Phase transitions for a planar quadratic contact process

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Abstract

We study a two dimensional version of Neuhauser’s long range sexual reproduction model and prove results that give bounds on the critical values $\lambda_f$ for that the process to survive from a finite set and $\lambda_e$ for the existence of a nontrivial stationary distribution. Our first result comes from a standard block construction, while the second involves a comparison with the “generic population model” of Bramson and Gray [3]. An interesting new feature of our work is the suggestion that, as in the one dimensional contact process, edge speeds characterize critical values. We are able to prove the following for our quadratic contact process when the range is large but suspect they are true for two dimensional finite range attractive particle systems that are symmetric with respect to reflection in each axis. There is a speed $c(\theta)$ for the expansion of the process each direction. If $c(\theta) > 0$ in all directions then $\lambda > \lambda_f$, while if at least one speed is positive then $\lambda > \lambda_e$. It is a challenging open problem to show that if some speeds is negative then the system dies out from any finite set.

1 Introduction

The contact process introduced by Harris [15] in 1974, is perhaps the simplest spatial model for the spread of a species. A site in $\mathbb{Z}^d$ can be occupied ($\xi_t(x) = 1$) or vacant ($\xi_t(x) = 0$). Occupied sites become vacant at rate 1, while vacant sites become occupied at rate $\lambda$ times the number of neighbors that are occupied, i.e., the death rate is constant and the birth rate is linear. All 0’s is an absorbing state. Due to monotonicity, if we let $\xi_t^1$ be the state of the process at time $t$ when we start with all sites occupied then $\xi_t^1$ converges to a limit $\xi_\infty^1$, called the upper invariant measure, which is the largest possible stationary distribution. For this and the other facts about the contact process, see Liggett’s book [17].

Let $\xi_t^A$ be the contact process started with 1’s on $A$ and 0 otherwise. In principle, the contact process could have two critical values:

- $\lambda_e = \inf\{\lambda : \lim_{t \to \infty} P(\xi_t^1(x) = 1) > 0\}$, the critical value for the existence of a nontrivial stationary distribution.

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\[ \lambda_f = \inf \{ \lambda : P(\xi_t^A \neq \emptyset \text{ for all } t) > 0 \text{ for some finite set } A \} \]

Results of Bezuidenhout and Gray \[1, 2\] imply that for finite range attractive processes \( \lambda_e \leq \lambda_f \).

In the case of the contact process, self-duality
\[ P(\xi_t^{[0]} \neq 0) = P(\xi_t^1(x) = 1) \]
implies that the critical values coincide: \( \lambda_e = \lambda_f \), a value we will call \( \lambda_c \). Let
\[ \tau^A = \inf \{ t : \xi_t^A \equiv 0 \} \]
be the time at which the process dies out. When \( \lambda = \lambda_c \), \( P(\tau^A < \infty) = 1 \) for all finite sets \( A \) and \( \xi_\infty^1 = \delta_0 \). From this, it follows fairly easily that the phase transition is continuous, i.e., \( P(\xi_\infty^1(x) = 1) \) is a continuous function of \( \lambda \). If \( \lambda > \lambda_c \), the complete convergence theorem implies that as \( t \to \infty \)
\[ \xi_t^A \to P(\tau^A < \infty) \delta_0 + P(\tau^A = \infty) \xi_\infty^1 \]
As a consequence of this result, all stationary distributions distributions have the form \( \theta \delta_0 + (1 - \theta) \xi_\infty^1 \). The distribution \( \delta_0 \) is stationary, but is unstable under perturbation. If we add spontaneous births at rate \( \beta \), then there is a unique stationary distribution \( \xi_\infty^\beta \) for the modified contact process, and as \( \beta \downarrow 0 \) we have
\[ P(\xi_\infty^\beta(x) = 1) \downarrow P(\xi_\infty^1(x) = 1). \]

In this paper, we will consider a variant of the contact process that has sexual reproduction, i.e., two individuals are needed to produce a new one. Many processes of this type have been studied, but many open problems remain. To emphasize that these processes are an important and natural generalization of the contact process with linear birth rates, we will call them quadratic contact processes. We use this term somewhat loosely. In the models we discuss, the birth rate will not always be a quadratic function of the number of occupied neighbors.

The oldest “quadratic contact process” is Toom’s NEC model, see \[23, 24\]. However, in its initial formulation it was a system in which each site could be in one of two states \( \xi(x) \in \{1, -1\} \). Let \( \zeta_n(x) \) is the majority opinion among \( x, x + (0, 1), x + (1, 0) \) at time \( n \). If \( \zeta_n(x) = +1 \), then \( \xi_{n+1} = +1 \) with probability \( 1 - p \) and \( \xi_{n+1}(x) = -1 \) with probability \( p \). If \( \zeta_n(x) = -1 \) then \( \xi_{n+1} = -1 \) with probability \( 1 - q \) and \( \xi_{n+1}(x) = 1 \) with probability \( q \). Toom proved that for small \( p \) and \( q \) there are two nontrivial stationary distributions. Note that in contrast to the Ising model, there is nonergodicity even when the system is asymmetric. For more on this process see \[1, 16, 20\]. In the last reference, the authors remark that “the nature of the NEC interfaces separating up from down spin regions has been investigated and found to be related to Kardar-Parisi-Zhang nonequilibrium dynamics in the biased case, and (equilibrium) Edwards-Wilkinson dynamics in the unbiased one.

Durrett and Gray (1985) reformulated Toom’s process as a continuous time growth model in which
\[
\begin{array}{c|c}
1 & \text{rate 1} \\
0 & \text{rate } \lambda \text{ if } \xi_t(x + (1, 0)) = \xi_t(x + (0, 1)) = 1
\end{array}
\]
If the initial configuration for this process has no 1’s outside a square then there will never be any births outside the square, so \( \lambda_f = \infty \). Using the contour method, they proved (in an unpublished work [10], see an announcement of results in [8])

(DG1) \( \lambda_e \leq 110 \).

(DG2) If \( p < p^* = 1 - p_c \), where \( p_c \) is the critical value for oriented bond percolation in \( d = 2 \), then the process starting from product measure with density \( p \) dies out.

(DG3) If \( \lambda > \lambda_e \) and \( \beta \) is such that \( 6 \beta^{1/4} \lambda^{3/4} < 1 \) then when we add spontaneous births at rate \( \beta \) there are two stationary distributions.

(DG2) is easy to prove. If there is an infinite path of 0’s starting from a site \( x \) in which each step is up or right then these 0’s can never become 1’s. These paths exist when the density of 0’s exceeds \( p_c \). The interest in this result is that it shows that the complete convergence theorem is false. The fact that product measures with density \( p < p^* \) converge to \( \delta_0 \) for any \( \lambda < \infty \) suggests that this model has a discontinuous phase transition, but proving this is a difficult problem.

Recently, Chatterjee and Durrett [4] have shown that the discrete time threshold two contact process on a random \( r \)-regular graph or on a homogeneous tree in which each vertex has degree \( r \) has a discontinuous phase transition if the degree \( r \geq 3 \). Varghese and Durrett [25] have used simulation and heuristic arguments to study two versions of the quadratic contact on random graphs generated by the configuration model. In their vertex centered case, which is the one relevant to this paper, they find a discontinuous transition for the Erdős-Renyi random graph, but on power law random graphs with degree distribution \( p_k \sim C k^{-\alpha} \) and \( \alpha < 3 \) the critical value is 0. When \( \lambda_c = 0 \), a simple argument implies that the transition is continuous.

Chen [5, 6] generalized Durrett and Gray’s model on \( \mathbb{Z}^2 \). Let \( e_1, e_2 \) be the two unit vectors and define:

| pair 1 | pair 2 | pair 3 | pair 4 |
|-------|-------|-------|-------|
| \( x - e_1, x - e_2 \) | \( x + e_1, x - e_2 \) | \( x + e_1, x + e_2 \) | \( x - e_1, x + e_2 \) |

Chen’s models are number by the pairs that can give birth: Model I (pair 1 = SW corner rule), Model IIa (pairs 1 and 2), Model IIb (pairs 1 and 3), Model III (pairs 1–3), and Model IV (any pair).

Chen [5] proved for Model IV that if \( 0 < p < p(\lambda) \) then

\[
P (0 \in \xi_t^p) \leq t^{-c \log_2 (1/p)}.
\]

If we add spontaneous births at rate \( \beta \) and let \( \xi_{\infty}^{0, \beta} \) be the limit as \( t \to \infty \) for the system starting from all 0’s (which exists by monotonicity) then for large \( \lambda \)

\[
\lim_{\beta \to 0} P (0 \in \xi_{\infty}^{0, \beta}) > 0.
\]

The second result says that the all 0’s state is unstable to perturbation. When the limit is 0, the all 0’s state is stable to perturbation. In this case if \( \lambda > \lambda_e \) then the perturbed system
will have two stationary distributions for small $\beta$. This is how (DG3) was proved. Chen [6] shows that this is true for Model III.

Durrett and Neuhauser [11] considered a model with deaths at rate 1, and births at rate $\beta \frac{k}{(2d)^2}$ at vacant sites with $k$ occupied nearest neighbors. The mean field equation, which is derived by assuming that all sites are independent, in this case is

$$\frac{du}{dt} = -u + \beta (1 - u) u^2$$

This ODE has $\beta_c = 4$ and $\beta_f = \infty$, i.e., there is a nontrivial fixed point for $\beta \geq 4$ but 0 is always locally attracting. They showed that in the limit of fast stirring both critical values converged to 4.5. This threshold is the point where the PDE

$$\frac{du}{dt} = u'' - u + \beta u(1 - u)$$

has traveling wave solutions $u(t, x) = w(x - ct)$ with $c > 0$. The largest fixed point of the mean field differential equation is $2/3$ at 4.5, but based on simulations they conjectured that the phase transition was continuous.

Neuhauser [21] considered the contact process with sexual reproduction in $d = 1$ with long-range interaction in continuous time. In her model, the spatial locations are $\epsilon Z$ and $\xi : \epsilon Z \to \{0, 1\}$ has the following dynamics:

(i) Particles die at rate $1$.

(ii) A pair of adjacent particles at $x$ and $x + \epsilon$ produces an offspring with rate $\lambda$, which is sent to a location $y$ with probability $k_{\epsilon}(x - y)$. $k_{\epsilon}$ is the offspring distribution kernel, derived from an exponentially decaying, symmetric probability kernel $k$ on $\mathbb{R}$.

(iii) The birth at $y$ is suppressed if $y$ is occupied.

For this process, she showed that:

**Theorem 1.** In the limit as $\epsilon \to 0$, starting from product measure, the density of particles, $u$, evolves as a solution to the integro-differential equation

$$\frac{\partial u}{\partial t} = -u + \lambda(1 - u)(k * u^2)$$  \hspace{1cm} (1)

In addition, (1) admits traveling wave solutions, and there is a nondecreasing function $c_k : (0, \infty) \to \mathbb{R} \cup \{-\infty\}$ giving the wave speed corresponding to $\lambda$ and $k$.

**Theorem 2.** If $c_k(\lambda) > 0$, then for small enough $\epsilon$, there is a nontrivial stationary distribution. Additionally, there is a constant $\lambda^*$ above which the wave speed is indeed strictly positive.

Neuhauser’s result plays an important part in our first proof, but the main motivation for this work came from research done by Guo, Evans, and Liu (in various permutations), see [12, 13, 14, 18, 19]. They considered a modification of Model IV in which particles hop
according to the simple exclusion process at rate $h$. Birth rates at a site are $1/4$ times the number of adjacent pairs of occupied sites, while deaths occur at rate $p$. Having $h > 0$ means that $p_f(h) > 0$. When $h$ is small $p_f(h) < p_e(h)$ in which case “both the vacuum and active steady state are stable.” When $p_e(h) > p_f(h)$ this model has a discontinuous phase transition.

They defined a speed $v(p, h, S)$ using simulation of the process in a strip with slope $S$ and argued that

$$p < p_e(h) \quad \text{if some } V(p, h, S) > 0$$

$$p < p_f(h) \quad \text{if all } V(p, h, S) > 0$$

It is an interesting problem to define the speeds rigorously for a class of attractive finite range process and to prove the relationships in the last display. Here, we avoid that problem by taking a long range limit to get an integro-differential equation for which the existence of speeds can be proved using results of Weinberger [26].

We consider the two-dimensional contact process with sexual reproduction in discrete time (the results generalize to all higher dimensions as well). Let $k : \mathbb{R}^2 \to [0, 1]$ be a probability density that invariant under reflections in either axis: i.e., $k(x_1, x_2) = k(-x_1, x_2)$ and $k(x_1, x_2) = k(x_1, -x_2)$. At time $n \in \{0, 1, 2, \ldots\}$ the state of a site $x$ on the lattice $\mathbb{Z}^2/L$ is given by $\xi^L_n(x)$, which can take on the values 1 (occupied) or 0 (vacant). Starting with an initial configuration of particles, $\xi^L_0$ on $\mathbb{Z}^2/L$, the process evolves in the following manner:

(i) At time $n$, given the configuration at the previous time $n-1$, with probability $\beta$, a vacant site $x$ on the lattice will generate a random variable $U_x$ with density $k$, choose the site $y$ closest to $x + U_x$, and then choose one of its four nearest neighbors $z$ at random. If both of the chosen sites are occupied, $x$ will also become occupied.

(ii) After all births have occurred, with probability $\eta$, each particle is killed, independently of the others.

If we take the limit of the particle system as $L \to \infty$ and ignore technical details, the density of 1’s at time $n$ should satisfy

$$u_{n+1} = (1 - \eta) \left[ u_n + \beta(1 - u_n) \left( k * u_n^2 \right) \right]. \quad (2)$$

In the situation in which $u_n(x) \equiv v_n$

$$v_{n+1} = (1 - \eta)v_n + \beta(1 - v_n)v_n^2. \quad (3)$$

When $\beta > 4\eta/(1 - \eta)$, there are three equilibrium solutions of (3): 0 and $\rho_u < 1/2 < \rho_s$ given by

$$\frac{1}{2} \left( 1 - \sqrt{1 - \frac{4\eta}{\beta(1 - \eta)}} \right)$$

Using results of Weinberger [26] we can prove existence of wave speeds under the assumption $\beta > 4\eta/(1 - \eta)$. That is, if $S^1$ is the circle of radius one in the plane, then there exists a function $c^*: S^1 \to \mathbb{R} \cup \{-\infty, \infty\}$, which gives the speed of propagation of plane waves solutions of (2) is in the direction $\theta$. 

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Theorem 3. If $c^*(\theta) > 0$ for all $\theta \in S^1$, then for large $L$ there is a nontrivial stationary distribution.

The proof of this result is a straightforward generalization of Neuhauser’s result in [21]. There are three ingredients: (i) a hydrodynamic limit which shows that as $L \to \infty$ the particle system converges to a solution of (2), (ii) a convergence result of Weinberger [26] for solutions of (2), and (iii) a block construction.

Our symmetry assumption implies that if the speed in some direction $(\theta_1, \theta_2)$ with $\theta_1 \theta_2 \neq 0$ is positive then there are a total of four directions obtained by reflection across the axes that also have positive speeds. From this, we see that if a large region of 0’s develops then the process will be able to fill in this hole and hence there should be a stationary distribution. This intuition can be made rigorous by using a comparison with a “generic population process” of Bramson and Gray [3].

Theorem 4. If there exists a $(\theta_1, \theta_2) \in S^1$ with $\theta_1 \theta_2 \neq 0$ and $c^*(\theta) > 0$, then for large $L$ there is a nontrivial stationary distribution.

Our application is more complicated than their proof of Toom’s eroder theorem. In their proof the bad regions are areas that contain 0’s, while in our proof the bad regions are areas where the density of 1’s is not large enough in some small box. In [3] the state outside a bad region is always the same, all 1’s. In this paper, we must use functions from Weinberger’s proof of the existence of wave speeds to control the decrease of the bad region.

By the same logic, if some speed is negative and the initial configuration is finite then it can be put inside a shrinking parallelogram so

Conjecture 1. If there is a $(\theta_1, \theta_2) \in S^1$ with $\theta_1 \theta_2 \neq 0$ and $c^*(\theta) < 0$, the system cannot survive starting from an initial configuration with only a finite number of particles.

In a number of situations, see e.g., Chapter 7 of Cox, Durrett, and Perkins [7], block constructions have been used to show that a process dies out. However, in that situation one shows that a large enough dead region will expand. Since our process is good at filling in holes, that conclusion is false here, and one will need a much different method to prove our conjecture.

2 Hydrodynamic Limit

In this section we will show that as $L \to \infty$ the particle system converges to the solution of the integrodifferential equation (2). The first thing to do is to explain what it means for a sequence of configurations $\xi_n : \mathbb{Z}^2 / L \to \{0, 1\}$ to converge to a function $u_n(x)$. To do this we will let $\gamma \in (0, 1)$ and partition the plane into squares with side $L^{-\gamma}$ whose corners are at the points $L^{-\gamma}(m, n)$ with $m, n \in \mathbb{Z}$. For any $x \in \mathbb{Z}^2 / L$, let $x^*$ be the bottom-left corner of the box containing $x$, and let

$$B^L(x) = x^* + [0, L^{-\gamma})^2. \quad (4)$$

Each such box contains $\sim L^{2-2\gamma}$ points.
Let \( S_n^L(x) \) be the number of particles in \( B^L(x) \) at time \( n \):

\[
S_n^L(x) = \sum_{y \in B^L(x)} \xi_n^L(y).
\]

Define \( \xi_n^L \sim u_n \) to mean that for all \( K \), as \( L \to \infty \)

\[
\sup_{x \in [-L,L]^2} \left| \frac{S_n^L(x)}{L^{2-2\gamma}} - u_n(x^*) \right| \to 0
\]  

(5)

Let \( R_n^L(x) \) = number of pairs of adjacent particles in \( B^L(x) \) at time \( n \):

\[
R_n^L(x) = \sum_{y \in B^L(x)} \zeta_n^L(y) \quad \text{where} \quad \zeta_n^L(y) = \xi_n^L(y) \cdot \frac{1}{2} \left[ \xi_n^L(y + e_1) + \xi_n^L(y + e_2) \right]
\]

We do this so each pair is only counted once. Define \( \zeta_n^L \sim u_n^2 \) to mean that for all \( K < \infty \), as \( L \to \infty \)

\[
\sup_{x \in [-L,L]^2} \left| \frac{R_n^L(x)}{L^{2-2\gamma}} - u_n(x^*)^2 \right| \to 0
\]  

(6)

**Theorem 5.** Suppose that \( u_0(x) : \mathbb{R}^2 \to [0,1] \) is continuous and that the sequence of initial configurations \( \xi_0^L \sim u_0 \) and \( \zeta_0^L \sim u_0^2 \). Then as \( L \to \infty \), \( \xi_n^L \sim u_n \) and \( \zeta_n^L \sim u_n^2 \), where \( u_n \) is the solution of

\[
u_{n+1} = (1 - \eta) \left[ u_n + \beta (1 - u_n) \left( k * u_n^2 \right) \right].
\]  

(7)

**Proof.** By induction it suffices to prove the result for \( n = 1 \). Let \( T_x \) be the translation by \( x \). In order to simplify the computation of expectations and variances, we will modify the birth step so that the first parent \( y \) will be chosen according to the probability kernel \( T_x \cdot k_L \), instead of \( T_x k_L \). As before, the second parent is chosen at random, with equal probability, from the nearest neighbors of the first parent. The next lemma shows that at time 1 the two processes are equal with high probability.

**Lemma 2.1.** If \( \xi_0^L(x) = \xi_0^L(x) \) for each \( x \in \mathbb{Z}^2/L \), then

\[
\mathbb{P} \left( \xi_1^L(x) \neq \hat{\xi}_1^L(x) \right) \to 0 \quad \text{as} \quad L \to \infty.
\]

**Proof.** For \( y \in \mathbb{Z}^2/L \), let \( \alpha_L(y) = k_L(y - x) \), \( \beta_L(y) = k_L(y - x^*) \), and let

\[
p_L = \sum_{y \in \mathbb{Z}^2/L} (\alpha_L(y) \land \beta_L(y))
\]

A standard argument from analysis shows that if \( f \in L^1(\mathbb{R}^2) \) and \( \delta = (\delta_1, \delta_2) \to (0,0) \) then \( \|T_\delta f - f\|_1 \to 0 \) (approximate \( f \) by a continuous function \( g \)). This implies that as \( L \to \infty \), \( p_L \to 1 \) uniformly for \( x \in \mathbb{Z}^2/L \).

To couple the two processes, use the same coin flips to see if births should occur. If a birth event occurs at site \( x \in \mathbb{Z}^2/L \) at time 1, flip a coin with probability \( p_L \) of heads. If heads comes up, then with probability

\[
\frac{[\alpha_L(y) \land \beta_L(y)]}{p_L}
\]
choose $y$ as the first parent for $x$ in both processes, and then make the same choice of second parent $z$. Otherwise, the particle in $\xi_L$ chooses its first parent $y$ with probability

$$(\alpha_L(y) - \beta_L(y)) + \frac{1}{1 - p_L},$$

the particle in $\hat{\xi}_L$ chooses its first parent $y$ with probability

$$(\beta_L(y) - \alpha_L(y)) + \frac{1}{1 - p_L},$$

and the choice of second parents are made independently. Once the births have been done, we use the same coin flips to decide the deaths and the proof is complete.

Let $\mathbb{P}_0$ denote the probability law for the process $\hat{\xi}_L$ with initial configuration $\hat{\xi}_L^0$. In order to write out the computations more compactly, introduce the notations $X_i = \xi_i^L(x)$, $Y_i = \xi_i^L(y)$, and $Z_i = \xi_i^L(z)$ for $i = 1, 2$ and $x, y, z \in \mathbb{Z}^2/L$, and set

$$K_L(x) = \sum_{y \in \mathbb{Z}^2/L} k_L(y - x^*) \xi_0^L(y) \cdot \frac{1}{4} \sum_{z \sim y} \xi_0^L(z).$$

so that

$$\mathbb{P}_0(X_1 = 1) = (1 - \eta) [X_0 + \beta (1 - X_0) \cdot K_L(x)] := p_x.$$ 

Since $X_1$ is Bernoulli, $\text{Var}_0(X_1) = p_x - p_x^2$.

The total number of particles alive in $B^L(x)$ at time 1 is

$$\hat{\mathcal{S}}_1^L(x) = \sum_{y \in B^L(x)} \hat{\xi}_1^L(y),$$

so the expected proportion of occupied sites in $B^L(x)$ at time 1 is

$$\mathbb{E}_0 \left( \frac{\hat{\mathcal{S}}_1^L(x)}{L^{2-2\gamma}} \right) = (1 - \eta) \left[ \frac{\hat{\mathcal{S}}_0^L(x)}{L^{2-2\gamma}} + \beta \left( 1 - \frac{\hat{\mathcal{S}}_0^L(x)}{L^{2-2\gamma}} \right) K_L(x) \right]. \quad (8)$$

For $y \neq z$ in $B^L(x)$,

$$\text{Cov}_0 (Y_1, Z_1) = \mathbb{E}_0 (Y_1 Z_1) - \mathbb{E}_0 (Y_1) \mathbb{E}_0 (Z_1)$$

$$= (1 - \eta)^2 \left[ Y_0 Z_0 + \beta \cdot K_L(x) \left( (1 - Y_0) Z_0 + Y_0 (1 - Z_0) \right) \right.$$

$$+ (\beta \cdot K_L(x))^2 (1 - Y_0) (1 - Z_0) \right] - p_y p_z = 0.$$ 

It is for this reason we modified our process so that all sites in $B_L(x)$ use the kernel $k_l(\cdot - x^*)$.

Since the covariance is 0,

$$\text{Var}_0 \left( \frac{\hat{\mathcal{S}}_1^L(x)}{L^{2-2\gamma}} \right) = \frac{1}{L^{4-4\gamma}} \sum_{y \in B^L(x)} (p_y - p_y^2)$$

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Noticing that
\[ p^2_x = (1 - \eta)^2 \left[ X^2 + 2\beta X_0(1 - X_0)K_L(x) + \beta^2(1 - X_0)^2K_L(x)^2 \right] \]
with \( X_0(1 - X_0) = 0 \) and \( K_L(x) \leq 1 \), the variance of \( \hat{S}_L(x)/L^{2-2\gamma} \) is
\[
\frac{1}{L^{4-4\gamma}} \left[ \hat{S}_L^2(x) + \beta \left( L^{2-2\gamma} - \hat{S}_L(x) \right) \right]
\]
\[
- \frac{1}{L^{4-4\gamma}} \left[ \hat{S}_L^2(x) + \beta^2 \left( L^{2-2\gamma} - \hat{S}_L(x) \right) \right] \leq C \cdot \frac{1}{L^{2-2\gamma}}.
\]
where \( C = \max\{1, \beta^2\} \) does not depend on \( L \). By Chebyshev’s inequality,
\[
\mathbb{P}_0 \left( \left| \hat{S}_L(x)/L^{2-2\gamma} - \mathbb{E}_0 \left( \hat{S}_L(x)/L^{2-2\gamma} \right) \right| \geq \delta \right) \leq \delta^{-2} \text{Var} \left( \hat{S}_L(x)/L^{2-2\gamma} \right) \leq \frac{C}{\delta^2 L^{2-2\gamma}}.
\]
There are \( 4L^{2\gamma} \cdot K^2 \) boxes of side length \( L^{-\gamma} \) in each \([-K, K]^2 \) box, so
\[
\mathbb{P}_0 \left( \sup_{x \in [-K,K]^2 \cap \mathbb{Z}^2/L} \left| \hat{S}_L(x)/L^{2-2\gamma} - \mathbb{E}_0 \left( \hat{S}_L(x)/L^{2-2\gamma} \right) \right| \geq \delta \right) \leq \frac{CL^{2\gamma}}{\delta^2 L^{2-2\gamma}}. \tag{9}
\]
For \( \gamma \in (0, \frac{1}{4}) \), this probability tends to 0 as \( L \to \infty \). For \( x \in \mathbb{R}^2 \) and \( x_L \in \mathbb{Z}^2/L \) such that \( x_L \to x \) as \( L \to \infty \),
\[ K_L(x_L) \to (k \ast u_0^2)(x). \]
By the assumptions of Theorem 5,
\[ \hat{S}_0^L(x)/L^{2-2\gamma} = S_0^L(x)/L^{2-2\gamma} \to u_0(x). \]
Together with (8), this implies that
\[ \mathbb{E}_0 \left( \frac{\hat{S}_L(x_L)}{L^{2-2\gamma}} \right) \to u_1(x) \quad \text{as} \quad L \to \infty. \]
Then by (9),
\[ \hat{S}_L(x_L)/L^{2-2\gamma} \to u_1(x) \]
and thus \( \xi^L_1 \sim u_1 \), and from Lemma 2.1 it follows that \( \xi^L_1 \sim u_1 \).

Our next step is to prove a result for pairs. Let
\[ \hat{R}_n^L(x) = \sum_{y \in B_n^L(x)} \hat{\xi}_n^L(y) \quad \text{where} \quad \hat{\xi}_n^L(y) = \hat{\xi}_n^L(y) \cdot \frac{1}{2} \left[ \hat{\xi}_n^L(y + e_1) + \hat{\xi}_n^L(y + e_2) \right] \]
and \( B_n^L(x) \) is the set of points \( y \) so that \( y + e_1 \) and \( y + e_2 \) are also in the box. As argued in the first part of the proof, given the initial configuration the \( \hat{\xi}_n^L(y) \) are independent so when \( y, z \in B_n^L(x) \) we have
\[ E \left( \hat{\xi}_n^L(y) \hat{\xi}_n^L(z) \right) = p_x^2. \]
The pairs \((\hat{\xi}^L_y, \hat{\xi}^L_z)\) for \(y \in B^L_y(x)\) and \(z = y + e_1\) or \(y + e_2\) are not independent when \(\{y, z\} \cap \{y', z'\} \neq \emptyset\) but one still has the estimate

\[ \text{Var}_0 \left( \frac{\hat{R}^L_t(x)}{L^{2-2\gamma}} \right) \leq \frac{C}{L^{2-2\gamma}} \]

so the previous proof can be repeated almost word for word to conclude that \(\hat{\zeta}^L_1 \sim u^2_1\) and by Lemma 2.1 \(\zeta^L_1 \sim u^2_1\).

With the hydrodynamic limit established, the proof of Theorem 3 is routine. Let \(\delta > 0\) be small enough so that \(\rho_s - 2\delta > \rho_u + 2\delta\). Theorem 6.2 in [26] implies that

**Lemma 2.2.** If \(K\) is large enough and \(N \geq N_K\) then \(u_0(x) > \rho_u + \delta\) on \([-K, K]^2\) implies that \(u_N(x) > \rho_s - \delta\) on \([-4K, 4K] \times [-K, K]\).

Let

\[ I_n = 2nK + [-K, K]^2. \]

We say that \(I_n\) is good at time \(t\) if we have \(S^L_t(x) \geq \rho_u + 2\delta\) for all \(x \in I_n\). Combining Theorem 5 with Lemma 2.2 gives

**Lemma 2.3.** Let \(\epsilon > 0\). If \(L\) is large enough and \(I_0\) is good at time 0 then with probability \(\geq 1 - \epsilon\), \(I_1\) and \(I_{-1}\) are good at time \(N\).

Since the process cannot move by more than distance 1 in one time step, the events involved in the last lemma have a finite range of interaction than only depends on \(N\). It follows from this and results in Durrett’s St. Flour notes that if \(\epsilon < \epsilon_N\) the good sites dominate a supercritical oriented percolation and hence there is a nontrivial stationary distribution.

### 3 Proof of Theorem 4

The result will be proved using Bramson and Gray’s comparison model [3]. The goal is to show that our quadratic contact process and their comparison process can be coupled so that regions in the quadratic contact process with a low density of particles are entirely contained in the vacant region of the comparison model. Then the existence of a nontrivial stationary distribution for the comparison model will imply the existence of a nontrivial stationary distribution for the quadratic contact model.

The comparison process of Bramson and Gray takes place on \(\mathbb{R}^2\), in continuous time. In the comparison process, at each time the plane is divided into two regions: a vacant and a nonvacant region. Let \(A_t\) denote the vacant region of the comparison process at time \(t\) and take \(A_0 = \emptyset\). Let \(\mathcal{P}\) be a Poisson process on \(\mathbb{R}^2 \times [0, \infty)\) with intensity \(\epsilon\). If \((x, t) \in \mathcal{P}\), a triangular vacant region centered at \(x \in \mathbb{R}^2\) of a certain fixed size is created at time \(t\). The edges of this region will be perpendicular to the fixed unit orientation vectors \(n_1, n_2, n_3\) and will move inward with rates \(a_1, a_2, a_3\), respectively, which may be negative.

In case two or more vacant regions overlap or collide, a new vacant region is formed with the same geometry, whose edges move outward at rate \(b > 0\) (the interaction rate) until
each edge catches up to the corresponding edges of all of the regions which produced the origi
nal overlap or collision. At that point, the edges of the overlap or collision region will
begin to move inward with rates $a_i$. We will explain this in more detail later. If $a_i > 0,$
for $i = 1, 2, 3$, Bramson and Gray have shown that the process has a nontrivial stationary
distribution when the error rate is small enough.

Coming back to the quadratic contact process, take $\xi_i$ to be three directions with positive
wave speeds $\alpha_i > 0, \; i = 1, 2, 3$. Since there is at least one positive speed, the offspring
distribution kernel has $\mathbb{Z}^2$ symmetry, and because the wave speed $c^*(\xi)$ is a lower semi-
continuous function in $\xi$ [26, Proposition 5.1], the existence of a positive speed in one direction
implies the existence of a positive speed in three directions, perpendicular to the edges of an
acute triangle in the plane.

We shall say that the box $B^L(x)$ is bad at time $n$ (see (4) for the definition of $B^L(x)$) if
the density of the particles inside $B^L(x)$ falls below an appropriate threshold $\alpha \in (\rho_u, \rho_s),$
to be specified below. Recall that $0 < \rho_u < \rho_s < 1$ are the nonzero fixed points of the operator

$$Q[u] = (1 - \eta) \left[ u + \beta (1 - u) \left( k * u^2 \right) \right],$$  \hspace{1cm} (10)

which exist when $\beta > 4\eta/(1 - \eta)$.

The collection of bad boxes at time $n$ will be contained in the vacant region of the comparison
process. Let $\mathcal{A}_n$ be the bad region (the collection of all bad boxes) of the contact
process on $\mathbb{Z}^2/L$ at time $n$. The process initially has all sites occupied, so that $\mathcal{A}_0 = \varnothing.$

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a function with the following properties (see Figure 1 for an example):

i) $\psi$ is continuous,

ii) $\psi$ is nonincreasing,

iii) $\psi(-\infty) \in (\rho_u, \rho_s)$ and $\psi(s) = 0$ for $s \geq 0$.

As in [26, Section 5], for $c \in \mathbb{R}$ and $\xi \in S^1$, we consider the operator

$$R_{c,\xi}[f](s) := \max \{ \psi(s), Q[f(x \cdot \xi + s + c)](0, 0) \}, \quad s \in \mathbb{R}.$$
Let $f_0 = \psi$, and

$$f_{n+1} = R_{c,\xi}[f_n], \quad n = 0, 1, \ldots$$

So, for each $n = 0, 1, \ldots$, $f_n$ is a function from $\mathbb{R}$ to $\mathbb{R}$. By [26, Lemma 5.1], for each $s \in \mathbb{R}$, the sequence $\{f_n(s)\}$ is nondecreasing, and, for each $n = 0, 1, \ldots$, the function $f_n$ is nonincreasing in its argument. Moreover, $\{f_n\}$ increases to a limiting function $f$ with $f(-\infty) = \rho_s$. If $f(\infty) = \rho_s$, then $c < c^*$, the wave speed in the direction $\xi$ ($c^*$ is independent of the initial choice of $\psi$ satisfying the above conditions). Thus

$$f \equiv \rho_s$$

(see [26, Section 5]). Let

$$c = \min\{\alpha_i/2 : i = 1, 2, 3\}. \quad (11)$$

For each of the directions $\xi_i$ with positive wave speed $\alpha_i$, $i = 1, 2, 3$, consider the sequence $\{f_{n,i}\}$ where $f_{0,i} = \psi$ and

$$f_{n+1,i} = R_{c,\xi_i}[f_{n,i}], \quad n = 0, 1, \ldots$$

By the above, the monotonicity of $Q$, and taking

$$\phi(s) = \min\{f_{n,i}(s) : i = 1, 2, 3\}$$

for large enough $n$, we see that there is a continuous nonincreasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that for each $i$, $Q[\phi]$ dominates the translate of $\phi$ by $c$ given in (11):

$$\phi(s - c) \leq Q[\phi(x \cdot \xi_i)](s) \quad (12)$$

for all $s \in \mathbb{R}$, and with

$$\lim_{n \to \infty} Q^n[\phi(x \cdot \xi_i)] \equiv \rho_s.$$

Now, set $\alpha = \phi(-\infty)$ and let

$$m = \min_i \sup \left\{ s : Q[\phi(x \cdot \xi_i)](s) = \alpha \right\} \quad \text{and} \quad M = \max_i \inf \left\{ s : Q[\phi(x \cdot \xi_i)](s) = 0 \right\}.$$
Set \( l = M - m \).

If an error occurs in the contact process (the detailed description of errors is below) such as a box \( B^L(x) \) that is good at time \( n - 1 \) becomes bad at time \( n \), in the comparison process this box is covered by a triangular region, \( D(x,n,t) \) (the first \( n \) indicating that the region forms at time \( n \) and the \( t \) indicating the shape of the region at time \( t \)), of an appropriate size, so that the center of the box is at the center of the triangle whose inscribed circle has radius

\[
 r = \left\lceil l + d(B) + c + d(k) \right\rceil,
\]

initially at time \( n \), where \( d(B) = L^{-\gamma}\sqrt{2} \) is the diameter of the box and

\[
 d(k) = \sup \left\{ |x - y| : x, y \in \mathbb{R}^2, k(x) \neq 0, k(y) \neq 0 \right\}
\]
is the diameter of the kernel. Let \( R \) be the radius of the circumscribed circle of \( D(x,n,n) \). Once such a region is created, the edges move inward, each with linear rate \( c \).

Set the interaction rate \( b := 2d(k) \). In the comparison process, should two or more triangles collide or overlap, a new collision or overlap region is formed at that time, which is the intersection of the maximal collection of the colliding or overlapping regions that has nonempty intersection. The edges of the new collision of overlap region then move outward at rate \( b \) in each direction until they have caught up to all of the respective edges of the regions that initiated the overlap or collision in each respective direction, and after that the edge of the overlap or collision region will again move inward with rate \( c \).

Bramson and Gray use a Poisson point process in their comparison model. Instead, we will consider a different point process of errors, \( \mathcal{P} \), derived from the quadratic contact process, which will be described below after introducing some definitions. In fact, the results of Bramson and Gray still hold with this underlying point process instead of the Poisson.

\( \mathcal{P} \) will be a point process of errors in the quadratic contact process. Two different types of errors are possible:

**Type I Error:** There is a region of good boxes, and the density in one of the boxes suddenly falls. Suppose that

\[
 B^L(x) \cap \tilde{A}_{n-1} = \emptyset.
\]

Figure 2: An overlap region is formed at some time \( t_0 \). The edges of the overlap region move outward, and by time \( t > t_0 \), they have caught up with the edges of the triangles initiating the overlap, so all edges are now moving in.
Thus, \( B^L(x) \) that was good at time \( n - 1 \) and all other boxes within distance \( d(k) \) are also good at \( n - 1 \) (since each triangle has a type of buffer region distance \( d(k) \) away from the edges with good boxes). If \( B^L(x) \) is bad at \( n \), this spontaneous error will be considered a Type I error.

**Type II Error:** There are already multiple low-density boxes in a region but with density above appropriate iterates of \( \phi \). In one box, the density falls below what is predicted by \( \phi \). If
\[
B^L(x) \cap \tilde{A}_{n-1} \neq \emptyset,
\]
let \( \{R_i(y_i, m_i, n - 1)\}_{i \in I} \), \( I \) a countable index set, be the maximal collection of triangular regions such that
\[
B^L(x) \subset \bigcap_{i \in I} R_i(y_i, m_i, n - 1),
\]
where \( R_i(y_i, m_i, n - 1) \) is a triangular region centered at \( y_i \in \mathbb{Z}^2/L \) that was created at time \( m_i \in \mathbb{N} \) (it could be one of the regions \( D \) or an overlap or a collision region).

For \( j = 1, 2, 3 \), let \( H_{i,j}(t) \) be the halfspace in \( \mathbb{R}^2 \) containing \( R(y_i, m_i, t) \) and with boundary containing the \( j^{th} \) side of the triangle. Then the results of Bramson and Gray imply that there exists a region \( R(y, m, n - 1) \) centered at \( y \) and created at time \( m \) such that
\[
\bigcap_{i \in I} R_i(y_i, m_i, t) \subset R(y, m, t) \text{ for all times } t \in [m, n - 1],
\]
and \( B^L(x) \subset R(y, m, n - 1) \).

Furthermore, the \( j^{th} \) edge and corresponding halfspace containing \( R \) and its \( j^{th} \) edge at time \( t \), \( H_j(t) \) either: (i) moves outward with rate \( \beta \), or (ii) moves inward with rate \( c \) and is such that
\[
\bigcup_{i \in I} H_{i,j}(t) \subset H_j(t).
\]

For \( x \in \bigcup_{i \in I} R_i(y_i, m_i, n) \), let
\[
\tilde{h}^{(i,j)}_n(x) = Q^{n-m_i} \left[ \phi \left( \xi_j \cdot (x - y_i) \right) \right],
\]
where \( Q^{n-m_i} \) is the application of \( Q \) \((n - m_i)\) times. Let
\[
h_n^{(i,j)}(x) = \inf_{i \in I} \tilde{h}^{(i,j)}_n(x)
\]
for \( x \in \mathbb{R}^2 \), and let
\[
h_n(x) = \max_{j=1,2,3} h_n^{(i,j)}(x).
\]

A type II error occurs at time \( n \) if for \( x \in \bigcup_{i \in I} R_i(y_i, m_i, n) \), the density of particles in \( B^L(x) \) is below \( h_n(x) \) at time \( n \).
3.1 Point process of errors

Next, we describe the point process, $\mathcal{P}$ on $\mathbb{R}^2 \times [0, \infty)$ used in the comparison process. It is derived from the quadratic contact process on $\mathbb{Z}^2 / L$. For each type I or type II error that occurs in the contact process, there is a single corresponding point in $\mathcal{P}$. If the error occurs in $B^{L}(x)$ at time $n$, let $(y, t) \in \mathcal{P}$ where $y \in x^* + [0, L^{-\gamma})^2$ is a single point in the box, chosen uniformly and at random from $x^* + [0, L^{-\gamma})^2$ and $t$ is chosen uniformly and at random from $[n - 1, n]$.

Although $\mathcal{P}$ is not quite a Poisson point process, it shares two key properties with the Poisson process, the only two used in Bramson and Gray’s proof, that are sufficient to demonstrate a nontrivial stationary distribution for large enough $L$. Namely (see [3], 2-1 and 2-2),

$$P (|B \cap \mathcal{P}| \geq 2) = O \left( \lambda(B)^2 \right)$$

(13)

as $\lambda(B) \to 0$, where $\lambda$ is Lebesgue measure on $\mathbb{R}^2 \times [0, \infty)$, for Borel sets $B$.

(13) is satisfied, as $\mathcal{P}$ is dominated by a Poisson process with parameter $\tilde{\epsilon}$ that satisfies

$$\epsilon L^{2\gamma} = 1 - e^{-\epsilon},$$

where $\epsilon$ is an upper bound on the probability of an error (type I or type II) in the contact process with $\epsilon L^{2\gamma} \to 0$ as $L \to \infty$ (see the proof of Theorem 6 below).

The second property is that for all small enough disjoint cubes

$$B_1, B_2, \ldots, B_m$$

in $\mathbb{R}^2 \times [0, \infty)$,

$$P \left( \bigcap_{j=1}^{m} \{B_j \cap \mathcal{P} \neq \emptyset\} \right) \leq \prod_{j=1}^{m} \left( 2\epsilon L^{2\gamma} \lambda(B_j) \right).$$

(14)

If $B_j = b_j \times [s_j, t_j]$, where $b_j$ is a cube in $\mathbb{R}^2$, it is sufficient to assume that

$$\lambda(b_j) < L^{-2\gamma}$$

for each $j$ and $t_j - s_j < 1$ for $j = 1, 2, \ldots, m$ (the time interval may also be open or half open). To see that (14) is then satisfied, first consider the case when $n \in (t_j - s_j)$ for all $j = 1, \ldots, m$, and some $n \in \mathbb{N}$.

Given the configuration of the quadratic contact process at time $n - 1$, births at different sites at time $n$ are independent of each other. The same holds for deaths. For each $j$,

$$P (B_j \cap \mathcal{P} \neq \emptyset) \leq \epsilon L^{2\gamma} \lambda(B_j).$$

Since the densities of particles in different boxes are independent, (14) holds.

We still assume that that

$$B_j = b_j \times [s_j, t_j],$$

(15)
where $b_j$ is a cube in $\mathbb{R}^2$, $\lambda(b_j) < L^{-2\gamma}$ for each $j$ and $t_j - s_j < 1$. If $n_j \in (t_j - s_j)$ for some $n_j \in \mathbb{N}$, split $B_j$ into two cubes:

$$B_{j,1} = b_j \times [s_j, n_j) \quad \text{and} \quad B_{j,2} = b_j \times [n_j, t_j].$$

We note that

$$P\left(\{B_{j,1} \cap \mathcal{P} \neq \emptyset\} \cap \{B_{j,2} \cap \mathcal{P} \neq \emptyset\}\right) \leq 2\epsilon L^{2\gamma} \max\{\lambda(B_{j,1}), \lambda(B_{j,2})\}$$

So we can suppose this does not occur and let $n_j = \lfloor s_j \rfloor$. Also suppose that the cubes are ordered in a way that $n_1 \leq n_2 \leq \ldots \leq n_m$.

(14) can be shown by induction on $m$. It clearly holds for $m = 1$. Let

$$A_j = \{B_j \cap \mathcal{P} \neq \emptyset\}.$$

Then

$$P\left(\bigcap_{j=1}^m A_j\right) = P\left(\bigcap_{j=1}^{m-1} A_j\right) \cdot P\left(\bigcap_{j=1}^{m-1} A_j\right) \leq \epsilon L^{2\gamma} \lambda(B_m) \prod_{j=1}^{m-1} (2\epsilon L^{2\gamma} \lambda(B_j)).$$

The second factor comes from the induction assumption and the first factor is from the upper bound on the error probability.

### 3.2 Comparison Result

**Theorem 6.** Let $\tilde{A}_n$ be the bad region of the contact process at time $n$ with death probability $\eta$, birth probability $\beta$, and finite offspring distribution kernel $k$ such that there is at least one direction with a positive wave speed.

Let $A_t$ be the vacant region of the comparison process with point process $\mathcal{P}$, orientation vectors $n_i = \xi_i$, speeds $a_i = c$, and interaction rate $b = 2d(k)$. Then the processes $A_n$ and $\tilde{A}_n$ can be jointly coupled so that

$$\tilde{A}_n \subset A_n$$

for all $n \in \{0, 1, 2, \ldots\}$. Hence, for large enough $L$, the quadratic contact process has a nontrivial stationary distribution.

**Proof.** We use an induction argument. Run the contact process from an initial configuration with every site occupied: $\tilde{A}_0 \subset A_0$ since both processes begin with empty vacant region.

Now, assuming $\tilde{A}_{n-1} \subset A_{n-1}$, we show that after one time step, we still have $\tilde{A}_n \subset A_n$. First, suppose that a type I error occurs. Let $x \in \mathbb{Z}^2/L$ and $B^L(x)$ the box containing $x$. Then if the box is good,

$$S_{n-1}^L(x) > \alpha \cdot m,$$
where \( m = L^{2-2\gamma} \) is the number of points in \( B(x) \).

From previous calculations,

\[
\mathbb{E}(S_n^L(x)) = \sum_{y \in B^L(x)} Q[u_{n-1}^L](y)
\]

and

\[
\mathbb{E}[S_n^L(x) \mid B(x) \text{ good at time } n-1] = \sum_{y \in B^L(x)} Q[u_{n-1}^L](y) \geq Q(\alpha) \cdot m > \alpha \cdot m
\]

We also have

\[
\text{Var}(S_n^L(x)) \leq cL^{2-2\gamma}.
\]

Now fix any \( \delta_1 \in (0, Q(\alpha) - \alpha) \).

By Chebyshev’s inequality,

\[
\mathbb{P}_{\xi_{n-1}^L} \left( \left| S_n^L(x) - \sum_{y \in B^L(x)} Q(u_{n-1}^L)(y) \right| \geq \delta_1 m \right) \leq \frac{cm}{\delta_1^2 m^2} = C_1 L^{2\gamma-2}
\]

This gives an upper bound on the probability of a good box spontaneously going bad, for a good box has density of particles \( \geq \alpha \) and is therefore expected at the next time step to have density of particles \( \geq Q(\alpha) \), so

\[
Q(\alpha) - \delta_1 > \alpha.
\]

Next, consider the other way that an error can occur: if an already existing bad region’s behavior is too different from what is expected, that is, a Type II error. In this case, ‘too far below’ will mean below \( Q[\phi(x \cdot \xi_i)] \). Choose \( \delta_2 \) so that for \( s < 0, Q[\phi(x \cdot \xi_i)](s) - \phi(s) < \delta_2 \).

\[
\phi(\xi_i \cdot x - c) \leq Q[\phi(\xi_i \cdot x)] \leq Q[u_{n-1}^L(x)] = \mathbb{E}(\xi_n^L(x))
\]

Again, for the probability of an error, we use Chebyshev’s inequality to obtain, for any \( x \) in the bad region,

\[
\mathbb{P}_{\xi_{n-1}^L} \left( \left| S_n^L(x) - \sum_{y \in B^L(x)} Q(u_{n-1}^L)(y) \right| \geq \delta_2 m \right) \leq \frac{cm}{\delta_2^2 m^2} = C_2 L^{2\gamma-2}
\]

Taking \( \delta = \min\{\delta_1, \delta_2\} \) and \( C = \max\{C_1, C_2\} \), we obtain an upper bound on the probability of an error occurring in any single box at a single time step, \( \tilde{\epsilon} := CL^{2\gamma-2} \).

The two processes are already coupled in the following way: if in the contact process, an error occurs at time \( n \) in some box, place a single point in the cube

\[
Q = B^L \times [n-1, n) \in \mathbb{R}^2 \times [0, \infty)
\]
uniformly at random. If no error occurs, leave the corresponding box empty in $\mathcal{P}$.

It follows directly from our definitions of the parameters that $\tilde{A}_n \subset A_n$ for each $n$. The rates have been set up so that all boxes with density $< \alpha$ in the contact process are covered by some triangular region. A triangular region automatically covers type I and type II errors. All other low density boxes can only be in the vicinity of the type I and II errors: within distance $d(K)$ times the age of the error. The interaction rate $b$ ensures that if there is a large cluster of errors, the overlap/collision region grows faster than the surrounding bad boxes can spread (only by $d(k)$ units per single time step), and from the perimeter of the bad regions, the positive wave speeds “propagate” the high density boxes into the former bad regions. The comparison process $A_n$ has a nontrivial stationary distribution. Hence, so does the contact process.

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