CONTINUOUS QUIVERS OF TYPE A (I)
THE GENERALIZED BARCODE THEOREM

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ABSTRACT. We generalize type A quivers to continuous type A quivers and prove basic results about pointwise finite-dimensional representations. In particular, we generalize Crawley-Boevey’s Barcode theorem to continuous quivers with alternating orientations: every pointwise finite-dimensional representation of a continuous type A quiver is the direct sum of pointwise one-dimensional indecomposable whose supports are intervals. We also classify the indecomposable projective representations. This is part of a longer work in which we study a generalization of the continuous cluster category (introduced by the first and last author in 2015) and a continuous generalization of mutation.

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INTRODUCTION

History. We generalize type A quivers to continuous quivers of type A and study its representations. These generalize representations of the real line which are the basis for the continuous cluster category of $\mathbb{I}_T$. The fundamental theorem for representations of $\mathbb{R}$ is called the Barcode theorem.

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proven in [C-B]. It states that every pointwise finite representation of $\mathbb{R}$ is a sum of indecomposable representations which are supported on intervals in $\mathbb{R}$. The discrete version of this theorem is the basic result in persistent homology used in topological data analysis [ZC]. The basic example is given by the Vietoris-Rips complex. (The earliest reference to using the complex this way that the authors could find are Carlsson, Ishkhanov, de Silva, and Zomorodian [CIdSZ], and Chazal and Oudot [CO], both in 2008.) This is the functor which assigns to each finite subset $X$ of $\mathbb{R}^n$ and every nonnegative real number $r$ the homology of the set of all points which are a distance $\leq r$ of a point in $X$. For any $r \leq s$ we get a map in homology $H_*(r) \to H_*(s)$. So, we get a (graded) representation of $\mathbb{R}$. The support intervals of the indecomposable components are half-closed $[a, b)$ and the collection of these intervals is called the barcode of $X$. For more general representations of $\mathbb{R}$ the support intervals can be any interval: $(a, b), (a, b], [a, b)$ or $[a, b)$. Carlsson, de Silva, and Morozov introduced zigzag persistent homology in [CGSM] and Botnan proved a similar decomposition theorem to Crawley-Boevey’s for zigzag persistence in [B].

Persistent homology has recently been used to study fractal dimension [AA et all] $S$ and has been shown to be effective in recovering some signals in noise [JS]. Persistent homology has been applied to 3D shape classifications [CC-SGMO], the study of plant root systems [EH], identification of breast cancer subtypes [NLC], and many other real world applications.

**This Paper.** In 2017 the third author taught a 5-day course on the material in [IT]. The second author asked, naively, about using all the intervals and other orientations. This work is partially the result of that answering these questions. In the present paper we consider an alternating orientation so long as $S$ does not have accumulation points and provide the Generalized BarCode Theorem.

**Theorem A** (Theorem 2.3.2). The representations $M_I$ are indecomposable and any pointwise one-dimensional indecomposable representation of $A_{\mathbb{R}}$ is isomorphic to $M_I$ for some interval $I \subseteq \mathbb{R}$. Let $V$ and $V'$ be indecomposable representations of a continuous type $A$ quiver. Then $V \cong V'$ if and only if supp $V = $ supp $V'$.

We allow for any alternating orientation so long as $S$ does not have accumulation points and provide the Generalized BarCode Theorem.

**Theorem B** (Theorem 2.4.13). Let $V$ be a pointwise finite-dimensional representation of a continuous type $A$ quiver. Then $V$ is a direct sum of interval indecomposables.

Instrumental in proving the theorems above is the following theorem about projective representations in the category of pointwise finite-dimensional representations, denoted $\text{Rep}^{pwf}_k(A_{\mathbb{R}})$.

**Theorem C** (Theorem 2.1.16). Let $s_0 \leq a < s_1$ with $s_0$ a sink and $s_1$ the next source. Let $P$ be a pointwise finite-dimensional representation of $A_{\mathbb{R}}$ with supp $P \subset [s_0, a]$.

1. Then $P$ is projective in $\text{Rep}^{pwf}_k(A_{\mathbb{R}})$ if and only if all maps $P(x, s_0) : P(x) \to P(s_0)$ are injective for all $x \in \text{supp} P$.

2. Every projective representation in $\text{Rep}^{pwf}_k(A_{\mathbb{R}})$ with support in $[s_0, a]$ is a finite direct sum of representations of the forms $P_b$ and $P_b'$ for $s_0 \leq b \leq a$. 
(2') Every projective representation in $\text{Rep}_{\text{pwf}}(A_{\mathbb{R}})$ with support in $(s_0,a]$ (i.e., $s_0 = -\infty$) is a possibly infinite direct sum of representations of the forms $P_b$ and $P_{b'}$ for $s_0 < b \leq a$.

Finally, in Section 3 we prove properties about the category of finitely presented representations (Definition 3.1.3) over any continuous quiver of type $A$. Some of the properties extended to pointwise finite-dimensional representations and bounded-dimensional representations (Definition 1.3.1), denoted $\text{Rep}_{b}^k(A_{\mathbb{R}})$.

**Theorem D** (Theorem 3.0.1). Let $A_{\mathbb{R}}$ be a continuous quiver of type $A$. Then the following hold.

1. For indecomposable representations $M_I$ and $M_J$ in $\text{Rep}_{\text{pwf}}(A_{\mathbb{R}})$, $\text{Rep}_{b}^k(A_{\mathbb{R}})$, or $\text{rep}_k(A_{\mathbb{R}})$, we have $\text{Hom}(M_I, M_J) \cong k$ or $\text{Hom}(M_I, M_J) = 0$ (Proposition 3.1.2).
2. Every morphism $f : V \to W$ in $\text{Rep}_{\text{pwf}}(A_{\mathbb{R}})$, $\text{Rep}_{b}^k(A_{\mathbb{R}})$, or $\text{rep}_k(A_{\mathbb{R}})$ has a kernel, a cokernel, and coinciding image and coimage in that category. (Lemma 3.1.4)
3. The category $\text{rep}_k(A_{\mathbb{R}})$ is Krull-Schmidt, but not artinian (Lemma 3.1.5, Proposition 3.1.6).
4. The global dimension of $\text{rep}_k(A_{\mathbb{R}})$ is 1 (Proposition 3.2.5).
5. The $\text{Ext}$ space of two indecomposables $M_I$ and $M_J$ in $\text{Rep}_{\text{pwf}}(A_{\mathbb{R}})$, $\text{Rep}_{b}^k(A_{\mathbb{R}})$, or $\text{rep}_k(A_{\mathbb{R}})$ is either isomorphic to $k$ or is 0 (Proposition 3.2.6).
6. While some Auslander-Reiten sequences exist (Proposition 3.3.2), some indecomposables have neither a left nor a right Auslander-Reiten sequence (Proposition 3.3.3).

**Future Work.** This is the beginning of a longer work in which a generalized version of the continuous cluster category and a continuous generalization of mutation are studied. The key differences in the continuous cluster categories are: (1) we are considering alternating orientation of $\mathbb{R}$ and (2) we have four types of intervals $[a, b)$, i.e. $(a, b)$, $(a, b]$, $[a, b]$ and $[a, b)$, as opposed to only the half-closed intervals $(a, b]$ considered in [IT].

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1. **Continuous Quivers of Type $A$**

We let $k$ denote a field for the entirety of this paper.

1.1. **Quiver of Continuous Type $A$: $A_{\mathbb{R}}$**. The goal of this section is to generalize the definition of type $A$ quivers to a continuous setting. The set $\mathbb{R}$ will serve as the vertices in our quiver. We will choose a set of sinks and sources, which will induce the orientation on the continuous quiver by indicating which vertices have paths to which others. The picture below gives an intuitive idea of the result of choosing a continuous type $A$ quiver and the definition follows.

![Continuous Quiver Diagram](image-url)

**Definition 1.1.1.** A quiver of continuous type $A$, denoted by $A_{\mathbb{R}}$, is a triple ($\mathbb{R}, S, \preceq$), where:

1. (a) $S \subset \mathbb{R}$ is a discrete subset, possibly empty, with no accumulation points.
   (b) Order on $S \cup \{\pm \infty\}$ is induced by the order of $\mathbb{R}$, and $-\infty < s < +\infty$ for $\forall s \in S$. 
Elements of \( S \cup \{\pm \infty\} \) are indexed by a subset of \( \mathbb{Z} \cup \{\pm \infty\} \) so that \( s_n \) denotes the element of \( S \cup \{\pm \infty\} \) with index \( n \). The indexing must adhere to the following two conditions:

1. There exists \( s_0 \in S \cup \{\pm \infty\} \).
2. If \( m \leq n \in \mathbb{Z} \cup \{\pm \infty\} \) and \( s_m, s_n \in S \cup \{\pm \infty\} \) then for all \( p \in \mathbb{Z} \cup \{\pm \infty\} \) such that \( m \leq p \leq n \) the element \( s_p \) is in \( S \cup \{\pm \infty\} \).

(2) New partial order \( \preceq \) on \( \mathbb{R} \), which we call the orientation of \( A_\mathbb{R} \), is defined as:

- \( p_1 \) The \( \preceq \) order between consecutive elements of \( S \cup \{\pm \infty\} \) does not change.
- \( p_2 \) Order reverses at each element of \( S \).
- \( p_3 \) If \( n \) is even \( s_n \) is a sink.
- \( p_3' \) If \( n \) is odd \( s_n \) is a source.

**Definition 1.1.2.** Let \( A_\mathbb{R} = (\mathbb{R}, S, \preceq) \) be a quiver of continuous type \( A \). Then the associated continuous path algebra \( kA_\mathbb{R} \) is the associative algebra over \( k \) (without unity) whose basis consists of pairs \((x, y)\), where \( y \preceq x \). Multiplication on the pairs is given by

\[
(w, x)(y, z) = \begin{cases} (w, z) & x = y \\ (w, z) & x \neq y \end{cases}
\]

**Remark 1.1.3.** The indexing requirements on \( S \) have the following immediate consequences.

- If \( S \) is empty then either (i) \( s_{-1} = -\infty \) and \( s_0 = +\infty \) or (ii) \( s_0 = -\infty \) and \( s_1 = +\infty \).
- If \( S \) is unbounded above (below) then \( +\infty = s_{+\infty} (-\infty = s_{-\infty}) \).
- If \( S \) is bounded above (below) then there is no \( s_{+\infty} (s_{-\infty}) \) in \( S \).

The rules for the partial order have the following consequences. If \( x < y \in \mathbb{R} \) and there is some \( s_n \in S \) such that \( x < s_n < y \) then \( x \npreceq y \) and \( y \npreceq x \). If \( x \leq y \in \mathbb{R} \) and there exists \( s_n, s_{n+1} \in S \) such that \( s_n \leq x \leq y \leq s_{n+1} \) then:

\[
\begin{align*}
x \preceq y & \text{ if } n \text{ is even} \\
y \preceq x & \text{ if } n \text{ is odd.}
\end{align*}
\]

**Example 1.1.4.** We provide four examples of \( S \) and the induced partial order \( \preceq \) on \( \mathbb{R} \).

1. A finite example: \( S = \{\frac{1}{2}, \pi\} \), \( \bar{S} = \{-\infty, \frac{1}{2}, \pi, +\infty\} \), \( s_{-2} = -\infty \), \( s_{-1} = \frac{1}{2} \), \( s_0 = \pi \), and \( s_1 = +\infty \).

2. A “half” unbounded example: \( S = \{2n : n \in \mathbb{N}\} \), \( s_{-1} = -\infty \), \( s_n = 2n \) when \( n \geq 0 \), and \( s_{+\infty} = +\infty \).

3. An unbounded example: \( S = \{\frac{n}{2} : n \in \mathbb{Z}\} \), \( s_{-\infty} = -\infty \), \( s_n = \frac{n}{2} \), and \( s_{+\infty} = +\infty \).

4. One of the two \( S = \emptyset \) possibilities: \( S = \emptyset \), \( s_0 = -\infty \), and \( s_1 = +\infty \). This causes \( \preceq \) to coincide with \( \leq \).

**Remark 1.1.5.** It is important to note that the choice which element of \( S \) becomes \( s_0 \) determines the entire indexing of \( S \) and thus the entire partial order \( \preceq \). Additionally, given a set \( S \) there are exactly two partial orders \( \preceq \) possible no matter which element of \( S \) is chosen to be \( s_0 \). The two partial orders are opposites of each other.
Remark 1.1.6. From now on, whenever we refer to $A_{\mathbb{R}}$, we are implicitly assuming some $S$ with indexing and $\leq$ have been set. By ‘the straight descending orientation’ we mean the one where $S = \emptyset$, $s_0 = -\infty$, and $s_1 = +\infty$ as in Example 1.1.4. This is the case where $\leq$ coincides with $\preceq$.

1.2. Representations of $A_{\mathbb{R}}$: $\text{Rep}_k(A_{\mathbb{R}})$.

Definition 1.2.1. A representation $V$ of $A_{\mathbb{R}}$ is a module over the path algebra $kA_{\mathbb{R}}$. Explicitly, one assigns to each real number $x$ a vector space $V(x)$ and to each pair $(x, y)$, where $y \leq x$, a linear transformation $V(x, y) : V(x) \rightarrow V(y)$ such that $V(y, z) \circ V(x, y) = V(x, z)$ whenever such a composition is defined. The support of a representation $V$ is the set of all $x \in \mathbb{R}$ such that $V(x) \neq 0$. We denote the support of a representation $V$ by $\text{supp} V$.

A simple representation at $x$ is a representation $V$ such that $V(x) \cong k$ and if $y \neq x$ then $V(y) = 0$. The linear map $V(x, x)$ is the identity and $V(y, z) = 0$ if $y \neq x$ or $z \neq x$.

Definition 1.2.2. A morphism $f : V \rightarrow W$ of representations of $A_{\mathbb{R}}$ is a morphism of $kA_{\mathbb{R}}$ modules. Explicitly, it is a collection of linear maps $f(x) : V(x) \rightarrow W(x)$, for all $x \in \mathbb{R}$, making the following squares commute for each pair $x, y \in \mathbb{R}$ where $y \leq x$:

$$
\begin{array}{ccc}
V(x) & V(y) \\
V(x, y) & V(y) \\
W(x) & W(y).
\end{array}
$$

Since we’re working with modules over an associative algebra, and associative algebras are in particular rings, the category of $k$-representations of $A_{\mathbb{R}}$, denoted $\text{Rep}_k(A_{\mathbb{R}})$, is abelian.

Propositions 1.2.3 and 1.2.4 can be proved almost the exact same way as they would for discrete quivers of type $A$.

Proposition 1.2.3. A morphism of representations $f : V \rightarrow W$ in $\text{Rep}_k(A_{\mathbb{R}})$ is an isomorphism if and only if $f(x)$ is an isomorphism for each $x \in \mathbb{R}$.

Proposition 1.2.4. Let $V$ and $V'$ be representations of $A_{\mathbb{R}}$ such that $V \cong V'$. Then $\text{supp} V = \text{supp} V'$.

1.3. The Subcategories $\text{Rep}_{k_{\text{pwf}}}(A_{\mathbb{R}})$ and $\text{Rep}_{k_{\text{b}}}(A_{\mathbb{R}})$. In this subsection we define the pointwise finite and bounded subcategories of $\text{Rep}_k(A_{\mathbb{R}})$. We provide examples of representations in each subcategory and highlight the differences between them.

Definition 1.3.1. The category of pointwise finite representations, denoted $\text{Rep}_{k_{\text{pwf}}}(A_{\mathbb{R}})$, is the full subcategory of $\text{Rep}_k(A_{\mathbb{R}})$ consisting of representations $V$ such that for all $x \in \mathbb{R}$, $\dim V(x) < \infty$.

The category of bounded representations, denoted $\text{Rep}_{k_{\text{b}}}(A_{\mathbb{R}})$, is the full subcategory of $\text{Rep}_{k_{\text{pwf}}}(A_{\mathbb{R}})$ whose objects are representations $V$ such that there exists $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, $\dim V(x) < n$.

It is important to note that the conditions in Definition 1.3.1 are not related to the support of any representation. I.e. there exist representations in both $\text{Rep}_{k_{\text{pwf}}}(A_{\mathbb{R}})$ and $\text{Rep}_{k_{\text{b}}}(A_{\mathbb{R}})$ with unbounded support. Such examples are provided below.

Example 1.3.2. We now give some examples of representations in $\text{Rep}_{k_{\text{pwf}}}(A_{\mathbb{R}})$ and $\text{Rep}_{k_{\text{b}}}(A_{\mathbb{R}})$. Each representation will be over $A_{\mathbb{R}}$ with the straight descending orientation (see Remark 1.1.6).

1. We give an example of a representation in $\text{Rep}_{k_{\text{b}}}(A_{\mathbb{R}})$ with unbounded support. A representation in $\text{Rep}_{k_{\text{b}}}(A_{\mathbb{R}})$ is $V$:

$$
V(x) = \begin{cases} 
k & x \geq 0 \\
0 & x < 0
\end{cases}
$$

$$
V(x, y) = \begin{cases} 
1_k & 0 \leq y \leq x \\
0 & \text{otherwise}
\end{cases}
$$
Notice that the support of $V$ is unbounded. This is fine. The dimension of all the $V(x)$ vector spaces is bounded above by 1.

(2) We now give an example of an infinite coproduct that is still in $\text{Rep}^\text{bf}_k(A_\mathbb{R})$. Let $M = \bigoplus M(z)$ where for each $z \in \mathbb{R}$, $M(z)$ be the following representation of $A_\mathbb{R}$:

$$M(z)(x) = \begin{cases} k & x = z \\ 0 & \text{otherwise} \end{cases} \quad M(z)(x, y) = \begin{cases} 1_k & x = y = z \\ 0 & \text{otherwise} \end{cases}$$

That is, $M(z)$ is the simple representation at $z$, which is clearly in $\text{Rep}^\text{bf}_k(A_\mathbb{R})$. However, $M = \bigoplus_{z \in \mathbb{R}} M(z)$ is also in $\text{Rep}^\text{bf}_k(A_\mathbb{R})$ since $\dim M(x) = 1$ for all $x \in \mathbb{R}$.

(3) We now give an example of a representation in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ but not in $\text{Rep}^\text{bf}_k(A_\mathbb{R})$. Let $W$ be the representation of $A_\mathbb{R}$ where $W(x)$ is $k^n$ where $n = 0$ if $x < 1$ and $n$ is the largest integer less than or equal to $x$ otherwise. I.e., $W(10.4) = k^{10}$. Let $W(x, y)$ be 0 if $y < 1$ or $x < 1$. Otherwise, $W(x, y)$ is the projection of the first $\dim W(x)$ coordinates of $k^{\dim W(x)}$ using the standard basis. For example, $W(10, 4)$ is the projection of $k^{10}$ onto the first 4 coordinates. While $W(x)$ is finite-dimensional for all $x \in \mathbb{R}$, there is no $n$ such that $\dim W(x) \leq n$ for all $x \in \mathbb{R}$.

Originally, the authors only attempted to prove a version of Theorem 2.4.13 for $\text{Rep}^\text{bf}_k(A_\mathbb{R})$. However, it was noted that nearly all the proof techniques relied on finite-dimensional vector spaces, not on the dimension of the vector spaces being bounded. In the category $\text{Rep}^\text{bf}_k(A_\mathbb{R})$ the authors discovered projective indecomposable objects that are not projective in $\text{Rep}^\text{bf}_k(A_\mathbb{R})$. Further study revealed these objects to also be projective in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$. See Section 2.1 for details on these new projective objects. These new projectives in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ are necessary to obtain a category of finitely generated representations (Definition 3.1.3, denoted $\text{rep}_k(A_\mathbb{R})$) which has all the reasonable properties one could expect from a continuous version of finitely generated representations.

In contrast to the apparent superiority of $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$, the category is simply too big to even have all projective covers. While pathological examples of representations without projective covers can be constructed in both $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ and $\text{Rep}^\text{bf}_k(A_\mathbb{R})$, the more well-behaved examples of representations without projective covers exist only in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$. See Example 2.1.17 in Section 2.1. Such a representation does not exist in $\text{Rep}^\text{bf}_k(A_\mathbb{R})$ and so this can be considered the first step towards finitely generated representations.

2. Generalized Barcode Theorem

2.1. Projectives. We will construct all pointwise finite-dimensional projective representations in the category $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$. There are two types of indecomposable projectives: projectives $P_c$ generated at one point as in Definition 2.1.3 which are quite similar to the projectives for finite quivers. These kinds of projectives are actually projective in both $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ and in $\text{Rep}_k(A_\mathbb{R})$. The new kind of projectives $P_c$ and $P_c$ will be projectives which have half open intervals as supports as in Definition 2.1.13. These representations are projective in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ but are not projective in $\text{Rep}_k(A_\mathbb{R})$. We start with the easy case of a projective generated at one point.

Definition 2.1.1. Given any point $c \in \mathbb{R}$ and any vector space $X$ over $k$, let $(PX)_c$ be the representation defined as follows.

$$(PX)_c(x) = \begin{cases} X & \text{if } x \leq c \\ 0 & \text{otherwise} \end{cases}$$

and $(PX)_c(x, y) = \text{id}_X$ if $y \leq x \leq c$. 


Lemma 2.1.2. For any representation \( V \) in \( \text{Rep}_k(A_\mathbb{R}) \) (not necessarily pointwise finite) and any \( k \)-vector space \( X \) we have:
\[
\text{Hom}((PX)_c, V) = \text{Hom}_k(X, V(c)),
\]
i.e., the functor which takes \( X \) to \((PX)_c\) is left adjoint to the evaluation functor \( V \mapsto V(c) \).

Proof. Given any morphism \( f : (PX)_c \to V \), let \( f_c : (PX)_c(c) = X \to V(c) \) be the restriction of \( f \) to the point \( c \). Then, for any \( x \leq c \), the commutativity of the diagram:
\[
\begin{array}{ccc}
(PX)_c & \xrightarrow{id_X} & X \\
\downarrow{f_c} & & \downarrow{f_x} \\
V(c) & \rightarrow & V(x)
\end{array}
\]
forces the map \( f_x : (PX)_c(x) \to X(x) \) to be equal to \( V(c,x) \circ f_c \). Conversely, any linear map \( g : X \to V(c) \) extends to a morphism \( \overline{g} : (PX)_c \to V \) by the same formula \( \overline{g}(x) = V(c,x) \circ g : (PX)_c(x) = X \to X(x)) \). \( \square \)

Theorem 2.1.3. For any vector space \( X \) and any \( c \in \mathbb{R} \), the representation \( (PX)_c \) is projective in \( \text{Rep}_k(A_\mathbb{R}) \).

Proof. Let \( p : V \to W \) be an epimorphism and let \( f : (PX)_c \to W \) be any morphism. Then \( p_c : V(c) \to W(c) \) is an epimorphism. So, the linear map \( f_c : X \to W(c) \) lifts to a map \( g : X \to V(c) \) which, by Lemma 2.1.2, extends to a morphism \( \overline{g} : (PX)_c \to V \). Since \( p \circ \overline{g} \) and \( f \) agree at \( c \), they are equal by Lemma 2.1.2. So, \( (PX)_c \) is projective. \( \square \)

It is clear that \( (PX)_c \) is indecomposable if and only if \( X \) is one-dimensional as it is in the following definition. In this case the indecomposable projective is denoted simply by \( P_c \).

Definition 2.1.4. Let \( c \in \mathbb{R} \). Define the representation \( P_c \) in \( \text{Rep}_k(A_\mathbb{R}) \) as:
\[
P_c(x) = \begin{cases} k & \text{if } x \leq c \\ 0 & \text{otherwise} \end{cases}, \quad P_c(x,y) = \begin{cases} 1_k & \text{if } y \leq x \leq c \\ 0 & \text{otherwise} \end{cases}
\]

The rest of this subsection is devoted to the construction of all pointwise finite-dimensional projective representation, including objects \( P_{(a,b)} \) for \( s_{2n-1} < a < s_{2n} \) \( b < s_{2n+1} \) with supports \( (a, s_{2n}) \) and \( [s_{2n}, b) \) respectively.

In order to describe these new types of projective representations in the category of pointwise finite-dimensional representations of \( A_\mathbb{R} \), we need to set up notation of “image filtration” (Definition 2.1.5) and “support intervals” (Definition 2.1.7). Recall \( s_n \) is a sink if \( n \) is even and a source if \( n \) is odd.

Definition 2.1.5. Let \( V \) be a pointwise finite-dimensional representation of \( A_\mathbb{R} \) such that \( \text{supp} V \subset [s_0, s_1] \). (If \( s_0 = -\infty \) or \( s_1 = +\infty \) then \( \text{supp} V \) is a subset of \( (s_0, s_1] \), \( [s_0, s_1) \), or \( (s_0, s_1) \), whichever applies.) Let \( b = s_0 \) or \( b \in (s_0, s_1) \).

Let \( b \) be the greatest lower bound of \( \text{supp} V \). When \( b \in \text{supp} V \) we set \( V^\bullet(b) = V(b) \). The image filtration of \( V^\bullet(b) \) is the set of distinct subspaces of the form \( V(x, b)(V(x)) \). Let \( V^\circ(b) \) be the colimit of the vector spaces \( V(x) \), for \( b < x < s_1 \), with the linear maps \( V(x,y) \), for \( b < y \leq x < s_1 \). Since each \( V(x) \) is finite-dimensional, \( V^\circ(b) \) is at most countably infinite dimensional. Denote by \( V^\circ(x, b) \) the colimit linear map from \( V(x) \) to \( V^\circ(b) \). The image filtration of \( V^\circ(b) \) is the set of distinct (finite-dimensional) subspaces of the form \( V^\circ(x, b)(V(x)) \).

When \( b \in \text{supp} V \) we take \( I \) to be \([b, c]\) or \([b, c)\). When \( b \notin \text{supp} V \) we take \( I \) to be \((b, c]\) or \((b, c)\).

For all such \( I \) and when \( b \in \text{supp} V \), let
\[
(1) \quad V_I^\bullet(b) := \bigcap_{x \in I} V(x, b)(V(x)) \subset V^\bullet(b)
\]
Whether or not \( b \in \text{supp} \, V \), we let
\[
V_{I \setminus \{b\}}^\circ (b) := \bigcap_{x \in I \setminus \{b\}} V^\circ(x, b)(V(x)) \subset V^\circ(b).
\]
Then \( V_I^\bullet (b) \) and \( V_I^\circ (b) \) are members of the image filtrations of \( V^\bullet (b) \) and \( V^\circ (b) \), respectively. In particular, there exists \( x_0 \in I \) such that \( V_I^\bullet (b) = V(x_0, b)(V(x_0)) \) or \( x_0 \in I \setminus b \) such that \( V_I^\circ (b) = V^\circ(x_0, b)(V(x_0)) \). Whenever \( b \in \text{supp} \, V \), \( V^\bullet (b) \) is finite-dimensional and so the image filtration is finite. Since \( V^\bullet (b) \) may not be finite-dimensional and the dimension of the vector spaces \( V(x) \) are not bounded the filtration on \( V^\circ (b) \) may be infinite but still countable with a minimal term. In fact, \( V(s_1, b)(b) \) and \( V^\circ(s_1, b) \) are the minimal objects in the filtrations of \( V^\bullet (b) \) and \( V^\circ (b) \), respectively.

**Remark 2.1.6.** (a) If \( I = [b, c] \ (c \in I) \) then
\[
V_I^\bullet (b) = V_{[b,c]}^\bullet (b) = V(c, b)(V(c))
\]
\[
V_{I \setminus \{b\}}^\circ (b) = V_{[b,c]}^\circ (b) = V^\circ(c, b)(V(c)).
\]
(b) If \( I = [b, c] \ (c \notin I) \) then whenever \( c \in \mathbb{R} \) we have
\[
V_I^\bullet (b) = V_{[b,c]}^\bullet (b) \supset V(c, b)(V(c))
\]
\[
V_{I \setminus \{b\}}^\circ (b) = V_{[b,c]}^\circ (b) \supset V^\circ(c, b)(V(c))
\]
but in both cases the subspaces may be different.

(c) For any \( x < z \) in \( I \), the term \( V(x, b)(V(x)) \) is redundant in the intersection \( \square \) since \( V(x, b)(V(x)) \supset V(z, b)(V(z)) \). Thus,
\[
V_I^\bullet (b) = \bigcap_{x \in I} V(x, b)(V(x)) = \bigcap_{y \in I, y \geq z} V(z, b)V(y, z)(V(y)) = V(z, b)V_{I \setminus [z,s]}^\bullet (z)
\]
\[ (c') \text{ For any } x < z \text{ in } I, \text{ the term } V^\circ(x, b)(V(x)) \text{ is redundant in the intersection } \square \text{ since } \]
\[ V^\circ(x, b)(V(x)) \supset V^\circ(z, b)(V(z)). \text{ Thus, } \]
\[
V_I^\circ (b) = \bigcap_{x \in I} V^\circ(x, b)(V(x)) = \bigcap_{y \in I, y \geq z} V^\circ(z, b)V^\circ(y, z)(V(y)) = V^\circ(z, b)V_{I \setminus [z,s]}^\circ (z)
\]

**Definition 2.1.7.** Let \( W \subset V^\bullet (b) \) be a subspace. Define \( I_W \) as:
\[
I_W = \{ x \geq b \in \text{supp} \, V \ | \ W \subset V(x, b)(V(x)) \}
\]
Such \( \{I_W\} \) are called support intervals for \( V^\bullet (b) \).

Let \( W \subset V^\circ (b) \) be a finite-dimensional subspace. Then we define \( I_W \) similarly for \( b \notin I \):
\[
I_W = \{ x > b \in \text{supp} \, V \ | \ W \subset V^\circ(x, b)(V(x)) \}
\]
These \( \{I_W\} \) are also called support intervals for \( V^\circ (b) \).

**Proposition 2.1.8.** (a) There is a 1-1 correspondence between support intervals for \( V^\bullet (b) \) and the terms in the image filtration of \( V^\bullet (b) \) given by \( I \mapsto I^\bullet_I \subset V^\bullet (b) \) and \( W \mapsto I_W \).

(a') There is also a 1-1 correspondence between the support intervals for \( V^\circ (b) \) and the terms in the image filtration of \( V^\circ (b) \) given by \( I \mapsto I^\circ_I \subset V^\circ (b) \) and \( W \mapsto I_W \).

**Proof.** We first prove (a). If \( W = V(x_0, b)(V(x_0)) \) then \( I_W \) contains \( x_0 \) and \( V_{I_W}^\bullet \) is the intersection of \( V(x_0, b)(V(x_0)) = W \) and the subspaces \( V(x, b)(V(x)) \) which all contains \( W \) by definition. So, \( V_{I_W}^\bullet = W \).

If \( I = I_W \) then \( W \subset I^\bullet_I \) by definition. If \( V(x, b)(V(x)) \) contains \( V_{I_W}^\bullet \), it contains \( W \). So, \( I^\bullet_I \subset I_W = I \). But \( I \subset I^\bullet_I \). So, \( I = I^\bullet_I \) for any support interval \( I \).

The proof of (a) as stated works for (a') if we replace \( V_I^\bullet \) with \( V_I^\circ \). \( \square \)
Remark 2.1.9. The image filtration of $V^\bullet(b)$ can be written:
\[
V^\bullet(b) \supseteq V(x_n, b)(V(x_n)) \supseteq V(x_{n-1}, b)(V(x_{n-1})) \supseteq \cdots \supseteq V(x_1, b)(V(x_1)).
\]
By Proposition 2.1.8 we see this is actually a filtration
\[
(*) \quad V^\bullet(b) \supseteq V_{I_n}^\bullet \supseteq \cdots \supseteq V_{I_1}^\bullet,
\]
where each $x_i$ in the first form is an element of $I_i$ in the second form.

For the image filtration of $V^\circ(b)$, we have the following equivalent forms, where each $x_i$ in the first form is an element of $I_i$ in the second form:
\[
V^\circ(b) \cdots \supseteq V^\circ(x_n, b)(V(x_n)) \supseteq V^\circ(x_{n-1}, b)(V(x_{n-1})) \supseteq \cdots \supseteq V^\circ(x_1, b)(V(x_1)).
\]
By Proposition 2.1.8 we see this is actually a filtration
\[
(\circ*) \quad V^\circ(b) \cdots \supseteq V_{I_n}^\circ \supseteq V_{I_{n-1}^\circ} \supseteq \cdots \supseteq V_{I_1}^\circ.
\]

Lemma 2.1.10. (a) Let $W' \subseteq W$ be consecutive terms in the image filtration $(*)$ of $V^\bullet(b)$. Then $I_W \subseteq I_{W'}$ and, for any element $x \in (I_{W'} - I_W)$, we have $V(x, b)(V(x)) = W'$ in $V^\bullet(b)$.

(a') Let $W' \subseteq W$ be consecutive terms in the image filtration $(\circ*)$ of $V^\circ(b)$. Then $I_W \subseteq I_{W'}$ and, for any element $x \in (I_{W'} - I_W)$, we have $V^\circ(x, b)(V(x)) = W'$ in $V^\circ(b)$.

Proof. We prove (a) first. Since $x \notin I_W$ it follows from Remark 2.1.6(c) that $W$ is not a subset of $V(x, b)(V(x))$ in $V^\bullet(b)$. Since $x \in I_{W'}$ it follows that $W' \subset V(x, b)(V(x))$. Thus $W' \subset V(x, b)(V(x)) \subseteq W$. Since $W, W'$ are consecutive in the image filtration, $V(x, b)(V(x)) = W'$ as claimed. The proof of (a) as stated works for (a') by replacing $V^\bullet$ with $V^\circ$.

Lemma 2.1.11. Let $W \subset V^\bullet(b)$ (resp. $W \subset V^\circ(b)$) be a finite-dimensional subspace and let $x_1 < x_2 \in I_W$. Let $I_1 = I_W \cap [x_1, s_1]$ and $I_2 = I_W \cap [x_2, s_1]$. (If $s_1 = +\infty$ then, for each $i$, $I_i = I_W \cap [x_i, s_1]$). Then $V_{I_2}^\bullet(x_1) \subset V(x_2, x_1)(V_{I_2}^\bullet(x_2))$ (resp. $V_{I_2}^\circ(x_1) \subset V(x_2, x_1)(V_{I_2}^\circ(x_2))$).

Proof. This is a special case of Remarks 2.1.6(c) and 2.1.6(c').

Lemma 2.1.12.

(a) For any interval $I$ of the form $[b, c]$ or $(b, c)$ and any element $v \in V^\bullet(b)$, there is a collection of elements $\{V_x^\bullet \in V(x)\}_{x \in I}$ so that $V_b^\bullet = v$ and $V(y, x)(V_x^\bullet) = V_x^\bullet$ for all $b \leq x \leq y \in I$.

(a') For any interval $I$ of the form $[b, c]$ or $(b, c)$ and any element $v \in V^\circ(b)$, there is a collection of elements $\{v_x \in V(x)\}_{x \in I}$ so that $v_b = v$ and $V(y, x)(v_x) = v_x$ and $V^\circ(x, b)(v_x) = v_b$ for all $b < x \leq y \in I$.

Proof. If $I = [b, c]$ then $V_{I_2}^\bullet(b) = V(c, b)(V(c))$ and we can choose one element $w \in V^\bullet(c)$ so that $V(c, b)(w) = v$ and let $v_x = V(c, x)(w)$ for all $x \in [b, c]$. Similarly, if $I = (b, c]$ then $V_{I_2}^\circ(b) = V^\circ(c, b)(V(c))$ and we make a similar choice.

Otherwise, $I = [b, c]$ or $I = (b, c)$ for some $c > b$. In this case, choose an increasing sequence of real numbers $b < x_1 < x_2 < x_3 < \cdots$ in $(b, c)$ converging to $c$. Let $J_0 = I$, for each $l > 0$, let $J_l = I \cap [x_l, c)$ and editorially, let $v_l \in V_{J_l}^\bullet(x_l)$ (resp. $v_l \in V_{J_l}^\circ(x_l)$) be chosen recursively as follows.

(1) Set $v_0 = v \in V_{I_2}^\bullet(b)$, (resp. $v_0 = v \in V_{I_2}^\circ(b)$).

(2) Given $v_k$ in $V_{J_k}^\bullet(x_k)$ (resp. in $V_{J_k}^\circ(x_k)$), by Lemma 2.1.11 there exists $v_{k+1}$ in $V_{J_{k+1}}^\bullet(x_{k+1})$ (resp. $V_{J_{k+1}}^\circ(x_{k+1})$) so that $V(x_{k+1}, x_k)(v_{k+1}) = v_k$ (resp. $V^\circ(x_{k+1}, x_k)(v_{k+1}) = v_k$).

After this sequence of elements $v_k$ is chosen, the vector $v_x$ for any $x \in I$ is given by $v_x = V(x, x)(v_x)$ for any $x < x$. This is well defined by condition (2) in the case of $V^\bullet(b)$ and by condition (2) combined with the universal property of $V^\circ(b)$ in that case.
Definition 2.1.13. Let $s_0$ be a sink or $-\infty$ and let $s_1 > s_0$ be the next source or $+\infty$. Let $s_0 < a < s_1$. For $I = [s_0, a]$ or $[s_0, a)$ let $P_I$; also written $P_a = P_{[s_0, a]}$ or $P_a = P_{(s_0, a)}$, denote the representation with support $I$ so that $P_I(x)$ is one-dimensional with generator $v_x$ for all $x \in I$ and $P_I(y)(v_y) = v_x$ for all $x < y \in I$.

For $a = s_1$, define $P_a$ as before. However, when $a = s_1$, $P_a$ is not defined this way. If $s_0 = -\infty$ then $P_a$ and $P_a$ are instead $P_{[s_0, a]}$ and $P_{(s_0, a)}$, respectively.

Proposition 2.1.14. $P_a$ and $P_a$ as in Definition 2.1.13 are projective in $\text{Rep}_{k}^{\text{pwf}}(A_{\mathbb{R}})$.

Proof. We first assume that $s_0 \in \mathbb{R}$. To show that $P_I$ is projective it suffices to show that any epimorphism $p : E \to P_I$ has a section. Let $W \subset E^*(s_0)$ be the smallest term in the image filtration of $E^*(s_0)$ which maps onto $P_I^*(s_0)$.

Claim: $I_W$ contains $I$ and thus $W \subset E_I^*(s_0)$.

Proof: For each $x \in I$, there is a $w \in E(x)$ so that $p(x)(w) = v_x \in P_I(x)$. But then $p_{w_0}E^*(x, s_0)(w) = P_I^*(x, s_0)(v_x) = v_{s_0} \neq 0$. So, $W \subset E^*(x, s_0)(E(x))$ which implies $x \in I_W$. Since this holds for all $x \in I$ we get that $I \subset I_W$.

By construction of $W$, there is a $w \in W \subset E_I^*(s_0)$ so that $p(w) = v_{s_0}$. By Lemma 2.1.12 there are elements $w_x \in E(x)$ for all $x \in I$ so that $E(x, y)(w_x) = w_y$ for all $y \leq x \in I$. Then, a section $s : P \to E$ is given by $s(x) = w_x$ for all $x \in I$.

If we instead assume $s_0 = -\infty$ then above we replace $E^*$ with $E^\circ$ and $P^*$ with $P^\circ$ where appropriate. By the universal property of colimits, the map on representations induces a map $E^\circ(-\infty) \to P_I^\circ(-\infty)$. Then the rest of the proof holds as stated. □

Lemma 2.1.15. Suppose $s_0 = -\infty$ and $P$ is a pointwise finite-dimensional representation with support $(s_0, a)$, for $a \leq s_1$, or $(s_0, a]$, for $a < s_1$. Either $P^\circ(s_0)$ is finite-dimensional or $P^\circ(s_0) \cong k^\infty$.

Proof. Since $P$ is pointwise finite-dimensional, if the dimension of $P(x)$ is bounded by some $n$ for all $x \in \mathbb{R}$ then $P^\circ(s_0)$ is also bounded by $n$.

Now suppose $P^\circ(s_0)$ is not finite-dimensional. For each $i > 0$, let $n_i = \dim P^\circ_i$. Let $e_i \in k^\infty$ denote the unit vector with a 1 in the $i$th coordinate. For a choice of basis of $P^\circ(s_0)$, we note that since each morphism is a monomorphism and the image filtration $(\circ s)$ of $P^\circ(s_0)$ has a minimal element, we may inductively choose a basis on $P^\circ(s_0)$. We do this by first choosing a basis of $P^\circ_1$, then completing it to a a basis of $P^\circ_2$, and so on. Since each $P^\circ_i$ is finite-dimensional this is well defined.

Since we have a consistent choice of bases, map the chosen basis of each $P^\circ_i$ to the collection $\{e_i\} \subset k^\infty$ in a consistent way. Since each $P^\circ_i \cong P(x_i)$ this induces a map $P^\circ(s_0) \to k^\infty$.

To see the map is surjective take any element $w$ of $k^\infty$; $w$ has finitely many nonzero coordinates. Thus it is some linear combination of finitely many $e_j$’s. Then there is a $P^\circ_i$ whose basis contains enough elements to surject on to the $e_j$’s. Thus there is an element $v$ in $P^\circ_i$ such that $v \mapsto w$ and so there is an element $\tilde{v} \in P^\circ(s_0)$ that maps to $w$. The map is injective since if $\tilde{v} \neq \tilde{v}'$ in $P^\circ(s_0)$ then there is a pair $v \neq v'$ in $P^\circ_i$ such that $v \mapsto \tilde{v}$ and $v' \mapsto \tilde{v}'$. We know $v$ and $v'$ map to different elements in $k^\infty$ so $\tilde{v}$ and $\tilde{v}'$ must also. Therefore, $P^\circ(s_0) \cong k^\infty$. □

The following theorem will give a characterization of one sided projective objects in $\text{Rep}_{k}^{\text{pwf}}(A_{\mathbb{R}})$.

Theorem 2.1.16. Let $s_0 \leq a < s_1$ with $s_0$ a sink and $s_1$ the next source. Let $P$ be a pointwise finite-dimensional representation of $A_{\mathbb{R}}$ with supp $P \subset [s_0, a]$.

1. Then $P$ is projective in $\text{Rep}_{k}^{\text{pwf}}(A_{\mathbb{R}})$ if and only if all maps $P(x, s_0) : P(x) \to P(s_0)$ are injective for all $x \in \text{supp}P$.

2. Every projective representation in $\text{Rep}_{k}^{\text{pwf}}(A_{\mathbb{R}})$ with support in $[s_0, a]$ is a finite direct sum of representations of the forms $P_b$ and $P_b$ for $s_0 \leq b \leq a$. 

(2') Every projective representation in $\text{Rep}^{\text{pwf}}_k(A_\mathbb{R})$ with support in $(s_0, a]$ (i.e., $s_0 = -\infty$) is a possibly infinite direct sum of representations of the forms $P_b$ and $P_{b'}$ for $s_0 < b \leq a$.

**Proof.** When $a = s_0$, statements (1) are (2) are trivially true and statement (2') does not apply.

(1) Suppose that there is some $x_0 \in [b, a] \subset [s_0, a]$ so that $P(x_0, s_0) : P(x_0) \to P(s_0)$ is not injective. Then we will show that $P$ is not projective. Indeed consider the quotient object $Q$ given $Q(x) = P(x)$ for all $x \geq x_0$ and $Q(x) = 0$ for all $x < x_0$. We have an epimorphism $\pi : P \to Q$.

Let $\tilde{Q}$ be the representation given by $\tilde{Q}(x) = P(x)$ for $x \geq x_0$ and $\tilde{Q}(x) = P(x_0)$ for all $x \leq x_0$ with $\tilde{Q}(y, x) = \text{Id}$, the identity, when $x, y \leq x_0$ and $\tilde{Q}(y, x) = P(y, x_0)$ when $x \leq x_0 < y$. Let $p : \tilde{Q} \to Q$ be the projection map. Claim: the quotient map $\pi : P \to Q$ does not lift to $\tilde{Q}$, i.e. there is no $\gamma : P \to \tilde{Q}$ such that $p \circ \gamma = \pi$. Proof of claim: Since $\pi_{x_0} = \text{Id} : P(x_0) = Q(x_0)$ and $p_{x_0} = \text{Id} : \tilde{Q}(x_0) \to Q(x_0)$ we would have $\gamma_{x_0} = \text{Id}$. But that gives a contradiction to the basic property of maps between representations: $\gamma_{x_0} \circ P(x_0, s_0) = \tilde{Q}(x_0, s_0) \circ \gamma_{x_0} = \text{Id}$, but $\gamma_{x_0} \circ P(x_0, s_0)$ is not injective by assumption. Therefore, $P$ is not projective.

Conversely, suppose that all morphisms $P(x, s_0)$ are monomorphisms. Choose a basis $B$ for $P(s_0)$ compatible with the image filtration. Then, a subset $B_i$ of $B$ is a basis for each subspace $P_{J_i}(s_0)$ in the image filtration of $P(s_0)$ where $J_i = I_{P_{J_i}(s_0)}$ are ordered by inclusion: $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$. Then $P_{J_i}(s_0) \supseteq P_{J_i}(s_0) \supseteq \cdots \supseteq P_{J_i}(s_0)$.

(2) By Lemma 2.1.12 every $v \in B_i - B_{i+1}$ lifts to a compatible system of elements $v_x \in P(x)$ for all $x \in J_i$. By Lemma 2.1.10 $P(x, s_0)(P(x)) = P_{J_i}(s_0)$ for all $x \in J_i - J_{i-1}$ (where $J_0 = \emptyset$). Since $P(x, s_0)$ is a monomorphism, the liftings $v_x \in P(x)$ for all $v \in B_i$ form a basis for $P(x)$ for all $x \in J_i - J_{i-1}$. For each $v \in B_i$, the lifting $v_x$ of $v$ generate a pointwise one-dimensional subrepresentation $Q_v$ of $P$ with support in $J_i$, i.e. $P_0$ has the form $P_0$ or $P_{b'}$ depending on whether $J_i = [s_0, b]$ or $[s_0, b)$. The $Q_v$, for $v \in B$, are disjoint and generate all of $P(x)$ for every $x \in [s_0, a]$. Thus $P$ is a direct sum of the $Q_v$, as claimed.

Below is an example of such a decomposition for (2).

\[ [s_0, b_1] = \text{supp}Q_{v_1} \quad \quad [s_0, b_2] = \text{supp}Q_{v_2} \quad \quad [s_0, b_3] = \text{supp}Q_{v_3} \quad \quad [s_0, b_4] = \text{supp}Q_{v_4} \quad \quad [s_0, b_5] = \text{supp}Q_{v_5} \]

With the exception of choosing a basis, we may apply all of the argument for statement (2) to statement (2'). If $P^o(s_0)$ is finite-dimensional we get a basis and apply the argument for (2). By Lemma 2.1.15 if $P^o(s_0)$ is infinite-dimensional then it is isomorphic to $k^\infty$ and by the proof of the same lemma we have a basis that respects the filtration. We then apply the argument for (2).

**Example 2.1.17.** Let $A_\mathbb{R}$ have the straight descending orientation and for each positive integer $n$ let $V_n$ be the following representation:

\[ V_n(x) = \begin{cases} k & n \leq x \\ 0 & \text{otherwise} \end{cases} \quad \quad V_n(x, y) = \begin{cases} 1_k & n \leq y \leq x \\ 0 & \text{otherwise} \end{cases} \]

Using Theorem 2.1.16 we see that the projective cover of each $V_n$ is the projective indecomposable with support $\mathbb{R} = (-\infty, +\infty)$.

Note that $V = \bigoplus V_n$ is still pointwise finite. One can check it is isomorphic to the representation $W$ (item (3) in Example 1.3.2). However, the projective cover is infinitely many copies of the indecomposable projective with support $(-\infty, +\infty)$, which is not pointwise finite-dimensional. Therefore, this rather tame example does not have a projective cover in $\text{Rep}^{\text{pwf}}_k(A_\mathbb{R})$. 


However, the dually constructed representation $V'$ (each $V'_n$ has support $(-\infty, n]$) is its own projective cover by Theorem 2.1.16 and so does have a projective cover. While $V$ and $V'$ exist in $\text{Rep}_k^{\text{proj}}(A_{\mathbb{R}})$, neither exists in $\text{Rep}_k^{\text{proj}}(A_{\mathbb{R}})$. So, this type of asymmetry does not happen in $\text{Rep}_k^{\text{proj}}(A_{\mathbb{R}})$.

2.2. Sufficient Conditions for Indecomposables. Here we give sufficient conditions for pointwise finite-dimensional representations to be indecomposable. In Section 2.4 we will show that these conditions are also necessary.

**Proposition 2.2.1.** Let $V$ be a representation of $A_{\mathbb{R}}$ such that

1. $\dim V(x) \leq 1$ for all $x \in \mathbb{R}$,
2. if $V(x) \neq 0 \neq V(z)$ and $x \leq y \leq z$ in $\mathbb{R}$ then $V(y) \neq 0$, and
3. if $V(x) \neq 0 \neq V(y)$ and $x \leq y$ then $V(y, x)$ is an isomorphism.

Then $V$ is indecomposable.

**Proof.** Suppose $V$ is not indecomposable for contradiction. Then $V \cong W_1 \oplus W_2$ with $W_1 \neq 0 \neq W_2$. Then $\text{supp} W_1 \cap \text{supp} W_2 = \emptyset$ and $\text{supp} W_1 \cup \text{supp} W_2 = \text{supp} V$. Since $W_1, W_2 \neq 0$ there exist $x_1 \in \text{supp} W_1$ and $x_2 \in \text{supp} W_2$. By symmetry we may assume $x_1 < x_2$.

Claim 1: There are only finitely many elements of $S$ in the open interval $(x_1, x_2)$.

Pf: This follows from the fact that $(x_1, x_2)$ is a bounded interval, i.e., $[x_1, x_2]$ is compact. If $S \cap [x_1, x_2]$ were infinite, it would contain a converging sequence (any infinite subset of a compact set contains a converging sequence). By definition, $S$ does not contain a converging sequence. So, $S \cap [x_1, x_2]$ is finite. A fortiori, $S \cap (x_1, x_2)$ is finite.

Claim 2: There exist $x_1 \in \text{supp} W_1$ and $x_2 \in \text{supp} W_2$ such that $S \cap (x_1, x_2) = \emptyset$.

Pf: Let $n = \#\{S \cap (x_1, x_2)\}$. If $n \geq 1$ we will find another pair $x'_1 < x'_2$ in the respective supports of $W_1, W_2$ so that $\#\{S \cap (x'_1, x'_2)\} < \#\{S \cap (x_1, x_2)\} = n$. This will imply that $n = 0$.

To find this second pair $x'_1, x'_2$ choose any element $s_k$ in $S \cap (x_1, x_2)$ which is nonempty by assumption that $n \geq 1$. Then $s_k$ is in the support of $W_1$ or $W_2$. In the first case, $x'_1 = s_k, x'_2 = x_2$ gives the desired pair. Indeed, in this case, $S \cap (x'_1, x'_2) \subset S \cap (x_1, x_2)$ since $s_k \in S \cap (x_1, x_2)$ but $s_k \notin S \cap (x'_1, x'_2)$. Also, $x'_1 = s_k$ is in the support of $W_1$ by assumption and $x'_2 = x_2$ is in the support of $W_2$. The second case is similar. In both cases, the value of $n$ can be reduced if it is positive. So, the minimal value of $n$ is 0.

By Claim 2 we may assume there are no elements of $S$ between $x_1$ and $x_2$, i.e. the $\prec$ orientation of $S$ is constant in the closed interval $[x_1, x_2]$ and either $V(x_2, x_1)$ or $V(x_1, x_2)$ is an isomorphism.

In the first case, we consider the projection $V \to W_1$ and in the second case we consider the other projection $f : V \to W_2$. By symmetry, we may take the first case, i.e. $V(x_2, x_1)$ is an isomorphism. Then we have the following commuting diagram:

$$
\begin{array}{ccc}
V(x_1) & \overset{V(x_2, x_1)}{\cong} & V(x_2) \\
\cong f_{x_1} & & f_{x_2} \\
W_1(x_1) & \longrightarrow & W_1(x_2) = 0
\end{array}
$$

Since $x_1 \in \text{supp} W_1$, it follows that $f_{x_1} : V(x_1) \to W_1(x_1)$ is an isomorphism. But $W_1(x_2) = 0$ since $x_2 \in \text{supp} W_2$ and $x_2 \notin \text{supp} W_1$. The commutativity of the diagram then gives a contradiction. Thus $V$ is indecomposable. □

**Definition 2.2.2.** For any interval $I$ in $\mathbb{R}$ let $M_I$ be the representation of $A_{\mathbb{R}}$ given as follows.

$$
M_I(x) = \begin{cases} 
  k & x \in I \\
  0 & \text{otherwise}
\end{cases} \quad M_I(x, y) = \begin{cases} 
  1_k & y \preceq x \text{ and } x, y \in I \\
  0 & \text{otherwise}
\end{cases}
$$

The conditions of Proposition 2.2.1 are clearly satisfied. So, $M_I$ is indecomposable. If a representation $V \cong M_I$ we call $V$ an **interval indecomposable** or **interval indecomposable representation**.
Corollary 2.2.3. Let $V$ be an indecomposable representation which is pointwise one-dimensional (satisfies the conditions of Proposition 2.2.1). Let $J \subseteq \text{supp} V$ be a connected subset and let $V_J$ be the restriction of $V$ to $J$, i.e. $V_J(x) = V(x) = k$ for all $x \in J$ and $V_J(x) = 0$ for all $x \notin J$, and $V_J(y, x) = V(y, x)$ for all $y, x \in J$. Then $V_J$ is an indecomposable representation.

Proof. We will show that conditions (1), (2) and (3) of Proposition 2.2.1 are satisfied by the representation $V_J$.

(1) By definition of $V_J$ it follows that $\dim_k V_J(x) \leq 1$.

(2) This follows since $J$ is connected subset of $\text{supp} V$.

(3) Suppose there is $x \preceq y$ with $x, y \in J$ such that $V_J(y, x) = V(y, x)$ not an isomorphism. Since $\dim_k V_J(x) \leq 1$, this is equivalent to $V_J(y, x) = 0$.

Let $I = \{ t \mid x \prec t \preceq y \text{ such that } V(y, t) \neq 0 \}$. Then $x \in (\text{supp} V) \setminus I = J_1 \cup J_2$ where $J_1 \cap J_2 = \emptyset$ and we may assume $x \in J_1$. Then $V_{J_1}$ is a subrepresentation of $V$ but is also a quotient of $V$ since the map $\pi : V \to V_{J_1}$ defined as $\pi_x = \text{Id}_{V_J(x)}$ for $x \in J_1$ and $\pi_x = 0$ for $x \notin J_1$ is a representation homomorphism using the fact that $V(t_2, t_1) = 0$ for all $t_2 \in (\text{supp} V) \setminus J_1$ and all $t_1 \in J_1$. Actually $\pi$ is a splitting for the inclusion $V_{J_1} \to V$, contradicting the assumption that $V$ is indecomposable. Therefore (3) holds for $V_J$.

So by Proposition 2.2.1 it follows that $V_J$ is indecomposable.

\square

2.3. Filtrations. In this section will provide some lemmas necessary for Section 2.4. In both this section and in Section 2.4 we will be using notation $\text{Hom}(\_, \_)$ for $\text{Hom}_{\text{Rep}_k^{\text{pwf}}(A_{\mathbb{R}})}(\_, \_)$ and $\text{End}(\_)$ for $\text{End}_{\text{Rep}_k^{\text{pwf}}(A_{\mathbb{R}})}(\_)$ where $\text{Rep}_k^{\text{pwf}}(A_{\mathbb{R}})$ is the full subcategory of $\text{Rep}_k(A_{\mathbb{R}})$ whose objects are all pointwise finite representations of $A_{\mathbb{R}}$.

Lemma 2.3.1. Let $V$ be an indecomposable pointwise one-dimensional representation. Then the endomorphism ring of $V$ is the field $k$.

Proof. Let $x_0 \in \text{supp} V$. By definition, $V(x_0) \cong k$. Choose a morphism $f(x_0) : V(x_0) \to V(x_0)$.

Claim: if $f(x_0) \neq 0$ this determines an isomorphism $V \cong V$.

Since $V(x_0) \cong k$, $f(x_0)$ is an isomorphism. If $y \in \text{supp} V$ such that $y \preceq x_0$ then $V(x_0, y)$ is an isomorphism. So, for all $y \preceq x_0$ in $\text{supp} V$, define

$$f(y) := V(x_0, y) \circ f(x_0) \circ (V(x_0, y))^{-1}.$$ 

Dually, for all $y \in \text{supp} V$ such that $x_0 \preceq y$ define

$$f(y) := (V(y, x_0))^{-1} \circ f(x_0) \circ V(y, x_0).$$

If there are no sinks and sources in $\text{supp} V$, except possibly the endpoints, we have an induced morphism $V \to V$ such that $f(x)$ is an isomorphism for all $x \in \mathbb{R}$ (by setting $f(x) = 0$ when $x \notin \text{supp} V$). By Proposition 1.2.3 $f$ is an isomorphism. Now, suppose there is a sink or source in the interior of $\text{supp} V$.

Let $s_n$ be a source such that $x_0 \preceq s_n$. By the paragraph above we already have $f(s_n)$. For each $y \preceq s_n$ for which we do not yet have an $f(y)$ we can use the technique above and define it without making choices. By a dual argument if $s_n \preceq x_0$ we can define $f(y)$ for all $y$ such that $s_n \preceq y$. Note that between any real number $x$ and $x_0$ there are only finitely many sinks and sources between $x$ and $x_0$ in the total oder of $\mathbb{R}$. By repeated use of this technique, we have an induced isomorphism $f(x) : V(x) \cong V(x)$ for all $x \in \mathbb{R}$. Thus, we have an induced isomorphism $f : V \cong V$. If $g : V \to V$ is a nonzero morphism then $g(x)$ is nonzero as before is an isomorphism that determines the rest of $g$. Then $g(x)$ and $f(x)$ are multiplication by nonzero scalars and there exists $t \in k$ such that $tg(x) = f(x)$. Therefore, $\text{End}(V) \cong k$.

\square

Theorem 2.3.2. Let $V$ and $V'$ be two indecomposable pointwise one-dimensional representations of $A_{\mathbb{R}}$. Then $\text{supp} V = \text{supp} V'$ if and only if $V \cong V'$. 

Proof. We first assume \( \text{supp } V = \text{supp } V' \). Let \( x_0 \in \text{supp } V = \text{supp } V' \). By definition, \( V(x_0) \cong k \cong V'(x_0) \). Choose an isomorphism \( f(x_0) : V(x_0) \cong V'(x_0) \) and apply the argument from Lemma 2.3.1. The reverse direction is a special case of Proposition 2.3.4.

**Definition 2.3.3.** Let \( X_1 \subset X_2 \subset \cdots \subset X_n \) be a filtration of a vector space \( X = X_n \). A basis \( B \) for \( X \) is said to respect the filtration if \( B \cap X_j \) is a basis for \( X_j \) for each \( j \). A direct sum decomposition \( X = \bigoplus Y_i \) of \( X \) is said to respect the filtration if each \( X_j \) is a direct sum of some of the \( Y_i \).

**Lemma 2.3.4.** For any \( b \in \mathbb{R} \), let \( \mathcal{V}_b \) be the full subcategory of \( \text{Rep}_{\mathbb{R}}^{	ext{prof}}(A_{\mathbb{R}}) \) whose objects are interval indecomposable \( V \) with \( b \in \text{supp}(V) \subset [b, \infty) \). Let \( \mathcal{W}_b := \text{add } \mathcal{V}_b \). Then:

1. The restriction map, \( \text{res} : \mathcal{V}_b \to \text{Rep}_k(\{b\}) \) given by \( \text{res}(V) = V(b) \) and \( \text{res}(f) = f(b) \) defines a monomorphism \( \text{Hom}_{\mathcal{V}_b}(V, V') \to \text{Hom}_k(V(b), V'(b)) \) for all \( V, V' \in \mathcal{V}_b \), i.e. restriction to \( b \) is a faithful functor on \( \mathcal{V}_b \).
2. The restriction map, \( \text{res} : \mathcal{W}_b \to \text{Rep}_k(\{b\}) \) is also a faithful functor on \( \mathcal{W}_b \).
3. There is a unique total ordering on the set of isomorphism classes of objects of \( \mathcal{V}_b \) so that:
   a. \( \text{Hom}_{\mathcal{V}_b}(V, V') = 0 \) and \( \dim_k \text{Hom}_{\mathcal{V}_b}(V, V') = 1 \) whenever \( V > V' \) and \( \text{dim}_k \text{Hom}_{\mathcal{V}_b}(V, V') = 0 \) whenever \( V < V' \) and \( \text{dim}_k \text{Hom}_{\mathcal{V}_b}(V, V') = 0 \) whenever \( V = V' \).
   b. Composition of nonzero maps \( V \to V' \to V'' \) is always nonzero.
4. Any \( W \in \mathcal{W}_b \) has a unique filtration \( 0 = W_0 \subset W_1 \subset \cdots \subset W_m \) so that each \( W_k/W_{k-1} \) lies in \( \text{add } \mathcal{V}_k \) where \( V_1 < V_2 < \cdots < V_m \). Evaluating at vertex \( b \) we get a filtration \( W_1(b) \subset W_2(b) \subset \cdots \subset W_m(b) = W(b) \) which we call the filtration of \( W(b) \) induced by the filtration of \( W \).
5. For any \( W \in \mathcal{W}_b \), any direct sum decomposition \( W(b) = \bigoplus X_i \) of \( W(b) \) into one-dimensional subspaces which respects the filtration of \( W(b) \) induced from the filtration of \( W \) extends to a direct sum decomposition of \( W \), i.e. \( W = \bigoplus Y_i \) so that \( Y_i(b) = X_i \) for all \( i \).

Proof. (1) Given \( V, V' \) in \( \mathcal{V}_b \), the support of one of them contains the support of the other. Let \( J = \text{supp } V \cap \text{supp } V' \subset [b, \infty) \). Then either \( J = \text{supp } V \) or \( J = \text{supp } V' \). Suppose \( J = \text{supp } V \). Since \( V \) is indecomposable \( J \) is connected and \( J \subset \text{supp } V' \). Any morphism \( f : V \to V' \) induces a morphism \( f_J : V_J \to V'_J \) by restricting to \( J \). By Theorem 2.3.2, \( V_J \cong V'_J \) is either \( V \) or \( V' \). Then we have, by Lemma 2.3.1, that \( f_J \) is a scalar times a fixed isomorphism \( V_J \cong V'_J \). In particular \( f = 0 \) if and only if \( f \) is zero at \( b \). So, evaluation at \( b \) is faithful. (2) follows immediately from (1).

(3) Given \( V, V' \) in \( \mathcal{V}_b \), suppose by symmetry that the support of \( V \) is properly contained in the support of \( V' \). Then, there is some \( m > b \) so that the support of \( V \) is contained in \([b, m] \). There are only finitely many elements of \( S \) inside this compact set. Without loss of generality we may assume that \( b \in S \). Let \( l \) be maximal so that \( s_l \) is in the support of \( V \). If \( s_l \) is a source, then \( V \) is a sub-representation of \( V' \). If \( s_l \) is a sink, then \( V \) is a quotient representation of \( V' \). In the first case, \( \text{Hom}(V, V') = k \) and \( \text{Hom}(V, V') = 0 \). In the second case, \( \text{Hom}(V, V') = 0 \) and \( \text{Hom}(V', V) = k \).

If there are nonzero morphisms \( V \to V' \to V'' \) then, by (1), evaluation at \( b \) gives isomorphisms \( V(b) \cong V'(b) \cong V''(b) \). So, the relation of having a nonzero morphism \( V \to V' \) is transitive, reflexive, antisymmetric and any two elements are related. So, this is a total ordering.

(4) Given \( W \in \mathcal{W}_b \) we have by definition a direct sum decomposition \( W = \bigoplus_{1 \leq i \leq m} (V_i)^{n_i} \) where we order the summands according to the total order given in (3). So, there exists a filtration \( 0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_m = W \) so that \( W_i/W_{i-1} = n_i V_i \). Since \( \text{Hom}(V_i, V_j) = 0 \) for \( i < j \), the sub-representation \( W_i \) is uniquely characterized as the trace of \( V_1 \oplus V_2 \oplus \cdots \oplus V_i \) in \( W \). So, the HN-filtration is unique.

(5) Let \( n = \dim W(b) \) and let \( G \) be the subgroup of \( GL(n, k) \) which preserves the filtration \( W_1(b) \subset W_2(b) \subset \cdots \subset W_m(b) \). This is a block upper triangular matrix group which acts transitively on the set of all bases which respect this filtration of \( W(b) \). Since \( \text{Hom}(V_i, V_j) = K \) for \( i \leq j \) and \( \text{Hom}(V_i, V_j) = 0 \) for \( i > j \), we have by (3) that the restriction map \( \text{Aut}(W) \to \text{Aut}(W(b)) = G \) is an isomorphism. Therefore, \( \text{Aut}(W) \) acts transitively on the set of all bases for the vector space \( W(b) \) which respect the given filtration of \( W(b) \).
Recall we are given a direct sum decomposition $W(b) = \bigoplus X_i$ into one-dimensional subspaces that respects the induced filtration and $W = \bigoplus Y_i$ a direct sum decomposition of $W$ into pointwise one-dimensional indecomposable representations. One such basis is given by choosing a generator $x_i \in X_i$ for each summand $X_i$ of $W(b) = \bigoplus X_i$. A second basis is given by choosing a generator $y_i \in Y_i(b)$ where $W = \bigoplus (V_i)^p = \bigoplus Y_i$, where each $Y_i$ is equal to some $V_i$, is the given decomposition of $W$ into indecomposable representations which are one-dimensional at $b$. Take $\varphi \in G$ which takes $(y_i)$ to $(x_i)$. Then $W = \bigoplus \varphi (Y_i)$ is the required decomposition of $W$ extending the chosen decomposition of $W(b)$.

Lemma 2.3.5 (Lemma Y). Given any two finite filtrations of a finite-dimensional vector space $X$, there exists a direct sum decomposition of $X$ into one-dimensional subspaces which respects both filtrations.

Proof. Given any two filtrations $V_1 \subset V_2 \subset \cdots \subset V_n = X$ and $W_1 \subset W_2 \subset \cdots \subset W_m = X$ of $X$ we have the following representation of an indecomposable component $M_{n+m-1}$:

$$M : \quad V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset X \xrightarrow{\varphi} W_{m-1} \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} W_2 \xrightarrow{\varphi} W_1.$$  

We have a direct sum decomposition $M = \bigoplus M_i$ where each $M_i$ is one-dimensional at the middle vertex. This gives a direct sum decomposition of $X$ into one-dimensional subspaces. Then it suffices to prove the following.

Claim: This decomposition $X = M(n) = \bigoplus M_i(n)$ respects both filtrations.

Proof: Since the maps in the representation $M$ are all monomorphisms, the same holds for each indecomposable component $M_i$. So, each component is nonzero at vertex $n$ (where $M(n) = X$). For any $1 \leq j < n$, consider the set $I_j$ of all indices $i$ so that $M_i(j) \neq 0$. Then the sum of all $M_i(n)$ for all $i \in I_j$ is equal to $V_j$. Thus $\bigoplus M_i(n)$ respects the first filtration $V_i$ of $X$. Similarly, $\bigoplus M_i(n)$ respects both filtrations. This proves the lemma. 

2.4. Necessary Conditions and Theorem. In this section we prove that the sufficient conditions in Proposition 2.2.1 are also necessary conditions. The two statements combined form a necessary foundation for the Generalized BarCode Theorem (Theorem 2.4.13).

Definition 2.4.1. Choose $A_k$, a continuous quiver of type $A$. The opposite quiver of $A_k$, denoted $A_k^{op}$, is the continuous quiver of type $A$ where $x \preceq y$ in $A_k^{op}$ if and only if $y \preceq x$ in $A_k$.

Let $V$ be a pointwise finite representation of $A_k$. The dual representation of $V$, denoted $DV$, is the pointwise finite representation of $A_k^{op}$ given by

$$DV(x) := D(V(x))$$

$$DV(y, x) := D(V(x, y))$$

Remark 2.4.2. In Definition 2.4.1 note that since $V$ is pointwise finite we have $DDV \cong V$.

Lemma 2.4.3. Let $V$ be any object of $\text{Rep}_k^{pwf}(A_k)$. Then the restriction $V_J$ of $V$ to any closed interval $J = [a, b]$ where $s_n \leq a < b \leq s_{n+1}$ for some $n \in \mathbb{Z}$, decomposes as $V_J = A \oplus B$ where $A$ has support in the open interval $(a, b)$ and $B$ is a finite direct sum of indecomposable one-dimensional representations which are nonzero at either $a$, $b$ or both.

Proof. Without loss of generality we may assume that $s_n$ is a sink and $s_{n+1}$ is a source. Let $K$ be the subrepresentation of $V_J$ given by $K(x) = \ker(V(x, n) : V(x) \to V(s_n))$ for all $x \in J = [s_n, s_{n+1}]$.

Then $P = V_J/K$ is a projective representation of $J$ since all morphisms $P_a \to P_n$ are monomorphisms by construction of $K$. So $V_J \cong K \oplus P$. It is easy to decompose $P$ as a direct sum of finitely many pointwise one-dimensional representations all of which are nonzero at $n$.

It remains to show that $K$ is a direct sum of a representation with support on the open interval $(s_n, s_{n+1})$ and a finite number of pointwise one-dimensional representations all nonzero at $s_{n+1}$. This is accomplished using the dual representation $DK$. Since $DK$ is a representation of the opposite quiver, the interval $J$ with $n$ as source and $s_{n+1}$ as sink, using exactly the same argument
as above we see that $DK = A \oplus B$ where $A_{n+1} = 0$ and $B$ is a projective representation of $J^{op}$.
Thus $K \cong DA \oplus DB$ where $DA$ has support in the open interval $(s_n, s_{n+1})$ and $DB$ is a finite
direct sum of one-dimensional representations all of which are nonzero at $s_{n+1}$.

Lemma 2.4.4. If $V$ is an indecomposable object of $\text{Rep}^{\text{prof}}_k(A_{\mathbb{R}})$ with support in
an interval $[s_n, s_{n+1}]$ for some $n \in \mathbb{Z}$, then $V$ is pointwise one-dimensional.

Proof. The support of $V$ must be an interval $J \subseteq [s_n, s_{n+1}]$. If $J$ contains either of its endpoints
then the previous lemma applies. It remains to consider the case when $J = (a, b)$ is open. Let $c \in (a, b)$.
Then applying the previous lemma to the intervals $[a, c]$ and $[c, b]$ we decompose $V_{[a,c]}$ and
$V_{[c,b]}$ into a direct sum of finitely many pointwise one-dimensional representations each of which
is nonzero at $c$. The other components of $V_{[a,c]}$ and $V_{[c,b]}$ given by the lemma must be zero since
they would be components of $V$. This is equivalent to a representation of a finite quiver of type $A_m$
with straight orientation. So, we can choose the decompositions of $V_{[a,c]}$ and $V_{[c,b]}$ so that they
give the same decomposition of $V_c$. This decomposes $V = V_{[a,b]}$ into a direct sum of pointwise
one-dimensional representations. Since $V$ is indecomposable there is only one component.

Lemma 2.4.5. Let $V$ be an indecomposable object of $\text{Rep}^{\text{prof}}_k(A_{\mathbb{R}})$. For any two integers $n < m$,
the restriction of $V$ to the closed interval $[s_n, s_m]$ is a direct sum of finitely many indecomposable
pointwise one-dimensional representations.

Proof. The proof is by induction on $m - n$. Suppose first that $m - n = 1$ and let $J = [s_n, s_m] =
[s_n, s_{n+1}]$. If $\text{supp}V \subseteq J$ then $V$ is pointwise one-dimensional by Lemma 2.4.3. If $\text{supp}V \not\subseteq J$ then,
by Lemma 2.4.3 the restriction of $V$ to $J$ is a direct sum of pointwise one-dimensional objects plus
a summand $A$ with support on $(s_n, s_{n+1})$. But such a summand would also be a summand of $V$ by
Lemma 2.4.6. Therefore, $A = 0$ and the Lemma holds for $m = n + 1$.

Now suppose $m \geq n + 2$ and take any integer $k$ so that $n < l < m$. By induction on $m - n$, $V_{[s_n, s_l]}$ and $V_{[s_l, s_m]}$ 
decompose into pointwise one-dimensional components. By Lemma 2.3.4 this gives two filtrations of $V_k$.
By Lemma Y, there is a direct sum decomposition of $V_n$ compatible with both filtrations. This extends to compatible direct sum decompositions of $V_{[s_n, s_l]}$ and $V_{[s_l, s_m]}$ 
which paste together to give a decomposition of $V_{[s_n, s_m]}$ into one-dimensional representations.

Lemma 2.4.6. Let $V$ be a representation in $\text{Rep}^{\text{prof}}_k(A_{\mathbb{R}})$ and let $V_{(-\infty, b]}$ be the restriction of $V$ to
the interval $(-\infty, b]$. Then any summand $W$ of $V_{(-\infty, b]}$ which is zero at $b$ is a summand of $V$.

Proof. Let $\pi : V_{(-\infty, b]} \to V_{(-\infty, b]}$ be the projection to $W$. Then $\pi_b : V(b) \to V(b)$ is zero. So,
$\pi$ and the zero morphism on $V_{[b, \infty)}$ agree on the overlap of their domains. So, their union is an
endomorphism of $V$. This endomorphism is evidently the projection to $W$ showing that $W$ is a
summand of $V$.

Construction 2.4.7. Let $A_{\mathbb{R}}$ be a continuous quiver of type $A$ whose sinks and sources are un-
bounded above. I.e., for each sink or source $s_n$ there is an $s_{n+1}$. Let $V$ be a pointwise finite-
dimensional representation of $A_{\mathbb{R}}$ such that, for all $n \in \mathbb{Z}$, the restriction $V_{[s_n, s_{n+1}]}$ contains no
direct summands whose support is contained entirely in $(s_n, s_{n+1})$ (i.e., $A = 0$ in the $A \oplus B$ decom-
position in Lemma 2.4.3).

Consider the restriction $V_{[s_{l-1}, s_l]}$. By assumption $V_{[s_{l-1}, s_l]}$ is a finite direct sum of indecomposables,
all of whose support includes $s_l$ or $s_{l-1}$. Let $V_{0,0}^l$ be the direct sum of all those summands
that include only $s_l$, not $s_{l-1}$. Now consider $V_{[s_{l-1}, s_{l+1}]}$. By assumption $V_{[s_{l-1}, s_{l+1}]}$ is a finite
direct sum of indecomposables, each of whose support contains $s_{l-1}, s_l$ or $s_{l+1}$. Let $V_{0,1}^l$ be the direct
sum of all such indecomposables whose support contains both $s_l$ and $s_{l+1}$, but not $s_{l-1}$. Let $V_{1,1}^l$ be the
direct sum of such indecomposables whose support contains $s_l$ but not $s_{l+1}$ or $s_{l-1}$. We ignore
those indecomposables whose support does not contain $s_l$. 

We can continue this process for all \( n \geq 0 \). For each \( n \geq 0 \) and \( 0 \leq i \leq n \) we define \( V^l_{i,n} \) in the following way. It is the direct sum of those summands of \( V_{[s_{i-1}, s_{i+n}]} \) whose support contains exactly sinks and sources \( s_{l+j} \) for \( 0 \leq j \leq n - i \). Note that this never includes \( s_{l-1} \). In particular, \( V^l_{0,n} \) is the direct sum of those interval indecomposable summands of \( V_{[s_{i-1}, s_{i+n}]} \) whose support contains \( s_l \) and \( s_{l+n} \). We have three examples below, two from the previous paragraph and also the summands we consider from \( V_{[s_{l-1}, s_{l+2}]} \).

\[
\begin{array}{ccc}
V^l_{0,0} & V^l_{0,1} \oplus V^l_{1,1} & V^l_{0,2} \oplus V^l_{1,2} \oplus V^l_{2,2} \\
\end{array}
\]

We note that if \( 1 \leq i \leq n \) then \( V^l_{i,n} = V^l_{i+1,n+1} \). Note also that the constructions can be made on \((-\infty, s_{l+1}]\) instead and those representations are denoted \( V^l_{l,n} \).

**Proposition 2.4.8.** Let \( A_R \) and \( V \) be as in Construction 2.4.7 for some \( s_l \). Then, for all \( n \geq 1 \) and \( 1 \leq i \leq n \), \( V^l_{i,n} \) and \( V^l_{l,n} \) are split subrepresentations of \( V \).

**Proof.** We know the representation \( V^l_{1,n} \) is a split subrepresentation of \( V_{[s_{l-1}, s_{l+n+1}]} \) as a consequence of Lemma 2.4.5. We know that \( V^l_{1,n}(s_{l-1}) = 0 \) and \( V^l_{1,n}(s_{l+n+1}) = 0 \). By two uses of Lemma 2.4.6 we see that \( V^l_{1,n} \) is a split subrepresentation of \( V \). Finally, recall that \( V^l_{i,n} = V^l_{i+1,n+1} \) when \( i \geq 1 \).

By a similar argument \( V^l_{l,n} \) is a split subrepresentation of \( V \).

**Lemma 2.4.9.** Let \( l \in \mathbb{Z} \) and \( V \) be a representation with support contained in \([s_{l-1}, +\infty)\). Assume that for all \( n \geq 0 \), any indecomposable summand of \( V_{[s_{l-1}, s_{l+n}]} \) has support at \( s_{l+n} \). Then a decomposition of \( V_{[s_{l-1}, s_{l+n}]} \) into interval indecomposables extends to a decomposition of \( V_{[s_{l-1}, s_{l+n+1}]} \).

**Proof.** Suppose \( V_{[s_{l-1}, s_{l+n}]} \cong \bigoplus_i M_i \) is a decomposition. Then each \( i \) includes \( s_{l+n} \).

If \( s_{l+n} \) is a sink then \( V_{[s_{l+n}, s_{l+n+1}]} \) is a monomorphism. Any interval indecomposable summands of \( V_{[s_{l+n}, s_{l+n+1}]} \) that do not have support at \( s_{l+n} \) are projective. In particular, they are split subrepresentations of the same restriction (combine Lemmas 2.4.3 and 2.4.6). Let \( U_{n+1} \) be the quotient of \( V_{[s_{l-1}, s_{l+n+1}]} \) by these projective interval indecomposables. Since \( (U_{n+1})_{[s_{l-1}, s_{l+n}]} = V_{[s_{l-1}, s_{l+n}]} \) and \( U_{n+1}(s_{l+n+1}) = s_{l+n+1} \) is an isomorphism we can extend the decomposition to \( U_{n+1} \). Since \( U_{n+1} \) is a decomposable summand of \( V_{[s_{l-1}, s_{l+n+1}]} \) and the other summand is decomposable by Theorem 2.1.19 we have extended our decomposition.

If \( s_{l+n} \) is a source then \( V_{[s_{l+n}, s_{l+n+1}]} \) is an epimorphism. Any interval indecomposable summands of \( V_{[s_{l+n}, s_{l+n+1}]} \) that do not have support at \( s_{l+n} \) are injective. They are split subrepresentations as before. We can now apply the same argument in the previous paragraph and extend the decomposition.

The assumptions in the following lemma are justified by Proposition 2.4.8.

**Lemma 2.4.10.** Let \( A_R \) and \( V \) be as in Construction 2.4.7. For all \( l \in \mathbb{Z} \), \( n \geq 1 \), and \( 1 \leq i \leq n \), assume \( V^l_{i,n} = 0 = V^l_{l,n} \). Then \( V \) contains a summand as in Lemma 2.4.9 but whose support does not contain \( s_{l-1} \).

**Proof.** For each \( n \geq 1 \) and a decomposition of \( V_{[s_{l-1}, s_{l+n}]} \) let \( K_n \) be the sum of interval summands whose support is nonzero at \( s_{l-1} \). By assumption, if \( s_{l-1} \leq x \leq y \leq s_{l+n} \) then \( \dim K_n(x) \geq \ldots \)
dim \( K_n(y) \). Note that \( \dim K_n(x) = \dim K_{n+1}(x) \) on \([s_{l-1}, s_{l+n}]\), though the decomposition of \( K_n \) is not assumed to extend exactly.

Therefore we have a function \([s_{l-1}, +\infty) \to \mathbb{N}\) that is weakly decreasing and whose initial value is finite. Therefore, the function must stabilize to some particular value. Let \( m \) be sufficiently large that \( \dim K_n(s_{l+n}) = \dim K_{n+1}(s_{l+n+1}) \) for all \( n \geq m \). Then, by assumption, every map \( V(s_{l+n}, s_{l+n+1}) \) for \( n \geq m \) is mono or epi. So we can use the same technique in Lemma 2.4.9 to extend a decomposition of \( V_{[s_{l-1}, s_{l+n+1}]} \) to all of \( V_{[s_{l-1}, +\infty)} \).

Then any summands of \( V_{[s_{l-1}, +\infty)} \) with bounded support that is nonzero at \( s_{l-1} \) are split subrepresentations of \( V_{[s_{l-1}, +\infty)} \) (Lemma 2.4.9). Denote those summands by \( U \) and the rest by \( W \). Then \( W \) satisfies the hypothesis of Lemma 2.4.9 and we have a decomposition of \( W \) already. In particular, we can write \( W \cong W_1 \oplus W_2 \) where the summands of \( W_1 \) are nonzero at \( s_{l-1} \) and the summands at \( W_2 \) are 0 at \( s_{l-1} \). Then by a further use of Lemma 2.4.6 we see \( W_2 \) is actually the summand of \( V \) that we desired.

\( \square \)

**Notation 2.4.11.** Let \( A_\mathbb{R} \) and \( V \) be as in Construction 2.4.7. For some \( l \), let \( W_2 \) be as in the end of the proof of Lemma 2.4.10. As seen in the proof, \( W_2 \) is a direct sum of interval indecomposables. Let \( V_{l,0,\infty}^i \) be the direct sum of those summands of \( W_2 \) who have support at \( s_l \).

**Remark 2.4.12.** Construction 2.4.7, Proposition 2.4.8, Lemma 2.4.9, Lemma 2.4.10, and Notation 2.4.11 can all be performed on \((−\infty, s_{l+1}]\) instead of \([s_{l-1}, +\infty)\). These representations will be denoted \( V_{l,i}^{0,\infty} \).

**Theorem 2.4.13.** Let \( A_\mathbb{R} \) be a continuous quiver of type \( A \) and \( V \) be a representation in \( \operatorname{Rep}_k^\text{pwf}(A_\mathbb{R}) \). Then \( V \) is the direct sum of interval indecomposables (Definition 2.2.2).

**Proof. Outline:** We complete this proof in four parts. In Part 1, we consider the indecomposable summands whose support is contained entirely between a sink and source. In Part 2, we consider the indecomposable summands whose support contains at least one but only finitely many sinks and sources. In Part 3, we consider the indecomposable summands whose support may contain infinitely many sinks and sources, but is bounded on exactly one side. Finally, in Part 4, we concern ourselves with indecomposable summands whose support is \( \mathbb{R} \). Since the case where \( A_\mathbb{R} \) has no sinks or sources in \( \mathbb{R} \) has been covered by Crawley-Boevey in [C-B], we assume that \( A_\mathbb{R} \) has at least one sink or source in \( \mathbb{R} \).

**Part 1:** Let \( s_n \) and \( s_{n+1} \) be an adjacent pair of sink, source or \( \pm\infty \); however, only one may be \( \pm\infty \) by assumption. We use the notation \([s_n, s_{n+1}]\) even if one of the endpoints is actually \( \pm\infty \). By Lemma 2.4.3 \( V_{[s_n, s_{n+1}]} \) decomposes to \( A_n \oplus B_n \) where the support of \( A_n \) is contained in \((s_n, s_{n+1})\). By Lemma 2.4.6 \( A_n \) is a direct summand of \( V \).

Thus, for all \( n \) where \( s_n \) or \( s_{n+1} \) is in \( \mathbb{R} \), we have such an \( A_n \). So we have that \( V \cong (\bigoplus A_n) \oplus U \).

By [C-B] each \( A_n \) decomposes into a direct sum of indecomposable representations.

**Part 2:** We now assume \( V \cong U \) as in the end of Part 1. If \( A_\mathbb{R} \) has finitely many sinks and sources then, by the proof of Lemma 2.4.5 \( V \) is a finite direct sum of indecomposable representations. So we shall now assume \( A_\mathbb{R} \) has infinitely many sinks and sources. Choose a sink or source \( s_l \in \mathbb{R} \). By Proposition 2.4.8 we know \( V_{l,n}^i \) is a split subrepresentation with bounded support for all \( 1 \leq i \leq n \). Thus, for all \( l \) such that \( s_l \in \mathbb{R} \) we obtain such direct summands, none of which are counted twice. Thus we have \( V \cong (\bigoplus_{i=1}^n V_{l,i}^i) \oplus U \).

**Part 3:** Now we assume \( V \cong U \) as in the end of Part 2. Then for each \( l \in \mathbb{Z} \) we apply Lemma 2.4.10 and obtain \( V_{l,0,\infty}^i \) as in Notation 2.4.11. By Remark 2.4.12 we also obtain \( V_{l,0,\infty}^i \) for each \( l \). Each \( V_{l,0,\infty}^i \) and \( V_{l,0,\infty}^i \) decompose into interval indecomposables and so we have \( V \cong (\bigoplus(V_{l,0,\infty}^i \oplus V_{l,0,\infty}^i)) \oplus U \).

**Part 4:** We assume \( V \cong U \) as in the end of Part 3. For any \( s_l \in \mathbb{R} \), we know \( V_{l,0,\infty}^i = 0 \) and \( V_{l,0,\infty}^i = 0 \). Choose some sink or source \( s_l \) in \( \mathbb{R} \) and let \( X = V_{[s_l, +\infty)} \) and \( Y = V_{(-\infty, s_l]} \). We can then
construct $X^d_{0,\infty}$ and $Y^0_{1,\infty}$. Since $V^d_{0,\infty} = 0$ and $V^0_{1,\infty} = 0$, we see $\dim X^d_{0,\infty}(s_l) = \dim Y^0_{1,\infty}(s_l)$. In particular, they are both finite.

Furthermore, $V(x, y)$ is an isomorphism for all $y \leq x$ in $\mathbb{R}$. Choose a decomposition of $V_{[s_l, s_{l+1}]}$ and use the technique in Lemma 2.4.9 to extend this decomposition to all of $X$ and all of $Y$. But together this yields a decomposition of $V$.

This will give us a bijection $V \cong \bigoplus_{\dim V(s_l)} M_{(-\infty, +\infty)}$. Thus, $V$ is a direct sum of indecomposable representations.

**Conclusion:** In Parts 1–3 we decomposed $V$ into $Z \oplus U$ and in Parts 2–4 we decomposed the previous Part’s $U$. In Parts 1–3 we showed that the $Z$ summand was a direct sum of indecomposables and in Part 4 we showed the final $U$ is a direct sum of indecomposables. Therefore, given any pointwise finite-dimensional representation $V$ of $A_\mathbb{R}$, it is the direct sum of indecomposable representations. If $V$ itself is indecomposable it appears as one of described indecomposable summands, depending on its support. □

**Remark 2.4.14.** The theorem above, with the aid of Theorem 2.1.16 completely classifies indecomposable projective objects in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ and $\text{Rep}^b_k(A_\mathbb{R})$. They come in three forms, up to isomorphism.

1. $P_a$ as in Definition 2.1.3
   
   
   \[ P_a(x) = \begin{cases} 
   k & x \leq a \\
   0 & \text{otherwise}
   \end{cases} \]

   \[ P_a(x, y) = \begin{cases} 
   1_k & y \leq x \leq a \\
   0 & \text{otherwise}
   \end{cases} \]

2. $P_a$ given by

   \[ P_a = \begin{cases} 
   k & x \leq a, x < a \\
   0 & \text{otherwise}
   \end{cases} \]

   \[ P_a(x, y) = \begin{cases} 
   1_k & y \leq x \leq a, y \leq x < a \\
   0 & \text{otherwise}
   \end{cases} \]

3. $P(a)$ given by

   \[ P(a) = \begin{cases} 
   k & x \leq a, a < x \\
   0 & \text{otherwise}
   \end{cases} \]

   \[ P(a, x, y) = \begin{cases} 
   1_k & y \leq x \leq a, a < x \leq y \\
   0 & \text{otherwise}
   \end{cases} \]

Note that unless $a$ is a source at least one of (2) or (3) will define the 0 representation. If $a$ is a sink then both (2) and (3) will be the 0 representation.

Additionally, it is worth noting that if $V$ is a subrepresentation of any sum of projective indecomposables then $V$ is also projective. This follows from Theorem 2.1.16(1). Therefore, $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ is hereditary.

**Example 2.4.15.** Let the set of sinks and sources $S = \{0, 1\}$, where $s_0 = 0$ is a sink and $s_1 = 1$ is a source. We provide a complete list of indecomposable projectives in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$ with this orientation. The values $a, b, c \in \mathbb{R}$ below are such that $a < 0 < b < 1 < c$.

\[
\begin{array}{ccccccccccc}
\{0\}, & (\infty, 0], & (a, 0], & [a, 0], & [0, b], & [0, 1], & [0, +\infty), & (1, +\infty), & (c, +\infty), & [c, +\infty) \\
\{P_0, P_{-\infty}, P_a, P_a, P_b, P_b, P_1, P_1, P_{c}, P_{c}\}
\end{array}
\]

**Remark 2.4.16.** We also have the indecomposable injective objects in $\text{Rep}^\text{pwf}_k(A_\mathbb{R})$.

1. $I_a$ given by:

   \[ I_a(x) = \begin{cases} 
   k & a \leq x \\
   0 & \text{otherwise}
   \end{cases} \]

   \[ I_a(x, y) = \begin{cases} 
   1_k & a \leq y \leq x \\
   0 & \text{otherwise}
   \end{cases} \]

2. $I_a$ given by

   \[ I_a = \begin{cases} 
   k & a \leq x, x < a \\
   0 & \text{otherwise}
   \end{cases} \]

   \[ I_a(x, y) = \begin{cases} 
   1_k & a \leq y \leq x, x \leq y < a \\
   0 & \text{otherwise}
   \end{cases} \]
2.5. More on $P_{(a)}$, $P_{a}$, and the Pointwise Finite Requirement. As mentioned in Section 2.1, the indecomposable projectives $P_{(a)}$ and $P_{a}$, whichever are nonzero, are not projective in $\text{Rep}_k(A_{\mathbb{R}})$. They are only projective in the smaller subcategory $\text{Rep}_{k}^{\text{pwf}}(A_{\mathbb{R}})$. We will prove this using a specific representation, denoted $\Psi$, that exists only in $\text{Rep}_k(A_{\mathbb{R}})$. I.e., it is not pointwise finite-dimensional. We will use that same representation to show why Theorem 2.4.13 can fail without the pointwise finite assumption.

Construction 2.5.1. We will denote the problematic representation by $\Psi$. First, let $a \in \mathbb{R}$ such that $a$ is not a sink. Let $p \in \mathbb{R}$ such that $p \leq a$ and $p \neq a$. By symmetry, suppose $p < a$. Let $\{x_i\}_{i=0}^{\infty}$ be a strictly increasing sequence converging to $a$ such that $x_0 > p$. Let $M = \bigoplus_{(x_i)} M_{[p,x_i]}$. Then the support of $M$ is $[p,a)$. Let $\pi : M(p) \to k$ be a surjection given by sending each $1$ in $M_{[p,x_i]}(p) = k$ to $1 \in k$.

Let $\Psi$ be given by

$$
\Psi(x) = \begin{cases} 
  k & x = p \\
  \Psi(x) & x \neq p 
\end{cases}
$$

$$
\Psi(x,y) = \begin{cases} 
  k & x = y = p \\
  \pi \circ M(x,y) & x \neq y = p \\
  M(x,y) & \text{otherwise}.
\end{cases}
$$

We see that $\Psi$ also has support $[p,a)$.

Proposition 2.5.2. Let $A_{\mathbb{R}}$ be a continuous quiver of type $A$. Let $p, a \in \mathbb{R}$ such that $a$ is not a sink, $p \leq a$, and $p < a$. Then there is no nontrivial morphism $P_{a} \to \Psi$, where $\Psi$ is from Construction 2.5.1.

Proof. Choose $x_m$ in the sequence from Construction 2.5.1. Let $f(x_m) : P_{a} \to \Psi(x)$ be a linear map. Since $P_{a} = k$, $f(x_m)$ is determined by $f(x_m)(1)$. Since $\Psi(x) = M(x)$ for $x \neq p$, we see

$$
f(1) = (0, \ldots, 0, r_m, r_{m+1}, \ldots, r_n, 0, 0, \ldots)
$$

Then for any linear map $f(x_{n+1}) : P_{a}(x_{n+1}) \to \Psi(x_{n+1})$ we know that

$$
f(x_m) \circ P_{a}(x_{n+1}, x_m) \neq \Psi(x_{n+1}, x_m) \circ f(x_{n+1})
$$

Therefore, there is no morphism of representations $P_{a} \to \Psi$. □

Proposition 2.5.3. Let $A_{\mathbb{R}}$ be a continuous quiver of type $A$ and $a \in \mathbb{R}$ such that $a$ is not a sink. Then each nonzero $P_{(a)}$ and $P_{a}$ is not projective in $\text{Rep}_k(A_{\mathbb{R}})$.

Proof. Let $p \in \mathbb{R}$ such that $p < a$ and $p \leq a$. The other case, where $p > a$ and $p \leq a$, is similar. Then, there is a nontrivial morphism of indecomposable representations $f : P_{a} \to M_{[p,a]}$. Let $\Psi$ be as in Construction 2.5.1. For each $x \in [p,a)$, let $f(x) : \Psi(x) \to M_{[p,a]}(x)$ be given by $\Psi(x,p)$. Since $\Psi(p) = M_{[p,a]}(x)$ for all $x \in [p,a)$, this is a well-defined morphism of representations. In particular, it is an epimorphism.

So now we have an epimorphism $P_{a} \to M_{[p,a]}$ and an epimorphism $\Psi \to M_{[p,a]}$. However, there is no nontrivial morphism $P_{a} \to \Psi$ in $\text{Rep}_k(A_{\mathbb{R}})$, by Proposition 2.5.2. Therefore, $P_{a}$ is not projective. □

Proposition 2.5.4. Let $A_{\mathbb{R}}$ be a continuous quiver of type $A$ and $\Psi$ as in Construction 2.5.1. Then $\Psi$ is not the direct sum of pointwise one-dimensional indecomposables.
Proof. We saw in the proof of Proposition 2.5.3 that there is an epimorphism \( \mathcal{P} \to M_{[p,a]} \). However, just as in the proof of Proposition 2.5.2 there are no nontrivial morphisms \( M_{[p,a]} \to \mathcal{P} \). Thus, \( M_{[p,a]} \) is not a direct summand of \( \mathcal{P} \). But if \( \mathcal{P} \) had a direct sum decomposition, one of the components must have support \([p,a)\). But that would mean the indecomposable is \( M_{[p,a)} \). Therefore, \( \mathcal{P} \) does not decompose into a direct sum of one-dimensional indecomposables. \( \square \)

2.6. Relation to the BarCode Theorem and Zigzags. Theorem 2.4.13 is, in some sense, a combination of the Crawley-Boevey’s BarCode theorem from [C-B] and Botnan’s decomposition theorem in [B]. Part of our argument actually follows the latter paper. The BarCode theorem handles representations on the continuum but only a straight orientation. By contrast, Botnan’s decomposition handles the infinite zigzag orientation but only in the discrete setting. While one might think to use Botnan’s paper explicitly with Crawley-Boevey’s result, this is not actually possible. The decomposition in Parts 2 and 3 of the proof of Theorem 2.4.13 would fail since our representations have supports that contain sinks and sources but whose bounds could be neither. Section 2.3 is essentially devoted to handling these types of representations so that they fit into the argument.

Both previous papers worked with pointwise finite-dimensional representations and each displayed a non-example for a representation that is not pointwise finite-dimensional. Theorem 2.4.13 adheres to exactly the same restrictions and the relevant non-example appears in Section 2.5 as Construction 2.5.1 and Proposition 2.5.3.

3. Finitely Generated Representations: \( \text{rep}_k(A_\mathbb{R}) \)

In this section we will prove results about the category of finitely generated representations, denoted \( \text{rep}_k(A_\mathbb{R}) \). Many of the properties one could reasonably expect to hold in a continuous version of \( \text{rep}_k(A_n) \) do, in fact, hold for \( \text{rep}_k(A_\mathbb{R}) \). The properties that change due to the nature of the continuum are Auslander-Reiten sequences and descending chains of subrepresentations. We provide an incomplete list of the properties that hold or do not hold in the form of a theorem and dedicate the rest of this section to proving each of the items in the theorem.

**Theorem 3.0.1.** Let \( A_\mathbb{R} \) be a continuous quiver of type A and denote by \( \text{rep}_k(A_\mathbb{R}) \) the category of finitely generated representations (Definition 3.1.3). Then the following hold.

1. For indecomposable representations \( M_I \) and \( M_J \) in \( \text{Rep}_k^{\text{pwf}}(A_\mathbb{R}) \), \( \text{Rep}_k^{b}(A_\mathbb{R}) \), or \( \text{rep}_k(A_\mathbb{R}) \), we have \( \text{Hom}(M_I, M_J) \cong k \) or \( \text{Hom}(M_I, M_J) = 0 \) (Proposition 3.1.4).
2. Every morphism \( f : V \to W \) in \( \text{Rep}_k^{\text{pwf}}(A_\mathbb{R}) \), \( \text{Rep}_k^{b}(A_\mathbb{R}) \), or \( \text{rep}_k(A_\mathbb{R}) \) has a kernel, a cokernel, and coinciding image and coimage in that category. (Lemma 3.1.5)
3. The category \( \text{rep}_k(A_\mathbb{R}) \) Krull-Schmidt, but not artinian (Lemma 3.1.5 Proposition 3.1.6).
4. The global dimension of \( \text{rep}_k(A_\mathbb{R}) \) is 1 (Proposition 3.2.6).
5. The Ext space of two indecomposables \( M_I \) and \( M_J \) in \( \text{Rep}_k^{\text{pwf}}(A_\mathbb{R}) \), \( \text{Rep}_k^{b}(A_\mathbb{R}) \), or \( \text{rep}_k(A_\mathbb{R}) \) is either isomorphic to \( k \) or is 0 (Proposition 3.2.6).
6. While some Auslander-Reiten sequences exist (Proposition 3.3.2), some indecomposables have neither a left nor a right Auslander-Reiten sequence (Proposition 3.3.4).

3.1. Requisites and Definition. In this subsection we define the category of finitely generated representations of a continuous type A and prove Theorem 3.0.1 (1) – (3).

**Notation 3.1.1.** We may use \( | \) instead of \( (, ) \), \([,] \), or \( [a,b] \) to write an interval. When this happens, we mean that the endpoint may or may not be included; either we are making no assumptions about endpoints or it is clear what choice is possible from context. I.e., for all \( a, b \in \mathbb{R} \), \( [a,b] \) can be one of four possibilities. However, when we write our intervals, we allow \( a = -\infty \) and \( b = +\infty \) so long as we obtain a subset of \( \mathbb{R} \). So, the notation \( [a,b] \) will never mean \([-\infty,b), [a,\infty), \) or \([-\infty,\infty) \).
Proposition 3.1.2. Let $V$ and $W$ be indecomposable representations in $\text{Rep}_{k}^{\text{pref}}(A_{\mathbb{R}})$. Then either $\text{Hom}(V, W) \cong k$ or $\text{Hom}(V, W) = 0$.

Proof. Suppose $\text{Hom}(V, W) \neq 0$ and choose a nontrivial $f : V \to W$. Then there is $x \in \mathbb{R}$ such that $f(x) : V(x) \to W(x)$ is not $0$. Since $V(x) \cong k \cong W(x)$ we see $f(x)$ is an isomorphism. For all $y \leq x, W(x, y) \circ f(x) = f(y) \circ V(x, y)$. If $V(y) \neq 0$ and $W(y) \neq 0$ then $f(y) = W(x, y) \circ f(x) \circ V(x, y)^{-1}$. For all $z$ such that $x \leq z$, $W(z, x) \circ f(z) = f(x) \circ V(z, x)$. Then again if the vector spaces are nontrivial we have $f(z) = W(z, x)^{-1} \circ f(x) \circ V(z, x)$.

So for the sink and source $s \leq z \leq s'$ we see each of $f(s)$ and $f(s')$ are either $0$ or determined by $x$. Since the set of sinks and sources is discrete with no accumulation points we can use our arguments in the previous paragraph repeatedly and see that each nontrivial $f(y)$ is determined by $f(x)$. Since $\text{Hom}(V(x), W(x)) \cong k$ and every nontrivial $f(y)$ is determined by $f(x)$, we see $\text{Hom}(V, W) \cong k$. □

Definition 3.1.3. We define $\text{rep}_{k}(A_{\mathbb{R}})$ as the full subcategory of $\text{Rep}_{k}^{\text{pref}}(A_{\mathbb{R}})$ whose objects are representations $V$ that are finitely generated by indecomposable projectives (listed in Remark 2.4.14).

Lemma 3.1.4. Let $f : V \to W$ be a morphism in $C$ where $C = \text{Rep}_{k}^{\text{pref}}(A_{\mathbb{R}})$, $\text{Rep}_{k}^{b}(A_{\mathbb{R}})$, or $\text{rep}_{k}(A_{\mathbb{R}})$.

- $f$ has a kernel in $C$,
- $f$ has a cokernel in $C$, and
- the image and coimage of $f$ coincide and lie in $C$.

Proof. First note that $f$ is a morphism in $\text{Rep}_{k}(A_{\mathbb{R}})$. By a dimension argument for $V(x)$, $W(x)$, $\ker f(x)$, and $\text{coker} f(x)$ at each $x \in \mathbb{R}$ the statement must be true for $C = \text{Rep}_{k}^{\text{pref}}(A_{\mathbb{R}})$ and $C = \text{Rep}_{k}^{b}(A_{\mathbb{R}})$.

Now suppose $C = \text{rep}_{k}(A_{\mathbb{R}})$. Since $\text{Rep}_{k}^{\text{pref}}(A_{\mathbb{R}})$ is abelian the image and coimage of $f$ coincide. Since $V \to \text{im} f$ and $V$ is finitely generated, so is $\text{im} f$. Similarly, since $W$ is finitely generated by some $\bigoplus_{i=1}^{n} P_i$ there is a surjection $\bigoplus_{i=1}^{n} P_i \twoheadrightarrow \text{coker} f$.

Suppose $g : \bigoplus Q_i \to V$ generates $V$. Then $\ker( f \circ g)$ is a subrepresentation of a projective; since $\text{Rep}_{k}^{\text{pref}}(A_{\mathbb{R}})$ is hereditary this means $\ker( f \circ g)$ is projective. Also $\ker( f \circ g)$ maps to $\ker f$. For any $0 \neq \bar{v} \in \ker f(x)$ there is $v \in V(x)$ from the inclusion. Then there is $\bar{v} \in \bigoplus Q_i(x)$ that maps to $v$.

Let $\bigoplus Q_i = \ker(f \circ g)$. Any projective submodule of a finitely generated projective is finitely generated, so $\bigoplus Q_i$ is finitely generated. We also know that since $\bar{v} \mapsto v \mapsto 0$, there exists $\bar{v} \in \bigoplus Q_i(x)$ that maps to $v$ and so maps to $\bar{v}$. Thus, $\bigoplus Q_i \to \ker f$ so $\ker f$ is also finitely generated. Therefore, $\ker f$, $\text{im} f$, and $\text{coker} f$ are all generated by finitely generated and so in $\text{rep}_{k}(A_{\mathbb{R}})$.

Lemma 3.1.5. Let $V$ be a representation in $\text{rep}_{k}(A_{\mathbb{R}})$. Then $V$ is isomorphic to a finite direct sum of interval indecomposables. Furthermore, $\text{rep}_{k}(A_{\mathbb{R}})$ is Krull-Schmidt.

Proof. Suppose $V$ is in $\text{rep}_{k}(A_{\mathbb{R}})$ and $\bigoplus_{i=1}^{n} Q_i \to V$ be a surjective morphism required by Definition 3.1.3.

Since $\dim Q_i(x) \leq 1$ for all $x \in \mathbb{R}$, $\dim V(x) \leq n$ for all $x \in \mathbb{R}$. That is, both $Q$ and $V$ are in $\text{Rep}_{k}^{b}(A_{\mathbb{R}})$. By Theorem 2.4.13 $V$ is a direct sum of (a priori possibly infinitely many) interval indecomposables.

Since each $Q_i$ is projective, the support of each $Q_i$ contains at most 3 sinks and sources (1 source and 2 sinks). Then, since $Q$ is a finite direct sum, the support of $Q$ itself contains finitely many sinks and sources. Since $Q$ surjects onto $V$, the support of $V$ must also contain only finitely many sinks and sources.

For contradiction, suppose $V$ is an infinite direct sum of indecomposables. Since $V$ is pointwise finite-dimensional and its support contains finitely many sinks and sources, infinitely many
summands must have support that does not contain a sink or a source; i.e. each of these indecomposable's support is bounded by an adjacent sink and source. Since there are only finitely many sinks and sources in the support of $V$, infinitely many must have support between the same adjacent sink and source.

For each $Q_i = P_a$ for some $a$ (classification in Remark 2.4.14), any indecomposable hit by $Q_i$ must contain $a$ in its support. Since $V$ is pointwise finite dimensional there can only be finitely many such indecomposables. Thus there must be some $Q_i = P_{(a \text{ or } b)}$.

If $Q_i = P_{(a}$ then any indecomposable $V_a$ hit by $Q_i$ has the property that $\text{glb supps} V_a \leq a$. If $Q_i$ hits infinitely many indecomposables there must be infinitely many with support of the form $(a, b_i)$ and the $b_i$ must converge on $a$. However, $V$ is also in $\text{Rep}_R^b(A_R)$ and so this is a contradiction as $\lim \dim V(x)$ as $x \to a$ from above would be $\infty$. The same argument holds if $Q_i = P_{a)$. Therefore, $V$ is the direct sum of finitely many indecomposables. Combined with Theorem 2.3.2 and Lemma 2.3.1 this shows $\text{rep}_k(A_R)$ is Krull-Schmidt. □

**Proposition 3.1.6.** The category $\text{rep}_k(A_R)$ is not Artinian.

**Proof.** Let $P_a$ be a projective indecomposable (Remark 2.4.14) such that $a$ is not in $S$. Let $b \in \tilde{S}$ such that $b \preceq a$; note $b \neq a$. Then, for every $b \preceq z \preceq a$ such that $b \neq z \neq a$, $P_z \subsetneq P_a$. Furthermore, for any two such $z, z'$ such that $z \preceq z'$, we have $P_z \subsetneq P_{z'} \subsetneq P_a$. Thus, we have an infinite (uncountable!) descending chain and so $\text{rep}_k(A_R)$ is not Artinian. □

**Example 3.1.7.** Let us return to the representation $M$ in Example 1.3.2. It is an uncountable sum and so not in the category $\text{rep}_k(A_R)$. In particular, any surjection onto $M$ by a sum of interval indecomposables would require the source representation to be an uncountable sum as well.

### 3.2. Properties of $\text{rep}_k(A_R)$

We now prove Theorem 3.0.1 (4) and (5).

**Proposition 3.2.1.** Let $A_R$ and $A'_R$ be different orientations such that the sinks and sources are unbounded above and below in both $A_R$ and $A'_R$. Then $\text{rep}_k(A_R) \cong \text{rep}_k(A'_R)$.

**Proof.** We’ll define a bijection $F : \mathbb{R} \to \mathbb{R}$ that induces a bijection on (isomorphism classes of) indecomposables and thus an equivalence of categories. Recall $S$ is the set of sinks and sources of $A_R$ and $S'$ is the set of sinks and sources of $A'_R$. First define the bijection on $S \to S'$ to be $s_n \mapsto s'_n$. Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $s_n < x < s_{n+1}$. Then $x = t \cdot s_n + (1-t)s_{n+1}$ for some $t \in (0,1)$. Let $F(x) = t \cdot F(s_n) + (1-t)F(s_{n+1})$.

This induces a bijection on indecomposables as it is a bijection on $\mathbb{R}$. In particular, if $x \preceq y$ then $F(x) \preceq F(y)$. If $\text{Hom}(M_{[a,b]}, M_{[c,d]}) \cong k$ in $\text{rep}_k(A_R)$ then $a \preceq c$ and $b \preceq d$. Since $F(a) \preceq F(c)$ and $F(b) \preceq F(d)$, the $\text{Hom}$-set from $M_{[F(a), F(b)]}$ to $M_{[F(c), F(d)]}$ is also isomorphic to $k$. Thus we have an equivalence on the indecomposables. Since both categories are Krull-Schmidt we have an equivalence of categories. □

**Proposition 3.2.2.** Let $P$ and $Q$ be projective indecomposables in $\text{rep}_k(A_R)$ and $I$ and $J$ be injective indecomposables in $\text{rep}_k(A_R)$.

- Any morphism $f : P \to Q$ is either $\theta$ or $\text{mono}$.
- Any morphism $g : I \to J$ is either $\theta$ or $\text{epi}$.

**Proof.** We will prove the first statement; the second is dual. Let $f : P \to Q$ be a map of indecomposable projectives. By Theorem 2.1.16 and Remark 2.4.14 the image $\text{im } f$ in $Q$ is a submodule and so projective. Since $P$ surjects on to $\text{im } f$ it is a split subrepresentation of $P$. However, $P$ is indecomposable so $\text{im } f = 0$ or $\text{im } f \cong P$. □

Below, for each indecomposable representation $V$ in $\text{rep}_k(A_R)$ we create two projective representations $P_0(V)$ and $P_1(V)$. In Proposition 3.2.5 we prove that $P_1(V) \to P_0(V) \to V$ is the minimal projective presentation of $V$. 
**Construction 3.2.3.** Let $V$ be an indecomposable in $\text{rep}_k(A_\mathbb{R})$ with support $|a, b|$. If $V$ is projective let $P_0(V) = V$ and $P_1(V) = 0$.

Now suppose $V$ is not projective. Recall $S$ is the set of sinks and sources of $A_\mathbb{R}$ in $\mathbb{R}$. Since $V$ is finitely generated $|a, b| \cap S$ is finite. We let $P_0(V)$ be the direct sum of the following indecomposable projectives.

- $P_s$ for all sources $s$ in $(a, b)$.
- $P_{a}$ if $a \notin |a, b|$ and there exists $x \leq a$ in $|a, b|$.
- $P_{a}$ if $a \in [a, b]$ and there exists $x \leq a, x \neq a$ in $|a, b|$.
- $P_{b}$ if $b \notin [a, b]$ and there exists $x \leq b$ in $|a, b|$.
- $P_{b}$ if $b \in [a, b]$ and there exists $x \leq b, x \neq b$ in $|a, b|$.

We let $P_1(V)$ be the direct sum of the following indecomposable projectives.

- $P_s$ for all sources $s$ in $(a, b)$.
- $P_{a}$ if $a \notin |a, b|$ and there exists $a \leq x$ in $|a, b|$.
- $P_{a}$ if $a \in |a, b|$.
- $P_{b}$ if $b \notin [a, b]$ and there exists $b \leq x$ in $|a, b|$.
- $P_{b}$ if $b \in [a, b]$.

If $a$ or $b$ is a sink and in $|a, b|$ then the summand $P_{a}$ or $P_{b}$ is 0, respectively. We see that both $P_0(V)$ and $P_1(V)$ are nontrivial and finitely generated, so in $\text{rep}_k(A_\mathbb{R})$.

**Proposition 3.2.4.** Let $V$, $P_1(V)$, and $P_0(V)$ be as in Construction 3.2.3 Then there is an injective morphism $P_1(V) \hookrightarrow P_0(V)$ whose cokernel is $V$.

**Proof.** If $V$ is projective the statement is trivially true. Now suppose $V$ is not projective. There are finitely many sinks and sources, totally ordered. So on those summands we let the maps be defined in the following way where $\pm$ means scalar multiplication by $\pm 1$:

\[
\begin{array}{cccccccc}
\vdots & - & + & P_{s2n} & - & + & P_{s2n+2} & - & + & \ldots & - & + & \ldots \\
\vdots & P_{a} & P_{a} & P_{a} & P_{a} & P_{a} & \cdots & P_{a} & P_{a} & \cdots & P_{a} & P_{a} & \cdots \\
\end{array}
\]

Since there is no accumulation of elements of $S$ in $\mathbb{R}$, a projective indecomposable at $a$ can only appear as a summand of $P_0(V)$ or $P_1(V)$, but not both. The similar statement is true for $b$. Thus, only one type of projective summand of each $a$ or $b$ may appear in $P_0(V)$ and $P_1(V)$. Denote whichever summands appear, if any, by $P_{a}$ and $P_{b}$.

If $P_{a}$ appears in $P_1(V)$ then there is a nontrivial map from $P_{a}$ to $P_{s2n+1}$ or $P_{b}$, depending on whether or not $(a, b)$ contains any sources. If this is the case, use scalar multiplication by $-1$. In the similar case for $b$, use scalar multiplication by $+1$.

If $P_{a}$ appears in $P_0(V)$ then there is a nontrivial map from $P_{s2n}$ or $P_{b}$ to $P_{a}$, depending on whether or not $(a, b)$ contains any sinks. If this is the case, use scalar multiplication by $+1$. In the similar case for $b$, use scalar multiplication by $-1$.

Instead of proving that this map is injective with cokernel $V$, we instead note that the kernel of the surjection $P_0(V) \twoheadrightarrow V$ is $P_1(V)$. This is equivalent. \qed

**Proposition 3.2.5.** The following hold:

- For any indecomposable $V$ in $\text{rep}_k(A_\mathbb{R})$, $P_1(V) \hookrightarrow P_0(V) \twoheadrightarrow V$ is the minimal projective resolution and presentation of $V$.
- All representations in $\text{rep}_k(A_\mathbb{R})$ are finitely presented.
- The global dimension of $\text{rep}_k(A_\mathbb{R})$ is 1.

**Proof.** We see $P_1(V)$ is superfluous in $P_0(V)$ and $P_1(V) \hookrightarrow P_0(V) \twoheadrightarrow V$ is exact by Proposition 3.2.4. Thus the sequence is the minimal projective resolution and presentation of $V$. Furthermore, noting
that the reversal of orientation \(\preceq\) on \(\mathbb{R}\) gives the opposite category, we see the global dimension of \(\text{rep}_k(A_\mathbb{R})\) is 1. \(\square\)

**Proposition 3.2.6.** Let \(V\) and \(W\) be indecomposables in \(\text{rep}_k(A_\mathbb{R})\). If \(\text{Ext}^1(W, V) \neq 0\) then \(\text{Ext}^1(W, V) \cong k\).

**Proof.** Let \(V\) and \(W\) be indecomposables in \(\text{rep}_k(A_\mathbb{R})\). By Proposition 3.2.5, the projective resolution of \(V\) is \(P_1(V) \to P_0(V) \to V\). By definition \(\text{Ext}^i(V, W)\) is the \(i\)th homology group in the chain

\[
0 \to \text{Hom}(P_0(V), W) \to \text{Hom}(P_1(V), W) \to 0.
\]

Suppose \(\text{Ext}^1(W, V) \neq 0\).

Index the projectives in \(P_0(V)\) that nontrivially map to \(W\) from 1 to \(m\), denoted \(P_1, \ldots, P_m\), such that if \(P_a = P_i\) and \(P_b = P_{i+1}\) for \(a, b \in \mathbb{R}\) then \(a < b\). Then \(\text{Hom}(P_0(V), W) \cong k^m\). Let \(f : (x_1, \ldots, x_m)\) be a nontrivial map \(P_0(V) \to W\) and \(i : P_1(V) \to P_0(V)\) the inclusion. Index the projectives in \(P_1(V)\) that nontrivially map to \(W\) from 1 to \(n\), similarly to the projectives in \(P_0(V)\), denoted \(Q_1, \ldots, Q_n\).

Then \(Q_1\) maps to \(P_1\) and \(P_2\) or just \(P_1\). If \(Q_1\) only maps to \(P_1\) then the projective \(Q_2\) maps to both \(P_1\) and \(P_2\). If \(Q_1\) maps to both \(P_1\) and \(P_2\) then \(Q_2\) maps to \(P_2\) and \(P_3\). Thus, the composition \(f \circ i\) will be one of four forms:

- \(\{x_1, x_1 \oplus x_2, \ldots, x_{i-1} \oplus x_i\}\),
- \(\{x_1 \oplus x_2, \ldots, x_{i-1} \oplus x_i, x_i\}\),
- \(\{x_1, x_1 \oplus x_2, \ldots, x_{i-1} \oplus x_i, x_i\}\),
- \(\{x_1 \oplus x_2, \ldots, x_{i-1} \oplus x_i\}\).

In any case, basic linear algebra shows us that \(\text{Hom}(P_0(V), W) \to \text{Hom}(P_1(V), W)\) is surjective or injective and the difference in dimensions is either 0 or 1. Therefore \(\dim \text{Ext}^1(W, V)\) is 0 or 1. \(\square\)

### 3.3. Existence of Some Auslander-Reiten Sequences

In this subsection we will show that for any orientation of a continuous type \(A\) quiver, the category \(\text{rep}_k(A_\mathbb{R})\) contains some Auslander-Reiten sequences but not all Auslander-Reiten sequences (Theorem 3.3.1(6)). However, we will not provide a complete classification of Auslander-Reiten sequences in this paper. Such a classification will be provided in the sequel to this paper.

We recall the definition of an almost-split sequence, commonly called an Auslander-Reiten sequence. Such short exact sequences were originally defined by Auslander and Reiten in [AR].

**Definition 3.3.1.** Let \(\mathcal{A}\) be an abelian category and \(0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0\) a short exact sequence in \(\mathcal{A}\). The short exact sequence is an almost split sequence, or Auslander-Reiten sequence if the following conditions hold:

- \(f\) is not a section and \(g\) is not a retraction.
- \(U\) and \(W\) are indecomposable.
- If \(h : U \to X\) is a nontrivial morphism of indecomposables and \(U \not\cong X\) then \(h\) factors through \(f\).
- If \(h : X \to W\) is a nontrivial morphism of indecomposables and \(X \not\cong W\) then \(h\) factors through \(g\).

In the following proposition, recall that \(S\) is the set of sinks and sources in a continuous quiver of type \(A\) and that \(\bar{S}\) includes \(\pm \infty\).

**Proposition 3.3.2.** Let \(s_n, s_{n+1} \in \bar{S}\) and \(a, b \in \mathbb{R}\) such that \(s_n < a < b < s_{n+1}\). One of the following is a short exact sequence and in particular an Auslander-Reiten sequence.
• If \( s_n \) is a sink then the Auslander-Reiten sequence is

\[
0 \longrightarrow M_{[a,b]} \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} M_{[a,b]} \oplus M_{(a,b)} \xrightarrow{\begin{bmatrix} 1 & -1 \end{bmatrix}} M_{(a,b)} \longrightarrow 0
\]

• If \( s_n \) is a source then the Auslander-Reiten sequence is

\[
0 \longrightarrow M_{(a,b)} \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} M_{(a,b)} \oplus M_{[a,b]} \longrightarrow 0
\]

Proof. We note the two cases are symmetric and prove the first. We see the first map is injective, the second is surjective, and that the sequence is exact at \([a, b] \oplus (a, b)\). Thus, the sequence is a short exact sequence.

Denote the map \( M_{[a,b]} \rightarrow M_{[a,b]} \oplus M_{(a,b)} \) in the sequence by \( h_1 \oplus h_2 \). By Proposition 2.2.1 we know both \( M_{[a,b]} \) and \( M_{(a,b)} \) are indecomposable. Let \( V \) be another indecomposable representation in \( \text{rep}_k(A_2) \). By definition the support of \( V \) is an interval \([c, d]\). If there exists \( x \in [c, d] \) such that \( x < a \) then any morphism \( f : M_{[a,b]} \rightarrow V \) must be 0. Additionally, if there exists \( x \in [a, b] \) such that \( x \geq d \) and \( x \notin [c, d] \) then any \( f : M_{[a,b]} \rightarrow V \) must be 0. Thus, any morphism \( M_{[a,b]} \oplus M_{(a,b)} \rightarrow M_{(a,b)} \) must 0 and morphism \( M_{(a,b)} \rightarrow M_{[a,b]} \oplus M_{(a,b)} \) must be 0.

Claim: If \( V \not\cong M_{[a,b]} \) and \( f : M_{[a,b]} \rightarrow V \) is a nonzero morphism then there exists either a nonzero morphism \( g_1 : M_{[a,b]} \rightarrow V \) or \( g_2 : M_{(a,b)} \rightarrow V \) such that \( g_1 \circ h_1 = f \). Proof of claim: If \( V \not\cong M_{[a,b]} \) then, by the conditions in the previous paragraph combined with Theorem 2.3.2 either \( b \in [c, d] \) or \( a \notin [c, d] \). If \( b \in [c, d] \) then \( g_1 \) is a nonzero morphism and so \( g_1 \circ h_1 = f \). If \( a \notin [c, d] \) then \( g_2 \) is a nonzero morphism and so \( g_2 \circ h_2 = f \). In either case, \( f \) factors through \( M_{[a,b]} \oplus M_{(a,b)} \).

Finally, if \( [c, d] = [a, b] \) then by Theorem 2.3.2 \( f \) is an isomorphism. By a dual argument, a morphism from an indecomposable \( W \) to \( M_{(a,b)} \) that is not an isomorphism factors through \( M_{[a,b]} \oplus M_{(a,b)} \). Therefore, the given sequence is an Auslander-Reiten sequence. \( \square \)

We give an example of a representation with no left or right Auslander-Reiten sequences in the form of a proposition.

**Proposition 3.3.3.** Let \( M_a \) be the indecomposable representation with support \( \{a\} \) where \( a \) is neither a sink nor a source. Then there is no Auslander-Reiten sequences of either of the following forms:

\[
0 \longrightarrow M_a \longrightarrow B \longrightarrow C \longrightarrow 0
\]

\[
0 \longrightarrow A \longrightarrow B \longrightarrow M_a \longrightarrow 0.
\]

Proof. Suppose \( s_{2n} < a < s_{2n+1} \), where \( s_{2n} \) is a sink and \( s_{2n+1} \) is a source. The other case is similar. For any indecomposable \( M_I \), if \( \text{Hom}(M_I, M_{[a]}) \cong k \) then \( I = [c, a] \). We note that, for each \( x \in (s_0, a) \), \( \text{Hom}(M_{[x,a]}, M_{[a]}) \cong k \) for all \( i \geq 0 \).

Let \( M_I \) be some indecomposable such that \( \text{Hom}(M_I, M_{[a]}) \cong k \). For any \( x \in (s_0, a) \) such that \( c < x \) we have \( \text{Hom}(M_I, M_{[x]}) \cong k \). Since the Hom space between any two indecomposables is either \( k \) or 0 (Proposition 3.1.2), all nontrivial maps \( M_I \rightarrow M_{[a]} \) factor through every indecomposable \( M_{[x,a]} \) for \( x \in (s_0, a) \) and \( x > c \). Thus, it is not possible to have an Auslander-Reiten sequence in \( \text{rep}_k(A_2) \) of the form \( 0 \rightarrow A \rightarrow B \rightarrow M_{(a)} \rightarrow 0 \). By a dual argument, the other form is not possible, either. \( \square \)

4. Future Material

In the next paper, Continuous Quivers of Type A (II), we will completely classify all Auslander-Reiten sequences that appear in \( \text{rep}_k(A_2) \). Furthermore, we will define a continuous analog of the
Auslander-Reiten quiver, which we will call the Auslander-Reiten Space, for both rep(k(A_R)) and the bounded derived category: \(D^b(rep_k(A_R))\). We will show the Auslander-Reiten space exhibits many of the same properties as an Auslander-Reiten quiver, such as how to find extensions and Auslander-Reiten sequences. We will also completely classify all Auslander-Reiten triangles in \(D^b(rep_k(A_R))\) and prove which continuous type A quivers are derived-equivalent.

The papers after (II) will address an orbit category based on the definitions in [BMRRT], where cluster categories were defined as orbit categories, and in [IT], where the first and last author introduced the notion of a continuous cluster category. From there we will define continuous clusters, a generalization of clusters from [IT]. We will also show there is an association between continuous clusters and laminations of the hyperbolic plane. This extends the previous association to discrete laminations. Finally, we will define a continuous generalization of mutation that, when restricted to discrete or even transfinite mutation (introduced in [BG]) behaves as before.

The generalization of the discrete type A setting to a continuous setting allows for the embedding of any discrete structure into the continuous structure. The authors hope that the ability to work in the embedded model, as opposed to the discrete model in isolation, may lead to new techniques and results about known structures and new connections to existing areas of mathematics not yet known.

References

[AA et all] H. Adams, M. Aminian, E. Farnell, M Kirby, J. Mirth, R. Neville, C. Peterson, P. Shipman, and C. Shonkwiler. A fractal dimension for measures via persistent homology. Abel Symposia, 2019.

[AR] M. Auslander and I. Reiten, Representation theory of Artin algebras. III. Almost split sequences, Communications in Algebra, 1975, 3 (3): 239–294

[BG] K. Baur and S. Gratz, Transfinite mutations in the completed infinity-gon, Journal of Combinatorial Series A, Volume 155, April 2018, 321 – 359

[B] M.B. Botnan, Interval Decomposition of Infinite Zigzag Persistence Modules, Proceedings of the American Mathematical Society, 2017, Vol.145, pp.3571–3577

[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, Tilting theory and cluster combinatorics, Advances in Mathematics, Volume 204 (2006), 572 – 618

[CdSM] G. Carlsson, V. de Silva, and D. Morozov, Zigzag persistent homology and real-valued functions, Proceedings of the twenty-fifth annual symposium on computational geometry, 2009, 247–256

[CdSZ] G. Carlsson, T. Ishkhanov, V. de Silva, and A. Zomorodian, On the local behavior of spaces of natural images, International Journal of Computer Vision, 76:1–12, 2008

[CC-SGMO] F. Chazal, D. Cohen-Steiner, L. J. Guibas, F. Mémoli, and S. Y. Oudot. Gromov-Hausdorff Stable Signatures for Shapes using Persistence, Computer Graphics Forum (proc. Symposium on Geometry Processing) (2009), pp. 1393–1403.

[CO] F. Chazal and S. Oudot, Towards persistence-based reconstruction in Euclidean spaces, In Proceedings of the 24th Annual Symposium on Computational Geometry, pages 232–241. ACM, 2008

[CR] X.W. Chen and C.M. Ringel, Hereditary Triangulated Categories, Journal of Noncommutative Geometry, 12 (2018) 1 – 20.

[C-B] W. Crawley-Boevey, Decomposition of pointwise finite-dimensional persistence modules, Journal of Algebra and its Applications, 25 June 2015, Vol.14(5)

[EH] H. Edelsbrunner and J. L. Harer. Computational topology: an introduction. AMS Bookstore, 2010

[GGG] J. E. Grabowski and S. Gratz with M. Groechenig, Cluster algebras of infinite rank, J. London math. Soc (2) 89 (2014) 337 – 363

[IT] K. Igusa and G. Todorov, Continuous Cluster Categories I, Algebras and Representation Theory (2015), 18:65 – 101.

[JS] J. Jacquette and B. Schweinhart, Fractal Dimension Estimation with Persistent Homology: A Comparative Study, 2019. arXiv:1907.11182

[NLC] M. Nicolau, A. J. Levine, and G. Carlsson, Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival, Proceedings of the National Academy of Sciences 108.17 (2011), pp. 7265–7270

[S] B. Schweinhart, Fractal Dimensiona and the Persistent Homology of Random Geometric Complexes, arXiv:1808.02196
[ZC] A. Zomorodian and G. Carlsson, *Computing Persistent Homology*, Discrete and Computational Geometry 33:249–274 (2005)