TOPOLOGY OF CHARACTER VARIETIES OF ABELIAN GROUPS

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Abstract. Let $G$ be a complex reductive algebraic group (not necessarily connected), let $K$ be a maximal compact subgroup, and let $\Gamma$ be a finitely generated Abelian group. We prove that the conjugation orbit space $\text{Hom}(\Gamma, K)/K$ is a strong deformation retract of the GIT quotient space $\text{Hom}(\Gamma, G)/G$. As a corollary, we determine necessary and sufficient conditions for the character variety $\text{Hom}(\Gamma, G)/G$ to be irreducible when $G$ is connected and semisimple. For a general connected reductive $G$, analogous conditions are found to be sufficient for irreducibility, when $\Gamma$ is free abelian.

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1. Introduction

The description of the space of commuting elements in a compact Lie group is an interesting algebro-geometric problem with applications in mathematical physics, remarkably in supersymmetric Yang-Mills theory and mirror symmetry (7, 28, 47, 48). Some special cases of this problem and related questions have recently received attention as can be seen, for example, in the articles [1, 4, 5, 16, 22, 36, 42].

Let $K$ be a compact Lie group and view the $\mathbb{Z}$-module $\mathbb{Z}^r$ as a free Abelian group of rank $r > 0$. The space of commuting $r$-tuples of elements in $K$ can be naturally identified with the set $\text{Hom}(\mathbb{Z}^r, K)$ of group homomorphisms from $\mathbb{Z}^r$ to $K$, by evaluating a homomorphism on a set of free generators for $\mathbb{Z}^r$. From both representation-theoretic and geometric viewpoints, one is interested in homomorphisms up to conjugacy, so the quotient space $X_{\Gamma}(K) := \text{Hom}(\mathbb{Z}^r, K)/K$, where $K$ acts by conjugation, is the main object to consider. Since every compact Lie group is isomorphic to a matrix group, it is not difficult to see that this orbit space is a semialgebraic space, but many of its general properties remain unknown.

In this article, we also consider the analogous space for a complex reductive affine algebraic group $G$. More generally, in many of our results we replace the free Abelian group $\mathbb{Z}^r$ by an arbitrary finitely generated Abelian group $\Gamma$.

In this context, it is useful to work with the geometric invariant theory (GIT) quotient space, denoted by $\text{Hom}(\Gamma, G)//G$, and usually called the $G$-character variety of $\Gamma$ (see Section 2 for details). The character varieties $X_{\Gamma}(G) := \text{Hom}(\Gamma, G)//G$ are naturally affine algebraic sets, not necessarily irreducible.

Here is our first main result:

**Theorem 1.1.** Let $G$ a complex reductive algebraic group, $K$ a maximal compact subgroup of $G$, and $\Gamma$ a finitely generated Abelian group. Then there exists a strong deformation retraction from $X_{\Gamma}(G)$ to $X_{\Gamma}(K)$.

We remark that in [10], the analogous result was shown for a free (non-Abelian) group $F_r$ of rank $r$; that is, the free group character variety $\text{Hom}(F_r, G)//G$ deformation retracts to $\text{Hom}(F_r, K)/K$. In the same article, the special case $K = U(n)$, $G = \text{GL}(n, \mathbb{C})$ and $\Gamma = \mathbb{Z}^r$ of Theorem 1.1 was also shown.

More recently, Pettet and Souto in [36] have shown that, under the hypothesis of Theorem 1.1, $\text{Hom}(Z^r, G)$ deformation retracts to $\text{Hom}(Z^r, K)$. This motivated our related but independent proof of Theorem 1.1, which is very different from the argument in [10].

In fact, there are great differences between the free Abelian and free (non-Abelian) group cases. For instance, the deformation retraction from $\text{Hom}(F_r, G) \cong G^r$ to $\text{Hom}(F_r, K) \cong K^r$ is trivial, although the deformation from $X_{F_r}(G) = \text{Hom}(F_r, G)//G$ to $X_{F_r}(K) = \text{Hom}(F_r, K)/K$ is not. Moreover, in the case of free Abelian groups the deformation is not determined in general by the factor-wise deformation (as happens in the free group case), since by a recent result, any such deformation cannot preserve commutativity (see [15]). We conjecture that
the analogous result is valid for right angled Artin groups, a class of groups that interpolates between free and free Abelian ones (see Section 4.3).

Returning to the situation of a general finitely generated group $\Gamma$, the interest in the spaces $\text{Hom}(\Gamma, K)/K$ and $\text{Hom}(\Gamma, G)/G$ is also related to their differential-geometric interpretation. Consider a differentiable manifold $B$ whose fundamental group $\pi_1(B)$ is isomorphic to $\Gamma$ (when $\Gamma = \mathbb{Z}^r$, we can choose $B$ to be an $r$-dimensional torus). By fixing a base point in $B$ and using the standard holonomy construction in the differential geometry of principal bundles, one can interpret $\text{Hom}(\Gamma, K)$ as the space of pointed flat connections on principal $K$-bundles over $B$, and $\mathcal{X}_\Gamma(K) = \text{Hom}(\Gamma, K)/K$ as the moduli space of flat connections on principal $K$-bundles over $B$ (see [47]).

The use of differential and algebro-geometric methods to study the geometry and topology of these spaces was achieved with great success when $B$ is a closed surface of genus $g > 1$ (in this case $\pi_1(B)$ is non-Abelian), via the celebrated Narasimhan-Seshadri theorem and its generalizations, which deal also with non-compact Lie groups (see, for example [3, 26, 35, 43, 44]). Indeed, the character varieties $\mathcal{X}_\Gamma(G)$ introduced above can be interpreted as a moduli space of polystable $G$-Higgs bundles over a compact Kähler manifold with $\Gamma$ the fundamental group of the manifold, or central extension thereof (which yields an identification in the topological category, but not in the algebraic or complex analytic ones). Some of the multiple conclusions from this approach was the determination of the number of components for many spaces of the form $\text{Hom}(\pi_1(B), H)/H$ for a closed surface $B$ and real reductive (not necessarily compact or complex analytic) Lie group $H$ (see for example [8, 18]). We remark that these character varieties are also main players in mirror symmetry and the geometric Langlands programme (see, for example [19, 25, 29]).

In contrast, for the case of $\pi_1(B) = \mathbb{Z}^r$, the number of path components of $\text{Hom}(\mathbb{Z}^r, K)/K$ has only recently found a satisfactory answer (for general compact semisimple $K$ and arbitrary $r$), see [28].

Using this determination, and also a very recent result by A. Sikora [42], one of the main applications of Theorem 1.1 is the following theorem. By the classification of finitely generated Abelian groups, any such group $\Gamma$ can be written as $\Gamma = \mathbb{Z}^r \oplus \Gamma'$ where $r$ is called the rank of $\Gamma$ and $\Gamma'$ is the finite group of torsion elements (all of those having finite order). We say that $\Gamma$ is free if $\Gamma'$ is trivial.

**Theorem 1.2.** Let $G$ be a semisimple connected algebraic group over $\mathbb{C}$, and let $r$ be the rank of the Abelian group $\Gamma$. Then $\mathcal{X}_\Gamma(G)$ is an irreducible variety if and only if:

1. $\Gamma$ is free, and
2. Either $r = 1$, or $r = 2$ and $G$ is simply connected, or $r \geq 3$ and $G$ is a product of $\text{SL}(n, \mathbb{C})$’s and $\text{Sp}(n, \mathbb{C})$’s.

An interesting consequence of this result is a sufficient condition for irreducibility of $\mathcal{X}_{\mathbb{Z}^r}(G)$ for a general connected reductive $G$ (see Corollary 5.14), generalizing the result in [12, Prop. 2.6].
Theorem 1.2 should be considered in relation to an analogous problem in a different context. Let $C_{r,n}$ be the algebraic set of commuting $r$-tuples of $n \times n$ complex matrices. Determining the irreducibility of $C_{r,n}$ is a surprisingly difficult linear algebra problem, related to the determination of canonical forms for similarity classes of general $r$-tuples of matrices (see [20]). In fact, as a consequence of deep theorems by Gerstenhaber and Guralnick (see [21, 23]) $C_{r,n}$ was shown to be reducible when $r,n \geq 4$, and when $r = 3$ and $n \geq 32$; moreover, $C_{r,n}$ is irreducible when $r = 1, 2$ (for all $n$) and when $r = 3$ and $n \leq 8$. The remaining cases: $r = 3$ and $n$ strictly between 8 and 32, are still open (as far as we know). These results yield corresponding statements for $\text{Hom}(\mathbb{Z}^r, G)$, when $G = \text{GL}(n, \mathbb{C})$ or $G = \text{SL}(n, \mathbb{C})$.

We finish the Introduction with a summary of the article. The proof of Theorem 1.1 is divided into three main steps. The first step, carried out in Section 2, consists in obtaining the identifications:

\[
\text{Hom}(\Gamma, K)/K = G(\text{Hom}(\Gamma, G))/G,
\]
\[
\text{Hom}(\Gamma, G)//G = \text{Hom}(\Gamma, G)^{ps}/G.
\]

Here, for a subset $X \subset \text{Hom}(\Gamma, G)$, we let $G(X) := \{gxg^{-1} : g \in G, x \in X\}$, and the superscript $ps$ refers to the subset of representations with closed orbits (called polystable). These identifications hold, in fact, for any finitely generated group $\Gamma$. For the second step, in Section 3, we restrict to a fixed Abelian $\Gamma$. It consists in showing that one can replace the polystable representations by “representations” into $G_{ss}$, the semisimple part of $G$, and that we have $G(\text{Hom}(\Gamma, G)) = \text{Hom}(\Gamma, G(K))$, where $G(K) := \{gkg^{-1} : g \in G, k \in K\}$.

Finally in Section 4, the proof is completed by constructing a strong deformation retraction from $G_{ss}$ to $G(K)$ with a certain number of desired properties. This last part of the proof is inspired by the methods and results of [36], although is self-contained. Note also that (except for the partial results on right angled Artin groups) we do not need to use their generalized Jordan decomposition, since in our GIT quotient framework, we can work directly with polystable representations.

In Section 5, besides proving Theorem 1.2, we also study two different characterizations of a special component $\mathfrak{X}_{Z^r}(G)$, usually called the identity component. This, together with known results on the compact group case, provides a final application of Theorem 1.1 to the simple connectivity and to the cohomology ring of the character varieties $\mathfrak{X}_{Z^r}(G)$, in a few examples, such as when $G$ is $\text{SL}(n, \mathbb{C})$ or $\text{Sp}(n, \mathbb{C})$.

2. Quotients and Character Varieties

An affine algebraic group is called reductive if it does not contain any non-trivial closed connected unipotent normal subgroup (see [6, 27] for generalities on algebraic groups). Since we do not consider Abelian varieties in this paper, we will abbreviate the term affine algebraic group to simply algebraic group.
Let $G$ be a complex reductive algebraic group (we include the possibility that $G$ is disconnected), and let $X$ be a complex affine $G$-variety, that is, we have an action $G \times X \to X$, $(g, x) \mapsto g \cdot x$ which is a morphism between affine varieties. We use the term \textit{affine variety} to mean an affine algebraic set, not necessarily irreducible. All of our algebraic groups and varieties will be considered over $\mathbb{C}$. In particular, the group $G$ is also a Lie group.

This action induces a natural action of $G$ on the ring $\mathbb{C}[X]$ of regular functions on $X$. The subring of $G$-invariant functions $\mathbb{C}[X]^G$ is finitely generated since $G$ is reductive, and defines the affine GIT (geometric invariant theory) quotient as the corresponding affine variety, usually denoted as:

\begin{equation}
X/G := \text{Spec}_{\text{max}}(\mathbb{C}[X]^G).
\end{equation}

See [12] or [34] for the details of these constructions and an introduction to GIT. If $x \in X$, we denote by $[x]$ or by $G \cdot x$ its $G$-orbit in $X$, and by $\overline{[x]}$ its \textit{extended orbit} in $X$ which is, by definition, the union of the orbits whose closure intersects $\overline{[x]} = G \cdot x$. As shown in [12], $X/G$ is exactly the space of classes $[x]$. In the context of GIT, one usually considers the Zariski topology. However, being interested also in the usual topology on Lie groups, we will always equip our variety $X$ with a natural embedding into an affine space $\mathbb{C}^N$. This induces on $X$ a natural Euclidean topology. When we need to distinguish the two kinds of topological closures, we use a label in the overline; we will mean the Euclidean topology, when no explicitly reference is made.

We observe that every orbit $[x] = G \cdot x$ is the image of the action map, which is algebraic, so $[x]$ is constructible and thus contains a Zariski open subset $O \subset [x]$, which is dense in its Zariski closure $\overline{[x]}_Z$. Hence, $O$ is also open and dense, in the Euclidean topology, inside $\overline{[x]}_Z$. Therefore, the Euclidean and Zariski closures of orbits coincide: $\overline{[x]}_E = \overline{[x]}_Z$.

2.1. \textbf{The polystable quotient.} If the orbit of $x$ is closed, $[x] = \overline{[x]}$, we say that $x$ is \textit{polystable}. Denote the subset of polystable points by $X^{ps} \subset X$. Since for any $g \in G$, $[g \cdot x] = [x]$, it is clear that $X^{ps}$ is a $G$-space. It is not, in general, an affine variety.

We first show that, considering Euclidean topologies, there is a natural homeomorphism between the \textit{polystable quotient} $X^{ps}/G$ and the GIT quotient $X/G$.

Consider the canonical projection $\pi_{ps} : X^{ps} \to X^{ps}/G$, and the GIT projection $\pi_G : X \to X/G$. Define also $\mathcal{I}_{ps} : X^{ps}/G \hookrightarrow X/G$ to be the canonical map which sends a $G$-orbit $[x]$ to its extended $G$-orbit $\overline{[x]}$. These maps form the following diagram, whose commutativity is clear:

\begin{equation}
\begin{array}{ccc}
X^{ps} & \hookrightarrow & X \\
\pi_{ps} \downarrow & & \downarrow \pi_G \\
X^{ps}/G & \overset{\mathcal{I}_{ps}}{\longrightarrow} & X/G.
\end{array}
\end{equation}
In [32, 31], \( \pi_G \) is shown to be a closed mapping on \( G \)-invariant Euclidean closed sets (see also [39], page 141).

**Theorem 2.1.** Let \( G \) be a complex reductive algebraic group, and \( X \) a complex affine \( G \)-variety (with the natural Euclidean topology coming from \( X \subset \mathbb{C}^N \)). Then \( \mathcal{I}_{ps} \) is a homeomorphism.

**Proof.** With respect to the Euclidean topologies, and the induced topologies on the quotients, the map \( \mathcal{I}_{ps} \) is continuous. This follows from the identification of \( \mathcal{I}_{ps} \) with the composition of continuous maps

\[
X^{ps}/G \xrightarrow{\iota_{ps}} X/G \xrightarrow{\pi_{ps}} X/G.
\]

We will show that \( \mathcal{I}_{ps} \) is a bijection and its inverse is continuous. It is a standard fact (see [12, 33]) that every extended orbit equivalence class contains a unique closed orbit, and this closed orbit lies in the closure of all of the others in that class. In particular, for every \([x] \in X/G\) there exists a representative \( x^{ps} \in X \) whose orbit is closed and \( x^{ps} \in [x] \). So, \( x^{ps} \in X^{ps} \) and \([x^{ps}] \in X^{ps}/G\). Thus, \( \mathcal{I}_{ps}([x^{ps}]) = [x] \) which shows \( \mathcal{I}_{ps} \) is surjective.

Since two orbits \([x_1]\) and \([x_2]\) are contained in \([x]\) if and only if \([x_1] \cap [x_2] \neq \emptyset\), no two distinct closed orbits are identified in the GIT quotient \( \mathfrak{X}_r(G) \). Thus \( \mathcal{I}_{ps} \) is injective.

Finally, let us show that \( \mathcal{I}_{ps} \) is a closed map. Suppose that \( C \subset X^{ps}/G \) is a closed set. Then, by definition \( C' := \pi_{ps}^{-1}(C) \subset X^{ps} \) is \( G \)-invariant. By commutativity of the diagram (2.2) we have

\[
\mathcal{I}_{ps}(C) = \pi_G(\pi_{ps}^{-1}(C)) = \pi_G(C').
\]

Let \( C'' \) be the Euclidean closure of \( C' \) in \( X \). By definition of subspace topologies, since \( C' \) is closed in \( X^{ps} \), we have \( C' = C'' \cap X^{ps} \).

Now let us show that \( C'' \), the set of limit points of sequences in \( C' \), is \( G \)-invariant. Let \( x_0 \in C'' \). If \( x_0 \in C' \), then clearly \( g \cdot x_0 \) is in \( C' \). Otherwise, there is a sequence \( \{x_n\} \subset C' \) converging to \( x_0 \). But \( G \) acts by polynomials and hence is continuous in the Euclidean topology. Therefore, \( g \cdot x_n \) limits to \( g \cdot x_0 \), and \( g \cdot x_n \) in \( C'' \) for all \( n \) and \( g \) since \( C' \) is \( G \)-invariant. Therefore, \( g \cdot x_0 \) is in the limit set and hence in \( C'' \).

Obviously, \( \pi_G(C'') \subset \pi_G(C'') \). Let us show the reverse inclusion. If \([x] \in \pi_G(C'' \in X/G) \) is an extended orbit for some \( x \in C'' \), then, as above, there is a \( x^{ps} \in X^{ps} \) whose orbit is closed and \( x^{ps} \in [x] \). In particular, \([x^{ps}] = [x] \). This implies \([x] \cap [x^{ps}] = [x] \cap G \cdot x^{ps} \neq \emptyset\), so there exists \( g \in G \) such that \( g \cdot x^{ps} \in [x] \). Thus, since \( C'' \) is \( G \)-invariant and (Euclidean) closed, \( g^{-1} \cdot (g \cdot x^{ps}) = x^{ps} \in C'' \).

Note that here we are using the fact, observed above, that the Euclidean and Zariski closures of orbits coincide. Therefore \( x^{ps} \in C' = C'' \cap X^{ps} \), and so \([x] = [x^{ps}] \in \pi_G(C') \). We conclude that indeed \( \pi_G(C') = \pi_G(C'') \).
Since $C''$ is closed, $G$-invariant in $X$ and $\pi_G$ is a closed map for $G$-invariant sets as mentioned above, we conclude that $\mathcal{I}_{ps}(C) = \pi_G(C'') = \pi_G(C''')$ is a closed set. This shows that $\mathcal{I}_{ps}$ is a closed map. Being bijective, it is also an open map. So, the inverse of $\mathcal{I}_{ps}$ is continuous, and hence $\mathcal{I}_{ps}$ is a homeomorphism. □

**Remark 2.2.** Since $X/G$ is a complete metric space, the same holds for $X^{ps}/G$ as well. The metric can be explicitly given as follows. Let $\{f_1, \ldots, f_N\}$ generate the ring of invariants $\mathbb{C}[X]^G$, define $F = (f_1, \ldots, f_N)$ to be the mapping $X \to \mathbb{C}^N$, and let $\| \cdot \|$ be the Euclidean metric on $X$. For $[v], [w] \in X/G$ define $d([v], [w]) = \|F(v) - F(w)\|$. Thus $d$ is well-defined since $F$ is $G$-invariant; and $d$ is non-negative, symmetric and satisfies the triangle inequality because $\| \cdot \|$ does. It is not definite however. Since $d([v], [w]) = 0$ if the Zariski closure of $[v]$ and $[w]$ intersect (even if they are not equal orbits), this problem is exactly fixed upon restricting to $X^{ps}/G$.

### 2.2. Polystable and compact quotients for character varieties

Let $\Gamma$ be a finitely generated group, $H$ be a Lie group, and consider the representation space $\text{Hom}(\Gamma, H)$, the space of homomorphisms from $\Gamma$ to $H$. For example, when $\Gamma = F_r$ is a free group on $r$ generators, the evaluation of a representation on a set of free generators provides a homeomorphism with the Cartesian product

$$\text{Hom}(F_r, H) \cong H^r,$$

where we consider the compact-open topology on $\text{Hom}(F_r, H)$ with $F_r$ given the discrete topology.

In general, by choosing generators $\gamma_1, \ldots, \gamma_r$ for $\Gamma$ (for some $r$), we have a natural epimorphism $F_r \to \Gamma$. This allows one to embed $\text{Hom}(\Gamma, H) \subset \text{Hom}(F_r, H) \cong H^r$ and consider on $\text{Hom}(\Gamma, H)$ the Euclidean topology induced from the manifold $H^r$.

The Lie group $H$ acts on $\text{Hom}(\Gamma, H)$ by conjugation of representations; that is, $h \cdot \rho = h\rho h^{-1}$ for $h \in H$ and $\rho \in \text{Hom}(\Gamma, H)$. Let $\text{Hom}(\Gamma, H)/H$ be the quotient space by this action, and let $\mathcal{X}_\Gamma(H)$ denote the identification space $(\text{Hom}(\Gamma, H)/H)/\sim$, where two orbits are equivalent if and only if they are members of a chain of orbits whose closures pair-wise intersect.

Two main classes of examples are important for us, for both of which the induced topology on $\mathcal{X}_\Gamma(H)$ will be Hausdorff. When $H = K$ is a compact Lie group, this is the usual orbit space (which is semi-algebraic and compact) since all such orbits are closed, so $\mathcal{X}_\Gamma(H) = \text{Hom}(\Gamma, K)/K$. When $H = G$ is a reductive algebraic group over $\mathbb{C}$ each equivalence class is indeed an extended $G$-orbit in the sense defined just before subsection 2.1 (see [31]). So, in this case the quotient $\mathcal{X}_\Gamma(H)$ can be identified with the GIT quotient considered in (2.1):

$$\mathcal{X}_\Gamma(G) = \text{Hom}(\Gamma, G)/G := \text{Spec}_{\max}(\mathbb{C}[\text{Hom}(\Gamma, G)]^G),$$

and is called the $G$-character variety of $\Gamma$. In either case, the spaces $\mathcal{X}_\Gamma(H)$ will be semi-algebraic sets and thus CW-complexes in the natural Euclidean topologies we consider in this paper.
Recall that $\text{Hom}(\Gamma, G)^{ps}$ denotes the subset of $\text{Hom}(\Gamma, G)$ consisting of representations whose $G$-orbit is closed. Applying the map $I_{ps}$ from Proposition 2.1 to character varieties, we obtain the following proposition.

**Proposition 2.3.** Let $G$ be a complex reductive algebraic group, and $\Gamma$ be a finitely generated group. Then, the natural map $I_{ps} : \text{Hom}(\Gamma, G)^{ps}/G \to X_{\Gamma}(G)$ is a homeomorphism.

Let now $K$ be a fixed maximal compact subgroup of a complex reductive algebraic group $G$. Then $G$ is the Zariski closure of $K$. Over $\mathbb{C}$, all reductive algebraic groups arise as the Zariski closure of compact Lie groups. In particular, as discussed in [39], we may assume $K \subset O(n, \mathbb{R})$ is a real affine variety by the Peter-Weyl theorem, and thus $G \subset O(n, \mathbb{C})$ is the complex points of the real variety $K$.

Using the fact that $G$ is also isomorphic to the unique complexification $K_{\mathbb{C}}$ of $K$, one can show the following.

**Proposition 2.4.** Let $\Gamma$ be a finitely generated group and $G = K_{\mathbb{C}}$. Then the inclusion mapping $\text{Hom}(\Gamma, K) \hookrightarrow \text{Hom}(\Gamma, G)$ induces an injective mapping $\iota : \text{Hom}(\Gamma, K)/K \hookrightarrow \text{Hom}(\Gamma, G)/G$ such that $\iota(\text{Hom}(\Gamma, K)/K)$ is a CW-subcomplex of $\text{Hom}(\Gamma, G)/G$.

**Proof.** Since $G$ is the complex points of the real variety $K$, the real points of $G$ coincide with $K$. In the same way, the set of real points of the affine variety $\text{Hom}(\Gamma, G) \subset G^r$ is precisely $\text{Hom}(\Gamma, K)$. Since $\text{Hom}(\Gamma, K)$ is compact and stable under $K$, the result is a direct consequence of [17, Thm. 4.3].

Given a representation $\rho \in \text{Hom}(\Gamma, G)$, the subset $\rho(\Gamma) = \{\rho(\gamma), \gamma \in \Gamma\} \subset G$ is the group algebraically generated by $\rho(\gamma_1), \ldots, \rho(\gamma_r)$, where $\gamma_1, \ldots, \gamma_r$ are the generators of $\Gamma$.

**Lemma 2.5.** Let $\Gamma$ be a finitely generated group and $G = K_{\mathbb{C}}$. Then $\text{Hom}(\Gamma, K) \subset \text{Hom}(\Gamma, G)^{ps}$.

**Proof.** Suppose $\rho \in \text{Hom}(\Gamma, K)$. Then the Euclidean closure of $\rho(\Gamma)$ in $K$ is a compact subgroup $J$ of $K$. This implies that the Zariski closure of $\rho(\Gamma)$ coincides with the Zariski closure of $J$ in $G$ and hence it is a reductive algebraic group (precisely equal to the complexification of $J$). Thus, $\overline{\rho(\Gamma)}^Z$ is a linearly reductive group (all its linear representations are completely reducible). In [38, Thm. 3.6], it is shown that $\overline{\rho(\Gamma)}^Z$ is linearly reductive if and only if the $G$-orbit of $\rho$ in $\text{Hom}(\Gamma, G)$ is closed. We conclude $G \cdot \rho$ is closed, that is, $\rho \in \text{Hom}(\Gamma, G)^{ps}$. We note that Richardson’s result in [38] is stated for $r$-tuples of elements in $G$. The case of closed $G$-invariant subsets of $G^r$, such as our $\text{Hom}(\Gamma, G)$, is an easy consequence (see also [15]).

Define the mapping $\iota_K$ as the composition

$\text{Hom}(\Gamma, K)/K \hookrightarrow \text{Hom}(\Gamma, G)^{ps}/K \rightarrow \text{Hom}(\Gamma, G)^{ps}/G$. 


We want now to describe the image of $\iota_K$.

**Proposition 2.6.** The following diagram is commutative:

```
\[
\begin{array}{ccc}
\Hom(\Gamma, G)/G & \xrightarrow{\pi} & \Hom(\Gamma, G)/\!/G \\
I_{ps} \downarrow & & \downarrow \iota_K \\
\Hom(\Gamma, G)^{ps}/G & \xrightarrow{I_{ps}} & \Hom(\Gamma, G)^{\!/ps}/G \\
\end{array}
\]
```

Consequently, $\iota_K(\mathfrak{X}_\Gamma(K)) \cong \iota(\mathfrak{X}_\Gamma(K))$ as CW complexes.

**Proof.** Since all mappings in the diagram are composites of natural inclusions and projections, they are continuous. The top triangle of maps is commutative by definition of $I_{ps}$. Note that $\iota$ is the cellular inclusion from Proposition 2.4. Then Proposition 2.4 implies that all $G$-equivalent $K$-valued representations are in fact $K$-equivalent (else $\iota$ would not be injective). Therefore, since $\Hom(\Gamma, K) \subset \Hom(\Gamma, G)^{ps}$, we conclude that $\iota_K$ is also injective. Therefore, the bottom triangle of maps is commutative. \hfill $\square$

Now define

$$G(\Hom(\Gamma, K)) := \{gp_\rho^{-1} \mid g \in G \text{ and } \rho \in \Hom(\Gamma, K)\}.$$  

Clearly, $G(\Hom(\Gamma, K)) \subset \Hom(\Gamma, G)^{ps}$ as a $G$-subspace since all $K$-representations have closed orbits by Lemma 2.5 and conjugates of representations with closed $G$-orbits likewise have closed $G$-orbits (since $G \cdot (gp_\rho^{-1}) = G \cdot \rho$).

**Proposition 2.7.** $\iota_K : \Hom(\Gamma, K)/K \rightarrow \Hom(\Gamma, G)^{ps}/G$ is an embedding and $\iota_K(\Hom(\Gamma, K)/K) = G(\Hom(\Gamma, K))/G$.

**Proof.** Proposition 2.6 implies that $\iota_K$ is a continuous injection. We now show it is onto $G(\Hom(\Gamma, K))/G$. First, let $[\rho]_K \in \mathfrak{X}_\Gamma(K)$. For any $\rho' \in \iota_K([\rho]_K)$, $\rho'$ is a $G$-conjugate of a $K$-conjugate of $\rho$ and so $\rho' = g{kpk^{-1}}g^{-1} \in G(\Hom(\Gamma, K))$; thus $\iota_K([\rho]_K) \in G(\Hom(\Gamma, K))/G$. Conversely, let $[\rho]_G \in G(\Hom(\Gamma, K))/G$. Then by definition, there exists $g \in G$ such that $g\rho g^{-1} \in \Hom(\Gamma, K)$. Thus $[g\rho g^{-1}]_K \in \Hom(\Gamma, K)/K$ and $\iota_K([g\rho g^{-1}]_K) = [g\rho g^{-1}]_G = [\rho]_G$; we conclude $G(\Hom(\Gamma, K))/G \subset \iota_K(\mathfrak{X}_\Gamma(K))$. Therefore, $\iota_K(\Hom(\Gamma, K)/K) = G(\Hom(\Gamma, K))/G$.

From Proposition 2.3, $\Hom(\Gamma, G)^{ps}/G \cong \Hom(\Gamma, G)/G$ and so is Hausdorff (the latter is an algebraic subset of some $\mathbb{C}^N$, with the Euclidean subspace topology). On the other hand, $\Hom(\Gamma, K)/K$ is compact (a closed subset of the compact $K^N$ is a compact set, and a compact quotient of a compact space is compact). Since a continuous injection from a compact space to a Hausdorff space is an embedding, we are done. \hfill $\square$

We record the following fact for later use.
Proposition 2.8. $\mathfrak{X}_\Gamma(G \times H) \cong \mathfrak{X}_\Gamma(G) \times \mathfrak{X}_\Gamma(H)$ for any reductive algebraic $C$-groups $G, H$.

Proof. Write $\rho \in \text{Hom}(\Gamma, G \times H)$ as $\rho = (\rho_G, \rho_H)$. Clearly, $\text{Hom}(\Gamma, G \times H) \cong \text{Hom}(\Gamma, G) \times \text{Hom}(\Gamma, H)$ and under this identification, the $G \times H$-action separates into an independent $G$-action on $\text{Hom}(\Gamma, G)$ and $H$-action on $\text{Hom}(\Gamma, H)$. Thus, as orbit spaces $\text{Hom}(\Gamma, G \times H)/(G \times H) \cong \text{Hom}(\Gamma, G)/G \times \text{Hom}(\Gamma, H)/H$. Moreover, since the GIT quotients are determined by orbit closures, we conclude our result simply by noting that $(G \times H) \cdot \rho = (G \cdot \rho_G) \times (H \cdot \rho_H) = G \cdot \rho_G \times H \cdot \rho_H$. \hfill $\square$

3. Finitely generated Abelian groups

From now on, we let $\Gamma$ be a finitely generated Abelian group. By the classification of finitely generated Abelian groups, there are integers $s, t \geq 0$ and $n_1, \ldots, n_t > 1$ such that

$$\Gamma \cong \mathbb{Z}^s \oplus \bigoplus_{i=1}^t \mathbb{Z}_{n_i},$$

where $\mathbb{Z}_m$ denotes the cyclic group $\mathbb{Z}/m\mathbb{Z}$. In general, $\text{Hom}(\Gamma, G)$ is an algebraic subvariety of $G^{s+t}$, given by $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_{s+t}))$ where $\{\gamma_1, \ldots, \gamma_s\}$ generate $\mathbb{Z}^s$ and $\gamma_{s+j}$ is a generator of $\mathbb{Z}_{n_j}$ for $j = 1, \ldots, t$. Recall that an element in $G$ is called semisimple if for any finite dimensional rational representation of $G$, the element $g$ acts completely reducibly. Let $G_{ss}$ denote the set of semisimple elements in $G$. Then define

$$\text{Hom}(\Gamma, G_{ss}) = \{\rho \in \text{Hom}(\Gamma, G) \mid \rho(\gamma_j) \in G_{ss}, \quad j = 1, \ldots, s+t\}.$$

This is an abuse of notation (since $G_{ss}$ is not a group) but a harmless one, in view of the next result. Since $G_{ss}$ is preserved by conjugation, $G$ acts on $\text{Hom}(\Gamma, G_{ss})$ by simultaneous conjugation. In what follows we will often abbreviate $r = s + t$. Recall that a diagonalizable group is an algebraic group isomorphic to a closed subgroup of a torus (see [3]).

Proposition 3.1. Let $\Gamma$ be a finitely generated Abelian group and let $H_\rho = \rho(\Gamma)^{\mathbb{Z}}$ be the Zariski closure of $\rho(\Gamma) \subset G$. Then, the following are equivalent:

1. $\rho \in \text{Hom}(\Gamma, G_{ss})$
2. $\rho \in \text{Hom}(\Gamma, G)^{ps}$
3. $H_\rho$ is a diagonalizable group
4. $H_\rho$ is a reductive group

In particular, $\text{Hom}(\Gamma, G)^{ps}/G = \text{Hom}(\Gamma, G_{ss})/G$.

Proof. In [38], it is shown that the Zariski closure of $\rho(\Gamma)$ in $G$ is linearly reductive if and only if the $G$-orbit of $\rho$ in $\text{Hom}(\Gamma, G) \subset G^r$ is closed. Since being reductive is equivalent to linearly reductive (in characteristic 0), this shows the equivalence between (2) and (4) (which is in fact valid for any $\Gamma$ not necessarily Abelian).

Now let $\gamma_1, \ldots, \gamma_r$ be a generating set for $\Gamma$ and let $a_j := \rho(\gamma_j) \in G$, $j = 1, \ldots, r$ for a fixed $\rho \in \text{Hom}(\Gamma, G)$. Since $\Gamma$ is Abelian, $\rho(\Gamma) \subset G$, the subgroup
generated by \{a_1,...,a_r\}, is an Abelian subgroup of \(G\). So, the Zariski closure \(H_\rho\) of \(\rho(\Gamma)\) is an Abelian algebraic group (since commutation relations are polynomial). Now suppose that \(\rho \in \text{Hom}(\Gamma, G_{ss})\) which means by definition that \(a_j \in G_{ss}\), and consider a linear embedding of \(G, \psi : G \to \text{GL}(n, \mathbb{C})\). Then, the matrices \(\psi(a_1), \ldots, \psi(a_j)\) can be simultaneously conjugated by an element in \(\text{GL}(n, \mathbb{C})\) to lie in some maximal torus \(T\) of \(\text{GL}(n, \mathbb{C})\). Because \(T\) is Zariski closed in \(\text{GL}(n, \mathbb{C})\), this means that for every \(g \in H_\rho\) we will have \(\psi(g) \in T\). Recall that the multiplicative Jordan decomposition is preserved by homomorphisms: for \(g = g_sg_u \in G\) with \(g_s \in G_{ss}\) and \(g_u\) unipotent, we have \(\psi(g_s) = (\psi(g))_s\) and \(\psi(g_u) = (\psi(g))_u\). Thus, for \(g \in H_\rho\), we have \(\psi(g) = \psi(g)_s \cdot \psi(g)_u \in T\) which implies \(\psi(g)_u = \psi(g_u) = 1\), and since \(\psi\) is injective \(g_u = 1\). So, \(H_\rho\) consists only of semisimple elements of \(G\) and by [3] this means that \(H_\rho\) is a diagonalizable group, and hence reductive. Thus, (1) implies (3) and so (4). Conversely, let \(H_\rho\) be reductive. Since it is also Abelian (as \(\Gamma\) is Abelian) then, again by [6] it consists of semisimple elements. In particular, \(\rho(\gamma_j) \in G_{ss}\). So (4) implies (1) as well. \(\Box\)

**Remark 3.2.** The fact that \(\Gamma\) is Abelian is crucial in Proposition 3.1. Indeed, if \(\Gamma\) is not Abelian, neither of the inclusions \(\text{Hom}(\Gamma, G_{ss}) \subset \text{Hom}(\Gamma, G)^{ps}\) or \(\text{Hom}(\Gamma, G_{ss}) \supset \text{Hom}(\Gamma, G)^{ps}\) is true in general. Here are simple counter-examples. Let \(\Gamma = F_2\), the free group of rank 2, and \(G = \text{SL}(2, \mathbb{C})\). Let \(\rho = (g, h) \in \text{Hom}(\Gamma, G)\) with \(g = \left(\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right)\) and \(h = \left(\begin{array}{cc} y & 1 \\ 0 & y^{-1} \end{array} \right)\), where \(x, y \in \mathbb{C} \setminus \{0\}\) are each not \(\pm 1\). Then both \(g\) and \(h\) are semi-simple, so \(\rho \in \text{Hom}(\Gamma, G_{ss})\). However, by conjugating with \(\left(\begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right)\) and taking a limit as \(t\) goes to 0 we see that the tuple \((g, h)\) limits to \(\left(\left(\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right), \left(\begin{array}{cc} y & 0 \\ 0 & y^{-1} \end{array} \right)\right)\), which is not in the \(G\)-orbit of \((g, h)\). So, \(\rho \notin \text{Hom}(\Gamma, G)^{ps}\). Now suppose that \(\Gamma = F_3\), the free group on 3 generators and let \(\rho = (g, h, k) \in G^3\) with \(G = \text{SL}(2, \mathbb{C})\) again. Consider

\[
g = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \quad h = \left(\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right), \quad k = \left(\begin{array}{cc} y & z \\ z & w \end{array} \right)
\]

satisfying \(yw - z^2 = 1\) and \((x - x^{-1})z \neq 0\). If \(\rho\) is an irreducible representation, then its orbit is closed; and if it were reducible, then \((h, k)\) could be made simultaneously upper-triangular. However, a simple computation shows this to be impossible. Thus, \(\rho \in \text{Hom}(F_3, G)^{ps}\), but \(g \notin G_{ss}\), so \(\rho \notin \text{Hom}(F_3, G_{ss})\).

Next, define \(G(K) = \bigcup_{g \in G} gKg^{-1}\) to be the set of all \(G\)-conjugates of the group \(K\), and consequently define

\[
\text{Hom}(\Gamma, G(K)) = \{ \rho \in \text{Hom}(\Gamma, G) \mid \rho(\gamma_1), \ldots, \rho(\gamma_r) \in G(K) \}.
\]

Clearly, \(G(\text{Hom}(\Gamma, K)) \subset \text{Hom}(\Gamma, G(K))\).
Lemma 3.3. If \( \Gamma \) is a finitely generated Abelian group, then
\[
G(\text{Hom}(\Gamma, K)) = \text{Hom}(\Gamma, G(K)).
\]

Proof. Since \( G(\text{Hom}(\Gamma, K)) \subset \text{Hom}(\Gamma, G(K)) \) for any \( \Gamma \), it suffices to prove the reverse inclusion for finitely generated Abelian groups. This argument follows the one in [36, pg. 16]. Let \( \Gamma \) be generated by \( \{\gamma_1, \ldots, \gamma_r\} \) and let \( \rho \in \text{Hom}(\Gamma, G(K)) \). Let \( A \) be the Zariski closure of the group generated by \( \{\rho(\gamma_1), \ldots, \rho(\gamma_r)\} \). Since \( G(K) \) only consists of semisimple elements, the proof of Lemma 3.1 tells us that \( A \) is Abelian and consists of only semisimple elements. Since \( A \) is algebraic it has a maximal compact subgroup \( B \). Since it is Abelian, \( B \) is unique. Since each \( \rho(\gamma_i) \) is conjugate to an element in \( K \), each is in some maximal compact subgroup. Therefore, each of them is in the unique maximal compact \( B \). However, all maximal compact subgroups are conjugate, so there exists a \( g \in G \) so that \( gBg^{-1} \subset K \) which in turn implies that \( g \rho(\gamma_i) g^{-1} \subset K \) for all \( 1 \leq i \leq r \). By definition, this implies \( \rho \in G(\text{Hom}(\Gamma, K)) \) and as such \( G(\text{Hom}(\Gamma, K)) \supset \text{Hom}(\Gamma, G(K)) \). \( \square \)

Summarizing the last two sections, we have shown that when \( \Gamma \) is a finitely generated Abelian group, we can replace the right inclusion, with the left inclusion in the following diagram:

\[
\begin{align*}
\text{Hom}(\Gamma, G_{ss})/G & \cong \text{Hom}(\Gamma, G)/G, \\
\text{Hom}(\Gamma, G(K))/G & \cong \text{Hom}(\Gamma, K)/K.
\end{align*}
\]

We will now show that there is a \( G \)-equivariant strong deformation retraction \( \text{Hom}(\Gamma, G(K)) \hookrightarrow \text{Hom}(\Gamma, G_{ss}) \).

4. Deformation Retraction of Character Varieties

Recall that a strong deformation retraction (SDR) from a topological space \( M \) to a subspace \( N \subset M \) is a continuous map \( \phi : [0, 1] \times M \rightarrow M \) such that (1) \( \phi_0 \) is the identity on \( M \), (2) \( \phi_t(n) = n \) for all \( n \in N \) and \( t \in [0, 1] \), and (3) \( \phi_1(M) \subset N \). In short, it is a homotopy relative to \( N \) between the identity on \( M \) and a retraction mapping to \( N \). We are going to construct an explicit \( G \)-equivariant strong deformation retraction from \( G_{ss} \) to \( G(K) \).

Following Pettet-Souto [36], we start with a deformation in the case when \( G = \text{SL}(n, \mathbb{C}) \). Let \( \Delta_n \) be the subgroup of diagonal matrices in \( \text{SL}(n, \mathbb{C}) \), which is a maximal torus, identified in the usual way with a subgroup of \( (\mathbb{C}^*)^n \). Consider the following deformation retraction from \( \Delta_n \) to the subset \( \Delta_n \cap \text{SU}(n) \):

\[
\sigma : [0, 1] \times \Delta_n \rightarrow \Delta_n \\
(t, g) \mapsto \sigma_t(g)
\]

where, for \( g = \text{diag}(z_1, \ldots, z_n) \in \Delta_n \), and \( z_1, \ldots, z_n \in \mathbb{C}^* \),

\[
\sigma_t(g) := \text{diag}(|z_1|^{-t}z_1, \ldots, |z_n|^{-t}z_n).
\]
The strong deformation retraction properties of $\sigma_t$ are easily established. Note that $\sigma_t$ is a homomorphism for every $t \in [0, 1]$.

Suppose that $g \in \text{SL}(n, \mathbb{C})_{ss}$ is semisimple, which means it is diagonalizable. Since all maximal tori are conjugate, there is $h \in \text{SL}(n, \mathbb{C})$ (depending on $g$) so that $hgh^{-1} \in \Delta_n$. Define the following map:

$$(4.2) \quad \delta : [0, 1] \times \text{SL}(n, \mathbb{C})_{ss} \to \text{SL}(n, \mathbb{C})$$

by letting

$$(\delta_t(g)) := h^{-1}\sigma_t(hgh^{-1})h.$$ 

We have the following properties of $\delta$.

**Lemma 4.1.** The map $\delta$ satisfies:

1. $\delta_t$ is well defined; that is, it does not depend on the choice of $h$;
2. $\delta_t(g) \in \text{SL}(n, \mathbb{C})_{ss}$ for all $t \in [0, 1]$ and all $g \in \text{SL}(n, \mathbb{C})_{ss}$;
3. $\delta$ is a strong deformation retraction from $\text{SL}(n, \mathbb{C})_{ss}$ to the set of $\text{SL}(n, \mathbb{C})$ conjugates of $\text{SU}(n)$.

Moreover, for every $t \in [0, 1]$, we have:

4. $\delta_t$ is $\text{SL}(n, \mathbb{C})$-equivariant;
5. if $g_1g_2 = g_2g_1$ then $\delta_t(g_1)\delta_t(g_2) = \delta_t(g_2)\delta_t(g_1)$;
6. $\delta_t(g^m) = (\delta_t(g))^m$ for all $m \in \mathbb{N}$, and all $g \in \text{SL}(n, \mathbb{C})_{ss}$

**Proof.** All these properties follow from simple computations. For properties (1) to (5) the reader may consult [36], or Appendix A.1. Let us show (6). If $g \in \text{SL}(n, \mathbb{C})_{ss}$ is semisimple, and $h \in \text{SL}(n, \mathbb{C})$ is chosen so that $hgh^{-1} \in \Delta_n$, then $(hgh^{-1})^m = hgh^{-1}$ is also in $\Delta_n$. So,

$$\delta_t(g^m) = h^{-1}\sigma_t(hgh^{-1})h = h^{-1}(\sigma_t(hgh^{-1}))^m h = (\delta_t(g))^m$$

as wanted. Note that we used the homomorphism property of $\sigma_t$. \qed

Let $m \in \mathbb{N}$. For any group $G$, one can define the $m$th power map $p_m : G \to G$, by $p_m(g) = g^m$. If $g, h \in G$ and $h^m = g$ we say that $h$ is an $m$th root of $g$.

For later convenience, we here record the following fact about the power map on $\text{SL}(n, \mathbb{C})$. Recall that $\Delta_n \subset \text{SL}(n, \mathbb{C})$ denotes the subgroup of diagonal matrices in $\text{SL}(n, \mathbb{C})$.

**Lemma 4.2.** Let $p_m : \text{SL}(n, \mathbb{C}) \to \text{SL}(n, \mathbb{C})$ be the $m$th power map. Then $p_m^{-1}(\Delta_n) = \Delta_n$.

**Proof.** One inclusion is clear. If $g \in \Delta_n$ then $g^m \in \Delta_n$ so that $g \in p_m^{-1}(\Delta_n)$ by definition. For the converse, assume that $g \in p_m^{-1}(\Delta_n)$, which means that $g^m \in \Delta_n$, so $g^m$ is semisimple. Let $g = gs gu$ be the multiplicative Jordan decomposition, so that $gs$ is semisimple, $su$ is unipotent and $gs su = su gs$. Then $g^m = (gs su)^m = g^m_s g^m_u$. Since powers of diagonalizable and unipotent matrices remain respectively diagonalizable and unipotent, $g^m_s$ is semisimple and $g^m_u$ is unipotent. By the uniqueness of the Jordan decomposition, we conclude that $g^m = g^m_s$ and $g^m_u = 1$. Since the exponential map is a diffeomorphism between nilpotent and unipotent
matrices in $\text{SL}(n, \mathbb{C})$, we have $g_u^m = \exp(m \exp^{-1}(g_u)) = 1$ which implies that $\exp^{-1} g_u = 0$ and so $g_u = 1$. Therefore, $g = g_s$ is also semisimple. Since $g$ and $g^m$ share the same eigenvectors (a linear algebra argument) and $g^m \in \Delta_n$, we see that $g$ is also in $\Delta_n$. □

### 4.1. Deformation retraction for general $G$.

Now let $G$ be a complex reductive algebraic group, not necessarily connected, and let $G_0$ be its identity component. Let $1$ denote also the identity in $G$. Then $G_0$ is normal in $G$, and indeed there is a short exact sequence

$$1 \to G_0 \to G \xrightarrow{\pi} F \to 1$$

where $F := G/G_0 \cong \pi_0(G)$ is a finite group. As an algebraic set, $G$ is isomorphic to the Cartesian product $G_0 \times F$, and we can write

$$G = \prod_{f \in F} G_f$$

where $G_f$ denotes the connected component $G_f := \pi^{-1}(f)$ for some $f \in F$ (in particular $G_0 = \pi^{-1}(1_f)$).

However, in general $G$ is not the direct product of $F$ and $G_0$ as groups. On the other hand, we have a simple relation between semisimple elements in $G$ and in $G_0$, which will enable us to deduce an appropriate deformation retraction for such a general $G$. Recall that $G_{ss}$ denotes the set of semisimple elements in $G$.

**Lemma 4.3.** Let $f \in F$ have order $m$, so that $f^m = 1_F$. If $g \in G_{ss} \cap G_f$, then $g^m \in (G_0)_{ss}$. In particular, if $N$ is the order of the group $F$, $G_{ss}^N \subset (G_0)_{ss}$.

**Proof.** Let $g \in G_{ss} \cap G_f$. Then $\pi(g^m) = (\pi(g))^m = f^m = 1$, so by exactness of the sequence, $g^m \in G_0$. Since $g$ is semisimple, it is clear that all its powers are also semisimple. So, $g^m \in (G_0)_{ss}$. The second statement follows from the fact that if $N = \#F$, then $f^N = 1_F$, for all $f \in F$. □

Consider now an affine algebraic embedding of groups $G \subset \text{SL}(n, \mathbb{C})$. Let $K$ and $T_0$ be respectively, a maximal compact subgroup of $G$ and a maximal torus of $G_0$ such that $T_0 \cap K$ is a maximal torus in $K_0 = K \cap G_0$. Up to conjugating in $\text{SL}(n, \mathbb{C})$ we may assume that $T_0 \subset \Delta_n$ and that $K \subset \text{SU}(n)$.

**Lemma 4.4.** We have the inclusion $G_{ss} \subset \text{SL}(n, \mathbb{C})_{ss}$.

**Proof.** Given $g \in G_{ss}$, suppose that $g \in G_f$ with $f^m = 1_F$. Then, by Lemma 4.3, $g^m \in (G_0)_{ss}$. So, because $G_0$ is connected and all maximal tori are conjugate inside it, there exists $h \in G_0$ so that $h g^m h^{-1} \in T_0 \subset \Delta_n$. This means that $h g h^{-1} \in p_m^{-1}(T_0) \subset p_m^{-1}(\Delta_n) = \Delta_n$, by Lemma 4.2. So $h g h^{-1}$ is semisimple in $\text{SL}(n, \mathbb{C})$, and the same holds for $g$. □

This Lemma allows us to restrict the map $\delta$ in equation (4.2) to $G_{ss}$, and so we define a new map $\delta : [0, 1] \times G_{ss} \to \text{SL}(n, \mathbb{C})_{ss}$, still denoted by $\delta$. In particular, for every $t \in [0, 1]$, $\delta_t$ preserves commutativity (ie, if $g_1 g_2 = g_2 g_1$ with $g_1, g_2 \in G_{ss}$ then $\delta_t(g_1) \delta_t(g_2) = \delta_t(g_2) \delta_t(g_1)$) and torsion (ie, $g^N = 1$ implies that $\delta_t(g^N) = (\delta_t(g))^N = 1$ for all $m \in \mathbb{N}$, and all $G_{ss}$) automatically, by items (5) and
Lemma 4.5. We have:

1. For all \( t \in [0, 1] \) and all \( g \in G_{ss} \), \( \delta_t(g) \in G_{ss} \);
2. \( \delta_1(g) \in G(K) \) for all \( g \in G_{ss} \).

Proof. This was shown in [36]. But see also Appendix A.2 for a self-contained independent proof.

Finally, define \( \delta^r : [0, 1] \times G_{ss}^r \to G_{ss}^r \) by \( \delta^r_t(g_1, \ldots, g_r) = (\delta_t(g_1), \ldots, \delta_t(g_r)) \) and let \( G \) act on \( G_{ss}^r \) by simultaneous conjugation in each factor.

Corollary 4.6. \( \delta^r : [0, 1] \times G_{ss}^r \to G_{ss}^r \) is a \( G \)-equivariant deformation retraction onto \( G(K)^r \) that preserves torsion and commutativity.

Proof. The result follows immediately from the the last lemmas.

Remark 4.7. A version of Corollary 4.6 also appears in [36]. The fact that \( \delta^r \) preserves torsion is only verified here, and the fact that it applies to disconnected groups \( G \) was independently discovered by us and the authors of [36] (see also Appendix A).

4.2. Proof of the Main Theorem. Recall that Theorem 1.1 states that \( \text{Hom}(\Gamma, G)/G \) strongly deformation retracts to \( \text{Hom}(\Gamma, K)/K \) for any reductive algebraic \( \mathbb{C} \)-group and any finitely generated Abelian group \( \Gamma \).

Proof. [Proof of Theorem 1.1] Suppose that \( \Gamma \) is generated by \( r \) elements, as has been our convention. Proposition 2.3 and Lemma 3.1 imply that \( \mathcal{X}_r(G) \cong \text{Hom}(\Gamma, G_{ss})/G \), and Proposition 2.7 and Lemma 3.3 imply that \( \mathcal{X}_r(K) \cong \text{Hom}(\Gamma, G(K))/G \). In Subsection 4.1 we proved that the mapping \( \delta^r_t : \text{Hom}(\Gamma, G_{ss}) \to \text{Hom}(\Gamma, G_{ss}) \) is a \( G \)-equivariant strong deformation retraction onto \( \text{Hom}(\Gamma, G(K)) \). Therefore, we have a strong deformation retraction from \( \text{Hom}(\Gamma, G_{ss})/G \) onto \( \text{Hom}(\Gamma, G(K))/G \). Therefore, \( \mathcal{X}_r(G) \cong \text{Hom}(\Gamma, G_{ss})/G \) strongly deformation retracts onto \( \mathcal{X}_r(K) \cong \text{Hom}(\Gamma, G(K))/G \).

Remark 4.8. Theorem 1.1 corrects and generalizes the proof of Proposition 7.1 in [16]. In that paper, the authors consider free Abelian groups as an example to situate the main theorems for free groups (non-Abelian) of that work. The statement and its proof are correct in some cases, for example when \( G \) is the general linear group or special linear group.

Since a deformation retraction is in particular a homotopy equivalence, the spaces \( \mathcal{X}_r(G) \) and \( \mathcal{X}_r(K) \) have the same homotopy type. So, Theorem 1.1 implies they have the same homotopy groups \( \pi_n(\mathcal{X}_r(K)) \cong \pi_n(\mathcal{X}_r(G)) \), and cohomology rings \( H^*(\mathcal{X}_r(K)) \cong H^*(\mathcal{X}_r(G)) \) for all values of \( * \). Moreover, these results do not depend on the choices made to define \( \delta \) since different choices lead to
homeomorphic spaces. In particular, these spaces have the same number of path components. Recall that we are using the Euclidean topology on both $\mathcal{X}_\Gamma(G)$ and $\mathcal{X}_\Gamma(K)$. Also, note that the path components in these spaces are the same as the connected components, since the two notions are equivalent in the context of semi-algebraic sets (which are always locally path connected), such as $\mathcal{X}_\Gamma(K)$.

The following is immediate.

**Corollary 4.9.** There is a natural bijection between the connected components of $\mathcal{X}_\Gamma(G)$ and those of $\mathcal{X}_\Gamma(K)$. Corresponding components have the same homotopy type, and the deformation retraction preserves each component of $\mathcal{X}_\Gamma(G)$.

In particular, for every representation, there is another one in the same component, but taking values in $K$.

**Corollary 4.10.** Let $G = K_C$ be a complex reductive algebraic group and let $\Gamma$ be finitely generated Abelian group. If $C_C \subset \mathcal{X}_\Gamma(G)$ denotes a path component, then there exists $\rho \in \text{Hom}(\Gamma, K)$ such that $[\rho] \in C_C$.

**Proof.** Theorem 1.1 implies that for every component $C_C \subset \mathcal{X}_\Gamma(C)$, there exists a component $C \subset \mathcal{X}_\Gamma(K)$ such that $C \subset C_C$. The result follows. \[ \square \]

Let us write $|X|$ for the number of path components of a locally path-connected space.

**Lemma 4.11.** Let $G$ be a connected reductive algebraic group and $X$ an affine $G$-variety. Then $|X| = |X/G|$.

**Proof.** Since the quotient map $\pi : X \to X/G$ is continuous, a given path component is mapped to a single path component. Being also surjective, we obtain $|X| \geq |X/G|$. Suppose, by contradiction that inequality holds. Then, at least two path components, say $A, B \subset X$, are being identified by the action of $G$ or by the further GIT equivalence. This means that there are $a \in A$, $b \in B$, and a sequence $\{g_n\} \subset G$ such that $\lim_{n \to \infty} g_n \cdot a = b$ (or possibly with the roles of $a$ and $b$ reversed).

Note this includes the possibility that $b$ is in the $G$-orbit of $a$ by considering the constant sequence. Returning to the argument, we conclude $b \in \overline{G \cdot a}$. Since $G$ is connected, the orbit $G \cdot a = \{g \cdot a : g \in G\}$ is connected, and thus $\overline{G \cdot a}$ is also connected. Therefore, $\overline{G \cdot a} \subset \overline{A} = A$ since $A$ is closed. This gives a contradiction as $A$ and $B$ are disjoint in $X$. \[ \square \]

**Remark 4.12.** A similar argument shows that if $X$ is a $G$-space with $G$ connected then $|X| = |X/G|$.

In particular, using Corollary 4.9 we have proved the following proposition.

**Proposition 4.13.** Let $G$ be a connected complex reductive algebraic group with maximal compact subgroup $K$, and let $\Gamma$ be a finitely generated Abelian group. Then, $|\text{Hom}(\Gamma, G)| = |\mathcal{X}_\Gamma(G)| = |\mathcal{X}_\Gamma(K)| = |\text{Hom}(\Gamma, K)|$. 
4.3. Right angled Artin groups. We end this section with a conjecture, based on a remark mentioned in [36]. Let us define a right-angled Artin group (RAAG) with torsion to be a finitely generated group which admits a finite presentation with generators $\gamma_1, \cdots, \gamma_r$ and where every relation is either $\gamma_i \gamma_j = \gamma_j \gamma_i$, or a torsion relation of the form $\gamma_i^m = 1$ (for some $i, j$ and $m \geq 2$). These groups include free products of cyclic groups and Abelian groups as extremes. Combining the main result of [16] with Theorem 1.1, we have shown that there is strong deformation retraction from $X_F(G)$ to $X_F(K)$ when $\Gamma$ is either a free product of infinite cyclic groups (a free group), or when $\Gamma$ is Abelian.

Conjecture 4.14. Let $G$ be a complex reductive algebraic group and let $K$ be a maximal compact subgroup. Let $\Gamma$ be a right-angled Artin group with torsion. There is strong deformation retraction from $X_F(G)$ to $X_F(K)$.

Providing further evidence for this conjecture, we now use our main theorem and the main result of [36] to prove two theorems. First, we need a lemma. Recall that a weak deformation retraction between a space $X$ and a subspace $A$ is a continuous family of mappings $F_t : X \to X$, $t \in [0, 1]$, such that $F_0$ is the identity on $X$, $F_t(X) \subset A$, and $F_t(A) \subset A$ for all $t$.

Lemma 4.15. Let $\Gamma$ be a finitely generated Abelian group, and let $G$ be a complex reductive algebraic group with maximal compact $K$. Then, there exists a $G$-equivariant weak deformation retraction from $\text{Hom}(\Gamma, G)$ onto $\text{Hom}(\Gamma, G_{ss})$ that fixes $K$ during the retraction.

Proof. In [36] it is shown that for every $g \in G$ and every $\epsilon > 0$, there is a continuous $G$-equivariant mapping $g \mapsto (g^\epsilon_s, g^\epsilon_u)$ such that:

(1) $g = g^\epsilon_s g^\epsilon_u = g^\epsilon_u g^\epsilon_s$,
(2) if $h$ commutes with $g$ it also commutes with $g^\epsilon_s$ and $g^\epsilon_u$,
(3) $g^\epsilon_s \in G_{ss}$, and
(4) if $g \in H$ where $H$ is an algebraic subgroup, then $g^\epsilon_s, g^\epsilon_u \in H$.
(5) the spectral radius of $g^\epsilon_u - 1$ is at most $\epsilon$.

This is called the approximate Jordan decomposition, since $g^\epsilon_u$ is not unipotent, but is approximately so by (5). Using this, they show that for sufficiently small $\epsilon > 0$, $F_t(g) = g^\epsilon_u \text{Exp}((1 - t)\text{Log}(g^\epsilon_s))$ is a $G$-equivariant weak deformation retraction from $G$ to $G_{ss}$ that preserves commutativity and point-wise fixes $K$ for all $t$. Note that $\text{Exp}$ and $\text{Log}$ are the usual power series functions in terms of matrices, which makes sense in this context since we are choosing an embedding of $G$ as a matrix group. Returning to the proof, we consequently obtain a $G$-equivariant weak deformation retraction from $\text{Hom}(\mathbb{Z}^r, G)$ to $\text{Hom}(\mathbb{Z}^r, G_{ss})$ that keeps $\text{Hom}(\mathbb{Z}^r, K)$ point-wise fixed for all $t$.

To prove the lemma, it then suffices to prove that $F_t$ preserves the algebraic torsion relations for all $t$. So suppose $g^m = 1$ for some $m$. Then since $H = \langle 1 \rangle$ is an algebraic subgroup, property (4) of approximate Jordan decomposition gives $(g^m)^\epsilon_s = 1$ and $(g^m)^\epsilon_u = 1$. Therefore, $F_t(g^m) = 1$. On the other hand, since $g^\epsilon_s$
and \( g_s^m \) commute, we have \( 1 = g^m = (g_s^m)(g_u^m) \), and so \((g_s^m)^m = 1\) if \((g_u^m)^m = 1\).
Now considering the subgroup \( H = \langle g \rangle = \{1, g, \ldots, g^{m-1}\} \), which is algebraic since it is finite (it has at most \( m \) elements, as \( g^m = 1 \)), property (4) again shows that \( g_s^m \) is in \( H \). Thus, \((g_s^m)^m = 1\) since for all \( h \in H \), \( h = g^k \) for some \( k \), and so \( h^m = (g^m)^k = 1 \).
Moreover, \( G \)-equivariance and the fact that \( g_s^k \) and \( g_u^k \) commute together imply that \( g_s^m \) commutes with \( F_i(g) \). Thus,
\[
g_s^k \exp((1-t)\log(g_u^m)) = \exp((1-t)\log(g_u^m))g_s^k
\]
for all \( t \). We conclude:
\[
F_i(g)^m = (g_s^m)(\exp((1-t)\log(g_u^m)))^m = 1^m \exp((1-t)\log((g_u^m)^m)) = \exp((1-t)\log(1)) = 1.
\]
Putting these observations together leads to \( F_i(g)^m = 1 = F_i(g)^m \), as desired. \( \square \)

**Theorem 4.16.** Let \( \Gamma \) be a RAAG with torsion, \( G \) be a complex reductive algebraic group and let \( K \) be a maximal compact subgroup of \( G \). Then \( \mathfrak{X}_\Gamma(G) \) strongly deformation retracts onto \( \text{Hom}(\Gamma, G(K))/G \) which fixes the subspace \( \text{Hom}(\Gamma, K)/K \).

**Proof.** By the above lemma there is a \( G \)-equivariant weak deformation retraction from \( G \) to \( G_{ss} \) that fixes \( K \), preserves torsion, and preserves commutativity. Putting this together with the \( G \)-equivariant strong deformation retraction from \( G_{ss} \) to \( G(K) \) that also fixes \( K \), preserves torsion, and preserves commutativity from Theorem 1.1 gives a \( G \)-equivariant weak deformation retraction from \( G \) to \( G(K) \) that fixes \( K \) and also preserves torsion and commutativity.

Let \( \Gamma \) be generated by \( r \) elements. The relations are either torsion relations or commutativity relations. Applying the weak deformation retraction from \( G \) to \( G(K) \) factor-wise to \( \text{Hom}(\Gamma, G) \subset G^r \) gives a \( G \)-equivariant weak deformation retraction onto \( \text{Hom}(\Gamma, G(K)) \) that fixes \( \text{Hom}(\Gamma, K) \) for all time.

Therefore, we obtain a weak deformation retraction from \( \text{Hom}(\Gamma, G)/G \) onto \( \text{Hom}(\Gamma, G(K))/G \) which contains \( G(\text{Hom}(\Gamma, K))/G \cong \text{Hom}(\Gamma, K)/K \) as a fixed subspace.

Note that for each \( t \) in \([0, 1] \) both maps \( F_i \) in Lemma 1.15 and \( \delta_t \) in Lemma 1.5 are homeomorphisms, and so since they are \( G \)-equivariant, they send closed orbits to closed orbits. And for the \( t = 1 \) cases, by continuity and Proposition 2.1 the limit of closed orbits in the polystable quotient corresponds to a closed orbit.

Thus, restricting this weak deformation retraction to the subspace of closed orbits then determines a weak retraction from \( \mathfrak{X}_\Gamma(G) \) onto \( \text{Hom}(\Gamma, G(K))/G \) which contains \( \text{Hom}(\Gamma, K)/K \) as a fixed subspace.

Since this weak retraction establishes that the inclusion mapping \( \text{Hom}(\Gamma, G(K))/G \hookrightarrow \mathfrak{X}_\Gamma(G) \) induces an isomorphism on homotopy groups, and \( \text{Hom}(\Gamma, G(K))/G \) is a cellular sub-complex of \( \text{Hom}(\Gamma, G)/G \) given they are
semi-algebraic sets, Whitehead’s Theorem (see [24]) implies there is a strong
defformation retraction from $\text{Hom}(\Gamma, G)/G$ onto $\text{Hom}(\Gamma, G(K))/G$ which contains
$\text{Hom}(\Gamma, K)/K$ as a sub-complex. □

**Remark 4.17.** Given Lemmas 3.1 and 3.3, the above theorem includes our main
result as a special case since Abelian groups are RAAG’s with torsion.

**Theorem 4.18.** Let $\Gamma$ be a RAAG with torsion, $G$ be a complex reductive al-
gebraic group and let $K$ be a maximal compact subgroup of $G$. If there exists
a $K$-equivariant weak retraction from $\text{Hom}(\Gamma, G(K))$ to $\text{Hom}(\Gamma, K)$, then $\mathfrak{X}_\Gamma(G)$
strongly deformation retracts onto $\mathfrak{X}_\Gamma(K)$.

**Proof.** The general Kempf-Ness theory (see [17]) implies that for any finitely
generated $\Gamma$, $\text{Hom}(\Gamma, G)K$-equivariantly strongly deformation retracts onto a $K$-stable
subspace $\mathcal{KN} \subset \text{Hom}(\Gamma, G)^{ps}$ such that $\mathcal{KN}/K \cong \mathfrak{X}_\Gamma(G)$. Thus, $\text{Hom}(\Gamma, G)/K$ is
weakly homotopic to $\mathfrak{X}_\Gamma(G)$.

Theorem 4.16 implies that there exists a $K$-equivariant weak deformation re-
traction from $\text{Hom}(\Gamma, G)$ onto $\text{Hom}(\Gamma, G(K))$. However, by hypothesis there exists
a $K$-equivariant weak retraction from $\text{Hom}(\Gamma, G(K))$ onto $\text{Hom}(\Gamma, K)$. Putting
this together we obtain a weak retraction from $\text{Hom}(\Gamma, G)/K$ to $\mathfrak{X}_\Gamma(K)$; and so
$\text{Hom}(\Gamma, G)/K$ is weakly homotopic to $\mathfrak{X}_\Gamma(K)$.

Thus, $\mathfrak{X}_\Gamma(G)$ is weakly homotopic to $\mathfrak{X}_\Gamma(K)$. The computation of the Kempf-
Ness set in [16] implies that $\mathfrak{X}_\Gamma(K)$ is contained in $\mathcal{KN}$.

Thus, by Corollary 4.10 in [17] we conclude the result. □

**Remark 4.19.** Pettet and Souto in [36] show there is a weak retract from
$\text{Hom}(\mathbb{Z}, G(K))$ to $\text{Hom}(\mathbb{Z}, K)$, but they do not show it preserves torsion or that
it is $K$-equivariant. This shows that their result alone does not imply ours. Given
the above theorem, it is natural to ask if one can make their weak retraction $K$-
equivariant and obtain a special case of our main theorem, but as the following
example shows, the existence of SDR from a $G$-space $X$ to a $G$-subspace $Y$, and
even a SDR on quotients $X/G$ to $Y/G$, together are not enough to imply the
existence of a $G$-equivariant SDR.

Here is the example: Take a CW space $Z$ that is acyclic but not contractible.
Let $X$ be the suspension of $Z$, and let $G$ be the order 2 group that acts on the
suspension by switching the two cones. Notice that $Z$ is the fixed point set.
Then $X$ is contractible since the suspension of an acyclic space is contractible (by
Whitehead’s Theorem), and the orbit space is contractible since it is a cone. But $X$
is not $G$-equivariantly contractible because the fixed point set is not contractible.

5. **Applications to Irreducibility and Topology of $\mathfrak{X}_\Gamma(G)$.**

We now investigate other interesting consequences of Theorem 4.18 related to
the topology and irreducibility of the character varieties $\mathfrak{X}_\Gamma(G)$.

We first consider the free Abelian group of rank $r$, $\Gamma = \mathbb{Z}^r$. For these groups, and
for a connected and semisimple compact Lie group $K$, the cases when $\text{Hom}(\mathbb{Z}, K)$
is connected have been determined (see [28]).
This leads us to consider also connected *semisimple* algebraic groups $G$; by definition, these do not contain any non-trivial closed connected Abelian normal subgroups, and are clearly reductive. As usual, we assume that the trivial group is not semisimple. Consequently, every connected semisimple group has rank at least 1.

**Theorem 5.1.** Let $G$ be a semisimple connected algebraic group over $\mathbb{C}$. Then $\mathfrak{X}_{\mathbb{Z}^r}(G)$ is connected if and only if $r = 1$, $r = 2$ and $G$ is simply connected, or $r \geq 3$ and $G$ is a product of $\text{SL}(n, \mathbb{C})$’s and $\text{Sp}(n, \mathbb{C})$’s.

**Proof.** Let $K$ be a maximal compact subgroup of $G$. Then $K$ is connected and semisimple. According to [28], $\text{Hom}(\mathbb{Z}^r, K)$ is connected if and only if $r = 1$, $r = 2$ and $G$ is simply connected, or $r \geq 3$ and $G$ is a product of $\text{SU}(n)$’s and $\text{Sp}(n)$’s. By Proposition 4.13, $\text{Hom}(\mathbb{Z}^r, K)$ is connected if and only if $\mathfrak{X}_{\mathbb{Z}^r}(K)$ is connected. However, by Corollary 4.9, $\mathfrak{X}_{\mathbb{Z}^r}(K)$ is connected if and only if $\mathfrak{X}_{\mathbb{Z}^r}(G)$ is connected, and the result follows. $\square$

5.1. **The identity component.** There is an extremely useful characterization of $\mathfrak{X}_{\mathbb{Z}^r}(G)$, recently obtained by A. Sikora [42], analogous to work on the compact case by T. Baird [5] (see also [2]).

To describe it, denote by $T$ the maximal torus of $G$, by $N_G(T)$ its normalizer in $G$, and by $W = N_G(T)/T$ its Weyl group. For any $r \in \mathbb{N}$, the Weyl group $W$ acts on $T^r$ by simultaneous conjugation. Consider the natural morphisms of affine varieties

$$
\begin{align*}
\text{Hom}(\mathbb{Z}^r, T) & \hookrightarrow \text{Hom}(\mathbb{Z}^r, G) \\
\| \downarrow \pi & \\
T^r & \xrightarrow{\varphi} \mathfrak{X}_{\mathbb{Z}^r}(G).
\end{align*}
$$

(5.1)

Here, the top row is the natural inclusion, $\pi$ the canonical projection, and $\varphi$ makes the diagram commute. The morphism $\varphi$ factors through the action of the Weyl group, and Sikora (see [42]) showed the following theorem. Denote by $\mathfrak{X}^0_{\mathbb{Z}^r}(G) \subset \mathfrak{X}_{\mathbb{Z}^r}(G)$ the image of $\varphi$.

**Theorem 5.2.** [See [42]] Let $G$ be a complex reductive algebraic group. Then $\mathfrak{X}^0_{\mathbb{Z}^r}(G)$ is an irreducible component of $\mathfrak{X}_{\mathbb{Z}^r}(G)$ and there is a bijective birational normalization morphism

$$
\chi : T^r / W \rightarrow \mathfrak{X}^0_{\mathbb{Z}^r}(G).
$$

Moreover, $\mathfrak{X}_{\mathbb{Z}^r}(\text{SL}(n, \mathbb{C}))$ and $\mathfrak{X}_{\mathbb{Z}^r}(\text{Sp}(n, \mathbb{C}))$ are irreducible normal varieties for any $r, n \geq 1$.

**Remark 5.3.** Similar reasoning shows that $\text{Hom}^0(\mathbb{Z}^r, K)/K$, the identity component of the compact character variety, is bijective to $(T \cap K)^r / W$. This statement appears already in [3, Remarks 3 & 4]. Moreover, this implies $\text{Hom}^0(\mathbb{Z}^r, K)/K$ is homeomorphic to $(T \cap K)^r / W$ since they are both compact and Hausdorff.
The character variety $\mathfrak{X}_\Gamma(G)$ has a natural base point, which is the trivial representation defined by $\rho_e(\gamma) := e$, the identity in $G$, for all $\gamma \in \Gamma$ (more precisely, the base point is the class $[\rho_e]$ of $\rho_e$). Let us denote the component of $\mathfrak{X}_\Gamma(G)$ containing this base point by $\mathfrak{X}_\Gamma^e(G)$, and the corresponding component of $\text{Hom}(\Gamma, G)$ by $\text{Hom}^e(\Gamma, G)$. Both of these components are referred to as the identity component.

**Definition 5.4.** Denote by $W_r(G) \subset \text{Hom}(\mathbb{Z}^r, G) \subset G^r$ the subset of commuting $r$-tuples $\rho = (g_1, \ldots, g_r) \in G^r$ such that there is a maximal torus $T \subset G$ with $g_1, \ldots, g_r \in T$.

**Remark 5.5.** One may ask whether one can simultaneously conjugate commuting elements in a reductive group to a single maximal torus. This works for $\text{SL}(n, \mathbb{C})$, but fails in general. For instance consider $\text{diag}(-1, -1, 1)$ and $\text{diag}(1, -1, -1)$ in $\text{SO}(3)$. They commute, and the first is in $\text{SO}(2)$ and the second is in $\{1\} \times \text{SO}(2)$, but the two elements cannot be simultaneously conjugated by $\text{SO}(3)$ into the same maximal torus of $\text{SO}(3)$.

Recall the definition of $\mathfrak{X}_{\mathbb{Z}^r}(G)$ before Theorem 5.2. From Equation (5.1), we deduce that $W_r(G) = \pi^{-1}(\mathfrak{X}_{\mathbb{Z}^r}(G))$ where $\pi : \text{Hom}(\mathbb{Z}^r, G) \to \mathfrak{X}_\Gamma(G)$ is the quotient morphism. Therefore, $W_r(G)/G = \mathfrak{X}_{\mathbb{Z}^r}^e(G)$.

Since $\mathfrak{X}_{\mathbb{Z}^r}^0(G)$ is irreducible and contains the identity it is connected, and thus by definition $\mathfrak{X}_{\mathbb{Z}^r}^0(G) \subset \mathfrak{X}_{\mathbb{Z}^r}^e(G)$.

**Lemma 5.6.** $W_r(G) \subset \text{Hom}^e(\mathbb{Z}^r, G)$.

**Proof.** If $\rho \in W_r(G)$, let $T$ be such that $\rho = (g_1, \ldots, g_r) \in T^r$. Since $T$ is connected, just let $\gamma_1, \ldots, \gamma_r$ be paths inside $T$ joining $g_1$ to $e \in G$, ..., $g_r$ to $e$.

These paths will form, component-wise, a path of commuting $r$-tuples inside $T^r$ joining $\rho$ to $(e, \ldots, e)$. So, $\rho$ is in $\text{Hom}^e(\mathbb{Z}^r, G)$.

Notice that $\mathfrak{X}_{\mathbb{Z}^r}(G) = \mathfrak{X}_{\mathbb{Z}^r}^0(G)$ if we also had $\text{Hom}^e(\mathbb{Z}^r, G) \subset W_r(G)$, by Lemma 5.6. This leads naturally to the following problem.

**Problem 5.7.** Determine which $r$ and $G$ give the equality $\mathfrak{X}_{\mathbb{Z}^r}(G) = \mathfrak{X}_{\mathbb{Z}^r}^0(G)$.

In [42] it is shown that this equality holds when $r = 1, 2$ and $G$ is connected reductive (see also [37]), or for any $r$ when $G$ equals $\text{SL}(n, \mathbb{C})$, $\text{GL}(n, \mathbb{C})$ or $\text{Sp}(n, \mathbb{C})$. More generally, we prove below that this equality holds, for any $r$, when the derived group of $G$ is a product of $\text{SL}(n, \mathbb{C})$’s and $\text{Sp}(n, \mathbb{C})$’s (see Corollary 5.14).

### 5.2. Irreducibility of $\mathfrak{X}_\Gamma(G)$ for semisimple connected $G$.

Theorem 5.1 lists the cases where $\mathfrak{X}_\Gamma(G)$ is connected with $G$ semisimple connected and $\Gamma$ free Abelian. To generalize the result to the case of a general Abelian $\Gamma$, we need the following lemma.

**Lemma 5.8.** If $\Gamma$ is a finitely generated Abelian group which is not free (i.e., it has some torsion), then $\mathfrak{X}_\Gamma(G)$ is not path-connected.
Proof. Since $G$ is connected the conclusion follows from Proposition 4.13 if $\text{Hom}(\Gamma, G)$ is disconnected. Say $\Gamma$ has non-trivial torsion. Then, it can be written in the form $\Gamma = \mathbb{Z}_d \oplus \Gamma'$, for some $d \in \mathbb{N} \setminus \{0, 1\}$, and denote by $\delta \in \Gamma$ a generator of the $\mathbb{Z}_d$-summand, so that $\delta^d = 1$. Denote by $\rho_0 \in \text{Hom}(\Gamma, G)$ the trivial representation $\rho_0(\gamma) = e \in G$ for all $\gamma \in \Gamma$. In order to define a general representation $\rho$ of $\Gamma$ into $G$, we just need to assign the value at $\delta$, say $\rho(\delta) = g \in G$, satisfying $g^d = e$, together with $\rho' \in \text{Hom}(\Gamma', G)$, as long as $g$ commutes with every $\rho'(\gamma')$ for $\gamma' \in \Gamma'$.

Assume now that $G$ is faithfully embedded as an algebraic representation in $\text{GL}(n, \mathbb{C})$, in such a way that a maximal torus $T \subset G$ is a subtorus of $\Delta_n$, the torus of diagonal elements in $\text{GL}(n, \mathbb{C})$.

We can always arrange for this since a maximal torus is a connected maximal Abelian subgroup and thus contained in a connected maximal Abelian subgroup of $\text{GL}(n, \mathbb{C})$, a maximal torus, and all maximal tori in $\text{GL}(n, \mathbb{C})$ are conjugate. So let

$$\varphi : T \to \Delta_n$$

$$h \mapsto (\varphi_1(h), \ldots, \varphi_n(h))$$

be the corresponding group homomorphism. Since $\varphi$ is an embedding, at least one of the components is nontrivial; without loss of generality let it be $\varphi_1 : T \to \mathbb{C}^*$. Thus $\varphi_1$ is an algebraic character of $T$ and thus has the form $t_1^{m_1} \cdots t_n^{m_n}$. This means, since the character is non-trivial, that $\varphi_1$ is surjective, and so there is $h_1 \in T$ such that $\varphi_1(h_1) = e^{2\pi i}$. Now, the assignment $\rho_1(\delta) = h_1$ and $\rho_1(\gamma) = e$ for all $\gamma \in \Gamma'$ defines a representation $\rho_1 \in \text{Hom}(\Gamma, G)$, since $\rho_1(\delta^d) = h_1^d = 1 \in \mathbb{C}^*$ (and $e$ commutes with everything).

Suppose that there was a continuous path $\rho_t$, $t \in [0, 1]$ from $\rho_0$ to $\rho_1$. Then, as a function of $t$, $(\varphi_1 \circ \rho_t)(\delta) : [0, 1] \to \mathbb{C}^*$ is continuous, and has values on $\mathbb{Z}_d \subset \mathbb{C}^*$ the set of $d$-th roots of unity, since $(\varphi_1 \circ \rho_t)(\delta^d) = ((\varphi_1 \circ \rho_t)(\delta))^d = 1$. Since a continuous map sends the connected interval $[0, 1]$ to a connected set, but the image of $(\varphi_1 \circ \rho_t)(\delta)$ is disconnected (as $(\varphi_1 \circ \rho_0)(\delta) = 1$ and $(\varphi_1 \circ \rho_1)(\delta) = e^{2\pi i} \neq 1$), this contradiction shows that $\text{Hom}(\Gamma, G)$ is not connected.

In the special case of Abelian reductive groups, we are able to compute exactly the number of path components. For related results on counting components, see Corollary 3.4 in [2].

**Lemma 5.9.** Let $T$ be a complex reductive Abelian connected group of dimension $m$ (or its maximal compact subgroup), and let $\Gamma = \mathbb{Z}^* \oplus \bigoplus_{j=1}^{t} \mathbb{Z}_{n_j}$. Then $|\text{Hom}(\Gamma, T)| = \prod_{j=1}^{t} n_j^m$.

**Proof.** Given two finitely generated Abelian groups $A, B$, and an Abelian group $T$, we have $\text{Hom}(A \oplus B, T) \cong \text{Hom}(A, T) \times \text{Hom}(B, T)$. Given two Lie groups $T_1, T_2$ we also have $\text{Hom}(\Gamma, T_1 \times T_2) \cong \text{Hom}(\Gamma, T_1) \times \text{Hom}(\Gamma, T_2)$. If $T$ is a complex Abelian reductive connected group of dimension $m$, then $T$ is isomorphic to a
complex torus, so \( T \cong (\mathbb{C}^*)^m \) with maximal compact subgroup \((S^1)^m\). Either way, \( \text{Hom}(\mathbb{Z}, T) \cong T \). We obtain:

\[
\text{Hom}(\Gamma, T) \cong T^s \times \text{Hom}(\bigoplus_{j=1}^t \mathbb{Z}_{n_j}, (\mathbb{C}^*)^m) \cong T^s \times \prod_{j=1}^t (\text{Hom}(\mathbb{Z}_{n_j}, \mathbb{C}^*))^m.
\]

Finally, because \( T \) is connected and \( |\text{Hom}(\mathbb{Z}, \mathbb{C}^*)| = n \) (for any \( n \in \mathbb{N} \), the set of \( n \)-th roots of unity has cardinality \( n \)), we obtain the desired formula. \( \square \)

Now we are ready to extend Theorem 5.1 as follows.

**Theorem 5.10.** Let \( G \) be a semisimple connected algebraic group over \( \mathbb{C} \) and \( \Gamma \) a finitely generated Abelian group of rank \( r \). Then \( \mathfrak{X}_\Gamma(G) \) is path connected if and only if:

1. \( \Gamma \) is free, and
2. \( r = 1 \), \( r = 2 \) and \( G \) is simply connected, or \( r \geq 3 \) and \( G \) is a product of \( \text{SL}(n, \mathbb{C})'s \) and \( \text{Sp}(n, \mathbb{C})'s \).

**Proof.** Corollary 5.1 shows the statement for \( \Gamma = \mathbb{Z}^r \). Now, suppose that \( \Gamma \) is not free. Then \( \mathfrak{X}_\Gamma(G) \) cannot be connected, by the preceding Lemma 5.8, so the result follows. \( \square \)

Finally we can show Theorem 1.2 which says when \( G \) is semisimple and connected and \( \Gamma \) is finitely generated and Abelian, then \( \mathfrak{X}_\Gamma(G) \) is an irreducible variety if and only if \( \Gamma \) is free and either \( r = 1 \), \( r = 2 \) and \( G \) is simply connected, or \( r \geq 3 \) and \( G \) is a product of \( \text{SL}(n, \mathbb{C})'s \) and \( \text{Sp}(n, \mathbb{C})'s \).

**Proof.** [Proof of Theorem 1.2] If a variety is irreducible, it is path-connected [40, 41]. So, by Theorem 5.10, the given conditions are necessary.

Conversely, assume that \( \Gamma \) is free, and either \( r = 1 \), \( r = 2 \) and \( G \) is simply connected, or \( r \geq 3 \) and \( G \) is a product of \( \text{SL}(n, \mathbb{C})'s \) and \( \text{Sp}(n, \mathbb{C})'s \). In [46], the case \( r = 1 \) is shown to be irreducible; in [37] the \( r = 2 \) and \( G \) simply-connected case is shown to be irreducible. Finally, in the very recent pre-print [42] the cases \( r \geq 3 \) and \( G = \text{SL}(n, \mathbb{C}) \) or \( G = \text{Sp}(n, \mathbb{C}) \) are shown to be irreducible. However, by Proposition 2.8 if \( G \) is a product of \( \text{SL}(n, \mathbb{C})'s \) and \( \text{Sp}(n, \mathbb{C})'s \) then \( \mathfrak{X}_\mathbb{Z}^r(G) \) decomposes into a product of \( \mathfrak{X}_\mathbb{Z}^r(\text{SL}(n, \mathbb{C}))'s \) and \( \mathfrak{X}_\mathbb{Z}^r(\text{Sp}(n, \mathbb{C}))'s \). Since the product of irreducible varieties is irreducible, we conclude that the last case is irreducible too. \( \square \)

**Remark 5.11.** As is shown in [11], a complex affine algebraic set \( V \) is an irreducible algebraic variety if and only if \( V \) contains a connected dense open subset which is smooth. Thus, in the context of this last Theorem, we conclude that the smooth points of \( \mathfrak{X}_\mathbb{Z}^r(G) \) are path-connected without exhibiting a smooth path between smooth points.

We now can settle a question implicit in [42].

**Corollary 5.12.** Let \( G \) be a complex exceptional Lie group. If \( r \geq 3 \), then \( \mathfrak{X}_\mathbb{Z}^r(G) \) is not irreducible, nor path-connected. If \( r = 2 \), then \( \mathfrak{X}_\mathbb{Z}^r(G) \) is irreducible (respectively, path-connected) if and only if \( G \) is \( E_8, F_4, \) or \( G_2 \).
The complex exceptional Lie groups are simple, and hence semisimple and connected. Moreover, being simple and exceptional implies they are not products of $\text{SL}(n, \mathbb{C})$’s and $\text{Sp}(n, \mathbb{C})$’s. It is also known that $E_8, F_4, \text{and } G_2$ are simply-connected, while $E_6$ and $E_7$ are not. Therefore, Theorems 5.10 and 1.2 give the result. □

5.3. Irreducibility of $X_G(G)$ for connected reductive $G$. Suppose now that $G$ is a connected reductive group. One can represent any element $g \in G$ as $g = th$ where $t$ belongs to $Z^0$, the connected component of the center of $G$, and $h$ is in $DG$, the derived group of $G$ (see [33, page 200]). By definition $DG = [G, G]$ is the group of commutators of elements in $G$, and it is well known that $DG$ is a connected semisimple algebraic group.

Moreover, the canonical morphism $\tilde{G} := Z^0 \times DG \to G$, defined by the product is what is called a central isogeny (see [10, Cor 5.3.3]). This means that the kernel of the morphism above is a finite subgroup $F$ of the center of $G$. Recall also that, because $G$ is reductive, the radical of $G$ coincides with $Z^0$ and this is a torus, hence isomorphic to some $(\mathbb{C}^*)^d$, $d \geq 0$. So, we have an exact sequence

$$1 \to F \to \tilde{G} = (\mathbb{C}^*)^d \times DG \to G \to 1.$$ 

Suppose now $\Gamma = \langle \gamma_1, ..., \gamma_r \rangle \cong \mathbb{Z}^r$, a free Abelian group, and let $\rho \in \text{Hom}(\Gamma, G)$. Using the sequence above, for every $a_i = \rho(\gamma_i)$ there exists $\tilde{a}_i = (t_i, h_i) \in \tilde{G} = Z^0 \times DG$ such that $a_i = t_i h_i$. Since $\{t_1, ..., t_r\}$ is in the center of $G$, and since $\{a_1, ..., a_r\}$ all commute with each other, we conclude that for all $1 \leq i, j \leq r$ $t_i t_j h_i h_j = a_i a_j = a_j a_i = t_i t_j h_i h_j$, and thus $h_i h_j = h_j h_i$. Therefore, we obtain $\tilde{\rho} = (\tilde{a}_1, ..., \tilde{a}_r) \in \text{Hom}(\Gamma, \tilde{G})$. Now because $F$ is Abelian (a finite subgroup of $Z$), $\text{Hom}(\Gamma, F) = F^r$ is indeed an Abelian group itself.

Since $F$ is central, there is a natural action of $\text{Hom}(\Gamma, F)$ on $\text{Hom}(\Gamma, \tilde{G})$ and indeed, one can easily check that

$$\text{Hom}(\Gamma, G) = \text{Hom}(\Gamma, \tilde{G})/\text{Hom}(\Gamma, F)$$

$$\cong \left(\text{Hom}(\Gamma, (\mathbb{C}^*)^d) \times \text{Hom}(\Gamma, DG)\right)/\text{Hom}(\Gamma, F)$$

as affine algebraic varieties.

Since $\text{Hom}(\mathbb{Z}^r, (\mathbb{C}^*)^d) = (\mathbb{C}^*)^{dr}$ is irreducible, the Cartesian product of irreducible varieties is irreducible, and the quotient of an irreducible variety by a reductive group (this includes the finite group case) is irreducible, formula (5.2) immediately implies the following.

**Lemma 5.13.** Let $G$ be a connected reductive group. If $\text{Hom}(\mathbb{Z}^r, DG)$ is irreducible, then the same holds for $\text{Hom}(\mathbb{Z}^r, G)$ and $X_{\mathbb{Z}^r}(G)$.

Now, we present a partial generalization our main theorem to the case when $G$ is connected and reductive.

**Corollary 5.14.** Let $G$ be a connected reductive group. Suppose either $r = 1$, $r = 2$ and $DG$ is simply connected, or $r \geq 3$ and $DG$ is a product of $\text{SL}(n, \mathbb{C})$’s and $\text{Sp}(n, \mathbb{C})$’s. Then $X_{\mathbb{Z}^r}(G)$ is irreducible.
Proof. As above, writing $G$ as $((\mathbb{C}^*)^d \times DG)/F$, we have:
\[
\mathfrak{x}_{\mathbb{Z}^r}(G) = \Hom(\mathbb{Z}^r, G) \mod G \\
= [(\Hom(\mathbb{Z}^r, (\mathbb{C}^*)^d) \times \Hom(\mathbb{Z}^r, DG))/\Hom(\mathbb{Z}^r, F)] \mod G \\
\cong [(\mathbb{C}^*)^{dr} \times (\Hom(\mathbb{Z}^r, DG)/\Hom(\mathbb{Z}^r, F)] \\
\cong [(\mathbb{C}^*)^{dr} \times \mathfrak{x}_{\mathbb{Z}^r}(DG)] / \Hom(\mathbb{Z}^r, F)
\]

In the equation above, the fact that the action of the finite central group $\Hom(\mathbb{Z}^r, F)$ commutes with the conjugation action by $G$ justifies both the interchange of the two quotients, and the fact that $G$ only acts non-trivially on the factor $\Hom(\mathbb{Z}^r, DG)$. Note also that $G = DG \cdot Z^0$ acts on $DG$ by conjugation in the natural way: writing $g = g_D \cdot g_0$ with $g_D \in DG$ and $g_0 \in Z^0$, we have $g \cdot h := ghg^{-1} = g_D h(g_D)^{-1}$. This justifies the identification of quotient spaces $\Hom(\mathbb{Z}^r, DG)/\mod G = \mathfrak{x}_{\mathbb{Z}^r}(DG)$.

Now by Theorem 1.2 in either of the three assumed situations, $\mathfrak{x}_{\mathbb{Z}^r}(DG)$ is irreducible. So, both the Cartesian product $(\mathbb{C}^*)^{dr} \times \mathfrak{x}_{\mathbb{Z}^r}(DG)$ and $\mathfrak{x}_{\mathbb{Z}^r}(G)$ (its quotient by $\Hom(\mathbb{Z}^r, F)$) are irreducible. \qed

Remark 5.15. Note that Corollary 5.14 only applies to free Abelian groups $\Gamma$, and moreover, it is not clear whether the converse direction holds. Namely, if $\mathfrak{x}_{\mathbb{Z}^r}(G)$ is irreducible, it does not necessarily follow that the derived group $DG$ is a product of special linear and symplectic groups (see also Remark 5.17).

5.4. Further applications to algebra and topology. Let $C_{r,n}$ denote the algebraic subvariety of $\mathbb{C}^{rn^2}$ consisting of commuting $r$-tuples of $n \times n$ complex matrices. It is known that $C_{r,n}$ is irreducible when $r \leq 2$ (for any $n$) and that it is reducible (not irreducible) when $r, n \geq 4$ (see [21, 23]). The irreducibility for some cases when $r = 3$ is still an open question, as far as we know.

Proposition 5.16. The representation variety $\Hom(\mathbb{Z}^r, \SL(n, \mathbb{C}))$ is irreducible if and only if $C_{r,n}$ is irreducible.

Proof. Denote an element of $C_{r,n}$ by $A = (A_1, \cdots, A_r)$ so that $A_1, \cdots, A_r$ are $n \times n$ matrices satisfying $A_i A_j = A_j A_i$ for all $i, j = 1, \ldots, r$. The following simple construction relates $C_{r,n}$ with $\Hom(\mathbb{Z}^r, \SL(n, \mathbb{C}))$. Consider the map
\[
\psi : C_{r,n} \to \mathbb{C}^r \\
(A_1, \cdots, A_r) \mapsto (\det A_1, \cdots, \det A_r).
\]

By evaluating a homomorphism $\rho \in \Hom(\mathbb{Z}^r, \SL(n, \mathbb{C}))$ on a set of commuting generators of $\mathbb{Z}^r$, it is clear that we have an isomorphism of varieties
\[
\Hom(\mathbb{Z}^r, \SL(n, \mathbb{C})) \cong \psi^{-1}(1, \cdots, 1).
\]

Note that $C_{r,n}$ is invariant by dilation, that is, if $(A_1, \cdots, A_r) \in C_{r,n}$, then $(\lambda A_1, \cdots, \lambda A_r) \in C_{r,n}$, for any scalar $\lambda \in \mathbb{C}$. It follows that $\psi$ is surjective $\psi(C_{r,n}) = \mathbb{C}^r$. Now suppose that $C_{r,n}$ is irreducible. Then, by Bertini’s theorem ([40], vol I, p. 139) there is a Zariski open subset $U \subset \mathbb{C}^r$ such that $\psi^{-1}(y)$ is...
irreducible for \( y \in U \). It is not obvious a priori that \((1, \cdots, 1) \in U\), but we can observe the fact that all fibers of \( \psi \) of the form \( \psi^{-1}(z_1, \cdots, z_r) \), with \( z_1 \cdots z_r \neq 0 \), are indeed isomorphic as algebraic varieties. This, together with the fact that \( U \) is dense, shows that \( \text{Hom}(Z^r, \text{SL}(n, \mathbb{C})) \cong \psi^{-1}(1, \cdots, 1) \) is irreducible. Conversely, suppose \( \text{Hom}(Z^r, \text{SL}(n, \mathbb{C})) \) is irreducible. Then, the representation variety \( \text{Hom}(Z^r, \text{GL}(n, \mathbb{C})) \) is also irreducible. Indeed, \( \text{SL}(n, \mathbb{C}) \) is the derived group of \( \text{GL}(n, \mathbb{C}) \), so this follows from Lemma 5.13. Now note that \( \text{Hom}(Z^r, \text{GL}(n, \mathbb{C})) \) is naturally an open dense subset in \( C_{\mathbb{R}} \) of \( \text{GL}(n, \mathbb{C}) \) on a set of generators of \( Z \) on the whole of \( C \).\[\]Remark 5.17. In general, if \( V \) is an irreducible \( G \)-variety, then \( V//G \) is irreducible. Since, by Theorem 1.2, \( \mathfrak{X}_{Z^r}(G) = \text{Hom}(Z^r, G)//G \) is always irreducible, for \( G = \text{SL}(n, \mathbb{C}) \), the above provides a non-trivial example showing the converse is generally not true. Note also that the argument above shows that both representation varieties \( \text{Hom}(Z^r, \text{SL}(n, \mathbb{C})) \) and \( \text{Hom}(Z^r, \text{GL}(n, \mathbb{C})) \) are either simultaneously irreducible or reducible.\[\]

**Corollary 5.18.** The ring \( \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))] \) contains zero divisors for \( r, n \geq 4 \) but the invariant subring \( \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))]_{\text{SL}(n, \mathbb{C})} \) does not.\[\]

**Proof.** The variety \( C_{\mathbb{R}} \) of commuting \( r \)-tuples of \( n \times n \) matrices is reducible when \( r, n \geq 4 \) by [21] [23]. From Proposition, it follows that the same happens for \( \text{Hom}(Z^r, \text{SL}(n, \mathbb{C})) \), which means that the ring \( \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))] \) contains zero divisors. By Theorem 2, \( \mathfrak{X}_{Z^r}(\text{SL}(n, \mathbb{C})) = \text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))/\text{SL}(n, \mathbb{C}) \) is always irreducible, so the invariant ring \( \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))]_{\text{SL}(n, \mathbb{C})} \) has no zero divisors. \[\]

**Remark 5.19.** Note this implies that given two non-zero polynomials \( f, g \in \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))] \) satisfying \( fg = 0 \) and one of them, say \( f \), is invariant under \( \text{SL}(n, \mathbb{C}) \), then \( g \notin \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))]_{\text{SL}(n, \mathbb{C})} \). Moreover, under these assumptions, letting \( R: \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))] \to \mathbb{C}[\text{Hom}(Z^r, \text{SL}(n, \mathbb{C}))]_{\text{SL}(n, \mathbb{C})} \) be the Reynold’s operator (see [14]), then \( R(g) = 0 \) since \( 0 = R(0) = R(fg) = f R(g) \).\[\]

Another consequence of our main Theorem concerns the fundamental group of \( \mathfrak{X}_{Z^r}(G) \).\[\]

**Corollary 5.20.** Let \( G \) be a connected reductive complex algebraic group, and let \( \mathfrak{X}_{Z^r}(G) \subset \mathfrak{X}_{Z^r}(G) \) be the identity component. Then,\[\]

1. If \( G \) is simply connected, then \( \mathfrak{X}_{Z^r}(G) \) is simply connected;\[\]
2. If \( G \) is a product of \( \text{SL}(n, \mathbb{C})'s \) and \( \text{Sp}(n, \mathbb{C})'s \), then \( \mathfrak{X}_{Z^r}(G) \) is simply connected.
Proof. (1) Let $K$ be a maximal compact subgroup of $G$. In [22], it is shown that the fundamental group of the identity component of $\text{Hom}(\mathbb{Z}^k, K)$ is $\pi_1(K)^r$. Thus if $G$ is simply-connected, then so is $K$ and consequently the identity component of $\text{Hom}(\mathbb{Z}^k, K)$ is simply-connected. In [3] it is shown that if a connected compact group acts on a simply connected space $X$, then $X/K$ is also simply-connected. We therefore conclude that $\mathcal{X}_{\mathbb{Z}^r}(K)$ is simply-connected. Theorem 1.1 then implies that $\mathcal{X}_{\mathbb{Z}^r}(G)$ is simply-connected as well.

(2) Suppose $G$ is a product of $\text{SL}(n, \mathbb{C})$’s and $\text{Sp}(n, \mathbb{C})$’s. From Theorem 1.2 $\mathcal{X}_{\mathbb{Z}^r}(G) = \mathcal{X}_{\mathbb{Z}^r}(G)$, so the result follows from (1) and the fact that both $\text{SL}(n, \mathbb{C})$ and $\text{Sp}(n, \mathbb{C})$ are simply connected.

We finish with a result about the rational cohomology of $\mathcal{X}_{\mathbb{Z}^r}(G)$. For that we introduce some notation. Consider $n, r \geq 1$ and $rn$ variables $\alpha_i^j$, $1 \leq i \leq r$ and $1 \leq j \leq n$, and let $\Lambda_\mathbb{Q}[\alpha_1^1, ..., \alpha_n^n]$ be the exterior algebra over $\mathbb{Q}$ on these variables. Let the symmetric group on $n$ letters, $S_n$, act on this algebra by simultaneously permuting the lower indices in the symbols $\alpha_i^j$ (i.e., $\sigma \in S_n$ acts by $\alpha_i^j \mapsto \alpha_{\sigma(i)}^j$).

Theorem 5.21. The cohomology ring $H^*(\mathcal{X}_{\mathbb{Z}^r}(\text{GL}(n, \mathbb{C})); \mathbb{Q})$ is isomorphic to the invariant ring $\Lambda_\mathbb{Q}[\alpha_1^1, ..., \alpha_n^n]^{S_n}$.

Proof. As shown in [16], $\mathcal{X}_{\mathbb{Z}^r}(U(n)) \cong T^r/W$ where $W$ is the Weyl group of $U(n)$ and $T \cong (S^1)^n$ is a $n$-torus (maximal torus in $U(n)$). Therefore, $H^*(\mathcal{X}_{\mathbb{Z}^r}(\text{GL}(n, \mathbb{C})); \mathbb{Q}) \cong H^*(T^r/W; \mathbb{Q})$ by Theorem 1.1 (see also Theorem 5.2 and Remark 5.3). Since $W$ is finite and $T^r$ is compact, results in [9] imply $H^*(T^r/W; \mathbb{Q}) \cong H^*(T^r; \mathbb{Q})^W$. Then in [24] it is shown that $H^*(T^r; \mathbb{Q}) = H^*((S^1)^n; \mathbb{Q})$ is ring isomorphic to the exterior algebra $\Lambda_\mathbb{Q}[\alpha_1^1, ..., \alpha_n^n]$ where $\alpha_i^j$ can be explicitly described as the generators of $H^1(S^1; \mathbb{Q})$ induced by $S^1 \hookrightarrow T = (S^1)^n \subset U(n) \subset \text{GL}(n, \mathbb{C}) \hookrightarrow \text{GL}(n, \mathbb{C})^r$, with $i$ and $j$ labeling the inclusions into the corresponding factors. Note that the Weyl group of $U(n)$ (or $\text{GL}(n, \mathbb{C})$) is isomorphic to $S_n$ and its action on $T^r$ is precisely the action described above. Putting these facts together, we conclude that the cohomology ring $H^*(\mathcal{X}_{\mathbb{Z}^r}(\text{GL}(n, \mathbb{C})); \mathbb{Q})$ is isomorphic to the invariant ring $\Lambda_\mathbb{Q}[\alpha_1^1, ..., \alpha_n^n]^{S_n}$.

Similar results hold for other groups $G$ in the cases when $\mathcal{X}_{\mathbb{Z}^r}(G)$ is connected; that is, for $G = \text{SL}(n, \mathbb{C})$ and $\text{Sp}(n, \mathbb{C})$.

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APPENDIX A. PROOFS OF LEMMATA 4.1 AND 4.5

A.1. Proof of Lemma 4.1 We first show that $\delta$ is well-defined. Let $h_1$ and $h_2$ be such that $d_i := h_i g h_i^{-1} \in \Delta_n$. Then $(h_1 h_2^{-1}) d_2 (h_1 h_2^{-1})^{-1} = d_1$. Two elements in a torus are conjugate if and only if they are conjugate by an element of the Weyl group $W$ (see [30]). Thus, $w := h_1 h_2^{-1} \in W$. Recall the definition of $\delta_t$ in (4.2) and that $\delta_t|_{\Delta_n}$ coincides with $\sigma_t$ as defined in (4.1) and so, it is $W$-equivariant for all $t$. Thus:

$$
\delta_t(g) = h_1^{-1} \sigma_t(h_1 g h_1^{-1}) h_1 = h_1^{-1} \sigma_t(d_1) h_1 \\
= h_1^{-1} \sigma_t(w d_2 w^{-1}) h_1 \\
= h_1^{-1} w \sigma_t(d_2) w^{-1} h_1 \\
= h_2^{-1} \sigma_t(h_2 g h_2^{-1}) h_2
$$

as required.

Secondly, $\delta_t(g) \in \text{SL}(n, \mathbb{C})_s$ for all $t \in [0, 1]$ and $g \in \text{SL}(n, \mathbb{C})_s$ since conjugates of elements in $\Delta_n$ are semisimple. Clearly, $\sigma$ is continuous, which implies $\delta$ is continuous as well.

Next, we show $\delta_t : \text{SL}(n, \mathbb{C})_s \rightarrow \text{SL}(n, \mathbb{C})_s$ is conjugation equivariant. Let $l \in \text{SL}(n, \mathbb{C})$ and suppose that $h g h^{-1} \in \Delta_n$. Then, $h l^{-1}$ diagonalizes $l g l^{-1}$ (as $(h l^{-1}) l g l^{-1} (h l^{-1})^{-1} = h g h^{-1}$ is diagonal), and since $\delta$ is well-defined, we compute:

$$
\delta_t(l g l^{-1}) = (h l^{-1})^{-1} \sigma_t(h g h^{-1}) h l^{-1} \\
= l h^{-1} \sigma_t(h g h^{-1}) h l^{-1} \\
= l \delta_t(g) l^{-1},
$$

as wanted.

To prove that $\delta$ is a strong deformation retraction, it remains to prove (a) for all $g \in \text{SL}(n, \mathbb{C})_s$, $\delta_0(g) = g$ and $\delta_1(g) = h k h^{-1}$ for some $k \in \text{SU}(n)$ and $h \in \text{SL}(n, \mathbb{C})$, and (b) for any $g \in \text{SL}(n, \mathbb{C})$ and $k \in \text{SU}(n)$, then $\delta_t(g k g^{-1}) = g k g^{-1}$ for all $t$.

Item (a) is immediate since the definition of $\delta_t$ immediately implies $\delta_0$ is the identity on semisimple elements and that $\delta_1$ maps into $\text{SL}(n, \mathbb{C})$-conjugates of $\text{SU}(n)$. Now we prove item (b). Note that $\delta_t$ is the identity on $\text{SU}(n)$ since whenever $k = h^{-1} \text{diag}(\ldots, z_j, \ldots) h$ is unitary, we have $|z_j| = 1$ for all $1 \leq j \leq n$ (showing $|z_j|^{-t} = 1$ for all $t$). Thus, $\delta_t(g k g^{-1}) = g \delta_t(k) g^{-1} = g k g^{-1}$ by equivariance of $\delta_t$. This completes the proof that $\delta$ is a strong deformation retraction.

Lastly, we prove $\delta_t$ preserves commutativity. Suppose $g_1 g_2 = g_2 g_1$. However, since they are diagonalizable, and are in $\text{SL}(n, \mathbb{C})$, we know there is an element $h \in \text{SL}(n, \mathbb{C})$ which simultaneously triangulizes them (see [13]). However, since conjugation preserves semisimplicity and non-diagonal upper triangular matrices are not semisimple, we conclude that $h$ simultaneously diagonalizes $g_1$ and $g_2$.
Then $\delta_i(g_1)$ and $\delta_i(g_2)$ commute if and only if $h\delta_i(g_1)h^{-1}$ and $h\delta_i(g_2)h^{-1}$ commute. It suffices to show that $\delta_i(hg_1h^{-1})$ and $\delta_i(hg_2h^{-1})$ commute for all $t$ by equivariance. However, both $\delta_i(hg_1h^{-1})$ and $\delta_i(hg_2h^{-1})$ are in $\Delta_n$ for all $t$, and thus they commute.

A.2. Proof of Lemma 4.5. Keeping the notation of Section 4, we start with a simple lemma.

**Lemma A.1.** Let $H \cong (\mathbb{C}^*)^k$ be a subtorus of $\Delta_n \cong (\mathbb{C}^*)^{n-1}$ (so $k \leq n - 1$). Then $p_m^{-1}(H) \subset \Delta_n$ is the disjoint union of a finite number of affine subvarieties, all of which are isomorphic to $H$ (as varieties, but not as groups). More precisely, every connected component of $p_m^{-1}(H)$ is a coset $gH$, for some $g \in p_m^{-1}(H)$.

**Proof.** $gH$ are all in $p_m^{-1}(H)$ for all $g$ in $p_m^{-1}(H)$ since the inverse image is contained in $\Delta_n$ (it is Abelian). In fact, $p_m^{-1}(H)$ is a subgroup for this reason. Therefore it is reductive since $p_m$ is algebraic (and thus continuous), and inverse images of Zariski closed sets under algebraic maps are Zariski closed. However, Zariski closed subgroups of a torus are reductive since the radical must be a torus, and therefore $p_m^{-1}(H)$ is a reductive subgroup and so has finitely many components. Since cosets are disjoint and homeomorphic, each is closed in $\Delta_n$ since $H$ is closed in $\Delta_n$. On the other hand, each is open in the subspace topology and so are components. Only the identity coset is a group in and of itself, the rest are algebraic subsets (closed cosets are algebraic).

Now we conclude the proof of Lemma 4.5.

**Proof.** (1) Let $g \in G_{ss}$ and assume $g \in G_f$ with $f^m = 1$. Then, as in Lemma 4.4 there is $h \in G_0$ so that $hg^m h^{-1} \in T_0 \subset \Delta_n$. By definition of $\delta$ and property (6) of Lemma 4.1, we have $(\delta_t(g))^m = \delta_t(g^m) = h^{-1}\sigma_t(hg^m h^{-1} h)$. So,

$$\sigma_t(hg^m h^{-1}) = h(\delta_t(g))^m h^{-1} = (h\delta_t(g)h^{-1})^m. \quad (A.1)$$

Next, we prove $\sigma_t$ preserves $T_0$ for all $t \in [0, 1]$. Indeed, this deformation is nothing but the usual polar deformation restricted to the torus $T_0$, which therefore stays in $T_0$; see page 117 in [30]. The main idea is this: the torus $T_0$ is algebraically cut out of the diagonals $\Delta_n$, however since $T_0$ is a group and $\text{diag}(e^{i\theta_1}, ..., e^{i\theta_n}) \in K_0$ we know that all positive multiples of $\text{diag}(r_1, ..., r_n)$ are in $T_0$ which implies all positive multiples satisfy the relations. This then implies that $\text{diag}(r_1^{1-t}, ..., r_n^{1-t})$ satisfy the relations for all $t$, which in turn implies $\text{diag}(r_1^{1-t} e^{i\theta_1}, ..., r_n^{1-t} e^{i\theta_n})$ is in $T_0$ for all $t$.

Now, we use the fact that $\sigma_t$ preserves $T_0$ for all $t \in [0, 1]$ to conclude (from Equation (A.1)) that for all $t$, $(h\delta_t(g)h^{-1})^m \in T_0$, and therefore $h\delta_t(g)h^{-1} \in p_m^{-1}(T_0)$ for all $t$. By continuity of $\delta_t$ with respect to the parameter $t$, we conclude that $h\delta_t(g)h^{-1}$ is always in the same connected component of $p_m^{-1}(T_0)$ for all $t$; denote this component by $T_1$. Now, by Lemma A.1 every connected component of $p_m^{-1}(T_0)$ is of the form $gT_0$ for some $\tilde{g} \in p_m^{-1}(T_0)$. 




Since \( hgh^{-1} = h\delta_0(g)h^{-1} \in T_1 \subset p^{-1}_m(T_0) \), we may take \( \tilde{g} = hgh^{-1} \in G \). Therefore, the coset \( T_1 = \tilde{g}T_0 = hgh^{-1}T_0 \subset G \). More generally, this means that any connected component of \( p^{-1}_m(T_0) \) which contains a point of \( G \), is completely contained inside \( G \). So, we conclude that \( h\delta_t(g)h^{-1} \in G \), which means that \( \delta_t(g) \in G \) for all \( t \).

Finally \( \delta_t(g) \in G_{ss} \) since it is a \( m \)-th root of a semisimple \( \delta_t(g^m) \), and \( g^m_u = 1 \) implies \( g_u = 1 \) ([6], page 87).

(2) If we let \( t = 1 \) in the formulas above, since conjugates of semisimple elements are semisimple ([6], page 85), we get \( h\delta_1(g)h^{-1} \in G_{ss} \cap SU(n) \subset K \), which means by definition, that \( \delta_1(g) \in G(K) \).

\[ \square \]

References

[1] Alejandro Adem, Frederick R. Cohen, and José Manuel Gómez. Commuting elements in central products of special unitary groups. *Proc. Edinb. Math. Soc. (2)*, 56(1):1–12, 2013.
[2] Alejandro Adem and José Manuel Gómez. On the structure of spaces of commuting elements in compact lie groups. *Configuration Spaces: Geometry, Topology and Combinatorics, Publ. Scuola Normale Superiore*, 14 (CRM Series), 2013 (Birkhauser).
[3] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.
[4] Thomas Baird, Lisa Jeffrey, and Paul Selick. The space of commuting n-tuples in su(2). arXiv:0911.4953v3, 2009.
[5] Thomas John Baird. Cohomology of the space of commuting n-tuples in a compact Lie group. *Algebr. Geom. Topol.*, 7:737–754, 2007.
[6] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
[7] Armand Borel, Robert Friedman, and John W. Morgan. Almost commuting elements in compact Lie groups. *Mem. Amer. Math. Soc.*, 157(747):x+136, 2002.
[8] Steven B. Bradlow, Oscar García-Prada, and Peter B. Gothen. Surface group representations and \( U(p,q) \)-Higgs bundles. *J. Differential Geom.*, 64(1):111–170, 2003.
[9] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.
[10] Brian Conrad. Reductive group schemes (sga3 summer school, 2011), 2011. Available at math.stanford.edu/~conrad/papers/luminysga3.pdf.
[11] Daryl Cooper and Jason Fox Manning. Non-faithful representations of surface groups into \( \text{sl}(2,c) \) which kill no simple closed curve. arXiv:1104.4492.
[12] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
[13] M. P. Drazin. Some generalizations of matrix commutativity. *Proc. London Math. Soc. (3)*, 1:222–231, 1951.
[14] Jean-Marc Drézet. Luna’s slice theorem and applications. In *Algebraic group actions and quotients*, pages 39–89. Hindawi Publ. Corp., Cairo, 2004.
[15] Carlos Florentino and Ana Cristina Casimiro. Stability of affine G-varieties and irreducibility in reductive groups. *Internat. J. Math.*, 23(8):1250082, 30, 2012.
[16] Carlos Florentino and Sean Lawton. The topology of moduli spaces of free group representations. *Math. Ann.*, 345(2):453–489, 2009.
[17] Carlos Florentino and Sean Lawton. Character varieties and the moduli of quiver representations. In *the tradition of Ahlfors-Bers, Papers from the 5th Ahlfors-Bers Colloquium*
held at Rice University, Contemporary Mathematics, American Mathematical Society, to appear.

[18] O. García-Prada and I. Mundet i Riera. Representations of the fundamental group of a closed oriented surface in $\text{Sp}(4, \mathbb{R})$. *Topology*, 43(4):831–855, 2004.

[19] O. García-Prada, P. B. Gothen, and I. Mundet i Riera. The hitchin-kobayashi correspondence, higgs pairs and surface group representations. *arXiv:0909.4487*, 2009.

[20] I. M. Gel’fand and V. A. Ponomarev. Remarks on the classification of a pair of commuting linear transformations in a finite-dimensional space. *Funkcional. Anal. i Priložen.*, 3(4):81–82, 1969.

[21] Murray Gerstenhaber. On dominance and varieties of commuting matrices. *Ann. of Math. (2)*, 73:324–348, 1961.

[22] José Manuel Gómez, Alexandra Pettet, and Juan Souto. On the fundamental group of $\text{Hom}(\mathbb{Z}^k, G)$. *Math. Z.*, 271(1-2):33–44, 2012.

[23] Robert M. Guralnick. A note on commuting pairs of matrices. *Linear and Multilinear Algebra*, 31(1-4):71–75, 1992.

[24] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[25] Tamás Hausel and Michael Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153(1):197–229, 2003.

[26] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[27] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[28] V. G. Kac and A. V. Smilga. Vacuum structure in supersymmetric Yang-Mills theories with any gauge group. In *The many faces of the superworld*, pages 185–234. World Sci. Publ., River Edge, NJ, 2000.

[29] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.*, 1(1):1–236, 2007.

[30] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[31] D. Luna. Sur certaines opérations différentiables des groupes de Lie. *Amer. J. Math.*, 97:172–181, 1975.

[32] Domingo Luna. Slices étales. In *Sur les groupes algébriques*, pages 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. Soc. Math. France, Paris, 1973.

[33] James S. Milne. Basic theory of affine group schemes, 2012. Available at www.jmilne.org/math/.

[34] Shigeru Mukai. *An introduction to invariants and moduli*, volume 81 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.

[35] M. S. Narasimhan and C. S. Seshadri. Holomorphic vector bundles on a compact Riemann surface. In *Differential Analysis, Bombay Colloq., 1964*, pages 249–250. Oxford Univ. Press, London, 1964.

[36] Alexandra Pettet and Juan Souto. Commuting tuples in reductive groups and their maximal compact subgroups. *submitted–see http://www-personal.umich.edu/~apettet/research.html*, 2011.

[37] R. W. Richardson. Commuting varieties of semisimple Lie algebras and algebraic groups. *Compositio Math.*, 38(3):311–327, 1979.

[38] R. W. Richardson. Conjugacy classes of $n$-tuples in Lie algebras and algebraic groups. *Duke Math. J.*, 57(1):1–35, 1988.

[39] Gerald W. Schwarz. The topology of algebraic quotients. In *Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988)*, volume 80 of *Progr. Math.*, pages 135–151. Birkhäuser Boston, Boston, MA, 1989.
[40] Igor R. Shafarevich. Basic algebraic geometry. 1. Springer-Verlag, Berlin, second edition, 1994. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.

[41] Igor R. Shafarevich. Basic algebraic geometry. 2. Springer-Verlag, Berlin, second edition, 1994. Schemes and complex manifolds, Translated from the 1988 Russian edition by Miles Reid.

[42] Adam S. Sikora. Character varieties of abelian groups. arXiv:1207.5284, 2012.

[43] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. Inst. Hautes Études Sci. Publ. Math., (79):47–129, 1994.

[44] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. II. Inst. Hautes Études Sci. Publ. Math., (80):5–79 (1995), 1994.

[45] Juan Souto. A remark on the homotopy equivalence of SU_n and SL_nC. arXiv:1008.0816v1, 2010.

[46] Robert Steinberg. Regular elements of semisimple algebraic groups. Inst. Hautes Études Sci. Publ. Math., (25):49–80, 1965.

[47] William P. Thurston. Three-dimensional geometry and topology. Vol. 1, volume 35 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[48] Edward Witten. Supersymmetric index in four-dimensional gauge theories. Adv. Theor. Math. Phys., 5(5):841–907, 2001.

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