A discussion of stochastic dominance and mean-risk optimal portfolio problems based on mean-variance-mixture models

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Abstract
The classical Markowitz mean-variance model uses variance as a risk measure and calculates frontier portfolios in closed form by using standard optimization techniques. For general mean-risk models such closed form optimal portfolios are difficult to obtain. In this note, we obtain closed form expression for frontier portfolios under mean-risk criteria when risk is modelled by any finite law-invariant convex measures of risk and when return vectors follow the class of normal mean-variance mixture (NMVM) distributions. To achieve this goal, we first present necessary as well as sufficient conditions for stochastic dominance within the class of one dimensional NMVM models and then we apply them to portfolio optimization problems. Our main result in this paper states that when return vectors follow the class of NMVM distributions the associated mean-risk frontier portfolios can be obtained by optimizing a Markowitz mean-variance model with an appropriately adjusted return vector.

Keywords: Frontier portfolios; Mean-variance mixtures; Risk measures; Stochastic dominance; Mean-risk criteria

JEL Classification: G11

1 Motivation
In risk management and portfolio optimization problems it is essential to model the underlying asset returns by proper distributions. The classical Markowitz portfolio optimization approach is based on the assumption that asset returns are normally distributed. However, there has been overwhelming empirical evidences that the normal distribution as a model for asset returns must be rejected. Empirical asset returns exhibit semi-heavy tails, i.e. the kurtosis of the empirical data is higher than the kurtosis of the normal distribution. Generalized hyperbolic (GH) distributions allow a more realistic description of asset returns than the normal distributions. The GH distributions were introduced in [6] and they belong to the class of Normal mean-variance mixture (NMVM) models.

An $n$–dimensional random vector $X$ has normal mean-variance mixture distribution with a mixing distribution $Z$ if for any given $Z = z \in \mathbb{R}_+ =: (0, +\infty)$ the random vector $X$ satisfies
the mixing distribution $Z$ in (1) is independent from $N_n$.

In this paper we study portfolio optimization problems based on $n$ risky assets. We assume that the return vector $X$ of these $n$ risky assets satisfy (1). The portfolio set is given by $\mathbb{R}^n$ and for each portfolio $\omega \in \mathbb{R}^n$ the corresponding portfolio return is given by $\omega^T X$, where $T$ denotes a transpose.

We assume that all the random variables in this paper are defined in an atomless probability space $(\Omega, \mathcal{F}, P)$. We denote by $L^0 = L^0(\Omega, \mathcal{F}, P)$ the space of all finite valued random variables and by $L^p = L^p(\Omega, \mathcal{F}, P), 1 \leq p < +\infty$, the space of random variables with finite $p$ moments. $L^\infty(\Omega, \mathcal{F}, P)$ denotes the space of bounded random variables. For vectors $x = (x_1, x_2, \cdots , x_n)^T$ and $y = (y_1, y_2, \cdots , y_n)^T$ we denote by $x \cdot y = x^T y = \sum_{i=1}^{n} x_i y_i$ the scalar product of them and by $||x|| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ the Euclidean norm of $x$. As in [17], for each positive integer $p$ we denote by $L^p_n = (L^p)^n$ the space of all $n$-dimensional random vectors with all the components belong to $L^p$. In this paper, unless otherwise stated we assume that the mixing distribution satisfies $Z \in L^1$. We also assume that $Z$ has probability density function $f_Z(z)$. This latter assumption on $Z$ is not necessary for some of the results of this paper. However, since our main goal in this paper is to study closed form solutions for frontier portfolios for GH models, the assumption on the existence of $f_Z(z)$ for the mixing distribution $Z$ is imposed to keep our calculations simple.

With $L^p_n, p \geq 1$, defined as above, returns $\xi$ of any $n$ assets are represented by the elements in $L^p_n$, i.e., $\xi \in L^p_n$. For any given portfolio $\omega \in \mathbb{R}^n$, the associated loss is then given by $-\omega^T \xi$. The risk of this loss is measured by the real number $\rho(-\omega^T \xi)$ for some given measure of risk $\rho$. Our main goal in this paper is to study the following portfolio optimization problem

\[
\begin{align*}
\min_{\omega} & \quad \rho(-\omega^T \xi), \\
\text{s.t.} \quad & E(-\omega^T \xi) = r, \\
& \omega^T e = 1,
\end{align*}
\]

for a class of risk measures $\rho$ when the return vectors $\xi$ have the distribution as $X$ in (1). The solutions $\omega^*$ of the problem (2), if they exist, are called frontier portfolios. In our work we obtain closed form solutions for the above problem (2) when the risk measure $\rho$ is a finite law-invariant convex risk measure, see Section 3 below for this.

As mentioned earlier, while our objective in this paper is to study the problem (2) for as large class of risk measures $\rho$ as possible, our initial motivation of this paper was to discuss closed form solutions of the problem (2) for the two most popular regulatory risk measures Value-at-Risk (VaR) and the Conditional Value-at-risk (CVaR).

The risk measure $CVaR_\alpha$ (here $\alpha$ denotes significance level from now on) is well defined on $L^1$ and it belongs to the class of coherent risk measures that was introduced in the landmark papers [4, 5], that is, it satisfies (a) subadditivity (b) monotonicity (c) positive homogeneity
(d) translation invariance. Other than these the CVaR has many other favorable properties. But the key properties of CVaR that help us in our analysis in our paper are its law invariance property, i.e., \( CVaR_\alpha(X) = CVaR_\alpha(Y) \) whenever \( X \overset{d}{=} Y \), and its consistency with the second order stochastic dominance property, see Lemma 3.15 below and the paragraph that precedes to it. We remark here that not all the coherent risk measures are law invariant (see the second paragraph of page 292 of [10] for an example for this) and consistency with the second order stochastic dominance is also strong requirement (especially if the risk measure is not law-invariant) for a convex risk measure. Another important property of CVaR that is essential for our discussions in this paper is its continuity in the \( L_p \) space for any \( p \geq 1 \), see Theorem 4.1 of [27] and page 11 of [17] and the references there for this. The risk measure VaR is well defined on \( L_0 \) and, unlike CVaR, it is not convex risk measure. It is consistent with the first order stochastic dominance property but it is not consistent with the second order stochastic dominance property. Due to these, optimization problems associated with it is challenging. The main goal of our paper is to obtain closed form solution for frontier portfolios under the mean-CVaR criteria when the returns of risky assets are modelled as (1).

There are quite popular models within (1) in financial modelling. If \( Z \) follow Generalized Inverse Gaussian (GIG) distributions, the distributions of \( X \) follow multi-dimensional generalized hyperbolic (mGH) distributions. These class of distributions were introduced in the paper [6] and their numerous financial applications were discussed in the subsequent papers [1], [15], [16], [32], [33], and [39]. For example, the paper [32] uses the Variance-Gamma distributions for modelling share market returns. The paper [16] advocates the use of hyperbolic distributions in financial modelling. The paper [1] finds that the Generalized Hyperbolic Skew Student’s \( t \) distribution matches empirical financial data very well. In the multivariate financial data setting, the paper [26] calibrate the mGH model and its subclasses to both multi-variate stock and multi-variate exchange rate returns. In their likelihood-ratio test the Gaussian model is always rejected against the general mGH models. The paper [1] applies the multivariate NIG (Normal Inverse Gaussian) distribution in risk management and demonstrates that this model provides a much better fit to the empirical distribution of hedge fund returns than the Normal distribution. In general, the GH distributions can model asymmetry and heavy tail behaviour of empirical financial data returns very well (compared to the classical Normal models), see [24], [9], [26] and the references there for this.

A GIG distribution \( Z \) has three parameters \( \lambda, \chi, \) and \( \psi \) and its density is given by

\[
 f_{\text{GIG}}(z; \lambda, \chi, \psi) = \begin{cases} 
 \frac{\chi^{-\lambda}(\psi)^{\frac{3}{2}}}{2K_{\lambda}(\sqrt{\chi\psi})} z^{\lambda-1} e^{-\frac{z^2 + \psi}{2\chi}}, & z > 0, \\
 0, & z \leq 0,
\end{cases}
\]

(3)

where \( K_\lambda(x) = \frac{1}{2} \int_0^\infty y^{\lambda-1} e^{-\frac{y^2 + \psi}{2\chi}} dy \) is the modified Bessel function of the third kind with index \( \lambda \) for \( x > 0 \). The parameters in (3) satisfy \( \chi > 0 \) and \( \psi \geq 0 \) if \( \lambda < 0 \); \( \chi > 0 \) and \( \psi > 0 \) if \( \lambda = 0 \); and \( \chi \geq 0 \) and \( \psi > 0 \) if \( \lambda > 0 \). The moments of \( Z \) are given in page 11 of [23]. When \( \chi > 0, \psi > 0 \), for any \( k \in \mathbb{N} \) we have

\[
 EZ^k = \frac{(\chi/\psi)^{\frac{k}{2}} K_{\lambda+k}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})},
\]

(4)
which are finite numbers and therefore the mixing distributions $Z = GIG(\lambda, \chi, \psi)$ satisfy the stated conditions of our model (1) above for most of the parameters $\lambda, \chi, \psi$.

With $Z \sim GIG$ in (1), the density function of $X$ has the following form

$$f_X(x) = \frac{(\sqrt{\chi/\psi})^\lambda (\psi + \gamma^T\Sigma^{-1}\gamma) N_2 - \lambda}{(2\pi)^2 |\Sigma|^\frac{1}{2} K_\lambda(\sqrt{\chi/\psi})} \times \frac{K_{\lambda - \frac{\alpha}{2}}(\sqrt{\psi + Q(x))} (\psi + \gamma^T\Sigma^{-1}\gamma) e^{(x-\mu)^T\Sigma^{-1}(x-\mu)}}{K_{\lambda - \frac{(\chi + Q(x)) (\psi + \gamma^T\Sigma^{-1}\gamma)}{|\Sigma|^\frac{1}{2}}} N_2 - \lambda},$$  

(5)

where $Q(x) = (x-\mu)^T\Sigma^{-1}(x-\mu)$ denotes the Mahalanobis distance.

The GH distributions contain many special cases that are quite attractive for financial modelling. For example,

a) The case $\lambda = \frac{n+1}{2}$ corresponds to multivariate hyperbolic distribution, see [16] and [9] for applications of this case in financial modelling.

b) When $\lambda = -\frac{1}{2}$, the distribution of $X$ is called Normal Inverse Gaussian (NIG) distribution, see [6] for more details of this distribution.

c) If $\chi = 0$ and $\lambda > 0$, the distribution of $X$ is a Variance Gamma (VG) distribution, see [32] for its applications.

d) If $\psi = 0$ and $\lambda < 0$, the distribution of $X$ is called generalized hyperbolic Student’s $t$ distribution.

In the Markowitz’s Portfolio theory (MPT), variance is used as a risk measure and the corresponding frontier portfolios are identified by using mean-variance optimization criteria. Despite its groundbreaking role in the modern portfolio theory, the MPT comes with some limitations. This is partly due to the shortcomings of variance as a risk measure: variance can not distinguish skewness, kurtosis, upside and downside risks of empirical financial data well for example. Other popular alternative risk measures to variance are VaR and CVaR. The risk measure VaR, which became popular in the financial industry partly due to its handy and easy-to-understand representation of potential losses, measures the smallest relative loss level whose probability of being exceeded is less than a prefixed significance level. It suffers from several shortcomings however also. As mentioned earlier, it is not sub-additive for non-elliptical distributions and it is generally a non-convex function of portfolio weights, see [4] and the references there. Therefore portfolio optimization problems associated with it normally comes with multiple local extrema making the optimization problem computationally expensive. The risk measure CVaR does not suffer from these shortcomings due to its convexity.

Next we introduce some definitions. For any random variable $H$ with cumulative distribution function $F_H(x)$ (we assume all the random variables are continuous in this note), the value at risk at level $\alpha \in [0, 1]$ is defined as

$$VaR_\alpha(H) = F_H^{-1}(\alpha) = \min\{x \in \mathbb{R} : P(H \leq x) \geq \alpha\}, \alpha \in (0, 1],$$

$$VaR_0(H) = \inf\{x : P(H \leq x) > 0\},$$

and the conditional value at risk (CVaR) at significance level $\alpha$ is defined as

$$CVaR_\alpha(H) = \frac{1}{1 - \alpha} \int_0^1 VaR_\alpha(H) ds, \alpha \in (0, 1],$$

$$CVaR_1(H) = VaR_1(H).$$

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For the properties and various applications of these risk measures see [21], [38], [2], [17].

As stated earlier, our interest in this paper is to study solutions of the following type of optimization problems
\[
\min_{\omega \in D} CVaR_{\alpha}(-\omega^T X)
\]
for any domain \( D \subset \mathbb{R}^n \) of the portfolio set \( \mathbb{R}^n \). The domain \( D \) can equal to \( S \) given by
\[
S = \{ \omega \in \mathbb{R}^n : 0 \leq \omega_i \leq 1, \ 1 \leq i \leq n, \ \sum_{i=1}^n \omega_i = 1 \},
\]
for example. Then \( S \) represents the set of portfolios with short-sales restrictions.

In this paper, we are particularly interested in the closed form solutions of the following standard optimization problems associated with \( CVaR \) for any significance level \( \alpha \in (0, 1) \)
\[
\min_{\omega} CVaR_{\alpha}(-\omega^T X),
E(-\omega^T X) = r,
\omega^T e = 1,
\]
where \( r \) is any given real number and \( e \) is the column vector of ones of dimension \( n \). We call the set of solutions of the problem \( \text{(8)} \) for all \( r \in \mathbb{R} \), the set of frontier portfolios associated with the risk measure \( CVaR_{\alpha}(\cdot) \) in this paper.

The problem \( \text{(8)} \) were studied for normal returns in [41]. The paper [3] generalized CVaR to a spectral risk measure in a static setting. The paper [11] studied optimization problems with spectral risk measures (which is called weighted value-at-risk in this paper). The paper [44] studied multi-step dynamic version of CVaR and solved the corresponding mean-CVaR problems by using the techniques in [41]. More recently, robust implementation of the problems associated with CVaR were studied in [28] and [40].

In Markowitz mean-variance portfolio optimization framework the variance is used as a risk measure. In this case, the optimization problem is
\[
\min_{\omega} \text{Var}(-\omega^T X),
E(-\omega^T X) = r,
\omega^T e = 1.
\]
The closed form solution of \( \text{(9)} \) is standard and can be found in any standard textbooks that discusses the capital asset pricing model, see page 64 of [25] for example. Here we write down the solution of \( \text{(9)} \). Let \( \mu = EX \) denote the mean vector and \( V = \text{Cov}(X) \) denote the co-variance matrix of \( X \). For each \( r \in \mathbb{R} \), the solution of \( \text{(9)} \) is given by
\[
\omega^*_r = \omega^*_r(\mu, V) = \frac{1}{d^3}[d^2(V^{-1} e) - d^1(V^{-1} \mu)] + \frac{r}{d^4}[d^3(V^{-1} \mu) - d^1(V^{-1} e)],
\]
where
\[
d^1 = e^T V^{-1} \mu, \ d^2 = \mu^T V^{-1} \mu, \ d^3 = e^T V^{-1} e, \ d^4 = d^2 d^3 - (d^1)^2.
\]

**Remark 1.1.** Note that since \( V^{-1} \) is positive definite we have \((d^1 \mu - d^2 e)^T V^{-1}(d^1 \mu - d^2 e) > 0 \) and \( d^2 > 0 \) (we assume \( \mu = EX \neq 0 \) in this paper). Since
\[
(d^1 \mu - d^2 e)^T V^{-1}(d^1 \mu - d^2 e) = d^2(d^2 d^3 - (d^1)^2) = d^2 d^4 > 0,
\]
we have \( d^4 > 0 \) in [10].
For the problem (8) above a closed form solution was not provided in the past literature to the best of our knowledge. In this paper, as a main result, we will show that the solution of the problem (8) above is given by in a form similar to (10), see our theorem 3.3 below.

To achieve this goal, we first need to study necessary conditions for stochastic dominance within the class of one dimensional mean-variance mixture models. These necessary conditions and some of the results developed in [36] will help us to achieve our goal in this paper.

We first introduce some notations. Let $A_1, A_2, \cdots, A_n$, denote the column vectors of $A$. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows

$$x = T(\omega) = \omega^TA.$$  

Then we have

$$\omega^TX \stackrel{d}{=} x^T\mu_0 + x^T\gamma_0Z + ||x||\sqrt{Z}N(0,1),$$  

where $\mu_0 = (\mu_0^1, \mu_0^2, \cdots, \mu_0^n)^T$ and $\gamma_0 = (\gamma_0^1, \gamma_0^2, \cdots, \gamma_0^n)^T$ are coefficients of the linear combinations $\mu = \sum_{i=1}^{n} \mu_0^iA_i$ and $\gamma = \sum_{i=1}^{n} \gamma_0^iA_i$.

We define

$$W := \mu_0 + \gamma_0Z + \sqrt{Z}N_n$$

and we call $W$ the NMVM vector associated with $X$. For any domain $D$ of portfolios $\omega$, we denote by

$$R_D = T(D)$$

the image of $D$ under the transformation $T$ given by (12). Let $A_1^r, A_2^r, \cdots, A_n^r$ denote the row vectors of $A$, then $R_S$ is a convex region in $\mathbb{R}^n$ with vertices $A_1^r, A_2^r, \cdots, A_n^r$. The optimal solutions $\omega^*$ of

$$\min_{\omega \in D} CVaR_\alpha(-\omega^TX)$$

is related to the optimal solutions $x^*$ of

$$\min_{x \in R_D} CVaR_\alpha(-x^TW)$$

by $x^* = (\omega^*)^TA$ for any domain $D$ of portfolios.

In our paper, we mostly work in the $x-$co-ordinate system rather than the $\omega-$co-ordinate system and discuss the solutions of the (17) type problems.

For any given two portfolios $\omega_1, \omega_2 \in \mathbb{R}^n$, we have the corresponding $x_1 = T\omega_1$ and $x_2 = T\omega_2$. We would like to be able to compare the values of $CVaR_\alpha(-x_1^TW)$ and $CVaR_\alpha(-x_2^TW)$. From (vi) of page 14 of [29] (also see Theorem 4.A.3. of [45]), we have that $x_1^TW$ second order stochastically dominates $x_2^TW$ if and only if $CVaR_\alpha(-x_1^TW) \leq CVaR_\alpha(-x_2^TW)$ for any $\alpha \in (0,1)$. This inspires us to study the stochastic dominance property within the class of one dimensional NMVM models.

## 2 Stochastic dominance

As stated earlier, stochastic dominance (SD) has important relation with the properties of the CVaR risk measure. In this section we investigate some necessary conditions for SD within the class of one dimensional NMVM models. For each $k \in \mathbb{N}$ (here $\mathbb{N}$ denotes the set of positive integers from now on) and for any random variable $H \in L^k$ we define $G_H^{(k)}(x) = ||(x - H)^+||_k$.
for all $x \in \mathbb{R}$, where $x^+$ denotes the positive part of $x$. For any $k \in \mathbb{N}$ and for any two random variables $H, Q \in L_k$, we say that $H$ $(k+1)$'th order stochastically dominates $Q$ if

$$G_H^{(k)}(x) \leq G_Q^{(k)}(x), \forall x \in \mathbb{R}. \quad (18)$$

As in [39], we use the notation $H \succeq_{(k+1)} Q$ to denote that $H$ $(k+1)$'th order stochastically dominates $Q$.

According to Proposition 6 of [39], the function $G_H^{(k)}(x)$ is an increasing convex function with $\lim_{x \to -\infty} G_H^{(k)}(x) = 0$ and $G_H^{(k)}(x) \geq x - EH$ for all $x \in \mathbb{R}$. It was also shown in the same Proposition that $x - EH$ is a right asymptotic line of the function $G_H^{(k)}(x)$, i.e., $\lim_{x \to +\infty} \left( G_H^{(k)}(x) - (x - EH) \right) = 0$.

The value of $G_H^{(k)}(x)$ at $x = EH$ is called central semideviation of $H$ and it is denoted by $\delta_H^{(k)} = G_H^{(k)}(EH)$. Namely

$$\delta_H^{(k)} = \left\| (EH - H)^+ \right\|_k = \left\| E \left( (EH - H)^+ \right) \right\|_k^{1/2}. \quad (19)$$

Proposition 4 of [36] shows that $\delta_H^{(k)}$ is a convex function, i.e., $\delta_H^{(k)}(tH + (1-t)Q) \leq t\delta_H^{(k)} + (1-t)\delta_Q^{(k)}$ for any $t \in [0,1]$ and any $H, Q \in L_k$ and Corollary 2 of the same paper shows that if $H \succeq_{(k+1)} Q$ then

$$EH - \delta_H^{(m)} \geq EQ - \delta_Q^{(m)}, \quad EH \geq EQ, \quad (20)$$

for all $m \geq k$ as long as $H \in L_m$. The above relation (20) plays important role in our discussions in this paper.

The stochastic dominance property defined through the relation (18) is related to the cumulative distribution functions (CDF) of random variables. In fact, many past papers define stochastic dominance through CDF, see [12], [13], [35], [36], and the references there for example.

Let $H$ be a random variable and let $F_H(x) = P(H \leq x)$ be its CDF. We assume that $H$ has probability density function, while this is not necessary, for convenience of notations and denote it by $h(x)$. Denote $F_H^{(0)}(x) = h(x), F_H^{(1)}(x) = F_H(x)$, and define

$$F_H^{(k)}(x) = \int_{-\infty}^{x} F_H^{(k-1)}(s)ds, \quad (21)$$

for each $k \in \mathbb{N}$. In this note, we call $F_H^{(k)}(\cdot)$ the $k$'th order cumulative distribution function of $H$ ($k$'th order CDF for short) and we use the notation $k-CDF$ to denote $k$'th order cumulative distribution functions of random variables.

By using induction method, Proposition 1 of [36] shows that

$$F_H^{(k+1)}(x) = \frac{1}{k!} E[(x - H)^+]^k, \quad (22)$$

for any $k \in \mathbb{N}$ as long as $H \in L^k$. Similarly, for random variable $Q \in L^k$ with CDF given by $F_Q(x) = P(Q \leq x)$ and density function given by $q(x)$, we define $F_Q^{(k)}(x)$ as in (21) for each $k \in \mathbb{N}$. Due to (22), the stochastic dominance property defined in (18) is equal to the following condition

$$F_H^{(k)}(x) \leq F_Q^{(k)}(x), \forall x \in \mathbb{R}, \quad (23)$$

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for any \( k \in \mathbb{N} \).

The condition [23], which needs to be checked for all the real numbers \( x \), shows that stochastic dominance is an infinite dimensional problem and hence necessary and sufficient conditions for SD are difficult to obtain.

For some simple random variables necessary and sufficient conditions for stochastic dominance are well known. For example, if \( H \sim N(\mu_1, \sigma_1) \) and \( Q \sim N(\mu_2, \sigma_2) \), then \( H \succeq (2) Q \) if an only if \( \mu_1 \geq \mu_2 \) and \( \sigma_1 \leq \sigma_2 \), see Theorem 6.2 of [31] and also Theorem 3.1 of [14] for instance. In the next subsection we will show that in fact \( X \succeq (k+1) Y \) is equivalent to \( \mu_1 \geq \mu_2 \) and \( \sigma_1 \leq \sigma_2 \) for each \( k \in \mathbb{N} \). Our proof of this result gives a new approach for the proof of the results in Theorem 6.2 of [31] and also Theorem 3.1 of [14], but it also shows that \( H \succeq (k+1) Q \) implies \( \mu_1 \geq \mu_2 \) and \( \sigma_1 \leq \sigma_2 \) for each \( k \in \mathbb{N} \), see Proposition 2.9 below.

For general class of random variables, this type of necessary and sufficient conditions are difficult to construct. For the reminder of this section, we calculate \( k-CDF \) for certain type of random variables explicitly for any \( k \in \mathbb{N} \). These results are related to stochastic dominance property through [23].

Next we calculate the \( k \)th order CDFs of 1. Normal, 2. Elliptical, and 3. NMVM random variables for any \( k \in \mathbb{N} \). Below we start with calculating k-CDF for normal random variables.

1. When \( H \sim N(0, 1) \), we denote \( F_H^{(k)}(x) \) by \( \phi^{(k)}(x) \). From [22] we have

\[
\phi^{(k)}(x) = \frac{1}{(k-1)!} E[(x - N)^+ | k-1],
\]

where \( N \) is the standard normal random variable. We show the following Lemma

**Lemma 2.1.** For any \( k \geq 2 \), we have

\[
\phi^{(k)}(x) = \frac{1}{k-1} [x \phi^{(k-1)}(x) + \phi^{(k-2)}(x)],
\]

where \( \phi^{(0)}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2} \) and \( \phi^{(1)}(x) = \Phi(x) \).

**Proof.** We use induction. When \( k = 2 \), we have

\[
\phi^{(2)}(x) = \int_{-\infty}^{x} \phi^{(1)}(s)ds = s\phi^{(1)}(s)\bigg|_{-\infty}^{x} - \int_{-\infty}^{x} s\phi^{(0)}(s)ds = x\phi^{(1)}(x) + \phi^{(0)}(x),
\]

where we have used \( \lim_{s \to -\infty} [s\phi^{(1)}(s)] = \lim_{s \to -\infty} \phi^{(1)}(s)/s = 0 \) which follows from L’Hopital’s rule. Assume [25] is true for \( k \) and we show that it is also true for \( k + 1 \). To this end, we first integrate both sides of [25] and then apply [25]. We obtain

\[
\phi^{(k+1)}(x) = \int_{-\infty}^{x} \phi^{(k)}(s)ds = \frac{1}{k-1} \int_{-\infty}^{x} s\phi^{(k-1)}(s)ds + \frac{1}{k-1} \phi^{(k-1)}(x)
\]

\[
= \frac{1}{k-1} [x \phi^{(k)}(x)] - \frac{1}{k-1} \phi^{(k+1)}(x) + \frac{1}{k-1} \phi^{(k-1)}(x)
\]

\[
= \frac{1}{k-1} [x \phi^{(k)}(x)] - \frac{1}{k-1} \phi^{(k+1)}(x) + \frac{1}{k-1} \phi^{(k-1)}(x),
\]

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Lemma 2.3. For any $s$ shows that
\[ \phi^{(k)}(x) = \frac{1}{k} [x \phi^{(k-1)}(x) + \phi^{(k-2)}(x)], \]
and this completes the proof. \qed

Remark 2.2. The relation \[\phi^{(k)}(0) = \begin{cases} \frac{1}{s} & k = 2i, \\ \frac{1}{\pi} \cdot \frac{1}{\sqrt{2}} & k = 2i + 1. \end{cases} \] leads us to
\[\phi^{(k)}(0) = \frac{1}{s} \cdot \frac{1}{\pi} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{k}, \quad k = 2i, \quad k = 2i + 1. \] (28)

We observe that $\phi^{(k)}(0)$ is a decreasing sequence that goes to zero. From the relation \[\phi^{(k)}(x) \geq \frac{x}{s} \phi^{(k-1)}(x) \] and therefore when $x \geq k - 1$, we have $\phi^{(k)}(x) \geq \phi^{(k-1)}(x)$. This shows that $\phi^{(k)}(x)$ and $\phi^{(k-1)}(x)$ intersects at some point $x > 0$.

Next, by using \[\phi^{(3)}(x) = x \phi^{(2)}(x) + \phi^{(1)}(x), \] we can obtain the following expression for $\phi^{(k)}(x)$

Lemma 2.3. For any $k \geq 2$ we have
\[\phi^{(k)}(x) = p_{k-1}(x) \Phi(x) + q_{k-2}(x) \phi^{(0)}(x), \] (29)
where $p_{k-1}(x)$ is a $k - 1$th order polynomial that satisfies
\[p_j(x) = \frac{x}{j} p_{j-1}(x) + \frac{1}{j} p_{j-2}(x), \quad j \geq 2, \quad p_1(x) = x, \quad p_0(x) = 1, \] (30)
and $q_{k-2}(x)$ is a $k - 2$th order polynomial that satisfies
\[q_i(x) = \frac{x}{i+1} q_{i-1}(x) + \frac{1}{i+1} q_{i-2}(x), \quad i \geq 2, \quad q_1(x) = \frac{x}{2}, \quad q_0(x) = 1. \] (31)

Proof. When $k = 2$, from \[\phi^{(2)}(x) = x \Phi(x) + \phi^{(0)}(x), \] it is easy to see $\phi^{(2)}(x) = x \Phi(x) + \phi^{(0)}(x) = p_1(x) \Phi(x) + q_0(x) \phi^{(0)}(x)$. When $k = 3$, we can easily calculate $\phi^{(3)}(x) = \frac{x^2 + 1}{2} \phi^{(2)}(x) + \frac{x}{2} \phi^{(0)}(x) = p_2(x) \Phi(x) + q_1(x) \phi^{(0)}(x)$ again by using \[\phi^{(2)}(x) = x \Phi(x) + \phi^{(0)}(x). \] Now assume \[\phi^{(k)}(x) \] is true for all $2, 3, \cdots, k$ and we would like to prove it for $k + 1$. By \[\phi^{(k)}(x) \] we have
\[\phi^{(k+1)}(x) = \frac{x}{k} \phi^{(k)}(x) + \frac{1}{k} \phi^{(k-1)}(x)\]
\[= \frac{x}{k} [p_{k-1}(x) \Phi(x) + q_{k-2}(x) \phi^{(0)}(x)] + \frac{1}{k} [p_{k-2}(x) \Phi(x) + q_{k-3}(x) \phi^{(0)}(x)]\]
\[= \frac{x}{k} p_{k-1}(x) \Phi(x) + \frac{1}{k} p_{k-2}(x) \Phi(x) + \frac{x}{k} q_{k-2}(x) \phi^{(0)}(x) + \frac{1}{k} q_{k-3}(x) \phi^{(0)}(x)\]
\[= p_k(x) \Phi(x) + q_{k-1}(x) \phi^{(0)}(x), \] (32)
where $p_k(x) = \frac{x}{k} p_{k-1}(x) + \frac{1}{k} p_{k-2}(x)$ and $q_{k-1}(x) = \frac{x}{k} q_{k-2}(x) + \frac{1}{k} q_{k-3}(x)$. Clearly $p_k(x)$ is a $k'$th order polynomial and $q_{k-1}(x)$ is a $(k - 1)'$th order polynomial. \qed
Lemma 2.4. The function \( y(x) = \phi^{(k)}(x) \) satisfies
\[
\begin{align*}
y'' + xy' - (k-1)y &= 0, \\
y(0) &= \phi^{(k)}(0), \quad y'(0) = \phi^{(k-1)}(0),
\end{align*}
\tag{33}
\] and the polynomial solution \( y(x) = \sum_{j=0}^{+\infty} a_j x^j \) of (33) is given by
\[
a_{j+3} = \frac{(k-1) - (j+1)}{(j+2)(j+3)} a_{j+1}, \quad j = 0, 1, \ldots,
\]
\[
a_2 = \frac{k-1}{2} a_0, \quad a_0 = \phi^{(k)}(0), \quad a_1 = \phi^{(k-1)}(0).
\]

Proof. The equation (25) can be written as \((k-1)\phi^{(k)}(x) = x\phi^{(k-1)}(x) + \phi^{(k-2)}(x)\). With \( y =: \phi^{(k)}(x) \) observe that \( y' = \phi^{(k-1)}(x) \) and \( y'' = \phi^{(k-2)}(x) \). Therefore the equation in (33) holds. To find its polynomial solution we plug \( y(x) = \sum_{j=0}^{+\infty} a_j x^j \) into (33) and obtain a new polynomial that equals to zero. Then all the co-efficients of this new polynomial are zero. This gives us the expressions for \( a_j \).

Remark 2.5. Observe that \( a_{k+1} = 0 \) and hence \( a_{k+2j+1} = 0 \) for all \( j \geq 0 \). We have
\[
a_{k+2j} = (-1)^j \frac{3 \cdot 5 \cdots (2j-3) \cdot (2j-1)}{(k+2j)!} \frac{1}{\sqrt{2\pi}},
\]
and
\[
a_0 = \phi^{(k)}(0), a_1 = \phi^{(k-1)}(0), \cdots, a_j = \frac{1}{j!} \phi^{(k-j)}(0), \cdots, a_k = \frac{1}{k!} \phi^{(0)}(0) = \frac{1}{k!} \frac{1}{\sqrt{2\pi}}.
\]

In the case that \( H \sim N(\mu, \sigma^2) \), we denote \( F_H^{(k)}(x) \) by \( \varphi^{(k)}(x; \mu, \sigma^2) \). By using (24), we can easily obtain
\[
\varphi^{(k)}(x; \mu, \sigma^2) = \sigma^{k-1} \phi^{(k)} \left( \frac{x - \mu}{\sigma} \right).
\tag{34}
\]

By letting \( y(x) = \varphi^{(k)}(x; \mu, \sigma^2) \), one can easily show that it satisfies the following equation
\[
\sigma^2 y'' + (x - \mu) y' - (k-1)y = 0,
\]
with the initial conditions
\[
y(0) = \sigma^{k-1} \phi^{(k)} \left( \frac{-\mu}{\sigma} \right), \quad y'(0) = \sigma^{k-2} \phi^{(k-1)} \left( \frac{-\mu}{\sigma} \right).
\]

2. When \( H \sim \mu + \sigma \sqrt{Z} N(0,1) \), where \( Z \in L^{k-1} \) is a positive random variable independent from \( N(0,1) \), we denote \( F^{(k)}(z) \) by \( \varphi_c(x; \mu, \sigma) \). From (24), we have
\[
\varphi_c^{(k)}(x; \mu, \sigma) = \sigma^{k-1} \int_0^{+\infty} \frac{z^{k-1}}{2^k} \phi^{(k)} \left( \frac{x - \mu}{\sigma \sqrt{z}} \right) f_Z(z) dz.
\tag{35}
\]
Proposition 2.6. For each $k \geq 2$, when $Z \in L^{k-1}$ we have

\[
\frac{d\varphi_k^{(k)}(x; \mu, \sigma)}{d\mu} = -\sigma^{k-2} \int_0^{+\infty} z^{\frac{k-2}{2}} \phi^{(k-1)}(\frac{x - \mu}{\sigma\sqrt{z}}) f_Z(z) dz < 0,
\]

\[
\frac{d\varphi_k^{(k)}(x; \mu, \sigma)}{d\sigma} = \sigma^{k-2} \int_0^{+\infty} z^{\frac{k-2}{2}} \phi^{(k-2)}(\frac{x - \mu}{\sigma\sqrt{z}}) f_Z(z) dz > 0,
\]

and therefore if $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$, then $H = \mu_1 + \sigma_1 Z N$ k’th order stochastically dominates $Q = \mu_2 + \sigma_2 Z N$ for each $k \geq 2$.

Proof. The first part of (36) is direct. To see the second part note that

\[
\frac{d\varphi_k^{(k)}(x; \mu, \sigma)}{d\sigma} = (k-1)\sigma^{k-2} \int_0^{+\infty} z^{\frac{k-1}{2}} \phi^{(k)}(\frac{x - \mu}{\sigma\sqrt{z}}) f_Z(z) dz + \sigma^{k-1} \int_0^{+\infty} z^{\frac{k-1}{2}} (-\frac{x - \mu}{\sigma^2\sqrt{z}}) \phi^{(k-1)}(\frac{x - \mu}{\sigma\sqrt{z}}) f_Z(z) dz
\]

\[
= \sigma^{k-2} \int_0^{+\infty} z^{\frac{k-1}{2}} [(k-1)\phi^{(k)}(\frac{x - \mu}{\sigma\sqrt{z}}) - (\frac{x - \mu}{\sigma\sqrt{z}}) \phi^{(k-1)}(\frac{x - \mu}{\sigma\sqrt{z}})] f_Z(z) dz.
\]

From (25) we have

\[
(k-1)\phi^{(k)}(\frac{x - \mu}{\sigma\sqrt{z}}) - (\frac{x - \mu}{\sigma\sqrt{z}}) \phi^{(k-1)}(\frac{x - \mu}{\sigma\sqrt{z}}) = \phi^{(k-2)}(\frac{x - \mu}{\sigma\sqrt{z}}).
\]

Then we plug (38) into (37). This gives us the second equation in the Proposition. \(\square\)

Next we show the following result

Proposition 2.7. Let $H \sim \mu_1 + \sigma_1 Z N(0,1)$ and $Q \sim \mu_2 + \sigma_2 Z N(0,1)$ be two elliptical random variables with $\mu_1, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0$, and $Z$ is any positive random variable with $EZ^k < \infty$ for all positive integers $k$. Then for each $k \geq 2$, $X \succeq (k) Y$ if and only if $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$.

Proof. The relation $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$ implies $X \succeq (k) Y$ for each $k \geq 2$ follows from Proposition 2.6 above. To see the other direction, note that $EH = \mu_1$ and $EQ = \mu_2$ and so the relation $\mu_1 \geq \mu_2$ follows from Theorem 1 of [36]. To see the other relation $\sigma_1 \leq \sigma_2$, note that the central semi-deviations of $H$ and $Q$ are $\delta_H^{(j)} = \sigma_1 \|(ZN)^+\|_j$ and $\delta_Q^{(j)} = \sigma_2 \|(ZN)^+\|_j$. Corollary 2 of [36] implies

\[
\mu_1 - \sigma_1\|(ZN)^+\|_j \geq \mu_2 - \sigma_2\|(ZN)^+\|_j, \forall j \geq k.
\]

Now, since $(ZN)^+$ is an unbounded random variable we have $\lim_{j \to \infty} \|(ZN)^+\|_j = \infty$. Dividing both sides of (39) by $\|(ZN)^+\|_j$ and letting $j \to \infty$ we obtain $-\sigma_1 \geq -\sigma_2$. This completes the proof. \(\square\)

3. When $H \sim \mu + \gamma Z + \sigma\sqrt{Z} N(0,1)$, we denote $F^{(k)}(x)$ by $\varphi^{(k)}_m(x; \mu, \gamma, \sigma)$. From (24), we have

\[
\varphi^{(k)}_m(x; \mu, \gamma, \sigma) = \sigma^{k-1} \int_0^{+\infty} z^{\frac{k-1}{2}} \phi^{(k)}(\frac{x - \mu - \gamma z}{\sigma\sqrt{z}}) f_Z(z) dz
\]

By direct calculation and by using (25), we obtain the following result
Proposition 2.8. For each positive integer \(k \geq 2\), when \(Z \in L^{k-1}\) we have

\[
\frac{d\varphi_m^{(k)}(x; \mu, \gamma, \sigma)}{d\mu} = -\sigma^{-k-2} \int_0^{+\infty} z^{k-2} \phi^{(k-1)}(\frac{x - \mu - \gamma z}{\sqrt{\sigma}z}) f_Z(z) dz < 0,
\]

\[
\frac{d\varphi_m^{(k)}(x; \mu, \gamma, \sigma)}{d\gamma} = -\sigma^{-k-2} \int_0^{+\infty} z^{k-2} \phi^{(k-1)}(\frac{\eta - \mu - \gamma z}{\sqrt{\sigma}z}) f_Z(z) dz < 0,
\]

\[
\frac{d\varphi_m^{(k)}(x; \mu, \gamma, \sigma)}{d\sigma} = \sigma^{-k-2} \int_0^{+\infty} z^{k-2} \phi^{(k-1)}(\frac{\eta - \mu - \gamma z}{\sqrt{\sigma}z}) f_Z(z) dz > 0,
\]

and therefore for any \(H \sim \mu_1 + \gamma_1 Z + \sigma_1 \sqrt{Z} N(0,1)\) and \(Q \sim \mu_2 + \gamma_2 Z + \sigma_2 \sqrt{Z} N(0,1)\) we have \(H \succeq (k) Q\) whenever \(\mu_1 \geq \mu_2, \gamma_1 \geq \gamma_2, \sigma_1 \leq \sigma_2\).

Proof. The first two equations of (41) follow from taking the corresponding derivatives of (40). The third equation is obtained by taking the derivative of (40) with respect to \(\sigma\) and by using the relation (25) similar to the proof of Proposition 2.6.

The above Proposition gives sufficient conditions for SD for NMVM models with general mixing distribution \(Z\) as long as it is integrable. When, the mixing distribution \(Z\) is bounded, i.e., \(0 \leq a \leq Z \leq b\) for two real numbers \(a, b\), we can obtain a weaker sufficient conditions for SD.

Proposition 2.9. Suppose \(H \sim \mu_1 + \gamma_1 Z + \sigma_1 \sqrt{Z} N(0,1)\), \(Q \sim \mu_2 + \gamma_2 Z + \sigma_2 \sqrt{Z} N(0,1)\), and \(Z \in [a, b]\) for two real numbers \(0 \leq a \leq b\). If \(\mu_1 + z \gamma_1 \geq \mu_2 + z \gamma_2\) for all \(z \in [a, b]\) and \(\sigma_1 \leq \sigma_2\), then \(H \succeq (k) Q\) for any \(k \geq 2\).

Proof. For each \(z \in [a, b]\), since \(\mu_1 + z \gamma_1 \geq \mu_2 + z \gamma_2\) and \(\sigma_1 \sqrt{z} \leq \sigma_2 \sqrt{z}\), from Proposition 2.6 we have

\[
(\sigma_1 \sqrt{z})^{k-1} \phi^{(k)}(\frac{x - \mu_1 - \gamma_1 z}{\sigma_1 \sqrt{z}}) \leq (\sigma_2 \sqrt{z})^{k-1} \phi^{(k)}(\frac{x - \mu_2 - \gamma_2 z}{\sigma_2 \sqrt{z}}),
\]

for all \(z \in [a, b]\). Then from (40) we have \(\varphi_m^{(k)}(x; \mu_1, \gamma_1, \sigma_1) \leq \varphi_m^{(k)}(x; \mu_2, \gamma_2, \sigma_2)\). Therefore \(H \succeq (k) Q\) for any \(k \geq 2\).

Remark 2.10. The sufficient condition for SD in Proposition 2.9 is weaker than the sufficient condition in Proposition 2.9 above. To see this, note that the numbers \(\mu_1 = 1, \gamma_1 = 10, \sigma_1 = 1\) and \(\mu_2 = 2, \gamma_2 = 1, \sigma_2 = 1\) with \([a, b] = [10, 100]\) satisfy the necessary condition of Proposition 2.9 above but not the necessary condition of Proposition 2.8.

Next we discuss some necessary conditions for SD within the class of one dimensional NMVM models. For this purpose, we need to carry on further discussions on the SD characterizations of the one-dimensional NMVM models. First we introduce some notations. Let \(m(\cdot)\) denote any Borel function from \((0, +\infty)\) to the real line \(\mathbb{R}\). Denote \(Z_m =: m(Z)\). For any real numbers \(a_1, b_1, a_2, b_2\), define \(\bar{X}_m = a_1 + b_1 Z_m + \sqrt{Z} N\) and \(\bar{Y}_m = a_2 + b_2 Z_m + \sqrt{Z} N\). We first prove the following Lemma.

Lemma 2.11. Consider the model (4) and assume that \(Z \in L^p\) for all positive integers \(p\). Then for any bounded Borel function \(m\), we have

\[
\lim_{k \to +\infty} \frac{||\bar{X}_m^+||_k}{||\bar{Y}_m^+||_k} = 1.
\]

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Proof. Observe that $\bar{X}_m - \bar{Y}_m = a_1 - a_2 + (b_1 - b_2)Z_m$ are bounded random variables. By using $(a + b)^+ \leq a^+ + b^+$ for any real numbers and the triangle inequality for norms, we have

$$\| (\bar{X}_m)^+ \|_k = \| (\bar{X}_m - \bar{Y}_m)^+ \|_k \leq \| (\bar{X}_m - \bar{Y}_m)^+ \|_k + \| (Y_m)^+ \|_k. \quad (43)$$

From this it follows that

$$\| (\bar{X}_m)^+ \|_k \leq 1 + \| (\bar{X}_m - \bar{Y}_m)^+ \|_k. \quad (44)$$

Since $(\bar{X}_m - \bar{Y}_m)^+$ are bounded random variables, we have $\sup_{k \geq 1} \| (\bar{X}_m - \bar{Y}_m)^+ \|_k < \infty$ and since $(Y_m)^+$ are unbounded random variables we have $\lim_{k \to \infty} \| Y_m \|_k \to \infty$. Therefore from (44) we conclude that

$$\lim_{k \to \infty} \| (\bar{X}_m)^+ \|_k \leq 1. \quad (45)$$

In a similar way, we have

$$\| (\bar{Y}_m)^+ \|_k = \| (\bar{Y}_m - \bar{X}_m + \bar{X}_m)^+ \|_k \leq \| (\bar{Y}_m - \bar{X}_m)^+ \|_k + \| (\bar{X}_m)^+ \|_k. \quad (46)$$

From this we obtain

$$\| (\bar{X}_m)^+ \|_k \geq \frac{\| (\bar{X}_m - \bar{Y}_m)^+ \|_k}{\| (Y_m)^+ \|_k} = \frac{1}{\| (X_m - Y_m)^+ \|_k/\| (X_m)^+ \|_k + 1}. \quad (47)$$

Since $(\bar{X}_m)^+$ are unbounded random variables we have $\lim_{k \to \infty} \| (\bar{X}_m)^+ \|_k = \infty$ and therefore $\| (X_m - Y_m)^+ \|_k/\| (X_m)^+ \|_k \to 0$ as $k \to \infty$. Therefore from (47) we conclude that

$$\lim_{k \to \infty} \| (\bar{X}_m)^+ \|_k \geq 1. \quad (48)$$

Now, from (45) and (48) we obtain (42). \hfill \Box

Next, for any bounded Borel function $m$ and any real numbers $a_1, a_2, b_1, b_2$, and $c_1 > 0, c_2 > 0$, let $X_m = a_1 + b_1Z_m + c_1\sqrt{Z}N$ and $Y_m = a_2 + b_2Z_m + c_2\sqrt{Z}N$.

Lemma 2.12. Consider the model (1) and assume that $Z \in L^p$ for all positive integers $p$. Then for any bounded Borel function $m$ and for each $k \in \mathbb{N}$, the relation $\bar{X}_m \geq_{(k+1)} \bar{Y}_m$ implies $a_1 + b_1EZ_m \geq a_2 + b_2EZ_m$ and $c_1 \leq c_2$.

Proof. Since $EX_m = a_1 + b_1EZ_m$ and $E\bar{Y}_m = a_2 + b_2EZ_m$, the relation $a_1 + b_1EZ_m \geq a_2 + b_2EZ_m$ follows from Theorem 1 of [36]. To show $c_1 \leq c_2$, we use Corollary 2 of the same paper [36]. First observe that for any integer $j \geq k$ the central semi-deviations of order $j$ are equal to

$$\delta_{X_m}^{(j)} = c_1\| (\frac{b_1}{c_1}EZ_m - \frac{b_1}{c_1}Z_m + ZN)^+ \|_j, \quad \delta_{Y_m}^{(j)} = c_2\| (\frac{b_2}{c_2}EZ_m - \frac{b_2}{c_2}Z_m + ZN)^+ \|_j. \quad (49)$$

Denote $D_j =: \| (\frac{b_1}{c_1}EZ_m - \frac{b_1}{c_1}Z_m + ZN)^+ \|_j$ and $E_j =: \| (\frac{b_2}{c_2}EZ_m - \frac{b_2}{c_2}Z_m + ZN)^+ \|_j$ and observe that Lemma 2.11 implies $\lim_{j \to \infty} D_j/E_j = 1$. Also since $(\frac{b_1}{c_1}EZ_m - \frac{b_1}{c_1}Z_m + ZN)^+$ and $(\frac{b_2}{c_2}EZ_m - \frac{b_2}{c_2}Z_m + ZN)^+$ are unbounded random variables we have $\lim_{j \to \infty} D_j = +\infty$ and $\lim_{j \to \infty} E_j = +\infty$. Corollary 2 of [36] implies

$$a_1 + b_1EZ_m - c_1D_j \geq a_2 + b_2EZ_m - c_2E_j, \quad (50)$$

for any $j \geq k$. Now dividing both sides of (50) by $E_j$ and letting $j \to \infty$ we obtain $-c_1 \geq -c_2$. This shows that $c_1 \leq c_2$. \hfill \Box

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Now for any real numbers \( b_1, b_2 \), and any \( c_1 > 0, c_2 > 0 \), let \( \hat{X}_m = b_1 Z_m + c_1 \sqrt{Z} N \) and \( \hat{Y}_m = b_2 Z_m + c_2 \sqrt{Z} N \). We have the following Corollary.

**Corollary 2.13.** For any positive valued and bounded Borel function \( m \) and for each \( k \in \mathbb{N} \), we have

\[
\hat{X}_m \succeq_{(k+1)} \hat{Y}_m \iff b_1 \geq b_2, \ c_1 \leq c_2.
\]  

**Proof.** If \( \hat{X}_m \succeq_{(k+1)} \hat{Y}_m \), then from Lemma 2.12 we have \( b_1 E Z_m \geq b_2 E Z_m \) and \( c_1 \leq c_2 \). This is equivalent to \( b_1 \geq b_2 \) and \( c_1 \leq c_2 \) as \( E Z_m > 0 \). To show the other direction, observe that for any \( Y_m =: \gamma Z_m + \sigma \sqrt{Z} N \) with \( \gamma \in \mathbb{R}, \ \sigma > 0 \), we have

\[
\varphi_{Y_m}^{(k)}(x; \gamma, \sigma) = \frac{\sigma^{k-1}}{\sigma} \int_{0}^{+\infty} z^{k-1} \phi(k-1) \left( \frac{x - \gamma Z_m}{\sigma \sqrt{z}} \right) f(z) dz.
\]  

Now following the same idea as in the proof of Proposition 2.8, we can show that

\[
\frac{d\varphi_{Y_m}^{(k)}(x; \gamma, \sigma)}{d\gamma} = -\alpha^{k-2} \int_{0}^{+\infty} z^{k-2} z_m \phi(k-1) \left( \frac{x - \gamma Z_m}{\sigma \sqrt{z}} \right) f(z) dz < 0,
\]

\[
\frac{d\varphi_{Y_m}^{(k)}(x; \gamma, \sigma)}{d\sigma} = \alpha^{k-2} \int_{0}^{+\infty} z^{k-2} \phi(k-1) \left( \frac{x - \gamma Z_m}{\sigma \sqrt{z}} \right) f(z) dz > 0.
\]

This shows that \( b_1 \geq b_2 \) and \( c_1 \leq c_2 \) implies \( \hat{X}_m \succeq_{(k+1)} \hat{Y}_m \). \qed

**Remark 2.14.** Recall that, from our Proposition 2.7, for Normal random variables \( a, b \) \( N(b_1, c_1) \) and \( N(b_2, c_2) \) we have \( N(b_1, c_1) \succeq_{(k+1)} N(b_2, c_2) \) if and only if \( b_1 \geq b_2 \) and \( c_1 \leq c_2 \). The above Corollary 2.13 shows that similar property holds for the class of random variables \( b Z_m + c \sqrt{Z} N : a, b \in \mathbb{R} \) for arbitrarily fixed mixing distribution \( Z \) and for any positively valued bounded Borel function \( m \). Namely, we have

\[
b_1 Z_m + c_1 \sqrt{Z} N \succeq_{(k)} b_2 Z_m + c_2 \sqrt{Z} N \iff b_1 \geq b_2, \ c_1 \leq c_2.
\]  

It is not clear if similar type of characterizations as in (54) also holds for the class of random variables \( b Z + c \sqrt{Z} N : a, b \in \mathbb{R} \) for arbitrarily fixed mixing random variable \( Z \).

## 3 Closed form solutions for frontier portfolios

In the previous section we discussed some sufficient as well as some necessary conditions for SD within one dimensional NMVM models. In this section we apply these results for the study of frontier portfolios, i.e., solutions of the problem (2).

First we discuss some conditions on the risk measure \( \rho \) for the problem (2). Recall that a risk measure \( \rho \) on \( L^p, p \in [1, +\infty) \), is a map from \( L^p \) to \((-\infty, +\infty]\) with \( \rho(c) < +\infty \) for all \( c \in R \). A risk measure \( \rho \) on \( L^p \) satisfies \( L^p \)-Fatou property if

\[
\liminf_{n \to \infty} \rho(X_n) \geq \rho(X),
\]

for any \( \{X_n\}_{n \geq 1} \in L^p \) with \( X_n \to X \) in \( L^p \) norm. A risk measure \( \rho \) on \( L^p, p \in [1, +\infty) \), is consistent with the second order stochastic dominance (SSD-consistent from now on) if \( H \succeq_{(2)} Q \)
implies $\rho(H) \geq \rho(Q)$ for any $H, Q \in L^p$. The paper [34] calls a risk measure consistent if it is translation invariant and satisfies the SSD-consistency. Proposition B.1. of [34] shows that any law-invariant convex risk measure on $L^p, p \in [1, +\infty)$, with the Fatou property is a consistent risk measure. As stated in the paragraph preceding Proposition B.1. of [34], any real-valued law-invariant convex risk measure always satisfies the Fatou property and they are $L^p-$continuous, see [43] for further details. For the discussions of this section, we need the $L^p-$continuity, SSD-consistency, and law-invariance property of the risk measures. Therefore, in this section we assume that our risk measures $\rho$ are real-valued, law invariant, and convex.

Remark 3.1. We remark here that the solution for the problem (2) exists when the risk measure $\rho$ is finite-valued, law-invariant, and convex risk measure. To see this, note that the map $\omega \rightarrow \rho(-\omega^T \xi)$ is a continuous function and $\rho(-\omega^T \xi)$ is bounded from below as explained in the Appendix B of [22].

The above Remark 3.1 shows that the problem (2) always have solution if the risk measure $\rho$ is finite-valued law-invariant convex risk measure. Our goal in this section is to give closed form solution for the problem (2) under the stated conditions above on the risk measure $\rho$. To this end, we first recall the Markovitz mean-variance optimization problem

$$\min Var(-\omega^T \xi),$$

$$s.t. E(-\omega^T \xi) = r,$$

$$\omega^T e = 1.$$  

(55)

For return vectors $X$ given by (1), we have the associated random vectors

$$W = \mu_0 + \gamma_0 Z + \sqrt{Z} N_n,$$

(56)

where $\mu_0$ and $\gamma_0$ are given as in (13), such that $\omega^T X = x^T W = x^T \mu_0 + x^T \gamma_0 Z + ||x|| \sqrt{Z} N(0, 1)$ with $x^T = T(\omega) = \omega^T A$. Therefore, we have the following relations

$$E(-\omega^T X) = E(-x^T W) = -x^T \mu_0 - x^T \gamma_0 EZ,$$

$$Var(-\omega^T X) = Var(x^T W) = (x^T \gamma_0)^2 Var(Z) + ||x||^2 EZ.$$  

(57)

Observe that the relation $\omega^T e = 1$ can be expressed as $\omega^T AA^{-1} e = x^T A^{-1} e = x^T e_A = 1$, where we denoted $e_A = A^{-1} e$. Therefore in the x-coordinate system, the above optimization problem (55) can be written as

$$\min \left( (x^T \gamma_0)^2 Var(Z) + ||x||^2 EZ \right),$$

$$s.t. -x^T (\mu_0 - \gamma_0 EZ) = r,$$

$$x^T e_A = 1.$$  

(58)

The optimization problem (55) is a quadratic optimization problem and its closed form solution can be obtained easily by Lagrangian method. As mentioned earlier, its solution is given by (10).

In this section, we will show that the optimal portfolio in our mean-risk portfolio optimization problem (2) also takes a similar form as in (10). Therefore, we first introduce the following
notations. For any random vector \( \theta \) with mean vector \( \mu_\theta = E\theta \) and co-variance matrix \( \Sigma_\theta = \text{Cov}(\theta) \) we introduce the following expressions

\[
\omega_\theta^* = \omega_\theta^*(\mu_\theta, \Sigma_\theta) = \frac{1}{d_\theta} [d_\theta^2 (\Sigma_\theta^{-1} e) - d_\theta^1 (\Sigma_\theta^{-1} \mu_\theta)] + \frac{r}{d_\theta} [d_\theta^3 (\Sigma_\theta^{-1} \mu_\theta) - d_\theta^1 (\Sigma_\theta^{-1} e)],
\]

(59)

where

\[
d_1 = e^T \Sigma_\theta^{-1} \mu_\theta, \quad d_2 = \mu_\theta^T \Sigma_\theta^{-1} \mu_\theta, \quad d_3 = e^T \Sigma_\theta^{-1} e, \quad d_4 = d_2 d_3 - (d_1)^2.
\]

(60)

**Remark 3.2.** The right-hand-side of (59) only involves the mean vector \( \mu_\theta \) and the co-variance matrix \( \Sigma_\theta \) of the random vector \( \theta \). Therefore \( \omega_\theta^* = \omega_\eta^* \) as long as \( \theta \) and \( \eta \) have the same mean vectors and co-variance matrices. Here we used the random variable \( \theta \) in the definition of \( \omega_\theta^* \) to imply that the portfolio \( \omega_\theta^* \) is the frontier portfolio under the mean-variance criteria with return vector \( \theta \).

Note that the above expression (59) is well defined as long as \( \mu_\theta \neq 0 \) and \( \Sigma_\theta \) is positive definite (all the eigenvalues are strictly positive numbers) as explained in Remark 1.1 above.

When the risk measure variance is replaced by a more general risk measure \( \rho \) in (55), the solution of the problem (55) is not known in closed form to the best of our knowledge. Our Theorem 3.3 below shows that, frontier portfolios, under any real-valued, law invariant, and convex risk measure, exist and they are in the form as in (59) when the return vectors are given by (1).

**Theorem 3.3.** Let \( \rho \) be any finite-valued law-invariant convex risk measure on \( L^p, p \geq 1 \). Let the return vector be given by (1) with \( Z \in L^p \). Then the solution of the following problem

\[
\min_{\omega} \rho(-\omega^T X),
\]

s.t. \( E(-\omega^T X) = r, \omega^T e = 1 \),

(61)

is given by \( \omega_\theta^* \) as in (59) for any random vector \( \theta \) with mean \( \mu_\theta = \mu + \gamma E Z \) and co-variance matrix \( \Sigma_\theta = \Sigma \).

**Remark 3.4.** Observe that \( \mu_X = EX = \mu + \gamma E Z = \mu_\theta \). However the covariance matrix \( \text{Cov}(X) = \gamma \gamma^T \text{Var}(Z) + A^T A \gamma E Z \) is different from \( \Sigma_\theta = A^T A \). Therefore, the optimal solution of (59) is different from the optimal solution of (61). But these two solutions are similar in the sense that they differ only in the co-variance matrices \( \Sigma_X \) and \( \Sigma_\theta \).

**Remark 3.5.** The message of the Theorem 3.3 above is that the mean-risk frontier portfolios under any real-valued law invariant convex risk measure \( \rho \) for return vectors \( X \) as in (1), can be obtained by solving a Markowitz mean-variance optimal portfolio problem with an appropriately adjusted return vector \( \theta \) as in the Theorem 3.3 above.

The risk measure \( CVaR_\alpha \) for each fixed \( \alpha \in (0, 1) \) is a finite valued law-invariant convex risk measure. Therefore the result of the above Theorem 3.3 holds for this risk measure also. We state this in the following Corollary.
Corollary 3.6. Let $X$ be a return vector given by (7) with $Z \in L^p$ for some positive integer $p$. Then for each fixed $\alpha \in (0, 1)$, closed form solutions of the following optimization problem

$$
\begin{align*}
\min_{\omega} & \quad CVaR_\alpha(-\omega^T X), \\
\text{s.t.} & \quad E(-\omega^T X) = r, \\
& \quad \omega^T e = 1,
\end{align*}
$$

(62)

are given by $\omega^*_\theta$ as in (59) for any random vector $\theta$ with $\mu_\theta = EY = \mu + \gamma E\Sigma$ and $\Sigma_\theta = A^T A$.

Remark 3.7. The Corollary 3.6 above shows, in particular, that when the return vectors are given by (7), there exist portfolios that are frontier to the problem (62) for all the significance level $\alpha \in (0, 1)$ simultaneously.

Remark 3.8. We remark that our Theorem 3.3 above can be applied to simplify the calculations of optimal values of $CVaR_\alpha$ for portfolios of Normal returns, a topic which was discussed in Theorem 2 of [41] (see also [42]). To this end, first observe that if $Z = 1$ in (7), we have $X \sim N_n(\mu + \gamma, \Sigma)$. Namely our model (7) becomes a Normal random vector. For rotational simplicity, in our discussions below we denote this Normal random vector by $N(\gamma, \Sigma)$ instead of $N_n(\mu + \gamma, \Sigma)$. Observe that with all the other model parameters $\gamma$ and $\Sigma$ fixed, the expression (59) becomes a function of the expected return level $r$. Therefore for convenience we denote $\omega_r := \omega^*_\theta$ for $\theta \triangleq N_n(\gamma, \Sigma)$ in expression (59). We write this $\omega_r$ in a different form as

$$
\omega_r = k_1(r)\Sigma^{-1}e + k_2(r)\Sigma^{-1}\gamma,
$$

(63)

where

$$
k_1(r) = d_0^2/d_0^4 - rd_0^3/d_0^4, \quad k_2(r) = rd_0^3/d_0^4 - d_0^2/d_0^4.
$$

(64)

With these notations we have $\omega_r^* n \sim N(\omega_r^* \gamma, \sigma^2)$, where

$$
\sigma^2 := \omega_r^T \Sigma \omega_r = k_1^2(r)e^T \Sigma^{-1}e + 2k_1^1(r)k_2(r)e^T \Sigma^{-1}\gamma + k_2^2(r)\gamma^T \Sigma^{-1}\gamma.
$$

(65)

From [37] (see also equation (2) of [30]), for a Normal random variable $H = N(\delta, \sigma^2)$ we have

$$
CVaR_\alpha(H) = \delta + \frac{1}{\sigma} \varphi\left(\frac{z_\alpha - \delta}{\sigma}\right)/(1 - \Phi\left(\frac{z_\alpha - \delta}{\sigma}\right)) \sigma^2,
$$

(66)

where $z_\alpha$ is the $\alpha$-quantile of the standard Normal random variable $N(0, 1)$. For any return level $r$ define

$$
h_\alpha(r) := r + \frac{1}{\sigma_r} \varphi\left(\frac{z_\alpha - r}{\sigma_r}\right)/(1 - \Phi\left(\frac{z_\alpha - r}{\sigma_r}\right)) \sigma_r^2,
$$

(67)

where $\sigma_r$ is given by (63). Now, for any domain $D$ of portfolios define $D_\gamma := \{\omega^T \gamma : \omega \in D\}$. Assume $D$ satisfies the property that for any $r' \in D_\gamma$ and any portfolio $\omega$ with $\omega^T \gamma = r'$ we have $\omega \in D$. Then we clearly have

$$
\min_{\omega \in D} CVaR_\alpha(\omega^T \theta) = \min_{r \in D_\gamma} h_\alpha(r),
$$

(68)

where $\theta \triangleq N_n(\gamma, \Sigma)$. For example $D$ can be $D = \{\omega : \omega^T \gamma \geq r'\}$ for some given return level $r'$. Note that the right-hand-side of (68) is a minimization problem of a real valued function on a subset of the real line which is a simpler problem than finding the minimum value of $F_\alpha(\omega, a)$ in (4) of [47] as it is done in Theorem 2 of the same paper.
The proof of the Theorem 3.3 above needs some preparation. We first prove the following Lemma. First, observe that from (40) for any $H = a + bZ + c\sqrt{Z}N$ we have

$$E(H - k)^+ = \int_0^{+\infty} c\sqrt{z}\phi^{(2)}(\frac{k - a - bz}{c\sqrt{z}})f(z)dz.$$  

(69)

Define

$$\Psi = \Psi_H(u, c) =: \int_0^{+\infty} c\sqrt{z}\phi^{(2)}(\frac{u}{c\sqrt{z}})f(z)dz.$$  

(70)

It is easy to check that

$$\frac{d\Psi}{du} = \int_0^{+\infty} \phi^{(1)}(\frac{u}{c\sqrt{z}})f(z)dz > 0, \quad \frac{d\Psi}{dc} = \int_0^{+\infty} \sqrt{z}\phi^{(0)}(\frac{u}{c\sqrt{z}})f(z)dz > 0.$$  

(71)

Therefore $\Psi_H(u, c)$ is an increasing function of $u$ and also of $c$. We use these facts in the proof of the following Lemma.

**Lemma 3.10.** Let $Z$ be any random variable that satisfies $\beta \geq Z \geq \alpha > 0$ for some finite numbers $\alpha$ and $\beta$. Denote $\eta = EZ$ and consider the NMVM models $Q = a_1 + b_1Z + c_1\sqrt{Z}N$ and $R = a_2 + b_2Z + c_2\sqrt{Z}N$, where $a_1, a_2, b_1, b_2$ are any real numbers and $c_1 > 0$, $c_2 > 0$. Then we have $Q \succeq (2) R$ if and only if $a_1 + b_1EZ \geq a_2 + b_2EZ$ and $c_1 \leq c_2$. 

**Proof.** If $Q \succeq (2) R$ then $a_1 + b_1EZ \geq a_2 + b_2EZ$ follows from Theorem 1 of [36] and $c_1 \leq c_2$ follows from Lemma 2.12 above. To show the other direction we need to show $I_1 = E(Q - k)^+ \leq I_2 =: E(R - k)^+$ for any real number $k$. From (69) we have

$$I_1 = \int_{\eta}^{\beta} c_1\sqrt{z}\phi^{(2)}(\frac{k - a - b_1z}{c_1\sqrt{z}})f(z)dz + \int_{\alpha}^{\eta} c_1\sqrt{z}\phi^{(2)}(\frac{k - a_1 - b_1z}{c_1\sqrt{z}})f(z)dz,$$

$$I_2 = \int_{\eta}^{\beta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a - b_2z}{c_2\sqrt{z}})f(z)dz + \int_{\alpha}^{\eta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a_2 - b_2z}{c_2\sqrt{z}})f(z)dz.$$  

(72)

We have

$$I_2 - I_1 = \int_{\eta}^{\beta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a - b_2z}{c_2\sqrt{z}}) - \phi^{(2)}(\frac{k - a_2 - b_2z}{c_2\sqrt{z}})f(z)dz$$

$$+ \int_{\alpha}^{\eta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a - b_2z}{c_2\sqrt{z}}) - c_1\sqrt{z}\phi^{(2)}(\frac{k - a_1 - b_1z}{c_1\sqrt{z}})f(z)dz$$

$$+ \int_{\eta}^{\beta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a_1 - b_1z}{c_2\sqrt{z}}) - \phi^{(2)}(\frac{k - a_1 - b_1z}{c_2\sqrt{z}})f(z)dz$$

$$+ \int_{\alpha}^{\eta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a_1 - b_1z}{c_2\sqrt{z}}) - c_1\sqrt{z}\phi^{(2)}(\frac{k - a_1 - b_1z}{c_1\sqrt{z}})f(z)dz$$

$$\geq \int_{\eta}^{\beta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a - b_2z}{c_2\sqrt{z}}) - \phi^{(2)}(\frac{k - a_2 - b_2z}{c_2\sqrt{z}})f(z)dz$$

$$+ \int_{\alpha}^{\beta} c_2\sqrt{z}\phi^{(2)}(\frac{k - a_1 - b_1z}{c_2\sqrt{z}}) - \phi^{(2)}(\frac{k - a_1 - b_1z}{c_2\sqrt{z}})f(z)dz.$$  

(73)
The above inequality follows because the two terms
\[
\int_\eta^\beta c_2 \sqrt{z} \phi^{(2)} \left( \frac{k - a_1 - b_1 z}{c_2 \sqrt{z}} \right) - c_1 \sqrt{z} \phi^{(2)} \left( \frac{k - a_1 - b_1 z}{c_1 \sqrt{z}} \right) |f(z)| dz
\]
and
\[
\int_\alpha^\eta c_2 \sqrt{z} \phi^{(2)} \left( \frac{k - a_1 - b_1 z}{c_2 \sqrt{z}} \right) - c_1 \sqrt{z} \phi^{(2)} \left( \frac{k - a_1 - b_1 z}{c_1 \sqrt{z}} \right) |f(z)| dz
\]
are positive numbers as the function \( \Psi(\mu, c) \) is an increasing function in the argument \( c \) as explained in the paragraph preceding to this Lemma. We need to show that the term
\[
I = \int_\eta^\beta c_2 \sqrt{z} \phi^{(2)} \left( \frac{k - a_2 - b_2 z}{c_2 \sqrt{z}} \right) - \phi^{(2)} \left( \frac{k - a_1 - b_1 z}{c_2 \sqrt{z}} \right) |f(z)| dz + \int_\alpha^\eta c_2 \sqrt{z} \phi^{(2)} \left( \frac{k - a_2 - b_2 z}{c_2 \sqrt{z}} \right) - \phi^{(2)} \left( \frac{k - a_1 - b_1 z}{c_2 \sqrt{z}} \right) |f(z)| dz. (74)
\]
is a positive number also. By middle value Theorem we have
\[
I = \int_\eta^\beta \phi^{(1)}(\ell_1(z))[(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz + \int_\alpha^\eta \phi^{(1)}(\ell_2(z))[(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz, (75)
\]
where \( \ell_1(z) \) is between \( \frac{k - a_2 - b_2 z}{c_2 \sqrt{z}} \) and \( \frac{k - a_1 - b_1 z}{c_2 \sqrt{z}} \) when \( z \in [\eta, \beta] \) and also \( \ell_1(z) \) is between \( \frac{k - a_2 - b_2 z}{c_2 \sqrt{z}} \) and \( \frac{k - a_1 - b_1 z}{c_2 \sqrt{z}} \) when \( z \in [\alpha, \eta] \). Observe that without the terms \( \phi^{(1)}(\ell_1(z)) \) and \( \phi^{(1)}(\ell_2(z)) \) in (75) we would have
\[
\int_\eta^\beta [(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz + \int_\alpha^\eta [(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz
\]
\[(76)
= (a_1 - a_2) + (b_1 - b_2)EZ \geq 0.
\]

Now, define \( g(z) = (a_1 - a_2) + (b_1 - b_2)z \). Then \( g(z) \) is an increasing or decreasing linear function of \( z \) depending on \( b_1 \geq b_2 \) or \( b_1 \leq b_2 \). Therefore either \( g(z) \geq 0 \) on \( [\eta, \beta] \) and \( g(z) \leq 0 \) on \( [\alpha, \eta] \) or \( g(z) \leq 0 \) on \( [\eta, \beta] \) and \( g(z) \geq 0 \) on \( [\alpha, \eta] \). If \( g(z) \geq 0 \) on \( [\eta, \beta] \) and \( g(z) \leq 0 \) on \( [\alpha, \eta] \), then \( \ell_1(z) \geq \ell_2(z) \) and therefore \( \phi^{(1)}(\ell_1(z)) \geq \phi^{(1)}(\ell_2(z)) \). This means that the function \( g(z) \) is multiplied by a larger valued function \( \phi^{(1)}(\ell_1(z)) \) when it is positive valued in \( [\eta, \beta] \) compared to it is multiplied by a smaller valued function \( \phi^{(1)}(\ell_1(z)) \) while it takes negative values in \( [\alpha, \eta] \). This shows that we have
\[
\int_\eta^\beta \phi^{(1)}(\ell_1(z))[(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz \geq \int_\alpha^\eta \phi^{(1)}(\ell_2(z))[(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz.
\]
This in turn implies \( I \geq 0 \). On the other hand if \( g(z) \leq 0 \) on \( [\eta, \beta] \) and \( g(z) \geq 0 \) on \( [\alpha, \eta] \) then \( \ell_1(z) \leq \ell_2(z) \) and therefore \( \phi^{(1)}(\ell_1(z)) \leq \phi^{(1)}(\ell_2(z)) \) which implies
\[
\int_\alpha^\eta \phi^{(1)}(\ell_2(z))[(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz \geq \int_\eta^\beta \phi^{(1)}(\ell_1(z))[(a_1 - a_2) + (b_1 - b_2)z] |f(z)| dz.
\]
This shows \( I \geq 0 \). This completes the proof. □

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Remark 3.11. We should mention that the assumption on $Z$ being bounded mixing random variable in the above Lemma 3.10 is needed as it guarantees that the functions $\frac{k-a_i-b_i}{c_i\sqrt{z}}, i = 1, 2,$ in the proof of this Lemma are bounded functions. Since $\lim_{x \to +\infty} \phi^{(2)}(x) \to +\infty,$ boundedness of these functions are needed in the proof of this Lemma.

Remark 3.12. The above Lemma 3.10 shows that when the mixing random variable is bounded $Z \in L^\infty$ and when

$$a_1 + b_1 EZ = a_2 + b_2 EZ,$$  (77)

we have $Q \succeq (2) R$ if and only if $c_1 \leq c_2.$

Remark 3.13. Observe that our Proposition 2.9 follows from Lemma 3.10 above also.

Remark 3.14. For any given model (1) define $Z^m = Z[1/m, m]$ for any positive integer $m \geq 2.$ Define

$$X^m =: \mu + \gamma Z^m + \sqrt{Z^m} AN_n,$$  (78)

for all $m \geq 2.$ For any finite valued law invariant convex risk measure $\rho$ on $L^p, p \geq 1,$ consider the following optimization problems

$$\min_{\omega} \rho(-\omega^T X^m),$$
$$\text{s.t. } E(-\omega^T X^m) = r,$$
$$\omega^T e = 1.$$  (79)

for each positive integer $m$ and for any real number $r.$ Let $\omega^*_m$ denote the solution of the following optimization problem

$$\min_{\omega} \omega^T \Sigma \omega,$$
$$\text{s.t. } E(\omega^T X^m) = r,$$
$$\omega^T e = 1,$$  (80)

for each $m \geq 2.$ Then $\omega^*_m$ takes the form $\omega^*_m = \omega_0(\mu + \gamma EZ^m, \Sigma)$ as in (59). The above Lemma 3.10 shows that $\omega^*_m X^m \succeq (2) \omega^T X^m$ for all $\omega$ with $E(-\omega^T X^m) = r, \omega^T e = 1.$ Therefore

$$\rho(-\omega^*_m X^m) \leq \rho(-\omega^T X^m)$$

for all the other portfolios $\omega$ with $E(-\omega^T X^m) = r, \omega^T e = 1.$ This shows that $\omega^*_m = \omega_0(\mu + \gamma EZ^m, \Sigma)$ is a solution for (79) for each $m \geq 2.$

Now recall our Corollary 3.6 above. This Corollary shows that the portfolio $\omega^*_0$ in this Corollary is the frontier portfolio for the problem (62) simultaneously for all the significance level $\alpha \in (0, 1).$ This has some implications and before we discuss them we first recall a result that connects the SSD relation among random variables to the CVaR values of the random variables. We state this result in a Lemma below without proof. This result can be found in (vi) of page 14 of [29], or in Theorem 4.4.3 of [45], or in [20].

Lemma 3.15. For any two portfolios $\omega_1, \omega_2 \in R^n$ with $x_1 = T(\omega_1)$ and $x_2 = T(\omega_2),$ we have

$$x_1^T W \succeq (2) x_2^T W \Leftrightarrow CVaR_\alpha(-x_1^T W) \leq CVaR_\alpha(-x_2^T W), \forall \alpha \in (0, 1).$$  (81)
Proposition 3.16. Assume $Z \in L^p$ and consider the following two NMVM models $Q = a_1 + b_1 Z + c_1 \sqrt{Z} N$ and $R = a_2 + b_2 Z + c_2 \sqrt{Z} N$, where $a_1, a_2, b_1, b_2$ are any real numbers and $c_1 > 0$, $c_2 > 0$. The following condition

$$a_1 + b_1 EZ \geq a_2 + b_2 EZ \quad \text{and} \quad c_1 \leq c_2,$$  \hspace{1cm} (82)

is sufficient for $Q \succeq_{(2)} R$.

Proof. We divide into two cases. Assume $a_1 + b_1 EZ > a_2 + b_2 EZ$ first. Define $Z^m = Z1_{[1/m,m]}$ as in Remark 3.17 above for each positive integer $m \geq 2$. Then by the dominated convergence theorem we have $EZ^m \to EZ$. Therefore there exists a positive integer $m_0$ such that we have

$$a_1 + b_1 EZ^m \geq a_2 + b_2 EZ^m$$  \hspace{1cm} (83)

for all $m \geq m_0$. Define $Q^m = a_1 + b_1 Z^m + c_1 \sqrt{Z^m} N$ and $R^m = a_2 + b_2 Z^m + c_2 \sqrt{Z^m} N$. Observe that $Q^m \to Q$ and $R^m \to R$ in $L^p$. From Lemma 3.10 we have $Q^m \succeq_{(2)} R^m$ for all $m \geq m_0$. Since for each $\alpha \in (0, 1)$ the risk measure $CVar_\alpha$ is SSD consistent we have

$$CVar_\alpha(-Q^m) \leq CVar_\alpha(-R^m), \quad \forall m \geq m_0,$$  \hspace{1cm} (84)

for each $\alpha \in (0, 1)$. Since for each fixed $\alpha \in (0, 1)$ the risk measure $CVar_\alpha(\cdot)$ is continuous on $L^p$, by taking limits to both sides of (84) as $m \to +\infty$ we obtain

$$CVar_\alpha(-Q) \leq CVar_\alpha(-R),$$  \hspace{1cm} (85)

for each $\alpha \in (0, 1)$. Then $Q \succeq_{(2)} R$ follows from Lemma 3.15 above. Now assume $a_1 + b_1 EZ = a_2 + b_2 EZ$. Define $Z^m$ as above and let $\delta_m =: (a_1 + b_1 EZ^m) - (a_2 + b_2 EZ^m)$. Since $EZ^m \to EZ$ we have $\delta_m \to 0$ as $m \to +\infty$. Define the following two NMVM models $Q^m = a_1 + b_1 Z^m + c_1 \sqrt{Z^m} N$ and $R^m = \delta_m + a_2 + b_2 Z^m + c_2 \sqrt{Z^m} N$. Observe that $Q^m \to Q$ and $R^m \to R$ in $L^p$. From Lemma 3.10 we have $Q^m \succeq_{(2)} R^m$ for all $m \geq 2$. Now by following the same arguments as in the case $a_1 + b_1 EZ > a_2 + b_2 EZ$ above we obtain $Q \succeq_{(2)} R$. This completes the proof. \hfill \Box

Remark 3.17. Observe that part of the claim on stochastic dominance in our Proposition 2.8 also follows from our Proposition 3.16 above.

Remark 3.18. Our above Proposition 3.16 shows that the condition (82) is sufficient for $Q \succeq_{(2)} R$. However, it is not clear if $Q \succeq_{(2)} R$ also implies (82) under the condition that $Z \in L^p$ for each fixed $p \geq 1$. By contrast, if $Z \in L^\infty$ then by Lemma 3.10 the condition $Q \succeq_{(2)} R$ implies (82). If $Z \in L^p$ for all $p \geq 1$, then from our Lemma 2.12 the condition $Q \succeq_{(2)} R$ implies (82).

Proof of Theorem 3.3. Define the random vectors $X^m$ as in Remark 3.14 above. Let $\omega^*_m = \omega_0(\mu + \gamma EZ^m, \Sigma)$ be the optimal portfolio discussed in the same Remark. Since $\mu + \gamma EZ^m \to \mu + \gamma EZ$ we have that $\omega^*_m$ converges to $\omega^* =: \omega_0(\mu + \gamma EZ, \Sigma)$ in the Euclidean norm $\| \cdot \|$. (Here the random vector $\theta$ is understood to be any random vector with $E\theta = \mu + \gamma EZ$ and $\text{Cov}(\theta) = \Sigma$ now). Also observe that $X^m \to X$ under $L^p$ norm $\| \cdot \|_p$. Therefore we have $(\omega^*_m)^T X^m \to (\omega^*)^T X$ in $L^p$. Then by the continuity of $\rho$ in $L^p$ we have

$$\rho(-(\omega^*_m)^T X^m) \to \rho(-(\omega^*)^T X).$$ \hspace{1cm} (86)
Now from the optimality of \( \omega_m^* \) we have
\[
\rho(-\omega_m^* X^m) \leq \rho(-\omega^T X^\omega), \tag{87}
\]
for any portfolio \( \omega \) in the corresponding domain. Also due to the continuity of \( \rho \) in \( L^p \) we have \( \rho(-\omega^T X^\omega) \to \rho(-\omega^T X) \). By taking limit to the both sides of (87) we obtain
\[
\rho(-\omega^* X) \leq \rho(-\omega^T X), \tag{88}
\]
for any \( \omega \) in the corresponding domain. This completes the proof. \( \square \)

**Remark 3.19.** We remark here that the proof of our Theorem 3.3 above can also be obtained by using our Proposition 3.16 above directly without relying on the continuity of \( \rho \). Therefore in fact the conclusion of our Theorem 3.3 holds for any risk measure \( \rho \) on \( L^p \) that is law-invariant and SSD-consistent.

**Corollary 3.20.** Let \( x^* \) denote the image of \( \omega^*_0 \) in our Theorem 3.3 above and let \( x \) represent the images of the other portfolios that satisfy \(-\omega^T (\mu + \gamma EZ) = r, \omega^T e = 1 \) under the transformation (12). Then we have
\[
(x^*)^T \mu_0 + (x^*)^T \gamma_0 EZ_m + ||x^*||\sqrt{Z}N \geq (2) x^T \mu_0 + x^T \gamma_0 EZ_m + ||x||\sqrt{Z}N, \tag{89}
\]
for all \( x \).

**Remark 3.21.** The portfolio \( \omega^*_0 \) in our Theorem 3.3, which we denote by \( \omega^*_1 \), has the following form
\[
\omega^*_1 = k_1(r) \Sigma^{-1} e + k_2(r) \Sigma^{-1} \mu + [k_2(r)EZ] \Sigma^{-1} \gamma, \tag{90}
\]
where \( k_1(r) \) and \( k_2(r) \) are given by (64) with
\[
d^1 = e^T \Sigma^{-1} (\mu + \gamma EZ), d^2 = (\mu + \gamma EZ)^T \Sigma^{-1} (\mu + \gamma EZ), d^3 = e^T \Sigma^{-1} e, d^4 = d^2 d^3 - (d^1)^2.
\]
By simple calculations we obtain
\[
\omega^*_1 X = a(r) + b(r) Z + \sigma(r) \sqrt{Z} N(0, 1),
\]
where
\[
a(r) = k_1(r) e^T \Sigma^{-1} \mu + k_2(r) (\mu + \gamma EZ)^{-1} \Sigma^{-1} \mu,
b(r) = k_1(r) e^T \Sigma^{-1} \gamma + k_2(r) (\mu + \gamma EZ)^{-1} \Sigma^{-1} \gamma,
\sigma(r) = k_1(r) \sqrt{e^T \Sigma e} + k_2(r) \sqrt{(\mu + \gamma EZ)^T \Sigma^{-1} (\mu + \gamma EZ)}.
\]
As applications of our results above, next we give some examples.

**Example 3.22.** Assume that \( X \sim GH_n(\lambda, \chi, \psi, \mu, \Sigma, \gamma) \). The corresponding mixing distribution is \( Z \sim GIG(\lambda, \chi, \psi) \), see page 11 of [23] (note that they have used the parameters \( \chi = \delta^2 \) and \( \psi = \gamma^2 \)). We have
\[
EZ = \frac{\sqrt{\chi/\psi} K_{\lambda+1}(\sqrt{\chi \psi})}{K_{\lambda}(\sqrt{\chi \psi})}. \tag{91}
\]
22
For any finite law-invariant convex risk measure $\rho$, frontier portfolios for the problem (61) is given by

$$\omega^\star_\theta = \omega^\star_\theta (\mu + \gamma EZ, \Sigma)$$  \hspace{1cm} (92)

as in (59), where $\theta$ is a random vector with $E\theta = \mu + \gamma EZ$ and $\Sigma = A^T A$. This shows also that (see our Corollary 3.6 above) $\omega^\star_\theta$ is a frontier portfolio under the risk measure $CVaR_\alpha$ simultaneously for all the significance level $\alpha \in (0,1)$.

**Example 3.23.** Assume our mixing distribution $Z$ is given by $Z \sim H(\delta, \gamma)$, where $H(\delta, \gamma)$ is a positively valued random variable with moment generating function given by the expression (3.4) of the paper [7]. For any portfolio $\omega$, the distribution of $\omega^T R$ follows $z$ distribution, see Section 3 of [7] for the details of these class of distributions. $z$ distributions are NMVM models as it was shown in Theorem 3.1 of [7]. They are useful to model distributions of log income as discussed in the third paragraph at page 151 of [7]. Both of our Theorem 3.3 and Corollary 3.6 applies to the case of $z$ distributions.

**Example 3.24.** Consider the following type of risk measures

$$\hat{\rho}(X) = \sup_{\lambda \in \mathcal{M}} \int_{(0,1]} CVaR_\beta(X) \lambda(d\beta),$$  \hspace{1cm} (93)

where $\mathcal{M}$ is any subset of the set $\mathcal{M}_1((0,1])$ of probability measures on $(0,1]$, see Section 5.1 and the preceding sections of [19] for more details of these risk measures. For any portfolio $\omega$ and any return vector as in (1), we have

$$\hat{\rho}(-\omega^T X) = \sup_{\lambda \in \mathcal{M}} \int_{(0,1]} CVaR_\beta(-\omega^T X) \lambda(d\beta).$$  \hspace{1cm} (94)

From our Theorem 3.3 above we conclude that the solution of the following optimization problem

$$\min_{\omega} \hat{\rho}(-\omega^T X),$$

$$\text{s.t. } E(-\omega^T X) = r,$$

$$\omega^T e = 1,$$  \hspace{1cm} (95)

is given by $\omega^\star_\theta$ as in (59) for any random vector $\theta$ with $\mu_\theta = E\theta = \mu + \gamma EZ$ and $\Sigma_\theta = A^T A$. We should mention here that Proposition 5.13 of [8] characterizes a class of risk measures that are representable in the above form. Therefore the conclusion of this Example also holds for the risk measures discussed in the Proposition 5.13 of [8] also.

**Example 3.25.** Consider the semi-deviation risk measure

$$Dev_q(Y) := -EY + \|Y - EY\|_q, Y \in L^p_\alpha.$$  \hspace{1cm} (96)

that was discussed in Example 2.22 of [46]. It is law-invariant coherent risk measure and when $q \leq p$ it is $\| \cdot \|_p$ continuous as explained in the same example of [46]. Therefore our Theorem 3.3 applies to the case of this risk measure also.
4 Conclusion

This paper gives closed form expression for frontier portfolios under the mean-risk criteria when the underlying return vectors follow the class of NMVM distributions. Closed form solution for frontier portfolios under the Markowitz mean-variance framework is well known. For general mean-risk models closed form expressions for frontier portfolios are difficult to obtain. In our paper, we showed that when return vectors follow the class of NMVM models the associated frontier portfolios under the mean-risk (with any finite law-invariant convex risk measure) criteria can be obtained by solving the frontier portfolios of Markovitz mean-variance model with an appropriately adjusted return vector.

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