Antiferromagnetic Order and $\pi$-Triplet Pairing in the Fulde-Ferrell-Larkin-Ovchinnikov State

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The antiferromagnetic Fulde-Ferrell-Larkin-Ovchinnikov (AFM-FFLO) state of coexisting $d$-wave FFLO superconductivity and incommensurate AFM order is studied on the basis of Bogoliubov-de Gennes (BdG) equations. We show that the incommensurate AFM order is stabilized in the FFLO state by the appearance of the Andreev bound state localized around the zeros of the FFLO order parameter. The AFM-FFLO state is further enhanced by the induced $\pi$-triplet superconductivity (pair density wave). The AFM order occurs in the FFLO state even when it is neither stable in the normal state nor in the BCS state. The order parameters of the AFM order, $d$-wave superconductivity, and $\pi$-triplet pairing are investigated by focusing on their spatial structures. Roles of the spin fluctuations beyond the BdG equations are discussed. Their relevance to the high-field superconducting phase of CeCoIn$_5$ is discussed.

KEYWORDS: FFLO state, antiferromagnetism, $\pi$-triplet pairing, CeCoIn$_5$

1. Introduction

The Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state in superconductors was predicted in the 1960s by Fulde and Ferrell,\textsuperscript{1} and Larkin and Ovchinnikov.\textsuperscript{2} In contrast to the Bardeen-Cooper-Schrieffer (BCS) state, Cooper pairs have a finite total momentum in the FFLO state, which leads to the spontaneous breaking of the spatial symmetry. Although this novel superconducting state with an exotic symmetry has been attracting much interest, the experimental search for this state had been fruitless for nearly 40 years. Under these circumstances, the discovery of a new high-field superconducting (HFSC) phase in CeCoIn$_5$,\textsuperscript{3,4} which is a likely candidate for the FFLO state, triggered many theoretical and experimental studies.\textsuperscript{5} This recent interest on the FFLO superconductivity/superfluidity extends further in various related fields, such as organic superconductors,\textsuperscript{6-10} cold atom gases,\textsuperscript{11,12} astrophysics, and nuclear physics.\textsuperscript{13}

The HFSC phase of CeCoIn$_5$ has been interpreted widely within the concept of the FFLO state.\textsuperscript{5,14-24} However, recent observations of the AFM order in the HFSC phase call for a reexamination of this conclusion.\textsuperscript{25-27} It is expected that this AFM order will be closely related to the AFM quantum critical point observed in CeCoIn$_5$.\textsuperscript{28,29} Therefore, the nuclear magnetic resonance (NMR)\textsuperscript{25,27} and neutron scattering\textsuperscript{26} measurements may have uncovered a novel superconducting state in this strongly correlated electron system. In particular, the neutron scattering measurement has explored the properties of the AFM order and found that the wave vector of the AFM order is incommensurate $q_{IC} = \vec{Q} + \delta_{IC}$ with $\delta_{IC}$ along the $[1,-1,0]$ direction for magnetic fields in the $ab$-plane of the tetragonal lattice.\textsuperscript{26} The AFM staggered moment $\vec{M}_{AF}$ is directed to the $c$-axis.

Some scenarios have been proposed for the HFSC phase of CeCoIn$_5$ from the theoretical point of view. We have investigated the AFM order arising from the nodal quasiparticles in the inhomogeneous Larkin-Ovchinnikov state.\textsuperscript{30} The SDW order triggered by the emergence of $\pi$-triplet pairing\textsuperscript{31-34} has been investigated in the BCS state\textsuperscript{31,32,34} and in the homogeneous Fulde-Ferrell state.\textsuperscript{33} In this study, we examine the AFM-FFLO state, in which the AFM order appears in the inhomogeneous FFLO state, on the basis of BdG equations. The typical phase diagram in the $H$-$T$ plane and the spatial structure of the AFM-FFLO state are investigated in detail.

2. Formulation

Our theoretical analysis is based on the microscopic model,

$$
H = -t \sum_{\langle i,j \rangle, \sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + t' \sum_{\langle\langle i,j \rangle\rangle, \sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + U \sum_i n_{\uparrow i} n_{\downarrow i} + V \sum_{\langle i,j \rangle} n_{\uparrow i} n_{\downarrow j} + J \sum_{\langle i,j \rangle} \vec{S}_{i} \cdot \vec{S}_{j} - g_{B}H \sum_i \vec{S}_{i}. \quad (1)
$$

where $\vec{S}_{i}$ is the spin operator and $n_{\uparrow i}$ is the number operator at site $i$. To describe the quasi-two-dimensional electronic structure of CeCoIn$_5$, we assume a square lattice, in which the bracket $\langle i,j \rangle$ ($\langle\langle i,j \rangle\rangle$) denotes the summation over the nearest-neighbor sites (next-nearest-neighbor sites). The on-site repulsive interaction is given by $U$, and $V$ and $J$ stand for the attractive and AFM exchange interactions, respectively, between nearest-neighbor sites. We assume $V$ to stabilize the $d$-wave superconducting state within the mean field BdG equations and choose $J > 0$ for the AFM correlation in CeCoIn$_5$. We assume $U = 1.15$, $V = -0.4$, and $J = 0.53$ throughout this paper. The $d$-wave superconductivity and the significant AFM spin fluctuation can be self-consistently described using many-body theories.
such as the fluctuation exchange (FLEX) approximation for the Hubbard or periodical Anderson model.\textsuperscript{35,36} We here assume the interactions $V$ and $J$ to describe these features in the inhomogeneous LO phase on the basis of BdG equations. Although the BdG equations neglect the AFM spin fluctuation beyond the mean field approximation, they are suitable for studying the qualitative features of the inhomogeneous superconducting and/or magnetic state. We will discuss the roles of AFM spin fluctuation at the end of this paper.

With the last term in eq. (1), we include the Zeeman coupling due to the applied magnetic field. The $g$-factor is assumed to be $g_{B} = 2$. We assume the magnetic field to lie parallel to the [100]-axis, which we choose to be the quantized axis of spin, with $\vec{H} = H\hat{z}$. In this paper, we identify the $a$-, $b$-, and $c$-axes of the tetragonal lattice with the $x$, $y$, and $z$-axes for the spin, respectively. We choose the unit of energy such that $t = 1$ and $t'/t = 0.25$. The chemical potential enters as $\mu = \mu_{0} + (1/2 U + 4V)n_{0}$, where $n_{0}$ is the number density for $U = V = J = H = 0$. We choose $\mu_{0} = -1.05$ so as to reproduce the incommensurate AFM order observed in the neutron scattering measurement for CeCoIn$_{5}$.\textsuperscript{26} Then, we obtain the number density $n \sim 0.77$.

The BdG equations are formulated in a standard manner. We take into account the Hartree term arising from $U$, $V$, and $J$. The BdG equations are self-consistently solved for the mean fields of the spin $\langle \vec{S} \rangle$, charge $\langle n \rangle$, and superconductivity $\Delta_{i,j}^{\sigma\sigma'} = < c_{i\sigma} c_{j\sigma'}^{\dagger} >$. Since the magnetic moment is nearly opposite between the nearest-neighbor sites, the order parameter of the AFM order is described by the staggered moment defined as

$$\vec{M}_{\text{AF}}(i) = (-1)^{m+n} < \vec{S}_{i} >,$$

with $i = (m,n)$. Two components of the pairing field $\Delta_{i,j}^{\sigma\sigma'}$ play dominant roles in the following results. The first component is the $d$-wave spin singlet pairing whose order parameter is described as

$$\Delta_{i,j}^{d}(i) = \Delta_{i,i+a}^{1} + \Delta_{i,i-a}^{1} - \Delta_{i,i+b}^{1} - \Delta_{i,i-b}^{1}$$

with $\vec{a}$ and $\vec{b}$ being the unit vectors along the $a$- and $b$-axes, respectively. The second component is the equal spin $\pi$-triplet pairing whose order parameter is described by the generalized d-vector $d_{a,b}(i)$ as

$$\Delta_{i,i+\vec{a}}^{2\sigma} = \pm (-1)^{m+n+1/2} < -\sigma d_{\vec{a}}^{\sigma}(i) + id_{\vec{b}}^{\sigma}(i) >$$

$$\Delta_{i,i+\vec{b}}^{2\sigma} = \pm (-1)^{m+n+1/2} < -\sigma d_{\vec{b}}^{\sigma}(i) + id_{\vec{a}}^{\sigma}(i) >$$

The $d$-wave spin singlet pairing is finite in the entire superconducting region, while the $\pi$-triplet pairing appears in the state of coexisting AFM order and superconductivity.\textsuperscript{51-53,37-40} Because of the linear coupling between the magnetic, spin singlet pairing, and $\pi$-triplet pairing order parameters, two finite order parameters among them induce the other order parameter. Our model eq. (1) includes the interactions leading to the magnetic order and spin singlet $d$-wave superconductivity, while the pairing interaction for the $\pi$-triplet pairing is negligible. Therefore, the $\pi$-triplet pairing does not belong to the dominant orders, but is induced as a secondary order parameter by the AFM order.

The mean field Hamiltonian is obtained as

$$H = -t \sum_{<i,j>,\sigma} c_{i,\sigma}^{\dagger} c_{j,\sigma} + t' \sum_{<<i,j>>,\sigma} c_{i,\sigma}^{\dagger} c_{j,\sigma}$$

$$+ \sum_{i} \left( \frac{1}{2} U < n_{i} > + V \sum_{\delta} < n_{i+\delta} > - \mu \right) n_{i}$$

$$+ \sum_{i} \left( -2U < \vec{S}_{i} > + J \sum_{\delta} < \vec{S}_{i+\delta} > - g_{B} \vec{H} \right) \vec{S}_{i}$$

$$+ \frac{1}{2} \sum_{i,\delta,\sigma,\sigma'} \left| \Delta_{i,i+\delta}^{\sigma\sigma'} \right|^{2} c_{i,\sigma}^{\dagger} c_{i+\delta,\sigma'}^{\dagger} c_{i,\sigma} c_{i+\delta,\sigma'} + c.c.,$$

where the summation of $\delta$ is taken over $\delta = \pm a_{x}, \pm b_{x}$. The pairing field is obtained as $\Delta_{i,i+\delta}^{1} = -(V - J/4) < c_{i\uparrow}^{\dagger} c_{i+\delta,\uparrow}^{\dagger} > - J/2 < c_{i\downarrow}^{\dagger} c_{i+\delta,\downarrow}^{\dagger} >$ and $\Delta_{i,i+\delta}^{2\sigma} = -(V + J/4) < c_{i\sigma}^{\dagger} c_{i+\delta,\sigma}^{\dagger} >$. The thermodynamic average $< >$ is obtained on the basis of the mean field Hamiltonian eq. (6). The self-consistent equations for $< \vec{S}_{i} >$, $< n_{i} >$, and $\Delta_{i,i+\delta}^{\sigma\sigma'}$ for each $i$ yield the BdG equations.

The free energy is obtained as

$$F = -\frac{1}{2} \sum_{\alpha} \log[1 + \exp(-E_{\alpha}/T)]$$

$$+ \sum_{i} \left( \frac{1}{4} U < n_{i} >^{2} + \frac{1}{2} V \sum_{\delta} < n_{i+\delta} > < n_{i+\delta} > \right)$$

$$+ \sum_{i} \left( -2U < \vec{S}_{i} >^{2} + \frac{1}{2} J \sum_{\delta} < \vec{S}_{i+\delta} > \cdot < \vec{S}_{i+\delta} > \right)$$

$$+ \frac{1}{2} \sum_{i,\delta,\sigma,\sigma'} \Delta_{i,i+\delta}^{\sigma\sigma'} < c_{i+\delta,\sigma}^{\dagger} c_{i,\sigma} + c.c.,$$

where $E_{\alpha}$ is the energy of Bogoliubov quasiparticles obtained by the mean field Hamiltonian eq. (6). The stable phase is determined by minimizing the free energy for the self-consistent solutions of the BdG equations.

### 3. Results

#### 3.1 Phase diagram

We first show the phase diagram against the temperature and magnetic field. The normal, uniform BCS, purely FFLO, and AFM-FFLO states are shown in Fig. 1. At high magnetic fields, the phase transition from the normal state to a superconducting state is of the first order, consistent with the experimental results for CeCoIn$_{5}$.\textsuperscript{51,41,42} We have shown that both the on-site repulsion $U$ and AFM interaction $J$ are required to reproduce the first order phase transition to the FFLO state.\textsuperscript{43} This indicates that both the AFM spin fluctuation and the local electron correlation play an essential role in the phase diagram of CeCoIn$_{5}$. These features are qualitatively understood on the basis of the Fermi liquid theory.\textsuperscript{44} Although the size of the numerical calculation (40 x 40 lattices) is not sufficiently large for distinguishing the first and second order phase transitions from the
BCS state to the FFLO state (BCS-FFLO transition), the BCS-FFLO transition is expected to be of the second order, as shown for a similar model.\textsuperscript{30} The second order BCS-FFLO transition is described by the nucleation of the FFLO nodal plane.\textsuperscript{30, 44, 45} Because of our tractable system size limitation, maximally two nodal planes fit into our calculation, as shown in Fig. 2.

An important finding obtained from Fig. 1 is the appearance of the AFM-FFLO state. The Neél temperature $T_N$ in the BCS and normal states is less than 0.01, which is the lower limit of our calculation, while we obtain a much higher Neél temperature $T_N$ of $\sim 0.02$ in the FFLO state. Thus, the AFM order is favored in the FFLO state rather than in the BCS and normal states.

Two mechanisms stabilize the AFM order in the FFLO state. The first mechanism is the appearance of Andreev bound states around the spatial nodes of the modulated superconducting order parameter in the FFLO state. The $\pi$-phase shift of the order parameter introduced in the FFLO state produces a large local density of states (DOS) and triggers the AFM order.\textsuperscript{30} This is the reason why the AFM order is favored in the FFLO state rather than in the normal and BCS states. The appearance of the Andreev bound states is a characteristic feature of the inhomogeneous Larkin-Ovchinnikov state and is absent in the homogeneous Fulde-Ferrell state. The role of the Andreev bound states in the FFLO nodal structure is much more pronounced than that of the single vortex that has been investigated by Ogata.\textsuperscript{46} This is because the FFLO nodal plane is a two-dimensional object, while the vortex is an one-dimensional object.

The second mechanism is the linear coupling between the AFM order and the $\pi$-triplet pairing that favors the AFM order in the spin singlet superconducting state. It has been shown that this coupling stabilizes the AFM state in the BCS state\textsuperscript{31, 32} and that in the Fulde-Ferrell state.\textsuperscript{33} In contrast to the previous studies,\textsuperscript{31, 32} the $\pi$-triplet pairing state without the AFM order is hardly stabilized in our model. However, the coupling to the $\pi$-triplet pairing significantly stabilizes the AFM order in the Larkin-Ovchinnikov state. The thin dashed line in Fig. 1 shows the fictitious Neél temperature in the FFLO state, if the $\pi$-triplet pairing is neglected. We observe the substantial increase in $T_N$ due to the admixed $\pi$-triplet pairing. Thus, the $\pi$-triplet pairing plays a quantitatively important role in the stability of the AFM-FFLO state even when the amplitude of $\pi$-triplet pairing is small.

We now turn to the question why the AFM order appears discontinuously at the normal-FFLO transition as well as at the BCS-FFLO transition. The former discontinuity is simply caused by the first order phase transition to the FFLO state. The order parameter of $d$-wave superconductivity appears discontinuous at the normal-FFLO transition and the AFM state is directly coupled to this order. The discontinuity at the BCS-FFLO transition looks more surprising because the phase transition is of the second order. The key lies in the appearance of the Andreev bound states that we have mentioned above. The second order phase transition from the BCS state to the FFLO state is associated with the nucleation of domain walls in the superconducting order parameter.\textsuperscript{30, 44, 45} The Andreev bound states localized around such domain walls are the source of the AFM instability. Since the Andreev bound states in the isolated domain wall are nearly independent of the adjacent domain walls, within the mean field approach, the Neél temperature is nearly independent of the density of domain walls. In other words, the AFM order occurs immediately when domain walls are nucleated at the BCS-FFLO transition.\textsuperscript{30} Note that the spatially averaged magnetic moment is continuous at the second order BCS-FFLO transition, although the Neél temperature is discontinuous there. The phase boundary near the BCS-FFLO transition obtained here by the BdG equations would be altered by the spin fluctuation, as discussed in §4. However, the spin fluctuations do not alter the result that the AFM order is confined in the FFLO state. This feature of the phase diagram is consistent with the experiment of CeCoIn$_5$ that revealed the magnetic order in the HFSC phase, but neither in the normal state nor in the low field superconducting phase.\textsuperscript{26}

3.2 Spatial structure of AFM-FFLO state

We here investigate the spatial structures of the AFM-FFLO state shown in Fig. 1. Figure 2(a) depicts the order parameter of the $d$-wave spin singlet superconductivity $\Delta^d(\hat{r})$, while Fig. 2(b) shows the AFM staggered moment $\vec{M}_{AF}(\hat{r})$ for $\hat{r} = (m, n)$. In Fig. 2(a), we assume that the modulation vector of FFLO superconductivity is parallel to the magnetic field $\vec{q}_{\text{FFLO}} \parallel \vec{H}$, in accordance with ref. 22. We find that the AFM staggered moment is perpendicular to the applied magnetic field, consistent with the neutron scattering measurement.\textsuperscript{28} Since we neglect the spin-orbit coupling, the AFM state with $\vec{M}_{AF}(\hat{r}) \parallel \hat{y}$ is degenerate with that with $\vec{M}_{AF}(\hat{r}) \parallel \hat{y}$. We choose the former in the following results.

We observe that the AFM staggered moment is local-
The spin singlet pairing, the generalized d-vector for the π-triplet pairing changes the sign at the zeros of $M_{AF}(\vec{i})$ and those of $\Delta^d(\vec{i})$. The modulation wave vectors of the π-triplet pairing are mainly $\vec{q}_{FFLO}$ along the $a$-axis and $\delta_{IC}$ along the $b$-axis.

Finally, we show the spatially inhomogeneous magnetization along the magnetic field $M^z(\vec{i})$ (Fig. 4). The magnetization mainly arises from the polarization of the Andreev bound states. Therefore, we clearly observe the magnetization localized around the spatial nodes of the spin singlet pairing field. In addition, a weak modulation of magnetization appears along the nodal lines owing to the incommensurate AFM order. It is shown that the magnetization is enhanced at the intersection points between the nodal planes of the incommensurate AFM order and the spin singlet pairing.

Fig. 2. (Color online) Spatial dependence of (a) the spin singlet pairing field with $d$-wave symmetry $\Delta^d(\vec{i})$ and (b) the AFM staggered moment $M_{AF}(\vec{i})$ for $\vec{i} = (m, n)$. We assume $T = 0.01$ and $H = 0.112$. The direction of the magnetic field is shown by an arrow.

Fig. 3. (Color online) Spatial dependence of the order parameter for the π-triplet pairing. We plot $-d^a_n(\vec{i}) + id^b_n(\vec{i}) = (-1)^{m+n}(\Delta^a_n,\vec{i},\vec{i}+\vec{a} - \Delta^b_n,\vec{i},\vec{i}-\vec{a})$. The directions of the FFLO modulation $\vec{q}_{FFLO}$ and incommensurability $\delta_{IC}$ are shown by arrows. The parameters are the same as those in Fig. 2.

Fig. 4. (Color online) Spatial dependence of the magnetization $M^z(\vec{i}) = \langle S^z_{\vec{i}} \rangle$ parallel to the magnetic field. The parameters are the same as those in Fig. 2.
3.3 Neutron scattering

We here propose an experiment that can unambiguously identify the AFM-FFLO state. Figure 5 shows the Fourier transformation of the magnetic moment perpendicular to the magnetic field,

\[ S^z(q) = \frac{1}{N} \sum_i S^z_i \exp(i\mathbf{q} \cdot \mathbf{r}_i). \]  

(8)

We observe two sharp peaks in \( S^z(q) \) at the incommensurate wave vectors \( \mathbf{q}_{IC} = \mathbf{Q} \pm \delta_{1C} \), where \( \delta_{1C}/2\pi = (0, 0.075) \) in our calculation. In addition to these main peaks, the satellite peaks appear at \( q = q_{IC} + q_{st} \) with \( q_{st} \parallel \mathbf{H} \perp \delta_{1C} \) in the AFM-FFLO state. These satellite peaks are the characteristic feature of the AFM-FFLO state and unambiguously show the spontaneous translation symmetry breaking along the magnetic field. Therefore, if the spin structure in Fig. 5 is observed by a neutron scattering measurement, that would give unambiguous evidence for the AFM order in the FFLO state. The neutron scattering measurements have revealed a sharp peak at \( q = q_{1C} \),\(^{26}\) however, the presence of satellite peaks shown in Fig. 5 has not yet been explored experimentally.

4. Roles of Spin Fluctuation

We turn to the roles of the spin fluctuations neglected in the BdG equations. First, the curvature of the BCS-FFLO transition line is changed by spin fluctuations near the AFM quantum critical point. We have investigated the FFLO superconductivity in the two-dimensional Hubbard model on the basis of the FLEX approximation. The resulting BCS-FFLO transition line shows a concave curvature,\(^{36}\) as observed for the HFSC phase of CeCoIn\(_5\).\(^{19}\) This is mainly due to the quasiparticle lifetime renormalization by the spin fluctuation that suppresses the FFLO state in the high-temperature region.\(^{36}\) This is in sharp contrast to the BdG equation that shows the convex BCS-FFLO transition line as in Fig. 1.

Second, we discuss the mechanism that stabilizes the incommensurate AFM order in the FFLO state. Spin susceptibility in the spatially uniform state is enhanced at \( q = \mathbf{Q} + \delta_{1C} \) nearly independent of the direction of \( \delta_{1C} \) in the \( ab \)-plane on the tetragonal lattice. Note that the incommensurate AFM order in the uniform phases, such as the normal and BCS states. On the other hand, the appearance of the significant in-plane anisotropy due to the modulated FFLO order parameter suppresses the in-plane anisotropy and favors the AFM order. This mechanism may play a role in stabilizing the AFM order in the HFSC phase of CeCoIn\(_5\).

Finally, the spin fluctuations play another role just above the BCS-FFLO transition line. The continuous phase transition from the BCS state to the FFLO state is described by the nucleation of domain walls. The density of domain walls approaches zero near the BCS-FFLO transition line. Then, the spatial dimensionality of the AFM order is reduced from three to quasi-two-dimensions. The reduced dimension enhances the fluctuations and suppresses the long-range order at finite temperatures when we neglect the broken SU(2)symmetry due to the spin-orbit coupling. The AFM order is suppressed by this effect just above the BCS-FFLO transition line. The density of domain walls rapidly increases with growing magnetic field from the BCS-FFLO transition line, and then, this effect of spin fluctuation is suppressed.

Taking into account the roles of spin fluctuations, we obtain the schematic phase diagram in Fig. 6(b). The AFM order is confined in the FFLO state, but it is suppressed around the BCS-FFLO transition line. The pure FFLO state is stabilized just above the BCS-FFLO transition line. The two phase transition lines, namely, the BCS-FFLO transition and the AFM order, are close to each other. Therefore, it may be difficult to distinguish these transition lines in an experiment. The CeCoIn\(_5\) at ambient pressure seems to be the case.

5. Summary and Discussion

We investigated the incommensurate AFM order in the \( d \)-wave spin singlet superconducting state on the basis of the BdG equations. It has been shown that the AFM order coexists with the FFLO superconducting state even when the AFM order occurs neither in the normal state nor in the BCS state. In other words, the AFM phase...
can be confined in the FFLO phase at high fields, consistent with the experimental results for CeCoIn$_5$.\(^\text{25,26}\) Magnetic instability is enhanced in the inhomogeneous Larkin-Ovchinnikov state because the $\pi$-phase shift of the pairing field gives rise to the Andreev bound states and produces the large local DOS at zero energy. The mixing with the $\pi$-triplet pairing also enhances the AFM order in the FFLO state.

The structures of the AFM order, namely, the directions of the AFM magnetic moment $\vec{M}_{\text{AF}}(\vec{r})$ and incommensurability $\tilde{\delta}_{\text{IC}}$, are consistent with the recent neutron scattering measurement for the HFSC phase of CeCoIn$_5$.\(^\text{26}\) Both $\vec{M}_{\text{AF}}(\vec{r})$ and $\tilde{\delta}_{\text{IC}}$ are perpendicular to the magnetic field. It has been shown that the amplitude of the incommensurability $|\tilde{\delta}_{\text{IC}}|$ is independent of the density of spatial nodes of the modulated superconducting order parameter.\(^\text{30}\) This is also consistent with the experimental result.\(^\text{26}\) These results indicate that the HFSC phase of CeCoIn$_5$ is the AFM-FFLO state in which the incommensurate AFM order coexists with the FFLO superconductivity. To obtain unambiguous evidence for this scenario, we propose another neutron scattering experiment that can detect the spontaneous translation symmetry breaking along the magnetic field.

Observations under pressure may be a further way to explore the HFSC phase of CeCoIn$_5$. Since the AFM order is expected to be suppressed by the pressure as in the other Ce-based heavy fermions,\(^\text{49}\) the AFM-FFLO state is gradually suppressed by the pressure, as schematically shown in Fig. 7. This is in sharp contrast to the pure FFLO state that is enhanced by the pressure as it goes away from the quantum critical point.\(^\text{36}\) Thus, the AFM order should be distinguished by the BCS-FFLO transition under pressure below $P < P_c$, as shown in Fig. 7(b). When the pressure exceeds the critical value $P_c$, the AFM-FFLO phase vanishes, as shown in Fig. 7(c). The experimental data for the pressure dependence are consistent with the enhanced FFLO phase,\(^\text{19}\) but the magnetic order in the low-pressure region has not yet been studied.

Finally, we discuss the experimental results for CeCoIn$_5$. One of the key experiments is that on the pressure dependence of the $H$-$T$-phase diagram\(^\text{19}\) mentioned above. It has been shown that the HFSC phase of CeCoIn$_5$ is enhanced by the pressure. This experimental result is hardly understood by regarding the HFSC phase as an AFM ordered state in the uniform superconducting state or in the simple Abrikosov vortex state. Therefore, the other quantum condensed state likely emerges in the HFSC phase of CeCoIn$_5$, such as the FFLO superconductivity investigated in this paper or the $\pi$-triplet pairing (pair density wave) proposed by other authors.\(^\text{31,32,34}\)

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