QUANTUM PHASE SHIFT AND NEUTRINO OSCILLATIONS IN A STATIONARY, WEAK GRAVITATIONAL FIELD

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A new method based on Synge’s world function is developed for determining within the WKB approximation the gravitationally induced quantum phase shift of a particle propagating in a stationary spacetime. This method avoids any calculation of geodesics. A detailed treatment is given for relativistic particles within the weak field, linear approximation of any metric theory. The method is applied to the calculation of the oscillation terms governing the interference of neutrinos considered as the superposition of two eigenstates having different masses. It is shown that the neutrino oscillations are not sensitive to the gravitomagnetic components of the metric as long as the spin contributions can be ignored. Explicit calculations are performed when the source of the field is a spherical, homogeneous body. A comparison is made with previous results obtained in Schwarzschild spacetime.

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1. Introduction
A lot of papers has been devoted to the effects of gravitation on the neutrino oscillations (see, e.g., Refs. [10]. As far as we know, all the proposed derivations of the neutrino oscillation formula within the WKB approximation require some integrations along null or timelike geodesics. For this reason, even in the simple case of the Schwarzschild metric treated in the linear approximation, the calculations become heavy for non-radial propagations and the results remain limited to the case where the neutrinos are created outside the matter generating the gravitational field. The purpose of the present paper is to provide a new method which avoids any determination of geodesics. Based on Synge’s world function as defined in Ref. [10] this method has shown its usefulness in the study of light rays (see Refs. [11][13]) and can be applied to neutrinos and more generally to all kinds of particles, as long as
the spin contributions may be ignored.\footnote{Note that the gravitational contributions to the spin precession vanish in the Schwarzschild spacetime (see, e.g., Refs.\textsuperscript{14} and \textsuperscript{15}).}

We begin by giving a general relation between the quantum phase of a particle freely propagating in a stationary spacetime and Synge’s world function. Then, restricting our attention to the case of relativistic particles, we obtain the formulas yielding the travel time and the quantum accumulated phase within the framework of the weak field, linear approximation of any metric theory of gravity. It must be noted that our method is not suitable for the cases where multiple-path propagation effects may occur since the definition of Synge’s world function requires the existence of a single geodesic path between two given points-events.

We apply our method to the determination of the oscillations of neutrinos considered as superpositions of two eigenstates with different masses. We assume that the different eigenstates of the neutrinos are described by wave packets and that only the components of these wave packets having the same energy contribute to the interference (see, e.g., Refs.\textsuperscript{16,17} and \textsuperscript{7}). For practical calculations we consider that the metric perturbations are given by the terms of order $G/c^2$ and $G/c^3$ within the standard post-Newtonian formalism, $G$ being the Newtonian gravitational constant.

We obtain explicit expressions for a gravitational field generated by a spherical, homogeneous body. Our results hold even in the case of non-radial propagations and for sources located inside the central body. Finally, we compare our formula giving the accumulated quantum phase with the results found in Ref.\textsuperscript{7} for the exterior Schwarzschild metric.

We suppose that spacetime is covered by a global quasi Minkowskian coordinate system $(x^\mu)$. We put $x^0 = ct$, $t$ being a timelike coordinate and we use the notation $x = (x^i)$. The signature adopted for the metric tensor $g_{\mu\nu}$ is $(+−−−)$.

## 2. Phase shift in a weak stationary gravitational field

### 2.1. Quantum phase shift and world function

Let us consider a free particle with a mass $m$ propagating from a point $x_A$ to a point $x_B$. The quantum phase of this particle accumulated between $x_A$ and $x_B$ is given by

$$\Phi_{AB} = \frac{mc}{\hbar} s_{AB},$$

where $s_{AB}$ is the length of the timelike geodesic $\Gamma_{AB}$ joining $x_A$ and $x_B$ (see, e.g., Ref.\textsuperscript{18}). Let us introduce Synge’s world function $\Omega(x_A, x_B)$, defined as

$$\Omega(x_A, x_B) = \frac{1}{2} \int_0^1 g_{\mu\nu}(x^\alpha(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda,$$

where the integral is taken over $\Gamma_{AB}$ and $\lambda$ is the unique affine parameter along $\Gamma_{AB}$ which satisfies the boundary conditions $\lambda_A = 0$ and $\lambda_B = 1$. Since $s^2_{AB} = 2\Omega(x_A, x_B)$
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(c.f. Refs. [10] and [12], $\Phi_{AB}$ and $\Omega(x_A, x_B)$ are linked by the relation

$$\Phi_{AB} = \frac{mc}{\hbar} \sqrt{2\Omega(x_A, x_B)}.$$  (3)

As a consequence, the knowledge of the world function corresponding to a given metric enables to determine the accumulated phase $\Phi_{AB}$. In addition, the world function enables to perform the calculation of the energy-momentum vector $p_\alpha = mc u_\alpha$ of the particle at $x_A$ and $x_B$, $u_\alpha$ denoting the covariant components of the unit 4-velocity vector $dx^\alpha/ds$ along $\Gamma_{AB}$. Indeed, it may be inferred from the definition of the world function that the values of $u_\alpha$ at $x_A$ and $x_B$ are given by (see Refs. [10] and [12])

$$(u_\alpha)_A = -\frac{1}{\sqrt{2\Omega(x_A, x_B)}} \frac{\partial \Omega}{\partial x_\alpha^A}, \quad (u_\alpha)_B = \frac{1}{\sqrt{2\Omega(x_A, x_B)}} \frac{\partial \Omega}{\partial x_\alpha^B}. \quad (4)$$

2.2. Phase shift in a stationary spacetime

Henceforth, we restrict our attention to stationary spacetimes. We choose the coordinate system $x^\mu$ so that the metric components do not depend on $x^0$. The world function then has the form $\Omega(x^0_B - x^0_A, x_A, x_B)$ and the component $u_0$ is a constant of the motion, i.e.

$$u_0 = E/mc^2, \quad E = \text{const.} \quad (5)$$

Let $x$ be a point through which the particle is passing. The energy of the particle as locally measured by a stationary observer staying at $x$ is the quantity given by

$$E_{loc}(x) = \frac{E}{\sqrt{g_{00}(x)}}. \quad (6)$$

It is easily deduced from Eqs. (4) written for $\alpha = 0$ and from Eq. (5) that $x^0_B - x^0_A$, $x_A$, $x_B$ and $E$ are linked by the relation

$$\sqrt{2\Omega(x^0_B - x^0_A, x_A, x_B)} = \frac{mc^2}{E} \hat{\Omega}(x^0_B - x^0_A, x_A, x_B), \quad (7)$$

where the following notation is used

$$\hat{\Omega}(x^0_B - x^0_A, x_A, x_B) \equiv \frac{\partial \Omega(x^0_B - x^0_A, x_A, x_B)}{\partial (x^0_B - x^0_A)}. \quad (8)$$

Solving Eq. (7) for $x^0_B - x^0_A$ gives the (coordinate) travel time of the particle between its point of emission and its point of reception as a function of $x_A$, $x_B$ and $E$ for a given mass $m$. In what follows we shall denote this function by $T(x_A, x_B, E; m)$, so that $x^0_B - x^0_A$ is given by

$$x^0_B - x^0_A = cT(x_A, x_B, E; m). \quad (9)$$

Substituting for $x^0_B - x^0_A$ from Eq. (9) into the right-hand side (R.H.S.) of Eq. (7) yields $\sqrt{2\Omega}$ as a function of $x_A$, $x_B$, $E$ and $m$. Consequently, Eq. (3) reads

$$\Phi_{AB} = \frac{m^2c^3}{\hbar E} \Omega(cT(x_A, x_B, E; m), x_A, x_B). \quad (10)$$
Noting that Eq. (1) is equivalent to (see, e.g., Ref. 15)
\[ \Phi_{AB} = \frac{1}{\hbar} \int_{\Gamma_{AB}} p_\mu dx^\mu \] (11)
and that \( p_0 \) is a constant of the motion given by \( p_0 = E/c \), the accumulated phase may be written in the form
\[ \Phi_{AB} = \frac{E}{hc} (x^0_B - x^0_A) + \Psi_{AB}, \] (12)
where \( \Psi_{AB} = \frac{1}{\hbar} \int_B^A p_i dx^i \) is a function of \( x_A, x_B, E \) and \( m \). Indeed, inserting Eqs. (9) and (10) into Eq. (12), it is easily seen that
\[ \Psi_{AB} = \frac{E}{hc} \left[ \frac{m^2 c^4}{E^2} \Omega(cT(x_A, x_B), E, m, x_A, x_B) - cT(x_A, x_B), E, m) \right]. \] (13)

We shall see below that \( \Psi_{AB} \) is the relevant quantity for determining the stationary oscillations of neutrinos. As a consequence, Eq. (13) constitutes the basic formula of the present work and we shall henceforth concentrate our attention on the determination of its right-hand side.

2.3. Weak field, linear approximation

Let us now specialize in the weak field, linear approximation. We assume that there exists a coordinate system in which the metric may be written in the form
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(G^2), \] (14)
where \( \eta_{\mu\nu} = \text{diag} (1, -1, -1, -1) \) and the quantities \( h_{\mu\nu} \) are considered as perturbation terms of the first order in \( G \). It is shown in Ref. 11 that the world function is then given by
\[ \Omega(x_A, x_B) = \frac{1}{2} \left[ \eta_{\mu\nu} + \int_0^1 h_{\mu\nu}(x(0)(\lambda))d\lambda \right] (x^\mu_B - x^\mu_A)(x^\nu_B - x^\nu_A) + O(G^2), \] (15)
the integration being performed along the straight line of parametrized equations \( x^\alpha_0(\lambda) = (x^\alpha_B - x^\alpha_A)\lambda + x^\alpha_A \). Since spacetime is assumed to be stationary, Eq. (15) may be written as
\[ \Omega(x_A, x_B) = \frac{1}{2} (x^0_B - x^0_A)^2 [1 + 2A(x_A, x_B)] + (x^0_B - x^0_A)R_{AB}B(x_A, x_B) \]
\[ - \frac{1}{2} \frac{R_{AB}^2}{2C(x_A, x_B)} + O(G^2), \] (16)
where \( R_{AB} = |x_B - x_A| \) and
\[ A(x_A, x_B) = \frac{1}{2} \int_0^1 h_{00}(x(0)(\lambda))d\lambda, \] (17)
\[ B(x_A, x_B) = \frac{(x^1_B - x^1_A)}{R_{AB}} \int_0^1 h_{01}(x(0)(\lambda))d\lambda, \] (18)
\[ C(x_A, x_B) = \frac{(x^1_B - x^1_A)(x^2_B - x^2_A)}{2R_{AB}^2} \int_0^1 h_{ij}(x(0)(\lambda))d\lambda, \] (19)
with
\[ x_{(0)}(\lambda) = (x_0 - x_A)\lambda + x_A. \]  
(20)

For the sake of brevity, we shall henceforth omit the symbol \( O(G^2) \) standing for the post-post-Minkowskian terms.

Differentiating the R.H.S. of Eq. (16) with respect to \( x_0^B - x_0^A \), and then substituting the obtained expression into Eq. (8) yield
\[ \dot{\Omega}(x_0^B - x_0^A, x_A, x_B) = (x_0^B - x_0^A) \left[ 1 + 2A(x_A, x_B) \right] + R_{AB}B(x_A, x_B). \]  
(21)

Squaring each side of Eq. (7), and then using Eqs. (16) and (21) give an equation of the second degree in \( x_0^B - x_0^A \). Solving this equation leads to
\[ cT(x_A, x_B, E; m) = R_{AB} \left\{ \frac{E}{\sqrt{E^2 - m^2c^4}} \left[ 1 - \left( 1 - \frac{m^2c^4}{E^2 - m^2c^4} \right) A(x_A, x_B) \right] - C(x_A, x_B) \right\}. \]  
(22)

Finally, inserting Eqs. (21) and (22) into Eq. (13) gives for \( \Psi_{AB} \)
\[ \Psi_{AB} = -R_{AB} \sqrt{E^2 - m^2c^4} \left[ 1 - \frac{E^2}{E^2 - m^2c^4} A(x_A, x_B) - C(x_A, x_B) \right]. \]  
(23)

It may be seen that the expression of \( cT(x_A, x_B, E; m) \) given by Eq. (22) is valid only if the term \( 1 - m^2c^4/E^2 \) is significantly far from 0. So we shall henceforth assume that the condition
\[ mc^2 \ll E \]  
(24)
is satisfied. Since it may be shown that the constant \( E \) is linked to the velocity \( v_x \) of the particle as measured by a stationary observer at \( x \) by the relation (cf., e.g., Ref. 19)
\[ E = mc^2 \sqrt{\frac{g_{00}(x)}{1 - v^2_x / c^2}}, \]  
(25)
we can consider that our formulas hold for relativistic particles.

In a general stationary spacetime, the infinitesimal spatial distance \( ds \) between two points \( (x^\alpha) \) and \( (x^\alpha + dx^\alpha) \) as measured by a stationary observer is determined by (cf. Ref. 20)
\[ ds^2 = -\left( g_{ij} - \frac{g_{00}g_{ij}}{g_{00}} \right) dx^i dx^j. \]  
(26)

\[ ^b \text{Let } m = 0 \text{ in Eq. (22). It is easily seen that the expression of the function } T(x_A, x_0, E; 0) \text{ is independent of } E \text{ and coincides with the expression of the time transfer function giving the travel time of a photon between points } x_A \text{ and } x_0 \text{ (see, e.g., Refs. 11 and 13).} \]
So an intrinsic distance $D_{AB}$ between $x_A$ and $x_B$ may be defined by

$$D_{AB} = \int_{x_A}^{x_B} d\sigma,$$  \hspace{1cm} (27)

where the integral is taken along the geodesic path $\Gamma_{AB}^*$ relative to the metric (26) joining $x_A$ and $x_B$. Within the linear approximation $\Gamma_{AB}^*$ is a curve given by a position function of the form $x(\lambda) = x(0) + \xi(\lambda)$, where $x(0)$ is defined by Eq. (20) and $\xi(\lambda)$ is a vector function of order $G$ satisfying the boundary conditions $\xi(0) = 0, \xi(1) = 0$. Taking these boundary conditions into account, it is easily seen that Eqs. (26) and (27) yield

$$D_{AB} = R_{AB}[1 - C(x_A, x_B)] + O(G^2).$$  \hspace{1cm} (28)

This formula generalizes the expression of the geodesic radial distance in Schwarzschild spacetime used in some discussions (see, e.g., Refs. [3] and [7]). Taking Eq. (28) into account, Eq. (23) reads now

$$\Psi_{AB} = -D_{AB} \sqrt{\frac{E^2 - m^2c^4}{\hbar c}} \left[ 1 - \frac{E^2}{E^2 - m^2c^4} A(x_A, x_B) ight]$$

$$- \frac{E}{\sqrt{E^2 - m^2c^4}} B(x_A, x_B).$$  \hspace{1cm} (29)

Owing to the shortness of the formula (29), we shall systematically use $D_{AB}$ instead of $R_{AB}$ in what follows.

3. Application to the neutrino oscillations

3.1. General formulas

Let us apply the previous results to the oscillation of neutrinos emitted at point $x_A$ and detected at point $x_B$. We consider that the state of a neutrino is a superposition of two eigenstates created with the same initial phase. The masses corresponding to these eigenstates are denoted by $m_1$ and $m_2$ and supposed to be distinct. It results from Eq. (12) that the quantity governing the stationary interference is the oscillation term

$$I \propto \exp \left[ \frac{E_2 - E_1}{\hbar c} \left( x_0^0 - x_0^0 \right) + \left( \Psi_{AB} \right)_{E_2,m_2} - \left( \Psi_{AB} \right)_{E_1,m_1} \right],$$  \hspace{1cm} (30)

with obvious notations.

For the reason invoked in Introduction, we shall henceforth suppose that $E_1 = E_2$. On this assumption, the phase difference involved in Eq. (30) is simply the quantity $\Delta \Psi_{AB}$ defined as

$$\Delta \Psi_{AB} = \left( \Psi_{AB} \right)_{E,m_2} - \left( \Psi_{AB} \right)_{E,m_1}. $$  \hspace{1cm} (31)
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This last expression is equivalent to the phase difference considered in Ref. [4]. Using Eq. (29), Eq. (31) yields

\[ \Delta \Psi_{AB} = D_{AB} \frac{(m_2^2 - m_1^2)c^3}{\hbar(\sqrt{E^2 - m_1^2c^4} + \sqrt{E^2 - m_2^2c^4})} \times \left[ 1 + \frac{E^2}{\sqrt{E^2 - m_1^2c^4}} + \frac{E^2}{\sqrt{E^2 - m_2^2c^4}} \right] A(x_A, x_B). \]  

(32)

We note that the gravitational quantity \( B(x_A, x_B) \) does not occur in Eq. (32). According to Eq. (18), this feature shows that the gravitomagnetic components \( h_{0i} \) of the metric do not contribute to the neutrino oscillations.

We can apply the formula (32) when the stationary field is treated in the slow-motion, post-Newtonian limit of metric theories of gravity, provided only the metric perturbations of order \( 1/c^2 \) and \( 1/c^3 \) are taken into account. Using the parametrized post-Newtonian formalism and assuming that the coordinates \( (x^\mu) \) correspond to a standard post-Newtonian gauge as defined in Ref. [21], the relevant terms in the metric are

\[ h_{00} = -\frac{2}{c^2} U, \quad h_{ij} = -\frac{2\gamma}{c^2} U \delta_{ij}, \]  

(33)

where \( U \) is the Newtonian gravitational potential. Then

\[ A(x_A, x_B) = -\frac{1}{c^2} \int_0^1 U(x_{(0)}(\lambda))d\lambda, \]  

(34)

where \( x_{(0)}(\lambda) \) is defined by (20). Introducing the energy of the neutrinos as measured by a stationary observer at \( x_B \), i.e. the quantity \( E_B = E_{\text{loc}}(x_B) \), and then substituting for \( A(x_A, x_B) \) from Eq. (33) into Eq. (32) yield

\[ \Delta \Psi_{AB} = D_{AB} \left( \frac{m_2^2 - m_1^2c^3}{2\hbar E_B} \left( 1 + \frac{(m_1^2 + m_2^2)c^4}{4E_B^2} + \frac{(m_1^2 + m_2^2)c^4}{8E_B^4} + \cdots \right) \right) 
+ \left[ 1 + \frac{3(m_1^2 + m_2^2)c^4}{4E_B^2} + \cdots \right] \frac{U_B}{c^2} - \frac{1}{c^2} \int_0^1 U(x_{(0)}(\lambda))d\lambda \right] A(x_A, x_B). \]  

(35)

where \( D_{AB} \) is now given by

\[ D_{AB} = R_{AB} \left[ 1 + \frac{\gamma}{c^2} \int_0^1 U(x_{(0)}(\lambda))d\lambda \right] \]  

(36)

since \( C(x_A, x_B) = \gamma A(x_A, x_B) \).

The formula (35) has the virtue of expressing \( \Delta \Psi_{AB} \) in terms of intrinsic quantities. It will be easy to compare it with the results obtained in previous works for neutrinos propagating in a radial direction. However, the geodesic distance \( D_{AB} \) has not been already introduced in the case of the propagation along a non-radial path and the constant of the motion \( E \) instead of \( E_{\text{loc}} \) has been often employed. So, it is
useful to write Eq. (35) in terms of $R_{AB}$ and $E$. We find

$$\Delta \Psi_{AB} = R_{AB} \frac{(m_2^2 - m_1^2)c^3}{2\hbar E} \left\{ 1 + \frac{(m_1^2 + m_2^2)c^4}{4E^2} + \frac{(m_1^4 + m_1^2m_2^2 + m_2^4)c^8}{8E^4} + \cdots \right\}$$

$$+ \left\{ \gamma - 1 + \frac{(\gamma - 3)(m_1^2 + m_2^2)c^4}{4E^2} + \cdots \right\} \frac{1}{c^2} \int_0^1 U(x(0)(\lambda))d\lambda \right\}. \quad (37)$$

Under this form, the gravitational contribution of order $(m_2^2 - m_1^2)c^4/E^2$ is found to be absent in general relativity, which predicts $\gamma = 1$. This statement generalizes a conclusion previously drawn in Refs. $3, 5, 7$ and $8$ for radial propagations in the Schwarzschild spacetime. Nevertheless, we must be cautious since this conclusion is inferred from Eq. (37) which involves $R_{AB}$, a coordinate dependent quantity. On the other hand, it is worthy of note that the intrinsic distance $D_{AB}$ occurring in Eq. (35) implicitly introduces gravity even in general relativity.

Some works assume that the phase difference $\Delta \Phi_{AB}$ defined as

$$\Delta \Phi_{AB} = (\Phi_{AB})_{E,m_2} - (\Phi_{AB})_{E,m_1} \quad (38)$$

is the relevant quantity for determining the neutrino oscillations (see, e.g., Ref. $9$). So it may be of interest to mention the expression which could be deduced from Eq. (10) under the present assumptions. A straightforward calculation shows that

$$\Delta \Phi_{AB} = D_{AB} \frac{(m_2^2 - m_1^2)c^3}{\hbar E_B} \left\{ 1 + \frac{(m_1^2 + m_2^2)c^4}{2E_B^2} + \frac{3(m_1^4 + m_1^2m_2^2 + m_2^4)c^8}{8E_B^4} + \cdots \right\}$$

$$+ \left\{ 1 + \frac{3(m_1^2 + m_2^2)c^4}{2E_B^2} + \cdots \right\} \left[ \frac{U_B}{c^2} - \frac{1}{c^2} \int_0^1 U(x(0)(\lambda))d\lambda \right\}. \quad (39)$$

We remark that $\Delta \Phi_{AB}$ is approximatively twice greater than $\Delta \Psi_{AB}$, a feature already pointed out in Ref. $4$. In addition, it may be noted that the gravitational contribution of order $(m_2^2 - m_1^2)c^4/E^2$ in Eq. (39) is just equal to twice the contribution of the same order appearing in Eq. (35).

### 3.2. Neutrinos in the field of a spherically symmetric body

According to $35$ and $39$, an explicit calculation of the oscillation terms $\Delta \Psi_{AB}$ or $\Delta \Phi_{AB}$ only requires an integration of the Newtonian potential $U$ along the line defined by (20). As an example, we consider the gravitational field generated by an isolated, spherically symmetric body having a mass $M$ and a radius $r_0$. For the sake of simplicity, this body is assumed to be homogeneous. We shall only examine the case where the path of neutrinos is entirely outside the body and the case where the emission is located inside the body. We do not treat the case where the neutrinos are emitted outside the body but are going through the body. Indeed, the present formalism using the world function is not adapted to such a configuration, in which multiple-path propagations may occur.

\[\text{Note that the gravitational contribution of order } (m_2^2 - m_1^2)c^8/E^4 \text{ was calculated in Refs. } 4 \text{ and } 22 \text{ under the same assumptions.}\]
3.2.1. Trajectories entirely outside the body

In this case, $U(r) = GM/r$ at any point of the straight line defined by Eq. (20). As a consequence

$$\int_0^1 U(x_{(0)}(\lambda))d\lambda = \frac{GM}{R_{AB}} \ln \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \quad (40)$$

where $r_A = |x_A|$ and $r_B = |x_B|$ (cf., e.g., Ref. [11] and Refs. therein). Inserting Eq. (40) into Eq. (35) yields

$$\Delta \Psi_{AB} = D_{AB} \frac{(m_2^2 - m_1^2)c^3}{2\hbar E_B} \left[ 1 + \frac{(m_2^2 + m_3^2)c^4}{4E_B^2} + \frac{(m_1^2 + m_2^2m_3^2 + m_4^2)c^8}{8E_B^4} + \cdots \right] + \frac{GM(m_2^2 - m_1^2)c^3}{2\hbar E_B} \left[ 1 + \frac{3(m_1^2 + m_2^2)c^4}{4E_B^2} + \cdots \right] \times \left[ \frac{R_{AB}}{r_B} - \ln \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right]. \quad (41)$$

The expression for a radial propagation outside the central body is obtained by setting $R_{AB} = |r_B - r_A|$ in Eq. (41).

In the following subsections it will be useful to introduce the zeroth-order “distance of closest approach” between the straight line passing through $x_A$ and $x_B$ and the origin $O$ of coordinates $x^i$, namely

$$r_c = \frac{|x_A \times x_B|}{R_{AB}} \quad (42)$$

So let us express the relevant quantities in terms of $r_A$, $r_B$ and $r_c$. Some calculations show that

$$\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} = \frac{r_B + N_{AB} \cdot x_B}{r_A + N_{AB} \cdot x_A} \quad (43)$$

where $N_{AB} = (x_B - x_A)/R_{AB}$. Then an elementary geometric reasoning shows that Eq. (43) reads

$$\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} = \frac{r_B + \epsilon \sqrt{r_B^2 - r_c^2}}{r_A + \epsilon \sqrt{r_A^2 - r_c^2}} \quad (44)$$

where

$$\epsilon = \text{sign}(x_A \cdot N_{AB}), \quad \epsilon' = \text{sign}(x_B \cdot N_{AB}). \quad (45)$$

Moreover, with the above definition of $\epsilon$, $R_{AB}$ is given by

$$R_{AB} = \epsilon' \sqrt{r_B^2 - r_c^2} - \epsilon \sqrt{r_A^2 - r_c^2}. \quad (46)$$

Using Eqs. (44) and (46), the last line in Eq. (41) may be rewritten as

$$\frac{R_{AB}}{r_B} - \ln \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} = \frac{\epsilon' \sqrt{r_B^2 - r_c^2} - \epsilon \sqrt{r_A^2 - r_c^2}}{r_B} - \ln \frac{r_B + \epsilon' \sqrt{r_B^2 - r_c^2}}{r_A + \epsilon \sqrt{r_A^2 - r_c^2}} \quad (47)$$

It may be noted that the expression for a radial propagation is now obtained by setting $r_c = 0$. 

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3.2.2. Emission located inside the massive body and reception located outside

In order to simplify the evaluation of the integral in the R.H.S. of (41), it is useful to introduce the new parameter \( \bar{x} \) defined by

\[
\bar{x} = \lambda + \frac{N_{AB} \cdot x_A}{R_{AB}},
\]

the range of \( \bar{x} \) being \( \bar{x}_0 \leq \bar{x} \leq \bar{x}_o \), where \( \bar{x}_0 = N_{AB} \cdot x_A/R_{AB} \) and \( \bar{x}_o = N_{AB} \cdot x_0/R_{AB} \).

Denoting by \( \lambda_0 \) the value of \( \lambda \) corresponding to the intersection of the segment connecting \( x_A \) and \( x_0 \) with the sphere \( r = r_0 \), we have

\[
\int_{\lambda_0}^{\lambda} U(x, \lambda) d\lambda = \int_{\lambda_0}^{\lambda} \frac{GM}{2r_0} \left[ 3 - \frac{|x_0(\lambda)|^2}{r_0^2} \right] d\bar{x} + \int_{\lambda_0}^{\lambda} \frac{GM}{|x_0(\lambda)|} d\bar{x}.
\]

Using \( \lambda_0 = \sqrt{r_0^2 - r_c^2}/R_{AB} \), a lengthy but straightforward calculation leads to

\[
\Delta \Psi_{AB} = D_{AB} \left[ \left( \frac{m_1^2 - m_2^2}{2hE_0} \right)^2 + \frac{(m_1^2 + m_2^2)c^4}{4E_0^2} + \frac{(m_1^2 + m_2^2m_2^2 + m_2^4)c^8}{8E_0^2} + \ldots \right]
\]

\[
\times \left[ \frac{R_{AB}}{r_B} - \sqrt{r_0^2 - r_c^2} - \epsilon \sqrt{r_0^2 - r_c^2} \right]
\]

\[
\times \left( \frac{4}{3} - \frac{r_0^2 + r_c^2 - \epsilon \sqrt{r_0^2 - r_c^2}}{6r_0^2} \right)
\]

\[
- \ln \left( \frac{r_B + \sqrt{r_0^2 - r_c^2}}{r_0 + \sqrt{r_0^2 - r_c^2}} \right),
\]

where \( R_{AB} \) may be replaced by the R.H.S. of Eq. (46) with \( \epsilon' = 1 \).

3.3. Comparison with previous results for non-radial propagation

Among all the above-mentioned papers, the most coherent study of the gravitational quantum phase shift for a non-radial timelike geodesic in an exterior Schwarzschild spacetime is given in Ref. [7]. In what follows, we want to compare the expression of \( \Delta \Psi_{AB} \) found in this work for a non-radial trajectory with our formula [23] written outside a spherically symmetric massive body. To do this, we have to calculate \( R_{AB} \) as a function of \( r_A, r_B \) and the true value \( r_p \) of the radial coordinate at the pericenter of the trajectory. We can proceed to this calculation by determining the relation between \( r_p \) and \( r_c \). It will be assumed that \( r_p \gg GM/c^2 \).

In isotropic spherical coordinates \((x^0, r, \vartheta, \varphi)\), the metric [38] may be written in the form

\[
ds^2 = \left( 1 - \frac{2GM}{c^2r} \right) (dx^0)^2 - \left( 1 + \frac{2\gamma GM}{c^2r} \right) (dr^2 + r^2d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2).
\]

It is well known that the trajectory is in a plane passing through the origin \( O \). As a consequence, the axes may be chosen in such a way that \( \vartheta = \pi/2 \) during the

\[\text{dSome authors carry out their calculations along null geodesics (see, e.g., Refs. 14 and 23).}\]
propagation. Then, using the Euler-Lagrange equations for the geodesics of (49), it is easily seen that the path of a particle of mass $m$ is determined by an equation as follows

$$\left(\frac{dr}{d\varphi}\right)^2 = r^2 \left[\frac{r^2}{b^2} + \frac{2GM}{c^2 b} \left(\frac{E^2}{E^2 - m^2 c^4}\right) \frac{b}{r^2} - 1\right],$$  

(50)

where $b$ is the impact parameter of the particle, defined as the Euclidean distance which could be measured by an inertial observer at rest at infinity between the asymptote to the ray and the line parallel to this asymptote passing through the center $O$. The pericenter $P$ being defined as the point where $dr/d\varphi = 0$, the value of $r$ at this point is immediately deduced from Eq. (50). It is easily seen that $b$ and $r_p$ are related by

$$b = r_p \sqrt{1 + \left(\gamma + \frac{E^2}{E^2 - m^2 c^4}\right) \frac{2GM}{c^2 r_p}}. $$  

(51)

Since we suppose that $r_p \gg GM/c^2$, Eq. (51) reduces to

$$b = r_p \left(\gamma + \frac{E^2}{E^2 - m^2 c^4}\right) \frac{GM}{c^2}. $$  

(52)

Let us now proceed to the calculation of the impact parameter $b$ as a function of $r_A$, $r_B$ and $r_c$. Returning to the quasi Cartesian isotropic coordinates used in Eqs. (33), and introducing the vector notation $\mathbf{p} = (p_1, p_2, p_3)$, where the quantities $p_i$ are the spacelike covariant components of the energy-momentum 4-vector defined in Sect. II, it may be seen from the geodesic equations that the vector $\mathbf{L}$ defined as

$$\mathbf{L} = -\mathbf{x} \times \mathbf{p}$$

remains constant along any geodesic path. For a freely falling particle able to go to infinity, the magnitude of $\mathbf{L}$ is such that $|\mathbf{L}| = \lim_{|x| \to \infty} |\mathbf{x} \times \mathbf{p}| = bE/c^2|(dx/dt)_\infty|$ since $\mathbf{p} \to -(E/c^2)(dx/dt)_\infty$ when $|x| \to \infty$. But it follows from Eq. (25) that $|(dx/cdt)_\infty| = \sqrt{1 - m^2 c^4/E^2}$. As a consequence the impact parameter $b$ may be identified to the constant of the motion given by

$$b = \frac{c}{\sqrt{E^2 - m^2 c^4}} |\mathbf{x} \times \mathbf{p}|. $$  

(53)

According to Eqs. (3), (4) and (12), we have $(\mathbf{p})_A = \hbar \nabla_{x_A} \Psi_{AB}$. As a consequence, taking Eq. (23) into account, we get

$$\langle \mathbf{p} \rangle_B = -\frac{1}{c} \sqrt{E^2 - m^2 c^4} \times \left[\mathbf{N}_{AB} + \left(\gamma + \frac{E^2}{E^2 - m^2 c^4}\right) \frac{GM}{c^2 r_A r_B} (r_A + r_B) \mathbf{n}_{AB} - \frac{R_{AB}}{c^2} (1 + \mathbf{n}_A \cdot \mathbf{n}_B)\right], $$  

(54)

where $\mathbf{n}_A = \mathbf{x}_A/r_A$ and $\mathbf{n}_B = \mathbf{x}_B/r_B$.

This equation generalizes the relation between the conserved angular momentum $\mathbf{L}$ and the impact parameter for a photon (see, e.g., Ref. 24).
Using Eq. (54), and then taking into account Eq. (42), Eq. (55) gives an expression as follows for the impact parameter

\[ b = r_c \left[ 1 + \left( \gamma + \frac{E^2}{E^2 - m^2c^4} \right) \frac{GM}{c^2} \frac{r_a + r_b}{r_a r_b (1 + \mathbf{n}_a \cdot \mathbf{n}_b)} \right]. \]

Comparing now Eq. (55) with Eq. (52) shows that \( r_p \) and \( r_c \) are linked by the relation

\[ r_p = r_c + \left( \gamma + \frac{E^2}{E^2 - m^2c^4} \right) \frac{GM}{c^2 r_c} \left[ \frac{r_p^2 (r_a + r_b)}{r_a r_b (1 + \mathbf{n}_a \cdot \mathbf{n}_b)} - r_c \right]. \]

For shortness, we limit ourselves to the case \( r_b > r_a \), which implies \( \epsilon' = 1 \). Noting that Eq. (42) implies

\[ r_a - r_b (\mathbf{n}_a \cdot \mathbf{n}_b) = -\epsilon \frac{R_{AB}}{r_a} \sqrt{r_a^2 - r_c^2}, \]

\[ r_b - r_a (\mathbf{n}_a \cdot \mathbf{n}_b) = \frac{R_{AB}}{r_b} \sqrt{r_b^2 - r_c^2}, \]

it may be seen that Eq. (53) becomes

\[ r_p = r_c + \left( \gamma + \frac{E^2}{E^2 - m^2c^4} \right) \frac{GM}{c^2 r_c} \left( r_a - r_c \right) \sqrt{r_a^2 - r_c^2} - \epsilon (r_b - r_c) \sqrt{r_a^2 - r_c^2}. \]

Now, inverting Eq. (57) in order to find \( r_c \) as a function of \( r_a, r_b \) and \( r_p \), and then substituting the result into Eq. (50) yield

\[ R_{AB} = \sqrt{r_a^2 - r_c^2} - \epsilon \sqrt{r_a^2 - r_p^2} + \left( \gamma + \frac{E^2}{E^2 - m^2c^4} \right) \frac{GM}{c^2} \left( \frac{r_a - r_p}{r_a + r_p} - \epsilon \frac{r_a - r_p}{r_a + r_b} \right). \]

Substituting for \( R_{AB} \) from Eq. (58) into Eq. (23) gives for \( \Psi_{AB} \):

\[ \Psi_{AB} = -\frac{\sqrt{E^2 - m^2c^4}}{\hbar c} \left\{ \sqrt{r_a^2 - r_b^2} - \epsilon \sqrt{r_a^2 - r_p^2} + \left( \gamma + \frac{E^2}{E^2 - m^2c^4} \right) \frac{GM}{c^2} \right\} \times \frac{r_a - r_b}{r_a + r_b + \epsilon \sqrt{r_a^2 - r_b^2}} \left[ \frac{r_b - r_p}{r_b + r_p} - \epsilon \frac{r_a - r_p}{r_a + r_b} + \frac{r_a + \sqrt{r_a^2 - r_b^2}}{r_a + \epsilon \sqrt{r_a^2 - r_p^2}} \right]. \]

The change of radial coordinates defined as

\[ r = \rho - \frac{\gammaGM}{c^2} \]

transforms the metric (49) into

\[ ds^2 = \left( 1 - \frac{2GM}{c^2 \rho} \right) - \left( 1 + \frac{2\gamma GM}{c^2 \rho} \right) d\rho^2 - \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

up to terms of order \( G^2 \). This metric coincides of course with the linearized Schwarzschild metric when \( \gamma = 1 \).
Noting that
\[ \sqrt{r^2 - r_p^2} = \sqrt{\rho^2 - \rho_p^2} - \gamma \frac{GM}{c^2} \frac{\sqrt{\rho - \rho_v}}{\rho + \rho_v} \]
for any \( r \geq r_p \), and then neglecting terms of order \( m^4 c^8 / E^4 \), Eq. (59) finally gives
\[ \Psi_{AB} = \Psi^*_{AB} + \frac{m^2 c^3}{2 \hbar E} \left\{ \sqrt{\rho^2_B - \rho_p^2} - \epsilon \sqrt{\rho^2_A - \rho_p^2} \right\}, \]
where \( \Psi^*_{AB} \) is a quantity which does not depend on the mass \( m \) and consequently disappears from the expression of \( \Delta \Psi_{AB} \):
\[ \Psi^*_{AB} = -\frac{E}{\hbar c} \left\{ \sqrt{\rho^2_B - \rho_p^2} - \epsilon \sqrt{\rho^2_A - \rho_p^2} + \frac{GM}{c^2} \left\{ \sqrt{\frac{\rho_B - \rho_v}{\rho_B + \rho_v} - \epsilon \sqrt{\frac{\rho_A - \rho_v}{\rho_A + \rho_v}} - (\gamma - 1) \ln \frac{\rho_B + \sqrt{\rho^2_B - \rho_p^2}}{\rho_A + \epsilon \sqrt{\rho^2_A - \rho_p^2}} \right\} \right\} \]
(62)

Putting \( \gamma = 1 \), we recover Eqs. (51) and (52) in Ref. 7 according as \( \epsilon = 1 \) or \( \epsilon = -1 \).

4. Conclusion

Synge’s world function constitutes a powerful tool for determining the quantum phase shift of freely falling particles in any stationary, weak gravitational field considered in the linear approximation. The method works provided the spin effects may be neglected and multiple-path effects are not relevant.

It is shown by the formulas (35) and (37) that the neutrino oscillations are not sensitive to the gravitomagnetic components of the metric and that the post-Newtonian parameter \( \gamma \) is involved in the gravitational phase shift only via the expression of the geodesic distance \( D_{AB} \).

Explicit formulas are written for neutrinos propagating in the field of a homogeneous, spherically symmetric body, see Eqs. (41) and (48). The calculations are straightforward even for neutrinos emitted inside the body and traveling along non-radial geodesics. It is shown that the results previously obtained in Schwarzschild spacetime are direct consequences of our more general formulas. Moreover, it must be pointed that our results are given in terms of the position vectors \( \mathbf{x}_A \) and \( \mathbf{x}_B \), in contrast with the formulas obtained in Ref. 7 where \( r_p \) is not explicitly calculated.

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