AN INEQUALITY FOR ELLIPTIC ISOMETRIES IN CAT(κ) SPACES

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Abstract. In this paper, we first prove an optimal inequality of Alexandrov angles for elliptic isometries of finite order on any CAT(κ) space, and the equality case leads to an isometrically embedded circle in the completion of the space of directions, which implies that there is a 2-flat in the tangent cone rotated by the induced action of the isometry. Then we prove that if the symmetry group of a Platonic solid, regular orthoplex or hypercube acts on a set of vertices in any CAT(κ) space in a way corresponds to the set of vertices of the solid, then any pair of points corresponding to an edge will make an angle at the circumcenter bounded below by that of the edge.

1. Introduction

In CAT(κ) space elliptic isometries one usually deals with are those of finite order. For such an isometry, we would like to know what angle it turns a point that is not fixed. The simplest example is a rotation of order n on a plane. In this case every point except the fixed point is turned by an angle 2π/n, measured at the fixed point. For an orthogonal transformation with finite order in $\mathbb{R}^k$, one can show that any non-fixed point is turned by an angle at least 2π/n. For n ≥ 3, this lower bound is achieved if there is a 2-dimensional invariant subspace such that the action restricted on it is a rotation of angle 2π/n. It is natural to expect the same for a general complete CAT(0) space, i.e. 2π/n is a lower bound for the angle.

It seems that the above question have received little attention so far. The only result which the author can find is the following estimate by Caprace and Monod using a very short and elementary argument: Let $X$ be a complete CAT(0) space, and $g$ be an elliptic isometry of finite order n on $X$. For any point $x$ not fixed by $g$, denote by $c$ the closest point to $x$ in $\text{Fix}(g)$. (Then $c$ has to be the circumcenter of the orbit of $x$.) In the middle of their proof of Alexandrov angle rigidity ([CM09], Proof of Proposition 6.8), they showed that $\angle_c(x, g \cdot x) \geq 1/n$. This lower bound is much smaller than the one we want. (However, it must be noted that the focus of that paper is not on metric geometric side but on group theoretic side, thus an optimal bound was not needed therein.)

In this paper, we will not only show that 2π/n is a lower bound for an elliptic isometry in a CAT(0) space, but we will show a more general result...
for finite order elliptic isometries on a CAT(κ) space. We suppose n ≥ 3, since for n = 2 the inequality is trivial and is actually an equality.

**Theorem I.** Let g be any elliptic isometry with finite order n ≥ 3 on a complete CAT(κ) space X, and x be a point not fixed by g. If κ > 0 assume that the orbit of x by g has radius less than \( \pi/(2\sqrt{\kappa}) \). Then

\[
\angle_c(x, g \cdot x) \geq \frac{2\pi}{n}
\]

where c is the circumcenter of the orbit of x.

Moreover, analogous to the situation in \( \mathbb{R}^k \), we have an invariant 2-flat in the tangent cone when this bound is achieved.

**Theorem II.** If n ≥ 3 and \( \angle_c(x, g \cdot x) = \frac{2\pi}{n} \), then the concatenation of the geodesic segments \([g^i \cdot x, g^{i+1} \cdot x]\) in \( S_p(X) \) forms an isometrically embedded circle of length \( 2\pi \), where \( x \) is the direction of geodesic from p to x. Also the tangent cone at c contains a 2-flat on which g induces a rotation of angle \( 2\pi/n \).

Using these results and the technique in [LS97], we will prove inequalities of Alexandrov angles similar to the one above for the symmetry groups of Platonic solids, regular orthoplex and regular hypercube acting on a set of vertices in any CAT(κ) space. The case of the regular n-simplex, is actually a special case of the main theorem in [LS97].

**Theorem III.** Let W be a regular Platonic solid or a regular hypercube or a regular orthoplex with vertices \( x_i \), and G be the (orientation preserving) symmetric group of W. Suppose that G acts on a CAT(κ) space X by isometries. Let \( x_i \) be points in X on which the induced action of G is equivariant with that on \( x_i \). Suppose that the set of points \( \{x_i\} \) has radius less than \( \pi/(2\kappa) \) if \( \kappa > 0 \). For any edge of W with endpoints \( x_i \) and \( x_j \), the Alexandrov angle \( \angle_c(x_i, x_j) \) is no less than the corresponding (Euclidean) angle \( \angle_c(A, B) \), where c and A are the circumcenters of the set \( \{x_i\} \) and W respectively.

2. Proof of the lower bound

We recall the definition of space of directions ([BH99] Def. II.3.18).

**Definition 2.1.** For any \( p \in X \), consider all the non-trivial geodesics issuing from \( p \). Define an equivalent relation on the set by \( \gamma \sim \gamma' \) iff the Alexandrov angle between them \( \angle_p(\gamma, \gamma') = 0 \). The equivalent classes under this relation form a metric space with Alexandrov angle \( \angle(\cdot, \cdot) \) as the metric. This metric space is the space of directions at \( p \) and is denoted as \( S_p(X) \).

By a theorem of Nikolaev ([BH99] Theorem II.3.19), as long as X is a metric space with curvature bounded above by some \( \kappa \), then the completion of \( S_p(X) \) is a CAT(1) space.

The following is a simple lemma. We include a proof which will be referred to later.
Lemma 2.2. Let $\alpha$ be a closed curve on the unit 2-sphere. If $\alpha$ is shorter than $2\pi$, then there is a point $m$ on the sphere such that $\alpha$ is contained in the hemisphere $B(m, \pi/2)$, where the metric is the angle metric on the sphere.

Proof. Take two points $x, x'$ on the curve $\alpha$ that divide it into two curves of equal length. As $d(x, x') < \pi$, there is a unique geodesic segment joining $x$ and $x'$, and we let $m$ be the midpoint of this segment. Then $m$ satisfies the desired condition. Otherwise, there would be a point $z$ in the hemisphere $B(m, \pi/2)$, and let $\alpha_1$ be the rotated images of $z$ and $\alpha_1$. Now $d(z, z') = 2d(z, m) = \pi$, but there is a path on $\alpha_1 \cup \alpha_1'$ joining $z$ to $z'$ with length equal that of $\alpha_1$, which is shorter than $\pi$, a contradiction. \hfill \Box

The main ingredient of our proof is the majorization theorem of Reshetnyak.

Theorem 2.3 (Reshetnyak majorization theorem [AKP]). Any closed curve $\alpha$ in a CAT($\kappa$) space $U$ with length less than $2\pi/\sqrt{\kappa}$ is majorized by a convex region $D$ in $M^2(\kappa)$, i.e. there is a distance non-increasing map from $D$ to $U$ such that its restriction to the boundary $\partial D$ is mapped to $\alpha$ preserving the length.

For $\kappa \leq 0$, define a map $\exp_{p}^{-1} : X \setminus \{p\} \to \overline{S_p(X)}$ which maps every point $x$ different from $p$ to the direction represented by the geodesic segment $[p, x]$; for $\kappa > 0$, define $\exp_{p}^{-1}$ similarly but with domain $B(p, \pi/\sqrt{\kappa}) \setminus \{p\}$.

Let $K$ be a compact set in $X$. If $\kappa > 0$ assume that $K$ has radius less than $\pi/(2\sqrt{\kappa})$. By Theorem B of [LS97] any bounded set of radius less than $\pi/(2\sqrt{\kappa})$ in a complete CAT($\kappa$) space has a unique circumcenter ; while if $\kappa \leq 0$ the uniqueness of circumcenter of any bounded set is automatic. Let $p$ be a point in $X$ such that $K \neq \{p\}$, and let $r = \sup_{x \in K} d(p, x)$.

Proposition 2.4. If the image of $\partial B(p, r) \cap K$ under $\exp_{p}^{-1}$ is contained in $B(m, \pi/2)$ for some point $m \in \overline{S_p(X)}$, then $p$ is not the circumcenter of $K$.

Proof. As $m$ is in the completion of the space of directions $S_p(X)$, with slight modification if necessary, $m$ is represented by a geodesic $\gamma([0, s_1])$, where $\gamma(0) = p$, $s_1 > 0$, and $s_1 < \pi/(2\sqrt{\kappa}) - r$ if $\kappa > 0$, so that $\gamma \subset B(x, \pi/(2\sqrt{\kappa}))$ for any point $x \in K$. Then $s \mapsto d(x, \gamma(s))$ is convex for any $x \in K$.

We claim that there exists some point $\gamma(s_2)$ such that $K \subset B(\gamma(s_2), r)$, implying that $p$ is not the circumcenter of $K$. Suppose the claim is false, then for every $\gamma(s_1/n)$ there is a point $x_n \in K$ with $d(x_n, \gamma(s_1/n)) \geq r$. Then by convexity of $s \mapsto d(x_n, \gamma(s))$ the function is strictly increasing on $[s_1/n, s_1]$. Since $K$ is compact, a subsequence of $x_n$, denoted again by $s_n$, converges to a point $x_0$, then the functions $s \mapsto d(x_n, \gamma(s))$ converges uniformly to $s \mapsto d(x_0, \gamma(s))$. So $r \geq d(x_0, p) \geq \lim_{n \to \infty} d(x_n, \gamma(s_1/n)) \geq r$, i.e. $d(x_0, p) = r$, and $s \mapsto d(x_0, \gamma(s))$ is strictly increasing on $[0, s_1]$. But
from the first variation formula for CAT($\kappa$) space \cite{BH99}, Corollary II.3.6),
\[
\cos \angle_p(\gamma(\cdot), x_0) = \lim_{s \to 0} \frac{d(p, x_0) - d(\gamma(s), x_0)}{s} > 0,
\]
a contradiction. Hence the claim. \hfill \Box

With the above results we show Theorem \ref{thm1}.

**Proof.** Note that Theorem B of \cite{LS97} asserts that the orbit of $x$ has a unique
circumcenter for the case $\kappa > 0$, so the point $c$ is well-defined. Suppose on
the contrary that $\angle_c(x, g \cdot x) < 2\pi/n$. Joining $\exp^{-1}(g^i \cdot x)$ successively
by geodesic segments in $\overline{S_c(X)}$, we obtain a closed $n$-gon in $\overline{S_c(X)}$ with
length less than $2\pi$. Apply the Reshetnyak majorization theorem, we get
a distance non-increasing map from a convex region on the unit 2-sphere
to $\overline{S_c(X)}$ such that the boundary of the convex region is mapped to the
$n$-gon both of which have the same length. By Lemma \ref{lem2.0} there exists a
point $\overline{m}$ on the 2-sphere such that $B(\overline{m}, \pi/2)$ covers the convex region, and
from the construction of $\overline{m}$ in the proof it follows that $\overline{m}$ is in the convex
region. Let $m$ be the image of $\overline{m}$ in $\overline{S_c(X)}$. As the map does not increase
distance, the $n$-gon is in $B(m, \pi/2)$. Proposition \ref{prop2.3} implies that $c$ is not
the circumcenter, a contradiction. \hfill \Box

Next we show Theorem \ref{thm1} by using Theorem \ref{thm1} to derive a contradiction.

**Proof.** We claim that joining any two consecutive segments gives a local
geodesic. For this it suffices to show that for any $m \in [x, g \cdot x]$, $d(m, g \cdot m) = 2\pi/n$.
Suppose not, then take $m \in [x, g \cdot x]$ such that $d(m, g \cdot m) < 2\pi/n$. The
segments $[g^i \cdot m, g^{i+1} \cdot m]$ form a curve shorter than $2\pi$, so using Reshetnyak
majorization theorem as before this curve is contained in an open ball of
radius $\pi/2$. Since $\overline{S_c(X)}$ is CAT(1), the orbit of $m$ has a unique circumcenter
$q$ in $\overline{S_c(X)}$, which is fixed by $g$. The distance between $q$ and the orbit of $\chi$
equals $d(q, \chi)$, and by Proposition \ref{prop2.3} this distance is at least $\pi/2$, otherwise
$c$ could not be the circumcenter of the orbit of $x$, so there exists $\chi' \in [x, m]$ with
$d(\chi', q) = \pi/2$.

Applying Theorem \ref{thm1} to $\overline{S_c(X)}$ and the orbit of $m$, we see that $\angle_q(m, g \cdot m) \geq 2\pi/n$. Then
\[
\angle_q(\chi', m) + \angle_q(m, g \cdot \chi') = \angle_q(g \cdot \chi', g \cdot m) + \angle_q(m, g \cdot \chi') \\
\geq \angle_q(m, g \cdot m) \geq \frac{2\pi}{n},
\]
while
\[
d(\chi', m) + d(m, g \cdot \chi') \leq d(\chi', m) + d(m, g \cdot \chi) + d(g \cdot \chi, g \cdot \chi') \\
= d(\chi', m) + d(m, g \cdot \chi) + d(\chi, \chi') \\
= d(\chi, g \cdot \chi) = \frac{2\pi}{n}.
\]
Construct the comparison triangles $\triangle(q, \chi', m)$ of $\triangle(q, \chi', m)$ and $\triangle(q, m, g \cdot \chi')$ of $\triangle(q, m, g \cdot \chi')$ on the unit 2-sphere on the opposite sides of $[q, m]$. Since $d(q, m) < \pi/2$, the segments $[\chi', m]$ and $[m, g \cdot \chi']$ do not form a geodesic, so
\[
d(\chi', g \cdot \chi') < d(\chi', m) + d(m, g \cdot \chi') \leq \frac{2\pi}{n},
\]
As $d(q, \chi') = d(q, g \cdot \chi') = \pi/2$, in the triangle $\triangle(q, \chi', g \cdot \chi')$ we have
\[
\angle_q(\chi', m) + \angle_q(m, g \cdot \chi') \geq \frac{2\pi}{n}.
\]
Hence a contradiction. (Note that here if we used $\chi$ instead of $\chi'$, then $d(q, \chi) \geq \pi/2$ only implies $\angle_q(\chi, g \cdot \chi) \geq d(\chi, g \cdot \chi)$, and the argument would not work.)

The claim means that the concatenation of $[g^i \cdot \chi, g^{i+1} \cdot \chi]$ is a local geodesic in the CAT(1) space $S_c(X)$, thus it is an isometric embedding of a circle of length $2\pi$.

The Euclidean cone over this circle is a 2-flat in the tangent cone. Since $g$ acts on the circle by rotation of $2\pi/n$, this gives the same rotation by $g$ on the tangent cone.

Finally, we give an inequality on distances between 3 consecutive orbit points in CAT(0) space using the above inequality. We suppose the order of $g$ is at least 4, since if the order is 3, the orbit points must form an equilateral triangle.

**Corollary 2.5.** Let $X$ be a complete CAT(0) space. Suppose $g$ has order $n \geq 4$. For any point $x$ not fixed by $g$,
\[
d(g^2 \cdot x, x) \leq 2 \cos \left( \frac{\pi}{n} \right) d(g \cdot x, x),
\]
with equality holds iff $g^i \cdot x$ are vertices of an isometrically embedded flat regular $n$-gon.

**Proof.** To simplify notation we denote $g^i \cdot x$ and $g^2 \cdot x$ by $y$ and $z$. Consider the two comparison triangles $\triangle(\bar{\tau}, \bar{\pi}, \bar{y})$ and $\triangle(\bar{\pi}, \bar{y}, \bar{\tau})$ on the flat plane with $\bar{\tau}$ and $\bar{\pi}$ as their common vertices, and place $\bar{\tau}$ and $\bar{\pi}$ on opposite sides of edge $[\bar{\tau}, \bar{\pi}]$. These two triangles are congruent isosceles triangles.

In the case $\angle_{\bar{\tau}}(\bar{\pi}, \bar{y}) > \pi/2$, we have strict inequality
\[
d(g^2 \cdot x, x) \leq d(g^2 \cdot x, c) + d(c, x) = d(\bar{\tau}, \bar{\pi}) + d(\bar{\tau}, \bar{\pi})
\[
< \sqrt{2}d(\bar{\tau}, \bar{y}) = \sqrt{2}d(g \cdot x, x) \leq 2 \cos \left( \frac{\pi}{n} \right) d(g \cdot x, x).
\]
In the case $\angle(x, y) \leq \pi/2$, draw segment $[x, z]$ intersecting segment $[c, y]$ at the point $p$. By Theorem I we get
\[ \angle(x, y) \geq \frac{\pi}{n}, \]

hence
\[ \angle(x, y) = \frac{1}{2} \angle(x, y) \geq \frac{\pi}{n}, \]

and
\[ d(p, x) = d(z, p) = d(x, y) \cos \angle(x, y) \leq \cos \frac{\pi}{n} d(x, y). \]

Let $p$ be on geodesic segment $[c, y]$ such that $d(c, p) = d(c, y)$. It follows that
\[ d(g^2 \cdot x, x) \leq d(g^2 \cdot x, p) + d(p, x) \leq d(z, p) + d(p, x) \]

\[ \leq 2 \cos \left( \frac{\pi}{n} \right) d(x, y) = 2 \cos \left( \frac{\pi}{n} \right) d(g \cdot x, x). \]

Hence the inequality.

If the equality holds, by (2.3) and (2.4) $\angle(x, y) = \pi/n$, so from (2.1) and (2.2) we have $\angle(x, y) = \angle(c, x, y) = 2 \pi/n$. Then the Flat Triangle Theorem implies that $\triangle(c, x, g \cdot x)$ is flat, so is $\triangle(c, g \cdot x, g^2 \cdot x)$. Moreover, from (2.4) $d(g^2 \cdot x, x) = d(g^2 \cdot x, p) + d(p, x)$, so $[g^2 \cdot x, p] \cup [p, x]$ is a geodesic, which implies that $\triangle(x, g \cdot x, g^2 \cdot x)$ is flat. Hence $\angle(g, x, g^2 \cdot x) = (n-2)\pi/n$, thus the sum of the angles of the $n$-gon equals $(n-2)\pi$, so by a corollary of the Flat Quadrilateral Theorem ([BH99] Exercise II.2.12(1)), the convex hull of the $n$-gon is isometric to a flat regular $n$-gon. The converse is clear. \( \square \)

3. Regular polyhedra

We will now derive results for a finite set of points with symmetry of any regular polyhedron similar to Theorem I. We will employ the tools for tangent cones developed by Lang and Schroeder [LS97] and we will also need the inequality in Theorem II. The main theorem we will prove is the following:

**Theorem 3.1.** Let $W$ be a regular Platonic solid or a regular hypercube or a regular orthoplex with vertices $\overline{c}$, and $G$ be the (orientation preserving) symmetric group of $W$. Suppose that $G$ acts on a $\text{CAT}(\kappa)$ space $X$ by isometries. Let $x_i$ be points in $X$ on which the induced action of $G$ is equivariant with that on $\overline{c}$. Suppose that the set of points $\{x_i\}$ has radius less than $\pi/(2\kappa)$ if $\kappa > 0$. For any edge of $W$ with endpoints $\overline{x_i}$ and $\overline{x_j}$, the Alexandrov angle $\angle(x_i, x_j)$ is no less than the corresponding (Euclidean) angle $\angle(\overline{x_i}, \overline{x_j})$, where $c$ and $\overline{c}$ are the circumcenters of the set $\{x_i\}$ and $W$ respectively.

We note that if $W$ is a regular $n$-simplex, then the above theorem is a special case of the main theorem of Lang and Schroeder ([LS97] Theorem A).
Project the points $x_i$ to the completed space of directions $\overline{S_c(X)}$, then embed $\overline{S_c(X)}$ to its tangent cone, which is the Euclidean cone $C_0(\overline{S_c(X)})$. Let $v_i$ be the images of points $x_i$ in the tangent cone, and let $o$ be the origin of the tangent cone. Then $\overline{S_c(X)}$ is the unit sphere centered at $o$, and $v_i$ are at a distance 1 away from $o$.

**Lemma 3.2.** The origin $o$ is the circumcenter of the points $v_i$.

**Proof.** Assume otherwise, then the segment from $o$ to the circumcenter of the points $v_i$ would make an angle less than $\pi/2$ with segments from $o$ to $v_i$. With slight perturbation if necessary, this would give a geodesic segment from $c$ which shortens the distance to $x_i$ along it, hence $c$ would not be the circumcenter of $x_i$, a contradiction. □

Thus it suffices to prove the theorem for tangent cones. In the following, we let $Y$ be a metric space such that the Euclidean cone $C_0(Y)$ is a complete CAT(0) space.

Recall that the “scalar product” on $C_0(Y)$ is defined as

$$\langle v, w \rangle := \|v\|\|w\| \cos \angle_0(v, w)$$

with the concavity property

$$\langle \gamma(t), w \rangle \geq (1-t) \langle u, w \rangle + t \langle v, w \rangle$$

resulting from the CAT(0) inequality, where $\gamma : [0,1] \to C_0(Y)$ is a geodesic segment with $\gamma(0) = u$ and $\gamma(1) = v$.

**Proposition 3.3 [LS97] Proposition 2.4.** Let $v_1, \ldots, v_n \in C_0(Y)$, and $(v, \lambda), (v', \lambda') \in C$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$. Then

$$\langle v, v' \rangle \geq \sum_{i,j=1}^n \lambda_i \lambda'_j \langle v_i, v_j \rangle.$$ 

For a convex hull $K$ of $n$ points $v_i$, a correspondance $C \subset K \times \Delta_{n-1}$ with the set of $n$-tuples $\Delta_{n-1} = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) : 0 \leq \lambda_i \leq 1, \sum_{i=1}^n \lambda_i = 1\}$ is defined in [LS97] as follows

1. $(v_1, e_i) \in C$, where $e_i$ is the $i$-th unit vector in the standard basis of $\mathbb{R}^n$.

2. For any $(v, \lambda), (v', \lambda') \in C$, let $\gamma : [0,1] \to C_0(Y)$ be a geodesic from $v$ to $v'$, then $(\gamma(t), (1-t)\lambda + t\lambda') \in C$ for all $t \in [0,1]$. The projections of this correspondance to $G$ and $\Delta_n$ are surjective but may not be injective.

If a finite group $G$ acts on the points $v_i$, then it induces an action on $C$ by $g \cdot (v, (\lambda_1, \ldots, \lambda_n)) = (g \cdot v, (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$ where $\sigma$ is a permutation on the indices induced by $g$ such that $\sigma(j) = i$ when $g \cdot v_i = v_j$. It can be seen that $(g \cdot v, (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})) \in C$ for any $g \in G$ and $(v, (\lambda_1, \ldots, \lambda_n)) \in C$.

Suppose 0 is the circumcenter of the points $v_i$, then it is in the closure $\overline{K}$ of the convex hull, so for any $\epsilon < 0$ there exists a point $p \in K$ such that $\|p\| < \epsilon$, thus $C$ contains $(p, (\lambda_1, \ldots, \lambda_n))$. Since the action of $G$ on $C_0(Y)$ stabilized the set $\{v_i\}$, it must fix the unique circumcenter 0. Hence $\|g \cdot p\| = \|p\|$ for any $g$. 

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Let \( G = \{g_j\}_1^n \). Consider \( g_j \cdot (p, (\lambda_1, \cdots, \lambda_n)) = (g_j \cdot p, (\lambda_{\sigma_j(1)}, \cdots, \lambda_{\sigma_j(n)}) \).

In the collection of \( k \)-th coordinates \( \sigma_j(k) \), every \( \lambda_i \) appears the same number of times, so the average
\[
\frac{1}{m} \sum_{j=1}^{m} (\lambda_{\sigma_j(1)}, \cdots, \lambda_{\sigma_j(n)}) = \left( \frac{1}{n}, \cdots, \frac{1}{n} \right)
\]

Now define successively \( p_1 = g_1 \cdot p \), \( p_{j+1} = g_k(\frac{1}{k+1}) \) where \( g_k : [0,1] \to C_0(Y) \) is a geodesic from \( p_k \) to \( g_{k+1} \cdot p \). Then \( (p_k, \frac{1}{k} \sum_{j=1}^{k} (\lambda_{\sigma_j(1)}, \cdots, \lambda_{\sigma_j(n)})) \in C \), so \( (p_m, (\frac{1}{n}, \cdots, \frac{1}{n})) \in C \). If \( \|p_k\|, \|g_{k+1}\| < \epsilon \), then the CAT(0) inequality on \( \triangle(0, p_k, g_{k+1} \cdot p) \) implies that any point on the segment \( [p_k, g_{k+1}] \) has norm less than \( \epsilon \). Therefore we have \( \|p_k\| < \epsilon \) for all \( k \). Using Proposition \( 3.3 \) to \( (p_m, (\frac{1}{n}, \cdots, \frac{1}{n})) \), we have \( \epsilon^2 > \|p_m\| \geq \sum_{i,j=1}^{n} (v_i, v_j) \) Thus
\[
(3.1) \quad 0 \geq \sum_{i,j=1}^{n} (v_i, v_j)
\]

Let \( G \) be the symmetric group of \( W \) acting on \( C(Y) \) by isometries. If \( g \cdot [x_i, x_j] = [x_i, x_j] \) for (unoriented) segments \([x_i, x_j]\) and \([x_i, x_j]\) in \( W \), then \( g \cdot [v_i, v_j] = [v_i, v_j] \) for segments \([v_i, v_j]\) and \([v_i, v_j]\) in \( C_0(Y) \), so \( (v_i, v_j) = (v_i, v_j) \).

We will consider different cases of \( W \).

3.1. \textbf{W is a \( k \)-dimensional orthoplex.} Consider the \( k \)-dimensional orthoplex in \( \mathbb{R}^k \) with vertices \( \{\pm e_i\}_{i=1}^k \), where \( \{e_i\} \) is the standard basis. There are two orbits of chords under the symmetry group action of the orthoplex, one consists of pairs of opposite vertices \( (e_i, -e_i) \), another consists of the edges. For the 2\( k \) points in \( C_0(Y) \) acted on in the same way by the group, label the points corresponding to \( e_i \) as \( v_i \), and those corresponding to \( -e_i \) as \( v_{k+i} \).

From the inequality \( 3.1 \)
\[
0 \geq \sum_{i,j=1}^{2k} (v_i, v_j) = 2k + 2k(2k - 2) (v_1, v_2) + 2k (v_1, v_{n+1})
\]
\[
= 2k (1 + (2k - 2) \cos \angle_o(v_1, v_2) + \cos \angle_o(v_1, v_{n+1}))
\]
\[
\geq 2k (1 + (2k - 2) \cos \angle_o(v_1, v_2) + (-1))
\]
\[
= 2k (2k - 2) \cos \angle_o(v_1, v_2)
\]
Hence \( \angle_o(v_1, v_2) \geq \frac{\pi}{2} = \angle_c(e_1, e_2) \).

3.2. \textbf{W is a \( k \)-dimensional hypercube.} Consider the \( k \)-dimensional hypercube in \( \mathbb{R}^k \) with \( 2^k \) vertices \( (x_1, \cdots, x_k) \) where each \( x_i \) is either \( \pm 1 \). There are \( k \) orbits of chords under the symmetry group action of the hypercube; each class consists of pairs of vertices differing in the same number of coordinates. Edges of the hypercubes are pairs of vertices differing in only one coordinate. These edges make an angle \( \arccos(1 - 2/k) \) at the center.
Assume there are $2^k$ points in $C_0(Y)$ acted on in the same way by the group. For any two points that corresponding to two vertices of the hypercube differing in $i$ coordinates, let $a_i$ be the distance between them and $\alpha_i$ be the angle they make at the circumcenter.

We will need the following simple inequality.

**Lemma 3.4.** Let $X$ be a complete CAT(0) space, $\{x_i\}$ be a finite set of points in $X$, $c$ be the circumcenter of $\{x_i\}$, and $\angle_c(x_1, x_2) \geq \alpha$. If all the points $x_i$ are at an equal distance from $c$, then

$$d(x_1, x_2)^2 \geq \text{diam}(\{x_i\})^2(1 - \cos \alpha)/2.$$

**Proof.** Let $d(x_i, c) = r$. Using the comparison triangle, we see that

$$d(x_1, x_2)^2 \geq 2r^2(1 - \cos \alpha).$$

Since $\text{diam}(\{x_i\}) \leq 2r$, we have the result. \qed

We will do an induction on the dimension $k$. We will prove that for $i \leq k$, $\alpha_1 \geq \arccos(1 - 2/k)$. The case $k = 2$ has already been proved in Corollary 2.5. Assume that the assertion is true for $k - 1$. For $2 \leq m < k$, the symmetry group of the $m$-hypercube is embedded in that of the $k$-hypercube as a subgroup acting on a $m$-dimensional subspace of $\mathbb{R}^k$. By induction hypothesis, the vertices of a $m$-dimensional face of the hypercube satisfy the angle inequality for the $m$-hypercube, so from the above lemma, we have $a_m^2 \leq ma_1^2$.

From the inequality 3.1

$$0 \geq \sum_{i,j=1}^{2^k} \langle v_i, v_j \rangle = 2^k + \sum_{j=1}^{k} 2^k(k \choose j) \cos \alpha_j$$

$$= 2^k + \sum_{j=1}^{k} 2^k(k \choose j) \left(1 - \frac{a_j^2}{2}\right)$$

$$\geq 2^k + \sum_{j=1}^{k} 2^k(k \choose j) \left(1 - \frac{j a_1^2}{2}\right)$$

$$= 2^k(2^k - k \cdot 2^{k-2}a_1^2)$$

Hence $a_1^2 \geq 4/k$, and so $\alpha_1 \geq \arccos(1 - 2/k)$, which is the corresponding angle in the $k$-dimensional hypercube.

3.3. *W is an icosahedron.* There are three orbits of chords in the regular icosahedron, the representatives of which are $(x_0, x_1)$, $(x_0, x_2)$, $(x_0, x_3)$, as shown in Figure 1. Denote the lengths of $(x_0, x_1)$, $(x_0, x_2)$, $(x_0, x_3)$ in $C_0(Y)$ as $a_1, a_2, a_3$ respectively, and the angles they make with the circumcenter as $\alpha_1, \alpha_2, \alpha_3$ respectively. The stabilizer group $C_5$ of $x_{h_0}$ has an orbit $\{x_{h_1}, x_{h_2}, x_{h_3}, x_{h_4}, x_{h_5}\}$, and the corresponding vertices are shown in Figure 3. Then we have $a_2 = d(x_{h_1}, x_{h_3}) \leq 2 \cos(\pi/5)a_1$ by Corollary 2.5.
Figure 1. Representatives of chord orbits in an icosahedron

Figure 2. The pentagon \(x_{h1}x_{h2}x_{h3}x_{h4}x_{h5}\)

From the inequality 3.1

\[
0 \geq \sum_{i,j=1}^{12} \langle v_i, v_j \rangle = 12 + 60 \cos \alpha_1 + 60 \cos \alpha_2 + 12 \cos \alpha_3
\]

\[
= 12 + 60 \left( 1 - \frac{a_1^2}{2} \right) + 60 \left( 1 - \frac{a_2^2}{2} \right) + 12 \cos \alpha_3
\]

\[
\geq 12 + 60 \left( 1 - \frac{\alpha_1^2}{2} \right) + 60 \left( 1 - \frac{(2 \cos(\pi/5) a_1)^2}{2} \right) + 12(-1)
\]

\[
= 120 - 30 a_1^2 \left( 4 \cos^2 \left( \frac{\pi}{5} \right) + 1 \right)
\]

Hence \(a_1^2 \geq \frac{4}{1 + 4 \cos^2 \left( \frac{\pi}{5} \right)} = \frac{1}{\sqrt{5}}\), and so \(\alpha_1 \geq \arccos(1/\sqrt{5})\), which is the corresponding angle in the regular icosahedron.

3.4. **W is a dodecahedron.** There are six orbits of chords in the regular icosahedron, the representatives of which are \((x_0, x_1), (x_0, x_2), (x_0, x_3), (x_0, x_4)\), \((x_0, x_5)\), \((x_0, x_6)\), as shown in Figure 1. Denote the lengths of the corresponding chords \((x_0, x_i)\) as \(\alpha_i\), and the angles they make with the circumcenter as \(\alpha_i\).

In Figure 4, the set of vertices \(\{x_{h1}, x_{h2}, x_{h3}, x_{h4}, x_{h5}\}\) is stabilized by the subgroup \(C_5\), hence \(a_2 = d(x_{h1}, x_{h3}) \leq 2 \cos(\pi/5) a_1\).

The same holds for the set of vertices \(\{x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}\}\), as shown in Figure 5, hence \(a_5 = d(x_{i1}, x_{i3}) \leq 2 \cos(\pi/5) a_2 \leq 4 \cos^2(\pi/5) a_1\).

In Figure 6, since all the chords of the quadrilateral \(\{x_{j0}, x_{j1}, x_{j2}, x_{j3}\}\) have equal length, we may apply the parallelogram inequality for CAT(0) space to it and get...
\[ a_3^2 + a_4^2 = d(x_{j_0}, x_{j_2})^2 + d(x_{j_1}, x_{j_3})^2 \]
\[ \leq d(x_{j_0}, x_{j_1})^2 + d(x_{j_1}, x_{j_2})^2 + d(x_{j_2}, x_{j_3})^2 + d(x_{j_3}, x_{j_0})^2 \]
\[ = 4a_2^2 \leq 16 \cos \frac{\pi}{5} a_1^2 \]
From the inequality 3.1

\[ 0 \geq \sum_{i,j=1}^{20} \langle v_i, v_j \rangle \]

\[ = 20(1 + 3 \cos \alpha_1 + 6 \cos \alpha_2 + 3 \cos \alpha_3 + 3 \cos \alpha_4 + 3 \cos \alpha_5 + \cos \alpha_6) \]

\[ \geq 20(1 + 3 \cos \alpha_1 + 6 \cos \alpha_2 + 3 \cos \alpha_3 + 3 \cos \alpha_4 + 3 \cos \alpha_5 + (-1)) \]

\[ = 20 \left( 3 \left(1 - \frac{a_1^2}{2}\right) + 6 \left(1 - \frac{a_2^2}{2}\right) + 3 \left(2 - \frac{a_3^2}{2} - \frac{a_4^2}{2}\right) + 3 \left(1 - \frac{a_5^2}{2}\right) \right) \]

\[ \geq 20 \left( 18 - \frac{3}{2} a_1^2 - 12 \cos^2 \left(\frac{\pi}{5}\right) a_1^2 - 24 \cos^2 \left(\frac{\pi}{5}\right) a_1^2 - 24 \cos^4 \left(\frac{\pi}{5}\right) a_1^2 \right) \]

Then \( a_1^2 \geq 2 - (2\sqrt{5})/3 \), so \( \alpha_1 \geq \arccos(1 - a_1^2/2) = \arccos(\sqrt{5}/3) \), which is the corresponding angle in the regular dodecahedron.

Hence we have Theorem 3.1.

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