Scalar fields on $p$AdS

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Abstract

We propose equations of motion of real-valued scalar fields on $p$AdS, which is a $p$-adic version of AdS space-time alternative to the Bruhat-Tits tree. The analytical expressions of the Green’s functions are given for a particular parameter, and the limiting behaviors are studied for general cases. Difference between $p$AdS and the Bruhat-Tits tree is also pointed out in the calculation of correlation functions by the AdS/CFT method.

1 Introduction

There are at least 2 reasons to study physics over $p$-adic numbers $\mathbb{Q}_p$. The first one is all experimental data are rational numbers $\mathbb{Q}$ indicating that any field (mathematics) including $\mathbb{Q}$ is possibly used in physics. The second one is due to lack of quantum gravity, properties of space-time at large scales may be incorrect at small scales, such as the Archimedean property [1–3]. $\mathbb{Q}_p$ is such an alternative, including $\mathbb{Q}$ and space-time over which is non-Archimedean. It is widely used in physics [1, 2, 4–10]. The application of $p$-adic numbers in the anti-de Sitter/conformal field theory correspondence (AdS/CFT) [11–13] begins when the Bruhat-Tits tree (BTtree) is treated as a $p$-adic version of AdS space-time in [14, 15]. More properties of the BTtree are studied based on their work [16–20].

$p$AdS [14] is another $p$-adic version of AdS space-time which can reproduce the BTtree after being coarse-grained. We wonder is there any difference between them if applying the AdS/CFT method. Thus our paper is devoted to further studies on this $p$AdS. Section 2 gives a very brief introduction to the work of [14]. In section 3, we propose equations of motion (EOM) of scalar fields on $p$AdS and work out the corresponding Green’s functions along with some properties studied. Difference between the BTtree and $p$AdS is also pointed out in this section with the help of Witten diagrams [13, 21, 22]. The last section contains our main results and the possible future studies.

2 $p$-adic AdS/CFT

In [14], the BTtree is treated as a $p$-adic version of AdS space-time with the Euclidean time. The place $\{\cdots, p^{-1}, p^0, p^1, \cdots\}$ (positional notation in mathematics) of $p$-adic numbers is regarded as the holographic dimension. The action, EOM of a scalar field on the BTtree with a point source fixed at
vertex $a_0$ and the corresponding Green’s function are found to be

$$S = \sum_{<ab>_a} \frac{1}{2} (\phi_a - \phi_b)^2 + \sum_a \left( \frac{1}{2} m^2 \phi_a^2 - \delta(a, a_0) \phi_a \right)$$

$$\sum_{<ab>_a} (\phi_a - \phi_b) + m^2 \phi_a = \delta(a, a_0) = \begin{cases} 1, & a = a_0 \\ 0, & a \neq a_0 \end{cases}$$

where $a(b)$ denotes a vertex on the BTtree, $\phi_a(\phi_b)$ represents the field on vertices, $<ab>$ means the sum is over the nearest neighboring vertices, $m^2 \zeta_p(\Delta - 1) \zeta_p(-\Delta) = -1$ where $\zeta_p(s)(1 - p^{-s}) = 1$ and $d(a, a_0)$ gives the number of edges between $a$ and $a_0$. According to the spirit of the AdS/CFT, correlation functions are calculated at tree level.

Considering that coordinate of holographic dimension on the BTtree only takes discrete values, which is significantly different from the case in usual AdS over real numbers, another $p$-adic version of AdS space-time $p$AdS $\equiv \mathbb{Q}_p \times \mathbb{Q}_p$ is purposed at the end of [14], where coordinate of holographic dimension takes values in the whole $\mathbb{Q}_p$. “Chordal distance” is introduced ($u$ distance) as

$$u(x, y) = \frac{|x_1 - y_1, x_2 - y_2|^2}{|x_1 y_1|},$$

where subscript 1 corresponds to the holographic dimension and $| \cdot |_a$ is the supremum norm of $|x_1 - y_1, x_2 - y_2|_a = sup(\{|x_1 - y_1|, |x_2 - y_2|\})$. $| \cdot |_a$ denotes $p$-adic absolute value in this paper. Based on this $u$ distance, $p$AdS is represented by a tree (Fig. 1), which is crucial for our work. Each vertex on this tree represents a ball in $p$AdS. In this figure we have $x, y, z, u, v \in p$AdS and $x, y \in a_2 \subset a_1$. We call the dimension where blue edges extend along “level”. The BTtree plane is the 1st level with level coordinate $p^0$. Generally, $u$ distance between any two points separated at the $n$-th level (but still in the same ball at the 1st level) with level coordinate $p^{2(n-1)}$ is $u = |p^{2(n-1)}| = p^{2(1-n)}$ which is always not larger than 1. It gives $u(x, z) = |p^0| = p^{-0}$ and $u(x, y) = |p^2| = p^{-2}$ in Fig. 1. As for points belonging to different balls at the
1st level, for example in this figure, the \( u \) distance is

\[
  u(x, u) = p^{d(a_1, b)} = p^1 \quad \text{and} \quad u(x, v) = p^{d(a_1, c)} = p^2 ,
\]

where \( d(a_1, b) \) gives the number of edges between \( a_1 \) and \( b \) at the 1st level. \( u \) distance in such case is always larger than 1. Consequently, if we treat points whose \( u \) distance between them is not larger than 1 as a single point, only structure at the 1st level left, which means the BTtree can be obtained by coarse-graining pAdS.

Letting \( a_n \) represent a ball at the \( n \)-th level, which contains balls \( b_{ni} \) at the \((n+1)\)-th level \( (\sum_i b_{ni} = a_n) \), “measure” \( \mu \) is introduced as

\[
  \mu(a_1) = \frac{1}{\zeta_p(1)} , \quad \mu(b_{ni}) = \mu(b_{nj}) \quad \text{and} \quad \sum_i \mu(b_{ni}) = \mu(a_n) .
\]

Let \( u_n \) and \( \mu_n \) represent the \( u \) distance between any two points separated at the \( n \)-th level (but still in the same ball at the 1st level) and the measure of a ball at the \( n \)-th level. Series \( \{u_n\} \) and \( \{\mu_n\} \) satisfy

\[
  u_n = p^{2(1-n)} , \quad \mu_n = \frac{\mu_1}{p(p-1)} p^{2(2-n)} \quad \text{where} \ n \geq 2 \quad \text{and} \quad \mu_1 = \frac{1}{\zeta_p(1)} .
\]

## 3 Scalar fields on pAdS

In this section we propose one of those EOM’s which can reproduce the EOM on the BTtree \([1]\) by coarse-graining pAdS and give the corresponding Green’s functions. After studying several properties of the Green’s functions, we present the difference between pAdS and the BTtree in the AdS/CFT aspect. The ansatz and the Green’s function when \( u \geq p \) are given in section 3.1. Analytical expressions of the Green’s functions when \( \alpha = 2 \) are summarized in section 3.2. pAdS and the BTtree are compared in section 3.3.

### 3.1 EOM and ansatz

We propose the action and EOM of a scalar field \( \phi(x) \) on pAdS with a point source sitting at \( x_0 \) as

\[
  S = \int_{x \in \text{pAdS}} dx \left( \frac{1}{4} \int_{u(x,y) \leq p} dy \frac{(\phi(x) - \phi(y))^2}{u(x,y)^a} + \frac{1}{2} m^2 \phi(x)^2 - \delta(x, x_0) \phi(x) \right) + \int_{u(x,y) \leq p} dy \frac{\phi(x) - \phi(y)}{u(x,y)^a} + m^2 \phi(x) = \delta(x, x_0) \quad \text{where} \ \alpha > 0 ,
\]

where \( dy \) is the measure \([1]\). Let \( b_i , \ i = 1, 2, \cdots, p + 1 \) denote the nearest neighboring balls of \( a \) at the 1st level. Integrating both sides of this EOM with \( \int_{x \in a} dx \) gives

\[
  \sum_i \left( \frac{\mu_1}{p^a} \phi(a) - \frac{\mu_1}{p^a} \phi(b_i) \right) + m^2 \phi(a) = \delta(a, a_0)
\]

where \( \phi(a) \equiv \int_{x \in a} dx \phi(x) \) and \( \phi(b_i) \equiv \int_{x \in b_i} dx \phi(x) \) .

\[
  \delta(a, a_0) = \int_{x \in a} dx \delta(x, x_0) \quad \text{is the same as that in} \ [1] . \ \text{After redefining} \ \phi(x) \ \text{and} \ m^2 , \ \text{it will be exactly the same as} \ [1] .
\]

To solve EOM \([6]\), consider a trivial ansatz with spherical symmetry

\[
  \phi(x) = f(u(x, x_0)) .
\]

Since \( u \) distance only takes discrete values, other points in pAdS form a series of spherical shells around the source point \( x_0 \). Refer to Fig.\([2]\) where \( \phi(x) \) is a constant when \( x \in a_n - a_{n+1} \) or \( x \in b(c) \). Combining
this ansatz, the Green’s function at the 1st level (1) and (3), the solution of EOM (6) when can be written out immediately, which is

\[ \alpha \]

3.2 Green’s functions and critical \( \alpha \)

Here \( m \) and integrating both sides of EOM (6) with \( n \sum 1 \), we can write down the expression of \( x \) and \( u \) belonging to when they belong to different balls at the 1st level. As for the case of \( u(x, x_0) \leq 1 \), we can write down the expression of \( \phi(x) \) as

\[ \phi(x) = \int_{y \in a_n - a_{n+1}} dy\phi(y) = \frac{\phi(a_n) - \phi(a_{n+1})}{\mu_n - \mu_{n+1}} \text{ when } x \in a_n - a_{n+1}. \]  (10)

3.2 Green’s functions and critical \( \alpha \)

Referring to Fig. 2 a point source is fixed at \( x_0 \in \cdots \subset a_{n+1} \subset a_n \subset a_{n-1} \subset \cdots \subset a_2 \subset a_1 \). \( b_i \), \( i = 1, 2, \cdots, p + 1 \) still denote the nearest neighboring balls of \( a_1 \) at the 1st level. Letting \( \int_{x \in a_i} dx\phi(x) = \phi_i \) and integrating both sides of EOM (6) with \( \int_{x \in a_n - a_{n+1}} dx \), after several tedious derivations, finally we obtain the recurrence relation of \( \{\phi_n+1\} \) as

\[ \phi_{n+3} = -\frac{B_n}{B_{n+1}}\phi_{n+1} + \frac{B_n + B_{n+1}}{B_{n+1}} + \frac{1}{u_{n+1}^{p+1}} - \frac{1}{u_{n+2}^{p+1}}\phi_{n+2} \]  (11)

where \( B_n \equiv \frac{\mu_{n+1}^{p+1} + \mu_{n+1}^{p+1} + m^2 + \mu_{n+1}^{p+1} + \sum_{i=2}^{n} \mu_{n+1}^{p+1} - \mu_{n+1}^{p+1}}{\mu_{n+1}^{p+1} - \mu_{n+2}}. \)

\[ \sum_{i=2}^{n} \frac{\mu_{n+1}^{p+1} - \mu_{n+1}^{p+1}}{\mu_{n+1}^{p+1}} = 0 \text{ when } n = 1. \]  

It seems that for \( \alpha \in \mathbb{N} \), where \( \mathbb{N} \) denotes the set of natural numbers, \( \{\phi_{n+1}\} \) can be solved analytically, such as

\[ p^{2n}\phi_{n+1} = c_1 + c_2 \sum_{k=0}^{n-1} \frac{p^{2k}}{1 - bp^{2k}} \text{ when } \alpha = 2, \]  (12)
Fig. 3. 2D and 3D $G(x, x_0)$ versus log$_p u(x, x_0)$ figures for different $\alpha$'s when $u(x, x_0) \leq 1$ (log$_p u \leq 0$). We set $p = 2$ and $\Delta = 3$. The point source $x_0$ sits at log$_p u(x, x_0) = -\infty$. In 2D figure, curves those go to $+\infty$ as log$_p u \to -\infty$ correspond to cases of $0 < \alpha \leq 2$. Others those go to constants correspond to cases of $\alpha > 2$.

where $c_1$, $c_2$ and $b$ are constants. Rewriting $\phi(x)$ as $G(x, x_0)$ and putting (9), (10) and (12) together, we summarize the analytical expressions of $G(x, x_0)$ when $\alpha = 2$ as

$$G(x, x_0) = \begin{cases} \frac{c_3}{\mu_1 p^2} u^- \Delta, & u \geq p \\ \frac{c_4}{\mu_1 - \mu_2} u - \Delta, & u = 1 \\ \frac{c_5}{\mu_1 p} \left( c_1 + c_2 \sum_{k=0}^{n-1} \frac{p^{2k}}{1 - bp^{2k}} - c_2 \frac{1}{p^{2-1} - bp^{2n}} \right), & u = p^{-2n} \end{cases}$$

(13)

For general $\alpha$'s, we numerically plot two figures when $x$ and the point source $x_0$ are in the same ball at the 1st level ($u(x, x_0) \leq 1$). See Fig. 3. In 2D figure, it can be found that there is a critical value $\alpha = 2$ whose curve is a straight line when log$_p u(x, x_0)$ is small enough. And this numerical result can be concluded as

$$G(x, x_0) = \begin{cases} +\infty, & 0 < \alpha \leq 2 \\ \text{constant}, & \alpha > 2 \end{cases}$$

(14)

This critical $\alpha$ can be confirmed analytically. Considering the recurrence relation of $\{\phi_{n+1}\}$ (11), when $n \to +\infty$ (log$_p u \to -\infty$ or $x \to x_0$), it can be worked out that

$$g \equiv \lim_{n \to +\infty} -\frac{B_n}{B_{n+1}} = \begin{cases} -p^{-2}, & 0 < \alpha < 1 \\ -p^{-2\alpha}, & \alpha \geq 1 \end{cases}$$

$$h \equiv \lim_{n \to +\infty} \frac{B_n + B_{n+1} + \frac{1}{u_{n+1}} - \frac{1}{u_{n+2}}}{B_{n+1}} = \begin{cases} 1 + p^{-2}, & 0 < \alpha < 1 \\ p^{-2} + p^{-2\alpha}, & \alpha \geq 1 \end{cases}$$

(15)

In such large $n$ limit, $\phi_{n+3} = g\phi_{n+1} + h\phi_{n+2}$ can be solved. Together with the relation between $\phi_{n+1}$ and $G(x, x_0)$ (10), finally we have

$$G(x, x_0) = \begin{cases} c_3 + c_4 p^{2n}, & 0 < \alpha \leq 1 \\ c_5 + c_6 p^{2(2-\alpha)n}, & \alpha \geq 1 \text{ and } \alpha \neq 2 \\ c_7 + c_8 n, & \alpha = 2 \end{cases}$$

(16)

which gives the same critical value $\alpha = 2$. $c_i$, $i = 3$ to 8 are constants. Although there may be some problems when we take the limit $n \to +\infty$ partly to obtain $\phi_{n+3} = g\phi_{n+1} + h\phi_{n+2}$ from (11), we still accept this result since it agrees with that of numerical calculation.
3.3 $p$AdS/CFT versus BTtree/CFT: small loop diagrams

Let’s consider the calculation of the 2-point function in CFT by $p$AdS/CFT method (the AdS/CFT method with AdS represented by $p$AdS). Firstly at tree level, repeating exactly the same procedures as those in [14], it is no surprise that we obtain the similar result (17) since the Green’s functions on these two space-times are actually the same when $u \geq p$.

$$A_{pAdS}^{tree} = A_{BTtree}^{tree} \mu_1^2/p^\alpha.$$  \hspace{1cm} (17)

The coefficient $\mu_1^2/p^\alpha$ comes from the integral over balls and $u$ distance between neighboring balls at the 1st level.

Secondly, let’s turn to 1-loop calculation using the Witten diagram in Fig. 4. Consider the situation the loop is so small ($u(x, y) \leq 1$) that the loop structure disappears after coarse-graining. Such small loop diagram ($A_{pAdS}^{small 1-loop}$) will be missing in the calculation by the BTtree/CFT method (the AdS/CFT method with AdS represented by the BTtree). Let $a$ represent a ball at the 1st level and $K$ denote the bulk-boundary Green’s function which is the regularization of the bulk-bulk Green’s function when one point approaches the boundary. Rewriting $K(P_1, x)$ as $K(P_1, a)$ when $x \in a$, we can obtain

$$A_{pAdS}^{1-loop} = A_{BTtree}^{1-loop} \mu_1^2 + A_{pAdS}^{small 1-loop}$$

$$= A_{BTtree}^{1-loop} \mu_1^2 + \lambda^2 \sum_{a \in BTtree} K(P_1, a)K(P_2, a) \int_{x,y \in a} dxdyG(x,y)^2$$

$$\frac{\lambda^2 p^2}{4} \sum_{a \in BTtree} K(P_1, a)K(P_2, a),$$  \hspace{1cm} (18)

where $\sum KK$ is divergent.

Finally, we conclude that the existence of small loop Witten diagrams only on $p$AdS illustrates one difference between $p$AdS and the BTtree.

4 Discussion

In this paper, we (i) propose EOM [8] of a real-valued scalar field on $p$AdS; (ii) find out the analytical expressions of the Green’s functions [13] for $\alpha = 2$; (iii) give the limiting behaviors of $G(x, x_0)$ [16] when $x \to x_0$; (iv) point out one difference between $p$AdS and the BTtree is that small loop Witten diagrams only exist on $p$AdS [18].

There are at least 2 possible future studies. One is to consider spinor fields. Different from AdS over real numbers, space-time inside each ball at the 1st level in $p$AdS is flat. Based on this fact, maybe we
can find spinor structure in such flat region before obtaining any gravitational theory on pAdS. EOM of a free scalar field can be decomposed into the following two equations

\[
\int_{u<p} dy \phi(x) - \phi(y) u(x, y)^{\alpha} + \left(\frac{p + 1}{p^\alpha} + \frac{m^2}{\mu_1}\right)(\mu_1 \phi(x) - \phi(a)) = 0 \quad \text{when } x \in a
\]

\[
\sum_i \left(\frac{\mu_1}{p^\alpha} \phi(a) - \frac{\mu_1}{p^\alpha} \phi(b_i)\right) + m^2 \phi(a) = 0.
\]

The 1st one describes the motion in small region \(u \leq 1\) (the motion in the same ball at the 1st level) and the 2nd one describes the motion in large region \(u \geq p\) (the motion between different balls at the 1st level). \(b_i\)'s still denote the nearest neighboring balls of \(a\) at the 1st level. Maybe spinor structure can be found by taking the square root of the 1st equation.

The other is to search for the gravitational theory by studying the \(u\) distance. Let \(l(x, y)\) represent the geodesic distance in an Euclidean space-time over real numbers. According to some results of [14] and Table I we think \(u\) distance should be identified as a \(p\)-adic version of square of the geodesic distance \((u \sim l^2)\). In this table, \(t_E\) is the Euclidean time and \(z\) is the holographic dimension in space-time over real numbers. Integral \(\int y x (\cdots)\) should be calculated along the geodesic connecting \(x\) and \(y\). Referring to the equation of \(l(x, y)\), which should be derived in general relativity, maybe we can find the equation of \(u(x, y)\). We hope the latter can lead us to the gravitational theory on \(\mathbb{Q}_p \times \mathbb{Q}_p\) from another direction, which is different from the analysis on graphs [18].

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