Estimating the Shannon entropy and (un)certainty relations for design-structured POVMs

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Complementarity relations between various characterizations of a probability distribution are at the core of information theory. In particular, lower and upper bounds for the entropic function are of great importance. In applied topics, we often deal with situations, where the sums of certain powers of probabilities are known. The main question is how to convert the imposed restrictions into two-sided estimates on the Shannon entropy. It is addressed in two different ways. The more intuitive of them is based on truncated expansions of the Taylor type. Another method is based on the use of coefficients of the shifted Chebyshev polynomials. We propose here a family of polynomials for estimating the Shannon entropy from below. As a result, estimates are more uniform in the sense that errors do not become too large in particular points. The presented method is used for deriving uncertainty and certainty relations for positive operator-valued measures assigned to a quantum design. Quantum designs are currently the subject of active researches due to potential use in quantum information science. It is shown that the derived estimates are applicable in quantum tomography and detecting steerability of quantum states.

Keywords: Complementarity relation, quantum design, Shannon entropy, Chebyshev polynomial

I. INTRODUCTION

The concept of entropy is fundamental in statistical physics and information theory [1]. Irrespective of the context of their use, entropic functions are hardly exposed to measure immediately. It is not obvious that the total uncertainty about a multipartite system corresponds to the sum of the entropies of its parts [2]. Hence, we are interested in ways to connect entropic quantities with those characteristics that are easier to observe in practice. Results of such a kind constitute an essential part of information sciences, including quantum area [3, 4]. Inequalities between the entropy and the index of coincidence are one of the examples naturally raised in several ways [5]. Connecting inequalities between several information measures can also be used to choose properly a single notion appropriate in a particular context. The authors of [6] reviewed this issue with respect to quantum cryptography. In this paper, the problem of entropy characterization will be treated within uncertainty and certainty relations for positive operator-valued measures (POVMs) assigned to a quantum design.

Information entropies are useful to characterize genuine uncertainties in various situations [7]. They give a natural way to pose uncertainty relations in the presence of quantum memory [8]. For discrete observables, the entropic approach to uncertainty relations was developed in [9, 10]. Basic results found in this area are reviewed in [11, 12]. For entropic uncertainty relations with continuous variables, see [13, 14] and references therein. In quantum information science, special types of measurements have found a lot of attention. Mutually unbiased bases [15] and symmetric informationally complete measurements [16] are especially important examples. Quantum designs, also called complex projective designs, have been considered for several reasons [17, 18]. Characterizing uncertainties in design-structured POVMs is one of the questions raised in this connection. The authors of [19] formulated uncertainty relations for such POVMs in terms of the Rényi and Tsallis entropies. Rényi formulation was improved in [20]. The results of [19, 20] also give entropic inequalities to detect a special kind of nonlocal correlations known as quantum steerability.

As the Rényi entropy cannot increase with growth of its order, uncertainty relations derived in [19, 20] imply estimates on the corresponding Shannon entropy from below. However, such estimates are sufficiently far from optimality. The question of deriving uncertainty relations in terms of the Shannon entropy is beyond the methods of [19, 20]. The aim of this work is to address this question in detail. Moreover, the developed approach naturally leads to estimates on the corresponding Shannon entropy from above. Thereby, certainty relations for design-structured POVMs are formulated. The paper is organized as follows. In Section II, the methods to get two-sided estimates on the Shannon entropy are considered. Here, we propose polynomials whose coefficients are not determined according to Taylor’s scheme. Section III is devoted to uncertainty and certainty relations for POVMs assigned to a quantum design. These relations are formulated and compared with previous results within several examples. Applications for estimating the von Neumann entropy and steering inequalities are briefly mentioned. Section IV concludes the paper. Some auxiliary material is presented in two appendices.
II. ON TWO-SIDED ESTIMATING ON THE SHANNON ENTROPY

In this section, we present two-sided estimates on the Shannon entropy in terms of the power sums of probabilities. The Shannon entropy of probability distribution \( p = \{ p_j \} \) is defined as \( H_1(p) := -\sum_j p_j \ln p_j \). \( \text{(2.1)} \)

For a pair of discrete random variables, the conditional entropy of \( X \) given \( Z \) reads as \( H_1(X|Z) := \sum_z p(z) H_1(p_{X|z}) \), \( \text{(2.2)} \) where \( p_{X|z} \) consists of conditional probabilities \( p(x|z) \).

The authors of \( \textbf{5} \) introduced information diagrams that allow them to study relations between \( \text{(2.1)} \) and the index of coincidence:

\[ I^{(2)}(p) := \sum_j p_j^2. \]
\( \text{(2.3)} \)

It is natural to generalize \( \text{(2.3)} \) as

\[ I^{(s)}(p) := \sum_j p_j^s. \]
\( \text{(2.4)} \)

Such indices were briefly discussed in \( \textbf{3} \). We will use them with positive integer \( s \). For \( s = 0 \), the index \( \text{(2.4)} \) is equal to the number of nonzero probabilities. The Tsallis \( s \)-entropy is defined as \( \textbf{23} \)

\[ H_s(p) := \frac{1}{1-s} \left( \sum_j p_j^s - 1 \right) = \frac{I^{(s)}(p) - 1}{1-s}. \]
\( \text{(2.5)} \)

In the limit \( s \to 1 \), the latter reduces to \( \text{(2.1)} \). For a discussion of basic properties of \( \textbf{23} \) and other entropic functions, see section 2.7 of \( \textbf{24} \).

Suppose that several quantities of the form \( \text{(2.4)} \) are known exactly. Hence, available values of the Shannon entropy have to be restricted. The main question is how to characterize allowed entropic values explicitly. So, one way to express two-sided estimates on the entropic function is in terms of indices of the form \( \text{(2.4)} \). In this work, we aim to build a polynomial \( q(x) \) of degree \( n \) such that

\[ x \ln x \leq q(x) = \sum_{s=1}^n q_n^{(s)} x^s \]
\( \text{(2.6)} \)

for all \( x \in [0,1] \). The inequality \( \text{(2.6)} \) allows us to estimate the Shannon entropy from below. Also, we integrate both the sides of the inequality

\[ 1 + \ln \hat{x} \leq 1 + \sum_{s=1}^n q_n^{(s)} \hat{x}^{s-1} \]
\( \text{(2.7)} \)

from \( \hat{x} = x \) to \( \hat{x} = 1 \), whence

\[ -x \ln x \leq 1 + \sum_{s=1}^n q_n^{(s)} \frac{x^s}{s} - x - \sum_{s=1}^n q_n^{(s)} \frac{x^s}{s}. \]
\( \text{(2.8)} \)

This inequality will be used to estimate the Shannon entropy from above. In effect, both the relations \( \text{(2.6)} \) and \( \text{(2.8)} \) should be applied with rescaled probabilities.

The main question is to find desired polynomials. At first glance, spline functions may seem an appropriate tool in this situation. However, this approach is hardly applicable to the case of interest. Using suitable truncation of Taylor series is a natural way with rich history. Deficiencies of this approach come from the fact that truncated series gave very accurate results near the point of expansion but insufficient ones in peripheral regions \( \textbf{23} \). Nevertheless, truncated expansions of the Taylor type are still a powerful tool for studying inequalities of interest. Instead, there are power expansions whose coefficients are not determined according to the Taylor scheme. In numerical analysis, such expansions are often based on the use of a suitable family of orthogonal polynomials. The significance of Chebyshev polynomials in various problems of applied analysis was clearly demonstrated by Lanczos \( \textbf{25} \). Before addressing his ideas, we consider truncated expansions of the Taylor type.
Putting $1 - x = z \in [0, 1]$ and using well-known expansions, we have
\[
\ln x = \ln(1 - z) = -\sum_{r=1}^{\infty} \frac{z^r}{r} \leq -\sum_{r=1}^{n-1} \frac{z^r}{r}.
\] (2.9)

Multiplying the latter by $-x \leq 0$, one gets
\[
f_n(x) \leq -x \ln x,
\] (2.10)

where
\[
f_n(x) = x \sum_{r=1}^{n-1} \frac{(1 - x)^r}{r} = \sum_{s=1}^{n} a_n^{(s)} x^s.
\] (2.11)

It is immediate to check that
\[
a_n^{(1)} = \sum_{r=1}^{n-1} \frac{1}{r}, \quad a_n^{(s)} = (-1)^{s-1} \sum_{r=s-1}^{n-1} \frac{1}{r} \binom{r}{s-1} \quad (2 \leq s \leq n).
\] (2.12)

To estimate $(-x \ln x)$ from above, we could combine $x \ln x \leq -f_n(x)$ with (2.6) and (2.8). The more direct way is to truncate the expansion of $(z - 1) \ln(1 - z)$. In any case, the result reads as
\[
x \ln x \leq h_n(x),
\] (2.13)

where
\[
h_n(x) = (1 - x)\left(1 - \sum_{r=1}^{n-1} \frac{(1 - x)^r}{r(r+1)}\right) = \sum_{s=0}^{n} b_n^{(s)} x^s.
\] (2.14)

The coefficients are expressed as $b_n^{(0)} = \frac{1}{n}$,
\[
b_n^{(1)} = \sum_{r=2}^{n-1} \frac{1}{r}, \quad b_n^{(s)} = \frac{(-1)^{s-1}}{s} \sum_{r=s-1}^{n-1} \frac{1}{r} \binom{r}{s-1} \quad (2 \leq s \leq n).
\] (2.15)

As obtained due to the Taylor scheme, the estimates (2.10) and (2.13) demonstrate a typical behavior. They give a very good approximation from their own side in some left neighborhood of the point $x = 1$. In contrast, inaccuracies become comparatively large in a right neighborhood of the point $x = 0$. They are most obvious for (2.13) due to $h_n(0) = 1/n$. Thus, there is a certain asymmetry with respect to estimating in left and right regions of the range $x \in [0, 1]$. This asymmetry can be reduced due to the methods described in [23]. Although the inequalities (2.10) and (2.13) hold for $n = 1$, it is not used in the following.

Using orthogonal polynomials for approximation, expansions with rigid coefficients are best known. That is, adding a next member of the chosen orthogonal family into our expansion does not affect the sum we have obtained before. As a reverse side of this fact, we have to deal with oscillating approximations. One can only minimize an integral error, say, in the sense of least squares. Such expansions are not suitable to fit the function of interest from above or below solely. As was emphasized by Lanczos (see, e.g., chapter VII of [23]), the condition of rigidity of coefficients received an exceptional attention for rather historical reasons. To reach more effective approximations, adding a next member should be combined with changing all the coefficients. So, we have arrived at expansions with flexible coefficients.

In section VII.12 of his book [23] Lanczos described the so-called $\tau$-method to solve differential equations. The next paragraph of [23] is devoted to a reformulation with canonical polynomials. Let function $x \mapsto y(x)$ obey a certain ordinary differential equation. Its power fitting can be found as an approximate solution of this equation. When the original equation is not solvable in polynomials, it is modified by adding an inhomogeneous term. Lanczos proposed to add a term proportional to some shifted Chebyshev polynomial. This procedure generally includes a certain freedom. The following example gives a useful power expansion of $y(x) = x \ln x$. For the equation
\[
xy'(x) - y(x) = x
\] (2.16)

with the boundary condition $y(1) = 0$, one adds $n$th shifted Chebyshev polynomial $T^*_n(x)$ on the right. Some facts about these polynomials are recalled in Appendix A. The procedure results in the sum
\[
\frac{(-1)^n}{2n^2} \left(x - 1 + \sum_{s=2}^{n} c_n^{(s)} \frac{x^s - x}{s-1}\right),
\] (2.17)
Proposition 1

The main result of this section is posed as follows. With growth of $-\infty$ immediately leads to $x = 0$, though the initial function is negative for all $x \in (0, 1)$. The constant term in (2.17) is determined by $T_n^*(x)$ on the right, whereas any term $x$ is a solution of homogeneous equation. Adding to (2.17) the linear term $(\omega(1-x)/(2n^2))$, for $n \geq 2$ we finally write

$$g_n(x) = \frac{(-1)^n}{2n^2} \sum_{s=2}^n \omega(x) \frac{x^s - x}{s - 1}$$

(2.18)

so that $g_n(0) = 0$ and $g_n(1) = 0$. Polynomials of the form (2.18) are the first main finding of this work. They will be used to estimate $x \ln x$ from above.

Let us consider the first derivative at the least points of the interval $x \in [0, 1]$. It follows from (B1) that $g_n'(1) = 1$ for even $n \geq 2$ and $g_n'(1) = 1 - 1/n^2$ for odd $n \geq 3$. To express $g_n'(0)$ as an expression with definite sign, we use (B2) and (B3), whence

$$g_n'(0) = -\frac{4}{n^2} \sum_{r=1}^{\lfloor n/2 \rfloor} \frac{4r^2}{n - 2r} \quad (n \text{ odd}) ,
$$

(2.19)

$$g_n'(0) = -\frac{4}{n^2} \sum_{r=1}^{\lfloor n/2 \rfloor} \frac{(2r - 1)^2}{n - 2r + 1} \quad (n \text{ even}) .
$$

(2.20)

To fit the original function, the line of $g_n(x)$ should go closely to the line of $y(x)$. As $y'(x) = 1 + \ln x \to -\infty$ for $x \to +0$, the graph of $y(x)$ tends to enter the abscissa axis normally. Thus, the value $g_n'(0)$ is expected to be negative. On the other hand, this value is certainly finite, whence the slope of tangential line is far enough from the vertical. In some right vicinity of the point $x = 0$, our approximation has to be relatively poor, since derivatives are inevitably finite for any polynomial. With growth of $n$, the absolute value of $g_n'(0)$ increases with improving an approximation. We see from $-\infty < g_n'(0) < 0$ and $g_n(0) = y(0)$ that $g_n(x)$ exceeds $y(x)$ in a right neighborhood of the point $x = 0$. The main result of this section is posed as follows.

**Proposition 1** Let polynomials $g_n(x)$ be defined by (2.18). For $x \in [0, 1]$ and $n \geq 2$, it holds that

$$x \ln x \leq g_n(x) .
$$

(2.21)

**Proof.** We aim to prove nonnegativity of the difference $\gamma_n(x) = g_n(x) - x \ln x$. It is zero for $x = 0$ and $x = 1$. The task is to show (2.21) for points between 0 and 1. First, the function $y(x) = x \ln x$ obeys (2.16). Combining (2.18) with (A4) immediately leads to

$$x g_n'(x) - g_n(x) = \frac{(-1)^n}{2n^2} \left[T_n^*(x) - (-1)^n x\right] .
$$

(2.22)

Due to (2.16) and (2.22), one obtains

$$\frac{d}{dx} \gamma_n(x) = \frac{x \gamma_n'(x) - \gamma_n(x)}{x^2} = \frac{-(-1)^n T_n^*(x) - 1}{2n^2 x^2} .
$$

(2.23)

Putting $x = \cos^2 \theta/2$ with $\theta$ between 0 and $\pi$, we have $T_n^*(x) = T_n^*(\cos \theta) = \cos n\theta$. So, the right-hand side of (2.23) cannot be positive. For $x \in (\varepsilon, 1)$ with small $\varepsilon > 0$, the ratio $\gamma_n(x)/x$ is a smooth decreasing function vanishing at the right least point of the interval. By monotonicity, this ratio is nonnegative for all $x \in (\varepsilon, 1)$. Making $\varepsilon > 0$ arbitrarily small completes the proof. ■

Substituting $n = 2$ gives $g_2(x) = -f_2(x) = x^2 - x$ so that $x \ln x \leq g_2(x)$ follows from (2.19). For other values of $n$, functions of the form (2.18) do not relate to the Taylor scheme. The coefficients $c_n^{(s)}$ increase considerably with growth of $n$. It is generally suitable to consider only moderate values of $n$. With $n = 15$, the difference between the sides of (2.21) is less than one thousandth of the maximum of $|x \ln x|$ in the interval $x \in [0, 1]$. At fixed $n$, the polynomial $g_n(x)$ improves an approximation for relatively small $x$ without introducing considerable errors in a left vicinity of the point $x = 1$.

Due to (2.21), we estimate the function of interest from below as

$$\sum_{s=1}^n \tilde{a}_n^{(s)} x^s \leq -x \ln x .
$$

(2.24)
Here, the coefficients are expressed in terms of coefficients of nth Chebyshev polynomial by the formulas

\[
\tilde{a}_n^{(s)} = \frac{(-1)^n}{2n^2} \sum_{s=2}^{n} \frac{a_{ns}}{s-1}, \quad \tilde{a}_n^{(s)} = \frac{(-1)^{n+1}}{2n^2} \frac{a_{ns}}{s-1} \quad (2 \leq s \leq n). \tag{2.25}
\]

For \( n = 2 \), the left-hand side of (2.24) is equal to \( f_2(x) = x - x^2 \). It is not the case for \( n \geq 3 \). Applying (2.6) and (2.8) with \( g_n(x) \) finally leads to

\[-x \ln x \leq \sum_{s=0}^{n} \tilde{b}_n^{(s)} x^s, \tag{2.26}
\]

where

\[
\tilde{b}_n^{(0)} = 1 - \sum_{s=1}^{n} \frac{\tilde{a}_n^{(s)}}{s}, \quad \tilde{b}_n^{(1)} = \tilde{a}_n^{(1)} - 1, \quad \tilde{b}_n^{(s)} = \frac{\tilde{a}_n^{(s)}}{s} \quad (2 \leq s \leq n). \tag{2.27}
\]

The result (2.26) allows us to estimate the Shannon entropy from above. For \( n = 2 \), the inequality (2.26) coincides with (2.13). For other values of \( n \), coefficients of polynomials in the formulas (2.24) and (2.26) do not correspond to the Taylor scheme. Hence, the estimates by means of (2.24) and (2.26) are not very good near the point \( x = 1 \). For points sufficiently close to 1 from the left, we prefer (2.10) and (2.13). In regions peripheral to \( x = 1 \), the results (2.24) and (2.26) provide more accurate estimation. This distinction is especially clear near the point \( x = 0 \). Nevertheless, any estimate by polynomials will hardly be very precise for points close to zero from the right. In effect, the function of interest is not analytic in this point.

Thus, there is an asymmetry in estimating the function \( x \mapsto -x \ln x \) in regions close, respectively, to one of the two least points of the range \( x \in [0, 1] \). It is also validated by some numerical inspection. To make estimates better, one shall move actual points of approximation to the right, where expected errors are less. It can be made due to estimating the maximal probability from above (20). Let the index of the form (2.4) be given for some integer \( n \geq 2 \), and let \( L \) be the number of nonzero probabilities. It then holds that

\[
\max_j p_j \leq Y_{L-1}^{(n)}(I^{(n)}), \tag{2.28}
\]

where \( Y_{L-1}^{(n)}(I^{(n)}) \) denotes the maximal real root of the algebraic equation

\[
(1 - Y)^n + (L - 1)^{n-1} Y^n = (L - 1)^{n-1} I^{(n)}. \tag{2.29}
\]

When other parameters are fixed, the quantity \( Y_{L-1}^{(n)}(I^{(n)}) \) is increasing and concave with respect to \( I^{(n)} \) [20].

For \( n \geq 5 \), we are generally unable to write \( Y_{L-1}^{(n)}(I^{(n)}) \) analytically using radicals. Instead, it can be calculated numerically with any desired accuracy [20]. For \( n = 2, 3, 4 \), one can express the answer in a closed analytic form. The case \( n = 2 \) is answered by [20]

\[
Y_{L-1}^{(2)}(I^{(2)}) = \frac{1}{L} \left( 1 + \sqrt{L - 1} \sqrt{LI^{(2)} - 1} \right). \tag{2.30}
\]

Due to our findings, the following statement takes place.

**Proposition 2** Let \( I^{(s)}(p) \) be given for all \( s = 2, \ldots, n \). Then the Shannon entropy satisfies

\[
\sum_{s=1}^{n} a_n^{(s)} Y^{1-s} I^{(s)}(p) - \ln Y \leq H_1(p) \leq \sum_{s=0}^{n} b_n^{(s)} Y^{1-s} I^{(s)}(p) - \ln Y, \tag{2.31}
\]

where \( Y = Y_{L-1}^{(n)}(I^{(n)}) \) and the coefficients \( a_n^{(s)} \) and \( b_n^{(s)} \) are defined by (2.12) and (2.15), respectively. It also holds that

\[
\sum_{s=1}^{n} a_n^{(s)} Y^{1-s} I^{(s)}(p) - \ln Y \leq H_1(p) \leq \sum_{s=0}^{n} b_n^{(s)} Y^{1-s} I^{(s)}(p) - \ln Y, \tag{2.32}
\]

where the coefficients are defined by (2.27) and (2.27).
Proof. The points of approximation $x_j = p_j/\Upsilon$ lie in the interval $x \in [0,1]$ due to (2.28). Substituting $p_j = \Upsilon x_j$ into (2.31) results in the expression

$$H_1(p) = -\ln \Upsilon - \sum_{j=1}^{L} x_j \ln x_j.$$  \hfill (2.33)

Using (2.10) and (2.13) in all the points $x_j$, we obtain after summation that

$$\sum_{s=1}^{n} a_n^{(s)} \frac{I^{(s)}(p)}{Y^s} \leq -\sum_{j=1}^{L} x_j \ln x_j \leq \sum_{s=0}^{n} b_n^{(s)} \frac{I^{(s)}(p)}{Y^s}.$$ \hfill (2.34)

Combining (2.33) with (2.34) completes the proof of (2.31). Similarly, the two-sided estimate (2.32) follows from (2.24) and (2.26). \hfill \blacksquare

The statement of Proposition 2 provides two-sided estimates on the Shannon entropy, when several indices of the form (2.4) are given exactly. In the next section, we will use (2.31) and (2.32) to derive uncertainty and certainty relations for POVMs assigned to several quantum designs of degree 3, 5, and 7. They are quantum counterparts of some spherical designs in three dimensions listed in [29]. Quantum designs have found a lot of attention due to potential applications in emerging technologies of information processing. They also allow us to show a good accuracy of new estimates for a sufficient number of outcomes. Utility of both the formulas (2.31) and (2.32) will be exemplified as well. Finally, we use (2.21) to give an inequality between the Shannon entropy and Tsallis entropies of integer degree $s \geq 2$, viz.,

$$H_1(p) \geq \frac{(-1)^n}{2n^2} \sum_{s=2}^{n} c_n^{(s)} H_s(p).$$ \hfill (2.35)

It is important as holding irrespectively to the number of events.

### III. UNCERTAINTY AND CERTAINTY RELATIONS FOR DESIGN-STRUCTURED POVMs

This section is devoted to complementarity relations for POVMs assigned to a quantum design. When several indices of the form (2.4) are given, uncertainty and certainty relations directly follow from the results of the previous section. Let us recall briefly the required material concerning quantum designs. Dealing with rays in $d$-dimensional Hilbert space $\mathcal{H}$, one selects $K$ unit vectors $|\phi_k\rangle$ with the following property. For all real polynomials $P_t$ of degree at most $t$ it holds that [17]

$$\frac{1}{K^2} \sum_{j,k=1}^{K} P_t \left( |\langle \phi_j | \phi_k \rangle|^2 \right) = \int \int d\mu(\psi) d\mu(\psi') P_t \left( |\langle \psi | \psi' \rangle|^2 \right).$$  \hfill (3.1)

By $\mu(\psi)$, one denotes the unique unitarily invariant probability measure on the corresponding complex projective space induced by the Haar measure. Then unit vectors $|\phi_k\rangle$ are said to form a quantum $t$-design. Quantum designs have useful formal properties posed as follows. Let $\Pi_{sym}^{(t)}$ be the projector onto the symmetric subspace of $\mathcal{H}^{\otimes t}$. It holds that [17]

$$\frac{1}{K} \sum_{k=1}^{K} |\phi_k\rangle \langle \phi_k|^\otimes t = D_d^{(t)} \Pi_{sym}^{(t)},$$ \hfill (3.2)

where $D_d^{(t)}$ denotes the inverse of $\text{tr}(\Pi_{sym}^{(t)})$, namely,

$$D_d^{(t)} = \left( \frac{d + t - 1}{t} \right)^{-1} = \frac{t! (d-1)!}{(d+t-1)!}.$$ \hfill (3.3)

At the given $t$, we can rewrite (3.2) for all positive integers $s \leq t$. Substituting $s = 1$ leads to the formula

$$\frac{d}{K} \sum_{k=1}^{K} |\phi_k\rangle \langle \phi_k| = \mathbb{1}_d.$$ \hfill (3.4)
Thus, unit vectors $|\phi_k\rangle$ allow us to build to a resolution of the identity in $\mathcal{H}$. Also, there may be several resolutions assigned to the given $t$-design. We will call them design-structured POVMs. The case of single one deals with the complete set $\mathcal{E}$ consisting of operators

$$E_k = \frac{d}{K} |\phi_k\rangle\langle\phi_k|.$$  \hfill (3.5)

Sometimes, the set of $M$ rank-one POVMs $\{\mathcal{E}^{(m)}\}_{m=1}^M$ can be assigned to the given quantum design. Each of POVMs consists of $\ell$ operators of the form

$$E_j^{(m)} = \frac{d}{\ell} |\phi_j^{(m)}\rangle\langle\phi_j^{(m)}|.$$  \hfill (3.6)

The integers $\ell$ and $M$ are connected by $K = \ell M$.

If the state of interest is described by density matrix $\rho$, then the probability of $j$th outcome is equal to

$$p_j(\mathcal{E}^{(m)}; \rho) = \frac{d}{\ell} \langle \phi_j^{(m)} | \rho | \phi_j^{(m)} \rangle.$$  \hfill (3.7)

Substituting these probabilities into (2.11) gives the entropy $H_1(\mathcal{E}^{(m)}; \rho)$. It follows from (3.2) that

$$\frac{1}{K} \sum_{k=1}^K (\phi_k | \rho | \phi_k)^s = \mathcal{D}_d^{(s)} \text{tr} (\rho \otimes \Pi^{(s)}_{\text{sym}}).$$  \hfill (3.8)

Combining (3.7) with (3.8) then gives

$$\sum_{m=1}^M \sum_{j=1}^\ell p_j(\mathcal{E}^{(m)}; \rho)^s = \left( \frac{d}{\ell} \right)^s \sum_{k=1}^K (\phi_k | \rho | \phi_k)^s = K \ell^{-s} d^s \mathcal{D}_d^{(s)} \text{tr} (\rho \otimes \Pi^{(s)}_{\text{sym}}),$$  \hfill (3.9)

where $s = 2, \ldots, t$. This fact can be used instead of (3.1) to verify quantum designs [19]. When a single POVM is assigned, one has $\ell = K$ and

$$\sum_{k=1}^K p_k(\mathcal{E}; \rho)^s = K^{1-s} d^s \mathcal{D}_d^{(s)} \text{tr} (\rho \otimes \Pi^{(s)}_{\text{sym}}).$$  \hfill (3.10)

The authors of [19, 27] resolved the question of how to express $\text{tr} (\rho \otimes \Pi^{(s)}_{\text{sym}})$ as a sum of monomials of the moments of $\rho$. On the other hand, such expressions allow us to get several moments of $\rho$ from quantities of the form

$$\bar{\beta}_t^{(s)}(\rho) = \ell^{1-s} d^s \mathcal{D}_d^{(s)} \text{tr} (\rho \otimes \Pi^{(s)}_{\text{sym}}),$$  \hfill (3.11)

$$\bar{\beta}^{(s)}(\rho) = K^{1-s} d^s \mathcal{D}_d^{(s)} \text{tr} (\rho \otimes \Pi^{(s)}_{\text{sym}}).$$  \hfill (3.12)

For $s = 0, 1$, we, respectively, have $\bar{\beta}^{(0)}(\rho) = K$ and $\bar{\beta}^{(1)}(\rho) = 1$. The term (3.12) is obtained from (3.11) for $\ell = K$.

For a pure state $\rho = |\psi\rangle\langle\psi|$, the trace in (3.11) is equal to 1 so that

$$\bar{\beta}_t^{(s)} (|\psi\rangle\langle\psi|) = \ell^{1-s} d^s \mathcal{D}_d^{(s)}.$$  \hfill (3.13)

The formulas (3.9) and (3.10) are the base to pose entropic uncertainty relations for design-structured POVMs. Uncertainty relations in terms of generalized entropies were considered in [19, 20]. Findings of the previous section allow one to obtain uncertainty and certainty relations in terms of the Shannon entropy.

**Proposition 3** Let $M$ rank-one POVMs $\mathcal{E}^{(m)}$, each with $\ell$ elements of the form (3.6), be assigned to a quantum $t$-design $\{|\phi_k\rangle\}_{k=1}^K$ in $d$ dimensions. It then holds that

$$\sum_{s=1}^t a_t^{(s)} \mathcal{Y}^{1-s} \bar{\beta}_t^{(s)}(\rho) - \ln \mathcal{Y} \leq \frac{1}{M} \sum_{m=1}^M H_1(\mathcal{E}^{(m)}; \rho) \leq \sum_{s=0}^t b_t^{(s)} \mathcal{Y}^{1-s} \bar{\beta}_t^{(s)}(\rho) - \ln \mathcal{Y},$$  \hfill (3.14)

$$\sum_{s=1}^t \bar{a}_t^{(s)} \mathcal{Y}^{1-s} \bar{\beta}_t^{(s)}(\rho) - \ln \mathcal{Y} \leq \frac{1}{M} \sum_{m=1}^M H_1(\mathcal{E}^{(m)}; \rho) \leq \sum_{s=0}^t \bar{b}_t^{(s)} \mathcal{Y}^{1-s} \bar{\beta}_t^{(s)}(\rho) - \ln \mathcal{Y},$$  \hfill (3.15)

where $\mathcal{Y} = \min \{ M \mathcal{Y}_{K-1}^{(t)} (\bar{\beta}^{(t)}(\rho)), 1 \}$. 
Proof. For the case of $M$ POVMs, the left-hand side of (3.9) is actually the sum of $s$-indices for all POVMs. For all $k = 1, \ldots, K$, one has
\[
\frac{d}{K} \langle \phi_k | \rho | \phi_k \rangle = p_k (E; \rho) \leq \Upsilon_{K-1}^{(t)} (\bar{\beta}^{(t)} (\rho)) ,
\]
whence
\[
\frac{d}{t} \langle \phi_j^{(m)} | \rho | \phi_j^{(m)} \rangle = p_j (E^{(m)}; \rho) \leq M \Upsilon_{K-1}^{(t)} (\bar{\beta}^{(t)} (\rho))
\]
(3.16)
due to $K = \ell M$. To each of the $M$ entropies $H_1 (E^{(m)}; \rho)$, we apply the two-sided estimate (2.31) with maximal power $t$ and the defined $\Upsilon$ so that
\[
\sum_{s=1}^{t} a_t^{(s)} \Upsilon^{1-s} I^{(s)} (p^{(m)}) - \ln \Upsilon \leq H_1 (E^{(m)}; \rho) \leq \sum_{s=0}^{t} b_t^{(s)} \Upsilon^{1-s} I^{(s)} (p^{(m)}) - \ln \Upsilon .
\]
(3.17)
It is seen from (3.9) that
\[
\sum_{m=1}^{M} I^{(s)} (p^{(m)}) = K \ell^{-s} d_{\Upsilon}^{(s)} \text{tr} (\rho^{\otimes s} \gamma_{\text{sym}}^{(s)}) = M \beta_t^{(s)} (\rho) .
\]
(3.18)
Summing (3.17) with respect to $m$ and substituting (3.18), we get (3.14) multiplied by common factor $M$. Similar reasons allow us to derive (3.15) from (2.32).

Thus, we have obtained two families of complementarity relations for the Shannon entropy averaged over all $M$ POVMs $E^{(m)}$. When single POVM $E$ with $K$ elements (3.5) is assigned to the given $t$-design, the relations (3.14) and (3.15), respectively, reduce to
\[
\sum_{s=1}^{t} a_t^{(s)} \Upsilon^{1-s} \bar{\beta}^{(s)} (\rho) - \ln \Upsilon \leq H_1 (E; \rho) \leq \sum_{s=0}^{t} b_t^{(s)} \Upsilon^{1-s} \bar{\beta}^{(s)} (\rho) - \ln \Upsilon ,
\]
(3.19)
\[
\sum_{s=1}^{t} \tilde{a}_t^{(s)} \Upsilon^{1-s} \bar{\beta}^{(s)} (\rho) - \ln \Upsilon \leq H_1 (E; \rho) \leq \sum_{s=0}^{t} \tilde{b}_t^{(s)} \Upsilon^{1-s} \bar{\beta}^{(s)} (\rho) - \ln \Upsilon .
\]
(3.20)
We shall below consider several examples concerning mainly (3.19) and (3.20). They are a reflection of sufficiently strong restrictions imposed on measurement statistics for design-structured POVMs. Hence, complementarity relations for various entropic functions follow. Uncertainty relations in terms of generalized entropies were considered in [19, 20].

In a sense, the above discussion complements the analysis by adding relations in terms of the Shannon entropy. For a pure state, the left-hand sides of (3.19) and (3.15) lead to
\[
\frac{1}{M} \sum_{m=1}^{M} H_1 (E^{(m)}; | \psi \rangle \langle \psi |) \geq \sum_{s=1}^{t} a_t^{(s)} (\Upsilon \ell)^{1-s} d_{\Upsilon}^{(s)} - \ln \Upsilon ,
\]
(3.21)
\[
\frac{1}{M} \sum_{m=1}^{M} H_1 (E^{(m)}; | \psi \rangle \langle \psi |) \geq \sum_{s=1}^{t} \tilde{a}_t^{(s)} (\Upsilon \ell)^{1-s} d_{\Upsilon}^{(s)} - \ln \Upsilon ,
\]
(3.22)
where $\Upsilon = \min \{ M \Upsilon_{K-1}^{(t)} (K_{K-1}^{(t)} d_{\Upsilon}^{(t)}), 1 \}$.

The formulas (3.21) and (3.22) are interesting due to the following. In many applications of uncertainty relations, we ask for restrictions that hold for all states. Such state-independent relations are often reached with substituting a pure state. In typical cases, we deal with several mutually unbiased bases or POVMs assigned to a quantum design. Here, a total set of used vectors is overcomplete so that generated probabilities are distributed more uniformly for sufficiently mixed states. Of course, entropic functions then take larger values on average. This behavior coincides with increasing $\Upsilon_{K-1}^{(t)} (\beta)$ with respect to $\beta$ [20]. Quantities of the form (3.11) and (3.12) are clearly maximal for pure states. Combining this with (3.19) shows that a diapason of probability values shortens with a growth of state mixedness. So, the choice of a pure state leads to lower entropic bounds that hold for all states. The fact was too shown for Rényi-entropy uncertainty relations [20]. Due to the applied methods of derivation, the validity of (3.21) and (3.22) for all states is not easy to gain analytically. Nevertheless, we will see this physically natural behavior with all the examples. By the way, the last term in the right-hand sides of both (3.21) and (3.22) exhibits such a behavior.
In any case, the desired fact can be checked by inspection in each concrete example of design-structured POVMs. At the end of this section, we will apply (3.21) and (3.22) as state-independent formulations. The state-independent upper bound is merely \( \ln \ell \) for each of \( M \) POVMs and \( \ln K \) for a single POVM.

Let us compare the above finding with relations previously published in the literature. Here, we quote some results of the papers \([5, 28]\). One of the results of Harremoës and Topsøe \([5]\) can be reformulated as follows \([28]\): For probability distribution \( p \) with \( K \) probabilities, it holds that

\[
H_1(p) \geq \ln(k + 1) + k \ln \left( \frac{k + 1}{k} \right) - k(k + 1) \ln \left( \frac{k + 1}{k} \right) f^{(2)}(p) ,
\]

(3.23)

where \( k \) is integer and \( 1 \leq k \leq K - 1 \). To increase the lower bound, we take \( k \) depending on the actual value of \( f^{(2)}(p) \). The choice \( k = K - 1 \) is optimal only for states sufficiently close to the maximally mixed one. In the following examples, the right-hand side of (3.23) is maximized with respect to integer \( k \).

Let us proceed to examples of applications to concrete quantum designs in two dimensions. A short description of these designs in terms of components of the Bloch vector is given in \([19]\). This vector comes to one of the vertices forming some polyhedron. To characterize the qubit density matrix, its minimal eigenvalue \( \lambda \) is used. It is instructive to visualize the two-sided estimates (3.14) and (3.15) together with (3.23). To avoid bulky legends on figures, the following notation will be utilized. By “LT-estimate” and “UT-estimate,” we mean the left- and right-hand sides of (3.14), respectively. They are based on approximation by polynomials, whose coefficients are due to the Taylor scheme. The terms “LCh-estimate” and “UCh-estimate,” respectively, refer to the left- and right-hand sides of (3.15). Due to (2.21), such estimates use polynomials with coefficients linked to coefficients of the shifted Chebyshev polynomials. The right-hand side of (3.23) following from information diagrams will be referred to as “ID-estimate.” We will mainly focus on the case of single assigned POVMs.

Let us begin with the 3-design with \( K = 6 \) vertices forming an octahedron. In Figure 1, we plot the three lower estimates and the two upper ones as functions of \( \lambda \). Of course, the corresponding Shannon entropy varies in sufficiently restricted diapason. The two-sided estimate (3.19) is better only for states very close to the maximally mixed one. We also notice that ID-estimate almost coincides with the left-hand side of (3.19) but is better for almost all \( \lambda \). Thus, the result (3.23) well converts the actual value of (3.23) into estimating the Shannon entropy from below. In fact, the proposed estimates are not very good for degree 3. Nevertheless, the formula (3.20) shows considerably better results for pure states and states with low mixedness. For pure states, the difference between LCh-estimate and ID-estimate is more than one fifth of the difference between UCh-estimate and LCh-estimate. The latter describes a diapason in which the Shannon entropy varies. Overall, LCh- and UCh-estimates are better in a sufficiently wide range of \( \lambda \). For the maximally mixed state, the five curves are all converging at one point.

In the example with the octahedron, the 3-design of interest is formed by eigenstates of the Pauli matrices. Here, one deals with the complete set of three mutually unbiased bases in two dimensions. Entropic uncertainty relations for such bases were extensively studied. Using (3.23), the authors of \([28]\) studied entropic uncertainty relations for a
set of mutually unbiased bases. For $M = 3$ and $d = 2$, their main result reads as

$$
\frac{1}{3} \sum_{m=1}^{3} H_1(\mathcal{E}^{(m)}; \rho) \geq 2 - \frac{\text{tr}(\rho^2)}{3} \ln 4.
$$

(3.24)

In this example, the right-hand side of (3.24) will be referred to as ID-estimate. The five estimates on the average Shannon entropy are presented in Figure 2. In this example, ID-estimate provides a better lower bound. For pure states, the difference between ID-estimate and LCh-estimate is almost one fifth of the difference between UCh-estimate and ID-estimate. It was already noticed that our estimates seem to be insufficient for degree 3. Another origin of comparatively poor results of (3.15) is due to the fact that $3 \chi_5^{(3)}(\tilde{\beta}^{(3)}(\rho)) > 1$ for states with sufficiently low mixedness. A more accurate way to estimate the average maximal probability from above is based on the inequality

$$
\frac{1}{M} \sum_{m=1}^{M} \max_j p_j(\mathcal{E}^{(m)}; \rho) \leq \chi_{\ell-1}^{(t)}(\tilde{\beta}_{\ell}^{(t)}(\rho)).
$$

(3.25)

The latter should be used here with $t = 3$ and $\ell = 2$. Unfortunately, methods of deriving (3.14) and (3.15) are such that one is hardly able to use just (3.25). The right-hand side of (3.23) has a simple analytical structure such that results of the form (3.24) follow. This example shows that the proposed two-sided estimates provide better results in application to a single entropy. As was already mentioned, only the sum of $s$-indices up to $s = t$ is given by (3.24).

The following example is the 5-design with $K = 12$ vertices forming an icosahedron. The three lower bounds and two upper bounds versus $\lambda$ are shown in Figure 3. Of course, allowed changes of the corresponding Shannon entropy are enough limited in their value. Close to the maximally mixed state, the two-sided estimate (3.19) is better than (3.20). Moreover, we see a wider region in which the result (3.19) leads to stronger bounds. For states with lower mixedness, the left-hand sides of (3.19) and (3.20) are both considerably stronger than the lower estimate obtained from the information diagrams. For pure states, the difference between LCh-estimate and ID-estimate is almost 1.4 times the difference between UCh-estimate and LCh-estimate. In other words, ID-estimate here is far from actual entropic values. It is insufficient even in comparison with (3.19). We could expect this, since ID-estimate uses only the index of coincidence (2.3), whereas new estimates take into account indices of the form (2.4) for $s = 2, 3, 4, 5$.

Similarly to the previous examples, all the five curves converge at one point for the maximally mixed state.

Let us consider also the 5-design with $K = 30$ vertices forming an icosidodecahedron. On the average, allowed values of the Shannon entropy are larger in view of an increased number of outcomes. The five curves shown in Figure 4 curves form a picture quite similar to the previous example. The interval $\lambda \in [0, 1/2]$ is divided into two approximately equal parts. The former is where the two-sided estimate (3.20) gives better results, and the latter is a domain for the use of (3.19). As before, for pure states ID-estimate is sufficiently far from optimality. In effect,
the difference between LCh-estimate and ID-estimate is comparable with the difference between UCh-estimate and LCh-estimate. Nevertheless, this distinction is less than in the previous case with 12 vertices. Close to the maximally mixed state, all of the five curves become coinciding.

Finally, we test the discussed estimates with the 7-design obtained from a deformed snub cube. The regular snub cube has 60 equal edges, and its 24 vertices form a 3-design. Moving these vertices slightly, one can obtain a 7-design [29]. The resulting 7-design was first found by McLaren [30]. In Figure 5, we plot the estimates of interest for the McLaren design. A range for possible values of the Shannon entropy is more narrow than in the previous examples. Another consequence of increasing \( t \) is that ID-estimate becomes insufficient for states beyond a vicinity of the maximally mixed one. For pure states, the difference between LCh-estimate and ID-estimate is almost three times the difference between UCh-estimate and LCh-estimate. All five curves tend to coincide on the right. The two-sided estimate (3.19) gives better results for states with low mixedness. Otherwise, the two-sided estimate (3.19) is preferable. Distinctions between these estimates are less than in the previous examples.
Design-structured POVMs can be used for estimating the von Neumann entropy of a quantum state, viz.,

\[ H_1(\rho) := -\text{tr}(\rho \ln \rho). \]  

(3.26)

Let us restrict a consideration to a single assigned POVM that gives quantities of the form (3.12) for \( s = 2, \ldots, t \). Then the trace \( \text{tr}(\rho^2) \) is gained from \( \bar{\beta}(2) \), the trace \( \text{tr}(\rho^3) \) is gained from \( \bar{\beta}(3) \) and the known \( \text{tr}(\rho^2) \), and so on. In this way, we find such traces with integer powers of \( \rho \) up to \( \text{tr}(\rho^t) \). We can interpret (3.26) as the Shannon entropy calculated with eigenvalues of \( \rho \). It then follows from (2.31) and (2.32) that

\[
\sum_{s=1}^{t} a_t^{(s)} \Lambda^{1-s} \text{tr}(\rho^s) - \ln \Lambda \leq H_1(\rho) \leq b_t^{(0)} \Lambda d + \sum_{s=1}^{t} b_t^{(s)} \Lambda^{1-s} \text{tr}(\rho^s) - \ln \Lambda,
\]

(3.27)

\[
\sum_{s=1}^{t} \tilde{a}_t^{(s)} \Lambda^{1-s} \text{tr}(\rho^s) - \ln \Lambda \leq H_1(\rho) \leq \tilde{b}_t^{(0)} \Lambda d + \sum_{s=1}^{t} \tilde{b}_t^{(s)} \Lambda^{1-s} \text{tr}(\rho^s) - \ln \Lambda,
\]

(3.28)

where \( \Lambda = \Upsilon_{d-1}^{(t)}(\text{tr}(\rho^t)) \). Here, the quantity \( \Lambda \) estimates the eigenvalues of \( \rho \) from above. When \( t \geq d \), measurement statistics with many trials lead to determining eigenvalues of \( \rho \). Otherwise, only ranges of available values could be mentioned. Yet, the formulas (3.27) and (3.28) allow us to estimate promptly the von Neumann entropy of input state from both the sides. For example, Figure 6 shows the von Neumann entropy and its estimates for the qubit case with \( t = 5 \). The 5-design can be formed by icosahedron vertices marked on the Bloch sphere. Except for states with low mixedness, an accuracy of estimation is very good.

The above uncertainty relations for design-structured POVMs can be used to formulate quantum steering inequalities. This question was already addressed in [19, 20]. So, we will discuss steering inequalities very shortly. For more details, see the review [31] and references therein. Alice and Bob share a bipartite quantum state \( \rho_{AB} \) and repeat this any number of times [32]. Alice performs on her subsystem a measurement chosen from the set of POVMs \( \{ \mathcal{F}^{(m)} \}_{m=1}^{M} \). Hence, the actual state of Bob’s subsystem is conditioned on Alice’s result. This conditioned state is subjected to a measurement chosen accordingly from the set \( \{ \mathcal{E}^{(m)} \}_{m=1}^{M} \). The conditional entropies \( H_1(\mathcal{E}^{(m)}|\mathcal{F}^{(m)}) \) are calculated due to (2.42) with the use of generated probabilities and classical side information from Alice. The authors of [33] examined the question of how to derive quantum steering inequalities on the base of entropic uncertainty relations. One of the conditions implies that state-independent uncertainty relations should be used. Other conditions are clearly valid for the standard entropic functions. As was already mentioned, validity of the lower entropic bounds (3.21) and (3.22) for all states should be checked in each concrete example. If they actually provide a state-independent
formulation, then we have the steering inequalities

\[ \frac{1}{M} \sum_{m=1}^{M} H_1(\mathcal{E}^{(m)}|\mathcal{F}^{(m)}) \geq \sum_{s=1}^{t} a_{t}^{(s)}(Y \ell)^{1-s} d_s D_d^{(s)} - \ln Y, \]  

(3.29)

\[ \frac{1}{M} \sum_{m=1}^{M} H_1(\mathcal{E}^{(m)}|\mathcal{F}^{(m)}) \geq \sum_{s=1}^{t} \tilde{a}_{t}^{(s)}(Y \ell)^{1-s} d_s D_d^{(s)} - \ln Y, \]  

(3.30)

where \( Y = \min\{M Y^{(t)}_{K-1}(K^{1-t}d^t D^t_d), 1\} \). Violation of any steering inequalities means steerability of the tried state. It was previously discussed that the two-sided estimates (3.14) and (3.15) give better results in the case of single assigned POVM. Similarly, the same feature characterizes the steering inequalities (3.29) and (3.30).

IV. CONCLUDING REMARKS

We considered methods to derive two-sided estimates on the Shannon entropy by means of polynomial functions. The first way is based on truncated expansions of the Taylor type. It leads to results that are intuitively clear and easy to prove. The second way uses expansions with flexible coefficients. The significance of such expansions in applied analysis was emphasized by Lanczos [25]. As a result, we have arrived at a family of polynomials whose coefficients are connected with the shifted Chebyshev polynomials. Overall, this way provides more balanced estimates. On the other hand, the validity of obtained estimates is not obvious. It turned out that polynomials of moderate degree are often sufficient to reach enough good results. Any advances to reveal the nature of polynomial estimates with flexible coefficients will be welcome.

Recently, quantum designs are actively studied due to constraints of the form (3.9), and their corollaries are useful in quantum information processing. The derived two-sided estimates were applied to formulate uncertainty and certainty relations for design-structured POVMs. A quality of new complementarity relations is characterized by comparing with the lower entropic bounds following from information diagrams. In general, new relations lead to stronger inequalities in application to the case of a single assigned POVM. Also, an actual degree of the corresponding polynomial should not be very low. It is natural due to the increased number of restrictions imposed on generated probabilities. So, the proposed approach allows us to take into account several given indices of the form (2.4). It was also exemplified that further improvements seem to be achievable. Maybe information diagrams with more indices could be used for these purposes. However, such diagrams will be complicated to examine.
Appendix A: Some facts about Chebyshev polynomials

The Chebyshev polynomials of the first kind are defined in terms of the ordinary generating function (see, e.g., item 22.9.9 in Table 22.9 of [34])

$$\sum_{n=0}^{\infty} T_n(\xi)\tau^n = \frac{1 - \tau \xi}{1 - 2\tau \xi + \tau^2}. \tag{A1}$$

In the main text, we refer to the shifted Chebyshev polynomials defined as

$$T_n^*(x) = T_n(2x - 1), \tag{A2}$$

where $x \in [0, 1]$. The representation reads as (see, e.g., section VII.8 in [25])

$$T_n^*(x) = \sum_{s=0}^{n} c_n^{(s)} x^s, \quad c_n^{(s)} = (-1)^{n+s} 2^{2s-1} \left[ \frac{n+s}{n-s} - \left( \frac{n+s-1}{n-s} \right) \right]. \tag{A3}$$

In particular, we write the two coefficients

$$c_n^{(0)} = (-1)^n, \quad c_n^{(1)} = (-1)^{n+1} 2^n. \tag{A4}$$

Exact values of first coefficients can be found, e.g., in Table VII of the appendix of [25].

Appendix B: On the first derivative of (2.18)

Differentiating (2.18) with respect to $x$ and substituting $x = 1$, one has

$$g'_n(1) = \frac{(-1)^n}{2n^2} 2^{2n-2} \tau n \left[ -c_n^{(0)} - c_n^{(1)} \right] = \frac{(-1)^n - 1 + 2n^2}{2n^2}, \tag{B1}$$

where we used $T_n^*(1) = T_n(1) = 1$. The first derivative at the point $x = 0$ satisfies

$$(-1)^{n+1} 2^n g'_n(0) = \sum_{s=2}^{n} c_n^{(s)} s - 1 = \int_0^1 \frac{T_n^*(x) - c_n^{(0)} - c_n^{(1)} x}{x^2} \, dx. \tag{B2}$$

Multiplying (B2) by $\tau^n$ and summing with respect to $n$, we have

$$\int_0^1 \frac{1 - \tau \xi}{1 - 2\tau \xi + \tau^2} - \sum_{n=0}^{\infty} \left( c_n^{(0)} + c_n^{(1)} x \right) \tau^n \frac{dx}{x^2}, \tag{B3}$$

where $\xi = 2x - 1$. Substituting (A4), usual calculus allows us to rewrite (B3) as

$$\sum_{n=0}^{\infty} \tau^n \sum_{s=2}^{n} c_n^{(s)} s - 1 = \frac{4(\tau - \tau^2)}{(1 + \tau)^3} \ln \frac{1 + \tau}{1 - \tau}. \tag{B4}$$

The expansion of (B4) into power series starts with the term $\propto \tau^2$ and results in

$$\sum_{n=0}^{\infty} \tau^n \sum_{s=2}^{n} c_n^{(s)} s - 1 = (-8) \sum_{q=1}^{\infty} \frac{\tau^q}{q} \sum_{m=1}^{\infty} \frac{\tau^m}{m} \tag{B5}.$$

We further find the coefficient of $\tau^n$ in (B5). Combining this with (B2) leads to (2.19) and (2.20).

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