Bulk versus Brane Emissivities of Photon Fields: For the case of Higher-Dimensional Schwarzschild Phase

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Abstract

The emission spectra for the spin-1 photon fields are computed when the spacetime is a $(4+n)$-dimensional Schwarzschild phase. For the case of the bulk emission we compute the spectra for the vector mode and scalar mode separately. Although the emissivities for the scalar mode is larger than those for the vector mode when $n$ is small, the emissivities for the vector mode photon become dominant rapidly with increasing $n$. For the case of the brane emission the emission spectra are numerically computed by making use of the complex potential method. Comparison of the total bulk emissivities with total brane emissivities indicates that the effect of the field spin makes the bulk emission to be rapidly dominant with increasing $n$. However, the bulk-to-brane relative emissivity per degree of freedom always remains smaller than unity. The importance for the spin-2 graviton emission problem is discussed.

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I. INTRODUCTION

The assumption for the existence of the extra dimensions has a long history\[1, 2\]. Recent quantum gravity such as string theories\[3\] and brane-world scenarios\[4\] also adopts this assumption. Especially modern brane-world scenarios predict the emergence of the TeV-scale gravity, which opens the possibility to make tiny black holes by high-energy scattering in the future collider\[5\]. In this reason much attention is paid recently to the higher-dimensional black holes\[6\].

The most well-known quantum gravity effect of the black hole is a Hawking radiation\[7\], which makes the black hole different from the black body. The Hawking radiation for the higher-dimensional black holes was extensively studied for last few years to support the experimental significance in the future colliders. Emparan, Horowitz and Myers (EHM) argued that the brane-world black holes radiate mainly on the brane\[8\]. To support their argument roughly EHM used a simple setting like an higher-dimensional black body.

More exact calculation on the absorption and emission problems for the higher-dimensional non-rotating black holes was performed in Ref.\[9, 10\]. The authors of Ref.\[9\] carried out the numerical calculation in the background of the (4+n)-dimensional Schwarzschild black hole. In Ref.\[10\] different numerical technique was adopted and the Hawking radiation by the (4+n)-dimensional charged black hole was explored. The numerical results of Ref.\[9, 10\] support the EHM argument, black holes radiate mainly on the brane, if n is not too large.

More recently, there was a suggestion that EHM argument should be examined carefully when the black holes have angular momenta\[11\]. When the fields are scattering with the rotating black hole, there is a factor called superradiance\[12\], which does not exist in the case of the non-rotating black hole. The authors in Ref.\[11\] argued that EHM argument may be wrong due to the existence of the superradiance modes. The condition for the existence of the superradiance modes was derived for the bulk fields\[13\] and brane-localized fields\[14\]. In Ref.\[15\] numerical calculation was performed for the bulk and brane-localized scalar fields in the background of the five-dimensional rotating black holes with two different angular momenta. According to Ref.\[15\] the numerical calculation shows that the energy amplification for the bulk scalar is very small (roughly order of $10^{-9}\%$) while that for the brane scalar is order of unity. This big difference indicates that the effect of the superradiance
is negligible for the case of the bulk scalar. Therefore, the standard claim, \textit{black holes radiate mainly on the brane}, still holds although the effect of the superradiance is taken into account.

There is an another factor we should check carefully when the Hawking radiation for the higher-dimensional black holes is studied. This is an effect for the higher-spin particles like graviton. Since the graviton is not generally localized on the brane unlike the usual standard model particles, the EHM argument should be carefully re-checked in the graviton emission. When the spacetime background is a \((4 + n)\)-dimensional Schwarzschild metric

\begin{equation}
 ds^2 = -h dt^2 + h^{-1} dr^2 + r^2 d\Omega_{n+2}^2
\end{equation}

where \(h = 1 - (r_H/r)^{n+1}\) and the angular part \(d\Omega_{n+2}^2\) is a spherically symmetric line element in a form

\begin{equation}
 d\Omega_{n+2}^2 = d\theta_{n+1}^2 + \sin^2 \theta_{n+1} \left[ d\theta_n^2 + \sin^2 \theta_n \left( \cdots + \sin^2 \theta_2 \left( d\theta_1^2 + \sin^2 \theta_1 d\varphi^2 \right) \cdots \right) \right],
\end{equation}

the radial equations for the bulk graviton were derived in Ref.[16] by extending the well-known Regge-Wheeler method[17]. Using these radial equations, the absorption and emission spectra for the scalar, vector, and tensor modes of the bulk graviton were computed recently[18]. It was shown that the total emissivity for the bulk graviton increases rapidly compared to the spin-0 bulk scalar when the extra dimensions exist. The ratio of the total emissivities between the bulk graviton and bulk scalar, for example, becomes 5.16\%, 147.7\%, 595.2\% and 3417\% when \(n = 0, 1, 2\) and 6 respectively. This tremendous increase of the emission rate for the bulk graviton makes us believe that the missing energy is not negligible in the future collider experiment although it is not larger than the visible one. In order to compare the graviton emissivities on the brane and in the bulk we should compute the emission rate for the brane-localized graviton. In order to explore the absorption and emission problems for the brane graviton the axial and polar perturbations were studied in Ref.[19] when the spacetime background is a \(4d\) induced metric from Eq.(1.1)

\begin{equation}
 ds_4^2 = -h dt^2 + h^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\end{equation}

In these perturbations there is a difficulty arising due to the fact that the metric (1.3) is not a vacuum solution of the \(4d\) Einstein field equation. Thus in Ref.[19] the metric (1.3) was regarded as the non-vacuum solution. This yields an additional difficulty on the treatment of the energy-momentum tensor in the given perturbation.
In order to explore the effect of the field spin in the absorption and emission problems we choose the spin-1 photon field in the background of the \((4 + n)\)-dimensional Schwarzschild phase (1.1). The brane emission rates for the spin-1 field were already computed in the background of non-rotating black hole in Ref. [9] and rotating black hole in the last reference of Ref. [14]. In this paper we will compute the bulk emission rate too to analyze the validity of the EHM argument in the Hawking evaporation of the spin-1 fields. In Sec. II we compute the absorption and emission spectra for the bulk photon fields. For the comparison we compute the spectra for the vector mode photon and scalar mode photon separately. In Sec. III we compute the emission spectra for the brane-localized photon fields by making use of the induced metric (1.3). In the next section the emission rates for the bulk photon fields are compared to those for the brane-localized photon fields. It is shown that the brane emission is a bit dominant when \(n \leq 4\). However, the dominance is changed into the total bulk emission when \(n \geq 6\). However the bulk-to-brane emissivity per degree of freedom always remains smaller than unity, which supports the EHM argument. In Sec. V a brief conclusion is given.

II. BULK PHOTON

The electromagnetic perturbation in the background of the higher-dimensional spherically symmetric black holes was discussed in detail in Ref. [20]. The most remarkable fact is that the perturbations are classified into the scalar- and vector-type modes according to their tensorial behavior on the spherical section of the background metric. The radial equations for the modes are expressed as a Schrödinger-like equation in the form

\[
\left[\left(\hbar \frac{d}{dr}\right)^2 + \omega^2\right] R = V_{BL}R
\]

(2.1)

where the effective potential \(V_{BL}\) is

\[
V_{BL} = \frac{\hbar}{r^2} \left[\ell(\ell + n + 1) + \frac{n(n + 2)}{4} + \sigma_n(1 - h)\right]
\]

(2.2)
with \( \ell \geq 1 \). The \( n \)-dependent parameter \( \sigma_n \) is also dependent on the modes for the bulk photon in the following

\[
\sigma_n = \begin{cases} 
\frac{n(n+4)}{4} & \text{for vector photon mode} \\
-\frac{n(3n+4)}{4} & \text{for scalar photon mode} \\
\frac{(n+2)^2}{4} & \text{for bulk scalar field}
\end{cases}
\]  

(2.3)

Defining the dimensionless parameters \( x \equiv \omega r \) and \( x_H \equiv \omega r_H \), one can rewrite Eq.(2.1) as following:

\[
x^2(x^{n+1} - x_H^{n+1})^2 \frac{d^2 R}{dx^2} + (n + 1)x^{n+1}x(x^{n+1} - x_H^{n+1}) \frac{dR}{dx} + \left[x^{2n+4} - (x^{n+1} - x_H^{n+1}) \left[\ell(\ell + n + 1) + \frac{n(n+2)}{4}\right]x^{n+1} + \sigma_n x_H^{n+1}\right] R = 0.
\]  

(2.4)

Since Eq.(2.4) is real, it is easy to show that if \( R \) is a solution of Eq.(2.4), its complex conjugate \( R^* \) is solution too. The Wronskian between them is

\[
W[R^*, R]_x \equiv R^* \frac{dR}{dx} - R \frac{dR^*}{dx} = C \frac{x^{n+1}}{x^{n+1} - x_H^{n+1}}
\]  

(2.5)

where \( C \) is an integration constant.

Now, we consider the solution of Eq.(2.4), \( G_{n,\ell}^{BL} \), which is convergent in the neighborhood of the near-horizon \( x \sim x_H \). Since \( x = x_H \) is a regular singular point of the radial equation (2.4), we can express \( G_{n,\ell}^{BL} \) as a convergent series in the form:

\[
G_{n,\ell}^{BL}(x, x_H) = \sum_{N=0}^{\infty} d_{\ell,N}(x - x_H)^{N+\rho_n}
\]  

(2.6)

where \( \rho_n = -ix_H/(n+1) \). Making use of Eq.(2.5), it is easy to derive the Wronskian between \( G_{n,\ell}^{BL*} \) and \( G_{n,\ell}^{BL} \):

\[
W[G_{n,\ell}^{BL*}, G_{n,\ell}^{BL}]_x = -2i\left|g_{n,\ell}\right|^2 \frac{x^{n+1}}{x^{n+1} - x_H^{n+1}}
\]  

(2.7)

where \( g_{n,\ell} \equiv d_{\ell,0} \). Inserting Eq.(2.6) into (2.4), one can directly derive the recursion relation for the coefficients \( d_{\ell,N} \). The recursion relation is, of course, \( n \)-dependent and lengthy. Therefore, we will not present it explicitly.

Next we consider the solutions of Eq.(2.4), \( F_{n,\ell(\pm)}^{BL} \), which are convergent in the asymptotic regime:

\[
F_{n,\ell(\pm)}^{BL}(x, x_H) = (\pm i)^{\ell+1+n/2}xe^{\mp ix}(x - x_H)^{\pm \rho_n} \sum_{N=0}^{\infty} \tau_{N(\pm)} x^{-(N+1)}
\]  

(2.8)
with $\tau_{0(\pm)} = 1$. The solutions $F_+$ and $F_-$ are, therefore, the ingoing and outgoing waves respectively. The Wronskian between them is

$$W[F_{BL}^{n,\ell(+)}(x), F_{BL}^{n,\ell(-)}(x)] = 2i\frac{x_{n+1}}{x_{n+1} - x_H}.$$  \hspace{1cm} (2.9)

As in the previous case the recursion relation for $\tau_{N(\pm)}$ can be derived explicitly by inserting (2.8) into (2.4).

We assume the real scattering solution $R_{n,\ell}$ behaves in the near-horizon and asymptotic regimes as following:

$$R_{n,\ell}(x) \rightarrow x \sim g_{n,\ell}(x - x_H)\rho_{n}[1 + O(x - x_H)]$$  \hspace{1cm} (2.10)

$$R_{n,\ell}(x) \rightarrow \infty \sim 2\frac{n}{\sqrt{\pi}}\Gamma\left(\frac{1 + n}{2}\right)Q_{n,\ell}$$

where $S_{n,\ell}(x_H)$ is a scattering amplitude and $Q_{n,\ell}$ is a quantity related to the multiplicities for the modes defined

$$Q_{n,\ell} = \begin{cases} 
\frac{\ell(\ell+n+1)(\ell+n+1)!}{(n+2)(\ell+1)!n!} & \text{for vector mode photon} \\
\frac{(2\ell+n+1)(\ell+n)!}{\ell(n+2)!} & \text{for scalar mode photon} \\
\frac{(2\ell+n+1)(\ell+n)!}{\ell(n+1)!} & \text{for bulk scalar field}
\end{cases}$$  \hspace{1cm} (2.11)

Introducing a phase shift $\delta_{n,\ell}$ as $S_{n,\ell} \equiv e^{2i\delta_{n,\ell}}$, one can rewrite the second equation of Eq.(2.10) in a form

$$R_{n,\ell}(x) \rightarrow \infty \sim \frac{2\frac{n}{\sqrt{\pi}}\Gamma\left(\frac{1 + n}{2}\right)}{Q_{n,\ell}e^{i\delta_{n,\ell}}}$$

$$\times \sin \left[ x + i\rho_{n}\ln|x - x_H| - \frac{\pi}{2}\left(\ell + \frac{n}{2}\right) + \delta_{n,\ell} \right] + O\left(\frac{1}{x}\right).$$  \hspace{1cm} (2.12)

The first equation of Eq.(2.10) guarantees that the Wronskian $W[R_{n,\ell}^*, R_{n,\ell}]$ is exactly same with Eq.(2.7). However, Eq.(2.12) implies

$$W[R_{n,\ell}^*, R_{n,\ell}] = -i\frac{2^n}{\pi}\Gamma^2\left(\frac{1 + n}{2}\right)Q_{n,\ell}^2e^{-2\beta_{n,\ell}}\sinh 2\beta_{n,\ell}\frac{x_{n+1}}{x_{n+1} - x_H}$$  \hspace{1cm} (2.13)

where $\beta_{n,\ell} = \text{Im}[\delta_{n,\ell}]$. Thus, from Eq.(2.7) and (2.13) it is easy to derive a relation

$$|g_{n,\ell}|^2 = \frac{2^{n-2}}{\pi}\Gamma^2\left(\frac{1 + n}{2}\right)Q_{n,\ell}^2\left(1 - e^{-4\beta_{n,\ell}}\right).$$  \hspace{1cm} (2.14)
Eq. (2.14) enables us to compute the transmission coefficient $1 - |S_{n,\ell}|^2 = 1 - e^{-4\beta_{n,\ell}}$ once the coefficient $g_{n,\ell}$ is known.

Now, we would like to explain how to compute $g_{n,\ell}$ numerically. For the explanation it is convenient to introduce a new radial solution $\tilde{\phi}_{n,\ell}(x)$, which differs from $R_{n,\ell}(x, x_H)$ in its normalization in such a way that

$$\tilde{\phi}_{n,\ell}(x, x_H) \equiv \frac{R_{n,\ell}(x, x_H)}{g_{n,\ell}} \sim (x - x_H)^{\rho_n} [1 + O(x - x_H)].$$

(2.15)

Since $F^{BL}_{n,\ell(\pm)}$ in Eq. (2.8) are two linearly independent solutions of the radial equation Eq. (2.4), one can simply put

$$\tilde{\phi}_{n,\ell}(x, x_H) = f^{(-)}_{n,\ell}(x_H)F^{BL}_{n,\ell(+)}(x, x_H) + f^{(+)}_{n,\ell}(x_H)F^{BL}_{n,\ell(-)}(x, x_H)$$

(2.16)

where $f^{(\pm)}_{n,\ell}$ are called jost functions. Using Eq. (2.9) one can easily compute the jost functions from $\tilde{\phi}_{n,\ell}$ as following

$$f^{(\pm)}_{n,\ell}(x_H) = \pm \frac{x^{n+1} - x_{n+1}^H}{2i x^{n+1}} W[F^{BL}_{n,\ell(\pm)}, \tilde{\phi}_{n,\ell}]_x.$$  

(2.17)

Inserting the explicit expressions of $F^{BL}_{n,\ell(\pm)}$ presented in Eq. (2.8) into Eq. (2.16) and comparing it with the second equation of Eq. (2.10), one can derive the following two relations

$$S_{n,\ell}(x_H) = \frac{f^{(+)}_{n,\ell}(x_H)}{f^{(-)}_{n,\ell}(x_H)}$$

(2.18)

$$f^{(-)}_{n,\ell}(x_H) = \frac{2^{\frac{n}{2}-1}}{\sqrt{\pi} g_{n,\ell}(x_H)} \Gamma \left( \frac{1 + n}{2} \right) Q_{n,\ell}.$$  

Combining Eq. (2.14) and (2.18) makes the greybody factor (or transmission coefficient) of the black hole to be

$$1 - |S_{n,\ell}|^2 = \frac{1}{|f^{(-)}_{n,\ell}|^2}. $$

(2.19)

Thus, the partial absorption cross sections for each modes become

$$\sigma^{BL}_{n,\ell} = 2^{n+1} \pi (n+1)/2 \Gamma \left( \frac{3 + n}{2} \right) Q_{n,\ell} \frac{r_H^{n+2}}{x_H^{n+2} |f^{(-)}_{n,\ell}|^2}, $$

(2.20)

Applying the Hawking formula, one can compute the bulk emission rate, i.e. the energy emitted to the bulk per unit time and unit energy interval, as following

$$\frac{d^2 \Gamma^{BL}_{D}}{d\omega dt} = \left[ 2^{n+2} \pi (n+3)/2 \Gamma \left( \frac{3 + n}{2} \right) \right]^{-1} f_n \frac{\omega^{n+3} \sigma^{BL}_n(\omega)}{e^{\omega/T_H} - 1} $$

(2.21)
where \( \sigma_{abs}^{BL} = \sum_{\ell} \sigma_{n,\ell}^{BL} \), \( T_H = (n + 1)/4\pi r_H \) and

\[
f_n = \begin{cases} 
  n + 2 & \text{for photon modes} \\
  1 & \text{for bulk scalar field}
\end{cases}
\] (2.22)

The jost functions \( f_{n,\ell}^{(\pm)}(x_H) \) can be computed numerically by adopting the analytic continuation. In order to apply the continuation we need a solution of Eq. (2.4), which is convergent in the neighborhood of \( x = b \), where \( b \) is an arbitrary point. Thus the expression of the solution is

\[
\tilde{R}_{n,\ell}(x) = (x - x_H)^{\rho_n} \sum_{N=0}^{\infty} D_N(x - b)^N.
\] (2.23)

The recursion relation for the coefficients \( D_N \) can be explicitly derived by inserting Eq. (2.23) into (2.4). Since it is too lengthy, we will not present the explicit expression. Using a solution (2.23), one can increase the convergent region for the near-horizon solution from the near-horizon regime and decrease the convergent region for the asymptotic solution from the asymptotic regime. Repeating the procedure eventually makes the two solutions which have common convergent region. Then one can compute the jost functions by making use of these two solutions and Eq. (2.17).

Fig. 1 is a plot of the emission spectrum \( d^2\Gamma/d\omega dt \) for the spin-1 photon when there is no extra dimension. For a comparison the spectrum for the spin-0 scalar field is plotted together. Fig. 1 shows that the emission rate for the scalar field is much larger than that for the photon field when \( n = 0 \).

However, the situation is drastically changed when the extra dimensions exist. In Fig. 2 the emission spectra for the scalar photon mode, vector photon mode, and bulk scalar field are plotted when \( n = 1 \) (Fig. 2(a)), \( n = 2 \) (Fig. 2(b)), \( n = 4 \) (Fig. 2(c)) and \( n = 6 \) (Fig. 2(d)). Fig. 2 indicates that the emission rate in general increases with increasing \( n \) regardless of the type of fields. However, the increasing rates of the emission spectra for the photon modes are much larger than that for the bulk scalar field.

Table I shows the total emission rate, i.e. \( \int d\omega d^2\Gamma/d\omega dt \) for the scalar mode, vector mode and bulk scalar when \( n = 0, 1, 2, 4 \) and 6. The table indicates that the total emissivity for the photon field is only 23% of that for the spin-0 field when there is no extra dimension. However, this ratio factor increases rapidly with increasing \( n \). For example, this factor becomes 172%, 321%, 499% and 790% when \( n = 1, 2, 4 \) and 6.
Another interesting feature is that for comparatively small $n$ the emission rate for the scalar mode photon is larger than that for the vector mode photon. However, the increasing rate for the vector mode is much larger compared to that for the scalar mode with increasing $n$. As a result, the total emissivity for the vector mode becomes dominant more and more in the photon emission spectrum for large $n$. For example, the emissivity for the vector mode is roughly three times than that for the scalar mode when $n = 6$.

![Fig. 1](image)

**Fig. 1:** Plot of the emission spectra for the spin-1 photon and spin-0 scalar fields when there is no extra dimension. This figure indicates that the emission rate for the photon is much smaller than that for the scalar when $n = 0$.

|                  | 4d        | 5d        | 6d       | 8d        | 10d       |
|------------------|-----------|-----------|----------|-----------|-----------|
| scalar mode photon | $6.728 \times 10^{-5}$ | $1.088 \times 10^{-3}$ | $4.77 \times 10^{-3}$ | $3.64 \times 10^{-2}$ | 0.2451 |
| vector mode photon | $6.728 \times 10^{-5}$ | $7.21 \times 10^{-4}$ | $3.82 \times 10^{-3}$ | $5.73 \times 10^{-2}$ | 0.708 |
| bulk scalar field | $2.975 \times 10^{-4}$ | $1.05 \times 10^{-3}$ | $2.68 \times 10^{-3}$ | $1.876 \times 10^{-2}$ | 0.1206 |
| photon / bulk scalar | 0.46 | 1.72 | 3.21 | 4.99 | 7.90 |

Table I: Relative Emission Rates for Bulk Photon and Scalar Fields
FIG. 2: Plot of the emission spectra for the bulk photon modes when $n = 1$ (a), $n = 2$ (b), $n = 4$ (c) and $n = 6$ (d). The spectra for the bulk scalar fields are plotted together for a comparison. The figures indicate that the emission rates in general increase with increasing $n$ regardless of the type of fields. However, they also indicate that as $n$ increases, the emission rates for the photon fields become more and more dominant compared to those for the bulk scalar fields.

III. BRANE PHOTON

In this section we would like to compute the emissivity of the spin-1 photon on the brane by the induced metric

$$ds^2 = -h dt^2 + h^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (3.1)

In fact, this was already computed in Ref.[9] by different numerical method. We would like to re-calculate the emission rates of the brane-localized photon field by incorporating our
numerical method into the complex potential method introduced in Ref. [21] to compare the results with those for the bulk photon.

The radial equation for the scalar \((s = 0)\), fermion \((s = 1/2)\), and vector \((s = 1)\) can be expressed as a following master equation [14] :

\[
\Lambda^2 Y + P \Lambda Y - QY = 0 \tag{3.2}
\]

where \(\Lambda_{\pm} = d/dr_{\pm} \pm i\omega\), \(\Lambda^2 \equiv \Lambda_+\Lambda_-\), \(d/dr_{\pm} \equiv h d/dr\) and

\[
P = \frac{d}{dr_{\pm}} \ln \left( \frac{r^2}{h} \right)^{-s} \tag{3.3}
\]

\[
Q = \frac{h}{r^2} \left[ A_{\ell s} + (2s + n + 1)(ns + s + 1)(1 - h) \right]
\]

with \(A_{\ell s} = \ell(\ell + 1) - s(s + 1)\). Defining \(Y = fR\) with \((1/f)d df/dr = -P/2\). one can change Eq.(3.2) into the Schrödinger-like expression with a complex potential in the following:

\[
\Lambda^2 R = V_s R \tag{3.4}
\]

where

\[
V_s = i\omega P + \frac{P^2}{4} + \frac{1}{2} \frac{dP}{dr_{\pm}} + Q
\]

\[
= \frac{h}{r^2} \left[ A_{\ell s} + q_{ns}(1 - h) + \frac{s^2}{4h} \{(n + 1)(1 - h) - 2h\}^2 + sh \right] + \frac{i\omega s}{r} \{(n + 1)(1 - h) - 2h\}
\]

and \(q_{ns} = (2s + n + 1)(ns + s + 1) - (s/2)(n + 1)(n + 4)\).

Defining the dimensionless parameters \(x = \omega r\) and \(x = \omega r_H\), we can rewrite Eq.(3.4) in the following form:

\[
x^2 (x^{n+1} - x_H^{n+1})^2 \frac{d^2 R}{dx^2} + (n + 1)x_H^{n+1} x \left( x^{n+1} - x_H^{n+1} \right) \frac{dR}{dx}
\]

\[
- \left[ isx^{n+2} \{(n + 3)x_H^{n+1} - 2x^{n+1}\} + A_{\ell s} x^{n+1} \left( x^{n+1} - x_H^{n+1} \right) + q_{ns} x_H^{n+1} \left( x^{n+1} - x_H^{n+1} \right) \right.
\]

\[
+ \frac{s^2}{4} \left\{ (n + 3)x_H^{n+1} - 2x^{n+1} \right\}^2 - x^{2n+4} + s \left( x^{n+1} - x_H^{n+1} \right)^2 \right] R = 0.
\]

Using Eq.(3.6) one can easily show that if \(R_1\) and \(R_2\) are solutions of Eq.(3.6), the Wronskian between them reduces to

\[
W[R_1, R_2]_x \equiv R_1 \frac{dR_2}{dx} - R_2 \frac{dR_1}{dx} = C \frac{x^{n+1}}{x^{n+1} - x_H^{n+1}} \tag{3.7}
\]
where $C$ is an integration constant.

The solution of Eq. (3.6) which is convergent in the near-horizon regime can be written as

$$G^{BR}_{n,\ell}(x, x_{H}) = \omega^{s-1+ix_{H}}x_{H}^{\frac{n}{2}(n-1)+1} \left[ (n+1)x_{H}^{n} \right]^{-\frac{1}{2}+ix_{H}} \sum_{N=0}^{\infty} d_{\ell,N}(x-x_{H})^{N+\rho_{n}}$$

with $d_{\ell,0} = 1$ and

$$\rho_{n} = \pm \left[ \frac{s^{2}}{4} - \frac{x_{H}^{2}}{(n+1)^{2}} + is \frac{x_{H}}{n+1} \right]^{1/2}.$$

The sign of $\rho_{n}$ is chosen by $\text{Im} \rho_{n} < 0$. The multiplication constant in Eq. (3.8) is chosen for the later convenience. The recursion relation for the coefficient $d_{\ell,n}$ can be derived explicitly by inserting Eq. (3.8) into (3.6).

The ingoing and outgoing solutions which are convergent in the asymptotic regime are respectively

$$F^{BR}_{n,\ell(+)}(x, x_{H}) = \omega^{-s}x^{s+1}e^{-ix} \sum_{N=0}^{\infty} \tau_{N(+)x}^{s}(N+1)$$

$$F^{BR}_{n,\ell(-)}(x, x_{H}) = \omega^{s}x^{-s+1}e^{ix} \sum_{N=0}^{\infty} \tau_{N(-)x}^{s}(N+1).$$

Of course, the recursion relations for $\tau_{N(\pm)}$ can be explicitly derived by making use of the radial equation (3.6). Eq. (3.7) guarantees that the Wronskian between $F^{BR}_{n,\ell(+)}$ and $F^{BR}_{n,\ell(-)}$ is

$$W[F^{BR}_{n,\ell(+)}; F^{BR}_{n,\ell(-)}] = 2i \frac{x_{H}^{n+1}}{x_{H}^{n+1} - x_{H}^{n+1}}.$$  

In order to compute the absorption and emission spectra we define the near-horizon and asymptotic solutions in the following:

$$R_{NH}(x) = A_{n}G^{BR}_{n,\ell}(x, x_{H})$$

$$R_{\infty}(x) = F^{BR}_{n,\ell(+)}(x, x_{H}) + B_{n}F^{BR}_{n,\ell(-)}(x, x_{H}).$$

Using (3.11), one can compute the coefficients $A_{n}$ and $B_{n}$

$$A_{n} = -\frac{2ix_{H}^{n+1}}{x_{H}^{n+1} - x_{H}^{n+1}} \frac{1}{W[F^{BR}_{n,\ell(-)}; G^{BR}_{n,\ell}]}$$

$$B_{n} = \frac{W[F^{BR}_{n,\ell(+)}; G^{BR}_{n,\ell}] x}{W[F^{BR}_{n,\ell(-)}; G^{BR}_{n,\ell}]}.$$

The method of the complex potential introduced in Ref. [21] makes that the transmission coefficient reduces to $|A_{n}|^{2}/r_{H}^{2}$. Thus the absorption cross section for the photon field
becomes
\[ \sigma_{n,\ell}^{BR} = \frac{\pi(2\ell + 1)}{(\omega r_H)^2}|A_n|^2. \] (3.14)

Applying the Hawking formula, one can compute the brane emission rate \( d^2\Gamma_D^{BR}/d\omega dt \), i.e. the energy emitted on the brane per unit time and unit energy interval:
\[ \frac{d^2\Gamma_D^{BR}}{d\omega dt} = \hat{f}_s(2\pi^2)^{-1}\omega^3\sigma_{abs}^{BR}(\omega) e^{\omega/T_H} - 1 \] (3.15)

where \( \sigma_{abs}^{BR} = \sum_\ell \sigma_{n,\ell}^{BR} \), \( T_H = (n + 1)/4\pi r_H \) and \( \hat{f}_s = 2 \) (or 1) for photon field (or brane scalar field).

\[ \frac{d^2\Gamma_D^{BR}}{d\omega dt} = \frac{\pi^2}{\omega^3} \sigma_{abs}^{BR}(\omega) e^{\omega/T_H} - 1 \]

FIG. 3: The \( n \)-dependence of the emission spectra for the brane-localized spin-1 photon fields. With increasing \( n \) the emission rates tend to increase rapidly.

In Fig. 3 the \( n \)-dependence of the emission spectra on the brane for the spin-1 photon fields is plotted when \( n = 1, 2, 4, \) and 6. As it is well-known, the existence of the extra dimensions in general enhances the emissivity. Fig. 3 indicates that the increasing rate of the emissivity is very rapid with increasing \( n \). In the next section we will compare the bulk and brane emissivities for the spin-1 photon fields.


IV. BULK VERSUS BRANE

In Ref. [8] EHM argued that the higher-dimensional black holes mainly radiate on the brane. This argument was confirmed in Ref. [9], where the Hawking emissivities of the bulk and brane-localized scalar fields were numerically compared in the background of the $(4 + n)$-dimensional Schwarzschild black hole. If one defines the total emission rates

\[ \tilde{\Gamma}^{BL}_D \equiv \int d\omega \frac{d^2\Gamma^{BL}_D}{dtd\omega}, \quad \tilde{\Gamma}^{BR}_D \equiv \int d\omega \frac{d^2\Gamma^{BR}_D}{dtd\omega}, \]

(4.1)

the ratio $\tilde{\Gamma}^{BL}_D / \tilde{\Gamma}^{BR}_D$ in the scalar emission is summarized in Table II. The Table II indicates that the emission on the brane is dominant when $D \leq 11$ for the case of scalar fields.

| $D$  | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|------|---|---|---|---|---|---|----|----|
| $\tilde{\Gamma}^{BL}_D / \tilde{\Gamma}^{BR}_D$ | 1.0 | 0.40 | 0.24 | 0.22 | 0.24 | 0.33 | 0.52 | 0.93 |

Table II: Relative Bulk-to-Brane Emission Rates for Scalar Fields

Now, we would like to summarize the result of the spin-1 photon emission rate in the background of the $(4 + n)$-dimensional Schwarzschild phase. In Fig. 4 we plot the emission spectra for the brane-localized and bulk photons together for a comparison when $n = 1$ (Fig. 4(a)), $n = 2$ (Fig. 4(b)), $n = 4$ (Fig. 4(c)) and $n = 6$ (Fig. 4(d)). Fig. 4 indicates that the emission on the brane is dominant when $n$ is not too large. However, the emission rates for the bulk vector mode photon rapidly increase with increasing $n$. Therefore, the peak point of the emission spectrum for the vector mode becomes roughly same with that for the brane-localized photon when $n = 6$.

For more precise comparison the relative bulk-to-brane emission rates are summarized in Table III. Table III indicates that the brane emission is a bit dominant when $D \leq 8$. But the dominance is changed into the total bulk emission when $D \geq 10$. Since the bulk photon has $n + 2$ polarization states while brane photon has 2 helicities, it is also possible to compute the bulk-to-brane emissivity per degree of freedom (d.o.f.) which is shown in Table III. In spite of the rapid increase of the emission rates for the bulk photon with increasing $n$, this ratio always remains smaller than unity, which strongly supports the EHM argument.

Comparing Table III with Table II, one can realize that the effect of nonzero spin tends to make the bulk emission to be dominant more easily. Thus it seems to be very interesting to examine the graviton emission problem, where the effect of the spin may be more strong.
FIG. 4: Plot of the emission spectra for the brane-localized and bulk photons when \( n = 1 \) (a), \( n = 2 \) (b), \( n = 4 \) (c) and \( n = 6 \) (d). The figures indicate that the emission on the brane is a bit dominant when \( n \) is not too large. However, the emission rates for the bulk vector mode photon increase very rapidly with increasing \( n \) compared to other fields. Thus the dominance of the brane emission is no longer true when \( D \geq 10 \).

|                | \( D = 4 \) | \( D = 5 \) | \( D = 6 \) | \( D = 7 \) | \( D = 8 \) | \( D = 10 \) |
|----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Scalar Mode / Brane | 0.50        | 0.30        | 0.25        | 0.24        | 0.26        | 0.51        |
| Vector Mode / Brane  | 0.50        | 0.20        | 0.20        | 0.26        | 0.42        | 1.49        |
| Bulk / Brane         | 1.0         | 0.50        | 0.44        | 0.50        | 0.68        | 2.0         |
| (Bulk / Brane) per d.o.f. | 1.0        | 0.33        | 0.22        | 0.20        | 0.23        | 0.50        |

Table III: Relative Bulk-to-Brane Emission Rates for Photon Fields
V. CONCLUSION

In this paper we computed the absorption and emission spectra for the bulk and brane-localized spin-1 photon fields when the spacetime background is a $(4 + n)$-dimensional Schwarzschild phase. For the case of the bulk photon field we compute the spectra for the scalar mode and vector mode separately. The total emissivity for the scalar mode photon is dominant compared to that for the vector mode photon for comparatively small $n$. However, the increasing rate of the emission rate for the vector mode is much larger than that for the scalar mode with increasing $n$. As a result, the total emissivity for the vector mode photon becomes dominant more and more in the bulk emission rate for large $n$. For example, the vector-to-scalar relative emissivities become 66.3%, 80.1%, 157.4% and 288.9% when $n = 1, 2, 4$ and 6 respectively.

The absorption and emission spectra for the brane-localized photon field are also computed using the complex potential method introduced in Ref. [21]. The total emissivity for the brane-localized spin-1 field tends to increase with increasing $n$ like the case of the bulk photon field.

Comparing the total emissivity for the bulk photon to that for the brane-localized photon, we can see that the latter is a little bit larger than the former when $n$ is small like a scalar field. Of course, the former becomes dominant more and more with increasing $n$. Comparision of Table III with Table II indicates that the effect of the non-zero spin tends to make the bulk emission to be dominant more easily. The dominance for the bulk photon seems to be mainly due to the fact that the bulk photon has larger d.o.f. than the brane photon. Thus we compute the bulk-to-brane relative emissivity per d.o.f., which is shown in Table III. In spite of rapid increase of the bulk emissivity, this ratio always remains smaller than unity, which supports the EHM argument. However, the rapid increase of the total bulk emissivity indicates that the missing energy in the future collider experiment is not negligible compared to the visible one. Thus, one should carefully consider the portion of the missing energy when the experiment in the future colliders is setting up. In order to explore the missing energy more exactly it is important to examine the graviton emission problem.

The emission rates for the bulk graviton were discussed in detail in Ref. [18]. However, the rates for the brane graviton are not computed completely. In fact, the emission spectra
for the axial mode graviton were computed in Ref. [19]. Since the induced metric (1.3) is not a vacuum solution of 4d Einstein field equation, the author in Ref. [19] tried to determine the components of the energy-momentum tensor by making use of the general principles. However, the energy-momentum tensor is not fully determined from the general principles and therefore an assumption was used to fix the tensor.

To compute the emission rates for the brane-localized graviton without invoking the assumption, we think we should rely on the complex potential method again. For this case there seem to be two non-trivial tasks. Firstly, we should check whether Eq. (3.2) is a really master equation including the spin-2 graviton fields. As Ref. [9] has shown, Eq. (3.2) holds for the scalar, fermion and electromagnetic fields. However, it is not shown explicitly that Eq. (3.2) holds for the spin-2 graviton field\(^1\). Secondly, in Ref. [21] the greybody factor for the graviton field was computed by making use of the Hawking-Hartle theorem [23] when there is no extra dimension. It seems to be formidable job to extend the theorem when there are extra dimensions because the induced metric is not a vacuum solution of the 4d Einstein equation. We hope to discuss this issue in the near future.

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\(^{1}\) However, Ref. [22] used Eq. (3.2) for the computation of the quasinormal modes of the gravitational perturbation.
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