TENSOR PRODUCTS FOR GELFAND-SHILOV AND PILIPOVIĆ DISTRIBUTION SPACES

JOACHIM TOFT

Abstract. We show basic properties on tensor products for Gelfand-Shilov distributions and Pilipović distributions. This also includes the Fubini’s property of such tensor products. We also apply the Fubini property to deduce some properties for short-time Fourier transforms of Gelfand-Shilov and Pilipović distributions.

0. Introduction

An important issue in mathematics concerns tensor products. When considering the functions $f_j$ defined on $\Omega_j \subseteq \mathbb{R}^{d_j}$, $j = 1, 2$, and with values in $\mathbb{C}$, their tensor product $f_1 \otimes f_2$ is the function from $\Omega_1 \times \Omega_2$ to $\mathbb{C}$ given by the formula

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2), \quad x_j \in \Omega_j, \; j = 1, 2.$$ 

Let $f_j, \varphi_j \in S(\mathbb{R}^{d_j})$, $f = f_1 \otimes f_2, \varphi \in S(\mathbb{R}^{d_1+d_2})$, and let $\psi_1$ and $\psi_2$ be given by

$$\psi_1(x_1) = \langle f_2, \varphi(x_1, \cdot) \rangle \quad \text{and} \quad \psi_2(x_2) = \langle f_1, \varphi(\cdot, x_2) \rangle,$$

(For notations, see [7] and Section 1) Then it follows that

$$\langle f, \varphi_1 \otimes \varphi_2 \rangle = \langle f_1, \varphi_1 \rangle \langle f_2, \varphi_2 \rangle,$$  

and that the Fubini’s property

$$\langle f, \varphi \rangle = \langle f_1, \psi_1 \rangle = \langle f_2, \psi_2 \rangle$$

holds.

The formulae (0.2) and (0.3) are essential when searching for extension of tensor products to distributions. By the analysis in [7] Chapter V and VII, we have the following.

Theorem 0.1. Let $f_j \in \mathcal{S}'(\mathbb{R}^{d_j}), \varphi \in \mathcal{S}(\mathbb{R}^{d_1+d_2})$ and let $\psi_1$ be given by (0.1), $j = 1, 2$. Then $\psi_j \in \mathcal{S}(\mathbb{R}^{d_2}), j = 1, 2$, and there is a unique $f \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$ such that for every $\varphi_1 \in \mathcal{S}(\mathbb{R}^{d_1})$ and $\varphi_2 \in \mathcal{S}(\mathbb{R}^{d_2})$, (0.2) and (0.3) hold.

2010 Mathematics Subject Classification. 46A32, 46Fxx, 46M05.

Key words and phrases. Ultradistributions, Fubini.
The existence of a distribution $f$ in the previous theorem which satisfies (0.2) can also be deduced by a general and abstract result on tensor products for nuclear spaces (see [15, Chapter 50]). On the other hand, in order to reach the Fubbini property (0.3), it seems that more structures are needed.

A more specific approach in the lines of the ideas in [15] is indicated in [8, 12], where $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ are described by suitable series expansions of Hermite functions. By following such an approach, the situations are essentially reduced to questions on tensor products of weighted $\ell^2$ spaces, and both properties (0.2) and (0.3) follow from such an approach.

In Sections 2 and 3 we show that Theorem 0.1 holds in the context of Gelfand-Shilov spaces, Pilipović spaces and their distribution (dual) spaces. In particular, we prove that the following results hold true.

**Theorem 0.2.** Let $s_j, \sigma_j > 0$, $f_j \in (S^{\sigma_j}_{s_j})'(\mathbb{R}^{d_j})$, $\varphi \in S^{\sigma_1, \sigma_2}_{s_1, s_2}(\mathbb{R}^{d_1+d_2})$ and let $\psi_j$ be given by (0.1), $j = 1, 2$. Then $\psi_j \in S^{\sigma_j}_{s_j}(\mathbb{R}^{d_j})$, $j = 1, 2$, and there is a unique $f \in (S^{\sigma_1, \sigma_2}_{s_1, s_2})'(\mathbb{R}^{d_1+d_2})$ such that for every $\varphi_1 \in S^{\sigma_1}_{s_1}(\mathbb{R}^{d_1})$ and $\varphi_2 \in S^{\sigma_2}_{s_2}(\mathbb{R}^{d_2})$, (0.2) and (0.3) hold.

The same holds true with $\Sigma_{s_1}^{\sigma_1}$, $(\Sigma_{s_1}^{\sigma_1})'$, $\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2}$ and $(\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2})'$ in place of $S^{\sigma_j}_{s_j}$, $(S^{\sigma_j}_{s_j})'$, $S^{\sigma_1, \sigma_2}_{s_1, s_2}$ and $(S^{\sigma_1, \sigma_2}_{s_1, s_2})'$, respectively, at each occurrence.

**Theorem 0.3.** Let $s \in \mathbb{R}_0$ and let $f_j \in \mathcal{H}_s'(\mathbb{R}^{d_j})$, $\varphi \in \mathcal{H}_s(\mathbb{R}^{d_1+d_2})$ and let $\psi_j$ be given by (0.1), $j = 1, 2$. Then $\psi_j \in \mathcal{H}_s(\mathbb{R}^{d_j})$, $j = 1, 2$, and there is a unique $f \in \mathcal{H}_s'(\mathbb{R}^{d_1+d_2})$ such that for every $\varphi_1 \in \mathcal{H}_s(\mathbb{R}^{d_1})$ and $\varphi_2 \in \mathcal{H}_s(\mathbb{R}^{d_2})$, (0.2) and (0.3) hold.

The same holds true with $\mathcal{H}_{0, s}$ and $\mathcal{H}_{0, s}'$ in place of $\mathcal{H}_s$ and $\mathcal{H}_s'$, respectively, at each occurrence.

The distribution $f$ in Theorem 0.1, Theorem 0.2 or in Theorem 0.3 is called the tensor product of $f_1$ and $f_2$ and is denoted by $f_1 \otimes f_2$ as before.

We remark that Gelfand-Shilov spaces of functions and distributions appear naturally when discussing analyticity and well-posedness of solutions to partial differential equations (cf. [2, 3]). Pilipović spaces of functions and distributions often agree with Fourier-invariant Gelfand-Shilov spaces, and possess convenient mapping properties with respect to the Bargmann transform. They therefore seem to be suitable to have in background on problems in partial differential equations which have been transformed by the Bargmann transform (see [5, 14] for more details).

Since the spaces in Theorems 0.2 and 0.3 are unions and intersections of nuclear spaces, the existence of $f$ satisfying (0.2) may be deduced by the abstract analogous results in [15]. Some parts of Theorem 0.2 are also proved in [8].
In Section 2 we give a proof of Theorem 0.2 by using the framework in [7] for the proof of Theorem 0.1. In Section 3 we use that Pilipović spaces and their distribution spaces can be described by unions and intersections of Hilbert spaces of Hermite series expansions. In similar ways as in [12], this essentially reduce the situation to deal with questions on tensor products of weighted $\ell^2$ spaces.

In the end of Section 2 we also give example on how to apply the Fubbini property (0.3) to deduce certain relations for short-time Fourier transforms (which often called coherent state transform in physics) of Gelfand-Shilov distributions (see Example 2.5). In Section 3 we also discuss such questions for Pilipović spaces which are not Gelfand-Shilov distributions (cf. Remark 3.3).

1. Preliminaries

In this section we recall some basic facts. We start by giving the definition of Gelfand-Shilov spaces. Thereafter we recall some of their properties.

1.1. Gelfand-Shilov spaces. We start by recalling some facts about Gelfand-Shilov spaces (cf. [6]). Let $0 < h, s_j, \sigma_j \in \mathbb{R}, j = 1, \ldots, n$, be fixed, $d = d_1 + \cdots d_n$, where $d_j \geq 0$ are integers, and let $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$ and $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n_+$.

For multi-indices of multi-indices we let

$$\alpha!^s = \alpha_1!^{s_1} \cdots \alpha_n!^{s_n}, \quad x^\alpha = x^{\alpha_1} \cdots x^{\alpha_n},$$

$$D^\alpha_x = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} \quad \text{and} \quad |\alpha| = |\alpha_1| + \cdots + |\alpha_n|$$

when $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{d_1} \times \cdots \times \mathbb{N}^{d_n}$, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$.

For any $f \in C^\infty(\mathbb{R}^d)$, we let

$$\|f\|_{S_{x,h}^\sigma} \equiv \sup \left( \frac{\|x^\alpha \partial_x^\beta f\|_{L^\infty(\mathbb{R}^d)}}{h^{|\alpha|+|\beta|} \alpha!^s \beta!^\sigma} \right), \quad (1.1)$$

where the supremum is taken over all $\alpha_j, \beta_j \in \mathbb{N}^{d_j}, j = 1, \ldots, d$. Then $f \mapsto \|f\|_{S_{x,h}^\sigma}$ defines a norm on $C^\infty(\mathbb{R}^d)$. The space $S_{x,h}^\sigma(\mathbb{R}^d)$ is the Banach space which consist of all $f \in C^\infty(\mathbb{R}^d)$ such that $\|f\|_{S_{x,h}^\sigma}$ is finite. In the case $d_1 = d_1 \geq 1$, $d_2 = \cdots = d_n = 0$, $s = s_1$, $\sigma = \sigma_1$ and $x_1 = x$, (1.1) is interpreted as

$$\|f\|_{S_{x,h}^\sigma} \equiv \sup_{\alpha, \beta \in \mathbb{N}^d} \left( \frac{\|x^\alpha \partial_x^\beta f(x)\|_{L^\infty(\mathbb{R}^d)}}{h^{|\alpha|+|\beta|} \alpha!^s \beta!^\sigma} \right). \quad (1.1)'$$
The Gelfand-Shilov spaces \( S^\sigma_s(\mathbb{R}^d) \) and \( \Sigma^\sigma_s(\mathbb{R}^d) \) are defined as the inductive and projective limits respectively of \( S^\sigma_{s,h}(\mathbb{R}^d) \). This implies that

\[
S^\sigma_s(\mathbb{R}^d) = \bigcup_{h>0} S^\sigma_{s,h}(\mathbb{R}^d),
\]

\[
\Sigma^\sigma_s(\mathbb{R}^d) = \bigcap_{h>0} S^\sigma_{s,h}(\mathbb{R}^d),
\]

and that the topology for \( S^\sigma_s(\mathbb{R}^d) \) is the strongest possible one such that the inclusion map from \( S^\sigma_{s,h}(\mathbb{R}^d) \) to \( S^\sigma_s(\mathbb{R}^d) \) is continuous, for every choice of \( h > 0 \). The space \( \Sigma^\sigma_s(\mathbb{R}^d) \) is a Fréchet space with seminorms \( \| \cdot \|_{\Sigma^\sigma_s,h} \), \( h > 0 \). Moreover,

\[
\Sigma^\sigma_s(\mathbb{R}^d) \neq \{0\} \iff s_j + \sigma_j \geq 1 \text{ and } (s_j, \sigma_j) \neq \left(\frac{1}{2}, \frac{1}{2}\right), j = 1, \ldots, n,
\]

and

\[
S^\sigma_s(\mathbb{R}^d) \neq \{0\} \iff s_j + \sigma_j \geq 1, \ j = 1, \ldots, n.
\]

There are various kinds of characterisations of the spaces \( S^\sigma_s(\mathbb{R}^d) \) and \( \Sigma^\sigma_s(\mathbb{R}^d) \), e.g. in terms of the exponential decay of their elements. Later on it will be useful that \( f \in S^\sigma_s(\mathbb{R}^d) \) (respectively \( f \in \Sigma^\sigma_s(\mathbb{R}^d) \)), if and only if

\[
|\partial^\sigma f(x)| \lesssim h^{[\sigma]} e^{-r(|x_1|^\frac{1}{\alpha_1} + \cdots + |x_n|^\frac{1}{\alpha_n})}
\]

for some \( h, r > 0 \) (respectively for every \( h > 0, \varepsilon > 0 \)).

If \( \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n \) and \( s, \sigma \in \mathbb{R}^n_+ \), then

\[
\Sigma^\sigma_s(\mathbb{R}^d) \hookrightarrow S^\sigma_s(\mathbb{R}^d) \hookrightarrow \Sigma^{\sigma+\varepsilon \mathbf{1}}_{s+1}(\mathbb{R}^d) \hookrightarrow \mathcal{F}(\mathbb{R}^d)
\]

for every \( \varepsilon > 0 \). If in addition \( s_j + \sigma_j \geq 1 \) for every \( j \), then the last two inclusions in (1.3) are dense, and if in addition \( (s_j, \sigma_j) \neq \left(\frac{1}{2}, \frac{1}{2}\right) \) for every \( j \), then the first inclusion in (1.3) is dense.

The Gelfand-Shilov distribution spaces \( (S^\sigma_s(\mathbb{R}^d))' \) and \( (\Sigma^\sigma_s(\mathbb{R}^d))' \) are the projective and inductive limit respectively of \( (S^\sigma_s)'(\mathbb{R}^d) \). This means that

\[
(S^\sigma_s)'(\mathbb{R}^d) = \bigcap_{h>0} (S^\sigma_{s,h})'(\mathbb{R}^d),
\]

\[
(\Sigma^\sigma_s)'(\mathbb{R}^d) = \bigcup_{h>0} (\Sigma^\sigma_{s,h})'(\mathbb{R}^d),
\]

If in addition \( d_1 = d \geq 1, \ d_2 = \cdots = d_s = 0, \ s = s_1 = \sigma \), then we set \( (S^\sigma_s)'(\mathbb{R}^d) \) = \( (S^\sigma_s)'(\mathbb{R}^d) \) and \( (\Sigma^\sigma_s)'(\mathbb{R}^d) \) = \( (\Sigma^\sigma_s)'(\mathbb{R}^d) \). We remark that the analysis in [10] shows that \( (S^\sigma_s)'(\mathbb{R}^d) \) is the dual of \( S^\sigma_s(\mathbb{R}^d) \).
and that \((\Sigma^s_\sigma)'(\mathbb{R}^d)\) is the dual of \(\Sigma^s_\sigma(\mathbb{R}^d)\) (also in topological sense). By the inequalities \(n!k! \leq (n + k)! \leq 2^{n+k}n!k!\) it follows that

\[
S^s_\sigma(\mathbb{R}^d) = \mathcal{S}^s(\mathbb{R}^d), \quad \Sigma^s_\sigma(\mathbb{R}^d) = \Sigma^s(\mathbb{R}^d),
\]

\[
(S^s_\sigma)'(\mathbb{R}^d) = (\mathcal{S}^s)'(\mathbb{R}^{d_1+d_2}), \quad (\Sigma^s_\sigma)'(\mathbb{R}^d) = (\Sigma^s_\sigma)'(\mathbb{R}^{d_1+d_2}).
\]

Corresponding relations in (1.3) for Gelfand-Shilov distributions are

\[
\mathcal{S}^s(\mathbb{R}^d) \hookrightarrow (\Sigma^s_{\sigma+1})'(\mathbb{R}^d) \hookrightarrow (\mathcal{S}^s_\sigma)'(\mathbb{R}^d)
\]

when \(s_j + \sigma_j \geq 1, j = 1, \ldots, n\), and

\[
(\mathcal{S}^s_\sigma)'(\mathbb{R}^d) \hookrightarrow (\Sigma^s_\sigma)'(\mathbb{R}^d)
\]

when \(s_j + \sigma_j \geq 1\) and \((s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2}), j = 1, \ldots, n\).

The Gelfand-Shilov spaces possess several convenient mapping properties. For example they are invariant under translations, dilations, and to some extent (partial) Fourier transformations. For any \(f \in L^1(\mathbb{R}^d)\), its Fourier transform is defined by

\[
(\hat{f}(\xi)) = \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} \, dx.
\]

If instead \(f \in L^1(\mathbb{R}^{d_1+\cdots+d_n})\), then the partial Fourier transform of \(f\) with respect to \(k \in \{1, \ldots, n\}\) is given by

\[
(\mathcal{F}_k f)(x_1, \ldots, \xi_k, \ldots, x_n) = \int_{\mathbb{R}^{d_k}} f(x_1, \ldots, x_n) e^{-i(x_k, \xi_k)} \, dx_k, \quad x_j, \xi_j \in \mathbb{R}^{d_j}.
\]

**Remark 1.1.** Let \(d = d_1 + \cdots + d_n\), \(j \in \{1, \ldots, n\}\), \(s, \sigma \in \mathbb{R}_+^n\), \(r_{k,j} (s, \sigma) = (r_{k,j,1}, \ldots, r_{k,j,n})\), \(k = 1, 2\), where

\[
\begin{align*}
\{ s_l, \quad l \neq j \\
\{ s_l, \quad l = j
\end{align*}
\]

and \(r_{2,j,l} = \{ s_l, \quad l \neq j \}

Then the following follows from the general theory of Schwartz functions and Gelfand-Shilov functions and their distributions (see e.g. [2167]):

1. the definition of \(\mathcal{F}_j\) extends to a homeomorphism on \(\mathcal{S}'(\mathbb{R}^d)\) and restricts to a homeomorphism on \(\mathcal{S}(\mathbb{R}^d)\);
2. the definition of \(\mathcal{F}_j\) extends uniquely to a homeomorphism from \((\mathcal{S}^s_\sigma)'(\mathbb{R}^d)\) to \((\mathcal{S}^r_{1,j}(s,\sigma))'(\mathbb{R}^d)\), and from \((\Sigma^s_\sigma)'(\mathbb{R}^d)\) to \((\Sigma^r_{1,j}(s,\sigma))'(\mathbb{R}^d)\);
3. \(\mathcal{F}_j\) restricts to homeomorphisms from \(\mathcal{S}^s_\sigma(\mathbb{R}^d)\) to \(\mathcal{S}^r_{1,j}(s,\sigma)(\mathbb{R}^d)\), and from \(\Sigma^s_\sigma(\mathbb{R}^d)\) to \(\Sigma^r_{1,j}(s,\sigma)(\mathbb{R}^d)\).
1.2. Pilipović spaces. Next we make a review of Pilipović spaces. These spaces can be defined in terms of Hermite series expansions. We recall that the Hermite function of order \( \alpha \in \mathbb{N}^d \) is defined by
\[
h_\alpha(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{|\alpha|}!)^{-\frac{1}{2}}e^{-\frac{|x|^2}{2}}(\partial^\alpha e^{-|x|^2}).
\]
It follows that
\[
h_\alpha(x) = ((2\pi)^\frac{d}{2}\alpha!)^{-1}e^{-\frac{|x|^2}{2}}p_\alpha(x),
\]
for some polynomial \( p_\alpha \) on \( \mathbb{R}^d \), which is called the Hermite polynomial of order \( \alpha \). The Hermite functions are eigenfunctions to the Fourier transform, and to the Harmonic oscillator \( H_d \equiv |x|^2 - \Delta \) which acts on functions and (ultra-)distributions defined on \( \mathbb{R}^d \). More precisely, we have
\[
H_d h_\alpha = (2|\alpha| + d)h_\alpha, \quad H_d \equiv |x|^2 - \Delta.
\]
It is well-known that the set of Hermite functions is a basis for \( \mathcal{S}(\mathbb{R}^d) \) and an orthonormal basis for \( L^2(\mathbb{R}^d) \) (cf. [12]). In particular, if \( f \in L^2(\mathbb{R}^d) \), then
\[
\|f\|_{L^2(\mathbb{R}^d)}^2 = \sum_{\alpha \in \mathbb{N}^d} |c_h(f, \alpha)|^2,
\]
where
\[
f(x) = \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha)h_\alpha, \quad (1.4)
\]
is the Hermite series expansion of \( f \), and
\[
c_h(f, \alpha) = (f, h_\alpha)_{L^2(\mathbb{R}^d)} \quad (1.5)
\]
is the Hermite coefficient of \( f \) of order \( \alpha \in \mathbb{R}^d \).

In order to define the full scale of Pilipović spaces, their order \( s \) should belong to the extended set
\[
\mathbb{R}_s = \mathbb{R}_+ \bigcup \{ b_\sigma ; \sigma \in \mathbb{R}_+ \},
\]
of \( \mathbb{R}_+ \), with extended inequality relations as
\[
s_1 < b_\sigma < s_2 \quad \text{and} \quad b_{\sigma 1} < b_{\sigma 2}
\]
when \( s_1 < \frac{1}{2} \leq s_2 \) and \( \sigma_1 < \sigma_2 \). (Cf. [14].)

For \( r > 0 \) and \( s \in \mathbb{R}_s \) we set
\[
\vartheta_{r,s}(\alpha) \equiv \begin{cases} 
e^{-r|\alpha|\frac{1}{2r}}, & s \in \mathbb{R}_+, \\ r^{|\alpha|}\alpha!^{-\frac{1}{2}}, & s = b_\sigma \end{cases}, \quad (1.6)
\]
and
\[
\vartheta'_{r,s}(\alpha) \equiv \begin{cases} e^{r|\alpha|\frac{1}{2r}}, & s \in \mathbb{R}_+, \\ r^{|\alpha|}\alpha!^{\frac{1}{2r}}, & s = b_\sigma \end{cases}, \quad (1.7)
\]
Definition 1.2. Let $s \in \mathbb{R}_\flat = \mathbb{R}_\flat \cup \{0\}$, and let $\vartheta_{r,s}$ and $\vartheta'_{r,s}$ be as in (1.6) and (1.7).

(1) $H_0(\mathbb{R}^d)$ consists of all Hermite polynomials, and $H'_0(\mathbb{R}^d)$ consists of all formal Hermite series expansions in (1.4);
(2) if $s \in \mathbb{R}_\flat$, then $H_s(\mathbb{R}^d)$ ($H'_s(\mathbb{R}^d)$) consists of all $f \in L^2(\mathbb{R}^d)$ such that
$$|c_h(f, h_\alpha)| \lesssim \vartheta_{r,s}(\alpha)$$
holds true for some $r \in \mathbb{R}_+$ (for every $r \in \mathbb{R}_+$);
(3) if $s \in \mathbb{R}_\flat$, then $H'_s(\mathbb{R}^d)$ ($H'_s(\mathbb{R}^d)$) consists of all formal Hermite series expansions in (1.4) such that
$$|c_h(f, h_\alpha)| \lesssim \vartheta'_{r,s}(\alpha)$$
holds true for every $r \in \mathbb{R}_+$ (for some $r \in \mathbb{R}_+$).

The spaces $H_s(\mathbb{R}^d)$ and $H'_s(\mathbb{R}^d)$ are called Pilipović spaces of Roumieu respectively Beurling types of order $s$, and $H'_s(\mathbb{R}^d)$ and $H'_s(\mathbb{R}^d)$ are called Pilipović distribution spaces of Roumieu respectively Beurling types of order $s$.

Remark 1.3. Let $\mathcal{H}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ be the Fourier invariant Gelfand-Shilov spaces of order $s \in \mathbb{R}_+$ and of Rourmeu and Beurling types respectively (see [14] for notations). Then it is proved in [9], [10] that
$$H_0^s(\mathbb{R}^d) = \Sigma_s(\mathbb{R}^d) \neq \{0\}, \quad s > \frac{1}{2},$$
$$H_0^s(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}, \quad s \leq \frac{1}{2},$$
$$H_s(\mathbb{R}^d) = \mathcal{H}_s(\mathbb{R}^d) \neq \{0\}, \quad s \geq \frac{1}{2}$$
and
$$H_s(\mathbb{R}^d) \neq \mathcal{H}_s(\mathbb{R}^d) = \{0\}, \quad s < \frac{1}{2}. $$

Next we recall the topologies for Pilipović spaces. Let $s \in \mathbb{R}_\flat$, $r > 0$, and let $\|f\|_{H_{s,r}}$ and $\|f\|_{H'_{s,r}}$ be given by
$$\|f\|_{H_{s,r}} \equiv \sup_{\alpha \in \mathbb{N}^d} |c_h(f, h_\alpha)/\vartheta_{r,s}(\alpha)|, \quad s \in \mathbb{R}_\flat, \quad (1.8)$$
and
$$\|f\|_{H'_{s,r}} \equiv \sup_{\alpha \in \mathbb{N}^d} |c_h(f, h_\alpha)/\vartheta'_{r,s}(\alpha)|, \quad s \in \mathbb{R}_\flat. \quad (1.9)$$
when $f$ is a formal expansion in (1.4). Then $H_{s,r}(\mathbb{R}^d)$ consists of all expansions (1.4) such that $\|f\|_{H_{s,r}}$ is finite, and $H'_{s,r}(\mathbb{R}^d)$ consists of all expansions (1.4) such that $\|f\|_{H'_{s,r}}$ is finite. It follows that both
$\mathcal{H}_{s,r}(\mathbb{R}^d)$ and $\mathcal{H}_{s,r}'(\mathbb{R}^d)$ are Banach spaces under the norms $f \mapsto \|f\|_{\mathcal{H}_{s,r}}$ and $f \mapsto \|f\|_{\mathcal{H}_{s,r}'}$, respectively.

We let the topologies of $\mathcal{H}_s(\mathbb{R}^d)$ and $\mathcal{H}_s'(\mathbb{R}^d)$ be the inductive respectively projective limit topology of $\mathcal{H}_{s,r}(\mathbb{R}^d)$ with respect to $r > 0$. In the same way, the topologies of $\mathcal{H}_s'(\mathbb{R}^d)$ and $(\mathcal{H}_s)'(\mathbb{R}^d)$ are the projective respectively inductive limit topology of $\mathcal{H}_{s,r}(\mathbb{R}^d)$ with respect to $r > 0$.

Suppose instead $s = 0$. For any integer $N \geq 0$, we set

$$\|f\|_{(0,N)} = \sup_{|\alpha| \leq N} |c_\alpha(f)|, \quad f \in \mathcal{H}_0'(\mathbb{R}^d).$$

The topology for $\mathcal{H}_0'(\mathbb{R}^d)$ is defined by the semi-norms $\| \cdot \|_{(0,N)}$.

We also let $\mathcal{H}_{0,N}(\mathbb{R}^d)$ be the vector space which consists of all $f \in \mathcal{H}_0'(\mathbb{R}^d)$ such that $c_\alpha(f) = 0$ when $|\alpha| > N$, and equip this space with the topology, defined by the norm $\| \cdot \|_{(0,N)}$. The topology of $\mathcal{H}_0(\mathbb{R}^d)$ is then defined as the inductive limit topology of $\mathcal{H}_{0,N}(\mathbb{R}^d)$ with respect to $N \geq 0$.

It follows that all the spaces in Definition 1.2 are complete, and that $\mathcal{H}_0(\mathbb{R}^d)$ and $\mathcal{H}_0'(\mathbb{R}^d)$ are Fréchet space with semi-norms $f \mapsto \|f\|_{\mathcal{H}_{s,r}}$ and $f \mapsto \|f\|_{\mathcal{H}_{s,r}'}$, respectively.

The following characterisations of Pilipović spaces can be found in [14]. The proof is therefore omitted.

**Proposition 1.4.** Let $s \in \mathbb{R}_+ \cup \{0\}$ and let $f \in \mathcal{H}_0'(\mathbb{R}^d)$. Then $f \in \mathcal{H}_0(\mathbb{R}^d)$ if and only if $f \in C^\infty(\mathbb{R}^d)$ and satisfies $H_0^N f(x) \lesssim h^N |N|^{2s}$ for every $h > 0$ (for some $h > 0$).

Finally we remark that the Pilipović spaces of functions and distributions possess convenient mapping properties under the Bargmann transform (cf. [14]).

2. Tensor Product for Gelfand-Shilov Spaces

In this section we start by proving Theorem 0.2. Thereafter we deduce a multi-linear version of this result.

For the proof of Theorem 0.2 we first need the following analogy of Lemma 4.1.3 in [17].

**Lemma 2.1.** Let $s_1, s_2, \sigma_1, \sigma_2 > 0$, $\varphi, \psi \in S_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$. Then the Riemann sum

$$\sum_{k \in \mathbb{Z}^d} \varphi(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d, \quad d = d_1 + d_2,$$

converges to $(\varphi \ast \psi)(x)$ in $S_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$ as $\varepsilon \to 0$.

The same holds true if each $S_{s_1,s_2}^{\sigma_1,\sigma_2}$ and $(S_{s_1,s_2}^{\sigma_1,\sigma_2})'$ are replaced by $\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2}$ and $(\Sigma_{s_1,s_2}^{\sigma_1,\sigma_2})'$, respectively.
Proof. We may assume that $\varepsilon > 0$, and consider first the case when $\varphi$ and $\psi$ are real-valued. For multi-indices we use the convention

$$
\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{d_1+d_2}, \quad \beta = (\beta_1, \beta_2) \in \mathbb{N}^{d_1+d_2}, \quad \text{and} \quad \alpha! = \alpha_1! \alpha_2!
$$

when $\alpha_j, \beta_j \in \mathbb{N}^{d_j}$ and $s = (s_1, s_2) \in \mathbb{R}^2$, $j = 1, 2$. Set

$$
R_{\varepsilon, \alpha, \beta}(x) = x^\alpha D_x^\beta \left( \varphi(x - y) \psi(y) \, dy - \sum_{k \in \mathbb{Z}^d} \varphi(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d \right).
$$

By the mean-value theorem we have for some $\rho_k = \rho_k(x, y) \in Q_{d,1}$, $k \in \mathbb{Z}^d$ that

$$
|R_{\varepsilon, \alpha, \beta}(x)| = \left| \int_{\mathbb{R}^d} x^\alpha (D_x^\beta \varphi)(x - y) \psi(y) \, dy - \sum_{k \in \mathbb{Z}^d} x^\alpha (D_x^\beta \varphi)(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d \right|
$$

$$
= \left| \sum_{k \in \mathbb{Z}^d} \left( \int_{\varepsilon k + Q_{d,s}} x^\alpha (D_x^\beta \varphi)(x - y) \psi(y) \, dy - x^\alpha (D_x^\beta \varphi)(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d \right) \right|
$$

$$
\leq \sum_{k \in \mathbb{Z}^d} \left| x^\alpha (D_x^\beta \varphi)(x - \varepsilon k - \varepsilon \rho_k) \psi(\varepsilon k + \varepsilon \rho_k) - x^\alpha (D_x^\beta \varphi)(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d \right|
$$

$$
\leq \sum_{k \in \mathbb{Z}^d} \sup_{z \in Q_{d,s}} \left| |D_z| \left( x^\alpha (D_x^\beta \varphi)(x - \varepsilon k - z) \psi(\varepsilon k + z) \right) \right| \varepsilon^{d+1} \leq J_1 + J_2,
$$

where

$$
J_1 = \sum_{\gamma \leq \alpha} \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \left( \frac{\alpha}{\gamma} \right) \sup_{y \in \varepsilon k + Q_{d,s}} \left| (x - y)^\gamma D_x^\beta \varphi(x - y) y^{\alpha - \gamma} \psi(y) \right| \varepsilon^{d+1}
$$

and

$$
J_2 = \sum_{\gamma \leq \alpha} \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \left( \frac{\alpha}{\gamma} \right) \sup_{y \in \varepsilon k + Q_{d,s}} \left| (x - y)^\gamma D_x^\beta \varphi(x - y) y^{\alpha - \gamma} D_{y_j} \psi(y) \right| \varepsilon^{d+1}.
$$
Since \((m + 1)! \leq 2^m m!\) when \(m \geq 0\) is an integer, and \(\varphi, \psi \in S_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1 + d_2})\), we get

\[
J_j \lesssim h^{\alpha+\beta} |\sigma| |\sigma| d \sum_{\gamma \leq \alpha} \sum_{k \in \mathbb{Z}^d} \left( \frac{\alpha}{\gamma} \right) \gamma^r |\beta|^r \sup_{y \in k + Q_{d, \varepsilon}} \left| y^{\alpha - \gamma} e^{-2r(|y|^{\frac{1}{d_1}} + |y|^{\frac{1}{d_2}})} \right| \epsilon^{d+1}
\]

\[
\lesssim \epsilon h^{\alpha+\beta} |\sigma| |\sigma| \sum_{\gamma \leq \alpha} \sum_{k \in \mathbb{Z}^d} \left( \frac{\alpha}{\gamma} \right) \gamma^r |\beta|^r (\alpha - \gamma)! \epsilon^{r(|\varepsilon k_1^{\frac{1}{d_1}} + |\varepsilon k_2^{\frac{1}{d_2}}|)} \epsilon^{d}
\]

\[
\lesssim \epsilon (2^r h)^{\alpha+\beta} |\sigma| |\sigma| \sum_{k \in \mathbb{Z}^d} \epsilon^{-r(|\varepsilon k_1^{\frac{1}{d_1}} + |\varepsilon k_2^{\frac{1}{d_2}}|)} \epsilon^{d} \lesssim \epsilon (2^r h)^{\alpha+\beta} |\sigma| |\sigma|
\]

for \(j = 1, 2\), for some positive constants \(h\) and \(r\). This implies that for some \(h > 0\) we have

\[
\sup_{\alpha, \beta \in \mathbb{N}^d} \left( \frac{\|R_{\epsilon, \alpha, \beta}\|_{L^\infty}}{h^{\alpha+\beta} |\sigma| |\sigma|} \right) \leq C \epsilon
\]

for some positive constants \(C\) and \(h\) which are independent of \(\varepsilon\).

Since the right-hand side tends to zero when \(\varepsilon > 0\) tends to zero, the stated convergence follows in this case.

The general case follows from the previous case, after writing \(\varphi = \varphi_1 + i \varphi_2\) and \(\psi = \psi_1 + i \psi_2\) with \(\varphi_j\) and \(\psi_j\) being real-valued, \(j = 1, 2\), giving that \(\varphi * \psi\) is a superposition of \(\varphi_{j_1} * \psi_{j_2}, j_1, j_2 \in \{1, 2\}\), and using the fact that \(\varphi_j \in S_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1 + d_2})\) when \(\varphi \in S_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1 + d_2})\). □

We may now prove the following result related to [7, Theorem 4.1.2]

**Lemma 2.2.** Let \(s_1, s_2, \sigma_1, \sigma_2 > 0\), \(\varphi, \psi \in S_{s_1, s_2}(\mathbb{R}^{d_1 + d_2})\) and let \(f \in (S_{s_1, s_2})'(\mathbb{R}^{d_1 + d_2})\). Then

\[
(f * \varphi) * \psi = f * (\varphi * \psi).
\]

The same holds true if each \(S_{s_1, s_2}\) and \((S_{s_1, s_2})'\) are replaced by \(\Sigma_{s_1, s_2}\) and \((\Sigma_{s_1, s_2})'\), respectively.

**Proof.** We use the same notations in the previous proof. Since the Riemann sum in Lemma 2.1 converges to \(\varphi * \psi\) in \(S_{s_1, s_2}\), we get

\[
(f * (\varphi * \psi))(x) = \lim_{\varepsilon \to 0} \left\langle f, \sum_{k \in \mathbb{Z}^d} \varphi(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d \right\rangle
\]

\[
= \lim_{\varepsilon \to 0} \left( \sum_{k \in \mathbb{Z}^d} (f * \varphi)(x - \varepsilon k) \psi(\varepsilon k) \varepsilon^d \right).
\]

Here the second equality follows by the fact that

\[
y \mapsto \sum_{k \in \mathbb{Z}^d} \varphi(x - y - \varepsilon k) \psi(\varepsilon k)
\]

converges in \(S_{s_1, s_2}\).
We have that $f \ast \varphi$ is smooth, and for some $r_0 > 0$ we have
\[
|(f \ast \varphi)(x - \varepsilon k)\psi(\varepsilon k)| \leq e^{r(|x_1 - \varepsilon k_1|^{1/2} + |x_2 - \varepsilon k_2|^{1/2})}e^{-2r_0(|\varepsilon k_1|^{1/2} + |\varepsilon k_2|^{1/2})}
\]
for every $r > 0$. This gives
\[
|(f \ast \varphi)(x - \varepsilon k)\psi(\varepsilon k)| \leq C_x e^{-r_0(|\varepsilon k_1|^{1/2} + |\varepsilon k_2|^{1/2})},
\]
for some constant $C_x$ which only depends on $x$ and $r_0$. It follows that
\[
\sum_{k \in \mathbb{Z}^d} (f \ast \varphi)(x - \varepsilon k)\psi(\varepsilon k)\varepsilon^d
\]
is a Riemann sum which converges to
\[
\int (f \ast \varphi)(x - y)\psi(y) \, dy = ((f \ast \varphi) \ast \psi)(x).
\]
Hence (2.2) holds, and the result follows.

By the previous lemma it is now straight-forward to prove the following.

**Lemma 2.3.** Let $s, \sigma \in \mathbb{R}_+^n$, $d = d_1 + \cdots + d_n$ and suppose $f \in (S_s^\sigma)'(\mathbb{R}^d)$ satisfies $\langle f, \varphi_1 \otimes \cdots \otimes \varphi_n \rangle = 0$ for every $\varphi_j \in S_{s_j}^{\sigma_j}(\mathbb{R}^{d_j})$, $j = 1, \ldots, n$. Then $f = 0$.

The same holds true if each $S_{s_j}^{\sigma_j}$, $(S_{s_j}^{\sigma_j})'$, $S_s^\sigma$ and $(S_s^\sigma)'$ are replaced by $\Sigma_{s_j}^{\sigma_j}$, $(\Sigma_{s_j}^{\sigma_j})'$, $\Sigma_s^\sigma$ and $(\Sigma_s^\sigma)'$, respectively.

**Proof.** We only prove the result in the Roumieu case. The Beurling case follows by similar arguments and is left for the reader. We use the same notations as in the previous proofs.

First suppose $n = 2$. Let $\varphi \in S_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$, $\phi_j \in S_{s_j}^{\sigma_j}(\mathbb{R}^{d_j})$ be such that
\[
\int_{\mathbb{R}^{d_j}} \phi_j(x_j) \, dx_j = 1,
\]
and let
\[
\phi_\varepsilon = |\varepsilon|^{-(d_1+d_2)}(\phi_1 \otimes \phi_2)(\varepsilon^{-1} \cdot),
\]
when $\varepsilon$ is real. Then the assumptions implies that $\tilde{f} \ast \phi_\varepsilon = 0$ for every $\varepsilon$. Here $\tilde{f}$ is defined by $\tilde{f}(x) = f(-x)$. By Lemma 2.2 we get
\[
\langle f, \varphi \rangle = \lim_{\varepsilon \to 0} \langle f, \phi_\varepsilon \ast \varphi \rangle = \lim_{\varepsilon \to 0} (\tilde{f} \ast (\phi_\varepsilon \ast \varphi))(0) = \lim_{\varepsilon \to 0} ((\tilde{f} \ast \phi_\varepsilon) \ast \varphi)(0),
\]
and the result follows for $n = 2$.

For general $n \geq 2$, the result follows from the case $n = 2$ and induction. The details are left for the reader.

**Proof of Theorem 0.2.** We only prove the result in the Roumieu cases. The Beurling cases follow by similar arguments and are left for the reader.

By straight-forward computations it follows that
\[
\varphi \mapsto \langle f_1, \psi_1 \rangle \quad \text{and} \quad \varphi \mapsto \langle f_2, \psi_2 \rangle
\]
define continuous linear forms $g_1$ and $g_2$ on $S_{s_1,s_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{d_1+d_2})$. Hence $g_1, g_2 \in (S_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$. It is clear that both $g_1$ and $g_2$ in place of $f$ satisfy (0.2), and the existence of $f$ follows.

If $f \in (S_{s_1,s_2}^{\sigma_1,\sigma_2})'(\mathbb{R}^{d_1+d_2})$ is arbitrary such that (0.2) holds, then

$$\langle f - g_j, \varphi_1 \otimes \varphi_2 \rangle = \langle f_1, \varphi_1 \rangle \langle f_2, \varphi_2 \rangle - \langle f_1, \varphi_1 \rangle \langle f_2, \varphi_2 \rangle = 0,$$

and Lemma 2.3 shows that $f = g_1 = g_2$. This gives the uniqueness of $f$, as well as (0.3).

In order to consider corresponding multi-linear situation of Theorem 0.2 we let $S_n$ be the permutation group of $\{1, \ldots, n\}$, and let inductively

$$\varphi_{n,\tau}(x_{\tau(1)}, \ldots, x_{\tau(n)}) = \varphi(x_1, \ldots, x_n), \quad x_j \in \mathbb{R}^{d_j}, \quad \tau \in S_n, \quad (2.3)$$

and

$$\varphi_{j,\tau}(x_{\tau(1)}, \ldots, x_{\tau(j)}) = \langle f_{\tau(1)+1}, \varphi_{j+1,\tau}(x_{\tau(1)}, \ldots, x_{\tau(j)}, \cdot) \rangle \quad (2.4)$$

when $f_j$ for $j = 1, \ldots, n$ are suitable distributions and $\varphi$ is a suitable function. Then Theorem 0.2 can be reformulated as follows. It is also convenient to set

$$s_{j,\tau} = (s_{\tau(1)}, \ldots, s_{\tau(j)}), \quad \sigma_{j,\tau} = (\sigma_{\tau(1)}, \ldots, \sigma_{\tau(j)}) \quad (2.5)$$

and $d_{j,\tau} = d_{\tau(1)} + \cdots + d_{\tau(j)}$,

when $j = 1, \ldots, n$ and $s, \sigma \in \mathbb{R}_+^n$.

**Theorem 2.4.** Let $\tau \in S_2$, $d = d_1 + d_2$, $s, \sigma \in \mathbb{R}^2_+$, $s_{j,\tau}$ and $\sigma_{j,\tau}$ be as in (2.5), $f_j \in (S_{s_j}^\sigma)'(\mathbb{R}^{d_j})$, $\varphi \in S_{s_j}^\sigma(\mathbb{R}^d)$ and let $\varphi_{j,\tau}$ be given by (2.3) and (2.4), $j = 1, 2$. Then $\varphi_{j,\tau} \in S_{s_j}^{\sigma_j}(\mathbb{R}^{d_j})$, and there is a unique distribution $f$ in $(S_{s_j}^\sigma)'(\mathbb{R}^d)$ such that for every $\varphi_j \in S_{s_j}^{\sigma_j}(\mathbb{R}^{d_j})$, $j = 1, \ldots, n$, and $\varphi_2 \in S_{s_2}^{\sigma_2}(\mathbb{R}^{d_2})$,

$$\langle f, \varphi_1 \otimes \varphi_2 \rangle = \prod_{k=1}^2 \langle f_k, \varphi_k \rangle \quad \text{and} \quad \langle f, \varphi \rangle = \langle f_{\tau(1)}, \varphi_{1,\tau} \rangle \quad (2.6)$$

hold.

The same holds true with $\Sigma_{s_j}^{\sigma_j}$, $(\Sigma_{s_j}^{\sigma_j})'$, $\Sigma_s$ and $(\Sigma_s)'$ in place of $S_{s_j}^{\sigma_j}$, $(S_{s_j}^\sigma)'$, $S_s^\sigma$ and $(S_s^\sigma)'$, respectively, at each occurrence.

Here the second equality in (2.6) is the same as the Fubbbini property (0.3). The multi-linear version of the previous theorem is the following, and follows by similar arguments as for its proof. The details are left for the reader.

**Theorem 2.4.** Let $\tau \in S_n$, $d = d_1 + \cdots + d_n$, $s, \sigma \in \mathbb{R}_+^n$, $d_{j,\tau}$, $s_{j,\tau}$ and $\sigma_{j,\tau}$ be as in (2.5), $f_j \in (S_{s_j}^\sigma)'(\mathbb{R}^{d_j})$, $\varphi \in S_{s_j}^\sigma(\mathbb{R}^d)$ and let $\varphi_{j,\tau}$ be given by (2.3) and (2.4), $j = 1, \ldots, n$. Then $\varphi_{j,\tau} \in S_{s_j}^{\sigma_j}(\mathbb{R}^{d_j})$, and there is
a unique distribution $f$ in $(S_H^\sigma)'(\mathbb{R}^d)$ such that for every $\varphi_j \in S_H^\sigma_j(\mathbb{R}^d)$, $j = 1, \ldots, n$,

$$\langle f, \varphi_1 \otimes \cdots \otimes \varphi_n \rangle = \prod_{k=1}^n \langle f_k, \varphi_k \rangle \quad \text{and} \quad \langle f, \varphi \rangle = \langle f_{\tau(1)}, \varphi_{1,\tau} \rangle$$ (2.6)'

hold.

The same holds true with $\Sigma s_j$ and $\Sigma s_j$ in place of $S_H^\sigma_j$, $(S_H^\sigma_j)'$, $S_H^\sigma$ and $(S_H^\sigma)'$, respectively, at each occurrence.

**Example 2.5.** Let $s, \sigma > 0$. An important object in time-frequency and micro-local analysis concerns the short-time Fourier transform. If $\phi \in S_H^\sigma(\mathbb{R}^d) \setminus \{0\}$ is fixed, then the short-time Fourier transform of $f \in S_H^\sigma(\mathbb{R}^d)$ is defined by

$$V_\phi f(x, \xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(y) \overline{\phi(y-x)} e^{-i(y, \xi)} \, dy.$$ (2.6)

It follows that

$$V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \langle f, \phi(\cdot - x)e^{-i(\cdot, \xi)} \rangle$$ (2.7)

and

$$V_\phi f(x, \xi) = \mathcal{F}(f \cdot \overline{\phi(\cdot - x)})(\xi)$$ (2.8)

for such choices of $\phi$ and $f$.

We notice that the right-hand side of (2.7) also makes sense as a smooth function on $\mathbb{R}^{2d}$ if the assumption on $f$ is relaxed into $f \in (S_H^\sigma)'(\mathbb{R}^d)$. For such $f$ we therefore let (2.7) define the short-time Fourier transform of $f$ with respect to $\phi$. Since the map which takes $\phi$ into $y \mapsto \phi(y-x)e^{i(y, \xi)}$ is continuous and smooth with respect to $(x, \xi)$ from $S_H^\sigma(\mathbb{R}^d)$ to itself it follows that $V_\phi f$ is smooth. By [14, Proposition 2.2] it follows that $V_\phi f$ belongs to $(S_H^{\sigma,s})'(\mathbb{R}^{2d})$. Consequently,

$$V_\phi f \in (S_H^{\sigma,s})'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}).$$

Let $U$ be the operator which takes any $F(x, y)$ into $F(y, y-x)$ and recall that $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y)$ with respect to the $y$ variable. Then the right-hand side of (2.8) equals

$$(\mathcal{F}_2(U(f \otimes \overline{\phi}))) (x, \xi).$$ (2.9)

We notice that the right-hand side makes sense as an element in $(S_H^{\sigma,s})'(\mathbb{R}^{2d})$ for any $f, \phi \in (S_H^\sigma)'(\mathbb{R}^d)$ in view of Remark [1.1] which may be used to extend the definition of the short-time Fourier transform to even more general situations.

We claim that the right-hand sides of (2.7) and (2.8) agree when $f \in (S_H^\sigma)'(\mathbb{R}^d)$ and $\phi \in S_H^\sigma(\mathbb{R}^d)$. 

13
In fact, let \( \psi \in S_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) and set
\[
\varphi(x, \xi, y) \equiv \psi(x, \xi)\phi(y - x)e^{-i(y, \xi)} \in S_{s,\sigma}^{\sigma,s}(\mathbb{R}^{3d}),
\]
\[
F \equiv 1_{\mathbb{R}^{2d}} \otimes f = 1_{\mathbb{R}^{d}} \otimes 1_{\mathbb{R}^{d}} \otimes f \in (S_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{3d}),
\]
\[
\varphi_1(x, y) \equiv \int_{\mathbb{R}^{d}} \varphi(x, \xi, y) \, d\xi = \int_{\mathbb{R}^{d}} \psi(x, \xi)\phi(y - x)e^{-i(y, \xi)} \, dx d\xi,
\]
\[
\varphi_2(x, \xi) \equiv \langle f, \psi(x, \xi)\phi(\cdot - x)e^{-i(\cdot, \xi)} \rangle = \psi(x, \xi) \cdot V_\phi f(x, \xi),
\]
and let \( g \) be the right-hand side of \((2.8)\). By the Fubbini property at the right-hand of \((2.6)\) we get
\[
\langle F, \varphi \rangle = \langle 1_{\mathbb{R}^{d}} \otimes f, \varphi_1 \rangle = \langle g, \psi \rangle
\]
and
\[
\langle F, \varphi \rangle = \langle 1_{\mathbb{R}^{2d}}, \varphi_2 \rangle = \langle V_\phi f, \psi \rangle.
\]
Since \( \psi \) was arbitrarily chosen, it follows that \( g = V_\phi f \) in \((S_{s,\sigma}^{\sigma,s})'(\mathbb{R}^{2d})\), and the claim follows.

3. Tensor product of Pilipović spaces

In this section we discuss the tensor map on Pilipović spaces. Especially we prove Theorem \(1.2\). Thereafter we deduce a multi-linear version of this result.

First we show that the tensor map possess natural mapping properties on Pilipović spaces.

**Proposition 3.1.** Let \( s \in \mathbb{R}_+ \). Then the following is true:

1. the map \((f_1, f_2) \mapsto f_1 \otimes f_2\) from \(\mathcal{S}(\mathbb{R}^{d_1}) \times \mathcal{S}(\mathbb{R}^{d_2}) \to \mathcal{S}(\mathbb{R}^{d_1+d_2})\), restricts to a continuous map from \(\mathcal{H}(\mathbb{R}^{d_1}) \times \mathcal{H}(\mathbb{R}^{d_2}) \to \mathcal{H}(\mathbb{R}^{d_1+d_2})\);
2. the map \((f_1, f_2) \mapsto f_1 \otimes f_2\) from \(\mathcal{S}(\mathbb{R}^{d_1}) \times \mathcal{S}(\mathbb{R}^{d_2}) \to \mathcal{S}(\mathbb{R}^{d_1+d_2})\), restricts to a continuous map from \(\mathcal{H}_0(\mathbb{R}^{d_1}) \times \mathcal{H}_0(\mathbb{R}^{d_2}) \to \mathcal{H}_0(\mathbb{R}^{d_1+d_2})\).

**Proof.** We only prove (1) and in the case \( s > 0 \). The case \( s = 0 \) and (2) follow by similar arguments and are left for the reader. If
\[
f_j = \sum_{\alpha_j \in \mathbb{N}^{d_j}} c_{\alpha_j}(f_j) h_{\alpha_j},
\]
then
\[
f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha h_\alpha, \quad c_\alpha = c_{\alpha_1}(f_1)c_{\alpha_2}(f_2), \quad \alpha = (\alpha_1, \alpha_2), \quad \alpha_j \in \mathbb{N}^{d_j}, \quad j = 1, 2.
\]
If \( s \in \mathbb{R}_+ \), then
\[
|c_{\alpha_j}(f_j)| \lesssim e^{-c|\alpha_j|^{\frac{1}{d_j}}},
\]
for some \( c > 0 \). This gives
\[
|c_\alpha| \lesssim e^{-c(|\alpha_1|^{\frac{1}{d_1}}+|\alpha_2|^{\frac{1}{d_2}})} \lesssim e^{-c|\alpha|^{\frac{1}{d}}/(1+2^\frac{1}{d})}, \quad \alpha = (\alpha_1, \alpha_2),
\]
for some \( c > 0 \).
and it follows that \( f \in \mathcal{H}_s(\mathbb{R}^d) \).

If instead \( s = b_\sigma \), for some \( \sigma > 0 \), then
\[
|c_{\alpha_j}(f_j)| \lesssim r^{i|\alpha_j|\frac{1}{d}}, \quad j = 1, 2,
\]
for some \( r > 0 \). Hence, if \( \alpha = (\alpha_1, \alpha_2) \), we get
\[
|c_\alpha| \lesssim r^{i|\alpha_1|\alpha_2|\frac{1}{d}} \leq ((2^\frac{d}{2} + 1)r)^{|\alpha|\frac{1}{d}},
\]
for every \( r > 0 \), and it follows that \( f \in \mathcal{H}_s(\mathbb{R}^d) \) in this case as well.

From these estimates it also follows that the map \((f_1, f_2) \rightarrow f_1 \otimes f_2\) is continuous from \( \mathcal{H}_s(\mathbb{R}^{d_1}) \times \mathcal{H}_s(\mathbb{R}^{d_2}) \) to \( \mathcal{H}_s(\mathbb{R}^{d_1+d_2}) \), and the result follows.

\[ \square \]

**Proof of Theorem 0.3** Let \( d = d_1 + d_2 \). We shall deal with the Hermite sequence representations of the elements in the Pilipović spaces. Such approach is performed in [12], when deducing tensor product and kernel results for tempered distributions. We only prove the results when \( f_j \in \mathcal{H}'_s(\mathbb{R}^d) \) and \( s > 0 \). The cases when \( f_j \in \mathcal{H}'_{0,s}(\mathbb{R}^d) \) or \( s = 0 \) follow by similar arguments and are left for the reader.

First we prove the uniqueness. Suppose that both \( f, g \in \mathcal{H}'_s(\mathbb{R}^d) \) satisfy
\[
\langle f, \phi_1 \otimes \phi_2 \rangle = \langle g, \phi_1 \otimes \phi_2 \rangle = \langle f_1, \phi_1 \rangle \langle f_2, \phi_2 \rangle, \quad (3.1)
\]
and let \( c_\alpha(f) \) and \( c_\alpha(g) \) be their Hermite coefficients of order \( \alpha \mathbb{N}^d \). By choosing \( \phi_1 = h_{\alpha_1} \) and \( \phi_2 = h_{\alpha_2} \), (3.1) implies \( c_\alpha(f) = c_\alpha(g) \) when \( \alpha = (\alpha_1, \alpha_2) \). Consequently, \( f = g \), and the uniqueness follows.

We have
\[
f_j = \sum_{\alpha_j \in \mathbb{N}^{d_j}} c_{\alpha_j}(f_j) h_{\alpha_j},
\]
where \( c_{\alpha_j}(f_j) \) for every \( \alpha_j \in \mathbb{N}^{d_j} \) are unique and equal to \( \langle f_j, h_{\alpha_j} \rangle \), \( j = 1, 2 \).

Now let \( f \) be the element in \( \mathcal{H}'_0(\mathbb{R}^d) \), \( d = d_1 + d_2 \) with expansion
\[
f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha h_\alpha,
\]
where
\[
c_\alpha = c_{\alpha_1}(f_1) c_{\alpha_2}(f_2), \quad \alpha = (\alpha_1, \alpha_2), \quad \alpha_j \in \mathbb{N}^{d_j}, \quad j = 1, 2.
\]

We claim that \( f \in \mathcal{H}'_s(\mathbb{R}^d) \).

In fact, if \( s \in \mathbb{R}_+ \), then
\[
|c_{\alpha_j}(f_j)| \lesssim e^{i|\alpha_j|\frac{1}{d}} \quad (3.2)
\]
for every \( \varepsilon > 0 \), and it follows that
\[
|c_\alpha| \lesssim e^{i(|\alpha_1|\frac{1}{d} + |\alpha_2|\frac{1}{d})} \leq e^{2\varepsilon|\alpha|\frac{1}{d}}, \quad \alpha = (\alpha_1, \alpha_2),
\]
for every \( \varepsilon > 0 \). This is the same as \( f \in \mathcal{H}'_s(\mathbb{R}^d) \).
Theorem 3.2 follows by similar arguments. The details are left for the reader.

If \( j \in \mathbb{N} \), then (2.3) holds. If \( f \in H_s'(\mathbb{R}^d) \) in this case as well.

If \( \varphi_j \in H_0(\mathbb{R}^d) \) and \( \varphi \in H_0(\mathbb{R}^d) \), then (0.2) and (0.3) follow by straight-forward computations, using the fact that the set of Hermite functions is an orthonormal basis of \( L^2 \). For general \( \varphi_j \in H_s(\mathbb{R}^d) \) and \( \varphi \in H_s(\mathbb{R}^d) \), \( j = 1, 2 \), the result now follows from dominating convergence, using the fact that \( H_0(\mathbb{R}^d) \) is dense in \( H_s(\mathbb{R}^d) \).

In order to formulate a multi-linear version of Theorem 0.3 we first reformulate the result as follows.

**Theorem 3.2.** Let \( \tau \in S_2 \), \( d = d_1 + d_2 \), \( s \in \mathbb{R} \), \( d_j, \tau \) be as in (2.5), \( f_j \in (H_s)'(\mathbb{R}^d) \), \( \varphi \in H_s(\mathbb{R}^d) \) and let \( \varphi_{j,\tau} \) be given by (2.3) and (2.4), \( j = 1, 2 \). Then \( \varphi_{j,\tau} \in H_s(\mathbb{R}^{d_1 + d_2}) \), and there is a unique distribution \( f \) in \( (H_s)'(\mathbb{R}^d) \) such that for every \( \varphi_j \in H_s(\mathbb{R}^d) \), \( j = 1, 2 \),

\[
\langle f, \varphi_1 \otimes \varphi_2 \rangle = \prod_{k=1}^2 \langle f_k, \varphi_k \rangle \quad \text{and} \quad \langle f, \varphi \rangle = \langle f_{\tau(1)}, \varphi_{1,\tau} \rangle
\]

hold.

The same holds true with \( H_{0,s} \) and \( H_{0,s}' \) in place of \( H_s \) and \( H_s' \), respectively, at each occurrence.

The multi-linear version of the previous theorem is the following, and follows by similar arguments. The details are left for the reader.

**Theorem 3.2.** Let \( \tau \in S_2 \), \( d = d_1 + \cdots + d_n \), \( s \in \mathbb{R} \), \( d_j, \tau \) be as in (2.5), \( f_j \in (H_s)'(\mathbb{R}^d) \), \( \varphi \in H_s(\mathbb{R}^d) \) and let \( \varphi_{j,\tau} \) be given by (2.3) and (2.4), \( j = 1, \ldots, n \). Then \( \varphi_{j,\tau} \in H_s(\mathbb{R}^{d_1 + \cdots + d_n}) \), and there is a unique distribution \( f \) in \( (H_s)'(\mathbb{R}^d) \) such that for every \( \varphi_j \in H_s(\mathbb{R}^d) \), \( j = 1, \ldots, n \),

\[
\langle f, \varphi_1 \otimes \cdots \otimes \varphi_n \rangle = \prod_{k=1}^n \langle f_k, \varphi_k \rangle \quad \text{and} \quad \langle f, \varphi \rangle = \langle f_{\tau(1)}, \varphi_{1,\tau} \rangle
\]

hold.

The same holds true with \( H_{0,s} \) and \( H_{0,s}' \) in place of \( H_s \) and \( H_s' \), respectively, at each occurrence.

**Remark 3.3.** Only certain parts of the properties in Example 2.5 carry over to Pilipović spaces of functions and distributions, in the case when these spaces do not agree with Gelfand-Shilov spaces of functions and distributions. (See Remark 1.3.) In order to deal with such questions, it
it convenient to consider the image of such spaces under the Bargmann transform, which is defined by

\[(\mathcal{U}_d f)(z) = \pi^{-\frac{d}{4}} \langle f, \exp\left(-\frac{1}{2}\left(\langle z, z \rangle + | \cdot |^2 \right) + \sqrt{2}\langle z, \cdot \rangle \right) \rangle,\]

when \( f \) is a suitable (ultra-)distribution (cf. [1,14]).

In fact, let \( \mathcal{A}_s(\mathbb{C}^d) \) (\( \mathcal{A}_{0,s}(\mathbb{C}^d) \)) be the set of all \( F \in A(\mathbb{C}^d) \), the set of entire functions on \( \mathbb{C}^d \), which satisfies

\[|F(z)| \lesssim e^{r \log(z)} \frac{1}{1 - 2s}\]

when \( s < \frac{1}{2} \) and

\[|F(z)| \lesssim e^{r|z|^\frac{2s}{1-2s}}\]

when \( s = b_\sigma \), for some \( r > 0 \) (for every \( r > 0 \)). Also let \( \mathcal{A}_{0,1/2}(\mathbb{C}^d) \) be the set of all \( F \in A(\mathbb{C}^d) \) such that \( |F(z)| \lesssim e^{r|z|^2} \) for all \( r > 0 \). Then it is proved in [5,14] that \( \mathcal{U}_d \) is bijective from \( \mathcal{H}_s(\mathbb{R}^d) \) to \( \mathcal{A}_s(\mathbb{C}^d) \) when \( s \in \mathbb{R}_0 \) and \( s < \frac{1}{2} \), and from \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) to \( \mathcal{A}_{0,s}(\mathbb{C}^d) \) when \( s \in \mathbb{R}_0 \) and \( s \leq \frac{1}{2} \).

By straight-forward computations we have

\[(\mathcal{U}_d (f(\cdot - x_0)))(z) = e^{\sqrt{2}(z,x_0) + \frac{1}{2}|x_0|^2} (\mathcal{U}_d f)(z + \sqrt{2}x_0)\]

and

\[(\mathcal{U}_d (fe^{-i \langle \cdot, \xi_0 \rangle}))(z) = e^{-\sqrt{2}(z,\xi_0) + \frac{1}{2}|\xi_0|^2} (\mathcal{U}_d f)(z + i\sqrt{2}\xi_0).\]

Consequently, by Remark 1.3 and the mapping properties of the Pilipović spaces above under the Bargmann transform, it follows that the following is true:

1. if \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}'_s(\mathbb{R}^d) \) are invariant under translations and modulations, if and only if \( s \geq b_1 \);
2. if \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) and \( \mathcal{H}'_{0,s}(\mathbb{R}^d) \) are invariant under translations and modulations, if and only if \( s > b_1 \).

In particular, the short-time Fourier transform

\[V_\phi f(x, \xi) = \langle f, \overline{\phi(\cdot - x)}e^{-i \langle \cdot, \xi \rangle} \rangle\]

makes sense as a smooth function when \( s \geq b_1 \), \( f \in \mathcal{H}'_s(\mathbb{R}^d) \) and \( \phi \in \mathcal{H}_s(\mathbb{R}^d) \), and when \( s > b_1 \), \( f \in \mathcal{H}'_{0,s}(\mathbb{R}^d) \) and \( \phi \in \mathcal{H}_{0,s}(\mathbb{R}^d) \).

On the other hand, for \( s < \frac{1}{2} \), it seems to be difficult to guarantee that (2.8) is true in general, since the map \( U \) in Example 2.5 seems not to be well-defined on Pilipović spaces which fail to be Gelfand-Shilov spaces.
REFERENCES

[1] V. Bargmann \textit{On a Hilbert space of analytic functions and an associated integral transform}, Comm. Pure Appl. Math., \textbf{14} (1961), 187–214.

[2] M. Cappiello, L. Rodino \textit{SG-pseudodifferential operators and Gelfand-Shilov spaces}, Rocky Mountain J. Math., \textbf{36} (2006), 1117–1148.

[3] M. Cappiello, J. Toft \textit{Pseudo-differential operators in a Gelfand-Shilov setting}, Math. Nachr. \textbf{290} (2017), 738–755.

[4] J. Chung, S.-Y. Chung, D. Kim \textit{Characterizations of the Gelfand-Shilov spaces via Fourier transforms}, Proc. Amer. Math. Soc. \textbf{124} (1996), 2101–2108.

[5] C. Fernandez, A. Galbis, J. Toft \textit{The Bargmann transform and powers of harmonic oscillator on Gelfand-Shilov subspaces}, RACSAM \textbf{111} (2017), 1-13.

[6] I. M. Gelfand, G. E. Shilov, \textit{Generalized functions, II-III}, Academic Press, New York London, 1968.

[7] L. Hörmander \textit{The Analysis of Linear Partial Differential Operators}, vol I–III, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983, 1985.

[8] Z. Lozanov Crvenković, D. Perišić \textit{Hermite expansions of elements of Gelfand Shilov spaces in quasianalytic and non quasianalytic case}, Novi Sad J. Math. \textbf{37} (2007), 129–147.

[9] S. Pilipović \textit{Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions}, SIAM J. Math. Anal. \textbf{17} (1986), 477–484.

[10] S. Pilipović \textit{Structural theorems for periodic ultradistributions}, Proc. Amer. Math. Soc. \textbf{98} (1986), 261–266.

[11] S. Pilipović \textit{Tempered ultradistributions}, Boll. U.M.I. \textbf{7} (1988), 235–251.

[12] M. Reed, B. Simon \textit{Methods of modern mathematical physics, I, II}, Academic Press, London New York, 1979.

[13] J. Toft \textit{The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators}, J. Pseudo-Differ. Oper. Appl. \textbf{3} (2012), 145–227.

[14] J. Toft \textit{Images of function and distribution spaces under the Bargmann transform}, J. Pseudo-Differ. Oper. Appl. \textbf{8} (2017), 83–139.

[15] F. Treves \textit{Topological vector spaces, distributions and kernels}, Academic Press, New York-London, 1967.

Department of Mathematics, Linnaeus University, Växjö, Sweden
E-mail address: joachim.toft@lnu.se