Quantitative Weighted Bounds for the $q$-Variation of Singular Integrals with Rough Kernels

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Abstract

In this paper, we study the quantitative weighted bounds for the $q$-variational singular integral operators with rough kernels, a stronger nonlinearity than the maximal truncations. The main result is for the truncated singular integrals itself

$$\|V_q\{T_{\Omega, \varepsilon}\}_{\varepsilon > 0} \|_{L^p(w) \to L^p(w)} \lesssim \|\Omega\|_{L^{\infty}(w)}^{1+1/q} \{w\}_{A_p},$$

it is the best known quantitative result for this class of operators. In the course of establishing the above estimate, we obtain several quantitative weighted bounds which are of independent interest.

Keywords Variation inequality · Singular integral operator · Quantitative weighted bounds · Rough kernel

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1 Introduction and Statement of Main Results

1.1 Background and Main Result

Let $F = \{F_t : t \in I \subset \mathbb{R}\}$ be a family of Lebesgue measurable functions defined on $\mathbb{R}^n$. For $x$ in $\mathbb{R}^n$, the value of the $q$-variation function $V_q(F)$ of the family $F$ at $x$ is defined by

$$V_q(F)(x) := \sup \left( \sum_{k \geq 1} |F_{t_k}(x) - F_{t_{k-1}}(x)|^q \right)^{\frac{1}{q}}, \quad q \geq 1,$$

where the supremum runs over all increasing subsequences $\{t_k : k \geq 0\} \subset I$. Specifically, when $q = \infty$,

$$V_\infty(F)(x) = \sup_{k \geq 1} |F_{t_k}(x) - F_{t_{k-1}}(x)|,$$

where the supremum runs over all increasing subsequences $\{t_k : k \geq 0\} \subset I$. It is trivial that $\sup_{t \in \mathbb{R}} |F_t(x)| \leq V_\infty(F)(x) + |F_{t_0}(x)|$ for any $t_0 \in I$.

Suppose $\mathcal{A} = \{A_t\}_{t \in I}$ is a family of operators on $L^p(\mathbb{R}^n) \ (1 \leq p \leq \infty)$, the associated strong $q$-variation operator $V_q\mathcal{A}$ is defined as

$$V_q\mathcal{A}(f)(x) = V_q(\{A_t f(x)\}_{t \in I}).$$

There are two elementary but important observations that motivate the development of variational inequalities in ergodic theory and harmonic analysis. The first one is that from the fact that $V_q(F)(x) < \infty$ with finite $q$ implies the convergence of $F_t(x)$ as $t \to t_0$ whenever $t_0$ is an adherent point for $I$, it is easy to observe that $A_t(f)$ converges a.e. for $f \in L^p$ whenever $V_q\mathcal{A}$ for finite $q$ is of weak type $(p, p)$ with $p < \infty$. The second one is that $q$-variation function dominates pointwisely the maximal function: for any $q \geq 1$,

$$A^*(f)(x) \leq A_{t_0}f(x) + V_q(\mathcal{A}f)(x), \quad (1.2)$$

where $A^*$ is the maximal operator defined by $A^*(f)(x) := \sup_{t \in I} |A_t(f)(x)|$ and $t_0 \in I$ is any fixed number.

Because of the first observation, Bourgain [1] proved in ergodic theory the first variational estimate which had originated from the regularity of Brownian motion in martingale theory [28, 38] since the pointwise convergence may not hold for any non-trivial dense subclass of functions in some ergodic models and thus the maximal Banach principle does not work, see [22, 23] for further results in this direction. Because of the second observation, many maximal inequalities in harmonic analysis, such as maximal singular integrals, maximal operators of Radon type, the Carleson–Hunt theorem, Cauchy type integrals, dimension free Hardy–Littlewood maximal
estimates etc, have been strengthened to variational inequalities [2, 4, 5, 24, 33–37, 43]. There are also many other publications on vector-valued and weighted norm estimates coming to enrich the literature on this subject (cf. e.g., [12, 16, 17, 24, 27]).

In this paper, we continue to study $q$-variational estimates but focus on the quantitative weighted bounds for singular integrals with rough kernels.

For $\varepsilon > 0$, suppose that $T_{\Omega, \varepsilon}$ is the truncated homogeneous singular integral operator defined by

$$T_{\Omega, \varepsilon} f(x) = \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y|^{n}} f(x - y) dy,$$

where $y' = y/|y|$, $\Omega \in L \log^+ L(S^{n-1})$ is homogeneous of degree zero and satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(y') d\sigma(y') = 0.$$

Denote the family $\{T_{\Omega, \varepsilon}\}_{\varepsilon > 0}$ of operators by $T_{\Omega}$. For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, the Calderón–Zygmund singular integral operator $T_{\Omega}$ with homogeneous kernel can be defined by

$$T_{\Omega} f(x) = \lim_{\varepsilon \to 0^+} T_{\Omega, \varepsilon} f(x), \text{ a.e. } x \in \mathbb{R}^n.$$
the sharp weighted estimates for smooth truncations to the case of sharp truncations. The latter result was generalized to matrix weight by Duong et al. [14]. However, the (sharp) quantitative weighted version of Theorem A, that is for operators without regularity, has not been considered up to now except the special case of \( q = \infty \) due to Di Plinio et al. [11] and Lerner [30]. One main goal of the present paper is to address this question, and to provide the quantitative weighted variational inequality for \( \mathcal{T}_\Omega \).

The arguments in [25, 39] show that strong \( q \)-variational estimates with respect to the martingale sequence in \( E_N f(x) = \frac{1}{|Q|} \int_Q f \) (here \( Q \) is the unique dyadic interval of sidelength \( 2^N \) containing \( x \)) fail whenever \( q < 2 \) and the strong \( q \)-variational estimates of \( \mathcal{T}_\Omega \) depends on \( \{E_N\} \). Thus, throughout the paper we will only consider the case \( q > 2 \) for strong \( q \)-variations.

**Theorem 1.1** Let \( \Omega \in L^\infty(S^{n-1}) \) satisfying (1.4) and \( \mathcal{T}_\Omega \) be defined as above. Then the variational inequality

\[
\| V_q \mathcal{T}_\Omega(f) \|_{L^p(w)} \lesssim \| \Omega \|_{L^\infty(w)}^{1+1/q} \|w\| A_p \|f\|_{L^p(w)}
\]  

holds for \( q > 2 \), \( 1 < p < \infty \) and \( w \in A_p \), where the implicit constant \( c_{p,q,n} \) is independent of \( f \) and \( w \), and \( (w)_{A_p} := \max[[w]_{A_\infty}, [w^{1-p'}]_{A_\infty}] \), \( \{w\} A_p := [w]_{A_p}^{1/p} \max\{[w]^{1/p}_{A_\infty}, [w^{1-p'}]^{1/p}_{A_\infty}\} \), and the constant \( [w] A_p \), \( 1 < p \leq \infty \), is given in (2.1) and (2.2).

**Remark 1.2** Our quantitative estimate in (1.7) for \( q = \infty \) matches the best known results in [11, 30] for \( T^*_\Omega \).

**1.2 Approach and Main Methods**

We now state the methods and techniques for proving our main result. Some estimates are of independent interest, and hence we will single them out as theorems or propositions. As usual, we shall prove Theorem 1.1 by verifying separately the corresponding inequalities for the long and short variations as follows (see e.g., [24, Lemma 1.3]).

**Theorem 1.3** Let \( q > 2 \). Let \( \Omega \in L^\infty(S^{n-1}) \) satisfying (1.4). Then for \( 1 < p < \infty \) and \( w \in A_p \),

\[
\| V_q (\{T_{\Omega,2^k} f\}_{k \in \mathbb{Z}}) \|_{L^p(w)} \lesssim \| \Omega \|_{L^\infty(w)}^{1+1/q} \{w\} A_p \|f\|_{L^p(w)}.
\]  

**Theorem 1.4** Let \( q \geq 2 \). Let \( \Omega \in L^\infty(S^{n-1}) \) satisfying (1.4). Then for \( 1 < p < \infty \) and \( w \in A_p \),

\[
\| \mathcal{S}_q (\mathcal{T}_\Omega f) \|_{L^p(w)} \lesssim \| \Omega \|_{L^\infty(w)}^{1/q} \{w\} A_p \|f\|_{L^p(w)},
\]  

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where
\[
S_q(\mathcal{T}_f) := \left( \sum_{k \in \mathbb{Z}} [V_q(\{(\mathcal{T}_f)_{2^k \leq |\cdot| < 2^{k+1})\})^q]^{1/q} \right).
\]

**Remark 1.5** It is worthy to point out the above two results are of independent interest due to the presence of different quantitative behaviors. However for \( q < 2 \), we don’t know whether or not the \( L^p \) boundedness of the short variations \( S_q(\mathcal{T}_f) \) holds.

The proof of Theorem 1.3 is given in Sect. 6 and the proof of Theorem 1.4 in Sect. 7, while the important tools for the proof of these two theorems are established in Sects. 3, 4 and 5.

### 1.2.1 Methods for Proving Theorem 1.3

As a standard way to study variational inequalities for rough singular integrals (see e.g., [7, 10] originating from [13]), the first step is to exploit the Cotlar type decomposition of \( \mathcal{T}_{2^k} \). Let \( \phi \in C_0^\infty(\mathbb{R}^n) \) be a non-negative radial function, supported in \( \{|x| \leq 1/4\} \) with \( \int \phi(x)\,dx = 1 \), and denote \( \phi_k(x) := 2^{-kn}\phi(2^{-k}x) \). Let \( K(\Omega)(x) := \Omega(x/|x|)|x|^{-n} \), and \( \delta_0 \) be the Dirac measure at 0. Then for a reasonable function \( f \), we decompose \( \mathcal{T}_{2^k} f \) as
\[
\mathcal{T}_{2^k} f = \phi_k * \mathcal{T}_f - \phi_k * (K\chi_{|\cdot| \leq 2^k} * f) + (\delta_0 - \phi_k) * \mathcal{T}_{2^k} f,
\]
which divides \( \mathcal{T}_f = \{\mathcal{T}_{2^k}\}_{k \in \mathbb{Z}} \) into three families:
\[
\mathcal{T}_{1}^1(f) := \{\phi_k * \mathcal{T}_f\}_{k \in \mathbb{Z}}, \quad \mathcal{T}_{2}^2(f) := \{\phi_k * (K\chi_{|\cdot| \leq 2^k} * f)\}_{k \in \mathbb{Z}}, \quad \mathcal{T}_{3}^3(f)
\]
\[
:= \{(\delta_0 - \phi_k) * \mathcal{T}_{2^k} f\}_{k \in \mathbb{Z}}.
\]

Thus, it suffices to verify the weighted \( L^p \) estimate for the variation of \( \mathcal{T}_{1}^i(f), i = 1, 2, 3 \).

The second usual step to deal with variational estimates for rough singular integrals is to exploit the almost orthogonality principle based on Littlewood–Paley decomposition and interpolation. However, this step is not obvious at all in the present setting for quantitative weighted estimates. Indeed, the standard Littlewood–Paley decomposition seems to be insufficient. We will exploit the one associated to a sequence of natural numbers \( N = \{N(j)\}_{j \geq 0} \) used in [21]; moreover, it is quite involved to gain the sharp kernel estimates involving the smoothing version of \( \mathcal{T}_f \) which are necessary to obtain the quantitative weighted estimates. Now let us comment on the proof term by term in details.

**Comments on the estimate of** \( \mathcal{T}_{1}^1(f) \). Let us first formulate below the desired estimate of this term which is of independent interest since this variational estimate strengthens the result of Lerner [30]. Denote by \( \Phi(g) = \{\phi_j * g\}_{j \in \mathbb{Z}} \).
Theorem 1.6 \hspace{0.1cm} Let $q > 2$. Let $\phi$ be as used in (1.10) and $T_\Omega$ be given as in (1.3) with $\Omega \in L^\infty(S^{n-1})$ satisfying (1.4). Then for $1 < p < \infty$ and $w \in A_p$,

$$
\| V_q \Phi (T_\Omega f) \|_{L^p(w)} \lesssim \| \Omega \|_{L^\infty(w)}^{1 + 1/q} \{ w \}_A \| f \|_{L^p(w)}. \tag{1.11}
$$

The proof of Theorem 1.6 is given in Section 5.

To obtain (1.11), we apply the Littlewood–Paley decomposition of $T_\Omega$ in [21] to get $T_\Omega = \sum_{j=0}^{\infty} T_j^N$, where each $T_j^N$ is a $\omega_j$-Dini Calderón–Zygmund operator of convolution type. This yields

$$
\| V_q \Phi (T_\Omega f) \|_{L^p(w)} \leq \sum_{j=0}^{\infty} \| V_q \Phi (T_j^N f) \|_{L^p(w)}.
$$

Then to control the summation of the right-hand side of the above inequality, it suffices to show for each term the unweighed estimate with rapid decay (with respect to $j$) and the weighted norm $L^p$ estimate for all $1 < p < \infty$ and then to apply the Stein–Weiss interpolation—Lemma 2.2. The rapid decay estimate will follow from the $L^2$ estimate of $T_j^N$; the involved part lies in the estimates in terms of the weighted constant and the Dini norm. The latter is not only necessary for further application but also is of independent interest, we formulate it in the following theorem.

Theorem 1.7 \hspace{0.1cm} Let $q > 2$. Let $T$ be a $\omega$-Dini Calderón–Zygmund operator of convolution type with the kernel $K$ satisfying the following cancellation condition: for any $0 < \varepsilon$, $R < \infty$,

$$
\int_{\varepsilon < |x| < R} K(x) \, dx = 0. \tag{1.12}
$$

Then for any $1 < p < \infty$ and $w \in A_p$,

$$
\| V_q \Phi (T f) \|_{L^p(w)} \lesssim \left( \| \omega \|_{Dini}^{1/q} \{ w \}_A \| \omega \|_{Dini}^{1/2} \| C_K + \| T \|_{L^2 \rightarrow L^2} \{ w \}_A \| f \|_{L^p(w)} \right),
$$

where $\omega$ satisfies the Dini condition (2.3) and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the $A_p$ constant by a smaller one.

The proof of Theorem 1.7 is given in Section 5.

Firstly, it would be impossible to get the weighted bound in (1.13) via iteration by showing the weighted estimate of $V_q \Phi$ since the one of $T$ is already linear (see, e.g., [21, 29]).

We provide the proof of (1.13) based on the Cotlar type decomposition

$$
\phi_k * T f = K_k^\ast f + \phi_k * K_k \ast f + (\phi_k - \delta_0) * K_k^\ast f, \tag{1.14}
$$
where $K_k = K \chi_{|\cdot| \leq 2^k}$ and $K^k = K \chi_{|\cdot| \geq 2^k}$. Now, the sharp weighted bound of $V_q(\{K^k * f\}_{j \in \mathbb{Z}})$ has been obtained in [9] (see also [6, 14]) with bounds $\|\omega\|_{\text{Dini}} + C_K + \|T\|_{L^2 \to L^2}$. Regarding the second term, at a first glance, it should be easier to handle than the second term in (1.10) since the kernel $K$ enjoys some regularity; but it turns out that they are in the same level of complexity as they will follow from Proposition 1.8 below. Finally, the third term in (1.14) is hard to deal with which will follow from Proposition 1.9.

We first control the variation norm by a stronger norm. That is,

$$V_q(\{\phi_k * K^k * f\}_{k}) = \|\phi_k * K^k * f\|_{V_q} \leq 2\|\phi_k * K^k * f\|_{\ell_q} =: L_q(\{\phi_k * K^k * f\}_{k}).$$

Then all the desired estimates for the left quantity involving $V_q$ will follow from that involving $L_q$.

We establish the following result.

**Proposition 1.8** Let $K$ be a kernel satisfying the mean value zero property (1.12) and the size condition (2.5). Then the following assertions hold.

1. Weak type $(1, 1)$ boundedness:

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * K^k * f|^2 \right)^{1/2} \right\|_{L^{1, \infty}} \lesssim C_K \|f\|_{L^1}.$$  \hfill (1.15)

2. Weak type $(1, 1)$ boundedness of the grand maximal truncated operator (see (2.7)):

$$\left\| M(\sum_{k \in \mathbb{Z}} |\phi_k * K^k * f|^2)^{1/2} \right\|_{L^{1, \infty}} \lesssim C_K \|f\|_{L^1}.$$  \hfill (1.16)

3. Sharp weighted norm inequalities: for all $1 < p < \infty$, and $w \in A_p$,

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * K^k * f|^2 \right)^{1/2} \right\|_{L^p(w)} \lesssim C_K \{w\}_{A_p} \|f\|_{L^p(w)}.$$  \hfill (1.17)

The proof of Proposition 1.8 is given in Section 3.

Surely, by the fact that $\ell^2 \subset \ell^q$ for $q \geq 2$, the desired estimate for the second term in the decomposition (1.14) can be deduced from the Proposition 1.8. Moreover, it is worth highlighting that in Proposition 1.8, we do not need any regularity assumption on the kernel $K$.

Again, we start with the comments on the estimate of the third term in the decomposition (1.14). We prove the following result.

**Proposition 1.9** Let $q > 2$. Let $K$ be a kernel satisfying the mean value zero property (1.12) and the size condition (2.5) and the Dini condition (2.6). Let $C(K, \omega, q) = C_K + \|\omega\|_{\text{Dini}} + \|\omega\|^{1/q}_{\text{Dini}} \|\omega\|^{1/q'}_{\text{Dini}} + \|\omega\|^{1/q'}_{\text{Dini}} \|\omega\|^{1/2}_{\text{Dini}}$, where $\omega$ satisfies the Dini condition (2.3). Then the following assertions hold.
(1) Weak type $(1,1)$ boundedness:

$$
\left\| \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k * f|^q \right)^{1/q} \right\|_{L^{1,\infty}} \lesssim C(K, \omega, q) \| f \|_{L^1}. \quad (1.18)
$$

Weak type $(1,1)$ boundedness of the grand maximal truncated operator:

$$
\left\| M_1 \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k * f|^q \right)^{1/q} \right\|_{L^{1,\infty}} \lesssim C(K, \omega, q) \| f \|_{L^1}. \quad (1.19)
$$

(2) Sharp weighted norm inequalities: for all $1 < p < \infty$, and $w \in A_p$,

$$
\left\| \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k * f|^q \right)^{1/q} \right\|_{L^p(w)} \lesssim C(K, \omega, q) \| w \|_{A_p} \| f \|_{L^p(w)}. \quad (1.20)
$$

This proposition will be shown in Sect. 4.

We point out that at the moment we are unable to prove the above result for $q = 2$ since our argument in showing the weak type $(1,1)$ estimate of the localized maximal operator (1.19) depends heavily on the same estimate of $q$-variation operator associated to the Hardy–Littlewood averages.

In this process, together with the weak type $(1,1)$ estimates of variational Calderón–Zygmund operators [14], we also obtain the following result as a byproduct.

**Corollary 1.10** Let $q > 2$ and $T$ be as in Theorem 1.7. Then

$$
\| V_q \Phi(T f) \|_{L^{1,\infty}} \lesssim (\| \omega \|_{Dini} + \| \omega \|_{Dini}^{1/q} + \| \omega \|_{Dini}^{1/q'}) \| f \|_{L^1},
$$

where $\omega$ satisfies the Dini condition (2.3).

**Comments on the estimate of $T^2_{\Omega}(f)$**. About $T^2_{\Omega}(f)$, we start with the comments on the estimate of the second term in the decomposition (1.14) since the latter involving regular kernel $K$ seems to be easier to handle. In particular, Proposition 1.8 applies well to $T^2_{\Omega}(f)$ and thus the almost orthogonality technique can be avoided, improving the corresponding argument in [7].

**Comments on the estimate of $T^3_{\Omega}(f)$**. For the rough kernel $K_{\Omega}$ in $T^3_{\Omega}(f)$, we have to decompose it by using the Littlewood–Paley decomposition from [21], repeat the argument of Proposition 1.9, require a slightly more involved proof and then exploit the almost orthogonality principle.

### 1.2.2 Methods for proving Theorem 1.4

Following from the sharp weighted boundedness of the Hardy–Littlewood maximal operator (see (7.1)),

$$
\| S^\infty(T_{\Omega} f) \|_{L^p(w)} \leq c_{p,n} \| \Omega \|_{L^\infty(\{ w \})_{A_p}} \| f \|_{L^p(w)}.
$$
Thus by interpolation, it suffices to prove that
\[ \|S_2(Tf)\|_{L^p(w)} \leq c_{p,n} \|\Omega\|_{L^\infty(w)}^{1/2} \|w\|_{A_p} \|f\|_{L^p(w)}. \] (1.21)

To verify this, we write
\[ S_2(Tf)(x) = \left( \sum_{k \in \mathbb{Z}} |V_{2,k}(f)(x)|^2 \right)^{1/2} = \left( \sum_{k \in \mathbb{Z}} \|\{T_{k,t}f(x)\}_{t \in [1,2]}\|_{V_2^2}^2 \right)^{1/2}, \] (1.22)
where \( T_{k,t}f(x) = v_{k,t} * f(x) \) for \( t \in [1,2] \), and \( v_{0,t}(x) = \Omega(x/|x|)|x|^{-n} \chi_{t \leq |x| \leq 2} \) and \( v_{k,t}(x) = 2^{-kn} v_{0,t}(2^{-k}x) \) for \( k \in \mathbb{Z} \). Next, by using the Littlewood–Paley decomposition as in [21], we further have \( T_{k,t} = \sum_{j=0}^\infty T_{N,k,t,j} \), whose definition will be given in Section 7. Therefore, by the Minkowski inequality, we get
\[ S_2(Tf)(x) \leq \sum_{j=0}^\infty S_{2,j}^N(f)(x), \]
where
\[ S_{2,j}^N(f)(x) := \left( \sum_{k \in \mathbb{Z}} \|\{T_{N,k,t,j}f(x)\}_{t \in [1,2]}\|_{V_2^2}^2 \right)^{1/2}. \]

Hence, to prove (1.21), it again suffices to verify for \( S_{2,j}^N(f) \) the unweighted norm with rapid decay in \( j \) and the weighted norm estimate and then to apply the Stein–Weiss interpolation—Lemma 2.2.

**Notation.** From now on, \( p' = p/(p-1) \) represents the conjugate number of \( p \in [1,\infty) \); \( X \lesssim Y \) stands for \( X \leq CY \) for a constant \( C > 0 \) which is independent of the essential variables living on \( X \) & \( Y \); and \( X \approx Y \) denotes \( X \lesssim Y \lesssim X \).

## 2 Preliminaries

### 2.1 Muckenhoupt Weights

Let \( w \) be a non-negative locally integrable function defined on \( \mathbb{R}^n \). For \( 1 < p < \infty \), we say that \( w \in A_p \) if there exists a constant \( C > 0 \) such that
\[ [w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'}dx \right)^{p-1} \leq C. \] (2.1)
We will adopt the following definition for the $A_\infty$ constant for a weight $w$ introduced by Fujii [15] and later by Wilson [42]:

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w \chi_Q)(x) \, dx. \quad (2.2)$$

Here $w(Q) := \int_Q w(x) \, dx$, and the supremum above is taken over all cubes with edges parallel to the coordinate axes. When the supremum is finite, we will say that $w$ belongs to the $A_\infty$ class. $A_\infty := \bigcup_{p \geq 1} A_p$. Recall in Theorem 1.1, $(w)_{A_p} := \max\{[w]_{A_\infty}, [w^{1-p'/p}]_{A_\infty}\}$. Using the facts that $[w^{1-p'/p}]_{A_p} = [w]_{A_p}^{1/p}$ and $[w]_{A_\infty} \leq C[w]_{A_p}$ (see [19]), one easily checks that $(w)_{A_p} \leq [w]_{A_p}^{\max(1,1/(p-1))}$ (see [21]).

### 2.2 $\omega$-Dini Calderón–Zygmund Operators of Convolution Type

A modulus of continuity $\omega$ if it is increasing and subadditive in the sense that

$$u \leq t + s \Rightarrow \omega(u) \leq \omega(t) + \omega(s).$$

$\omega$ satisfies the classical Dini condition

$$\|\omega\|_{Dini} := \int_0^1 \omega(t) \frac{dt}{t} < \infty. \quad (2.3)$$

For any $c > 0$ the integral can be equivalently (up to a $c$-dependent multiplicative constant) replaced by the sum over $2^{-j/c}$ with $j \in \mathbb{N}$.

Let $T$ be a bounded linear operator on $L^2(\mathbb{R}^n)$ represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x - y) f(y) \, dy, \ \forall x \notin \text{supp} f. \quad (2.4)$$

We say that $T$ is an $\omega$-Dini Calderón–Zygmund operators of convolution type if the kernel $K$ satisfies the following size and smoothness conditions:

for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$|K(x)| \leq \frac{C_K}{|x|^n}; \quad (2.5)$$

for any $x, y \in \mathbb{R}^n$ with $2|y| \leq |x|$,

$$|K(x - y) - K(x)| \leq \frac{\omega(|y|/|x|)}{|x|^n}. \quad (2.6)$$
2.3 Criterion for Sharp Weighted Norm Estimate

The grand maximal truncated operator $M_U$, associated to a sub-linear operator $U$, is defined by

$$M_U f(x) = M_U f(x) = \sup_{Q \ni x} \text{ess sup} |U(f \chi_{R^n \setminus Q})(\xi)|,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$.

**Lemma 2.1** ([29]) Assume that both $U$ and $M_U$ be of weak type $(1, 1)$, then for every compactly supported $f \in L^p(\mathbb{R}^n)$ ($1 < p < \infty$) and $w \in A_p$,

$$\|Uf\|_{L^p(w)} \lesssim K \{w\} A_p \|f\|_{L^p(w)},$$

where $K = \|U\|_{L^1 \rightarrow L^1} + \|M_U\|_{L^1 \rightarrow L^1}$.

2.4 The Stein–Weiss Interpolation Theorem with Change of Measures

The following interpolation result with change of measures due to Stein and Weiss plays an important role in dealing with rough singular integrals. Again, we will use frequently this tool.

**Lemma 2.2** ([40]) Assume that $1 \leq p_0, p_1 \leq \infty$, that $w_0$ and $w_1$ be positive weights, and $T$ be a sublinear operator satisfying

$$T : L^{p_i}(w_i) \rightarrow L^{p_i}(w_i), \quad i = 0, 1,$$

with quasi-norms $M_0$ and $M_1$, respectively. Then

$$T : L^p(w) \rightarrow L^p(w),$$

with quasi-norm $M \leq M_0^\frac{\lambda}{\lambda} M_1^{1-\lambda}$, where

$$\frac{1}{p} = \frac{\lambda}{p_0} + \frac{1-\lambda}{p_1}, \quad w = w_0^{\frac{p_0}{p}} w_1^{\frac{p_1}{p}}.$$

3 Proof of Proposition 1.8

We first give a useful Lemma which plays an important role in the proof of Proposition 1.8.

**Lemma 3.1** For any fixed $k \in \mathbb{Z}$, let $\mu_k$ be a function such that $\text{supp } \mu_k \subset \{x : |x| \lesssim 2^k\}$. If there exist constants $\gamma_1, \gamma_2, \gamma_3$ such that $\mu_k$ satisfy

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\mu_k * f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \gamma_1 \|f\|_{L^2},$$

(3.1)
and
\[ |\mu_k(x)| \lesssim \frac{\gamma^2}{2kn}, \quad k \in \mathbb{Z}, \quad (3.2) \]
and for \(0 < 2|y| \leq |x|\),
\[ \sum_{k \in \mathbb{Z}} |\mu_k(x - y) - \mu_k(x)| \lesssim \gamma^3 \frac{|y|^\nu}{|x|^{n+\nu}}. \quad (3.3) \]

Then for \(1 < p < \infty\), and \(w \in A_p\),
\[ \left\| \left( \sum_{k \in \mathbb{Z}} |\mu_k * f|^2 \right)^{1/2} \right\|_{L^p(w)} \lesssim (\gamma_1 + \gamma_2 + \gamma_3) \|f\|_{L^p(w)}. \quad (3.4) \]

**Proof** To prove (3.4), by Lemma 2.1, it suffices to verify that
\[ \left\| \left( \sum_{k \in \mathbb{Z}} |\mu_k * f|^2 \right)^{1/2} \right\|_{L^1, \infty} \lesssim (\gamma_1 + \gamma_2 + \gamma_3) \|f\|_{L^1}, \quad (3.5) \]
and
\[ \|M(\sum_{k \in \mathbb{Z}} |\mu_k * f|^2)^{1/2}\|_{L^1, \infty} \lesssim (\gamma_1 + \gamma_2 + \gamma_3) \|f\|_{L^1}. \quad (3.6) \]

By virtue of (3.1)–(3.3), we know \(\{\mu_k * f\}\) is a vector Calderón–Zygmund operator. By using a Calderón–Zygmund decomposition and a trivial computation, it is easy to verify that (3.5) holds. Next, we verify (3.6). Take \(x\) and \(\xi\) in any fixed cube \(Q\). Let \(B_x = B(x, 2\sqrt{n\ell(Q)})\), then \(3Q \subset B_x\). By using the triangle inequality, we get

\[
\left( \sum_{k \in \mathbb{Z}} |\mu_k * (f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|^2 \right)^{1/2} \\
\leq \left( \sum_{k \in \mathbb{Z}} |\mu_k * (f \chi_{\mathbb{R}^n \setminus B_x})(\xi)|^2 \right)^{1/2} - \left( \sum_{k \in \mathbb{Z}} |\mu_k * (f \chi_{\mathbb{R}^n \setminus B_x})(x)|^2 \right)^{1/2} \\
+ \left( \sum_{k \in \mathbb{Z}} |\mu_k * f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|^2 \right)^{1/2} \\
+ \left( \sum_{k \in \mathbb{Z}} |\mu_k * (f \chi_{\mathbb{R}^n \setminus B_x})(x)|^2 \right)^{1/2} =: I + II + III.
\]

For the term \(I\), since \(x\) and \(\xi\) in \(Q\), then \(|x - \xi| \leq \sqrt{n\ell(Q)}\), and \(|x - y| \geq 2\sqrt{n\ell(Q)}\), then we get \(2|x - \xi| \leq |x - y|\), by (3.3) we have
\[
I \leq \int_{\mathbb{R}^n \setminus B_x} \sum_{k \in \mathbb{Z}} |\mu_k(\xi - y) - \mu_k(x - y)| |f(y)| \, dy \\
\leq \int_{\mathbb{R}^n \setminus B_x} \frac{\gamma_3 |x - \xi|}{|x - y|^{n+1}} |f(y)| \, dy \lesssim \gamma_3 Mf(x).
\]

For the term \( II \), since \( \xi \) in \( Q \) and \( y \in B_x \setminus 3Q \), then \(|y - \xi| \geq 3\ell(Q)\) and \(|y - \xi|^{-n} \lesssim \ell(Q)^{-n} \) and \(|x - y| \leq 2\sqrt{n} \ell(Q)\), then by the size condition (3.2), we get

\[
II \leq \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |\mu_k(\xi - y)||f|_{B_x \setminus 3Q}(y) \, dy \\
\lesssim \gamma_2 \int_{|x-y| \leq 2\sqrt{n} \ell(Q)} |\xi - y|^{-n} |f(y)| \, dy \lesssim \gamma_2 Mf(x).
\]

For the last term, recalling that \( \text{supp} \mu_k \subset \{x : |x| \leq 2^k\} \) and that \( B_x = \{y : |y-x| \leq 2\sqrt{n} \ell(Q)\} \), we have that \( \mu_k * f |_{\mathbb{R}^n \setminus B_x} = \int_{2\sqrt{n} \ell(Q) < |x-y| \leq 2^k} \mu_k(x-y) f(y) \, dy \). Thus,

\[
III = \left( \sum_{2\sqrt{n} \ell(Q) \leq 2^k} |\mu_k * f |_{\mathbb{R}^n \setminus B_x}(x)|^2 \right)^{1/2} \\
\leq \left( \sum_{k \in \mathbb{Z}} |\mu_k * f(x)|^2 \right)^{1/2} + \left( \sum_{2\sqrt{n} \ell(Q) \leq 2^k} |\mu_k * f |_{B_x}(x)|^2 \right)^{1/2}.
\]

By using the size estimate (3.2), we have

\[
\left( \sum_{2\sqrt{n} \ell(Q) \leq 2^k} |\mu_k * f |_{\mathbb{R}^n \setminus B_x}(x)|^2 \right)^{1/2} \\
\lesssim \sum_{2\sqrt{n} \ell(Q) \leq 2^k} \frac{\gamma_2}{2^{2kn}} \int_{|x-y| \leq 2\sqrt{n} \ell(Q)} |f(y)| \, dy \lesssim \gamma_2 Mf(x),
\]

which gives

\[
III \leq \left( \sum_{k \in \mathbb{Z}} |\mu_k * f(x)|^2 \right)^{1/2} + \gamma_2 Mf(x).
\]

Combining the estimates of \( I, II \) and \( III \), we get

\[
\mathcal{M}(\sum_{k \in \mathbb{Z}} |\mu_k * f(x)|^2)^{1/2} \lesssim \left( \sum_{k \in \mathbb{Z}} |\mu_k * f(x)|^2 \right)^{1/2} + (\gamma_2 + \gamma_3) Mf(x).
\]
Then the weak type \((1, 1)\) of the Hardy–Littlewood maximal function \(M\) and the estimate in (3.5) imply
\[
\|M(\sum_{k \in \mathbb{Z}} |\mu_k f|^2)^{1/2}\|_{L^{1, \infty}} \lesssim (\gamma_1 + \gamma_2 + \gamma_3) \|f\|_{L^1},
\]
which shows that (3.6) holds. The proof of Lemma 3.1 is complete. \(\square\)

Now we return to the proof of Proposition 1.8. We point out that to estimate \(\{\phi_k \ast K_k\}\), we do not need any regularity condition on \(K\), we only assume \(K\) satisfies the size condition (2.5) and the cancellation condition (1.12). We first verify the size estimate of \(\phi_k \ast K_k(x)\) for any fixed \(k \in \mathbb{Z}\). Since \(\text{supp} \phi_k \subset \{x : |x| \leq 2^k/4\}\), we have \(\text{supp} \phi_k \ast K_k \subset \{x : |x| \leq 2^{k+1}\}\). Then from (2.5) and (1.12) we have for some \(\theta \in (0, 1)\)
\[
|\phi_k \ast K_k(x)| = \left| \int_{|z| \leq 2^k} (\phi_k(x - z) - \phi_k(x)) K(z) \, dz \right| 
\lesssim \frac{C_K}{2^k} \int_{|z| \leq 2^k} |(\nabla \phi)_k(x - \theta z)| \frac{1}{|z|^{n-1}} \, dz 
\lesssim \frac{C_K}{2^k(n+1)} \int_{|z| \leq 2^k} \frac{1}{|z|^{n-1}} \, dz \chi_{|x| \leq 2^{k+1}}(x) 
\lesssim \frac{C_K}{2^{kn}} \chi_{|x| \leq 2^{k+1}}(x). 
\tag{3.7}
\]

Now, we estimate \(\nabla \phi_k \ast K_k(x)\). Recall that \(\phi \in C^\infty_0(\mathbb{R}^n)\) with \(\int \phi = 1\) and \(\text{supp} \phi \subset \{x : |x| \leq 1/4\}\). It is easy to verify that \(\int \nabla \phi = 0\) and \(\text{supp} (\nabla \phi) \subset \{x : |x| \leq 1/4\}\) with \(\nabla \phi \in C^\infty_0(\mathbb{R}^n)\). Since \(\text{supp} (\nabla \phi)_k \subset \{x : |x| \leq 2^k/4\}\) and \(\text{supp} K_k \subset \{x : |x| \leq 2^k\}\) then we get \(\text{supp} (\nabla \phi)_k \ast K_k \subset \{x : |x| \leq 2^{k+1}\}\). Then repeating the argument of (3.7), we get
\[
|\nabla \phi_k \ast K_k(x)| = \frac{1}{2^k}|(\nabla \phi)_k \ast K_k(x)| \lesssim \frac{C_K}{2^{k(n+1)}} \chi_{|x| \leq 2^{k+1}}(x). 
\tag{3.8}
\]
Thus for \(|x| \geq 2|y|\), and for some \(\theta \in (0, 1)\),
\[
\sum_{k \in \mathbb{Z}} |\phi_k \ast K_k(x - y) - \phi_k \ast K_k(x)| \lesssim \sum_{k \in \mathbb{Z}} |\nabla \phi_k \ast K_k(x - \theta y)| |y|.
\]
Since \(|x| \geq 2|y|\), then \(x/2 \leq |x - \theta y| \leq 3x/2\), we get \(|\nabla \phi_k \ast K_k(x)| \leq \frac{C_K}{2^{k(n+1)}} \chi_{|x| \leq 2^{k+1}}\). Thus for \(|x| \geq 2|y|\), we get
\[
\sum_{k \in \mathbb{Z}} |\phi_k \ast K_k(x - y) - \phi_k \ast K_k(x)| \lesssim \sum_{k \in \mathbb{Z}} \frac{C_K}{2^{k(n+1)}} \chi_{|x| \leq 2^{k+1}} |y| 
\lesssim C_K \sum_{k \in \mathbb{Z}} 2^{-k(n+1)} \chi_{2^k \geq |x|/2} |y| 
\lesssim C_K |y| \sum_{2^k \geq |x|/2} 2^{-k(n+1)}
\]
\[ \lesssim C_K \frac{|y|}{|x|^{n+1}}. \]

On the other hand, it is easy to verify that
\[ |\phi_k * K_k(x)| \lesssim \frac{C_K}{2k^n} \chi_{|x| \leq 2^{k+1}} \lesssim C_K \frac{2^k}{(2^k + |x|)^{n+1}}, \]
and
\[ |\phi_k * K_k(x + h) - \phi_k * K_k(x)| \lesssim C_K \frac{|h|^\eta}{(2^k + |x|)^{n+1}}, \quad |h| \leq 2^k \]
for some \( \eta \in (0, 1) \). In addition, since \( \phi_k * K_k * 1 = 0 \), by applying the \( T1 \) Theorem for square functions (see [8]) we have
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * K_k * f|^2 \right)^{1/2} \right\|_{L^2} \lesssim C_K \| f \|_{L^2}. \tag{3.9}
\]
Then applying Lemma 3.1 to \( \{\mu_k\} = \{\phi_k * K_k\} \), we get
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * K_k * f|^2 \right)^{1/2} \right\|_{L^{1, \infty}} \lesssim C_K \| f \|_{L^1}, \quad \| \mathcal{M} (\sum_{k \in \mathbb{Z}} |\phi_k * K_k * f|^2)^{1/2} \|_{L^{1, \infty}} \lesssim C_K \| f \|_{L^1}
\]
and then by Lemma 2.1, for \( 1 < p < \infty \) and \( w \in A_p \),
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * K_k * f|^2 \right)^{1/2} \right\|_{L^p(w)} \lesssim C_K \{w\}_{A_p} \| f \|_{L^p(w)}.
\]
This verifies (1.15)–(1.17). The proof of Proposition 1.8 is complete. \( \square \)

4 Proof of Proposition 1.9

Step 1. First, we would like to prove that
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k * f|^q \right)^{1/q} \right\|_{L^2} \lesssim (C_K + \|\omega\|_{Dini} + \|\omega\|_{Dini}^{1/q'} \|\omega\|_{Dini}^{1/2} \|\omega\|_{Dini}^{2/q'}) \| f \|_{L^2}. \tag{4.1}
\]
We first use Fourier transform and Plancherel theorem to deal with the \( L^2 \)-norm of \( (\sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k * f|^2)^{1/2} \). For \( |2^k \xi| < 1 \), write
\[
\hat{K}^k(\xi) = \int_{2^k < |x| < |\xi|^{-1}} K(x) e^{-2\pi i x \cdot \xi} dx + \int_{|\xi|^{-1} < |x|} K(x) e^{-2\pi i x \cdot \xi} dx =: I_{1,k}(\xi) + I_{2,k}(\xi).
\]

For the term \( I_{1,k}(\xi) \), by the size estimate (2.5) and cancellation (1.12), we have

\[
|I_{1,k}(\xi)| = \left| \int_{2^k < |x| \leq |\xi|^{-1}} K(x) (e^{-2\pi i x \cdot \xi} - 1) dx \right| \lesssim 2\pi |\xi| \int_{|x| \leq |\xi|^{-1}} |x||K(x)| dx \lesssim C_K
\]
uniformly in \( k \).

We now consider \( I_{2,k}(\xi) \). Let \( z = \frac{\xi}{2|\xi|^2} \) so that \( e^{2\pi iz \cdot \xi} = -1 \) and \( 2|z| = |\xi|^{-1} \). By changing variables \( x = x' - z \), we rewrite \( I_{2,k}(\xi) \) as

\[
I_{2,k}(\xi) = -\int_{|\xi|^{-1} < |x' - z|} K(x' - z) e^{-2\pi i x' \cdot \xi} dx'.
\]

Taking the average of the above inequality and the original definition of \( I_{2,k} \) gives

\[
I_{2,k}(\xi) = \frac{1}{2} \int_{|\xi|^{-1} < |x|} K(x) e^{-2\pi i x \cdot \xi} dx - \frac{1}{2} \int_{|\xi|^{-1} < |x - z|} K(x - z) e^{-2\pi i x \cdot \xi} dx.
\]

Now we split

\[
I_{2,k}(\xi) = \frac{1}{2} \int_{|\xi|^{-1} < |x|} (K(x) - K(x - z)) e^{-2\pi i x \cdot \xi} dx + \frac{1}{2} \int_{\mathbb{R}^n} K(x - z)(\chi_{[|\xi|^{-1} < |x|]} - \chi_{[|\xi|^{-1} < |x - z|]}) e^{-2\pi i x \cdot \xi} dx.
\]

For the first term, by noting that \( 2|z| = |\xi|^{-1} \) and using (2.6) we get

\[
\left| \int_{|\xi|^{-1} < |x|} (K(x) - K(x - z)) e^{-2\pi i x \cdot \xi} dx \right| \leq \int_{|\xi|^{-1} < |x|} \frac{1}{|x|^n} \omega \left( \frac{|z|}{|x|} \right) dx
\]

\[
\leq \int_{|\xi|^{-1} < |x|} \frac{1}{|x|^n} \omega \left( \frac{|\xi|^{-1}}{2|x|} \right) dx = \int_{|\xi|^{-1} < r} \omega \left( \frac{|\xi|^{-1}}{2r} \right) dr \int_{S^{n-1}} d\sigma(\theta) \lesssim \|\omega\|_{Dini}.
\]

For the second term, we note that \( \chi_{[|\xi|^{-1} < |x|]} - \chi_{[|\xi|^{-1} < |x - z|]} \) is nonzero if and only if \( |\xi|^{-1} - |z| \leq |x - z| \leq |\xi|^{-1} + |z| \). Thus by (2.5)
Combining the above estimates, we get $|I_{2,k}(\xi)| \lesssim C_K + \|\omega\|_{Dini}$, which, together with the estimate of $|I_{1,k}(\xi)|$, gives

$$|\widehat{K^k}(\xi)| \lesssim C_K + \|\omega\|_{Dini}. \quad (4.2)$$

For $|2^k \xi| > 1$, let $z = \frac{\xi}{2|\xi|^2}$ so that $e^{2\pi i z \cdot \xi} = -1$ and $2|z| = |\xi|^{-1}$. Similarly to $I_{2,k}(\xi)$, we get

$$\widehat{K^k}(\xi) = \frac{1}{2} \int_{2^k < |x|} (K(x) - K(x-z)) e^{-2\pi i x \cdot \xi} dx + \frac{1}{2} \int_{\mathbb{R}^n} K(x-z) (\chi_{[2^k, \infty)}(x) - \chi_{[2^k, \infty)}(x-z)) e^{-2\pi i x \cdot \xi} dx$$

$$=: J_{1,k}(\xi) + J_{2,k}(\xi).$$

For the term $J_{1,k}$, by noting that $2|z| = |\xi|^{-1}$ and using (2.6) we get

$$|J_{1,k}(\xi)| \leq \frac{1}{2} \int_{2^k < |x|} \frac{1}{|x|^n} \omega \left( \frac{|z|}{|x|} \right) dx \leq \frac{1}{2} \int_{2^k < |x|} \frac{1}{|x|^n} \omega \left( \frac{|\xi|^{-1}}{2|x|} \right) dx$$

$$= \frac{1}{2} \int_{2^k < r} \omega \left( \frac{|\xi|^{-1}}{2r} \right) \frac{dr}{r} \int_{S^{n-1}} d\sigma(\theta)$$

$$\lesssim \omega^{1/2} ((2^k \xi)^{-1}) \int_{2^k < r} \omega \left( \frac{2^k}{2r} \right)^{1/2} \frac{dr}{r} \int_{S^{n-1}} d\sigma(\theta)$$

$$\lesssim \|\omega\|_{Dini} \omega^{1/2} ((2^k \xi)^{-1}).$$

For the term $J_{2,k}$, we note that $\chi_{2^k < |x|} - \chi_{2^k < |x-z|}$ is nonzero if and only if $2^k - |z| \leq |x - z| \leq 2^k + |z|$. Thus by (2.5)

$$|J_{2,k}(\xi)| \lesssim \int_{2^k - |z| \leq |x-z| \leq 2^k + |z|} |K(x-z)| dx \lesssim \int_{2^k - |z| \leq |x| \leq 2^k + |z|} \frac{C_K}{|x|^n} dx$$

$$\lesssim \frac{1}{2^k} \int_{2^k - |z| \leq |x| \leq 2^k + |z|} \frac{C_K}{|x|^{n-1}} dx \lesssim C_K |2^k \xi|^{-1}.$$
Since $\hat{\phi} \in S(\mathbb{R}^n)$ and $\hat{\phi}(0) = 1$, then $|1 - \hat{\phi}_k(\xi)| \lesssim \min(|2^k \xi|, 1)$. By the Plancherel Theorem, (4.2) and (4.3), we get

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) * K^k * f|^2 \right)^{1/2} \right\|_{L^2}^2 \leq \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |(\delta_0 - \phi_k)(\xi)|^2 |\hat{K}^k(\xi)|^2 |\hat{f}(\xi)|^2 d\xi$$

$$\lesssim \int_{\mathbb{R}^n} ((C_K + \|\omega\|_{\text{Dini}})^2 \sum_{|2^k \xi| \leq 1} |2^k \xi|^2 + C_K^2 \sum_{|2^k \xi| \geq 1} |2^k \xi|^{-2} + \|\omega^{1/2}\|_{\text{Dini}}^2 \sum_{|2^k \xi| \geq 1} \omega(|2^k \xi|^{-1})) |\hat{f}(\xi)|^2 d\xi$$

$$\lesssim (C_K + \|\omega\|_{\text{Dini}} + \|\omega^{1/2}\|_{\text{Dini}} \|\omega\|_{\text{Dini}}^{1/2})^2 \|f\|_{L^2}^2. \quad (4.4)$$

On the other hand, we claim that for any fixed $k \in \mathbb{Z}$,

$$|(\delta_0 - \phi_k) * K^k(x)| \lesssim \omega(2^k / |x|) \frac{1}{|x|^{n}} \chi_{|x| > 3.2^k/4} + \frac{C_K}{|x|^{n}} \chi_{3.2^k \leq |x| \leq 5 \cdot 2^k}, \quad (4.5)$$

which will be proved in Step 2. Then we can get

$$\left\| \sup_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) * K^k * f| \right\|_{L^2} \lesssim (C_K + \|\omega\|_{\text{Dini}}) \|Mf\|_{L^2} \lesssim (C_K + \|\omega\|_{\text{Dini}}) \|f\|_{L^2}. \quad (4.6)$$

Interpolating between (4.4) and (4.6), we get for $q \geq 2$

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) * K^k * f|^q \right)^{1/q} \right\|_{L^2} \lesssim (C_K + \|\omega\|_{\text{Dini}} + \|\omega\|^{1/q}_{\text{Dini}} \|\omega^{1/2}\|_{\text{Dini}}^{2/q}) \|f\|_{L^2}.$$  

We get (4.1).

**Step 2.** We now estimate $|(\delta_0 - \phi_k) * K^k(x)|$ for any fixed $k \in \mathbb{Z}$.

$$|(\delta_0 - \phi_k) * K^k(x)| = \left| \int_{\mathbb{R}^n} \phi_k(x - z)(K^k(z) - K^k(x)) \, dz \right|$$

$$\leq \left| \int_{\mathbb{R}^n} \phi_k(x - z)(K(z) - K(x)) \chi_{|z| > 2^k} \, dz \right|$$

$$+ \left| \int_{\mathbb{R}^n} \phi_k(x - z)K(x)(\chi_{|z| > 2^k} - \chi_{|x| > 2^k}) \, dz \right|$$

$$=: A_{k,1}(x) + A_{k,2}(x).$$
We first consider $A_{k,1}$. By noting that supp $\phi_k \subset \{ x : |x| \leq 2^k / 4 \}$ and that $|z| > 2^k$, we have $|x| \geq \frac{3|z|}{4} > 3 \cdot 2^k / 4$. Then by the regularity condition of $K$ in (2.6),

\[
A_{k,1}(x) \lesssim \int_{|z|>2^k} |\phi_k(x-z)| \frac{\omega(|x-z|/|x|)}{|x|^n} \, dz \lesssim \frac{\omega(2^k/|x|)}{|x|^n} \chi_{|x|>3 \cdot 2^{k/4}}(x). \tag{4.7}
\]

For the term $A_{k,2}$, we first note that $\chi_{|z|>2^k} \chi_{|x|>2^k}$ is nonzero if and only if $\frac{3}{4} \cdot 2^k \leq |x| \leq \frac{5}{4} \cdot 2^k$ since $|x-z| \leq 2^k / 4$. Thus,

\[
A_{k,2}(x) \lesssim |K(x)| \chi_{\frac{3}{4} \cdot 2^k \leq |x| \leq \frac{5}{4} \cdot 2^k}(x) \int_{\mathbb{R}^n} |\phi_k(x-z)| \, dz \lesssim \frac{C_K}{|x|^n} \chi_{\frac{3}{4} \cdot 2^k \leq |x| \leq \frac{5}{4} \cdot 2^k}(x).
\]

Combining the two cases, we get that for any fixed $k \in \mathbb{Z}$,

\[
| (\phi_k - \delta_0) * K^k(x) | \lesssim \omega(2^k/|x|) \frac{1}{|x|^n} \chi_{|x|>3 \cdot 2^{k/4}} + \frac{C_K}{|x|^n} \chi_{\frac{3}{4} \cdot 2^k \leq |x| \leq \frac{5}{4} \cdot 2^k}, \tag{4.8}
\]

which verifies (4.5) and implies that

\[
\sum_{k \in \mathbb{Z}} | (\phi_k - \delta_0) * K^k(x) | \lesssim (\| \omega \|_{Dini} + C_K) \frac{1}{|x|^n}. \tag{4.9}
\]

We now estimate $\| \{(\phi_k - \delta_0) * K^k(x-y) - (\phi_k - \delta_0) * K^k(x)\}_k \|_{\ell^q}$ for $0 < |y| \leq |x| / 2$. To begin with, we claim that for $|y| \leq \frac{2^k}{2},$

\[
| (\phi_k - \delta_0) * K^k(x-y) - (\phi_k - \delta_0) * K^k(x) | \lesssim \left( \frac{\omega(|y|/|x|)}{|x|^n} + \frac{|y|^\theta}{|x|^{n+\theta}} \right) \chi_{|x| \geq 2^k/2}(x) + \frac{C_K}{|x|^n} \chi_{2^k - |y| \leq |x| \leq 2^k + |y|}(x). \tag{4.10}
\]

Assuming (4.10) for the moment, we write

\[
\left( \sum_{k \in \mathbb{Z}} | (\phi_k - \delta_0) * K^k(x-y) - (\phi_k - \delta_0) * K^k(x) |^q \right)^{1/q} \leq \left( \sum_{\frac{2^k}{2} \leq |y|} + \sum_{\frac{2^k}{2} \geq |y|} | (\phi_k - \delta_0) * K^k(x-y) - (\phi_k - \delta_0) * K^k(x) |^q \right)^{1/q} =: II_1 + II_2.
\]
For the term $I_{1}$, by noting that $|x| \geq 2|y| > 0$ and by using (4.8) we get

$$I_{1} \lesssim \left( \sum_{\frac{2k}{\tau} \geq |y|} \left| \left( (\phi_{k} - \delta_{0}) * K^{k}(x) \right) \left( (\phi_{k} - \delta_{0}) * K^{k}(x - y) \right) \right|^q \right)^{1/q}$$

$$\lesssim \sum_{\frac{2k}{\tau} \geq |y|} C_{K} \left( \frac{|y|^{\theta}}{|x|^{\sigma}} \chi_{|x| \geq 2k/\tau} \right)^{q} + \sum_{\frac{2k}{\tau} \geq |y|} C_{K} \chi_{|y| \leq |x| \leq 2k + |y|}$$

$$\lesssim C_{K} \left( \frac{|y|^{\theta/2}}{|x|^{\sigma/2}} \right) + \omega(\frac{|y|/|x|}{|x|^{\sigma}}) \left( \sum_{\frac{2k}{\tau} \geq |y|} \left( \frac{|y|/2^{k}}{|x|} \right) \right) + \sum_{\frac{2k}{\tau} \geq |y|} C_{K} \chi_{|y| \leq |x| \leq 2k + |y|}$$

Combining the estimates of $I_{1}$ and $I_{2}$, we get

$$\left( \sum_{k \in \mathbb{Z}} \left| (\phi_{k} - \delta_{0}) * K^{k}(x - y) - (\phi_{k} - \delta_{0}) * K^{k}(x) \right|^q \right)^{1/q}$$

$$\lesssim C_{K} \left( \frac{|y|^{\theta/2}}{|x|^{\sigma/2}} + \omega(\frac{|y|/|x|}{|x|^{\sigma}}) \right) + \sum_{\frac{2k}{\tau} \geq |y|} C_{K} \chi_{|y| \leq |x| \leq 2k + |y|}$$

Now we return to give the proof of (4.10). We write

$$|(\phi_{k} - \delta_{0}) * K^{k}(x - y) - (\phi_{k} - \delta_{0}) * K^{k}(x)|$$

$$\leq \left| \int_{\mathbb{R}^{n}} \phi_{k}(x - z) \left( K^{k}(x - y) - K^{k}(x) \right) dz \right|$$
\[ + \left| \int_{\mathbb{R}^n} \phi_k(x - z)(K^k(z) - K^k(z)) \, dz \right| \]
\[ =: III_1 + III_2. \]

For the term \(III_1\), recall that \(|y| \leq \frac{2^k}{z}\). By noting the facts that \(0 < |y| \leq \frac{2^k}{z}\) and \(|x - y| \geq 2^k\) imply \(|x| > \frac{2^k}{z}\) and that \(\chi_{|x - y| > 2^k} - \chi_{|x| > 2^k}\) is nonzero if and only if \(2^k - |y| \leq |x| \leq 2^k + |y|\), we have

\[ III_1 \lesssim \int_{\mathbb{R}^n} |\phi_k(x - z)||K(x - y) - K(x)||\chi_{|x - y| > 2^k} \, dz \]
\[ + \int_{\mathbb{R}^n} |\phi_k(x - z)||K(x)||\chi_{|x - y| > 2^k} - \chi_{|x| > 2^k} \, dz \]
\[ \lesssim \frac{\omega(|y|/|x|)}{|x|^n} \chi_{|x| > 2^k} \frac{\alpha x}{z} \sum_{Q} \frac{\omega(|y|/|x|)}{|x|^n} \chi_{2^k - |y| \leq |x| \leq 2^k + |y|}. \]

Similarly, for the term \(III_2\), we get

\[ III_2 \lesssim \int_{\mathbb{R}^n} |\phi_k(x - z)||\omega(|y|/|x|)| \frac{\chi_{|x| > 2^k}}{|x|^n} \, dz + C_k \int_{\mathbb{R}^n} |\phi_k(x - z)| \frac{\chi_{|x| > 2^k}}{|x|^n} \, dz. \]

Since \(|x - z| \leq 2^k/4\) and \(|z| > \frac{2^k}{z}\), we get \(3|z|/2 \geq |x| \geq |z|/2 > 2^k/4\). Thus, there exists some \(\eta \in (0, 1)\),

\[ III_2 \lesssim \frac{\omega(|y|/|x|)}{|x|^n} \chi_{|x| > 2^k/4} + C_k \frac{|y|^q}{|x|^n+\eta} \chi_{|x| > 2^k/4}. \]

Combining the estimates of \(III_1\) and \(III_2\), we get (4.10).

**Step 3.** By Lemma 2.1, to prove (1.20), we need to verify that

\[ \left\| \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f|^q \right)^{1/q} \right\|_{L^{1, \infty}} \]
\[ \lesssim (C_k + \|\omega\|_{Dini} + \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q'} \|\omega\|_{Dini}^{1/2} \|\omega\|_{Dini}^{2/q}) \|f\|_{L^1} \quad (4.12) \]

and

\[ \left\| \mathcal{M} \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f|^q \right)^{1/q} \right\|_{L^{1, \infty}} \lesssim (C_k + \|\omega\|_{Dini}) \]
\[ + \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q'} + \|\omega\|_{Dini}^{1/2} \|\omega\|_{Dini}^{2/q}). \|f\|_{L^1}. \quad (4.13) \]

The two above inequalities just are (1.18) and (1.17).

To verify (4.12), we apply the Calderón-Zygmund decomposition to \(f\) at height \(\alpha\) to obtain that there is a disjoint family of dyadic cubes \(\{Q\}\) with total measure \(\sum Q |Q| \lesssim \alpha^{-1} \|f\|_{L^1}\), and that \(f = g + h\), with \(\|g\|_{L^\infty} \lesssim \alpha\), and \(\|g\|_{L^1} \lesssim \|f\|_{L^1}\), \(h = \sum_{Q} h_Q\), where \(\text{supp}(h_Q) \subseteq Q\), \(\int_{\mathbb{R}^n} h_Q(x) \, dx = 0\) and \(\sum \|h_Q\|_{L^1} \lesssim \|f\|_{L^1}\).
By Chebychev’s inequality and (4.1), we get

\[
\left| \left\{ x : \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast g(x)|^q \right)^{1/q} > \alpha \right\} \right| \\
\lesssim (C_K + \|\omega\|_{Dini} + \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/2}) \alpha^{-2} \|f\|_{L^1}^2
\]

Then it suffices to estimate \((\sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast h(x)|^q)^{1/q}\) away from \(\bigcup \tilde{Q}\) with \(\tilde{Q} = 2Q\). Let \(y_Q\) denote the center of \(Q\). Since \(\int_{\mathbb{R}^n} h_Q(x) \, dx = 0\), then we can write

\[
(\phi_k - \delta_0) \ast K^k \ast h(x) = \sum_{Q} (\phi_k - \delta_0) \ast K^k \ast h_Q(x)
\]

\[
= \sum_{Q} \int_{Q} (\phi_k - \delta_0) \ast K^k (x - y) h_Q(y) \, dy
\]

\[
= \sum_{Q} \int_{Q} [(\phi_k - \delta_0) \ast K^k (x - y) - (\phi_k - \delta_0) \ast K^k (x - y_Q)] h_Q(y) \, dy.
\]

Now by Chebychev’s inequality and (4.11),

\[
\alpha \left| \left\{ x \notin \bigcup \tilde{Q} : \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast h(x)|^q \right)^{1/q} > \alpha \right\} \right| \\
\leq \sum_{Q} \int_{x \notin \tilde{Q}} \int_{Q} |h_Q(y)| \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k (x - y) - (\phi_k - \delta_0) \ast K^k (x - y_Q)|^q \right)^{1/q} \, dy \, dx
\]

\[
\lesssim \sum_{Q} \int_{Q} |h_Q(y)| \int_{|x - y_Q| \geq 2^j} \left( \frac{C_k |y - y_Q|^n}{|x - y_Q|^{n+\eta/2}} + \frac{\omega(|y - y_Q|/|x - y_Q|)^{1/2}}{|x - y_Q|^n} \|\omega\|_{Dini}^{1/q} \right) \, dx \, dy
\]

\[
+ \sum_{2^j \geq |y - y_Q|} \int_{|x - y_Q| \leq 2^j} \left( \frac{C_k |y - y_Q|^n}{|x - y_Q|^{n+\eta/2}} + \frac{\omega(|y - y_Q|/|x - y_Q|)^{1/2}}{|x - y_Q|^n} \right) \, dx \, dy
\]

\[
\lesssim \sum_{Q} \int_{Q} |h_Q(y)| \sum_{j=1}^{\infty} \int_{2^j \epsilon(Q) < |x - y_Q| \leq 2^{j+1} \epsilon(Q)} \frac{C_k |y - y_Q|^n}{|x - y_Q|^{n+\eta/2}} \, dx \, dy
\]

\[
+ \omega(|y - y_Q|/|x - y_Q|)^{1/2} \|\omega\|_{Dini}^{1/q} \, dx \, dy
\]

\[
+ \sum_{2^j \geq |y - y_Q|} \int_{|x - y_Q| \leq 2^j} \left( \frac{C_k |y - y_Q|^n}{|x - y_Q|^{n+\eta/2}} + \frac{\omega(|y - y_Q|/|x - y_Q|)^{1/2}}{|x - y_Q|^n} \right) \, dx \, dy.
\]
If \(|y - y_Q| \leq 2^k/2\), then \(|x - y_Q| \geq 2^k - |y - y_Q| \geq 2^k/2\), thereby \(|x - y_Q|^{-1} \leq 2.2^{-k}\),

\[\alpha \| x \notin \bigcup \tilde{Q} : \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast h(x)|^q \right)^{1/q} > \alpha \right\] 

\[
\lesssim \sum_{Q} \int_{Q} |h_Q(y)| \sum_{j=1}^{\infty} \int_{2^j \ell(Q) < |x - y_Q| \leq 2^{j+1} \ell(Q)} \frac{C_k \ell(Q)^{n/2}}{2^j \ell(Q)^{(n+\ell)/2}} + \omega(2^{-j}) \frac{1}{\ell(Q)^n} \|\omega\|_{Dini}^{1/q} \, dx \, dy 
\]

\[
+ C_k \sum_{Q} \int_{Q} |h_Q(y)| \sum_{\frac{2^k}{2} \geq |y - y_Q|} 2^{-k} \int_{2^k - |y - y_Q| \leq r \leq 2^k + |y - y_Q|} \, dr \int_{\mathbb{R}^{n-1}} d\sigma(d\theta) \, dy 
\]

\[
\lesssim C_k \sum_{Q} \int_{Q} |h_Q(y)| \, dy \sum_{j=1}^{\infty} (2^{-\eta j/2} + \omega(2^{-j}) \frac{1}{\ell(Q)^n} \|\omega\|_{Dini}^{1/q} \right) 
\]

\[
+ C_k \sum_{Q} \int_{Q} |h_Q(y)| \sum_{\frac{2^k}{2} \geq |y - y_Q|} \frac{|y - y_Q|}{2^k} \, dy 
\]

\[
\lesssim (C_k + \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q}) \sum_{Q} \|h_Q\|_{L^1} 
\]

\[
+ C_k \sum_{Q} \int_{Q} |h_Q(y)||y - y_Q| \sum_{\frac{2^k}{2} \geq |y - y_Q|} 2^{-k} \, dy 
\]

\[
\lesssim (C_k + \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q}) \sum_{Q} \|h_Q\|_{L^1} + C_k \sum_{Q} \int_{Q} |h_Q(y)| \, dy 
\]

\[
\lesssim (C_k + \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q}) \|f\|_{L^1} 
\]

This finishes the proof of (4.12).

Now, we verify (4.13). Let \(Q\) be a cube, and take \(x, \xi \in Q\). Let \(B(x) = B(x, 2\sqrt{n} \ell(Q))\), then \(3Q \subset B_x\). By the triangle inequality, we get

\[
\left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast (f \chi_{\mathbb{R}^n \setminus 3Q}(\xi))|^q \right)^{1/q} 
\]

\[
\leq \left| \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast (f \chi_{\mathbb{R}^n \setminus B_x}(\xi))|^q \right)^{1/q} \right| - \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast (f \chi_{\mathbb{R}^n \setminus B_x}(x))|^q \right)^{1/q} 
\]

\[
+ \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast (f \chi_{B_x \setminus 3Q}(\xi))|^q \right)^{1/q} + \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast (f \chi_{B_x \setminus 3Q}(x))|^q \right)^{1/q} 
\]

\[=: I + II + III.\]
For the term $I$, by using the triangular inequality,

\[
I \lesssim \left( \sum_{|x-\xi| \geq \frac{2k}{q}} \left( \int_{\mathbb{R}^n \setminus B_k} \left| (\phi_k - \delta_0) * K^k(\xi - y) - (\phi_k - \delta_0) * K^k(x - y) \right| |f(y)| \, dy \right)^q \right)^{1/q}
\]
\[
+ \left( \sum_{|x-\xi| \geq \frac{2k}{q}} \left( \int_{\mathbb{R}^n \setminus B_k} \left| (\phi_k - \delta_0) * K^k(\xi - y) - (\phi_k - \delta_0) * K^k(x - y) \right| |f(y)| \, dy \right)^q \right)^{1/q}
\]
\[
=: I_1 + I_2.
\]

By using the size estimate (4.8) and the fact that $\frac{1}{2} |x - y| \leq |\xi - y| \leq \frac{3}{2} |x - y|$ (since $2|x - \xi| \leq |x - y|$), we get

\[
I_1 \lesssim \left( \sum_{|x-\xi| \geq \frac{2k}{q}} \left( \int_{\mathbb{R}^n \setminus B_k} \frac{\omega(2^k/|x - y|)}{|x - y|^n} \chi_{|x - y| > \frac{1}{2} 2k} |f(y)| \, dy \right)^q \right)^{1/q}
\]
\[
+ C_K \int_{\mathbb{R}^n \setminus B_k} \sum_{|x-\xi| \geq \frac{2k}{q}} \frac{|f(y)|}{|x - y|^n} \chi_{\frac{1}{2} 2k \leq |x - y| \leq \frac{3}{2} 2k} \, dy
\]
\[
\lesssim \left( \sum_{|x-\xi| \geq \frac{2k}{q}} \left( \sum_{j \geq 1} \int_{2^j \sqrt{\ell}(Q) \leq |x - y| \leq 2^{j+1} \sqrt{\ell}(Q)} \omega(\frac{2^k}{2^j \ell(Q)}) \frac{|f(y)|}{(2^j \ell(Q))^n} \, dy \right)^q \right)^{1/q}
\]
\[
+ C_K \int_{\mathbb{R}^n \setminus B_k} \frac{|x - \xi|}{|x - y|^n + 1} |f(y)| \, dy
\]
\[
\lesssim \left( \sum_{|x-\xi| \geq \frac{2k}{q}} \omega\left(\frac{2^k}{\ell(Q)}\right) \right)^{1/q} \sum_{j \geq 1} \omega^{1/q'}(2^{-j}) Mf(x) + C_K Mf(x)
\]
\[
\lesssim (\|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q'} + C_K) Mf(x).
\]

For the term $I_2$, by (4.10), we get

\[
I_2 \lesssim \left( \sum_{|x-\xi| \leq \frac{2k}{q}} \left( \int_{\mathbb{R}^n \setminus B_k} \left( \frac{\omega(|x-\xi|/|x - y|)}{|x - y|^n} + C_K \frac{|x - \xi|}{|x - y|^n} \right) \chi_{|x - y| \geq 2k/4} |f(y)| \, dy \right)^q \right)^{1/q}
\]
\[
+ C_K \left( \sum_{|x-\xi| \leq \frac{2k}{q}} \left( \int_{\mathbb{R}^n \setminus B_k} \frac{|f(y)|}{|x - y|^n} \chi_{2^k - |x - y| \leq 2^k + |x - \xi|} \, dy \right)^q \right)^{1/q}
\]
\[
\lesssim (\|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q'} + C_K) Mf(x) + \tilde{I}_2.
\]

To estimate $\tilde{I}_2$, it suffices to consider the form

\[
C_K \left( \sum_{|x-\xi| \leq \frac{2k}{q}} \left( \int_{s_k \leq |x - y| \leq s_{k+1}} \frac{1}{|x - y|^n} |f(y)| \, dy \right)^q \right)^{1/q},
\]
where \(|s_{k+1} - s_k| \leq 2|x - \xi|\), and \(\frac{2^k}{2} \leq s_k \leq \frac{3}{2} \cdot 2^k\). Using the hypothesis that \(|x - \xi| < \ell(Q)\) and the kernel estimate we can bound the above by a dimensional constant times

\[
C_K \left( \sum_{|x - \xi| \leq \frac{2^k}{2}} \left( s_k^{-n} \int_{s_k \leq |x - y| \leq s_{k+1}} |f(y)| \, dy \right)^q \right)^{1/q}.
\]

The above \(\ell^q\) norm can be written as

\[
\left( \sum_{|x - \xi| \leq \frac{2^k}{2}} \left( s_k^{-n} \int_{|x - y| \leq s_{k+1}} |f(y)| \, dy - s_k^{-n} \int_{|x - y| \leq s_k} |f(y)| \, dy \right)^q \right)^{1/q}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} \left( s_{k+1}^{-n} \int_{|x - y| \leq s_{k+1}} |f(y)| \, dy - s_k^{-n} \int_{|x - y| \leq s_k} |f(y)| \, dy \right)^q \right)^{1/q}
\]

\[
+ \left( \sum_{|x - \xi| \leq \frac{2^k}{2}} \left( (s_k^{-n} - s_{k+1}^{-n}) \int_{|x - y| \leq s_{k+1}} |f(y)| \, dy \right)^q \right)^{1/q}
\]

\[
\lesssim V_q \mathcal{A}(|f|)(x) + Mf(x) \left( \sum_{|x - \xi| \leq \frac{2^k}{2}} \left( (s_k^{-n} - s_{k+1}^{-n}) / s_{k+1}^{-n} \right)^q \right)^{1/q},
\]

where \(V_q \mathcal{A}(|f|)(x) = V_q \{A_t(|f|)(x)\}_{t > 0}\), and

\[
A_t(f)(x) = \frac{1}{|B_t|} \int_{B_t} f(x + y) \, dy, \quad x \in \mathbb{R}^n, \ t > 0.
\]

Here \(B_t\) denotes the open ball in \(\mathbb{R}^n\) of center at the origin and radius \(t\). Also note that

\[
\left( \sum_{|x - \xi| \leq \frac{2^k}{2}} \left( (s_k^{-n} - s_{k+1}^{-n}) / s_{k+1}^{-n} \right)^q \right)^{1/q} \lesssim \sum_{|x - \xi| \leq \frac{2^k}{2}} \frac{s_{k+1} - s_k}{2^k} \lesssim \sum_{|x - \xi| \leq \frac{2^k}{2}} \frac{|x - \xi|}{2^k} \leq c_n.
\]

Thus we get

\[
\widetilde{I}_2 \lesssim C_K (V_q \mathcal{A}(|f|)(x) + Mf(x)).
\] (4.14)

Thus combining the estimates of \(I_1\) and \(I_2\), we get

\[
I \leq (\|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q'})^q + C_K Mf(x) + C_K V_q \mathcal{A}(|f|)(x).
\]
For the term $II$, by using (4.9) and the fact that $|x - y| \simeq |\xi - y|$ (since $2|x - \xi| \leq |x - y|$), we get

$$II \leq \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k (\xi - y)||f|^q \chi_{\mathbb{R}^n \setminus B_{y}}(y) \, dy$$

(4.15)

$$\lesssim (\|\omega\|_{Dini} + C_K) \int_{3\ell(Q) \leq |x-y| \leq 2\sqrt{n}(Q)} |\xi - y|^n f(y) \, dy \lesssim (\|\omega\|_{Dini} + C_K) Mf(x).$$

For the term $III$, since $\text{supp} \, (\phi_k - \delta_0) * K^k \subset \{x : |x| \geq \frac{3}{4} \cdot 2^k\}$, we obtain that

$$III \leq \left( \sum_{\frac{3}{4} \cdot 2^k \geq 2\sqrt{n}(Q)} |(\phi_k - \delta_0) * K^k * f \chi_{\mathbb{R}^n \setminus B_{x}}(x)|^q \right)^{1/q}$$

$$+ \left( \sum_{\frac{3}{4} \cdot 2^k \leq 2\sqrt{n}(Q)} |(\phi_k - \delta_0) * K^k * f \chi_{\mathbb{R}^n \setminus B_{x}}(x)|^q \right)^{1/q}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) * K^k * f(x)|^q \right)^{1/q}$$

$$+ \left( \sum_{\frac{3}{4} \cdot 2^k \leq 2\sqrt{n}(Q)} |(\phi_k - \delta_0) * K^k * f \chi_{\mathbb{R}^n \setminus B_{x}}(x)|^q \right)^{1/q}.$$ 

By using the size estimate (4.8) we get

$$|\phi_k - \delta_0) * K^k * f \chi_{\mathbb{R}^n \setminus B_{x}}(x)|$$

$$\lesssim \int_{|x-y| \geq 2\sqrt{n}(Q)} \omega(2^k/|x-y|) + C_K \frac{2^k}{2j \ell(Q)} |f(y)| \, dy$$

$$\lesssim \sum_{j \geq 1} \int_{2j \sqrt{n}(Q) \leq |x-y| \leq 2j+1 \sqrt{n}(Q)} \omega(\frac{2^k}{2j \ell(Q)}) + C_K \frac{2^k}{2j \ell(Q)} |f(y)| \, dy$$

$$\lesssim \left( \sum_{j \geq 1} \omega(2^{-j})^{1/q'} \omega(\frac{2^k}{\ell(Q)})^{1/q} + C_K \frac{2^k}{\ell(Q)} \right) Mf(x)$$

$$\lesssim \left( \|\omega\|_{Dini}^{1/q'} \omega(\frac{2^k}{\ell(Q)})^{1/q} + C_K \frac{2^k}{\ell(Q)} \right) Mf(x),$$

which implies

$$\left( \sum_{\frac{3}{4} \cdot 2^k \geq 2\sqrt{n}(Q)} |(\phi_k - \delta_0) * K^k * f \chi_{\mathbb{R}^n \setminus B_{x}}(x)|^q \right)^{1/q}$$

$$\lesssim \left( \|\omega\|_{Dini}^{1/q} \|\omega\|_{Dini}^{1/q'} + C_K \right) Mf(x).$$
Thus, we obtain that

\[
III \lesssim \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f(x)|^q \right)^{1/q} + \left( \|\omega\|_{Dini}^{1/q} \|\omega^{1/q'}\|_{Dini} + C_K \right) Mf(x).
\]

Combining the estimates of \(I, II\) and \(III\), we get

\[
M\left( \sum_{j \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f(x)|^q \right)^{1/q} \lesssim \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f(x)|^q \right)^{1/q} + \left( C_K + \|\omega\|_{Dini}^{1/q} \|\omega^{1/q'}\|_{Dini} \right) Mf(x)
\]

\[
+ C_K V_q \mathcal{A}(\|f\|)(x).
\]

Then by the weak type \((1, 1)\) of \(M\) (see [41]) and \(V_q \mathcal{A}(\|f\|)\)(see [23]), and (4.12), we get

\[
\|M\left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f(x)|^q \right)^{1/q} \|_{L^{1,\infty}} \lesssim \left( C_K + \|\omega\|_{Dini}^{1/q} \|\omega^{1/q'}\|_{Dini} \right) \|f\|_{L^1},
\]

which gives the proof of (4.13). The proof of Proposition 1.9 is complete. \(\square\)

5 Proofs of Theorems 1.6 and 1.7

Proof of Theorem 1.7 Recalling that for any fixed \(k \in \mathbb{Z}\), we denote by

\[K_k(x) = K(x)\chi_{|x| \leq 2^k}, \quad K^k(x) = K(x)\chi_{|x| > 2^k}.
\]

Then for any fixed \(k \in \mathbb{Z}\), write

\[
\phi_k \ast K(x) = \phi_k \ast K_k(x) + (\phi_k - \delta_0) \ast K^k(x) + K^k(x),
\]

where \(\delta_0\) is the Dirac measure at 0. Thus by the triangle inequality, we get

\[
V_q\{\phi_k \ast K \ast f\}_{k \in \mathbb{Z}} \leq V_q\{\phi_k \ast K_k \ast f\}_{k \in \mathbb{Z}} + V_q\{(\phi_k - \delta_0) \ast K^k \ast f\}_{k \in \mathbb{Z}} + V_q\{K^k \ast f\}_{k \in \mathbb{Z}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} |\phi_k \ast K \ast f|^q \right)^{1/q} + \left( \sum_{k \in \mathbb{Z}} |(\phi_k - \delta_0) \ast K^k \ast f|^q \right)^{1/q}
\]

\[
+ V_q\{K^k \ast f\}_{k \in \mathbb{Z}}.
\]

Since the following results have been established in [9, Theorem 1.5], [14, Prosition 2.1] and [6, Theorem 1.3],

\[
\|V_q\{K^k \ast f\}_{k \in \mathbb{Z}}\|_{L^{1,\infty}} \lesssim \left( \|\omega\|_{Dini} + C_K + \|T\|_{L^2 \to L^2} \right) \|f\|_{L^1}.
\]

\(\square\)
and for $1 < p < \infty$, $w \in A_p$,

$$
\|V_q \{K^k * f \}_{k \in \mathbb{Z}}\|_{L^p(w)} \lesssim (\|\omega\|_{Dini} + C_K + \|T\|_{L^2 \to L^2}) \{w\} A_p \|f\|_{L^p(w)}. \quad (5.2)
$$

Then combining Propositions 1.8 and 1.9, we finish the proof of Theorem 1.7. ☐

**Proof of Theorem 1.6** Recall the definition of the operator $T_\Omega$ given in the introduction. It can be written as

$$
T_\Omega f = \sum_{k \in \mathbb{Z}} T_k f = \sum_{k \in \mathbb{Z}} v_k * f, \quad v_k = \frac{\Omega(x^j)}{|x|^n} \chi_{[2^k < |x| \leq 2^{k+1}]}. \quad (5.3)
$$

We consider the following partition of unity. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $\text{supp} \varphi \subset \{x: |x| \leq \frac{1}{100}\}$ and $\int \varphi dx = 1$, and so that $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. Let us also define $\psi$ by

$$
\hat{\psi}(\xi) = \hat{\varphi}(\xi) - \hat{\varphi}(2\xi). \quad (5.4)
$$

Then, with this choice of $\psi$, it follows that $\int \psi dx = 0$. We write $\varphi_j(x) = \frac{1}{2^{jn}} \varphi(\frac{x}{2^j})$, and $\psi_j(x) = \frac{1}{2^{jn}} \psi(\frac{x}{2^j})$. We now define the partial sum operators $S_j$ by $S_j(f) = f * \varphi_j$. Their differences are given by

$$
S_j(f) - S_{j+1}(f) = f * \psi_j. \quad (5.4)
$$

Since $S_j f \to f$ as $j \to -\infty$, for any sequence of integers $N = \{N(j)\}_{j=0}^\infty$, with $0 = N(0) < N(1) < \cdots < N(j) \to \infty$, we have the identity

$$
T_k = T_k S_k + \sum_{j=1}^\infty T_k (S_k - N(j) - S_k - N(j-1)). \quad (5.5)
$$

In this way,

$$
T_\Omega = \sum_{j=0}^\infty \tilde{T}_j = \sum_{j=0}^\infty T_j^N, \quad (5.6)
$$

where

$$
\tilde{T}_0 := T_0^N := \sum_{k \in \mathbb{Z}} T_k S_k \quad (5.7)
$$

and, for $j \geq 1$,

$$
\tilde{T}_j := \sum_{k \in \mathbb{Z}} T_k (S_k - j - S_k - (j-1)),

T_j^N := \sum_{k \in \mathbb{Z}} T_k (S_k - N(j) - S_k - N(j-1)) = \sum_{i=N(j)+1}^{N(j)} \tilde{T}_i. \quad (5.8)
$$
Therefore,
\[ \| V_q \{ \phi_l \ast T \Omega f \}_{l \in \mathbb{Z}} \|_{L^p(w)} \leq \sum_{j=0}^{\infty} \| V_q \{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}} \|_{L^p(w)}. \]

To prove Theorem 1.6, we claim that the following inequalities hold for \( 1 < p < \infty \) and \( w \in A_p \):
\[ \| V_q \{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}} \|_{L^p} \lesssim \| \Omega \|_{L^{\infty}} 2^{-\theta N(j-1)} (1 + N(j))^{1+1/q} \| f \|_{L^p} \quad (5.9) \]
and
\[ \| V_q \{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}} \|_{L^p(w)} \lesssim \| \Omega \|_{L^{\infty}} (1 + N(j))^{1+1/q} \{ w \}_A \| f \|_{L^p(w)}. \quad (5.10) \]

Assume the above two claims at the moment. Set \( \varepsilon := \frac{1}{2} c_n/(w)_{A_p} \), \( c_n \) is small enough (see [21, Corollary 3.18]). By (5.10), we have, for this choice of \( \varepsilon \),
\[ \| V_q (\{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}}) \|_{L^p(w^{1+\varepsilon})} \lesssim \| \Omega \|_{L^{\infty}} (1 + N(j))^{1+1/q} \{ w^{1+\varepsilon} \}_{A_p} \| f \|_{L^p(w^{1+\varepsilon})} \quad (5.11) \]

Now we are in position to apply the interpolation theorem with change of measures by Stein and Weiss—Lemma 2.2. We apply it to \( T = V_q \{ \phi_l \ast T_j^N \}_{l \in \mathbb{Z}} \) with \( p_0 = p_1 = p \), \( w_0 = w^0 = 1 \) and \( w_1 = w^{1+\varepsilon} \), so that by \( \lambda = \varepsilon/(1 + \varepsilon) \), (5.9) and (5.11), one has for some \( \theta, \gamma > 0 \) such that
\[ \| V_q \{ \phi_l \ast T_j^N \}_{l \in \mathbb{Z}} \|_{L^p(w)} \rightarrow L^p(w) \]
\[ \lesssim \| V_q \{ \phi_l \ast T_j^N \}_{l \in \mathbb{Z}} \|^{\varepsilon/(1+\varepsilon)}_{L^p(w)} \| V_q \{ \phi_l \ast T_j^N \}_{l \in \mathbb{Z}} \|^{(1+\varepsilon)/(1+\varepsilon)}_{L^p(w^{1+\varepsilon})} \]
\[ \lesssim \| \Omega \|_{L^{\infty}} (1 + N(j))^{1+1/q} 2^{-\theta N(j-1)\varepsilon/(1+\varepsilon)} \{ w \}_A \]
\[ \lesssim \| \Omega \|_{L^{\infty}} (1 + N(j))^{1+1/q} 2^{-\gamma N(j-1)/(w)A} \{ w \}_{A}. \]

Thus
\[ \| V_q \{ \phi_l \ast T \Omega \}_{l \in \mathbb{Z}} \|_{L^p(w)} \rightarrow L^p(w) \]
\[ \lesssim \| \Omega \|_{L^{\infty}} \{ w \}_A \sum_{j=0}^{\infty} (1 + N(j))^{1+1/q} 2^{-\gamma N(j-1)/(w)A} \]
We now choose $N(j) = 2^j$ for $j \geq 1$. Then, using $e^{-x} \leq 2x^{-2}$, we have

$$
\sum_{j=0}^{\infty} (1 + N(j))^{1+1/q} 2^{-yN(j-1)/(w)A_p} \\
\quad \lesssim \sum_{j: 2^j \leq (w)A_p} 2^{j(1+1/q)} + \sum_{j: 2^j \geq (w)A_p} 2^{j(1+1/q)} \left( \frac{(w)A_p}{2^j} \right)^2 \lesssim (w)^{1+1/q},
$$

by summing the two geometric series in the last step. This implies

$$
\| V_q \{ \phi_I * T^j \Omega f \}_{I \in \mathbb{Z}} \|_{L^p(w)} \lesssim \| \Omega \|_{L^\infty} (w)^{1+1/q} \| f \|_{L^p(w)}.
$$

and hence the proof of Theorem 1.6 is complete under the assumptions that (5.9) and (5.10) hold.

Now we turn to proving the claims (5.9) and (5.10). The following inequality can be found in [21, Lemma 3.7],

$$
\| T^j f \|_{L^2} \lesssim \| \Omega \|_{L^\infty} 2^{-\alpha N(j-1)} \| f \|_{L^2}, \tag{5.12}
$$

for some $0 < \alpha < 1$ independent of $T_\Omega$ and $j$. Then by (see [10, 24])

$$
\| V_q \{ \phi_I * f \}_{I \in \mathbb{Z}} \|_{L^2} \lesssim \| f \|_{L^2}, \tag{5.13}
$$

we get

$$
\| V_q \{ \phi_I * T^j \Omega f \}_{I \in \mathbb{Z}} \|_{L^2} \lesssim \| T^j \Omega f \|_{L^2} \lesssim \| \Omega \|_{L^\infty} 2^{-\alpha N(j-1)} \| f \|_{L^2}. \tag{5.14}
$$

The operator $T^j$ is a $\omega$-Calderón–Zygmund operator with the kernel $K^j$ (see [21, Lemma 3.10]) satisfying

$$
|K^j(x)| \lesssim \frac{\| \Omega \|_{L^\infty}}{|x|^n} \tag{5.15}
$$

and for $2|y| \leq |x|$,

$$
|K^j(x - y) - K^j(x)| \lesssim \frac{\omega_j(|y|/|x|)}{|x|^n}, \tag{5.16}
$$

where $\omega_j(t) \leq \| \Omega \|_{L^\infty} \min(1, 2^{N(j)}t)$, and $\| \omega_j \|_{Dini} \lesssim \| \Omega \|_{L^\infty}(1 + N(j))$. For $r > 1$, the Dini norm of $\omega_j^{1/r}$ is estimated as

$$
\| \omega_j^{1/r} \|_{Dini} \lesssim \| \Omega \|_{L^\infty}^{1/r} \int_0^{2^{-N(j)}} 2^{N(j)/r} t^{1/r} \frac{dt}{t} + \| \Omega \|_{L^\infty}^{1/r} \int_{2^{-N(j)}}^1 \frac{dt}{t} \\
\quad \lesssim \| \Omega \|_{L^\infty}^{1/r}(1 + N(j)). \tag{5.17}
$$
Applying Theorem 1.7 to $T = T_j^N$, we get for $1 < p < \infty$ and $w \in A_p$

$$
\|V_q \{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}} \|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1+1/q} \{w\}_{A_p} \| f \|_{L^p(w)},
$$

which gives the proof of (5.10). Taking $w = 1$, we get for $1 < p < \infty$,

$$
\|V_q \{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}} \|_{L^p} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1+1/q} \| f \|_{L^p}.
$$

(5.18)

Interpolating between (5.14) and (5.18), we get

$$
\|V_q \{ \phi_l \ast T_j^N f \}_{l \in \mathbb{Z}} \|_{L^p} \lesssim \|\Omega\|_{L^\infty} 2^{-\theta N(j-1)} (1 + N(j))^{1+1/q} \| f \|_{L^p}
$$

which establishes the proof of (5.9). The proof of Theorem 1.6 is complete. \hfill \Box

6 Proof of Theorem 1.3

Recalling that for any fixed $k \in \mathbb{Z}$,

$$
T_{\Omega,2^k} f(x) = \int_{|x-y|>2^k} \frac{\Omega(x-y)}{|x-y|^n} f(y) \, dy,
$$

and $K_\Omega(x) = \frac{\Omega(x')}{|x'|^n}$. Following [13], for any fixed $k \in \mathbb{Z}$, we write

$$
T_{\Omega,2^k} f(x) = \phi_k \ast T_\Omega f - \phi_k \ast K_\Omega \chi_{|\cdot| \leq 2^k} \ast f + (\delta_0 - \phi_k) \ast T_{\Omega,2^k} f,
$$

which splits $T_\Omega = \{T_{\Omega,2^k}\}_{k \in \mathbb{Z}}$ into three families:

$$
T_\Omega^1(f) := \{\phi_k \ast T_\Omega f\}_{k \in \mathbb{Z}}, \quad T_\Omega^2(f) := \{\phi_k \ast (K_\Omega \chi_{|\cdot| \leq 2^k}) \ast f\}_{k \in \mathbb{Z}}, \quad T_\Omega^3(f) := \{(\delta_0 - \phi_k) \ast T_{\Omega,2^k} f\}_{k \in \mathbb{Z}}.
$$

Thus, it suffices to estimate the weighted $L^p$ norm of $T_\Omega^i(f), i = 1, 2, 3$.

**Part 1** Let us first consider $T_\Omega^1(f)$. By Theorem 1.6, we get that for $1 < p < \infty$ and $w \in A_p$,

$$
\|V_q \{ T^1_\Omega(f) \} \|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} (w)_{A_p}^{1+1/q} \{w\}_{A_p} \| f \|_{L^p(w)}.
$$

**Part 2** For the term $T_\Omega^2(f)$, by the Minkowski inequality, we get for $1 < p < \infty$ and $w \in A_p$,

$$
\|V_q \{ T^2_\Omega(f) \} \|_{L^p(w)} \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k \ast K_\Omega \chi_{|\cdot| \leq 2^k} \ast f|^q \right)^{1/q} \right\|_{L^p(w)}.
$$
It is easy to verify that $K_\Omega$ satisfies the cancellation condition (1.12) and the size estimate $|K_\Omega(x)| \leq \frac{\|\Omega\|_{L^{\infty}}}{|x|^{\rho}}$. Then applying Proposition 1.8 to $K = K_\Omega$, we get

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * K_\Omega \chi_{|x| \leq 2^k} * f|^q \right)^{1/q} \right\|_{L^p(w)} \lesssim \|\Omega\|_{L^{\infty}(w)} A_p \|f\|_{L^p(w)},
\]

by $\ell^2 \subset \ell^q$ for $q > 2$.

**Part 3** We now estimate $T_\Omega^3(f)$. We claim that for $1 < p < \infty$ and $w \in A_p$,

\[
\|V_q(T_\Omega^3(f))\|_{L^p(w)} \lesssim \|\Omega\|_{L^{\infty}(w)} A_p \|f\|_{L^p(w)},
\]

which, together with the estimates of $T_\Omega^1(f), T_\Omega^2(f)$ and $T_\Omega^3(f)$, gives

\[
\|V_q T_\Omega(f)\|_{L^p(w)} \lesssim \|\Omega\|_{L^{\infty}(w)} A_p \|f\|_{L^p(w)}.
\]

This finishes the proof of Theorem 1.3.

We now provide the proof of (6.1). Recall that $T_\Omega^3(f) := \{ (\delta_0 - \phi_k) * f \}_{k \in \mathbb{Z}}$. For $k \in \mathbb{Z}$, we define $v_k$ as $v_k(x) = \frac{\Omega(x')}{|x|^{\rho}} \chi_{2^k < |x| \leq 2^{k+1}}(x)$. Then we can write $T_\Omega, 2^k f(x) = \sum_{s \geq 0} v_{k+s} * f(x)$. Recall that $S_j(f) = f * \varphi_j$ and their differences are given by $S_j(f) - S_{j+1}(f) = f * \psi_j$, where $\varphi_j$ and $\psi_j$ are defined as in Section 5. Denote by $T_k f(x) = v_k * f(x)$. Similarly, for any sequence of integers $N = \{N(j)\}_{j=0}^{\infty}$, with $0 = N(0) < N(1) < \cdots < N(j) \to \infty$, we also have the identity.

\[
T_{k+s} = T_{k+s} S_{k+s} + \sum_{j=1}^{\infty} T_{k+s} (S_{k+s-N(j)} - S_{k+s-N(j-1)}) =: \sum_{j=0}^{\infty} T_{k+s,j}^N,
\]

where $T_{k+s,0} = T_{k+s} S_{k+s}$ and for $j \geq 1$,

\[
T_{k+s,j}^N = \sum_{i=N(j-1)+1}^{N(j)} T_{k+s} (S_{k+s-i} - S_{k+s-(i-1)}) =: \sum_{i=N(j-1)+1}^{N(j)} T_{k+s,i}.
\]

Then

\[
(\delta_0 - \phi_k) * v_{k+s} * f(x) = \sum_{j=0}^{\infty} (\delta_0 - \phi_k) * T_{k+s,j}^N f(x).
\]

So

\[
\|V_q(T_\Omega^3(f))\|_{L^p(w)} \leq \sum_{j=0}^{\infty} \left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) * T_{k+s,j}^N f|^q \right)^{1/q} \right\|_{L^p(w)}.
\]
We first estimate the $L^2$-norm of $(\sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast \sum_{s \geq 0} T_{k+s,j}^N f|^q)^{1/q}$.

**Lemma 6.1** We have for $j \geq 0$, there is a positive $0 < \tau < 1$ such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast \sum_{s \geq 0} T_{k+s,j}^N f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \| \Omega \|_{L^\infty} 2^{-\tau N(j-1)} \| f \|_{L^2}.
$$

**Proof** By the Plancherel Theorem and (6.3), we get for $j \geq 1$,

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast \sum_{s \geq 0} T_{k+s,j}^N f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \sum_{s \geq 0} \sum_{i=N(j-1)+1}^{N(j)} \left( \int_{\mathbb{R}^n} |(\delta_0 - \phi_k)(\xi)|^2 |T_{k+s,i} f(\xi)|^2 d\xi \right)^{1/2}.$$ 

Since $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\phi}(0) = 1$, $|1 - \hat{\phi}(\xi)| \lesssim \min(1, |\xi|)$. Also by $\hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\psi}(0) = 0$, then $|\hat{\psi}(\xi)| \lesssim \min(1, |\xi|)$. Therefore, $|1 - \hat{\phi_k}(\xi)| \lesssim \min(1, |2^k \xi|^\alpha)$, $|\hat{\psi}(2^{k+s-i} \xi)| \lesssim \min(|2^{k+s-i} \xi|^\gamma, 1)$, $|\hat{v}_{k+s}(\xi)| \lesssim \| \Omega \|_{L^\infty} |2^{k+s} \xi|^{-\beta}$ (see [13]) for some $0 < \beta$, $\gamma$, $\alpha < 1$, $\alpha + \gamma - \beta > 0$, $\gamma < \beta$ and $\alpha < \beta$,

$$\sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k)(\xi)|^2 |T_{k+s,i} f(\xi)|^2 \lesssim \| \Omega \|_{L^\infty} 2^{-2(\beta - \alpha)i} 2^{-2(\beta - \gamma)s |\hat{f}(\xi)|^2}.$$ 

Therefore, by summing the geometric series and the Plancherel theorem, we get for some $\tau \in (0, 1)$,

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast \sum_{s \geq 0} T_{k+s,j}^N f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \| \Omega \|_{L^\infty} 2^{-\tau N(j-1)} \| f \|_{L^2}.$$ 

For $j = 0$, $|1 - \hat{\phi}(2^k \xi)| \lesssim \min(|2^k \xi|, 1)$, $|\hat{\phi}(2^k \xi)| \lesssim 1$, $|\hat{v}_{k+s}(\xi)| \lesssim \| \Omega \|_{L^\infty} |2^{k+s} \xi|^{-\beta}$ for some $0 < \beta < 1$ (see [13]). Therefore, by the Plancherel theorem, we get

$$\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast \sum_{s \geq 0} T_{k+s,0}^N f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \sum_{s \geq 0} \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |1 - \hat{\phi_k}(\xi)|^2 |\hat{v}_{k+s}(\xi)|^2 |\hat{\phi}(2^k \xi)|^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$
\[
\lesssim \sum_{s \geq 0} \left( \int_{\mathbb{R}^n} \|\Omega\|^2_{L^\infty} \left( \sum_{|2^k \xi| \leq 1} |2^{k+s} \xi|^{-2\beta} + \sum_{|2^k \xi| \geq 1} |2^{k+s} \xi|^{-2\beta} \right) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\
\lesssim \|\Omega\|_{L^\infty} \|f\|_{L^2}.
\]

The proof of Lemma 6.1 is complete by \( \ell^2 \subset \ell^q \) for \( q > 2 \).

Denote \( T_{k,j}^{N} := \sum_{s \geq 0} T_{k+s,j}^{N} \), and let \( K_{k,j}^{N} \) be the kernel of \( T_{k,j}^{N} \) given by

\[
K_{k,j}^{N} := \sum_{s \geq 0} v_{k+s} * (\varphi_{k+s-N(j)} - \varphi_{k+s-N(j-1)}) \tag{6.6}
\]

for \( j \geq 1 \) and for \( j = 0 \),

\[
K_{k,0}^{N} := \sum_{s \geq 0} v_{k+s} * \varphi_{k+s} \tag{6.7}
\]

In the following, we will verify that \( K_{k,j}^{N} \) is a \( \omega \)-Dini Calderón–Zygmund kernel satisfying (2.5) and (2.6).

**Lemma 6.2** For \( j \geq 0 \) and \( k \in \mathbb{Z} \). Then we have the size estimate

\[
|K_{k,j}^{N}(x)| \lesssim \frac{\|\Omega\|_{L^\infty}}{|x|^n} \chi_{|x| \geq \frac{3}{4} 2^k}(x) \tag{6.8}
\]

and the regularity estimate

\[
|K_{k,j}^{N}(x-h) - K_{k,j}^{N}(x)| \lesssim \omega_j\left(\frac{|h|}{|x|}\right) \frac{|h|}{|x|^n} \chi_{|x| \geq \frac{3}{4} 2^k}(x), \quad 0 < |h| < \frac{|x|}{2}, \tag{6.9}
\]

where \( \omega_j(t) \leq \|\Omega_j\|_{L^\infty} \min(1, 2^{N(j)} t) \).

**Proof** In order to get the required estimates for the kernel \( K_{k,j}^{N} \), we first study the kernel of each \( H_{k,s,N(j)} \) which is defined by

\[
H_{k,s,N(j)} := v_{k+s} * \varphi_{k+s-N(j)}. \nonumber
\]

First, we estimate \( |H_{k,s,N(j)}(x)| \). Since \( \text{supp} \ \varphi \subset \{x : |x| \leq \frac{1}{100}\} \), a simple computation gives that

\[
|H_{k,s,N(j)}(x)| \leq \int_{2^{k+s} \leq |y| \leq 2^{k+s+1}} \frac{|\Omega(y)|}{|y|^n} |\varphi_{k+s-N(j)}(x-y)| dy \\
\lesssim \frac{\|\Omega\|_{L^\infty}}{|x|^n} \chi_{\frac{3}{4} 2^{k+s} \leq |x| \leq \frac{5}{4} 2^{k+s}}. \tag{6.10}
\]
From the triangular inequality, and \( N(j - 1) < N(j) \), we obtain that the kernel
\[
K^{N}_{k,j} := \sum_{s \geq 0} (H_{k,s,N(j)} - H_{k,s,N(j-1)}) \]
satisfies
\[
|K^{N}_{k,j}(x)| \lesssim \sum_{s \geq 0} \frac{\|\Omega\|_{L^\infty}}{|x|^n} \chi_{\frac{3}{4} \cdot 2^{k+s+1}} \lesssim \frac{\|\Omega\|_{L^\infty}}{|x|^n} \chi_{|x| \geq \frac{3}{4} \cdot 2^{k+s}}. \tag{6.11}
\]

On the other hand, we compute the gradient. Again by the support of \( \varphi \), we have
\[
\nabla H_{k,s,N(j)}(x) = \nu_{k+s} \ast \nabla \varphi_{k+s-N(j)}(x) = \frac{1}{2^{k+s-N(j)}} \int_{2^{k+s} < |y| \leq 2^{k+s+1}} \frac{\Omega(y)}{|y|^n} (\nabla \varphi)_{k+s-N(j)}(x - y) dy.
\]
Since \( |x - y| \leq \frac{2^{k+s}}{100} \cdot 2^{k+s} < |y| \leq 2^{k+s+1} \), then \( |x| \simeq |y| \) and \( \frac{3}{4} \cdot 2^{k+s} \leq |x| \leq \frac{5}{4} \cdot 2^{k+s} \). Thus, we get
\[
|\nabla H_{k,s,N(j)}(x)| \lesssim \|\Omega\|_{L^\infty} \frac{2^{N(j)}}{|x|^n+1} \chi_{\frac{3}{4} \cdot 2^{k+s+1}} \lesssim \|\Omega\|_{L^\infty} \frac{2^{N(j)}}{|x|^n+1} \chi_{|x| \geq \frac{3}{4} \cdot 2^{k+s}}. \tag{6.12}
\]
Therefore, if \( |h| < \frac{|x|}{2} \), we get \( |x - \theta h| \simeq |x| \) and
\[
|K^{N}_{k,j}(x - h) - K^{N}_{k,j}(x)| \leq |\nabla K^{N}_{k,j}(x - \theta h)||h|
\lesssim \sum_{s \geq 0} \frac{\|\Omega\|_{L^\infty}}{|x|^n} \chi_{\frac{3}{4} \cdot 2^{k+s}} \cdot |h|
\lesssim \|\Omega\|_{L^\infty} \frac{2^{N(j)}|h|}{|x|^n+1} \chi_{|x| \geq \frac{3}{4} \cdot 2^{k}}. \tag{6.13}
\]
If \( |h| < \frac{|x|}{2} \), combining (6.11) and (6.13), we get for \( j \geq 0 \) and \( k \in \mathbb{Z} \),
\[
|K^{N}_{k,j}(x - h) - K^{N}_{k,j}(x)| \lesssim \omega_j \left( \frac{|h|}{|x|} \right) \chi_{|x| \geq \frac{3}{4} \cdot 2^{k}},
\]
where \( \omega_j(t) \leq \|\Omega\|_{L^\infty} \min(1, 2^{N(j)}t) \). The proof of Lemma 6.2 is complete. \( \Box \)

We now continue the proof of (6.1). By noting that \( \text{supp } K^{N}_{k,j} \subset \{ x : |x| \geq \frac{3}{4} \cdot 2^{k} \} \) and that \( K^{N}_{k,j} \) satisfies mean value zero, (6.8) and (6.9), repeating the argument of Proposition 1.9 only with \( K^k \) replaced by \( K^{N}_{k,j} \), we can also get for \( 1 < p < \infty \) and \( w \in A_p \)
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast T^{N}_{k,j} f|^q \right)^{1/q} \right\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1+1/q} \| f \|_{L^p(w)}. \tag{6.14}
\]
Taking $w = 1$ in (6.14) we get for $1 < p < \infty$,
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast T_{N,j}^N f|^q \right)^{1/q} \right\|_{L^p} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1 + 1/q} \|f\|_{L^p}.
\] (6.15)

Interpolating between (6.5) and (6.15), we get
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast T_{N,j}^N f|^q \right)^{1/q} \right\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} 2^{-\tau N(j-1)} (1 + N(j))^{1 + 1/q} \|f\|_{L^p(w)}.
\] (6.16)

Similar to the proof of Theorem 1.6, based on (6.14) and (6.16), applying the interpolation theorem with change of measures (Lemma 2.2), we get
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |(\delta_0 - \phi_k) \ast T_{N,j}^N f|^q \right)^{1/q} \right\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty(w)} \left\{ \omega \right\}_{A_p}^{1 + 1/q} \|f\|_{L^p(w)},
\]
which gives (6.1). The proof of Theorem 1.3 is complete.

\[\square\]

7 Proof of Theorem 1.4

Recall that we can write
\[
\begin{align*}
S_q(T\Omega f)(x) &= \left( \sum_{k \in \mathbb{Z}} [V_{q,k}(f)(x)]^q \right)^{1/q}; \\
V_{q,k}(f)(x) &= \left( \sup_{2^k \leq t_0 < \cdots < t_\lambda < 2^{k+1}} \sum_{l=0}^{\lambda-1} |T_{k,t_l+1} f(x) - T_{k,t_l} f(x)|^q \right)^{1/q},
\end{align*}
\]
where
\[
T_{k,t_l} f(x) = \int_{t_l \leq |x-y| \leq 2^{k+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.
\]
Observe that
\[
S_{\infty}(T\Omega f)(x) \leq \|\Omega\|_{L^\infty} Mf(x).
\]

Then by the sharp weighted boundedness of the Hardy–Littlewood maximal operator $M$ (see Hytönen–Pérez [19, Corollary 1.10], the original version was due to Buckley [3]),
\[
\|Mf\|_{L^p(w)} \leq c_n \cdot p' \cdot \left[ w \right]_{A_p}^{1/p} \left[ w^{1-p'} \right]_{A_\infty}^1 \|f\|_{L^p(w)}, \quad 1 < p < \infty,
\]
we get
\[\|S_\infty(T_{\Omega} f)\|_{L^p(w)} \leq c_n \cdot p' \cdot \|\Omega\|_{L^\infty\{w\}} A_p \|f\|_{L^p(w)}.\]  
(7.1)

Now we claim that
\[\|S_2(T_{\Omega} f)\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty\{w\}}^{1/2} A_p \|f\|_{L^p(w)}.\]  
(7.2)

In fact, interpolating between (7.1) and (7.2), we get for \(q \geq 2,\)
\[\|S_q(T_{\Omega} f)\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty\{w\}}^{1/q} A_p \|f\|_{L^p(w)}.\]  
(7.3)

Now we turn to verify (7.2). Recall that \(S_j(f) = f \ast \varphi_j\) and their differences are given by
\[S_j(f) - S_{j+1}(f) = f \ast \psi_j,\]  
(7.4)

where \(\varphi_j\) and \(\psi_j\) are defined as in Section 5. Similarly, for any sequence of integers \(N = \{N(j)\}_{j=0}^\infty,\) with \(0 = N(0) < N(1) < \cdots < N(j) \to \infty,\) we also have the identity
\[T_{k,t} = T_{k,t,0} + \sum_{j=1}^\infty T_{k,t} (S_{k-N(j)} - S_{k-N(j-1)}) = \sum_{j=0}^\infty T_{N_{k,t,j}},\]  
(7.5)

where \(T_{k,t,0} := T_{k,t,0} := T_{k,t}s_k\) and for \(j \geq 1,\)
\[T_{N_{k,t,j}} = \sum_{i=N(j-1)+1}^{N(j)} T_{k,t} (S_{k-i} - S_{k-(i-1)}) = \sum_{i=N(j-1)+1}^{N(j)} T_{k,t,i}.\]  
(7.6)

Therefore, by the Minkowski inequality, we get
\[S_2(T_{\Omega} f)(x) = \left(\sum_{k \in \mathbb{Z}} \left| \sum_{j=0}^\infty T_{N_{k,t,j}} f(x) \right|^2 \right)^{1/2} \leq \sum_{j=0}^\infty \left(\sum_{k \in \mathbb{Z}} \left| T_{k,t,j} f(x) \right|^2 \right)^{1/2} =: \sum_{j=0}^\infty S_{2,j}^N(f)(x).\]  
(7.7)

Then for \(1 < p < \infty\) and \(w \in A_p,\)
\[\|S_2(T_{\Omega} f)\|_{L^p(w)} \leq \sum_{j=0}^\infty \|S_{2,j}^N(f)\|_{L^p(w)}.\]

**Part 1.** We first give the \(L^2\)-norm of \(S_{2,j}^N(f)(x).\)
Lemma 7.1 There is a positive $0 < \tau < 1$ such that
\[
\|S_{2,j}^N(f)\|_{L^2} \lesssim \|\Omega\|_{L^\infty} 2^{-\tau N(j-1)}(N(j) + 1)^{1/2} \|f\|_{L^2}, \quad j \geq 0.
\] (7.8)

Proof For $t \in [1, 2]$, we define $v_{k,t}$ as $v_{k,t}(x) = \frac{\Omega(x')}{|x'|} \chi_{2^k t < |x| \leq 2^{k+1}}(x)$. Then $T_{k,t}$ can be expressed as $T_{k,t} f(x) = v_{k,t} * f(x)$, $t \in [1, 2]$. Recall that
\[
S_{2,j}^N(f)(x) = \left( \sum_{k \in \mathbb{Z}} \|\{T_{k,t,j}^N f(x)\}_{t \in [1,2]}\|_{V_2}^2 \right)^{1/2}.
\]

Note that
\[
\|a\|_{V_2} \leq C \|a\|_{L^2}^{1/2} \|a'\|_{L^2}^{1/2},
\] (7.9)
where $a' = \{ \frac{d}{dt} a_t : t \in \mathbb{R} \}$ (see [24]). We have applied (7.9) to $a = \{T_{k,t,j}^N f(x)\}_{t \in [1,2]}$, then we have
\[
\|\{T_{k,t,j}^N f(x)\}_{t \in [1,2]}\|_{V_2}^2 \leq C \|T_{k,t,j}^N f(x)\|_{L^2([1,2])} \| \frac{d}{dt} T_{k,t,j}^N f(x)\|_{L^2([1,2])}
= C \left( \int_1^2 |T_{k,t,j}^N f(x)| \frac{2 \, dt}{t} \right)^{1/2} \left( \int_1^2 \frac{d}{dt} |T_{k,t,j}^N f(x)| \frac{2 \, dt}{t} \right)^{1/2}.
\]

Then by the above inequalities, we get
\[
S_{2,j}^N(f)(x)^2 = \sum_{k \in \mathbb{Z}} \|\{T_{k,t,j}^N f(x)\}_{t \in [1,2]}\|_{V_2}^2 \leq C \sum_{k \in \mathbb{Z}} \left( \int_1^2 |T_{k,t,j}^N f(x)| \frac{2 \, dt}{t} \right)^{1/2} \left( \int_1^2 \frac{d}{dt} |T_{k,t,j}^N f(x)| \frac{2 \, dt}{t} \right)^{1/2}.
\]

This, along with the Cauchy–Schwarz inequality, yields
\[
\|S_{2,j}^N(f)\|_{L^2}^2 \leq \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,j}^N f|^2 \frac{dt}{t} \right)^{1/2} \left( \int_1^2 \sum_{k \in \mathbb{Z}} \frac{d}{dt} |T_{k,t,j}^N f|^2 \frac{dt}{t} \right)^{1/2}.
\] (7.10)
By the Plancherel theorem and (7.6), we get for $j \geq 1$,
\[
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,j} f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2} = \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,j} f(\xi)|^2 \frac{dt}{t} \frac{d\xi}{\xi} \right)^{\frac{1}{2}} \leq \sum_{i=N(j-1)+1}^{N(j)} \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,i} f(\xi)|^2 \frac{dt}{t} \frac{d\xi}{\xi} \right)^{\frac{1}{2}}.
\]

(7.11)

Since $|\hat{\psi}(2^k \xi)| \lesssim \min(|2^k \xi|, 1)$, $|\hat{v}_{k,t}(\xi)| \lesssim \|\Omega\|_{L^\infty} |2^k \xi|^{-\beta}$ for some $0 < \beta < 1$ (see [13]),
\[
\sum_{k \in \mathbb{Z}} |T_{k,t,i} f(\xi)|^2 = \sum_{k \in \mathbb{Z}} |\hat{v}_{k,t}(\xi)|^2 |\hat{\psi}(2^k \xi)|^2 |\hat{f}(\xi)|^2 = \lesssim \|\Omega\|^2_{L^\infty} \left( \sum_{|2^k \xi| \leq 2^i} |2^k \xi|^2 |2^k \xi|^{-2\beta} + \sum_{|2^k \xi| \geq 2^i} |2^k \xi|^{-2\beta} \right) |\hat{f}(\xi)|^2 \lesssim \|\Omega\|^2_{L^\infty} 2^{-2\beta i} |\hat{f}(\xi)|^2.
\]

Then by summing the geometric series and the Plancherel theorem, we get
\[
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,j} f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2} \lesssim \|\Omega\|_{L^\infty} 2^{-\beta N(j-1)} \|f\|_{L^2}.
\]

(7.12)

For $j = 0$, $|\hat{v}_{k,t}(\xi)| \lesssim \|\Omega\|_{L^\infty} \min(|2^k \xi|, |2^k \xi|^{-\beta})$ for some $0 < \beta < 1$ and $|\hat{\phi}(2^k \xi)| \lesssim 1$,
\[
\sum_{k \in \mathbb{Z}} |T_{k,t,0} f(\xi)|^2 = \sum_{k \in \mathbb{Z}} |\hat{v}_{k,t}(\xi)|^2 |\hat{\phi}(2^k \xi)|^2 |\hat{f}(\xi)|^2 = \lesssim \|\Omega\|^2_{L^\infty} \left( \sum_{|2^k \xi| \leq 1} |2^k \xi|^2 + \sum_{|2^k \xi| \geq 1} |2^k \xi|^{-2\beta} \right) |\hat{f}(\xi)|^2 \lesssim \|\Omega\|^2_{L^\infty} |\hat{f}(\xi)|^2.
\]

Then by the Plancherel theorem, we get
\[
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,j} f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2} \lesssim \|\Omega\|_{L^\infty} \|f\|_{L^2}.
\]

Combining the above estimates for $j > 0$ and $j = 0$, we get
\[
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} |T_{k,t,j} f|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2} \lesssim 2^{-\beta N(j-1)} \|\Omega\|_{L^\infty} \|f\|_{L^2}.
\]

(7.13)
On the other hand, we recall the elementary fact: for any Schwartz function $h$,
\[
\frac{d}{dt}[v_{k,t} \ast h(x)] = \frac{d}{dt} \left[ \int_{\mathbb{S}^{n-1}} \Omega(y') \int_{2^k t}^{2^{k+1} t} \frac{1}{r} h(x - ry') dr d\sigma(y') \right]
= \frac{1}{t} \int_{\mathbb{S}^{n-1}} \Omega(y') h(x - 2^k t y') d\sigma(y').
\]

Then by $\int_{\mathbb{S}^{n-1}} \Omega(y') d\sigma(y') = 0$ and $t \in [1, 2]$, we get
\[
\left| \frac{d}{dt}[v_{k,t} \ast h] \wedge(\xi) \right| = \left| \hat{h}(\xi) \right| \frac{1}{t} \left| \int_{\mathbb{S}^{n-1}} \Omega(y')(e^{2\pi i 2^k t y' \cdot \xi} - 1) d\sigma(y') \right|
\lesssim \| \Omega \|_{L^\infty} |\hat{h}(\xi)| \min(|2^k \xi|, 1).
\tag{7.14}
\]

Therefore, by the Plancherel theorem, we get for $j \geq 1$
\[
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,j} f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2} \lesssim \sum_{i=N(j)-1}^{N(j)} \left( \int_1^2 \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,i} f(\xi) \right|^2 \frac{dt}{t} d\xi \right)^{1/2}.
\]

By (7.14) and $|\hat{\psi}(2^{-i} \xi)| \lesssim \min(|2^{-i} \xi|, |2^{-i} \xi|^{-\beta})$ for some $0 < \beta < 1$,
\[
\sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,i} f(\xi) \right|^2
= \sum_{k \in \mathbb{Z}} \left| \left( \frac{d}{dt} v_{j,t} \ast \psi_{k-i} \ast f \right) \wedge(\xi) \right|^2
\lesssim \| \Omega \|_{L^\infty}^2 \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^{-i} \xi)|^2 |\hat{f}(\xi)|^2
\lesssim \left( \sum_{|2^k \xi| \leq 2^i} |2^{-i} \xi|^2 + \sum_{|2^k \xi| > 2^i} |2^{-i} \xi|^{-2\beta} \right) \| \Omega \|_{L^\infty}^2 |\hat{f}(\xi)|^2
\lesssim \| \Omega \|_{L^\infty}^2 |\hat{f}(\xi)|^2.
\]

Then by the Plancherel theorem, we get for $j \geq 1$
\[
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,j} f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2} \lesssim N(j) \| \Omega \|_{L^\infty} \| f \|_{L^2}.
\tag{7.15}
\]

For $j = 0$,
\[
\sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,0} f(\xi) \right|^2 |\hat{f}(\xi)|^2 = \sum_{k \in \mathbb{Z}} \left| \left( \frac{d}{dt} v_{k,t} \ast \varphi_k \ast f \right) \wedge(\xi) \right|^2.
\]
Then by (7.14), we have

\[ \left| \left( \frac{d}{dt} v_{k,t} \ast \varphi_k \ast f \right)(\xi) \right|^2 \leq \| \Omega \|^2_{L^\infty} \min(|2^k \xi|, 1)^2 |\varphi_k \ast f(\xi)|^2 \leq \| \Omega \|^2_{L^\infty} \min(|2^k \xi|, 1)^2 |\varphi(2^k \xi)|^2 |\hat{f}(\xi)|^2. \]

By \( |\varphi(2^k \xi)| \lesssim \min(1, |2^k \xi|^{-\beta}) \), then

\[ \left| \left( \frac{d}{dt} v_{k,t} \ast \varphi_k \ast f \right)(\xi) \right|^2 \lesssim \| \Omega \|^2_{L^\infty} \min(|2^k \xi|, 1)^2 \min(1, |2^k \xi|^{-2\beta}) |\hat{f}(\xi)|^2. \]

Then by the above inequalities, we get

\[ \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,0}^{N,j} f(\xi) \right|^2 |\hat{f}(\xi)|^2 \lesssim \| \Omega \|^2_{L^\infty} \min(|2^k \xi|, 1)^2 \min(1, |2^k \xi|^{-2\beta}) |\hat{f}(\xi)|^2 \]

Then by the Plancherel theorem and the above inequality, we get

\[ \left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,0}^{N,j} f(\xi) \right|^2 dt \right)^{1/2} \right\|_{L^2} \lesssim \| \Omega \|_{L^\infty} \| f \|_{L^2}. \] (7.16)

Combining the case of \( j > 0 \) and \( j = 0 \), we get

\[ \left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{k,t,j}^{N} f(\xi) \right|^2 dt \right)^{1/2} \right\|_{L^2} \lesssim (1 + N(j)) \| \Omega \|_{L^\infty} \| f \|_{L^2}. \] (7.17)

This along with (7.10) and (7.13), we get for \( 0 < \beta < 1 \),

\[ \| S_{2,j}^N(f) \|_{L^2} \lesssim 2^{-\beta N(j-1)} (N(j) + 1)^{1/2} \| f \|_{L^2}, \]

which proves Lemma 7.1. \( \square \)

**Part 2.** Next, we give the \( L^p(w) \)-norm of \( S_{2,j}^N(f)(x) \). For \( j \geq 1 \), we denote by \( S_{k,j} := S_{k-N(j-1)} - S_{k-N(j)} \). For \( j = 0 \), we denote by \( S_{k,0} := S_{k} \). We have the following observation:
\[
S^N_{2,j} (f)(x) = \left( \sum_{k \in \mathbb{Z}} \sup_{[t_l, t_{l+1}] \subseteq [1, 2]} \sum_{l=1}^{\lambda-1} |T_{k,t_l} f(x) - T_{k,t_{l+1}} f(x)|^2 \right)^{1/2} \\
=: \left( \sum_{k \in \mathbb{Z}} \sup_{t_l < \cdots < t_k \in [1, 2]} \sum_{l=1}^{\lambda-1} |T_{k,t_l,t_{l+1}} f(x)|^2 \right)^{1/2},
\]

where the operator \( T_{k,t_l,t_{l+1}} \) is given by

\[
T_{k,t_l,t_{l+1}} f(x) := \int_{2^k t_l < |x-y| \leq 2^k t_{l+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy
= v_{k,t_l,t_{l+1}} * f(x), \quad [t_l, t_{l+1}] \subseteq [1, 2].
\]

Denote by

\[
K_{k,l,j} := v_{k,t_l,t_{l+1}} * (\varphi_{k-N(j)} - \varphi_{k-N(j-1)}) \tag{7.18}
\]

the kernel of \( T_{k,t_l,t_{l+1}} S_{k,j} \) for \( j \geq 1 \) and by

\[
K_{k,l,0} := v_{k,t_l,t_{l+1}} * \varphi_k \tag{7.19}
\]

the kernel of \( T_{k,t_l,t_{l+1}} S_{k,0} \) for \( j = 0 \). Then

\[
S^N_{2,j} (f)(x) = \left( \sum_{k \in \mathbb{Z}} \sup_{[t_l, t_{l+1}] \subseteq [1, 2]} \sum_{l=1}^{\lambda-1} |K_{k,l,j} * f(x)|^2 \right)^{1/2}, \tag{7.20}
\]

In the following, we give the kernel estimates.

**Lemma 7.2** For every \( x \in \mathbb{R}^n \setminus \{0\}, \ j \geq 0 \) and \( k \in \mathbb{Z}, \)

\[
\sup_{[t_l, t_{l+1}] \subseteq [1, 2]} \sum_{l=1}^{\lambda-1} |K_{k,l,j}(x)| \lesssim \|\Omega\|_{L^\infty} \frac{2^k}{|x|^{n+1}} \chi_{2^k-1 \leq |x| \leq 2^{k+2}}. \tag{7.21}
\]

If \( 0 < |y| \leq \frac{|x|}{2}, \)

\[
\sup_{[t_l, t_{l+1}] \subseteq [1, 2]} \sum_{l=1}^{\lambda-1} |K_{k,l,j}(x-y) - K_{k,l,j}(x)| \lesssim \frac{\omega_j\left(\frac{|y|}{|x|}\right)}{|x|^n} \chi_{2^k-1 \leq |x| \leq 2^{k+2}}, \tag{7.22}
\]

where \( \omega_j(t) \leq \|\Omega\|_{L^\infty} \min(1, 2^N(j)t). \)
Proof Let $x \in \mathbb{R}^n \setminus \{0\}$. Since supp $\varphi \subset \{x : |x| \leq \frac{1}{100}\}$, we get that

\[
\sup_{t_1 < \cdots < t_k} \left| \sum_{l=1}^{\lambda-1} v_{k,t_1,t_{l+1}} * \varphi_{k-N(j)}(x) \right| = \sup_{t_1 < \cdots < t_k} \left| \sum_{l=1}^{\lambda-1} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} \chi_{2^k t_1 < |y| < 2^k t_{l+1}} \varphi_{k-N(j)}(x-y) dy \right| \tag{7.23}
\]

Then we get

\[
\sup_{t_1 < \cdots < t_k} \left| \sum_{l=1}^{\lambda-1} v_{k,t_1,t_{l+1}} * \varphi_{k-N(j)}(x) \right| \lesssim \|\Omega\|_{L^\infty} \frac{2^k}{|x|^{n+1}} \chi_{2^{k-1} \leq |x| \leq 2^{k+2}}. \tag{7.24}
\]

On the other hand, we compute the gradient. Again by taking into account the support of $\varphi$, we obtain that

\[
\sup_{t_1 < \cdots < t_k} \left| \nabla \left( \sum_{l=1}^{\lambda-1} v_{k,t_1,t_{l+1}} * \varphi_{k-N(j)}(x) \right) \right| = \sup_{t_1 < \cdots < t_k} \left| \sum_{l=1}^{\lambda-1} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} \chi_{2^k t_1 < |y| < 2^k t_{l+1}} 2^{-(k-N(j))(n+1)} \nabla \varphi \left( \frac{x-y}{2^k \chi_{2^k}} \right) dy \right| \tag{7.25}
\]

\[
\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(y')|}{|y|^n} \chi_{2^k \leq |y| < 2^{k+2}} 2^{-(k-N(j))(n+1)} \left| \nabla \varphi \left( \frac{x-y}{2^k \chi_{2^k}} \right) \right| dy \lesssim \|\Omega\|_{L^\infty} \frac{2^{N(j)}}{|x|^{n+1}} \chi_{2^{k-1} \leq |x| \leq 2^{k+2}}.
\]

Therefore, by the gradient estimate, for $|y| \leq \frac{1}{2} |x|$ we have

\[
\sup_{t_1 < \cdots < t_k} \left| \sum_{l=1}^{\lambda-1} v_{k,t_1,t_{l+1}} * \varphi_{k-N(j)}(x-y) - v_{k,t_1,t_{l+1}} * \varphi_{k-N(j)}(x) \right| \lesssim \|\Omega\|_{L^\infty} \frac{2^{N(j)}}{|x|^{n}} \chi_{2^{k-1} \leq |x| \leq 2^{k+2}}. \tag{7.26}
\]
Then using (7.23) and (7.26), we get

\[
\sup_{t_1 < \cdots < t_\lambda} \sum_{l=1}^{\lambda-1} |v_{k,t_l,t_{l+1}} \ast \varphi_{k-N(j)}(x - y) - v_{k,t_l,t_{l+1}} \ast \varphi_{k-N(j)}(x)| \\
\lesssim \frac{1}{|x|^n} \omega_j \left( \frac{|y|}{|x|} \right) \chi_{2^{k-1} \leq |x| \leq 2^{k+2}},
\]

(7.27)

where \( \omega_j(t) \leq \|\Omega\|_{L^\infty} \min(1, 2^{N(j)} t) \) (for \( j = 0 \), the subtraction is not even needed).

From the triangle inequality, and \( N(j-1) < N(j) \) it follows that the kernel \( v_{k,t_l,t_{l+1}} \ast (\varphi_{k-N(j)} - \varphi_{k-N(j-1)}) \) satisfies the same estimates (7.24) and (7.27) which proves Lemma 7.2. \( \square \)

**Lemma 7.3** Let \( S^N_{2,j} \) be defined as in (7.7). Then we get

\[
\|S^N_{2,j}(f)\|_{L^1,\infty} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1/2} \|f\|_{L^1}.
\]

(7.28)

**Proof.** We perform the Calderón-Zygmund decomposition of \( f \) at height \( \alpha \), thereby producing a disjoint family of dyadic cubes \( \{Q\} \) with total measure \( \sum_Q |Q| \lesssim \alpha^{-1} \|f\|_{L^1} \) and allowing us to write \( f = g + h \) as in the proof of Theorem 1.7. From Lemma 7.1,

\[
\|S^N_{2,j}(g)\|_{L^1,\infty} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1/2} \|f\|_{L^1}.
\]

It suffices to estimate \( S^N_{2,j}(h) \) away from \( \bigcup \tilde{Q} \) where \( \tilde{Q} \) is a large fixed dilate of \( Q \). Denote

\[
S^N_{2,j,k} h(x) = \left( \sup_{t_1 < \cdots < t_\lambda} \sum_{l=1}^{\lambda-1} |K_{k,t_l,k} h(x)|^2 \right)^{1/2}.
\]

(7.29)

Then from the definition of \( S^N_{2,j} \) in (7.20) we get \( S^N_{2,j,k}(h)(x) \leq \left( \sum_{k \in \mathbb{Z}} |S^N_{2,j,k} h(x)|^2 \right)^{1/2} \). Since

\[
\sum_{k \in \mathbb{Z}} |S^N_{2,j,k} h(x)|^2 \leq \sum_{k \in \mathbb{Z}} |S^N_{2,j,k} h(x)| \sup_{k \in \mathbb{Z}} |S^N_{2,j,k} h(x)|
\]

and

\[
\sum_{k \in \mathbb{Z}} |S^N_{2,j,k} h(x)|^2 \geq \alpha^2 = (1 + N(j))^{1/2} \alpha (1 + N(j))^{-1/2} \alpha,
\]

\( \omega_j \) is a fixed dilate of \( Q \).
then we get

\[ \left\{ x \notin \tilde{Q} : \left( \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)|^2 \right)^{1/2} > \alpha \right\} \]

\[ = \{ x \notin \tilde{Q} : \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)|^2 > \alpha^2 \} \]

\[ \subset \{ x \notin \tilde{Q} : \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > \alpha (1 + N(j))^{1/2} \} \]

\[ \cup \{ x \notin \tilde{Q} : \sup_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > (1 + N(j))^{-1/2} \alpha \} \]

By the above inequalities we get

\[ \alpha \left\{ x \notin \tilde{Q} : S_{2,j}^N h(x) > \alpha \right\} \]

\[ \leq \alpha \left\{ x \notin \tilde{Q} : \left( \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)|^2 \right)^{1/2} > \alpha \right\} \]

\[ \leq \alpha \left\{ x \notin \tilde{Q} : \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > (1 + N(j))^{1/2} \alpha \right\} \]

\[ + \alpha \left\{ x \notin \tilde{Q} : \sup_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > (1 + N(j))^{-1/2} \alpha \right\} \]

\[ =: I + II. \]

For the term \( I \), by Chebyshev’s inequality we have

\[ \alpha \left\{ x \notin \tilde{Q} : \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > (1 + N(j))^{1/2} \alpha \right\} \]

\[ \leq (1 + N(j))^{-1/2} \sum_Q \int_{(\tilde{Q})^c} \sum_{k \in \mathbb{Z}} S_{2,j,k}^N h_Q(x) \, dx. \]

Denote by \( y_Q \) the center of \( Q \). For \( x \notin \tilde{Q} \) and \( y \in Q \), we get \( 2|y - y_Q| \leq |x - y| \), then by \( |y - y_Q| \leq \ell(Q) \) and using the vanishing mean value of \( h_Q \) and (7.22),

\[ \sum_{k \in \mathbb{Z}} S_{2,j,k}^N h_Q(x) \leq \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |h_Q(y)| \frac{\omega_j \left( \frac{|y-y_Q|}{|x-y|} \right)}{|x-y|^n} \chi_{2^{k-1} \leq |x-y| < 2^{k+1}} \, dy \]

\[ \leq \int_{\mathbb{R}^n} |h_Q(y)| \frac{\omega_j \left( \frac{\ell(Q)}{|x-y|} \right)}{|x-y|^n} \, dy. \]
Hence
\[
\alpha \left| \left\{ x \notin \cup \widetilde{Q} : \sum_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > (1 + N(j))^{1/2} \alpha \right\} \right| \lesssim (1 + N(j))^{-1/2} \sum_Q \int_{\mathbb{R}^n} |h_Q(y)| \int_{|x-y| \geq 2\ell(Q)} \frac{\omega_j(t \ell(Q))}{|x-y|^n} \, dx \, dy
\]
\[
\lesssim (1 + N(j))^{-1/2} \|\omega_j\|_{Dini} \sum_Q \|h_Q\|_{L^1} \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1/2} \|f\|_{L^1},
\]
where the last inequality follows from \(\|\omega_j\|_{Dini} \leq \|\Omega\|_{L^\infty} (1 + N(j))\). On the other hand, for the term \(I I\), by (7.21), we have
\[
\sup_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| \leq \|\Omega\|_{L^\infty} Mh(x).
\]
Then by the weak type (1, 1) of \(M\) (see [41]),
\[
\alpha \left| \left\{ x \notin \cup \widetilde{Q} : \sup_{k \in \mathbb{Z}} |S_{2,j,k}^N h(x)| > (1 + N(j))^{-1/2} \alpha \right\} \right| \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1/2} \|f\|_{L^1}.
\]
Combining the estimates of (7.30) and (7.31), we get
\[
\alpha \left| \left\{ x \notin \cup \widetilde{Q} : S_{2,j}^N h(x) > \alpha \right\} \right| \lesssim \|\Omega\|_{L^\infty} (1 + N(j))^{1/2} \|f\|_{L^1}.
\]
The proof of Lemma 7.3 is complete. \(\square\)

**Lemma 7.4** For \(j \geq 0\), let \(S_{2,j}^N\) be defined as in (7.7). Then we get
\[
M_{S_{2,j}^N} f(x) \lesssim S_{2,j}^N (f)(x) + \|\omega_j\|^2_{Dini} Mf(x), \tag{7.32}
\]
where \(\omega_j(t) \leq \|\Omega\|_{L^\infty} \min(1, 2^{N(j)} t)\).

**Proof** Let \(Q\) be a cube, and take \(x, \xi \in Q\). Let \(B(x) = B(x, 2\sqrt{n} \ell(Q))\), then \(3Q \subset B_x\). By the triangular inequality, we get
\[
|S_{2,j}^N (f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \leq |S_{2,j}^N (f \chi_{\mathbb{R}^n \setminus B_x})(\xi)| - S_{2,j}^N (f \chi_{\mathbb{R}^n \setminus B_x})(x)
\]
\[
+ S_{2,j}^N (f \chi_{B_x \setminus 3Q})(\xi) + S_{2,j}^N (f \chi_{B_x \setminus B_x})(x)
\]
\[=: I + I I + I I I.\]

We begin by estimating term \(I\).
\[
I \leq \left( \sum_{2^l \leq \ell(Q)} + \sum_{2^l > \ell(Q)} \right) \left( \sup_{t_1 < \cdots < t_\ell} \sum_{[t_i, t_{i+1}] \subset [1, 2]} \int_{\mathbb{R}^n \setminus B_x} |K_{k,l}(x) - K_{k,l}(x-y)| \|f(y)\| \, dy \right)^{1/2}.
\]
For $2^k \leq \ell(Q)$, since $|x - \xi| \leq 2|x - y|$, we can get $|x - y| \simeq |\xi - y|$. Therefore by (7.21)

$$
\sum_{2^k \leq \ell(Q)} \sup_{t_1, \ldots, t_k} \sum_{l=1}^{\lambda-1} \int_{\mathbb{R}^n \setminus B_x} |K_{k,l,j}(\xi - y) - K_{k,l,j}(x - y)||f(y)|\,dy \\
\lesssim \|\Omega\|_{L^\infty} \sum_{2^k \leq \ell(Q)} \int_{|x - y| \geq 2\sqrt{n}\ell(Q)} \frac{2^k}{|x - y|^{n+1}}|f(y)|\,dy \\
\lesssim \|\Omega\|_{L^\infty} Mf(x).
$$

For $2^k > \ell(Q)$, by (7.22) we get

$$
\left( \sum_{2^k > \ell(Q)} \left( \sup_{[t_1, t_{i+1}] \subset [1,2]} \sum_{l=1}^{\lambda-1} \int_{\mathbb{R}^n \setminus B_x} |K_{k,l,j}(\xi - y) - K_{k,l,j}(x - y)||f(y)|\,dy \right)^2 \right)^{1/2} \\
\lesssim \left( \sum_{2^k > \ell(Q)} \left( \int_{|x - y| \geq 2\sqrt{n}\ell(Q)} \frac{\omega_j(|x - \xi|/|x - y|)}{|x - y|^n} \chi_{2^k \leq |x - y| \leq 2^{k+2}}|f(y)|\,dy \right)^2 \right)^{1/2} \\
\lesssim \left( \sum_{2^k > \ell(Q)} \omega_j^2(\ell(Q)/2^k) \right)^{1/2} Mf(x) \lesssim \|\omega_j^2\|_{Dim}^{1/2} Mf(x).
$$

Combining the estimates of $\sum_{2^k \leq \ell(Q)}$ and $\sum_{2^k > \ell(Q)}$, we get

$$I \lesssim (\|\omega_j^2\|_{Dim}^{1/2} + \|\Omega\|_{L^\infty}) Mf(x).
$$

For the term $II$, by using (7.21) and noting that $|x - y| \simeq |\xi - y|$ (since $3|x - \xi| \leq |x - y|$), we have

$$II \lesssim \sum_{k \in \mathbb{Z}} \sum_{[t_1, t_{i+1}] \subset [1,2]} \sum_{l=1}^{\lambda-1} \int_{\mathbb{R}^n} |K_{k,l,j}(\xi - y)||f\chi_{B_x \setminus 3Q}(y)|\,dy \\
\lesssim \|\Omega\|_{L^\infty} \int_{3\ell(Q) \leq |x - y| \leq 2\sqrt{n}\ell(Q)} \frac{1}{|x - y|^n} \sum_{k \in \mathbb{Z}} \chi_{2^k \leq |x - y| \leq 2^{k+2}}|f(y)|\,dy \\
\lesssim \|\Omega\|_{L^\infty} Mf(x).
$$

We now turn to the term $III$. Note that

$$S_{2,j}^N(f \chi_{\mathbb{R}^n \setminus B_x})(x) = \left( \sum_{k \in \mathbb{Z}} \sup_{[t_1, t_{i+1}] \subset [1,2]} \sum_{l=1}^{\lambda-1} |K_{k,l,j} \ast f\chi_{\mathbb{R}^n \setminus B_x}(x)|^2 \right)^{1/2},
$$

where for $j \geq 1$,

$$K_{k,l,j} \ast f(x) = v_{k,t_l,t_{l+1}} \ast (\phi_{k-N(j)} - \phi_{k-N(j-1)}) \ast f(x).$$
and for $j = 0$,

$$K_{k,l,j} * f(x) = v_{k,l,j} * \varphi_k * f(x).$$

Since \( \text{supp} v_{k,l,j} * \varphi_k \subset \{ x : |x| \leq 2^{k+1} \} \) and \( \text{supp} v_{k,l,j} * \varphi_k - N(j) \subset \{ x : |x| \leq 2^{k+1} \} \), from (7.21) we get

$$III = \left( \sum_{k \in \mathbb{Z}} \sup_{t_1 < \cdots < t_l} \left\{ \sum_{j = 1}^{\lambda} \left| \int_{|x-y| > 2^k} K_{k,l,j}(x-y) f(y) dy \right| \right\} \right)^{1/2}.$$

Combining the estimates of $I$, $II$ and $III$, we get

$$S^N_{2,j}(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi) \lesssim S^N_{2,j}(f)(x) + (\| \Omega \|_{L^\infty} + \| \omega_j^{1/2} \|_{Dini} ) M f(x).$$

which leads to

$$M_{S^N_{2,j}} f(x) \lesssim S^N_{2,j} f(x) + (\| \Omega \|_{L^\infty} + \| \omega_j^{1/2} \|_{Dini} ) M f(x).$$

The proof of Lemma 7.4 is complete. \( \Box \)

Then by Lemma 7.3, Lemma 7.4 and the weak type (1, 1) of $M$ (see [41]) we get

$$\| M_{S^N_{2,j}} f \|_{L^1} \lesssim \| \Omega \|_{L^\infty} (1 + N(j))^{1/2} \| f \|_{L^1} \tag{7.33}$$

since $\| \omega_j^{1/2} \|_{Dini} \lesssim \| \Omega \|_{L^\infty} (1 + N(j))^{1/2}$.

Since $S^N_{2,j}$ satisfies (7.28) and (7.33), therefore by applying Lemma 2.1 to $U = S^N_{2,j}$, we get that for $1 < p < \infty$ and $w \in A_p$,

$$\| S^N_{2,j}(f) \|_{L^p(w)} \lesssim \| \Omega \|_{L^\infty} (1 + N(j))^{1/2} \{ w \} A_p \| f \|_{L^p(w)}. \tag{7.34}$$

Taking $w = 1$ in the above inequality gives

$$\| S^N_{2,j}(f) \|_{L^p} \lesssim \| \Omega \|_{L^\infty} (1 + N(j))^{1/2} \| f \|_{L^p}. \tag{7.35}$$
Interpolating between \((7.8)\) and \((7.35)\) shows that there exists some \(\theta \in (0, 1)\)
\[
\|S_{N,j}^2(f)\|_{L^p} \lesssim \|\Omega\|_{L^\infty} 2^{-\theta N(j-1)} (1 + N(j))^{1/2} \|f\|_{L^p}.
\] (7.36)
Combining \((7.34)\) and \((7.36)\), and by using the interpolation theorem with change of measures-Lemma 2.2, we obtain that
\[
\|S_2(T_{\Omega f})\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\} A_p(w)^{1/2} \|f\|_{L^p(w)},
\]
which gives the proof of \((7.2)\). The proof of Theorem 1.4 is complete. \(\square\)

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References

1. Bourgain, J.: Pointwise ergodic theorems for arithmetic sets. Inst. Hautes Études SCI Publ. Math. 69, 5–41 (1989)
2. Bourgain, J., Mirek, M., Stein, E.M., Wróbel, B.: On dimension-free variational inequalities for averaging operators in \(\mathbb{R}^d\). Geom. Funct. Anal. 28, 58–99 (2018)
3. Buckley, S.M.: Estimates for operator norms on weighted spaces and reverse Jensen inequalities. Trans. Am. Math. Soc. 340, 253–272 (1993)
4. Campbell, J., Jones, R., Reinhold, K., Wierdl, M.: Oscillation and variation for the Hilbert transform. Duke Math. J. 105, 59–83 (2000)
5. Campbell, J., Jones, R., Reinhold, K., Wierdl, M.: Oscillation and variation for singular integrals in higher dimensions. Trans. Am. Math. Soc. 355, 2115–2137 (2002)
6. Cheng, W., Ma, T.: Quantitative and sharp weighted estimates for \(q\)-variations of singular operators. Acta Math. Sin. (Chin. Ser.) 62, 279–286 (2019)
7. Chen, Y., Ding, Y., Hong, G., Liu, H.: Weighted jump and variational inequalities for rough operators. J. Funct. Anal. 274, 2446–2475 (2018)
8. Christ, M., Journé, J.-L.: Polynomial growth estimates for multilinear singular integral operators. Acta Math. 159, 51–80 (1987)
9. Clara de Franca Silva, F., Zorin-Kranich, P.: Sparse domination of sharp variational truncations. http://arxiv.org/abs/1604.05506
10. Ding, Y., Hong, G., Liu, H.: Jump and variation inequalities for rough operators. J. Fourier Anal. Appl. 23, 679–711 (2017)
11. Di Plinio, F., Hytönen, T., Li, K.: Sparse bounds for maximal rough singular integrals via the Fourier transform. Ann. Inst. Fourier 70, 1871–1902 (2020)
12. Do, Y., Muscalu, C., Thiele, C.: Variational estimates for paraproducts. Rev. Mat. Iberoam. 28, 857–878 (2012)
13. Duoandikoetxea, J., Rubio de Francia, J.: Maximal and singular integral operators via Fourier transform estimates. Invent. Math. 84, 541–561 (1986)
14. Duong, X.T., Li, J., Yang, D.: Variation of Calderón-Zygmund operators with matrix weight. Commun. Contemp. Math. 23, 2050062 (2021)
15. Fujii, N.: Weighted bounded mean oscillation and singular integrals. Math. Jpn. 22, 529–534 (1977/1978)
16. Guillemin, T., Torrea, J.: Dimension free estimates for the oscillation of Riesz transforms. Isr. J. Math. 141, 125–144 (2004)
17. Hong, G.: The behaviour of square functions from ergodic theory in \(L^\infty\). Proc. Am. Math. Soc. 143, 4797–4802 (2015)
18. Hytönen, T., Lacey, M.T., Pérez, C.: Sharp weighted bounds for the \(q\)-variation of singular integrals. Bull. Lond. Math. Soc. 45, 529–540 (2013)
19. Hytönen, T., Pérez, C.: Sharp weighted bounds involving \(A_\infty\). Anal. PDE 6, 777–818 (2013)
20. Hytönen, T., Lacey, M., Parissis, I.: A variation norm Carleson theorem for vector-valued Walsh-Fourier series. Rev. Mat. Iberoam. 30, 979–1014 (2014)
21. Hytönen, T., Roncal, L., Tapiola, O.: Quantitative weighted estimates for rough homogeneous singular integrals. Isr. J. Math. 218(1), 133–164 (2017)
22. Jones, R., Kaufman, R., Rosenblatt, J., Wierdl, M.: Oscillation in ergodic theory. Ergod. Theory Dyn. Syst. 18, 889–935 (1998)
23. Jones, R., Rosenblatt, J., Wierdl, M.: Oscillation in ergodic theory: higher dimensional results. Isr. J. Math. 135, 1–27 (2003)
24. Jones, R., Seeger, A., Wright, J.: Strong variational and jump inequalities in harmonic analysis. Trans. Am. Math. Soc. 360, 6711–6742 (2008)
25. Jones, R., Wang, G.: Variation inequalities for the Fejér and Poisson kernels. Trans. Am. Math. Soc 356, 4493–4518 (2004)
26. Krause, B., Zorin-Kranich, P.: Weighted and vector-valued variational estimates for ergodic averages. Ergod. Theory Dyn. Syst. 38, 244–256 (2018)
27. Le Merdy, C., Xu, Q.: Strong q-variation inequalities for analytic semigroups. Ann. Inst. Fourier (Grenoble) 62, 2069–2097 (2012)
28. Lépingle, D.: La variation d’ordre p des semi-martingales. Z. Wahrsch. Verw. Gebiete. 36, 295–316 (1976)
29. Lerner, A.K.: On pointwise estimates involving sparse operators. N. Y. J. Math. 22, 341–349 (2017)
30. Lerner, A.K.: A note on weighted bounds for rough singular integrals. C. R. Acad. Sci. Paris Ser. I(356), 77–80 (2018)
31. Ma, T., Torrea, J., Xu, Q.: Weighted variation inequalities for differential operators and singular integrals. J. Funct. Anal. 268, 376–416 (2015)
32. Ma, T., Torrea, J., Xu, Q.: Weighted variation inequalities for differential operators and singular integrals in higher dimensions. Sci. China Math. 60, 1419–1442 (2017)
33. Mas, A., Tolsa, X.: Variation for the Riesz transform and uniform rectifiability. J. Eur. Math. Soc. 16, 2267–321 (2014)
34. Mirek, M., Trojan, B.: Discrete maximal functions in higher dimensions and applications to ergodic theory. Am. J. Math. 138, 1495–1532 (2016)
35. Mirek, M., Stein, E.M., Trojan, B.: $\ell_p(Z^d)$-estimates for discrete operators of Radon type: variational estimates. Invent. Math. 209, 665–748 (2017)
36. Mirek, M., Trojan, B., Zorin-Kranich, P.: Variational estimates for averages and truncated singular integrals along the prime numbers. Trans. Am. Math. Soc. 69, 5403–5423 (2017)
37. Oberlin, R., Seeger, A., Tao, T., Thiele, C., Wright, J.: A variation norm Carleson theorem. J. Eur. Math. Soc. 14, 421–464 (2012)
38. Pisier, G., Xu, Q.: The strong $p$-variation of martingales and orthogonal series. Probab. Theory Relat. Fields 77, 497–514 (1988)
39. Qian, J.: The $p$-variation of partial sum processes and the empirical process. Ann. Prob. 77, 1370–1383 (1998)
40. Stein, E.M., Weiss, G.: Interpolation of operators with change of measures. Trans. Am. Math. Soc. 87, 159–172 (1958)
41. Stein, E.M.: Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Univ. Press, Princeton (1993)
42. Wilson, J.M.: Weighted inequalities for the dyadic square function without dyadic $A_1$. Duke Math. J. 55, 99–150 (1987)
43. Zorin-Kranich, P.: Variation estimates for averages along primes and polynomials. J. Funct. Anal. 268, 210–238 (2015)

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