Bayesian Experimental Design with Application to
Dynamical Vehicle Models

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Abstract—In this paper we propose two novel experimental design techniques for designing maximally informative experiments to estimate the parameters of nonlinear dynamical vehicle models. The two techniques include a batch design and a sequential design technique which seek to maximise the expected Shannon information gain of the parameter distribution using either an online or offline approach (respectively). We apply and compare the techniques in both simulation and real-world experiments with a wheeled vehicle. In our simulation experiments, both of our proposed designs provide superior Shannon information gains relative to an unoptimised benchmark technique. In our real-world experiments, our sequential design technique achieves superior expected Shannon information gains relative to our batch design technique and the benchmark technique.

Index Terms—Learning and Adaptive Systems, Wheeled Robots, Dynamics, Experimental Design

I. INTRODUCTION

The rapid rise of robotic vehicle automation and control has driven the need for accurate mathematical models describing their motion. Many of the physical parameters of a vehicle such as the tyre cornering stiffness are difficult to measure and have to be inferred from sensors such as GPS and inertial measurement units. The amount of information we gain about the unknown parameters is largely dependant on how the system is excited. Designing an experiment so that it is maximally informative about the unknown parameters is often the goal of experimental design. This paper presents novel experimental design techniques for parameter estimation applied to dynamical vehicle systems.

There are two main competing frameworks in experimental design: frequentist and Bayesian. In the frequentist framework, an experiment is designed based on some functional of the Fisher information matrix [1]. This is often used in robotics to design optimally exciting trajectories [2]–[4]. On the other hand, Bayesian approaches design experiments based on the posterior distribution [5]. For nonlinear models, the optimal experiment is dependant on the yet to be estimated true model parameters; whilst it is possible in frequentist approaches to average the experiment design over prior knowledge of the parameters, in Bayesian approaches a priori information is naturally incorporated into the design selection process [5]. Thus, Bayesian approaches seem well suited to experiment design for dynamical vehicle models due to their systematic utilisation of prior knowledge. It should be noted that in the special case of Gaussian distributions, the Bayesian and frequentist solutions are equivalent (see [6]).

Experimental design can be approached sequentially or in batch. Batch algorithms aim to select the optimal set of experimental inputs given our prior knowledge, whilst sequential algorithms aim to adaptively select experimental inputs during the conduct of the experiment based on the current knowledge of the parameters [7]. Batch design techniques for parameter estimation are common in robotics [4], [8]–[10]; but the potentially elegant sequential design techniques have not been investigated. In this paper we provide three key contributions:

1) We propose a sequential design technique for selecting the maximally informative experiment for parameter estimation;
2) We illustrate that our sequential design technique can be applied to nonlinear dynamical vehicle models;
3) We conduct and report real-world experiments demonstrating the practical implementation and performance of our sequential design technique.

This paper is structured as follows. In Section II, we outline the experimental design problem in the context of a dynamical vehicle system. In Section III, we propose our batch and sequential design algorithms to solve the experimental design problem. Simulation results are discussed in Section IV and real-world results are discussed in Section V. Lastly we discuss and conclude our results in Section VI.

II. VEHICLE DYNAMICS AND PROBLEM DEFINITION

In this paper, we consider the standard single track vehicle shown in Fig. 1 with a first order steering response. A common representation of this class of vehicles is the following system of continuous-time ordinary differential equations,

\[
\dot{X} = v_x \cos \psi - v_y \sin \psi
\]

\[
\dot{Y} = v_x \sin \psi + v_y \cos \psi
\]

\[
\dot{v}_x = \frac{1}{m} (F_{xf}(U_m) \cos \delta - F_{yf} \sin \delta + F_{xr}(U_m)) + v_y \omega
\]

\[
\dot{v}_y = \frac{1}{m} (F_{xf}(U_m) \sin \delta + F_{yf} \cos \delta + F_{yr}) + v_x \omega
\]

\[
\dot{\psi} = \omega
\]

\[
\dot{\omega} = \frac{\ell}{I} (F_{xf}(U_m) \sin \delta + F_{yf} \cos \delta - F_{yr})
\]

\[
\dot{\delta} = (K_s U_s - \delta)/\tau_s
\]
where the states and parameters are defined in Table I. Here, $U_s$ is the pulse width of the pulse width modulated (PWM) steering servo input, $F_{xf}$ and $F_{yr}$ are the front and rear lateral tyre forces (respectively), and $F_{xf}(U_m)$ and $F_{yr}(U_m)$ are the front and rear longitudinal tyre forces (respectively) as a function of the PWM motor input pulse width $U_m$. The tyre forces are defined in the Appendix using the parameters $\mu$, $\alpha$, $v$, $x$, $yr$, $\delta$, $\tau$, $I$, $F$, $\psi$, $\omega$, $m$, $b$, $K_s$, and $\nu$. See [11] for the derivation of this model.

### Table I

**Single Track Vehicle Notation**

| Symbol | Definition |
|--------|------------|
| $X$, $Y$ | Cartesian coordinates of vehicle in world frame. |
| $v_x$, $v_y$ | Longitudinal and lateral body velocities. |
| $F_{xf}$, $F_{yr}$ | Front and rear lateral tyre forces. |
| $\mu_x$ | Rolling resistance coefficient. |
| $\mu_y$ | Lateral rolling friction coefficient. |
| $C_{\alpha}$ | Cornering stiffness of the wheels. |
| $\delta$ | Steering angle. |
| $\ell$ | Distance from C.G. to front and rear axle. |
| $\psi$ | Heading angle. |
| $\omega$ | Heading rate. |
| $m$ | Vehicle mass. |
| $b$ | Damping coefficient. |
| $I$ | Vehicle moment on inertia. |
| $K_s$ | Steering response gain. |
| $\tau$ | Steering response time constant. |

Discretising the vehicle model in (1)–(7) and adding process noise to represent the model inaccuracies, we can reshape the model into the following discrete-time state space equation,

$$x_{k+1} = f(x_k, \theta, d_k) + \nu_k \quad \text{for } k \geq 0,$$

where $x_k = [X, Y, v_x, v_y, \psi, \omega, \delta]^\top$ is the vector of latent states, $f(\cdot)$ is a nonlinear function dependant on the discretisation method used, $\theta \in \mathbb{R}^p$ is the vector of unknown model parameters, $d_k = [U_m, U_s]^\top \in \mathcal{D}$ is the vector of experimental inputs in the design space $\mathcal{D}$, and $\nu_k \in \mathbb{R}^n$ is assumed zero mean additive noise generated from the multivariate normal distribution, $\mathcal{N}(0, Q)$, with covariance matrix $Q \in \mathbb{R}^{n \times n}$. We assume the latent states $x_k$ of the vehicle can be partially observed through noisy measurements, namely,

$$y_k = h(x_k) + \omega_k,$$

where $y_k \in \mathbb{R}^q$ are the observed measurements, $h(\cdot)$ is a possibly nonlinear observation function and $\omega_k \in \mathbb{R}^q$ is assumed zero mean additive measurement noise generated from the multivariate normal distribution, $\mathcal{N}(0, R)$, with covariance matrix $R \in \mathbb{R}^{q \times q}$. From the measurements, we assume that both the latent states and unknown parameters are identifiable. See [12] and [13] for assessing identifiability of dynamical systems using information-theoretic approaches.

We seek an experimental design to precisely estimate the parameters $\theta$ in the sense of describing input output behaviour of the system. To do so, we note that from an experiment of length $N$, we have the observations $y_{1:N} \triangleq \{y_1, \ldots, y_N\}$. The information provided from the observations about the unknown parameters is conditional on how the system is excited with the experimental inputs $d_{0:N-1} \triangleq \{d_0, \ldots, d_{N-1}\}$. Such information can be measured by a utility function $U(d_{0:N-1}, y_{1:N})$ which, as purposed by Lindley in [14], is defined as the gain in Shannon information (negative entropy) *a priori to a posteriori* about $\theta$. Assuming the prior parameter distribution does not depend on the design $d_{0:N-1}$, which is generally the case, then the utility can be defined as the posterior Shannon information about $\theta$,

$$U(d_{0:N-1}, y_{1:N}) = \int p(\theta|y_{1:N}, d_{0:N-1}) \log p(\theta|y_{1:N}, d_{0:N-1})d\theta,$$  \hspace{1cm} (10)

where $p(\theta|y_{1:N}, d_{0:N-1})$ is the posterior parameter distribution. As the measure $U(d_{0:N-1}, y_{1:N})$ depends on $y_{1:N}$ (unknown *a priori*), we take the expectation as follows:

$$U(d_{0:N-1}) = \int U(d_{0:N-1}, y_{1:N})p(y_{1:N}|d_{0:N-1})dy_{1:N},$$  \hspace{1cm} (11)

where $U(d_{0:N-1})$ is the expected utility function. Thus, the objective of experimental design is to select the experimental inputs $d^*_{0:N-1}$ which maximise the expected utility, namely,

$$d^*_{0:N-1} = \arg\max_{d_{0:N-1} \in \mathcal{D}^N} U(d_{0:N-1}).$$  \hspace{1cm} (12)

One of the key challenges in experimental design is solving the optimisation problem in (12) as it quickly becomes computationally intractable as $N$ increases. A second key challenge is evaluating the integral in (11) which usually cannot be computed in closed form. In the next section, we propose new methods to address these two challenges.

### III. Proposed Experimental Design Techniques

In this section, the first approach we propose is a batch design technique where we constrain the experimental inputs to a parametric waveform to significantly reduce the dimensionality of the optimisation problem. The second approach is a sequential design technique where we iteratively select the current experimental input given all previous inputs and observations. Sequentially optimising over a single input simplifies the optimisation problem, however, we lose consideration of the effects of future inputs on the dynamical model.
A. Proposed Batch Experimental Design Technique

For batch experimental design, optimising the experimental inputs \(d_{0:N−1}\) is feasible only for very small experimental lengths \(N\) [15], [16]. For large \(N\), it is common in the experimental design literature on dynamical systems to use square wave type signals to reduce the dimensionality of the optimisation problem [17]–[19]. For our proposed batch experimental design technique, we constrain the experimental inputs to PRBS which has the advantage of having an optimal crest factor and being a deterministic signal [20].

In practice, PRBS signals can sometimes be hard to realise and may require slew rate constraints (for example, see [17]). However, because our inputs are pulse widths of PWM signals, we assume no slew rate constraints. We can now define the experimental input sequence as,

\[
d_{0:N−1}(f_c) \triangleq c + A \cdot \text{PRBS}(N, f_c),
\]

where \(c\) is the signal offset, \(A\) is the signal amplitude and the function \(\text{PRBS}(N, f_c)\) generates a PRBS signal of length \(N\) with a clock frequency of \(f_c\). The clock period \(1/f_c\) of a PRBS defines the minimum time the signal can stay at one value. We use the ‘idinput’ function from the System Identification Toolbox in MATLAB for our PRBS function. We assume that \(c\) and \(A\) are constant and use \(f_c\) to uniquely define the experimental input sequence \(d_{0:N−1}\). Our goal now becomes finding the optimal clock frequency \(f_c\) that maximises the Shannon information gain about \(\theta\), namely,

\[
f_c^* = \arg\max_{f_c \in \mathcal{F}} \mathcal{U}(d_{0:N−1}(f_c)).
\]

where \(\mathcal{U}(d_{0:N−1}(f_c))\) is given by (11). Because the integral in (11) cannot be solved analytically, we propose to approximate it via Monte Carlo integration, that is,

\[
\hat{\mathcal{U}}(d_{0:N−1}(f_c)) \triangleq \frac{1}{M} \sum_{i=1}^{M} \mathcal{U}(d_{0:N−1}(f_c), y_{1:N}^{(i)}),
\]

where the Monte Carlo samples \(y_{1:N}^{(i)}\) for \(i = 1, \ldots, M\) are generated by first sampling the parameters according to \(\theta^{(i)} \sim p(\theta)\) and then sampling \(y_{1:N}^{(i)}\) from the distribution \(p(y_{1:N}(\theta^{(i)}, d_{0:N−1}(f_c)))\). The optimisation problem we then seek to solve is:

\[
f_c^* = \arg\max_{f_c \in \mathcal{F}} \hat{\mathcal{U}}(d_{0:N−1}(f_c)),
\]

We measure the utility \(\mathcal{U}(d_{0:N−1}(f_c), y_{1:N})\) in (15) as the posterior Shannon information about \(\theta\) as illustrated in (10). Taking the posterior distribution as Gaussian, this can be evaluated as follows:

\[
\mathcal{U}(d_{0:N−1}(f_c), y_{1:N}) = C_p - 0.5 \log(\det P_{\theta,N}),
\]

where \(C_p \triangleq -0.5 \rho \log(1 + 2\pi^2 \rho), \rho\) is the dimension of \(\theta\), and \(P_{\theta,N}\) is the covariance matrix of the posterior parameter distribution. The covariance matrix \(P_{\theta,N}\) is the only term dependant on \(f_c\) and \(y_{1:N}\) in (17) and so the optimisation problem (16) is equivalent to Bayesian D-optimality [21] which seeks to minimise the expected value of \(\log(\det P_{\theta,N})\). We use a dual unscented Kalman Filter (D-UKF) [22] to provide a Gaussian approximation of the parameter distribution and state distribution at each time step. In the D-UKF, the unknown parameters are assigned the process and observation models:

\[
\theta_{k+1} = \theta_k \quad \text{and} \quad y_{k+1} = h(f(x_k, \theta_k, d_k)) + \epsilon_{k+1},
\]

where \(\epsilon_{k+1} \in \mathbb{R}^q\) is measurement noise assumed generated from the multivariate normal distribution, \(\mathcal{N}(0, Z)\), with covariance matrix \(Z \in \mathbb{R}^{q \times q}\).

Starting with an initial Gaussian state distribution \(\mathcal{N}(\hat{x}_0, P_{x,0})\) and an initial parameter distribution \(\mathcal{N}(\hat{\theta}_0, P_{\theta,0})\), the D-UKF first uses the state distribution mean \(\hat{x}_0\) in (18) to update the parameter distribution. The D-UKF then uses the updated parameter distribution mean with the state process and observation models (8)–(9) to update the state distribution. This repeats at each time step to give us an estimate of the state distribution \(\mathcal{N}(\hat{x}_k, P_{x,k})\) and the parameter distribution \(\mathcal{N}(\hat{\theta}_k, P_{\theta,k})\) for use in (17).

B. Multisine Batch Experimental Design Technique

Experimental design for robotics has classically relied on the concepts of sufficiently informative experiments and persistence of excitation. For a linear system of order \(n\), an experiment is sufficiently informative if it is persistently exciting of order \(2n\) (cf. [20]), which typically implies constructing an input signal using as many sinusoids as the number of unknown parameters. For nonlinear systems, however, designing sufficiently informative experiments is a nontrivial problem. A popular technique is to use a finite Fourier series, also known as a multisine, and optimise the frequencies to maximise the log of the determinant of the Fisher information matrix [2], [23]. Thus, we propose an alternative batch design technique that uses a multisine signal instead of a PRBS for comparison.

One key benefit of multisine signals is that they may be more plant-friendly than PRBS signals (e.g., when there are slew rate limits on inputs/actuators). Multisines may also be beneficial in the presence of static nonlinearities due to their high entropy compared to PRBS signals (cf. [24]).

Using a multisine to define our design sequence, we get,

\[
d_k = \kappa + \sum_{i=1}^{C} A_i \sin(2\pi f_0 k T_s + \phi_i), \quad k = 0, \ldots, N−1
\]

where \(\kappa\) is the offset of the signal, \(C\) is the number of sinusoidal components, \(A_i\) are the amplitudes of each component, \(f_0\) is the fundamental frequency and the frequency resolution of the multisine, \(T_s\) is the sampling time and \(\phi_i\) are the phases of the sinusoids. Here, each of the sinusoids are harmonics of the fundamental frequency \(f_0\) to ensure periodicity. For simplicity, we assume the amplitudes \(A_i\) of each sinusoid are equal, i.e. the power in each frequency component is equal. In Schoukens and Ljung [25], it is suggested to scale the amplitude of the multisine to cover the full amplitude range of interest. Additionally, it is also recommended that at least one, and preferably a few periods of the multisine should be used in an experiment. Thus, we use three periods of the multisine and set \(f_0 = 3/(T_s N)\), where \(T_s N\) is total time of the experiment.

The phases of the multisine should also be selected such that the crest factor of the signal is minimised. A simple and
computationally inexpensive method to reduce the crest factor is to use the Schroeder phase choices [26], that is, \( \phi_i = \phi_1 - i(i - 1)/N \), for \( i = 1, \ldots, C \), where \( \phi_1 \) is the phase of the first sinusoidal component. The last variable to select in the multisine is the number of sinusoidal components \( C \). Thus, we can optimise the design sequence by following (14)–(16) with the maximising argument being \( C \) in place of \( f_c \).

**C. Proposed Sequential Experimental Design Technique**

As an alternative to our batch design technique (16), we now propose a sequential design technique to sequentially select the current experimental input given the previous inputs and observations. At time \( k \), our goal is thus to find the experimental input which maximises the expected utility, i.e.,

\[
d_k^* = \underset{d \in D}{\text{argmax}} U(d|y_{1:k}, d_{0:k-1}).
\]

Again, the expected utility is not easily calculated, so we approximate it with Monte Carlo integration, i.e.,

\[
\hat{U}(d|y_{1:k}, d_{0:k-1}) \triangleq \frac{1}{M} \sum_{i=1}^{M} U(d, \hat{y}_{k+1}^{(i)}|y_{1:k}, d_{0:k-1}).
\]

Here, \( \hat{y}_{k+1}^{(i)} \) for \( i = 1, \ldots, M \) are possible future observations generated by first sampling the parameters \( \theta_{k}^{(i)} \) from \( p(\theta_{k}|y_{1:k}, d_{0:k-1}) \) and then sampling from the distribution \( p(y_{k+1}|y_{1:k}, \theta_{k}^{(i)}, d, d_{0:k-1}) \). We now seek to solve:

\[
d_k^* = \underset{d \in D}{\text{argmax}} \hat{U}(d|y_{1:k}, d_{0:k-1}).
\]

We again use the posterior Shannon information to evaluate the utility function, namely,

\[
\hat{U}(d, \hat{y}_{k+1}|y_{1:k}, d_{0:k-1}) \triangleq C_{\rho} - 0.5 \log (\text{det} \bar{P}_{\theta, k+1}),
\]

where \( \bar{P}_{\theta, k+1} \) is the covariance matrix of the possible posterior distribution, which we calculate using a D-UKF. The complete design selection process is illustrated in Algorithm 1.

**Algorithm 1 Sequential Design Technique**

At time \( k \), we have \( \{\hat{x}_k, P_{x,k}\} \) and \( \{\theta_k, P_{\theta,k}\} \).

for \( d \in D \) do

for \( i = 1, \ldots, M \) do

Sample possible response \( \hat{y}_{k+1}^{(i)} \).

Compute \( \bar{P}_{\theta, k+1}^{(i)} \) using a D-UKF.

Compute the utility \( \hat{U}(d, \hat{y}_{k+1}^{(i)}|y_{1:k}, d_{0:k-1}) \) using (22).

end for

Compute \( \hat{U}(d|y_{1:k}, d_{0:k-1}) \) using (20).

end for

Select design point which maximises the expected utility: \( d_k^* = \underset{d \in D}{\text{argmax}} \hat{U}(d|y_{1:k}, d_{0:k-1}) \)

D. Computational Complexity of Proposed Methods

Experiment design inherently involves optimisation over design sequences. Without constraints, batch optimisation of the design sequence becomes computationally infeasible for large \( N \). In this paper, we have constrained the design sequence to both PRBS and multisine signals which reduces the number of design choices from \( N \) (all design points) to one (either the PRBS clock frequency or the number of sinusoidal components in the multisine). Nonetheless, other constraints on the design sequence that reduce the dimensionality of the optimisation problem can be employed. For instance, our proposed sequential design technique avoids the parametric parameterisation of the input signal by optimising over \( N = 1 \) at each time step.

**IV. SIMULATION STUDY**

In this section, we present our simulation study comparing the performance of our sequential, PRBS batch and multisine batch design techniques. We first outline our experimental setup including the prior distributions and input constraints together with the implementations of our sequential, PRBS batch and multisine batch design techniques. Lastly, we present and discuss our simulation results.

A. Prior Distributions and Input constraints

For our vehicle model in simulation, we discretise the continuous time model in (1)–(7) using the fourth order Runge-Kutta discretisation method with a sampling period of 0.01s. This length of 0.01s coincides with the measurement sampling rate in our simulations and is the minimal choice due to the computational requirements of running our online sequential design technique in real-time. For our discretised model, we add process noise with a covariance matrix of

\[
Q = \text{diag}([0 0 10^{-6} 10^{-6} 0 10^{-6} 10^{-8}])
\]

where \( \text{diag}(X) \) is a diagonal matrix with the elements of the vector \( X \) on its diagonal. Our choice of the size of \( Q \) represents a small model inaccuracy. The size is also experimentally dependant on the sampling period (0.01s).

We assume that the states \( v_x, v_y \) and \( \omega \) are observed through noisy measurements described by a diagonal covariance matrix, \( R \), with diagonal (variance) elements 0.1. This \( R \) matrix was selected to match the error variance in the VICON motion capture cameras used in our real-world experiments (see Section V). We set the true initial vehicle state vector to \([0, 0, 0, 0, 0, 0, 0, 0]^T\) and assume the initial state distribution

![Fig. 2. Block diagram of the sequential design process.](image-url)
used in the D-UKFs is Gaussian with a mean set to the true initial state and a covariance matrix set to,
\[ P_{x,0} = \text{diag}([0.01 \ 0.01 \ 4 \ 4 \ 0.01 \ 4 \ 1]). \] (24)
Here, the size of the diagonals in \( P_{x,0} \) represent a moderate uncertainty about the state, with a higher uncertainty about the velocity states compared to the position states.

Overall, the six parameters that are assumed unknown are \( \theta = [\mu_x, b, K_m, I_c, C_o, \mu_y]^T \). In this study, we assume a true parameter vector of \( \theta_{\text{true}} = [0.2, 1, 0.3, 0.13, 1200, 10]^T \). We set \( m = 1.6kg \), \( \ell = 0.13m \), \( K_c = 0.002348 \) and \( \tau_s = 0.0625 \) which are used to match the real world experiments detailed later (see Section V). We use a normal distribution for the prior with mean \( \theta_{\text{true}} \) and covariance matrix
\[ P_{\theta,0} = \text{diag}([11 \ 11 \ 2.8 \ 0.45 \ 1.1 \times 10^5 \ 25]). \] (25)
Here, the diagonal values of \( P_{\theta,0} \) were selected to represent a large uncertainty about each of the unknown parameters.

To estimate the unknown model parameters, we split our real-world experiments into two separate cases: a longitudinal case where we fix the steering angle to \( 0^\circ \) and estimate the parameters \( \mu_x, b \) and \( K_m \), and a lateral case where we fix the motor input to \( U_m = 1585\mu s \) and estimate the parameters \( I_c, C_o \) and \( \mu_y \). Splitting the vehicle model into longitudinal and lateral dynamics is commonly used to separate and simplify the parameter estimation problem (cf. [27], [28]).

For the longitudinal case, the vehicle model is affine in the unknown parameters. Thus, the posterior parameter distribution generated from the UKF algorithm will coincide with the standard Kalman filter (KF) solution.

In both longitudinal and lateral cases, we assume that the motor \( U_m \) and steering servo \( U_s \) inputs are constrained to,
\[ 1560\mu s \leq U_m \leq 1600\mu s \text{ and } 1110\mu s \leq U_s \leq 1660\mu s. \] (26)

B. Sequential Design Simulation Setup

For our sequential design technique in the longitudinal and lateral cases, we assume that \( \theta_{\text{true}} \) is kept constant at \( 1385\mu s \). Again, the steering input \( U_s \) is kept constant at \( 1385\mu s \). For the lateral case, we use a normal distribution for the prior with mean \( \theta_{\text{true}} \) and covariance matrix
\[ P_{\theta,0} = \text{diag}([11 \ 11 \ 2.8 \ 0.45 \ 1.1 \times 10^5 \ 25]). \] (25)
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B. Sequential Design Simulation Setup

For our sequential design technique in the longitudinal and lateral cases, we use \( M = 50 \) to approximate the expected utility in (20). We also assume a discrete design space with 10 linearly spaced levels between the constraints detailed in (26). This design space assumptions is used to reduce the computational load of the technique.

C. PRBS Batch Design Simulation Setup

For the longitudinal case, we use a PRBS for the motor input \( U_m \) with \( c = 1580\mu s \) and \( A = 20\mu s \). The steering input \( U_s \) is kept constant at \( 1385\mu s \) which is approximately a steering angle of \( 0^\circ \). For the lateral case, we use a PRBS for the steering input \( U_s \) with \( c = 1385\mu s \) and \( A = 282\mu s \). For both the longitudinal and lateral cases, we constrain the clock frequency of the PRBS signal to \( 0.5Hz \leq f_c \leq 50Hz \) and discolour the frequency space between these limits using \( L = 150 \) linearly spaced clock frequency levels.

The expected utility of a clock frequency is approximated using (15) with \( M = 500 \) and we find that the clock frequency that maximises the expected utility for the longitudinal case is \( 5.1Hz \). For the lateral case, we use the true values of \( \mu_x = 0.044, b = 1.15 \) and \( K_m = 0.18 \), assuming that they were estimated from the longitudinal case, and find that the clock frequency that maximises the expected utility is \( 11.13Hz \).

D. Multisine Batch Design Simulation Setup

For the longitudinal case, we use a multisine for the motor input \( U_m \) with \( \kappa = 1580\mu s \). Again, the steering input \( U_s \) is kept constant at \( 1385\mu s \). For the lateral case, we use a multisine for the steering input \( U_s \) with \( h = 1385\mu s \). For both the longitudinal and and lateral cases, experiments are run with a length of \( N = 10000 \) and \( T_s = 0.01 \). Thus, the fundamental frequency of the multisines is set to \( f_0 = 0.3 \) Hz. The amplitudes of the multisines in the longitudinal and lateral cases are scaled to cover the motor input and steering servo input amplitude ranges defined in (26).

We perform a brute force optimisation for the optimal number of sinusoidal components \( C \) over \( C \in \{1, 2, \ldots, 50\} \). The expected utility of each \( C \) value is approximated using (15) with \( M = 500 \). For the longitudinal and lateral cases, we find optimal values of \( C = 6 \) and \( C = 11 \), respectively.

E. Simulation Results

For both the longitudinal and lateral cases, we perform 500 Monte Carlo (MC) trials of:
\[ \bullet \] Our proposed sequential design technique;
\[ \bullet \] Our proposed PRBS batch design technique;
\[ \bullet \] Our multisine batch design technique; and,
\[ \bullet \] A random design technique.
Here, the random design technique simply selects a random experimental input from the design space - it is used as a benchmark for the sequential and batch techniques. All MC trials have an experimental length of 10s with a sampling time of 0.01s. Performance for each trial is measured using the Shannon information gain from the prior to posterior parameter distribution.

Figs. 3 and 4 illustrate boxplots of the performance of each technique for the longitudinal and lateral cases respectively. For the longitudinal experiments, the median Shannon information gain for the 500 simulations using each technique is:
\[ \bullet \] 17.9 nats for the sequential design technique,
\[ \bullet \] 17.66 nats for the PRBS design technique,
For both the longitudinal and lateral cases, these results suggest that our PRBS batch design technique achieves a higher median Shannon information gain than the multisinse batch design technique. Thus, it seems that the higher power PRBS signal choice may be superior to the higher entropy multisinse signal choice. Because of this, in our real-world experiments, we remove the multisinse batch technique and focus on comparing our sequential and PRBS batch design techniques. A further discussion regarding our simulation results will be provided in Section VI.

V. REAL WORLD EXPERIMENTS

A. Vehicle System

In our real-world experiments, we use a remote controlled (RC) electric four wheeled driven, front wheel steered car that allows for PWM inputs to the speed controller and steering servo (see Fig. 5). The gain $K_s$ and time constant $\tau_s$ of the steering response was estimated using a slow motion camera to be 0.002348 and 0.0625s respectively. The mass and distance from the centre of gravity (C.G.) to the front and rear axles have been measured as 1.6kg and 0.13m, respectively.

For the real-world experiments, we can observe the longitudinal velocity $v_x$, lateral velocity $v_y$ and heading rate $\omega$ through noisy measurements sampled at 0.01s from a VICON motion capture system in a 3m x 4m area. The length of each experiment is set to approximately 1.3 seconds due to the small experimental area. The variance of the measurement errors from the VICON system were estimated as 0.1m/s for the longitudinal and lateral velocity; and 0.1rad/s for the heading rate. We therefore use the same $R$ covariance matrix as in the simulation study and a process noise covariance matrix of $Q = \text{diag } \left(\begin{bmatrix} 0 & 0 & 0.01 & 0.01 & 0.01 \end{bmatrix}\right)$. (27)

We now present our real-world experiments and results.

The $Q$ matrix is larger in the real-world study compared to the simulation study to account for the additional model inaccuracies from using a real system. The initial state estimate is set to $[0,0,0,0,0,0]^T$. However, for the lateral case, we start the vehicle with a forward speed of 1.6m/s to approximately keep the forward velocity in steady state for the experiment. Thus, we set the initial state estimate of $v_x$ to 1.6m/s. The covariance of the initial state is unchanged from the simulation study where it was defined in (24). We also use the same initial parameter distribution that was used in the simulation study, that is, $\theta_0 = [0.2,1,0.3,0.13,1200,10]^T$ and a covariance matrix given by (25).

B. Sequential and Batch Design Real-World Setup

All parameters for the sequential design technique and the random design technique remain unchanged from the simulation experiments. Due to the computational limits of performing our sequential technique in real-time, we are unable to perform the Monte Carlo integration as illustrated in (20), and instead use the approximation $\hat{U}(d|y_{1:k},d_{0,k-1}) \approx U(d, \bar{y}_{k+1}|y_{1:k},d_{0,k-1})$, where $\bar{y}_{k+1}$ is the mean of the distribution $p(y_{k+1}|y_{1:k},\theta_k,d_k)$. Specifically, we no longer average the utility over $\theta$ and $\bar{y}_{k+1}$ resulting instead in a local optimisation around their means. This approach is often unsuitable for uncertain parameters, however, due to the uncertainty being sequentially updated throughout an experiment, the validity of this assumption increases over time.

For the setup of our batch design technique, we have to run a simulation of the batch design technique again with the new system and experimental parameters that have changed from the simulation study. For this, we again assume a discrete design space with 10 linearly spaced levels between the constraints detailed in (26). We set $M = 500$ for the Monte Carlo integration and calculate that the PRBS clock frequency which maximises the expected utility is $f_c = 2.81\text{Hz}$ for the longitudinal case and $f_c = 10.8\text{Hz}$ for the lateral case.

C. Real-World Results

Following our simulation study, we perform separate experiments for both longitudinal and lateral cases. In each case, we
perform seven experiments using our sequential design technique, seven experiments using our batch design technique; and seven experiments using a random design technique. For each experiment, we record the Shannon information gain about the unknown parameters. We compare the results from the longitudinal and lateral cases separately.

1) Longitudinal Case: Fig. 6 displays the Shannon information gains achieved in each of the seven experiments for the different design techniques for the longitudinal case. The average Shannon information gains for the 7 longitudinal experiments performed for each design technique are,

- 11.35 nats for our sequential design technique;
- 11.02 nats for our batch design technique; and,
- 10.95 nats for the random design technique.

We perform a simple model validation test with the models generated using each design technique. These models are generated from the experiments that achieved the median Shannon information gain for each design technique. However, due to the short experiment length and the motor input constraints, the velocities of the estimated models drifted away from the validation data set over time.

2) Lateral Case: For the lateral case, we use the longitudinal parameter estimates from the highest performing experiment. The longitudinal parameter estimates are as follows: 0.31 for the rolling resistance coefficient, 2.61Ns/m for the damping coefficient and 0.43 for the motor gain. These parameter values are assumed to be known constants in the lateral case. Fig. 7 displays the Shannon information gains achieved in the seven experiments for each design technique for the lateral case. In these lateral case experiments, the average Shannon information gains for each technique are,

- 5.58 nats for our sequential design technique
- 4.43 nats for our batch design technique
- 4.41 nats for the random design technique.

We again perform a model validation test with the models generated using each design technique. Fig. 8 compares the lateral velocities of the estimated models against measured lateral velocities in a validation data set generated using a 4Hz square wave. The RMS of the error between the measured lateral velocity and the lateral velocity of the models generated from the sequential, batch and random design techniques are 0.108m/s, 0.14m/s and 0.192m/s (respectively).

VI. DISCUSSION AND CONCLUSIONS

We have compared the performance of our sequential design technique, our batch design techniques and a random design technique in simulation and in real-world experiments for a dynamical vehicle model. In simulation, our sequential design technique was superior to the two batch design techniques and the random design technique achieving a higher median Shannon information gain. Additionally, both of the batch design techniques outperformed the random design technique.

Similar results were revealed in our real-world experiments, with our sequential design technique producing the highest average Shannon information gain, our PRBS batch design technique producing the second highest, and the random design technique producing the lowest. However, there was one randomly designed experiment in our real-world results displayed in Fig. 7 that achieved a higher Shannon information gain than all batch experimental designs. This was likely due to our short experiment length increasing the probability of a random design producing an informative experiment. With longer experiment lengths, such as the 10 seconds used in simulation, we would expect this event to occur less frequently.

In addition to delivering higher average Shannon information gains than the batch and random design techniques, our sequential design technique also yielded models with lower RMSE’s in its predicted lateral velocity. Overall, it seems that our sequential design technique outperforms the batch design techniques due to its ability to recursively update the prior distribution throughout an experiment. Future work
could investigate whether better designs are selected using a multistep look-ahead version of our sequential design technique.

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APPENDIX

Assuming there is no longitudinal wheel slip (no spinning without traction) and that the driving force is applied equally to the front and rear wheels, we calculate the longitudinal front and rear wheel forces using,

\[
F_{xf}(U_m) = F_{xr}(U_m) = \frac{F_d(U_m)}{2} - F_r - F_b,
\]

where \(F_{xf}(U_m)\) is the driving force \(F_r\), the rolling resistance and \(F_b\) is a damping force added to account for some of the unmodelled losses in the motor. Assuming the motor response is fast and will have negligible effects on the vehicles motion, we approximate the driving force with \(F_d(U_m) = K_m \cdot U_m\) where \(K_m\) is the motor gain and \(U_m\) is the motor input. The damping force is given by \(F_b = b \cdot \sqrt{v_x^2 + v_y^2}\) where \(b\) is the damping coefficient. We approximate the rolling resistance via

\[
F_r = F_z \cdot \mu_s \cdot \text{sgn} v_x
\]

(28)

where \(F_z\) is the normal force acting on the wheel. Around the discontinuity in (28), it becomes difficult to solve the ordinary differential equations describing the vehicles motion, therefore, we use the continuous approximation of \(F_r = F_z \cdot \frac{2}{\pi} \cdot \tan(\pi v_x / 2)\). For the normal force \(F_z\) we assume that the vehicles mass is split evenly between the front and rear, that is, \(F_z = \frac{m \cdot g}{2}\). We model the tyre dynamics using the Fiala tyre model [29] which uses piecewise linear equations to approximate the lateral and rear tyre forces as

\[
F_{yl}(\alpha) = \begin{cases} -C_\alpha \cdot \tan \alpha + \frac{C_\alpha^2}{3 \mu_y} \cdot \tan \alpha |\tan \alpha|, & |\alpha| < \alpha_{sl} \\ -\mu_y F_z \cdot \text{sgn} \alpha, & |\alpha| \geq \alpha_{sl} \end{cases}
\]

where \(i = \{f, r\}\) and \(\alpha_{sl} = \arctan \left( \frac{3 \mu_y F_z}{C_\alpha} \right)\) with \(\mu_y\) being the lateral friction coefficient of the wheels which is assumed to be equal for the front and rear wheels. Here, the front and rear slip angles \(\alpha_f\) and \(\alpha_r\) (respectively) are calculated using,

\[
\alpha_f = \arctan \left( \frac{V_y + \omega \ell}{V_x} - \delta \right), \quad \alpha_r = \arctan \left( \frac{V_y - \omega \ell}{V_x} \right).
\]