ALMOST PERIODIC SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH EXPONENTIAL DICHOTOMY DRIVEN BY LÉVY NOISE

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Abstract. In this paper, we study almost periodic solutions for semilinear stochastic differential equations driven by Lévy noise with exponential dichotomy property. Under suitable conditions on the coefficients, we obtain the existence and uniqueness of bounded solutions. Furthermore, this unique bounded solution is almost periodic in distribution under slightly stronger conditions. We also give two examples to illustrate our results.

1. Introduction

Since it was introduced in the 1920s by Bohr and Bochner, almost periodicity has been extensively studied in differential equations and dynamical systems. Following the pioneering work of Favard [11], the separation condition was adopted by Amerio [1], Fink [12], Seifert [24] et al to study the almost periodic solutions to differential equations. Apart from the separation method, there are also other methods to investigate almost periodic solutions, e.g. the stability or Lyapunov function method in dynamical systems by Miller [19], Yoshizawa [29] et al, the skew-product flow method by Sacker and Sell [23], Shen and Yi [25] et al, the fixed point method by Coppel [7] et al.

From 1980s, the almost periodic solutions to stochastic differential equations driven by Gaussian noise have been well developed, see Arnold and Tudor [3], Bezandry and Diagana [4, 6], Da Prato and Tudor [8] and Halanay [14], Tudor [27], among others.

Lévy processes are stochastic processes with independent and stationary increments, and have many applications ranging from physics, biology, to finance etc. There are many well known examples of Lévy processes, such as Wiener processes, Poisson processes and stable processes. We refer the reader to Sato [22] for the theory of Lévy processes, to Applebaum [2] and Peszat and Zabczyk [21] for finite and infinite dimensional stochastic differential equations with Lévy noise perturbations, respectively. Wang and Liu [25] investigated almost periodic solutions in square-mean sense to stochastic differential equations perturbed by Lévy noise. Indeed, as indicated in [15, 18], it appears that almost periodicity in distribution sense is a more appropriate concept relatively to solutions of stochastic differential equations. Recently, Liu and Sun [16] and Sun [26] studied almost automorphic in distribution solutions of stochastic differential equations with Lévy noise.

In this paper, we study the existence and uniqueness of bounded solution that is almost periodic in distribution to semilinear stochastic differential equations with exponential dichotomy property driven by infinite dimensional Lévy processes which may admit large jumps.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and results of Lévy processes and almost periodic processes. In Sections 3, we investigate the existence and uniqueness of bounded solution for semilinear stochastic differential equations with exponential dichotomy property driven by Lévy noise. Moreover, the unique bounded solution is almost periodic in distribution under suitable conditions. In Section 4, we give two examples to illustrate the theoretical results obtained in this paper.

2000 Mathematics Subject Classification. 60H10, 34C27, 60G51, 34G20.

Key words and phrases. Almost periodicity, stochastic differential equations, exponential dichotomy, Lévy process.

This work is part of the author’s Ph.D thesis which is completed on April 2013. This work is partially supported by NSFC Grants 11271151, 11101183.
2. Preliminaries

Throughout the rest of this paper, we assume that \( (\mathbb{H}, \| \cdot \|) \) and \( (U, \cdot |_{U}) \) are real separable Hilbert spaces. We denote by \( L(U, \mathbb{H}) \) the space of all bounded linear operators from \( U \) to \( \mathbb{H} \). Note that \( L(U, \mathbb{H}) \) is a Banach space, and we denote the norm by \( \| \cdot \|_{L(U, \mathbb{H})} \). Let \( (\Omega, \mathcal{F}, \mathbf{P}) \) be a probability space, and \( L^2(\mathbb{P}, \mathbb{H}) \) stand for the space of all \( \mathbb{H} \)-valued random variables \( Y \) such that

\[
\mathbb{E}\|Y\|^2 = \int_{\Omega} \|Y\|^2 \, d\mathbf{P} < \infty.
\]

For \( Y \in L^2(\mathbb{P}, \mathbb{H}) \), let

\[
\|Y\|_2 := \left( \int_{\Omega} \|Y\|^2 \, d\mathbf{P} \right)^{1/2}.
\]

Then \( L^2(\mathbb{P}, \mathbb{H}) \) is a Hilbert space equipped with the norm \( \| \cdot \|_2 \). The Lévy process we consider is \( U \)-valued.

2.1. Lévy process and Lévy-Itô decomposition.

**Definition 2.1.** A \( U \)-valued stochastic process \( L = (L(t), t \geq 0) \) is called Lévy process if:

1. \( L(0) = 0 \) almost surely;
2. \( L \) has independent and stationary increments;
3. \( L \) is stochastically continuous, i.e. for all \( \epsilon > 0 \) and for all \( s > 0 \)
   \[
   \lim_{t \to s} P(|L(t) - L(s)|_U > \epsilon) = 0.
   \]

Let \( L \) be an Lévy process. We recall some definitions and Lévy-Itô decomposition theorem; see [2, 21, 22] for details.

**Definition 2.2.**

1. A Borel set \( B \) in \( U \setminus \{0\} \) is bounded below if \( 0 \notin \overline{B} \), the closure of \( B \).
2. \( \nu(\cdot) = \mathbb{E}(N(1, \cdot)) \) is called the intensity measure associated with \( L \), where \( \Delta L(t) = L(t) - L(t-) \) for each \( t \geq 0 \) and
   \[
   N(t, B)(\omega) := \sharp\{0 \leq s \leq t : \Delta L(s)(\omega) \in B\} = \sum_{0 \leq s \leq t} \chi_B(\Delta L(s)(\omega))
   \]
   with \( \chi_B \) being the indicator function for any Borel set \( B \) in \( U \setminus \{0\} \).
3. \( N(t, B) \) is called Poisson random measure if \( B \) is bounded below, for each \( t \geq 0 \).
4. For each \( t \geq 0 \) and \( B \) bounded below, we define the compensated Poisson random measure by
   \[
   \tilde{N}(t, B) = N(t, B) - t\nu(B).
   \]

**Proposition 2.3** (Lévy-Itô decomposition). If \( L \) is a \( U \)-valued Lévy process, then there exist \( a \in U \), a \( U \)-valued Wiener process \( W \) with covariance operator \( Q \), so called \( Q \)-Wiener process, and an independent Poisson random measure \( N \) on \( \mathbb{R}^+ \times (U \setminus \{0\}) \) such that, for each \( t \geq 0 \)

\[
L(t) = at + W(t) + \int_{|x|_U < 1} x\tilde{N}(t, dx) + \int_{|x|_U \geq 1} xN(t, dx).
\]

Here the Poisson random measure \( N \) has the intensity measure \( \nu \) which satisfies

\[
\int_{U} (|y|_U^2 \wedge 1)\nu(dy) < \infty
\]

and \( \tilde{N} \) is the compensated Poisson random measure of \( N \).

In this paper, Wiener processes we consider are \( Q \)-Wiener processes; see [9] for details. For simplicity, we assume that the covariance operator \( Q \) of \( W \) is of trace class, i.e. \( \text{Tr}Q < \infty \). Assume that \( L_1 \) and \( L_2 \) are two independent, identically distributed Lévy processes with decompositions as in Proposition 2.3 with \( a, Q, W, N \). Let

\[
L(t) = \begin{cases} 
L_1(t), & \text{for } t \geq 0, \\
-L_2(-t), & \text{for } t \leq 0.
\end{cases}
\]
Then $L$ is a two-sided Lévy process. We assume that the two sided Lévy process $L$ is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}})$. By the Lévy-Itô decomposition, it follows that $\int_{|x| \geq 1} \nu(dx) < \infty$. Then we denote $b := \int_{|x| \geq 1} \nu(dx)$ throughout this paper. We note that the process $\hat{L} := (\hat{L}(t) = L(t+s) - L(s))$ for some $s \in \mathbb{R}$ is also a two-sided Lévy process with the same law as $L$.

2.2. Square-mean almost periodic processes.

**Definition 2.4.** [6] A stochastic process $Y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be $L^2$-continuous if for any $s \in \mathbb{R}$,

$$\lim_{t \to s} \mathbb{E}\|Y(t) - Y(s)\|^2 = 0.$$ 

It is $L^2$-bounded if $\sup_{t \in \mathbb{R}} \|Y(t)\|_2 < \infty$.

Note that if an $\mathbb{H}$-valued process is $L^2$-continuous, then it is necessarily stochastically continuous.

**Definition 2.5.** [28] (1) An $L^2$-continuous stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean almost periodic if for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and an $L^2$-continuous stochastic process $y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \mathbb{E}\|x(t + s_n) - y(t)\|^2 = 0.$$ 

The collection of all square-mean almost periodic stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is denoted by $AP(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

(2) A continuous function $g : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L(U, L^2(\mathbb{P}, \mathbb{H}))$, $(t, Y) \mapsto g(t, Y)$ is said to be uniformly square-mean almost periodic if for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a continuous function $\tilde{g} : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L(U, L^2(\mathbb{P}, \mathbb{H}))$ such that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}, Y \in \mathbb{K}} \mathbb{E}\|g(t + s_n, Y) - \tilde{g}(t, Y)\|^2_{L(U, L^2(\mathbb{P}, \mathbb{H}))} = 0$$

for every bounded or compact set $\mathbb{K} \subset L^2(\mathbb{P}, \mathbb{H})$.

(3) A function $F : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \times U \to L^2(\mathbb{P}, \mathbb{H})$, $(t, Y, x) \mapsto F(t, Y, x)$ is said to be uniformly Poisson square-mean almost periodic if $F$ is continuous in the following sense

$$\int_U \mathbb{E}\|F(t, Y, x) - F(t', Y', x)\|^2 \nu(dx) \to 0 \quad \text{as } (t', Y') \to (t, Y)$$

and that for every sequence of real numbers $\{s'_n\}$ and every bounded or compact set $\mathbb{K} \subset L^2(\mathbb{P}, \mathbb{H})$, there exist a subsequence $\{s_n\}$ and a continuous function $\tilde{F} : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \times U \to L^2(\mathbb{P}, \mathbb{H})$ in the above sense, with $\int_U \mathbb{E}\| \tilde{F}(t, Y, x)\|^2 \nu(dx) < \infty$, such that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}, Y \in \mathbb{K}} \int_U \mathbb{E}\|F(t + s_n, Y, x) - \tilde{F}(t, Y, x)\|^2 \nu(dx) = 0$$

for every bounded or compact set $\mathbb{K} \subset L^2(\mathbb{P}, \mathbb{H})$.

In the sequel, (uniformly Poisson) square-mean almost periodicity is also called (uniformly Poisson) $L^2$-almost periodicity.

**Lemma 2.6.** [4] $AP(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ is a Banach space when it is equipped with the norm

$$\|Y\|_\infty := \sup_{t \in \mathbb{R}} \|Y(t)\|_2 = \sup_{t \in \mathbb{R}}(\mathbb{E}\|Y(t)\|^2)^{1/2},$$

for $Y \in AP(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

**Proposition 2.7.** [4] Let $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $(t, Y) \mapsto f(t, Y)$ be square-mean almost periodic on each compact set of $L^2(\mathbb{P}, \mathbb{H})$, and assume that $f$ satisfies Lipschitz condition in the following sense:

$$\mathbb{E}\|f(t, Y) - f(t, Z)\|^2 \leq LE\|Y - Z\|^2$$

for all $Y, Z \in L^2(\mathbb{P}, \mathbb{H})$ and $t \in \mathbb{R}$, where $L > 0$ is independent of $t$. Then for any square-mean almost periodic process $Y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$, the stochastic process $F : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ given by $F(t) := f(t, Y(t))$ is square-mean almost periodic.
Proposition 2.8. If $F, F_1, F_2 : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \times U \to L^2(\mathbb{P}, \mathbb{H})$ are uniformly Poisson square-mean almost periodic functions on each compact set of $L^2(\mathbb{P}, \mathbb{H})$, then

1. $F_1 + F_2$ is Poisson square-mean almost periodic.
2. $\lambda F$ is Poisson square-mean almost periodic for every scalar $\lambda$.
3. For any compact subset $K \subset L^2(\mathbb{P}, \mathbb{H})$, there exists a constant $M = M(K) > 0$ such that
   \[
   \sup_{t \in \mathbb{R}, Y \in K} \mathbb{E}\|F(t, Y, x)\|^2 \nu(dx) \leq M.
   \]

Proposition 2.9. Let $F : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \times U \to L^2(\mathbb{P}, \mathbb{H})$, $(t, Y, x) \mapsto F(t, Y, x)$ be uniformly Poisson square-mean almost periodic on any compact subsets of $L^2(\mathbb{P}, \mathbb{H})$, and assume that $F$ satisfies the Lipschitz condition in the follow sense:
\[
\int_U \mathbb{E}\|F(t, Y, x) - F(t, Z, x)\|^2 \nu(dx) \leq L \mathbb{E}\|Y - Z\|^2
\]
for all $Y, Z \in L^2(\mathbb{P}, \mathbb{H})$ and $t \in \mathbb{R}$, where $L > 0$ is independent of $t$. Then for any square-mean almost periodic process $Y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$, the function $D : \mathbb{R} \times U \to L^2(\mathbb{P}, \mathbb{H})$ given by $D(t, x) := F(t, Y(t), x)$ is Poisson square-mean almost periodic.

2.3. Almost periodicity in distribution. Let $\mathcal{P}(\mathbb{H})$ be the space of all Borel probability measures on $\mathbb{H}$ endowed with the $\beta$ metric:
\[
\beta(\mu, \nu) := \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : \|f\|_{BL} \leq 1 \right\}, \quad \mu, \nu \in \mathcal{P}(\mathbb{H}),
\]
where $f$ are Lipschitz continuous real-valued functions on $\mathbb{H}$ with the norms
\[
\|f\|_{BL} = \|f\|_L + \|f\|_\infty, \quad \|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}, \quad \|f\|_\infty = \sup_{x \in \mathbb{H}} |f(x)|.
\]
See [11] §11.3 for $\beta$ metric and related properties. Recall that a sequence $\{\mu_n\} \subset \mathcal{P}(\mathbb{H})$ is said to weakly converge to $\mu$ if $\int f d\mu_n \to \int f d\mu$ for all $f \in C_b(\mathbb{H})$, the space of all bounded continuous real-valued functions on $\mathbb{H}$. It is well known that the $\beta$ metric is a complete metric on $\mathcal{P}(\mathbb{H})$ and that a sequence $\{\mu_n\}$ weakly converges to $\mu$ if and only if $\beta(\mu_n, \mu) \to 0$ as $n \to \infty$.

Definition 2.10. An $\mathbb{H}$-valued stochastic process $Y(t)$ is said to be almost periodic in distribution if its law $\mu(t)$ is a $\mathcal{P}(\mathbb{H})$-valued almost periodic mapping, i.e. for every sequence of real numbers $\{s_n\}$, there exist a subsequence $\{s_{n_k}\}$ and a $\mathcal{P}(\mathbb{H})$-valued continuous mapping $\tilde{\mu}(t)$ such that
\[
\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \beta(\mu(t + s_n), \tilde{\mu}(t)) = 0.
\]

Remark 2.11. Recall that a sequence of random variables $\{X_n\}$ is said to converge in distribution to the random variable $X$ if the corresponding laws $\{\mu_n\}$ weakly converge to the law $\mu$ of $X$, i.e. $\beta(\mu_n, \mu) \to 0$.

Since $L^2$ convergence implies convergence in distribution for a sequence of random variables, a square-mean almost periodic stochastic process is necessarily an almost periodic in distribution one; but the converse is not true.

3. Main Results

Consider the following semilinear stochastic differential equation
\[
(3.1) \quad dY(t) = AY(t)dt + f(t, Y(t))dt + g(t, Y(t))dW(t)
+ \int_{|x|<1} F(t, Y(t), x)\tilde{N}(dt, dx)
+ \int_{|x|\geq1} G(t, Y(t), x)N(dt, dx), \quad t \in \mathbb{R},
\]
where $A$ is an infinitesimal generator which generates a $C^0$-semigroup $\{T(t)\}_{t \geq 0}$ on $\mathbb{H}$, $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $g : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L(U, L^2(\mathbb{P}, \mathbb{H}))$, $F : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \times U \to L^2(\mathbb{P}, \mathbb{H})$, $G : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \times U \to L^2(\mathbb{P}, \mathbb{H})$ are continuous functions; $W$ and $N$ are the Lévy-Itô decomposition components of the two-sided Lévy process.
Definition 3.1. The semigroup \( \{T(t)\}_{t \geq 0} \) is said to satisfy the exponential dichotomy property if there exist projection \( P \) and constants \( K, \omega > 0 \) such that \( T(t) \) commutes with \( P \) for each \( t \geq 0 \), \( \ker(P) \) is invariant with respect to \( T(t) \), \( T(t) : R(J) \to R(J) \) is invertible and the following holds

\[
\|T(t)Px\| \leq K \exp(-\omega t)\|x\|, \quad t \geq 0; \quad \|T(t)Jx\| \leq K \exp(\omega t)\|x\|, \quad t \leq 0,
\]

where \( J = I - P \) and \( T(t) = (T(-t))^{-1} \) for \( t \leq 0 \).

Definition 3.2. An \( \mathcal{F}_t \)-progressively measurable stochastic process \( \{Y(t)\}_{t \in \mathbb{R}} \) is called a mild solution of (3.1) if it satisfies the corresponding stochastic integral equation

\[
Y(t) = T(t - r)Y(r) + \int_r^t T(t - s)f(s, Y(s-))ds + \int_r^t T(t - s)g(s, Y(s-))dW(s)
\]

\[
+ \int_r^t \int_{|u| < 1} T(t - s)F(s, Y(s-), x)\tilde{N}(ds, dx)
\]

\[
+ \int_r^t \int_{|u| \geq 1} T(t - s)G(s, Y(s-), x)\tilde{N}(ds, dx),
\]

for all \( t \geq r \) and each \( r \in \mathbb{R} \).

In this section, we require the following basic assumptions.

(H1) The semigroup \( T(t) \) generated by \( A \) satisfies the exponential dichotomy property, that is, (3.2) holds.

(H2) Assume that \( f, g \) are uniformly square-mean almost periodic and \( F, G \) are uniformly Poisson square-mean almost periodic.

(H3) Assume that \( f, g, F \) and \( G \) satisfy Lipschitz conditions in \( Y \) uniformly with respect to \( t \), that is, for all \( Y, Z \in \mathcal{L}^2(\mathbb{P}, \mathcal{H}) \) and \( t \in \mathbb{R} \),

\[
\mathbb{E}\|f(t, Y) - f(t, Z)\|^2 \leq LE\|Y - Z\|^2,
\]

\[
\mathbb{E}\|g(t, Y) - g(t, Z)\|_{\mathbb{L}^1(\mathcal{L}(\mathbb{P}, \mathcal{H}))}^2 \leq LE\|Y - Z\|^2,
\]

(3.4) \[
\int_{|u| < 1} \mathbb{E}\|F(t, Y, x) - F(t, Z, x)\|^2\nu(dx) \leq LE\|Y - Z\|^2,
\]

(3.5) \[
\int_{|u| \geq 1} \mathbb{E}\|G(t, Y, x) - G(t, Z, x)\|^2\nu(dx) \leq LE\|Y - Z\|^2,
\]

for some constant \( L > 0 \) independent of \( t \).

Lemma 3.3. Assume (H1)-(H3). If (3.1) has an \( \mathcal{L}^2 \)-bounded mild solution, then this solution is \( \mathcal{L}^2 \)-continuous.

Proof. Since \( T(t) \) is a \( C^0 \)-semigroup, it follows from [20] Chapter 1, Theorem 2.2 that there exist positive constants \( M, \delta \) such that \( \|T(t)\| \leq Me^{\delta t} \) for all \( t \geq 0 \). If \( Y(t) \) is an \( \mathcal{L}^2 \)-bounded solution of (3.1), i.e. (3.3) holds, then it follows from the (3.2), Itô’s isometry, the properties of the integral for the Poisson random measure and Cauchy-Schwarz inequality, we have for \( t \geq r \)

\[
\mathbb{E}\|Y(t) - Y(r)\|^2
\]

\[
\leq 5\mathbb{E}\|T(t - r)Y(r) - Y(r)\|^2
\]

\[
+ 5\mathbb{E}\left\| \int_r^t T(t - s)f(s, Y(s-))ds \right\|^2
\]

\[
+ 5\mathbb{E}\left\| \int_r^t T(t - s)g(s, Y(s-))dW(s) \right\|^2
\]

\[
+ 5\mathbb{E}\left\| \int_r^t \int_{|u| < 1} T(t - s)F(s, Y(s-), x)\tilde{N}(ds, dx) \right\|^2
\]

\[
+ 5\mathbb{E}\left\| \int_r^t \int_{|u| \geq 1} T(t - s)G(s, Y(s-), x)\tilde{N}(ds, dx) \right\|^2
\]
Note that \(\|T(t) - I\|^2 \rightarrow 0\) when \(t \rightarrow r\), it follows from the Lipschitz property of \(f\) in \(s\) and the \(L^2\)-boundedness of \(Y(\cdot)\) that
\[
\sup_{s \in \mathbb{R}} \|f(s, Y(s))\|_2 \leq \sup_{s \in \mathbb{R}} \|f(s, Y(s)) - f(s, 0)\|_2 + \sup_{s \in \mathbb{R}} \|f(s, 0)\|_2 \leq \sqrt{T}\sup_{s \in \mathbb{R}} \|Y(s)\|_2 + \sup_{s \in \mathbb{R}} \|f(s, 0)\|_2 < \infty;
\]
similarly
\[
\sup_{s \in \mathbb{R}} \|g(s, Y(s))\|_2 < \infty.
\]
By the Lipschitz property and Poisson almost periodicity of \(F\), and the \(L^2\)-boundedness of \(Y(\cdot)\), we have
\[
\sup_{s \in \mathbb{R}} \int_{|x| < 1} \mathbb{E}\|F(s, Y(s), x)\nu(dx)\|^2 \leq \sup_{s \in \mathbb{R}} \int_{|x| < 1} \mathbb{E}\|F(s, Y(s), x) - F(s, 0, x)\|^2\nu(dx)
\]
\[
+ \sup_{s \in \mathbb{R}} \int_{|x| < 1} \mathbb{E}\|F(s, 0, x)\|^2\nu(dx)
\]
\[
\leq L\sup_{s \in \mathbb{R}} \|Y(s)\|^2 + M(0) < \infty;
\]
similarly,
\[
\sup_{s \in \mathbb{R}} \int_{|x| \geq 1} \mathbb{E}\|G(s, Y(s), x)\nu(dx)\|^2 < \infty.
\]
Therefore,
\[
\mathbb{E}\|Y(t) - Y(r)\|^2 \rightarrow 0 \quad t \rightarrow r.
\]
That is, \(Y(\cdot)\) is \(L^2\)-continuous.

\[\textbf{Theorem 3.4.} \text{ Let } (H1)-(H3) \text{ be satisfied. Then } [3.1] \text{ has a unique } L^2 \text{-bounded mild solution, provided}
\]
\[
1 + \frac{2b}{\omega^2} + \frac{2}{\omega < \frac{1}{16K^2L}}.
\]

\[\text{Proof.} \text{ Firstly, note that if } Y : \mathbb{R} \rightarrow L^2(\mathcal{P}, \mathbb{H}) \text{ is bounded, i.e. } \|Y\|_\infty < \infty, \text{ then } Y(t) \text{ is a mild solution of (3.1) if and only if it satisfies the following integral equation}
\]
\[
Y(t) = \int_{-\infty}^{t} T(t-s)Pf(s, Y(s-))ds + \int_{-\infty}^{t} T(t-s)Pg(s, Y(s-))dW(s)
\]
\[
+ \int_{-\infty}^{t} \int_{|x| < 1} T(t-s)PF(s, Y(s-), x)\tilde{N}(ds, dx)
\]
\[
+ \int_{-\infty}^{t} \int_{|x| \geq 1} T(t-s)PG(s, Y(s-), x)N(ds, dx)
\]
\[ - \int_t^{+\infty} T(t-s) Jf(s, Y(s-)) ds - \int_t^{+\infty} T(t-s) Jg(s, Y(s-)) dW(s) \]

\[ - \int_t^{+\infty} \int_{|x| < 1} T(t-s) JF(s, Y(s-), x) \tilde{N}(ds, dx) \]

(3.7)

\[ - \int_t^{+\infty} \int_{|x| \geq 1} T(t-s) JG(s, Y(s-), x) N(ds, dx). \]

Consider the nonlinear operator \( S \) acting on the Banach space \( C_b(\mathbb{R}; \mathcal{L}^2(\mathbb{P}, \mathbb{F})) \) which is given by

\[
(SY)(t) := \int_{-\infty}^t T(t-s) Pf(s, Y(s-)) ds + \int_{-\infty}^t T(t-s) Pg(s, Y(s-)) dW(s) \\
+ \int_{-\infty}^t \int_{|x| < 1} T(t-s) PF(s, Y(s-), x) \tilde{N}(ds, dx) \\
+ \int_{-\infty}^t \int_{|x| \geq 1} T(t-s) PG(s, Y(s-), x) N(ds, dx) \\
- \int_t^{+\infty} T(t-s) Jf(s, Y(s-)) ds - \int_t^{+\infty} T(t-s) Jg(s, Y(s-)) dW(s) \\
- \int_t^{+\infty} \int_{|x| < 1} T(t-s) JF(s, Y(s-), x) \tilde{N}(ds, dx) \\
- \int_t^{+\infty} \int_{|x| \geq 1} T(t-s) JG(s, Y(s-), x) N(ds, dx) =: (S_1Y)(t) + (S_2Y)(t) + (S_3Y)(t) + (S_4Y)(t),
\]

where

\[
(S_1Y)(t) := \int_{-\infty}^t T(t-s) Pf(s, Y(s-)) ds - \int_t^{+\infty} T(t-s) Jf(s, Y(s-)) ds, \\
(S_2Y)(t) := \int_{-\infty}^t T(t-s) Pg(s, Y(s-)) dW(s) - \int_t^{+\infty} T(t-s) Jg(s, Y(s-)) dW(s), \\
(S_3Y)(t) := \int_{-\infty}^t \int_{|x| < 1} T(t-s) PF(s, Y(s-), x) \tilde{N}(ds, dx) \\
- \int_t^{+\infty} \int_{|x| < 1} T(t-s) JF(s, Y(s-), x) \tilde{N}(ds, dx), \\
(S_4Y)(t) := \int_{-\infty}^t \int_{|x| \geq 1} T(t-s) PG(s, Y(s-), x) N(ds, dx) \\
- \int_t^{+\infty} \int_{|x| \geq 1} T(t-s) JG(s, Y(s-), x) N(ds, dx).
\]

Since \( T(\cdot) \) satisfies the exponential dichotomy property, \( f, g \) are uniformly \( \mathcal{L}^2 \)-almost periodic, \( F, G \) are uniformly Poisson \( \mathcal{L}^2 \)-almost periodic, and they satisfy the Lipschitz property, it follows that the operator \( S \) maps \( C_b(\mathbb{R}; \mathcal{L}^2(\mathbb{P}, \mathbb{F})) \) to itself. If \( S \) is contraction mapping, then the Banach fixed point theorem yields that (3.1) admits a unique \( \mathcal{L}^2 \)-bounded and \( \mathcal{L}^2 \)-continuous mild solution.

Now we will show that \( S \) is a contraction mapping on \( C_b(\mathbb{R}; \mathcal{L}^2(\mathbb{P}, \mathbb{F})) \). For \( Y_1, Y_2 \in C_b(\mathbb{R}; \mathcal{L}^2(\mathbb{P}, \mathbb{F})) \) and \( t \in \mathbb{R} \) we have

\[
E \| (SY_1)(t) - (SY_2)(t) \|^2 = E \left\| \int_{-\infty}^t T(t-s) Pf(s, Y_1(s-)) - f(s, Y_2(s-)) ds \\
+ \int_{-\infty}^t T(t-s) Pf(s, Y_1(s-)) - f(s, Y_2(s-)) ds \right\| E \left\| T(t-s) Pf(s, Y_2(s-)) - f(s, Y_2(s-)) dW(s) \right\|
\]
\[ + \int_{-\infty}^{t} \int_{|x| < 1} T(t-s)P[F(s, Y_1(s-), x) - F(s, Y_2(s-), x)]\tilde{N}(ds, dx) \\
+ \int_{-\infty}^{t} \int_{|x| \geq 1} T(t-s)P[G(s, Y_1(s-), x) - G(s, Y_2(s-), x)]N(ds, dx) \\
- \int_{t}^{+\infty} T(t-s)J[f(s, Y_1(s-)) - f(s, Y_2(s-))]ds \\
- \int_{t}^{+\infty} T(t-s)J[g(s, Y_1(s-)) - g(s, Y_2(s-))]dW(s) \\
- \int_{t}^{+\infty} \int_{|x| < 1} T(t-s)J[F(s, Y_1(s-), x) - F(s, Y_2(s-), x)]\tilde{N}(ds, dx) \\
- \int_{t}^{+\infty} \int_{|x| \geq 1} T(t-s)J[G(s, Y_1(s-), x) - G(s, Y_2(s-), x)]N(ds, dx) \| ^2 \\
\leq 8E \left\| \int_{-\infty}^{t} T(t-s)P[f(s, Y_1(s-)) - f(s, Y_2(s-))]ds \right\|^2 \\
+ 8E \left\| \int_{-\infty}^{t} T(t-s)P[g(s, Y_1(s-)) - g(s, Y_2(s-))]dW(s) \right\|^2 \\
+ 8E \left\| \int_{-\infty}^{t} \int_{|x| < 1} T(t-s)P[F(s, Y_1(s-), x) - F(s, Y_2(s-), x)]\tilde{N}(ds, dx) \right\|^2 \\
+ 8E \left\| \int_{-\infty}^{t} \int_{|x| \geq 1} T(t-s)P[G(s, Y_1(s-), x) - G(s, Y_2(s-), x)]N(ds, dx) \right\|^2 \\
+ 8E \left\| \int_{t}^{+\infty} T(t-s)J[f(s, Y_1(s-)) - f(s, Y_2(s-))]ds \right\|^2 \\
+ 8E \left\| \int_{t}^{+\infty} T(t-s)J[g(s, Y_1(s-)) - g(s, Y_2(s-))]dW(s) \right\|^2 \\
+ 8E \left\| \int_{t}^{+\infty} \int_{|x| < 1} T(t-s)J[F(s, Y_1(s-), x) - F(s, Y_2(s-), x)]\tilde{N}(ds, dx) \right\|^2 \\
+ 8E \left\| \int_{t}^{+\infty} \int_{|x| \geq 1} T(t-s)J[G(s, Y_1(s-), x) - G(s, Y_2(s-), x)]N(ds, dx) \right\|^2 .
\]

Similar to the proof of [H Theorem 3.2], it follows from the Cauchy-Schwarz inequality that we have the following estimates for the 1st and the 5th term on the right hand side of the above inequality

\[
E \left\| \int_{-\infty}^{t} T(t-s)P[f(s, Y_1(s-)) - f(s, Y_2(s-))]ds \right\|^2 \leq \frac{K^2L}{\omega^2} \sup_{s \in \mathbb{R}} E \| Y_1(s) - Y_2(s) \|^2 ,
\]

\[
E \left\| \int_{t}^{+\infty} T(t-s)J[f(s, Y_1(s-)) - f(s, Y_2(s-))]ds \right\|^2 \leq \frac{K^2L}{\omega^2} \sup_{s \in \mathbb{R}} E \| Y_1(s) - Y_2(s) \|^2 .
\]

The 2nd and the 6th term can be estimated by

\[
E \left\| \int_{-\infty}^{t} T(t-s)P[g(s, Y_1(s-)) - g(s, Y_2(s-))]dW(s) \right\|^2 \\
\leq K^2 \left( \int_{-\infty}^{t} e^{-2\omega(t-s)}E \| (g(s, Y_1(s-)) - g(s, Y_2(s-)))Q^{1/2} \|_{L(U, L^2(p, \mathbb{R}))}^2 ds \right) \\
\leq \frac{K^2L}{2\omega} \sup_{s \in \mathbb{R}} E \| Y_1(s) - Y_2(s) \|^2
\]
and

\[ E \left\| \int_t^{+\infty} T(t-s)J[g(s,Y_1(s)-) - g(s,Y_2(s-))]dW(s) \right\|^2 \leq K^2 \left( \int_1^{+\infty} e^{2\omega(t-s)}E\|g(s,Y_1(s)-) - g(s,Y_2(s-))\|_{L^2}^2 \right)^{1/2} \|T(t,s)\|_{L^1} ds \]

\[ \leq \frac{K^2L}{2\omega} \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2. \]

For the 3rd term, we have

\[ E \left\| \int_t^{\infty} \int_{|x|<1} T(t-s)P[F(s,Y_1(s)-) - F(s,Y_2(s-),x)]N(ds,dx) \right\|^2 \leq 2E \left\| \int_t^{\infty} \int_{|x|<1} T(t-s)P[F(s,Y_1(s)-) - F(s,Y_2(s-),x)]N(ds,dx) \right\|^2 \]

\[ + 2E \left\| \int_t^{\infty} \int_{|x| \geq 1} T(t-s)P[F(s,Y_1(s)-) - F(s,Y_2(s-),x)]\nu(dx)ds \right\|^2 \]

\[ \leq \frac{K^2L}{\omega} \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2 \]

\[ + 2K^2E \left( \int_t^{\infty} \int_{|x| \geq 1} e^{-\omega(t-s)/2} \|G(s,Y_1(s)-) - G(s,Y_2(s-),x)\|_{L^2}^2 \nu(dx)ds \right)^2 \]

\[ \leq \frac{K^2L}{\omega} \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2 + 2K^2 \int_{-\infty}^{t} e^{-\omega(t-s)} ds \cdot b \]

\[ \cdot \int_{-\infty}^{t} \int_{|x| \geq 1} e^{-\omega(t-s)} E\|G(s,Y_1(s)-) - G(s,Y_2(s-),x)\|_{L^2}^2 \nu(dx)ds \]

\[ \leq \left( \frac{K^2L}{\omega} + \frac{2K^2Lb}{\omega^2} \right) \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2. \]

Similarly, the 7th and the 8th term can be estimated by

\[ E \left\| \int_t^{+\infty} \int_{|x|<1} T(t-s)J[F(s,Y_1(s)-) - F(s,Y_2(s-),x)]N(ds,dx) \right\|^2 \leq \frac{K^2L}{2\omega} \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2 \]
and 

\[
E \left\| \int_t^{\infty} \int_{|x| \geq 1} T(t-s)J[G(s,Y_1(s)-), x) - G(s,Y_2(s)-), x)]N(ds, dx) \right\|^2 \\
\leq 2E \left\| \int_t^{\infty} \int_{|x| \geq 1} T(t-s)J[G(s,Y_1(s)-), x) - G(s,Y_2(s)-), x)]\tilde{N}(ds, dx) \right\|^2 \\
+ 2E \left\| \int_t^{\infty} \int_{|x| \geq 1} T(t-s)J[G(s,Y_1(s)-), x) - G(s,Y_2(s)-), x)]\nu(dx)ds \right\|^2 \\
\leq \left( \frac{K^2L}{\omega} + \frac{2K^2Lb}{\omega^2} \right) \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2.
\]

So for any \( t \in \mathbb{R} \),

\[
E\|(SY_1)(t) - (SY_2)(t)\|^2 \leq \left[ \frac{16K^2L}{\omega^2}(1 + 2bL) + \frac{32K^2L}{\omega} \right] \sup_{s \in \mathbb{R}} E \|Y_1(s) - Y_2(s)\|^2,
\]

that is,

\[
(3.8) \quad \|(SY_1)(t) - (SY_2)(t)\|^2 \leq \eta \cdot \sup_{s \in \mathbb{R}} \|Y_1(s) - Y_2(s)\|^2,
\]

where \( \eta := \frac{16K^2L}{\omega^2}(1 + 2bL) + \frac{32K^2L}{\omega} \). Since

\[
(3.9) \quad \sup_{s \in \mathbb{R}} \|Y_1(s) - Y_2(s)\|^2 \leq \left( \sup_{s \in \mathbb{R}} \|Y_1(s) - Y_2(s)\|^2 \right)^2,
\]

it follows from (3.8) and (3.9) that for arbitrary \( t \in \mathbb{R} \),

\[
\|S(Y_1)(t) - S(Y_2)(t)\|_2 \leq \sqrt{\eta} \|Y_1 - Y_2\|_\infty.
\]

Thus

\[
\|SY_1 - SY_2\|_\infty = \sup_{t \in \mathbb{R}} \|S(Y_1)(t) - S(Y_2)(t)\|_2 \leq \sqrt{\eta} \|Y_1 - Y_2\|_\infty.
\]

By (3.6) and the fact \( \eta < 1 \), \( S \) is a contraction mapping on \( C_b(\mathbb{R}; \mathcal{L}^2(\mathbb{P}, \mathbb{H})) \). Therefore, there exists a unique \( v \in C_b(\mathbb{R}; \mathcal{L}^2(\mathbb{P}, \mathbb{H})) \) with \( Sv = v \), which is the unique \( \mathcal{L}^2 \)-bounded and \( \mathcal{L}^2 \)-continuous solution of (3.1). \( \square \)

**Theorem 3.5.** Let (H1)-(H3) be satisfied. Then the unique \( \mathcal{L}^2 \)-bounded mild solution of (3.1) obtained in Theorem 3.4 is almost periodic in distribution, provided

\[
(3.10) \quad \frac{1 + 2b}{\omega^2} + \frac{2}{\omega} < \frac{1}{32K^2L}.
\]

**Proof.** For given sequence of real numbers \( \{s_n\} \), since \( f, g \) are almost periodic and \( F,G \) are Poisson almost periodic, there exist a subsequence \( \{s_n\} \) of \( \{s_n\} \) and and \( \mathcal{L}^2 \)-continuous stochastic process \( \tilde{f}, \tilde{g} \) and continuous functions \( F, \tilde{G} \) in the sense of Definition 2.5 (3), such that

\[
(3.11) \quad \lim_{n \to \infty} \sup_{t \in \mathbb{R}, Y \in K} E\|f(t + s_n, Y) - \tilde{f}(t, Y)\|^2 = 0,
\]

\[
(3.12) \quad \lim_{n \to \infty} \sup_{t \in \mathbb{R}, Y \in K} E\|(g(t + s_n, Y) - \tilde{g}(t, Y))Q^{1/2}\|_{L(U, \mathcal{L}^2(\mathbb{P}, \mathbb{H}))}^2 = 0,
\]

\[
(3.13) \quad \lim_{n \to \infty} \sup_{t \in \mathbb{R}, Y \in K} \int_{|x| < 1} E\|F(t + s_n, Y, x) - \tilde{F}(t, Y, x)\|^2 \nu(dx) = 0,
\]

and

\[
(3.14) \quad \lim_{n \to \infty} \sup_{t \in \mathbb{R}, Y \in K} \int_{|x| \geq 1} E\|G(t + s_n, Y, x) - \tilde{G}(t, Y, x)\|^2 \nu(dx) = 0
\]

hold for any bounded set \( K \subset \mathcal{L}^2(\mathbb{P}, \mathbb{H}) \).
Let $\tilde{Y}(\cdot)$ be the solution of the following stochastic integral equation\footnote{Note that $f, \tilde{g}, \tilde{F}, \tilde{G}$ are uniformly bounded in $t$ when $Y = 0$ and Lipschitz in $Y$ with the same Lipschitz constants as that of $f, g, F, G$, so by Theorem \[3,4\] we know this integral equation admits a unique $L^2$-bounded solution.}

$$
\tilde{Y}(t) := \int_{-\infty}^{t} T(t - s)Pf(s, \tilde{Y}(s-))ds + \int_{-\infty}^{t} T(t - s)P\tilde{g}(s, \tilde{Y}(s-))dW(s)
$$

$$
+ \int_{-\infty}^{t} \int_{|x| < 1} T(t - s)P\tilde{F}(s, \tilde{Y}(s-), x)\tilde{N}(ds, dx)
$$

$$
+ \int_{-\infty}^{t} \int_{|x| \geq 1} T(t - s)P\tilde{G}(s, \tilde{Y}(s-), x)N(ds, dx)
$$

$$
- \int_{t}^{+\infty} T(t - s)Jf(s, \tilde{Y}(s-))ds - \int_{t}^{+\infty} T(t - s)\tilde{g}(s, \tilde{Y}(s-))dW(s)
$$

$$
- \int_{t}^{+\infty} \int_{|x| < 1} T(t - s)J\tilde{F}(s, \tilde{Y}(s-), x)\tilde{N}(ds, dx)
$$

$$
- \int_{t}^{+\infty} \int_{|x| \geq 1} T(t - s)J\tilde{G}(s, \tilde{Y}(s-), x)N(ds, dx)
$$

For each $\sigma \in \mathbb{R}$, denote $W_n(\sigma) := W(\sigma + s_n) - W(s_n)$, $N_n(\sigma, x) := N(\sigma + s_n, x) - N(s_n, x)$, $\tilde{N}_n(\sigma, x) := \tilde{N}(\sigma + s_n, x) - \tilde{N}(s_n, x)$. Then $W_n$ is a $Q$-Wiener process with the same distribution as $W$, and $N_n$ is a Poisson random measure with the same distribution as $N$ and the corresponding compensated Poisson random measure being $\tilde{N}_n$. Let $\sigma = s - s_n$, then we have

$$
Y(t + s_n) = \int_{-\infty}^{t} T(t - \sigma)Pf(\sigma + s_n, Y(\sigma + s_n-))d\sigma
$$

$$
+ \int_{-\infty}^{t} T(t - \sigma)Pg(\sigma + s_n, Y(\sigma + s_n-))dW_n(\sigma)
$$

$$
+ \int_{-\infty}^{t} \int_{|x| < 1} T(t - \sigma)PF(\sigma + s_n, Y(\sigma + s_n-), x)\tilde{N}_n(d\sigma, dx)
$$

$$
+ \int_{-\infty}^{t} \int_{|x| \geq 1} T(t - \sigma)PG(\sigma, Y(\sigma + s_n-), x)N_n(d\sigma, dx)
$$

$$
\int_{t}^{+\infty} T(t - \sigma)Jf(\sigma + s_n, Y(\sigma + s_n-))d\sigma
$$

$$
- \int_{t}^{+\infty} T(t - \sigma)J\tilde{F}(\sigma + s_n, Y(\sigma + s_n-), x)\tilde{N}_n(d\sigma, dx)
$$

$$
- \int_{t}^{+\infty} \int_{|x| \geq 1} T(t - \sigma)JG(\sigma, Y(\sigma + s_n-), x)N_n(d\sigma, dx).
$$

Consider the stochastic process $Y_n(\cdot)$ which satisfies the following stochastic integral equation

$$
Y_n(t) = \int_{-\infty}^{t} T(t - \sigma)Pf(\sigma + s_n, Y_n(\sigma-))d\sigma
$$

$$
+ \int_{-\infty}^{t} T(t - \sigma)Pg(\sigma + s_n, Y_n(\sigma-))dW(\sigma)
$$

$$
+ \int_{-\infty}^{t} \int_{|x| < 1} T(t - \sigma)PF(\sigma + s_n, Y_n(\sigma-), x)\tilde{N}(d\sigma, dx)
$$

$$
+ \int_{-\infty}^{t} \int_{|x| \geq 1} T(t - \sigma)PG(\sigma, Y_n(\sigma-), x)N(d\sigma, dx) \]
exists, is unique and periodic and satisfy the same Lipschitz condition as that of $f, g$.

The functions $f(\cdot + s_n, \cdot)$ and $g(\cdot + s_n, \cdot)$ are uniformly square-mean almost periodic and Lipschitz in $\mathbb{R}$ with the same Lipschitz constants as that of $f, g$, and $F(\cdot + s_n, \cdot), G(\cdot + s_n, \cdot)$ are uniformly Poisson square-mean almost periodic and satisfy the same Lipschitz condition as that of $F, G$. So similar to Theorem \ref{thm}, this $Y_n(\cdot)$ exists, is unique and $\mathcal{L}^2$-bounded.

It follows from Itô’s isometry and the properties of the integral for Poisson random measures that

$$E\|Y_n(t) - \tilde{Y}(t)\|^2$$

$$\leq 8E \left\| \int_{-\infty}^{t} T(t - \sigma)P[f(\sigma + s_n, Y_n(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-))]d\sigma \right\|^2$$

$$+ 8E \left\| \int_{t}^{+\infty} T(t - \sigma)J[f(\sigma + s_n, Y_n(\sigma-))]dW(\sigma) \right\|^2$$

$$+ 8E \left\| \int_{-\infty}^{t} T(t - \sigma)P[g(\sigma + s_n, Y_n(\sigma-)) - \tilde{g}(\sigma, \tilde{Y}(\sigma-))]dW(\sigma) \right\|^2$$

$$+ 8E \left\| \int_{t}^{+\infty} T(t - \sigma)J[g(\sigma + s_n, Y_n(\sigma-))]dW(\sigma) \right\|^2$$

$$=: I_1 + I_2 + I_3 + I_4.$$

By the Cauchy-Schwarz inequality we have the following estimate for $I_1$:

$$I_1 \leq 16E \left\| \int_{-\infty}^{t} T(t - \sigma)P[f(\sigma + s_n, Y_n(\sigma-)) - f(\sigma + s_n, \tilde{Y}(\sigma-))]d\sigma \right\|^2$$

$$+ 16E \left\| \int_{-\infty}^{t} T(t - \sigma)P[f(\sigma + s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-))]d\sigma \right\|^2$$

$$+ 16E \left\| \int_{t}^{+\infty} T(t - \sigma)J[f(\sigma + s_n, Y_n(\sigma-)) - f(\sigma + s_n, \tilde{Y}(\sigma-))]d\sigma \right\|^2$$

$$+ 16E \left\| \int_{t}^{+\infty} T(t - \sigma)J[f(\sigma + s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-))]d\sigma \right\|^2.$$
\[
\begin{align*}
\leq & \ 16 \mathbb{E} \left( \int_{-\infty}^{t} \| T(t-\sigma)P \cdot \| f(\sigma+s_n, Y_n(\sigma-)) - f(\sigma+s_n, \tilde{Y}(\sigma-)) \| d\sigma \right)^2 \\
+ & \ 16 \mathbb{E} \left( \int_{-\infty}^{t} \| T(t-\sigma)P \cdot \| f(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-)) \| d\sigma \right)^2 \\
+ & \ 16 \mathbb{E} \left( \int_{t}^{+\infty} \| T(t-\sigma)J \| \cdot \| f(\sigma+s_n, Y_n(\sigma-)) - f(\sigma+s_n, \tilde{Y}(\sigma-)) \| d\sigma \right)^2 \\
+ & \ 16 \mathbb{E} \left( \int_{t}^{+\infty} \| T(t-\sigma)J \| \cdot \| f(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-)) \| d\sigma \right)^2 \\
\leq & \ 16 K^2 \int_{-\infty}^{t} e^{-\frac{\omega(t-\sigma)}{2}} d\sigma \\
\times & \ \int_{-\infty}^{t} e^{-\frac{\omega(t-\sigma)}{2}} \mathbb{E} \| f(\sigma+s_n, Y_n(\sigma-)) - f(\sigma+s_n, \tilde{Y}(\sigma-)) \|^2 d\sigma \\
+ & \ 16 K^2 \int_{-\infty}^{t} e^{-\frac{\omega(t-\sigma)}{2}} d\sigma \\
\times & \ \int_{-\infty}^{t} e^{-\frac{\omega(t-\sigma)}{2}} \mathbb{E} \| f(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-)) \|^2 d\sigma \\
+ & \ 16 K^2 \int_{t}^{+\infty} e^{-\frac{\omega(t-\sigma)}{2}} d\sigma \\
\times & \ \int_{t}^{+\infty} e^{-\frac{\omega(t-\sigma)}{2}} \mathbb{E} \| f(\sigma+s_n, Y_n(\sigma-)) - f(\sigma+s_n, \tilde{Y}(\sigma-)) \|^2 d\sigma \\
+ & \ 16 K^2 \int_{t}^{+\infty} e^{-\frac{\omega(t-\sigma)}{2}} d\sigma \\
\times & \ \int_{t}^{+\infty} e^{-\frac{\omega(t-\sigma)}{2}} \mathbb{E} \| f(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-)) \|^2 d\sigma \\
\leq & \ \frac{16 K^2 L}{\omega} \int_{-\infty}^{t} e^{-\omega(t-\sigma)} \mathbb{E} \| Y_n(\sigma-) - \tilde{Y}(\sigma-) \|^2 d\sigma \\
+ & \ \frac{16 K^2 L}{\omega} \int_{t}^{+\infty} e^{\omega(t-\sigma)} \mathbb{E} \| Y_n(\sigma-) - \tilde{Y}(\sigma-) \|^2 d\sigma + E_1^n(t) \\
\leq & \ \frac{32 K^2 L}{\omega^2} \sup_{\sigma \in \mathbb{R}} \mathbb{E} \| Y_n(\sigma-) - \tilde{Y}(\sigma-) \|^2 + E_1^n(t),
\end{align*}
\]

where
\[
E_1^n(t) := \frac{16 K^2}{\omega} \int_{-\infty}^{t} e^{-\omega(t-\sigma)} \mathbb{E} \| f(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-)) \|^2 d\sigma \\
+ \ \frac{16 K^2}{\omega} \int_{t}^{+\infty} e^{\omega(t-\sigma)} \mathbb{E} \| f(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{f}(\sigma, \tilde{Y}(\sigma-)) \|^2 d\sigma.
\]

By (3.11) and the $L^2$-boundedness of $\tilde{Y}(\cdot)$, we have $\lim_{n \to \infty} \sup_{t \in \mathbb{R}} E_1^n(t) = 0$.

For $I_2$, by Itô’s isometry we have:
\[
I_2 \leq 16 \mathbb{E} \left( \int_{-\infty}^{t} T(t-\sigma)P[g(\sigma+s_n, Y_n(\sigma-)) - g(\sigma+s_n, \tilde{Y}(\sigma-))]dW(\sigma) \right)^2 \\
+ 16 \mathbb{E} \left( \int_{-\infty}^{t} T(t-\sigma)P[g(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{g}(\sigma, \tilde{Y}(\sigma-))]dW(\sigma) \right)^2 \\
+ 16 \mathbb{E} \left( \int_{t}^{+\infty} T(t-\sigma)J[g(\sigma+s_n, Y_n(\sigma-)) - g(\sigma+s_n, \tilde{Y}(\sigma-))]dW(\sigma) \right)^2 \\
+ 16 \mathbb{E} \left( \int_{t}^{+\infty} T(t-\sigma)J[g(\sigma+s_n, \tilde{Y}(\sigma-)) - \tilde{g}(\sigma, \tilde{Y}(\sigma-))]dW(\sigma) \right)^2.
\]
\[ + 16 E \left\| \int_{t}^{+\infty} \left( T(t - \sigma)J[g(\sigma + s_n, \tilde{Y}(\sigma -)) - \tilde{g}(\sigma, \tilde{Y}(\sigma -))] \right) \, dW(\sigma) \right\|^2 \]
\[ \leq 16 E \int_{-\infty}^{t} \left\| T(t - \sigma)P \right\|^2 \cdot \| g(\sigma + s_n, Y_n(\sigma -)) - g(\sigma + s_n, \tilde{Y}(\sigma -)) \|^2 \| Q^{1/2} \|^2 \, d\sigma \]
\[ + 16 E \int_{t}^{+\infty} \| T(t - \sigma)J \|^2 \cdot \| g(\sigma + s_n, \tilde{Y}(\sigma -)) - \tilde{g}(\sigma, \tilde{Y}(\sigma -)) \|^2 \| Q^{1/2} \|^2 \, d\sigma \]
\[ + 16 E \int_{t}^{+\infty} \| T(t - \sigma)J \|^2 \cdot \| g(\sigma + s_n, Y_n(\sigma -)) - g(\sigma + s_n, \tilde{Y}(\sigma -)) \|^2 \| Q^{1/2} \|^2 \, d\sigma \]
\[ \leq 16 K^2 L \int_{-\infty}^{t} e^{-2\omega(t - \sigma)} \cdot E\| Y_n(\sigma -) - \tilde{Y}(\sigma -) \|^2 \, d\sigma \]
\[ + 16 K^2 L \int_{t}^{+\infty} e^{2\omega(t - \sigma)} \cdot E\| Y_n(\sigma -) - \tilde{Y}(\sigma -) \|^2 \, d\sigma + E_2^n(t) \]
\[ \leq \frac{16 K^2 L}{\omega} \sup_{\sigma \in \mathbb{R}} E\| Y_n(\sigma) - \tilde{Y}(\sigma) \|^2 + E_2^n(t) \]

with

\[ E_2^n(t) := 16 K^2 \int_{-\infty}^{t} e^{-2\omega(t - \sigma)} \cdot E\| g(\sigma + s_n, \tilde{Y}(\sigma -)) - \tilde{g}(\sigma, \tilde{Y}(\sigma -)) \|^2 \| Q^{1/2} \|^2 \, d\sigma \]
\[ + 16 K^2 \int_{t}^{+\infty} e^{2\omega(t - \sigma)} \cdot E\| g(\sigma + s_n, \tilde{Y}(\sigma -)) - \tilde{g}(\sigma, \tilde{Y}(\sigma -)) \|^2 \| Q^{1/2} \|^2 \, d\sigma. \]

Similar to \( E_2^n \), we have

\[ E_2^n(t) \to 0 \quad \text{as } n \to \infty \]

uniformly in \( t \in \mathbb{R} \).

By the properties of the integral for Poisson random measures and \ref{3.1} we have

\[ I_3 \leq 16 E \left\| \int_{-\infty}^{t} \int_{|x| < 1} T(t - \sigma)P[F(\sigma + s_n, Y_n(\sigma -), x) \right. \]
\[ - F(\sigma + s_n, \tilde{Y}(\sigma -), x)] \tilde{N}(d\sigma, dx) \left\|^2 \right. \]
\[ + 16 E \left\| \int_{t}^{+\infty} \int_{|x| < 1} T(t - \sigma)P[F(\sigma + s_n, \tilde{Y}(\sigma -), x) \right. \]
\[ - \tilde{F}(\sigma, \tilde{Y}(\sigma -), x)] \tilde{N}(d\sigma, dx) \left\|^2 \right. \]
\[ + 16 E \left\| \int_{t}^{+\infty} \int_{|x| < 1} T(t - \sigma)J[F(\sigma + s_n, Y_n(\sigma -), x) \right. \]
\[ - F(\sigma + s_n, \tilde{Y}(\sigma -), x)] \tilde{N}(d\sigma, dx) \left\|^2 \right. \]
\[ + 16 E \left\| \int_{t}^{+\infty} \int_{|x| < 1} T(t - \sigma)J[F(\sigma + s_n, \tilde{Y}(\sigma -), x) \right. \]
\[ - \tilde{F}(\sigma, \tilde{Y}(\sigma -), x)] \tilde{N}(d\sigma, dx) \left\|^2 \right. \]
\[ \leq 16 K^2 \int_{-\infty}^{t} \int_{|x| < 1} e^{-2\omega(t - \sigma)} E\| F(\sigma + s_n, Y_n(\sigma -), x) \|^2 \]
where

\begin{align*}
-F(\sigma + s_n, \tilde{Y}(\sigma-), x) & \leq 2\nu(dx)d\sigma \\
+16K^2 & \int_{-\infty}^{t} \int_{|x|\leq 1} e^{-2\omega(t-\sigma)} E F(\sigma + s_n, \tilde{Y}(\sigma-), x) d\sigma \\
- \tilde{F}(\sigma, \tilde{Y}(\sigma-), x) & \leq 2\nu(dx)d\sigma \\
+16K^2 & \int_{t}^{+\infty} \int_{|x|\leq 1} e^{2\omega(t-\sigma)} E F(\sigma + s_n, Y_n(\sigma-), x) d\sigma \\
- F(\sigma + s_n, \tilde{Y}(\sigma-), x) & \leq 2\nu(dx)d\sigma \\
+16K^2 & \int_{t}^{+\infty} \int_{|x|\leq 1} e^{2\omega(t-\sigma)} E F(\sigma + s_n, \tilde{Y}(\sigma-), x) d\sigma \\
- \tilde{F}(\sigma, \tilde{Y}(\sigma-), x) & \leq 2\nu(dx)d\sigma \\
\leq & 16K^2 L \int_{-\infty}^{t} e^{-2\omega(t-\sigma)} \cdot E\|Y_n(\sigma-) - \tilde{Y}(\sigma-)\|^2 d\sigma \\
+ & 16K^2 L \int_{t}^{+\infty} e^{2\omega(t-\sigma)} \cdot E\|Y_n(\sigma-) - \tilde{Y}(\sigma-)\|^2 d\sigma + \mathcal{E}_3^n(t) \\
\leq & \frac{16K^2 L}{\omega} \sup_{\sigma \in \mathbb{R}} E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 + \mathcal{E}_3^n(t),
\end{align*}

where

\begin{align*}
\mathcal{E}_3^n(t) & := 16K^2 \int_{-\infty}^{t} \int_{|x|\leq 1} e^{-2\omega(t-\sigma)} E F(\sigma + s_n, \tilde{Y}(\sigma-), x) d\sigma \\
& \quad - F(\sigma, \tilde{Y}(\sigma-), x) \|2\nu(dx)d\sigma \\
+16K^2 & \int_{t}^{+\infty} \int_{|x|\leq 1} e^{2\omega(t-\sigma)} E F(\sigma + s_n, \tilde{Y}(\sigma-), x) d\sigma \\
& \quad - \tilde{F}(\sigma, \tilde{Y}(\sigma-), x) \|2\nu(dx)d\sigma.
\end{align*}

By \cite{[3.13]} and the \mathcal{L}^2-boundedness of \( \tilde{Y}(\cdot) \), we have \( \lim_{n \to \infty} \sup_{t \in \mathbb{R}} \mathcal{E}_3^n(t) = 0 \).

Now let us estimate the last term \( I_4 \). It follows from the properties of the integral of Poisson random measures, \cite{[3.5]} and the Cauchy-Schwarz inequality that

\begin{align*}
I_4 & \leq 16E \left\| \int_{-\infty}^{t} \int_{|x|\leq 1} T(t-\sigma)P[G(\sigma + s_n, Y_n(\sigma-), x) \\
& \quad - G(\sigma + s_n, \tilde{Y}(\sigma-), x)]N(d\sigma, dx) \right\|^2 \\
+ & 16E \left\| \int_{-\infty}^{t} \int_{|x|\leq 1} T(t-\sigma)P[G(\sigma + s_n, \tilde{Y}(\sigma-), x) \\
& \quad - \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]N(d\sigma, dx) \right\|^2 \\
+ & 16E \left\| \int_{t}^{+\infty} \int_{|x|\leq 1} T(t-\sigma)J[G(\sigma + s_n, Y_n(\sigma-), x) \\
& \quad - G(\sigma + s_n, \tilde{Y}(\sigma-), x)]N(d\sigma, dx) \right\|^2 \\
+ & 16E \left\| \int_{t}^{+\infty} \int_{|x|\leq 1} T(t-\sigma)J[G(\sigma + s_n, \tilde{Y}(\sigma-), x) \\
& \quad - \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]N(d\sigma, dx) \right\|^2.
\end{align*}
\[
- \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]N(d\sigma, dx)\|^2 \\
\leq 32E\left[ \frac{1}{t} \right] \int_{-\infty}^{t} \int_{|u| \geq 1} T(t-\sigma)P[G(\sigma + s_n, Y_n(\sigma-), x) \\
- G(\sigma + s_n, \tilde{Y}(\sigma-), x)]N(d\sigma, dx)\|^2 \\
+ 32E\left[ \frac{1}{t} \right] \int_{-\infty}^{t} \int_{|u| \geq 1} T(t-\sigma)P[G(\sigma + s_n, Y_n(\sigma-), x) \\
- G(\sigma + s_n, \tilde{Y}(\sigma-), x)]\nu(dx)d\sigma\|^2 \\
+ 32E\left[ \frac{1}{t} \right] \int_{-\infty}^{t} \int_{|u| \geq 1} T(t-\sigma)P[G(\sigma + s_n, \tilde{Y}(\sigma-), x) \\
- \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]N(d\sigma, dx)\|^2 \\
+ 32E\left[ \frac{1}{t} \right] \int_{-\infty}^{t} \int_{|u| \geq 1} T(t-\sigma)J[G(\sigma + s_n, Y_n(\sigma-), x) \\
- G(\sigma + s_n, \tilde{Y}(\sigma-), x)]N(d\sigma, dx)\|^2 \\
+ 32E\left[ \frac{1}{t} \right] \int_{-\infty}^{t} \int_{|u| \geq 1} T(t-\sigma)J[G(\sigma + s_n, \tilde{Y}(\sigma-), x) \\
- \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]\nu(dx)d\sigma\|^2 \\
+ 32E\left[ \frac{1}{t} \right] \int_{-\infty}^{t} \int_{|u| \geq 1} T(t-\sigma)J[G(\sigma + s_n, \tilde{Y}(\sigma-), x) \\
- \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]\nu(dx)d\sigma\|^2 \\
\leq 32K^2L \int_{-\infty}^{t} e^{-2\omega(t-\sigma)}E\|Y_n(\sigma-) - \tilde{Y}(\sigma-)\|^2 d\sigma \\
+ 32K^2b \int_{-\infty}^{t} e^{-\omega(t-\sigma)}d\sigma \cdot \int_{-\infty}^{t} \int_{|u| \geq 1} e^{-\omega(t-\sigma)}E\|G(\sigma + s_n, Y_n(\sigma-), x) \\
- G(\sigma + s_n, \tilde{Y}(\sigma-), x)]\nu(dx)d\sigma \\
+ 32K^2 \int_{-\infty}^{t} \int_{|u| \geq 1} e^{-2\omega(t-\sigma)}E\|G(\sigma + s_n, \tilde{Y}(\sigma-), x) \\
- \tilde{G}(\sigma, \tilde{Y}(\sigma-), x)]\nu(dx)d\sigma
\[ +32K^2 b \int_{-\infty}^{t} e^{-\omega(t-\sigma)} d\sigma \]

\[ \cdot \int_{-\infty}^{t} \int_{|x| \geq 1} e^{-\omega(t-\sigma)} E\|G(\sigma + s_n, \tilde{Y}(\sigma), x) - \hat{G}(\sigma, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma \]

\[ +32K^2 L \int_{t}^{+\infty} e^{2\omega(t-\sigma)} E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 d\sigma \]

\[ +32K^2 b \int_{t}^{+\infty} e^{\omega(t-\sigma)} d\sigma \cdot \int_{t}^{+\infty} \int_{|x| \geq 1} e^{\omega(t-\sigma)} E\|G(\sigma + s_n, Y_n(\sigma), x) - G(\sigma + s_n, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma \]

\[ +32K^2 b \int_{t}^{+\infty} e^{\omega(t-\sigma)} d\sigma \cdot \int_{t}^{+\infty} \int_{|x| \geq 1} e^{\omega(t-\sigma)} E\|G(\sigma + s_n, \tilde{Y}(\sigma), x) - G(\sigma, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma \]

\[ \leq 32K^2 L \int_{-\infty}^{t} e^{-2\omega(t-\sigma)} \cdot E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 d\sigma \]

\[ + \frac{32K^2 b L}{\omega} \int_{-\infty}^{t} e^{-\omega(t-\sigma)} \cdot E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 d\sigma \]

\[ + \frac{32K^2 b L}{\omega} \int_{t}^{+\infty} e^{2\omega(t-\sigma)} \cdot E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 d\sigma \]

\[ + \frac{32K^2 b L}{\omega} \int_{t}^{+\infty} e^{\omega(t-\sigma)} \cdot E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 d\sigma + E^n(t) \]

where

\[ E^n(t) := \]

\[ 32K^2 \int_{-\infty}^{t} \int_{|x| \geq 1} e^{-2\omega(t-\sigma)} E\|G(\sigma + s_n, \tilde{Y}(\sigma), x) - \hat{G}(\sigma, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma \]

\[ + \frac{32K^2 b}{\omega} \cdot \int_{-\infty}^{t} \int_{|x| \geq 1} e^{-\omega(t-\sigma)} E\|G(\sigma + s_n, \tilde{Y}(\sigma), x) - \hat{G}(\sigma, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma \]

\[ + 32K^2 \int_{t}^{+\infty} \int_{|x| \geq 1} e^{2\omega(t-\sigma)} E\|G(\sigma + s_n, \tilde{Y}(\sigma), x) - \hat{G}(\sigma, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma \]

\[ + \frac{32K^2 b}{\omega} \cdot \int_{t}^{+\infty} \int_{|x| \geq 1} e^{\omega(t-\sigma)} E\|G(\sigma + s_n, \tilde{Y}(\sigma), x) - \hat{G}(\sigma, \tilde{Y}(\sigma), x)\|^2 \nu(dx) d\sigma . \]

It follows from (3.14) and the $C^2$-boundedness of $\tilde{Y}(\cdot)$ that $E^n(t) \to 0$ uniformly in $t \in \mathbb{R}$ as $n \to \infty$.

By the above estimates of $I_1-I_4$, we have

\[ E\|Y_n(t) - \tilde{Y}(t)\|^2 \leq E^n(t) + \left( \frac{32K^2 L}{\omega^2} + \frac{64K^2 b L}{\omega} + \frac{64K^2 b L}{\omega^2} \right) \sup_{\sigma \in \mathbb{R}} E\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2 \]

with $E^n(t) = \sum_{i=1}^{4} E^n_i(t)$. So

\[ \sup_{t \in \mathbb{R}} E\|Y_n(t) - \tilde{Y}(t)\|^2 \leq \sup_{t \in \mathbb{R}} E^n(t) \]
Then, the inequalities (3.6) and (3.10) become
\[
\sup_{\sigma \in \mathbb{R}} \mathbb{E}\|Y_n(\sigma) - \tilde{Y}(\sigma)\|^2.
\]
It follows from (3.10) and the fact \( \lim_{n \to \infty} \sup_{t \in \mathbb{R}} \mathcal{E}^n(t) = 0 \) that
\[
\sup_{t \in \mathbb{R}} \mathbb{E}\|Y_n(t) - \tilde{Y}(t)\|^2 \to 0 \quad \text{as} \quad n \to \infty.
\]
\( t \in \mathbb{R} \), Since \( Y(t+s_n) \) and \( Y_n(t) \) share the same distribution for each \( t \in \mathbb{R} \), this yields that \( Y(t+s_n) \to \tilde{Y}(t) \) in distribution uniformly in \( t \in \mathbb{R} \) as \( n \to \infty \). The proof is complete. \( \square \)

4. Applications

In this section, we give two examples to illustrate our results in this paper.

Example 4.1. Let us consider an ordinary differential equation perturbed by a two-sided Lévy noise:
\[
dy = (Ay + f(t,y))dt + g(t,y)dW + \int_{|x|<1} F(t,y,x)N(dt,dx) + \int_{|x|\geq1} G(t,y,x)N(dt,dx),
\]
where
\[
A := \begin{pmatrix} 8 & 0 \\ 0 & -6 \end{pmatrix},
\]
\[
f(t,y) := \begin{pmatrix} 0 \\ \cos \sqrt{2t} \sin \sqrt{3t} \end{pmatrix},
\]
\[
g(t,y) := \begin{pmatrix} 0 \\ \frac{1}{12} \sin(y + \cos \sqrt{3t} + \cos \sqrt{2t}) \end{pmatrix},
\]
\[
F(t,y,x) := \begin{pmatrix} 0 \\ \frac{1}{18}y \end{pmatrix},
\]
\[
G(t,y,x) := \begin{pmatrix} 0 \\ \frac{\sin^2(\sqrt{3t})}{3 + \cos \sqrt{2t} + \cos \sqrt{5t}} \end{pmatrix},
\]
where \( W \) is a one-dimensional Brownian motion and \( N \) is a Poisson random measure in \( \mathbb{R} \) independent of \( W \). It is clear that (H1) and (H2) hold, that is, the semigroup by \( A \) satisfies the exponential dichotomy property with \( K = 1 \) and \( \omega = 6 \); \( f, g \) are almost periodic in \( t \) and \( F,G \) are Poisson square-mean almost periodic in \( t \). The Lipschitz constants of \( f, g, F, G \) in \( y \) can be chosen as \( 1/8, 1/12, 1/10, 1/9 \), respectively, so the Lipschitz conditions in (H3) are satisfied with \( L = (1/8)^2 = 1/64 \) provided
\[
\int_{|x|<1} \frac{1}{100} \nu(dx) \leq \frac{1}{64} \quad \text{and} \quad \int_{|x|\geq1} \frac{1}{81} \nu(dx) \leq \frac{1}{64},
\]
i.e.
\[
\nu((-1,1)) < \frac{25}{16} \quad \text{and} \quad b \leq \frac{81}{64}.
\]
Then, the inequalities (3.4) and (3.10) become
\[
\frac{1 + 2b}{36} + \frac{2}{6} < 4 \quad \text{and} \quad \frac{1 + 2b}{36} + \frac{2}{6} < 2,
\]
i.e. \( b < 59/2 \). By Theorem 3.3, 4.1 admits a unique \( L^2 \)-bounded mild solution if \( \nu((-1,1)) < 25/16, b \leq 81/64 \); furthermore, by Theorem 3.5 this unique \( L^2 \)-bounded solution is almost periodic in distribution under the same conditions.

Example 4.2. Let \( \Omega \subset \mathbb{R}^n \), be a bounded subset whose boundary \( \partial \Omega \) is of class \( C^2 \) and being locally on one side of \( \Omega \).

Consider the parabolic stochastic partial differential equation:
\[
du(t,\xi) = \{A(\xi)u(t,\xi) + f(t,u(t,\xi))\}dt + g(t,u(t,\xi))dW(t,\xi)
\]
\[
+ h(t,u(t,\xi),Z(t,\xi))dZ(t,\xi)
\]
\[
\sum_{i,j=1}^{n} r_{ij}(\xi)a_{ij}(t,\xi)D_{ij}u(t,\xi) = 0, \quad t \in \mathbb{R}, \quad \xi \in \partial \Omega,
\]
Here $D_i := \frac{\partial}{\partial u_i}$, $f, g, h$ are continuous functions with additional properties which would be specified below, $W$ is a $Q$-Wiener process on $L^2(\Omega)$ with $\text{Tr} Q < \infty$, and $Z$ is a Lévy pure jump process on $L^2(\Omega)$ which is independent of $W$, $\nu(\xi) = (n_1(\xi), a_2(\xi), \ldots, n_n(\xi))$ is the outer unit normal vector, the family of operators $A(\xi)$ are formally given by

$$ A(\xi) = \sum_{i,j=1}^{n} \frac{\partial}{\partial u_i} \left( a_{ij}(\xi) \frac{\partial}{\partial u_j} \right) + a_0(\xi), \quad t \in \mathbb{R}, \quad \xi \in \Omega, $$

and $a_0, a_{ij}(i, j = 1, 2, \ldots, n)$ satisfy the following conditions:

(i) The coefficients $(a_{ij})_{i,j=1}^{n}$ are symmetric, that is, $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$. Moreover, $a_{ij} \in L^2(\mathbb{P}, C(\bar{\Omega})) \cap L^2(\mathbb{P}, C^1(\bar{\Omega})) \cap \mathbb{L}^2(\mathbb{P}, L^2(\Omega))$ for all $i, j = 1, \ldots, n$, and $a_0 \in L^2(\mathbb{P}, L^2(\Omega))$.

(ii) There exists $\epsilon_0 > 0$ such that

$$ \sum_{i,j=1}^{n} a_{ij}(\xi) \eta_i \eta_j \geq \epsilon_0 |\eta|^2, $$

for all $\xi \in \Omega$ and $\eta \in \mathbb{R}^n$.

Denote $H = L^2(\Omega)$. For each $t \in \mathbb{R}$ define an operator $A$ on $L^2(\mathbb{P}, H)$ by

$$ D(A) = \{ u \in W^{2,2}(\Omega) : \sum_{i,j=1}^{n} n_i(\cdot) a_{ij}(\cdot) D_i u = 0 \text{ on } \partial\Omega \} $$

and $AY = A(\xi)u(\xi)$ for all $Y \in D(A)$.

Under above assumptions, the existence of a semigroup $T(t)$ satisfying (H1) is obtained, see [17] for details.

Then the parabolic stochastic partial differential equation can be written as an abstract evolution equation

\begin{equation}
\begin{aligned}
dY &= (AY + F(t, Y))dt + G(t, Y)dW + \int_{|z|_{L_2} < 1} \mathcal{H}(t, Y, z) \tilde{N}(dt, dz) \\
&\quad + \int_{|z|_{L_2} \geq 1} \mathcal{H}(t, Y, z) \tilde{N}(dt, dz)
\end{aligned}
\end{equation}

on the Hilbert space $\mathbb{H}$, where

$$ F(t, Y) := f(t, u), \quad G(t, Y)dW := g(t, u)dW $$

$$ h(t, u, Z)dZ := \int_{|z|_{L_2} < 1} \mathcal{H}(t, Y, z) \tilde{N}(dt, dz) + \int_{|z|_{L_2} \geq 1} \mathcal{H}(t, Y, z) N(dt, dz) $$

with

$$ Z(t, \xi) = \int_{|z|_{L_2} < 1} z \tilde{N}(t, dz) + \int_{|z|_{L_2} \geq 1} z N(t, dz), \quad \mathcal{H}(t, Y, z) = h(t, u, z)z. $$

Here we assume for simplicity that the Lévy pure jump process $Z$ is decomposed as above by the Lévy-Itô decomposition theorem.

We assume that the continuous functions $f(t, u)$ and $g(t, u)$ are Lipschitz with respect to $u$ and uniformly almost periodic, then the functions $F$ and $G$ in [11] are Lipschitz with respect to $Y$ and uniformly almost periodic. Let the continuous function $h(t, u)$ be Lipschitz with respect to $u$ and uniformly Poisson almost periodic, and the intensity measure $\nu$ of the Poisson process $Z$ be such that $\mathcal{H}$ is Lipschitz in $Y$ and uniformly Poisson square-mean almost periodic in the sense of [8.3] and [8.5]. In particular, when $\nu$ is a finite measure, the required Lipschitz condition for $\mathcal{H}$ holds. That is, (H3) holds.

Since (H1)-(H3) are satisfied, the parabolic stochastic partial differential equation has a unique $L^2$-bounded mild solution $u \in L^2(\mathbb{P}, H)$, when $K$ and the Lipschitz constants for $f, g, h$ are appropriately small; furthermore, this unique $L^2$-bounded solution is almost periodic in distribution, when the Lipschitz constants for $f, g, h$ are smaller.
References

[1] L. Amerio, Soluzioni quasi-periodiche, o limitate di sistemi differenziali nonlineari quasi-periodiche, o limitatìs 39 (1955), 97–119.
[2] D. Applebaum, Lévy Process and Stochastic Calculus. Second edition. Cambridge University Press, 2009. xxx+460 pp.
[3] L. Arnold and C. Tudor, Stationary and almost periodic solutions of almost periodic affine stochastic differential equations. Stochastics Stochastics Rep. 64 (1998), 177–193.
[4] P. H. Bezandry and T. Diagana, Existence of almost periodic solutions to some stochastic differential equations. Appl. Anal. 86 (2007), 819–827.
[5] P. H. Bezandry and T. Diagana, Square-mean almost periodic solutions nonautonomous stochastic differential equations. Electron. J. Diff. Eqns. 117 (2007), 1–10.
[6] P. H. Bezandry and T. Diagana, Almost Periodic Stochastic Processes. Springer, New York, 2011. xvi+235 pp.
[7] W. A. Coppel, Almost periodic properties of ordinary differential equations, Ann. Math. Pura. Appl. 76 (1967), 27–50.
[8] G. Da Prato and C. Tudor, Periodic and almost periodic solutions for semilinear stochastic equations. Stochastic Anal. Appl. 13 (1995), 13–33.
[9] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions. Cambridge University Press, 1992. xviii+454 pp.
[10] R. M. Dudley, Real Analysis and Probability, 2nd edition, Cambridge University Press, 2003.
[11] J. Favard, Sur les equations differentielles lineaires a coefficients presque-periodiques, Acta Math. 51 (1928), 31–81.
[12] A. M. Fink, Semi-separated conditions for almost periodic solutions, J. Differential Equations 11 (1972), 245–251.
[13] A. M. Fink, Almost Periodic Differential Equations. Lecture Notes in Math., vol. 377, Springer-Verlag, Berlin, Heidelberg, and New York, 1974.
[14] A. Halanay, Periodic and almost periodic solutions to affine stochastic systems. Proceedings of the Eleventh International Conference on Nonlinear Oscillations (Budapest, 1987), 94–101, János Bolyai Math. Soc., Budapest, 1987.
[15] M. Kamenskii, O. Mellah, and P. Raynaud de Fitte, Weak averaging of semilinear stochastic differential equations with almost periodic coefficients, arXiv: 1210.7412v2
[16] Z. Liu and K. Sun, Almost automorphic solutions for stochastic differential equations driven by Lévy noise. J. Funct. Anal. 226 (2014), 1115–1149.
[17] L. Maniar and R. Schnaubelt, Almost periodicity of inhomogeneous parabolic evolution equations. Lecture Notes in Pure and Applied Mathematics, New York, 2003, pp. 299–318.
[18] O. Mellah and P. Raynaud de Fitte, Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients. Electron. J. Differential Equations, Vol. 2013 (2013), No. 91, pp. 1–7.
[19] R. K. Miller, Almost periodic differential equations as dynamical systems with applications to the existence of a.p. solutions, J. Differential Equations 1 (1965), 337–345.
[20] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences 44, Springer, 1983.
[21] S. Peszat and J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise. Cambridge University Press, 2007. xii+419 pp.
[22] K. I. Sato, Lévy Processes and Infinite Divisibility. Cambridge University Press (1999).
[23] R. J. Sacker and G.R. Sell, Lifting properties in skew-product flows with applications to differential equations, Memoirs Amer. Math. Soc. 11 (1977) no. 190.
[24] G. Seifert, Almost periodic solutions for almost periodic systems of linear differential equations, J. Differential Equations 2 (1966), 305–319.
[25] W. Shen and Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows. Mem. Amer. Math. Soc. 136 (1998), no. 647, x+93 pp.
[26] K. Sun, Almost automorphic solutions for stochastic differential equations driven by Lévy noise with exponential dichotomy, preprint.
[27] C. Tudor, Almost periodic solutions of affine stochastic evolution equations. Stochastics Stochastics Rep. 38 (1992), 251–266.
[28] Y. Wang and Z. Liu, Almost periodic solutions for stochastic differential equations with Lévy noise. Nonlinearity 25 (2012), 2803–2821.
[29] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions, Appl. Math. Sciences No.14, Springer-Verlag, New York, 1975.

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