Discovering and Certifying Lower Bounds for the
Online Bin Stretching Problem

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Abstract

There are several problems in the theory of online computation where tight lower bounds on the competitive ratio are unknown and expected to be difficult to describe in a short form. A good example is the Online Bin Stretching problem, in which the task is to pack the incoming items online into bins while minimizing the load of the largest bin. Additionally, the optimal load of the entire instance is known in advance.

The contribution of this paper is twofold. We use the Coq proof assistant to formalize the Online Bin Stretching problem and provide a program certifying lower bounds of this problem. Because of the size of the certificates, previously claimed lower bounds were never formally proven. To the best of our knowledge, this is the first use of a formal verification toolkit to certify a lower bound for an online problem.

We also provide the first non-trivial lower bounds for Online Bin Stretching with 6, 7 and 8 bins, and increase the best known lower bound for 3 bins. We describe in detail the algorithmic improvements which were necessary for the discovery of the new lower bounds, which are several orders of magnitude more complex.
1 Introduction

The problem ONLINE BIN STRETCHING has been introduced by Azar and Regev [1] as a semi-online generalization of the ONLINE BIN PACKING problem. Specifically, the task consists of packing various-size elements (items) arriving in an online fashion into \( m \) different bins. The problem belongs to the category of semi-online problems as there is a guarantee (known beforehand) that all the input items can be packed into \( m \) bins of a given size \( g \). The objective is to minimize the load of the largest bin. The performance measure (here named the stretching factor) of an online algorithm is the maximum for all inputs of the load of the largest bin divided by \( g \).

Note that this setting is equivalent to the classical scheduling problem ONLINE MAKESPAN MINIMIZATION where the optimal makespan of the instance is known in advance to the algorithm.

Lower bounds for ONLINE BIN STRETCHING

The earliest lower bound for ONLINE BIN STRETCHING is due to Kellerer et. al. [2], dating even before the introductory paper of Azar and Regev [1]. Kellerer et. al. [2] show a lower bound of \( 4/3 \) for the case of 2 bins, while Azar and Regev [1] extend it to an arbitrary number of bins. In the special case of 2 bins, it is known that this lower bound is tight, as there is an algorithm with stretching factor \( 4/3 \) [2].

On the algorithmic front, the earliest algorithm for any number of bins [1] achieves a stretching factor of 1.625. More efficient algorithms have since been proposed, and the current best algorithms designed by Böhm et. al. [3] have a stretching factor of 1.5 for any number of bins and 11/8 = 1.375 for exactly 3 bins.

The original lower bound of \( 4/3 \) for any number of bins \( m \) is depicted in Figure 1. Each node of the tree corresponds to a state of the online process: the online algorithm has packed the current items in the bins, and the next item of the instance is provided. Each child of a node represents a possible choice for the online algorithm in which all bin sizes are less than 4. Every leaf node contains a proof of existence of a packing that fits all items into \( m \) bins of capacity 3.

Since the earliest publication of the lower bound above in 1997 (for two bins, [2]), significant effort has been spent by several research groups in order to discover a new lower bound with a better ratio. Despite those efforts, no better lower bound is known for general \( m \).

Positive progress has been made for cases with small fixed values of \( m \). Gabay, Brauner and Kotov [4] present a lower bound of \( 19/14 \approx 1.357 \) for \( m = 3 \) using an extensive computer search, essentially an implementation of the minimax algorithm in this setting.

As the binary result (true or false) of such a computer program should not be blindly trusted, as it is prone to human error, they produced a decision tree, similar to Figure 1 in order to prove the result. Printing out this tree required a 6-page appendix and it can therefore still be verified by a human, but such a task is quite tedious. This lower bound has been independently generalized to \( m = 4 \) machines by Gabay et. al. [5] and Böhm et. al [6], and the strategy was already too large to be printed on paper. Subsequent research by Böhm et. al. [6] leads to the current state described in Table 1. The first contribution of this paper is to extend these results as presented in Table 2. Specifically, the lower bound of 19/14,
which was already established for $m \in \{2, 3, 4, 5\}$, is now established for the settings $m \in \{6, 7, 8\}$. For $m = 3$, which is the only setting for which a lower bound larger than $19/14$ is known, we have also improved it from $45/33$ to $112/82$ (recall that the best upper bound here equals $11/8 = 112.75/82$). It should be noted that the size of the trees involved has dramatically increased, going from a few thousands of nodes to billions of nodes. This is the consequence of several major improvements in the computer program, which were previously described in the PhD thesis of one of the authors [7], and which we detail in this paper.

Certified algorithms

Due to the enormous increase in the size of the strategy output, the aforementioned researchers had to resort to a separate program, which can be called a checker, in order to verify the validity of the tree. Therefore, the lower bounds proved so far depend on the correctness of this checker program. It should be noted that the trees are not actually stored as explicitly, but rather use a DAG structure in order to avoid duplicate subtrees.

This method of computing lower bounds falls into the definition of certifying algorithms, which were introduced by Blum and Kannan in [8]. Such an algorithm can be defined as providing a certificate, or a witness in addition to the classic output: given an input $x$, it computes the output $y$ and provides a witness $w$. The certifying algorithm is accompanied by a checker program, which is typically much simpler, and which can verify, given $x$, $y$, and $w$, that $y$ is a valid solution. In our context, the witness corresponds to the tree describing the strategy. Such a strategy has for instance been adopted in the algorithmic library LEDA [9] concerning the
maximum cardinality matching problem on graphs. The remaining drawback of this approach is that the checker program still has to be correct in order to trust the solution $y$. While the program was arguably simple, Alkassar et. al. [10] used the automatic verifier VCC and the interactive theorem prover Isabelle [11] in order to build a formal proof of the correctness of the checker program. Surprisingly, there was a bug in the checker, which could make it accept a wrong solution for some ill-formed witness. For a complete survey on the domain of certifying algorithms, we refer the reader to [12]. Subsequent works on checker verification can be found in [13, 14, 15].

The lesson that one can learn from this example is that it is arguably dangerous to base a result on the output of a non-trivial program, even if this program seems simple, such as the checker of the online bin stretching lower bounds. The second contribution of this paper is therefore to provide a certified checker. Specifically, we use the proof assistant Coq [16] to formalize the Online Bin Stretching problem. Then, we build a checker in the Gallina language used by Coq. We prove that if this checker returns true given a strategy tree, then the corresponding lower bound is valid. Finally, we run the program on the existing trees in order to certify their validity. This program has been developed to be easy to be reused in the future in order to certify new lower bounds. To the best of our knowledge, this is the first time that a proof assistant software is used to certify such a lower bound found by computer search. We hope that this contribution will help to establish the standard of formally verifying results that cannot be properly checked by a human.

It should be noted that we do not provide any certified result if the computer search procedure does not find a lower bound. As the item sizes are constrained in the computer search, there may exist a lower bound requiring other item sizes.

The rest of the paper is organized as follows. In Section 2, we formally define the problem. In Section 3, we describe in detail the program that we used to improve the best known lower bounds via computer search. In Section 4, we propose a formalization of the lower bound property in Coq, prove that this property matches

| Value of $m$ | 2   | 3   | 4   | 5   | any |
|--------------|-----|-----|-----|-----|-----|
| Upper bound  | 4/3 | 1.375 | 1.5 | 1.5 | 1.5 |
| Lower bound  | 4/3 | 45/33 | 19/14 | 19/14 | 4/3 |
| Tree nodes   | 1.333 | 1.3636 | 1.357 | 1.357 | 1.333 |
| Value of $m$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
| Lower bound  | 4/3 | 112/82 | 19/14 | 19/14 | 19/14 | 19/14 | 19/14 |
| Tree nodes   | 1.333 | 1.3658 | 1.357 | 1.357 | 1.357 | 1.357 | 1.357 |

Table 1: Previously known lower bounds and number of nodes in the tree describing them.

Table 2: Current known lower bounds and number of nodes in the tree describing them.
our definition, and detail the results obtained on the best known lower bounds. Note that we do not detail here the Coq proofs nor the checker, as they are not necessary to prove the result. The complete code is available online at [17] and [18].

2 Bin stretching as a two-player game

In this section, we formally define Online Bin Stretching with integer-sized items as an equivalent two-player game with the two players named Algorithm and Adversary.

The bin stretching game (BSG) will be parameterized by three positive integers \( m, t \) and \( g \). Before proceeding formally, we wish to note that \( m \) stands for the number of bins (machines) in the instance, \( t \) stands for the target of the Algorithm and \( g \) corresponds to the guarantee that the Adversary must satisfy.

Definition 1. The bin stretching game \( \text{BSG}(m, t, g) \) is a two player game defined as follows: During each round (indexed by \( i \)), the player Adversary chooses a positive integer \( e_i \), corresponding to the size of the next item of the input sequence. After that, the player Algorithm chooses a bin index \( b_i \) between 0 and \( m - 1 \), into which he packs this item. The player Adversary wins if and only if there exists a round \( j \) such that:

1. (Hitting the target.) Algorithm loads a bin to capacity \( t \), i.e., there exists a bin index \( b \) such that \( \sum_{i \leq j} (1_{b_i=b} \cdot e_i) \geq t \), where \( 1_{b_i=b} \) equals 1 if \( b_i = b \) and 0 otherwise.

2. (The guarantee.) There exists a packing of the items \( \{e_i, i \leq j\} \) into \( m \) bins with capacity at most \( g \).

Note that after some amount of rounds (at most during the \( gm \)-th round), Adversary cannot win as any subsequent packing will have a load of at least \( g + 1 \). Thus, whenever the player Adversary is unable to present any item, we note the game state as winning for the player Algorithm.

We will often refer to a specific state of the game in progress, and we wish to describe two possible representations of such a state.

Definition 2. A bin configuration is a state of the game before the player Adversary makes a move (presents an item). Such a configuration can be represented as a pair \( (I, Lo) \), where \( Lo \) is an \( m \)-tuple of loads of the bins and \( I \) is a list of items such that these items can be packed into \( m \) bins forming exactly the \( m \)-tuple \( Lo \).

We also define the extended representation of a bin configuration as the \( m \)-tuple \([E_1, E_2, \ldots, E_m] \), where each \( E_i \) is a list of items which are currently packed into bin \( i \).

For example, suppose \( m = 3 \) and the following items were presented by the player Adversary so far: \([1, 1, 2, 3, 4]\). Then, one possible bin configuration might be \(([1, 1, 2, 3, 4], [5, 4, 2]) \) with the extended representation being \(([4, 1], [3, 1], [2]) \).

A careful reader will observe there can be another extended representation of the same bin configuration, namely \(([3, 2], [4], [1, 1]) \). However, it is true that, from the point of view of both Algorithm and Adversary, the game state is the same.
– the loads are the same and the sequence of items is also. There is nothing in
the second representation that either player can use to their benefit compared to
the first representation. Thus, it is correct to treat them as variants of a single bin
configuration.

The property of existence of a winning strategy for player Adversary is clearly
the main goal of our efforts. We define it formally, so that we can refer to it later,
during our verification efforts:

**Definition 3.** If the player Adversary has a winning strategy, we say that BSG(m, t, g)
satisfies the property $\text{LowerBoundBS}$. This implies that no online algorithm can solve
the Online Bin Stretching problem with a stretching factor smaller than $t/g$.

If the player Adversary has a winning strategy for the extended game with a
starting bin configuration $(I, Lo)$, we say that BSG(m, t, g) satisfies the property
$\text{LowerBoundBS}(I, Lo)$.

If $Lo$ is composed only of zeros, we then have by definition:

$\text{LowerBoundBS}(\text{nil}, Lo) \implies \text{LowerBoundBS}$.

### 2.1 Notation

**Sizes** Most of the time, it is convenient for us to use the variable of the item $e$
interchangeably as its size. When it is useful to distinguish these two concepts, we
also use the function $s(e)$ for the size of item $e$. Extending the notation, we also
use $s(A)$ for the current load of bin $A$ (sum of the items packed within) and $s(T)$
for the total load of a set, list, or tuple of bins $T$.

**Set operations on bin configurations** The bin configuration $(I, Lo)$ consists
of the $m$-tuple of loads $Lo$ and list (multiset) of items $I$. We use set-theoretical
notation to refer to individual bins within $Lo$; for example, we may choose an
arbitrary bin $A \in Lo$ or consider $Lo$ without the largest bin $B_1$, which we would
denote as $Lo \setminus B_1$.

### 3 Computing new lower bounds via computer search

Our implemented algorithm is a parallel, multi-computer implementation of the
classical minimax game search algorithm. We now describe a pseudocode of its
sequential version. The main procedure of the minimax algorithm is the procedure
**Sequential** as stated below, which recursively calls the evaluation subroutines
**EvaluateAdversary** and **EvaluateAlgorithm**. The peculiarities of our algo-

The one of the differences between our algorithm and the algorithm of Gabay et. al.
[5] is that our bin stretching game from Definition 1 contains no concept of payoff,
and thus our algorithm makes no use of alpha-beta pruning – indeed, as either
Algorithm or Adversary has a winning strategy from each bin configuration,
there is no need to use this type of pruning.
Algorithm 1 Procedure \texttt{EvaluateAdversary}

Input is a bin configuration \( C = (I, Lo) \).

1: if the configuration is cached (Section 3.2), return the value found in cache.
2: Create a list \( L \) of items which can be sent as the next step of the player Adversary (Section 3.1).
3: for every item size \( i \) in the list \( L \) do
4:   Recurse by running \texttt{EvaluateAlgorithm}(\( C, i \)).
5:   if \texttt{EvaluateAlgorithm}(\( C, i \)) returns 0 (the configuration is winning for player Adversary), stop the cycle and return 0.
6: if the evaluation reaches this step, store the configuration in the cache and return 1 (player Algorithm wins).

Algorithm 2 Procedure \texttt{EvaluateAlgorithm}

Input is a bin configuration \( C = (I, Lo) \) and item \( i \).

1: Prune the tree using known algorithms (Section 3.3.1).
2: for any one of the \( m \) bins do
3:   if \( i \) can be packed into the bin so that its load is at most \( t - 1 \) then
4:     Create a configuration \( C' \) that corresponds to this packing.
5:     Run \texttt{EvaluateAdversary}(\( C' \)).
6:     if \texttt{EvaluateAdversary}(\( C' \)) returns 1, return 1 as well.
7: if we reach this step, no placement of \( i \) results in victory of Algorithm; return 0.

Algorithm 3 Procedure \texttt{Sequential}

Input is a bin configuration \( C = (I, Lo) \).

1: Fix parameters \( m, t, g \).
2: Run \texttt{EvaluateAdversary}(\( C \)).
3: if \texttt{EvaluateAdversary}(\( C \)) returns 0 then
4:   return success (a lower bound exists).
5: else
6:   return failure.
3.1 Verifying the offline optimum guarantee

When we evaluate a turn of the Adversary, we need to create the list \( L = \{0, 1, \ldots, y\} \subseteq \{0, 1, \ldots, g\} \) of items that Adversary can actually send while satisfying the Online Bin Stretching guarantee. In other words, we compute the value \( y \) representing the maximum item size that the adversary can send while satisfying the guarantee. We do this operation inside the procedure \text{MaxFeas}, which we describe in Section 3.1.1.

3.1.1 Procedure \text{MaxFeas}

If we wish to directly compute the maximum feasible value \( y \) which can be sent from the configuration \((I, Lo)\) where \( |I| = n \), we can do so by calling the dynamic program \text{DynprogMax} (Section 3.1.2). The complexity of \text{DynprogMax} is \( O(n \cdot g^m) \) in the worst-case scenario (if we ignore potential slowdowns via hashing).

This is polynomial when \( m \) is a constant, but already for \( m = 3 \) and especially when \( 4 \leq m \leq 8 \) such a call per game state becomes prohibitively expensive. Therefore, we first compute estimates \( LB \) and \( UB \) on the value \( y \) such that \( LB \leq y \leq UB \). Ideally, our faster estimates can get to the ideal value directly, making the dynamic programming call unnecessary.

We first initialize the upper bound from previous computations. The upper bound will be set as \( UB := \min(y', m \cdot g - V) \), where \( y' \) is the maximum feasible value that was computed in the previous vertex of Adversary’s turn, and \( V \) is the total size of all items in the instance. The second term \( m \cdot g - V \) is therefore the sum of all items that can yet arrive in this instance.

Online Best Fit To find the first lower bound on \( y \) quickly, we employ an online bin packing algorithm Online Best Fit. This algorithm maintains a packing of items \( I \) to \( m \) bins of size \( g \) during the evaluation of the algorithm \text{Sequential}, packing each item as it is selected by the player Adversary. The algorithm Online Best Fit packs each item \( i \) into the most-loaded bin where the item fits.

Once the algorithm \text{Sequential} selects a different item \( i' \) and evaluates a different branch of the game tree, Online Best Fit removes \( i \) from its bin and inserts \( i' \) to the most-loaded bin where \( i' \) fits.

As Online Best Fit maintains just one packing, which may not be optimal, it can happen that Online Best Fit is unable to pack the next item \( i \) even though \( i \) is a feasible item. In that case, we mark the packing as inconsistent and do not use the lower bound from Online Best Fit until its online packing becomes feasible again.

If the packing maintained by Online Best Fit is still feasible, we return as the lower bound value \( LB \) the amount of unused space on the least-loaded bin.

The main advantage of Online Best Fit is that it takes at most \( O(m) \) time per each step, and especially for the earlier stages of the evaluation its returned value can match the value of \( y \).

Checking the cache Next, if a gap still remains between \( LB \) and \( UB \), we try to tighten it by calling a procedure \text{Query} which queries the cache of feasible and infeasible item multisets. The procedure has a ternary answer – either an item
multiset $\mathcal{I} \cup \{j\}$ was previously computed to be feasible, or it was computed to be infeasible, or this item set is not present in the cache at all.

We update $LB$ to be the largest value which is confirmed to be feasible, and update $UB$ to be 1 less than the smallest value confirmed to be infeasible.

**Best Fit Decreasing** If the values $LB$ and $UB$ are still unequal, we employ a standard offline bin packing algorithm called **Best Fit Decreasing**. **Best Fit Decreasing** takes items from $\mathcal{I}$ and first sorts them in decreasing order of their sizes. After that it considers each item one by one in this order, packing it into a bin where it “fits best” – where it minimizes the empty space of a bin. We can also interpret it as first sorting the items in decreasing order and then applying the algorithm **Online Best Fit** defined above.

As for its complexity, **Best Fit Decreasing** takes in the worst case $O(m \cdot I)$ time. It does not need to sort items in $\mathcal{I}$, as the internal representation of $\mathcal{I}$ keeps the items sorted.

As with **Online Best Fit**, the lower bound $LB$ will updated to the maximum empty space over all $m$ bins, after **Best Fit Decreasing** has ended packing. Such an item can always be sent without invalidating the **Online Bin Stretching** guarantee.

### 3.1.2 Procedure DynprogMax

Procedure **DynprogMax** is a sparse modification of the standard dynamic programming algorithm for **Knapsack**. Given a multiset $\mathcal{I}$, $|\mathcal{I}| = n$ on input, our task is to find the largest item $y$ which can be packed together with $\mathcal{I}$ into $m$ bins (knapsacks) of capacity $g$ each.

Instead of initializing the entire DP table, our sparse approach uses a queue-based algorithm that generates a queue $Q_i$ of all valid $m$-tuples that can arise by packing the first $i$ items. We do not need to remember where the items are packed, only the sorted loads of the bins represented by the $m$-tuple.

To generate a queue $Q_{i+1}$, we initialize it to be an empty queue. Next, we traverse the old queue $Q_i$ and add the new item $\mathcal{I}[i+1]$ to all bins as long as it fits, creating up to $m$ new tuples that need to be added to $Q_{i+1}$.

Unsurprisingly, we wish to make sure that we do not add the same tuple several times during one step. We can use an auxiliary $\{0, 1\}$ array for this purpose, but we have ultimately settled on a hash-based approach.

We use a small array $A$ of 64-bit integers (of approximately $2^{10} \cdot 2^{13}$ elements). When considering a tuple $t'$ that arises from adding $i$ to one of the bins in the tuple $t$, we first compute the hash $h(t')$ of the tuple $t'$. Since we use Zobrist hashing (see Section 3.2), this operation takes only constant time.

Next, we consider adding $t'$ to the queue $Q_{i+1}$. We use the first $10 - 13$ bits of $h(t')$ (let $f$ denote their value) and add $t'$ to $Q_{i+1}$ when $A[f] \neq h(t')$ – in other words, when the small array $A$ contains something other than the hash of $t'$ at the position $f$. We update $A[f]$ to contain $h(t')$ and continue.

While our hashing technique clearly can lead to duplicate entries in the queue, note that this does not hurt the correctness of our algorithm, only its running time in the worst case.
We continue adding new items to the tuples until we do \( n \) steps and all items are packed. In the final pass of the queue, we look at the empty space \( e \) in the least-loaded bin. The output of \( \text{DYNPROGMAX} \) and the value of \( y \) is the maximum value of \( e \) over all tuples in the final pass.

Ignoring the collisions of the hashing scheme (which can happen but will not play a big role if we compute the expected running time based on our randomized hashing function), the time complexity of the procedure \( \text{MAXFEAS} \) is quite high in the worst case: \( \mathcal{O}(|I| \cdot g^n) \).

Nonetheless, we are convinced that our approach is much faster than implementing \( \text{MAXFEAS} \) using integer linear programming or using a CSP solver (which has been done in [5]) and contributes to the fact that we can solve much larger instances.

### 3.2 Caching

Our minimax algorithm employs extensive use of caching. We cache solutions of the dynamic programming procedure \( \text{MAXFEAS} \) as well as any evaluated bin configuration \( C \) (as a hash) with its value.

**Hash table properties** We store a large hash table of bin configurations with a 64-bit integer hash. The hash table is addressed by a prefix of the hash, usually between 20 – 30 bits (depending on the computer used).

We solve collisions by a simple linear probing scheme of a fixed length (say 4). In it, when a value needs to be inserted to an occupied position, we check the following 4 slots for an empty space and we insert the value there, should we find it. If all 4 slots are occupied, we replace one value at random.

**Hash function** Our hash function is based on Zobrist hashing [19], which we now describe.

For each bin configuration, we count occurrences of items, creating pairs \((i, f)\) belonging to \(\{1, \ldots, g\} \times \{0, 1 \ldots, m \cdot g\}\), where \(i\) is the item type and \(f\) its frequency (the number of items of this size packed in all \(m\) bins).

As for the loads of the \(m\) bins, we maintain that they are sorted in descending order. We also think of them as ordered pairs \((j, g)\), with \(j\) being the position of the bin in the ordering (e.g. 1 – largest, \(m\) – smallest) and \(g\) the actual value of the load.

For example, we can think of bin configuration \(((3, 3, 2), \{1, 1, 1, 2, 3\})\) as a set of load pairs \((1, 3), (2, 3), (3, 2)\) along with pairs for items: \((1, 3), (2, 1), (3, 1), (4, 0), (5, 0)\) and so on.

At the start of our program, we associate a 64-bit number with each pair \((i, f)\). We also associate a 64-bit number for each possible load of one bin. These two sets of numbers are stored as a matrix of size \(g \times (m \cdot g)\) and a matrix of size \((l – 1) \times m\).

The Zobrist hash function is then simply a XOR of all associated numbers for a particular bin configuration.

The main advantage of this approach is fast computation of new hash values. Suppose that we have a bin configuration \(m\) with hash \(H\). After one round of the player \textsc{Adversary} and one round of the player \textsc{Algorithm}, a new bin configuration \(B'\) is formed, with one new item placed.
Calculating the hash $H'$ of $B'$ can be done in time $O(m)$, provided we remember the hash $H$; the new hash is calculated by applying XOR to $H$, the new associated values, and the previous associated values which have changed.

**Caching of the procedure MAXFEAS** We use essentially the same approach for caching results in the procedure MAXFEAS, except only the $m$-tuple of loads needs to be hashed.

We also remark upon the values being cached in the procedure MAXFEAS. At first glance, it seems that it might be best to store the value of $y$ with each input multiset $I$. However, this is a very bad idea, as we would lose upon a lot of symmetry.

Indeed, if we set $i$ to be any item from the list $I$, we would lose out on the fact that we know a lower bound on the largest value that can be sent for a multiset $I \setminus \{i\} \cup \{y\}$ – namely $s(i)$, the value we know is compatible.

Instead, it is much better to cache binary feasibilities or infeasibilities for a specific multiset $I$. We use these results to improve the values of $LB$ and $UB$ for other calls of procedure MAXFEAS.

### 3.3 Tree pruning

Alongside the extensive caching described in Subsection 3.2, we also prune some bin configurations where it is possible to prove that a simple online algorithm is able to finalize the packing. Such a bin configuration is then clearly won for player ALGORITHM, as it can follow the output of the online algorithm.

#### 3.3.1 Algorithmic pruning

Recall that in the game $\text{BSG}(m, t, g)$, the player ALGORITHM is trying to pack all items into $m$ bins with load at most $t - 1$. If the search algorithm can quickly deduce that a bin configuration leads to a successful packing, we can immediately evaluate the configuration as winning for the player ALGORITHM and thus prune the tree.

We can lift several such winning tests – so-called *good situations* for the player ALGORITHM – from the algorithmic results of Böhm, Sgall, van Stee and Vesely [6]. However, since the number of bins $m$ rises from 3 in [4] up to 8, the situations can not always be directly generalized.

We now state the new situations that we have generalized from [6] for $\text{BSG}(m, t, g)$ with $m \geq 4$. For $m = 3$, we use the good situations directly from [6].

For the following, we set $\alpha$ to be the extra space that the player ALGORITHM can use without losing, namely $\alpha = (t - 1) - g$. We also make use of the notation from Section 2.1.

**Good Situation 1.** Given a bin configuration $(I, Lo)$ such that the total load of all but the last bin is at least $(m - 1) \cdot g - \alpha$, there exists an online algorithm that packs all remaining items into $m$ bins of capacity $t - 1$.

*Proof.* If the total amount packed is $(m - 1)g - \alpha$, the remaining volume for the instance is $mg - (m - 1)g - \alpha = g + \alpha = t - 1$, which will always fit on the last bin. \[\square\]
**Good Situation 2.** Given a bin configuration \((I, Lo)\) such that there exist two bins \(A, B\) such that:

1. \(s(Lo \setminus \{A, B\}) \geq (m - 2)g - 2\alpha - 1\),
2. there exists a bin \(C \in Lo \setminus \{A, B\}\) with load below \(\alpha\).

then there exists an online algorithm that packs all remaining items into \(m\) bins of capacity \(t - 1\).

**Proof.** We pack the remaining input first into \(A\) until an item cannot fit – we place that item into \(C\), where it always fits. After the item is packed into \(C\), the load of \(Lo \setminus \{B\}\) satisfies

\[
s(Lo \setminus \{B\}) \geq (m - 2)g - 2\alpha - 1 + (g + \alpha + 1) \geq (m - 1)g - \alpha,
\]

which means Good Situation \([I]\) is reached.

**Good Situation 3.** Consider a bin configuration \((I, Lo)\), and let \(B_m\) stand for the least-loaded bin in \(Lo\) and \(B_{m-1}\) for the second-least-loaded bin. Define the following sizes:

1. Let \(s\) be the sum of loads of all bins excluding the last two.
2. Let \(r\) (the last bin load requirement) be the smallest number for which the following holds: If the load of \(B_m\) is raised to \(r\), GS\([I]\) is reached (after reordering the bins).
3. Let \(o\) (the overflow) be defined as \(t - r\).

Then, if \(r \leq t - 1\) and if any bin \(A \in \{B_{m-1}, B_m\}\) satisfies

\[
\alpha \geq s(A) \geq (m - 1) \cdot g - \alpha - o - s,
\]

there exists an online algorithm that packs all remaining items into \(m\) bins of capacity \(t - 1\).

**Proof.** We first observe that if the bin \(B_{m-1}\) is raised to \(r\), GS\([I]\) is also reached. This follows from the definition of \(r\), which checks for the condition of GS\([I]\) that sums the load on all except the least-loaded bin. The definition of \(r\) implies that \(B_m\) overtakes \(B_{m-1}\), as otherwise GS\([I]\) holds immediately. Raising \(B_{m-1}\) to \(r\) instead (and keeping \(B_m\) as the least-loaded bin) leads to the same calculation, and thus the same conclusion of reaching GS\([I]\).

Let \(A\) be the bin satisfying the bound \(\alpha \geq s(A) \geq (m - 1) g - \alpha - o - s\) and let \(B\) be the other bin from \(\{B_{m-1}, B_m\}\). The algorithm packs greedily into \(B\). If \(B\) reaches the threshold load \(r\), then GS\([I]\) is reached. Assuming the threshold is not reached, there exists a currently unpacked item of size at least \(o\) that does not fit into \(B\). As \(s(A) \leq \alpha\), the item \(o\) can be packed into \(A\). Summing up loads on \(Lo \setminus \{B\}\), we include \(s\) (for all bins except \(B_{m-1}\) and \(B_m\)), \(o\) and the lower bound on \(s(A)\), which sums up to at least

\[
s + o + (m - 1) \cdot g - \alpha - o - s \geq (m - 1) \cdot g - \alpha,
\]

which is sufficient for GS\([I]\).  

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3.3.2 Adversarial pruning

The algorithmic pruning of Section 3.3.1, consisting of 5 situations from the literature for $m = 3$ and 3 generic ones for $m \geq 4$, is a crucial component of our computational approach – at least according to informal experiments done during our programming efforts. In contrast, we have only a few tools to quickly detect that a bin configuration is winning for the player \textsc{Adversary}. More specifically, we use the following two criteria:

**Large item heuristic** Once any bin has load at least $t - g$, an item of size $g$ packed into that bin would cause it to reach load $t$, which is a victory for the player \textsc{Adversary}. Suppose that the $k$-th bin reaches load $l \geq t - g$. We compute the size of the smallest item $i$ such that

1. $i + l \geq t$;
2. For any bin $B_t$ with $t \in [(k + 1), m]$ it holds that $s(B_t) + 2l \geq t$; in other words, \textsc{Algorithm} cannot pack two items of size $l$ into any bin starting from the $(k + 1)$-st.

Finally, we check if \textsc{Adversary} can send $m - k + 1$ copies of the item of size $l$. If so, it is a winning bin configuration for this player and we prune the tree.

Notice that there may be multiple different values of $l$ for one bin configuration; for instance, in the setting of $t = 19$, $g = 14$, for 5 bins with loads $(10, 10, 10, 9, 1)$, we should check whether we can send 2 items of size 10 or 3 items of size 9. Therefore, in the implementation, we compute for each bin its own candidate value of $l$ and then check whether at least one is feasible using the dynamic programming test described in Section 3.1.

**Five/nine heuristic** We use a specific heuristic for the case of $t = 19$, $g = 14$, as it is a good candidate for a general lower bound. This heuristic was experimentally observed to slightly compress the size of the output tree in this setting.

This heuristic comes into play once there is a bin of load at least 5 and once all bins are non-empty (even load 1 is sufficient). The item sizes 5 and 9 are complementary in the sense that one of each can fit together in the optimal packing of capacity 14, but the two of them cannot be packed together into a bin that already has load at least 5.

A pair of items of size 9 also cannot fit together into any other bin – as all the bins have already load at least 1.

Finally, if there are many bins of load at least 5 and the guarantees allow an input consisting of sufficiently many items of size 14, we may again reach a bin of load at least 19.

We apply this heuristic only when it is true that at all times, $m$ items of size 9 can arrive on input without breaking the adversarial guarantee. While the condition is true, all bins must have load strictly below 10, or a load of 19 is reached immediately.

Our heuristic considers repeatedly sending items of size 5. If at any point there are only $p$ bins left with load strictly less than 5 and at the same time $p + 1$ items of size 14 can arrive on input, the configuration is winning for the player \textsc{Adversary}. On the other hand, if at any point there is a bin of load at least
10 and the invariant that $m$ items of size 9 can still arrive holds, we are also in a winning state for Adversary.

If it is true that by repeatedly sending items of size 5 we eventually reach at least one of the aforementioned two situations, we mark the initial bin configuration as winning for the player Adversary.

**A note on performance** While both of our heuristics reduce the number of tasks in our tree and the number of considered vertices, we were unable to evaluate them in every single vertex of the game tree without a performance penalty. Even the large item heuristic, which can be implemented with just one additional call to the dynamic programming procedures of Section 3.1 slows the program down considerably.

This is likely due to the fact that caching outputs of the dynamic programming calls of Section 3.1 lead to some vertices that do not need to call any dynamic programming procedure, while with our heuristics they are forced to call at least one.

### 3.4 Monotonicity

One of the new heuristics that enables us to go from a lower bound of $19/14$ on 5 bins to 8 bins is iterating on lower bounds by monotonicity. We define it as follows:

**Definition 4.** A winning strategy for Adversary has monotonicity $k$ if it is true that for any two items $e_i, e_{i+1}$ such that $e_{i+1}$ is sent immediately after $e_i$, we have $e_{i+1} \geq e_i - k$.

Using this concept, we can iterate over $k$ from 0 (non-decreasing instances) to $g - 1$ (full generality) to find the smallest value of monotonicity which leads to a lower bound, if any.

A potential downside of iterating over monotonicity is that it can introduce an $g$-fold increase in elapsed time in the case that no lower bound exists. Additionally, it is quite likely that monotonicity becomes less useful as the value of $g$ increases, as the item of relative size 1 gets smaller and smaller.

Still, solving decision trees of low monotonicity is much faster than solving the full tree, and we have empirically observed that lower bounds of lower monotonicity are fairly common; see Tables 3 and 4 for our empirical results.

**Monotonicity caveat** It is important to remark that when looking for a lower bound for a specific monotonicity value, it is *not true anymore* that a bin configuration is sufficient to describe one state of the bin stretching game. To see this, consider monotonicity 1. If the first three input items are 1, 2, 3, the next item needs to be of size 2 or larger. However, if the three input items are 1, 3, 2 (which is permissible for monotonicity 1), the next item on input can be of size 1 and above. This means that the two states are not equivalent, even though their bin configuration is the same.

To remedy this, we internally extend the definition of the bin configuration by also marking which item arrived last in the input sequence, which is sufficient for a fixed value of the monotonicity.
3.5 Parallelization

Up until now, we have described a single-threaded minimax algorithm with caching and pruning. To get the computing power necessary for results above 5 bins, we have implemented the minimax search as a parallel program for a computer cluster. We now describe the particulars of this implementation.

**Tasks** Our evaluation of the game tree proceeds in the following way: first, we start evaluating the game tree on the main computer (which we internally call *queen*) until a vertex corresponding to ADVERSARY’s next move meets a certain threshold (for instance, sufficient depth). After that, we designate this adversarial vertex as a *task*.

Alongside the queen, we have processes whose job is to evaluate the tasks – we call them the *workers*. Workers which run on the same machine will have a common cache that they access via atomic primitives in order to maintain consistency. Workers on separate machines do not share information.

Due to the mixed environment of standard Unix threads and MPI processes, we also have a single *overseer* per each physical machine. This overseer handles the MPI communication as well as spawning the individual worker threads.

The tasks are all generated in advance by the queen. After that, their bin configurations are synchronized with all overseers running. The queen then assigns tasks to overseers online, namely by assigning a batch of 250-500 tasks to an overseer. The overseer reports each value of a finished task immediately to the queen. When an overseer is finished processing a batch, it requests and receives a new one.

We have selected this communication strategy for two reasons:

1. To minimize congestion in the processing phase through the fact that the bin configurations are synchronized beforehand and only identifiers are shared in the online assignment phase.
2. To allow the queen to evaluate and prune unfinished tasks and therefore avoid some unnecessary processing by the workers.

**Task selection** As mentioned above, an important decision to be made by the lower bound algorithm designer is where to split a vertex of the game tree into a task and send it to be processed in the parallel environment.

Based on our experiments, it seems that maintaining a right balance of the number of tasks as well as their running time is crucial to good performance. When the tasks are too shallow, the performance of the algorithm is dominated by the elapsed time of the most difficult task in the list, which diminishes the gains coming from the parallel implementation.

On the other hand, if there are millions of tasks, the algorithms will still work correctly but we might lose performance from diminishing advantages of individual caching as well as due to pruning happening later in the process.

At the outset of our computational efforts, we have only used *task depth* as the principal guideline – when $k$ items arrived on input (with $k$ usually in the range of $\{4, 5, 6\}$), we mark the bin configuration as a new task.

Experimenting with running time has shown us that the presence of larger items speeds up the overall evaluation of the lower bound. One possible reason may be
that large items come with a lower bound on the upcoming item size, provided
monotonicity is set to be small.

Therefore, we have ultimately settled on a mixed task threshold function which
takes into account both the task depth $k$ and also the task load $l$, which is the sum
of sizes of all items arrived so far in the instance. We split off a task when its task
load is above $l$, and failing that when its task depth is below $k$. After some practical
experiments, we have settled on setting $k \in \{5, 6, 7\}$ and $l$ to be around $20 - 40$
of the optimal bin capacity $g$.

**Initial strategy**  Our implementation also allows us to pre-select some initial
strategy for the player ADVERSARY in advance. This way we can use our (so far
limited) intuitive understanding of what is a good initial move and decrease the
time needed to evaluate the whole tree.

A particularly good strategy for the lower bound of $19/14$ ($t = 19, g = 14$) seems
to be sending an item of size 5 as the first item, followed by several (5 in the case
of $m = 8$) items of size 1. This adversarial strategy leads to a lower bound instance
for 6, 7 and 8 bins.

We have therefore implemented a way to pre-select items to be sent in the first
few rounds of the game. Given such a list of items, we compute all possible moves
of the player ALGORITHM and create a queue of bin configurations that we each
evaluate sequentially.

The fact that already this linear, non-adaptive strategy of sending 5,1,1, . . . is
enough to get a lower bound of 19/14 for 8 bins was a pleasant surprise to us. We
believe this fact is due to the size of the sequence being already non-trivial (the
item 5 alone occupies slightly more than 25% of one stretched bin).

A natural extension is to allow the user to input a partial game tree (an adaptive
strategy for the player ADVERSARY) and have the algorithm evaluate it sequentially;
this can be easily added to our implementation once we learn more about which
items should be the among the first to send.

**Technology**  We have settled on using a combination of OpenMPI [20] and the
standard thread library as provided by the C++ programming language. In our
setting, OpenMPI is used to provide inter-computer communication API for sending
and receiving tasks as described above. We employ the standard Unix threads to
spawn the worker processes themselves; this way they can easily share one large
cache for evaluated bin configurations.

We have originally considered using only OpenMPI processes for both inter-
computer communication as well as memory sharing on one physical computer;
this functionality is present in the latest version of the MPI standard, MPI-3.0.
However, after implementing the shared memory functionality, we have noticed
some slowdown of the worker processes when the shared memory was large (more
than 1 gigabyte). This forced us into the heterogeneous model that we use right
now.

### 3.6 Results

Tables 3 and 4 summarize our results; we include previous results for completeness.
Note that there may be a lower bound of size say $\frac{41}{30}$ even though none was found
when computing \( t = 41, g = 30 \); the same bound can be achieved, possibly, by setting \( t = 164, g = 120 \), which is beyond the capabilities of our current program. In addition, it is unreasonable to formally certify the correctness of the program result when no lower bound is found, so negative outputs should not be considered as definitive results. We nevertheless also report in Table 3 some candidate fractions for which the program terminates without finding a lower bound. This allows some insights on values leading to negative results and the computing time required to explore the whole solution space. We can see for instance that the fractions \( \frac{30}{22} \) and \( \frac{56}{41} \) did not lead to a lower bound although the decimal value of \( \frac{112}{82} \) is not smaller.

| Fraction | Decimal | L. b. | Mon. | Linear | Parallel |
|----------|---------|-------|------|--------|----------|
| 19/14    | 1.3571  | Yes   | 0    | 2s.    |          |
| 22/16    | 1.375   | No    |      | 2s.    |          |
| 26/19    | 1.3684  | No    |      | 3s.    |          |
| 30/22    | 1.336   | No    |      | 6s.    |          |
| 33/24    | 1.375   | No    |      | 5s.    |          |
| 34/25    | 1.36    | Yes   | 1    | 15s.   |          |
| 41/30    | 1.356   | No    |      |        |          |
| 45/33    | 1.356   | Yes   | 1    | 1min. 48s. |          |
| 55/40    | 1.375   | No    |      | 3min. 6s. |          |
| 56/41    | 1.3659  | No    |      | 30min. 7s. |          |
| 82/60    | 1.36    | No    |      | 21m. 49s. |          |
| 86/63    | 1.36507 | Yes   | 6    | 29s.   |          |
| 112/82   | 1.3659  | Yes   | 8    | 3h. 21m. 31s. |          |

Table 3: The results and performance of our linear and parallel computations for Online Bin Stretching with three bins. The results above the horizontal line were previously shown in [5] and [21]. The column L. b. indicates whether a lower bound was found when starting with the given stretching factor \( t/g \) as seen in column Fraction. The column Mon. shows the lowest monotonicity that our program needs to find a lower bound. In the case of negative results, time measurements were done only using full generality, i.e. with monotonicity \( g - 1 \). The linear results were computed on a server with an AMD Opteron 6134 CPU and 64496 MB RAM. The size of the hash table was set to \( 2^{25} \). The parallel results were computed using OpenMPI on a heterogeneous cluster with 109 worker processes running. The output of the program was not generated during the time measurements.
Table 4: The results produced by our minimax algorithm for more than 3 bins. Tested on the same machine and with the same parameters as in Table 3, both for linear and parallel computations. The result † was computed subsequently in a parallel environment with 64 threads and 512 MB of shared cache.

In columns Mon. and Parallel, we list in brackets monotonicity and elapsed time of computation for an input having an item of size 5 at the start. Monotonicity is measured only starting with the second item.

| Bins | Fraction | Decimal | L. b. | Linear Mon. (5) Elapsed time | Parallel (5) |
|------|----------|---------|-------|------------------------------|--------------|
| 4    | 19/14    | 1.3571  | Yes   |                             |              |
| 5    | 19/14    | 1.3571  | Yes   | 2 (1)                       | 10s.         |
| 4    | 30/22    | 1.36    | No    |                             | 19s.         |
| 4    | 34/25    | 1.36    | No    |                             | 48s.         |
| 4    | 45/33    | 1.36    | No    |                             | 1h. 1m 40s. †|
| 6    | 19/14    | 1.3571  | Yes   | 0 (0)                       | 11s.         |
| 7    | 19/14    | 1.3571  | Yes   | 1 (0)                       | 2m. 13s. (16s.)|
| 8    | 19/14    | 1.3571  | Yes   | Unk. (1)                    | (1h. 14s.)   |

4 Certification

We describe in this section how we certify the results obtained using the computer search of Section 3 via the Coq proof assistant system. The aim of this section, as explained in the introduction, is to formalize the lower bound property – a predicate being true if there exists a valid lower bound – in Coq, prove that this property as we define it matches the intended meaning, and comment on the technical challenges that were needed to be overcome.

We first describe the Coq formalization of the problem previously defined in Section 2. In Section 4.1 we define the relevant types and preliminary functions. In Section 4.2 we describe the core of our formalization. Specifically, we first define the function updating the bin configuration after the addition of a given item in some bin. We then define an inductive predicate that recognizes a winning strategy for Adversary, given a bin configuration. We finally use this predicate to define the main predicate LowerBoundCoq. We then show in Section 4.3 that this formalization is correct: if the Coq predicate LowerBoundCoq is true, then the property LowerBoundBS is true. Indeed, the goal of the Coq script is to prove that LowerBoundCoq is true for given values of $m$, $t$, $g$ defining the game $BSG(m, t, g)$. In Section 4.3, we therefore show that this result actually implies a lower bound on this game, i.e., implies LowerBoundBS. Finally, in Section 4.4 we present the results obtained on the files generated by the program described in Section 3 and detail some features that had to be implemented in order to handle the large file sizes involved.

The code is available online at [18], and also contains a program which translates an adversary strategy expressed using the widespread GraphViz format into a file which can be directly processed by our Coq script. Then, future lower bounds can be easily certified using the same script.

The first part of this section can serve as an introduction to formalization of
an online lower bound into the language of Coq, and we hope it will be interesting even to readers with primarily theoretical focus.

4.1 Preliminaries

We start with defining variables $m, t, g$, which correspond to the parameters of Bin Stretching from Section 3—$m$ is the number of bins, $t$ is the target capacity (how much the player Algorithm is allowed to pack), and $g$ is the guarantee capacity (how much the player Adversary is allowed to pack). We also require that $m$ is strictly positive, although for any intended use all three variables will be positive.

Variables $m, t, g : \text{nat}$.
Hypothesis $\text{Pos}m : m > 0$.

In order to distinguish some objects in our properties by a specific name, much like in a programming language, we define a few key data types. The type $\text{BinExtended}$ represents the list of items present in a given bin, and is then implemented as a list of integers. The type $\text{BinLoads}$ represents the current load of all bins, and is then also implemented as a list of integers, one per bin. The $\text{BinsExtended}$ type corresponds to a bin configuration in its extended representation (see Definition 2), and is internally represented as a list of types $\text{BinExtended}$, one per bin.

Definition $\text{BinExtended} := \text{list nat}$.
Definition $\text{Bin Loads} := \text{list nat}$.
Definition $\text{Bins Extended} := \text{list BinExtended}$.

For any list of integers, the property $\text{Iszero}$ is true if and only if the list contains only zeros and at most $m$ items. It represents the starting loads of the bins, where some bins may be omitted.

The simplest recursive functions in Coq can be defined using the Fixpoint keyword, which can be used only when the recursion is applied to a simple inductive object, such as a list. The following basic functions are defined in this manner. The function $\text{BinSum}$ returns the load of a bin, given the list of items present in this bin. The function $\text{MaxBinSum}$ returns the load of the highest (most-loaded) bin, given the bin configuration, and the function $\text{MaxBinValue}$ returns the load of the highest bin given only the vector of loads. Note that $\text{nil}$ represents the empty list and $x::s$ represents the list of head $x$ and tail $s$.

Definition $\text{Iszero} \ l := (\text{length} \ l <= m) \land (\forall \ e, \ \text{In} \ e \ l \rightarrow e = 0)$.

Fixpoint $\text{BinSum} \ (B : \text{BinExtended}) := \text{match} \ B \ \text{with}$
  | $\text{nil}$    ⇒  0
  | $x :: s$    ⇒  $x + \text{BinSum} \ s$
end.

Fixpoint $\text{MaxBinSum} \ (P : \text{BinsExtended}) := \text{match} \ P \ \text{with}$
  | $\text{nil}$    ⇒  0
end.

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\[ x :: s \Rightarrow \max (\text{BinSum} \ x) (\text{MaxBinSum} \ s) \]

\[ \text{Fixpoint} \ \text{MaxBinValue} (\text{\textbackslash codeloads}: \text{BinLoads}) := \text{match} \ \text{\textbackslash codeloads} \ \text{with} \]
\[ | \text{nil} \Rightarrow 0 | x :: s \Rightarrow \max x (\text{MaxBinValue} \ s) \]
\[ \text{end}. \]

A \text{Fixpoint} can also be defined over natural numbers, where the natural numbers themselves – defined with the successor function \( S \ k \) as in the Peano arithmetic, defined to be equal to \( k + 1 \) – serve as the inductive structure. We use this type of recursion to define the function \text{AddToBin} that models adding an item to a bin. The function takes three parameters: \( \text{Lo} \) of type \text{BinLoads} and two integers \( e \) and \( b \). This function increases the load of the \( b \)-th bin by a value equal to \( e \). If \( b \) is larger than the length of \( \text{Lo} \), a new item of value \( e \) is appended to \( \text{Lo} \).

\[ \text{Fixpoint} \ \text{AddToBin} (\text{\textbackslash Lo}: \text{BinLoads}) (e: \text{nat}) (b: \text{nat}) := \text{match} \ \text{\textbackslash Lo}, b \ \text{with} \]
\[ | \text{nil}, b \Rightarrow [e] | x :: s, 0 \Rightarrow (x+e) :: s | x :: s, (S k) \Rightarrow x :: (\text{AddToBin} \ s e k) \]
\[ \text{end}. \]

### 4.2 Defining the main predicates

We now define a few properties specific to the \text{ONLINE BIN STRETCHING} problem.

The predicate \text{SequencePacked}, given a list of item sizes (integers) \( I \) and an element \( P \) of type \text{BinsExtended}, is true if the configuration \( P \) uses at least all the items of the list (sequence) \( I \). It uses two functions which are part of the Coq standard library. The function \text{count_occ Nat.eq_dec x y} returns the number of occurrences of the element \( y \) in the list \( x \) and the function \text{concat} concatenates a list of lists of elements.

\[ \text{Definition} \ \text{SequencePacked} (I: \text{list nat}) (P: \text{BinsExtended}) := \forall e, \text{count_occ Nat.eq_dec} I \ e \ \leq \ \text{count_occ Nat.eq_dec} \ (\text{concat} \ P) \ e. \]

The predicate \text{GuaranteePacking}, given the same parameters as the predicate \text{SequencePacked}, is true if \text{SequencePacked} is true, the length of \( P \) is equal to \( m \) and no bin has load larger than \( g \). Such a packing \( P \) is then a certificate that the items described in \( I \) can be packed in \( m \) bins of capacity \( g \).

\[ \text{Definition} \ \text{GuaranteePacking} (I: \text{list nat}) (P: \text{BinsExtended}) := \text{SequencePacked} \ I \ P \land \text{length} P = m \land \text{MaxBinSum} P \leq g. \]

The main predicate used in the formulation is \text{OnlineInfeasible}, which is a parametric predicate with three variables: an integer \( X \), a list \( I \), and \( \text{Lo} \), which is one value of \text{BinLoads}. The list \( I \) corresponds to items being sent on input initially,
and the state of bins $\text{Lo}$ corresponds to one algorithmic packing of items from $I$ into $m$ bins. The auxiliary variable $X$ is not necessary in the definition, but it allows the Coq prover to easily assume an induction hypothesis when inductively proving properties of $\text{OnlineInfeasible}$.

We employ the Coq syntax to distinguish only two cases when this predicate is true, naming them $\text{Overflow}$ and $\text{Deadend}$. In the case of $\text{Overflow}$, one bin of $\text{Lo}$ is loaded to at least the value $t$, and at the same time it still holds that there exists an optimal packing for the current input list $I$.

The other case when $\text{OnlineInfeasible}$ is true, which we call $\text{Deadend}$, occurs when we can recursively reach the state $\text{OnlineInfeasible}$ being true by presenting a new item $e'$ (a positive integer) on input, for all choices of adding $e'$ into any of the bins. For practical reasons, we use a non-negative integer $e$ as the variable used for the new item, and define $e' = e + 1$, which can be written as “$S e$” in the Coq code. This is one of the simplest ways to define a positive (nonzero) integer.

The full statement of $\text{OnlineInfeasible}$ is as follows:

$$
\text{Inductive OnlineInfeasible : nat $\rightarrow$ list nat $\rightarrow$ BinLoads $\rightarrow$ Prop} :=
| \text{Overflow X I Lo : } \exists P, \text{GuaranteePacking I P} \rightarrow \text{OnlineInfeasible X I Lo}
| \text{Deadend X I Lo : } \text{length Lo} \leq m \rightarrow \exists e', \forall b < m : \text{OnlineInfeasible X ( (S e') :: I) (AddToBin Lo (S e) b) } \rightarrow \text{OnlineInfeasible (S X) I Lo}.
$$

The syntax implies the following equivalence.

$$
\forall X, Lo, I: \text{OnlineInfeasible} (X+1) I Lo \iff (t \leq \text{MaxBinValue Lo} \land \exists P : \text{GuaranteePacking I P})
\lor (\text{length Lo} \leq m \land \exists e' > 0, \forall b < m : \text{OnlineInfeasible X (e'::I) (AddToBin Lo e' b)})

\text{The final predicate defined is:}
$$

$$
\text{Definition LowerBoundCoq := exists s, Iszero s \land OnlineInfeasible (m \cdot g + 2) s.}
$$

The value $m \cdot g + 2$ in the definition of the predicate is there as a simple upper bound of the number of inductive steps sufficient for any correct proof (recall that no more than $mg$ items can arrive in any valid input for ONLINE BIN STRETCHING).

### 4.3 Correctness of the Coq formulation

The Coq proof assistant can now be used to simplify our proving efforts, using our computed instances as a proof strategy to verify that the proposition $\text{LowerBoundCoq}$ is true. What remains to be formally proven outside of the Coq system is to show that the propositions from Section 4 actually match what we actually wish to compute, namely a lower bound on the bin stretching game.

First, let us restate the winning properties for bin stretching from Section 2:
Definition 3. If the player Adversary has a winning strategy, we say that BSG($m, t, g$) satisfies the property LowerBoundBS. This implies that no online algorithm can solve the Online Bin Stretching problem with a stretching factor smaller than $t/g$.

If the player Adversary has a winning strategy for the extended game with a starting bin configuration $(I, Lo)$, we say that BSG($m, t, g$) satisfies the property LowerBoundBS($I, Lo$).

The final piece of the puzzle is to show the following theorem, which immediately implies the correctness of the Coq formulation (Corollary 1).

Theorem 1. For any $I, Lo, X$, the proposition OnlineInfeasible $X I Lo$ implies LowerBoundBS($I, Lo$).

Corollary 1. LowerBoundCoq implies LowerBoundBS.

Proof of Theorem 1. We prove this result by reverse induction on the sum of the items of $I$. Let $I$ and $Lo$ be two lists of integers and $X$ be an integer.

We make use of the decreasing parameter $X$ and the fact that $e' > 0$ in the definition of OnlineInfeasible, to prove the following property in Coq (validated in supplementary code as Theorem OI_length):

$$∀X, I, Lo, \text{OnlineInfeasible } X I Lo \implies (\text{BinSum } I \leq mg).$$

Therefore, for a list $I$ with a large sum, OnlineInfeasible($X, I, Lo$) is false for any value of $X$ and $Lo$, making Theorem 1 (an implication) true.

Moving on, we fix the value of $X$ and $Lo$. Let $L ≥ 0$ and suppose by induction that for all $I$ whose items sum to more than $L$, for all $Lo$, the proposition (OnlineInfeasible $X I Lo$) implies the property LowerBoundBS($I, Lo$).

Consider any $I$, a list whose items sum to exactly $L$, and any $Lo$ such that we have OnlineInfeasible($X, I, Lo$).

We want to show the property LowerBoundBS($I, Lo$). Using Equation 1 and the proposition OnlineInfeasible($X, I, Lo$), we have two cases.

First, if $(t ≤ \text{MaxBinValue} \text{ \backslash codloads } ∧ \exists P, \text{ GuaranteePacking I P})$ holds, then one bin of $Lo$ has load at least $t$ and there exists a packing of the items of $I$ into $m$ bins with load at most $g$. Therefore, the property LowerBoundBS($I, Lo$) is true.

Otherwise, there exists a number $e' > 0$ such that the following property holds.

$$\text{length } Lo ≤ m \quad ∧ \quad ∀b < m, \text{OnlineInfeasible } (X-1) (e'::I) (\text{AddToBin } Lo e' b) \quad (2)$$

Note that the value of $X - 1$ can be replaced by $X$ as the following is true (and proved in Coq as Theorem OI_Succ): If the predicate OnlineInfeasible is true when parameterized by $X - 1$, then it is also true when parameterized by $X$.

Consider any possible move $b$ for ALGORITHM after Adversary played $e'$. Using Equation 2 and the induction hypothesis, we know that the property LowerBoundBS($e' :: I, (\text{AddToBin } Lo e' b)$) holds. So, after ALGORITHM played $b$, Adversary has a winning strategy.

As this is true for all possible moves $b$ of ALGORITHM, we have the property LowerBoundBS($I, Lo$), which completes the proof. □
4.4 Verification of a winning strategy for Adversary

We now detail how we used the results obtained in Section 3 in order to prove the property \texttt{LowerBoundCoq} for a given game \texttt{BSG}(m, t, g). We rely on a file, computed by the aforementioned program, which describes a winning strategy for \texttt{ADVERSARY} which moves he makes after each possible move of \texttt{ALGORITHM}, as well as the packing solutions on winning states. The format is based on the tree structure illustrated in Figure 1 with several improvements described below. In order to verify that this file is a correct representation of a lower bound, we implement in Coq a function that performs multiple checks, which we call \texttt{Check} in this section. In essence, this function is analogous to the verifier program discussed in Section 1.

The crucial difference is that \texttt{Check} is certified: a theorem, proven in Coq, states that, if \texttt{Check} returns \texttt{true}, then the predicate \texttt{LowerBoundCoq} is valid for the game \texttt{BSG}(m, t, g). Then, by Corollary 1, \texttt{LowerBoundBS} is also valid for the game \texttt{BSG}(m, t, g).

Although we do not detail the complete Coq script here, which exceeds 2000 lines [18], we would like to emphasize that the format used to store the lower bound as well as the function \texttt{Check} that verifies it are not implemented in a naive manner because of the file sizes involved. The features implemented, which therefore complicate the Coq script proving the correctness of \texttt{Check}, include the following.

\textbf{DAG encoding} The naive tree decomposition of a winning strategy for \texttt{ADVERSARY} details every decision that has to be made, but may contain a large number of duplicate subtrees. Indeed, several nodes of the tree correspond to the same list of items and loads of bins (up to irrelevant permutations). Therefore, we use a DAG structure to store these duplicates. As constant-access data structures are not available in Coq, we use a single list of trees \texttt{R} to denote all the existing duplicate subtrees. When examining the possible outcomes from a node according to the decision of \texttt{ALGORITHM}, there are then three possibilities: it corresponds to a direct child of this node (if such a subtree is unique), it corresponds to a tree in the list \texttt{R}, or one bin exceeds the target load. Note that trees of the list \texttt{R} can themselves refer to subsequent trees of the same list, and we are then able to prove our results by induction. It remains to implement a fast way to check that an item is present in the next part of the list \texttt{R}. We use for this purpose an AVL tree dictionary indexed by a pair of lists describing the current items and bin loads. To assess the importance of removing these duplicates, notice on Table 5 that it decreased the largest graph size by three orders of magnitude.

\textbf{Last layer compression} Often, the last items that are sent by the adversary are independent from the decisions made by \texttt{ALGORITHM}. However, they can represent a large portion of the nodes in the normal DAG representation. Hence, we store such a situation only as a single node with a list of upcoming items, instead of the full tree. This corresponds directly to storing only one node when the \textit{large item heuristic} of Section 3.3.2 is successful. The number of nodes obtained in such a compressed DAG (cDAG) is represented in Table 5 and leads to a decrease by a factor of 5.
Binary integers Coq proofs often rely on the Peano arithmetic, where a natural integer is represented in a unary way by being either 0 or the successor of an integer. In order to decrease the time and resources required to prove our results, we perform the computations using a binary integer representation. We have therefore implemented two analogous functions, which we can name `Check_binary` and `Check_unary`, working respectively on binary and unary integers. We prove that these functions give the same result and that if `Check_unary` returns true, then `LowerBoundCoq` is valid. Therefore, we can run the function `Check_binary` while using unary arithmetic in most of our proofs.

| Value of $m$ | 3     | 4     | 5     | 6     | 7     | 8     |
|-------------|-------|-------|-------|-------|-------|-------|
| Lower bound | 112/82 | 19/14 | 19/14 | 19/14 | 19/14 | 19/14 |
| Tree nodes  | 186k  | 433   | 3908  | 3.8M  | 231M  | 2.5G  |
| DAG nodes   | 103k  | 236   | 1271  | 38k   | 186k  | 1.6M  |
| cDAG nodes  | 37k   | 102   | 408   | 7k    | 61k   | 598k  |
| Time        | 38s   | 1s    | 2s    | 12s   | 4m30  | 2h    |

Table 5: Size of the uncompressed and compressed DAGs and (approximate) time needed to load the trees and certify each lower bound. The running times were computed on a machine with the Intel Core i5-6600 CPU and 32 GB of RAM.

With these features implemented, we have been able to certify all the lower bound results previously published and presented in this paper. The amount of time necessary to run each Coq script is reported in Table 5.

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