GAUSS SUMS IN ALGEBRA AND TOPOLOGY

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Abstract. We consider Gauss sums associated to functions $T \to \mathbb{R}/\mathbb{Z}$ which satisfy some sort of “quadratic” property and investigate their elementary properties. These properties and a Gauss sum formula from the nineteenth century due to Dirichlet give the Milgram Gauss sum formula computing the signature mod 8 of a non–singular bilinear form over $\mathbb{Q}$. Brown derived some results on the signature mod 8 of non–singular integral forms. Kirby and Melvin gave a formula for a generalization of this invariant to possibly non–singular forms and we further generalize it here. The Milgram Gauss sum formula and these formulas allow us to reprove Brown’s result without resort to Witt group calculations. Assuming a bit of algebraic topology, we reprove a theorem of Morita’s computing the signature mod 8 of an oriented Poincaré duality space from the Pontrjagin square without using Bockstein spectral sequences. Since we work with forms which may be singular, we also obtain a version of Morita’s theorem for Poincaré spaces with boundary. Finally we apply our results to the bilinear form $Sq^1 x \cup y$ on $H^1(M; \mathbb{Z}/2\mathbb{Z})$ of an orientable 3–manifold.

1. Introduction.

Gauss sums have a long and venerable history. A general version has a finite set $T$, a function $\psi : T \to \mathbb{R}/\mathbb{Z}$ and the associated Gauss sum

$$G(\psi) = \sum_{t \in T} e^{2\pi i \psi(t)} .$$

Even more general notions would replace $T$ by a measure space and the finite sum by an integral. The Gauss sum problem is to evaluate $G(\psi)$ with the first examples going back to Gauss [6, Art.356].

In the generality of a function on a finite set, it is difficult to say very much useful except that the problem can be divided into a magnitude and a phase. We can consider the magnitude or norm, $N(\psi) = |G(\psi)|$, and when $N(\psi) \neq 0$, define the phase $\beta(\psi) \in \mathbb{R}/\mathbb{Z}$ by

$$G(\psi) = N(\psi) \cdot e^{2\pi i \beta(\psi)} .$$

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Here are two general constructions. Given $\psi_i$ on $T_i$, $i = 1, 2$, define the orthogonal sum $\psi_1 \perp \psi_2 : T_1 \times T_2 \to \mathbb{R}/\mathbb{Z}$ by $(\psi_1 \perp \psi_2)(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2)$. Check that

\[(1.1) \quad G(\psi_1 \perp \psi_2) = G(\psi_1) \cdot G(\psi_2) .\]

For the second construction, observe $\mathbb{Z}$ acts on the functions by $(a \cdot \psi)(x) = a \cdot \psi(x)$ for $a \in \mathbb{Z}$. For $a = -1$,

\[(1.2) \quad G(-\psi) = \overline{G(\psi)} .\]

We review and extend some examples for which the magnitude and phase can be calculated and apply these results to some problems in algebra and topology.

Start with an easy example: $T$ is a finite group and $\psi$ is a homomorphism. If $\psi$ is trivial, the answer is immediate: $N(\psi) = |T|$ and $\beta(\psi) = 0$ since every term in the sum is 1. For the non–trivial case do the sum in two ways and compare the answers: specifically, fix an element $c \in T$ with $\psi(c) \neq 0$ and observe

\[\sum_{t \in T} e^{2\pi i \psi(t)} = \sum_{t \in T} e^{2\pi i \psi(t+c)} \]

because we are summing over the same set and

\[\sum_{t \in T} e^{2\pi i \psi(t+c)} = e^{2\pi i \psi(c)} \sum_{t \in T} e^{2\pi i \psi(t)} \]

since $\psi$ is a homomorphism. Since $e^{2\pi i \psi(c)} \neq 1$, $N(\psi) = 0$.

Our main interest is the case where $T$ is a finite abelian group and $\psi$ has some sort of “quadratic” property, which we now review. A function $\psi : T \to \mathbb{Q}/\mathbb{Z}$ is called a quadratic function provided

\begin{align*}
(1.3) & \quad \psi(ax) = a^2 \psi(x) \text{ for all integers } a \text{ and all } x \in T \\
(1.4) & \quad b(x, y) = \psi(x + y) - \psi(x) - \psi(y) : T \times T \to \mathbb{Q}/\mathbb{Z} \text{ defines a bilinear form.}
\end{align*}

We call $\psi$ a quadratic enhancement of $b$ and $b$ the associated bilinear form to $\psi$. If $\psi$ just satisfies 1.4, we say $\psi$ is an enhancement of $b$. Condition 1.4 is equivalent to the condition

\[(1.5) \quad \psi(x_1 + x_2 + x_3) = \psi(x_1 + x_2) + \psi(x_1 + x_3) + \psi(x_2 + x_3) - \left(\psi(x_1) + \psi(x_2) + \psi(x_3)\right) \]

for all $x_i \in T$. Any function $\psi : T \to \mathbb{R}/\mathbb{Z}$ satisfying 1.4 will be called 2–linear. The associated bilinear form, $b_\psi$, is defined by 1.4: note that
the bilinear form is symmetric. The set of functions from $T$ to $\mathbb{R}/\mathbb{Z}$ is a group and the set of 2–linear functions is a subgroup. The orthogonal sum of 2–linear functions is 2–linear.

Any symmetric bilinear form has enhancements. Given $b: T \times T \to \mathbb{Q}/\mathbb{Z}$ define $\mathcal{W}(b)$ to be $T \oplus \mathbb{Q}/\mathbb{Z}$ with group structure $(t_1, r_1) + (t_2, r_2) = (t_1 + t_2, r_1 + r_2 - b(t_1, t_2))$. It is straightforward to check that $\mathcal{W}(b)$ is an abelian group and that $\iota: \mathbb{Q}/\mathbb{Z} \to \mathcal{W}(b)$ defined by $\iota(r) = (0, r)$ is an injective homomorphism. Since $\mathbb{Q}/\mathbb{Z}$ is divisible, $\iota$ has a section, $\Psi: \mathcal{W}(b) \to \mathbb{Q}/\mathbb{Z}$. Define $\psi(t) = \Psi(t, 0)$ and check that $\psi$ is an enhancement of $b$. Conversely, given an enhancement of $b$, define $\Psi(t, m) = \psi(t) + m$ and check that it is a section, so the set of enhancements of $b$ corresponds bijectively to the set of sections of $\iota$. Let $\Gamma(b)$ denote the space of sections of $\iota$ or equivalently the space of enhancements of $b$.

The set $\Gamma(b)$ is acted on by the group of homomorphisms, $T^* = \text{Hom}(T, \mathbb{Q}/\mathbb{Z})$: given one enhancement of $b$, say $\psi$, and $h \in T^*$ define $\psi_h$ by $\psi_h(t) = \psi(t) + h(t)$ and check that $\psi_h$ is an enhancement of $b$. If $\psi_1$ and $\psi_2$ are enhancements of $b$, check that $\psi_1 - \psi_2 \in T^*$. Pick an element $\psi \in \Gamma(b)$ and define $T^* \to \Gamma(b)$ using the action on $\psi$: this function is a bijection.

A homomorphism, $h: T_1 \to T_2$ is an isometry between two 2–linear functions provided $\psi_2(h(x)) = \psi_1(x)$: if $h$ is an isomorphism we say $\psi_1$ and $\psi_2$ are isometric, written $\psi_1 \cong \psi_2$. The map $h$ is an isometry in the usual sense between the associated bilinear forms. As an example, if $a \in \mathbb{Z}$ is relatively prime to $|T|$ then multiplication by $a$ on $T$ gives an isometry between $a^2 \cdot \psi$ and $\psi$ if $\psi$ is quadratic.

Any 2–linear function satisfies $\psi(0) = 0$ and $b(x, x) = \psi(x) + \psi(-x)$ (compute $b(x, -x)$ using bilinearity). Induction shows that

$$\psi(ax) = \frac{a^2 + a}{2} \cdot \psi(x) + \frac{a^2 - a}{2} \cdot \psi(-x).$$

Furthermore, $\psi^g: T \to \mathbb{Q}/\mathbb{Z}$ defined by $\psi^g(x) = \psi(x) - \psi(-x)$ is a homomorphism and $\psi$ is quadratic if and only if $\psi^g$ is identically 0. Check that $\psi^g$ vanishes on elements in $T$ of order 2 and that $\psi^g$ behaves well under our two constructions: $(\psi_1 \perp \psi_2)^g = \psi_1^g + \psi_2^g$ and $(a \cdot \psi)^g = a \cdot \psi^g$.

It follows that, if $ax = 0$ then $a\psi(x) = a\psi(-x) = 0$. Hence $2a\psi(x) = 0$ and any 2–linear function automatically lands in $\mathbb{Q}/\mathbb{Z}$. If $a$ is odd it further follows that $a^2\psi(x) = 0$ and hence $a\psi(x) = 0$. It follows that $\psi = \perp_p \psi_p$, where $\psi_p$ denotes $\psi$ restricted to the $p$–torsion subgroup and $\psi_p$ takes values in $\mathbb{Z} [\frac{1}{p}] / \mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Finally, it follows that the group of 2–linear functions is finite.
The action of $T^*$ on enhancements has the following effect. Check that $(\psi_h)^q = \psi^q + 2h$ so if $\psi$ is quadratic, $\psi_h$ is also quadratic if and only if $h$ takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Since $\psi^q$ vanishes on elements of order 2, there exists a homomorphism such that $\psi^q = 2h$ and then $\psi_{-h}$ is quadratic. If $\psi$ is a quadratic enhancement of $b$ then $\psi_h$ is also quadratic if and only if $h$ takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

If $K \subset T$ is a subgroup, define $K^\perp = \{ t \in T \mid b_\psi(t, k) = 0 \ \forall \ k \in K \}$. We call $\psi$ and $b_\psi$ non–singular provided $T^\perp = \{ 0 \}$. Note $(T_1 \perp T_2)^\perp = T_1^\perp \oplus T_2^\perp$. If $K$ satisfies $\psi|_K = 0$, then $K \subset K^\perp$ and $\psi$ induces a well–defined function

$$\psi_{K^\perp/K} : K^\perp/K \to \mathbb{Q}/\mathbb{Z}.$$  

Check that $(K^\perp/K)^\perp = (T^\perp)/(T^\perp \cap K)$ so $\psi_{K^\perp/K}$ is tame (resp. non–singular) if $\psi$ is.

Any bilinear form $b$ defines an adjoint homomorphism $Ad(b) : T \to \text{Hom}(T, \mathbb{Q}/\mathbb{Z})$ and $T^\perp$ is the kernel of $Ad(b)$. This means that non–singular forms have the property that for any $h \in T^*$ there exists a unique $c \in T$ such that $h(t) = b(t, c)$ for all $t \in T$. In the singular case, given any $h \in T^*$ with $h|_{T^\perp}$ trivial, there exist $c \in T$, not unique, such that $h(t) = b(t, c)$ for all $t \in T$.

**Definition 1.7.** For $x \in T/T^\perp$ we use the notation $\psi_x$ to denote the enhancement $\psi_x(t) = \psi(t) + b(x_1, t)$ for all $t \in T$, where $x_1 \in T$ maps to $x$.

Check that $\psi_x$ is independent of the choice of $x_1$ as is $\psi(x_1)$, which will be denoted $\psi(x)$.

**Lemma 1.8.** $\psi_x(t) = \psi(t + x_1) - \psi(x_1)$ so $\beta(\psi_x) = \beta(\psi) - \psi(x)$.

For any 2–linear function, $\psi$ restricted to $T^\perp$ is a homomorphism: denote it by $\psi^a$. We say $\psi$ is tame provided $\psi^a$ is trivial. If $\psi$ is tame then it induces a non–singular form on $T/T^\perp$, which we denote by $\psi^\text{red}$. The orthogonal sum of two tame 2–linear functions is tame. If $\psi$ is tame, so is $a \cdot \psi$ for any $a \in \mathbb{Z}$ which is relatively prime to $|T|$. If $\psi$ is quadratic $\psi^a$ takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ and whenever $2x \in T^\perp$, $\psi^a(2x) = 0$.

Suppose $\psi$ is tame. Then $\psi_h$ is tame if and only if $h$ vanishes on $T^\perp$.

**Remarks 1.9.** Since $\mathbb{Q}/\mathbb{Z}$ is injective there is always an extension $h$ of $\psi^a$ to all of $T$ and for any such $h$, $\psi_{-h}$ is tame. If $\psi$ is tame the function $x \in T/T^\perp \mapsto \psi_x$ defines a bijection between $T/T^\perp$ and the tame enhancements of $b$. 


Remarks 1.10. For any abelian group $T$ and $n \in \mathbb{Z}$, let $nT = \{x \in T \mid nx = 0\}$ and let $n \cdot T = \{x \in T \mid x = ny\}$. If $\psi$ is tame and quadratic the function $x \in 2(T/T^\perp) \mapsto \psi_x$ defines a bijection between $2(T/T^\perp)$ and the tame quadratic enhancements of $b$.

Theorem 1.11. Every symmetric bilinear form $b$ has a tame quadratic enhancement.

Proof. We have seen $b$ has enhancements. Using (1.9) construct a tame one $\psi_t$. Pass to $\psi_t^{\text{red}}$ and use (1.10) to construct a quadratic $\hat{\psi} : T/T^\perp \to \mathbb{Q}/\mathbb{Z}$ and check that the composition $\psi : T \to T/T^\perp \xrightarrow{\psi_t} \mathbb{Q}/\mathbb{Z}$ is a tame quadratic enhancement of $b$. □

For subgroups $K \subset T$ with $\psi|_K = 0$, the Gauss sums for $\psi$ and $\psi_{K^\perp/K}$ are related. This has been noticed before, [2], [3], [11], [12], [14] and many others, but apparently only for non–singular quadratic functions.

Theorem 1.12. Let $\psi : T \to \mathbb{R}/\mathbb{Z}$ be 2–linear and suppose $\psi|_K = 0$ for some subgroup $K$. Then

$$G(\psi) = |K| \cdot G(\psi|_{K^\perp/K}).$$

We emphasize that neither tame nor quadratic is assumed.

Proof. The usual proof assuming non-singular still works. Pick coset representatives $\alpha_i$ for $T/K^\perp$ and $\gamma_j$ for $K^\perp/K$ and let $\kappa_k$ run over the elements of $K$. Then every element of $T$ can be written uniquely as $\alpha_i + \gamma_j + \kappa_k$ and $\psi(\alpha_i + \gamma_j + \kappa_k) = b(\alpha_i, \gamma_j) + b(\alpha_i, \kappa_k) + \psi(\alpha_i) + \psi(\gamma_j)$. If we fix $\alpha_i$ and $\gamma_j$ and sum over $\kappa_k$ then the only term which depends on $\kappa_k$ is $b(\alpha_i, \kappa_k)$ which is a homomorphism. Hence the sum is 0 if $b(\alpha_i, \kappa_k)$ is non–zero for some $\kappa_k$. But by definition of $K^\perp$, $b(\alpha_i, \kappa_k)$ is non–trivial except for the $\alpha_i$ in the 0 coset, say $\alpha_0$, and in this case the sum is just $|K| \cdot e^{2\pi i \psi(\alpha_0 + \gamma_j)}$. □

A standard observation permits us to evaluate the magnitude of $G(\psi)$.

Theorem 1.13. If $\psi$ is 2–linear then $N(\psi) = 0$ if $\psi$ is not tame and $N(\psi) = \sqrt{|T^\perp| \cdot |T|}$ if $\psi$ is tame.

Proof. If $\psi$ is not tame then there is a $c \in T^\perp$ such that $\psi(c) \neq 0$ and $\psi(t + c) = \psi(t) + \psi(c)$ for all $t \in T$. As in the homomorphism case it follows that $N(\psi) = 0$. If $\psi$ is tame, check that the evident inclusion $\Delta : T \subset T \perp T/T^\perp$ with 2–linear function $\psi \perp -\psi^{\text{red}}$ satisfies
Theorem 1.14. Suppose \( N(T) \) and \( \psi \) \( \perp -\psi^{\text{red}} \) restricted to \( \Delta(T) \) vanishes so \( G(\psi \perp -\psi^{\text{red}}) = N(T) \). But \( G(\psi \perp -\psi^{\text{red}}) = G(\psi) \cdot G(-\psi^{\text{red}}); G(\psi) = N(T^\perp) \cdot G(\psi^{\text{red}}); \) and \( G(-\psi^{\text{red}}) = G(\bar{\psi}^{\text{red}}). \) □

We can evaluate \( \beta(\psi_h) \) in terms of \( \beta(\psi) \) if both are tame.

**Theorem 1.14.** Suppose \( \psi \) is 2–linear and tame and that \( h : T \to Q/Z \) is a homomorphism. Then \( \psi_h \) is tame if and only if \( h(T^\perp) = 0 \). If \( q_h \) is tame let \( c \in T/T^\perp \) be the unique element corresponding to \( h \), so \( \psi_h = \psi_c \). The value of \( \psi(c_1) \in Q/Z \) is the same for all elements reducing to \( c \), so let us denote that common value by \( \psi(c) \). Then

\[
\beta(\psi_h) = \beta(\psi) - \psi(c).
\]

**Proof.** Since \( \psi_h = \psi_c \), the displayed formula holds by Lemma 1.8 □

**Remark 1.15.** We check that a potential source of new functions in fact yields nothing new. Let \( \psi \) be a tame enhancement and define a new function

\[
\Delta_\beta : T/T^\perp \to R/Z
\]

by \( \Delta_\beta(x) = \beta(\psi) - \beta(\psi_x) \) for all \( x \in T/T^\perp \). Then (1.14) says \( \Delta_\beta = \psi^{\text{red}} \).

It follows from a beautiful argument due to Frank Connolly, [11, p.393], that, in the quadratic case, \( \psi \perp \psi \perp \psi \perp \psi \) is isometric to \( -\psi \perp -\psi \perp -\psi \perp -\psi \). It follows that for any tame, quadratic 2–linear function, \( \beta(\psi) \in Z/8Z \subset Q/Z \) and it then follows from (1.14) that \( \beta(\psi) \in Q/Z \) for any tame 2–linear function.

Connolly’s result can be made more precise: let \( T_p \) denote the \( p \)-torsion subgroup of \( T \) and let \( \psi_p = \psi|_{T_p} \). As usual, it suffices to understand \( \psi_p \).

**Theorem 1.16.** Assume \( \psi_p \) is quadratic 2–linear. If \( p \equiv 1 \mod 4 \), \( \psi_p \cong -\psi_p \); if \( p \equiv 3 \mod 4 \), \( \psi_p \perp \psi_p \cong -\psi_p \perp -\psi_p \); if \( p \equiv 2 \), \( \psi_2 \perp \psi_2 \perp \psi_2 \perp \psi_2 \cong -\psi_2 \perp -\psi_2 \perp -\psi_2 \perp -\psi_2 \).

Here is a version of Connolly’s argument. Choose \( k \) so that \( p^k \) is an exponent for \( T_p \) and hence \( \psi_p \) takes values in \( Z/p^kZ \subset Q/Z \), \( p \) odd or in \( Z/2^{k+1}Z \subset Q/Z \). Find integers \( x_i \) such that \( x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv -1 \mod p^\ell \) with \( \ell \geq k \) \( p \) odd or \( \ell > k + 1 \) if \( p = 2 \). Additionally assume \( x_3 = x_4 = 0 \) if \( p \) is odd and further assume \( x_2 = 0 \) if \( p \equiv 1 \mod
4. Assuming this done, Connolly’s matrix \( M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\
-x_2 & x_1-x_4 & x_3 \\
x_3-x_4-x_1 & x_2 \\
x_4 & x_3-x_2-x_1 \end{bmatrix} \) corresponds to an evident linear map from \( T_p \oplus T_p \oplus T_p \oplus T_p \) to itself. Check \( M \cdot M^T = (x_1^2 + x_2^2 + x_3^2 + x_4^2)I \) where \( I \) is the 4 by 4 identity matrix. Hence \( M \) defines a bijection which can be checked to give an isometry between \( \psi_p \perp \psi_p \perp \psi_p \perp \psi_p \) and \(-\psi_p \perp -\psi_p \perp -\psi_p \perp -\psi_p \).

If \( x_3 = x_4 = 0 \) or \( x_2 = x_3 = x_4 = 0 \) the evident square submatrix gives the required isometry.

Connolly produced the \( x_i \) by an appeal to Lagrange’s four–squares theorem. Here is a different approach which uses only quadratic reciprocity. If we could solve

\[
(1.17) \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv -1 \mod p^\ell
\]

for all large \( \ell \) we would be done.

If \((x_1, x_2, x_3, x_4)\) satisfies (1.17) for \( p \) odd and \( \ell \geq 1 \), then we can find \( a \) such that \((x_1 + ap \ell)^2 + x_2^2 + x_3^2 + x_4^2 \equiv -1 \mod p^{\ell+1} \). If \( p = 2 \) and if \((x_1, x_2, x_3, x_4)\) satisfies (1.17) for \( p = 2 \) and \( \ell \geq 3 \), then we can find \( a \) so that \((x_1 + a2^{\ell-1})^2 + x_2^2 + x_3^2 + x_4^2 \equiv -1 \mod 2^{\ell+1} \). This construction is just Hensel’s lemma for this simple case.

Note (1,1,1,2) satisfies (1.17) for \( p = 2, \ell = 3 \). If \( p \equiv 1 \mod 4 \), then \(-1\) is a quadratic residue so we can solve \( x_1^2 \equiv -1 \mod p \). If \( p \equiv 3 \mod 4 \), \( x_1^2 + x_2^2 \equiv -1 \mod p \) has solutions since there are \( \frac{p-1}{2} \) distinct values for \(-1 - x_2^2 \) and only \( \frac{p-1}{2} \) non–residues.

Brown [2] studied the case in which \( T \) is a \( \mathbb{Z}/2\mathbb{Z} \) vector space. Any 2–linear function on a \( \mathbb{Z}/2\mathbb{Z} \) vector space is quadratic and Brown’s functions were assumed non–singular, although tame would have sufficed for many of his results. For example, Brown gave a different argument for \( \beta(\psi) \in \mathbb{Z}/8\mathbb{Z} \). An element \( \omega \in T \) is characteristic provided \( b(x, x) = b(\omega, x) \) for all \( x \in T \). Brown’s argument is to observe that characteristic elements exist and then \( \psi_\omega = -\psi \). By (1.14), if \( \psi \) is tame, \( \beta(\psi) = \psi(\omega) + \beta(\psi_\omega) \in \mathbb{Q}/\mathbb{Z} \), so \( 2\beta(\psi) = \psi(\omega) \in \mathbb{Z}/4\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} \).

For later use, recall the Gauss sum formula of Dirichlet [3] that we require.

**Theorem 1.18.** If \( m > 0 \) then

\[
\sum_{s=0}^{m-1} e^{2\pi is^2/m} = \begin{cases} 
(1 + i)\sqrt{m} & m \equiv 0 \mod 4 \\
\sqrt{m} & m \equiv 1 \mod 4 \\
0 & m \equiv 2 \mod 4 \\
i\sqrt{m} & m \equiv 3 \mod 4
\end{cases}
\]
Some elementary Galois theory allows us to extend (1.6). Let \( \psi: T \to \mathbb{Q}/\mathbb{Z} \) be a quadratic function on a finite \( p \)-group of order \( p^r \) and let \( a \in \mathbb{Z} \) be prime to \( p \). As we saw above, if \( a_1 \equiv a_2 \cdot s^2 \) mod \( p^{r+1} \), then \( \beta(a_1 \cdot \psi) = \beta(a_2 \cdot \psi) \). For odd \( p \) the integers mod \( p^{r+1} \) divide into two sets, the \( \mathbb{Q} \) is even, then the right hand side of (1.20) must again be integer, so

\[
\beta(a \cdot \psi) = a \cdot \beta(\psi) + \frac{\ell_2(a) \cdot e}{2}
\]

and for \( p = 2 \)

\[
\beta(a \cdot \psi) = \beta(\psi) + \frac{\ell_2(a) \cdot e}{2}
\]

**Proof.** It suffices to do the non-singular case since we can work with \( \psi_{\text{red}} \). Let \( \zeta \) be a primitive \( p^r \) root of unity where \( p^r \) annihilates \( \psi(t) \) for all \( t \in T \). Let \( \omega = e^{2\pi i / p^r} \).

Using (1.16) we see

\[
G(\psi) = p^r \omega^\sigma \quad \text{where } \beta(\psi) = \frac{\sigma}{8}
\]

Now the left hand side of (1.20) lies in \( \mathbb{Q}[\zeta] \) and hence so must the right. The integers relatively prime to \( p \) map onto the Galois group of \( \mathbb{Q}[\zeta] \) over \( \mathbb{Q} \). The map sends the integer \( a \) to \( \zeta^a \) so we need to study the effect of the Galois automorphism \( \gamma_a: \zeta \mapsto \zeta^a \) on \( G(\psi) \).

For \( p \) odd, \( \mathbb{Q}[\zeta] \cap \mathbb{Q}[\omega] = \mathbb{Q} \), so \( \pm 1 \) are the only powers of \( \omega \) in \( \mathbb{Q}[\zeta] \). For \( p \equiv 1 \mod 4 \) Dirichlet (1.18) shows \( \sqrt{p} \in \mathbb{Q}[\zeta] \) and one can check that \( \gamma_\eta(\sqrt{p}) = -\sqrt{p} \) for some \( \eta \) in the Galois group.

If \( e \) is even, then the right hand side of (1.20) must be an integer, so \( \beta(\eta \cdot \psi) = \beta(\psi) \). If \( e \) is odd, then the right hand side of (1.20) must be \( \sqrt{p} \) times an integer, so \( \beta(\eta \cdot \psi) = \beta(\psi) + \frac{1}{2} \). Additionally \( \beta(\psi) \) is either 0 or \( \frac{1}{2} \). Note both cases are covered by the formula (1.19)_p.

For \( p \equiv 3 \mod 4 \) Dirichlet (1.18) shows \( i\sqrt{p} \in \mathbb{Q}[\zeta] \) and \( \gamma_\eta(i\sqrt{p}) = -i\sqrt{p} \). If \( e \) is even, then the right hand side of (1.20) must again be
an integer, so $\beta(\eta \cdot \psi) = \beta(\psi)$. If $e$ is odd, then the right hand side of (1.20) must be \(\sqrt{p}\) times an integer, so $\beta(\eta \cdot \psi) = \beta(\psi) + \frac{1}{2}$ again and (1.19)$_p$ holds in this case too. Additionally $p \equiv 3 \mod 4$, $\beta(\psi) = \pm \frac{1}{4}$ if $e$ is odd and $0$ or $\frac{1}{2}$ if $e$ is even.

For $p = 2$, there are three multiplications to be worked out. Which class an integer $a$ belongs to can be determined by reducing $a$ mod 8. The numbers are $-1$ and $\pm 3$ mod 8. The affect of $-1$ we know: $\beta(-\psi) = -\beta(\psi)$. The wrinkle when we multiply by $\pm 3$ is that these Galois actions send $\sqrt{2}$ to $-\sqrt{2}$. If $e$ is even then this does not matter and $\beta(\eta \cdot \psi) = \eta \cdot \beta(\psi)$ When $e$ is odd we get $\beta(\eta \cdot \psi) = \eta \cdot \beta(\psi) + \frac{1}{2}$.

So for $p = 2$ (1.20) holds.

\[ \square \]

2. Some algebra applications.

**Theorem 2.1.** Given a symmetric bilinear form $B$ on a rational vector space, $V$, define $Q: V \rightarrow \mathbb{Q}$ by $Q(v) = \frac{B(v,v)}{2}$. Call a lattice integral if $B(v_1,v_2) \in \mathbb{Z}$ for all $v_1, v_2 \in L$. Pick a lattice $L \subset V$ such that $Q(x)$ is integral for all $x \in L$. Define $L^\# = \{ v \in V \mid B(v, \ell) \in \mathbb{Z} \forall \ell \in L \}$ and check that $L$ integral implies $L \subset L^\#$. If $L$ is non–singular, check that $L^\# / L$ is finite. Check that $Q$ induces a non–singular quadratic function $\psi_L: L^\# / L \rightarrow \mathbb{Q}/\mathbb{Z}$ which enhances the symmetric bilinear form $b_L: L^\# / L \times L^\# / L \rightarrow \mathbb{Q}/\mathbb{Z}$ induced by $B$. The Milgram Gauss Sum Formula \[11\] says

\[ (2.1) \quad \beta(\psi_L) = \frac{\sigma(B)}{8}, \]

where $\sigma(B)$ denotes the signature of $B$.

We prove the formula using some straightforward manipulations and Dirichlet’s Gauss sum formula. The basic outline, except for the appeal to Dirichlet, is in Milnor and Husemoller [12] who attribute it to Knebusch. A proof for det $B$ odd was given earlier by Blij [1].

**Proof.** Call a lattice $L$ acceptable if $Q(x) \in \mathbb{Z}$ for all $x \in L$. If $L_1$ and $L_2$ are acceptable lattices, so is $L_1 \cap L_2$. To show all acceptable lattices give the same answer, it suffices to show $\beta(\psi_{L_2}) = \beta(\psi_{L_1})$ under the additional assumption that $L_1 \subset L_2$, and hence $L_1 \subset L_2 \subset L_2^\# \subset L_1^\#$.

Let $T = L_1^\# / L_1$ and apply (1.12) to $K = L_2 / L_1 \subset T$. Check $K^\perp = L_2^\# / L_1$ so $K_{L_1} = L_2^\# / L_2$.

Over $\mathbb{Q}$, $B$ can be diagonalized and there are acceptable diagonal lattices, so it suffices to show $\beta((2m)) = \frac{1}{8}$ if $m > 0$. Now $G((2m)) = \sum_{s=0}^{2m-1} e^{2\pi i s \frac{2m}{m}}$. Dirichlet (1.18) in case $4m$ says $\sum_{s=0}^{4m-1} e^{2\pi i \frac{s^2}{4m}} = (1 +
\( (s + 2m)^2 = s^2 \mod 4m, \ (1 + i)\sqrt{4m} = 2 \cdot G(\langle 2m \rangle) \) and the result follows.

**Theorem 2.2.** Suppose as above that \( B \) is a non–singular bilinear form on a rational vector space \( V \) and \( L \) is an integral lattice with \( L^\# \) defined as in (2.1). Check \( B \) still induces a symmetric, bilinear form \( b_L \) on \( L^\#/L \) which is still non–singular. To apply the Milgram Gauss sum formula to \( L^\#/L \) we need additionally that \( B(v, v) \in 2\mathbb{Z} \) for all \( v \in L \). If \( L \) does not satisfy this condition we can proceed as follows. There exists a characteristic element \( \nu \in L\): i.e. \( B(v, v) \equiv B(\nu, v) \mod 2 \) for all \( v \in L \). The element \( \nu \) is not unique but pick one. Then set \( Q(x) = B(x, x) - B(\nu, x)^2 \) for all \( x \in L^\# \). Check that \( Q \) induces a quadratic function \( \psi_L : L^\#/L \rightarrow \mathbb{Q}/\mathbb{Z} \) and that \( \psi_L \) enhances \( b_L \).

\[
\beta(\psi_\nu) = \frac{\sigma(B)}{8} - \frac{B(\nu, \nu)}{8} \in \mathbb{Q}/\mathbb{Z}.
\]

**Proof.** Let \( L_1 = 2L \subset L \). Check that the function \( Q \) defined above induces a function \( \psi_{L_1} : L_1^\#/L_1 \rightarrow \mathbb{Q}/\mathbb{Z} \) which is an enhancement of \( b_{L_1} \). The function \( \psi_{L_1} \) is almost certainly not quadratic, but just as in the proof of (2.1) prove that \( \beta(\psi_\nu) = \beta(\psi_{L_1}) \). Note \( \psi_{L_1} + B(\omega, \nu/2) = \psi \) on \( L_1^\#/L_1 \) where \( \psi \) is the quadratic function induced by \( \frac{B(x, x)}{2} \) on \( L_1^#/L_1 \). Use (2.1) to calculate \( \beta(\psi) = \frac{\sigma(B)}{8} \) and use (1.14) to deduce that \( \beta(\psi_\nu) = \beta(\psi) - \psi(\nu/2) = \beta(\psi) - \frac{B(\nu, \nu)}{8} \).

**Remarks 2.3.** If \( \det B \) is odd, then \( \nu \) is unique up to sums with elements of the form \( 2x \). A classical argument shows \( B(\nu, \nu) \equiv B(\nu + 2x, \nu + 2x) \mod 8 \) and one checks that \( \psi_\nu \) does not depend on the choice of \( \nu \) either. For \( \det B \) odd (2.2) is a result of Blij [1]. The general case appears in Brumfiel&Morgan [3].

If \( \det B \) is even however, there are different choices for \( \nu \) which give different enhancements and different values of \( B(\nu, \nu) \). Indeed, it is a theorem of Brumfiel&Morgan [3] and Wall [16] that any quadratic enhancement of a non–singular symmetric bilinear form can be obtained as \( \psi_L \) for an appropriate \( B \) and \( \nu \).

We next turn to Brown [2] for other ways to obtain quadratic functions. Given a symmetric bilinear form, \( B : V \times V \rightarrow \mathbb{Z} \), define

\[
\psi_B : V \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}
\]
by $\psi_B(x) = \frac{B(x, x)}{4}$. This function is 2–linear, quadratic and the associated bilinear form is $B_2: (V \otimes \mathbb{Z}/2\mathbb{Z}) \times (V \otimes \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ obtained by reducing $B$ mod 2. There is an obvious generalization: let $V_m = V \otimes \mathbb{Z}/m\mathbb{Z}$ and define

$$\psi_{B,m}: V_m \to \mathbb{Z}/2m\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

with $\psi_{B,m}(x) = \frac{B(x, x)}{2m}$. In order for $\psi_{B,m}$ to be defined on $V_m$ it is necessary and sufficient that $m$ be even. This quadratic enhancement, even for $m = 2$, need not be tame. It is non–singular if and only if $\det B$ is relatively prime to $m$. A further generalization is to recall that for any quadratic $\psi: T \to \mathbb{Q}/\mathbb{Z}$ the function $B \otimes \psi: V \otimes T \to \mathbb{Q}/\mathbb{Z}$ defined by $(B \otimes \psi)(v \otimes x) = B(v, v)\psi(x)$ is also quadratic [12, p.111]. (One should also check that it really is defined. Note that the formula $B = \langle 1 \rangle$ implies that $\psi$ is quadratic so the formula does not work for non–quadratic enhancements.) If $m$ is even, let $1_{m}: \mathbb{Z}/m\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ be defined by $1_m(1) = \frac{1}{2m}$. Then $B \otimes 1_m = \psi_{B,m}$ so this generalizes Brown’s construction.

**Theorem 2.4.** If $B$ is a symmetric form over $\mathbb{Z}$ with determinant $\pm 1$ and if $\psi$ is tame quadratic, then

$$\beta(B \otimes \psi) = \sigma(B) \cdot \beta(\psi) \in \mathbb{Q}/\mathbb{Z}.$$  

*Proof.* As Brown remarks, [11,12] shows that $\beta(B \otimes \psi)$ only depends on the Witt class of $B$. Now the Witt ring of $\mathbb{Z}$ is infinite cyclic generated by the form $\langle 1 \rangle$. But $\langle 1 \rangle \otimes \psi = \psi$. □

**Theorem 2.5.** Another application of Dirichlet’s Gauss sum formula as in (2.1) shows that for positive even $m$, $\beta(1_m) = \frac{1}{8}$, so we get Brown’s theorem, $\beta(\psi_{B,2}) = \frac{\sigma(B)}{8}$.

**Theorem 2.6.** If $m$ is odd, define $1_m: \mathbb{Z}/m\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ by $1_m(1) = \frac{1}{m}$. If $m > 0$ and $m \equiv 1 \pmod{4}$, $\beta(1_m) = 0$ and if $m > 0$, $m \equiv 3 \pmod{4}$, $\beta(1_m) = \frac{1}{4}$.

In a bit, we will give another proof of (2.4) that bypasses the Witt ring calculation, but before that, we generalize a formula of Melvin and Kirby [7, p.522] for computing $\beta(\psi_{B,2})$ and use it to compute $\beta(B \otimes \psi)$ in favorable situations.

Given $\psi$, fix a positive integer $m$ so that all the values of $\psi$ applied to elements of $T$ lie in $\mathbb{Z}/m\mathbb{Z}$. When computing $\beta(B \otimes \psi)$ any two integer matrices which are the same mod $m$ clearly yield the same result. We can also split $\psi$ into its $p$–primary pieces and work with one prime at a
time since \((B \otimes \psi)_p = B \otimes \psi_p\), so let \(m = p^k\). We say \(B\) is \(p^r\)-similar to \(C\), written \(B \sim_{pr} C\), if there is an integral entry matrix \(M\) with \(\det M\) prime to \(p\) so that \(C \equiv M^{tr} BM \mod p^r\). We note that if \(B \sim_mC\) then \(M\) gives an isometry between \(B \otimes \psi\) and \(C \otimes \psi\), so \(\beta(B \otimes \psi) = \beta(C \otimes \psi)\) or neither is tame.

Call any integral matrix of the form \(H_{m_1,m_2} = \left(\begin{array}{cc} 2m_1 & 1 \\ 1 & 2m_2 \end{array}\right)\) pseudo–hyperbolic. Call any form which is an orthogonal sum of rank one forms and pseudo–hyperbolics reduced. A form which is an orthogonal sum of rank one forms will be called diagonal.

The usual Gram–Schmidt process can be applied mod \(p^r\) to yield the following algorithm for finding a reduced matrix \(p^r\)-similar to \(B\). If there is a diagonal entry \(\alpha\) in \(B\) generated by \(x\) with \(\alpha\) relatively prime to \(p\), the usual Gram–Schmidt process defines a projection \(p^r: V \otimes \mathbb{Q} \to V \otimes \mathbb{Q}\) by \(p(v) = v - \frac{b(x,v)}{b(x,x)} x\) and then observes that \(B\) is isometric to \(\langle \alpha \rangle \perp \operatorname{Im}(p)\). The only denominators in the matrix \(B'\) for \(\operatorname{Im}(p)\) are divisors of \(b(x,x)\) so we can find a positive integer \(r \equiv 1 \mod p^r\) so that \(rB'\) is integral and \(\langle \alpha \rangle \perp (rB')\) is congruent mod \(m\) to \(B\). Note \(B\) and \(\langle \alpha \rangle \perp (rB')\) are \(p^r\)-similar. Continue until we get to a new form \(B' = D_0 \perp B_0\) where \(D_0\) is diagonal, \(B_0\) has all diagonal entries divisible by \(p\) and \(B'\) is \(p^r\)-similar to \(B\).

Next suppose that some entry in \(B_0\) is prime to \(p\). This means there are basis elements \(x, y\) such that \(B(x,y)\) is prime to \(p\). If \(p\) is odd change the basis to \(x + y, x - y\) and note that there are now diagonal entries prime to \(p\).

If \(p = 2\) this does not work, but we can change the basis element \(y\) to \(ay\), \(a\) odd so that in this new basis \(B(x,y) = 1\). One can now orthogonally split off a pseudo–hyperbolic. Since before we split off the pseudo–hyperbolic all squares were even afterwards all diagonal entries will still be even.

We can continue these reductions until we have found a matrix \(\bar{B}_0 = R_0 \perp B_1\) with \(\bar{B}_0\) \(p^r\)-similar to \(B\), with \(R_0\) reduced and with every entry of \(B_1\) divisible by \(p\). Divide \(B_1\) by \(p\) and continue. Eventually we will obtain an orthogonal sum

\[
C_0 \perp pC_1 \perp \cdots \perp p^wC_w
\]

where \(C\) is \(p^r\)-similar to \(B\) and each \(C_i\) is reduced with \(\det C_i\) prime to \(p\). If we take \(r\) so that \(p^r\) is at least the largest power of \(p\) dividing \(\det B\) then \(w\) and the sizes of each of the \(C_i\) are determined by \(B\) since the \(p\) torsion in cokernel of \(C\) determines the \(w\) and the size of the \(C_i\)
and for this large an \( r \) the \( p \) torsion in the cokernel of \( C \) is isomorphic to the \( p \) torsion in the cokernel of \( B \).

For \( \det A \) prime to \( p \) define a mod 8 integer \( \sigma_p(A) \) by

\[
\sigma_p(A) \equiv \begin{cases} 
\text{rank } A & \text{if } p \text{ is odd} \\
N_1 - N_{-1} + 3N_3 - 3N_{-3} & \text{if } p = 2
\end{cases} \pmod{8}
\]

where \( N_i \) is the number of diagonal entries congruent to \( i \pmod{8} \) in any reduced matrix 8–similar to \( A \). It is not obvious that \( \sigma_p(A) \) is well–defined for \( p = 2 \) but this is checked in the proof of the next result.

**Theorem 2.8.** For each \( p \) and \( \psi: T \to \mathbb{Q}/\mathbb{Z} \) with \( \psi \) tame and quadratic, \( T \) a \( p \)-group with \( |T/T^\perp| = p^r \), and \( B \) a symmetric integral form with \( \det B \) prime to \( p \),

\[
\beta(B \otimes \psi) = \sigma_p(B) \cdot \beta(\psi) + \frac{\ell_p(\det B) \cdot e}{2}.
\]

**Proof.** We may pass to \( \psi^{\text{red}} \) so without loss of generality assume \( \psi \) in also non–singular. Note \( B \otimes \psi \) is also non–singular. We start by proving the formula under the additional assumptions that \( B \) is reduced and we have define \( \sigma_p(B) \) using \( B \) as our reduced form if \( p = 2 \). The desired formula is additive for orthogonal sum so it suffices to prove the result for \( \langle a \rangle \) plus the pseudo–hyperbolics if \( p = 2 \).

Since \( a \cdot \psi = \langle a \rangle \otimes \psi \) the formula for rank one forms is just a restatement of the formula in [1,19].

Next let \( H \) be a pseudo–hyperbolic. Working mod 8 one can show that

\[
\langle -1 \rangle \perp \begin{pmatrix} 2m_1 & 1 \\ 1 & 2m_2 \end{pmatrix} \sim 8 \langle 2m_1 - 1 \rangle \perp \langle 2m_2 - 1 \rangle \perp \langle 1 - 2m_1 - 2m_2 \rangle.
\]

Compute that

\[
\beta(H \otimes 1_2) = 0 \cdot \beta(1_2) + \frac{\ell_2(\det H) \cdot e}{2}.
\]

Now work mod \( 2^r \), \( r > 3 \), and suppose \( \langle -1 \rangle \perp \begin{pmatrix} 2m_1 & 1 \\ 1 & 2m_2 \end{pmatrix} \sim 2^r \langle a_1 \rangle \perp \langle a_2 \rangle \perp \langle a_3 \rangle \). Calculating \( \beta(H \otimes 1_2) \) using this decomposition shows \( 1 + a_1 + a_2 + a_3 \equiv 0 \pmod{8} \). It then follows that, for any tame \( \psi \),

\[
\beta(H \otimes \psi) = \beta(\psi) \big( 1 + a_1 + a_2 + a_3 \big) + \frac{\ell_2(a_1 \cdot a_2 \cdot a_3) \cdot e}{2} = \frac{\ell_2(\det H) \cdot e}{2}.
\]
But $\sigma_2(H) = 0$ by definition. Hence the formula holds for pseudo–hyperbolics.

Note next that if $B \sim_{p^r} C$, then $\ell_p(\det B) = \ell_p(\det C)$, provided $r \geq 3$ if $p = 2$.

If $p$ is odd but $B$ is not necessarily reduced, chose a reduced $C$ with $C$ $p^r$–similar to $B$ and with $p^r \cdot \psi(t) = 0$ for all $t \in T$. Then $\beta(B \otimes \psi) = \beta(C \otimes \psi)$ and for odd primes clearly $\sigma_p(B) = \sigma_p(C)$ so the formula holds for odd primes.

Now suppose $C_1$ and $C_2$ are two matrices $8$–similar to $B$. Then $\beta(B \otimes 1_4) = \beta(C_1 \otimes 1_4) = \frac{\sigma_2(C_1)}{8}$, so $\sigma_2(C_1) \equiv \sigma_2(C_2) \mod 8$ and we let $\sigma_2(B)$ denote this common value. For any $\psi$ and $B$ as in theorem, choose a reduced $C$ $2^r$–similar to $B$ with $2^r \cdot \psi(t) = 0$ for all $t \in T$ and just as in the odd case $\beta(B \otimes \psi) = \beta(C \otimes \psi)$. We have just checked that $\sigma_2(B) = \sigma_2(C)$ and $\ell_2(\det B) = \ell_2(\det C)$.

We can now prove (2.4) directly from 2.8. It suffices to prove (2.4) to prove (2.4) a prime at a time. For $p \equiv 1 \mod 4$, $\beta(\psi)$ is a multiple of $\frac{1}{2}$ and $\ell_p(1) = 0$. Since rank $B \equiv \sigma(B) \mod 2$, (2.4) holds. For $p \equiv 3 \mod 4$, $\beta(\psi)$ is a multiple of $\frac{1}{2}$: recall rank $B \equiv \sigma(B) \mod 4$ if $\det B > 0$ and rank $B \equiv \sigma(B) + 2 \mod 4$ if $\det B < 0$. Since $\ell_p(-1) = 1$ (2.4) holds again.

For $p = 2$ we adopt a different approach. Recall that we calculated $\beta(B \otimes 1_4) = \frac{\sigma_2(B)}{8}$. But we can calculate $\beta(B \otimes 1_4)$ directly using the Milgram Gauss sum formula. First note that $2V$ is an acceptable lattice in $V \otimes \mathbb{Q}$ and since $\det B = \pm 1$, $(2V)^* = \frac{1}{2}V$. Calculate that $V^*/V \cong V \otimes \mathbb{Z}/4\mathbb{Z}$ and under this isomorphism the enhancement on $V^*/V$ is the one on $B \otimes 1_4$. Since $\det B = \pm 1$, $\ell_2(\det B) = 0$ so $\sigma(B) \equiv \sigma_2(B) \mod 8$.

Kirby and Melvin [7] introduced this reduction for $p^r = 4$ including the removal of pseudo–hyperbolics by orthogonally summing with $1 \perp (-1)$. If there are any diagonal entries of the form (2) they observe that $\psi_{B,2}$ is not tame; otherwise it is. Then they compute that

$$\beta(\psi_{B,2}) = \frac{n_1 - n_3 + 4\epsilon_H}{8} \in \mathbb{Q}/\mathbb{Z}$$

where $n_1$ is the number of diagonal entries congruent to 1 mod 4; $n_3$ is the number of diagonal entries congruent to 3 mod 4; and $\epsilon_H$ is the number of pseudo–hyperbolics congruent to $\begin{pmatrix} 2 & \pm 1 \\ \pm 1 & 2 \end{pmatrix}$ mod 4.

For a proof, let $C \sim_4 B$ where $C = R_0 \perp 2R_1 \perp \cdots$ and $B \otimes 1_2 = R_0 \otimes 1_2 \perp R_1 \otimes (2 \cdot 1_2)$. Now $n_2$ is the number of diagonal summands of $R_1$. If $n_2 > 0$ the second summand is not tame and so neither is
Given two quadratic enhancements on \( \mathbb{Z} \) vector spaces. We see no hope for defining a product in general, but with \( \det \) not hard to verify that if \( \beta(B,2) = \beta(R_0 \otimes 1_2) = \frac{n_1-n_3+4(N_3-N_3)}{8} + \frac{\ell_2(\det R_0)}{2} \) and one checks that \( \ell_2(\det R_0) + N_3 + N_{-3} \) is \( \epsilon_H \mod 2 \).

Even though we can compute \( \sigma_2 \) by working mod 4 it is worth considering the mod 8 calculation since for \( \det B \) odd we need only do the reduction until all the diagonal elements are even, we need not actually split off the pseudo–hyperbolics to compute \( \sigma_2 \). As an extreme example, if \( B \) is a symmetric matrix over \( \mathbb{Z} \) with all diagonal entries even and with \( \det B \) odd, then \( \sigma_2(B) = 0 \).

**Remark 2.9.** Given a symmetric matrix over \( B \) with even diagonal entries we can find matrices \( S \) such that \( S = S^{tr} \).

The function \( \psi: V \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \) given by \( \psi(x) = \frac{e^{2\pi ix}}{2} \) is a quadratic enhancement of \( B \): in fact \( \psi = B \otimes 1_2 \). If \( \det B \) is odd, it follows that \( \beta(\psi) = \beta(\psi_{B,2}) = \frac{\ell_2(\det B)}{2} \). If \( S \) is a Seifert matrix for a knot \( S^{4k-3} \) in \( S^{4k-1} \), then \( \beta(\psi) \) is the Arf invariant of the knot, \( \det B = \Delta(-1) \) is the value of the Alexander polynomial of the knot evaluated at \( -1 \) and \( \beta(\psi) = \frac{\ell_2(\Delta(-1))}{2} \) is a theorem of Levine [9, p.544]. In [14] we employed a slightly different variant of the Gram–Schmidt process to classify arbitrary non–singular quadratic functions on a \( p \) group: Thm. 3.5 (p.268) asserts that any non–singular \( \psi_p \) on a finite abelian \( p \) group is an orthogonal sum.

**Theorem 2.10.**

\[
\psi_p = R_1 \otimes 1_p \perp \cdots \perp R_k \otimes 1_{p^k}
\]

where each \( R_i \) is reduced and has \( \det R_i \) prime to \( p \). Furthermore, \( \text{rank } R_i \) depends only on \( T \) and

\[
\beta(\psi_p) = \sum_{s=1}^{k} \beta(R_s \otimes 1_{p^s}) = \sum_{s=1}^{k} \left( \sigma_p(R_s) + \frac{\ell_p(\det R_s) \cdot s}{2} \right).
\]

**Remark 2.11.** Brown [2] defined a product on enhancements on \( \mathbb{Z}/2\mathbb{Z} \) vector spaces. We see no hope for defining a product in general, but given two quadratic enhancements on \( \mathbb{Z}/p\mathbb{Z} \) vector spaces, say \( \psi_1 = R \otimes 1_p \) and \( \psi_2 = S \otimes 1_p \), define \( \psi_1 \cdot \psi_2 \) to be \( (R \otimes S) \otimes 1_p \). Check that this product is well defined. It clearly is commutative and it is not hard to verify that if \( \beta(\psi_1) = \frac{\alpha_1}{8} \) then \( \beta(\psi_1 \cdot \psi_2) = \frac{\alpha_1 \cdot \alpha_2}{8} \). When \( p = 2 \) this is Brown’s product and Brown’s theorem.
Given a form $B$ and a quadratic enhancement $\psi$ we can work out $(B \otimes \psi)_p$ by finding a matrix $p^r$–similar to one as in (2.7) and a quadratic enhancement $\psi_p$ decomposed as in (2.10), it is clear that $(B \otimes \psi)_p$ is an orthogonal sum of terms of the form $(C_i \otimes R_j) \otimes (p^i \cdot 1_{p^j})$. Hence, to describe $B \otimes \psi$ or even just $p^i \cdot \psi_p$ it suffices to describe $p^i \cdot 1_{p^r}$. Let $T = \mathbb{Z}/p^r \mathbb{Z}$ denote the domain of $p^i \cdot \psi_p$. This form has $T^\perp = p^{r-i}T$.

If $p$ is odd $p^i \cdot 1_{p^r}$ is tame and $(p^i \cdot 1_{p^r})^{\mathrm{red}} = 1_{p^{r-i}}$. (Or trivial if $i \geq r$).

If $p = 2$ this still works except if $r = i$: the form $2^r \cdot 1_{2^r}$ is not tame, although $H_{m,n} \otimes (2^r \cdot 1_{2^r}) = 2^r \cdot (H_{m,n} \otimes 1_{2^r})$ is trivial, hence tame.

Hence $2^i \cdot \psi_2$ is tame if and only if $R_i$ has no diagonal summands. If $p^i \cdot \psi_p$ is tame, we have

\begin{equation}
\beta((B \otimes \psi)_p) = \sum_{i<j} \sigma_p(C_i) \cdot \sigma_p(R_j) + \frac{\sigma_p(C_i) \cdot \ell_p(R_j) + \ell_p(C_i) \cdot \sigma_p(R_j)}{2} \cdot (j - i)
\end{equation}

The result can be packaged a little better.
Let \( \sigma'_p(A) = \sigma_p(A) + 4 \cdot \ell_p(\det A) \). Then
\[
(2.14)' \quad \beta((B \otimes \psi)_p) = \sum_{i \neq j \mod 2} \sum_{i < j} \frac{\sigma_p(C_i) \cdot \sigma_p(R_j)}{8} + \frac{\sigma'_p(C_i) \cdot \sigma'_p(R_j)}{8}
\]

For \( p = 2 \) the argument for invariance fails, as does the result: \( 1_4 \perp 1_2 \) is isometric to \( \langle 3 \rangle \otimes 1_4 \perp \langle 3 \rangle \otimes 1_2 \) but \( \beta(1_4) = \frac{1}{8} \neq \beta(\langle 3 \rangle \otimes 1_4) = \frac{3}{8} \).

But just make any choice of \( C_i \) and \( R_j \) and then compute. The only problem occurs for terms of the form \( C_i \otimes R_i \) (same subscripts): if \( N_1 + N_3 + N_5 + N_7 \) for \( C_i \) is not 0 and if \( N_1 + N_3 + N_5 + N_7 \) for \( R_i \) is not 0, then \( B \otimes \psi \) will not be tame. Otherwise \( B \otimes \psi \) will be tame both \((2.14)\) and \((2.14)'\) hold without change. This generalizes the Kirby\&Melvin calculation: their \( \psi = 1_2 \) so \( R_1 = \langle 1 \rangle \) and \( C = C_0 \perp 2C_1 \perp \cdots \). For \( R_1, N_1 + N_3 + N_5 + N_7 = 1 \) so to be tame we must have \( N_1 + N_3 + N_5 + N_7 = 0 \) for \( C_1 \), which translates to no \( \langle 2 \rangle \)'s in \( C \). Moreover \( \sigma'_2(R_1) = 1 \) and \( \sigma'_2(C_0) = n_1 - n_3 + 4\epsilon_H \).

3. Topology applications.

**Theorem 3.1. (A theorem of Morita [13])** Let \( X \) be a 4k dimensional, oriented, connected Poincaré duality space without boundary. The Pontrjagin square
\[
\mathcal{P} : H^{2k}(X; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z}
\]
is a quadratic enhancement of the cup product pairing \( H^{2k}(X; \mathbb{Z}/2\mathbb{Z}) \times H^{2k}(X; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \). Brown conjectured and Morita proved that
\[
\beta(\mathcal{P}) = \frac{\sigma(X)}{8} \in \mathbb{Q}/\mathbb{Z}
\]
where \( \sigma(X) \) denotes the signature of \( X \).

For the proof, let \( K \) denote the image of the torsion in \( H^{2k}(X; \mathbb{Z}) \) in \( H^{2k}(X; \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \). An observation of Massey [10] which he claims was well-known at the time says that, in our notation,
\[
K^\perp / K = (H^{2k}(X; \mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}/2\mathbb{Z}.
\]

Recall that on \( H^{2k}(X; \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \subset H^{2k}(X; \mathbb{Z}/2\mathbb{Z}) \) the Pontrjagin square is just the cup product square reduced mod 4. This shows that \( \mathcal{P} \) vanishes on \( K \) and that the enhancement induced by \( \mathcal{P} \) on \( K^\perp / K \) is the mod 4 quadratic enhancement of the cup product form \( B \) on \( H^{2k}(X; \mathbb{Z})/\text{torsion} \). Now [2.4] completes the proof. \( \square \)
Theorem 3.2. Suppose $X$ is a $4k$ dimensional, oriented, connected Poincaré duality space with boundary. The Pontrjagin square

$$\mathcal{P}: H^{2k}(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/4\mathbb{Z}$$

is a quadratic enhancement of the cup product pairing. If $H^{2k}(\partial X; \mathbb{Z})$ is torsion free, then

$$\beta(\mathcal{P}) = \frac{\sigma(X)}{8} \in \mathbb{Q}/\mathbb{Z}$$

Several differences arise in the bounded case. The first is that the cup product pairing has an annihilator: if $T = H^{2k}(X, \partial X; \mathbb{Z}/2\mathbb{Z})$, $T^\perp$ is the image of $H^{2k-1}(\partial X; \mathbb{Z}/2\mathbb{Z})$ in $H^{2k}(X, \partial X; \mathbb{Z}/2\mathbb{Z})$. A theorem of Thomas [15] calculates that the compositions

$$H^{2k-1}(\partial X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{2k}(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mathcal{P}} H^{4k}(X, \partial X; \mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$$

are equal. Note that if $H^{2k}(\partial X; \mathbb{Z})$ is torsion–free, $Sq^1 x = 0$ so $\mathcal{P}$ is tame in this case. (But not in general: any oriented boundary for an $RP^{2k-1}$ is going to have $\mathcal{P}$ not tame.) We can identify $T/T^\perp$ with the image of $H^{2k}(X, \partial X; \mathbb{Z}/2\mathbb{Z})$ in $H^{2k}(X, \mathbb{Z}/2\mathbb{Z})$.

Let $I_X$ denote the image of $H^{2k}(X, \partial X; \mathbb{Z})$ in $H^{2k}(X, \mathbb{Z})$. Since $H^{2k}(\partial X; \mathbb{Z})$ is torsion–free, $I_X \otimes \mathbb{Z}/2\mathbb{Z} \subset T/T^\perp$. Let $K \subset T/T^\perp$ denote the image of the torsion of $I_X$ in $T/T^\perp$.

A straightforward generalization of Massey’s observation shows that $K^\perp = I_X$. Let $B_X$ denote the matrix for the form on $I_X/torsion$ so $\sigma(X) = \sigma(B_X)$. Since $H^{2k}(\partial X; \mathbb{Z})$ is torsion–free, det $B_X = \pm 1$ and just as in 3.1 $\beta(\mathcal{P}) = \beta(\psi_{B_X,4})$. 2.4 completes the proof. 

The function $x \cup Sq^1 x$ arises in other contexts. It is the squaring homomorphism associated to the bilinear form

$$R: H^k(M; \mathbb{Z}/2\mathbb{Z}) \times H^k(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$$

defined by $R(x, y) = \langle Sq^1 x \cup y, [M] \rangle$ on any oriented $2k+1$ dimensional manifold. The form $R$ is symmetric.

Let $M^3$ be a closed 3 manifold with a fixed Spin structure. In [8, p.209], we showed how to quadratically enhance the linking form $\ell$ on $H^2(M; \mathbb{Z})$. Let $\psi$ denote this enhancement. We further showed that $\beta(\psi)$ is just Rochlin’s $\mu$–invariant of $M$ mod 8. Spin structures on $M$ are acted on by $H^1(M; \mathbb{Z}/2\mathbb{Z})$. For $x \in H^1(M; \mathbb{Z}/2\mathbb{Z})$, define $\bar{\mu}(x)$ to be the difference of the $\mu$ invariant for $M$ with its given Spin structure minus the $\mu$ invariant for $M$ with Spin structure obtained by acting via $x$, all reduced mod 8. We showed that $\bar{\mu}$ is a quadratic enhancement of $R$. (See the remark after formula 4.9, p.213 [8]).
Now in general $R$ is singular: in fact $R^\perp$ is precisely the kernel of $S_Q^1$. We see that the image of $H^1(M;\mathbb{Z})$ in $H^1(M;\mathbb{Z}/2\mathbb{Z})$ acts trivially on the quadratic enhancement on $\ell$ so we get an action of $H^1(M;\mathbb{Z}/2\mathbb{Z})/H^1(M;\mathbb{Z})$ on this set. This group is identified via the integral Bockstein $\delta$ with $2H^2(M;\mathbb{Z})$ and $R$ naturally induces a bilinear form on $2H^2(M;\mathbb{Z})$: the enhancement $\hat{\mu}$ also extends to $2H^2(M;\mathbb{Z})$.

In (1.15) we remarked that $2H^2(M;\mathbb{Z})$ acts on the quadratic enhancements of the linking form, and a comparison of (1.15) and the enhancement $\hat{\mu}$ shows that $\hat{\mu}(x) = \Delta_\beta(\delta x)$ for any $x \in H^1(M;\mathbb{Z}/2\mathbb{Z})$. So the quadratic enhancement on $R$ is just the quadratic enhancement of the linking form restricted to $2H^2(M;\mathbb{Z})$.

**Theorem 3.3.** If the torsion subgroup of $H_1(M^3;\mathbb{Z})$ is a $\mathbb{Z}/2\mathbb{Z}$ vector space, then the $\mu$ invariant mod 8 is $\beta(\hat{\mu})$ since $\hat{\mu} = \psi$.

**Remark 3.4.** In general, $\hat{\mu}$ is tame unless $4(H^2(M^3;\mathbb{Z}))$ contains an $x$ with $\ell(x,x) = \pm \frac{1}{4}$, in which case $\hat{\mu}$ is not tame: e.g. the lens spaces $L(4,\pm 1)$.

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