Stability of Cubic and Quartic Functional Equations in Non-Archimedean Spaces

M. Eshaghi Gordji · M.B. Savadkouhi

Abstract We prove generalized Hyers-Ulam–Rassias stability of the cubic functional equation $f(kx+y) + f(kx-y) = k[f(x+y) + f(x-y)] + 2(k^3-k)f(x)$ for all $k \in \mathbb{N}$ and the quartic functional equation $f(kx+y) + f(kx-y) = k^2[f(x+y) + f(x-y)] + 2k^2(k^2-1)f(x) - 2(k^2-1)f(y)$ for all $k \in \mathbb{N}$ in non-Archimedean normed spaces.

Keywords Generalized Hyers–Ulam–Rassias stability · Cubic functional equation · Quartic functional equation · Non-Archimedean space · p-adic

Mathematics Subject Classification (2000) 39B22 · 39B82 · 46S10

1 Introduction

A classical question in the theory of functional equations is the following: “When is it true that a function which approximately satisfies a functional equation $\epsilon$ must be close to an exact solution of $\epsilon$?”

If the problem accepts a solution, we say that the equation $\epsilon$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [28] in 1940.

We are given a group $G$ and a metric group $G'$ with metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \epsilon$ for all $x \in G$?

Ulam’s problem was partially solved by Hyers [13] in 1941. Let $E_1$ be a normed space, $E_2$ a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$
\|f(x+y) - f(x) - f(y)\| \leq \epsilon \quad (x, y \in E_1),
$$

M. Eshaghi Gordji (✉) · M.B. Savadkouhi
Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran
e-mail: madjid.eshaghi@gmail.com
M.B. Savadkouhi
e-mail: bavand.m@gmail.com
where $\epsilon > 0$ is a constant. Then the limit $T(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$ exists for each $x \in E_1$ and $T$ is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \epsilon$$  \hspace{1cm} (1.1)

for all $x \in E_1$. Also, if for each $x$ the function $t \mapsto f(tx)$ from $\mathbb{R}$ to $E_2$ is continuous on $\mathbb{R}$, then $T$ is linear. If $f$ is continuous at a single point of $E_1$, then $T$ is continuous everywhere in $E_1$. Moreover (1.1) is sharp.

In 1978, Th.M. Rassias [23] formulated and proved the following theorem, which implies Hyers’ theorem as a special case. Suppose that $E$ and $F$ are real normed spaces with $F$ a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p)$$  \hspace{1cm} (1.2)

for all $x, y \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{\epsilon \|x\|^p}{(1 - 2^{p-1})}$$

for all $x \in E$. The case of the existence of a unique additive mapping had been obtained by $T$. The terminology Hyers–Ulam stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [8, 11, 14] and [15]. In 1994, P. Găvruta [9] provided a further generalization of Th.M. Rassias’ theorem in which he replaced the bound $\epsilon (\|x\|^p + \|y\|^p)$ in (1.2) by a general control function $\phi(x, y)$ for the existence of a unique linear mapping.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping, see [24, 25]. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. In [4], Czerwik proved the generalized Hyers–Ulam stability of the quadratic functional equation. Borelli and Forti [2] generalized the stability result as follows (cf. [20, 21]): Let $G$ be an Abelian group, and $X$ a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^i+1} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the property $\|f(x) - Q(x)\| \leq \Phi(x, x)$ for all $x \in G$.

Jun and Kim [16] introduced the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$$  \hspace{1cm} (1.3)