The combinatorics of $k$-marked Durfee symbols

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June 16, 2008

Abstract. Andrews recently introduced $k$-marked Durfee symbols which are connected to moments of Dyson’s rank. By these connections, Andrews deduced their generating functions and some combinatorial properties and left their purely combinatorial proofs as open problems. The primary goal of this article is to provide combinatorial proofs in answer to Andrews’ request. We obtain a relation between $k$-marked Durfee symbols and Durfee symbols by constructing bijections, and all identities on $k$-marked Durfee symbols given by Andrews could follow from this relation. In a similar manner, we also prove the identities due to Andrews on $k$-marked odd Durfee symbols combinatorially, which resemble ordinary $k$-marked Durfee symbols with a modified subscript and with odd numbers as entries.

Keywords: rank, the moment of rank, the symmetrized moment of rank, Durfee symbols, $k$-marked Durfee symbols, odd Durfee symbols, $k$-marked odd Durfee symbols

AMS Classifications: 05A17, 05A19, 11P83

1 Introduction

We will adopt the terminology on partitions in Andrews [2]. A partition $\lambda$ of a positive integer $n$ is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. Then $\lambda_i$ are called the parts of $\lambda$, $\lambda_1$ is its largest part. The number of parts of $\lambda$ is called the length of $\lambda$, denoted by $l(\lambda)$. The weight of $\lambda$ is the sum of parts of $\lambda$, denoted by $|\lambda|$.

The rank of a partition $\lambda$ introduced by Dyson [11] is defined as the largest part minus the number of parts, which is usually denoted by $r(\lambda) = \lambda_1 - l(\lambda)$. Let $N(m; n)$ denote the number of partitions of $n$ with rank $m$, we have


Theorem 1.1 (Dyson) The generating function for \( N(m; n) \) is given by
\[
\sum_{n=0}^{+\infty} N(m; n)q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{+\infty} (-1)^{n-1} q^{n(3n-1)/2+|m|n} (1 - q^n), \quad |q| < 1. \tag{1.1}
\]
where \((a; q)_n = \prod_{j=0}^{n-1}(1 - aq^j)\) and \((a; q)_{\infty} = \lim_{n \to \infty}(a; q)_n\). The identity (1.1) was first discovered by Dyson [11] in 1944 and first proved by Atkin and Swinnerton-Dyer [5]. Later Dyson [12] gave a simple combinatorial argument of it. We refer to [7, p.63] for more details.

Definition 1.2 For a nonnegative integer \( n \), a Durfee symbol of \( n \) is a two-row array with a subscript
\[
(\alpha, \beta)_D = \left( \begin{array}{cccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_s \\
\beta_1 & \beta_2 & \cdots & \beta_t \\
\end{array} \right)_D \tag{1.2}
\]
where \( D \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0, \ D \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_t > 0 \) and \( n = \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{t} \beta_i + D^2 \).

For example, there are 5 Durfee symbols of 4.
\[
\left(\begin{array}{ccc}1 & 1 & 1 \\
1 & 1 \end{array}\right)_1, \quad \left(\begin{array}{cc}1 & 1 \\
1 & 1 \end{array}\right)_1, \quad \left(\begin{array}{cc}1 & 1 \\
1 & 1 \end{array}\right)_1, \quad \left(\begin{array}{c}1 \\
1 \end{array}\right)_2.
\]
The difference of the lengths of \( \alpha \) and \( \beta \) is called rank of Durfee symbols \((\alpha, \beta)_D\). We use \( D_1(m; n) \) to denote the number of Durfee symbols of \( n \) with rank \( m \). Andrews [4, Section 3] showed that by constructing a bijection

Theorem 1.3 (Andrews) The number of ordinary partitions of \( n \) with rank equal to \( m \) is equal to the number of Durfee symbols of \( n \) with rank equal to \( m \), that is
\[
N(m; n) = D_1(m; n). \tag{1.3}
\]
Andrews [4] introduced the \( k \)th symmetrized rank moment \( \eta_k(n) \), defined by
\[
\eta_k(n) = \sum_{m=-\infty}^{+\infty} \left( m + \left\lfloor \frac{k-1}{2} \right\rfloor \right) N(m, n), \tag{1.4}
\]
which are linear combinations of the \( k \)th rank moments \( N_k(n) \)
\[
N_k(n) = \sum_{m=-\infty}^{+\infty} m^k N(m, n), \tag{1.5}
\]
considered by Atkin and Garvan [6]. To give a combinatorial explanation of (1.4), Andrews [4] introduced \( k \)-marked Durfee symbols, which can be thought of as the generalized Durfee symbols.
Definition 1.4 A k-marked Durfee symbol of $n$ is composed of $k$ pairs of partitions and a subscript, which is defined as

$$\eta = \left( \begin{array}{c} \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \\ \beta^k, \beta^{k-1}, \ldots, \beta^1 \end{array} \right)_{D},$$

where $\alpha^i$ (resp. $\beta^i$) represents a partition and $\sum_{i=1}^{k} (|\alpha^i| + |\beta^i|) + D^2 = n$. Furthermore, the partitions $\alpha^i$ and $\beta^i$ must satisfy the following three conditions where $\alpha^i_1$ (resp. $\beta^i_1$) is the largest part of the partition $\alpha^i$ (resp. $\beta^i$) and $\alpha^i_l(\alpha^i)$ (resp. $\beta^i_l(\beta^i)$) is the smallest part of the partition $\alpha^i$ (resp. $\beta^i$).

1. For $1 \leq i < k$, $\alpha^i$ must be non-empty partition, while $\alpha^k$ and $\beta^i$ could be empty;
2. $\beta^i_{i-1} \leq \alpha^i_{i-1} \leq \beta^i_{l(\beta^i)}$ for $2 \leq i \leq k$;
3. $\beta^k_1, \alpha^k \leq D$.

Clearly, 1-marked Durfee symbol is just Durfee symbol.

Let $\eta = \left( \begin{array}{c} \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \\ \beta^k, \beta^{k-1}, \ldots, \beta^1 \end{array} \right)_{D}$ be a k-marked Durfee symbol. The pair of partitions $\left( \begin{array}{c} \alpha^i \\ \beta^i \end{array} \right)$ is called the $i$th vector of $\eta$. We define $\rho_i(\eta)$, the $i$th rank of $\eta$ by

$$\rho_i(\eta) = \begin{cases} l(\alpha^i) - l(\beta^i) - 1 & \text{for } 1 \leq i < k, \\ l(\alpha^k) - l(\beta^k) & \text{for } i = k. \end{cases}$$

For example, $\left( \begin{array}{c} 4_3 \\ 5_3 \\ 3_2 \\ 2_2 \\ 2_1 \end{array} \right)$ is a 3-marked Durfee symbol of 55 where $\alpha^3 = (4, 4)$, $\alpha^2 = (3, 3, 2)$, $\alpha^1 = (2)$, and $\beta^3 = (5)$, $\beta^2 = (3, 2)$, $\beta^1 = (2)$. The first rank is $-1$, the second rank is $0$, and the third rank is $1$.

Let $D_k(m_1, m_2, \ldots, m_k; n)$ denote the number of $k$-marked Durfee symbols of $n$ with $i$th rank equal to $m_i$ and $D_k(n)$ denote the number of $k$-marked Durfee symbols of $n$, it’s clear to see that

$$D_k(n) = \sum_{m_1, \ldots, m_k = -\infty}^{+\infty} D_k(m_1, m_2, \ldots, m_k; n). \quad (1.6)$$

In his recent work [4], Andrews used the connections between $k$-marked Durfee symbols and the symmetrized rank moments (1.4) to find identities relating the generating function for $D_k(m_1, m_2, \ldots, m_k; n)$, as well as to deduce some combinatorial properties and Ramanujan-type congruences for $k$-marked Durfee symbols. At the end of the paper, Andrews proposed a variety of serious questions which fall into 3 basic groups: combinatorial, asymptotic and congruential. The recent works [8, 9] by Kathrin Bringmann,
Frank Garvan, and Karl Mahlburg focused on the study relating asymptotical and congruential properties of \( k \)-marked Durfee symbols. They used the automorphic properties to prove the existence of infinitely many congruences for \( k \)-marked Durfee symbols. The primary goal of this article is to answer Andrews’ request on combinatorics (Problems 1-4 and 6-9 on page 39 of [4]). We will give combinatorial proofs of the identities relating the generating function for \( D_k(m_1, m_2, \ldots, m_k; n) \) and combinatorial properties for \( k \)-marked Durfee symbols. To be specific, we first derive the following partition identity (1.7) by constructing bijections, which gives a relation between \( k \)-marked Durfee symbols and Durfee symbols. We then show that all identities on \( k \)-marked Durfee symbols given by Andrews ([4, Problems 1-4]) could follow from this identity. We then use the similar method to study the identities of Andrews on \( k \)-marked odd Durfee symbols ([4, Problems 6-9]), which resemble ordinary Durfee symbols with a modified subscript and with odd numbers as entries.

\[ D_k(m_1, m_2, \ldots, m_k; n) = \sum_{j=0}^{\infty} \binom{j+k-2}{k-2} N \left( \sum_{i=1}^{k} |m_i| + 2j + k - 1; n \right). \]  

The paper is organized as follows. In Section 2, we consider the relation between \( k \)-marked Durfee symbols and Durfee symbols and prove Theorem 1.5. To this end, we introduce a special class of \( k \)-marked Durfee symbols, which we call \( k \)-marked strict shifted Durfee symbols since each of their vectors except for the \( k \)th vector is a two-line strict shifted plane partition. We deduce the desired relation by building the connections between \( k \)-marked strict shifted Durfee symbols, Durfee symbols, and \( k \)-marked Durfee symbols respectively. In Section 3, we give combinatorial proofs of the identities due to Andrews on \( k \)-marked Durfee symbols with the help of Theorem 1.5. In particular, the symmetry of \( k \)-marked Durfee symbols ([4, Corollary 12]) could be thought of as a direct consequence of Theorem 1.5. Section 4 is devoted to the study of \( k \)-marked odd Durfee symbols.

## 2 \( k \)-marked strict shifted Durfee symbols

In this section, we will establish the relations between \( k \)-marked strict shifted Durfee symbols, Durfee symbols, and \( k \)-marked Durfee symbols respectively, and then deduce Theorem 1.5. We begin by defining \( k \)-marked strict shifted Durfee symbols.

To define these objects, we need recall the concept of strict shifted plane partitions introduced by Andrews [3]. Mills, Robbins and Rumsey [14] showed that strict shifted plane partitions whose row lengths are equal to row leaders are bijective to cyclically symmetric plane partitions.

A strict shifted plane partition of \( n \) is an array \( \pi = (a_{ij}) \) of positive integers defined only for \( j \geq i \), which has non-increasing rows and strictly decreasing columns, such that
the sum of its elements is \( n \). Such an array can be written

\[
\begin{array}{ccccccc}
  a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1\mu_1} \\
  a_{22} & a_{23} & a_{24} & \cdots & a_{2\mu_2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{rr} & \cdots & a_{r\mu_r}
\end{array}
\]

where \( a_{ij} \geq a_{i,j+1} \) and \( a_{ij} > a_{i+1,j} \), \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \) and \( \sum_{i,j} a_{ij} = n \).

We can regard a two-lined strict shifted partition of \( n \) as a pair of partitions \((\alpha, \beta)\) of \( n \), where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \), \( \beta = (\beta_1, \beta_2, \ldots, \beta_s) \), \( r > s \), and \( \alpha_{i+1} > \beta_i \) for \( i = 1, 2, \ldots, s \).

For example \((3,3,2,2,1)\) is a two-lined strict shifted partition.

A \( k \)-marked Durfee symbol \( \eta = (\alpha^k, \alpha^{k-1}, \ldots, \alpha^1) \) is said to be strict shifted if all of its vectors except for the \( k \)th vector \((\alpha^k, \beta^k)\) are two-lined strict shifted partitions. For example, 3-marked Durfee symbol in \((2,3)\) is strict shifted. Let \( \mathcal{D}^n_k(m_1, m_2, \ldots, m_k; n) \) denote the number of \( k \)-marked strict shifted Durfee symbols of \( n \) with \( i \)th rank equal to \( m_i \).

We now build a connection between \( k \)-marked strict shifted Durfee symbols of \( n \) and Durfee symbols of \( n \).

**Theorem 2.1** Given \( k \) nonnegative integers \( m_1, m_2, \ldots, m_k \), there is a bijection \( \Phi \) between the set of \( k \)-marked strict shifted Durfee symbols of \( n \) with \( i \)th rank equal to \( m_i \) and the set of Durfee symbols of \( n \) with rank equal to \( \sum_{i=1}^{k} m_i + k - 1 \).

**Proof.** The map \( \Phi \): Let \( \eta = (\alpha^k, \alpha^{k-1}, \ldots, \alpha^1) \) counted by \( \mathcal{D}^n_k(m_1, \ldots, m_k; n) \), we then obtain a Durfee symbol \((\gamma, \delta) \) \( \mathcal{D} \) when remove all subscripts of \( \eta \). Obviously, the resulting Durfee symbol \((\gamma, \delta) \) \( \mathcal{D} \) is enumerated by \( \mathcal{D}(\sum_{i=1}^{k} m_i + k - 1; n) \).

The reverse map \( \Phi^{-1} \): Let \((\gamma, \delta) \) \( \mathcal{D} \) be counted by \( \mathcal{D}(\sum_{i=1}^{k} m_i + k - 1; n) \), we will construct a \( k \)-marked Durfee symbol \( \eta = (\alpha^k, \alpha^{k-1}, \ldots, \alpha^1) \) whose \( i \)th rank equal to \( m_i \). Let

\[
\begin{pmatrix}
  \gamma_1 & \gamma_2 & \cdots & \gamma_l \\
  \delta_1 & \delta_2 & \cdots & \delta_s
\end{pmatrix}_D
\]

where \( D \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_l \) and \( D \geq \delta_1 \geq \delta_2 \geq \cdots \geq \delta_s \), we assume that \( \delta_j = 0 \) for \( j \geq s + 1 \). Note that \( l - s = \sum_{i=1}^{k} m_i + k - 1 \).

We now split \((\gamma, \delta)\) to generate the \( k \)th vector \((\alpha^k, \beta^k)\) of \( \eta \). Let \( j \) be largest nonnegative integer such that \( \delta_j \geq \gamma_{m_k+j+1} \), that is for any \( i \geq j + 1 \), we have \( \delta_i < \gamma_{m_k+i+1} \). Let

\[
\begin{pmatrix}
  \alpha^k \\
  \beta^k
\end{pmatrix} =
\begin{pmatrix}
  \gamma_1 & \gamma_2 & \cdots & \gamma_{m_k+j} \\
  \delta_1 & \delta_2 & \cdots & \delta_j
\end{pmatrix}_D^\prime,
\]

where \( \alpha^k, \beta^k \) are partitions of \( m_1, \ldots, m_k \), respectively.
and
\[
\begin{pmatrix}
\gamma' \\
\delta'
\end{pmatrix}
= \begin{pmatrix}
\gamma_1' & \gamma_2' & \cdots & \gamma_{\nu'}' \\
\delta_1' & \delta_2' & \cdots & \delta_{\nu'}'
\end{pmatrix},
\]
where \(\gamma_i' = \gamma_{m_k+j+i}, \delta_i' = \delta_{j+i} \) for \(i \geq 1\). Obviously, \(l(\alpha^k) - l(\beta^k) = m_k\). Furthermore, \((\gamma', \delta')\) is a strict shifted partition from the fact that for any \(i \geq 1\), \(\delta_{i+j} < \gamma_{m_k+i+j+1}\) and
\[
l' - s' = \sum_{i=1}^{k-1} m_i + k - 1.
\]

We continue to split \((\gamma', \delta')\) to construct the \((k-1)\)th vector \((\alpha^{k-1}, \beta^{k-1})\) of \(\eta\). Let \(j\) be largest nonnegative integer such that \(\gamma_j' \geq \gamma_{m_k-j+2}\), we then let
\[
\begin{pmatrix}
\alpha^{k-1} \\
\beta^{k-1}
\end{pmatrix}
= \begin{pmatrix}
\gamma_1' & \gamma_2' & \cdots & \gamma_{m_k-j+1}' \\
\delta_1' & \delta_2' & \cdots & \delta_{m_k-j+1}'
\end{pmatrix},
\]
and
\[
\begin{pmatrix}
\gamma'' \\
\delta''
\end{pmatrix}
= \begin{pmatrix}
\gamma_1'' & \gamma_2'' & \cdots & \gamma_{m_k-j+2}'' \\
\delta_1'' & \delta_2'' & \cdots & \delta_{m_k-j+2}''
\end{pmatrix},
\]
where \(\gamma_i'' = \gamma_{m_k-j+i+1}, \delta_i'' = \delta_{j+i} \) for \(i \geq 1\). Clearly, \(l(\alpha^{k-1}) - l(\beta^{k-1}) = m_k-1 + 1\) and \((\alpha^{k-1}, \beta^{k-1})\) is a strict shifted partition for \((\gamma', \delta')\) is strict shifted. Observe that \(\delta_{j+i} < \gamma_{m_k-j+i+2}\) for \(i \geq 1\), so \((\gamma'', \delta'')\) is also strict shifted and \(l'' - s'' = \sum_{i=1}^{k-2} m_i + k - 2\).

Repeat the above process to generate \((\alpha^{k-2}, \beta^{k-2}), \ldots, (\alpha^1, \beta^1)\) respectively and let \(D' = D\), it’s straightforward to see that the \(k\)-marked Durfee symbol \(\eta = \begin{pmatrix}
\alpha_k & \alpha_{k-1} & \ldots & \alpha_1 \\
\beta_k & \beta_{k-1} & \ldots & \beta_1
\end{pmatrix}_D\) is counted by \(D'_k(m_1, m_2, \ldots, m_k; n)\).

We now illustrate the reverse map \(\Phi^{-1}\) by going through an example in details. Take \(m_1 = 1, m_2 = 1, m_3 = 0\), and let
\[
\begin{pmatrix}
\gamma \\
\delta
\end{pmatrix}_D = \begin{pmatrix}
6 & 6 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 \\
5 & 5 & 4 & 2 & 1 & 1 & 1
\end{pmatrix},
\]
we first split \((\gamma, \delta)\) to get \((\alpha^3, \beta^3)\), note that the divisional part \(\delta_j\) is the smallest part satisfying \(\delta_j \geq \gamma_{j+1}\).
\[
\begin{pmatrix}
\alpha^3 \\
\beta^3
\end{pmatrix}
= \begin{pmatrix}
6 & 6 & 3 \\
5 & 5 & 4
\end{pmatrix},
\]
\[
\begin{pmatrix}
\gamma' \\
\delta'
\end{pmatrix}
= \begin{pmatrix}
3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\
2 & 1 & 1 & 1
\end{pmatrix},
\]
we then split \((\gamma', \delta')\) to generate \((\alpha^2, \beta^2)\), the divisional part \(\delta_j'\) is the smallest part satisfying \(\delta_j' \geq \gamma_{j+2}\). The remaining part of \((\gamma', \delta')\) is just \((\alpha^1, \beta^1)\).
\[
\begin{pmatrix}
\alpha^2 \\
\beta^2
\end{pmatrix}
= \begin{pmatrix}
3 & 3 & 3 & 2 & 2 & 1 & 1 & 1
\end{pmatrix},
\]
\[
\begin{pmatrix}
\alpha^1 \\
\beta^1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1
\end{pmatrix}.
\]
Thus we get
\[
\eta = \begin{pmatrix}
6_3 & 6_3 & 3_3 & 3_2 & 3_2 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_2 & 1_1 & 1_1 \\
5_3 & 5_3 & 4_3 & 2_2 & 1_2 & 1_2 & 1_2
\end{pmatrix}_6.
\]

By Theorems 2.1 and 1.3 we then deduce the following partition identity:

\[(2.8)\]
Corollary 2.2 For \( m_i \geq 0 \) and \( k \geq 1 \), we have

\[
D_{k}^{ss}(m_1, m_2, \ldots, m_k; n) = N \left( \sum_{i=1}^{k} m_i + k - 1; n \right). 
\] (2.9)

To establish the relation between \( k \)-marked strict shifted Durfee symbols and \( k \)-marked Durfee symbols, we need define a statistic on \( k \)-marked Durfee symbols. In the same way, we first define this statistic on a pair of partitions.

For \((\gamma, \delta)\), the part \( \delta_i \) is said to be balanced if \( \gamma_{i+1} \leq \delta_i \) and the number of parts greater than \( \delta_i \) in \( \gamma \) (\( \gamma_1 \) is not counted) is equal to the number of unbalanced parts before \( \delta_i \) in \( \delta \). For example, let \((\gamma, \delta) = (4 \ 3 \ 3 \ 1 \ 1, 3 \ 2 \ 2)\), the first part 3 of \( \delta \) is balanced while the third part 2 is not balanced, although it satisfies the first condition \( \gamma_{i+1} \leq \delta_i \), there are two parts greater than 2 (4 is not counted) in \( \gamma \), while there is one unbalanced part (the second part 2) before the third part 2 in \( \delta \).

It should be pointed out that for any unbalanced part \( \delta_i \), the number of parts greater than \( \delta_i \) in \( \gamma \) (\( \gamma_1 \) is not counted) is greater than the number of unbalanced parts before \( \delta_i \) in \( \delta \). We will state this in the following proposition:

Proposition 2.3 Let \((\gamma, \delta)\) be a pair of partitions, for any part \( \delta_j \) of \( \delta \), let \( d_j \) denote the difference between the number of parts greater than \( \delta_j \) in \( \gamma \) (except for \( \gamma_1 \)) and the number of unbalanced parts before \( \delta_j \) in \( \delta \), we then have \( d_j \geq 0 \).

For the above example, we see that \( d_1 = 0, d_2 = 2, d_3 = 1 \).

Proof. If \( \gamma_{j+1} > \delta_j \), it’s clear to see that \( d_j \geq 1 \); If \( \gamma_{j+1} \leq \delta_j \), we consider the following two cases:

Case 1 When \( \delta_{j-1} < \gamma_j \), and at this time, there are \( j - 1 \) parts greater than \( \delta_j \) in \( \gamma \) (\( \gamma_1 \) is not counted) and the number of unbalanced parts before \( \delta_j \) in \( \delta \) is less than or equal to \( j - 1 \), so \( d_j \geq 0 \).

Case 2 When \( \delta_{j-1} \geq \gamma_j \), let \( t \) be largest nonnegative integer less than \( j \) such that \( \delta_t < \gamma_{t+1} \). Here we assume that \( \delta_0 = 0 \) and then \( t \) must exist. From Case 1, we know that \( d_{t+1} \geq 0 \). We use the induction to prove that \( d_t \geq 0 \) for \( t + 2 \leq i \leq j \).

We first prove that \( d_{t+2} \geq 0 \). Note that \( \gamma_{t+1} > \delta_{t+1} \geq \gamma_{t+2} \). If \( \delta_{t+2} < \gamma_{t+2} \), then \( d_{t+2} \geq d_{t+1} \geq 0 \); If \( \delta_{t+2} \geq \gamma_{t+2} \), then \( d_{t+2} \geq d_{t+1} - 1 \). In particular, when \( d_{t+1} = 0 \), then \( \delta_{t+1} \) is balanced, thus \( d_{t+2} = d_{t+1} = 0 \). So \( d_{t+2} \geq 0 \).

Assume that \( d_i \geq 0 \) and \( \gamma_p > \delta_i \geq \gamma_{p+1} \). We will prove that \( d_{i+1} \geq 0 \). If \( \delta_{i+1} \geq \gamma_{p+1} \), the similar argument on the case for \( \delta_{t+2} \) could show that \( d_{i+1} \geq 0 \). If \( \delta_{i+1} < \gamma_{p+1} \), and then \( d_{i+1} \geq d_i \geq 0 \).
We use $b(\gamma, \delta)$ to denote the number of balanced parts in $(\gamma, \delta)$. Clearly, $0 \leq b(\gamma, \delta) \leq l(\delta)$. Let $\eta = \left( \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \right)_D$ be a $k$-marked Durfee symbol, we define $nb_i(\eta)$, called the $i$th balanced number by

$$nb_i(\eta) = \begin{cases} b(\alpha^i, \beta^i) & \text{for } 1 \leq i < k, \\ 0 & \text{for } i = k. \end{cases}$$

For $\left( \begin{array}{cccc} 4_3 & 4_3 & 3_2 & 2_2 \\ 3_2 & 2_2 & 2_1 & 1_1 \\ 5_3 & 2_1 & 1_1 \end{array} \right)_5$, we have $nb_1 = 1$, $nb_2 = 2$, $nb_3 = 0$.

We next state a theorem concerning strict shifted partitions.

**Theorem 2.4** Given two nonnegative integers $r, m$, there is a bijection $\psi$ between the set of pairs of partitions $(\alpha, \beta)$ of $n$ with $\beta_1 \leq \alpha_1$ where there are $r$ balanced parts and the difference of the lengths of $\alpha$ and $\beta$ equals to $m$ and the set of strict shifted partitions $(\bar{\alpha}, \bar{\beta})$ of $n$ where the difference of the lengths of $\bar{\alpha}$ and $\bar{\beta}$ equals to $m + 2r$.

**Proof.** The map $\psi$: Let $(\alpha, \beta)$ be a pair of partitions with $r$ balanced parts and $l(\alpha) - l(\beta) = m$. The strict shifted partitions $(\bar{\alpha}, \bar{\beta})$ is constructed as follows. $\bar{\alpha}$ is composed of all parts of $\alpha$ and all balanced parts of $\beta$. $\bar{\beta}$ consists of all unbalanced parts of $\beta$. Take an example, let $\left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{cccc} 6 & 5 & 5 & 3 \\ 4 & 4 & 3 \end{array} \right)$, where the underlined parts in $\beta$ are balanced. According to the above construction, we then get $\left( \begin{array}{c} \bar{\alpha} \\ \bar{\beta} \end{array} \right) = \left( \begin{array}{cccc} 6 & 5 & 5 & 5 \\ 4 & 4 & 3 & 3 \\ 3 & 3 & 2 & \end{array} \right)$. It’s clear to see that $|\alpha| + |\beta| = |\bar{\alpha}| + |\bar{\beta}|$, $l(\bar{\alpha}) - l(\bar{\beta}) = l(\alpha) + r - (l(\beta) - r) = m + 2r$. By Proposition 2.3 one can also easily know that $(\bar{\alpha}, \bar{\beta})$ is strict shifted.

The reverse map $\psi^{-1}$: Let $(\bar{\alpha}, \bar{\beta})$ be a strict shifted partition where the difference of the lengths of $\bar{\alpha}$ and $\bar{\beta}$ is $m + 2r$, that is $l(\bar{\alpha}) - l(\bar{\beta}) = m + 2r$. We now construct a pair of partitions $(\alpha, \beta)$ with $r$ balanced parts and $l(\alpha) - l(\beta) = m$.

First of all, attach subscript $g_i$ for each part $\tilde{\alpha}_i$ of $\bar{\alpha}$, where $g_i$ denotes the difference between the number of parts before $\tilde{\alpha}_i$ in $\bar{\alpha}$ ($\tilde{\alpha}_1$ is not counted) and the number of parts greater than or equal to $\tilde{\alpha}_i$ in $\beta$. We let $g_1 = 0$.

For example, if $\left( \begin{array}{c} \bar{\alpha} \\ \bar{\beta} \end{array} \right) = \left( \begin{array}{cccc} 6 & 5 & 5 & 5 \\ 4 & 4 & 3 & 3 \\ 3 & 3 & 2 & 1 \end{array} \right)$, attach the subscripts for all parts of $\bar{\alpha}$ to get $\left( \begin{array}{cccc} 6_0 & 5_0 & 5_1 & 5_2 \\ 4 & 4 & 3 & 3 \\ 3_0 & 3_1 & 3_2 & 2_3 \end{array} \right)$.

One could easily know that $g_2 = 0$ and $g_i \geq 0$ for any $3 \leq i \leq l(\bar{\alpha})$ from the fact that $(\bar{\alpha}, \bar{\beta})$ is strict shifted. Let $\tilde{\alpha}_i$ be the smallest part in all of parts of $\alpha$ with subscript equal to $i$, we have the following conclusion:
Lemma 2.5 For $0 \leq i \leq m + 2r - 2$, $\alpha_{t_i}$ exists, and $\alpha_{t_0} \geq \alpha_{t_1} \geq \cdots \geq \alpha_{t_{m+2r-2}}$.

In the above example, $m + 2r = 6$ and $\alpha_{t_0} = 3$, $\alpha_{t_1} = 3$, $\alpha_{t_2} = 3$, $\alpha_{t_3} = 2$, $\alpha_{t_4} = 1$.

Proof. We use the induction to show that the sequence of subscripts $\{g_1, g_2, \ldots, g_{l(\alpha)}\}$ consists of all nonnegative integers less than $m + 2r - 1$. Obviously, 0 is in this sequence. Assume that $i$ is in this sequence, that is there is a part $\alpha_p$ such that $g_p = i$, we now prove that $i + 1$ is also in this sequence. By the induction hypothesis, we know the subscript of the part $\alpha_p$ is $i$, that is $\beta_{p-1} < \alpha_p \leq \beta_{p-1}$. Let $l(\beta) = s$ and note that $i \leq m + 2r - 3$, we have $s + i + 3 \leq s + m + 2r = l(\alpha)$. If $\alpha_{s+i+3} \leq \beta_s$, then the subscript of $\alpha_{s+i+3}$ is $i + 1$; Otherwise, there must exist $p + 1 \leq j \leq s + i + 2$ such that $\beta_{j-1} < \alpha_j \leq \beta_{j-1}$. This is because that $\alpha_{p+1} \leq \beta_{p-1}$ and $\alpha_{s+i+2} > \beta_s$. Hence, the subscript of $\alpha_j$ is $i + 1$. Therefore, $\alpha_{t_i}$ exists for $0 \leq i \leq m + 2r - 2$ and $\alpha_{t_0} \geq \alpha_{t_1} \geq \cdots \geq \alpha_{t_{m+2r-2}}$ when note that given a part $\alpha_p$ with subscript $i$, we could always find a part after $\alpha_p$ whose subscript is $i + 1$.

Let $\gamma$ be a partition having $r$ parts whose parts are $\alpha_{t_0}, \alpha_{t_2}, \ldots, \alpha_{t_r-1}$ respectively. Take $r = 2$ in the above example, $\gamma = (3, 3)$. We now construct the partitions $(\alpha, \beta)$. The partition $\alpha$ consists of all parts in $\bar{\alpha}$, while not in $\gamma$. $\beta$ is composed of all parts both in $\bar{\alpha}$ and $\gamma$. In the above example, we therefore get
\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \begin{pmatrix}
6 & 5 & 5 & 3 & 2 & 1 \\
4 & 4 & 3 & 2 & 2
\end{pmatrix}.
\]

It's obvious to see that $|\bar{\alpha}| + |\bar{\beta}| = |\alpha| + |\beta|$, $l(\alpha) - l(\beta) = l(\bar{\alpha}) - l(\gamma) - [l(\bar{\beta}) + l(\gamma)] = m$. We now show that $(\alpha, \beta)$ has exactly $r$ balanced parts. From Lemma 2.5 and the definition of $\gamma$, we know that for each part $\beta_t$ from $\gamma$ in $\beta$, the number of parts greater than $\beta_t$ in $\alpha$ equals the number of parts from $\beta$ greater than or equal to $\beta_t$ in $\beta$. Thus we just need to prove that the parts from $\bar{\beta}$ in $\beta$ are unbalanced. We use induction on the part from $\bar{\beta}$ in $\beta$. We first verify the largest part $\bar{\beta}_1$ of $\bar{\beta}$ is unbalanced. Suppose that there are $t$ parts $\gamma_1, \gamma_2, \ldots, \gamma_t$ from $\gamma$ greater than $\bar{\beta}_1$, then $\gamma_i = \bar{\alpha}_{t+i-1}$, $i = 1, 2, \ldots, t$ and $\alpha_2 = \bar{\alpha}_{t+2}$. We claim that $\alpha_2 = \bar{\alpha}_{t+2} > \bar{\beta}_1$. Recall that the part $\gamma_t$ is the smallest part whose subscript is $i$. If $\alpha_2 = \bar{\alpha}_{t+2} \leq \bar{\beta}_1$, the subscript of $\bar{\alpha}_{t+2}$ is less than $t + 1$, this contradicts to the definition of $\gamma_t$. Clearly, these $t$ parts from $\gamma$ are balanced, and $\bar{\beta}_1$ is not balanced. We now consider the part $\bar{\beta}_j$ from $\bar{\beta}$ in $\beta$. Assume that all parts from $\bar{\beta}$ before $\bar{\beta}_j$ in $\beta$ are not balanced and there are $t$ parts from $\gamma$ before $\bar{\beta}_j$. We next justify $\alpha_{t+1} > \beta_j$ and then by the hypothesis, we know that there are $j - 1$ unbalanced parts before $\bar{\beta}_j$, while there are at least $j$ parts larger than $\bar{\beta}_j$ in $\alpha$, so $\beta_j$ is unbalanced. Since there are $t$ parts from $\gamma$ before $\bar{\beta}_j$, then $\alpha_{t+j+1} = \bar{\alpha}_{t+j+1}$ and if $\alpha_{t+j+1} = \alpha_{j+1} \leq \beta_j$, then the subscript of $\alpha_{t+j+1}$ is less than $t$, which contradicts to the definition of the parts of $\gamma$, so $\alpha_{j+1} > \beta_j$ and we therefore complete the proof.

The next theorem gives a relation between $k$-marked strict shifted Durfee symbols and $k$-marked Durfee symbols.

Theorem 2.6 Given $2k$ nonnegative integers $m_1, m_2, \ldots, m_k$, and $t_1, t_2, \ldots, t_k$ where $t_k = 0$. There is a bijection $\Psi$ between the set of $k$-marked Durfee symbols of $n$ with
ith rank equal to \( m_i \) and ith balanced number equal to \( t_i \) and the set of \( k \)-marked strict shifted Durfee symbols of \( n \) with ith rank equal to \( m_i + 2t_i \).

**Proof.** Let \( \eta = \left( \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \right) \) be a \( k \)-marked Durfee symbol with ith rank equal to \( m_i \) and ith balanced number equal to \( t_i \). We now apply the bijection \( \psi \) in Theorem 2.4 on each vector \((\alpha^i,\beta^i)\) of \( \eta \) except for kth vector \((\alpha^k,\beta^k)\), to generate \( (\bar{\alpha}^i,\bar{\beta}^i) \). From Theorem 2.4, we know that \( (\bar{\alpha}^i,\bar{\beta}^i) \) is strict shifted and \( l(\bar{\alpha}^i) - l(\bar{\beta}^i) = l(\alpha^i) - l(\beta^i) + 2t_i \). Let \( \bar{\eta} = \left( \alpha^k, \bar{\alpha}^{k-1}, \ldots, \bar{\alpha}^1 \right) \) which has the same subscript and the same kth vector with \( \eta \). It’s obvious to see that \( \bar{\eta} \) is a \( k \)-marked strict shifted Durfee symbol with ith rank equal to \( m_i + 2t_i \).

By Theorem 2.6, one can derive the following identity readily.

**Corollary 2.7** For \( m_i \geq 0 \) and \( k \geq 2 \), we have

\[
\mathcal{D}_k(m_1, m_2, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1}=0}^{+\infty} \mathcal{D}^{ss}_k(m_1 + 2t_1, \ldots, m_{k-1} + 2t_{k-1}, m_k; n). \tag{2.10}
\]

Combine Corollaries 2.1 and 2.7 to get:

**Theorem 2.8** For \( m_i \geq 0 \) and \( k \geq 2 \), we have

\[
\mathcal{D}_k(m_1, m_2, \ldots, m_k; n) = \sum_{i=1}^{k} N \left( \sum_{i=1}^{k} m_i + 2 \sum_{i=1}^{k-1} t_i + k - 1; n \right). \tag{2.11}
\]

The following compact form of Theorem 2.8 can be easily obtained upon utilizing the fact that the number of solutions to \( t_1 + t_2 + \cdots + t_{k-1} = j \) in nonnegative integers is \( \binom{j+k-2}{k-2} \).

**Theorem 2.9** For \( m_i \geq 0 \) and \( k \geq 2 \), we have

\[
\mathcal{D}_k(m_1, m_2, \ldots, m_k; n) = \sum_{j=0}^{+\infty} \binom{j+k-2}{k-2} N \left( \sum_{i=1}^{k} m_i + 2j + k - 1; n \right). \tag{2.12}
\]

We next generalize Theorem 2.9 to give Theorem 1.5 which holds for any integer \( m_i \). To do this, we prove the following conclusion by constructing a simple bijection \( \Theta \).

**Theorem 2.10** For \( k \geq 1 \) and \( 1 \leq p \leq k \), we have

\[
\mathcal{D}_k(m_1, \ldots, m_p, \ldots, m_k; n) = \mathcal{D}_k(m_1, \ldots, -m_p, \ldots, m_k; n). \tag{2.13}
\]
Proof. Let \( \eta = \left( \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \right) \) be a \( k \)-marked Durfee symbol with \( i \)th rank equal to \( m_i \). We will construct another \( k \)-marked Durfee symbol \( \tilde{\eta} \) with \( i \)th rank equal to \( \tilde{m}_i \) such that \( \tilde{m}_p = -m_p \) and \( \tilde{m}_i = m_i \) for \( i \neq p \).

Define

\[
\tilde{\eta} = \left( \alpha^k, \ldots, \alpha^{p+1}, \bar{\alpha}^p, \alpha^{p-1}, \ldots, \alpha^1 \right)_D,
\]

where \( \alpha^k = \beta^k \) and \( \bar{\beta}^k = \alpha^k \) for \( p = k \); When \( p \neq k \), \( \bar{\alpha}^p \) consists of all parts of \( \beta^p \) and the largest part \( \alpha^p \) of \( \alpha^p \). \( \bar{\beta}^p \) consists of all parts of \( \beta^p \) except for the largest part \( \alpha^p \). It’s clear to see that \( \tilde{m}_p = l(\bar{\alpha}^p) - l(\bar{\beta}^p) = l(\beta^p) + 1 - [l(\alpha^p) - 1] = -[l(\alpha^p) - l(\beta^p) - 1] = -m_p \) for \( p \neq k \) and \( \tilde{m}_k = l(\bar{\alpha}^k) - l(\bar{\beta}^k) = -[l(\alpha^k) - l(\beta^k)] = -m_k \), so \( \tilde{\eta} \) is desired.

By Theorem 2.10 we could generalize Theorem 2.8 to give the following theorem which is useful to prove a relationship between \( k \)-marked Durfee symbols and the symmetrized rank moment given by Andrews (see Theorem 3.3).

**Theorem 2.11** For \( k \geq 2 \), we have

\[
D_k(m_1, m_2, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1} = 0}^{+\infty} N \left( \sum_{i=1}^{k} |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1; n \right).
\]  
(2.14)

Theorem 1.5 is a compact form of Theorem 2.11 which immediately follows from Theorems 2.9 and 2.10.

### 3 Andrews’ identities on \( k \)-marked Durfee symbols

In this section, we aim to show the identities on \( k \)-marked Durfee symbols given by Andrews with the help of Theorem 1.5. Recall that \( D_k(m_1, m_2, \ldots, m_k; n) \) denotes the number of \( k \)-marked Durfee symbols of \( n \) with \( i \)th rank equal to \( m_i \). Andrews considered the following generating function for \( D_k(m_1, m_2, \ldots, m_k; n) \):

\[
R_k(x_1, \ldots, x_k; q) = \sum_{m_1, \ldots, m_k = -\infty}^{+\infty} \sum_{n=0}^{+\infty} D_k(m_1, \ldots, m_k; n)x_1^{m_1} \cdots x_k^{m_k} q^n, \ |q| < 1.
\]  
(3.15)

By applying the \( k \)-fold generalization of Watson’s transformation between a very-well-poised \( {}_8\phi_7 \)-series and a balanced \( {}_4\phi_3 \)-series [11, p.199, Theorem 4], Andrews gave the generating function \( R_k(x_1, x_2, \ldots, x_k; q) \) in the following theorem.

**Theorem 3.1 (Corollary 11, Andrews [4])**

\[
R_k(x_1, x_2, \ldots, x_k; q) = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{+\infty} (-1)^{n-1} q^{3n(n-1)/2+kn} \frac{(1 + q^n)(1 - q^n)^2}{\prod_{j=1}^{k}(1 - x_j q^n)(1 - \frac{q^n}{x_j})}.
\]  
(3.16)
Proof. We will reformulate this identity as the partition identity (1.7) in Theorem 1.5. The key step is to give a partition interpretation of the right side hand of (3.16). We will show that it is the generating function for the summation on the right side of (1.7).

First, the right hand side of (3.16) can be written as the difference of the following two terms:

\[
\frac{1}{(q; q)_\infty} \frac{1}{\prod_{j=2}^{n} (1 - x_j q^n)(1 - \frac{q^n}{x_j})} - \frac{1}{(q; q)_\infty} \sum_{n=1}^{+\infty} (-1)^n q^{n(3n-1)/2 + kn} (1 - x_1)(1 - x_1^{-1})(1 + q^n) \frac{1 - x_1}{(1 - x_1 q^n)(1 - x_1^{-1} q^n)} (1 - x_1 q^n)(1 - x_1^{-1} q^n) = 2 + \sum_{m_1 = -\infty}^{+\infty} x_1^{m_1} [q^{n|m_1|} - q^{n(|m_1|-1)}],
\]

We next expand each term of the above two terms, note that

\[
(1 - x_1)(1 - x_1^{-1})(1 + q^n) = (1 - x_1) + (1 - x_1^{-1}) = 2 + \sum_{m_1 = -\infty}^{+\infty} x_1^{m_1} [q^{n|m_1|} - q^{n(|m_1|-1)}],
\]

Given \(k\) integers \(m_1, \ldots, m_k\), it’s clear to see that the coefficients of \(x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}\) on the series expansion of the right hand side of (3.16) are

\[
\frac{1}{(q; q)_\infty} \sum_{n=1}^{+\infty} (-1)^n q^{n(3n-1)/2 + kn} [q^{n|m_1|} - q^{n(|m_1|-1)}] \sum_{t_2, \ldots, t_k = 0}^{+\infty} q^{n(\sum_{i=2}^{k} |m_i| + 2\sum_{i=2}^{k} t_i)} = \frac{1}{(q; q)_\infty} \sum_{n=1}^{+\infty} (-1)^n q^{n(3n-1)/2 + kn} (1 - q^n) q^{n(|m_1|-1)} \sum_{j=0}^{+\infty} \left( \frac{j + k - 2}{k - 2} \right) q^{n(\sum_{i=2}^{k} |m_i| + 2j + k - 1)} = \sum_{j=0}^{+\infty} \left( \frac{j + k - 2}{k - 2} \right) \frac{1}{(q; q)_\infty} \sum_{n=1}^{+\infty} (-1)^n q^{n(3n-1)/2} (1 - q^n) q^{n(\sum_{i=1}^{k} |m_i| + 2j + k - 1)} = \sum_{j=0}^{+\infty} \left( \frac{j + k - 2}{k - 2} \right) N(\sum_{i=1}^{k} |m_i| + 2j + k - 1; n)q^n
\]

where the penultimate identity follows from Theorem 1.1 and we then obtain the following combinatorial interpretation:

\[
\frac{1}{(q; q)_\infty} \sum_{n=1}^{+\infty} (-1)^n q^{n(3n-1)/2 + kn} (1 + q^n)(1 - q^n)^2 \prod_{j=1}^{n} (1 - x_j q^n)(1 - \frac{q^n}{x_j}) = \sum_{m_1, \ldots, m_k = -\infty}^{+\infty} \sum_{n=0}^{+\infty} \left[ \sum_{j=0}^{+\infty} \left( \frac{j + k - 2}{k - 2} \right) N(\sum_{i=1}^{k} |m_i| + 2j + k - 1; n) \right] x_1^{m_1} \cdots x_k^{m_k} q^n. \tag{3.17}
\]
Combining (3.15) and (3.17), we reach our conclusion that the identity (3.16) can be restated as the partition identity (1.7). Thus, we have obtained a combinatorial proof of (3.16) based on Theorem 1.5.

Recently, Bringmann, Lovejoy, and Osbur defined a two-parameter generalization of $k$-marked Durfee symbols in [10]. They deduced the generating function [10, Theorem 2.2] for the two-parameter generalization of $k$-marked Durfee symbols using the similar argument of Andrews, which reduces to the identity (3.16) when $d = e = 0$.

From the generating function $R_k(x_1, x_2, \ldots, x_k; q)$ in Theorem 3.1, Andrews immediately found the following symmetry of $k$-marked Durfee symbols.

**Theorem 3.2 (Corollary 12, Andrews [4])** $D_k(m_1, m_2, \ldots, m_k; n)$ is symmetric in $m_1, m_2, \ldots, m_k$.

**Proof.** This symmetry can also immediately follow from Theorem 1.5.

In fact, the composite of the bijections on Section 2 provides a bijection for this symmetry. We take an example to explain this process. Let

$$\eta = \begin{pmatrix} 6_3 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_1 & 1_1 \\ 5_3 & 3_2 & 3_2 & 1_2 & 1_2 & 1_1 & 1_1 & 1_1 \end{pmatrix}$$

counted by $D_3(-2, 1, 0; 68)$,

we aim to construct a 3-marked Durfee symbol $\bar{\eta}$ counted by $D_3(1, -2, 0; 68)$. We will first combine all subscripts of $k$-marked Durfee symbol $\eta$ to get a Durfee symbol, and then split this Durfee symbol over again to get our desired $k$-marked Durfee symbol $\bar{\eta}$.

First, applying the bijection $\Theta$ in Theorem 2.10 into $\eta$ to get $\eta'$ enumerated by $D_3(2, 1, 0; 68)$,

$$\eta' = \begin{pmatrix} 6_3 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_1 & 1_1 \\ 5_3 & 3_2 & 3_2 & 1_2 & 1_2 & 1_1 & 1_1 & 1_1 \end{pmatrix},$$

we now utilize the bijection $\Psi$ in Theorem 2.6 on $\eta'$ to get a $k$-marked strict shifted Durfee symbol. Observe that there are two balanced parts in the second vector of $\eta'$, and there is no balanced part in other vectors of $\eta'$. So we will get a $k$-marked strict shifted Durfee symbol $\eta''$ which counted by $D_3^{ss}(2, 1 + 2 \times 2, 0; 68)$,

$$\eta'' = \begin{pmatrix} 6_3 & 3_2 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_1 & 1_1 & 1_1 \\ 5_3 & 1_2 & 1_2 & 1_2 & 1_2 & 1_1 & 1_1 & 1_1 \end{pmatrix},$$

applying the bijection $\Phi$ in Theorem 2.1 to get $\eta'''$ which counted by $D_1(9; 68)$

$$\eta''' = \begin{pmatrix} 6 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$
First, apply the reverse map $\Phi^{-1}$ in Theorem 2.1 on $\eta'''$ to get $\bar{\eta}''$ which counted by $D_{3s}^*(1, 2 \times 2, 0; 68)$,
\[
\bar{\eta}'' = \left( \begin{array}{cccccccc}
6_3 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_1 & 1_1 \\
5_3 & 1_2 & & & & & & \\
\end{array} \right)_6,
\]
using the reverse map $\Psi^{-1}$ in Theorem 2.6 on $\bar{\eta}''$, we get $k$-marked Durfee symbol $\bar{\eta}'$ counted by $D_3(1, 2, 0; 68)$
\[
\bar{\eta}' = \left( \begin{array}{cccccccc}
6_3 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_1 & 1_1 \\
5_3 & 3_2 & 3_2 & 1_2 & 1_2 & 1_1 & & \\
\end{array} \right)_6,
\]
Finally, we obtain the desired $k$-marked Durfee symbol $\bar{\eta}$ counted by $D_3(1, -2, 0; 68)$ when applying the bijection $\Theta$ in Theorem 2.10 on $\bar{\eta}'$.
\[
\bar{\eta} = \left( \begin{array}{cccccccc}
6_3 & 3_2 & 3_2 & 3_2 & 2_2 & 2_2 & 1_2 & 1_1 \\
5_3 & 3_2 & 2_2 & 2_2 & 1_2 & 1_2 & 1_1 & \\
\end{array} \right)_6.
\]

By the generating function $R_k(x_1, x_2, \ldots, x_k; q)$ and the generating function for $\eta_{2k}(n)$, Andrews showed that the number of $(k + 1)$-marked Durfee symbols of $n$ equals the symmetrized $(2k)$-th moment function at $n$ in [4], that is

**Theorem 3.3 (Corollary 13, Andrews [4])** For $k \geq 1$,
\[
D_{k+1}(n) = \eta_{2k}(n).
\]

**Proof.** Recall that
\[
\eta_{2k}(n) = \sum_{m=-\infty}^{+\infty} \frac{(m + k - 1)}{2k} N(m; n) = \sum_{m=1}^{+\infty} \left[ \left( \frac{m + k - 1}{2k} \right) + \left( \frac{m + k}{2k} \right) \right] N(m; n),
\]
where the second equality follows from the rank symmetry $N(-m; n) = N(m; n)$ and the fact $\left( \frac{m+k-1}{2k} \right) = \left( \frac{m+k}{2k} \right)$.

\[
\begin{align*}
D_{k+1}(n) &= \sum_{m_1, \ldots, m_{k+1}=-\infty}^{+\infty} \sum_{m_{k+1}=-\infty}^{+\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n) \\
&= \sum_{m_1, \ldots, m_{k+1}=-\infty}^{+\infty} \sum_{t_1, \ldots, t_k=0}^{+\infty} N \left( \sum_{i=1}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k; n \right),
\end{align*}
\]
where the second equality follows from Theorem 2.11. So it suffices to show that the number of solutions to $|m_1| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = m - k$ where $m_i$ is integer, and $t_i$ is nonnegative integer equals to $\left( \frac{m+k-1}{2k} \right) + \left( \frac{m+k}{2k} \right)$. 
Let \( c(n) \) denote the number of solutions to \(|m_1| + |m_2| + \cdots + |m_{k+1}| + 2t_1 + 2t_2 + \cdots + 2t_k = n \) where \( m_i \) is integer and \( t_i \) is nonnegative integer. It's easy to know that the generating function for \( c(n) \) is

\[
\sum_{n=0}^{+\infty} c(n) q^n = \frac{1}{(1 - q)^{k+1}} \times \frac{1}{(1 - q^2)^k} = (1 + q) \times \frac{1}{(1 - q)^{2k+1}} = (1 + q) \sum_{n=0}^{+\infty} \binom{2k + n}{n} q^n = \sum_{n=0}^{+\infty} \left( \begin{array}{c} 2k + n \\ 2k \end{array} \right) q^n + \sum_{n=1}^{+\infty} \left( \begin{array}{c} 2k + n - 1 \\ 2k \end{array} \right) q^n.
\]

Comparing coefficients of \( q^n \) in the above expression, we obtain

\[
c(n) = \left( \begin{array}{c} 2k + n \\ 2k \end{array} \right) + \left( \begin{array}{c} 2k + n - 1 \\ 2k \end{array} \right).
\]

Thus we reach our conclusion.

By partial fraction expansion, Andrews [4] also gave the following relationship between the generating function for \( k \)-marked Durfee symbols and the generating function for Durfee symbols, which plays an important role in the study of Ramanujan-type congruences for \( k \)-marked Durfee symbols.

**Theorem 3.4 (Theorem 7, Andrews [4])**

\[
R_k(x_1, x_2, \ldots, x_k; q) = \sum_{i=1}^{k} \frac{R_1(x_i; q)}{\prod_{j=1, j \neq i}^{k} (x_i - x_j)(1 - \frac{1}{x_i x_j})}.
\]  

**Proof.** Similarly, we will restate this identity as the partition identity (1.7) in Theorem 1.5. We first consider the series expansion of the right hand side of (3.19). To do this, we need to work in a larger ring: the field of iterated Laurent series \( \mathbb{K} \ll x_k, x_{k-1}, \ldots, x_1 \gg = \mathbb{K}((x_k))(x_{k-1}) \cdots ((x_1)) \) where \( \mathbb{K} = \mathbb{C}(q) \), in which all series are regarded first as Laurent series in \( x_1 \), then as Laurent series in \( x_2 \), and so on. For more detailed account of the properties of this field, with other applications, see [15] and [16].

Every element of \( \mathbb{K} \ll x_k, x_{k-1}, \ldots, x_1 \gg \) has a unique iterated Laurent series expansion. In particular, the series expansion of \( 1/(1 - \frac{1}{x_i x_j}) \) is:

\[
\frac{1}{1 - \frac{1}{x_i x_j}} = -\frac{x_i x_j}{1 - x_i x_j} = -\sum_{l=1}^{+\infty} x_i^l x_j^l,
\]

(3.20)
The series expansions of $1/(x_i - x_j)$ will be especially important. If $j < i$, then
\[
\frac{1}{x_i - x_j} = \frac{x_i^{-1}}{1 - x_j/x_i} = \sum_{l=0}^{\infty} x_i^{-l-1} x_j^l.
\]

However, if $j > i$ then this expansion is not valid and instead we have the expansion:
\[
\frac{1}{x_i - x_j} = \frac{-x_j^{-1}}{1 - x_i/x_j} = -\sum_{l=0}^{\infty} x_i^l x_j^{-l-1}.
\]

Thus for $j < i$, the series expansion of $\frac{1}{(x_i - x_j)(1 - x_i^{-1} x_j^{-1})}$ is
\[
\frac{1}{(x_i - x_j)(1 - x_i^{-1} x_j^{-1})} = -\sum_{l=1}^{\infty} x_i^l x_j^{-l-1} \sum_{m=0}^{\infty} x_i^{m-1} x_j^m = -\sum_{m_j=1}^{\infty} x_j^{m_j} \sum_{t_j=0}^{\infty} x_i^{2t_j - m_j + 1}, \quad (3.21)
\]
and for $j > i$, we have the following expansion:
\[
\frac{1}{(x_i - x_j)(1 - x_i^{-1} x_j^{-1})} = \sum_{l=1}^{\infty} x_i^l x_j^{-l-1} \sum_{m=0}^{\infty} x_i^{-m} x_j^{-m-1} = \sum_{m_j=0}^{\infty} x_j^{m_j} \sum_{t_j=0}^{\infty} x_i^{m_j + 2t_j + 1}. \quad (3.22)
\]

We now consider the series expansion of the $i$th term of the right hand side of (3.19).
\[
\frac{R_1(x_i; q)}{\prod_{j \neq i}^{k} (x_i - x_j)(1 - \frac{1}{x, x_j})} = \sum_{n=0}^{\infty} \sum_{m_i=-\infty}^{\infty} \mathcal{D}_1(m_i; n) x_i^{m_i} q^n
\]
\[
D_1(x_i; q) = \prod_{j \neq i}^{k} (x_i - x_j)(1 - \frac{1}{x_i, x_j}). \quad (3.23)
\]

Observe that the numerator in the above term is a series expansion in $x_i$ and $q$ and by the series expansions (3.21) and (3.22), we obtain a series expansion of (3.23), in which the exponents of $x_1, x_2, \ldots, x_{i-1}$ must be positive and the coefficients of $x_1^{m_1} \cdots x_k^{m_k} q^n$ for $m_1, \ldots, m_{i-1} \geq 1$ are
\[
(-1)^{i-1} \sum_{t_i} D_1 \left( \sum_{j=1}^{i-1} m_j + m_i - \sum_{j=i+1}^{k} |m_j| - 2|t_i| - (k - 1); n \right), \quad (3.24)
\]
where the sum ranges over all sequences $t_i = (t_1, \ldots, \hat{t}_i, \ldots, t_k)$ (omitting $t_i$) where $0 \leq t_j < |m_j|$ for $j < i$ and $t_j$ could be arbitrary nonnegative integer for $j > i$. Define $|t_i| = \sum_{j \neq i} t_j$.

Thus, we obtain a series expansion of the right hand of (3.19):
\[
\sum_{i=1}^{k} \frac{R_1(x_i; q)}{\prod_{j \neq i}^{k} (x_i - x_j)(1 - \frac{1}{x_i, x_j})} = \sum_{i=1}^{k} \sum_{m_1, \ldots, m_{i-1}=1}^{\infty} \sum_{m_i=-\infty}^{\infty} \sum_{n=0}^{\infty} x_1^{m_1} \cdots x_k^{m_k} q^n \quad (3.25)
\]
\[
\times \left[ (-1)^{i-1} \sum_{t_i} D_1 \left( \sum_{j=1}^{i-1} m_j + m_i - \sum_{j=i+1}^{k} |m_j| - 2|t_i| - (k - 1); n \right) \right].
\]
Lemma 3.5

Let \( \mathbf{m}_i = (m_1, \ldots, m_k) \) where \( m_j \geq 1 \) for \( j < i \), \( m_i \leq 0 \), and others could be arbitrary integers. Define \( \mathbf{x}^m = x_1^{m_1} \cdots x_k^{m_k} \). Obviously, the term \( \mathbf{x}^m q^n \) would be appeared in the series expansions of the first \( t \) terms of \((3.25)\). We next use the induction to prove the coefficients of \( \mathbf{x}^m q^n \) in the series expansion of \((3.25)\) equal to

\[
\sum_{j=0}^{+\infty} \binom{j+k-2}{k-2} N \left( \sum_{i=1}^{k} |m_i| + 2j + k - 1; n \right).
\]

(3.26)

Let \( \mathcal{T}_i(m_1, \ldots, m_{i-1}) \) denote the set of all sequences \( \mathbf{t}_i \) of nonnegative integers (omitting the \( i \)th vector \( t_i \)) such that \( t_j \) less than \( |m_j| \) for \( j < i \). The following two lemmas are useful in our argument.

**Lemma 3.5**

1. The number of sequences \( \mathbf{t}'_{p+1} \in \mathcal{T}_{p+1}(m_1, \ldots, m_p) \) equals the number of sequences \( \mathbf{t}_p \in \mathcal{T}_p(m_1, \ldots, m_p) \) where the \( (p+1) \)th vector \( t_{p+1} < |m_p| \) such that \( |t_p| = |t'_p| \).

2. The number of sequences \( \mathbf{t}_p \in \mathcal{T}_p(m_1, \ldots, m_{p-1}) \) where the \( (p+1) \)th vector \( t_{p+1} \geq |m_p| \) is equal to the number of sequences \( \mathbf{t}'_p \in \mathcal{T}_p(m_1, \ldots, m_{p-1}) \) such that \( |t_p| = |t'_p| + |m_p| \).

**Proof.** Given a sequence \( \mathbf{t}_p = (t_1, \ldots, t_p, \ldots, t_k) \) (omitting \( t_p \)) where \( 0 \leq t_j < |m_j| \) for \( j < i \) and others could be arbitrary nonnegative integers.

1. If \( t_{p+1} < |m_p| \), we define \( \mathbf{t}'_{p+1} = (t'_1, \ldots, t'_{p+1}, \ldots, t'_k) \) where \( t'_j = t_j \) for \( j \neq p, p+1 \) and \( t'_p = t_{p+1} \). Obviously, \( \mathbf{t}'_{p+1} \in \mathcal{T}_{p+1}(m_1, \ldots, m_p) \) and \( |t'_p| = |t'_{p+1}| \).

2. If \( t_{p+1} \geq |m_p| \), we define \( \mathbf{t}'_p = (t'_1, \ldots, t'_p, \ldots, t'_k) \) where \( t'_j = t_j \) for \( j \neq p, p+1 \) and \( t'_{p+1} = t_{p+1} - |m_p| \), it’s clear to see that \( \mathbf{t}'_p \in \mathcal{T}_p(m_1, \ldots, m_{p-1}) \) and \( |t_p| = |t'_p| + |m_p| \).

Furthermore, one can easily see that the above two processes are reservable.

We now consider the coefficients of \( \mathbf{x}^{m_1} q^n \) in the series expansion of \((3.25)\). It’s known that only the series expansion of the first term of \((3.25)\) contains the term \( \mathbf{x}^{m_1} q^n \), and the coefficients of \( \mathbf{x}^{m_1} q^n \) are

\[
\sum_{\mathbf{t}_1} D_1 \left( m_1 - \sum_{i=2}^{k} |m_i| - 2|t_1| - (k - 1); n \right)
= \sum_{j=0}^{+\infty} \binom{j+k-2}{k-2} N \left( \sum_{i=1}^{k} |m_i| + 2j + k - 1; n \right),
\]

where the equality follows from the fact that \( m_1 \leq 0 \), \( D_1(-m; n) = N(m; n) \) and the number of solutions to \( t_2 + t_3 + \cdots + t_k = j \) in nonnegative integers is \( \binom{j+k-2}{k-2} \).
Assume that the coefficients of $x^{m_p q^n}$ in the series expansion of (3.25) equal to (3.26), we now show that the coefficients of $x^{m_{p+1} q^n}$ are also equal to (3.26). Observe that $x^{m_p q^n}$ appears in the series expansions of the first $p$ terms of (3.25) and the term $x^{m_{p+1} q^n}$ appears in the series expansions of the first $(p+1)$ terms. Furthermore, the coefficients of $x^{m_p q^n}$ and $x^{m_{p+1} q^n}$ are the same in the series expansions of the first $(p-1)$ terms. Therefore, if we verify the sum of the coefficients of $x^{m_p q^n}$ in the series expansions of the $p$th term and $(p+1)$th term of (3.25) equal to the coefficients of $x^{m_p q^n}$ in the series expansions of the $p$th term, we could reach our conclusion by the induction hypothesis.

By (3.24), it’s known that the coefficients of $x^{m_p q^n}$ (where $m_p \leq 0$) in the series expansion of the $p$th term are

$$(-1)^{p-1} \sum_{t_p} D_1 \left( \sum_{j=1}^{p-1} m_j - \sum_{j=p}^{k} |m_j| - 2|t_p| - (k-1); n \right),$$

and the coefficients of $x^{m_{p+1} q^n}$ (where $m_p \geq 1$) in the series expansions of the $p$th term and $(p+1)$th term are

$$(-1)^{p-1} \sum_{t_p} D_1 \left( \sum_{j=1}^{p} m_j - \sum_{j=p+1}^{k} |m_j| - 2|t_p| - (k-1); n \right)$$

$$+ (-1)^{p} \sum_{t_{p+1}} D_1 \left( \sum_{j=1}^{p} m_j - \sum_{j=p+1}^{k} |m_j| - 2|t_{p+1}| - (k-1); n \right),$$

which equal to (3.27) by Lemma 3.5. Thus we get our conclusion, and by the definition of $R_k(x_1, x_2, \ldots, x_k; q)$, we could recast (3.19) as the partition identity (1.7).

4 $k$-marked odd Durfee symbols

This section is devoted to solving the problems raised by Andrews ([4, Problems 6-9]) on $k$-marked odd Durfee symbols. We begin this section by defining odd Durfee symbols which resemble ordinary Durfee symbols with a modified subscript and with odd numbers as entries.

**Definition 4.1** An odd Durfee symbol of $n$ is a two-row array with subscript

$$(\alpha, \beta)_D = \left( \begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \beta_1 & \beta_2 & \cdots & \beta_t \end{array} \right)_D$$

where $\alpha_i$ and $\beta_i$ are all odd numbers, $2D + 1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0$, $2D + 1 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_t > 0$, and $n = \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{t} \beta_i + 2D^2 + 2D + 1$.

The odd rank of an odd Durfee symbol is defined as the number of parts of $\alpha$ minus the number of parts of $\beta$, let $\mathcal{D}_0^0(m; n)$ denote the number of odd Durfee symbols of $n$ with odd rank $m$, we then have
Theorem 4.2 The generating function for $D^0_1(m; n)$ is given by
\[
\sum_{n=0}^{+\infty} D^0_1(m; n)q^n = 1/(q^2; q^2)_{\infty}\sum_{n=0}^{+\infty} (-1)^n q^{3n^2 + 3n + 1 + |m|(2n+1)}.  \tag{4.29}
\]
This result can easily follow by comparing the coefficients of $z^m$ in (4.30) given by Andrews [4, (8.4)-(8.5)]:
\[
\sum_{n=1}^{+\infty} \sum_{m=-\infty}^{+\infty} D^0_1(m; n)z^m q^n = \sum_{n\geq 0} \frac{q^{2n(n+1)+1}}{(zq; q^2)_{n+1}(z^{-1}q; q^2)_{n+1}}
= \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n q^{3n^2 + 3n + 1}}{1 - zq^{2n+1}},  \tag{4.30}
\]
where the first equality follows by direct combinatorial argument. The second equality is given by [17, p.66].

Andrews also [4] defined the $k$th symmetrized odd rank moment by
\[
\eta^0_k(n) = \sum_{m=-\infty}^{+\infty} \left( m + \left\lfloor \frac{k+1}{2} \right\rfloor \right) D^0_1(m; n),  \tag{4.31}
\]
and introduced $k$-marked odd Durfee symbols, whose definition is almost identical to that of $k$-marked Durfee symbols (Definition 1.4).

Definition 4.3 A $k$-marked odd Durfee symbol of $n$ is composed of $k$ pairs of partitions into odd parts with the subscript, which is defined as
\[
\eta^0 = \left( \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \right)_D,
\]
where $\alpha^i$ (resp. $\beta^i$) are all partitions with odd parts and $\sum_{i=1}^{k} (|\alpha^i| + |\beta^i|) + 2D^2 + 2D + 1 = n$. Furthermore, the partitions $\alpha^i$ and $\beta^i$ must satisfy almost the same conditions with $k$-marked Durfee symbols expect for the third term in Definition 1.4 where for the $k$th vector $(\alpha^k, \beta^k)$ of $k$-marked odd Durfee symbol, $\beta^1_k, \alpha^1_k \leq 2D + 1$.

Following $k$-marked Durfee symbol, Andrews defined the $i$th odd rank for $k$-marked odd Durfee symbol. For a $k$-marked odd Durfee symbol $\eta^0$, we define $\rho_i(\eta^0)$, the $i$th odd rank of $\eta^0$ by
\[
\rho_i(\eta^0) = \begin{cases} l(\alpha^i) - l(\beta^i) - 1 & \text{for } 1 \leq i < k, \\
l(\alpha^k) - l(\beta^k) & \text{for } i = k. \end{cases}
\]
Let $D^0_k(m_1, m_2, \ldots, m_k; n)$ denote the number of $k$-marked odd Durfee symbols of $n$ with $i$th odd rank equal to $m_i$ and $D^0_k(n)$ denote the number of $k$-marked odd Durfee symbols of $n$. Define $R^0_k(x_1, x_2, \ldots, x_k; q)$ by

$$R^0_k(x_1, x_2, \ldots, x_k; q) = \sum_{m_1, \ldots, m_k = -\infty}^{+\infty} \sum_{n=0}^{+\infty} D^0_k(m_1, m_2, \ldots, m_k; n)x_1^{m_1}x_2^{m_2} \cdots x_k^{m_k} q^n.$$  

Andrews deduced the following four identities on $k$-marked odd Durfee symbols which are much similar with $k$-marked Durfee symbols.

**Theorem 4.4** (Corollary 27, Andrews [4])

$R^0_k(x_1, x_2, \ldots, x_k; q) = 1/q^2 (1-q^{2n+2}) \sum_{j=0}^{\infty} \frac{(1-q^{3n+2j+4})}{\prod_{j=1}^{k}(1-x_j q^{n+j})}.$  

**Theorem 4.5** (Corollary 28, Andrews [4]) $D^0_k(m_1, m_2, \ldots, m_k; n)$ is symmetric in $m_1, m_2, \ldots, m_k$.

**Theorem 4.6** (Corollary 29, Andrews [4]) For $k \geq 1$,

$$D^0_{k+1}(n) = \eta^0_{2k}(n).$$  

**Theorem 4.7** (Theorem 25, Andrews [4])

$$R^0_k(x_1, x_2, \ldots, x_k; q) = \sum_{i=1}^{k} \frac{R^0_1(x_i; q)}{\prod_{j=1}^{k}(x_i - x_j)(1 - \frac{q^n}{x_i x_j})}.$$  

We now give a brief expository of how to prove these four conclusions combinatorially. First of all, it’s straightforward to see that the bijection $\Phi$ in Theorem 2.1, $\Psi$ in Theorem 2.6 and the bijection $\Theta$ in Theorem 2.10 on Section 2 are valid for $k$-marked odd Durfee symbols, one then easily deduces the same result for $k$-marked odd Durfee symbols as Theorem 1.5.

**Theorem 4.8** For $k \geq 2$, we have

$$D^0_k(m_1, m_2, \ldots, m_k; n) = \sum_{j=0}^{+\infty} \binom{j+k-2}{k-2} D^0_1 \left( \sum_{i=1}^{k} |m_i| + 2j + k - 1; n \right).$$  

Thus, Theorems 4.5, 4.6 and 4.7 can be deduced from Theorem 4.8 by the precisely same progressions as Theorems 3.2, 3.3 and 3.4 on Section 3. To prove Theorem 4.4, it suffices to prove that the right side hand of (4.32) is the generating function for the
summation on the right side of (4.35), which can be easily derived by Theorem 4.2 following the same progression as Theorem 3.1.

**Acknowledgments.** I would like to thank Guoce Xin and Yue Zhou for helpful discussions, and I am grateful to George E. Andrews for valuable comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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