STABILITY OF THE HECKE ALGEBRA OF WREATH PRODUCTS

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Abstract. The Hecke algebras $\mathcal{H}_{n,k}$ of the group pairs $(S_{kn}, S_k \wr S_n)$ can be endowed with a filtration with respect to the orbit structures of the elements of $S_{kn}$ relative to the action of $S_{kn}$ on the set of $k$-partitions of $\{1, \ldots, kn\}$. We prove that the structure constants of the associated filtered algebra $\mathcal{F}_{n,k}$ is independent of $n$. The stability property enables the construction of a universal algebra $\mathcal{F}$ to govern the algebras $\mathcal{F}_{n,k}$. We also prove that the structure constants of the algebras $\mathcal{H}_{n,k}$ are polynomials in $n$. For $k = 2$, when the algebras $(\mathcal{F}_{n,2})_{n \in \mathbb{N}}$ are commutative, these results were obtained in [1], [2] and [9].

1. Introduction

1.1. The asymptotic study of sub-algebras of the group algebras $\mathbb{C}[S_n]$ goes back to the work of Farahat and Higman [4]. They study the centers $\mathcal{Z}_n = Z(\mathbb{C}[S_n])$ of the group algebras $\mathbb{C}[S_n]$. To prove Nakamura’s conjecture, they prove a stability result for the structure coefficients of the algebras $\mathcal{Z}_n$. From the perspective of [6] and [11], we can describe the method of Farahat and Higman as follows:

(1) Construct a conjugacy invariant numerical function on the symmetric group to measure the complexity of a given element.
(2) Endow the centers $\mathcal{Z}_n$ of the group algebras $\mathbb{C}[S_n]$ with a suitable filtration invariant under conjugation.
(3) (Stability property) Prove that the structure constants of the associated filtered algebra $\mathcal{Z}_n$ are independent of $n$, hence obtain a universal algebra $\mathcal{Z}$ that governs all the algebras $\mathcal{Z}_n$.

The term stability is introduced in [11]. Wang used this strategy and proved that the family $(G \wr S_n)_{n \in \mathbb{N}}$, with $G$ finite, satisfies the stability property, [12]. Recently, Wan and Wang applied the same strategy to the family $(GL_n(q))_{n \in \mathbb{N}}$, [11]. Using the filtration induced by the reflection length they proved that the family $(GL_n(q))_{n \in \mathbb{N}}$ also satisfies the stability property. Following [11], we proved that the family $(Sp_{2n}(q))$ satisfies the stability property with respect to the reflection length induced from $GL_{2n}(q)$, [8].

Farahat and Higman’s approach produces a master algebra which maps surjectively onto individual algebras, but there are no direct relation between the individual algebras. Molev and Olshanski construct the individual algebras out of partial permutations, with natural maps between the consecutive algebras. Consequently, the master algebra is constructed as a projective limit. Using partial isomorphism method, Ivanov and Kerov calculated the structure constants of $\mathcal{Z}_n$ and proved that the family $(S_n)_{n \in \mathbb{N}}$ satisfies the stability property, [3]. Méliot also used the partial isomorphism method, [7], and proved that the family $GL_n(q)$ satisfies the stability property. Likewise, Tout proved that the structure constants of the family $S_k \wr S_n$ are polynomials in $n$, [10].

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2 http://journals.tubitak.gov.tr/math/issues/mat-10-34-4/mat-34-4-1-1009-43.pdf
1.2. The central algebra \( Z(\mathbb{C}[G]) \) can be viewed as the Hecke algebra of group pair, which enables one to generalize the previous work to a wider a class of algebras. In fact, recall that when \( G \) is a finite group and \( H \) its a subgroup, then the Hecke algebra \( \mathcal{H}(G, H) \) attached to the pair \((G, H)\) is defined as the sub-algebra of the complex valued functions on \( G \) that are invariant along \( H \)-double cosets. If two \( \mathbb{C} \)-valued functions \( \phi \) and \( \psi \) on \( G \) are contained in the \( \mathcal{H}(G, H) \) then their multiplication \( \phi \cdot \psi \) is defined by the rule

\[
(\phi \cdot \psi)(x) := \sum_{y \in G} \phi(y)\psi(y^{-1}x) = \sum_{yz = x} \phi(y)\psi(z).
\]

The algebra \( Z(\mathbb{C}[G]) \) is isomorphic to the Hecke algebra \( \mathcal{H}(\text{diag}(G) \setminus G \times G/\text{diag}(G)) \) where \( \text{diag}(G) = \{(g, g) \in G \times G : g \in G\} \), cf. [3, Th.1.5.22]. As a result, the class of double-coset algebras is more general than the class of the centers of group algebras. One can pose the question of stability property for the family of double-coset algebras. In the literature, the only double-coset algebra so far considered is \((S_{2n}, B_n)\), where \( B_n \) is the centralizer of \( \tau_n = (1, 2)(3, 4) \cdots (2n-1, 2n) \). It turns out that the family of the double-coset algebras of the pair \((S_{2n}, B_n)\) also satisfies the stability property. This result was obtained by Aker and Can in [1], and by Can and the author in [2] as generalizations of the Farahat and Higman method. Tout proved the same stability as a generalization of the partial isomorphism method of Ivanov-Kerov, [9]. We note that the double-coset algebra of the pair \((S_{2n}, B_n)\) is commutative.

1.3. We consider the family \((\mathcal{H}_{n,k})_{n \in \mathbb{N}}\) of algebras \( \mathcal{H}_{n,k} = \mathbb{C}[(S_k \wr S_n) \setminus (S_k \wr S_n)] \), where \( k \) is a fixed positive integer greater or equal to 2. These algebras are non-commutative unless \( k = 2 \). We show that there is a natural filtration on the non-commutative Hecke algebra \( \mathcal{H}_{n,k} \), and the structure constants of the induced filtered algebra \( \mathcal{F}_n \) is independent of \( n \). In particular, \( \mathcal{H}_{n,k} \) satisfies the stability property. Taking \( k = 2 \) we reproduce the case of the pair \((S_{2n}, B_n)\) mentioned above. We use the Farahat-Higman approach but our proof differs from the one presented in [1]. The proof in [1] relies on a twist of the automorphism \( g \mapsto g^{\kappa_n} \), which does not have a direct generalization to the case \( k \neq 2 \). Instead we focus on the action of \( S_{kn} \) on the set of \( k \)-partitions of the set \( \{1, 2, \cdots, kn\} \) and reduce the proof of the stability property to proving that the "orbits" of certain elements must intersect "transversely". In the rest of the introduction we briefly explain these terms.

The group \( S_{kn} \) acts transitively on the set of \( k \)-partitions of the set \([kn] := \{1, \cdots, kn\} \). The wreath product \( S_k \wr S_n \) can be identified with the stabilizer of the \( k \)-partition

\[
\{1, 2, \cdots, k\}, \cdots, \{k(n-1) + 1, \cdots, kn\}
\]

of \([kn] \). For \( g \in S_{kn} \), we define a graph \( G_g \) (Def. 4.4), called the type of \( g \), on the set of \( n \)-vertices. The isomorphism class of the graph \( G_g \) completely determines the \( S_k \wr S_n \)-double coset of \( g \in S_{kn} \). We define the modified type \( G^*_g \) as the graph obtained from \( G_g \) by removing the connected components that consist of a single vertex. The modified type of an element does not change under the embedding \( S_{kn} \hookrightarrow S_{k(n+1)} \). The set of modified types with \( n \)-vertices is denoted by \( G_{n,k} \) and the union of modified types is denoted by \( G_{\infty,k} \). Let \( M \in G_{n,k} \) be a modified type. The sizes of the connected components of \( M \) define a partition \( \lambda_M \) of \( n \). If \( \lambda_M = (\lambda_1, \cdots, \lambda_t) \), the weight \( ||M|| \) of the graph \( M \) is defined as the integer \( \sum_{i=1}^t (\lambda_i - 1) \). The algebra \( \mathcal{H}_n \) is generated by the double coset sums

\[
S_M(n) := \sum_{\substack{g \in S_{kn} \\text{ such that } G^*_g \supseteq M}} g.
\]
The multiplication in the algebra $\mathcal{H}_n$ can be written as
\[
S_M(n) \cdot S_N(n) = \sum_{||L|| \leq ||M|| + ||N||} c_{M,N}^{L}(n) \cdot S_L(n),
\]
where $M, N, L \in G_{\infty,k}$. We are ready to state the main result of this paper.

**Theorem 1.1.** The functions $c_{M,N}^{L}(n)$ are polynomials in $n$. If the equality $||L|| = ||M|| + ||N||$ holds then the function $c_{M,N}^{L}(n)$ is independent of $n$.

1.4. The paper is organized as follows. In section 2 we define the group $S_k \wr S_n$ as a subgroup of $S_n$ and introduce the necessary notation. In section 3 we review the basics of the double-coset algebras and introduce the abstract set-up that captures the properties satisfied by the family of $k$-partitions. Each positive integer $p$ and its positive integer $q$ implies that it can be covered by $\Gamma$. In other words, the group $\mathcal{H}_{\infty}$ consists of elements that permutes $\Gamma$-parts. Now we introduce the Hecke algebra of the pair of $S_k \wr S_n$, which will be our main interest in the rest of the article. We start with introducing the necessary notation.

The set of positive integers is denoted by $\mathbb{N}_+$. For every positive integer $n$ in $\mathbb{N}_+$, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. If $g$ is a permutation of the set of positive integers $\mathbb{N}_+$, then the support of $g$ is defined to be the set of the positive integers $\mathbb{N}(g) = \{r \in \mathbb{N}_+ : g(r) \neq r\}$ that are moved by $g$. Let $A$ be a subset of positive integers $\mathbb{N}_+$. The set of permutations $g$ of $A$ with finite support $\mathbb{N}(g)$ is denoted by $S_A$. If the subset $A$ is equal to $[n]$ for some positive integer $n$, then by abuse of notation we will stick to the usual notation and write $S_n$ rather than $S_{[n]}$. Likewise, if $A = \mathbb{N}_+$ then we will write $S_{\infty}$ rather than $S_{\mathbb{N}_+}$. Elements of $S_{\infty}$ are called finitary permutations of $\mathbb{N}$.

Let $k$ be a fixed positive integer greater or equal to 2. A $k$-partition of an arbitrary set $X$ is a collection of disjoint $k$-elemental subsets of $X$ where the union covers $X$. For every positive integer $i$ we introduce the set
\[
\Gamma_i = \{ k(i - 1) + 1, \ldots, ki \}.
\]
The collection $(\Gamma_i)_{i \in \mathbb{N}_+}$ is a $k$-partition of positive integers $\mathbb{N}_+$. The sets of the form $\Gamma_i$ will be called $\Gamma$-part. Each positive integer $r$ is contained in a unique $\Gamma$-part. If $r \in \Gamma_i$, we say that the $\Gamma$-part of $r$ is $i$, in which case we will write $p(r) = i$. Two positive integers $r$ and $s$ will be called partners if $p(r) = p(s)$, i.e. if they are contained in the same $\Gamma_i$ for some positive integer $i$. The subgroup $H_{\infty}$ of $S_\infty$ is defined to be group of finitary permutations that preserve the partner relationship. More precisely, a permutation $h$ in $S_{\infty}$ is an element of $H_{\infty}$ if and only if the equality $p(h(r)) = p(h(s))$ for all positive integers $r$ and $s$. Notice that the integer $k$ is not used in the notation. But this will not cause any ambiguity since we will only be dealing with a fixed $k$.

The group $H_{\infty}$ can be described in a more suggestive way. In fact, the group of finitary permutations $S_\infty$ acts on the set of $k$-partitions of $\mathbb{N}_+$. The point stabilizer of the $k$-partition $(\Gamma_i)_{i \in \mathbb{N}_+}$ is equal to the group $H_{\infty}$. In other words, the group $H_{\infty}$ consists of elements that permutes $\Gamma$-parts. Now we generalize the definition of $H_{\infty}$ to subsets of $\mathbb{N}_+$. Let $A$ be a subset of positive integers $\mathbb{N}_+$ that it can be covered by $\Gamma$-parts. We define $H_A$ as the group of permutations $g$ satisfying the two
conditions: 1. The support $\mathbb{N}(g)$ is contained $A$; 2. The permutation $g$ stabilizes the $k$-partition $(\Gamma_i)_{i \in \mathbb{N}_+}$. In other words,

$$H_A = H_\infty \cap S_A.$$  

As before, if $A = [kn]$ then by abuse of notation we write $H_n$ rather than $H_{[kn]}$.

**Remark 2.1.** Let $n$ be a fixed positive integer. We will define two subgroups $S_n$ and $Y_n$ of $H_n$ for which the internal semi-direct product $Y_n \rtimes S_n$ is equal to $H_n$. We start with the definition of $S_n$. For each pair of distinct positive integers $i$ and $j$ that are less than or equal to $n$, we define the permutation $\tau_{ij}$ of $[kn]$ by setting

$$\tau_{ij} = \prod_{r=1}^{k}(k(i-1)+r, k(j-1)+r),$$

where $(k(i-1)+r, k(j-1)+r)$ denotes the transposition that interchanges the positive integers $k(i-1)+r$ and $k(j-1)+r$. The permutation $\tau_{ij}$ interchanges the elements of $\Gamma_i$ and $\Gamma_j$ monotone increasingly, and it acts as identity elsewhere. The subgroup of $S_{nk}$ generated by the permutations $\tau_{ij}$ is denoted by $S_n$. The association sending the transposition $(ij) \in S_n$ to the permutation $\tau_{ij} \in S_n$ defines an isomorphism between $S_n$ and $S_n$, which justifies the notation. Secondly, we define the group $Y_n$ as the Young subgroup $S_{\Gamma_1} \times \cdots \times S_{\Gamma_n}$ of $S_{kn}$. The group $Y_n$ is a subgroup of $H_n$. It can be shown easily that $S_n$ normalizes $Y_n$ and $H_n = Y_n \rtimes S_n$. The internal semi-direct product decomposition implies that every element $h \in H_n$ can be written as a product

$$h = h_Y h_S,$$

where $h_Y \in Y_n$ and $h_S \in S_n$. In general, let $A$ be a subset of positive integers $\mathbb{N}_+$ which is equal to union of $\Gamma_{j}$, $j \in J$, for some subset $J \subset \mathbb{N}$. Discussing as above, one can show that there is an isomorphism $H_A \simeq H_{[J]} = S_k \wr S_J$. If $J$ is finite then the cardinality of $H_A$ is equal to $|J|!(k!)^{|J|}$.

**Example 2.2.** Let $k = 3$ and $n = 7$. The permutation $g = (1, 8, 18, 21, 6, 10, 13, 2, 11, 3, 12)(4, 16, 19)(5, 17, 20)$ is an element of $S_{21}$. The positive integers 1 and 2 are contained in $\Gamma_1 = \{1, 2, 3\}$ and thus $p(1) = p(2) = 1$. On the other hand $p(g(1)) = p(8) = 3$ as $8 \in \Gamma_3 = \{7, 8, 9\}$, and $p(g(2)) = p(11) = 4$. It follows that $g \notin H_7$.

**Lemma 2.3.** The group $H_n$ (resp. $H_\infty$) is isomorphic to the wreath product $S_k \wr S_n$ (resp. $S_k \wr S_\infty$) for every $n \in \mathbb{N}_+$. The cardinality of $H_n$ is equal to $n!(k!)^n$.

**Proof.** Directly follows from the previous Remark. $\square$

**Definition 2.4.** Let $g$ be an arbitrary permutation in $S_{kn}$. The ordinary $H_\infty$-support $[g]_H^o$ and the (completed) $H_\infty$-support $[g]_H$ of $g$ are defined as follows:

$$[g]_H^o := \{ \Gamma_i | g(\Gamma_i) \neq \Gamma_j, \forall j \in \mathbb{N} \}, \quad \text{and} \quad [g]_H := \bigcup_{\Gamma_i \in [g]_H^o} \Gamma_i.$$  

The $H$-support $[g]_H$ is a finite union $\Gamma$-parts. As a result the group $H_{[g]_H}$ makes sense, cf. Eq. 2. Note that the permutation $g$ may act on $([g]_H)^c$ non-trivially.

**Example 2.5.** We reconsider the permutation $g = g = (1, 8, 18, 21, 6, 10, 13, 2, 11, 3, 12)(4, 16, 19)(5, 17, 20)$ in $S_{21}$ defined in Example 2.2. The $H$-support of $g$ is

$$[g]_H = \{ \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5 \},$$

while $g(\Gamma_6) = \Gamma_7$ and $g(\Gamma_7) = \Gamma_2$. Observe that there is no inclusion relation between the support $\mathbb{N}(g)$ and $H$-support $[g]_H$ of $g$.  

4
3. Hecke Algebra of a Pair \((G, H)\)

In this section we review the basics of Hecke algebras of attached to finite groups.

Let \(G\) be a finite group and \(H\) be a subgroup of \(G\). Denote the set of \(H\)-double cosets in \(G\) by \(\mathcal{D}\). Then the Hecke algebra \(\mathcal{H} = \mathcal{H}(G, H)\) of the pair \((G, H)\) is defined to be the \(\mathbb{C}\)-sub-algebra of \(\mathbb{C}[G]\) generated by the elements

\[
S_\lambda = \frac{1}{|H|} \sum_{g \in \lambda} g,
\]

where \(\lambda\) runs over the set \(\mathcal{D}_\eta\) of the \(H_\eta\)-double cosets. It turns out that the set \(\{S_\lambda \mid \lambda \in \mathcal{D}\}\) is in fact a basis for \(\mathcal{H}\). This means, if \(\lambda\) and \(\mu\) are fixed double cosets then there exist a unique complex number \(c_{\lambda,\mu}^\eta\) for each double coset \(\eta\) in \(\mathcal{D}\) such that

\[
S_\lambda \cdot S_\mu = \sum_{\eta \in \mathcal{D}} c_{\lambda,\mu}^\eta S_\eta.
\]

The constants \(c_{\lambda,\mu}^\eta\) are called **structure constants** and they determine the multiplicative structure on \(\mathcal{H}\) uniquely.

### 3.1. Reverted action and structure constants in terms of centralizers

In this subsection we show that the calculation of the structure constants can be interpreted in terms of \(H\)-centralizers of certain elements. We start with fixing three \(H\)-double cosets \(\lambda, \mu, \eta\) in \(G\). If \(A\) is a subset of \(\eta\), then *fiber* of the cartesian product \(\lambda \times \mu\) in \(A\) is defined by setting

\[
V(\lambda \times \mu : A) := \{(g_1, g_2) \in \lambda \times \mu \mid g_1 g_2 \in A\}.
\]

The next lemma reduces the study of the structure constants to the study of the fibers of the form \(V(\lambda \times \mu : g^H)\), where \(g^H\) denotes the \(H\)-conjugacy class of \(g\).

**Lemma 3.1.** If \(g\) is an element of the double coset \(\eta\), then

\[
c_{\lambda,\mu}^\eta = \frac{|V(\lambda \times \mu : g)|}{|H|} = \frac{|V(\lambda \times \mu : g^H)|}{|H||g^H|} = \frac{|V(\lambda \times \mu : g^H)||C_H(g)|}{|H|^2},
\]

where \(C_H(g)\) denotes the centralizer of \(g\) in \(H\).

**Proof.** The first equality directly follows from the definition of the structure constants and the fact that \(|V(\lambda \times \mu : z)|\) is equal to the number of times \(z\) occurs in the product \((\sum_{g_1 \in \lambda} g_1)(\sum_{g_2 \in \lambda} g_2)\).

The second equality is a consequence of the identity

\[
V(\lambda \times \mu : A \cup B) = V(\lambda \times \mu : B) \cup V(\lambda \times \mu : B),
\]

which holds for disjoint subsets \(A, B\) of \(\eta\). The final equation is consequence of the formula

\[
|g^H| = \frac{|H|}{|C_H(g)|}.
\]

\(\Box\)

We now consider the reverted action of \(H \times H\) on the group \(G \times G\), which is defined by the rule

\[
(h_1, h_2) \cdot_{H \times H} (g_1, g_2) = (h_1 g_1 h_2^{-1}, h_2 g_2 h_1^{-1}),
\]

where \(h_1, h_2 \in H\) and \(g_1, g_2 \in G\). The fiber set \(V = V(\lambda \times \mu : g^H)\) is closed under \(H \times H\) action, where \(g\) is an arbitrary element in the double coset \(\eta\). It is equal to disjoint union of \(H \times H\) orbits,
say equal to the union of the orbits \( \Lambda_1, \cdots, \Lambda_r \). Let \((g_{1i}, g_{2i})\) be the representative of the orbit \( L_i \), for \( i = 1, \ldots, r \). Using the orbit formula for group actions, we deduce

\[
|\Lambda_i| = |H \times H \cdot (g_{1i}, g_{2i})| = \frac{|H \times H|}{|Stab_{H \times H}(g_{1i}, g_{2i})|}.
\]

Combining last the equality of Lemma 3.1 with Eq. (3.1) and bearing in mind the fact that the union of \( \Lambda_i, i = 1, \ldots, r \), is equal to the fiber set \( V \) yield the following formula for the structure constants:

\[
C^n_{\lambda, \mu} = \sum_{i=1}^{r} \frac{|\Lambda_i| |C_H(g_{1i}, g_{2i})|}{|H|^2} = \sum_{i=1}^{r} \frac{|C_H(g_{1i}, g_{2i})|}{|Stab_{H \times H}(g_{1i}, g_{2i})|}.
\]

**Remark 3.2.** A pair \((h_1, h_2) \in H \times H\) is contained in the stabilizer of a pair \((g_1, g_2)\) if and only if \( h_1 g_1 h_2^{-1} = g_1 \) and \( h_2 g_2 h_1^{-1} = g_2 \), which is equivalent to \( h_2 = g_1^{-1} h_1 g_1 \) and \( h_1 = g_2^{-1} h_2 g_2 \). In particular, given \((g_1, g_2)\), the elements \( h_1 \) and \( h_2 \) determine each other uniquely whenever \((h_1, h_2)\) stabilizes \((g_1, g_2)\).

**Lemma 3.3.** The projection map \( \pi : (h_1, h_2) \mapsto h_1 \) defines a bijection between \( Stab_{H \times H}(g_1, g_2) \) and the intersection \( C_H(g_1 g_2) \cap g_1 H g_1^{-1} \).

**Proof.** Injectivity of \( \pi \) is a consequence of Remark 3.2. We prove that \( \pi \) is a surjection onto the intersection \( C_H(g_1 g_2) \cap g_1 H g_1^{-1} \). To show that \( \pi \) sends elements into \( C_H(g_1 g_2) \cap g_1 H g_1^{-1} \), we pick an arbitrary element \((h_1, h_2)\) from the stabilizer of the pair \((g_1, g_2)\). From the previous remark we get \( h_1 = g_1 h_2 g_1^{-1} \) and hence \( h_1 \in g_1 H g_1^{-1} \). Combining the equalities \( h_1 g_1 h_2^{-1} = g_1 \) and \( h_2 g_2 h_1^{-1} = g_2 \) yields

\[
h_1 g_1 h_2^{-1} h_2 g_2 h_1^{-1} = g_1 g_2,
\]

which proves that \( h_1 \) is an element of the centralizer \( C_H(g_1 g_2) \). Thus \( h_1 \) is an element of the intersection \( C_H(g_1 g_2) \cap g_1 H g_1^{-1} \). Conversely, assume that \( h_1 \) is an element of the intersection \( C_H(g_1 g_2) \cap g_1 H g_1^{-1} \). Since conjugation is an automorphism, there exists a unique element \( h_2 \in H \) such that \( h_1 = g_1 h_2 g_1^{-1} \). We claim that \((h_1, h_2)\) stabilizes \( Stab_{H \times H}(g_1, g_2) \). The equality \( h_1 = g_1 h_2 g_1^{-1} \) can be written as \( h_1 g_1 h_2^{-1} = g_1 \). So we just need to show that \( h_2 g_2 h_1^{-1} = g_2 \). Substituting \( h_2 = g_1^{-1} h_1 g_1 \) and using the fact that \( h_1 \) commutes with \( g_1 g_2 \) we deduce

\[
h_2 g_2 h_1^{-1} = (g_1^{-1} h_1 g_1) g_2 h_1^{-1} = g_1^{-1} g_1 g_2 = g_2.
\]

This finishes the proof of the surjectivity.

**Corollary 3.4.** The structure constant \( c^n_{\lambda, \mu} \) is a non-negative integer and given by the formula

\[
c^n_{\lambda, \mu} = \sum_{i=1}^{r} \frac{|C_H(g_{1i}, g_{2i})|}{|C_H(g_{1i}, g_{2i}) \cap g_1 H g_1^{-1}|},
\]

where the elements \( g_{1i}, g_{2i} \) are as above.

**Proof.** Use Lemma 3.3 and Eq. (3.1). \( \square \)

**3.2 Family of Hecke algebras.** In this subsection we fix two ascending chains \((G_n)_{n \in \mathbb{N}_+}\) and \((H_n)_{n \in \mathbb{N}_+}\) of finite groups such that \( H_n \subset G_n \) for every positive integer \( n \). The set of \( H_n \)-double cosets in \( G_n \) is denoted by \( D_n \), The union of the groups \( G_n \) (resp. \( H_n \)) will be denoted by \( G_\infty \) (resp. \( H_\infty \)). We view the group \( H_\infty \) as a subgroup of \( G_\infty \) and denote the \( H_\infty \)-double cosets in \( G_\infty \) is denoted by \( D_\infty \).

**Definition 3.5.** The family \((G_n, H_n)_{n \in \mathbb{N}_+}\) is called saturated, if for all positive integers \( m \leq n \) and for all \( H_n \)-double coset \( \lambda_n \), the intersection \( \lambda_n \cap G_m \) is either empty or equal to a single \( H_m \)-double coset.
Now we consider a saturated family \((G_n, H_n)_{n \in \mathbb{N}_+}\) and fix two positive integers \(m \leq n\). We claim that the map
\[
\varphi_m^n : H_m g H_m \mapsto H_n g H_n
\]
is an injection from \(\mathcal{D}_m\) into \(\mathcal{D}_n\). Indeed, assume that \(H_m g H_m = H_n g' H_n\) for some \(g, g' \in G_m\). Since the family \((G_n, H_n)_{n \in \mathbb{N}_+}\) is saturated, the non-empty intersection
\[
(H_m g H_m) \cap G_m = (H_n g' H_n) \cap G_m
\]
is equal to a single \(H_m\)-double coset. But the intersection contains both \(g\) and \(g'\), which means that \(H_m\)-double cosets of \(g\) and \(g'\) are equal. For a saturated family, we identify \(\mathcal{D}_m\) with its image in \(\mathcal{D}_n\) whenever \(m \leq n\). The set \(\mathcal{D}\) of \(H_\infty\)-double cosets can be identified with the union of \(\mathcal{D}_n\). If \(\lambda \in \mathcal{D}\) set \(\lambda(n) := \lambda \cap G_n\). As a result of being saturated, \(\lambda(n)\) is either empty or an \(H_n\)-double coset in \(G_n\). Using this observation one can write \(\mathcal{D}_n = \{\lambda(n) \mid \lambda \in \mathcal{D}, \lambda(n) \neq \emptyset\}\). If \(\lambda(n) \neq \emptyset\) set
\[
S_\lambda(n) = \frac{1}{|H_n|} \sum_{g \in \lambda(n)} g.
\]
The set \(\{S_\lambda(n) \mid \lambda \in \mathcal{D}, \lambda(n) \neq \emptyset\}\) is a basis of the algebra \(\mathcal{H}(G_n, H_n)\) and thus there exists non-negative integers \(c_{\lambda,\mu}^\eta(n)\) such that
\[
S_\lambda(n) \cdot S_\mu(n) = \sum_{\eta \in \mathcal{D}} c_{\lambda,\mu}^\eta(n) S_\eta(n).
\]
The functions \(c_{\lambda,\mu}^\eta(n)\) are called the **structural coefficients** of the family \((G_n, H_n)\).

**Definition 3.6.** A saturated family \((G_n, H_n)_{n \in \mathbb{N}_+}\) is called *admissible* if for some \(g \in \eta\), the fiber set
\[
V(\lambda \times \mu : g^H) = \{(g_1, g_2) \in \lambda \times \mu : g_1 g_2 \in g^H\}
\]
adopts finitely many \(H_\infty \times H_\infty\) orbits.

**Corollary 3.7.** Let \((G_n, H_n)_{n \in \mathbb{N}}\) be an admissible family and \(\lambda, \mu, \eta\) be three \(H_\infty\)-double coset. Let \(m\) be the minimal positive integer such that the intersections \(\lambda(m), \mu(m), \) and \(\eta(m)\) are all non-empty. There exist elements \(g_1, \ldots, g_r\) in \(\lambda(m)\), and \(g_{21}, \ldots, g_{2r}\) in \(\mu(m)\) such that and
\[
c_{\lambda,\mu}^\eta(n) = \sum_{i=1}^r \frac{|C_{H_n}(g_{1i} g_{2i})|}{|C_{H_n}(g_{1i} g_{2i}) \cap H_n g_{1i}^{-1} |}.
\]

**Proof.** Follows from the previous discussion and Corollary 3.4. \(\square\)

4. \(H_n\)-double cosets and the polynomiality of the structural coefficients

In this section we prove that the family \((S_{kn}, H_n)_{n \in \mathbb{N}_+}\) is admissible and the structural coefficients are polynomials in \(n\). We start with a parametrization the \(H_n\)-double cosets.

**4.1. \(H_n\)-double cosets.** We first define a red-blue graph to temporarily parameterize the double-cosets.

**Definition 4.1.** Let \(g\) be a permutation in \(S_{kn}\). The red-blue graph \(G_g = (V_g, E_g)\) is the unique graph satisfying the following: The vertex set \(V_g\) of the graph \(G_g\) consists of the integer pairs of the form \((i, g(i))\), where \(i\) runs over the set \(\{1, \ldots, kn\}\). The edge set \(E_g\) consists of red edges and blue edges. Let \((r, g(r))\) and \((s, g(s))\) be two vertices in \(V_g\). There is a red edge between \((r, g(r))\) and \((s, g(s))\) if and only if \(p(r) = p(s)\); there is a blue edge between \((r, g(r))\) and \((s, g(s))\) if and only if \(p(g(r)) = p(g(s))\). Notice that, when \(k = 2\), the graph \(G_g\) is same as the graph defined in [6, Ch.VII, Sc.2].

**Lemma 4.2.** The isomorphism class of the red-blue graph \(G_g\) determines \(H_n g H_n\) completely.
Proof. The proof is a direct generalization of [6, 2.1(i), Ch.VII, Sc.2]. Let \( g, g' \) be two elements of \( S_{kn} \) and assume that \( G_g \) and \( G_{g'} \) are isomorphic. There is a bijection between the vertex sets

\[
V_g \rightarrow V_{g'}
\]

\[(r, g(r)) \rightarrow (h(r), h'(g(r)))\]

that preserves the red edges and blue edges. Since the pair \( (h(r), h'(g(r))) \) is an edge of \( G_{g'} \), by definition of the vertex set \( V_{g'} \), it follows that \( g'h(r) = h'g(r) \), for all positive integers \( r \) in \([kn]\). Since the red edges are mapped to red edges, \( p(r) = p(s) \) implies \( p(h(r)) = p(h(s)) \), hence \( h \in H_n \). Similarly, blue edges are permuted among themselves, \( p(g(r)) = p(g(s)) \) implies \( p(hg(r)) = p(hg(s)) \). As a consequence the permutation \( h' \) is also contained in \( H_n \). The equality \( g'h(r) = h'g(r) \) now implies that the elements \( g \) and \( g' \) are contained in the same \( H_n \)-double coset.

It is straightforward to prove that the elements contained in the same double coset have isomorphic red-blue graphs.

\[
\text{Figure 1. The red-blue graph } G_g \text{ when } g = (1, 8, 18, 21, 6, 10, 13, 2, 11, 3, 12)(4, 16, 19)(5, 17, 20), \text{ where } k = 3 \text{ and } n = 7.
\]

Remark 4.3. Observe that, the graph obtained from \( G_g \) by erasing the blue edges (resp. red edges) is equal to disjoint union of complete graphs on \( k \)-vertices.

Now we define a second graph, \( G_g = (V, E^g) \), whose isomorphism class completely determines the isomorphism class of the red-blue graph \( G_g \). In fact, the graph \( G_g \) is obtained from \( G_g \) by collapsing the the \( k \)-complete red-graphs into a single vertex \( v_i \). The graph \( G_g \) can be described as follows.

Definition 4.4. The vertices of the graph \( G_g = (V, E^g) \) are denoted by \( v_1, \cdots, v_n \). For \( u = 1, \cdots, n, \) and \( \{r, s\} \subset \Gamma_u, r \neq s, \) there is an edge \( e^g_{r,s} = e^g_{s,r} \):

\[
e^g_{r,s} = e^g_{s,r} = \{\{v_i, r\}\{v_j, s\}\}
\]

where \( r \in g(\Gamma_i), s \in g(\Gamma_j). \) Notice that \( e^g_{r,s} \) can be written in the following form:

\[
e^g_{r,s} = e^g_{s,r} = \{\{v_{p(g^{-1}(r))}, r\}, \{v_{p(g^{-1}(s))}, s\}\}.
\]
We denote the set of edges $e^g_{rs}$ where $r, s \in \Gamma_u$ by $E^g_u$. The isomorphism class of $G_g$ is denoted by $\tilde{G}_g$. The set of isomorphism classes of graphs of the form $G_g$ for $g \in S_{kn}$ is denoted by $\mathcal{G}_{n,k}$.

**Example 4.5.** Collapsing the complete 3-graphs with red edges of the graph presented above into points we obtain the graph $G_g$, where $g = (1, 8, 18, 21, 6, 10, 13, 2, 11, 3, 12)(4, 16, 19)(5, 17, 20)$.

![Graph with red edges](image)

We note that the isomorphism classes of the graph $G_g$ and red-blue graph $\mathbf{G}_g$ are recoverable from each other uniquely. We also note that the vertices of $G_g$ corresponds to the red-connected components of $\mathbf{G}_g$. Likewise, edges of $G_g$ corresponds to the blue edges of $\mathbf{G}_g$. According to the definition, $v_i$ and $v_j$ are connected by an edge if and only if there exist $r, s$ with $p(r) = p(s)$ such that $g^{-1}(r) \in \Gamma_i$ and $g^{-1}(s) \in \Gamma_j$. From this observation, one deduces the following:

**Remark 4.6.** The set $\Gamma_i$ is not element of $[g]_B^{-}$ if and only if $\{v_i\}$ is a connected component of $G_g$. Equivalently, the vertex $v_i$ is an isolated vertex of $G_g$ if and only if $g(\Gamma_i) = \Gamma_j$ for some $j \in [n]$; which is tantamount to say that $\Gamma_i$ is contained in the complement $[kn] - [g]_B$.

We reformulate Lemma 4.2.

**Corollary 4.7.** There is a bijection

$$
B_n \setminus S_{kn} / B_n \rightarrow \mathcal{G}_{n,k}
$$

$$
B_{ng} B_n \rightarrow \tilde{G}_g
$$

for all $n, k > 0$. The graph $G_g$ will be called the coset type of $g$.

4.2. **Minimal double-coset representatives.**

**Definition 4.8.** Let $M \in \mathcal{D}$ be an $H$-double coset. An element $g \in M$ is called a **minimal representative** if it satisfies the following:

(a) The support $N(g) \subseteq [g]_H$ i.e.

$$
g(\Gamma_i) = \Gamma_j \quad \Rightarrow \quad i = j \quad \& \quad g(s) = s, \forall s \in \Gamma_i.
$$

(b) If $p(g(s)) = p(s)$ then $g(s) = s$.

**Lemma 4.9.** Every double coset admits a minimal representative. If $g$ is a minimal representative, then $g^{-1}$ is a minimal representative, and $[g]_H = [g^{-1}]_H$. Minimal elements are closed under conjugation by elements of $H$.

**Proof.** We just prove the existence of minimal representatives as the others are formal consequences of the definition of minimal representative. Let $g$ be a permutation in $S_{kn}$.

(a) Assume that $\Gamma_i$ is mapped to $\Gamma_j$ for some positive integers $i, j$ in $[n]$. This means $\Gamma_i$ and $\Gamma_j$ are both contained in the support $N(g)$ of $g$. Consider the permutation $\tau_{ij}g$. The permutation $\tau_{ij}g$ maps $\Gamma_i$ to itself. Moreover, its support $N(\tau_{ij}g)$ is contained in the support
Let $h$ be the unique permutation satisfying $h_{i|\Gamma_i} = \tau_{ij}g_{i|\Gamma_i}$, and $h_{i|\Gamma_i} = id$. The permutation $h$ is contained in $H$ and the element $h^{-1}g$ is identity on $\Gamma_i$. By construction $N(h^{-1}g) \subseteq N(g)$. The process terminates in finitely many steps. Resulting element satisfies the property (a) of Definition 4.8.

(b) Next we assume that the permutation $g$ satisfies the property (a). Let $s$ be a positive integer $s \in N(x)$. We write $g(s) = r$ and assume that $p(r) = p(s)$. The element $(rs)g$ satisfies the property (a) and $N((rs)g) \subseteq N(g) - s$. This process should terminate in finitely many steps. Resulting element satisfies property (b).

\[ \square \]

Remark 4.10. Let $g$ be a permutation. The above proof shows that one can find a minimal representative in the cosets $Hg$ and $gH$.

Example 4.11. Consider the element $g = (1, 8, 18, 21, 6, 10, 13, 2, 11, 3, 12)(4, 16, 19)(5, 17, 20)$. Following the procedure presented in the previous proof, we multiply $g$ on the left by the permutation $h = (5, 20, 17)(4, 19, 16)(6, 21, 18)$, and obtain the minimal representative $(1, 8, 6, 10, 13, 2, 11, 3, 12)$. We also note that $h = \tau_{2,6}\tau_{2,3}.$

4.3. Admissibility of the family $(S_{kn}, H_n)_{n \in \mathbb{N}}$. We denote the natural embedding of $S_{kn}$ into $S_{k(n+t)}$ with $\cdot^t$. The embedding $\cdot^t$ maps $H_n$ into $H_{n+t}$. If $g$ is a permutation in $S_{kn}$, then

\[ G_{\cdot^t} = G_g \sqcup \circ_k \sqcup \cdots \sqcup \circ_k \]

where $\circ_k$ is the graph with a single vertex with $k(k-1)/2$ loops.

Lemma 4.12. The family $(S_{kn}, H_n)$ is saturated.

Proof. Let $g_1$ and $g_2$ be two permutations in $S_{kn}$. Assume that the permutations $g_1^t$ and $g_2^t$ are contained in same $H$-double coset for some positive integer $t$. By Eq. (4.3) we have

\[ G_{g_1^t} = G_{g_1} \sqcup \circ_k \sqcup \cdots \sqcup \circ_k, \quad G_{g_2^t} = G_{g_2} \sqcup \circ_k \sqcup \cdots \sqcup \circ_k. \]

By Corollary 4.7 the graphs $G_{g_1^t}$ and $G_{g_2^t}$ are isomorphic. This implies that the graphs $G_{g_1}$ and $G_{g_2}$ are isomorphic. A second use of Corollary 4.7 finishes the proof.

\[ \square \]

Proposition 4.12. Let $M$, $N$, $L$ be three $H$-double cosets. If $g$ is a minimal representative for $L$ then the fiber set

\[ V = V(M \times N : gH) \]

admits finitely $H \times H$ orbits.

Proof. We will show that there exists a positive integer $ku$ such that every orbit has a representative in $V$ contains an element in $S_{ku} \times S_{ku}$. Assume that the pair $(g_1, g_2)$ is contained in $V$. By Remark 4.10 there exists a permutation $h_1 \in H$ such that $h_1g_1$ is a minimal representative of $M$. The pair

\[ (h_1g_1, g_2h_1^{-1}) = (h_1, id) \cdot (g_1, g_2) \]

is contained in the $H \times H$-orbit of $(g_1, g_2)$. On the other hand, by Lemma 4.9 we know that the permutation $h_1g_1g_2h_1^{-1}$ is a minimal representative of $L$. Thus, without loss of generality we may assume that $g_1$ and $g_1g_2$ are minimal representatives of their $H$-double cosets. The union

\[ T = [g_1]_H \cup [g_1g_2]_H \]

contains the supports $\mathbb{N}(g_1)$ and $\mathbb{N}(g_1g_2)$, because $g_1$ and $g_1g_2$ are both minimal representatives. Since $H$-supports can be covered by the sets of the form $\Gamma_i$, we may assume that the set $T$ is equal to the union of $\Gamma_{i_1}, \cdots, \Gamma_{i_t}$. The permutation $\tau = \tau_{i_1} \cdots \tau_{i_t}$ maps $T$ onto $|kt|$. This means
\( \tau g_1 \tau^{-1} \) and \( \tau g_1 g_2 \tau^{-1} \) are both elements of \( S_{kt} \). Since \( \tau g_2 \tau^{-1} = \tau g_1^{-1} \tau g_1 g_2 \tau^{-1} \) it follows that \( \mathbb{N}(\tau g_2 \tau^{-1}) \subset [kt] \). As a result we have
\[
(\tau x \tau^{-1}, \tau y \tau^{-1}) \in V \cap S_{kt} \times S_{kt}.
\]
This means the orbit of \((x, y)\) has a representative in \( S_{kt} \). The cardinality of the set \( T \) is \( kt \) and it is bounded by \( ku := |[g_1]|_H + |[g_1 g_2]|_H \). But the cardinalities \([g_1]|_H, [g_1 g_2]|_H \) are constant on double-cosets. Hence, every orbit has a representative in the finite set \( S_{ku} \times S_{ku} \). Thus, there exist finitely many orbits.

4.4. Centeralizers of the minimal representatives. In this subsection we investigate the intersection
\[
H_n \cap gH_n g^{-1}
\]
and prove that the structural coefficients \( c_{M,N}^{L}(n) \) are polynomials in \( n \).

Remark 4.14. Let \( g \) be a permutation in \( S_n \) and let \( C_{S_n}(g) \) be the centralizer of \( g \). The following hold:

1. If \( g' \) is in the centralizer \( C_{S_n}(g) \) then \( g' \) maps \( N(g) \) to \( N(g) \).
2. The centralizer \( C_{S_n}(g) \) admits the following direct-sum product:
\[
C_{S_n}(g) = C_{N(g)}(g) \oplus S_{|\mathbb{N}|-N(g)}.
\]

Our next result is analogous to the first part of Remark 4.14.

Lemma 4.15. Let \( g \in S_{kn} \) be a minimal representative of its \( H_n \)-double coset. If
\[
h \in H_n \cap gH_n g^{-1},
\]
then \( h \) maps \([g]\) bijectively onto \([g]\).

Proof. Let \( h, h' \) be two permutations in \( H_n \) and assume that \( h = gh'g^{-1} \). Assume that \( s \) is a positive integer contained in the \( H \)-support \([g]_H \). We shall show that \( h(s) \in [g]_H \). To this end we assume that \( h(s) \notin [g]_H \) and derive a contradiction. By Lemma 4.8, the permutation \( x^{-1} \) is a minimal representative. So there exists a positive integer \( r \) such that \( p(r) = p(g^{-1}(s)) \) but \( p(g(r)) \neq p(g(g^{-1}(s))) = p(s) \). We claim that \( p(h'(r)) \neq p(h'(g^{-1}(s))) \).

We first consider \( h'(g^{-1}(s)) \). Note that \( h' = g^{-1} hg \) and the minimality of \( g \) implies \( N(g^{-1}) = N(g) \in [g]_H \). The assumption \( h(s) \notin [g]_H \) implies \( g^{-1}(hg(g^{-1}(s))) = g^{-1}(h(s)) = h(s) \). In short
\[
(h'(g^{-1}(s))) = h(s) \notin [g]_B \quad \text{and} \quad p(h'(g^{-1}(s))) = p(h(s)).
\]

Next we consider \( h'(r) \). The two different cases are investigated separately:

1. First we assume that the integer \( hg(r) \) is contained in \([g]_H \). This implies that the integer \( h'(r) = g^{-1}hg(r) \) is contained in the \( H \)-support \([g]_H \). On the other hand \( h(s) \notin [g]_H \) as noted in the Eq. (4.4). Hence the integers \( p(h'(r)) \) and \( p(h'(g^{-1}(s))) \) can not be equal. This contradicts with the fact that \( h' \in H_n \).

2. Next we assume that \( hg(r) \) is not contained in \([g]_{H_n} \). The minimality of \( g^{-1} \) and the equality \( N(g) = N(g^{-1}) \) together imply that \( hg(r) \) is fixed by \( g^{-1} \). Consequently we have \( h'(r) = g^{-1}hg(r) = hg(r) \). Our initial assumption \( p(g(r)) \neq p(s) \) yields
\[
\begin{align*}
p(h'(r)) &= p(hg(r)) \\
\text{(as } h \in H_n \text{) } &\neq p(h(s)) \\
\text{(Eq. (4.4)) } &= p(h'(g^{-1}(s))).
\end{align*}
\]
But \( h' \in H_n \) and \( p(r) = p(g^{-1}(s)) \). A contradiction.
Corollary 4.16. If \( g \) is a minimal representative contained in \( S_{kn} \), then

\[
H_n \cap gH_ng^{-1} = (H_{[g]H} \cap gH_{[g]H}g^{-1}) \oplus H_{[kn]-[g]H}.
\]

Proof. Let \( g \) be a minimal representative, and \( h_1 \) be an element of the intersection \( H_n \cap gH_ng^{-1} \). Then there exists \( h_2 \in H_n \) such that \( h_1 = gh_2g^{-1} \). Writing \( h_2 = g^{-1}h_1g \) and applying the previous Lemma we deduce that \( h_2 \) maps \( [g^{-1}]_H \) to itself. Since \( [g^{-1}]_H = [g]_H \), cf. Lemma 4.9, we can write \( h_2 \) as a product \( h_{21}h_{22} \), where \( h_{21} \in H_{[g]H} \) and \( h_{22} \in H_{[kn]-[g]H} \). Consequently

\[
h_1 = gh_2g^{-1} = gh_21h_{22}g^{-1} = gh_21g^{-1}gh_22g^{-1} = gh_21g^{-1}h_{22}.
\]

The elements \( h_1 \) and \( h_{22} \) are in \( H \), thus \( gh_21g^{-1} \) is in \( H \). As \( g \) and \( h_{21} \) are permutations of \( [g]_H \) it follows that \( gh_21g^{-1} \in H_{[g]H} \cap gH_{[g]H}g^{-1} \). Putting these altogether we get that \( h_1 \) is an element of the intersection \( (H_{[g]H} \cap gH_{[g]H}g^{-1}) \oplus H_{[kn]-[g]H} \). The converse inclusion can be shown easily. As a result we get the following decomposition:

\[
H_n \cap gH_ng^{-1} = (H_{[g]H} \cap gH_{[g]H}g^{-1}) \oplus H_{[kn]-[g]H},
\]

\( \square \)

Now we generalize the second part of Remark 4.14.

Corollary 4.17. If \( g \) is a minimal representative in \( S_{kn} \), then

\[
C_{H_n}(g) = C_{H_n}(g) \oplus H_{[kn]-[g]H}.
\]

Proof. Let \( h \) be contained in the centralizer \( C_{H_n}(g) \). The equality \( ghg^{-1} = h \) implies \( h \in H_n \cap gH_ng^{-1} \). Now the result follows by arguing as in the proof of Corollary 4.16 and using the fact that \( H_{[kn]-[g]H} \) is contained in the centralizer of \( g \). \( \square \)

In order to prove the polynomiality of the structure coefficients we need yet another direct-sum decomposition. Recall from the formula given in Corollary 3.7 that the structural coefficients are finite sums of the quotients of the form

\[
\frac{|C_{H_n}(g_1g_2)|}{|C_{H_n}(g_1g_2) \cap g_1H_n g_1^{-1}|}.
\]

By the proof of Proposition 4.13 we may assume that \( g_1 \) and \( g_1g_2 \) are minimal representatives of their \( H \)-double cosets. We will derive the polynomiality result by showing that the quotient in Eq. (4.4) is a polynomial in \( n \). To this end we will combine two direct-sum decompositions given in Eq. (4.4) and Eq. (4.17). So let's assume that \( g_1 \) and \( g_2 \) are two permutations in \( S_{kn} \) and \( g_1 \) and \( g_1g_2 \) are minimal representatives. Let \( T \) denote the union \( [g_1g_2]_H \cup [g_1]_H \) and set

\[
T_1 = [g_1g_2]_H - [g_1]_B, \quad T_2 = [g_1g_2]_H \cap [g_1]_H, \quad \text{and} \quad T_3 = [g_1]_H - [g_1g_2]_H.
\]

Corollary 4.18. For every positive integer \( n \geq m \), the intersection \( C_{H_n}(g_1g_2) \cap g_1H_n g_1^{-1} \) admits the following direct-sum decomposition:

\[
C_{H_n}(g_1g_2) \cap g_1H_n g_1^{-1} = (C_{H_n}(g_1g_2) \cap g_1H_T g_1^{-1}) \oplus H_{[kn]-T}.
\]

Proof. Let \( h \) be an element of the intersection \( C_{H_n}(g_1g_2) \cap g_1H_n g_1^{-1} \). Combining Eq. (6) and Eq. (4.4) we deduce that

\[
h([g_1]_H) = [g_1]_H \quad h([g_1g_2]_H) = [g_1g_2]_H.
\]

As a result, the permutation \( h \) maps \( T \) to itself. One can now prove the existence of the direct-sum decomposition arguing as in Corollary 4.16 and using using the fact that \( h \) fixes \( T \) set-wise. \( \square \)
Corollary 4.19. If \( g_1 \) and \( g_1g_2 \) are minimal representatives of their \( H \)-double cosets then the quotient
\[
\frac{|C_{H_n}(g_1g_2)|}{|C_{H_n}(g_1g_2) \cap g_1H_ng_1^{-1}|}
\]
is a polynomial in \( n \).

Proof. Using the decomposition given in Eq. (4.17) and Eq. (6) we have the equality
\[
(15) \quad \frac{|C_{H_n}(g_1g_2)|}{|C_{H_n}(g_1g_2) \cap g_1H_ng_1^{-1}|} = \frac{|C_{H_{T_1 \cup T_2}}(g_1g_2) \cap g_1H_{T_1 \cup T_2}g_1^{-1}|}{|C_{H_{T_1 \cup T_2 \cup T_3}}(g_1g_2) \cap g_1H_{T_1 \cup T_2 \cup T_3}g_1^{-1}|} \cdot \frac{|H_{[kn]} - (T_1 \cup T_2)|}{|H_{[kn]} - (T_1 \cup T_2 \cup T_3)|}
\]
If we denote the cardinality \( |T_i| \) by \( kt_i \) then Eq. (4.4) can be written as
\[
\frac{|C_{H_n}(g_1g_2)|}{|C_{H_n}(g_1g_2) \cap g_1H_ng_1^{-1}|} = \zeta_{g_1,g_2} \cdot \frac{(k!)^{n-t_1-t_2}(n-t_1-t_2)!}{(k!)^{n-t_1-t_2-t_3}(n-t_1-t_2-t_3)!}
\]
where \( \zeta_{g_1,g_2} \) denotes the first multiplicand of the RHS of the Eq. (4.4), which is independent of \( n \). In other words
\[
\frac{|C_{H_n}(g_1g_2)|}{|C_{H_n}(g_1g_2) \cap g_1H_ng_1^{-1}|} = \zeta_{g_1,g_2} \cdot (k!)^t_3(n-t_1-t_2)(n-t_1-t_2-1) \cdots (n-t_1-t_2-t_3+1).
\]

We are ready to show that the structure coefficients are polynomial functions.

Proposition 4.20. The structure coefficient \( c^{L}_{M,N}(n) \) is a polynomial in \( n \), for all \( H \)-double cosets \( M, N, \) and \( L \).

Proof. Let \( M, N \) and \( K \) be three \( H \)-double cosets. Since the family \( (S_{kn}, B_n) \) is admissible by Proposition 4.13, we may apply Corollary 3.7. So the structure coefficient \( c^{L}_{M,N}(n) \) is of the following form
\[
c^{L}_{M,N}(n) = \sum_{i=1}^{r} \frac{|C_{H_n}(g_1;g_2)|}{|C_{H_n}(g_1;g_2) \cap g_{1i}H_ng_{1i}^{-1}|},
\]
where \( g_{1i} \) and \( g_{1i;g_2} \) are minimal representatives of their \( H \)-double cosets. Since each summand of the RHS is a polynomial in \( n \) by Corollary 4.19, so is \( c^{L}_{M,N}(n) \). \( \square \)

Remark 4.21. If \( [g_1]_H \subset [g_1g_2]_H \) then \( t_3 = 0 \), which implies that the index
\[
\frac{|C_{H_n}(g_1g_2)|}{|C_{H_n}(g_1g_2) \cap g_1H_ng_1^{-1}|}
\]
is independent of \( n \) and equal to \( \zeta_{g_1,g_2} \).

5. The Modified Coset Type

We have already seen that the embedding \( \overset{\dagger}{\iota} : S_{kn} \rightarrow S_{kn+kt} \) maps \( H_n \) into \( H_{n+t} \). If we denote the graph with one vertex and \( k(k-1)/2 \) loops by \( \circ_k \), then
\[
G_{g^{\overset{\dagger}{\iota}}} = G_g \sqcup \circ_k \sqcup \cdots \sqcup \circ_k^t \text{ many}.
\]
This means the attached graph \( G_g \) of a finitary permutation \( g \in S_\infty \) depends on the actual \( S_{kn} \) that realizes \( g \) as an element of \( S_\infty \). To remove the dependency we introduce the modified coset type of an element. Analogous definitions are given in the other works, see [1, Sec. 3.1], and [11, Sec. 3.1].
Definition 5.1. Let $g$ be a finitary permutation and assume that $g \in S_{kn}$ for some positive integer $n$. The modified coset type of the permutation $g$ is the graph $M_g$ that is obtained from $G_g$ by removing the isolated vertices of $G_g$. The set of isomorphism classes of modified coset types is denoted by $\mathcal{M}$.

From now on a graph $M$ will be denoted as a pair $(V,E)$, where the first (resp. second) component indicates the set of vertices (resp. edges) of $M$. The proof of the next lemma is straightforward.

Lemma 5.2. (1) Let $g$ be a finitary permutation in $S_\infty$. The modified coset type $M_g$ of $g$ is independent of the $S_{kn}$ in which $g$ is realized.

(2) The map sending the element $g$ to its modified coset type $M_g$ defines a bijection $H_\infty \backslash S_\infty / H_\infty \longrightarrow \mathcal{M}$, between the $H_\infty$-double cosets in $S_\infty$ and the set of modified coset types.

(3) Let $M = (V,E)$ be a modified type. The group $S_{kn}$ admits an element of $M$ if and only if $|V| \leq n$.

Let $M = (V,E)$ be an arbitrary graph. We will denote the number of connected components of $M$ by $w_M$ or $w_M(V,E)$ depending on the convenience. The connected components of $M$ can be denoted by $C_i = (V_i,E_i)$, where $i$ sums over the set $\{1,\ldots,w_M\}$. Denote the number of vertices of $C_i$ by $c_i$. Without loss of generality we may assume that $c_i$’s are in non-increasing order. As a result we obtain a partition $(c_1,\ldots,c_{w_M})$, which we denote by $\lambda_M$. The weight $||M||$ of the graph $M$ is defined by setting $||M|| := w_M \sum_{i=1}^{w_M} (c_i - 1) = |V| - w_M$.

We note that the modification of an arbitrary graph makes sense and for an arbitrary graph $M$ the equality $||M^\circ|| = ||M||$ holds.

With this notation we are able to state the main theorem of this article.

Theorem 5.3 (Stability theorem). Let $M, N$ and $L$ be three modified types. Then

$||M|| + ||N|| < ||L|| \implies c_{M,N}^L(n) = 0,$

and

$||M|| + ||N|| \geq ||L|| \implies c_{M,N}^L(n) \geq 0.$

In case of the equality $||L|| = ||M|| + ||N||$, the polynomial $c_{M,N}^L(n)$ is constant, i.e. independent of $n$.

Let $Z$ be the ring of functions $f : Z \rightarrow Z$ and for each modified coset type $X_M$ be an indeterminate which are algebraically independent over $Z$. Introduce the $Z$-algebra $\mathcal{F}_{\infty,k} := Z[X_M : M \in \mathcal{M}_k]$ where the multiplication of the indeterminates are defined as follows.

$X_M \cdot X_N = \sum_{||L|| = ||M|| + ||N||} c_{M,N}^L(n) X_L.$

By the first assertion of the stability theorem the sum is finite and hence make sense. The stability theorem then reads as $\mathcal{F}_{\infty,k}$ is an associative filtered algebra whose structure coefficients are non-negative integers.
6. DISCRETE GRAPH EVOLUTION

Throughout this section we fix an abstract set $V$ of vertices $v_1, \ldots, v_n$. Let $M = (V, E)$ be an arbitrary graph. The connected components of $M$ are denoted by $C_i = (V_i, E_i)$, for $i = 1, \ldots, w_M$. Let $e$ be an edge of $M$. The vertices that the edge $e$ joins are called the end-points of the edge $e$. The set of end-points of $e$ is denoted by $V(e)$. The definition of end-points directly generalizes to arbitrary subsets of the edge set $E$. If $D$ is a subset of $E$ then the set of end-points of $D$ is denoted by $V(D)$. The graph on the set of vertices $V(D)$ whose edge set is $D$ is denoted by $G(D)$. If $v$ is a vertex of $M$ then the connected component that contains the vertex $v$ is denoted by $C_M(v)$. In general, if $W$ is a subset of $V$ then the graph $C_M(W)$ is the union

$$C_M(W) = \bigcup_{v \in W} C_M(v).$$

The relative size $s_M(W)$ of $W$ relative to $M$ is defined as the positive integer

$$s_M(W) = \# \{i \in \mathbb{N} \mid V_i \cap W \neq \emptyset \} = w(C_M(W)),$$

the minimal number of connected components of $M$, whose union contains the vertex set $W$. Let $D$ be a set of edges of the graph $M$ and $G(D)$ denote the sub-graph of $M$ whose vertex set is $V(D)$. The relative size $s_M(D)$ of $D$ relative to $M$ is defined as the positive integer

$$s_M(D) = \# \{i \in \mathbb{N} \mid E_i \cap D \neq \emptyset \} = w(C_M(V(D))) = s_M(V(D)),$$

the relative size the of the set of end-points of $D$. The non-relative size $s(D)$ of $D$ is defined as the positive integer

$$s(D) = w(G(D)),$$

the number of connected components of the sub-graph $G(D)$. Note that the non-relative size of an edge set implicitly assumes the existence of an ambient graph. However, this does not provide any problem as the sub-graph $G(D)$ is completely determined by the edge set $D$ alone. We note the the relative size of an edge set is bounded by its non-relative size. That is

$$s_M(D) \leq s(D).$$

This inequality is a consequence of the fact that vertices that are connected by an edge in $D$ are also connected in the ambient graph $M$. Finally we note that the non-relative site is a sub-additive function in the following sense: If $E_1$ and $E_2$ are two edge sets then

$$s(E_1 \cup E_2) \leq s(E_1) + s(E_2),$$

where the equality does not necessarily hold, even if the edge sets $E_1$ and $E_2$ are disjoint.

**Example 6.1.** Consider the graph $M = G_g$ attached to $g = (1, 8, 18, 6, 10, 13, 2, 11, 3, 12)(4, 16)(5, 17)$, where $k = 3$, and $n = 7$. 

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We denote the connected components of $M$ from left to right by $C_1$, $C_2$ and $C_3$. If $W = \{v_1, v_2, v_3, v_5\}$ then the graph $C_M(W)$ is the disjoint union of $C_1$ and $C_2$, because $C_M(v_i) = C_i$ for $i = 1, 2, 3$, and $C_M(v_5) = C_2$. This also means that the relative size $s_M(W)$ of $W$ with respect to $M$ is equal to 2. Let $D$ be the edge set $\{e_{7,8}, e_{2,3}, e_{10,14}\}$. Then the set of end-points of $D$ is equal to $W$, hence $s_M(D) = 2$. Number of connected components of $G(D)$ is 3. As a result, the non-relative size $s(D)$ of the edge set $D$ is 3, which is an example of the strict inequality $s_M(D) < s(D)$.

**Definition 6.2.** Let $M_0 = (V, E_0)$ and $M_1 = (V, E_1)$ be two graphs on the set of vertices $V$. The graph $M_1$ is evolved from $M_0$ through the edge replacement pairs $(E_{0i}, E_{1i})_{i=1}^t$, if the collection $E_{01}, \ldots, E_{0t}$ defines a set partition of the vertex set $E_0$ of $M_0$, and the collection $E_{11}, \ldots, E_{1t}$ defines a set-partition of the edge set $E_1$ of $M_1$ such that

$$V(E_{0i}) = V(E_{1i})$$

for all $i = 1, \ldots, t$. The sequence graphs

$$M_0 = G_0, \ldots, G_t = M_1,$$

is called the evolution chain, where the $G_i$ is obtained from $G_{i-1}$ by replacing the edges $E_{0i}$ by $E_{1i}$, for $i = 1, \ldots, t$. The number $t$ is called the length of the evolution.

The change in the edge set $E_{G_i}$ of the graph $G_i$ as $i$ increases can be depicted as follows:

$$E_{G_{i-1}} = E_{11} \sqcup \cdots \sqcup E_{1(i-1)} \sqcup E_{0i} \sqcup E_{0(i+1)} \sqcup \cdots \sqcup E_{0t},$$

$$E_{G_i} = E_{11} \sqcup \cdots \sqcup E_{1(i-1)} \sqcup E_{1i} \sqcup E_{0(i+1)} \sqcup \cdots \sqcup E_{0t}.$$  

**Lemma 6.3.** Let $M_0$ and $M_1$ be two graphs and assume that $M_1$ is evolved from $M_0$ through the edge replacement pairs $(E_{0i}, E_{1i})_{i=1}^t$. The weight of the graphs in the evolution chain obey the inequality

$$||G_i|| \leq ||G_{i-1}|| + (s(E_{0i}) - 1).$$
In particular

$$||M_1|| \leq ||M_0|| + \left( \sum_{j=1}^{t} s(E_{0j}) \right) - t.$$  

Proof. Let \( i \) be a fixed positive integer in \([t]\), and denote the connected components of \( G_{i-1} \) by \( C_j = (V_j, E_j) \), for \( j = 1, \ldots, w \). By \( W_i \) we mean the end-point sets \( V(E_{0i}) \) and \( V(E_{1i}) \), which are equal by assumption. Some of the connected components of \( G_{i-1} \) contains vertices contained in \( W_i \). Without loss of generality we may assume that \( C_1, \ldots, C_t \) are the connected components that contains an element in \( W_i \). In other words, the graph \( C_{H_{i-1}}(W_i) \) is equal to union of the graphs \( C_1, \ldots, C_t \). Hence we deduce

$$l = s_{G_{i-1}}(W_i) = s_{G_{i-1}}(E_{0i}) \leq s(E_{0i}).$$

The graph \( G_{i-1} \) can be written as a disjoint union of graphs

$$G_{i-1} = C_{G_{i-1}}(W_i) \sqcup C_{i+1} \sqcup \cdots \sqcup C_r.$$  

Since the end-points of \( E_{0i} \) and \( E_{1i} \) are all contained in \( W_i \), the process of passing from \( G_{i-1} \) to \( G_i \) only effects the graph \( C_{G_{i-1}}(W_i) \). Thus the graph \( G_i \) admits the decomposition

$$G_i = K_i \sqcup C_{i+1} \sqcup \cdots \sqcup C_r,$$

where \( K_i \) is the graph obtained from \( C_{G_{i-1}}(W_i) \) by replacing the edges \( E_{0i} \) with the edges \( E_{1i} \). The decomposition in Eq.(6) implies

$$||G_i|| = ||K_i|| + ||C_{i+1}|| + \cdots + ||C_r||.$$  

The vertex set \( V_{K_i} \) of the graph \( K_i \) is equal to the vertex set of \( C_{G_{i-1}}(W_i) \). Thus the definition of graph weight yields

$$||C_{G_{i-1}}(W_i)|| - ||K_i|| = |w(C_{G_{i-1}}(W_i)) - w(K_i)| = |l - w(K_i)|.$$

Using this equality and combining Eqs. (6) and (6) we deduce

$$||G_i|| - ||G_{i-1}|| = |l - w(K_i)| \leq l - 1 \leq s_{E_{0i}} - 1.$$

The first inequality follows because \( w(C_{H_{i-1}}(W_i)) = 1 \) and \( w(K_i) \) is bounded below by 1, where the second inequality is a consequence of Eq.(6).

Next Lemma will play a crucial role in the proof of the stability of the family \((S_{kn}, H_n)_{n \in \mathbb{N}^+}\).

Corollary 6.4. Let \( M_0 \) and \( M_1 \) be two graphs and assume that \( M_1 \) is evolved from \( M_0 \) through the edge replacement pairs \((E_{0i}, E_{1i})_{i=1}^{t}\). If the inequality

$$||M_1|| \leq ||M_0|| + \left( \sum_{j=1}^{t} s(E_{0j}) \right) - t$$

is an equality then the connected components of \( M_0 \) remain connected in \( M_1 \).

Proof. Our proof relies on the proof of Lemma 6.3. By Eq.(6) and Eq.(6) the connected components \( C_{i+1}, \ldots, C_r \) of the graph \( G_{i-1} \) remain connected in \( G_i \). So we need to prove that the connected components \( C_1, \ldots, C_t \) of \( G_{i-1} \) remain connected in \( G_i \). The vertex set of the graph \( K_i \) is equal to the union of the connected components \( C_1, \ldots, C_t \). So it suffices to prove that \( K_i \) is connected. Assume that (6.4) is an equality. Thus Eq.(6) is an equality, for every \( i = 1, \ldots, t \). In particular \( w(K_i) = 1 \), for every \( i = 1, \ldots, t \), as a result \( K_i \) is connected. \(\square\)
In the next section we will use Lemma 6.3 to show that the Hecke algebras \( \mathcal{H}_n \) admits a filtration. In the course of implementing Lemma 6.3 we will need some more terminology concerning the attached graphs, which we will introduce now. So we turn our attention to the graphs of the form \( G_g \) for \( g \in S_{kn} \). Recall that the graph \( G_g = (V,E_g) \) is defined as follows. The vertex set is by definition \( V = \{v_1, \ldots, v_n\} \) and

\[
E_g := \bigcup_{u=1}^l E_{g,u}
\]

where

\[
E_{g,u}^0 := \{e_{r,s}^g = e_{s,r}^g | r, s \in \Gamma_u, r \neq s\}.
\]

The end-points of the edge \( e_{r,s}^g = e_{s,r}^g \) are \( v_{p(g^{-1}(r))} \) and \( v_{p(g^{-1}(s))} \). Thus we may write

\[
e_{r,s}^g = \{v_{p(g^{-1}(r))}, r\} \cup \{v_{p(g^{-1}(s))}, s\}.
\]

The edges \( e_{r,s}^g \) and \( e_{r',s'}^g \) are equal if and only if \( \{r, s\} = \{r', s'\} \). Note that \( p(g^{-1}(r)) = i \) just means that \( g^{-1}(r) \in \Gamma_i \). A vertex \( v_i \) is an end-point of \( e_{r,s}^g \) if and only if \( g^{-1}(r) \in \Gamma_i \) or \( g^{-1}(s) \in \Gamma_i \).

**Lemma 6.5.** Let \( g \) be an arbitrary permutation in \( S_{kn} \) and \( u \) be an element of \([n]\). The set of end-points \( V(E_u^0) \) of the edge set \( E_{u}^0 \) is given as follows:

\[
V(E_u^0) = \{v_i \in V : \Gamma_i \cap g^{-1}(\Gamma_u) \neq \emptyset\}.
\]

More generally, if \( J \subset \{1, \ldots, n\} \) then

\[
V(\bigcup_{u \in J} E_u^0) = \{v_i \in V : \Gamma_i \cap g^{-1}(\bigcup_{u \in J} \Gamma_u) \neq \emptyset\}.
\]

**Proof.** Directly follows from the definitions. \( \square \)

**Lemma 6.6.** Let \( g \) be an arbitrary permutation in \( S_{kn} \) and \( C \) be a connected component of \( G_g \). Let \( J_C \) be the set of indices \( u \in \{1, \ldots, n\} \) such that the vertex \( v_u \) is contained in \( C \). Then there exists a unique subset \( g(J_C) \) of \( \{1, \ldots, n\} \) such that

\[
g(\bigcup_{u \in J_C} \Gamma_u) = \bigcup_{u \in g(J_C)} \Gamma_u.
\]

**Proof.** We first observe that \( J_C = \{u \in [n] \mid v_u \in C\} \). Now let us assume that \( r \) is an element of \( g(\bigcup_{u \in J} \Gamma_u) \). Then \( p(g^{-1}(r)) \in J \) and hence the vertex \( v_{p(g^{-1}(r))} \) is an element of the connected component \( C \). We shall show that partners of \( r \) also contained in the set \( g(\bigcup_{u \in J} \Gamma_u) \). Let \( s \) be a partner of \( r \). Then there is an edge \( e_{s,r}^g \) that joins the vertices \( v_{p(g^{-1}(s))} \) and \( v_{p(g^{-1}(r))} \). Since \( C \) is a connected component, the vertex \( v_{p(g^{-1}(s))} \) is an element of \( J_C \). From this we conclude that \( p(g^{-1}(s)) \in J \). In other words, \( s \in g(\bigcup_{u \in J_C} \Gamma_u) \). \( \square \)

**Lemma 6.7.** In every \( H \)-double coset one can find a minimal representative \( g \) such that \( g(J_C) = J_C \) for every connected component \( C \) of the graph \( G_g \). Such an element will be referred as an element with closed connected components.

**Proof.** Let \( g \) be a minimal element. There is an element \( h \) of \( H \) that maps

\[
\bigcup_{u' \in g(J_C)} \Gamma_{u'} \longrightarrow \bigcup_{u \in J_C} \Gamma_u
\]

bijectively. Up to multiplication with an element of the Young subgroup \( S_{\Gamma_1} \times \cdots \times S_{\Gamma_n} \) one may assume \( hg(r) = r \) whenever \( p(hg(r)) = p(r) \). The element \( hg \) is minimal and satisfies the claim of the lemma. \( \square \)
Remark 6.8. Let $g$ be an element with closed connected components. We note that the equalities

$$g\left( \bigcup_{u \in J_C} \Gamma_u \right) = \bigcup_{u \in J_C} \Gamma_u = g^{-1}\left( \bigcup_{u \in J_C} \Gamma_u \right)$$

hold for every connected component $C$ of $G_g$. It is also clear that $|J_C|$ is the number of vertices in $C$, which is a direct consequence of the fact that $J_C$ is the set of indices $u$ for which $v_u$ is a vertex in $C$.

Lemma 6.9. If $J$ is an arbitrary subset of $\{1, 2, \ldots, n\}$, then

$$s(\bigcup_{u \in J} E_u^g) \leq |J|.$$ 

Proof. Let $v_i, v_j \in V(E_u^g)$. Then there exists $r, s \in \Gamma_u$ such that $g^{-1}(r) \in \Gamma_i$ and $g^{-1}(s) \in \Gamma_j$. But this means $v_i$ and $v_j$ are joined by the edge $e_{r,s}^g$. This proves that $G(s(E_u^g))$ is connected. The general case follows from the sub-additivity of $s(\gamma)$, cf. Eq.(6). \qed

7. Proof of stability

In this section we prove the stability theorem.

Theorem (Stability theorem). Let $M, N, L$ be three modified types. Then

$$||L|| > ||M|| + ||N|| \implies c_{M,N}^L(n) = 0,$$

and

$$||L|| \leq ||M|| + ||N|| \implies c_{M,N}^L(n) \geq 0.$$ 

In case of the equality $||L|| = ||M|| + ||N||$, the polynomial $c_{M,N}^L(n)$ is constant, i.e. independent of $n$.

Let $g_1, g_2$ be two permutations in $S_{mk}$. Assume that the permutations $g_1$ and $g_1 g_2$ are minimal representatives of their $B$-double cosets. If necessary by replacing $g_1$ with $hg_1$ and $g_2$ with $g_2 h^{-1}$ for some suitably chosen $h$ in $H$, we may assume that $g_1$ and $g_1 g_2$ are minimal representatives, and $g_1$ has closed connected components. Our aim is to show that $G_{g_1 g_2}$ is evolved from $G_{g_2}$. Let $C_1, \cdots, C_t$ be the connected components of $G_{g_1}$. If we denote the number of vertices of $C_i$ by $l_i$, then

$$||G_{g_1}|| = \sum_{i=1}^t (l_i - 1).$$

(24)

We note that by Remark 6.8 the set equality

$$\bigcup_{u \in J_C} g_1(\Gamma_u) = \bigcup_{u \in J_C} \Gamma_u,$$

holds for every connected component $C$ of $G_{g_1}$, since $g_1$ is an element with closed connected components. For each connected component $C$ of $G_g$ we will define edge sets $E_C^{g_2}$ and $E_C^{g_1 g_2}$ that will serve as the edge replacement pairs of the evolution of $M_{g_2}$ into $M_{g_1 g_2}$. In fact, let

$$E_C^{g_2} := \bigcup_{u \in J_C} E_u^{g_2} \quad \text{and} \quad E_C^{g_1 g_2} := \bigcup_{u \in J_C} E_u^{g_1 g_2}$$
Since the sets $J_C$ covers $[n]$ as $C$ runs over the connected components of $G_{g_1}$, it follows that

$$E_{g_2} = \bigcup_{i=1}^{t} E_{C_i}^{g_2} \quad \& \quad E_{g_1 g_2} = \bigcup_{i=1}^{t} E_{C_i}^{g_1 g_2}.$$ 

**Lemma 7.1.** If $C$ is a connected component of $G_{g_1}$ then the equality

$$V(E_{g_2}^{C}) = V(E_{g_1 g_2}^{C}).$$

between the end-points of the edge sets $E_{g_2}^{C}$ and $E_{g_1 g_2}^{C}$ holds.

**Proof.** By Lemma 6.5 and Eq. (7) we have

$$V(E_{g_2}^{C}) = V\left(\bigcup_{j \in J_C} E_{j}^{g_2}\right)$$

(By Lemma 6.5) = \{v_i \in V : \Gamma_i \cap g_2^{-1}\left(\bigcup_{u \in J_C} \Gamma_u\right) \neq \emptyset\}

= \{v_i \in V : \Gamma_i \cap (g_1 g_2)^{-1} g_1 \left(\bigcup_{u \in J_C} \Gamma_u\right) \neq \emptyset\}

(By Eq. (7)) = \{v_i \in V : \Gamma_i \cap (g_1 g_2)^{-1} \left(\bigcup_{u \in J_C} \Gamma_u\right) \neq \emptyset\}

(By Lemma 6.5) = V\left(\bigcup_{j \in J_C} E_{j}^{g_1 g_2}\right)

= V(E_{g_1 g_2}^{C}).$$

\[\square\]

By the Lemma we see that $G_{g_1 g_2}$ is evolved from $G_{g_2}$ through the edge replacement pairs $(E_{C_i}^{g_2}, E_{C_i}^{g_1 g_2})_{i=1}^{t}$. Since the edge set $E_{C_i}^{g_2}$ is equal to the union of $E_{u}^{g_2}$ where $u \in J_{C_i}$, Lemma 6.9 implies that

$$s(E_{C_i}^{g_2}) \leq |J_{C_i}| = l_i.$$ 

Invoking Lemma 6.3 we deduce the following:

$$||G_{g_1 g_2}|| \leq ||G_{g_2}|| + \sum_{i=1}^{t} (s(E_{C_i}^{g_2}) - 1)$$

(By Eq. (7)) \leq ||G_{g_2}|| + \sum_{i=1}^{t} (l_i - 1)

(By Eq. (7)) = ||G_{g_2}|| + ||G_{g_1}||.

This proves the first statement of the stability theorem which states that $e_{M,N}^L = 0$ whenever $||L||$ exceeds the sum $||M|| + ||N||$.

**Proposition 7.2.** Let $g_1, g_2$ be two permutations in $S_{kn}$. Assume that $g_1$ is a minimal representative with closed connected components and $g_1 g_2$ is a minimal representative. If the equality

$$||G_{g_1 g_2}|| = ||G_{g_1}|| + ||G_{g_2}||$$

holds then the following hold:

1. $[g_2]_H \subseteq [g_1 g_2]_H$.
2. $N(g_2) \subseteq [g_1 g_2]_H$.
3. $[g_1]_H \subseteq [g_1 g_2]_H$. 

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Before proving the Proposition we note the following. In the light of Remark 4.21, the inclusion $[g_1]_H \subseteq [g_1g_2]_H$ implies that $c^K_{M,N}(n)$ is independent of $n$, hence finishes the proof of the stability theorem. Note also that $\mathcal{N}(g_1) \subset [g_1]_H$ and $\mathcal{N}(g_1g_2) \subset [g_1g_2]_H$ because $g_1$ and $g_1g_2$ are minimal representatives. However, it is not necessarily true that $\mathcal{N}(g_2) \subset [g_2]_H$.

Proof. (1) Assume that for some positive integer $j$ the set $\Gamma_j$ is a subset of $[g_2]_H$. Then by Remark 4.6, the vertex $v_j$ is not an isolated vertex of $G_{g_2}$. If $C$ denotes the connected component of $v_j$ then $|C| \geq 2$. By Remark 6.4, the set of vertices of the connected component $C$ completely contained in a connected component $C'$ of $G_{g_1g_2}$. As a result $|C'| \geq 2$. Thus the vertex $v_j$ is not an isolated vertex of $G_{g_1g_2}$. Using Remark 4.6 once again, we see that $\Gamma_j \subset [g_1g_2]_H$. This proves the inclusion $[g_2]_H \subset [g_1g_2]_H$.

(2) Next we prove $\mathcal{N}(g_2) \subseteq [g_1g_2]_H$. Using the previous item it suffices to show that $\mathcal{N}(g_2) - [g_2]_H \subset [g_1g_2]_H$. So let $r$ be an integer contained in the difference $\mathcal{N}(g_2) - [g_2]_H$. For simplicity we write $p(r) = i$, hence we have $r \in \Gamma_i$. Since $H$-supports contain elements that are partners as a whole, it follows that $\Gamma_i$ is a subset of the complement $[g_2]_H^c$. By definition of $H$-support $g(\Gamma_i) = \Gamma_j$ for some positive integer $j$. Since $r$ is contained in the complement $[g_1g_2]_H^c$ it follows that $\Gamma_i$ is a subset of the complement $[g_1g_2]_H^c$. By the minimality, the permutation $g_1g_2$ acts on the complement $[g_1g_2]_H^c$ as identity. But this means

$$\Gamma_i = g_1g_2(\Gamma_i) = g_1(\Gamma_j).$$

Since $g_1$ is a minimal representative it follows that $i = j$ and $g_1$ acts on $\Gamma_i$ as identity. As a result $g_2$ acts on $\Gamma_i$ as identity, in particular $g_2(r) = r$. But we picked the element $r$ from $\mathcal{N}(g_2) - [g_2]_H$, a contradiction. So either $\mathcal{N}(g_2) - [g_2]_H = \emptyset$ or our assumption $r \notin [g_1g_2]_H$ is wrong. Both cases imply that $\mathcal{N}(g_2) \subset [g_1g_2]_H$.

(3) It suffices to show that $g_1$ acts on $([g_1g_2]_H)^c$ as identity. But this is clear since $g_2$ and $g_1g_2$ act on $([g_1g_2]_H)^c$ as identity. 

\[\Box\]

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