FUNCTIONAL BOSONIZATION OF NON-RELATIVISTIC FERMIONS IN (2 + 1) DIMENSIONS

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Received (received date)
Revised (revised date)

We analyze the universality of the bosonization rules in non-relativistic fermionic systems in (2 + 1)d. We show that, in the case of linear fermionic dispersion relations, a general fermionic theory can be mapped into a gauge theory in such a way that the fermionic density maps into a magnetic flux and the fermionic current maps into a transverse electric field. These are universal rules in the sense that they remain valid whatever the interaction considered. We also show that these rules are universal in the case of non-linear dispersion relations provided we consider only density-density interactions. We apply the functional bosonization formalism to a non-relativistic and non-local massive Thirring-like model and evaluate the spectrum of collective excitations in several limits. In the large mass limit, we are able to exactly calculate this spectrum for arbitrary density-density and current-current interactions. We also analyze the massless case and show that it has no collective excitations for any density-density potential in the Gaussian approximation. Moreover, the presence of current interactions may induce a gapless mode with a linear dispersion relation.

1. Introduction

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The method of bosonization has proven to be a very useful technique in understanding the non-perturbative regime of strongly correlated fermionic systems in one spatial dimension. Its analysis has shown that the bosonization rules are universal, in the sense that they are model independent. Another important property is that the technique displays a powerful duality mechanism, which is evident in the connection of the non-perturbative (resp. perturbative) regime of the \((1 + 1)\)-dimensional massive Thirring model to the perturbative (resp. non-perturbative) one of the corresponding bosonized sine-Gordon model.2

In recent years, many efforts have been done in order to generalize bosonization to higher dimensions, with special emphasis on \((2 + 1)\) fermionic systems, due to their relevance for low-dimensional condensed-matter physics as, for instance, the quantum Hall effect and high-\(T_c\) superconductivity.

In the context of condensed-matter systems the first attempts to understand the bosonization of fluctuations around the Fermi surface in higher dimensions was made by A. Luther, followed by F. D. M. Haldane, who considered the two-dimensional system as a collection of infinite one-dimensional systems parametrized by an angle. Also, a functional bosonization method has been applied, essentially, to the evaluation of single-particle Green’s functions of interacting fermionic systems. The fermionic interaction is decoupled by making a Hubbard–Stratonovich transformation using a bosonic auxiliary field \(\phi(x)\). The Green’s function is calculated in a fixed \(\phi(x)\) background and then is averaged using an effective action that depends only on the Hubbard–Stratonovich field. This approach is suitable to implement several approximation schemes and also to systematically analyze the case of fermionic systems with non-linear dispersion relations. Bosonization of Fermi liquids in higher dimensions has been also considered by A. H. Castro Neto and E. Fradkin.

As for quantum field theory, two different approaches have been proposed to bosonize systems in higher dimensions, namely, the canonical and the functional methods. The canonical method was the one employed to obtain the first results for the bosonization of free massless fermions in two dimensions. In this case, bosonization is achieved by introducing a vector gauge field, for which the bosonized action takes the form of a non-local Maxwell–Chern–Simons model.

Functional bosonization, as a calculational tool, is undoubtedly a clear contribution to the understanding of strongly correlated electrons in higher dimensions. However, it is known in \((1 + 1)d\) systems that there are general properties that do not explicitly depend on the details of the one-particle Green’s function, but do depend on global features such as symmetries and the universality of the fermion–boson mapping. For instance, it has been shown recently that the conductance of finite incommensurate Peierls–Fröhlich systems is universal due to a global chiral anomaly and to the universality of the \((1 + 1)d\) bosonization rules. For this reason, it is relevant to ask oneself whether there exist, in higher dimensions, similar universal rules.
A first step in this direction has been taken by E. Fradkin and F. A. Schaposnik, who have analyzed the bosonization of a massive Thirring model in 2 + 1 dimensions. They have shown that in the infinite-mass limit the fermionic current maps into a topological current, and the bosonized action is a local Maxwell–Chern–Simons one. Since then, a great deal of work has been done towards the understanding of the bosonization structure in 2 + 1 dimensions. While the canonical method is useful to investigate the mapping among the fundamental fields of the fermionic and the bosonic models, the functional method is appropriate to establish the general framework for the higher-dimensional bosonization of current correlation functions.

In a recent paper, it has been shown for relativistic systems that the mapping between a fermionic current and a topological bosonic one is, in fact, universal. This means that this mapping may be performed for any kind of current interaction. In the present article we extend these results to the case of non-relativistic interacting fermions. We show that this universality remains valid in the case of non-relativistic fermions with a linearized dispersion relation. In the case of a non-linear dispersion relation, the boson–fermion mapping is universal, provided we only consider density–density interactions. As an example, we apply this technique to the case of a non-relativistic Thirring-like model with arbitrary density and current interactions. We compute the bosonized action in the Gaussian approximation and evaluate the spectrum of collective excitations of the model for arbitrary potentials. We find that in the large mass limit the system has collective excitations whose dispersion relation has a gap depending on the particular relation between the potentials. In the massless case, we show that the model has no collective excitations when considering only density interactions. However, a gapless collective mode appears when current–current interactions are taken into account.

The present article is organized as follows. In Section 2 we analyze the case of a linearized fermionic dispersion relation. In §2.1 we define a non-relativistic Thirring-like model and in §2.2 we review the functional bosonization technique. The Gaussian approximation for the Thirring-like model is developed in §2.3 and in §2.4 the two-particle correlation functions are evaluated. The spectrum of collective excitations for particular limits of the model are described in Section 3. Finally, in Section 4 we discuss the universality of the bosonization rules for interacting fermions with a non-linear dispersion relation and in Section 5 we present our conclusions.

2. A non-relativistic Thirring-like model in \((2 + 1)d\)

2.1. The model

Let us present a non-relativistic (and non-local) Thirring-like model in two spatial dimensions. The non-relativistic character of the model is displayed in the kinetic part of the action as well as in the interaction terms. In order to develop
the functional bosonization technique presented in the following section, we find the Lagrangian formalism more suitable. For this reason, we present the action of the non-relativistic Thirring-like model using the imaginary-time ($\tau$) formalism at zero temperature (Euclidean metric) in the following form:\(^{a}\)

$$S = S_0 + S_{\text{int}},$$

with

$$S_0 = \int_0^\infty d\tau \int d^2x \, \bar{\psi} (\gamma_\mu \partial_\mu + m) \psi$$

and

$$S_{\text{int}} = \int d\tau_1 d\tau_2 d^2x d^2y \, [V_0(x, y) J_0(x) J_0(y) + V(x, y) J_i(x) J_i(y)];$$

where the fermion field $\psi$ is written as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The interaction piece of the action has been written in terms of currents $J_\mu$ defined as

$$J_\mu = \bar{\psi} \gamma_\mu \psi \left\{ J_0 = \psi^\dagger \psi \\ J_i = \psi^\dagger \gamma_0 \gamma_i \psi \right\},$$

where we have adopted the representation

$$\gamma_0 = -i\sigma_3, \quad \gamma_1 = v_1 \sigma_1, \quad \gamma_2 = v_2 \sigma_2,$$

in which $\sigma_1, \sigma_2$ and $\sigma_3$ are the Pauli matrices, $v_1$ and $v_2$ could be identified with the components of some “Fermi velocity”, while $V_0(x, y)$ and $V(x, y)$ are symmetric bilocal arbitrary potentials describing density–density interactions and current–current interactions, respectively. The density–density potential represents a two-body scattering interaction and may take the form, for instance, of the usual non-relativistic Coulomb interaction. Although in a non-relativistic system the current–current potential is much weaker than the density–density one, we have introduced it here for two reasons. First, we have considered the potential $V(x, y)$ in order to turn the model more general, so that we could cover a greater set of situations by choosing, after the calculations, different potentials (note that in the case of $V_0 = V = g^2 \delta(x - y)$ we recover the usual relativistic local Thirring model). Second, in the last few years several models of fermions coupled with gauge fields were considered in the description of strongly correlated electrons.\(^{17}\) In particular, the Chern–Simons dynamics

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\(^{a}\)In what follows we will use the following notation: Greek indices run from zero to two while Latin indices may take the values one and two.
for the gauge field, incorporated in the Chern–Simons–Landau–Ginzburg theory for
the fractional quantum Hall effect \cite{18}, induces long-distance current–current inter-
actions upon integration on the transverse gauge fields, which could be modeled by
\( V(x, y) \) \cite{18}.

It is not difficult to show that the kinetic term of the action can be formally
manipulated almost in the same way that a relativistic one. In fact, the algebra of
the above defined \( \gamma \) matrices,

\[ \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu}, \tag{6} \]

induces a metric tensor \( g_{\mu\nu} = \text{diag}(1, v_1, v_2) \). With this metric, the square of a
three-vector \( k = (\omega, k_1, k_2) \) reads \( k^2 = k_\mu k^\mu = \omega^2 + v_1^2 k_1^2 + v_2^2 k_2^2 \). Thus, we can work
with a general Euclidean metric \( g_{\mu\nu} \) and perform, at the end of the calculations, an
analytic continuation of the metric coefficients \( v_1 \) and \( v_2 \). It is interesting to note
that in the relativistic case of \( v_1 = v_2 = 1 \) this procedure is nothing but the rigorous
definition of a Wick rotation. Moreover, we shall find interesting topological results
that do not depend on metric, so that without lost of generality we can work with
\( v_1 = v_2 = 1 \) and make, at the end of the calculation, the corresponding replacements.

The model presented in this section was extensively analyzed in \((1 + 1) d\) in the
context of the study of many-body physics of quantum wires \cite{21,22,23,24}. It is the
purpose of the present paper to extend its analysis to \( 2 + 1 \) dimensions.

2.2. The bosonization technique

In this section we present a functional bosonization technique in \( 2 + 1 \) dimensions
which shall be applied to the particular case of the model described in the previous
subsection. This technique is suitable for the calculation of current correlation
functions, as it essentially maps a conserved fermionic current into a topological
bosonic one. In \( 1 + 1 \) dimensions the bosonic field is a scalar one, provided the
currents map as

\[ J^F_\mu = \bar{\psi}\gamma_\mu \psi \longrightarrow J^B_\mu = \epsilon_{\mu\nu\rho} \partial_\nu \varphi. \tag{7} \]

Note that \( J^F_\mu \) is a conserved current \( (\partial_\mu J^F_\mu = 0) \) due to the global \( U(1) \) invariance of
the theory; in other words, it is conserved due to the field equations of motion. On
the other hand, the conservation of the bosonic version of the current \( (\partial_\mu J^B_\mu = 0) \)
is automatic and does not depend on the equations of motion. In this sense we say
that the current is bosonized into a topological one.

If one wishes to apply the bosonization technique in \( 2 + 1 \) dimensions in an
analogous fashion, one has to remember the fact that the only way to build up a
topological current in \( 2 + 1 \) dimensions is by employing a vector field, instead of
a scalar one. Thus, we are looking for a bosonization scheme that maps fermionic
currents into bosonic ones in the following way:

\[ J^F_\mu = \bar{\psi}\gamma_\mu \psi \longrightarrow J^B_\mu = \epsilon_{\mu\nu\rho} \partial_\nu A_\rho. \tag{8} \]
Note that, again in this case, $\partial_\mu J^B_\mu = 0$ follows automatically, due to the totally antisymmetric Levi-Civita tensor $\epsilon_{\mu\nu\rho}$ in $(2 + 1)d$. It is interesting to remark that the fermionic current is invariant under a local $U(1)$ gauge transformation of the fields given by

$$\psi \rightarrow e^{i\alpha(x)}\psi$$
$$\bar{\psi} \rightarrow \bar{\psi}e^{-i\alpha(x)},$$

conversely, the bosonic topological current also has this invariance, since, when the vector field transforms as

$$A_\mu \rightarrow A_\mu + i\partial_\mu \alpha(x),$$

then

$$J'_\mu = \epsilon_{\mu\nu\rho} \partial_\nu (A_\rho + i\partial_\rho \alpha(x)) = J_\mu,$$

due to the antisymmetry of $\epsilon_{\mu\nu\rho}$. Therefore, we expect that, after the bosonization process, the bosonized theory to be a gauge theory.

In the functional formalism, the current correlation functions are evaluated from the generating functional

$$Z[s] = \int D\bar{\psi}D\psi e^{-S[\bar{\psi},\psi] - i \int d^3x \; s_\mu J^F_\mu},$$

by functionally differentiating equation (13) with respect to the source $s_\mu$:

$$\langle J^F_\mu(x_1) \ldots J^F_\mu(x_n) \rangle_S = i^n \frac{\delta \ln Z[s]}{\delta s_\mu(x_n) \ldots \delta s_\mu(x_1)}.$$  

The aim of this program is to find a bosonic representation to the generating functional of current correlation functions in the form

$$Z[s] = \int DA_\mu e^{-S_{bos}[A_\mu]} - i \int d^3x \; s_\mu J^B_\mu,$$

so that

$$\langle J^B_\mu(x_1) \ldots J^B_\mu(x_n) \rangle_{S_{bos}} = i^n \frac{\delta \ln Z[s]}{\delta s_\mu(x_n) \ldots \delta s_\mu(x_1)}.$$  

By comparing eqs. (14) and (16) we see that the theories described by $S[\psi]$ and $S_{bos}[A_\mu]$ are equivalent in the sense that

$$\langle J^F_\mu(x_1) \ldots J^F_\mu(x_n) \rangle_S = \langle J^B_\mu(x_1) \ldots J^B_\mu(x_n) \rangle_{S_{bos}}.$$  

In order to write a formal expression for the bosonized action and currents, we follow the path-integral approach of refs. [11], [12], [16]. We can write the generating functional for interacting massive fermions as
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\[ Z[s] = \int D\psi D\bar{\psi} e^{-\int d^3x \bar{\psi}(\partial + im + is\cdot)\psi - S^I[J]}, \]  

(18)

where we use the notation \( \mathbf{s} = \gamma_\mu s_\mu \) and \( \mathbf{\varphi} = \gamma_\mu \partial_\mu \); \( S^I[J] \) is an arbitrary interaction action depending only on fermionic currents. The non-relativistic Thirring model presented above is a particular case corresponding to quadratic current interactions. In what follows we shall consider any type of current interactions (not necessarily quadratic). The integration of the fermionic variables is a very difficult task due to the interaction part of the action which may contain non-quadratic fermionic terms. The usual trick in the case of local quartic interactions is to perform a Hubbard–Stratonovich transformation in order to obtain a quadratic fermionic action, at the cost of introducing a new auxiliary field. However, for an arbitrary interaction, such a transformation is not available. Here, we shall obtain a quadratic fermionic action for arbitrary interacting potentials, provided that they only depend on the fermionic current. The technique that we will use was presented for the first time in ref. 16.

We begin by reexpressing the exponential of the fermionic interaction \( S^I[J] \) in terms of its functional Fourier transform,

\[ e^{-S^I[J]} = \int DA_\mu e^{-S[a]} - i \int d^3x J_\mu a^\mu. \]  

(19)

Therefore, for the generating functional of eq.(18), we get

\[ Z^{\text{int}}[s] = \int DA_\mu D\psi D\bar{\psi} e^{-\int d^3x \bar{\psi}(\partial + im + is\cdot)\psi - S[a]}. \]  

(20)

In this way, we can interpret the fermionic interaction as an effective interacting fermionic system minimally coupled to a dynamical quantum vector field \( a_\mu \) whose action is \( S[a] \). Note that for a generic interaction \( S^I[J] \) which is not quadratic in the current \( J_\mu \) the computation of \( S[a] \) is, in general, not possible. However, we will see that our final result will be given only in terms of the known quantity \( S^I[J] \), without relying on the explicit computation of its Fourier transform \( S[a] \). In other words, all that will be needed is the reasonable assumption that the Fourier representation (19) does in fact exist. In particular, for a local density–density two-body interaction, \( S[a] \) is quadratic and (19) coincides with the usual Hubbard–Stratonovich transformation.

The next step towards bosonization is to decouple the external source \( s_\mu \) from the fermion fields. This is accomplished by first making the following change of variables (with trivial Jacobian)

\[
\begin{align*}
\psi & \quad \rightarrow \quad e^{i\alpha(x)}\psi, \\
\bar{\psi} & \quad \rightarrow \quad \bar{\psi} e^{-i\alpha(x)}, \\
D\psi D\bar{\psi} & \quad \rightarrow \quad D\psi D\bar{\psi},
\end{align*}
\]  

(21)
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The resulting generating functional $Z[s]$ does not depend on $\alpha$; therefore, we can integrate over $\alpha$ modifying only a global multiplicative factor that will not enter the correlation functions. We thus have

$$Z[s] = \int D\psi D\bar{\psi} D\alpha D\alpha' e^{-\int d^3x \bar{\psi}(i\partial + s + a + b\alpha + m)\psi - S[a]}.$$  \hspace{1cm} (23)

Let us now introduce the field $b_\mu$ through the condition

$$\partial_\mu \alpha = b_\mu.$$  \hspace{1cm} (24)

In order to integrate over $b_\mu$ (instead of $\alpha$) in the path-integral (23) we must impose that $b_\mu$ be a pure-gauge field. This can be done by inserting the delta functional $\delta(\varepsilon_{\mu\nu\rho} f_{\nu\rho}[b])$ in the expression for $Z[s]$, $f_{\mu\nu} = \partial_\nu b_\rho - \partial_\rho b_\nu$ being the $b$ field strength. Thus,

$$Z[s] = \int D\psi D\bar{\psi} D\alpha Db_\mu \delta(\varepsilon_{\mu\nu\rho} f_{\nu\rho}[b]) e^{-\int d^3x \bar{\psi}(i\partial + s + a + b + m)\psi - S[a]}.$$  \hspace{1cm} (25)

The source $s_\mu$ now decouples from the fermions by shifting the field through $b_\mu$

$$b_\mu \rightarrow b_\mu - s_\mu - a_\mu,$$  \hspace{1cm} (26)

so that we obtain

$$Z[s] = \int D\alpha D\alpha' Db_\mu \delta(\varepsilon_{\mu\nu\rho} f_{\nu\rho}[b - s - a]) e^{-\int d^3x \bar{\psi}(i\partial + s + b + m)\psi - S[a]}.$$  \hspace{1cm} (27)

The fermionic variables now may be integrated out since their integral is quadratic and is decoupled from the source $s_\mu$. The resulting fermionic determinant may be written as an exponential factor, through the property \( \ln \det \hat{O} = \text{Tr} \ln \hat{O} \), and we get

$$Z[s] = \int D\alpha D\alpha' Db_\mu \delta(\varepsilon_{\mu\nu\rho} f_{\nu\rho}[b - s - a]) e^{\text{Tr} \ln(\partial + i\bar{\psi} + m) - S[a]}.$$  \hspace{1cm} (28)

Exponentiating the delta functional by means of a Lagrange multiplier $A_\mu$, the generating functional becomes

$$Z[s] = \int D\alpha D\alpha' D\alpha' D\alpha'' D\alpha'' \delta(\varepsilon_{\mu\nu\rho} f_{\nu\rho}[b - s - a]) e^{-K_B[A]} e^{-i \int d^3x \varepsilon_{\mu\nu\rho} A_\rho \partial_\nu (s_\mu + a_\mu) - S[a]},$$  \hspace{1cm} (29)

where

$$e^{-K_B[A]} = \int D\bar{b}_\mu e^{\text{Tr} \ln(\partial + i\bar{\psi} + m) + \text{Tr} \ln(\partial + i\bar{\psi} + m)} e^{i \int d^3x \varepsilon_{\mu\nu\rho} A_\rho \partial_\nu (s_\mu + a_\mu) - S[a]}.$$  \hspace{1cm} (30)

isolates the $b$-dependent part of the generating functional. We now see that the exponential of the bosonizing free action $K_B$ is obtained from the transverse Fourier transform of the exponential of the effective action $\text{Tr} \ln(\partial + i\bar{\psi} + m)$. 

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Integrating now over $a_\mu$ and using again the representation (19), we finally obtain

$$Z[s] = \int \mathcal{D}A_\mu \ e^{-S_{bos}[A_\mu]} - i \int d^3x \ s_\mu \varepsilon_{\mu\nu\rho} \partial_\nu A_\rho, \quad (31)$$

where

$$S_{bos}[A_\mu] = K_B[A] + S^I[\epsilon \partial A]. \quad (32)$$

As anticipated, we see that our final result relies only on the known action $S^I$, the unknown action $S[a]$ being only needed at the intermediate steps. We also see that the bosonized theory is a gauge theory. This statement is simple to verify. The interaction part of the action depends only on the current $J_\mu$ that is automatically gauge invariant as shown in (12). The gauge invariance of the kinetic term $K_B$ can be understood from equation (30). If we make a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$, the last term in the exponential of (30) transforms as

$$\int d^3x \ \varepsilon_{\mu\nu\rho} (A_\mu + \partial_\mu \alpha) \partial_\nu A_\rho = \int d^3x \ \varepsilon_{\mu\nu\rho} \partial_\nu A_\rho - \int d^3x \ \alpha \varepsilon_{\mu\nu\rho} \partial_\nu \partial_\nu \partial_\rho \quad (33)$$

where we have integrated by parts the last term. Of course, this term vanishes, due to the antisymmetry of $\varepsilon_{\mu\nu\rho}$, from which results the gauge invariance of $K_B$.

Comparing eqs. (18) and (31), we can read off the bosonization rules for the fermionic action as well as for the fermionic current. We see that the bosonic field has the property of a gauge field, and that the bosonic current takes the form

$$J_\mu = \varepsilon_{\mu\nu\rho} \partial_\nu A_\rho \ \left\{ \begin{array}{c} J_0 = B \\ J_i = \epsilon_{ij} E_j \end{array} \right. \quad (34)$$

Therefore, the fermionic density bosonizes to a “magnetic” field $B$ and the fermionic current to a transverse “electric” field $E_i$, both related to the $A_\mu$-field. These are the main results of this section and they are summarized in Table 1.

Another interesting and important result is that the kinetic term and the interaction term of the fermionic action are bosonized independently. That is to say, the bosonization process does not mix these terms. Last but not least, the equivalence

| Fundamental Fields | Fermionic $\psi = (\psi_1 \ \psi_2)$ | Bosonic $A_\mu$ |
|--------------------|----------------------------------|-----------------|
| Conserved Current  | $\psi \gamma^0 \gamma_\mu \psi$  | $\varepsilon_{\mu\nu\rho} \partial_\nu A_\rho$ |
| Kinetic Term       | $\int d^3x \ \psi (\partial + m) \psi$ | $K_B(A_\mu)$ |
| Interaction Term   | $S^I(\psi \gamma_\mu \psi)$ | $S^I(\varepsilon_{\mu\nu\rho} \partial_\nu A_\rho)$ |

Table 1. Bosonization rules for fermions with linearized dispersion relation in 2 + 1 dimensions
between the fermionic and the bosonic current is not only exact but it is also a solid universal result, in the sense that it remain the same in the interacting quantum theory. We have shown that we can bosonize any term involving the fermionic current by simply replacing $J_\mu$ by the bosonic topological current $\varepsilon_{\mu\nu\rho}\partial_{\nu}A_{\rho}$.

The universality for such a large class of interaction terms shown in this section is strongly dependent on the fact that the kinetic theory has only first order derivatives. We shall come back to this point in §4.2.

2.3. The Gaussian approximation

As we have seen, the results depicted in table I are exact and universal. In other words, they do not depend on any particular detail of the fermionic interaction provided that it depends only on the fermionic vector bilinear. In this sense, we have exactly bosonized the current and a general class of interactions. However, the kinetic fermionic action is bosonized into $K_B[A_\mu]$ given by eq. (30). Thus, in order to obtain an explicit expression for this action we have to evaluate the transverse Fourier transform of the exponential of the fermionic determinant. Of course, this is a very difficult task and no exact result is known in two spatial dimensions. Nevertheless, expressions (30), (31) and (32) constitute the starting point for different approximation schemes not always available in the fermionic version. For example, the semiclassical approximation is very easily performed in a bosonic theory, while it is very hard to handle in the fermionic counterpart.

A very interesting observation about the action $K_B$ was made in ref. 16. It was shown that there is a non-linear and non-local field transformation that maps the bosonized action $K_B$ into a pure local Chern–Simons form. The underlying topological structure of the bosonized action is then made explicit and, in some sense, this transformation is a convenient form to distinguish the topological contents of a fermionic theory from the non-topological dynamical contributions. However, although this formal observation is a clear positive contribution towards the understanding of the bosonized action, the practical applications of such a structure as a reliable computational tool remain undeveloped.

Therefore, it is necessary to develop an approximation scheme for the evaluation of current correlation functions. In ref. 12, the limit in which the “mass” parameter $m$ is taken to infinity was considered. This is equivalent to a low-energy expansion of the fermionic theory, since the energy fluctuations must be much smaller than the typical energy of the gap, $2m$. In this limit, the bosonized kinetic term takes the form of a local Maxwell–Chern–Simons term:

$$K_B[A_\mu] = \int d^3x \frac{i}{2} \varepsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{4m}F_{\mu\nu}F_{\mu\nu} + \cdots$$

This is a quadratic action, simplifying in this way the evaluation of current correlation functions. It is worth stressing that in this limit the bosonized action is exact, in the sense that the next quartic term is proportional to $1/m^2$.

In ref. 23, a more general approximation was discussed, the so-called Gauss-
sian approximation, in which one takes into account the full quadratic part of the
fermionic determinant, which may be written as

$$\text{Tr} \ln(\partial + i\partial + m) = T_{PC}(b) + T_{PV}(b)$$

(36)

with

$$T_{PC}(b) = -\frac{1}{4} \int d^3 x \ F_{\mu\nu}(b) F(-\partial^2) \ F_{\mu\nu}(b),$$

(37)

$$T_{PV}(b) = -\frac{i}{2} \int d^3 x \ b_\mu G(-\partial^2) \ \epsilon_{\mu\nu\lambda} \partial_\nu b_\lambda,$$

(38)

where $T_{PC}$ and $T_{PV}$ come from the parity-conserving and parity-violating pieces of
the vacuum-polarization tensor, respectively.

A standard one-loop calculation for the Fourier transforms of the functions $F$ and $G$ in (36) yields

$$\tilde{F}(k^2) = \frac{|m|}{4\pi k^2} \left[ 1 - \frac{1}{k^2} \frac{k^2}{4m^2} \text{arcsin}(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}} \right],$$

$$\tilde{G}(k^2) = \frac{m}{2\pi |k|} \text{arcsin}(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}},$$

(39)

where here and in what follows we shall always denote momentum-space represen-
tations by putting a tilde over the corresponding coordinate-space representation
quantity.

In order to evaluate $K_B$, it is necessary to calculate the functional transverse
Fourier transform of (36). Since this expression is quadratic, it is a simple task to
evaluate the functional integral provided we have properly fixed the gauge. We thus
obtain

$$K_B = \int d^3 x \left\{ \frac{1}{4} F_{\mu\nu} C_1 F_{\mu\nu} - \frac{i}{2} A_\mu C_2 \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \right\},$$

(40)

where

$$C_1 = \frac{F}{-\partial^2 F^2 + G^2},$$

(41)

$$C_2 = \frac{G}{-\partial^2 F^2 + G^2}.$$
approximation is in fact the exact one for any “mass”, provided we use it to calculate two-point correlations functions. For higher order correlation functions it is not at all easy to justify this approximation for small \(m\).

Turning now to the interaction term of the model, its bosonization is performed by just replacing \(J_0 \rightarrow \frac{1}{2} \epsilon_{ij} F_{ij}\) and \(J_i \rightarrow \frac{1}{2} \epsilon_{i\mu\nu} F_{\mu\nu}\) in (3). We thus obtain

\[
S_I[A_\mu] = \int d^3x d^3y \left\{ \epsilon_{ij} F_{ij}(x) V_{(0)}(x, y) \epsilon_{lm} F_{lm}(y) + F_{i0}(x) V_{(1)}(x, y) F_{i0}(y) \right\} .
\]

(43)

Finally, from eqs. (40) and (43) we can write the bosonized action for the non-relativistic Thirring model in \(2 + 1\) dimensions in the Gaussian approximation as

\[
S_{bos}[A] = \int d^3 x \left\{ \frac{1}{4} F_{\mu\nu} C_1 F_{\mu\nu} - i \frac{1}{2} A_\mu C_2 \epsilon_{\mu\nu\lambda} \partial_\lambda A_\lambda \right\} \\
+ \int d^3 x d^3 y \left\{ \frac{1}{4} \epsilon_{ij} F_{ij}(x) V_0(x, y) \epsilon_{lm} F_{lm}(y) + F_{i0}(x) V(x, y) F_{i0}(y) \right\} .
\]

(44)

2.4. Current–density correlation functions

In order to calculate current–current correlation functions, we need to evaluate the generating functional \(Z[s]\) given by \((31)\) and \((32)\). We shall use for \(S_{bos}\) the Gaussian approximation obtained in \((44)\).

Since we are dealing with a gauge theory, to functional integrate the generating functional we have to properly fix the gauge in \((44)\). We gain more insight of the problem by choosing the Coulomb gauge \(\vec{\nabla} \cdot \vec{A} = 0\) and reexpressing the gauge-fixed action in terms of \(A_0\) and the associated magnetic field \(B = \vec{\nabla} \times \vec{A}\).

We note that, since \(\vec{\nabla} \cdot \vec{A} = 0\), we can always write the components of the field \(\vec{A}\) as

\[
A_i = \epsilon_{ij} \partial_j \phi,
\]

(45)

where \(\phi(x)\) is an arbitrary scalar field. The relation between \(B\) and \(\vec{A}\) implies, together with \((17)\), that \(B\) is none other that the Laplacian of the field \(\phi\) (apart from a sign), so that the vector potential is expressed in terms of \(B\) using the Laplacian Green’s function:

\[
A_i = -\epsilon_{ij} \frac{\partial_j}{\nabla^2} B.
\]

(46)

In this way we can write the generating functional in the Coulomb gauge as a function of \(A_0\) and \(B\):

\[
Z[s] = \int DA_0 DB e^{-S_0(A_0, B) - i \int d^3 x s_0 B + s_i \left( \epsilon_{ij} \partial_j A_0 - \frac{\partial_0 \phi}{\nabla^2} B \right)} ,
\]

(47)
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\[ S_{\text{bos}} = \frac{1}{2} \int d^3x d^3y \left[ B(x) \left\{ C_1 \left(1 + \frac{\partial_0^2}{\nabla^2}\right) + \left(V_0 + V \frac{\partial_0^2}{\nabla^2}\right)\right\} (x - y)B(y) \right. \\
- \left. \frac{1}{2}A_0(x) \left\{ \nabla^2 (C_1 + V) \right\} (x - y)A_0(y) - iA_0(x)C_2(x - y)B(y) \right], \quad (48) \]

where \(C_1\) and \(C_2\) are given by (41) and (42) and \(V_0(x - y)\) and \(V(x - y)\) are the density–density and current–current interaction potentials, respectively.

At this stage of the calculation, it is convenient to write the action in momentum space. Thus, by Fourier transforming (48), we obtain

\[ S_{\text{bos}} = \frac{1}{2} \int \frac{d\omega d\vec{k}}{(2\pi)^3} \tilde{B}^* \left\{ \tilde{C}_1 \left(1 + \frac{\omega^2}{|\vec{k}|^2}\right) + \left(\tilde{V}_0 + \tilde{V} \frac{\omega^2}{|\vec{k}|^2}\right)\right\} \tilde{B} \\
- \frac{1}{2} \int \frac{d\omega d\vec{k}}{(2\pi)^3} \tilde{A}_0^* \left\{ \vec{k}^2 \left(\tilde{C}_1 + \tilde{V}\right)\right\} \tilde{A}_0 \\
- i \int \frac{d\omega d\vec{k}}{(2\pi)^3} \tilde{A}_0(\omega, \vec{k})\tilde{C}_2(-\omega, -\vec{k}). \quad (49) \]

Integrating out the \(\tilde{A}_0\) and \(\tilde{B}\) fields, we find

\[ Z[s] = e^{-\int d\omega d\vec{k} \, \tilde{s}_\mu(\omega, \vec{k}) \Pi_{\mu\nu}(\omega, \vec{k}) \tilde{s}_\nu(-\omega, -\vec{k})}, \quad (50) \]

where the vacuum polarization tensor \(\Pi_{\mu\nu}\) takes the form

\[ \Pi_{\mu\nu}(\omega, \vec{k}) = \mathcal{P}(\omega, \vec{k}) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + i \mathcal{Q}(\omega, \vec{k}) \epsilon_{\mu\nu\rho} k_\rho, \quad (51) \]

with

\[ \mathcal{P}(\omega, \vec{k}) = (\omega^2 + k^2) \left(\tilde{V} + \tilde{C}_1\right) \Delta(\omega, k) \quad (52) \]
\[ \mathcal{Q}(\omega, \vec{k}) = \tilde{C}_2 \Delta(\omega, k) \quad (53) \]

and

\[ \Delta(\omega, \vec{k}) = \frac{1}{\left(\tilde{V} + \tilde{C}_1\right) \left\{ \omega^2 \left(\tilde{C}_1 + \tilde{V}\right) + \vec{k}^2 \left(\tilde{C}_1 + \tilde{V}_0\right)\right\} + \tilde{C}_2^2}. \quad (54) \]

The vacuum polarization tensor for the interacting theory \(\Pi_{\mu\nu}\) has the correct symmetry properties. In fact,

\[ \Pi_{\mu\nu}(\omega, \vec{k}) = \Pi_{\nu\mu}(\omega, \vec{k}) = \Pi_{\nu\mu}(-\omega, -\vec{k}); \quad (55) \]

also, it must be transverse, as a consequence of gauge invariance:
\[ k_\mu \Pi_{\mu\nu} = \Pi_{\nu\mu} k_\mu = 0. \]  

(56)

This condition is automatically satisfied due to the tensor structure of \( \Pi_{\mu\nu} \) (see eq. (51)).

It is interesting to note that we can rewrite the parity-conserving part of the vacuum polarization tensor as

\[
\mathcal{P}(\omega, \vec{k}) = \frac{\omega^2 + k^2}{\left( \omega^2 \left( 1 + \frac{\tilde{V}}{C_1} \right) + \vec{k}^2 \left( 1 + \frac{\tilde{V}_0}{C_1} \right) \right) + \frac{C_2^2}{C_1^2} \frac{1}{1 + \frac{\tilde{V}}{C_1}}}. 
\]

(57)

This formula should be compared with the one.spatial-dimensional result \(^2\). We see that the effect of the kinematics in two space dimensions, given by \( \tilde{C}_1 \) and \( \tilde{C}_2 \), is twofold. The last term in the denominator (proportional to \( \tilde{C}_2 \)) is due to a parity-breaking effect and we shall show that it is responsible for opening gaps in the excitation spectrum (see Section 3). The parity-conserving effects renormalize the potentials as \( \tilde{V}^R = \frac{\tilde{V}}{C_1} \) and \( \tilde{V}_0^R = \frac{\tilde{V}_0}{C_1} \). Note that this is a dynamical renormalization (since \( \tilde{C}_1 \) depends on \( \omega \)), which changes the structure of the interactions. However, we shall see that, in the large \( m \) limit \( \tilde{C}_1 \) is a constant and the net effect is just a coupling-constant renormalization.

Taking into account this structure for \( \Pi_{\mu\nu} \), it is now simple to write an expression for the current expectation value as we have, by definition,

\[
\langle J_\mu \rangle = \frac{\delta \ln Z[s]}{\delta s_\mu}; 
\]

(58)

from eq. (50), it is expressed in terms of \( \Pi_{\mu\nu} \) in the form

\[
\langle J_\mu \rangle = \Pi_{\mu\nu} s_\nu = \Pi_{\mu0} s_0 + \Pi_{\mu i} s_i. 
\]

(59)

Gauge invariance implies that the components of \( \Pi_{\mu\nu} \) are not independent (see eq. (54)). In particular, we have

\[
\Pi_{\mu0} = -\frac{1}{\omega} \Pi_{\mu i} k_i. 
\]

(60)

Using this relation and defining an external electric field as \( \vec{E}^{ext} = \vec{\nabla} s_0 - \partial_0 \vec{s} \), we find

\[
\langle J_\mu \rangle = i \frac{1}{\omega} \Pi_{\mu j} \vec{E}_j^{ext}. 
\]

(61)

It is interesting to have an explicit equation for the density fluctuations (in the linear response approximation) induced by an arbitrary external electromagnetic field. We obtain from (51) and (61).
\[ \langle \rho \rangle = \frac{1}{\omega} \Pi_{ij} \tilde{E}^{\text{ext}}_j \]
\[ = -\frac{i}{\omega^2 + \vec{k}^2} \mathcal{P}(\omega, \vec{k}) k_i \tilde{E}^{\text{ext}}_i + \frac{1}{\omega} \mathcal{Q}(\omega, \vec{k}) \epsilon_{ij} k_j \tilde{E}^{\text{ext}}_j \]
\[ = -\tilde{\Delta}(\omega, \vec{k})(\tilde{V} + \tilde{C}_1)\mathcal{F} \left( \nabla \cdot \tilde{E}^{\text{ext}} \right) - \frac{i}{\omega} \tilde{C}_2 \Delta(\omega, \vec{k}) \mathcal{F} \left( \nabla \times \tilde{E}^{\text{ext}} \right), \quad (62) \]

where \( \mathcal{F}(\ldots) \) denotes Fourier transform.

Noting that the external classical field \( \tilde{E}^{\text{ext}} \) satisfies Maxwell’s equations, we finally obtain

\[ \langle \rho \rangle = -\tilde{\Delta}(\omega, \vec{k}) \left\{ \tilde{\rho}^{\text{ext}}(\tilde{V} + \tilde{C}_1) + \tilde{C}_2 \tilde{B}^{\text{ext}} \right\}. \quad (63) \]

We see that a density-charge excitation in the system may arise as a consequence of external charges and/or external magnetic fields. The general relation between a magnetic field and density comes, in general, from the parity-breaking terms of the model. In the fermionic version this is seen as a gap in the spectrum and, in the bosonic picture, as a Chern–Simons-like action. Equation \((63)\) is a generalization of the Aharonov–Casher calculation \(^{28}\) to interacting systems submitted to an arbitrary electromagnetic field.

Similarly to \( \langle \rho \rangle \), we can write an expression for \( \langle J_i \rangle \):

\[ \langle J_i \rangle = i \left\{ \frac{\mathcal{P}(\omega, \vec{k})}{\omega} \tilde{E}^{\text{ext}}_i + i \mathcal{Q}(\omega, \vec{k}) \epsilon_{ij} \tilde{E}^{\text{ext}}_j \right\} - \frac{k_i \mathcal{P}(\omega, \vec{k})}{\omega(\omega^2 + \vec{k}^2)} \tilde{\rho}^{\text{ext}}. \quad (64) \]

In the case of a divergenceless external electric field, we find

\[ \langle J_i(\omega, \vec{k}) \rangle = \sigma_{ij}(\omega, \vec{k}) \tilde{E}^{\text{ext}}_j(\omega, \vec{k}), \quad (65) \]

where we have introduced a conductivity tensor

\[ \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix}, \quad (66) \]

for whose elements we have

\[ \sigma_{xx}(\omega, \vec{k}) = \sigma_{yy}(\omega, \vec{k}) = \frac{\mathcal{P}(\omega, \vec{k})}{\omega}, \quad (67) \]
\[ \sigma_{xy}(\omega, \vec{k}) = \frac{\mathcal{Q}(\omega, \vec{k})}{\omega}. \quad (68) \]

In the same way we have calculated expressions for the current mean value, we can write expressions for the two-point current correlation function, as it takes the simple form

\[ \langle J_{\mu}(x) J_{\nu}(y) \rangle = \frac{\delta \ln Z[s]}{\delta s(y) \delta s(x)} = \Pi_{\mu\nu}(x - y). \quad (69) \]
In particular, using (51), the density–density correlation function may be written as

$$\langle \rho \rho \rangle = \Pi_{00} = \vec{k}^2 (\tilde{V} + \tilde{C}_1) \tilde{\Delta}(\omega, \vec{k}).$$  \hfill (70)

The excitation spectrum is given by the analytic properties of $\tilde{\Delta}$. In the next section we shall develop this point in detail.

Similarly, the spatial components of the two-point current correlation function also take a simple form:

$$\langle J_1 J_1 \rangle = (\tilde{V} + \tilde{C}_1) (\omega^2 + k_1^2) \tilde{\Delta}(\omega, \vec{k}),$$ \hfill (71)

$$\langle J_2 J_2 \rangle = (\tilde{V} + \tilde{C}_1) (\omega^2 + k_2^2) \tilde{\Delta}(\omega, \vec{k}),$$ \hfill (72)

$$\langle J_1 J_2 \rangle = -\left\{ (\tilde{V} + \tilde{C}_1) k_1 k_2 + \tilde{C}_2 \omega \right\} \tilde{\Delta}(\omega, \vec{k}).$$ \hfill (73)

3. Spectrum of collective excitations

3.1. The large $m$ limit

We have seen from the density–density correlation function (eq. (71)) that the spectrum of collective excitations can be read off from the singularities of $\tilde{\Delta}(\omega, \vec{k})$ in the complex $\omega$ plane. Let us analyze here the case of a large fermionic gap where the simplest case to be considered is the infinite-gap limit. Performing the $m \to \infty$ limit in eq. (39), we obtain

$$\lim_{m \to \infty} \tilde{F}(k) = 0,$$ \hfill (74)

$$\lim_{m \to \infty} \tilde{G}(k) = \frac{1}{4\pi},$$ \hfill (75)

implying $\tilde{C}_1 = 0$ and $\tilde{C}_2 = (1/2\pi)^3$. Replacing these values into eq. (74), we obtain

$$\tilde{\Delta}(\omega, \vec{k}) = \frac{1}{V^2} \frac{\omega^2 + \vec{k}^2 \omega}{\omega^2 + \frac{1}{(2\pi)^3 V^2}}.$$ \hfill (76)

In the infinite-gap limit in the absence of current–current interactions, $\tilde{\Delta}$ does not depend on $\omega$, which implies the nonexistence of any kind of density excitation. Moreover, the conductivity of the system is $\sigma_{xx} = \sigma_{yy} = 0$, while $\sigma_{xy} = 1/2$ (in units of $e^2/h$). This result is completely independent of the density–density potential $V_0$. The presence of a transverse current comes from the fact that the fermionic system is not invariant under a parity transformation due to the term proportional to $m \bar{\psi} \psi$. As a consequence, the system behaves as an insulator ($\sigma_{xx} = 0$) whatever the potential $V_0$: although there is a transverse current, it does not dissipate energy since it is perpendicular to the applied electric field. It is important to stress that
this is the same behavior of free fermions with a large gap, as the effective action for
the external electromagnetic field is a pure topological Chern–Simons one. Through
this simple exercise, we thus verify that the same system but with an arbitrary two-
body interaction $(\rho - \rho)$ remains topological in the infinite-gap limit.

Turning back to the case in which the current–current interaction is present,
still in the infinite gap limit, we realize that it plays a crucial role. The potential $V$
induces a pole in the density–density correlation function associated to a collective
excitation with the dispersion relation

$$\omega = \pm \sqrt{\frac{\tilde{V}_0}{V} \tilde{k}^2 + \frac{\tilde{C}_2}{V^2}}$$

(77)

and residue

$$\operatorname{Res}(\tilde{\Delta}) = \frac{1}{2 \sqrt{\tilde{V}_0 \tilde{V} \tilde{k}^2 + \tilde{C}_2}}.$$  (78)

Consider, for example, the case of a local current–current interaction with coupling
constant $g^2$ and a Coulomb density–density interaction $\tilde{V}_0 = e^2/|k|$. In this case,
the collective excitations have a gap proportional to $1/g^2$, so that, the stronger the
current interactions, the weaker the energy necessary to excite the collective modes.
A direct consequence of the existence of this gap is that the conductivity is not
affected by these excitations. It is simple to verify that also with a strong current
interaction, $\sigma_{xx} = 0$ and $\sigma_{xy} = 1/2$. This topological property breaks down if the
gap disappears as it happens with a potential $\tilde{V}_0 = 1/|k|^\alpha$ with $\alpha \geq 2$.

A different situation emerges when the fermionic gap $2m$ is large but not infinity.
We now keep a $1/m$-contribution from eq. (39), so that $\tilde{\Delta}$ is given by eq. (54) with
$\tilde{C}_1 = \frac{1}{12 \pi m}$ and $\tilde{C}_2 = (1/2\pi)^3$. In these conditions, the spectrum of collective
excitations contains a mode with dispersion relation

$$\omega = \pm \sqrt{\frac{\tilde{V}_0 + \tilde{C}_1}{V + \tilde{C}_1} \tilde{k}^2 + \frac{\tilde{C}_2}{(V + \tilde{C}_1)^2}}$$

(79)

and residue

$$\operatorname{Res}(\tilde{\Delta}) = \frac{1}{2 \sqrt{(V + \tilde{C}_1)(\tilde{V}_0 + \tilde{C}_1) \tilde{k}^2 + \frac{\tilde{C}_2}{(V + \tilde{C}_1)}}}.$$  (80)

As expected, this case has a similar behavior to the previous one ($m \to \infty$) with the
only difference that the potentials $\tilde{V}_0$ and $\tilde{V}_1$ are renormalized by the constant $\tilde{C}_1$.
In fact, the parity-conserving contribution of the dynamics of the free fermions acts
as local density–density and current–current interactions with coupling constant $\tilde{C}_1$
renormalizing in this way the “true” potentials. As a final remark, note that these
results are exact for any interaction. In particular, eq. (79) represents the exact
dispersion relation of the collective modes for any arbitrary static potentials $\tilde{V}_0(\vec{k})$ and $\tilde{V}(\vec{k})$.

3.2. The massless case

In this section we analyse the $m \to 0$ limit of the non-relativistic Thirring model. Making $m \to 0$ in (39) we find

$$\lim_{m \to 0} \tilde{F}(\omega, \vec{k}) = \frac{1}{16\sqrt{k^2 - \omega^2}},$$

$$\lim_{m \to 0} \tilde{G}(\omega, \vec{k}) = 0,$$

implying that

$$\lim_{m \to 0} \tilde{C}_1(\omega, \vec{k}) = \frac{16}{\sqrt{k^2 - \omega^2}},$$

$$\lim_{m \to 0} \tilde{C}_2(\omega, \vec{k}) = 0.$$ (84)

This result leads to

$$\hat{\Delta} = \frac{1}{16\sqrt{k^2 - \omega^2} - \tilde{V}_0 k^2 \omega + \tilde{V}}.$$ (85)

Here, several comments are in order. First of all, expression (85) depends essentially on eq. (82), which is not well established and it seems to be regularization dependent. In ref. 16, we argue that if we use a regularization that does not introduce any additional symmetry breaking on the system, the correct result is that of eq. (82). Fortunately, although the value of $\hat{\Delta}$ depends on the particular value of $\lim_{m \to 0} \tilde{G}$, it was shown in ref. 16 that the collective-excitation spectrum does not depend on it, being insensible to the parity-breaking term of the fermionic determinant.

To understand the excitation spectrum, let us simplify expression (85) by considering a pure density–density interaction ($\tilde{V} = 0$). In this case, eq. (85) reduces to

$$\hat{\Delta} = \frac{1}{16\sqrt{k^2 - \omega^2} + \tilde{V}_0 \omega + \tilde{V}}.$$ (86)

This equation for $\hat{\Delta}$ has no poles. Instead, it has a cut for $k^2 < \omega^2$ whose discontinuity

$$\delta^+ \hat{\Delta} \equiv -i \lim_{\epsilon \to 0} \left\{ \hat{\Delta}(\omega + i\epsilon, \vec{k}) - \hat{\Delta}(\omega - i\epsilon, \vec{k}) \right\}$$ (87)

is given by
\[ \delta^+ \tilde{\Delta} = \frac{16 \sqrt{\omega^2 - \vec{k}^2}}{16^2 (\omega^2 - \vec{k}^2) + k^4 V_0^2} \Theta(\omega^2 - \vec{k}^2) \Theta(\omega), \]

where \( \Theta \) is the Heaviside function. This result shows that the model in the zero mass limit (in the Gaussian approximation) has no collective excitations. Instead, there is a continuum of modes that contribute to any physical observable. For instance, from eq. (63) it is possible to calculate the density fluctuations induced by an external charge. However, this density profile cannot be interpreted as a superposition of density waves with a definite dispersion relation.

Let us consider now the effect of a current–current interaction on this system. For simplicity, let us consider \( V_0 = 0 \) and \( \tilde{V} = g^2 \) in (85). We immediately realize that we have two regimes of momenta. For \( \vec{k}^2 < \omega^2 \) we have a cut in the spectrum whose discontinuity is given by

\[ \delta^+ \tilde{\Delta} = \frac{16 \sqrt{\omega^2 - \vec{k}^2}}{16^2 (\omega^2 - \vec{k}^2) + g^4 \omega^2} \Theta(\omega^2 - \vec{k}^2) \Theta(\omega). \]  

Moreover, for high momenta \( \vec{k}^2 > \omega^2 \), the current-current potential induces a gapless mode with linear dispersion relation \( \omega = v|\vec{k}| \) with \( v = \frac{1}{1 + g^2/16} < 1 \).

4. Non-relativistic spinless electrons in (2 + 1) \( d \)

4.1. The current generating functional

In this section we consider a spinless non-relativistic fermionic system in two spatial dimensions with kinetic action given by

\[ S_0 = \int d^2 x dt \left\{ i \partial_t \psi^\ast(x) \left( i \partial_t + \frac{1}{2m} \nabla^2 - \mu \right) \psi(x) \right\}. \]

This action has a global \( U(1) \) symmetry and as a consequence it has conserved Noether currents,

\[ J_0 = \psi^\ast \psi, \]
\[ J_i = \psi^\ast \nabla_i \psi, \]

in such a way that \( \partial_t J_0 + \nabla \cdot \vec{J} = 0 \).

In principle, any functional of the currents \( S_I(J_0, J_i) \), could be considered for the interaction part of the action, thus we can write quite generally

\[ S[\psi^\ast, \psi] = S_0[\psi^\ast, \psi] + S_I[J_0, J_i] \]

The idea is to develop the bosonization technique discussed in the last section to evaluate density-current correlation functions. The main point of this technique is to couple the system to an external electromagnetic field and use it as a source to
evaluate current expectation values. In the present case, this is not so simple to accomplish, due to the non-linear dispersion relation \((\omega = \vec{k}^2 / 2m)\). More explicitly, let us couple the action (93) to a vector gauge field \((s_0, \vec{s})\). The result is

\[
S = \int d^2x dt \psi^* \left\{ i\partial_t + s_0 + \frac{(\vec{\nabla} + \vec{s})^2}{2m} - \mu \right\} \psi + S_I(J_0, \vec{J}, \vec{\vec{s}J}_0). \tag{94}
\]

The main difference with respect to the linear case is that here the interaction action is modified due to the derivative operator that appears in the current. In other words, \(J_i\) is not a local gauge-invariant current and therefore it picks up a factor \(s_i\) which couples to \(J_0\). By functional differentiating (94) with relation to \((s_0, s_i)\), we find

\[
\frac{\delta S}{\delta s_0} = J_0, \tag{95}
\]

\[
\frac{\delta S}{\delta s_i} = J_i + \frac{J_0}{2m} s_i + \frac{\delta S_I}{\delta J_i} J_0. \tag{96}
\]

We see from (95) that in the limit \(s_\mu \to 0\) the potential \(s_0\) may be interpreted as a source of \(J_0\). However, from (96) it can be deduced that in the same limit \(s_i\) only acts as a source of \(J_i\) provided

\[
\left. \frac{\delta S_I[J_0, J_i]}{\delta J_i} \right|_{J_0} = 0, \tag{97}
\]

implying that, in order to use the functional bosonization technique developed in the present article for the evaluation of current correlation functions, we must consider only density–density interactions.

4.2. The bosonization technique

Consider the generating functional

\[
Z[s] = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-S[\psi^*, \psi, s_\mu]}, \tag{98}
\]

where

\[
S[\psi^*, \psi, s_\mu] = \int d^2x dt \psi^* \left\{ i\partial_t + s_0 + \frac{(\vec{\nabla} + \vec{s})^2}{2m} - \mu \right\} \psi + S_I[J_0]. \tag{99}
\]

In order to integrate the fermions, we first rewrite the interaction action as

\[
e^{-S_I[J_0]} = \int \mathcal{D}\phi e^{-\vec{S}[\phi]-i \int d^3x J_0 \phi}, \tag{100}
\]

so that

\[
Z[s] = \int \mathcal{D}\psi \mathcal{D}\psi^* \mathcal{D}\phi e^{-S[\psi^*, \psi, s_\mu, \phi]} \tag{101}
\]
Table 2. Bosonization rules for fermions with quadratic dispersion relation in 2 + 1 dimensions

where now we have

\[ S = \int d^2x dt \psi^* \left\{ i\partial_t + s_0 + i\phi + \frac{(\vec{\nabla} + s)^2}{2m} - \mu \right\} \psi + \bar{S}[\phi]. \] (102)

The decoupling of the source \( s_\mu \) from the fermions and expressing the generating functional as a function of a gauge field \( A_\mu \) follow from the same formal mathematical steps that one takes from eq. (22) to eq. (32), obtaining

\[ Z[s] = \int DA_\mu e^{-S_{bos}[A_\mu]} - i \int d^3x \ s_0 B + s_i \epsilon_{ij} E_j, \] (103)

where

\[ S_{bos}[A_\mu] = K_B[A] + S_I[B] \] (104)

and

\[ e^{-K_B[A]} = \int Db_\mu e^{\text{Tr} \ln \left( i\partial_t + b_0 + \frac{(\vec{\nabla} + b)^2}{2m} - \mu \right) + i \int d^3x \epsilon_{\mu\nu\rho} A_\mu \partial_\nu b_\rho}. \] (105)

In this way, for a non-relativistic gas with density–density interactions, we find bosonization rules very similar to the ones encountered for the fermionic system with linear dispersion relation. In fact, the exponential of the bosonized free fermion action is the transverse Fourier transform of the fermionic determinant and the bosonized interaction action has the same form as the fermionic interacting action when substituting the density \( \rho \) by the magnetic field \( B \). Moreover, from the coupling to an external source \( s_\mu \), it is simple to read off the bosonization rules for the currents:

\[ J_\mu = \epsilon_{\mu\nu\rho} \partial_\nu A_\rho \left\{ \begin{array}{c} J_0 = B \\ J_i = \epsilon_{ij} E_j \end{array} \right. . \] (106)

This means that we can map any fermionic theory in 2\textit{d} into a gauge theory by associating charges to magnetic fluxes and currents to transverse electric fields. These results are summarized in Table 2.
Eq. (105) for $K_B$ is an exact formal result and can be used to develop different kind of approximation schemes. As an example, let us consider here the Gaussian approximation.

In the Gaussian approximation the fermionic determinant takes the form

$$\text{Tr} \ln \left( i \partial_t + b_0 + \frac{1}{2m} \left( \vec{\nabla} + \vec{b} \right)^2 - \mu \right) = n \int d^3x \; b_0(x) - \int d^3xd^3y \; b_\mu(x) K_{\mu\nu}(x-y) b_\nu(y), \quad (107)$$

where $n$ is the mean density of electrons. The first (linear) term is a tadpole contribution and can be dropped out simply by measuring densities from $n$. The second term, the non-relativistic vacuum polarization tensor, gets the contribution from two types of diagrams, the bubble and the tadpole diagrams arising form the interaction $\vec{b}^2 \psi^\dagger \psi$. It is simple to evaluate $K_B$, since the transverse Fourier transform is quadratic. Thus, after a proper gauge fixing is introduced, we can perform the functional integral, obtaining

$$K_B[A] = \frac{1}{4} \int d^3k \; \tilde{B}^* K_{00}^{-1}(\omega, \vec{k}) \tilde{B} + \frac{1}{4} \int d^3k \; \tilde{E}_i^* \left( \frac{k_i k_j}{k_i k_j K_{ij} - k^2 \text{Tr} K} \right) \tilde{E}_j. \quad (108)$$

We have used the fact that the vacuum polarization tensor is symmetric in this approximation, which implies that the free system does not break parity. For this reason, a cross-term $B E_i$ is absent from $K_B$. However, it may happen that configurations with strong magnetic fields, that explicitly break parity, could induce a Chern–Simons-type action. In refs. 32 and 33, this behavior of the functional determinant was deduced for long distances; the resulting action is non-quadratic, making the evaluation of the transverse Fourier transform a very difficult task. However, it is possible to perform it in the limit of a strong magnetic field, in which it is possible to project the system onto the first Landau level 36.

Turning back to the Gaussian approximation, we note that it has only been performed in the bosonized action of the free fermionic term. The interaction part is exactly bosonized by just replacing $\rho$ by $B$. Considering a two-body scattering interaction $\frac{1}{4} \rho V \rho$, the full bosonized action reads

$$S_{\text{bos}}[A] = \frac{1}{4} \int d^3k \; \tilde{B}^* \left( 1 + \frac{K_{00} \tilde{V}(\vec{k})}{K_{00}} \right) \tilde{B} + \frac{1}{4} \int d^3k \; \tilde{E}_i^* \left( \frac{k_i k_j}{k_i k_j K_{ij} - k^2 \text{Tr} K} \right) \tilde{E}_j. \quad (109)$$

Note that we were able to transform a fermionic interacting action in an action that describes dynamical electric fields and magnetic fluxes. In order to calculate current correlation functions we must fix the gauge. In the Coulomb gauge, the above action reads
\[ S_{\text{bos}}[\hat{A}] = \frac{1}{4} \int d^3k \; \hat{B}^* \left( \frac{1 + K_{00} \tilde{V}(k)}{K_{00}} \right) \hat{B} - \frac{1}{4} \int d^3k \; \hat{A}_0^* \left( \frac{\tilde{k}^4}{k_i k_j K_{ij} - k^2 \text{Tr} K} \right) \hat{A}_0. \]

(110)

It is now a simple task to evaluate, for example, the density–density correlation function:

\[ \langle \rho(\omega, \vec{k}) \rho(-\omega, -\vec{k}) \rangle_{S} = \langle B(\omega, \vec{k}) B(-\omega, -\vec{k}) \rangle_{S_{\text{bos}}} = \frac{K_{00}}{1 + K_{00} \tilde{V}(k)}. \]

(111)

It is interesting to note, that, in this case, the Gaussian approximation coincides with the random-phase approximation (RPA). Of course, equation (105) is a natural framework to systematically improve this approximation.

The current–current correlation function reads

\[ \langle J_i(\omega, \vec{k}) J_k(-\omega, -\vec{k}) \rangle_{S} = \epsilon_{ik} \epsilon_{km} \langle E_i(\omega, \vec{k}) E_k(-\omega, -\vec{k}) \rangle_{S_{\text{bos}}} = \langle \tilde{E}^2 \rangle \delta_{ik} - \langle E_i E_k \rangle \\
= \left( k^2 \delta_{ik} - k_i k_k \right) \langle A_0 A_0 \rangle + \frac{\omega^2 k_i k_k}{k^2} \langle B B \rangle \\
= \left( k^2 \delta_{ik} - k_i k_k \right) \frac{k_i k_n K_{ln} - k^2 \text{Tr} K}{k^4} + \frac{\omega^2 k_i k_k}{k^2} \frac{K_{00}}{1 + K_{00} \tilde{V}(k)}. \]

(112)

Finally, we can evaluate the density–current correlation function, obtaining

\[ \langle \rho(\omega, \vec{k}) J_k(-\omega, -\vec{k}) \rangle_{S} = \epsilon_{km} \langle B(\omega, \vec{k}) E_k(-\omega, -\vec{k}) \rangle_{S_{\text{bos}}} = \frac{\omega k_k}{k^2} \langle B B \rangle \\
= \frac{\omega k_k}{k^2} \frac{K_{00}}{1 + K_{00} \tilde{V}(k)}. \]

(113)

For the sake of consistency let us check whether current conservation \( k_i J_i + \omega \rho = 0 \) holds in the above current correlation functions. Multiplying eq. (112) by \( k_i \) and using eq. (113), we obtain

\[ k_i \langle J_i(\omega, \vec{k}) J_k(-\omega, -\vec{k}) \rangle_{S} = \frac{\omega^2 k_k}{k^2} \frac{K_{00}}{1 + K_{00} \tilde{V}(k)} = \omega \langle \rho(\omega, \vec{k}) J_k(-\omega, -\vec{k}) \rangle_{S}, \]

(114)

as it must be. Let us remark that the current conservation in the fermionic version is guaranteed thanks to the global \( U(1) \) invariance, while in the bosonic counterpart it is guaranteed by the topological character of the current.

5. Conclusions
In this paper we have addressed the question of the universality of current bosonization rules in \((2 + 1)\text{-}d\) non-relativistic fermions with linear and non-linear dispersion relations. We have shown that in the linear dispersion relation case, the fermionic action bosonizes to a gauge theory. The bosonization process does not mix the kinetic and interacting parts of the action, making it possible to analyze them individually. We have found that the kinetic part of the fermionic action maps into an effective action \(K_B[A]\) that is given by the logarithm of the transverse Fourier transform of the fermionic determinant. Conversely, the interacting part of the action bosonizes by simply replacing the fermionic current \(J_\mu = \epsilon_{\mu\nu\rho} \partial_\nu A_\rho\). These rules are universal, in the sense that they do not depend on the particular model. Moreover, as the bosonized current is topological, it is metric independent and therefore this rule does not depend on any local property of the model.

In the case of a non-linear dispersion relation, we have found that the universal rules can be applied if we only consider density–density interactions. The current–current interaction destroys universality in this case. The reason for this behavior resides in the fact that for non-linear dispersion relations, the fermionic currents (for instance \(j_i = \psi^\ast \nabla_i \psi\)) are not gauge invariant. For this reason, we believe that it could be possible to find universal bosonization rules in the general case, provided we are able to reformulate the bosonization technique in terms of gauge-invariant objects.

We have applied the functional bosonization formalism to a non-relativistic Thirring-like model and have evaluated the spectrum of collective excitations in various limits. In the large-mass limit, we were able to exactly calculate this spectrum for arbitrary density–density and current–current interactions. We have found collective excitations whose dispersion relation has a gap depending on the particular relation between the potentials. An interesting observation is that the parity-conserving part of the vacuum polarization tensor (and of course, the expression for the collective excitations) is similar to the one-dimensional Tomonaga–Luttinger model, up to a coupling constant renormalization. The physical interpretation is that the kinetic energy of fermions in two spatial dimensional with a large gap (mass \(m\)) is frozen. Consequently, the only effect of kinematics is to renormalize the coupling constants. Of course, in \((2 + 1)\) dimensions there is, in addition to the parity-conserving term, a parity-breaking term \(C_2\), which has no equivalent in \((1+1)\) dimension and is the responsible for opening the gap in the excitation spectrum. It is necessary to stress here that these are exact results without any kind of approximation other than the consideration of a large gap responsible for the insulating face.

We have also analyzed the massless case in a Gaussian approximation in order to evaluate the bosonized action and we have shown that in this approximation there are no collective excitations for any density–density potential. However, the presence of a current interaction may induce a gapless collective mode with linear dispersion relation.
Another important application of bosonization is to improve the usual approximation schemes in many body physics. Since we are bosonizing the interactions exactly, all the approximations are made on the bosonized kinetic fermionic action. In order to improve the Gaussian approximation, we need a better understanding of the structure of the transverse Fourier transform of the fermionic determinant. A first step towards this aim was made in refs. 16 and 34 where a topological Chern–Simons structure was identified. In this context, the problem of going beyond the quadratic approximation in $K_B$ is that of evaluating the Fourier transform of an interacting gauge theory. An interesting method to decouple this type interaction was recently proposed in ref. 35.

Summarizing, we believe that the existence of universal rules for bosonization in two spatial dimensions could help us improve our understanding of strongly correlated systems. In particular, the topological character of the current and the topological content of the bosonized action could play a similar rôle to the one that chiral symmetry plays in one dimension, possibly allowing, for instance, the calculation of universal properties of two dimensional systems. This kind of application is now under development and will be reported elsewhere. Of course, a definite improvement on the comprehension of the structure of $K_B$ would be to discover a fermion–boson mapping between fields (and not between currents) in two dimensions. This is one of the most important open questions within this subject.

Acknowledgements

We would like to acknowledge Profs. S. P. Sorella and L. E. Oxman for useful discussions. D. G. B. also acknowledges Prof. E. Fradkin for useful comments and for his kindly hospitality at UIUC, where part of this work was completed.

D. G. B. is partially supported by State University of Rio de Janeiro, the Brazilian agency CNPq through a post-doctoral fellowship and NSF, grant number NSF DMR98-17941 at UIUC.

The Conselho Nacional de Desenvolvimento Científico e Tecnológico, the Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro, and SR2-UERJ are gratefully acknowledged for financial support.

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