Possible solution to the main cosmological constant problem

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Abstract

A modified-gravity-type model of two hypothetical massless vector fields is presented. These vector fields are gravitationally coupled to standard matter and an effective cosmological constant. Considered in a cosmological context, the vector fields dynamically cancel an arbitrary cosmological constant, and flat Minkowski spacetime appears as the limit of attractor-type solutions of the field equations. Asymptotically, the field equations give rise to a standard Friedmann-Robertson-Walker universe and standard Newtonian gravitational dynamics of small systems.

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I. INTRODUCTION AND SUMMARY

The main cosmological constant problem (CCP No. 1 or CCP1, for short) lies in the apparent conflict between certain theoretical expectations and experimental facts (see, e.g., Ref. [1] for an extensive review). The key theoretical expectation is that the zero-point energy of the quantum fields in the equilibrium vacuum state naturally produces an unsuppressed effective cosmological constant $\Lambda$ in the classical gravitational field equation. The key experimental fact is the observed negligible value of $\Lambda$. The qualifications ‘unsuppressed’ and ‘negligible’ refer to the known energy scales of elementary particle physics. Hence, CCP1 motivates us to discover the mechanism which cancels the gravitational effects of this zero-point energy, without fine-tuning the theory.

Several years ago, Dolgov [2] proposed a remarkable solution to CCP1 by having an evolving massless vector field which dynamically cancels the effective cosmological constant $\Lambda$. The original Dolgov model, however, runs into two obstacles. The first obstacle [3] is that the steadily increasing vector field ruins the Newtonian gravitational dynamics of a localized matter distribution, e.g., the matter of the Solar System. The second obstacle, already noted by Dolgov himself, is that the expansion of the asymptotic Universe is too fast, inexorably diluting any standard-matter component initially present.

Inspired by the $q$–theory approach [4, 5] to CCP1, two extended vector-field models have been constructed, which circumvent each obstacle separately [6, 7]. The question is whether or not there exists a further extended vector-field model which deals with both obstacles simultaneously. The present article answers this question affirmatively by doing the obvious, namely, by combining the two previous extended models.

With this final extended vector-field model, we have a possible solution of the main cosmological constant problem and no unwanted side effects. Indeed, the final extended vector-field model cancels an arbitrary (Planck-scale) cosmological constant $\Lambda$ without fine-tuning, while maintaining the standard local Newtonian gravitational dynamics and providing for an acceptable late-universe Hubble expansion (there may still be an inflationary phase in the very early universe [8]). But, this is only a ‘possible’ solution, because it is not clear if such massless vector fields exist in reality (having a consistent quantum theory and a mechanism to guarantee their masslessness). A further caveat on this ‘possible’ solution is mentioned in Endnote [9] which is called in Sec. [V.D].

Taking for granted that CCP1 has been solved in principle, the next problem (CCP2) is to explain the small but nonzero value measured in the actual nonequilibrium Universe. This problem lies outside the scope of the present article. Some relevant remarks can be found in the recent review [5], which contains, moreover, a brief summary of $q$–theory.
The outline of this article is straightforward: first, the model is defined (Sec. II), then, the homogeneous background solution is determined (Sec. III) and found to correspond to an attractor-type solution (Sec. IV), and, finally, the local gravitational dynamics of small-scale systems is shown to be Newtonian (Sec. V). Two appendices give mathematical proofs for the existence of attractor solutions in related but simpler vector-field models, namely, the original Dolgov model [2] and the model of our second article [7].

All calculations contained in this article are analytic. The main results are, first, the exact solution (3.8) with constants (3.9) and Λ–cancellation (3.10) and, second, the vanishing modification (5.19a) of the weak-field gravity theory (5.18) for a localized matter distribution in a perfect-equilibrium background.

II. TWO-VECTOR-FIELD MODEL

The model is presented in Sec. II A, together with appropriate cosmological Ansätze for the fields. The reduced field equations are given in Secs. II B and II C.

A. Action and Ansätze

Consider a model of two massless vector fields, $A_\alpha(x)$ and $B_\alpha(x)$. This model is governed by the following effective action ($\hbar = c = 1$):

$$S_{\text{eff}}[g, A, B, \phi] =$$
$$- \int d^4x \sqrt{-\det(g)} \left( \frac{1}{2} (E_{\text{Planck}})^2 R[g] + \epsilon(F_A, F_B) + \Lambda + L_M[g, \phi] \right), \quad (2.1a)$$

$$E_{\text{Planck}} \equiv (8\pi G)^{-1/2}, \quad (2.1b)$$

with the Ricci scalar $R(x)$ of the metric $g_{\alpha\beta}(x)$, the effective cosmological constant $\Lambda$, a generic massless matter field $\phi(x)$ with a standard Lagrange density $L_M(x)$, and a function $\epsilon(F_A, F_B)$ of the following two auxiliary variables $F_A(x)$ and $F_B(x)$:

$$F_A[g, A] \equiv (Q_{3A})^2 - \frac{1}{2} R A_\alpha A^\alpha, \quad Q_{3A}[g, A] \equiv \nabla^\alpha A_\alpha, \quad (2.2a)$$

$$F_B[g, B] \equiv (Q_{3B})^2 - \frac{1}{2} R B_\alpha B^\alpha, \quad Q_{3B}[g, B] \equiv \nabla^\alpha B_\alpha, \quad (2.2b)$$

where $\nabla_\alpha$ denotes the covariant derivative (later also written as a semicolon in front of the relevant spacetime index). As will become clear at the very end of this article (Sec. V D), the coupling constant $G$ entering the reduced Planck energy (2.1b) can be identified with Newton’s gravitational coupling constant $G_N$, first measured by Cavendish. The magnitude
of the cosmological constant is considered to be of the order of the Planck energy, $|\Lambda| \sim (E_{\text{Planck}})^4$. For the moment, we set the standard term $\mathcal{L}_M$ in (2.1a) to zero [possible zero-point-energy contributions from the $\phi$ field have already been included in $\Lambda$, assuming a proper (relativistic) regularization tracing back to the fundamental microscopic theory].

Combining the *Ansätze* of our previous work \cite{6,7}, we take the following special function:

$$\epsilon(F_A, F_B) = (E_{\text{Planck}})^4 \left( a \frac{F_A}{F_B} + b \frac{F_B}{F_A} \right),$$

with numerical constants $a = \pm 1$ and $b = -a$. In explicit calculations later on, we will use the values $a = -b = 1$. The function (2.3) possesses the following symmetry properties:

$$F_A \frac{\partial \epsilon}{\partial F_A} + F_B \frac{\partial \epsilon}{\partial F_B} = 0,$$

$$F_A^2 \frac{\partial^2 \epsilon}{\partial F_A^2} + 2F_A F_B \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} + F_B^2 \frac{\partial^2 \epsilon}{\partial F_B^2} = 0,$$

which will turn out to be crucial for the preservation of standard Newtonian gravity on small scales (Sec. III D).

In this article, we start by considering a spatially flat, homogeneous, and isotropic universe. The corresponding Robertson–Walker (RW) metric in suitable spacetime coordinates is given by:

$$\left( g_{\alpha\beta}(x_1, x_2, x_3, t) \right) = \left( \text{diag}[1, -a^2(t), -a^2(t), -a^2(t)] \right),$$

where $a(t)$ is the scale factor as a function of cosmic time $t$. The usual Hubble parameter is defined by $H \equiv (da/dt)/a$.

The following *Ansätze* \cite{2} for the background vector fields are consistent with the homogeneous and isotropic background metric (2.5):

$$A_\alpha(x_1, x_2, x_3, t) = A_0(t) \delta_\alpha^0,$$

$$B_\alpha(x_1, x_2, x_3, t) = B_0(t) \delta_\alpha^0,$$

involving only two functions of $t$.

**B. Reduced vector-field equations**

The variational principle for the vector fields of action (2.1a) gives the following two equations:

$$\left( \nabla_\alpha Q_{3A} + \frac{1}{2} R A_\alpha \right) \frac{\partial \epsilon}{\partial F_A} + Q_{3A} \nabla_\alpha \left( \frac{\partial \epsilon}{\partial F_A} \right) = 0,$$

$$\left( \nabla_\alpha Q_{3B} + \frac{1}{2} R B_\alpha \right) \frac{\partial \epsilon}{\partial F_B} + Q_{3B} \nabla_\alpha \left( \frac{\partial \epsilon}{\partial F_B} \right) = 0.$$
The Ansätze (2.5) and (2.6) reduce the eight partial differential equations (2.7a) and (2.7b) to the following two ordinary differential equations (ODEs):

\[
\left( \ddot{A}_0 + 3H \dot{A}_0 - 6H^2 A_0 \right) \frac{\partial \epsilon}{\partial F_A} + \left( \dot{A}_0 + 3HA_0 \right) \frac{d}{dt} \left( \frac{\partial \epsilon}{\partial F_A} \right) = 0 , \quad (2.8a)
\]

\[
\left( \ddot{B}_0 + 3H \dot{B}_0 - 6H^2 B_0 \right) \frac{\partial \epsilon}{\partial F_B} + \left( \dot{B}_0 + 3HB_0 \right) \frac{d}{dt} \left( \frac{\partial \epsilon}{\partial F_B} \right) = 0 , \quad (2.8b)
\]

where the overdot stands for differentiation with respect to the cosmic time \( t \). Similarly, the \( q \)-theory-type variables from (2.2) become

\[
Q^{3A}(t) = \frac{d}{dt} A_0(t) + 3H A_0(t) , \quad (2.9a)
\]

\[
Q^{3B}(t) = \frac{d}{dt} B_0(t) + 3H B_0(t) . \quad (2.9b)
\]

Observe that having \( A_0 \propto t \) and \( H \propto 1/t \) makes \( Q^{3A} \) in (2.9a) into a genuine (spacetime-independent) \( q \)-theory variable [4] and similarly for \( Q^{3B} \) in (2.9b).

C. Generalized FRW equations

The energy-momentum tensor of the vector fields is calculated by varying the action with respect to the metric tensor \( g_{\alpha\beta} \). This tensor is found to be given by

\[
T_{\alpha\beta} = \left( \epsilon(F_A,F_B) - 2F_A \frac{\partial \epsilon}{\partial F_A} - 2F_B \frac{\partial \epsilon}{\partial F_B} \right) g_{\alpha\beta}
+ \frac{\partial \epsilon}{\partial F_A} \left( R_{\alpha\beta}A^2 - R A_\alpha A_\beta \right) + \frac{\partial \epsilon}{\partial F_B} \left( R_{\alpha\beta}B^2 - R B_\alpha B_\beta \right)
- \nabla_\alpha \nabla_\beta \left( A^2 \frac{\partial \epsilon}{\partial F_A} + B^2 \frac{\partial \epsilon}{\partial F_B} \right) + g_{\alpha\beta} \nabla^2 \left( A^2 \frac{\partial \epsilon}{\partial F_A} + B^2 \frac{\partial \epsilon}{\partial F_B} \right) . \quad (2.10)
\]

At this time, we also introduce a contribution to the total energy-momentum tensor from the standard-matter sector of the theory, that is, we consider having \( L_M \neq 0 \) in the original effective action (2.1a). In the cosmological context, the standard-matter component is described by a homogenous relativistic fluid. Note that this physical setup is not altogether unrealistic, as the masses of the standard-model particles are negligible for temperatures \( T \sim E_{\text{Planck}} \gg 10^2 \text{ GeV} \).

From the previous Ansätze (2.3), (2.5), and (2.6), the generalized Friedmann–Robertson–Walker (FRW) equations and the standard-matter energy-conservation equation are

\[
3H^2 = (E_{\text{Planck}})^{-2} \left[ \Lambda + \rho(A,B) + \rho_M \right] , \quad (2.11a)
\]

\[
2 \dot{H} + 3H^2 = (E_{\text{Planck}})^{-2} \left[ \Lambda - P(A,B) - w_M \rho_M \right] , \quad (2.11b)
\]

\[
\dot{\rho}_M = -3(1 + w_M) H \rho_M , \quad (2.11c)
\]
where the last equation describes the adiabatic evolution of a perfect relativistic fluid with a homogeneous energy density $\rho_M(t)$ and pressure $P_M(t) = w_M \rho_M(t)$ for constant equation-of-state parameter $w_M = 1/3$. The vector-field energy density (from $T^{0^0}_0 = \rho$) and isotropic pressure (from $T_{ij}^i = -P \delta_{ij}$) appearing in (2.11) are given by

$$
\rho(A, B) = \epsilon(F_A, F_B) + 3 \left( \dot{H} + 3 H^2 \right) \left( A_0^2 \frac{\partial \epsilon}{\partial F_A} + B_0^2 \frac{\partial \epsilon}{\partial F_B} \right),
$$

$$
P(A, B) = -\epsilon(F_A, F_B) + \left( \dot{H} + 3 H^2 \right) \left( A_0^2 \frac{\partial \epsilon}{\partial F_A} + B_0^2 \frac{\partial \epsilon}{\partial F_B} \right) - 2 H \frac{d}{dt} \left( A_0^2 \frac{\partial \epsilon}{\partial F_A} + B_0^2 \frac{\partial \epsilon}{\partial F_B} \right) - \frac{d^2}{dt^2} \left( A_0^2 \frac{\partial \epsilon}{\partial F_A} + B_0^2 \frac{\partial \epsilon}{\partial F_B} \right),
$$

where the symmetry property (2.4a) has been taken into account. As (2.12a) contains a term $\dot{H}$, for example, it is clear that (2.11a) is not the standard Friedmann equation.

### III. ASYMPTOTIC SOLUTION

It is a straightforward exercise to determine the asymptotic ($t \to \infty$) solution from the reduced field equations as given in Sec. II. In a first reading, it is possible to skip the technical details and to jump ahead to Sec. III C, which contains the main physics result of this section.

#### A. Dimensionless ODEs

As in our previous articles [6, 7], we introduce dimensionless variables by rescaling with appropriate powers of the reduced Planck energy $E_{Planck}$ without additional numerical factors. Specifically, we replace

$$
\{ \Lambda, \epsilon, t, H \} \rightarrow \{ \lambda, e, \tau, h \},
$$

$$
\{ Q_{3A}, Q_{3B}, A_0, B_0, \rho_M \} \rightarrow \{ q_{3A}, q_{3B}, v, w, r_M \}.
$$

The following dimensionless ODEs for the vector fields $v(\tau)$ and $w(\tau)$ and the Hubble parameter $h(\tau)$ result from the previous vector-field, generalized Friedmann, and matter
energy-conservation equations:

\[ 0 = \left( \ddot{v} + 3h \dot{v} - 6h^2 v \right) \frac{\partial e}{\partial f_A} + \left( \dot{v} + 3h v \right) \frac{d}{d\tau} \left( \frac{\partial e}{\partial f_A} \right), \]  

(3.2a)

\[ 0 = \left( \ddot{w} + 3h \dot{w} - 6h^2 w \right) \frac{\partial e}{\partial f_B} + \left( \dot{w} + 3h w \right) \frac{d}{d\tau} \left( \frac{\partial e}{\partial f_B} \right), \]  

(3.2b)

\[ 0 = 3h^2 - \lambda - \bar{e} - r_M - 3 \left( \dot{h} + 3h^2 \right) g_{AB} - 3h \frac{d}{d\tau} g_{AB}, \]  

(3.2c)

\[ 0 = \dot{r}_M + 4hr_M, \]  

(3.2d)

where the overdot now stands for differentiation with respect to the dimensionless cosmic time \( \tau \). In addition, we have the following definitions:

\[ e = f_A/f_B - f_B/f_A, \]  

(3.3a)

\[ \bar{e} = e - 2f_A \left( \frac{\partial e}{\partial f_A} \right) - 2f_B \left( \frac{\partial e}{\partial f_B} \right) = e, \]  

(3.3b)

\[ f_A = (\dot{v} + 3h v)^2 + 3(\dot{h} + 2h^2) v^2, \]  

(3.3c)

\[ f_B = (\dot{w} + 3h w)^2 + 3(\dot{h} + 2h^2) w^2, \]  

(3.3d)

\[ g_{AB} = v^2 \left( \frac{\partial e}{\partial f_A} \right) + w^2 \left( \frac{\partial e}{\partial f_B} \right). \]  

(3.3e)

A further FRW equation, given by the dimensionless version of (2.11b), can be shown to be consistent with the above ODEs.

Using the symmetry property (2.4a) in (3.2a) and (3.2b), it can be shown that boundary conditions at \( \tau = \tau_0 \) with \( v(\tau_0)/w(\tau_0) = \dot{v}(\tau_0)/\dot{w}(\tau_0) \) give proportional \( v(\tau) \) and \( w(\tau) \) solutions: \( v(\tau) = [v(\tau_0)/w(\tau_0)] w(\tau) \).

**B. Expansion coefficients**

The asymptotic solution of the differential Eqs. (3.2), for \( \lambda \) of arbitrary sign, is given by the following series:

\[ v(\tau) = \alpha_0 \tau + \alpha_1 + \alpha_2 \tau^{-1} + O(\tau^{-2}), \]  

(3.4a)

\[ w(\tau) = \beta_0 \tau + \beta_1 + \beta_2 \tau^{-1} + O(\tau^{-2}), \]  

(3.4b)

\[ h(\tau) = \gamma_0 \tau^{-1} + \gamma_1 \tau^{-2} + \gamma_2 \tau^{-3} + O(\tau^{-4}), \]  

(3.4c)

\[ r_M(\tau) = \delta_0 \tau^{-2} + \delta_1 \tau^{-3} + \delta_2 \tau^{-4} + O(\tau^{-5}), \]  

(3.4d)
with leading-order coefficients:

\[ \alpha_0 \equiv 1, \]  
\[ \beta_0 = \pm \sqrt{\frac{\lambda}{2} + \sqrt{1 + \left(\frac{\lambda}{2}\right)^2}}, \]  
\[ \gamma_0 = \frac{1}{2}, \]  
\[ \delta_0 = \frac{3}{4}, \]

next-to-leading-order coefficients:

\[ \alpha_1 = -2 \alpha_0 \gamma_1, \]  
\[ \beta_1 = -2 \beta_0 \gamma_1, \]  
\[ \gamma_1 = \gamma_1, \]  
\[ \delta_1 = 3 \gamma_1, \]

and next-to-next-to-leading-order coefficients:

\[ \alpha_2 = 0, \]  
\[ \beta_2 = 0, \]  
\[ \gamma_2 = 2 (\gamma_1)^2, \]  
\[ \delta_2 = 9 (\gamma_1)^2. \]

These vector and metric fields have only one arbitrary constant, \( \gamma_1 \), which we interpret as being due to the time-shift invariance of the equations (\( \tau \to \tau + \text{const} \)). The general (attractor-type) solution of the three second-order ODEs and the single first-order ODE in (3.2) will have seven arbitrary constants (see Sec. IV D).

Different starting values of \( v(\tau), w(\tau), h(\tau), \text{ and } r_M(\tau) \), at large enough \( \tau = \tau_{\text{start}} \) and in an appropriate domain, give different values of \( \gamma_1 \). Excluded starting values are those with \( \{v(\tau_{\text{start}}), \dot{v}(\tau_{\text{start}})\} = \{0, 0\} \) and/or \( \{w(\tau_{\text{start}}), \dot{w}(\tau_{\text{start}})\} = \{0, 0\} \) and/or \( r_M(\tau_{\text{start}}) = 0 \).

**C. Dynamic cancellation of \( \Lambda \)**

The calculational details of this section and the next should not make us forget that the vector fields of the model cancel the effective cosmological constant \( \Lambda \) exactly and without fine-tuning.
Indeed, the field equations (2.8) and (2.11) give nonzero vector-field components, a Hubble parameter, and a matter energy density of the form (3.4) for coefficient $\gamma_1 = 0$,

\begin{align}
A_0(t) &= \alpha_0 (E_{\text{Planck}})^2 t, \\
B_0(t) &= \beta_0 (E_{\text{Planck}})^2 t, \\
H(t) &= \gamma_0 t^{-1}, \\
\rho_M(t) &= \delta_0 (E_{\text{Planck}})^2 t^{-2},
\end{align}

where the overall normalization of $A_0$ and $B_0$ is irrelevant, as only the ratio of the vector-field components enters the action (2.1a) according to Eqs. (2.2), (2.3), and (2.6). The coefficients $\alpha_0$, $\beta_0$, $\gamma_0$, and $\delta_0$ in (3.8) are not put in by hand but appear dynamically. Specifically, the following values have been calculated in Sec. III B:

\begin{align}
(\beta_0/\alpha_0)^2 &= \frac{1}{2} \Lambda/(E_{\text{Planck}})^4 + \sqrt{1 + \frac{1}{4} \Lambda^2/(E_{\text{Planck}})^8}, \\
\gamma_0 &= 1/2, \\
\delta_0 &= 3/4.
\end{align}

These particular fields give an exact cancellation of $\Lambda$ appearing on the right-hand side of the generalized FRW Eqs. (2.11),

\begin{align}
\Lambda + \epsilon \left( F_A, F_B \right)_{\text{eul}} &= \Lambda + \epsilon \left( (Q_{3A})^2, (Q_{3B})^2 \right) \\
&= \Lambda + (E_{\text{Planck}})^4 \left[ (\alpha_0/\beta_0)^2 - (\beta_0/\alpha_0)^2 \right] = 0, \tag{3.10}
\end{align}

where the definitions (2.2), (2.3), and (2.9) have been used for $F_{A,B}$, $\epsilon$, and $Q_{3A,B}$, respectively. With the nullification (3.10), the FRW Eqs. (2.11) are solved to order $t^0$.

Including the higher-order terms of the asymptotic solution (3.4), also called the perfect-equilibrium solution later on, we can evaluate the effective vacuum energy density of what may be called the microscopic dark-energy component, that is, the energy density not from standard matter but from the initial (‘bare’) cosmological constant, the vector fields, and the modified gravity. A convenient definition for a spatially flat RW universe is as follows:

\begin{align}
\rho_{V\text{-micro}}(t) &\equiv 3 (E_{\text{Planck}})^2 H(t)^2 - \rho_M(t), \tag{3.11}
\end{align}

which was simply denoted $\rho_V$ in Ref. [7]. The result from the asymptotic solution (3.4) is

\begin{align}
\rho_{V\text{-micro}}(t) \bigg|_{\text{asym sol.}} &= O(t^{-5}), \tag{3.12a}
\end{align}
which implies

\[
\lim_{t \to \infty} \rho_{V \text{-micro}}(t) / \rho_{M}(t) \bigg|_{\text{asymp. sol.}} = 0.
\]  

Result (3.12b) traces back to the special properties of the $\varepsilon$–function (2.3) and was absent for the simpler models of Refs. [6, 7], which exhibited the behavior $\rho_{V \text{-micro}}(t) \propto t^{-2}$. Assuming the relevance of our model function (2.3) to physics, the implication is that a new mechanism is needed to explain the observed finite remnant vacuum energy density of order (meV)$^4$.

Expanding on the last remarks of the previous paragraph, it is not difficult to see what the implications are for the present energy-density ratio of dark energy and matter. For the sake of the argument, use $|\tilde{\rho}_{V \text{-micro}}(t)| = t^{-4}$, which may still be an overestimate as quantum-dissipative effects can be expected to produce an exponential decrease (cf. Ref. [5] and paper [15] quoted therein). A present cold-dark-matter energy density of the order of the critical energy density gives $\rho_{CDM}(t_0) \sim t_{\text{Planck}}^2 t_0^{-2}$, for $t_{\text{Planck}} \equiv 1/E_{\text{Planck}} \sim 10^{-42}$ s and $t_0 \sim (cH_0)^{-1} \sim 10^{17}$ s. The present energy-density ratio would then be completely negligible, $|\tilde{\rho}_{V \text{-micro}}(t_0)| / \rho_{CDM}(t_0) \sim (t_{\text{Planck}}/t_0)^2 \sim 10^{-118}$. In fact, the ratio would already be extremely small near the electroweak crossover: $|\tilde{\rho}_{V \text{-micro}}(t_{ew})| / \rho_{M}(t_{ew}) \sim (E_{ew}/E_{\text{Planck}})^4 \sim 10^{-60}$, for $E_{ew} \sim \text{TeV}$ and $t_{ew} \sim E_{\text{Planck}}/(E_{ew})^2$, as derived from the spatially-flat Friedmann equation with $\rho_{M} \sim (E_{ew})^4$. With negligible $\tilde{\rho}_{V \text{-micro}}(t)$ from the microscopic variables ($A_\alpha$, $B_\alpha$, and effectively $\Lambda$), further contributions to the vacuum energy density $\rho_{V \text{-macro}}(t)$ may come from phase transitions and mass effects of the macroscopic standard-model fields. As discussed in Ref. [5], the resulting $\rho_{V \text{-macro}}(t)$ may decrease stepwise, approximately as $t_{\text{Planck}}^{-2} t^{-2}$.

In conclusion, the exact solution (3.8) is of paramount importance, especially if it is an attractor-type solution. This attractor-type behavior will be discussed in the next section.

IV. ATTRACTOR-TYPE SOLUTIONS

The present section is a direct follow-up of the previous one and is also rather technical. In order to get an idea of the attractor-type behavior, it is possible, in a first reading, to consider only Sec. IV C.

A. Mathematical considerations

The model of interest has an action-density term $\varepsilon(F_A, F_B)$ as given by (2.3). For completeness, two simpler models are discussed in the appendices: in App. A, the original Dolgov model [2] with just a $(Q_1)^2$ term in the action density and, in App. B, our previous model [7] with a single $F_A$ term as defined in (2.2a).
It turns out, however, that the first-order system of differential equations for the \( \epsilon(F_A, F_B) \) model does not have the relatively simple structure as found in the appendices, specifically, Eqs. (A12a) and (B11a). Physically, the extra complications may be due to the fact the \( \epsilon(F_A, F_B) \) model is really an \( f(R) \) modified-gravity theory, which entails higher-derivative field equations [in our case, (2.11b) has third-order derivatives of \( A_0(t) \), \( B_0(t) \), and \( H(t) \)].

One possible way forward would be to rewrite this particular modified-gravity theory as a scalar-tensor theory (more precisely, a scalar-vector-tensor theory). Instead, we prefer to adopt a low-tech (read brute-force) approach by pushing the explicit solutions as far as possible. This approach suffices to show the attractor-type behavior, even though it lacks mathematical rigor compared to the approach in the appendices. In fact, what would be needed here is the mathematical proof that the infinite sums in the expressions of Sec. IV D converge, but we will simply assume this to be the case, as has been done in most of the literature on the subject (cf. Ref. [2, b]). Still, awaiting this rigorous proof and the precise knowledge of the attractor domain, we will only speak about ‘attractor-type solutions’ of the \( \epsilon(F_A, F_B) \) model rather than ‘the attractor solution’ \( \textit{tout court} \).

B. ODEs

The complete system of differential equations from Sec. III A can be written as follows:

1. \( 0 = \ddot{v} + 3 h \dot{v} - 6 h^2 v + (\dot{v} + 3 h v) \frac{d}{d\tau} \ln \left| \frac{\partial e}{\partial f_A} \right|, \) \hspace{1cm} (4.1a)
2. \( 0 = \ddot{w} + 3 h \dot{w} - 6 h^2 w + (\dot{w} + 3 h w) \frac{d}{d\tau} \ln \left| \frac{\partial e}{\partial f_B} \right|, \) \hspace{1cm} (4.1b)
3. \( 0 = 3 h^2 - \lambda - e - r_M - 3 (\dot{h} + 3 h^2) g_{AB} - 3 h \dot{g}_{AB}, \) \hspace{1cm} (4.1c)
4. \( 0 = 2 \dot{h} + 3 h^2 - \lambda - e + \frac{1}{3} r_M + (\dot{h} + 3 h^2) g_{AB} - 2 h \dot{g}_{AB} - \ddot{g}_{AB}, \) \hspace{1cm} (4.1d)
5. \( 0 = \dot{r}_M + 4 h r_M, \) \hspace{1cm} (4.1e)

where \( e \), \( f_A \), \( f_B \), and \( g_{AB} \) have already been defined in (3.3). We will now give several explicit analytic solutions of these ODEs.
C. Particular class of exact solutions

The differential system (4.1) has the following class of exact solutions for \( \tau > \tau_0 \):

\[
\begin{align*}
v(\tau) &= (\tau - \tau_0) C_1 + \frac{C_3}{C_2 (\tau - \tau_0)^{3/2}}, \\
w(\tau) &= (\tau - \tau_0) C_2 + C_4 \frac{C_3}{C_1 (\tau - \tau_0)^{3/2}}, \\
h(\tau) &= \frac{1}{2} \frac{1}{(\tau - \tau_0)}, \\
r_M(\tau) &= \frac{3}{4} \frac{1}{(\tau - \tau_0)^2},
\end{align*}
\]

with a real constant \( \tau_0 \in \mathbb{R} \), nonvanishing real constants \( C_1, C_2 \in \mathbb{R}\{0\} \), a real constant \( C_3 \in \mathbb{R} \), and a discrete constant \( C_4 \in \{-1, +1\} \). The constants \( \tau_0, C_3, \) and \( C_4 \) are arbitrary. The real ratio \( C_1/C_2 \) is determined by the input cosmological constant \( \lambda \) via a quartic equation,

\[
\frac{C_1}{C_2} \equiv R_C,
\]

\[
(R_C)^4 + \lambda (R_C)^2 = 1,
\]

as follows from, e.g., the generalized Friedmann equation (4.1c). Hence, the number of free parameters in (4.2) is four: \( \tau_0, (C_1 C_2), C_3, \) and \( C_4 \). The physically relevant parameters are, however, only the ratio \( C_3/(C_1 C_2) \) and the relative sign \( C_4 \).

Observe that all solutions in (4.2) give for the effective vacuum energy density of the microscopic degrees of freedom an exactly vanishing result,

\[
r_{V-micro}(\tau) \bigg|_{C_1/C_2=R_C, \tau_0, C_1, C_2, C_3, C_4} = 0,
\]

with definition \( r_{V-micro}(\tau) \equiv 3 h(\tau)^2 - r_M(\tau) \) from (3.11) and \( R_C \) the positive or negative real solution of (4.3b). Result (4.4) also holds for the special case \( C_3 = 0 \), which corresponds to the perfect-equilibrium solution (3.8) with constants (3.9) and an arbitrary time-shift.

For the case of \( C_3 \neq 0 \) and \( C_4 = -1 \), the rescaled solutions \( v(\tau)/C_1 \) and \( w(\tau)/C_2 \) in (4.2) are different at finite values of \( \tau \), specifically, \( v(\tau)/C_1 - w(\tau)/C_2 \propto (\tau - \tau_0)^{-3/2} \). Still, both of these functions \( v(\tau)/C_1 \) and \( w(\tau)/C_2 \) approach the same asymptotic solution, the one from above, the other from below. This is precisely the attractor-type behavior discussed in Sec. IV A and the two appendices (see also Refs. [4, (c)] and [6, 7] for related numerical results).
D. Series and attractor-type behavior

A generalized Ansatz for a nontrivial solution of (4.1) at $\tau \geq \tau_1 > 0$ is as follows:

$$\tau^{-1} v(\tau) = \left[ v_1/\tau_1 \right] + (\tau - \tau_1)^2 + \frac{(\tau - \tau_1) \left[ \dot{v}_1/\tau_1 - v_1/\tau_1^2 \right] + (\tau - \tau_1)^2}{1 + (\tau - \tau_1)^2} + \sum_{n=1}^{\infty} a_n \left( \frac{(\tau - \tau_1)^2}{\tau^3} \right)^n, \quad (4.5a)$$

$$(\beta_0 \tau)^{-1} w(\tau) = \left[ w_1/(\beta_0 \tau_1) \right] + (\tau - \tau_1)^2 + \frac{(\tau - \tau_1) \left[ \dot{w}_1/(\beta_0 \tau_1) - w_1/(\beta_0 \tau_1^2) \right] + (\tau - \tau_1)^2}{1 + (\tau - \tau_1)^3} + \sum_{n=1}^{\infty} b_n \left( \frac{(\tau - \tau_1)^2}{\tau^3} \right)^n, \quad (4.5b)$$

$$2 \tau h(\tau) = \left[ 2 \tau_1 h_1 \right] + (\tau - \tau_1)^2 + \frac{(\tau - \tau_1) \left[ 2 \tau_1 \dot{h}_1 + 2 h_1 \right] + (\tau - \tau_1)^2}{1 + (\tau - \tau_1)^3} + \sum_{n=1}^{\infty} c_n \left( \frac{(\tau - \tau_1)^2}{\tau^3} \right)^n, \quad (4.5c)$$

$$\left(4/3\right) \tau^2 r_M(\tau) = \left[ \left(4/3\right) \tau_1^2 r_{M1} \right] + (\tau - \tau_1)^2 + \frac{(\tau - \tau_1) \left[ (8/3) (1 - 2 \tau_1 h_1) \tau_1 r_{M1} \right] + (\tau - \tau_1)^2}{1 + (\tau - \tau_1)^3} + \sum_{n=1}^{\infty} d_n \left( \frac{(\tau - \tau_1)^2}{\tau^3} \right)^n, \quad (4.5d)$$

with $\beta_0$ given by (3.5). The seven constant parameters $v_1$, $\dot{v}_1$, $w_1$, $\dot{w}_1$, $h_1$, $\dot{h}_1$, and $r_{M1}$ in (4.5) represent the initial values of the functions and their first derivatives at $\tau = \tau_1$:

$$v(\tau_1) = v_1, \quad \dot{v}(\tau_1) = \dot{v}_1, \quad (4.6a)$$

$$w(\tau_1) = w_1, \quad \dot{w}(\tau_1) = \dot{w}_1, \quad (4.6b)$$

$$h(\tau_1) = h_1, \quad \dot{h}(\tau_1) = \dot{h}_1, \quad (4.6c)$$

$$r_M(\tau_1) = r_{M1}, \quad (4.6d)$$

where $r_M(\tau)$ requires only a single boundary condition value as its ODE is first-order, the other ODEs being second-order. These initial values must be sufficiently close to those of the perfect-equilibrium solution, given by (4.2) with $C_1 = 1$ and $C_3 = \tau_0 = 0$. 13
Inserting the expansions (4.5) into (4.1) gives values for the coefficients $a_n$, $b_n$, $c_n$, and $d_n$ in terms of the initial conditions $v_1$, ..., $r_{M1}$. The expressions for these coefficients are rather bulky (even for $\tau \gg \tau_1$) and, here, we only indicate the dependence on the initial conditions for the first few coefficients,

\begin{align*}
a_1 &= a_1(\tau_1, h_1, \dot{h}_1), \\
b_1 &= b_1(\tau_1, h_1, \dot{h}_1), \\
c_i &= c_i(\tau_1, h_1, \dot{h}_1), \quad \text{for } i = 1, \ldots, 5, \\
d_1 &= d_1(\tau_1, h_1, \dot{h}_1), \\
a_j &= a_j(\tau_1, v_1, \dot{v}_1, h_1, \dot{h}_1), \quad \text{for } j = 2, \ldots, 7, \\
b_j &= b_j(\tau_1, w_1, \dot{w}_1, h_1, \dot{h}_1), \quad \text{for } j = 2, \ldots, 7, \\
d_k &= d_k(\tau_1, h_1, \dot{h}_1, r_{M1}), \quad \text{for } k = 2, \ldots, 5.
\end{align*}

In the limit of large cosmic times (that is, large on the scale of the Planck time, $\tau \gg \tau_1$), the corresponding solution takes the following form:

\begin{align*}
v(\tau) &= \tau - 1 - c_1 + O(\tau^{-5}), \\
w(\tau) &= \beta_0 (\tau - 1 - c_1) + O(\tau^{-5}), \\
h(\tau) &= \frac{1}{2} \left[ \sum_{n=0}^{4} \frac{(1 + c_1)^n}{\tau^n} + O(\frac{1}{\tau^5}) \right], \\
r_M(\tau) &= \frac{3}{4} \frac{1}{\tau^2} \left[ \sum_{n=0}^{4} (n + 1) \frac{(1 + c_1)^n}{\tau^n} + O(\frac{1}{\tau^5}) \right].
\end{align*}

Extrapolating this result, we obtain the asymptotic (perfect-equilibrium) solution,

\begin{align*}
v_{\text{asymp}}(\tau) &= \tau - \tilde{\tau}_1, \\
w_{\text{asymp}}(\tau) &= \beta_0 (\tau - \tilde{\tau}_1), \\
h_{\text{asymp}}(\tau) &= \frac{1}{2} (\tau - \tilde{\tau}_1)^{-1}, \\
r_{M\text{asymp}}(\tau) &= \frac{3}{4} (\tau - \tilde{\tau}_1)^{-2},
\end{align*}

where $\tilde{\tau}_1 \equiv 1 + c_1$. Observe that, apart from the overall time-shift $\tilde{\tau}_1$, the obtained asymptotic solution is independent of the initial conditions (4.6) encoded in the Ansatz (4.5).

The tentative conclusion is that different initial conditions give different solutions, which, however, approach the same asymptotic solution (4.9). Hence, there is an attractor-type
behavior. But, as explained in Sec. IV A, this conclusion needs to be proven rigorously and the proper attractor domain needs to be determined.

V. SECOND-ORDER PERTURBATIONS

We, now, turn to localized perturbations of the metric tensor field and the two vector fields. Denoting the four spacetime coordinates \((x_1, x_2, x_3, t)\) collectively as \(x\), we consider the tensor field
\[
g_{\alpha\beta}(x) = g_{\alpha\beta}(t) + \hat{h}_{\alpha\beta}(x),
\]
with the metric \(g_{\alpha\beta}(t)\) of the flat RW spacetime (2.5) and \(\hat{h}_{\alpha\beta}(x)\) the perturbation \((|\hat{h}_{\alpha\beta}| \ll 1)\). On small scales, the relevant background metric is the standard Minkowski metric \(\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)\). In addition, we consider the two vector fields
\[
A_\alpha(x) = A_\alpha(t) + \delta A_\alpha(x),
\]
\[
B_\alpha(x) = B_\alpha(t) + \delta B_\alpha(x),
\]
with \(A_\alpha(t) = A_0(t) \delta^0_\alpha, B_\alpha(t) = B_0(t) \delta^0_\alpha, |\delta A_\alpha| \ll |A_0|,\) and \(|\delta B_\alpha| \ll |B_0|\).

This section is highly technical and, in a first reading, it is possible to skip ahead to Sec. V D with the main physics result of this section.

A. Variation of the vector-field Lagrange density

The Lagrange density of the vector fields is given by \(L_{A,B} = \Lambda + \epsilon(F_A, F_B)\), where the effective cosmological constant \(\Lambda\) has been included for convenience. To second order, the perturbed Lagrange density reads
\[
L_{A,B}^{(\text{perturb.})} = L_{A,B}^{(0)} + L_{A,B}^{(1)} + L_{A,B}^{(2)},
\]
with
\[
L_{A,B}^{(0)} = \Lambda + \epsilon(F_A, F_B),
\]
\[
L_{A,B}^{(1)} = \frac{\partial \epsilon}{\partial F_A} \delta^{(1)} F_A + \frac{\partial \epsilon}{\partial F_B} \delta^{(1)} F_B,
\]
\[
L_{A,B}^{(2)} = \frac{\partial \epsilon}{\partial F_A} \delta^{(2)} F_A + \frac{\partial \epsilon}{\partial F_B} \delta^{(2)} F_B + \frac{1}{2} \frac{\partial^2 \epsilon}{\partial F_A^2} \left(\delta^{(1)} F_A\right)^2
+ \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta^{(1)} F_A \delta^{(1)} F_B + \frac{1}{2} \frac{\partial^2 \epsilon}{\partial F_B^2} \left(\delta^{(1)} F_B\right)^2.
\]
The first- and second-order variations of $Q_A$ and $F_A$ are

$$
\delta^{(1)} Q_{3A} = \delta A_\alpha^\alpha + \hat{h}^{\alpha\beta} A_{\alpha;\beta} - g^{\alpha\beta} \delta^{(1)} \Gamma^\gamma_{\alpha\beta} A_\gamma ,
$$

(5.5a)

$$
\delta^{(2)} Q_{3A} = \hat{h}^{\alpha\beta} \delta A_{\alpha;\beta} - g^{\alpha\beta} \delta^{(1)} \Gamma^\gamma_{\alpha\beta} \delta A_\gamma - \hat{h}^{\alpha\beta} \delta^{(1)} \Gamma^\gamma_{\alpha\beta} A_\gamma - g^{\alpha\beta} \delta^{(2)} \Gamma^\gamma_{\alpha\beta} A_\gamma ,
$$

(5.5b)

$$
\delta^{(1)} F_A = 2Q_{3A} \delta^{(1)} Q_{3A} - \frac{1}{2} R \left( 2A^\alpha \delta A_\alpha + \hat{h}^{\alpha\beta} A_\alpha A_\beta \right) - \frac{1}{2} A^2 \delta^{(1)} R ,
$$

(5.5c)

$$
\delta^{(2)} F_A = 2Q_{3A} \delta^{(2)} Q_{3A} + (\delta^{(1)} Q_{3A})^2 - \frac{1}{2} R \left( \delta A^\alpha \delta A_\alpha + 2\hat{h}^{\alpha\beta} A_\alpha \delta A_\beta \right)
$$

$$
- \frac{1}{2} \delta^{(1)} R \left( 2A^\alpha \delta A_\alpha + \hat{h}^{\alpha\beta} A_\alpha A_\beta \right) - \frac{1}{2} A^2 \delta^{(2)} R .
$$

(5.5d)

Replacing $A_\alpha$ and $\delta A_\alpha$ in (5.5) by $B_\alpha$ and $\delta B_\alpha$ gives the first- and second-order variations of $Q_B$ and $F_B$.

For future use, we rewrite $L_{A,B}^{(1)}$ and $L_{A,B}^{(2)}$ in dimensionless form,

$$
L_{A,B}^{(1)} = Q^2_{3A0} \frac{\partial \varepsilon}{\partial F_A} \delta^{(1)} f_A + Q^2_{3B0} \frac{\partial \varepsilon}{\partial F_B} \delta^{(1)} f_B ,
$$

(5.6a)

$$
L_{A,B}^{(2)} = Q^2_{3A0} \frac{\partial \varepsilon}{\partial F_A} \delta^{(2)} f_A + Q^2_{3B0} \frac{\partial \varepsilon}{\partial F_B} \delta^{(2)} f_B + \frac{1}{2} Q^4_{3A0} \frac{\partial^2 \varepsilon}{\partial F_A^2} \left( \delta^{(1)} f_A \right)^2 + \frac{1}{2} Q^4_{3B0} \frac{\partial^2 \varepsilon}{\partial F_B^2} \left( \delta^{(1)} f_B \right)^2 ,
$$

(5.6b)

where $\delta^{(1)} f_{A,B}$ and $\delta^{(2)} f_{A,B}$ correspond to $\delta^{(1)} F_{A,B}$ and $\delta^{(2)} F_{A,B}$ expressed in terms of dimensionless variables $\overline{v}_\alpha, \overline{w}_\alpha$ and $\hat{v}_\alpha, \hat{w}_\alpha$. These dimensionless variables are defined as follows:

$$
\overline{v}_\alpha(t) \equiv \frac{1}{Q_{3A0} A_\alpha(t) , \quad \overline{w}_\alpha(t) \equiv \frac{1}{Q_{3B0} B_\alpha(t) ,
$$

(5.7a)

$$
\hat{v}_\alpha(x) \equiv \frac{1}{Q_{3A0}} \delta A_\alpha(x) , \quad \hat{w}_\alpha(x) \equiv \frac{1}{Q_{3B0}} \delta B_\alpha(x) ,
$$

(5.7b)

$$
\hat{z}_\alpha \equiv \hat{v}_\alpha - \hat{w}_\alpha ,
$$

(5.7c)

with dimensional constants $Q_{3A0}$ and $Q_{3B0}$. In (5.7c), we have added the definition of $\hat{z}_\alpha$, which will be used extensively in the next subsections. Note also that, in the above definitions, the background fields are distinguished by a bar and the perturbation fields by a hat.
B. Equations for the vector-field perturbations

The equations of the vector-field perturbations are

\[
\partial_\alpha \left( \bar{\nabla}_3 \left[ Q_{3A0}^2 \frac{\partial^2 \epsilon}{\partial F_A^2} \delta f_A + Q_{3B0}^2 \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta f_B \right] + \frac{\partial \epsilon}{\partial F_A} \delta q_{3A} \right) + \frac{1}{2} \frac{\partial \epsilon}{\partial F_A} \left( R \bar{\nabla}_\alpha + R \hat{\nabla}_\alpha \right) + \frac{1}{2} R \left( Q_{3A0}^2 \frac{\partial^2 \epsilon}{\partial F_A^2} \delta f_A + Q_{3B0}^2 \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta f_B \right) \bar{\nabla}_\alpha = 0, 
\]

(5.8a)

\[
\partial_\alpha \left( \bar{\nabla}_3 \left[ Q_{3B0}^2 \frac{\partial^2 \epsilon}{\partial F_B^2} \delta f_B + Q_{3A0}^2 \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta f_A \right] + \frac{\partial \epsilon}{\partial F_B} \delta q_{3B} \right) + \frac{1}{2} \frac{\partial \epsilon}{\partial F_B} \left( R \bar{\nabla}_\alpha + R \hat{\nabla}_\alpha \right) + \frac{1}{2} R \left( Q_{3B0}^2 \frac{\partial^2 \epsilon}{\partial F_B^2} \delta f_B + Q_{3A0}^2 \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta f_A \right) \bar{\nabla}_\alpha = 0, 
\]

(5.8b)

with \( \delta f_{A,B} \equiv \delta^{(1)} f_{A,B}, \delta q_{3A,3B} \equiv \delta^{(1)} q_{3A,3B} \), and \( \delta R \equiv \delta^{(1)} R \). Furthermore, we have \( \bar{\nabla}_3 = \bar{v}_0 + 3 h \bar{v}_0 \) and \( \bar{\nabla}_3 = \bar{w}_0 + 3 h \bar{w}_0 \). Note that the above equations for the perturbations \( \hat{\nabla}_\alpha \) and \( \hat{\nabla}_\alpha \) carry a third derivative of the metric perturbation \( \hat{h}_{\alpha \beta} \), since \( \delta f_A \) and \( \delta f_B \) contain \( \delta R \), which already has a second derivative of \( \hat{h}_{\alpha \beta} \).

Using the background vector-field Eqs. (2.7a) and (2.7b), the perturbation Eqs. (5.8a) and (5.8b) can be reduced to

\[
\partial_\alpha \left( \frac{\partial q_{3A}}{q_{3A}} \right) + \frac{1}{2} R \left( R \bar{\nabla}_\alpha + \left[ \delta R - R \frac{\delta q_{3A}}{q_{3A}} \right] \bar{\nabla}_\alpha \right) = 0, 
\]

(5.9a)

\[
\partial_\alpha \left( \frac{\partial q_{3B}}{q_{3B}} \right) + \frac{1}{2} R \left( R \hat{\nabla}_\alpha + \left[ \delta R - R \frac{\delta q_{3B}}{q_{3B}} \right] \hat{\nabla}_\alpha \right) = 0, 
\]

(5.9b)

with definitions

\[
\delta \Omega_A \equiv \left( \frac{\partial \epsilon}{\partial F_A} \right)^{-1} \left( \frac{\partial^2 \epsilon}{\partial F_A^2} \delta F_A + \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta F_B \right), 
\]

(5.10a)

\[
\delta \Omega_B \equiv \left( \frac{\partial \epsilon}{\partial F_B} \right)^{-1} \left( \frac{\partial^2 \epsilon}{\partial F_B^2} \delta F_B + \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \delta F_A \right). 
\]

(5.10b)

Taking \( \bar{\nabla}_\alpha(t) = \bar{\nabla}_\alpha(t) = \zeta_\alpha(t) \) for \( \zeta_\alpha(t) = (\zeta(t), 0, 0, 0) \), we find \( \bar{\nabla}_3 = \bar{\nabla}_3 = \zeta + 3 h \zeta \). Subtracting (5.9b) from (5.9a) then gives

\[
\partial_\alpha \left( \Xi \left( 2 \bar{\nabla}_3 \nabla^\beta \zeta_\beta - R \zeta^\beta \zeta_\beta \right) + \frac{1}{2} \frac{1}{\bar{q}_3^2} R \zeta_\alpha - \frac{1}{2} \frac{1}{\bar{q}_3^2} R \zeta_\alpha \nabla^\beta \zeta_\beta \right) = 0, 
\]

(5.11)

where \( \zeta_\alpha \) has already been defined in (5.7) and

\[
\Xi \equiv \frac{(Q_{3A})^2}{(\bar{q}_3)^2} \left[ \left( \frac{\partial \epsilon}{\partial F_A} \right)^{-1} \frac{\partial^2 \epsilon}{\partial F_A^2} - \left( \frac{\partial \epsilon}{\partial F_B} \right)^{-1} \frac{\partial^2 \epsilon}{\partial F_B^2} \right] 
\]

\[
\equiv \frac{(Q_{3B})^2}{(\bar{q}_3)^2} \left[ \left( \frac{\partial \epsilon}{\partial F_B} \right)^{-1} \frac{\partial^2 \epsilon}{\partial F_B^2} - \left( \frac{\partial \epsilon}{\partial F_A} \right)^{-1} \frac{\partial^2 \epsilon}{\partial F_A^2} \right]. 
\]

(5.12)
Notice that \( \hat{z}_\alpha \) in (5.11) is not coupled to the metric perturbation \( \hat{h}_{\alpha\beta} \): \( \hat{z}_\alpha \) depends only on the functions \( \zeta(t) \) and \( H(t) \) from the background fields, together with the initial conditions for \( \hat{v}_\alpha \) and \( \hat{w}_\alpha \). This result follows from the symmetry properties of the function \( \epsilon(F_A, F_B) \).

Substituting the \( \epsilon \) function (2.3) into (5.16), we find

\[
\Xi = -\frac{1}{(q_3)^2 - (1/2) R \zeta^2}.
\]  

(5.13)

In the perfect-equilibrium state with Hubble parameter \( H(t) = 1/2 \ t^{-1} \) (implying \( R = 0 \)) and constant values of \( Q_{3A} \) and \( Q_{3B} \) (as mentioned in the last sentence of Sec. II B), Eqs. (5.11) and (5.13) give the following final equation for \( \hat{z}_\alpha \equiv \hat{v}_\alpha - \hat{w}_\alpha \):

\[
\nabla_\alpha \nabla^\beta \hat{z}_\beta \bigg|_{\text{equil. background}} = \partial_\alpha \left[ t^{-3/2} \partial^3 \left( t^{3/2} \hat{z}_\beta \right) \right] = 0.
\]  

(5.14)

For perturbation fields which are analytic and of finite support (\( \hat{v}_\alpha = \hat{w}_\alpha = 0 \) for \( t \in [0, T] \) and \( |\vec{x}| \geq R \)), the solution is trivial and

\[
\hat{z}_\alpha \bigg|_{\text{local perturb.}} = 0.
\]  

(5.15)

In other words, the two linear vector-field perturbations turn out to be equal, \( \delta A_\alpha(x) = \delta B_\alpha(x) \), which is the same result as obtained in Ref. [6] by different methods. The explanation of (5.15) is simple: the localized perturbation fields \( \hat{v}_\alpha \) and \( \hat{w}_\alpha \) obey the same equation and their boundary conditions over an exterior region are also the same (zero, in fact).

C. Energy-momentum tensor of the vector-field perturbations

The linear perturbation of the energy-momentum tensor of the vector fields is given by the following expression (only the arguments \( \hat{h}, \hat{v}, \) and \( \hat{w} \) are shown explicitly on the left-hand side):

\[
\Theta_{\alpha\beta}[\hat{h}, \hat{v}, \hat{w}] = (\Lambda + \epsilon) \hat{h}_{\alpha\beta} + \frac{1}{2} (\mu_{A0} - \mu_{B0}) \left( (2 \bar{q}_3 \nabla^\lambda \hat{z}_\lambda - R \zeta^\lambda \hat{z}_\lambda) g_{\alpha\beta} + 2 R_{\alpha\beta} \zeta^\lambda \hat{z}_\lambda - R \left( \zeta_\alpha \hat{z}_\beta + \zeta_\beta \hat{z}_\alpha \right) \right) \\
+ \left( \nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2 \right) \left( 2 \mu_{B0} \zeta^\lambda \hat{z}_\lambda + \nu_{B0} \left( 2 \bar{q}_3 \nabla^\lambda \hat{z}_\lambda - R \zeta^\lambda \hat{z}_\lambda \right) \zeta^2 \right) \\
+ \frac{1}{2} (\nu_{A0} - \nu_{B0}) \left( 2 \bar{q}_3 \nabla^\lambda \hat{z}_\lambda - R \zeta^\lambda \hat{z}_\lambda \right) \left( R_{\alpha\beta} \zeta^2 - R \zeta_\alpha \zeta_\beta \right),
\]  

(5.16)

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with definitions

\[
\begin{align*}
\mu_A &\equiv (Q_{3A})^2 \frac{\partial \epsilon}{\partial F_A}, \\
\mu_B &\equiv (Q_{3B})^2 \frac{\partial \epsilon}{\partial F_B}, \\
\nu_A &\equiv (Q_{3A})^2 \left( (Q_{3A})^2 \frac{\partial^2 \epsilon}{\partial F_A^2} + (Q_{3B})^2 \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \right), \\
\nu_B &\equiv (Q_{3B})^2 \left( (Q_{3B})^2 \frac{\partial^2 \epsilon}{\partial F_B^2} + (Q_{3A})^2 \frac{\partial^2 \epsilon}{\partial F_A \partial F_B} \right),
\end{align*}
\]

so that \(\mu_{A0} + \mu_{B0} = 0\) and \(\nu_{A0} + \nu_{B0} = 0\) for the special function \((2.3)\) and the perfect-equilibrium background fields with a Ricci-flat spacetime \((R = 0)\).

Manifestly, \(\Theta_{\alpha\beta}[^{\hat{h}},^{\hat{v}},^{\hat{w}}]\) does not contain derivatives of the metric perturbation \(\hat{h}_{\alpha\beta}\) and depends only on the difference between the vector-field perturbations, \(\hat{z}_\alpha\) as defined by \((5.7c)\). These results rely on the symmetry properties \((2.4a)\) and \((2.4b)\) of the special \(\epsilon(F_A, F_B)\) function \((2.3)\) and on the fact that \(F_A\) and \(F_B\) are quadratic with respect to the vector fields and that the background fields evolve identically as mentioned below \((5.10b)\).

D. Standard local Newtonian dynamics

With the results of the previous two subsections, we can, at last, turn to the physical question of interest: the gravitational self-interaction of small (noncosmological) systems. This has been discussed extensively in our previous article \([6]\), so we can be brief.

The linear equation for the weak gravitational field from a localized matter distribution is then

\[
\begin{align*}
\Box \hat{h}_{\alpha\beta} + 16 \pi G S_{\alpha\beta} &= 0, \\
S_{\alpha\beta} &\equiv T_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \eta^{\gamma\delta} T_{\gamma\delta}, \\
T_{\alpha\beta} &= T_{\alpha\beta}^{(\text{matter})} + \Theta_{\alpha\beta},
\end{align*}
\]

where the harmonic gauge, \(\partial_\alpha \hat{h}^{\alpha}_{\beta} = (1/2) \partial_\beta \hat{h}^{\alpha}_{\alpha}\), has been used to simplify the standard derivative term on the left-hand side of \((5.18a)\), with d’Alembertian \(\Box \equiv \eta^{\alpha\beta} \partial_\alpha \partial_\beta\). The only new contribution appears as the second term on the right-hand side of \((5.18c)\) and has been given in \((5.10)\).

Several comments are in order. First, note that the background fields \(\mathbf{v}_\alpha(\tau)\) and \(\mathbf{w}_\alpha(\tau)\) are such that the \(\Lambda + \epsilon\) term in \((5.16)\) vanishes for the perfect-equilibrium background; see, in particular, the derivation \((3.10)\). Second, recall that the energy-momentum tensor \(\Theta_{\alpha\beta}\) of the perturbations depends only on the metric perturbation \(\hat{h}_{\alpha\beta}\) (but not its derivatives) and the difference of the vector-field perturbations. Specifically, the behavior is as follows,
in a symbolic notation: \[ \Theta_{\alpha\beta} [\hat{h}, \hat{v}, \hat{w}] = \Theta_{\alpha\beta} [\hat{h}, (\hat{v} - \hat{w}), (\nabla + \nabla^2 + \nabla^3) (\hat{v} - \hat{w})] \]. The main input for this result is that the normalized background vector fields evolve identically, \( \overline{v}_\alpha = \overline{w}_\alpha = \zeta_\alpha (t) \) for \( t \to \infty \). But this is precisely what was found in Sec. III. The evolution of \( \hat{z}_\alpha \equiv \hat{v}_\alpha - \hat{w}_\alpha \) is, therefore, not affected by the metric perturbation \( \hat{h}_{\alpha\beta} \) (at least, to the linear order in perturbation theory considered). Moreover, (5.15) states that \( \hat{z}_\alpha \) vanishes due to the boundary conditions at infinity (the energy density of the matter perturbation being localized in space and time).

With \( \Lambda + \epsilon = 0 \) and \( \hat{z}_\alpha = 0 \) nullifying (5.16), the conclusion is that the nonstandard term in (5.18) drops out,

\[
\left[ \Theta_{\alpha\beta} [\hat{h}, \hat{v}, \hat{w}] \right]_{\text{local perturb.}}^{\text{equil. background}} = 0,
\]

and that the linear weak-gravity field equation (in harmonic gauge) equals the one of general relativity,

\[
[\Box \hat{h}_{\alpha\beta} + 16 \pi G S_{\alpha\beta}^{\text{matter}}]_{\text{local perturb.}}^{\text{equil. background}} = 0,
\]

with the standard-matter source term

\[
S_{\alpha\beta}^{\text{matter}} \equiv T_{\alpha\beta}^{\text{matter}} - \frac{1}{2} \eta_{\alpha\beta} \eta_{\gamma\delta} T_{\gamma\delta}^{\text{matter}}.
\]

As mentioned before, these results hold for perfect-equilibrium background fields [given by (2.5), (2.6), and (3.8) in dimensional form or (4.9) in dimensionless form], which have dynamically canceled the cosmological constant \( \Lambda \) (see Sec. III.C). Recall that the main cosmological constant problem, CCP1 as formulated in Sec. II, is precisely concerned with the dynamic cancellation of \( \Lambda \) in the equilibrium state of the quantum vacuum. The study of small self-gravitating systems in a nonequilibrium background (even if this background rapidly approaches the equilibrium state, as discussed in the penultimate paragraph of Sec. III.C), lies outside the scope of the present article [9].

Equation (5.19b) shows, in particular, that the standard Newtonian law of gravity (i.e., the Poisson equation) holds for local nonrelativistic matter distributions such as the Solar System or the Galaxy. This implies that the constant \( G \) in (5.19b), which traces back to the original action (2.1), can be identified with Newton’s gravitational coupling constant,

\[
G = G_N = 6.6743(7) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2},
\]

where the numerical value has been taken from the CODATA-2006 compilation [13]. The corresponding numerical value of the gravitational energy scale defined in (2.15) is then the usual one, \( E_{\text{Planck}} \approx 2.44 \times 10^{18} \text{ GeV} \).

The present article, just as its predecessor [6], only considers the linear theory of small self-gravitating systems. It remains to be seen whether or not the present setup reproduces locally the standard nonlinear theory, i.e., general relativity.
ACKNOWLEDGMENTS

It is a pleasure to thank the referee for helpful remarks.

Appendix A: Attractor solution in a model with a \((Q_1)^2\) term

1. ODEs and new variables

The original Dolgov model \([2]\), with a single massless vector field \(A_\alpha(x)\) and a positive cosmological constant \(\Lambda\), is defined by the action \((2.1a)\), setting \(B_\alpha(x) \equiv 0\) and having a vacuum-energy-density term \(\epsilon_D\) based on a different contraction of the vector-field derivatives,

\[\epsilon_D = (Q_1)^2,\tag{A1a}\]

\[(Q_1)^2 \equiv A_{\alpha;\beta} A^{\alpha;\beta} = \left(\frac{dA_0}{dt}\right)^2 + 3 H^2 A_0^2,\tag{A1b}\]

where the last step in \((A1b)\) holds for the RW metric \((2.5)\) and the isotropic Ansatz \((2.6a)\).

With the dimensionless variables \((3.1)\), the basic equations are given by the reduced vector-field and FRW equations:

\[\ddot{v} + 3 h \dot{v} - 3 h^2 v = 0,\tag{A2a}\]

\[3 h^2 = \lambda - \dot{v}^2 - 3 h^2 v^2,\tag{A2b}\]

\[2 \dot{h} + 3 h^2 = \lambda - \dot{v}^2 - 3 h^2 v^2 - 2 \dot{h} v^2 - 4 h v \dot{v} + 2 \dot{v}^2,\tag{A2c}\]

for \(\lambda > 0\). It is possible to set \(\lambda = 1\) by an appropriate rescaling of \(\tau\) and \(1/h\), but we prefer to keep \(\lambda\) explicit, in order to facilitate comparison with the models of App.\([3]\) and Sec.\([IV]\).

The system of differential equations can now be rewritten as follows:

\[\ddot{v} + 3 h \dot{v} - 3 h^2 v = 0,\tag{A3a}\]

\[3 (1 + v^2) h^2 - \lambda + \dot{v}^2 = 0,\tag{A3b}\]

\[(1 + v^2) \dot{h} - \dot{v}^2 + 2 h \dot{v} v = 0.\tag{A3c}\]

Next, introduce new variables (using the natural logarithm \(\ln\)):

\[y_1 \equiv \dot{v}, \quad y_2 \equiv h v,\tag{A4a}\]

\[s \equiv \ln(a) - \ln(a_{\text{start}}),\tag{A4b}\]
where \( a = a(\tau) \) is the scale factor of the flat RW metric and \( a_{\text{start}} \) its value at \( \tau = \tau_{\text{start}} \).

Writing (A3) in terms of the new variables (A4) gives the following *autonomous* system of differential equations:

\[
y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2),
\]

(A5)

where the prime stands for differentiation with respect to \( s \) and

\[
F_1(y_1, y_2) = -3(y_1 - y_2),
\]

(A6a)

\[
F_2(y_1, y_2) = \frac{y_1}{\lambda - y_1^2} (\lambda - y_1^2 + 3y_1y_2 - 6y_2^2).
\]

(A6b)

Recall that the system (A5) is called autonomous because the independent variable \( s \) does not occur explicitly (see, e.g., Refs. [14, 15] for background material).

### 2. Critical points

A critical point \((y_{10}, y_{20})\) of system (A5) is defined as follows:

\[
F_1(y_1, y_2) \bigg|_{y_{10}, y_{20}} = F_2(y_1, y_2) \bigg|_{y_{10}, y_{20}} = 0.
\]

(A7)

A straightforward calculation gives two such critical points,

\[
y_{10}^\pm = \pm \sqrt{\lambda} / 2, \quad y_{20}^\pm = \pm \sqrt{\lambda} / 2,
\]

(A8)

corresponding to the asymptotic solutions

\[
v_{\text{asymp}}^\pm = \pm \left(\sqrt{\lambda} / 2\right) \tau, \quad h_{\text{asymp}} = \tau^{-1},
\]

(A9)

in terms of the original variables.\(^1\)

### 3. Stability analysis: Linearization

Make the following shift of variables:

\[
y_1 = y_{10} + Y_1, \quad y_2 = y_{20} + Y_2.
\]

(A10)

\(^1\) There is also a critical point \((0,0)\), which corresponds to de Sitter spacetime with \( v = 0 \) and \( h^2 = \lambda/3 \) if (A2b) is used as a constraint equation. This critical point is not of interest to us now and further discussion of this case will be omitted. In addition, it can be shown that the critical point \((0,0)\) is not asymptotically stable.
Then, (A5) becomes
\[
\frac{dY_1}{ds} = -3Y_1 + 3Y_2 \equiv G_1, \quad (A11a)
\]
\[
\frac{dY_2}{ds} = \frac{y_{10} + Y_1}{\lambda - (y_{10} + Y_1)^2} \times \left( \lambda - (y_{10} + Y_1)^2 + 3(y_{10} + Y_1)(y_{20} + Y_2) - 6(y_{20} + Y_2)^2 \right) \equiv G_2. \quad (A11b)
\]

In order to prove that the critical points \((y_{10}, y_{20})\) from (A8) are asymptotically stable solutions, it suffices to consider small \(Y_1\) and \(Y_2\): \(|Y_1| \ll |y_{10}|\) and \(|Y_2| \ll |y_{20}|\). We, then, find the following vector equation:
\[
\frac{d}{ds} Y(s) = A \cdot Y(s) + f(Y_1, Y_2), \quad (A12a)
\]
with the vectors
\[
Y(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}, \quad f(Y_1, Y_2) = \begin{pmatrix} f_1(Y_1, Y_2) \\ f_2(Y_1, Y_2) \end{pmatrix}, \quad (A12b)
\]
and the constant matrix
\[
A = \frac{1}{3} \begin{pmatrix} -9 & +9 \\ +1 & -9 \end{pmatrix}. \quad (A13)
\]

The eigenvalues of \(A\) are both negative (\(\sigma_1 = -2, \sigma_2 = -4\)). The vector component \(f_1\) is zero and \(f_2\) is quadratic in \(Y_1\) or \(Y_2\) to leading order:
\[
f_1(Y_1, Y_2) = 0, \quad (A14a) \\
f_2(Y_1, Y_2) = O\left(Y_1^2, Y_2^2, Y_1 Y_2\right), \quad (A14b)
\]
so that the following bounds hold:
\[
\lim_{y_{1,10} \to 0} \frac{f_1(Y_1, Y_2)}{\sqrt{Y_1^2 + Y_2^2}} = \lim_{y_{1,10} \to 0} \frac{f_2(Y_1, Y_2)}{\sqrt{Y_1^2 + Y_2^2}} = 0. \quad (A14c)
\]

With these results, the Poincaré–Lyapunov theorem (Theorem 7.1 in Ref. [14]; see also Theorem 66.2 in Ref. [15]) proves that the critical points \((y_{10}, y_{20})\) from (A8) are asymptotically stable (attractor) solutions.

4. Stability analysis: Lyapunov function

For completeness, we also give another proof which directly starts from (A11). This proof relies on the construction of an appropriate Lyapunov function \(V\), in order to be able to
apply the second Lyapunov stability theorem. The construction proceeds in three steps.

The first step is to define the Lyapunov candidate function $V[s, Y_1, Y_2]$ with properties $V[s, 0, 0] = 0$ and $V[s, Y_1, Y_2] > 0$ for $(Y_1, Y_2) \neq (0, 0)$. Specifically, take the following quadratic function:

$$V[s, Y_1, Y_2] = Y_1^2 + Y_2^2 + (Y_1 - Y_2)^2,$$  \hspace{1cm} (A15)

which has the required properties and no explicit dependence on $s$.

The second step is to calculate the orbital derivative,

$$L_s V \equiv \frac{\partial V}{\partial s} + \frac{\partial V}{\partial Y_1} G_1 + \frac{\partial V}{\partial Y_2} G_2,$$  \hspace{1cm} (A16)

where the first derivative on the right-hand side vanishes for the choice (A15) and where $G_1$ and $G_2$ are defined by the right-hand sides of (A11). The explicit result for $L_s V$ is

$$L_s V = -\frac{2}{4 - (1 + 2 Y_1)^2} \left[ 19 (Y_1 - Y_2)^2 + 8 Y_2^2 + 24 (Y_1^2 Y_2 - Y_1^3 + Y_2^3) \\
-28 Y_1^4 + 56 Y_1^3 Y_2 - 60 Y_1^2 Y_2^2 + 48 Y_1 Y_2^3 \right],$$  \hspace{1cm} (A17)

with Taylor expansion

$$L_s V = \left. L_s V \right|_{(Y_1, Y_2) \neq (0, 0)} < 0,$$  \hspace{1cm} (A18)

The third and last step is to demonstrate that (A17) implies the following inequality for a sufficiently small domain of $Y_1$ and $Y_2$ around $Y_1 = Y_2 = 0$:

$$L_s V \bigg|_{(Y_1, Y_2) \neq (0, 0)} < 0,$$  \hspace{1cm} (A19)

where the strict inequality holds with the point $Y_1 = Y_2 = 0$ excluded. Result (A19) implies that the function (A15) is a genuine Lyapunov function.

The result (A19) for Lyapunov function (A15) now establishes the fact that the solution $Y(s) = 0$ is asymptotically stable (Theorem 8.2 in Ref. [14] and Theorem 25.2 in Ref. [15]).

The precise mathematical definition of this asymptotic attractor behavior can be found in, e.g., Sec. 5.2 of Ref. [14] (for a general discussion, see also Sec. 35 of Ref. [15]). In short, arbitrary starting values $(v_0(s_0), \dot{v}_0(s_0))$ in a sufficiently small domain give a solution $(v(s), \dot{v}(s))$ which asymptotically approaches the solution (A9) as ‘time’ $s$ runs towards infinity.
Appendix B: Attractor solution in a model with an $F_A$ term

1. ODEs and new variables

In Ref. [7], we considered a single-vector-field model with the combination $[\Lambda + \zeta_0 (Q_3)^2 + \kappa R^2 A^2]$ in the action density for $\Lambda > 0$. For the case of $\zeta_0 = 1$ and $\kappa = -1/2$, this corresponds to having a vacuum-energy-density term $\epsilon = F_A$ in the action (2.1a), with $B_\alpha (x) \equiv 0$ and $F_A$ defined by (2.2a).

The resulting dimensionless ODEs read:

\begin{align*}
\ddot{v} + 3 h \dot{v} - 6 h^2 v &= 0, \quad (B1a) \\
3 h^2 &= \lambda - (\dot{v} + 3 h v)^2 + 3 h v (h v + 2 \dot{v}) + r_M, \quad (B1b) \\
2 \dot{h} + 3 h^2 &= \lambda - (\dot{v} + 3 h v)^2 - \left[(4 \dot{h} + 9 h^2) v^2 - 2 \dot{v}^2 - 2 v \ddot{v} - 4 h v \dot{v}\right] - \frac{1}{3} r_M, \quad (B1c) \\
\dot{r}_M + 4 h r_M &= 0, \quad (B1d)
\end{align*}

for $\lambda > 0$. With $h(\tau) \equiv \dot{a}(\tau)/a(\tau)$, the solution of (B1d) is known to be $r_M(\tau) \propto 1/a(\tau)^4$.

The system of differential equations can then be rewritten as follows:

\begin{align*}
\ddot{v} + 3 h \dot{v} - 6 h^2 v &= 0, \quad (B2a) \\
3 \left(1 + 2 v^2\right) h^2 - \lambda + \dot{v}^2 - r_{M\text{start}} \left(a_{\text{start}}/a\right)^4 &= 0, \quad (B2b) \\
(1 + 2 v^2) \dot{h} - \dot{v}^2 + 4 h v \dot{v} + \frac{2}{3} r_{M\text{start}} \left(a_{\text{start}}/a\right)^4 &= 0, \quad (B2c)
\end{align*}

where $a = a(\tau)$ is the scale factor of the flat RW metric, $a_{\text{start}}$ its value at $\tau = \tau_{\text{start}}$, and $r_{M\text{start}}$ the corresponding starting value of $r_M$.

Next, introduce new variables:

\begin{align*}
y_1 &\equiv \dot{v}, \quad y_2 \equiv h v, \quad (B3a) \\
s &\equiv \ln(a) - \ln(a_{\text{start}}). \quad (B3b)
\end{align*}

Writing (B2) in terms of the new variables (B3) gives the following nonautonomous system of differential equations:

\begin{align*}
y_1' = F_1(s, y_1, y_2), \quad y_2' = F_2(s, y_1, y_2), \quad (B4)
\end{align*}

where the prime stands for differentiation with respect to $s$ and

\begin{align*}
F_1(s, y_1, y_2) &= -3 (y_1 - 2 y_2), \quad (B5a) \\
F_2(s, y_1, y_2) &= y_1 + y_2 \frac{3 y_2^2 - 12 y_1 y_2 - 2 r_{M\text{start}} \exp[-4 s]}{\lambda - y_1^2 + r_{M\text{start}} \exp[-4 s]}. \quad (B5b)
\end{align*}
2. Critical points

A critical point \((y_{10}, y_{20})\) of system (B4) is defined as follows:

\[
\lim_{s \to \infty} F_1(s, y_1, y_2) \bigg|_{y_{10}, y_{20}} = \lim_{s \to \infty} F_2(s, y_1, y_2) \bigg|_{y_{10}, y_{20}} = 0.
\]

(B6)

A straightforward calculation gives two such critical points,

\[
y_{10}^\pm = \pm \sqrt{2 \lambda/5}, \quad y_{20}^\pm = \pm \sqrt{\lambda/10},
\]

(B7)

corresponding to the asymptotic solutions

\[
v_{\text{asymp}}^\pm = \pm \sqrt{2 \lambda/5} \tau, \quad h_{\text{asymp}} = (1/2) \tau^{-1},
\]

(B8)
in terms of the original variables.\(^2\)

3. Stability analysis

Make the following shift of variables:

\[
y_1 = y_{10} + Y_1, \quad y_2 = y_{20} + Y_2.
\]

(B9)

Then, (B4) becomes

\[
\frac{dY_1}{ds} = -3 Y_1 + 6 Y_2,
\]

(B10a)

\[
\frac{dY_2}{ds} = y_{10} + Y_1 + (y_{20} + Y_2)
\]

\[
\times \frac{3 (y_{10} + Y_1)^2 - 12 (y_{10} + Y_1)(y_{20} + Y_2) - 2 r_{M\text{start}} \exp[-4 s]}{-4 s}
\]

\[
\lambda - (y_{10} + Y_1)^2 + r_{M\text{start}} \exp[-4 s].
\]

(B10b)

In order to prove that the critical points \((y_{10}, y_{20})\) from (B7) are asymptotically stable solutions, it suffices to consider small \(Y_1\) and \(Y_2\): \(|Y_1| \ll |y_{10}|\) and \(|Y_2| \ll |y_{20}|\). We then, find the following vector equation:

\[
\frac{d}{ds} Y(s) = A \cdot Y(s) + B(s) \cdot Y(s) + f(s, Y_1, Y_2),
\]

(B11a)

with the vectors

\[
Y(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}, \quad f(s, Y_1, Y_2) = \begin{pmatrix} f_1(s, Y_1, Y_2) \\ f_2(s, Y_1, Y_2) \end{pmatrix},
\]

(B11b)

\(^2\) It is obvious that \((0, 0)\) is also a critical point, independent of the value of \(r_{M\text{start}}\). See Ftn. \(\Box\) for further comments.
and the matrices
\[
A = \frac{1}{3} \begin{pmatrix} -9 & +18 \\ -1 & -18 \end{pmatrix}, \quad B(s) = \frac{20 \alpha(s)}{3 (3 \lambda + 5 \alpha(s))} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
in terms of the auxiliary variable
\[
\alpha(s) \equiv r_{M_{\text{start}}} \exp[-4 s].
\]
The eigenvalues of $A$ are both negative ($\sigma_1 = -4$, $\sigma_2 = -5$) and the matrix $B(s)$ vanishes as $s \to +\infty$. The vector component $f_1$ is zero and $f_2$ is quadratic in $Y_1$ or $Y_2$ to leading order,
\[
f_1(s, Y_1, Y_2) = 0, \quad f_2(s, Y_1, Y_2) = O \left( \frac{5 \alpha(s) - 13 \lambda}{(5 \alpha(s) + 3 \lambda)^2} Y_1^2, \frac{1}{5 \alpha(s) + 3 \lambda} Y_2^2, \frac{25 \alpha(s) + 27 \lambda}{(5 \alpha(s) + 3 \lambda)^2} Y_1 Y_2 \right),
\]
so that the following bounds hold:
\[
\lim_{Y_1, Y_2 \to 0} \frac{f_1(s, Y_1, Y_2)}{\sqrt{Y_1^2 + Y_2^2}} = \lim_{Y_1, Y_2 \to 0} \frac{f_2(s, Y_1, Y_2)}{\sqrt{Y_1^2 + Y_2^2}} = 0.
\]
With these results, the Poincaré–Lyapunov theorem (Theorem 7.1 in Ref. [14]; see also the discussion below Theorem 66.2 in Ref. [15]) proves that the critical points $(y_{10}, y_{20})$ from (B7) are asymptotically stable (attractor) solutions. The above discussion provides the details for the result announced in the Note Added of Ref. [7].

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[9] A related potential problem of a nonequilibrium background may be the small but nonzero mass of the graviton. This can be most easily seen by considering Eq. (3.4a) of Ref. [6], where the nonderivative $\hat{h}$ terms then have nonvanishing prefactors in square brackets due to the nonequilibrium background. [A nonderivative $\hat{h}$ term also appears in (5.16) of the present article, namely, the first term on the right-hand side, as $(\Lambda + \epsilon)$ need no longer vanish.] The potential problem, now, is the unacceptable modification of general relativity due to the vanDam–Veltman–Zakharov effect for the linear theory of gravitons with arbitrarily small mass [10]. However, it has been argued [11] that this effect is an artifact of the linear approximation (see also Sec. IV.C.2 in Ref. [12] for further discussion and references). In our case, a realistic universe with a finite age would not give an absolutely perfect equilibrium background and the graviton-mass issue needs to be resolved. Alternatively, there is the possibility that the vector-field model can be adapted so as to avoid these nonderivative $\hat{h}$ terms altogether. This behavior can perhaps result from a special type of interaction between the standard matter fields and the massless vector fields $A_\alpha$ and $B_\alpha$.

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