THE KERNEL UNIPOTENT CONJECTURE AND THE VANISHING OF MASSEY PRODUCTS FOR ODD RIGID FIELDS

JÁN MINÁČ AND NGUYỄN DUY TÂN
(WITH AN APPENDIX BY IDO EFRAT, JÁN MINÁČ AND NGUYỄN DUY TÂN)

ABSTRACT. A major difficult problem in Galois theory is the characterization of profinite groups which are realizable as absolute Galois groups of fields. Recently the Kernel $n$-Unipotent Conjecture and the Vanishing $n$-Massey Conjecture for $n \geq 3$ were formulated. These conjectures evolved in the last forty years as a byproduct of the application of topological methods to Galois cohomology. We show that both of these conjectures are true for odd rigid fields. This is the first case of a significant family of fields where both of the conjectures are verified besides fields whose Galois groups of $p$-maximal extensions are free pro-$p$-groups. We also prove the Kernel Unipotent Conjecture for Demushkin groups of rank 2, and establish various filtration results for free pro-$p$-groups, provide examples of pro-$p$-groups which do not have the kernel $n$-unipotent property, compare various Zassenhaus filtrations with the descending $p$-central series and establish new type of automatic Galois realization.

1. INTRODUCTION

In 1928, W. Krull in [Kr] introduced a topology for a general Galois group. Thus each Galois group is a profinite topological group. If $F$ is a field and $F_{\text{sep}}$ is its separable closure, then $G_F = \text{Gal}(F_{\text{sep}}/F)$ - the Galois group of $F_{\text{sep}}$ over $F$ - is called the absolute Galois group of $F$. What special properties do absolute Galois groups have among all profinite groups?

In the classical papers [AS1, AS2] published in 1927, E. Artin and O. Schreier developed a theory of real fields, and they showed in particular that the only non-trivial finite subgroups of absolute Galois groups are groups of order 2. In [Be], E. Becker developed some parts of Artin-Schreier theory by replacing separable closures of fields by maximal $p$-extensions of fields. Conjectures [1.1] and [1.3] below describe conjecturally rather important special properties of absolute Galois groups and their maximal pro-$p$-quotients. In [MT1] we explained how these conjectures evolved over a number of years.

Let $G$ be a profinite group and let $\mathbb{F}_p$ be a field with $p$ elements considered as a discrete $G$-module with trivial action. For each $i \in \mathbb{N} \cup \{0\}$, let $H^i(G, \mathbb{F}_p)$ be the $i$-th cohomology of $G$ with $\mathbb{F}_p$-coefficients. Let $a_1, \ldots, a_n \in H^1(G, \mathbb{F}_p)$. In Section 7 we recall the definition of an $n$-fold Massey product

$$\langle a_1, \ldots, a_n \rangle \subseteq H^2(G, \mathbb{F}_p),$$

when it is defined. So an $n$-fold Massey product is not always defined, and when it is defined, in general it is not a single-valued function of $a_1, \ldots, a_n$ but it is a subset of $H^2(G, \mathbb{F}_p)$. Nevertheless the significance of $n$-Massey products for Galois theory is considerable. An important theorem is Dwyer’s theorem [Dwy, Theorem 2.4], quoted as Theorem 7.2 below.

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in our paper. It shows that certain lifts of unipotent representations are possible if and only if \( \langle \alpha_1, \ldots, \alpha_n \rangle \) is defined and \( 0 \in \langle \alpha_1, \ldots, \alpha_n \rangle \). If this is so, then we say that \( \langle \alpha_1, \ldots, \alpha_n \rangle \) vanishes. Assume that a prime \( p \) and an integer \( n \in \mathbb{N} \) are given. If, for each \( \langle \alpha_1, \ldots, \alpha_n \rangle \) which is defined, we have that \( \langle \alpha_1, \ldots, \alpha_n \rangle \) vanishes, then we say that \( G \) has the vanishing \( n \)-fold Massey product property with respect to \( \mathbb{F}_p \).

In searching for other significant Galois groups which could play an analogous role as dihedral groups of order 8, play in the quotients of \( G_F(2) \) by its third descending 2-central series, the authors of [GLMS] found certain groups \( G_1 \) and \( G_2 \) of order 32 and 64 respectively. In 2004 J. Labute, during his visit to the first author, pointed out the significance of Massey products for Galois theory, and he pointed out two remarkable works [Mo1] and [Vo]. In 2011 the first author, while working with I. Efrat, observed the relevance of Massey products with work in [GLMS]. Then in 2013, I. Efrat [Ef1] investigated a property of profinite groups, what we later christened as the kernel \( n \)-unipotent property below. In [MT1] it was observed that \( G_2 \simeq U_4(F_2) \) and \( G_1 \) is isomorphic to a subgroup of \( U_4(F_2) \). Moreover it was shown that \( G_1 \) and \( G_2 \) play an important role in vanishing triple Massey products with respect to \( p = 2 \). Finally the following conjecture was formulated in [MT1].

**Conjecture 1.1.** Let \( p \) be a prime number and \( n \geq 3 \) an integer. Let \( F \) be a field, which contains a primitive \( p \)-th root of unity if \( \text{char}(F) \neq p \). Then the absolute Galois group \( G_F \) of \( F \) has the vanishing \( n \)-fold Massey product property with respect to \( \mathbb{F}_p \).

This conjecture is a significant conjecture in Galois theory, and in each case when it can be established for any particular family of fields, any \( n \in \mathbb{N} \), \( n \geq 3 \) and any prime \( p \), it has remarkable consequences for the automatic realization of Galois groups and the structure of absolute Galois groups. For further motivations, results and applications, see [MT1].

In [MT1] we show that Conjecture 1.1 is true for \( n = 3, p = 2 \), and all fields. This is the first case when the validity of this conjecture is established for all fields. This results extends a previous result of M. J. Hopkins and K. G. Wickelgren in [HW], where they prove this result for global fields of characteristic not 2. In [MT1] we also prove Conjecture 1.1 for all local fields and all \( n \geq 3 \), and all primes \( p \). In [MT2] we prove Conjecture 1.1 for algebraic number fields \( n = 3 \) and all primes \( p \).

In [MT1] the Kernel \( n \)-Unipotent Conjecture recalled below, was also formulated and discussed. In Section 2, we recall the definition of the \( p \)-Zassenhaus filtration \( G_{(n)} \) for any profinite group \( G \) and prime number \( p \). Recall that for a unital commutative ring \( \Lambda \), \( U_n(\Lambda) \) is the group of all upper-triangular unipotent \( n \times n \)-matrices with entries in \( \Lambda \). The following very interesting property of profinite groups was first studied in [Ef1].

**Definition 1.2.** Let \( G \) be a pro-\( p \)-group and let \( n \geq 1 \) be an integer. We say that \( G \) has the kernel \( n \)-unipotent property if

\[
G_{(n)} = \bigcap \ker(\rho: G \to U_n(\mathbb{F}_p)),
\]

where \( \rho \) runs over the set of all representations (continuous homomorphisms) \( G \to U_n(\mathbb{F}_p) \).

It is easy to see that for \( n = 1 \) and 2, every pro-\( p \)-group \( G \) has the kernel \( n \)-unipotent property. In Appendix written jointly with Ido Efrat, we show that for each \( n \geq 3 \) there exists a finitely generated pro-\( p \)-group \( G \) such that \( G \) does not have the kernel \( n \)-unipotent property. In analogy with transcendental numbers, although we assume that many pro-\( p \)-groups do not have the kernel \( n \)-unipotent property, one has to find a suitable family in order to able to check this. In a subsequent paper [MT3], we show that every pro-\( p \) Demushkin
group has the kernel \( n \)-unipotent property for \( n = 3, 4 \). In Section 5, we also show that pro-\( p \) Demushkin groups of rank 2 have the kernel \( n \)-unipotent property for all \( n \geq 3 \) (see Proposition \( 5.4 \)). It is shown in [Ef1] Theorem A] that every free pro-\( p \)-group has the kernel \( n \)-property for all \( n \geq 3 \). This theorem was in [Ef1] deduced from a more general theorem called Theorem A’ in [Ef1]. (In Section 2, we provide an alternative direct short proof (see Theorem 2.6 part a). That such a proof is possible was announced earlier in [MT1 Section 8]. Recently, Efrat in [Ef2] also obtained such a proof independently from us.)

It was shown that for \( G = G_F(p) \), where \( F \) is a field containing a primitive \( p \)-th root of unity, \( G \) has the kernel \( 3 \)-unipotent property. (See [MS2, VI, EM1] for the case \( p = 2 \) and [EM2 Example 9.5 (1)] for the case \( p > 2 \)) These are deep results closely related to the well-known Merkurjev-Suslin theorem ([MeSu]).

**Conjecture 1.3** (Kernel \( n \)-Unipotent Conjecture). Let \( F \) be a field containing a primitive \( p \)-th root of unity and let \( G = G_F(p) \). Let \( n \geq 3 \) be an integer. Then \( G \) has the kernel \( n \)-unipotent property.

A connection between Conjectures 1.1 and 1.3 is via Dwyer’s theorem quoted as Theorem 7.2 below. Namely, in order to obtain \( n \)-unipotent representations of \( G = G_F(p) \) we need \( n \)-Massey products vanishing for suitable \( \alpha_1, \ldots, \alpha_{n-1} \in H^1(G, \mathbb{F}_p) \).

The main results in our paper are Theorems 5.2 and 8.1. In them we establish both the Vanishing \( n \)-Massey Conjecture and the Kernel \( n \)-Unipotent Conjecture for \( p \)-rigid fields \( (p > 2) \) and for all \( n \geq 3 \). We shall now recall the definition of rigid fields.

Assume \( p > 2 \). This assumption is made throughout the paper except when we explicitly consider \( p = 2 \). Let \( F \) be a field, which we assume to contain a fixed primitive \( p \)-th root of unity \( \zeta_p \). For each \( \alpha \in F^\times = F \setminus \{0\} \), we have an element \( \chi_\alpha \in \text{Hom}(G_F, \mathbb{F}_p) = H^1(G_F, \mathbb{F}_p) \) defined by \( \chi_\alpha = \zeta_p^{\chi_\alpha(r)} \sqrt{\alpha} \) for all \( \alpha \in G_F \).

Then \( F \) is \( p \)-rigid if and only if \( \chi_a \cup \chi_b = 0 \in H^2(G_F, \mathbb{F}_p) \) implies that \( \chi_a, \chi_b \in H^1(G_F, \mathbb{F}_p) \) are linearly dependent over \( \mathbb{F}_p \). (This definition coincides with the one given in Definition 4.1.) Since we assume that \( p > 2 \), we sometimes call such a field an odd rigid field. Besides fields \( F \) with \( G_F(p) \) free pro-\( p \)-groups or Demushkin pro-\( p \)-groups, \( p \)-rigid fields play a fundamental role in current studies of Galois groups of maximal \( p \)-extensions. (See e.g., [CMQ], [EK], [Wa1], [Wa2] and [Wa3].)

The structure of this paper is as follows. In Section 2 we will present another (short and direct) proof for a result, which was first proved by I. Efrat ([Ef1]), that every free pro-\( p \)-group has the kernel \( n \)-unipotent property for all \( n \) (Theorem 2.6 part a). At the same time, we obtain analogous new results for other filtrations, such as the descending central series and the descending \( p \)-central series. The result for the descending \( p \)-central series is interesting because it provides us with first steps toward an analogy to the Kernel \( n \)-Unipotent Conjecture when we replace the \( p \)-Zassenhaus filtration by the descending \( p \)-central series. We also obtain discrete versions of these results (see Theorem 2.7). In Section 3 we provide some results on unipotent matrices which we shall later use for constructing unipotent representations. In Section 4 we recall some basic facts on the structure of Galois groups of maximal \( p \)-extensions of odd \( p \)-rigid fields. Using results obtained in Section 3 and the results recalled in Section 4, we prove the Kernel \( n \)-Unipotent Conjecture for odd rigid fields in Section 5. In this section we also show that every Demushkin group with rank at most 2 has the kernel \( n \)-unipotent property for all \( n \). In Section 6, for a given \( n \geq 3 \), we provide an example of a torsion-free pro-\( p \)-group which does not have the kernel \( n \)-unipotent property. We also compare the \( p \)-th and \((p + 1)\)-th terms in the \( p \)-Zassenhaus filtration with the third term in
the descending $p$-central series for $G_F(p)$, here $F$ is a field containing a primitive $p$-th root of unity (see Proposition 6.2). In Section 7, we review some basic facts on Massey products.

In Section 8 we prove the Vanishing $n$-Massey Conjecture for any odd rigid field and for any $n$. See Theorem 8.5. This result is a consequence of a more general result, Theorem 8.1. The latter theorem has its own interest because it deals with general fields which contain a primitive $p$-th root of unity if their characteristic is different from $p$, and it also provides an explanation of the well-known result of Artin-Schreier result mentioned at the beginning of the Introduction. In this last section, we also derive a new type of automatic Galois realization (see Corollary 8.4). In Appendix written jointly with Ido Efrat, for each $n \geq 3$ we provide examples of pro-$p$-groups which do not satify the kernel $n$-unipotent property.

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2. Kernel conjecture for free-$p$-groups

Recall that for a profinite group $G$ and a prime number $p$, the descending central series $(G_i)$, the descending $p$-central series $(G^{(i)})$ and the $p$-Zassenhaus filtration $(G^{(i)})$ of $G$ are defined inductively by

$$G_1 = G, \quad G_{i+1} = [G_i, G],$$

by

$$G^{(1)} = G, \quad G^{(i+1)} = (G^{(i)})^p[G^{(i)}, G],$$

and by

$$G^{(1)} = G, \quad G^{(n)} = G_{(n/p)}^p \prod_{i+j=n} [G^{(i)}, G^{(j)}],$$

where $\lceil n/p \rceil$ is the least integer which is greater than or equal to $n/p$. (Here for closed subgroups $H$ and $K$ of $G$, the symbol $[H, K]$ means the smallest closed subgroup of $G$ containing the commutators $[x, y] = x^{-1}y^{-1}xy, x \in H, y \in K$. Similarly, $H^p$ means the smallest closed subgroup of $G$ containing the $p$-th powers $x^p, x \in H$.)

Let $\Lambda$ be $\mathbb{Z}_p$ or $\mathbb{Z}/p^r\mathbb{Z}$ with $r \in \mathbb{N}$. Let $S$ be a free pro-$p$-group on a finite set of generators $x_1, \ldots, x_d$. Then we have the Magnus homomorphism from the completed group algebra $\Lambda[[S]]$ to the ring $\Lambda\langle\langle X_1, \ldots, X_d \rangle\rangle$ of the formal power series in $d$ non-commuting variables $X_1, \ldots, X_d$ over $\Lambda$ (equipped with the topology of coefficient-wise convergence).

$$\psi: \Lambda[[S]] \rightarrow \Lambda\langle\langle X_1, \ldots, X_d \rangle\rangle, x_i \mapsto 1 + X_i.$$

One basic result is the following

Lemma 2.1. The Magnus homomorphism $\psi$ is a (continuous) isomorphism.

Proof. See, for example, [Se2, Chapter I, Proposition 7] or [Laz, Chapter 6]. □

A multi-index $I = (i_1, \ldots, i_k)$ is called of height $d$ if $1 \leq i_r \leq d$. Its length $|I|$ is $k$. If multi-indices $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_l)$ are given, we denote by

$$IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l),$$
the concatenation of $I$ and $J$. Also for a multi-index $I = (i_1, \ldots, i_k)$ of height $d$, we set $X_I = X_{i_1} \cdots X_{i_k}$.

The Magnus expansion (relative to $\Lambda$) $\psi(s)$ of $s \in S$ is given by

$$\psi(s) = 1 + \sum_I e_{I, \Lambda}(s) X_I,$$

where $I$ runs over all multi-indices of height $d$. So for each $I$, we have a map $e_{I, \Lambda} : S \to \Lambda$. For $I = \emptyset$, we define $e_{\emptyset, \Lambda}(s) = 1$, for all $s \in S$.

The next lemma records some quite well-known facts.

**Lemma 2.2.** Let $\sigma, \tau$ be elements in $S$. Let $n \geq 2$ be an integer.

(a) For any multi-index $I$ of height $n$, one has

$$e_{I, \Lambda}(\sigma \tau) = \sum_{I_1, I_2 = I} e_{I_1, \Lambda}(\sigma) e_{I_2, \Lambda}(\tau).$$

(b) We have $\sigma \in S_n$ if and only if $e_{I, \mathbb{Z}_p}(\sigma) = 0$ for every multi-index $I$ with $1 \leq |I| < n$.

(c) We have $\sigma \in S(n)$ if and only if $e_{I, \mathbb{F}_p}(\sigma) = 0$ for every multi-index $I$ with $1 \leq |I| < n$.

(d) We have $\sigma \in S^{(n)}$ if and only if $\nu(e_{I, \mathbb{Z}_p}(\sigma)) \geq n - |I|$ for every multi-index $I$ with $1 \leq |I| < n$. Here $\nu$ is the $p$-adic valuation on $\mathbb{Z}_p$ ($\nu(p) = 1$).

**Proof.** (a) It just follows from the fact that $\psi$ is a homomorphism.

(b) See e.g., [Mo2 Proposition 8.15].

(c) See e.g., [Vo2 Lemma 2.19] and/or [Ef1 Proposition 6.2]. The key point here is that by the Jennings-Brauer theorem (see [DDMS Theorem 12.9]), the $p$-Zassenhaus filtration on $S$ coincides with the filtration induced from the $I_{\mathbb{F}_p}(S)$-adic filtration on $\mathbb{F}_p[[S]]$ (here $I_{\mathbb{F}_p}(S)$ is the augmentation ideal of $\mathbb{F}_p[[S]]$). Namely, we have $S_{(n)} = \{ s \in S \mid s - 1 \in I_{\mathbb{F}_p}(S) \}$.

(d) See e.g., [Ko2 Theorem 7.14] together with Lemma 2.1 or [Ko1 Satz 1].

The following lemma was proved in [Ef1] Lemma 7.4.

**Lemma 2.3.** Let $I = (i_1, \ldots, i_k)$ be a multi-index. We define a map

$$\rho = \rho_{I, \Lambda} : S \to \mathbb{U}_{k+1}(\Lambda),$$

by

$$\rho(\sigma)_{\mu\nu} = e_{(i_1, \ldots, i_{\nu-1}), \Lambda}(\sigma),$$

for $\sigma \in S$ and $\mu < \nu$ (the other entries being obvious). Then $\rho$ is a continuous group homomorphism.

**Proof.** It follows from Lemma 2.2a. □

For $1 \leq i, j \leq n$, let $e_{ij}$ be the $n \times n$ matrix with the 1 in $\Lambda$ in the position $(i, j)$ and 0 elsewhere. It is well-known that the set of elements $1 + e_{i,i+1}$, $i = 1, \ldots, n$, generate the group $\mathbb{U}_n(\Lambda)$. (See [We, page 455].)

**Lemma 2.4.** Let $n, r, s \geq 1$ are integers. Then we have

$$\mathbb{U}_n(\mathbb{Z}_p)_{(n)} = \mathbb{U}_n(\mathbb{F}_p)_{(n)} = \mathbb{U}_r(\mathbb{Z}/p^s\mathbb{Z})^{(r+s-1)} = 1.$$
Let $S$ be a free pro-$p$-group on $r-1$ generators $x_1,\ldots, x_{r-1}$. We consider the multi-index $I = (1,\ldots, r-1)$ of length $r-1$. Then by Lemma 2.3, we have a continuous homomorphism

$$\rho = \rho_{I} : S \to \mathcal{U}_{r}(\mathbb{Z}/p^{\infty})$$

which is defined by

$$\rho(\sigma)_{\mu \nu} = e_{(\mu,\ldots,\nu-1)}(\mathbb{Z}/p^{\infty})(\sigma),$$

for $\sigma \in S$ and $\mu < \nu$ (the other entries being obvious). In particular $\rho(x_i) = 1 + e_{i,i+1}$, for $i = 1,\ldots, r-1$. Thus $\rho$ is surjective, and hence $\rho(S)^{(r+s-1)} = \mathcal{U}_{r}(\mathbb{Z}/p^{s})^{(r+s-1)}$.

Now let $\sigma$ be any element in $S^{(r+s-1)}$. For each sub-multi-index $J \subseteq I$, we have $v_{p}(\rho_{I}Z_{p}(\sigma)) + \lvert J \rvert \geq r+s-1$ by Lemma 2.2, part d). Thus $v_{p}(\rho_{I}Z_{p}(\sigma)) \geq s$, and hence $\rho_{I}Z_{p}(\sigma) = 0$. It implies that $\rho(\sigma) = 1$. Therefore $\mathcal{U}_{r}(\mathbb{Z}/p^{s})^{(r+s-1)} = \rho(S)^{(r+s-1)} = 1$.

**Lemma 2.5.** Let $G$ be a pro-$p$-group.

(a) Every continuous homomorphism $\rho : G \to \mathcal{U}_{n}(\mathbb{Z}_{p})$ is trivial on $G_{n}$.

(b) Every continuous homomorphism $\rho : G \to \mathcal{U}_{n}(\mathbb{F}_{p})$ is trivial on $G_{n}$.

(c) For each $k = 1,\ldots, n-1$, every continuous homomorphism $\rho : G \to \mathcal{U}_{k+1}(\mathbb{Z}/p^{n-k})$ is trivial on $G_{n}$.

**Proof.** (a) This follows from the fact that $\mathcal{U}_{n}(\mathbb{Z}_{p})_{n} = 1$.

(b) This follows from the fact that $\mathcal{U}_{n}(\mathbb{F}_{p})_{n} = 1$.

(c) This follows from the fact that $\mathcal{U}_{k}(\mathbb{Z}/p^{n+1-k})^{(n)} = 1$. 

Part a) of the following theorem was first proved by I. Efrat in [Ef1, Theorem A]. (In [MT1], we mentioned that we found a short direct proof of it but we did not write details of this proof.) After completion of the previous version of our paper, we received from I. Efrat, the preprint [Ef2], where he also found a direct proof of this fact, and he also characterized independently from us the $n$-th term $S_{n}$ of the descending $p$-central series of a free pro-$p$-group $S$ under a condition that $n < p$ (see [Ef2, Version 1, Theorem in Introduction]). In our correspondence with Efrat, we clarified that our use of Koch’s Theorem 7.14 in [Ko2] was correct. In the new version, Efrat was able to remove the extra hypothesis $n < p$. (See [Ef2, Version 2, Theorem in Introduction].) As we mentioned in the introduction, this characterization of $S_{n}$ is important for a possible formulation of an analogue of the Kernel $n$-Unipotent Conjecture for the descending $p$-central series rather than the $p$-Zassenhaus filtration of absolute Galois groups.

In addition to [Ef2], the discrete variant of the version of Theorem 2.6 below is also obtained. Efrat also points out in [Ef2, Introduction] that in Theorem 2.2(b) we recover the result of Grün [Gr1]. (See also [Ef2, Example 6.2].) We refer to the very interesting history of this result to [CM, Chapter 2, Section 7] and [Ro, Section 6]. In order to illustrate historical developments we have added additional references following I. Efrat; namely to the crucial work of W. Magnus [Mag1] and [Mag2] related to some filtration of free groups. Magnus’s paper [Mag1] initiated this research. In our proof we use only [Wi2]. The paper of Grün is not easy to read and it contains some gaps. Again following Efrat we refer the reader to a nice exposition of Grün’s work (see [Ro]). One should also mention that the results in [Ko1] form extensions of the previous results of [Sko], who proved these results for odd primes $p$, while Koch was able to extend them for all primes. Lazard’s thesis [Laz] contains a systematical approach to filtrations on groups obtained from filtrations on rings building on the pioneering work of [Mag2], [Wi2] and a number of others. In Chapter 5,
in particular, Lazard deals with both the Zassenhaus filtration and the descending \( p \)-central series. Zassenhaus’s original dimension groups were considered in [Zas].

The mathematical content of [Ef2] and our results in Section 2 were obtained independently from each other and they seem to nicely complement to each other.

**Theorem 2.6.** Let \( S \) be a free pro-\( p \)-group and \( n \geq 1 \). Then

(a) \( S^{(n)} \) is the intersection of all kernels of linear representations \( \rho : S \to \mathbb{U}_n(\mathbb{F}_p) \).

(b) \( S_n \) is the intersection of all kernels of linear representations \( \rho : S \to \mathbb{U}_n(\mathbb{Z}_p) \).

(c) \( S^{(n)} \) is the intersection of all kernels of linear representations \( \rho : S \to \mathbb{U}_{k+1}(\mathbb{Z}/p^{n-k}\mathbb{Z}) \), where \( k = 1, \ldots, n - 1 \).

**Proof.** It suffices to consider that the case \( S \) is finitely generated by a limit argument ([RZ, Lemma 3.3.11]).

a) By Lemma 2.5, one has

\[
S^{(n)} \subseteq \bigcap_{k=1}^{n-1} \ker(\rho : S \to \mathbb{U}_n(\mathbb{F}_p)).
\]

Now let \( \sigma \) be any element in \( S \setminus S^{(n)} \). We shall show that there exists a group representation \( \rho : S \to \mathbb{U}_n(\mathbb{F}_p) \) such that \( \sigma \not\in \ker \rho \) and then we are done.

In fact, by Lemma 2.1 c), there exists a multi-index \( I = (i_1, \ldots, i_k) \) of length \( k \) with \( 1 \leq k \leq n - 1 \) such that \( \epsilon_{I, F_p}(\sigma) \neq 0 \). By Lemma 2.3, the map \( \rho : S \to \mathbb{U}_{k+1}(\mathbb{F}_p) \) defined by

\[
\rho(\tau)_{\mu
u} = \epsilon_{(i_\mu \cdots i_\nu \cdots), F_p}(\tau) \quad (\mu < \nu),
\]

is a continuous group homomorphism. Since \( k + 1 \leq n \), we can embed \( \mathbb{U}_{k+1}(\mathbb{F}_p) \hookrightarrow \mathbb{U}_n(\mathbb{F}_p) \) and obtain a homomorphism still denoted by \( \rho \),

\[
\rho : S \to \mathbb{U}_{k+1}(\mathbb{F}_p) \hookrightarrow \mathbb{U}_n(\mathbb{F}_p).
\]

Then \( \rho(\sigma)_{1,k+1} = \epsilon_{I, F_p}(\sigma) \neq 0 \). Therefore \( \sigma \not\in \ker \rho \), as desired.

b) We proceed in the same way as in part a).

c) We proceed in the same way as in part a). However, for the convenience of the reader, we include a full proof here. By Lemma 2.5, one has

\[
S^{(n)} \subseteq \bigcap_{k=1}^{n-1} \bigcap_{\rho} \ker(\rho : S \to \mathbb{U}_{k+1}(\mathbb{Z}/p^{n-k}\mathbb{Z})).
\]

Now let \( \sigma \) be any element in \( S \setminus S^{(n)} \). By Lemma 2.1 d), there exists a multi-index \( I = (i_1, \ldots, i_k) \) of length \( k \) with \( 1 \leq k \leq n - 1 \) such that \( \nu(\epsilon_{I, \mathbb{Z}_p}(\sigma)) < n - k \). This implies that \( \epsilon_{I, \mathbb{Z}/p^{n-k}\mathbb{Z}}(\sigma) \neq 0 \). By Lemma 2.3, the map \( \rho : S \to \mathbb{U}_{k+1}(\mathbb{Z}/p^{n-k}\mathbb{Z}) \) defined by

\[
\rho(\tau)_{\mu
u} = \epsilon_{(i_\mu \cdots i_\nu \cdots), \mathbb{Z}/p^{n-k}\mathbb{Z}}(\tau) \quad (\mu < \nu),
\]

is a continuous group homomorphism. We have \( \rho(\sigma)_{1,k+1} = \epsilon_{I, \mathbb{Z}/p^{n-k}\mathbb{Z}}(\sigma) \neq 0 \). Therefore \( \sigma \not\in \ker \rho \), as desired. \( \square \)

Now let \( S \) be an abstract (discrete) free group on \( d \) generators \( x_1, \ldots, x_d \). Let \( \Lambda \) be \( \mathbb{Z} \) or \( \mathbb{Z}/p^r\mathbb{Z} \) with \( r \in \mathbb{N} \). We also have the Magnus homomorphism

\[
\psi : \Lambda[S] \to \Lambda\langle \langle X_1, \ldots, X_d \rangle \rangle, x_i \mapsto 1 + X_i.
\]

Lemmas 2.1, 2.5 have their obvious counterparts in this case. These lemmas still hold true in the discrete setting by some obvious changes, for example, by replacing \( \mathbb{Z}_p \) by \( \mathbb{Z} \), completed
group algebras by group algebras. Note also that the Magnus homomorphism \( \psi \) is injective (see [Mo2] Lemma 8.1 and [Ko2] Lemma 4.4). The counterpart of Lemma 2.2(b) is ensured by a result of Witt [Wi2, Satz 11], see also [Mo2] Proposition 8.5. By proceeding in the same way as in the proof of Theorem 2.6 we immediately obtain the following discrete version of Theorem 2.6.

**Theorem 2.7.** Let \( S \) be a finitely generated free group and \( n \geq 1 \). Then the following statements are true.

(a) \( S_{(n)} \) is the intersection of all kernels of linear representations \( \rho : S \to \text{U}_n(\mathbb{F}_p) \).

(b) \( S_n \) is the intersection of all kernels of linear representations \( \rho : S \to \text{U}_n(\mathbb{Z}) \).

(c) \( S^{(n)} \) is the intersection of all kernels of linear representations \( \rho : S \to \text{U}_{k+1}(\mathbb{Z}/p^{n-k}\mathbb{Z}) \), where \( k = 1, \ldots, n - 1 \).

3. Some Results in Unipotent Representations of Finite Groups

Let \( K \) be a field of characteristic \( p > 0 \) and let \( s \geq 1 \) be an integer. Let \( X \) be the square matrix of size \( n := 1 + p^s \) in \( \text{U}_n(K) \) having zero everywhere except for 1’s in the positions \((i, i+1), 1 \leq i \leq n - 1\). We denote by \( K[X] \) the \( K \)-algebra generated by \( X \) in the full matrix algebra \( \text{Mat}_n(K) \).

**Lemma 3.1.** Let the notation be as above. Then the following are true.

1. \( X^n = 0 \) but \( X^{n-1} \neq 0 \).
2. For \( f(X) = a_lX^l + a_{l+1}X^{l+1} + \cdots \in K[X] \) with \( a_l \neq 0 \), \( f(X) \) is a unit in \( K[X] \) if and only if \( l = 0 \).
3. Every \( K \)-algebra automorphism \( \phi : K[X] \to K[X] \) is determined by the value of \( \phi(X) \) and has the form \( \phi(X) = Xf(X) \) with \( f(X) \) a unit in \( K[X] \). Conversely, if \( f(X) \) is a unit in \( K[X] \) then \( \phi(X) = Xf(X) \) defines a \( K \)-algebra automorphism \( \phi \) of \( K[X] \).

**Proof.** (1) and (2) are straightforward.

(3) Assume that \( \phi : K[X] \to K[X] \) is a \( K \)-algebra automorphism. Then we write

\[
\phi(X) = a_lX^l + a_{l+1}X^{l+1} + \cdots =: X^l f(X) \in K[X],
\]

where \( a_l \neq 0 \). Then \( f(X) \) is a unit in \( K[X] \). It remains to show that \( l = 1 \). Since \( X \) is not a unit, \( \phi(X) \) is not a unit either. Hence \( l \geq 1 \). But \( l \) cannot be \( \geq 2 \). Otherwise we would have

\[
\phi(X^{n-1}) = X^{l(n-1)} f(X)^{n-1} = 0,
\]

because \( l(n-1) \geq 2(n-1) > n \) as \( n = p^s + 1 \geq 3 \). This is a contradiction since \( \phi \) is injective and \( X^{n-1} \neq 0 \). Therefore \( l = 1 \), as desired.

Conversely, assume that \( f(X) = a_0 + a_1X + \cdots, a_0 \neq 0 \), is a unit in \( K[X] \). Let \( \varphi : K[X] \to K[X] \) be a \( K \)-algebra endomorphism defined by \( \varphi(X) = Xf(X) \). Then \( \varphi(X^i) = X^i f(X)^i \) and the matrix \( M \) of \( \varphi \) with respect to the base \( \{1, X, \ldots, X^{n-1}\} \) of the \( K \)-vector space \( K[X] \) is a lower triangular matrix with \( 1, a_0, a_0^2, \ldots, a_0^{n-1} \) on the diagonal. Since \( \det(M) = \prod_{i=1}^{n-1} a_0^i \neq 0 \), \( \varphi \) is an isomorphism as a \( K \)-linear map. We denote by \( \psi \) its inverse as a \( K \)-linear map. Then we can check that \( \psi \) is in fact a \( K \)-algebra homomorphism. Therefore, \( \varphi \) is a \( K \)-algebra automorphism. \( \square \)

The following Lemma 3.2 admits a simple direct proof which we shall omit.

**Lemma 3.2.** The centralizer of \( X \) in \( \text{Mat}_n(K) \) is \( K[X] \).
The following result is a generalization of [Ja, Lemma 4.2]. In this lemma only the case of \( \text{char}(K) = 2 \) was considered.

**Lemma 3.3.** Any \( K \)-algebra automorphism \( \varphi \) of \( K[X] \) is induced by conjugation with some upper triangular matrix \( A \in \text{GL}_n(K) \). For a given \( \varphi \) there is a unique such \( A \) which has only zeros in the last column except for a 1 in the \((n,n)\) position. Any other matrix inducing \( \varphi \) has the form \( AB \) for some unit \( B \) in \( K[X] \).

**Proof.** Let \( \varphi(X) = Xf(X), \) where \( f(X) \) is a unit in \( K[X] \). Let \( v_1, v_2, \ldots, v_n \) be the basis of \( K^n \) such that
\[ Xv_1 = 0; Xv_i = v_{i-1}, \forall 2 \leq i \leq n. \]
Define \( A \) to be the matrix such that
\[ Av_i = f(X)^{n-i}v_i. \]
Since \( f(X) \) is a unit, \( A \) is invertible. And for \( i \geq 2 \), we have
\[ AXv_i = Av_{i-1} = f(X)^{n-i+1}v_{i-1} = f(X)^{n-i+1}Xv_i = Xf(X)f(X)^{n-i}v_i = Xf(X)Av_i. \]
The first and the last term are equal also for \( i = 1 \). In fact,
\[ Xf(X)Av_1 = Xf(X)f(X)^{n-1}v_1 = f(X)f(X)^{n-1}Xv_1 = 0. \]
Therefore \( AX = Xf(X)A \) and thus
\[ AXA^{-1} = \varphi(X). \]

When \( f(X)^{n-i}v_i \) is expressed in terms of \( v_j \), only \( v_j \) with \( j \leq i \) can have nonzero coefficients. (The shape of the polynomial \( f(X)^{n-j} \) is not important for this observation.) Thus \( A \) is upper triangular. Moreover, since \( Av_n = v_n \), the last column of \( A \) has only zero entries except for a 1 at the \((n,n)\) position. Also note that if \( f(X) = 1 + aX + \cdots \) then \( A \) is a unipotent upper triangular matrix.

Now assume that \( A_0 \) also induces \( \varphi \). Then \( A_0XA_0^{-1} = AXA^{-1} \). Hence \( A^{-1}A_0X = XA^{-1}A_0 \). Thus \( A^{-1}A_0B = B \), i.e., \( A_0 = AB \) with \( B \) a unit in \( K[X] \). Assume further that the last column of \( A_0 \) contains only zero except for a 1 at the \((n,n)\) position. Then \( A_0v_n = v_n \). Hence \( Bv_n = A^{-1}A_0v_n = A^{-1}v_n = v_n \).

Writing \( B = a_0 + a_1X + \cdots \) implies that \( a_0 = 1 \) and \( a_i = 0 \) for all \( 1 \leq i \leq n - 1 \). Hence \( B = 1 \) and \( A_0 = AB = A \). \( \square \)

**Lemma 3.4.** Let the notation be as above. Let \( B = 1 + X \). Let \( k \) be a positive integer. Then
1. There exists a unique matrix \( A \) in \( \mathbb{U}_n(K) \) with only zero entries in the last column except at the \((n,n)\) position such that
\[ ABA^{-1} = B^{1+p^k}. \]
2. If \( p = 2 \) then there exists a unique matrix \( A \) in \( \mathbb{U}_n(K) \) with only zero entries in the last column except at the \((n,n)\) position such that
\[ ABA^{-1} = B^{-(1+2^k)}. \]
Proof. We define a $K$-algebra automorphism $\varphi(X)$ of $K[X]$ by
\[
\varphi(X) := \begin{cases} 
X(1 + X^{p^k - 1} + X^{p^k}) & \text{if we are in Part (1)}, \\
X(1 + X^{2^k - 1} + X^{2^k})(1 + X)^{-(1+2^k)} & \text{if we are in Part (2)}. 
\end{cases}
\]
Then
\[
\varphi(1 + X) = \begin{cases} 
1 + X(1 + X^{p^k - 1} + X^{p^k}) = (1 + X)^{1+p^k} & \text{if we are in Part (1)}, \\
1 + X(1 + X^{2^k - 1} + X^{2^k})(1 + X)^{-(1+2^k)} = (1 + X)^{-(1+2^k)} & \text{if we are in Part (2)}. 
\end{cases}
\]
By Lemma 3.3, there exists an upper triangular matrix $A$ such that
\[
AXA^{-1} = \varphi(X),
\]
or equivalently,
\[
A(1 + X)A^{-1} = \varphi(1 + X).
\]
Also from the construction of $A$ as in the proof of Lemma 3.3, we see that $A$ is a unipotent matrix. \hfill \Box

When $p = 2$, we also have the following result.

**Lemma 3.5.** Let the notation be as above. Assume that $p = 2$. Then there exists a unique matrix $A$ in $U_n(K)$ with only zero entries in the last column except at the $(n, n)$ position such that
\[
ABA^{-1} = B^{-1}.
\]

**Proof.** This is proved in [Ja, page 154]. \hfill \Box

We will compute the order of matrix $A$ found in Lemma 3.4 Part (1). If $k > s$ then $(1 + X)^{1+p^k} = (1 + X)(1 + X^{p^k}) = 1 + X$ and $A = I$ by the uniqueness of $A$. We consider the case $k \leq s$ in the following proposition.

**Proposition 3.6.** Let the notation be as in Lemma 3.4 Part (1). Assume that $k \leq s$. Then the order of $A$ is $p^{s+1-k}$.

First we need the following elementary lemma.

**Lemma 3.7.** Let $a, b, l$ be three integers with $l \geq 1$. Assume that
\[
a \equiv b \text{ mod } p^l \text{ and } a \not\equiv b \text{ mod } p^{l+1}.
\]
Then $a^p \equiv b^p \text{ mod } p^{l+1}$ and $a^p \not\equiv b^p \text{ mod } p^{l+2}$. \hfill \Box

**Proof of Proposition 3.6** Since $\varphi(1 + X) = A(1 + X)A^{-1} = (1 + X)^{1+p^k}$, we obtain $\varphi^p(1 + X) = A^{p^l}(1 + X)A^{-p^l} = (1 + X)^{(1+p^k)p^l}$, for all $l \geq 0$.

On the other hand, since $(1 + p^k) \equiv 1 \text{ mod } p^k$ and $(1 + p^k) \not\equiv 1 \text{ mod } p^{k+1}$, we obtain
\[
(1 + p^k)^{p^{s+1-k}} \equiv 1 \text{ mod } p^s \text{ and } (1 + p^k)^{p^{s-k}} \not\equiv 1 \text{ mod } p^s.
\]
Thus $(1 + X)^{(1+p^k)p^{s+1-k}} = 1 + X$ and $(1 + X)^{(1+p^k)p^{s-k}} \not= 1 + X$ since $1 + X$ is of order $p^{s+1}$. Hence
\[
A^{p^{s+1-k}}(1 + X)A^{-p^{s+1-k}} = (1 + X)^{(1+p^k)p^{s+1-k}} = 1 + X,
\]
and
\[
A^{p^{s-k}}(1 + X)A^{-p^{s-k}} = (1 + X)^{(1+p^k)p^{s-k}} \not= 1 + X.
\]
Therefore $A^{p^{r+k+1}} = 1$ (by the uniqueness statement applied to $\phi^{p^{r+1-k}}$ and $A^{p^{r+k}}$), and $A^{p^{r+k}} \neq 1$. Hence the order of $A$ is $p^{r+1-k}$. □

Let $p$ be a prime. The extra-special group of order $p^3$ and exponent $p^2$ is the group

$$M_{p^3} = \langle x, y \mid x^{p^2} = y^p = 1, yxy^{-1} = x^{1+p} \rangle.$$ 

**Corollary 3.8.** Let $p$ be an odd prime number and $K$ a field of characteristic $p$. Let $M$ be the cyclic group $\mathbb{Z}/p^2\mathbb{Z}$ or $M_{p^3}$. The smallest number $n$ such that $M$ can be embedded in $\mathbb{U}_n(K)$ is $n = p + 1$.

**Proof.** Note that every element in $\mathbb{U}_p(K)$ is of exponent $p$. Hence $M$ cannot be embedded in $\mathbb{U}_r(K)$ with $r \leq p$.

If $M = \mathbb{Z}/p^2\mathbb{Z}$, we define an embedding of $M$ into $\mathbb{U}_{p+1}(K)$ by sending 1 to the matrix $B = 1 + X$.

If $M = M_{p^3}$, then we define a group homomorphism $\varphi: M \to \mathbb{U}_{p+1}(K)$ by sending $x$ to matrix $B$ and $y$ to matrix $A$, where $A$ is a matrix in $\mathbb{U}_{p+1}(K)$ such that $ABA^{-1} = B^{1+p}$. The existence and uniqueness of $A$ is assured by Lemma 3.4. Furthermore the (group) order of $A$ is $p$. Then $\varphi$ is well-defined and $\varphi$ is an injection. □

4. **Galois Groups of maximal $p$-extensions of rigid fields**

Let $p$ be an odd prime. Let $F$ be a field containing a (fixed) primitive $p$-th root of unity. For each $a \in F^\times$, let $\chi_a \in H^1(G_F, \mathbb{F}_p) = H^1(G_F(p), \mathbb{F}_p)$ be the character associating to $a$ via the Kummer map $F^\times \to H^1(G_F, \mathbb{F}_p) = H^1(G_F(p), \mathbb{F}_p)$.

**Definition 4.1.** An element $a \in F \setminus F^p$ is called $p$-rigid if $\chi_a \cup \chi_b = 0$ implies $b \in a^iF^p$ for some $i \geq 0$. The field $F$ is called $p$-rigid if every element in $F \setminus F^p$ is $p$-rigid.

For $h > 0$, let $\mu_{p^h}$ be the group of $p^h$ roots of unity. We also set $\mu_{p^\infty}$ to be the group of all roots of unity of order $p^m$ for some $m \geq 0$. Finally we set $k \in \mathbb{N} \cup \{\infty\}$ to be the maximum of all $h \in \mathbb{N} \cup \{\infty\}$ such that $\mu_{p^h} \subseteq F$. Then we have the following theorem, see [CMO, Theorem 4.10] and also [Wa3, Theorem 2] for part (2).

**Theorem 4.2.** Suppose $F$ is a $p$-rigid field and let $G = G_F(p)$. Then we have the following.

1. $$G/G^{(n)} = \begin{cases} (\prod I \mathbb{Z}/p^{n-1}\mathbb{Z}) \times \mathbb{Z}/p^{n-1}\mathbb{Z} & \text{if } k < \infty, \\
\prod I \mathbb{Z}/p^{n-1}\mathbb{Z} & \text{if } k = \infty. \end{cases}$$

2. $$G = \begin{cases} (\prod I \mathbb{Z}_p) \times \mathbb{Z}_p & \text{if } k < \infty, \\
\prod I \mathbb{Z}_p & \text{if } k = \infty. \end{cases}$$

Moreover when $k < \infty$ there exists a generator $\sigma$ of the outer factor $\mathbb{Z}/p^{n-1}\mathbb{Z}$ in (1) and of the outer factor $\mathbb{Z}_p$ in (2) such that for each $\tau$ from the inner factor $\prod I \mathbb{Z}/p^{n-1}\mathbb{Z}$ in (1) and each $\tau$ from the inner factor $\prod I \mathbb{Z}_p$ in (2) we have

$$\sigma \tau \sigma^{-1} = \tau^{p^k+1}.$$
Corollary 4.3. Let $F$ be a $p$-rigid field and $G = G_F(p)$. Let $n \geq 2$ and let $s$ be the integer such that $p^{s-1} < n \leq p^s$. Then $G_{(n)} = G^p$ and

$$G / G_{(n)} = \begin{cases} \prod_I \mathbb{Z} / p^s \mathbb{Z} \times \mathbb{Z} / p^s & \text{if } k < \infty, \\ \prod_I \mathbb{Z} / p^s \mathbb{Z} & \text{if } k = \infty. \end{cases}$$

Moreover when $k < \infty$ there exists a generator $\sigma$ of the outer factor $\mathbb{Z} / p^s \mathbb{Z}$ such that for each $\tau$ from the inner factor $\prod_I \mathbb{Z} / p^s \mathbb{Z}$ we have

$$\sigma \tau \sigma^{-1} = \tau p^{s+1}.$$ 

Proof. The first statement $G_{(n)} = G^p$ is proved at the end of the paper [CMQ]. The second statement follows from Theorem 4.2 and $G^{(s+1)} = G^p$ [CMQ, Remark 4.2]. \hfill $\Box$

5. Kernel Unipotent Conjecture over Odd Rigid Fields

Proposition 5.1. Let $k \geq 1$ be an integer. Let $H$ be a pro-$p$-group. Assume that $H$ satisfies one of the following conditions.

1. $H \simeq (\prod_I \mathbb{Z} / p^{s+1} \mathbb{Z}).$
2. $H \simeq (\prod_I \mathbb{Z} / p^{s+1} \mathbb{Z}) \ltimes (\mathbb{Z} / p^{s+1} \mathbb{Z}) =: U \rtimes V$,
   and there exists a generator $\sigma \in V$ for $V$ such that $\sigma \tau \sigma^{-1} = \tau p^{s+1}$, for all $\tau \in U$.
3. If $p = 2$ and $H \simeq (\prod_I \mathbb{Z} / p^{s+1} \mathbb{Z}) \ltimes (\mathbb{Z} / p^{s+1} \mathbb{Z}) =: U \times V$,
   and there exists a generator $\sigma \in V$ for $V$ such that $\sigma \tau \sigma^{-1} = \tau^{-(2^k+1)}$, for all $\tau \in U$.
4. If $p = 2$ and $H \simeq (\prod_I \mathbb{Z} / p^{s+1} \mathbb{Z}) \ltimes (\mathbb{Z} / p^{s+1} \mathbb{Z}) =: U \times V$,
   and there exists a generator $\sigma \in V$ for $V$ such that $\sigma \tau \sigma^{-1} = \tau^{-1}$, for all $\tau \in U$.

Let $n = p^s + 1$. Then for every $u \in H$, $u \neq 1$, there exists a representation $H \rightarrow \mathbb{U}_n(\mathbb{F}_p)$ such that $\rho(u) \neq 1$.

Proof. Let $B := 1 + X$ be as in Lemmas 3.4 and 3.5.

1. We write $u = (u_i)_I \in U$ and let $C_i$ be a copy of $\mathbb{Z} / p^{s+1} \mathbb{Z}$ at the $i$-th coordinate in $U$. Then there exists $i_0$ such that $u_{i_0}$ is not the identity element in $C_{i_0}$. Let $\tau$ be a generator of $C_{i_0}$ and let us write $u_{i_0} = \tau^a$ with $p^{s+1} \mid a$. We define a representation $\rho: H \rightarrow \mathbb{U}_n(\mathbb{F}_p)$ by: $\rho(\tau) \mapsto B$, $\rho(C_i) = 1$ for all $i \neq i_0$. Since $B^{p^{s+1}} = 1$, $\rho$ is a well-defined homomorphism. Moreover $\rho(u) = \rho(u_{i_0}) = B^a \neq 1$.

2)-(4): We write $x = uv$, with $u \in U$, $v \in V$. For convenience, we write $\varphi(B) = B^{1+p^k}$, $B^{-(1+p^k)}$, or $B^{-1}$ if we are in Part (2), Part (3), or Part (4) respectively.

Case 1: $v \neq 1$. We define a group homomorphism $\rho : H \rightarrow \mathbb{U}_n(\mathbb{F}_p)$ as follow: $\rho(\tau) = 1$ for all $\tau \in U$ and $\rho(\sigma) = B$. This is a well-defined homomorphism since it is clear that
\[ B1B^{-1} = 1 = \varphi(1) \] and that \( Bp^{s+1} = 1 \). Writing \( v = \sigma^a \) for some \( a \in \mathbb{Z} \) such that \( p^{s+1} \nmid a \) \((v \neq 1 \text{ in } V)\). Since \( B \) is of order \( p^{s+1} \), \( \rho(x) = \rho(v) = B^a \neq 1 \), as desired.

**Case 2:** \( v = 1 \). Then \( u \neq 1 \).

By Lemmas 3.4 and 3.5 there exists \( A \in \mathbb{U}_n(\mathbb{F}_p) \) such that \( ABA^{-1} = \varphi(B) \). We can define a representation \( \rho : U \times V \to \mathbb{U}_n(\mathbb{F}_p) \) such that \( \rho(x) = \rho(u) \neq 1 \) as follows. We write \( u = (u_i)_1 \in U \) and let \( C_i \) be a copy of \( \mathbb{Z}/p^{s+1}\mathbb{Z} \) at the \( i \)-th coordinate in \( U \). Then there exists \( i_0 \) such that \( u_{i_0} \) is not the identity element in \( C_{i_0} \). Let \( \tau \) be a generator of \( C_{i_0} \), and let us write \( u_{i_0} = \tau^d \) with \( p^{s+1} \nmid a \). We define \( \rho \) by: \( \rho(\tau) \mapsto B^i \), \( \rho(C_i) = 1 \) for all \( i \neq i_0 \) and \( \rho(\sigma) = A \).

Since \( ABA^{-1} = \varphi(B) \) and hence \( AB^rA^{-1} = \varphi(B)^r = \varphi(B^r) \) for all \( r \), we conclude that \( \rho \) is a well-defined homomorphism. Moreover \( \rho(u) = \rho(u_{i_0}) = B^a \neq 1 \). 

**Theorem 5.2.** Let \( p \) be a prime number and let \( F \) be a rigid field. Let \( G = G_F(p) \) be the Galois group of the maximal \( p \)-extension of \( F \). Then for any natural number \( n \),

\[ G_{(n)} = \bigcap \ker(\rho : G \to \mathbb{U}_n(\mathbb{F}_p)), \]

where \( \rho \) runs over the set of all continuous homomorphisms \( G \to \mathbb{U}_n(\mathbb{F}_p) \).

**Proof.** By Corollary 4.3 for any integer \( s \geq 0 \) we have \( G_{(p^s+1)} = G_{(p^{s+2})} = \cdots = G_{(p^{s+1})} \). Thus it is enough to consider the case \( n = p^s + 1 \). Then the statement follows from Corollary 4.3 and Proposition 5.1. 

**Demushkin groups of rank 2.** (In this subsection we also consider \( p = 2 \).) Recall (see e.g., [La, Se1]) that a pro-\( p \)-group \( G \) is said to be a Demushkin group if

1. \( \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) < \infty \),
2. \( \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = 1 \),
3. the cup product \( H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p) \) is a non-degenerate bilinear form.

We call \( d := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) \) the rank of \( G \). When the rank \( d = 2 \) then \( G \) has the following presentation (see [Se1, page 147]): \( G \) is isomorphic to a pro-\( p \)-group on two generators \( x, y \) subject to one relation in one of the following types

- (Type 1) \( yxy^{-1} = x \),
- (Type 2) \( yxy^{-1} = x^{1+q} \) for some \( q \) with \( (q = p^k \geq p \text{ if } p > 2) \) and \( (q = 2^k \geq 4 \text{ if } p = 2) \),
- (Type 3) \( yxy^{-1} = x^{-1} \) and \( p = 2 \),
- (Type 4) \( yxy^{-1} = x^{-(1+m)} \) for some \( m = 2^k \geq 4 \) and \( p = 2 \).

Let \( F \) be a local field of residue characteristic different from \( p \). Assume that \( F \) contains a primitive \( p \)-th root of unity. Then the Galois group \( G_F(p) \) of the maximal \( p \)-extension of \( F \) is a Demushkin group of rank 2.
Lemma 5.3. Let $G$ be a Demushkin pro-$p$-group generated by $x, y$ and subject to one relation as above. Let $n \geq 2$ and let $s$ be a unique integer such that $p^{s-1} < n \leq p^s$. Then $G_{(n)} = G^{p^s}$ and

$$G/G_{(n)} = \begin{cases} \mathbb{Z}/p^s\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z} & \text{if we are in Type 1}, \\ \mathbb{Z}/p^s\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z} & \text{if we are in Type 2}, \\ \mathbb{Z}/2^s\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z} & \text{if we are in Type 3}, \\ \mathbb{Z}/2^s\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z} & \text{if we are in Type 4}. \end{cases}$$

Moreover, let $\sigma$ be the image of $x$ in the outer factor $\mathbb{Z}/p^s\mathbb{Z}$, then for each $\tau$ from the inner factor $\mathbb{Z}/p^s\mathbb{Z}$ we have

$$\sigma \tau \sigma^{-1} = \tau^{1+p^k} \text{ if we are in Type 2},$$

$$\sigma \tau \sigma^{-1} = \tau^{-1} \text{ if we are in Type 3},$$

$$\sigma \tau \sigma^{-1} = \tau^{-(1+2^k)} \text{ if we are in Type 4}.$$

Proof. If we are in Types (1)-(2), one can see that $[G, G] \subseteq G^p$ and if we are in Type 4, one has $[G, G] \subseteq G^4$. Hence $G$ is powerful ([DDMS, Chapter 3, Definition 3.1]). Hence in these cases, $G_{(n)} = G^{p^n}$ by [DDMS, Exercise 4, p. 289]. If we are in Type 3, one can show that $G_{(n)} = G^{2^n}$ by induction on $n$.

Then the remaining statements are straightforward.

Proposition 5.4. Let $G$ be a pro-$p$-group. Assume that $G$ is a Demushkin group of rank 2. Then for any natural number $n$,

$$G_{(n)} = \bigcap \ker(\rho : G \to \mathbb{U}_n(F_p)),$$

where $\rho$ runs over the set of all continuous homomorphisms $G \to \mathbb{U}_n(F_p)$.

Proof. This proof is similar to that of Theorem 5.2 using Lemma 5.3 instead of Corollary 4.3.

Remark 5.5. If the rank of a Demushkin group $G$ is 1 then $G$ is a cyclic group of order 2, and hence the Kernel $n$-Unipotent Conjecture is true for $G$.

Remark 5.6. Every Demushkin group $G$ of rank 2 is realizable as $G_F(p)$ for some field $F$. In fact, if its relation is of Type 1, then $G \simeq G_F(2)$ with $F = \mathcal{C}((X))(Y))$ (see [Wa1, Corollary 3.9, part (2)]). Now assume that the relation is of Type 2. By Dirichlet’s theorem on primes in arithmetic progressions, there is a prime number $\ell$ such that $\ell = p^k + 1 + p^{k+1}a$, for some $a \in \mathbb{Z}$. Then by [Se2, Exercise 2, page 98], the Galois group $G_{Q_2}(p)$ is isomorphic to a Demushkin group on two generators $x, y$ with one relation

$$yxy^{-1} = x^{1+p^k}.$$

Similarly, assume that we are given a pro-$2$ Demushkin group $G$ with generators $x, y$ such that the relation is of Type 4:

$$yxy^{-1} = x^{-(1+2^k)}.$$

Then by choosing a prime number $\ell$ such that $\ell = 2^k - 1 + 2^{k+1}a$, for some $a \in \mathbb{Z}$, we see that the Galois group $G_{Q_2}(2)$ is isomorphic to $G$ (see [Se2, Exercise 2, page 98]). For the case that our given pro-$2$ Demushkin group is of Type 3, we refer the reader to [JW] Table 5.2, Example 5]. Unfortunately, in this table there is a reference to Remark 2.6 which is missing.
in the paper in [W]. However, for our purposes, it suffices to observe that the required field is \( M((T)) \). Where \( M \) is a maximal algebraic extension of \( \mathbb{R}(X)(\sqrt{-1+X^2}) \), subject to the condition that \( \sqrt{-1} \notin M \).

6. Comparison between filtrations

Let \( G \) be a pro-

\( p \)-group. Let us consider two filtrations on \( G \): the descending \( p \)-central series \( (G^{(i)}) \) and the \( p \)-Zassenhaus filtration \( (G^{(i)}) \) of \( G \). In general we always have \( G^{(i)} \subseteq G^{(j)} \) for all \( i \) and all \( p \) and \( G^{(3)} = G^{(3)} \) if \( p = 2 \). From the very definition, we get \( G^{(p+1)} \subseteq G^{(3)} \). Indeed, we have

\[
G^{(3)} = (G^{(2)})^p[G^{(2)}, G] \quad \text{and} \quad G^{(p+1)} = (G^{(2)})^p \prod_{i+j=p+1} [G^{(i)}, G^{(j)}].
\]

Since \( G^{(2)} = G^{(2)} \) and \([G^{(i)}, G^{(p+1-i)}] \subseteq [G^{(2)}, G] = [G^{(2)}, G] \) for all \( i = 1, \ldots, p \), we get \( G^{(p+1)} \subseteq G^{(3)} \). For convenience, we introduce another filtration \( G_{<n>} \), called the kernel filtration, on group \( G \) which is defined by

\[
G_{<n>} := \bigcap_{\rho} \ker(\rho: G \to \mathbb{U}_{p+1}(\mathbb{F}_p)),
\]

where the intersection is taken over the collection of all (continuous) group homomorphisms \( \rho: G \to \mathbb{U}_{p+1}(\mathbb{F}_p) \). Note that by [MT] Lemma 3.6, one always has

\[
G^{(n)} \subseteq G_{<n>}.
\]

The kernel conjecture can be restated as the following:

**Conjecture 6.1.** Let \( F \) be a field of characteristic \( \neq p \), containing a primitive \( p \)-th root of unity. Let \( G = G_F(p) \) be the Galois group of the maximal pro-

\( p \)-extension of \( F \). Then for any \( n \), one has

\[
G^{(n)} = G_{<n>}.
\]

Inspired by Corollary 3.8 and the kernel conjecture, we have the following result which is interesting in its own right.

**Proposition 6.2.** Let \( p \) be prime and let \( F \) be a field of characteristic \( \neq p \), which contains a primitive \( p \)-th root of 1. Let \( G = G_F(p) \) be the Galois group of the maximal \( p \)-extension of \( F \).

1. We have

\[
G^{(p+1)} \subseteq G_{<p+1>} \subseteq G^{(3)}.
\]

2. Assume further that \( G \) is not isomorphic to the trivial group 1 or the cyclic group \( \mathbb{Z}/2\mathbb{Z} \) then \( p + 1 \) is the smallest integer \( n \) with the property that \( G^{(n)} \subseteq G^{(3)} \).

**Proof.** a) By [EM1 Main Theorem], \( G^{(3)} \) is the intersection of the kernels of all homomorphisms from \( G \) to \( \mathbb{Z}/p^2\mathbb{Z} \) or to \( M_{p^2} \). Then Corollary 3.8 implies that

\[
G^{(3)} \supseteq \bigcap_{\rho} \ker(\rho: G \to \mathbb{U}_{p+1}(\mathbb{F}_p)) = G_{<p+1>}.
\]

b) We first show that if \( G \) is not isomorphic to 1 or \( \mathbb{Z}/2\mathbb{Z} \) then \( G^{[3]} := G/G^{(3)} \) is of exponent exactly \( p^2 \). It is clear that every element of \( G^{[3]} \) is of order at most \( p^2 \), so it remains to show that \( G^{[3]} \) contains an element of order \( p^2 \).
First we treat the case \( p > 2 \). Since \( G \) is a non-trivial pro-\( p \)-group, one has \( \text{Hom}(G, \mathbb{F}_p) \cong \text{Hom}(G/G^{(2)}, \mathbb{F}_p) \neq 1 \) (see for example [Se2, Corollary 2, page 19]). In particular we can take an element \( x \in G \setminus G^{(2)} \). Then \( x \) has order exactly \( p^2 \) in \( G^{[3]} \) because \( x^{p^2} \in G^{(3)} \) and by [EM1, Proposition 12.3] (see also [BLMS, Theorem A.3]) we know that every element of \( G/G^{(3)} \) of order \( p \) is in fact in \( G^{(2)}/G^{(3)} \).

Now we consider the case \( p = 2 \) and assume that \( G^{[3]} = G/G^{(3)} \) is of exponent (at most) \( 2 \). This implies in particular that \( G^{[3]} \) is abelian and is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^J\). By the classification of abelian \( W \)-groups [MS1, Theorems 3.12 and 3.13] (note that the \( W \)-groups defined there coincide with groups of the form \( G_F^{[3]} \), then \( G^{[3]} \), which is also denoted by \( G_F \) in [MS1], is the trivial group or isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). One can check directly that this means that \( F \) is \( F(2) \), or \( F \) has a unique quadratic extension \( L = L(2) = F(2) \). (Here for a field \( F \), we denote \( F(2) \) the maximal 2-extension of \( F \).) Hence \( G \) itself is trivial or isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). This contradicts our assumption. Hence \( G^{[3]} \) is of exponent \( 4 \).

On the other hand, from the definition of the \( p \)-Zassenhaus filtration, we see that \( G/G(p) \) is of exponent (at most) \( p \). Therefore there cannot exist any surjections from

\[
G/G(p) \rightarrow G/G^{(3)}.
\]

In particular, \( G^{(3)} \) does not contain \( G(p) \) and hence also does not contain \( G(n) \) for all \( 1 \leq n < p \).

**Remark 6.3.** a) The proof of Proposition 6.2 (part b) shows that for a non-trivial pro-\( p \)-group \( G \), which is also not isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), if \( G/G^{(3)} \) is of exponent \( p \) then \( G \neq G_F(p) \) for any field \( F \) containing a primitive \( p \)-th root of unity. Note also that if \( G/G^{(3)} \) is of exponent \( p \) then \( G^{(3)} \) in fact contains \( G_F(p) \).

b) There are examples of pro-\( p \)-groups \( G \) such that \( G \) does not satisfy the conclusion in part a) of Proposition 6.2 (see Appendix). Then such pro-\( p \)-groups also cannot be realized as \( G_F(p) \) for any field \( F \) containing a primitive \( p \)-th root of unity.

7. **REVIEW OF MASSEY PRODUCTS IN THE COHOMOLOGY OF PROFINITE GROUPS**

Let \( p \) be a prime number. Let \( G \) be a profinite group. We consider \( \mathbb{F}_p \) as a trivial discrete \( G \)-module. Let \( ^\ast\mathcal{C}(G, \mathbb{F}_p, \delta) \) be the standard inhomogeneous continuous cochain complex of \( G \) with coefficients in \( \mathbb{F}_p \) [NSW Ch. I, §2]. We write \( H^1(G, \mathbb{F}_p) \) for the corresponding cohomology groups.

We shall assume that \( a_1, \ldots, a_n \) are elements in \( H^1(G, \mathbb{F}_p) \).

**Definition 7.1.** A collection \( M = (a_{ij}), 1 \leq i < j \leq n + 1, (i, j) \neq (1, n + 1) \), of elements of \( C^1(G, \mathbb{F}_p) \) is called a defining system for the \( nth \) order Massey product \( \langle a_1, \ldots, a_n \rangle \) if the following conditions are fulfilled:

1. \( a_{i,i+1} \) represents \( a_i \).
2. \( \delta a_{ij} = \sum_{l=i+1}^{j-1} a_{il} \cup a_{lj} \) for \( i + 1 < j \).

Then \( \sum_{k=2}^n a_{1k} \cup a_{k,n+1} \) is a 2-cocycle. Its cohomology class in \( H^2(G, \mathbb{F}_p) \) is called the value of the product relative to the defining system \( M \), and is denoted by \( \langle a_1, \ldots, a_n \rangle_M \).

The product \( \langle a_1, \ldots, a_n \rangle \) itself is the subset of \( H^2(G, \mathbb{F}_p) \) consisting of all elements which can be written in the form \( \langle a_1, \ldots, a_n \rangle_M \) for some defining system \( M \).
Theorem 7.2. Let $\alpha_1, \ldots, \alpha_n$ be elements of $H^1(G, \mathbb{F}_p)$. There is a one-one correspondence $M \leftrightarrow \rho_M$ between defining systems $M$ for $\langle \alpha_1, \ldots, \alpha_n \rangle$ and group homomorphisms $\rho_M : G \to \bar{U}_{n+1}(\mathbb{F}_p)$ with $(\rho_M)_{i+1} = -\alpha_i$ for $1 \leq i \leq n$.

Moreover $\langle \alpha_1, \ldots, \alpha_n \rangle_M = 0$ in $H^2(G, \mathbb{F}_p)$ if and only if the dotted arrow exists in the following commutative diagram

\[
\begin{array}{c}
G \\
\rho_M \\
\end{array} \quad \begin{array}{c}
\cdots \\
\cdots \\
\end{array} \quad \begin{array}{c}
0 \\
\mathbb{F}_p \\
\bar{U}_{n+1}(\mathbb{F}_p) \\
\end{array} \quad \begin{array}{c}
\rho_M \\
\bar{U}_{n+1}(\mathbb{F}_p) \\
\mathbb{F}_p \\
\end{array} \quad \begin{array}{c}
1 .
\end{array}
\]

Explicitly, the one-one correspondence in Theorem 7.2 is given by: For a defining system $M = (a_{ij})$ for $\langle \alpha_1, \ldots, \alpha_n \rangle$, $\rho_M : G \to \bar{U}_{n+1}(\mathbb{F}_p)$ is given by letting $(\rho_M)_{ij} = -a_{ij}$.

Corollary 7.3. Let $\rho : G \to \bar{U}_{n+1}(\mathbb{F}_p)$ be a group homomorphism, then the $n$-fold Massey product $\langle -\rho_1, \ldots, -\rho_{n+1} \rangle$ is defined and contains $0$.

8. Vanishing of Massey Products over Odd Rigid Fields

Theorem 8.1. Let $p$ be a prime number and $n \geq 3$ an integer. Let $F$ be a field. Assume that $F$ contains a primitive $p$-th root of unity if char$(F) \neq p$ and assume further that if $p = 2$ then $-1$ is in $F^2$. Let $G$ be the absolute Galois group $G_F$ of $F$ or its maximal pro-$p$ quotient $G_{F}(p)$. Then for any $\chi \in H^1(G_F, \mathbb{F}_p)$, the $n$-fold Massey product $\langle \chi, \ldots, \chi \rangle$ is defined and contains $0$.

It is enough to consider the case $G = G_{F}(p)$. Also if char$(F) = p$ then by a result of Witt ([Witt], [Sel] Chapter II, §2, Corollary 1), we know that $G_{F}(p)$ is a free pro-$p$-group. Hence, $\langle \chi, \ldots, \chi \rangle = 0$.

So we may assume that char$(F) \neq p$, and let us fix a primitive $p$-th root of unity $\xi$. Then $\chi = \chi_a$ for some $a \in F^\times$, where $\chi_a \in H^1(G_F, \mathbb{F}_p) = H^1(G_F(p), \mathbb{F}_p)$ is the character associating to $a$ via the Kummer map $F^\times \to H^1(G_F, F_p) = H^1(G_{F}(p), \mathbb{F}_p)$.

Also it suffices to consider the case $n = p^s$ for some integer $s$ since if the $m$-fold Massey product $\langle \chi_a, \ldots, \chi_a \rangle$ is defined, then for all $2 \leq n < m$, all $n$-fold Massey products $\langle \chi_a, \ldots, \chi_a \rangle$ are defined and contain $0$. Also we can assume $a \not\in F^p$.

For each integer $r \geq 1$, we choose a primitive $p^r$-th root of unity $\zeta_p^r$ in the way that $\zeta_p^{p^{r+1}} = \zeta_p^{p^r}$.

Proof. Case 0: First we assume that $\zeta_p^{p^{r+1}}$ is in $F$. Then the polynomial $X^{p^{r+1}} - a$ is irreducible over $F$ since $a \not\in (F^\times)^p$. Let $L = F(a^{1/p^{r+1}})$. Then $L/F$ is a cyclic extension of order $p^{r+1}$.
whose Galois group generated by \( \sigma \), where \( \sigma \) defined by \( \sigma(a^{1/p+1}) = \zeta_{p^{i+1}}a^{1/p^{i+1}} \). We define a representation \( \varphi \) from \( \text{Gal}(L/F) \rightarrow \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \) by \( \varphi(\sigma) = B + X \), where \( X \) is the matrix as in the beginning of Section 3. Then \( \varphi \) is well-defined and in fact it is an isomorphism. Let \( \rho : \text{Gal}(F) \rightarrow \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \) be the composite \( \text{Gal}(F) \rightarrow \text{Gal}(L/F) \xrightarrow{\varphi} \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \). We claim that \( \rho_i,i+1 = \chi_a \), for \( i = 1, \ldots, p^s \). Indeed, since both maps \( \rho \) and \( \chi_a \) factor through the quotient \( \text{Gal}(L/F) \), it suffices to check on the generator \( \sigma \) of \( \text{Gal}(L/F) \). Since \( \sigma(a^{1/p}) = \sigma(a^{1/p^{i+1}})^{p^s} = a^{1/p}\zeta_{p^s} \), one has
\[
\varphi_{i,i+1}(\sigma) = 1 = \chi_a(\sigma).
\]
Therefore \( \rho_{i,i+1} = \chi_a \). It implies that \( \langle -\chi_a, \ldots, -\chi_a \rangle \) contains 0 and hence \( \langle \chi_a, \ldots, \chi_a \rangle \) also contains 0.

From now on, we assume that \( \zeta_{p^{i+1}} \) is not in \( F \). Let \( p^k \) be the largest index such that \( \zeta_{p^k} \) is in \( F, 1 \leq k \leq s \).

**Case 1:** Assume that the polynomial \( X^{p^{i+1}} - a \) is irreducible over \( F(\zeta_{p^{i+1}}) \). Then the extension \( L := F(\zeta_{p^{i+1}}, a^{1/p^{i+1}}) \) is Galois over \( F \). Define two automorphisms \( \sigma, \tau \in \text{Gal}(L/F) \) as follows:
\[
\tau : a^{1/p^{i+1}} \mapsto \zeta_{p^{i+1}}a^{1/p^{i+1}} \text{ and } \tau : \zeta_{p^{i+1}} \mapsto \zeta_{p^{i+1}};
\]
\[
\sigma : \zeta_{p^{i+1}} \mapsto \zeta_{p^{i+1}}^{1+p^k} \text{ and } \sigma : a^{1/p^{i+1}} \mapsto a^{1/p^{i+1}}.
\]
Then we have
\[
\sigma \tau \sigma^{-1}(\zeta_{p^{i+1}}) = \zeta_{p^{i+1}};
\]
\[
\sigma \tau \sigma^{-1}(a^{1/p^{i+1}}) = \sigma(\zeta_{p^{i+1}})^{a^{1/p^{i+1}}} = \zeta_{p^{i+1}}^{1+p^k} a^{1/p^{i+1}} = \tau^{1+p^k}(a^{1/p^{i+1}}).
\]
Therefore \( \sigma \tau \sigma^{-1} = \tau^{1+p^k} \). In the group \( \text{Gal}(L/F) \), the order of \( \tau \) is \( p^{s+1-k} \) and that of \( \sigma \) is \( p^{s+1-k} \) and the group \( \text{Gal}(L/F) \) is presented as
\[
\text{Gal}(L/F) = \langle \sigma, \tau : \tau^{p^{s+1}} = \sigma^{p^{s+1-k}} = 1, \sigma \tau \sigma^{-1} = \tau^{1+p^k} \rangle.
\]
Let \( B = 1 + X \) and \( A \) in \( \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \) be the matrices as in Lemma 3.4 so that \( ABA^{-1} = B^{1+p^k} \). We define a homomorphism \( \varphi \) from \( \text{Gal}(L/F) \) to \( \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \) by letting \( \varphi(\tau) = B \) and \( \varphi(\sigma) = A \). Then \( \varphi \) is indeed well-defined because (*) holds, \( \text{ord}(B) = p^{s+1} \) and \( \text{ord}(A) = p^{s+1-k} \) by Proposition 3.6. (In fact, \( \varphi \) is an isomorphism to its image.) Let \( \rho : \text{Gal}(F) \rightarrow \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \) be the composite \( \text{Gal}(F) \rightarrow \text{Gal}(L/F) \xrightarrow{\varphi} \mathbb{U}_{p^{i+1}}(\mathbb{F}_p) \). We claim that \( \rho_{i,i+1} = \chi_a \), for each \( i = 1, \ldots, p^s \). Indeed, since both maps \( \rho \) and \( \chi_a \) factor through the quotient \( \text{Gal}(L/F) \), it suffices to check on the generators of \( \text{Gal}(L/F) \). Since \( \tau(a^{1/p}) = (\tau(a^{1/p^{i+1}}))^{p^s} = \zeta_p a^{1/p} \), we get
\[
\varphi_{i,i+1}(\tau) = 1 = \chi_a(\tau).
\]
On the other hand, since \( \sigma(a^{1/p}) = a^{1/p} \) and the nearby diagonal entries of the matrix \( A \) are 0, we get
\[
\varphi_{i,i+1}(\sigma) = 0 = \chi_a(\sigma).
\]
Therefore \( \rho_{i,i+1} = \chi_a \). This implies that \( \langle -\chi_a, \ldots, -\chi_a \rangle \) contains 0 and hence \( \langle \chi_a, \ldots, \chi_a \rangle \) also contains 0.
Case 2: Now assume that $X^{p^i+1} - a$ is reducible over $F(\zeta_{p^i+1})$. Then this implies that $a$ is in $F(\zeta_{p^i+1})^p$ if $p > 2$ and it is in $-4F(\zeta_{2^{i+1}})^4 \subseteq F(\zeta_{2^{i+1}})^2$ if $p = 2$. So in any case, we always have $a$ in $F(\zeta_{p^i+1})^p$. Since $F(\zeta_{p^i+1})/F$ is a cyclic extension (of degree $p^{i+1-k}$), a $p$-th root $a^{1/p}$ of $a$ has to be in $F(\zeta_{p^i+1})$. By Kummer theory, we get $a \in \langle [\zeta_{p^i}] \rangle \subseteq F^e/F^{ep}$. Thus $a = \zeta_{p^i}^m \mod F^{ep}$ for some $0 < m < p$. Hence $\chi_a = m\chi_{\zeta_{p^i}}$. By the linearity of Massey products (see e.g. [Fe, Lemma 6.2.4]), to show that $\langle \chi_{a_1}, \ldots, \chi_{a_k} \rangle$ is defined and contains $0$, it suffices to show that $\langle \chi_{\zeta_{p^i}}, \ldots, \chi_{\zeta_{p^i}} \rangle$ is defined and contains $0$, i.e., we can assume from the beginning that $a = \zeta_{p^i}^m$.

Now let $L = F(\zeta_{p^{i+1+k}})$. Then $L/F$ is a cyclic extension of order $p^{i+1}$ whose Galois group is generated by $\sigma$, where $\sigma(\zeta_{p^{i+1+k}}) = \zeta_{p^{i+1+k}} \in \zeta_{p^{i+1+k}} = \zeta_{p^{i+1+k}}^{1/p^i}$. We define a representation $\varphi$ from $\text{Gal}(L/F) \to \mathbb{U}_{p^{i+1}}(\mathbb{F}_p)$ by $\varphi(\sigma) = B = 1 + X$. Then $\varphi$ is well-defined. Let $\rho: \text{Gal}(F) \to \mathbb{U}_{p^{i+1}}(\mathbb{F}_p)$ be the composite $\text{Gal}(F) \to \text{Gal}(L/F) \to \mathbb{U}_{p^{i+1}}(\mathbb{F}_p)$. We claim that $\rho_{i+1} = \chi_a$ for each $i = 1, \ldots, p^i$. Indeed, since both maps $\rho$ and $\chi_a$ factor through the quotient $\text{Gal}(L/F)$, it suffices to check on the generator $\sigma$ of $\text{Gal}(L/F)$. Since $\sigma(a^{1/p}) = a^{1/p^i}$, one has

$$\varphi_{i+1}(\sigma) = 1 = \chi_a(\sigma).$$

Therefore $\rho_{i+1} = \chi_a$. This implies that $\langle -\chi_a, \ldots, -\chi_a \rangle$ contains $0$ and hence $\langle \chi_a, \ldots, \chi_a \rangle$ also contains $0$.

**Example 8.2.** Let $p$ be a prime number and $G = \mathbb{Z}/p\mathbb{Z}$. Let $\chi \in H^1(G, \mathbb{F}_p)$ be the identity map. In [MT1, Example 4.6], it is shown that if $p > 2$ then the $p$-fold Massey product $\langle \chi, \ldots, \chi \rangle$ is defined but does not contain $0$.

Now assume that $p = 2$, we claim that the 4-fold Massey product $\langle \chi, \chi, \chi, \chi \rangle$ is not defined. For a contradiction, suppose that the 4-fold Massey product $\langle \chi, \ldots, \chi \rangle$ is defined, then there exists a representation $\rho: G \to \mathbb{U}_5(\mathbb{F}_2)$ such that $\rho_{i+1} = \chi_i$ for $i = 1, \ldots, 4$. Let $\bar{B} := \rho(\bar{1}) \in \mathbb{U}_5(\mathbb{F}_2)$. Then all entries of $\bar{B}$ at the positions $(i, i+1)$, $i = 1, \ldots, 4$, are equal to $1$. Hence $B^2 \neq 1$, this contradicts the fact that $\bar{B}$ is the image of an element of order 2.

**Remark 8.3.** Theorem 8.1 is a generalization of [MT1 Proposition 4.5]. In [MT1] we use the latter result to provide an explanation to a part of the well-known Artin-Schreier’s theorem [AS1, AS2] (respectively, Becker’s theorem [Be]) which says that the absolute Galois group $G_F$ (respectively, its maximal pro-$p$-quotient $G_F(p)$) of any field $F$ cannot have an element of odd prime order.

We now use Theorem 8.1 to give an explanation to the full Artin-Schreier theorem (respectively, Becker’s theorem) which further says that $G_F$ (respectively, $G_F(2)$) cannot have elements of finite order greater than 2. In fact, it suffices to show that the absolute Galois group $G_F = \text{Gal}(F_{\text{sep}}/F)$ cannot be of order 4. First note that Theorem 8.1 and Example 8.2 imply that for any field $L$ if $G_L \simeq \mathbb{Z}/2\mathbb{Z}$ then $-1 \not\in L^2$. (Not surprisingly $L^2$ means the set of all squares in $L$.)

Now suppose that $|G_F| = 4$. Let $H$ be a subgroup of $G_F$ of order 2 and let $K$ be the fixed field $(F_{\text{sep}})^H$. Then $G_K \simeq \mathbb{Z}/2\mathbb{Z}$. Thus $-1$ is not in $K^2$, in particular $-1$ is not in $F^2$. Let $L := F(\sqrt{-1})$. Then $G_L$ is of order 2. Therefore $G_L \simeq \mathbb{Z}/2\mathbb{Z}$, hence $-1 \not\in L^2$, which is impossible.
For $1 \leq k \leq s$, we define the group

$$M_{p,k,s} = \langle \sigma, \tau \mid \tau^{p+1} = \sigma^{p+1-k} = 1, \sigma \tau \sigma^{-1} = \tau^{1+p^k} \rangle \simeq \mathbb{Z} / p^{s+1} \mathbb{Z} \rtimes \mathbb{Z} / p^{s+1-k} \mathbb{Z}.$$ 

We have a natural epimorphism $\lambda: M_{p,k,s} \to \mathbb{Z} / p^s \mathbb{Z}$ defined by $\tau \mapsto \bar{1}$, $\sigma \mapsto \bar{0}$. Note that when $k = s = 1$, $M_{p,1,1} \simeq M_{p,3}$. The following result is a corollary of the proof of Theorem 8.1. This result is also a generalization of [EM1, Proposition 10.2].

**Corollary 8.4.** Let $p > 2$ be an odd prime and $s \geq 1$ an integer. Let $G = G_F$ or $G_F(p)$ for a field $F$ containing a primitive $p$-th root of unity. Let $\psi: G \to \mathbb{Z} / p^s \mathbb{Z}$ be an epimorphism. Then

1. If $F$ contains a primitive $p^{s+1}$-th root of unity, then $\psi$ factors through the natural map $\mathbb{Z} / p^{s+1} \to \mathbb{Z} / p^s$.
2. If $F$ does not contain a primitive $p^{s+1}$-th root of unity, then $\psi$ factors through one of the epimorphisms:
   (a) the natural map $\mathbb{Z} / p^{s+1} \to \mathbb{Z} / p^s$;
   (b) the map $\lambda: M_{p,k,s} \to \mathbb{Z} / p^s \mathbb{Z}$ defined by $\tau \mapsto \bar{1}$, $\sigma \mapsto \bar{0}$, where $k$ is the largest integer such that $F$ contains a primitive $p^k$-th root of unity.

**Theorem 8.5.** Let $n \geq 3$ be an integer and let $p$ be an odd prime number. Let $F$ be a $p$-rigid field, which contains a primitive $p$-th root of unity. Then for any $\alpha_1, \ldots, \alpha_n \in H^1(G_F, \mathbb{F}_p)$, the $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ contains 0 whenever it is defined.

**Proof.** For each $i$, $\alpha_i = \chi_{a_i}$ for some element $a_i \in F^\times$. If one of the $a_i$’s is in $(F^\times)^p$ then the corresponding character $\chi_{a_i}$ is trivial and hence the result follows.

We assume now that all $a_i$’s are not in $(F^\times)^p$. Since $\langle \chi_{a_1}, \ldots, \chi_{a_n} \rangle$ is defined, $\chi_{a_1} \cup \chi_{a_2} = 0$. This implies that $a_2 = a_k^p F^p$ for some $1 \leq k < p$ since $F$ is $p$-rigid. Hence $\chi_{a_2} = k \chi_{a_1}$. By the linearity of Massey products (see e.g. [Fr, Lemma 6.2.4]), it is enough to consider the case $k = 1$, i.e., $\chi_{a_2} = \chi_{a_1}$. Similarly, it is enough to consider the case that all $\chi_{a_i}$’s are equal. But in this case Theorem 8.1 applies and hence the result follows. \(\square\)
APPENDIX

by Ido Efrat, Ján Mináč and Nguyễn Duy Tân.

In this appendix we construct, for each $n \geq 3$, examples of pro-$p$ groups which do not satisfy the kernel $n$-unipotent property. These examples are not realizable as maximal pro-$p$ Galois groups of fields containing a root of unity of order $p$. They also show that several other related results on the structure of such maximal pro-$p$ Galois groups are not valid for general torsion-free pro-$p$ groups (see the remarks below).

Let us keep the notation being as in Section 6. We fix an integer $N \geq 1$. Let $S$ be a free pro-$p$ group on generators $x_1, \ldots, x_N$. Let $R_0$ be the closed normal subgroup of $S$ generated by

$$r_{ij} = [x_1, x_2][x_i, x_j]^{-1}, \quad 1 \leq i < j \leq N,$$

and let $R$ be any closed normal subgroup of $S$ such that $RS(3) = R_0S(3)$. Let $G = S/R$ and denote the coset of $x_i$ in $G$ by $\bar{x}_i$. We note that $G(3) = RS(3)/R$. We also fix a set $\mathcal{L}$ of finite groups, and set

$$G_{\mathcal{L}} = \bigcap_{H \in \mathcal{L}} \bigcap_{r \in \mathcal{L}} \text{Ker}(r),$$

where the second intersection is over all group homomorphisms $r : G \to H$.

**Proposition A.1.** Suppose that $N > |H|$ for every $H \in \mathcal{L}$. Then:

- (a) $[\bar{x}_1, \bar{x}_2] \not\in G(3)$;
- (b) $[\bar{x}_1, \bar{x}_2] \in G_{\mathcal{L}}$;
- (c) $G_{\mathcal{L}} \not\subseteq G(n)$ for every $n \geq 3$.

**Proof.** (a) Let $T$ be the subgroup of $S$ generated by the commutators $[x_i, x_j]$, $1 \leq i < j \leq N$. By [Vo, Proposition 1.3.2], the cosets of these commutators in $S(2)/S(3)$ are $\mathbb{Z}/p\mathbb{Z}$-linearly independent. Therefore we can define a homomorphism $\varphi : TS(3)/S(3) \to \mathbb{Z}/p\mathbb{Z}$ by $\varphi([x_i, x_j]S(3)) = 1$ for $1 \leq i < j \leq N$. Then $\varphi(r_{ij} \mod S(3)) = 0$ for each $1 \leq i < j \leq N$, and hence $RS(3)/S(3) \subseteq \ker(\varphi)$. On the other hand $[x_1, x_2] \mod S(3)$ is not in $\ker(\varphi)$. Therefore $[x_1, x_2] \not\in RS(3)$, and hence $[\bar{x}_1, \bar{x}_2] \not\in G(3)$.

(b) Let $H \in \mathcal{L}$ and let $\rho : G \to H$ be a group homomorphism. Since $N > |H|$, there exist $1 \leq i < j \leq N$ such that $\rho(\bar{x}_i) = \rho(\bar{x}_j)$. Since $[\bar{x}_1, \bar{x}_2] = [\bar{x}_i, \bar{x}_j]$ in $G$, we have $\rho([\bar{x}_1, \bar{x}_2]) = \rho([\bar{x}_i, \bar{x}_j]) = [\rho(\bar{x}_i), \rho(\bar{x}_j)] = 1$, as desired.

(c) Use (a), (b), and the inclusion $G(n) \subseteq G(3)$. \qed

**Remarks A.2.**

1. Proposition A.1(c) with $\mathcal{L} = \{\mathbb{U}_n(\mathbb{F}_p)\}$ gives $G(n) \not\subseteq G^{(n)}$ for $n \geq 3$ and $N > |\mathbb{U}_n(\mathbb{F}_p)|$. Thus $G$ does not satisfy the kernel $n$-unipotent condition for any $n$ such that $n \geq 3$ and $N > p^{(n-1)n/2}$.

2. In particular, suppose that $N > p^3$. Then $G$ does not satisfy the kernel 3-unipotent condition. By [EM2, Theorem D] (see also [Ef1, Example 12.2]), this implies that $G$ is not realizable as the maximal pro-$p$ Galois group $G_F(p)$ of any field $F$ containing a root of unity of order $p$.

3. Let $S_i, G_i, i = 1, 2, \ldots$, be again the lower central filtrations of $S, G$, respectively. Take $R = R_0S_3$. Then $RS(3) = R_0S(3)$. As $R \leq S_2$ we have $G/G_2 \cong S/S_2 \cong \mathbb{Z}_p^N$. Also, $G_2 = G_2/G_3 = ([\bar{x}_1, \bar{x}_2]) \cong \mathbb{Z}_p$. Thus in this case $G$ is an extension of a torsion-free group by a torsion-free group, and therefore is also torsion-free.
Let $p > 2$, let $L = \{ \mathbb{Z}/p^2\mathbb{Z}, M_p^3 \}$, and suppose that $N > p^3$. Since $G^{(3)} \subset G^{(3)}$, Proposition A.1(c) implies that $G_L \subsetneq G^{(3)}$. In view of (3), this shows that the property of absolute Galois groups given in [EM1] Main Theorem does not hold in general for torsion-free pro-$p$ groups. Similarly, taking $p = 2$ and $L = \{ \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, D_4 \}$, we obtain that the property of absolute Galois groups given in [MS2] Corollary 2.18 (see also [EM1 Corollary 11.3]) does not hold in general for torsion-free pro-2 groups. Finally, taking $p > 2$ and $L = \{ \mathbb{Z}/p\mathbb{Z}, H_p \}$, we obtain the same for [EM2 Theorem D].

Elements of absolute Galois groups, and the corresponding absolute $p$-adic Galois groups, provide us an example of a pro-$p$ group not satisfying the conclusion in Proposition 6.2, part (1).

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DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O. BOX 653, BE’ER-SHEVA 84105 ISRAEL
E-mail address: efrat@math.bgu.ac.il

DEPARTMENT OF MATHEMATICS, WESTERN UNIVERSITY, LONDON, ONTARIO, CANADA N6A 5B7
E-mail address: minac@uwo.ca

DEPARTMENT OF MATHEMATICS, WESTERN UNIVERSITY, LONDON, ONTARIO, CANADA N6A 5B7 AND INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET, 10307, HANOI - VIETNAM
E-mail address: dnguy25@uwo.ca