Nonlinear vibrations of a cylindrical shell in a fractional viscoelastic medium with combinational internal resonances of the second order

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Abstract. Non-linear damped vibrations of a cylindrical shell embedded into a fractional derivative medium are investigated numerically for the case of the combinational internal resonance of the second order, using two different numerical methods with further comparison of the results obtained. The damping properties of the surrounding medium are described by the fractional derivative Kelvin-Voigt model utilizing the Riemann-Liouville fractional derivatives. A good agreement in results is declared.

1. Introduction
The analysis of forced nonlinear vibrations of thin cylindrical shell is the important area of applied mechanics, since shell elements are widely used in industry and technology [1]. The phenomena of internal resonance and energy exchange are very interesting fields for many researchers, because in low damping vibrations the energy interchange occurs between different vibratory modes [2]. It will suffice to mention the state-of-the-art articles [2-3] and the monograph [4] involving the extensive review of literature in the field of internal resonances in different mechanical systems. Various types of the internal resonance: one-to-one, two-to-one, three-to-one, as well as a variety of combinational resonances, when three and more natural modes interact, have been discussed. The enumerated internal resonances were investigated in various mechanical systems with multiple degree-of-freedom, as well as in strings, beams, plates, and shells.

Free damped nonlinear vibrations of cylindrical shells in a fractional derivative medium have been studied in [5], wherein the procedure resulting in decoupling linear parts of nonlinear equations has been proposed with the further utilization of the generalized method of multiple scales for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales.

It has been shown that the phenomenon of the internal resonance between vibrational subsystems of the cylindrical shell under consideration could be very critical, since in the circular cylindrical shell of such a type the two-to-one [5], one-to-one, three-to-one [6] internal resonances, as well as combinational internal resonances [7] could occur, which are governed by the order of smallness of viscosity. All possible cases of the internal resonance have been recently revealed in [7], which belong to the resonances of the constructive type, since all of them depend on the geometrical dimensions of the shell under consideration and its mechanical characteristics, that is why such resonances could not be ignored and eliminated for a particularly designed shell. It has been shown that the energy exchange could occur between two or three subsystems at a time: normal vibrations of the shell, its
torsional vibrations and shear vibrations along the shell axis. Such an energy exchange, if it takes place for a rather long time, could result in crack formation in the shell, and finally to its failure. The energy exchange has been illustrated pictorially by the phase portraits, wherein the phase trajectories of the phase fluid motion had been depicted [5-7].

In the present paper, we are going to analyze the nonlinear vibrations of a fractionally damped cylindrical shell under the conditions of a combinational internal resonance of the second order, when a certain natural frequency of vibrations $\Omega_1$ is equal to the difference between the frequency $\Omega_2$ and twice the frequency of $\Omega_3$, using two different numerical methods [8]. In the first method, the generalized displacements of a coupled set of nonlinear ordinary differential equations of the second order are estimated using the numerical solution of nonlinear multi-term fractional differential equations by the procedure based on the reducing of the problem to a system of fractional differential equations. According to the second method, the amplitudes and phases of nonlinear vibrations are estimated from the governing nonlinear differential equations describing amplitude-and-phase modulations for the case of the combinational internal resonance using the Runge-Kutta fourth order method.

2. Problem formulation

Let us consider the dynamic behavior of a simply-supported thin cylindrical shell with geometrical nonlinearities, vibrations of which in a viscoelastic fractional derivative medium are described in the cylindrical coordinates by the following three differential equations written in the dimensionless form [5]:

\[
\begin{aligned}
\frac{1}{2}u_{xx} + \frac{1-\nu}{2} \beta_i^2 u_{\varphi \varphi} + \frac{1+\nu}{2} \beta_i^2 w_{\varphi \varphi} - \nu \beta_i^2 w_{\varphi} + w_{\varphi} \left( w_{xx} + \frac{1-\nu}{2} \beta_i^2 w_{\varphi \varphi} \right) \\
+ \frac{1+\nu}{2} \beta_i^2 w_{x \varphi} = \ddot{u} + \omega_j \left( \frac{d}{dt} \right)^{\gamma} u,
\end{aligned}
\]

(1)

\[
\begin{aligned}
\beta_i^2 v_{\varphi \varphi} + \frac{1-\nu}{2} v_{xx} + \frac{1+\nu}{2} \beta_i^2 w_{\varphi \varphi} - \nu \beta_i^2 w_{\varphi} + \beta_i^2 w_{\varphi} \left( \beta_i^2 w_{\varphi \varphi} + \frac{1-\nu}{2} w_{xx} \right) \\
+ \frac{1+\nu}{2} \beta_i^2 w_{x \varphi} = \ddot{v} + \omega_j \left( \frac{d}{dt} \right)^{\gamma} v,
\end{aligned}
\]

(2)

\[
\begin{aligned}
\frac{\beta_i^2}{12} \left( w_{xxxx} + 2 \beta_i^2 w_{x \varphi \varphi \varphi} + \beta_i^2 w_{\varphi \varphi \varphi \varphi} \right) - \nu \beta_i^2 u_x - \beta_i^2 v_{\varphi} + \beta_i^2 w \\
+ \frac{1}{2} \nu \beta_i \left( w_{\varphi \varphi} \right)^2 + \frac{1}{2} \beta_i \left( w_{\varphi} \right)^2 - w_{\varphi} \left( u_{xx} + \frac{1-\nu}{2} \beta_i^2 u_{\varphi \varphi} + \frac{1+\nu}{2} \beta_i^2 v_{\varphi \varphi} \right) \\
- \beta_i \left( w_{\varphi \varphi} \left( \beta_i^2 w_{\varphi \varphi} + \frac{1-\nu}{2} v_{\varphi} + \frac{1+\nu}{2} \beta_i u_{\varphi} \right) - w_{xx} \left( u_x + \nu \beta_i^2 v_{\varphi} - \nu \beta_i w \right) \\
- \beta_i^2 w_{\varphi \varphi} \left( u_x + \beta_i^2 v_{\varphi} - \beta_i w \right) - (1-\nu) \beta_i w_{x \varphi} \left( \beta_i u_{\varphi} + v_{\varphi} \right) \\
= -\ddot{w} - \omega_j \left( \frac{d}{dt} \right)^{\gamma} w,
\end{aligned}
\]

(3)

subjected to the initial conditions

\[
\begin{aligned}
&u \big|_{t=0} = v \big|_{t=0} = w \big|_{t=0} = 0 , \quad \dot{u} \big|_{t=0} = \dot{v} \big|_{t=0} = \dot{w} \big|_{t=0} = 0 ,
\end{aligned}
\]

(4)

as well as the boundary conditions
where the $x$-axis is directed along the axis of the cylinder, $r$ and $\phi$ are the polar radius and angle, respectively, $u = u(x, \phi, t)$, $v = v(x, \phi, t)$, and $w = w(x, \phi, t)$ are the displacements functions of points located in the shell’s middle surface in three mutually orthogonal directions $x, \phi, r$, $\beta_i = l/R$ and $\beta_\phi = h/l$ are parameters defining the dimensions of the shell, $h$ is the thickness, $l$ is the length, $R$ is the radius, $t$ is time, an over dot denotes the time-derivative, lower indices label the derivatives with respect to the corresponding coordinates, $(d/dt)^\gamma$ is the fractional order of the operator of differentiation which is equivalent to the Riemann-Liouville fractional derivative of the $\gamma$-order [9]

$$D^\gamma F = \frac{d}{dt} \int_0^t (t - t') \frac{F(t' - t')}{\Gamma(1 - \gamma)} \, dt'.$$

To solve the partial differential equations (1)-(3) subjected to the initial (4) and boundary (5) conditions, we will use the approach suggested in [5], which involves the decoupling of linear parts of Eqs. (1)-(3) with further utilization of the generalized method of multiple time scales [7] to expand the amplitude functions into power series in terms of the small parameter. Assuming thereafter that the vibrational process occurs in such a way that only three natural modes corresponding to the generalized displacements $X_{imm}$ $(i=1,2,3)$ predominate and are coupled by the combinational resonance, resulting in the energy transfer between the coupled modes.

Thus, utilizing the fractional derivative Kelvin-Voigt model and the procedure proposed in [3,4] for decoupling the linear parts of differential equations of motion of the cylindrical shell, the system of equations of three predominating modes of vibrations, which could be coupled by some conditions of internal resonance, have been derived in [3,4] and written in the following dimensionless form:

$$\ddot{X}_{1mn} + \alpha_1 D^\gamma X_{1mn} + \Omega_{1mn}^2 X_{1mn} = -\sum_{i=1}^3 F_{imm} l_{imm}^1,$$

$$\ddot{X}_{2mn} + \alpha_2 D^\gamma X_{2mn} + \Omega_{2mn}^2 X_{2mn} = -\sum_{i=1}^3 F_{imm} l_{imm}^2,$$

$$\ddot{X}_{3mn} + \alpha_3 D^\gamma X_{3mn} + \Omega_{3mn}^2 X_{3mn} = -\sum_{i=1}^3 F_{imm} l_{imm}^3,$$

where $m$ and $n$ are integers, $\Omega_{1mn}$, $\Omega_{2mn}$ and $\Omega_{3mn}$ are the eigenvalues of the matrix $S_{ij}^{mn}$ and $l_{imm}^1$, $l_{imm}^2$ and $l_{imm}^3$ are eigenvectors of the same matrix $S_{ij}^{mn}$ with the elements

$$S_{ij}^{mn} = \begin{bmatrix}
S_{11}^{mn} & S_{12}^{mn} & S_{13}^{mn} \\
S_{21}^{mn} & S_{22}^{mn} & S_{23}^{mn} \\
S_{31}^{mn} & S_{32}^{mn} & S_{33}^{mn}
\end{bmatrix} = \begin{bmatrix}
\left(\pi^m n^2 + \frac{1}{2} \beta_i^2 n^2 \right) & \frac{1}{2} \beta_i \pi mn & \frac{\pi^m n^2 + \beta_i n^2}{\beta_i^2} \\
\frac{1}{2} \beta_i \pi mn & \left(\frac{1}{2} \pi^m n^2 + \beta_i^2 n^2 \right) & \beta_i n \\
\frac{\pi^m n^2}{\beta_i \pi mn} & \beta_i n & \frac{\beta_i^2}{12} \left(\pi^m n^2 + \beta_i n^2 \right)^2 + \beta_i
\end{bmatrix}$$
3. Methods of solution

Utilizing the procedure described in detail in [7], we will solve equations (7)-(9) using two different methods for the case of the difference combinational internal resonance \( \Omega_1 = \Omega_2 - 2\Omega_3 \).

The first method resides in the discretization of all derivatives in equations (7)-(9). For this purpose, we will follow the procedures suggested by Diethelm in [10,11].

Let us introduce the following notation:

\[
Y_1 = X_1, \quad Y_2 = D'X_1 = D'Y_1, \quad Y_3 = DX_1 = DY_1, \\
\dot{X}_1 = DDX_1 = DY_3 = -\sum_{i,m}F_{i,m}l_{i,m} - \alpha_1 Y_2 - \Omega^2 Y_1
\] (11)

The first time-derivative could be discretized using the trapezoidal rule

\[
DY_i = Y_{i+1} - Y_{i-1} = \frac{1}{2}h(Y_{i+1} + Y_{i-1}),
\] (12)

therefore

\[
Y_{i+1} - \frac{1}{2}hY_{i-1} = \alpha_1 Y_{i+1} + Y_{i-1} = s_{2,i+1}
\] (13)

The fractional derivative could be discretized using the Diethelm’s method [10,11] according to the definition

\[
D^\gamma Y_1 = \frac{1}{\Gamma(1 - \gamma)} \left( \sum_{k=0}^{\infty} \frac{Y_{1-k} + \gamma Y_{1-k-1}}{\gamma} \right) = \frac{Y_1}{\gamma},
\] (14)

whence it follows

\[
\gamma \chi_i Y_2 - \frac{1}{\gamma} \alpha_{bi} Y_{i-1} = \left( \sum_{k=0}^{\infty} \frac{Y_{i-k} + \gamma Y_{i-k-1}}{\gamma} \right),
\] (15)

Assuming that \( s_{1,i+1} = \left( \sum_{k=0}^{\infty} \frac{Y_{1-k} + \gamma Y_{1-k-1}}{\gamma} \right) \) and applying the described above procedure for other two equations, equations (7)-(9) could be transformed in the following matrix form:

\[
\begin{bmatrix}
-\gamma \alpha_{hi} & \gamma \chi_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{h}{2} \Omega^2_i & \frac{h}{2} \chi_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma \alpha_{hi} & \gamma \chi_i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\frac{h}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{h}{2} \Omega^2_i & \frac{h}{2} \chi_i & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma \alpha_{hi} & \gamma \chi_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{h}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{h}{2} \Omega^2_i & \frac{h}{2} \chi_i & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
Y_{i1} \\
Y_{i2} \\
Y_{i3} \\
Y_{i4} \\
Y_{i5} \\
Y_{i6} \\
Y_{i7} \\
Y_{i8} \\
Y_{i9} \\
\end{bmatrix} = \begin{bmatrix}
s_{1,i+1} \\
s_{2,i+1} \\
s_{3,i+1} \\
s_{4,i+1} \\
s_{5,i+1} \\
s_{6,i+1} \\
s_{7,i+1} \\
s_{8,i+1} \\
s_{9,i+1} \\
\end{bmatrix}
\] (16)
It is quite straightforward to solve (16) for $Y$ column by finding the inverse of the left matrix and multiplying it by $s$ (the right column which represents the already known values or initial conditions from the previous step).

The second method is based on the generalized multiple time scales approach proposed by Rossikhin and Shitikova [12], resulting in a set of six nonlinear differential equations in terms of amplitudes $a_i$ and phases $\varphi_i$, which for the case of the combinational internal resonance $\Omega_i = \Omega_2 - 2\Omega_3$, have the form

\begin{align*}
(a_i^+) + s_i a_i &= \Omega_1^{-1} c_{12} a_1 a_3^2 \sin \delta, \\
(a_2^+) + s_2 a_2 &= -\Omega_2^{-1} c_{2} a_2 a_3^2 \sin \delta, \\
(a_3^+) + s_3 a_3 &= -\Omega_3^{-1} c_{3} a_3 a_3^2 \sin \delta, \\
\phi_1 &= -\frac{1}{2} \sigma_1 - \frac{1}{2} \Omega_1^{-1} a_1^2 - \frac{1}{2} \Omega_2^{-1} a_2^2 - \frac{1}{2} \Omega_3^{-1} a_3^2 \cos \delta = 0, \\
\phi_2 &= -\frac{1}{2} \sigma_2 - \frac{1}{2} \Omega_2^{-1} a_2^2 - \frac{1}{2} \Omega_3^{-1} a_3^2 \cos \delta = 0, \\
\phi_3 &= -\frac{1}{2} \sigma_3 - \frac{1}{2} \Omega_3^{-1} a_3^2 \cos \delta = 0, \\
\end{align*}

where $\delta = \varphi_2 + \varphi_3 - \varphi_1$ is the phase difference, $s_i = \mu_i \tau_i^{-1} \Omega_i^{-1} \sin \psi_i$, $\sigma_i = \mu_i \tau_i^{-1} \Omega_i^{-1} \cos \psi_i$, $\psi = \pi \gamma / 2$, and all coefficients depending on the shell’s parameters are given in [7].

Equations (17) could be solved numerically using the Runge-Kutta fourth-order algorithm after rewriting (17) as

\begin{align*}
\dot{a}_1 &= -s_1 a_1 + \Omega_1^{-1} c_{12} a_1 a_3^2 \sin \delta, \\
\dot{a}_2 &= -s_2 a_2 - \Omega_2^{-1} c_{2} a_2 a_3^2 \sin \delta, \\
\dot{a}_3 &= -s_3 a_3 - \Omega_3^{-1} c_{3} a_3 a_3^2 \sin \delta, \\
\dot{\phi}_1 &= \frac{1}{2} \sigma_1 + \frac{1}{2} \Omega_1^{-1} a_1^2 + \frac{1}{2} \Omega_2^{-1} a_2^2 + \frac{1}{2} \Omega_3^{-1} a_3^2 \cos \delta, \\
\dot{\phi}_2 &= \frac{1}{2} \sigma_2 + \frac{1}{2} \Omega_2^{-1} a_2^2 + \frac{1}{2} \Omega_3^{-1} a_3^2 \cos \delta, \\
\dot{\phi}_3 &= \frac{1}{2} \sigma_3 + \frac{1}{2} \Omega_3^{-1} a_3^2 \cos \delta. \\
\end{align*}

4. Numerical results

The relation between the coefficients $\beta_2$ and $\beta_1$ representing shell’s parameters under the combinational internal resonance $\Omega_i = \Omega_2 - 2\Omega_3$ is shown graphically in figure 1. Figure 2 illustrates the interdependency between the shell’s parameters and the natural frequencies satisfying the condition of the combinational resonance under investigation. From figure 2(A) it is seen that the parameters $\beta_2$ and $\beta_1$ are exponentially interrelated, while from figures 2(B) and 2(C) it is evident that magnitudes of $\beta_2$ are linearly related to every frequency of combinational internal resonance $\Omega_i = \Omega_2 - 2\Omega_3$, and $\beta_1$ are exponentially related to them.

The numerical solution of (16) using the multi-step method has been carried out for the dimensionless parameters given in figure 1, and the results are shown in figure 3 which represents the time-dependence estimation of the generalized displacements $X_1$, $X_2$, and $X_3$ during vibrations at the
fractional parameter $\gamma = 0, 0.5, \text{ and } 1$. Figures 3 show the extensive energy exchange between the coupled modes of vibrations.

The solution of equations (18) has been carried out numerically using the Runge-Kutta forth-order algorithm, and the results are shown in figure 4, wherein the envelopes of the amplitudes of the coupled modes of nonlinear vibrations are presented at $\gamma = 0, 0.5, \text{ and } 1$ for the shell parameters $\beta_1=3.619, \beta_2=0.082816,$ and $\nu=0.33$.

![Figure 1](image1)

**Figure 1.** Shell’s parameters under the condition of the combinational internal resonance $\Omega_1=\Omega_2-2\Omega_3$.

![Figure 2](image2)

**Figure 2.** Interdependency between parameters $\beta_1, \beta_2$ and frequencies $\Omega_1, \Omega_2, \Omega_3$: (A) $\beta_1 - \beta_2$ dependence at $m_1=5, n_1=2, m_2=1, n_2=5, m_3=1, n_3=3$; (B) $\beta_1$-dependence of frequencies $\Omega_1, \Omega_2, \Omega_3$, and (C) $\beta_2$-dependence of frequencies.

### 5. Conclusion

In the present paper, nonlinear force driven vibrations of thin cylindrical shell in a viscoelastic medium have been studied, when the motion of the cylindrical shell is described by a set of three coupled nonlinear differential equations subjected to the conditions of the combinational internal resonance, resulting in the interaction of three orthogonal modes, corresponding to the mutually orthogonal displacements, using two different numerical methods. To describe the nonlinear damped vibrations of the thin shell under consideration, the fractional derivative Kelvin-Voigt model is used, because its prediction is in a good compliance with experimental data. The nonlinear set of resolving equations has been obtained in terms of the generalized displacements and in terms of the amplitudes and phases. The two systems have been solved numerically by two different methods.

Within the first method, the generalized displacements of a coupled set of nonlinear ordinary differential equations are estimated using numerical solution of nonlinear multi-term fractional differential equations by the procedure based on the reduction of the problem to a system of fractional
differential equations. According to the second method, the amplitudes and phases of nonlinear vibrations are estimated from the governing nonlinear differential equations describing amplitude-and-phase modulations for the case of the combinational internal resonance. A good agreement between the results obtained by the two methods has been found.

Figure 3. Time-dependence of the generalized displacements $X_1$, $X_2$, and $X_3$ calculated via the Diethelm’s multi-term method for $\Omega_1=4.8488$, $\Omega_2=24.17$, and $\Omega_3=9.5539$.

Figure 4. Envelopes of amplitudes of vibrations according to equations (18).
Relationships (16) and (17) allow one to trace the free vibrational process, resulting in the energy exchange between the coupled modes. The interrelation between the parameters $\beta_2$ and $\beta_1$, which define the geometrical parameters of the shell, and the shell’s natural frequencies $\Omega_1$, $\Omega_2$, and $\Omega_3$, which could fall within the condition of the internal combinational resonance $\Omega_1=\Omega_2-2\Omega_3$, has been revealed in the form of linear and exponential relations.

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