Static elastic shells in Einsteinian and Newtonian gravity

Christiane Maria Losert-Valiente Kroon *
Institut für Theoretische Physik,
Universität Wien,
Boltzmanngasse 5, A-1090 Wien,
Austria.

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Abstract

We study the behaviour of a specific system of relativistic elasticity in its own gravitational field: a static, spherically symmetric shell whose wall is of arbitrary thickness consisting of hyperelastic material. We give the system of field equations and boundary conditions within the framework of the Einsteinian theory of gravity. Furthermore, we analyze the situation in the Newtonian theory of gravity and obtain an existence result valid for small gravitational constants and pointwise stability by using the implicit function theorem. If one replaces the elastic material with a fluid, one finds that stable states can not exist.

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1 Introduction

Let us take any static spherically symmetric shell consisting of an elastic material. Picture the shell to be a sphere from which a sphere of smaller radius is cut out around the center of mass of the bigger sphere and replace the resulting hole by vacuum. The elastic material is assumed to be isotropic and homogeneous. At first, ignore the own gravitational field of the elastic shell. When “switching on” the shell’s self-gravitating field one will observe a deformation of the static elastic shell. In order to describe this process one can make use of either of the following two pictures: the spatial description (Euler picture) or the material description (Lagrange picture). In any of the pictures the undeformed body, \( B \), is represented by a three-dimensional differentiable manifold that is endowed with a flat metric (body metric), \( G_{AB} \), \( A, B = 1, 2, 3 \). The spacetime, \( M \), a four-dimensional manifold, is equipped with the spacetime metric \( g_{\alpha\beta} \), \( \alpha, \beta = 0, 1, 2, 3 \). We assume that there exists a natural state or relaxed state of the static elastic shell, i.e. one should understand this state as to be strain- and hence stressfree. In the spatial description we choose local coordinates \( y^\alpha \) on \( M \) and let the motion of the body be given by a surjective mapping \( z \) — the so-called deformation map — from the spacetime onto the body so that \( Y^A = z^A(y^\alpha) \) are coordinates on \( B \). The field equations describing the behaviour of the static elastic shell viewed from the point of the spatial description are the Einstein field equations with elastic matter source on \( M \). For the material description we have to assume that the spacetime has a foliation consisting of spacelike hypersurfaces. The motion of the body is given by the deformation map \( f \) from the undeformed body onto a submanifold in spacetime, namely \( f(B) \times \mathbb{R} \). The tensor field \( g_{\alpha\beta} \) solves the Einstein vacuum field equations on \( M/f(B) \times \mathbb{R} \) and the Einstein field equations with elastic matter source composed with the deformation map \( f \) on \( f(B) \times \mathbb{R} \). For the static case, the deformation map \( z^A \) of the spatial description is the inverse to the deformation map \( f^a \) — of the material description.

*E-mail address: christianelosert@gmx.at
From the experimental point of view it is clear that stable states for the above described configuration exist for elastic materials but do not exist in the case of fluids. It would be of interest to derive this from the theory of elastomechanics. In [3] an existence proof for general static, self-gravitating elastic bodies in the Newtonian theory of gravity was given which includes the static elastic shell as a special case. Here we follow a different approach to derive an existence result and, in addition, explicitly write down the field equations and boundary conditions for the static elastic shell in both the Einsteinian and Newtonian theory.

In the following section we will introduce some important quantities of elasticity that will simplify the tackling of our setting. We will make some crucial assumptions on the material under consideration. We obtain a nice expression for the energy-momentum tensor in terms of typical objects of elastomechanics and give the Einstein field equations and boundary conditions in the material description. In section 3 we consider our system of equations in the Newtonian limit. Still, the Newtonian field equation and boundary conditions cannot be solved explicitly for general elastic matter. Therefore, section 4 is concerned with an analytical approach to prove that for small gravitational constants and in the case of pointwise stability stable states of the static elastic shell exist. In the appendix, we solve the linearised system of Newtonian equations and illustrate the behaviour of the static elastic shell by means of three concrete examples.

## 2 Field equations and boundary conditions in the Einsteinian theory of gravity

### 2.1 Energy-momentum tensor

We derive the field equations, namely the Einstein equations, from a Lagrangian principle in the spatial description. We aim to present the energy-momentum tensor using quantities within the framework of the theory of elastomechanics. Let us start with the general definition of the energy-momentum tensor:

**Definition.** The energy-momentum tensor is defined by

\[ t_{\alpha\beta} := 2 \frac{\delta L}{\delta g_{\alpha\beta}} - L g_{\alpha\beta}, \]

where \( L \) is the Lagrangian density.

In our case, the Lagrangian density is given by

\[ L = n (\rho c^2 + w), \]

where \( n \) denotes the particle number density. The first term in brackets gives the energy density of the relaxed state and the second term, \( w \), is called the stored energy function density. It is a very important quantity of elastomechanics and we will enlighten this in the few following paragraphs. In general, the stored energy function depends on the deformation map, its derivatives with respect to the coordinates on spacetime and the coordinates on spacetime:

\[ w = w[z^A, \frac{\partial z^B}{\partial y^\alpha}, y^\beta]. \]

One can rewrite the energy-momentum tensor using the expression for the Lagrangian density such that one gets:

\[ t_{\alpha\beta} = Lu_\alpha u_\beta + S_{\alpha\beta}, \]

where \( u^\alpha \) is the 4-velocity — a timelike, future pointing vector field fulfilling \( u^\alpha \frac{\partial z^A}{\partial y^\alpha} = 0 \) and \( g_{\alpha\beta} u^\alpha u^\beta = -1 \). The term \( S_{\alpha\beta} \) is called the stress tensor and satisfies \( S_{\alpha\beta} u^\beta = 0 \). The stored energy function determines the terminology of elastic materials, for example via the following definition:
Definition. If there exists a stored energy function such that the stress tensor takes the form

\[ S_{\alpha\beta} = 2n \frac{\partial w}{\partial g^{\alpha\beta}}, \]

then the material is said to be hyperelastic.

All information about the specific classes of materials considered is contained in the stored energy function. Reminding of the form of the Lagrangian density one finds that we are actually dealing with hyperelastic material. Next, we want to introduce another important object of elasticity that is known as the Cauchy-Green strain tensor. Its name is justified when considering the fact that in spherical symmetry the \((R, R)\) component of the strain tensor \((H_{AB})\) measures the compression, \((H_{RR}) < 0\), as well as the stretching, \((H_{RR}) > 0\), of a material. In general, we have:

Definition. The strain tensor is defined as

\[ H^{AB} := g^{\alpha\beta} \frac{\partial z^A}{\partial y^\alpha} \frac{\partial z^B}{\partial y^\beta}, \]

and

\[ H^A_B = H^{AC} G_{CB}. \]

Covariance of \(w\) under spatial diffeomorphism gives us

\[ w = w[z^A, H^{BC}]. \]

Let us now assume that the stored energy function depends only on some invariants, \(J_i\), of the strain tensor \(H^{AB}\):

\[ w = w[J_i]. \]

We make use of a certain choice of invariants\(^1\) of the strain tensor, \(H^{AB}\), namely:

\[ J_1 = \text{tr}(H^{AB}), \]
\[ J_2 = \frac{1}{2} \left[ \text{tr}(H^{AB})^2 - \text{tr}(H^{AC} H^{CB}) \right], \]
\[ J_3 = \det(H^{AB}) = n^{-2}. \]

At natural points \((y^\alpha)_{|0}, (Y^A)_{|0}, (z^B)_{|0})\) the strain tensor \(H^{AB}\) takes the following expression

\[ H^{AB}_{|0} := H^{AB} \left( \frac{\partial z^C}{\partial y^\alpha} |_0, g^{\mu\nu} (y^\beta |_0) \right) = G^{AB} (Y^C |_0), \]

in other words, the state of the static elastic shell at natural points is strainfree. In the relaxed state the chosen invariants of the strain tensor reduce to the following constants

\[ J_1|_0 = J_2|_0 = 3, \quad J_3|_0 = 1. \]

The stored energy function has to vanish and have a minimum in a locally relaxed state of the matter. From the expansion of the stored energy function at natural points we see that for isotropic, homogeneous materials there are constants \(\lambda\) and \(\mu\) — the Lamé constants — such that

\[ \frac{\partial^2 w}{\partial H^{AB} \partial H^{CD}} = \frac{\lambda}{4 \rho_0} H_{AB} H_{CD} + \frac{\mu}{2 \rho_0} H_{C(A} H_{B)D}. \]

We reduce our analysis to isotropic and homogeneous material. Furthermore, we require our system to fulfill pointwise stability. In the case of isotropic, homogeneous elastic material — see, for example, \[\text{[3]}\] — this condition can be expressed as

\[ \mu > 0, \quad 3\lambda + 2\mu > 0. \]

\(^1\)For the verification of the fact that the square root of the determinant of the strain tensor is equal to the inverse number density we refer to \[\text{[4]}\].
Now, we will use all of the aforementioned assumptions and definitions to rewrite the energy-momentum tensor in terms of objects typical of the theory of elasticity. Due to the form of the boundary conditions it is more convenient to turn to the material description. Therefore, we derive the energy-momentum tensor in the material description, $T^{\alpha\beta}$, from the one of the spatial description, $t_{\alpha\beta}$, and then compose it with the deformation map $f^\alpha$. The energy-momentum tensor in the material description reads

$$ T^{\alpha\beta} = (J_3)^{-1} \left[ (\rho_0 c^2 + w)u^\alpha u^\beta - 2 \frac{\partial w}{\partial \xi_1} G^{AB} + (J_1 G^{AB} - H^{AB}) \frac{\partial w}{\partial \xi_2} + J_3 \tilde{H}^{AB} \frac{\partial f^\alpha}{\partial X^A} \frac{\partial f^\beta}{\partial X^B} \right], $$

where the quantities $J_i - i = 1, 2, 3$ are $w, u, \alpha$ and $H_{AB}$ are to be understood as the analogous objects in the material description to the ones we treated earlier in this section in the spatial description. The symbol $\tilde{H}^{AB}$ is meant to be the inverse of $H_{AB}$. Note that $\tilde{H}^{AB}$ is different to $H^{AB}$. The latter results of rising the indices of the strain tensor in the material description $H_{AB}$ with $G^{AB}$.

### 2.2 Field equations

Having obtained a nice expression for the energy-momentum tensor in the material description we now turn our attention towards the equations describing the process of deformation of the static elastic shell taking into account the influence if the shell’s self-gravitating field. The required field equations in the material description are the Einstein vacuum equations outside the deformed body and the Einstein field equations with an elastic matter source composed with the deformation map on the deformed body. That is, on $\mathbb{M}/f(\mathbb{B}) \times \mathbb{R}$ we have

$$ G^{\alpha\beta}(X^A) = 0, $$

and on $f(\mathbb{B}) \times \mathbb{R}$

$$ G^{\alpha\beta}(f^\gamma(X^A)) = \kappa T^{\alpha\beta}(f^\gamma(X^A)), $$

where $G^{\alpha\beta}$ is the Einstein curvature tensor, $\kappa := \frac{8\pi G}{c^4}$, $G$ is the gravitational constant, $c$ is the speed of light and $T^{\alpha\beta}$ is the energy-momentum tensor.

We choose coordinates $X^A = (R, \theta, \phi)$ on the body $\mathbb{B}$ and introduce on spacetime $\mathbb{M}$ the coordinates $x^a = (ct, x^3)$ where $x^a = f^\alpha(X^A) = (r = F(R), \vartheta = \theta, \varphi = \phi) = F(R)$ being a monotone function. The exterior Schwarzschild metric\(^2\) is matched to the outer boundary of the shell, that is where $R = R_o$ is the outer radius. The hollow region in the centre of the shell (where $R \leq R_i$, $R_i$ is the inner radius) is described by a flat metric. The body metric reads

$$ G_{AB} = \text{diag}(1, R^2, R^2 \sin^2 \theta). $$

We divide the spacetime in three regions corresponding, respectively, to the hollow centre, the deformed body itself and the Schwarzschildian exterior of the shell:

\[
\begin{align*}
1 & \quad g_{\alpha\beta} = \text{diag}(-C, 1, F^2(R), F^2(R) \sin^2 \theta), \\
2 & \quad g_{\alpha\beta} = \text{diag}(-A[F(R)], B[F(R)], F^2(R), F^2(R) \sin^2 \theta), \\
3 & \quad g_{\alpha\beta} = \text{diag}\left((-1 - \frac{2GM}{c^2 F(R)}), \left(1 - \frac{2GM}{c^2 F(R)}\right)^{-1}, F^2(R), F^2(R) \sin^2 \theta\right),
\end{align*}
\]

where $C$ is a positive constant and $M$ is the central mass.

We require asymptotic flatness, that is

\[
\begin{align*}
g_{tt}[F(R)] & \rightarrow -1 \quad \text{as } F \rightarrow \infty, \\
g_{rr}[F(R)] & \rightarrow 1 \quad \text{as } F \rightarrow \infty.
\end{align*}
\]

\(^2\)This makes sense since we will require asymptotic flatness and we, therefore, take into account the Birkhoff’s theorem — see [11].
In a slight abuse of notation we write \( T_{\alpha}^{\rho} = \varepsilon_{\alpha}^{\beta} T^{\beta \rho} \), and get

\[
T_{i}^{t} = -\left[ \frac{dF \sqrt{B F^2}}{dR \sqrt{R^2}} \right]^{-1} (\rho c^2 + w),
\]

\[
T_{r}^{r} = -2 R^2 \frac{dF}{dR} \sqrt{B} \left( \frac{\partial w}{\partial J_1} + 2 \frac{F^2}{R^2} \frac{\partial w}{\partial J_2} + \frac{F^4}{R^4} \frac{\partial w}{\partial J_3} \right),
\]

\[
T_{\theta}^{\phi} = T_{\phi}^{\theta} = -2 \frac{dF}{dR} \sqrt{B} \left[ \frac{\partial w}{\partial J_1} + \left( \frac{dF}{dR} \right)^2 B + \frac{F^2}{R^2} \right] \frac{\partial w}{\partial J_2} + \left( \frac{dF}{dR} \right)^2 B \frac{F^2}{R^2} \frac{\partial w}{\partial J_3} \right].
\]

Keeping these expressions in mind, and after having computed the Einstein curvature tensor we write the Einstein equations on \( f(\mathbb{B}) \times \mathbb{R} \) in the following way:

\[
\kappa T_{i}^{t} = -\frac{\partial B}{\partial F} B^2 F - \frac{1}{2} F^2 + \frac{1}{2} B F^2, \quad (1)
\]

\[
\kappa T_{r}^{r} = -\frac{\partial A}{A B F} - \frac{1}{2} F^2 + \frac{1}{2} B F^2, \quad (2)
\]

\[
\kappa T_{\theta}^{\phi} = \kappa T_{\phi}^{\theta} = -\frac{\partial B}{2 B^2 F} + \frac{\partial A}{2 A B F} + \frac{\partial B}{2 A B F} - \frac{\partial A}{4 A^2 B} F^2 = 0. \quad (3)
\]

If the first and second Einstein equations — \( 1 \) and \( 2 \) — hold then one can, instead of equation \( 3 \), consider — see, for example, \( 11 \) — the only part of the conservation law that is not fulfilled identically, namely:

\[
\nabla_{\alpha} T^{\alpha}_{r} = 0,
\]

which can be written as

\[
\left[ \frac{\partial A}{A} \left( \frac{dF}{dR} \right)^2 B + 2 \frac{\partial B}{\partial F} \left( \frac{dF}{dR} \right)^2 + 2 B \frac{dF}{dR} \left( \frac{dF}{dR} \right)^2 - \frac{\partial B}{\partial F} \frac{dF}{dR} - \frac{2 B}{2 B} \frac{dF}{dR} - \frac{4 F}{R^2} \right] \frac{\partial w}{\partial J_1} + 2 B \frac{dF}{dR} \frac{\partial w}{\partial J_2} + \frac{4 B}{R} \frac{dF}{dR} \frac{\partial w}{\partial J_3} + \frac{4 B}{R} \frac{dF}{dR} \frac{\partial w}{\partial J_3} = 0.
\]

For convenience, we choose equation \( 11 \) to substitute for equation \( 3 \) in our further investigations and consider equation \( 11 \) together with the first and second Einstein field equation, \( 1 \) and \( 2 \), the demanded system of field equations.

### 2.3 Boundary conditions

Clearly, additional equations will have to be fulfilled at the inner and outer boundaries of the deformed body in order to satisfy the standard matching conditions — see, for example, \( 7 \). The fundamental boundary conditions of our system have to be derived from the condition that the first fundamental form, \( g_{\alpha \beta} \), and the second fundamental form, \( K_{\alpha \beta} \), of the three metrics \( \hat{g}_{a b}, \hat{g}_{a b}, \hat{g}_{a b} \) \( (a, b = 1, 2, 3 \text{ spatial indices}) \) have to coincide at the inner and outer boundaries of the
static elastic shell. We consider the hypersurface $\mathcal{H} : F(R) = const$ and the normal vector field of $\mathcal{H}$, $n^\alpha$, with

$$n^\alpha \frac{\partial}{\partial x^\alpha} = (g^{rr})^{\frac{1}{2}} \frac{\partial}{\partial r}.$$  

The induced metric $h_{\alpha\beta}$ on $\mathcal{H}$ is Lorentzian-like, $n_\alpha$ being spatial. Therefore, we have

$$h_{\alpha\beta} dx^\alpha dx^\beta = g_{rr} c^2 dt^2 + F(R)^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  

where $\mathcal{L}$ denotes the Lie derivative. Considering the hypersurface $\mathcal{H}_i : F(R) = F(R_i)$ it follows that

$$A[F(R_i)] = constant,$$  

$$B[F(R_i)] = 1.$$  

(5)  

(6)

The boundary conditions derived from the matching conditions at the hypersurface $\mathcal{H}_o : F(R) = F(R_o)$ read

$$A[F(R_o)] = 1 - \frac{2GM}{c^2 F(R_o)},$$  

$$B[F(R_o)] = A[F(R_o)]^{-1}.$$  

(7)  

(8)

As a consequence of the 3+1 decomposition of the manifold we also have to look on the constraint equations:

$$(^3 R[h] + (K^a_a)^2 - K_{ab}K^{ab} = 2T_{ab}h^a h^b),$$  

$$D^b(K_{ab} - K^c_{ac}h_{ab}) = (\phi^*)^d a(T_{db} n^b),$$  

(9)  

(10)

where $^{(3)}R[h]$ is the Ricci scalar with respect to the three-dimensional metric $h$, $D^d M_{ab} := h^d e h^a h^b \nabla^d M_{a'b'}$ is the projective derivative, and $(\phi^*)^a b$ is the pull-back of the embedding.

The momentum constraint equations (10) are fulfilled identically, whereas the Hamilton constraint equation (9) leads to the following boundary condition:

$$T_{rr} n^r |_{\partial F(R)} = 0,$$

which is equivalent to

$$\frac{\sqrt{B} F^2}{R^2} T_{rr} n^r |_{\partial B} = 0,$$

(11)

where $n_R$ is the normal vector of the hypersurface $R = constant$.

Henceforth, we are able to give the entire system of field equations — (1), (2) and (4) — and boundary conditions — (5), (7) and (11) — of the static elastic shell in its own gravitational field within the framework of the Einsteinian theory of gravity. Let us now investigate the situation in the Newtonian theory of gravity.

### 3 Newtonian limit

Our aim is to derive the system of the Newtonian equations from the results we obtained in the latter section. From the first Einstein equation (11) we derive the following form of the field $B[F(R)]$, namely:

$$B[F(R)] = \left( 1 - \frac{2Gm[F(R)]}{c^2 F(R)} \right)^{-1},$$

where

$$m[F(R)] = \frac{4\pi}{c^2} \int_{R_i}^{R} F^2(\bar{R})(\rho_0 c^2 + w[F(\bar{R})]) d\bar{R}.$$
We insert the latter expression for the unknown field $B[F(R)]$ as well as the following form of the unknown field $A[F(R)]$, namely

$$A[F(R)] = e^2 \exp \frac{2U[F(R)]}{e^2},$$

— where $U$ can be viewed as a potential — into the system of field equations (11), (12), (14) and boundary conditions (13). In order to obtain the system of field equations and boundary conditions in the Newtonian theory of gravity we take the limit $c \to \infty$. In this Newtonian limit the first two Einstein equations (11) and (12) reduce to the Poisson equation

$$\triangle U = 4\pi G(\rho)_{\text{newt}},$$

and the conservation law (14) in the Newtonian limit gives the load equation

$$\frac{2}{F} \left( (T_{r}^r)_{\text{newt}} - (T_{\theta}^\theta)_{\text{newt}} \right) + \left( \frac{dF}{dR} \right)^{-1} \left( \frac{\partial}{\partial R} (T_{r}^r)_{\text{newt}} \right) = -\frac{\partial U}{\partial F} (\rho)_{\text{newt}},$$

where

$$(T_{r}^r)_{\text{newt}} = -2R^2 \left( \frac{dF}{dR} \right)^{-1} \left[ \frac{\partial w}{\partial J_1} + 2\frac{F^2}{R^2} \frac{\partial w}{\partial J_2} + \frac{F^4}{R^4} \frac{\partial w}{\partial J_3} \right],$$

$$(T_{\theta}^\theta)_{\text{newt}} = (T_{r}^\varphi)_{\text{newt}} = -2 \left( \frac{dF}{dR} \right)^{-1} \left[ \frac{\partial w}{\partial J_1} + \left( \frac{F^2}{R^2} \right) \frac{\partial w}{\partial J_2} + \frac{F^2}{R^2} \frac{dF}{dR} \frac{\partial w}{\partial J_3} \right],$$

$$(\rho)_{\text{newt}} = \frac{R^2}{F^2} \left( \frac{dF}{dR} \right) \rho_0.$$

The remaining boundary conditions in the Newtonian limit are

$$\frac{F^2}{R^2} (T_{r}^r)_{\text{newt}} \big|_{\partial B} = 0.$$  

Equations (12), (13) and (14) form the complete system of Newtonian equations for the static elastic shell in its self-gravitating field.

### 4 Main Theorem

The resulting equations of the latter section — (12), (13), (14) — are too complicated to be solved explicitly for general elastic materials. Therefore, we will resort to analytical methods in the sequel. In order to use the machinery of the implicit function theorem we write our field equation and boundary conditions as a map between Sobolev spaces. Integrating the Poisson equation (12) and inserting it into the load equation (13) we find the resulting Newtonian field equation to be an integro-differential equation of the following form:

$$\frac{2}{F} \left( (T_{r}^r)_{\text{newt}} - (T_{\theta}^\theta)_{\text{newt}} \right) + \left( \frac{dF}{dR} \right)^{-1} \left( \frac{\partial}{\partial R} (T_{r}^r)_{\text{newt}} \right) = -\frac{4\pi G R^2 \rho_0}{F^4} \left( \frac{dF}{dR} \right)^{-1} \int_{R_i}^{R} F^2(\bar{R}) d\bar{R}.$$

The latter together with the boundary condition

$$\frac{F^2}{R^2} (T_{r}^r)_{\text{newt}} \big|_{\partial B} = 0$$

leads to modelling the characteristic mapping, $\mathcal{F}$, corresponding to the static elastic shell in the context of the Newtonian material description. It is obtained from the system of field equation and boundary conditions in the Newtonian theory of gravity, that is

**Definition.** $S(\text{tatic})E(\text{lastic})S(\text{hell})$ map:

$$\mathcal{F} : W^{2,2}((R_i, R_o) \times \mathbb{R}) \to W^{2,2}((R_i, R_o) \times \mathbb{R}) \times W^{2,2}((R_i, R_o) \times \mathbb{R})$$

$$[F(R), \mathcal{G}] \mapsto \mathcal{F}[F(R), \mathcal{G}] = (\hat{E}[F(R)] - \mathcal{G} \hat{e}[F(R)], \hat{b}[F(R)]),$$

where $F(R), \mathcal{G}$ are the fields.

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[1] Einstein equations
[2] Poisson equation
where

**Static Elasticity Operator**

\[ \hat{E}[F(R)] := \left[ \frac{4}{3} \frac{d^2 F}{dr^2} + \frac{4}{3} \frac{dF}{dr} \right] + \frac{4}{3} \frac{\partial w}{\partial J_1} + 2 \frac{dF}{dr} \frac{\partial w}{\partial J_2} + \left( \frac{\partial w}{\partial J_3} \right)^2 - \left( \frac{4}{3} \frac{d^2 F}{dr^2} + \frac{4}{3} \frac{dF}{dr} \right)^2 \]

**Force Operator**

\[ \hat{e}[F(R)] := \frac{4 \pi p_0^2}{F^2} \int_{R_i} F^2(R) dR, \]

**Boundary Operator**

\[ \hat{b}[F(R)] := 2 \frac{dF}{dr} \frac{\partial w}{\partial J_1} + 2 \frac{dF}{dr} \frac{\partial w}{\partial J_2} + \frac{4}{3} \frac{\partial w}{\partial J_3} \mid_{\partial B}. \]

In the above definition \( W^{p,k} \) stands for the standard Sobolev spaces of functions having \( p \) weak derivatives in \( L^k \) — see, for example, [2].

**Remark.** The Sobolev spaces were chosen out of technical reasons. The first factor in the range of \( F \) rises from considerations concerning the field equation, the second factor in the range from considerations concerning the boundary conditions. The weight associated with the weak derivative in the first factor of the range comes from the appearance of second derivatives in the static elasticity operator while that of the second factor is dictated by a lemma in [10]. We will make use of this lemma later.

Before giving our main result we will recall one of our crucial assumptions and we will consider some technical lemmata that will be used to prove our main theorem.

**Assumption 1.** We require our system to satisfy pointwise stability. For isotropic linear elasticity pointwise stability — see, for example [3] — is fulfilled when

\[ \mu > 0 \quad 3\kappa = 3\lambda + 2\mu > 0, \]

where \( \lambda \) and \( \mu \) denote the Lamé constants.

**Remark.** The constant \( \kappa \) is called the bulk modulus. The inequality \( 3\kappa > 0 \) allows negative \( \lambda \) with \( -\frac{2}{3} \mu \) as a lower bound (auxetic materials).

**Lemma 1.** \( \mathcal{F} \in C^1 \left( U, W^{0,2}(\{ R_i, R_o \} \times \mathbb{R}) \right) \times W^{\frac{4}{3},2}(\{ R_i, R_o \} \times \mathbb{R}) \), where \( U \) is an open subset of the Sobolev space \( W^{2,2}(\{ R_i, R_o \} \times \mathbb{R}) \).

**Proof.** To check if the SES map is \( C^1 \), we consider the special case of \( m = 0 \) and \( p = 2 \) in the assumptions made in a lemma in [10]. We find that one easily derives the validity of the assertion for the pair of the following operators, namely, the static elasticity operator and the boundary operator. To show that the force operator is \( C^1 \), we make use of a corollary in [1] that reads as follows

**Corollary 1.** If \( \mathcal{F} : U \subset W^{2,2}(\{ R_i, R_o \} \times \mathbb{R}) \rightarrow W^{0,2}(\{ R_i, R_o \} \times \mathbb{R}) \times W^{\frac{4}{3},2}(\{ R_i, R_o \} \times \mathbb{R}) \) is \( C^1 \)-Gâteaux then it is \( C^1 \) and the two derivatives coincide.

Essentially, the force operator corresponds to \( \frac{\partial}{\partial F} U[F(R)] \). The computation of the Gâteaux derivative of \( \frac{\partial}{\partial F} U[F(R)] \) gives us

\[ \frac{d}{d\tau} \left( \frac{\partial}{\partial F} U[F(\tau)](R) \right) \bigg|_{\tau = 0} = D \left( \frac{\partial}{\partial F} U[F(0)](R) \right) \cdot \chi, \]
Lemma 2.

The Gâteaux derivative of $\frac{\partial}{\partial R} U[F(R)]$ reads

\[
\frac{d}{dt}\left(\frac{\partial}{\partial F} U[F(\tau(R))]\right)_{\tau=0} = \frac{4\pi \rho \mu_0}{3} (1 + \frac{R_1^3}{R^3}) \chi = D\left(\frac{\partial}{\partial F} U[F(0)(R)]\right) \cdot \chi.
\]

It is an element of the space of linear continuous maps $\mathcal{L}(W^{2,2}((R_i, R_o) \times \mathbb{R}), W^{0,2}((R_i, R_o) \times \mathbb{R}))$.

Now, $R \mapsto D\left(\frac{\partial}{\partial F} U[F(0)]\right)$ is continuous for $R \in [R_i, R_o]$ since the linear operator defined by the directional derivative is clearly bounded.

Thus, the force operator is $C^1$-Gâteaux, from where it follows that the force operator $\hat{e}[F(R)]$ is $C^1$. Therefore, we derive the SES map $\mathcal{F}[F(R), \mathcal{G}]$ — as defined above — is $C^1$. \(\square\)

Lemma 2. $\mathcal{F}[F_0 = R, G_0 = 0] \equiv (0, 0)$.

Proof. For $\mathcal{F}[F = F_0, G = G_0] \equiv (0, 0)$ we find $F_0$ and $G_0$ to be such that $F_0 = R$ and $G_0 = 0$ as — inter alia — the invariants of $(H^A_B)_{newt}$ on which $w$ depends are constant for $F_0 = R$ and $G_0 = 0$. We have

\[\mathcal{F}[F_0 = R, G_0 = 0] \equiv 0.\]

Lemma 3. $D_F \mathcal{F}[F_0, G_0]$ is an isomorphism from $W^{2,2}((R_i, R_o) \times \mathbb{R})$ onto $W^{0,2}((R_i, R_o) \times \mathbb{R}) \times W^{2,2}((R_i, R_o) \times \mathbb{R})$.

Proof. We divide the proof of this lemma in three parts:

First task. From the Taylor expansion of $\mathcal{F}$ — which is supposed to be at least $C^{r+1}$ — around $R$ for $\delta F$ sufficiently small we see that

\[\mathcal{F}[F_0 + \delta F, G_0] = D_F \mathcal{F}(F_0 = R, G_0 = 0)\delta F + o(||\delta F||^2).
\]

The components of the linearised SES map read

\[
\begin{align*}
\tilde{E}^{lin}[\delta F] &= \left(\frac{d^2}{dR^2}(\delta F) + 2 \frac{d}{dR}(\delta F) - \frac{2(\delta F)}{R^2}\right)(\lambda + 2\mu), \\
\tilde{e}^{lin}[\delta F] &= \frac{4\pi \rho \mu_0}{3R^2}(R^3 - R_i^3), \\
\tilde{b}^{lin}[\delta F] &= \left[\frac{d}{dR}(\delta F)(\lambda + 2\mu) + \frac{2}{R}\lambda(\delta F)\right]_{|_{\partial \Omega}},
\end{align*}
\]

hence $\tilde{E}^{lin}[\delta F, \mathcal{G}] := (\tilde{E}^{lin}[\delta F] - \mathcal{G}^{lin}[\delta F], \tilde{b}^{lin}[\delta F])$.

Neglecting higher orders, we are now able to concentrate on $\mathcal{F}[F_0 + \delta F, G_0]$ to see if $D\mathcal{F}(F_0 = R, G_0 = 0)$ is an isomorphism.

Second task. Next, we will show that $\mathcal{F}[F_0 + \delta F, G_0]$ is injective.

We compute the solution of $\tilde{E}^{lin}[\delta F] = 0$. The latter reads

\[
R^2 \frac{d^2}{dR^2}(\delta F) + 2R \frac{d}{dR}(\delta F) - 2(\delta F) = 0,
\]

which is an ordinary differential equation of the Eulerian type.

The solution of the above equation reads,

\[
\delta F(R) = u_1 R + u_2 R^{-2},
\]
where \( u_k \in \mathbb{R} \) for \( k = 1, 2 \). Inserting the solution \( \delta F \) in the linearised boundary conditions

\[
\left( \frac{d}{dR} (\delta F)(\lambda + 2\mu) + \frac{2(\delta F)}{R}\lambda \right)|_{\partial B} = 0,
\]

we derive

\[
\left( u_1 (3\lambda + 2\mu) - u_2 \frac{4\mu}{R^3} \right)|_{\partial B} = 0.
\]

We can write the latter equations in matricial form as

\[
A \vec{u} = 0, \tag{15}
\]

where

\[
A = \begin{pmatrix}
3\lambda + 2\mu & -4\mu \\
3\lambda + 2\mu & -4\mu
\end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

The determinant of the coefficients matrix of this system of linear equations, \( \det A \), is

\[
\det A = 12\kappa\mu \left( \frac{1}{R_i^3} - \frac{1}{R_o^3} \right),
\]

where \( 3\kappa = 3\lambda + 2\mu \).

Assume for the moment that the above system \( \text{lin} [\delta F] = 0 \) has solutions other than the trivial solution, that is equivalent to the fact that \( \det A = 0 \). If this is the case it follows from pointwise stability that \( R_i = R_o \). However, \( R_i = R_o \) is not an allowed solution. Thus, \( \det A \neq 0 \) and the system \( \text{lin} [\delta F] = 0 \) has only the trivial solution. Therefore, \( \text{lin} F [\delta F] = 0 \) has no kernel for \( G = 0 \) except for the trivial solution. That is, the mapping \( \text{lin} \ F [\delta F] \) is injective for \( G = 0 \).

**Third task.** Now, we will show that \( \text{lin} F [\delta F] \) is surjective for \( G = 0 \).

Let us write

\[
\frac{d^2}{dR^2} (\delta F) + \frac{2}{R} \frac{d}{dR} (\delta F) - 2 \frac{\delta F}{R^2} = \phi(R), \tag{16}
\]

and suppose that

\[
\phi(R) \in W^{0,2}((R_i, R_o)) \times W^{\frac{1}{2},2}((R_i, R_o)).
\]

The solution \( (\delta F)_h(R) \) of the above differential equation, \( \text{eq} \), consists of the sum over the solution \( (\delta F)_h(R) \) of the homogenous differential equation, that is \( \phi(R) = 0 \), and a particular solution \( (\delta F)_p(R) \) of the inhomogenous equation. The solution to the homogenous case as already computed before is

\[
(\delta F)_h(R) = u_1 R + u_2 R^{-2}.
\]

One way of obtaining a particular solution to \( \text{eq} \) is by means of the following ansatz

\[
(\delta F)_p(R) = v(R) R.
\]

After a few calculations we get

\[
v(R) = \int_{R_i}^{R} \frac{1}{R^3} \left( \int_{R_i}^{R} \frac{\phi(\tilde{R})}{\tilde{R}} d\tilde{R} + R_i^3 \frac{d}{dR} v(R_i) \right) d\tilde{R}. \tag{17}
\]

Since we are searching for any particular solution, it is possible to choose the constants of integration in a way that the last term in the above equation vanishes. Therefore, the solution of equation \( \text{eq} \) reads

\[
(\delta F)_p(R) = u_1 R + u_2 R^{-2} + R \int_{R_i}^{R} \frac{1}{R^3} \left( \int_{R_i}^{R} \frac{\phi(\tilde{R})}{\tilde{R}} d\tilde{R} \right) d\tilde{R}. \tag{18}
\]
Concerning \((\delta F)\), using the fundamental theorem of calculus it is easy to see that

\[
\frac{d}{dR}(\delta F)_{\gamma}(R) + \frac{\gamma}{R}(\delta F)_{\gamma}(R)\rvert_{\partial B} = 0,
\]

where \(\gamma = 2\lambda(\lambda + 2\mu)^{-1}\), we can determine the form of the constants \(u_1\) and \(u_2\) in terms of \(v(R_i)\) and \(v(R_o)\):

\[
u_2 = \frac{\gamma}{\gamma - 2} \left( R_o - R_i \right) \left( \frac{d}{dR}v(R_i) - R_o \frac{d}{dR}v(R_o) + (1 + \gamma)(v(R_i) - v(R_o)) \right),
\]

\[
u_1 = \frac{1}{1 + \gamma} \left( u_2(\gamma - 2)R_i^{-3} + R_i \frac{d}{dR}v(R_i) + \gamma v(R_i) \right).
\]

(19)

Let us write

\[
(\delta F)_{\gamma}(R) = (\delta F)_{\gamma_1}(R) + (\delta F)_{\gamma_2}(R) + (\delta F)_{\gamma_3}(R),
\]

where

\[
(\delta F)_{\gamma_1}(R) = u_1(\phi(R_i), \phi(R_o)) R, \\
(\delta F)_{\gamma_2}(R) = u_2(\phi(R_i), \phi(R_o)) R^{-2}, \\
(\delta F)_{\gamma_3}(R) = v(\phi(R)) R.
\]

Using the fundamental theorem of calculus it is easy to see that

\[
(\delta F)_{\gamma_1}(R) \in W^{2,2}\left((R_i, R_o)\right), \quad \text{and} \quad (\delta F)_{\gamma_2}(R) \in W^{2,2}\left((R_i, R_o)\right).
\]

Concerning \((\delta F)_{\gamma_3}(R)\), the following is true

\[
\frac{d}{dR}(\delta F)_{\gamma_3}(R) \in C^0\left((R_i, R_o)\right), \\
\left( \int_{R_i}^{R_o} \frac{d}{dR} (\delta F)_{\gamma_3}(R)^2 dR \right)^{\frac{1}{2}} < \infty.
\]

The same arguments for the second weak derivative of \((\delta F)_{\gamma_3}(R)\) lead to the result that

\[
(\delta F)_{\gamma_3}(R) \in W^{2,2}\left((R_i, R_o)\right).
\]

We conclude that the solution \((\delta F)_{\gamma}(R)\) of the inhomogenous differential equation \((\delta F)_{\gamma}(R) = 0\) with \(\phi(R) \in W^{0,2}\left((R_i, R_o)\right) \times W^{\frac{1}{2},2}\left\{(R_i, R_o)\right\}\) is an element of the Sobolev space as required, namely

\[
(\delta F)_{\gamma}(R) \in W^{2,2}\left((R_i, R_o)\right).
\]

Hence, \(\hat{F}\) is surjective for \(G = 0\).

**Remark.** At this point let us say something about the possibility of taking the limit \(R_i \to 0\), which yields a static elastic sphere. It can be checked that

\[
\phi = O(R^{-p}),
\]

where \(p < 1, \phi \in W^{0,2}\left((R_i, R_o)\right) \times W^{\frac{1}{2},2}\left\{(R_i, R_o)\right\}\). Therefore we conclude from \((\delta F)_{\gamma}(R) = 0\) that

\[
v(R) = \int_{R_i}^{R} R^{-4} \left( \int_{R_i}^{R} \frac{3-p}{R} dR \right)
= \frac{1}{4-p} \left( \frac{R_i^{1-p} - R_o^{1-p}}{1-p} + \frac{R_i^{1-p} - R_o^{1-p}}{3R_i^{1-p}} - \frac{R_i^{1-p} - R_o^{1-p}}{3} \right) = O(R_i^0),
\]

\(\hat{F}\) is surjective for \(G = 0\).
and
\[ v(R_i) = O(R_i^q), \quad R_i v(R_i) = O(R_i^q), \]
where \( q > 0 \). Thus, it follows from \ref{2} that
\[ u_2 \to 0 \quad \text{as} \quad R_i \to 0, \]
and the constant \( u_1 \) is bounded for \( R_i \to 0 \). The solution of the linearised system of equations for the static elastic sphere reads
\[ (\delta F)(R) \mid R \to 0 = u_1(R_i \to 0)R + R \int_0^R \tilde{R}^{-3} \left( \int_0^{\tilde{R}} \phi(R) \right) d\tilde{R}. \]

Again, the linearised mapping \( lin[\delta F] \) for \( G = 0 \) is both injective and surjective. Therefore,
\[ D_F F(F_0, G_0) : W^{2,2}((R_i, R_o), \mathbb{R}) \to W^{0,2}((R_i, R_o), \mathbb{R}) \times W^{2,2}((R_i, R_o), \mathbb{R}) \]
is an isomorphism. \( \square \)

Now we are ready to state and prove our main result.

**Theorem.** Let
\[ F : W^{2,2}((R_i, R_o) \times \mathbb{R}) \to W^{0,2}((R_i, R_o) \times \mathbb{R}) \times W^{2,2}((R_i, R_o) \times \mathbb{R}) \]
\[ [F(R), G] \to \mathcal{F}[F(R), G] = (\tilde{E}[F(R)] - \tilde{G}[F(R)], \tilde{b}[F(R)]) \]
where
\[
\tilde{E}[F(R)] := \left[ 2 \frac{d^2 F}{dR^2} + \frac{4dF}{R^2} \right] F \frac{\partial w}{\partial J_1} + 2 \frac{dF}{dR} \frac{\partial^2 w}{\partial J_2 \partial \tilde{R}} + \frac{4d^2 F}{R^2} \frac{F^2}{2} - \frac{4F^3}{R^4} \frac{\partial w}{\partial J_2} + \frac{4dF}{dR} \frac{F^2}{R^2} \frac{\partial^2 w}{\partial J_2 \partial \tilde{R}} + \frac{2d^2 F}{dR} F^2 \frac{\partial^2 w}{\partial J_3 \partial \tilde{R}} + \frac{4d^2 F}{dR^2} F^2 \frac{\partial w}{\partial J_3} + \frac{2d^2 F}{dR^2} F^2 \frac{\partial^2 w}{\partial J_3 \partial \tilde{R}} \right] \bigg|_{R}.
\]
\[
\tilde{G}[F(R)] := \frac{4\pi \rho_0^2}{F^2} \int_{R_i}^R F^2(R) d\tilde{R},
\]
\[
\tilde{b}[F(R)] := 2 \frac{dF}{dR} \frac{\partial w}{\partial J_1} + 2 \frac{F^2}{R^2} \frac{\partial w}{\partial J_2} + \frac{F^4}{R^4} \frac{\partial w}{\partial J_3} \bigg|_{R}.
\]

Then there exists a neighbourhood \( N_G \) of \( G_0, N_G \subset \mathbb{R} \) and a neighbourhood \( N_F \) of \( F_0, N_F \subset W^{2,2}((R_i, R_o) \times \mathbb{R}) \) and a map \( \tilde{F} \in C^1(N_G, N_F) \) such that
\[
(i) \quad \mathcal{F}[\tilde{F}(R, G), G] = (0, 0) \quad \forall G \in N_G,
\]
\[
(ii) \quad \mathcal{F}[F(R), G] = (0, 0), \quad [F(R), G] \in N_G \times N_F, \quad \text{implies} \ F(R) = \tilde{F}(R, G),
\]
\[
(iii) \quad \tilde{F}(R, G) = \left( D_F \mathcal{F}[\tilde{F}(R, G), G] \right)^{-1} \circ D_G \mathcal{F}[\tilde{F}(R, G), G], \quad \text{where} \ G \in N_G.
\]

**Proof.** The proof of the above theorem is a direct consequence from Lemma 1, Lemma 2 and Lemma 3 as well as from the implicit function theorem — see \[2]. \( \square \)
5 Concluding remarks

From the above theorem it is now clear that as long as pointwise stability is fulfilled, for example perfect fluids will not satisfy this condition, stable states of the static elastic shell in its own gravitational field exist if the body is sufficiently small.

The present work can be extended in several directions. It would be of great interest to provide the analogous analytic proof for the existence of stable states in the theory of Einsteinian gravity. Furthermore, one could undertake some numerical investigations picking out some specific realistic material. It would also be possible to replace the vacuum inside the hollow centre of the shell by some matter, for example air.

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A Solution to the system of linearised equations

The Newtonian system of field equation and boundary conditions is too complicated to be solved for general material, still the linearised equations can easily be solved analytically. In addition, we will list a few examples for which we investigate the behaviour of the static elastic shell or the static elastic sphere as a special case taking into consideration various models of matter. We consider the system of linearised equations

\[
\frac{d^2}{dR^2}(\delta F) + \frac{2}{R} \frac{d}{dR}(\delta F) - \frac{2(\delta F)}{R^2} = \phi(R),
\]

\[
\left( \frac{d}{dR}(\delta F) + \frac{\gamma}{R}(\delta F) \right)_{\partial B} = 0,
\]

where

\[
\phi(R) = K(R - \frac{R_i^3}{R^2}), \quad K = \frac{4\pi \rho_0}{3(\lambda + 2\mu)}, \quad \text{and} \quad \gamma = \frac{2\lambda}{\lambda + 2\mu}.
\]

As we know from section 4, the solution to the above system for a general \(\phi(R)\) — see equations (18), (17) and (19) — reads

\[
(\delta F)(R) = \left( u_1(\phi(R_i), \phi(R_o)) + v(\phi(R)) \right) R + u_2(\phi(R_i), \phi(R_o)) R^{-2}, \tag{20}
\]

where

\[
v(R) = \int_{R_i}^{R} \frac{1}{R^4} \left( \int_{R_i}^{\bar{R}} \frac{3}{R} \phi(\bar{R}) \frac{d}{d\bar{R}} \phi(\bar{R}) \right) d\bar{R},
\]

\[
u_2 = -\frac{R_i^3 R_o^3}{(\gamma - 2)(R_i^6 - R_o^6)} \left( R_i \frac{d}{dR} v(R_i) - R_o \frac{d}{dR} v(R_o) + (1 + \gamma)(v(R_i) - v(R_o)) \right),
\]

\[
u_1 = -\frac{1}{1 + \gamma} \left( u_2(\gamma - 2) R_i^{-3} + R_i \frac{d}{dR} v(R_i) + \gamma v(R_i) \right). \tag{22}
\]

Inserting \(\phi(R) = K(R - \frac{R_i^3}{R^2})\) into (21) we obtain

\[
v(R) = K \left( \frac{R^2}{10} + \frac{R_i^3}{2R} - \frac{R_i^5}{10R^5} - \frac{R_i^2}{2} \right).
\]
And the following holds,
\[
v(R_i) = \frac{d}{dR} v(R_i) = 0,
\]
\[
v(R_o) = \mathcal{K} \left( \frac{R_o^2}{10} + \frac{R_o^3}{2R_o} - \frac{R_o^5}{10R_o^2} - \frac{R_o^2}{2} \right),
\]
\[
\frac{d}{dR} v(R_o) = \mathcal{K} \left( \frac{R_o^3}{5} - \frac{R_o^3}{2R_o} + \frac{3R_o^5}{10R_o^2} \right).
\]

With these results it is clear that
\[
u_2 = \mathcal{K} \frac{R_o^3}{(\gamma - 2)(R_o^3 - R_i^3)} \left( \frac{3 + \gamma}{10} R_o^2 - \frac{\gamma - 2}{10} R_i^2\right) + \frac{2}{2} \left( \frac{1 + \gamma}{2} R_o^2 + \frac{\gamma R_i^2}{2} \right),
\]
\[
u_1 = -\mathcal{K} \frac{R_o^3}{(1 + \gamma)(R_o^3 - R_i^3)} \left( \frac{3 + \gamma}{10} R_o^2 - \frac{\gamma - 2}{10} R_i^2\right) + \frac{2}{2} \left( \frac{1 + \gamma}{2} R_o^2 + \frac{\gamma R_i^2}{2} \right).
\]

After some simplifying computations we have, in the end, the solution to the system of linearised equations, namely
\[
(\delta F)(R) = G \frac{4\pi \rho_o^2}{3(\lambda + 2\mu)} \left( \frac{R_o^3}{10} + \bar{u}_1 R + \bar{u}_2 R^{-2} + \frac{R_i^3}{2} \right),
\]
where
\[
\bar{u}_1 = -\frac{R_o^3}{3\kappa (R_o^3 - R_i^3)} \left( \frac{5\lambda + 6\mu}{10} R_o^2 - \frac{3(5\lambda + 2\mu)}{10} R_i^2 + \frac{R_i^3}{R_o^3} \right),
\]
\[
\bar{u}_2 = \frac{R_o^3}{4\mu (R_o^3 - R_i^3)} \left( \frac{5\lambda + 6\mu}{10} R_o^2 - \frac{3(5\lambda + 2\mu)}{10} R_i^2 + \frac{R_i^3}{R_o^3} \right),
\]
and \(3\kappa = 3\lambda + 2\mu\). Note that for \(R_i = R_o\), \(\bar{u}_1 = \bar{u}_2 = 0\).

Next, we want to give some common examples of elastic materials and investigate in which regions the material is stretched and where it is compressed. The \((R,R)\) component of the strain tensor \(H_{AB}\) measures the compression, \((H_{R,R}) < 0\), as well as the stretching, \((H_{R,R}) > 0\), of the material. For the linearised system of equations of the static elastic shell, we have
\[
(H^R_{R})(R) = \frac{d}{dR}(\delta F)(R) = \frac{G 4\pi \rho_o^2}{3(\lambda + 2\mu)} \left( \frac{3R_o^3}{10} + \bar{u}_1 - 2\bar{u}_2 R^{-3} \right).
\]

**Example 1: Ideal cork.** We consider a material where the Lamé constant \(\lambda = 0\). This is a model for ideal cork\(^3\). For such a material
\[
\bar{u}_1(\lambda = 0) = -\frac{3}{10} R_o^3 \frac{1-b^5}{1-b^3} < 0, \quad \bar{u}_2(\lambda = 0) = \frac{3}{10} R_o^3 \frac{b^5(1-b^2)}{1-b^3} > 0,
\]
where \(R_i = bR_o\), \(b \in [0,1]\). If \(b = 0\) we have the special case of a static elastic sphere — see example 3. We apply Désargues’ rule of signs\(^4\) to the following polynomial:
\[
R^3 + \frac{10}{3} \bar{u}_1 R^3 - \frac{20}{3} \bar{u}_2 = 0,
\]
and see that in this case we have at most one positive zero. Next, we consider \((H^R_{R})(R_i)\) and \((H^R_{R})(R_o)\) and investigate their signs:
\[
(H^R_{R})(R_i) \overset{\lambda=0}{=} 2 \frac{G 4\pi \rho_o^2 R_o^2 b^3 - 1}{1-b^2}, \quad (H^R_{R})(R_o) \overset{\lambda=0}{=} 2 \frac{G 4\pi \rho_o^2 R_o^2 b^3 (b^2 - 1)}{1-b^3} < 0.
\]

\(^3\)See, for example, the following homepage: \texttt{http://home.att.net/\sim\ numericas/answer/physics.htm}.

\(^4\)Désargues’ rule of signs states that for a given polynomial the number of sign changes of the coefficients of the polynomial gives the maximum number of positive roots — see e.g. \(\square\).
Thus, we know that for this particular material, there exists no positive zeros and the material is compressed in the whole shell. Alternatively to the latter discussion, we can investigate what happens to the solution \((\delta F)(R)\) at the inner radius, \(R_i\), and at the outer radius \(R_o\):

\[
(\delta F)(R_i) \xrightleftharpoons[=]{\lambda=0} \frac{G\pi \rho_0^2}{3\mu} \left( \frac{3R_i^3}{5} + \bar{u}_1(\lambda = 0)R_i + \bar{u}_2(\lambda = 0)R_i^{-2} \right)
= \frac{G\pi \rho_0^2 R_i^3 b}{10(1-b^3)}(-2b^5 + 3b^2 - 1) > 0,
\]

\[
(\delta F)(R_o) \xrightleftharpoons[=]{\lambda=0} \frac{G\pi \rho_0^2}{3\mu} \left( \frac{R_o^3 + 5R_o^3}{10} + \bar{u}_1(\lambda = 0)R_o + \bar{u}_2(\lambda = 0)R_o^{-2} \right)
= \frac{G\pi \rho_0^2 R_o^3}{10(1-b^3)}(-10b^6 + 11b^3 + 6b^5 - 4b^4) < 0.
\]

That means that under the influence of the gravitational field of the shell itself, the inner radius increases and the outer radius decreases. Therefore, the body is compressed.

**Example 2: Ideal rubber.** Here, we want to investigate the model for ideal rubber\(^5\), where \(\mu \ll 1\). We have

\[
\bar{u}_1(\mu \ll 1) = -\frac{R_i^2}{6(1-b^3)}(-3b^5 + 2b^3 + 1) < 0,
\]

\[
\bar{u}_2(\mu \ll 1) = -\frac{\lambda R_o^5}{8\mu(1-b^3)}b^3(2b^3 - 3b^2 + 1) < 0 \text{ if } \lambda < 0,
\]

\[
\bar{u}_2(\mu \ll 1) = -\frac{\lambda R_o^5}{8\mu(1-b^3)}b^3(2b^3 - 3b^2 + 1) > 0 \text{ if } \lambda > 0.
\]

From

\[
\gamma_{lin}^R \left( \begin{array}{c} H \\ R \end{array} \right) (R_i) \xrightleftharpoons[<1]{\mu} \frac{G\pi \rho_0^2 R_i^3}{3\mu(1-b^3)}(2b^3 - 3b^2 + 1) < 0,
\]

\[
\gamma_{lin}^R \left( \begin{array}{c} H \\ R \end{array} \right) (R_o) \xrightleftharpoons[<1]{\mu} \frac{G\pi \rho_0^2 R_o^3 b^3}{3\mu(1-b^3)}(2b^3 - 3b^2 + 1) < 0,
\]

we see that for \(\lambda > 0\) the material is compressed in the whole shell.

**Example 3: Elastic sphere.** Specialising on the elastic sphere\(^6\), we let \(R_i \to 0\) and see that

\[
\bar{u}_1(R_i \to 0) = -\frac{R_i^2}{16}(1 + \frac{2\kappa^2}{3\kappa}) < 0, \quad \text{and} \quad \bar{u}_2(R_i \to 0) = 0,
\]

where \(c_2^2 = \lambda + 2\mu\).

**Remark.** The constant \(c_2\) is the speed of the propagation of the progressive, strongly elliptic wave — see, for example, [8].

Example 3 has exactly one positive zero. In the case of the sphere the solution reduces to

\[
(\delta F)_{R_i \to 0}(R) = \frac{4\pi \rho_0^2}{30c_2^2} \left( R^3 - RR^2(1 + \frac{2\kappa^2}{3\kappa}) \right)
\]

Considering the latter equation, we see that within a sphere of radius \(R = R_o\sqrt{\frac{1}{3}(1 + c_2^2)}\) the material is compressed since

\[
\gamma_{lin}^R \left( \begin{array}{c} H \\ R \end{array} \right) \xrightleftharpoons[<0]{\mu} R_i \to 0,
\]

whereas outside this sphere the material is stretched:

\[
\gamma_{lin}^R \left( \begin{array}{c} H \\ R \end{array} \right) \xrightleftharpoons[>0]{\mu} R_i \to 0.
\]

\(^5\)See [http://home.att.net/~numericana/answer/physics.htm].

\(^6\)This special case has already been considered in [8] and [8].
Note that if \( \lambda = 0 \) (ideal cork) we have the following

\[
\left( \frac{\text{lin}}{\text{lin}} H R \right) (R_o) \xrightarrow{R \to 0} 0.
\]

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