Nonperturbative calculational method
in quantum field theory

Vladimir Dzhunushaliev *
Dept. Phys. and Microel. Engineer.,
Kyrgyz-Russian Slavic University
Bishkek, Kievskaya Str. 44, 720021, Kyrgyz Republic

Douglas Singleton†
Physics Dept., CSU Fresno,
M/S MH37 Fresno, CA 93740-8031, USA
and
Dept. of Theo. Phys., PFUR,
6 Miklukho-Maklaya St., Moscow 117198, Russia

Tatyana Nikulicheva
Dept. Phys. and Microel. Engineer.,
Kyrgyz-Russian Slavic University
Bishkek, Kievskaya Str. 44, 720021, Kyrgyz Republic

Abstract
An approximate procedure for performing nonperturbative calculations in quantum field theories is presented. The focus will be quantum non-Abelian gauge theories with the goal of understanding some of the open questions of these theories such as the confinement phenomenon and glueballs. One aspect of this nonperturbative method is the breaking down of the non-Abelian gauge group into smaller pieces. For example $SU(2) \rightarrow U(1) + \text{coset}$ or $SU(3) \rightarrow SU(2) + \text{coset}$. The procedure also uses some aspects of an old method by Heisenberg to calculate the n-point Green’s function of a strongly interacting, non-linear theory. Using these ideas we will give approximate calculations of the 2 and 4-points Green’s function of the theories considered.

1 Introduction
Modern perturbative quantum field theories (QFT) are based on the concept of a particle or field quanta. This perturbative paradigm has given rise to the most accurate description of some aspects of the physical world. The standard example being the theoretical calculation of the anomalous magnetic moment of the electron where theory and experiment are in agreement to the order of $10^{-9}$. Mathematically the elementary particles that arise in perturbative quantum field theory are harmonic excitations of fundamental fields. The quanta are defined through the creation and annihilation operators, $a^\dagger$ and $a$, of the “second-quantized” theory. For instance, in quantum electrodynamics (QED), the entire electromagnetic field can be seen as a collection of superposed quanta, each with an energy $\omega_k$. The Hamiltonian operator of the electromagnetic field (omitting the zero-point energy) can be written

$$\hat{H} = \sum_k \hat{N}_k \omega_k,$$

where

$$\hat{N}_k = a_k^\dagger a_k,$$

is the “number operator”, i.e. giving the number of quanta with a specific four-momentum $k$ when operating on a free state

$$\hat{N}_k |...n_k...\rangle = n_k |...n_k...\rangle.$$

*e-mail: dzhun@hotmail.kg
†e-mail: dougs@csufresno.edu
where \( n_k \) is a positive integer, the number of quanta with that particular momentum. The energy in the electromagnetic field is thus the eigenvalue of the Hamiltonian (1). The reasoning for fermion fields is the same, but then the number of quanta in any given state can be only 0 or 1 due to Fermi statistics.

Usually, in quantum field theories (for example, in QED) “fields” and “particles” are used interchangeably. This is because the almost universal usage of Feynman diagrams gives the false impression that a field is always equivalent to a “cloud” of quanta. The Feynman diagram technique (based on propagators and vertices) can be justified as an approximation for mildly nonlinear theories (i.e., in the weak coupling limit) but breaks down for strongly coupled Abelian and non-Abelian theories.

The physical reason of the appearance of quanta is that in linear theories (for example, in electrodynamics) a general solution can be obtained by making the Fourier expansion:

\[
A_{\mu}^{\text{linear}} = \int d^3k \sum_{\lambda=0}^{3} \left[ a_k(\lambda) \epsilon_{\mu}(k,\lambda)e^{-ik\cdot x} + a_k^\dagger(\lambda) \epsilon_{\mu}^*(k,\lambda)e^{ik\cdot x} \right],
\]  

where \( \epsilon_{\mu} \) is the polarization vector.

However, for a theory based on a non-Abelian group, like quantum chromodynamics (QCD), this can no longer be done in the strong coupling limit, due to the nonlinear nature of Yang-Mills equations

\[
A_{\mu}^{\text{QCD}} \neq \int d^3k \sum_{\lambda=0}^{3} \left[ a_k(\lambda) \epsilon_{\mu}(k,\lambda)e^{-ik\cdot x} + a_k^\dagger(\lambda) \epsilon_{\mu}^*(k,\lambda)e^{ik\cdot x} \right].
\]

Formally one can change this inequality to an equality but this is only a good approximation in the limit when the coupling is weak – \( g_s \ll 1 \). Due to the nonlinear character of non-Abelian gauge theories the superposition principle does not hold in general. There are plane wave solutions in non-Abelian gauge theories [1] however one can not superpose different plane waves as is done in (1.3). Thus, color fields can be represented by harmonic oscillators (gluons) only in the limit when the strong interaction coupling constant tends to zero, \( g_s \to 0 \) (or equivalently when \( Q^2 \to \infty \) because of asymptotic freedom). There are certain nonperturbative field configurations (for example, the hypothesized color electric flux tube that is thought to stretch between quarks and lead to the dual superconducting picture of confinement) in which the field distribution can not be explained as a cloud of quanta. The fields are there, but their quanta – gluons and quarks – are relevant only when probed at sufficiently short distances (or large momentum). One way of looking at this situation is that the fields of these nonperturbative configurations are split into ordered fields (for example, the fields inside flux tube stretched between quark and antiquark) and disordered fields (the fields outside the flux tube). These field can not be interpreted as a (perturbative) cloud of quanta. In such situations the fields play a primary role over the particles [2].

The need for nonperturbative techniques in strongly interacting, nonlinear quantum field theories is an old problem that has been around since the beginning of the study of quantum fields. Much effort has gone into trying to resolve this puzzle. The different approaches that have been tried include: (i) lattice QCD [3]-[18], (ii) the dual Meissner effect in the QCD-vacuum [19]-[21], (iii) instantons [22]-[23], (iv) path integration [24]-[29], (v) analytic calculations [30], (vi) Dyson-Schwinger equations [31] (vii) and various other methods [32]-[37]. Despite this the problem is not yet fully resolved. In the 1950’s Heisenberg [48]-[49] studied a nonlinear spinor field and worked out nonperturbative techniques for quantizing the nonlinear spinor field. In this method one writes down an infinite system of equations which connects together all the n-point Green’s functions of the theory (this can be compared to the infinite number of Feynman diagrams which must, in principle, be calculated for a given process in perturbative quantum field theory). In order to solve this system of equations one must find some physically reasonable approximations for cutting off this system of equations at some point so that one reduces the infinite system of equations into a finite system. Some preliminary work [50]-[53] has been done in applying these nonperturbative quantization techniques, in conjunction with ideas from Abelian projection, to study quantum SU(2) and SU(3) gauge theories, and also to examine the possible application of these techniques to the high-Tc superconductors of condensed matter physics. The advantage of this approach is that it is nonperturbative. The problem is to find physically reasonable motivations for cutting off the infinite equations set for Green’s functions.

The preliminary work in this nonperturbative approach was devoted to a group of classical singular solutions of the Yang-Mills field equations, and their quantization [50]-[53]. In these papers some singular classical solutions were found, and they were then quantized using a nonperturbative method with the approximation that some of the degrees of freedom of the non-Abelian gauge theory could be treated as almost classical degrees of freedom (this was the approximation which cut off the infinite system of equations). After the quantization procedure was applied the bad asymptotic behavior (the divergence of some components of the Yang-Mills fields) of the original classical solution was modified into acceptable asymptotic behavior. Physically the nonperturbative quantization smoothed out the bad classical behavior of the solutions.
In later work [52] the role of the other, non-classical degrees of freedom was investigated. In Ref. [54] the above approach was applied to the SU(2) non-Abelian gauge theory. The assumption was made that the SU(2) gauge field $A_\mu$ could be split into components which had different qualitative behavior: one component, $A_\mu$, was taken to be in an ordered phase, while the remaining two components, $A_{1,2}$, were in a disordered phase. Physically the ordered phase described the U(1) electric field in a hypothesized flux tube stretched between sources (i.e. quarks) of this field, and the disordered phase described a condensate which “pushes out” (i.e. exhibits the Meissner effect) the ordered phase. The calculations of this [54] indicated that such a scenario did occur. In this paper the dynamics of the disordered phase were “frozen” or taken to be nondynamical.

In the present paper we would like to bring together and extend several results of these analytical calculations in nonperturbative QFT. Two general types of calculations will be presented: (i) the decomposition of the SU(3) gauge field components into SU(2) + coset components. This will lead to flux tube solutions which are important in the dual superconducting model of confinement. (ii) The simplification of the SU(3) quantized fields using a scalar field. This will lead to a simple model for a glueball.

2 Heisenberg quantization method

2.1 Initial Heisenberg idea for the quantization of a nonlinear spinor field

Heisenberg’s basic idea is that the n-point Green’s functions can be found from some infinite set of coupled differential equations, which are derived from the field equations for the field operators. As an example we show how this method of quantization works for a spinor field with a nonlinear self interaction [48] [49]. The basic equation for the spinor field is:

$$\gamma^\mu \partial_\mu \hat{\psi}(x) - l^2 \mathfrak{Im}[\hat{\psi}(x)(\hat{\psi}(x)\hat{\psi}(x))] = 0$$  \hspace{1cm}(2.1)

where $\gamma^\mu$ are Dirac matrices; $\hat{\psi}(x)$, $\hat{\psi}(x)$ are the operators of the spinor field and its adjoint respectively; $\mathfrak{Im}[\hat{\psi}(\hat{\psi}\hat{\psi})] = \hat{\psi}(\hat{\psi}\hat{\psi})$ or $\hat{\psi}\gamma^5(\hat{\psi}\gamma^5\hat{\psi})$ or $\hat{\psi}\gamma^\mu\gamma^5(\hat{\psi}\gamma_\mu\hat{\psi})$.

The constant $l$ has units of length, and sets the scale for the strength of the interaction. Heisenberg emphasized that the 2-point Green’s function, $G_2(x_2, x_1)$, in this theory differs strongly from the propagator in a linear theory in its behavior on the light cone: in the nonlinear theory $G_2(x_2, x_1)$ oscillates strongly on the light cone in contrast to the propagator of the linear theory which has a $\delta$-like singularity. Heisenberg defined $\tau$ functions:

$$\tau(x_1 x_2...|y_1 y_2...) = \langle 0|T[\hat{\psi}(x_1)\hat{\psi}(x_2)...\hat{\psi}(y_1)\hat{\psi}(y_2)...]|\Phi\rangle  \hspace{1cm}(2.2)$$

where $T$ is the time ordering operator; $|\Phi\rangle$ is a state for the system described by Eq. (2.1). Eq. (2.2) establishes a one-to-one correspondence between the system state, $|\Phi\rangle$, and the set of functions $\tau$. This state can be defined using the infinite function set of Eq. (2.2). Using equation (2.1) and (2.2) we obtain the following infinite system of equations for the $\tau$'s:

$$l^{-2} \gamma^\mu_{(r)} \frac{\partial}{\partial x^\nu_{(r)}} \tau(x_1...x_n|y_1...y_n) = \mathfrak{Im}[\tau(x_1...x_n x_r|y_1...y_n y_r)] +$$

$$\delta(x_r - y_1)\tau(x_1...x_{r-1} x_{r+1}...x_n|y_2...y_{r-1} y_{r+1}...y_n) +$$

$$\delta(x_r - y_2)\tau(x_1...x_{r-1} x_{r+1}...x_n|y_1 y_2...y_{r-1} y_{r+1}...y_n) + ...$$  \hspace{1cm}(2.3)

Eq. (2.3) represents one of an infinite set of coupled equations which relate various orders (given by the index $n$) of the $\tau$ functions to one another. To make some head way toward solving this infinite set of equations Heisenberg employed the Tamm-Dankoff method whereby he only considered $\tau$ functions up to a certain order. This effectively turned the infinite set of coupled equations into a finite set of coupled equations.

Heisenberg used the procedure sketched above to study the Dirac equation with a nonlinear coupling. Here we apply this procedure to nonlinear, bosonic field theories such as QCD in the low energy limit. In particular we will apply this method to several solutions of the Yang-Mills field equations which have divergent fields and infinite energy. By making certain assumptions analogous to the Tamm-Dankoff cut-off, the unphysical asymptotic behavior of these classical Yang-Mills solutions will be “smoothed” out.

2.2 Heisenberg quantization for QCD

In this subsection we will apply a version of Heisenberg’s quantization method sketched above to QCD. The classical SU(3) Yang-Mills equations are:

$$\partial_\mu F^{\mu\nu} = 0  \hspace{1cm}(2.4)$$
where \( \mathcal{F}^{B}_{\mu \nu} = \partial_{\mu} A^{\nu}_{\mu} - \partial_{\nu} A^{\mu}_{\mu} + g f^{BCD} A^{C}_{\mu} A^{D}_{\mu} \) is the field strength; \( B, C, D = 1, \ldots, 8 \) are the SU(3) color indices; \( g \) is the coupling constant; \( f^{BCD} \) are the structure constants for the SU(3) gauge group. In quantizing the system given in Eqs. 2.4 - 2.5 via Heisenberg’s method one first replaces the classical fields by field operators \( \hat{A}^{B}_{\mu} \rightarrow \hat{A}^{B}_{\mu} \). This yields the following differential equations for the operators

\[
\partial_{\mu} \hat{F}^{B \mu \nu} = 0. \tag{2.5}
\]

These nonlinear equations for the field operators of the nonlinear quantum fields can be used to determine expectation values for the field operators \( \hat{A}^{B}_{\mu} \), where \( \langle \cdots \rangle = \langle Q \mid \cdots | Q \rangle \) and \( | Q \rangle \) is some quantum state. One can also use these equations to determine the expectation values of operators that are built up from the fundamental operators \( \hat{A}^{B}_{\mu} \). For example, the “electric” field operator, \( \hat{E}^{B}_{\mu} = \partial_{\mu} \hat{A}^{B}_{\mu} - \partial_{\nu} \hat{A}^{B}_{\nu} + g f^{BCD} A^{C}_{\mu} A^{D}_{\mu} \) giving the expectation \( \langle \hat{E}^{B} \rangle \). The simple gauge field expectation values, \( \langle A_{\mu}(x) \rangle \), are obtained by average Eq. 2.5 over some quantum state \( |Q\rangle \)

\[
\langle Q \mid \partial_{\mu} \hat{F}^{B \mu \nu} \mid Q \rangle = 0. \tag{2.6}
\]

One problem in using these equations to obtain expectation values like \( \langle A^{B}_{\mu} \rangle \), is that these equations involve not only powers or derivatives of \( \langle A^{B}_{\mu} \rangle \) \( \text{i.e.} \) terms like \( \partial_{\alpha}\langle A^{B}_{\mu} \rangle \) or \( \partial_{\alpha} \partial_{\beta} \langle A^{B}_{\mu} \rangle \) but also contain terms like \( G^{BC}_{\mu \nu} = \langle A^{B}_{\mu} \hat{A}^{C}_{\nu} \rangle \). Starting with Eq. 2.6 one can generate an operator differential equation for the product \( \hat{A}^{B}_{\mu} \hat{A}^{C}_{\nu} \) thus allowing the determination of the Green’s function \( G^{BC}_{\mu \nu} \)

\[
\langle Q \mid \hat{A}^{B}_{\mu}(x) \partial_{\mu} \hat{F}^{B \mu \nu}(x) \mid Q \rangle = 0. \tag{2.7}
\]

However this equation will in turn contain other, higher order Green’s functions \[50\] - \[52\]. Repeating these steps leads to an infinite set of equations connecting Green’s functions of ever increasing order. This construction, leading to an infinite set of coupled, differential equations, does not have an exact, analytical solution and so must be handled using some approximation.

Operators which are involved in equation 2.6 are only well determined if there is a Hilbert space of quantum states. Thus we need to look into the question of the definition of the quantum states \( |Q\rangle \) in the above construction. The resolution to this problem is as follows: There is an one-to-one correspondence between a given quantum state \( |Q\rangle \) and the infinite set of quantum expectation values over any product of field operators, \( G^{mn...}_{\mu \nu...} \langle x_{1}, x_{2} \ldots \rangle = \langle Q | A^{m}_{\mu}(x_{1}) A^{n}_{\nu}(x_{2}) \ldots | Q \rangle \). So if all the Green’s functions – \( G^{mn...}_{\mu \nu...} \langle x_{1}, x_{2} \ldots \rangle \) – are known then the quantum states, \( |Q\rangle \) are known, \text{i.e.} the action of \( |Q\rangle \) on any product of field operators \( \hat{A}^{m}_{\mu}(x_{1}) \hat{A}^{n}_{\nu}(x_{2}) \ldots \) is known. The Green’s functions are determined from the above, infinite set of equations (following Heisenberg’s idea).

Another problem associated with products of field operators like \( \hat{A}^{m}_{\mu}(x) \hat{A}^{n}_{\nu}(x) \) which occur in Eq. 2.6 is that the two operators occur at the same point. For non-interacting field it is well known that such products have a singularity. In this paper we are considering interacting fields so it is not known if a singularity would arise for such products of operators evaluated at the same point. Heisenberg in his investigations of a quantized nonlinear spinor field repeatedly underscored that in a quantum field theory with strong interaction the singularities of propagators can be essentially smoothed out. In this situation the problem of the renormalization of nonrenormalizable QFT will be crucially changed. This problem appears when we have the product of propagators of some quanta, but the nonrenormalizable theories have to be considered on the nonperturbative level which probably gives rise to smoothed out, nonsingular Green’s functions. Thus one might suppose that the nonperturbative quantization methods may resolve the problem of the quantization of nonrenormalizable QFT. Physically it is hypothesized that there are situations in interacting field theories where these singularities do not occur \( \text{e.g.} \) for flux tubes in Abelian or non-Abelian theory quantities like the “electric” field inside the tube, \( \langle E_{z}^{0} \rangle < \infty \), and energy density \( \varepsilon(x) = \langle E_{z}^{2} \rangle < \infty \) are nonsingular. Here we take as an assumption that such singularities do not occur.

### 3 The calculation of 2 and 4-points Green’s functions

The quantization of equations 2.4 - 2.7 in the spirit of section 2 evidently is very hard and deriving exact results is probably impossible. In order to do some calculations we give an approximate method which leads to the 2 and 4-points Green’s functions only. In order to derive the equations describing the quantized field we
average the Lagrangian over a quantum state \(|Q|\)

\[
\langle Q \mid \mathcal{L}_{SU(3)}(x) \mid Q \rangle = \langle \mathcal{L}_{SU(3)} \rangle =
\frac{1}{2} \left\langle \left( \partial_{\mu} \bar{A}_B^B(x) \right) \left( \partial^{\nu} \bar{A}_B^B(x) \right) - \left( \partial_{\mu} \bar{A}_B^B(x) \right) \left( \partial^{\nu} \bar{A}_B^B(x) \right) \right\rangle +
\frac{1}{2} g f^{BCD} \left\langle \left( \partial_{\mu} \bar{A}_B^B(x) - \partial^{\nu} \bar{A}_B^B(x) \right) \bar{A}_C^C(x) \bar{A}_D^D(x) \right\rangle +
\frac{1}{4} g^2 f^{BC_1 D_1} f^{BC_2 D_2} \left\langle \bar{A}_C^C_1(x) \bar{A}_D^D_1(x) \bar{A}_C^C_2(x) \bar{A}_D^D_2(x) \right\rangle.
\] (3.1)

Now we will detail the kind of physical situations we wish to describe. The model given here is similar to stationary turbulence when there are time dependent fluctuations in any point of the liquid but all averaged quantities are time independent. For a QFT this means that all Green’s functions are time independent and there is a correlation between quantum fields in different points at one moment

\[
\langle A_{\mu_1}^{B_1}(x_1) \cdots A_{\mu_n}^{B_n}(x_2) \rangle \neq 0,
\]

\[
t_1 = \cdots = t_n,
\]

\[
\vec{r}_1 \neq \cdots \neq \vec{r}_n.
\]

In linear and perturbative QFT this is not the case because the interaction is carried by quanta which move with a speed less than or equal to the speed of light. In these theories the correlation between quantum fields in different points at the same time is zero.

\[
\langle A_{\mu_1}^{B_1}(x_1) \cdots A_{\mu_n}^{B_n}(x_2) \rangle = 0,
\]

\[
t_1 = \cdots = t_n,
\]

\[
\vec{r}_1 \neq \cdots \neq \vec{r}_n.
\]

In this sense one can say that nonperturbative QFT in some physical situations is very close to turbulence, i.e., in nonperturbative QFT there may exist extended objects where quantized fields at all points are correlated between themselves (example from QCD are flux tubes and glueballs). Such objects fall into two categories

1. The averaged value of all quantized fields are zero

\[
\langle A_{\mu}^B(x) \rangle = 0.
\] (3.8)

But the square of these fields are nonzero

\[
\left\langle \left( A_{\mu}^B(x) \right)^2 \right\rangle \neq 0
\] (3.9)

2. Some components of the quantized fields are nonzero and some zero, and the square of some components is nonzero

\[
\langle A_{\mu_1}^{B_1}(x) \rangle \neq 0 \text{ for some } B_1 \in 1, 2, \cdots, 8
\] (3.10)

\[
\langle A_{\mu_2}^{B_2}(x) \rangle = 0 \text{ for remaining } B_2 \in 1, 2, \cdots, 8
\] (3.11)

\[
\left\langle \left( A_{\mu_3}^{B_3}(x) \right)^2 \right\rangle \neq 0 \text{ for some } B_3 \in 1, 2, \cdots, 8
\] (3.12)

The most natural case is when \(A_{\mu_1}^{B_1}\) belongs to a small subgroup of SU(3) gauge group (for example, to SU(2)) and \(A_{\mu_2}^{B_2}\) are the coset components and \(B_3 = B_2\). For example, \(B_1 = 1, 2, 3\) and \(B_2 = 4, 5, 6, 7, 8\).

In the first case the quantized fields are in a completely disordered phase. In the second case one has both ordered and disordered phases. In this paper we present some calculations which give illustrate each case. The first case of the completely disordered phase will be illustrated by a glueball configuration; the second case of both ordered and disordered phases will be illustrated by a flux tube solution.

### 4 The first case: completely disordered phase – glueball

#### 4.1 The vector to scalar field replacement approximation – \(A_{\mu}^B \rightarrow \phi^B\)

In any quantum field theory the Green’s functions contain the full information about the quantized fields. In this section we will present equations which describe only the 2 and 4-points Green’s functions for SU(3) gauge
theory (i.e. QCD) using the approximation where all the components of the gauge potential are in a disordered phase. To begin we average the SU(3) Lagrangian using some approximate expressions for the 2 and 4-points Green’s functions. The SU(3) Lagrangian is

\[
\mathcal{L}_{SU(3)} = \frac{1}{4} \mathcal{F}_{\mu\nu}^A \mathcal{F}_{\mu\nu}^A
\]  

(4.1)

where \( \mathcal{F}_{\mu\nu}^A = \partial_\mu \mathcal{A}_\nu^A - \partial_\nu \mathcal{A}_\mu^A + gf^{BCD} \mathcal{A}_\mu^B \mathcal{A}_\nu^C \mathcal{A}_\nu^D \) is the field strength operator; \( B, C, D = 1, \ldots, 8 \) are the SU(3) color indices; \( g \) is the coupling constant; \( f^{BCD} \) are the structure constants for the SU(3) gauge group; \( \mathcal{A}_\mu^B \) is the gauge potential operator. In order to derive equations describing the quantized field we average the Lagrangian over a quantum state \( \langle Q \rangle \)

\[
\langle Q | \mathcal{L} | Q \rangle = \langle \mathcal{L} \rangle = \frac{1}{2} \left( \left( \partial_\mu \mathcal{A}_\nu^B \right) \left( \partial^\nu \mathcal{A}^{B\nu}_\nu \right) - \left( \partial_\mu \mathcal{A}^{B\mu}_\mu \right) \left( \partial^\nu \mathcal{A}^{B\nu}_\nu \right) \right) + \frac{1}{2} g f^{BCD} \left( \left( \partial_\mu \mathcal{A}_\nu^B - \partial_\nu \mathcal{A}_\mu^B \right) \mathcal{A}_\mu^C \mathcal{A}_\nu^D \right) + \frac{1}{4} g^2 f^{BCD} f^{EF } \langle \mathcal{A}_\mu^C \mathcal{A}_\nu^D \mathcal{A}_\mu^E \mathcal{A}_\nu^F \rangle \]  

(4.2)

This expression contains the following 2, 3 and 4-points Green’s functions: \( \langle (\partial A)^2 \rangle, \langle (\partial A) A^2 \rangle \) and \( \langle A^4 \rangle \). Defining the 2-point Green’s function

\[
\langle \mathcal{A}^B_\alpha (x) \mathcal{A}^C_\beta (y) \rangle = G^{BC}_{\alpha \beta} (x, y),
\]  

(4.3)

the first term on the r.h.s of equation (4.2) becomes

\[
\left. \left( \partial_\mu \mathcal{A}_\nu^B \right) \left( \partial^\nu \mathcal{A}^{B\nu}_\nu \right) \right|_{y \to x} = \eta^{\alpha\beta} \partial_\mu \partial^\mu G^{AB}_{\alpha \beta} (x, y) \]  

(4.4)

where we denote \( x^\mu \) as \( x \) and \( y^\mu \) as \( y \). For simplicity we consider the case with \( x^0 = y^0 \). For this Green’s function we will make the “one-function approximation” \( \hat{g} \)

\[
G^{AB}_{\alpha \beta} (x, y) \approx -\eta_{\alpha\beta} f^{ACD} f^{BCD} \phi^D (x) \phi^E (y)
\]  

(4.5)

where \( \phi^A (x) \) is the scalar field which describes the 2-point Green’s function. Other authors have used a similar technique of replacing a vector, gauge field, by some combination of scalar fields. For example Corrigan and Fairlie \( \hat{f} \) and Wilczek \( \hat{g} \) were able to transform the Yang-Mills equations into those of a massless scalar \( \lambda \phi^4 \) theory by writing the SU(2) gauge fields in terms of of some combination of effective scalar fields. Physically this approximation means that the quantum properties of the field \( \mathcal{A}_\mu^B \) can be approximately described by a scalar field \( \phi^B (x) \), i.e. in this approximation the Lorentz character and Lorentz index \( \mu \) of the vector gauge fields are treated in a simple way through \( \hat{g} \). Taking into account this approximation we have

\[
\left. \left( \partial_\mu \mathcal{A}_\nu^B \right) \left( \partial^\nu \mathcal{A}^{B\nu}_\nu \right) \right|_{y \to x} = -\eta^{\nu\mu} f^{BAC} f^{RAD} \left( \partial_\mu \phi^C \right) \left( \partial^\nu \phi^D \right) = -12 \left( \partial_\mu \phi^A \right) \left( \partial^\nu \phi^A \right)
\]  

(4.6)

and

\[
\left. \left( \partial_\mu \mathcal{A}_\nu^B \right) \left( \partial^\nu \mathcal{A}^{B\nu}_\nu \right) \right|_{y \to x} = -3 \left( \partial_\mu \phi^A \right) \left( \partial^\nu \phi^A \right),
\]  

(4.7)

Next we make the assumption that the odd Green’s functions average to zero

\[
\left. \left( \partial_\mu \mathcal{A}^C_\alpha \mathcal{A}^D_\gamma (y) \mathcal{A}^D_\gamma (z) \right) \right|_{y \to x} = 0
\]  

(4.8)

This gives

\[
\left. \left( \partial_\mu \mathcal{A}^C_\alpha (x) \mathcal{A}^D_\gamma (y) \mathcal{A}^D_\gamma (z) \right) \right|_{y \to x} = 0
\]  

(4.9)

which finally leads to

\[
\left. \left( \partial_\nu \mathcal{A}^B_\mu - \partial_\mu \mathcal{A}^B_\nu \right) \mathcal{A}^C_\nu \mathcal{A}^{D\nu}_\nu \right|_{y \to x} = 0
\]  

(4.10)

Last for the quartic term on the r.h.s of equation (4.2) we make the approximation that

\[
\left. \left( \partial_\mu \mathcal{A}^B_\nu (y) \mathcal{A}^C_\beta (z) \mathcal{A}^B_\nu (u) \right) \right|_{y \to x} = 0
\]  

(4.11)
This assumption is that the 4-point Greens function is simply the sum of products of the 2-point Green's function. In this approximation the l.h.s of (4.11) is

\[ \langle \hat{A}_\nu^B(x) \hat{A}_\rho^C(x) \hat{A}^{D\mu}(x) \hat{A}^{R\nu}(x) \rangle = \]
\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) \left( f^{BE_1 P_1} f^{CE_1 Q_1} \phi^{P_1}(x) \phi^{Q_1}(x) \right) \left( f^{DE_2 P_2} f^{RE_2 Q_2} \phi^{P_2}(x) \phi^{Q_2}(x) \right) \eta_{\mu\nu} \eta^{\rho\sigma} + \]
\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) \left( f^{BE_1 P_1} f^{DE_1 P_1} \phi^{P_1}(x) \phi^{Q_1}(x) \right) \left( f^{CE_2 P_2} f^{RE_2 Q_2} \phi^{P_2}(x) \phi^{Q_2}(x) \right) \eta_{\mu\sigma} \eta^{\rho\nu} + \]
\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) \left( f^{BE_1 P_1} f^{RE_1 Q_1} \phi^{P_1}(x) \phi^{Q_1}(x) \right) \left( f^{CE_2 P_2} f^{DE_2 Q_2} \phi^{P_2}(x) \phi^{Q_2}(x) \right) \eta_{\mu\nu} \eta^{\rho\sigma} \]

(4.12)

We have taken the spacetime variables of the scalar field, \( \phi \), such that \( x = y = z = u \). In the above expression \( \lambda_{1,2}(P_{1,2}, Q_{1,2}) \) are some parameters depending on the values of the indices \( P_{1,2}, Q_{1,2} \)

\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) = \begin{cases} 
\lambda_1, & \text{if all indices } P_{1,2}, Q_{1,2} = 1, 2, 3, \\
\lambda_2, & \text{if all indices } P_{1,2}, Q_{1,2} = 4, 5, 6, 7, 8, \\
1, & \text{otherwise}
\end{cases} \]

(4.13)

where \( \lambda_{1,2} \) are constants. Equation (4.13) essentially introduces three different coupling strengths (\( \lambda_1, \lambda_2 \), and 1) in place of the original single coupling. This step is necessary in order to insure that the solutions we find are at least quasi-stable. This will be explained in more detail when we discuss the numerical solutions of this system in the next section. From the definition of the index \( \lambda_{1,2}(P_{1,2}, Q_{1,2}) \) one can see that this approximate quantization procedure acts differently for the scalar field components which belong to the small subgroup \( SU(2) \subset SU(3) \) versus the coset \( SU(3)/SU(2) \). The first term in eq. (4.12) can be written as

\[ \left( f^{BE_1 P_1} f^{CE_1 Q_1} \phi^{P_1}(x) \right) \left( f^{DE_2 P_2} f^{RE_2 Q_2} \phi^{P_2}(x) \phi^{Q_2}(x) \right) = \]
\[ \lambda_1 \left( f^{BE_1 a} f^{CE_1 b} \phi^a \phi^b \right) \left( f^{DE_2 c} f^{RE_2 d} \phi^c \phi^d \right) + \lambda_2 \left( f^{BE_1 m} f^{CE_1 n} \phi^m \phi^n \right) \left( f^{DE_2 P} f^{RE_2 Q} \phi^P \phi^Q \right) + \text{(other terms)} \]

(4.14)

A straightforward calculation using the antisymmetry property of the structure constants shows that

\[ f^{ABC} f^{ADR} \left( f^{BE_1 P_1} f^{CE_1 Q_1} \phi^{P_1}(x) \right) \left( f^{DE_2 P_2} f^{RE_2 Q_2} \phi^{P_2}(x) \right) = 0, \]

(4.15)

\[ f^{ABC} f^{ADR} \left( f^{BE_1 a} f^{CE_1 b} \phi^a \phi^b \right) \left( f^{DE_2 c} f^{RE_2 d} \phi^c \phi^d \right) = 0, \]

(4.16)

\[ f^{ABC} f^{ADR} \left( f^{BE_1 m} f^{CE_1 n} \phi^m \phi^n \right) \left( f^{DE_2 P} f^{RE_2 Q} \phi^P \phi^Q \right) = 0. \]

(4.17)

which consequently yields

\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) f^{ABC} f^{ADR} \left( f^{BE_1 P_1} f^{CE_1 Q_1} \phi^{P_1}(x) \right) \left( f^{DE_2 P_2} f^{RE_2 Q_2} \phi^{P_2}(x) \phi^{Q_2}(x) \right) \eta_{\mu\nu} \eta^{\rho\sigma} = 0. \]

(4.18)

A similar calculation for the second term in eq. (4.12) shows that

\[ f^{ABC} f^{ADR} \left( f^{BE_1 P_1} f^{DE_1 Q_1} \phi^{P_1}(x) \phi^{Q_1}(x) \right) \]
\[ \left( f^{CE_2 P_2} f^{RE_2 Q_2} \phi^{P_2}(x) \phi^{Q_2}(x) \right) = \frac{27}{8} \left( \phi^a \phi^a + \phi^m \phi^m \right)^2, \]

(4.19)

\[ f^{ABC} f^{ADR} \left( f^{BE_1 a} f^{DE_1 b} \phi^a \phi^b \right) \left( f^{CE_2 c} f^{RE_2 d} \phi^c \phi^d \right) = \frac{27}{8} \left( \phi^a \phi^a \right)^2, \]

(4.20)

\[ f^{ABC} f^{ADR} \left( f^{BE_1 m} f^{DE_1 n} \phi^m \phi^n \right) \left( f^{CE_2 P} f^{RE_2 Q} \phi^P \phi^Q \right) = \frac{27}{8} \left( \phi^m \phi^m \right)^2. \]

(4.21)

Consequently

\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) f^{ABC} f^{ADR} \left( f^{BE_1 P_1} f^{DE_1 Q_1} \phi^{P_1}(x) \phi^{Q_1}(x) \right) \left( f^{CE_2 P_2} f^{RE_2 Q_2} \phi^{P_2} \phi^{Q_2} \right) = \]
\[ \frac{27}{8} \lambda_1 \left( \phi^a \phi^a \right)^2 + \frac{27}{8} \lambda_2 \left( \phi^m \phi^m \right)^2 + \frac{27}{4} \left( \phi^a \phi^a \right) \left( \phi^m \phi^m \right). \]

(4.22)

Consequently

\[ \lambda_{1,2}(P_{1,2}, Q_{1,2}) f^{ABC} f^{ADR} \left( f^{BE_1 P_1} f^{DE_1 Q_1} \phi^{P_1}(x) \phi^{Q_1}(x) \right) \left( f^{CE_2 P_2} f^{DE_2 Q_2} \phi^{P_2} \phi^{Q_2} \right) = \]
\[ \frac{27}{8} \lambda_1 \left( \phi^a \phi^a \right)^2 + \frac{27}{8} \lambda_2 \left( \phi^m \phi^m \right)^2 + \frac{27}{4} \left( \phi^a \phi^a \right) \left( \phi^m \phi^m \right). \]

(4.23)
Combining these results gives the following expression for the quartic term

\[ f^{ABCD} f_{1234} = \frac{81}{2} \lambda_1 (\phi^a \phi^a)^2 + \frac{81}{2} \lambda_2 (\phi^m \phi^m)^2 + 81 (\phi^a \phi^a) (\phi^m \phi^m). \]  \hspace{1cm} (4.24)

Bringing together all the results gives the following effective Lagrangian which describes the 2 and 4-points Green’s functions

\[ \mathcal{L}_{\text{eff}} = -\frac{9}{2} \left( \partial_{\mu} \phi^A \right) \left( \partial^{\mu} \phi^A \right) + \frac{g^2}{4} \left[ \frac{81}{2} \lambda_1 (\phi^a \phi^a)^2 + \frac{81}{2} \lambda_2 (\phi^m \phi^m)^2 + 81 (\phi^a \phi^a) (\phi^m \phi^m) \right]. \]  \hspace{1cm} (4.25)

This Lagrangian can be put in the standard form by making the following redefinitions: \( \phi^a \rightarrow 2\phi^a/(3g) \) and \( \lambda_{1,2} \rightarrow \lambda_{1,2}/2 \) which yields

\[ \frac{g^2}{4} \mathcal{L}_{\text{eff}} = -\frac{1}{2} \left( \partial_{\mu} \phi^A \right)^2 + \frac{\lambda_1}{4} (\phi^a \phi^a - \phi_0^a \phi_0^a)^2 - \frac{\lambda_1}{4} (\phi_0^a \phi_0^a)^2 + \frac{\lambda_2}{4} (\phi^m \phi^m - \phi_0^m \phi_0^m)^2 + (\phi^a \phi^a) (\phi^m \phi^m). \]  \hspace{1cm} (4.26)

By splitting up the kinetic energy term into parts for \( \phi^a \) and \( \phi^m \) one sees that this Lagrangian is of the form of two, massless scalar fields with \( \lambda \phi^2 \) self interaction and an interaction term between the two scalar field (i.e. the last term in (4.20)). Now in (4.20) it was shown that radiative corrections could introduce a symmetry breaking (i.e. negative) mass term into an scalar Lagrangian similar to that in (4.20). This effect is called dimensional transmutation. For the pure scalar case one can not rigourously justify a radiatively generated symmetry breaking term since the scale at which the symmetry breaking occurs lies outside the region where perturbation theory is valid. Nevertheless it was postulated that a nonperturbative calculation would yield a similar negative mass, symmetry breaking term. In the present paper we will also make this assumption and add negative mass terms to the two scalar fields so that the Lagrangian can be written as

\[ \frac{g^2}{4} \mathcal{L}_{\text{eff}} = -\frac{1}{2} \left( \partial_{\mu} \phi^A \right)^2 + \frac{\lambda_1}{4} (\phi^a \phi^a - \phi_0^a \phi_0^a)^2 - \frac{\lambda_1}{4} (\phi_0^a \phi_0^a)^2 + \frac{\lambda_2}{4} (\phi^m \phi^m - \phi_0^m \phi_0^m)^2 + (\phi^a \phi^a) (\phi^m \phi^m). \]  \hspace{1cm} (4.27)

where \( \phi_0^A = (\phi_0^a, \phi_0^m) \) are some constants. The mass terms come from the cross terms of the second and fourth terms above. Next we drop the constant terms in the Lagrangian which gives

\[ \frac{g^2}{4} \mathcal{L}_{\text{eff}} = -\frac{1}{2} \left( \partial_{\mu} \phi^A \right)^2 + \frac{\lambda_1}{4} (\phi^a \phi^a - \phi_0^a \phi_0^a)^2 + \frac{\lambda_2}{4} (\phi^m \phi^m - \phi_0^m \phi_0^m)^2 + (\phi^a \phi^a) (\phi^m \phi^m). \]  \hspace{1cm} (4.28)

In the next subsection we will investigate numerically if this Lagrangian leads to a solution which is well behaved in some finite range, and which therefore could be viewed as a glueball. As a first step we write down the field equations that give an effective scalar field description of QCD

\[ \partial_{\mu} \partial^\mu \phi^a = -\phi^a \left[ 2\phi^m \phi^m + \lambda_1 (\phi^a \phi^a - \phi_0^a \phi_0^a) \right], \]  \hspace{1cm} (4.29)

\[ \partial_{\mu} \partial^\mu \phi^m = -\phi^m \left[ 2\phi^a \phi^a + \lambda_2 (\phi^m \phi^m - \phi_0^m \phi_0^m) \right]. \]  \hspace{1cm} (4.30)

In ref. (4.20) a similar coupled scalar field system was investigated and shown to have soliton solutions for some specific choice of the potential terms. In the next subsection it will be shown that the Lagrangian in (4.20) has solutions are well behaved in some finite range. We will interpret these solutions as glueballs since the original starting Lagrangian was the pure SU(3) gauge theory. The importance of the quantization procedure that led to the Lagrangian in (4.20) is evident. For classical, pure SU(3) theory one has no finite energy solutions – there are no classical glueballs. Our approximate, nonperturbative quantization method transformed the original pure Yang-Mills into an effective interacting scalar Lagrangian of (4.20). In the next section solutions which have a soliton-like character in some finite range are investigated, and it is suggested that this solutions may give a model for glueballs.

### 4.2 Soliton/Glueball solution for the \( \phi^A \) field

We begin our investigation of the field equations (4.20) (4.30) by taking the following ansatz for the scalar fields

\[ \phi^a(r) = \frac{\phi_0^a}{\sqrt{6}}, \quad a = 1, 2, 3, \]  \hspace{1cm} (4.31)

\[ \phi^m(r) = \frac{f(r)}{\sqrt{10}}, \quad m = 4, 5, 6, 7, 8. \]  \hspace{1cm} (4.32)
This ansatz implies that the components $\phi^a$ and the components $\phi^m$ have different behavior from one another. Substituting equations (4.31) (4.32) into equations (4.29) (4.30) gives the following coupled, nonlinear equations

$$\phi'' + \frac{2}{x}\phi' = \phi \left[ f^2 + \lambda_1 \left( \phi^2 - m^2 \right) \right],$$

$$f'' + \frac{2}{x}f' = f \left[ \phi^2 + \lambda_2 \left( f^2 - \mu^2 \right) \right].$$

(4.33)

(4.34)

The primes denote differentiation with respect to $r$, we have redefined $\lambda_{1,2}/2 \rightarrow \lambda_{1,2}$, and $2\phi_0^m\phi_0^m = m^2$ and $2\phi_0^m = \mu^2$; $m, \mu$ are constants which will be calculated by solving equations (4.33) and (4.34). We have not found analytical solutions to these equations and so we investigated them numerically. The preliminary numerical investigation showed that these equations do not have regular solutions, $f(x), \phi(x)$, for an arbitrary choice of the $m, \mu$ parameters. However, certain discrete values of these parameters appear to give solutions which are well behaved at least for some finite range. Thus we will solve equations (4.33) (4.34) as a nonlinear eigenvalue problem for eigenstates $\phi(x), f(x)$ and eigenvalues $m, \mu$, i.e. we will calculate the parameters $m, \mu$ such that the solutions $\phi(r)$ and $f(r)$ are regular solutions for some finite range.

The solutions of (4.34) will depend on the initial conditions of the ansatz functions ($\phi(0), f(0), \phi'(0), f'(0)$) and on $\lambda_{1,2}$. We will rescale the solutions by dividing equations (4.33) (4.34) by $\phi(0)$, and then introducing the dimensionless radius $x = r\phi(0)$. Finally by redefining $\phi(x)/\phi(0) \rightarrow \phi(x), f(x)/\phi(0) \rightarrow f(x)$ and $m/\phi(0) \rightarrow m, \mu/\phi(0) \rightarrow \mu$ we arrive at the rescaled equations

$$\phi'' + 2x\phi' = \phi \left[ f^2 + \lambda_1 \left( \phi^2 - m^2 \right) \right],$$

$$f'' + 2x f' = f \left[ \phi^2 + \lambda_2 \left( f^2 - \mu^2 \right) \right].$$

(4.35)

(4.36)

where now the primes denote differentiation with respect to $x$. We will look for solutions to (4.35) (4.36) which are finite at the origin, and which approach finite values as $x \rightarrow \infty$. In particular we would like the asymptotic values to approach $\phi(\infty) = m$ and $f(\infty) = 0$. We have not been able to numerically integrate the equations out to arbitrary $x$, but we have found solutions with $\phi(x_F) = m$ and $f(x_F) = 0$, where $x_F$ is some finite $x$ coordinate. To implement the numerical solution of (4.35) (4.36) we used the following boundary conditions

$$\phi(0) = 1, \quad \phi'(0) = 0,$$

$$f(0) = f_0, \quad f'(0) = 0.$$  

(4.37)

(4.38)

These boundary conditions will give solutions which have soliton-like profiles for the region $0 < x < x_F$. We now rewrite equation (4.36) in the following form

$$- \left( f'' + \frac{2}{x} f' \right) + f V_{eff} = (\lambda_2 \mu^2) f.$$  

(4.39)

where we have introduced an effective potential

$$V_{eff} = (\phi^2 + \lambda_2 f^2).$$  

(4.40)

Thus equation (4.39) is now in the form of the Schrödinger equation. It will have regular solutions provided $V_{eff}$ has a deep enough “potential well”. Taking into account the desired boundary conditions of $\phi(x)$ and $f(x)$ for $x \rightarrow x_F$ (i.e. $\phi(x_F) \rightarrow m$ and $f(x_F) \rightarrow 0$) we want $V_{eff}(x_F) > V_{eff}(0)$ or $m^2 > 1 + \lambda_2 f_0^2$. We would like to set $x_F = \infty$, but numerically we have only been able to integrate the equations (4.35) (4.36) out to some finite $x_F$. If $m^2$ is too close to $1 + \lambda_2 f_0^2$ the “potential well” may be too shallow and no solution exist. However, assuming $m^2$ is large enough a regular solution will exist and the “energy levels”, $\lambda_2 \mu^2$, will be quantized.

### 4.3 Numerical solution

We will use the following numerical method for solving equations (4.35) (4.36): input some arbitrary zeroth order approximation for the function $f(x)$ (which is denoted $f_0(x)$) and solve equation (4.35) for the zeroth order $\phi (\phi_0)$

$$\phi''_0 + \frac{2}{x}\phi'_0 = \phi_0 \left[ f_0^2 + \lambda_1 \left( \phi_0^2 - m_0^2 \right) \right]$$

(4.41)

where $m_0$ is the zeroth order approximation for the parameter $m$. The boundary conditions are given in (4.37). This regular solution $\phi_0(x)$ is then used to calculate the next order $f(x)$ ansatz function. Taking $g_0(x)$ and substituting it into equation (4.36) we now have an equation for the next order in $f(x)$.

$$f'' + \frac{2}{x} f' = f_1 \left[ \phi_0^2 + \lambda_2 \left( f_1^2 - \mu_1^2 \right) \right].$$

(4.42)
with the boundary conditions of equation (4.38) (for the present numerical calculations we take \( f_0(0) = \sqrt{0.6} \)). Numerically solving (4.42) then gives the first approximation \( f_1(x) \) which is then substituted into equation (4.35)

\[
\phi''_1 + \frac{2}{x} \phi'_1 = \phi_1 \left[ f_1^2 + \lambda_1 \left( \phi_1^2 - m_1^2 \right) \right].
\]
(4.43)

This equation is then solved for the first approximation in \( \phi_1(x) \) which is then put back into the next approximate equation for \( f(x) \) and so on i.e.

\[
\phi''_i + \frac{2}{x} \phi'_i = \phi_i \left[ f_i^2 + \lambda_i \left( \phi_i^2 - m_i^2 \right) \right]
\]
(4.44)

and

\[
f''_i + \frac{2}{x} f'_i = f_i \left[ \phi_i^2 + \lambda_2 \left( f_i^2 - \mu_i^2 \right) \right].
\]
(4.45)

At each step we find values \( m_i^2 \) and \( \mu_i^2 \) which approach the true eigenvalues values \( m^*^2 \) and \( \mu^*^2 \).

4.3.1 A more detailed description of the numerical calculations

For the numerical solution of equation (4.41) we make an initial arbitrary choice of the zeroth order approximation for \( f(x) \) as

\[
f_0(x) = \frac{\sqrt{0.6}}{\cosh^2 \frac{x}{4}}.
\]
(4.46)

The reason for this choice is that the \( \cosh \) function in (4.46) has a soliton-like shape, and we are looking for soliton-like solutions. Typical solutions for arbitrary values of \( m_0 \) are presented in Fig.1. From the figure one sees that when \( m_0 < m_0^* \) (\( m_0^* \) is the unknown parameter which is conjectured to give a regular solution) the solution \( \phi_0(x) \) is singular. Near to the singularity the equation (4.41) has the form

\[
\phi''_0 \approx \lambda_1 \phi_0^3
\]
(4.47)

and the following approximate solution

\[
\phi_0(x) \approx \sqrt{\frac{2}{\lambda_1}} \frac{1}{x_0 - x}
\]
(4.48)

where \( x_0 \) is some constant depending on \( m_0 \). On the other hand for \( m_0 > m_0^* \) a typical solution is shown in Fig.2. The corresponding asymptotical equation is

\[
\phi''_0(x) + \frac{2}{x} \phi'_0 \approx - \left( \lambda_1 m^2 \right) \phi_0
\]
(4.49)

which has the following approximate solution

\[
\phi_0(x) \approx \phi_\infty \frac{\sin \left( x\sqrt{\lambda_1 m^2} + \alpha \right)}{x}
\]
(4.50)
where $\phi_\infty$ and $\alpha$ are constants. We conjecture that there may exist some value, $m_0^\ast$, which yields an exceptional, non-singular solution. Within some numerical accuracy such a solution is presented in Fig.1. For this value $m_0^\ast$ the equation (4.14) has the following asymptotical behavior

$$\phi''_0(x) + \frac{2}{x} \phi'_0 \approx 2\lambda_1 (m_0^\ast)^2 (\phi_0 - m_0^\ast)$$

and the corresponding asymptotical solution is

$$\phi_0(x) \approx m_0^\ast + \beta e^{-\left(x\sqrt{2\lambda_1 (m_0^\ast)^2}\right)}$$

where $\phi_\infty$ and $\beta$ are some constants.

The next step is finding the first approximation for the $f_1(x)$ function. The equation is

$$f''_1 + \frac{2}{x} f' = f_1 \left[ \phi''_0 + \lambda_2 (f_1 - \mu_1^2) \right].$$

From the previous calculations we use the exceptional regular solution $\phi_0(x)$ with the asymptotical behavior (4.12). Then the numerical investigation shows that for an arbitrary $\mu$ there are two different singular solutions for $f_1(x)$ which are presented in Fig.2. As with equation (4.17) the singular behavior of the function $f_1(x)$ is given by

$$f_1(x) \approx \sqrt{\frac{2}{\lambda_2} \frac{1}{x-x_0}} \quad \text{by} \quad \mu_1 < \mu_1^\ast,$$

$$f_1(x) \approx -\sqrt{\frac{2}{\lambda_2} \frac{1}{x-x_0}} \quad \text{by} \quad \mu_1 > \mu_1^\ast.$$ 

This again may indicate that there is some intermediate special value for $\mu_1$ (i.e. $\mu_1 = \mu_1^\ast$) for which the solution, $f_1^\ast(x)$, is regular and has the following asymptotical behavior

$$f_1^\ast(x) \approx f_\infty e^{-x\sqrt{\phi''_\infty - \lambda_2 \mu^2}} \frac{1}{x}$$

where $f_\infty$ is some parameter. The next step is substituting the first approximation $f_1^\ast(x)$ into equation (4.15) to find the regular, exceptional solution $\phi_1^\ast(x)$ with eigenvalue $m_1^\ast$; then $\phi_1^\ast(x)$ is substituted into equation (4.16) to find the regular exceptional solution $f_2^\ast(x)$ with eigenvalue $\mu = \mu_2^\ast$ and so on.

The result of these calculations is presented in Fig’s. 3 and Table 1. We see that there is the convergence $\phi_1^\ast(x) \rightarrow \phi^\ast(x)$, $f_1^\ast(x) \rightarrow f^\ast(x)$, $m_1^\ast \rightarrow m^\ast$ and $\mu_1^\ast \rightarrow \mu^\ast$; $f^\ast(x), \phi^\ast(x)$ are the eigenstates and $m^\ast, \mu^\ast$ are eigenvalues of the nonlinear eigenvalue problem (4.13) (4.14).

From the figures it can be seen that the numerical integration of the coupled, nonlinear equations was only carried out to a finite $x$ (roughly $x = 20$). Beyond this it became increasingly difficult to adjust the mass values, $m^\ast, \mu^\ast$ in order to obtain the desired soliton-like behavior. At this stage it is not certain whether the solution can be made to be well behaved out to arbitrary $x$. However, the analogy between our system and the Schrödinger Equation given in (1.39) indicates that there should be some parameters which give regular
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Table 1: The iterative parameters $m_i^*$ and $\mu_i^*$.

|    | 1          | 2          | 3          | 4          |
|----|------------|------------|------------|------------|
| $m_i^*$ | 1.590537877... | 1.57911774... | 1.565224661... | 1.567908386... |
| $\mu_i^*$ | 1.222955... | 1.2287... | 1.22902... | 1.229019... |

solutions out to $x = \infty$. However based on the numerical results it is at present not certain as to the asymptotic behavior of the ansatz functions.

This numerical method was tested by applying it to the standard, analytic soliton solution of $\lambda \phi^4$ theory in 1 spatial dimensions. Details of this can be found in Appendix B.

Finally, we remark that according to Derrick’s Theorem [61] our solution, if it turns out to be regular for $0 < x < \infty$, it will not be stable. For pure scalar field theories as in [428] absolutely stable solutions exist only in 1 or 2 spatial dimensions. Thus our solution is at best quasi-stable. This is in fact the reason for choosing different couplings (i.e. $\lambda_1, \lambda_2, 1$) in (4.13). This resulted in the potential for the interacting scalar fields of (4.28) having global and local minima. Our solution is in one of the local minima. Thus although it is not absolutely stable it should be quasi-stable so long as there is a large enough barrier between the local and global minima. The can be accomplished by requiring that $\lambda_1$ and $\lambda_2$ be significantly different. This however raises the problem of why one should develop two different scales from a theory that originally only had a single scale. We will leave this open question for future work.

The final remark is that we can add the constant $\lambda_2 (\phi_0^m \phi_0^m)^2$ (which have not any influence on the field equations (4.29) (4.30)) to the Lagrangian (4.28) we will have the finite energy of glueball. The profiles of the energy with respect to $\lambda_1, \lambda_2$ and with different $\phi_0$ are presented in fig’s 5 6.

![Figure 5: The profile of the glueball energy with respect to $\lambda_1$.](image)

![Figure 6: The profile of the glueball energy with respect to $\lambda_2$.](image)

5 The second case: ordered and disordered phases – flux tube.

5.1 Reduction to a small subgroup

In this section we will describe the second case where one has both an ordered and disordered phase. We reiterate that we have not found an exact analytical method for solving the full equations (2.5) (2.6) for all the Green’s functions. Our basic approach in this case is to give some physically reasonable scheme for cutting off the infinite set of equations for the Green’s functions. In addition our approximate calculations will involve the decomposition of the initial gauge group into a smaller gauge group: $SU(3) \rightarrow SU(2)$. Physically we will find that this reduction splits the initial degrees of freedom (the $SU(3)$ gauge potential components) into $SU(2)$ and coset components. The $SU(2)$ components will represent the ordered phase while the coset components, which again will be represented via an effective scalar field, will represent the disordered phase. After these approximations one can perform analytical calculations which suggest the formation of a color electric flux tube. This is of interest since an important feature of the confinement phenomenon in the dual superconducting picture of the QCD vacuum is the formation of electric flux tubes between the confined quarks.

First we define the reduction of $SU(3)$ to $SU(2)$ following the conventions of ref. [21]. Starting with the $SU(3)$ gauge group with generators $T^B$, we define the $SU(3)$ gauge fields, $A_\mu = A_\mu^B T^B$. $SU(2)$ is a subgroup of
SU(3) and SU(3)/SU(2) is a coset. Then the gauge field $A_\mu$ can be decomposed as

$$A_\mu = A_\mu^{B=T} = a_\mu^a T^a + A_\mu^m T^m,$$

where the indices $a,b,c\ldots$ belongs to the subgroup SU(2) and $m,n,\ldots$ to the coset SU(3)/SU(2); $B$ are SU(3) indices. Based on this the field strength can be decomposed as

$$F_{\mu\nu}^B = F_{\mu\nu}^a + F_{\mu\nu}^m$$

where

$$F_{\mu\nu}^a = h_{\mu\nu}^a + \Phi_{\mu\nu}^a \in SU(2),$$

$$h_{\mu\nu}^a = \partial_\mu a_\nu^a - \partial_\nu a_\mu^a + g_{\mu\nu}^{abc} a_\mu^b a_\nu^c \in SU(2),$$

$$\Phi_{\mu\nu}^a = g_{\mu\nu}^{mn} A_\mu^m A_\nu^n \in SU(2),$$

$$F_{\mu\nu}^m = F_{\mu\nu}^m + G_{\mu\nu}^m \in SU(3)/SU(2),$$

$$F_{\mu\nu}^m = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m + g f^{mnp} A_\mu^n A_\nu^p \in SU(3)/SU(2),$$

$$G_{\mu\nu}^m = g f^{mnk} (A_\mu^k A_\nu^l - A_\nu^k A_\mu^l) \in SU(3)/SU(2)$$

where $f^{ABC}$ are the structure constants of SU(3), $e^{abc} = f^{abc}$ are the structure constants of SU(2). The SU(3) Yang-Mills field equations can be decomposed as

$$d_\nu (h_{\mu\nu}^a + \Phi_{\mu\nu}^a) = -g f^{mnk} A_\mu^m (F_{\nu\mu}^n + G_{\nu\mu}^n),$$

$$D_\nu (F_{\mu\nu}^m + G_{\mu\nu}^m) = -g f^{mnk} [A_\nu^m (h_{\nu\mu}^k + \Phi_{\nu\mu}^k) - a_\nu^m (F_{\mu\nu}^n + G_{\mu\nu}^n)],$$

where $d_\nu \cdots = \partial_\nu \cdots + f^{abc} a_\nu^b \cdots$ is the covariant derivative on the subgroup SU(2) and $D_\nu \cdots = \partial_\nu \cdots + f^{mnp} A_\nu^m \cdots \partial_\nu^p$. The Heisenberg quantization procedure gives us

$$d_\nu (\hat{h}_{\mu\nu}^a + \hat{\Phi}_{\mu\nu}^a) = -g f^{mnk} \hat{A}_\mu^m (\hat{F}_{\nu\mu}^n + \hat{G}_{\nu\mu}^n),$$

$$D_\nu (\hat{F}_{\mu\nu}^m + \hat{G}_{\mu\nu}^m) = -g f^{mnk} [\hat{A}_\nu^m (\hat{h}_{\nu\mu}^k + \hat{\Phi}_{\nu\mu}^k) - \hat{a}_\nu^m (\hat{F}_{\mu\nu}^n + \hat{G}_{\mu\nu}^n)]$$

### 5.2 Basic assumptions

It is evident that an exact quantization is impossible for eqs. (5.10) (5.11). Thus we look for some simplification in order to obtain equations which can be analyzed. Our basic aim is to cut off the set of infinite equations using some simplifying assumptions. For this purpose we will propose simple but physically reasonable ansätze for the 2 and 4-points Green’s functions – $\langle A_\mu^m(y)A_\nu^n(x) \rangle$, $\langle a_\mu(0)A_\nu^n(y)A_\mu^m(z)A_\nu^n(u) \rangle$ and $\langle A_\mu^m(y)A_\nu^n(z)A_\nu^n(u) \rangle$ – which are involved in averaged Lagrangian (5.2). As was mentioned earlier we assume that there are two phases

1. The gauge field components $a_\mu^a$ ($a=1,2,3$ $a_\mu^a \in SU(2)$) belonging to the small subgroup SU(2) are in an ordered phase. Mathematically this means that

$$\langle a_\mu^a(x) \rangle = (a_\mu^a(x))_{cl}.$$

The subscript means that this is a classical field. Thus we are treating these components as effectively classical gauge fields in the first approximation.

2. The gauge field components $A_\mu^m$ ($m=4,5,\ldots, 8$ and $A_\mu^m \in SU(3)/SU(2)$) belonging to the coset SU(3)/SU(2) are in a disordered phase (i.e. they form a condensate), but have non-zero energy. In mathematical terms this means that

$$\langle A_\mu^m(x) \rangle \neq 0$$

Later we will postulate a specific, and physically reasonable form for the non-zero term.

3. There is no correlation between the ordered (classical) and disordered (quantum) phases

$$\langle f(a_\mu^a)g(A_\nu^n) \rangle = f(a_\mu^a) \langle g(A_\nu^n) \rangle$$

Later we will give a specific form for this for the “mixed” 4-point Green’s function,

$$\langle a_\mu(0)a_\nu^n(y)A_\mu^m(z)A_\nu^n(u) \rangle.$$
5.3 Derivation of an effective Lagrangian

Our quantization procedure will deviate from the Heisenberg method in that we will take the expectation of the Lagrangian rather than for the equations of motions. Thus we will obtain an effective Lagrangian rather than approximate equations of motion. The Lagrangian we obtain from the original SU(2) Yang-Mills-Higgs system which has monopole solutions. The averaged Lagrangian is

\[ \mathcal{L} = -\frac{1}{4} \left( \langle F_{\mu \nu}^a F^{a \mu \nu} \rangle - \frac{1}{4} \left( \langle F_{\mu \nu}^a F^{a \mu \nu} \rangle + \langle F_{\mu \nu}^m F^{m \mu \nu} \rangle \right) \right. \]  

\[ (5.17) \]

here \( F_{\mu \nu}^a \) and \( F_{\mu \nu}^m \) are defined by equations \( (5.3)-(5.9) \).

5.3.1 Calculation of \( \langle F_{\mu \nu}^a F^{a \mu \nu} \rangle \)

We begin by calculating the first term on the r.h.s. of equation \( (5.17) \)

\[ \langle F_{\mu \nu}^a F^{a \mu \nu} \rangle = \langle h_{\mu \nu}^a h^{a \mu \nu} \rangle + 2 \langle h_{\mu \nu}^a \Phi_{a \mu \nu} \rangle + \langle \Phi_{a \mu \nu} \Phi_{a \mu \nu} \rangle. \]

\[ (5.18) \]

Immediately we see that the first term on the r.h.s. of this equation is the SU(2) Lagrangian as we assume that \( a_{\mu}^a \) and \( h_{\mu \nu}^a \) are effectively classical quantities and consequently

\[ \langle h_{\mu \nu}^a h^{a \mu \nu} \rangle \approx h_{\mu \nu}^a h^{a \mu \nu}. \]

\[ (5.19) \]

The second term in equation \( (5.18) \) is

\[ \langle h_{\mu \nu}^a \Phi_{a \mu \nu} \rangle = g f^{amn} \langle \left( \partial_\alpha a_{\nu}^b - \partial_\nu a_{\mu}^b \right) A^{m \mu} A^{n \nu} \rangle + g f^{abc} f^{amn} \langle a_{b \mu}^a a_{c \nu}^b A^{m \mu} A^{n \nu} \rangle. \]

\[ (5.20) \]

Using assumptions 1 and 3 from the previous section these terms become

\[ \langle a_{\alpha}^a (x) A_{\mu}^{m} (y) A_{\nu}^{b} (z) \rangle = a_{\alpha}^a (x) \langle A_{\mu}^{m} (y) A_{\nu}^{b} (z) \rangle = \eta_{\mu \nu} a_{\alpha}^a (x) \mathcal{G}^{mn} (y, z) \]

\[ (5.21) \]

and

\[ \langle a_{\alpha}^a (x) a_{\beta}^b (y) A_{\mu}^{m} (z) A_{\nu}^{h} (u) \rangle = a_{\alpha}^a (x) a_{\beta}^b (y) \eta_{\mu \nu} \mathcal{G}^{mn} (z, u) \]

\[ (5.22) \]

The function \( \mathcal{G}^{mn}(x, y) \) is the 2-point correlator (Green’s function) for the disordered phase. Because of the bosonic character of the coset gauge fields \( \mathcal{G}^{mn}(x, y) \) must be symmetric under exchange of these fields. Also by assumption 2 of the last section this expectation should be non-zero. We take the form for this 2-point correlator to be

\[ \langle A_{\mu}^{m} (y) A_{\nu}^{b} (x) \rangle = -\frac{1}{3} \eta_{\mu \nu} f^{mpb} f^{npc} \phi^b (y) \phi^c (x) = -\eta_{\mu \nu} \mathcal{G}^{mn} (y, x) \]

\[ (5.23) \]

with

\[ \mathcal{G}^{mn} (y, x) = \frac{1}{3} f^{mpb} f^{npc} \phi^b (y) \phi^c (x) \]

\[ (5.24) \]

here \( \phi^a \) is a real SU(2) triplet scalar fields. Thus we have replaced the coset gauge fields by an effective scalar field, which will be the scalar field in our effective SU(2)-scalar system. Aside from the factor of \(-\frac{1}{3}\) this is similar to the approximation made for the 2-point correlator in the last section in \( (1.3) \). The factor of \(-\frac{1}{3}\) is introduced so that the effective scalar field, \( \phi \), will have the correct coefficient for the kinetic energy term. If we were to take the scalar field to be constant (\( \phi^a (x) \approx const. \)) then \( (5.23) \) and \( (5.24) \) would represent an effective mass-like, condensation term of the coset gauge fields. With this we find that the middle term vanishes

\[ \langle h_{\mu \nu}^a \Phi_{a \mu \nu} \rangle = g f^{mpb} \left( f^{amn} \left( \partial_\mu a_{\nu}^b (x) - \partial_\nu a_{\mu}^b (x) \right) \mathcal{G}^{mn} (x, x) + \right. \]

\[ g f^{amn} f^{abc} a_{b \mu}^a (x) a_{c \nu}^b (x) \mathcal{G}^{mn} (x, x) \right) = 0 \]

\[ (5.25) \]

The last term which is quartic in the coset gauge fields will be considered at the end. Up to this point the SU(2) part of the Lagrangian is

\[ \langle F_{\mu \nu}^a F^{a \mu \nu} \rangle = \langle h_{\mu \nu}^a h^{a \mu \nu} \rangle + g^2 f^{mpb} f^{amn} \langle A_{\mu}^{m} A_{\nu}^{b} A_{\mu}^{m} A_{\nu}^{b} \rangle. \]

\[ (5.26) \]
5.3.2 Calculation of $\langle F^m_{\mu\nu} F^{mn\mu} \rangle$

Next we work on the coset part of the Lagrangian

$$\langle F^m_{\mu\nu} F^{mn\mu} \rangle =
\left[ (\partial_\mu A_\nu^m - g f^{mnb} A^b_\nu A^a_\mu) - (\partial_\nu A_\mu^m - g f^{mnb} A^a_\mu A^b_\nu) + g f^{mnp} A^a_\mu A^b_\nu \right]^2 =
2 \left[ (\partial_\mu A_\nu^m - g f^{mnb} A^b_\nu A^a_\mu)^2 \right] -
2 \left[ (\partial_\nu A_\mu^m - g f^{mnb} A^a_\mu A^b_\nu)^2 \right] -
2 \left[ (\partial_\mu A_\nu^m - g f^{mnb} A^a_\nu A^b_\mu) (\partial_\nu A_\mu^m - g f^{mn'b} A'^{a'}_{\nu'} A_{\mu'}^{b'}) \right] +
4 g \left[ (\partial_\mu A_\nu^m - g f^{mn'b} A'^{a'}_{\nu'} A_{\mu'}^{b'}) f^{mnp} A^a_\mu A^b_\nu \right] +
g^2 f^{mnp} f^{m'n'} \left( A_\mu^a A^n_{a'} A_{\nu'}^{b'} A_{\mu'}^{b'} \right).
$$

First we calculate

$$2 \left[ (\partial_\mu A_\nu^m)^2 \right] = 2 \left. \left[ (\partial_\mu A_\nu^m) (y) \partial_\mu A^m_\nu (x) \right] \right|_{y=x} = 2 \delta_{\mu \nu} \partial_\mu f^{mnp} \partial_\nu f^{mnp} \partial_\mu \phi^b \partial_\nu \phi^b = -\frac{8}{3} \partial_\mu \phi^a \partial_\mu \phi^a.
$$

(5.27)

Analogously

$$-2 \left. \left[ (\partial_\mu A_\nu^m) \partial_\mu A^m_\nu (x) \right] \right|_{y=x} = \frac{2}{3} \eta^a_\mu f^{mnp} \partial_\mu \phi^b \partial_\nu \phi^c = \frac{2}{3} \partial_\mu \phi^a \partial_\mu \phi^a.
$$

(5.29)

The next term is

$$-4 g \left[ (\partial_\mu A_\nu^m) f^{amn} A_{\mu'}^a A^{\nu'} \right] = -4 g f^{amn} a_{\mu'} A^{\nu'} =
-4 g f^{amn} a_{\mu'} A^{\nu'} (x) \left. \right|_{y=x} =
\frac{4}{3} \delta^{\mu}_{\mu'} \delta^{\nu}_{\nu'} f^{amn} \partial_\mu \phi^b \partial_\nu \phi^c x = \frac{16}{3} \delta^{\mu}_{\mu'} \delta^{\nu}_{\nu'} \partial_\mu \phi^b \partial_\nu \phi^c
$$

$$+ \frac{8}{3} \eta^{abc} d^{abc} \partial_\mu \phi^b \partial_\nu \phi^c.
$$

(5.30)

where the identity $f^{amn} f^{cnp} f^{bpm} = \frac{1}{2} e^{abc}$ was used. Analogously

$$4 g \left[ f^{amn} A_\nu^a (y) \partial_\mu A^m_\nu (x) \right] \left. \right|_{y=x} = \frac{4}{3} \delta^{\mu}_{\mu'} \delta^{\nu}_{\nu'} f^{amn} f^{bnp} f^{cpm} \partial_\mu \phi^b \partial_\nu \phi^c
$$

(5.31)

Using (5.28) the next term is

$$2 g^2 \left[ f^{d'm} A_\nu^a a^d_A d A^m_{\mu'} A^{d'} \right] = 2 g^2 f^{d'mn} a^d_m A^{m'n'} A^{d'} =
-\frac{2}{3} \eta^b e f^{d'm} f^{d'n'} f^{bpm} \partial_\mu \phi^b \partial_\nu \phi^c = -\frac{8}{3} g^2 e^{abc} d^{abc} \partial_\mu \phi^b \partial_\nu \phi^c.
$$

(5.32)

here $E^{d'b} = f^{d'n'm} f^{d'nm} f^{bnp} f^{cnp}$ and its components are

$$E^{aaaa} = E^{1111} = E^{2222} = E^{3333} = \frac{1}{4},
$$

$$E^{aabb} = E^{1122} = E^{1133} = E^{2211} = E^{2233} = E^{3311} = E^{3322} = \frac{1}{4},
$$

$$E^{abab} = E^{1211} = E^{1222} = E^{2111} = E^{2122} = E^{3111} = E^{3133} = \frac{1}{4}.
$$

(5.33)

We now note that $E^{d'b} a^d_m A^{m'n'} A^{d'} = \frac{1}{4} (e^{abc} d^{abc} a^d_m A^{m'n'} A^{d'}) + a^d_m A^{d'} = a^d_m A^{d'} a^d_m A^{d'} = a^d_m A^{d'} = a^d_m A^{d'}.$ Thus (5.32) becomes

$$2 g^2 \left[ f^{d'mn} A_\nu^a d A^{m'n'} A^{d'} \right] = -\frac{2 g^2}{3} e^{abc} d^{abc} a^d_m A^{d'} a^d_m A^{d'} = \frac{1}{4}.
$$

(5.34)
Analogously
\[-2g^2 \langle f^{mn} A^n_m A^d_d f^{d'm' n'} A^{n'}_{m'} A^{d'}_{d'} \rangle = g^2 \epsilon^{abc} \epsilon^{d'e'} A^b_\mu A^c_\nu A^{d'}_\mu A^{e'}_\nu \]
\[+ a^b_\mu A^c_\nu A^{d'}_\mu A^{e'}_\nu \]
(5.35)

Next we consider terms with three coset fields. The term involving the derivative is
\[f^{mnp} (\partial_{\mu} A^n_m (y)) A^{n'}_{\mu}(x) A^{p'}_{\nu}(z) \]
(5.36)

Since the gauge fields must be symmetric under exchange, and because of the antisymmetry of the of \( f^{mnp} \) this term vanishes. Next the terms involving three coset fields and one \( SU(2) \) field we will approximate as
\[f^{mnp} t^{n'm'} m'' n'' p'' (A^n_m A^{n'}_{m'} A^{p'}_{p''}) \approx \frac{1}{3} \epsilon_{mnp} \epsilon^{m'm'n'n''p'} (A^n_m A^{n'}_{m'} A^{p'}_{p''}) + \]
\[f^{mnp} t^{n'm'} m'' n'' p'' (A^n_m A^{n'}_{m'} A^{p'}_{p''}) \]
(5.37)

By the second assumption in the previous section, \( A^n_m (x) = 0 \), this term also vanishes. Thus
\[f^{mnp} t^{n'm'} m'' n'' p'' (A^n_m A^{n'}_{m'} A^{p'}_{p''}) = 0 \]
(5.38)

The full averaged Lagrangian is
\[-\frac{1}{4} \langle F^{A^\mu \nu} F^{A^\mu \nu} \rangle = - \frac{1}{4} A^{\alpha \beta} h^{\alpha \beta} + \frac{1}{2} \epsilon^{abc} \epsilon^{d' e'} \frac{g^2}{2} A^a_\mu A^b_\nu A^{d'}_\mu A^{e'}_\nu \]
\[+ \frac{1}{2} a^b_\mu A^c_\nu A^{d'}_\mu A^{e'}_\nu - \frac{1}{4} g^2 f^{A^\mu \nu} f^{A^\mu \nu} + \frac{1}{2} a^b_\mu A^c_\nu A^{d'}_\mu A^{e'}_\nu \]
(5.39)

where we have collected the quartic terms from eq. (5.20) and (5.38) together into
\[f^{A^\mu \nu} f^{A^\mu \nu} + \frac{1}{2} A^a_\mu A^b_\nu A^{d'}_\mu A^{e'}_\nu \]

5.3.3 The quartic term

In this section we show that the quartic term \(- f^{A^\mu \nu} f^{A^\mu \nu} + A^a_\mu A^b_\nu A^{d'}_\mu A^{e'}_\nu \) from eq. (5.20) becomes an effective \( \lambda \phi^4 \) interaction term for the effective scalar field introduced in eq. (5.24). Just as in eq. (5.24) where a quadratic gauge field expression was replaced by a quadratic effective scalar field expression, here we replace the quartic gauge field term by a quartic scalar field term
\[\langle A^n_m (x) A^n_m (y) A^p_p (z) A^p_p (u) \rangle = \left( E^{mnpq}_{1,abcd} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu + E^{mnpq}_{2,abcd} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu + E^{mnpq}_{3,abcd} \rho_\alpha \rho_\beta \eta_\mu \eta_\nu \right) \]
\[\phi^a(x) \phi^b(y) \phi^c(z) \phi^d(u) \]
(5.40)

Here \( E^{mnpq}_{1,abcd}, E^{mnpq}_{2,abcd}, E^{mnpq}_{3,abcd} \) are constants. Because of the bosonic character of the gauge fields in (5.40) the indices of these constants in conjunction with the indices of the \( \eta_{\alpha \beta} \)’s must reflect symmetry under exchange of the fields. The simplest choice that satisfies this choice is this
\[\langle A^n_m (x) A^n_m (y) A^p_p (z) A^p_p (u) \rangle = ( \delta^{mn} \delta^{pq} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu + \delta^{mp} \delta^{nq} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu + \delta^{mq} \delta^{np} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu ) \]
\[\epsilon_{abcd} \phi^a(x) \phi^b(y) \phi^c(z) \phi^d(u) \]
(5.41)

This choice of taking the constants from eq. (5.40) to be products of Kronecker deltas and fixing \( a = b = c = d \) for the lower indices, satisfies the bosonic character requirement for the gauge fields, and is equivalent to the reduction used for the quartic term in (5.40). Evaluating eq. (5.41) at one spacetime point \( (i.e. x = y = z = u) \) and contracting the indices to conform to quartic term in eq. (5.40) gives
\[\langle A^n_m (x) A^n_m (z) A^p_p (x) A^p_p (z) \rangle = ( \delta^{mn} \delta^{pq} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu + \delta^{mp} \delta^{nq} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu + \delta^{mq} \delta^{np} \eta_\alpha \eta_\beta \eta_\mu \eta_\nu ) \]
\[\left( \phi^a(x) \phi^a(z) \right)^2 \]
(5.42)
where the constants $\epsilon_{abcd}$ are chosen that at the point $x = y = z = u$

$$\epsilon_{abcd} = \frac{1}{3} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}).$$

(5.43)

This expression can be further simplified to

$$\langle A^m_\mu A^n_\nu A^p_\rho A^q_\sigma \rangle = (4\delta^{mn} \delta^{pq} + 16\delta^{mp} \delta^{nq} + 4\delta^{mq} \delta^{np})(\phi^a(x)\phi^a(x))^2.$$  

(5.44)

Substituting this into the original quartic term of eq. (5.51) yields

$$f^{Amp} f^{A'n'p'} \langle A^n_\mu A^p_\nu A^{n'}_\mu A^{p'}_\nu \rangle = (4f^{Amp} f^{A'n'p} + 4f^{Amp} f^{A'np}) (\phi^a(x)\phi^a(x))^2 =$$

$$12 f^{Amp} f^{Amp} (\phi^a(x)\phi^a(x))^2$$

where the antisymmetry property of the structure constants has been used. Using the explicit expression for the structure constants ($f^{123} = 1$; $f^{147} = f^{246} = f^{257} = f^{345} = f^{516} = f^{637} = \frac{1}{2}$; $f^{458} = f^{678} = \frac{\lambda}{2}$ plus those related to these by permutations), and recalling that the index $A$ runs from 1 – 8 while the indices $n, p$ run from 4 – 8 one can show that $f^{Amp} f^{Amp} = 12$ ($f^{123}$ and related constants do not contribute to this expression). Combining these results transforms the quartic term in eq. (5.51) as

$$\frac{1}{4} \int d^4x f^{Amp} f^{A'n'p'} \langle A^n_\mu A^p_\nu A^{n'}_\mu A^{p'}_\nu \rangle = 36g^2 (\phi^a(x)\phi^a(x))^2 = \frac{\lambda}{4} (\phi^a(x)\phi^a(x))^2$$

(5.46)

One can show that the simplification (4.11) is also applicable here and will lead to the same result. This has transformed the quartic gauge field term of the coset fields into a quartic interaction term for the effective scalar field. Substituting this result back into the averaged Lagrangian of eq. (5.51) we find

$$-\frac{1}{4} \int F_{\mu\nu}^A F^{A\mu\nu} = -\frac{1}{4} h_{\mu\nu} h^{\mu\nu} + \frac{1}{2} \left( \partial_\mu \phi^a + \frac{g}{2} \epsilon^{abc} a^b_\mu \phi^c \right)^2 + \frac{g^2}{2} a^b_\mu \epsilon^{abc} a^c_\nu \phi^c - \frac{\lambda}{4} (\phi^a(x)\phi^a(x))^2$$

(5.47)

The original pure SU(3) gauge theory has been transformed into an SU(2) gauge theory coupled to an effective triplet scalar field. This is similar to the Georgi-Glashow Lagrangian except for the presence of the term $\frac{g^2}{2} a^b_\mu \epsilon^{abc} a^c_\nu \phi^c$ and the absence of a negative mass term for $\phi^a$ of the form $m^2 (\phi^a(x)\phi^a(x))$.

The Georgi-Glashow Lagrangian is known to have topological monopole solutions [83] which have the form

$$\phi^a = \frac{x^a f(r)}{gr^2} ; \quad a_0^0 = 0 ; \quad a_i^0 = \frac{\epsilon_{abd} a^d [1 - h(r)]}{gr^2}$$

(5.48)

where $f(r)$ and $h(r)$ are functions determined by the field equations. For this form of the scalar and SU(2) gauge fields the term, $\frac{g^2}{2} a^b_\mu \epsilon^{abc} a^c_\nu \phi^c$, vanishes from the Lagrangian in eq. (5.47) by the antisymmetry of $\epsilon_{abd}$ and the symmetry of $x^a x^b$. Thus for the monopole ansatz of eq. (5.48) the Lagrangian in (5.47) becomes equivalent to the Georgi-Glashow Lagrangian minus only the mass term for the scalar field.

In the present work we simply postulate that the effective scalar field develops a negative mass term of the form $-\frac{m^2}{2} (\phi^a \phi^a)$ which is added by hand to the Lagrangian of (5.47) to yield

$$-\frac{1}{4} h_{\mu\nu} h^{\mu\nu} + \frac{1}{2} \left( \partial_\mu \phi^a + \frac{g}{2} \epsilon^{abc} a^b_\mu \phi^c \right)^2 + \frac{m^2}{2} (\phi^a \phi^a) - \frac{\lambda}{4} (\phi^a(x)\phi^a(x))^2 +$$

$$\frac{g^2}{2} a^b_\mu \epsilon^{abc} a^c_\nu \phi^c$$

(5.49)

The scalar field now has the standard symmetry breaking form and this effective Lagrangian has finite energy ‘t Hooft-Polyakov solutions (the last term should not alter the monopole construction since it vanishes under the ansatz in (5.48)).

At the end of this section we have to note the following problem. The covariant derivatives $D_\mu a^a_\nu = \partial_\mu a^a_\nu + gc^{abc} a^b_\mu a^c_\nu$ and $D_\mu \phi^a = \partial_\mu \phi^a + \frac{g}{2} \epsilon^{abc} a^b_\mu \phi^c$ are different. One can avoid this problem if we suppose that there is a correlation between ordered phase $a^a_\mu$ and disordered phase $A^a_\mu$

$$\langle a^a_{\mu_1} (x_1) \cdots a^a_{\mu_n} (x_n) A^m_\alpha (y) A^b_\beta (z) \rangle = k_n a^{\alpha a}_{\mu_1} (x_1) \cdots a^{\alpha a}_{\mu_n} (x_n) \langle A^m_\alpha (y) A^b_\beta (z) \rangle.$$  

(5.50)
If we suppose that \( k_1 = 2 \) and \( k_2 = 4 \) then the rhs of equations (5.50) and equations (5.51) will be multiplied on 2 and \( x \) equations (5.52)-(5.56) will be multiplied on 4. Consequently the full Lagrangian will be

\[
-\frac{1}{4} \left< F_{\mu \nu}^{A} F^{A}_{\mu \nu} \right> = -\frac{1}{4} h^{a}_{\mu \nu} h^{a \mu \nu} + \frac{1}{2} \left( \partial_{\mu} \phi^{a} + g \epsilon^{abc} a_{\mu}^{b} \phi^{c} \right)^{2} + \frac{m^{2}}{2} \left( \phi^{a} \phi^{a} \right)
\]

This Lagrangian we will investigate numerically on the next sections.

Our final result given in (5.49) depends on several crucial assumptions (e.g. the existence of the negative mass term \( -\frac{m^{2}}{2} \left( \phi^{a} \phi^{a} \right) \)). In the next section we will investigate a non-topological flux tube solution to this effective Lagrangian.

### 5.4 Flux tube equations

We will start from the SU(2) Yang - Mills - Higgs field equations (5.49) with broken gauge symmetry

\[
\mathcal{D}_{\mu} h^{a \mu \nu} = ge^{abc} \phi^{b} \mathcal{D}^{c} \phi^{c} - (m^{2})^{ab} a^{b \mu},
\]

\[
\mathcal{D}_{\mu} D^{a} \phi^{a} = -\lambda \phi^{a} \left( \phi^{a} \phi^{a} - \phi_{\infty}^{2} \right)
\]

where \( h^{a \mu \nu} = \partial_{\mu} a^{a}_{\nu} - \partial_{\nu} a^{a}_{\mu} + ge^{abc} a^{b}_{\mu} a^{c}_{\nu} \) is the field tensor for the SU(2) gauge potential \( a^{a}_{\mu} \); \( a, b, c = 1, 2, 3 \) are the color indices; \( D_{\nu} \{ \cdots \}^{a} = \partial_{\nu} \{ \cdots \}^{a} + ge^{abc} a^{b}_{\nu} \{ \cdots \}^{c} \) is the gauge derivative; \( \phi^{a} \) is the Higgs field; \( \lambda, g \) and \( \phi_{\infty} \) are some constants; \( (m^{2})^{ab} \) is a masses matrix which destroys the gauge invariance of the Yang - Mills - Higgs theory, here we choose \( (m^{2})^{ab} = \text{diag} \{ m_{1}^{2}, m_{2}^{2}, 0 \} \). With a standard SU(2)-Higgs symmetry breaking Lagrangian the masses developed would be identical, and would be related to the gauge coupling and vacuum expectation of the field \( -m_{a} \simeq \phi_{\infty} g \). In the present case we are postulating that the masses of the two gauge bosons are different. This was done in order to find electric flux tube solutions for the system (5.50)-(5.56). For equal masses electric flux tubes did not exist for the ansatz used. In numerically solving (5.51)-(5.56) the two mass come out close to one another, but are not equal. This splitting of the gauge bosons mass over and above the Higgs mechanism explicitly breaks the gauge invariance, and at present we have no good justification for this step other than the fact that it leads to the existence of electric flux tubes of the kind that are very important in confinement.

We will use the following ansatz

\[
a_{1}^{a}(\rho) = \frac{f(\rho)}{g}; \quad a_{2}^{a}(\rho) = \frac{v(\rho)}{g}; \quad \phi^{3}(\rho) = \frac{\phi(\rho)}{g}
\]

here \( z, \rho, \varphi \) are cylindrical coordinates. Substituting this into the Yang - Mills - Higgs equations (5.49) gives us

\[
f''' + \frac{f'}{x} = f \left( \phi^{2} + v^{2} - m_{1}^{2} \right),
\]

\[
v'' + \frac{v'}{x} = v \left( \phi^{2} - f^{2} - m_{2}^{2} \right),
\]

\[
\phi'' + \frac{\phi'}{x} = \phi \left[ -f^{2} + v^{2} + \lambda \left( \phi^{2} - \phi_{\infty}^{2} \right) \right]
\]

where we redefined \( \phi/\phi_{0} \rightarrow \phi, \ f/\phi_{0} \rightarrow f, \ v/\phi_{0} \rightarrow v, \ \phi_{\infty}/\phi_{0} \rightarrow \phi_{\infty}, \ m_{1,2}/\phi_{0} \rightarrow m_{1,2}, \ \rho \phi_{0} \rightarrow x; \ \phi_{0} = \phi(0) \). Similar equations with the presence of \( A_{\varphi}^{a} \) and without masses \( m_{1,2} \) were investigated in ref’s [4] [5] where it was shown that the corresponding solutions becomes singular either on a finite distance from the axis \( \rho = 0 \) or on the infinity. The solutions with the masses \( m_{1,2} \) and a magnetic field \( H_{z} \) were obtained in ref. [6] and the result is that the following versions of the flux tube exist: (1) the flux tube filled with electric/magnetic fields on the background of an external constant magnetic/electric field; (2) the Nielsen-Olesen flux tube dressed with transversal color electric and magnetic fields.

### 5.5 Numerical algorithm

As in section 4 we will solve the equations set (5.50)-(5.57) with the iterative procedure which is described in more detail in ref. [5]. Eq. (5.56) for the \( s^{th} \) step has the following form

\[
v_{s}'' + \frac{v_{s}'}{x} = v_{s} \left( \phi_{s-1}^{2} - f_{s-1}^{2} - m_{2}^{2} \right),
\]
here the functions $f_{i-1}$ and $\phi_{i-1}$ were defined at the $(i-1)^{th}$ step and the null approximation for the functions $f(x)$ and $\phi(x)$ is $\phi_0(x) = 1.3 - 0.3/\cosh^2(x/4)$ and $f_0(x) = 0.2/\cosh^2(x)$. Again these forms are chosen because they are already similar to the form of solution which is being sought. The numerical investigation for eq. (5.58) shows that there is a number $m_{2,i}$ for which: for $m_{2,i} > m_{2,i}^*$ the solution is singular $v_i(x) \rightarrow -\infty$ by $x \rightarrow \infty$ and for $m_{2,i} < m_{2,i}^*$ the solution is also singular $v_i(x) \rightarrow +\infty$ by $x \rightarrow \infty$. In Fig. 7 the singular solutions are presented. The asymptotical form of the equation for the singular solution is

$$v''_{i,s} + \frac{v'_{i,s}}{x} \approx v_{i,s} \left( \phi_{i-1}^2 - m_{2,i}^2 \right)$$

(5.59)

and corresponding asymptotical singular solutions are

$$v_i \approx \pm a \pm e^{\frac{x}{m_{2,i} - m_{2,i}^*}}$$

(5.60)

where $a_{\pm}$ are some constants. This indicates that there may exist some value, $m_{2,i} = m_{2,i}^*$, for which the solution $v_i(x)$ is regular. The number $m_{2,i}^*$ is found with the method of the iterative approximation as for $m_i^*$ and $\mu_i^*$ in the glueball case. It is necessary that the functions $f_{i-1}(x) \rightarrow 0$ and $\phi_{i-1}(x) \rightarrow \phi_{\infty,i-1}^*$ as $x \rightarrow \infty$ in order to have acceptable solutions.

Eq. (5.58) at this step has the following form

$$f''_i + \frac{f'_i}{x} = f_i \left( \phi_{i-1}^2 + v_i^2 - m_{i,i}^2 \right),$$

(5.61)

here the function $\phi_{i-1}$ was defined on $(i-1)^{th}$ step and the function $v_i$ is the solution of eq. (5.58). The numerical investigation for eq. (5.61) shows that there is a number $m_{1,i}^*$ for which: for $m_{1,i} > m_{1,i}^*$ the solution is singular $f_i(x) \rightarrow -\infty$ by $x \rightarrow \infty$ and for $m_{1,i} < m_{1,i}^*$ the solution is also singular $f_i(x) \rightarrow +\infty$ by $x \rightarrow \infty$. The asymptotic singular behavior for $f_i(x)$ is the same as in the previous case. This implies that there may be some particular value $m_{1,i} = m_{1,i}^*$ for which the solution $f_i(x)$ is regular. The number $m_{1,i}^*$ is found with the method of iterative approximation. It is necessary that the functions $v_i(x) \rightarrow 0$ and $\phi_{i-1}(x) \rightarrow \phi_{\infty,i}^*$ as $x \rightarrow \infty$ in order to have acceptable solutions.

The next step in the $i^{th}$ iteration is solving of eq. (5.57)

$$\phi''_i + \frac{\phi'_i}{x} = \phi_i \left[ -f_i^2 + v_i^2 + \lambda \left( \phi_i^2 - \phi_{\infty,i}^2 \right) \right]$$

(5.62)

here the functions $f_i$ and $v_i$ are the solutions of eqns. (5.58) (5.61). The numerical investigation for this equation shows that there is a number $\phi_{\infty,i}^*$ for which: for $\phi_{\infty,i} > \phi_{\infty,i}^*$ the function $\phi_i(x)$ oscillates with decreasing amplitude, for $\phi_{\infty,i} < \phi_{\infty,i}^*$ the function $\phi_i(x)$ oscillates with decreasing amplitude, for $\phi_{\infty,i} < \phi_{\infty,i}^*$ the solution $\phi_i(x) \rightarrow +\infty$ by $x \rightarrow x_0$. The singular solution in this case is similar to the solution presented on Fig. 6. The number $\phi_{\infty,i}^*$ is intermediate between the values that give the singular and oscillating solutions. For this value it is possible that a regular solution exists. To have acceptable behavior for the solution $\phi_i(x)$ it is necessary that the functions $v_i(x) \rightarrow 0$ and $f_i(x) \rightarrow 0$ as $x \rightarrow \infty$.

For the initial values we chose the following values: $f(0) = 0.2$, $v(0) = 0.5$ and $\phi(0) = 1.0$, and $f'(0) = v'(0) = \phi'(0) = 0$. The set of equations (5.58)-(5.57) has one independent parameter $\lambda$, which we set to $\lambda = 0.2$.
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|   | 1          | 2          | 3          | 4          |
|---|------------|------------|------------|------------|
| $m^*_{1,i}$ | 1.194482... | 1.20214... | 1.20106... | 1.2011...  |
| $m^*_{2,i}$ | 1.1405457... | 1.15170... | 1.15108... | 1.150935... |
| $\phi^*_{\infty,i}$ | 1.236086... | 1.23189... | 1.23253... | 1.23260... |

Table 2: The iterative parameters $m^*_{(1,2),i}$ and $\phi^*_{\infty,i}$.

to perform the numerical solution of the equations. The other parameters of the equations ($m^*_{(1,2),i}$ and $\phi^*_{\infty,i}$) are determined in the process of solving the equations.

The iterative process described above gives $m^*_{(1,2),i}$ and $\phi^*_{\infty,i}$ presented on Table 2. The functions $f_i(x)$, $v_i(x)$ and $\phi_0(x)$ for $i = 1, 2, 3, 4$ are presented in figs. 8, 9 and 10. As in the glueball analysis we have only numerically integrated the solution out to finite $x$ so it is not clear if this solution is truly well behaved as $x \to \infty$. For the range of $x$ where we have found the solution it behaves as desired.

In terms of the ansatz functions the color electric and magnetic fields are

\[
E^3(x) = F^3_{t\mathbf{z}} = \frac{f(x)v(x)}{g},
\]

\[
E^1_\rho(x) = F^1_{t\rho} = -\frac{f'(x)}{g},
\]

\[
H^2_\phi(x) = x\epsilon_{\varphi\rho\mathbf{z}}F^{2\rho\mathbf{z}} = -\frac{xv'(x)}{g}.
\]

These shapes of these fields is presented in fig. 11.
From eqs. (5.55)-(5.57) one can see that the asymptotic behavior of the regular solutions \( f^*(x) \), \( v^*(x) \) and \( \phi^*(x) \) is

\[
\begin{align*}
  f^*(x) &= f_0 e^{-x/\sqrt{\phi_\infty^2 - m_1^2}} + \cdots, \\
  v^*(x) &= v_0 e^{-x/\sqrt{\phi_\infty^2 - m_2^2}} + \cdots, \\
  \phi^*(x) &= \phi_\infty^* - \phi_0 e^{-x/\sqrt{2\lambda \phi_\infty^2}} + \cdots.
\end{align*}
\]

where \( f_0, v_0 \) and \( \phi_0 \) are constants.

From the Lagrangian in (5.49) the energy density \( \epsilon(x) \) in terms of the ansatz functions is given by

\[
2g^2 \epsilon(x) = f'^2(x) + v'^2(x) + \phi'^2(x) + 2(f(x)\phi^2(x) + f^2(x)\phi^2(x) + m_1^2 f^2(x) - m_2^2 v^2(x) + \frac{\lambda}{2} (\phi^2 - \phi_\infty^2)^2
\]

Its shape is shown in Fig. 12. The exponentially decreasing asymptotic forms indicate that the energy density should go to zero at the infinity, and thus the total energy of the system should also be finite. Since we have only been able to integrate the solution to finite \( x \) we can not say absolutely that the total energy is finite. In the figs. 13 and 14 the profiles of the linear energy density (the integral of the energy density over the \( \rho \) coordinate) is shown for different \( \lambda \) and \( \phi_0 \). The linear energy density is given by

\[
w = \int 2\pi \rho \epsilon(\rho) d\rho
\]

Figure 12: The energy density \( \epsilon(x) \).

Figure 13: The profile \( \epsilon(\lambda) \) with the different \( \phi_0 \).

Figure 14: The profile \( \epsilon(\phi_0) \) with the different \( \lambda \).
The flux of the longitudinal electric field is

\[ \Phi = \int E_z^3 ds = 2\pi \int_0^\infty \rho f(\rho) v(\rho) d\rho = \frac{2\pi}{g} \int_0^\infty x f(x) v(x) dx < \infty. \quad (5.71) \]

In the figs. the profiles of the electric flux for different \( \lambda \) and \( \phi_0 \) is given. Again the electric flux was calculated only out to some finite \( x \). Eq. (5.71) assumes that the ansatz functions remain well behaved (i.e. approach zero exponentially as \( x \to \infty \)) for larger \( x \).

![Figure 15: The profile \( \Phi(\lambda) \) with the different \( \phi_0 \).](image1)

![Figure 16: The profile \( \Phi(\lambda) \) with the different \( v_0 \).](image2)

![Figure 17: The profile \( \Phi(\phi_0) \) with the different \( \phi_0 \).](image3)

![Figure 18: The profile \( \Phi(\phi_0) \) with the different \( v_0 \).](image4)

We see that the flux \( \Phi \) depends on the parameters \( \lambda, v(0) \) and \( \phi(0) \) since the solutions \( f(x) \) and \( v(x) \) depend on these parameters. This is in contrast with such topological solutions as ’t Hooft-Polyakov monopole and Nielsen-Olesen flux tube which have fixed fluxes. This is the indication of the fact that the colored flux tube solution is not the consequence of a first order differential equation (the Bogomolnyi Equation \[70\]) as is the case for the topological solutions. The colored flux tube is a dynamical object not topological one. This raises questions about its stability, since this solution is then no longer guaranteed to be stable via topological arguments.

### 6 Discussion and conclusions

Now we summarize the nonperturbative quantization method of this paper and enumerate some features of nonperturbative quantum field theory:

- The existence and character of stationary, extended objects, such as flux tubes or glueballs must be explained using nonperturbative QFT. Such extended objects can not be understood within perturbative QFT as they are not simply a cloud of quanta. These objects share some similarity to a stationary turbulent liquid: there are fluctuations in any point within the liquid and simultaneously there exist averaged quantities (velocity, for example) which are correlated at any point as a consequence of the equations of motion for the turbulent liquid.
It is well known that the ordinary commutations relations like
\[ [\hat{\pi}(x), \hat{\phi}(y)] = i\delta(x - y) \] (6.1)
are correct only for linear field theory. The algebra of strongly interacting fields are more complicated and unknown. The algebra of nonperturbative field operators may be less singular than the algebra of linear fields in the sense that the Green’s functions for the nonperturbative field operators may be more smooth and well behaved. As a consequence the fields at different spatial points but at one time will be correlated in contrast with the second (linear) case. The physical explanation for this is that in the linear case the quantized field at different points can be correlated only by exchanging quanta which move with some speed, but in the nonperturbative case these fields are correlated through nonlinear equations (similar to the correlation in a turbulent liquid).

Nonperturbative QFT may be important for string theory. String theory says nothing about any possible inner structure of the string. By studying flux tube solutions such as those presented here one may obtain some insight at making a guess as to possible inner structure for strings. Here we would like to cite Polyakov [71]: “...It might be possible to visualize the superstrings as flux lines of some unknown gauge theory ...”

As a final remark one can think to apply the present nonperturbative method to other strongly coupled, nonlinear systems such as gravity. In this way one could see if some insight into quantum gravity might not be gleaned.

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A The flux tube in Euclidean spacetime

In the Euclidean spacetime the equations (5.55)-(5.57) can be simplified. Making the transition to Euclidean spacetime via \( f(x) \rightarrow if(x) \) we have the following equations
\[
\begin{align*}
f'' + \frac{f'}{x} &= f \left( \phi^2 + v^2 - m_1^2 \right), \\
v'' + \frac{v'}{x} &= v \left( \phi^2 + f^2 - m_2^2 \right), \\
\phi'' + \frac{\phi'}{x} &= \phi \left[ f^2 + v^2 + \lambda \left( \phi^2 - \phi_\infty^2 \right) \right].
\end{align*}
\]
(A.1) (A.2) (A.3)

For simplicity one can consider the case \( f(x) = v(x) \) and \( m_1 = m_2 = m \) which then leads to only two equations
\[
\begin{align*}
f'' + \frac{f'}{x} &= f \left( \phi^2 + f^2 - m^2 \right), \\
\phi'' + \frac{\phi'}{x} &= \phi \left[ 2f^2 + \lambda \left( \phi^2 - \phi_\infty^2 \right) \right].
\end{align*}
\]
(A.4) (A.5)

The numerical method for obtaining a solution is similar to the methods applied in sections 4, 5 and the results are presented in figs. 19, 20 and Table 3. In fig. 21 the profile of the color electric and magnetic fields are presented. In figs. 22, 23 we show the flux \( \Phi \) of color electric field
\[
\Phi = \int E^3 ds = 2\pi \int_0^\infty \rho f(\rho) v(\rho) \rho d\rho = \frac{2\pi}{g} \int_0^\infty x f(x) v(x) dx.
\]
(A.6)

through the tube with respect to parameters \( \lambda \) and \( \phi_0 \).

Table 3: The iterative parameters \( m_i^* \) and \( \phi_\infty^* \).

|   | 1          | 2                   | 3                   | 4                   |
|---|------------|---------------------|---------------------|---------------------|
| \( m_i^* \) | 0.9436768025... | 0.79033...         | 0.81214284...      | 0.8139867...        |
| \( \phi_\infty^* \) | 0.9175...   | 1.2248...          | 1.1186...          | 1.12185...          |
B The numerical calculations of the 1D \( \lambda \phi^4 \) soliton

As a test of the numerical method of solving the nonlinear equations (4.33) (4.34) we will apply this method to a known solution namely the \( \lambda \phi^4 \) soliton in 1D. The equation for this system is

\[
\frac{d^2 y}{dx^2} = y'' = y \left( 1 - y^2 \right), \tag{B.1}
\]

and the solution is

\[
y(x) = \sqrt{2} \cosh x. \tag{B.2}
\]

We rewrite (B.1) in the form of the Schrödinger equation

\[-y'' + y V_{\text{eff}} = -\lambda y \tag{B.3}\]

where \( V_{\text{eff}} = -y^2 \) and \( \lambda = 1 \). We will solve this equation by an iterative procedure. We start with the equation

\[-y_1'' + y_1 (-y_0^2) = -\lambda_1 y_1 \tag{B.4}\]

where \( y_1(x) \) and \( \lambda_1 \) are the first approximations of the solutions. For the numerical solution we choose the null approximation as

\[
y_0 = \frac{\sqrt{2}}{\cosh \left( \frac{x}{2} \right)}. \tag{B.5}\]

A typical solution for arbitrary values of the parameter \( \lambda_1 \) is presented in fig. 24. This picture shows that there is a value \( \lambda^*_1 \) for which the solution is exceptional one. One can find this exceptional solution choosing the appropriate value of the “energy level” \( \lambda^*_1 \). After which an exceptional solution \( y_1^*(x) \) is substituted into equation for the second approximation \( y_2(x) \)

\[-y_2'' - y_2 \left( y_1^* \right)^2 = -\lambda_2 y_2 \tag{B.6}\]
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Figure 22: The flux of the color electric field with respect to $\lambda$ with different $\phi_0$.

Figure 23: The flux of the color electric field with respect to $\phi_0$ and with different $\lambda$.

Figure 24: The singular solutions for the soliton equation.

Figure 25: The iterative functions $y_{1,2,3}$.

and so on. The result is presented in Table 4 and fig. 24. One can see that $\lambda_i^* \to 1$ and $y_i^*(x)$ is convergent to $y^*(x)$.

|   | 1       | 2       | 3       |
|---|---------|---------|---------|
| $\lambda_i^*$ | 1.406929666... | 1.148564915... | 1.03968278... |

Table 4: The iterative values of the parameter $\lambda$.

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