Multi-agent Assortment Optimization in Sequential Matching Markets

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Two-sided markets have become increasingly important during the last years, mostly because of their numerous applications in housing, labor and dating. Decentralized customer-supplier matching platforms face several challenges, specially due to the tradeoff between recommending suitable suppliers to customers and avoiding congestion among attractive suppliers.

In this work, we address the challenge of market congestion via the multi-agent assortment optimization problem in the two-sided sequential matching model introduced by Ashlagi et al. (2022). The setting is the following: we (the platform) offer a menu of suppliers to each customer. Then, every customer selects, simultaneously and independently, to match with a supplier or to remain unmatched. Each supplier observes the subset of customers that selected them, and choose either to match a customer or to leave the system. Therefore, a match takes place if both a customer and a supplier sequentially select each other. Each agent’s behavior is probabilistic and determined by a discrete choice model. Our goal is to choose an assortment family that maximizes the expected revenue of the matching. Given the hardness of the problem, we show a $1 - 1/e$-approximation factor for the heterogeneous setting where customers follow general choice models and suppliers follow a general choice model whose demand function is monotone and submodular. Our approach is flexible enough to allow for different assortment constraints and for a revenue objective function. Furthermore, we design an algorithm that beats the $1 - 1/e$ barrier and, in fact, is asymptotically optimal when suppliers follow the classic multinomial-logit choice model and are sufficiently selective. We finally provide other results and further insights. Notably, in the unconstrained setting where customers and suppliers follow multinomial-logit models, we design a simple and efficient approximation algorithm that appropriately randomizes over a family of nested-assortments. Also, we analyze various aspects of the matching market model that lead to several operational insights, such as the fact that matching platforms can benefit from allowing the more selective agents to initiate the matchmaking process.

**Key words**: Matching Markets, Sequential Matching, Discrete Choice Models, Approximation Algorithms

1. **Introduction**

Two-sided marketplaces have increasingly grown in popularity in the recent years. This is the result of the accelerating growth of the sharing economy that has become ubiquitous in our everyday life
and is estimated to reach a total combined revenue of $335 billions by 2025 (Osztovits A. and B. 2014). These platforms can be found in numerous areas such as labor (Upwork, TaskRabbit), accommodation (Airbnb, HomeAway), dating (Match.com, eHarmony) and ride-sharing (Uber, Lift, Netlift). Because of this, two-sided markets have been extensively studied over the last decade, e.g., (Rochet and Tirole 2003, Rysman 2009, Filistrucchi et al. 2014, Duch-Brown 2017), and more specifically for the applications mentioned above, e.g., (Hitsch et al. 2010, Horton 2010, Fradkin 2015, Banerjee et al. 2015, Marx and Schummer 2019).

Broadly speaking, there are two types of agents in these platforms: customers and suppliers. After the platform elicits the preferences of these agents, a transaction or match occurs when a customer and a supplier select each other. Unlike traditional marketplaces, popular platforms such as Airbnb are increasingly considering agents’ preferences when designing their systems (DeAmicis 2015, Tu et al. 2014, Horton 2017, Shi and Zhang 2019). This decision is not arbitrary given that accounting for the preferences of both sides could reduce the frictions or collisions between agents’ choices (Fradkin 2015, Halaburda et al. 2016, Arnosti et al. 2018, Kanoria and Saban 2021). Congestion among agents’ choices appears in two-sided markets that are decentralized, since the platform can only control what the agents observe but not their choices and interactions. In this context, the platform can clearly benefit from knowing the preferences of customers and suppliers, since by limiting the information that the agents observe, the platform can significantly reduce congestion among popular options and, consequently, improve the final outcome; we refer the interested to (Kanoria and Saban 2021, Shi 2022) for a detailed discussion. Let us clarify this challenge. Consider the case when customers are looking for suppliers in a specific platform. Then, the platform must address the following trade-off: on the one side, if customers consider some suppliers very attractive (with respect to the option of looking in a different platform, i.e., the outside option), then the platform must appropriately balance the information that customers observe to avoid that everyone chooses these suppliers, otherwise, the resulting outcome of the market may be suboptimal. On the other hand, if customers value the available suppliers less than the outside option, then the platform must show enough options to “convince” them to stay in the market and simultaneously avoid congestion among their choices.

Assortment optimization (Talluri and Van Ryzin 2004, Kök et al. 2008) appears as a natural tool to control recommendations in the market. The extensive literature on assortment optimization has mostly focused on single-customer settings, in which, the goal is to maximize the revenue by selecting an appropriate subset of options from which the customer observes and probabilistically chooses a single item. In this context, discrete choice models play a key role to define the customer’s preferences. However, as we show in Section 4.5.3, single-customer assortment optimization approaches fail to address market congestion. The adaptation of the classic assortment
optimization problem to a decentralized matching market is not immediate. \cite{Ashlagi2022} propose a stylized sequential matching model in which the platform accounts for the preferences of both sides: customers and suppliers. The objective of the platform is to select an assortment of suppliers for each customer in order to maximize the number of pairs customer-supplier that sequentially select each other. The authors show that even when the problem is composed by agents with simple choice models, the problem is strongly NP-hard. Hence, designing approximation algorithms becomes a natural approach. One of the main technical challenges in this problem is to characterize how attractive the suppliers are. For homogeneous customers, \cite{Ashlagi2022} characterizes two regimes: (i) when all the suppliers are more attractive than the outside option and (ii) when all the suppliers are less attractive than the outside option. However, their homogeneous setting does not account for different valuations that the agents may have for the same option. Also, their analysis is limited to the unconstrained setting where the platform has no limit on how many recommendations can be included in the assortments. In reality, this is not desirable since platforms may want to facilitate the process to customers by providing a limited number of recommendations, which saves time and effort for both sides.

In this work, we study the two-step sequential matching model of \cite{Ashlagi2022} in a heterogeneous setting where customers and suppliers have general choice models. Overall, our main contributions are as follows: (1) we provide strong provable guarantees for the setting with heterogeneous agents by appropriately adapting powerful submodular optimization tools; (2) our approach is sufficiently general to capture a wide class of choice models and constraints, including the logit-based models and random utility-based models; (3) we show improved approximation factors for the case where suppliers have multinomial-logit choice models: A preference structure widely studied in assortment optimization; (4) we discuss further results and operational implications of the model, for example: (a) we provide a simple algorithm that randomizes over a family of nested assortments for the setting in which customers and suppliers’ preferences are governed by multinomial-logit models and there are no constraints on the assortments; (b) we also analyze the natural question of whether the platform should prioritize suppliers or customers.

1.1. Problem Formulation

We study the multi-agent assortment optimization problem in the sequential matching model originally introduced by \cite{Ashlagi2022}. Formally, consider a two-sided market represented by a set $\mathcal{C}$ with $n$ customers and a set $\mathcal{S}$ with $m$ suppliers. The compatibility between customers and suppliers can be captured then by a bipartite graph in which each edge represents the compatibility between two agents. For simplicity, we assume that every supplier is compatible with every customer; however, our results easily extend to the case in which suppliers are compatible to a proper
subset of customers. Our objective is to present an assortment of suppliers \( S_i \subseteq S \) to each customer \( i \in C \) in order to maximize the expected revenue of the matching between customers and suppliers that is obtained as a result of a two-step sequential matching process. Following the terminology in the discrete choice literature, we assume that the choices made by each customer and each supplier are determined by a system of choice probabilities. Formally, each customer \( i \in C \) and each supplier \( j \in S \) have a system of choice probabilities \( \phi_i : S \cup \{0\} \times 2^S \rightarrow [0,1] \) and \( \phi_j : C \cup \{0\} \times 2^C \rightarrow [0,1] \), respectively, where 0 denotes the outside option. This means, for instance, that given a subset of options \( S \subseteq S \), a customer \( i \in C \) chooses \( j \in S \cup \{0\} \) with probability \( \phi_i(j, S) \) and for \( j \notin S \cup \{0\} \) we have \( \phi_i(j, S) = 0 \). In other words, these terms define a probability distribution over \( S \cup \{0\} \), namely, \( f_i(S) + \phi_i(0, S) = 1 \), where \( f_i(S) = \sum_{j \in S} \phi_i(j, S) \) is called the demand function of customer \( i \). Similarly, given a subset of customers \( C \subseteq C \), we denote by \( f_j(C) \) the demand function of supplier \( j \).

Two-step Sequential Matching. First, we select an assortment of suppliers \( S_i \subseteq S \) for each customer \( i \in C \). Then, every customer \( i \in C \) chooses–simultaneously, independently and irrevocably–with probability \( \phi_i(j, S_i) \) a supplier \( j \in S_i \) or leaves the system with probability \( \phi_i(0, S_i) \). For every supplier \( j \in C \), let \( C_j \) be the random subset of customers (possibly empty) that chose \( j \) after the first stage. In the second step of the process, each supplier \( j \in S \) selects–simultaneously, independently and irrevocably–a single customer \( i \) from \( C_j \) with probability \( \phi_j(i, C_j) \) or decides to remain unmatched with probability \( \phi_j(0, C_j) \). Thus, a match occurs when both a customer and a supplier sequentially select each other.

We now formalize the optimization problem. As we mentioned above, we would like to obtain an assortment family, \( S = (S_1, \ldots, S_n) \) that maximizes the expected revenue of the final matching. Let \( r_j \in \mathbb{R}_+ \) be the revenue obtained by the platform when supplier \( j \in C \) is matched. Let us denote by \( \mathcal{I}_i \subseteq 2^S \) the family of feasible assortments for customer \( i \in C \) and \( \mathcal{I} = \{ S \in 2^C \times S : S = (S_1, \ldots, S_n), S_i \in \mathcal{I}_i \text{ for all } i \in C \} \) the collection of all feasible assortment profiles. We denote by \( M_S \) the random variable that indicates the revenue of the final matching and whose distribution depends on the assortment family \( S \in \mathcal{I} \). Therefore, the revenue maximization problem corresponds to

\[
\max \{ \mathbb{E}[M_S] : S \in \mathcal{I} \}. \tag{1}
\]

In the remainder of the manuscript, we denote by \( \text{OPT} \) the optimal value of Problem (1) and \( S^* \) an optimal assortment family.
1.1.1. Challenges, Goals and Assumptions. The model described above generalizes the classic assortment optimization problem in which there is a single customer and the transaction occurs immediately after the customer’s action (Talluri and Van Ryzin 2004). In this classic setting, numerous discrete choice models have been considered to model customers’ behavior (Kök et al. 2008). One of the most widely used and studied is the multinomial logit (MNL) choice model (McFadden 1974, Talluri and Van Ryzin 2004), in which each supplier has an attractive value and the probability that the customer chooses a supplier from the assortment is proportional to the total value of the assortment plus the value of the outside option. Under this model, the classic assortment optimization problem can be solved in polynomial time by checking the revenue-ordered assortments (Talluri and Van Ryzin 2004). However, this is not true in general; for example see (Aouad et al. 2018, Berbeglia and Jorel 2020). We refer the interested reader to Section 1.2 for more details.

Given the tractability of the classic assortment optimization problem under the standard MNL model, one would hope that something similar occurs in the two-sided sequential market described above. Unfortunately, this is not the case as shown in (Ashlagi et al. 2022). First, one has to carefully account for suppliers preferences over customers, which technically means that the structure of the choice probabilities will determine the complexity of the optimization problem. Second, there exists a tradeoff between: (i) recommending potentially good suppliers to customers and (ii) avoiding congestion among customers’ choices. Specifically, Ashlagi et al. (2022) prove that Problem (1) is strongly NP-hard even in the simple case when customers are homogeneous with the same MNL choice model and suppliers choose uniformly over their potential customers. Due to this hardness result, our focus is to design polynomial-time approximation algorithms. We say that a polynomial time algorithm achieves an approximation factor $\alpha \in (0, 1]$ for Problem (1) if it outputs a feasible solution $S \in \mathcal{I}$ such that $\mathbb{E}[M_S] \geq \alpha \cdot \text{OPT}$, where the expectation is also taken over the possible randomization of the algorithm.

The model described above is sufficiently general and, in fact, we do not impose any specific structure on the customers’ choice models. We only make the standard assumption that we have access to an optimization oracle for the single-customer revenue optimization problem.

Assumption 1 (Static Oracle). For every customer $i \in \mathcal{C}$, there exists a polynomial-time algorithm that solves the single-customer assortment optimization problem. Specifically, for given non-negative values $\theta_{i,j}$ for all $j \in \mathcal{S}$, the algorithm returns a set $S_i \in \arg \max \left\{ \sum_{j \in \mathcal{S}} \theta_{i,j} \cdot \phi_i(j, S) : S \in \mathcal{I}_i \right\}$ in polynomial time.

$^1$In other words, the suppliers (or products) on the other side do not have preferences over the single customer.
This assumption is not restrictive since several single-customer assortment optimization problems can be solved efficiently, for example, for logit-based models (Talluri and Van Ryzin 2004, Rusmevichientong et al. 2010, Davis et al. 2013, Gallego and Topaloglu 2014) and the Markov-chain choice model (Blanchet et al. 2016). Moreover, our results easily extend to the case in which we have access to an approximate oracle instead of an exact oracle. Assumption 1 has also been previously considered in the literature of one-sided and two-sided online assortment optimization, see e.g. (Golrezaei et al. 2014, Goyal et al. 2020, Ma and Simchi-Levi 2020, Aouad and Saban 2023).

On the suppliers’ side, we impose the following mild assumption for their choice models.

**Assumption 2 (Suppliers’ Demand Function).** The demand function of any supplier \( j \in S \) is a monotone submodular set function. Formally, for every \( j \in S \), the function \( f_j : 2^C \to [0, 1] \) satisfies the following two properties: (i) for every \( C \subseteq C' \subseteq C \), we have \( f_j(C) \leq f_j(C') \); (ii) for every \( i \in C \) and \( C \subseteq C' \subseteq C \setminus \{i\} \), we have \( f_j(C \cup \{i\}) - f_j(C) \geq f_j(C' \cup \{i\}) - f_j(C') \).

We emphasize that this assumption is satisfied by a large class of choice models. First, any choice model that satisfies the standard substitutability axiom has a monotone demand function. Furthermore, the demand function of the Markov-chain model (Desir et al. 2015) and the wide class of random utility based models (Thurstone 1927, Luce et al. 1965, McFadden 1974, Baltas and Doyle 2001), which includes MNL, have a submodular demand function; see e.g., (Berbeglia and Joret 2020) for a proof. We also remark that, even though we assume that suppliers’ actions are determined by choice models, for our main technical results we can consider a general aggregate matching probability set function \( f_j : 2^C \to [0, 1] \) for each supplier \( j \in S \) that is monotone, submodular and does not necessarily correspond to the demand function of a choice model; we briefly discuss further extensions in Section 4.4.

We now define the Multinomial-Logit choice model.

**Definition 1 (MNL).** Consider a supplier \( j \in S \) with known values \( w_{j,i} \geq 0 \) for every customer \( i \in C \) and \( w_{j,0} = 1 \) their value for the outside option. Then, supplier \( j \) follows the MNL choice model if for any \( C \subseteq 2^C \),

\[
\phi_j(i, C) = \frac{w_{j,i}}{1 + w_j(C)} \quad \text{for } i \in C,
\]

where \( w_j(C) := \sum_{\ell \in C} w_{j,\ell} \). Similarly, we define the MNL choice model for a customer \( i \in C \) with known values \( v_{i,j} \geq 0 \) for every supplier \( j \in S \) and \( v_{i,0} = 1 \) her value for the outside option.

\(^2\) This axiom states that the probability of choosing a specific option does not increase when the assortment is enlarged.

\(^3\) Without loss of generality, we normalized the value of the outside option.

\(^4\) The value \( w_{j,i} \) can be interpreted as how attractive customer \( i \) is to supplier \( j \) and \( w_{j,0} \) is how attractive the outside option is, e.g., a competing platform.
As we mentioned above, Ashlagi et al. (2022) introduced Problem (1). In particular, Ashlagi et al. (2022) focus their analysis on the following case: revenues $r_j = 1$ for all $j \in S$, unconstrained setting with $I_i = 2^S$ for all $i \in C$, customers with the same MNL choice model, i.e., $v_{i,j} = v_j$ for all $i \in C$, $j \in S$ and suppliers with a Uniform model in which $w_{j,i} = 1/q_j$ for all $j \in C$, $i \in C$ where $q_j > 0$ is a given parameter. Note that in this setting, Assumptions [1] and [2] are satisfied, therefore, our results apply to their setting.

Our goal in this work is to provide strong constant approximation guarantees in the general setting under Assumptions [1] and [2]. Moreover, we will show improved guarantees in certain settings, specifically, we will study the small probability regime. To simply put, this regime captures the instances where suppliers have significantly lower demand value when only one customer is presented in the assortment compared to the largest possible demand value. Formally, we borrow the definition given in (Aouad and Saban 2023).

**Definition 2 (ε-Small Probability Regime).** Given $\epsilon \in (0,1)$, we define the $\epsilon$-small probability regime as the family of instances for which the demand functions are such that $f_j(\{i\}) \leq \epsilon \cdot f_j(C)$ for all $i \in C$ and $j \in S$.

Note that this regime captures markets in which customers are not sufficiently attractive for suppliers, or in other words, suppliers are picky. Large markets in which suppliers cannot easily distinguish their options are an example of this regime. The small probability assumption has been studied in the literature of online matching as the small bid regime, see e.g. (Mehta et al. 2007, Mehta and Panigrahi 2012), in one-sided online assortment optimization as the large inventory regime, e.g. (Golrezaei et al. 2014, Goyal et al. 2020) and in two-sided online assortment optimization (Aouad and Saban 2023).

**Notation.** In the remainder, we continue to reserve indices $i$ for customers and $j$ for suppliers. We use bold to denote vector variables and standard italic for unidimensional variables, for example, $S = (S_1, \ldots, S_n) \in \mathcal{I}$. We use $\cdot$ to denote the unidimensional product and $\langle \cdot, \cdot \rangle$ for the dot product. For two vectors $z, z'$ we write $z \vee z'$, for the vector whose $e$-th component is $\max\{z_e, z'_e\}$.

### 1.2. Related Work

Assortment optimization has been extensively studied in the revenue management literature. The classic model focuses on one-sided markets in which there is a single customer who chooses an item from a given menu. The customer’s behavior is determined by an underlying choice model. This problem was introduced by Talluri and Van Ryzin (2004) who consider the MNL framework as a discrete choice model. Talluri and Van Ryzin (2004) show that the problem can be solved in polynomial time by checking the revenue-ordered assortments. However, the problem becomes NP-hard
for other choice models (Bront et al. 2009, Rusmevichientong et al. 2014, Aouad et al. 2018). An important class of choice models are those that satisfy the regularity axiom. Berbeglia and Joret (2020) show that the revenue-ordered assortments strategy provides provable guarantees for a general regular discrete choice model. Other examples of discrete choice models that have been studied in the assortment optimization problem are the Mixed Multinomial Logit (Rusmevichientong et al. 2014), the Nested Logit (Davis et al. 2014), the Rank-based (Aouad et al. 2018) and the Markov chain-based (Blanchet et al. 2016) models. For other choice models that satisfy the regularity axiom, we refer to (Berbeglia and Joret 2020) and, for a survey on assortment optimization, we refer to (Kök et al. 2008). Assortment optimization under capacity constraints has also been studied (Rusmevichientong et al. 2010, Davis et al. 2013, Désir et al. 2015). In particular, under the MNL model the problem with capacity constraints is polynomial-time solvable (Rusmevichientong et al. 2010), whereas under the Markov chain model, the problem becomes APX-hard (Désir et al. 2015).

Reducing market congestion has been widely studied in the recent years. Congestion appears in matching platforms when many attractive agents, partners or competitors are present. If the platform does not carefully design market interventions, then the final outcome may be suboptimal in the sense that the majority of the matches will be concentrated in a smaller set of popular agents. In this context, the literature has focused on the design of different tools to reduce congestion such as matching recommendations, signalling the available capacity of the providers (Fradkin 2015, Horton 2017, Li and Netessine 2020), limiting the visibility (Halaburda et al. 2016, Arnosti et al. 2018) and choice (Immorlica et al. 2021), controlling which agents can initiate contact (Kanoria and Saban 2021, Ashlagi et al. 2020), among others; we refer the interested reader to (Shi 2022b), and the references therein, for a detailed discussion on matchmaking strategies that aim to improve congestion. Shi (2022b) studies the problem of improving market’s efficiency through the lens of communication complexity. Specifically, the author aims to study the minimum amount of interactions needed to arrive to a “good market outcome”. For instance, Shi (2022b) argues that the side whose preferences are harder to describe (the side who is more selective or pickier) should initiate the matchmaking process. Kanoria and Saban (2021) analyze the impact on agent’s welfare and the platform’s outcome when simple interventions are considered, such as limiting which side of the market reaches out first or withholding certain information.

There is a growing literature on assortment planning in two-sided markets. The assortment optimization problem in sequential matchings was introduced in (Ashlagi et al. 2022) who focus on the specific unconstrained setting with homogeneous customers following the MNL choice model and suppliers with uniform choice models. Under these conditions, Ashlagi et al. (2022) show that Problem (1) is strongly NP-hard and provide constant-factor approximation algorithms. The authors separate the analysis in two cases: \( v_j \geq 1 \) and \( v_j \leq 1 \), where \( v_j \) is the value that a customer has over
supplier \( j \). For the former, they provide a \((1 - e^{-1/24})/8\)-approximation factor by appropriately solving a concave relaxation. For the second case, they consider a \textit{bucketing} technique to design an auxiliary linear programming problem whose fractional solution can be rounded to obtain an approximately \((e - 1)/(61,952e)\)-approximation factor. By appropriately combining these analyses, they show a \((e - 1)/(123,904e)\)-approximation factor for the general case with no restriction on the \( v_j \)'s. Assortment optimization in two-sided markets has also been studied in dynamic settings. In the context of dating apps, Rios et al. (2022) focus on the use of behavioral information to improve the platform’s assortment decisions with the goal of optimizing matching rates. Recently, Aouad and Saban (2023) study an online two-sided assortment optimization model. Specifically, the authors design a greedy algorithm that guarantees a \(1/2\)-competitive ratio with respect to the optimal clairvoyant benchmark and show that no online algorithm can improve this factor. Under known i.i.d. arrivals, they show that there exists an online algorithm that guarantees a \(1 - 1/e\) competitive ratio which relies on a linear program relaxation of the optimal value. Furthermore, they show improved competitive ratios when the instances of the problem fall in the \(\epsilon\)-small probability regime and when suppliers follow the multinomial-logit model or the nested-logit model. Finally, in a recent work, Shi (2022a) uses assortment planning to design an optimal match recommendation policy that maximizes the total market surplus.

1.3. Our Contributions and Results

In this work, we present constant factor guarantees for the multi-agent assortment optimization Problem (1), a general heterogenous setting of the model introduced in (Ashlagi et al. 2022).

In Section 2, we focus on the most general setting in which customers can have any choice model that satisfies Assumption 1 and suppliers’ demand functions that satisfy Assumption 2. Our main result, Theorem 1, shows that there exists a polynomial time randomized algorithm that achieves a \(1 - 1/e\)-approximation factor. This algorithm relies on an appropriate continuous submodular relaxation over the space of distribution of assortment families. Given the properties of the suppliers’ demand function, we are able to adapt the well-known continuous greedy for submodular maximization (Vondrák 2008) that is able to approximately solve the continuous relaxation. This algorithm relies on efficiently solving a separable assortment optimization problem, which is possible thanks to Assumption 1. Roughly speaking, in every iteration, our method greedily updates the choice probabilities by adding a fraction of the choice probabilities determined by the current assortment family solution obtained from the subproblem. At the same time, given this family, we iteratively construct a distribution over the assortments for each customer. Since our algorithm iterates a polynomial number of times, then the support of each distribution is of polynomial size. Therefore, we can use the final distribution of each customer to sample an assortment. More
importantly, the expected value of this solution is at least $1 - 1/e$ fraction of the true optimum of Problem $\Pi$. As we show in Proposition 1, this guarantee is the best possible, even when suppliers have rank-based preferences in the $\epsilon$-small probability regime.

Recall that our Assumption 1 is not restrictive since several single-customer assortment optimization problems can be efficiently solved to optimality. For instance, the unconstrained single-customer assortment optimization under the MNL model can be solved in polynomial time by checking a family of nested assortments (Talluri and Van Ryzin 2004). This setting is of particular importance, since the model studied in (Ashlagi et al. 2022) considers customers with the same MNL model, no constraints on the assortments and suppliers with a uniform model whose demand function is monotone and submodular. Therefore, our $1 - 1/e$-approximation guarantee applies to their setting and further improves their $\approx 10^{-5}$ factor. Moreover, our algorithm can be easily adapted to the setting in which we have access to an efficient approximate oracle instead of an exact oracle as we observe in Remark 1.

Broadly speaking, our algorithm can be interpreted as a randomized offline version of the greedy algorithm proposed by Aouad and Saban (2023). In fact, one could be tempted to apply their online algorithm to our offline setting. However, their competitive ratio guarantee is against the clairvoyant optimum that is determined by the observed customer arrival pattern. From this online optimum one can construct a feasible solution to our problem, but it is not necessarily the optimal one. In Section 2.1 we present an adaptation of the greedy algorithm proposed in (Aouad and Saban 2023). Our method generates a random permutation of the set of customers and simulates that order as an arrival sequence for the greedy algorithm. We show that this random-order greedy algorithm achieves a sub-optimal $1/2$-approximation guarantee.

In Section 3, we focus our attention on the setting in which customers still have general preferences that satisfy Assumption 1 but suppliers have heterogeneous MNL choice models. Given the concavity of the suppliers’ demand function, we propose a concave relaxation of Problem $\Pi$ that can be efficiently solved by an adaptation of the classic Frank-Wolfe algorithm. In Theorem 3, we show that in the $\epsilon$-small probability regime, our algorithm achieves a $(1 - \epsilon)$-approximation guarantee. In other words, when suppliers are sufficiently picky (their values for the customers are sufficiently small), our algorithm constructs a solution whose objective value is asymptotically optimal. Finally, we show that in a general regime, our adaptation of the Frank-Wolfe algorithm achieves a $1/4$-approximation factor.

In Section 4 we present other results and our operational insights. First, we show that in the unconstrained setting with customers and suppliers having MNL choice models, a simple algorithm that randomizes over a family of nested assortments guarantees a $1/4$-approximation factor. This result is of particular relevance because: (1) our proposed method is efficient as it solves a
polynomial-sized concave relaxation; (2) randomizing over a nested subset family resembles the classic results in the assortment optimization literature under the MNL choice model. In Section 4.2, we focus on the setting when suppliers have nested-logit choice models for which we are able to obtain improved guarantees in the $\epsilon$-small probability regime. In Section 4.3, we show that when suppliers have independent choice models, then Assumption 1 guarantees that Problem (11) can be solved in polynomial-time. Later, in Section 4.4, we briefly discuss how our algorithms can be adapted to the case in which Assumption 2 is not satisfied. Specifically, we observe that: (1) if the demand functions are non-monotone submodular, then we can adapt the continuous greedy algorithm proposed in (Feldman et al. 2011) to obtain a $1/e$-approximation factor; (2) if the demand functions are monotone but weakly submodular, then the continuous greedy method that we propose still guarantees meaningful approximation guarantees.

Finally, in Section 4.5, we discuss several operational implications of the model. Recall that in the sequential matching model, the customer’s side initiates the matchmaking process, however, certain platforms may prefer that the opposite side starts choosing. Given this, our main goal is to understand the conditions for which is preferable that one side initiates the matchmaking process over the opposite side. For simplicity, we focus on the MNL-MNL model. First, in Proposition 5, we show that if both sides are of equal size, then the ratio between the optimal values of both variants of the problem (customers initiating versus suppliers initiating) can be, in the worst-case, arbitrarily close to zero. In Proposition 6, we show that even in a simple setting, it is better to let the agents that are more selective (whose opposite side is less attractive) to initiate the matchmaking process. When both sides are of different size, we show that it is better to initiate with the larger side, even if they are less selective (Proposition 7). We also analyze a customer-centric algorithm that does not account either for the interaction among customers’ preferences or for suppliers’ preferences. We show in Proposition 8 that the approximation guarantee of this approach can be arbitrarily close to zero. This last result highlights the importance of taking into account the preferences of both sides of the market when designing the assortment profiles.

2. Analysis for the General Setting

In this section, we provide constant approximation factors for the general setting when Assumptions 1 and 2 are satisfied. Specifically, our main result is the following.

**Theorem 1.** Under Assumptions 1 and 2 there exists a randomized polynomial-time algorithm that achieves (in expectation) a $(1 - 1/e)$-approximation factor for Problem (11).

We remark that Theorem 1 applies to the heterogeneous setting in which customers and suppliers could possibly have different choice models. Theorem 1 relies on an appropriate continuous submodular relaxation of Problem (11). Before formalizing this relaxation, let us study the objective
function of our problem; for this, we follow a similar notation to that of (Ashlagi et al. 2022). For any \( j \in S \) and assortment family \( S \in \mathcal{I} \), let \( Y_j^S \) be the indicator random variable that takes value 1 when supplier \( j \) is matched at the end of the process and 0 otherwise. Also, for \( j \in S \) and assortment family \( S \in \mathcal{I} \), let \( C_j^S \) be the random set of customers who selected supplier \( j \) in the first step of the sequential matching process. Observe that, conditioned on \( C_j^S \subseteq C \), the indicator variable \( Y_j^S \) equals 1 with probability \( f_j(C_j^S) \in [0, 1] \) and \( Y_j^S = 0 \) otherwise, where \( f_j(.) \) is the demand function of supplier \( j \) defined as the probability of \( j \) choosing some customer from \( C_j^S \). Therefore, for every \( j \in S \) and \( S \in \mathcal{I} \), we have

\[
E[Y_j^S] = E\left[E\left[Y_j^S|C_j^S\right]\right] = E\left[f_j(C_j^S)\right]
\]

where the expectation is over the probabilistic choices of the customers. Note also that once all customers pick a supplier or the outside option, subsets \( C_1^S, \ldots, C_m^S, C_0^S \), where the last set denotes the customers who select the outside option, form a partition of the set of customers \( C \). Namely, for all \( \bigcup_{j \in S \cup \{0\}} C_j^S = C \) and \( C_j^S \cap C_j'^S = \emptyset \) for all \( j, j' \in C \cup \{0\} \) such that \( j \neq j' \).

Recall that \( r_j \geq 0 \) corresponds to the revenue obtained by the platform when supplier \( j \) is matched. Now, we can express the objective function of our problem as follows

\[
E[M_S] = E\left[\sum_{j \in S} r_j \cdot Y_j^S \right] = E\left[\sum_{j \in S} r_j \cdot f_j(C_j^S)\right]
\]

\[
= \sum_{(C_0, C_1, \ldots, C_m): \forall \ell \neq \ell', C_{\ell} \cap C_{\ell'} = \emptyset} \left( \sum_{j \in S} r_j \cdot f_j(C_j) \right) \cdot \mathbb{P}(C_0^S = C_0, C_1^S = C_1, \ldots, C_m^S = C_m)
\]

\[
= \sum_{(C_0, C_1, \ldots, C_m): \forall \ell \neq \ell', C_{\ell} \cap C_{\ell'} = \emptyset} \left( \sum_{j \in S} r_j \cdot f_j(C_j) \right) \cdot \prod_{i \in C_0} \phi_i(0, S_i) \prod_{i \in C_1} \phi_i(1, S_i) \cdots \prod_{i \in C_m} \phi_i(m, S_i)
\]

\[
= \sum_{j \in S} r_j \cdot \sum_{C_j \subseteq C} f_j(C_j) \prod_{i \in C_j} \phi_i(j, S_i) \sum_{(C_0, \ldots, C_j, \ldots, C_m): \forall \ell \neq \ell'} \prod_{i \in C_\ell} \phi_i(\ell, S_i)
\]

\[
= \sum_{j \in S} r_j \cdot \sum_{C_j \subseteq C} f_j(C_j) \prod_{i \in C_j} \phi_i(j, S_i) \prod_{i \in C \setminus C_j} (1 - \phi_i(j, S_i)),
\]

\[
= \sum_{j \in S} r_j \cdot \sum_{C \subseteq C} f_j(C) \prod_{i \in C} \phi_i(j, S_i) \prod_{i \in C \setminus C} (1 - \phi_i(j, S_i)), \quad (2)
\]

where the third equality is the explicit expectation over all possible partitions, the following equality follows from the independent choices of customers, the fifth equality is obtained by reordering the sum, the sixth expression results from fixing \( C_j \) and summing over all partitions of the remaining customers and finally we note that \( C_j \) does not depend on \( j \). Expression (2) is crucial for the analysis of our algorithm. Indeed, this expression corresponds to the sum of the multilinear extensions of
the demand functions \( f_j : 2^C \to [0, 1] \). Let us briefly recap the definition of the multilinear extension of a set function.

**Multilinear Extension.** Consider a ground set of elements \( \mathcal{U} \) and a generic non-negative set function \( f : 2^\mathcal{U} \to \mathbb{R}_+ \). The multilinear extension \( F : [0,1]^\mathcal{U} \to \mathbb{R}_+ \) of function \( f(\cdot) \) is defined for any fractional vector \( \mathbf{z} \in [0,1]^\mathcal{U} \) as the expected value of \( f(A) \), where \( A \) is the random set generated by drawing independently each element \( e \in \mathcal{U} \) with probability \( z_e \). Formally,

\[
F(\mathbf{z}) = \mathbb{E}_{A \sim \mathbf{z}}[f(A)] = \sum_{A \subseteq \mathcal{U}} f(A) \prod_{e \in A} z_e \prod_{e \in \mathcal{U} \setminus A} (1 - z_e). \tag{3}
\]

Observe that \( F \) is in fact an extension of \( f \), since for any subset \( A \subseteq \mathcal{U} \), we have \( f(A) = F(1_A) \), where \( 1_A \in \{0,1\}^\mathcal{U} \) is the indicator vector of \( A \), i.e., \( 1_A(e) = 1 \) if \( e \in A \) and zero otherwise.

**Fact 1** ([Calinescu et al. 2011]). Let \( f \) be a monotone submodular function and \( F \) its multilinear extension. By monotonicity of \( f \), we have \( \frac{\partial F}{\partial z_e} \geq 0 \) for any \( e \in \mathcal{U} \). This implies that for any \( \mathbf{z} \leq \mathbf{z}' \) coordinate-wise, \( F(\mathbf{z}) \leq F(\mathbf{z}') \). On the other hand, by submodularity of \( f \), then for any pair of vectors \( \mathbf{z}, \mathbf{z}' \in [0,1]^\mathcal{U} \), we have that \( F(\mathbf{z} \vee \mathbf{z}') - F(\mathbf{z}) \leq \sum_{e \in \mathcal{U}} \mathbb{E}_{A \sim \mathbf{z}}[f(A \cup \{e\})] - f(A) \cdot z'_e \leq \langle \nabla F(\mathbf{z}), \mathbf{z}' \rangle \), where the expectation is over the product distribution defined by \( \mathbf{z} \).

Thus, we can write the objective function of Problem (1) as a linear combination of several multilinear extensions. For each \( j \in \mathcal{S} \), denote by \( F_j : [0,1]^C \to [0,1] \) the multilinear extension of the demand function \( f_j : 2^C \to [0,1] \). Expression (2) is equivalent to

\[
\mathbb{E}[\mathbf{M}_S] = \sum_{j \in \mathcal{S}} r_j \cdot F_j(\Phi_j(\mathbf{S})), \tag{4}
\]

where \( \Phi_j(\mathbf{S}) = (\phi_i(j, S_i))_{i \in \mathcal{C}} \) for all \( j \in \mathcal{S} \). Then, our main relaxation is the following:

\[
\max \sum_{j \in \mathcal{C}} r_j \cdot F_j(\mathbf{z}_j) \tag{5}
\]

\[\text{s.t. } (\mathbf{z}, \tau) \in \mathcal{P}.\]

where \( \mathbf{z}_j = (z_{i,j})_{i \in \mathcal{C}} \) for each \( j \in \mathcal{S} \) and

\[
\mathcal{P} = \left\{ (\mathbf{z}, \tau) \in [0,1]^C \times [0,1]^C \times [0,1]^2^C : z_{i,j} = \sum_{S \in \mathcal{I}_i : S \ni j} \tau_{i,S} \cdot \phi_i(j, S) \quad \text{for all } i \in \mathcal{C}, j \in \mathcal{S} \right\}.
\]

We can interpret each variable \( \tau_{i,S} \) as the probability that we show assortment \( S \in \mathcal{I}_i \) to customer \( i \in \mathcal{C} \) and \( z_{i,j} \) as the probability that customer \( i \) sees and chooses \( j \).

\(^5\) Recall that for \( j \in \mathcal{S} \) we have \( \phi_i(j, S_i) = 0 \) whenever \( j \notin S_i \).
Lemma 1. Problem (5) is a relaxation of Problem (1).

Proof. Given an optimal solution $S^*$ of Problem (1), we construct the following feasible solution of Problem (5): For every $i \in C$, consider $\tau_{i,S^*_i} = 1$ and $\tau_{i,S} = 0$ for every $S \neq S^*_i$. For every $i \in C$ and $j \in S^*_i$, consider $z_{i,j} = \phi_i(j, S^*_i)$ and 0 otherwise. Note that $z_{i,j} = \sum_{S \in \mathcal{I}_i : S \ni j} \tau_{i,S} \cdot \phi_i(j, S)$, then clearly $z$ and $\tau$ satisfy the constraints in (6). In fact, the objective value in Problem (5) equals OPT which concludes the proof. □

Note that Problem (5) corresponds to the maximization of a linear combination of multilinear extensions over polytope $P$. Since the demand functions of suppliers are monotone and submodular, we can exploit the properties of the multilinear extension and, consequently, appropriately adapt the well-known continuous greedy algorithm, originally design to approximately solve the problem of maximizing a monotone submodular function over a single matroid constraint (Vondrák 2008, Calinescu et al. 2011). Roughly speaking, the continuous greedy algorithm updates the current solution by summing up a fraction of a feasible solution obtained from a linear maximization subproblem. In other words, this algorithm can be thought as, in every iteration, greedily adding small fractions of several elements at the same time. More importantly, the continuous greedy algorithm returns a fractional feasible solution whose objective value is at least $1 - 1/e$ fraction of the optimal value of the continuous relaxation. We provide more details on the discrete-time version of the original continuous greedy algorithm in Appendix D.

Our goal is to design an adaptation of the continuous greedy algorithm in which the updates correspond to customers’ choice probabilities. In its continuous-time version, we can think of two points $z \in [0,1]^{C \times S}$ and $\tau \in [0,1]^{C \times 2^S}$ making a continuous trajectory (with respect to time $t \in [0,1]$) as follows: Start at $z(0) = 0$ and $\tau(0) = 0$, then the trajectory is determined by the following equations

$$
\frac{dz}{dt} = \Phi(S(t)),
\frac{d\tau}{dt} = \mathbb{1}_{S(t)},
$$

where $\Phi(S(t)) \in [0,1]^{C \times S}$ is the vector whose $(i,j)$-th component is $\phi_i(j, S_i(t))$, $\mathbb{1}_{S(t)} \in \{0,1\}^{C \times 2^S}$ is the indicator vector that is equal 1 in component $(i,S_i(t))$ for $i \in C$ and zero otherwise, and $S(t)$ at time $t$ is defined as an optimal solution of

$$
S(t) \in \arg \max \left\{ \sum_{i \in C} \sum_{j \in S_i} r_{ij} \cdot \mathbb{E}_{C \sim z(t)}[f_j(C_i^t \cup \{i\}) - f_j(C_j^t)] \cdot \phi_i(j, S_i) : S_i \in \mathcal{I}, \text{ for all } i \in C \right\}. \quad (7)
$$

In this problem, $\mathbb{E}_{C \sim z(t)}[f_j(C_i^t \cup \{i\}) - f_j(C_j^t)]$ is the expected marginal contribution of customer $i$ in the demand of supplier $j$, where the expectation is over the independent customer choices...
(product distribution) made according to probabilities \( z(t) \). We can interpret this quantity as the \( i \)-th component of the gradient of the multilinear extension \( F_j \) evaluated in the current point \( z_j(t) = (z_{i,j}(t))_{i \in \mathcal{I}} \) \(^6\) Note that, given Assumption 1 Problem (1) can be solved efficiently as it is separable into \( n \) different problems, one for each customer \( i \in C \) where the weights are \( \theta_{i,j} = r_j \cdot \mathbb{E}_{C \sim z(t)}[f_j(C_j^i \cup \{i\}) - f_j(C_j^i)] \). We formalize the discretized version of our method in Algorithm 1.

**Algorithm 1** Discretized Continuous Greedy with Choice Updates

**Input:** Choice models \( \Phi \) and feasibility assortment family \( \mathcal{I} \), revenues \( r_j \), and \( \delta = 1/n^2 \).

**Output:** Assortment family \( S \).

1: \( z^0 = 0, \tau^0_{i,S} = 0 \) for all \( i \in C, S \in \mathcal{I}_i \).

2: for \( t = 0, \delta, 2\delta, \ldots, 1 - \delta \) do

3: \( S^t \in \arg \max \left\{ \sum_{i \in C} \sum_{j \in S_i} r_j \cdot \mathbb{E}_{C \sim z^t}[f_j(C^i_j \cup \{i\}) - f_j(C^i_j)] \cdot \phi_i(j, S) : S_i \in \mathcal{I}_i, \text{ for all } i \in C \right\} \)

4: \( z^{t+\delta} = z^t + \delta \cdot \Phi(S^t) \)

5: \( \tau^t_{i,S^t_i} \leftarrow \tau^t_{i,S^t_i} + \delta, \text{ for all } i \in C. \)

6: Independently for each \( i \in C \), sample an assortment \( S_i \) from distribution \( \tau^1_i \).

We now provide the proof of the continuous time version of the algorithm and we defer the proof of the discretized version to the Appendix.

**Proof sketch of Theorem 1.** Let us define for any \( z \)

\[
F(z) = \sum_{j \in S} r_j \cdot F_j(z_j).
\]

and let \( S^* \) be an optimal solution of Problem (1). The variation of the objective function with respect to time satisfies the following

\[
\frac{dF}{dt}(z(t)) = \sum_{j \in S} r_j \cdot \frac{dF_j}{dt}(z_j(t)) = \sum_{j \in S} r_j \cdot (\nabla F_j(z_j(t)), \frac{dz_j}{dt})
\]

\[
= \sum_{j \in S} \sum_{i \in C} r_j \cdot \nabla_i F_j(z_j(t)) \cdot \phi_i(j, S_i(t))
\]

\[
\geq \sum_{j \in S} \sum_{i \in C} r_j \cdot \mathbb{E}_{C \sim z^t}[f_j(C^i_j \cup \{i\}) - f_j(C^i_j)] \cdot \phi_i(j, S_i(t))
\]

\[
\geq \sum_{j \in S} \sum_{i \in C} r_j \cdot \mathbb{E}_{C \sim z^t}[f_j(C^i_j \cup \{i\}) - f_j(C^i_j)] \cdot \phi_i(j, S^*_i)
\]

\[
\geq \sum_{j \in S} r_j \cdot F_j(z_j(t) \lor \Phi_j(S^*)) - \sum_{j \in S} r_j \cdot F_j(z_j(t))
\]
\[ \sum_{j \in S} r_j \cdot F_j(\Phi_j(S^*)) - \sum_{j \in V} r_j \cdot F_j(z_j(t)) = \text{OPT} - F(z(t)). \]

where the second equality is due to the chain rule, the next equality is due to the definition of \( \frac{dz_j}{dt} \), the first inequality is due to Fact \( \mathbb{F} \), the next inequality follows from the optimality of \( S(t) \) and the feasibility of \( S^* \) in the subproblem, the third inequality is due to Fact \( \mathbb{F} \), the following inequality is due to monotonicity. The solution of this ordinary differential equation is such that for all \( t \in [0,1] \)

\[ F(z(t)) \geq (1 - e^{-t}) \cdot \text{OPT}. \]

In particular for \( t = 1 \), we obtain that

\[ F(z(1)) \geq (1 - e^{-1}) \cdot \text{OPT}. \]  

(8)

Now we prove that sampling according to \( \tau(1) \) provides a solution to the problem with the same approximation factor in expectation. Recall that

\[ \frac{d\tau}{dt} = \mathbb{I}_{S(t)} \quad \text{and} \quad \frac{dz_j}{dt} = \Phi(S(t)), \]

so if we integrate over \( t \in [0,1] \) we obtain

\[ \tau(1) = \int_0^1 \mathbb{I}_{S(t)} \, dt, \quad \text{and} \quad z(1) = \int_0^1 \Phi(S(t)) \, dt \]

which means that \( \tau(1) \) is a convex combination of assortments \( S(t) \). In fact, for each \( i \in C \) we have

\[ \sum_{S \in \mathbb{I}_i} \tau_{i,S}(1) = \int_0^1 \mathbb{I}_{S(t)} \, dt = \int_0^1 1 \, dt = 1 \]

Therefore, by sampling an assortment \( S_i \) for each \( i \in C \) according to \( \tau_i(1) \), we get for each \( i \in C, j \in S \)

\[ \mathbb{E}_{S_i \sim \tau_i(1)}[\phi_i(j,S)] = \sum_{S \in \mathbb{I}_i \cap S \ni j} \tau_{i,S}(1) \phi_i(j,S) = \sum_{S \in \mathbb{I}_i \cap S \ni j} \int_0^1 \mathbb{I}_{S(t)} \, dt \cdot \phi_i(j,S) = \int_0^1 \phi_i(j,S_i(t)) \, dt = z_{i,j}(1). \]

Finally, since the rounding was done independently for each \( i \in C \), we obtain

\[ \mathbb{E}_{S \sim \tau(1)}[F(\Phi(S))] = \sum_{j \in S} r_j \cdot \mathbb{E}_{S \sim \tau(1)}[F_j(\Phi_j(S))] \]

\[ = \sum_{j \in S} r_j \cdot \sum_{C \subseteq C} f_j(C) \prod_{i \in C} \mathbb{E}_{S_i \sim \tau_i(1)}[\phi_i(j,S)] \prod_{i \in C \setminus i \ni j} (1 - \mathbb{E}_{S_i \sim \tau_i(1)}[\phi_i(j,S)]) \]

\[ = \sum_{j \in S} r_j \cdot \sum_{C \subseteq C} f_j(C) \prod_{i \in C} z_{i,j}(1) \prod_{i \in C \setminus i \ni j} (1 - z_{i,j}(1)) \]

\[ = F(z(1)) \geq (1 - 1/e) \cdot \text{OPT}, \]

which concludes the proof. \( \square \)
Remark 1. Note that the proof above can be easily adapted to the case in which we have an approximate oracle in Assumption 1 instead of an exact oracle. To be more precise, if for every \( i \in C \), the algorithm in Assumption 1 is a polynomial-time \( \beta_i \)-approximation algorithm, then Algorithm 1 guarantees a \( 1 - \exp(-\min_{i \in C} \{ \beta_i \}) \)-factor for Problem (1). For example, this captures the case when \( I_i \) corresponds to a single cardinality constraint and \( \phi_i \) is the Markov Chain choice model, for which there is a \( 1/2 \)-approximation algorithm for the single-customer assortment optimization problem (Désir et al. 2015). Similarly, for the nested logit model there exists a \( 1/2 \)-approximation algorithm for the single-customer problem with cardinality constraints (Gallego and Topaloglu 2014).

Following the ideas in (Aouad and Saban 2023), we can in fact prove that the guarantee in Theorem 1 is the best possible.

**Proposition 1.** Unless \( P = NP \), for any \( \delta, \epsilon > 0 \), there is no \( (1 - 1/e + \delta) \)-approximation algorithm for Problem (1) when suppliers have rank-based preferences in the \( \epsilon \)-small probability regime.

For completeness, we provide a proof of this proposition in the Appendix.

### 2.1. A Simpler Greedy Algorithm

One of the main concerns with Algorithm 1 is its running time, which is polynomial but in every step, the expected marginals of the multilinear extension have to be estimated. Therefore, in this section, our goal is to design an efficient algorithm for the general setting. One could be tempted to use the Greedy algorithm proposed in (Aouad and Saban 2023), however, its \( 1/2 \)-competitive ratio in the online setting does not directly translate into a \( 1/2 \)-approximation factor in our offline setting. This is because the Greedy algorithm achieves a \( 1/2 \) competitive ratio with respect to the best possible objective value in the given arrival order, therefore, the sequence of assortments shown by the clairvoyant optimum is a feasible solution to our problem, but not necessarily the optimal one.

In this section, we study an adaptation of the Greedy algorithm proposed in (Aouad and Saban 2023) which essentially draws a random permutation of the customers and simulates the Greedy algorithm for that online order. We formalize this method in Algorithm 2. Note that given Assumption 1, we can implement Step 4 efficiently.

**Theorem 2.** Algorithm 2 guarantees 1/2-approximation factor for Problem (1).

**Proof of Theorem 2.** Fix a permutation. With a slight abuse of notation, we write \( t \in \{1, \ldots, n\} \) for the \( t \)-th customer in the given permutation. Let \( S_1, \ldots, S_n \) and \( S^*_1, \ldots, S^*_n \) be the assortments
Algorithm 2 Random Order Greedy

Input: Choice models $\phi$ and feasibility assortment family $\mathcal{I}$, revenues $r_j$.

Output: Assortment family $S$.

1: Sample uniformly at random a permutation $i_1, \ldots, i_n$ over the set of customers $\mathcal{C}$.
2: For each $j \in \mathcal{S}$, set $C_j^0 = \emptyset$.
3: for $t = 1, \ldots, n$ do
   4: \quad $S_t \in \arg \max \left\{ \sum_{j \in \mathcal{S}} r_j \cdot [f_j(C_j^{t-1} \cup \{i_t\}) - f_j(C_j^{t-1})] \cdot \phi_{i_t}(j, S) : S \in \mathcal{I}_t \right\}$
   5: \quad Simulate the choice of customer $i_t$ by sampling $\ell \in S_t \cup \{0\}$ w.p. $\phi_{i_t}(\ell, S_t)$.
   6: \quad if $\ell \neq 0$, update $C^{t}_\ell \leftarrow C^{t-1}_\ell \cup \{i_t\}$ and $C^t_j \leftarrow C_j^{t-1}$ for all $j \in \mathcal{S} \setminus \{\ell\}$. Otherwise, $C^t_\ell \leftarrow C_j^{t-1}$ for all $j \in \mathcal{S}$.

output by Algorithm 2 and an optimal solution, respectively. \(^7\) Let $\Phi^{1:t-1} \in [0, 1]^{\mathcal{C} \times \mathcal{S}}$ be the vector of choice probabilities determined by the assortments presented to the first $t-1$ customers, i.e., $\Phi^{1:t-1}_{i,j} = \phi_i(j, S_t)$ for $i = 1, \ldots, t-1$, $j \in S_t$ and zero in the rest of the components. Let us condition on the choices made by the first $t-1$ customers, i.e., we condition on the random set of customers $C_j^{t-1}$ that chose supplier $j \in \mathcal{C}$ up to the end of iteration $t-1$ according to the choice probabilities in $\Phi^{1:t-1}$. Let $\mathbb{1}_{C^{t-1}}$ be the indicator vector of $C^{t-1}$ whose $(i,j)$-th component is 1 if $C_j^{t-1} \ni i$ and zero otherwise. Finally, let $\Phi^*_t \in [0, 1]^{\mathcal{C} \times \mathcal{S}}$ be the vector of the choice probabilities determined only by the optimal assortment of $t$-th customer, i.e., $\Phi^*_t = \phi_i(j, S^*_t)$ for all $j \in S^*_t$ and zero in the rest of the components.

Given the above notation, we have the following:

\[
F(\mathbb{1}_{C^{t-1}} \lor \Phi^*_t) - F(\mathbb{1}_{C^{t-1}}) = \sum_{j \in \mathcal{S}} r_j \cdot \left[ F_j(\mathbb{1}_{C^{t-1}} \lor \Phi^*_t) - F_j(\mathbb{1}_{C^{t-1}}) \right] \\
= \sum_{j \in \mathcal{S}} r_j \cdot \left[ f_j(C_j^{t-1} + t) \cdot \phi_{i_t}(j, S^*_t) + f_j(C_j^{t-1}) \cdot (1 - \phi_{i_t}(j, S^*_t)) - f_j(C_j^{t-1}) \right] \\
= \sum_{j \in \mathcal{S}} r_j \cdot \left[ f_j(C_j^{t-1} + t) - f_j(C_j^{t-1}) \right] \cdot \phi_{i_t}(j, S^*_t) \\
\leq \sum_{j \in \mathcal{C}} r_j \cdot \left[ f_j(C_j^{t-1} + t) - f_j(C_j^{t-1}) \right] \cdot \phi_{i}(j, S_t) \\
= F(\mathbb{1}_{C^{t-1}} \lor \Phi_t) - F(\mathbb{1}_{C^{t-1}})
\]

where the first equality is due to the definition of the function $F(.) = \sum_{j \in \mathcal{S}} r_j \cdot F_j(.)$, the second equality is due to the random choice of customer $t$ given assortment $S^*_t$, the first inequality is due to the choice of greedy, the last equality is due to the reverse computation made in the second equality and $\Phi_t \in [0, 1]^{\mathcal{C} \times \mathcal{S}}$ is the choice probability vector where the $(t,j)$-th component equals $\phi_{i}(j, S)$ for all $j \in \mathcal{S}$.

\(^7\) Recall that the optimal assortments do not depend on the permutation.
\[ \phi_t(j, S_t) \text{ for all } j \in S_t \text{ and zero in the remaining components. Taking expectation over } C_{t-1} \sim \Phi_{1:t-1} \]

we get

\[ F(\Phi_{1:t-1} \cup \Phi_t^*) - F(\Phi_{1:t-1}) \leq F(\Phi_{1:t} \cup \Phi_t^*) - F(\Phi_{1:t-1}) = F(\Phi_{1:t} - F(\Phi_{1:t-1}). \tag{9} \]

Summing over \( t = 1, \ldots, n \) on the left-hand side of (9) results in

\[ \sum_{t=1}^{n} F(\Phi_{1:t-1} \cup \Phi_t^*) - F(\Phi_{1:t-1}) \geq F(\Phi_{1:n} \cup \Phi^*) - F(\Phi_{1:n}) \geq F(\Phi^*) - F(\Phi_{1:n}) \]

where in the first inequality we applied submodularity and the next inequality is due to the monotonicity of the function. Summing over \( t = 1, \ldots, n \) on the right-hand side of (9) we obtain

\[ \sum_{t=1}^{n} F(\Phi_{1:t}) - F(\Phi_{1:t-1}) = F(\Phi_{1:n}) - F(\Phi_{1:0}) = F(\Phi_{1:n}) \]

where we define \( \Phi_{1:0} = 0 \). Therefore, we conclude that

\[ F(\Phi^*) \leq 2 \cdot F(\Phi_{1:n}), \]

and since the optimal solution does not depend on the permutation we can take expectation over the random permutations which finishes the proof. \( \square \)

3. The General-MNL Case

In this section, we study Problem (1) for the specific case when suppliers’ preferences are governed by MNL choice models. Given the structure of their demand function, we are able to design an adaptation of the Frank-Wolfe algorithm which is asymptotically optimal in the \( \varepsilon \)-small probability regime. However, in the general regime, we show that this algorithm achieves a sub-optimal approximation factor. In the remainder of this section, we assume that suppliers have MNL choice models according to Definition 1 which are determined by attractive values \( w_{j,i} \geq 0 \) for all \( j \in C \) and \( i \in C \). For this model, the demand function of supplier \( j \in C \) is

\[ f_j(C) = \sum_{i \in C} \phi_j(i, C) = \frac{w_j(C)}{1 + w_j(C)}. \]

where \( C \subseteq C \), the outside option is normalized \( w_{j,0} = 1 \) and \( w_j(C) = \sum_{i \in C} w_{j,i} \). Given the above, let us consider the following relaxation

\[ \max \sum_{j \in S} r_j \cdot \frac{z_j}{1 + z_j} \]

s.t. \[ \sum_{i \in C} w_{j,i} \cdot \sum_{S \in I_i, S \ni j} \tau_{i,S} \cdot \phi_i(j; S) = z_j, \text{ for all } j \in S \]

\[ \sum_{S \in I_i} \tau_{i,S} = 1, \text{ for all } i \in C, \]

\[ z_j, \tau_{i,S} \geq 0 \text{ for all } i \in C, S \in I_i, j \in S. \]
Lemma 2. Problem (10) is a relaxation of Problem (1) when suppliers have MNL choice models.

Proof. Consider an optimal solution of our problem $S^*$ and let $\Phi^*$ be the corresponding vector of customers’ choice probabilities. Then

$$\text{OPT} = \mathbb{E}[M_{S^*}] = \mathbb{E}_{\Phi^*} \left[ \sum_{j \in S} r_j \cdot f_j(C) \right] = \sum_{j \in S} \mathbb{E}_{\Phi^*} \left[ r_j \cdot \frac{w_j(C)}{1 + w_j(C)} \right] \leq \sum_{j \in S} r_j \cdot \frac{\mathbb{E}_{\Phi^*}[w_j(C)]}{1 + \mathbb{E}_{\Phi^*}[w_j(C)]}$$

where the last inequality follows by using Jensen’s inequality on the concave function $x/(1 + x)$. Note also that $\mathbb{E}_{\Phi^*}[w_j(C)]$ is the multilinear extension of a linear function which is linear. More formally, we have

$$\mathbb{E}_{\Phi^*}[w_j(C)] = \sum_{C \in C} \sum_{i \in C} w_{j,i} \prod_{k \in C} \phi_k(j, S^*_i) \prod_{k \in C \setminus C} (1 - \phi_k(j, S^*_i)) = \sum_{i \in C} w_{j,i} \cdot \phi_i(j, S^*_i).$$

We conclude the proof by constructing the following feasible solution to Problem (10): Consider $z_j = \sum_{i \in C} w_{j,i} \cdot \phi_i(j, S^*_i)$, $\tau_i,S^*_i = 1$ and $\tau_i,S = 0$ for every $i \in C$ and $S \neq S^*_i$. \qed

Since the objective function in (10) is a separable concave function, then Problem (10) can be solved efficiently as long as maximizing a linear objective over the feasible region can be done efficiently. This in fact the case due to Assumption $\square$. Given this, we can adapt the classic Frank-Wolfe algorithm to approximately solve our problem. We formally present our method in Algorithm 3 which in every iteration maintains a feasible solution computed as a convex combination between the incumbent point and the current ascent direction. Note that Assumption $\square$ guarantees that Step 3 can be executed efficiently.$^8$

**Algorithm 3** MNL Frank-Wolfe with Choice Updates

**Input:** Choice models $\phi_i$ and feasibility subset family $I_i$ for every $i \in C$, and $T \gg \sqrt{mRW}$ where

$$R = \max_j \{r_j\} \text{ and } W = \sum_j (\sum_i w_{j,i})^2.$$ 

**Output:** Assortment family $S$ and $z^T$.

1. $z^0 = 0, \tau_i,S = 0$ for all $i \in C$, $S \in I_i$ and $\tau_i,0 = 1$ for all $i \in C$.
2. for $t = 1, 2, \ldots, T$ do
3. $S^{t-1} = \arg \max \left\{ \sum_{i \in C} \sum_{j \in S_i} r_j \cdot \frac{1}{(1 + z^t_{j,i})^2} \cdot \phi_i(j, S) : S_i \in I_i, \text{ for all } i \in C \right\}$
4. $z^t_j = (1 - \gamma_t) \cdot z^{t-1}_j + \gamma_t \cdot \sum_{i \in C} w_{j,i} \cdot \phi_i(j, S^{t-1}_i)$ for all $j \in S$.
5. $\tau^t_{i,S^{t-1}_i} = (1 - \gamma_t) \cdot \tau^{t-1}_{i,S^{t-1}_i} + \gamma_t$ for all $i \in C$ and $\tau^t_{i,S} = (1 - \gamma_t) \cdot \tau^{t-1}_{i,S}$ for all $i \in C, S \neq S^*_i$.
6. For each $i \in C$, sample independently an assortment $S_i$ from distribution $\tau^T_{i,S}$.

In the next lemma, we show that the points are feasible at any step of Algorithm 3.$^9$

---

$^8$ This convex combination depends on the step size $\gamma_t$ which we specify in the proof.

$^9$ Here $1/(1 + z_j)$ corresponds to the $j$-th component of the gradient of the objective function.
Lemma 3. In any given step $t = 0, \ldots, T$, the points $z^t$ and $\tau^t$ are feasible in Problem (10).

More importantly, as we show in the following proposition, the objective value of the fractional solution output by Algorithm 3 asymptotically converges to the optimal value of Problem (10), up to a vanishing additive error.

Proposition 2. Algorithm 3 outputs a feasible solution $(z^T, \tau^T)$ such that

$$\sum_{j \in S} r_j \cdot \frac{z_j^T}{1 + z_j^T} \geq \text{OPT}_{(10)} - o(1)$$

where $\text{OPT}_{(10)}$ is the optimal value of Problem (10).

The proof follows the outline of the convergence proof of the classic Frank-Wolfe algorithm and we defer it to the Appendix.

3.1. Asymptotic Optimality in the $\epsilon$-Small Probability Regime

In this section, we assume that the instance of the problem is in the $\epsilon$-small probability regime with $\epsilon \in (0, 1)$. Specifically, given Definition 2, we have that the demand function of each supplier $j \in S$ is such that for every customer $i \in C$

$$f_j(i) \leq \epsilon \cdot \max_{C \subseteq C} f_j(C) \leq \epsilon$$

where the second inequality is because $f_j(C) \leq 1$ for any $C \subseteq C$. Moreover, $f_j(i) = w_{j,i}/(1 + w_{j,i})$ implies that $w_{j,i} \leq \epsilon/(1 - \epsilon)$. For this class of instances, our main result is the following:

Theorem 3. Algorithm 3 guarantees a $(1 - \epsilon)$-approximation factor in the $\epsilon$-small probability regime with $\epsilon \in (0, 1)$.

The key property to prove Theorem 3 is shown in the next lemma. Roughly speaking, the lemma shows that the concave objective function studied in Problem (10) is $(1 - \epsilon)$ close to the original objective function.

Lemma 4. Assume suppliers have MNL choice models with values $w_{ji} \leq \epsilon/(1 - \epsilon)$ for all $j \in S$, $i \in C$ and $\epsilon \in (0, 1)$. Then, for any vector $d \in [0, 1]^{C \times S}$, we have

$$\sum_{j \in S} r_j \cdot F_j(d_j) \geq (1 - \epsilon) \cdot \sum_{j \in S} r_j \cdot \frac{\sum_{i \in C} w_{j,i} d_{i,j}}{1 + \sum_{i \in C} w_{j,i} d_{i,j}}.$$  \hspace{1cm} (11)

where $d_j = (d_{i,j})_{i \in C}$ for all $j \in S$.

We defer the proof of the lemma above to Appendix B. Note that for a given assortment family $S$, we can consider $d = \Phi(S)$ whose $(i, j)$-th component is $\phi_i(j, S_i)$. For this vector, we have $F(d) = \sum_{j \in S} r_j \cdot F_j(d_j) = \mathbb{E}[M_S]$. 

Proof of Theorem 3. Consider the vector $\tau^T$ output by Algorithm 3 and let $S$ be the random assortment family sample from it. Then, we obtain

$$E_{S \sim \tau}[M_S] = \sum_{j \in S} \sum_{C \subseteq C} f_j(C) \prod_{i \in C} \left( \sum_{S \in I_i : S \ni j} \tau_{i,S}^T \cdot \phi_i(j,S) \right) \prod_{i \notin C} \left( 1 - \sum_{S \in I_i : S \ni j} \tau_{i,S}^T \cdot \phi_i(j,S) \right)$$

$$\geq (1 - \epsilon) \cdot \sum_{j \in S} r_j \cdot \frac{\sum_{i \in C} w_{j,i} \cdot \sum_{S \in I_i : S \ni j} \tau_{i,S}^T \cdot \phi_i(j,S)}{1 + \sum_{i \in C} w_{j,i} \cdot \sum_{S \in I_i : S \ni j} \tau_{i,S}^T \cdot \phi_i(j,S)}$$

$$= (1 - \epsilon) \cdot \sum_{j \in S} r_j \cdot \frac{z_j^T}{1 + z_j^T} \geq (1 - \epsilon) \cdot \text{OPT} - o(1) \geq (1 - \epsilon) \cdot \text{OPT} - o(1)$$

where the first equality is due the independent choices made by customers (same as the last part of the proof of Theorem 1), the next inequality follows from Lemma 4, the next equality from Lemma 3, the second inequality follows from Proposition 2 and the last one from Lemma 2. □

In large markets, we can think of $\epsilon$ being sufficiently small as suppliers are not able to distinguish between different customers. For this case, the result in Theorem 3 states that Algorithm 3 is asymptotically optimal.

3.2. Approximation Guarantee of Algorithm 3 in any Instance Regime

In this section, we focus on the approximation guarantee of Algorithm 3 when there is no restriction on the type of instances. Formally, we show the following result:

THEOREM 4. Algorithm 3 achieves a $1/4$-approximation factor.

To prove this theorem, we first show the following lemma.

LEMMA 5. Assume suppliers follow MNL choice models with values $w_{j,i} \geq 0$. Consider a new instance in which suppliers’ values over customers are capped as follows $\tilde{w}_{j,i} = \min \left\{ w_{j,i}, \frac{\epsilon}{1-\epsilon} \right\}$ for all $j \in S$, $i \in C$ where $\epsilon \in (0,1)$. Then, for any assortment family $S$, the following inequalities hold

$$E[M_S] \leq E[\tilde{M}_S] \leq \frac{1}{\epsilon} \cdot E[M_S],$$

where $\tilde{M}_S$ and $M_S$ are the random variables that indicate the total revenue of the final matching in the new instance and old instance, respectively, under the given assortment family $S$.

This lemma is analogous to Lemma 4.1 in (Ashlagi et al. 2022). Since Ashlagi et al. (2022) prove it only for the Uniform model (a special case of the MNL model) and for $\epsilon = 1/2$, we provide a proof of Lemma 5 in Appendix B for completeness.

Note that $\tau^T$ has a polynomial number of non-zero entries as long as the parameters $W$ and $R$ are polynomially (in the size of the input) bounded.
Proof of Theorem 4. Let \( \epsilon \in (0,1) \). First, we construct a new instance of the problem where suppliers’ values over customers are capped \( \tilde{w}_{j,i} = \min\{w_{j,i}, \epsilon/(1-\epsilon)\} \). Let OPT and \( \hat{\text{OPT}} \) be the optimal values of the original and the new instance, respectively. From Lemma 5, we know that
\[
\hat{\text{OPT}} \geq \epsilon \cdot \text{OPT}.
\]

Now, we can use the result in Theorem 3 to show that, in the new instance, Algorithm 3 outputs a random assortment family \( S \) sampled from a distribution \( \tau^T \) such that
\[
E_{S \sim \tau^T}[M_S] \geq (1-\epsilon) \cdot \hat{\text{OPT}}.
\]

By putting the previous two inequalities together, and Lemma 5, we obtain
\[
E_{S \sim \tau}[M_S] \geq \epsilon \cdot (1-\epsilon) \cdot \text{OPT}.
\]

We conclude the proof by noting that the factor \( \epsilon \cdot (1-\epsilon) \) is maximized in \( \epsilon = 1/2 \). For \( \epsilon = 1/2 \), we need to cap the values at 1, i.e., \( \tilde{w}_{j,i} = \min\{w_{j,i}, 1\} \) which essentially tells us that we need to distinguish between those suppliers that value more the customers than the outside option (recall that \( w_{j,0} = 1 \)) and those who value the outside option more. \( \square \)

4. Further Insights and Other Results

4.1. Randomization over Nested Assortments for the Unconstrained MNL-MNL Model

In this section, we study Problem (1) in the unconstrained setting when both suppliers and customers follow MNL choice models. This is a direct generalization of the original problem studied by Ashlagi et al. (2022). Specifically, we show the following:

Theorem 5. Assume that both suppliers and customers have MNL choice models and that there are no assortment constraints, i.e., \( \mathcal{I}_i = 2^S \) for every customer \( i \in \mathcal{C} \). Then, there exists a randomized polynomial-time algorithm that achieves 1/4-approximation factor for Problem (1).

The main advantage of the algorithm in Theorem 5 is that results from: first, solving a polynomial-sized concave relaxation and, second, from sampling an assortment for each customer from a family of at most \( m + 1 \) nested assortments, where \( m \) is the number of suppliers.

As in Definition 1, recall that the value of customer \( i \) for each supplier \( j \) is given by \( v_{i,j} \geq 0 \) and the value for the outside option is normalized \( v_{i,0} = 1 \). Given this, for our relaxation, we consider the following feasible region
\[
\mathcal{R} = \left\{ y \in \mathbb{R}^{\mathcal{C} \times (S \cup \{0\})} : \begin{array}{l}
  y_{i,0} + \sum_{j \in S} v_{i,j} y_{i,j} = 1 \\
  \text{for all } i \in \mathcal{C}, \\
  y_{i,j} \leq y_{i,0} \\
  \text{for all } i \in \mathcal{C}, j \in S
\end{array} \right\}
\]
\[ y_{i,j}, y_{i,0} \geq 0 \quad \text{for all } i \in \mathcal{C}, j \in \mathcal{S} \] \quad \text{(12)}

This feasible region has been previously studied for the single-customer assortment optimization problem, see e.g. [Davis et al. 2013, Sumida et al. 2021]. We now show that given a point \( y \in \mathcal{R} \), we can construct, for each customer, a lossless distribution over at most \( m + 1 \) nested assortments.

**Lemma 6.** Let \( y \in \mathcal{R} \) be any point. For each customer \( i \in \mathcal{C} \), there exists a distribution \( \tau_i \) over at most \( m + 1 \) different nested assortments such that \( \mathbb{P}_{\tau_i}(i \text{ chooses supplier } j) = v_{i,j}y_{i,j} \) for every \( i \in \mathcal{C}, j \in \mathcal{S} \).

**Proof.** Fix customer \( i \in \mathcal{C} \). For the given point \( y \), order the components in the \( i \)-th row in decreasing order: \( y_{i,0} \geq y_{i,j_1} \geq \cdots \geq y_{i,j_m} \). Then, construct the following sets:

\[
S_{i,0} = \emptyset, \quad S_{i,1} = \{j_1\}, \quad S_{i,2} = \{j_1, j_2\}, \ldots, \quad S_{i,m} = \mathcal{S}
\]

Note that this construction is possible, since there are no constraints on the assortments. Moreover, these assortments are nested which is a common feature in the literature of assortment optimization.

We now construct a distribution \( \tau_i \). For each \( \ell \in \{0, \ldots, m-1\} \) define

\[
\tau_{i,S_{i,\ell}} = (y_{i,\ell} - y_{i,\ell+1}) \cdot (1 + v_i(S_{i,\ell}))
\]

where \( v_i(S_{i,\ell}) = \sum_{j \in S_{i,\ell}} v_{i,j} \) and also consider \( \tau_{i,S_{i,m}} = y_{i,m} \cdot (1 + v_i(S_{i,m})) \). For any other subset \( S \neq S_{i,\ell} \) define \( \tau_{i,S} = 0 \). Given the ordering on the values, we have that \( \tau_{i,S} \geq 0 \) for every subset \( S \subseteq \mathcal{S} \). More importantly, we have

\[
\sum_{S \subseteq \mathcal{S}} \tau_{i,S} = \sum_{\ell=0}^{m-1} (y_{i,j_\ell} - y_{i,j_{\ell+1}}) \cdot (1 + v_i(S_{i,\ell})) + y_{i,j_m} \cdot (1 + v_i(S_{i,m})) = y_{i,0} + \sum_{\ell=1}^{m} v_{i,j_\ell}y_{i,j_\ell} = 1.
\]

where in the last equality we used that \( y \in \mathcal{R} \). Finally, let us compute the probability that a customer \( i \) chooses supplier \( j_\ell \) with \( \ell \in \{1, \ldots, m\} \)

\[
\mathbb{P}(i \text{ chooses } j_\ell) = \sum_{S \ni S_{i,\ell}} \mathbb{P}(i \text{ chooses } j_\ell \text{ in } S) \cdot \mathbb{P}(i \text{ sees assortment } S)
\]

\[
= \sum_{S \ni S_{i,\ell}} \tau_{i,S} \cdot \frac{v_{i,j_\ell}}{1 + v_i(S)} = \sum_{\sigma = \ell}^{m-1} (y_{i,j_\sigma} - y_{i,j_{\sigma+1}}) v_{i,j_\sigma} + y_{i,j_m} v_{i,j_m} = v_{i,j_\ell}y_{i,j_\ell}
\]

where the first equality is due to, once we conditioned on the assortment \( S \), the choice of \( i \) is independent from everything else. The third equality is because \( j_\ell \) appears in the \( \ell \)-th assortment and onward (recall they are nested). \( \Box \)
Note that the construction given in Lemma \[6\] has been previously considered in the literature of single-customer assortment optimization, see e.g. \cite{Topaloglu2013, Gallego2019}. However, to the best of our knowledge, it has not been used before as a randomized rounding tool.

For Theorem 5, we consider the following upper bound.

\[
\begin{align*}
\max & \quad \sum_{j \in S} r_j \cdot \frac{z_j}{1 + z_j} \\
\text{s.t.} & \quad z_j = \sum_{i \in C} w_{j,i} v_{i,j} y_{i,j} \\
& \quad y \in \mathcal{R}.
\end{align*}
\]  

**Lemma 7.** Assume customers follow MNL choice models and no constraints on the assortments exist, then Problem (13) is equivalent to Problem (10).

**Proof.** Consider an optimal solution \( z^* \) and \( \tau^* \) of Problem (10). Define for each \( i \in C \)

\[
y_{i,j} = \sum_{S \ni j} \tau^*_{i,S} \cdot \frac{1}{1 + v_i(S)} \quad \text{for } j \in S,
\]

and

\[
y_{i,0} = \sum_{S \subseteq S} \tau^*_{i,S} \cdot \frac{1}{1 + v_i(S)}.
\]

Clearly, \( y_{i,0} \geq y_{i,j} \) for every \( i \in C, j \in S \). Also,

\[
y_{i,0} + \sum_{j \in S} v_{i,j} y_{i,j} = \sum_{S \subseteq S} \tau^*_{i,S} \cdot \frac{1}{1 + v_i(S)} + \sum_{j \in S} \sum_{S \ni j} \tau^*_{i,S} \cdot \frac{v_{i,j}}{1 + v_i(S)}
\]

\[
= \sum_{S \subseteq S} \tau^*_{i,S} \cdot \frac{1}{1 + v_i(S)} + \sum_{S \subseteq S} \tau^*_{i,S} \cdot \sum_{j \in S} \frac{v_{i,j}}{1 + v_i(S)}
\]

\[
= \sum_{S \subseteq S} \tau^*_{i,S} = 1
\]

where in the last equality we used the feasibility of \( \tau^* \). Note as well that

\[
\sum_{i \in C} w_{j,i} v_{i,j} y_{i,j} = \sum_{i \in C} w_{j,i} \sum_{S \ni j} \tau^*_{i,S} \cdot \frac{v_{i,j}}{1 + v_i(S)} = z_j
\]

due to the feasibility of \( z^* \). This concludes the first direction. For the opposite direction, consider an optimal solution \( z^* \) and \( y^* \) of Problem (13). We construct \( \tau \) as in Lemma 6. Given this, we already know that \( \sum_{S \subseteq S} \tau^*_{i,S} = 1 \) for every \( i \in C \). Moreover, as shown in the proof of Lemma 6

\[
v_{i,j} y^*_{i,j} = \sum_{S \ni j} \tau^*_{i,S} \cdot \frac{v_{i,j}}{1 + v_i(S)}.
\]

Therefore,

\[
z_j^* = \sum_{i \in C} w_{j,i} v_{i,j} y^*_{i,j} = \sum_{i \in C} w_{j,i} \sum_{S \ni j} \tau^*_{i,S} \cdot \frac{v_{i,j}}{1 + v_i(S)},
\]

which implies that \( z^* \) and \( \tau \) are feasible in Problem (10). We conclude the proof by noting that both optimal objective values are equal.  \( \Box \)
Algorithm 4 Unconstrained MNL-MNL

Input: Non-negative values $v_{i,j}$ and $w_{j,i}$ for every $i \in \mathcal{C}$, $j \in \mathcal{S}$.

Output: Assortment family $\mathcal{S}$.

1: Obtain an optimal solution $y^*$ by solving Problem (13).
2: Obtain distribution $\tau$ as constructed in Lemma 6 using point $y^*$.
3: For each $i \in \mathcal{C}$, independently sample $S_i$ according to distribution $\tau_i$.

Note that we can efficiently solve Problem (13) using standard ascent algorithms from concave optimization. Our main method for this section is formalized in Algorithm 4.

Proof of Theorem 5. The expected objective value of the random assortment family output by Algorithm 4 is such that

$$\mathbb{E}_{\mathcal{S} \sim \tau}[\mathbf{M}_\mathcal{S}] \geq \frac{1}{4} \cdot \text{OPT}(13) = \frac{1}{4} \cdot \text{OPT}(10) \geq \frac{1}{4} \cdot \text{OPT},$$

where the first inequality can be shown in the same way as shown in the proof of Theorem 4, the next equality is due to Lemma 7 and the last inequality is due to Lemma 2. □

Remark 2. Note that we could have constructed a continuous submodular relaxation instead of the concave relaxation in (13), which would result in a $(1 - 1/e)$-approximation factor. However, solving the continuous submodular relaxation is computationally more challenging than solving (13).

4.2. The Nested-Logit Choice Model: Near-optimality in the $\epsilon$-Small Probability Regime

One natural question is whether we can obtain improved guarantees under the $\epsilon$-small probability regime for suppliers’ preferences that are beyond MNL choice models. For example, in the online setting, Aouad and Saban (2023) showed that an approximation factor better than $1 - 1/e$ can be obtained for the nested-logit model in the i.i.d. online arrival model.

The Nested-Logit Model. This model was introduced in (Williams 1977) which considers correlation between customers that belong to the same nest and, subsequently, alleviates the IIA property of the MNL model. Formally, a supplier $j \in \mathcal{S}$ follows the $\nu$-Nested Logit model ($\nu$-NL) if there exists a partition $\mathcal{C}_1, \ldots, \mathcal{C}_K$ of the set of customers $\mathcal{C}$, where each $\mathcal{C}_k$ is called a nest and $\nu = (\nu_1, \ldots, \nu_K)$ are such that $\nu_k \in (0,1]$ for all $k \in [K]$. Supplier $j$ has a value $w_{j,i}$ for each customer $i \in \mathcal{C}$ and her choice behavior is as follows: For a given subset of customers $\mathcal{C} \subseteq \mathcal{C}$, first, it chooses a nest $k \in [K]$ with probability

$$\frac{w_j(\mathcal{C}_k)^{\nu_k}}{1 + \sum_{q \in [K]} w_j(\mathcal{C}_q)^{\nu_q}}$$

where the outside option is normalized to 1, $\mathcal{C}_k = \mathcal{C} \cap \mathcal{C}_k$ and $w_j(\mathcal{C}_k) = \sum_{i \in \mathcal{C}_k} w_{j,i}$. After choosing a nest, then supplier $j$ chooses a customer $i$ from that nest with probability proportional to the
weight of the nest, i.e., \(w_{j,i}/w_j(C_k)\). Therefore, for any \(i \notin C\), the probability of \(j\) choosing \(i\) is zero, as expected. Given this, the demand function of supplier \(j\) is given by

\[
f_j(C) = \sum_{k \in [K]} \sum_{i \in C_k} \frac{w_{j,i}}{w_j(C_k)} \cdot \frac{(w_j(C_k))^\nu_k}{1 + \sum_{q \in [K]} (w_j(C_q))^\nu_q} = \frac{\sum_{k \in [K]} (w_j(C_k))^\nu_k}{1 + \sum_{k \in [K]} (w_j(C_k))^\nu_k}. \tag{14}
\]

Parameter \(\nu_k \in (0,1]\) is called the dissimilarity coefficient of the nest \(k\). In particular when \(\nu_k = 1\) for all \(k \in [K]\), we recover the standard MNL model. More generally, when \(\nu_k \in (0,1]\) for all \(k \in [K]\), the demand function is monotone and submodular as it is the composition of a concave function with a linear function. The value of the parameter \(\nu_k\) models the competition among customers in nest \(k\). In particular, when \(\nu_k\) is small, the contribution of an extra customer in the nest is marginal as individuals of that nest are more similar. Under the \(\epsilon\)-small probability regime with \(\epsilon \in (0,1)\), we have that for any \(i \in C\) and \(j \in S\)

\[
f_j(\{i\}) = \frac{w_{j,i}^{\nu_k}}{1 + w_{j,i}^{\nu_k}} \leq \epsilon \cdot f_j(C) \leq \epsilon
\]

which implies that \(w_{j,i} \leq (\epsilon/(1 - \epsilon))^{1/\nu_k} \) where \(k\) is the nest that \(i\) belongs to. With this in mind, our result is the following:

**Proposition 3.** Assume every suppliers has a \(\nu\)-NL model with \(\nu_k \in (0,1]\) for all \(k \in [K]\). Denote by \(\overline{\nu} = \max_k \nu_k\). Then, under the \(\epsilon\)-small probability regime with \(\epsilon \in (0, \frac{1}{n^{\overline{\nu}+1}}]\), there exists a \((1 - o(1))\)-approximation polynomial-time algorithm.

We defer the proof of Proposition 3 to Appendix C.1. As in Theorem 3, Proposition 3 shows that, in a large market with sufficiently picky suppliers, an appropriate concave relaxation leads to a solution whose objective value is asymptotically optimal.

### 4.3. The General-Independent Model

The simplest suppliers’ choice model that we can analyze in Problem (1) is the independent model. As stated next in Proposition 4, Problem (1) in this setting can be efficiently solved as long as Assumption 1 is satisfied. Formally, if supplier \(j \in S\) follows the independent model, then for any subset of customers \(C \subseteq \mathcal{C}\), the probability that \(j\) chooses \(i\) in \(C\) is given by \(\phi_j(i,C) = \phi_{j,i}\), where \(\phi_{j,i} \in [0,1]\) are parameters such that \(\sum_{i \in \mathcal{C}} \phi_{j,i} \leq 1\) for all \(j \in S\). Note that, for this model, the demand function of supplier \(j\) corresponds to

\[
f_j(C) = \sum_{i \in \mathcal{C}} \phi_{j,i}, \quad C \subseteq \mathcal{C}.
\]

**Proposition 4.** Assume suppliers follow the independent choice model. Then, given Assumption 1, Problem (1) can be solved in polynomial time.
Proof. The proof follows by noting that the multilinear extension of a linear function is linear. More precisely, consider Expression 2 with the demand function of the independent model, then we have for any assortment family $S$

$$
E[M_S] = \sum_{j \in S} \sum_{C \subseteq C} r_j \cdot \left( \sum_{i \in C} \phi_{j,i} \right) \prod_{i \in C} \phi_i(j, S_i) \prod_{i \in C \setminus C} (1 - \phi_i(j, S_i))
$$

$$
= \sum_{j \in S} r_j \cdot \sum_{i \in C} \phi_{j,i} \cdot \phi_i(j, S_i) \cdot \mathbb{1}_{j \in S_i}
$$

$$
= \sum_{i \in C} \sum_{j \in S_i} r_j \cdot \phi_{j,i} \cdot \phi_i(j, S_i).
$$

Therefore, when suppliers have independent choice models, Problem 1 is equivalent to

$$
\max \left\{ \sum_{i \in C} \sum_{j \in S_i} r_j \cdot \phi_{j,i} \cdot \phi_i(j, S_i) : S_i \in \mathcal{I}_i \text{ for all } i \in \mathcal{C} \right\}
$$

which can be efficiently solved due to Assumption 1.

4.4. Beyond Submodularity and Monotonicity

So far we have assumed that the demand function of each supplier is a monotone submodular function. As we mentioned after Assumption 1, we can think of $f_j$ as an aggregate hitting probability function which is not necessarily derived from a choice model. We now briefly discuss the case in which the functions $f_j$’s are either non-monotone or non-submodular.

First, for the case in which every $f_j$ is non-monotone but still submodular, then a simple variation of the continuous greedy algorithm is proposed in (Feldman et al. 2011) called measured continuous greedy. More importantly, the authors show that this algorithm achieves a $1/e$-approximation factor with respect to the optimal value of the relaxation defined by the multilinear extension. As we did in Section 2, we can adapt the measured continuous greedy to include choice updates, which would lead to $1/e$-approximation for Problem 1 under Assumption 1.

Second, if every $f_j$ is monotone but non-submodular, we can still study the case in which the functions are $\eta$-weakly submodular, see e.g. (Das and Kempe 2011, Bian et al. 2017, Chen et al. 2018). Specifically, a set function $f : 2^\mathcal{U} \to \mathbb{R}_+$ is $\eta$-weakly submodular for $\eta \in (0, 1]$ if for any sets $A, B \subseteq \mathcal{U}$ we have

$$
\sum_{e \in B} f(A \cup \{e\}) - f(A) \geq \eta \cdot [f(A \cup B) - f(A)].
$$

Note that for $\eta = 1$ the function is submodular. It is known that the continuous greedy algorithm guarantees a $(1 - e^{-\eta})$-approximation factor with respect to the optimal value of the continuous relaxation defined by the multilinear extension. Therefore, we can use Algorithm 1 for monotone $\eta$-weakly submodular demand functions to obtain a $(1 - e^{-\eta})$-approximation factor as long as Assumption 1 is met.
4.5. Operational Insights

In the following, we briefly discuss operational implications of the sequential matching market model that we study and the benefits of assortment optimization as an algorithmic tool to address congestion in matching markets.

In Sections 4.5.1 and 4.5.2, our main goal is to investigate the conditions for which the platforms should let suppliers initiate the matchmaking process versus customers initiate the process. Specifically, note that the model we study assumes that customers initiate the sequential matching, however, an analogous model can be proposed in which suppliers choose first and customers respond back. We aim to analyze how the ratio between the optimal values of both variants of the model depends on two aspects of the market: (1) the attractiveness of the agents (customers and suppliers); (2) imbalance on the size of each side. For the former, we show that the optimal matching size increases when the more selective agents (i.e., whose opposite side is less attractive) initiate the process. For the latter, we show that the optimal matching size increases when we let the larger side to initiate the sequential matching, even if one side is less attractive than the other side.

Finally, in Section 4.5.3, we consider the model in which customers initiate the process and we study the performance of a simple algorithm to construct the assortment for each customer without taking into account the preferences of other customers and suppliers’ preferences. We show that the performance of this algorithm is arbitrarily close to zero.

For simplicity, in all of the results of this section, we consider the unconstrained setting in which suppliers and customers have MNL choice models.

4.5.1. The Impact of Agents’ Attractiveness. Assume that both sides have the same size, \(|C| = |S| = n\) and \(r_j = 1\) for all \(j \in S\). First, we show that there is a worst-case instance in which the optimal matching sizes differ by a \(O(n)\) multiplicative factor.

Proposition 5. In the worst case, the ratio between the optimal value when customers initiate the matchmaking process and the optimal value when suppliers initiate the process can be arbitrarily close to zero.

The instance that we consider in this proposition is composed by very selective suppliers (i.e., customers’ side is not attractive) and less selective customers. Given that \(|C| = |S| = n\), if customers initiate the process, then there is no assortment family that guarantees that each supplier gets chosen by a significant number of customers (in expectation). This means that suppliers get less attractive options, and since they are very selective, then the resulting matching size is \(O(1)\). On the other hand, when suppliers initiate the process, we can control what suppliers see. Therefore, by
showing large enough assortments, suppliers will pick some customer with a constant probability, which means that each customer gets in expectation one supplier. Since customers are not selective, this approach results in final matching of $O(n)$ size.

We now analyze the impact of selectivity in instances that are not necessarily worst-case. For this, we will study a simple setting that will highlight the differences in the optimal value when either customers or suppliers initiate the matchmaking process.

**Proposition 6.** Let $w, v \in (0, 1]$ be two constants with respect to $n$. Assume that $w_{j,i} = w$ and $v_{i,j} = v$ for all $i \in \mathcal{C}$, $j \in \mathcal{S}$. If $w \leq v/3$, then for sufficiently large $n$ the optimal value when suppliers initiate the matchmaking process is at least the optimal value when customers initiate the process.

Note that in this result we are assuming that both sides are less attractive than the outside options, i.e., $w_{j,i}, v_{i,j} \in (0, 1]$. Roughly speaking, the proposition above shows that since customers are less attractive to suppliers than suppliers are to customers ($w \leq v/3$), then the matchmaking process results in a larger matching when suppliers start choosing. As in the previous proposition, this is due to the fact that it is better to control the recommendations of the more selective side. This assortment design observation aligns with previous results in the literature of market design, see e.g. [Kanoria and Saban 2021, Shi 2022b]. We defer the proofs of Propositions 5 and 6 to Appendix C.

**4.5.2. The Impact of the Size of Each Side.** We now study the optimal outcomes when both sides have different sizes. Recall that $n = |\mathcal{C}|$ is the number of customers and $m = |\mathcal{S}|$ is the number of suppliers. Assume $r_j = 1$ for all $j \in \mathcal{S}$.

**Proposition 7.** Let $w, v \in (0, 1]$ two constants with respect to $m$. Assume that $v_{i,j} = v$ and $w_{j,i} = w$ for all $j \in \mathcal{S}$, $i \in \mathcal{C}$. If $n \geq m/w$ and $v \leq 1/3$, then for sufficiently large $m$ the optimal value when customers initiate the matchmaking process is at least the optimal value when suppliers initiate the process.

Roughly speaking, without conditions on the ratio between $w$ and $v$, if the customers’ side is large enough, then the optimal expected matching is larger when customers initiate the process. Therefore, we observe that if there are enough customers, then the platform may benefit from letting them start choosing, even if suppliers are more selective than customers (e.g., as in Proposition 6). We defer the proof of Proposition 7 to Appendix C.

**4.5.3. Customer-centric Assortment Optimization.** In this section, we briefly analyze the performance of a method that does not account for suppliers’ preferences and the rest of the customers. Specifically, we consider Algorithm 5. Note that Algorithm 5 solves the classic single-customer assortment optimization problem for each customer $i \in \mathcal{C}$. As we see in the following proposition, the performance of this method can be arbitrarily bad.
Algorithm 5 Single-customer Assortment Optimization

**Input:** Values $v_{i,j} \geq 0$, revenues $r_j \geq 0$.

**Output:** Assortment family $S$.

1. Separately, for each customer $i \in C$, obtain $S_i$ as follows:

$$S_i \in \arg \max \left\{ \sum_{j \in S} r_j \cdot \frac{v_{i,j}}{1 + \sum_{\ell \in S} v_{i,\ell}} : S \in \mathcal{I}_i \right\}$$

**Proposition 8.** The worst case approximation guarantee of Algorithm 5 is zero.

The counterexample presented in this proposition shows that when we do not account for suppliers’ preferences, then customers’ requests can get “highly concentrated” on popular suppliers which results in a suboptimal solution since at most one of those customers will be served by the supplier. Consequently, to address these congestion challenges, the preference structure of both sides of the market need to be considered as in Algorithm 2.

### 5. Conclusion

This work presented a general framework for the multi-agent assortment optimization problem in sequential matching markets. While the related literature focuses on the MNL-Uniform model, we significantly expanded it to heterogeneous agents with general choice models. Moreover, our approach allows us to consider the maximization of the expected revenue over different type of constraints. Under mild assumptions, we provide an optimal $(1 - 1/e)$-approximation algorithm which consists of an adaptation of the continuous greedy algorithm with choice updates. We also design a more efficient algorithm which is variant of the greedy algorithm proposed in (Aouad and Saban 2023). This algorithm achieves a sub-optimal $1/2$-approximation factor. We are able to beat the $1 - 1/e$ barrier for the setting in which suppliers’ preferences have MNL models and the instances fall into the $\epsilon$-small probability regime. More precisely, we provide a variant of the Frank-Wolfe algorithm which is asymptotically optimal, however, with a sub-optimal approximation factor in general instances. Finally, we show other results and further insights. In particular, we study the MNL-MNL unconstrained model, for which a simple randomized nested-assortment algorithm achieves a $1/4$-multiplicative factor. Moreover, our operational insights indicate that platforms should allow agents that are more selective (whose opposite side is less attractive) to initiate the matchmaking process, since in this way platforms can control the recommendations that the agents see and avoid market congestion, resulting in better market outcomes. We also observed that the number of agents in each side plays a significant role at the moment of deciding which side starts choosing.
As a future research direction, we think that it would be interesting, and promising from a practical perspective, to look at models that allow for a second round of matches, namely, remaining unmatched suppliers can be matched.

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**Appendix A: Missing Proofs in Section 2**

In the following, we prove the approximation guarantee of Algorithm 1.

**Proof of Theorem 1.** To ease the exposition, in the following we consider $r_j = 1$ for all $j \in S$; the proof carries over for the case when the revenues $r_j$ are different. In this proof, we follow the outline of the proof for the approximation guarantee of the standard continuous greedy algorithm, specifically Lemma 3.3 in [Calinescu et al. 2011]. We assume that we have access to the exact value of the expected marginal values; otherwise, we can always estimate it to a desired close multiplicative factor as explained in [Calinescu et al. 2011] with a polynomial number of samples.

Consider $z^{t+\delta}$ and $z^t$ solutions from two consecutive iterations. Let $C_j^{t+\delta}$ the subset of customers who chose supplier $j$ according to choice probabilities $z_j^{t+\delta}$. Similarly, define $C_j^t$. Also, let $D_j^t$ be the random subset of customers who chose supplier $j$ according to choice probabilities given by $\delta \Phi_j(S^t)$ determined by the solution found in Step 3. Due to monotonicity of the demand functions,

$$F(z^{t+\delta}) = \sum_{j \in S} \mathbb{E}[f_j(C_j^{t+\delta})] \geq \sum_{j \in S} \mathbb{E}[f_j(C_j^t \cup D_j^t)],$$

where the expectation is over the independent choices of customers. More formally, for each $j \in S$ the set $C_j^{t+\delta}$ contains customers with independent probabilities $z_j^t + \delta \Phi_j(S^t)$, while the set $C_j^t \cup D_j^t$ contains customers with smaller independent probabilities $1 - (1 - z_{i,j}^t)(1 - \phi_i(j, S_j^t))$ for each $i \in C$. 
Given the above, we have

\[
F(z^{i+\delta}) - F(z') \geq \mathbb{E} \left[ \sum_{j \in S} f_j(C_i^t \cup D_i) - f_j(C_j^i) \right] \geq \sum_{j \in S} \sum_{i \in C} \mathbb{P}(D^t = \{i\}) \cdot \mathbb{E}[f_j(C_i^t \cup \{i\}) - f_j(C_j^i)]
\]

\[
= \sum_{j \in S} \sum_{i \in C} \delta \cdot \phi_j(j, S_i^t) \prod_{k \neq i} (1 - \delta \cdot \phi_k(j, S_k)) \cdot \mathbb{E}[f_j(C_i^t \cup \{i\}) - f_j(C_j^i)]
\]

\[
\geq \delta \cdot (1 - \delta)^{|C|} \cdot \sum_{j \in S} \sum_{i \in C} \phi_j(j, S_i^t) \cdot \mathbb{E}[f_j(C_i^t \cup \{i\}) - f_j(C_j^i)]
\]

\[
\geq \delta \cdot (1 - \delta)^{|C|} \cdot \sum_{j \in S} \sum_{i \in C} \phi_j(j, S_i^t) \cdot \mathbb{E}[f_j(C_i^t \cup \{i\}) - f_j(C_j^i)]
\]

\[
= \delta \cdot (1 - \delta)^{|C|} \cdot \sum_{j \in S} f_j(z^i \cup \Phi_j(S^*)) - F_j(z')
\]

\[
\geq \delta \cdot (1 - n\delta) \cdot (\text{OPT} - F(z^i)) \geq \delta \cdot \text{OPT} - F(z'),
\]

where \(\text{OPT} = (1 - n\delta) \cdot \text{OPT}\) and \(|C| = n\). The third inequality follows by noting that \(\phi_k(\cdot, \cdot) \leq 1\), the next inequality is due to optimality of Step 3, the fifth inequality is due to Fact 1 and the last one from monotonicity. Then, by solving the recurrence above we get

\[
\text{OPT} - F(z^i) \leq (1 - \delta)^{1/\delta} \cdot \text{OPT} \leq \frac{1}{e} \cdot \text{OPT},
\]

so by taking \(\delta = O(1/n^2)\) we can show that

\[
F(z^i) \geq (1 - 1/e - o(1)) \cdot \text{OPT}.
\]

The remaining part of the proof (sampling assortments) follows from the sketch proof given before. □

Before showing Proposition 1 we define the rank-based choice model.

**Definition 3 (Rank-based Choice Model).** A supplier \(j \in S\) is said to have a rank-based choice model if there exists a multinomial distribution \((\lambda_1^j, \ldots, \lambda_K^j)\) over \(K\) ranked lists \(L_1^j, \ldots, L_K^j\) where each list is a subset of customers with a strict order. For a given subset \(C \subseteq \mathcal{C}\), supplier \(j\) samples a ranked list \(L_k^j\) according to the multinomial distribution and chooses the top rank customer among those in \(C \cap L_k^j\); if empty, then the supplier remains unmatched. The demand function in this case corresponds to

\[
f_j(C) = \sum_{k=1}^{K} \lambda_k \cdot \min\{|C \cap L_k^j|, 1\}.
\]

**Proof of Proposition 1.** Let be \((\mathcal{N}, \mathcal{M}, \{u_j\}_{j \in \mathcal{M}})\) be an instance of the offline welfare maximization problem, see e.g. [Lehmann et al. 2006, Vondrák 2008], where \(\mathcal{N}\) is the set of items, \(\mathcal{M}\) the set of agents and \(\{u_j\}_{j \in \mathcal{M}}\) the coverage valuations. For these valuations we assume there exists a universe of elements \(\mathcal{E}\) and a subset system \(\mathcal{U} = \{U_i\}_{i \in \mathcal{N}}\). Each item \(i \in \mathcal{N}\) is associated with a set \(U_i\) and each valuation \(u_j\) is defined by a permutation \(\sigma_j : \mathcal{N} \rightarrow \mathcal{N}\) over the sets in \(\mathcal{U}\) such that \(u_j(C) = |\bigcup_{i \in C} U_{\sigma_j(i)}|\). Now we construct an instance of our problem:
For each item \( i \in \mathcal{N} \) define a customer \( i \), i.e. \( \mathcal{C} = \mathcal{N} \). Each customer \( i \) is: (i) defined by a choice model \( \phi_i \) with no outside option, i.e., \( \phi_i(0, S) = 0 \) whenever \( S \neq \emptyset \); (ii) associated to the set \( U_i \).

For each agent \( j \in \mathcal{M} \) we define a supplier \( j \), i.e., \( \mathcal{S} = \mathcal{M} \). Each supplier \( j \) has rank-based preferences defined as follows: Construct one preference list per element in \( U \), i.e., \( L_j^1, \ldots, L_j^{|U|} \), where \( L_j^k = \{ i : k \in U_{\sigma_j(i)} \} \) is ranked in arbitrary order and composed by customers who “cover” element \( k \in U \) with respect to permutation \( \sigma_j \). Finally, the choice model of supplier \( j \) is determined by a uniform distribution over the list \( L_j^1, \ldots, L_j^{|U|} \), i.e., for any set of customers \( C \) we have

\[
f_j(C) = \frac{1}{|U|} \sum_{k \in U} \min\{|C \cap L_k^i|, 1\} = \frac{1}{|U|} \sum_{k \in U} \mathbb{1}_{\{C \cap L_k^i \neq \emptyset\}}.
\]

Note then that for any \( C \subseteq \mathcal{N} \) and \( j \in \mathcal{M} \) we have

\[
u_j(C) = \bigcup_{i \in C} U_{\sigma_j(i)} = \sum_{k \in U} \mathbb{1}_{\{k \in U_{\cup_C U_{\sigma_j(i)}}\}} = \sum_{k \in U} \mathbb{1}_{\{C \cap L_k^i \neq \emptyset\}} = |U| \cdot f_j(C).
\]

Consider an optimal solution of the offline welfare maximization problem \( C_j^* \) for each \( j \in \mathcal{M} \). Recall that the sets \( C_j^* \) define a partition over \( \mathcal{N} \). We now construct a feasible solution to our problem: Let \( S_i = \{ j \} \) be the assortment of customer \( i \) where \( j \) is such that item \( i \in C_j^* \). Therefore, each customer chooses with probability 1 the only supplier in the assortment, so we obtain

\[
\text{OPT} \geq \mathbb{E}[\mathbb{M}_S] = \sum_{j \in S} f_j(C_j^*) = \frac{1}{|U|} \sum_{j \in \mathcal{M}} u_j(C_j^*) = \frac{1}{|U|} \cdot \text{OPT'},
\]

where \( \text{OPT'} \) is the optimal value of the offline welfare maximization problem. Assume by contradiction that there exists an \( 1 - 1/e + \delta \)-approximation algorithm for our problem. Let \( S \) be the feasible solution output by that algorithm. We now construct the following randomized algorithm for the offline welfare maximization problem. Sample the choices made by the customers and let \( C_j \) be the set of requests that supplier \( j \) received. Assign the \( C_j \) of items to agent \( j \) in the original instance. Then note that the expected value of this algorithm is

\[
\mathbb{E}_{C \sim \Phi} \left[ \sum_{j \in \mathcal{M}} u_j(C) \right] = |U| \mathbb{E}_{C \sim \Phi} \left[ \sum_{j \in S} f_j(C) \right] \geq |U| \cdot (1 - 1/e + \delta) \cdot \text{OPT} \geq (1 - 1/e + \delta) \cdot \text{OPT'},
\]

where \( \Phi \) is the vector of choice probabilities determined by assortment family \( S \) and the expectation above is over customers’ choices. The above contradicts the hardness of the offline welfare maximization problem, see e.g. (Khot et al. 2008, Kapralov et al. 2013). The \( \epsilon \)-regime analysis follows the same idea than in (Aouad and Saban 2023).
Appendix B: Missing Proofs in Section 3

Proof of Lemma 1. Let us proceed by induction. For \( t = 0 \), we know that \( z^0 = 0 \) and \( \tau_{i,S} = 0 \) for all \( i \in \mathcal{C} \) and \( S \in \mathcal{I} \), such that \( \mathcal{S} \neq \emptyset \), thus the first family of constraints of the feasible region is satisfied. Also, we have initially \( \tau_{i,\emptyset} = 1 \) for all \( i \in \mathcal{C} \), thus the second family of constraints is also satisfied. Now, let us assume that at the end of step \( t - 1 \), points \( z^{t-1} \) and \( \tau^{t-1} \) are feasible. Consider \( z^t \) and \( \tau^t \) the updates at the end of the step \( t \). We check the second family of constraints:

For any \( i \in \mathcal{C} \) we have

\[
\sum_{S \in \mathcal{I}_i} \tau_{i,S}^t = \sum_{S \in \mathcal{I}_i; S \neq S_i^t} \tau_{i,S}^{t-1} + \tau_{i,S_i^t}^t
\]

\[
= (1 - \gamma_t) \sum_{S \in \mathcal{I}_i; S \neq S_i^t} \tau_{i,S}^{t-1} + (1 - \gamma_t) \tau_{i,S_i^t}^{t-1} + \gamma_t
\]

\[
= (1 - \gamma_t) \sum_{S \in \mathcal{I}_i} \tau_{i,S}^{t-1} + \gamma_t = 1
\]

where in the last equality we used that \( \tau^{t-1} \) is feasible. Now we check the first family of constraints:

For any \( j \in \mathcal{S} \)

\[
z_j^t = (1 - \gamma_t) \cdot z_j^{t-1} + \gamma_t \cdot \sum_{i \in \mathcal{C}} w_{j,i} \cdot \phi_i(j, \mathcal{S}_i^t)
\]

\[
= (1 - \gamma_t) \sum_{i \in \mathcal{C}} w_{j,i} \cdot \sum_{S \in \mathcal{I}_i; S \neq j} \tau_{i,S}^{t-1} \phi_i(j, S) + \gamma_t \cdot \sum_{i \in \mathcal{C}} w_{j,i} \cdot \phi_i(j, \mathcal{S}_i^t)
\]

\[
= (1 - \gamma_t) \sum_{i \in \mathcal{C}} w_{j,i} \cdot \sum_{S \in \mathcal{I}_i; S \neq j} \tau_{i,S}^{t-1} \cdot \phi_i(j, S) + (1 - \gamma_t) \sum_{i \in \mathcal{C}} w_{j,i} \cdot \tau_{i,S_i^t}^{t-1} \cdot \phi_i(j, \mathcal{S}_i^t)
\]

\[
+ \gamma_t \sum_{i \in \mathcal{C}} w_{j,i} \cdot \phi_i(j, \mathcal{S}_i^t)
\]

\[
= \sum_{i \in \mathcal{C}} w_{j,i} \cdot \sum_{S \in \mathcal{I}_i; S \neq j} \tau_{i,S}^t \cdot \phi_i(j, S) + \sum_{i \in \mathcal{C}} w_{j,i} \cdot \tau_{i,S_i^t}^t \cdot \phi_i(j, \mathcal{S}_i^t)
\]

\[
= \sum_{i \in \mathcal{C}} w_{j,i} \cdot \sum_{S \in \mathcal{I}_i; S \neq j} \tau_{i,S}^t \cdot \phi_i(j, S). \square
\]

Proof of Proposition 2. Let us define \( g : \mathbb{R}_+^S \rightarrow \mathbb{R}_+ \) as the following function

\[
g(z) = \sum_{j \in \mathcal{S}} r_j \cdot \frac{z_j}{1 + z_j}
\]

which is continuous, concave and differentiable. Moreover, by the Mean Value Theorem, for any pair \( z, z' \) there exists a point \( \xi \in \mathbb{R}_+^S \) such that

\[
g(z') = g(z) + \langle \nabla g(z), z' - z \rangle + (z' - z)^T \nabla^2 g(\xi)(z' - z).
\]

Since the Hessian of \( g \) is a diagonal matrix with values \(-r_j/(1 + \xi_j)^3\) for each \( j \in \mathcal{S} \), we can obtain the following lower bound

\[
g(z') \geq g(z) + \langle \nabla g(z), z' - z \rangle - R \cdot ||z' - z||^2
\]

(15)
where \( R = \max_{j \in S} r_j \). By taking \( z' = z^{t+1} \) and \( z = z^t \) in (15) and using the update in Step 4 of Algorithm 1, we get

\[
g(z^{t+1}) - g(z^t) \geq \gamma_t \cdot \langle \nabla g(z^t), z^{t+1} - z^t \rangle - R \cdot \|z^{t+1} - z^t\|^2 \geq \gamma_t \cdot \langle \nabla g(z^t), \tilde{w}^t - z^t \rangle - \gamma_t^2 \cdot R \cdot \|\tilde{w}^t - z^t\|^2
\]

where \( \tilde{w}^t \in \mathbb{R}^m_+ \) is the vector whose \( j \)-th component equals \( \sum_{i \in \mathcal{C}} w_{j,i} \cdot \phi_t(j, S^t_i) \) and \( S^t_i \) is the solution found in Step 3 for customer \( i \in \mathcal{C} \). Note also due to the diameter of the feasible region, we know that \( \|\tilde{w}^t - z^t\|^2 \leq W \) where \( W = \sum_{j \in S} \left( \sum_{i \in \mathcal{C}} w_{j,i} \right)^2 \). Therefore,

\[
g(z^{t+1}) - g(z^t) \geq \gamma_t \cdot \langle \nabla g(z^t), \tilde{w}^t - z^t \rangle - \gamma_t^2 R W
\]

By taking \( \gamma_t = \min \left\{ \frac{\langle \nabla g(z^t), \tilde{w}^t - z^t \rangle}{2\sqrt{mRW}}, 1 \right\} \) where \( m = |\mathcal{S}| \), we have

\[
g(z^{t+1}) - g(z^t) \geq \min \left\{ \frac{\langle \nabla g(z^t), \tilde{w}^t - z^t \rangle^2}{2\sqrt{mRW}}, \frac{\langle \nabla g(z^t), \tilde{w}^t - z^t \rangle}{4m} \right\} - \min \left\{ \frac{\langle \nabla g(z^t), \tilde{w}^t - z^t \rangle^2}{4m}, RW \right\}
\]

\[
= \frac{\langle \nabla g(z^t), \tilde{w}^t - z^t \rangle^2}{4\sqrt{mRW}},
\]

where in the second inequality we used that \( \langle \nabla g(z^t), \tilde{w}^t - z^t \rangle \leq \|\nabla g(z^t)\| \cdot \|\tilde{w}^t - z^t\| \leq \sqrt{mRW} \).

On the other hand, by concavity of \( g \) we know that for any \( t \in [T] \),

\[
g(z^*) - g(z^t) \leq \langle \nabla g(z^t), z^* - z^t \rangle \leq \langle \nabla g(z^t), \Phi(S^t) - z^t \rangle
\]

where in the second inequality we use optimality of Step 3 in Algorithm 1. By putting the previous two inequalities together we get,

\[
g(z^{t+1}) - g(z^t) \geq \frac{\|g(z^*) - g(z^t)\|^2}{4\sqrt{mRW}}
\]

which is equivalent to

\[
h(z^t) - h(z^{t+1}) \geq \frac{h(z^t)^2}{4\sqrt{mRW}}
\]

where we defined \( h(z) = g(z^*) - g(z) \). Solving this classic recurrence, we get

\[
h(z^t) \leq \frac{4\sqrt{mRW}}{t+2}
\]

which in \( t = T \) implies that

\[
g(z^T) \geq g(z^*) - \frac{4\sqrt{mRW}}{T+2}.
\]

Therefore, after sufficiently enough iterations \( T \gg \sqrt{mRW} \), the error term becomes negligible. Finally, note that due to Lemma 3 the points \( z^T \) and \( \tau^T \) are feasible. \( \square \)
Proof of Lemma 4. Fix \( j \in S \) and consider \( i \in C \) such that \( d_{i,j} > 0 \). Then, conditioned on the event that \( i \) chose \( j \) (which happens with probability \( d_{i,j} \)), the expected probability (with respect to the product distribution) that \( j \) selects \( i \) back can be lower bounded as follows

\[
E_{C \sim \mathbf{d}_{i,j}} \left[ \phi_j(i, C) \big| \text{i chose } j \right] = E_{C \sim \mathbf{d}_{i,j}} \left[ \frac{w_{j,i}}{1 + w_{j,i} + \sum_{k \in C, k \neq i} w_{j,k}} \right] \bigg| \text{i chose } j
\]

\[
\geq \frac{w_{j,i}}{1 + w_{j,i} + \sum_{k \in C, k \neq i} w_{j,k}} \geq \frac{1}{1 + \epsilon/\epsilon} + \sum_{k \in C, k \neq i} w_{j,k} d_{k,j}
\]

\[
= (1 - \epsilon) \cdot \frac{w_{j,i}}{1 + \epsilon/\epsilon} + \sum_{k \in C, k \neq i} w_{j,k} d_{k,j}
\]

where \( \mathbf{d}_{i,j} \) is the vector without the \((i, j)\)-th component. The first equality follows from the assumption that \( i \) chose \( j \), the second equality follows from the independent choices of the customers, the following inequality is due to Jensen’s inequality, the second inequality is due to the assumption \( w_{j,i} \leq \epsilon/(1 - \epsilon) \) and the last inequality is because the expression decreases when we include more terms to the sum in the denominator. Therefore, the demand function of supplier \( j \) can be lower bounded as follows

\[
E_{C \sim \mathbf{d}}[f_j(C)] = E_{C \sim \mathbf{d}} \left[ \sum_{i \in C} \phi_j(i, C) \right] = E_{C \sim \mathbf{d}} \left[ \sum_{i \in C} \phi_j(i, C) \cdot 1_{\{i \in C\}} \right]
\]

\[
= \sum_{i \in C} E_{C \sim \mathbf{d}_{i,j}} \left[ \phi_j(i, C) \cdot 1_{\{i \in C\}} \right] \Pr(\text{i chooses } j)
\]

\[
= \sum_{i \in C} E_{C \sim \mathbf{d}_{i,j}} \left[ \phi_j(i, C) \right] \cdot d_{i,j} \geq (1 - \epsilon) \cdot \sum_{i \in C} \frac{w_{j,i}}{1 + \sum_{k \in C} w_{j,k} d_{k,j}} \cdot d_{i,j}.
\]

Finally, by multiplying by \( r_j \) and summing over \( j \in S \), we obtain the desired inequality. \( \square \)

Proof of Lemma 5. Fix supplier \( j \in S \). Since suppliers have MNL choice models, then the demand function of supplier \( j \) is \( f_j(C) = w_j(C)/(1 + w_j(C)) \), where \( w_j(C) := \sum_{i \in C} w_{j,i} \) and \( C \subseteq C \).

Define \( C_j^1 = \{i \in C: w_{j,i} > \epsilon/(1 - \epsilon)\} \) and denote by \( C_j^S \) the random set of customers that selected supplier \( j \) under assortment family \( S \). First, observe that

\[
E[f_j(C_j^S)] = E \left[ \frac{w_j(C_j^S)}{1 + w_j(C_j^S)} \right] = E \left[ \frac{w_j(C_j^S \setminus C_j^1) + w_j(C_j^1)}{1 + w_j(C_j^S \setminus C_j^1) + w_j(C_j^1)} \right] > E \left[ \frac{\bar{w}_j(C_j^S)}{1 + \bar{w}_j(C_j^S)} \right],
\]
where the expectation is taken over the randomness of $C_j$ and the inequality follows by noting that the function $f(z) = z/(1+z)$ is increasing in $z$ and $w_{j,i} > \frac{\epsilon}{1 - \epsilon} = \tilde{w}_{j,i}$ for all $i \in C_j$. Therefore, we conclude the first inequality

$$\mathbb{E}[M_s] = \sum_{j \in S} r_j \cdot \mathbb{E}[f_j(C_j^S)] \geq \sum_{j \in S} r_j \cdot \mathbb{E} \left[ \frac{\sum_{i \in C_j} \tilde{w}_{j,i}}{1 + \sum_{i \in C_j} \tilde{w}_{j,i}} \right] = \mathbb{E}[\tilde{M}_s].$$

To prove the other inequality note that

$$\mathbb{E}[f_j(C_j^S)] = \mathbb{E} \left[ \frac{w_j(C_j^S)}{1 + w_j(C_j^S)} \right] = \mathbb{E} \left[ \frac{w_j(C_j^S \setminus C_j^1) + w_j(C_j^1)}{1 + w_j(C_j^S \setminus C_j^1) + w_j(C_j^1)} \right] \leq \frac{1}{\epsilon} \cdot \mathbb{E} \left[ \frac{\tilde{w}_j(C_j^S)}{1 + \tilde{w}_j(C_j^S)} \right],$$

where the inequality results from observing that

$$\frac{w_j(C_j^S \setminus C_j^1) + w_j(C_j^1)}{1 + w_j(C_j^S \setminus C_j^1) + w_j(C_j^1)} \leq \frac{1}{\epsilon} \cdot \frac{w_j(C_j^S \setminus C_j^1) + \sum_{i \in C_j^1} \frac{\epsilon}{1 - \epsilon} = \frac{1}{\epsilon} \cdot \frac{\tilde{w}_j(C_j^S)}{1 + \tilde{w}_j(C_j^S)}.$$

By multiplying by $r_j$ and summing over $j \in S$, we conclude that $\mathbb{E}[M_s] \leq \frac{1}{\epsilon} \cdot \mathbb{E}[\tilde{M}_s]$. □

**Appendix C: Missing Proofs in Section 4**

**C.1. Missing Proofs in Section 4.2**

Similar to our approach in Section 3 we consider the following relaxation

$$\max \quad \sum_{j \in S} r_j \cdot \frac{\sum_{k \in [K]} z_{j,k}}{1 + \sum_{k \in [K]} z_{j,k}} \tag{16}$$

s.t. \quad $\sum_{i \in C_k} \tau_{i,S} \cdot \phi_i(j;S) = z_{j,k}$, \quad for all $j \in S, k \in [K]$

$$\sum_{S \in I_i} \tau_{i,S} = 1, \quad \text{for all } i \in C,$$

$$z_j, \tau_{i,S} \geq 0 \quad \text{for all } i \in C, S \in I_i, j \in S.$$

Note that the objective is concave and separable as for each $j \in S$, the values in $z_j = (z_{j,1}, \ldots, z_{j,K})$ do not appear in the objective of a different supplier $j' \neq j$. More importantly, by Jensen’s inequality we can show the following:

**Lemma 8.** Problem (16) is a relaxation of Problem (11) when suppliers have $\nu$-NL choice models.

Given the above, we can easily adapt Algorithm 3 to obtain the following result:

**Proposition 9.** The adaptation of Algorithm 3 to the $\nu$-NL choice model outputs a feasible solution $(z_T^T, \tau_T^T)$ such that

$$\sum_{j \in S} r_j \cdot \frac{\sum_{k \in [K]} (z_{j,k}^T)^{\nu_k}}{1 + \sum_{k \in [K]} (z_{j,k}^T)^{\nu_k}} \geq \text{OPT}_{16} - o(1)$$

where $\text{OPT}_{16}$ is the optimal value of Problem (16).
Therefore, our goal now is to prove an equivalent version of Lemma 4.

**Lemma 9.** Assume suppliers have $\nu$-NL choice models with values $w_{ji} \leq \epsilon/(1 - \epsilon)$ for all $j \in S$, $i \in C$ and $\epsilon \in (0, 1/(n^{2p} + 1)]$ where $\nu = \max_k \nu_k$. Then, for any vector $d \in [0, 1]^{C \times S}$, we have

$$
\sum_{j \in S} r_j \cdot F_j(d_j) \geq (1 - o(1)) \cdot \sum_{j \in S} r_j \cdot \frac{\sum_{k \in [K]} \left( \sum_{i \in C_k} w_{ji} d_{i,j} \right)^{\nu_k}}{1 + \sum_{k \in [K]} \left( \sum_{i \in C_k} w_{ji} d_{i,j} \right)^{\nu_k}}.
$$

where $F_j$ is the multilinear extension of the demand function $f_j$ and $d_j = (d_{i,j})_{i \in C}$ for all $j \in S$.

The proof of this lemma is similar to the proof of Lemma 4.

**Proof.** Fix $j \in S$ and consider $i \in C$ such that $d_{i,j} > 0$. Then, conditioned on the event that $i$ chose $j$ (which happens with probability $\nu_k$), the expected probability (with respect to the product distribution) that $j$ selects $i \in C_k$ back (as given in the first equality of Expression (14)) can be lower bounded as follows

$$
E \left[ \phi_j(i, C_k) | i \text{ chose } j \right]
= E \left[ \frac{w_{ji,i}}{w_{ji,i} + w_j(C_k \backslash \{i\})} \cdot \frac{(w_{ji,i} + w_j(C_k \backslash \{i\}))^{\nu_k}}{1 + (w_{ji,i} + w_j(C_k \backslash \{i\}))^{\nu_k} + \sum_{q \neq k} (w_j(C_q))^{\nu_q}} | i \text{ chose } j \right]
= E \left[ \frac{w_{ji,i}}{(w_{ji,i} + w_j(C_k \backslash \{i\}))^{1-\nu_k}} \cdot \frac{1}{1 + \frac{1}{n} + \sum_{q \neq k} (w_j(C_q))^{\nu_q}} \right]
\geq \left[ \frac{w_{ji,i} + E \left[ w_j(C_k \backslash \{i\}) \right]^{1-\nu_k}}{1 + \frac{1}{n} + \sum_{q \neq k} (w_j(C_q))^{\nu_q}} \right]
= \left[ \frac{w_{ji,i} + \sum_{\ell \in C_k \setminus \{i\}} w_{j,i \ell} d_{j,i \ell}}{1 + \frac{1}{n} \cdot w_{j,i \ell}} \cdot \frac{(w_{ji,i} + \sum_{\ell \in C_k \setminus \{i\}} w_{j,i \ell} d_{j,i \ell})^{1-\nu_k}}{1 + \sum_{q \neq k} (w_{j,i \ell})^{\nu_q}} \right]
\geq \left[ \frac{w_{ji,i} + \sum_{\ell \in C_k \setminus \{i\}} w_{j,i \ell} d_{j,i \ell}}{1 + \frac{1}{n} \cdot w_{j,i \ell}} \cdot \frac{(w_{ji,i} + \sum_{\ell \in C_k \setminus \{i\}} w_{j,i \ell} d_{j,i \ell})^{1-\nu_k}}{1 + \sum_{q \neq k} (w_{j,i \ell})^{\nu_q}} \right]
\geq \left[ \frac{\frac{1}{n^2} + \sum_{\ell \in C_k \setminus \{i\}} w_{j,i \ell} d_{j,i \ell}}{1 + \frac{1}{n} \cdot w_{j,i \ell} ^{\nu_q}} \cdot \frac{(w_{ji,i} + \sum_{\ell \in C_k \setminus \{i\}} w_{j,i \ell} d_{j,i \ell})^{1-\nu_k}}{1 + \sum_{q \neq k} (w_{j,i \ell})^{\nu_q}} \right].
$$
where the expectation is over $C \sim d_{ij}$ and $d_{ij}$ is the vector without the $(i,j)$-th component. The first equality follows from the assumption that $i$ chose $j$. The first inequality is due to

$$w_j(C_k) \leq \left| C_k \right| \cdot \left( \frac{\epsilon}{1 - \epsilon} \right)^{1/\nu_k} \leq n \cdot \left( \frac{1}{n^2} \right)^{\hat{\nu}/\nu_k} \leq \frac{1}{n}$$

since $\epsilon/(1 - \epsilon)$ is increasing in $\epsilon$ and $\epsilon \leq 1/(n^{2\hat{\nu}} + 1)$, moreover, $(1/n^2)^{\hat{\nu}/\nu_k} \leq (1/n^2)^{\nu_k/\nu_k}$ because $\nu_k \leq \hat{\nu}$. The next equality is because of independence of the customers’ choices and the partition structure of the nests. The second inequality is due to Jensen’s inequality and $\mathbb{E}[w_j(C_q)\alpha] \leq \mathbb{E}[w_j(C_q)]^\alpha$ for $\alpha \leq 1$. The next equality uses that

$$\mathbb{E}[w_j(C_q)] = \sum_{\ell \in C_q} w_{j,\ell} \cdot \mathbb{P}(\ell \text{ chooses } j) = \sum_{\ell \in C_q} w_{j,\ell} \cdot d_{\ell,j},$$

The third inequality is because we added one more term in the denominator of the second fraction. The fourth inequality is because $w_{j,i} \geq w_{j,i}d_{i,j}$. In the fifth inequality we use that $w_{j,i} \leq (\epsilon/(1 - \epsilon))^{1/\nu_k} \leq 1/n^2$. In the final inequality we added one more term in the denominator of the first fraction.

Therefore, the demand function of supplier $j$ can be lower bounded as follows

$$\mathbb{E}_{C \sim d}[f_j(C)] = \mathbb{E}_{C \sim d} \left[ \sum_{i \in C} \phi_j(i, C) \right] = \mathbb{E}_{C \sim d} \left[ \sum_{k \in [K]} \sum_{i \in C_k} \phi_j(i, C) \cdot 1_{\{i \in C\}} \right]$$

$$= \sum_{k \in [K]} \sum_{i \in C_k} \mathbb{E}_{C \sim d, i \in C} \left[ \phi_j(i, C) \cdot 1_{\{i \in C\}} \right] \mathbb{P}(i \text{ chooses } j)$$

$$= \sum_{k \in [K]} \sum_{i \in C_k} \mathbb{E}_{C \sim d, i \in C} \left[ \phi_j(i, C) \right] \cdot d_{i,j}$$

$$\geq \left( 1 - \frac{1}{n} - \frac{1}{n^2} \right) \cdot \sum_{k \in [K]} \sum_{i \in C_k} \frac{w_{j,i}}{\left( \sum_{\ell \in C_k} w_{j,\ell}d_{j,\ell} \right)^{\nu_k}} \cdot \left( \sum_{q \in [K]} \left( \sum_{\ell \in C_q} w_{j,\ell}d_{j,\ell} \right)^{\nu_k} \cdot d_{i,j} \right)$$

$$= (1 - o(1)) \cdot \sum_{k \in [K]} \left( \sum_{i \in C_k} w_{j,i}d_{i,j} \right)^{\nu_k} \cdot d_{i,j}$$

Finally, by multiplying by $r_j$ and summing over $j \in S$, we obtain the desired inequality. \hfill \Box

**Proof of Proposition 3.** The proof is analogous to the proof of Theorem 3. \hfill \Box

### C.2. Missing Proofs in Section 4.5

For simplicity, in all of the results of this section, we consider the problem of maximizing the cardinality of the matching, i.e., $r_j = 1$ for all $j \in S$. Before we provide the proofs, we need to
define the necessary notation for the setting in which suppliers initiate the matchmaking process. First, this process can be analogously defined as it was made for customers in the Introduction. We denote by $C = (C_1, \ldots, C_m)$ an assortment family of customers which is presented to suppliers. We analogously define $M_C$ the random variable indicating the revenue obtained in the final matching when suppliers start choosing. Therefore, the main problem that we focus for this setting is

$$\max\{\mathbb{E}[M_C]: C_j \in I_j \text{ for all } j \in S\},$$

where $C_j$ is the family of feasible assortments for supplier $j \in S$. For the remainder of this section, we denote by $OPT'$ the optimal value of this problem. For simplicity, we consider the unconstrained setting in both sequential models, i.e., $I_i = 2^S$ and $I_j = 2^C$. From Jensen’s inequality, when customers follow an MNL choice model, we have the following bound for a given assortment family $C$ and $M_C$:

$$\mathbb{E}[M_C] \leq \sum_{i \in C} \frac{\sum_{j \in S} v_{i,j} \cdot \phi_j(i, C_j)}{1 + \sum_{j \in S} v_{i,j} \cdot \phi_j(i, C_j)}. \quad (18)$$

Moreover, if customers’ values over suppliers are such that $v_{i,j} \leq 1$ for all $i \in C$, $j \in S$, we also have (equivalent to Lemma 4 with $\epsilon = 1/2$)

$$\mathbb{E}[M_C] \geq \frac{1}{2} \cdot \sum_{i \in C} \frac{\sum_{j \in S} v_{i,j} \cdot \phi_j(i, C_j)}{1 + \sum_{j \in S} v_{i,j} \cdot \phi_j(i, C_j)}. \quad (19)$$

Inequalities (18) and (19), will be crucial for the results in this section. Finally, we keep the notation for the process in which customers start choosing: for a given assortment family $S$ of suppliers that will be presented to the customers, $M_S$ corresponds to the random variable indicating the revenue obtained in the final matching. Recall that $OPT$ denotes the optimal value of the problem when customers initiate the process.

**Proof of Proposition 5.** Assume $|S| = |C| = n$. Consider customers and suppliers that follow MNL choice models with the following values: (1) there exists a constant $\alpha > 0$ such that $\sum_{j \in S} v_{i,j} \geq \alpha n$; (2) $w_{j,i} = 1/n$ for every $j \in S$, $i \in C$.

First, let us lower bound $OPT'$. For this, we will study the following assortment family $C = (C, \ldots, C)$, i.e., we show all the customers to every supplier. Then,

$$OPT' \geq \mathbb{E}[M_C] \geq \frac{1}{4} \cdot \sum_{i \in C} \frac{\sum_{j \in S} v_{i,j} \cdot \phi_j(i, C)}{1 + \sum_{j \in S} v_{i,j} \cdot \phi_j(i, C)} = \frac{1}{4} \cdot \sum_{i \in C} \frac{\sum_{j \in S} v_{i,j}}{2n + \sum_{j \in S} v_{i,j}} \geq \frac{1}{4} \cdot \sum_{i \in C} \frac{\alpha}{2 + \alpha} = \frac{\alpha n}{4(2 + \alpha)}$$

where the first inequality follows from Inequality (19) (applied to any scores $v_{i,j}$’s so the inequality has a factor 1/4 instead of 1/2 as in the proof of Theorem 4) and the first equality follows from noting that

$$\phi_j(i, C) = \frac{w_{j,i}}{1 + \sum_{i \in C} w_{j,i}} = \frac{1}{2n}. $$
In the second part of this proof, we will give an upper bound to OPT.

\[
\text{OPT} = \mathbb{E}[M_{S^*}] \leq \sum_{j \in \mathcal{S}} \frac{\sum_{i \in C} w_{j,i} \phi_i(j, S^*_i)}{1 + \sum_{i \in C} w_{j,i} \phi_i(j, S^*_i)} = \sum_{j \in \mathcal{S}} \frac{\frac{1}{n} \sum_{i \in C} \phi_i(j, S^*_i)}{1 + \frac{1}{n} \sum_{i \in C} \phi_i(j, S^*_i)}.
\]

Also, since \(w_{j,i} \leq 1\), Lemma 4 with \(\epsilon = 1/2\) implies

\[
\text{OPT} = \mathbb{E}[M_{S^*}] \geq \frac{1}{2} \sum_{j \in \mathcal{S}} \frac{\sum_{i \in C} w_{j,i} \phi_i(j, S^*_i)}{1 + \sum_{i \in C} w_{j,i} \phi_i(j, S^*_i)} = \frac{1}{2} \sum_{j \in \mathcal{S}} \frac{\frac{1}{n} \sum_{i \in C} \phi_i(j, S^*_i)}{1 + \frac{1}{n} \sum_{i \in C} \phi_i(j, S^*_i)}
\]

We want to maximize the expression on the right. Define \(g(z_1, \ldots, z_j) = \sum_{j \in \mathcal{S}} \frac{z_j}{n + z_j}\). Note that \(f(.)\) is a Schur-concave function. The worst case for our claim is when the expression on the right is maximized. This happens when “few” customers pick the outside option (in expectation), therefore we can assume that in expectation all customers do not pick the outside option, i.e.,

\[
\sum_{j \in \mathcal{S}} \sum_{i \in C} \phi_i(j, S^*_i) = \sum_{i \in C} \phi_i(j, S^*_i) = \sum_{i \in C} 1 = n
\]

This implies that \(\sum_{j \in \mathcal{S}} z_j = n\). Since \(f\) is Schur-concave, then it is maximized when \(z_1 = \cdots = z_{|\mathcal{S}|} = 1\); recall \(|\mathcal{S}| = |\mathcal{C}| = n\). This condition means that every supplier gets chosen (in expectation) by one customer. For instance, one instance of these characteristics is the following: \(v_{i,0} = 1\) and \(v_{i,j} \gg 1\) for all \(i \in \mathcal{C}, j \in \mathcal{S}\). Therefore, we obtain

\[
\text{OPT} \leq \sum_{j \in \mathcal{S}} \frac{\frac{1}{n} \sum_{i \in C} \phi_i(j, S^*_i)}{1 + \frac{1}{n} \sum_{i \in C} \phi_i(j, S^*_i)} = |\mathcal{S}| \cdot \frac{1}{n + 1} = \frac{n}{n + 1}
\]

Finally, the desired result follows by noting that

\[
\frac{\text{OPT}}{\text{OPT}} \leq \frac{n/(n + 1)}{\alpha n/(8 + 4\alpha)} \xrightarrow{n \to \infty} 0.
\]

This result for OPT is intuitive because of the selectivity of suppliers, \(w_{j,i} = 1/n\). We would need that the expected number of customers that choose each supplier is \(O(n)\). In this way, each supplier will pick one customer back with constant probability and we would get a matching of expected size \(O(n)\).

**Proof of Proposition 6.** First, let us upper bound OPT. As in the proof of Proposition 5

\[
\text{OPT} = \mathbb{E}[M_{S^*}] \leq \sum_{j \in \mathcal{S}} \frac{\sum_{i \in C} w_{j,i} \phi_i(j, S^*_i)}{1 + \sum_{i \in C} w_{j,i} \phi_i(j, S^*_i)} = \sum_{j \in \mathcal{S}} \frac{w \sum_{i \in C} \phi_i(j, S^*_i)}{1 + w \sum_{i \in C} \phi_i(j, S^*_i)}.
\]

We want to maximize the expression on the right. Define \(f(z_1, \ldots, z_{|\mathcal{S}|}) = \sum_{j \in \mathcal{S}} \frac{w z_j}{1 + w z_j}\). Note that \(f(.)\) is a Schur-concave function which is maximized when \(z_1 = \cdots = z_{|\mathcal{S}|}\). Let us analyze if there exists such as solution

\[
\phi_i(j, S^*_i) = \frac{v_{i,j}}{1 + \sum_{\ell \in S^*_i} v_{i,\ell}} = \frac{v}{1 + |S^*_i| v}
\]
Since we want to make \( \sum_{i \in C} \phi_i(j, S_i^*) \) as close to 1 as possible, then let us consider \( S_i^* = S \) for all \( i \in C \). Therefore, \( \phi_i(j, S_i^*) = v/(1 + nv) \) and
\[
\sum_{i \in C} \phi_i(j, S_i^*) = \frac{v}{1 + nv} |\{ i \in C : S_i^* \ni j \}| = \frac{nv}{1 + nv},
\]
where the second equality comes from observing that each supplier \( j \) is present in every assortment. The last expression is close to 1 for sufficiently large \( n \), since \( v \) is a constant. We conclude that
\[
\text{OPT} \leq \sum_{j \in S} \frac{w \sum_{i \in C} \phi_i(j, S_i^*)}{1 + w} = \frac{n v w}{1 + nv + wv} \approx n \cdot \frac{w}{1 + w}.
\]
Now, let us study \( \text{OPT}' \). For this note that since \( v_{i,j} \in [0, 1] \), then for any assortment family \( C \) because of Inequality \([19]\) we have
\[
\text{OPT}' \geq \mathbb{E}[M_C] \geq \frac{1}{2} \sum_{i \in C} \frac{\sum_{j \in S} v_{i,j} \cdot \phi_j(i, C_j)}{1 + v \sum_{j \in S} \phi_j(i, C_j)} = \frac{1}{2} \sum_{i \in C} \frac{v \sum_{j \in S} \phi_j(i, C_j)}{1 + v \sum_{j \in S} \phi_j(i, C_j)}.
\]
Let us consider \( C_j = C \) for all \( j \in S \). Therefore,
\[
\phi_j(i, C_j) = \frac{w_{j,i}}{1 + \sum_{k \in C} w_{j,k}} = \frac{w}{1 + nw}
\]
and
\[
\sum_{j \in S} \phi_j(i, C_j) = \frac{w}{1 + nw} |\{ j \in S : C_j \ni i \}| = \frac{nw}{1 + nw}.
\]
We conclude that
\[
\text{OPT}' \geq \frac{1}{2} \sum_{i \in C} \frac{v \sum_{j \in S} \phi_j(i, C_j)}{1 + v \sum_{j \in S} \phi_j(i, C_j)} = \frac{n}{2} \cdot \frac{nw}{1 + nw + nwv} \approx n \cdot \frac{v}{2(1 + v)}.
\]
We conclude the proof by noting that when \( n \to \infty \)
\[
\frac{\text{OPT}}{\text{OPT}'} \leq \frac{\mathbb{E}[M_C]}{\text{OPT}'} \leq \frac{w/(1 + w)}{v/2(1 + v)} = 2 \cdot \frac{w/(1 + w)}{v/(1 + v)} \leq 2 \cdot \frac{1 + v}{3 + v} \leq 1
\]
where the second inequality comes from \( u \leq v/3 \) and the last inequality follows from \( v \leq 1 \). \( \square \)

**Proof of Proposition 4.** First, let us lower bound \( \text{OPT} \). As in the proof of Proposition 3 we can obtain a lower bound with a feasible solution. Consider \( S_i = S \) for all \( i \in C \), then \( \phi_i(j, S_i) = v/(1 + mw) \) for all \( i \in C \) and \( j \in V \).
\[
\text{OPT} \geq \mathbb{E}[M_S] = \frac{1}{2} \sum_{j \in S} \frac{\sum_{i \in C} w_{j,i} \phi_i(j, S_i)}{1 + \sum_{i \in C} w_{j,i} \phi_i(j, S_i)} = \frac{1}{2} \sum_{j \in S} \frac{wv/(1 + mw)}{1 + wv/(1 + mw)} \geq \frac{m}{2} \cdot \frac{mv}{1 + 2mw},
\]
where in the last inequality we use that \( n \geq m/w \).

Now, let us upper bound \( \text{OPT}' \).
\[
\text{OPT}' = \mathbb{E}[M_C^*] \leq \sum_{i \in C} \frac{\sum_{j \in S} v_{i,j} \phi_j(i, C_j^*)}{1 + \sum_{j \in S} v_{i,j} \phi_j(i, C_j^*)} = \sum_{i \in C} \frac{v \sum_{j \in S} \phi_j(i, C_j^*)}{1 + v \sum_{j \in S} \phi_j(i, C_j^*)}.
\]
As in the previous proofs, we want to maximize the expression on the right. Define \( f(z_1, \ldots, z_n) = \sum_{i \in \mathcal{C}} \frac{w_i}{1 + z_i} \). Note that \( f(.) \) is a Schur-concave function which is maximized when \( z_1 = \ldots = z_n \). Let us analyze if there exists such a solution. Observe that

\[
\phi_j(i, C_j^*) = \frac{w_{j,i}}{1 + \sum_{k \in C_j^*} w_{j,k}} = \frac{w}{1 + |C_j^*|w}.
\]

Since we want to make \( \sum_{j \in \mathcal{S}} \phi_j(i, C_j^*) \) as close to 1 as possible, then let \( \mathcal{C}' \subseteq \mathcal{C} \) a subset of customers such that \( |\mathcal{C}'| = m \) and consider \( C_j^* = \mathcal{C}' \) for all \( j \in \mathcal{S} \). Recall that the maximum expected number of matches is upper bounded by \( \min\{n, m\} = m \) since agents select only one option. Therefore,

\[
\phi_j(i, C_j^*) = \frac{w}{1 + mw} \sum_{j \in \mathcal{C}' \forall i} = \frac{mw}{1 + mw}.
\]

Since only customers in \( \mathcal{C}' \) will get chosen, then

\[
\text{OPT}' \leq \sum_{i \in \mathcal{C}'} \frac{v}{1 + v} \sum_{j \in \mathcal{S}} \phi_j(i, C_j^*) = m \cdot \frac{v mw}{1 + mw + v mw}.
\]

We conclude the proof by noting that

\[
\frac{\text{OPT}}{\text{OPT}'} \geq \frac{m^2 v^2 / (1 + 2 mvw)} {m^2 vw / (1 + mw + v mw)} = \frac{1 + mw + vmw}{2w(1 + 2 mvw)} \xrightarrow{m \to \infty} \frac{1 + v}{4v} \geq 1
\]

where the last inequality follows from \( v \leq 1/3 \).

**Proof of Proposition 8.** Assume \( |\mathcal{C}| = |\mathcal{S}| = n \). Suppose every customer follow the same MNL choice model, i.e. \( v_{ij} = v_j \), with the following scores: there is a single supplier \( j' \) such that \( v_{j'} = n \) and \( v_j = \epsilon = 1/(n - 1) \) for all \( j \neq j' \). For suppliers preferences, assume that every \( j \in \mathcal{S} \) follows a MNL choice model with values \( w_{j,i} = 1 \). Finally, assume \( r_j = 1 \) for all suppliers \( j \in \mathcal{S} \).

In this instance, Algorithm 5 will output \( S_i = \mathcal{S} \) for every customer \( i \in \mathcal{C} \), since weights are all the same and the function \( \sum_{j \in \mathcal{S}} v_j / (1 + \sum_{j \in \mathcal{S}} v_j) \) is maximized when \( S = \mathcal{S} \) due to monotonicity. Now, let us evaluate the objective value of the assortment family \( S = (\mathcal{S}, \ldots, \mathcal{S}) \). Due to Jensen’s inequality we obtain

\[
\mathbb{E}[M_S] \leq \sum_{j \in \mathcal{S}} \frac{n^2}{1 + \sum_{i \in \mathcal{C}} \phi_i(j, \mathcal{S})} = \frac{n^2}{2 + n + n^2} + \frac{ne}{2 + n + ne} \leq 1 + \frac{1}{n}
\]

where the first term corresponds to \( j' \) and the second to the sum over \( j \neq j' \). Also, note that for all \( i \in \mathcal{C} \) we have \( \phi_i(j, \mathcal{S}) = v_j / (1 + \sum_{i \in \mathcal{S}} v_i) = n / (2 + n) \) for \( j = j' \) and \( \phi_i(j, \mathcal{S}) = \epsilon / (2 + n) \) for all \( j \neq j' \).

On the other hand, consider the following assortment family \( S' \): for customer \( i = 1 \) show assortment \( S'_1 = \{j'\} \) and for the rest of the customers \( i \neq 1 \) show assortment \( S'_i = \mathcal{S} \setminus \{j'\} \). Now, we will show that \( \mathbb{E}[M_{S'}] \geq n / 6 \). First, note that for every \( i \neq 1 \) and \( j \neq j' \),

\[
\phi_i(j, \mathcal{S}\setminus \{j'\}) = \frac{v_j}{1 + \sum_{i \in \mathcal{S}\setminus \{j'\}} v_i} = \frac{\epsilon}{1 + \epsilon \cdot (n - 1)}.
\]
which implies that for every $j \neq j'$ we have

$$\sum_{i \in C} \phi_i(j, S_i) = \sum_{i \neq 1} \frac{\epsilon}{1 + \epsilon \cdot (n - 1)} = \frac{(n - 1) \cdot \epsilon}{1 + \epsilon \cdot (n - 1)} = \frac{1}{2}.$$ 

On the other hand, for customer $i = 1$ we have $\phi_1(j', \{j'\}) = n/(n + 1)$. Therefore, Lemma 4 with $\epsilon = 1/2$ implies

$$\mathbb{E}[M_{S'_1}] \geq \frac{1}{2} \left( \frac{\phi_1(j', \{j'\})}{1 + \phi_1(j', \{j'\})} + \sum_{j \neq j'} \frac{\sum_{i \in C} \phi_i(j, S'_i)}{1 + \sum_{i \in C} \phi_i(j, S'_i)} \right) = \frac{1}{2} \left( \frac{n}{1 + 2n + \frac{n - 1}{3}} \right) \geq \frac{n}{6}$$

where the last inequality follows from $n/(1 + 2n) \geq 1/3$. Finally, the guarantee of Algorithm 5 can be upper bounded as follows

$$\frac{\mathbb{E}[M_{S}]}{\text{OPT}} \leq \frac{\mathbb{E}[M_{S}]}{\mathbb{E}[M_{S'}]} = \frac{1 + 1/n}{n/6} \xrightarrow{n \to \infty} 0. \quad \square$$

Appendix D: The Continuous Greedy Algorithm

Notably, Vondrák [Vondrák 2008] introduced the discretized continuous greedy algorithm that achieves a tight $1 - 1/e$ approximate solution. At a high level, the algorithm works as follows: let $F$ be the multilinear extension of a monotone submodular function $f : 2^U \to \mathbb{R}_+$ defined over a space of elements $U$. Let $\mathcal{P}$ be a generic polytope for which maximizing a linear function over $\mathcal{P}$ can be done in polynomial time. The continuous greedy algorithm discretizes the interval $[0, 1]$ into points $\{0, \delta, 2\delta, \ldots, 1\}$ for a given $\delta > 0$. Starting at $y_0 = 0$, for each $\tau \in \{\delta, 2\delta, \ldots, 1\}$ the algorithm obtains direction $z_\tau = \arg \max_{z \in \mathcal{P}} \langle \nabla F(y_{\tau - \delta}), z \rangle$. Then, the algorithm takes a step in the direction of $z_\tau$ by setting $y_{\tau,e} \leftarrow y_{\tau,e} - \delta z_{\tau,e}$ for all components $e \in U$. Finally, it outputs a fractional solution $y_1$ that satisfies $F(y_1) \geq (1 - 1/e - O(\delta)) \cdot \max_{y \in \mathcal{P}} F(y)$. If $\mathcal{P}$ is the convex hull of the independent sets of a matroid $\mathcal{M}$, then there exists a randomized rounding algorithm that given $y_1$ outputs a feasible set $A$ (base of the matroid) such that $f(A) \geq (1 - 1/e) \cdot \max \{ f(B) : B \in \mathcal{M} \}$.