Extreme phase and rotated quadrature measurements

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Abstract. We determine the extreme points of the convex set of covariant phase observables. Such extremals describe the best phase parameter measurements of laser light — the best in the sense that they are free from classical randomness due to fluctuations in the measuring procedure. We also characterize extreme fuzzy rotated quadratures.

PACS numbers: 03.65.–w

Submitted to: Physica Scripta

1. Introduction

Covariant phase observables constitute a simple and elegant solution to the quantum phase problem of a single-mode optical field (see, [21] and references therein). They describe coherent state phase (parameter) measurements which can be realized, for example, by using quantum optical homodyne or heterodyne detection. Since there exist infinite number of covariant phase observables, it is of great interest to classify the most precise and informative ones.

The set of covariant phase observables is convex. This means that, given two phase observables, one can form a random mixture of them. This mixture describe a new phase measurement. One the other hand, if a covariant phase observable $E$ can be represented as a nontrivial convex combination of two phase observables, one can equally measure these two phase observables and then mix their statistics to get the statistics of $E$.

The aim of this study, is to find such phase observables, so-called pure or extreme observables, which do not allow (nontrivial) convex decompositions. Pure phase observables then represent the best phase measurements in the sense that they are free from any classical randomness due to fluctuations in the measuring procedure (see, [8]).
Similarly, as in the case of phase observables, we determine the extreme points of
the convex set of fuzzy rotated quadratures. The rotated quadratures are important in
quantum optics, since they can be measured by balanced homodyne detection.

The structure of this article is the following: in section 2 we define coherent state
phase measurements (of laser light) and the associated phase observables. We also
consider the structure of such observables. The canonical phase observable is introduced
in section 2.1. A necessary and sufficient condition for extremality of a phase observable
is given in section 3. In section 4, we define fuzzy rotated quadratures and find extremal
quadratures.

2. Phase measurements

The quantum theory of a single-mode optical field is based on the Hilbert space
\( \mathcal{H} \) spanned by the photon number states \( \{|n\rangle \mid n = 0, 1, 2, \ldots \} \). We define the
usual lowering, raising, and number operators,
\[
a := \sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle \langle n|,
N := a^*a = \sum_{n=0}^{\infty} n |n\rangle \langle n|,
\]
respectively.

Coherent states \( |z\rangle := e^{-|z|^2/2} \sum_{n=0}^{\infty} z^n/\sqrt{n!} |n\rangle \), \( z \in \mathbb{C} \), describe the laser light;
here \( |z| \in [0, \infty) \) is the energy or intensity parameter and \( \arg z \in [0, 2\pi) \) is the phase
parameter. The number operator shifts the phase, that is, \( e^{i\theta} |z\rangle = \frac{1}{2\pi} \int_X e^{(m-n)\theta} d\theta |m\rangle \langle n| \)
for all \( z \in \mathbb{C} \), \( \theta \in [0, 2\pi) \), and \( X \in \mathcal{B}[0,2\pi) \).

A normalized positive operator measure (POM) \( E : \mathcal{B}[0,2\pi) \to \mathcal{L}(\mathcal{H}) \) is a phase
(parameter) measurement of laser light if
\[
\langle ze^{-i\theta}|E(X)|ze^{-i\theta}\rangle = \langle z|E(Xe^{i\theta})|z\rangle
\]
for all \( z \in \mathbb{C} \), \( \theta \in [0,2\pi) \), and \( X \in \mathcal{B}[0,2\pi) \). It is easy to show [19] that a POM \( E \) is a
phase measurement if and only if it is phase shift covariant, that is, if
\[
e^{i\theta N} E(X) e^{-i\theta N} = E(Xe^{i\theta})
\]
holds for all \( X \) and \( \theta \). Hence, we say that a POM \( E : \mathcal{B}[0,2\pi) \to \mathcal{L}(\mathcal{H}) \) is a (covariant)
phase observable if it is phase shift covariant.

The structure of phase observables is well known, see e.g. [9, 17, 4]. Any phase
observable \( E \) is of the form
\[
E(X) = \sum_{m,n=0}^{\infty} c_{m,n} \frac{1}{2\pi} \int_X e^{(m-n)\theta} d\theta |m\rangle \langle n|
\]
where the (unique) phase matrix \( (c_{m,n})_{m,n=0}^{\infty} \) is positive semidefinite and \( c_{m,m} = 1 \)
for all \( m \). As a positive semidefinite matrix, \( (c_{m,n}) \) has a Kolmogorov decomposition (see,
\( \| \| \mathcal{H} \| \) is the set of bounded operators on \( \mathcal{H} \), \( \mathcal{B}(\Omega) \) is the Borel \( \sigma \)-algebra of any topological space
\( \Omega \), and \( + \) means the addition modulo \( 2\pi \). A mapping \( E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H}) \) is a POM if and only if
\( X \mapsto \langle \psi|E(X)\psi\rangle \) is a probability measure for any vector state \( \psi \in \mathcal{H} \).
new Hilbert space $\mathcal{H}_{(\eta_n)}$ as the closure of $\text{lin}\{\eta_n \mid n = 0, 1, \ldots\}$, one sees that a certain uniqueness can be reached as follows [12]: if $(\varphi_n)$ is another sequence giving $(c_{m,n})$ and $\mathcal{H}_{(\varphi_n)}$ as above, then there exists a unitary operator $U : \mathcal{H}_{(\eta_n)} \to \mathcal{H}_{(\varphi_n)}$ such that $U\eta_n = \varphi_n$ for all $n$. Especially, the dimension of $\mathcal{H}_{(\eta_n)}$ depends only on $(c_{m,n})$ and we may define the rank of $(c_{m,n})$ (or $E$) as $\dim \mathcal{H}_{(\eta_n)}$. We denote it by $\text{rank } E$.

In what follows, we consider always a minimal Kolmogorov decomposition of a phase matrix $(c_{m,n})$, that is, a unit vector sequence $(\eta_n)$ of a Hilbert space $\mathcal{K}$ such that $c_{m,n} = \langle \eta_m | \eta_n \rangle$ for all $m, n$ and vectors $\eta_n$ span $\mathcal{K}$. Then $\text{rank } E = \dim \mathcal{K}$.

2.1. The canonical phase measurement

The canonical phase observable $E_{\text{can}}$ is determined by the phase matrix with the elements $c_{n,m} \equiv 1$ [8, 18]. Its minimal Kolmogorov decomposition is given by a constant vector sequence $\eta_n \equiv \eta \in \mathcal{H}$ so that $\mathcal{K} = \mathbb{C}\eta \cong \mathbb{C}$. Hence, $\text{rank } E_{\text{can}} = \dim \mathbb{C}\eta = 1$.

The canonical phase observable is associated to the polar decomposition of the lowering operator $a$, that is,

$$a = \int_0^{2\pi} e^{i\theta} dE_{\text{can}}(\theta) \sqrt{N}.$$ 

Moreover, $E_{\text{can}}$ is (up to a unitary equivalence) the only phase observable which generates number shifts [18]. This suggests that the number operator $N$ and the canonical phase $E_{\text{can}}$ form a canonical pair as the position and momentum observables.

For any phase observable $E$, let $g^E_z$ be the probability density of the coherent state phase measurement, that is,

$$\langle z | E(X) | z \rangle \equiv \frac{1}{2\pi} \int_X g^E_z(\theta) d\theta.$$ 

Now the canonical measurement $E_{\text{can}}$ gives the highest peak:

$$g^E_z(\text{arg } z) \leq g^{E_{\text{can}}}_z(\text{arg } z).$$

In addition, $g^{E_{\text{can}}}_z$ tends to the $2\pi$-periodic Dirac $\delta$-distribution in the classical limit $|z| \to \infty$ and for sufficiently large energies $|z|$, we have the approximative uncertainty relation

$$\Delta|z| E_{\text{can}} \Delta|z| N \approx \frac{1}{2},$$

where $\Delta|z|$ are the square roots of (minimum) variances [10, 18, 21]. All these facts demonstrate the canonicity of $E_{\text{can}}$ (for more properties of $E_{\text{can}}$, see the list in page 51 of [21]).

3. Extreme phase measurements

The set of phase observables is convex meaning that, for any two phase observables $E_1$ and $E_2$, one can form a (random mixture) phase observable $E = \lambda E_1 + (1 - \lambda) E_2$ where
$0 \leq \lambda \leq 1$. A phase observable $E$ is extreme or pure if it does not allow nontrivial convex decompositions, that is, if $E = \lambda E_1 + (1 - \lambda)E_2$ implies that $E_1 = E_2 = E$. If $E$ is extreme then the coherent state phase statistics $g_z^E$ cannot be obtained by measuring other phase observables in the coherent state $|z\rangle$ and then mixing their statistics. The following theorem [12, 15] characterizes extreme phase observables.

Let $E$ be a phase observable associated to unit vectors $\eta_n \in \mathcal{K}$ which span $\mathcal{K}$.

**Theorem 1** $E$ is extreme if and only if, for any bounded operator $A : \mathcal{K} \to \mathcal{K}$,

$$\langle \eta_n | A \eta_n \rangle = 0 \quad \text{for all } n \in \{0, 1, 2, \ldots\},$$

implies that $A = 0$.

The next theorem [15] shows that there exist infinite number of extreme phase observables.

**Theorem 2** There exist extreme phase observables of any rank $\in \{1, 2, \ldots, \infty\}$.

Since the canonical phase $E_{\text{can}}$ is of rank 1, it is automatically extreme [12]. Other rank 1 phase observables are unitarily equivalent to $E_{\text{can}}$, that is, they are of the form $U^* E_{\text{can}} U$ where the unitary operator $U$ commutes with the representation $\theta \mapsto e^{i\theta N}$ of $U(1)$, that is, $U$ is diagonal in the number basis. Indeed, if $E$ is of rank 1, the Hilbert space $\mathcal{K}$ (associated to the minimal Kolmogorov decomposition) can be chosen to be $\mathbb{C}$. Thus, the unit vector sequence $(\eta_n)$ is just a sequence of complex numbers $e^{i \alpha_n}$, $\alpha_n \in [0, 2\pi)$, and $U = \sum_{n=0}^{\infty} e^{i \alpha_n} |n\rangle \langle n|$.

Recently, we have proved [7] the following stronger result:

**Theorem 3** The canonical phase $E_{\text{can}}$ is extreme in the convex set of all POMs $\mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$.

This condition supports the canonicity of $E_{\text{can}}$; this result has been known to be true for spectral measures $\|$.\[\]

**Remark 4** There is no realistic direct measurement scheme for $E_{\text{can}}$ but some other phase observables, so-called phase space phase observables [17], can be measured. Let $D(z) := e^{z a^* - a}$, $z \in \mathbb{C}$, be the displacement operator and $T := \sum_{n=0}^{\infty} \lambda_n |n\rangle \langle n|$ where $\lambda_n \geq 0$ for all $n$ and $\sum_{n=0}^{\infty} \lambda_n = 1$. A phase space phase observable $E_T$ is defined by

$$E_T(X) := \frac{1}{\pi} \int_X \int_0^{\infty} D(re^{i\theta}) T D(re^{i\theta})^* r \, dr \, d\theta.$$

In principle, any phase space phase observable can be measured by using an eight-port homodyne detector [20, 14]. Indeed, $E_{\langle 0|0 \rangle}$ has been measured by Walker and Carroll [22]. It can be shown [2] that the rank of any $E_T$ is $\infty$ and $E_T$ is not extreme. This suggests that better phase measurement schemes could be found in future.

§ In the case of the canonical phase, $\eta_n \equiv \eta$ and $\mathcal{K} = \mathbb{C}\eta$. For any $A = a |\eta\rangle \langle \eta|$ the condition $\langle \eta | A \eta \rangle = a = 0$ implies that $A = 0$.

|| For spectral measures this is obvious since projections are extremals in the convex set of effects.
4. Extreme rotated quadratures

Define the quadrature operators
\[ Q := (a^* + a) / \sqrt{2} \]
\[ P := (a^* - a)i / \sqrt{2} \]
which, in the coordinate representation \( \mathcal{H} \cong L^2(\mathbb{R}) \), are the usual position and momentum operators
\[(Q\psi)(x) = x\psi(x) \quad \text{and} \quad (P\psi)(x) = -i\psi'(x)/dx,\]
respectively (in units where \( \hbar = 1 \)).

For any \( \theta \in [0, 2\pi) \), define the rotated quadrature operators \( Q_\theta \) and \( P_\theta \) by
\[ Q_\theta := R(\theta)QR(\theta)^*, \]
\[ P_\theta := R(\theta)PR(\theta)^* \]
where \( R(\theta) := e^{i\theta N} \). Note that \( P_\theta = Q_{\theta + \pi/2} \) and \( R(\pi/2) \) is the Fourier-Plancherel operator. The rotated quadratures can be measured by balanced homodyne detection [20, 13]. Next we define fuzzy rotated quadratures as the solutions of a covariance system.

Fix \( \theta \in [0, 2\pi) \) and choose a rotated momentum representation of \( \mathcal{H} \cong L^2(\mathbb{R}) \) such that
\[(Q_\theta\varphi)(p) = i\varphi'(p)/dp, \]
\[(P_\theta\varphi)(p) = p\varphi(p). \]
A POM \( F_\theta : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H}) \) is a fuzzy rotated quadrature if
\[ e^{iqP_\theta}F_\theta(X)e^{-iqP_\theta} = F_\theta(X + q) \]
for all \( q \in \mathbb{R} \) and \( X \in \mathcal{B}(\mathbb{R}) \). Any \( F_\theta \) is of the form
\[ \langle \varphi | F_\theta(X)|\psi \rangle = \frac{1}{2\pi} \int_X \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(p-p')x} \langle \eta_p | \eta_{p'} \rangle \varphi(p)\psi(p')dpdp'dx \]
for all integrable \( \varphi, \psi \in L^2(\mathbb{R}) \), where \( p \mapsto \eta_p \in \mathcal{H} \) is a (non-unique measurable) family of unit vectors [9, 10, 12]. Let \( \mathcal{K} \) be the closure of the image of the mapping
\[ L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} \varphi(p)\eta_p dp \in \mathcal{H}. \]
One may assume that \( \eta_p \in \mathcal{K} \) for all \( p \in \mathbb{R} \) [12]. Define the (unique) rank of \( F_\theta \) as
\[ \text{rank } F_\theta := \dim \mathcal{K}. \]

It is easy to see that, for a fixed \( \theta \), fuzzy rotated quadratures form a convex set. Similarly as in the case of phase, we have the following theorems [12]:

**Theorem 6** \( F_\theta \) is extreme if and only if, for any bounded operator \( A : \mathcal{K} \rightarrow \mathcal{K} \),
\[ \langle \eta_p | A\eta_p \rangle = 0 \quad \text{for almost all } p \in \mathbb{R}, \]
implies that \( A = 0 \).

¶ The choice \( \mathcal{K} \) gives a minimal Kolmogorov decomposition for a certain positive measurable field of operators. It is unique up to a unitary transformation [12].
Theorem 7 There exist extreme fuzzy rotated quadratures of any rank \( \in \{1, 2, \ldots, \infty\} \).

Moreover, any \( P_{\theta} \) of the rank 1 is a spectral measure, extreme, and unitarily equivalent to the sharp quadrature observable \( Q_{\theta} \) (for which \( \langle \eta_p | \eta_{p'} \rangle \equiv 1 \)), that is,

\[
\langle \varphi | P_{\theta}(X) \psi \rangle = \frac{1}{2\pi} \int_X \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(p-p')x} e^{i(\alpha_p-\alpha_{p'})} \overline{\varphi(p)} \psi(p') dp dp' dx
\]

where \( \alpha_p \in [0, 2\pi) \) and the unitary operator \( U \) is given by \( (U \psi)(p) := e^{i\alpha_p} \psi(p) \). As a spectral measure, \( Q_{\theta} \) is extremal in the convex set of all POMs \( \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H}) \). If the rank \( P_{\theta} > 1 \) then \( P_{\theta} \) cannot be a spectral measure [12].

Remark 8 If, in addition to the covariance condition [5], a fuzzy rotated quadrature \( P_{\theta} \) satisfies the invariance condition

\[
e^{ipQ_{\theta}} P_{\theta}(X) e^{-ipQ_{\theta}} = P_{\theta}(X)
\]

for all \( p \) and \( X \), then it is the following convolution:

\[
P_{\theta}(X) = \int_{\mathbb{R}} \rho(X-x) d\Pi_{Q_{\theta}}(x)
\]

where \( \rho \) is a probability measure on \( \mathbb{R} \) and \( \Pi_{Q_{\theta}} \) is a spectral measure of \( Q_{\theta} \) [3]. Hence, \( P_{\theta} \) is then a postprocessing of \( Q_{\theta} \) [2].

5. Discussion

Since there is no phase shift covariant spectral measures (self-adjoint operators) [17], the quantum phase problem is a true example of the case where the conventional formulation of quantum mechanics, where observables are self-adjoint operators, cannot be sufficient. We have seen that some properties of \( Q_{\theta} \) and \( E_{\text{can}} \) correspond each other except that \( E_{\text{can}} \) is not a spectral measure and thus a conventional observable. It should be stressed that, since \( E_{\text{can}} \) is extreme in the set of all observables, it cannot be considered as a noisy measurement of any spectral measure [7]. This underlines the canonicity of \( E_{\text{can}} \).

The results of this paper can be generalized for (almost) any observables, that is, for POMs. Various classes of observables correspond to the solutions of covariance systems with particular symmetry groups associated to them [8, 11]. But quite rarely covariance systems admit spectral measure solutions. As Holevo suggests in [11], the canonical quantization must be generalized to the context of covariance systems. In the same paper, he solves covariance systems in the case of type I symmetry groups. The most used symmetry groups in physics are of type I, so that the characterization is quite extensive.

As we have seen, a covariance system may have infinite number of solutions (covariant observables), so that it is important to find the physically most reasonable ones. Since covariant POMs form a convex set, its extremals are good candidates for
these observables (they describe pure measurements). In [1, 5, 6, 12], extremals are characterized for rather broad classes of covariance systems.

The final problem is to find ‘canonical’ observable(s) from the set of extremals. As shown in this paper, it is possible for phase observables although there is no projection valued phase observables at all.

Acknowledgments

The author thanks Pekka Lahti for the carefully reading of the manuscript.

References

[1] C. Carmeli, T. Heinosaari, J.-P. Pellonpää and A. Toigo, "Extremal covariant positive operator valued measures: The case of a compact symmetry group", J. Math. Phys. 49, 063504 (16 pp) (2008).
[2] C. Carmeli, T. Heinosaari, J.-P. Pellonpää and A. Toigo, "Optimal covariant measurements: the case of a compact symmetry group and phase observables", J. Phys. A: Math. Theor. 42, 145304 (18 pp) (2009).
[3] C. Carmeli, T. Heinonen, and A. Toigo, "Position and momentum observables on \( \mathbb{R} \) and on \( \mathbb{R}^3 \)", J. Math. Phys. 45, 2526-2539 (2004).
[4] G. Cassinelli, E. De Vito, P. Lahti, and J.-P. Pellonpää, "Covariant localizations in the torus and the phase observables", J. Math. Phys. 43, 693-704 (2002).
[5] G. Chiribella and G. M. D’Ariano, "Extremal covariant positive operator valued measures", J. Math. Phys. 45, 4435-4447 (2004).
[6] G. M. D’Ariano, "Extremal covariant quantum operations and positive operator valued measures", J. Math. Phys. 45, 3620-3635 (2004).
[7] T. Heinosaari and J.-P. Pellonpää, "The canonical phase measurement is pure", Phys. Rev. A (R), in press, arXiv:0909.4166
[8] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (North Holland, Amsterdam-NY, 1982).
[9] A. S. Holevo, "Generalized imprimitivity systems for Abelian groups", Sov. Math. (Iz. VUZ) 27, 53-80 (1983).
[10] A. S. Holevo, "Covariant measurements and imprimitivity systems", Lecture Notes in Mathematics 1055, 153-172 (1984).
[11] A. S. Holevo, "On a generalization of canonical quantization", Math. USSR Izvestya 28, 175-188 (1987).
[12] A. S. Holevo and J.-P. Pellonpää, "Extreme covariant observables for type I symmetry groups", Found. Phys. 39, 625-641 (2009).
[13] J. Kiukas and P. Lahti, "On the moment limit of quantum observables, with an application to the balanced homodyne detection", J. Mod. Opt. 55, 1175-1198 (2008).
[14] J. Kiukas and P. Lahti, "A note on the measurement of phase space observables with an eight-port homodyne detector", J. Mod. Opt. 55, 1891-1898 (2008).
[15] J. Kiukas and J.-P. Pellonpää, "A note on infinite extreme correlation matrices”, Linear Algebra Appl. 428, 2501-2508 (2008).
[16] P. J. Lahti and M. Maczynski, "Coherent states and number-phase uncertainty relations", Int. J. Theor. Phys. 37, 265-272 (1998).
[17] P. J. Lahti and J.-P. Pellonpää, "Covariant phase observables in quantum mechanics", J. Math. Phys. 40, 4688-4698 (1999).
[18] P. J. Lahti and J.-P. Pellonpää, "Characterizations of the canonical phase observable", J. Math. Phys. 41, 7352-7381 (2000).

[19] P. J. Lahti and J.-P. Pellonpää, "The Pegg-Barnett formalism and covariant phase observables", Phys. Scr. 66, 66-70 (2002).

[20] U. Leonhardt, Measuring the Quantum State of Light (Cambridge University Press, 1997).

[21] J.-P. Pellonpää, Covariant Phase Observables in Quantum Mechanics (Annales Universitatis Turkuensis A I, number 288, 188 pages, 2002, University of Turku, Finland). PhD-thesis. Available in: https://oa.doria.fi/handle/10024/5808

[22] N. G. Walker and J. E. Carroll, "Multiport homodyne detection near the quantum noise limit", Opt. Quant. Electron. 18, 355-363 (1986).