Perturbative Prediction for Parton Fragmentation into Heavy Hadron

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Abstract:
By expanding functions of parton fragmentation into a heavy hadron in the inverse of the heavy quark mass $m_Q$ we attempt to factorize them into perturbative- and nonperturbative parts. In our approach the nonperturbative parts can be defined as matrix elements in heavy quark effective theory, the shape of the functions is predicted by perturbative QCD. In this work we neglect effect at order of $m_Q^{-2}$ and calculate the perturbative parts at one-loop level for heavy quark- and gluon fragmentation. We compare our results from leading log approximation with experimental results from $e^+e^-$ colliders and find a deviation below or at 10% level. Adding effect of higher order in $\alpha_s$ it can be expected to reduce the deviation. The size of matrix elements appearing at the order we consider for several types of heavy hadrons is determined.
1. Introduction

Parton fragmentation functions are in general nonperturbative objects in the QCD factorization theorem [1] for predictions of inclusive productions of single hadron. For a light hadron the fragmentation happens at the energy scale of $\Lambda_{QCD}$, which is several hundreds MeV. Hence it is purely a long-distance process. For the hadron being a quarkonium, which consists mainly of a heavy quark $Q$ and its antiquark $\bar{Q}$, there is in the fragmentation not only long-distance effect but also certain short-distance effect because heavy quarks are involved and they provide a large energy-scale, the mass $m_Q$ of heavy quarks. This short-distance effect can be well described by perturbative QCD, while the long-distance effect can be parameterized with matrix elements of local operators defined in nonrelativistic QCD[2]. With these facts an amount of functions for parton fragmentation into a quarkonium is calculated(See [3-5] for an incomplete list of references). A question naturally arises that can we predict fragmentation function for a hadron containing a single quark $Q$? Here the large mass of the heavy quarks implies certain perturbative effect as it does in the case of a quarkonium. In this work we attempt to answer the question.

Recently our understanding of physics related to hadrons containing single heavy quark has grown rapidly. Such achievement is based on the development of an effective theory for heavy quarks, i.e. HQET, by starting from QCD(For HQET see reviews in [6]). The basic observation leading to HQET is that the heavy quark inside a heavy hadron carries the most momentum of the hadron. In this work we will refer heavy hadrons as those containing single heavy quark. With this observation one can decompose the momentum of the heavy quark into a large component which is roughly the momentum of the heavy hadron, and a small component which is at order of $\Lambda_{QCD}$. By integrating out the dynamical freedom carrying the small component HQET is obtained, in which predictions can be expanded in $m_Q^{-1}$, especially, at leading order every heavy hadron has the same mass—$m_Q$. The difference between masses of different heavy hadrons is at order of $m_Q^{-1}$. Many applications of HQET have been done for weak decays of heavy hadrons. Application of HQET to heavy quark fragmentation, i.e., heavy hadron production, appeared first in [7], where an expansion is obtained, the expansion parameter is the difference between the masses of the heavy hadron and of the heavy quark. The expansion is nonperturbative and formal, it does not tell in detail how the fragmentation depends on the energy fraction carried by the heavy hadron.

For heavy quark fragmentation a heavy quark is the initial parton which is off-shell with an invariant mass larger than $m_Q$. One can image the fragmentation as a two-step process. Before combining light quarks and glue to form a heavy hadron the heavy quark can emit or absorb light quarks and glue. After such emissions it combines light quarks or
glue to form the hadron and these light quarks and glue should have momenta at order of \( \Lambda_{QCD} \), as the formation is a long-range process. The most momentum of the heavy quark at this time will be carried by the hadron and the invariant mass of this heavy quark is at the order of the hadron mass. This heavy quark can approximately be treated as on-shell. Such picture suggests that the fragmentation function may be written in factorized form in which a part is for the process from the off-shell- to the on-shell quark and another part is for the transition into the heavy hadron. The first part can be treated perturbatively because of the large scale which is \( m_Q \) at least. The second can be parametrized by using HQET. For other initial partons the fragmentation can be thought as a heavy quark is first produced and then the formation follows. The production again can be handled perturbatively.

In this work we use the diagram expansion method to perform the factorization mentioned above. Such method was first used in deeply inelastic scattering to analyse twist-4 effect[8]. This method can be thought as extension of Wilson’s operator expansion to cases where the expansion is not applicable, for cases where Wilson’s operator expansion is applicable both methods delivery same results. For readers unfamiliar with this method we refer to [8,9,10]. With the method we obtain fragmentation functions as an expansion in \( m_Q^{-1} \), in each order the nonperturbative part is contained in matrix elements defined in HQET. In this work we will neglect all effect which is suppressed by \( m_Q^{-2} \). For fragmentation function \( D_{H/a}(z) \), where \( a \) stands for the initial parton , \( H \) for the heavy hadron and \( z \) is the momentum fraction carried by \( H \), the expansion may be written as:

\[
D_{H/a}(z) = \hat{D}_a(z) < 0|O_H|0 > + O\left(\frac{1}{m_Q^2}\right).
\]  

(1.1)

In Eq.(1.1) \( O_H \) is an operator is defined in HQET, its matrix element represents nonperturbative physics. The function \( \hat{D}_a(z) \) can be calculated perturbatively. We will calculate \( \hat{D}_Q(z) \), \( \hat{D}_G(z) \) up to order of \( \alpha_s \). The effect at order of \( m_Q^{-1} \) is included in the matrix element.

Our work is organized as the following: In Sect.2 we use the diagram expansion method for tree-level diagram and HQET to obtain the factorized form as in Eq.(1.1) for \( a = Q \). In doing so, we neglect the difference between masses of heavy hadrons and of heavy quarks. It should be be noted that at first look one can keep this difference, but one will have some problems at higher order of \( \alpha_s \). We will explain the problems in detail. In Sect. 3 we proceed to calculate \( \hat{D}_Q(z) \) and \( \hat{D}_G(z) \) at order of \( \alpha_s \). In Sect.4 we compare our results with experiment at \( e^+e^- \) colliders and determine the value of several matrix elements \( < 0|O_H|0 > \) by using Z-decays. Sect.5 is the summary of our work.

In this work we will use Feynman gauge and \( d \)-dimensional regularization. In this regularization infrared(I.R.) singularities appear as poles at \( d = 4 \). We take the normalization
of a state as:

\[ < p' | p > = 2p^0 (2\pi)^2 \delta^3(p' - p). \]

(1.2)

2. The Factorization and Tree-Level Results for Heavy Quark Fragmentation

We start in this section our analysis from definitions of fragmentation functions. These definitions are first given in [11]. Such definitions are conventionally written in light-cone coordinate system. In this coordinate system a \( d \)-vector \( p \) is expressed as \( p^\mu = (p^+, p^-, 0_T) \), with \( p^+ = (p^0 + p^{d-1})/\sqrt{2} \), \( p^- = (p^0 - p^{d-1})/\sqrt{2} \). We introduce a vector \( n \) with \( n^\mu = (0, 1, 0, \cdots, 0) = (0, 1, 0_T) \) in this system. The function of heavy quark fragmentation into a heavy hadron \( H \) carries momentum \( P \) is defined as[11]:

\[
D_{H/Q}(z) = z^d - 3 \frac{1}{4\pi} \int dx^- e^{-ix^- p^+} \frac{1}{3} \text{Tr}_{\text{color}} \frac{1}{2} \text{Tr}_{\text{Dirac}} [n \cdot \gamma < 0| Q(0) \\
\cdot \text{Pexp} \{-ig_s \int_0^\infty d\lambda n \cdot G^T(\lambda n) \} a_H^\dagger(P)a_H(P) \\
\cdot \text{Pexp} \{ig_s \int_0^\infty d\lambda n \cdot G^T(\lambda n) \} \bar{Q}(x^- n)|0> ],
\]

where \( G^a_\mu(x) = G^a_\mu(x) T^a \), \( G^a_\mu(x) = \) the gluon field and \( T^a(a = 1, \cdots, 8) \) are the \( SU(3) \)-color matrices. The subscript \( T \) denotes the transpose. \( Q(x) \) stands for the Dirac-field of heavy quark. \( a_H^\dagger(P) \) is the creation operator for the hadron \( H \) and \( P^\mu = (P^+, P^-, 0_T) \). For hadrons with nonzero spin the summation over the spin is understood. The hadron carries a fraction \( z \) of the momentum \( p \) of the heavy quark as the initial parton, i.e., \( P^+ = zp^+ \). The definition is a unrenormalized version. Ultraviolet divergences will appear in \( D_{H/Q}(z) \) and call for renormalization. The renormalization is discussed in [11]. We will use modified \( MS \)-scheme.

At tree-level there is only one diagram in the diagram expansion, which is given in Fig.1. We divide this diagram with a horizontal broken line into a upper- and a lower-parts. The upper part contains nonperturbative part and we represent it as a black box, corresponding to the nonperturbative object \( \Gamma_{ij}(q, P) \):

\[
\Gamma_{ij}(q, P) = \int d^4x e^{-iq \cdot x} < 0| Q_i(0)a_H^\dagger(P)a_H(P)\bar{Q}_j(x)|0>,
\]

(2.3)

where \( i \) and \( j \) stand for Dirac- and color-indices. Because of color-symmetry \( \Gamma(q, P) \) is diagonal in color-space. The contribution of Fig.1 can be written as

\[
D_{H/Q}(z) = \frac{z^d - 3}{24\pi} \int \frac{d^4q}{(2\pi)^4} \text{Tr}[\gamma \cdot n \Gamma(q, P)] 2\pi \delta(n \cdot k - n \cdot q).
\]

(2.4)
If we take $H$ to be a quark $Q$, the black box becomes a quark line, and $\Gamma(q, P)$ becomes $(2\pi)^4\delta^4(q - P)(\gamma \cdot q + m_Q)$, we obtain \( D_{Q/Q}(z) = \delta(1 - z) \). The assumption that the hadron carries the most momentum of the heavy quark implies that the dominant $x$-dependence of the matrix element in Eq.\((2.3)\) is $e^{ix \cdot P}$, the correction to this dependence can be expanded in $\Lambda_{QCD}/m_Q$. To proceed further we use HQET to write the field $Q(x)$ as:

\[
Q(x) = e^{-im_Qv \cdot x} \{ h_v(x) + \frac{1}{2m_Q}i\gamma \cdot D_T h_v(x) + O(\frac{1}{m_Q^2}) \} \tag{2.5}
\]

where

\[
D^\mu_T = D^\mu - v \cdot Dv^\mu, \quad v^\mu = \frac{P^\mu}{M_H} \tag{2.6}
\]

and $m_Q$ is the pole mass of the heavy quark. In Eq.\((2.6)\) $D^\mu$ is the covariant derivative. The effective Lagrangian for the field $h_v(x)$ reads:

\[
L_{\text{eff}} = \bar{h}_v i v \cdot D h_v + \frac{1}{2m_Q} \bar{h}_v (i\gamma \cdot D_T)^2 h_v + O(\frac{1}{m_Q}). \tag{2.7}
\]

The $x$-dependence of $h_v(x)$ in a matrix element is controlled by the scale at order of $\Lambda_{QCD}$ and can be expanded. With Eq.\((2.5)\) and neglecting $M_H - m_Q$ the matrix element in Eq.\((2.3)\) can be written as an expansion:

\[
<0|Q_i(0)a^\dagger_H(P)a_H(0)\bar{Q}_j(x)|0> = e^{ix \cdot P} \left\{ <0|(h_v)_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0)|0> - \frac{1}{2}x^\mu <0|\langle \partial_\mu h_v \rangle_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0) + (h_v)_i(0)a^\dagger_H a_H(\partial_\mu \bar{h}_v)_j(0)|0> + \frac{1}{2m_Q} <0|(i\gamma \cdot D_T h_v)_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0) + (h_v)_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0)|0> + \cdots \right\}. \tag{2.8}
\]

The $\cdots$ stand for terms which will lead to contributions at order of $m_Q^2$ or higher orders. The matrix elements in r.h.s. of Eq.\((2.8)\) are matrices in Dirac-indices $i$ and $j$. The structure of these matrices can be determined by using symmetries of parity($P$), of time-reversal($T$). The matrix element in the second and third line can be written as the form:

\[
<0|\langle \partial_\mu h_v \rangle_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0) + (h_v)_i(0)a^\dagger_H a_H(\partial_\mu \bar{h}_v)_j(0)|0> = C_{ij}^\mu <0|\text{Tr} \left\{ \left( (i\gamma \cdot D_T)^2 h_v \right)_i(0)a^\dagger_H a_H h_v(0) + \text{h.c.} \right\} |0>, \tag{2.9}
\]

\[
<0|(i\gamma \cdot D_T h_v)_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0) + (h_v)_i(0)a^\dagger_H a_H(\bar{h}_v)_j(0)|0> = B_{ij} <0|\text{Tr} \left\{ (i\gamma \cdot D_T h_v)_i(0)a^\dagger_H a_H h_v(0) + \text{h.c.} \right\} |0>,
\]

where $C_{ij}^\mu$ and $B_{ij}$ are matrix elements labeled by Dirac-indices. Because of $\gamma \cdot v h_v = h_v$ the expectation value in r.h.s. of the second equation is zero. In the first equation we used the
equation of motion. The dimension of the expectation value in r.h.s. of the first equation is 3 in \( m_Q \), while the expectation value in the first line of Eq.(2.8) is 1. Hence the contribution from the second line in Eq.(2.8) in final results will be at order of \( m_Q^{-2} \). We define the operator \( O_H \) as:

\[
O_H = \frac{1}{12M_H} \text{Tr}\{h_{\nu}(0)a_H^{\dagger}(P)a_H(P)\bar{h}_{\nu}(0)\}. \tag{2.10}
\]

With this operator the nonperturbative object \( \Gamma(q, P) \) becomes:

\[
\Gamma(q, P) = (2\pi)^4\delta^4(q - P)(\gamma \cdot q + m_Q) < 0|O_H|0 > + O(\frac{1}{m_Q^2}). \tag{2.11}
\]

Finally we obtain

\[
D_{H/Q}(z) = \delta(1 - z) < 0|O_H|0 > + O(\frac{1}{m_Q^2}). \tag{2.12}
\]

In Eq.(2.12) we have neglected all terms which are at order higher than \( m_Q^{-1} \). The obtained function is singular at \( z = 1 \). If we go to higher orders it becomes more singular, where derivatives of \( \delta \)-function appear. This means that the detail of the shape around \( z = 1 \) can not be predicted in our approach, but physical predictions still can be made by noting that they are convolutions of fragmentation functions with other functions in \( z \), the fragmentation functions are distributions. With the fragmentation function given in Eq.(2.12) the integral with a test function \( f(z) \) is approximated by:

\[
\int_0^1 dz f(z)D_{H/Q}(z) = f(1) < 0|O_H|0 > + O(\frac{1}{m_Q^2}). \tag{2.13}
\]

The situation here may look like the case of inclusive decays of B mesons[12], where one also encounters similar expansion as in Eq.(2.12) in which terms from higher orders are more singular. In [12] the accurate shape of decay-spectra was interested and the study there is corresponding to study the integral in our case:

\[
\int_{1-z_\Lambda}^1 dz f(z)D_{H/Q}(z) \tag{2.14}
\]

for \( z_\Lambda \) at order of \( \Lambda_{QCD}/m_Q \). Because the order of \( z_\Lambda \) our results in this work can not be applied to observables which are related to a integral like that in Eq.(2.14). The observables studied later are related to the integral in Eq.(2.13).

One can perform similar analysis as above for other partons. Based on such analyses we propose to write the functions of parton fragmentation into a heavy hadron as the factorized form:

\[
D_{H/a}(z) = \hat{D}_a(z) < 0|O_H|0 > + O(\frac{1}{m_Q^2}). \tag{2.15}
\]
In this form the hadron dependence is only contained in the matrix elements, the \( z \)-dependence is predicted completely by perturbative QCD. It should be noted that the matrix element \( \langle 0|O_H|0 \rangle \) also contains effect at order of \( m_Q^{-1} \), because it is defined in HQET in Eq.(2.7) with the accuracy at order of \( m_Q^{-1} \). In our approach here the function \( \hat{D}_a(z) \) is just the fragmentation function \( D_{Q/a}(z) \) of a parton \( a \) into a heavy quark \( Q \). This fact also implies any subtraction for calculating \( \hat{D}_a(z) \) at higher order of \( \alpha_s \) is not needed. Usually certain subtractions are needed to extract perturbative parts in a factorized form. To examine this we have calculated the matrix element \( \langle 0|O_H|0 \rangle \) by taking \( H = Q \) in HQET at one-loop level. Indeed, the matrix element does not receive any one-loop correction. This also shows that the \( \mu \)-dependence of the matrix element is suppressed at least by \( m_Q^{-2} \) or by \( \alpha_s^2 \).

In the above approach we have neglect the difference \( M_H - m_Q \) in the kinematic as in the case with a quarkonium, where the binding energy is neglected in the kinematic. It seems that such difference can be kept in the approach. With the difference HQET given in Eq.(2.6) is not suitable for our purpose. The reason is: The heavy quark in Fig.1 carries the momentum \( q = m_Q v + k_1 \). In HQET employed above we neglect \( k_1 \) at leading order of \( m_Q^{-1} \). The mass \( M_H \) is usually larger than \( m_Q \), so the fragmentation function is only non zero at \( z = M_H/m_Q \) which is larger than 1. This is in conflict with the definition given in Eq.(2.1), which says because of the conservation of momenta that \( D_{H/Q}(z) = 0 \) for \( z > 1 \). This problem may be solved at first loo by employing HQET with a residual mass. As already noted in [13] that the decomposition of quark momentum \( q = m_Q v + k_1 \) where \( k_1 \) is the small component, is arbitrary, one can also decompose \( q \) as:

\[
q = (m'_Q + \epsilon_m)v + k'_1
\]

(2.16)

where \( \epsilon_m \) is the residual mass at order \( \Lambda_{QCD} \) and \( k'_1 \) is the small component. With this decomposition one can obtain in analogy to Eq.(2.5):

\[
Q(x) = e^{-i(m'_Q + \epsilon_m) v \cdot x} \{ h'_v(x) + O(\frac{1}{m'_Q}) \}.
\]

(2.17)

The effective Lagrangian for the field \( h'_v \) reads:

\[
L_{\text{eff}} = \bar{h}'_v (iv \cdot D + \epsilon_m) h'_v + O(\frac{1}{m'_Q})
\]

(2.18)

Repeating the above steps one gets:

\[
D_{H/Q} = z_0^3 \delta(z - z_0) < 0|O_H'|0 > + O(\frac{1}{m_Q^2}), \quad z_0 = \frac{M_H}{m_Q + \epsilon_m}.
\]

(2.19)
The operator $O'_H$ is obtained by replacing $h_v$ in $O_H$ with $h'_v$. Choosing $\epsilon_m > M_H - m'_Q$ the function is only nonzero at $z = z_0 < 1$. However, the mass $m'_Q$ is not the pole mass, and the pole mass $m_Q$ is the sum $m'_Q + \epsilon_m$. Hence, the choice of $\epsilon_m$ is not possible. It seems that the effect from the difference $M_H - m_Q$ can not be handled in perturbative theory. To study this effect one may only employ nonperturbative methods or try to sum contributions of a series of higher-dimensional operators in [7].

In the case of heavy quark distribution in a heavy hadron the difference can be kept in our approach. The analysis is similar as that leading to Eq.(2.11), the corresponding diagram is just this by reversing Fig.1 and $p^+ = zP^+$. One obtains

$$f_{Q/H}(z) = \delta(z - \frac{m_Q}{M_H}) + O\left(\frac{1}{m_Q^2}\right)$$ (2.18)

where the matrix element corresponding to $<0|O_H|0>$ equals one plus corrections at order of $m_Q^{-2}$. The quark line here represents on-shell quark, therefore the problems mentioned above will not appear.

3. Results For Fragmentation at One-Loop Level

In this section we present a calculation of fragmentation function for a heavy quark into a heavy quark and for a gluon into a heavy quark at order of $\alpha_s$. This is the perturbative part in the fragmentation into a heavy hadron in Eq.(2.13). In the case of quark fragmentation contribution from every diagram at one-loop level contains a I.R. singularity, there is a delicate cancellation of the singularity between contributions from different diagrams. We show here in detail how this works.

At one-loop level there are four diagrams contributing to heavy quark fragmentation. They are given in Fig.2A–2D. Contribution from each diagram is not only ultraviolet divergent but also I.R. divergent. However, final result is free from I.R. singularity. With the Feynman rule given in [11], the contribution from Fig.2A is:

$$D_A(z) = \frac{z^{d-3}}{24\pi^2} \mu^\varepsilon \int \left(\frac{dk}{2\pi}\right)^d \text{Tr}\left\{ -ig_sT^a\gamma^\mu \frac{i}{\gamma \cdot (q-k) - m_Q + i0^+} \gamma \cdot n (ig_sT^a n_\mu) \right\} \cdot 2\pi i \delta(n \cdot (p-q)) \cdot \frac{i}{k^2 + i0^+} \cdot \frac{i}{n \cdot (p-q+k) + i0^+}$$ (3.1)

where $\mu$ is the renormalization scale, $\varepsilon = 4 - d$, $q^+ = zp^+$ and $q^2 = m_Q^2$. The contribution
from Fig.2B is:

\[
D_B(z) = \frac{z^{d-3}}{24\pi} \mu^\varepsilon \int \frac{dk}{2\pi^d} \text{Tr} \left\{ -ig_s T^a \gamma^\mu \frac{i}{\gamma \cdot (q + k) - m_Q + i0^+} \gamma \cdot n (-ig_s T^a n_\mu) \right. \\
\left. \cdot (\gamma \cdot q + m_Q) \right\} \cdot \frac{-i}{n \cdot (p - q) - i0^+} \cdot (-1)(2\pi)\delta(k^2)(2\pi)\delta(n \cdot (p - q - k)). \tag{3.2}
\]

Both terms are I.R. divergent. In \( D_A(z) \) the divergence appears at \( k^+ \sim 0 \), while in \( D_B(z) \) the divergence is because the on-shell gluon can carry very small energy. These divergences can be regularized in dimensional regularization and are represented by the terms as \( \varepsilon^{-1} \), where \( \varepsilon_I = d - 4 \). Performing the loop integration in Eq.(3.1) and Eq.(3.2) we obtain:

\[
D_A(z) = -\frac{2}{3\pi} g_s^2 \delta(1 - z) \cdot \frac{\pi^{d-2}}{(2\pi)^{d-2}} \cdot \Gamma \left( \frac{\varepsilon}{2} \right) \left( \frac{\mu}{m_Q} \right)^\varepsilon \\
\cdot \left\{ \frac{1}{\varepsilon_I} - 1 + \varepsilon_I + O(\varepsilon^2) \right\},
\]

\[
D_B(z) = \frac{2z}{3\pi} g_s^2 \cdot \frac{\pi^{d-2}}{(2\pi)^{d-2}} \cdot \Gamma \left( \frac{\varepsilon}{2} \right) \left( \frac{\mu}{m_Q} \right)^\varepsilon \\
\cdot \left\{ \frac{1}{\varepsilon_I} \delta(1 - z) + \frac{1}{(1 - z)_+} + \varepsilon_I \left( \frac{\ln(1 - z)}{1 - z} \right)_+ + O(\varepsilon^2) \right\}. \tag{3.3}
\]

The \( \Gamma \)-function \( \Gamma \left( \frac{\varepsilon}{2} \right) \) with \( \varepsilon = 4 - d \) represents U.V. divergence. The +-prescription is as usual. From Eq.(3.3) the sum \( D_A(z) + D_B(z) \) is free from the I.R. pole \( \varepsilon_I^{-1} \). After renormalization the sum is:

\[
(D_A(z) + D_B(z))^{(R)} = \frac{2}{3\pi} \alpha_s(\mu) \left\{ [\delta(1 - z) + \frac{z}{(1 - z)_+}] \ln \frac{\mu^2}{m_Q^2} + 2\delta(1 - z) - 2z \left( \frac{\ln(1 - z)}{1 - z} \right)_+ \right\}. \tag{3.4}
\]

The contribution from Fig.2C is just the one-loop correction to Fig.1. Because the quark-line is for a on-shell quark, the correction is to external line. This contribution contains also I.R. singularity. After renormalization the contribution is:

\[
D_C(z)^{(R)} = \frac{\alpha_s(\mu)}{3\pi} \delta(1 - z) \cdot \left\{ \frac{2}{\varepsilon_I} + \gamma - \ln(4\pi) - 2 + \frac{1}{2} \ln \frac{m_Q^2}{\mu^2} \right\}. \tag{3.5}
\]

The last contribution is from Fig.2D, it reads:

\[
D_D(z) = \frac{z^{d-3}}{24\pi} \mu^\varepsilon \int \frac{dk}{2\pi^d} \text{Tr} \left\{ (-ig_s T^a \gamma^\mu) \frac{i}{\gamma \cdot (q + k) - m_Q} \gamma \cdot n (-ig_s T^a n_\mu) \right. \\
\left. \cdot (\gamma \cdot q + m_Q) \right\} \cdot (-1)(2\pi)\delta(k^2)(2\pi)\delta(n \cdot (p - q - k)). \tag{3.6}
\]
Performing the $k$-integration and renormalization we obtain:

\[
D_D(z)^{(R)} = \frac{2\alpha_s(\mu)}{3\pi} \left\{ \ln \frac{\mu^2}{m_Q^2} - 2 \ln(1 - z) - 1 \right\} - \frac{2\alpha_s(\mu)}{3\pi} \left\{ \frac{2}{\varepsilon_I} - \ln(4\pi) + \gamma \right\} \delta(1 - z) + \frac{2z}{(1 - z)_+}. \tag{3.7}
\]

The total contribution to the one-loop correction of $D_Q/Q$ is the sum: $2(D_A(z) + D_B(z) + D_C(z)^{(R)} + D_D(z)^{(R)})$. In this sum the I.R. divergence in Eq.(3.7) cancels that in Eq.(3.5). Therefore the sum is free from I.R. singularity. With these results the function \(\hat{D}_Q(z)\) in Eq.(2.13) is:

\[
\hat{D}_Q(z) = D_{Q/Q}(z) = \delta(1 - z) + \frac{2\alpha_s(\mu)}{3\pi} \left\{ \frac{2}{\varepsilon_I} - \ln(4\pi) + \gamma \right\} \delta(1 - z) + \frac{2z}{(1 - z)_+} \tag{3.8}
\]

Now we turn to gluon fragmentation into a heavy hadron. The definition of gluon fragmentation function \(D_{H/G}(z)\) can also be found in [11]. Upto the order of $m_Q^{-1}$ we consider there is only one diagram drawn in Fig.3. It should be pointed out that there are more diagrams at higher orders, in which the lower part is connected with the black box not only with the quark lines as in Fig.3 but also with some gluon lines. This is also the case for heavy quark fragmentation if we go beyond the order of $m_Q^{-1}$. Repeating the procedure in Sect.2 we can obtain

\[
D_{H/G}(z) = \hat{D}_G(z) < 0|O_H|0 > + O\left(\frac{1}{m_Q^2}\right) \tag{3.9}
\]

as proposed in Eq.(2.13). The function \(\hat{D}_G(z)\) is just the fragmentation function \(D_{Q/G}(z)\) for a gluon into a heavy quark. From Fig.3 the contribution reads

\[
\frac{-z^{d-3}}{16\pi(d - 2)p^\mu} \int \left( \frac{dk}{2\pi} \right)^d 2\pi \delta(k^2 - m_Q^2) \cdot 2\pi \delta(n \cdot (p - q - k)) \cdot \frac{1}{(k + q)^2} \text{Tr} \left[ \gamma^\mu (\gamma \cdot k - m_Q) \gamma^\nu (\gamma \cdot q + m_Q) \right] \tag{3.10}
\]

\[
\cdot (p \cdot n g_{\mu\rho} - n_\mu (k + q)_{\rho})(p \cdot n g_{\nu\sigma} - n_\nu (k + q)_{\sigma}) g^{\rho\sigma}.
\]

We obtain after integration of the loop momentum $k$ and renormalization:

\[
\hat{D}_G(z) = D_{Q/G}(z) = \frac{\alpha_s(\mu)}{4\pi} \left( 1 - 2z + 2z^2 \right) \ln \frac{\mu^2}{m_Q^2}. \tag{3.11}
\]
It should be pointed out that the function of heavy quark fragmentation into a heavy quark has also been calculated in [14]. In [14] the fragmentation function has been extracted by calculating cross-sections at $e^+e^-$ collider and the result is formulated in a compact form with the $+$-prescription. With the property of the $+$-prescription one can show that the result in [14] agrees with that in Eq.(3.18).

For the later purpose we give here some moments for heavy quark fragmentation into a heavy quark. The moment is defined as

$$M_Q^{(N)}(\mu) = \int_0^1 dz z^{N-1} D_{Q/Q}(z, \mu).$$  \hspace{1cm} (3.12)

The moments for heavy quark fragmentation into a heavy hadron can be obtained as

$$M_{H/Q}^{(N)}(\mu) = <0|O_H||0> \cdot M_Q^{(N)}(\mu)$$ in our approach. The first two moments reads

$$M_Q^{(1)}(\mu) = 1 + O(\alpha_s^2(\mu)),$$

$$M_Q^{(2)}(\mu) = 1 + \frac{2\alpha_s(\mu)}{3\pi} \left(-\frac{4}{3} \ln \frac{\mu^2}{m_Q^2} - \frac{17}{9}\right) + O(\alpha_s^2(\mu)).$$  \hspace{1cm} (3.13)

These results will be used in the next section.

4. Comparison with experiment at $e^+e^-$ Collider

In this section we will compare our results with experimental results obtained from $e^+e^-$ collider. Before confronting to experimental results, we would like to make two comments:

1). If the method for factorization used here works for heavy quark fragmentation, it can also be applied directly to single heavy hadron production without concept of fragmentation. That means that one can expand the inclusive cross-section for production in term of $m_Q^{-1}$, where the same $\Gamma(q, P)$ in Eq.(2.3) appears at leading order. In this work we are unable to carry out such program. We will still use the results from QCD factorization theorem and neglect higher twist effect. In some cases the theoretical analysis will be much simple if one uses the results from QCD factorization theorem and fragmentation functions.

2). It is confused in the literature about formulation in terms of moments of heavy quark fragmentation function for the statement that heavy hadron carries the most momentum of the heavy quark. One encounters such formulation for the statement

$$M_{H/Q}^{(2)} = \int_0^1 dz z D_{H/Q}(z) = 1 - O(\frac{\Lambda_{QCD}}{m_Q}).$$  \hspace{1cm} (4.1)
It is easy to see that the formulation is wrong. For the second moment one can exactly show that:

$$\sum_H M^{(2)}_{H/Q} = 1. \tag{4.2}$$

With the formulation in Eq.(4.1) the sum-rule in Eq.(4.2) cannot be held. Therefore the formulation is wrong. The correct formulation for the statement is

$$\frac{M^{(2)}_{H/Q}}{M^{(1)}_{H/Q}} = 1 - O(\frac{\Lambda_{QCD}}{m_Q}). \tag{4.3}$$

In our approach the correction terms in Eq.(4.3) begin at order of $m_Q^{-2}$, and the factor 1 receives also radiative corrections staring at order of $\alpha_s$.

If functions for parton fragmentation into a heavy hadron is known or extracted from experiment, one can calculate the moments of the functions which is usually at large energy-scale. On the other hand one can calculate the moments of the functions using theoretical results as those in last sections, where one should take the energy-scale $\mu$ to be $m_Q$ to avoid large logarithmic contribution, and then use the evolution equation to predict the moments at the large energy-scale for comparison with the experimental results. Unfortunately, unlike parton distributions, the functions are not known well experimentally. However, for a comparison one can calculate directly with the theoretical predictions of the functions some physical observables, which are well measured in experiment. For this purpose we consider the inclusive process

$$e^+ + e^- \rightarrow H + X. \tag{4.4}$$

Denoting the beam energy as $E_{beam}$ the variable $x_H$ referring to the hadron $H$ is defined as:

$$x_H = \frac{E_H}{E_{beam}} = \frac{2E_H}{\sqrt{s}} \tag{4.5}$$

where $E_H$ is the energy carried by the hadron $H$. With the QCD factorization theorem the differential cross section can be written:

$$\frac{d\sigma(e^+ + e^- \rightarrow H + X)}{dx_H} = \sum_a \int_{x_H}^1 \frac{dz}{z} h_a(\frac{x_H}{z}, \mu) D_{H/a}(z, \mu) \tag{4.5}$$

where $a$ stands for all possible partons. In Eq.(4.5) the function $h_a$ is perturbative part and is known upto one-loop level in $\overline{MS}$-scheme[15]. We will work at one-loop level and hence we take only the contribution from the parton fragmentation where the parton $a$ is
the heavy quark \( Q \). For the fragmentation function we take the results obtained in the last section. The function \( h_Q \) is

\[
h_Q(y, \mu) = \sigma_Q(s) \left\{ \frac{3}{3\pi} \left\{ \frac{1}{(1-y)^+} + \frac{3}{2} \delta(1-y) \right\} \ln \frac{s}{\mu^2} + \frac{3}{2} \right\}
\]

where \( \sigma_Q(s) \) is the total cross-section for \( e^+ e^- \rightarrow Q + \bar{Q} \) at leading order of coupling constants in the standard model. The expectation value of an \( y \) observable \( O(x_H) \) as a function of \( x_H \) can be now calculated as

\[
< O(x_H) > = \int_0^1 dx O(x) \int_{x_H}^1 \frac{dz}{z} D_{H/Q}(z, \mu) h_Q(x_H/z, \mu) \int_0^1 dx \int_{x_H}^1 \frac{dz}{z} D_{H/Q}(z, \mu) h_Q(x_H/z, \mu) \]

If one takes the heavy quark fragmentation function in last sections and neglect the effect at order of \( m_Q^2 \), an interesting consequence is that the measured value of observable \( O(x_H) \) does not depend on the type of heavy hadrons. It should be noted that in experiment one can also measure \( < O(x_Q) > \) by averaging the mean \( < O(x_H) > \) for various hadrons \( H \). The mean \( < O(x_H) > \) is predicted by replacing \( H \) with \( Q \) in Eq.(4.7). In our approach we have:

\[
< O(x_Q) > = < O(x_H) > + O\left( \frac{1}{m_Q^2} \right) = < O(x_{H'}) > + O\left( \frac{1}{m_Q^2} \right) = \cdots \quad (4.8)
\]

In experiment, the well studied observable is \( O(x_H) = x_H \). Recent measurement at \( \sqrt{s} = M_Z \) by ALEPH[16,17] gives:

\[
< x_b > = 0.715 \pm 0.020, \quad < x_{H_b} > = 0.696 \pm 0.016 \quad (4.9)
\]

where \( H_b \) is the observed hadron in the process (4.4) and it can only be \( B^0 \) or \( B^+ \). These results give certain support for Eq.(4.8). A re-analysis of ARGUS data at \( \sqrt{s} = 10.6 \)GeV in [18] also shows:

\[
< x_D > \approx < x_{D^0} > \approx < x_{\Lambda_c} > \quad (4.10)
\]

at \( \sqrt{s} = 10.6 \)GeV. It is interesting to check whether Eq.(4.8) holds for othe type of observables in experiment or not. In the following we will concentrate on \( < x_Q > \). There are two
of ways to predict $< x_Q >$. One way is only to take the perturbative results in Eq.(4.6) and in Eq.(3.13) to calculate $< x_Q >$ in Eq.(4.7). We obtain

$$
< x_Q > = < x_H > = \frac{M_Q^{(2)}(\mu) \int_0^1 dy h_Q(y, \mu)}{M_Q^{(1)}(\mu) \int_0^1 dy h_Q(y, \mu)}
$$
$$
= 1 + \frac{\alpha_s(\mu)}{2\pi} \left\{ -\frac{16}{9} \ln \frac{s}{m_Q^2} + \frac{88}{27} \right\} + O(\alpha_s^2).
$$

In Eq.(4.9) there is a large logarithmic contribution. This and those at higher orders can make the perturbative series unreliable if one only take first two or three orders to make numerical predictions. Such logarithmic contributions can however be summed with renormalization group equations. In our case we can take $\mu = \sqrt{s}$ in Eq.(4.7) so that the large logarithmic $\ln \frac{s}{\mu^2}$ in $h_Q$ disappears. For $M_Q^{(N)}(s)$ in Eq.(4.11) we first calculate them with the result in the last section at energy-scale $\mu = m_Q$ and then use renormalization group equation to obtain $M_Q^{(N)}(s)$. Here one comment is in order. Since we have already one-loop results, one can use renormalization group equations for the moments at two-loop level to sum not only the leading log contributions but also next-to-leading log contributions. But the equations at two-loop level are unknown. The corresponding equations for parton distributions are known at two-loop level. At one-loop level there is a simple relation between these two sets of equations. At two-loop level this relation is not proven to be hold. Therefore we take only the renormalization group equation at one-loop to sum the leading log contributions and tree-level results for $M_Q^{(N)}(m_Q)$, i.e., we take the leading log approximation. The renormalization group equation for $M_Q^{(N)}(s)$ reads:

$$
\mu \frac{dM_Q^{(N)}(\mu)}{d\mu} = \frac{\alpha_s(\mu)}{2\pi} \gamma_{QQ}^{(N)} M_Q^{(N)}(\mu) + \cdots.
$$

In Eq.(4.10) $\gamma_{QQ}^{(N)}$ can be expanded in $\alpha_s(\mu)$ and only the leading term is known. The leading term for $\gamma_{QQ}^{(1)}$ and $\gamma_{QQ}^{(2)}$ can also be read from Eq.(3.13). We take only the leading term. The $\cdots$ stands for the contribution from the moments of gluon fragmentation function because of operator mixing. Including it the next-to-leading log contributions will be summed. In our approach at one-loop level this term should be neglected for consistence. With these in mind we obtain

$$
< x_Q > = < x_H > = \left( \frac{\alpha_s(m_Q)}{\alpha_s(\sqrt{s})} \right)^{-\frac{16}{9}} (1 + O(\alpha_s))
$$

where we used one-loop $\beta$-function for $\alpha_s$ and 5 as flavor number. The terms neglected at order of $\alpha_s$ are those terms: a). The terms at order $\alpha_s$ in $h_Q(y, \mu = \sqrt{s})$, b). The
terms in $M_Q^{(2)}(\mu = m_Q)$ at order $\alpha_s$ and c). The next-to-leading term in $\gamma^{(N)}_{QQ}$, two-loop effect in $\beta$-function and effect from the moments of the gluon fragmentation function in Eq.(4.12). To make numerical predictions from Eq.(4.13) and Eq.(4.11) we use two-loop $\beta$-function for determining $\alpha_s(\mu)$ at different scales. For $\mu \geq m_b$ we take 5 as flavor number and $\Lambda^{(5)} = 200$MeV. With these numbers we obtain $\alpha_s(M_Z) = 0.116$ which is close to the experimental value measured at $\mu = M_Z$. For $\mu < m_b$ we take 4 as flavor number and $\Lambda^{(4)} = 400$MeV. With these we obtain $\alpha_s(m_\tau) = 0.368$ which is also close the experimental value at $\mu = m_\tau$ where $m_\tau$ is the mass of $\tau$-lepton. Our input parameters for pole masses of heavy quark and for running $\alpha_s$ are:

\[
m_b = 5.0\text{GeV}, \quad m_c = 1.6\text{GeV}, \quad \Lambda^{(5)} = 200\text{MeV}, \quad \Lambda^{(4)} = 400\text{MeV}. \tag{4.14}
\]

In experiment there are also data for $< x_Q >$ measured at $\sqrt{s} = 29$GeV. Several groups have measured $< x_b >$ and $< x_c >$, where b- and c- quark were identified with their inclusive lepton-decays or with charged multiplicity measurements. The results from different groups and from different methods are summarized in [19]. We average these results from different groups and from different methods by meaning of unconstrained averaging as described in [20]. It should be pointed out that these groups have not only measured $< x_Q >$ but also tried to reconstruct the variable $z$ and obtained $< z_Q >$. However, such reconstruction relied of Monte-Carlo models for fragmentation. We will only make comparison with $< x_Q >$. The experimental values which we will compare with our predictions are:

\[
\begin{align*}
< x_b > &= 0.715 \pm 0.020, \quad < x_c > = 0.508 \pm 0.011, \quad \text{for } \sqrt{s} = M_Z, \\
< x_b > &= 0.754 \pm 0.034, \quad < x_c > = 0.585 \pm 0.036, \quad \text{for } \sqrt{s} = 29\text{GeV}, \\
< x_c > &= 0.640 \pm 0.009, \quad \text{for } \sqrt{s} = 10.6\text{GeV},
\end{align*} \tag{4.15}
\]

where the value for $< x_c >$ at $\sqrt{s} = 10.6$GeV is from the re-analysis of ARGUS data in [18]. The value for $< x_c >$ at $\sqrt{s} = M_Z$ is obtained from [21,22] where actually the values for $< x_{D^*} >$ are measured, we average them and take this value as $< x_c >$ according to Eq.(4.8).

With the input parameters in Eq.(4.14) we obtain from the perturbative result in Eq.(4.11) the following numbers:

\[
\begin{align*}
< x_b > &= 0.867, \quad < x_c > = 0.791, \quad \text{for } \sqrt{s} = M_Z, \\
< x_b > &= 0.933, \quad < x_c > = 0.843, \quad \text{for } \sqrt{s} = 29\text{GeV}, \\
< x_c > &= 0.906, \quad \text{for } \sqrt{s} = 10.6\text{GeV},
\end{align*} \tag{4.16}
\]

where $\mu$ in Eq.(4.11) was taken to be $\sqrt{s}$. Comparing with experimental values there are large deviations. The reason is probably because large corrections from higher orders in $\alpha_s$.
in which large logarithmic contributions exist. Therefore, the predictions above may not be reliable. With Eq.(4.13) where the leading log contributions are summed we obtain:

\[
\begin{align*}
<x_b> &= 0.772, \quad <x_c> = 0.563, \quad \text{for } \sqrt{s} = M_Z,\\
<x_b> &= 0.836, \quad <x_c> = 0.615, \quad \text{for } \sqrt{s} = 29\text{GeV},\\
<x_c> &= 0.674, \quad \text{for } \sqrt{s} = 10.6\text{GeV},
\end{align*}
\]

Comparing experimental values for \(<x_b>\) the deviation from the values given above is 8% at \(\sqrt{s} = M_Z\) and 10% at \(\sqrt{s} = 29\text{GeV}\). For the case with c-quark the deviation is 10% at \(\sqrt{s} = M_Z\) and is about 5% at other energy scales. Our predictions here are fairly in good agreement with experiment. Our predicted values are all larger than experimental values. The sources for the deviations mentioned above can be various. Higher twist effect neglected in Eq.(4.5) and in Eq.(4.7) can be one of them. An important source is the higher order correction in \(\alpha_s\). In Eq.(4.13) we neglected this correction. It should be noted that the correction in Eq.(4.13) has the form \(a\alpha_s(\sqrt{s}) + b\alpha_s(m_Q)\). Because \(\alpha_s(m_Q)\) is rather large, especially for c-quark, this correction can be large. If we add the corrections from a) and b) discussed after Eq.(4.13), the deviation at \(\sqrt{s} = M_Z\) is reduced to 5% for c-quark and to 4% for b-quark. At other energy scales the reduction is not so significant as that at \(\sqrt{s} = M_Z\), because this correction becomes smaller as \(\sqrt{s}\) decreases. Another possible source is the running \(\alpha_s\) at different energy scale, especially, the value of \(\alpha_s\) at lower energy scales, and also possible nonperturbative effect appearing at these scales for running \(\alpha_s(\mu)\). However, a detail study is needed here.

The last question we will study here is how large is the matrix element \(<0|O_H|0>\) defined in Eq.(2.9) for a given hadron. This matrix element should be calculated with nonperturbative methods, e.g., with lattice QCD. It can also be extracted from experimental results. It should be noted that this matrix element is universal, i.e., it does not depend on a specific process. We will use experimental data obtained in Z-decays to extract it for D mesons. However, information from experiment is not enough for estimating these matrix elements uniquely, certain assumptions must be made. For the estimation we do not use the concept of fragmentation. For the inclusive decay

\[
Z \rightarrow H_c + X
\]

where \(H_c\) stands for \(D^0, D^{*0}, D^+\) and \(D^{*+}\). One can write the branching ratio as

\[
\text{Br}(Z \rightarrow H_c + X) = \frac{\Gamma_{c\bar{c}}}{\Gamma_Z} \cdot P(c \rightarrow H_c) + \frac{\Gamma_{b\bar{b}}}{\Gamma_Z} \cdot P(b \rightarrow H_c) + R_{\text{in}}
\]

where we neglected the process of the gluon splitting into \(c\bar{c}\). The term \(R_{\text{in}}\) is the contribution from excited states of \(H_c\) which are first produced and then decay into \(H_c\) inclusively.
With the method here for the factorization the probability \( P(c \to H_c) \) is just the matrix element:

\[
P(c \to H_c) = < 0|O_{H_c}|0 >. \tag{4.20}\]

The probability \( P(b \to H_c) \) can be expected to be the same as \( P(c \to H_c) \) if we consider that the b-quark decays first through weak interaction into a c-quark and then the c-quark is transmitted into the hadron \( H_c \). For \( D^{*+} \) the ratio of these two probabilities is extracted in experiment which is close to 1\(^21\):

\[
\frac{P(c \to D^{*})}{P(b \to D^{*})} = 1.03 \pm 0.21. \tag{4.21}\]

We assume that this ratio is one for all \( H_c \).

For the excited state \( D^{*} \) we neglect the contribution from \( R_{\text{in}} \), so we have:

\[
\text{Br}(Z \to D^{*} + X) = \frac{\Gamma_{\bar{c}c} + \Gamma_{\bar{b}b}}{\Gamma_Z} \cdot < 0|O_{D^{*}}|0 >. \tag{4.22}\]

There is no information for \( \text{Br}(Z \to D^{*0} + X) \). The matrix element \( < 0|O_{D^{*0}}|0 > \) can not be determined with Eq.(4.22). If we assume isospin symmetry for light quarks, the matrix element is same as \( < 0|O_{D^{*+}}|0 > \). With the experimental value for \( \text{Br}(Z \to D^{*+} + X) \) in [20] and isospin symmetry we have:

\[
< 0|O_{D^{*0}}|0 >=< 0|O_{D^{*+}}|0 > \approx 0.22. \tag{4.23}\]

For \( D^{+} \) we take only the decay \( D^{*+} \to D^{+} + X \) into account for \( R_{\text{in}} \), according to [20] the decay has a chance of 31.7\%. With that we have

\[
\text{Br}(Z \to D^{+} + X) = \frac{\Gamma_{\bar{c}c} + \Gamma_{\bar{b}b}}{\Gamma_Z} \cdot \{ < 0|O_{D^{+}}|0 > + 31.7\% < 0|O_{D^{*+}}|0 > \}. \tag{4.24}\]

Taking experimental value we obtain

\[
< 0|O_{D^{+}}|0 > \approx 0.39. \tag{4.25}\]

For \( D^{0} \) we take only the contribution for \( R_{\text{in}} \) from the two decays \( D^{*0}, \ D^{*+} \to D^{0} + X \). The branching ratio for these two decays is 100\% and 68.3\% respectively[20]. With these the branching ration \( \text{Br}(Z \to D^{0} + X) \) can be written:

\[
\text{Br}(Z \to D^{0} + X) = \frac{\Gamma_{\bar{c}c} + \Gamma_{\bar{b}b}}{\Gamma_Z} \cdot \{ < 0|O_{D^{0}}|0 > + < 0|O_{D^{*0}}|0 > + 68.3\% < 0|O_{D^{*+}}|0 > \}. \tag{4.26}\]
It is interesting to note that with isospin symmetry as assumed before the matrix element with $D^0$ is the same as that in Eq.(4.25) and this branching ratio is predictable with Eq.(4.26). We obtain
\[ \text{Br}(Z \rightarrow D^0 + X) \approx 20.1\% \quad (4.27) \]
which is close to the experimental value $20.7 \pm 2.0\% [20]$, or using the experimental value we obtain
\[ < 0 | O_{D^0} | 0 > \approx 0.40 \quad (4.28) \]
which is in consistence with the assumed isospin symmetry.

Naively one would expect the matrix element $< 0 | O_D | 0 >$ to be 3 times of $< 0 | O_{D^*} | 0 >$ because $D$ and $D^*$ are spin-0 and spin-1 particles respectively. From our estimation above this relation is not hold. The reason for this is that the spin counting can not be applied here because the matrix element $< 0 | O_{H_c} | 0 >$ is the probability for the inclusive transition $c \rightarrow H_c + X$, where the unobserved state $X$ can not be vacuum or a given state and can have any possible orbital angular momentum. Further, the unobserved state $X$ for $D$ can be different than that for $D^*$.

There is not data available for b-flavored hadrons, so their matrix elements can not be estimated as we did for $< 0 | O_{H_c} | 0 >$. However the difference between $< 0 | O_{H_c} | 0 >$ and $< 0 | O_{H_b} | 0 >$, where $H_b$ stands for $B$ or $B^*$ mesons, is at order of $m_c^{-1}$ and of $m_b^{-1}$. Therefore one can take the value of $< 0 | O_{H_c} | 0 >$ for the corresponding $< 0 | O_{H_b} | 0 >$ as an approximation.

5. Summary

In this work we studied parton fragmentation into a heavy hadron. We factorized the process into a perturbative part and a nonperturbative part. The perturbative part is just the parton fragmentation function into a heavy quark at the order we consider. The nonperturbative part is a matrix element defined in HQET, which is universal. The $z$-dependence of fragmentation functions is predicted purely by perturbative theory. In this work we predicted this dependence for heavy quark- and gluon- fragmentation at one-loop level in QCD. With these results we calculated the mean value of the ratio between the energy carried by a heavy hadron or a heavy quark and the beam energy at $e^+e^-$ colliders. Comparing experiment we find that there is a deviation at 10% level between our predictions with the leading log approximation and experimental values. The sources for this deviation can be several, the important source may be higher order effect in $\alpha_s$ as discussed in the last
section. The value of the matrix element for \( D \) and \( D^* \) are determined with experimental data from \( Z \)-decays and one of the branching ratios for \( Z \rightarrow H_c + X \) can be predicted in our approach.

It should be pointed out that our procedure for factorization may directly be applied to heavy hadron production with the concept of fragmentation. In this work we do not carry out such program for specific process and leave it for future work.

**Note added:**

After the work is finished, the author is informed by Prof. O. Biebel of OPAL group about recent measurement of the total branching of \( c \rightarrow D^* \). The latest preliminary value of the measured branching \( f(c \rightarrow D^* + X) \) is \( 0.221 \pm 0.014 \pm 0.013 \)[23]. In the approach of our work the branching \( f(c \rightarrow D^* + X) \) is just the matrix element \( \langle 0 | O_{D^*} | 0 \rangle \). The value obtained in this work in Eq.(4.23) is close to the value measured by OPAL.

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**Figure Caption**

**Fig.1:** The Feynman diagram for contribution to heavy quark fragmentation at tree-level. The vertical broken line is the Cutkosky cut, the double line presents the line operator in Eq.(2.1).

**Fig.2A–2D:** The Feynman diagrams for one-loop contributions to heavy quark fragmentation into a heavy quark.

**Fig.3:** The Feynman diagram for contribution to gluon fragmentation into a heavy hadron.
Fig. 1
