OPTIMAL CUR MATRIX DECOMPOSITIONS

CHRISTOS BOUTSIDIS† AND DAVID P. WOODRUFF‡

Abstract. The CUR decomposition of an \( m \times n \) matrix \( A \) finds an \( m \times c \) matrix \( C \) with a subset of \( c < n \) columns of \( A \), together with an \( r \times n \) matrix \( R \) with a subset of \( r < m \) rows of \( A \), as well as a \( c \times r \) low-rank matrix \( U \) such that the matrix CUR approximates the matrix \( A \), that is, \( \| A - CUR \|_F^2 \leq (1 + \varepsilon)\| A - A_k \|_F^2 \), where \( \| \cdot \|_F \) denotes the Frobenius norm and \( A_k \) is the best \( m \times n \) matrix of rank \( k \) constructed via the SVD. We present input-sparsity-time and deterministic algorithms for constructing such a CUR decomposition where \( c = O(k/\varepsilon) \) and \( r = O(k/\varepsilon) \) and rank(\( U \)) = \( k \). Up to constant factors, our algorithms are simultaneously optimal in \( c, r \), and rank(\( U \)).

Key words. Randomized Algorithms, Numerical Linear Algebra, Low-rank Approximations.

AMS subject classifications. 15B52, 15A18, 11K45

1. Introduction. Given as inputs a matrix \( A \in \mathbb{R}^{m \times n} \) and integers \( c < n \) and \( r < m \), the CUR factorization of \( A \) finds \( C \in \mathbb{R}^{m \times c} \) with \( c \) columns of \( A \), \( R \in \mathbb{R}^{r \times n} \) with \( r \) rows of \( A \), and \( U \in \mathbb{R}^{n \times r} \) such that \( A = CUR + E \). Here, \( E = A - CUR \) is the residual error matrix. Compare this to the SVD factorization (let \( k < \text{rank}(A) \)), \( A = U_k \Sigma_k V_k^T + A_{p-k} \). The SVD residual error \( A_{p-k} \) is the best possible, under some rank constraints (see Section 3.2). The matrices \( U_k \in \mathbb{R}^{m \times k} \) and \( V_k \in \mathbb{R}^{n \times k} \) contain the top \( k \) left and right singular vectors of \( A \), while \( \Sigma_k \in \mathbb{R}^{k \times k} \) contains the top \( k \) largest singular values of \( A \). In CUR, \( C \) and \( R \) contain actual columns and rows of \( A \), a property which is desirable for feature selection and data interpretation [36]. This last property makes CUR attractive in a wide range of applications [36].

From an algorithmic perspective, the challenge is to construct \( C, U, \) and \( R \) quickly to minimize the approximation error \( \| A - CUR \|_F^2 \). Definition 1.1 states the precise optimization problem:

Definition 1.1 (The CUR Problem). Given \( A \in \mathbb{R}^{m \times n} \) of rank \( \rho = \text{rank}(A) \), rank parameter \( k < \rho \), and accuracy parameter \( 0 < \varepsilon < 1 \), construct \( C \in \mathbb{R}^{m \times c} \) with \( c \) columns from \( A \), \( R \in \mathbb{R}^{r \times n} \) with \( r \) rows from \( A \), and \( U \in \mathbb{R}^{n \times r} \), with \( c, r \), and rank(\( U \)) being as small as possible, in order to reconstruct \( A \) within relative-error:

\[
\| A - CUR \|_F^2 \leq (1 + \varepsilon)\| A - A_k \|_F^2.
\]

Here, \( A_k = U_k \Sigma_k V_k^T \in \mathbb{R}^{m \times n} \) is the best rank \( k \) matrix obtained via the SVD of \( A \).

Despite the significant amount of work and progress on CUR, spanning both the numerical linear algebra community [47, 48, 29, 28, 33, 41, 39, 30] and the theoretical computer science community [17, 22, 20, 21, 23, 36, 5, 27, 49], there are several important questions which still remain unanswered:

1. **Optimal CUR:** Are there any \((1 + \varepsilon)\)-error CUR algorithms selecting the optimal number of columns and rows?
2. **Rank \( k \) CUR:** Are there any \((1 + \varepsilon)\)-error CUR algorithms constructing \( U \) with optimal rank?
3. **Input-sparsity-time CUR:** Are there any \((1 + \varepsilon)\)-error algorithms for CUR running in time proportional to the number of the non-zero entries of \( A \)?
4. **Deterministic CUR:** Are there any deterministic, polynomial-time, \((1 + \varepsilon)\)-error CUR algorithms?

* An extended abstract appeared at the 46th ACM Symposium on Theory of Computing (STOC).
† Yahoo! Labs, New York, NY. Email: boutsidis@yahoo-inc.com
‡ IBM Research, Almaden, CA. Email: dpwoodru@us.ibm.com
Questions 1 and 4 were, for example, asked in Question 12 in the IITK list of “Open Problems in Data Streams and Related Topics” [37]. This article settles all of these questions via presenting two novel CUR algorithms. Our first algorithm is randomized and runs in “input-sparsity-time” (see Section 6); our second algorithm is deterministic and runs in polynomial time (see Section 7). Both algorithms achieve a relative-error bound with $c = O(k/\epsilon)$ columns, $r = O(k/\epsilon)$ rows, and rank($U$) = $k$. Additionally, a matching lower bound is proven (see Section 8), indicating that, up to constant factors, both algorithms select the optimal number of columns and rows and construct $U$ with the optimal rank($U$).

We also present an optimal randomized CUR algorithm in Section 5. This algorithm might be slower than the algorithm in Section 7, especially on sparse matrices, but it is simpler to describe and analyze, hence we use it as a stepping stone for the algorithms in Section 6 and Section 7. However, this algorithm might be faster when the input matrix is dense.

1.1. Outline. We summarize the main contributions of this paper in Subsection 1.2. In this subsection, we provide only a high level description of our results, hiding low level details and ideas. We give a summary of our techniques on how to obtain optimal CUR algorithms in Subsection 1.3. In Algorithm 1, we present a so-called proto-algorithm for an optimal, relative-error, rank $k$ CUR. Our algorithms in later sections (Algorithm 2 in Section 5, Algorithm 3 in Section 6, and Algorithm 4 in Section 7) are specific instances of this proto-algorithm. As we explain in Subsection 1.3, each instance corresponds to a specific implementation of the various steps in the proto-algorithm in order to obtain the desired properties; for example, to design a deterministic algorithm all steps in the proto-algorithm should be implemented in a deterministic way, etc. Section 2 summarizes results from prior literature and puts our CUR algorithms in context (see Table I). To design our CUR algorithms in Sections 5, 6, and 7 we need several “subset selection tools” from prior literature, which we summarize in Section 3, as well as new tools, which we present in Section 4. Finally, we give a lower bound for a CUR algorithm in Section 8.

1.2. Summary of contributions. This work has two main technical contributions answering all four open problems related to CUR that we described in Section 1.

The first contribution is an input-sparsity-time, relative-error CUR algorithm selecting an optimal number of columns and rows and constructing $U$ with optimal rank. The theorem below presents our main result.

**Theorem 1.2.** [Restatement of Theorem 6.1 in Section 6] Given $A \in \mathbb{R}^{m \times n}$ of rank $\rho$, a target rank $1 \leq k < \rho$, and $0 < \epsilon < 1$, there exists a randomized algorithm to select at most $c = O(k/\epsilon)$ columns and at most $r = O(k/\epsilon)$ rows from $A$ to form matrices $C \in \mathbb{R}^{m \times c}$, $R \in \mathbb{R}^{r \times n}$ and $U \in \mathbb{R}^{c \times r}$ with rank ($U$) = $k$ such that, with constant probability of success,

$$\|A - CUR\|_F^2 \leq (1 + O(\epsilon))\|A - A_k\|_F^2.$$ 

The matrices $C$, $U$, and $R$ can be computed in time

$$O(nnz(A) \log n + (m + n) \cdot \text{poly} (\log n, k, 1/\epsilon)).$$
To the best of our knowledge, this is the first algorithm to achieve a relative-
error CUR decomposition with $O(k/\epsilon)$ number of rows and columns. This number of
rows and columns is asymptotically optimal: Theorem 8.1 proves a matching lower
bound. The previous best such relative-error CUR algorithm [49] selects $c = O(k/\epsilon)$
columns and $r = O(k/\epsilon^2)$ rows from $A$. The first ever relative-error CUR algorithm
selects $c = O(k \log k/\epsilon^2)$ columns and $r = O(c \log c/\epsilon^2)$ rows [23]. Additionally, the
algorithms in [23, 49] construct $U$ with rank($U$) = $\omega(k)$, while we obtain rank($U$) = $k$,
which is also optimal, up to a constant 2, according to the lower bound in Theorem 8.1.

The running time of the algorithm in Theorem 1.2 is proportional to the number of the
non-zero entries of $A$. Recent progress on sketching-based numerical linear
algebra [11, 38, 40] has provided very accurate approximation algorithms that run in
time proportional to the number of the non-zero entries of the input matrix (input-
sparisity-time). Although [11, 38, 40] study several important problems, including
low-rank matrix approximation and least-squares regression, the question of whether
an input-sparsity-time CUR algorithm exists remained unexplored. To the best of
our knowledge, the only prior work addressing CUR on sparse matrices is in [4, 46].
Although high quality implementations are provided, the theoretical results are lack-
ing.

An important open problem in [23, 1, 37] is whether there exists a polynomial-
time deterministic relative-error CUR algorithm. We address this in Theorem 7.1.
The second main contribution of this work, our deterministic, relative-error CUR
algorithm runs in time $O(mn^3k/\epsilon)$. Additionally, it constructs $C$ with $c = O(k/\epsilon)$
columns, $R$ with $r = O(k/\epsilon)$ rows, and $U$ of rank $k$. The work in [31] provides
a relative-error, deterministic algorithm, based on volume sampling, constructing $C$
with $O(k/\epsilon)$ columns of $A$ such that $\|A - CC^TA\|^2_F \leq (1 + \epsilon)\|A - A_k\|^2_F$. It is not
obvious how to extend this to a rank $k$, column-based, relative-error decomposition,
which is a harder instance of the problem. The best deterministic algorithm for a
rank $k$, column-based, decomposition achieves $\|A - CC^TA\|^2_F \leq (2 + \epsilon)\|A - A_k\|^2_F$,
when $C$ has $O(k/\epsilon)$ columns of $A$ [6].

1.3. Overview of techniques. In this section, we give a high level overview of our
approach and explain the key points of our CUR algorithms. Our starting point
is the following tool from prior work which connects matrix factorizations and column
subset selection.

**Lemma 1.3 (Lemma 3.1 in [6])**. Let $A = AZZ^T + E \in \mathbb{R}^{m \times n}$ be a low-rank
matrix factorization of $A$, with $Z \in \mathbb{R}^{n \times k}$, and $Z^TZ = I_k$. Let $S \in \mathbb{R}^{n \times c}$ ($c \geq k$) be
any matrix such that rank($Z^TS$) = rank($Z$) = $k$. Let $C = AS \in \mathbb{R}^{m \times c}$. Then,

$$
\|A - CC^TA\|^2_F \leq \|A - \Pi_{C,k}^E(A)\|^2_F \leq \|E\|^2_F + \|ES(Z^TS)\|^2_F.
$$

Here, $\Pi_{C,k}^E(A) = CX_{opt} \in \mathbb{R}^{m \times n}$, where $X_{opt} \in \mathbb{R}^{c \times n}$ has rank at most $k$ and $CX_{opt}$
is the best rank $k$ approximation to $A$ in the column span of $C$.

If $S$ samples columns from $A$, i.e., $C = AS$ consists of columns of $A$, then, using
this lemma, one obtains a bound for a column-based, low-rank matrix approximation
of $A$ (this bound depends on the specific choice of $Z$). The crux of our approach
in obtaining an optimal, relative-error, rank $k$ CUR is to use this lemma twice and
adaptively, one time to sample columns from $A$ and the other time to sample rows.
Here, by adaptively we mean that the two sampling processes are not independent
from each other: the rows sampled in the second application of the lemma depend on
the columns sampled in the first application. Next, we present precisely an overview of this approach.

Assume that for an appropriate matrix $Z_1 \in \mathbb{R}^{n \times k}$ we can find $S_1 \in \mathbb{R}^{n \times c_1}$ that samples $c_1 = O(k)$ columns from $A$ such that after applying Lemma 1.3 with $A$ and $Z_1$ gives $(C_1 = AS_1; E_1 = A - AZ_1Z_1^\dagger)$:

$$\|A - C_1C_1^\dagger A\|_F^2 \leq \|E_1\|_F^2 + \|E_1S_1(Z_1^\dagger S_1)^\dagger\|_F^2 \leq O(1)\|A - AZ_1Z_1^\dagger\|_F^2 \leq O(1)\|A - A_k\|_F^2,$$

where in the third inequality we further assumed that

$$\|E_1\|_F^2 = \|A - AZ_1Z_1^\dagger\|_F^2 \leq O(1)\|A - A_k\|_F^2,$$

i.e., $Z_1$ approximates, within a constant factor, the best rank $k$ SVD of $A$ (one obvious choice for $Z_1$ is the matrix $V_k$ from the rank $k$ SVD of $A$; however, since this choice is costly we will use methods which approximate the SVD - see Section 3.3). The bound in the second inequality also requires that the term $\|E_1S_1(Z_1^\dagger S_1)^\dagger\|_F$ is sufficiently small, specifically $\|E_1S_1(Z_1^\dagger S_1)^\dagger\|_F^2 \leq O(1)\|E_1\|_F^2$. This can be achieved by choosing the sampling matrix $S_1$ carefully (we discuss below what choices of the matrix $S_1$ are appropriate).

Once we have this matrix $C_1$ with $O(k)$ columns of $A$, that approximates the best rank $k$ matrix within a constant factor, we can use the adaptive sampling method of [14] (see Lemma 3.9 in our paper) and additionally sample $O(k/\varepsilon)$ columns to obtain a matrix $C$ with $c = O(k) + O(k/\varepsilon)$ columns for which

$$\|A - CC^\dagger A\|_F^2 \leq \|A - CX_{opt}\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2. \quad (1.1)$$

Here, the adaptive sampling step turns a constant factor approximation to a relative-error one by sampling an additional of $O(k/\varepsilon)$ columns. $X_{opt} \in \mathbb{R}^{c \times n}$ has rank at most $k$ and $CX_{opt}$ is the best rank $k$ approximation to $A$ in $\text{span}(C)$. (See Section 3.5.1 for the exact computation of $X_{opt}$). This adaptive sampling step to turn a constant factor approximation to a relative-error one by increasing the number of columns slightly has been previously used in the near-optimal algorithms in [6]. This $C$ would be the matrix with columns of $A$ in the CUR factorization that we aim.

The main idea now is to use again Lemma 1.3 and sample rows from $A$, i.e., apply the lemma to $A^\top$. If $C$ had orthonormal columns, then it could immediately play the role of $Z$ in Lemma 1.3. In that case, $E = A^\top - A^\top CC^\top$, and we already have a bound for that term from the discussion above. However, $C$ is not orthonormal. But we could hope that we can find such an orthonormal matrix with a similar bound as in Eqn. 1.1. An obvious choice is to take $Z$ to be an orthonormal basis of $C$. However, this choice is not desirable because in that case $Z$ would have dimensions $m \times c$; hence, in order to satisfy the rank assumption in Lemma 1.3, we would need to sample at least $c$ rows from $Z \in \mathbb{R}^{m \times c}$. Recall that $c = O(k/\varepsilon)$, hence $O(k/\varepsilon)$ rows would be needed to be sampled in this step. This is not desirable because we aim to obtain a CUR decomposition with only $O(k/\varepsilon)$ rows, and to achieve that, we cannot afford to select $O(k/\varepsilon)$ rows at this step. We can only afford $O(k)$ rows (because otherwise the adaptive sampling step in the next step, which seems unavoidable, would sample $O(k/\varepsilon^2)$ rows from $A$). So, what we really want is some matrix $Z$ in the column span of $C$ such that $ZZ^\top A$ approximates $A$ as good as $CX_{opt}$ and at the same time the
number of columns of $Z$ is $O(k)$. Towards this end, we construct (see Lemma 3.12) $Z_2 \in \mathbb{R}^{m \times k}$ ($Z_2 \in \text{span}(C)$) with
\[
\|A^T - A^T Z_2 Z_2^T \|_F^2 \leq (1 + O(\varepsilon)) \|A - CX_{opt}\|_F^2.
\] (1.2)

Now, we want to apply Lemma 1.3 to $A^T$ and $Z_2 \in \mathbb{R}^{m \times k}$. Assume that we can find $S_2 \in \mathbb{R}^{m \times r_1}$, i.e., $R_1 = (A^T S_2)^T \in \mathbb{R}^{r \times n}$, with $r_1 = O(k)$ rows, such that $(E_2^† = A^T - A^T Z_2 Z_2^T Z_2)$:
\[
\|A - AR_1^† R_1\|_F^2 \leq \|E_2\|_F^2 + \|E_2 S_2(Z_2^T S_2)^T\|_F^2
\]
\[
= \|A - Z_2 Z_2^T A\|_F^2 + \|(A - Z_2 Z_2^T A) S_2(Z_2^T S_2)^T\|_F^2
\]
\[
\leq O(1) \|A - Z_2 Z_2^T A\|_F^2
\] (1.3)

The last inequality in Equation 1.3 also requires that $\|E_2 S_2(Z_2^T S_2)^†\|_F^2 = O(1) \|E_2\|_F^2$, which is possible after constructing the matrix $S_2$ appropriately.

So far, we have $C$ and a subset of rows of $R$ in the optimal CUR that we would like to construct. We need two additional steps: (i) we use the adaptive sampling method for CUR that appeared in [49] and further sample another $r_2 = O(k/\varepsilon)$ rows in $R_1$, i.e., construct $R \in \mathbb{R}^{r \times n}$ with $r = O(k + k/\varepsilon)$ rows, and (ii) for $Z_2 Z_2^T A R^1 R$, we replace $Z_2$ with $C \Theta$, for an appropriate $\Theta$ (such a $\Theta$ always exists because $Z_2$ is in the span of $C$) and take $U = \Theta Z_2^T A R^† \in \mathbb{R}^{c \times r}$. So overall,

$$\text{CUR} = Z_2 Z_2^T A R^1 R,$$

and the bound is,
\[
\|A - Z_2 Z_2^T A R^1 R\|_F^2 \leq \|A - Z_2 Z_2^T A\|_F^2 + \frac{\text{rank}(Z_2 Z_2^T A)}{r_2} \|A - AR_1^† R_1\|_F^2
\]
\[
= \|A - Z_2 Z_2^T A\|_F^2 + O(\varepsilon) \|A - AR_1^† R_1\|_F^2
\]
\[
\leq (1 + O(\varepsilon)) \|A - Z_2 Z_2^T A\|_F^2
\]
\[
\leq (1 + O(\varepsilon)) \|A - CX_{opt}\|_F^2
\]
\[
\leq (1 + O(\varepsilon)) \|A - A_k\|_F^2
\]

The first inequality is from the adaptive sampling argument (see also Lemma 3.10). The equality is from our choice of $r_2$. The second inequality is from Eqn. 1.3. The third inequality is from Eqn. 1.2. The last inequality is from Eqn. 1.1.

From all the above derivations, we now identify four sufficient conditions that give an optimal, relative-error, rank $k$ CUR. We call these conditions primitives. Designing matrices $C$, $U$, and $R$ satisfying those basic primitives, an optimal, relative-error, rank $k$ CUR is secured.

### 1.3.1. CUR Primitives

To construct an optimal, relative-error, and rank-$k$ CUR, we relied on four basic primitives:

1. There is $C \in \mathbb{R}^{m \times c}$, $X_{opt} \in \mathbb{R}^{c \times n}$ with a relative-error bound to $A - A_k$ and $c = O(k/\varepsilon)$: Equation 1.1.
2. There is $Z_2 \in \mathbb{R}^{m \times k}$ ($Z_2 \in \text{span}(C)$ and $Z_2^T Z_2 = I_k$) with a relative-error bound to $A - CX_{opt}$: Eqn. 1.2.
3. There is $R_1 \in \mathbb{R}^{r \times n}$ with a constant factor error to $A - Z_2 Z_2^T A$ and $r_1 = O(k)$: Equation 1.3.
Algorithm 1 A proto-algorithm for an optimal, relative-error, rank-$k$ CUR

Input: $A \in \mathbb{R}^{m \times n}$; rank parameter $k < \text{rank}(A)$; accuracy parameter $0 < \varepsilon < 1$.
Output: $C \in \mathbb{R}^{m \times c}$ with $c = O(k/\varepsilon)$; $R \in \mathbb{R}^{r \times n}$ with $r = O(k/\varepsilon)$; $U \in \mathbb{R}^{c \times r}$ with $\text{rank}(U) = k$.

1. Construct $C$ with $O(k + k/\varepsilon)$ columns
   1: Approximate SVD: find $Z_1 \in \mathbb{R}^{n \times O(k)}$ that approximates $V_k \in \mathbb{R}^{n \times k}$ from the SVD of $A$. Primitive 1
   2: Leverage-scores sampling: Sample $O(k \log k)$ columns from $A$ with probabilities from $Z_1$. Primitive 1
   3: BSS sampling: Downsample these columns to $O(k)$ columns. Primitive 1
   4: Adaptive sampling: Sample $O(k/\varepsilon)$ additional columns. Primitive 1

2. Construct $R$ with $O(k + k/\varepsilon)$ rows
   1: Restricted low-rank approximation: find $Z_2 \in \mathbb{R}^{m \times O(k)}$ with $Z_2 \in \text{span}(C)$. Primitive 2
   2: Leverage-scores sampling: Sample $O(k \log k)$ rows from $A$ with probabilities from $Z_2$. Primitive 3
   3: BSS sampling: Downsample these rows to $O(k)$ rows. Primitive 3
   4: Adaptive sampling: Sample $O(k/\varepsilon)$ additional rows. Primitive 4

3. Construct $U$ of rank $k$
   1: Let $Z_2 Z_2^T A R^\dagger R$; then, replace $Z_2 = C \Theta$, for an appropriate $\Theta$, and use

$$U = \Theta Z_2^T A R^\dagger.$$ 

4. There is an adaptive sampling algorithm for CUR, i.e., an algorithm that turns a constant factor CUR with $O(k/\varepsilon)$ columns and $O(k)$ rows to a relative-error CUR by sampling only $O(k/\varepsilon)$ additional rows.

The user can pick $C, Z_2, R_1$ in the above conditions the way she likes. Below, we discuss various implementation choices which lead to desirable CUR algorithms.

1.3.2. CUR proto-algorithm. To address primitive (1), we combine known ideas for column subset selection including leverage-scores sampling [23], BSS sampling [6] (i.e., deterministic sampling similar to the method of Batson, Spielman, and Srivastava [3]) and adaptive sampling [15] (see Section 3.4). To find $Z_1$, we use techniques for approximating the SVD (see Section 3.3). To address primitive (2) we use methods to find low-rank approximations within a subspace (see Section 3.5). To address primitive (3), again we employ leverage-score sampling [23] and BSS sampling [6], as in primitive (1). Finally, to address primitive (4), we use an adaptive sampling method [49] which turns a constant factor CUR to relative error by sampling an additional $O(k/\varepsilon)$ rows. We summarize this in Algorithm 1, which we call proto-algorithm for an optimal, relative-error, rank $k$ CUR. To obtain a deterministic or an input-sparsity-time CUR, we need to implement the corresponding steps in this proto-algorithm in the appropriate setting. We discuss those issues below.

1.3.3. Input-sparsity-time CUR. To design an algorithm that runs in time proportional to the number of non-zero entries of $A$, we need to implement all these primitives in input-sparsity-time, or equivalently, all the steps in the proto-algorithm
should be implemented in input-sparsity-time. It is already known in the literature how to implement approximate SVD in input-sparsity-time (see Section 6). The leverage-scores sampling step - given that one knows the probabilities - can be implemented fast as well (see Lemma 3.7). For the rest of the steps of the proto-algorithm we develop new tools. First, we design an input-sparsity-time version of the BSS sampling step (see Lemma 4.3). To do that, we combine the method from [6] with ideas from the sparse subspace embedding literature [11]. Second, we develop input-sparsity-time versions of the adaptive sampling algorithms of [15, 49] (see Lemma 4.4 and Lemma 4.5). To do that, we combine existing ideas in [15, 49] with the Johnson-Lindenstrauss lemma (see Lemma 3.17). Third, we develop an input-sparsity-time algorithm to find a relative-error (to the best possible) low-rank matrix within a given subspace (see Lemma 3.12). To do that, we combine ideas for subspace-restricted matrix approximations (see Section 3.5.1) with ideas from [34, 11]. Fourth, we present a method to compute a rank $k$ matrix $U$ in input sparsity time. To do that, we took the original construction of $U$, which is a series of products of various matrices, and reduced the dimensions of some of those matrices using the sparse subspace embeddings in [11] (see Lemma 6.8). The crux in this part is to “view” the computation of $U$ as the solution of a generalized matrix approximation problem (see Section 3.8) and then apply sketching ideas to this problem.

1.3.4. Deterministic CUR. To design a deterministic CUR, we need to implement the CUR proto-algorithm in a deterministic way. All steps can be implemented deterministically with tools from prior work (Section 7). The only piece missing is a deterministic version of adaptive sampling [15, 49], which we obtain by derandomizing the adaptive sampling algorithms in [15, 49] (Section 4.1.1).

2. Related work. Randomized algorithms for CUR [17, 20, 23, 36, 27, 49] are an instance of a large body of work on approximation algorithms for matrix computations [25, 16, 17, 18, 19, 20, 15, 44, 42, 10, 32, 11, 7, 50]. Those algorithms provide a new paradigm and a complementary perspective in speeding up basic kernels in numerical linear algebra, such as matrix multiplication [16], least-squares regression [44], and low-rank matrix approximation [7]. The application areas of those algorithms are broad, as discussed in the survey article [32].

Before discussing the details of some of the available CUR algorithms in [17, 20, 23, 36, 27, 49], we briefly mention a similar problem which constructs factorizations of the form $A = CX + E$, where $C$ contains columns of $A$ and $X$ has rank at most $k$. Unlike CUR, there are optimal algorithms for this problem [6, 31], in both the spectral and the Frobenius norm. Indeed, to obtain a relative-error optimal CUR in this paper we use a sampling method from [6], which allows to select $O(k)$ columns and rows. For a more detailed discussion of this CX problem, which is also known as CSSP (Column Subset Selection Problem) see [8, 6, 31].

Drineas and Kannan brought CUR factorizations to the theoretical computer science community in [17]. Their main algorithm is randomized and samples columns and rows from $A$ with probabilities proportional to their Euclidian length. The running time of this algorithm is linear in $m$ and $n$ and proportional to a small-degree polynomial in $k$ and $1/\varepsilon$, for some $\varepsilon > 0$, but the approximation bound is weak (see Theorem 3.1 in [17]): with $c = O(k/\varepsilon^2)$ columns and $r = O(k/\varepsilon)$ rows the bound is $\|A - \text{CUR}\|_F^2 \leq \|A - A_k\|_F^2 + \varepsilon \|A\|_F^2$. Drineas, Kannan, and Mahoney [20] built upon [17] but the error remained additive (see Theorem 5 in [20]): with $c = O(k/\varepsilon^4)$ columns and $r = O(k/\varepsilon^2)$ rows the error is $\|A - \text{CUR}\|_F^2 \leq \|A - A_k\|_F^2 + \varepsilon \|A\|_F^2$. The first relative-error CUR algorithm appeared in [23] (see Theorem 2 of [23]). The al-
algorithms presented in [36] and [23], respectively. We first review the results of Mahoney et al. [36] and then present our new algorithm.

**Algorithm 1:**

1. Compute the leverage scores of A.
2. Sample columns of A according to the leverage scores.
3. Compute the CUR decomposition of the sampled matrix.

**Algorithm 2:**

1. Compute the leverage scores of A.
2. Compute the subspace sampling of A.
3. Compute the CUR decomposition of the subsampled matrix.

These algorithms provide a trade-off between the quality of the CUR decomposition and the computational cost. The first algorithm is more efficient in terms of computational cost but may lead to a less accurate decomposition. The second algorithm provides better accuracy but requires more computation.

We now explain why the two relative-error CUR algorithms in Table 2.1 [23, 49] require more columns and rows in C and R, and larger rank(U), than our optimal CUR algorithms. First, both algorithms [23, 49] use Lemma 1.3 twice and adaptively, as we do (this is not explicitly stated in those articles but this claim can be validated after a careful comparison of their proofs with ours). Drineas et al. [23] implement the first step of the proto-algorithm in Algorithm 1 via leverage-scores sampling only. At that time, the BSS sampling was unknown and the adaptive sampling step wouldn’t be particularly helpful. For the second step in the proto-algorithm, they simply choose $Z_2$ as an orthonormal basis for C, hence $r = O(c \log c / \varepsilon^2)$ rows are required, where $c = O(k \log k / \varepsilon^2)$ are the columns selected in the first step. Given the above limitations, in the third step of the proto-algorithm, U unavoidably has rank $O(k \log k / \varepsilon^2)$. Wang and Zhang on the other hand [23], motivated by the near-optimal algorithm of Boutsidis et al. [6], take advantage of BSS and adaptive sampling to sample only $c = O(k / \varepsilon)$ columns in the first step of the proto-algorithm. However, they also use $Z_2$ as an orthonormal basis for C; hence $r = O(k / \varepsilon^2)$ and rank(U) = $O(k / \varepsilon)$. The $r = O(k / \varepsilon^2)$ term here is because the row-selection adaptive sampling step requires $O(p / \varepsilon)$ rows, where $p = \text{rank}(Z_2^T Z_2^T A)$. We now explain why the two relative-error CUR algorithms in Table 2.1 [23, 49] require more columns and rows in C and R, and larger rank(U), than our optimal CUR algorithms. First, both algorithms [23, 49] use Lemma 1.3 twice and adaptively, as we do (this is not explicitly stated in those articles but this claim can be validated after a careful comparison of their proofs with ours). Drineas et al. [23] implement the first step of the proto-algorithm in Algorithm 1 via leverage-scores sampling only. At that time, the BSS sampling was unknown and the adaptive sampling step wouldn’t be particularly helpful. For the second step in the proto-algorithm, they simply choose $Z_2$ as an orthonormal basis for C, hence $r = O(c \log c / \varepsilon^2)$ rows are required, where $c = O(k \log k / \varepsilon^2)$ are the columns selected in the first step. Given the above limitations, in the third step of the proto-algorithm, U unavoidably has rank $O(k \log k / \varepsilon^2)$. Wang and Zhang on the other hand [23], motivated by the near-optimal algorithm of Boutsidis et al. [6], take advantage of BSS and adaptive sampling to sample only $c = O(k / \varepsilon)$ columns in the first step of the proto-algorithm. However, they also use $Z_2$ as an orthonormal basis for C; hence $r = O(k / \varepsilon^2)$ and rank(U) = $O(k / \varepsilon)$. The $r = O(k / \varepsilon^2)$ term here is because the row-selection adaptive sampling step requires $O(p / \varepsilon)$ rows, where $p = \text{rank}(Z_2^T Z_2^T A)$. We now explain why the two relative-error CUR algorithms in Table 2.1 [23, 49] require more columns and rows in C and R, and larger rank(U), than our optimal CUR algorithms. First, both algorithms [23, 49] use Lemma 1.3 twice and adaptively, as we do (this is not explicitly stated in those articles but this claim can be validated after a careful comparison of their proofs with ours). Drineas et al. [23] implement the first step of the proto-algorithm in Algorithm 1 via leverage-scores sampling only. At that time, the BSS sampling was unknown and the adaptive sampling step wouldn’t be particularly helpful. For the second step in the proto-algorithm, they simply choose $Z_2$ as an orthonormal basis for C, hence $r = O(c \log c / \varepsilon^2)$ rows are required, where $c = O(k \log k / \varepsilon^2)$ are the columns selected in the first step. Given the above limitations, in the third step of the proto-algorithm, U unavoidably has rank $O(k \log k / \varepsilon^2)$. Wang and Zhang on the other hand [23], motivated by the near-optimal algorithm of Boutsidis et al. [6], take advantage of BSS and adaptive sampling to sample only $c = O(k / \varepsilon)$ columns in the first step of the proto-algorithm. However, they also use $Z_2$ as an orthonormal basis for C; hence $r = O(k / \varepsilon^2)$ and rank(U) = $O(k / \varepsilon)$. The $r = O(k / \varepsilon^2)$ term here is because the row-selection adaptive sampling step requires $O(p / \varepsilon)$ rows, where $p = \text{rank}(Z_2^T Z_2^T A)$. We now explain why the two relative-error CUR algorithms in Table 2.1 [23, 49] require more columns and rows in C and R, and larger rank(U), than our optimal CUR algorithms. First, both algorithms [23, 49] use Lemma 1.3 twice and adaptively, as we do (this is not explicitly stated in those articles but this claim can be validated after a careful comparison of their proofs with ours). Drineas et al. [23] implement the first step of the proto-algorithm in Algorithm 1 via leverage-scores sampling only. At that time, the BSS sampling was unknown and the adaptive sampling step wouldn’t be particularly helpful. For the second step in the proto-algorithm, they simply choose $Z_2$ as an orthonormal basis for C, hence $r = O(c \log c / \varepsilon^2)$ rows are required, where $c = O(k \log k / \varepsilon^2)$ are the columns selected in the first step. Given the above limitations, in the third step of the proto-algorithm, U unavoidably has rank $O(k \log k / \varepsilon^2)$. Wang and Zhang on the other hand [23], motivated by the near-optimal algorithm of Boutsidis et al. [6], take advantage of BSS and adaptive sampling to sample only $c = O(k / \varepsilon)$ columns in the first step of the proto-algorithm. However, they also use $Z_2$ as an orthonormal basis for C; hence $r = O(k / \varepsilon^2)$ and rank(U) = $O(k / \varepsilon)$. The $r = O(k / \varepsilon^2)$ term here is because the row-selection adaptive sampling step requires $O(p / \varepsilon)$ rows, where $p = \text{rank}(Z_2^T Z_2^T A)$. We now explain why the two relative-error CUR algorithms in Table 2.1 [23, 49] require more columns and rows in C and R, and larger rank(U), than our optimal CUR algorithms. First, both algorithms [23, 49] use Lemma 1.3 twice and adaptively, as we do (this is not explicitly stated in those articles but this claim can be validated after a careful comparison of their proofs with ours). Drineas et al. [23] implement the first step of the proto-algorithm in Algorithm 1 via leverage-scores sampling only. At that time, the BSS sampling was unknown and the adaptive sampling step wouldn’t be particularly helpful. For the second step in the proto-algorithm, they simply choose $Z_2$ as an orthonormal basis for C, hence $r = O(c \log c / \varepsilon^2)$ rows are required, where $c = O(k \log k / \varepsilon^2)$ are the columns selected in the first step. Given the above limitations, in the third step of the proto-algorithm, U unavoidably has rank $O(k \log k / \varepsilon^2)$. Wang and Zhang on the other hand [23], motivated by the near-optimal algorithm of Boutsidis et al. [6], take advantage of BSS and adaptive sampling to sample only $c = O(k / \varepsilon)$ columns in the first step of the proto-algorithm. However, they also use $Z_2$ as an orthonormal basis for C; hence $r = O(k / \varepsilon^2)$ and rank(U) = $O(k / \varepsilon)$. The $r = O(k / \varepsilon^2)$ term here is because the row-selection adaptive sampling step requires $O(p / \varepsilon)$ rows, where $p = \text{rank}(Z_2^T Z_2^T A)$. We now explain why the two relative-error CUR algorithms in Table 2.1 [23, 49] require more columns and rows in C and R, and larger rank(U), than our optimal CUR algorithms. First, both algorithms [23, 49] use Lemma 1.3 twice and adaptively, as we do (this is not explicitly stated in those articles but this claim can be validated after a careful comparison of their proofs with ours). Drineas et al. [23] implement the first step of the proto-algorithm in Algorithm 1 via leverage-scores sampling only. At that time, the BSS sampling was unknown and the adaptive sampling step wouldn’t be particularly helpful. For the second step in the proto-algorithm, they simply choose $Z_2$ as an orthonormal basis for C, hence $r = O(c \log c / \varepsilon^2)$ rows are required, where $c = O(k \log k / \varepsilon^2)$ are the columns selected in the first step. Given the above limitations, in the third step of the proto-algorithm, U unavoidably has rank $O(k \log k / \varepsilon^2)$. Wang and Zhang on the other hand [23], motivated by the near-optimal algorithm of Boutsidis et al. [6], take advantage of BSS and adaptive sampling to sample only $c = O(k / \varepsilon)$ columns in the first step of the proto-algorithm. However, they also use $Z_2$ as an orthonormal basis for C; hence $r = O(k / \varepsilon^2)$ and rank(U) = $O(k / \varepsilon)$. The $r = O(k / \varepsilon^2)$ term here is because the row-selection adaptive sampling step requires $O(p / \varepsilon)$ rows, where $p = \text{rank}(Z_2^T Z_2^T A)$.
Finally, there are several interesting results on CUR developed within the numerical linear algebra community [47, 48, 29, 28, 33, 41, 39, 30, 4, 46]. For example, [47, 48, 29, 28] discuss the so-called skeleton approximation, which focuses on the spectral norm version of the CUR problem via selecting exactly \( k \) columns and \( k \) rows. The algorithms there are deterministic, run in time proportional to the time to compute the rank \( k \) SVD of \( A \), and achieve bounds of the order,

\[
\|A - \text{CUR}\|_2 \leq O(\sqrt{k(n-k)} + \sqrt{k(m-k)})\|A - A_k\|_2.
\]

3. Column subset selection tools from existing literature. This section summarizes known techniques and related results from the literature that we use throughout this work.

3.1. A perturbation bound for column-based matrix reconstruction. We start with a restatement of Lemma 1.3. Recall that \( A \) is the matrix that we would like to approximate with a rank \( k \) matrix \( CX \), where \( C \) contains columns of \( A \), or a rank \( k \) matrix CUR where \( C \) contains columns of \( A \) and \( R \) contains rows of \( A \). For an appropriate sampling matrix \( S \), we can write \( C = AS \). The following lemma provides a general perturbation bound for such sampling matrices \( S \), under some rank assumption. We will use this lemma extensively when analyzing our CUR algorithms.

**Lemma 3.1 (Lemma 3.1 in [6]).** Let \( A = AZZ^T + E \in \mathbb{R}^{m \times n} \), with \( Z \in \mathbb{R}^{n \times k} \), and \( Z^TZ = I_k \). Let \( S \in \mathbb{R}^{n \times r} \) be any matrix such that \( \text{rank}(Z^TS) = \text{rank}(Z) = k \). Let \( C = AS \in \mathbb{R}^{m \times r} \). Then,

\[
\|A - CC^TA\|_F^2 \leq \|A - \Pi_{C,k}(A)\|_F^2 \leq \|A - C(Z^TS)^TZ^T\|_F^2 \leq \|E\|_F^2 + \|ES(Z^TS)^T\|_F^2.
\]

Here,

\[
\Pi_{C,k}(A) = CX_{\text{opt}} \in \mathbb{R}^{m \times n},
\]

where \( X_{\text{opt}} \in \mathbb{R}^{r \times n} \) has rank at most \( k \), \( CX_{\text{opt}} \) is the best rank \( k \) approximation to \( A \) in \( \text{span}(C) \), and \( (Z^TS)^T \) denotes the Moore-Penrose pseudoinverse of \( Z^TS \).

3.2. SVD and the pseudo-inverse. The singular value decomposition (SVD) appears often throughout the paper. The SVD of \( A \in \mathbb{R}^{m \times n} \) of rank \( r \leq \min\{m, n\} \) is

\[
A = \begin{pmatrix} U_k & U_{\rho-k} \end{pmatrix} \begin{pmatrix} \Sigma_k & 0 \\ 0 & \Sigma_{\rho-k} \end{pmatrix} \begin{pmatrix} V_k^T \\ V_{\rho-k}^T \end{pmatrix},
\]

with singular values \( \sigma_1(A) \geq \ldots \geq \sigma_k(A) \geq \sigma_{k+1}(A) \geq \ldots \geq \sigma_\rho(A) > 0 \). The matrices \( U_k \in \mathbb{R}^{m \times k} \) and \( U_{\rho-k} \in \mathbb{R}^{m \times (\rho-k)} \) contain the left singular vectors of \( A \); and, similarly, the matrices \( V_k \in \mathbb{R}^{n \times k} \) and \( V_{\rho-k} \in \mathbb{R}^{n \times (\rho-k)} \) contain the right singular vectors. \( \Sigma_k \in \mathbb{R}^{k \times k} \) and \( \Sigma_{\rho-k} \in \mathbb{R}^{(\rho-k) \times (\rho-k)} \) contain the singular values of \( A \). It is well-known that \( A_k = U_k \Sigma_k V_k^T \) minimizes \( \|A - X\|_F \) over all matrices \( X \in \mathbb{R}^{m \times n} \) of rank at most \( k \leq \rho \). We use \( A_{\rho-k} = A - A_k = U_{\rho-k} \Sigma_{\rho-k} V_{\rho-k}^T \). Also, \( \|A\|_F = \sqrt{\sum_{i=1}^\rho \sigma_i^2(A)} \) and \( \|A\|_2 = \sigma_1(A) \). The best rank \( k \) approximation to \( A \) satisfies: \( \|A - A_k\|_2 = \sigma_{k+1}(A) \); \( \|A - A_k\|_F = \sqrt{\sum_{i=k+1}^\rho \sigma_i^2(A)} \).
3.2.1. Pseudo-inverse. \(A^\dagger = V_A \Sigma_A^{-1} U_A^T \in \mathbb{R}^{n \times m}\) denotes the so-called Moore-Penrose pseudo-inverse of \(A \in \mathbb{R}^{m \times n}\) (here \(\Sigma_A^{-1}\) is the inverse of \(\Sigma_A\)), i.e., the unique \(n \times m\) matrix satisfying all four properties: \(A = AA^\dagger A\), \(A^\dagger AA^\dagger = A^\dagger\), \((AA^\dagger)^T = AA^\dagger\), and \((A^\dagger A)^T = A^\dagger A\). By the SVD of \(A\) and \(A^\dagger\), it is easy to verify that, for all \(i = 1, \ldots, \rho = \text{rank}(A) = \text{rank}(A^\dagger)\): \(\sigma_i(A^\dagger) = \frac{1}{\sigma_i(A)}\). Finally, for any \(A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times \ell}\), \((AB)^\dagger = B^\dagger A^\dagger\) if any one of the following three properties hold: (1) \(A^T A = I_n\); (2) \(B^T B = I_\ell\); or, (3) \(\text{rank}(A) = \text{rank}(B) = n\).

3.3. Fast approximate low-rank matrix approximations. The SVD provides the best rank \(k\) matrix \(A_k\) to approximate \(A\); however, it is somewhat costly to compute. In this work, to speedup our CUR algorithms, which extensively make use of the SVD, we use low-rank matrix factorization algorithms that are approximately as good as the SVD but can be implemented considerably faster (e.g., in linear time or in input sparsity time). We describe three such algorithms below. Indeed, we omit the details of the algorithms per se, instead we discuss the approximation bounds and the corresponding running times.

3.3.1. Deterministic approximate SVD. Recent work in [26] describes a deterministic algorithm for relative-error approximate low-rank matrix factorization that requires sub-cubic arithmetic operations.

**Lemma 3.2** (Theorem 3.1 in [26]). Given \(A \in \mathbb{R}^{n \times n}\) of rank \(\rho\), a target rank \(1 \leq k < \rho\), and \(0 < \epsilon \leq 1\), there exists a deterministic algorithm that computes \(Z \in \mathbb{R}^{n \times k}\) with \(Z^T Z = \text{I}_k\) and

\[
\|A - AZZ^T\|_F^2 \leq (1 + \epsilon) \|A - A_k\|_F^2.
\]

The proposed algorithm requires \(O(mnk^2\epsilon^{-2})\) arithmetic operations. We denote this procedure as

\[
Z = \text{Deterministic SVD}(A, k, \epsilon).
\]

**Proof.** Theorem 4.1 in [26] describes an algorithm that given \(A, k\) and \(\epsilon\) constructs \(Q_k \in \mathbb{R}^{k \times n}\) such that \(\|A - AQ_k^T Q_k\|_F^2 \leq (1 + \epsilon) \|A - A_k\|_F^2\). To obtain the desired factorization, we just need an additional step to orthonormalize the columns of \(Q_k^T\), which takes \(O(nk^2)\) time. So, assume that \(Q_k^T = ZR\) is a qr factorization of \(Q_k^T\) with \(Z \in \mathbb{R}^{n \times k}\) and \(R \in \mathbb{R}^{k \times k}\). Then,

\[
\begin{align*}
\|A - AZZ^T\|_F^2 &\leq \|A - AQ_k^T(R^T)Z\|_F^2 \\
&= \|A - AQ_k^T R (R^T)^{-1} Q_k\|_F^2 \\
&= \|A - AQ_k^T Q_k\|_F^2 \\
&\leq (1 + \epsilon) \|A - A_k\|_F^2.
\end{align*}
\]

\(\square\)

3.3.2. Randomized linear-time approximate SVD. The following result corresponds to a standard randomized algorithm for speeding up the SVD.

**Lemma 3.3** (Lemma 3.4 in [6]). Given \(A \in \mathbb{R}^{m \times n}\) of rank \(\rho\), a target rank \(2 \leq k < \rho\), and \(0 < \epsilon \leq 1\), there exists a randomized algorithm that computes \(Z \in \mathbb{R}^{n \times k}\) with \(Z^T Z = \text{I}_k\) and

\[
\mathbb{E} \left[ \|A - AZZ^T\|_F^2 \right] \leq (1 + \epsilon) \|A - A_k\|_F^2.
\]
The proposed algorithm requires $O((mnk^{-1})$ arithmetic operations. We denote this procedure as

$$Z = \text{RandomizedSVD}(A, k, \varepsilon).$$

3.3.3. Randomized input-sparsity-time approximate SVD. Clarkson and Woodruff [11] described a randomized algorithm that runs in time proportional to the number of non-zero entries in $A$ plus low-order terms.

**Lemma 3.4** (Theorem 47 in [12]). Given $A \in \mathbb{R}^{m \times n}$ of rank $\rho$, a target rank $1 \leq k < \rho$, and $0 < \varepsilon \leq 1$, there exists a randomized algorithm that computes $Z \in \mathbb{R}^{n \times k}$ with $Z^T Z = I_k$ and with probability at least 0.99,

$$\|A - AZZ^T\|_F^2 \leq (1 + \varepsilon) \|A - A_k\|_F^2.$$  

The proposed algorithm requires $O(nnz(A) \cdot \tilde{O}(nk^2 \varepsilon^{-4} + k^3 \varepsilon^{-5})$ arithmetic operations. We denote this procedure as

$$Z = \text{SparseSVD}(A, k, \varepsilon).$$

**Proof.** Theorem 47 in [12] describes an algorithm that constructs $L \in \mathbb{R}^{m \times k}$ with orthonormal columns, diagonal $D \in \mathbb{R}^{k \times k}$ and $W \in \mathbb{R}^{n \times k}$ with orthonormal columns such that $\|A - LDW^T\|_F^2 \leq (1 + \varepsilon) \|A - A_k\|_F^2$. We can just use $Z = W$, because $\|A - AWW^T\|_F^2 \leq \|A - LDW^T\|_F^2$. \qed

3.4. Column subset selection techniques. We now summarize the various tools from existing literature that we use in order to sample columns and/or rows from matrices.

3.4.1. Deterministic BSS sampling. The lemma below is a generalization of the spectral sparsification method of Batson, Spielman, and Strivastava (BSS) [3]; it was the main ingredient of the near-optimal algorithm in [6].

**Lemma 3.5** (Dual Set Spectral-Frobenius Sparsification. Lemma 3.6 in [6]). Let $V = \{v_1, \ldots, v_n\}$ be a decomposition of the identity, where $v_i \in \mathbb{R}^k$ ($k < n$) and $\sum_{i=1}^n v_i v_i^T = I_k$; let $A = \{a_1, \ldots, a_n\}$ be an arbitrary set of vectors, where $a_i \in \mathbb{R}^\ell$. Then, given an integer $r$ such that $k < r \leq n$, there exists a set of weights $s_i \geq 0$ ($i = 1 \ldots n$), at most $r$ of which are non-zero, such that

$$\lambda_k \left( \sum_{i=1}^n s_i v_i v_i^T \right) \geq \left( 1 - \sqrt{\frac{k}{r}} \right)^2,$$

$$\text{Tr} \left( \sum_{i=1}^n s_i a_i a_i^T \right) \leq \text{Tr} \left( \sum_{i=1}^n a_i a_i^T \right) = \sum_{i=1}^n \|a_i\|_2^2.$$  

Equivalently, if $V \in \mathbb{R}^{n \times k}$ is a matrix whose rows are the vectors $v_i^T$, $A \in \mathbb{R}^{n \times \ell}$ is a matrix whose rows are the vectors $a_i^T$, and $S \in \mathbb{R}^{n \times r}$ is the sampling matrix containing the weights $s_i > 0$, then:

$$\sigma_k \left(V^T S\right) \geq 1 - \sqrt{\frac{k}{r}}, \quad \|A^T S\|_F^2 \leq \|A^T\|_F^2.$$  

The weights $s_i$ can be computed in $O(nrk^2 + nr\ell)$ time. We denote this procedure as

$$S = \text{BssSampling}(V, A, r).$$
3.4.2. Randomized sampling. Next, we introduce a randomized method for sampling rows from tall-skinny matrices.

**Definition 3.6 (Random Sampling with Replacement [43]).** Let \( \mathbf{X} \in \mathbb{R}^{n \times k} \) with \( n > k \); and \( \mathbf{x}_i^T \in \mathbb{R}^{1 \times k} \) denote the \( i \)-th row of \( \mathbf{X} \) and \( 0 < \beta \leq 1 \). For \( i = 1, \ldots, n \), if \( \beta = 1 \), then \( p_i = (x_i^T x_i) / \| \mathbf{X} \|_F^2 \), otherwise compute some \( p_i \geq \beta (x_i^T x_i) / \| \mathbf{X} \|_F^2 \) with \( \sum_i p_i = 1 \). Let \( r \) be an integer with \( 1 \leq r \leq n \). Construct a sampling matrix \( \mathbf{\Omega} \in \mathbb{R}^{n \times r} \) and a rescaling matrix \( \mathbf{D} \in \mathbb{R}^{r \times r} \) as follows. Initially, \( \mathbf{\Omega} = \mathbf{0}_{n \times r} \) and \( \mathbf{D} = \mathbf{0}_{r \times r} \). Then, for every column \( j = 1, \ldots, r \) of \( \mathbf{\Omega}, \mathbf{D} \), independently, and with replacement, pick an index \( i \) from the set \( \{1, 2, \ldots, n\} \) with probability \( p_i \) and set \( \mathbf{\Omega}_{ij} = 1 \) and \( \mathbf{D}_{jj} = 1 / \sqrt{p_i} \). To denote this \( O(n k + n + r \log(r)) \) time procedure we write

\[
[\mathbf{\Omega}, \mathbf{D}] = \text{RandSampling}(\mathbf{X}, r, \beta).
\]

**Lemma 3.7 (Originally proved in [43]).** Let \( \mathbf{V} \in \mathbb{R}^{n \times k} \) with \( n > k \) and \( \mathbf{V}^T \mathbf{V} = \mathbf{I}_k \). Let \( 0 < \beta \leq 1, 0 < \delta \leq 1 \) and \( 4k \ln(2k / \delta) < r \leq n \). Let \( [\mathbf{\Omega}, \mathbf{D}] = \text{RandSampling}(\mathbf{V}, r, \beta) \). Then, for all \( i = 1, \ldots, k \), and with probability at least \( 1 - \delta \),

\[
1 - \sqrt{\frac{4k \ln(2k / \delta)}{\beta r}} \leq \sigma_i^2(\mathbf{V}^T \mathbf{\Omega} \mathbf{D}) \leq 1 + \sqrt{\frac{4k \ln(2k / \delta)}{\beta r}}.
\]  

**Proof.** This is Theorem 2 of [35] with \( \mathbf{S} = \mathbf{I} \), the identity matrix. \( \blacksquare \)

**Lemma 3.8.** For any \( \beta, r, \mathbf{X} \in \mathbb{R}^{n \times k} \) and \( \mathbf{Y} \in \mathbb{R}^{m \times n} \), let \( [\mathbf{\Omega}, \mathbf{D}] = \text{RandSampling}(\mathbf{X}, r, \beta) \); then, with probability at least 0.9: \( \| \mathbf{Y} \mathbf{\Omega} \mathbf{D} \|_F^2 \leq 10 \| \mathbf{Y} \|_F^2 \).

**Proof.** Eqn. (36) in [21] gives \( \mathbb{E} [\| \mathbf{Y} \mathbf{\Omega} \mathbf{D} \|_F^2] = \| \mathbf{Y} \|_F^2 \); apply Markov’s inequality to wrap up. \( \blacksquare \)

3.4.3. Adaptive sampling. Adaptive sampling aims at sampling columns from the input matrix in an adaptive fashion based on information of the residual error from approximating the matrix with the already sampled columns. The following lemma implements one round of adaptive sampling and will be used extensively in our analysis to improve constant factor column-based matrix approximations to relative-error ones.

**Lemma 3.9 (Theorem 2.1 of [14]).** Given \( \mathbf{A} \in \mathbb{R}^{m \times n} \) and \( \mathbf{V} \in \mathbb{R}^{m \times c_1} \) (with \( c_1 \leq n, m \)), define the residual

\[
\mathbf{B} = \mathbf{A} - \mathbf{V} \mathbf{V}^T \mathbf{A} \in \mathbb{R}^{m \times n}.
\]

For \( i = 1, \ldots, n \), and some fixed constant \( \alpha > 0 \), let \( p_i \) be a probability distribution such that for each \( i \):

\[
p_i \geq \alpha \| \mathbf{b}_i \|_2^2 / \| \mathbf{B} \|_F^2,
\]

where \( \mathbf{b}_i \) is the \( i \)-th column of the matrix \( \mathbf{B} \). Sample \( c_2 \) columns from \( \mathbf{A} \) in \( c_2 \) i.i.d. trials, where in each trial the \( i \)-th column is chosen with probability \( p_i \). Let \( \mathbf{C}_2 \in \mathbb{R}^{m \times c_2} \) contain the \( c_2 \) sampled columns and let \( \mathbf{C} = [\mathbf{V} \quad \mathbf{C}_2] \in \mathbb{R}^{m \times (c_1 + c_2)} \) contain the columns of \( \mathbf{V} \) and \( \mathbf{C}_2 \). Then, for any integer \( k > 0 \),

\[
\mathbb{E} [\| \mathbf{A} - \Pi_{\mathbf{C}_2,k}^F(\mathbf{A}) \|_F^2] \leq \| \mathbf{A} - \mathbf{A}_k \|_F^2 + \frac{k}{\alpha \cdot c_2} \| \mathbf{A} - \mathbf{V} \mathbf{V}^T \mathbf{A} \|_F^2.
\]
We denote this procedure as

\[ C_2 = \text{AdaptiveCols}(A, V, \alpha, c_2). \]

Given \( A \) and \( V \), the above algorithm requires \( O(c_1 mn + c_2 \log c_2) \) arithmetic operations to find \( C_2 \).

A recent result generalizes the above adaptive sampling lemma to sampling rows from a matrix. This new adaptive sampling lemma is particularly useful in designing CUR matrix decompositions.

**Lemma 3.10 (Theorem 4 in [49])**. Given \( A \in \mathbb{R}^{m \times n} \) and \( V \in \mathbb{R}^{m \times c} \) such that

\[ \text{rank}(V) = \text{rank}(VV^\dagger A) = \rho, \]

with \( \rho \leq c \leq n \), we let \( R_1 \in \mathbb{R}^{r_1 \times n} \) consist of \( r_1 \) rows of \( A \) and define the residual

\[ B = A - AR_1\Pi \in \mathbb{R}^{m \times n}. \]

For \( i = 1, \ldots, n \) let \( p_i \) be a probability distribution such that for each \( i : \)

\[ p_i = \| b_i \|_2^2 / \| B \|_F^2, \]

where \( b_i \) is the \( i \)-th column of \( B \). Sample \( r_2 \) rows from \( A \) in \( r_2 \) i.i.d. trials, where in each trial the \( i \)-th column is chosen with probability \( p_i \). Let \( R_2 \in \mathbb{R}^{r_2 \times n} \) contain the \( r_2 \) sampled columns and let \( R = [R_1^T, R_2^T]^T \in \mathbb{R}^{(r_1+r_2)\times n} \). Then,

\[ \mathbb{E} \left[ \| A - VV^\dagger A R_1 R^\dagger \|_F^2 \right] \leq \| A - VV^\dagger A \|_F^2 + \frac{\rho}{r_2} \| A - AR_1 R_1 \|_F^2. \]

We denote this procedure as

\[ R_2 = \text{AdaptiveRows}(A, V, R_1, r_2). \]

Given \( A, V, R_1 \), the above algorithm requires \( O(r_1 mn + r_2 \log r_2) \) arithmetic operations to find \( R_2 \).

We now discuss connections between Lemma 3.10 and Lemma 3.9. As it is discussed in Remark 6 in [49], setting \( V = A_k \) in Lemma 3.10 and using the identity \( A_k \Pi_k A_k = A_k \), gives

\[ \mathbb{E} \left[ \| A - A_k R^\dagger R \|_F^2 \right] \leq \| A - A_k \|_F^2 + \frac{k}{r_2} \| A - AR_1 R_1 \|_F^2. \]

Applying this result to the transpose of \( A \) and switching from rows \( R_1 \in \mathbb{R}^{r_1 \times n} \), \( R_2 \in \mathbb{R}^{r_2 \times n} \), and \( R \in \mathbb{R}^{r \times n} \) to columns \( C_1 \in \mathbb{R}^{m \times c_1} \), \( C_2 \in \mathbb{R}^{m \times c_2} \), and \( C \in \mathbb{R}^{m \times c} \), respectively, we have,

\[ \mathbb{E} \left[ \| A - CC^\dagger A_k \|_F^2 \right] \leq \| A - A_k \|_F^2 + (k/c_2) \| A - C_1 C_k A \|_F^2. \]

Since, \( \| A - CC^\dagger A \|_F^2 \leq \| A - CC^\dagger A_k \|_F^2 \), and \( \| A - CC^\dagger A \|_F^2 \leq \| A - \Pi_{C, k}^c A \|_F^2 \), and \( \| A - \Pi_{C, k}^c A \|_F^2 \leq \| A - CC^\dagger A_k \|_F^2 \), we obtain

\[ \mathbb{E} \left[ \| A - CC^\dagger A \|_F^2 \right] \leq \mathbb{E} \left[ \| A - CC^\dagger A_k \|_F^2 \right] \leq \| A - A_k \|_F^2 + \frac{k}{c_2} \| A - C_1 C_k A \|_F^2. \]

This result is identical to Lemma 3.9 with \( \alpha = 1 \) and \( V = C_1 \) containing columns of \( A \).
3.5. Low-rank approximations within a subspace. In this section we discuss algorithms to find low-rank approximations to matrices constrained to a given subspace.

3.5.1. The best rank $k$ matrix $\Pi_{V,k}^F(A)$ within a subspace $V$. Let $A \in \mathbb{R}^{m \times n}$, let $k < n$ be an integer, and let $V \in \mathbb{R}^{m \times c}$ with $k < c < n$. $\Pi_{V,k}^F(A) \in \mathbb{R}^{m \times n}$ is the best rank $k$ approximation to $A$ in the column span of $V$. Equivalently, we can write $\Pi_{V,k}^F(A) = VX_{\text{opt}}$, where

$$X_{\text{opt}} = \arg\min_{X \in \mathbb{R}^{c \times n} : \text{rank}(X) \leq k} ||A - VX||_F^2.$$ 

In order to compute $\Pi_{V,k}^F(A)$ given $A$, $V$, and $k$, we will use the following algorithm:

1. $V = Y\Psi$ is a qr decomposition of $V$ with $Y \in \mathbb{R}^{m \times c}$ and $\Psi \in \mathbb{R}^{c \times c}$. This step requires $O(mc^2)$ arithmetic operations.
2. $\Xi = Y^TA \in \mathbb{R}^{c \times n}$. This step requires $O(mnc)$ arithmetic operations.
3. $\Xi_k = \Delta_k\Sigma_k V_k^T \in \mathbb{R}^{c \times n}$ is a rank $k$ SVD of $\Xi$ with $\Delta \in \mathbb{R}^{c \times k}$, $\Sigma_k \in \mathbb{R}^{k \times k}$, and $V_k \in \mathbb{R}^{n \times k}$. This step requires $O(mc^2)$ arithmetic operations.
4. Return $Y\Delta\Delta^TY^T \in \mathbb{R}^{m \times n}$ of rank at most $k$.

$Y\Delta\Delta^TY^T \in \mathbb{R}^{m \times n}$ is a rank $k$ matrix that lies in the column span of $V$. The next lemma is a simple corollary of Lemma 4.3 in [10].

**Lemma 3.11.** Given $A \in \mathbb{R}^{m \times n}$, $V \in \mathbb{R}^{m \times c}$ and an integer $k$, $Y\Delta\Delta^TY^T$ and $QU_k\Sigma_k V_k^T$ satisfy: $||A - Y\Delta\Delta^TY^T A||_F^2 \leq ||A - Y\Delta\Sigma_k V_k^T||_F^2 = ||A - \Pi_{V,k}^F(A)||_F^2$. The above algorithm requires $O(mnc + nc^2)$ arithmetic operations to construct $Y, \Psi$, and $\Delta$. We will denote the above procedure as $[Y, \Psi, \Delta] = \text{BestSubspaceSVD}(A, V, r)$.

**Proof.** The equality was proven in Lemma 4.3 in [10]. To prove the inequality, notice that for any matrix $X$:

$$||A - Y\Delta\Delta^TY^T A||_F^2 \leq ||A - Y\Delta\Sigma_k V_k^T||_F^2 = ||A - \Pi_{V,k}^F(A)||_F^2.$$

Also, $(Y\Delta)^T = \Delta^TY^T = \Delta^TY^T$, because both matrices are orthonormal. \(\square\)

3.5.2. An approximate rank $k$ matrix within a subspace. The previous lemma provides a method for constructing the best rank $k$ matrix within a given subspace. This method, however, is somewhat costly if one wants to design algorithms that run in time proportional to the number of non zeros entries of $A$. To address this, we use the lemma below, which finds a rank $k$ matrix that is almost as good as the best rank $k$ matrix within $\text{span}(V)$.

**Lemma 3.12.** Let $A \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{m \times c}$. We further assume that for some rank parameter $k < c$ and accuracy parameter $0 < \epsilon < 1$,

$$||A - \Pi_{V,k}^F(A)||_F^2 \leq (1 + \epsilon)||A - A_k||_F^2.$$ 

In words, we assume that in the given subspace $V$, there exists a rank $k$ matrix that approximates the best rank $k$ matrix from the SVD within a relative error. Let $V = Y\Psi$ be a qr decomposition of $V$ with $Y \in \mathbb{R}^{m \times c}$ and $\Psi \in \mathbb{R}^{c \times c}$. Let $\Xi = Y^TAW^T \in \mathbb{R}^{c \times \xi}$, where $W^T \in \mathbb{R}^{n \times \xi}$ is a sparse subspace embedding matrix with $\xi = O(c^2/\epsilon^2)$ (see Definition 3.13). Let $\Delta \in \mathbb{R}^{c \times k}$ contain the top $k$ left singular vectors of $\Xi$, i.e.,
\[ \Xi_k = \Delta \Sigma_k \tilde{V}_k^T \in \mathbb{R}^{c \times n}, \text{ is a rank } k \text{ SVD of } \Xi, \text{ with } \Delta \in \mathbb{R}^{c \times k}, \Sigma_k \in \mathbb{R}^{k \times k}, \text{ and } \tilde{V}_k \in \mathbb{R}^{n \times k}. \text{ Then, with probability at least } 0.99, \]
\[ \|A - Y \Delta \Delta^T Y^T A\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2. \]

\( Y, \Psi, \text{ and } \Delta \) can be computed in \( O(\text{nnz}(A) + mc\xi) \) time. We denote the construction of \( Y, \Psi \) and \( \Delta \) as
\[ [Y, \Psi, \Delta] = \text{ApproxSubspaceSVD}(A, V, k, \varepsilon). \]

**Proof.** This result was proven inside the proof of Theorem 1.5 in [34]. Specifically, the error bound proven in [34] is for the transpose of \( A \) (also \( Y, \Delta \) are denoted with \( U, V \) in [34]). The only requirement for the embedding matrix \( W \) (denoted with \( P \) in the proof of Theorem 1.5 in [34]) is to be a subspace embedding for \( Y^T A \), in the sense that \( W \) is a subspace embedding for \( A \) in Lemma 3.14. Since our choice of \( W \) with \( \xi = O(\varepsilon^2/\varepsilon^3) \) satisfies this requirement we omit the details of the proof. The running time of the algorithm is \( O(\text{nnz}(A) + mc\xi) \) because one can compute (i) \( Y \) in \( O(mc\xi) \) time; (ii) \( \Xi \) in \( O(\text{nnz}(A) + mc\xi) \) time; and (iii) \( \Delta \) in \( O(\varepsilon^2 \min\{c, \xi\}) \) time. The failure probability 0.01 is due to Lemma 3.14. \( \Box \)

### 3.6. Sparse subspace embeddings.

Next, we discuss the so-called sparse subspace embedding matrices of Clarskon and Woodruff [12]. Those are linear transformations for dimensionality reduction that preserve both the sparsity of the input points as well as their geometry.

**Definition 3.13. [Sparse Subspace Embedding [12]]** We call \( W \in \mathbb{R}^{\xi \times n} \) a sparse subspace embedding of dimension \( \xi \) if it is constructed as follows, \( W = \Psi Y \), with
- \( h : [n] \rightarrow [\xi] \) is a random map so that for each \( i \in [n], h(i) = \xi', \) for \( \xi' \in [\xi] \) w.p. \( 1/\xi \),
- \( \Psi \in \mathbb{R}^{\xi \times n} \) is a binary matrix with \( \Psi_{h(i),i} = 1 \), and all remaining entries 0.
- \( Y \in \mathbb{R}^{n \times n} \) is a random diagonal matrix, with each diagonal entry independently chosen to be +1 or −1, with equal probability.

For any matrix \( A \) with \( n \) rows, computing \( WA \) requires \( O(\text{nnz}(A)) \) time.

**Lemma 3.14 ([12, 38, 40]).** Let \( A \in \mathbb{R}^{n \times d} \) have rank \( \rho \) and let \( W \in \mathbb{R}^{\xi \times n} \) be a randomly chosen sparse subspace embedding with dimension \( \xi = \Omega(\rho^2\varepsilon^{-2}) \), for some \( 0 < \varepsilon < 1 \). Then, with probability at least 0.99, and for all vectors \( y \in \mathbb{R}^d \) simultaneously,
\[ (1 - \varepsilon)\|Ay\|_2^2 \leq \|WAy\|_2^2 \leq (1 + \varepsilon)\|Ay\|_2^2. \]

**Lemma 3.15 (Lemma 40 in [12]).** Let \( A \in \mathbb{R}^{n \times d} \) and let \( W \in \mathbb{R}^{\xi \times n} \) be a randomly chosen sparse subspace embedding with dimension \( \xi = \Omega(\varepsilon^{-2}) \), for some \( 0 < \varepsilon < 1 \). Then, with probability at least 0.99,
\[ (1 - \varepsilon)\|A\|_F^2 \leq \|WA\|_F^2 \leq (1 + \varepsilon)\|A\|_F^2. \]

**Lemma 3.16 ([12]).** Let \( A \in \mathbb{R}^{n \times d} \) have rank \( \rho \) and \( B \in \mathbb{R}^{n \times \omega} \). Let \( W \in \mathbb{R}^{\xi \times n} \) be a randomly chosen sparse subspace embedding with \( \xi = \Omega(\rho^2\varepsilon^{-2}) \). Then, with probability at least 0.99, \( \|AX_{\text{opt}} - B\|_F^2 \leq \|AX_{\text{opt}} - B\|_F^2 \), where
\[ X_{\text{opt}} \in \arg \min_{X \in \mathbb{R}^{d \times \omega}} \|WX - WB\|_F^2. \]
16

BOUTSIDIS AND WOODRUFF

and,

\[ \mathbf{X}_{\text{opt}} \in \arg\min_{\mathbf{X} \in \mathbb{R}^{d \times \omega}} \| \mathbf{A} \mathbf{X} - \mathbf{B} \|_F^2. \]

Proof. Theorem 36 in [12] shows this bound under the assumption that the event of Lemma 22 in [12] occurs. In the proof of Lemma 22 in [12], it is shown that a sparse subspace embedding defined with \( \xi = \Omega(\rho^2 \varepsilon^{-2}) \) satisfies the bound, where the polylogarithmic factors in the dimension \( \xi \) were removed in [38, 40]. □

3.7. Johnson Lindestrauss transform. Here, we review the Johnson Lindestrauss transform, a standard method for dimension reduction that preserves the geometry of a set of points.

Lemma 3.17. (Theorem 1 in [2] for fixed \( \varepsilon = 1/2 \)) Let \( \mathbf{B} \in \mathbb{R}^{m \times n} \). Given \( \varepsilon, \beta > 0 \), let

\[ s = \frac{4 + 2 \beta}{(1/2)^2 - (1/2)^3} \log n. \]

Construct a matrix \( \mathbf{S} \in \mathbb{R}^{s \times m} \), each element of which is a random variable which takes values \( \pm 1/\sqrt{s} \) with equal probability. Let \( \tilde{\mathbf{B}} = \mathbf{S} \mathbf{B} \). Then, if \( \mathbf{b}_i \) and \( \tilde{\mathbf{b}}_i \) denote the \( i \)th column of \( \mathbf{B} \) and \( \tilde{\mathbf{B}} \), respectively, with probability at least \( 1 - n^{-\beta} \), and for all \( i = 1, \ldots, n \),

\[ (1 - \frac{1}{2}) \| \mathbf{b}_i \|_2^2 \leq \| \tilde{\mathbf{b}}_i \|_2^2 \leq (1 + \frac{1}{2}) \| \mathbf{b}_i \|_2^2. \]

Given \( \mathbf{B} \), it takes \( O(mn(B) \log n) \) arithmetic operations to construct \( \tilde{\mathbf{B}} \). We will denote this procedure as \( \tilde{\mathbf{B}} = JLT(\mathbf{B}, \beta) \).

3.8. Generalized rank-constrained matrix approximations. Let \( \mathbf{A} \in \mathbb{R}^{m \times n} \), \( \mathbf{C} \in \mathbb{R}^{m \times c} \), \( \mathbf{R} \in \mathbb{R}^{r \times n} \), and \( k \leq c, r \) be an integer. Consider the following optimization problem,

\[ \mathbf{U}_{\text{opt}} \in \arg\min_{\mathbf{U} \in \mathbb{R}^{c \times r}, \text{rank}(\mathbf{U}) \leq k} \| \mathbf{A} - \mathbf{C} \mathbf{U} \mathbf{V}^T \mathbf{R} \|_F^2. \]

Then, the solution \( \mathbf{U}_{\text{opt}} \in \mathbb{R}^{c \times r} \) with \( \text{rank}(\mathbf{U}_{\text{opt}}) \leq k \) that has the minimum \( \| \mathbf{U}_{\text{opt}} \|_F \) out of all possible feasible solutions is given via the following formula,

\[ \mathbf{U}_{\text{opt}} = \mathbf{C}^T \left( \mathbf{U}_C \mathbf{U}_C^T \mathbf{A} \mathbf{V} \mathbf{V}_R^T \right)_k \mathbf{R}^T. \]

\( \left( \mathbf{U}_C \mathbf{U}_C^T \mathbf{A} \mathbf{V} \mathbf{V}_R^T \right)_k \in \mathbb{R}^{m \times n} \) of rank at most \( k \) denotes the best rank \( k \) matrix to \( \mathbf{U}_C \mathbf{U}_C^T \mathbf{A} \mathbf{V} \mathbf{V}_R^T \in \mathbb{R}^{m \times n} \). This result was proven in [24] (see also [45] for the spectral norm version of the problem).

4. New column subset selection tools. To design our CUR algorithms in Sections 5, 6, and 7 we combine the subset selection tools from the previous section with novel tools that we describe below. The results in the present section could be of independent interest.
4.1. New tools for deterministic CUR decompositions.

4.1.1. Deterministic adaptive sampling. First, in the next lemma we derandomize the recent row/column adaptive sampling method of Wang and Zhang [49] (stated as Lemma 3.10 in our article). The argument uses pairwise-independence, and may be known, though we could not find a reference.

**Lemma 4.1.** Given \( A \in \mathbb{R}^{m \times n} \) and \( V \in \mathbb{R}^{m \times c} \) such that

\[
\text{rank}(V) = \text{rank}(VV^\dagger A) = \rho,
\]

with \( \rho \leq c \leq n \), we let \( R_1 \in \mathbb{R}^{\rho \times n} \) consist of \( r_1 \) rows of \( A \). There exists a deterministic algorithm to construct \( R_2 \in \mathbb{R}^{\rho \times n} \) with \( r_2 \) rows such that for

\[
R = [R_1^T, R_2^T]^T \in \mathbb{R}^{(r_1 + r_2) \times n},
\]

we have

\[
\|A - VV^\dagger AR_1\|_F^2 \leq \|A - VV^\dagger A\|_F^2 + \frac{4\rho}{r_2}\|A - AR_1\|_F^2.
\]

We denote this procedure as

\[
R_2 = \text{AdaptiveRowsD}(A, V, R_1, r_2).
\]

Given \( A, V, R_1 \), it takes \( O(m^2(mc^2 + nr^2 + mn(c + r))) \) time to find \( R_2 \). Here \( c = c_1 + c_2, r = r_1 + r_2 \).

**Proof.** This lemma corresponds to a derandomization of Theorem 5 of [49]. Note that the matrix \( C \) in their proof corresponds to our matrix \( V \), and despite their notation, as they state in their theorem statement and prove, the matrix \( C \) need not be a subset of columns of \( A \). To prove their theorem, they actually restate it as Theorem 15 in their paper, switching the role of rows and columns (so they will sample a subset \( C_2 \) of columns, given a subset \( C_1 \) of columns and an arbitrary matrix \( R \)), which we will now derandomize. Our lemma then follows by taking transposes.

In that theorem the authors define a distribution \( p \) on \( n \) columns \( b_1, \ldots, b_n \) where \( B = A - C_1C_1^T A \) for a matrix \( C_1 \in \mathbb{R}^{m \times c_1} \) consisting of \( c_1 \) columns of \( A \). In their proof they show that for \( p_i = \|b_i\|_2^2/\|B\|_F^2 \), if one samples \( c_2 \) columns of \( A \) in i.i.d. trials where in each trial the \( i \)-th column is chosen with probability \( p_i \), then if \( C_2 \in \mathbb{R}^{m \times c_2} \) consists of the \( c_2 \) sampled columns, and \( C = [C_1, C_2] \in \mathbb{R}^{m \times (c_1 + c_2)} \) then

\[
\mathbb{E}_{C_2}\|A - CC^\dagger AR_1\|_F^2 \leq \|A - AR_1\|_F^2 + \frac{\rho}{c_2}\|A - C_1C_1^TA\|_F^2.
\]

To derandomize this, we first discretize the distribution \( p \), defining a new distribution \( q \) (and along the way, a distribution \( r \)). Let \( i^* \in [n] \) be such that \( p_{i^*} \geq p_i \) for all \( i \neq i^* \). Then, let

\[
r_{i^*} = p_{i^*} + \sum_{i \neq i^*} \frac{p_i}{2},
\]

and \( r_i = p_i/2 \) for \( i \neq i^* \). Hence, \( r \) is a distribution. Now round each \( r_i, i \neq i^* \), up to the nearest integer multiple of \( 1/(4n) \) by adding \( \gamma_i \), and let \( q_i \) be the resulting value. For \( q_{i^*} \), let \( q_{i^*} = r_{i^*} - \sum_{i \neq i^*} \gamma_i \). Then \( \sum_i q_i = 1 \) and \( 0 \leq q_i \leq 1 \) for all \( i \neq i^* \). Now consider \( q_{i^*} \). We have,

\[
0 \leq \sum_{i \neq i^*} \gamma_i \leq n \cdot \frac{1}{4n} = \frac{1}{4}.
\]
On the other hand

\[ r_{i^*} = p_{i^*} + \sum_{i \neq i^*} \frac{p_i}{2} = \sum_{i} \frac{p_i}{2} = \frac{1}{2} \]

Hence \( q_{i^*} = r_{i^*} - \sum_{i \neq i^*} \gamma_i \geq \frac{1}{4} \). Also \( q_{i^*} \leq r_{i^*} \leq 1 \). It follows that \( q \) is a distribution.

Notice that for all \( i \neq i^* \), \( q_i \geq r_i / 2 \), while also, \( q_{i^*} \geq p_{i^*} / 4 \) since \( p_{i^*} \leq 1 \). Hence, all \( q_i \) satisfy \( q_i \geq p_i / 4 \).

Next we follow the argument in the proof of Theorem 15 of [49] replacing distribution \( p \) with \( q \), pointing out the differences. For \( \ell \in \{c_2 \} \), we define the vector random variable

\[ x_{j,(\ell)} = \frac{v_{i,j}}{q_i} b_i, \]

where \( v_{i,j} \) is the \((i,j)\)-th entry of \( V_{\mathbf{AR}^\dagger \mathbf{R},\rho} \in \mathbb{R}^{n \times \rho} \), where \( V_{\mathbf{AR}^\dagger \mathbf{R},\rho} \in \mathbb{R}^{n \times \rho} \) denotes the matrix of the top \( \rho \) right singular vectors of \( \mathbf{AR}^\dagger \mathbf{R} \). We have that

\[ E[x_{j,(\ell)}] = \sum_{i=1}^{n} q_i \frac{v_{i,j}}{q_i} b_i = \mathbf{B} v_j. \]

Moreover,

\[ E[\|x_{j,(\ell)}\|^2] = \sum_{i=1}^{n} q_i \frac{v_{i,j}^2}{q_i} \|b_i\|^2 \leq \sum_{i \neq i^*} \frac{v_{i,j}^2 \|b_i\|^2}{4\|b\|_F^2} + \sum_{i \neq i^*} \frac{v_{i,j}^2 \|b_i\|^2}{4\|\mathbf{B}\|_F^2} \leq 4\|\mathbf{B}\|_F^2, \tag{4.1} \]

using that for all \( j \), \( \sum_{i} v_{i,j}^2 = 1 \). This is slightly weaker than the corresponding upper bound of \( E[\|x_{j,(\ell)}\|^2] \leq \|\mathbf{B}\|_F^2 \) shown for distribution \( \rho \) in [49], but the constant 4 makes little difference for our purposes, while the fact that \( q \) is discrete will help with our derandomization.

Now let

\[ x_j = \frac{1}{c_2} \sum_{\ell=1}^{c_2} x_{j,(\ell)} \]

By linearity of expectation,

\[ E[x_j] = E[x_{j,(\ell)}] = \mathbf{B} v_j. \]

Next we bound \( E[\|x_j - \mathbf{B} v_j\|_F^2] \), and here we observe that pairwise independence of the trials is enough to bound this quantity:

\[ E[\|x_j - \mathbf{B} v_j\|^2] = E[\|x_j - E[x_j]\|^2] = \frac{1}{c_2} E[\|x_{j,(\ell)} - E[x_{j,(\ell)}]\|^2] = \frac{1}{c_2} E[\|x_{j,(\ell)} - \mathbf{B} v_j\|^2], \]

where the second equality uses pairwise-independence so that the sum of the variances is equal to the variance of the sum. That is, the variance \( E[\|x_j - E[x_j]\|^2] \) is \( 1/c_2 \) times the variance \( E[\|x_{j,(\ell)} - E[x_{j,(\ell)}]\|^2] \) since \( x_j \) is the average of the \( x_{j,(\ell)} \).

The remainder of the proof of Theorem 15 in [49] only uses their slightly stronger bound than (4.1). With our weaker bound they obtain

\[ E_{C_2} \|\mathbf{A} - CC^\dagger \mathbf{AR}^\dagger \mathbf{R}\|_F^2 \leq \|\mathbf{A} - \mathbf{AR}^\dagger \mathbf{R}\|_F^2 + \frac{4\rho}{c_2}\|\mathbf{B}\|_F^2, \]
satisfies: for any integer $R$ holds for any $V$.

In our work). In fact, we prove a slightly weaker version of the result where their bound would not have the factor of 4. In particular there exists a choice of $C_2$ for which for the corresponding $C$ of

$$
\| A - CC^I AR^I R \|_F^2 \leq \| A - AR^I R \|_F^2 + \frac{4\rho}{c_2} \| B \|_F^2.
$$

Now, $q$ is a discrete distribution and the trials need only be pairwise independent. Let $h : [c_2] \rightarrow [4n]$ be drawn from a pairwise-independent hash function family $F$. Then $F$ need only have size $O(n^2)$ [9]. For each trial $\ell \in [c_2]$, we compute $h(\ell)$. Since the probabilities $q_i$, $i = 1, \ldots, n$ are integer multiples of $1/(4n)$ and $\sum q_i = 1$, we pick the largest $i \in [n]$ for which $h(\ell)/(4n) \leq \sum_{i=1}^n q_i$. The probability of picking $i$ in each trial is therefore $q_i$. For each $h \in F$, we compute the corresponding column samples $C_2$ and compute $\| A - CC^I AR^I R \|_F^2$. We choose the $C_2$ resulting in the smallest such value. Computing this value can be done in $O(mc^2 + nr^2 + mnc + mn)$ time, and since $|F| = O(n^2)$, the overall time is $O(n^2(mc^2 + nr^2 + mnc + mn))$. Here $c = c_1 + c_2, r = r_1 + r_2$.

Switching back to our notation (by taking transposes), the overall running time is $O(m^2(mc^2 + nr^2 + mnc + mn))$.\]

A simple corollary to the previous lemma is a deterministic algorithm that corresponds to a derandomization of the adaptive sampling procedure of [14] (stated as Lemma 3.9 in our work). In fact, we prove a slightly weaker version of the result in [14] since we require $V$ to have columns of $A$; on the other hand, the result in [14] holds for any $V$. We remark that a direct derandomization of Theorem 2.1 in [14] is possible using the same argument, but we opt to omit it for simplicity.

**Lemma 4.2.** Given $A \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{m \times c_1}$ (with $c_1 \leq n, m$) containing columns of $A$, define the residual $B = A - VV^T A \in \mathbb{R}^{m \times n}$. There exists a deterministic algorithm to construct $C_2 \in \mathbb{R}^{m \times c_2}$ containing $c_2$ columns of $A$, such that $C = [V \ C_2] \in \mathbb{R}^{m \times (c_1 + c_2)}$, i.e., the matrix that contains the columns of $V$ and $C_2$, satisfies: for any integer $k > 0$,

$$
\| A - \Pi_{C,k} F(A) \|_F^2 \leq \| A - A_k \|_F^2 + \frac{4k}{c_2} \| A - VV^T A \|_F^2.
$$

We denote this procedure as

$$
C_2 = AdaptiveColsD(A, V, c_2).
$$

Given $A$ and $V$, the algorithm requires $O(mn^3 c)$ arithmetic operations to find $C_2$. Here $c = c_1 + c_2$.

**Proof.** Let us denote with $V'$ the matrix $V$ in Lemma 4.1 and reserve $V$ for the matrix $V$ in this lemma. Setting $V' = A_k$ in Lemma 4.1 and using $A_kA_k^T A = A_k$:

$$
\| A - A_k R^I R \|_F^2 \leq \| A - A_k \|_F^2 + \frac{4k}{r_2} \| A - AR^I R \|_F^2.
$$

Applying this result to the transpose of $A$ and switching from rows $R_1 \in \mathbb{R}^{r_1 \times n}$, $R_2 \in \mathbb{R}^{r_2 \times n}$, and $R^T \times n$ to columns $V = C_1 \in \mathbb{R}^{m \times c_1}, C_2 \in \mathbb{R}^{m \times c_2}$, and $C \in \mathbb{R}^{m \times c}$, respectively, we have, we have, $\| A - CC^I A_k \|_F^2 \leq \| A - A_k \|_F^2 + \frac{4k}{c_2} \| A - C^I C_1 A_k \|_F^2$. Since, $\| A - CC^I A \|_F^2 \leq \| A - CC^I A_k \|_F^2$, and $\| A - CC^I A \|_F^2 \leq \| A - \Pi_{C,k} F(A) \|_F^2$, and $\| A - \Pi_{C,k} F(A) \|_F^2 \leq \| A - CC^I A_k \|_F^2$, we conclude the proof as follows:

$$
\| A - CC^I A \|_F^2 \leq \| A - CC^I A_k \|_F^2 \leq \| A - A_k \|_F^2 + \frac{4k}{c_2} \| A - CC^I A_k \|_F^2.
$$

To summarize: $C_2' = AdaptiveRowsD(A^T, A_k^T, C_1^T, c_2)$, with $C_2 \in \mathbb{R}^{m \times c_2}$.\]
4.2. New tools for input-sparsity-time CUR decompositions.

4.2.1. Input-sparsity time BSS sampling. Next, we develop an “input-sparsity-time” version of the so-called “BSS sampling” algorithm [3]; to be precise, we develop an input-sparsity-time version of an extension of the original BSS algorithm that appeared in [6].

**Lemma 4.3 (Input-Sparsity-Time Dual-Set Spectral-Frobenius Sparsification).** Let $V = \{v_1, \ldots, v_n\}$ be a decomposition of the identity, where $v_i \in \mathbb{R}^k$ ($k < n$) and $\sum_{i=1}^n v_i v_i^T = I_k$; let $A = \{a_1, \ldots, a_n\}$ be an arbitrary set of vectors, where $a_i \in \mathbb{R}^\ell$. Let $W \in \mathbb{R}^{\xi \times \ell}$ be a randomly chosen sparse subspace embedding with $\xi = O(n^2/\epsilon^2) < \ell$, for some $0 < \epsilon < 1$ (see Definition 3.13). Consider a new set of vectors $B = \{Wa_1, \ldots, Wa_n\}$, and some integer $r$ such that $k < r \leq n$. Let the output of this be a set of weights $s_i \geq 0$ ($i = 1, \ldots, n$), at most $r$ of which are non-zero. Then, with probability at least $0.98$,

$$\lambda_k \left( \sum_{i=1}^n s_i v_i v_i^T \right) \geq \left( 1 - \sqrt{\frac{k}{r}} \right)^2,$$

$$\text{Tr} \left( \sum_{i=1}^n s_i a_i a_i^T \right) \leq \frac{1 + \epsilon}{1 - \epsilon} \cdot \text{Tr} \left( \sum_{i=1}^n a_i a_i^T \right) = \frac{1 + \epsilon}{1 - \epsilon} \cdot \sum_{i=1}^n \|a_i\|_2^2.$$

Equivalently, if $V \in \mathbb{R}^{n \times k}$ is a matrix whose rows are the vectors $v_i^T$, $A \in \mathbb{R}^{n \times \ell}$ is a matrix whose rows are the vectors $a_i^T$, $B = AW^T \in \mathbb{R}^{n \times \ell}$ is a matrix whose rows are the vectors $a_i^T W^T$, and $S \in \mathbb{R}^{n \times r}$ is the sampling matrix containing the weights $s_i > 0$, then with probability at least 0.98,

$$\sigma_k \left( V^T S \right) \geq 1 - \sqrt{k/r} \quad \|A^T S\|_F^2 \leq \frac{1 + \epsilon}{1 - \epsilon} \cdot \|A\|_F^2.$$

The weights $s_i$ can be computed in $O \left( \text{nnz}(A) + rk^2 + n\xi \right)$ time. We denote this procedure as

$$S = \text{BssSamplingSparse}(V, A, r, \epsilon).$$

**Proof.** The algorithm constructs $S$ as follows,

$$S = \text{BssSampling}(V, B, r).$$

The lower bound for the smallest singular value of $V$ is immediate from Lemma 3.5. That lemma also ensures,

$$\|B^T S\|_F^2 \leq \|B^T\|_F^2,$$

i.e.,

$$\|WA^T S\|_F^2 \leq \|WA^T\|_F^2.$$
Since $W$ is a subspace embedding, from Lemma 3.14 we have that with probability at least 0.99 and for all vectors $y \in \mathbb{R}^n$ simultaneously,

$$(1 - \varepsilon) \|A^T y\|_2^2 \leq \|WA^T y\|_2^2.$$

Apply this $r$ times for $y \in \mathbb{R}^n$ being columns from $S \in \mathbb{R}^{n \times r}$ and take a sum on the resulting inequalities,

$$(1 - \varepsilon) \|A^T S\|_F^2 \leq \|WA^T S\|_F^2.$$

Now, from Lemma 3.15 we have that with probability at least 0.99,

$$\|WA^T\|_F^2 \leq (1 + \varepsilon) \|A^T\|_F^2.$$

Combining all these inequalities together, we conclude that with probability at least 0.98,

$$\|A^T S\|_F^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \cdot \|A^T\|_F^2.$$

\[\boxempty\]

4.2.2. Input-sparsity time adaptive sampling. In this section, we develop “input-sparsity-time” versions of the adaptive sampling algorithms discussed in this paper. The lemma below corresponds to such a fast version of the adaptive sampling algorithm of Lemma 3.9.

**Lemma 4.4.** Given $A \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{m \times c_1}$ (with $c_1 \leq n, m$), there exists a randomized algorithm to construct $C_2 \in \mathbb{R}^{m \times c_2}$ containing $c_2$ columns of $A$, such that the matrix $C = [V \ C_2] \in \mathbb{R}^{m \times (c_1 + c_2)}$ containing the columns of $V$ and $C_2$ satisfies:

for any integer $k > 0$, and with probability $0.9 - \frac{1}{n}$

$$\|A - \Pi_{C,k}^F(A)\|_F^2 \leq \|A - A_k\|_F^2 + \frac{30k}{c_2} \|A - V V^\dagger A\|_F^2.$$

We denote this procedure

$$C_2 = \text{AdaptiveColsSparse}(A, V, c_2).$$

Given $A$ and $V$, the algorithm takes $O(n \text{nnz}(A) \log n + mc_1 \log n + mc_1^2)$ time to find $C_2$.

**Proof.** Define the residual $B = A - V V^\dagger A \in \mathbb{R}^{m \times n}$. From Lemma 3.17, let

$$B = JLT(B, 1).$$

The lemma gives us $\frac{1}{2} \|b_i\|_2^2 \geq \|b_i\|_2^2 \geq \frac{1}{3} \|b_i\|_2^2$ for all $i$. So from Lemma 3.9, after using the following distribution for the sampling,

$$p_i = \frac{\|b_i\|_2^2}{\|B\|_F^2} \geq \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{\|b_i\|_2^2}{\|B\|_F^2} = \frac{1}{3} \cdot \frac{\|b_i\|_2^2}{\|B\|_F^2},$$

we obtain,

$$\mathbb{E} \left[\|A - \Pi_{C,k}^F(A)\|_F^2\right] \leq \|A - A_k\|_F^2 + \frac{3k}{c_2} \|A - V V^\dagger A\|_F^2.$$
The expectation is taken with respect to the randomness in constructing \( C_2 \), so
\[
\mathbb{E} \left[ \| A - \Pi_{C,k}^F(A) \|_F^2 - \| A - A_k \|_F^2 \right] \leq \frac{3k}{c^2} \| A - VV^\top A \|_F^2.
\]
The following relation is immediate,
\[
\| A - \Pi_{C,k}^F(A) \|_F^2 - \| A - A_k \|_F^2 > 0,
\]
so, from Markov’s inequality, with probability at least 0.9
\[
\| A - \Pi_{C,k}^F(A) \|_F^2 - \| A - A_k \|_F^2 \leq \frac{30k}{c^2} \| A - VV^\top A \|_F^2,
\]
which implies,
\[
\| A - \Pi_{C,k}^F(A) \|_F^2 \leq \| A - A_k \|_F^2 + \frac{30k}{c^2} \| A - VV^\top A \|_F^2.
\]

The running time follows by a careful implementation of the random projection step inside the routine in Lemma 3.17. First, we compute the matrices \( SA \) and \( SV \) in \( O(\text{nnz}(A) \log n) \) and \( O(mc_1 \log n) \) arithmetic operations, respectively. Computing \( V^\top \) requires \( O(mc_2^2) \) operations, and \( (SV) V^\top \) another \( O(mc_1 \log n) \) operations. Finally, computing \( (SVV^\top) A \) takes \( O(\text{nnz}(A) \log n) \) operations and computing \( SA - SVV^\top A \) another \( O(n \log n) \) operations. So, all these steps can be implemented in time \( O(\text{nnz}(A) \log n + mc_1 \log n + mc_2^2) \).

Next, we develop an input-sparsity-time version of the adaptive sampling algorithm of Lemma 3.10.

**Lemma 4.5.** Given \( A \in \mathbb{R}^{m \times n} \) and \( V \in \mathbb{R}^{m \times c} \) such that
\[
\text{rank}(V) = \text{rank}(VV^\top A) = \rho,
\]
with \( \rho \leq c \leq n \), let \( R_1 \in \mathbb{R}^{r_1 \times n} \) consist of \( r_1 \) rows of \( A \). There exists a randomized algorithm to construct \( R_2 \in \mathbb{R}^{r_2 \times n} \) with \( r_2 \) rows such that for \( R = [R_1^\top, R_2^\top]^\top \in \mathbb{R}^{(r_1 + r_2) \times n} \), we have that with probability at least 0.9 \(-1/n\),
\[
\|A - VV^\top A R^\top R\|_F^2 \leq \|A - VV^\top A\|_F^2 + \frac{30\rho}{r_2} \|A - AR^\top R_1\|_F^2.
\]

We denote this procedure as
\[
R_2 = \text{AdaptiveRowsSparse}(A, V, R_1, r_2).
\]

Given \( A, V, R_1 \), the algorithm takes
\[
O(\text{nnz}(A) \log n + nr_1 \log n + nr_2^2)
\]
arithmetic operations to find \( R_2 \).

**Proof.** First, there is an immediate generalization of Theorem 15 in [49] to the case when the sampling probabilities satisfy \( p_i \geq \alpha \frac{\|b_i\|_2^2}{\|B\|_F^2} \) for some \( \alpha < 1 \), rather than just for \( \alpha = 1 \) (we omit the details). This leads to the following version of Lemma 3.10 described in our work: if the probabilities in that lemma satisfy \( p_i \geq \alpha \frac{\|b_i\|_2^2}{\|B\|_F^2} \) for some \( \alpha < 1 \), then the bound is
\[
\mathbb{E} \left[ \|A - VV^\top A R^\top R\|_F^2 \right] \leq \|A - VV^\top A\|_F^2 + \frac{\rho}{\alpha r_2} \|A - AR^\top R_1\|_F^2.
\]
Given this bound, the proof continues by repeating the ideas that we used in Lemma 4.4. Although the proof is similar to that in Lemma 4.4, we include a proof for completeness.

Define the residual

\[ B = A - AR_1^R \in \mathbb{R}^{m \times n}. \]

From Lemma 3.17, let

\[ \tilde{B} = JLT(B, 1). \]

By doing this we have \( \frac{3}{2} \| b_i \|_2^2 \geq \| \tilde{b}_i \|_2^2 \geq \frac{1}{2} \| b_i \|_2^2 \). So, after using the following distribution for the sampling,

\[ p_i = \frac{\| \tilde{b}_i \|_2^2}{\| B \|_F^2} \geq \frac{2}{3} \cdot \frac{\| b_i \|_2^2}{\| B \|_F^2}, \]

we obtain,

\[ \mathbb{E} \left[ \| A - VV^\dagger AR^1_1R_1 \|_F^2 \right] \leq \| A - VV^\dagger A \|_F^2 + \frac{3\rho}{r_2} \| A - AR^1_1R_1 \|_F^2. \]

The expectation is taken with respect to the randomness in constructing \( R_2 \), so

\[ \mathbb{E} \left[ \| A - VV^\dagger AR^1_1R_1 \|_F^2 - \| A - VV^\dagger A \|_F^2 \right] \leq \frac{3\rho}{r_2} \| A - AR^1_1R_1 \|_F^2. \]

We have

\[ \| A - VV^\dagger AR^1_1R_1 \|_F^2 - \| A - VV^\dagger A \|_F^2 > 0 \]

since

\[ \| A - VV^\dagger AR^1_1R_1 \|_F^2 = \| A - VV^\dagger A + VV^\dagger A - VV^\dagger AR^1_1R_1 \|_F^2 \]

\[ = \| A - VV^\dagger A \|_F^2 + \| VV^\dagger A - VV^\dagger AR^1_1R_1 \|_F^2. \]

Hence, by Markov’s inequality, with probability at least 0.9

\[ \| A - VV^\dagger AR^1_1R_1 \|_F^2 - \| A - VV^\dagger A \|_F^2 \leq \frac{30\rho}{r_2} \| A - AR^1_1R_1 \|_F^2, \]

which implies,

\[ \| A - VV^\dagger AR^1_1R_1 \|_F^2 \leq \| A - VV^\dagger A \|_F^2 + \frac{30\rho}{r_2} \| A - AR^1_1R_1 \|_F^2. \]

The running time follows by a careful implementation of the random projection step inside the routine in Lemma 3.17. First, we compute \( SA \) in \( O(nnz(A) \log n) \) arithmetic operations. Computing \( R^1_1 \) requires \( O(nr_1^2) \) operations. Then, computing \( (SA) V^\dagger \) \( R_1 \) requires another \( O(nr_1 \log n) \) operations. Finally, computing \( SA - SAR^1_1R_1 \) requires \( O(n \log n) \) operations. So, all these steps can be implemented in \( O(nnz(A) \log n + nr_1 \log n + nr_1^2) \) time. \( \square \)
Algorithm 2 A randomized, linear-time, optimal, relative-error, rank-$k$ CUR

**Input:** $A \in \mathbb{R}^{m \times n}$; rank parameter $k < \text{rank}(A)$; accuracy parameter $0 < \varepsilon < 1$.  
**Output:** $C \in \mathbb{R}^{m \times c}$ with $c = O(k/\varepsilon)$; $R \in \mathbb{R}^{r \times n}$ with $r = O(k/\varepsilon)$; $U \in \mathbb{R}^{c \times r}$ with rank($U$) = $k$.

1. **Construct $C$ with $O(k + k/\varepsilon)$ columns**
   1. $Z_1 = \text{RandomizedSVD}(A, k, 1)$; $Z_1 \in \mathbb{R}^{n \times k}$ ($Z_1^T Z_1 = I_k$); $E_1 = A - AZ_1 Z_1^T \in \mathbb{R}^{m \times n}$.
   2. $[\Omega_1, D_1] = \text{RandSampling}(Z_1, h_1, 1)$; $h_1 = 16k \ln(20k)$; $\Omega_1 \in \mathbb{R}^{n \times h_1}$; $D_1 \in \mathbb{R}^{h_1 \times h_1}$.  
      $M_1 = Z_1^T \Omega_1 D_1 \in \mathbb{R}^{k \times h_1}$.  
      $M_1 = U_{M_1} \Sigma_{M_1} V_{M_1}^T$ with rank($M_1$) = $k$ and $V_{M_1} \in \mathbb{R}^{h_1 \times k}$.
   3. $S_1 = \text{BssSampling}(V_{M_1}, (E_1 \Omega_1 D_1)^T, c_1)$, with $c_1 = 4k$.  
      $S_1 \in \mathbb{R}^{h_1 \times c_1}$ containing rescaled columns of $A$.
   4. $C_1 = A \Omega_1 D_1 S_1 \in \mathbb{R}^{m \times c_1}$ containing rescaled columns of $A$.
   5. $C = [C_1 \ C_2] \in \mathbb{R}^{m \times c}$ containing $c = c_1 + c_2 = 4k + \frac{1620k}{\varepsilon}$ rescaled columns of $A$.

2. **Construct $R$ with $O(k + k/\varepsilon)$ rows**
   1. $[Y, \Psi, \Delta] = \text{BestSubspaceSVD}(A, C, k)$; $Y \in \mathbb{R}^{m \times c}$, $\Psi \in \mathbb{R}^{c \times c}$, $\Delta \in \mathbb{R}^{c \times k}$; $B = Y \Delta$.
      $B = Z_2 D$ is a qr of $B$ with $Z_2 \in \mathbb{R}^{m \times k}$ ($Z_2^T Z_2 = I_k$), $D \in \mathbb{R}^{k \times k}$, and $E_2 = \Lambda^T - A^T Z_2 Z_2^T$.
   2. $[\Omega_2, D_2] = \text{RandSampling}(Z_2, h_2, 1)$; $h_2 = 8k \ln(20k)$; $\Omega_2 \in \mathbb{R}^{n \times h_2}$; $D_2 \in \mathbb{R}^{h_2 \times h_2}$
      $M_2 = Z_2^T \Omega_2 D_2 \in \mathbb{R}^{k \times h_2}$.  
      $M_2 = U_{M_2} \Sigma_{M_2} V_{M_2}^T$ with rank($M_2$) = $k$ and $V_{M_2} \in \mathbb{R}^{h_2 \times k}$.
   3. $S_2 = \text{BssSampling}(V_{M_2}, (E_2 \Omega_2 D_2)^T, r_1)$, with $r_1 = 4k$ and $S_2 \in \mathbb{R}^{h_2 \times r_1}$.
      $R_1 = (A^T \Omega_2 D_2 S_2)^T \in \mathbb{R}^{r_1 \times n}$ containing rescaled rows from $A$.
   4. $R_2 = \text{AdaptiveRows}(A, Z_2, R_1, r_2)$, with $r_2 = \frac{1620k}{\varepsilon}$ and $R_2 \in \mathbb{R}^{r_2 \times n}$ with rows of $A$.
      $R = [R_1^T \ R_2^T] \in \mathbb{R}^{(r_1 + r_2) \times n}$ containing $r = 4k + \frac{1620k}{\varepsilon}$ rescaled rows of $A$.

3. **Construct $U$ of rank $k$**
   1. Construct $U \in \mathbb{R}^{c \times r}$ with rank at most $k$ as follows (all those formulas are equivalent),
      $$U = \Psi^{-1} \Delta D^{-1} Z_2^T A R \dagger = \Psi^{-1} \Delta D^{-1} (C \Psi^{-1} \Delta D^{-1})^T A R \dagger$$
      $$= \Psi^{-1} \Delta D^{-1} (C \Psi^{-1} \Delta D^{-1}) \dagger A R \dagger$$
      $$= \Psi^{-1} \Delta D^{-1} \Delta^T \Psi C \dagger A R \dagger$$
      $$= \Psi^{-1} \Delta^T \Psi C \dagger A R \dagger$$
5. Linear-time randomized CUR. In this section, we present and analyze a randomized CUR algorithm that runs in linear time\footnote{“Linear time” here we mean running time proportional to $mnk\varepsilon^{-1}$.}~\footnote{To be precise, the algorithm we analyze here constructs $C$ and $R$ with rescaled columns and rows from $A$. To convert this to a truly CUR decomposition, keep the un-rescaled versions of $C$ and $R$ and introduce the scaling factors in $U$. The analysis carries over to that CUR version unchanged.}. The goal of this section is not to design the fastest possible CUR algorithm. Instead, we focus on simplicity. A potentially faster randomized algorithm, especially for sparse matrices $A$, is presented in Section 6. The algorithm of this section might be faster though for dense matrices, depending on the various parameters in the algorithm and the dimensions of the matrix. The analysis of the algorithm in this section serves as a stepping stone for the input-sparsity-time and the deterministic CUR algorithms presented later. Indeed, we provide detailed proofs for the results in this section; in later sections, to prove similar results, we often quote proofs from this section outlining the differences.

We start with the algorithm description, which closely follows the CUR proto-algorithm in Algorithm 1. Then, we give a detailed analysis of the running time complexity of the algorithm. Finally, in Theorem 5.1 we analyze the approximation error $\|A - CUR\|_F^2$.

5.1. Algorithm description. Algorithm 2 takes as input an $m \times n$ matrix $A$, rank parameter $k < \text{rank}(A)$, and accuracy parameter $0 < \varepsilon < 1$. These are precisely the inputs of the CUR problem in Definition 1.1. It returns matrix $C \in \mathbb{R}^{m \times c}$ with $c = O(k/\varepsilon)$ columns of $A$, matrix $R \in \mathbb{R}^{r \times n}$ with $r = O(k/\varepsilon)$ rows of $A$, as well as matrix $U \in \mathbb{R}^{c \times r}$ with rank at most $k$. Algorithm 2 follows closely the CUR proto-algorithm in Algorithm 1. In more detail, Algorithm 2 makes specific choices for the various steps of the proto-algorithm that can be implemented in linear time. Algorithm 2 runs in three steps: (i) in the first step, an optimal number of columns are selected in $C$; (ii) in the second step, an optimal number of rows are selected in $R$; and (iii) in the third step, an intersection matrix with optimal rank is constructed and denoted by $U$. The algorithm itself refers to several other algorithms, which we analyze in detail in different sections. Specifically, $\text{RandomizedSVD}$ is described in Lemma 3.3; $\text{RandSampling}$ in Lemma 3.7; $\text{BssSampling}$ in Lemma 3.5; $\text{AdaptiveCols}$ in Lemma 3.10. $\text{BestSubspaceSVD}$ in Lemma 3.11; and $\text{AdaptiveRows}$ in Lemma 3.10.

5.2. Running time analysis. Next, we give a detailed analysis of the arithmetic operations of Algorithm 2.

1. We need $O(mnk/\varepsilon + k^4 \ln k + \frac{k}{\varepsilon} \log \frac{k}{\varepsilon})$ time to find $c = 4k + \frac{1620k}{\varepsilon}$ columns of $A$ in $C \in \mathbb{R}^{m \times c}$.

   (a) We need $O(mnk)$ time to compute $Z_1$ (from Lemma 3.3), and $O(mnk)$ time to form $E_1$.

   (b) We need $O(kn + k \ln(k) \log(k \ln(k)))$ time to construct $\Omega_1, D_1$ (from Lemma 3.7).

   We need $O(k^3 \ln k)$ time to construct $V_{M_1}$.

   (c) We need $O(mk \ln k + k^4 \ln k)$ time to construct $S_1$ (from Lemma 3.5).

   We need $O(m + k)$ time to construct $C_1$.

   (d) We need $O(mnk/\varepsilon + \frac{k}{\varepsilon} \log \frac{k}{\varepsilon})$ time to construct $C_2$ (from Lemma 3.9).

   We need $O(m + k/\varepsilon)$ time to construct $C$.

2. We need $O(mnk/\varepsilon + k^4 \ln k + \frac{k}{\varepsilon} \log \frac{k}{\varepsilon})$ time to find $r = 4k + \frac{1620k}{\varepsilon}$ rows of $A$ in $R \in \mathbb{R}^{r \times n}$.

   (a) We need $O(mnk)$ to construct $Y$, $\Psi$, and $\Delta$ (from Lemma 3.11).
We need $O(mk^2)$ time to construct $Z_2$, and $O(mnk)$ time to form $E_2$.

(b) We need $O(kn + k\ln(k) \log(k \ln(k)))$ time to construct $\Omega_2, D_2$ (from Lemma 3.7).

(c) We need $O(nk \ln k)$ time to construct $V M_2$.

3. We need $O(mk^2/\varepsilon + m^2k/\varepsilon + mk^2/\varepsilon^2 + k^3/\varepsilon^3)$ time to construct $U$. First, we compute $C^T, R^T$, and $\Psi^{-1}$; then, we compute $U$ as follows:

$$U = \left(\Psi^{-1} \left(\Delta \left(\Delta^T \left(\Psi C^T\right)\right)\right)\right) \cdot (A R^T).$$

The total asymptotic running time of the algorithm is

$$O(n^2k/\varepsilon + m^2k/\varepsilon + mk^2/\varepsilon^2 + k^3/\varepsilon^3 + k\ln k + \frac{k}{\varepsilon} \log \frac{k}{\varepsilon}).$$

5.3. Error bounds. The theorem below presents our main quality-of-approximation result regarding Algorithm 2. We prove the theorem in Section 5.3.2.

**Theorem 5.1.** The matrices $C, U$, and $R$ in Algorithm 2 satisfy with probability at least $0.2$,

$$\|A - CUR\|_F^2 \leq (1 + 20\varepsilon) \|A - A_k\|_F^2.$$

5.3.1. Intermediate results. To prove Theorem 5.1 in Section 5.3.2, we need several intermediate results, some of which might be of independent interest.

First, we argue that the sampling of columns implemented via the matrices $\Omega_1, D_1$, and $S_1$ “preserves” the rank of $Z_1^T$. This is necessary in order to prove that $C_1 = A \Omega_1 D_1 S_1$ gives a “good” column-based, low-rank approximation to $A$. We make this statement precise in Lemma 5.3.

**Lemma 5.2.** The matrices $Z_1, \Omega_1, D_1, S_1$ in Algorithm 2 satisfy with probability at least $0.9$,

$$\text{rank}(Z_1^T \Omega_1 D_1 S_1) = k.$$

**Proof.** It suffices to show that $\sigma_k \left(Z_1^T \Omega_1 D_1 S_1\right) > 0$. Recall some notation that was introduced in the algorithm: $M_1 = Z_1^T \Omega_1 D_1 \in \mathbb{R}^{k \times k}$. From Lemma 3.7 with $V = Z_1$, $\Omega = \Omega_1$, $D = D_1$, and $r = h_1 = 16k \ln(20k)$, we obtain that with probability at least $0.9$,

$$\sigma_k^2(Z_1^T \Omega_1 D_1) \geq \frac{1}{2}. \tag{5.1}$$

This implies that $\text{rank}(M_1) = k$; hence, the SVD of $M_1$ is $M_1 = U_{M_1} \Sigma_{M_1} V_{M_1}^T$, with $U_{M_1} \in \mathbb{R}^{k \times k}$, $\Sigma_{M_1} \in \mathbb{R}^{k \times k}$, and $V_{M_1} \in \mathbb{R}^{h_1 \times k}$. Now recall that in step 1-d in the algorithm we constructed the matrix $S_1 \in \mathbb{R}^{h_1 \times 2k}$ as follows, $S_1 = \text{BssSampling}(V_{M_1}, E_1 \Omega_1 D_1, 2k)$. From Lemma 3.5, we have that

$$\sigma_k \left(V_{M_1}^T S_1\right) \geq \frac{1}{2}. \tag{5.2}$$
We manipulate the second term above as follows, 

\[ \text{rank}(Z_1^T \Omega_1 D_1 S_1) = \text{rank}(U_M, \Sigma_M, V_M^T, S_1) = \text{rank}(V_M^T, S_1) = k. \]

Next, we argue that the matrix \( C_1 \) in the algorithm offers a constant-factor, column-based, low-rank approximation to \( A \). Notice that \( C_1 \) contains \( O(k) \) columns of \( A \).

**Lemma 5.3.** The matrix \( C_1 \) in Algorithm 2 satisfies with probability at least 0.7,

\[ \| A - C_1 C_1^T A \|_F^2 \leq 1620 \cdot \| A - A_k \|_F^2. \]

**Proof.** We would like to apply Lemma 3.1 with \( Z = Z_1 \in \mathbb{R}^{n \times k} \) and \( S = \Omega_1 D_1 S_1 \in \mathbb{R}^{n \times c_1} \). First, we argue that the rank assumption of the lemma is satisfied for our specific choice of \( S \). In Lemma 5.2 we proved that with probability at least 0.9: \( \text{rank}(Z_1^T \Omega_1 D_1 S_1) = k \). So, for \( C_1 = A \Omega_1 D_1 S_1 \) (\( E_1 = A - AZ_1 \in \mathbb{R}^{m \times n} \)),

\[
\| A - C_1 C_1^T A \|_F^2 \leq \| A - \Pi_{C_1,k}(A) \|_F^2 \\
\leq \| A - C_1(Z_1 \Omega_1 D_1 S_1) \|_F^2 Z_1^T \|_F^2 \\
\leq \| E_1 \|_F^2 + \| E_1 \Omega_1 D_1 S_1(Z_1 \Omega_1 D_1 S_1) \|_F^2.
\]

We manipulate the second term above as follows,

\[
\| E_1 \Omega_1 D_1 S_1(Z_1 \Omega_1 D_1 S_1) \|_F^2 \leq \| E_1 \Omega_1 D_1 S_1 \|_F^2 \| (Z_1 \Omega_1 D_1 S_1) \|_2^2 \\
\leq \| E_1 \Omega_1 D_1 S_1 \|_F^2 \| (U_M, \Sigma_M, V_M^T, S_1) \|_2^2 \\
\leq \| E_1 \Omega_1 D_1 S_1 \|_F^2 \| \left( V_M^T, S_1 \right) \|_2 \| (U_M, \Sigma_M^T) \|_2^2 \\
\leq \| E_1 \Omega_1 D_1 S_1 \|_F^2 \| \left( V_M^T, S_1 \right) \|_2 \| (U_M, \Sigma_M^T) \|_2^2 \\
\leq \| E_1 \Omega_1 D_1 S_1 \|_F^2 \cdot \frac{1}{\sigma_k^2(V_M^T, S_1)} \cdot \frac{1}{\sigma_M^2(U_M, \Sigma_M)} \\
\leq \| E_1 \Omega_1 D_1 S_1 \|_F^2 \cdot 8 \\
\leq \| E_1 \Omega_1 D_1 \|_F^2 \cdot 8 \\
\leq 80 \| E_1 \|_F^2.
\]

(a) follows by the strong spectral submultiplicativity property of matrix norms. (b) follows by replacing \( Z_1 \Omega_1 D_1 = M_1 = U_M, \Sigma_M, V_M^T \) (c) follows by the fact that \( U_M, \Sigma_M \) is a full rank \( k \times k \) matrix. (d) follows by the spectral submultiplicativity property of matrix norms. (e) follows by the connection of the spectral norm of the pseudo-inverse with the singular values of the matrix to be pseudo-inverted. (f) follows by Equations 5.1 and 5.2 (here there is a 0.1 failure probability - in the same probability event \( \text{rank}(Z_1^T \Omega_1 D_1 S_1) = k \)). (g) follows by Lemma 3.5. (h) follows by Lemma 3.8 (here there is a 0.1 failure probability). So, overall with probability at least 0.8,

\[ \| E_1 \Omega DS_1(Z_1 \Omega DS_1) \|_F^2 \leq 80 \| E_1 \|_F^2. \]
hence, with the same probability,
\[ \|A - C_1C_1^\dagger A\|_F^2 \leq \|E_1\|_F^2 + 80\|E_1\|_F^2. \]

From Lemma 3.3 we obtain: \( E \left( \|E_1\|_F^2 \right) \leq 2\|A - A_k\|_F^2 \). This implies that with probability at least 0.9: \( \|E_1\|_F^2 \leq 20\|A - A_k\|_F^2 \); hence, with probability at least 0.7,
\[ \|A - C_1C_1^\dagger A\|_F^2 \leq 1620\|A - A_k\|_F^2. \]

Next, we argue that the matrix \( C \) in the algorithm offers a relative-error, column-based, low-rank approximation to \( A \).

**Lemma 5.4.** The matrix \( C \) in Algorithm 2 satisfies with probability at least 0.6,
\[ \|A - CC^\dagger A\|_F^2 \leq \|A - \Pi_{C,k}^F(A)\|_F^2 \leq (1 + 10\epsilon) \cdot \|A - A_k\|_F^2. \]

**Proof.** From Lemma 3.9,
\[ E \left( \|A - \Pi_{C,k}^F(A)\|_F^2 \right) \leq \|A - A_k\|_F^2 + \frac{\epsilon}{1620} \cdot \|A - C_1C_1^\dagger A\|_F^2. \]

Since \( \|A - A_k\|_F^2 \) is considered a constant with respect to the expectation operator, we can write
\[ E \left( \|A - \Pi_{C,k}^F(A)\|_F^2 - \|A - A_k\|_F^2 \right) \leq \frac{\epsilon}{1620} \cdot \|A - C_1C_1^\dagger A\|_F^2. \]

Notice that
\[ \|A - \Pi_{C,k}^F(A)\|_F^2 - \|A - A_k\|_F^2 > 0, \]
so we can apply Markov’s inequality and obtain that with probability at least 0.9,
\[ \|A - \Pi_{C,k}^F(A)\|_F^2 - \|A - A_k\|_F^2 \leq 10\frac{\epsilon}{1620} \cdot \|A - C_1C_1^\dagger A\|_F^2, \]

hence, with probability at least 0.9,
\[ \|A - \Pi_{C,k}^F(A)\|_F^2 \leq \|A - A_k\|_F^2 + 10\frac{\epsilon}{1620} \cdot \|A - C_1C_1^\dagger A\|_F^2. \]

In Lemma 5.3 we proved that for the matrix \( C_1 \) in the main algorithm and with probability at least 0.7,
\[ \|A - C_1C_1^\dagger A\|_F^2 \leq 1620 \cdot \|A - A_k\|_F^2. \]

Combining the last two bounds, we are in a probability event that fails with probability at most 0.4 and guarantees that
\[ \|A - \Pi_{C,k}^F(A)\|_F^2 \leq (1 + 10\epsilon) \cdot \|A - A_k\|_F^2. \]

Finally, by the definition of \( \Pi_{C,k}^F(A) \) we have that
\[ \|A - CC^\dagger A\|_F \leq \|A - \Pi_{C,k}^F(A)\|_F^2. \]
Next, we show that there is a rank $k$ matrix in $\text{span}(C)$ that achieves a similar relative-error bound as the relative-error bound achieved by $CC^TA$ in the previous lemma.

**Lemma 5.5.** The matrices $Y, \Delta$ in Algorithm 2 satisfy with probability at least 0.6:

$$\|A - Y\Delta\Delta^TYA\|_F^2 \leq (1 + 10\varepsilon)\|A - A_k\|_F^2.$$ 

**Proof.** This result is immediate from Lemma 3.11 and Lemma 5.4. □

The following two lemmas prove similar results to Lemmas 5.2 and 5.3 but for the matrix $R_1$.

**Lemma 5.6.** The matrices $Z_2, \Omega_2, D_2, S_2$ in Algorithm 2 satisfy with probability at least $0.9$, 

$$\text{rank}(Z_2^T\Omega_2D_2S_2) = k.$$ 

**Proof.** The proof is identical to the proof of Lemma 5.2. □

**Lemma 5.7.** The matrix $R_1$ in Algorithm 2 satisfies with probability at least $0.7$, 

$$\|A - AR_1^TR_1\|_F^2 \leq 1620 \cdot \|A - A_k\|_F^2.$$ 

**Proof.** The proof is identical to the proof of Lemma 5.3 with $A, C_1$ replaced by $A^T$ and $R_1^T$. □

**5.3.2. Proof of Theorem 5.1.** We are now ready to prove Theorem 5.1. Our construction of $C, U$, and $R$ implies that

$$\text{CUR} = Z_2Z_2^TAR_1^TR_1,$$

so below we analyze the error

$$\|A - Z_2Z_2^TAR_1^TR_1\|_F^2.$$ 

From Lemma 3.10 (with $V = Z_2$) and our choice of $r_2 = 1620k/\varepsilon$ in that Lemma,

$$\mathbb{E}\left[\|A - Z_2Z_2^TAR_1^TR_1\|_F^2\right] \leq \|A - Z_2Z_2^TA\|_F^2 + \frac{\varepsilon}{1620} \|A - AR_1^TR_1\|_F^2.$$ 

The expectation is taken with respect to the construction of $R_2$, so $\|A - Z_2Z_2^TA\|_F^2$ is a constant with respect to the expectation operator. Hence, we can write

$$\mathbb{E}\left[\|A - Z_2Z_2^TAR_1^TR_1\|_F^2 - \|A - Z_2Z_2^TA\|_F^2\right] \leq \frac{\varepsilon}{1620} \|A - AR_1^TR_1\|_F^2.$$ 

Notice that

$$\|A - Z_2Z_2^TAR_1^TR_1\|_F^2 - \|A - Z_2Z_2^TA\|_F^2 > 0,$$

so we can apply Markov’s inequality and obtain that with probability at least 0.9,

$$\|A - Z_2Z_2^TAR_1^TR_1\|_F^2 - \|A - Z_2Z_2^TA\|_F^2 \leq \frac{10\varepsilon}{1620} \|A - AR_1^TR_1\|_F^2.$$
Equivalently, with probability at least 0.9

\[ \| A - Z_2 Z_2^T A R^T R \|_F^2 \leq \| A - Z_2 Z_2^T A \|_F^2 + \frac{10\varepsilon}{1620} \| A - AR^T R\|_F^2. \]

We further manipulate this bound as follows:

\[
\| A - Z_2 Z_2^T A R^T R \|_F^2 \overset{(a)}{\leq} \| A - Z_2 Z_2^T A \|_F^2 + \frac{10\varepsilon}{1620} \| A - AR^T R\|_F^2 \\
\overset{(b)}{=} \| A - BB^T A \|_F^2 + \frac{10\varepsilon}{1620} \| A - AR^T R\|_F^2 \\
\overset{(c)}{=} \| A - BB^T A \|_F^2 + 10\varepsilon \| A - A_k \|_F^2 \\
\overset{(d)}{=} \| A - Y\Delta\Delta^T YA \|_F^2 + 10\varepsilon \| A - A_k \|_F^2 \\
\overset{(e)}{\leq} (1 + 10\varepsilon) \| A - A_k \|_F^2 + 20\varepsilon \| A - A_k \|_F^2,
\]

(b) follows by the fact that \( Z_2 Z_2^T = BB^T \) (to see this, \( BB^T = Z_2 D(Z_2 D)^T = Z_2 D D^{-1} Z_2 = Z_2 Z_2^T \)). (c) follows by Lemma 5.7 (there is a 0.3 failure probability to this bound). (d) follows by the fact that \( B = Y\Delta \) and \( B^T = (Y\Delta)^T = \Delta^T Y^\top \), because both matrices are orthonormal. (e) follows by Lemma 5.5 (there is a 0.4 failure probability to this bound). So, overall we obtain that with probability at least 0.2,

\[ \| A - Z_2 Z_2^T A R^T R \|_F^2 \leq \| A - A_k \|_F^2 + 20\varepsilon \| A - A_k \|_F^2, \]

which shows that with the same probability,

\[ \| A - CUR \|_F^2 \leq \| A - A_k \|_F^2 + 20\varepsilon \| A - A_k \|_F^2. \]

6. Input-Sparsity-Time CUR. In this section, we present and analyze a CUR algorithm that runs in input-sparsity time\(^3\). We start with the algorithm description, which closely follows the CUR proto-algorithm in Algorithm 1. Then, we give a detailed analysis of the running time complexity of the algorithm. Finally, in Theorem 6.1 we analyze the approximation error \( \| A - CUR \|_F^2 \).

6.1. Algorithm description. Algorithm 3 takes as input an \( m \times n \) matrix \( A \), rank parameter \( k < \text{rank}(A) \), and accuracy parameter \( 0 < \varepsilon < 1 \). These are precisely the inputs of the CUR problem in Definition 1.1. It returns matrix \( C \in \mathbb{R}^{n \times c} \) with \( c = O(k/\varepsilon) \) columns of \( A \), matrix \( R \in \mathbb{R}^{r \times n} \) with \( r = O(k/\varepsilon) \) rows of \( A \), as well as matrix \( U \in \mathbb{R}^{c \times r} \) with rank at most \( k \). Algorithm 3 follows closely the CUR proto-algorithm in Algorithm 1. In more detail, Algorithm 3 makes specific choices for the various steps of the proto-algorithm that can be implemented in input-sparsity-time. Algorithm 3 runs in three steps: (i) in the first step, an optimal number of columns are selected in \( C \); (ii) in the second step, an optimal number of rows are selected in \( R \); and (iii) in the third step, an intersection matrix with optimal rank is constructed and denoted as \( U \). The algorithm itself refers to several other algorithms, which we analyze in detail in different sections. Specifically, \( \text{SparseSVD} \) is described in Lemma 3.4;

\(^3\)To be precise, the algorithm we analyze here constructs \( C \) and \( R \) with rescaled columns and rows from \( A \). To convert this to a truly CUR decomposition, keep the un-rescaled versions of \( C \) and \( R \) and introduce the scaling factors in \( U \). The analysis carries over to that CUR version unchanged.
Algorithm 3: An input-sparsity-time, optimal, relative-error, rank-$k$ CUR

Input: $A \in \mathbb{R}^{m \times n}$; rank parameter $k < \text{rank}(A)$; and accuracy parameter $0 < \varepsilon < 1$.

Output: $C \in \mathbb{R}^{m \times c}$ with $c = O(k/\varepsilon)$; $R \in \mathbb{R}^{r \times n}$ with $r = O(k/\varepsilon)$; $U \in \mathbb{R}^{c \times r}$ with $\text{rank}(U) = k$.

1. **Construct $C$ with $O(k + k/\varepsilon)$ columns**
   1. Construct $C$ with $O(k + k/\varepsilon)$ columns.
   2. Let $Z_1 = \text{SparseSV}(A, k, 1)$; $Z_1 \in \mathbb{R}^{n \times k}$ ($Z_1^T Z_1 = I_k$).
   3. $\Omega_1, D_1 \in \text{RandSampling}(Z_1, h_1, 1)$; $h_1 = 16 kn(20k); \Omega_1 \in \mathbb{R}^{n \times h_1}; D_1 \in \mathbb{R}^{h_1 \times h_1}$.
   4. $M_1 = Z_1^T \Omega_1 D_1 \in \mathbb{R}^{k \times h_1}$. $M_1 = U_M \Sigma_M V_M^T$ with $\text{rank}(M_1) = k$ and $V_M, \Sigma_M \in \mathbb{R}^{h_1 \times k}$.
   5. $S_1 = \text{RandSamplingSparse}(V_M, (A - AZ_1 Z_1^T) \Omega_1 D_1)^T$, $c_1, 0.5$, with $c_1 = 4k$. $S_1 \in \mathbb{R}^{h_1 \times c_1}$ (see the running time analysis section for a detailed implementation of this step).
   6. $C_1 = A \Omega_1 D_1 S_1 \in \mathbb{R}^{m \times c_1}$ containing rescaled columns of $A$.
   7. $C = [C_1 C_2] \in \mathbb{R}^{m \times c}$ containing $c = c_1 + c_2 = 4k + O(\frac{4820k}{\varepsilon})$ rescaled columns of $A$.

2. **Construct $R$ with $O(k + k/\varepsilon)$ rows**
   1. Construct $R$ with $O(k + k/\varepsilon)$ rows.
   2. Let $Y, \Psi, \Delta \in \text{ApproxSubspaceSV}(A, C, k)$; $Y \in \mathbb{R}^{m \times c}$, $\Psi \in \mathbb{R}^{c \times c}$, $\Delta \in \mathbb{R}^{c \times k}$; $B = Y \Delta$.
   3. $B = Z_2 D$ is a qr decomposition of $B$ with $Z_2 \in \mathbb{R}^{m \times k}$ ($Z_2^T Z_2 = I_k$), $D \in \mathbb{R}^{k \times k}$.
   4. $\Omega_2, D_2 \in \text{RandSampling}(Z_2, h_2, 1)$; $h_2 = 8 kn(20k); \Omega_2 \in \mathbb{R}^{m \times h_2}; D_2 \in \mathbb{R}^{h_2 \times h_2}$.
   5. $M_2 = Z_2^T \Omega_2 D_2 \in \mathbb{R}^{k \times h_2}$. $M_2 = U_M \Sigma_M V_M^T$ with $\text{rank}(M_2) = k$ and $V_M, \Sigma_M \in \mathbb{R}^{h_2 \times k}$.
   6. $S_2 = \text{RandSamplingSparse}(V_M, (A^T - A^T Z_2 Z_2^T) \Omega_2 D_2)^T$, $r_1, 0.5$, with $r_1 = 4k$. $S_2 \in \mathbb{R}^{h_2 \times r_1}$ (see the running time analysis section for a detailed implementation of this step).
   7. $R_1 = (A^T \Omega_2 D_2 S_2)^T \in \mathbb{R}^{r_1 \times n}$ containing rescaled rows from $A$.
   8. $R_2 = \text{AdaptiveRowsSparse}(A, Z_2, R_1, r_2)$; $r_2 = \frac{4820k}{\varepsilon}$; $R_2 \in \mathbb{R}^{r_2 \times n}$ with rows of $A$.
   9. $R = [R_1^T R_2^T]^T \in \mathbb{R}^{(r_1 + r_2) \times n}$ containing $r = k + O(\frac{4820k}{\varepsilon})$ rescaled rows of $A$.

3. **Construct $U$ of rank $k$**
   1. Let $W \in \mathbb{R}^{\xi \times m}$ be a randomly chosen sparse subspace embedding with $\xi = \Omega(k^2 \varepsilon^{-2})$. Then,
   
   $U = \Psi^{-1} \Delta D^{-1} (WC \Psi^{-1} \Delta D^{-1})^\dagger W AR^\dagger = \Psi^{-1} \Delta \Delta^T (WC)^\dagger W AR^\dagger$
RandSampling in Lemma 3.7; BssSamplingSparse in Lemma 4.3; AdaptiveColsSparse in Lemma 4.4. ApproxSubspaceSVD in Lemma 3.12; and AdaptiveRowsSparse in Lemma 4.5.

It is worth pointing out the differences of Algorithm 3 with Algorithm 2. In a nutshell, Algorithm 3 replaces the steps in Algorithm 2 that cannot be implemented in input-sparsity-time with similar steps that enjoy this property. In some more detail, in step 1–1, Algorithm 3 uses SparseSVD instead of the standard SVD method, in steps 1–3 and 2–3, Algorithm 3 uses BssSamplingSparse instead of the standard version of BssSampling, in step 1–4 Algorithm 3 uses AdaptiveColsSparse instead of the standard AdaptiveCols procedure, and in step 2–4 Algorithm 3 uses AdaptiveRowsSparse instead of AdaptiveRows.

6.2. Running time analysis. Next, we give a detailed analysis of the arithmetic operations of Algorithm 3.

1. We need \( O(\text{nnz}(A) \log n + m \cdot \text{poly}(\log n, k, 1/\varepsilon)) \) time to find \( c = 4k + \frac{4820k}{\varepsilon} \) columns of \( A \) in \( C \in \mathbb{R}^{m \times c} \),
   \( a) \) We need \( O(\text{nnz}(A)) + \tilde{O}(nk^2 \varepsilon^{-4} + k^3 \varepsilon^{-5}) \) time to compute \( Z_1 \) (from Lemma 3.4).
   \( b) \) We need \( O(kn + k \ln(k) \log(k \ln(k))) \) time to construct \( \Omega_1, D_1 \) (from Lemma 3.7).
   \( c) \) We need \( O(k^3 \ln k) \) time to construct \( V_{M_1} \).
   \( d) \) We need \( O(\text{nnz}(A) + k^3 \log^2(k) \varepsilon^{-2} + k^3 \log^2(k) \varepsilon^{-2} + k^4 \log(k)) \) time for \( S_1 \) (from Lemma 4.3).
   We need \( O(m + k) \) time to construct \( C_1 \).
   \( e) \) We need \( O(\text{nnz}(A) \log n + mk \log n + mk^2) \) time to construct \( C_2 \) (from Lemma 4.4).
   \( f) \) We need \( O(m + k / \varepsilon) \) time to construct \( C \).

2. We need \( O(\text{nnz}(A) \log n + n \cdot \text{poly}(\log n, k, 1/\varepsilon)) \) time to find \( r = 4k + \frac{4820k}{\varepsilon} \) rows of \( A \) in \( Y \in \mathbb{R}^{r \times n} \),
   \( a) \) We need \( O(\text{nnz}(A) + mk^3 / \varepsilon^3) \) time to construct \( \mathbf{Y}, \Psi, \text{ and } \Delta \) (from Lemma 3.11).
   \( b) \) We need \( O(km + k \ln(k) \log(k \ln(k))) \) time to construct \( \Omega_2, D_2 \) (from Lemma 3.7).
   \( c) \) We need \( O(k^3 \ln k) \) time to construct \( V_{M_2} \).
   \( d) \) We need \( O(\text{nnz}(A) + k^3 \log^2(k) \varepsilon^{-2} + k^3 \log^2(k) \varepsilon^{-2} + k^4 \log(k)) \) time to construct \( S_2 \) (Lemma 4.3).
   \( e) \) We need \( O(n + k) \) time to construct \( R_1 \).
   \( f) \) We need \( O(\text{nnz}(A) \log n + nk \log n + nk^2) \) time to construct \( R_2 \) (from Lemma 4.5).
   \( g) \) We need \( O(n + k / \varepsilon) \) time to construct \( R \).

3. We need \( O(\text{nnz}(A) + nk^3 / \varepsilon^4 + k^4 / \varepsilon^n) \) time to construct \( U \). First, we compute \( WC \) and \( WA \) in \( O(\text{nnz}(A)) \) time. Then, we compute \((WC)^\dagger, R_1, \text{ and } \Psi^{-1};\)

---

\[4\] Here, we actually do not explicitly form \( (A - AZ_1Z_1^\dagger)\Omega D \). We only form \( A\Omega D \) and \( Z_1^\dagger\Omega D \) in \( O(1) \) time. Then, the algorithm of Lemma 4.3 multiplies \( (A - AZ_1Z_1^\dagger)\Omega D \) from the left with a sparse subspace embedding matrix \( W \in \mathbb{R}^{\xi \times m} \) with \( \xi = O(k^2 \log^2 k \varepsilon^{-2}) \). Computing \( WA \) takes \( O(\text{nnz}(A)) \) time. Then computing, \((WA)Z_1 \) and \((WAZ_1)Z_1^\dagger \) takes another \( O(\xi mk) + O(\xi nk) \) time, respectively. Finally, the sampling algorithm on \( W(A - AZ_1Z_1^\dagger)\Omega D \) is \( O(k^4 \log k + mk \log k) \) time.
finally, we compute $U$ as follows:

$$U = \left( \Psi^{-1} \left( \Delta \left( \Delta^T \left( \Psi(WC)^\dagger \right) \right) \right) \right) \cdot \left( (WA)^\dagger R \right).$$

The total assymptotic running time of the algorithm is

$$O(nnz(A) \log n + (m + n) \cdot poly(\log n, k, 1/\varepsilon)).$$

6.3. Error bounds. The theorem below presents our main quality-of-approximation result regarding Algorithm 3. We prove the theorem in Section 6.3.2.

**Theorem 6.1.** The matrices $C, U,$ and $R$ in Algorithm 3 satisfy with probability at least $0.16 - 2/n$,

$$\|A - CUR\|_F^2 \leq (1 + \varepsilon)^2 \cdot \|A - A_k\|_F^2.$$

6.3.1. Intermediate Results. To prove Theorem 6.1 in Section 6.3.2, we need several intermediate results, some of which might be of independent interest.

First, we argue that the sampling of columns implemented via the matrices $\Omega_1, D_1,$ and $S_1$ “preserves” the rank of $Z_1^T$. This is necessary in order to prove that $C_1 = A\Omega_1 D_1 S_1$ gives a “good” column-based, low-rank approximation to $A$. We make this statement precise in Lemma 6.3.

**Lemma 6.2.** The matrices $Z_1, \Omega_1, D_1, S_1$ in Algorithm 3 satisfy with probability at least $0.9$,

$$\text{rank}(Z_1^T \Omega_1 D_1 S_1) = k.$$

*Proof.* The proof is identical to the proof of Lemma 5.2. □

Next, we argue that the matrix $C_1$ in the algorithm offers a constant-factor, column-based, low-rank approximation to $A$. Notice that $C_1$ contains $O(k)$ columns of $A$.

**Lemma 6.3.** The matrix $C_1$ in Algorithm 3 satisfies with probability at least $0.69$,

$$\|A - C_1 C_1^\dagger A\|_F^2 \leq 4820 \cdot \|A - A_k\|_F^2.$$

*Proof.* The proof is very similar to the proof of Lemma 5.3 so we only highlight the differences. Indeed, the only difference is in step (g) in the manipulations of the term

$$\|E_1 \Omega_1 D_1 S_1 (Z_1 \Omega_1 D_1 S_1)^\dagger\|_F^2.$$

In that step, due to Lemma 4.3 and our choice of $\varepsilon = 0.5$ we will have an extra multiplicative term $3$ in the bound, so the final bound instead of 80 will be 240. Here, we will also have an extra failure probability 0.01. So the final bound for $\|A - C_1 C_1^\dagger A\|_F^2$ will have a constant 4820, as claimed. □

Next, we argue that the matrix $C$ in the algorithm offers a relative-error, column-based, low-rank approximation to $A$.

**Lemma 6.4.** The matrix $C$ in Algorithm 3 satisfies with probability at least $0.59 - 1/n$:

$$\|A - CC^\dagger A\|_F^2 \leq \|A - \Pi_{C,k}^F(A)\|_F^2 \leq (1 + 30\varepsilon) \cdot \|A - A_k\|_F^2.$$
Proof. From Lemma 4.4 and with probability at least 0.9 - \frac{1}{n},
\|A - \Pi_{C,F}^{C,k}(A)\|_F^2 \leq \|A - A_k\|_F^2 + 10 \cdot \|\frac{3\varepsilon}{4820} \cdot \|A - C_1 C_1^\dagger A\|_F^2.

In Lemma 6.3 we proved that for the matrix \(C_1\) in the algorithm and with probability at least 0.69,
\|A - C_1 C_1^\dagger A\|_F^2 \leq 4820 \cdot \|A - A_k\|_F^2.

Combining the last two bounds, we are in a probability event that fails with probability at most 0.41 + \frac{1}{n} and guarantees that
\|A - \Pi_{C,F}^{C,k}(A)\|_F^2 \leq (1 + 30\varepsilon) \cdot \|A - A_k\|_F^2.

Finally, by the definition of \(\Pi_{C,F}^{C,k}(A)\) we have that
\|A - CC_1^\dagger A\|_F \leq \|A - \Pi_{C,F}^{C,k}(A)\|_F^2.

Next, we show that there is a rank \(k\) matrix in \(\text{span}(C)\) that achieves a similar relative-error bound as the relative-error bound achieved by \(CC_1^\dagger A\) in the previous lemma.

Lemma 6.5. The matrices \(Y, \Delta\) in Algorithm 3 satisfy with probability at least 0.58 - \frac{1}{n}:
\|A - Y \Delta \Delta^T YA\|_F^2 \leq (1 + 30\varepsilon) \|A - A_k\|_F^2.

Proof. The result follows by combining Lemma 3.12 and Lemma 6.4.

The following two lemmas prove similar results to Lemmas 6.2 and 6.3 but for the matrix \(R_1\).

Lemma 6.6. The matrices \(Z_2, \Omega_2, D_2, S_2\) in Algorithm 3 satisfy with probability at least 0.9:
\[ \text{rank}(Z_2^T \Omega_2 D_2 S_2) = k. \]

Proof. The proof is identical to the proof of Lemma 6.2.

Lemma 6.7. The matrix \(R_1\) in Algorithm 3 satisfies with probability at least 0.69:
\[ \|A - AR_1^\dagger R_1\|_F^2 \leq 4820 \cdot \|A - A_k\|_F^2. \]

Proof. The proof is identical to the proof of Lemma 6.3 with \(A, C_1\) replaced by \(A^T\) and \(R_1^T\).

6.3.2. Proof of Theorem 6.1. We are now ready to prove Theorem 6.1. Notice that we construct \(U \in \mathbb{R}^{c \times r}\) of rank at most \(k\) as follows,
\[ U = \Psi^{-1} \Delta D^{-1} Y_{opt}, \quad (6.1) \]
where $Y_{opt} \in \mathbb{R}^{k \times r}$ is a matrix of rank at most $k$ constructed via the following formula,

$$Y_{opt} = (WC\Psi^{-1}\Delta D^{-1})^\dagger U_{WC\Psi^{-1}\Delta D^{-1}}^T (WAV_R V_R^T) V_R V_R^T R^\dagger = (WC\Psi^{-1}\Delta D^{-1})^\dagger WAV_R \tilde{V}_R^\dagger \tilde{R} = (WC\Psi^{-1}\Delta D^{-1})^\dagger WAR^\dagger.$$  

Here, $W \in \mathbb{R}^{\xi \times m}$ is a randomly chosen sparse subspace embedding (see Section 3.6). For this choice of $U$, we will analyze the error $\|A - CUR\|_F$. First, notice that (see Section 3.8):

$$Y_{opt} \in \text{argmin}_{Y \in \mathbb{R}^{k \times r}} \|WAV_R V_R^T - WC\Psi^{-1}\Delta D^{-1} Y\|_F.$$  

Define

$$\tilde{X}_{opt} = Y_{opt} R \in \mathbb{R}^{k \times n},$$  

which is also equivalent to the following,

$$\tilde{X}_{opt} = \text{argmin}_{X \in \mathbb{R}^{k \times n}} \|WAV_R V_R^T - WC\Psi^{-1}\Delta D^{-1} X\|_F = (WC\Psi^{-1}\Delta D^{-1})^\dagger WAV_R V_R^T.$$  

Also, define $X_{opt} \in \mathbb{R}^{k \times n}$ as follows

$$X_{opt} = \text{argmin}_{X \in \mathbb{R}^{k \times n}} \|AV_R V_R^T - C \Psi^{-1}\Delta D^{-1} X\|_F = (C \Psi^{-1}\Delta D^{-1})^\dagger AV_R V_R^T = Z_2^T \tilde{A} V_R V_R^T.$$  

The proof of the theorem is immediate after combining Lemma 6.8 and Lemma 6.9 below.

**Lemma 6.8.** The matrices $C$, $U$, and $R$ in Algorithm 3 satisfy with probability at least 0.99:

$$\|A - CUR\|_F^2 \leq (1 + \varepsilon)\|A - Z_2Z_2^T R\|_F^2.$$  

**Proof.**

\[\begin{align*}
\|A - CUR\|_F^2 &\leq ||A - C\Psi^{-1}\Delta D^{-1}Y_{opt}R\|_F^2 \\
&\leq \|A - C\Psi^{-1}\Delta D^{-1}\tilde{X}_{opt}\|_F^2 \\
&\leq \|AV_R V_R^T - C\Psi^{-1}\Delta D^{-1}\tilde{X}_{opt} + A - AV_R V_R^T\|_F^2 \\
&\leq \|AV_R V_R^T - C\Psi^{-1}\Delta D^{-1}\tilde{X}_{opt}\|_F^2 + \|A - AV_R V_R^T\|_F^2 \\
&\leq (1 + \varepsilon)\|AV_R V_R^T - C\Psi^{-1}\Delta D^{-1}X_{opt}\|_F^2 + \|A - AV_R V_R^T\|_F^2 \\
&\leq \varepsilon\|AV_R V_R^T - C\Psi^{-1}\Delta D^{-1}X_{opt}\|_F^2 + \|A - AV_R V_R^T\|_F^2 \\
&\leq \varepsilon\|AV_R V_R^T - C\Psi^{-1}\Delta D^{-1}X_{opt}\|_F^2 + \|A - C\Psi^{-1}\Delta D^{-1}(Z_2^T AR^\dagger R)\|_F^2 \\
&\leq \varepsilon\|AV_R V_R^T - C\Psi^{-1}\Delta D^{-1}(C\Psi^{-1}\Delta D^{-1})^\dagger AV_R V_R^T\|_F^2 + \|A - C\Psi^{-1}\Delta D^{-1}(Z_2^T AR^\dagger R)\|_F^2 \\
&\leq \varepsilon\|A - C\Psi^{-1}\Delta D^{-1}(C\Psi^{-1}\Delta D^{-1})^\dagger A\|_F^2 + \|A - C\Psi^{-1}\Delta D^{-1}(Z_2^T AR^\dagger R)\|_F^2 \\
&\leq \varepsilon\|A - Z_2Z_2^T A\|_F^2 + \|A - Z_2Z_2^T AR^\dagger R\|_F^2 \\
&\leq \varepsilon\|A - Z_2Z_2^T AR^\dagger R\|_F^2 + \|A - Z_2Z_2^T AR^\dagger R\|_F^2 \\
&\leq (1 + \varepsilon)\|A - Z_2Z_2^T AR^\dagger R\|_F^2 \\
\end{align*}\]
(α) follows from Eqn. 6.1; (β) follows from Eqn. 6.2; (γ) follows because
\[ A\hat{V}_R V_R^T - A V_R V_R^T = 0_{n \times n}; \]
(δ) follows by the Pythagorean Theorem (to see this notice that from Eqn. 6.3 we can write \( X_{opt} = MV_R^T \) - for an appropriate \( M \); also, \( A - AV_R V_R^T = A \left( I_n - V_R V_R^T \right) \));
(ε) follows by Lemma 3.16 (there is a failure probability 0.01;(η) follows from Eqn. 6.4;
(θ) follows by the Pythagorean Theorem; (i) follows from Eqn. 6.4; (κ) follows because \( V_R V_R^T \) is a projector matrix and can be dropped without increasing the Frobenius norm; (μ) follows by the optimality of \( Z_2 \).

**Lemma 6.9.** The matrices \( R \) and \( Z_2 \) in Algorithm 3 satisfy with probability at least 0.17 - \( \frac{2}{n} \),
\[ \| A - Z_2 Z^T_2 A R^T R \|_F^2 \leq \| A - A_k \|_F^2 + 60\varepsilon \| A - A_k \|_F^2. \]

**Proof.** From Lemma 4.5 (with \( V = Z_2 \)) and our choice of \( r_2 = 4820k/\varepsilon \) in that Lemma, with probability at least 0.9 - 1/n
\[ \| A - Z_2 Z^T_2 A R^T R \|_F^2 \leq \| A - Z_2 Z^T_2 A \|_F^2 + \frac{30\varepsilon}{4820} \| A - AR^T_1 R_1 \|_F^2 \]
Furthermore,
\[ \| A - Z_2 Z^T_2 A \|_F^2 + \frac{30\varepsilon}{4820} \| A - AR^T_1 R_1 \|_F^2 \overset{(b)}{=} \| A - BB^T A \|_F^2 + \frac{30\varepsilon}{4820} \| A - AR^T_1 R_1 \|_F^2 \leq \| A - BB^T A \|_F^2 + 30\varepsilon \| A - A_k \|_F^2 \overset{(c)}{=} \| A - Y \Delta \Delta^T Y A \|_F^2 + 30\varepsilon \| A - A_k \|_F^2 \overset{(d)}{=} (1 + 30\varepsilon) \| A - A_k \|_F^2 + 30\varepsilon \| A - A_k \|_F^2 \]

(b) follows by the fact that \( Z_2 Z^T_2 = BB^T \) (to see this, \( BB^T = Z_2 D(Z_2 D)^T = Z_2 DD^{-1} Z_2 = Z_2 Z^T_2 \), by our specific choice of those matrices). (c) follows by Lemma 6.7 (there is a 0.31 failure probability to this bound). (d) follows by the fact that \( B = Y \Delta \) and
\[ B^T = (Y \Delta)^T = \Delta^T \hat{Y}^T = \Delta^T \hat{Y}^T, \]
because both matrices are orthonormal. (e) follows by Lemma 6.5 (there is a 0.42 + \( \frac{1}{n} \) failure probability to this bound). So, overall we obtain that with probability at least 0.17 - \( \frac{2}{n} \),
\[ \| A - Z_2 Z^T_2 A R^T R \|_F^2 \leq \| A - A_k \|_F^2 + 60\varepsilon \| A - A_k \|_F^2. \]

\[ \square \]

7. **Deterministic CUR.** In this section, we present and analyze a deterministic, polynomial-time CUR algorithm\(^5\). We start with the algorithm description, which closely follows the CUR proto-algorithm in Algorithm 1 (indeed, in this case we do not need the leverage-scores sampling step in the proto-algorithm). Then, we give a detailed analysis of the running time complexity of the algorithm. Finally, in Theorem 7.1 we analyze the approximation error \( \| A - \text{CUR} \|_F^2 \).

\(^5\)To be precise, the algorithm we analyze here constructs \( C \) and \( R \) with rescaled columns and rows from \( A \). To convert this to a truly CUR decomposition, keep the un-rescaled versions of \( C \) and \( R \) and introduce the scaling factors in \( U \). The analysis carries over to that CUR version unchanged.
7.1. Algorithm description. Algorithm 4 takes as input an \( m \times n \) matrix \( A \), rank parameter \( k < \text{rank}(A) \), and accuracy parameter \( 0 < \varepsilon < 1 \). These are precisely the inputs of the CUR problem in Definition 1.1. It returns matrix \( C \in \mathbb{R}^{m \times c} \) with \( c = O(k/\varepsilon) \) columns of \( A \), matrix \( R \in \mathbb{R}^{r \times n} \) with \( r = O(k/\varepsilon) \) rows of \( A \), as well as matrix \( U \in \mathbb{R}^{c \times r} \) with rank at most \( k \). Algorithm 4 follows closely the CUR proto-algorithm in Algorithm 1. In more details, Algorithm 4 makes specific choices for the various steps of the proto-algorithm that can be implemented deterministically in polynomial time. Algorithm 4 runs in three steps: (i) in the first step, an optimal number of columns are selected in \( R \); (ii) in the second step, an optimal number of rows are selected in \( R \); and (iii) in the third step, an intersection matrix with optimal rank is constructed and denoted as \( U \). The algorithm itself refers to several other algorithms, which we analyze in detail in different sections. Specifically, DeterministicSVD is described in Lemma 3.2; BssSampling in Lemma 3.5; AdaptiveColsD in Lemma 4.2. BestSubspaceSVD in Lemma 3.11; and AdaptiveRowsD in Lemma 4.1. All those algorithms are deterministic.

7.2. Running time analysis. Next, we give a detailed analysis of the arithmetic operations of Algorithm 4.

1. We need \( O(mn^3k/\varepsilon) \) time to find \( c = 4k + \frac{10k}{\varepsilon} \) columns of \( A \) in \( C \in \mathbb{R}^{m \times c} \).
   (a) We need \( O(mnk^2) \) time to compute \( Z_1 \) (from Lemma 3.2), and \( O(mnk) \) time to form \( E_1 \).
   (b) We need \( O(mn + nk^3) \) time to construct \( S_1 \) (from Lemma 3.5).
   (c) We need \( O(mnk/\varepsilon) \) time to construct \( C_2 \) (from Lemma 4.2).
   (d) We need \( O(m + k) \) time to construct \( C \).

2. We need \( O(mn^3k/\varepsilon) \) time to find \( r = 4k + \frac{5k}{\varepsilon} \) rows of \( A \) in \( R \in \mathbb{R}^{r \times n} \).
   (a) We need \( O(mnk/\varepsilon) \) to construct \( Y, \Psi, \) and \( \Delta \) (from Lemma 3.11).
   (b) We need \( O(mnk^2) \) time to compute \( Z_2, \Psi \), and \( O(mnk) \) time to form \( E_2 \).
   (c) We need \( O(n + k) \) time to construct \( R_1 \).
   (d) We need \( O(mnk/\varepsilon) \) time to construct \( R_2 \) (from Lemma 4.1).
   (e) We need \( O(n + k) \) time to construct \( R \).

3. We need \( O(mn^3k/\varepsilon + n^2k/\varepsilon + mk^2/\varepsilon^2 + k^3/\varepsilon^3) \) time to construct \( U \). First, we compute \( C^\dagger, R^\dagger, \) and \( \Psi^{-1} \); then, we compute \( U \) as follows:

\[
U = (\Psi^{-1} \left( \Delta \left( \Delta^T (\Psi C^\dagger) \right) \right)) \cdot (AR^\dagger).
\]

The total asymptotic running time of the algorithm is \( O(mn^3k/\varepsilon) \).

7.3. Error bounds. The theorem below presents our main quality-of-approximation result regarding Algorithm 4. We prove the theorem in Section 7.3.2.

**Theorem 7.1.** The matrices \( C, U, \) and \( R \) in Algorithm 4 satisfy,

\[
\|A - CUR\|_F^2 \leq \|A - A_k\|_F^2 + 8\varepsilon\|A - A_k\|_F^2.
\]

7.3.1. Intermediate results. To prove Theorem 7.1 in Section 7.3.2, we need several intermediate results, some of which might be of independent interest.

First, we argue that the sampling of columns implemented via the matrix \( S_1 \) “preserves” the rank of \( Z_1^\dagger \). This is necessary in order to prove that \( C_1 = AS_1 \).
Algorithm 4 A deterministic, poly-time, optimal, relative-error, rank-k CUR

| Step | Description |
|------|-------------|
| 1.   | Construct $C$ with $O(k + k/\varepsilon)$ columns |
| 2.   | Construct $R$ with $O(k + k/\varepsilon)$ rows |
| 3.   | Construct $U$ of rank at most $k$ as follows (all those formulas are equivalent), |

$$U = \Psi^{-1} \Delta D^{-1} Z_2^T A R^\dagger = \Psi^{-1} \Delta D^{-1} (C \Psi^{-1} \Delta D^{-1})^T A R^\dagger = \Psi^{-1} \Delta D^{-1} (C \Psi^{-1} \Delta D^{-1})^\dagger A R^\dagger = \Psi^{-1} \Delta D^{-1} \Delta^T \Psi C^\dagger A R^\dagger = \Psi^{-1} \Delta^T \Psi C^\dagger A R^\dagger.$$

This gives a “good” column-based, low-rank approximation to $A$. We make this statement precise in Lemma 7.3.

**Lemma 7.2.** The matrices $Z_1, S_1$ in Algorithm 4 satisfy,

$$\operatorname{rank}(Z_1^T S_1) = k.$$

**Proof.** From Lemma 3.5 we obtain $\sigma_k \left( Z_1^T S_1 \right) \geq \frac{1}{\varepsilon}$, which implies $\operatorname{rank}(Z_1^T S_1) = k$. \[
\]

Next, we argue that the matrix $C_1$ in the algorithm offers a constant-factor, column-based, low-rank approximation to $A$. Notice that $C_1$ contains $O(k)$ columns.
of $A$.

**Lemma 7.3.** The matrix $C_1$ in Algorithm 4 satisfies

$$\|A - C_1 C_1^\dagger A\|_F^2 \leq 10 \cdot \|A - A_k\|_F^2.$$

**Proof.** We would like to apply Lemma 3.1 with $Z = Z_1 \in \mathbb{R}^{n \times k}$ and $S = S_1 \in \mathbb{R}^{n \times c_1}$. First, we argue that the rank assumption of the lemma is satisfied for our specific choice of $S$. In Lemma 7.2 we proved that $\text{rank} (Z_1^T S_1) = k$. So, for $C_1 = AS_1$ (also recall $E_1 = A - AZ_1 Z_1^T \in \mathbb{R}^{m \times n}$),

$$\|A - C_1 C_1^\dagger A\|_F^2 \leq \|A - \Pi_{C_1,k}^\delta (A)\|_F^2 \leq \|A - C_1 (Z_1 S_1)^\dagger Z_1^T\|_F^2 \leq \|E_1\|_F^2 + \|E_1 S_1 (Z_1 S_1)^\dagger\|_F^2.$$

We manipulate the second term,

$$\|E_1 S_1 (Z_1 S_1)^\dagger\|_F^2 \overset{(a)}{\leq} \|E_1 S_1\|_F^2 \|Z_1 S_1\|_F^2 \overset{(e)}{=} \|E_1 S_1\|_F^2 \cdot \sigma_k^{-2} (Z_1^T S_1) \overset{(f)}{\leq} \|E_1 S_1\|_F^2 \cdot 4 \overset{(g)}{\leq} \|E_1\|_F^2 .$$

(a) follows by the strong spectral submultiplicativity property of matrix norms. (e) follows by the connection of the spectral norm of the pseudo-inverse with the singular values of the matrix to be pseudo-inverted. (f) follows because $\sigma_k (Z_1^T S_1) \geq \frac{1}{2}$. (g) follows by Lemma 3.5. So,

$$\|E_1 S_1 (Z_1 S_1)^\dagger\|_F^2 \leq 4\|E_1\|_F^2,$$

hence,

$$\|A - C_1 C_1^\dagger A\|_F^2 \leq \|E_1\|_F^2 + 4\|E_1\|_F^2.$$

From Lemma 3.2: $\|E_1\|_F^2 \leq 2\|A - A_k\|_F^2$; hence,

$$\|A - C_1 C_1^\dagger A\|_F^2 \leq 10\|A - A_k\|_F^2.$$

Next, we argue that the matrix $C$ in the algorithm offers a relative-error, column-based, low-rank approximation to $A$.

**Lemma 7.4.** The matrix $C$ in Algorithm 4 satisfies,

$$\|A - CC^\dagger A\|_F^2 \leq \|A - \Pi_{C,k}^\delta (A)\|_F^2 \leq (1 + 4\varepsilon) \cdot \|A - A_k\|_F^2.$$

**Proof.** From Lemma 4.2,

$$\|A - \Pi_{C,k}^\delta (A)\|_F^2 \leq \|A - A_k\|_F^2 + \frac{4\varepsilon}{10} \cdot \|A - C_1 C_1^\dagger A\|_F^2.$$

Combine this with Lemma 7.3 to wrap up. \(\square\)
Next, we show that there is a rank $k$ matrix in $\text{span}(C)$ that achieves a similar relative-error bound as the relative-error bound achieved by $CC^TA$ in the previous lemma.

**Lemma 7.5.** The matrices $Y, \Delta$ in Algorithm 4 satisfy,

$$\|A - Y\Delta\Delta^TYA\|_F^2 \leq (1 + 4\varepsilon)\|A - A_k\|_F^2.$$  

**Proof.** This result is immediate from Lemma 3.11 and Lemma 7.4. \(\square\)

The following two lemmas prove similar results to Lemmas 7.2 and 7.3 but for the matrix $R_1$.

**Lemma 7.6.** The matrices $Z_2, S_2$ in Algorithm 4 satisfy,

$$\text{rank}(Z_2^TS_2) = k.$$  

**Proof.** The proof is identical to the proof of Lemma 7.2. \(\square\)

**Lemma 7.7.** The matrix $R_1$ in Algorithm 4 satisfies,

$$\|A - AR_1\|_F^2 \leq 10 \cdot \|A - A_k\|_F^2.$$  

**Proof.** The proof is identical to the proof of Lemma 7.3 with $A, C_1$ replaced by $A^T$ and $R_1^T$. \(\square\)

### 7.3.2. Proof of Theorem 7.1

We are now ready to prove Theorem 7.1. Our construction of $C, U$, and $R$ implies $\text{CUR} = Z_2Z_2^TAR^TR_1$, so below we analyze the error $\|A - Z_2Z_2^TAR^TR_1\|_F^2$. From Lemma 4.1 (with $V = Z_2$) and our choice of $r_2 = 10k/\varepsilon$ in that Lemma,

$$\|A - Z_2Z_2^TAR^TR_1\|_F^2 \leq \|A - Z_2Z_2^TAR^TR_1\|_F^2$$  

We further manipulate this bound as follows:

$$\|A - Z_2Z_2^TAR^TR_1\|_F^2 \leq \|A - Z_2Z_2^TAR^TR_1\|_F^2$$  

We further manipulate this bound as follows:

$$\|A - Z_2Z_2^TAR^TR_1\|_F^2 \leq \|A - Z_2Z_2^TAR^TR_1\|_F^2$$  

(b) follows by the fact that $Z_2Z_2^T = BB^\dagger$ (to see this, $BB^\dagger = Z_2D(Z_2D)^\dagger = Z_2DD^{-1}Z_2 = Z_2Z_2^T$). (c) follows by Lemma 7.7 (d) follows by the fact that $B = Y\Delta$ and $B^\dagger = (Y\Delta)^\dagger = \Delta^TY = \Delta^TY^T$, because both matrices are orthonormal. (e) follows by Lemma 7.5. So, overall we obtain,

$$\|A - Z_2Z_2^TAR^TR_1\|_F^2 \leq \|A - A_k\|_F^2 + 8\varepsilon\|A - A_k\|_F^2,$$

which shows that,

$$\|A - \text{CUR}\|_F^2 \leq \|A - A_k\|_F^2 + 8\varepsilon\|A - A_k\|_F^2.$$
8. Lower bound. In this section we will prove that a relative-error CUR is not possible unless \( C \) has \( \Omega(k/\varepsilon) \) columns of \( A \), \( R \) has \( \Omega(k/\varepsilon) \) rows of \( A \), and \( U \) has rank \( \Omega(k) \). We start with an outline of our approach. In Lemma 8.2 below we prove that there is a symmetric matrix \( A \in \mathbb{R}^{t \times t} \), such that, for any integer \( k \) (the rank parameter) and \( \varepsilon > 0 \), no subset of \( o(k/\varepsilon) \) columns of \( A \) span a \((1 + \varepsilon)\) approximation to \( A \). By symmetry (see Corollary 8.3), this also implies that no subset of \( o(k/\varepsilon) \) rows of \( A \) span a \((1 + \varepsilon)\) approximation to \( A \). Those two observations, along with an observation on the minimum possible rank for \( U \), give the lower bound. The theorem below is the main result in this section.

**Theorem 8.1.** Consider the matrix \( A \in \mathbb{R}^{t \times t} \) in Lemma 8.2. Consider a factorization \( \text{CUR} \), with \( C \in \mathbb{R}^{t \times c} \) containing \( c \) columns of \( A \), \( R \in \mathbb{R}^{r \times t} \) containing \( r \) rows of \( A \) and \( U \in \mathbb{R}^{c \times r} \), such that

\[
\|A - \text{CUR}\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2.
\]

Then, for any \( \varepsilon < 1/3 \) and any \( k \geq 1 \):

\[
c = \Omega(k/\varepsilon),
\]

and

\[
r = \Omega(k/\varepsilon),
\]

and

\[
\text{rank}(U) \geq k/2.
\]

**Proof.** Assume that \( C, U, R \) are as described in the statement of the theorem, with

\[
\|A - \text{CUR}\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2.
\]

Assume further that either \( c = o(k/\varepsilon) \) or \( r = o(k/\varepsilon) \) or \( \text{rank}(U) < k/2 \). We show that if any of these three assumptions is valid we obtain a contradiction; hence, we conclude that to achieve a relative error bound \( c = \Omega(k/\varepsilon) \) and \( r = \Omega(k/\varepsilon) \) and \( \text{rank}(A) \geq k/2 \).

If \( c = o(k/\varepsilon) \), then the columns of the matrix \( \text{CUR} \) are in the column space of \( C \), and so this implies that the columns of \( C \) span a \((1 + \varepsilon)\) approximation to \( A \) (i.e., there is a matrix \( C \in \mathbb{R}^{t \times c} \) with \( c = o(k/\varepsilon) \) columns of \( A \) and \( X = UR \) such that, \( \|A - CX\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2 \) ), contradicting that no subset of \( o(k/\varepsilon) \) columns of \( A \) span a \((1 + \varepsilon)\) approximation (i.e., contradicting Lemma 8.2).

If \( r = o(k/\varepsilon) \), then the rows of the matrix \( \text{CUR} \) are in the row space of \( R \), and so this implies that the rows of \( R \) span a \((1 + \varepsilon)\) approximation to \( A \) (i.e., there is a matrix \( R \in \mathbb{R}^{r \times t} \) with \( r = o(k/\varepsilon) \) rows of \( A \) and \( Y = UC \) such that, \( \|A - YR\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2 \) ), contradicting that no subset of \( o(k/\varepsilon) \) rows of \( A \) span a \((1 + \varepsilon)\) approximation (i.e., contradicting Corollary 8.3).

To continue the proof we need the details for the specific construction of the adversarial matrix \( A \). Those details are given in the proof of Lemma 8.2 but we repeat them here for completeness. For \( \alpha > 0 \) and integer \( n > 1 \), consider the matrix

\[
D = [e_1 + \frac{\alpha}{\sqrt{k}}e_2, e_1 + \frac{\alpha}{\sqrt{k}}e_3, \ldots, e_1 + \frac{\alpha}{\sqrt{k}}e_n, 1] \in \mathbb{R}^{(n+1) \times n},
\]
where, for \( i = 1 : n + 1 \), \( e_i \in \mathbb{R}^{n+1} \) are the standard basis vectors. \( D \) looks like the following matrix,

\[
D = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\frac{\alpha}{\sqrt{k}} & 1 & 1 & \cdots & 1 \\
\frac{\alpha}{\sqrt{k}} & \frac{\alpha}{\sqrt{k}} & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\alpha}{\sqrt{k}} & \frac{\alpha}{\sqrt{k}} & \frac{\alpha}{\sqrt{k}} & \cdots & 1 \\
\end{pmatrix} \in \mathbb{R}^{(n+1) \times n}.
\]

Let \( B \in \mathbb{R}^{m \times \ell} \) with \( m = (n+1)k \) and \( \ell = nk \) be constructed by repeating \( D \) \( k \) times along its main diagonal,

\[
B = \begin{pmatrix} D & \cdots & D \\ \vdots & \ddots & \vdots \\ D & \cdots & D \end{pmatrix} \in \mathbb{R}^{k(n+1) \times kn}.
\]

Let \( A \in \mathbb{R}^{t \times t} \) with \( t = (2n+1)k \) be the following matrix,

\[
A = \begin{pmatrix} B & B^T \end{pmatrix} \in \mathbb{R}^{(2kn+k) \times (2kn+k)}.
\]

If \( \text{rank}(U) < k/2 \), then \( \text{rank}(\text{CUR}) < k/2 \), hence when we approximate \( A \) with CUR, there will be an error of at least \( (3k/2)\|D\|_F^2 \), because \( A \) contains \( 2k \) blocks of \( D \) and \( D^T \) along its main diagonal. So,

\[
\frac{\|A - \text{CUR}\|_F^2}{\|A - A_k\|_F^2} \geq \frac{3k}{\ell(1 + \frac{2\alpha^2}{k})} = \frac{3k(n + n\alpha^2)}{\ell(1 + \frac{2\alpha^2}{k})} = \frac{3}{2} \cdot \frac{1 + \frac{\alpha^2}{1 + \frac{2\alpha^2}{k}}}{1 + \frac{2\alpha^2}{k}} = \frac{3}{2} \left( 1 - \frac{\alpha^2}{1 + \frac{2\alpha^2}{k}} \right) = \frac{3}{2} \left( \frac{k + \alpha^2}{k + 2\alpha^2} \right).
\]

In the above, we used

\[
\|A - A_k\|_F^2 = \ell(1 + \frac{2\alpha^2}{k}),
\]

which we showed in the proof of Lemma 8.2. In the proof of Lemma 8.2 we also chose \( \alpha = 10^{-10} \).

So, for example,

\[
\left( \frac{k + \alpha^2}{k + 2\alpha^2} \right) \geq 8/9;
\]

hence, the bound becomes,

\[
\frac{\|A - \text{CUR}\|_F^2}{\|A - A_k\|_F^2} \geq 1 + \frac{1}{3},
\]

contradicting that there is a relative error bound with \( \varepsilon < 1/3 \). \( \blacksquare \)

Lemma 8.2. There is a symmetric matrix \( A \in \mathbb{R}^{t \times t} \) such that, for any \( k \) and \( \varepsilon > 0 \), no subset of \( c = o(k/\varepsilon) \) columns of \( A \) span a \((1 + \varepsilon)\) approximation to \( A \), i.e., there is no \( C \in \mathbb{R}^{t \times c} \) with \( c = o(k/\varepsilon) \) columns of \( A \) and \( X \in \mathbb{R}^{c \times t} \) such that

\[
\|A - CX\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2.
\]
Proof. For $\alpha > 0$ and integer $n > 1$, consider the matrix
\[
D = \begin{bmatrix} e_1 + \frac{\alpha}{\sqrt{k}} e_2, e_1 + \frac{\alpha}{\sqrt{k}} e_3, \ldots, e_1 + \frac{\alpha}{\sqrt{k}} e_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times n},
\]
where, for $i = 1 : n + 1$, $e_i \in \mathbb{R}^{n+1}$ are the standard basis vectors. $D$ looks like the following matrix,
\[
D = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\
\frac{\alpha}{\sqrt{k}} & \frac{\alpha}{\sqrt{k}} & \frac{\alpha}{\sqrt{k}} & \cdots & \frac{\alpha}{\sqrt{k}} \\
\vdots & & & & \ddots \\
\frac{\alpha}{\sqrt{k}} & & & & \frac{\alpha}{\sqrt{k}} \
\end{bmatrix} \in \mathbb{R}^{(n+1) \times n}.
\]
This matrix is popular in proving lower bounds for column-based low rank matrix factorizations [13, 6]. Let $B \in \mathbb{R}^{m \times \ell}$ with $m = (n + 1)k$ and $\ell = nk$ be constructed by repeating $D$ $k$ times along its main diagonal,
\[
B = \begin{bmatrix} D \\
\vdots \\
D \end{bmatrix} \in \mathbb{R}^{k(n+1) \times kn}.
\]
Let $A \in \mathbb{R}^{t \times t}$ with $t = (2n + 1)k$ be the following matrix,
\[
A = \begin{bmatrix} B & B^T \end{bmatrix} \in \mathbb{R}^{(2kn+k) \times (2kn+k)}.
\]
This matrix $A$ is the hard instance for the lower bound (for some $\alpha$ that will be specified later). Below, we are interested in estimating what is the best - smallest - error that can occur when approximating $A$ with, let’s say, $c$ columns from $A$. In particular, we will show that a relative error approximation is not possible unless $c = \Omega(\sqrt{k}/\varepsilon)$ columns of $A$ are selected.
Let $C \in \mathbb{R}^{t \times c}$ contain $c$ columns of $A$ and let it be choice of $c$ columns that result to the smallest possible error $\|A - CC^\dagger A\|_F^2$. We prove a lower bound $\gamma$ of this form
\[
\|A - CC^\dagger A\|_F^2 \geq \gamma.
\]
Bound for $\|A - A_k\|_F^2$. First, for the matrix $D$ described above (these observations are also made in [13, 6]):
\[
D^T D = 1_n 1_n^T + \alpha^2/k I_n, \quad \sigma_1^2(D) = n + \alpha^2/k, \quad \text{and} \quad \sigma_i^2(D) = \alpha^2/k \quad \text{for} \quad i > 1.
\]
Since $B$ is block diagonal containing $k$ blocks of $D$ we obtain:
\[
\sigma_i^2(B) = n + \alpha^2/k, \quad \text{for} \quad i \leq k;
\]
\[
\sigma_i^2(B) = \alpha^2/k, \quad \text{for} \quad i > k.
\]
Hence, for any $k \geq 1$ we obtain
\[
\|A - A_k\|_F^2 = k(n + \alpha^2/k) + (2\alpha k - k)(\alpha^2/k) = \ell + \alpha^2 + 2\alpha + \alpha^2 = \ell + 2\alpha^2/k.
\]

**Bound for $\|A - CC^TA\|_F^2$.** This “optimum” matrix $C \in \mathbb{R}^{t \times c}$ is a matrix of the following form,
\[
C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},
\]
where $C_1 \in \mathbb{R}^{m \times c_1}$ contains $c_1$ columns from $B$ and $C_2 \in \mathbb{R}^{\ell \times c_2}$ contains $c_2$ columns from $B^T$, with $c = c_1 + c_2$. Also,
\[
\|A - CC^TA\|_F^2 = \| \begin{pmatrix} B \\ B^T \end{pmatrix} - \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} C_1^TB \\ C_2^TB^T \end{pmatrix} \|_F^2
\]
\[
= \|B - C_1C_1^TB\|_F^2 + \|B^T - C_2C_2^TB^T\|_F^2
\]
The last Equation in Section 9 in [6] shows precisely the following lower bound,
\[
\|B - C_1C_1^TB\|_F^2 \geq \frac{\alpha^2}{k}(\ell - c_1) \left(1 + \frac{k}{c_1 + \alpha^2}\right).
\]

Now, let $C_2 \in \mathbb{R}^{\ell \times c_2}$ contain any $c_2$ columns from $B^T$. We now compute an upper bound for $\|B^T - C_2C_2^TB^T\|_F^2$. $C_2$ contains $c_2$ columns from $B^T$, equivalently $C_2^T$ contains $c_2$ rows from $B$. Recall that $B$ contains $k$ copies of $D \in \mathbb{R}^{n \times n}$ along its main diagonal. Let us denote those copies $D_i$, for $i = 1 : k$. Let $c_2 = r_1 + r_2 + \cdots + r_k$, where $r_i$ is the number of rows selected from each $D_i$. Let also $R_i \in \mathbb{R}^{r_i \times n}$ contain these rows from the corresponding $D_i$. Then,
\[
\|B^T - C_2C_2^TB^T\|_F^2 = \sum_{i=1}^k \|D_i - D_iR_i^TR_i\|_F^2.
\]

We further manipulate this term as follows,
\[
\sum_{i=1}^k \|D_i - D_iR_i^TR_i\|_F^2 \geq \sum_{i=1}^k \frac{\alpha^2}{k}(n - r_i) = \frac{\alpha^2}{k} \sum_{i=1}^k (n - r_i) = \frac{\alpha^2}{k}(kn - c_2) = \frac{\alpha^2}{k}(\ell - c_2).
\]

We are now ready to prove the lower bound,
\[
\frac{\|A - CC^TA\|_F^2}{\|A - A_k\|_F^2} \geq \frac{\alpha^2}{k}(\ell - c_1) \left(1 + \frac{k}{c_1 + \alpha^2}\right) + \frac{\alpha^2}{k}(\ell - c_2) = \frac{\ell - c_1}{2\ell} \left(1 + \frac{k}{c_1 + \alpha^2}\right) + \frac{\ell - c_2}{2\ell}.
\]
By using \( \ell = nk \geq c_1/\varepsilon \), \( \ell = nk \geq c_2/\varepsilon \) and \( n = \omega(1/\varepsilon^2) \) we further manipulate the bound as follows,

\[
\frac{\|A - CC^\dagger A\|_F^2}{\|A - A_k\|_F^2} \geq \frac{\ell - c_1}{2\ell} \left(1 + \frac{k}{c_1 + \alpha^2}\right) + \frac{\ell - c_2}{2\ell} \geq 
\]

\[
\frac{1}{2} \left(1 - o(\varepsilon)\right) \left(1 + \frac{k}{c_1 + \alpha^2}\right) + \frac{1}{2} \left(1 - o(\varepsilon)\right) = 1 + \frac{k}{c_1 + \alpha^2} - o(\varepsilon)
\]

Using, \( c_1 < c \) we obtain,

\[
\frac{\|A - CC^\dagger A\|_F^2}{\|A - A_k\|_F^2} \geq 1 + \frac{k}{c + \alpha^2} - o(\varepsilon)
\]

The number of columns \( c \) satisfies: \( c > 1 \). Now, choose \( \alpha \) to be a sufficiently small positive constant, e.g. \( \alpha = 10^{-10} \). For this choice of \( \alpha \):

\[
\frac{k}{c + \alpha^2} \geq \frac{k}{2c}.
\]

Hence,

\[
\frac{\|A - CC^\dagger A\|_F^2}{\|A - A_k\|_F^2} \geq 1 + \frac{k}{c + \alpha^2} - o(\varepsilon) \geq 1 + \frac{k}{2c} - o(\varepsilon)
\]

So, to obtain a relative-error bound, we need at least \( c = \Omega(k/\varepsilon) \) columns. \( \square \)

The following result is an immediate corollary to Lemma 8.2.

**Corollary 8.3.** There is a symmetric matrix \( A \in \mathbb{R}^{t \times t} \) such that, for any \( k \) and \( \varepsilon > 0 \), no subset of \( o(k/\varepsilon) \) rows of \( A \) span a \((1 + \varepsilon)\) approximation to \( A \), i.e., there is no \( R \in \mathbb{R}^{c \times t} \) with \( r = o(k/\varepsilon) \) rows of \( A \) and \( Y \in \mathbb{R}^{t \times r} \) such that

\[
\|A - YR\|_F^2 \leq (1 + \varepsilon)\|A - A_k\|_F^2.
\]

**Remark.** To prove Lemma 8.2, we extend slightly the lower bound in [6], which is stated for a non-symmetric matrix. Here, we aim for a lower bound for a symmetric matrix. We should mention that a similar lower bound for a symmetric matrix appeared in Lemma 6.2 in [31]. We chose not to use this result because it was not clear to us how to address the lower bound for the rank of the intersection matrix \( U \) in Theorem 8.1.

**Acknowledgement.** David Woodruff would like to thank the Simons Institute at Berkeley where part of this work was done. He would also like to acknowledge the XDATA program of the Defense Advanced Research Projects Agency (DARPA), administered through Air Force Research Laboratory contract FA8750-12-C0323, for supporting part of this work.

**REFERENCES**

[1] List of open problems in sublinear algorithms: Problem 12: Deterministic cur-type decompositions. http://sublinear.info/12.

[2] D. Achlioptas. Database-friendly random projections: Johnson-lindenstrauss with binary coins. *J. Comput. Syst. Sci.*, 66(4):671–687, 2003.
[3] J. Batson, D. Spielman, and N. Srivastava. Twice-ramanujan sparsifiers. In Proceedings of the 41st annual ACM symposium on Theory of computing, pages 255–262. ACM, 2009.
[4] M. W. Berry, S. A. Pulatova, and G. Stewart. Algorithm 844: Computing sparse reduced-rank approximations to sparse matrices. ACM Transactions on Mathematical Software (TOMS), 31(2):252–269, 2005.
[5] J. Bien, Y. Xu, and M. W. Mahoney. Cur from a sparse optimization viewpoint. arXiv preprint arXiv:1011.0413, 2010.
[6] C. Boutsidis, P. Drineas, and M. Magdon-Ismail. Near optimal column based matrix reconstruction. SIAM Journal on Computing (SICOMP), 2013.
[7] C. Boutsidis and A. Gittens. Improved matrix algorithms via the subsampled randomized hadamard transform. SIAM Journal on Matrix Analysis and Applications (SIMAX), 2013.
[8] C. Boutsidis, M. W. Mahoney, and P. Drineas. An improved approximation algorithm for the column subset selection problem. In Proceedings of the Nineteenth Annual ACM -SIAM Symposium on Discrete Algorithms (SODA), pages 968–977, 2009.
[9] L. Carter and M. N. Wegman. Universal classes of hash functions. J. Comput. Syst. Sci., 18(2):143–154, 1979.
[10] K. Clarkson and D. Woodruff. Numerical linear algebra in the streaming model. In Proceedings of the 41st annual ACM symposium on Theory of computing (STOC), 2009.
[11] K. L. Clarkson and D. P. Woodruff. Low rank approximation and regression in input sparsity time. In In STOC, 2013.
[12] K. L. Clarkson and D. P. Woodruff. Low rank approximation and regression in input sparsity time. ArXiV report: http://arxiv.org/pdf/1207.6365v4.pdf, 2013.
[13] A. Deshpande and L. Rademacher. Efficient volume sampling for row/column subset selection. In Proceedings of the 42th Annual ACM Symposium on Theory of Computing (STOC), 2010.
[14] A. Deshpande, L. Rademacher, S. Vempala, and G. Wang. Matrix approximation and projective clustering via volume sampling. In Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1117–1126, 2006.
[15] A. Deshpande and S. Vempala. Adaptive sampling and fast low-rank matrix approximation. In RANDOM - APPROX, 2006.
[16] P. Drineas and R. Kannan. Fast Monte-Carlo algorithms for approximate matrix multiplication. In Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science, pages 452–459, 2001.
[17] P. Drineas and R. Kannan. Pass efficient algorithms for approximating large matrices. In Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 223–232, 2003.
[18] P. Drineas, R. Kannan, and M. Mahoney. Fast Monte Carlo algorithms for matrices I: Approximating matrix multiplication. SIAM Journal of Computing, 36(1):132–157, 2006.
[19] P. Drineas, R. Kannan, and M. Mahoney. Fast Monte Carlo algorithms for matrices II: Computing a low-rank approximation to a matrix. SIAM Journal of Computing, 36(1):158–183, 2006.
[20] P. Drineas, R. Kannan, and M. Mahoney. Fast Monte Carlo algorithms for matrices III: Computing a compressed approximate matrix decomposition. SIAM Journal of Computing, 36(1):184–206, 2006.
[21] P. Drineas, M. Mahoney, and S. Muthukrishnan. Polynomial time algorithm for column-row based relative-error low-rank matrix approximation. Technical Report 2006-04, DIMACS, March 2006.
[22] P. Drineas and M. W. Mahoney. On the nyström method for approximating a gram matrix for improved kernel-based learning. The Journal of Machine Learning Research, 6:2153–2175, 2005.
[23] P. Drineas, M. W. Mahoney, and S. Muthukrishnan. Relative-error cur matrix decompositions. SIAM Journal Matrix Analysis and Applications, 30(2):844–881, 2008.
[24] S. Friedland and A. Torokhti. Generalized rank-constrained matrix approximations. SIAM Journal on Matrix Analysis and Applications, 29(2):656–659, 2007.
[25] A. Frieze, R. Kannan, and S. Vempala. Fast Monte-Carlo algorithms for finding low-rank approximations. In Proceedings of the 39th Annual IEEE Symposium on Foundations of Computer Science, pages 370–378, 1998.
[26] M. Ghashami and J. Phillips. Relative errors for deterministic low-rank matrix approximations. In SODA, 2013.
[27] A. Gittens and M. W. Mahoney. Revisiting the nyström method for improved large-scale machine learning. arXiv preprint arXiv:1303.1849, 2013.
[28] S. Goreinov, E. Tyrtyshnikov, and N. Zamarashkin. A theory of pseudoskeleton approximations.
[29] S. Goreinov, N. Zamarashkin, and E. Tyrtyshnikov. Pseudo-skeleton approximations by matrices of maximal volume. *Mathematical Notes*, 62(4):515–519, 1997.

[30] M. Gu and L. Miranian. Strong rank revealing Cholesky factorization. *Electronic Transactions on Numerical Analysis*, 17:76–92, 2004.

[31] V. Guruswami and A. K. Sinop. Optimal column-based low-rank matrix reconstruction. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1207–1214. SIAM, 2012.

[32] N. Halko, P. Martinsson, and J. Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Review*, 53(2):217–288, 2011.

[33] T. Hwang, W. Lin, and D. Pierce. Improved bound for rank revealing LU factorizations. *Linear algebra and its applications*, 261(1-3):173–186, 1997.

[34] R. Kannan, S. Vempala, and D. P. Woodruff. Nimble algorithms for cloud computing. *arXiv preprint arXiv:1304.3162*, v3, 2013.

[35] M. Magdon-Ismail. Row Sampling for Matrix Algorithms via a Non-Commutative Bernstein Bound. *Arxiv preprint arXiv:1008.0587*, 2010.

[36] M. W. Mahoney and P. Drineas. CUR matrix decompositions for improved data analysis. *Proceedings of the National Academy of Sciences*, 106(3):697–702, 2009.

[37] A. McGregor. Open problems in data streams and related topics. In *ITK Workshop on Algorithms For Data Streams*, 2006.

[38] X. Meng and M. W. Mahoney. Low-distortion subspace embeddings in input-sparsity time and applications to robust linear regression. In *In STOC*, pages 91–100. ACM, 2013.

[39] L. Miranian and M. Gu. Strong rank revealing LU factorizations. *Linear algebra and its applications*, 367:1–16, 2003.

[40] J. Nelson and H. L. Nguyen. Osnap: Faster numerical linear algebra algorithms via sparser subspace embeddings. In *In FOCS*, 2013.

[41] C. Pan. On the existence and computation of rank-revealing LU factorizations. *Linear Algebra and its Applications*, 316(1-3):199–222, 2000.

[42] V. Rokhlin and M. Tygert. A fast randomized algorithm for overdetermined linear least-squares regression. *Proceedings of the National Academy of Sciences*, 105(36):13212, 2008.

[43] M. Rudelson and R. Vershynin. Sampling from large matrices: An approach through geometric functional analysis. *JACM: Journal of the ACM*, 54, 2007.

[44] T. Sarlos. Improved approximation algorithms for large matrices via random projections. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, 2006.

[45] K. C. Sou and A. Ranzer. On generalized matrix approximation problem in the spectral norm. *Linear Algebra and its Applications*, 436(7):2331–2341, 2012.

[46] G. Stewart. Four algorithms for the efficient computation of truncated QR approximations to a sparse matrix. *Numerische Mathematik*, 83:313–323, 1999.

[47] E. Tyrtyshnikov. Mosaic-skeleton approximations. *Calcolo*, 33(1):47–57, 1996.

[48] E. Tyrtyshnikov. Incomplete cross approximation in the mosaic-skeleton method. *Computing*, 64(4):367–380, 2000.

[49] S. Wang and Z. Zhang. Improving CUR matrix decomposition and the Nyström approximation via adaptive sampling. *Journal of Machine Learning Research*, 2013.

[50] A. Zouzias and N. Freris. Randomized extended kaczmarz for solving least-squares. *SIAM Journal on Matrix Analysis and its Applications*, 2013.