usual number-theoretic Möbius function and $\zeta(s)$ is the Riemann zeta function. The inverse of $P_G(s)$ is then called the probabilistic zeta function of $G$; see [1] and [14].

Note that if $\mu_G(H) \neq 0$ then $H$ is an intersection of maximal subgroups of $G$, cf. [11]. This implies $P_G(s) = P_{G/\text{Frat}(G)}(s)$, where $\text{Frat}(G)$ denotes the Frattini subgroup of $G$ - the intersection of the maximal subgroups of $G$. Hence, one can only hope to get back information of $G/\text{Frat}(G)$ from the knowledge of $P_G(s)$.

One natural question asks what we can say about $G$ and $H$ whenever $P_G(s) = P_H(s)$. It’s known that if $G$ is a simple group, then $H/\text{Frat}(H) \cong G$, cf. [5] [17]. When $G$ is not simple, the problem becomes much harder. Patassini makes a significant progress by obtaining the following results.

Theorem 1. [18] Let $G$ and $H$ be finite groups with $P_G(s) = P_H(s)$. Then $G$ and $H$ have the same non-Frattini abelian chief factors.

2010 Mathematics Subject Classification. 20D06.

Key words and phrases. Finite groups; Probabilistic zeta function.
Theorem 2. Let $G$ and $H$ be finite groups whose nonabelian composition factors are alternating groups $\text{Alt}(k)$ where either $5 \leq k \leq 4.2 \cdot 10^{16}$ or $k \geq (e^{15} + 2)^3$. If $P_G(s) = P_H(s)$ then $G$ and $H$ have the same non-Frattini chief factors.

Using the same method, we prove in this paper the following theorem.

Theorem 3. Let $G$ and $H$ be finite groups such that $P_G(s) = P_H(s)$. Assume that the nonabelian composition factors of $G$ and $H$ are $\text{PSL}(2, p)$, for some prime $p \geq 5$. Then $G$ and $H$ have the same non-Frattini chief factors.

From the proof of Theorem 3, we obtain the following consequence.

Corollary 4. Let $G$ and $H$ be finite groups such that $P_G(s) = P_H(s)$. Assume that the nonabelian composition factors of $G$ and $H$ are either $\text{PSL}(2, p)$, for some prime $p \geq 5$ or alternating group $\text{Alt}(n)$ with $n$ satisfying the hypothesis of Theorem 2. Then $G$ and $H$ have the same non-Frattini chief factors.

Notations. In this paper, groups are always finite. Given a finite Dirichlet series $F(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$ and a prime number $r$, we denote by $F(r)(s)$ the Dirichlet series obtained from $F(s)$ by deleting the $a_n/n^s$ with $n$ divisible by $r$. Given a group $H$, we write $\pi(H)$ for the set of the prime divisors of $|H|$.

Acknowledgements. This research is supported by the DFG Sonderforschungsbereich 701 “Spectral Structures and Topological Methods in Mathematics” at Bielefeld University. The author also acknowledges Professor Gareth Jones for useful discussions.

2. Preliminaries

Given a normal subgroup $N$ of $G$, it’s shown in [2, Section 2.2] that

$$P_G(s) = P_{G/N}(s)P_{G,N}(s),$$

where

$$P_{G,N}(s) = \sum_{n \in \mathbb{N}} \frac{b_n(G, N)}{n^s},$$

with $b_n(G, N) = \sum_{|G:H| = n} \mu_G(H)$. By taking a chief series

$$\Sigma : 1 = G_k < \cdots < G_1 < G_0 = G,$$

and iterating equation (2) we can express $P_G(s)$ as a product of Dirichlet polynomials indexed by the non-Frattini chief factors in $\Sigma$:

$$P_G(s) = \prod_{G_i/G_{i+1} \notin \text{Frat}(G/G_{i+1})} P_{G/G_{i+1}, G_i/G_{i+1}}(s).$$

It was proved in [4] that the factors in (3) are independent of the choice of the series $\Sigma$. Moreover, it also describes how those factors look like as follows.
Let $A$ be a minimal normal subgroup of $G$. The monolithic primitive group associated to $A$ is defined as

$$L_A := \begin{cases} A \rtimes G/C_G(A) & \text{if } A \text{ is abelian,} \\ G/C_G(A) & \text{otherwise.} \end{cases}$$

Note that $A \cong \text{soc}(L_A)$. Define

$$\widetilde{P}_{L_A}(s) = P_{L_A}(s),$$

$$\widetilde{P}_{L_A,i}(s) = P_{L_A}(s) - \frac{(1 + q_A + \cdots + q_A^{i-2})\gamma_A}{|A|^s}, \text{ for } i > 1,$$

where $\gamma_A = |C_{\text{Aut}(A)}(L_A)|$ and $q_A = |\text{End}_{L_A}(A)|$ if $A$ is abelian, $q_A = 1$ otherwise. If $A = H/K$ is a non-Frattini chief factor of $G$ then $P_{G/K,H/K}(s) = \widetilde{P}_{L_A,A}(s)$ where $\widetilde{P}_{L_A,A}(s)$ is one of the $\widetilde{P}_{L_A,i}(s)$ for a suitable choice $i$, cf. [6, Theorem 17]. Notice that if $A$ is abelian then

$$P_{L_A,A}(s) = 1 - \frac{c(A)}{|A|^s},$$

where $c(A)$ is the number of complements of $A$ in $L_A$, cf. [10]. Assume that $A \cong S_n^a$ is nonabelian. Let $X_A$ be the subgroup of $\text{Aut}(S_A)$ induced by the conjugation action of the normalizer in $L_A$ of a simple component of $S_n^a$. Note that $X_A$ is an almost simple group with socle $S_A$, cf. [4, Section 2]. One has from [12, Theorem 5] the following proposition.

**Proposition 5.**

$$\widetilde{P}_{L_A,A}^{(r)}(s) = P_{L_A,A}^{(r)}(s) = P_{X_A,S_A}^{(r)}(ns - n + 1)$$

for every prime divisor $r$ of the order of $S$.

### 3. Proof of Theorem 3

Analogous to [19, Proposition 28], we have the following crucial result.

**Proposition 6.** Let $X$ be an almost simple group with socle $\text{PSL}(2,p)$, and $Z$ an almost simple group such that $P_{X,\text{PSL}(2,p)}^{(r)} = P_{Z,\text{soc}(Z)}^{(r)}(s)$ for every $r \in \pi(\text{PSL}(2,p))$. Then $\text{soc}(Z) \cong \text{PSL}(2,p)$.

**Proof.** Let $n$ be the minimal index of a proper subgroup of $X$ which supplements $\text{PSL}(2,p)$. Since $P_{X,\text{PSL}(2,p)}^{(p)}(s) = P_{Z,\text{soc}(Z)}^{(p)}(s)$, it follows from [3, Proposition 3.5] that $a_n(X,\text{PSL}(2,p)) = a_n(Z,\text{soc}(Z)) < 0$. Thus $n$ is the minimal index of a proper subgroup of $Z$. Since $Z$ is an almost simple, it is a primitive group of degree $n$.

If $p = 5$ then $n = 5$ and $\text{PSL}(2,5) \cong \text{Alt}(5)$. The result follows from [19, Proposition 28]. If $p = 7$ and $X = \text{PSL}(2,7)$ then $n = 7$ and (cf. [13, Section 7])

$$P_{X,\text{PSL}(2,7)}(s) = P_{\text{PSL}(2,7)}(s) = 1 - \frac{14}{7^s} - \frac{8}{8^s} + \frac{21}{21^s} + \frac{28}{28^s} + \frac{56}{56^s} - \frac{84}{84^s}.$$
It follows from GAP [9] that soc($Z$) $\cong$ PSL(2, 7). If $p = 11$ and $X = \text{PSL}(2, 11)$ then $n = 11$ and (cf. [15, Section 7])

$$P_{\text{PSL}(2,11)}(s) = 1 - \frac{22}{11^s} - \frac{12}{12^s} + \frac{66}{66^s} + \frac{220}{110^s} + \frac{132}{132^s} + \frac{165}{165^s} - \frac{220}{220^s} - \frac{990}{330^s} + \frac{660}{660^s}.$$ 

By GAP [9], the possibility for $Z$ that $Z = M_{11}$. However by considering $P^{(2)}_{\text{PSL}(2,11)}(s) = P^{(2)}_{\text{Z,soc}(Z)}(s)$ and noting that $M_{11}$ has a maximal subgroup of index 55 (cf. [3]), we obtain a contradiction. This implies soc($Z$) $\cong$ PSL(2, 11).

Assume now that $p > 11$. Then $n = p+1$. Since $P^{(2)}_{X,\text{PSL}(2,p)}(s) = P^{(2)}_{\text{Z,soc}(Z)}(s)$, [7] Lemma 2.7 implies that $p$ divides $|Z|$. Thus $Z$ contains a $p$-cycle, since $p > (p+1)/2$. It follows from [15] that either $Z = M_{24}$ or soc($Z$) = Alt$(p+1)$ or soc($Z$) = PSL$(2, p)$. If $Z = M_{24}$, then by considering $P^{(2)}_{X,\text{PSL}(2,23)}(s) = P^{(2)}_{\text{Z,soc}(Z)}(s)$ and noting that $M_{24}$ has a maximal subgroup of index 7·11·23, one gets a contradiction. Assume that soc($Z$) = Alt$(p+1)$.

By Bertrand’s postulate (cf. [20]), there exists a prime $l$ such that $(p+1)/2 < l < p − 1$. Thus $l$ divides $|X|$ (cf. [4, Proposition 2.3]), which is a contradiction. As a conclusion, we have that soc($Z$) $\cong$ PSL$(2, p)$.

Proof of Theorem 3. The proof is analogous to that of [19, Theorem 3]. We present here for the sake of completeness of the paper.

By Theorem 1, $G$ and $H$ have the same non-Frattini abelian chief factors. Thus we may assume that $G$ and $H$ have no non-Frattini abelian chief factors.

Let $\mathcal{CF}(G)$ be the set of the non-Frattini chief factors of $G$. For each $A \in \mathcal{CF}(G)$, the polynomial $\tilde{P}_{\pi, A}(s)$ is irreducible, cf. [16, Theorem 4]. Hence,

$$P_G(s) = \prod_{A \in \mathcal{CF}(G)} \tilde{P}_{\pi, A}(s)$$

is a factorization of $P_G(s)$ into irreducible factors. Thus, there is a bijection between the sets $\mathcal{CF}(G)$ and $\mathcal{CF}(H)$ such that $A \cong S_n^A \in \mathcal{CF}(G)$ and $B \cong S_n^B \in \mathcal{CF}(H)$ are associated if and only if $\tilde{P}_{\pi, A}(s) = \tilde{P}_{\pi, B}(s)$. Since $\tilde{P}_{\pi, A}(s) = \tilde{P}_{\pi, B}(s)$ for every $r \in \pi(A)$. It follows that $n_A = n_B$, cf. [19, Proposition 27]. Thus $\tilde{P}_{\pi, A}(s) = \tilde{P}_{\pi, B}(s)$ for every $r \in \pi(A)$, cf. Proposition 5. Proposition 6 implies that $S_A \cong S_B$. Therefore $A \cong S_n^A \cong S_n^B \cong B$ as desired.

Remark 7. One could expect that this paper can be extended to PSL$(n, q)$ for some prime power $q$. The difficulty arises in Proposition 6 when one tries to find a prime $r$ such that $Z$ contains a cycle of length $r$. The candidates are primitive prime divisors of $q^n − 1$. However, we do not know how large they are compared to $q^n − 1/2(q − 1)$ to ensure that $Z$ contains a cycle of such prime length. The method in this paper (from [19]) cannot be used to recognize all non-Frattini chief factors in general, even if one can luckily show that $Z$ contains a cycle, since then soc($Z$) is “almost always” alternating or PSL$(n, q)$, cf. [13].
References

[1] Boston, N. A probabilistic generalization of the Riemann zeta function. In Analytic number theory, Vol. 1 (Allerton Park, IL, 1995), vol. 138 of Progr. Math. Birkhäuser Boston, Boston, MA, 1996, pp. 155–162.

[2] Brown, K. S. The coset poset and probabilistic zeta function of a finite group. J. Algebra 225 (2000), 989–1012.

[3] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A., and Wilson, R. A. Atlas of finite groups. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[4] Damian, E., and Lucchini, A. Finite groups with p-multiplicative probabilistic zeta function. Communications in Algebra 35 (2007), 3451–3472.

[5] Damian, E., and Lucchini, A. The probabilistic zeta function of finite simple groups. J. Algebra 313, 2 (2007), 957–971.

[6] Detomi, E., and Lucchini, A. Crowns and factorization of the probabilistic zeta function of a finite group. J. Algebra 265, 2 (2003), 651–668.

[7] Dung, D. H., and Lucchini, A. A finiteness condition on the coefficients of the probabilistic zeta function. Int. J. Group Theory 2, 1 (2013), 167–174.

[8] Dung, D. H., and Lucchini, A. Rationality of the probabilistic zeta functions of finitely generated profinite groups. J. Group Theory 17, 2 (2014), 317–335.

[9] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.7.7, 2015.

[10] Gaschütz, W. Die Eulersche Funktion endlicher auflösbarer Gruppen. Illinois J. Math. 3 (1959), 469–476.

[11] Hall, P. The eulerian functions of a groups. Q. J. Math. 7 (1936), 134–151.

[12] Jiménez-Seral, P. Coefficients of the probabilistic function of a monolithic group. Glasg. Math. J. 50, 1 (2008), 75–81.

[13] Jones, G. A. Primitive permutation groups containing a cycle. Bull. Aust. Math. Soc. 89, 1 (2014), 159–165.

[14] Mann, A. Positively finitely generated groups. Forum Math. 8, 4 (1996), 429–459.

[15] Patassini, M. The probabilistic zeta function of $PSL(2,q)$, of the Suzuki groups $^2B_2(q)$ and of the Ree groups $^2G_2(q)$. Pacific J. Math. 240, 1 (2009), 185–200.

[16] Patassini, M. On the irreducibility of the Dirichlet polynomial of a simple group of Lie type. Israel J. Math. 185 (2011), 477–507.

[17] Patassini, M. Recognizing the characteristic of a simple group of Lie type from its probabilistic zeta function. J. Algebra 332 (2011), 480–499.

[18] Patassini, M. Recognizing the non-Frattini abelian chief factors of a finite group from its probabilistic zeta function. Comm. Algebra 40, 12 (2012), 4494–4508.

[19] Patassini, M. On the irreducibility of the Dirichlet polynomial of an alternating group. Trans. Amer. Math. Soc. 365, 8 (2013), 4041–4062.

[20] Ramanujan, S. A proof of bertrand’s postulate. J. Indian Math. Soc. 11 (1919), 181–182.

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