Singular asymptotics for the Clarkson-McLeod solutions of the fourth Painlevé equation

Jun Xia*, Shuai-Xia Xu† and Yu-Qiu Zhao*

Abstract

We consider the Clarkson-McLeod solutions of the fourth Painlevé equation. This family of solutions behave like \( \kappa D_{\alpha - \frac{1}{2}}^{\frac{1}{2}}(\sqrt{2}x) \) as \( x \to +\infty \), where \( \kappa \) is an arbitrary real constant and \( D_{\alpha - \frac{1}{2}}(x) \) is the parabolic cylinder function. Using the Deift-Zhou nonlinear steepest descent method, we obtain the singular asymptotics of the solutions as \( x \to -\infty \) when \( \kappa (\kappa - \kappa^*) > 0 \) for some real constant \( \kappa^* \). The connection formulas are also explicitly evaluated. This proves and extends Clarkson and McLeod’s conjecture that when the parameter \( \kappa > \kappa^* > 0 \), the Clarkson-McLeod solutions have infinitely many simple poles on the negative real axis.

2020 mathematics subject classification: 30E15; 33E17; 34E05; 41A60
Keywords and phrases: The fourth Painlevé equation; Clarkson-McLeod solutions; singular asymptotics; connection formulas; Riemann-Hilbert problems; Deift-Zhou nonlinear steepest descent method

1 Introduction and statement of results

We study the asymptotics of the solutions \( q(x) \) of the fourth Painlevé (PIV, [6, 10]) equation

\[
\frac{d^2 q}{dx^2} = \frac{1}{2q} \left( \frac{dq}{dx} \right)^2 + \frac{3}{2} q^3 + 4xq^2 + (2x^2 - 4\alpha + \beta) q - \frac{\beta^2}{2q},
\]

(1.1)

with the parameters \( \alpha \in \mathbb{R}, \beta = 0 \) and satisfying the boundary condition

\[
q(x) \to 0 \quad \text{as} \quad x \to +\infty.
\]

(1.2)

In the pioneering works of Clarkson and McLeod [4] and Bassom et al. [2], it is proven that any real solution of (1.1) satisfying the boundary condition (1.2) has the following asymptotic behavior

\[
q(x; \kappa) \sim \kappa D_{\alpha - \frac{1}{2}}^{\frac{1}{2}}(\sqrt{2}x), \quad x \to +\infty
\]

(1.3)

for some constant \( \kappa \), where \( D_{\nu}(x) \) is the parabolic cylinder function with order \( \nu \); cf. [10] Chapter 12. Conversely, for any real constant \( \kappa \), there exists a unique solution of (1.1) asymptotic to \( \kappa D_{\alpha - \frac{1}{2}}^{\frac{1}{2}}(\sqrt{2}x) \) as \( x \to +\infty \). These solutions \( q(x; \kappa) \) are now known as the Clarkson-McLeod solutions of the fourth Painlevé equation. It is worth mentioning that a parameter \( k \) is used in [2, 4] such that \( \kappa = 2^{3/2}k^2 \).

For the asymptotics of the Clarkson-McLeod solutions as \( x \to -\infty \), there has been the following conjecture.

\*Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China.

†Institut Franco-Chinois de l’Energie Nucléaire, Sun Yat-sen University, Guangzhou 510275, China.
Conjecture (Clarkson-McLeod [4])

There exists a constant $\kappa^* > 0$ such that:

(a) When $0 < \kappa < \kappa^*$, as $x \to -\infty$,

$$q(x; \kappa) \sim c_n 2^{\alpha + 1} x^{2\alpha - 1} e^{-x^2}$$

(1.4)

if $\alpha - \frac{1}{2} = n \in \mathbb{N}$, and

$$q(x; \kappa) \sim -\frac{2\pi}{3} + (-1)^{[\alpha + \frac{1}{2}]} \frac{4d}{\sqrt{3}} \sin \left( \frac{x^2}{\sqrt{3}} - \frac{4d^2}{\sqrt{3}} \ln(-\sqrt{2}x) + a + O \left( \frac{1}{x^2} \right) \right) + O \left( \frac{1}{x} \right)$$

(1.5)

if $\alpha - \frac{1}{2} \notin \mathbb{Z}$, where the constants $c_n$, $d$, $a$ are dependent on $\kappa$.

(b) When $\kappa = \kappa^*$, $q(x; \kappa)$ behaves like $-2x$ as $x \to -\infty$.

(c) When $\kappa > \kappa^*$, $q(x; \kappa)$ has a pole on the negative real axis.

In case (a), the asymptotic formula (1.4) has been proven in [2, 4] for $\alpha - \frac{1}{2} = n \in \mathbb{N}$, and the values of $c_n$ and $\kappa^*$ were explicitly evaluated as

$$c_n = \frac{\kappa}{2\sqrt{2} - 2\sqrt{2\pi} n! \kappa}, \quad \kappa^* = \frac{1}{\sqrt{\pi} n!}.$$  

(1.6)

If $\alpha - \frac{1}{2} \notin \mathbb{Z}$, the value of $\kappa^*$ was conjectured to be

$$\kappa^* = \frac{1}{\sqrt{\pi} \Gamma \left( \frac{\alpha + \frac{1}{2}}{2} \right)}.$$  

The asymptotic formula (1.5) was later justified by Abdullayev [1] using the integral equation method and by Its and Kapaev [9] via the isomonodromy method, respectively. The connection formulas for the dependence on $\kappa$ of $a$ and $d$ in (1.5) were explicitly evaluated in [9, 13]. However, to the best of our knowledge, the asymptotics of $q(x; \kappa)$ as $x \to -\infty$ in cases (b) and (c) are still to be explored.

The present paper is devoted to the studies of the asymptotics as $x \to -\infty$ of the Clarkson-McLeod solutions corresponding to case (c) of the Clarkson-McLeod conjecture. We derive the singular asymptotics for this family of Clarkson-McLeod solutions as $x \to -\infty$ with explicit expressions of the connection formulas.

**Theorem 1.1.** Assume that $\alpha \in \mathbb{R}$, $\alpha - \frac{1}{2} \notin \mathbb{Z}$, $\beta = 0$, $\kappa^*$ be given by (1.6), and let $q(x; \kappa)$ be a real solution of (1.1) satisfying the asymptotic behavior (1.3) as $x \to +\infty$ with real parameter $\kappa$ such that $\kappa (\kappa - \kappa^*) > 0$, then $q(x; \kappa)$ has the following asymptotic behavior as $x \to -\infty$

$$q(x; \kappa) = -\frac{2}{3} x + \frac{2x}{2 \cos \left( \frac{\sqrt{3}}{3} x^2 - b \ln \left( 2\sqrt{3}x^2 \right) + \psi \right) + 1} + O \left( \frac{1}{x} \right),$$

(1.7)

where

$$\begin{cases} 
 b = -\frac{1}{2\pi} \ln(|\rho|^2 - 1), \\
 \psi = -\frac{2\pi}{3} \alpha - \arg \Gamma \left( -bi + \frac{1}{2} \right) - \arg \rho,
\end{cases}$$

(1.8)

and the connection between $\rho$ and $\kappa$ is given by

$$\rho = 1 - \frac{2\pi^3}{e^{2\pi \alpha} \Gamma \left( \frac{1}{2} - \alpha \right)} \kappa.$$  

(1.9)

The error term in the asymptotic expansion is uniform for $x$ bounded away from the singularities appearing on the right-hand side of (1.7).
Remark 1.2. The existence of a family of solutions satisfying the asymptotic behavior \( (1.3) \) was first established by Bassom \textit{et al.} in \cite{2} for the case \( \kappa > 0 \) and subsequently proven by Its and Kapaev in \cite{9} for general \( \kappa \in \mathbb{R} \) with \( \kappa \neq 0 \). The remaining case \( \kappa = 0 \) is corresponding to the trivial solution \( q(x; 0) = 0 \).

From the asymptotic formula \( (1.7) \), we obtain the following asymptotic approximation of the location of the large negative poles of the solution \( q(x; \kappa) \).

**Corollary 1.3.** Under the assumptions of Theorem 1.1, the solution \( q(x; \kappa) \) has infinitely many simple poles on the negative real axis. Moreover, we have the following asymptotic approximation of the locations of large negative poles of \( q(x; \kappa) \)

\[
a_n^\pm = -(2\pi)^{\frac{1}{2}} \left[ \sqrt{n} + \frac{\ln n}{4\pi \sqrt{n}} + \frac{\ln(12\pi) - \psi \pm \frac{2\pi}{3}}{4\pi \sqrt{n}} + O\left(\frac{\ln^2 n}{n^{3/2}}\right) \right], \quad n \to \infty, \tag{1.10}
\]

with \( b \) and \( \psi \) given in \( (1.8) \).

Let us consider the case \( \kappa^* > 0 \). For \( \kappa > \kappa^* \), the existence of a negative pole of the solution \( q(x; \kappa) \) of the fourth Painlevé equation \( (1.1) \) satisfying the asymptotic behavior \( (1.3) \) was first conjectured by Clarkson and McLeod \cite{4}. For \( \kappa > \kappa^* \) and \( \kappa < 0 \), it was shown numerically by Reeger and Fornberg in \cite{11} Figures 5 and 7] that when \( \alpha = 0 \), the solution \( q(x; \kappa) \) have infinitely many poles on the negative real axis. Corollary 1.3 rigorously confirms and extends both the conjecture of Clarkson and McLeod, and the numerical results of Reeger and Fornberg, in that there are infinitely many poles of \( q(x; \kappa) \) on the negative real \( x \)-axis for general parameter \( \alpha \).

Further numerical analysis of \( (1.1) \) is worthwhile to demonstrate the accuracy of the asymptotic results. Analogous to \( (1.7) \), singular asymptotics of a family of solutions of homogeneous and inhomogeneous second Painlevé equation have been established earlier in \cite{9} and \cite{8}, respectively.

In Theorem 1.1, the restriction \( \alpha - \frac{1}{2} \not\in \mathbb{Z} \) has been brought in for technical reasons; cf. \cite{2,10}. However, the asymptotics of the Clarkson-McLeod solution \( q(x; \kappa) \) as \( x \to -\infty \) for real \( \kappa \) such that \( \kappa (\kappa - \kappa^*) \leq 0 \), along with the exceptional case \( \alpha - \frac{1}{2} \in \mathbb{Z} \), can also be derived by using the Deift-Zhou nonlinear steepest descent method and we will report those results elsewhere.

The rest of the present paper is arranged as follows. In Section 2, we recall the Riemann-Hilbert (RH) problem for the PIV equation \( (1.1) \). The nonlinear steepest descent analysis of the RH problem are performed in Section 3. The main results will then be proved in the final section, Section 4. Proof of Theorem 1.1 will be given in Section 4.1 and proof of Corollary 1.3 in Section 4.2. For the convenience of the reader, we collect in the Appendix the Airy, Bessel and parabolic cylinder parametrices used in the RH analysis.

## 2 Riemann-Hilbert problem for PIV equation

We recall the RH problem for the PIV equation \( (1.1) \) in this section. More details can be found in \cite{9} Section 2] and \cite{3} Chapter 5.1].

### RH problem for PIV

Let \( \Sigma = \cup_{k=1}^{8} \gamma_k \), where \( \gamma_k = \{ \xi \in \mathbb{C} : \arg \xi = k\pi/4 \} \). Then, \( \Psi(\xi) := \Psi(\xi; x) \) satisfies the following RH problem.

1. \( \Psi(\xi) \) is analytic for \( \xi \in \mathbb{C} \setminus \Sigma \), where \( \Sigma \) is illustrated in Figure 1.
(2) $\Psi(\xi)$ satisfies the jump conditions

$$\Psi_+(\xi) = \Psi_-(\xi) \begin{cases} S_k, & \xi \in \gamma_k, \ k = 1, \cdots, 7, \\ S_8e^{-2\pi i(\alpha-\beta)\sigma_3}, & \xi \in \gamma_8, \end{cases}$$

where $\Psi_+$ and $\Psi_-$ denote the limits of the function $\Psi$ on the ray $\gamma_k$ from the left and the right hand side, respectively. Here $\sigma_3$ is one of the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The Stokes matrices $S_k$’s are of the form

$$S_{2j-1} = \begin{pmatrix} 1 & s_{2j-1} \\ 0 & 1 \end{pmatrix}, \quad S_{2j} = \begin{pmatrix} 1 & 0 \\ s_{2j} & 1 \end{pmatrix}, \quad j = 1, 2, 3, 4.$$ 

The constants $s_k$’s are known as the Stokes multipliers. They are constrained by

$$s_{k+4} = -s_k e^{(-1)^k2\pi i(\alpha-\beta)}, \quad k = 1, 2, 3, 4, \quad \text{ (2.1)}$$

$$[(1 + s_1 s_2)(1 + s_3 s_4) + s_1 s_4] e^{-\pi i(\alpha-\beta)} - (1 + s_2 s_3) e^{\pi i(\alpha-\beta)} = -2i \sin \pi \alpha. \quad \text{ (2.2)}$$

(3) $\Psi(\xi)$ satisfies the following asymptotic behavior as $\xi \to \infty$

$$\Psi(\xi) = \left( I + \frac{\Psi_1}{\xi} + O\left(\frac{1}{\xi^2}\right) \right) e^{\theta(\xi)\sigma_3}, \quad \theta(\xi) = \frac{1}{8} \xi^4 + \frac{1}{2} s^2 \xi^2 + (\alpha - \beta) \ln \xi, \quad \text{ (2.3)}$$

where the branch of $\ln \xi$ is chosen such that $\arg \xi \in (0, 2\pi)$.

(4) $\Psi(\xi)$ has the following asymptotic behavior near $\xi = 0$

$$\Psi(\xi) = \Psi_0(\xi)\xi^{\alpha\sigma_3} E \quad \text{ (2.4)}$$

for the generic case $\alpha - \frac{1}{2} \notin \mathbb{Z}$, where $\Psi_0(\xi)$ is analytic and invertible, the branch of $\xi^\alpha$ is chosen such that $\arg \xi \in (0, 2\pi)$ and

$$E = \begin{cases} E_0, & \det E_0 = 1, \quad \xi \in \Omega_1, \\ E_0S_1 \cdots S_{k-1}, & \xi \in \Omega_k, \quad k = 2, \cdots, 8. \end{cases} \quad \text{ (2.5)}$$
The solution \( q(x) \) of the PIV equation (1.1) is then determined by the solution to the above RH problem for \( \Psi(\xi) \) via the formula

\[
q(x) = (\Psi_1)_12(\Psi_1)_{21},
\]

(2.6)

where \( \Psi_1 = \Psi_1(x) \) is the coefficient in (2.3) and \( M_{ij} \) denotes the \((i,j)\)-th entry of a matrix \( M \).

The above RH formulation is valid no matter \( \beta = 0 \) or not. Nevertheless, if \( \beta \neq 0 \), the PIV equation (1.1) possesses no Clarkson-McLeod solutions. We focus on the asymptotic analysis of the Clarkson-McLeod solutions from now on, and consider the case \( \beta = 0 \).

From Its and Kapaev [9, Equation (2.23)], it is seen that for any real solution of (1.1), the Stokes multipliers must satisfy the conditions

\[
\bar{s}_0 = s_0, \quad \bar{s}_1 = -s_3 e^{2\pi i \alpha}.
\]

(2.7)

For any real solution satisfying asymptotic behavior (1.3), it follows from [9, Equation (2.42)] that the associated Stokes multipliers further fulfill the following conditions

\[
s_2 = 0, \quad s_1 + s_3 = 0, \quad s_* \neq 1, \quad (1 - s_*) e^{\pi i \alpha} \in \mathbb{R},
\]

(2.8)

where \( s_* \) is defined by

\[
s_* = 1 + s_0 s_1.
\]

(2.9)

The conditions \( s_2 = 0 \) and \( s_1 + s_3 = 0 \) in (2.8) imply that the Stokes matrices satisfy

\[
S_2 = S_6 = I, \quad S_1 = S_3^{-1}.
\]

Moreover, from [9, Equation (3.15)], the connection matrix \( E_0 \) takes the form

\[
E_0 = p^{\sigma_3} \left( \begin{array}{cc}
1 & 0 \\
\frac{s_0 e^{2\pi i \alpha}}{e^{2\pi i \alpha} + 1} & 1
\end{array} \right),
\]

(2.10)

where \( p \) is an arbitrary nonzero constant.

In view of [9, Equation (3.42)], we have the following explicit relation between the parameter \( \kappa \) in (1.3) and the composite Stokes multiplier \( s_* \)

\[
\kappa = \frac{e^{\pi i \alpha} \Gamma \left( \frac{1}{2} - \alpha \right)}{2\pi i} (1 - s_*).
\]

(2.11)

The relation (2.11), together with (1.6), (2.7), and (2.8), implies that the conditions on \( \kappa \) can be equivalently expressed in terms of the composite Stokes multiplier \( s_* \) as shown in the following table:

| \( s_* \) | \( \kappa > 0 \) | \( \kappa < 0 \) |
|---|---|---|
| \( |s_*| < 1 \) | \( 0 < \kappa < \kappa^* \) | \( \kappa^* < \kappa < 0 \) |
| \( |s_*| = 1, \ s_* \neq 1 \) | \( \kappa = \kappa^* \) | \( \kappa = \kappa^* \) |
| \( |s_*| > 1 \) | \( \kappa < 0 \) or \( \kappa > \kappa^* \) | \( \kappa < \kappa^* \) or \( \kappa > 0 \) |

Table 1: The correspondence between \( s_* \) and \( \kappa \)
3 Nonlinear steepest descent analysis

In this section, we consider the case $|s_*| > 1$. We shall perform the Deift-Zhou nonlinear steepest descent analysis of the RH problem $\Psi$ for the PIV equation (1.1) as $x \to -\infty$.

Assume now that $x < 0$. We begin with the following re-scaling transformation

$$\Phi(z) = (-x)^{-\frac{3}{2}} \Psi \left( (-x)^{\frac{3}{2}} z; x \right).$$

As a result, $\Phi(z)$ satisfies the following RH problem.

**RH problem for $\Phi(z)$**

1. $\Phi(z)$ is analytic for $z \in \mathbb{C} \setminus \{ \Sigma \setminus (\gamma_2 \cup \gamma_6) \}$; cf. Figure 1 for the contour.

2. $\Phi(z)$ fulfills the following jump relations

$$\Phi_+ = \Phi_- \begin{cases} S_k, & z \in \gamma_k, \ k = 1, 3, 4, 5, 7, \\ S_8 e^{-2\pi i \sigma_3}, & z \in \gamma_8. \end{cases}$$

3. At infinity, $\Phi(z)$ has the following asymptotic behavior

$$\Phi(z) = \left( I + \frac{\Phi_1}{z} + O \left( \frac{1}{z^2} \right) \right) z^{\alpha \sigma_3} e^{\frac{x^2}{8} \left( z^4 - 4z^2 \right) \sigma_3},$$

where the branch of $z^{\alpha}$ is chosen such that $\arg z \in (0, 2\pi)$.

4. $\Phi(z)$ has the same asymptotic behavior as $\Psi(z)$ at $z = 0$; see (2.4) and (2.5).

Simultaneously, it follows from (2.6) and (3.1) that

$$q(x; \kappa) = -x (\Phi_1)_{12} (\Phi_1)_{21},$$

where $\Phi_1 = \Phi_1(x)$ is the coefficient in the expansion (3.2).

3.1 Normalization and deformations of the jump curves

To normalize the asymptotic behavior of $\Phi(z)$ at infinity, we introduce the $g$-function

$$g(z) = \frac{1}{8} \left( z^2 - \frac{8}{3} \right)^{\frac{3}{2}},$$

where the branch of the power is taken such that $\arg(z \pm \sqrt{\frac{8}{3}}) \in (-\pi, \pi)$. A straightforward computation gives

$$g(z) = \frac{1}{8} z^4 - \frac{1}{2} z^2 + \frac{1}{3} + O(z^{-2}), \quad \text{as} \quad z \to \infty.$$ 

It is easy to see that $g(z)$ has four saddle points, namely, the points satisfying $g'(z) = 0$,

$$z_{1,\pm} = \pm \sqrt{\frac{2}{3}}, \quad z_{2,\pm} = \pm \sqrt{\frac{8}{3}}.$$ 

The second transformation is now defined as

$$U(z) = e^{\frac{x^2}{8} \sigma_3} \Phi(z) z^{-\alpha \sigma_3} e^{-x^2 g(z) \sigma_3}.$$ 

As a consequence, $U(z)$ solves the following RH problem.
RH problem for $U(z)$

(1) $U(z)$ is analytic for $z \in \mathbb{C} \setminus \{\Sigma \setminus \left(\gamma_2 \cup \gamma_6\right)\}$; cf. Figure 1 for the contour.

(2) $U(z)$ satisfies $U_+(z) = U_-(z)J_U(z)$, where

$$J_U(z) = \begin{cases} 
(1 \quad sk_2e^{2x^2g(z)}) \\
(0 \quad 1) \\
(-s_0|z|^{-2\alpha} - e^{-2x^2(g_+(z) + g_-(z))} \quad 0) \\
(e^{2x^2(g_-(z) - g_+(z))} \quad -s_0|z|^{-2\alpha} - e^{-2x^2(g_+(z) + g_-(z))}) \\
(0 \quad e^{2x^2(g_+(z) - g_-(z))}) \\
\end{cases},$$

$z \in \gamma_k, \ k = 1, 3, 5, 7$,

(3) $U(z)$ is normalized at infinity, that is,

$$U(z) = I + O(z^{-1}), \quad \text{as } z \to \infty.$$

(4) $U(z)$ possesses the following asymptotic behavior as $z \to 0$

$$U(z) = U_0(z)z^{\alpha_\sigma_3}E_0z^{-\alpha_\sigma_3}e^{-x^2g(z)\sigma_3},$$

where $\arg z \in (0, \frac{\pi}{4})$ and $U_0(z)$ is analytic in a neighborhood of $z = 0$. The behavior of $U(z)$ in other regions is determined by (3.6) and the jump relations satisfied by $U(z)$.

Next, we transform the above RH problem into a RH problem formulated on the anti-Stokes curves of $g(z)$, as depicted in Figure 2. To this end, first we note that the above RH problem for $U(z)$ can be rewritten as the RH problem posed on the curves shown in Figure 3 where we have used the notations $\tilde{S}_k, k = 1, 3, 4, 5, 7, 8$ to denote the corresponding jump matrices $J_U(z)$.

Figure 2: The anti-Stokes curves of the exponent $g(z)$
This transformation is based on the following factorizations

\[
\tilde{S}_4 = \begin{pmatrix}
1 & -s_0^{-1}e^{2\pi i\alpha}e^{2x^2g_+(z)} \\
0 & s_0^{-1}e^{2\pi i\alpha}e^{-2x^2g_-(z)}
\end{pmatrix}
= \tilde{S}_U_1\tilde{S}_P_+\tilde{S}_U_2,
\]

(3.7)

\[
\tilde{S}_8 = \begin{pmatrix}
e^{x^2(g_-(z)-g_+(z))} & 0 \\
s_0|z|^{-2\alpha}e^{-x^2g_+(z)+g_-(z)} & e^{x^2(g_+(z)-g_-(z))}
\end{pmatrix}
= \tilde{S}_U_3\tilde{S}_P_+\tilde{S}_U_4,
\]

(3.8)

\[
(\tilde{S}_3\tilde{S}_4\tilde{S}_5)^{-1} = \begin{pmatrix}
s_0e^{2\pi i\alpha} & s_1(e^{-2\pi i\alpha} + s_4)e^{2\alpha} \\
0 & (1 + s_0s_1e^{2\pi i\alpha})e^{x^2(g_+(z)-g_-(z))}
\end{pmatrix}
= \tilde{S}_{L_1}\tilde{S}_P\tilde{S}_{L_2},
\]

(3.9)

and

\[
\tilde{S}_7\tilde{S}_8\tilde{S}_1 = \begin{pmatrix}
1 & s_0s_1e^{2\pi i\alpha}e^{x^2(g_-(z)-g_+(z))} \\
0 & s_0|z|^{-2\alpha}
\end{pmatrix}
= \tilde{S}_{L_3}\tilde{S}_P\tilde{S}_{L_4}.
\]

(3.10)
It should be mentioned that in the above factorizations, we have used the property
\[ g_+ (z) + g_- (z) = 0, \quad z \in [z_{2,-}, z_{2,+}], \]
and the complex conjugate relation
\[ \bar{s}_s = 1 + s_0 s_1 e^{2\pi i \alpha}, \quad (3.11) \]
which follows from (2.1), (2.7) and (2.9).

Based on these matrix factorizations, we obtain an equivalent RH problem formulated on the curves shown in Figure 4, where we have used the same notations to stand for the analytic extensions of jump matrices \( \tilde{S}_{U_k}(z) \) and \( \tilde{S}_{L_k}(z) \), \( k = 1, 2, 3, 4 \). In the next step, to deform the jump curves into the anti-Stokes lines of \( g(z) \) as shown in Figure 2, we blow up the four lens. As a consequence, we arrive at the following RH problem for \( T(z) \).

Figure 4: The second deformation of the jump curves of the RH problem

```
```

Figure 5: The final jump curves \( \Sigma_T \) of the RH problem for \( T(z) \)

**RH problem for \( T(z) \)**

1. \( T(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma_T \), where \( \Sigma_T = \bigcup_{k=1}^{20} \pi_k \) is depicted in Figure 5.
(2) $T(z)$ satisfies the jump relations $T_+(z) = T_-(z)J_k(z)$ for $z \in \pi_k$, where

\[
J_1(z) = \begin{pmatrix} 1 & -s_0^{-1}z \cdot 2ae^{-2z^2g(z)} \\ 0 & 1 \end{pmatrix}, \quad J_2(z) = \begin{pmatrix} 1 & s_0s_0^{-1}z \cdot 2ae^{-2z^2g(z)} \\ 0 & 1 \end{pmatrix},
\]

\[
J_3(z) = \begin{pmatrix} 1 & 0 \\ -s_0^{-2}e^{-2z^2g(z)} & 1 \end{pmatrix}, \quad J_4(z) = \begin{pmatrix} 1 & 0 \\ e^{-2z^2g(z)} & 1 \end{pmatrix},
\]

\[
J_5(z) = \begin{pmatrix} 1 & 0 \\ s_0^{-2}e^{-2z^2g(z)} & 1 \end{pmatrix}, \quad J_6(z) = \begin{pmatrix} 1 & 0 \\ s_0^{-2}e^{-2z^2g(z)} & 1 \end{pmatrix},
\]

\[
J_7(z) = J_9(z) = \begin{pmatrix} 1 & s_0^{-1}z \cdot 2ae^{-2z^2g(z)} \\ 0 & 1 \end{pmatrix}, \quad J_8(z) = \begin{pmatrix} 1 & 0 \\ -s_0e^{2\pi iz} & 1 \end{pmatrix},
\]

\[
J_{10}(z) = \begin{pmatrix} 1 & -s_0^{-1} \cdot 2ae^{-2z^2g(z)} \\ 0 & 1 \end{pmatrix}, \quad J_{11}(z) = \begin{pmatrix} 1 & 0 \\ s_0e^{2\pi iz} & 1 \end{pmatrix},
\]

\[
J_{12}(z) = \begin{pmatrix} 1 & 0 \\ e^{2\pi iz} & 1 \end{pmatrix}, \quad J_{13}(z) = \begin{pmatrix} 1 & 0 \\ s_0e^{2\pi iz} & 1 \end{pmatrix},
\]

\[
J_{14}(z) = \begin{pmatrix} 1 & 0 \\ s_0e^{2\pi iz} & 1 \end{pmatrix}, \quad J_{15}(z) = \begin{pmatrix} 1 & 0 \\ -s_0^{-2}e^{2\pi iz} & 1 \end{pmatrix},
\]

\[
J_{16}(z) = \begin{pmatrix} 1 & 0 \\ s_0^{-2}e^{-2z^2g(z)} & 1 \end{pmatrix}, \quad J_{17}(z) = J_{20}(z)^{-1} = \begin{pmatrix} 0 & s_0^{-1}z^{2\alpha} \\ 1 & 0 \end{pmatrix},
\]

and

\[
J_{19}(z) = J_{18}(z)^{-1} = \begin{pmatrix} 0 & s_0^{-1}z^{2\alpha} \\ 1 & 1/(s_0^{-1}z^{2\alpha}) \end{pmatrix}.
\]

(3) $T(z) = I + O(z^{-1})$ as $z \to \infty$.

(4) $T(z)$ has the following asymptotic behavior near the origin

\[
T(z) = T_0(z)z^{\alpha\sigma_3}E_0z^{-\alpha\sigma_3}e^{-z^2g(z)}\sigma_3J_3(z),
\]

where arg $z \in (0, \pi/2)$ and $T_0(z)$ is analytic in a neighborhood of $z = 0$. The behavior of $T(z)$ in other regions is determined by (3.12) and the jump relations satisfied by $T(z)$.

Using the lower and upper triangular structure of the jump matrices, the sign of Re $g(z)$ on the anti-Stokes curves (cf. Figure 2) and the property that Re $g(z) > 0$ on the imaginary axis, it follows that the jump matrices for $T$ tend to the identity matrix exponentially fast as $x \to -\infty$, except the ones on the segment $[z_{2,-}, z_{2,+}]$. In the next subsections, we shall construct the global parametrix with jumps on the segment $[z_{2,-}, z_{2,+}]$ and the local parametrices near the saddle points $z_{1,\pm} = \pm \sqrt{2/3}$, $z_{2,\pm} = \pm \sqrt{8/3}$ and the origin.

3.2 Global parametrix on $[z_{2,-}, z_{2,+}]$

Orienting the line segment $[z_{2,-}, z_{2,+}]$ rightward, we are now in a position to solve the following RH problem for a $2 \times 2$ matrix-valued function $P(\infty)(z)$. 

10
RH problem for $P^{(\infty)}(z)$

1. $P^{(\infty)}(z)$ is analytic for $z \in \mathbb{C} \setminus [z_{2,-}, z_{2,+}]$.
2. $P^{(\infty)}(z)$ satisfies the jump relations
   \[ P_+^{(\infty)}(z) = P_-^{(\infty)}(z)J_\infty(z), \]
   \[ J_\infty(z) = \begin{cases} 
   \left( \begin{array}{cc}
   0 & -s_0^{-1}|z|^{2\alpha} \\
   s_0|z|^{-2\alpha} & 0 
   \end{array} \right), & z \in [z_{2,-}, z_{1,-}], \\
   \left( \begin{array}{cc}
   0 & s_0^{-1}(|s_*|^2 - 1)|z|^{2\alpha} \\
   s_0(1 - |s_*|^2)^{-1}|z|^{-2\alpha} & 0 
   \end{array} \right), & z \in [z_{1,-}, z_{1,+}], \\
   \left( \begin{array}{cc}
   0 & -s_0^{-1}|z|^{2\alpha} \\
   s_0|z|^{-2\alpha} & 0 
   \end{array} \right), & z \in [z_{1,+}, z_{2,+}]. 
   \end{cases} \]

3. $P^{(\infty)}(z)$ have at most singularities of order $\frac{3}{2}$ at $z = z_{1,\pm}$, respectively.

4. As $z \to \infty$, we have $P^{(\infty)}(z) = I + O(z^{-1})$.

A solution of the above RH problem is given by
   \[ P^{(\infty)}(z) = H(z)s_0^{-\frac{\alpha}{2}}D_\infty^{-\sigma_3}X(z)D(z)^{\sigma_3}s_0^{\frac{\alpha}{2}}, \]
where $X(z)$ is given by
   \[ X(z) = \frac{1}{2} \begin{pmatrix} \omega + \omega^{-1} & i(\omega - \omega^{-1}) \\ -i(\omega - \omega^{-1}) & \omega + \omega^{-1} \end{pmatrix}, \quad \omega = \omega(z) = \left( \frac{z - \sqrt{\frac{2}{3}}}{z + \sqrt{\frac{2}{3}}} \right)^{1/4}, \]
the Szegő function
   \[ D(z) = \left( \frac{3}{8} \right)^{\frac{\alpha}{2}} \left( z + \left( z^2 - \frac{8}{3} \right)^{\frac{1}{2}} \right)^{\alpha} z^{-\alpha} \left( \frac{(2 - \sqrt{3})z - i \left( z^2 - \frac{8}{3} \right)^{\frac{1}{2}} - \frac{8}{3}}{(2 - \sqrt{3})z + i \left( z^2 - \frac{8}{3} \right)^{\frac{1}{2}} - \frac{8}{3}} \right)^{\nu}, \]
and
   \[ D_\infty = \lim_{z \to \infty} D(z) = 2^{-\frac{\alpha}{2}} 3^{\frac{\alpha}{2}} e^{\frac{\pi \mu}{3}}. \]
The branches of the functions in (3.15) and (3.16) are chosen such that
   \[ \arg z \in (-\pi, \pi), \quad \arg \left( z \pm \sqrt{\frac{2}{3}} \right) \in (-\pi, \pi), \quad \arg \left( z^2 - \frac{8}{3} \right)^{\frac{1}{2}} \in (-\pi, \pi) \]
and
   \[ \arg \left( \frac{(2 - \sqrt{3})z - i \left( z^2 - \frac{8}{3} \right)^{\frac{1}{2}} - \frac{8}{3}}{(2 - \sqrt{3})z + i \left( z^2 - \frac{8}{3} \right)^{\frac{1}{2}} - \frac{8}{3}} \right) \in (-\pi, \pi). \]
The exponent $\nu$ in (3.16) is defined by
   \[ \nu = -\frac{1}{2\pi i} \ln(|s_*|^2 - 1) - \frac{1}{2} =: \nu_0 - \frac{1}{2} \quad \text{with} \quad \nu_0 \in i\mathbb{R}, \]
(3.18)
noting that $|s_e| > 1$ in the case we are considering.

The factor $H(z)$ in (3.14) is brought in to meet the matching conditions (3.22) and (3.36) below. We seek for a meromorphic function of the form

$$H(z) = I + \frac{A}{z - \sqrt{\frac{2}{3}}} + \frac{B}{z + \sqrt{\frac{2}{3}}}$$

(3.19)

with

$$\det H(z) = 1,$$

(3.20)

where the constant matrices $A$ and $B$ are to be determined. Moreover, since $P(\infty)(z)$ satisfies the symmetric relation $P(\infty)(z) = \sigma_3 P(\infty)(-z) \sigma_3$, we also require that $H(z) = \sigma_3 H(-z) \sigma_3$. Therefore, $A$ and $B$ are subject to the constraint

$$A = -\sigma_3 B \sigma_3.$$

(3.21)

Remark 3.1. $H(z)$ in (3.19) brings extra poles $z = \pm \sqrt{2/3}$ to the global parametrix (3.14) for $P(\infty)(z)$. Such obstacles also arose in deriving singular asymptotics for the PII transcendents [3, 8]. In [3], Bothner and Its developed a certain dressing technique to transform the RH problem to another one without poles. Similar matching technique was used earlier in [15] to derive a uniform asymptotic approximation of the Pollaczek polynomials, and then in [14] for an asymptotic study of a system of Szegő class polynomials.

### 3.3 Local parametrices near $z_{1,\pm}$

In this subsection, we construct two parametrices $P^{(1,\pm)}(z)$ satisfying the same jump conditions as $T(z)$ on the contours $\Sigma_T$ (see Figure 5), respectively in the neighborhoods $U(z_{1,\pm}, \delta)$ of the saddle points $z_{1,\pm} = \pm \sqrt{2/3}$ and matching with $P(\infty)(z)$ on the boundaries $\partial U(z_{1,\pm}, \delta)$.

**RH problem for $P^{(1, +)}(z)$**

(1) $P^{(1, +)}(z)$ is analytic for $z \in U(z_{1, +}, \delta) \setminus \Sigma_T$, where $U(z_{1, +}, \delta) = \{z \in \mathbb{C} : |z - z_{1, +}| < \delta\}$.

(2) $P^{(1, +)}(z)$ shares the same jump conditions as $T(z)$ on $U(z_{1, +}, \delta) \cap \Sigma_T$.

(3) On the boundary of the disc $\partial U(z_{1, +}, \delta) = \{z \in \mathbb{C} : |z - z_{1, +}| = \delta\}$,

$$P^{(1, +)}(z) = (I + O(|x|^{-2})) P(\infty)(z), \text{ as } x \to -\infty. \quad (3.22)$$

In order to construct a solution to the above RH problem, first we define the conformal mapping

$$\varphi(z) = \begin{cases} 2\sqrt{-\frac{\sqrt{3}i}{6}} - g(z), & \text{Im } z > 0, \\ 2\sqrt{-\frac{\sqrt{3}i}{6}} + g(z), & \text{Im } z < 0, \end{cases} \quad (3.23)$$

where the branches of the square roots are specified choosing

$$\varphi(z) = e^{\frac{2\pi}{3}} 2 \cdot 3^{-\frac{1}{2}} \left(z - \sqrt{\frac{2}{3}} \right) (1 + o(1)), \text{ as } z \to \sqrt{\frac{2}{3}}. \quad (3.24)$$
Let $\Phi^{(PC)}$ be the parabolic cylinder parametrix given in Appendix A.2 with the parameter $\nu$ defined by (3.18). Then, the solution to above RH problem can be constructed as follows:

$$P^{(1,+)}(z) = E^{(1,+)}(z)\Phi^{(PC)}(|x|\varphi(z)) \left( \frac{s_0}{h_0} \right)^{\frac{\alpha}{2}} \left\{ \begin{array}{ll}
-\frac{s_0}{\sigma_1}z^{-\alpha\sigma_3}e^{-\nu x^2(z)\pi^2}, & \text{Im } z > 0, \\
\frac{s_0}{\sigma_3}e^{2\nu (\alpha + \nu) \pi x^2(z)\sigma_3}, & \text{Im } z < 0,
\end{array} \right.$$  

(3.25)

where $h_0$ is defined in (A.3) and $E^{(1,+)}(z)$ is given by

$$E^{(1,+)}(z) = W^{(+)}(z) \left( \frac{s_0}{h_0} \right)^{-\frac{\alpha}{2}} |x|^{\nu\sigma_3}e^{it/2x^2\sigma_3} \left( 1 - \frac{1}{|x|^{\nu \sigma_3}} e^{-\nu x^2(z)\pi^2} \right) \left( |x|\varphi(z) \right) \left( \begin{array}{c}
0 \\
1
\end{array} \right),$$  

(3.26)

with

$$W^{(+)}(z) = \left\{ \begin{array}{ll}
P^{(\infty)}(z)z^{\alpha\sigma_3}(-\sigma_1)s_0^{\frac{3}{2}\sigma_3}\varphi(z)^{\nu\sigma_3}, & \text{Im } z > 0,
\end{array} \right.$$  

(3.27)

Here, the branch of the function $\varphi(z)^\nu$ is chosen by requiring $\arg z < 0, (0, 2\pi)$. This leads to the jump relations

$$\begin{align*}
(\varphi(z)^\nu)_+ = (\varphi(z)^\nu)_- e^{2\pi i\nu}, & \quad z \in [z_{1+}, z_{2+}], \\
(\varphi(z)^\nu)_+ = (\varphi(z)^\nu)_-, & \quad z \in [z_{2-}, z_{1+}],
\end{align*}$$  

(3.28)

Using (3.13) and (3.28), it is readily verified that $E^{(1,+)}(z)$ is holomorphic in the deleted neighborhood $U(z_{1+}, \delta) \setminus \{z_{1+}\}$. The analyticity of $E^{(1,+)}(z)$ at the isolated point $z_{1+} = \sqrt{\frac{2}{3}}$ will be guaranteed by a proper choice of the constant matrices $A$ and $B$ in (3.19).

Indeed, by computing the Laurent expansion of $E^{(1,+)}(z)$ at $z_{1+} = \sqrt{\frac{2}{3}}$ using (3.14), (3.15), (3.16) and (3.24), we have

$$E^{(1,+)}(z) = \left( \frac{A}{\tau} + I + \sqrt{\frac{3}{8}} B + O(\tau) \right) \left( I - \frac{c}{\sqrt{3}\tau} \left( e^{\frac{\pi i}{\pi} s_0 D^2_{\infty}} - e^{-\frac{\pi i}{\pi} s_0^{-1} D^2_{\infty}} \right) \right) M_1(z),$$  

(3.29)

where $\tau = z - z_{1+}, M_1(z)$ is analytic near $z = z_{1+}$ and $c = c(x)$ is given by

$$c = \frac{i \sqrt{6} e^{i\phi}}{2 + e^{i\phi}}, \quad \phi = -\frac{\sqrt{3}}{3} x^2 + iv_0 \ln (2\sqrt{3}x^2) + \frac{2\pi \alpha}{3} + \arg \left( v_0 + \frac{1}{2} \right) + \arg s_\tau,$$  

(3.30)

with $v_0$ given in (3.18).

To ensure that $E^{(1,+)}(z)$ is holomorphic at $z_{1+} = \sqrt{\frac{2}{3}}$, it is seen from (3.29) that $A$ and $B$ must fulfill the following algebraic equations

$$A = \frac{c}{\sqrt{3}} \left( I + \sqrt{\frac{3}{8}} B \right) \left( e^{\frac{\pi i}{\pi} s_0 D^2_{\infty}} - e^{-\frac{\pi i}{\pi} s_0^{-1} D^2_{\infty}} \right),$$  

(3.31)

$$0 = A \left( e^{\frac{\pi i}{\pi} s_0 D^2_{\infty}} - e^{-\frac{\pi i}{\pi} s_0^{-1} D^2_{\infty}} \right).$$  

(3.32)

Equation (3.32) follows directly from (3.31) since the second matrix on the right-hand side of (3.31) is nilpotent. A combination of the equation (3.31) with the symmetric condition (3.21) gives us the explicit expressions of $A$ and $B$, namely

$$A = \sqrt{\frac{2}{3}} \left( \frac{c}{c + \sqrt{2}} - e^{\frac{\pi i}{\pi} s_0^{-1} D^2_{\infty}} - e^{-\frac{\pi i}{\pi} s_0 D^2_{\infty}} \right), \quad B = \sqrt{\frac{2}{3}} \left( \frac{c}{c - \sqrt{2}} - e^{\frac{\pi i}{\pi} s_0 D^2_{\infty}} - e^{-\frac{\pi i}{\pi} s_0^{-1} D^2_{\infty}} \right).$$  

(3.33)
Having determined $A$ and $B$, straightforward verification shows that the determinant condition (3.20) holds.

It should be mentioned that we assume in (3.33) that $x$ lies outside of the zero sets of the functions $\sqrt{2} \pm e(x)$, which consist of two sequences of points $\{x_n\}$ and $\{y_n\}$ for $n \in \mathbb{N}$, defined respectively by the equations

$$-\frac{\sqrt{3}}{3} x_n^2 + iv_0 \ln \left(2\sqrt{3}x_n^2\right) + \frac{2\pi \alpha}{3} + \arg \Gamma \left(v_0 + \frac{1}{2}\right) + \arg s_\ast + \frac{2\pi}{3} + 2n\pi = 0,$$

and

$$-\frac{\sqrt{3}}{3} y_n^2 + iv_0 \ln \left(2\sqrt{3}y_n^2\right) + \frac{2\pi \alpha}{3} + \arg \Gamma \left(v_0 + \frac{1}{2}\right) + \arg s_\ast - \frac{2\pi}{3} + 2n\pi = 0.$$  

As we will see later, these points are the singularities appeared in the leading term of the asymptotic formula (1.7). More precisely, $\{x_n\}$ and $\{y_n\}$ are approximate to the simple poles of $q(x; \kappa)$ on the negative real axis, such that $x_n \sim a_n^+ \text{ and } y_n \sim a_n^-$ as $n \to \infty$; see equation (1.10) in Corollary 1.3.

Finally, a combination of (3.14), (3.19), (3.25) and (A.4) gives us the matching condition (3.22).

**RH problem for $P^{(1,-)}(z)$**

1. $P^{(1,-)}(z)$ is analytic for $z \in U(z_{1,-}, \delta) \setminus \Sigma_T$, where $U(z_{1,-}, \delta) = \{z \in \mathbb{C} : |z - z_{1,-}| < \delta\}$.

2. $P^{(1,-)}(z)$ satisfies the same jump conditions as $T(z)$ on $U(z_{1,-}, \delta) \cap \Sigma_T$.

3. On the boundary $\partial U(z_{1,-}, \delta) = \{z \in \mathbb{C} : |z - z_{1,-}| = \delta\}$,

$$P^{(1,-)}(z) = \left(I + O \left(|x|^{-2}\right)\right) P^{(\infty)}(z), \text{ as } x \to -\infty.$$  

Similar to the construction of $P^{(1,+)}(z)$, we introduce a conformal mapping

$$\zeta(z) = \begin{cases} 2\sqrt{\frac{-\sqrt{3}i}{6} + g(z)}, & \text{Im } z > 0, \\ 2\sqrt{\frac{-\sqrt{3}i}{6} - g(z)}, & \text{Im } z < 0, \end{cases}$$

where the branches of the square roots are chosen such that

$$\zeta(z) = e^{\frac{3\pi i}{4}} \cdot 3^{-\frac{1}{4}} \left(z + \sqrt{\frac{2}{3}}\right) (1 + o(1)), \text{ as } z \to -\sqrt{\frac{2}{3}}.$$  

The solution to the above RH problem can be constructed in terms of the parabolic cylinder function as follows:

$$P^{(1,-)}(z) = E^{(1,-)}(z) \Phi^{(PC)}(\lambda|x|\zeta(z)) \left(\frac{\frac{s_\ast}{h_0}}{e^\nu}\right)^{\frac{s_\ast}{2}} e^{2\pi iv_\tau}\left(s_0 e^{2\pi i\alpha} \right)^{\frac{s_3}{2}} z^{-\alpha\sigma_1} e^{-x^2 g(z) \sigma_3}, \text{ Im } z > 0,$$

$$\left(s_0 e^{2\pi i\alpha} \right)^{-\frac{s_3}{2}} \sigma_3 \sigma_1^\dagger z^{-\alpha\sigma_3} e^{-x^2 g(z) \sigma_3}, \text{ Im } z < 0,$$

where $\Phi^{(PC)}$ is the parabolic cylinder parametrix given in Appendix A.2 $h_0$ and $\nu$ are defined in (A.3) and (3.18), respectively. Here, $E^{(1,-)}(z)$ is given by

$$E^{(1,-)}(z) = W^{(-)}(z) \left(\frac{s_\ast}{h_0}\right)^{\frac{-s_3}{2}} \left|x\right|^\nu e^{\frac{ivg_3}{6} \sigma_3} \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)^{\frac{s_3}{2}} \left|x\right|\zeta(z)$$

$$\left(\frac{1}{\frac{s_3}{2} \zeta(z)}\right)^{\frac{s_3}{2}} \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)^{\frac{s_3}{2}} \left|x\right|\zeta(z)$$

$$\left(\frac{1}{\frac{s_3}{2} \zeta(z)}\right)^{\frac{s_3}{2}} \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)^{\frac{s_3}{2}} \left|x\right|\zeta(z)$$

$$\left(\frac{1}{\frac{s_3}{2} \zeta(z)}\right)^{\frac{s_3}{2}} \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)^{\frac{s_3}{2}} \left|x\right|\zeta(z)$$

$$\left(\frac{1}{\frac{s_3}{2} \zeta(z)}\right)^{\frac{s_3}{2}} \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right)^{\frac{s_3}{2}} \left|x\right|\zeta(z)$$
To find a solution to the above RH problem, we define the conformal mapping condition (3.43).

In this subsection, we seek two parametrices $P(U\sigma_3 \sigma_3 \sigma_3 \sigma_3 (s_0 e^{2\pi \iota \alpha}) \sigma_3) e^{-2\pi \iota \nu \sigma_3} \zeta(z)^{\nu \sigma_3}, \quad \text{Im } z > 0, \quad (3.41)

The branch of the function $\zeta(z)^{\nu}$ is chosen such that $\arg(\zeta(z)) \in (-\pi, \pi)$. This implies that

\[
\begin{cases}
    (\zeta(z)^{\nu})_+ = (\zeta(z)^{\nu})_-, & z \in [z_{2,-}, z_{1,-}],
    \\
    (\zeta(z)^{\nu})_+ = (\zeta(z)^{\nu})_-, & z \in [z_{1,-}, z_{2,+}].
\end{cases}
\]

(3.42)

Using the jump relations (3.13) and (3.42), it is straightforward to check that $W(-) (z)$ is holomorphic in $U(z_{1,-}, \delta)$. Furthermore, combining (3.14), (3.19), (3.39) with the asymptotic behavior (3.4), we obtain the matching condition (3.36).

### 3.4 Local parametrices near saddle points $z_{2,\pm}$

In this subsection, we seek two parametrices $P(2,\pm)(z)$ satisfying the same jump conditions as $T(z)$ on the curves $\Sigma_T$ (see Figure 5) in the neighbourhoods $U(z_{2,\pm}, \delta)$ of the saddle points $z_{2,\pm} = \pm \sqrt{8/3}$, matching with $P(\infty)(z)$ on the boundaries $\partial U(z_{2,\pm}, \delta)$.

**RH problem for $P(2,\pm)(z)$**

1. $P(2,\pm)(z)$ is analytic for $z \in U(z_{2,+}, \delta) \setminus \Sigma_T$, where $U(z_{2,+}, \delta) = \{z \in \mathbb{C} : |z - z_{2,\pm}| < \delta\}$.
2. $P(2,\pm)(z)$ satisfies the same jump conditions as $T(z)$ on $U(z_{2,+}, \delta) \cap \Sigma_T$.
3. On the boundary $\partial U(z_{2,+}, \delta) = \{z \in \mathbb{C} : |z - z_{2,+}| = \delta\}$, we have
   
   \[ P(2,\pm)(z) = (I + O(|x|^{-2})) P(\infty)(z), \quad \text{as } x \to -\infty. \]

(3.43)

To find a solution to the above RH problem, we define the conformal mapping

\[ \eta(z) = \left( \frac{3}{2} g(z) \right)^{\frac{2}{3}}, \]

where the branch is chosen such that

\[ \eta(z) = 2^{k/3} 3^{-i/3} \left( z - \sqrt{8/3} \right) (1 + o(1)), \quad \text{as } z \to \sqrt{8/3}. \]

(3.44)

Then, the solution to the above RH problem can be explicitly constructed in terms of the Airy function

\[ P(2,\pm)(z) = E(2,\pm)(z) \Phi(\text{Ai}) \left( |x|^{\frac{2}{3}} \eta(z) \right) (s_0 e^{2\pi \iota \alpha})^{-\frac{3}{2}} \sigma_1 e^{\mp \pi \iota \alpha \sigma_3} z^{-\alpha \sigma_3} e^{-x^2 g(z) \sigma_3}, \quad \pm \text{Im } z > 0, \]

(3.46)

where $\Phi(\text{Ai})$ denotes the standard Airy parametrix (see Appendix A.1 below), and $E(2,\pm)(z)$ is given by

\[ E(2,\pm)(z) = P(\infty)(z) z^{\alpha \sigma_3} e^{\pm \pi \iota \alpha \sigma_3} \sigma_1 (s_0 e^{2\pi \iota \alpha})^{\frac{3}{2}} \frac{1}{\sqrt{2}} \left( \frac{1}{1 - i} \right) |x|^{\frac{2}{3}} \eta(z)^{\frac{2}{3}}, \quad \pm \text{Im } z > 0. \]

(3.47)

Here, the branch of the function $\eta(z)^{\frac{2}{3}}$ is chosen such that $\arg(\eta(z)) \in (-\pi, \pi)$. This implies that on the segment $[z_{2,-}, z_{2,+}]$, we have

\[ \left( \eta(z)^{\frac{2}{3}} \right)_+ = \left( \eta(z)^{\frac{2}{3}} \right)_-. e^{\mp \pi \sigma_3}. \]

(3.48)

It then follows from (3.13) and (3.48) that $E(2,\pm)(z)$ is analytic in the neighborhood $U(z_{2,+}, \delta)$. Finally, combining (3.14) and the asymptotic behavior (3.2) with (3.46), we get the matching condition (3.43).
To proceed, we define the conformal mapping $P$ on the curves $\Sigma_T$. In this subsection, we seek a parametrix

**3.5 Local parametrix near the origin**

The RH problem for $P^{(2,-)}(z)$

1. $P^{(2,-)}(z)$ is analytic for $z \in U(z_{2,-}, \delta) \setminus \Sigma_T$, where $U(z_{2,-}, \delta) = \{z \in \mathbb{C} : |z - z_{2,-}| < \delta\}$.
2. $P^{(2,-)}(z)$ shares the same jump conditions as $T(z)$ on $U(z_{2,-}, \delta) \cap \Sigma_T$.
3. On the boundary $\partial U(z_{2,-}, \delta) = \{z \in \mathbb{C} : |z - z_{2,-}| = \delta\}$, we have
   
   $$P^{(2,-)}(z) = (I + O(|z|^{-2})) P^{(\infty)}(z), \quad x \rightarrow -\infty.$$  

(3.49)

Similarly, the solution to the above RH problem can also be built out of the Airy function

$$P^{(2,-)}(z) = E^{(2,-)}(z) \Phi^{(Ai)} \left( \frac{|z|^\frac{2}{3}}{\eta(-z)} \right) \left( s_0 e^{2\pi i \alpha} \right)^{-\frac{\pi}{4}} \sigma_2 z^{-\alpha \sigma_3} e^{-x^2 g(z)\sigma_3},$$  

(3.50)

where $\Phi^{(Ai)}$ again denotes the standard Airy parametrix given in Appendix A.1, $\eta(z)$ is defined in (3.44) and $E^{(2,-)}(z)$ is given by

$$E^{(2,-)}(z) = P^{(\infty)}(z) z^{\alpha \sigma_3} \sigma_2 \left( s_0 e^{2\pi i \alpha} \right)^{\frac{\pi}{4}} \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \\ 1 \end{array} \right) |z|^\frac{\pi}{4} \left( \eta(-z) \right)^{\frac{\pi}{4}}.$$  

(3.51)

It is straightforward to check that $E^{(2,-)}(z)$ is analytic in the neighborhood $U(z_{2,-}, \delta)$. The matching condition (3.49) follows from (3.14), (A.2) and (3.50).

**3.5 Local parametrix near the origin**

In this subsection, we seek a parametrix $P^{(0)}(z)$ satisfying the same jump conditions as $\Sigma_T$ on the curves $\Sigma_T$ (see Figure 5) in the neighborhood $U(0, \delta)$ of the origin and matching with $P^{(\infty)}(z)$ on the boundary $\partial U(0, \delta)$.

The RH problem for $P^{(0)}(z)$

1. $P^{(0)}(z)$ is analytic for $z \in U(0, \delta) \setminus \Sigma_T$, where $U(0, \delta) = \{z \in \mathbb{C} : |z| < \delta\}$.
2. $P^{(0)}(z)$ satisfies the same jump conditions as $T(z)$ on $U(0, \delta) \cap \Sigma_T$.
3. On the boundary $\partial U(0, \delta) = \{z \in \mathbb{C} : |z| = \delta\}$, we have
   
   $$P^{(0)}(z) = (I + O(|z|^{-2})) P^{(\infty)}(z), \quad x \rightarrow -\infty.$$  

(3.52)

4. $P^{(0)}(z)$ has the same asymptotic behavior as $T(z)$ near the origin; see (3.12).

To proceed, we define the conformal mapping

$$\lambda(z) = \pm ig(z) = \pm \frac{1}{8} iz \left( z^2 - \frac{8}{3} \right)^{\frac{3}{2}}, \quad \pm \text{Im} \ z > 0,$$

(3.53)

which has the following behavior at $z = 0$

$$\lambda(z) = 2^{\frac{3}{2}} 3^{-\frac{3}{2}} z (1 + o(1)), \quad z \rightarrow 0.$$  

(3.54)

Let $\Phi^{(Bes)}(z)$ be the Bessel parametrix with parameter $\alpha$ as given in Appendix A.3. We then define

$$P^{(0)}(z) = E^{(0)}(z) \Phi^{(Bes)} \left( x^2 \lambda(z) \right) C(z) \left[ s_1(e^{-2\pi i \alpha} + s_*) \right]^{-\frac{\pi}{4}} z^{-\alpha \sigma_3} e^{-x^2 g(z)\sigma_3},$$  

(3.55)
where \( \text{arg } z \in (0, 2\pi) \), \( C(z) \) is a piecewise constant matrix defined in regions \( \Lambda_k \) described in Figure 9

\[
C(z) = \begin{cases} 
I, & z \in \Lambda_1 \cup \Lambda_8, \\
\left( \begin{array}{cc} 1 & 0 \\ -e^{-2\pi i \alpha} & 1 \end{array} \right), & z \in \Lambda_2, \\
\left( \begin{array}{cc} e^{-\pi i \alpha} & 0 \\ -e^{\pi i \alpha} & e^{\pi i \alpha} \end{array} \right), & z \in \Lambda_3, \\
e^{-\pi i \alpha \sigma_3}, & z \in \Lambda_4, \\
e^{\pi i \alpha \sigma_3}, & z \in \Lambda_5, \\
\left( \begin{array}{cc} e^{\pi i \alpha} & 0 \\ -e^{\pi i \alpha} & e^{-\pi i \alpha} \end{array} \right), & z \in \Lambda_6, \\
\left( \begin{array}{cc} 1 & 0 \\ e^{2\pi i \alpha} & 1 \end{array} \right), & z \in \Lambda_7,
\end{cases}
\]

and \( E^{(0)}(z) \) is given by

\[
E^{(0)}(z) = P^{(\infty)}(z)z^{\alpha_3} s_1(e^{-2\pi i \alpha} + s_*) \sigma_3^{\frac{\alpha_3}{2}} Q(z) e^{-\frac{1}{2}\pi i \sigma_3 \frac{1}{\sqrt{2}} (1 \, i \, 1)} \quad (3.56)
\]

with

\[
Q(z) = \begin{cases} 
e^{\frac{1}{2}\pi i \alpha_3}, & \text{Im } z > 0, \ |z| < \delta, \\
\sigma_3 \sigma_1 e^{\frac{1}{2}\pi i \alpha_3}, & \text{Im } z < 0, \ |z| < \delta.
\end{cases} \quad (3.57)
\]

Using \((3.13), (3.56)\) and \((3.57)\), we see that \( E^{(0)}(z) \) is analytic in the deleted neighborhood \( U(0, \delta) \setminus \{0\} \). Inserting \((3.14)\) into \((3.56)\) shows that \( E^{(0)}(z) \) is bounded at \( z = 0 \). Therefore, \( E^{(0)}(z) \) is analytic in the neighborhood \( U(0, \delta) \). Moreover, the matching condition \((3.52)\) follows from \((3.55), (3.56)\) and \((A.6)\).

From \((3.55)\) and \((A.5)\), it is straightforward to verify that the function \( P^{(0)}(z) \) constructed in \((3.55)\) satisfies the same jump relations as \( T(z) \) on \( \Sigma_T \cap U(0, \delta) \). Recalling the definition of the connection matrix in \((2.10)\), we can rewrite the asymptotic behavior \((3.12)\) in the form

\[
T(z) = \hat{T}_0(z) z^{\alpha_3} \left( \begin{array}{cc} 1 & \frac{1}{1+e^{-2\pi i \alpha}} \\ 0 & -e^{-2\pi i \alpha} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{array} \right) s_1(e^{-2\pi i \alpha} + s_*) \sigma_3^{\frac{\alpha_3}{2}} z^{-\alpha_3} e^{-x^2 g(z) \sigma_3}, \quad (3.58)
\]

where \( \hat{T}_0(z) \) is analytic in a neighborhood of \( z = 0 \). Comparing \((3.58)\) with \((3.55)\) and \((A.8)\), we see that \( P^{(0)}(z) \) satisfies the asymptotic behavior \((3.12)\) as \( z \to 0 \).

### 3.6 Final transformation

The final transformation is defined by

\[
R(z) = \begin{cases} 
T(z) \left[ P^{(1, \pm)}(z) \right]^{-1}, & z \in U(z_1, \pm, \delta) \setminus \Sigma_T, \\
T(z) \left[ P^{(2, \pm)}(z) \right]^{-1}, & z \in U(z_2, \pm, \delta) \setminus \Sigma_T, \\
T(z) \left[ P^{(0)}(z) \right]^{-1}, & z \in U(0, \delta) \setminus \Sigma_T, \\
T(z) \left[ P^{(\infty)}(z) \right]^{-1}, & \text{elsewhere.}
\end{cases} \quad (3.59)
\]

Then, \( R(z) \) satisfies the following RH problem.
RH problem for $R(z)$

1. $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_R$, where the contour $\Sigma_R$ is illustrated in Figure 6.

2. On the contour $\Sigma_R$, we have $R_+(z) = R_-(z)J_R(z)$, where

$$J_R(z) = \begin{cases} 
 p(1,\pm)(z)p(\infty)(z)^{-1}, & z \in \partial U(z_{1,\pm}, \delta), \\
 p(2,\pm)(z)p(\infty)(z)^{-1}, & z \in \partial U(z_{2,\pm}, \delta), \\
 p(0)(z)p(\infty)(z)^{-1}, & z \in \partial U(0, \delta), \\
 p(\infty)(z)J_T(z)p(\infty)(z)^{-1}, & z \in \pi_k, \quad k = 1, \ldots, 16.
\end{cases} \tag{3.60}$$

3. As $z \to \infty$, we have

$$R(z) = I + \frac{R_1(x)}{z} + O \left(\frac{1}{z^2}\right).$$ \tag{3.61}

In view of the matching conditions (3.22), (3.36), (3.43), (3.49), and (3.52), it is readily seen that as $x \to -\infty$

$$J_R(z) = \begin{cases} 
 I + O(|x|^{-2}), & z \in \partial U(0, \delta) \cup \partial U(z_{1,\pm}, \delta) \cup \partial U(z_{2,\pm}, \delta), \\
 I + O(e^{-c_1|z|^2}), & z \in \pi_k, \quad k = 1, \ldots, 16,
\end{cases} \tag{3.62}$$

where $c_1$ is a positive constant.

Consequently, we have

$$R(z) = I + O(|x|^{-2}) \quad \text{as} \quad x \to -\infty,$$ \tag{3.63}

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

4 Proof of the main results

4.1 Proof of Theorem 1.1

By (3.3), the PIV solutions $q(x; \kappa)$ can be expressed in terms of $\Phi_1$ in (3.2). Tracing back the series of transformations performed in Section 3

$$\Phi \mapsto U \mapsto T \mapsto R,$$
we have that for large $z$
\[e^{\frac{1}{3}x^2\sigma_3}\Phi(z)z^{-\alpha_3}e^{-x^2g(z)}\sigma_3 = R(z)P^{(\infty)}(z).\] (4.1)

We obtain the following expansion by using (3.14), (3.19) and (3.33)
\[P^{(\infty)}(z) = I + \frac{P_1^{(\infty)}(x)}{z} + O\left(\frac{1}{z^2}\right), \text{ as } z \to \infty,\] (4.2)

where
\[P_1^{(\infty)}(x) = \begin{pmatrix} 0 & -\frac{\sqrt{2}(c+i\sqrt{2/3})}{c+\sqrt{2}}s_0D\infty \\ \frac{\sqrt{2}(c+i\sqrt{2/3})}{c+\sqrt{2}}s_0D\infty & 0 \end{pmatrix}.\] (4.3)

Substituting the expansions (3.2), (3.61) and (4.2) into (4.1), we find
\[\Phi_1(x) = e^{-\frac{1}{3}x^2\sigma_3}\left(P_1^{(\infty)}(x) + R_1(x) \right)e^{\frac{1}{3}x^2\sigma_3}.\] (4.4)

In virtue of the error estimation (3.63), we get
\[R_1(x) = O\left(x^{-2}\right), \text{ as } x \to -\infty.\] (4.5)

Thus, we have
\[\Phi_1(x) = e^{-\frac{1}{3}x^2\sigma_3}\left[P_1^{(\infty)}(x) + O\left(x^{-2}\right) \right]e^{\frac{1}{3}x^2\sigma_3}.\] (4.6)

Substituting the asymptotic approximation (4.6) into (3.3), in view of (4.3), we obtain
\[q(x; \kappa) = -2x - 4i\sqrt{\frac{2}{3}}\frac{c - i\sqrt{2/3}}{c^2 - 2}x + O(x^{-1}), \text{ as } x \to -\infty,\] (4.7)

where $c$ is defined in (3.30) and the error term is uniform for $x$ bounded away from the zeros of $c^2 - 2$. Recalling the definition of $c$ in (3.30), we may write
\[\frac{c - i\sqrt{2/3}}{c^2 - 2} = i\frac{\sqrt{2}}{4\sqrt{3}}\left(2 + \frac{3}{2\cos \phi + 1}\right).\] (4.8)

By inserting (4.8) and the expression (3.30) of $\phi$ into (4.7), we arrive at the asymptotic expansion (1.7). Finally, the connection formulas (1.8) follow from the relation (2.11) and the definition of $\nu_0$ in (3.18). This completes the proof of Theorem 1.1.

4.2 Proof of Corollary 1.3

To derive the desired expansions for the poles, we first recall the following result for the zeros of real functions given in [7]:

**Lemma 4.1.** In the interval $[z_h - \varrho, z_h + \varrho]$, suppose $f(x) = h(x) + \varepsilon(x)$, where $f(x)$ is continuous, $h(x)$ is differentiable, $h(z_h) = 0$, $m = \min |h'(x)| > 0$, and
\[E_\varepsilon = \max |\varepsilon(x)| < \min\{|h(z_h - \varrho)|, |h(z_h + \varrho)|\}.\]

Then there exists a zero $z_f$ of $f(x)$ in the interval such that $|z_f - z_h| \leq E_\varepsilon/m$. 

19
Now we are in a position to prove Corollary 1.3. First, use the same argument as in Section 4.1, we have the asymptotic approximation

\[
\frac{x}{q(x; \kappa)} = \frac{3}{4} \frac{\cos \vartheta + 1}{1 - \cos \vartheta} + O \left( \frac{1}{x^2} \right), \quad x \to -\infty, \tag{4.9}
\]

where \( \vartheta = \vartheta(x) = \frac{1}{\sqrt{3}} x^2 - b \ln(2\sqrt{3}x^2) + \psi \), with \( b \) and \( \psi \) being given by (1.8). The error term in (4.9) is uniform for \( \vartheta \) bounded away from \( 2n\pi \) for integers \( n \), as \( x \to -\infty \). It is seen that

\[
x \sim -\frac{3}{4} \vartheta^{1/2}
\]

and we may take \( \vartheta \) as the large parameter. Applying Lemma 4.1, we see that for large integers \( n \), there exist zeros \( x = a_n^\pm \) of \( x/q(x; \kappa) \), corresponding respectively to \( \vartheta \sim 2n\pi \pm 2\pi/3 \). More precisely, there exist poles \( x = a_n^\pm \) of \( q(x; \kappa) \) such that

\[
\vartheta(a_n^\pm) - (2n\pi \pm 2\pi/3) = O(1/n), \quad n \to \infty,
\]

from which (1.10) follows.

Acknowledgements

The work of Shuai-Xia Xu was supported in part by the National Natural Science Foundation of China under grant numbers 11571376 and 11971492, and by the Natural Science Foundation for Distinguished Young Scholars of Guangdong Province of China (Grant No.2022B1515020063). Yu-Qiu Zhao was supported in part by the National Natural Science Foundation of China under grant numbers 11571375 and 11971489.

Appendix A  Local parametrix models

A.1  Airy parametrix

Let \( w = e^{2\pi i/3} \), we define

\[
\Phi^{(Ai)}(s) = M \begin{cases}
(Ai(s) \quad Ai(ws)) e^{-i\frac{2}{3}\sigma_3}, & s \in I, \\
(Ai(s) \quad Ai'(ws)) e^{-i\frac{2}{3}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & s \in II, \\
(Ai(s) \quad Ai'(ws)) e^{-i\frac{2}{3}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in III, \\
(Ai(s) \quad Ai'(ws)) e^{-i\frac{2}{3}\sigma_3}, & s \in IV,
\end{cases}
\]

where \( Ai(s) \) is the Airy function (cf. [10, Chapter 9]),

\[
M = \sqrt{2\pi e^{\frac{1}{6} \pi i}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},
\]

and the regions I-IV are shown in Figure 7. It is easy to check that \( \Phi^{(Ai)}(s) \) solves the following RH problem (cf. [5, Chapter 7]):
RH problem for $\Phi^{(Ai)}(s)$

1. $\Phi^{(Ai)}(s)$ is analytic for $s \in \mathbb{C} \setminus \bigcup_{k=1}^{4} \Sigma_k$.

2. $\Phi^{(Ai)}(s)$ satisfies the jump relations $\Phi^+_k(s) = \Phi^-_k(s)J_k$, $s \in \Sigma_k$, $k = 1, 2, 3, 4$, where
   
   \[
   J_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
   \]

3. $\Phi^{(Ai)}(s)$ satisfies the following asymptotic behavior as $s \to \infty$:
   \[
   \Phi^{(Ai)}(s) = s^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + O\left(s^{-\frac{3}{2}}\right) \right) e^{-\frac{2}{3}s^2\sigma_3}. \tag{A.2}
   \]

Figure 7: The jump contours and regions for $\Phi^{(Ai)}$

A.2 Parabolic cylinder parametrix

Let

\[
D(s) = 2^{-\frac{\sigma_3}{2}} \begin{pmatrix} D_{\nu-\frac{1}{2}}(is) & D_\nu(s) \\ D'_{\nu-\frac{1}{2}}(is) & D'_\nu(s) \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{2}(\nu+1)} & 0 \\ 0 & 1 \end{pmatrix},
\]

where $D_\nu$ is the standard parabolic cylinder function with parameter $\nu$ (cf. [10, Chapter 12]). Denote

\[
H_0 = \begin{pmatrix} 1 & 0 \\ h_0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & h_1 \\ 0 & 1 \end{pmatrix}, \quad H_{n+2} = e^{i\pi(\nu+\frac{1}{2})\sigma_3} H_n e^{-i\pi(\nu+\frac{1}{2})\sigma_3}, \quad n = 0, 1,
\]

with

\[
h_0 = -i \frac{\sqrt{2\pi}}{\Gamma(\nu + 1)}, \quad h_1 = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu}, \quad 1 + h_0 h_1 = e^{2\pi i \nu}. \tag{A.3}
\]

We define

\[
\Phi^{(PC)}(s) = \begin{cases}
D(s), & \text{arg } s \in (-\pi/4, 0); \\
D(s)H_0, & \text{arg } s \in (0, \pi/2); \\
D(s)H_1, & \text{arg } s \in (\pi/2, \pi); \\
D(s)H_2, & \text{arg } s \in (\pi, 3\pi/2); \\
D(s)H_3, & \text{arg } s \in (3\pi/2, 7\pi/4).
\end{cases}
\]

Then $\Phi^{(PC)}(s)$ solves the following RH problem (cf. [3, 6]).
RH problem for $\Phi^{(PC)}(s)$

(1) $\Phi^{(PC)}(s)$ is analytic for all $s \in \mathbb{C} \setminus \bigcup_{k=1}^{5} \Sigma_k$, where $\Sigma_k = \{ s \in \mathbb{C} : \arg s = \frac{k\pi}{4} \}$, $k = 1, 2, 3, 4$ and $\Sigma_5 = \{ s \in \mathbb{C} : \arg s = -\frac{\pi}{4} \}$; see Figure 8.

(2) $\Phi^{(PC)}(s)$ satisfies the jump conditions as indicated in Figure 8.

(3) $\Phi^{(PC)}(s)$ has the following asymptotic behavior at infinity

$$
\Phi^{(PC)}(s) = \begin{pmatrix} 0 & 1 \\ 1 & -s \end{pmatrix} 2^{\frac{\sigma_3}{4}} \left( \frac{1 + O\left(\frac{1}{s}\right)}{s} + O\left(\frac{1}{s^3}\right) \right) e^{\left(\frac{2}{4} - \nu \ln s\right)\sigma_3}. 
$$

(A.4)

Figure 8: The jump contours and jump matrices for $\Phi^{(PC)}$

A.3 A Bessel model parametrix

We start with the following RH problem.

RH problem for $\Phi^{(Bes)}(s)$

(1) $\Phi^{(Bes)}(s)$ is analytic for all $s \in \mathbb{C} \setminus \bigcup_{k=1}^{8} \Gamma_k$, where $\Gamma_k = \{ s \in \mathbb{C} : \arg s = k\pi/4 \}$ are depicted in Figure 9.

(2) $\Phi^{(Bes)}(s)$ satisfies the following jump conditions

$$
\Phi^{(Bes)}_+(s) = \Phi^{(Bes)}_-(s) \begin{cases} 
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , & s \in \Gamma_1 \cup \Gamma_5, \\
\begin{pmatrix} 1 & e^{-2\pi i \alpha} \\ e^{2\pi i \alpha} & 1 \end{pmatrix} , & s \in \Gamma_2 \cup \Gamma_6, \\
\begin{pmatrix} e^{\pi i \alpha} & 0 \\ 0 & e^{-\pi i \alpha} \end{pmatrix} , & s \in \Gamma_3 \cup \Gamma_7, \\
\begin{pmatrix} 1 & 0 \\ e^{2\pi i \alpha} & 1 \end{pmatrix} , & s \in \Gamma_4 \cup \Gamma_8.
\end{cases}
$$

(A.5)
The asymptotic behavior of \( \Phi^{(\text{Bes})}(s) \) at infinity is different in each quadrant. As \( s \to \infty \),

\[
\Phi^{(\text{Bes})}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left( I + O\left(s^{-1}\right)\right) e^{\frac{\pi i s}{4}} e^{-i s \sigma_3} \begin{cases} e^{\frac{\pi i s}{4}}, & s \in \Lambda_1 \cup \Lambda_2, \\ e^{\frac{\pi i s}{4}}, & s \in \Lambda_3 \cup \Lambda_4, \\ e^{\frac{\pi i s}{4}} \sigma_3 \sigma_3, & s \in \Lambda_5 \cup \Lambda_6, \\ e^{-\frac{\pi i s}{4}} \sigma_3 \sigma_3, & s \in \Lambda_7 \cup \Lambda_8, \end{cases}
\] (A.6)

According to [12], the above RH problem can be constructed in terms of the modified Bessel function \( I_{\alpha \pm \frac{1}{2}}(s) \) and \( K_{\alpha \pm \frac{1}{2}}(s) \).

\[
\Phi^{(\text{Bes})}(s) = \begin{pmatrix} \sqrt{\pi s} \frac{1}{2} I_{\alpha + \frac{1}{2}}(se^{-\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}} s^{\frac{1}{2}} K_{\alpha + \frac{1}{2}}(se^{-\frac{\pi i}{2}}) \\ -i \sqrt{\pi s} \frac{1}{2} I_{\alpha - \frac{1}{2}}(se^{-\frac{\pi i}{2}}) & -\frac{1}{\sqrt{\pi}} s^{\frac{1}{2}} K_{\alpha - \frac{1}{2}}(se^{-\frac{\pi i}{2}}) \end{pmatrix} e^{-\frac{1}{2} \alpha \pi i \sigma_3}
\] (A.7)

for \( s \in \Lambda_2 \), where \( s^{1/2} \) takes the principal branch. While the explicit expressions of \( \Phi^{(\text{Bes})}(z) \) in other sectors are determined by (A.7) and the jump relation (A.5).

Using the series expansion of the modified Bessel function [10, (10.25.2)]

\[
I_{\nu}(s) = \left( \frac{1}{2} \right) ^{\nu} \sum_{k=0}^{\infty} \frac{\left( \frac{1}{4} s^2 \right)^k}{k! \Gamma(\nu + k + 1)} \text{ for } \arg s \in (-\pi, \pi),
\]

and the relation [10] (10.27.4)

\[
K_{\nu}(s) = \frac{\pi I_{-\nu}(s) - I_{\nu}(s)}{2 \sin(\pi \nu)}, \quad \nu \notin \mathbb{Z},
\]

it is seen from (A.7) that

\[
\Phi^{(\text{Bes})}(s) = B(s) s^{\alpha \sigma_3} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-\frac{\pi i s}{2}}, \quad s \in \Lambda_2,
\] (A.8)

where \( \alpha - \frac{1}{2} \notin \mathbb{Z} \) and \( B(s) \) is an entire function in \( s \). The behavior of \( \Phi^{(\text{Bes})}(s) \) near the origin in the other sectors \( \Lambda_k \) can be determined by (A.8) and the jump relations (A.5).

Figure 9: The jump contours and regions for \( \Phi^{(\text{Bes})} \)
References

[1] A. S. Abdullayev, Justification of asymptotic formulas for the fourth Painlevé equation, *Stud. Appl. Math.*, **99** (1997), 255-283.

[2] A. P. Bassom, P. A. Clarkson, A. C. Hicks and J. B. McLeod, Integral equations and exact solutions for the fourth Painlevé equation, *Proc. R. Soc. A*, **437** (1992), 1-24.

[3] T. Bothner and A. Its, The nonlinear steepest descent approach to the singular asymptotics of the second Painlevé transcendent, *Phys. D*, **241** (2012), 2204-2225.

[4] P. A. Clarkson and J. B. McLeod, Integral equations and connection formulae for the Painlevé equations, *Painlevé transcendents, their asymptotics and physical applications*, D. Levi and P. Winternitz (eds.), 1992, 1-31.

[5] P. Deift, *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach*, Courant Lecture Notes, vol. 3, New York University, 1999.

[6] A. S. Fokas, A. R. Its, A. A. Kapaev and V. Y. Novokshenov, *Painlevé transcendents: The Riemann-Hilbert approach*, Math. Surv. Monog., Vol. 128, Amer. Math. Soc., Providence, RI, 2006.

[7] H. W. Hethcote, Error bounds for asymptotic approximations of zeros of transcendental functions, *SIAM J. Math. Anal.*, **1** (1970), 147-152.

[8] W. Y. Hu, Singular asymptotics for solutions of the inhomogeneous Painlevé II equation, *Nonlinearity*, **32** (2019), 3843-3872.

[9] A. R. Its and A. A. Kapaev, Connection formulae for the fourth Painlevé transcendent; Clarkson-McLeod solution, *J. Phys. A*, **31** (1998), 4073-4113.

[10] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B.V. Saunders (eds.), *NIST digital library of mathematical functions*, 2020.

[11] J. A. Reeger and B. Fornberg, Painlevé IV with both parameters zero: a numerical study, *Stud. Appl. Math.*, **130** (2013), 108-133.

[12] M. Vanlessen, Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight, *J. Approx. Theory*, **125** (2003), 198-237.

[13] R. Wong and H. Y. Zhang, On the connection formulas of the fourth Painlevé transcendent, *Anal. Appl.*, **4** (2009), 419-448.

[14] J.-R. Zhou, S.-X. Xu and Y.-Q. Zhao, Uniform asymptotics of a system of Szegő class polynomials via the Riemann-Hilbert approach, *Anal. Appl.*, **9** (2011), 447-480.

[15] J.-R. Zhou and Y.-Q. Zhao, Uniform asymptotics of the Pollaczek polynomials via the Riemann-Hilbert approach, *Proc. R. Soc. Lond. Ser. A*, **464** (2008), 2091-2112.