DEFERRED CESÀRO CONULL FK SPACES

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Abstract. In this paper, we study the (strongly) deferred Cesàro conull FK-spaces and we give some characterizations. We also apply these results to summability domains.

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1. Introduction

The classification of conservative matrices as conull or coregular was defined by Wilansky in [13]. The mentioned classification was extended to all FK spaces by Yurimyae [21] and Snyder [17]. Some results of Sember [15] were improved by Bennett [3] for conull FK spaces. Then, İnce [13] studied (strongly) Cesàro conull FK spaces, Dağadur [8] continued to work on $C\lambda$-conull FK spaces and to give some characterizations.

In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequences $x$ by

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$ and $\lim_{n \to \infty} q(n) = \infty$. $D_{p,q}$ is clearly regular for any choice of $\{p(n)\}$ and $\{q(n)\}$.

$w$ denotes the spaces of all complex valued sequences, any vector subspace $X$ of $w$ is a sequence space. A sequence space $X$ with a complete, matrizable, locally convex topology $\tau$ is called FK space if the inclusion map $i : (X, \tau) \to w$, $i(x) = x$ is continuous when $w$ is endowed with the topology of coordinatewise convergence. An FK space whose topology is normable is called a BK space.

By $c$, $l_\infty$ we denote the spaces of all convergent sequences and bounded sequences, respectively. These are FK spaces under $\|x\| = \sup_n |x_n|$. $bv = \{x \in w : \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty\}$ the spaces of all summable sequences of bounded variation; $cs = \{x \in w : \sup_k |\sum_{n=1}^{k} x_n| < \infty\}$, the spaces of all summable sequences. $l = \{x \in w : \sum_{n} |x_n| < \infty\}$, the spaces of all absolute summable sequences.

Throughout this study $e$ denotes the sequences of ones; $\delta^j$ ($j = 1, 2, \ldots$) the sequence with the one in the $j$-th position; $\phi$ the linear span of $\delta^j$'s. The linear span of $\phi$ and $e$ is denoted by $\phi_1$. A sequence $x$ in a locally convex sequence space $X$ is said the property AK if $x^{(n)} \to x$ in $X$ where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k$. An FK space $X$
is called conservative if \( c \subset X \). Also, an FK space \( X \) is called semi-conservativce if \( X^f \subset c \).

We recall (see [10] and [11]) that the \( \sigma \)-dual of a subset \( X \) of \( w \) is defined to be

\[
X^\sigma = \left\{ x \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k y_j \text{ exists for all } y \in X \right\}
\]

\[
= \left\{ x \in w : x.y \in \sigma s \text{ for all } y \in X \right\},
\]

where \( x.y = (x_n y_n) \).

Following Yurimyae [21] and Snyder [17] we say that an FK space \((X, \tau)\) containing \( \phi_1 \) is a conull space if \( e - e^{(n)} \to 0 \) (weakly) in \( X \). It is strongly conull space if \( e - e^{(n)} \to 0 \) in \( X \). \((X, \tau)\) is a K-space containing \( \phi_1 \) is a conull space if \( e - e^{(n)} \to 0 \) (weakly) in \( X \). Bennett [3] gave a relationship between (strongly) conull and (weak) wedge FK-spaces. Also, in [13], an FK space \((X, \tau)\) containing \( \phi_1 \) is a Cesàro conull space if \( e - e^{(n)} \to 0 \) (weakly) in \( X \), and it is strongly Cesàro conull space if \( e - e^{(n)} \to 0 \) in \( X \).

2. Deferred Cesàro conull FK spaces

In this section, the concept of deferred Cesàro conullity for an FK space \( X \) containing \( \phi_1 \) is defined, and several theorems on this subject are given.

The sequence space

\[
\sigma^2_{p}[s] := \left\{ x \in w : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \right\}
\]

is BK spaces with the norm

\[
\| x \| = \sup_{n} \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \right|.
\]

The proof follows the same lines as in (see [3], [7], [10] and [11]), so we omit the details.

We defined \( d \)-dual of a subset \( X \) of \( w \) as

\[
X^d = \left\{ x \in w : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j y_j \text{ exists for all } y \in X \right\}
\]

\[
= \left\{ x \in w : x.y \in \sigma^2_{p}[s] \text{ for all } y \in X \right\}.
\]

Let \( X \) be a set of sequences. Then \( X \subset X^{\nu} \) for \( \nu = f, \sigma, d \).

A sequence \( x \) in a locally convex sequence space \( X \) is said the property \( \sigma^2_{p}[K] \) if

\[
\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \text{ in } X.
\]
Definition 2.1. An FK space $X$ is called deferred semiconservative if $X^j \subset \sigma_{q}^n[s]$. This means that $X \supset \phi$ and
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) \right\}
\]
is convergent for each $f \in X'$. 

Definition 2.2. Let $X$ be an FK space containing $\phi_1$ and
\[(2.1) \quad \zeta^n := e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \]
\[= \left(0, 0, \ldots, 0, \frac{1}{q(n) - p(n)}, \frac{2}{q(n) - p(n)}, \frac{3}{q(n) - p(n)}, \ldots, \frac{q(n) - p(n) - 1}{q(n) - p(n)}, 1, 1, \ldots \right).\]

If $\zeta^n \to 0$ in $X$ then $X$ is called strongly deferred Cesàro conull, where $e^{(k)} := \sum_{j=1}^{k} \delta^j$. If the convergence holds in the weak topology in (2.1) then $X$ is called deferred Cesàro conull FK space. Hence $X$ is deferred Cesàro conull iff
\[
f(e) = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j), \quad \forall f \in X'.
\]

It is clear from the above definitions that if $X$ is deferred Cesàro conull FK space then it is deferred semiconservative space.

Theorem 2.3. Let $X$ be an FK space with $\phi_1 \subset X$ and $\left\{ \frac{p(n)}{q(n) - p(n)} \right\}$ be a bounded sequence. If $X$ is Cesàro conull, then it is deferred Cesàro conull.

Proof. Let $X$ be Cesàro conull. Then for each $f \in X'$, we have
\[
f(e) = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j).\]

Let $s_k(f) = \sum_{j=1}^{k} f(\delta^j)$. So, $(s_k(f))$ is Cesàro summable to $f(e)$. Since $\left\{ \frac{p(n)}{q(n) - p(n)} \right\}$ is bounded, by Theorem 4.1 of [1] it is deferred Cesàro summable to same value. Hence, $X$ is deferred Cesàro conull space.

Theorem 2.4. Let $X$ be an FK space with $\phi_1 \subset X$ and $q(n) = n$. If $X$ is deferred Cesàro conull, then it is Cesàro conull.

Proof. Let $X$ be deferred Cesàro conull. Then
\[
f(e) = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j), \quad \forall f \in X'.
\]

Let $s_k(f) = \sum_{j=1}^{k} f(\delta^j)$. So, $(s_k(f))$ is deferred Cesàro summable to $f(e)$. Since $q(n) = n$, by Theorem 6.1 of [1], it is Cesàro summable to same value. Thus $X$ is Cesàro conull space.

Combining Theorems 2.3 and 2.4 we get the following
Theorem 2.5. Let $X$ be an FK space with $\phi_1 \subset X$ and $\left\{ \frac{p(n)}{n-p(n)} \right\}$ be a bounded sequence. Then $X$ is Cesàro conull if and only if it is deferred Cesàro conull.

Theorem 2.6. Let $X$ be a deferred semiconservative FK space. Then $X$ is conull if and only if it is deferred Cesàro conull.

Proof. Only the sufficiency part needs to be proved. Assume that $X$ is deferred Cesàro conull. A few calculation yields that

$$f(e) - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) = \chi(f) + \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} f(\delta^j).$$

where $\chi(f) = f(e) - \sum_{j=1}^{\infty} f(\delta^j)$. Since $X$ is deferred semiconservative we have $(f(\delta^j)) \in \sigma_{p}[s]$; so, the second term on the right hand side in (2.2) tends to zero as $n \to \infty$. The left hand side must tend to zero as $n \to \infty$ because of deferred Cesàro conullity. This implies that $\chi(f) = 0$, i.e. $X$ is conull. $\Box$

Since every conservative FK space is deferred semiconservative FK space, the following corollary is clear.

Corollary 2.7. Let $X$ be a conservative FK space. Then $X$ is conull if and only if it is deferred Cesàro conull.

By taking $X = c_A$, we immediately get the following

Corollary 2.8. Let $A$ be a conservative matrix. Then $c_A$ is conull if and only if it is deferred Cesàro conull.

Corollary 2.9. Let $X$ be a deferred Cesàro conull FK space but not conull space. Then $X \neq c_A$, for any conservative matrix $A$.

Definition 2.10. Let $(X, \tau)$ be a K space containing $\phi$. $(X, \tau)$ is called deferred Cesàro wedge space if

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \delta^k = \left(0, \ldots, 0, \frac{1}{q(n) - p(n)}, \frac{1}{q(n) - p(n)}, \ldots, \frac{1}{q(n) - p(n)}, 0, \ldots \right) \to 0 \ (n \to \infty) \text{ in } X;$$

and $(X, \tau)$ is called weak deferred Cesàro wedge space, if

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \delta^k \to 0 \ (n \to \infty) \ (weakly) \text{ in } X.$$

Let $\{p(n)\}$ and $\{q(n)\}$ be sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$ and $\lim_{n \to \infty} q(n) = \infty$. Then we have the following implications:

- Deferred Cesàro Conull
  - Strongly deferred Cesàro conull
  - Weakly deferred Cesàro conull
  - Deferred Cesàro wedge
None of the above implications can be reversed.

Now we examine the relation between (strongly) deferred Cesàro conull and (weak) deferred Cesàro wedge FK spaces. To see this, consider the one-to-one and onto mapping

\[ S : w \rightarrow w, \quad Sx = \left( x_1, x_1 + x_2, \ldots, \sum_{k=1}^{n} x_k, \ldots \right), \]

and

\[ S^{-1}x = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}, \ldots) \quad [3] \text{ and } [5]. \]

**Lemma 2.11.** Let \((X, \tau)\) be an FK space. Then

i) \(X\) is strongly deferred Cesàro conull if and only if \(S^{-1}(X)\) is deferred Cesàro wedge space.

ii) \(X\) is deferred Cesàro conull if and only if \(S^{-1}(X)\) is weak deferred Cesàro wedge space.

**Proof.** The FK topology of \(X\) can be given a sequence of seminorms \(\{r(n)\}\). Then \(S^{-1}(X)\) can be topologized by \(t_n(x) = r_n(Sx), \quad (n = 1, 2, \ldots)\), so that it too becomes an FK space as well [12].

(ii) Necessity. Let \(X\) be a deferred Cesàro conull. Observe that \(S : (S^{-1}(X), \tau') \rightarrow (X, \tau)\) is a topological isomorphism [12]. Since \(X\) is deferred Cesàro conull, \(\zeta^n \rightarrow 0\) (weakly) in \(X\). Because \(S^{-1} : (X, \tau) \rightarrow (S^{-1}(X), \tau')\) is continuous, it is weakly continuous. So,

\[ S^{-1} \left( e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j \right) = \frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} \delta^j \rightarrow 0 \text{ (weakly)} \]

in \(S^{-1}(X)\). So \(S^{-1}(X)\) is weak deferred Cesàro wedge space.

Sufficiency. It is enough to observe that \(S : (S^{-1}(X), \tau') \rightarrow (X, \tau)\) is weakly continuous and then

\[ S \left( \frac{1}{q(n) - p(n)} \sum_{j=p(n)+1}^{q(n)} \delta^j \right) = e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j. \]

\[ \square \]

**Theorem 2.12.** Let \(X\) be an FK space with \(\phi_1 \subset X\). If \(X\) is not a deferred Cesàro conull then it cannot be a \(\sigma_p^n[K]\) space.

**Proof.** Let \(X\) be a \(\sigma_p^n[K]\) space. Then we obtain \(\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \rightarrow x \quad (n \rightarrow \infty)\) for each \(x \in X\). In particular, let \(x = e \in X\). Therefore, \(X\) is deferred Cesàro conull, so, it is not \(\sigma_p^n[K]\) space.

By taking \(X = c_A\) in the Theorem 2.12, we immediately get the following

**Corollary 2.13.** Let \(c_A\) be a conservative space. If \(c_A\) is not a deferred Cesàro conull, then it cannot be a \(\sigma_p^n[K]\) space.

**Theorem 2.14.** Let \(X\) be a conservative FK space and consider the propositions below:

i) \(X\) is a \(\sigma_p^n[K]\) space

ii) \(X\) is strongly deferred Cesàro conull
iii) $X$ is deferred Cesàro conull
iv) $e \in \bar{\phi}$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) holds.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obtained from the definitions.

(iii) $\Rightarrow$ (iv). If $f = 0$, on $\phi$, then $f(e) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) = 0$.

Hence, from Hahn Banach Theorem, $e \in \bar{\phi}$ is obtained. □

Using the fact that the space $z^{-1}.X = \{ x : z.x \in X \}$ is an FK space [19] one can get immediately the following:

**Proposition 2.15.** Let $(X, q)$ be an FK-$\sigma_p^q[K]$ space and $z \in w$, then $z^{-1}.X$ is also a $\sigma_p^q[K]$ space.

Taking $X = \sigma_p^q[s]$ in Proposition 2.15 we get

\[
z^{-1}.\sigma_p^q[s] = \{ x : z.x \in \sigma_p^q[s] \}
= \left\{ x : \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j x_j \text{ exists} \right\}
:= z^d.
\]

So we have

**Theorem 2.16.** If $z \in w$, then $z^d$ is a $\sigma_p^q[K]$ space.

**Theorem 2.17.** If $z \in \sigma_p^q[s]$, then $z^d$ is strongly deferred Cesàro conull FK space.

Proof. If $z \in \sigma_p^q[s]$, then $e \in z^{-1}\sigma_p^q[s] = z^d$. From Theorem 2.16, $z^d$ is a $\sigma_p^q[K]$ space. Hence, we have $e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \to 0$, $n \to \infty$. This completes the proof. □

**Theorem 2.18.** i) An FK space that contains a (strongly) deferred Cesàro conull FK space must be a (strongly) deferred Cesàro conull FK space.

ii) A closed subspace, containing $\phi_1$, of a (strongly) deferred Cesàro conull FK space is a (strongly) deferred Cesàro conull FK space

iii) A countable intersection of (strongly) deferred Cesàro conull FK spaces is a (strongly) deferred Cesàro conull FK spaces.

The proof is easily obtained from the elementary properties of FK-spaces [18].

**Corollary 2.19.** If $c$ is closed in $X$, $X$ is not deferred Cesàro conull.

Proof. $c = c_{L}$ is not deferred Cesàro conull. From Theorem 2.18 we obtain desired result. □

**Theorem 2.20.** If $X$ be a deferred Cesàro conull space, then $l_\infty \cap X$ is nonseperable subspace of $l_\infty$.

Proof. $c$ is not deferred Cesàro conull, and hence, by Theorem 2.18(ii) implies that $c \cap X$ is not closed in $X$. By Theorem 8 of [4], desired result is obtained. □
3. SOME DISTINGUISHED SUBSPACES OF $X$

We shall now study the subspaces $D^q_pW$, $D^q_pS$ and $D^q_pF^+$ of an FK-space $X$.

**Definition 3.1.** Let $X$ be an FK space $\supset \phi$. Then

$$D^q_pW := D^q_pW(X) = \left\{ x \in X : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \text{ (weakly) in } X \right\}$$

$$= \left\{ x \in X : f(x) = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j) \text{ for all } f \in X \right\},$$

$$D^q_pS := D^q_pS(X) = \left\{ x \in X : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \right\}$$

$$= \left\{ x \in X : x \text{ has } \sigma^q_p[K] \text{ in } X \right\}$$

$$= \left\{ x \in X : x = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \delta^j \right\}.$$ 

Thus $X$ is an $\sigma^q_p[K]$-space if and only if $D^q_pS = X$.

$$D^q_pF^+ := D^q_pF^+(X)$$

$$= \left\{ x \in X : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \text{ is weakly Cauchy in } X \right\}$$

$$= \left\{ x \in X : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j) \text{ exists for all } f \in X' \right\}$$

$$= \left\{ x \in X : \{x_n f(\delta^n)\} \in \sigma^q_p[s] \text{ for all } f \in X' \right\} = (X^f)^d.$$ 

$$D^q_pF = D^q_pF^+ \cap X$$

**Theorem 3.2.** Let $X$ be an FK space $\supset \phi$, $z \in w$. Then

i) $z \in D^q_pW$ if and only if $z^{-1}.X$ is deferred Cesàro conull; in particular $e \in D^q_pW$ if and only if $X$ is deferred Cesàro conull.

ii) $z \in D^q_pS$ if and only if $z^{-1}.X$ is strongly deferred Cesàro conull; in particular $e \in D^q_pS$ if and only if $X$ is strongly deferred Cesàro conull.

iii) $z \in D^q_pF^+$ if and only if $z^{-1}.X$ is deferred semiconservative FK space; in particular $e \in D^q_pF^+$ if and only if $X$ is deferred semiconservative FK space.
Proof. (i) Necessity. Let \( f \in (z^{-1}.X)' \). By theorem 4.4.10 of [18], \( f \in (z^{-1}.X)' \) iff \( f(x) = \alpha x + g(zx) \), \( \alpha \in \phi, g \in X' \). So we have

\[
(3.1) \quad f \left( e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j \right) = \alpha \left( e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j \right) \\
+ g \left( z \left( e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j \right) \right) \\
= \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \alpha_j + g \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j \delta^j \right). 
\]

Since \( \alpha \in \phi \), the sum \( \sum_{j=1}^{\infty} \alpha_j \) exists, so we have

\[
\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k+1} \alpha_j \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. 
\]

By hypothesis for all \( g \in X' \), \( g \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j \delta^j \right) \rightarrow 0 \) as \( n \rightarrow \infty \). Sufficiency is trivially.

(ii) Necessity. Consider Theorem 4.3.6 of [18] to obtain the seminorms of \( z^{-1}.X \). Observe that

\[
t_i(\zeta^n) = \begin{cases} 
0, & i \leq p(n) \\
\frac{i-p(n)}{q(n)-p(n)}, & p(n) < i \leq q(n) \\
1, & i \geq q(n).
\end{cases}
\]

Hence, for each \( i \), \( r(\zeta^n) \rightarrow 0 \) as \( n \rightarrow \infty \). Also,

\[
h(\zeta^n) = r(z, \zeta^n) = t \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j \delta^j \right). 
\]

Now the result follows at once.

(iii) Let \( z^{-1}.X \) be deferred semiconservative. Then \( (z^{-1}.X)^f \subset \sigma_p^q[s] \). Hence \( f \in (z^{-1}.X)' \). So, \( f(x) = \alpha x + g(zx) \), \( \alpha \in \phi, g \in (z^{-1}.X)' \) and \( f(\delta^n) = \alpha_n x + g(z_n \delta^n) \). Thus, since \( \alpha \in \phi \subset \sigma_p^q[s] \) then \( f(\delta) \in \sigma_p^q[s] \) if and only if \( \{z_n g(\delta^n)\} \subset \sigma_p^q[s] \), i.e. \( z \in D_p^q E^+ \).

\[ \square \]

4. SUMMABILITY DOMAINS AND APPLICATIONS

In this section we give simple conditions for a summability domain \( Y_A \) to be (strongly) deferred Cesàro conull. We shall be concerned with matrix transformations \( y = Ax \), where \( x, y \in w \), \( A = (a_{ij})_{i,j=1}^{\infty} \) is an infinite matrix with complex coefficients, and

\[
y_i = \sum_{j=1}^{\infty} a_{ij} x_j \quad (i = 1, 2, \ldots). 
\]

The sequence \( \{a_{ij}\}_{j=1}^{\infty} \) is called the \( i \)-th row of \( A \) and is denoted by \( a^i \), \( (i = 1, 2, \ldots) \); similarly, the \( j \)-th column of the matrix \( A \). \( \{a_{ij}\}_{i=1}^{\infty} \) is denoted by \( a^j \), \( (j = 1, 2, \ldots) \). For an FK space \( Y \), we consider the summability domain \( Y_A \) defined by

\[
Y_A = \{ x \in w : Ax \text{ exists and } Ax \in Y \}. 
\]
Then $Y_A$ is an FK space under the seminorms $t_n(x) = |x_n|$, ($n = 1, 2, \ldots$);

$$h_n(x) = \sup_m \sum_{j=1}^{m} a_{ij}x_j \quad (n = 1, 2, \ldots) \quad \text{and} \quad (r \circ A)(x) = r(Ax) \quad [18].$$

**Theorem 4.1.** Let $Y$ be an FK space and $A$ be a matrix such that $\phi_1 \subset Y_A$. Then $Y_A$ is a deferred Cesàro conull space if and only if

$$A \left[ e - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \right] \rightarrow 0 \quad \text{(weakly) in } Y.$$  

**Proof.** Necessity. Let $Y_A$ be a deferred Cesàro conull space. Then $\forall f \in Y'_A$, (4.1)

$$f(\zeta^n) = 0 \quad (n \rightarrow \infty).$$  

Let $f(x) = g(Ax)$, for $g \in Y'_A$. So by Theorem 4.4.2 of [18]; $f \in Y'_A$. Because of $f(\zeta^n) = g(A\zeta^n)$, the desired result is obtained from (4.1).

Sufficiency. Let $f \in Y'_A$. Again by Theorem 4.4.2 of [18] $\forall f \in Y'_A$, iff

$$f(x) = \sum_{k=1}^{\infty} \alpha_kx_k + g(Ax),$$

for all $x \in Y_A$, where $\alpha \in w_A^\beta = \{ x : \sum_{n=1}^{\infty} x_ny_n \text{ convergent for all } y \in w_A \}$ and $g \in Y'$. Hence we have

$$f(\zeta^n) = \frac{1}{q(n)-p(n)} \sum_{j=p(n)+1}^{\infty} \sum_{j=k+1}^{\infty} \alpha_j + g(A(\zeta^n)).$$

By hypothesis $e \in Y_A \subset w_A$. Then $\alpha \in w_A^\beta \subset e^\beta = cs$ which implies that

$$\lim_{n \rightarrow \infty} \frac{1}{q(n)-p(n)} \sum_{j=p(n)+1}^{\infty} \sum_{j=k+1}^{\infty} \alpha_j = 0.$$

Also the second term on the right hand side of (4.2) tends to zero. This completed the proof. 

**Theorem 4.2.** $z \in w$, $Y$ is an FK-space and $A$ is a matrix such that $\phi \subset Y_A$ i.e. the columns of $A$ belong to $Y$. Then the following propositions are equivalent in $Y_A$.

1) $z \in D_k^W$, \hfill 
2) $Az - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} A_z^{(k)} \rightarrow 0 \quad \text{(weakly) in } Y,$ \hfill 
3) $Y_{Az}$ is deferred Cesàro conull, \hfill 
4) $g(Az) = \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_jg(a^j)$ for each $g \in Y'$, where

$$Az = (a_{ij}z_j) \quad , \quad A_z^{(k)} = \sum_{j=1}^{k} z_ja^j \quad \text{and} \quad (A_z^{(k)})_i = \sum_{j=1}^{k} a_{ij}z_j.$$

**Proof.** By previous theorem (i) $\equiv$ (iii) . To show that (iii) $\equiv$ (iv) , let $A_z := B$. Then by Theorem 4.1 $Y_B$ is a deferred Cesàro conull if and only if $\forall g \in Y'$,

$$g(Be) = \lim_{n \rightarrow \infty} \frac{1}{q(n)-p(n)} \sum_{j=p(n)+1}^{q(n)} g(Be^{(k)}).$$
Thus, since

\[
Be = \left[ \sum_{j=1}^{\infty} b_{ij} \right] = \left[ \sum_{j=1}^{\infty} a_{ij}z_j \right] = Az,
\]

then we have

\[
(Be^{(k)})_i = \left[ \sum_{j=1}^{\infty} a_{ij}z_j \right] \quad \text{and} \quad Be^{(k)} = \sum_{j=1}^{k} a_j z_j.
\]

So for each \( g \in Y' \), we obtained

\[
g(Az) = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j g(a^j).
\]

Since \( A z^{(k)} = \sum_{j=1}^{k} z_j a^j \), the proof of (ii) \( \equiv \) (iii) is clear. \( \square \)

**Theorem 4.3.** \( z \in w \), \( (Y, r) \) is an FK-space and \( A \) is a matrix such that \( \phi \subset Y_A \) i.e. the columns of \( A \) belong to \( Y \). Then the following propositions are equivalent in \( Y_A \).

\[
i) \ z \in D_{p,S}^{\phi}, \\
ii) \ Az - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1} q(n) A z^{(k)} \to 0 \ \text{in} \ Y, \\
iii) \ Y_{Az} \text{ is strongly deferred Cesàro conull,} \\
iv) \ Az = \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1} q(n) \sum_{j=1}^{k} z_j a^j \text{ convergence in } Y, \text{ where } a^j \text{ is the } k^{th} \text{ column of } A.
\]

**Proof.** By Theorem 3.2, we obtain (i) \( \equiv \) (iii). Since \( A z^{(k)} = \sum_{j=1}^{k} z_j a^j \), we have (ii) \( \equiv \) (iv). Let \( z \in D_{p,S}^{\phi} \). Since \( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1} q(n) \sum_{j=1}^{k} z_j a^j \to 0, \ n \to \infty \), and \( A : Y_A \to Y \) is continuous so \( A(z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1} q(n) z^k) \to 0, \ n \to \infty \). This gives (i) \( \Rightarrow \) (ii).

By Theorem 4.3.8. of [18] \( (w_A, t \cup h) \) is an AK-space, so it is \( \sigma_p^q[K] \)-space. Since \( z \in Y_A \subset w_A \), for each \( i \), we obtain

\[
t_i \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0 \quad \text{and} \quad h_i \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0.
\]

By hypothesis

\[
(r \circ A) \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) = r \left( Az - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} A z^{(k)} \right) \to 0,
\]

which proves theorem. \( \square \)

**Corollary 4.4.** Let \( (Y, q) \) is an FK-space and \( A \) is a matrix such that \( \phi_1 \subset Y_A \). Then \( Y_A \) is strongly deferred Cesàro conull if and only if

\[
A \left[ e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \right] \to 0 \ \text{in} \ Y.
\]

**Proof.** Take \( z = e \) in Theorem 4.3 (ii) \( \equiv \) (iii)) . \( \square \)
Our next result follows immediately from definitions.

**Theorem 4.5.** Let $Y$ be a locally convex FK-space in which every weakly convergent sequence is strongly convergent. Then for the $Y$, $D^0_p S = D^0_p W$.

**Example 4.6.** Theorem 4.3 applies when $Y = l$, bv and bv0.

The following result is obtained by Theorem 4.5.

**Theorem 4.7.** Let $Y$ be an FK space such that weakly convergent sequences are convergent in the FK topology and let $A$ be a matrix. Then $Y_A$ is deferred Cesàro conull space if and only if it is strongly deferred Cesàro conull space.

**Corollary 4.8.** Let $\phi_1 \subset c_A$. Then $c_A$ is strongly deferred Cesàro conull if and only if

$$\limsup_n \sup_i \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0.$$

**Proof.** Take $Y = c$ in Corollary 4.4. $c_A$ is strongly deferred Cesàro conull if and only if

$$A \left( e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)} \right) \to 0 \text{ in } c.$$

So, we get

$$\lim_n \left\| A \left( e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j \right) \right\|_{\infty} = 0 \quad \iff \quad \limsup_n \sup_i \left| A e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} a_{ij} \right| = 0$$

$$\iff \limsup_n \sup_i \left| \sum_{j=1}^{\infty} a_{ij} - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \left( \sum_{j=1}^{\infty} a_{ij} - \sum_{j=k+1}^{\infty} a_{ij} \right) \right| = 0$$

$$\iff \limsup_n \sup_i \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0.$$

This proves the result. \( \square \)

**Corollary 4.9.** Let $\phi \subset l_A$. Then $l_A$ is (strongly) deferred Cesàro conull space if and only if

$$\lim \sum_{i=1}^{\infty} \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0.$$ 

**Proof.** Let $l_A$ be deferred Cesàro conull space. By Theorem 3.2(i) $e \in D^0_p W$. By Theorem 4.7 $e \in D^0_p W$ if and only if

$$\left( A e - \sum_{k=p(n)+1}^{q(n)} A e^{(k)} \right) \to 0 \text{ (weakly)}, n \to \infty \text{ in } l.$$
Thus,

$$\left\| A \left( e - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} \delta^j \right) \right\|_l \to 0, n \to \infty.$$ 

So we have

$$\lim_{n} \left\| \left( A e - \sum_{k=p(n)+1}^{q(n)} a_{ij} \right) \right\|_l = 0$$

$$\iff \lim_{n} \sum_{i=1}^{\infty} \left( A e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} a_{ij} \right) = 0$$

$$\iff \lim_{n} \sum_{i=1}^{\infty} a_{ij} - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \left( \sum_{j=1}^{\infty} a_{ij} - \sum_{j=k+1}^{\infty} a_{ij} \right) = 0$$

$$\iff \lim_{n} \sum_{i=1}^{\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} = 0.$$ 

This proves the corollary. 

**Corollary 4.10.** Let $\phi \subset (bv)_A$. Then $(bv)_A$ is (strongly) deferred Cesàro conull space if and only if

$$\lim_{n} \left\{ \sum_{i=1}^{\infty} \left( \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} a_{ij} \right) \right\} = 0.$$

Proof. Let $(bv)_A$ is deferred Cesàro conull space. By Theorem 4.2, $e \in D^q_pW$. Since weakly and strongly convergence are equivalent in $bv$, $D^q_pS = D^q_pW$ in $bv_A$ by Theorem 4.7. So, $e \in D^q_pS$. Thus

$$\left( A e - \sum_{k=p(n)+1}^{q(n)} A e^{(k)} \right) \to 0, n \to \infty \text{ in } bv.$$ 

So we

$$\lim_{n} \left\| \left( A e - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} a_{ij} \right) \right\|_{bv} = 0$$

$$\iff \lim_{n} \left\| \left( \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} a_{ij} \right) \right\|_{bv} = 0.$$
Corollary 4.11. Let $\phi \subset (l_\infty)_A$. Then $(l_\infty)_A$ is deferred Cesàro conull space if and only if

\[ \sup_{i,n} \left| \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} a_{ij} \right| < \infty \]

i) for any given $\varepsilon > 0$ and increasing sequences $p(n_s)$ and $q(n_s)$ of positive integers, there exists $L$ such that

\[ \sup_{i} \min_{1 \leq s \leq L} \left| \frac{1}{q(n_s) - p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=1}^{k} a_{ij} \right| < \varepsilon \]

Proof. Let us take $Y = l_\infty$ in Theorem 4.12. Then $(l_\infty)_A$ is deferred Cesàro conull space if and only if

\[ A e - \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} a_{ij} \to 0 \text{ (weakly)} , n \to \infty \text{ in } l_\infty. \]

This means that

\[ \sup_{n} \left| \left( \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} a_{ij} \right) \right| \infty < \infty \iff \sup_{i,n} \left| \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} a_{ij} \right| < \infty \]

and there exists $L$ such that

\[ \sup_{i} \min_{1 \leq s \leq L} \left| \frac{1}{q(n_s) - p(n_s)} \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=1}^{k} a_{ij} - \sum_{j=1}^{\infty} a_{ij} \right| < \varepsilon \]

for $\varepsilon > 0$ and increasing sequences $p(n_s)$, $q(n_s)$ of positive integers (see [9], p. 281). Moreover

\[ \left| \sum_{k=p(n_s)+1}^{q(n_s)} \sum_{j=1}^{k} a_{ij} - \sum_{j=1}^{\infty} a_{ij} \right| \]
Thus we get
\[
\sup_{i} \min_{1 \leq s \leq L} \left| \frac{1}{q(n_s)} - \frac{p(n_s)}{p(n_s)+1} \sum_{k=p(n_s)+1}^{\infty} \sum_{j=k+1}^{\infty} a_{ij} \right| < \varepsilon.
\]
The proof is completed. □

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