CHANGING GEARS:
ISOSPECTRALITY VIA EIGENDERIVATIVE TRANSPLANTATION

PETER DOYLE AND PETER HERBRICH

Abstract. We introduce a new method for constructing isospectral quantum graphs that is based on transplanting derivatives of eigenfunctions. We also present simple digraphs with the same reversing zeta function, which generalizes the Bartholdi zeta function to digraphs.

1. Introduction

Quantum graphs are singular one-dimensional manifolds equipped with self-adjoint operators, whose spectra allow for explicit computations. Following [BE09], let $G = (V,E,l)$ be a finite metric graph with vertices $V = \{v_1,v_2,\ldots,v_n\}$, edges $E = \{e_1,e_2,\ldots,e_m\}$, and edge lengths $l = (l_1,l_2,\ldots,l_m) \in \mathbb{R}_+^m$. In particular, $e_i$ is parameterized by $x_i \in [0,l_i]$, which determines an orientation of the edges. However, the differential operator of interest acts as $\Delta_i = -\frac{d^2}{dx_i^2}$ on $C_0^\infty(0,l_i)$ and is thus invariant under transformations of the form $x_i' = l_i - x_i$, which allows to regard edges as undirected. The self-adjoint extensions of the corresponding symmetric operator with initial domain $\oplus_{i=1}^m H^2_0(0,l_i)$ can be parameterized in terms of boundary conditions at the vertices [KS99a], which turn the metric graph into a quantum graph. Unless otherwise stated, we consider Kirchhoff-Neumann conditions, which require that for each vertex $v_i$, functions on adjacent edges take the same value at $v_i$, and the sum of their outgoing derivatives at $v_i$ vanishes. In particular, leaf vertices carry Neumann boundary conditions. It is well-known that finite quantum graphs have discrete spectrum [BK13]. Quantum graphs are called isospectral if their spectra coincide, including multiplicities.

Recently, it was discovered [OB12] that the graphs in Figure 1A have the same set of eigenvalues when viewed as either edge-weighted combinatorial graphs or quantum graphs with edge weights or lengths $(a,a,b,b,c,c)$, respectively. These graphs first appeared in [MM03] where a certain line graph construction was applied to the $7_1$ pair of Dirichlet isospectral planar domains in [BCDS94]. The second author [Her15] has generalized this construction to manifolds with mixed Dirichlet-Neumann boundary conditions and revealed its connection to the graph-theoretic characterization of the famous Sunada method [Sun85] given in [Her11].

It is worth mentioning that the main arguments in [OB12] are based on the widespread misconception that eigenfunctions on quantum graphs with Kirchhoff-Neumann conditions are determined by their values at the vertices [KS99b, SS06, BS07]. For example, if $a$, $b$, and $c$ are integer multiples of some $r > 0$, then each of the graphs in Figures 1A and 1B has countably many eigenfunctions which are supported on its central cycle of length $a+b+c$ and vanish at all vertices. Instead, [OB12] asserts that if $f_i \colon [0,l_i] \to \mathbb{R}$ denotes the restriction
of an eigenfunction to the edge $e_i$ with eigenvalue $\lambda \geq 0$, then

$$f_i(x_i) = \frac{1}{\sin(\sqrt{\lambda}l_i)} \left(f_i(0) \sin(\sqrt{\lambda}(l_i - x_i)) + f_i(l_i) \sin(\sqrt{\lambda}x_i)\right).$$

Similarly, [KS99b] suggests to ignore edges $e_i$ satisfying $\sqrt{\lambda}l_i = k\pi$ for some $k \in \mathbb{Z}_+$ when determining whether $\lambda$ is an eigenvalue, while such edges contribute the boundary conditions

$$f_i(l_i) = (-1)^k f_i(0) \quad \text{and} \quad f_i'(l_i) = (-1)^k f_i'(0).$$

In Section 2, we introduce a method for constructing isospectral quantum graphs that avoids explicit computations and thus bypasses the above-mentioned shortcomings. It is a derivative of Buser’s transplantation method [Bus86], which itself can be viewed as the combinatorial incarnation of the Sunada method [Sun85]. In contrast to the latter, our method can produce pairs of quantum graphs without common covers. In Section 3 we relate our method to random walks on combinatorial graphs. In addition, we present related pairs of non-regular simple digraphs that have the same reversing zeta function as introduced in [Her14]. Thus, these digraphs exhibit a noteworthy degree of spectral indistinguishability.

2. Gear graphs and eigenderivative transplantation

**Definition.** An $n$-gear is a quantum graph with $2n$ vertices that is comprised of a polygon with $n$ sides of lengths $(l_1, l_2, \ldots, l_n) \in \mathbb{R}_+^n$ as well as $n$ leaf edges of lengths $(l_1, l_2, \ldots, l_n)$, called teeth, such that the $i$th tooth is adjacent to the $i$th polygon side. The dual $n$-gear is obtained by attaching each tooth at the other vertex of its corresponding polygon side.

Figures 1A and 1B show pairs of mutually dual 3-gears with lengths $(l_1, l_2, l_3) = (a, b, c)$. We note that an $n$-gear has $n$ leaf vertices and $n$ polygon vertices, the latter of which have degrees 2, 3, or 4. Let $G$ be an $n$-gear. We parameterize the edges of $G$ such that its polygon is an oriented cycle with respect to the induced orientation, and corresponding polygon sides and teeth are head-to-head or tail-to-tail. Figure 1C indicates the 4 possible neighborhoods of a polygon vertex. In order to produce pairs as in Figure 2, we introduce an auxiliary
weight \( w > 0 \) and consider the vertex conditions described in Figure [I] where oriented edges and functions on them are denoted by the same symbol. The corresponding operator is self-adjoint with respect to the measure obtained by weighting the Lebesgue measures on the teeth by \( w \). Namely, if \( f \) has restrictions \((p_i)_{i=1}^n\) and \((t_i)_{i=1}^n\), then we consider

\[
\|f\|_w^2 = \langle f, f \rangle_w = \sum_{i=1}^n \int_0^{t_i} p_i^2(x) + w t_i^2(x) dx.
\]

If \( \varphi \) has restrictions \((\pi_i)_{i=1}^n\) and \((\tau_i)_{i=1}^n\), then integration by parts gives

\[
\langle \Delta f, \varphi \rangle_w = -\sum_{i=1}^n \int_0^{t_i} p_i''(x) \pi_i(x) + w t_i''(x) \tau_i(x) dx
\]

\[
= \sum_{i=1}^n \int_0^{t_i} p_i'(x) \pi_i(x) + w t_i'(x) \tau_i(x) dx - \sum_{i=1}^n p_i'(x) \pi_i(x) + w t_i'(x) \tau_i(x) \biggr|_0^{t_i}.
\]

If \( f \) and \( \varphi \) obey the vertex conditions described in Figure [I], then the latter sum vanishes, which can be seen by collecting terms vertex-by-vertex and considering the 4 cases shown in Figure [IC] separately. In particular, \( \langle \Delta f, \varphi \rangle_w = \langle f, \Delta \varphi \rangle_w \), and there exists an \( \langle \cdot, \cdot \rangle_w \)-orthonormal basis of eigenfunctions with eigenvalue sequence \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \).

We proceed with the core argument, which shows that this sequence is contained in the spectrum of the corresponding operator on the dual \( n \)-gear of \( G \). If the neighborhood of a polygon vertex looks like one in the upper half of Figure [IC] with edges \( p_j \) and \( p_k \), then the neighborhood of the corresponding polygon vertex of the dual \( n \)-gear looks like the respective one in the lower half, with edges \( \tilde{p}_j \) and \( \tilde{p}_k \), and vice versa. This will allow us to introduce a linear function, called transplantation, between the spans of non-constant eigenfunctions on \( G \) and those on its dual, which is locally given by

\[
(3) \quad \tilde{p}_i = p_i' + w t_i' \quad \text{and} \quad \tilde{t}_i = p_i' - t_i', \quad \text{or vice versa.}
\]

Let \( f \) be an eigenfunction on \( G \) with eigenvalue \( \lambda > 0 \). Assume that \( G \) has a polygon vertex of degree 3 as in the upper right corner of Figure [IC]. Let \( p_j \), \( t_j \), \( p_k \), and \( t_k \) denote the corresponding restrictions of \( f \) so that \( p_j'' = \lambda p_j \), \( t_j'' = \lambda t_j \), \( p_k'' = \lambda p_k \), \( t_k'' = \lambda t_k \), as well as

\[
t_j'(0) = 0, \quad t_k'(0) = 0, \quad p_j(l_j) = t_j(l_j) = p_k(0), \quad \text{and} \quad p_j'(l_j) + w t_j'(l_j) = p_k'(0).
\]

We show that [3] gives rise to a function \( \tilde{f} \) that is well-defined at the vertex \( \tilde{p}_j(l_j) \), obeys the desired vertex conditions at \( \tilde{p}_j(l_j) \) and \( \tilde{t}_j(l_j) \), and is an eigenfunction on \( \tilde{p}_j \) and \( \tilde{t}_j \). Namely,

\[
(4) \quad \tilde{p}_j'' = p_j'' + w t_j'' = \lambda p_j' + w \lambda t_j' = \lambda \tilde{p}_j.
\]

Likewise, \( \tilde{t}_j'' = \lambda \tilde{t}_j \), \( \tilde{p}_k'' = \lambda \tilde{p}_k \), and \( \tilde{t}_k'' = \lambda \tilde{t}_k \). More interestingly,

\[
\tilde{p}_j(l_j) = p_j(l_j) - t_j'(l_j) = \lambda(p_j(l_j) - t_j(l_j)) = 0, \\
\tilde{p}_j(l_j) = p_j'(l_j) + w t_j'(l_j) = p_k'(0) = \tilde{p}_k(0) = \tilde{t}_k(0), \\
\tilde{p}_j'(l_j) = \lambda(p_j(l_j) + w t_j(l_j)) = \lambda(1 + w)p_k(0) \\
= \lambda(p_k(0) + w t_k(0)) + w \lambda(p_k(0) - t_k(0)) = \tilde{p}_k(0) + w \tilde{t}_k(0).
\]

Similar arguments apply to the remaining 3 cases in Figure [IC]. In particular, [3] gives rise to a globally well-defined eigenfunction \( \tilde{f} \) on the dual \( n \)-gear of \( G \). For the sake of simplicity, we henceforth assume that \( f \) has restrictions \((p_i)_{i=1}^n\) and \((t_i)_{i=1}^n\), and that \( \tilde{f} \) has restrictions \((\tilde{p}_i)_{i=1}^n\) and \((\tilde{t}_i)_{i=1}^n\). The general case just differs by a redistribution of tildes.
It remains to show that for any eigenvalue \( \lambda > 0 \), the transplantation given by (3) restricts to an injective function between the \( \lambda \)-eigenspaces. To this end, we write (3) in matrix form
\[
\begin{pmatrix}
\tilde{p}_i \\
\tilde{t}_i
\end{pmatrix} = \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p'_i \\
 t'_i
\end{pmatrix}, \quad \text{which leads to} \quad \begin{pmatrix} p_i \\
 t_i
\end{pmatrix} = \frac{1}{\lambda (1 + w)} \begin{pmatrix} 1 & w \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{p}_i \\
 \tilde{t}_i
\end{pmatrix}.
\]
Since every \( n \)-gear has \( \lambda = 0 \) as a simple eigenvalue, we have shown the following.

**Theorem 1.** Mutually dual \( n \)-gears are isospectral for every weight \( w > 0 \).

It is worth mentioning that normalizing (3) by the factor \((\lambda (1 + w))^{-1/2}\) leads to a linear isometry between the \( \lambda \)-eigenspaces. More precisely, we have
\[
\tilde{p}_i^2 + \tilde{t}_i^2 = (p'_i + w t'_i)^2 + w(p'_i - t'_i)^2 = (1 + w)(p_i^2 + w t_i^2).
\]
In particular, integration by parts gives
\[
\frac{1}{1 + w}\|\tilde{f}\|_w^2 = \sum_{i=1}^n \int_{l_i} p'_i(x)p'_i(x) + w t'_i(x)t'_i(x)dx
\]
\[
= \sum_{i=1}^n \lambda p_i^2(x) + w \lambda t_i^2(x)dx + \sum_{i=1}^n p'_i(x)p_i(x) + w t'_i(x)t_i(x)\bigg|_{0}^{l_i} = \lambda \|f\|_w^2.
\]
As above, the latter sum vanishes since \( f \) obeys the desired vertex conditions. Yet, the definition (3) has the advantage over its normalized version as it is the same on all eigenspaces.

Finally, we explain how mutually dual \( n \)-gears give rise to pairs as in Figure 2. Roughly speaking, each eigenspace decomposes orthogonally under the action of the respective graph’s isometry group, and the subspace of invariant elements and its orthogonal complement are transplanted separately. We exemplify the method with the help of the graphs in Figure 2(A). Each of them features a \( \mathbb{Z}_2 \)-action, given by swapping parallel edges that make up a side of its central 3-gon, and a \( \mathbb{Z}_3 \)-action, which moves all but these edges. Thus, each eigenspace is a unitary representation of \( \mathbb{Z}_2 \times \mathbb{Z}_3 \). Eigenfunctions on which \( \mathbb{Z}_3 \) acts by multiplication by \( e^{2\pi i/3} \) or its square vanish on the central 3-gon. Also, their derivatives on the 3 parallel edges of length \( b \) sum to zero at the common polygon vertex, equally for length \( c \). For such eigenfunctions, the graphs in Figure 2(A) essentially reduce to the same 3 subgraphs each of which has 2 vertices and 3 edges, and we can transplant trivially from subgraphs to subgraphs. Similarly, eigenfunctions that are odd with respect to the \( \mathbb{Z}_2 \)-action are supported on the central 3-gon, and we can transplant trivially from 3-gon to 3-gon. Hence, it suffices to consider the spaces of \( \mathbb{Z}_2 \times \mathbb{Z}_3 \)-invariant \( \lambda \)-eigenfunctions on the graphs. However, these spaces are linearly isometric to the \( \lambda \)-eigenspaces of the 3-gears in Figure 2(B) with \( w = 3/2 \), where the isometry is given by merging the parallel edges of the graphs in Figure 2A.
3. Combinatorial eigenderivative transplantation

Long before quantum graphs were introduced as model systems in quantum chaos [KS97, KS99b], they had been studied under different names in chemistry and biology, see [Kuc02] and references therein. Notably, [vB85] reduces the spectral analysis of quantum graphs with Kirchhoff-Neumann conditions and commensurable edge lengths to that of combinatorial graphs with associated row-stochastic matrices. In contrast to [KS99b, SS06, BS07, OB12], [vB85] rigorously treats the so-called Dirichlet eigenvalues [BK13], for which (1) fails.

In the context of n-gears, this leads to a combinatorial eigenderivative transplantation. We consider pairs of mutually dual n-gears with commensurable edge lengths \((l_1, l_2, \ldots, l_m)\). Since scaling all edges by some factor leads to eigenvalues scaled by that same factor, we may assume that \((l_1, l_2, \ldots, l_m) \in \mathbb{Z}_+^m\). As is well-known [BK13], adding or removing a vertex of degree 2 carrying Kirchhoff-Neumann conditions leaves the set of eigenfunctions, and therefore the spectrum, unchanged. We thus replace each edge of length \(l_i\) by a path consisting of \(l_i\) edges of length 1, that is, we subdivide all edges into edges of unit length.

In the style of [vB85], we first consider Dirichlet eigenvalues \(\lambda = (k\pi)^2\) with \(k \in \mathbb{Z}\). By virtue of (2), the Neumann conditions at the leaf vertices propagate towards the polygons so that any \(\lambda\)-eigenfunction has vanishing first derivatives not only at the leaf vertices but also at the polygon vertices in the direction of the respective leaf vertex. In particular, any \(\lambda\)-eigenfunction is uniquely determined by its restriction to the polygon, which is, in fact, a \(\lambda\)-eigenfunction of the circle with circumference \(l_1 + l_2 + \ldots + l_m\). On the other hand, any \(\lambda\)-eigenfunction of this circle gives rise to unique \(\lambda\)-eigenfunctions on the n-gears.

We turn to eigenvalues \(\lambda\) for which (1) holds. In particular, any \(\lambda\)-eigenfunction \(f\) on one of the subdivided n-gears is determined by its values at the vertices. We differentiate (1) and set \(l_i = 1\) to obtain

\[
\sin(\sqrt{\lambda}) f_i'(0) = \sqrt{\lambda} \left(-\cos(\sqrt{\lambda}) f_i(0) + f_i(1)\right).
\]

If \(v\) is a vertex with neighbors \(P\) on the polygon and neighbors \(T\) on teeth, then the vertex condition at \(v\) described in Figure \(\ref{fig:hip}\) is given by

\[
\cos(\sqrt{\lambda}) f(v) = \frac{1}{|P| + w |T|} \left( \sum_{v' \in P} f(v') + \sum_{v' \in T} w f(v')\right).
\]

Thus, the values of \(f\) at the vertices give rise to a \(\cos(\sqrt{\lambda})\)-eigenvector of the row-stochastic matrix \(M\) that corresponds to the random walk on the vertices where edges belonging to teeth are taken \(w\) times as likely as edges belonging to the polygon. In fact, the \(\lambda\)-eigenspace is isomorphic to the \(\cos(\sqrt{\lambda})\)-eigenspace of \(M\). For the sake of brevity, we call \(M\) the Markov matrix of the n-gear, and denote the right-hand side of (6) by \(M[f](v)\). We note that \(M\) is irreducible and has period 2 or 1, depending on whether the subdivided n-gear is bipartite or not. Moreover, \(M = D_1^{-1} AD_2\) where \(D_1\) and \(D_2\) are invertible diagonal matrices, and \(A\) is the adjacency matrix of this graph. In particular, \(M\) is similar to a symmetric matrix and therefore has spectrum in \([-1, 1]\), where 1 is the simple Perron-Frobenius eigenvalue, and \(-1\) is an eigenvalue precisely if \(M\) has period 2, in which case it is also simple. In the following, we give an alternative proof of Theorem \(\ref{thm:isospectral}\). Since the eigenvalues of a quantum graph depend continuously on its edge lengths [BK13 Theorem 3.1.2], it suffices to show the following.

**Theorem 2.** The Markov matrices of mutually dual n-gears with integral edge lengths are isospectral for every weight \(w > 0\).
Let \( f \) be a function on the vertices of a subdivided \( n \)-gear. In view of (3), we define the outward and inward derivatives of \( f \) at the vertex \( v \) along the edge shared with vertex \( v' \) as

\[
\begin{align*}
f'_{[v,v']}(v) &= -M[f](v) + f(v') \quad \text{and} \quad \tilde{f}'_{[v,v']}(v) = -f'_{[v,v']}(v) = M[f](v) - f(v').
\end{align*}
\]

This definition makes any function satisfy the combinatorial version of the desired vertex conditions. Namely, if \( v \) has neighbors \( P \) and \( T \) on the polygon and teeth, respectively, then

\[
(7) \quad \sum_{v' \in P} f'_{[v,v']}(v) + \sum_{v' \in T} w f'_{[v,v']}(v) = -(|P| + |T|)M[f](v) + \left( \sum_{v' \in P} f(v') + \sum_{v' \in T} w f(v') \right) = 0.
\]

In particular, if \( v \) has the sole neighbor \( v' \), then \( f'_{[v,v']}(v) = 0 \), and if \( v \) has degree 2 and neighbors \( v' \) and \( v'' \), then \( f'_{[v,v']}(v) = f'_{[v,v'']}(v) \). For the sake of simplicity, we assume that the underlying \( n \)-gear has oriented edges \( (p_i)_{i=1}^n \) and \( (t_i)_{i=1}^n \) as in Figure 1C, the general case is obtained by redistributing tildes. Similarly to Section 2, we denote the restrictions of \( f \) to corresponding previously-introduced paths by the same symbol, where each of the original \( n \) polygon vertices appears in as many paths as its degree. We orient these paths as their underlying edges, and define the derivative along \( p_i = [v_0, v_1, \ldots, v_k] \) as

\[
p'_i(v_j) = \begin{cases} f'_{[v_j,v_{j+1}]}(v_j) & \text{if } j < l_i, \\
f'_{[v_{j-1},v_j]}(v_j) & \text{if } j > 0,
\end{cases}
\]

similarly for the restrictions \( (t_i)_{i=1}^n \). This allows to transplant an arbitrary function \( f \) via (3) to obtain a function \( \tilde{f} \) with restrictions \( (\tilde{p}_i)_{i=1}^n \) and \( (\tilde{t}_i)_{i=1}^n \) on the subdivided dual \( n \)-gear, which is well-defined by reason of (7) and virtually the same argument that showed continuity in the quantum graph setting. In order to show that this transplantation maps eigenfunctions of \( M \) to eigenfunctions of its counterpart \( \tilde{M} \), assume that \( M[f] = \mu f \) for some \( \mu \in [-1, 1] \). If \([v', v, v'']\) is part of one of the paths \((p_i)_{i=1}^n\) or \((t_i)_{i=1}^n\), say \( p_i \), then

\[
p'_i(v') + \tilde{p}'_i(v'') = -M[f](v') + f(v) - f(v) + M[f](v'') = \mu(-f(v') + M[f](v) - M[f](v) + f(v'')) = 2\mu p'_i(v).
\]

This entails the combinatorial version of (4), namely, \( \tilde{M}[^\tilde{f}](\tilde{v}) = \mu \tilde{f}(\tilde{v}) \) at all vertices \( \tilde{v} \) that were introduced with the subdivision. We therefore turn to vertices of the underlying \( n \)-gear. Let \( \tilde{v} \) be the leaf vertex with neighbor \( \tilde{v}_t \) on the path \( \tilde{t}_j \) in the lower right corner of Figure 1C, and \( v \) be the polygon vertex with neighbors \( v_p \) and \( v_t \) on \( p_j \) and \( t_j \) in the upper one. Then

\[
\tilde{M}[^\tilde{f}](\tilde{v}) = \tilde{t}_j(\tilde{v}_t) = p'_j(v_p) - t'_j(v_t) = -M[f](v_p) + f(v) - f(v) + M[f](v_t) = \mu(-f(v_p) + M[f](v) - M[f](v) + f(v_t)) = \mu(p'_j(v) - t'_j(v)) = \mu \tilde{f}(\tilde{v}).
\]

Similarly, let \( \tilde{v} \) and \( v \) be the polygon vertices with neighbors \( \tilde{v}_{t_k}, \tilde{v}_{p_k}, \tilde{v}_{p_j}, \tilde{v}_{t_j}, v_{p_j}, v_{p_k}, \) and \( v_{p_k} \) on \( \tilde{t}_k, \tilde{p}_k, \tilde{p}_j, t_j, p_j, \) and \( p_k, \) respectively, and \( v_{t_k} \) be the neighbor of the leaf vertex on \( t_k \). Then

\[
(2 + w)\tilde{M}[^\tilde{f}](\tilde{v}) = \tilde{p}_j(\tilde{v}_{p_j}) + \tilde{p}_k(\tilde{v}_{p_k}) + w \tilde{t}_k(\tilde{v}_{t_k}) = p'_j(v_p) + w t'_j(v_t) + p'_k(v_p) + w t'_k(v_k) = \mu(-f(v_p) - w f(v_t) + (1 + w)f(v_p)) + f(v) + w f(v) - (1 + w)f(v) = \mu(M[f](v) - f(v_p)) + w(M[f](v) - f(v_t)) + (1 + w)(f(v_p) - M[f](v))) = \mu(p'_j(v) + w t'_j(v) + (1 + w)p'_k(v)) = \mu(\tilde{p}_j(\tilde{v}) + \tilde{p}_k(\tilde{v}) + w \tilde{t}_k(\tilde{v})) = (2 + w)\mu \tilde{f}(\tilde{v}).
\]
In order to determine the kernel of the transplantation, we assume that \( f = 0 \). Since \( w \neq -1 \), we have \( p_i = t_i = 0 \) for all \( i \). Thus, if \( [v, v'] \) is part of one of the paths \((p_i)_{i=1}^{n}\) or \((t_i)_{i=1}^{n}\), then

\[
0 = f_{[v, v']}(v) = -M[f](v) + f(v') = -\mu v + f(v'),
\]

which leads to \( f(v') = \mu f(v) \).

Hence, \( f(v) = \mu f(v) \) for each vertex \( v \) on the central polygon of length \( l = l_1 + l_2 + \ldots + l_n \). If \( f(v) = 0 \) for one such vertex \( v \), then \( f \) vanishes on the entire polygon, and through \( M[f] = \mu f \) on the entire subdivided \( n \)-gear. On the other hand, if \( f(v) \neq 0 \) for a vertex \( v \) on the polygon, then \( |\mu| = 1 \), that is, \( \mu = \pm 1 \). Hence, the transplantation is injective on the span of \( \mu \)-eigenvectors with \( \mu \neq \pm 1 \). Since subdivided mutually dual \( n \)-gears are either both bipartite or both non-bipartite, we have proven Theorem 2.

In order to derive an explicit conjugator for \( M \) and \( \tilde{M} \), we note that their \( \pm \)-eigenspaces are given by functions that satisfy \( f(v) = \pm f(v') \) whenever \( v \) and \( v' \) are neighbors. Thus, if \( M[f] = \pm f \), then

\[
f_{[v, v']}(v) = -f_{[v', v]}(v) = -M[f](v) + f(v') = \mp f(v) + f(v') = 0.
\]

Hence, the \( \pm \)-eigenspaces of \( M \) are annihilated by the transplantation. Let \( J_+ \) be a rank-1 matrix that maps the 1-eigenspace of \( M \) onto that of \( \tilde{M} \), and annihilates all other eigenspaces. If \(-1\) is an eigenvalue of \( M \), we choose \( J_- \) analogously, otherwise we define \( J_- = 0 \). Writing \( M = D_1^{-1} A D_2 \) as above, the eigenspaces of \( M \) are orthogonal with respect to the inner product given by the diagonal matrix \( D_1 D_2 \). In particular, we can choose \( J_+ = J_{2n} D_1 D_2 \) where \( J_{2n} \) denotes the \( 2n \times 2n \) all-ones matrix. If \( T \) denotes the matrix that corresponds to the transplantation, then \( C = T + J_+ + J_- \) is invertible and satisfies \( \tilde{M} C = CM \) by construction. In addition, the arguments at the end of Section 2 equally apply to the subdivided versions of the graphs in Figure 2 which shows that their Markov matrices are isospectral.

Lastly, we mention further presences of the eigenderivative transplantation method in terms of conjugacy. In fact, the method became apparent to us when we discovered corresponding conjugators for the matrices \( A^1(\sqrt{X}) \) and \( A^2(\sqrt{X}) \) in [OB12], which arise when one assumes (11) on all edges, and which characterize eigenvalues through the transcendental equations \( \det(A^i(\sqrt{X})) = 0 \). Another characterization of the eigenvalues of a quantum graph is given by the scattering approach, which yields an exact trace formula [KS99b, BE09]. It can be shown that mutually dual \( n \)-gears have conjugated edge \( S \)-matrices, giving yet another isospectrality proof for the pairs in Figure 2.

Finally, we consider the digraphs \( G \) and \( \tilde{G} \) in Figure 3. Note that the teeth of \( \tilde{G} \) are head-to-tail with their corresponding polygon side. We let \( I_{12} \) and \( J_{12} \) denote the \( 12 \times 12 \)
On discrete subgroups of the two by two projective linear group over \( \mathbb{Q}_p \)-adic fields, respectively. For \( G \), we denote its adjacency matrix by \( A_G \), its out-degree matrix by \( D^\text{out}_G \), and its in-degree matrix by \( D^\text{in}_G \), the latter two of which have the row sums of \( A_G \) and \( A_G^T \) on their diagonals. For \( z = (x, y, \alpha, \beta, \gamma, \delta) \in \mathbb{C}^6 \), we let

\[
L_G(z) = xI_{12} + yJ_{12} + \alpha A_G + \beta A_G^T + \gamma D^\text{out}_G + \delta D^\text{in}_G,
\]

and similarly for \( L_{\tilde{G}}(z) \). The homogeneous polynomial \( \det(L_G(z))_{|y=0} \) can be viewed as a generalized characteristic polynomial. It determines, and is determined by, the reversing zeta function \([\text{Her}14]\), which generalizes the Bartholdi zeta function \([\text{Bar}99]\) to digraphs, which in turn generalizes the famous Ihara-Selberg zeta function \([\text{Iha}66]\). The matrix

\[
T = \begin{pmatrix}
\alpha^3 & 0 & 0 & 0 & 0 & 2\alpha^2\gamma & \alpha^3 & 0 & 0 & 0 & 0 \\
0 & 2\alpha^2\gamma & \alpha^3 & 0 & 0 & 0 & 0 & \alpha^2\gamma & \alpha^3 & 0 & 0 \\
0 & 0 & 2\alpha^2\gamma & \alpha^3 & 0 & 0 & 0 & 0 & \alpha^2\gamma & \alpha^3 & 0 \\
0 & 0 & 0 & \alpha^2\gamma & \alpha^3 & 0 & 0 & 0 & 0 & \alpha^2\gamma & \alpha^3 \\
0 & 0 & 0 & 0 & \alpha^2\gamma & \alpha^3 & 0 & 0 & 0 & 0 & \alpha^2\gamma & \alpha^3 \\
0 & 0 & 0 & 0 & 0 & \alpha^2\beta & 0 & 0 & 0 & 0 & \alpha^2\beta \\
0 & \alpha^2\beta & 0 & 0 & 0 & 0 & \alpha^2\beta & 0 & 0 & 0 & \alpha^2\beta \\
0 & \alpha^2\gamma & \alpha^2\beta & 0 & 0 & 0 & \alpha^2\beta & 0 & 0 & 0 & \alpha^2\beta \\
0 & 0 & 0 & \beta^3 & 0 & 0 & 0 & 0 & 0 & \beta^3 & 0 \\
0 & 0 & 0 & 0 & \beta^2\gamma & \alpha^2\beta & 0 & 0 & 0 & \beta^2\gamma & \alpha^2\beta \\
0 & 0 & 0 & 0 & 0 & \alpha^2\gamma & \alpha^2\beta & 0 & 0 & 0 & \alpha^2\gamma & \alpha^2\beta
\end{pmatrix}
\]

satisfies \( L_{\tilde{G}}(z)T = TL_G(z) \) and has determinant \((2\alpha^3)^6 - (2\alpha^2\gamma)^6)\alpha^8\beta^{10} \), which can be seen by adding its last 6 columns to its first 6 ones. In particular, \( \det(L_G(z)) = \det(L_{\tilde{G}}(z)) \), meaning \( G \) and \( \tilde{G} \) are zeta-equivalent and have zeta-equivalent complements \([\text{Her}14]\). The pattern of non-zero entries in \( T \) is reminiscent of combinatorial eigendervative transplantation. The conjugator \( T \) can be readily generalized to all pairs coming from dual \( n \)-gears all of whose polygon vertices have degree 3, where teeth have to be oriented as in Figure 3. However, the graphs in Figure 13 do not lead to zeta-equivalent non-isomorphic digraphs.

**References**

[Bar99] Laurent Bartholdi, *Counting paths in graphs*, Enseign. Math. (2) 45 (1999), no. 1-2, 83–131.

[BCDS94] Peter Buser, John Conway, Peter Doyle, and Klaus-Dieter Semmler, *Some planar isospectral domains*, Internat. Math. Res. Notices (1994), no. 9, 391ff., approx. 9 pp. (electronic).

[BE09] Jens Bolte and Sebastian Endres, *The trace formula for quantum graphs with general self adjoint boundary conditions*, Ann. Henri Poincaré 10 (2009), no. 1, 189–223.

[BK13] Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs, vol. 186, American Mathematical Society, Providence, RI, 2013.

[BS07] Ram Band and Uzy Smilansky, *Resolving the isospectrality of the dihedral graphs by counting nodal domains*, Eur. Phys. J. Special Topics 145 (2007), 171–179.

[Bus86] Peter Buser, *Isospectral Riemann surfaces*, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 2, 167–192.

[Her11] Peter Herbrich, *On inaudible properties of broken drums - Isospectrality with mixed Dirichlet-Neumann boundary conditions*, arXiv:1111.6789 (2011).

[Her14] , *Zeta-equivalent digraphs: Simultaneous cospectrality*, arXiv:1412.4763.

[Her15] , *Line graphs and the transplantation method*, arXiv:1504.02339.

[Iha66] Yasutaka Iihara, *On discrete subgroups of the two by two projective linear group over \( p \)-adic fields*, J. Math. Soc. Japan 18 (1966), 219–235.
Tsampikos Kottos and Uzy Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett. 79 (1997), 4794–4797.

V. Kostrykin and R. Schrader, *Kirchhoff’s rule for quantum wires*, J. Phys. A 32 (1999), no. 4, 595–630.

Tsampikos Kottos and Uzy Smilansky, *Periodic orbit theory and spectral statistics for quantum graphs*, Ann. Physics 274 (1999), no. 1, 76–124.

Peter Kuchment, *Graph models for waves in thin structures*, Waves Random Media 12 (2002), no. 4, R1–R24.

Patrick McDonald and Robert Meyers, *Isospectral polygons, planar graphs and heat content*, Proc. Amer. Math. Soc. 131 (2003), no. 11, 3589–3599 (electronic).

Idan Oren and Ram Band, *Isospectral graphs with identical nodal counts*, J. Phys. A 45 (2012), no. 17, 135203, 12.

Talia Shapira and Uzy Smilansky, *Quantum graphs which sound the same*, NATO Sci. Ser. II Math. Phys. Chem. 213 (2006), 17–29.

Toshikazu Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. (2) 121 (1985), no. 1, 169–186.

Joachim von Below, *A characteristic equation associated to an eigenvalue problem on $c^2$-networks*, Linear Algebra Appl. 71 (1985), 309–325.

Department of Mathematics, Dartmouth College, Hanover, NH, USA

E-mail address: peter.g.doyle@dartmouth.edu, peter.herbrich@dartmouth.edu