Thermodynamic Limit for the Ising Model on the Cayley Tree

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While the Ising model on the Cayley tree has no spontaneous magnetization at nonzero temperatures in the thermodynamic limit, we show that finite systems of astronomical sizes remain magnetically ordered in a wide temperature range, if the symmetry is broken by fixing an arbitrary single (bulk or surface) spin. We compare the behavior of the finite size magnetization of this model with that of the Ising model on both the Sierpinski Gasket, and the one-dimensional linear chain. This comparison reveals the analogy of the behavior of the present model with the Sierpinski Gasket case.

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The zero-field partition function of the Ising model on the Cayley tree is identical to that of the one-dimensional linear Ising chain [1], and thus it does not lead to any singularities of the zero-field thermodynamic response functions. This model manifests thermodynamic singular behavior only in response to the magnetic field, and it has been well established [1, 2, 3, 4] that the susceptibility diverges in a wide range of temperatures, while spontaneous magnetization remains zero at any finite temperature in the thermodynamic limit.

While the above facts were established three decades ago, the exact analytical expression for the zero field magnetization has been derived only recently [5, 6]. Similar to the expression for the zero-field partition function, the obtained expression for magnetization is equivalent to the expression for the zero-field partition function, the thermodynamic limit.

It turns out that the nonlinear coupled recursion relations for the partition function of any two consecutive generations of arbitrary sizes (therefore also for the thermodynamic limit) cannot be iterated to yield a closed form expression in the general nonzero field case [1, 2, 3, 4, 8]. The nonlinear coupled recursion relations (2) cannot be iterated to yield corresponding closed form expressions for systems of arbitrary sizes (therefore also for the thermodynamic limit). Following Eggarter [8], we further consider a single $n$-generation branch of a Cayley tree, composed of two $(n-1)$-generation branches connected to a single initial site, with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle n \rangle} S_i S_j - H \sum_i S_i,$$

where $J$ is the coupling constant, $H$ is the external magnetic field, $S_i = \pm 1$ is the spin at site $i$, and $\langle n \rangle$ denotes summation over the nearest-neighbor pairs. The $n$-generation branch consists of $N_n = 2^{n+1} - 1$ spins, the 0-generation branch being a single spin. The recursion relations for the partition function of any two consecutive generation branches were found by Eggarter [8] to be

$$Z_{n+1}^\pm = y^{\pm 1} \left[ x^{\pm 1} Z_n^+ + x^{\mp 1} Z_n^- \right]^2,$$

where $x \equiv \exp(\beta J)$, $y \equiv \exp(\beta H)$, $\beta \equiv 1/k_B T$ is the reciprocal of the product of the Boltzmann constant $k_B$ and temperature $T$, and $Z_n^+$ and $Z_n^-$ denote the branch partition functions restricted by fixing the initial spin (connecting the two $(n-1)$-generation branches) into the $\{+\}$ and $\{-\}$ position, respectively.

It turns out that the nonlinear coupled recursion relations [2] cannot be iterated to yield a closed form expression in the general nonzero field case [1, 2, 3, 4, 8]. However, only recently the present authors have arrived [6] at the exact analytical expression for the zero-field magnetization and susceptibility, by considering the recursion relations for the field derivatives of the partition function, which can be iterated in the limit $H \rightarrow 0$ to yield corresponding closed form expressions for systems of arbitrary sizes (therefore also for the thermodynamic limit)

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limit). The exact expression for restricted magnetization, for arbitrary tree generation level \( n \), was found to be:

\[
< m >^\pm_n = \frac{(2 t)^{n+1} - 1}{(2^{n+1} - 1)(2 t - 1)}, \tag{3}
\]

where \( t = \tanh(\beta J) \). The same expression was (also only recently) independently obtained by Mélin et al. \(^4\), using a different approach.

It turns out that the zero-field magnetization (as well as all the other odd field derivatives of the partition function) is identically zero in strictly zero field, as the expressions corresponding to the two possible orientations of the initial spin differ only in sign. However, this is also true for an arbitrary Ising system in (strictly) zero field, as every spin configuration has a mirror image (obtained by flipping all the spins) with exactly the same energy and inverted sign of the configurational magnetization, and to remove this degeneracy one first has to break the symmetry. The usual procedure of breaking the symmetry by retaining an infinitesimal field, then taking the thermodynamic limit, and finally taking the zero field limit, is not suitable here because of the fact that susceptibility diverges in a wide temperature region, rather than in a single (critical) point.

Instead, here it seems most suitable to break the symmetry simply by fixing the initial spin in the upward position (therefore disregarding all configurations corresponding to downward initial spin orientation), and in what follows we shall first examine the analytical results corresponding to this situation. It can be argued, however, that the initial spin has a privileged role in this lattice, and we shall here also examine numerically the consequence of fixing an arbitrary spin (on the surface, or in the bulk), on the thermodynamic behavior of extremely large (but finite) systems. It will be shown in the rest of this paper that fixing any one spin, which is equivalent to imposing an infinitely strong local field of infinitesimal range (acting on a single spin), leads to symmetry breaking, causing magnetic ordering of systems of astronomical sizes in a wide temperature range.

By fixing the initial spin, one can consider as the order parameter the restricted magnetization

\[
< m >^\pm_n = \frac{1}{N_n} \frac{1}{Z_n^\pm} \frac{\partial Z_n^\pm}{\partial \beta H} = \frac{(2 t)^{n+1} - 1}{(2^{n+1} - 1)(2 t - 1)}. \tag{4}
\]

As \( t < 1 \) for \( T \neq 0 \), this expression clearly demonstrates that there is no magnetic ordering in the thermodynamic limit at any nonzero temperature, confirming earlier results \(^1\) \(^2\) \(^3\) \(^4\) \(^5\). However, comparing with the corresponding expression for the one-dimensional Ising model (which may also be considered as a Cayley tree with the branching number \( B=1 \))

\[
< m >^\pm_n = \frac{(t)^{n+1} - 1}{(n + 1)(t - 1)}, \tag{5}
\]

one sees that the generation level is found in place of the number of particles (these two being the same in the case of the chain). If one takes the Avogadro’s number (\( \sim 10^{23} \)), and the estimated number of hadrons in the observable Universe (\( \sim 10^{80} \), as the two basic reference points in our description of the physical world (as opposed to the mathematical definition of infinity), it turns out that the slight difference between expressions \(^4\) \(^5\) and \(^3\) leads to strikingly different behavior of the two systems. More precisely, it will be shown in this work that the Cayley tree remains ordered in a wide temperature region for systems by far exceeding, in number of particles, the size of the observable Universe. This behavior is reminiscent of that of the Ising model on the Sierpinski Gasket, and may provide some further insight into the thermodynamics of model systems on hierarchical and fractal lattices in general, the latter nowadays being commonly accepted as frequent representatives of our physical world.

Let us begin this (extreme) finite size analysis by explicitly plotting magnetization of the Cayley tree and the Ising linear chain, as given by expressions \(^4\) \(^5\), for various system sizes \( N < 2^{10+1} - 1 \sim 10^{309} \) \((N = 2^{n+1} - 1 \) for the Cayley tree, and \( N = n + 1 \) for the Ising linear chain), as shown in Figs. 1a) and 1b), respectively. It may be interesting to mention here that the extremely simple form of expression \(^5\) is deceptive in the sense that it requires special numerical handling when such large system sizes are considered (for \( N = 2^{10+1} - 1 \) two thousand digit precision was needed by the Maple VI software for algebraic manipulation, for numerical calculation of magnetization at low temperatures).

One can see from Fig. 1a that even for these inconceivably massive systems (with more then 200 orders of magnitude more particles than the estimated number of hadrons in the observable Universe), magnetization of the Cayley tree remains substantially removed from its limiting value of zero, while from Fig. 1b it is seen that the magnetization of the Ising linear chain for the same system sizes far more rapidly collapses to zero, and cannot be distinguished from its limiting zero value on the scale of the graph.

We have also performed similar calculations for the Ising model on the Sierpinski Gasket, for which it was established by Liu \(^7\) that the decay of correlations is extremely slow, and the mean square magnetization \( \langle m^2 \rangle \) approaches its limiting zero value in such a slow fashion that the thermodynamic limit represents a poor approximation for both laboratory size systems, and systems of astronomical dimensions. In this case the lattice at stage \( n \) is constructed by assembling three triangular structures of stage \( n - 1 \), and rather then considering \( \langle m^2 \rangle \) as was done in reference \(^2\), one can break the symmetry by fixing the three vortex spins in the current construction stage, and consider the corresponding restricted magnetization \( < m >^\pm_n \). In Fig. 2 we show the comparison of the points \( T_m \), where the restricted magnetization falls to half of its maximum value, for the Ising model on the
Sierpinski gasket, the Cayley tree, and the linear chain.

It is seen that the magnetization of the Cayley tree demonstrates behavior similar to the Sierpinski gasket, with an extremely slow decay of ordering with the number of particles (observe the double logarithmic scale on the abscissa), in complete contrast with the linear Ising chain. It should be noted that while both in the case of the linear chain and the Sierpinski gasket the considered system sizes also imply astronomic linear dimensions, in the case of the Cayley tree the system linear dimension remains miniscule (for the largest considered systems of $N = 2^{10}+1-1$ particles, the largest distance between any two spins remains only $2^{10} = 1024$ lattice units). In the case of the Cayley tree one cannot therefore talk of “long range correlations”, rather, the extremely slow decay of magnetic ordering with temperature may be attributed to the short system span and the fact that every spin “sees” a huge number of other spins a short distance away (perhaps a term “wide range correlations” would be more appropriate). In the case of other fractal and hierarchical lattices, it remains to be seen to what extent the relationship between “long range” and “wide range” correlations may contribute to the magnetic ordering of finite systems. In particular, the question remains whether real porous samples with magnetic properties that may well be microscopically explained by model systems which do not display spontaneous magnetization in the thermodynamic limit, still may exhibit magnetic ordering at measurable temperatures, due to both their intricate structure and their finite size.

As mentioned earlier, it can be argued that the initial spin occupies a privileged role in the Cayley tree, and here we address the question of whether fixing any other spin is sufficient to break the symmetry, and cause ordering of massive systems in a wide temperature range. Let us therefore consider the situation of building the Cayley tree in the regular fashion (using recursion relations (2)) up to level $p$, fixing a single spin at that level, and then applying modified recursion relations (corresponding to asymmetric restricted partition functions) up to level $q$. From here on, we shall term the branch containing the fixed spin as the “left” branch, denoting the corresponding restricted partition functions by $w_n^-$, in parallel with the regular “right” branches (that do not contain any fixed spins), with the “regular” restricted partition functions $Z_n^+$. Choosing $p = 0$ corresponds to a single surface spin being fixed, $p = q$ corresponds to fixing the initial spin, and $0 < p < q$ corresponds to fixing a single bulk
spin at level \( p \).

Formally differentiating equation (2) with respect to field, the recursion relations for the field derivatives of the restricted partition functions up to level \( p \) are found to be

\[
\frac{\partial Z^{\pm}_{n+1}}{\partial h} = y^{\pm 1} \left[ \pm (Z^{+}_{n} x^{\pm 1} + Z^{-}_{n} x^{-1})^2 + 2 (Z^{+}_{n} x^{\pm 1} + Z^{-}_{n} x^{-1}) \left( \frac{\partial Z^{+}_{n}}{\partial h} x^{\pm 1} + \frac{\partial Z^{-}_{n}}{\partial h} x^{-1} \right) \right] (6)
\]

where we have used notation \( h \equiv \beta H \) for the reduced field.

At all following levels (including \( p+1 \)) the “right” partition functions will differ from the “left” partition functions, and the general recursion relation is given by

\[
w^{\pm}_{n+1} = y^{\pm 1} \left[ (w^{+}_{n} x^{\pm 1} + w^{-}_{n} x^{-1}) (Z^{+}_{n} x^{\pm 1} + Z^{-}_{n} x^{-1}) \right],
\]

where for the derivatives we have

\[
\frac{\partial w^{\pm}_{n+1}}{\partial h} = y^{\pm 1} \left[ \pm (w^{+}_{n} x^{\pm 1} + w^{-}_{n} x^{-1}) (Z^{+}_{n} x^{\pm 1} + Z^{-}_{n} x^{-1}) + \left( \frac{\partial w^{+}_{n}}{\partial h} x^{\pm 1} + \frac{\partial w^{-}_{n}}{\partial h} x^{-1} \right) (Z^{+}_{n} x^{\pm 1} + Z^{-}_{n} x^{-1}) \right] + \left( \frac{\partial Z^{+}_{n}}{\partial h} x^{\pm 1} + \frac{\partial Z^{-}_{n}}{\partial h} x^{-1} \right) \left( w^{+}_{n} x^{\pm 1} + w^{-}_{n} x^{-1} \right).
\]

(7)

(8)

We now take one “left” branch of level \( p \) (by setting \( w^{+}_{p} = Z^{+}_{p} \) and \( w^{-}_{p} = 0 \)), and one “right” branch of level \( p \) (using both \( Z^{+}_{p} \) and \( Z^{-}_{p} \)), to form the “left” branch structures at level \( p+1 \) (if \( p = q \) we do not continue with forming the \( p+1 \) level structure, rather just retain \( Z^{+}_{p} \) as the final partition function). The recursion relations (7) and (8) for the restricted partition functions an their derivatives, respectively, at this step take explicit form

\[
w^{+}_{p+1} = y^{\pm 1} Z^{+}_{p} \left[ Z^{+}_{p} x^{\pm 2} + Z^{-}_{p} \right],
\]

(9)

and

\[
\frac{\partial w^{+}_{p+1}}{\partial h} = y^{\pm 1} \left[ \pm Z^{+}_{p} (Z^{+}_{p} x^{\pm 2} + Z^{-}_{p}) + \frac{\partial Z^{+}_{p}}{\partial h} x^{\pm 2} + \frac{\partial Z^{-}_{p}}{\partial h} \right].
\]

(10)

In the following steps (\( p+1 < n \leq q \)), the “left” branches are combined with (regular) “right” branches to form new “left” branches through general recursion relations (7) and (8), while the “right” branches are all the time calculated using (2) and (6).

Evidently, the set of equations (2), (6–10) is too complex to be iterated analytically into a closed form expression, for arbitrary \( p \) and \( q \). On the other hand, using algebraic manipulation software such as Maple or Mathematica, it is a simple exercise to perform these iterations numerically for arbitrary size and temperature, with arbitrary precision (depending only on the available computer speed and memory, where present day personal computers are more then well suited for the job). We have performed such calculations for various values of \( p \), for \( q \leq 1024 \) (corresponding to systems of up to \( 2^{10^{10}+1} \approx 10^{309} \) particles), in all cases it was found that fixing any spin indeed does break the symmetry, inducing ordering of enormous systems in a wide temperature range, while fixing the initial spin does produce a somewhat stronger effect then fixing a single surface or bulk spin. In fact, it is found that temperature \( T_{m} \), where magnetization falls off to half of its maximum value, monotonously increases with \( p \), for any given generation level \( q \). This effect may be attributed to the fact that looking from a given position, the average distance to all the other spins monotonously decreases as the position shifts from surface towards the center. Larger distance from the point of fixing a spin implies a weaker effect, in accordance with the numerically observed behavior. From the figure, it follows that the difference between the values of \( T_{m} \) obtained by fixing the initial spin and by fixing a single surface spin diminishes with system size, and that the conclusions about ordering of extremely large systems in a wide temperature range continues to hold (as well as the comparison of the current model with the Sierpinski gasket and the chain), if one chooses to break the symmetry by fixing an arbitrary spin.

In conclusion, as the Cayley tree in itself (with its infinite dimension and nonzero fraction of surface nodes) represents a highly unphysical system in the thermodynamic limit, modeling real world physical phenomena (such as, for instance, blood vessels or the nervous system) requires more attention focus on the finite size behavior. In this work it is shown that the thermodynamic limit represents a questionable approximation for the Ising model on the Cayley tree, as the magnetic or-
FIG. 3: Temperature $T_m$ where magnetization falls to half of its maximum value, for the Ising system on the Cayley tree, as a function of system size $N = 2^q - 1$, on the double logarithmic scale. The upper curve (open circles) corresponds to fixing the initial spin ($p = q$), and the bottom curve (full circles) to fixing a single surface spin ($p = 0$). For all the other values of $0 < p < q$, $T_m$ is found to lie between the two shown curves, monotonously increasing with $p$.

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