Bilinear expansion of Schur functions in Schur $Q$-functions: a fermionic approach

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Abstract

An identity is derived expressing Schur functions as sums over products of pairs of Schur $Q$-functions, generalizing previously known special cases. This is shown to follow from their representations as vacuum expectation values (VEV’s) of products of either charged or neutral fermionic creation and annihilation operators, Wick’s theorem and a factorization identity for VEV’s of products of two mutually anti-commuting sets of neutral fermionic operators.

1 Introduction

Fermionic methods are central to Sato’s construction of $\tau$-functions for the KP infinite integrable hierarchy, as well as the BKP hierarchy \cite{3, 8, 11}. In this work, we make use of the relations between charged and neutral fermionic operators to derive a bilinear identity relating Schur functions \cite{9}, which are the basic building blocks for solutions of the KP hierarchy \cite{6, 11}, to Schur’s $Q$-functions, which play a similar rôle with respect to the BKP hierarchy \cite{3, 12}.

An identity was derived in \cite{1}, expressing determinants of submatrices of skew matrices as sums over products of the Pfaffians of their principal minors. Geometrically, this may be interpreted as a bilinear relation between the Plücker map, which embeds Grassmannians of $k$-dimensional subspaces of a given vector space into the projectivization of the space of exterior $k$-forms, and the Cartan map \cite{2}, which embeds the Grassmannian of maximal isotropic subspaces with respect to a complex scalar product into the projectivization of the irreducible spinor modules. The result in \cite{1} was based on Cartan’s construction of bilinear forms on Clifford modules, with values in spaces of homogeneous exterior forms. The main result derived here is Theorem 5.1, Section 5), which may be viewed as a

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function theoretic realization of this identity, with the determinant identified, through the Jacobi-Trudi identity \[9\] with the Schur function, and the Pfaffians, with Schur’s Q-functions.

Section 2 recalls the definition of creation and annihilation operators on fermionic Fock space as linear generators of an infinite dimensional Clifford algebra. Two mutually anticommuting subalgebras generated by neutral fermions are defined, and a key factorization Lemma 2.3 given for vacuum state expectation values (VEV’s) of products of linear elements. The representation of Schur functions and Schur Q-functions as VEV’s of products of creation and annihilation operators conjugated by elements of the infinite abelian groups that generate KP and BKP flows is recalled in Section 3. Section 4 introduces the notion of polarizations associated to integer partitions, and the associated products of neutral fermion operators determined by binary sequences. The main result is Theorem 5.1, Section 5, which gives an expression for Schur functions, evaluated on the odd KP flow variables, as sums over products of pairs of Schur Q-functions, generalizing certain previously known special cases \[9\] to arbitrary Schur functions.

2 Fermionic Fock space

2.1 Charged and neutral fermions

For a separable Hilbert space \( \mathcal{H} \), with orthonormal basis \( \{ e_j \}_{j \in \mathbb{Z}} \), the corresponding fermionic Fock space \( \mathcal{F} \) is defined as the semi-infinite wedge product space \[6, 11\]:

\[ \mathcal{F} = \Lambda^{\infty/2} \mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n, \]  

(2.1)

with elements denoted as ket vectors \( |w\rangle \). The sector \( \mathcal{F}_n \) with fermionic charge \( n \in \mathbb{Z} \) has an orthonormal basis, denoted \( \{ |\lambda; n\rangle \} \), labelled by pairs \( (\lambda, n) \) of an integer partition \( \lambda \) and the integer \( n \), defined by:

\[ |\lambda; n\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots, \]  

(2.2)

where the infinite sequence of integers \( \{l_i\}_{i \in \mathbb{N}^+} \), called particle positions, are related to the parts \( (\lambda_1 \geq \lambda_2 \cdots, \lambda_{\ell(\lambda)}, 0, \ldots) \) of \( \lambda \) by

\[ l_i := \lambda_i - i + n, \quad i \in \mathbb{N}^+. \]  

(2.3)

The length \( \ell(\lambda) \) of the partition \( \lambda \) is the number of positive parts \( \{\lambda_i\}_{i=1, \ldots, \ell(\lambda)} \) and the finite sequence is completed by adding an infinite sequence of 0’s following these. Its weight is

\[ |\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i. \]  

(2.4)

The particle positions \( \{l_i\}_{i \in \mathbb{N}^+} \), form a strictly decreasing sequence that saturates, after \( \ell(\lambda) \) terms, to all subsequent consecutively decreasing integers.
The algebra of fermionic operators on $\mathcal{F}$ form an irreducible representation

$$\Gamma : \mathcal{C}_{H+H^*}(Q) \to \text{End}(\mathcal{F})$$

$$\Gamma : \xi \mapsto \Gamma_\xi$$

(2.5)
of the Clifford algebra $\mathcal{C}_{H+H^*}(Q)$ on $H+H^*$ corresponding to the (complex) scalar product $Q$ defined by

$$Q(u + \mu, v + \nu) := \mu(v) + \nu(u), \quad u, v \in H, \; \mu, \nu \in H^*.$$  

(2.6)

These are realized as endomorphisms of $\mathcal{F}$, with the linear elements acting by exterior and interior multiplication:

$$\Gamma_v = v \wedge, \quad \Gamma_\mu = i_\mu, \quad v \in H, \; \mu \in H^*.$$  

(2.7)

Denoting the dual basis for $H^*$ as $\{e^j\}_{j \in \mathbb{Z}}$, with

$$e^j(e_k) = \delta_{jk},$$

(2.8)

$\mathcal{C}_{H+H^*}(Q)$ is generated by the scalars and linear elements, with the representations of the basis elements denoted

$$\psi_j := e_j \wedge, \quad \psi^\dagger_j := i_{e^j}, \quad j \in \mathbb{Z}.$$  

(2.9)

These are referred to as (charged) fermionic creation and annihilation operators, respectively, and satisfy the anticommutation relations:

$$[\psi_j, \psi_k]_+ = [\psi^\dagger_j, \psi^\dagger_k]_+ = 0, \quad [\psi_j, \psi^\dagger_k]_+ = \delta_{jk}.$$  

(2.10)

The vacua $|n\rangle$ in each charge sector $\mathcal{F}_n$ is the basis element corresponding to the trivial partition $\lambda = \emptyset$:

$$|n\rangle := |\emptyset; n\rangle = e_{n-1} \wedge e_{n-2} \wedge \cdots.$$  

(2.11)

Elements of the dual space $\mathcal{F}^*$ are denoted as bra vectors $\langle w |$, with the dual basis $\{\langle \lambda; n |\}$ for $\mathcal{F}_n^*$ defined by the pairing

$$\langle \lambda; n | \mu; m \rangle = \delta_{\lambda \mu} \delta_{nm}.$$  

(2.12)

For KP $\tau$-functions, we need only consider the $n = 0$ charge sector $\mathcal{F}_0$, and generally drop the charge $n$ symbol, denoting the basis elements simply as

$$|\lambda\rangle := |\lambda; 0\rangle.$$  

(2.13)

For $j > 0$, $\psi_{-j}$ and $\psi^\dagger_{j-1}$ (resp. $\psi^\dagger_{-j}$ and $\psi_{j-1}$) annihilate the right (resp. left) vacua:

$$\psi_{-j}|0\rangle = 0, \quad \psi^\dagger_{j-1}|0\rangle = 0, \quad \forall j > 0,$$

(2.14)

$$\langle 0 | \psi^\dagger_{-j} = 0, \quad \langle 0 | \psi_{j-1} = 0, \quad \forall j > 0.$$  

(2.15)
Neutral fermions $\phi^+_j$ and $\phi^-_j$ are defined \cite{3} as
\[
\phi^+_j := \frac{\psi_j + (-1)^j \psi^-_j}{\sqrt{2}}, \quad \phi^-_j := \frac{i \psi_j - (-1)^j \psi^+_j}{\sqrt{2}}, \quad j \in \mathbb{Z}
\] (2.16)
(where $i = \sqrt{-1}$), and satisfy
\[
[\phi^+_j, \phi^-_k]_+ = 0, \quad [\phi^+_j, \phi^+_k]_+ = [\phi^-_j, \phi^-_k]_+ = (-1)^j \delta_{j+k,0}.
\] (2.17)
In particular,
\[
(\phi^+_0)^2 = (\phi^-_0)^2 = \frac{1}{4}.
\] (2.18)
Acting on the vacua $|0\rangle$ and $|1\rangle$, we have
\[
\phi^+_j |0\rangle = \phi^-_j |0\rangle = \phi^+_j |1\rangle = \phi^-_j |1\rangle = 0, \quad \forall j > 0, \quad \forall j > 0,
\] (2.19)
\[
\langle 0 | \phi^+_j = \langle 0 | \phi^-_j = \langle 1 | \phi^+_j = \langle 1 | \phi^-_j = 0, \quad \forall j > 0,
\] (2.20)
\[
\phi^+_0 |0\rangle = -i \phi^-_0 |0\rangle = \frac{1}{\sqrt{2}} \psi_0 |0\rangle = \frac{1}{\sqrt{2}} |1\rangle,
\] (2.21)
\[
\langle 0 | \phi^+_0 = i \langle 0 | \phi^-_0 = \frac{1}{\sqrt{2}} \langle 0 | \psi^*_0 = \frac{1}{\sqrt{2}} \langle 1 |.
\] (2.22)
The pairwise expectation values are:
\[
\langle 0 | \phi^+_j \phi^+_k |0\rangle = \langle 0 | \phi^-_j \phi^-_k |0\rangle = \begin{cases} (-1)^k \delta_{j,-k} & \text{if } k > 0, \\ \frac{1}{2} \delta_{j,0} & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases}
\] (2.23)
\[
\langle 0 | \phi^+_j \phi^-_k |0\rangle = -\langle 0 | \phi^-_j \phi^+_k |0\rangle = \frac{i}{2} \delta_{j,k,0}.
\] (2.24)

2.2 Fermionic Wick theorem

For an even number of fermionic operators $(w_1, \ldots, w_{2L})$ that anticommute:
\[
[w_j, w_k]_+ = 0, \quad 1 \leq j, k \leq 2L,
\] (2.25)
the matrix with elements $\langle 0 | w_j w_k |0\rangle$ is skew symmetric, and Wick’s theorem implies that the vacuum state expectation value $\langle 0 | w_1 \cdots w_{2L} |0\rangle$ of the product is given by the Pfaffian
\[
\langle 0 | w_1 \cdots w_{2L} |0\rangle = \text{Pf} \left( \langle 0 | w_j w_k |0\rangle \right)_{1 \leq j, k \leq 2L}.
\] (2.26)

On the other hand, if the odd elements $w_1, w_3, \ldots$ are linear combinations of creation operators $\{ \psi_j \}_{j \in \mathbb{Z}}$ and the even ones $w_2, w_4, \ldots$ linear combinations of annihilation operator $\{ \psi^*_j \}_{j \in \mathbb{Z}}$, Wick’s theorem implies
\[
\langle 0 | w_1 \cdots w_{2L} |0\rangle = \det \left( \langle 0 | w_j w_k |0\rangle \right)_{j=1,3,\ldots; k=2,4,\ldots}.
\] (2.27)
2.3 Current components and a factorization lemma

The positive current components of charged fermions, defined as

\[ J_n = \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+n}^\dagger, \quad n \in \mathbb{N}^+, \quad (2.28) \]

commute amongst themselves

\[ [J_n, J_m] = 0, \quad n, m \in \mathbb{N}^+, \quad (2.29) \]

and generate the KP flows \([6, 11]\).

The neutral fermion current components \(J^B_n\) and \(\hat{J}^B_n\) are defined as

\[
J^B_+ := \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^{j+1} \phi_j^+ \phi_{j-n}^+, \quad J^-_n := \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^{j+1} \phi_j^- \phi_{j-n}^-; \quad n \in \mathbb{N}^+. \quad (2.30)
\]

The even components \(\{J^B_{2p}, J^-_{2p}\}\) all vanish, while the odd ones mutually commute:

\[
[J^B_{2p-1}, J^B_{2p+1}] = 0, \quad [J^-_{2p-1}, J^-_{2p+1}] = 0, \quad [J^B_{2p-1}, J^-_{2q-1}] = 0, \quad p, q \in \mathbb{N}^+, \quad (2.31)
\]

and both generate BKP flows \([3, 4, 6]\).

By (2.14), the positive current components \(J_n\) annihilate the vacuum \(|0\rangle\):

\[ J_n|0\rangle = 0, \quad n \in \mathbb{N}^+ \quad (2.32) \]

and, by (2.19), the neutral current components \(J^B_n\) and \(\hat{J}^B_n\) annihilate the vacua \(|0\rangle\) and \(|1\rangle\):

\[ J^B_+|0\rangle = 0, \quad J^-_n|0\rangle = 0, \quad J^B_+|1\rangle = 0, \quad J^-_n|1\rangle, \quad n \in \mathbb{N}^+. \quad (2.33) \]

It also follows from (2.28), (2.30) and

\[
\psi_j = \frac{\phi_j^+ - i\phi_j^-}{\sqrt{2}}, \quad \psi_{-j} = (-1)^j \frac{\phi_j^+ + i\phi_j^-}{\sqrt{2}}, \quad (2.34)
\]

that:

**Lemma 2.1.** For odd \(n = 2p - 1\),

\[ J_{2p-1} = J^B_{2p-1} + J^-_{2p-1}, \quad p \in \mathbb{N}^+. \quad (2.35) \]

The following factorization property of VEV’s is proved in \([6]\) and \([12]\).

**Lemma 2.2.** If \(a^+\) and \(a^-\) are (finite or infinite) sums of monomials in the \(\{\phi^+_i\}_{i \neq 0}\) of even and odd degrees, respectively, and \(\hat{a}^+\) and \(\hat{a}^-\) are sums of monomials in the \(\{\phi^-_i\}_{i \neq 0}\) of even and odd degrees, respectively, then

\[
\langle 0|(a^+ + \phi^+_0 a^-)(\hat{a}^+ + \phi^+_0 \hat{a}^-)|0\rangle = \langle 1|(a^+ + \phi^-_0 a^-)(\hat{a}^+ + \phi^-_0 \hat{a}^-)|1\rangle
\]

\[ = \langle 0|a^+|0\rangle \langle 0|\hat{a}^+|0\rangle. \quad (2.36) \]
As an immediate corollary, it follows that:

**Lemma 2.3 (Factorization).** If \( (u_1^-, \ldots, u_n^+) \) and \( (u_1^-, \ldots, u_m^-) \) are linear combinations of the operators \( \{ \phi_i^+ \}_{i \in \mathbb{Z}} \) and \( \{ \phi_i^- \}_{i \in \mathbb{Z}} \) respectively, the VEV of their product can be factorized as:

\[
\langle 0 | u_1^+ \cdots u_n^+ u_1^- \cdots u_m^- | 0 \rangle = \begin{cases} 
\langle 0 | u_1^+ \cdots u_n^+ | 0 \rangle \langle 0 | u_1^- \cdots u_m^- | 0 \rangle & \text{if } n \text{ and } m \text{ are even} \\
0 & \text{if } n \text{ and } m \text{ have different parity} \\
2i \langle 0 | u_1^+ \cdots u_n^+ \phi_0^+ | 0 \rangle \langle 0 | u_1^- \cdots u_m^- \phi_0^- | 0 \rangle & \text{if } n \text{ and } m \text{ are odd}.
\end{cases}
\] (2.37)

In particular, for odd \( n \) and \( m \), \( \langle 0 | u_1^+ \cdots u_n^+ \phi_0^+ | 0 \rangle \) and \( \langle 0 | u_1^- \cdots u_m^- \phi_0^- | 0 \rangle \) vanish unless the terms in the products \( u_1^+ \cdots u_n^+ \) and \( u_1^- \cdots u_m^- \) that are linear in \( \phi_0^+ \) and \( \phi_0^- \), respectively, are nonzero.

### 3 Schur functions and Schur Q-functions via fermions.

#### 3.1 Fermionic representation of KP flows and Schur functions.

Let \( t = (t_1, t_2, t_3, \ldots) \) denote the infinite sequence of KP flow parameters, and define the abelian group \( \Gamma_+ \) of KP flows \( \{ \gamma_+(t) := e^{\sum_{i=1}^{\infty} t_i \Lambda^i} \} \), where \( \Lambda \in \text{End}(\mathcal{H}) \) is the upward shift element

\[
\Lambda(e_i) = e_{i-1}. \tag{3.1}
\]

These act on \( \mathcal{F} \) via the fermionic representation

\[
\tilde{\gamma}_+(t) := e^{\sum_{n=1}^{\infty} J_n t_n}. \tag{3.2}
\]

and, by (2.32), they stabilize the vacuum element

\[
\tilde{\gamma}_+(t) | 0 \rangle = | 0 \rangle. \tag{3.3}
\]

We define \( \psi_j(t), \psi_j^\dagger(t) \) to be the conjugates of \( \psi_j \) and \( \psi_j^\dagger \) by \( \tilde{\gamma}_+(t) \).

\[
\psi_j(t) := \tilde{\gamma}_+(t) \psi_j(\tilde{\gamma}_+(t))^{-1}, \quad \psi_j^\dagger(t) := \tilde{\gamma}_+(t) \psi_j^\dagger(\tilde{\gamma}_+(t))^{-1}. \tag{3.4}
\]

It follows that these satisfy the same anticommutation relations (2.10) as \( \{ \psi_j, \psi_j^\dagger \}_{j \in \mathbb{Z}} \):

\[
[\psi_j(t), \psi_k(t)]_+ = [\psi_j^\dagger(t), \psi_k^\dagger(t)]_+ = 0, \quad [\psi_j(t), \psi_k^\dagger(t)]_+ = \delta_{jk}. \tag{3.5}
\]

Let \( (\alpha|\beta) \) be the Frobenius notation for a partition \( \lambda \) with \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and \( \beta = (\beta_1, \ldots, \beta_r) \) the “arm” and “leg” lengths, respectively, along the principal diagonal of the corresponding Young diagram, and \( r \) the Frobenius rank (i.e., the number of elements on the principal diagonal). Viewed as functions of the normalized power sum symmetric functions \( \{ x_a \}_{a \in \mathbb{N}} \) in some auxiliary set of variables \( \{ x_a \}_{a \in \mathbb{N}} \)

\[
t_i := \frac{1}{i!} p_i = \frac{1}{i} \sum_{a=1}^{\infty} x_a^i. \tag{3.6}
\]
we may express the corresponding Schur function \( s_{(\alpha|\beta)}(t) \) [9], viewed as functions of the normalized power sums \( t \), defined in (3.6), as the following vacuum state expectation values [6,11]

\[
s_{(\alpha|\beta)}(t) = (-1)^{\sum_{j=0}^{r} \beta_j} \frac{1}{2^r (r-1)!} \langle 0 | \psi_{\alpha_1}(t) \cdots \psi_{\alpha_r}(t) \psi_{-\beta_{r-1}}(t) \cdots \psi_{-\beta_1}(t) | 0 \rangle
\]

\[
= (-1)^{\sum_{j=1}^{r} \beta_j} \prod_{j=1}^{r} \left( \psi_{\alpha_j}(t) \psi_{-\beta_j}(t) \right) | 0 \rangle,
\]

where, by (3.5),

\[
[\psi_{\alpha_k}(t), \psi_{-\beta_j}(t)]_+ = 0, \quad \forall j, k \in \mathbb{Z}.
\]  

**Remark 3.1. Giambelli identity.** Applying Wick's theorem (2.27) to the right hand side of (3.7) gives the Giambelli identity [9]

\[
s_{(\alpha|\beta)}(t) = \det \left( (-1)^{\beta_j} \langle 0 | \psi_{\alpha_k}(t) \psi_{-\beta_j}(t) | 0 \rangle \right)_{1 \leq j, k \leq r}
\]

expressing \( s_{(\alpha|\beta)}(t) \) as the determinant of the \( r \times r \) matrix formed from the hook partition Schur functions for each pair of Frobenius indices.

### 3.2 Fermionic representation of BKP flows and Schur Q-functions

Denoting the set of odd flow variables as

\[
t_B = (t_1, t_3, t_5, \ldots),
\]

these may be viewed as determining a subset \( \{t'\} \) of the \( t \)'s, where

\[
t' := (t_1, 0, t_3, 0, t_5, \ldots).
\]

Following [3,4,12], we define two mutually commuting abelian groups of BKP flows \( \Gamma^B = \{ \gamma^B(t_B^+) \} \) and \( \Gamma^B = \{ \gamma^B(t_B) \} \), with Clifford representations

\[
\hat{\gamma}^{B+}(t_B) := e^{\sum_{p=0}^{\infty} J_{2p+1} t_{2p+1}}, \quad \hat{\gamma}^{B-}(t_B) := e^{\sum_{q=1}^{\infty} J_{2q-1} t_{2q-1}}.
\]

and note that, by (2.33), these stabilize both the vacua \( |0\rangle \) and \( |1\rangle \)

\[
\hat{\gamma}^{B+}(t_B) |0\rangle = |0\rangle, \quad \hat{\gamma}^{B+}(t_B) |1\rangle = |1\rangle,
\]

\[
\hat{\gamma}^{B-}(t_B) |0\rangle = |0\rangle, \quad \hat{\gamma}^{B-}(t_B) |1\rangle = |1\rangle.
\]

Defining

\[
\phi_j^+(t_B) := \hat{\gamma}^{B+}(t_B) \phi_j^+ \left( \hat{\gamma}^{B+}(t_B) \right)^{-1}, \quad \phi_j^-(t_B) := \hat{\gamma}^{B-}(t_B) \phi_j^- \left( \hat{\gamma}^{B-}(t_B) \right)^{-1},
\]

it follows that these satisfy the same anticommutation relations \( (2.17) \) as \( \{ \phi_j^+, \phi_k^- \}_{j,k \in \mathbb{Z}} \):

\[
[\phi_j^+(t_B), \phi_k^-(t_B)]_+ = 0, \quad [\phi_j^+(t_B), \phi_k^+(t_B)]_+ = [\phi_j^-(t_B), \phi_k^-(t_B)]_+ = (-1)^j \delta_{j+k,0}.
\]

From eq. (2.35), we have
Lemma 3.1.
\[
\psi_j(t') = \frac{\phi_j^+(t_B) - i\phi_j^-(t_B)}{\sqrt{2}}, \quad \psi_j^+(t') = (-1)^j \frac{\phi_j^+(t_B) + i\phi_j^-(t_B)}{\sqrt{2}}.
\] (3.16)

Remark 3.2. It follows from eq. (2.33) that (2.19) - (2.21) are still valid if and for completeness,
\[
\phi_0^+(t_B)|0\rangle = \frac{1}{2}|1\rangle, \quad \phi_0^-(t_B)|0\rangle = \frac{i}{2}|1\rangle.
\] (3.17)

From (3.16), it follows that (3.7) may equivalently be expressed as:

Lemma 3.2.
\[
s_{(\alpha|\beta)}(t') = (-1)^{\frac{1}{2}r(r+1)2^{-r}}(0|\phi_{\alpha_1}^+(t_B) - i\phi_{\alpha_1}^-(t_B)) \cdots (\phi_{\alpha_r}^+(t_B) - i\phi_{\alpha_r}^-(t_B))
\times (\phi_{\beta_{r+1}}^+(t_B) + i\phi_{\beta_{r+1}}^-(t_B)) \cdots (\phi_{\beta_{2r-1}}^+(t_B) + i\phi_{\beta_{2r-1}}^-(t_B)) |0\rangle.
\] (3.18)

To define Schur’s Q-functions \(Q_\alpha\) we begin, following [9], by defining an infinite skew symmetric matrix \((Q_{ij})_{i,j\in\mathbb{N}}\), whose entries are symmetric functions of the infinite sequence of indeterminates \(x = (x_1, x_2, \ldots)\), via the following formula:
\[
Q_{ij}(x) := \begin{cases} q_i(x)q_j(x) + 2\sum_{k=1}^{\infty}(-1)^k q_{i+k}(x)q_{j-k}(x) & \text{if } (i, j) \neq (0, 0), \\ 0 & \text{if } (i, j) = (0, 0), \end{cases}
\] (3.19)

where the \(q_i(x)\)’s are defined by the generating function:
\[
\prod_{i=1}^{\infty} \frac{1 + zx_i}{1 - zx_i} = \sum_{i=0}^{\infty} z^i q_i(x).
\] (3.20)

In particular
\[
Q_{(j,0)}(x) = -Q_{(0,j)}(x) = q_j(x) \quad \text{for } j \geq 1.
\] (3.21)

For a strict partition \(\alpha\) of even cardinality \(r\) (including a possible zero part \(\alpha_r = 0\)), let \(M_\alpha(x)\) denote the \(r \times r\) skew symmetric matrix with entries
\[
(M_\alpha(x))_{ij} := Q_{\alpha_i,\alpha_j}(x), \quad 1 \leq i, j \leq r.
\] (3.22)

The Schur Q-function is defined as its Pfaffian [9]
\[
Q_\alpha(x) := \text{Pf}(M_\alpha(x))
\] (3.23)
and, for completeness,
\[
Q_\emptyset := 1.
\] (3.24)

Equivalently, these may be viewed as functions of the odd (normalized) power sum symmetric functions \(t_B = (t_1, t_3, \ldots)\)
\[
t_{2i-1} := \frac{1}{2i-1} p_{2i-1}(x) = \frac{1}{2i-1} \sum_{a=1}^{\infty} x_a^{2i-1}, \quad i = 1, 2, \ldots.
\] (3.25)
which we denote
\[ \tilde{q}_j(t_B) := q_j(x), \quad \tilde{Q}_{ij}(t_B) := Q_{ij}(x). \quad (3.26) \]
We then have the fermionic VEV formulae \[3\] [10] [12]:
\[ \tilde{Q}_{\alpha}(1/2 t_B) = 2 r^2 \langle 0 | \phi_{\alpha_1}^+ (t_B) \cdots \phi_{\alpha_r}^+ (t_B) | 0 \rangle, \]
\[ (3.27) \]
\[ = 2 r^2 \langle 0 | \phi_{\alpha_1}^- (t_B) \cdots \phi_{\alpha_r}^- (t_B) | 0 \rangle, \]
\[ (3.28) \]
which follow from the Pfaffian form \((2.26)\) of Wick’s theorem.

### 3.3 Examples: “doubles”, hook partitions and an \( r = 2 \) case

Consider a set of Frobenius indices
\[ \alpha = (\alpha_1, \ldots, \alpha_r), \quad (3.29) \]
with \( \alpha_i > \alpha_{i+1}, \ \alpha_r \geq 0 \) (or strict partitions, with \( \alpha_r = 0 \) allowed as a part). Following [9], let \( \text{DP} \) denote the set of strict partitions (with all parts \( \geq 1 \)). Associated to \( \alpha \) we define the strict partition
\[ I(\alpha) = (I_1(\alpha), \ldots, I_r(\alpha)) \in \text{DP} \quad (3.30) \]
whose parts are obtained form the \( \alpha_i \)'s by shifting upward by 1:
\[ I_i(\alpha) := \alpha_i + 1. \quad (3.31) \]
If a partition \( \lambda \) is related to a strict partition \( I = (I_1, \ldots I_r) \) in such a way that its Frobenius indices are
\[ \lambda = (I_1, \ldots I_r | I_1 - 1, \ldots I_r - 1)), \quad (3.32) \]
it is called [9] the double of \( I \), and denoted \( \lambda = D(I) \).

**Example 3.1.** It is known (see [9], Chapter III, Section 8, example 10 (b)) that
\[ s_{D(\alpha)}(t') = \begin{cases} 2^{-r} \left( \tilde{Q}_\alpha \left( \frac{1}{2} t_B \right) \right)^2 & \text{if } r \text{ is even,} \\ 2^{-r} \left( \tilde{Q}_{\alpha,0} \left( \frac{1}{2} t_B \right) \right)^2 & \text{if } r \text{ is odd,} \end{cases} \quad (3.33) \]
where \( D(\alpha) \) is the double of \( \alpha \). To prove this using fermionic VEV's, we use:
\[ (-1)^{\beta_j+1} \psi_{\alpha_j}(t') \psi_{-\beta_j-1}(t') = -\frac{1}{2} \left( \phi_{\alpha_j}^+ (t_B) - i \phi_{\alpha_j}^- (t_B) \right) \left( \phi_{\beta_j+1}^+ (t_B) + i \phi_{\beta_j+1}^- (t_B) \right), \]
\[ = \frac{1}{2} \left( \phi_{\beta_j+1}^+ (t_B) \phi_{\alpha_j}^+ (t_B) + \phi_{\beta_j+1}^- (t_B) \phi_{\alpha_j}^- (t_B) \right) + \frac{i}{2} \left( \phi_{\beta_j+1}^+ (t_B) \phi_{\alpha_j}^- (t_B) - \phi_{\beta_j+1}^- (t_B) \phi_{\alpha_j}^+ (t_B) \right), \]
\[ (3.34) \]
so for \( \alpha_j = \beta_j + 1 \), we have
\[ (-1)^{\alpha_j} \psi_{\alpha_j}(t') \psi_{-\alpha_j}^\dagger (t') = -i \phi_{\alpha_j}^+ (t_B) \phi_{\alpha_j}^- (t_B). \quad (3.35) \]
From eqs. (3.7) and (3.35) it follows that

\[
\begin{align*}
\sum_{Q} s_{D(\alpha)}(t') &= (-i)^r \langle 0 | \left( \prod_{m=1}^{r} \phi^+_{\alpha_j}(t_B) \phi^-_{\alpha_j}(t_B) \right) | 0 \rangle \\
&= (i)^r (-1)^{\frac{r(r+1)}{2}} \langle 0 | \left( \phi^+_{\alpha_1}(t_B) \cdots \phi^+_{\alpha_r}(t_B) \phi^-_{\alpha_1}(t_B) \cdots \phi^-_{\alpha_r}(t_B) \right) | 0 \rangle. \quad (3.36)
\end{align*}
\]

Applying Lemma 2.3, eqs. (3.27) and (3.28) gives (3.33).

**Example 3.2.** It is also known (see [9], Chapter III, Section 8, example 10 (a, iv) ) that

\[
\sum_{Q} s_{(j|k)}(t') = \frac{1}{2} \left( \tilde{Q}_j(\frac{1}{2}t_B) \tilde{Q}_{k+1}(\frac{1}{2}t_B) - \tilde{Q}_{j,k+1}(\frac{1}{2}t_B) \right) \\
= \frac{1}{2} \left( \tilde{Q}_{(j,0)}(\frac{1}{2}t_B) \tilde{Q}_{(k+1,0)}(\frac{1}{2}t_B) - \tilde{Q}_{j,k+1}(\frac{1}{2}t_B) \tilde{Q}_{0}(\frac{1}{2}t_B) \right). \quad (3.37)
\]

To derive this fermionically, we apply (3.7), which gives:

\[
\begin{align*}
\sum_{Q} s_{(j|k)}(t') &= -\frac{1}{2} \langle 0 | (\phi^+_j(t_B) - i \phi^-_j(t_B)) (\phi^+_k(t_B) + i \phi^-_k(t_B)) | 0 \rangle \\
&= \frac{1}{2} \langle 0 | \phi^+_k(t_B) \phi^+_j(t_B) | 0 \rangle + \frac{1}{2} \langle 0 | \phi^-_k(t_B) \phi^-_j(t_B) | 0 \rangle - \frac{1}{2} \langle 0 | \phi^+_k(t_B) \phi^-_j(t_B) | 0 \rangle \\
&+ \langle 0 | \phi^-_k(t_B) \phi^+_j(t_B) | 0 \rangle - \frac{1}{2} \langle 0 | \phi^-_k(t_B) \phi^-_j(t_B) | 0 \rangle - \frac{1}{2} \langle 0 | \phi^+_k(t_B) \phi^-_j(t_B) | 0 \rangle \\
&= \frac{1}{2} \left( \tilde{Q}_{(j,0)}(\frac{1}{2}t_B) \tilde{Q}_{(k+1,0)}(\frac{1}{2}t_B) - \tilde{Q}_{j,k+1}(\frac{1}{2}t_B) \tilde{Q}_{0}(\frac{1}{2}t_B) \right). \quad (3.38)
\end{align*}
\]

where Lemma 2.3 has been used in the third equality, and eqs. (3.17), (3.27) and (3.28) in the last.

**Example 3.3.** Consider a partition \( \lambda = (\alpha_1, \alpha_2|\beta_1, \alpha_2 - 1) \) of Frobenius rank \( r = 2 \) in which \( \alpha_1 > \beta_1 + 1 > \alpha_2 \) and \( \beta_2 = \alpha_2 - 1 \). Expanding the RHS of eq. (3.18) for this case, collecting the four types of terms:

\[
\begin{align*}
\langle 0 | \phi^+_1(t_B) \phi^+_2(t_B) \phi^-_{\beta_1+1}(t_B) \phi^-_{\beta_2}(t_B) | 0 \rangle, & \quad \langle 0 | \phi^+_1(t_B) \phi^-_{\alpha_2}(t_B) \phi^-_{\beta_1+1}(t_B) \phi^-_{\alpha_2}(t_B) | 0 \rangle, \\
\langle 0 | \phi^+_1(t_B) \phi^+_2(t_B) \phi^-_{\beta_1+1}(t_B) \phi^+_1(t_B) | 0 \rangle, & \quad \langle 0 | \phi^-_{\alpha_1}(t_B) \phi^+_2(t_B) \phi^-_{\beta_1+1}(t_B) \phi^+_1(t_B) | 0 \rangle,
\end{align*}
\]

applying Lemma 2.3 and using eqs. (3.27) and (3.28) gives

\[
\begin{align*}
\sum_{Q} s_{(\alpha_1,\alpha_2|\beta_1,\alpha_2-1)}(t') &= 1 - \frac{1}{2} \left( \tilde{Q}_{(\alpha_1,\alpha_2)}(\frac{1}{2}t_B) \tilde{Q}_{(\beta_1+1,\alpha_2)}(\frac{1}{2}t_B) - \frac{1}{4} \tilde{Q}_{(\alpha_1,\beta_1+1,\alpha_2,0)}(\frac{1}{2}t_B) \tilde{Q}_{(\alpha_2,0)}(\frac{1}{2}t_B) \right). \quad (3.40)
\end{align*}
\]

Following some preparatory definitions in Section 4, Theorem 5.1. Section 5 provides a generalization of these identities expressing \( s_{(\alpha,\beta)}(t') \), for arbitrary partitions \((\alpha|\beta), \) as sums over products of Schur \(Q\)-functions.
4 Polarizations and binary markings

Let \((\alpha|\beta)\) be a partition of Frobenius rank \(r\). Denote the union and intersection of \(\alpha\) with \(I(\beta)\) as

\[
S := \alpha \cap I(\beta), \quad T := \alpha \cup I(\beta),
\]

and their cardinalities (or lengths, when viewed as strict partitions) as

\[
s := \#(S) = \ell(S), \quad t := \#(T) = \ell(T) = 2r - s.
\]

**Definition 4.1. Polarizations.** For any partition \((\alpha|\beta)\) a *polarization* is a pair of strict partitions

\[
\mu := (\mu^+, \mu^-)
\]

(with 0’s allowed as parts), such that the following conditions are satisfied

\[
\mu^+ \cap \mu^- = S = \alpha \cap I(\beta), \quad \mu^+ \cup \mu^- = T = \alpha \cup I(\beta).
\]

Let \(P(\alpha, \beta)\) denote the set of all polarizations corresponding to a partition \((\alpha|\beta)\). The following Lemma shows that the cardinality of \(P(\alpha, \beta)\) is \(2^{2r-2s}\).

**Lemma 4.1.** The number of distinct polarizations \((\mu^+, \mu^-)\) corresponding to a pair of strict partitions \(S \subset T\), with \(T\) of cardinality \(2r - s\) and \(S\) of cardinality \(s\) is \(2^{2r-2s}\).

**Proof.** The total number of elements in \(T\) is \(2r - s\), and the number of these that are in \(S\), and hence in both \(\mu^+\) and \(\mu^-\) is \(s\). The remaining \(2(r-s)\) elements are either in one or the other of the two strict partitions \(\mu^\pm\), but not both, and hence there are \(2^{2(r-s)}\) distinct ways to select the polarization \((\mu^+, \mu^-)\).

Let

\[
m^+(\mu) := \#(\mu^+), \quad m^- (\mu) = \#(\mu^-),
\]

denote the cardinalities of \(\mu^+\) and \(\mu^-\). Their sum is

\[
m^+(\mu) + m^- (\mu) = 2r,
\]

and therefore, they are either both even or both odd. Denoting the cardinalities of the intersections \(\alpha \cap \mu^-\) and \(I(\beta) \cap \mu^-\)

\[
\pi(\mu) := \#(\alpha \cap \mu^-), \quad \tilde{\pi}(\mu) := \#(I(\beta) \cap \mu^-),
\]

it follows that

\[
\pi(\mu) + \tilde{\pi}(\mu) = m^- (\mu) + s.
\]

**Definition 4.2. Binary markings.** For any integer \(j\) between 0 and \(2^{2r} - 1\), we have the associated binary sequence

\[
\epsilon(j) := (\epsilon_1(j), \ldots, \epsilon_{2r}(j)),
\]
where \( \epsilon_k(j) = + \) if the \( k \)th element in the binary representation of \( j \) is 0 and \( \epsilon_k(j) = - \) if the \( k \)th element is 1. We call the product

\[
\phi^{(j)}(\alpha, \beta) := \phi_{\alpha_1}^{(j)} \cdots \phi_{\alpha_r}^{(j)} \phi_{\beta_{r+1}}^{(j)} \cdots \phi_{\beta_{r+s+1}}^{(j)}
\]  

(4.10)

the \( j \)th binary marking of the sequence \( (\alpha, I(\beta)) \).

If any two factors in (4.10) coincide, the product vanishes. The number of these is \( 2^{2r-s}(2^s-1) \), and the number of nonvanishing \( \phi^{(j)}(\alpha, \beta) \)'s is \( 2^{2r-s} \). For the latter, there are \( s \) pairs \( (m, n) \) of type \( (\phi_{\alpha_m}^+, \phi_{\beta_{n+1}}^-) \) and \( (\phi_{\alpha_m}^-, \phi_{\beta_{n+1}}^+) \) where \( \alpha_m = \beta_n + 1 \in S \). By reordering the product (4.10) so that all the \( \phi^+ \)'s appear to the left, and the \( \phi^- \)'s appear to the right, with all the subscripts in each group in decreasing order, every binary sequence \( \epsilon(j) \) for which \( \phi^{(j)}(\alpha, \beta) \neq 0 \) determines a unique polarization \( \mu(j) = (\mu^+(j), \mu^-(j)) \) such that

\[
\phi^{(j)}(\alpha, \beta) =: \pm \phi_{\mu_1^+}^+ \cdots \phi_{\mu_{m(\mu)}^+}^+ \phi_{\mu_1^-}^- \cdots \phi_{\mu_{m(\mu)}^-}^-.
\]  

(4.11)

Of these, the polarization determined by

\[
\epsilon(2^r - 1) = (+, \ldots, +, -, \ldots, -),
\]  

(4.12)

is

\[
(\mu^+(2^r - 1), \mu^-(2^r - 1)) := (\alpha, I(\beta)),
\]  

(4.13)

and this will be referred to as the canonical polarization.

The \( 2^{2r-s} \) binary sequences \( \epsilon(j) \) for which \( \phi^{(j)}(\alpha, \beta) \neq 0 \) may be divided into \( 2^{2r-2s} \) equivalence classes \([\epsilon(j)]\), each containing \( 2^s \) elements, for which the binary markings are equal, within a sign, and hence the polarization is the same:

\[
[\epsilon(j)] = [\epsilon(\tilde{j})] \quad \text{if and only if} \quad \mu(j) = \mu(\tilde{j}).
\]  

(4.14)

There is thus a bijection between the set \( \mathcal{P}(\alpha, \beta) \) of polarizations and the set of equivalence classes \([\epsilon(j)]\) of binary sequences which define (within a sign) the same nonvanishing binary markings \( \phi^{(j)}(\alpha, \beta) \). The \( 2^s \) elements of each equivalence class \([\epsilon(j)]\) are related by interchanging any number of the \( s \) pairs \( (m, n) \) of type

\[
(\phi_{\alpha_m}^+, \phi_{\beta_{n+1}}^-) \leftrightarrow (\phi_{\alpha_m}^-, \phi_{\beta_{n+1}}^+),
\]  

(4.15)

where \( \alpha_m = \beta_n + 1 \in S \).

The set of all \( \alpha_m \)'s and \( \beta_n + 1 \)'s appearing in (4.10) with a + superscript, written in decreasing order, is the strict partition \( \mu^+(j) \) forming the first part of the polarization \( \mu(j) \) and the set of all \( \alpha_m \)'s and \( \beta_n + 1 \)'s in appearing with a – superscript, also written in decreasing order, is the second part \( \mu^-(j) \).

In every equivalence class \( \epsilon(j) \), there is a unique element \( \epsilon_{\alpha_m}(j_0) \), for which

\[
\epsilon_{\alpha_m}(j_0) = +, \quad \forall \alpha_m \in S.
\]  

(4.16)

which will be referred to as the canonical representative.
Remark 4.1. There is one canonical representative $\epsilon(j_0)$ in each equivalence class $[\epsilon(j)]$, so the number of these is equal to the number $2^{s-2s}$ of (nontrivial) equivalence classes, and they are in bijective correspondence with the polarizations $\mu \in \mathcal{P}(\alpha, \beta)$.

Let $\sigma(j)$ denote the parity of the number of $\alpha_m$’s in the binary marking (4.10) that are in $S$ and appear with a superscript $-$, which equals the number of exchanges (4.15) of elements in the product (4.10) defining $\phi(j)(\alpha, \beta)$ needed to convert it to $\phi(j_0)(\alpha, \beta)$

$$\phi(j)(\alpha, \beta) = \sigma(j)\phi(j_0)(\alpha, \beta),$$

(4.17)

In particular, $\sigma(j_0) = 1$.

Definition 4.3. Polarization sign. We define the sign of the polarization $\mu = \mu(j)$, denoted $\text{sgn}(\mu)$ by

$$\phi(j_0)(\alpha, \beta) =: \text{sgn}(\mu) \phi^+_{\mu^+_1} \cdots \phi^+_{\mu^+_s(\mu)} \phi^-_{\mu^-_1} \cdots \phi^-_{\mu^-_s(\mu)}.$$

(4.18)

It follows that

$$\phi(j)(\alpha, \beta) = \sigma(j)\text{sgn}(\mu) \phi^+_{\mu^+_1} \cdots \phi^+_{\mu^+_s(\mu)} \phi^-_{\mu^-_1} \cdots \phi^-_{\mu^-_s(\mu)}.$$

(4.19)

Example 4.1. The partition $\lambda = ((2, 0), (1, 0))$ has

$$\alpha = (2, 0), \quad I(\beta) = (2, 1), \quad S = (2), \quad T = (2, 1, 0), \quad r = 2, \quad s = 1.$$

(4.20)

The elements $j_0$ are 2, 3, 6 and 7. The $2^2 = 4$ distinct associated polarizations are

$$\mu(2) = \mu(8) = (2, 1, 0), \quad \mu(3) = \mu(9) = ((2, 0), (2, 1)), \quad \mu(6) = \mu(12) = ((2, 1), (2, 0)), \quad \mu(7) = \mu(13) = ((2), (2, 1, 0)).$$

(4.21)

The values of $\sigma(j)$ and $\text{sgn}(\mu(j))$ for these are:

$$\begin{align*}
\sigma(2) &= +, \quad \sigma(8) = -, \quad \text{sgn}(\mu(2)) = \text{sgn}(\mu(8)) = +, \\
\sigma(3) &= +, \quad \sigma(9) = -, \quad \text{sgn}(\mu(3)) = \text{sgn}(\mu(9)) = +, \\
\sigma(6) &= +, \quad \sigma(12) = -, \quad \text{sgn}(\mu(6)) = \text{sgn}(\mu(12)) = -, \\
\sigma(7) &= +, \quad \sigma(13) = -, \quad \text{sgn}(\mu(7)) = \text{sgn}(\mu(13)) = +.
\end{align*}$$

(4.22)

The vanishing binary markings $\phi(j)(\alpha, \beta)$ correspond to: $j = 0, 1, 4, 5, 10, 11, 14, 15$.

Definition 4.4. Supplemented partitions. If $\mu$ is a strict partition of cardinality $r$ (with 0 allowed as a part), define the associated supplemented partition $\hat{\mu}$ to be

$$\hat{\mu} := \begin{cases} 
\mu, & \text{if } r \text{ is even,} \\
(\mu, 0), & \text{if } r \text{ is odd.}
\end{cases}$$

(4.23)

Note that $\hat{\mu}$ is always of even cardinality, but not necessarily strict since, if $r$ is odd, it may possibly have two 0 parts ($\mu_r = 0, 0$) at the end. If $m^\pm(\mu)$ are the cardinalities of $\mu^\pm$, we denote by $\hat{m}^\pm(\mu)$ the cardinalities of $\hat{\mu}^\pm$. 
5 Schur functions as sums over products Schur Q-functions

Our main result expresses any Schur function, \( s(\alpha|\beta)(t') \), evaluated at \( t' \), as a sum over products of Schur Q-functions.

**Theorem 5.1.** For any partition \( (\alpha|\beta) = (\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r) \), we have

\[
    s(\alpha|\beta)(t') = \frac{(-1)^{\frac{1}{2}r(r+1)+s}}{2^{2r-s}} \sum_{\mu \in P(\alpha, \beta)} \sgn(\mu)(-1)^{\pi(\mu)}\frac{1}{2}\hat{\mu}^+(\frac{1}{2}t_B)\hat{\mu}^-(\frac{1}{2}t_B). \tag{5.1}
\]

**Proof.** Using Lemma 3.2 expanding the product in (3.18) and using (4.3) and (4.17) gives

\[
    s(\alpha|\beta)(t') = \frac{(-1)^{\frac{1}{2}r(r+1)+s}}{2^r} \sum_{j=0}^{2r-1} (-1)^{\pi(\mu(j))}\hat{m}(-\mu(j)) \sigma(j)\langle 0|\phi(j)(t_B)(\alpha, \beta)|0\rangle
    \times \langle 0|\phi^+_{\mu_1}(t_B)\cdots\phi^+_{\mu_{m^+}(\mu)}(t_B)\phi^-_{\mu_1}(t_B)\cdots\phi^-_{\mu_{m^-(\mu)}(t_B)}|0\rangle. \tag{5.2}
\]

If \( m^+(\mu) \) and \( m^-(\mu) \) are both even, by Lemma 2.3 and eq. (3.27) we have

\[
    \langle 0|\phi^+_{\mu_1}(t_B)\cdots\phi^+_{\mu_{m^+}(\mu)}(t_B)\phi^-_{\mu_1}(t_B)\cdots\phi^-_{\mu_{m^-(\mu)}(t_B)}|0\rangle
    = \langle 0|\phi^+_{\mu_1}(t_B)\cdots\phi^+_{\mu_{m^+}(\mu)}(t_B)|0\rangle\langle 0|\phi^-_{\mu_1}(t_B)\cdots\phi^-_{\mu_{m^-(\mu)}(t_B)}|0\rangle
    = 2^{-r}\hat{Q}_{\mu^+}(\frac{1}{2}t_B)\hat{Q}_{\mu^-}(\frac{1}{2}t_B) = 2^{-r}\hat{Q}_{\mu^+}(\frac{1}{2}t_B)\hat{Q}_{\mu^-}(\frac{1}{2}t_B), \tag{5.3}
\]

and

\[
    \hat{m}(-\mu) = (-1)^{\frac{1}{2}\hat{m}^-(\mu)} = (-1)^{\frac{1}{2}\hat{m}^-(\mu)}. \tag{5.4}
\]

If \( m^+(\mu) \) and \( m^-(\mu) \) are both odd, by Lemma 2.3 and eq. (3.28) we have

\[
    \langle 0|\phi^+_{\mu_1}(t_B)\cdots\phi^+_{\mu_{m^+}(\mu)}(t_B)\phi^-_{\mu_1}(t_B)\cdots\phi^-_{\mu_{m^-(\mu)}(t_B)}|0\rangle
    = 2i\langle 0|\phi^+_{\mu_1}(t_B)\cdots\phi^+_{\mu_{m^+}(\mu)}(t_B)|0\rangle\langle 0|\phi^-_{\mu_1}(t_B)\cdots\phi^-_{\mu_{m^-(\mu)}(t_B)}|0\rangle
    = 2^{-r}i\hat{Q}_{\mu^+}(\frac{1}{2}t_B)\hat{Q}_{\mu^-}(\frac{1}{2}t_B), \tag{5.5}
\]

and

\[
    \hat{m}(-\mu) = (-1)^{\frac{1}{2}\hat{m}^-(\mu)+1} = (-1)^{\frac{1}{2}\hat{m}^-(\mu)}. \tag{5.6}
\]

In both cases, substituting these in (5.2) gives (5.1).

**Remark 5.1.** Note that half the \( 2^{2(r-s)} \) terms in the sum (5.1) are the same as the other half (under the interchange \( (\mu^+, \mu^-) \leftrightarrow (\mu^-, \mu^+) \)), leaving only \( 2^{2r-2s-1} \) distinct terms (except for the case \( s = r \), where there is just one).
5.1 Examples.

Example 5.1 (cf. Example 3.1). Consider the case of a “double”
\[ D(\alpha) = (\alpha_1, \ldots, \alpha_r | \alpha_1 - 1, \cdots, \alpha_r - 1), \]  
for which \( s = r \). The only binary sequence of type \( \epsilon(j_0) \) (i.e. for which none of the elements \( \alpha_m \) corresponds to an upper index \( - \)) is:
\[ \epsilon(2^r - 1) = (+, \cdots, +, - \cdots -), \]  
which gives the canonical polarization
\[ \mu = \mu(2^r - 1) = ((\alpha_1, \ldots, \alpha_r), (\alpha_1, \ldots, \alpha_r)). \]  
For this case we have
\[ \text{sgn}(\mu) = +1, \quad \pi(\mu) = r, \quad \hat{m}^-(\mu) = \begin{cases} r & \text{if } r \text{ is even} \\ r + 1 & \text{if } r \text{ is odd} \end{cases}. \]  
Therefore
\[ (-1)^{\frac{1}{2}(r+1)+s} \frac{1}{2^{2r-s}} \text{sgn}(\mu)(-1)^{\pi(\mu)+\frac{1}{2}\hat{m}^-(\mu)} = \frac{1}{2^r}, \]  
whether \( r \) is even or odd, and eq. (5.1) gives
\[ s_{D(I(\alpha))}(t') = \begin{cases} 2^{-r} \left( \tilde{Q}_\alpha \left( \frac{1}{2} t_B \right) \right)^2 & \text{if } r \text{ is even}, \\ 2^{-r} \left( \tilde{Q}_{(\alpha,0)} \left( \frac{1}{2} t_B \right) \right)^2 & \text{if } r \text{ is odd}, \end{cases} \]  
in agreement with eq. (3.33).

Example 5.2 (cf. Example 3.2). Consider a hook partition \( \lambda = (\alpha_1 | \beta_1) \) with \( \alpha_1 > \beta_1 + 1 \). Then \( S = \emptyset, (r,s) = (1,0) \). Therefore there are four admissible values for \( \mu \):
\[ (\mu^+(0), (\mu^-)(0)) = ((\alpha_1, \beta_1 + 1), (\emptyset)), \quad (\mu^+(1), (\mu^-)(1)) = ((\alpha_1), (\beta_1 + 1)), \]  
\[ (\mu^+(2), (\mu^-)(2)) = ((\beta_1 + 1), (\alpha_1)), \quad (\mu^+(3), (\mu^-)(3)) = ((\emptyset), (\alpha_1, \beta_1 + 1)). \]  
Assuming \( \alpha_1 > \beta_1 + 1 \) to fix the order in the notations and the correct sign factor, the values of \( \text{sgn}(\mu), \pi(\mu) \) and \( \hat{m}^-(\mu) \) for each case is:
\[ \text{sgn}(\mu(0)) = +1, \quad \pi(\mu(0)) = 0, \quad \hat{m}^-(\mu(0)) = 0, \]  
\[ \text{sgn}(\mu(1)) = +1, \quad \pi(\mu(1)) = 0, \quad \hat{m}^-(\mu(1)) = 2, \]  
\[ \text{sgn}(\mu(2)) = -1, \quad \pi(\mu(2)) = 1, \quad \hat{m}^-(\mu(2)) = 2, \]  
\[ \text{sgn}(\mu(3)) = +1, \quad \phi(\mu(3)) = 1, \quad \hat{m}^-(\mu(3)) = 2. \]  
Substituting in eq. (5.1) gives
\[ s_{(\alpha_1|\beta_1)}(t') = \frac{1}{2} \tilde{Q}_{(\alpha_1,0)} \tilde{Q}_{(\beta_1+1,0)} - \frac{1}{2} \tilde{Q}_{(\alpha_1,\beta_1+1)} \tilde{Q}_\emptyset, \]  
which is the same as (3.37).
Example 5.3 (cf. Examples 3.2, 3.3 for \( r = 1, 2 \)). For \( r \geq 1 \), we have the following generalization of Example 5.2. Assume the Frobenius indices satisfy \( \beta_j + 1 = \alpha_k \) for all pairs except possibly one, say \( \alpha_1 > \beta_1 + 1 > \alpha_2 \), as in Example 3.3. Let

\[
\lambda = (\alpha_1, \alpha_2, \ldots, \alpha_r | \beta_1, \alpha_2 - 1, \ldots, \alpha_r - 1),
\]

(5.16)

with \( \alpha_1 > \beta_1 + 1 > \alpha_2 \) if \( r > 1 \). Then \( s = r - 1 \) and there are four possible polarizations \( \{(\mu^+(i), \mu^-(i))\}_{i=1,...,4} \):

\[
\begin{align*}
(\mu^+(2^r - 1), \mu^-(2^r - 1)) & = ((\alpha_1, \alpha_2, \ldots, \alpha_r), (\beta_1 + 1, \alpha_2, \ldots, \alpha_r)), \\
(\mu^+(2^{2r-1} + 2^{r-1} - 1), \mu^-(2^{2r-1} + 2^{r-1} - 1)) & = ((\beta_1 + 1, \alpha_2, \ldots, \alpha_r), (\alpha_1, \alpha_2, \ldots, \alpha_r)), \\
(\mu^+(2^{2r-1} + 2^{r-1} - 1), \mu^-(2^{2r-1} + 2^{r-1} - 1)) & = ((\beta_1, \alpha_3, \ldots, \alpha_r), (\alpha_1, \beta_1 + 1, \alpha_2, \ldots, \alpha_r)), \\
(\mu^+(2^{r-1} - 1), \mu^-(2^{r-1} - 1)) & = ((\alpha_1, \beta_1 + 1, \alpha_2, \ldots, \alpha_r), (\alpha_2, \ldots, \alpha_r)).
\end{align*}
\]

(5.17)

The corresponding values of \( \text{sgn}(\mu), \pi(u) \) and \( \hat{m}^-(\mu) \) are:

\[
\begin{align*}
\text{sgn}(\mu(2^r - 1)) & = 1, \quad \pi(\mu(2^r - 1)) = r - 1, \quad \hat{m}^-(\mu(2^r - 1)) = \hat{r}, \\
\text{sgn}(\mu(2^{2r-1} + 2^{r-1} - 1)) & = -1, \quad \pi(\mu(2^{2r-1} + 2^{r-1} - 1)) = r, \quad \hat{m}^-(\mu(2^{2r-1} + 2^{r-1} - 1)) = \hat{r}, \\
\text{sgn}(\mu(2^{2r-1} + 2^{r-1} - 1)) & = (-1)^{r-1}, \quad \pi(\mu(2^{2r-1} + 2^{r-1} - 1)) = r, \quad \hat{m}^-(\mu(2^{2r-1} + 2^{r-1} - 1)) = r + 1, \\
\text{sgn}(\mu(2^{r-1} - 1)) & = (-1)^{r-1}, \quad \pi(\mu(2^{r-1} - 1)) = r - 1, \quad \hat{m}^-(\mu(2^{r-1} - 1)) = r - 1.
\end{align*}
\]

(5.18)

Substituting these in eq. (5.1) gives

\[
\begin{align*}
s(\alpha_1, \ldots, \alpha_r | \beta_1, \alpha_2 - 1, \ldots, \alpha_r - 1)(t') = \frac{1}{2^r} (\hat{Q}_{\mu^+(2^r - 1)}(\frac{1}{2} t_B) \hat{Q}_{\mu^-(2^r - 1)}(\frac{1}{2} t_B)) \\
- \hat{Q}_{\mu^+(2^{r-1} - 1)}(\frac{1}{2} t_B) \hat{Q}_{\mu^-(2^{r-1} - 1)}(\frac{1}{2} t_B)),
\end{align*}
\]

(5.19)

in agreement with (3.37) for \( r = 1 \) and (3.40) for \( r = 2 \).

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