THE THRESHOLD FOR SUBGROUP PROFILES TO AGREE IS $\Omega(\log n)$.

JAMES B. WILSON

Abstract. For primes $p, e > 2$ there are at least $p^{e-3/e}$ groups of order $p^{2e+2}$ that have equal multisets of isomorphism types of proper subgroups and proper quotient groups, isomorphic character tables, and power maps. This obstructs recent speculation concerning a path towards efficient isomorphism tests for general finite groups. These groups have a special purpose polylogarithmic-time isomorphism test.

1. Introduction

A recent breakthrough result by Babai has pushed the complexity of isomorphism testing of finite graphs on $n$ vertices to an upper bound of $n^{O((\log n)^c)}$ for some $c \geq 1$ [1]. This brings the complexity of graph isomorphism within range of the present complexity for isomorphism testing of groups of order $n$. That complexity is bounded by $n^{O(\mu(n))}$ where given the prime factorization $n = p_1^{e_1} \cdots p_s^{e_s}$,

$$\mu(p_1^{e_1} \cdots p_s^{e_s}) = \max\{e_1, \ldots, e_s\}.$$  

Pultr and Hedrlín [15] constructed reductions that imply that group isomorphism reduces to graph isomorphism in time polynomial in $n$; see also [14]. When $\mu(n)$ is bounded then group isomorphism is in polynomial time in $n$. For each $c > 1$, as $n \to \infty$, the number of integers $n$ for which $\mu(n) \leq c$ tends to $1/\zeta(c)$, e.g. 60% of integers are square-free and 99% have $\mu(n) \leq 8$. Yet for $n = p^d$ the group isomorphism problem has the complexity of $n^{O(\log p \cdot n)}$ which makes it an obstacle to the improvement of graph isomorphism.

In that vein recent speculation by Gowers [10] and Babai [1, p. 81] has revisited the idea of using a portion of the subgroups of a finite group to determine isomorphism types of finite groups. Algorithms for testing isomorphism have been successfully using such ideas as heuristics for some time; cf. fingerprinting in [8]. Yet, proving efficiency based on these heuristics has been obstructed by knowledge of examples of Rottländer, and others, which show that lattices are not enough to characterize isomorphism [16].

Circumventing existing counter-examples, Gowers introduced a threshold criterion. In [10] he asked if as the number $d(G) = \min\{d : G = \langle x_1, \ldots, x_d \rangle\}$
grows toward the upper bound of \( \log_2 |G| \) (actually \( \mu(|G|) + 1 \) \[11\]), is the isomorphism type of \( G \) determined by the subgroups that are \( k \)-generated, for a \( k \) much smaller than \( d(|G|) \)? If true it would improve isomorphism testing to \( O(|G|^{2k}) \) steps. Glauberman-Grabowski \[9\] gave examples \( G \) where \( k \geq \sqrt{2 \log_3 |G| - 5/2} \). We will give examples of groups \( G \) of odd order \( |G| = p^e \) for which we need \( k = \ell - 2 \). So in general we need \( k \geq \log_3 |G| - 2 \).

Fix primes \( p, e > 2 \). The Heisenberg group over a field \( \mathbb{F}_{p^e} \) of order \( p^{e^2} \) is:

\[
H = H(\mathbb{F}_{p^e}) = \left\{ \begin{bmatrix} 1 & \alpha & \gamma \\ 1 & \beta & 1 \\ & & 1 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{F}_{p^e} \right\}.
\]

By the **subgroup profile** of a group \( G \) we mean the partition of the proper subgroups of \( G \) into isomorphism classes. Likewise define the **quotient-group profile**. We prove:

**Theorem 1.1.** The groups \( N \leq H' \) of index \( p^{2e+2} \) in \( H \) are normal in \( H \) and each \( H/N \) has the same subgroup and quotient-group profile and has \( d(H/N) = 2e \). Yet,

\[
\mathcal{G}_{p,e} = \{ H/N : N \leq H', |H : N| = p^{2e+2} \}
\]

has at least \( p^{e^3/e} \) isomorphism classes.

1.1. **Further invariants.** More can be said about the similarities in the groups of Theorem 1.1: Brauer had asked if non-isomorphic groups could have isomorphic character tables together with exponent structure. Dade offered the first counter-examples \[7\]. The groups in \( \mathcal{G}_{p,e} \) also have isomorphic character tables and together with \( p \)-th power maps \[13\]. Indeed, these examples have the largest possible character tables with that property, specifically of size \( \frac{p}{p^e} \times \frac{n}{p^e} \) (the largest any character table can be is \( n \times n \)).

All noncentral conjugacy classes in \( H/N \) have the same size. The groups are both directly and centrally indecomposable and with the same algebraic type of indecomposability (an invariant introduced in \[13\] Theorem 4.41; \[18\] Theorem 8 that links indecomposability to isomorphism types of local commutative rings and local Jordan algebras).

Barnes-Wall \[3\] show that the lattice of a nilpotent group of class 2 and exponent \( p \) determines the isomorphism type of the group (which corrects an errant remark of the author). The groups in \( \mathcal{G}_{p,e} \) have maximum sized lattices with \( |G|^{\Theta(|G|)} \) subgroups, chains of length \( \log_p |G| \) and antichains of length \( |G|^{\Theta(|G|)} \). We have no tools to compare such large lattices.

Despite similarities, isomorphism in \( \mathcal{G}_{p,e} \) is easy to test.

**Theorem 1.2** \([6,13]\). (a) There is a deterministic algorithm that, given a black-box group \( G \), determines if \( G \cong H/N \) for some \( N < H \) and if so returns a surjection \( H \to G \). The timing is polynomial in \( e + p \).

(b) There is a deterministic algorithm that given groups \( G_1, G_2 \in \mathcal{G}_{p,e} \), decides if \( G_1 \cong G_2 \). The timing is polynomial in \( e + p \).
We leave discussion of computational models for groups to the references just cited, which we note involve algorithms that apply to a broader class of problems. Narrowed to our specific setting where $p$ and $e$ are prime, the precise complexity is $O(p + e^\omega \log_2^2 p)$ where $2 \leq \omega < 3$ is the exponent of feasible matrix multiplication. The leading $p$ can be replaced by $\log_2^2 p$ at the cost of a Las Vegas polynomial-time algorithm. It takes $\Omega((e \log_2^2 p)^2)$ bits to input the groups we consider by any of the standard methods including matrices, presentations, or permutations.

Some of the steps in the proof of Theorem 1.1 can be extracted as special cases of results in [5, 6, 13]. However, there is an increased need to provide an easier introduction into the methods represented in those works. We opt to make this exposition largely self-contained and we rely in as much as possible on proofs based in linear algebra.

Preliminaries. We assume all groups in this note are finite. The Frattini subgroup $\Phi(G)$ is the intersection of the maximal subgroups of $G$. The exponent of a group $G$ is the least positive integer $m$ such that for every $g \in G$, $g^m = 1$. The commutator subgroup $H'$ is the smallest normal subgroup whose quotient is abelian, equivalently the subgroup generated by commutators $[x, y] = x^{-1}xy = x^{-1}y^{-1}xy$, and $G^p$ is the subgroup generated by $p$-th powers. The genus of $G$ is $d(\Phi(G))$. We require the following:

**Theorem 1.3** (Burnside Basis Theorem). For a $p$-group $G$, $\Phi(G) = G'G^p$ and $G/\Phi(G) \cong \mathbb{Z}_p^{d(G)}$.

2. A formula for subgroup profiles

We prove a formula that, under some hypotheses, calculates the subgroup profiles in $p$-groups. This allows us to construct groups that produce the same profile without need to directly compare the groups.

**Theorem 2.1.** Let $G$ be a $p$-group in which $\text{Aut}(G)$ acts transitively on maximal subgroups and such that for a maximal subgroup $M$ of $G$, $d(G) = 1 + d(M)$. Then for every $J < G$, the size of $\mathcal{J}(J) = \{K < G : K \cong J\}$ is

$$\sum_{f=0}^{d(M)} \frac{p^{1+d(M)} - 1}{p^f - 1} \left| \left\{ K \leq M : \begin{array}{l} K \cong J, \\ |M : K\Phi(M)| = p^f \end{array} \right\} \right|.$$

In particular the profile map $J \mapsto |\mathcal{J}(J)|$ depends only the isomorphism type of a maximal subgroup of $G$.

**Lemma 2.2.** In a $p$-group $G$ with a maximal subgroup $M$ having $d(G) = 1 + d(M)$, it follows that $\Phi(G) = \Phi(M)$.

**Proof.** Using the Burnside Basis Theorem on $G$ and on $M$ we calculate:

$$1 = \frac{|G : \Phi(G)| \cdot |\Phi(G)|}{|G : M| \cdot |M : \Phi(M)| \cdot |\Phi(M)|} = \frac{p^{d(G)}|\Phi(G)|}{p^{1+d(M)}|\Phi(M)|} = \frac{|\Phi(G)|}{\Phi(M)}.$$

As $\Phi(M) = M'M^p \leq G'G^p = \Phi(G)$, we find that $\Phi(M) = \Phi(G)$.

\[\blacksquare\]
Proof of Theorem 2.1. Fix $J < G$. We use an $\text{Aut}(G)$-invariant partition:

$$\mathcal{J}(J) = \bigcup_{f=1}^{d(G)} \mathcal{J}(J, f), \quad \mathcal{J}(J, f) = \{K \in \mathcal{J}(J) : |G : K\Phi(G)| = p^f\}.$$ 

Let $\mathcal{M}$ be the set of maximal subgroups of $G$.

Fix $f$ and define a bipartite graph between the two sets $\mathcal{J}(J, f)$ and $\mathcal{M}$, such that $(K, X) \in \mathcal{J}(J, f) \times \mathcal{M}$ is an edge if, and only if, $K \leq X$. The action of $\text{Aut}(G)$ on this graph permutes the vertices of $\mathcal{M}$ transitively. In particular, the degree of every vertex $X \in \mathcal{M}$ the same as the degree of $\mathcal{M}$.

Apply Lemma 2.2 to conclude that $\Phi(G) = \Phi(M)$. Thus, for every $K \leq M$, $K\Phi(G) = K\Phi(M)$ and so

$$\deg M = |\{K \leq M : K \cong J, |G : K\Phi(G)| = p^f\}|$$

$$= |\{K \leq M : K \cong J, |M : K\Phi(M)| = p^{f-1}\}|.$$ 

Next we compute the degree of $K \in \mathcal{J}(J, f)$, i.e. the size of the set:

$$\{X \in \mathcal{M} : K \leq X\} = \{X \in \mathcal{M} : K\Phi(G) \leq X\}.$$ 

Since $G/\Phi(G) \cong \mathbb{Z}_p^{d(G)}$ and $|G : K\Phi(G)| = p^f$ it follows that:

$$(G/\Phi(G))/(K\Phi(G)/\Phi(G)) \cong \mathbb{Z}_p^f.$$ 

In particular the number maximal subgroups of $G$ containing $K$ equals the number of hyperplanes in an $f$-dimensional $\mathbb{Z}_p$-vector space.

At this point we count the number of edges in our graph in two ways.

$$\frac{p^{d(G)} - 1}{p - 1} \deg M = \sum_{X \in \mathcal{M}} \deg X = \sum_{K \in \mathcal{J}(J, f)} \deg K = |\mathcal{J}(J, f)| \frac{p^f - 1}{p - 1}.$$ 

Thus $|\mathcal{J}(J, f)| = \frac{p^{d(G)} - 1}{p - 1} \deg M$. The claim follows. 

3. Making $p$-groups with matrices

We are interested in quotients of groups of $(3 \times 3)$-matrices, but it will be easier to discuss properties of a larger class of groups. For that we use a general constructions of $p$-groups that has roots in studies of Brahma and Baer [2,4]. Fix a set $\{L_1, \ldots, L_t\}$ of $(r \times s)$-matrices and define the following group of matrices. Here and throughout empty blocks in matrices are presumed to be 0.

(3.1) $B(L_1, \ldots, L_t) = \left\{ \begin{bmatrix} 1 & a & c \\ I_r & L_1b^f & \cdots & L_tb^f \\ & & I_t \\ \\ a \in \mathbb{Z}_p^r \\ b \in \mathbb{Z}_p^s \\ c \in \mathbb{Z}_p^t \end{bmatrix} : \begin{array}{c} \end{array} \right\}$. 

The dimension of $\langle L_1, \ldots, L_t \rangle$ is the genus $g$ of the group. If $r, s, g \approx n/3$ then this construction already defines $p^{n^2/3+O(n^2)}$ isomorphism types of groups of order $p^n$, which is approximately a square-root of all the possible groups of order $p^n$. So despite humble appearance, this family is extremely complex. Our most important examples will be the Heisenberg groups.

As our fields $\mathbb{F}_q$ are finite, there exists an $\omega \in \mathbb{F}_q$ such that $\mathbb{F}_q = \mathbb{Z}_p(\omega)$. In particular, $\{1, \omega, \ldots, \omega^{e-1}\}$ is a basis for $\mathbb{F}_q$ as a $\mathbb{Z}_p$-vector space. Define $m(\omega)_{ij}^{(k)} \in \mathbb{Z}_p$ as the constants such that:

$$\omega^i \cdot \omega^j = \sum_{k=0}^{e-1} m(\omega)_{ij}^{(k)} \omega^k.$$ 

Also let $M(\omega)^{(k)} \in M_e(\mathbb{Z}_p)$ be such that $[M(\omega)^{(k)}]_{(i+1)(j+1)} = m(\omega)_{ij}^{(k)}$.

**Example 3.2.** If $\mathbb{F}_{pe} = \mathbb{Z}_p(\omega)$ then $H(\mathbb{F}_q) \cong B(M(\omega)^{(0)}, \ldots, M(\omega)^{(e-1)})$.

In the following section we will use the groups $B(L_1, L_2)$ to give an alternative description of the groups $H/N \in \mathcal{G}_{p,e}$. We will prove:

**Theorem 3.3.** The groups in $\mathcal{G}_{p,e}$ are isomorphic to the groups $B(L_1, L_2)$ where $\{L_1, L_2\}$ is a linearly independent set of $(e \times e)$-matrices with $L_1$ invertible and $L_1^{-1}L_2$ has an irreducible minimum polynomial of degree $e$.

### 3.1. Subgroups by row, column, and matrix elimination.

One way to explore the subgroups of the groups $B(L_1,\ldots,L_g)$ is to restrict the range of values of $a$ or $b$ in the formula given in (3.1). For instance, suppose we restrict the coordinate $a_i = 0$. The result is that the values in the $i$-th row of each matrix $L_1, \ldots, L_g$ can be ignored within that subgroup. Hence the subgroup we get is isomorphic to the group we obtain by first removing the $i$-th row of each matrix in $\{L_1, \ldots, L_g\}$ and then using the construction of (3.1) to create a group on these smaller matrices. Removing one row produces a maximal subgroup, two rows a subgroup of index $p^2$, and so on. The similar idea applies to columns. Reversing the process and inserting rows or columns creates subgroup embeddings.

Restricting values of $c$ may result in a subset that is not closed to multiplication. An easy way to avoid that concern is to eliminate entries $c_i$ only once the corresponding matrix $L_i = 0$.

**Example 3.4.** Using row, column, and matrix insertion, we embed $H(\mathbb{Z}_3)$ into $H(\mathbb{F}_9)$. In this example we let $\mathbb{F}_9 = \mathbb{Z}_3[x]/(x^2 + 1)$. We partition the matrices to help identify the row or column insertions.

$$H(\mathbb{Z}_p) \cong B([1]) \hookrightarrow B([1], [0])$$

$$\hookrightarrow B([1][0], [0][1])$$

$$\hookrightarrow B \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \cong H(\mathbb{F}_{p^2}).$$
We emphasize that this approach is not guaranteed to explore every subgroup, but it is nevertheless a good place to begin.

Next we can construct a family of groups each having a maximal subgroup of a fixed isomorphism type. As in [12, p. 70], for a polynomial \( a(t) = a_0t^0 + \cdots + a_{e-1}t^{e-1} + t^e \in \mathbb{Z}_p[t] \), the companion matrix will be:

\[
C(a(t)) = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
-1 & \cdots & a_0 & \cdots & a_{e-1}
\end{bmatrix}.
\]

**Lemma 3.5.** For a polynomial \( a(t) \) of degree \( e \), the group \( G = B(I_e, C(a(t))) \) has a maximal subgroup \( M \) whose isomorphism type depends only on \( p \) and \( e \) and \( d(G) = 1 + d(M) \).

**Proof.** We delete the last row of \( I_e \) and \( C(a(t)) \) to obtain:

\[
M = B \left( \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix} \right) \left( \begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
a_0 & \cdots & a_{e-1}
\end{bmatrix} \right) = G.
\]

Evidently \(|M : \Phi(M)| = p^{2e-1} = p^{d(G)-1}| \).

**3.2. Quotient groups by linear combinations.** Next we will want to explore some of quotient groups of \( B(L_1, \ldots, L_g) \). One can see in (3.1) that for each subset \( \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, g\} \), there is a natural surjection \( B(L_1, \ldots, L_g) \to B(L_{i_1}, \ldots, L_{i_s}) \) and that is indeed a group homomorphism. This is an analogue to the way we created subgroups in the previous section.

Likewise, fix scalars \( (a_1, \ldots, a_g) \). Then there is a surjective homomorphism \( B(L_1, \ldots, L_g) \to B(a_1L_1 + \cdots + a_gL_g) \). More generally given a \((g' \times g)\)-matrix \( A \), there is a surjective homomorphism:

\[
B(L_1, \ldots, L_g) \to B \left( \sum_{j=1}^g A_{1j}L_j, \ldots, \sum_{j=1}^g A_{g'j}L_j \right).
\]

**3.3. Notable isomorphisms.** There are also direct ways to create groups isomorphic to \( B(L_1, \ldots, L_g) \). For example, for invertible matrices \( X \in \mathbb{M}_n(\mathbb{Z}_p) \) and \( Y \in \mathbb{M}_m(\mathbb{Z}_p) \),

\[
B(L_1, \ldots, L_g) \cong B(XL_1Y^t, \ldots, XL_gY^t).
\]
We may also permute the order of the matrices. In particular we can always insist the first matrix have largest rank and that it be expressed in the form \([t_0 0]\) through Gaussian elimination. Thus the groups \(B(L_1)\) can be classified up to isomorphism by the rank of \(L_1\).

More generally, for each \(i, j \in \{1, \ldots, g\}\), and \(s \in \mathbb{Z}_p^\times\),

\[
B(L_1, \ldots, L_g) \cong B(L_1, \ldots, L_i + sL_j, \ldots, L_g) \cong B(L_1, \ldots, sL_i, \ldots, L_g).
\]

Thus, if \(\{L'_1, \ldots, L'_g\}\) is another basis for \(\langle L_1, \ldots, L_g\rangle\), then \(B(L_1, \ldots, L_g) \cong B(L'_1, \ldots, L'_g)\). There can be further isomorphisms between these groups, but these will suffice for our present discussion.

Using these observations we can make even more complex embeddings.

**Lemma 3.6.** For every \(a(t), b(t) \in \mathbb{Z}_p[t]\) with \(\deg b(t) =: f\) and less than \(e := \deg a(t)\), there is an embedding \(B(I_f, C(b(t))) \rightarrow B(I_e, C(a(t)))\).

We emphasize that \(b(t)\) has no relation to \(a(t)\) other than having lower degree. So there is no algebraic reason to guess at the possible embedding of \(B(I_f, C(b(t)))\) into \(B(I_e, C(a(t)))\). Yet with matrices it is an easy calculation.

**Proof.** Fix \(b_0, \ldots, b_{e-2} \in \mathbb{Z}_p\).

\[
\begin{align*}
(3.7) & \quad \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \cdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ \cdots & \ddots & 1 \\ 1 & 0 \end{bmatrix}, \\
(3.8) & \quad \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \cdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ \cdots & \ddots & 1 \\ b_0 & b_{e-2} & b_{e-2} \end{bmatrix}.
\end{align*}
\]

Thus, setting \(b(t) = b_0 t^0 + \cdots + b_{e-2} t^{e-2} + t^{e-1}\), we obtain the following embedding. For the isomorphism we are using the identity \(B(L_1Y^t, L_2Y^t) \cong B(L_1, L_2)\) following the calculation of (3.7) and (3.8).

\[
B(I_{e-1}, C(b(t))) \rightarrow B \left( \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ \cdots & \ddots & 1 \\ b_0 & b_{e-2} & b_{e-2} \end{bmatrix} \right) 
\cong B \left( \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ \cdots & \ddots & 1 \\ \cdots & \ddots \\ \cdots & \ddots & 1 \\ b_0 & \cdots & b_{e-2} \end{bmatrix} \right) 
\rightarrow B(I_e, C(a(t))). \quad \Box
\]

In light of Theorem 3.3 Lemma 3.6 shows that the groups \(H/N \in \mathcal{G}_{p,e}\) each have \(p^{\Omega(e)}\) isomorphism types of proper subgroups.
Evidently there are isomorphisms \( \iota \) \( \theta \), none of these isomorphisms is natural in the category of groups. In particular \( \theta \) is not part of the operations of a group.

\[ (4.3) \]

Theorem 4.1. The role of commutation. The first principle in nilpotent group theory is to treat groups like rings by invoking commutation \( [x, y] = x^{-1}x^y = x^{-1}y^{-1}xy \) as a skew-commutative multiplication. This very nearly distributes over the usual product, in the following way.

\[ [xy, z] = [x, z]^y[y, z], \quad [x, yz] = [x, z][x, y]^z. \]

With \( q = p^e \) and \( H = H(\mathbb{F}_q) \), commutation takes the following form.

\[ (4.3) \]

This shows the following two groups are abelian.

\[ H' = [H, H] = \left\{ \begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \gamma \in \mathbb{F}_q \right\}, \quad H/H' \cong \{(\alpha, \beta) : \alpha, \beta \in \mathbb{F}_q^m \}. \]

Evidently there are isomorphisms \( \iota : H/H' \to \mathbb{F}^2_{q^2} \) and \( \hat{\iota} : H' \to \mathbb{F}_q^2 \). None of these isomorphisms is natural in the category of groups. In particular neither \( H/H' \) nor \( H' \) is an obvious \( \mathbb{F}_q \)-vector space as scalar multiplication is not part of the operations of a group.
Normal subgroups are now easily described.

**Lemma 4.4.** For \( h \in H - H' \), \([h, H] = H'\); thus, if \( M \) is normal in \( H \) then either \( H' \leq N \) or \( N \leq H' \). In either case, \((H/N)' = H'/N/N\).

### 4.2. Quotients of \( H \)

To inspect the quotients of \( H \) we use a method to “linearize” a nilpotent group which is in some sense the reversal of the constructions we gave in Section 3. Early versions of this approach were applied by Brahana and Baer [2, 4].

By the identities (4.2) imply that commutation factors through \( H/H' \times H/H' \to H' \) and thereby affords a biadditive map \([.]_+ : (F_q^2, +) \times (F_q^2, +) \to (F_q, +)\):

\[
[(\alpha, \beta), (\alpha', \beta')]_+ = \alpha\beta' - \alpha'\beta.
\]

To distinguish between the various roles of \([.]\) we let \([.]\) denote group commutation and \([.]_+\) the biadditive mapping that commutation produces.

**Remark 4.6.** The expression in (4.5) is obviously \( F_q \)-bilinear. However, the relationship of \([.]_+\) to the commutation map \([.] : H/H' \times H/H' \to H'\) is only as abelian groups, made explicitly through the (unnatural) choice of \((i, i)\) above. So geometric information about \( F_q \)-bilinear maps cannot be directly applied in our situation.

Now Lemma 4.4 shows that for \( N < H'\), \((H/N)' = H'/N\). So the commutation of the quotient \( H/N \) will accordingly afford a new biadditive map

\[
[.]_+ : (F_q^2, +) \times (F_q^2, +) \to (F_q, +) \xrightarrow{\pi} \mathbb{Z}_p^g
\]

where \( \pi \) is given as the homomorphism \( \langle F_q, + \rangle \cong H' \to H'/N \cong \mathbb{Z}_p^g \). The genus of \( H/N \) is the value \( g \).

Let us look closely at the case of genus \( g = 1 \). Fix \( \pi : \langle F_q, + \rangle \to \mathbb{Z}_p \). Choose a basis \( \{\alpha_1, \ldots, \alpha_e\} \) for \( \langle F_q, + \rangle \) as a \( \mathbb{Z}_p \)-vector space and such that \( \pi(\alpha_i) = 1 \) if \( i = 1 \) and 0 otherwise. Define

\[
L_{ij} = \pi((\alpha_i, 0), (0, \alpha_j)) = \pi(\alpha_i\alpha_j).
\]

Regarded as a map of \( \mathbb{Z}_p \)-vector spaces we see:

\[
[(\alpha, \beta), (\alpha', \beta')]_+^{H/N} = \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & L \end{bmatrix} \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}.
\]

As we vary \( H/N \) amongst groups of arbitrary genus \( 1 \leq g \leq e \) we describe \([.]_+ = [.]_+^{H/N} \) by a linearly independent set of invertible matrices \( L_1, \ldots, L_g \in M_e(\mathbb{Z}_p) \) such that

\[
[(\alpha, \beta), (\alpha', \beta')]_+ = \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & L_1 & L_2 & \cdots & L_g \end{bmatrix} \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}.
\]

This demonstrates the following correspondence.

**Theorem 4.7** (Brahana correspondence). A group \( H/N \) whose commutation is described by matrices \( (L_1, \ldots, L_g) \) has an isomorphism to the group \( B(L_1, \ldots, L_g) \). In particular all quotients \( H/N \) of genus 1 are isomorphic.
Proof. If \( g = 1 \) we can assume \( L_1 = I_e \) and so \( H/N \cong B(I_e) \). The required isomorphism \( B(L_1, \ldots, L_g) \to H/N \) is as follows.

\[
\begin{bmatrix}
1 & a & c \\
I_e & L_1b^t & \cdots & L_gb^t \\
\multicolumn{3}{c|}{I_g}
\end{bmatrix} \quad \mapsto \begin{bmatrix}
1 & \iota^{-1}(a) & \iota^{-1}(c) \\
1 & \iota^{-1}(b) & 1
\end{bmatrix} \mod N.
\]

\[\square\]

**Corollary 4.9.** The groups in \( G_{p,e} \) have the equal quotient group profiles.

Proof. Fix \( N \leq H' \) with \(|H : N| = p^{2e+2} \). As \( p^{2e+2} = |H : H'| \cdot |H' : N| = p^{2e}|H' : N| \) we find \(|H' : N| = p^2 \), and so \( H/N \) has genus 2. As in Lemma 4.4, if \( N < K < H \) and \( K/N \) is normal in \( H/N \), then \( K < H' \) or \( H' \leq K \). If \( H' \leq K \) then \((H/K)/(K/N) \cong H/K \cong \mathbb{Z}_p \) where \( p^f = |H : K| \). This does not depend on the choice of \( N \). The number of choices for \( K \) is the number of subgroups in \( \mathbb{Z}_p^2 \) of index \( f \), which again does not depend on \( N \). Otherwise \( N < K < H' \) and so \(|H' : K| = p \). Thus \((H/N)/(K/N) \cong H/K \) has genus 1. So by Theorem 4.1, its isomorphism type is fixed and independent of \( N \). Finally, \( H'/N \cong \mathbb{Z}_p^2 \), so there are exactly \( p + 1 \) choices of \( K \) with \( N < K < H' \). This is independent of \( N \). \[\square\]

**4.3. Distributive products.** To prove Theorem 4.1 we need a brief detour to discuss distributive products. Take \( A \subset M_r(\mathbb{Z}_p) \times M_s(\mathbb{Z}_p) \). It follows that \( M_{r \times s}(\mathbb{Z}_p) \) decomposes into subspaces as follows.

\[
\left\langle F^tX - XF^* : X \in M_{r \times s}(\mathbb{Z}_p) \right\rangle \oplus \{X : \forall (F, F^*) \in A, F^tX = XF^*\}.
\]

We write \( \mathbb{Z}_p^r \otimes_A \mathbb{Z}_p^s \) for the right-hand subspace. The projection \( \pi_A \) from \( M_{r \times s}(\mathbb{Z}_p) \) onto \( \mathbb{Z}_p^r \otimes_A \mathbb{Z}_p^s \) allows us to define a distributive tensor product

\[
\otimes = \otimes_A : \mathbb{Z}_p^r \times \mathbb{Z}_p^s \to \mathbb{Z}_p^r \otimes_A \mathbb{Z}_p^s \quad u \otimes v = \pi_A(u'v).
\]

Notice for \((F, F^*) \in A, \pi_A(F^tX) = \pi_A(XF^*)\) and so we find:

\[
uF \otimes v = \pi_A(F^tuv^*) = \pi_A(u'vF^*) = u \otimes (vF^*).
\]

Consider an example with

\[
A = \left\{ \left[ \begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array} \right], \begin{array}{cc}
\delta & -\beta \\
-\gamma & \alpha
\end{array} \right] : \alpha, \beta, \gamma, \delta \in \mathbb{F}_q \right\}.
\]

(4.10)
For \((\alpha, \beta), (\gamma, \delta) \in \langle F_q^2, + \rangle\),
\[
(\alpha, \beta) \otimes (\gamma, \delta) = (1, 0) \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \otimes (1, 0) \begin{bmatrix} \gamma & \delta \\ 0 & 0 \end{bmatrix} = (1, 0) \begin{bmatrix} \gamma & \delta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & \alpha \\ 0 & -\beta \end{bmatrix} = (1, 0) (\alpha \delta - \beta \gamma, 0).
\]
Therefore \((\alpha, \beta) \otimes (\gamma, \delta) \mapsto \alpha \delta - \beta \gamma\) defines an isomorphism
\[
\langle F_q^2, + \rangle \otimes \langle F_q^2, + \rangle \cong \langle F_q, + \rangle
\]
and furthermore \(\otimes_A : \langle F_q^2, + \rangle \times \langle F_q^2, + \rangle \rightarrow \langle F_q^2, + \rangle \otimes \langle F_q^2, + \rangle\) is equivalent to \([\cdot]_H^\perp\). That the commutation of the Heisenberg group is a tensor product over a matrix ring is at the core of how Theorem 4.11 is possible.

In general for a distributive product \(* : Z_p^r \times Z_p^s \rightarrow Z_p^l\) a pair \((F, F^*) \in M_v(Z_p) \times M_s(Z_p)\) is an adjoint if it satisfies, for all \(u \in Z_p^r\) and \(v \in Z_p^s\),
\[
(uF) \circ v = u \circ (vF^*).
\]
(This is the same notion of adjoints we find in texts on linear algebra, cf. [12] p. 143, but we use it on arbitrary products not just inner products.) The adjoint identity is linear and so it defines a subspace:
\[
\text{Adj}(\circ) = \{(F, F^*) : \forall u \in Z_p^r, \forall v \in Z_p^s, (uF) \circ v = u \circ (vF^*)\}.
\]
Under the product \((F, F^*)(G, G^*) = (FG, G^* F^*)\) this makes \(\text{Adj}(\circ)\) into a ring. In fact \(\text{Adj}(\circ)\) is the largest ring \(A\) over which the product \(\circ\) factors through the tensor \(\otimes_A\), more precisely:

**Theorem 4.11** (Adjoint-tensor Galois correspondence [5 Theorem 2.11]). Fix a distributive product \(* : Z_p^r \times Z_p^s \rightarrow Z_p^l\) and \(A \subset M_v(Z_p) \times M_s(Z_p)\). Then \(A \subset \text{Adj}(\circ)\) if, and only if, there is a homomorphism \(\hat{\circ} : Z_p^r \otimes_A Z_p^s \rightarrow Z_p^l\) such that \(u \circ v = \hat{\circ}(u \otimes v)\).

Now we refocus on the goal of Theorem 4.11.

**Lemma 4.12.** If \(F_{p^e} \subset A \subset M_v(Z_p)\) and \(e\) prime, then \(A = F_{p^e}\) or \(M_v(Z_p)\).

**Proof.** Let \(V = Z_p^e\) be an \(A\)-module. As \(F_{p^e}\) is contained in \(A\), \(V\) is also an \(F_{p^e}\)-vector space, and it is 1-dimensional. Thus, as an \(A\)-module \(V\) is simple. Now \(A\) is also faithfully represented on \(V\). Thus by Jacobson’s Density Theorem [12] p. 262, \(A\) is \(\text{End}_Z(V) \cong M_f(Z)\), where \(Z \cong F_{p^e}\) is the center of \(A\). Furthermore, \(e = fs\). As \(e\) is prime either \(f = 1\) and \(A = F_{p^e}\), or else \(f = e\) and \(s = 1\) which makes \(A = M_v(Z_p)\).

**Lemma 4.13.** If \(H/N\) has genus \(g > 1\) then \(\text{Adj}([\cdot]_H^\perp) = \text{Adj}([\cdot]_{H/N}^\perp) \cong M_2(F_q)\).

**Proof.** We start by observing some necessary adjoints. The adjoint-tensor Galois correspondence shows \(\text{Adj}([\cdot]_H^\perp) \subset \text{Adj}([\cdot]_{H/N}^\perp)\). In our example we found \(M_2(F_q) \cong \text{Adj}([\cdot]_{H}^\perp)\).
Next we know that the commutation in $H/N$ is given by a set $\{L_1, \ldots, L_g\}$ of linearly independent invertible matrices. So the linear equations to solve to describe $\text{Adj}(\ast)_{+}^{H/N}$ are the following. For each $1 \leq i \leq g$,

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} 0 & L_i \\ -L_i & 0 \end{bmatrix} = \begin{bmatrix} 0 & L_i \\ -L_i & 0 \end{bmatrix} \begin{bmatrix} F_{11}^* & F_{12}^* \\ F_{21}^* & F_{22}^* \end{bmatrix}^t$$

For $i = 1$ we get $F_{11}^* = L_1 F_{11}^t L_1^{-t}$, $F_{12}^* = -L_1 F_{12}^t L_1^{-t}$, $F_{21}^* = -L_1 F_{21}^t L_1^{-t}$, and $F_{22}^* = L_1 F_{11}^t L_1^{-t}$. Now $L_2$ adds the further constraint that $F_{ij} L_1^{-1} L_2 = L_1^{-1} L_2 F_{ij}$.

Now consider the algebra

$$A = \{ F \in \mathbb{M}_e(\mathbb{F}_p) : FL_1^{-1} L_2 = L_1^{-1} L_2 F \}.$$

By the previous inclusion we know that $\mathbb{F}_q \subseteq A \subseteq \mathbb{M}_e(\mathbb{F}_p)$. If $A = \mathbb{M}_e(\mathbb{F}_p)$ then $L_2$ commutes with every matrix and thus $L_2$ is a scalar matrix. However, $L_2$ and $L_1 = I_e$ are linearly independent. So $L_2$ cannot be scalar. As a result $A \neq \mathbb{M}_e(\mathbb{F}_p)$. By Lemma 4.12 $A = \mathbb{F}_q$. That is,

$$\text{Adj}(\ast)_{+}^{H/N} \subseteq \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & -\beta \end{bmatrix}, \begin{bmatrix} \delta & -\gamma \\ -\delta & \alpha \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{F}_q \right\}.$$

So indeed $\text{Adj}(\ast)_{+}^{H} = \text{Adj}(\ast)_{+}^{H/N}$. □

**Proof Theorem 2.3** Fix a group $B(L_1, L_2)$. If $B(L_1, L_2)$ is a quotient of $H$ then so is $B(L_1)$. By Corollary 4.9 $B(L_1) \cong B(I_e)$. Therefore we may assume $L_1 = I_e$ and let $a(t)$ be the minimum polynomial of $L_2$. As $\{L_1, L_2\}$ are linearly independent we know that $L_2$ cannot be a scalar matrix and so $a(t)$ has degree at least 2.

Now let $C(L_2) = \{ F \in \mathbb{M}_e(\mathbb{Z}_p) : FL_2 = L_2 F \}$. Following the calculation of the adjoint ring above we know that

$$\text{Adj}(\ast)_{+}^{H/N} = \left\{ \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \begin{bmatrix} F_{11}^t & -F_{12}^t \\ -F_{21}^t & F_{22}^t \end{bmatrix} : F_{ij} L_2 = L_2 F_{ij} \right\} \cong \mathbb{M}_2(C(L_2)).$$

Thus, if $B(L_1, L_2)$ is a quotient of $H$ then $C(L_2) \cong \mathbb{F}_q$. Since $\mathbb{Z}_p[L_2] \cong \mathbb{Z}_p[t]/(a(t)) \subseteq C(L_2)$ it follows that $\mathbb{Z}_p[L_2]$ is a subfield of $\mathbb{F}_q$. As $a(t)$ has degree greater than 1 and $\mathbb{F}_q$ has no intermediate fields, it follows that $\mathbb{Z}_p[L_2] = \mathbb{F}_q$. Thus $a(t)$ is an irreducible polynomial of degree $e$.

Conversely if $L_2$ is conjugate to $C(a(t))$ then $\text{Adj}(\ast)_{+}^{H/N} \cong \mathbb{M}_2(\mathbb{F}_q) \cong \text{Adj}(\ast)_{+}^{H}$). By Adjoint-Tensor Galois correspondence, the commutation in $B(I_e, L_2)$ factors through the tensor product over $\text{Adj}(\ast)_{+}^{H}$ which is the commutation of $H$. Therefore $B(I_e, L_2)$ is a quotient of $H$. □

**4.4. Automorphisms of Heisenberg groups.** Now we need to consider the automorphisms of $H$, assuming $p > 2$. Each automorphism is described by three constituents:

1. a homomorphism $\tau : (\mathbb{F}_q^2, +) \rightarrow (\mathbb{F}_q^2, +)$,
(2) an invertible matrix \[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\] over \(\mathbb{F}_q\), and

(3) a field automorphism \(\alpha \mapsto \bar{\alpha}\) of \(\mathbb{F}_q\).

The corresponding automorphism is as follows.

\[
\begin{pmatrix} 1 & \alpha' \gamma' \\ 1 & \beta' \end{pmatrix} \mapsto \begin{pmatrix} 1 & (\alpha' \alpha + \beta' \gamma) + \tau(\alpha', \beta') \\ 1 & \alpha' \beta + \beta' \delta \end{pmatrix}.
\]

Remark 4.15. Classic knowledge of automorphisms of Heisenberg groups over \(\mathbb{Z}_q, \mathbb{Z}, \mathbb{R}\) and \(\mathbb{C}\) is largely inapplicable here. For \(\mathbb{R}\) and \(\mathbb{C}\) the automorphisms are presumed to be smooth, ours have no such restrictions. As we cautioned in Remark 4.6, in the case of \(\mathbb{F}_q\) over \(\mathbb{F}_q\), the restriction of \(\phi\) to another minimal right ideal \(\{[\gamma, \delta] : \gamma, \delta \in \mathbb{F}_q\}\) is \(\mathbb{F}_q\)-bilinear but \([,] : H/H' \times H/H' \to H'\) is only biadditive. So only the cases of \(\mathbb{Z}\) and \(\mathbb{Z}_p\) are immediate by standard geometric methods.

We have just seen that the commutation of Heisenberg groups is actually a special type of distributive product, a tensor product. This means instead of acting on a biadditive map we can act on a ring \(\text{Adj}([,]) \cong \mathbb{M}_2(\mathbb{F}_q)\).

**Theorem 4.16 (Skolem-Noether [12, p. 237])**. The ring automorphisms of \(\mathbb{M}_2(\mathbb{F}_q)\) are \(X \mapsto T^{-1}XT\) where \(T\) is an invertible \(2 \times 2\) matrix and \(\alpha \mapsto \bar{\alpha}\) is a field automorphism of \(\mathbb{F}_q\) applied to each entry of \(X\).

**Proof.** First the automorphism \(\phi\) will send \(\alpha I_2 \mapsto \bar{\alpha} I_2\) which gives us the field automorphism \(\sigma\). Replacing \(\phi\) with \(\phi(X^{-1})\) we now have an \(\mathbb{F}_q\)-linear automorphism. Therefore it maps the minimal right ideal \(\{[\alpha, \beta] : \alpha, \beta \in \mathbb{F}_q\}\) to another minimal right ideal \(\{[0, 0] : \gamma, \delta \in \mathbb{F}_q\}\), or for some \(\nu \in \mathbb{F}_q\), \(\{[\gamma, \delta] : \gamma, \delta \in \mathbb{F}_q\}\).

Each of these is a 2-dimensional vector spaces over \(\mathbb{F}_q\) so that transformation can be given by an invertible square matrix \(T\).

**Lemma 4.17.** There is an epimorphism \(\text{Aut}(H) \to \text{Aut}(\mathbb{M}_2(\mathbb{F}_q))\). The kernel consists of those automorphisms that are the identity on \(H/H'\).

**Proof.** Let \(\phi : H \to H\) be an automorphism. Since \(\phi([h, k]) = [\phi(h), \phi(k)]\), \(\phi\) factors through \(\mathbb{Z}_p^{2e} \cong H/H' \to H/H' \cong \mathbb{Z}_p^{2e}\). So we let \(T\) be the matrix representing that transformation. Also we let \(\hat{T}\) be the matrix describing the restriction of \(\phi\) to \(\mathbb{Z}_p^e \cong H' \to H' \cong \mathbb{Z}_p^e\). Notice \((T, \hat{T})\) satisfy

\[
((\alpha, \beta)T, (\alpha', \beta')T)_{+} = [(\alpha, \beta), (\alpha', \beta')]_{+} \hat{T}.
\]

Now take \((F, F^*) \in \text{Adj}([,])_{+}\). It follows that

\[
[(\alpha, \beta)T^{-1}FT, (\alpha', \beta')]_{+} = [(\alpha, \beta)T^{-1}F, (\alpha', \beta')T^{-1}]_{+} \hat{T}
= [(\alpha, \beta), (\alpha', \beta')]T^{-1}F^*T_{+}.
\]

In this way Aut\((H)\) acts on \(\text{Adj}([,])_{+} \cong \mathbb{M}_2(\mathbb{F}_q)\). We saw that commutation in Aut\((H)\) is the same as the tensor product with \(\text{Adj}([,])_{+}\), so every automorphism of \(\text{Adj}([,])_{+}\) determines an automorphism of \(H\).
Proof of Theorem 4.1. In the Brahma correspondence we saw that every nonabelian quotient $H/N$ is determined up to isomorphism by the matrices $(L_1, \ldots, L_g)$ which also define $[, ]^H/N$. Fix an isomorphism $\phi : H/N_1 \to H/N_2$. Since $(H/N_1)/(H/N_1)' \cong H/H' \cong \langle \mathbb{F}_q^2, + \rangle$, we see $\phi$ determines a matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2e}(\mathbb{Z}_p)$. Using the fact that $\langle \mathbb{F}_q^2, + \rangle \cong \langle \mathbb{F}_q, + \rangle \otimes M_{2e}(\mathbb{F}_q)$, we define $\Gamma : H \to H$ as follows.

From our proof of Lemma 4.17 we notice $\Gamma$ is an automorphism of $H$ if, and only if, $T^{-1} \text{Adj}([, ])|_H = \text{Adj}([, ])|_{H/N}$. Since $\phi$ is an isomorphism $H/N_1 \to H/N_2$ we know that $T^{-1} \text{Adj}([, ])|_H = \text{Adj}([, ])|_{H/N_2}$. By Lemma 4.13 we know $\text{Adj}([, ])|_H = \text{Adj}([, ])|_{H/N}$.

Corollary 4.18. The set $G_{p, e}$ has at least $p^{e-3}/e$ isomorphism types.

Proof. The number of subgroups $N < H'$ of index $p^2$ is the number of subspaces of codimension 2 in a vector space $H' \cong \mathbb{Z}_p^{2e}$. That number is $\frac{(p^e-1)(p^e-p)}{(p^2-1)(p-1)}$. Meanwhile the action by $\text{Aut}(H)$ on $H'$ has size $e(p^e - 1)$; see (4.14). So the number of orbits is at least $p^{e-3}/e$.

5. PROOF OF THEOREM 1.1

Lemma 5.1. For every $N \leq H'$, $\text{Aut}(H/N)$ acts transitively on the maximal subgroups of $H/N$.

Proof. The group $\text{SL}(2, \mathbb{F}_{p^e})$ acts transitively on hyperplanes of $\mathbb{Z}_p^{2e}$ and those coincide with the maximal subgroups of $H$. Following (4.11), this action lifts to $\text{Aut}(H)$ and is furthermore the identity on $H'$. Thus for $N < H'$, this action transfers to $H/N$. Lastly, observe that $\Phi(H/N) = H'/N = \Phi(H)/N$, so the maximal subgroups of $H/N$ are the groups $X/N$ where $X$ is maximal in $H$.

Proof of Theorem 7.1. Corollary 4.18 & 4.9 establish that the set $G_{p, e}$ has at least $p^{e-3}/e$ isomorphism types and that these groups all have the same quotient group profile. Finally, Lemmas 5.1 & 3.5 allow us to invoke Theorem 2.1 to conclude the proof.

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THE THRESHOLD FOR SUBGROUP PROFILES TO AGREE IS $\Omega(\log n)$. 15

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Department of Mathematics, Colorado State University, Fort Collins, CO 80523.

E-mail address: James.Wilson@ColoState.Edu