Jordan-Wigner transformation and qubits with nontrivial exchange rule

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Abstract

Well-known (spinless) fermionic qubits may need more subtle consideration in comparison with usual (spinful) fermions. Taking into account a model with local fermionic modes, formally only the ‘occupied’ states $|1\rangle$ could be relevant for antisymmetry with respect to particles interchange, but ‘vacuum’ state $|0\rangle$ is not. Introduction of exchange rule for such fermionic qubits indexed by some ‘positions’ may look questionable due to general super-selection principle. However, a consistent algebraic construction of such ‘super-indexed’ qubits is presented in this work. Considered method has some relation with construction of super-spaces, but it has some differences with standard definition of supersymmetry sometimes used for generalizations of qubit model.

1 Introduction

Analogues of fermionic creation and annihilation (ladder) operators were suggested by Richard Feynman for description of quantum computers already in the very first works [1, 2]. However, Jordan-Wigner transformation [3] is necessary to make such operators anticommuting for different qubits. Such approach was used later in so-called fermionic quantum computation [4].

Representation of fermionic ladder operators in such a way formally requires some consequent indexing (order) for description of Jordan-Wigner transformation. The order does not manifest itself directly in algebraic properties of ladder operators, but transformations of states formally depend on such indexes in rather nonlocal way.
States of physical bosons and fermions can be described in natural way by symmetric and antisymmetric tensors respectively, but fermionic quantum computation is rather relevant with more subtle exchange behavior of some quasi-particles.

Formally, qubits in state $|0\rangle$ corresponds to ‘empty modes’ and only qubits in state $|1\rangle$ treated as ‘occupied modes’ could be relevant to fermionic exchange principle for qubits marked by some indexes. Consistent mathematical model of such ‘super-indexed’ states is suggested in presented work.

More detailed description of such states is introduced in Sec. 2.1 together with formal definition of signed exchange rule and ‘super-indexed’ qubits denoted further as $S$-qubits. The different kinds of operators acting on $S$-qubits are constructed in Sec. 2.2.

Algebraic models of $S$-qubits are discussed in Sec. 3. The ‘non-trivial’ non-commutative part of such model is similar with exterior algebra recollected in Sec. 3.1. However, ‘trivial’ commutative elements could not be naturally presented in such a way and more complete model is suggested in Sec. 3.2. The Clifford algebras initially used in Sec. 2.2 for applications to gates and operators become the basic tools here. The $S$-qubits are introduced as minimal left ideals of the Clifford algebras. Finally, some comparison with possible alternative models of qubits related with super-spaces are outlined in Sec. 3.3.

## 2 Super-indexed qubits

\textit{Flesh flourisht of fermison with frumentee noble.}
\textit{(Alliterative Morte Arthure)}

### 2.1 States

Let us introduce special notation for qubits marked by some set of indexes $\mathcal{I}$ with basic states denoted as

$$\hat{\mu}^a \hat{\nu}^b \cdots = \hat{\mu}^a \hat{\nu}^b \cdots, \quad a, b, \ldots \in \mathcal{I}, \quad \mu, \nu, \ldots \in \{0, 1\}.$$  \quad (1)

All indexes in the sequence $a, b, \ldots$ must be different. The $\mathcal{I}$ can be associated with some nodes in multi-dimensional lattices, more general graphs or other
configurations without natural ordering. Thus, the qubits in Eq. (1) may be rearranged in different ways.

An idea about basic states of qubits as ‘occupation numbers’ of anti-commuting ‘local fermionic modes’ (LFM) [4] can be formalized by introduction of equivalence relation between elements Eq. (1) with different order of the indexes defined for any neighboring pair by signed exchange rule

$$\hat{b} \hat{a} \ket{\mu} \ket{\nu} \simeq (-1)^{\mu \nu} \hat{a} \hat{b} \ket{\nu} \ket{\mu}, \quad \forall a \neq b \in I, \quad \mu, \nu \in \{0, 1\}. \tag{2}$$

Due to such rules terms $\ket{0}$ with ‘attached’ indexes can be exchanged (‘commute’) with any state $\hat{b} \ket{\psi} = \alpha \hat{a} \ket{0} + \beta \hat{b} \ket{1}$, but two $\ket{1}$ require change of the sign for such a swap (‘anti-commute’). For standard notation and qubits without special indexes the exchange rule could be implemented by signed swap operator

$$\hat{S}_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

States Eq. (1) with equivalence relation Eq. (2) define basis of some linear space $S$ denoted here as ‘super-indexed’ qubits or $S$-qubits.

The equivalence relation Eq. (2) can be extended on arbitrary permutation $\pi$. Such operator is denoted further $\hat{\pi}_\pm$ and notation $\hat{\pi}$ is reserved for usual permutation. The construction of $\hat{\pi}_\pm$ does not depend on decomposition of $\pi$ on adjacent transpositions, i.e., swaps of $S$-qubits considered above. Such consistency becomes more natural from algebraic models below in Sec. 3 and ‘physical’ interpretation with LFM.

It can be also proved for arbitrary state by direct check for the basis. Let us consider for a given basic state different sequences of transpositions produced the same permutation. It is necessary to show that the sign does not depend on the decomposition of the permutation into the sequence. Let us consider restriction $(\pi_1)$ of permutation on subset of indexes corresponding to $S$-qubits with unit value. For the only nontrivial case the restriction of swap on such subset corresponds to exchange of two units with change of sign. So, for any decomposition the basic vector may change the sign only if the permutation $\pi_1$ is odd. Thus, $\hat{\pi}_\pm$ is the same for any decomposition of $\hat{\pi}$ on transpositions defined by signed exchange rule Eq. (2).
The relation $\hat{S}_\pm$ can be considered as a formalization of swap with two LFM denoted as ‘$\Leftrightarrow$’ in [4]. It could be expressed as composition of usual exchange of qubits ‘$\leftrightarrow$’ and ‘swap defect’ operator [4]

$$
\hat{D} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
$$

Perhaps, the term ‘fermionic qubits’ might be not very justified for the model considered here, because the property Eq. (2) would correspond to fermion for $\hat{1}$ (‘occupied,’ $n = 1$) and boson for $\hat{0}$ (‘empty,’ $n = 0$).

Thus, $S$-qubits could be considered as quasi-particles (‘fermions’) with combined statistics, because exchange rule instead of ($\pm 1$) multiplier for bosons or fermions should use swap defect operator

$$
\begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix} \hat{D} \begin{pmatrix}
\hat{b} \\
\hat{a}
\end{pmatrix} \mapsto (-1)^{\hat{n}_a \hat{n}_b} \begin{pmatrix}
\hat{b} \\
\hat{a}
\end{pmatrix},
$$

where a formal representation $\hat{D} = (-1)^{\hat{n}_a \hat{n}_b}$ is used, where $\hat{n}_a$ and $\hat{n}_b$ are analogues of occupation number operators defined for usual qubit as

$$
\hat{n} |\nu\rangle = \nu |\nu\rangle, \quad \nu \in \{0, 1\}, \quad \hat{n} = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
$$

The result of a swap Eq. (5) is defined in simple ‘product’ form Eq. (2) for the basis, but for arbitrary states the expressions are less trivial.

The $S$-qubit also could be compared with an element of super vector space, but due to some subtleties outlined in Sec. 3.3 such approach should be discussed elsewhere.

The scalar product of $S$-qubits states can be naturally defined for the equivalent sequences of indexes $S$ in both (‘bra’ and ‘ket’) parts

$$
\langle S | \hat{S} \Psi \rangle = \langle \Psi | \hat{S} \Phi \rangle = \langle S \Psi | \Phi \rangle.
$$

The definition Eq. (7) does not depend on a sequence $S$. Indeed, let us consider permutations of indexes $\pi: S \mapsto S'$. For the basic states a permutation may only introduce ($\pm 1$) multiplier and the scalar product Eq. (7) does not change. It can be also checked directly for arbitrary states

$$
\langle S' | \hat{S}' \Psi \rangle = \langle \Psi | \hat{S}' \pi^\dagger \pi \hat{S} \Phi \rangle = \langle \Psi | \Phi \rangle = \langle S \Psi | \hat{S} \Phi \rangle.
$$
2.2 Operators

Description of quantum gates with annihilation and creation (‘ladder’) operators was initially suggested by R. Feynman [1, 2]. However, despite of formal resemblance with Pauli exclusion principle for fermions

\[ \hat{a}^\dagger |0\rangle = |1\rangle, \quad \hat{a}|1\rangle = |0\rangle, \quad \hat{a}|0\rangle = \hat{a}^\dagger |1\rangle = 0, \]

they do not satisfy canonical anticommutation relation (CAR) for different qubits. Sometimes, usual qubits are compared with so-called ‘hardcore’ bosons, but it is not discussed here. It is considered instead, how ladder operators with CAR can be introduced for $S$-qubits, see Eq. (14) below.

The creation operator $a^\dagger_a$ can be defined for basic states taking into account exchange rule Eq. (2) of $S$-qubits

\[ a^\dagger_a |L \ldots,0, \ldots\rangle = |1, \ldots, \ldots\rangle, \quad a^\dagger_a |L \ldots,1, \ldots\rangle = 0, \]

where $L$ and $R$ correspond to arbitrary sequences before and after index ‘$a$’ respectively. The conjugated annihilation operator $a_a = (a^\dagger_a)^\dagger$ in simpler case with index ‘$a$’ in the first position can be written

\[ a_a |0, \ldots\rangle = 0, \quad a_a |1, \ldots\rangle = |0, \ldots\rangle. \]

A sign for application of operator $a_a$ to arbitrary position should be found using rearrangement of indexes together with signed exchange rule Eq. (2).

Let us denote $\pm_a$ a sign derived from Eq. (2) for expressions such as

\[ \pm_a |L \ldots,1, \ldots\rangle = \pm_a |L \ldots,\ldots\rangle, \]

where $\pm_a = (-1)^{\#L}$ with $\#L = \sum_{l \in L} n_l$ is number of units in sequence $L$ of positions before ‘$a$’. for simplicity $L$ and $R$ are omitted further.

Finally, Eqs. (9,10) can be rewritten

\[ a^\dagger_a |L \ldots,0, \ldots\rangle = \pm_a |L \ldots,1, \ldots\rangle, \quad a^\dagger_a |L \ldots,1, \ldots\rangle = 0, \]

\[ a_a |L \ldots,0, \ldots\rangle = 0, \quad a_a |L \ldots,1, \ldots\rangle = \pm_a |L \ldots,0, \ldots\rangle. \]

For consequent indexes $a = 0, \ldots, m - 1$ Eq. (12) is in agreement with usual Jordan-Wigner transformation [3, 4], i.e.,

\[ a_a |n_0, \ldots, n_{a-1}, 1, n_{a+1}, \ldots\rangle = (-1)^{\sum_{k=0}^{a-1} n_k} |n_0, \ldots, n_{a-1}, 0, n_{a+1}, \ldots\rangle \]

\[ a_a |n_0, \ldots, n_{a-1}, 0, n_{a+1}, \ldots\rangle = 0, \]
and $a_a^\dagger$ is Hermitian conjugation

$$a_a^\dagger|n_0, \ldots, n_{a-1}, 0, n_{a+1}, \ldots\rangle = (-1)^{\sum_{k=0}^{a-1} n_k} |n_0, \ldots, n_{a-1}, 1, n_{a+1}, \ldots\rangle$$

$$a_a^\dagger|n_0, \ldots, n_{a-1}, 1, n_{a+1}, \ldots\rangle = 0.$$  

(13b)

The $a_a$ and $a_a^\dagger$ defined in such a way satisfy canonical anticommutation relations

$$\{a_a, a_b\} = \{a_a^\dagger, a_b^\dagger\} = 0, \quad \{a_a, a_b^\dagger\} = \delta_{ab}1.$$  

(14)

Let us now introduce Clifford algebra $\mathfrak{Cl}(2m)$ with $2m$ generators using operators Eq. (13)

$$e_a = i(a_a^\dagger + a_a), \quad e'_a = a_a - a_a^\dagger.$$  

(15)

Operators of so-called Majorana fermionic modes coincides with Eq. (15) up to imaginary unit multiplier [4].

With earlier definitions of annihilation and creation operators it can be expressed for basis as

$$e_a|\ldots, a_{a+0}0, \ldots\rangle = \pm a^\dagger|\ldots, a_{a+1}, 0, \ldots\rangle,$$

$$e'_a|\ldots, a_{a+0}0, \ldots\rangle = \mp a^\dagger|\ldots, a_{a+1}, 0, \ldots\rangle.$$  

(16)

where $\mp_a = -(\pm_a)$.

For consequent indexes $j = 0, \ldots, m-1$ the Eq. (16) again correspond to Jordan-Wigner formalism with definition of complex Clifford algebra $\mathfrak{Cl}(2m, \mathbb{C})$ by tensor product of Pauli matrices [3, 5]

$$e_j = i \hat{\sigma}^x \otimes \ldots \otimes \hat{\sigma}^x \otimes \hat{1} \otimes \cdots \otimes \hat{1},$$

$$e'_j = i \hat{\sigma}^y \otimes \ldots \otimes \hat{\sigma}^y \otimes \hat{1} \otimes \cdots \otimes \hat{1}.$$  

(17)

However, Eq. (17) directly introduces order of indexes unlike more abstract definitions of operators such as Eq. (12) and Eq. (16) respecting structure of $S$-qubits without necessity of predefined order.

The linear combinations of all possible products with operators $e_a, e'_a$ (or $a_a, a_a^\dagger$) for given set $\mathcal{I}$ with $m_{\mathcal{I}}$ indexes generate Clifford algebra $\mathfrak{Cl}(2m_{\mathcal{I}}, \mathbb{C})$ with dimension $2^{2m_{\mathcal{I}}}$. Thus, an arbitrary linear operator on space $S$ can be
represented in such a way, but unitarity should be also taken into account for construction of quantum gates on $S$-qubits.

An alternative notation $e_{a'} = e'_a$, $a' \in \mathcal{I}' \sim \mathcal{I}$ unifying two sets of generators from Eq. (15) into the single collection with doubled set of indexes $2\mathcal{I} = \mathcal{I} \cup \mathcal{I}'$ is also used further for brevity. Definition of Clifford algebra $\mathcal{C}(2m, \mathbb{C})$ can be written with such a set as

$$\{e_a, e_b\} = -2\delta_{ab}1, \quad a, b \in 2\mathcal{I}. \quad (18)$$

and conjugation of elements as

$$e^\dagger_a = -e_a, \quad a \in 2\mathcal{I}. \quad (19)$$

The elements Eq. (15) generate $\mathcal{C}(2m, \mathbb{C})$ isomorphic with whole algebra of $2^m \times 2^m$ complex matrices. The unitary gates may be expressed as exponents of Hermitian elements with pure imaginary multipliers discussed below.

Let us consider for some sequence $L$ with $l$ indexes from $2\mathcal{I}$ a product of $l$ generators

$$e_L = e_{a_1} \cdots e_{a_l}, \quad a_1, \ldots, a_l \in 2\mathcal{I}. \quad (20)$$

Linear subspaces $\mathcal{C}^{(l)}$ is introduced as a span of such products, $e_L \in \mathcal{C}^{(l)}$. The notation $\mathcal{C}^0$ and $\mathcal{C}^1$ is reserved here for standard decomposition of $\mathcal{C}$ as $\mathbb{Z}_2$-graded algebra with two subspaces corresponding to linear span of all possible products with even and odd $l$ respectively [10]

$$\mathcal{C}(n) = \mathcal{C}^0(n) \oplus \mathcal{C}^1(n). \quad (21)$$

The square of element $e_L$ can be expressed as

$$e_{L}^2 = (-1)^{\varsigma_l}, \quad \varsigma_l = \frac{l(l + 1)}{2} \mod 2. \quad (22)$$

All such elements are unitary with respect to conjugation operation [5]

$$e_{L}^\dagger = (-1)^{\varsigma_l} e_L, \quad e_{L}^\dagger e_L = 1. \quad (23)$$

The construction of Hermitian basis is also straightforward

$$(i^{\alpha} e_L)^\dagger = (-i)^{\alpha} e_L^\dagger = (-i)^{\alpha} (-1)^{\varsigma_l} e_L = i^{\alpha} e_L. \quad (24)$$
An exponential representation of unitary operators is simply derived from Eq. (24) for arbitrary compositions of basic elements, e.g., for $h_l \in \mathcal{O}^{(l)}$

$$u(\tau) = \exp(-i h_l \tau), \quad \imath_i = i \cdot i^8 = i^{8+1}, \quad \tau \in \mathbb{R}. \quad (25)$$

Due to property $\varsigma_{l+4} = \varsigma_l$, multipliers can be given in the table,

| $l \mod 4$ | 0 | 1 | 2 | 3 |
|-------------|---|---|---|---|
| $\varsigma_l$ | 0 | 1 | 1 | 0 |
| $-\imath_i$ | -i | 1 | 1 | -i |

Expression of unitary group $U(2^m)$ using families of quantum gates can be derived using approach with exponents due to correspondence between Lie algebras and Lie groups. The method initially was suggested for construction of universal set of quantum gates [6, 7, 8]. Clifford algebra $C^{\ell}(2^m)$ with Lie bracket operation defined as a standard commutator

$$[a, b] = ab - ba \quad (27)$$

can be used for representation of Lie algebra of special unitary group $su(2^m)$ and group $SU(2^m)$ can be expressed as exponents of elements from $\mathcal{O}(2m)$.

In the exponential representation analogue of one-gates for $S$-qubits with $l = 1, 2$ can be expressed as

$$u_j = \exp(h_1 e_j + h_2 e'_j + h_3 e_j e'_j), \quad h_1, h_2, h_3 \in \mathbb{R}. \quad (28)$$

It can be also rewritten

$$u_j = q_0 + q_1 e_j + q_2 e'_j + q_3 e_j e'_j, \quad q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}. \quad (29)$$

Analogue of two-gates for $S$-qubits with indexes $j, k \in \mathcal{I}$ can be declared by analogue exponents with linear combination of different products including $e_j$, $e'_j$, $e_k$, $e'_k$ with coefficients are either real ($l = 1, 2$) or pure imaginary ($l = 3, 4$).

Similar exponents with more general elements from linear subspaces $\mathcal{O}^{(l)}$ for $l = 2$ and $l = 1, 2$ (with arbitrary combinations of indexes from $2\mathcal{I}$) generate ‘non-universal’ subgroups isomorphic to $\text{Spin}(2m)$ and $\text{Spin}(2m+1)$ respectively [5, 8], but inclusion element with $l = 3$ is enough to generate unitary group $SU(2^m)$ [8].
Due to physical reasons for some models only terms with even number of generators must be used [4]. Formally, such terms belong to even subalgebra $\mathcal{O}^0$ that may be again treated as a Clifford algebra due to standard isomorphism $\mathcal{O}(n - 1) \cong \mathcal{O}^0(n)$ [10]

$$\mathcal{O}(n - 1) \to \mathcal{O}^0(n), \quad \epsilon_L \mapsto \begin{cases} \epsilon_L, & \epsilon_L \in \mathcal{O}^0(n - 1), \\ \epsilon_L \epsilon_n, & \epsilon_L \in \mathcal{O}^1(n - 1). \end{cases} \quad (30)$$

Thus, a model with even number of generators in Hamiltonians and universal subset of quantum gate with $l = 2, 4$ [4] is also described by Clifford algebra due to isomorphism $\mathcal{O}^0(2m) \cong \mathcal{O}(2m - 1)$.

3 Algebraic models of $S$-qubits

3.1 Exterior algebra

Let us consider a vector space $V$ with basis $x_j, j = 0, \ldots, m - 1$. The exterior (Grassmann) algebra is defined as

$$\Lambda(V) = \bigoplus_{k=0}^{m} \Lambda^k(V), \quad (31)$$

where $\Lambda^0(V)$ are scalars, $\Lambda^1(V) = V$ are vectors, and $\Lambda^k(V), k > 1$ are antisymmetric $k$-forms (tensors) with basis

$$x_{j_1} \wedge \cdots \wedge x_{j_k}, \quad j_1 < \cdots < j_k \quad (32)$$

where ‘$\wedge$’ denotes antisymmetric (exterior) product $x_j \wedge x_k = -x_k \wedge x_j$, $x \wedge x = 0, \forall x \in V$.

The dimension of whole space $\Lambda(V)$ is $2^m$ and any basic state Eq. (1) of $S$-qubits could be mapped into $\Lambda(V)$

$$\hat{\left| n_{j_1}, \ldots, n_{j_m} \right\rangle} \mapsto \bigwedge_{\substack{j \in I \\n_j = 1}} x_j. \quad (33)$$

Such a method inserts into exterior product only $x_j$ with indexes $j$ satisfying $n_j = 1$. However, Eq. (33) is one-to-one map and arbitrary form $\Omega \in \Lambda(V)$ corresponds to some state $\hat{\left| \Omega \right\rangle}$ up to appropriate normalization.
The creation and annihilation operators in such representation correspond to a known construction of Clifford algebra using space of linear transformations on $\Lambda(V)$ [5] and may be expressed for basis Eq. (32)

\begin{align}
\hat{a}_j^\dagger : x_{j_1} \wedge \cdots \wedge x_{j_k} &\quad \mapsto \quad x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_k}, \\
\hat{a}_j : x_{j_1} \wedge \cdots \wedge x_{j_k} &\quad \mapsto \quad \sum_{l=1}^{k} (-1)^l x_{j_1} \wedge \cdots \wedge (\delta_{j,l} 1) \wedge \cdots \wedge x_{j_k},
\end{align}

where $1$ is unit of algebra $\Lambda(V)$ and notation $1 \wedge \omega = \omega \wedge 1 = \omega$, $\omega \in \Lambda(V)$ is supposed in Eq. (34). Such operators satisfy Eq. (14) and respect map Eq. (32) due to consistency of Eq. (34a) with Eq. (9) and Eq. (34b) with Eq. (12b).

The generators of complex Clifford algebra $\mathcal{C}(2m, \mathbb{C})$ can be expressed with earlier defined pair of generators Eq. (15) for each index and for real case elements $e'_j$ might be used to produce $\mathcal{C}(m, \mathbb{R})$ [5].

The considered representation of $S$-qubits with exterior algebra $\Lambda(V)$ despite of one-to-one correspondence Eq. (33) for complete basis may be not very convenient for work with ‘reduced’ expressions such as Eq. (2), because only qubits with state $|1\rangle$ map into different $x_a$, but any sequence of qubits in state $|0\rangle$ formally corresponds to unit scalar $1 \in \Lambda^0(V)$. An approach with Clifford algebras discussed next helps to avoid such a problem.

### 3.2 Clifford algebras and spinors

For Clifford algebra $\mathcal{C} = \mathcal{C}(2m, \mathbb{C})$ the space of spinors has dimension $2^m$ and it can be represented as minimal left ideal [9]. The left ideal $\ell \subset \mathcal{C}$ by definition has a property

$$ \mathfrak{c} \ell \subset \ell : \quad \forall \mathfrak{l} \in \ell, \ \mathfrak{c} \in \mathcal{C}. $$

By definition, the (nonzero) minimal left ideal does not contain any other (nonzero) left ideal.

The notation with single set of indexes $a \in \mathcal{I}$ and $m$ pairs of generators $e_a$ and $e'_a$ is again used below. Annihilations and creation operators corresponding Eq. (15) are also useful further

$$ a_a = \frac{e_a + i e'_a}{2i}, \quad a_a^\dagger = \frac{e_a - i e'_a}{2i}. $$
The minimal left ideal $\ell$ is generated by all possible products with an appropriate element $\ell_\emptyset$

$$\ell = \left\{ c \ell_\emptyset : c \in C, \ell_\emptyset = \prod_{a \in I} \ell_a^0 \right\}$$  \hspace{1cm} (37)

where

$$\ell_a^0 = \frac{1 + i e_a e'_a}{2} = a_a a_a^\dagger$$  \hspace{1cm} (38)

are $N$ commuting projectors $(\ell_a^0)^2 = \ell_a^0$. Due to identity $\ell_a^0 = i e_a e'_a \ell_a^0$ for any index $a$ it can be written

$$e'_a \ell_\emptyset = -i e_a \ell_\emptyset.$$  \hspace{1cm} (39)

Let us apply definition of $\ell$ Eq. (37) to linear decomposition of $c$ on terms with products of generators $e_a$ and $e'_a$. Any element of $\ell$ in Eq. (37) may be rewritten as a linear combination of terms without $e'_a$ due to Eq. (39). Thus, $\ell$ has dimension $2^m$ with products at most $m$ different generators $e_a$ on $\ell_\emptyset$ as a basis. Let us also introduce notation

$$\ell_a^1 = a_a^\dagger \ell_a^0 = a_a^\dagger$$  \hspace{1cm} (40)

then a basis of spinor space $\ell$ can be rewritten in agreement with Eq. (1)

$$(\ell_a^\mu \ell_b^\nu \cdots) \ell_\emptyset \leftrightarrow \begin{pmatrix} a \ b \ 
\mu \ \nu \ \cdots \ 
\end{pmatrix}, \ a, b, \ldots \in I, \ \mu, \nu, \ldots \in \{0, 1\},$$  \hspace{1cm} (41)

there all indexes $a, b, \ldots$ must be different.

In representation Eq. (17) with consequent indexes $a = 0, \ldots, m - 1$ the elements $\ell_a^0$ correspond to $2^m \times 2^m$ diagonal matrices with units and zeros described by equation

$$\ell_a^0 \leftrightarrow \underbrace{1 \otimes \cdots \otimes 1}_{a} \otimes \ell_0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-a-1},$$  \hspace{1cm} (42)

where $\ell_0 = |0\rangle \langle 0|$. Therefore, $\ell_\emptyset$ corresponds to a $2^m \times 2^m$ diagonal matrix with unit only in the very first position

$$\ell_\emptyset \leftrightarrow \ell_0 \otimes \cdots \otimes \ell_0 = \begin{pmatrix} 0, \ldots, 0 \ 
0, \ldots, 0 \ 
\end{pmatrix}.$$  \hspace{1cm} (43)

In such a case a product $c \ell_\emptyset$ in definition of ideal $\ell$ Eq. (37) corresponds to a $2^m \times 2^m$ matrix with only nonzero first column. It can be used for representation of a vector with $2^m$ components.
For arbitrary sequences of indexes from a set $\mathcal{I}$ an analogue of Eq. (2) also holds
\[ \ell^a_\mu \ell^b_\nu = (-1)^{\mu\nu} \ell^b_\nu \ell^a_\mu, \quad a \neq b \in \mathcal{I}, \quad \mu, \nu \in \{0, 1\}. \] (44)

The inequality of indexes $a \neq b$ is essential, because the super-commutativity Eq. (44) does not hold for $a = b$ if $\mu \neq \nu$
\[ \ell^a_0 \ell^a_0 = \ell^a_0, \quad \ell^a_1 \ell^a_1 = \ell^a_0 \ell^a_1 = 0, \quad \ell^a_1 \ell^a_0 = \ell^a_1, \quad \ell^a_0 \ell^a_1 = \ell^a_1 \ell^a_0. \] (45)

Anyway, all indexes of $S$-qubits Eq. (1) are different by definition and inequality in Eq. (45) does not affect considered representation Eq. (41).

However, some other expressions for operators or scalar product Eq. (7) may require more general combinations of indexes. It is discussed below.

For arbitrary element $\ell \in \mathcal{I}$ the operators $a_a, a^\dagger_a$ can be naturally defined via left multiplication
\[ a_a : \ell \mapsto a_a \ell, \quad a^\dagger_a : \ell \mapsto a^\dagger_a \ell. \]

With respect to map Eq. (41) it corresponds to Eq. (12). Let us check that.

Operators $a_a, a^\dagger_a$ commute with $\ell^a_0$ and anticommute with $\ell^a_1$ for $a \neq b$. For equivalent indexes quite natural expressions follow from definitions
\[ a_a \ell^a_0 = 0, \quad a_a \ell^a_1 = \ell^a_0, \quad a^\dagger_a \ell^a_0 = \ell^a_1, \quad a^\dagger_a \ell^a_1 = 0. \] (46)

Let us rewrite map Eq. (41) with shorter notation for basic states $n$
\[ |n\rangle = |n_{j_a} n_{j_b}, \ldots \rangle \longleftrightarrow \ell_n = \prod_{j \in \mathcal{I}} \ell^j_{n_j}. \] (41')

It may be also expressed in an alternative form
\[ \ell_n = \prod_{j \in \mathcal{I}} (a^\dagger_j)^{n_j} \ell^j_0 = \left( \prod_{\substack{j \in \mathcal{I} \\ n_j=1}} a^\dagger_j \right) \ell_\emptyset. \] (47)

resembling Eq. (33) for Grassmann algebra. Due to Eq. (37) products of operators $a^\dagger_a$ are mapped by Eq. (47) into elements of left ideal of Clifford algebra, cf Eq. (35).
With respect to map Eq. (41) actions of $a_0^\dagger$ are in agreement with Eq. (9) and $a_n$ satisfy an analogue of Eq. (10). Thus, operators $a_n, a_n^\dagger$ and their linear combinations $e_n, e_n^\dagger$ are corresponding to Eq. (12) and Eq. (16) respectively.

The scalar product Eq. (7) also can be naturally expressed. Let us find conjugations of $\ell_0^a$ and $\ell_1^a$

$$\ell_0^{a\dagger} = a_0^a a_0^\dagger = \ell_0^a, \quad \ell_1^{a\dagger} = a_n.$$

The equation

$$\ell_0^{a\dagger} \ell_0^a = \delta_{jk} \ell_0^a, \quad j, k = 0, 1$$

can be checked directly

$$\ell_0^{a\dagger} \ell_0^a = \ell_1^{a\dagger} \ell_1^a = \ell_0^a, \quad \ell_0^{a\dagger} \ell_1^a = \ell_1^{a\dagger} \ell_0^a = 0$$

together with appropriate expressions for products Eq. (41’)

$$\ell_n \ell_n = \ell_\varnothing, \quad \ell_n^\dagger \ell_n' = 0.$$  

Let us also use notation $\ell_\Psi$ for representation of arbitrary $|\Psi\rangle$, i.e., linear combinations of basic states. Then scalar product Eq. (7) can be written using Eq. (50)

$$\ell_\Psi^\dagger \ell_\Phi = \langle \Psi^\dagger | \Phi \rangle \ell_\varnothing = \langle \Psi^\dagger | \Phi \rangle \ell_\varnothing = \langle \Psi^\dagger | \Phi \rangle \ell_\Psi^\dagger \ell_\Phi = 2^{-m} \langle \Psi | \Phi \rangle.$$

where super-index $\mathcal{I}$ denotes set of indexes used in Eqs. (41, 41’) and it can be dropped, because all indexes are naturally taken into account in such algebraic expressions with appropriate order.

Special notation can be used for ‘scalar part’ of an element

$$c \in \mathcal{O}(2m, \mathbb{C}), \quad c = c1 + \cdots, \quad \text{Sc}(c) = c.$$  

Eq. (51) can be rewritten now to express the scalar product directly

$$\text{Sc}(\ell_\Psi^\dagger \ell_\Phi) = \text{Sc}(\langle \Psi^\dagger | \Phi \rangle \ell_\varnothing) = \langle \Psi^\dagger | \Phi \rangle \text{Sc}(\ell_\varnothing) = 2^{-m} \langle \Psi | \Phi \rangle.$$  

Let us introduce an analogue of density operator. For pure state it can be defined

$$\varrho_\Psi = \ell_\Psi^\dagger \ell_\Psi,$$  

with natural property

$$\varrho_\Psi \ell_\Phi = \ell_\Psi \ell_\Psi^\dagger \ell_\Phi = \ell_\Psi \langle \Psi | \Phi \rangle \ell_\varnothing = \langle \Psi | \Phi \rangle \ell_\Psi \ell_\varnothing = \langle \Psi | \Phi \rangle \ell_\Psi.$$  

Arbitrary operators can be expressed using linear combinations with pairs of basic states

$$|n\rangle \langle n| \longleftrightarrow \varrho_{n,n} = \ell_n \ell_n^\dagger, \quad \varrho_{n,n} \ell_\Phi = \ell_n \langle n | \Phi \rangle \ell_\varnothing = \Phi_n \ell_n.$$  

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3.3 Super vector spaces

A super vector space \([11, 12]\) is \(\mathbb{Z}_2\)-graded vector space

\[ V = V_0 \oplus V_1, \quad 0, 1 \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}. \]  

(57)

The complex super vector space is denoted \(\mathbb{C}^{d_0|d_1}\), where \(d_i\) is dimension of \(V_i\). The elements \(v \in V_0, p(v) = 0\) and \(w \in V_1, p(w) = 1\) are called even and odd respectively.

The \(\mathbb{Z}_2\)-graded (super) tensor product for such elements can be defined using sign rule

\[ v \otimes u = (-1)^{p(v) p(u)} u \otimes v. \]  

(58)

Roughly speaking, one S-qubit could be compared with element of \(\mathbb{C}^{1|1}\), but such approach encounters difficulties for more S-qubits. Indeed, \(\mathbb{Z}_2\)-graded tensor product Eq. (58) in definition Eq. (1) should use different copies of initial space. Such approach may be quite natural in definition of \(\mathbb{Z}_2\)-graded tensor product of algebras and can be used for construction of Clifford algebras [10, 13].

However, it is not quite clear, how to implement similar idea for construction of S-qubits from \(\mathbb{C}^{1|1}\), because implementation of \(m\) different copies of S-qubit may require to use bigger spaces such as \(\mathbb{C}^{m|m}\).

Let us consider basis of \(V = \mathbb{C}^{m|m}\): \(e_k^0 \in V_0, \ e_k^1 \in V_1, \ k = 0, \ldots, m - 1\). States of qubits \(\alpha e_k^0 + \beta e_k^1\) belong to different 2D subspaces of \(V\) and their tensor product for \(k = 0, \ldots, m - 1\) is the linear subspace with dimension \(2^m\) of the ‘whole’ tensor product \(V^\otimes m\) with dimension \((2m)^m\). However, it may look as not very natural choice.

For more trivial cases \(\mathbb{C}^{d_0|0}\) and \(\mathbb{C}^{0|d_1}\), the tensor product of super vector spaces could be treated as symmetric and antisymmetric tensors respectively, but in such a case all vector spaces in product are usually considered as identical. Similar approach with identical copies of \(\mathbb{C}^{d_0|d_1}\) is also quite common in supersymmetry. Thus, superspace is only briefly mentioned here for comparison with other models of S-qubits and the term super-indexed is used earlier to emphasize the difference with known supersymmetric model of qubits [14].

It should be mentioned also, that in the spinor model of S-qubits discussed in Sec. 3.2 the elements \(\ell^a_\mu\) formally do not belong to superalgebra. Despite the super-commutation rule is valid for different super-indexes Eq. (44), it can be violated for the same one, Eq. (45).
4 Conclusion

Jordan-Wigner transformation maps some operators Eq. (a) acting ‘locally’ on \( n \) qubits (or spin-\( \frac{1}{2} \) systems) into \( n \) fermionic creation and annihilation (ladder) operators. The ‘nonlocal’ construction of such a map supposes introduction of some formal ordering on the set of qubits. Such ordering may be natural for some simple models such as 1D chain. All ladder operators in fermionic system are formally equivalent and unnatural order produces technical difficulties for more general models such as multidimensional lattices and more general graphs.

Antisymmetric algebra may be formally used for equal (unordered) description of ladder operators, but it does not answer a question about inequality of states. To address such a problem in this work was suggested model of ‘super-indexed’ \( S \)-qubits. Equivalence relation necessary for agreement with Jordan-Wigner transformation and anti-commutativity of ladder operators is signed exchange rule for \( S \)-qubits Eq. (2).

Algebraic model of \( S \)-qubits with such property was also discussed. The model uses Clifford algebras and spinors. Such approach is different with analogue constructions more common in supersymmetric models only briefly discussed in subsection above.

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