1/k-HOMOGENEOUS LONG SOLENOIDS

JAN P. BOROŃSKI, GARY GRUENHAGE, AND GEORGE KOZLOWSKI

Abstract. We study nonmetric analogues of Vietoris solenoids. Let \( \Lambda \) be an ordered continuum, and let \( \vec{p} = (p_1, p_2, \ldots) \) be a sequence of positive integers. We define a natural inverse limit space \( S(\Lambda, \vec{p}) \), where the first factor space is the nonmetric “circle” obtained by identifying the endpoints of \( \Lambda \), and the \( n \)-th factor space, \( n > 1 \), consists of \( p_1p_2\cdots p_{n-1} \) copies of \( \Lambda \) laid end to end in a circle. We prove that for every cardinal \( \kappa \geq 1 \), there is an ordered continuum \( \Lambda \) such that \( S(\Lambda, \vec{p}) \) is \( \frac{1}{\kappa} \)-homogeneous; for \( \kappa > 1 \), \( \Lambda \) is built from copies of the long line. Our example with \( \kappa = 2 \) provides a nonmetric answer to a question of Neumann-Lara, Pellicer-Covarrubias and Puga-Espinosa from 2005, and with \( \kappa = 1 \) provides an example of a nonmetric homogeneous circle-like indecomposable continuum. Finally, we employ a cohomology argument to prove that for each ordered continuum \( \Lambda \), as \( \vec{p} \) varies there are \( 2^\omega \)-many nonhomeomorphic spaces \( S(\Lambda, \vec{p}) \).

1. Introduction

The present paper is concerned with nonmetric analogues of Vietoris solenoids. Recall that a 1-dimensional (Vietoris) solenoid is a compact and connected space (i.e. continuum) given as the inverse limit of circles, with \( n \)-fold covering maps as bonding maps. (The circle is a trivial example of a solenoid with \( n = 1 \).) Solenoids were first considered by Vietoris \cite{28} for \( n = 2 \), defined as what would now be called the mapping torus of a homeomorphism of the Cantor set. Van Dantzig \cite{6} defined and extensively studied \( n \)-adic solenoids for all integers \( n \geq 2 \). He gave two constructions, one via the mapping torus and another by using a nested sequence of solid tori in \( \mathbb{R}^3 \), and mentioned in passing that the latter construction could also be used to construct spaces for arbitrary products \( b_n = \prod_{i=1}^n n_i \) of integers \( n_i \geq 2 \) rather than powers \( n^\nu \). See \cite{12} for references to van Dantzig’s further work and an extensive discussion of solenoidal groups and \( a \)-adic solenoids, where \( a \) is a sequence of integers \( \geq 2 \). When Steenrod \cite{26} used a 2-adic solenoid as the inverse limit of circles in an example, he credited Vietoris for having defined the same space, which suggests that this modern representation had already passed into folklore. McCord \cite{17} introduced higher dimensional analogues of Vietoris solenoids, where the factor spaces in the inverse limit are connected, locally pathwise connected, and semi-locally simply connected spaces, and each bonding map is a regular covering map. Smale \cite{23} used the construction of solenoids in terms of a descending sequence of

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solid tori, to show that they can arise as hyperbolic attractors in smooth dynamical systems. His results were extended to higher dimensions by R.F. Williams [29]. Solenoids have been a major theme of research in algebraic and general topology, topological algebra, dynamical systems and the theory of foliations, as part of a wider class of the so-called matchbox manifolds (see e.g. [5] for recent results).

It is well known that each Vietoris solenoid $S$ is homogeneous and circle-like, and if $S$ is not the circle that it is also indecomposable. Recall that a space $S$ is homogeneous if for any two points $x$ and $y$ in $S$ there is a homeomorphism $h : S \to S$ such that $h(x) = y$. A continuum is indecomposable if it cannot be given as the union of two proper subcontinua. $S$ is said to be circle-like if for any open cover $U$ of $S$ there is a finite open subcover $\{U_1, \ldots, U_t\}$ that forms a circular chain; i.e. $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$(mod $t$). Hagopian and Rogers [10] classified homogeneous circle-like metric continua. According to their classification the following are the only such continua: Knaster’s pseudo-arc [14], Vietoris solenoids, and solenoids of pseudo-arcs [20]. A related, important circle-like metric continuum is R.H. Bing’s pseudo-circle [1]. Fearnley [8] and Rogers [21] independently showed that the pseudo-circle is not homogeneous (see also [27]). Later, Kennedy and Rogers [15] proved that the pseudo-circle is uncountably nonhomogeneous; i.e. it has uncountably many topological types of points. This fact motivated a lot of interest in degree of homogeneity of circle-like continua, that can be encapsulate in the following question:

**Question.** If a circle-like continuum is not homogeneous, how nonhomogeneous can it be?

To make the above question more precise, given a continuum $Y$ by Homeo($Y$) denote its homeomorphism group. We say that $Y$ is $\frac{1}{k}$-homogeneous if the action of Homeo($Y$) on $Y$ has exactly $k$ orbits, where $k$ is a cardinal number. So homogeneous spaces are $\frac{1}{1}$-homogeneous and the smaller the $k$ is the more homogeneous $Y$ is. Consequently, the pseudo-circle is $\frac{1}{\kappa}$-homogeneous, for an uncountable cardinal $\kappa$. At present time, however, it is an open question as whether $\kappa$ is equal to the cardinality of real numbers, and the answer may depend on the axioms of set theory. The last few years provided examples of $\frac{1}{k}$-homogeneous circle-like metric continua for the case when $k$ is a natural number. Neumann-Lara, Pellicer-Covarrubias and Puga-Espinosa [18] gave an example of a decomposable circle-like $\frac{1}{2}$-homogeneous continuum. They asked (Question 4.10 [18]) if there exists an indecomposable $\frac{1}{2}$-homogeneous circle-like continuum. Such an example (in fact a class of examples) was constructed by Pyrih and Vejnar [19], and independently by the first author [2]. Jiménez-Herández, Minc and Pellicer-Covarrubias [13] constructed a family of $\frac{1}{n}$-homogeneous solenoidal continua for every integer $n > 2$. Topologically inequivalent examples were also given by the first author [3].

In this note we provide a nonmetric positive answer to the question from [18], and then we go on to construct a $\frac{1}{k}$-homogeneous example with the same properties for any cardinal $k$, finite or infinite. For $k = 1$, the construction yields a nonmetric homogeneous circle-like indecomposable continuum. Each of our examples depends on an ordered continuum $\Lambda$ and a sequence $\vec{p}$ of positive integers. In the final section

of this paper, we employ a cohomology argument to prove that for each ordered continuum \( \Lambda \), as \( \vec{p} \) varies there are \( 2^\omega \)-many homeomorphism types.

Our notation for ordinals, cardinals, and ordinal arithmetic follows [19]. So, e.g., \( \omega \) is the least infinite ordinal and also denotes the least infinite cardinal, \( \omega_1 \) is the least uncountable ordinal and cardinal, etc. Also, an ordinal is the set of its predecessors, e.g., \( \omega = \{ n : n < \omega \} = \{ n : n \in \omega \} \) is the set of natural numbers, and \( \omega_1 \) is the set of countable ordinals. The set of positive integers is denoted by \( \mathbb{N} \).

2. The finite case

All of our examples have the following form. Take a compact connected linearly ordered space \( \Lambda \), and let \( \Sigma \) be the “circle” obtained by identifying the endpoints of \( \Lambda \), which we denote by \( \partial \Lambda \). Let \( \Sigma^{(n)} \) be \( n \) copies of \( \Lambda \) laid end to end, with the right endpoint of the \( i \)th copy identified with the left endpoint of the \( i + 1 \)st copy, \( i < n \), and the right endpoint of the \( n \)th copy identified with the left endpoint of the first copy. Let the identified endpoints be labeled \( \infty_0, \infty_1, \ldots, \infty_{n-1} \). Also, for \( x \in \Lambda \setminus \partial \Lambda \), let \( \infty_i + x \) denote its copy in the \( i \)th copy of \( \Lambda \) as a subset of \( \Sigma^{(n)} \).

If \( m \) and \( n \) are positive integers, there is a natural mapping \( \phi^m_n : \Sigma^{(mn)} \to \Sigma^{(n)} \) defined by \( \phi^m_n(\infty_i) = \infty_j \) and \( \phi^m_n(\infty_i + x) = \infty_j + x \), where \( x \in \Lambda \setminus \partial \Lambda \) and \( j = i \mod n \). Note that \( \phi^m_n \) is an \( m \)-fold covering map, precisely

\[
(\phi^m_n)^{-1}(\infty_j + x) = \{ \infty_{j+kn} + x : k = 0, 1, \ldots, m-1 \},
\]

and the same formula without \( x \) holds as well.

Let \( \vec{p} = (p_1, p_2, \ldots) \) be a sequence of positive integers \( \geq 2 \). Let

\[
S(\Lambda, \vec{p}) = \lim_{\mathbb{N}} \{ \phi^p_{k(n)}(\Sigma^{(k(n))}) \cap \Sigma^{(n)}, n \in \mathbb{N} \},
\]

where \( k(1) = 1 \) and \( k(n) = p_1 p_2 \cdots p_{n-1} \) for \( n > 1 \).

**Theorem 2.1.** For each \( \vec{p}, S(\Lambda, \vec{p}) \) is an indecomposable circle-like continuum.

**Proof.** **Claim 1.** \( S(\Lambda, \vec{p}) \) is circle-like.

It is well known that an inverse limit of circle-like continua is circle-like. Since each factor space \( \Sigma^{(k(n))} \) is circle-like so is \( S(\Lambda, \vec{p}) \).

**Claim 2.** \( S(\Lambda, \vec{p}) \) is indecomposable.

By contradiction suppose that there are two proper subcontinua \( C \) and \( G \) of \( S(\Lambda, \vec{p}) \) such that \( C \cup G = S(\Lambda, \vec{p}) \). Let \( C_n \) and \( G_n \) be the projections onto \( \Sigma^{(k(n))} \) of \( C \) and \( G \) respectively such that \( \Sigma^{(k(n))} \notin \{ C_n, G_n \} \). Since \( C_n \) (as well as \( G_n \)) is a proper subcontinuum of \( \Sigma^{(k(n))} \) it is an arc (perhaps nonmetric) that is a proper subset of \( \Sigma^{(k(n))} \). Note that \( (\phi^{p_{k(n)}}_{k(n)})^{-1}(C_n) \) and \( (\phi^{p_{k(n)}}_{k(n)})^{-1}(G_n) \) each consists of \( p_n \) disjoint homeomorphic copies of \( C_n \) and \( G_n \) respectively. Since \( \pi_{n+1}(C) \) is connected it follows that it misses a component of \( (\phi^{p_{k(n)}}_{k(n)})^{-1}(C_n) \). The same is true about \( \pi_{n+1}(G) \) with respect to \( (\phi^{p_{k(n)}}_{k(n)})^{-1}(G_n) \). Consequently \( \pi_{n+1}(C \cup G) = \pi_{n+1}(C) \cup \pi_{n+1}(G) \neq \Sigma^{(n+1)} \) contradicting surjectivity of \( \pi_{n+1} \). \( \square \)

Recall that the long line is the space obtained by putting a copy of the open unit interval \((0, 1)\) in between \( \alpha \) and \( \alpha + 1 \) for each \( \alpha \in \omega_1 \), and giving it the natural
order topology. We will refer to this line as the “standard” long line. The long line of length \( \kappa \) can be similarly defined for any ordinal \( \kappa \) (though it shouldn’t be considered “long” if \( \kappa < \omega_1 \)). Note that long lines are locally compact. The closed long line of length \( \kappa \) adds \( \kappa \) as a compactifying point. We’ll often use interval notation to denote these lines and subintervals thereof; e.g., \([0, \omega_1)\) is the standard long line, and \([0, \kappa)\) is the closed long line of length \( \kappa \).

Before stating our next result, let us describe some natural autohomeomorphisms of \( \Sigma^{(n)} \) when constructed from the ordered continuum \( \Lambda \) as previously described. Note that for each \( k = 0, 1, \ldots, n - 1 \), the “rotation” \( R_k \) of \( \Sigma^{(n)} \) that maps \( \infty_i + x \) to \( \infty_j + x \), where \( j = i + k \mod n \), is an autohomeomorphism of \( \Sigma^{(n)} \). Also, given a homeomorphism \( S: \Lambda \to \Lambda \) which leaves the endpoints fixed, the map \( \hat{S}_n \) of \( \Sigma^{(n)} \) which applies \( S \) to the interior of each of the \( n \) copies of \( \Lambda \) and leaves the points \( \infty_i, i = 0, 1, \ldots, n - 1 \) fixed is another autohomeomorphism. It is easily checked that these autohomeomorphisms commute with the bonding maps.

**Theorem 2.2.** Let \( \Lambda = [0, \omega_1) \) be the standard closed long line. Then for each sequence \( \vec{p} \) of positive integers, \( S(\Lambda, \vec{p}) \) is \( \frac{1}{\omega} \)-homogeneous.

**Proof.** Recall that \( \Sigma \) is obtained by indentifying the endpoints of \([0, \omega_1)\); let \( \infty \) denote the collapsed point \( \{0, \omega_1\} \). By a reasoning similar to [2], we shall show that the two orbits are given by \( O_1 = \{ \vec{s} \in S(\Lambda, \vec{p}) : s_1 \neq \infty \} \) and \( O_2 = \{ \vec{s} \in S(\Lambda, \vec{p}) : s_1 = \infty \} \).

First suppose \( \vec{s}, \vec{w} \in O_1 \). Choose \( \alpha < \omega_1 \) larger than both \( s_1 \) and \( w_1 \). Since \([0, \alpha] \) is a metric arc, there is an autohomeomorphism of \([0, \alpha] \) which maps \( s_1 \) to \( w_1 \) and keeps the endpoints fixed. Clearly this autohomeomorphism extends to an autohomeomorphism \( S \) of \( \Lambda \) which maps \( s_1 \) to \( w_1 \). Let \( H_1 = \hat{S}_1 \). For \( n > 1 \), note that \( s_n \) and \( w_n \) are points corresponding to \( s_1 \) and \( w_1 \), resp., in one of the copies of \( \Lambda \) making up \( \Sigma^{(k(n))} \), though they need not both lie in the same copy. So we let \( H_n = \hat{S}_{k(n)} \) followed by the appropriate rotation to take \( s_n \) to \( w_n \). Since these homeomorphisms commute with the bonding maps, the sequence \( H_1, H_2, \ldots \) defines an autohomeomorphism of the inverse limit space which maps \( \vec{s} \) to \( \vec{w} \).

If \( \vec{s}, \vec{w} \in O_2 \), then for each \( n, s_n \) and \( w_n \) are \( \infty_i \) and \( \infty_j \), for some \( i, j < n \), whence an appropriate rotation of \( \Sigma^{(k(n))} \) will send \( s_n \) to \( w_n \). So again we obtain an autohomeomorphism of the inverse limit space sending \( \vec{s} \) to \( \vec{w} \).

Finally, note that every point in \( O_1 \) is a point of first countability in \( S(\Lambda, \vec{p}) \), while no point of \( O_2 \) is \( G_\delta \). Thus no autohomeomorphism sends a point of \( O_1 \) to \( O_2 \). It follows that these two sets are precisely the orbits. \( \square \)

Let \( \Lambda_1 = [0, \omega_1) \), and let \( \Lambda_1^- = \Lambda_1 \setminus \{\omega_1\} \). In other words, \( \Lambda_1^- \) is the standard long line. Now let \( \Lambda_2^\circ = \mathbb{Z} \times \Lambda_1^- \) with the lexicographic order, where \( \mathbb{Z} \) is the set of integers. Note that the point \((n + 1, 0) = l.u.b. \{(n, x) : x \in \Lambda_1\} \) and so \((n + 1, 0) \) compactifies \( \{n\} \times \Lambda \). So \( \Lambda_2^\circ \) is a locally compact connected LOTS (linearly ordered topological space) with no first or last point. We may think of \( \Lambda_2^\circ \) as the real line with each open interval \((n, n + 1), n \in \mathbb{Z} \), replaced by the open long line \((0, \omega_1) \). For convenience, we denote the point \((n, x) \) in \( \Lambda_2^\circ \) by \( n + x \) and the point \( n + 0 = (n, 0) \) by \( n \). Now let \( \Lambda_2 = \Lambda_2^\circ \cup \{\infty, -\infty\} \) be the two point compactification of \( \Lambda_2^\circ \).
Let $\Lambda_0$, let $\Lambda_{n+1}$ be obtained from $\Lambda_n$ just like $\Lambda_2$ was obtained from $\Lambda_1$. I.e., $\Lambda_{n+1}$ is the two point compactification of $Z \times \Lambda_n^{-}$ with the lexicographic order, where $\Lambda_n^{-}$ is $\Lambda_n$ minus its right endpoint. It will be helpful to consider the following translation map on $\Lambda_n$’s. For $x$ in $\Lambda_n^+$, let $i + x$ denote the point $(i, x)$ in $\Lambda_2$. Now on $\Sigma$, $T_k$ is the map which sends $i + x$ to $(i + k) + x$ and $\infty$ is fixed. On $\Sigma^{(n)}$, $T_k$ does the same on each copy of $\Lambda_2$ and keeps the other points (i.e., $\infty_0, ..., \infty_{n-1}$) fixed. $T_k$ is defined analogously in the inductive step, where $\Lambda_{n+1}$ is the two point compactification of $Z \times \Lambda_n^{-}$.

We want to prove that $S(\Lambda_n, \bar{p})$ is $\frac{1}{n+1}$-homogeneous. It will be helpful to first prove the following lemma. Recall that it is well known that 1-dimensional Vietoris solenoids (inverse limits of circles with $p$-fold covering maps as bonding maps) have a base that consists of open sets homeomorphic to $\mathbb{C} \times (0, 1)$, where $\mathbb{C}$ is a Cantor set. It is easy to see that the non-metric solenoids we consider will have a similar property, where $(0, 1)$ is replaced by a basic open set (i.e., an arc) in $\Sigma$. For completeness sake we sketch a proof of this fact (in a more general form that has essentially the same proof).

**Lemma 2.3.** Let $S = \lim_{\leftarrow} \{\phi_n, \Sigma^{(n)}, n \in \mathbb{N}\}$ be an inverse limit of locally connected spaces in which the bonding maps are finite-to-one (but at least 2-to-one) covering maps. Suppose also that each point $x \in \Sigma$ is contained in an open set $O$ which for all $n$ is evenly covered by the map $\phi_n, 1 = \phi_1 \circ \phi_2 \circ ... \circ \phi_{n-1} : \Sigma^{(n)} \to \Sigma$. Then any point $x \in S$ has a local open base of sets $U$ homeomorphic to $U_1 \times \mathbb{C}$, where $U_1 = \pi_1(U)$ is connected and $\mathbb{C}$ is the Cantor set.

**Proof.** Let $\bar{p} \in S$, and suppose $U$ is an open subset of $S$ that contains $\bar{p}$. Without loss of generality, we may assume $U = \pi^{-1}_i(V)$ for some connected open subset $V$ of $X_i$ containing $p_i$. We may also assume that $U_1 = \pi_1(U) = \phi_{i-1,1}(V)$ is evenly covered by $\phi_{i,1}$ for all $n$. Let $C = \pi^{-1}_i(p_i)$. Then $C$ is closed, and contained in $\Pi_{j>i}(\phi_{j-1,i}(p_i))$, a product of finite sets. Hence $C$ is totally disconnected. Since $|\phi_{j-1,i}(x)| \geq 2$ for any $x$ and any $j > i$, it is easy to check that $C$ has no isolated points. Thus $C$ is a Cantor set.

Define $h : \pi^{-1}_i(V) \to V \times C$ by $h(\bar{x}) = (x_i, q_i)$, where $q_i \in C$ is such that for each $j > i$, $q_j$ and $x_j$ are in the same slice of $V$ with respect to $\phi_{j-1,i}$. It is straightforward to check that $h$ is a homeomorphism. Since $V \cong U_1$, we are done. 

The solenoids constructed in this paper satisfy the conditions of Lemma 2.3 where $O$ can be taken to be any proper arc contained in $\Sigma$. The following corollary will help us show that certain points are not in the same orbit class.

**Corollary 2.4.** Let $S$ be as in Lemma 2.3 and let $\bar{x}, \bar{y} \in S$. If there is an autohomeomorphism of $S$ taking $\bar{x}$ to $\bar{y}$, then any neighborhood $U_1$ of $y_1$ contains a homeomorphic copy of some neighborhood $V_1$ of $x_1$.

**Proof.** Let $h$ be an autohomeomorphism of $S$ taking $\bar{x}$ to $\bar{y}$, and let $U_1$ be a neighborhood of $y_1$. By Lemma 2.3, there is a neighborhood $W$ of $\bar{y}$ such that $W \cong W_1 \times \mathbb{C}$, where $W_1 = \pi_1(W)$ is connected and contained in $U_1$. Let $V$ be a neighborhood of $x$ such that $V \cong V_1 \times \mathbb{C}$, where $V_1 = \pi_1(V)$ is connected, and
Theorem 2.5. For each positive integer $\kappa \geq 1$ and sequence $\vec{p}$ of integers $\geq 2$, $S(\Lambda, \vec{p})$ is $\frac{1}{\kappa}$-homogeneous.

Proof. The proof is by induction. Theorem 2.2 takes care of the case $\kappa = 1$. It will be helpful to look at the case $\kappa = 2$ before describing the inductive step. So, let $S = S(\Lambda_2, \vec{p})$, and let $\infty$ denote the point $\partial \Lambda$ in $\Sigma$. We claim that the three orbits are

1. $O_1 = \{ \vec{x} \in S : x_1 = \infty \}$
2. $O_2 = \{ \vec{x} \in S : x_1 = k, k \in \mathbb{Z} \}$
3. $O_3 = \{ \vec{x} \in S : x_1 \notin \{ \infty \} \cup \mathbb{Z} \}$

If $\vec{x} \in O_3$, then $x_1$ corresponds topologically to a point $(> 0)$ in the long line $\Lambda$, and so does $x_n$ for all $n$ (indeed the $x_n$'s all correspond to the same point in $\Lambda$). Thus $\vec{x}$ is a point of first countability. Furthermore, $x_1$ has a neighborhood contained in the long line, so $\vec{x}$ has a neighborhood consisting of points of first countability.

If $\vec{x} \in O_2$, then $x_1$ corresponds to the compactifying point $\omega_1$ in $[0, \omega_1]$, and so does each $x_n$. So no coordinate is $G_\delta$, and it easily follows that $\vec{x}$ is not a $G_\delta$ point in $S$.

If $\vec{x} \in O_1$, then $x_1 = \infty$ and for $n > 1$, $x_n = \infty_i$ for some $i = 0, 1, \ldots, n - 1$. Note that $\infty_i = \lim_{k \to \infty} \infty_{i-1} + k = \lim_{k \to \infty} \infty_i + k$. So each $x_n$ is $G_\delta$, and it follows that $\vec{x}$ is $G_\delta$ and hence a point of first countability. But every neighborhood of $\vec{x}$ contains a point with first coordinate $k$ for some $k \in \mathbb{Z}$, i.e., a non-$G_\delta$-point in $O_2$.

It follows from the above discussion that no point in $O_i$, $i = 1, 2, 3$, is in the orbit of a point in $O_j$, $j \neq i$. So it remains to prove that for each i, if $\vec{x}, \vec{y} \in O_i$, then $\vec{y}$ is in the orbit of $\vec{x}$.

Suppose $\vec{x}, \vec{y} \in O_1$, i.e., $x_1 = y_1 = \infty$. Let $H_1 : \Sigma \to \Sigma$ be the identity. Suppose $n \geq 2$. Then there are $i, j < k(n)$ such that $x_n = \infty_i$ and $y_n = \infty_j$. Let $H_n : \Sigma^{(k(n))} \to \Sigma^{(k(n))}$ be the rotation $R_k$, where $k = j - i \mod k(n)$. Then $(H_n)_{n \in \mathbb{N}}$ defines an autohomeomorphism of $S$ that maps $\vec{x}$ to $\vec{y}$.

Suppose $\vec{x}, \vec{y} \in O_2$. Then $x_1 = y_1 = q$ for some $p, q \in \mathbb{Z}$, so let $H_1 = T_k$, where $k = q - p$. Suppose $n \geq 2$. There are $i, j < k(n)$ such that $x_n = \infty_i + p$, $y_n = \infty_j + q$. Let $H_n = R_l \circ T_k$, where $l = j - i \mod k(n)$. Note that $T_k$ will take $x_n = \infty_i + p$ to $y_n = \infty_i + q$, and then the rotation $R_l$ takes $y_n$ to $y_n$; hence $H_n(x_n) = y_n$. Since both $R_l$ and $T_k$ commute with the bonding maps, so does $H_n$. So again, $(H_n)_{n \in \mathbb{N}}$ defines an autohomeomorphism of $S$ that maps $\vec{x}$ to $\vec{y}$.

Finally, suppose $\vec{x}, \vec{y} \in O_3$. Then $x_1 = p + w, y_1 = q + z$ for some $p, q \in \mathbb{Z}$ and $w, z \in L \setminus \{0\}$. There is an autohomeomorphism $A$ of $\Lambda$ which maps $w$ to $z$. Let $A$ be the autohomeomorphism of $\Sigma$ as we described previously, i.e., which applies $A$ in every copy of $\Lambda$, and then let $H_1 = T_k \circ A$, where $k = q - p$. If $n \geq 2$, then $x_n = \infty_i + p + w, y_n = \infty_j + q + z$ for some $i, j < k(n)$. Then let $H_n = R_l \circ T_k \circ A$, where $l = j - i \mod k(n)$. Similar reasoning to the previous case shows that these $H_n$’s define an autohomeomorphism of $S$ that maps $\vec{x}$ to $\vec{y}$. 
To prove that the $\kappa + 2$-orbit classes we will define for $S(\Lambda_{\kappa+1}, \vec{p})$ really are different requires us to define the following “types” of points: let $p$ be a point in some $\Lambda_\kappa$ or $\Lambda_\kappa/\partial \Lambda_\kappa$. Then $p$ is of

1. Type 1 if $p$ has a neighborhood $N$ such that every point of $N$ is a point of first countability;
2. Type 2 if $p$ is not $G_\delta$, and has a neighborhood $N$ such that $p$ is the only non-$G_\delta$ point in $N$;
3. Type 3 if $p$ is a limit point of Type 2 points, and has a neighborhood $N$ such that $p$ is the only limit of Type 2 points in $N$.

$(\kappa + 1)$ Type $\kappa + 1$, for $\kappa \geq 3$, if $p$ is a limit point of Type $\kappa$ points, and has a neighborhood $N$ such that $p$ is the only limit of Type $\kappa$ points in $N$.

Now suppose $\Lambda_\kappa$ satisfies:

(i) $\Lambda_\kappa$ has points of Type $i$ for $i = 1, 2, ..., \kappa + 1$, and each point of $\Lambda_\kappa$ is one of these types;
(ii) The endpoints of $\Lambda_\kappa$ are the only points in $\Lambda_\kappa$ of Type $\kappa + 1$;
(iii) If $-\infty < x, y < \infty$ are the same type in $\Lambda_\kappa$, then there is an autohomeomorphism $h$ of $\Lambda_\kappa$ mapping $x$ to $y$ and keeping the endpoints fixed.

It is easily checked that $\Lambda_3$ satisfies the above conditions. Now we show that if $\Lambda_\kappa$ satisfies these conditions, then $S = S(\Lambda_{\kappa+1}, \vec{p})$ has orbit classes $O_i = \{ \vec{x} \in S : x_1 \text{ has Type } i \}$ for $i = 1, 2, ..., \kappa + 2$.

Recall that $\Lambda_{\kappa+1}$ is the two point compactification of $\Lambda_{\kappa+1}^0 = \mathbb{Z} \times \Lambda_\kappa^-$ with the lexicographic order, where $\Lambda_\kappa^- = \Lambda_\kappa \setminus \{ \max \Lambda \}$. $\Lambda_{\kappa+1}^0$ can be thought of as the real line with each open interval $(k, k+1)$ replaced by $\Lambda_\kappa \setminus \partial \Lambda$. By condition (ii) above, the points $k = (k, \min \Lambda)$ are the only Type $\kappa + 1$ points in $\Lambda_{\kappa+1}^0$, and note that this makes the endpoints of $\Lambda_{\kappa+1}$ the only Type $\kappa + 2$ points in that space, and hence the point $\infty$ of the corresponding quotient space $\Sigma$ the only Type $\kappa + 2$ point there.

Let $\vec{x} \in O_i$ and $\vec{y} \in O_j$ with $i \neq j$. We show that there is no autohomeomorphism of $S$ taking $\vec{x}$ to $\vec{y}$. W.l.o.g., $i > j$. By Corollary 2.4, every neighborhood of $y_1$ contains a copy of some neighborhood of $x_1$. By an easy induction, every neighborhood of $x_i$ contains non-$G_\delta$ points. So we have a contraction if $j = 1$, since $y_1$ then has a neighborhood of all $G_\delta$ points. Suppose $j > 0$. Another easy induction shows that every neighborhood of $x_i$ contains infinitely many points of Type $j$. So again we have a contradiction since $y_1$ has a neighborhood $N$ with only one Type $j$ point. It follows that no autohomeomorphism of $S$ maps $\vec{x}$ to $\vec{y}$.

Suppose now that $\vec{x}$ and $\vec{y}$ are in the same $O_a$. If $a = \kappa + 2$, then $x_1 = y_1 = \infty$, and every $x_m$ (resp. $y_m$) for $m \geq 2$ is $\infty_i$ (resp. $\infty_j$) for some $i, j < m$. Thus the appropriate rotation $R_k$ will map $x_m$ to $y_m$ and commute with the bonding maps; it follows that there is an autohomeomorphism of $S$ mapping $\vec{x}$ to $\vec{y}$. If $a = \kappa + 1$, then $x_1$ and $y_1$ correspond to points in $\mathbb{Z}$ in $\Lambda_{\kappa+1}^\gamma$. This case is easily taken care of by the same argument as for $O_2$ in the case $\kappa = 2$.

If $a < \kappa + 1$, the reasoning is similar to that of $O_3$ above. To wit, $x_1 = p + w, y_1 = q + z$ for some $p, q, \in \mathbb{Z}$ and $w, z \in \Lambda_\kappa \setminus \partial \Lambda$, where $w$ and $z$ are of the same type. By assumption $(iii)$ on $\Lambda_\kappa$, there is an autohomeomorphism $A$ of $\Lambda_\kappa$ which maps $w$ as
to \(z\) and fixes the endpoints. Let \(H_1 = \hat{A}_1\) be the autohomeomorphism of \(\Sigma\) which applies \(A\) in every copy of \(\Lambda_\kappa\) making up \(\Lambda_\kappa^{\kappa+1}\), and then let \(H_1 = T_k \circ \hat{A}\), where \(k = q - p\). If \(m \geq 2\), then \(x_m = \infty_i + p + w, y_m = \infty_j + q + z\) for some \(i, j < k(m)\). Then let \(H_m = R_l \circ T_k \circ \hat{A}\), where \(l = j - i \mod k(m)\). Similar reasoning to previous cases shows that these \(H_m\)'s define an autohomeomorphism of \(S\) that maps \(\vec{x}\) to \(\vec{y}\).

Finally, we need to prove that \(\Lambda_\kappa^{\kappa+1}\) satisfies the inductive conditions. Condition (i) is easily seen, and condition (ii) was already mentioned. It remains to check (iii).

Suppose \(x, y \in \Lambda_\kappa^{\kappa+1}\) are of the same type. If that type is \(\kappa + 1\), then \(x = p, y = q\) for some \(p, q \in \mathbb{Z}\), so we can let \(h = T_k\), where \(k = q - p\). If the type is \(< \kappa + 1\), then \(x = p + w, y = q + z\) for some \(p, q \in \mathbb{Z}\) and \(w, z \in \Lambda_\kappa^\kappa\), where \(w\) and \(z\) are of the same type. By assumption (iii) on \(\Lambda_n\), there is an autohomeomorphism \(A\) of \(\Lambda_n\) which maps \(w\) to \(z\) and fixes the endpoints. Let \(h = T_k \circ A\), where \(A\) is the map which applies \(A\) to each copy of \(\Lambda_\kappa\) making up \(\Lambda_\kappa^{\kappa+1}\) and keeps other points fixed. Then \(h\) is an autohomeomorphism of \(\Lambda_\kappa^{\kappa+1}\) which maps \(x\) to \(y\) and keeps the endpoints fixed.

We finish this section with our example for \(\kappa = 1\); i.e. a nonmetric homogeneous circle-like indecomposable continuum. Recall that, as mentioned in Introduction, all Vietoris solenoids are homogeneous, and metric homogeneous circle-like continua were classified in [20], but there do not seem to be any results in the literature that would explicitly prove the classification incomplete in the nonmetric case. In addition, in [9] Gutek and Hagopian asked if there exists a nonmetrizable circle-like homogeneous indecomposable continuum having only arcs for nondegenerate proper subcontinua. Our example satisfies all but the last mentioned property, and it should be quite clear that all of its proper subcontinua are homeomorphic to an order-homogeneous nonmetric arc.

**Example 2.6.** There is an ordered continuum \(\Lambda\) such that \(S(\Lambda, \vec{p})\) is a nonmetric homogeneous indecomposable circle-like continuum.

**Proof.** Let \(\Lambda\) be any nonmetric ordered continuum which is order-homogeneous, i.e., \(\Lambda \cong [x, y]\) for any \(x < y \in \Lambda\). Such spaces (with additional properties not relevant here) have been constructed by, for example, K.P. Hart and J. van Mill [11]. Using methods of this section, it is easy to see that the corresponding spaces \(\Sigma^{(n)}\) and \(S(\Lambda, \vec{p})\) are homogeneous. \(S(\Lambda, \vec{p})\) is nonmetric because it admits a continuous surjection onto the nonmetric first coordinate, and it is indecomposable and circle-like by Theorem 2.1.

In addition to the above example, in private communication, Michel Smith informed us that he conjectures the results in [24] could be used to exhibit other nonmetric homogeneous circle-like indecomposable continua.

3. \(1/\kappa\) Homogeneous for Infinite \(\kappa\)

In the section, we show that for any infinite cardinal \(\kappa\), there is a \(1/\kappa\) homogeneous indecomposable circle-like continuum. These continua are also of the form \(S(\Lambda, \vec{p})\) for some ordered continuum \(\Lambda\), but now \(\Lambda\) is going to be simply the long
line of some ordinal length. We can also think of these $\Lambda$ as being obtained by putting copies of the standard long line end to end some ordinal number of times.

Ordinal multiplication will be useful here, so we recall some basics (see, e.g., the first chapter of [16] for an excellent sketch of ordinal arithmetic). If $\alpha$ and $\beta$ are ordinals, then $\alpha \cdot \beta$ is the ordinal whose order type is that of the ordinal $\alpha$ (recall an ordinal may be thought of as the set of its predecessors) laid end to end $\beta$ times. Formally, we can define $\alpha \cdot \beta$ as the ordinal whose order type is that of $\beta \times \alpha$ with the lexicographic order. Multiplication is not commutative, e.g., $\omega \cdot 2$ is equal to two copies of the standard long line end to end; it is the same as $\omega + \omega$. However, $2 \cdot \omega$ is the ordinal 2 laid end to end $\omega$ times; note that the resulting order type is $\omega$, so $2 \cdot \omega = \omega$.

Ordinal exponentiation is defined inductively. $\gamma^0 = 1$, $\gamma^{\beta+1} = \gamma^\beta \cdot \gamma$, and if $\alpha$ is a limit ordinal, $\gamma^\alpha = \sup\{\gamma^\beta : \beta < \alpha\}$. So, e.g., $\omega^2 = \omega \cdot \omega$, $\omega^3 = \omega^2 \cdot \omega$, etc., and $\omega^n = \sup\{\omega^n : n < \omega\}$. In particular, note that $\omega^n$ is a countable ordinal.

The following lemma will be useful to determine a lower bound for the number of orbit classes of some $S(\Lambda, \vec{p})$’s where $\Lambda$ is a long line of some ordinal length.

**Lemma 3.1.** Let $\delta = \omega_1 \cdot \omega^n$, where $\alpha$ is some ordinal, and consider the closed long line $[0, \delta]$. Then for any $x < \delta$, the interval $[x, \delta]$ cannot be homeomorphically embedded in $[0, y]$ for any $y < \delta$.

**Proof.** By induction on $\alpha$. If $\alpha = 0$, then $\delta = \omega_1 \cdot \omega^0 = \omega_1 \cdot 1 = \omega_1$, and the result is well-known and easy to prove (e.g., $\omega_1$ is not $G_3$ in $[0, \omega_1]$, but any point $< \omega_1$ is $G_3$). So suppose $\alpha > 0$ and the result holds for any $\beta < \alpha$.

**Case 1.** $\alpha$ is a limit ordinal. Suppose by way of contradiction that $h : [x, \delta] \to [0, y]$ is a homeomorphic embedding, where $y < \delta$. Choose $\beta < \alpha$ such that $\gamma = \omega_1 \cdot \omega^\beta$ is greater than $\max\{x, y\}$. Then $h$ embeds $[x, \gamma]$ homeomorphically into $[0, y]$ with $y < \gamma$, contradicting the induction hypothesis.

**Case 2.** $\alpha = \beta + 1$. Let $\gamma = \omega_1 \cdot \omega^\beta$, and note that $\delta = \omega_1 \cdot \omega^n = \omega_1 \cdot \omega^{\beta+1} = \omega_1 \cdot \omega^\beta \cdot \omega = \gamma \cdot \omega$. So $[0, \delta]$ is the same as countably many copies of $[0, \gamma]$ laid end to end in order type $\omega$, and $\delta = \sup\{\gamma \cdot n : n < \omega\}$.

Suppose $h : [x, \delta] \to [0, y]$ is a homeomorphic embedding, where $y < \delta$. Let $n < \omega$ be least such that $\gamma \cdot n > h(\delta)$. Choose $z$ with $x \leq z < \delta$ such that $h(z) \geq \gamma \cdot (n - 1)$, and then let $m < \omega$ be least such that $\gamma \cdot m > z$. Then $h$ embeds $[z, \gamma \cdot m]$ homeomorphically into $[\gamma \cdot (n - 1), h(\delta)]$, and $h(\delta) < \gamma \cdot n$. Since $[z, \gamma \cdot m] \cong [x', \gamma]$ for some $x' < \gamma$, and $[\gamma \cdot (n - 1), \gamma \cdot n] \cong [0, \gamma]$, this contradicts the induction hypothesis. □

**Corollary 3.2.** Let $\Lambda$ be a closed long line of some ordinal length. If $\vec{x}, \vec{y} \in S(\Lambda, \vec{p})$, $x_1 = \omega_1 \cdot \omega^n$, and $y_1 = \omega_1 \cdot \omega^\beta$, with $\alpha \neq \beta$, then no autohomeomorphism of $S(\Lambda, \vec{p})$ maps $\vec{x}$ to $\vec{y}$.

**Proof.** W.l.o.g., $\alpha > \beta$. By Corollary 3.1 every neighborhood of $y_1$ must contain a homeomorphic copy of a neighborhood of $x_1$. But this would contradict Lemma 3.1. □

The next result shows that for any infinite cardinal $\kappa$, there is a $1/\kappa$ homogeneous space of the form $S(\Lambda, \vec{p})$; in fact, taking $\Lambda$ to be the closed long line of length $\omega_1 \cdot \omega^\kappa$ works. If $\kappa = \omega$, then this says that taking countably many copies of the standard
long line and lining them up in type \( \omega \omega \) works. As noted above, \( \omega \omega \) is a countable ordinal, so \( \omega_1 \cdot \omega \omega \) is an ordinal strictly between the cardinals \( \omega_1 \) and \( \omega_2 \). So is \( \omega_1 \cdot \omega \omega \), but this can be simplified. Indeed, for any uncountable cardinal \( \kappa \), \( \omega \kappa = \kappa \).

(The reason: \( \alpha < \beta \) implies \( \omega ^\alpha < \omega ^\beta \), so \( \omega ^\kappa \) has to be at least \( \kappa \), and one may show by induction that \( \alpha < \kappa \) implies \( \omega ^\alpha < \kappa \), so it can’t be more than \( \kappa \).) So \( \omega_1 \cdot \omega \omega = \omega_1 \cdot \omega_1 = \omega_1 ^2 \), and hence the long line of this length is the same as the standard long line laid end to end \( \omega_1 \) times. If \( \kappa \geq \omega_2 \), the formula can be simplified even more: \( \omega_1 \cdot \omega ^\kappa = \omega_1 \cdot \kappa = \kappa \). Now we will prove:

**Theorem 3.3.** Let \( \kappa \) be an infinite cardinal. If \( \Lambda \) is the closed long line of length \( \omega_1 \cdot \omega ^\kappa \), then \( S(\Lambda, \vec{p}) \) is \( 1/\kappa \) homogeneous. In particular, the long line of length \( \omega_1 \cdot \omega ^\omega \) yields a \( 1/\omega \) homogeneous continuum, the one of length \( \omega_2 ^\omega \) yields a \( 1/\omega_1 \) homogeneous continuum, and for \( \kappa \geq \omega_2 \), the long line of length \( \kappa \) yields a \( 1/\kappa \) homogeneous continuum.

**Proof.** Let \( \kappa \) be an infinite cardinal, let \( \Lambda \) be the closed long line of length \( \delta = \omega_1 \cdot \omega ^\kappa \).

We will show that \( S(\Lambda, \vec{p}) \) is \( 1/\kappa \) homogeneous; the “In particular...” then follows by the remarks in the preceding paragraph. That \( S(\Lambda, \vec{p}) \) has at least \( \kappa \) many orbit classes is immediate from Corollary 3.2.

It remains to show that there are no more than \( \kappa \) many orbit classes. It is easy to see that one may use the rotations \( R_j \) as previously defined to show that if \( x_1 = y_1 \), then \( \vec{x} \) and \( \vec{y} \) are in the same orbit class. Let \( NG \) be the set of all points \( x \in \Sigma \) that are either not \( G_\delta \) in \( \Sigma \) or are a limit of non-\( G_\delta \) points. For each \( x \in NG \),

\[ O_x = \{ \vec{x} \in S(\Lambda, \vec{p}) : x_1 = x \} \]

is either an orbit class or is properly contained in one (we don’t know which, but conjecture the former). Note that \( NG \) is closed in \( \Sigma \), and that \( \Sigma \setminus NG \) breaks up into \( \kappa \) many components each homeomorphic in a natural way to the standard open long line \((0, \omega_1)\). Indeed, note that if \( \alpha \in NG \), \( \alpha < \kappa \), then \( \alpha + \omega_1 \) is the least point in \( NG \) greater than \( \alpha \). It follows that \( \Sigma \setminus NG \) is equal to

\[ \bigcup \{ (\alpha, \alpha + \omega_1) : \alpha < \kappa, \alpha \in NG, \text{ or } \alpha = 0 \} \]

If \( x_1 \) and \( y_1 \) fall into the same maximal interval \((\alpha, \alpha + \omega_1)\) of \( \Sigma \setminus NG \), then there is an autohomeorphism \( h \) of \([\alpha, \alpha + \omega_1]\) sending \( x_1 \) to \( y_1 \) (and leaving \( \alpha \) and \( \alpha + \omega_1 \) fixed, as it must). Note that each \( x_n \) and \( y_n \) correspond to \( x_1 \) and \( y_1 \) in one of the \( k(n) \) copies of \( \Lambda \) making up \( \Sigma^{(k(n))} \). So for each \( n \), an autohomeomorphism of \( \Sigma^{(k(n))} \) which applies \( h \) to the interval \((\alpha, \alpha + \omega_1)\) in each copy of \( \Lambda \) in \( \Sigma^{(k(n))} \), and leaves other points fixed, followed by the appropriate rotation, will take \( x_n \) to \( y_n \) and commute with the bonding maps. The resulting homeomorphism of \( S(\Lambda, \vec{p}) \) takes \( \vec{x} \) to \( \vec{y} \).

Thus the following are either orbit classes or proper subsets of an orbit class:

(i) For each \( \alpha \in NG \), \( O_\alpha = \{ \vec{x} \in S(\Lambda, \vec{p}) : x_1 = \alpha \} \};

(ii) For each maximal connected interval \((\alpha, \alpha + \omega_1)\) in \( \Sigma \setminus NG \), \( P_\alpha = \{ \vec{x} \in S(\Lambda, \vec{p}) : x_1 \in (\alpha, \alpha + \omega_1) \} \).

As \( |NG| = \kappa \), \( S(\Lambda, \vec{p}) \) has at most \( \kappa \) many orbit classes. \( \square \)
Conjecture. The sets given in (i) and (ii) above are precisely the orbit classes of $S(\Lambda, \vec{p})$.

4. Cohomology of spaces with related linear ordering

In this section we show that for fixed $\Lambda$, as $\vec{p}$ varies we obtain $2^\omega$-many non-homeomorphic spaces $S(\Lambda, \vec{p})$. As in the metric case, this is a cohomology argument (see [17]). We obtain a theorem on the cohomology of certain quotients of ordered continua, and then apply this theorem to our specific situation to conclude that the first cohomology group of $S(\Lambda, \vec{p})$ with coefficients in $\mathbb{Z}$ is the group $\mathbb{Q}(\vec{p})$ of all rationals of the form $m/p_1 \cdots p_n$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. As there are $2^\omega$-many such groups, we have our desired result.

In what follows, all spaces will be compact Hausdorff, and $G$ will be an abelian group. $H^q$ will denote the Čech-Alexander-Spanier cohomology group in dimension $q$ with coefficients $G$. Expositions are found in Chapter 6 of Spanier [25] (where $H^q$ is written $\check{H}^q$), which serves as a reference for this section, and Eilenberg and Steenrod [7]. The proofs will be based on the well known properties of the theory rather than some specific construction.

Definitions.

(1) By an ordered continuum is meant a nondegenerate compact connected space $\Lambda$ which has a linear order and whose topology is the order topology. If $a, b \in \Lambda$, then $[a, b]$ is the set of all $x \in \Lambda$ such that $a \leq x \leq b$, and $\partial \Lambda$ is the set consisting of the first and last points of $\Lambda$.

(2) If $\Lambda'$ is another ordered continuum, then the assertion that a map $\theta: \Lambda \to \Lambda'$ is compliant means that $\theta$ maps the first point of $\Lambda$ to the first point of $\Lambda'$ and the last point of $\Lambda$ to the last point of $\Lambda'$.

(3) If $\Sigma$ is obtained from $\Lambda$ by collapsing $\partial \Lambda$ with quotient map $\pi$, a map $\phi: \Sigma \to S^1$ is standard means that there is a compliant map $\theta: \Lambda \to I$, and $\phi = \sigma \theta \pi^{-1}$, where $S^1$ will be treated as obtained from $\mathbb{I}$ by collapsing $\partial \mathbb{I}$ with quotient map $\sigma: \mathbb{I} \to S^1$.

Remark. Expressions like $\sigma \theta \pi^{-1}$ will be used only if they are single-valued and therefore define functions.

The following result is known (see e.g. Exercise 2 of Chapter 2 in [4]), but early references are obscure.

Lemma 4.1. If $\Lambda$ is an ordered continuum, then $\widetilde{H}^0(\Lambda) = 0$ for every integer $q$.

Proof. For all spaces, $\widetilde{H}^q = H^q = 0$ if $q < 0$. It follows from the connectedness of $\Lambda$ that $\widetilde{H}^0(\Lambda) = 0$. Let $q$ be a positive integer, and let $u \in H^q(\Lambda)$. For each $x \in \Lambda$ there is a closed neighborhood $N_x$ such that $i_{\Lambda, x}^* N_x(u) = 0$, where in general $i_{\Lambda, x}: A \subset X$ is the inclusion map. A finite number of these sets cover $\Lambda$, and one may find a finite number of points $a_0 < a_1 < \cdots < a_n$ of $\Lambda$ such that for $j = 1, \ldots, n$ the sets $[a_{j-1}, a_j]$ cover $\Lambda$ and further satisfy $i_{\Lambda, [a_{j-1}, a_j]}^* u = 0$. Proceed by induction on $n$, starting trivially with $n = 1$. For the inductive step $i_{\Lambda, [a_1, a_{n-1}]}^* u = 0$; let $\Lambda_1 = [a_1, a_{n-1}]$ and $\Lambda_2 = [a_{n-1}, a_n]$. A portion of the
Mayer–Vietoris sequence gives the exact sequence
\[ 0 = \tilde{H}^q(\{a_{n-1}\}) \to H^q(\Lambda) \xrightarrow{(\alpha_1, \alpha_2)} H^q(\Lambda_1) \oplus H^q(\Lambda_2), \]
where \(\alpha_1 = i^*_{\Lambda, [a_1, a_{n-1}]}\) and \(\alpha_2 = i^*_{\Lambda, [a_{n-1}, a_n]}\). Exactness implies that the map \((\alpha_1, \alpha_2)\) is injective, and since \(\alpha_1(u) = 0\) and \(\alpha_2(u) = 0\), it follows that \(u = 0\).

**Lemma 4.2.** If \(\Sigma\) be obtained from \(\Lambda\) by collapsing \(\partial\Lambda\), then there exist standard maps \(\phi: \Sigma \to S^1\) and if \(\phi\) and \(\phi'\) are standard maps, then \(\phi \simeq \phi'\).

**Proof.** Let \(\pi: \Lambda \to \Sigma\) be the quotient map. By Urysohn’s Lemma there is a map \(\theta: \Lambda \to I\) which maps the first point of \(\Lambda\) to 0 and the last point of \(\Lambda\) to 1 so that \(\phi = \sigma \theta \pi^{-1}\) is a standard map. If \(\phi\) and \(\phi'\) are defined by \(\theta\) and \(\theta'\) respectively, then by Tietze’s Theorem the maps \(\theta\) and \(\theta'\) are homotopic rel \(\partial\Lambda\), which induces a homotopy \(\phi \simeq \phi'\).

**Lemma 4.3.** If \(\theta: \Lambda \to I\) is a compliant map, then the map \(\theta_0: (\Lambda, \partial\Lambda) \to (I, \partial I)\) defined by \(\theta\) induces an isomorphism \(H^1(I, \partial I) \cong H^1(\Lambda, \partial\Lambda)\).

**Proof.** Consider the long exact ladder for \(\theta\) in reduced cohomology:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{H}^0(\partial I) & \xrightarrow{\delta^*} & H^1(I, \partial I) & \longrightarrow & H^1(I) & = 0 \\
& & \theta_0^\intercal \downarrow \approx & & \theta_0^\intercal \downarrow & & & \\
0 & \longrightarrow & \tilde{H}^0(\partial\Lambda) & \xrightarrow{\delta^*} & H^1(\Lambda, \partial\Lambda) & \longrightarrow & H^1(\Lambda) & = 0
\end{array}
\]

Since the map \(\theta_0: \partial\Lambda \to \partial I\) defined by \(\theta\) is a homeomorphism, the isomorphism on the left follows.

**Lemma 4.4.** If \(\Sigma\) is obtained from \(\Lambda\) by collapsing \(\partial\Lambda\) and the map \(\phi: \Sigma \to S^1\) is a standard map, then \(\phi^*: H^1(S^1) \to H^1(\Sigma)\) is an isomorphism.

**Proof.** Let \(\pi: \Lambda \to \Sigma\) be the quotient map, let \(\theta: \Lambda \to I\) be a compliant map with \(\phi = \sigma \theta \pi^{-1}\), let \(\Pi\) be the one point set which is the image of \(\partial\Lambda\), and let \(\phi_0: \Sigma, \Pi \to S^1, P\) be defined by \(\phi\). The portion of the long exact ladder in reduced cohomology for \(\phi\) has the portion

\[
\begin{array}{ccccccccc}
0 = \tilde{H}^0(P) & \longrightarrow & H^1(S^1, P) & \xrightarrow{\approx} & H^1(S^1) & \longrightarrow & 0 \\
& & \phi_0^\intercal \downarrow \approx & & \phi^* \downarrow & & & \\
0 = \tilde{H}^0(\Pi) & \longrightarrow & H^1(\Sigma, \Pi) & \xrightarrow{\approx} & H^1(\Sigma) & \longrightarrow & 0
\end{array}
\]

From this and Lemma 1.3 results the diagram

\[
\begin{array}{ccccccccc}
H^1(I, \partial I) & \xleftarrow{\approx} & H^1(S^1, P) & \xrightarrow{\approx} & H^1(S^1) \\
\downarrow \approx & & \phi_0^\intercal \downarrow & & \phi^* \downarrow & & & \\
H^1(\Lambda, \partial\Lambda) & \xleftarrow{\approx} & H^1(\Sigma, \Pi) & \xrightarrow{\approx} & H^1(\Sigma)
\end{array}
\]

where the two isomorphisms on the upper and lower left are induced by the relative homeomorphisms \((I, \partial I) \to (S^1, P)\) and \((\Lambda, \partial\Lambda) \to (\Sigma, \Pi)\) defined by \(\sigma\) and \(\pi\), respectively.
respectively. It follows that $\phi^*_0$ is an isomorphism, and then that $\phi^*$ is an isomorphism. \hfill $\Box$

The following lemma formalizes a technique which occasionally appears in the development of local degree theory for maps between manifolds.

**Lemma 4.5.** Let $X$ be a compact space and $U_1, \ldots, U_n$ nonempty pairwise disjoint open subsets of $X$ with union $U$. Suppose $f: X \to Y$ maps $X \setminus U$ to a one point subset $B$ of $Y$, and for $j = 1, \ldots, n$, $f_j: X \to Y$ agrees with $f$ on $U_j$ and maps $X \setminus U_j$ and all the other sets $U_k$ to $B$. Then for all positive integers $q$, $f^* = f_1^* + \cdots + f_n^*: H^q(Y) \to H^q(X)$.

**Proof.** Let $W$ be the space obtained from $X$ by collapsing $X \setminus U$ to a point $w$. Let $\pi: X \to W$ be the quotient map. For each $j = 1, \ldots, n$ let $W_j$ be the closure of the image $\pi(U_j)$, let $r_j: W \to W_j$ be the retraction which sends each $W_k$ for $k \neq j$ to $w$, and let $e_j: W_j \subset W$ be the inclusion map. The map $f$ induces a map $g: W \to Y$ such that $f = g\pi$, and for each $j$, $f_j = g\pi_j r_j\pi$.

It is a standard result that $H^q(W)$ is the direct sum of the groups $H^q(W_1), \ldots, H^q(W_n)$; in particular, the following equation holds:

$$f^* = \pi^* g^* = \pi^*(r_1^* e_1^* + \cdots + r_n^* e_n^*) g^* = \pi^* r_1^* e_1^* g^* + \cdots + \pi^* r_n^* e_n^* g^* = f_1^* + \cdots + f_n^* \hfill \Box$$

**Theorem 4.6.** Assume that

(a) $\Sigma$ and $\Sigma'$ are obtained from ordered continua $\Lambda$ and $\Lambda'$ by collapsing $\partial \Lambda$ and $\partial \Lambda'$ respectively;
(b) $a_0 < a_1 < \cdots < a_n$ are points of $\Lambda$ with $a_0$ the first point of $\Lambda$ and $a_n$ the last point of $\Lambda$;
(c) For $j = 1, \ldots, n$ there are compliant maps $\gamma_j: \Lambda_j \to \Lambda'$, where $\Lambda_j = [a_{j-1}, a_j]$;
(d) $f: \Sigma \to \Sigma'$ is defined by the equations $f(\pi(x)) = \pi'(\gamma_j(x))$ for all $x \in \Lambda_j$ for $j = 1, \ldots, n$, where $\pi: \Lambda \to \Sigma$ and $\pi': \Lambda' \to \Sigma'$ are the respective quotient maps.

Then for any standard maps $\phi: \Sigma \to S^1$ and $\phi': \Sigma \to S^1$ the map $f^*$ is multiplication by $n$ with respect to the isomorphisms $\phi^*$ and $\phi'^*$, as in the diagram

$$
\begin{array}{ccc}
H^1(\Sigma') & \xrightarrow{f^*} & H^1(\Sigma) \\
\phi'^* \approx & & \phi^* \approx \\
H^1(S^1) & \xrightarrow{n} & H^1(S^1)
\end{array}
$$

**Proof.** Let $\theta: \Lambda \to \Sigma$ and $\theta': \Lambda' \to \Sigma'$ be compliant maps with $\phi = \sigma \theta \pi^{-1}$ and $\phi' = \sigma \theta' (\pi')^{-1}$. For $j = 1, \ldots, n$ let $\tilde{\gamma}_j: \Lambda \to \Lambda'$ be a compliant map which satisfies the additional conditions that $\tilde{\gamma}_j | \Lambda_j = \gamma_j$ and $\tilde{\gamma}_j$ collapses each $\Lambda_k$ with $k \neq j$. 

...
to one of the endpoints of $\Lambda'$, and let $f_j: \Sigma \to \Sigma'$ be defined by the equations $f(\pi(x)) = \pi'(\gamma_j(x))$ for all $x \in \Lambda_j$.

For $j = 1, \ldots, n$ the map $\theta_j$ defined by $\theta_j = \theta_j \gamma_j$ is a compliant map, and with $\phi_j = \sigma \theta_j \pi$ all the maps $\phi$ and $\phi_1, \ldots, \phi_n$ are standard and therefore homotopic by Lemma 4.5, which implies that $\phi^* = \phi_1^* = \cdots = \phi_n^*$.

For each $j = 1, \ldots, n$

- $\phi^* f_j \pi = \phi^* \pi' \gamma_j$ follows from $f_j \pi = \pi' \gamma_j$ (defined by $\gamma_j$);
- $\phi^* \pi' \gamma_j = \sigma \theta^* \gamma_j$ follows from $\phi^* \pi' = \sigma \theta^* (\phi^*$ defined by $\theta^*)$;
- $\sigma \theta^* \gamma_j = \sigma \theta_j$ follows from $\theta^* \gamma_j = \theta_j$ (definition above) and finally,

$$\sigma \theta_j = \phi_j \pi (\phi_j$ defined by $\theta_j)$.

Hence $\phi^* f_j = \phi_j$ and therefore $f_j^* \phi^* = \phi_j^* = \phi^*$ for $j = 1, \ldots, n$. Since $\phi^*$ and $\phi^*$ are isomorphisms by Lemma 4.3, these equations imply that $1_{\Sigma^*} = (\phi^*)^{-1} f_j^* \phi^* = \cdots = (\phi^*)^{-1} f_n^* \phi^*$, and then that $f_1^* = \cdots = f_n^*$. Since $f^* = f_1^* + \cdots + f_n^*$ by Lemma 4.3, $f^* = n f_1^*$, and therefore $(\phi^*)^{-1} f^* \phi^* = (\phi^*)^{-1} n f_1^* \phi^* = n(\phi^*)^{-1} f_1^* \phi^* = n 1_{\Sigma^*}$.

Next we apply this theorem to our specific situation. As before, let $\Lambda$ be an ordered continuum and let $\Lambda^{(n)}$ be $n$ copies of $\Lambda$ laid end to end as $\Lambda_1 \sqcup \Lambda_2 \sqcup \cdots \sqcup \Lambda_n$ with the last point of $\Lambda_j$ identified with the first point of $\Lambda_{j+1}$ for $j = 1, \ldots, n - 1$. Let $\Sigma$ and $\Sigma^{(n)}$ be the corresponding quotient spaces obtained by identifying first and last points of $\Lambda$ and $\Lambda^{(n)}$ with respective quotient maps $\pi$ and $\pi^{(n)}$. Let $f: \Sigma^{(n)} \to \Sigma$ be the map which maps each image of $\Lambda_j$ in $\Sigma^{(n)}$ in the order preserving way to $\Lambda$ and passing to the quotient $\Sigma$: precisely, there is a commutative diagram

$$
\begin{array}{ccc}
\Lambda & \overset{\approx}{\leftarrow} & \Lambda_j \\
\pi \downarrow & & \downarrow \pi^{(n)|\Lambda_j} \\
\Sigma & \overset{f}{\leftarrow} & \Sigma^{(n)}
\end{array}
$$

By Theorem 4.6 for all compliant maps $\theta: \Lambda \to I$ and $\theta_0: \Lambda^{(n)} \to I$, the resulting standard maps $\phi$ and $\phi_0$ induce isomorphisms with respect to which the map $f^*$ is multiplication by $n$:

$$
\begin{array}{ccc}
H^1(\Sigma) & \overset{f^*}{\longrightarrow} & H^1(\Sigma^{(n)}) \\
\phi^* \big|_{\approx} & \overset{\approx}{\leftarrow} & \phi_0^* \big|_{\approx} \\
H^1(S^1) & \overset{n}{\longrightarrow} & H^1(S^1)
\end{array}
$$

From here it is easy to obtain our desired corollary below. The inverse limit space $S(\Lambda, \bar{p})$ is as in the previous sections, the group $Q(\bar{p})$ is as defined at the beginning of this section, and $\epsilon$ is the cardinality of the set of real numbers.

**Corollary 4.7.** Let $\Lambda$ be an ordered continuum, and $\bar{p} = (p_1, p_2, \ldots)$ a sequence of positive integers. Then the first Čech-Alexander-Spanier cohomology group $H^1(S(\Lambda, \bar{p}))$ of $S(\Lambda, \bar{p})$ with coefficients $\mathbb{Z}$ is isomorphic to $Q(\bar{p})$. Hence as $\bar{p}$ varies, there are $\epsilon$ many homeomorphism types among spaces of the form $S(\Lambda, \bar{p})$. 
Proof. For convenience, let \( p_0 = 1 \). For each \( n \in \mathbb{N} \) let \( \Lambda_n = \Lambda^{(p_0p_1\cdots p_n)} \) and let \( \Sigma_n = \Sigma^{(p_0p_1\cdots p_n)} \). The construction above produces a map \( f_n : \Sigma_n \to \Sigma_{n-1} \) for each positive integer \( n \) and the limit of the corresponding inverse sequence is \( S(\Lambda, \vec{p}) \).

The resulting isomorphisms \( \phi_n^* \) combine with the induced maps \( f_n^* \) on cohomology to produce the following diagram of direct sequences:

\[
\begin{align*}
H^1(\Sigma_0) & \xrightarrow{f_1^*} H^1(\Sigma_1) & \xrightarrow{f_2^*} H^1(\Sigma_2) & \xrightarrow{f_3^*} \cdots \\
\phi_0 \approx & \phi_1 \approx \phi_2 \approx \cdots \\
H^1(\mathbb{S}^1) & \xrightarrow{p_1} H^1(\mathbb{S}^1) & \xrightarrow{p_2} H^1(\mathbb{S}^1) & \xrightarrow{p_3} \cdots
\end{align*}
\]

Since \( G = \mathbb{Z} \) the group \( H^1(\mathbb{S}^1) \) is isomorphic with \( \mathbb{Z} \). Using one such isomorphism the direct sequence becomes a sequence of groups equal to \( \mathbb{Z} \) with maps equal to the appropriate multiplication. It is a standard result that the direct limit of the sequence is \( \mathbb{Q}(\vec{p}) \). By the continuity property of Čech-Alexander-Spanier cohomology for inverse sequences the first cohomology of the inverse limit is isomorphic to \( \mathbb{Q}(\vec{p}) \).

In \cite{McCord}, McCord defines two sequences \( \vec{p} \) and \( \vec{q} \) of prime numbers to be equivalent if a finite number terms can be deleted from each sequence so that every prime number appears the same number of times in each sequence and states that two sequences \( \vec{p} \) and \( \vec{q} \) are equivalent if and only if \( \mathbb{Q}(\vec{p}) \approx \mathbb{Q}(\vec{q}) \). This classification implies that the cardinal number of the set of all subgroups of \( \mathbb{Q} \) of the form \( \mathbb{Q}(\vec{p}) \) is equal to the cardinal \( \mathfrak{c} \) of the real line. Hence for each fixed \( \Lambda \), as \( \vec{p} \) varies there are \( \mathfrak{c} \) many homeomorphism types among inverse limits of the form \( S(\Lambda, \vec{p}) \). Actually, according to Rotman \cite{Rotman}, the concepts and results from abelian group theory which apply here go back to the 1914 dissertation of F. W. Levi, and discussion and proofs can be found in Rotman’s book .

\[\square\]

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(J. P. Boroński) Institute for Research and Applications of Fuzzy Modeling, National Supercomputing Center IT4Innovations, Division of the University of Ostrava, 30. dubna 22, 70103 Ostrava, Czech Republic – and – Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland
E-mail address: jan.boronski@osu.cz

(Gary Gruenhage) Department of Mathematics and Statistics, Auburn University, AL 36849, USA
E-mail address: garyg@auburn.edu

(George Kozlowski) Department of Mathematics and Statistics, Auburn University, AL 36849, USA
E-mail address: kozloga@auburn.edu