Principal components analysis of regularly varying functions

Piotr Kokoszka
Colorado State University and Stilian Stoev
University of Michigan and Qian Xiong
Colorado State University

The paper is concerned with asymptotic properties of the principal components analysis of functional data. The currently available results assume the existence of the fourth moment. We develop analogous results in a setting which does not require this assumption. Instead, we assume that the observed functions are regularly varying. We derive the asymptotic distribution of the sample covariance operator and of the sample functional principal components. We obtain a number of results on the convergence of moments and almost sure convergence. We apply the new theory to establish the consistency of the regression operator in a functional linear model.

Key words: Functional data, Principal components, Regular variation

1. Introduction

A fundamental technique of functional data analysis is to replace infinite dimensional curves by coefficients of their projections onto suitable, fixed or data–driven, systems, e.g. ?, ?, ?, ?. A finite number of these coefficients encode the shape of the curves and are amenable to various statistical procedures. The best systems are those that lead to low dimensional representations, and so provide the most efficient dimension reduction. Of these, the functional principal components (FPCs) have been most extensively used, with hundreds of papers dedicated to the various aspects of their theory and applications.

If \( X, X_1, X_2, \ldots, X_N \) are mean zero iid functions in \( \mathbb{L}^2 \) with \( \mathbb{E} \|X\|^2 < \infty \), then

\[
X_n(t) = \sum_{j=1}^{\infty} \xi_{nj} v_j(t), \quad \mathbb{E} \xi_{nj}^2 = \lambda_j. \tag{1.1}
\]

The FPCs \( v_j \) and the eigenvalues \( \lambda_j \) are, respectively, the eigenfunctions and the eigenvalues of the covariance operator \( C : \mathbb{L}^2 \to \mathbb{L}^2 \) defined by \( C(x)(t) = \int \text{Cov}(X(t), X(s))x(s)ds \). As such, the \( v_j \) are orthogonal. We assume they are normalized to unit norm. The \( v_j \) form an optimal orthonormal basis for dimension reduction measured by the \( \mathbb{L}^2 \) norm, see e.g. Theorem 11.4.1 in ?.

The \( v_j \) and the \( \lambda_j \) are estimated by \( \hat{v}_j \) and \( \hat{\lambda}_j \) defined by

\[
\int \hat{c}(t,s)\hat{v}_j(s)ds = \hat{\lambda}_j\hat{v}_j(t), \tag{1.2}
\]

where

\[
\hat{c}(t,s) = \frac{1}{N} \sum_{n=1}^{N} X_n(t)X_n(s). \tag{1.3}
\]

Like the \( v_j \), the \( \hat{v}_j \) are defined only up to a sign. Thus, strictly speaking, in the formulas that follow, the \( \hat{v}_j \) would need to be replaced with \( \hat{c}_j \hat{v}_j \), where \( \hat{c}_j = \text{sign} \langle \hat{v}_j, v_j \rangle \). As is customary, to lighten the notation, we assume that the orientations of \( \hat{v}_j \) and \( v_j \) match, i.e. \( \hat{c}_j = 1 \).

Under the existence of the fourth moment,

\[
\mathbb{E} \|X\|^4 = \left\{ \int X^2(t)dt \right\}^2 < \infty, \tag{1.4}
\]

\[1\]
and assuming $\lambda_1 > \lambda_2 > \ldots$, it has been shown that for each $j \geq 1$,
\[
\limsup_{N \to \infty} N E \| \hat{v}_j - v_j \|^2 < \infty, \quad \limsup_{N \to \infty} N E \left( \hat{\lambda}_j - \lambda_j \right)^2 < \infty, \tag{1.5}
\]
\[
N^{1/2}(\hat{\lambda}_j - \lambda_j) \xrightarrow{d} N(0, \sigma_j^2), \tag{1.6}
\]
\[
N^{1/2}(\hat{v}_j - v_j) \xrightarrow{d} N(0, C_j), \tag{1.7}
\]
for a suitably defined variance $\sigma_j^2$ and a covariance operator $C_j$. The above relations, especially (1.5), have been used to derive large sample justifications of inferential procedures based on the estimated FPCs $\hat{v}_j$. In most scenarios, one can show that replacing the $\hat{v}_j$ by the $v_j$ and the $\hat{\lambda}_j$ by the $\lambda_j$ is asymptotically negligible. Relations (1.5) were established by [7] and extended to weakly dependent functional time series by [8]. Relations (1.6) and (1.7) follow from the results of [9]. In case of continuous functions satisfying regularity conditions, they follow from the results of [2].

A crucial assumption for the relations (1.5)–(1.7) to hold is the existence of the fourth moment, i.e. (1.4), the iid assumption can be relaxed in many ways. Nothing is at present known about the asymptotic properties of the FPCs and their eigenvalues if (1.4) does not hold. Our objective is to explore what can be said about the asymptotic behavior of $\hat{C}$, $\hat{v}_j$, and $\hat{\lambda}_j$ if (1.4) fails. We would thus like to consider the case of $E\|X_n\|^2 < \infty$ and $E\|X_n\|^4 = \infty$. Such an assumption is however too general. From mid 1980s to mid 1990s similar questions were posed for scalar time series for which the fourth moment does not exist. A number of results pertaining to the convergence of sample covariances and the periodogram have been derived under the assumption of regularly varying tails, e.g. Davis and Resnick ([7], [8], [9], [10], [11], [12], [13]); many others are summarized in the monograph of [14]. The assumption of regular variation is natural because non–normal stable limits can be derived by establishing a connection to random variables in a stable domain of attraction, which is characterized by regular variation. This is the approach we take. We assume that the functions $X_n$ are regularly varying in the space $L^2$ with the index $\alpha \in (2, 4)$, which implies $E\|X_n\|^2 < \infty$ and $E\|X_n\|^4 = \infty$. Suitable definitions and assumptions are presented in Section 2.

The paper is organized as follows. The remainder of the introduction provides a practical motivation for the theory we develop. It is not necessary to understand the contribution of the paper, but, we think, it gives a good feel for what is being studied. The formal exposition begins in Section 2, in which notation and assumptions are specified. Section 3 is dedicated to the convergence of the sample covariance operator (the integral operator with kernel (1.3)). These results are then used in Section 4 to derive various convergence results for the sample FPCs and their eigenvalues. Section 5 shows how the results derived in previous sections can be used in a context of a functional regression model. Its objective is to illustrate the applicability of our theory in a well–known and extensively studied setting. It is hoped that it will motivate and guide applications to other problems of functional data analysis. All proofs which go beyond simple arguments are presented in Online material.

We conclude this introduction by presenting a specific data context. Denote by $P_i(t)$ the price of an asset at time $t$ of trading day $i$. For the assets we consider in our illustration, $t$ is time in minutes between 9:30 and 16:00 EST (NYSE opening times) rescaled to the unit interval (0, 1). The intraday return curve on day $i$ is defined by $X_i(t) = \log P_i(t) - \log P_i(0)$. In practice, $P_i(0)$ is the price after the first minute of trading. The curves $X_i$ show how the return accumulates over the trading day, see e.g. [7]; examples of are shown in Figure 1.

The first three sample FPCs, $\hat{v}_1$, $\hat{v}_2$, $\hat{v}_3$, are shown in Figure 2. They are computed, using (1.2), from minute-by-minute Walmart returns form July 05, 2006 to Dec 30, 2011, $N = 1, 378$ trading days. (This time interval is used for the other assets we consider.) The curves $\hat{X}_i = \sum_{j=1}^3 \hat{\xi}_{ij} \hat{v}_j$, with the scores $\hat{\xi}_{ij} = \int X_i(t) \hat{v}_j(t) dt$, visually approximate the curves $X_i$ well. One can thus expect that the $\hat{v}_j$ (with properly adjusted sign) are good estimators of the population FPCs $v_j$ in (1.1). Relations (1.5) and (1.7) show that this is indeed the case, if $E\|X\|^4 < \infty$. (The curves $X_i$ can be assumed to form a stationary time series in $L^2$, see [7].) We will now argue that the assumption of the finite fourth moment is not
Figure 1. Five consecutive intraday return curves, Walmart stock. The raw returns are noisy grey lines. The smoother black lines are approximations $\hat{X}_i(t) = \sum_{j=1}^3 \hat{\xi}_{ij} \hat{v}_j$.

Figure 2. The first three sample FPCs of intraday returns on Walmart stock.
realistic, so, with the currently available theory, it is not clear if the $i_j$ are good estimators of the $v_j$. If $E\|X\|^4 < \infty$, then $E\xi_{1,j}^4 < \infty$ for every $j$. Figure 3 shows the Hill plots of the sample score $\hat{\xi}_{ij}$ for two stocks and for $j = 1, 2, 3$. Hill plots for other blue chip stocks look similar. These plots illustrate several properties. 1) It is reasonable to assume that the scores have Pareto tails. 2) The tail index $\alpha$ is smaller than 4, implying that the fourth moment does not exist. 3) It is reasonable to assume that the tail index does not depend on $j$ and is between 2 and 4. With such a motivation, we are now able to formalize in the next section the setting of this paper.

2. Preliminaries

The functions $X_n$ are assumed to be independent and identically distributed in $L^2$, with the same distribution as $X$, which is regularly varying with index $\alpha \in (2, 4)$. By $L^2 := L^2(\mathcal{T})$, we denote the usual separable Hilbert space of square integrable functions on some compact subset $\mathcal{T}$ of an Euclidean space.

In a typical FDA framework, $\mathcal{T} = [0, 1]$, e.g. Chapter 2 of ?. Regular variation in finite–dimensional spaces has been a topic of extensive research for decades, see e.g. Resnick (?, ?) and ?. We shall need the concept of regular variation of measures on infinitely-dimensional function spaces. To this end, we start by recalling some terminology and fundamental facts about regularly varying functions.

A measurable function $L : (0, \infty) \to \mathbb{R}$ is said to be slowly varying (at infinity) if, for all $\lambda > 0$,

$$
\frac{L(\lambda u)}{L(u)} \to 1, \quad u \to \infty.
$$

Functions of the form $R(u) = u^\rho L(u)$ are said to be regularly varying with exponent $\rho \in \mathbb{R}$.

The notion of regular variation extends to measures and provides an elegant and powerful framework for establishing limit theorems. It was first introduced by ? and has been since extended to Banach and even metric spaces using the notion of $M_0$ convergence (see e.g. ?). Even though we will work only with Hilbert spaces, we review the theory in a more general context.

Consider a separable Banach space $B$ and let $B_\epsilon := \{ z \in B : \| z \| < \epsilon \}$ be the open ball of radius $\epsilon > 0$, centered at the origin. A Borel measure $\mu$ defined on $\mathbb{E}_0 := B \setminus \{ 0 \}$ is said to be boundedly finite if $\mu(A) < \infty$, for all Borel sets that are bounded away from $0$, that is, such that $A \cap B_\epsilon = \emptyset$, for some $\epsilon > 0$. Let $M_0$ be the collection of all such measures. For $\mu_n, \mu \in M_0$, we say that $\mu_n$ converge to $\mu$ in the $M_0$ topology, if $\mu_n(A) \to \mu(A)$, for all bounded away from $0$, $\mu$-continuity Borel sets $A$, i.e., such that $\mu(\partial A) = 0$, where $\partial A := \overline{A} \setminus A^o$ denotes the boundary of $A$. The $M_0$ convergence can be metrized such that $M_0$ becomes a complete separable metric space (Theorem 2.3 in ? and also Section 2.2. of ?). The following result is known, see e.g. Chapter 2 of ? and references therein.

**Proposition 2.1** Let $X$ be a random element in a separable Banach space $B$ and $\alpha > 0$. The following three statements are equivalent:

(i) For some slowly varying function $L$,

$$P(\|X\| > u) = u^{-\alpha} L(u) \quad (2.1)$$

and

$$P(u^{-1}X \in \cdot \mid \|X\| > u) \overset{M_0}{\to} \mu(\cdot), \quad u \to \infty, \quad (2.2)$$

where $\mu$ is a non-null measure on the Borel $\sigma$-field $\mathcal{B}(\mathbb{E}_0)$ of $\mathbb{E}_0 = B \setminus \{ 0 \}$.

(ii) There exists a probability measure $\Gamma$ on the unit sphere $S$ in $B$ such that, for every $t > 0$,

$$P(\|X\| > tu, \|X\| \in \cdot \mid \|X\| > u) \overset{w}{\to} t^{-\alpha} \Gamma(\cdot), \quad u \to \infty.$$

(iii) Relation (2.1) holds, and for the same spectral measure $\Gamma$ in (ii),

$$P(\|X\| \in \cdot \mid \|X\| > u) \overset{w}{\to} \Gamma(\cdot), \quad u \to \infty.$$
Figure 3. Hill plots (an estimate of $\alpha$ as a function of upper order statistics) for sample FPC scores for Walmart (left) and IBM (right). From top to bottom: levels $j = 1, 2, 3$. 
If any one of the equivalent conditions in Proposition 2.1 hold, we shall say that \( X \) is regularly varying with index \( \alpha \). The measures \( \mu \) and \( \Gamma \) will be referred to as exponent and angular measures of \( X \), respectively.

The measure \( \Gamma \) is sometimes called the spectral measure, but we will use the adjective “spectral” in the context of stable measures which appear in Section 3. It is important to distinguish the angular measure of a regularly varying random function and a spectral measure of a stable distribution, although they are related. We also note that we call \( \alpha \) the tail index, and \( -\alpha \) the tail exponent.

We will work under the following assumption.

**Assumption 2.1** The random element \( X \) in the separable Hilbert space \( H = L^2 \) has mean zero and is regularly varying with index \( \alpha \in (2, 4) \). The observations \( X_1, X_2, \ldots \) are independent copies of \( X \).

Assumption 2.1 is a coordinate free condition not related in any way to functional principal components. The next assumption relates the asymptotic behavior of the FPC scores to the assumed regular variation. It implies, in particular, that the expansion \( X(t) = \sum_{j=1}^{\infty} \xi_j v_j(t) \) contains infinitely many terms, so that we study infinite dimensional objects. We will see in the proofs of Proposition 3.1 and Theorem 3.2 that under Assumption 2.1 the limit

\[
Q_{nm} = \lim_{u \to \infty} \frac{P \left( \left\{ \sum_{j=n}^{\infty} \xi_j^2 \right\}^{1/2} \left\{ \sum_{j=m}^{\infty} \xi_j^2 \right\}^{1/2} > u \right)}{P \left( \sum_{j=1}^{\infty} \xi_j^2 > u \right)}
\]

exists and is finite. We impose the following assumption related to condition (2.2).

**Assumption 2.2** For every \( n, m \geq 1 \), \( Q_{nm} > 0 \).

Assumption 2.2 postulates, intuitively, that the tail sums \( \sum_{j=n}^{\infty} \xi_j^2 \) must have extreme probability tails comparable to that of \( \|X\|^2 \).

We now collect several useful facts that will be used in the following. The exponent measure \( \mu \) satisfies

\[
\mu(tA) = t^{-\alpha} \mu(A), \ \forall t > 0, \ A \in \mathcal{B}(\mathbb{B}_0).
\]

(2.3)

It admits the polar coordinate representation via the angular measure \( \Gamma \). That is, if \( x = r \theta \), where \( r := \|x\| \) and \( \theta = x/\|x\| \), for \( x \neq 0 \), we have

\[
\mu(dx) = \alpha r^{-\alpha-1} dr \Gamma(d\theta).
\]

(2.4)

This means that for every bounded measurable function \( f \) that vanishes on a neighborhood of \( 0 \), we have

\[
\int_{\mathbb{B}} f(x) \mu(dx) = \int_{\mathbb{B}} \int_0^\infty f(r \theta) \alpha r^{-\alpha-1} dr \Gamma(d\theta).
\]

There exists a sequence \( \{a_N\} \) such that

\[
NP(X \in a_N A) \to \mu(A),
\]

(2.5)

for any set \( A \) in \( \mathcal{B}(\mathbb{B}_0) \) with \( \mu(\partial A) = 0 \). One can take, for example,

\[
a_N = N^{1/\alpha} L_0(N),
\]

(2.6)

with a slowly varying function \( L_0 \) satisfying \( L_0^{-\alpha}(N)L(N^{1/\alpha} L_0(N)) \to 1 \).

We will work with Hilbert–Schmidt operators. A linear operator \( \Psi : H \to H \) is Hilbert–Schmidt if \( \sum_{j=1}^{\infty} \|\Psi(e_j)\|^2 < \infty \), where \( \{e_j\} \) is any orthonormal basis of \( H \). Every Hilbert–Schmidt operator is bounded. The space of Hilbert–Schmidt operators will be denoted by \( \mathcal{S} \). It is itself a separable Hilbert space with the inner product

\[
\langle \Psi_1, \Psi_2 \rangle_{\mathcal{S}} = \sum_{j=1}^{\infty} \langle \Psi_1(e_j), \Psi_2(e_j) \rangle.
\]
If $\Psi$ is an integral operator defined by $\Psi(x)(t) = \int \psi(t,s)x(s)ds$, $x \in L^2$, then $\|\Psi\|_S^2 = \int \psi^2(t,s)dtds$.

Relations (1.5) essentially follow from the bound
\[
E \left\| \hat{C} - C \right\|_S^2 \leq N^{-1}E\|X\|^4,
\]
where the subscript $S$ indicates the Hilbert–Schmidt norm. Under Assumption 2.1 such a bound is useless because, by (2.1), $E\|X\|^2 = \infty$. In fact, one can show that under Assumption 2.1, $E\|\hat{C}\|_S^2 = \infty$, so no other bound on $E\|\hat{C} - C\|_S^2$ can be expected. The following Proposition 2.2 implies however that under Assumption 2.1 the population covariance operator $C$ is a Hilbert-Schmidt operator, and $\hat{C} \in S$ with probability 1. This means that the space $S$ does provide a convenient framework.

**Proposition 2.2** Suppose $X$ is a random element of $L^2$ with $E\|X\|^2 < \infty$ and $\hat{C}$ is the sample covariance operator based on $N$ iid copies of $X$. Then $C \in S$ and $\hat{C} \in S$ with probability 1.

Like all proofs, the proof of Proposition 2.2 is presented in the on-line material.

### 3. Limit distribution of $\hat{C}$

We will show that $Nk_N^{-1}(\hat{C} - C)$ converges to an $\alpha/2$–stable Hilbert–Schmidt operator, for an appropriately defined regularly varying sequence $\{k_N\}$. Unless stated otherwise, all limits in the following are taken as $N \to \infty$.

Observe that for any $x \in H$,
\[
Nk_N^{-1}(\hat{C} - C)(x) = Nk_N^{-1}\left( N^{-1} \sum_{n=1}^{N} (X_n, x) X_n - E[\langle X_1, x \rangle X_1] \right)
\]
\[
= k_N^{-1}\left( \sum_{n=1}^{N} (X_n, x) X_n - NE[\langle X_1, x \rangle X_1] \right)
\]
\[
= k_N^{-1}\left( \sum_{n=1}^{N} (X_n \otimes X_n)(x) - NE[\langle X_1 \otimes X_1 \rangle](x) \right),
\]
where $(X_n \otimes X_n)(x) = \langle X_n, x \rangle X_n$. Since the $X_n \otimes X_n$ are Hilbert–Schmidt operators, the last expression shows a connection between the asymptotic distribution of $\hat{C}$ and convergence to a stable limit in the Hilbert space $S$ of Hilbert–Schmidt operators. We therefore restate below, as Theorem 3.1, Theorem 4.11 of ? which provides conditions for the stable domain of attraction in a separable Hilbert space. The Hilbert space we will consider in the following will be $S$ and the stability index will be $\alpha/2$, $\alpha \in (2, 4)$. However, when stating the result of Kuelbs and Mandrekar, we will use a generic Hilbert space $H$ and the generic stability index $p \in (0, 2)$. Recall that for a stable random element $S \in H$ with index $p \in (0, 2)$, there exists a spectral measure $\sigma_S$ defined on the unit sphere $S_H = \{z \in H : \|z\| = 1\}$, such that the characteristic functional of $S$ is given by
\[
E \exp\{i \langle x, S \rangle\} = \exp\left\{i \langle x, \beta_S \rangle - \int_{S_H} |\langle x, s \rangle|^p \sigma_S(ds) + iC(p, x) \right\}, \quad x \in H,
\]
where
\[
C(p, x) = \begin{cases} \tan \frac{\pi}{2p} \int_{S_H} |\langle x, s \rangle|^{p-1} \sigma_S(ds) & \text{if } p \neq 1, \\ \frac{2}{\pi} \int_{S_H} |\langle x, s \rangle| \log |\langle x, s \rangle| \sigma_S(ds) & \text{if } p = 1. \end{cases}
\]
We denote the above representation by $S \sim [p, \sigma_S, \beta_S]$. The $p$-stable random element $S$ is necessarily regularly varying with index $p \in (0, 2)$. In fact, its angular measure is precisely the normalized spectral measure appearing in (3.2), i.e.,
\[
\Gamma_S(\cdot) = \frac{\sigma_S(\cdot)}{\sigma_S(S_H)}
\]
? derived sufficient and necessary conditions on the distribution of \(Z\) under which
\[
b_N^{-1} \left( \sum_{i=1}^{N} Z_i - \gamma_N \right) \overset{d}{\rightarrow} S, \tag{3.3}
\]
where the \(Z_i\) are iid copies of \(Z\). They assume that the support of the distribution of \(S\), equivalently of the distribution of \(Z\), spans the whole Hilbert space \(H\). In our context, we will need to work with \(Z\) whose distribution is not supported on the whole space. Denote by \(L(Z)\) the smallest closed subspace which contains the support of the distribution of \(Z\). Then \(L(Z)\) is a Hilbert space itself with the inner product inherited from \(H\). Denote by \(\{e_j, j \in \mathbb{N}\}\) an orthonormal basis of \(L(Z)\). We assume that this is an infinite basis because we consider infinite dimensional data. (The finite dimensional case has already been dealt with by \(\gamma\).)

Introduce the projections
\[
\pi_m(z) = \sum_{j=m}^{\infty} \langle z, e_j \rangle e_j, \quad z \in H.
\]

**Theorem 3.1** Let \(Z_1, Z_2, \ldots\) be iid random elements in a separable Hilbert space \(H\) with the same distribution as \(Z\). Let \(\{e_j, j \in \mathbb{N}\}\) be an orthonormal basis of \(L(Z)\). There exist normalizing constants \(b_N\) and \(\gamma_N\) such that (3.3) holds if and only if
\[
\frac{P \left( \|\pi_m(Z)\| > tu \right)}{P \left( \|Z\| > u \right)} \rightarrow \frac{c_m}{c_1} u^{-p}, \quad u \rightarrow \infty, \tag{3.4}
\]
where for each \(m \geq 1\), \(c_m > 0\), and \(\lim_{m \rightarrow \infty} c_m = 0\), and where
\[
\frac{P \left( \|Z\| > u, Z/\|Z\| \in A \right)}{P \left( \|Z\| > u, Z/\|Z\| \in A^* \right)} \rightarrow \frac{\sigma_S(A)}{\sigma_S(A^*)}, \quad u \rightarrow \infty, \tag{3.5}
\]
for all continuity sets \(A, A^* \in \mathcal{B}(S_H)\) with \(\sigma_S(A^*) > 0\).

If (3.3) holds, the sequence \(b_N\) must satisfy
\[
b_N \rightarrow \infty, \quad \frac{b_N}{b_{N+1}} \rightarrow 1, \quad Nb_N^{-2} E \left( \|Z\|^2 I_{\{\|Z\| \leq b_N\}} \right) \rightarrow \lambda_p \sigma_S(S_H), \tag{3.6}
\]
where
\[
\lambda_p = \begin{cases} 
\frac{1}{\Gamma(3-p) \cos(p \pi/2)} & \text{if } p \neq 1 \\
\frac{2}{\pi} & \text{if } p = 1, \end{cases} \tag{3.7}
\]
and \(\Gamma(a) := \int_{0}^{\infty} e^{-x} x^{a-1} dx, a > 0\) is the Euler gamma function. Furthermore, the \(\gamma_N \in H\) may be chosen as
\[
\gamma_N = N E \left( Z I_{\{\|Z\| \leq b_N\}} \right). \tag{3.8}
\]

**Remark 3.1** The origin of the constant \(\lambda_p\) appearing in (3.6) can be understood as follows. Consider the simple scalar case \(H = \mathbb{R}\). Let \(Z\) be symmetric \(\alpha\)-stable with \(E[e^{iZx}] = e^{-c|x|^\alpha}, x \in \mathbb{R}\), where in this case, \(c = \sigma(S_H) \equiv \sigma(\{-1, 1\}) > 0\). Consider iid copies \(Z_i, i = 1, 2, \ldots\) of \(Z\) and observe that by the \(p\)-stability property
\[
\frac{1}{N^{1/\alpha}} \sum_{j=1}^{N} Z_j \overset{d}{\rightarrow} Z \equiv S,
\]
and hence (3.3) holds trivially with \(b_N := N^{1/\alpha}\) and \(\gamma_N := 0\).

On the other hand, by Proposition 1.2.15 on page 16 in \(?\), we have
\[
P(\|Z\| > x) \sim \frac{c(1-p)}{\Gamma(2-p) \cos(p \pi/2)} x^{-p}, \quad \text{as } x \rightarrow \infty.
\]
This along with an integration by parts and an application of Karamata’s theorem yield \(Nb_N^{-2} E[Z^2 I_{\{\|Z\| \leq b_N\}}] \rightarrow \lambda_p \sigma_S(S_H), \) giving the constant in (3.6).
Proposition 3.1 Conditions (3.4) and (3.5) in Theorem 3.1 hold if and only if $Z$ is regularly varying in $H$ with index $p \in (0, 2)$ and for each $m \geq 1, \mu_2(A_m) > 0$, where

\[ A_m = \left\{ z \in H : \| \pi_m(z) \| = \left\| \sum_{j=m}^{\infty} \langle z, e_j \rangle e_j \right\| > 1 \right\}. \]  

(3.9)

Our next objective is to show that if $X$ is a regularly varying element of a separable Hilbert space $H$ whose index is $\alpha > 0$, then the operator $Y = X \otimes X$ is regularly varying with index $\alpha/2$, in the space of Hilbert–Schmidt operators. If $y, z \in H$, then $y \otimes z$ is an element of $\mathcal{S}$ defined by $(y \otimes z)(x) = \langle y, x \rangle z, x \in H$. It is easy to check that $\| y \otimes z \|_\mathcal{S} = \| y \| \| z \|$. If $B_1, B_2 \subset H$, we denote by $B_1 \otimes B_2$ the subset of $\mathcal{S}$ defined as the set of operators of the form $x_1 \otimes x_2$, with $x_1 \in B_1, x_2 \in B_2$. Denote by $\mathcal{S}_H$ the unit sphere in $H$ centered at the origin, and by $\mathcal{S}_S$ such a sphere in $\mathcal{S}$.

The next result is valid for all $\alpha > 0$.

Proposition 3.2 Suppose $X$ is a regularly varying element with index $\alpha > 0$ of a separable Hilbert space $H$. Then the operator $Y = X \otimes X$ is a regularly varying element with index $\alpha/2$ of the space $\mathcal{S}$ of Hilbert–Schmidt operators.

Remark 3.2 The proof of Proposition 3.2 shows that the angular measure of $X \otimes X$ is supported on the diagonal $\{ \Psi \in \mathcal{S}_S : \Psi = x \otimes x \text{ for some } x \in \mathcal{S}_H \}$ and that $\Gamma_{X \otimes X}(B \otimes B) = \Gamma_X(B), \forall B \subset \mathcal{B}(\mathcal{S}_H)$.

The next result specifies the limit distribution of the sums of the $X_i \otimes X_i$ based on the results derived so far.

Theorem 3.2 Suppose Assumptions 2.1 and 2.2 hold. Then, there exist normalizing constants $k_N$ and operators $\psi_N$ such that

\[ k_N^{-1} \left( \sum_{i=1}^{N} X_i \otimes X_i - \psi_N \right) \xrightarrow{d} S, \]  

(3.10)

where $S \in \mathcal{S}$ is a stable random operator, $S \sim [\alpha/2, \sigma_S, 0]$, where the spectral measure $\sigma_S$ is defined on the unit sphere $\mathcal{S}_S = \{ y \in \mathcal{S} : \| y \|_\mathcal{S} = 1 \}$. The normalizing constants may be chosen as follows

\[ k_N = \left( \frac{\alpha}{4 - \alpha} \right)^{2/\alpha} a_N^2, \quad \psi_N = N E \left[ (X \otimes X) I_{\{ \| X \|^2 \leq k_N \}} \right], \]  

(3.11)

where $a_N$ is defined by (2.6).

The final result of this section specifies the asymptotic distribution of $\hat{C} - C$.

Theorem 3.3 Suppose Assumptions 2.1 and 2.2 hold. Then,

\[ N k_N^{-1} (\hat{C} - C) \xrightarrow{d} S - \frac{\alpha}{\alpha - 2} \int_{\mathcal{S}_H} (\theta \otimes \theta) \Gamma_X(d \theta), \]  

(3.12)

where $S \in \mathcal{S}$ and $\{ k_N \}$ are as in Theorem 3.2. ($k_N = N^{2/\alpha} L(N)$ for a slowly varying $L$.)

If the $X_i$ are scalars, then the angular measure $\Gamma_X$ is concentrated on $\mathcal{S}_H = \{-1, 1\}$, with $\Gamma_X(1) = p, \Gamma_X(-1) = 1 - p$, in the notation of 7. Thus $\int_{\mathcal{S}_H} \theta^2 \Gamma_X(d \theta) = 1$, and we recover the centering $\alpha/(\alpha - 2)$ in Theorem 2.2 of 7. Relation (3.12) explains the structure of this centering in a much more general context.

Theorem 3.3 readily leads to a strong law of large numbers which can be derived by an application of the following result, a consequence of Theorem 3.1 of 7.
Theorem 3.4 Suppose $Y_i, i \geq 1$, are iid mean zero elements of a separable Hilbert space with $E\|Y_i\|^\gamma < \infty$, for some $1 \leq \gamma < 2$. Then,

$$\frac{1}{N^{1/\gamma}} \sum_{i=1}^{N} Y_i \overset{p}{\to} 0 \quad \text{if and only if} \quad \frac{1}{N^{1/\gamma}} \sum_{i=1}^{N} Y_i^{\alpha/\gamma} \to 0.$$  

Set $Y_i = X_i \otimes X_i - E[X \otimes X]$. Then the $Y_i$ are iid mean zero elements of $S$ which, by Proposition 3.2, satisfy $E\|Y_i\|_S^2 < \infty$, for any $\gamma \in (0, \alpha/2)$. Theorem 3.3 implies that for any $\gamma \in (0, \alpha/2)$, $N^{-1/\gamma} \sum_{i=1}^{N} Y_i \overset{p}{\to} 0$. Thus Theorem 3.4 leads to the following corollary.

Corollary 3.1 Suppose Assumptions 2.1 and 2.2 hold. Then, for any $\gamma \in [1, \alpha/2)$, $N^{1-1/\gamma}\|\hat{C} - C\|_S \to 0$ with probability 1.

4. Convergence of eigenfunctions and eigenvalues

We first formulate and prove a general result which allows us to derive the asymptotic distributions of the eigenfunctions and eigenvalues of an estimator of the covariance operator from the asymptotic distribution of the operator itself. The proof of this result is implicit in the proofs of the results of Section 2 of ?, which pertain to the asymptotic normality of the sample covariance operator if $E\|X\|^4 < \infty$. The result and the technique of proof are however more general, and can be used in different contexts, so we state and prove it in detail.

Assumption 4.1 Suppose $C$ is the covariance operator of a random function $X$ taking values in $L^2$ such that $E\|X\|^2 < \infty$. Suppose $\hat{C}$ is an estimator of $C$ which is a.s. symmetric, nonnegative-definite and Hilbert–Schmidt. Assume that for some random operator $Z \in S$, and for some $r_N \to \infty$,

$$Z_N := r_N (\hat{C} - C) \overset{d}{\to} Z.$$  

In our setting, $Z \in S$ is specified in (3.12), and $r_N = N^{-\beta} L(N)$ for some $0 < \beta < 1/2$. More precisely,

$$r_N = Na_N^{-2}, \quad a_N = N^{1/\alpha} L_0(N), \quad \alpha \in (2, 4).$$  

We will work with the eigenfunctions and eigenvalues defined by

$$C(v_j) = \lambda_j v_j, \quad \hat{C}(\hat{v}_j) = \hat{\lambda}_j \hat{v}_j, \quad j \geq 1.$$  

Assumption 4.1 implies that $\hat{\lambda}_j \geq 0$ and the $\hat{v}_j$ are orthogonal with probability 1. We assume that, like the $v_j$, the $\hat{v}_j$ have unit norms. To lighten the notation, we assume that $\text{sign}(\hat{v}_j, v_j) = 1$. This sign does not appear in any of our final results, it cancels in the proofs. We assume that both sets of eigenvalues are ordered in decreasing order. The next assumption is standard, it ensures that the population eigenspaces are one dimensional.

Assumption 4.2 $\lambda_1 > \lambda_2, \ldots, > \lambda_p > \lambda_{p+1}$.

Set

$$T_j = \sum_{k \neq j} (\lambda_j - \lambda_k)^{-1} \langle Z, v_j \otimes v_k \rangle v_k.$$  

Lemma 6.2 in online material shows that the series defining $T_j$ converges a.s. in $L^2$.

Theorem 4.1 Suppose Assumptions 4.1 and 4.2 hold. Then,

$$r_N \{\hat{v}_j - v_j, \ 1 \leq j \leq p\} \overset{d}{\to} \{T_j, \ 1 \leq j \leq p\}, \quad \text{in} \ (L^2)^p,$$

and

$$r_N \{\hat{\lambda}_j - \lambda_j, \ 1 \leq j \leq p\} \overset{d}{\to} \{(Z(v_j), v_j), \ 1 \leq j \leq p\}, \quad \text{in} \ \mathbb{R}^p.$$
If $Z$ is an $(\alpha/2)$–stable random operator in $S$, then the $T_j$ are jointly $(\alpha/2)$–stable random functions in $L^2$, and $(Z(v_j), v_j)$ are jointly $(\alpha/2)$–stable random variables. This follows directly from the definition of a stable distribution, e.g. Section 6.2 of book. Under Assumption 2.1, $r_N = N^{1-2/\alpha}L_0^{-2}(N)$. Theorem 4.1 thus leads to the following corollary.

**Corollary 4.1** Suppose Assumptions 2.1, 2.2 and 4.2 hold. Then,

$$N^{1-2/\alpha}L_0^{-2}(N)\{\hat{v}_j - v_j, 1 \leq j \leq p\} \overset{d}{\to} \{T_j, 1 \leq j \leq p\}, \text{ in } (L^2)^p,$$

where the $T_j$ are jointly $(\alpha/2)$–stable in $L^2$, and

$$N^{1-2/\alpha}L_0^{-2}(N)\{\hat{\lambda}_j - \lambda_j, 1 \leq j \leq p\} \overset{d}{\to} \{S_j, 1 \leq j \leq p\}, \text{ in } \mathbb{R}^p,$$

where the $S_j$ are jointly $(\alpha/2)$–stable in $\mathbb{R}$.

Corollary 4.1 implies the rates in probability $\hat{v}_j - v_j = O_P(r_N^{-1})$ and $\hat{\lambda}_j - \lambda_j = O_P(r_N^{-1})$, with $r_N = N^{1-2/\alpha}L_0^{-2}(N)$. This means, that the distances between $\hat{v}_j$ and $\lambda_j$ and the corresponding population parameters are approximately of the order $N^{2/\alpha-1}$, i.e. are asymptotically larger that these distances in the case of $E|X|^4 < \infty$, which are of the order $N^{-1/2}$. Note that $2/\alpha - 1 \rightarrow -1/2$, as $\alpha \rightarrow 4$.

It is often useful to have some bounds on moments, analogous to relations (1.5). Since the tails of $\|T_j\|$ and $|S_j|$ behave like $t^{-\alpha/2}$, e.g. Section 6.7 of book, $E\|T_j\|^\gamma < \infty$, $0 < \gamma < \alpha/2$, with an analogous relation for $|S_j|$. We can thus expect convergence of moments of order $\gamma \in (0, \alpha/2)$. The following theorem specifies the corresponding results.

**Theorem 4.2** If Assumptions 2.1 and 2.2 hold, then for each $\gamma \in (0, \alpha/2)$, there is a slowly varying function $L_\gamma$ such that

$$\limsup_{N \rightarrow \infty} N^{\gamma(1-2/\alpha)}L_\gamma(N)E\|\hat{C} - C\|^\gamma_S < \infty$$

and for $j \geq 1$,

$$\limsup_{N \rightarrow \infty} N^{\gamma(1-2/\alpha)}L_\gamma(N)E|\hat{\lambda}_j - \lambda_j|^\gamma < \infty.$$

If, in addition, Assumption 4.2 holds, then for $1 \leq j \leq p$,

$$\lim_{N \rightarrow \infty} N^{\gamma(1-2/\alpha)}L_\gamma(N)E\|\hat{v}_j - v_j\|^\gamma < \infty.$$

Several cruder bounds can be derived from Theorem 4.2. In applications, it is often convenient to take $\gamma = 1$. Then $E\|\hat{C} - C\|_S \leq N^{2/\alpha - 1}L_1(N)$. By Potter bounds, e.g. Proposition 2.6 (ii) in book, for any $\epsilon > 0$ there is a constant $C_\epsilon$ such that for $x > x_\epsilon$, $L_1(x) \leq C_\epsilon x^\epsilon$. For each $\alpha \in (2, 4)$, we can choose $\epsilon$ so small that $-\delta(\alpha) := 2/\alpha - 1 + \epsilon < 0$. This leads to the following corollary.

**Corollary 4.2** If Assumptions 2.1 and 2.2 hold, then for each $\alpha \in (2, 4)$, there are constant $C_\alpha$ and $\delta(\alpha) > 0$ such that

$$E\|\hat{C} - C\|_S \leq C_\alpha N^{-\delta(\alpha)} \quad \text{and} \quad E\|\hat{\lambda}_j - \lambda_j\| \leq C_\alpha N^{-\delta(\alpha)}.$$

If, in addition, Assumption 4.2 holds, then for $1 \leq j \leq p$, $E\|\hat{v}_j - v_j\| \leq C_\alpha(j)N^{-\delta(\alpha)}$.

Corollary 4.2 implies that $E\|\hat{C} - C\|_S$, $E\|\hat{\lambda}_j - \lambda_j\|$ and $E\|\hat{v}_j - v_j\|$ tend to zero, for any $\alpha \in (2, 4)$.

**5. An application: functional linear regression**

One of the most widely used tools of functional data analysis is the functional regression model, e.g. book, book, book. Suppose $X_1, X_2, \ldots, X_N$ are explanatory functions, $Y_1, Y_2, \ldots, Y_N$ are response functions, and assume that

$$Y_i(t) = \int_0^1 \psi(t,s)X_i(s)ds + \varepsilon_i(t), \quad 1 \leq i \leq N,$$

(5.1)
where \( \psi(\cdot, \cdot) \) is the kernel of \( \Psi \in \mathcal{S} \). The \( X_i \) are mean zero iid functions in \( L^2 = L^2([0, 1]) \), and so are the error functions \( \varepsilon_i \). Consequently, the \( Y_i \) are iid in \( L^2 \). A question that has been investigated from many angles is how to consistently estimate the regression kernel \( \psi(\cdot, \cdot) \). An estimator that has become popular following the work of \( ? \) can be constructed as follows.

The population version of (5.1) is \( Y(t) = \int \psi(t, s)X(s)ds + \varepsilon(t) \). Denote by \( v_i \) the FPCs of \( X \) and by \( u_j \) those of \( Y \), so that

\[
X(s) = \sum_{i=1}^{\infty} \xi_i v_i(s), \quad Y(t) = \sum_{j=1}^{\infty} \zeta_j u_j(t).
\]

If \( \varepsilon \) is independent of \( X \), then, with \( \lambda_\ell = E[\xi_\ell^2] \),

\[
\psi(t, s) = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \frac{E[\xi_\ell \zeta_k]}{\lambda_\ell} u_k(t)v_\ell(s),
\]

with the series converging in \( L^2([0, 1] \times [0, 1]) \), equivalently in \( \mathcal{S} \), see Lemma 8.1 in \( ? \). This motivates the estimator

\[
\hat{\psi}_{KL}(t, s) = \sum_{k=1}^{K} \sum_{\ell=1}^{L} \frac{\hat{\sigma}_{\ell k}}{\lambda_\ell} \hat{u}_k(t)\hat{v}_\ell(s),
\]

where \( \hat{u}_k \) are the eigenfunctions of \( \hat{C}_Y \) and \( \hat{\sigma}_{\ell k} \) is an estimator of \( E[\xi_\ell \zeta_k] \). \( ? \) study the above estimator under the assumption that data are observed sparsely and with measurement errors. This requires two-stage smoothing, so their assumptions focus on conditions on the various smoothing parameters and the random mechanism that generates the sparse observations. Like in all work of this type, they assume that the underlying functions have finite fourth moments: \( E||X||^4 < \infty \), \( E||\varepsilon||^4 < \infty \), and so \( E||Y||^4 < \infty \). Our objective is to show that if the \( X_i \) satisfy the assumptions of Section 2, then

\[
\| \hat{\Psi}_{KL} - \Psi \|_L \xrightarrow{a.s.} 0,
\]

as \( N \to \infty \), and \( K, L \to \infty \) at suitable rates determined by the rate of decay of the eigenvalues. The norm \( \| \cdot \|_L \) is the usual operator norm. The integral operators \( \Psi \) and \( \hat{\Psi}_{KL} \) are defined by their kernels \( \psi(\cdot, \cdot) \) and \( \hat{\psi}_{KL}(\cdot, \cdot) \), respectively. We focus on moment conditions, so we assume that the functions \( X_i, Y_i \) are fully observed, and use the estimator

\[
\hat{\sigma}_{\ell k} = \frac{1}{N} \sum_{i=1}^{N} \hat{\xi}_{i\ell} \hat{\zeta}_{ik}, \quad \hat{\xi}_{i\ell} = \langle X_i, \hat{v}_\ell \rangle, \quad \hat{\zeta}_{ik} = \langle Y_i, \hat{u}_k \rangle.
\]

Since the regression operator \( \Psi \) is infinitely dimensional, we strengthen Assumption 4.2 to the following assumption.

**Assumption 5.1** The eigenvalues \( \lambda_\ell = E[\xi_\ell^2] \) and \( \gamma_\ell = E[\zeta_\ell^2] \) satisfy

\[
\lambda_1 > \lambda_2 > \ldots > 0, \quad \gamma_1 > \gamma_2 > \ldots > 0.
\]

Many issues related to the infinite dimension of the functional data in model (5.1) are already present when considering projections on the unobservable subspaces

\[
\mathcal{Y}_L = \text{span} \{ v_1, v_2, \ldots, v_L \}, \quad \mathcal{U}_K = \text{span} \{ u_1, u_2, \ldots, u_K \}.
\]

Therefore we first consider the convergence of the operator with the kernel

\[
\psi_{KL}(t, s) = \sum_{k=1}^{K} \sum_{\ell=1}^{L} \frac{\sigma_{\ell k}}{\lambda_\ell} u_k(t)v_\ell(s).
\]
Set \( \sigma_{\ell k} = E[\xi_i \xi_j k] \) and observe that

\[
\psi_{KL}(t, s) - \psi(t, s) = - \sum_{k > K \text{ or } \ell > L} \frac{\sigma_{\ell k}}{\lambda_\ell} u_k(t)v_\ell(s).
\]

Therefore

\[
\| \Psi_{KL} - \Psi \|_L^2 \leq \| \Psi_{KL} - \Psi \|_S^2 = \sum_{k > K \text{ or } \ell > L} \frac{\sigma_{\ell k}^2}{\lambda_\ell^2}.
\]

The condition

\[
\sum_{k=1}^\infty \sum_{\ell=1}^\infty \frac{\sigma_{\ell k}^2}{\lambda_\ell^2} < \infty,
\]

which is Assumption (A1) of ?, implies that the remainder term is asymptotically negligible. It is instructive to rewrite condition (5.4) in a different form. Observe that

\[
\sigma_{\ell k} = E[\xi_i \langle \Psi(X) + \varepsilon, u_k \rangle] = E[\xi_i \sum_{i=1}^\infty \xi_i \langle \Psi(v_i), u_k \rangle] = \lambda_\ell \langle \Psi(v_\ell), v_k \rangle.
\]

Therefore

\[
\sum_{k=1}^\infty \sum_{\ell=1}^\infty \frac{\sigma_{\ell k}^2}{\lambda_\ell^2} = \sum_{\ell=1}^\infty \frac{1}{\lambda_\ell^2} \sum_{k=1}^\infty \lambda_\ell^2 \langle \Psi(v_\ell), v_k \rangle^2 = \sum_{\ell=1}^\infty \| \Psi(v_\ell) \|^2 = \| \Psi \|_S^2.
\]

We see that condition (5.4) simply means that \( \Psi \) is a Hilbert–Schmidt operator, and so it holds under our general assumptions on model (5.1).

The last assumption implicitly restricts the rates at which \( K \) and \( L \) tend to infinity with \( N \). Under Assumption 5.1, the following quantities are well defined

\[
\alpha_j = \min \{ \lambda_j - \lambda_{j+1}, \lambda_{j-1} - \lambda_j \}, \quad j \geq 2, \quad \alpha_1 = \lambda_1 - \lambda_2,
\]

\[
\beta_j = \min \{ \gamma_j - \gamma_{j+1}, \gamma_{j-1} - \gamma_j \}, \quad j \geq 2, \quad \beta_1 = \gamma_1 - \gamma_2.
\]

**Assumption 5.2** The truncation levels \( K \) and \( L \) tend to infinity with \( N \) in such a way that for some \( \gamma \in (1, \alpha/2) \),

\[
\limsup_{N \to \infty} \lambda_L^{-3/2} L^{1/2} N^{1/\gamma - 1} < \infty,
\]

\[
\limsup_{N \to \infty} \lambda_L^{-1} \left( \sum_{j=1}^L \alpha_j^{-1} \right) N^{1/\gamma - 1} < \infty,
\]

\[
\limsup_{N \to \infty} \lambda_L^{-1} K^{1/2} N^{1/\gamma - 1} < \infty,
\]

\[
\limsup_{N \to \infty} \lambda_L^{-1} \left\{ \left( \sum_{k=1}^K \beta_k^{-1} \right) + \left( \sum_{k=1}^K \beta_k^{-2} \right)^{1/2} \right\} N^{1/\gamma - 1} < \infty.
\]

The conditions in Assumption 5.2 could be restated or unified; and could be replaced by slightly different conditions by modifying the technique of proof. The essence of this assumption is that \( K \) and \( L \) must tend to infinity sufficiently slowly, and the rate is influenced by index \( \alpha \); the closer \( \alpha \) is to 4, the larger \( \gamma \) can be taken, so \( K \) and \( L \) can be larger.

**Theorem 5.1** Suppose model (5.1) holds with \( \Psi \in S \), the \( X_i \) and the \( Y_i \) satisfying Assumptions 2.1 and 2.2, and square integrable \( \varepsilon_i \), \( E \| \varepsilon_i \| < \infty \). Then relation (5.2) holds under Assumptions 5.1 and 5.2.
Acknowledgements: The authors have been supported by NSF grant “FRG: Collaborative Research: Extreme Value Theory for Spatially Indexed Functional Data” (1462067 CSU, 1462368 Michigan). We thank Professor Jan Rosiński for directing us to the work of ?, and Mr. Ben Zheng for preparing the figures.
6. Proofs of the results stated in the paper

Throughout the proofs, we will use relatively well–known properties of slowly varying functions, which we collect in Lemma 6.1 for ease of reference. For the proofs and many more details, see e.g., ? and ?.

**Lemma 6.1** If \( L \) is a slowly varying function, then:

(i) \( L_1(u) = L(u^\rho), \ \rho > 0 \) and \( L_2(u) = |L(u)|^a, \ a \in \mathbb{R} \) are slowly varying.

(ii) (Potter bounds) For all \( \delta > 0 \), we have \( L(u) = o(u^\delta) \), as \( u \to \infty \).

(iii) (Karamata’s Theorem) For all \( \rho > -1 \) and \( \eta > 1 \), as \( u \to \infty \), we have

\[
\int_0^u x^\rho L(x) dx \sim \frac{u^{\rho+1} L(u)}{\rho + 1} \quad \text{and} \quad \int_u^\infty x^{-\eta} L(x) dx \sim \frac{u^{-(\eta-1)} L(u)}{\eta - 1},
\]

where \( a(u) \sim b(u) \) means \( a(u)/b(u) \to 1 \), as \( u \to \infty \).

6.1. Proofs of Proposition 2.2 and of the results of Section 3

**Proof of Proposition 2.2**

Since \( C \) is a covariance operator, it is nuclear (\( \sum_{j \geq 1} \lambda_j < \infty \)), e.g. Theorem 11.2.2 of ?, and so it is Hilbert–Schmidt (\( \sum_{j \geq 1} \lambda_j^2 < \infty \)).

We now verify that \( \hat{C} \) is a.s. a Hilbert-Schmidt operator. Observe that

\[
\|\hat{C}\|_S^2 = \iint \hat{c}^2(t,s) dt ds = \iint \left\{ \frac{1}{N} \sum_{n=1}^N X_n(t)X_n(s) \right\}^2 dt ds.
\]

It thus suffices to show that

\[
\iint \{X_n(t)X_n(s)\}^2 dt ds = \iint S_n^2(t,s) dt ds < \infty \quad \text{a.s.,}
\]

where

\[
S_n(t,s) = \sum_{j \geq 1} \xi_{nj} v_j(t) \sum_{j' \geq 1} \xi_{nj'} v_{j'}(s).
\]

Observe that

\[
\iint S_n^2(t,s) dt ds = \sum_{j, j' = 1}^{\infty} \sum_{i, i' = 1}^{\infty} \xi_{nj} \xi_{nj'} \xi_{ni} \xi_{ni'} \int v_j(t) v_i(t) dt \int v_{j'}(s) v_{i'}(s) ds.
\]

Therefore, by the orthonormality by the \( v_j \),

\[
\iint S_n^2(t,s) dt ds = \sum_{j, j' = 1}^{\infty} \xi_{nj} \xi_{nj'} \sum_{i, i' = 1}^{\infty} \xi_{ni} \xi_{ni'} = \left\{ \sum_{j \geq 1} \xi_{nj}^2 \right\}^2.
\]

Finally, observe that

\[
\sum_{j = 1}^{\infty} \xi_{nj}^2 = \int_0^1 X_n^2(t) dt = \|X\|^2 < \infty \quad \text{a.s.}
\]

because \( X \) is a random element of \( L^2 \).
Proof of Proposition 3.1

Set

\[ \Gamma(\cdot) = \frac{\sigma_S(\cdot)}{\sigma_S(S_H)} \]  

(6.1)

Recall that (6.1) specifies the relationship between the stable spectral measure \( \sigma_S \) and the angular measure \( \Gamma \) of a regularly varying distribution appearing in Proposition 2.1.

First we assume (3.4) and (3.5) hold. Take \( m = 1 \) in (3.4) and \( A^* = S_H \) in (3.5), we then have for every \( t > 0 \),

\[ \frac{P(\|Z\| > tu, Z/\|Z\| \in A)}{P(\|Z\| > u)} = \frac{P(\|Z\| > tu, Z/\|Z\| \in A^*)}{P(\|Z\| > u)} \frac{P(\|Z\| > tu)}{P(\|Z\| > u)} \]

\[ \xrightarrow{\sigma_S(A) \rightarrow \sigma_S(S_H)} \frac{\Gamma(A)}{\Gamma(A^*)} = \frac{\sigma_S(A)}{\sigma_S(A^*)}, \quad (u \to \infty) \]

for any continuity set \( A \) of \( \sigma_S \) (equivalently, of \( \Gamma \)). Thus condition (ii) in Proposition 2.1 holds, which implies that \( Z \) is regularly varying with index \( p \).

Next we assume that \( Z \) is regularly varying with index \( p \), and show that (3.4) and (3.5) will hold. Using condition (ii) in Proposition 2.1, we have

\[ \frac{P(\|Z\| > u, Z/\|Z\| \in A)}{P(\|Z\| > u)} = \frac{P(\|Z\| > u, Z/\|Z\| \in A^*)}{P(\|Z\| > u)} \frac{P(\|Z\| > u)}{P(\|Z\| > u)} \]

\[ \xrightarrow{\sigma_S(A) \rightarrow \sigma_S(S_H)} \frac{\Gamma(A)}{\Gamma(A^*)} = \frac{\sigma_S(A)}{\sigma_S(A^*)}, \quad (u \to \infty) \]

for all continuity sets \( A, A^* \in \mathcal{B}(S_H) \) with \( \sigma_S(A^*) > 0 \). Then, with the set \( A_m \) defined by (3.9),

\[ \frac{P(\|\pi_m(Z)\| > tu)}{P(\|Z\| > u)} = \frac{P(t^{-1}u^{-1}Z \in A_m)}{P(u^{-1}Z \in A_1)} \]

\[ = \frac{P(\|Z\| > u)}{P(\|Z\| > u)} \frac{P(t^{-1}u^{-1}Z \in A_m)}{P(u^{-1}Z \in A_1)} \frac{P(\|Z\| > tu)}{P(\|Z\| > u)} \]

\[ \xrightarrow{\mu_Z(A_m) \rightarrow \mu_Z(A_1)} = \frac{\mu_Z(A_m)}{\mu_Z(A_1)} \]

\[ \frac{c_m}{c_1}, \quad (u \to \infty) \]

where the above convergence follows from (2.2) provided we can show that \( A_m, m \geq 1 \) are continuity sets of the measure \( \mu_Z \). We do that next.

By the definition of \( A_m \) in (3.9) and since \( \pi_m \) is continuous and homogeneous, we have

\[ \partial A_m = \{ z \in H : \|\pi_m(z)\| = 1 \} \quad \text{and} \quad \partial(rA_m) = r\partial A_m = \{ z \in H : \|\pi_m(z)\| = r \}. \]

Furthermore, we have that \( r_1A_m \supset r_2A_m \) for all \( 0 < r_1 < r_2 \). This implies that \( A_m = \cup_{r>1} \partial(rA_m) \), where the sets \( \partial(rA_m) \) are all disjoint in \( r \). By the homogeneity of \( \mu_Z \), however, (recall (2.3)) it follows that \( \mu_Z(\partial(rA_m)) = r^{-p} \mu_Z(\partial A_m) \). In particular,

\[ \mu_Z(A_m) \geq \sum_i r_i^{-p} \mu_Z(\partial A_m) = \sum_i r_i^{-p} \mu_Z(\partial A_m), \]

for any sequence \( r_i > 1 \). If \( \mu_Z(\partial A_m) > 0 \), then by taking \( r_i \)'s such that \( \sum_i r_i^{-p} = \infty \), we obtain \( \mu_Z(A_m) = \infty \), which is not possible since \( A_m \) is bounded away from zero. We have thus shown that \( \mu_Z(\partial A_m) = 0 \), i.e., \( A_m \) is a continuity set of \( \mu_Z \) for all \( m \geq 1 \).
To complete the proof of (3.4), it remains to show that \( c_m = \mu_Z(A_m) \to 0 \), as \( m \to \infty \). Notice that \( A_m \supset A_{m+1} \) and thus \( \lim_{m \to \infty} \mu_Z(A_m) = \mu_Z(\cap_{m=1}^{\infty} A_m) \), since \( \mu(A_1) < \infty \). It is easy to see that \( \cap_{m=1}^{\infty} A_m = \emptyset \). Indeed, for each \( z \in H \), we have \( \|z\|^2 = \sum_{j=1}^{\infty} \langle z, e_j \rangle^2 < \infty \) and therefore

\[
\|\pi_m(z)\|^2 = \sum_{j=m}^{\infty} \langle z, e_j \rangle^2 \to 0, \quad \text{as } m \to \infty.
\]

If \( z \in \cap_{m \geq 1} A_m \), then \( \|\pi_m(z)\| > 1 \) for each \( m \geq 1 \), which is impossible.

**Proof of Proposition 3.2**

Since \( \|Y\|_{S} = \|X\|^2 \) and \( P(\|X\| > u) = u^{-\alpha} L(u) \), we conclude that

\[
P(\|Y\|_{S} > u) = u^{-\alpha/2} L(u^{1/2}).
\]

Notice that \( u \mapsto L(u^{1/2}) \) is a slowly varying function. Thus, by Proposition 2.1 (iii), to establish the regular variation of \( Y \) it remains to show that there must exist a probability measure \( \Gamma_Y \) on \( S_S \) such that

\[
P\left(\|Y\|_{S}^{-1} Y \in A | \|Y\|_{S} > u\right) \to \Gamma_Y(A), \quad u \to \infty,
\]

for every \( \Gamma_Y \)-continuity set \( A \). The operator \( Y \) takes values only in a small subset of \( S_S \), namely in

\[
S_S(1) = \{ \Psi \in S_S : \Psi = x \otimes x \text{ for some } x \in S_H \}.
\]

The set \( S_S(1) \) is closed in \( S_S \) and its Borel subsets have the form \( B \otimes B \), where \( B \) is a Borel subset of \( S_H \). We know that

\[
\Gamma(u)(B) := P(X/\|X\| \in B | \|X\| > u) \to \Gamma(B), \quad u \to \infty,
\]

for every \( \Gamma \)-continuity set \( B \in S_H \). Denote by \( \xi_u \) a random element of \( H \) taking values in \( S_H \) whose distribution is \( \Gamma(u) \). Then we have

\[
\xi_u \to \xi, \quad u \to \infty, \quad (6.3)
\]

where \( \xi \) has distribution \( \Gamma \). Furthermore, denote by \( \eta_u \) a random element of \( S \) taking values in \( S_S(1) \) whose distribution is

\[
P(\eta_u \in A) = \frac{P\left(\|Y\|_{S}^{-1} Y \in A, \|Y\|_{S} > u\right)}{P(\|Y\|_{S} > u)}, \quad A \in S_S(1).
\]

We want to identify a random element \( \eta \) such that

\[
\eta_u \to \eta, \quad u \to \infty, \quad (6.4)
\]

whose distribution will be the desired measure \( \Gamma_Y \).

We first verify that

\[
\eta_u \overset{d}{=} \xi_{u^{1/2}} \otimes \xi_{u^{1/2}}, \quad (6.5)
\]

Relation (6.5) is equivalent to

\[
P\left(\|Y\|_{S}^{-1} Y \in A, \|Y\|_{S} > u\right) = P(\xi_{u^{1/2}} \otimes \xi_{u^{1/2}} \in A), \quad \forall A \in S_S(1), \quad (6.6)
\]

Set \( A = B \otimes B \). Since \( \|Y\|_{S} = \|X\|^2 \), the left–hand side of (6.6) is

\[
P\left(\|Y\|_{S}^{-1} Y \in A, \|Y\|_{S} > u\right) = \frac{P\left(\|X\|^{-1} X \otimes (\|X\|^{-1} X) \in B \otimes B, \|X\| > u^{1/2}\right)}{P(\|X\| > u^{1/2})}
\]

\[
= \frac{P(\|X\|^{-1} X \in B, \|X\| > u^{1/2})}{P(\|X\| > u^{1/2})}
\]

\[
= \Gamma(u^{1/2})(B).
\]
while the right–hand side of (6.6) is
\[
P (ξ_{n1/2} ⊗ ξ_{n1/2} \in A) = P (ξ_{n1/2} \in B, ξ_{n1/2} \in B) = P (ξ_{n1/2} \in B) = \Gamma(u^{1/2}) (B).
\]
Therefore, (6.5) holds. It remains to show that
\[
η_u \overset{d}{=} ξ_{u1/2} ⊗ ξ_{u1/2} \overset{d}{=} ξ ⊗ ξ := η, \quad u \to \infty.
\]
The above relation holds because by (6.7) and (6.3),
\[
P (ξ_{u1/2} ⊗ ξ_{u1/2} \in A) = \Gamma(u^{1/2}) (B) \to \Gamma(B) = P (ξ \in B) = P (η \in A),
\]
provided B is a continuity set of Γ. Using the relation \(||y ⊗ z||_S = ||y|| \cdot ||z||\), it is easy to check that \(x_n ⊗ x_n \to x ⊗ x\) in S if and only if \(x_n \to x\) in H. Hence, \(\partial A = \partial B ⊗ \partial B\), so the continuity sets of the distribution of η have the form \(B ⊗ B\) with \(Γ(\partial B) = 0\).

**Proof of Theorem 3.2**

By Proposition 3.2, the operators \(X_i ⊗ X_i\) are iid regularly varying elements of \(S\), whose index of regular variation is \(α/2 ∈ (1, 2)\). In order to use Theorem 3.1, we first verify that \(μ_{X ⊗ X}(A_m) > 0\), cf. Proposition 3.1. This is where Assumption 2.2 comes into play. An orthonormal basis of \(L(X ⊗ X)\) is \(\{v_i ⊗ v_j, i, j ≥ 1\}\), where the \(v_j\) are the FPCs of \(X\). Set
\[
A_{n,m} = \{Ψ ∈ S : \left\| \sum_{i=n}^{∞} \sum_{j=m}^{∞} \langle Ψ, v_i ⊗ v_j \rangle_S v_i ⊗ v_j \right\|_S > 1 \}.
\]
We must thus verify that \(μ_{X ⊗ X}(A_{n,m}) > 0\). By (2.2),
\[
μ_{X ⊗ X}(A_{n,m}) = \lim_{u \to ∞} \frac{P(X ⊗ X ∈ uA_{n,m})}{P(\|X ⊗ X\|_S > u)}.
\]
Clearly
\[
P(\|X ⊗ X\|_S > u) = P(\|X\|^2 > u) = P \left( \sum_{j=1}^{∞} ξ_j^2 > u \right),
\]
which is the denominator of \(Q_{nm}\) in Assumption 2.2. Turning to the numerator, observe that \(X ⊗ X ∈ uA_{n,m}\) iff
\[
\left\| \sum_{i=n}^{∞} \sum_{j=m}^{∞} \langle X ⊗ X, v_i ⊗ v_j \rangle_S v_i ⊗ v_j \right\|_S > u.
\]
Direct verification, which uses the definition of the inner product in \(S\) and the orthonormality of the \(v_j\), shows that \(\langle X ⊗ X, v_i ⊗ v_j \rangle_S = ξ_i ξ_j\). It follows that \(X ⊗ X ∈ uA_{n,m}\) iff
\[
\left\| \sum_{i=n}^{∞} \sum_{j=m}^{∞} ξ_i ξ_j v_i ⊗ v_j \right\|^2_S > u^2.
\]
Using the definition of the Hilbert–Schmidt norm and the orthogonality of the \(v_j\) again, we see that the above inequality is equivalent to \(\sum_{i=n}^{∞} ξ_i^2 \sum_{j=m}^{∞} ξ_j^2 > u^2\), so \(P(X ⊗ X ∈ uA_{n,m})\) is equal to the numerator of \(Q_{nm}\).

It remains to show that the normalizing sequences can be chosen as specified in (3.11). It is easy to check that \(k_N \to ∞\) and \(k_N \to 1\). We will show that
\[
N k_N^{-2} E \left( \left\| X \right\|^4 I_{\left\{ \left\| X \right\|^2 ≤ k_N \right\}} \right) \to 1,
\]
(6.8)
which in view of (3.6) would yield (3.10), where the spectral measure of the limit $S$ is normalized so that $\lambda_\rho S(S) = 1$ with $\lambda_\rho$ in (3.7).

Observe that by the Tonelli-Fubini Theorem, we have

$$E\left[\|X\|^4 I_{\|X\|^2 \leq k_N}\right] = E\left[\int_0^\infty I_{x<\|X\|^4 \leq k_N^2} dx\right]$$

$$= \int_0^{k_N^2} [P(\|X\|^4 > x) - P(\|X\|^2 > k_N)] dx$$

$$= \int_0^{k_N^2} x^{-\alpha/4} L(x^{1/4}) dx - k_N^2 k_N^{-\alpha/2} L(k_N^{1/2}),$$

where we used the fact that $P(\|X\| > x) = x^{-\alpha} L(x)$. Now, by applying Karamata’s theorem (Lemma 6.1 (iii)) to the integral in the last expression, we obtain

$$E\left[\|X\|^4 I_{\|X\|^2 \leq k_N}\right] \sim \frac{1}{(1-\alpha/4)} k_N^{2-\alpha/2} L(k_N^{1/2}) - k_N^{2-\alpha/2} L(k_N^{1/2})$$

$$= \left(\frac{4}{(4-\alpha)} - 1\right) k_N^{2-\alpha/2} L(k_N^{1/2}) = \frac{\alpha}{(4-\alpha)} k_N^{2-\alpha/2} L(k_N^{1/2}),$$

(6.9)

as $k_N \to \infty$, where $c_N \sim d_N$ means that $c_N/d_N \to 1$.

In view of (2.5) by taking $A = \{x : \|x\| > 1\}$, we obtain

$$NP(\|X\| > a_N) = Na_N^{-\alpha} L(a_N) \to 1,$$

(6.10)

since $\mu$ is normalized so that $\mu(A) = 1$ and $\mu(\partial A) = 0$ by Proposition 2.2.2 of ?. Thus, multiplying (6.9) by $N k_N^{-2}$ and recalling (3.11), we obtain

$$N k_N^{-2} \frac{\alpha}{(4-\alpha)} k_N^{2-\alpha/2} L(k_N^{1/2}) = a_N^{-\alpha} L(c_\alpha a_N),$$

where $c_\alpha = (\alpha/(4-\alpha))^{1/\alpha}$. Since $L$ is a slowly varying function, we have $L(c_\alpha a_N) \sim L(a_N)$ as $a_N \to \infty$, and therefore by (6.10), we obtain (6.8). This completes the proof.

Proof of Theorem 3.3

Observe that by (3.1),

$$N k_N^{-1} \left(\hat{C} - C\right) = k_N^{-1} \left(\sum_{n=1}^N X_n \otimes X_n - \psi_N\right) + k_N^{-1} N E \left[(X \otimes X) I_{\|X\|^2 > k_N}\right],$$

(6.11)

with $k_N$ and $\psi_N$ as in Theorem 3.2. The first term converges to $S$, so we must verify the existence of the second term, show that it converges, and describe its limit. The issue is subtle because $k_N \to \infty$ implies that $k_N^{-1} N \left[(X \otimes X) I_{\|X\|^2 > k_N}\right] \to 0$ with probability 1, yet the expected value does not tend to zero even in the case of scalar observations, see Theorem 2.2 of ?. It is convenient to approach the problem in a slightly more general setting.

Suppose $Y$ is a regularly varying element of a separable Hilbert space whose index of regular variation is $p$, $p \in (1, 2)$. In our application, $Y = X \otimes X$, the Hilbert space is $S$ and $p = \alpha/2$. Denote by $\mu_Y$ the exponent measure of $Y$ and by $u_N$ a regularly varying sequence such that $NP(\|Y\| > u_N) \to 1$, so that

$$\mu_{N\gamma}(A) := \frac{P(Y \in u_N A)}{P(\|Y\| > u_N)} \to \mu_Y(A),$$

(6.12)
with the usual restrictions on the set $A$, cf. Proposition 2.1. Set

$$Y_N = u_N^{-1}NYI_{\{\|Y\| > u_N\}}$$

and observe that $E[|Y_N|]$ exists in the sense of Bochner. Indeed, by (2.1) and the Potter bounds (Lemma 6.1), we have

$$P(\|Y\| > u) = u^{-p}L(u) = o(u^{-p+\delta}), \quad \text{as } u \to \infty,$$

for an arbitrarily small $\delta > 0$. Since $p \in (1,2)$, by taking $p - \delta > 1$, we obtain $E[\|Y\|] = \int_0^\infty P(\|Y\| > y)dy < \infty$ and the expectation of $Y$ and hence $Y_N$ is well-defined.

Now set $M_N = E[|Y_N|]$. We want to identify $M \in H$ such that $\|M_N - M\| \to 0$. We will show that the above convergence holds with

$$M = \int_{B^c} y\mu_Y(dy), \quad (6.13)$$

where $B = \{y : \|y\| \leq 1\}$. Recall that $Y$ is regularly varying and by (2.4) its exponent and angular measures are related as follows

$$\mu_Y(dy) = pr^{-p-1}dr\Gamma_Y(d\theta), \quad (6.14)$$

where $r := \|y\|$ and $\theta := y/\|y\|$ are polar coordinates in $H$. Thus, in polar coordinates, we obtain

$$\int_{B^c} \|y\|\mu_Y(dy) = \int_1^\infty \int_{S} r\|\theta\|\Gamma_Y(d\theta) pr^{-p-1}dr = \left( p \int_1^\infty r^{-p}dr \right) \int_{S} \|\theta\|\Gamma_Y(d\theta) = \frac{p}{p-1}. \quad (6.15)$$

This shows that the Bochner integral in (6.13) is well defined and in fact equals

$$M = \frac{p}{p-1} \int_{S} \theta\Gamma_Y(d\theta).$$

In view of Remark 3.2, by taking $Y = X \otimes X$ and $p = \alpha/2$, we then obtain

$$M = \frac{\alpha}{\alpha - 2} \int_{S_H} (\theta \otimes \theta) \Gamma_X(d\theta),$$

which is the expression for the offset in (3.12).

Observe that by the definition (6.12) of $\mu_{N,Y}$, since $NP(\|Y\| > u_N) \to 1$, for any Bochner integrable mapping of the Hilbert space into itself, or to the real line,

$$NE[f(u_N^{-1}Y)] \sim \int f(y)\mu_{N,Y}(dy). \quad (6.16)$$

Therefore,

$$M_N = NE\left[u_N^{-1}YI_{B^c}(u_N^{-1}Y)\right] \sim \int_{B^c} y\mu_{N,Y}(dy).$$

Observe that $\mu_{N,Y}(B^c) = 1$, and by (6.14),

$$\mu_Y(B^c) = \int_1^\infty \int_{S} pr^{-p-1}dr\Gamma_Y(d\theta) = \sigma_Y(S) = 1.$$

Thus $\mu_{N,Y}$ and $\mu_Y$ are probability measures on $B^c$, and we want to show that

$$\int_{B^c} y\mu_{N,Y}(dy) \to \int_{B^c} y\mu_Y(dy).$$
Since $\mu_{N,Y}$ converges weakly to $\mu_Y$, it suffices to verify that
\[
\sup_{N \geq 1} \int_{B^c} \|y\|^{1+\delta} \mu_{N,Y}(dy) < \infty, \tag{6.17}
\]
for some $\delta > 0$ (this implies strong uniform integrability). Observe that by (6.16),
\[
\int_{B^c} \|y\|^{1+\delta} \mu_{N,Y}(dy) = NE \left[ \|u_N^{-1}Y\|^{1+\delta} I_{B^c}(u_N^{-1}Y) \right] = Nu_N^{-1-\delta} E_N(\delta), \tag{6.18}
\]
where
\[
E_N(\delta) = E \left[ \|Y\|^{1+\delta} I_{\{\|Y\| > u_N\}} \right].
\]
By the Tonelli–Fubini theorem, we have
\[
E_N(\delta) = E \left( \int_{u_N^{1+\delta}}^{\infty} P(\|Y\|^{1+\delta} > x) \, dx \right) = \int_{u_N^{1+\delta}}^{\infty} P(\|Y\|^{1+\delta} > x) \, dx
\]
Now, by picking $\delta > 0$ such that $\eta := p/(1 + \delta) > 1$ and applying the Karamata Theorem (Lemma 6.1(iii)), for the right-hand side of (6.18), we obtain
\[
Nu_N^{-1-\delta} E_N(\delta) \sim Nu_N^{-1-\delta} \frac{1}{\eta - 1} \left( u_N^{1/(1+\delta)} \right)^{1-p/(1+\delta)} L(u_N)
\]
\[
\sim \frac{1}{\eta - 1} Nu_N^{-p} L(u_N) = \frac{1}{\eta - 1} NP(\|Y\| > u_N) \rightarrow \frac{1}{\eta - 1},
\]
where the last convergence follows from the definition of the sequence $u_N$. This shows that the supremum in (6.17) is finite, which completes the proof.

6.2. Proofs of the results of Section 4

Proof of Theorem 4.1

The results of this section require Assumptions 4.1 and 4.2.

Before stating Theorem 4.1, we referred to Lemma 6.2 which ensures that the the series
\[
T_{j,N} = \sum_{k \neq j} (\lambda_j - \lambda_k)^{-1} \langle Z_N, v_j \otimes v_k \rangle v_k;
\]
\[
T_j = \sum_{k \neq j} (\lambda_j - \lambda_k)^{-1} \langle Z, v_j \otimes v_k \rangle v_k.
\]
converge a.s. in $L^2$. These series play a fundamental role in our arguments.

Lemma 6.2 Suppose $\Psi \in \mathcal{S}$. For $1 \leq j \leq p$, set
\[
g_j(\Psi) = \sum_{k \neq j} (\lambda_j - \lambda_k)^{-1} \langle \Psi, v_j \otimes v_k \rangle v_k.
\]
Then, the series defining $g_j(\Psi)$ converges in $L^2$. 

\[\text{imsart-bj ver. 2014/10/16 file: rv.tex date: December 10, 2018} \]
Proof: Since the $v_k$ are orthonormal, it is enough to check that
\[
\sum_{k \neq j} (\lambda_j - \lambda_k)^{-2} \langle \Psi, v_j \otimes v_k \rangle^2 < \infty.
\]
Since the system \( \{v_j \otimes v_k, j,k \geq 1\} \) forms an orthonormal basis in \( S \)
\[
\sum_{j,k \geq 1} \langle \Psi, v_j \otimes v_k \rangle^2 = \|\Psi\|_S^2 < \infty.
\]
Therefore,
\[
\sum_{k \neq j} (\lambda_j - \lambda_k)^{-2} \langle \Psi, v_j \otimes v_k \rangle^2 \leq \alpha_j^{-2} \|\Psi\|_S^2,
\]
with \( \alpha_j \) defined in (5.7).

We will use the following lemma, which is analogous to Lemma 1 in [?], whose fully analogous proof, based on algebraic manipulations, is omitted.

**Lemma 6.3** For any \( j \geq 1 \),
\[
\langle \hat{v}_j - v_j, v_j \rangle = -\frac{1}{2} \| \hat{v}_j - v_j \|^2.
\]
For any \( j,k \geq 1 \) such that \( j \neq k \) and \( \hat{\lambda}_j \neq \lambda_k \),
\[
\langle \hat{v}_j - v_j, v_k \rangle = r_N^{-1} (\hat{\lambda}_j - \lambda_k)^{-1} \langle Z_N, \hat{v}_j \otimes v_k \rangle.
\]

By Assumption 4.1, \( \| \hat{C} - C \|_S = O_P(r_N^{-1}) \). Using the well–known inequalities
\[
|\hat{\lambda}_j - \lambda_j| \leq \| \hat{C} - C \|_S, \quad \| \hat{v}_j - v_j \| \leq 2 \sqrt{2} \alpha_j \| \hat{C} - C \|_S,
\]
(see e.g. Lemmas 2.2 and 2.3 in [?]), we obtain the following Lemma.

**Lemma 6.4** For \( 1 \leq j \leq p \),
\[
\| \hat{C} - C \|_S = O_P(r_N^{-1}), \quad |\hat{\lambda}_j - \lambda_j| = O_P(r_N^{-1}), \quad \| \hat{v}_j - v_j \| = O_P(r_N^{-1}).
\]

**Lemma 6.5** For \( 1 \leq j \leq p \),
\[
\| r_N (\hat{v}_j - v_j) - T_j,N \| = O_P \left( r_N^{-1} \right).
\]

Proof: The same arguments apply to any fixed \( j \in \{1,2,\ldots,p\} \), so to reduce the number of indexes used, we present them for \( j = 1 \). Set
\[
d_{N,k} = \langle r_N (\hat{v}_1 - v_1) - T_{1,N}, v_k \rangle,
\]
where
\[
T_{1,N} = \sum_{\ell \geq 2} (\lambda_1 - \lambda_\ell)^{-1} \langle Z_N, v_1 \otimes v_\ell \rangle v_\ell.
\]
By Parseval’s identity,
\[
\| r_N (\hat{v}_j - v_j) - T_{j,N} \|^2 = \sum_{k=1}^{\infty} d_{N,k}^2.
\]
Focusing on the first term, \( k = 1 \), observe that
\[
\langle T_{1,N}, v_1 \rangle = \sum_{\ell \geq 2} (\lambda_1 - \lambda_\ell)^{-1} \langle Z_N, v_1 \otimes v_\ell \rangle \langle v_\ell, v_\ell \rangle = 0
\]
and, by Lemmas 6.3 and 6.4,
\[ \langle r_N(\hat{v}_1 - v_1), v_1 \rangle = -\frac{r_N}{2} \|\hat{v}_1 - v_1\|^2 = O_P(r_N^{-1}). \]

We conclude that \(d_{N,1}^2 O_P(r_N^{-2})\), and it remain to show that
\[ \sum_{k=2}^{\infty} d_{N,k}^2 = O_P(r_N^{-2}). \tag{6.19} \]

In the remainder of the proof it is assumed that \(k \geq 2\). Since
\[ \langle T_{1,N}, v_k \rangle = (\lambda_1 - \lambda_k)^{-1} \langle Z_N, v_1 \otimes v_k \rangle, \]
by Lemma 6.3,
\[ d_{N,k} = (\hat{\lambda}_1 - \lambda_k)^{-1} \langle Z_N, \hat{v}_1 \otimes v_k \rangle - (\lambda_1 - \lambda_k)^{-1} \langle Z_N, v_1 \otimes v_k \rangle. \]

Using a common denominator and rearranging the numerator, we obtain
\[ d_{N,k} = \frac{\langle (\lambda_1 - \lambda_k)Z_N(\hat{v}_1 - v_1) + (\lambda_1 - \hat{\lambda}_1)Z_N(v_1), v_k \rangle}{(\lambda_1 - \lambda_k)^2(\lambda_1 - \lambda_k)^2}. \]

It is convenient to decompose the sum in (6.19) as
\[ \sum_{k=2}^{\infty} d_{N,k}^2 = D_{N,1} + D_{N,2} + D_{N,3}, \]
where
\[ D_{N,1} = \sum_{k \geq 2} \frac{\langle Z_N(\hat{v}_1 - v_1), v_k \rangle^2}{(\hat{\lambda}_1 - \lambda_k)^2}, \]
\[ D_{N,2} = \sum_{k \geq 2} \frac{2(\lambda_1 - \hat{\lambda}_1) \langle Z_N(\hat{v}_1 - v_1), v_k \rangle \langle Z_N(v_1), v_k \rangle}{(\hat{\lambda}_1 - \lambda_k)^2(\lambda_1 - \lambda_k)}, \]
\[ D_{N,3} = \sum_{k \geq 2} \frac{(\lambda_1 - \hat{\lambda}_1)^2 \langle Z_N(v_1), v_k \rangle^2}{(\lambda_1 - \lambda_k)^2(\lambda_1 - \lambda_k)^2}. \]

Since \(\hat{\lambda}_1 - \lambda_k \geq \hat{\lambda}_1 - \lambda_2\), by Parseval’s identity,
\[ D_{N,1} \leq \frac{1}{(\hat{\lambda}_1 - \lambda_2)^2} \sum_{k \geq 2} \langle Z_N(\hat{v}_1 - v_1), v_k \rangle^2 \leq \frac{\|Z_N(\hat{v}_1 - v_1)\|^2}{(\hat{\lambda}_1 - \lambda_2)^2}. \]

By Lemma 6.4, the denominator converges in probability to \((\lambda_1 - \lambda_2)^2\), and the numerator is bounded above by \(\|Z_N\|^2\|\hat{v}_1 - v_1\|^2 = O_P(r_N^{-2})\).

A similar argument shows that
\[ |D_{N,2}| \leq \frac{2(\lambda_1 - \hat{\lambda}_1)}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_2)} |\langle Z_N(\hat{v}_1 - v_1), Z_N(v_1) \rangle|. \]

The denominator again converges to a positive constant. By the Cauchy–Schwarz inequality,
\[ |\langle Z_N(\hat{v}_1 - v_1), Z_N(v_1) \rangle| \leq \|Z_N(\hat{v}_1 - v_1)\| \|Z_N(v_1)\| \leq \|Z_N\|^2 \|\hat{v}_1 - v_1\|. \]
We see that $D_{N,2} = O_P(r_N^{-2})$.

The above method also shows that $D_{N,3} = O_P(r_N^{-2})$. 

**Proof of Theorem 4.1:** To prove the first relation, we use the decomposition

$$
r_N(\hat{v}_j - v_j) = T_{j,N} + (r_n(\hat{v}_j - v_j) - T_{j,N}).$$

By Lemma 6.5, it suffices to show that the $T_{j,n}$ converge jointly in distribution to the $T_j$. Consider the operator $g : S \to (L^2)^p$ defined by

$$g(\Psi) = [g_1(\Psi), g_2(\Psi), \ldots, g_p(\Psi)]^T,$$

with the functions $g_j$ defined in Lemma 6.2. The proof of Lemma 6.2 shows that $\|g_j(\Psi)\| \leq \alpha_j^{-1}\|\Psi\|_S$, so each $g_j$ is a continuous linear operator. Hence $g$ is continuous, and so $g(Z_N) \overset{d}{\to} g(Z)$. Since, $g_j(Z_N) = T_{j,N}$ and $g_j(Z) = T_j$, the required convergence follows.

Now we turn to the convergence of the eigenvalues. We will derive an analogous decomposition,

$$r_N(\hat{\lambda}_j - \lambda_j) = (Z_N(v_j), v_j) + \beta_N(j),$$

and show that for each $j = 1, 2, \ldots, p$, $\beta_N(j) = O_P(r_N^{-1})$. Since the projections

$$S \ni \Psi \mapsto (\Psi(v_j), v_j) = (\Psi, v_j \otimes v_j)_S$$

are continuous, the claim will follow.

Observe that

$$(\hat{\lambda}_j - \lambda_j)v_j = \hat{\lambda}_j v_j - \hat{\lambda}_j \hat{v}_j + \hat{\lambda}_j \hat{v}_j - \lambda_j v_j$$

$$= \hat{\lambda}_j (v_j - \hat{v}_j) + \hat{C}(\hat{v}_j) - C(v_j)$$

$$= (\hat{C} - C)(\hat{v}_j) + C(\hat{v}_j - v_j) - \hat{\lambda}_j (\hat{v}_j - v_j).$$

It follows that

$$r_N(\hat{\lambda}_j - \lambda_j)v_j = Z_N(\hat{v}_j) + r_N\left\{C(\hat{v}_j - v_j) - \hat{\lambda}_j (\hat{v}_j - v_j)\right\}.$$ 

We decompose the first term as $Z_N(\hat{v}_j) = Z_N(v_j) + Z_N(\hat{v}_j - v_j)$ and get (6.20) with

$$\beta_N(j) = (Z_N(\hat{v}_j - v_j), v_j) + r_N\left\{C(\hat{v}_j - v_j) - \hat{\lambda}_j (\hat{v}_j - v_j), v_j\right\}$$

$$= r_N\left\{[(\hat{C} - C) + C - \hat{\lambda}_j] (\hat{v}_j - v_j), v_j\right\}$$

$$= r_N\left\{[(\hat{C} - C) + (C - \lambda_j) - (\hat{\lambda}_j - \lambda_j)] (\hat{v}_j - v_j), v_j\right\}.$$ 

By Lemma 6.4,

$$\left\langle (\hat{C} - C)(\hat{v}_j - v_j), v_j \right\rangle = O_P(r_N^{-2})$$

and

$$\left\langle (\hat{\lambda}_j - \lambda_j)(\hat{v}_j - v_j), v_j \right\rangle = O_P(r_N^{-2}).$$

Since $C$ is symmetric

$$\langle (C - \lambda_j)(\hat{v}_j - v_j), v_j \rangle = (\hat{v}_j - v_j, (C - \lambda_j)(v_j)) = 0.$$ 

This shows that $\beta_N(j) = O_P(r_N^{-1})$, and completes the proof.
**Proof of Theorem 4.2**

We start with a simple lemma, custom formulated for our needs.

**Lemma 6.6** Suppose \( \{X_n\} \) and \( \{Y_n\} \) are sequences of nonnegative random variables and \( \{a_n\} \) is a convergent sequence of nonnegative numbers. Suppose \( X_n \leq Y_n + a_n \). If the \( Y_n \) are uniformly integrable, then so are the \( X_n \).

**Proof:** We will establish a more general result under the assumption that \( C := \sup_{n \in \mathbb{N}} a_n < \infty \). Recall that a sequence \( \{X_n\} \) is uniformly integrable if and only if the following two conditions hold

(i) We have \( \sup_{n \in \mathbb{N}} E|X_n| < \infty \).

(ii) For all \( \epsilon > 0 \), there exists a \( \delta > 0 \), such that \( \sup_{n \in \mathbb{N}} E(|X_n|1_A) < \epsilon \), for all events such that \( P(A) < \delta \) (see, e.g., Theorem 6.5.1 on page 184 in ?).

Since \( \{Y_n\} \) is uniformly integrable, we have \( \sup_{n \in \mathbb{N}} E|Y_n| < \infty \) and Condition (i) above follows from the triangle inequality and the boundedness of the sequence \( \{a_n\} \). To show that Condition (ii) holds, observe that by the triangle inequality

\[
\sup_{n \in \mathbb{N}} E(|X_n|1_A) \leq \sup_{n \in \mathbb{N}} E(|Y_n|1_A) + CP(A).
\]

Using the uniform integrability of \( \{Y_n\} \), for every \( \epsilon > 0 \), one can find \( \delta' > 0 \) such that the first term in the right-hand side of (6.21) is less than \( \epsilon/2 \), provided \( P(A) < \delta' \). By setting \( \delta := \min\{\delta', \epsilon/(2C)\} \), we also ensure that the second term therein is less than \( \epsilon/2 \) for all \( P(A) < \delta \leq \delta' \). This completes the proof of the uniform integrability of \( \{X_n\} \).

In the following, we assume that \( \gamma \) is a fixed number in \((0, \alpha/2)\). Theorem 6.1 of ? implies that, in the notation of Theorem 3.1, cf. (3.3),

\[
\lim_{N \to \infty} E\left\| b_N^{-1} \left( \sum_{i=1}^{N} Z_i - \gamma N \right) \right\|^\gamma = E \left\| S \right\|^\gamma.
\]

Applying the above result to (3.10), we obtain

\[
\lim_{N \to \infty} E \left\| S_N \right\|^\gamma = E \left\| S \right\|^\gamma,
\]

where

\[
S_N = k_N^{-1} \left( \sum_{i=1}^{N} X_i \otimes X_i - \psi_N \right).
\]

In the framework of Theorem 3.3, set

\[
M = \int_{\mathbb{B}_S} y \mu_X \otimes X(dy)
\]

and

\[
M_N = k_N^{-1} NE \left[ (X \otimes X) I_{\|X\|^2 \geq k_N} \right],
\]

so that (6.11) becomes

\[
N k_N^{-1} \left( \hat{C} - C \right) = S_N - M_N.
\]
with $S_N \xrightarrow{d} S$ and $\|M_N - M\|_S \to 0$. We now explain why we can conclude that
\[
E \|S_N - M_N\|_S^2 \to E \|S - M\|_S^2.
\] (6.23)

Since $S_N - M_N \xrightarrow{d} S - M$ in $S$, $\|S_N - M_N\|_S^2 \to \|S - M\|_S^2$ in $\mathbb{R}$. Convergence (6.23) will follow if we can assert that the nonnegative random variables $\|S_N - M_N\|_S^2$ are uniformly integrable. Since $\|S_N\|_S^2 \to \|S\|_S^2$ and (6.22) holds, Theorem 3.6 in ? implies that the random variables $\|S_N\|_S^2$ are uniformly integrable. Relation (6.23) thus follows from the inequality
\[
\|S_N - M_N\|_S^2 \leq C_\gamma \{\|S_N\|_S^2 + \|M_N\|_S^2\}
\]
and Lemma 6.6. Relation (6.23) implies the first relation in Theorem 4.2 with $L_\gamma(N) = L_0^{-\gamma}(N)$.

Since $|\hat{\lambda}_j - \lambda_j| \leq \|\hat{C} - C\|_S$ (see e.g. Lemma 2.2 in ?), the second relation follows from the first. Under Assumption 4.2, $\|\hat{v}_j - v_j\| \leq a_j\|\hat{C} - C\|_S$ (see e.g. Lemma 2.3 in ? or Lemma 4.3 in ?), so the third relation also follows from the first.

### 6.3. Proof of Theorem 5.1

Since $\|\Psi_{KL} - \Psi\|_\mathcal{L} \to 0$ by (5.3) and (5.4), it is enough to show that
\[
\left\|\hat{\Psi}_{KL} - \Psi_{KL}\right\|_\mathcal{L} \overset{a.s.}{\to} 0.
\] (6.24)

The operators $\Psi_{KL}$ and $\hat{\Psi}_{KL}$ have the following expansions:
\[
\hat{\Psi}_{KL}(x) = \sum_{k=1}^{K} \sum_{\ell=1}^{L} \frac{\hat{\sigma}_{\ell k}}{\lambda_\ell} \langle \hat{v}_\ell, x \rangle \hat{u}_k, \quad \Psi_{KL}(x) = \sum_{k=1}^{K} \sum_{\ell=1}^{L} \frac{\sigma_{\ell k}}{\lambda_\ell} \langle v_\ell, x \rangle u_k.
\]

Introduce the sample analogs of the subspaces $\mathcal{V}_L$ and $\mathcal{U}_K$,
\[
\hat{\mathcal{V}}_L = \text{span} \{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_L\}, \quad \hat{\mathcal{U}}_K = \text{span} \{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_K\},
\]
and consider the following projections:
\[
\pi^L = \text{projection onto } \mathcal{V}_L, \quad \hat{\pi}^L = \text{projection onto } \hat{\mathcal{V}}_L;
\]
\[
\pi^K = \text{projection onto } \mathcal{U}_K, \quad \hat{\pi}^K = \text{projection onto } \hat{\mathcal{U}}_K.
\]

Observe that
\[
\hat{\Psi}_{KL} = \hat{\pi}^K D_N \hat{C}^{-1} \hat{\pi}^L, \quad \Psi_{KL} = \pi^K D C^{-1} \pi^L,
\]
where
\[
D = E \left[ X \otimes Y \right], \quad D_N = \frac{1}{N} \sum_{i=1}^{N} X_i \otimes Y_i,
\]
and
\[
C = \sum_{j=1}^{\infty} \lambda_j v_j \otimes v_j, \quad \hat{C} = \sum_{j=1}^{\infty} \lambda_j \hat{v}_j \otimes \hat{v}_j, \quad C^{-1} = \sum_{j=1}^{\infty} \lambda_j^{-1} v_j \otimes v_j, \quad \hat{C}^{-1} = \sum_{j=1}^{\infty} \lambda_j^{-1} \hat{v}_j \otimes \hat{v}_j.
\]

Notice that for any $y = \pi^L(x)$ or $y = \hat{\pi}^L(x)$, $C^{-1}(y)$ and $\hat{C}^{-1}(y)$ exist.
For \( x \in L^2 \), consider the decomposition

\[
\left( \hat{\Psi}_{KL} - \Psi_{KL} \right)(x) = \hat{\pi}^K D_N \left( \sum_{j=1}^L \lambda_j^{-1} \langle \hat{v}_j, x \rangle \hat{v}_j \right) - \pi^K D \left( \sum_{j=1}^L \lambda_j^{-1} \langle v_j, x \rangle v_j \right)
\]

\[
= \hat{\pi}^K D_N \left( \sum_{j=1}^L \left( \lambda_j^{-1} - \lambda_j^{-1} \right) \langle \hat{v}_j, x \rangle \hat{v}_j \right) + \hat{\pi}^K D_N \left( \sum_{j=1}^L \lambda_j^{-1} \langle \hat{v}_j - v_j, x \rangle \hat{v}_j \right) + \hat{\pi}^K D_N \left( \sum_{j=1}^L \lambda_j^{-1} \langle v_j, x \rangle (\hat{v}_j - v_j) \right) + \left( \hat{\pi}^K D_N - \pi^K D \right) \left( \sum_{j=1}^L \lambda_j^{-1} \langle v_j, x \rangle v_j \right)
\]

\[
= : a_N(x) + b_N(x) + c_N(x) + d_N(x),
\]

where

\[
a_N(x) = \hat{\pi}^K D_N \left( \sum_{j=1}^L \left( \lambda_j^{-1} - \lambda_j^{-1} \right) \langle \hat{v}_j, x \rangle \hat{v}_j \right),
\]

\[
b_N(x) = \hat{\pi}^K D_N \left( \sum_{j=1}^L \lambda_j^{-1} \langle \hat{v}_j - v_j, x \rangle \hat{v}_j \right),
\]

\[
c_N(x) = \hat{\pi}^K D_N \left( \sum_{j=1}^L \lambda_j^{-1} \langle v_j, x \rangle (\hat{v}_j - v_j) \right),
\]

\[
d_N(x) = \left( \hat{\pi}^K D_N - \pi^K D \right) \left( \sum_{j=1}^L \lambda_j^{-1} \langle v_j, x \rangle v_j \right).
\]

Relation (6.24) will follow from Lemmas 6.8, 6.9, 6.10 and 6.13. The first two of these lemmas use the following result.

**Lemma 6.7** Under the assumptions of Theorem 5.1,

\[
\| \hat{\pi}^K D_N(\hat{v}_j) \| \leq \hat{\lambda}_j^{1/2} \left( \frac{1}{N} \sum_{i=1}^N \| Y_i \|^2 \right)^{1/2}.
\]
PROOF: For each integer \( \ell \), we have

\[
| \langle \hat{\pi}^K D_N(\hat{v}_j), \hat{u}_\ell \rangle | = \left| \left( \sum_{k=1}^{K} \frac{1}{N} \sum_{i=1}^{N} (X_i, \hat{v}_j) \langle Y_i, \hat{u}_k \rangle \hat{u}_k, \hat{u}_\ell \right) \right|
\]

\[
= \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, \hat{v}_j \rangle \langle Y_i, \hat{u}_\ell \rangle \right|
\]

\[
\leq \frac{1}{N} \left( \sum_{i=1}^{N} \langle X_i, \hat{v}_j \rangle^2 \right)^{1/2} \left( \sum_{i=1}^{N} \langle Y_i, \hat{u}_\ell \rangle^2 \right)^{1/2}
\]

\[
= \left( \langle \hat{\mathcal{C}}(\hat{v}_j), \hat{v}_j \rangle \right)^{1/2} \left( \langle \hat{\mathcal{C}}_Y(\hat{u}_\ell), \hat{u}_\ell \rangle \right)^{1/2}
\]

\[
= \hat{\lambda}_j^{1/2} \hat{\gamma}_\ell^{1/2}. \quad (\hat{\gamma}_\ell = \langle \hat{\mathcal{C}}_Y(\hat{u}_\ell), \hat{u}_\ell \rangle)
\]

Therefore,

\[
\| \hat{\pi}^K D_N(\hat{v}_j) \| = \sum_{\ell=1}^{\infty} \langle \hat{\pi}^K D_N(\hat{v}_j), \hat{u}_\ell \rangle^2 \leq \hat{\lambda}_j \sum_{\ell=1}^{\infty} \hat{\gamma}_\ell,
\]

and

\[
\sum_{\ell=1}^{\infty} \hat{\gamma}_\ell = \sum_{\ell=1}^{\infty} \left( \frac{1}{N} \sum_{i=1}^{N} \langle Y_i, \hat{u}_\ell \rangle^2 \right) = \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2.
\]

Hence the claim holds. \( \square \)

LEMMA 6.8 Under the assumptions of Theorem 5.1, \( \| a_N \|_L \xrightarrow{a.s.} 0 \).

PROOF: Observe that

\[
\| a_N(x) \| = \left\| \hat{\pi}^K D_N \left( \sum_{j=1}^{L} \hat{\lambda}_j^{-1} \hat{v}_j, x \right) \hat{v}_j \right\|
\]

\[
\leq \sum_{j=1}^{L} \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j \lambda_j} \| \hat{v}_j, x \| \| \hat{\pi}^K D_N(\hat{v}_j) \|.
\]

By Lemma 6.7, Lemma 2.2 of \( ? \) and the Cauchy-Schwarz inequality, we obtain the bound

\[
\| a_N(x) \| \leq \sum_{j=1}^{L} \lambda^{-1/2}_j \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \right)^{1/2} \| \hat{\mathcal{C}} - C \|_L
\]

\[
\leq \lambda^{-1/2}_L \lambda^{-1/2}_j \| x \| L^{1/2} \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \right)^{1/2} \| \hat{\mathcal{C}} - C \|_L.
\]

By Corollary 3.1, for \( N > N_1 \) (random),

\[
\hat{\lambda}_L \geq \lambda_L - \| \hat{\mathcal{C}} - C \|_L \geq \lambda_L/2.
\]

Then we have

\[
\| a_N \|_L \leq \sqrt{2} \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \right)^{1/2} \lambda^{-3/2}_L \lambda^{1/2}_L \| \hat{\mathcal{C}} - C \|_L.
\]
Corollary 3.1 implies that, for any $\gamma \in (1, \alpha/2)$, $N^{1-1/\gamma} \| \hat{C} - C \|_S \overset{a.s.}{\to} 0$, and by the strong law of large numbers
\[
\frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \overset{a.s.}{\to} E \| Y \|^2 \leq 2 \left( \| \Psi \|_2^2 E \| X \|^2 + E \| \epsilon \|^2 \right) < \infty.
\]
The claim thus follows from condition (5.9).

**Lemma 6.9** Under the assumptions of Theorem 5.1, $\| b_N \|_L \overset{a.s.}{\to} 0$.

**Proof:** Lemma 6.7 implies that
\[
\| b_N(x) \| = \left\| \hat{\pi}^K D_N \left( \sum_{j=1}^{L} \lambda_j^{-1} (\hat{v}_j - v_j, x) \hat{v}_j \right) \right\|
\leq \sum_{j=1}^{L} \lambda_j^{-1} \| \hat{v}_j - v_j, x \| \| \hat{\pi}^K D_N(\hat{v}_j) \|
\leq \sum_{j=1}^{L} \lambda_j^{-1} \hat{\lambda}_j^{1/2} \| x \| \| \hat{v}_j - v_j \| \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \right)^{1/2}.
\]
Lemma 2.3 of ? yields the relation
\[
\| \hat{v}_j - v_j \| \leq 2\sqrt{2} \alpha_j^{-1} \| \hat{C} - C \|_L,
\]
with the $\alpha_i$ defined in (5.7). Hence,
\[
\| b_N \|_L \leq 2\sqrt{2} \hat{\lambda}_1^{1/2} \left( \sum_{j=1}^{L} \alpha_j^{-1} \right) \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \right)^{1/2} \| \hat{C} - C \|_L.
\]
Since, for $N > N_2$ (random),
\[
\hat{\lambda}_1 \leq \lambda_1 + \| \hat{C} - C \|_L \leq \frac{3}{2} \lambda_1,
\]
we have
\[
\| b_N \|_L \leq 2\sqrt{3} \lambda_1^{1/2} \left( \sum_{j=1}^{L} \alpha_j^{-1} \right) \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|^2 \right)^{1/2} \| \hat{C} - C \|_L.
\]
By Corollary 3.1 and the strong law of large numbers, the claim follows from (5.10).

**Lemma 6.10** Under the assumptions of Theorem 5.1, $\| c_N \|_L \overset{a.s.}{\to} 0$. 

\[\text{imsart-bj ver. 2014/10/16 file: rv.tex date: December 10, 2018}\]
Proof: Observe that
\begin{equation*}
\|c_N(x)\| = \left\| \hat{\pi}^K D_N \left( \sum_{j=1}^L \lambda_j^{-1} (v_j, x) (\hat{v}_j - v_j) \right) \right\|
\end{equation*}
\begin{align*}
&\leq \left\| \hat{\pi}^K D_N \right\|_\mathcal{L} \left\| \sum_{j=1}^L \lambda_j^{-1} |(v_j, x)| \|(\hat{v}_j - v_j)\| \right. \\
&\leq \left\| \hat{\pi}^K D_N \right\|_\mathcal{L} \left( \sum_{j=1}^L \lambda_j^{-1} \alpha_j^{-1} |(v_j, x)| \right) \left\| \hat{C} - C \right\|_\mathcal{L} \\
&\leq \left\| \hat{\pi}^K D_N \right\|_\mathcal{L} \lambda_L^{-1} \|x\| \left( \sum_{j=1}^L \alpha_j^{-1} \right) \left\| \hat{C} - C \right\|_\mathcal{L}.
\end{align*}
Therefore,
\begin{equation*}
\|c_N\| \leq \left\| D_N \right\|_\mathcal{L} \lambda_L^{-1} \left( \sum_{j=1}^L \alpha_j^{-1} \right) \left\| \hat{C} - C \right\|_\mathcal{L}.
\end{equation*}
Since, by the law of large numbers, \(\left\| \hat{\pi}^K D_N \right\|_\mathcal{L} \stackrel{a.s.}{\to} 0\), the claim follows from condition (5.10).

To deal with the last term, we need additional lemmas.

Lemma 6.11 Under the assumptions of Theorem 5.1, \(N^{1-1/\gamma} \|D_N - D\|_S \stackrel{a.s.}{\to} 0\).

Proof: The decomposition
\begin{equation*}
\frac{1}{N} \sum_{i=1}^N X_i \otimes Y_i = \frac{1}{N} \sum_{i=1}^N X_i \otimes \Psi(X_i) + \frac{1}{N} \sum_{i=1}^N X_i \otimes \varepsilon_i
\end{equation*}
and the identities
\begin{equation*}
X_i \otimes \Psi(X_i) = \Psi(X_i \otimes X_i), \quad E[X \otimes \Psi(X)] = \Psi E[X \otimes X], \quad E[X \otimes \varepsilon] = 0
\end{equation*}
imply that
\begin{equation*}
\|D_N - D\|_S = \left\| \frac{1}{N} \sum_{i=1}^N X_i \otimes Y_i - E[X \otimes Y]\right\|_S
\end{equation*}
\begin{align*}
&\leq \left\| \Psi\right\|_S \left\| \hat{C} - C \right\|_S + \left\| \frac{1}{N} \sum_{i=1}^N X_i \otimes \varepsilon_i\right\|_S.
\end{align*}
For any \(1 \leq \gamma < 2\),
\begin{equation*}
\left\| \frac{1}{N^{1/\gamma}} \sum_{i=1}^N X_i \otimes \varepsilon_i\right\|_S \stackrel{a.s.}{\to} 0.
\end{equation*}
The above convergence follows from Theorem 4.1 of \(\gamma\) which implies that in any separable Banach space of Rademacher type \(\gamma\), \(1 \leq \gamma < 2\), \(N^{-1/\gamma} \sum_{i=1}^N Y_i \stackrel{a.s.}{\to} 0\), provided the \(Y_i\) are iid with \(E\|Y_i\|^\gamma < \infty\) and \(EY_i = 0\). In our case, the Banach space is the Hilbert space \(S\) (a Hilbert space has Rademacher type \(\gamma\) for any \(\gamma \leq 2\), see e.g. Theorems 3.5.2 and 3.5.7 of \(\gamma\)). Clearly, \(E[X_i \otimes \varepsilon_i] = 0\) and \(E\|X_i \otimes \varepsilon_i\|_S^\gamma = E\|X_i\|_S^\gamma \|\varepsilon_i\|_S^\gamma < \infty\). Another application of Corollary 3.1 completes the proof.

Lemma 6.12 Under the assumptions of Theorem 5.1, \(\lambda_L^{-1} \left\| \hat{\pi}^K D_N - \pi^K D \right\|_\mathcal{L} \stackrel{a.s.}{\to} 0\).
Lemma 6.13

By the triangle inequality,

$$\|\hat{\pi}^K D_N - \pi^K D\|_L \leq \|\hat{\pi}^K D_N - \hat{\pi}^K D\|_L + \|\hat{\pi}^K D - \pi^K D\|_L.$$ 

For the first term, we have

$$\|\hat{\pi}^K D_N - \hat{\pi}^K D\|_L = \sup_{\|x\| \leq 1} \left\| \sum_{k=1}^{K} \langle (D_N - D) (x), \hat{u}_k \rangle \hat{u}_k \right\|$$

$$\leq \sup_{\|x\| \leq 1} \left( \sum_{k=1}^{K} \| (D_N - D) (x), \hat{u}_k \| \right)$$

$$\leq K^{1/2} \| D_N - D \|_L.$$ 

Thus, $\lambda^{-1}_L \|\hat{\pi}^K D_N - \hat{\pi}^K D\|_L \xrightarrow{a.s.} 0$ by Lemma 6.11 and condition (5.11).

Turning to the second term, observe first that

$$D(x) = E[(X, x) Y] = \Psi(E[(X, x) X]) = \Psi(C(x)).$$

Setting $y = \Psi(C(x))$ we thus have

$$\pi^K D(x) = \sum_{k=1}^{K} \langle y, u_k \rangle u_k, \quad \hat{\pi}^K D(x) = \sum_{k=1}^{K} \langle y, \hat{u}_k \rangle \hat{u}_k.$$ 

Consequently, $\hat{\pi}^K D(x) - \pi^K D(x) = D_1(x) + D_2(x)$, where

$$D_1(x) = \sum_{k=1}^{K} \langle y, u_k - \hat{u}_k \rangle u_k, \quad D_2(x) = \sum_{k=1}^{K} \langle y, \hat{u}_k \rangle (u_k - \hat{u}_k).$$ 

Next,

$$\|D_1(x)\| \leq \|y\| \left\{ \sum_{k=1}^{K} \|u_k - \hat{u}_k\|^2 \right\}^{1/2} \leq 2\sqrt{2} \|y\| \left\| \hat{C}_Y - C_Y \right\| \left\{ \sum_{k=1}^{K} \frac{1}{\beta_k^2} \right\}^{1/2}$$

and

$$\|D_2(x)\| \leq \sum_{k=1}^{K} \| y, \hat{u}_k \| \| u_k - \hat{u}_k \| \leq 2\sqrt{2} \|y\| \left\| \hat{C}_Y - C_Y \right\| \sum_{k=1}^{K} \frac{1}{\beta_k}.$$ 

We see that condition (5.12) implies that $\lambda^{-1}_L \|\hat{\pi}^K D - \hat{\pi}^K D\|_L \xrightarrow{a.s.} 0$. 

\[\square\]

Lemma 6.13 Under the assumptions of Theorem 5.1, $\|d_N\|_L \xrightarrow{a.s.} 0$.

Proof: Observe that

$$\|d_N(x)\|^2 = \left\| (\hat{\pi}^K D_N - \pi^K D) \left( \sum_{j=1}^{L} \lambda_j^{-1} (v_j, x) v_j \right) \right\|^2$$

$$\leq \|\hat{\pi}^K D_N - \pi^K D\|_L^2 \left( \sum_{j=1}^{L} \lambda_j^{-2} (v_j, x)^2 \right)$$

$$\leq \|\hat{\pi}^K D_N - \pi^K D\|_L^2 \lambda^{-2}_L \left( \sum_{j=1}^{L} (v_j, x)^2 \right)$$

$$\leq \|\hat{\pi}^K D_N - \pi^K D\|_L^2 \lambda^{-2}_L \|x\|^2.$$
Consequently, $\|d_N\|_\mathcal{L} \leq \|\hat{\pi}^K D_N - \pi^K D\|_{\mathcal{L}} \lambda_*^{-1}$, so the claim follows from Lemma 6.12 and condition (5.9). \[\blacksquare\]