Fractional Green’s function for the time-dependent scattering problem in the
Space-time-fractional quantum mechanics

Jianping Dong

Department of Mathematics, College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

Integral form of the space-time-fractional Schrödinger equation for the scattering problem in the fractional quantum mechanics is studied in this paper. We define the fractional Green’s function for the space-time fractional Schrödinger equation and express it in terms of Fox’s H-function and in a computable series form. The asymptotic formula of the Green’s function for large argument is also obtained, and applied to study the fractional quantum scattering problem. We get the approximate scattering wave function with correction of every order.

1 INTRODUCTION

The fractional calculus \[1\], which is a generalization to the standard (integer) one, has been successfully applied in many science and engineering fields, such as anomalous transport, viscoelastic material, signal analysis and processing, control system \[4\]. In recent years, the fractional calculus enters the world of quantum mechanics. In the fractional quantum mechanics \[7\], the master equation is the fractional Schrödinger equation \[9\] instead of the standard one. The standard Schrödinger equation \[10\] was reformulated by Feynman and Hibbs \[12\] using the path integral approach considering the Gaussian probability distribution. The Lévy stochastic process is a natural generalization of the Gaussian process. The possibility of developing the path integral over the paths of the Lévy motion was discussed by Kac \[13\], who pointed out that the Lévy path integral generates the functional measure in the space of left (or right) continued functions having only discontinuities of the first kind. Then, Laskin \[7\] generalized Feynman path integral to Lévy one, and developed a space-fractional Schrödinger equation containing the Riesz fractional derivative \[2\]. Then, he constructed the fractional quantum mechanics and showed some properties of the space fractional quantum system \[8\].

Afterwards, Naber \[16\] constructed a time-fractional Schrödinger equation by introducing the Caputo fractional derivative \[1\] instead of the first order derivative over time to the standard Schrödinger equation to describe non-Markovian evolution in quantum physics. The Hamiltonian for the time fractional quantum system was found to be non-Hermitian and not local in time. Naber solved the time fractional Schrödinger equation for a free particle and for a potential well. Probability and the resulting energy levels are found to increase over time to a limiting value depending on the order of the time derivative. More recently, from the standard Schrödinger equation, Wang and Xu \[17\] established a Schrödinger equation with both space and time fractional derivatives, and solved the generalized Schrödinger equation for a free particle and for an infinite rectangular potential well. Then, a similar space-time-fractional Schrödinger equation is obtained by Dong and Xu \[18\] from the space-fractional Schrödinger equation. They expressed this fractional equation in a more simple form, and studied the time evolution behaviors of the space-time-fractional quantum system in the time-independent potential fields.

At present, the progresses for the space-time-fractional quantum system is fewer. Besides the results given in Refs. \[17\], \[18\] mentioned before, Jiang \[20\] developed a time-space fractional Schrödinger equation containing a nonlocal term, and obtained the time dependent solutions in terms of the H-function. All of these results are in the one-dimensional case. This paper focuses on the time-dependent scattering problem in the fractional quantum system described by the space-time-fractional Schrödinger equation, given by Dong and Xu, in the three-dimensional case. We will define the Green’s function for the time-dependent 3D space-time-fractional Schrödinger equation for the scattering problem, and the Green’s function will be calculated in terms of Fox’s H-function and in a computable series form. The asymptotic property of the Green’s function will also be discussed and applied to the fractional scattering problem.

1Email:Dongjp.sdu@gmail.com
\section{Green's Function of the Fractional Schrödinger Equation}

The space-time fractional Schrödinger equation \cite{9} obtained by Laskin reads (in two dimensions)
\begin{equation}
(i\hbar)^{\alpha} D_\alpha^t \psi(r, t) = \mathcal{H}_\alpha \psi(r, t),
\end{equation}
where \( \psi(r, t) \) is the time-dependent wave function, and
\[ \mathcal{H}_\alpha = M [-D_\alpha(\hbar\nabla)^\alpha + V(r, t)]. \]
Here, \( D_\alpha \) with physical dimension \([D_\alpha] = \text{[Energy]}^{1-\alpha} \times \text{[Length]}^\alpha \times \text{[Time]}^{-\alpha} \) is dependent on \( \alpha \) (we have \( D_\alpha = 1/2m \) for \( \alpha = 2 \), \( m \) denotes the mass of a particle) and \((\hbar\nabla)^\alpha\) is the quantum Riesz fractional operator \cite{2,7} defined by
\begin{equation}
(\hbar\nabla)^\alpha \psi(r, t) = \frac{1}{(2\pi\hbar)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} |\mathbf{p}|^\alpha \int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} \psi(r, t) d^3r.
\end{equation}
Note that by use of the method of dimensional analysis we have given a specific expression of \( D_\alpha \) in Ref. \cite{18} as \( D_\alpha = \bar{c}^{2-\alpha}/(\alpha m^{\alpha-1}) \), where \( \bar{c} \) denotes the characteristic velocity of the non-relativistic quantum system.

Now, we define a Green's function of the FSE by
\begin{equation}
[(i\hbar)^{\alpha} D_\alpha^t + \mathcal{D}(\hbar\nabla)^\alpha] G(r, t; r', t') = \delta(r - r')\delta(t - t'),
\end{equation}
with the causality condition
\begin{equation}
G(r, t; r', t') = 0, \quad \text{when } t < t'.
\end{equation}
Here, \( \mathcal{D} = M \cdot D_\alpha \). Then, the space-FSE \cite{1} becomes an integral equation,
\begin{equation}
\psi(r, t) = \psi_0(r, t) + \mathcal{D} \int G(r, t; r', t') V(r', t') \psi(r', t') d^3r' d^3r',
\end{equation}
in which \( \psi_0(r, t) \) satisfies the free-particle Schrödinger equation,
\begin{equation}
[i\hbar \frac{\partial}{\partial t} + \mathcal{D}(\hbar\nabla)^\alpha] \psi_0(r, t) = 0.
\end{equation}
By use of the method of separation of variables, the basic solution to Eq. \cite{7} can be easily obtained.
\begin{equation}
\psi_0(r, t) = e^{i(\mathbf{k}\cdot\mathbf{r} - Et)}/\hbar, \quad \text{a constant product factor is omitted}
\end{equation}
(8)

where \( E \) denotes the energy of the free particle, and \( \mathbf{k} = (k_x, k_y) \), in which \( k_x, k_y \) are arbitrary constants but satisfying \( |\mathbf{k}| = \sqrt{k_x^2 + k_y^2} = (E/\mathcal{D})^{1/\alpha} \). Replacing \( \mathbf{k} \) by momentum \( \mathbf{p} \), and \( E \) by \( \mathcal{D}|\mathbf{p}|^\alpha \) respectively, the free particle solution \( \psi_0(r, t) \) can be changed to the fractional plane wave solution \cite{14},
\begin{equation}
\psi_0(r, t) = e^{i(\mathbf{p}\cdot\mathbf{r} - E\mathcal{D}|\mathbf{p}|^\alpha)/\hbar}.
\end{equation}
(9)

Now we turn back to solve Eq. \cite{4}. Defining the Fourier-Laplace transform pair, with respect to \( \mathbf{r} \) and \( t \), of the Green's function \( G(r, t; r', t') \) as
\begin{equation}
\hat{G}(\mathbf{p}, s; r', t') = \int d^3r \int_{0}^{\infty} ds e^{-s/\hbar - i\mathbf{p}\cdot\mathbf{r}/\hbar} G(r, t; r', t'),
\end{equation}
\begin{equation}
G(r, t; r', t') = \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} \frac{ds}{2\pi i} \int d^3p e^{i\mathbf{p}\cdot\mathbf{r}/\hbar + is/\hbar} \hat{G}(\mathbf{p}, s; r', t').
\end{equation}
(10)
(11)
After taking the above Fourier-Laplace transform with respect to \( r \) and \( t \), Eq. (4) can be changed to
\[
(i\hbar)^\beta [s^\beta \hat{G}(p, s; r', t') + s^{-\beta} \hat{G}(p, 0; r', t')] - D[p] \hat{G}(p, \omega; r', t') = e^{-ip \cdot r'/\hbar} e^{-i\omega t'}/\hbar,
\]
in which \( \hat{G}(p, 0; r', t') \) is the Fourier transform of \( G(r, 0; r', t') \) with respect to \( r \). Then, taking account of the causality condition given by (5), we have \( G(r, 0; r', t') = 0 \), so that \( \hat{G}(p, 0; r', t') = 0 \). Now, we can obtain
\[
\hat{G}(p, \omega; r', t') = e^{-ip \cdot r'/\hbar} e^{-i\omega t'}/\hbar.
\]
Inverting the Fourier-Laplace transform gives
\[
G(r, t; r', t') = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{(s-t')/\hbar} \frac{e^{is\beta} - D[p]}{e^{is\beta}/\hbar} ds = \text{Res}_{s=s_0} \left( \frac{e^{is\beta} - D[p]}{e^{is\beta}/\hbar} \right)_{s=s_0}.
\]
Here, \( \text{Res}_{s=s_0} \) denotes the residue of \( \hat{G}(p, \omega; r', t') \) at the unique pole \( s_0 \), and \( s_0 = (D[p] + \beta) / (i\hbar) \). Now Eq. (14) becomes
\[
G(r, t; r', t') = \frac{N_1}{(2\pi)^3} \int_{0}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} e^{(r-r')/\hbar} e^{-i\eta p} |p|^{-\nu} d^3 p, \quad t > t',
\]
in which
\[
N_1 = \frac{1}{i\hbar^\beta 2\pi^2 \xi}, \quad \xi = D[1/\hbar(t - t')/\hbar, \nu = \alpha/\beta, \gamma = \alpha(\beta - 1)/\beta.
\]
To execute the above integration, we choose the spherical coordinates \((p, \theta, \varphi)\), with the positive direction of the \( p \)-axis along \( r - r' \). Then, \( p \cdot (r - r') = p|\mathbf{r} - \mathbf{r}'| \cos \theta \), in which \( p \) and \( |\mathbf{r} - \mathbf{r}'| \) denote the magnitudes of the vectors \( \mathbf{p} \) and \( \mathbf{r} - \mathbf{r}' \), respectively. Thus, Eq. (16) is converted into
\[
G(r, t; r', t') = \frac{N_1}{2\pi^2 \hbar^2 |\mathbf{r} - \mathbf{r}'|} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{\infty} e^{(r-r')/\hbar} e^{-i\eta p} |p|^{-\nu} \sin \theta dp
\]
(17)
in the last integral, substituting \( p \) for \( \xi^{1/\gamma} \) gives
\[
G(r, t; r', t') = N \cdot \frac{I(x)}{|\mathbf{r} - \mathbf{r}'|},
\]
in which
\[
I(x) = \int_{0}^{\infty} p^{1-\gamma} \sin(xp) e^{-ip} dp, \quad x \geq 0,
\]
(19)
(20)
Using the Mellin transform (23) and its inverse transform, \( I(x) \) can be expressed in terms of Fox's \( H \)-function. Taking Mellin transform to \( I(x) \) with respect to \( x \) yields
\[
I(s) = M \{ I(x), s \} = \int_{0}^{\infty} p^{1-\gamma} e^{-ip} \left( \int_{0}^{\infty} \sin(px)x^{s-1} dx \right) dp
\]
\[
= \Gamma(s) \sin \left( \frac{\pi s}{2} \right) \int_{0}^{\infty} p^{1-\gamma-1} e^{-ip} dp, \quad (\text{Re } s < 1).
\]
Thus, note that the formulas \[23\]
Then using the contour in Fig. 1, the above integral can be calculated,

\[
\int_0^{i\infty} p^{1-\gamma-1}\pi e^{-|p|} dp = \frac{1}{\nu} \Gamma\left(\frac{2-\gamma-s}{\nu}\right)e^{-\nu\pi}. \tag{22}
\]

Thus,

\[
\tilde{I}(s) = \frac{1}{\nu} \Gamma(s)\Gamma\left(\frac{2-\gamma-s}{\nu}\right)\sin\left(\frac{\pi s}{2}\right)e^{-\nu\pi} = \frac{1}{\nu}[\tilde{I}_1(s) - \tilde{I}_2(s)], \tag{23}
\]

where

\[
\tilde{I}_1(s) = \Gamma(s)\Gamma\left(\frac{2-\gamma-s}{\nu}\right)\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{2-\gamma-s}{2\nu}\right)\pi = \frac{\pi^2\Gamma(s)\Gamma\left(\frac{2-\gamma-s}{\nu}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}, \tag{24}
\]

\[
\tilde{I}_2(s) = \Gamma(s)\Gamma\left(\frac{2-\gamma-s}{\nu}\right)\sin\left(\frac{\pi s}{2}\right)\sin\left(\frac{2-\gamma-s}{2\nu}\right)\pi = \frac{\pi^2\Gamma(s)\Gamma\left(\frac{2-\gamma-s}{\nu}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}. \tag{25}
\]

Note that the formulas \[23\]

\[
\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}, \quad \text{and} \quad \cos(\pi z) = \frac{\pi}{1(1/2 + z)\Gamma(1/2 - z)}
\]

have been used here. Inverting the Mellin transform and comparing the expression with the definition of Fox’s \(H\)-function (see Eqs. (A1) and (A2) in the Appendix of this paper or Refs. \[24\]–\[26\]), we obtain

\[
I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{I}(s)x^{-s} ds = \frac{\pi^2}{\nu} \left[ H_{3,3}^{1,1} \left[ \begin{array}{c} (1-(2-\gamma)/\nu, 1/\nu), (0, 1/2), (1-2-\gamma)/(2\nu), 1/(2\nu) \\
(0, 1), (0, 1/2), (1/2-2-\gamma)/(2\nu), 1/(2\nu) \end{array} \right] - \right.
\]

\[
\left. \left( H_{3,3}^{1,1} \left[ \begin{array}{c} (1-2-\gamma)/\nu, 1/\nu), (0, 1/2), (1-2-\gamma)/(2\nu), 1/(2\nu) \\
(0, 1), (0, 1/2), (1-2-\gamma)/(2\nu), 1/(2\nu) \end{array} \right] \right) \right] \right] = \pi^2 \left[ H_1(x') - iH_2(x') \right], \tag{26}
\]
Note that the first property of the $H$-function in the Appendix has been used (or see Property 1.4 on page 12 of Ref. [24]). Finally, we get

$$G(r, t; r', t') = \frac{N\pi^2}{|r - r'|} \left[ H_1\left(\frac{|r - r'|^\nu}{\xi \hbar}\right) - iH_2\left(\frac{|r - r'|^\nu}{\xi \hbar}\right)\right]$$

$$= \frac{\mathcal{L}^{(\nu+1/2)/(\nu+1)} H_2(\gamma)}{2(b - t')^{(\nu+1)/(\nu+1)}} |r - r'|^\nu.$$

### 3 COMPUTABLE SERIES FORM OF THE GREEN’S FUNCTION

Using the series form of the Fox’s $H$-function (see Property 3 in the Appendix of this paper or §3.7 in Ref. [25]), a computable form of the Green’s Function can be obtained. For the $H$-function $H_1(x)$ given by Eq. (27), we have

$$H_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma((2 - \gamma)/\nu + k/\nu)}{\Gamma(1 + k/2)\Gamma(1/2 + (2 - \gamma)/(2\nu) + k/(2\nu)) \Gamma(1/2 - (2 - \gamma)/(2\nu) - k/(2\nu))} \frac{(-1)^k x^{k/\nu}}{k!}$$

$$= \sum_{k=0}^{\infty} \left(\frac{2 - \gamma + k}{\nu}\right) \frac{\sin(-k\pi/2) \cos((1/\nu + (k - \gamma)/(2\nu))\pi)}{\pi} (1/\nu) \frac{(-1)^k x^{k/\nu}}{k!}.$$

In a similar way, we obtain

$$H_2(x) = \frac{1}{\nu \pi} \sum_{n=0}^{\infty} \left(\frac{2n + 3 - \gamma}{\nu}\right) \frac{\sin(2n + 3 - \gamma/2\nu)}{2\nu} \frac{(-1)^n}{(2n + 1)!} x^{(2n+1)/\nu}.$$

Then, the Green’s Function given by Eq. (29) can be expressed in a series form as

$$G(r, t; r', t') = \frac{N}{\sqrt{|r - r'|}} \sum_{n=0}^{\infty} \left(\frac{2n + 3 - \gamma}{\nu}\right) \frac{\exp\left(\frac{2n + 3 - \gamma}{2\nu}\pi i\right)}{2\nu} (1/\nu) \frac{(-1)^n}{(2n + 1)!} \frac{|r - r'|^{2n}}{\xi^{2n}/\hbar}.$$

When $\beta = 1$, this formula reduces to the space-fractional case, see Eq.(a) in Ref. .

### 4 ASYMPTOTIC PROPERTY OF THE GREEN’S FUNCTION AND ITS APPLICATION TO THE SCATTERING PROBLEM

In quantum mechanics, the scattering theory [10][11] is often used to study the inner structure of a matter. The fractional quantum mechanics is a natural generalization to the standard quantum mechanics, so the research on the generalized quantum scattering problem under the framework of fractional quantum mechanics is meaningful. In the scattering problems, we usually consider the behavior of the particles far away from the scattering center, and assume the potential $V(r)$ is non-zero only in a small domain. Therefore, for $r$ and $r'$ in Eq. (6), we have $|r| >> |r'|$, or $|r - r'| \to \infty$. In this section,
we study the asymptotic properties of the Green’s function when \(|r - r'| \rightarrow \infty\). In this case, \(x = |r - r'|/((\xi\hbar)^{-1})\), defined by Eq. (20), also approaches infinity. Recalling Eq. (29), \(G(r, t; r', t')\) can be rewritten in terms of \(x\) as

\[
G(r, t; r', t') = \frac{Na^2}{|r - r'|} \cdot \left[ H_1(x') - iH_2(x') \right],
\]

in which the two H-functions \(H_1(x)\) and \(H_2(x)\) have been defined by Eqs. (27) and (28) respectively. It is easy to verify that the two H-functions \(H_1(x)\) and \(H_2(x)\) satisfy the conditions needed by Property 4 in the Appendix. Thus, for large \(x\), using the asymptotic formula (A10), we obtain

\[
H_1(x) = \frac{1}{\pi^2} \Gamma(2 - \gamma) \sin(\gamma\pi/2)x^{\nu-2} + \frac{M}{\pi i} \cos\left[\frac{\pi}{4} + (\nu - 1)\left(\frac{x}{\nu}\right)^{1/(\nu-1)}\right] x^{\nu/2} + o\left(x^{\nu/2}\right),
\]

\[
H_2(x) = \frac{1}{\pi^2} \Gamma(2 - \gamma) \sin(\gamma\pi/2)x^{\nu-2} - \frac{M}{\pi i} \sin\left[\frac{\pi}{4} + (\nu - 1)\left(\frac{x}{\nu}\right)^{1/(\nu-1)}\right] x^{\nu/2} + o\left(x^{\nu/2}\right),
\]

in which

\[
M = \frac{\nu^{(2\nu-3)/(2(\nu-1))}}{\sqrt{2\pi(\nu-1)}}.
\]

Then, from Eq. (29), we get the asymptotic formula of \(G(r, t; r', t')\) for large \(|r - r'|\),

\[
G(r, t; r', t') = \frac{Ae^{i\pi/4}|r - r'|^{\nu-1}}{2\xi^{2\nu}h^{3\nu+1}} \exp\left[i(\alpha - 1)\left(\frac{|r - r'|}{\alpha\xi\hbar}\right)^{\alpha/(\alpha-1)}\right] + o\left(|r - r'|^{\nu}\xi\hbar\right).
\]

When \(\alpha = 2, \beta = 1\), after some calculations, Eq. (36) reduces to

\[
G(r, t; r', t') = \frac{1}{\hbar} \left(\frac{m}{2\pi\hbar(t - t')^2}\right)^{3/2} \exp\left[i\frac{\hbar|r - r'|^2}{2(m(t - t'))}\right] + o\left(\frac{\sqrt{2m}|r - r'|}{\sqrt{(t - t')}\hbar}\right).
\]

which accords with the exact result in the standard quantum mechanics [27].

In the scattering problem, we can use merely the first term of the asymptotic formula (36) for \(G(r, t; r', t')\), then an approximate wave function for the scattering problem can be obtained. Let’s invoke the Born approximation [11][28]: Suppose the incoming plane wave is not substantially altered by the potential. Then, in Eq. (6), it makes sense to use

\[
\psi(r', t') \approx \psi_0(r', t') = e^{i(k \cdot r' - Et')/\hbar}.
\]

Then Eq. (6) becomes

\[
\psi(r, t) \approx \psi_0(r, t) + \frac{Ae^{i\pi/4}}{2\hbar^{3\nu+1}} \int \frac{|r - r'|^{\nu-1}}{\xi^{2\nu}} \exp\left[i(\alpha - 1)\left(\frac{|r - r'|}{\alpha\xi\hbar}\right)^{\alpha/(\alpha-1)}\right] + i(k \cdot r' - Et')/\hbar \right] V(r', t') d^3r' d't'.
\]

Furthermore, considering \(|r - r'| \approx |r| = r\), we can get

\[
\psi(r, t) \approx \psi_0(r, t) + \frac{Ae^{i\pi/4}}{2\hbar^{3\nu+1}} \int \frac{1}{\xi^{2\nu}} \exp\left[i(\alpha - 1)\left(\frac{r}{\alpha\xi\hbar}\right)^{\alpha/(\alpha-1)}\right] + i(k \cdot r' - Et')/\hbar \right] V(r', t') d^3r' d't'.
\]

The second term of the right side of the above formula gives the approximate scattering wave function.

We can also generate a series of higher-order corrections to the approximate wave function. From Eq. (6), we can build an iteration scheme for the wave function as

\[
\psi^{(n)}(r, t) = \psi_0(r, t) + \int G(r, t; r', t') V(r', t') \psi^{(n-1)}(r', t') d^3r' d't'.
\]

6
Note that $\phi^{(0)}(r, t) = \psi_0(r, t) = e^{i(k-r-E)t/\hbar}$, and $\phi^{(0)}$ is the $n$th-order corrections to the wave function. For a given potential function, using Eq. (41), the analytical approximate solutions of every order can be obtained. In a series form, we have

$$\phi = \phi_0 + \int GV\phi_0 + \iint GVGV\phi_0 + \iiiint GVGVGV\phi_0 + \cdots.$$ (42)

In each integrand only the incident wave function ($\phi_0$) appears, together with more and more powers of $GV$.

5 CONCLUSIONS

In this paper, the three-dimensional space-time-fractional Schrödinger equation with time-dependent potential is studied. We define the Green’s function of the STFSE for the time-dependent scattering problem in the fractional quantum mechanics, and the STFSE is converted into an integral form. We give the mathematical expression of the Green’s function in terms of Fox’s H-function and in a computable series form. The asymptotic property of the Green’s function for $|r-r'| \to \infty$ (or $|r| \gg |r'|$) was also given. In this case, the Green’s function acts like an exponential function (see Eq. (36)). Using this result, we obtained the approximate scattering wave function for the time-dependent fractional quantum scattering problem (see Eq. (40)). A series of higher-order corrections to the approximate wave function were also given in Eq. (42). These results are useful for the time-dependent scattering problem in the fractional quantum mechanics. All of these results contain those in the standard quantum mechanics as special cases.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 11147109), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20113218120030), and the Fundamental Research Funds for the Central Universities (Grant No. NS2012119).

APPENDIX: FOX’S H-FUNCTION AND SOME PROPERTIES

The Fox’s $H$-function [24,25] is defined by an integral of Mellin-Barnes type [29] as

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\right] = \frac{1}{2\pi i} \int_L \chi(s)z^{-s}ds,$$ (A1)

where

$$\chi(s) = \frac{\prod_{j=1}^p \Gamma(a_j + A_j s) \prod_{j=m+1}^q \Gamma(1 - a_j - A_j s)}{\prod_{j=1}^p \Gamma(b_j + B_j s) \prod_{j=m+1}^q \Gamma(1 - b_j - B_j s)}.$$ (A2)

The contour $L$ runs from $c - i\infty$ to $c + i\infty$ separating the poles of $\Gamma(1 - a_i - A_i s)$, $(i = 1, \cdots, n)$ from those of $\Gamma(b_j + B_j s)$, $(j = 1, \cdots, m)$. Here we present some properties of the $H$-function used in our paper. In order to give the results, the following definitions will be used,

$$\Delta = \sum_{j=1}^q B_j - \sum_{i=1}^p A_i; \quad \Delta^* = \sum_{i=1}^n A_i - \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j;$$

$$\delta = \prod_{j=1}^q (A_j)^{-A_j} \prod_{j=1}^q (B_j)^{B_j}; \quad \mu = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}.$$(A3)

The following properties of the $H$-function can be found in Refs. [24,26].

Property 1:
\[ \frac{1}{k} H_{p,q}^{m,n} \left( z \left| \begin{array}{c} (a_p, A_p) \\ (b_p, B_p) \end{array} \right. \right) = H_{p,q}^{m,n} \left( z \left| \begin{array}{c} (a_p, kA_p) \\ (b_p, kB_p) \end{array} \right. \right), \text{ for } k > 0 \]  

Property 2:

\[ z^\sigma H_{p,q}^{m,n} \left( z \left| \begin{array}{c} (a_p, A_p) \\ (b_p, B_p) \end{array} \right. \right) = H_{p,q}^{m,n} \left( z \left| \begin{array}{c} (a_p + \sigma A_p, A_p) \\ (b_p + \sigma B_p, B_p) \end{array} \right. \right), \text{ for } \sigma \in \mathbb{C} \]  

Property 3: Explicit Power Series Expansion

For \( \Delta > 0 \), \( z \neq 0 \) or \( \Delta = 0, |z| > \delta \), there holds the following expansion for the \( H \)-function [25],

\[ H_{p,q}^{m,n}(z) = \sum_{k=1}^{m} \sum_{j=0}^{\infty} \frac{\prod_{l=1}^{m} \Gamma(b_j - B_j s_{hk}) \prod_{i=0}^{n} \Gamma(1 - a_i + A_i s_{hk}) (-1)^j z^{s_{hk}} \Gamma(1 - a_i + A_i s_{hk})}{\prod_{i=0}^{n} \Gamma(a_i - A_i s_{hk}) \prod_{j=0}^{p} \Gamma(1 - b_j + B_j s_{hk})} k! B_n, \]  

where \( s_{hk} = (b_h + k)/B_h \), if the following conditions are satisfied:

1. The poles of the gamma functions \( \Gamma(1 - a_i - A_i s), (i = 1, \ldots, n) \) and those of \( \Gamma(b_j + B_j s), (j = 1, \ldots, m) \) do not coincide:

\[ A_i(b_j + l) \neq B_j(a_i - k - 1), (i = 1, \ldots, n; j = 1, \ldots, m; k, l = 0, 1, 2, \ldots). \]  

2. The poles of the gamma functions \( \Gamma(b_j + B_j s), (j = 1, \ldots, m) \) are simple:

\[ B_i(b_j + l) \neq B_j(b_i + k), (i \neq j; i, j = 1, \ldots, m; k, l = 0, 1, 2, \ldots). \]  

Property 4: Asymptotic Expansions at Infinity in the Case \( \Delta > 0, \Delta' = 0 \)

When the poles of the gamma functions \( \Gamma(1 - a_i - A_i s), (i = 1, \ldots, n) \) are simple:

\[ A_i(1 - A_i + l) \neq A_i(1 - a_i + k), (i \neq j; i, j = 1, \ldots, n; k, l = 0, 1, 2, \ldots), \]  

and the condition (A7) is satisfied, the \( H \)-function has the following asymptotic expansion [26],

\[ H_{p,q}^{m,n}(z) = \sum_{i=1}^{n} \left[ h_i z^{(a_i - 1)/A_i} + o \left( z^{(a_i - 1)/A_i} \right) \right] + A_i^{(\mu+1)/\Delta} \left( c_0 \exp \left[ (B + C z^{1/\Delta}) i \right] - d_0 \exp \left[ -(B + C z^{1/\Delta}) i \right] \right) + o \left( z^{(\mu+1)/\Delta} \right), \]

where

\[ h_i = 1 \prod_{j=1}^{m} \Gamma(b_j - B_j (a_i - 1)/A_i) \prod_{j=0}^{p} \Gamma(1 - a_i + A_i (a_i - 1)/A_i) \]

\[ A = \frac{A_0}{2\pi i \Delta} \left( \frac{\Delta}{\delta} \right)^{\mu+1/\Delta}, \quad B = \frac{(2\mu + 1)\pi}{4}, \quad C = \left( \frac{\Delta}{\delta} \right)^{1/\Delta}, \]

\[ A_0 = (2\pi)^{(p-q+1)/2} \Delta^{-\mu} \prod_{j=1}^{p} A_j^{-a_j+1/2} \prod_{j=1}^{q} B_j^{b_j-1/2}, \]

\[ c_0 = (2\pi)^{m+n-p} \exp \left[ \sum_{j=1}^{p} a_j - \sum_{j=1}^{m} b_j \right], \]

\[ d_0 = (-2\pi)^{m+n-p} \exp \left[ - \sum_{j=1}^{p} a_j - \sum_{j=1}^{m} b_j \right]. \]
References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[2] A.A. Kilbas, H.M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[3] M. Y. Xu and W. C. Tan, Intermediate processes and critical phenomena: Theory, method and progress of fractional operators and their applications to modern mechanics, Science in China (G series) 36(3) (2006) 225-238.
[4] B. J. West, M. Bologna and Paolo Grigolini, Physics of Fractal Operators, Springer, New York, 2003.
[5] V. E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Higher Education Press, Beijing, 2010.
[6] C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, and V. Feliu-Batlle, Fractional-order Systems and Controls: Fundamentals and Applications, Springer, London, 2010.
[7] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A 268 (2000) 298–305.
[8] N. Laskin, Fractional quantum mechanics, Phys. Rev. E 62 (2000) 3135–3145.
[9] N. Laskin, Fractional Schrodinger equation, Phys. Rev. E 66 (2002) 056108.
[10] F.S. Levin, An Introduction to Quantum Theory, Cambridge University Press, 2002.
[11] D.J. Griffiths, Introduction to Quantum Mechanics (2nd edition, Prentice Hall, 2004)
[12] R.P. Feynman and A. R. Hibbs, Quantum Mechanics and path Integrals, McGraw-Hill, New York, 1965.
[13] M. Kac. On Some Connections between Probability Theory and Differential and Integral Equations, in Second Berkeley Symposium on Mathematical Statistics and Probability (edited by Jerzy Neyman), University of California Press, Berkeley, California, 1951.
[14] N. Laskin, Fractals and quantum mechanics, Chaos 10 (2000) 780–790.
[15] N. Laskin, Levy flights over quantum paths, Comm. Nonlin. Sci. Num. Sim. 12(2), 2007.
[16] M. Naber, J. Math. Phys. 45(8), 2004. 3339-3352.
[17] S.W. Wang and M.Y. Xu, J. Math. Phys. 48 (2007) 043502.
[18] J.P. Dong, M.Y. Xu, J. Math. Anal. Appl., 344, 1005-1017 (2008).
[19] X.Y. Guo, M.Y. Xu, J. Math. Phys. 47 082104 (2006).
[20] X. Y. Jiang, Time-space fractional Schrodinger like equation with a nonlocal term, Eur. Phys. J. Special Topics , 193 (2011) 61–70.
[21] M. Masujima, Applied Mathematical Methods in Theoretical Physics, Wiley-VCH, Weinheim, 2005.
[22] J. A. Mark, S. F. Athanassios, Complex Variables: Introduction and Applications, 2nd edition, Cambridge University Press, Cambridge, 2003.
[23] A.D. Polyanin, A.V. Manzhirov, Handbook of Mathematics for Engineers and Scientists, Chapman and Hall/CRC, 2006.
[24] A.M. Mathai, R.K. Saxena and H.J. Haubold, The H-function:Theory and Applications, Springer, New York, 2009.
[25] A.M. Mathai and R.K. Saxena, The H-function with Applications in Statistics and Other Disciplines (Wiley Eastern, New Delhi,1978).
[26] A.A. Kilbas and M. Saigo, H-Transforms: Theory and Applications, CRC, Floroda, 2004.
[27] E.N. Economou, Green's Function in Quantum Physics, 3rd Edition, Springer-Verlag, Berlin, Heidelberg, New York, 2006.
[28] Z. Born, Z. Physik 38, 803 (1926).
[29] R. B. Paris, Asymptotics and Mellin-Barnes Integrals, Cambridge University Press, Cambridge, 2001.
$C_R$

$\omega_0$
