HIGHER DU BOIS AND HIGHER RATIONAL SINGULARITIES

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Abstract. We prove that the higher direct images $R^q f_* \Omega^p_{Y/S}$ of the sheaves of relative Kähler differentials are locally free and compatible with arbitrary base change for flat proper families whose fibers have $k$-Du Bois local complete intersection singularities, for $p \leq k$ and all $q \geq 0$, generalizing a result of Du Bois (the case $k = 0$). We then propose a definition of $k$-rational singularities extending the definition of rational singularities, and show that, if $X$ is a $k$-rational variety with either isolated or local complete intersection singularities, then $X$ is $k$-Du Bois. As applications, we discuss the behavior of Hodge numbers in families and the unobstructedness of deformations of singular Calabi-Yau varieties.

In an appendix, Morihiko Saito proves that, in the case of hypersurface singularities, the $k$-rationality definition proposed here is equivalent to a previously given numerical definition for $k$-rational singularities. As an immediate consequence, it follows that for hypersurface singularities, $k$-Du Bois singularities are $(k-1)$-rational. Independently, we have proved that the latter statement also holds for isolated local complete intersection singularities, and conjecture that it holds more generally for all local complete intersection singularities.

1. Introduction

The Hodge numbers are constant in a smooth family of complex projective varieties over a connected base. A powerful way of encoding this fundamental fact is Deligne’s theorem [Del68]: If $f : Y \to S$ is a smooth morphism of complex projective varieties, then the higher direct image sheaves $R^q f_* \Omega^p_{Y/S}$ of the relative Kähler differentials are locally free and compatible with base change. This theorem fails for families of varieties which have singular fibers (and in positive characteristic). For $Y$ a singular compact complex algebraic variety, Du Bois [DB81] showed that the Kähler-de Rham complex $\Omega^\cdot_Y$ should be replaced by the filtered de Rham or Deligne-Du Bois complex $\Omega^\cdot_Y$ whose graded pieces $\Omega^p_Y \in D^{b}_{coh}(Y)$ play the role of $\Omega^p_Y$ (see [PS08, §7.3]). Namely, the associated spectral sequence with $E_1^{p,q} = H^q(Y; \Omega^p_Y)$ degenerates at $E_1$ and computes $H^{p+q}(Y, \mathbb{C})$ together with the Hodge filtration associated to the mixed Hodge structure on $H^\cdot(Y)$. However, the associated Hodge numbers $h^{p,q} := \dim H^q(Y; \Omega^p_Y)$ do not behave well in families in general.

The filtered de Rham complex is related to the complex of Kähler differentials via the canonical comparison Kähler-to-Du Bois map $\phi^p : \Omega^p_Y \to \Omega^p_Y$. The maps $\phi^p$ are isomorphisms for all $p$ only when $Y$ is smooth. It is thus natural to consider the case when $\phi^p$ is a quasi-isomorphism in a certain range. Steenbrink [St83, §3] introduced the notion of Du Bois singularities, which play a role in the study of compactifications of moduli. By definition, $Y$ is Du Bois if $\phi^0$ is a quasi-isomorphism. Following [MOPW21] and [JJSY22a], we say that $Y$ is $k$-Du Bois if $\phi^p$ is a quasi-isomorphism for $0 \leq p \leq k$. Thus 0-Du Bois singularities are exactly the Du Bois singularities in Steenbrink’s terminology. A key property satisfied by Du Bois singularities is the following:

**Theorem 1.1** (Du Bois [DB81, Thm. 4.6], [DBJ74, Lem. 1]). Let $f : Y \to S$ be a flat proper family of complex algebraic varieties. Assume that some fiber $Y_s$ has Du Bois singularities. Then, possibly after replacing $S$ by a neighborhood of $s$, for all $q \geq 0$, the sheaves $R^q f_* \mathcal{O}_Y$ are locally free of finite type and compatible with base change.
The theorem can be interpreted (in particular) as giving a relation between the mixed Hodge structure $H^*(Y_0)$ of a singular fiber $Y_0$ with the limit mixed Hodge $H^*_{\text{lim}}$ associated to a one-parameter smoothing $X/\Delta$. In this version, the theorem (for dimension 2 slc hypersurface singularities) was independently established by Shah [Sha79], and plays a key role in the study of degenerations of K3 surfaces (e.g. [Sha80]) and related objects (e.g. [Laz10], [KLSV18]). Theorem 1.1 continues to have important consequences for the study of compact moduli of varieties of general type (see e.g. [Ko21], esp. §2.5 of loc. cit.).

Here, we prove the following generalization of Theorem 1.1 for the case of a local complete intersection (lici) morphism:

**Theorem 1.2.** Let $f : Y \to S$ be a flat proper family of complex algebraic varieties and let $s \in S$. Suppose that the fiber $Y_s$ has $k$-Du Bois lci singularities. Then, possibly after replacing $S$ by a neighborhood of $s$, the higher direct image sheaves $R^qf_*\Omega^p_{Y/S}$ of the relative Kähler differentials are locally free and compatible with base change for $0 \leq p \leq k$ and all $q \geq 0$.

**Remark 1.3.** We use the lci assumption to control the sheaves of Kähler and relative Kähler differentials in two ways. First, a result of [MP21] gives an estimate on the codimension of the singular locus for $k$-Du Bois lci singularities. Using this and some results of Greuel, we prove a key technical point: under the lci assumption, the sheaves $\Omega^p_{X/S}$ are flat over $S$ for $p \leq k$ (Theorem 2.4). In case $k = 0$, both the codimension estimate and the flatness are automatic and one recovers Theorem 1.2 as a special case.

As a corollary, we obtain:

**Corollary 1.4.** Let $f : Y \to S$ be a flat proper family of complex algebraic varieties over an irreducible base. For $s \in S$, suppose that the fiber $Y_s$ has $k$-Du Bois lci singularities. Then, for every fiber $t$ such that $Y_t$ is smooth, $\dim \operatorname{Gr}^p_k H^{p+q}(Y_t) = \dim \operatorname{Gr}^p_k H^{p+q}(Y_s)$ for every $q$ and for $0 \leq p \leq k$. Equivalently $h^{p,q}(Y_s) = h^{p,q}(Y_t)$ for all $p \leq k$. \hfill \Box

In the case of hypersurface singularities, results similar to Corollary 1.4 were obtained by Kerr-Laza [KL20] with further clarifications given by Saito (personal communications) based on [Sai16]. These type of results are for example relevant to the study of the moduli of cubic fourfolds [Laz10].

Another application of Theorem 1.2 is the following generalization of the results of Kawamata [Kaw92], Ran [Ran92], and Tian [Tia92] on the unobstructedness of deformations for nodal Calabi-Yau varieties in any dimension, where the special case of isolated hypersurface singularities was established in §6 of the first version of [FL22a]:

**Corollary 1.5.** Let $Y$ be a canonical Calabi-Yau variety (Definition 4.4) which is additionally a scheme with 1-Du Bois lci (not necessarily isolated) singularities. Then the functor $\operatorname{Def}(Y)$ is unobstructed.

A more well-known class of singularities are the rational singularities. By work of Steenbrink [Ste83], Kovács [Kov99], and others, a rational singularity is Du Bois. In the context of higher Du Bois singularities, it is natural to consider higher rational singularities. In the case of hypersurfaces $X$, the $k$-Du Bois singularities are characterized numerically by the condition $\alpha_X \geq k + 1$ ([MOPW21 Thm. 1.1], [JKSY22a Thm. 1]), where $\alpha_X$ is the minimal exponent invariant, a generalization of the log canonical threshold. Given that $\alpha_X > 1$ characterizes rational hypersurface singularities (Sai93), it is natural to define $k$-rational singularities numerically by the strict inequality $\alpha_X > k + 1$ ([KL20 §4]); this definition also occurs implicitly in [MOPW21] and [JKSY22a]. While results can be obtained using this numerical definition, it has the disadvantage of being restricted to hypersurfaces, and to be somewhat ad hoc. A more general definition of $k$-rational isolated singularities, based on local vanishing properties (see esp. [MP20]), was given in §3 of the first version of [FL22a] (see also [MP22] for further discussion).
In this paper, we propose a more intrinsic definition of \( k \)-rational singularities in general (Definition 3.12) and show that it agrees with the usual definition of rational singularities for \( k = 0 \) and with the definition of [FL22a, §3] (under mild assumptions; see Corollaries 3.15 and 3.17). Additionally, for hypersurface singularities, M. Saito proves that the new definition proposed here is indeed equivalent to the previous numerical definition mentioned above (Theorem A.1). The main advantage of the definition of higher rational singularities given here is that it naturally factors through the higher Du Bois condition. In analogy with the case \( k = 0 \), we conjecture that \( k \)-rational implies \( k \)-Du Bois in general. In [FL22a, §3] (and expanded in [FL22b]), we verified this conjecture under the assumption of isolated lci singularities. Here, we generalize this in both directions, for arbitrary isolated or lci singularities:

**Theorem 1.6.** Let \( X \) have either lci singularities (not necessarily isolated) or isolated singularities (not necessarily lci). If \( X \) is \( k \)-rational, then \( X \) is \( k \)-Du Bois.

**Remark 1.7.** Mustată and Popa gave an independent proof of Theorem 1.6 for the case of lci singularities ([MP22, Thm. B]).

The isolated complete intersection case (see [FL22b]) sheds light on the tight relationship between higher rational and higher Du Bois singularities. In particular, we note the following conjecture:

**Conjecture 1.8.** If \( X \) has lci singularities and \( X \) is \( k \)-Du Bois, then \( X \) is \( (k-1) \)-rational.

We proved this conjecture for an isolated hypersurface singularity in the first version of [FL22a, §3]. It is an immediate consequence of the following result:

**Proposition 1.9.** Let \((X, x)\) be an isolated hypersurface singularity and let \( \tilde{\alpha}_{X,x} = \tilde{\alpha}_X \) be the minimal exponent as defined by Saito [Sai93]. Then

(i) \( X \) is \( k \)-Du Bois \iff \( \tilde{\alpha}_X \geq k + 1 \).

(ii) \( X \) is \( k \)-rational \iff \( \tilde{\alpha}_X > k + 1 \). □

Here (i) follows from [JKSY22a, Thm. 1] and [MOPW21, Thm. 1.1.] and holds true for a general, not necessarily isolated, hypersurface singularity, and (ii) is proved in [FL22a, FL22b]. In Appendix A, M. Saito proves (ii) for the case of a general hypersurface singularity, based on the results of [JKSY22a]. Thus Conjecture 1.8 holds also for the case of general hypersurface singularities, not necessarily isolated (Cor. A.2). Mustată and Popa have proved (ii) of Proposition 1.9 and hence of Conjecture 1.8 in this case as well (cf. [MP22, Thm. E, Cor. F]). Additionally, in [FL22b], we establish Conjecture 1.8 for isolated lci singularities:

**Theorem 1.10.** If \( X \) has isolated lci singularities and \( X \) is \( k \)-Du Bois, then \( X \) is \( (k-1) \)-rational. □

Since \( k \)-rational singularities are milder than \( k \)-Du Bois singularities, one expects that more of the Hodge diamond is preserved in families with \( k \)-rational singularities. Indeed, this is the case as shown in [KL20, Cor. 4.2] (isolated hypersurface \( k \)-rational singularities) and [KLS22, Thm. 1] (arbitrary rational singularities). Here, we extend these results to \( k \)-rational lci singularities, and clarify the difference between \( k \)-Du Bois and \( k \)-rational in this context. Essentially, for \( k \)-rational singularities, in addition to the preservation given by Corollary 1.4, one gains Hodge symmetry in a certain range. Informally, we can say that the frontier Hodge diamond up to coniveau \( k \) is preserved for deformations of \( k \)-rational singularities.

**Corollary 1.11.** Let \( f : Y \to S \) be a flat proper family of complex algebraic varieties over an irreducible base. For \( s \in S \), suppose that the fiber \( Y_s \) has \( k \)-rational lci singularities. Then, for every fiber \( t \) such that \( Y_t \) is smooth, and for all \( p \leq k \),

\[
h^{p,q}(Y_s) = h^{q,p}(Y_s) = h^{n-p,n-q}(Y_s) = h^{p,q}(Y_t) = h^{q,p}(Y_t) = h^{n-p,n-q}(Y_t).
\]
Moreover, for all \( p \leq k \),
\[
\check{h}^{p,q}(Y_s) = \dim \text{Gr}^p_F \text{Im}((H^{p+q}(Y_s) \xrightarrow{\pi_s^*} H^{p+q}(\hat{Y}_s)),
\]
where \( \pi_s : \hat{Y}_s \to Y_s \) is an arbitrary projective resolution.

Remark 1.12. (1) The Du Bois complex for a (connected) curve \( C \) was computed in [DB81 Prop. 4.9]: let \( \bar{C} \) be the normalization of \( C \) and \( C^w \) the weak normalization, so that there is a factorization \( \bar{C} \to C^w \to C \). Then \( \check{h}^{1,0}(C) = p_0(C) = g(\bar{C}) \), while \( \check{h}^{0,1}(C) = p_a(C^w) \). Thus no Hodge number, except \( h^{0,0} = 1 \) is always preserved in flat proper families of curves. Note that \( C \) has Du Bois singularities \( \iff C^w \cong C \) and \( C \) has rational singularities \( \iff C \) is smooth. However, at least one of the Hodge symmetries \( \check{h}^{1,0} \neq \check{h}^{0,1} \) will fail for \( C \) Du Bois (depending on \( C \) of compact/non-compact type), reflecting the fact that there are no rational singularities in dimension 1.

(2) Fourfolds \( Y \) with ADE hypersurface singularities (such as those occurring in [Laz10]) are examples of 1-rational singularities. For such fourfolds, Corollary 1.11 gives that only \( h^{2,2}(Y) \) can vary in small deformations. Thus, any smoothing of \( Y \) will have finite monodromy (compare [KLS22 Cor. 1]).

A brief description of the contents of this paper is as follows. Section 2 deals with some basic results about Kähler differentials in the lci case. These include Theorem 2.4 regarding the flatness of the relative Kähler differentials and Proposition 2.6 on restricting to a generic hypersurface section. In Section 3, we give a quick review of the definition and the basic facts about higher Du Bois singularities (following [MOPW21], [JKSY22a], and [FL22a §3]) and define higher rational singularities. After these preliminaries, we establish Theorem 1.2. Our argument is close to the original argument ([DBJ74 Lemma 1]) used to establish Theorem 1.1 following a suggestion of J. Kollár. Finally, in Section 5, we prove Theorem 1.6 following the strategy of [Kov99], and deduce a consequence about the Hodge numbers of a smoothing along the lines of Corollary 1.4. An appendix section by M. Saito discusses Conjecture 1.8 in the hypersurface case.

Finally, beyond the conjectures and speculation we have already made, we emphasize the importance of extending these results wherever possible to the non-lci case.

Notations and Conventions. We work in the complex algebraic category (but see Remark 1.3 for some possible generalizations). Following our conventions from previous work [FL22a], we use \( X \) (and \( X' \)) in the local context (e.g. statements such as Theorem 2.4 are purely local, and hold in the analytic case as well), whereas \( Y \) and \( \pi : Y \to S \) are meant to be proper over \( \text{Spec } \mathbb{C} \) or \( S \) respectively (e.g. in Theorem 1.2 where the properness is essential). The scheme or analytic space \( X \) is an algebraic variety if it is reduced and irreducible. In this case, its singular locus \( X_{\text{sing}} \) is denoted by \( \Sigma \). We set \( n = \dim X \) and \( d = \dim \Sigma \).

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2. Some results on Kähler and relative Kähler differentials

The Kähler differentials are coherent sheaves that are determined by certain universal properties, including compatibility with base change. For smooth families \( Y/S \), the proof of the constancy of
the Hodge numbers uses in an essential way the semi-continuity of \( h^0(\Omega^p_{X/S}) \), which in turn depends on the flatness of \( \Omega^p_{X/S} \). Here, we generalize this key point, noting that, for an lci morphism, \( \Omega^p_{X/S} \) is flat over \( S \) (Theorem 2.1) for \( p \) satisfying a bound depending on the dimension \( d \) of the singular locus. Our argument depends essentially on the lci assumption, and it is a consequence of some depth estimates for \( \Omega^p_X \) for \( X \) with lci singularities due to Greuel.

A second result (Proposition 2.6) that follows by related arguments are higher adjunction type results regarding restrictions of Kähler differentials to generic hypersurface sections.

2.1. A flatness result. It is well known that Kähler \( p \)-differentials on singular spaces can have torsion. For instance this is already the case for \( \Omega^1_C \), where \( C \) is a nodal curve. However, we have the following [Gre75, Lemma 1.8]

**Theorem 2.1** (Greuel). Suppose that \( X \) is an lci singularity of dimension \( n \) and that \( \dim \Sigma \leq d \). Then, for all \( x \in X \), depth\( x \Omega^p X \geq n - p \) for \( p \leq n - d \). More generally, let \( f : X \to S \) be a flat lci morphism of relative dimension \( n \) over a smooth base \( S \) and let \( X_{\text{crit}} \) denote the critical locus of \( f \), i.e. the points of \( X \) where \( f \) is not a smooth morphism. If \( \dim S = m \) and the relative dimension of \( X_{\text{crit}} \) is at most \( d \) then, for every \( x \in X \), then depth\( x \Omega^p X/S \geq \dim X - p = n - m - p \) for \( p \leq n - d \).

**Corollary 2.2.** Suppose that \( f : X \to S \) is a flat lci morphism of relative dimension \( n \) over a smooth base \( S \) of dimension \( m \), and that \( \dim (X_t)_{\text{sing}} \leq d \), with \( n - d \geq 2k + 1 \) for some integer \( k \geq 1 \) and every point \( t \in S \). Let \( X_{\text{crit}} \) denote the critical locus of \( f \), i.e. the points of \( X \) where \( f \) is not a smooth morphism. Then, for every \( x \in X \) and \( p \leq n - d \),

\[
\text{depth}_x \Omega^p X/S \geq d + m + 2,
\]

and, for all \( p \leq k \) and every open subset \( U \) of \( X \), the restriction map

\[
H^0(U; \Omega^p X/S|U) \to H^0(U - X_{\text{crit}}; \Omega^p X/S|U - X_{\text{crit}})
\]

is an isomorphism.

**Proof.** By assumption, \( \dim X_{\text{crit}} \leq d + m \). Note that \( n - d \geq 2k + 1 \geq k \), and hence, if \( p \leq k \), then \( p \leq n - d \). Thus Theorem 2.1 implies that, for all \( x \in X \),

\[
\text{depth}_x \Omega^p X/S \geq n + m - p \geq n - k + m \geq d + m + k \geq d + m + 2 \geq \dim X_{\text{crit}} + 2.
\]

Then \( H^0(U; \Omega^p X/S|U) \to H^0(U - X_{\text{crit}}; \Omega^p X/S|U - X_{\text{crit}}) \) is an isomorphism, by a theorem due in the analytic case to Scheja [Sch64, Gre75, Satz 1.2].

**Proposition 2.3.** Suppose that \( f : X \to S \) is a flat lci morphism of relative dimension \( n \) over a smooth base \( S \) of dimension \( m \), and that \( \dim (X_t)_{\text{sing}} \leq d \), with \( n - d \geq 2k + 1 \) for some integer \( k \geq 1 \) and every point \( t \in S \). Suppose that \( X \subseteq A \times S \), where \( A \) is smooth, and let \( I \) be the defining ideal of \( X \) in \( A \times S \). Then there exists a filtration of \( \bigwedge^k \pi_A^* \Omega^1_A|X = \pi_A^* \Omega^k_A|X \) by coherent subsheaves \( \mathcal{K}^a \), \( 0 \leq a \leq k \), such that \( \mathcal{K}^0 = \pi_A^* \Omega^1_A|X \), depth\( x \mathcal{K}^a \geq d + m + 2 \) for all \( x \in X \), and, for all \( a \geq 0 \),

\[
\mathcal{K}^a / \mathcal{K}^{a+1} \cong \bigwedge^a (I/I^2) \otimes \Omega^{k-a}_{X/S}.
\]

**Proof.** The proof is by induction on \( k \). The case \( k = 1 \) is the conormal sequence

\[
0 \to I/I^2 \to \pi_A^* \Omega^1_A|X \to \Omega^1_{X/S} \to 0,
\]

where we let \( \mathcal{K}^1 \) be the image of \( I/I^2 \). In general, for all \( 0 \leq a \leq k \), let \( \mathcal{K}^a \) be the image of \( \bigwedge^a (I/I^2) \otimes \pi_A^* \Omega^{k-a}_{X/S}|_X \) in \( \pi_A^* \Omega^k_A|X \). Thus \( \{\mathcal{K}^a\} \) is the usual Koszul filtration, and the proposition is clear over all points of \( X - X_{\text{crit}} \). Moreover, \( \mathcal{K}^k \) is either 0 or \( \bigwedge^k (I/I^2) \). By hypothesis \( k < n =
rank $\Omega_{X/S}^1$. Let $r = \text{rank } I/I^2$. We can clearly assume that $r \geq 1$. If $r \geq k$, then $\mathcal{K}^k = \bigwedge^r(I/I^2)$. If $r < k$, then $\mathcal{K}^k = 0$, and the largest $p$ such that $\mathcal{K}^p \neq 0$ is $\mathcal{K}^*$, the image of $\bigwedge^r(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k$. In this case, the image of $\bigwedge^r(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k \rightarrow \bigwedge^r(K/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X$ maps to $0$. By the inductive hypothesis, the sequence

$$\bigwedge^r(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k \rightarrow \bigwedge^r(K/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X \rightarrow \bigwedge^r(K/I^2) \otimes \Omega_{K/S}^{k-r}$$

is exact, so there is an induced map $\varphi_r : \bigwedge^r(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X \rightarrow \pi_1^*\Omega_{X/S}^k | X$. Then the image of $\varphi_r$ is contained in the torsion free sheaf $\mathcal{K}^r$ and is equal to $\mathcal{K}^r$. Moreover $\varphi_r$ is injective over $X - X_{\text{crit}}$. By the inductive hypothesis on $k$,

$$\text{depth}_x \bigwedge^r(I/I^2) \otimes \Omega_{X/S}^{k-r} \geq d + m + 2,$$

and hence $\varphi_r = \mathcal{K}^r$, and $\varphi_r : \bigwedge^r(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X \rightarrow \mathcal{K}^r/\mathcal{K}^{r+1}$ is an isomorphism.

For a general $a$, there is a commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \bigwedge^a(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X & \longrightarrow & \bigwedge^a(I/I^2) \otimes \Omega_{X/S}^{k-a} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{K}^{a+1} & \longrightarrow & \mathcal{K}^a & \longrightarrow & \mathcal{K}^a/\mathcal{K}^{a+1} & \longrightarrow & 0,
\end{array}$$

where by definition $\mathcal{J} = \text{Ker} \{ \bigwedge^a(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X \rightarrow \bigwedge^a(I/I^2) \otimes \Omega_{X/S}^{k-a} \}$ and $\varphi_a$, which exists and is an isomorphism over $X - X_{\text{crit}}$, is yet to be constructed over all of $X$.

For a fixed $k$, arguing by descending induction on $a$, and starting either at $k$ or at $r$, we can assume that $\text{depth}_x \mathcal{K}^{a+1} \geq d + m + 2$. Moreover, $\mathcal{K}^{a+1} | X - X_{\text{crit}}$ is a subbundle of $\mathcal{K}^a | X - X_{\text{crit}}$, so the torsion subsheaf of $\mathcal{K}^a/\mathcal{K}^{a+1}$ is supported on $X_{\text{crit}}$. If $\sigma$ is a section of $\mathcal{K}^a$ over some open subset $U$ and $\sigma | V \in \mathcal{K}^{a+1}$ for an open subset $V = U - W$ of $U$, where $W$ is an analytic subset of $U$, then we may assume that $W = X_{\text{crit}}$. Since $\mathcal{K}^a$ is torsion free and $\text{depth}_x \mathcal{K}^{a+1} \geq d + m + 2$, $\sigma \in \mathcal{K}^{a+1}(U)$. Thus $\mathcal{K}^a/\mathcal{K}^{a+1}$ is torsion free.

The kernel $\mathcal{I}$ of $\bigwedge^a(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X \rightarrow \mathcal{K}^a/\mathcal{K}^{a+1}$ is equal to $\mathcal{J} = \text{Ker} \{ \bigwedge^a(I/I^2) \otimes \pi_1^*\Omega_{A_k}^k | X \rightarrow \bigwedge^a(I/I^2) \otimes \Omega_{X/S}^{k-a} \}$ over $X - X_{\text{crit}}$. Moreover, $\text{depth}_x \mathcal{J} \geq \text{depth}_x \bigwedge^a(I/I^2) \otimes \Omega_{X/S}^{k-a} \geq d + m + 2$. Thus $\mathcal{I} \subseteq \mathcal{J}$, and they are equal over $X - X_{\text{crit}}$. Since $\mathcal{K}^a/\mathcal{K}^{a+1}$ is torsion free, they are equal and there is an induced injective homomorphism $\varphi_a$ as in the diagram. Since $\varphi_a$ is an isomorphism over $X - X_{\text{crit}}$ and $\text{depth}_x \bigwedge^a(I/I^2) \otimes \Omega_{X/S}^{k-a} \geq d + m + 2$, $\varphi_a$ is an isomorphism. Finally, since both $\text{depth}_x \mathcal{K}^{a+1} \geq d + m + 2$ and $\text{depth}_x \mathcal{K}^a/\mathcal{K}^{a+1} \geq d + m + 2$, $\text{depth}_x \mathcal{K}^a \geq d + m + 2$ as well.

A key technical step in establishing Theorem 1.2, where the lci assumption seems crucial, is the following:

**Theorem 2.4.** Suppose that $f : X \rightarrow S$ is a flat lci morphism, where $S$ is an arbitrary base space. Let $s \in S$ and suppose that $\text{codim}_{X_s}(X_s)^{\text{sing}} \geq 2k + 1$, where $X_s = f^{-1}(s)$ is the fiber of $f$ over $s$. Then, possibly after replacing $S$ by a neighborhood of $s$, the sheaf of relative differentials $\Omega_{X/S}^p$ is flat over $S$ for all $p \leq k$. For $p = 0, 1$, $\Omega_{X/S}^p$ is flat over $S$ with no assumption on the dimension of the singular locus.

**Remark 2.5.** For $k \geq 2$, at least some assumption is needed for the hypotheses in Theorem 2.4. For example, if $X$ has simple normal crossings singularities, say $X$ is the product of a nodal curve $\text{Spec} \mathbb{C}[x, y]/(xy)$ with a smooth manifold of dimension $n - 1$ and $X$ is the standard smoothing over $\mathbb{A}^1$ given by $xy = t$, then $\text{depth}_x \Omega_{X}^1 = n$ at a singular point and $\Omega_{X}^2$ has torsion in the fiber over the singular fiber at $t = 0$, hence is not flat over $\mathbb{A}^1$.  


In the same spirit, with respect to Theorem [12], if $\pi: C \to \Delta$ is a family of smooth projective curves acquiring a single ordinary double point over 0, with the same local picture as above, the sheaf $\Omega^1_{C/\Delta}$ of relative Kähler differentials is flat over $\Delta$. However, if $C_0$ is the fiber over 0, the Hodge spectral sequence $H^q(C_0; \Omega^p_{C_0}) \Rightarrow H^{p+q}(C_0; \Omega^*_{C_0}) = H^{p+q}(C_0; \mathbb{C})$ does not degenerate at $E_1$. If $C_0$ is reducible, the values of $h^0(\Omega^1_{C_0})$ and $h^1(\Omega^1_{C_1})$ jump up at $t = 0$. Hence $R^1\pi_*\Omega^1_{C/\Delta}$ has torsion at 0, and in particular is not locally free.

**Proof of Theorem [2.4]** It suffices to consider the case $p = k$. Since every deformation is (locally) pulled back from the versal deformation, by standard properties of flatness and wedge product under base change we can assume that $S$ is smooth. First, with no assumption on the singular locus, $\mathcal{O}_X$ is flat over $S$ by assumption. To see that $\Omega^1_{X/S}$ is flat over $S$, again with no assumption on the singular locus, we have the conormal sequence

$$0 \to I/I^2 \xrightarrow{\partial} \pi^*\Omega^1_A|\chi \to \Omega^1_{X/S} \to 0.$$ 

Let $I_0$ be the ideal of $X$ in $A$. Then the conormal sequence for $X/S$ becomes the corresponding sequence for $\Omega^1_X$ after tensoring with $\mathcal{O}_{S,s}/m_s$. In particular, $\partial$ is injective. Hence $\Omega^1_{X/S} = \text{Coker } \partial$ is flat over $S$ by the local criterion of flatness [Mat80, (20.E) pp. 150–151].

For $k \geq 2$, by induction on $k$ and descending induction on $a$, $K^{a+1}$ and $K^a/K^{a+1}$ are flat over $S$ for all $a \geq 1$, and hence so is $K^a$. Let $K^a_0$ be the image of $\wedge^a(I_0/I_0^2) \otimes K^a|X$ in $K^a|X$. Proposition 2.3 applied to the case $S = \text{pt}$ implies that $K^a_0/K^{a+1}_0 \cong \wedge^a(I_0/I_0^2) \otimes K^a|X$. We have a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & K^{a+1} \otimes (\mathcal{O}_{S,s}/m_s) & \longrightarrow & K^a \otimes (\mathcal{O}_{S,s}/m_s) & \longrightarrow & K^a/K^{a+1} \otimes (\mathcal{O}_{S,s}/m_s) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \cong \\
0 & \longrightarrow & K^{a+1}_0 & \longrightarrow & K^a_0 & \longrightarrow & K^a/K^{a+1}_0 & \longrightarrow & 0
\end{array}
$$

Here, the top row is exact for $a \geq 1$ by the flatness of $K^a/K^{a+1}$ and induction on $k$. By descending induction on $a$, the above diagram implies that the natural map $K^a \otimes (\mathcal{O}_{S,s}/m_s) \to K^a_0$ is an isomorphism for all $a \geq 1$. In particular, for $a = 1$, we have the map $u: K^1 \to \pi^*\Omega^1_A|\chi$ with cokernel $\Omega^1_{X/S}$, and the reduction $\bar{u}$ of $u$ mod $m_s$ is the natural inclusion $K^1_0 \to \Omega^1_A|\chi$. Since $\bar{u}$ is injective, $\Omega^1_{X/S} = \text{Coker } \bar{u}$ is flat over $S$ by the local criterion of flatness as above. This completes the proof of Theorem 2.3. \hfill \square

**2.2. Kähler differentials and generic hyperplane sections.**

**Proposition 2.6.** Let $X$ be a reduced local complete intersection with $X_{\text{sing}} = \Sigma$ and let $H$ be a Cartier divisor on $X$ which is a generic element of a base point free linear system, with ideal sheaf $I = \mathcal{O}_X(-H)$. If $\text{codim } \Sigma \geq 2k - 1$, then

$$(*)_p \quad 0 \to I/I^2 \otimes \Omega^p_H \to \Omega^p_H|H \to \Omega^p_H \to 0$$

is exact for $p \leq k$.

**Proof.** We can assume that $X$ is affine and that $H = (f)$ for a function $f: X \to \mathbb{A}^1$ which has no critical points away from $\Sigma$. The sequence $(*)_p$ is always exact for $p = 0$ and for $p = 1$ because we have the exact sequence

$$I/I^2 \to \Omega^1_X|H \to \Omega^1_H \to 0,$$

with $I/I^2 \cong \mathcal{O}_H$ locally free and $H$ reduced (as it is generically reduced and loci).

By induction on $k$, it suffices to consider the case $p = k$. By the remarks above, we can assume $k \geq 2$. By the de Rham lemma [Gre75, Lemma 1.6], we have the following:
Lemma 2.7. Let $d = \dim \Sigma$. For $0 \leq p < n - d$, the following sequence is exact:

$$0 \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p+1}_X.$$

Equivalently, for $0 \leq p < n - d$,

$$df \wedge: \Omega^p_X / df \wedge \Omega^{p-1}_X \to \Omega^{p+1}_X$$

is injective.

Proof. Since $f$ is generic, the critical set $C(f)$ of $f$ (in the notation of \cite{Gre75}) is just $\Sigma$. Then the result follows immediately from \cite{Gre75} Lemma 1.6 (with $g = h$, $k = 1$, and $C(f,g) \cap N(h) = \Sigma$).

From the lemma, since $n - d = \text{codim} \Sigma \geq 2k - 1 > k$, because $k > 1$, there is an exact sequence

$$0 \to \Omega^{k-1}_X / df \wedge \Omega^{k-2}_X \to \Omega^k_X / df \wedge \Omega^{k-1}_X \to 0.$$

Tensoring with $\mathcal{O}_H$ gives an exact sequence

$$0 \to \text{Tor}^1_{\mathcal{O}_X}(\Omega^k_X / df \wedge \Omega^{k-1}_X, \mathcal{O}_H) \to \Omega^{k-1}_H \to \Omega^k_X / df \wedge \Omega^{k-1}_X \to 0.$$

So we have to show that, for a general choice of $f$, $\text{Tor}^1_{\mathcal{O}_X}(\Omega^k_X / df \wedge \Omega^{k-1}_X, \mathcal{O}_H) = 0$. Since $\mathcal{O}_H$ has the free resolution

$$0 \to \mathcal{O}_X \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_H \to 0,$$

it suffices to show that $\Omega^k_X / df \wedge \Omega^{k-1}_X$ has no $f$-torsion, i.e. that multiplication by $f$ is injective. Since there is an inclusion $\Omega^k_X / df \wedge \Omega^{k-1}_X \to \Omega^{k+1}_X$, it suffices to prove that $\Omega^{k+1}_X$ has no $f$-torsion. Let $\tau$ be an $f$-torsion local section of $\Omega^{k+1}_X$. Note that the support of $\tau$ must be contained in $\Sigma \cap V(f)$, since $\Omega^{k+1}_X | X - \Sigma$ is torsion free and $f$ is invertible on $X - V(f)$. By the assumption that $H$ is general, every component of $\Sigma \cap V(f)$ has dimension $\leq d - 1$. By \cite{Gre75} Lemma 1.8, since $k + 1 \leq 2k - 1 \leq \text{codim} \Sigma$ as long as $k \geq 2$, we have, for every $x \in X$,

$${\text{depth}}_{x} \Omega^{k+1}_X \geq n - (k + 1).$$

Using $n - d \geq 2k - 1$ gives $n - k - 1 \geq d + k - 2 \geq d$, since we are assuming $k \geq 2$. Thus

$${\text{depth}}_{x} \Omega^{k+1}_X \geq \dim(\Sigma \cap V(f)) + 1.$$

It follows from Scheja’s theorem (see e.g. \cite{Gre75} Lemma 1.8) that, for every open subset $U$ of $X$, the restriction map

$$H^0(U; \Omega^{k+1}_X | U) \to H^0(U - (\Sigma \cap V(f)); \Omega^{k+1}_X | U - (\Sigma \cap V(f))$$

is injective. In particular, $\tau = 0$. Hence $(*)_p$ is exact for $p \leq k$. \hfill $\square$

Note that we showed the following in the course of proving Proposition 2.6.

Corollary 2.8. With $X$ and $H$ as in the statement of Proposition 2.6 $\text{Tor}^1_{\mathcal{O}_X}(\Omega^p_X, \mathcal{O}_H) = 0$ for all $p \leq k + 1$. \hfill $\square$

3. $k$-Du Bois and $k$-rational singularities

We now review the notion of Du Bois and higher Du Bois singularities, and propose a general definition for higher rational singularities. After verifying that our definition agrees with the standard definition of rational singularities, as well as with previous definitions of higher rational singularities (under mild assumptions, expected to hold in general), we discuss the connection between higher rational and higher Du Bois singularities.
3.1. **The filtered de Rham complex.** On a smooth scheme $Y$ proper over $\text{Spec} \, \mathbb{C}$ or a compact Kähler manifold, the Hodge-de Rham spectral sequence with $E_1$ page $E_1^{p,q} = H^q(Y; \Omega_X^p)$ degenerates at $E_1$ and computes the Hodge structure on $H^{p+q}(Y, \mathbb{C})$. For $X$ not necessarily smooth or proper over $\text{Spec} \, \mathbb{C}$, Deligne showed that $H^*(X, \mathbb{C})$ carries a canonical mixed Hodge structure. Subsequently, Du Bois [DB81] introduced an object $\Omega^*_X$ in the filtered derived category whose graded pieces $\Omega^*_X = \text{Gr}^p_F \Omega^*_X$ are analogous to $\Omega^p_X$ in the smooth case (see [PS08] §7.3). Since $\Omega^*_X$ is defined locally in the étale topology, it agrees with $\Omega^p_X$ at smooth points. Near the singular locus, if $\pi: \tilde{X} \to X$ is a log resolution with $E \subseteq \tilde{X}$ the reduced exceptional divisor, $\Omega^*_X$ is closely related to $R\pi_*\Omega^*_X(\log E)(-E)$. More precisely, we have the following [PS08] Example 7.25:

**Theorem 3.1.** If $\pi: \tilde{X} \to X$ is a log resolution with reduced exceptional divisor $E$, and we define $\Omega^*_{X,\Sigma} = R\pi_*\Omega^*_X(\log E)(-E)$, where $\Sigma$ is the singular locus of $X$, then there is the distinguished triangle of relative cohomology

$$\Omega^p_{X,\Sigma} \to \Omega^p_X \to \Omega^p_\Sigma \to \Omega^p_{X,\Sigma}[1]. \quad \square$$

In the proper case, there is the following fundamental result of Du Bois, based on Deligne’s construction of the mixed Hodge structure on $Y$:

**Theorem 3.2** (Du Bois). If $Y$ is proper over $\text{Spec} \, \mathbb{C}$, then the spectral sequence with $E_1$ page $\mathbb{H}^q(Y; \Omega^p_Y) \Longrightarrow \mathbb{H}^{p+q}(Y; \Omega^*_Y) = H^{p+q}(Y; \mathbb{C})$ degenerates at $E_1$ and the corresponding filtration on $H^*(Y; \mathbb{C})$ is the Hodge filtration associated to the mixed Hodge structure on $H^*(Y; \mathbb{C})$. \quad \square

The result above allows us to define Hodge numbers in the singular case:

**Definition 3.3.** Let $Y$ be a compact complex algebraic variety. We define the *Hodge-Du Bois numbers*

$$\underline{h}^{p,q}(Y) := \dim \text{Gr}^p_F H^{p+q}(Y),$$

for $0 \leq p, q \leq n$. In particular, if $Y$ is smooth, $\underline{h}^{p,q}(Y) = h^{p,q}(Y)$. In general, by Theorem 3.2

$$\underline{h}^{p,q}(Y) = \dim \mathbb{H}^q(\Omega^*_X) = \sum_{0 \leq r \leq q} h^{p,r}_{\pi_*(\log E)}(Y),$$

where $h^{p,r}_{\pi_*(\log E)}(Y) := \dim \text{Gr}^p_F \text{Gr}^W_{p+r} H^{p+q}(Y)$ are the *Hodge-Deligne numbers* associated to the mixed Hodge structure on $Y$.

**Remark 3.4.** As the example of nodal curves shows (Remark 1.12), the Hodge-Du Bois diamond will not satisfy either of the Hodge symmetries: $\underline{h}^{p,q} = \underline{h}^{q,p}$ or $\underline{h}^{p,q} = \underline{h}^{n-p,n-q}$. Nonetheless, some vestige of these symmetries remains (in the form of inequalities) as those given by Lemma 3.22 below.

Another key consequence for us is the following:

**Corollary 3.5.** With notation as above,

(i) The natural map $H^i(Y; \mathbb{C}) \to \mathbb{H}^i(Y; \Omega^*_X/F^{k+1}\Omega^*_X)$ is surjective for all $i$ and $k$.

(ii) The spectral sequence with $E_1$ term

$$E_1^{p,q} = \begin{cases} \mathbb{H}^p(Y; \Omega^*_Y), & \text{for } p \leq k; \\ 0, & \text{for } p > k. \end{cases}$$

converging to $\mathbb{H}^{p+q}(Y; \Omega^*_X/F^{k+1}\Omega^*_X)$ degenerates at $E_1$. \quad \square

**Proof.** This is a consequence of the following general fact: If $(\mathcal{C}^*, d, \mathcal{F}^*\mathcal{C}^*)$ is a filtered complex, then the associated spectral sequence degenerates at the $E_1$ page $\iff$ the differential $d$ is strict with respect to the filtration $\mathcal{F}^*\mathcal{C}^*$. Then, assuming $E_1$-degeneration, an easy argument shows
that $H^i(C^\bullet, d) \to H^i(C^\bullet/F^{k+1}C^\bullet, d)$ is surjective for every $i$ and $k$. Since the induced filtration on $C^\bullet/F^{k+1}C^\bullet$ is also strict with respect to $d$, the corresponding spectral sequence for the complex $C^\bullet/F^{k+1}C^\bullet$ also degenerates at $E_1$. □

3.2. Du Bois and higher Du Bois singularities. There is a natural comparison map

$$\phi^* : (\Omega^\bullet_X, \sigma^\bullet) \to (\Omega^\bullet_X, F^\bullet \Omega^\bullet_X)$$

[PS08 p.175], where $\sigma^\bullet$ is the trivial or naive filtration on $\Omega^\bullet_X$. Following [MOPW21] and [JKSY22a], one defines:

Definition 3.6. Let $X$ be a complex algebraic variety. Then $X$ is $k$-Du Bois if the natural maps

$$\phi^p : \Omega^p_X \to \Omega^p_X$$

are quasi-isomorphisms for $0 \leq p \leq k$. Note that the case $k = 0$ coincides with the usual definition of Du Bois singularities [PS08 Def. 7.34].

Example 3.7. Let $f(z_1, \ldots, z_{n+1}) = z_1^{d_1} + \cdots + z_{n+1}^{d_{n+1}}$ define the weighted homogeneous singularity $X = V(f) \subseteq \mathbb{A}^{n+1}$. Then, by a theorem of Saito [Sai16 (2.5.1)] (and see also [FL22b]), $X$ is $k$-Du Bois at 0 $\iff \sum_{i=1}^{n+1} \frac{1}{d_i} \geq k + 1$. In particular, an ordinary double point of dimension $n$ is $k$-Du Bois for all $k \leq \left[\frac{n-1}{2}\right]$. Thus an ordinary double point of dimension 3 is 1-Du Bois, and in fact is the unique 1-Du Bois hypersurface singularity in dimension 3 (e.g. [NS95 Thm. 2.2]). More generally, it is expected that $k$-Du Bois singularities occur first in dimension $2k + 1$. This is true at least in the lci case as the following result shows.

Theorem 3.8 ([MP21], [MOPW21], [JKSY22a]). Let $X$ be a complex algebraic variety with lci singularities. Assume $X$ is $k$-Du Bois with $k \geq 1$. Then $X$ is normal and regular in codimension $2k$, i.e. $\text{codim } \Sigma \geq 2k + 1$.

Proof. The normality claim is [MP21 Cor. 13.5]. For hypersurface singularities, various dimension bounds (covering the claim of the theorem) were obtained in both [MOPW21] and [JKSY22a]. The general lci case follows from [MP21 Thm. F] (numerical characterization of $k$-Du Bois) and [MP21 Cor. 9.26] (bounds on $\text{dim } \Sigma$ in terms of the relevant numerical invariant). □

Corollary 3.9. Suppose that $X$ is an lci $k$-Du Bois singularity and that $f : X \to S$ is a flat morphism, where $S$ is arbitrary, with $X = X_s = f^{-1}(s)$ for some $s \in S$. Then possibly after replacing $S$ by a neighborhood of $s$, the sheaf of relative differentials $\Omega^p_{X/S}$ is flat over $S$ for all $p \leq k$.

Proof. This is immediate from Theorem 2.4 and Theorem 3.8. □

Remark 3.10. (i) Theorem 3.8 and Theorem 2.4 imply the following previously known results:

1. If $X$ has hypersurface $k$-Du Bois singularities, then $\Omega^p_X$ is torsion free for $0 \leq p \leq k$ ([JKSY22a Prop. 2.2]).
2. If $X$ has lci $k$-Du Bois singularities, then $\Omega^p_X$ is reflexive for $0 \leq p \leq k$ ([MP21 Cor. 13.5]).

(ii) For $p = 1$ and lci singularities, the situation is well understood by a result of Kunz [Kun86 Prop. 9.7]: $\Omega^1_X$ satisfies Serre’s condition $S_a \iff X$ satisfies $R_a$ (i.e. is regular in codimension $a$).
3.3. Higher rational singularities. The standard definition of a rational singularity involves the choice of a resolution (e.g. [KM98 Def. 5.8]). As we will explain below, it is possible to give an equivalent definition for rational singularities without reference to resolutions, but using instead the dualizing complex. In addition to being more intrinsic, it generalizes to higher rational singularities, and it factors naturally through the higher Du Bois condition.

For a complex algebraic variety $X$ of dimension $n$, let $\omega_X^\bullet$ denote the dualizing complex. Define the Grothendieck duality functor $D_X$ as follows:

$$D_X(-) := R\text{Hom}(-, \omega_X^\bullet)[-n].$$

In particular, $\omega_X^\bullet = D_X(O_X)[n]$. 

**Lemma 3.11.** For all $p$, there exists a natural sequence of maps in the derived category

$$\Omega^p_X \xrightarrow{\phi^p} \Omega_X^p \xrightarrow{\psi^p} D_X(\Omega_X^{n-p}).$$

**Proof.** By functoriality of the filtered de Rham complex, there is a map $\Omega^{n-p}_X \rightarrow R\pi_* \Omega_X^{n-p}$. Applying $D_X$ gives

$$D_X(R\pi_* \Omega_X^{n-p}) \rightarrow D_X(\Omega_X^{n-p}).$$

Since $\pi$ is proper, Grothendieck duality gives

$$D_X(R\pi_* \Omega_X^{n-p}) \cong R\pi_* D_X(\Omega_X^{n-p}) \cong R\pi_* \Omega_X^p.$$

Thus we get a sequence of maps

$$\Omega^p_X \xrightarrow{\psi^p \circ \phi^p} D_X(\Omega_X^{n-p})$$

as claimed. The map $\psi^p$ is easily seen to be independent of the choice of a resolution, by the usual factorization arguments. $\square$

**Definition 3.12.** The variety $X$ has $k$-rational singularities if the maps

$$\Omega^p_X \xrightarrow{\psi^p \circ \phi^p} D_X(\Omega_X^{n-p})$$

are quasi-isomorphisms for all $0 \leq p \leq k$.

**Example 3.13.** Let $f(z_1, \ldots, z_{n+1}) = z_1^{d_1} + \cdots + z_{n+1}^{d_{n+1}}$ define the weighted homogeneous singularity $X = V(f) \subseteq \mathbb{A}^{n+1}$. Then, by [FL22b], $X$ is $k$-rational at $0 \iff \sum_{i=1}^{n+1} \frac{1}{d_i} > k + 1$. In particular, an ordinary double point of dimension $n$ is $k$-rational $\iff k < \frac{n-1}{2}$. Thus an ordinary double point of dimension $3$ is not $1$-rational. On the other hand, ADE singularities in dimension $4$ are $1$-rational. Conjecturally, $k$-rational singularities occur in codimension at least $2(k+1)$ (compare Theorem 3.8).

The following lemma connects our definition to more standard ones:

**Lemma 3.14.** Suppose that $\dim \Sigma \leq d$. Then, for all $p < n - d$,

$$D_X(\Omega_X^{n-p}) \cong R\pi_* \Omega_X^p(\log E).$$

In particular, if codim $\Sigma \geq 2k + 1$ for some $k \geq 0$, then $D_X(\Omega_X^{n-p}) \cong R\pi_* \Omega_X^p(\log E)$ for all $p \leq k$.

**Proof.** By Theorem 3.1, there is the distinguished triangle of relative cohomology

$$\Omega_X^{n-p} \rightarrow \Omega_X^{n-p} \rightarrow \Omega_{X,\Sigma}^{n-p} \rightarrow.$$

11
Since $n-p > d$, $\Omega^n_{\Sigma} = 0$ and hence $\Omega^n_{\Sigma} = \Omega^{n-p}_{\Sigma} = R\pi_* \Omega^{n-p}_{\hat{X}}(\log E)(-E)$. Applying Grothendieck duality, it follows as in [MOP21] §2.2 that

$$D(X(\Omega^{n-p}_{\Sigma})) = D(X(R\pi_*(\Omega^{n-p}_{\hat{X}}(\log E)(-E))) = R\pi_* D(\Omega^{n-p}_{\hat{X}}(\log E)(-E)) = R\pi_* \Omega^p_{\hat{X}}(\log E).$$

The final statement is clear since $n - d \geq 2k + 1 \iff n - p \geq n - k \geq d + k + 1 > d.$

**Corollary 3.15.** $X$ is 0-rational $\iff X$ has rational singularities.

**Proof.** Since $X$ is reduced, $\dim \Sigma \leq n - 1$. Thus $X$ is 0-rational $\iff$ the natural map $\mathcal{O}_X \to R\pi_* \mathcal{O}_{\hat{X}}$ is an isomorphism $\iff$ $X$ has rational singularities in the usual sense.

**Remark 3.16.** Lemma 3.14 and Grothendieck duality give the identification $\Omega^n_X = R\pi_* \omega^E_X$, which by Grauert-Riemenschneider vanishing is in fact a single sheaf, the Grauert-Riemenschneider sheaf $\omega^E_X := \pi_* \omega^E_X$. Following [KK20], let us denote by $\omega_X := D_X(\Omega^n_X)$. Then the dual form of Definition 3.12 for $k = 0$ is: $X$ has rational singularities $\iff$ the composite map

$$(3.1) \quad \omega^E_X \to \omega^E_X \to \omega^E_X$$

is a quasi-isomorphism. This formulation occurs for instance in [KK20], and it is equivalent to that given by [KM98] Thm. 5.10(3)] (note that the quasi-isomorphism $\omega^E_X$ forces $X$ to be Cohen-Macaulay).

In [FL22a, §3], we defined $k$-rational singularities for an isolated singularity by the condition that $R\pi_*(\Omega_p^k)(\log E) = \Omega_{\hat{X}}^k$. This is equivalent to Definition 3.12 (under a mild assumption):

**Corollary 3.17.** In the above notation, suppose that $\text{codim} \Sigma \geq 2k + 1$.

(i) $X$ is $k$-rational $\iff$ the natural map $\Omega^n_X \to R\pi_*(\Omega^k_{\hat{X}}(\log E))$ is an isomorphism for all $p \leq k$.

(ii) $X$ is $k$-rational $\iff$ $X$ is $(k-1)$-rational and $\Omega^n_X \to R\pi_*(\Omega^k_{\hat{X}}(\log E))$ is an isomorphism.

In the lci case, the assumption on $R^0\pi_* \Omega^k_{\hat{X}}(\log E)$ is automatic:

**Lemma 3.18.** Suppose that $X$ has lci singularities and $\text{codim} \Sigma \geq 2k + 1$. Then, for all $p \leq k$, $\psi^p \circ \phi^p : \Omega^p_X \to R^0\pi_* \Omega^p_{\hat{X}}(\log E)$ is an isomorphism. Hence $X$ is $k$-rational $\iff$ for all $p \leq k$ and all $q > 0$, $R^q\pi_* \Omega^p_{\hat{X}}(\log E) = 0$.

**Proof.** This follows easily from Theorem 2.1 and Corollary 2.2 as $R^0\pi_* \Omega^p_{\hat{X}}(\log E)$ is torsion free for all $p$.

**Remark 3.19.** Suppose that $X$ has an isolated singularity $x$, so that $\Sigma = \{x\}$. Then by Theorem 3.1 there is the distinguished triangle of relative cohomology

$$\Omega^n_{X,x} \to \Omega^n_X \to \mathbb{C}[0] \to.$$ 

Hence $\Omega^n_{X,x} \to \Omega^n_X$ is an isomorphism for $p > 0$, so if $X$ is lci, by the same argument as that of Lemma 3.18 $\phi^p : \Omega^p_X \to R^0\pi_* \Omega^p_{\hat{X}}(\log E)(-E)$ is an isomorphism. For $p = 0$, if $X$ has an isolated singularity and is normal, then the map $\pi_* \mathcal{O}_{\hat{X}}(-E) \to \Omega^0_X$ has cokernel $\mathbb{C}[0]$ and factors through $m_x \subseteq \mathcal{O}_X$. Hence $\mathcal{O}_X \to R^0\pi_* \mathcal{O}_{\hat{X}}$ is an isomorphism. Thus, if $X$ has an isolated lci singularity and $\dim X \geq 2$, then $X$ is $k$-Du Bois $\iff$ for all $p \leq k$ and all $q > 0$, $R^q\pi_* \Omega^p_{\hat{X}}(\log E)(-E) = 0$. 

12
3.4. k-rational vs. k-Du Bois singularities. Steenbrink \cite{Ste83} proved that, if $X$ is an isolated rational singularity, then $X$ is Du Bois, and Kovács \cite{Kov99} generalized this, showing that that any rational singularity is Du Bois. Saito gave a different proof in \cite[Thm. 5.4]{Sai00}. The method of \cite{Ste83} generalizes to prove that isolated $k$-rational singularities are $k$-Du Bois (see \cite{FL22b}), and the method of \cite{Kov99} can be generalized to handle both isolated and lci singularities. Using the ideas of \cite{Kov99}, we shall show the following in Section 5:

**Theorem 3.20.** Suppose either that $X$ has isolated singularities or $X$ is lci. If $X$ is $k$-rational, then $X$ is $k$-Du Bois.

Assuming for the moment Theorem 3.20, we note that the $k$-rational assumption adds additional Hodge symmetries to the $k$-Du Bois condition: If $X$ is $k$-Du Bois, then $X$ is $k$-rational $\iff$ the map $\psi^p : \Omega^n_X \to \mathbb{D}_X(\Omega^{n-p}_X)$ is a quasi-isomorphism for all $p \leq k$. If $X$ is proper, these assumptions lead to a Hodge symmetry, Serre duality for the Hodge-Du Bois numbers:

**Corollary 3.21.** If $Y$ is a compact complex algebraic variety of dimension $n$ with lci $k$-rational singularities, then

$$h^{p,q}(Y) = \dim \text{Gr}^p_F H^{p+q}(Y) = \dim \text{Gr}^{n-p}_F H^{2n-(p+q)}(Y) = h^{n-p,n-q}(Y)$$

for $0 \leq p \leq k$. In particular, taking $p+q = n$ gives $h^{p,n-p} = h^{n-p,p}$.

**Proof.** Using Theorem 3.2 we get the following identifications

$$\text{Gr}^p_F H^{p+q}(Y) \cong H^p(Y;\Omega^q_X) \cong H^p(X;\mathbb{D}_Y(\Omega^{n-p}_Y)) \cong H^{n-q}(Y;\Omega^{n-p}_Y)^\vee,$$

where the middle isomorphism is given by the quasi-isomorphism $\psi^p$ (for $p \leq k$).

It remains to discuss the Hodge symmetry $h^{p,q} = h^{q,p}$, which is induced by the complex conjugation. We recall that the cohomology of a compact singular algebraic variety $Y$ carries a mixed Hodge structure $(H^i(Y), F^*, W^*)$. The Hodge-Deligne numbers $h^{p,r}_i = \text{Gr}^p_F \text{Gr}^W_{p+r} H^i(Y)$ satisfy the symmetry given by the conjugation: $h^{p,r}_i = h^{i-p,r}_i$ (as $\text{Gr}^W_{p+r} H^i(Y)$ is a pure Hodge structure).

Since $Y$ is compact, the weights on $H^i(Y)$ are at most $i$, and in fact between $2i-2n$ and $i$ if $i \geq n$. It follows that, for $i \leq n$, $h^{p,i-p} = \sum_{r=0}^{i-p} h^{p,r}$, However, the Hodge-Du Bois numbers do not satisfy this symmetry as they reflect only the Hodge filtration $F^*$. In fact, we note the following:

**Lemma 3.22.** Let $Y$ be a compact complex algebraic variety of dimension $n$. For $0 \leq p \leq i \leq n$,

$$\sum_{a=0}^{p} h^{i-a,a} \leq \sum_{a=0}^{p} h^{a,i-a}.$$ 

Furthermore, equality holds above for all $p \leq k$ $\iff$ $h^{p,i-p} = h^{i-p,p}$ for all $p \leq k$ $\iff$ $\text{Gr}^p_F W_{i-1} H^i(Y) = 0$ for all $p \leq k$.

**Proof.** Clearly $\sum_{a=0}^{p} h^{i-a,a} = \sum_{a=0}^{p} h^{a,i-a}$ for all $p \leq k$ $\iff$ $h^{p,i-p} = h^{i-p,p}$ for all $p \leq k$.

For a fixed $p$, we have

$$\sum_{a=0}^{p} h^{i-a,a} = \sum_{a=0}^{p} \sum_{q \leq a} h^{i-a,q} = \sum_{a=0}^{p} \sum_{q \leq a} h^{q,i-a} = \sum_{r \leq p} \sum_{r+s \leq i} h^{r,s}_i,$$

by the Hodge symmetries, whereas

$$\sum_{a=0}^{p} h^{a,i-a} = \sum_{a=0}^{p} \sum_{a+q \leq i} h^{a,q} = \sum_{r \leq p} \sum_{r+s \leq i} h^{r,s}_i.$$
Note that, if \( i - p \leq s \leq i \) and \( r + s \leq i \), then \( r \leq i - s \leq p \), so the second sum is greater that the first, giving the inequality. For a given \( p \), equality holds \( \iff h_{i}^{r,s} = 0 \) for \( r \leq p \) and \( s \leq i - p - 1 \). Moreover, \( h_{i}^{r,s} = 0 \) for \( r \leq p \) and \( s \leq i - p - 1 \) for some \( p \leq k \) \( \iff h_{i}^{r,s} = 0 \) for \( r \leq k \), \( r + s \leq i - 1 \). This is equivalent to: \( \text{Gr}^{p}_{F}W_{i-1}H^{j}(Y) = 0 \) for \( p \leq k \). \( \square \)

Recall that, for any resolution \( \pi : \hat{Y} \to Y \),

\[
W_{i-1}H^{j}(Y) = \text{Ker} \left( H^{i}(Y) \xrightarrow{\pi^{*}} H^{i}(\hat{Y}) \right)
\]

(e.g. [PS08, Cor. 5.42]). Using this, we obtain:

**Theorem 3.23.** If \( Y \) is a compact complex algebraic variety of dimension \( n \) with either isolated or lci k-rational singularities, then

\[
h^{p,q} = h^{q,p}.
\]

for \( 0 \leq p \leq k \) and \( 0 \leq q \leq n \).

**Proof.** In view of the discussion above, we define the discrepancy

\[
\delta_{i}^{p} := \dim \text{Gr}^{p}_{F}W_{i-1}H^{j}(Y) = \dim \text{Gr}^{p}_{F}\text{Ker} \left( H^{i}(Y) \xrightarrow{\pi^{*}} H^{i}(\hat{Y}) \right).
\]

By Lemma 3.22, the equality \( h^{p,i-p} = h^{i-p,p} \) holds for \( p \leq k \) \( \iff \delta_{i}^{p} = 0 \) for \( p \leq k \). The map \( \psi^{p} \) occurring in the definition of higher rationality (Definition 3.12 and Lemma 3.11) factors through the resolution \( \pi : \hat{Y} \to Y \), and at the level of cohomology corresponds to the \( \text{Gr}^{p}_{F} \) piece (see also Corollary 3.21) of the natural map

\[
\Psi^{i} : H^{i}(Y) \xrightarrow{\pi^{*}} H^{i}(\hat{Y}) \xrightarrow{\psi^{p}} H^{2n-i}(\hat{Y})^{\vee}(-n) \xrightarrow{(\pi^{*})^{\vee}} H^{2n-i}(Y)^{(\hat{Y})}(-n),
\]

where all spaces are endowed with the natural Hodge structures. On the graded piece \( \text{Gr}^{p}_{F}H^{i}(Y) = H^{i-p}(Y; \Omega^{p}_{\hat{Y}}) \), \( \text{Gr}^{p}_{F}\Psi^{i} \) is the map \( \psi^{p} : H^{i-p}(Y; \Omega^{p}_{\hat{Y}}) \to H^{n-i+p}(Y; \Omega_{\hat{Y}}^{n-p})^{\vee} \), which is an isomorphism if \( Y \) is both k-rational and k-Du Bois, and in particular if \( Y \) is k-rational and has either isolated singularities or lci singularities. By the strictness of morphisms of Hodge filtrations, if \( \text{Gr}^{p}_{F}\Psi^{i} \) is an isomorphism then \( \pi^{*} \) is injective on \( \text{Gr}^{p}_{F} \) and hence \( \delta_{i}^{p} = 0 \). Thus, k-rationality implies \( \delta_{i}^{p} = 0 \) for \( p \leq k \) and all \( i \), which in turn means \( h^{p,i-p} = h^{i-p,i} \) in this range. \( \square \)

4. **Proof of Theorem 1.2 and Corollary 1.5**

We turn now to the global setting of a deformation of a compact analytic space or proper scheme \( Y \), and to the question of the local freeness of \( \Omega_{Y/S}^{p,q} \).

**Theorem 4.1.** Let \( f : \mathcal{Y} \to \text{Spec } A \) be a proper morphism of complex spaces, where \( A \) is an Artin local \( \mathbb{C} \)-algebra, with closed fiber \( Y \). Let \( (\mathcal{F}^{*}, d) \) be a bounded complex of coherent sheaves on \( \mathcal{Y} \), flat over \( A \), where \( d : \mathcal{F}^{i} \to \mathcal{F}^{i+1} \) is \( A \)-linear, but not necessarily \( \mathcal{O}_{\mathcal{Y}} \)-linear. Finally suppose that the natural map \( H^{i}(\mathcal{Y}; \mathcal{F}^{*}) = \mathbb{R}^{i}f_{*}\mathcal{F}^{*} \to H^{i}(Y; f^{*}Y) \) is surjective for all \( i \). Then \( H^{i}(\mathcal{Y}; \mathcal{F}^{*}) \) is a finite \( A \)-module whose length satisfies:

\[
\ell(H^{i}(\mathcal{Y}; \mathcal{F}^{*})) = \ell(A) \dim H^{i}(Y; f^{*}Y).
\]

**Proof.** This is a minor variation on very standard arguments. Note that \( H^{i}(\mathcal{Y}; \mathcal{F}^{*}) \) is a finite \( A \)-module since it is the abutment of a spectral sequence with \( E_{1}^{p,q} = H^{q}(Y; f^{p}) \) and such that all of the differentials in the spectral sequence are \( A \)-module homomorphisms. Next we show that, for every finite \( A \)-module \( M \), the natural map

\[
\Psi_{M} : H^{i}(\mathcal{Y}; \mathcal{F}^{*}) \otimes_{A} M \to H^{i}(\mathcal{Y}; \mathcal{F}^{*} \otimes_{A} M)
\]

is surjective. The proof is via induction on \( \ell(M) \), the case \( \ell(M) = 1 \) being the hypothesis of the theorem. The inductive step follows from: given an exact sequence \( 0 \to M' \to M \to M'' \to 0 \)
such that $\Psi_{M'}$ and $\Psi_{M''}$ are surjective, then $\Psi_M$ is surjective. This follows from the commutative diagram

$$
\begin{array}{cccc}
\mathbb{H}^i(Y; F^\bullet) \otimes_A M' & \longrightarrow & \mathbb{H}^i(Y; F^\bullet) \otimes_A M & \longrightarrow & \mathbb{H}^i(Y; F^\bullet) \otimes_A M'' & \longrightarrow & 0 \\
\downarrow \Psi_{M'} & & \downarrow \Psi_{M} & & \downarrow \Psi_{M''} & & \\
\mathbb{H}^i(Y; F^\bullet \otimes_A M') & \longrightarrow & \mathbb{H}^i(Y; F^\bullet \otimes_A M) & \longrightarrow & \mathbb{H}^i(Y; F^\bullet \otimes_A M'')
\end{array}
$$

where the top row is exact since tensor product is right exact and the second is exact since $F^\bullet$ is $A$-flat.

To prove the theorem, we argue by induction on $\ell(A)$. The result is clearly true if $\ell(A) = 1$. For the inductive step, write $A$ as a small extension $0 \to I \to A \to A/I \to 0$. By flatness, there is an exact sequence

$$
\cdots \to \mathbb{H}^i(Y; F^\bullet \otimes_A I) \to \mathbb{H}^i(Y; F^\bullet) \to \mathbb{H}^i(Y; F^\bullet \otimes_A A/I) \to \cdots
$$

Then the above implies that $\mathbb{H}^i(Y; F^\bullet) \to \mathbb{H}^i(Y; F^\bullet \otimes_A A/I)$ is surjective for all $i$. Hence the long exact sequence breaks up into short exact sequences and thus

$$
\ell(\mathbb{H}^i(Y; F^\bullet)) = \ell(\mathbb{H}^i(Y; F^\bullet \otimes_A I)) + \ell(\mathbb{H}^i(Y; F^\bullet \otimes_A A/I))
$$

$$
= \dim \mathbb{H}^i(Y; F^\bullet | Y) + \ell(A/I) \dim \mathbb{H}^i(Y; F^\bullet | Y)
$$

$$
= \ell(A) \dim \mathbb{H}^i(Y; F^\bullet | Y).
$$

This concludes the inductive step. \qed

Now assume that $f: Y \to \text{Spec } A$ is a proper morphism, where $Y$ is a scheme of finite type over $\text{Spec } \mathbb{C}$ and $A$ is an Artin local $\mathbb{C}$-algebra, with closed fiber $Y$. Consider the complex $\Omega_Y^{\bullet}/\text{Spec } A$ and the quotient complex $\Omega_Y^{\bullet}/\text{Spec } A/\sigma \geq k+1 \Omega_Y^{\bullet}/\text{Spec } A$. For the closed fiber $Y$, we also have the complex $\Omega_Y^{\bullet}/\sigma \geq k+1 \Omega_Y^{\bullet}$.

**Lemma 4.2.** With notation as above, suppose that $Y$ is $k$-Du Bois. Then:

(i) For every $i$, the natural map

$$
\mathbb{H}^i(Y; \Omega_Y^{\bullet}/\text{Spec } A/\sigma \geq k+1 \Omega_Y^{\bullet}/\text{Spec } A) \to \mathbb{H}^i(Y; \Omega_Y^{\bullet}/\sigma \geq k+1 \Omega_Y^{\bullet})
$$

is surjective.

(ii) The spectral sequence with $E_1$ term

$$
E_1^{p,q} = \begin{cases} 
H^q(Y; \Omega_Y^p), & \text{for } p \leq k; \\
0, & \text{for } p > k.
\end{cases}
$$

converging to $\mathbb{H}^{p+q}(Y; \Omega_Y^{\bullet}/\sigma \geq k+1 \Omega_Y^{\bullet})$ degenerates at $E_1$. Hence, for every $i$,

$$
\dim \mathbb{H}^i(Y; \Omega_Y^{\bullet}/\sigma \geq k+1 \Omega_Y^{\bullet}) = \sum_{\substack{p+q=i \\
p \leq k}} \dim H^p(Y; \Omega_Y^q).
$$

**Proof.** By the $k$-Du Bois condition, the natural map $\Omega_Y^{\bullet}/\sigma \geq k+1 \Omega_Y^{\bullet} \to \Omega_Y^{\bullet}/F^{k+1} \Omega_Y^{\bullet}$ is a quasi-isomorphism of filtered complexes. Thus there are isomorphisms

$$
\mathbb{H}^i(Y; \Omega_Y^{\bullet}/\sigma \geq k+1 \Omega_Y^{\bullet}) \cong \mathbb{H}^i(Y; \Omega_Y^{\bullet}/F^{k+1} \Omega_Y^{\bullet})
$$

$$
H^q(Y; \Omega_Y^p) \cong H^q(Y; \Omega_Y^p) \quad (p \leq k).
$$

By Corollary 5.5(ii), the spectral sequence with $E_1$ page $\mathbb{H}^q(Y; \Omega_Y^p)$ for $p \leq k$ and 0 otherwise degenerates at $E_1$. Hence the same is true for the spectral sequence in (ii).
To prove (i), arguing as in [DBJ74], there is a commutative diagram

\[
\begin{array}{ccc}
H^i(Y; \mathbb{C}) & \longrightarrow & H^i(Y; \Omega^*_Y / \text{Spec } A) \\
\downarrow & & \downarrow \\
H^i(Y; \mathbb{C}) & \longrightarrow & H^i(Y; \Omega^*_Y) \\
\end{array}
\]

By Corollary 3.5(i), \( H^i(Y; \mathbb{C}) \rightarrow H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y) \) is surjective. Hence

\[
H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y) \rightarrow H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y)
\]
is surjective as well.

**Proof of Theorem 1.2.** By Corollary 3.9, possibly after shrinking \( S \), \( \Omega^p_Y / S \) is flat over \( S \). By a standard argument, it is enough to show the following: Let \( A \) be an Artin local \( \mathbb{C} \)-algebra and \( f : Y \rightarrow \text{Spec } A \) a proper morphism whose closed fiber \( \overline{Y} \) is isomorphic to \( Y_s \). Then \( R^if_*\Omega^p_Y / \text{Spec } A = H^i(Y; \Omega^p_Y / \text{Spec } A) \) is a free \( A \)-module, compatible with base change, and in fact it is enough to show that, for every \( i \) and quotient \( A \rightarrow A/I = \overline{A} \), the natural map \( H^i(Y; \Omega^p_Y / \text{Spec } A) \rightarrow H^i(Y; \Omega^p_Y / \text{Spec } \overline{A}) \) is surjective, where \( Y = Y \times_{\text{Spec } A} \text{Spec } \overline{A} \).

By Theorem 4.1 and Lemma 4.2,

\[
\ell(H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y / \text{Spec } A)) = \ell(A) \dim H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y)
\]

On the other hand, by analogy with Lemma 4.2(ii), there is the spectral sequence converging to \( H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y / \text{Spec } A) \) with \( E_1 \) page

\[
E_1^{p,q} = \begin{cases} 
H^q(Y; \Omega^p_Y), & \text{for } p \leq k; \\
0, & \text{for } p > k.
\end{cases}
\]

Thus \( \ell(H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y / \text{Spec } A)) \leq \sum_{p+q=i, p \leq k} \ell(H^q(Y; \Omega^p_Y)) \), with equality \( \iff \) the spectral sequence degenerates at \( E_1 \). Moreover, \( \ell(H^q(Y; \Omega^p_Y)) \leq \ell(A) \dim H^q(Y; \Omega^p_Y) \), with equality holding for all \( q \iff \) for every \( q \) and every quotient \( A \rightarrow A/I = \overline{A} \), the natural map \( H^q(Y; \Omega^p_Y / \text{Spec } A) \rightarrow H^q(Y; \Omega^p_Y / \text{Spec } \overline{A}) \) is surjective. Combining, we have

\[
\ell(H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y / \text{Spec } A)) \leq \sum_{p+q=i, p \leq k} \ell(H^q(Y; \Omega^p_Y)) \leq \sum_{p+q=i, p \leq k} \ell(A) \dim H^q(Y; \Omega^p_Y)
\]

\[
= \ell(A) \dim H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y) = \ell(H^i(Y; \Omega^*_Y / \sigma^{\geq k+1} \Omega^*_Y / \text{Spec } A)).
\]

Thus all of the inequalities must have been equalities, and in particular \( H^i(Y; \Omega^p_Y / \text{Spec } A) \rightarrow H^i(Y; \Omega^p_Y / \text{Spec } \overline{A}) \) is surjective for every quotient \( A \rightarrow \overline{A} \). This completes the proof, by the remarks in the first paragraph of the proof.

**Remark 4.3.** Instead of assuming that \( Y \) is a scheme of finite type proper over \( \text{Spec } \mathbb{C} \), it is enough to assume that \( Y \) is a compact analytic space with isolated singularities and that there exists a resolution of singularities of \( Y \) which satisfies the \( \partial \partial \)-lemma. It seems likely that there is a more general result assuming sufficiently strong conditions on every subvariety of \( Y_{\text{sing}} \).
We turn now to a proof of Corollary 1.5. First, we define canonical Calabi-Yau varieties following [FL22a, Definition 6.1]:

**Definition 4.4.** A canonical Calabi-Yau variety $Y$ is a scheme proper over Spec $\mathbb{C}$ (resp. a compact analytic space) which is reduced, Gorenstein, with canonical singularities, and such that $\omega_Y \cong O_Y$.

**Proof of Corollary 1.5.** It suffices by [Kaw92] to show that $Y$ has the $T^1$-lifting property. In fact, we show the following somewhat stronger statement: let $\alpha$ be an Artin local $\mathbb{C}$-algebra, $\pi : Y \to Spec A$ a deformation of $Y$ over $A$, and $I$ an ideal of $A$ with $\overline{A} = A/I$ with $\overline{Y} = Y \times_{Spec A} Spec \overline{A}$, the natural map

$$\text{Ext}^1(\Omega^1_{\overline{Y}/Spec A}, O_Y) \to \text{Ext}^1(\Omega^1_{\overline{Y}/Spec \overline{A}}, O_{\overline{Y}})$$

is surjective. Arguing as in [Kaw92, Lemma 3], we claim that $\omega_{Y/Spec A} \cong O_Y$. To see this, note that, as $Y$ is $1$-Du Bois it is $0$-Du Bois, and hence by Theorem 1.1 $R^i\pi_* O_Y = H^i(Y; O_Y)$ is a free $A$-module for all $i$, compatible with base change. In particular, $R^n\pi_* O_Y$ is a free rank one $A$-module. An application of relative duality then shows that $(R^n\pi_* O_Y)^\vee \cong R^{n-i}\omega_{Y/Spec A}$, and hence $R^{n-i}\omega_{Y/Spec A}$ is a free $A$-module for all $i$. Thus $R^n\pi_* \omega_{Y/Spec A}$ is a free rank one $A$-module and the natural map $R^n\pi_* \omega_{Y/Spec A} \otimes A O_Y \to \omega_{Y/Spec A}$ is an isomorphism. Hence $\omega_{Y/Spec A} \cong O_Y$.

By a similar application of relative duality, and since $R^n\pi_* \Omega^1_{\overline{Y}/Spec A} = H^i(Y; \Omega^1_{\overline{Y}/Spec A})$ is a free $A$-module for all $i$, there is an isomorphism of $A$-modules

$$\text{Ext}^1(\Omega^1_{\overline{Y}/Spec A}, O_Y) \cong \text{Ext}^1(\Omega^1_{\overline{Y}/Spec A}, \omega_{Y/Spec A}) \cong \text{Hom}_A(H^{n-1}(Y; \Omega^1_{\overline{Y}/Spec A}), A).$$

Then the $T^1$-lifting criterion follows from the fact that $H^{n-1}(Y; \Omega^1_{\overline{Y}/Spec A})$ is a free $A$-module and that the natural map $H^{n-1}(Y; \Omega^1_{\overline{Y}/Spec A}) \otimes A \overline{A} \to H^{n-1}(\overline{Y}; \Omega^1_{\overline{Y}/Spec \overline{A}})$ is an isomorphism. □

**Remark 4.5.** A similar argument shows that, if $Y$ is a compact analytic space with isolated lci $1$-Du Bois singularities such that $\omega_Y \cong O_Y$ and there exists a resolution of singularities of $Y$ satisfying the $\partial\partial$-lemma, then Def $(Y)$ is unobstructed.

5. PROOF OF THEOREM 3.20

For $X$ an algebraic variety, let $\Sigma_k = \Sigma_k(X)$ be the set of points where $X$ is not $k$-Du Bois. Thus, for all $k$, $\Sigma_k \subseteq \Sigma_{k+1} \subseteq \Sigma$. It is easy to see that $\Sigma_k$ is a closed subvarieties of $X$. In fact, completing the morphism $\phi^p : \Omega^p_X \to \Omega^p_X$ to a distinguished triangle

$$\Omega^p_X \to \Omega^p_X \to G^p \xrightarrow{+1} ,$$

by definition we have

$$\Sigma_k = \bigcup_{0 \leq p \leq k} \text{Supp} \mathcal{H}^i G^p.$$

We want to consider situations more general than $k$-rational:

**Definition 5.1.** Given the map $\phi^p : \Omega^p_X \to \Omega^p_X$, let $h^p : \Omega^p_X \to \Omega^p_X$ be a left inverse. More generally, we consider a set of $k+1$ left inverses $\{h^p\}_{p=0}^k$. If $X$ is $k$-Du Bois, then $\phi^p$ is an isomorphism for $p \leq k$ and so $\{h^p\}_{p=0}^k = \{(\phi^p)^{-1}\}_{p=0}^k$ is a set of left inverses. In this case, we shall always use $\phi^p$ to identify $\Omega^p_X$ with $\Omega^p_X$, and thus $h^p = 1$ for $p \leq k$.

Let $\alpha \in H^0(X; \Omega^1_X)$. Then $\alpha$ pulls back to some fixed hyperresolution and so defines a map $\alpha \wedge : \Omega^{p-1}_X \to \Omega^p_X$. Two left inverses $h^p$ and $h^{p-1}$ are compatible if, for all $p \leq k$ and all $\alpha \in H^0(X; \Omega^1_X)$,

$$h^p \circ (\alpha \wedge) = (\alpha \wedge) \circ h^{p-1}.$$
If \( X \) is \( k \)-Du Bois, then, for all \( p \leq k \), the left inverses \( h^p \) and \( h^{p-1} \) are isomorphisms, in fact the identity after identifying \( \Omega_X^p \) and \( \Omega_X^{p-1} \) with \( \Omega_X^p \) and \( \Omega_X^{p-1} \) respectively, and necessarily \( h^p \circ (\alpha \land) = (\alpha \land) \circ h^{p-1} \), hence \( h^p \) and \( h^{p-1} \) are compatible.

Finally, the set of \( k+1 \) left inverses \( \{h^p\}_{p=0}^k \) is compatible if, for all \( p \leq k \), \( h^p \) and \( h^{p-1} \) are compatible.

**Lemma 5.2.** If the map \( \Omega_X^p / \sigma^{\geq k+1} \rightarrow \Omega_X^p / F^{k+1} \) has a set of \( k+1 \) left inverses, then there exists a left inverse \( H_k : \Omega_X^p / F^{k+1} \rightarrow \Omega_X^p / \sigma^{\geq k+1} \) in the filtered derived category.

**Proof.** We argue by induction on \( k \). There is a diagram

\[
\begin{array}{cccc}
\Omega_X^p / F^k & \longrightarrow & \Omega_X^p / F^{k+1} & \longrightarrow & \Omega_X^p \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_X^p / \sigma^{\geq k} & \longrightarrow & \Omega_X^p / \sigma^{\geq k+1} & \longrightarrow & \Omega_X^p \\
\end{array}
\]

We can thus complete the diagram to find a filtered morphism \( H'_k : \Omega_X^p / F^{k+1} \rightarrow \Omega_X^p / \sigma^{\geq k+1} \) which yields a morphism of distinguished triangles. Then the composition \( \Omega_X^p / \sigma^{\geq k+1} \rightarrow \Omega_X^p / F^{k+1} \xrightarrow{H'_k} \Omega_X^p / \sigma^{\geq k+1} \) is a filtered isomorphism, so after modifying \( H'_k \) we can replace it with \( H_k \) which is a left inverse.

**Lemma 5.3.** If \( X \) is \( k \)-rational, then there exists a compatible set of \( k+1 \) left inverses \( \{h^p\}_{p=0}^k \).

**Proof.** The following diagram commutes up to a sign (all vertical maps are \( \alpha \land \)):

\[
\begin{array}{cccc}
\Omega_X^{p-1} & \longrightarrow & \Omega_X^{p-1} & \longrightarrow & R\pi_* \Omega_X^{p-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_X^p & \longrightarrow & \Omega_X^p & \longrightarrow & R\pi_* \Omega_X^p \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{D}(\mathcal{O}_X^{p-1}) & \longrightarrow & \mathbb{D}(\mathcal{O}_X^p) \\
\end{array}
\]

For \( p \leq k \), let \( h^p \) be the left inverse to the map \( \Omega_X^p \rightarrow \Omega_X^p \) defined by taking the inverse to the isomorphism \( \gamma^p = \psi^p \circ \phi^p : \Omega_X^p \rightarrow \mathbb{D}(\mathcal{O}_X^{p-1}) \) and composing it with the map \( \Omega_X^p \rightarrow \mathbb{D}(\mathcal{O}_X^{p-1}) \). Since \( \gamma^p \circ (\alpha \land) = (\alpha \land) \circ \gamma^p \), \( \gamma^p \circ (\alpha \land) = (\alpha \land) \circ \gamma^p \), and \( h^p \circ (\alpha \land) = (\alpha \land) \circ h^p \).

We next deal with the case where \( \dim \Sigma_k = 0 \):

**Proposition 5.4.** Suppose that \( \dim \Sigma_k = 0 \) and that there is a left inverse \( H_k : \Omega_X^p / F^{k+1} \rightarrow \Omega_X^p / \sigma^{\geq k+1} \) to the map \( \Omega_X^p / \sigma^{\geq k+1} \rightarrow \Omega_X^p / F^{k+1} \). Then \( \Omega_X^p / \sigma^{\geq k+1} \cong \Omega_X^p / F^{k+1} \).

**Proof.** This is the argument of Kovács [Kov99], [PS08, p. 186]: Let \( Y \) be some projective completion of \( X \), so that \( Y - X = D \). Let \( Z = D \cup \Sigma_k \subseteq Y \). By hypothesis, \( \Sigma_k \) is finite, hence \( Z \) is closed. Let \( U = X - \Sigma_k = Y - D - \Sigma_k \). Thus \( U \) is \( k \)-Du Bois. Now consider the commutative diagram

\[
\begin{array}{cccc}
\mathbb{H}^{i-1}(U; \Omega^*_U / \sigma^{\geq k+1}) & \longrightarrow & \mathbb{H}^i(Z; \Omega^*_Y / \sigma^{\geq k+1}) & \longrightarrow & \mathbb{H}^i(Y; \Omega^*_Y / \sigma^{\geq k+1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^i(U; \Omega^*_U / F^{k+1}) & \longrightarrow & \mathbb{H}^i(Z; \Omega^*_Y / F^{k+1}) & \longrightarrow & \mathbb{H}^i(Y; \Omega^*_Y / F^{k+1}) \\
\end{array}
\]

We claim that \( \mathbb{H}^i(Y; \Omega^*_Y / \sigma^{\geq k+1}) \rightarrow \mathbb{H}^i(Y; \Omega^*_Y / F^{k+1}) \) is surjective: this is the usual argument that \( H^i(Y; \mathbb{C}) \rightarrow \mathbb{H}^i(Y; \Omega^*_Y / F^{k+1}) \) is surjective and factors through \( H^i(Y; \mathbb{C}) \rightarrow \mathbb{H}^i(Y; \Omega^*_Y / \sigma^{\geq k+1}) \).
Then $\mathbb{H}^i_Z(Y; \Omega^*/\sigma^{\geq k+1}) \to \mathbb{H}^i_Z(Y; \Omega^*/F^{k+1})$ is surjective. We have compatible direct sum decompositions

$$\mathbb{H}^i_Z(Y; \Omega^*/\sigma^{\geq k+1}) \xrightarrow{\cong} \mathbb{H}^i_D(Y; \Omega^*/\sigma^{\geq k+1}) \oplus \mathbb{H}^i_{\Sigma_k}(Y; \Omega^*/\sigma^{\geq k+1})$$

$$\downarrow$$

$$\mathbb{H}^i_Z(Y; \Omega^*/F^{k+1}) \xrightarrow{\cong} \mathbb{H}^i_D(Y; \Omega^*/F^{k+1}) \oplus \mathbb{H}^i_{\Sigma_k}(Y; \Omega^*/F^{k+1})$$.

Hence $\mathbb{H}^i_{\Sigma_k}(Y; \Omega^*/\sigma^{\geq k+1}) \to \mathbb{H}^i_{\Sigma_k}(Y; \Omega^*/F^{k+1})$ is surjective. By excision, $\mathbb{H}^i_{\Sigma_k}(Y; \Omega^*/\sigma^{\geq k+1}) \cong \mathbb{H}^i_{\Sigma_k}(X; \Omega_X^*/\sigma^{\geq k+1})$ and similarly for $\mathbb{H}^i_{\Sigma_k}(Y; \Omega^*/F^{k+1})$. Thus,

$$\mathbb{H}^i_{\Sigma_k}(X; \Omega_X^*/\sigma^{\geq k+1}) \to \mathbb{H}^i_{\Sigma_k}(X; \Omega_X^*/F^{k+1})$$

is surjective. However, the existence of the left inverse $H_k$ gives a map $\mathbb{H}^i_{\Sigma_k}(X; \Omega^*_X/F^{k+1}) \to \mathbb{H}^i_{\Sigma_k}(X; \Omega^*_X/\sigma^{\geq k+1})$ such that the composed map $\mathbb{H}^i_{\Sigma_k}(X; \Omega^*_X/\sigma^{\geq k+1}) \to \mathbb{H}^i_{\Sigma_k}(X; \Omega^*_X/\sigma^{\geq k+1})$ is the identity. Hence

$$\mathbb{H}^i_{\Sigma_k}(X; \Omega^*_X/\sigma^{\geq k+1}) \to \mathbb{H}^i_{\Sigma_k}(X; \Omega^*_X/F^{k+1})$$

is also injective, and thus an isomorphism. Since $\Omega^*_X[X - \Sigma_k] \to \Omega^*_X[X - \Sigma_k]$ is an isomorphism, it then follows from the “Localization principle” of [PS08, p. 186] that $\Omega^*_X/\sigma^{\geq k+1} \cong \Omega^*_X/F^{k+1}$. □

The following corollary then deals with the case $\dim \Sigma = 0$ of Theorem 1.20 in fact under a somewhat weaker hypothesis.

**Corollary 5.5.** Under the hypotheses of the preceding proposition, suppose that there exists a left inverse $H_k: \Omega^*_X/F^{k+1} \to \Omega^*_X/\sigma^{\geq k+1}$ to the map $\Omega^*_X/\sigma^{\geq k+1} \to \Omega^*_X/F^{k+1}$. Then $X$ is $k$-Du Bois. In particular, if $X$ is $k$-rational and $\dim \Sigma_k = 0$, then $X$ is $k$-Du Bois.

**Proof.** The proof is by induction on $k$. There is a morphism of distinguished triangles

$$\Omega^*_X/\sigma^{\geq k} \longrightarrow \Omega^*_X/\sigma^{\geq k+1} \longrightarrow \Omega^k_X \quad \Omega^k_X \quad +1$$

$$\Omega^*/F^{k} \longrightarrow \Omega^*/F^{k+1} \longrightarrow \Omega^k_X \quad \Omega^k_X \quad +1$$

Here, the center vertical arrow is an isomorphism by hypothesis and the left vertical arrow is an isomorphism by induction. Thus, the right vertical arrow is an isomorphism as well. □

To handle the case where $\dim \Sigma_k > 0$, we consider the effect of passing to a general hyperplane section. There is the following general result of Navarro Aznar [GNAPG88, V(2.2.1)]:

**Proposition 5.6** (Navarro Aznar). If $X$ is an algebraic variety and $H$ is a general element of a base point free linear system on $X$, there is a distinguished triangle

$$\Omega^p_{H} \otimes O_H(-H) \xrightarrow{df} \Omega^p_X \otimes^L O_H \to \Omega^p_{H} \quad +1$$

Recall that we have defined the sequence

$$(*)_p \quad 0 \to I/I^2 \otimes \Omega^p_{H} \to \Omega^p_X|H \to \Omega^p_{H} \to 0,$$

Since the maps in the above distinguished triangle are clearly compatible with the augmentation maps $\phi^p$, we get:
Corollary 5.7. If \((*)_p\) is exact, then there is a morphism of distinguished triangles
\[
\begin{array}{c}
\Omega_H^{p-1} \otimes O_H(-H) \xrightarrow{df} \Omega_X^p \otimes O_H \\
\Omega_H^{p-1} \otimes O_H(-H) \xrightarrow{df} \Omega_X^p \otimes^L O_H
\end{array}
\]
In particular, if \(X\) is \((k-1)\)-Du Bois and \(H\), then \((*)_p\) is exact for \(p \leq k\) and hence there is a morphism of distinguished triangles as above for \(p \leq k\).

Proof. We only need to check the final statement. By Theorem 5.8 \(\mathrm{codim} \Sigma \geq 2k-1\). Thus, for a general \(H\), by Proposition 2.6 the sequence \((*)_p\) is exact for all \(p \leq k\).

The key step is then the following:

Theorem 5.8. Let \(X\) be a reduced locally complete intersection and suppose that there exists a compatible set of \(k+1\) left inverses \(\{h^i\}_{p=0}^k\). Then \(X\) is \(k\)-Du Bois.

Proof. We will assume that \(\Sigma_k(X) = \emptyset\) and derive a contradiction. As this is a local question, we can assume that \(X\) is affine. The proof is by induction both on \(k\) and on \(\dim \Sigma_k\). The cases \(k = 0\) (Kovács) and \(\dim \Sigma_k = 0\) have been proved, or follow by starting the induction at \(k = -1\).

Assume inductively that the theorem has been proved for all \(j < k\) and for all varieties \(H\) with \(\dim \Sigma_k(H) < \dim \Sigma_k\). In particular, \(X\) is \((k-1)\)-Du Bois, and hence \(\mathrm{codim} \Sigma \geq 2k-1\). Choose a general \(H\) as in Proposition 5.6. Then \(\dim \Sigma_k \cap H < \dim \Sigma_k\) and, if \(\dim \Sigma_k > 0\), then \(\Sigma_k \cap H = \emptyset\).

We claim that \(\Sigma_k(H) \neq \emptyset\). For otherwise \(H\) is \(k\)-Du Bois, hence \(\Omega_H^p \cong \Omega_H^{p+1}\) for all \(p \leq k\). Then \(\Omega_X^k \otimes^L O_H \cong \Omega_X^k \otimes^L O_H\) as follows from the morphism of distinguished triangles in Corollary 5.7.

On the other hand, we have the distinguished triangle
\[
\Omega_X^k \otimes^L O_H \to \Omega_X^k \otimes^L O_H \to \Omega_X^k \otimes^L O_H \to \Omega_X^k \otimes^L O_H \to 1.
\]

Since \(O_H\) has the projective resolution \(O_X \xrightarrow{f} O_X\), there is an exact sequence
\[
0 \to \operatorname{Tor}_1^{O_X}(O_H, G^k) \to \operatorname{Tor}_1^H(G^k \otimes^L O_H) \to \operatorname{Tor}_1^H(G^k) \otimes O_H \to 0.
\]

Thus \(\bigcup \operatorname{Supp} H^i(G^k \otimes^L O_H) = \Sigma_k \cap H \neq \emptyset\), and hence \(\Omega_X^k \otimes^L O_H \to \Omega_X^k \otimes^L O_H\) is not an isomorphism.

Then for all \(p \leq k\), we claim that we can find a compatible set of \(p+1\) left inverses \(\{h^i\}_{i=0}^p\). Again, we argue by induction on \(p\) and the case \(p = 0\) is clear. For the inductive step, assume that there exists a compatible set of \(p\) filtered left inverses \(\{h^i\}_{i=0}^p\) for \(H\). By the inductive hypothesis, \(H\) and \(X\) are \((p-1)\)-Du Bois. Also, by Corollary 5.7 \(\Omega_X^k \otimes^L O_H = \Omega_X^k \otimes O_H\). In what follows, we fix the function \(f\) and will omit the factor \(O_H(-H)\) as it is trivialized.

By the assumption of compatibility, there is a commutative diagram
\[
\begin{array}{c}
\Omega_X^{p-1} \otimes O_H \xrightarrow{df} \Omega_X^p \otimes^L O_H \\
\Omega_X^{p-1} \otimes O_H \xrightarrow{df} \Omega_X^p \otimes O_H
\end{array}
\]
Then there is an induced diagram
\[
\begin{array}{c}
\Omega_X^{p-1} \otimes O_H \xrightarrow{df} \Omega_X^{p-1} \otimes^L O_H \\
\Omega_X^{p-1} \otimes O_H \xrightarrow{df} \Omega_X^p \otimes O_H
\end{array}
\]

Here, the left hand square is commutative and the outer rectangle is commutative. Since all terms except for $\Omega^p_X \otimes^L \mathcal{O}_H$ are sheaves and the morphism $\Omega^{p-1}_X \otimes \mathcal{O}_H \to \Omega^{p-1}_H$ is surjective, a straightforward argument shows that the right hand square is commutative as well.

Then there is a diagram of distinguished triangles

$$
\begin{array}{c}
\Omega^{p-1}_H \cong \Omega^{p-1}_H \\
\downarrow = \\
\Omega^{p-1}_H \cong \Omega^{p-1}_H \\
\downarrow \\
\Omega^p_H \\
\end{array}
\xrightarrow{df^{\wedge}}
\begin{array}{c}
\Omega^p_X \otimes^L \mathcal{O}_H \\
\downarrow \\
\Omega^p_H \\
\end{array}
\to
\begin{array}{c}
\Omega^p_H \\
\downarrow \\
\Omega^p_H \\
\end{array}
\xrightarrow{+1}
$$

Thus, we can complete the diagram by finding $\tilde{h}^p$ which makes the diagram commute, and it is automatically a left inverse to the map $\Omega^p_H \to \Omega^p_H$. We claim that any such $\tilde{h}^p$ is compatible with $h^{p-1} = \text{Id}$: Since $\Omega^{p-1}_H \cong \Omega^{p-1}_H$, there is a commutative diagram with vertical arrows given by $\alpha \wedge$:

$$
\begin{array}{c}
\Omega^{p-1}_H \cong \Omega^{p-1}_H \\
\downarrow \\
\Omega^p_H \\
\end{array}
\xrightarrow{\tilde{h}^p}
\begin{array}{c}
\Omega^p_H \\
\downarrow \\
\Omega^p_H \\
\end{array}
$$

As $\tilde{h}^p$ is a left inverse, $\tilde{h}^p(\alpha \wedge \varphi) = \alpha \wedge \varphi = \alpha \wedge h^{p-1}(\varphi)$. This completes the inductive step for $p$. If follows that $\tilde{h}^p$ is compatible for all $p \leq k$.

But then, since $\dim \Sigma_k(H) < \dim \Sigma_k$, by the inductive hypothesis $\Sigma_k(H) = \emptyset$. This is a contradiction to the statement that $\Sigma_k(H) \neq \emptyset$.

The following is then the lci case of Theorem 3.20

**Corollary 5.9.** If $X$ is a $k$-rational algebraic variety with lci singularities, then $X$ is $k$-Du Bois. 

*Proof.* This follows from Lemma 5.3 and Theorem 5.8.

In fact, the proof also shows the following result of Mustață-Popa [MP21, Theorem 9.17]:

**Corollary 5.10.** If $X$ is an algebraic variety with $k$-Du Bois singularities, then a general complete intersection $H_1 \cap \cdots \cap H_n$ of $X$ is $k$-Du Bois.

As an application of Corollary 5.9 we have:

**Proposition 5.11.** Let $f : \mathcal{Y} \to S$ be a flat proper family of complex algebraic varieties of relative dimension $n$ over an irreducible base $S$. For $s \in S$, suppose that the fiber $Y_s$ has $k$-rational lci singularities. Then, for every fiber $t$ such that $Y_t$ is smooth, $\dim \text{Gr}^p_F H^{p+q}(Y_t) = \dim \text{Gr}^{n-p}_F H^{2n-p-q}(Y_s)$ for every $q$ and for $0 \leq p \leq k$. Equivalently, for such $p$ and $q$,

$$
\text{h}^{p,q}(Y_t) = \text{h}^{n-p,n-q}(Y_s) = \text{h}^{p,q}(Y_s).
$$

*Proof.* By Corollary 5.9 $Y_s$ is $k$-Du Bois. By Theorem 1.2 for $p \leq k$, $R^q f_* \Omega^p_Y$ is locally free in a neighborhood of $s$ and compatible with base change. Thus,

$$
\dim \text{Gr}^p_F H^{p+q}(Y_t) = \dim H^q(Y_t; \Omega^p_{Y_t}) = \text{h}^{n-p}(Y_s; \Omega^p_{Y_s}).
$$

Then $\dim \text{Gr}^p_F H^{p+q}(Y_t) = \dim \mathbb{H}^q(Y_s; D_Y(\Omega^{n-p}_{Y_s}))$. By Grothendieck duality, $\mathbb{H}^q(Y_s; D_Y(\Omega^{n-p}_{Y_s}))$ is dual to $\mathbb{H}^{n-q}(Y_s; \Omega^{n-p}_{Y_s})$. Computing dimensions gives the result.

In fact, combining the above with the Hodge symmetries given by Theorem 3.23 we obtain Corollary 1.11 announced in the introduction. This is modeled on [KLS22, Thm. 1] (case $k = 0$). Note however that loc. cit. does not assume lci singularities, and works in the analytic category.
Appendix A. Proof of Conjecture L8 for hypersurfaces

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We prove that two definitions of higher k-rational singularities for hypersurfaces coincide, see Theorem [A.1] below. (The case \( k = 0 \) was treated in [Sai93].) This implies a proof of Conjecture L8 for hypersurfaces using the converse of a theorem of Mustaţă, Olano, Popa, and Witaszek [MOPW21, Thm. 1.1] (see [JKSY22a, Thm. 1]).

**Theorem A.1.** Assume \( X \) is a reduced hypersurface of a smooth complex algebraic variety \( Y \). Then for \( k \in \mathbb{Z}_{>0} \), we have \( \tilde{\alpha}_{X} > k + 1 \) if and only if \( X \) has only \( k \)-rational singularities.

**Proof.** Assume \( \tilde{\alpha}_{X} > k + 1 \). We may assume that \( X \subset Y \) is defined by a function \( f \) shrinking \( X, Y \) if necessary. Since \( \tilde{\alpha}_{X} > k + 1 \), we have by [JKSY22a, Thm. 2] the canonical isomorphism

\[
\Omega_{X}^{d_{Y} - k} = \sigma_{> 0}(\Omega_{Y}^{d_{Y} - k} |_{X}, df \wedge),
\]

where \( d_{Y} := \text{dim} Y \). Applying the functor \( \mathbb{D} \), we get the isomorphisms

\[
\mathbb{D}(\Omega_{X}^{d_{Y} - k}) = \mathbb{D}(\sigma_{> 0}(\Omega_{Y}^{d_{Y} - k} |_{X}, df \wedge))
\]

\[
= C(f : \sigma_{< 0}(\Omega_{Y}^{d_{Y}} |_{X}, df \wedge) \Rightarrow \sigma_{< 0}(\Omega_{Y}^{d_{Y}} |_{X}, df \wedge)[d_{Y} - 1],
\]

since \( \mathbb{D}(L) = \mathcal{L} \otimes_{\mathcal{O}_{Y}} \mathcal{L}[d_{Y}] \) for a locally free \( \mathcal{O}_{Y} \)-module \( L \) in general.

It is well known that

\[
\mathcal{H}^{j}(\Omega_{Y}^{d_{Y}} |_{X}, df \wedge) = 0 \text{ if } j < \text{dim}_{Y} \text{Sing } X,
\]

\[
\tilde{\alpha}_{X} < \frac{1}{2} \text{dim}_{Y} \text{Sing } X,
\]

see for instance [JKSY22a, Prop. 1–2]. These imply the quasi-isomorphism

\[
\sigma_{< 0}(\Omega_{Y}^{d_{Y}} |_{X}, df \wedge) \cong \Omega_{Y}^{d_{Y}} / df \wedge \Theta_{Y}^{k-1},
\]

together with the injection

\[
\Omega_{Y}^{d_{Y}} / df \wedge \Theta_{Y}^{k-1} \hookrightarrow \Theta_{Y}^{k+1},
\]

which gives \( f \)-torsion-freeness of \( \Omega_{Y}^{d_{Y}} / df \wedge \Theta_{Y}^{k-1} \). We thus get the canonical isomorphism

\[
\mathbb{D}(\Omega_{X}^{d_{Y} - k})[-d_{X}] = \Omega_{X}^{k}.
\]

Here \( k \) can be replaced by any \( j \in [0, k-1] \). So \( X \) has only \( k \)-rational singularities.

Assume now \( X \) has only \( k \)-rational singularities. This means that the composition

\[
\Omega_{X}^{k} \rightarrow \mathcal{O}_{X}^{k} \rightarrow \mathbb{D}(\Omega_{X}^{d_{Y} - k})[-d_{X}]
\]

is an isomorphism (hence \( \Omega_{X}^{k} \) is a direct factor of \( \mathcal{O}_{X}^{k} \)) and the same holds with \( k \) replaced by any \( j \in [0, k-1] \). We will consider the morphisms obtained by applying the functor \( \mathbb{D} \) to these morphisms. We argue by induction on \( k \). Note that

\[
\tilde{\alpha}_{X} > k,
\]

since \( X \) has only \( (k-1) \)-rational singularities by definition. This implies that

\[
k + 1 < \text{dim}_{Y} \text{Sing } X,
\]

using (4), since \( \text{dim}_{Y} \text{Sing } X \geq 2 \). By the same argument as above, we then get that

\[
\mathbb{D}(\Omega_{X}^{k})[-d_{X}] = \sigma_{> 0}(\Omega_{Y}^{d_{Y} - k} |_{X}, df \wedge).
\]
On the other hand we have by \cite{Sai00} Thm. 4.2

\[(A12)\quad \Omega^k_X = \text{Gr}^F_{\varrho} \text{DR}(\mathbb{Q}_{h,X}[d_X])[k-d_X],\]

hence

\[(A13)\quad D(\Omega^k_X)[-d_X] = \text{Gr}^{d_X-k} \text{DR} D(\mathbb{Q}_{h,X}(d_X)[d_X])[d_X-k].\]

By the theory of Hodge ideals (see \cite{MP19, Sai16, JKS22b, JKS22a}) and using (9), we can get the isomorphism

\[(A14)\quad D(\Omega^k_X)[-d_X] \cong K^{(k)} \subset \sigma_{\geq 0}(\Omega^*_Y [d_Y-k]|_X, df \wedge),\]

where \(K^{(k)}: = \Omega^j_Y + d_Y-k|_X\) if \(j \neq k\), and \(K^{(k),k} := I_k(X)\Omega^d_Y / f \Omega^d_Y\) with \(I_k(X)\) the Hodge ideal.

If \(\alpha_X \in (k, k+1)\) so that \(I_k(X) \neq \mathcal{O}_Y\) (see for instance \cite{Sai16 Cor. 1}), then by (A11), (A14) the canonical morphism

\[(A15)\quad H^{k-d_X} D(\Omega^k_X) \to H^{k-d_X} D(\Omega^k_X)\]

is never surjective. Indeed, since \(\Omega^k_X\) is a direct factor of \(\Omega^k_X\), the mapping cone of a morphism \(\phi: \Omega^k_X \to \Omega^k_X\) is independent of \(\phi\) as long as it induces an isomorphism on \(X \setminus \text{Sing} X\). Note that \(H^0_{\text{Sing} X} \Omega^k_X = 0\), since the proof of Prop. 2.2 in \cite{JKS22a} holds also for \(q = p+1\) (where the last inequality in the proof of Prop. 2.2 becomes \(q+1 = p+2 < \text{codim}_Y \text{Sing} X\)). Hence the dual of the composition (8) cannot be an isomorphism.

Assume now \(\alpha_X = k+1\). (Here \(\Omega^k_X = \Omega^k_X\), see \cite{MOP21, JKS22a}.) The canonical isomorphism (A1) and the second morphism of (8) are induced by the canonical morphism of mixed Hodge modules

\[(A16)\quad \mathbb{Q}_{h,X}[d_X] \to D(\mathbb{Q}_{h,X}(d_X)[d_X]),\]

see \cite{JKS22a} 3.1. Note that this coincides with the composition of \(\mathbb{Q}_{h,X}[d_X] \to \rho_* \mathbb{Q}_{h,X}[d_X]\) with its dual, where \(\rho: \tilde{X} \to X\) is a desingularization. Let \((M', F), (M'', F)\) be the underlying filtered \(D_Y\)-modules of its kernel and cokernel respectively. Then the condition \(\alpha_X = k+1\) implies that

\[(A17)\quad \min \{ p \in \mathbb{Z} \mid F_p M' \neq 0 \} > d_X-k, \quad \min \{ p \in \mathbb{Z} \mid F_p M'' \neq 0 \} = d_X-k;\]

using \cite[(1.3.2–4)]{Sai16} and \cite[3.1]{JKS22a} together with the \(N\)-primitive decomposition, see for instance \cite[(2.2.4)]{KLS22}. We then see that the morphism (A16) cannot induce an isomorphism

\[(A18)\quad \text{Gr}^{d_X-k} \text{DR}(\mathbb{Q}_{h,X}[d_X]) \nrightarrow \text{Gr}^{d_X-k} \text{DR}(D(\mathbb{Q}_{h,X}(d_X)[d_X])).\]

This means that the composition (8) cannot be an isomorphism in view of (A12–A13). This is a contradiction. We thus get \(\alpha_X > k+1\). This finishes the proof of Theorem \(\text{A.1}\). \qed

Combining Theorem \(\text{A.1}\) with the converse of a theorem of Mustață, Olano, Popa, and Witaszek \cite[Thm. 1.1]{MOP21} (see \cite[Thm. 1]{JKS22a}), we can get a positive answer to Conjecture \(\text{I.8}\) for hypersurfaces as follows.

**Corollary A.2.** Assume \(X\) is a reduced hypersurface of a smooth complex algebraic variety \(Y\), and has only \(k\)-Du Bois singularities \((k \geq 1)\). Then \(X\) has only \((k-1)\)-rational singularities.

**Remark A.3.** In \cite{JKS22a} an assertion slightly stronger than the converse of \cite[Thm. 1.1]{MOP21} is proved (since it is not assumed that the isomorphism is induced by the canonical morphism). The assertion can be proved rather easily using the extension of \cite[Prop. 2.2]{JKS22a} to the case \(q = p+1\) as is explained after (A15).
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