BILATERAL SUMS RELATED TO RAMANUJAN-LIKE SERIES

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Abstract. We define bilateral series related to Ramanujan-like series for $1/\pi^2$. Then, we conjecture a property of them and give some applications.

1. Introduction

In [5] we constructed bilateral summations related to Ramanujan-type series for $1/\pi$ and applied them to prove a new kind of identities, which we called “the upside-down” counterpart. In this paper we consider bilateral sums related to Ramanujan-like series for $1/\pi^2$, conjecture a property of them and give some applications. We will need the following theorems

Theorem 1. Suppose that $f(x)$ is the sum over all integers $n$ of $g(n + x)$. Then, clearly $f(x)$ is a periodic function of period $x = 1$.

Theorem 2. Let $f : C \to C$ be a holomorphic function such that $f(x) = O(e^{cn|\text{Im}(x)|})$, where $c \geq 0$ is a constant, and suppose that $f(x)$ admits a Fourier series:

$$f(x) = \sum_{n=-\infty}^{+\infty} a_n e^{2\pi i n x}.$$ Then $|2n| > c \Rightarrow a_n = 0$.

Proof. It is known that the coefficients of a Fourier series are given by

$$a_n = \int_{a+it}^{a+1+it} f(x) e^{-2\pi i n x} dx.$$ Then for $n < 0$ let $t \to +\infty$ and for $n \geq 0$ let $t \to -\infty$. \hfill \Box

2. Ramanujan-like series for $1/\pi^2$

Let $s_0 = 1/2$, $s_3 = 1 - s_1$, $s_4 = 1 - s_2$. We recall that a Ramanujan-like series for $1/\pi^2$ is a series of the form

$$\sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right] (a + bn + cn^2)z^n = \frac{1}{\pi^2},$$

where $z$, $a$, $b$, and $c$ are algebraic numbers and the possible couples $(s_1, s_2)$ are $(1/2, 1/2), (1/2, 1/3), (1/2, 1/4), (1/2, 1/6), (1/3, 1/3), (1/3, 1/4), (1/3, 1/6), (1/4, 1/4), (1/4, 1/6), (1/6, 1/6), (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12)$. Up to date 11 convergent and 6 “divergent” formulas are known in this new family (see all in the Appendix). The value

$$\tau = \frac{c}{\sqrt{1 - z}},$$
plays an important roll in the theory [21 eq. 3.47].

3. Bilateral series related to Ramanujan-like series for $1/\pi^2$

We introduce the function

$$ f(x) = \prod_{i=0}^{4} \frac{\cos \pi x - \cos \pi s_i}{1 - \cos \pi s_i} \sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i)^{n+x}}{(1)^{n+x}} \right] [a + b(n + x) + c(n + x)^2](-z)^{n+x}. \quad (3) $$

Applying Theorem 1, we see that it is a periodic function of period $x = 1$ because $\cos \pi x - \cos \pi x = \cos \pi x$ and $(-1)^n \cos \pi x = \cos \pi(n + x)$. Besides it is analytic due to the factor with the cosinus, which cancels the poles at $x = -s_i$ and therefore all the other poles due to the periodicity of $f(x)$. On the other hand we conjecture that $f(x) = O(e^{\pi |\text{Im}(x)|})$. Hence by Theorem 2 we deduce that

$$ \prod_{i=0}^{4} \frac{\cos \pi x - \cos \pi s_i}{1 - \cos \pi s_i} \sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i)^{n+x}}{(1)^{n+x}} \right] [a + b(n + x) + c(n + x)^2](-z)^{n+x} = \frac{1}{\pi^2} (u_1 \cos 2\pi x + u_2 \cos 4\pi x + (1 - u_1 - u_2) + v_1 \sin 2\pi x + v_2 \sin 4\pi x), $$

where $u_1, u_2, v_1, v_2$ do not depend on $x$, and we can determine their values by giving values to $x$. We conjecture that for the bilateral series corresponding to a Ramanujan-like series for $1/\pi^2$ (that is, when $z, a, b, c$ are algebraic numbers), the values of $u_1, u_2, v_1$ and $v_2$ are rational numbers. In addition, for the case of alternating series we conjecture that $v_1 = v_2 = 0$. These bilateral series are “divergent”, because the sum to the left or the sum to the right “diverges”. However we turn them convergent by analytic continuation. We can write (3) in the following way:

$$ \sum_{n=0}^{\infty} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i + x)^n}{(1 + x)^n} \right] \left[ a + b(n + x) + c(n + x)^2 \right](-z)^{n+x} + x^5 \sum_{n=1}^{\infty} (-1)^n \left[ \prod_{i=0}^{4} \frac{(1 - x)^n}{(s_i - x)^n} \right] \frac{a + b(-n + x) + c(-n + x)^2}{(-n + x)^5} (-z)^{-n+x} = \frac{1}{\pi^2} \prod_{i=0}^{4} \frac{(1)_x}{(s_i)_x} u_1 \cos 2\pi x + u_2 \cos 4\pi x + (1 - u_1 - u_2) + v_1 \sin 2\pi x + v_2 \sin 4\pi x, $$

where we have splitted the summation in two sums and used the properties

$$ \frac{1}{(1)^{-n+x}} = (-1)^{n+1} \frac{(1-x)^n}{(1)_x} \frac{x}{n-x}, \quad (s_i)^{-n+x} = (-1)^n \frac{(s_i)_x}{(1 - s_i - x)_n}. $$

As the second sum is $O(x^5)$, formula (5) implies an expansion of the form

$$ \sum_{n=0}^{\infty} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i + x)^n}{(1 + x)^n} \right] \left[ a + b(n + x) + c(n + x)^2 \right](-z)^{n+x} = \frac{1}{\pi^2} - \frac{x^2}{2!} + j \pi^2 \frac{x^4}{4!} + O(x^5), $$

were $j \neq 0$. This expansion, for instance, gives a closed form for

$$ \sum_{n=0}^{\infty} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i + x)^n}{(1 + x)^n} \right] \left[ a + b(n + x) + c(n + x)^2 \right](-z)^{n+x} = \frac{1}{\pi^2} - \frac{x^2}{2!} + j \pi^2 \frac{x^4}{4!} + O(x^5), $$

were $j \neq 0$. This expansion, for instance, gives a closed form for
if $z < 0$ (alternating series), and an expansion of the form
\[
\sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} (s_i + x)^n \right] \frac{a + b(n + x) + c(n + x)^2}{(1 + x)^n} \cdot z^{n+x} = \frac{1}{\pi^2} - \frac{k}{2!} + \frac{j \pi^2 x^4}{4!} + O(x^5),
\]
if $0 < z < 1$ (convergent series of positive terms).

Hence, the conjecture that the coefficients of the Fourier expansion are rational is equivalent to the conjecture that $k$ and $j$ are rational stated in [2, eq. 3.48]. The relation of $\tau$ with $k$ and $j$ is [2, eq. 3.48]:
\[
\tau^2 = \frac{j}{12} + \frac{k^2}{4} + \frac{5k}{3} + 1 + (\cot^2 \pi s_1)(\cot^2 \pi s_2) + (1 + k)(\cot^2 \pi s_1 + \cot^2 \pi s_2).
\]

From (5) we deduce that
\[
\sum_{n \in \mathbb{Z}} \left[ \prod_{i=0}^{4} (s_i)^n \right] (a + bn + cn^2)z^n = \sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} (s_i)^n \right] (a + bn + cn^2)z^n,
\]
and as the function [13] has period $x = 1$, we see that for all integer $x$, we have
\[
\sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{i=0}^{4} (s_i)^n \right] (a + bn + cn^2)(1 - z)^{n+x} = (-1)^x \sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} (s_i)^n \right] (a + bn + cn^2)z^n.
\]

If $z > 0$, we can write
\[
\sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{i=0}^{4} (s_i)^n \right] (a + bn + cn^2)(1 - z)^{n+x} = e^{-\pi x} \sum_{n \in \mathbb{Z}} \left[ \prod_{i=0}^{4} (s_i)^n \right] (a + bn + cn^2)z^n.
\]

Taking into account (7) and (8), we have written the following Maple procedure:

```maple
bilater:=proc(s1,s2,z0,x,j)
local p,s0,s3,s4,z1,h,hh,r,u,v;
p:=(a,b)->pochhammer(a,b): s0:=1/2; s3:=1-s1; s4:=1-s2;
h:=n->p(s0,n)*p(s1,n)*p(s2,n)*p(s3,n)*p(s4,n)/p(1,n)^5:
if type(x,integer) then; hh:=n->h(n)*n^j:
r:=evalf((-1)^x*subs(z=z0,sum(hh(n)*z^n,n=0..infinity))):
else z1:=-z0; hh:=(j,n)->exp(-I*Pi*x)*h(n+x)*(n+x)^j;
u:=j->subs(z=z1,sum(hh(j,n)*(-z)^(-n+x),n=1..infinity)):
v:=j->subs(z=z1,sum(hh(j,-n)*(-z)^(-n+x),n=1..infinity)):
r:=evalf(u(j)+v(j)):
return r:
else;
end:
end:
```
This procedure calculates the bilateral sum
\[
\sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{j=0}^{4} \frac{(s_j)_{n+x}}{(1)_{n+x}} \right] (-z)^{n+x} (n + x)^j,
\]
and helps the reader to check the examples.

4. EXAMPLES OF BILATERAL SERIES

We give some examples of bilateral series related to Ramanujan-like series for \(1/\pi^2\).

**Example 1.** Formula (25) in the Appendix is
\[(9) \quad \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \frac{(3/2)_n}{(1)_{n+x}} 20(n + x)^2 + 8(n + x) + 1 = \frac{1}{\pi^2}.
\]
Taking \(v_1 = v_2 = 0\) and giving two values to \(x\) we get numerical approximations of \(u_1, u_2\). We observe that these values look rational and do not change using other values of \(x\).

Hence, we conjecture that
\[(10) \quad \frac{1}{8} \sum_{n \in \mathbb{Z}} (-1)^n \frac{(3/2)_n}{(1)_{n+x}} 20(n + x)^2 + 8(n + x) + 1 = \frac{1 - \frac{1}{2} \cos 2\pi x + \frac{3}{2} \cos 4\pi x}{\pi^2 \cos^3 \pi x},
\]
after replacing \(u_1, u_2\) with their guessed rational values.

**Example 2.** Formula (32) in the Appendix is
\[(11) \quad \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)_n}{(1)_{n+x}} (\frac{1}{4})_{n+x} (\frac{3}{4})_{n+x} (\frac{3}{4})_{n+x} (\frac{4}{6})_{n+x} (\frac{5}{6})_{n+x} \left( \frac{3}{4} \right)^{6n} (1930n^2 + 549n + 45) = \frac{384}{\pi}.
\]
From it we conjecture the bilateral form
\[(12) \quad \sum_{n \in \mathbb{Z}} (-1)^n \frac{(1/2)_n}{(1)_{n+x}} (\frac{1}{4})_{n+x} (\frac{3}{4})_{n+x} (\frac{3}{4})_{n+x} (\frac{4}{6})_{n+x} (\frac{5}{6})_{n+x} \left( \frac{3}{4} \right)^{6(n+x)} \times (1930(n + x)^2 + 549(n + x) + 45) = \frac{11 - 14 \cos 2\pi x + 6 \cos 4\pi x}{\pi^2 \cos \pi x (4 \cos^2 \pi x - 1)(4 \cos^2 \pi x - 3)},
\]
after identifying the coefficients.

**Example 3.** From the following formula:
\[(13) \quad \sum_{n=0}^{\infty} \frac{(1/2)^n (1/4)_n (3/4)_n}{(1)_{n+x} (1)_{n+x}} \left( \frac{3}{4} \right)^{6n} (120n^2 + 34n + 3) = \frac{32}{\pi^2},
\]
which is (24) in the Appendix, we get the bilateral form
\[
\frac{1}{32} \sum_{n \in \mathbb{Z}} (-1)^n \frac{(1/2)_n}{(1)_{n+x}} \left( \frac{3}{4} \right)_{n+x} \left( \frac{1}{16} \right)^{n+x} (120(n + x)^2 + 34(n + x) + 3) = e^{i\pi x} \frac{3 - \frac{7}{2} \cos 2\pi x + \frac{3}{2} \cos 4\pi x + \left( \frac{1}{2} \sin 2\pi x - \frac{1}{2} \sin 4\pi x \right) i}{\pi^2 \cos^3 \pi x (2 \cos^2 \pi x - 1)},
\]
after identifying the coefficients of the Fourier expansion from their numerical approximations. Observe that \( v_1 \) and \( v_2 \) are not null because the series in (12) is of positive terms. Observe also the factor \( e^{i\pi x} \), which comes from \((-1)^n/(-16)^{n+x}\)

**Example 4.** For the formula (29) in the Appendix we have the bilateral identity

\[
\sqrt{7} \sum_{n\in\mathbb{Z}} \frac{\left(\frac{1}{2}\right)_{n+x} \left(\frac{1}{8}\right)_{n+x} \left(\frac{3}{8}\right)_{n+x} \left(\frac{\pi}{8}\right)_{n+x} \left(\frac{\pi}{8}\right)_{n+x}}{(1)^{n+x}} = e^{i\pi x} \frac{29 - 79}{2} \cos 2\pi x + \frac{23}{2} \cos 4\pi x + \left(\frac{5}{2} \sin 2\pi x - \frac{3}{2} \sin 4\pi x\right) i
\]

evaluating the factor \( \pi^2 \text{csc}^2 \frac{\pi}{8} \text{csc}^2 \frac{3\pi}{8} \),

after identifying the coefficients.

**Example 5.** Looking at the formula (31) in the Appendix which has \( z = (3/\phi)^3 \), we had the intuition that another series with the other sign of the square root, namely \( z = -(3\phi)^3 \), could exist and we were right because we discovered it by using the PSLQ algorithm. Here we recover it in a different way by writing the bilateral identity

\[
\sum_{n\in\mathbb{Z}} \frac{\left(\frac{1}{2}\right)^3_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{\phi}{3}\right)_n}{(1)^{n+x}} (-1)^n (3\phi)^{3(n+x)} (c(n+x)^2 + b(n+x) + a) = 4\pi^2 \frac{u_1 \cos 2\pi x + u_2 \cos 4\pi x + (1 - u_1 - u_2)}{\cos^3 \pi x (\cos^2 \pi x - \frac{1}{4})}
\]

giving five values to \( x \) we get numerical approximations of \( a, b, c, u_1 \) and \( u_2 \), which we could identify:

\[
u_1 = \frac{17}{36}, \quad u_2 = \frac{3}{16}, \quad c = 2408 + \frac{216}{\phi}, \quad b = 1800 + \frac{162}{\phi}, \quad a = 333 + \frac{30}{\phi}.
\]

Replacing these values and taking \( x = 0 \), we arrive at the “divergent” formula (38), in the second list of the Appendix.

**Example 6.** From the “divergent” series (36) in the Appendix, namely

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^3_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)^{n+x}} (28n^2 + 18n + 3)(-1)^n 3^{3n} = \frac{6}{\pi^2},
\]

we have the following bilateral sum evaluation:

\[
\frac{1}{6} \sum_{n\in\mathbb{Z}} \frac{\left(\frac{1}{2}\right)^3_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{\phi}{3}\right)_n}{(1)^{n+x}} (28(n+x)^2 + 18(n+x) + 3) (-1)^n 27^{n+x} \]

\[
= \frac{5 + 4 \cos 2\pi x + 3 \cos 4\pi x}{4\pi^2 \cos^3 \pi x (4 \cos^2 \pi x - 1)},
\]

after identifying the coefficients of the Fourier terminating expansion from their numerical approximations obtained giving values to \( x \).
5. Applications of the bilateral series

If \(|z| > 1\), then letting \(x \to -1/2\) or \(x \to -s_1\), etc in (15), we obtain the evaluation of some convergent series. In general, taking the limit of (5) as \(x \to -s_j\), where \(j \in \{0,1,2,3,4\}\), we deduce the following identity:

\[
\sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} \frac{(s_j)_n}{(s_i + s_j)_n} \right] \left[ a + b(-n - s_j) + c(-n - s_j)^2 \right] z^{-n} = \frac{(-z)^{s_j}}{\pi^2} \lim_{x \to -s_j} \left[ \prod_{i=0}^{4} \frac{(1)_x}{(s_i)_x} \right] u_1 \cos 2\pi x + u_2 \cos 4\pi x + (1 - u_1 - u_2) + v_1 \sin 2\pi x + v_2 \sin 4\pi x \]

Another application is explained in [5], but only proved for bilateral sums related to Ramanujan-type series for \(1/\pi\).

Example 7. From the formula (15), and taking \(s_j = 1/2\) in (16), we get

\[
\sum_{n=0}^{\infty} \frac{28n^2 + 10n + 1}{6n + 1} \left( \frac{-1}{27} \right)^n = \frac{3}{\pi},
\]

which is a new convergent series for \(1/\pi\). In the same way, from the ‘divergent’ series (13/14), we get the convergent evaluation

\[
\sum_{n=0}^{\infty} \frac{(1/2)_n^5}{(1)^3_n (1/6)_n (1/6)_n} \frac{(2408 + 216) n^2 + (608 + 54) n + (35 + 3)}{6n + 1} \left( \frac{-1}{3\phi} \right)^{3n} = \frac{3\sqrt{\phi^3}}{\pi},
\]

where \(\phi\) is the fifth power of the golden ratio.

Example 8. From (10), and letting \(s_j \to -1/2\) in (16), we recover formula (33) in the Appendix. We observe that a formula with \(z\) implies another one with \(z^{-1}\) in the family \(s_1 = s_2 = 1/2\), and we will refer to this property as “duality”.

6. The mirror map

All the parameters of the bilateral sums related to Ramanujan-like series for \(1/\pi^2\) are algebraic, but the value of \(q\) (related to \(z\) by the mirror map) is not. Now we will define a function \(w(x)\) in which \(q\) is easily related to the coefficients of the Fourier expansion. First, we let \(u(x)\) and \(v(x)\) be the analytic functions

\[
u(x) = \prod_{i=0}^{4} \frac{\cos \pi x - \cos \pi s_i}{1 - \cos \pi s_i} \sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i)_{n+x}}{(1)_{n+x}} \right] (-z)^{n+x},
\]

and

\[
v(x) = \prod_{i=0}^{4} \frac{\cos \pi x - \cos \pi s_i}{1 - \cos \pi s_i} \sum_{n \in \mathbb{Z}} (-1)^n \left[ \prod_{i=0}^{4} \frac{(s_i)_{n+x}}{(1)_{n+x}} \right] (n + x)(-z)^{n+x}.
\]

Then, we define the analytic function

\[
w(x) = v(0)u(x) - u(0)v(x).
\]

We have the following Fourier expansion:

\[
w(x) = a_1 \cos 2\pi x + a_2 \cos 4\pi x + (1 - a_1 - a_2) + b_1 \sin 2\pi x + b_2 \sin 4\pi x,
\]
The above identity for the function $w(x)$ implies an expansion of the form

$$
\sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} \left( \frac{S_i}{n} \right) \right] n z^n \sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} \left( \frac{S_i}{n+x} \right) \right] (-z)^{n+x}
$$

if $z < 0$ (alternating series), and an expansion of the form

$$
\sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} \left( \frac{S_i}{n} \right) \right] n z^n \sum_{n=0}^{\infty} \left[ \prod_{i=0}^{4} \left( \frac{S_i}{n+x} \right) \right] (n+x)(-z)^{n+x}
$$

if $z > 0$ and $z \neq 1$ (series of positive terms). From the differential equations for Calabi-Yau threefolds \cite{3, 2, 1, 10}, we deduce that

$$
-\frac{1}{\pi} \frac{p_2}{p_1} = t, \quad q = e^{-\pi t} \quad \text{or} \quad q = e^{-\pi t}
$$

for (19) and (20) respectively, where $z = z(q)$ is the mirror map, and

$$
k = 2 \left( \frac{1}{\pi^2} \frac{p_4}{p_2} - \frac{5}{3} - \cot^2 \pi s_1 - \cot^2 \pi s_2 \right).
$$

The method used in \cite{1} is good for convergent Ramanujan series, but not when the series is “too divergent”. However the method based on bilateral series permits to calculate $q$ with many digits in all the cases, when we know the value of $z$.

**Example 9.** Taking $s_1 = 1/2$, $s_2 = 1/3$ and $z = 27\phi^{-3}$ (formula (31) in the Appendix), and using the formulas in \cite{1}, and $q = e^{-\pi t}$, we get

$$
t_1 = t(27\phi^{-3}) = 3.619403396730928522140860042453285904901.
$$

However for $z = (-3\phi)^3$ (formula (38) in the Appendix), we need to use the method based on bilateral sums. Then, we get that the value of $t$ corresponding to (38), is with 40 correct digits (we can calculate many more) equal to

$$
t_2 = t(-27\phi^{3}) = 1.23341216518959404372375611862890324841.
$$

We get the values of $\tau$ from \cite{2}, and they are $\tau_1 = 4/3 \cdot \sqrt{10}$ and $\tau_2 = 1/9 \cdot \sqrt{10}$. We thought that the values of $t_1$ and $t_2$ would be easily related but we have not succeeded in finding any relation.
Example 10. For $s_1 = s_2 = 1/2$, the series with $z$ and with $z^{-1}$ satisfy duality. For the formulas (23) and (35) in the Appendix, we have

$$t_1 = t(-2^{-10}) = 4.412809109031200738238212268698423552548,$$

$$t_2 = t(-2^{10}) = 1.25230243423118475460661614044396018505.$$

We get $\tau_1 = \tau(-2^{-10}) = \sqrt{41}$ and $\tau_2 = 1/16 \cdot \sqrt{41}$ from [2], and we observe that $t_1 t_2 = 2\sqrt{41} + 3\sqrt{41} - 3$.

Hence the values of $t_1$ and $t_2$ are nicely related.

7. Conclusion

We have conjectured that the terminating Fourier expansion of the function (3) corresponding to a Ramanujan-like series for $1/\pi^2$ has rational coefficients. A challenging problem is to prove it. That is, to obtain rigorously the bilateral form corresponding to any Ramanujan-like series for $1/\pi^2$. This would suppose to give another step towards understanding the family of formulas for $1/\pi^2$.

Below, we show how to solve the problem for the Ramanujan-type series for $1/\pi$. Consider, for example the Ramanujan series

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \frac{21460n + 1123}{882^{2n}} = \frac{3528}{\pi}. \quad (21)$$

To solve the problem of finding the bilateral form we need to determine the value of $u$ in

$$\sum_{n \in \mathbb{Z}} (-1)^n \left(\frac{1}{2}\right)_{n+x} \left(\frac{1}{4}\right)_{n+x} \left(\frac{3}{4}\right)_{n+x} \frac{21360(n + x) + 1123}{3528 \cdot 882^{2(n+x)}} = \frac{1 - u + u \cos 2\pi x}{\pi \cos \pi x (2 \cos^2 \pi x - 1)}. \quad (22)$$

Expanding the right side, comparing with [2, Expansion 1.1], and using [2, eq. 2.30-2.32], we rigorously obtain that $u = 10$, which is a rational number.

8. Appendix

In the following two lists $\phi$ means the fifth power of the golden ratio. That is

$$\phi = \left(\frac{1 + \sqrt{5}}{2}\right)^5.$$  

The notation $? = ?$ means that the formula remains unproved, and the notation " = " means that we get the equality by analytic continuation.

8.1. List of convergent formulas.

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \frac{21460n + 1123}{882^{2n}} = \frac{128}{\pi^2}, \quad (23)$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \frac{1}{2^{4n}} \frac{120n^2 + 34n + 3}{(120n^2 + 34n + 3)} = \frac{32}{\pi^2}, \quad (24)$$
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} (20n^2 + 8n + 1) = \frac{8}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{1}{2^{10n}} (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{1}{48^n} (252n^2 + 63n + 5) = \frac{48}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{1}{803n^2 + 693n + 29} = \frac{128\sqrt{5}}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{1}{74n} (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{\left(\frac{3}{4}\right)^n}{7n^2 + 27n + 3} = \frac{48}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{\left[32 - \frac{216}{\phi}\right]n^2 + (18 - \frac{162}{\phi})n + (3 - \frac{30}{\phi})}{3n^2} = \frac{3}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{\left(\frac{3}{4}\right)^n}{(1930n^2 + 549n + 45)} = \frac{384}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n (\frac{3}{4})_n (\frac{1}{4})_n (\frac{1}{4})_n (-1)^n}{(1)_n^5} \frac{\left(\frac{3}{5}\right)^n}{(532n^2 + 126n + 9)} = \frac{375}{4\pi^2},
\]

8.2. List of “divergent” formulas.

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{(1)_n^5} (10n^2 + 6n + 1)(-1)^n 4^n = \frac{4}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{(1)_n^5} (205n^2 + 160n + 32)(-1)^n 2^{10n} = \frac{16}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3 (\frac{3}{4})_n (\frac{3}{4})_n}{(1)_n^5} (28n^2 + 18n + 3)(-1)^n 3^{3n} = \frac{6}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{5}{2})_n (\frac{3}{4})_n (\frac{3}{4})_n}{(1)_n^5} (172n^2 + 75n + 9)(-1)^n \left(\frac{27}{16}\right)^n = \frac{48}{\pi^2},
\]
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n n}{(1)_n^5} \left[n^2 + (1800 + 162\phi^{-1})n + (333 + 30\phi^{-1})\right] = \frac{36}{\pi^2},
\]

(38)

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n n}{(1)_n^5} \left[(2408 + 216\phi)n^2 + (1800 + 162\phi)n + (333 + 30\phi)\right] = \frac{\pi^2}{36},
\]

(39)

Formulas (23, 24, 25, 30, 34), were proved by the author using the WZ-method. The upside-down of (37) was proved in [7, Example 42], and we can extract from it the WZ-pair that we need to prove (37). All the other formulas are conjectured. The conjectured formula (33) is joint with Gert Almkvist. In [1] we recovered all the known convergent series for $1/\pi^2$ and also two “divergent” ones: (31) and (37). However, the method used in [1] does not allow to discover “too divergent” series such as (35) or (38). We can check that the mosaic supercongruences pattern [3] hold for the formula (38). Inspired by these congruences, by the conjectured formulas [6, Conj. 1.1–1.6] and by [5], we observe that

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_5}{(1)_n^5} \left(-1\right)^{3n} \frac{(2408 + 216\phi^{-1})n^2 - (1800 + 162\phi^{-1})n + (333 + 30\phi^{-1})}{n^5} \equiv \frac{1125}{4} \sqrt{5} L_5(3) - 448\zeta(3),
\]

(40)

seems true. As the convergence is fast, we can use it to get many digits of $L_5(3)$. We discovered the formula (30) very recently. Related to it are the Zudilin-type supercongruences [9, 11]:

\[
\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n \left(\frac{7}{3}\right)_n \left(\frac{8}{3}\right)_n \left(\frac{10}{3}\right)_n \left(\frac{11}{3}\right)_n \left(\frac{13}{3}\right)_n \left(\frac{14}{3}\right)_n \left(\frac{16}{3}\right)_n \left(\frac{17}{3}\right)_n \left(\frac{19}{3}\right)_n \left(\frac{21}{3}\right)_n \left(\frac{22}{3}\right)_n \left(\frac{24}{3}\right)_n \left(\frac{25}{3}\right)_n \left(\frac{27}{3}\right)_n \left(\frac{28}{3}\right)_n \left(\frac{30}{3}\right)_n \left(\frac{31}{3}\right)_n \left(\frac{33}{3}\right)_n \left(\frac{34}{3}\right)_n \left(\frac{36}{3}\right)_n \left(\frac{37}{3}\right)_n \left(\frac{39}{3}\right)_n \left(\frac{40}{3}\right)_n \left(\frac{42}{3}\right)_n \left(\frac{43}{3}\right)_n \left(\frac{45}{3}\n
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n \left(\frac{7}{3}\right)_n \left(\frac{8}{3}\right)_n \left(\frac{10}{3}\right)_n \left(\frac{11}{3}\right)_n \left(\frac{13}{3}\right)_n \left(\frac{14}{3}\right)_n \left(\frac{16}{3}\right)_n \left(\frac{17}{3}\right)_n \left(\frac{19}{3}\right)_n \left(\frac{21}{3}\right)_n \left(\frac{22}{3}\right)_n \left(\frac{24}{3}\right)_n \left(\frac{25}{3}\right)_n \left(\frac{27}{3}\right)_n \left(\frac{28}{3}\right)_n \left(\frac{30}{3}\right)_n \left(\frac{31}{3}\right)_n \left(\frac{33}{3}\right)_n \left(\frac{34}{3}\right)_n \left(\frac{36}{3}\right)_n \left(\frac{37}{3}\right)_n \left(\frac{39}{3}\right)_n \left(\frac{40}{3}\right)_n \left(\frac{42}{3}\right)_n \left(\frac{43}{3}\right)_n \left(\frac{45}{3}\n
which is a convergent series for $\zeta(3)$.

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