Open String in the Presence of the pp-wave, Linear Dilaton and Kalb-Ramond Backgrounds

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Abstract

We study open strings, attached to a Dp-brane, in the presence of the pp-wave background along with a constant antisymmetric $B$-field as well as the linear dilaton. The noncommutativity structure of this system also will be investigated.

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1 Introduction

String theory in various backgrounds deeply has been studied. Some of these backgrounds admit a solvable string theory. One of them is the pp-wave spacetime [1] which is supported by a null, constant 5-from flux and can be obtained from the $AdS_5 \times S^5$ metric by taking the Penrose limit. The pp-wave background is a maximal supersymmetric space in which closed string theory is exactly solvable in the light-cone gauge [2, 3]. The other profitable background is the constant antisymmetric $B$-field which has been extensively studied in the literature. It leads to nontrivial physics on the branes. The noncommutativity of open string end points’ positions, which are attached to a D-brane [4], is a consequence of the mixed boundary condition in the $B$-field background. In addition, we have the linear dilaton field as a background which is the simplest background for non-critical string theory [5]. Among the various conformal field theories the linear dilaton CFT has some interesting applications in the string theory [6]. For example, the D-brane noncommutativity is investigated in various background fields like the dilaton [7, 8, 9, 10].

In this article we shall consider all of the above three background fields for the open string theory and investigate on the solvability of the theory. Beside, it has been demonstrated that in the light-cone formulation of strings in the pp-wave, the momentum space also becomes noncommutative, which render a full noncommutative phase space. This fact also motivated us to extend the problem by adding the mentioned background fields and see whether any new kind of quantum geometry arises.

2 Open string in a set of background fields

The pp-wave background consists of a plane wave metric, accompanied by a homogeneous R-R 5-form flux

\[ ds^2 = -f^2 X^i X^i (dX^+)^2 + 2dX^+ dX^- + dX^I dX^I, \quad I = 1, 2, \ldots, 8 \]
\[ F_5 = f dX^+ \wedge (dX^1 \wedge dX^2 \wedge dX^3 \wedge dX^4 + dX^5 \wedge dX^6 \wedge dX^7 \wedge dX^8). \]  

(1)

Now let consider an open string, attached to a Dp-brane, in the presence of the following background fields: the pp-wave metric, a constant Kalb-Ramond tensor field $B_{\mu\nu}$ and the dilaton field $\Phi$. In the light-cone formalism coordinates are decomposed as

\[ \{X^+, X^-\} \cup \{X^I|I = 1, 2, \ldots, p - 1\} \cup \{X^i|i = p + 1, \ldots, 9\}, \]  

where $X^\pm = \sqrt{2}(X^0 \pm X^p)$
and \( X^+ = x^+ + \alpha' p^+ \tau \). The string sigma-model action in the mentioned backgrounds is

\[
S = - \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \left[ g_{IJ} \left( \sqrt{-h} h^{ab} \partial_a X^I \partial_b X^J + m^2 X^I X^J \right) + \epsilon^{ab} B_{IJ} \partial_a X^I \partial_b X^J \right],
\]

(2)

where \( \Sigma \) is the string worldsheet which has the metric \( h_{ab} \) with \( h = \det h_{ab} \). The scalar curvature \( R^{(2)} \) is constructed from the metric \( h_{ab} \). The spacetime metric also is \( g_{\mu\nu} \) which is given by Eq. (1). The mass parameter \( m \), i.e. the mass of the worldsheet fields \( X^I \), is defined as \( m := \alpha' p^+ f \).

In the conventional case the dilaton usually is a general arbitrary function of the spacetime coordinates, but by considering a linear dilaton only the simplified equations are appeared. Thus, we suppose that the dilaton field has a linear form along the brane worldvolume, i.e. \( \Phi = a_\alpha X^\alpha \), where the parameters \( \{ a_\alpha | \alpha = 0, 1, \ldots, p \} \) are constant. Regarding the diffeomorphism invariance of the action (1) we are able to choose a conformally flat form for the worldsheet metric, i.e. \( h_{ab}(\sigma, \tau) = e^{\rho(\sigma, \tau)} \eta_{ab} \). Since the dilaton field removes the Weyl symmetry we are not allowed to set \( \rho(\sigma, \tau) \) equal to zero. We finish our setup by setting \( a_0 = a_p = 0 \) to avoid the presence of the coordinates \( X^\pm \) in the action.

Vanishing the variation of the action gives the equations of motion for the worldsheet fields \( X^I \) and \( \rho \) as in the following

\[
(\partial^2 - m^2) X^I + \frac{1}{2} \alpha' a^I \partial^2 \rho = 0,
\]

(3)

\[
a_I \partial^2 X^I = 0
\]

(4)

where \( \partial^2 = -\partial_\tau^2 + \partial_\sigma^2 \). Assuming a non-critical string theory, i.e. \( a^2 = a_I a^I \propto d - 26 \neq 0 \), these equations can be written as

\[
(\partial^2 - m^2) X^I + A^I_{\ J} X^J = 0,
\]

(5)

\[
\partial^2 \rho = \frac{2m^2 a_I X^I}{\alpha' a^2},
\]

(6)

where the matrix is defined by \( A_{IJ} := m^2 a_I a_J/a^2 \). The first equation reveals that the worldsheet fields \( X^I \)s effectively feel the potential \( V(X) = \frac{1}{2} A_{IJ} X^I X^J + V_0 \) where \( V_0 \) is the potential at the origin of the coordinates. We observe that presence of the linear dilaton and pp-wave background simultaneously is origin of this potential. However, vanishing of the variation of the action also defines the boundary conditions for the open string. For
example, for the open string end at $\sigma = 0$ we receive the equations
\begin{align}
(\partial_\sigma X^I - B^I_{\ j} \partial_\tau X^J) |_{\sigma=0} &= 0, \\
(a_I \partial_\sigma X^I) |_{\sigma=0} &= 0
\end{align}
for $X^I$s and $\rho$, respectively.

It is not very easy to solve Eq. (5) in general case. Therefore, we consider the situation in which the only nonzero components of the vector $a_I$ to be $a_1$ and $a_2$ which gives $\Phi = a_1 X^1 + a_2 X^2$. We also apply the following block diagonal form for the $B$-field
\begin{equation}
B = \begin{pmatrix}
0 & b & 0 & 0 \\
-b & 0 & 0 & 0 \\
0 & 0 & 0 & b' \\
0 & 0 & -b' & 0
\end{pmatrix},
\end{equation}
where the nonzero elements are $B_{12} = b$ and $B_{34} = b'$. Now, rewriting Eq. (5), we obtain
\begin{align}
\Delta_1 X^1 + kX^2 &= 0, \\
\Delta_2 X^2 + kX^1 &= 0
\end{align}
where the operators $\Delta_{\{1,2\}}$ and the constant $k$ are defined by
\begin{align}
\Delta_{\{1,2\}} := \partial^2 - m^2 a^2_{\{2,1\}} a^2, \\
k := m^2 \frac{a_1 a_2}{a^2}.
\end{align}

By combining Eqs. (10) we acquire the equations
\begin{align}
\Delta_2 \Delta_1 X^1 - k^2 X^1 &= 0, \\
X^2 &= -\frac{1}{k} \Delta_1 X^1.
\end{align}
The general solution of the partial differential Eq. (12), which is a rank four, with boundary condition (7) can be written in the feature

\begin{align}
X^1(\sigma, \tau) &= \left( x^1 \cos \omega_0 \tau + 2\alpha' p^1 \frac{\sin \omega_0 \tau}{\omega_0} \right) \cosh(\omega_0 b \sigma) \\
&- \frac{1}{\omega_0} \left(-2\alpha' p^2 \cos \omega_0 \tau + x^2 \omega_0 \sin \omega_0 \tau \right) \sinh(\omega_0 b \sigma) \\
&+ \sqrt{2\alpha'} \sum_{n \neq 0} e^{-i\omega_n \tau} \left( i \frac{\alpha_n}{\omega_n} \cos n \sigma + \frac{\alpha_n^2}{n} b \sin n \sigma \right).
\end{align}
Then Eq. (13) implies that
\[ X^2(\sigma, \tau) = -\frac{a_1}{a_2} X^1(\sigma, \tau). \]  

(15)

Here the frequencies are
\[ \omega_0 = \pm \frac{m}{\sqrt{1 + b^2}}, \]
\[ \omega_n = \text{sign}(n) \sqrt{m^2 + n^2}. \]  

(16)

Note that we can write Eqs. (10) as \( \Delta_2 \Delta_1 X^2 - k^2 X^2 = 0 \) which reveals that \( X^2 \) has a solution similar to Eq. (14) in which the indices 1 and 2 are interchanged. Comparing this solution for \( X^2 \) and Eq. (14) leads to \( b = 0 \). In this case, the mixed boundary conditions for \( X^1 \) and \( X^2 \) reduce to the Neumann boundary conditions. Thus, the mode expansion for \( X^1 \) is
\[ X^1(\sigma, \tau) = x^1 \cos m\tau + 2\alpha' \rho^1 \frac{\sin m\tau}{m} + \sum_{n \neq 0} e^{-i\omega_n \tau} \frac{\alpha_n^1}{\omega_n} \cos n\sigma, \]  

(17)

and again \( X^2(\sigma, \tau) = -\frac{a_1}{a_2} X^1(\sigma, \tau) \).

Now with a different approach we demonstrate that our setup is consistent only for \( b = 0 \). It is noteworthy that the dilaton term of the action cannot be treated just classically but it has a quantum worldsheet correction that modifies the energy-momentum tensor. The string action (2) with the linear dilaton \( \Phi = a_\mu X^\mu \) is a family of CFTs with the energy-momentum tensor
\[ T(z) = -\frac{1}{\alpha'} :g_{\mu\nu} \partial X^\mu \partial X^\nu : + a_\mu \partial^2 X^\mu, \]
\[ \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :g_{\mu\nu} \bar{\partial} X^\mu \bar{\partial} X^\nu : + a_\mu \bar{\partial}^2 X^\mu. \]  

(18)

Regarding the fact that \( X^- \) is related to the other coordinates, except \( X^+ \), and having the pp-wave metric (1), we receive
\[ T(z) = -\frac{1}{\alpha'} + \left( \frac{1}{2} \bar{\partial} X^K \bar{\partial} X_K + \frac{m^2}{8 \xi^2} X^K X_K + \bar{\partial} X^i \partial X_i \right) : + a_K \partial^2 X^K, \]
\[ \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} + \left( \frac{1}{2} \bar{\partial} X^K \bar{\partial} X_K + \frac{m^2}{8 \xi^2} X^K X_K + \bar{\partial} X^i \partial X_i \right) : + a_K \bar{\partial}^2 X^K, \]  

(19)

where \( K \in \{1, 2, 3, 4\} \) and \( i \in \{5, \cdots, p - 1\} \cup \{p + 1, \cdots, 9\} \). Note that we have assumed the only nontrivial dilaton coefficients to be \( a_1 \) and \( a_2 \). Since at the boundary momentum
does not flow we must have $T(z) = \tilde{T}(\bar{z})$ (at the boundary $z = \bar{z}$), hence we obtain the following conditions

$$a^k \alpha_n^l B_{kl} = 0$$
$$a^k p_0^l B_{kl} = 0$$
$$a^k x_0^l B_{kl} = 0, \quad k, l = 1, 2. \quad (20)$$

These equations impose $B_{12} = b = 0$.

3 Quantization

For the string coordinates $X^3$ and $X^4$ we can write the solutions as

$$X'_I(\sigma, \tau) = X'_0(\sigma, \tau) + X'_1(\sigma, \tau),$$
$$X'_0(\sigma, \tau) = \left( x'^{\prime} \cos \omega_0 \tau + 2\alpha' p'^{\prime} \frac{\sin \omega_0 \tau}{\omega_0} \right) \cosh(\omega_0 b' \sigma)$$
$$+ \frac{1}{\omega_0 b'} B'^{\prime \prime}_{I',J'} \left( -x'^{\prime} \omega_0 \sin \omega_0 \tau + 2\alpha' p'^{\prime} \cos \omega_0 \tau \right) \sinh(\omega_0 b' \sigma),$$
$$X'_1(\sigma, \tau) = \sqrt{2\alpha'} \sum_{n \neq 0} e^{-i\omega_n \tau} \left( \frac{i \alpha_n^{I'}}{\omega_n} \cos n \sigma + \frac{\alpha_n^{J'}}{n} B'^{\prime \prime}_{I',J'} \sin n \sigma \right), \quad (21)$$

where $X'_0$ is the zero-mode part and $X'_1$ indicates the oscillating part, and $\{I', J' = 3, 4\}$. Note that both signs of $\omega_0$ determine only one solution for $X^{I'}$. According to our setup we see that only the coordinates $X^3$ and $X^4$ contain the $B$-field elements. Therefore, only the plane $X^3X^4$ is noncommutative, which now we investigate it.

The canonical momentum corresponding to the open string coordinate $X^I$ is given by

$$P^I(\sigma, \tau) = \frac{1}{2\pi \alpha'} \left( \partial_\tau X^I - B_{I',J'} \partial_\sigma X^{I'} \right). \quad (22)$$

For the directions $X^3$ and $X^4$ the conjugate momenta also split into two pieces, i.e. zero-mode part and oscillating part

$$P'^I(\sigma, \tau) = P'_0(\sigma, \tau) + P'_1(\sigma, \tau),$$
$$P'_0(\sigma, \tau) = \frac{1}{2\pi \alpha'} \left[ M'^{I',J'} \left( -x'^{\prime} \omega_0 \sin \omega_0 \tau + 2\alpha' p'^{\prime} \cos \omega_0 \tau \right) \cosh(\omega_0 b' \sigma) \right.$$  
$$- \left. \frac{1}{b} (BM)^{I',J'} \left( 2\alpha' p'^{\prime} \sin \omega_0 \tau + x'^{\prime} \omega_0 \cos \omega_0 \tau \right) \sinh(\omega_0 b' \sigma) \right],$$
$$P'_1(\sigma, \tau) = \frac{1}{\pi \sqrt{2\alpha'}} \sum_{n \neq 0} e^{-i\omega_n \tau} \left( \frac{\alpha_n^{J'}}{\omega_n} M_{I',J'} \cos n \sigma + \frac{m^2}{n\omega_n} \alpha_n^{J'} B'^{\prime \prime}_{I',J'} \sin n \sigma \right). \quad (23)$$
The symmetric matrix $M$ is given by

$$M_{I'J'} = (1 + b^2)\delta_{I'J'}.$$  \hfill (24)

Again both signs of $\omega_0$ specify one value for each of the momentum components $P^3$ and $P^4$.

It is known that in a D-brane with the $B$-field background the spatial coordinates of the brane do not commute. Now for our setup we investigate it. For quantizing the open string theory we use the following symplectic form

$$\Omega = \int_0^\pi d\sigma \left( \sum_{I' = 3}^4 \sum_{J' = 3}^4 g_{I'J'} dP^{I'} \wedge dX^{J'} \right).$$ \hfill (25)

This can be justified by analysis of the constraint structure of the theory (e.g. see the Refs. [11] and [12]). Using Eqs. (21) and (23), this differential form finds the feature

$$\Omega = 4 \sum_{I' = 3}^4 \sum_{J' = 3}^4 \left( \frac{\sinh^2(\pi\omega_0 b)}{2\pi\omega_0 b} (BM)_{I'J'} dx^{I'} \wedge dx^{J'} - \frac{\sinh^2(\pi\alpha' b)}{2\pi\alpha' b^2} (BM)_{I'J'} dp^{I'} \wedge dp^{J'} + i \sum_{n = 1}^{\infty} \frac{M_{I'J'}(n)}{\omega_n} d\alpha^J_n \wedge d\alpha^J_{-n} \right)$$ \hfill (26)

where the symmetric matrices $M_{(n)I'J'}$ is defined by

$$M_{(n)I'J'} = \left( 1 + \frac{\omega_n^2 b^2}{n^2} \right) \delta_{I'J'}.$$ \hfill (27)

This symplectic form enables us to obtain the following nontrivial commutation relations

$$[x^{I'}, x^{J'}] = 2i\pi\alpha'(BM^{-1})^{I'J'},$$

$$[x^{I'}, p^{J'}] = iM^{I'J'} \frac{\pi\omega_0 b'}{\tanh(\pi\omega_0 b')} ,$$

$$[p^{I'}, p^{J'}] = i\frac{\pi\omega_0^2}{2\alpha'} (BM^{-1})^{I'J'},$$

$$[\alpha^I_n, \alpha^J_s] = \omega_n M^{I'J'}(n) \delta_{n+s,0} ,$$ \hfill (28)

The matrix $M_{(n)I'J'}$ is inverse of $M_{(n)I'J'}$, and the matrix $(BM^{-1})$ is antisymmetric. $M^{I'J'}_{(n)}$ may be interpreted as a mode-dependent open string metric for the oscillators.

The above results enable us to calculate intrinsic commutation relations for the open string coordinates and their conjugate momenta

$$[X^{I'}(\sigma, \tau), X^{J'}(\sigma', \tau)] = i2\pi\alpha'(BM^{-1})^{I'J'},$$ \hfill (29)
\[
\left[ P^{I'}(\sigma, \tau), P^{J'}(\sigma', \tau) \right] = i \frac{m^2}{2\pi\alpha'} B^{I'J'}, \quad (30)
\]

\[
\left[ X^{I'}(\sigma, \tau), P^{J'}(\sigma', \tau) \right] = i \delta^{I'J'} \delta(\sigma - \sigma'), \quad (31)
\]

where the first and the second equations are established on the brane, i.e. at \( \sigma = \sigma' = 0 \).

The left-hand sides of Eqs. (29) and (30) define the noncommutativity parameters associated with the space and momentum parts of the 4-dimensional phase space, respectively. The Eqs. (28)-(31) clarify that the open string zero modes as well as the string coordinates \( X^{I'} \) and the momenta \( P^{I'} \) feel a noncommutative phase space.

### 4 Conclusions

In this article the behavior of an open string, which is attached to a Dp-brane in the presence of the massless fields: Kalb-Ramond and the linear dilaton in the pp-wave background was investigated. We chose a suitable configuration of the background fields such that they to be appropriate for the light-cone gauge formalism. Presence of the linear dilaton effectively deforms the equations of motion, boundary conditions and, due to the lack of the Weyl invariance, introduces a new worldsheet field \( \rho \) in the theory.

Separation of the variables elaborates a quad differential equation for the string coordinates. Solving this equation and the boundary equations, we established that the noncommutativity is extremely influenced by the dilaton field. If there is a magnetic field along two specific directions of the brane, the positions of the string endpoints in that plane are expected to be noncommutative. However, it is possible to suppress the noncommutativity by turning on a linear dilaton field such that its vector components fall into that plane, which quenches the magnetic field. In fact, by adding the elements of the magnetic field, without altering the dilaton vector, one can turn on the noncommutativity in some other directions. The momentum components of the open string in these directions also take a noncommutative structure.

The equations of this paper also are valid for a general block diagonal magnetic field. It may be interesting to extend the problem beyond this consideration with more components, then find out whether new results reveal.
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