On a conjecture of R. Brück and some linear differential equations

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Abstract
In this paper, we mainly investigate the Brück conjecture concerning entire function \( f \) and its differential polynomial \( L_1(f) = a_k(z)f^{(k)} + \cdots + a_0(z)f \) sharing an entire function \( \alpha(z) \) with \( \sigma(\alpha) \leq \sigma(f) \), by using the theory of complex differential equation.

Keywords: Entire function, Brück conjecture, Difference polynomial

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Introduction and some results
It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna value distribution theory of meromorphic functions. For a meromorphic function \( f \) in the whole complex plane \( \mathbb{C} \), we shall use the following standard notations of the value distribution theory:

\[ T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots \]

(see Hayman 1964; Yang 1993; Yi and Yang 2003, 1995). We use \( S(r,f) \) to denote any quantity satisfying \( S(r,f) = o(T(r,f)) \), as \( r \to +\infty \), possibly outside of a set with finite measure. A meromorphic function \( a(z) \) is called a small function with respect to \( f \) if \( T(r,a) = S(r,f) \). In addition, we will use the notation \( \sigma(f) \), \( \mu(f) \) to denote the order and the lower order of meromorphic function \( f(z) \), which are defined by

\[
\sigma(f) = \limsup_{r \to +\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r \to +\infty} \frac{\log \log M(r,f)}{\log r},
\]

and

\[
\mu(f) = \liminf_{r \to +\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r \to +\infty} \frac{\log \log M(r,f)}{\log r},
\]

where \( M(r,f) = \max_{|z|=r} |f(z)| \). We also use \( \tau(f) \) to denote the type of an entire function \( f(z) \) with \( 0 < \sigma(f) = \sigma < +\infty \), which is defined to be (see Hayman 1964)

\[
\tau(f) = \limsup_{r \to +\infty} \frac{\log M(r,f)}{r^\sigma}.
\]
We use $\sigma_2(f)$ to denote the hyper-order of $f(z)$, which is defined to be (see Yi and Yang 2003, 1995)

$$\sigma_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r,f)}{\log r}.$$ 

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, for some $a \in \mathbb{C} \cup \{\infty\}$, if the zeros of $f(z) - a$ and $g(z) - a$ (if $a = \infty$, zeros of $f(z)$ and $g(z) - a$ are the poles of $f(z)$ and $g(z)$ respectively) coincide in locations and multiplicities we say that $f(z)$ and $g(z)$ share the value $a$ CM (counting multiplicities) and if coincide in locations only we say that $f(z)$ and $g(z)$ share $a$ IM (ignoring multiplicities).

Rubel and Yang (1977) proved the following result.

**Theorem 1.1** Rubel and Yang (1977). Let $f$ be a nonconstant entire function. If $f$ and $f'$ share two finite distinct values CM, then $f \equiv f'$.

In 1996, Brück proposed the following conjecture Brück (1996):

**Conjecture 1.1** Brück (1996). Let $f$ be a non-constant entire function. Suppose that $\sigma_2(f)$ is not a positive integer or infinite, if $f$ and $f'$ share one finite value $a$ CM, then

$$f' - a f - a = c$$

for some non-zero constant $c$.

Gundersen and Yang (1998) proved that Brück conjecture holds for entire functions of finite order and obtained the following result.

**Theorem 1.2** [Gundersen and Yang (1998), Theorem 1]. Let $f$ be a nonconstant entire function of finite order. If $f$ and $f'$ share one finite value $a$ CM, then $f' - a f - a = c$ for some non-zero constant $c$.

The shared value problems related to a meromorphic function $f$ and its derivative $f^{(k)}$ have been a more widely studied subtopic of the uniqueness theory of entire and meromorphic functions in the field of complex analysis (see Chen et al. 2014; Li and Yi 2007; Liao 2015; Mues and Steinmetz 1986; Zhang and Yang 2009; Zhang 2005; Zhao 2012).

Li and Cao (2008) improved the Brück conjecture for entire function and its derivation sharing polynomials and obtained the following result:

**Theorem 1.3** Li and Cao (2008). Let $Q_1$ and $Q_2$ be two nonzero polynomials, and let $P$ be a polynomial. If $f$ is a nonconstant entire solution of the equation

$$f^{(k)} - Q_1 = (f - Q_2)e^P,$$

then $\sigma_2(f) = \deg P$, where and in the following, $\deg P$ is the degree of $P$.

Mao (2009) studied the problem on Brück conjecture when $f^{(k)}$ is replaced by differential polynomial $L(f) = A_k f^{(k)} + \cdots + A_1 f' + A_0 f$ in Theorem 1.3.
Motivated by the above question, the main purpose of this article is to study the growth of solution of differential equation on entire function \( f \) and its linear differential polynomial

\[
L_1(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f,
\]

where \( k \) is a positive integer, \( a_k(z)(\not\equiv 0), a_{k-1}(z), \ldots, a_1(z) \) and \( a_0(z) \) are polynomials, and obtain the following theorems.

**Theorem 1.4**  \( \text{Mao (2009).} \) Let \( P(z) \) be a polynomial, \( A_k(z)(\not\equiv 0), \ldots, A_0(z) \) be polynomials, and \( f \) be an entire function of order

\[
\sigma(f) > 1 + \max_{0 \leq j < k-1} \left\{ \frac{\deg A_j - \deg A_k}{k - j}, 0 \right\}
\]

and hyper-order \( \sigma_2(f) < \frac{1}{2} \) if \( f \) and \( L(f) \) share \( P \) CM, then

\[
\frac{L(f) - P(z)}{f(z) - P(z)} = c,
\]

for some constant \( c \not\equiv 0 \), where, and in the sequel, \( \deg A_j \) denotes the degree of \( A_j(z) \), \( k \) is a positive integer.

Chang and Zhu (2009) further investigated the problem related to Brück conjecture and proved that Theorem 1.2 remains valid if the value \( a \) is replaced by a function \( a(z) \).

**Theorem 1.5**  \( \text{[Chang and Zhu (2009), Theorem 1].} \) Let \( f \) be an entire function of finite order and \( a(z) \) be a function such that \( \sigma(a) < \sigma(f) < +\infty \). If \( f \) and \( f' \) share \( a(z) \) CM, then \( \frac{f}{f'-a} = c \) for some non-zero constant \( c \).

Thus, an interesting subject arises naturally about this problem: \textit{what would happen when \( \sigma(a) < \sigma(f) < +\infty \) is replaced by \( 0 < \sigma(a) = \sigma(f) < +\infty \) in Theorems 1.2–1.5?}

**Conclusions**

Motivated by the above question, the main purpose of this article is to study the growth of solution of differential equation on entire function \( f \) and its linear differential polynomial

\[
L_1(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f,
\]

where \( k \) is a positive integer, \( a_k(z)(\not\equiv 0), a_{k-1}(z), \ldots, a_1(z) \) and \( a_0(z) \) are polynomials, and obtain the following theorems.

**Theorem 2.1**  \( \text{Let} \ f(z) \ \text{and} \ \alpha(z) \ \text{be two nonconstant entire functions and satisfy} \ 0 < \sigma(a) = \sigma(f) < +\infty \ \text{and} \ \tau(f) > \tau(a), \ \text{and let} \ P(z) \ \text{be a polynomial such that} \)

\[
\sigma(f) > \deg P + \max \left\{ \frac{\deg a_j - \deg a_k}{k - j}, 0 \right\}.
\]

If \( f \) is a nonconstant entire solution of the following differential equation

\[
L_1(f) - \alpha(z) = (f(z) - \alpha(z))e^{P(z)},
\]

where \( L_1(f) \) is stated as in (1). Then \( P(z) \) is a constant.

If \( L_1(f) \) is replaced by the following linear differential polynomial \( L_2(f) \)

\[
L_2(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f + \beta(z),
\]

where \( k \) is a positive integer, \( a_k(z)(\not\equiv 0), a_{k-1}(z), \ldots, a_1(z) \) and \( a_0(z) \) are polynomials, and \( \beta \) is an entire function satisfying either \( \sigma(\beta) < \mu(f) \) or \( 0 < \sigma(\beta) = \sigma(f) < +\infty \) and \( \tau(\beta) < \tau(f) \), then we obtain the following results.
**Theorem 2.2** Let \( f(z) \) and \( \alpha(z) \) be two nonconstant entire functions and satisfy \( 0 < \sigma(\alpha) = \sigma(f) < +\infty \) and \( \tau(f) > \tau(\alpha) \), and let \( P(z) \) be a polynomial satisfying (2). If \( f \) is a nonconstant entire solution of the following differential equation

\[
L_2(f) - \alpha(z) = (f(z) - \alpha(z))e^{P(z)},
\]

where \( L_2(f) \) is stated as in (4) and \( \beta \) is an entire function satisfying \( 0 < \sigma(\beta) = \sigma(f) < +\infty \) and \( \tau(\beta) < \tau(f) \). Then \( P(z) \) is a constant.

**Theorem 2.3** Let \( f(z) \) and \( \alpha(z) \) be two nonconstant entire functions and satisfy \( \sigma(\alpha) < \mu(f) \), and let \( P(z) \) be a polynomial satisfying (2). If \( f \) is a nonconstant entire solution of Eq. (5), where \( L_2(f) \) is stated as in (4) and \( \beta \) is an entire function satisfying \( \sigma(\beta) < \mu(f) \). Then \( \sigma(f) = \deg P \).

**Corollary 2.1** Let \( f(z) \) and \( \alpha(z) \) be two nonconstant entire functions and satisfy \( \sigma(\alpha) < \mu(f) \), and let \( P(z) \) be a polynomial satisfying (2). If \( f \) is a nonconstant entire solution of Eq. (3), where \( L_1(f) \) is stated as in (1). Then \( \sigma(f) = \deg P \).

**Some Lemmas**

To prove our theorems, we will require some lemmas as follows.

**Lemma 3.1** Laine (1993). Let \( f(z) \) be a transcendental entire function, \( \nu(r,f) \) be the central index of \( f(z) \). Then there exists a set \( E \subset (1, +\infty) \) with finite logarithmic measure, we choose \( z \) satisfying \( |z| = r \notin [0, 1] \cup E \) and \( |f(z)| = M(r,f) \), we get

\[
\frac{f^{(j)}(z)}{f(z)} = \left( \frac{\nu(r,f)}{z} \right)^j (1 + o(1)), \quad \text{for } j \in \mathbb{N}.
\]

**Lemma 3.2** He and Xiao (1988). Let \( f(z) \) be an entire function of finite order \( \sigma(f) = \sigma < +\infty \) and let \( \nu(r,f) \) be the central index of \( f \). Then

\[
\limsup_{r \to +\infty} \frac{\log \nu(r,f)}{\log r} = \sigma(f).
\]

And if \( f \) is a transcendental entire function of hyper order \( \sigma_2(f) \), then

\[
\limsup_{r \to +\infty} \frac{\log \log \nu(r,f)}{\log r} = \sigma_2(f).
\]

**Lemma 3.3** Mao (2009). Let \( f \) be a transcendental entire function and let \( E \subset [1, +\infty) \) be a set having finite logarithmic measure. Then there exists \( \{z_n = r_n e^{i\theta_n}\} \) such that \( |f(z_n)| = M(r_n,f), \theta_n \in [0, 2\pi), \lim_{n \to +\infty} \theta_n = \theta_0 \in [0, 2\pi) \), \( r_n \notin E \) and if \( 0 < \sigma(f) < +\infty \), then for any given \( \epsilon > 0 \) and sufficiently large \( r_n \)

\[
r_n^{\sigma(f)-\epsilon} < \nu(r_n, f) < r_n^{\sigma(f)+\epsilon}.
\]

If \( \sigma(f) = +\infty \), then for any given large \( M > 0 \) and sufficiently large \( r_n \), \( \nu(r_n, f) > r_n^M \).
Lemma 3.4 Laine (1993). Let \( P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0 \) with \( b_n \neq 0 \) be a polynomial. Then, for every \( \varepsilon > 0 \), there exists \( r_0 > 0 \) such that for all \( r = |z| > r_0 \) the inequalities
\[
(1 - \varepsilon)|b_n|r^n \leq |P(z)| \leq (1 + \varepsilon)|b_n|r^n
\]
hold.

Lemma 3.5 Let \( f(z) \) and \( A(z) \) be two entire functions with \( 0 < \sigma(f) = \sigma(A) < +\infty, 0 < \tau(A) < \tau(f) < +\infty \), then there exists a set \( E \subset [1, +\infty) \) that has infinite logarithmic measure such that for all \( r \in E \) and a positive number \( \kappa > 0 \), we have
\[
\frac{M(r,A)}{M(r,f)} < \exp\{-\kappa r^\sigma\}.
\]

Proof By definition, there exists an increasing sequence \( \{r_m\} \rightarrow +\infty \) satisfying \( (1 + \frac{1}{m})r_m < r_{m+1} \) and
\[
\lim_{m \to +\infty} \frac{\log M(r_m, f)}{r_m^\sigma} = \tau(f). \tag{6}
\]
For any given \( \beta (\tau(A) < \beta < \tau(f)) \), then there exists some positive integer \( m_0 \) such that for all \( m \geq m_0 \) and for any given \( \varepsilon (0 < \varepsilon < \tau(f) - \beta) \), we have
\[
\log M(r_m, f) > (\tau(f) - \varepsilon)r_m^\sigma. \tag{7}
\]
Thus, there exists some positive integer \( m_1 \) such that for all \( m \geq m_1 \), we have
\[
\left( \frac{m}{m + 1} \right)^\sigma > \frac{\beta}{(\tau(f) - \varepsilon)}. \tag{8}
\]
From (6–8), for all \( m \geq m_2 = \max\{m_0, m_1\} \) and for any \( r \in [r_m, (1 + \frac{1}{m})r_m] \), we have
\[
M(r,f) \geq M(r_m, f) > \exp\{(\tau(f) - \varepsilon)r_m^\sigma\}
\geq \exp\left\{ (\tau(f) - \varepsilon)\left( \frac{m}{m + 1} \right)^\sigma \right\} > \exp\{\beta r^\sigma\}. \tag{9}
\]
Set \( E = \bigcup_{m = m_2}^{\infty} [r_m, (1 + \frac{1}{m})r_m] \), then
\[
m_1E = \sum_{m = m_2}^{\infty} \int_{r_m}^{(1 + \frac{1}{m})r_m} \frac{dt}{t} = \sum_{m = m_2}^{\infty} \log \left( 1 + \frac{1}{m} \right) = +\infty.
\]
From the definition of type of entire function, for any sufficiently small \( \varepsilon > 0 \), we have
\[
M(r,A) < \exp\{\tau(A) + \varepsilon r^\sigma\}. \tag{10}
\]
By (9) and (10), set \( \kappa = \beta - \tau(A) - \varepsilon \), for all \( r \in E \), we have
\[
\frac{M(r,A)}{M(r,f)} < \exp\{-(\beta - \tau(A) - \varepsilon)r^\sigma\} = e^{-\kappa r^\sigma}.
\]
Thus, this completes the proof of this lemma. □

The proof of Theorem 2.1

Proof Since $P(z)$ is a polynomial, assume that $\deg P = m \geq 1$. Let

$$P(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $b_m, \ldots, b_0$ are constants and $b_m \neq 0, m \geq 1$. Thus, it follows from (3) and Lemma 3.4 that

$$|b_m|^m (1 + o(1)) = |P(z)| = \left| \log \frac{L_1(f(z))}{f(z)} - \frac{\sigma(z)}{f(z)} \right|, \quad (11)$$

Since $L_1(f) = a_k f^k + a_k f^{k-1} + \cdots + a_0 f$, from Lemma 3.1, then there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure, such that for some point $|z| = re^{i\theta} (\theta \in [0, 2\pi)), r \notin E_1$ and $M(r, f) = |f(z)|$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left\{ \frac{\nu(r, f)}{z} \right\}^j (1 + o(1)), \quad 1 \leq j \leq k.$$ 

Thus, it follows that

$$\frac{L_1(f(z))}{f(z)} = \alpha_k \left\{ \frac{\nu(r, f)}{z} \right\}^k (1 + o(1)) + \sum_{j=1}^{k} a_k \frac{\nu(r, f)}{z} \left( 1 + o(1) \right) + a_0$$

$$= \frac{\alpha_k}{z^k} (1 + o(1)) \left[ \frac{\nu(r, f)}{z} + \sum_{j=1}^{k} \frac{a_k}{\alpha_k} z^{-j} \nu(r, f) \right]. \quad (12)$$

From Lemma 3.3, there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f), \theta_n \in [0, 2\pi), \lim_{n \to \infty} \theta_n = \theta_0 \in [0, 2\pi), r_n \notin E_k$, then for any given $\varepsilon$ satisfying

$$0 < \varepsilon < \min_{1 \leq j \leq k} \frac{\sigma(f) - \deg P - \frac{d_{k-j}}{3k - j}}{3k - j},$$

where $d_{k-j} = \deg a_{k-j} - \deg a_k$, and sufficiently large $r_n$ we have

$$r_n^{\sigma(f) - \varepsilon} < \nu(r_n, f) < r_n^{\sigma(f) + \varepsilon}. \quad (13)$$

Since $a_0(z), \ldots, a_k(z)$ are polynomials, let $a_j(z) = \sum_{i=0}^{s_j} a_{j,i} z^i$, where $s_j = \deg a_j, j = 0, 1, \ldots, k$. Then, from Lemma 3.4 and (13), we have
where \( d_k = s_k - s_k \) and \( M \) is a positive constant. Since 
\[- j \sigma(f) + d_k + \deg P + (k - j) \varepsilon < -2k \varepsilon < 0, \]
from (14), it follows that
\[
\left| \frac{a_k}{a_k} - z^j v(r_n f)^{k-j} (1 + o(1)) \right| \leq M \left| \frac{t_k - j s_k}{t_k} \right| r_n^{(\sigma(f) - 2 \varepsilon)} - j \sigma(f) + d_k - j \deg P + (k - j) \varepsilon, \]
for \( r_n \rightarrow +\infty, \ r_n \notin E_1. \) 

Since \( 0 < \sigma(\omega) = \sigma(f) < +\infty \) and \( \tau(\omega) < \tau(f) < +\infty, \)
from Lemma 3.5, there exists a set \( E \subset [1, +\infty) \) that has infinite logarithmic measure such that for a sequence \( \{r_n\}_1 \in E_2 = E - E_1, \)
we have
\[
\frac{M(r, \omega)}{M(r, f)} < \exp[-k r_n^{\sigma(f)}] \rightarrow 0, \quad \text{as} \ n \rightarrow +\infty. \]

From (11), (12), (15), (16) and Lemma 3.2, we can get that
\[
|b_m| r_n^m (1 + o(1)) = |P(z)| = O(\log r_n), \quad \text{(17)}
\]
which is impossible. Thus, \( P(z) \) is not a polynomial, that is, \( P(z) \) is a constant.

Thus, this completes the proof of Theorem 2.1. \( \square \)

**The proof of Theorem 2.2**

**Proof** First of all, we rewrite (5) as
\[
\frac{L_2(f) - \alpha(z)}{f(z) - \alpha(z)} = \frac{L_2(f)}{f(z) - \alpha(z)} + \frac{\beta(\omega)}{f(z) - \alpha(z)} + o(1) = \varepsilon P(z), \quad \text{(18)}
\]
where \( L_2(f) \) is stated as in Theorem 2.1. Since \( 0 < \sigma(f) = \sigma(\omega) = \sigma(\beta) < +\infty, \)
\( \tau(\omega) < \tau(f) \) and \( \tau(\beta) < \tau(f), \)
from Lemma 3.5, there exists a set \( E \subset [1, +\infty) \) that has infinite logarithmic measure such that for a sequence \( \{r_n\}_1 \in E_3 = E - E_1, \)
we have
\[
\frac{M(r, \omega)}{M(r, f)} < \exp[-k r_n^{\sigma(f)}] \rightarrow 0, \quad \text{and} \quad \frac{M(r, \beta)}{M(r, f)} < \exp[-k r_n^{\sigma(f)}] \rightarrow 0, \quad \text{as} \ n \rightarrow +\infty. \]

Then by using the proceeding as in proof of Theorem 2.1, we prove that \( P(z) \) is a constant.
This completes the proof of Theorem 2.2. \( \square \)
**The proof of Theorem 2.3.**

**Proof**  From \(P(z)\) is a polynomial, we will consider two cases (i) \(\sigma(f) < +\infty\) and (ii) \(\sigma(f) = +\infty\).

**Case 1.** Suppose that \(\sigma(f) < +\infty\). Then \(\sigma(f) = 0\). Since \(\sigma(\alpha) < \mu(f)\), \(\sigma(\beta) < \mu(f)\), from Definitions of the order and the lower order, there exists infinite sequence \([z_n]_{1}^{\infty}\), we have

\[
\frac{|\alpha(z_n)|}{|f(z_n)|} \to 0, \text{ and } \frac{\beta(z_n)}{|f(z_n)|} \to 0, \text{ as } n \to \infty.
\]

Thus, by using the same argument as in Theorem 2.1, we can get that \(P(z)\) is a constant, that is, \(\deg P = 0\). Therefore, \(\sigma(f) = \deg P\).

**Case 2.** Suppose that \(\sigma(f) = +\infty\). Set \(F(z) = f(z) - \alpha(z)\). Since \(\sigma(\alpha) < \mu(f)\), it follows from (2) that

\[
\sigma(F) = +\infty, \quad \sigma_2(F) = \sigma_2(f),
\]

and

\[
\sigma(F) > \deg P + \max \left\{ \frac{\deg a_j - \deg a_k}{k-j}, 0 \right\}.
\]

Furthermore, we can rewrite (4) as

\[
a_k(z) \frac{F^{(k)}(z)}{F(z)} + \cdots + a_1(z) \frac{F'(z)}{F(z)} + a_0(z) + \frac{\gamma(z)}{F(z)} = e^{P(z)},
\]

where \(\gamma(z) = a_k \alpha^{(k)} + \cdots + a_1 \alpha + \beta - \alpha\). Since \(\sigma(\beta) < \mu(f)\), \(\sigma(\alpha) < \mu(f)\) and \(a_i(z), (i = 0, \ldots, k)\) are polynomials, we have

\[
\sigma(\gamma) \leq \max\{\sigma(\alpha), \sigma(\beta)\} < \mu(f) \leq \sigma(f).
\]

From Lemma 3.1, there exists a set \(E_4 \subset (1, +\infty)\) with finite logarithmic measure, we choose \(z\) satisfying \(|z| = r \notin [0, 1] \cup E_4\) and \(|F(z)| = M(r, F)|\) we get

\[
F^{(j)}(z) = \frac{v(r, F)}{z}^j (1 + o(1)), \text{ for } j = 1, 2, \ldots, k.
\]

Since \(\sigma(f) = +\infty\), then it follows from Lemma 3.3 that there exists \([z_n = r_n e^{i\theta_n}]\) with \(|F(z_n)| = M(r_n, F), \theta_n \in [0, 2\pi), \lim_{n \to \infty} \theta_n = \theta_0 \in [0, 2\pi), r_n \notin E_{5n}\) such that for any large constant \(K\) and for sufficiently large \(r_n\) we have

\[
v(r_n, F) \geq r_n^K.
\]

From \(M(r_n, F) = |F(z_n)|, F(z), \gamma(z)\) are entire functions and (18), by using definitions of the order and the lower order, we have

\[
\frac{|\gamma(z_n)}{F(z_n)} \to 0, \text{ as } r \to +\infty.
\]
Thus, it follows from (21), (23)–(25) that

\[ a_k \left( \frac{v(r_n, F)}{z_n} \right)^k (1 + o(1)) = e^{p(z_n)}. \]  

(26)

Let

\[ P(z) = b_0 z^m + b_{m-1} z^{m-1} + \cdots + b_0, \]

where \( b_m, \ldots, b_0 \) are constants and \( b_m \neq 0, m \geq 1 \). From Lemma 3.4, there exists sufficiently large positive number \( r_0 \) and \( n_0 \in \mathbb{N}_+ \), such that for sufficiently large positive integer \( n > n_0 \) satisfying \( |z_n| = r_n > r_0 \), we have for every \( \varepsilon' > 0 \)

\[ \log |b_m| + m \log |z_n| + \log |1 - \varepsilon'| \leq \log |P(z_n)| \leq |\log \log e^{P(z_n)}|. \]  

(27)

It follows from (26) that

\[ |\log \log e^{P(z_n)}| \leq |\log |a_k|| + \log \log v(r_n, F) + \log \log r_n + O(1) \]

\[ \leq \log \log v(r_n, F) + O(\log \log r_n). \]  

(28)

Thus, we have from (27), (28) and Lemma 3.2 that

\[ m = \deg P(z) \leq \sigma_2(F) = \sigma_2(f). \]  

(29)

On the other hand, since \( a_k \) is a polynomial, it follows from (27) and Lemma 3.4 that

\[ M(r_n, e^{P(z_n)}) \geq K_1 r_n^{d_k} \left( \frac{v(r_n, F)}{r_n} \right)^k, \]

where \( K_1 > 0 \) is a constant. Then we have

\[ v(r_n, F)^k \leq K_1^{-1} r_n^{-d_k} M(r_n, e^{P(z_n)}). \]  

(30)

Thus, it follows from (30) and Lemma 3.2 that

\[ \sigma_2(f) = \sigma_2(F) = \limsup_{r_n \to +\infty} \frac{\log \log v(r_n, F)}{\log r_n} = \limsup_{r_n \to +\infty} \frac{\log \log v(r_n, F)^k}{\log r_n} \]

\[ \leq \limsup_{r_n \to +\infty} \frac{\log K_1^{-1} r_n^{-d_k} M(r_n, e^{P(z_n)})}{\log r_n} = \sigma(\varepsilon^p). \]

Since \( P(z) \) is a polynomial, then \( \sigma(\varepsilon^p) = \deg P = m \). By combining (29), we have \( \sigma_2(f) = \deg P \).

Therefore, this completes the proof of Theorem 2.3. \( \square \)

**Authors’ contributions**

HYX and LZY completed the main part of this article. Both authors read and approved the final manuscript.

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Competing interests
The authors declare that they have no competing interests.

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