THE SUM OF TWO UNBOUNDED LINEAR OPERATORS: CLOSEDNESS, SELF-ADJOINTNESS AND NORMALITY

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Abstract. In the present paper we give results on the closedness and the self-adjointness of the sum of two unbounded operators. We present a new approach to these fundamental questions in operator theory. We also prove a new version of the Fuglede theorem where the operators involved are all unbounded. This new Fuglede theorem allows us to prove (under extra conditions) that the sum of two unbounded normal operators remains normal. Also a result on the normality of the unbounded product of two normal operators is obtained as a consequence of this new "Fuglede theorem". Some interesting examples are also given.

1. Introduction

We start with some standard notions and results about linear operators on a Hilbert space. We assume the reader is familiar with other results and definitions about linear operators. Some general references are [8, 9, 13, 15, 26, 28, 33].

All operators are assumed to be densely defined together with any operation involving them or their adjoints. Bounded operators are assumed to be defined on the whole Hilbert space.

If $A$ and $B$ are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then $B$ is called an extension of $A$, and we write $A \subset B$, if $D(A) \subset D(B)$ and if $A$ and $B$ coincide on $D(A)$. If $A \subset B$, then $B^* \subset A^*$.

The product $AB$ of two unbounded operators $A$ and $B$ is defined by

$$BA(x) = B(Ax) \text{ for } x \in D(BA)$$

where

$$D(BA) = \{ x \in D(A) : Ax \in D(B) \}.$$  

Since the expression $AB = BA$ will be often met, and in order to avoid possible confusions, we recall that by writing $AB = BA$, we mean that $ABx = BAx$ for all $x \in D(AB) = D(BA)$.

Recall that the unbounded operator $A$, defined on a Hilbert space $H$, is said to be invertible if there exists an everywhere defined (i.e. on the whole of $H$) bounded operator $B$ such that

$$BA \subset AB = I$$

where $I$ is the usual identity operator. This is the definition adopted in the present paper. It may be found in e.g. [2] or [3]. We insist on the inverse being defined everywhere since if it were not, that it is known from the literature that some of

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the results to be proved (for instance Corollary [H] may fail to hold. Of course, in some textbooks, they do not assume the inverse defined everywhere as in e.g. [L3].

An unbounded operator $A$ is said to be closed if its graph is closed; self-adjoint if $A = A^*$ (hence from known facts self-adjoint operators are automatically closed); normal if it is closed and $AA^* = A^*A$ (this implies that $D(AA^*) = D(A^*A)$).

The following lemma is standard (for a proof see eg. [L3]):

**Lemma 1.** Let $A$ be a densely defined operator. If $A$ is invertible, then it is closed.

When dealing with self-adjoint and normal operators, taking adjoints is compulsory, so we list some straightforward results about the adjoint of the sum and the product of unbounded operators.

**Theorem 1.** Let $A$ be an unbounded operator.

1. $(A + B)^* = A^* + B^*$ if $B$ is bounded, and $(A + B)^* \supset A^* + B^*$ if $B$ is unbounded.

2. $A + B$ is closed if $A$ is assumed to be closed and if $B$ is bounded.

3. $(BA)^* = A^*B^*$ if $B$ is bounded.

4. $A^*B^* \subset (BA)^*$ for any unbounded $B$ and if $BA$ is densely defined.

The following is also well-known

**Lemma 2.** The product $AB$ (in this order) of two closed operators is closed if one of the following occurs:

1. $A$ is invertible,

2. $B$ is bounded.

The following lemma is essential

**Lemma 3 ([L3]).** If $A$ and $B$ are densely defined and $A$ is invertible with inverse $A^{-1}$ in $B(H)$, then $(BA)^* = A^*B^*$.

We include a proof (not outlined in [L3]) to show the importance of assuming the inverse defined everywhere:

**Proof.** Since $A$ is invertible, $AA^{-1} = I$ (in [L3] we would have $AA^{-1} \subset I$ only) and hence

$$BAA^{-1} = B \implies (A^{-1})^* (BA)^* \subset [(BA)A^{-1}]^* \subset B^*.$$

But $(A^{-1})^* = (A^*)^{-1}$ and so

$$A^*(A^*)^{-1}(BA)^* = (BA)^* \subset A^*B^*.$$

Since we always have $A^*B^* \subset (BA)^*$ the result follows.

$\square$

It is worth noticing that similar papers on sums and products exist. The interested reader may look at [3], [7], [10], [11], [12], [14], [16], [17], [18], [19], [20], [21], [24], [22], [31] and [32], and further bibliography cited therein.

Let us briefly say a few words on how the paper is organized. In the main results section, we start by proving the first result on the closedness of the sum. Then a self-adjointness result is established. Also, it is proved that the adjoint of the sum is the sum of the adjoints. To treat the case of the normality of the sum of two normal operators, we prove a new version of the Fuglede theorem where the operators involved are unbounded. This last result can too be used to prove a result on the normality of the product of two unbounded normal operators.
2. Main Results

We start by the closedness of the sum. We have

**Theorem 2.** Let $A$ and $B$ be two unbounded operators such that $AB = BA$. If $A$ (for instance) is invertible, $B$ is closed and $D(BA^{-1}) \subset D(A)$, then $A + B$ is closed on $D(B)$.

**Proof.** First note that the domain of $A + B$ is $D(A) \cap D(B)$. But

$$AB = BA \implies A^{-1}B \subset BA^{-1} \implies D(B) = D(A^{-1}B) \subset D(BA^{-1}) \subset D(A),$$

hence the domain of $A + B$ is $D(B)$.

By Lemma 1, $A$ is automatically closed. Then we have

$$A + B = A + BAA^{-1} = A + ABA^{-1} = A(I + BA^{-1}) \text{ (since } D(BA^{-1}) \subset D(A)).$$

Since $A^{-1}$ is bounded (and $B$ is closed), $BA^{-1}$ is closed, hence by Theorem 1, $I + BA^{-1}$ is closed so that $A(I + BA^{-1})$ is also closed by Lemma 2 proving the closedness of $A + B$ on $D(B)$. \qed

**Remark.** Before going further in this paper and since conditions of the type $D(BA^{-1}) \subset D(A)$ will be often met, we give an example of a couple of two unbounded operators satisfying this latter condition. Let $A$ and $B$ be the two unbounded closed operators defined by

$$Af(x) = (x^2 + 1)^2 f(x) \text{ and } Bf(x) = (x^2 + 1)f(x)$$

on their respective domains

$$D(A) = \{f \in L^2(\mathbb{R}) : (x^2 + 1)^2 f \in L^2(\mathbb{R})\}$$

and

$$D(B) = \{f \in L^2(\mathbb{R}) : (x^2 + 1)f \in L^2(\mathbb{R})\}.$$

The operator $B$ is invertible with a bounded inverse given by

$$B^{-1}f(x) = \frac{1}{1 + x^2} f(x)$$

on the whole Hilbert space $L^2(\mathbb{R})$. Then

$$D(AB^{-1}) = \{f \in L^2(\mathbb{R}) : \frac{1}{1 + x^2} f \in L^2(\mathbb{R}), \frac{(1 + x^2)^2}{1 + x^2} f = (1 + x^2)f \in L^2(\mathbb{R})\} = D(B).$$

**Remark.** The hypothesis $D(BA^{-1}) \subset D(A)$ cannot merely be dropped. As a counterexample, let $A$ be any invertible closed operator with domain $D(A) \subsetneq H$ where $H$ is a complex Hilbert space. Let $B = -A$. Then $A + B = 0$ on $D(A)$ is not closed. Moreover, $AB = BA$ but

$$D(BA^{-1}) = D(AA^{-1}) = D(I) = H \not\subset D(A).$$

We now pass to the self-adjointness of the sum. We have

**Theorem 3** (cf. Lemma 4.15.1 in [26]). Let $A$ and $B$ be two unbounded self-adjoint operators such that $B$ (for instance) is also invertible. If $AB = BA$ and $D(AB^{-1}) \subset D(B)$, then $A + B$ is self-adjoint on $D(A)$. 


Proof. It is clear, thanks to \( D(A) = D(B^{-1}A) \subset D(AB^{-1}) \subset D(B) \), that the domain of \( A + B \) is \( D(A) \). We have
\[
B^{-1}BA + B \subset A + B
\]
evaluating at \( AB \) and hence
\[
(I + B^{-1}A)B \subset A + B.
\]
So we have
\[
(A + B)^* \subset [(I + B^{-1}A)B]^*
\]
\[
= B^*(I + B^{-1}A)^* \quad \text{ (by Lemma \ref{lemma3} since \( B \) is invertible)}
\]
\[
= B^*[I + (B^{-1}A)^*] \quad \text{ (by Theorem \ref{theorem1})}
\]
\[
= B^*[I + A^*(B^{-1})^*] \quad \text{ (since \( B^{-1} \) is bounded)}
\]
\[
= B(I + AB^{-1}) \quad \text{ (since \( A \) and \( B \) are self-adjoint)}
\]
\[
= B + BAB^{-1} \quad \text{ (for \( D(AB^{-1}) \subset D(B) \))}
\]
\[
= B + ABB^{-1}
\]
\[
= A + B.
\]
The following known fact
\[
A + B \subset (A + B)^*,
\]
then makes the "inclusion" an exact equality, establishing the self-adjointness of \( A + B \) on \( D(A) \).
\[\square\]

Remark. The condition \( D(AB^{-1}) \subset D(A) \) cannot just be dispensed with. As before, take \( A \) to be any unbounded and invertible self-adjoint operator and \( B = -A \).

Remark. Of course, writing \( AB = BA \) does not mean that \( A \) and \( B \) strongly commute, i.e. it does not signify that their spectral projections commute. See e.g. \cite{4, 6, 25, 27} and \cite{30}.

However, a result by Devinatz-Nussbaum-von Neumann (see \cite{4}) shows that if there exists a self-adjoint operator \( T \) such that \( T \subseteq T_1T_2 \), where \( T_1 \) and \( T_2 \) are self-adjoint, then \( T_1 \) and \( T_2 \) strongly commute. Thus we have

Proposition 1. Let \( A \) and \( B \) be two unbounded self-adjoint operators such that \( B \) (for instance) is also invertible. If \( AB = BA \), then \( A \) and \( B \) strongly commute.

Proof. By Lemma \ref{lemma3} we may write
\[
(AB)^* = B^*A^* = BA = AB,
\]
i.e. \( AB \) is self-adjoint. By the Devinatz-Nussbaum-von Neumann theorem, \( A \) and \( B \) strongly commute. \[\square\]

We can also give a result on the adjoint of the sum of two closed operators. This generalizes the previous one as we will be explaining in a remark below its proof. Besides it will be useful in the case of the sum of two normal operators. We have

Theorem 4. Let \( A \) and \( B \) be two unbounded invertible operators such that \( AB = BA \). If \( D(A^*(B^*)^{-1}) \subset D(B^*) \), then
\[
(A + B)^* = A^* + B^*.
\]
Proof. The idea of proof is akin to that of Theorem 3. First, we always have
\[ A^* + B^* \subset (A + B)^*. \]
Second, since \( A \) and \( B \) are both invertible, by Lemma 3 we have
\[ AB = BA \implies A^*B^* = B^*A^*. \]
Now write
\[ B^{-1}BA + B \subset A + B \]
as \( AB \)
so that
\[ (I + B^{-1}A)B \subset A + B \]
and hence
\[ (A + B)^* \subset (I + B^{-1}A)B]^* \]
\[ = B^*(I + B^{-1}A)^* \] (by Lemma 3 since \( B \) is invertible)
\[ = B^*[I + (B^{-1}A)^*] \] (by Theorem 11)
\[ = B^*[I + A^*(B^{-1})^*] \] (since \( B^{-1} \) is bounded)
\[ = B^* + B^*A^*(B^{-1})^{-1} \] (as \( D(A^*(B^*)^{-1}) \subset D(B^*) \))
\[ = B^* + A^*B^*(B^{-1})^{-1} \] (for \( A^*B^* = B^*A^* \))
\[ = A^* + B^*. \]
The proof is therefore complete. \( \square \)

Remark. We could have supposed that only \( B \) is invertible, but then we would have added the hypothesis \( B^*A^* \subset A^*B^* \). This latter observation tells us that Theorem 4 generalizes in fact Theorem 3.

Remark. The condition \( D(A^*(B^*)^{-1}) \subset D(B^*) \) cannot just be dispensed with. As before, take \( A \) to be any unbounded closed and invertible operator and \( B = -A \). Then \( D(A^*(B^*)^{-1}) \subset D(B^*) \) is not satisfied. At the same time observe that
\[ D(A^* - A^*) \neq D((A - A)^*) = D(0^*) = H, \]
where \( H \) is the whole Hilbert space.

In [20], we proved the following result

**Theorem 5.** Let \( A \) and \( B \) be two unbounded normal operators such that \( B \) is \( A \)-bounded with relative bound smaller than one. Assume that \( BA^* \subset A^*B \) and \( B^*A \subset AB^* \). Then \( A + B \) is normal on \( D(A) \).

To prove it, we had to use a theorem by Hess-Kato (see [11]), mainly for the closedness of \( A + B \) and to have \( (A + B)^* = A^* + B^* \). Thanks to Theorems 2 & 4 we may avoid the use of that theorem. Besides we are able here to prove a new version of the Fuglede theorem where all operators involved are unbounded which will allow us to establish the normality of the sum of two normal operators. We digress a bit to say that another all unbounded-operator-version of Fuglede-Putnam is the Fuglede-Putnam-Mortad theorem that may be found in [23], cf. [?, ?].

Here is the promised result
Theorem 6. Let $A$ and $B$ be two unbounded normal and invertible operators. Then
$$AB = BA \implies AB^* = B^*A \text{ and } BA^* = A^*B.$$ 

Proof. Since $B$ is invertible, we may write
$$AB = BA \implies B^{-1}A \subset AB^{-1}.$$ 

Since $B^{-1}$ is bounded and $A$ is unbounded and normal, by the classic Fuglede theorem we have
$$B^{-1}A \subset B^{-1}A^* \subset A^*B^{-1}.$$ 

Therefore,
$$A^*B \subset BA^*.$$ 

But $B$ is invertible, then by Lemma 5 we may obtain
$$AB^* \subset (BA^*)^* \subset (A^*B)^* = B^*A.$$ 

Interchanging the roles of $A$ and $B$, we shall get
$$B^*A \subset AB^* \text{ and } BA^* \subset A^*B.$$ 

Thus
$$B^*A = AB^* \text{ and } BA^* = A^*B.$$ 

□

Remark. A similar result holds with one operator assumed normal. The key point again is that the inverse is bounded and everywhere defined. So since $AB = BA$, we obtain $A^{-1}B^{-1} \subset B^{-1}A^{-1}$. Since these operators are everywhere defined, we get $A^{-1}B^{-1} = B^{-1}A^{-1}$. The rest follows by the bounded version of Fuglede theorem.

However, if we do not assume the bounded inverse defined everywhere, then $AB = BA$ does not imply that $A^{-1}B^{-1} = B^{-1}A^{-1}$. Here is a counterexample which appeared in [30]. It reads:

Let $S$ be the unilateral shift on the Hilbert space $\ell^2$. We may then easily show that both $S+S^*$ and $S-S^*$ are injective. Hence $A = (S+S^*)^{-1}$ and $B = i(S-S^*)^{-1}$ are unbounded self-adjoint such that $AB = BA$. Nonetheless
$$A^{-1}B^{-1} \neq B^{-1}A^{-1}$$
since otherwise $S$ and $S^*$ would commute!

As a first consequence of Theorem 4 we have

Theorem 7. Let $A$ and $B$ be two unbounded invertible normal operators with domains $D(A)$ and $D(B)$ respectively. If $AB = BA$, $D(A) \subset D(BA^*)$ and $D(AB^{-1}) = D(A(B^*)^{-1}) \subset D(B)$, then $A + B$ is normal on $D(A)$.

Proof. To prove that $A + B$ is normal, we must say why $A + B$ is closed and show that
$$(A + B)^*(A + B) = (A + B)(A + B)^*.$$ 

$A + B$ is closed thanks to Theorem 2. Also, since $A$ and $B$ are both normal, we obviously have
$$D(A(B^*)^{-1}) \subset D(B) \implies D(A^*(B^*)^{-1}) \subset D(B^*)$$
so that Theorem 3 applies and yields
$$(A + B)^* = A^* + B^*.$$
Since $AB = BA$, Theorem 6 implies that

$$AB^* = B^*A \text{ and } BA^* = A^*B.$$ 

Since $D(A) \subset D(BA^*)$, we have

$$D(A) \subset D(A^*B) (\subset D(B))$$

and

$$D(A) \subset D(BA^*) = D(A^*B) = D(AB) = D(BA) = D(B^*A).$$

All these domain inclusions allow us to have

1. $A^*(A + B) = A^*A + A^*B$ and $B^*(A + B) = B^*A + B^*B$.
2. $A(A^* + B^*) = AA^* + AB^*$ and $B(A^* + B^*) = BA^* + BB^*$.

Hence we may write

$$(A + B)^*(A + B) = A^*A + A^*B + B^*A + B^*B$$

and

$$(A + B)(A + B)^* = AA^* + AB^* + BA^* + BB^*.$$ 

Thus

$$(A + B)^*(A + B) = (A + B)(A + B)^*.$$ 

□

As another consequence of Theorem 6, we have the following result on the normality of the product of two unbounded normal operators.

**Corollary 1.** Let $A$ and $B$ be two unbounded invertible normal operators. If $BA = AB$, then $BA$ (and $AB$) is normal.

**Remark.** By a result of Devinatz-Nussbaum (see [5]), if $A$, $B$ and $N$ are normal where $N = AB = BA$, then $A$ and $B$ strongly commute. Hence we have as a consequence:

**Corollary 2.** Let $A$ and $B$ be two unbounded invertible normal operators. If $BA = AB$, then $A$ and $B$ strongly commute.

Now we prove Corollary 1.

**Proof.** We first note that $BA$ is closed thanks to Lemma 2 (or simply since $BA$ is invertible hence Lemma 1 applies!), hence so is $AB$. Lemma 3 then gives us $(AB)^* = B^*A^*$. By Theorem 6 we may then write

$$(AB)^*AB = B^*A^*AB$$

$$= B^*AA^*B$$

$$= AB^*A^*B$$

$$= AB^*BA^*$$

$$= ABB^*A^*$$

$$= (AB)(AB)^*,$$

establishing the normality of $AB$. □
Remark. In [22] we had the same result with the extra conditions $D(A), D(B) \subset D(BA)$. Here we have showed that the last two conditions are not essential. Hence Corollary 1 is an improvement of the result that appeared in [22].

Remark. Let us give an example that shows the importance of assuming $A$ and $B$ invertible. Let $B$ be the operator defined by

$$Bf(x) = -xf'(x) - f(x)$$

on its domain

$$D(B) = \{ f \in L^2(\mathbb{R}) : xf' \in L^2(\mathbb{R}) \}$$

where the derivative is taken in the distributional sense. Then $B$ is normal (it is in fact the adjoint of the operator defined by $xf'(x)$ on the domain $\{ f \in L^2(\mathbb{R}) : xf' \in L^2(\mathbb{R}) \}$). Set $A = B + I$ hence

$$Af(x) = -xf'(x)$$

on

$$D(A) = \{ f \in L^2(\mathbb{R}) : xf' \in L^2(\mathbb{R}) \}.$$

We may then easily check that

$$ABf(x) = BAf(x) = x^2 f''(x)$$

on their common domain

$$D(B^2) = \{ f \in L^2(\mathbb{R}) : xf', x^2 f'' \in L^2(\mathbb{R}) \}.$$

Hence $AB$ and $BA$ are not closed, hence they are not normal.

Now, proceeding as in [18] (where a similar operator was dealt with) we may show that via a form of the Mellin transform that $B$ is unitary equivalent to the multiplication operator $M$ defined by

$$Mf(x) = (x + \frac{1}{2}i)f(x)$$

on its domain

$$D(M) = \{ f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R}) \}.$$

But $M$ is known to be non invertible, so neither is $B$ nor is $A$.

3. Conclusion

Lemma 3 has played a very important role in the proofs of most of the results in the present paper. Of course, we could have used other similar known results in the literature. See for instance [11] and [29]. Theorem 6 also played an important role in the proof of Theorem 7. We could have also used the Fuglede-Putnam-Mortad theorem which appeared in [23]. We also think that Theorem 6 should have other applications somewhere else.

References

[1] J. v. Casteren and S. Goldberg, *The conjugate of the product of operators*, Studia Mathematica 38 (1970), 125-130.

[2] J. B. Conway, *A Course in Functional Analysis*, Springer, 1990 (2nd edition).

[3] A. van Daele, *On Pairs of Closed Operators*, Bull. Soc. Math. Belg. Ser. B 34 (1982), no. 1, 25-40.

[4] A. Devinatz, A. E. Nussbaum, J. von Neumann, On the Permutability of Self-adjoint Operators, *Ann. of Math. (2)*, 62 (1955), 199-203.
[5] A. Devinatz, A. E. Nussbaum, On the Permutability of Normal Operators, Ann. of Math. (2), 65 (1957), 144-152.
[6] B. Fuglede, Conditions for Two Selfadjoint Operators to Commute or to Satisfy the Weyl Relation, Math. Scand., 51/1 (1982), 163-178.
[7] F. Gesztesy, J. A. Goldstein, H. Holden, G. Teschl, Abstract Wave Equations and Associated Dirac-Type Operators, Annali di Matematica. DOI 10.1007/s10231-011-0200-7.
[8] I. Gohberg, S. Goldberg, M. A. Kaashoek, Basic Classes of Linear Operators, Birkhäuser Verlag, Basel, 2003.
[9] S. Goldberg, Unbounded Linear Operators, McGraw-Hill, 1966.
[10] K. Gustafson, On operator sum and product adjoints and closures, Canad. Math. Bull. 54 (2011), 456-463.
[11] P. Hess, T. Kato, Perturbation of Closed Operators and Their Adjoints, Comment. Math. Helv., 45 (1970) 524-529.
[12] P. E. T. Jorgensen, Unbounded Operators: Perturbations and Commutativity Problems, J. Funct. Anal. 39/3 (1980) 281-307.
[13] T. Kato, Perturbation Theory for Linear Operators, 2nd Edition, Springer, 1980.
[14] H. Kosaki, On Intersections of Domains of Unbounded Positive Operators, Kyushu J. Math., 60/1 (2006) 3-25.
[15] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxford G.T.M. 2, Oxford University Press 1997.
[16] B. Messirdi, M. H. Mortad, A. Azzouz, G. Djellouli, A Topological Characterization of the Product of Two Closed Operators, Colloq. Math., 112/2 (2008) 269-278.
[17] M. H. Mortad, An Application of the Putnam-Fuglede Theorem to Normal Products of Self-adjoint Operators, Proc. Amer. Math. Soc. 131 (2003), 3135-3141.
[18] M. H. Mortad, On some product of two unbounded self-adjoint operators, Integral Equations Operator Theory 64 (2009), 399-408.
[19] M. H. Mortad, On the Adjoint and the Closure of the Sum of Two Unbounded Operators, Canad. Math. Bull., 54/3 (2011) 498-505. DOI:10.4153/CMB-2011-041-7.
[20] M. H. Mortad, On the Normality of the Sum of Two Normal Operators, Complex Anal. Oper. Theory, 6/1 (2012), 105-112. DOI: 10.1007/s11785-010-0072-7.
[21] M. H. Mortad, On the closedness, the self-adjointness and the normality of the product of two unbounded operators, Demonstratio Math., 45/1 (2012), 161-167.
[22] M. H. Mortad, Kh. Madani, More on the Normality of the Unbounded Product of Two Normal Operators, Rend. Semin. Mat. Univ. Politec. Torino, (to appear). arXiv:1202.6142v1.
[23] M. H. Mortad, An All-Unbounded-Operator Version of the Fuglede-Putnam Theorem, Complex Anal. Oper. Theory, (to appear). DOI: 10.1007/s11785-011-0133-6.
[24] M. H. Mortad, Products of Unbounded Normal Operators, (submitted). arXiv:1202.6143v1.
[25] E. Nelson, Analytic vectors, Ann. of Math. (2), 70 (1959) 572-615.
[26] C. Putnam, Commutation Properties of Hilbert Space Operators, Springer, 1967.
[27] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Vol.1, Functional Analysis, Academic Press, 1972.
[28] W. Rudin, Functional Analysis, McGraw-Hill, 1991 (2nd edition).
[29] M. Schechter, The Conjugate of a Product of Operators, J. Functional Analysis 6 (1970), 26-28.
[30] K. Schmüdgen, J. Friedrich, On commuting unbounded selfadjoint operators II, Integral Equations Operator Theory, 7/6 (1984), 815-867.
[31] Z. Sebestyén, J. Stochel, On Products of Unbounded Operators, Acta Math. Hungar. 100/1-2 (2003) 105-129.
[32] F. H. Vasilescu, Anticommuting Selfadjoint Operators, Rev. Roumaine Math. Pures Appl. 28/1 (1983), 76-91.
[33] J. Weidmann, Linear operators in Hilbert spaces (translated from the German by J. Szücs), Springer-Verlag, GTM 68 (1980).
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