Some formulae for products of Fubini polynomials with applications

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Abstract

In this paper we evaluate sums and integrals of products of Fubini polynomials and have new explicit formulas for Fubini polynomials and numbers. As a consequence of these results new explicit formulas for $p$-Bernoulli numbers and Apostol-Bernoulli functions are given. Besides, integrals of products of Apostol-Bernoulli functions are derived.

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1 Introduction, Definitions and Notations

Let $\left\{\begin{array}{l} n \\ k \end{array}\right\}$ be the Stirling numbers of the second kind [18]. Fubini polynomials are defined by [29]

$$F_n(y) = \sum_{k=0}^{n} \left\{\begin{array}{l} n \\ k \end{array}\right\} k! y^k.$$  (1)

They have the exponential generating function

$$\frac{1}{1 - y (e^t - 1)} = \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!},$$  (2)

and are related to the geometric series [5]

$$\left(\frac{y}{1 - y}\right)^m = \sum_{k=0}^{\infty} y^k \left(\frac{y}{1 - y}\right)^m, \quad |y| < 1.$$  (3)

Because of this relation Fubini polynomials are also called geometric polynomials. In addition, the following recurrence relation holds for the Fubini polynomials [11]

$$F_{n+1}(y) = y \frac{d}{dy} [F_n(y) + y F_n(y)].$$  (4)

The $n$th Fubini number (ordered Bell number or geometric number) [10], [19], [29], $F_n$, is defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{\begin{array}{l} n \\ k \end{array}\right\} k!.$$  (5)
and counts all the possible set partitions of an \( n \) element set such that the order of the blocks matters. Besides with this combinatorial property, these numbers are seen in the evaluation of the following series

\[
\sum_{k=0}^{\infty} \frac{k^n}{2^k} = 2 F_n. \tag{5}
\]

In the literature numerous identities concerned with these polynomials and numbers were obtained \[5, 6, 7, 8, 12, 16\] and their generalizations are given \[6, 13, 14, 17\]. The main purpose of this paper is to generalize the binomial formulas \[11\].

\[
\sum_{k=0}^{n} \binom{n}{k} F_k = 2 F_n, \quad n > 0, \tag{6}
\]

\[
2 \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k = (-1)^n F_n + 1, \quad n \geq 0, \tag{7}
\]

and the integral representation \[22\]

\[
\int_{-1}^{0} F_n(y) \, dy = B_n, \quad n > 0. \tag{8}
\]

Here \( B_n \) is the Bernoulli numbers defined by the explicit formula

\[
B_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k!}{k + 1}. \tag{9}
\]

As applications of these generalizations we obtain explicit formulas for Apostol-Bernoulli functions and \( p \)-Bernoulli numbers and integrals of products of Apostol-Bernoulli functions. We use generating function technique in the proofs.

Now we state our results.

## 2 Sums of products of Fubini polynomials

In this section we define two variable Fubini polynomials and obtain some basic properties which give us new formulas for \( F_n(y) \). Moreover we shall consider the sums of products of two Fubini polynomials. The sums of products of various polynomials and numbers with or without binomial coefficients have been studied (e.g., \[2, 21, 24, 25, 28, 30\]).

Two variable Fubini polynomials are defined by means of the following generating function

\[
\sum_{n=0}^{\infty} F_n(x; y) \frac{t^n}{n!} = \frac{e^{xt}}{1 - y(e^t - 1)}. \tag{10}
\]

For some special cases of \[10\], we have

\[
F_n(0; y) = F_n(y) \quad \text{and} \quad F_n(0; 1) = F_n. \tag{11}
\]
We can rewrite (10) as
\[ \sum_{n=0}^{\infty} F_n(x; y) \frac{t^n}{n!} = \frac{e^{xt}}{1 - y(e^t - 1)} e^{xt} \]
\[ = \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \binom{n}{k} F_k(y) x^{n-k} \right] \frac{t^n}{n!}. \]
Comparing the coefficients of \( \frac{t^n}{n!} \) yields
\[ F_n(x; y) = \sum_{k=0}^{n} \binom{n}{k} F_k(y) x^{n-k}. \] (12)

From (10) we have
\[ \sum_{n=0}^{\infty} [F_n(x+1; y) - F_n(x; y)] \frac{t^n}{n!} = \frac{e^{xt} (e^t - 1)}{1 - y(e^t - 1)} \]
\[ = \frac{1}{y} \left[ \frac{e^{xt}}{1 - y(e^t - 1)} - e^{xt} \right] \]
\[ = \frac{1}{y} \sum_{n=0}^{\infty} [F_n(x; y) - x^n] \frac{t^n}{n!}. \]
Comparing the coefficients of \( \frac{t^n}{n!} \) gives
\[ yF_n(x + 1; y) = (1 + y) F_n(x; y) - x^n. \] (13)

Thus, setting \( x = 0 \) and \( x = -1 \) in (13) we find
\[ yF_n(1; y) = (1 + y) F_n(y), \quad n > 0, \]
\[ (1 + y) F_n(-1; y) = yF_n(y) + (-1)^n, \quad n \geq 0, \] (14) (15)
respectively. Combining these relations with (12) gives the equations (6) and (7) which were obtained by using Euler-Siedel matrix method in (11).

Now, we want to give the generalization of the binomial formula (6). Derivative of (10) can be written as
\[ \frac{\partial}{\partial t} \left( \frac{e^{xt}}{1 - y(e^t - 1)} \right) = \frac{xe^{xt}}{1 - y(e^t - 1)} + \frac{ye^t}{1 - y(e^t - 1)} e^{xt}. \]
Taking \( x = x_1 + x_2 - 1 \) leads
\[ \frac{\partial}{\partial t} \left( \frac{e^{xt}}{1 - y(e^t - 1)} \right) = \sum_{n=0}^{\infty} F_{n+1}(x_1 + x_2 - 1; y) \frac{t^n}{n!}, \]
\[ \frac{xe^{xt}}{1 - y(e^t - 1)} = (x_1 + x_2 - 1) \sum_{n=0}^{\infty} F_n(x_1 + x_2 - 1; y) \frac{t^n}{n!}. \]
and
\[ \frac{ye^t}{1 - y(e^t - 1)} \frac{e^{xt}}{1 - y(e^t - 1)} = y \left( \sum_{n=0}^{\infty} F_n(x_1; y) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} F_n(x_2; y) \frac{t^n}{n!} \right) \]
\[ = y \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} F_k(x_1; y) F_{n-k}(x_2; y) \frac{t^n}{n!}. \]

By equating the coefficients of \( \frac{t^n}{n!} \), we get
\[ y \sum_{k=0}^{n} \binom{n}{k} F_k(x_1; y) F_{n-k}(x_2; y) = F_{n+1}(x_1 + x_2 - 1; y) - (x_1 + x_2 - 1) F_n(x_1 + x_2 - 1; y). \]

For \( x_1 = x_2 = 0 \) in the above equation, using (15) give the sums of products of the Fubini polynomials.

**Theorem 1** For \( n \geq 0 \),
\[ (y + 1) \sum_{k=0}^{n} \binom{n}{k} F_k(y) F_{n-k}(y) = F_{n+1}(y) + F_n(y). \] (16)

When \( y = 1 \) this becomes
\[ 2 \sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k} = F_{n+1} + F_n. \] (17)

Now, we investigate the sums of products of the Fubini polynomials for different \( y \) values in the following theorem.

**Theorem 2** For \( n \geq 0 \) and \( y_1 \neq y_2 \),
\[ \sum_{k=0}^{n} \binom{n}{k} F_k(y_1) F_{n-k}(y_2) = y_2 F_n(y_2) - y_1 F_n(y_1) \frac{y_2 F_n(y_2) - y_1 F_n(y_1)}{y_2 - y_1}. \] (18)

**Proof.** The products of (10) can be written as
\[ \frac{e^{x_1 t}}{(1 - y_1(e^t - 1))} \frac{e^{x_2 t}}{(1 - y_2(e^t - 1))} \]
\[ = \frac{y_2}{y_2 - y_1} \frac{e^{(x_1 + x_2)t}}{1 - y_2(e^t - 1)} - \frac{y_1}{y_2 - y_1} \frac{e^{(x_1 + x_2)t}}{1 - y_1(e^t - 1)}. \] (19)

Using the same method as in the proof of Theorem 1 we have
\[ \sum_{k=0}^{n} \binom{n}{k} F_k(x_1; y_1) F_{n-k}(x_2; y_2) = \frac{y_2 F_n(x_1 + x_2; y_2) - y_1 F_n(x_1 + x_2; y_1)}{y_2 - y_1}. \]
For $x_1 = x_2 = 0$ in the above equation gives the desired equation.

As we know, for $y = 1$ Fubini polynomials reduce to Fubini numbers. We now point out (see [23]) that Fubini numbers arise for other value of $y$, too. If we take $y - 1$ in place of $y$ in [10] we have

$$F_n(x; y - 1) = (-1)^n F_n(1 - x; -y).$$

(20)

Setting $x = 0$ in the above equation and using the relation (14) we have the reflection formula

$$F_n(y) = (-1)^n \frac{y}{y + 1} F_n(-y - 1), \quad n > 0.$$  

(21)

Therefore, using [11] gives a new explicit formula for Fubini polynomials in the following theorem.

**Theorem 3** For $n > 0$ we obtain

$$F_n(y) = y \sum_{k=1}^{n} {n \choose k} (-1)^{n+k} k! (y + 1)^{k-1}.$$  

(22)

Note that, when $y = 1$, (22) reduce to [20] Thereom 4.2. Moreover, from (21) we get two conclusion as

$$F_{2k} \left( \frac{-1}{2} \right) = 0 \quad \text{and} \quad F_n(-2) = (-1)^n 2F_n.$$  

(23)

Thus, if we take $y_1 = -2$ and $y_2 = 1$ in (18) and use the second part of (23), we obtain the alternating sums of products of Fubini numbers.

**Corollary 4** For $n > 0$, we have

$$\sum_{k=0}^{n} {n \choose k} (-1)^k F_k F_{n-k} = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is odd} \\
\frac{4}{3} F_n & \text{if } n \text{ is even}.
\end{array} \right.$$  

(24)

Finally, we obtain a new explicit formula for Fubini polynomials and numbers in the following theorem.

**Theorem 5** For $y \neq -\frac{1}{2}$,

$$F_n(y) = \sum_{k=0}^{n} {n \choose k} k! y^k \frac{2^{n+1} (y + 1)^k + (-1)^{k+1}}{(2y + 1)^{k+1}}.$$  

(25)

When $y = 1$ this becomes

$$F_n = \sum_{k=0}^{n} {n \choose k} k! \frac{2^{n+2} + (-1)^{k+1}}{3^{k+1}}, \quad n \geq 0.$$  

(26)

When $y = -2$ this becomes

$$F_n = \sum_{k=0}^{n} (-1)^{n-k} {n \choose k} k! \frac{2^{k-1} [2^{n+k+1} + 1]}{3^{k+1}}, \quad n > 0.$$  

(27)
Proof. If we take \( \frac{1}{y^2 - 1} \) in place of \( y \) in (2) we arrive at
\[
\frac{1}{1 - \frac{1}{y^2 - 1} (e^{2t} - 1)} = \frac{y^2 - 1}{2y^2} \left[ \frac{1}{y - e^t} + \frac{1}{y + e^t} \right].
\] (28)
Each of the function in the above equation can be written as
\[
\frac{1}{1 - \frac{1}{y^2 - 1} (e^{2t} - 1)} = \sum_{n=0}^{\infty} 2^n F_n \left( \frac{1}{y^2 - 1} \right) \frac{t^n}{n!},
\] (29)
\[
\frac{1}{y - e^t} = \frac{y}{y - 1} \sum_{n=0}^{\infty} F_n \left( \frac{1}{y - 1} \right) \frac{t^n}{n!},
\] (30)
\[
\frac{1}{y + e^t} = \frac{y}{y + 1} \sum_{n=0}^{\infty} F_n \left( \frac{1}{y + 1} \right) \frac{t^n}{n!}.
\] (31)
By equating the coefficients of \( \frac{t^n}{n!} \), we have
\[
F_n (y) = 2^{n+1} (1 + y) F_n \left( \frac{y^2}{1 + 2y} \right) - (1 + 2y) F_n (-y).
\] (32)
Finally, using (1) in the right hand side of the above equation yields (26). □

3 Integrals of products of Fubini polynomials

The integrals of products of various polynomials and functions have been studied (e.g., [3, 6, 9, 23, 26]). In this section we deal with an integral for a product of two Fubini polynomials. First we need the following Lemma 6 and Lemma 7.

Lemma 6 For all \( k \geq 0 \) and \( n \geq 1 \) we have
\[
\int_{-1}^{0} y^k F_n \left( \frac{y^2}{1 + 2y} \right) dy = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \left[ \left( \begin{array}{c} k+1 \\ j+1 \end{array} \right) \right] B_{n+j},
\] (33)
where \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) is the Stirling numbers of the first kind ([18]).

Proof. We prove (33) by induction on \( k \). The case \( k = 0 \) of (33) is known from [3]. If we integrate both sides of (11) with respect to \( y \) from \(-1\) to \( 0\) and apply integration by parts, we have
\[
\int_{-1}^{0} F_{n+1} \left( \frac{y^2}{1 + 2y} \right) dy = \int_{-1}^{0} y \frac{d}{dy} \left[ F_n (y) + yF_n (y) \right] dy
\]
\[
= \left[ y (F_n (y) + yF_n (y)) \right]_{-1}^{0} - \int_{-1}^{0} \left[ F_n (y) + yF_n (y) \right] dy.
\]
So using (8) yields the case \( k = 1 \) of (33) as

\[
\int_{-1}^{0} yF_n(y) \, dy = -(B_{n+1} + B_n).
\]

Multiplying both sides of (3) with \( y \) and integrating it with respect to \( y \) from \(-1\) to \( 0 \) we obtain

\[
\int_{-1}^{0} yF_{n+1}(y) \, dy = \int_{-1}^{0} y^2 \frac{d}{dy} [F_n(y) + yF_n(y)] \, dy.
\]

Applying integration by parts and using (8) yields the case \( k = 2 \) of (33) as

\[
2 \int_{-1}^{0} y^2 F_n(y) \, dy = B_{n+2} + 3B_{n+1} + 2B_n.
\]

If we multiply both sides of (33) with \( y^k \) and integrating it with respect to \( y \) from \(-1\) to \( 0 \) we obtain

\[
\int_{-1}^{0} y^k F_{n+1}(y) \, dy = \int_{-1}^{0} y^{k+1} \frac{d}{dy} [F_n(y) + yF_n(y)] \, dy.
\]

Applying integration by parts to the right hand side of the above equation and considering

\[
\int_{-1}^{0} y^k F_n(y) \, dy = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k+1}{j+1} B_{n+j},
\]

we have

\[
\int_{-1}^{0} y^{k+1} F_{n+1}(y) \, dy = \frac{(-1)^{k+1}}{(k+1)!} \sum_{j=0}^{k} \binom{k+1}{j+1} B_{n+j+1} + \frac{(-1)^{k+1}}{(k+1)!} \sum_{j=0}^{k} (k+1) \binom{k+1}{j+1} B_{n+j}.
\]

Finally, the well known relations

\[
\binom{n+1}{k} = n \binom{n}{k} + \binom{n}{k-1} \quad \text{and} \quad \binom{n}{1} = (n-1)!,
\]

give that the statement is true for \( k + 1 \). \( \blacksquare \)
Lemma 7  For any non-negative integer $m$ and $j$,

$$
\sum_{k=j}^{m} {m \choose k} \frac{1}{j+1} (-1)^k = (-1)^m {m \choose j}.
$$

(34)

**Proof.** We rewrite this equation into matrix form by using the matrices

$$(S_1)_{i,j} = (-1)^{i+j} \frac{i+1}{j+1}, \quad (S_2)_{i,j} = \frac{i}{j}, \quad (B)_{i,j} = \binom{i}{j}.$$

These can be considered as infinite matrices so that the statement we are going to prove takes the form

$$S_2 S_1 = B^{-1},$$

as the elementwise inverse of the matrix $B$ is $(B)_{i,k}^{-1} = (-1)^{i+k} \frac{i}{k}$. The above equation is equivalent to

$$S_1 = B^{-1} S_2^{-1} = (S_2 B)^{-1}.$$

The matrix on the right hand side is easily decipherable. Elementwise it is

$$((S_2 B)^{-1})_{i,j} = \sum_{k=0}^{i} \binom{i}{k} \binom{k}{j}.$$

The latter sum simply equals to

$$\sum_{k=0}^{i} \frac{i}{k} \binom{k}{j} = \binom{i+1}{j+1},$$

as it is known (see [18, p. 251, formula (6.15)]). Hence our original statement equals to the matrix equation

$$(S_1)_{i,j}^{-1} = \binom{i+1}{j+1}.$$

This is nothing else but the reformulation of the fact that the second and signed first kind Stirling matrices are inverses of each other. \( \blacksquare \)

Now, we are ready to give the integrals of products of Fubini polynomials. Using (1) we have

$$\int_{-1}^{0} F_m(y) F_n(y) dy = \int_{-1}^{0} \sum_{k=0}^{m} {m \choose k} k! y^k F_n(y) dy.$$

Then, interchanging the sum and integral in the above equation and using (33) yield

$$\int_{-1}^{0} F_m(y) F_n(y) dy = \sum_{j=0}^{m} \sum_{k=j}^{m} {m \choose k} \frac{1}{j+1} (-1)^k B_{n+j}.$$ 

Finally, using Lemma [7] gives the following theorem.
Theorem 8 For all \( m \geq 0 \) and \( n \geq 1 \) we have
\[
\int_{-1}^{0} F_m(y) F_n(y) \, dy = (-1)^m \sum_{j=0}^{m} \binom{m}{j} B_{n+j}.
\] (35)

Using the representation (1) in (35) and integrating termwise one obtains
\[
\sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} \frac{(-1)^{k+j} k! j!}{k+j+1} = (-1)^m \sum_{j=0}^{m} \binom{m}{j} B_{n+j}.
\]
This double sum identity extends (35).

In order to give an application of Lemma 6, now we emphasize the summation in the right hand of (33). Rahmani [27] defined \( p \)-Bernoulli numbers as
\[
\sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = {}_2F_1(1,1; p+2; 1-e^t),
\]
where \( {}_2F_1(a,b;c;z) \) denotes the Gaussian hypergeometric function [1]. These numbers can be written in terms Stirling numbers of the first kind
\[
\sum_{j=0}^{p} (-1)^{j+1} \binom{p}{j} B_{n+j} = \frac{p!}{p+1} B_{n,p}, \quad n, p \geq 0.
\]

From the above equation, we have
\[
\sum_{j=0}^{p} (-1)^{j+1} \binom{p+1}{j+1} B_{n+j} = \frac{(p+1)!}{p+2} B_{n-1,p+1}, \quad n \geq 1, p \geq 0.
\] (36)

Moreover when \( n \) is odd or even we have
\[
(-1)^{j+1} B_{n+j} = B_{n+j} \quad \text{or} \quad (-1)^{j+1} B_{n+j} = -B_{n+j}, \quad n > 1,
\]
respectively. Therefore we have
\[
\sum_{j=0}^{p} \binom{p+1}{j+1} B_{n+j} = \begin{cases} 
\frac{(p+1)!}{p+2} B_{n-1,p+1} & \text{if } n \text{ is odd} \\
-\frac{(p+1)!}{p+2} B_{n-1,p+1} & \text{if } n \text{ is even} 
\end{cases}.
\]

Using the above equation, (35) can be written as
\[
\int_{-1}^{0} y^n F_n(y) \, dy = \begin{cases} 
(-1)^n \frac{p+n+1}{p+2} B_{n-1,p+1} & \text{if } n \text{ is odd} \\
(-1)^{n+1} \frac{p+n+1}{p+2} B_{n-1,p+1} & \text{if } n \text{ is even} 
\end{cases},
\] (37)
where \( n > 1, p \geq 0 \). On the other hand, using (1) in the left part of (37), a new explicit formula for \( p \)-Bernoulli numbers is obtained.
Theorem 9 For \( n > 1 \) and \( p > 0 \),

\[
B_{2n-1,p} = \frac{p+1}{p} \sum_{k=0}^{2n-1} \frac{2n-1}{k+1} \frac{(-1)^{k+1} (k+1)!}{k+p+1}
\]

and

\[
B_{2n,p} = \frac{p+1}{p} \sum_{k=0}^{2n} \frac{2n+1}{k+1} \frac{(-1)^{k} (k+1)!}{k+p+1}.
\]

4 Applications

Apostol-Bernoulli functions \( B_n(\lambda) \) have the following explicit expression

\[
B_n(\lambda) = \frac{n}{\lambda - 1}\sum_{k=0}^{n-1} \binom{n-1}{k} k! \left( \frac{1}{1-\lambda} \right)^k, \quad \lambda \in \mathbb{C}\setminus\{1\}.
\]  

(38)

Thus for \( \lambda \neq 1 \),

\[
B_0(\lambda) = 0, \quad B_1(\lambda) = \frac{1}{\lambda - 1}, \quad B_2(\lambda) = \frac{-2\lambda}{(\lambda - 1)^2}, \ldots \text{etc.}
\]

The functions \( B_n(\lambda) \) are rational functions in the second variable, \( \lambda \). These functions were introduced by Apostol [4] in order to evaluate the Lerch transcendent (also Lerch zeta function) for negative integer values of \( s \) and also were studied and generalized recently in a number of papers, under the name Apostol-Bernoulli numbers.

Comparing the (38) to (1), Apostol-Bernoulli functions can be expressed by Fubini polynomials as (17)

\[
B_{n+1}(\lambda) = \frac{n+1}{\lambda - 1} F_n \left( \frac{\lambda}{1-\lambda} \right), \quad \lambda \in \mathbb{C}\setminus\{1\}.
\]  

(39)

We can use this relation to obtain some new properties of \( B_n(\lambda) \). For example setting \( y = \frac{\lambda}{1-\lambda} \) in (22) we have

\[
\frac{B_{n+1}(\lambda)}{(n+1)} = (-1)^n \lambda \sum_{k=0}^{n} \binom{n}{k} k! \left( \frac{1}{\lambda - 1} \right)^{k+1}, \quad \lambda \neq 1, \ n \geq 0,
\]

which was obtained in [20, Thereom 4.3]. Similarly, from Thereom 11 we get the sums of products of Apostol-Bernoulli functions as given in [15, Corollary 1.3] by different method. Moreover, using the equation (25) of Theorem 5 gives a new explicit formula for Apostol-Bernoulli functions.

Corollary 10 For \( \lambda \neq \pm 1 \) and \( n \geq 0 \),

\[
\frac{B_{n+1}(\lambda)}{(n+1)} = \sum_{k=0}^{n} \binom{n}{k} k! \left( -\lambda \right)^k \frac{2^{n+1} \lambda^k + (\lambda - 1)^{k+1}}{(\lambda^2 - 1)^{k+1}}.
\]  

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To give a different application of the relation (39), first we deal with Lemma 6. Replacing \( y \) with \( \frac{\lambda}{1-\lambda} \) in (33), we have

\[
\int_{-\infty}^{0} \frac{\lambda^k}{(\lambda-1)^{k+1}} B_{n+1}(\lambda) \, d\lambda = \frac{n+1}{k!} \sum_{j=0}^{k} \left[ \frac{k+1}{j+1} \right] B_{n+j},
\]

where \( k \geq 0 \) and \( n \geq 1 \). Similarly, from Theorem 8 we obtain the integrals of products of Apostol-Bernoulli functions as given in the following corollary.

**Corollary 11** For all \( m \geq 0 \) and \( n \geq 1 \) we have

\[
\int_{-\infty}^{0} B_m(\lambda) B_n(\lambda) \, d\lambda = (-1)^m (m+1)(n+1) \sum_{j=0}^{m} \binom{m}{j} B_{n+j}.
\]

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