EXTENSION OF NEWTON-STEFFENSSEN METHOD
BY GEJJI-JAFARI DECOMPOSITION TECHNIQUE
FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper we extend Newton-Steffenssen method for
solving nonlinear equations, introduced by Sharma [J.R. Sharma, A
composite third order Newton-Steffenssen method for solving nonlin-
ear equations, Appl. Math. Comput. 169 (2005), 242-246] by using the
Gejji-Jafari decomposition technique. Several numerical examples are
given to illustrate the efficiency and performance of this new method.

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gence, simple root, decomposition method, numerical results.

1. Introduction

Solving nonlinear equations is one of the most important problems
in numerical analysis. To solve nonlinear equations, iterative methods
such as Newton’s method are usually used. Throughout this paper we
consider iterative methods to find a simple root \( \alpha \), of a nonlinear equa-
tion \( f(x) = 0 \), where \( f : I \subset R \rightarrow R \) for an open interval \( I \). Many
variants of the Newton’s method have been suggested in the litera-
ture by different techniques. One of them is Adomain decomposition
method which is used in [6] and other literatures. To implement Ado-
main decomposition method, one has to calculate Adomain polynomial,
which is another difficult task. Other techniques have also their limi-
tations. To overcome these difficulties, a new decomposition technique
is introduced by Gejji-Jafari in [1]. In this paper we use this technique
to extend Newton-Steffenssen method introduced by Sharma [2].

2. Gejji-Jafari decomposition method

Consider the nonlinear equation

\[ f(x) = 0. \] (2.1)
Throughout the paper we assume that $f(x)$ has a simple root at $\alpha$ and $\gamma$ is an initial guess close to $\alpha$. Let us transform the nonlinear equation (2.1) into the following canonical form:

$$x = c + N(x), \quad (2.2)$$

where $N(x)$ nonlinear operator and $c$ is a constant. The main idea of this technique is to look for a solution having the series form

$$x = \sum_{i=0}^{\infty} x_i, \quad (2.3)$$

The nonlinear operator $N$ can be decomposed as

$$N(x) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N \left( \sum_{j=0}^{i} x_j \right) - N \left( \sum_{j=0}^{i-1} x_j \right) \right\}. \quad (2.4)$$

From equations (2.3) and (2.4), equation (2.2) is equivalent to

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N \left( \sum_{j=0}^{i} x_j \right) - N \left( \sum_{j=0}^{i-1} x_j \right) \right\}. \quad (2.5)$$

Thus we have the following recurrence relation:

- $x_0 = c$
- $x_1 = N(x_0)$
- $x_2 = N(x_0 + x_1) - N(x_0)$
- $x_2 = N(x_0 + x_1 + x_2) - N(x_0 + x_1)$
- $\ldots$
- $x_{m+1} = N(x_0 + x_1 + \ldots + x_m) - N(x_0 + x_1 + \ldots + x_{m-1}). \quad (2.6)$

Then

$$x_0 + x_1 + \ldots + x_{m+1} = N(x_0 + x_1 + \ldots + x_m); \quad m = 1, 2, \ldots, \quad (2.7)$$

and

$$x = c + \sum_{i=1}^{\infty} x_i \quad (2.8)$$

In [1] it is proved that the series $\sum_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution of (2.2).
3. Extension of Newton-Steffenssen method

Consider the following coupled system:

\[ f(\gamma) + (x - \gamma) \left[ f'(\gamma) + \frac{g(x)}{(x - \gamma)} \right] = 0, \quad (3.1) \]

\[ g(x) = f(x) - f(\gamma) - f'(\gamma)(x - \gamma). \quad (3.2) \]

The equation (3.1) of the above system can be rewritten as

\[ x = \gamma - \frac{f(\gamma)(x - \gamma)}{f'(\gamma)(x - \gamma) + g(x)}. \quad (3.3) \]

Comparing Equations (2.2), Ist of (2.6) and (3.3), we have

\[ x_0 = c = \gamma. \quad (3.4) \]

and

\[ N(x) = - \frac{f(\gamma)(x - \gamma)}{f'(\gamma)(x - \gamma) + g(x)}. \quad (3.5) \]

Note that \( x \) is approximated by

\[ X_m = x_0 + x_1 + \ldots + x_m, \quad (3.6) \]

where \( \lim_{m \to \infty} X_m = x \).

For \( m = 0 \),

\[ x \approx X_0 = x_0 = c = \gamma. \quad (3.7) \]

For \( m = 1 \),

\[ x \approx X_1 = x_0 + x_1 = \gamma + N(x_0). \quad (3.8) \]

where \( N(x_0) \) is to be calculate. From (3.5) we have

\[ N(x_0) = - \frac{f(\gamma)(x_0 - \gamma)}{f'(\gamma)(x_0 - \gamma) + g(x_0)}. \quad (3.9) \]

Thus (3.8) becomes

\[ x \approx X_1 = x_0 + x_1 = \gamma - \frac{f(\gamma)(x_0 - \gamma)}{f'(\gamma)(x_0 - \gamma) + g(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}. \quad (3.10) \]

which yields the famous Newton’s method with second order convergence

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (3.11) \]

For \( m = 2 \),

\[ x \approx X_2 = x_0 + x_1 + x_2 = \gamma + N(x_0 + x_1). \quad (3.12) \]
where \( N(x_0 + x_1) \) is to be calculate. From (3.5), (3.2) and (3.10) we have

\[
N(x_0 + x_1) = \frac{f(\gamma)^2}{f'(\gamma)\{f(x_0 + x_1) - f(\gamma)\}}.
\]

(3.13)

Thus we have

\[
x \approx X_2 = x_0 + x_1 + x_2 = \gamma - \frac{f(\gamma)^2}{f'(\gamma)\{f(\gamma) - f(x_0 + x_1)\}},
\]

(3.14)

which gives the following well known Newton-Steffenssen method [2] with third order convergence

\[
x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)\{f(x_n) - f(y_n)\}},
\]

(3.15)

where \( y_n = x_n - \frac{f(x_n)}{f'(x_n)} \).

For \( m = 3 \),

\[
x \approx X_3 = x_0 + x_1 + x_2 + x_3 = \gamma + N(x_0 + x_1 + x_2).
\]

(3.16)

where \( N(x_0 + x_1 + x_2) \) is to be calculate. From (3.5), (3.2) and (3.14) we have

\[
N(x_0 + x_1 + x_2) = \frac{f(\gamma)^3}{f'(\gamma)\{f(\gamma) - f(x_0 + x_1)\}\{f(x_0 + x_1 + x_2) - f(\gamma)\}}.
\]

(3.17)

Thus we have

\[
x \approx X_3 = x_0 + x_1 + x_2 + x_3
\]

\[
= \gamma - \frac{f(\gamma)^3}{f'(\gamma)\{f(\gamma) - f(x_0 + x_1)\}\{f(\gamma) - f(x_0 + x_1 + x_2)\}},
\]

(3.18)

which suggests the following three-step iterative method

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)},
\]

\[
z_n = x_n - \frac{f(x_n)^2}{f'(x_n)\{f(x_n) - f(y_n)\}},
\]

\[
x_{n+1} = x_n - \frac{f(x_n)^3}{f'(x_n)\{f(x_n) - f(y_n)\}\{f(x_n) - f(z_n)\}}.
\]

(3.19)
Similarly we can obtain higher-order iterative methods. For general \( n \) it can be shown that the \((n - 1)\)-step iterative method is

\[
a_1 = x_n - \frac{f(x_n)}{f'(x_n)},
\]
\[
a_2 = x_n - \frac{f(x_n)^2}{f'(x_n)\{f(x_n) - f(a_1)\}},
\]
\[
a_3 = x_n - \frac{f(x_n)^3}{f'(x_n)\{f(x_n) - f(a_1)\}\{f(x_n) - f(a_2)\}},
\]
\[
\ldots
\]
\[
a(n-2) = x_n - \frac{f(x_n)^{n-2}}{f'(x_n)\{f(x_n) - f(a_1)\}\{f(x_n) - f(a_2)\}\{f(x_n) - f(a_3)\}\ldots\{f(x_n) - f(a(n-3))\}}
\]
\[
x_{n+1} = x_n - \frac{f(x_n)^{n-1}}{f'(x_n)\{f(x_n) - f(a_1)\}\{f(x_n) - f(a_2)\}\{f(x_n) - f(a_3)\}\ldots\{f(x_n) - f(a(n-2))\}}
\]

Now we prove that order of convergence of the iterative method (3.19) is four, which is shown by the following theorem:

**Theorem 3.1.** Let \( \alpha \in I \) be a simple zero of a sufficiently differentiable function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) in an open interval I. If \( x_0 \) is sufficiently close to \( \alpha \), then the three-step method defined by (3.19) has order fourth-order convergence.

**Proof.** By applying the Taylor series expansion theorem and taking account \( f(\alpha) = 0 \), we can write

\[
f(x_n) = e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9), \tag{3.21}
\]

where \( c_k = \frac{f^{(k)}(\alpha)}{k!}, k = 1, 2, \ldots \) and \( e_n \) be the error in \( x_n \) after \( n \) iterations i.e. \( e_n = x_n - \alpha \);

\[
f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + O(e_n^8)]. \tag{3.22}
\]

By considering the above relations, one can obtain

\[
y_n = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + \ldots + O(e_n^8). \tag{3.23}
\]

At this time, we should expand \( f(y_n) \) around the exact \( \alpha \) root by taking into consideration (3.23). Accordingly, we have

\[
f(y_n) = f'(\alpha)[c_2 e_n^2 + 2(-c_2^2 + c_3)e_n^3 + \ldots + O(e_n^{12})] \tag{3.24}
\]
From (3.21), (3.22) and (3.24) it can found that
\[ z_n = \alpha + c_2^2 e^3 + (3c_2 c_3 - 3c_3^2) e^4 + \ldots + O(e_n^8). \]  
(3.25)

Now expand \( f(z_n) \) around the exact \( \alpha \) root, we have
\[ f(z_n) = f'(\alpha)[c_2^2 e^3 + (3c_2 c_3 - 3c_3^2) e^4 + \ldots + O(e_n^8)]. \]  
(3.26)

Finally by virtue of (3.21), (3.22), (3.24) and (3.27) it can be obtained that
\[ e_{n+1} = c_3^3 e^4 + O(e_n^5). \]  
(3.27)

Similarly we can prove that iterative method (3.20) has \( n^{th} \)-order convergence.

4. Numerical Testing

Here we consider, the following eight test functions to illustrate the accuracy of new iterative method. Some of them are taken from [7] and some from [8]. The root of each nonlinear test function is also listed. All the computations reported here we have done using Mathematica 8. Scientific computations in many branches of science and technology demand very high precision degree of numerical precision. We consider the number of decimal places as follows: 10000 digits floating point (SetAccuracy=10000) with SetAccuracy Command. In examples considered in this article, the stopping criterion is the \( |f(x_n)| \leq \epsilon \) where \( \epsilon = 10^{-10000} \). The test non-linear functions are listed in Table-1.

Here we comparer performance of our new method (3.19) to the methods of Yun (YN) [3], of Chun (CN) [6] and Noor(NR) [5]. The results of comparison for the test function are provided in the Table 2. It can be seen that the resulting method from our class are accurate and efficient in terms of number of accurate decimal places to find the roots after some iterations.

**Table 1. Test functions and their roots.**

| Non-linear function | Roots |
|---------------------|-------|
| \( f_1(x) = \sin^2 x - x^2 + 1 \) | 1.4044916482153412260 |
| \( f_2(x) = x^2 - e^x - 3x + 2 \) | 0.25753028543986076046 |
| \( f_3(x) = (x - 1)^3 - 1 \) | 2 |
| \( f_4(x) = x^3 - 10 \) | 2.1544346900318837218 |
| \( f_5(x) = xe^x - \sin^2 x + 3\cos x + 5 \) | -1.2076478271309189270 |
| \( f_6(x) = e^{x^2+7x-30} - 1 \) | 3 |
| \( f_7(x) = x^2 + \sin x + x \) | 0 |
| \( f_8(x) = \sin(2 \cos x) - 1 - x^2 + e^{\sin x^3} \) | 1.3061752018468278250 |
Table 2. Comparison of different methods with the same total number of function evaluations (TNFE = 24)

| TestFunction | Guess | CH      | YN      | NR      |
|--------------|-------|---------|---------|---------|
| $f_1$        | -1.0  | 0.15403e-267 | 0.41715e-642 | 0.52856e-2 | 0.22708e-1576 |
|              | 2.0   | 0.35940e-1457 | 0.23037e-1530 | 0.11251e-3 | 0.13526e-2046 |
|              | 1.0   | 0.15403e-267 | 0.41715e-642 | 0.52856e-2 | 0.22708e-1576 |
| $f_2$        | 2.0   | 0.74095e-1843 | 0.42734e-1881 | 0.16354e-4 | 0.62927e-2949 |
|              | 2.5   | 0.70566e-1411 | 0.31383e-1450 | 0.19981e-3 | 0.18307e-1924 |
|              | -1.5  | 0.91945e-2055 | 0.25223e-2127 | 0.46951e-4 | 0.13343e-2794 |
| $f_3$        | 3.5   | 0.95901e-373  | 0.60165e-400  | 0.43860e-1 | 0.20884e-671  |
|              | 3.1   | 0.22428e-546  | 0.32703e-582  | 0.13254e-1 | 0.36490e-921  |
|              | 1.5   | 0.39254e+1    | 0.21668e+1    | 0.23818e+1 | 0.38950e+663  |
| $f_4$        | 1.5   | 0.610763-350  | 0.25884e-740  | 0.27421e-1 | 0.47059e-1750 |
|              | 1.2   | 0.38918e-2    | 0.12361e-298  | 0.85122e+0 | 0.71673e-925  |
|              | 1.0   | 0.749233+3    | 0.276466e+5   | 0.42601e+1 | 0.18324e-518  |
| $f_5$        | -2.0  | 0.74075e-140  | 0.16833e-150  | 0.10709e+1 | 0.60969e-322  |
|              | -1.5  | 0.19698e-1039 | 0.21190e-1089 | 0.26108e-2 | 0.52642e-1607 |
|              | -1.0  | 0.19824e-693  | 0.10164e-1074 | 0.38167e-2 | 0.49335e-2031 |
| $f_6$        | 3.5   | 0.45434e-5    | 0.99135e-6    | 0.13063e+2 | 0.13526e-22   |
|              | 3.2   | 0.11804e-200  | 0.14326e-215  | 0.92374e-1 | 0.69853e-422  |
|              | 2.9   | 0.20627e+114  | Indeterminate | 0.42942e+0 | 0.45296e+496  |
| $f_7$        | 0.3   | 0.51508e-2921 | 0.46046e-3026 | 0.14553e-6 | 0.13490e-3702 |
|              | 0.1   | 0.10912e-4558 | 0.46696e-4679 | 0.77208e-10| 0.16080e-5443 |
|              | -0.2  | 0.27239e-2693 | 0.16105e-2868 | 0.19173e-6 | 0.17394e-3843 |
| $f_8$        | 1.35  | 0.29081e-4253 | 0.15643e-4384 | 0.62249e-9 | 0.15434e-5050 |
|              | 1.31  | 0.37148e-8008 | 0.11024e-8139 | 0.20749e-16| 0.66987e-8948 |
|              | 1.29  | 0.56923e-5177 | 0.37152e-5314 | 0.56633e-11| 0.20166e-6195 |

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