On decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term

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Abstract: In this paper we consider a singular nonlocal viscoelastic problem with a nonlinear source term and a possible damping term. We proved that if the initial data enter into the stable set, the solution exists globally and decays to zero with a more general rate, and if the initial data enter into the unstable set, the solution with non-positive initial energy as well as positive initial energy blows up in finite time. These are achieved by using the potential well theory, the modified convexity method and the perturbed energy method.

Keywords: singular nonlocal viscoelastic problem; general decay; blow-up; potential well theory

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1 Introduction

In this paper, we investigate the following one-dimensional viscoelastic problem with a nonlocal boundary condition

$$\begin{cases}
 u_{tt} - \frac{1}{x} (xu_x)_x + \int_0^t g(t-s) \frac{1}{x} (xu_x(x,s))_x ds + au_t = |u|^{p-2} u, & x \in (0, \ell), t \in (0, \infty), \\
 u(\ell, t) = 0, & t \in [0, \infty), \\
 u(x,0) = u_0(x), & x \in [0, \ell],
\end{cases}$$

(1.1)

where $a \geq 0$, $\ell < \infty$, $p > 2$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$.

This type of evolution problems, with nonlocal constraints, are generally encountered in heart transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics. The nonlocal boundary conditions arise mainly when the data on the boundary cannot be measured directly, but their average values are known. We can refer to the works of Cahlon and Shi [4], Cannon [5], Choi and Chan [8], Ewing and Lin [9], Ionkin [10], Kamynin [11], Samarskii [27], and Shi and Shilor [28]. The first paper discussed second order partial differential equations with nonlocal integral conditions goes back to Cannon [5]. In fact, most of the works were about the classical solutions. Later, Mixed problems with classical and nonlocal (integral) boundary conditions related to parabolic and hyperbolic equations have been extensively established and several results concerning existence and uniqueness have been considered by Bouziani [3], Ionkin [10], Kamynin [11], Mesloub [18], Pulkina [26].
In the absence of the viscoelastic term (i.e., $g = 0$), Mesloub and Bouziani [16] studied the following equation

$$v_{tt} - \frac{1}{x}v_x - v_{xx} = f(x, t), \quad x \in (0, \ell), \quad t \in (0, T),$$

and obtained the existence and uniqueness of a strong solution. Later, Mesloub and Messaoudi [18] solved a three-point boundary-value problem for a hyperbolic equation with a Bessel operator and an integral condition based on an energy method. Then in [19] they considered a nonlinear one-dimensional hyperbolic problem with a linear damping term and established a blow-up result for large initial data and a decay result for small initial data.

In the presence of the viscoelastic term (i.e., $g \neq 0$), Mecheri et al. [15] studied the following equation

$$u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x, s))_xds + au_t = f(x, t), \quad 0 < x < 1, \quad t > 0,$$

for $a > 0$ and proved the existence and uniqueness of the strong solution. Then, Mesloub et al. [17] considered a nonlinear mixed problem for a viscoelastic equation with a dissipation term under a nonlocal boundary condition and obtained the existence and uniqueness of the weak solution based on the iteration processes. Later, the global existence, decay and blow-up of solutions of problem (1.1) (when $a = 0$) were established by Mesloub and Messaoudi in [20], where the authors studied the blow-up result with only negative initial energy. Recently, Wu [31] improved [20] by establishing the blow-up result with nonpositive initial energy as well as positive initial energy.

For the case of initial and boundary value problems for linear and nonlinear viscoelastic equations with classical conditions, many results have also been extensively studied. Cavalcanti et al. [6] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^mu = 0, \quad (x, t) \in \Omega \times (0, \infty),$$

for $a : \Omega \rightarrow \mathbb{R}^+$, a function, which may be null on a part of the domain $\Omega$. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with $\omega$ satisfying some geometry restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

the authors established an exponential rate of decay. Berrimi and Messaoudi [2] improved Cavalcanti’s result by introducing a different functional which allowed to weak the conditions on both $a$ and $g$. In particular, the function $a(x)$ can vanish on the whole domain $\Omega$ and consequently the geometry condition has disappeared. In [7], Cavalcanti et al. considered

$$u_{tt} - k_0\Delta u + \int_0^t \text{div}[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function $g$ and $a(x) + b(x) \geq \rho > 0$, for all $x \in \Omega$. They improved the result of [6] by establishing exponential stability for $g$ decaying exponentially and $h$ linear and polynomial stability for $g$ decaying polynomially and $h$ nonlinear. In
Berrimi and Messaoudi considered
\[ u_{tt} - \Delta u + \int_0^t g(t - \tau)\Delta u(\tau)d\tau = |u|^{p-2}u \]
in a bounded domain and \( p > 2 \). They established a local existence result and showed that, under weaker condition \( g'(t) \leq \xi g^r(t) \), the solution is global and decay in a polynomial or exponential fashion when the initial data is small enough. Then Messaoudi [23] improved this result by establishing a general decay of energy which is similar to the relaxation function under weaker condition that \( g'(t) \leq \xi(t)g(t) \). In regard of nonexistence, Messaoudi [21] considered
\[ u_{tt} - \Delta u + \int_0^t g(t - \tau)\Delta u(\tau)d\tau + a|u_t|^{m-2}u_t = |u|^{p-2}u \]
and established a blow up result for solutions with negative energy if \( p > m \) and a global existence result for \( p \leq m \). Then Messaoudi [22] improved this result by accommodating certain solutions with positive initial energy. Liu [13] obtained the similar blow-up result for the viscoelastic problem with strong damping and nonlinear source by using the potential well theory and convexity technique. For other related works, we refer the readers to [12, 14, 25, 21, 29, 30, 32, 33, 34] and the references therein.

Inspired by [1, 13, 20, 23], we intend to study the blow-up and decay properties of problem (1.1) in this paper. Our goal is to establish a decay result with a more general rate and a blow-up result with non-positive initial energy as well as positive initial energy. The main difficulties we encounter here arise from the simultaneous appearance of the singular nonlocal viscoelastic term, the possible damping term, as well as the nonlinear source term. We first show that if the initial data enter into the unstable set, the source term is enough to obtain blow-up result no matter \( a = 0 \) or \( a > 0 \). This is achieved by using the potential well theory and the modified convexity method. We then establish the decay result under the condition that \( g'(t) \leq -\xi(t)g^r(t) \), which is more general than that of [1, 23], by constructing some functionals and using the perturbed energy method.

The paper is organized as follows. In Section 2 we present some assumptions and known results and state the main results. Section 3 is devoted to the proof of the blow-up result. The decay result is proved in Sections 4.

2 Preliminaries and main results

In this section we first introduce some functional spaces and present some assumptions and known results which will be used throughout this work.

Let \( L_x^p = L_x^p(0, \ell) \) be the weighted Banach space equipped with the norm
\[ \|u\|_p = \left( \int_0^\ell x|u|^pdx \right)^{\frac{1}{p}}. \]
In particular, when \( p = 2 \), we denote \( H = L^2_x(0, \ell) \) to be the weighted Hilbert space of square integrable functions having the finite norm

\[
||u||_H = \left( \int_0^\ell x u^2dx \right)^{\frac{1}{2}}.
\]

We take \( V = V^{1,1}_x(0, \ell) \) to be the weighted Hilbert space equipped with the norm

\[
||u||_V = \left( ||u||^2_H + ||u_x||^2_H \right)^{\frac{1}{2}},
\]

and

\[ V_0 = \{ u \in V \text{ such that } u(\ell) = 0 \}. \]

For the relaxation function \( g \), we give the following assumptions:

(G1) \( g(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) is a non-increasing \( C^2 \) function such that
\[
g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0.
\]

(G2) There exists a positive differentiable function \( \xi(t) \) such that
\[
g'(t) \leq -\xi(t)g^r(t), \quad t \geq 0, 1 \leq r < \frac{3}{2},\quad (2.1)
\]

and \( \xi(t) \) satisfies, for some positive constant \( L \),
\[
\left| \frac{\xi'(t)}{\xi(t)} \right| \leq L, \quad \xi'(t) \leq 0, \quad \int_0^{+\infty} \xi(s)ds = +\infty, \quad \forall \ t > 0.
\]

Furthermore, when \( 1 < r < \frac{3}{2} \), for any fixed \( t_0 > 0 \), there exists a positive constant \( C_r \) depending only on \( r \), such that
\[
\frac{t}{\left( 1 + \int_{t_0}^t \xi(s)ds \right)^{\frac{1}{2(r-1)}}} \leq C_r, \quad \forall \ t \geq t_0.\quad (2.2)
\]

**Remark 1** Condition \( r < \frac{3}{2} \) is made to ensure that \( \int_0^\infty g^{2-r}(s)ds < \infty \).

**Remark 2** If \( \xi(t) \equiv \xi = \text{constant} \), (G2) recaptures that of \([1, 13, 20]\). If \( r \equiv 1 \), (G2) recaptures that of \([23, 24]\). Therefore, (G2) is a generalization of \([1, 13, 20, 23, 24]\). In particular, when \( \xi(t) \equiv \xi \) and \( 1 < r < \frac{3}{2} \), (2.2) holds naturally.

**Lemma 2.1** \([20]\), Poincaré-type inequality) For any \( v \) in \( V_0 \), we have
\[
\int_0^\ell xv^2(x)dx \leq C_p \int_0^\ell xv_x^2(x)dx,
\]
where \( C_p \) is some positive constant.
Lemma 2.2 ([20]) For any $v$ in $V_0$, $2 < p < 4$, we have
\[ \int_0^\ell x|v|^p dx \leq C_* \|v_x\|_2^p, \]
where $C_*$ is a constant depending on $\ell$ and $p$ only.

We state, without proof, a local existence result for problem (1.1). The proof can be easily established by adopting the arguments of [1], [17] and [19].

**Theorem 2.3** Suppose that (G1) holds and $2 < p < 3$. Then for any $u_0$ in $V_0$ and $u_1$ in $H$, problem (1.1) has a unique local solution
\[ u \in C(0, T_{\text{max}}; V_0) \cap C^1(0, T_{\text{max}}; H) \]
for $T_{\text{max}} > 0$ small enough.

**Remark 3** The condition $2 < p < 3$ is needed so that the embedding of $V_0$ in $L^2_x$ is Lipschitz (see [19, Lemma 5.2]).

Next we introduce the functionals for $I(t)$, $J(t)$ and $E(t)$:
\[
I(t) := I(u(t)) = \left(1 - \int_0^t g(s) ds\right) \int_0^\ell x u_x^2 dx + (g \circ u_x)(t) - \int_0^\ell x|u(t)|^p dx, \tag{2.3}
\]
\[
J(t) := J(u(t)) = \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \int_0^\ell x u_x^2 dx + \frac{1}{2} (g \circ u_x)(t) - \frac{1}{p} \int_0^\ell x|u(t)|^p dx, \tag{2.4}
\]
\[
E(t) := E(u(t)) = J(t) + \frac{1}{2} \int_0^\ell x u_x^2 dx, \tag{2.5}
\]
where
\[
(g \circ u_x)(t) = \int_0^t \int_0^\ell x g(t - s) |u_x(x, t) - u_x(x, s)|^2 ds dx.
\]

**Remark 4** A multiplication of equation (1.1) by $x u_t$ and integration over $(0, \ell)$ easily yields
\[
E'(t) = \frac{1}{2} (g' \circ u_x)(t) - \frac{1}{2} g(t) \int_0^\ell x u_x^2 dx - a \int_0^\ell x u_t^2 dx \leq -a \int_0^\ell x u_t^2 dx \leq 0, \quad \forall t \geq 0. \tag{2.6}
\]

We are now in a position to state our main results.

**Theorem 2.4** Assume that (G1) holds and $2 < p < 3$, let $u$ be the unique local solution to problem (1.1) and denote $d_1 = \frac{p - 2}{2p} \left(\frac{1}{C_*^2} r^2\right)^{\frac{p}{p - 2}}$. For any fixed $\delta < 1$, assume that $u_0, u_1$ satisfy
\[
E(0) < \delta d_1, \quad I(0) < 0. \tag{2.7}
\]
Suppose that
\[
\int_0^\infty g(s) ds \leq \frac{p - 2}{p - 2 + 1/(1 - \hat{\delta}^2 p + 2\delta(1 - \delta))} \tag{2.8}
\]
where $\hat{\delta} = \max\{0, \delta\}$. Then the solution of problem (1.1) blows up in a finite time $T^*$ in the sense that
\[
\lim_{t \to T^*^-} \|u\|_H^2 = +\infty.
\]
Theorem 2.5 Assume that (G1) holds and \( 2 < p < 3 \), let \( u \) be the unique local solution to problem (1.1). In addition, assume that \( u_0, u_1 \) satisfy
\[
E(0) < d_1, \quad I(0) > 0.
\] (2.9)

Then the solution \( u \) is global and satisfies
\[
\int_0^\ell xu_2^2dx \leq \frac{2p}{l(p-2)}E(t) \leq \frac{2p}{l(p-2)}E(0), \quad \forall \ t > 0.
\] (2.10)

Theorem 2.6 Under the assumptions of theorem 2.5, suppose further that (G2) holds.
Then for each \( t_0 > 0 \), there exist positive constants \( K \) and \( \kappa \) such that
\[
E(t) \leq \begin{cases} 
Ke^{-\kappa \int_{t_0}^t \xi(s)ds}, & r = 1, \\
K \left( 1 + \int_{t_0}^t \xi(s)ds \right)^{-\frac{1}{r-1}}, & 1 < r < \frac{3}{2}.
\end{cases}
\] (2.11)

Remark 5 Note that when \( 1 < r < \frac{3}{2} \), we obtain more general type of decays. If we choose \( \xi(t) \equiv \xi \), (2.11) gives the polynomial rate decay as \( E(t) \leq K(1 + t)^{-\frac{1}{r-1}} \), which coincides with the results of [1, 13, 20]. If we choose \( \xi(t) = (1 + t)^{-m} \) for \( 0 < m < 3 - 2r < 1 \) (which satisfies (2.2)), we have \( g(t) \leq \frac{C_0}{(1+t)^q} \) with \( q = \frac{1-m}{r-1} \) and (2.11) also gives the polynomial rate of decay as \( E(t) \leq \frac{C}{(1+t)^q} \). In particular, if we choose \( \xi(t) = \frac{2(r-1)}{(t+1)^{3-2r}} + \frac{1}{t+1} \), which satisfies (G2), then we have \( g(t) \leq \frac{C}{[(t+1)^{2(r-1)}+\ln(t+1)-1]^\frac{1}{r-1}} \) and a new type of decay as \( E(t) \leq \frac{K}{[(t+1)^{2(r-1)}+\ln(t+1)-1]^\frac{1}{r-1}} \) is established.

3 Blow-up of solutions

In this section, we prove a finite time blow-up result for initial data in the unstable set.

For \( t \geq 0 \), we define
\[
d(t) = \inf_{u \in V_0 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u)
\]
and
\[
\mathcal{N} = \{u \in V_0 \setminus \{0\} : I(u(t)) = 0\}.
\] (3.1)

Then we can prove the following lemma.

Lemma 3.1 For \( t \geq 0 \), we have
\[
0 < d_1 \leq d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u)
\]
and
\[
d(t) = \inf_{u \in \mathcal{N}} J(u).
\] (3.2)
Proof. Obviously, 
\[ d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u). \]

Since 
\[
J(\lambda u) = \frac{\lambda^2}{2} \left[a \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right] - \frac{\lambda p}{p} \int_0^\ell x|u|^pdx.
\]

We get 
\[
\frac{d}{d\lambda} J(\lambda u) = \lambda \left[a \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right] - \lambda^{p-1} \int_0^\ell x|u|^pdx.
\]

Let 
\[
\frac{d}{d\lambda} J(\lambda u) = 0,
\]

which implies 
\[
\tilde{\lambda}_1 = 0, \quad \tilde{\lambda}_2 = \left[a \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right]^{\frac{1}{p-2}} \left(\int_0^\ell x|u|^pdx\right)^{\frac{2}{p-2}}.
\]

An elementary calculation shows 
\[
\frac{d^2}{d\lambda^2} J(\lambda u) > 0 \quad \text{and} \quad \frac{d^2}{d\lambda^2} J(\lambda u) < 0.
\]

Using (G1) and Lemma 2.2, we get 
\[
\sup_{\lambda \geq 0} J(\lambda u) = J(\tilde{\lambda}_2 u) = \frac{p-2}{2p} \left[\frac{a \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right] \left(\int_0^\ell x|u|^pdx\right)^{\frac{2}{p-2}} \geq \frac{p-2}{2p} \left(\frac{l}{C^{2/p}}\right)^{\frac{2}{p-2}} = d_1 > 0,
\]

which implies that \(d(t) \geq d_1\).

To get (3.2), straightforward computations lead to 
\[
I(\tilde{\lambda}_2 u) = \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell x(\tilde{\lambda}_2 u)_x^2 dx + (g \circ (\tilde{\lambda}_2 u)_x)(t) - \int_0^\ell x|\tilde{\lambda}_2 u|^pdx
\]

\[
= \left[\frac{a \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right]^{\frac{2}{p-2}} \left[\left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right]^{\frac{p}{p-2}} \int_0^\ell x|u|^pdx\right.
\]

\[
- \left[\frac{a \left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right]^{\frac{2}{p-2}} \left[\left(1 - \int_0^\ell g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right]^{\frac{p}{p-2}} \int_0^\ell x|u|^pdx
\]
Next, we consider two cases:

On the other hand, from (3.6) and Lemma 2.2, we obtain

\[
\lambda_2(u) = \left[ \frac{\left(1 - \int_0^t g(s)ds \right) \int_0^t x u_2^2 dx + (g \circ u_x)(t)}{\int_0^t x |u|^p dx} \right]^{\frac{1}{p-2}} = 1.
\]

Therefore we have \( \lambda_2(u)u = u \) for all \( u \in \mathcal{N} \). Thus we complete the proof.

**Lemma 3.2** Under the same assumptions as in Theorem 2.4, one has \( I(u(t)) < 0 \) and

\[
d_1 < \frac{p-2}{2p} \left(1 - \int_0^t g(s)ds \right) \int_0^t x u_2^2 dx + (g \circ u_x)(t) \right) < \frac{p-2}{2p} \int_0^t x |u|^p dx,
\]

for all \( t \in [0, T_{\max}) \).

**Proof.** Using (2.6) and (2.7), we have \( E(t) \leq \delta d_1 \) for all \( t \in [0, T_{\max}) \). Furthermore, we can obtain \( I(u(t)) < 0 \) for all \( t \in [0, T_{\max}) \).

In fact, if it is not true, then there exists some \( t_0 \in [0, T_{\max}) \) such that \( I(t_0) \geq 0 \). Since \( I(0) < 0 \), it follows that there exists some \( \bar{t} \in (0, t_0] \) such that \( I(u(\bar{t})) = 0 \). Define

\[
t^* = \inf \left\{ \bar{t} \in (0, t_0] : \left(1 - \int_0^{\bar{t}} g(s)ds \right) \int_0^{\bar{t}} x u_2^2 dx + (g \circ u_x)(\bar{t}) \right) = \int_0^{\bar{t}} x |u(\bar{t})|^p dx \right\}. \tag{3.5}
\]

Then, we have \( I(u(t^*)) = 0 \) and

\[
\left(1 - \int_0^t g(s)ds \right) \int_0^t x u_2^2 dx + (g \circ u_x)(t) \right) < \int_0^t x |u|^p dx, \quad 0 \leq t < t^*. \tag{3.6}
\]

Next, we consider two cases:

Case 1: Suppose that \( \|u(t^*)\|_{H_2}^2 = 0 \), using the regularity of \( u(t) \), we have

\[
\lim_{t \to t^*} \|u(t)\|_{H_2}^2 = 0. \tag{3.7}
\]

On the other hand, from (3.6) and Lemma 2.2, we obtain

\[
\left(1 - \int_0^t g(s)ds \right) \int_0^t x u_2^2 dx + (g \circ u_x)(t) \right) < \int_0^t x |u|^p dx \leq C_* \|u_x\|_2^p, \quad 0 \leq t < t^*, \tag{3.8}
\]

and \( \|u(t)\|_{H_2}^2 \neq 0 \), for all \( t \in [0, t^*) \). Therefore we have

\[
\lim_{t \to t^*} \|u(t)\|_{H_2}^2 > \left( \frac{l}{C_*} \right)^{\frac{1}{p-2}},
\]
which contradicts to (3.7).

Case 2: Suppose that \( \|u(t^*)\|^2_H \neq 0 \). Applying Lemma 3.1 we see that \( d(t) \) is the infimum of \( J(u(t)) \) over all functions \( u \) in \( \mathcal{N} \) and \( J(u(t^*)) \geq d(t) \geq d_1 \), which contradicts to \( J(u(t^*)) \leq E(t^*) < d_1 \). Thus, we conclude that \( I(t) < 0 \) for all \( t \in [0, T_{max}) \).

To get (3.4), we use (3.6), Lemma 3.1 and the conclusion that \( I(t) < 0 \) for all \( t \in [0, T_{max}) \) and get

\[
d_1 \leq \frac{p-2}{2p} \left[ \left(1 - \int_0^t g(s) ds\right) \int_0^\ell x u_2^2 dx + (g \circ u_x)(t) \right] \left[ \left(1 - \int_0^t g(s) ds\right) \int_0^\ell x u_2^2 dx + (g \circ u_x)(t) \right] \frac{2}{p-2} \left( \int_0^\ell x |u|^p dx \right)^{\frac{2}{p-2}}
\]

\[
< \frac{p-2}{2p} \left[ \left(1 - \int_0^t g(s) ds\right) \int_0^\ell x u_2^2 dx + (g \circ u_x)(t) \right], \quad 0 \leq t < T_{max}.
\]

It follows from (3.6) and (3.9) that

\[
0 < d_1 < \frac{p-2}{2p} \left[ \left(1 - \int_0^t g(s) ds\right) \int_0^\ell x u_2^2 dx + (g \circ u_x)(t) \right] < \frac{p-2}{2p} \int_0^\ell x |u|^p dx, \quad 0 \leq t < T_{max}.
\]

Thus, we complete the proof.

**Lemma 3.3** ([12]) Let \( L(t) \) be a positive \( C^2 \) function, which satisfies, for \( t > 0 \), the inequality

\[
L(t) L''(t) - (1 + \zeta) L'(t)^2 \geq 0
\]

with some \( \zeta > 0 \). If \( L(0) > 0 \) and \( L'(0) > 0 \), then there exists a time \( T^* \leq \frac{L(0)}{\zeta L''(0)} \) such that

\[
\lim_{t \to T^*} L(t) = \infty.
\]

**Proof of Theorem 2.4** Assume by contradiction that the solution \( u \) is global. Then, we consider \( L : [0, T] \to \mathbb{R}_+ \) defined by

\[
L(t) = \int_0^\ell x u_2^2 dx + a \int_0^\ell x u_2^2 dx + a (T - t) \int_0^\ell x u_0^2 dx + b(t + T_0)^2,
\]

where \( T, b, T_0 \) are positive constants to be chosen later. Then \( L(0) > 0 \). Furthermore,

\[
L'(t) = 2 \int_0^\ell x uu_t dx + a \int_0^\ell x (u^2 - u_0^2) dx + 2b(t + T_0)
\]

\[
= 2 \int_0^\ell x uu_t dx + 2a \int_0^\ell x uu_s dx + 2b(t + T_0),
\]

and, consequently,

\[
L''(t) = 2 \int_0^\ell x uu_{tt} dx + 2 \int_0^\ell x u_t^2 dx + 2a \int_0^\ell x uu_t dx + 2b
\]

...
for almost every $t \in [0, T]$. Testing the equation (1.1) with $xu$ and plugging the result into the expression of $L''(t)$, we obtain

$$L''(t) = -2 \int_0^t xu^2 dx + 2 \int_0^t \int_0^t g(t-s)xu_x(x,t)u_x(x,s) ds dx$$

$$- 2a \int_0^t xu_t dx + 2 \int_0^t x|u|^p dx + 2 \int_0^t xu_t^2 dx + 2a \int_0^t xu_t dx + 2b$$

$$= 2 \left[ \int_0^t xu_t^2 dx - \left( 1 - \int_0^t g(s) ds \right) \int_0^t xu_t^2 dx \right.$$  

$$- \int_0^t \int_0^t g(t-s)xu_x(x,t) (u_x(x,t) - u_x(x,s)) ds dx$$

$$+ \int_0^t x|u|^p dx + b \right] + (p + 2) \left[ \eta(t) - \left( L(t) - a(T-t) \int_0^t xu_0^2 dx \right) \right.$$  

$$\left( \int_0^t xu_t^2 dx + a \int_0^t \int_0^t xu_s^2 ds dx + b \right) \right],$$

where

$$\eta(t) = \left( \int_0^t xu^2 dx + a \int_0^t \int_0^t xu^2 ds dx + b(t+T_0)^2 \right) \left( \int_0^t xu_t^2 dx + a \int_0^t \int_0^t xu_s^2 ds dx + b \right)$$

$$- \left[ \int_0^t xu_t dx + a \int_0^t \int_0^t xu_s dx dx + b(t+T_0) \right]^2.$$  

Using Schwarz’s inequality, we can easily get $\eta(t) \geq 0$ for every $t \in [0, T]$. As a consequence, we reach the following differential inequality

$$L(t)L''(t) - \frac{p+2}{4} L'(t)^2 \geq L(t)\Phi(t), \quad \text{a.e. } t \in [0, T],$$

where $\Phi : [0, T] \rightarrow \mathbb{R}_+$ is the map defined by

$$\Phi(t) = -p \int_0^t xu_t^2 dx - 2 \left( 1 - \int_0^t g(s) ds \right) \int_0^t xu_t^2 dx - a(p+2) \int_0^t \int_0^t xu_s^2 ds dx$$

$$- 2 \int_0^t \int_0^t g(t-s)xu_x(x,t) (u_x(x,t) - u_x(x,s)) ds dx + 2 \int_0^t x|u|^p dx - pb$$

$$= -2pE(t) + p(g \circ u_x)(t) + (p-2) \left( 1 - \int_0^t g(s) ds \right) \int_0^t xu_s^2 dx - pb$$

$$- 2 \int_0^t \int_0^t g(t-s)xu_x(x,t) (u_x(x,t) - u_x(x,s)) ds dx - a(p+2) \int_0^t \int_0^t xu_s^2 ds dx.$$
By (2.6), for all \( t \in [0, T] \) we may also write
\[
\Phi(t) \geq -2pE(0) + p(g \circ u_x)(t) + (p - 2) \left( 1 - \int_0^t g(s)ds \right) \int_0^t xu_x^2dx - pb \\
- 2 \int_0^t \int_0^t g(t-s)xu_x(x,t) \left( u_x(x,t) - u_x(x,s) \right) dsdx \\
+ a(p - 2) \int_0^t \int_0^t xu_x^2dxds.
\]
(3.13)

By using Young’s inequality, we have
\[
2 \int_0^t \int_0^t g(t-s)xu_x(x,t) \left( u_x(x,t) - u_x(x,s) \right) dsdx \\
\leq \frac{1}{\varepsilon} \int_0^t g(s) \int_0^t xu_x^2dsdx + \varepsilon(g \circ u_x)(t),
\]
(3.14)
for any \( \varepsilon > 0 \). Substituting (3.14) for the fifth term of the right hand side of (3.13), we obtain
\[
\Phi(t) \geq -2pE(0) + \left[ (p - 2) - \left( p - 2 + \frac{1}{\varepsilon} \right) \int_0^t g(s)ds \right] \int_0^t xu_x^2dx \\
+ (p - \varepsilon)(g \circ u_x)(t) + a(p - 2) \int_0^t \int_0^t xu_x^2dxds - pb.
\]
(3.15)

If \( \delta \leq 0 \), i.e., \( E(0) < 0 \), we choose \( \varepsilon = p \) in (3.15) and \( b \) small enough such that \( b \leq -2E(0) \). Together with (2.8), we obtain
\[
\Phi(t) \geq \left[ (p - 2) - \left( p - 2 + \frac{1}{p} \right) \int_0^t g(s)ds \right] \int_0^t xu_x^2dx + a(p - 2) \int_0^t \int_0^t xu_x^2dxds \\
+ p(-2E(0) - b) \\
\geq \left[ (p - 2) - \left( p - 2 + \frac{1}{p} \right) \int_0^t g(s)ds \right] \int_0^t xu_x^2dx + a(p - 2) \int_0^t \int_0^t xu_x^2dxds \\
\geq a(p - 2) \int_0^t \int_0^t xu_x^2dxds \\
\geq 0.
\]
(3.16)

If \( 0 < \delta < 1 \), i.e., \( E(0) < \delta d_1 \), we choose \( \varepsilon = (1 - \delta)p + 2\delta \) and \( b = 2(\delta d_1 - E(0)) > 0 \) in (3.15). Then we get
\[
\Phi(t) \geq -2p\delta d_1 + \left[ (p - 2) - \left( p - 2 + \frac{1}{(1 - \delta)p + 2\delta} \right) \int_0^t g(s)ds \right] \int_0^t xu_x^2dx \\
+ \delta(p - 2)(g \circ u_x)(t) + a(p - 2) \int_0^t \int_0^t xu_x^2dxds.
\]
By (2.8), we have
\[
(p - 2) - \left( p - 2 + \frac{1}{(1 - \delta)p + 2\delta} \right) \int_0^t g(s)ds \geq \delta(p - 2) \left( 1 - \int_0^t g(s)ds \right)
\]
and therefore, by (3.4) and (2.7) we get
\[
\Phi(t) \geq -2p\delta d_1 + \delta(p-2) \left[ \left(1 - \int_0^t g(s)ds \right) \int_0^\ell xu_2^2dx + (g \circ u_x)(t) \right] \\
+ a(p-2) \int_0^t \int_0^\ell xu_2^2dxds \\
\geq 2p(\delta d_1 - \delta d_1) + a(p-2) \int_0^t \int_0^\ell xu_2^2dxds \\
\geq 0. \tag{3.17}
\]

Therefore, combining (3.12), (3.16), and (3.17), we arrive at

\[L(t)L''(t) - \frac{p+2}{4}L'(t)^2 \geq 0, \quad \text{a.e. } t \in [0,T].\]

Let \(T_0\) be any number which depends only on \(p\), \(b\), \(\int_0^\ell xu_0^2dx\) and \(\int_0^\ell xu_1^2dx\) as

\[T_0 > \frac{(p-2 + 4a) \int_0^\ell xu_0^2dx + (p-2) \int_0^\ell xu_1^2dx}{2(p-2)b},\]

which fulfills the requirement of

\[L'(0) = 2 \int_0^\ell xu_0u_1dx + 2bT_0 > 0.\]

Then using Lemma 3.3 we obtain that \(L(t)\) goes to \(\infty\) as \(t\) tends to some \(T^*\) satisfying

\[T^* \leq \frac{4L(0)}{(p-2)L'(0)} = \frac{2(1 + aT) \int_0^\ell xu_0^2dx + 2bT_0^2}{(p-2) \int_0^\ell xu_0u_1dx + (p-2)bT_0}. \tag{3.18}\]

Finally, for fixed \(T_0\), we choose \(T\) as

\[T > \frac{4 \left( \int_0^\ell xu_0^2dx + bT_0^2 \right)}{2(p-2)bT_0 - (p-2 + 4a) \int_0^\ell xu_0^2dx - (p-2) \int_0^\ell xu_1^2dx}. \tag{3.19}\]

Combing (3.18) and (3.19), we get \(T > T^*\) and this contradicts to our assumption, which finishes our proof.

**Remark 6** We can see that, when \(a = 0\), Wu [31] established blow-up results under some restrictions on \(\int_0^\ell xu_0u_1dx\), which are no more needed in this paper. In fact, we use the potential well theory and the modified convexity method, which is different from that in Wu [31].

4 Decay of solutions

In this section we prove our decay result. For this purpose, we need the following lemmas.

**Lemma 4.1 (20 Lemma 4.1)** Under the same assumption as in Theorem 2.6, one has \(I(u(t)) > 0\) for all \(t \in [0,T_{\max})\).
Proof of Theorem 2.5. We can refer to [20, Lemma 4.2]. Next, we use the following “modified” functional

\[ F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \quad (4.1) \]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are positive constants and

\[ \Psi(t) = \xi(t) \int_0^t xu_\tau d\tau, \quad (4.2) \]

\[ \chi(t) = -\xi(t) \int_0^t xu_\tau \int_0^{t-s} g(t-s)(u(t) - u(s)) ds dx. \quad (4.3) \]

Lemma 4.2 For \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough, we have

\[ \alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \quad (4.4) \]

holds for two positive constants \( \alpha_1 \) and \( \alpha_2 \).

Proof. Straightforward computations lead to

\[
F(t) &= E(t) + \varepsilon_1 \xi(t) \int_0^t xu_\tau d\tau - \varepsilon_2 \xi(t) \int_0^t xu_\tau \int_0^{t-s} g(t-s)(u(t) - u(s)) ds dx \\
&\leq E(t) + \varepsilon_1 \xi(t) \int_0^t xu_\tau^2 d\tau + \varepsilon_2 \xi(t) \int_0^t xu_\tau^2 d\tau + \varepsilon_2 \xi(t) \int_0^t xu_\tau^2 d\tau \\
&\quad + \frac{\varepsilon_2}{2} \xi(t) \int_0^t x \left( \int_0^t g(t-s)(u(t) - u(s)) ds \right)^2 dx \\
&\leq E(t) + \varepsilon_1 \xi(t) \int_0^t xu_\tau^2 d\tau + \varepsilon_2 \xi(t) \int_0^t xu_\tau^2 d\tau + \varepsilon_2 \xi(t) \int_0^t xu_\tau^2 d\tau \\
&\quad + \frac{\varepsilon_2}{2} \xi(t) \int_0^t x \left( \int_0^t g(s) ds \int_0^t g(t-s)(u(t) - u(s))^2 ds dx \right) \\
&\leq E(t) + \frac{\varepsilon_1 + \varepsilon_2}{2} \xi(t) \int_0^t xu_\tau^2 d\tau + \frac{C_p \varepsilon_1}{2} \xi(t) \int_0^t xu_\tau^2 d\tau \\
&\quad + \frac{\varepsilon_2}{2} (1 - l) \xi(t) \int_0^t x g(t-s)(u(t) - u(s))^2 ds dx \\
&\leq E(t) + \frac{\varepsilon_1 + \varepsilon_2}{2} \xi(t) \int_0^t xu_\tau^2 d\tau + \frac{C_p \varepsilon_1}{2} \xi(t) \int_0^t xu_\tau^2 d\tau + \frac{\varepsilon_2}{2} (1 - l) C_p \xi(t)(g \circ u_x)(t) \\
&\leq \frac{1}{\alpha_1} E(t),
\]

and in the same way, we get

\[
F(t) \geq E(t) - \frac{\varepsilon_1 + \varepsilon_2}{2} \xi(t) \int_0^t xu_\tau^2 d\tau - \frac{C_p \varepsilon_1}{2} \xi(t) \int_0^t xu_\tau^2 d\tau - \frac{\varepsilon_2}{2} (1 - l) C_p \xi(t)(g \circ u_x)(t) \\
\geq \left[ \frac{1}{2} - \frac{\varepsilon_1 + \varepsilon_2}{2} \xi(t) \right] \int_0^t xu_\tau^2 d\tau + \left( \frac{1}{2} - \frac{C_p \varepsilon_1}{2} \xi(t) \right) \int_0^t xu_\tau^2 d\tau \\
&\quad + \left[ \frac{1}{2} - \frac{C_p \varepsilon_2}{2} (1 - l) \xi(t) \right] (g \circ u_x)(t) - \frac{1}{p} \int_0^t x |u|^p dx \\
&\geq \frac{1}{\alpha_2} E(t),
\]
for $\varepsilon_1$ and $\varepsilon_2$ small enough.

**Lemma 4.3** ([20] Lemma 4.5) Let $v \in L^\infty((0, T); H), v_x \in L^\infty((0, T); H)$ and $g$ be a continuous function on $[0, T]$ and suppose that $0 < \tau < 1$ and $r > 0$. Then there exists a constant $C > 0$ such that
\[
\int_0^t g(t-s)\|v_x(\cdot, t) - v_x(\cdot, s)\|_H^2 ds
\]
\[
\leq C \left( \sup_{0 < s < T} \|v(\cdot, s)\|_H^2 \right) \left( \int_0^t g^\tau(s) ds \right)^\frac{1}{\tau} \left( \int_0^t g^r(t-s)\|v_x(\cdot, t) - v_x(\cdot, s)\|_H^2 ds \right)^\frac{1}{r}.
\]

**Lemma 4.4** ([20] Lemma 4.6) Let $v \in L^\infty((0, T); H), v_x \in L^\infty((0, T); H)$ and $g$ be a continuous function on $[0, T]$ and suppose that $r > 0$. Then there exists a constant $C > 0$ such that
\[
\int_0^t g(t-s)\|v_x(\cdot, t) - v_x(\cdot, s)\|_H^2 ds
\]
\[
\leq C \left( t\|v_x(\cdot, t)\|_H^2 + \int_0^t \|v_x(\cdot, s)\|_H^2 ds \right)^\frac{1}{2} \left( \int_0^t g^r(t-s)\|v_x(\cdot, t) - v_x(\cdot, s)\|_H^2 ds \right)^\frac{1}{r}.
\]

**Lemma 4.5** Assume that $2 < p < 3$ and that (G1), (G2) and (2.31) hold. Then the functional $\Psi(t)$, defined by (1.2), satisfies
\[
\Psi'(t) \leq \left( 1 + \frac{a}{2\beta} + \frac{L}{2\alpha} \right) \xi(t) \int_0^t x u_t^2 dx - \left( \frac{l - a\beta C_p - \alpha C_p L}{2} \right) \xi(t) \int_0^t x u_x^2 dx
\]
\[
+ \frac{\xi(t)}{2} \left( \int_0^t g^{2-r}(s) ds \right) (g^r \circ u_x)(t) + \xi(t) \|u\|_{L^p}^p,
\] (4.5)
for all $\alpha, \beta > 0$.

**Proof.** By using the differential equation in (1.1), we easily see that
\[
\Psi'(t) = \xi(t) \int_0^t x u_t^2 dx + \xi(t) \int_0^t x u_t u_x dx + \xi'(t) \int_0^t x u_t dx
\]
\[
= \xi(t) \int_0^t x u_t^2 dx - \xi(t) \int_0^t x u_x^2 dx + \xi(t) \int_0^t x|u|^p dx - a\xi(t) \int_0^t x u_t dx
\]
\[
+ \xi(t) \int_0^t x u_x \int_0^t g(t-s) u_x(x, s) ds dx + \xi'(t) \int_0^t x u_t dx.
\] (4.6)

By Young’s inequality, (G1), (G2), Lemma 2.1 and direct calculations, we arrive at (see [20])
\[
\xi(t) \int_0^t x u_x \int_0^t g(t-s) u_x(x, s) ds dx
\]
\[
\leq \frac{\xi(t)}{2} \int_0^t x u_x^2 dx + \frac{\xi(t)}{2} \int_0^t x \left[ \int_0^t g(t-s) (|u_x(s) - u_x(t)| + |u_x(t)|) ds \right]^2 dx
\]
\[
\leq \frac{\xi(t)}{2} \int_0^t x u_x^2 dx + \frac{\xi(t)}{2} \left( 1 + \eta \right)^2 \int_0^t x u_x^2 dx
\]
\[
+ \frac{\xi(t)}{2} \left( 1 + \frac{1}{\eta} \right) \int_0^t g^{2-r}(s) ds \int_0^t \int_0^t x g^r(t-s)|u_x(s) - u_x(t)|^2 ds dx.
\] (4.7)
for any \( \eta > 0 \). We also have

\[
\xi'(t) \int_0^\ell x u u_t dx \leq \frac{\xi(t)}{2} \left( C_p \alpha \int_0^\ell x u_x^2 dx + \frac{1}{\alpha} \int_0^\ell x u_t^2 dx \right), \quad \forall \alpha > 0,
\]

and

\[
-\alpha \xi(t) \int_0^\ell x u u_t dx \leq \frac{\alpha C_p \xi(t)}{2} \int_0^\ell x u_x^2 dx + \frac{\alpha}{2} \xi(t) \int_0^\ell x u_t^2 dx.
\]

Combining (4.6)-(4.9), we arrive at

\[
\Psi'(t) \leq \left( 1 + \frac{L}{2\alpha} + \frac{\alpha}{2\beta} \right) \xi(t) \int_0^\ell x u_x^2 dx - \frac{\xi(t)}{2} \left[ 1 - (1 + \eta)(1 - t)^2 - aC_p\beta - \alpha C_p L \right] \int_0^\ell x u_x^2 dx
\]

\[
+ \frac{\xi(t)}{2} \left( 1 + \frac{1}{\eta} \right) \left( \int_0^\ell g^{2-r}(s) ds \right) (g^r \circ u_x)(t) + \xi(t) \|u\|_{L^p_x}.
\]

By choosing \( \eta = \frac{t}{1+t} \), (4.5) is established.

**Lemma 4.6** Assume \( 2 < p < 3 \) and that (G1), (G2) and \( (2.3) \) hold. Then the functional \( \chi(t) \), defined by (4.23), satisfies

\[
\chi'(t) \leq \xi(t) \left[ 1 + C^\alpha + 2(1 - t)^2 \right] \int_0^\ell x u_x^2 dx
\]

\[
+ \xi(t) \left[ \theta - \int_0^\ell g(s) ds + a\theta + \theta L \right] \int_0^\ell x u_x^2 dx
\]

\[
+ \left[ \frac{1}{2\theta} + 2\theta + \frac{C_p + (a + L)C_p}{4\theta} \right] \xi(t) \left( \int_0^\ell g^{2-r}(s) ds \right) (g^r \circ u_x)(t)
\]

\[
- \frac{C_p}{4\theta} \xi(t) g(0)(g^r \circ u_x)(t),
\]

for all \( \theta > 0 \).

**Proof.** Direct calculations give

\[
\chi'(t) = \xi(t) \int_0^\ell x u_x(t) \left( \int_0^t g(t - s)(u_x(t) - u_x(s)) ds \right) dx
\]

\[
- \xi(t) \int_0^\ell x \left( \int_0^t g(t - s)(u_x(t) - u_x(s)) ds \right) \left( \int_0^t g(t - s) u_x(s) ds \right) dx
\]

\[
- \xi(t) \int_0^\ell x |u|^{p-2} u \left( \int_0^t g(t - s)(u(t) - u(s)) ds \right) dx
\]

\[
- \xi(t) \int_0^\ell x u_t \int_0^t g'(t - s)(u(t) - u(s)) ds dx
\]

\[
+ a\xi(t) \int_0^\ell x u_t \int_0^t g(t - s)(u(t) - u(s)) ds dx
\]

\[
- \xi(t) \int_0^\ell x u_t^2 \int_0^t g(t - s) ds dx - \xi(t) \int_0^\ell x u_t \int_0^t g(t - s)(u(t) - u(s)) ds dx.
\]
We now estimate the right hand side of (4.11). For \( \theta > 0 \), similar as in [20], we have the estimates of the first to the fourth terms.

The first term

\[
\xi(t) \int_0^t x u_x(t) \left( \int_0^t g(t-s)(u_x(t)-u_x(s))ds \right) dx \\
\leq \theta \xi(t) \int_0^t x u_x^2 dx + \frac{1}{4\theta} \xi(t) \left( \int_0^t g^{2-r}(s)ds \right) (g' \circ u_x)(t). \tag{4.12}
\]

The second term

\[
\xi(t) \int_0^t x \left( \int_0^t g(t-s)(u_x(t)-u_x(s))ds \right) dx \\
\leq 2\theta (1 - t)^2 \xi(t) \int_0^t x u_x^2 dx + \left( 2\theta + \frac{1}{4\theta} \right) \xi(t) \left( \int_0^t g^{2-r}(s)ds \right) (g' \circ u_x)(t). \tag{4.13}
\]

The third term

\[
\xi(t) \int_0^t x |u|^{p-2} u \left( \int_0^t g(t-s)(u(t)-u(s))ds \right) dx \\
\leq \theta C^* \xi(t) \int_0^t x u_x^2 dx + \xi(t) \frac{C_p}{4\theta} \left( \int_0^t g^{2-r}(s)ds \right) (g' \circ u_x)(t), \tag{4.14}
\]

where \( C^* = \frac{C_p}{3-p} \left( \frac{2p}{(p-2)} E(0) \right)^{p-2} \). The fourth term

\[
- \xi(t) \int_0^t x u_t \int_0^t g'(t-s)(u(t)-u(s))ds dx \\
\leq \theta \xi(t) \int_0^t x u_x^2 dx - \frac{g(0)}{4\theta} C_p \xi(t) (g' \circ u_x)(t). \tag{4.15}
\]

For the fifth term, by Young’s inequality and Lemma [2,8] we have

\[
a \xi(t) \int_0^t x u_t \int_0^t g(t-s)(u(t)-u(s))ds dx \\
\leq a \theta \xi(t) \int_0^t x u_x^2 dx + \frac{aC_p}{4\theta} \xi(t) \left( \int_0^t g^{2-r}(s)ds \right) (g' \circ u_x)(t). \tag{4.16}
\]

For the seventh term

\[
- \xi'(t) \int_0^t x u_t \int_0^t g(t-s)(u(t)-u(s))ds dx \\
\leq \xi(t) \left| \xi'(t) \right| \left[ \theta \int_0^t x u_x^2 dx + \frac{C_p}{4\theta} \left( \int_0^t g^{2-r}(s)ds \right) (g' \circ u_x)(t) \right] \\
\leq \theta L \xi(t) \int_0^t x u_x^2 dx + \frac{C_p L}{4\theta} \xi(t) \left( \int_0^t g^{2-r}(s)ds \right) (g' \circ u_x)(t). \tag{4.17}
\]

A combination of (4.11)-(4.17) yields (4.10).
By using (2.6), (4.5), (4.10) and (4.18), we obtain

\[ \int_0^t g(s) \, ds \geq \int_0^{t_0} g(s) \, ds := g_0, \quad \forall \, t \geq t_0. \]  

(4.18)

Proof of Theorem 2.6. Since \( g \) is continuous and \( g(0) > 0 \), then for any \( t_0 > 0 \), we have

\[ F'(t) = E'(t) + \varepsilon_1 \Psi'(t) + \varepsilon_2 \chi'(t) \]

\[ = \frac{1}{2} (g' \circ u_x) (t) - \frac{1}{2} g(t) \int_0^t x u_x^2 \, dx - a \int_0^t x u_x^2 \, dx + \varepsilon_1 \Psi'(t) + \varepsilon_2 \chi'(t) \]

\[ \leq - \left[ a - \varepsilon_1 \left( 1 + \frac{a}{2\beta} + \frac{L}{2\alpha} \right) \xi(t) + \varepsilon_2 \xi(t) (g_0 - \theta(1 + L) - a\theta) \right] \int_0^t x u_x^2 \, dx \]

\[ + \varepsilon_1 \xi(t) \int_0^t x |u|^p \, dx + \left[ \frac{1}{2} - \varepsilon_2 \xi(0) \frac{C_p g(0)}{4\theta} \right] (g' \circ u_x)(t) \]

\[ - \left\{ \frac{\varepsilon_1}{2} (l - a\beta C_p - \alpha C_p L) - \varepsilon_2 \theta \left[ (1 + C^*) + 2(1 - l)^2 \right] \right\} \xi(t) \int_0^t x u_x^2 \, dx \]

\[ + \left\{ \frac{\varepsilon_1}{2l} + \varepsilon_2 \left[ \frac{1}{2\theta} + 2\theta + \frac{C_p + (a + L)C_p}{4\theta} \right] \right\} \xi(t) \left( \int_0^t g_{2-r}(s) \, ds \right) (g' \circ u_x)(t), \]

(4.19)

since \( 0 < \xi(t) \leq \xi(0) \).

When \( a > 0 \), we choose \( \alpha \) and \( \beta \) so small that

\[ l - a\beta C_p - \alpha C_p L > \frac{l}{2} \]

and then choose \( \theta \) small enough satisfying

\[ k_2 = \frac{\varepsilon_1 l}{4} - \varepsilon_2 \theta \left[ (1 + C^*) + 2(1 - l)^2 \right] > 0. \]

(4.20)

As far as \( \alpha, \beta \) and \( \theta \) are fixed, we then pick \( \varepsilon_1 \) and \( \varepsilon_2 \) so small that (4.14) and (4.20) remain valid and

\[ k_1 = \frac{a}{\xi(0)} - \varepsilon_1 \left( 1 + \frac{a}{2\beta} + \frac{L}{2\alpha} \right) + \varepsilon_2 (g_0 - \theta(1 + L) - a\theta) > 0, \]

\[ k_3 = \frac{1}{2} - \varepsilon_2 \frac{C_p g(0)}{4\theta} \frac{\xi(0)}{\xi(0)} - \left\{ \frac{\varepsilon_1}{2l} + \varepsilon_2 \left[ \frac{1}{2\theta} + 2\theta + \frac{C_p + (a + L)C_p}{4\theta} \right] \right\} \left( \int_0^t g_{2-r}(s) \, ds \right) > 0. \]

Therefore, using the assumption \( g'(t) \leq -\xi(t)g''(t) \) in (G2), we have, for some \( \sigma > 0 \),

\[ F'(t) \leq -\sigma \xi(t) \left[ \int_0^t x u_x^2 \, dx - \int_0^t x |u|^p \, dx + \int_0^t x u_x^2 \, dx + (g' \circ u_x)(t) \right], \quad \forall \, t \geq t_0. \]

(4.21)
When $a = 0$, we choose $\theta$, $\alpha$ so small that $g_0 - (1 + L)\theta > \frac{1}{2}g_0$, $l - \alpha C_p L > \frac{1}{2}$, and
\[
\frac{4\theta [1 + C^* + 2(1 - l)^2]}{l} < \frac{g_0}{2 + \frac{L}{\alpha}}.
\]
Whence $\theta$ and $\alpha$ are fixed, the choice of $\varepsilon_1$ and $\varepsilon_2$ satisfying
\[
\frac{4\theta [1 + C^* + 2(1 - l)^2]}{l} \varepsilon_2 < \varepsilon_1 < \frac{g_0 \varepsilon_2}{2 + \frac{L}{\alpha}}
\]
will make
\[
k_1 = -\varepsilon_1 \left(1 + \frac{L}{2\alpha}\right) \xi(0) + \varepsilon_2 \xi(0) \left(g_0 - \theta(1 + L)\right) > 0,
\]
\[
k_2 = \frac{\varepsilon_1}{2} (l - \alpha C_p L) - \varepsilon_2 \theta \left[(1 + C^* + 2(1 - l)^2]\right] > 0.
\]
We then pick $\varepsilon_1$ and $\varepsilon_2$ so small that (4.4), (4.22) and (4.23) remain valid and
\[
k_3 = \frac{1}{2} - \frac{\varepsilon_2 C_p g(0)}{4\theta} \xi(0) - \left\{\frac{\varepsilon_1}{2t} + \varepsilon_2 \left[\frac{1}{2t} + 2\theta + \frac{C_p + L C_p}{4\theta}\right] \left(\int_0^t g^{2-r}(s)ds\right)\right\} > 0.
\]
We can still get (4.21).

Next, as (4.21) is proved, we will give the following two cases according to the different ranges of $r$:

Case 1. $r = 1$.

By virtue of the choice of $\varepsilon_1$, $\varepsilon_2$ and $\theta$, we estimate (4.21) and obtain, for some constant $\alpha > 0$,
\[
F'(t) \leq -\alpha \xi(t) E(t), \quad \forall t \geq t_0.
\]
Hence, with the help of the left hand side inequality in (4.4) and (4.24), we find
\[
F'(t) \leq -\alpha \alpha_1 \xi(t) F(t), \quad \forall t \geq t_0.
\]
A simple integration of (4.25) over $(t_0, t)$ leads to
\[
F(t) \leq F(t_0) e^{(-\alpha \alpha_1) \int_{t_0}^t \xi(s)ds}, \quad \forall t \geq t_0.
\]
Therefore, (2.11) is established by virtue of (4.4) again.

Case 2. $1 < r < \frac{3}{2}$.

By using (2.1) we get
\[
g(t)^{1-r} \geq (r - 1) \int_{t_0}^t \xi(s)ds + g(t_0)^{1-r}.
\]
For $\forall 0 < \tau < 1$, we further have
\[
\int_0^\infty g^{1-\tau}(s)ds \leq \int_0^\infty \frac{1}{[(r - 1) \int_{t_0}^t \xi(s)ds + g(t_0)^{1-r}]^{\frac{1}{1-\tau}}}ds.
\]
For \( 0 < \tau < 2 - r < 1 \), we have \( \frac{1 - \tau}{r - 1} > 1 \). And using the fact that \( \int_0^{\infty} \xi(s)ds = +\infty \), we obtain
\[
\int_0^{\infty} g^{1-\tau}(s)ds < \infty, \quad \forall \ 0 < \tau < 2 - r.
\]
So Lemma 4.3 and (2.10) yield
\[
(g \circ u_x)(t) \leq C \left( E(0) \int_0^{\infty} g^{1-\tau}(s)ds \right)^{\frac{r-1}{r+1+r}} \leq C (g^r \circ u_x)^{\frac{r-1}{r+1+r}}.
\]
for some positive constant \( C \). Therefore, for any \( r_1 > 1 \), we arrive at
\[
E^{r_1}(t) \leq CE^{r_1-1}(0) \left( \int_0^t x u_1^2 dx - \int_0^t |u|^p dx + \int_0^t x u_2^2 dx \right) + C (g \circ u_x)^{r_1} \leq CE^{r_1-1}(0) \left( \int_0^t x u_1^2 dx - \int_0^t |u|^p dx + \int_0^t x u_2^2 dx \right) + C (g^r \circ u_x)^{\frac{r_1}{r+1+r}}. \quad (4.27)
\]
By choosing \( \tau = \frac{1}{2} \) and \( r_1 = 2r - 1 \) (hence \( \frac{r_1}{r+1+r} = 1 \)), estimate (4.27) gives, for some \( \Gamma > 0 \),
\[
E^{r_1}(t) \leq \Gamma \left[ \int_0^t x u_1^2 dx - \int_0^t |u|^p dx + \int_0^t x u_2^2 dx + (g^r \circ u_x)(t) \right]. \quad (4.28)
\]
By combining (4.21), (4.21) and (4.28), we obtain
\[
F'(t) \leq -\frac{\sigma}{1} \xi(t) E^{r_1}(t) \leq -\frac{\sigma}{1} \alpha_1^{r_1} F^{r_1}(t) \xi(t), \quad \forall t \geq t_0. \quad (4.29)
\]
A simple integration of (4.29) leads to
\[
F(t) \leq C_1 \left( 1 + \int_0^t \xi(s)ds \right)^{\frac{1}{r_1-1}}, \quad \forall t \geq t_0. \quad (4.30)
\]
Therefore,
\[
\int_{t_0}^{\infty} F(t)dt \leq C_1 \int_{t_0}^{\infty} \frac{1}{\left( 1 + \int_0^t \xi(s)ds \right)^{\frac{1}{r_1-1}}}dt.
\]
Since \( \frac{1}{r_1-1} > 1 \) and \( 1 + \int_0^t \xi(s)ds \to +\infty \) as \( t \to +\infty \), we get
\[
\int_{t_0}^{\infty} F(t)dt < \infty. \quad (4.31)
\]
In addition, by using (2.2) we have
\[
t F(t) \leq \frac{C_1 t}{\left( 1 + \int_0^t \xi(s)ds \right)^{\frac{1}{r_1-1}}} \leq C r.
\]
Therefore, we obtain
\[
\sup_{t \geq t_0} t F(t) < +\infty. \quad (4.32)
\]
Since $E(t)$ is bounded, we use (4.4), (4.31) and (4.32) to get
\[ \int_0^\infty F(t) \, dt + \sup_{t \geq 0} tF(t) < \infty. \]

Then, by using (2.10) and Lemma 4.4, we have
\[
\begin{align*}
(g \circ u_x)(t) &\leq C_2 \left( t\|u_x(\cdot, t)\|_{L^2}^2 + \int_0^t \|u_x(\cdot, s)\|^2_{L^2} \, ds \right)^{\frac{r-1}{r}} \left( \int_0^t g^r(t-s)\|u_x(\cdot, t) - u_x(\cdot, s)\|^2_{L^2} \, ds \right)^{\frac{1}{r}} \\
&\leq C_2 \left( tF(t) + \int_0^t F(s) \, ds \right)^{\frac{r-1}{r}} (g^r \circ u_x)^{\frac{1}{r}} \\
&\leq C_3 (g^r \circ u_x)^{\frac{1}{r}},
\end{align*}
\]
which implies that
\[
(g^r \circ u_x)(t) \geq C_4 (g^r \circ u_x)^r, \tag{4.33}
\]
for some constant $C_4 > 0$.

Consequently, a combination of (4.21) and (4.33) yields
\[
F'(t) \leq -C_5 \xi(t) \left[ \int_0^\ell x u_i^2 \, dx - \int_0^\ell x |u|^p \, dx + \int_0^\ell x u_j^2 \, dx + (g \circ u_x)^r \right], \quad \forall \, t \geq t_0,
\]
for some constant $C_5 > 0$.

On the other hand, as in [1], we can get
\[
E^r(t) \leq C_6 \left[ \int_0^\ell x u_i^2 \, dx - \int_0^\ell x |u|^p \, dx + \int_0^\ell x u_j^2 \, dx + (g \circ u_x)^r \right]
\]
for all $t \geq 0$ and some constant $C_6 > 0$. Combining the last two inequalities and (4.4), we obtain
\[
F'(t) \leq -C_7 \xi(t) F^r(t), \quad \forall \, t \geq t_0 \tag{4.34}
\]
for some constant $C_7 > 0$. A simple integration of (4.34) over $(t_0, t)$ gives
\[
F(t) \leq C_8 \left( 1 + \int_{t_0}^t \xi(s) \, ds \right)^{-\frac{1}{r-1}}, \quad \forall \, t \geq t_0.
\]

Therefore, (2.11) is obtained by virtue of (4.4) again.

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