MOVABLE CONES OF COMPLETE INTERSECTIONS OF MULTIDEGREE ONE ON PRODUCTS OF PROJECTIVE SPACES

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Abstract. We study Calabi-Yau manifolds which are complete intersections of hypersurfaces of multidegree 1 in an $m$-fold product of $n$-dimensional projective spaces. Using the theory of Coxeter groups, we show that the birational automorphism group of such a Calabi-Yau manifold $X$ is infinite and a free product of copies of $\mathbb{Z}$. Moreover, we give an explicit description of the boundary of the movable cone $\overline{\text{Mov}}(X)$. In the end, we consider examples for the general and non-general case and picture the movable cone and the fundamental domain for the action of $\text{Bir}(X)$.

1. Introduction

The Kawamata–Morrison cone conjecture predicts that the movable cone $\overline{\text{Mov}}(X)$ of a Calabi-Yau manifold $X$ is rational polyhedral up to the action of the birational automorphism group $\text{Bir}(X)$. Even though it has been proved in several instances (see [Kaw97], [Tot10], [PS12], [LP13], [Ogu14], [CO15], [Yá22], [Wan22], [HT18], [LW22], [ILW22] and references therein), the conjecture remains widely open. We refer to [LOP20] for a good survey on this problem.

The main focus of this article are the following varieties: let $\mathbb{P} := \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ be an $m$-fold product of projective spaces, with $n \geq 1$ and $nm - (n + 1) \geq 3$. Consider $X$ to be a general complete intersection subvariety given by the intersection of $n + 1$ ample divisors of multidegree $(1, \ldots, 1)$. By generality and adjunction we have that $X$ is a smooth Calabi-Yau variety. Moreover, $X$ has only finitely many minimal models $X = X_0, X_1, \ldots, X_m$ up to isomorphism and satisfies the Kawamata-Morrison cone conjecture for the movable cone (see [Wan22]). However, the structures of the movable cone $\overline{\text{Mov}}(X)$ and the birational automorphism group $\text{Bir}(X)$ are not known.

In this paper, we show that in contrast to the complete intersection Calabi-Yau manifolds considered in [CO15] and [Yá22], the birational automorphism group is not a Coxeter group in this case and give a complete description of $\text{Bir}(X)$.

Theorem 1.1. Let $\mathbb{P} = \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ be an $m$-fold product and consider $X$ a general complete intersection of $n + 1$ hypersurfaces of multidegree $(1, \ldots, 1)$ in $\mathbb{P}$.

1. The automorphism group $\text{Aut}(X)$ is finite and acts trivially on $N^1(X)_{\mathbb{R}}$. If $n \geq 3$ and $m \geq 3$, then $\text{Aut}(X)$ is trivial for $X$ very general.

2. If $n \geq 2$, the birational automorphism group $\text{Bir}(X)$ is isomorphic to the group \( \mathbb{Z} \ast \cdots \ast \mathbb{Z} \) up to the finite kernel $\text{Aut}(X)$.

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Recall that Wang showed that a general $X$ has exactly $m + 1$ minimal models $X = X_0, X_1, \ldots, X_m$ and that the cone $\text{Nef}^e(X_i)$ for $i = 0, \ldots, m$ is a rational polyhedral cone in $N^1(X_i)_{\mathbb{R}} \cong \mathbb{R}^m$ spanned by $m$ extremal rays (see [Wan22]). Using results from [Loo14], Wang concluded that $X$ satisfies the movable cone conjecture. We refine this statement by providing an explicit description of a fundamental domain $\Pi$ for the action of $\text{Bir}(X)$ on the effective movable cone $\text{Mov}^e(X)$.

**Theorem 1.2.** Let $\mathbb{P} = \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ be an $m$-fold product, and let $X$ be a general complete intersection of $n + 1$ hypersurfaces of multidegree $(1, \ldots, 1)$ in $\mathbb{P}$ with minimal models $X_0 = X, X_1, \ldots, X_m$. Then there exist birational maps $\varphi_{0,i}: X \to X_i$ for $i = 1, \ldots, m$ such that

$$
\Pi = \text{Nef}^e(X_0) \cup \bigcup_{i=1,\ldots,m} \varphi_{0,i}^* \text{Nef}^e(X_i) \subset \text{Mov}^e(X)
$$

is a fundamental domain for the action of $\text{Bir}(X)$ on $\text{Mov}^e(X)$.

Consequently, the fundamental domain $\Pi$ is the union of $m + 1$ polyhedral cones, where each pullback of $\text{Nef}^e(X_i)$ for $i = 1, \ldots, m$ is glued to a codimension one face of $\text{Nef}^e(X_0)$.

For our last main theorem we recall that a Coxeter group $W$ is, roughly speaking, a group generated by a finite number of reflections, and it can be represented as matrices acting on a real dimensional vector space whose dimension is equal to the number of generators. We can associate a rational polyhedral cone $D$ to the group $W$, the $W$-orbit of $D$ is a convex cone, called the Tits cone, and the cone $D$ is a fundamental domain for the action of $W$ on the Tits cone. In [CO15] and [Yà22] they identify $\text{Bir}(X)$ with a Coxeter group $W$, such that under this connection the closure of the Tits cone of $W$ corresponds to $\text{Mov}(X)$ and the fundamental domain $D$ corresponds to the nef cone $\text{Nef}(X)$. As $\text{Bir}(X)$ is not a Coxeter group in the case of Theorem 1.1, this argument cannot be used directly. Nevertheless, we find that its elements can be related to a Coxeter system $(W, S)$. More precisely, we show that, up to a permutation, any birational automorphism of $X$ can be represented as a product of matrices $t_1, \ldots, t_m$ associated to $(W, S)$ (see Subsection 2.1 for the definitions). Using this together with the description of the fundamental domain $\Pi$ from Theorem 1.2, we explicitly describe the boundary of $\text{Mov}(X)$ following [Yà22].

**Theorem 1.3.** Let $\mathbb{P}$ and $X$ be as above. For $i = 1, \ldots, m$, let $H_i$ be the pullback of a hyperplane class of the $i$-th factor $\mathbb{P}^n$ to $X \subset \mathbb{P}^n \times \cdots \times \mathbb{P}^n$. Then the boundary of $\text{Mov}(X)$ is the closure of the union of the $W$-orbits of the cones $\{a_\lambda v_\lambda + \sum_{k \neq i,j} a_k H_k \mid a_\lambda \geq 0, a_k \geq 0\}$, where $v_\lambda$ is:

- If $n \geq 3$, an eigenvector associated to the unique eigenvalue $\lambda > 1$ of $t_i t_j$; or
- If $n = 2$, $v_\lambda = 0$.

Another emphasis of this paper is to consider explicit examples. Whereas the movable cone is well described for a Calabi-Yau manifold $X$ with infinite $\text{Bir}(X)$ and Picard number $\rho(X) = 2$ (see [LP13]), the structure of the cone $\text{Mov}(X)$ is poorly understood in higher Picard numbers. A class of concrete examples may help to get a better understanding of the general situation and can be used for determining divisors on the boundary of $\text{Mov}(X)$ or $\text{Eff}(X)$ where the numerical dimension behaves interestingly.
In Section 4 we consider examples for different $n$ and $m$. On the one hand we provide examples for the general case where we picture the movable cone together with its chamber structure by the action of Bir($X$); and by using the description of the boundary from Theorem 1.3 (see Examples 4.2 and 4.3 in Subsection 4.1). On the other hand, in Subsection 4.2 we give examples where the automorphism group of $X$ is non-trivial which are not considered in [Wan22]. In particular, we notice that if the automorphism group does not act trivially on $N^1(X)_\mathbb{R}$, then the structure of the movable cone is significantly different to the general case. This is shown in Example 4.4 where we verify that the movable cone conjecture is satisfied in this case and provide a conjectural description of the pseudoeffective cone.

Having an explicit description of both the group of birational automorphisms of $X$ and of the movable cone $\overline{\text{Mov}}(X)$ has proven to be useful to compute the numerical dimension of divisors on the boundary of $\overline{\text{Mov}}(X)$ (see for example [Les22], [HS22], [Yá22]). One important detail of these computations is that $\overline{\text{Mov}}(X) = \overline{\text{Eff}}(X)$, so the numerical dimension of a divisor on the boundary can be computed by pulling it back to a nef divisor via a birational automorphism of $X$. In the case of Example 4.4, we have that the pseudoeffective cone is not equal to the movable cone, so this exact method cannot be used. In Proposition 4.7 we compute the numerical dimension of most of the expected boundary of $\overline{\text{Eff}}(X)$. See also Question 4.9 and the discussion preceding it.

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2. Preliminaries

2.1. Coxeter groups. For a complete introduction to the theory of Coxeter groups we refer to [Hum90]. Let $W$ be a finitely generated group and let $S = \{s_i\}_{i=1}^n$ be a finite set of generators. We say that the pair $(W, S)$ is a Coxeter system if there are integers $m_{ij} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that

- $W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \rangle$,
- $m_{ii} = 1$ for all $i$,
- $m_{ij} = m_{ji} \geq 2$ or $= \infty$.

A group is called a Coxeter group if there is a finite subset $S \subset W$ such that $(W, S)$ is a Coxeter system.

Notice that if $m_{ij} = \infty$ for all distinct $i$ and $j$, then

$$W \cong \mathbb{Z}/2\mathbb{Z} \ast \cdots \ast \mathbb{Z}/2\mathbb{Z}$$

$n$ times

and is sometimes referred as the universal Coxeter group of rank $n$.

To a given Coxeter system $(W, S)$ we can associate a real vector space $V$ of dimension $n = |S|$ with basis $\Delta = \{\alpha_i\}_{i=1}^n$, and a bilinear form $Q$. To define the bilinear form $Q$, for each pair $(i, j)$ such that $m_{ij} = \infty$ we choose real numbers $c_{ij}$, with the condition
that $c_{ij} = c_{ji} \geq 1$. Then the bilinear form is defined as

$$Q(\alpha_i, \alpha_j) = \begin{cases} -\cos \left( \frac{\pi}{m_{ij}} \right) & \text{if } m_{ij} < \infty \\ -c_{ij} & \text{if } m_{ij} = \infty, \end{cases}$$

and we can encode the choice of numbers $c_{ij}$ in a matrix $Q$ with entries $Q_{ij} = Q(\alpha_i, \alpha_j)$. Notice that by definition $m_{ii} = 1$, so $Q(\alpha_i, \alpha_i) = -\cos \pi = 1$.

For each generator $s_i$ of the Coxeter system $(W, S)$ we can assign to it an element $\tau_i \in \text{GL}(V)$ such that $\tau_i(w) = w - 2Q(w, \alpha_i)\alpha_i$ for $w \in V$.

**Proposition 2.1.** The homomorphism $\rho: W \to \text{GL}(V)$ defined by $s_i \mapsto \tau_i$ is a faithful geometric representation of the Coxeter group $W$.

**Proof.** For the case when we choose $c_{ij} = 1$ for $m_{ij} = \infty$, see [Hum90, Section 5.4 Corollary], and for the general case see [Vin71, Theorem 2(6)]. \qed

Given that the representation does depend on the choice of $Q$, we keep track of this data by denoting our Coxeter system $(W, S)_Q$.

In this article we are more interested of the action of $W$ on the dual space $V^*$. Define in $V^*$ the convex cone $D \subseteq V^*$ as the intersection of the half-spaces

$$D := \bigcap_{i=1}^n \{ f \in V^* \mid f(\alpha_i) \geq 0 \}.$$

Denote the dual representation of $W$ on $V^*$ as $\rho^*$. Then we define the Tits cone $T$ as the $W$-orbit of $D$, i.e,

$$T = \bigcup_{w \in W} \rho^*(w)(D).$$

**Proposition 2.2 ([Vin71, Theorem 2 and Proposition 8]).** The Tits cone $T$ is a convex cone, and $D$ is a fundamental domain for the action of $W$.

To connect the action of $W$ on $V$ and on $V^*$ we can identify $V^*$ with $V$ as follows. Let $\{\beta_i\}$ be the vectors on $V$ such that $Q(\alpha_i, \beta_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker Delta. Under this identification, the fundamental domain $D$ is the cone

$$D = \left\{ \sum_{i=1}^n a_i \beta_i \mid a_i \geq 0 \text{ for all } 1 \geq i \geq n \right\},$$

and the matrices associated to the generators of the dual action of $W$, $\rho^*(s_i)$, with respect to the basis $\{\beta_i\}$ correspond to the transpose of the matrices associated to $\tau_i$ with respect to the basis $\{\alpha_i\}$. We call $t_i$ the matrix representing $\rho^*(s_i)$ with respect to the basis $\{\beta_i\}$.

**Example 2.3.** Let $(W, \{s_1, s_2, s_3\})$ be a Coxeter system with $m_{ij} = \infty$ when $i \neq j$. Then $W \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Choose the bilinear form $Q$ with associated matrix

$$Q = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$
Then with respect to the basis \( \{ \alpha_1, \alpha_2, \alpha_3 \} \) the matrices associated to \( \tau_1, \tau_2 \) and \( \tau_3 \) are
\[
\left(\begin{array}{ccc}
-1 & 4 & 4 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad
\left(\begin{array}{ccc}
1 & 0 & 0 \\
4 & -1 & 4 \\
0 & 0 & 1
\end{array}\right), \quad
\left(\begin{array}{ccc}
1 & 0 & 0 \\
4 & 4 & -1 \\
0 & 0 & 1
\end{array}\right)
\]
respectively. Then the matrices \( t_1, t_2 \) and \( t_3 \) are
\[
\left(\begin{array}{ccc}
-1 & 0 & 0 \\
4 & 1 & 0 \\
4 & 0 & 1
\end{array}\right), \quad
\left(\begin{array}{ccc}
1 & 4 & 0 \\
0 & -1 & 0 \\
0 & 4 & 1
\end{array}\right), \quad
\left(\begin{array}{ccc}
1 & 0 & 4 \\
0 & 1 & 4 \\
0 & 0 & -1
\end{array}\right)
\]

Remark 2.4. When studying the Tits cone \( T \) it is convenient to work on \( \mathbb{P}V \). Choose \( V_1 \) an affine hyperplane such that each ray \( \mathbb{R}_{>0}\alpha_i \), for \( 1 \leq i \leq n \), intersects at one point and denote it \( \hat{\alpha}_i \). Let \( \varphi \) be the linear form such that \( V_1 = \{ v \in V \mid \varphi(v) = 1 \} \). Define
\[
\hat{v} = \frac{v}{\varphi(v)}, \text{ for } v \in V \setminus \{ \varphi(v) = 0 \}.
\]
These points, along with the hyperplane at infinity, form \( \mathbb{P}V \), and we will refer to points in \( \mathbb{P}V \) as directions of \( V \).

To study the boundary of the closure of the Tits cone, we need some additional properties of \( W \). We say that the Coxeter system \( (W, S)_Q \) is Lorentzian if the signature of the matrix \( Q \) is \( (n-1, 1) \). Notice that this property depends on the choice of matrix \( Q \) and it is not intrinsic to the Coxeter system \( (W, S) \).

The main consequence of this property is that given an infinite reduced word \( w = s_{k_1}s_{k_2}\cdots \in W \), any injective sequence \( \{ w_i \cdot \hat{x} \} \), where \( w_i = s_{k_1}s_{k_2}\cdots s_{k_i} \), converges to the same direction, regardless of \( \hat{x} \in \mathbb{P}V \). Here, by injective sequence we mean that \( w_i \cdot \hat{x} \neq w_j \cdot \hat{x} \) as elements in \( \mathbb{P}V \), when \( i \neq j \).

Theorem 2.5. Let \( w \) be an infinite reduced word in a Lorentzian Coxeter system \( (W, S)_Q \). Then every injective convergent sequence \( \{ w_i \cdot \hat{x} \} \) converges to the same unique direction \( \hat{\gamma}(w) \).

Proof. This result follow from [CL17, Theorem 2.5, Corollary 2.6, Theorem 2.8 and Corollary 2.9]. \( \square \)

Remark 2.6. It is interesting to notice that by [CL17, Theorem 2.5], after the identification of the dual space \( V^* \) with \( V \), all the accumulation points of \( \overline{\text{Mov}}(X) \) lie on the quadratic cone defined by the quadric associated to the inverse of the symmetric matrix \( Q \) of the Coxeter system \( (W, S)_Q \).

2.2. Calabi-Yau manifolds and main construction. We recall some definitions and results which will be used in this paper.

Definition 2.7. A Calabi-Yau manifold \( X \) of dimension \( n \) is a projective manifold with trivial canonical class \( K_X \) and \( h^1(X, \mathcal{O}_X) = 0 \).

Let \( X \) be a projective manifold, and let \( N^1(X) \) be the Néron–Severi group of \( X \). We denote the rank of \( N^1(X) \) by \( \rho(X) \). Inside \( N^1(X)_\mathbb{R} = N^1(X) \otimes \mathbb{R} \) we have
- the nef cone \( \text{Nef}(X) \), and its interior the ample cone \( \text{Amp}(X) \);
- the effective nef cone \( \text{Nef}^e(X) = \text{Nef}(X) \cap \text{Eff}(X) \);
\begin{itemize}
    \item the closure of the cone generated by movable divisors \( \overline{\text{Mov}}(X) \) and its interior \( \text{Mov}(X) \);
    \item the effective cone \( \text{Eff}(X) \) with closure \( \overline{\text{Eff}}(X) \), the pseudoeffective cone; and
    \item the effective movable cone \( \overline{\text{Mov}}^e(X) = \overline{\text{Mov}}(X) \cap \text{Eff}(X) \).
\end{itemize}
Furthermore, we denote by \( \text{Aut}(X) \) the automorphism group and by \( \text{Bir}(X) \) the birational automorphism group.

The Kawamata–Morrison cone conjecture which connects the convex geometry of the nef cone \( \text{Nef}(X) \) and the movable cone \( \overline{\text{Mov}}(X) \) of a Calabi-Yau manifold \( X \) with the action of \( \text{Aut}(X) \) and \( \text{Bir}(X) \), respectively, can be formulated as follows:

\textbf{Conjecture 2.8} ([Mor93],[Kaw97]). \textit{Let} \( X \) \textit{be a Calabi-Yau manifold.}

\begin{enumerate}
    \item There exists a rational polyhedral cone \( \Pi \) which is a fundamental domain for the action of \( \text{Aut}(X) \) on \( \text{Nef}(X) \cap \text{Eff}(X) \), in the sense that \( \text{Nef}(X) \cap \text{Eff}(X) = \bigcup_{g \in \text{Aut}(X)} g^* \Pi \) with \( \text{int} \Pi \cap \text{int} g^* \Pi = \emptyset \) unless \( g^* = \text{id} \).
    \item There exists a rational polyhedral cone \( \Pi' \) which is a fundamental domain for the action of \( \text{Bir}(X) \) on \( \overline{\text{Mov}}(X) \cap \text{Eff}(X) \).
\end{enumerate}

Next we explain Wang’s construction of Calabi-Yau manifolds as general complete intersections in a product of projective spaces. For more details and proofs we refer to [Wan22].

\textbf{Notation 2.9.} We write \( \widehat{P} = P_n^0 \times P_n^1 \times \ldots \times P_n^m \) for the \((m+1)\)-fold product of \( n \)-dimensional projective spaces and assume that \( mn - (n+1) \geq 3 \). We denote by \( \widehat{P}_i \) the \( m \)-fold product of \( P_n \)'s, where the \( i \)-th factor is missing, and by \( \widehat{P}_{i,j} \) the product where the \( i \)-th and the \( j \)-th factors are missing. If we denote by \( H_i \) the pull-back of a hyperplane class in \( P_n^i \), then \( \text{Pic}(\widehat{P}) \) is generated by \( H_0, \ldots, H_m \) and

\[ \text{Eff}(\widehat{P}) = \text{Nef}(\widehat{P}) = \text{Mov}(\widehat{P}) = \text{Cone}(H_0, \ldots, H_m) \]

where the last object is the convex cone in \( N^1(\widehat{P})_\mathbb{R} \cong \mathbb{R}^{m+1} \) generated by \( H_0, \ldots, H_m \).

We consider a general complete intersection \( X \) given by \( n+1 \) hypersurfaces \( D_j \) of multidegree \((1, \ldots, 1)\) in \( \widehat{P}_0 \). Then, by generality and adjunction, \( X \) is a smooth Calabi-Yau variety of dimension \( mn - n - 1 \). By the constraints on \( n \) and \( m \) we have that \( \dim X \geq 3 \).

\textbf{Remark 2.10.} As the general complete intersection of \( n \) hypersurfaces of multidegree \((1, \ldots, 1)\) in \( \widehat{P}_0 \) is a smooth Fano manifold, by intersecting \( n+1 \) of such hypersurfaces and obtaining \( X \), we know that

\[ \text{Nef}(X) \cong \text{Nef}(\widehat{P}_0) \]

and that the automorphism group \( \text{Aut}(X) \) is finite by [CO15, Theorem 3.1].

\textbf{Construction 2.11} ([Wan22, Construction 7.2]). \textit{Let} \( x^i = [x_0^i, \ldots, x_n^i] \) \textit{be the homogeneous coordinates of} \( P_n^i \) \textit{for} \( i = 0, \ldots, m \). \textit{We denote by}

\[ I = \{(r_1, \ldots, r_m) \mid 0 \leq r_k \leq n \text{ for each } k \} \]
We can write the \( n + 1 \) hypersurfaces \( D_j \) defining \( X \) as

\[
\sum_{r=(r_1,\ldots,r_m)\in I} a_j^r x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}.
\]

Moreover, setting

\[
b^r_{r_0} := a_j^r
\]

for each \( r = (r_1,\ldots,r_m) \in I \) and \( 0 \leq r_0 \leq n \) we can define a \((1,\ldots,1)\)-form

\[
F = \sum_{0 \leq r_k \leq n, 0 \leq k \leq m} b^r_{r_0} x_0^{r_0} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}
\]

in \( \mathbb{P} = \mathbb{P}_0^n \times \mathbb{P}_1^n \times \cdots \times \mathbb{P}_m^n \).

Now, differentiating \( F \) with respect to the \( n + 1 \) variables of a \( \mathbb{P}_i^n \) gives \( n + 1 \) hypersurfaces of multidegree \((1,\ldots,1)\) in \( \hat{\mathbb{P}}_i \) which define a complete intersection variety \( X_i \) with \( X_0 = X \). Moreover, if we choose \( X \) generically, the \( X_i \) are also smooth and define Calabi-Yau varieties by [Wan22, Section 7] with

\[
\text{Nef}(X_i) \cong \text{Nef}(\hat{\mathbb{P}}_i) \cong \text{Nef}(\mathbb{P}_0) \cong \text{Nef}(X_0).
\]

For \( i \neq j \), we denote the pullback of the hyperplane class of \( \mathbb{P}_j^n \) to \( X_i \) by \( H_j^i \). Then \( \text{Nef}(X_i) \) is the convex cone generated by the \( m \) elements in

\[
B_i = \{H_0^i,\ldots,H_{i-1}^i,H_{i+1}^i,\ldots,H_m^i\}
\]

which form a basis of \( N^1(X_i)_{\mathbb{R}} \cong \mathbb{R}^m \).

Two different \( X_i \) and \( X_j \) can be projected to the space \( \hat{\mathbb{P}}_{i,j} \) and have the same image there which we denote by \( W_{i,j} \). Indeed let \( A_{i,j} \) be the \((n+1) \times (n+1)\)-matrix obtained from \( F \) by first differentiating \( F \) with respect to the \( n + 1 \) variables of \( \mathbb{P}_i^n \) and afterwards differentiating these \( n + 1 \) forms with respect to the variables of \( \mathbb{P}_j^n \). Then \( A_{i,j} = A_{j,i}^t \)

and the image of \( X_i \) (resp. \( X_j \)) to \( \hat{\mathbb{P}}_{i,j} \) is the determinantal hypersurface

\[
W_{i,j} = (\det(A_{i,j}) = 0) \subset \hat{\mathbb{P}}_{i,j}
\]

of multidegree \((n+1,\ldots,n+1)\) and dimension

\[(m-1)n-1 = \dim X_i.\]

In the following, we denote the projection from \( X_i \) to \( W_{i,j} = W_{j,i} \) by \( \pi_{i,j} \). The projection \( \pi_{i,j}^l \) has connected fibers and is an isomorphism outside the subset of \( W_{i,j} \) consisting of points where the rank of \( A_{i,j} \) is \( \leq n-1 \). Thus, for a general choice of \( X \), the maps \( \pi_{i,j}^l \) are birational. Furthermore, the two projections \( \pi_{i,j}^l \) and \( \pi_{i,j}^r \) are simultaneously divisorial or small by [Wan22, Lemma 7.4].

Finally, for each pair of \( i \neq j \) there is an induced birational map \( \varphi_{i,j} \):

\[
\begin{array}{ccc}
X_i & \xrightarrow[\pi_{i,j}^r]{} & X_j \\
\downarrow[\pi_{i,j}^l] & & \downarrow[\pi_{j,i}^l] \\
W_{i,j} & \xrightarrow{\varphi_{i,j}} & W_{j,i} \\
\end{array}
\]
Moreover, if $F$ is chosen generically, then $\varphi_{i,j}$ is not an isomorphism over $W_{i,j}$ (see [Wan22, Assumption 7.5]) and the projections $\pi^i_{i,j}$ and $\pi^j_{i,j}$ are both small. In this case, $\varphi_{i,j}$ is the flop of $\pi^i_{i,j}$ (see [Wan22, Lemma 7.6]). The main theorem of [Wan22] concerning this class of Calabi-Yau manifolds is the following theorem.

**Theorem 2.12 ([Wan22, Theorem 1.3]).** Let $X \subset \mathbb{P}^n_1 \times \ldots \times \mathbb{P}^n_m$ be a general complete intersection of $n+1$ hypersurfaces of multidegree $(1, \ldots, 1)$, and let $X_i \subset \mathbb{P}_i$ for $i = 1, \ldots, m$ as defined above. Then $X$ has only finitely many minimal models $X_0 = X, X_1, \ldots, X_m$ up to isomorphism. Moreover, there exists a rational polyhedral cone $\Pi$ which is a fundamental domain for the action of $\text{Bir}(X)$ on $\text{Mov}^r(X)$.

### 3. Computing $\text{Bir}(X)$

Throughout this section we will assume that $\mathbb{P} = \mathbb{P}^n_0 \times \ldots \times \mathbb{P}^n_m$ with $nm - (n + 1) \geq 3$ and that $F$ is a general form of multidegree $(1, \ldots, 1)$ in $\mathbb{P}$ as in Notation 2.9. For $i = 0, \ldots, m$, let $X_i \subset \mathbb{P}_i$ be the Calabi-Yau variety introduced in the previous section. Recall that $\dim X_i = n(m - 1) - 1$ and $\rho(X_i) = m$.

Consider the Coxeter system $(W, S)_Q$, with $S = \{s_1, \ldots, s_m\}$, $W = \langle s_i \mid (s_is_j)^{m_{ij}} = 1 \rangle$. If $n = 1$, set $m_{ij} = 3$ for $i \neq j$, and if $n \geq 2$ let $m_{ij} = \infty$ for $i \neq j$. The matrix $Q$ is given by

\[
Q_{ij} = \begin{cases} 
1 & \text{if } i = j \\
-n/2 & \text{if } i \neq j.
\end{cases}
\]

Under this setup we have that for $n \geq 2$, $W \cong \mathbb{Z}/2\mathbb{Z} \ast \ldots \ast \mathbb{Z}/2\mathbb{Z}$.

The matrices $t_j$ from Subsection 2.1 associated to $(W, S)_Q$ are given by

\[
t_j = (e_1 \ldots e_{j-1} v_j e_{j+1} \ldots e_m),
\]

where $v_j = (n, \ldots, n, -1, n, \ldots, n)^T$ and $e_j$ is the $j$-th standard vector. Also define the matrices $\text{Per}_\sigma$, with $\sigma$ an element of the symmetric group $S_m$, as the matrix that permutes rows (respectively columns) by the permutation $\sigma$. With this definition $\text{Per}_\sigma \cdot \text{Per}_{\sigma'} = \text{Per}_{\sigma \sigma'}$.

The following result connects the matrices associated to the flops $\varphi_{i,j}: X_i \dasharrow X_j$ from Subsection 2.2 and the matrices $t_k$ obtained from the geometrical representation of this Coxeter system.

**Proposition 3.1.** For $i < j$, the matrix of $\varphi_{i,j}^*$ with respect to the bases $B_j$ and $B_i$ from Equation (2.2) is

\[
t_j \cdot \text{Per}_{(i+1,i+2,\ldots,j)}^{-1}.
\]

**Proof.** Recall the diagram for $\varphi_{i,j}$ from Subsection 2.2:

\[
\begin{array}{ccc}
X_i & \overset{\varphi_{i,j}}{\longrightarrow} & X_j \\
\pi^i_{i,j} \downarrow & & \downarrow \pi^j_{i,j} \\
W_{i,j} & & \\
\end{array}
\]
From the diagram and the definition of the map \( \varphi_{i,j} \) we see directly that for \( k \not\in \{i, j\} \) we have

\[
\varphi_{i,j}^* (H_k^j) = H_k^i
\]

and it remains to determine \( \varphi_{i,j}^* (H_i^j) \). We consider the push-forwards of \( H_i^j \) and \( H_j^i \) to \( W_{i,j} \). By the definition of the variety \( W_{i,j} \), \( (\pi_{i,j})_* H_i^j \) corresponds to the maximal minors of an \( n \times (n + 1) \) submatrix of \( A_{i,j} \) whose rows are general linear combinations of the rows of \( A_{i,j} \), whereas \( (\pi_{j,i})_* H_j^i \) corresponds to the maximal minors of an \( (n + 1) \times n \) submatrix of \( A_{i,j} \) whose columns are general linear combinations of the columns of \( A_{i,j} \). Thus, their sum is a complete intersection of \( W_{i,j} \) with a hypersurface of bidegree \( (n, \ldots, n) \), corresponding to the determinant of a \( n \times n \) submatrix of \( A_{i,j} \), that is

\[
(\pi_{i,j})_* H_i^j + (\pi_{j,i})_* H_j^i = n \left( \sum_{k=0}^{m} h_k \right)_{W_{i,j}},
\]

where \( h_k \) corresponds to the pullback of the hyperplane class of \( \mathbb{P}^n_k \) to \( \hat{\mathbb{P}}_{i,j} \). Consequently, on \( X_i \) we obtain

\[
\varphi_{i,j}^* (H_j^i) = -H_j^i + \sum_{k=0}^{m} nH_k^i.
\]

Recall from Section 2.2 that the set \( B_i = \{H_0^i, \ldots, H_{i-1}^i, H_{i+1}^i, \ldots, H_m^i\} \) is a basis of \( N^1(X_i)_{\mathbb{R}} \). Because \( i < j \), the element \( H_i^j \) is in the \( (i + 1) \)-th position of the basis \( B_j \), while the element \( H_j^i \) is in the \( j \)-th position of the basis \( B_i \). After permuting \( B_j \) by the cycle \( (i + 1, i + 2, \ldots, j)^{-1} \) the matrix of \( \varphi_{i,j}^* \) is given by the columns

\[
(e_1 \ldots e_{j-1} v_j e_{j+1} \ldots e_m),
\]

where \( v_j = (n, \ldots, n, -1, n, \ldots, n)^T \). This matrix corresponds to the matrix \( t_j \), which proves the statement. \( \square \)

From an easy computation we obtain the following result:

**Lemma 3.2.** Given \( t_i \) as defined above and \( \sigma \in S_m \), we have that

\[
\text{Per}_\sigma \cdot t_i = t_{\sigma(i)} \cdot \text{Per}_\sigma
\]

**Proposition 3.3.** For \( i > j \), the matrix of \( \varphi_{i,j} \) with respect to the bases \( B_j \) and \( B_i \) is

\[
t_{j+1} \cdot \text{Per}_{(j+1,j+2,\ldots,i)}.
\]

**Proof.** From the construction we have that \( (\varphi_{i,j}^*)^{-1} = (\varphi_{j,i}^*)^{-1} \cdot t_i \). From Proposition 3.1, \( (\varphi_{i,j}^*)^{-1} = \text{Per}_{(j+1,j+2,\ldots,i)} \cdot t_i \) because \( t_i^2 = \text{id} \). Applying Lemma 3.2 we obtain that \( \varphi_{i,j}^* = t_{j+1} \cdot \text{Per}_{(j+1,j+2,\ldots,i)} \).

\( \square \)
Proposition 3.4. Let $\psi_{i,j} : X \to X$ be the composition of flops

\[ X \xrightarrow{\varphi_{0,0}} X_0 \xrightarrow{\varphi_{0,i}} X_i \xrightarrow{\varphi_{i,j}} X_j \xrightarrow{\varphi_{j,0}} X. \]

Then $\psi_{i,j}^* = t_i t_j t_i \cdot \text{Per}_{(i,j)}$.

Proof. Assume that $i < j$. Then by Proposition 3.1

\[ \psi_{i,j}^* = \varphi_{0,i}^* \cdot \varphi_{i,j}^* \cdot (\varphi_{0,j}^*)^{-1} = t_i \cdot \text{Per}_{(1,\ldots,i-1)} \cdot t_j \cdot \text{Per}_{(i+1,\ldots,j-1)} \cdot \text{Per}_{(1,\ldots,j)} \cdot t_j. \]

Because $i < j$, by Lemma 3.2 we obtain $\text{Per}_{(1,\ldots,i-1)} \cdot t_j = t_j \cdot \text{Per}_{(1,\ldots,i-1)}$. Then

\[ \psi_{i,j}^* = t_i \cdot t_j \cdot \text{Per}_{(1,\ldots,i-1)} \cdot \text{Per}_{(i+1,\ldots,j-1)} \cdot \text{Per}_{(1,\ldots,j)} \cdot t_j. \]

We need to compute $(1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}(1,\ldots,j)$. If $k < i$, then

\[ (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}(1,\ldots,j)[k] = (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}[k+1] = (1,\ldots,i)^{-1}[k+1] = k. \]

If $i < k < j$, then

\[ (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}(1,\ldots,j)[k] = (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}[k+1] = (1,\ldots,i)^{-1}[k] = k. \]

If $j < k$, then

\[ (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}(1,\ldots,j)[k] = (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}[k] = (1,\ldots,i)^{-1}[k] = k. \]

Finally,

\[ (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}(1,\ldots,j)[i] = (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}[i+1] = (1,\ldots,i)^{-1}[j] = j. \]

\[ (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}(1,\ldots,j)[j] = (1,\ldots,i)^{-1}(i+1,\ldots,j)^{-1}[1] = (1,\ldots,i)^{-1}[1] = i. \]
Therefore, \((i, \ldots, i)^{-1}(i + 1, \ldots, j)^{-1}(1, \ldots, j) = (i, j)\).

Back to our computation, we get
\[
\psi_{i,j}^* = t_i \cdot t_j \cdot \text{Per}_{(i,\ldots,i)} \cdot \text{Per}_{(i+1,\ldots,j)} \cdot \text{Per}_{(1,\ldots,j)} \cdot t_j = t_i \cdot t_j \cdot \text{Per}_{(i,j)} \cdot t_j = t_it_jt_i \cdot \text{Per}_{(i,j)}.
\]

For \(i > j\), notice that \(\psi_{i,j}^* = (\psi_{i,j}^*)^{-1}\), and so \(\psi_{i,j}^* = \text{Per}_{(i,j)} \cdot t_j t_i \cdot t_it_j \). \(\square\)

**Proposition 3.5.** For \(i \neq j\) and \(1 \leq i, j \leq m\), and consider the birational automorphism \(\psi_{i,j}\). If \(n = 1\), then \(\psi_{i,j}^*\) has order 2, and if \(n \geq 2\), \(\psi_{i,j}^*\) has infinite order. In the case of \(n = 2\) we get that 1 is the only eigenvalue, and that \(\psi_{i,j}^*\) is not diagonalizable. When \(n \geq 3\), \(\psi_{i,j}^*\) is diagonalizable and the eigenvalues are \(\lambda, 1\) and \(\lambda^{-1}\) for some \(\lambda > 1\). Even more, the eigenvalue \(\lambda\) has multiplicity one.

**Proof.** Combining the results from Lemma 3.2 and Proposition 3.4 we get that
\[
\psi_{i,j}^* = t_it_jt_i \cdot \text{Per}_{(i,j)} = \text{Per}_{(i,j)} \cdot t_j t_i.
\]

Hence \((\psi_{i,j}^*)^2 = t_it_jt_jt_j = (t_it_j)^3\). Recall that for \(n = 1\), \(m_{ij} = 3\), so \((t_it_j)^3 = 1\).

For the case \(n \geq 2\) it is enough to compute the eigenvalues of \(t_it_j\). To do so, we follow the same computation done in [Ya22, Proposition 4.11].

We write the matrices \(t_i\) and \(t_j\) in terms of their columns
\[
t_i = (e_1 \ldots e_{i-1} \; v_i \; e_{i+1} \ldots e_m)
\]
and
\[
t_j = (e_1 \ldots e_{j-1} \; v_j \; e_{j+1} \ldots e_m),
\]
where \(v_k = (n, \ldots, n, 1, n, \ldots, n)^T\), the \(k\)th entry.

By computing the product \(t_it_j\) we get that the characteristic polynomial of it is
\[
cp_{t_it_j}(x) = (x - 1)^{m-2}(x^2 - (n^2 - 2)x + 1).
\]

If \(n = 2\), then \(cp_{t_it_j}(x) = (x - 1)^m\), but because \(t_it_j\) is not the identity matrix, we have that \(t_it_j\) has infinite order and it is not diagonalizable.

If \(n \geq 3\), then \(x^2 - (n^2 - 2)x + 1\) has two different real roots \(\lambda_1, \lambda_2\). Because the sum of them is a positive number and \(\lambda_1\lambda_2 = 1\), then one and only one of them must be greater than 1. \(\square\)

Next we describe the automorphism and birational automorphism group of \(X\).

**Proposition 3.6.** For a general \(X\), the automorphisms of \(X\) act trivially on \(\text{Mov}^c(X)\).

**Proof.** We proceed similarly as in [CO15, Theorem 3.3]. First we remark that for a general choice of \(n+1\) \((1, \ldots, 1)\)-forms in \(\mathbb{P}_0\), the corresponding variety \(X\) is a smooth Calabi-Yau variety. Moreover, we can assume that any \(n\) of the \(n+1\) forms define also a smooth variety which is then a Fano manifold \(Y\). Then \(X\) is a smooth anticanonical section of \(Y\) and thus, \(\text{Aut}(X)\) is finite by [CO15, Theorem 3.1].

The group \(\text{Aut}(X)\) preserves the ample cone of \(X\) and hence fixes the set \(\{H_1^0, \ldots, H_m^0\}\). Similarly as in the proof of [CO15, Theorem 3.3] we have
\[
H^0(\mathbb{P}_0, H_i) = H^0(X, H_i^0)
\]
and $\text{Aut}(X)$ is a finite subgroup of
\[ \text{Aut}(\widehat{\mathbb{P}}_0) = PGL_{n+1}(\mathbb{C})^m \rtimes S_m. \]

Let $G$ be the image of
\[ \text{Aut}(\mathbb{P}^n_0) \to PGL_{n+1}(\mathbb{C})^m \rtimes S_m \to S_m. \]

For an element $id \neq g \in G$, there exists a lift to an automorphism of $\widehat{\mathbb{P}}_0$ which restricts to an automorphism of $X$. By definition, the automorphism permutes some of the $\mathbb{P}^n$s. Thus, the ideal generated by the $(1, \ldots, 1)$-forms defining $X$ must contain a form which is invariant under this permutation which is impossible for a general choice of the defining equations. Consequently, $G = \{id\}$.

Thus, $\text{Aut}(X)$ can be identified with a finite subgroup of $PGL_{n+1}(\mathbb{C})^m$ and therefore it acts trivially on $\text{Mov}^e(X)$. □

When $n$ and $m$ are sufficiently large, we obtain the following stronger result:

**Proposition 3.7.** Assume that $n$ and $m$ are greater or equal than 3. Then for a very general $X$, the group $\text{Aut}(X)$ is trivial.

**Proof.** From the proof of Proposition 3.6 we have that $\text{Aut}(X)$ can be identified with a finite subgroup of $PGL_{n+1}(\mathbb{C})^m$. Let $g \in PGL_{n+1}(\mathbb{C})^m$ be a non-trivial element of finite order which induces an automorphism of $X$. Up to conjugacy, the co-action of $g$ is given by
\[ g^*(x^i_j) = c_{i,j}x^i_j \]
where each $c_{i,j}$ for $i = 1, \ldots, m$ and $j = 0, \ldots, n$ is a root of unity. Let $f_0, \ldots, f_n$ be the $(1, \ldots, 1)$-forms defining $X \subset \widehat{\mathbb{P}}_0$. As $X$ is invariant under $g$ and all defining equations $f_i$ have the same multidegree, there exists a matrix $M = (m_{i,j}) \in GL_{n+1}(\mathbb{C})$ of finite order such that
\[
M \begin{pmatrix} g^*f_0 \\ \vdots \\ g^*f_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix}
\]
with $g^*f_i := f_i \circ g$. Moreover, we may assume that the generating equations $f_i$ are chosen so that $M$ is a diagonal matrix whose diagonal elements $m_{i,i}$ are roots of unity. Now, we define the $(1, \ldots, 1)$-form $F$ in $\mathbb{P}$ as before
\[ F = \sum_{j=0}^n x^0_j f_j. \]

Let $\tilde{g}$ be an element in $PGL_{n+1}(\mathbb{C})^{m+1} = PGL_{n+1}(\mathbb{C}) \times PGL_{n+1}(\mathbb{C})^m$ which acts via $M^T$ on $\mathbb{P}_0^n$ and via $g$ on $\widehat{\mathbb{P}}_0$. Then, by the definition of $M$ and $F$, we have
\[ F \circ \tilde{g} = \sum_{j=0}^n (m_{j,j} x^0_j) g^*f_j = \sum_{j=0}^n x^0_j (m_{j,j} g^*f_j) = \sum_{j=0}^n x^0_j f_j = F. \]

A monomial
\[ x^{r_0}_0 x^{r_1}_1 \cdots x^{r_m}_m \]
of $F$ is mapped to
\[ m_{r_0,r_0} c_{1,r_1} \cdots c_{m,r_m} \cdot x^{r_0}_0 x^{r_1}_1 \cdots x^{r_m}_m \]
where \( 0 \leq r_i \leq n \) for all \( i = 0, \ldots, m \). If we set \( c_{0, r_0} = m_{r_0, r_0} \), then comparing coefficients of \( F \) and \( F \circ \tilde{g} \) in (3.2) implies that

\[
(3.3) \quad c_{0, r_0} c_{1, r_1} \cdots c_{m, r_m} = 1.
\]

for any tuple \((r_0, \ldots, r_m)\) describing a monomial of \( F \). As \( g \) is not the identity, we assume without loss of generality that \( g^* \) acts non-trivially on the last component of \( \mathbb{P} \). If for a fixed choice \((r_0, \ldots, r_m-1)\), every monomial with index \((r_0, \ldots, r_m-1, j)\) for \( j = 0, \ldots, n \) occurs in \( F \) with a non-zero coefficient, then (3.3) implies that

\[
c_{m,0} = c_{m,1} = \cdots = c_{m,n}
\]

which is a contradiction. Thus, if \( F \) admits a non-trivial automorphism, the number of monomial factors of \( F \) is at most \( n(n+1)^m \). Thus, the subset of \((1, \ldots, 1)\)-forms \( F \) which admit a non-trivial automorphism is a countable union of varieties of dimension at most

\[
(n+1)^m - 1 + \dim \text{PGL}_{n+1}(\mathbb{C})^m - 1 = (n+1)^m - (m+1)((n+1)^2 - 1) - 1.
\]

As the space of all possible \((1, \ldots, 1)\)-forms in \( \mathbb{P} \) has dimension \((n+1)^m + 1 - 1\), the codimension of this subvariety is

\[
(n+1)^m - 1 - [n(n+1)^m - 1 + (m+1)((n+1)^2 - 1)]) = (n+1)^m - (m+1)((n+1)^2 - 1).
\]

If \( n \geq 3 \) and \( m \geq 3 \), the expression above is a strictly increasing function in either \( n \) or \( m \). Given that for \( n = m = 3 \) the expression is equal to 4, we have that the codimension is \( \geq 4 \). Consequently, removing a countable union of subsets of codimension \( \geq 4 \), the very general \( F \) (and hence \( X \)) has a trivial automorphism group.

\[\square\]

Remark 3.8. The previous argument fails when at least one of \( m \) and \( n \) is strictly smaller than 3. Example 4.10 provides an example in the case of \( n = 4 \) and \( m = 2 \) where the group \( \text{Aut}(X) \) is not trivial.

Proposition 3.9. For a general \( X \), we have \( \text{Bir}(X) = \text{Aut}(X) \cdot \langle \psi_{i,j} \mid 1 \leq i < j \leq m \rangle \).

Proof. Let \( \alpha \in \text{Bir}(X) \). By [Kaw08, Theorem 1] every birational map between Calabi-Yau manifolds can be decomposed into a sequence of flops, up to an automorphism. By [Wan22, Corollary 7.6] any flop of a minimal model \( X_i \) is among the maps \( \varphi_{i,j} : X_i \rightarrow \rightarrow X_j \) described in Subsection 2.2. Hence we can write

\[
\alpha = \varphi_{i,0} \circ \varphi_{i-1;i} \circ \cdots \circ \varphi_{i_0;i_1} \circ \varphi_{i_1;i_0}
\]

and by introducing elements of the form \( \varphi_{0,i} \circ \varphi_{i,0} \) we obtain that

\[
\alpha = \psi_{i-1;i} \circ \cdots \circ \psi_{i_1;i_2} \circ \psi_{i_2;i_1}.
\]

Therefore, up to an automorphism of \( X \)

\[
\text{Bir}(X) = \langle \psi_{i,j} \mid 1 \leq i < j \leq m \rangle. \quad \square
\]

From Proposition 3.5 we know that when \( n \geq 2 \), the generators \( \psi_{i,j} \) have infinite order. We prove, even more, that the group generated by the \( \psi_{i,j} \) is free of rank \( \binom{m}{2} \).
Lemma 3.11. Let $w_1$ and $w_2$ be two reduced words, and let $t_1$ and $t_2$ be reduced words in terms of the elements $t_i$ associated to the Coxeter system $(W,S)\subseteq Q$, where $t_1, t_2, \ldots, t_r$ is also the freely reduced word. Then $k_1 = i_1$ and $k_2 = j_1$.

Proof. Let $w = \psi_{i_1,j_1}^* \psi_{i_2,j_2}^* \cdots \psi_{i_s,j_s}^* = t_{i_1} t_{i_2} \cdots t_{i_r} \cdot \text{Per}_\sigma$ be freely reduced. Define $\ell_\psi(w) := s$ and $\ell_t(w) := r$ the length of the word $w$ in terms of the generators $\psi_{i,j}$ and in terms of the elements $t_i$. Notice that being freely reduced means that for all $1 \leq l < s$, we cannot have simultaneously $i_{l+1} = j_l$ and $j_{l+1} = i_l$. We will prove the lemma by induction on length $\ell_\psi(w)$. Let $\psi_{i_1,j_1}^* = t_{i_1} t_{i_2} t_{i_3} \cdots t_{i_l} \cdot \text{Per}_\sigma$ and $\psi_{i_{l+1},j_{l+1}}^* = t_{i_{l+1}} t_{i_{l+2}} t_{i_{l+3}} \cdots t_{i_r} \cdot \text{Per}_\sigma$ be two generators, where $\sigma_1 = (i_1, j_1)$ and $\sigma_2 = (i_2, j_2)$. Then

$$
\psi_{i_1,j_1}^* \psi_{i_{l+1},j_{l+1}}^* = t_{i_1} t_{i_2} t_{i_3} \cdots t_{i_r} \cdot \text{Per}_\sigma \cdot \text{Per}_\sigma.
$$

We notice that if $i_2 \neq j_1$, then $\ell_t(\psi_{i_1,j_1}^* \psi_{i_{l+1},j_{l+1}}^*) = 6$, which means in particular that the first three factors of $\psi_{i_1,j_1}^* \psi_{i_{l+1},j_{l+1}}^*$ when written in terms of the $t_i$'s are $t_{i_1} t_{i_2} t_{i_3}$. In the case when $i_2 = j_1$ we have that

$$
\psi_{i_1,j_1}^* \psi_{i_{l+1},j_{l+1}}^* = t_{i_1} t_{i_2} t_{i_3} \cdots t_{i_r} \cdot \text{Per}_\sigma \cdot \text{Per}_\sigma.
$$

In both cases the first two factors are $t_{i_1} t_{i_2}$, so the statement holds.

Assume that the statement is true for all words of length $s$ and suppose that $w$ has length $s + 1$. Then

$$
w = \psi_{i_1,j_1}^* \cdot w',
$$

with $w' = \psi_{i_{l+2},j_{l+2}}^* \cdots \psi_{i_s,j_s}^* = t_{i_{l+2}} t_{i_{l+3}} \cdots t_{i_r} \cdot \text{Per}_\sigma$. Then

$$
w = t_{i_1} t_{i_2} t_{i_3} \cdots t_{i_{l+1}} \cdot \text{Per}_\sigma.
$$

Because we don’t have simultaneously that $i_2 = j_1$ and $j_2 = i_1$, then

$$
w = \begin{cases}
  t_{i_1} t_{i_2} t_{i_3} \cdots & \text{if } i_2 \neq j_1 \\
  t_{i_1} t_{i_2} t_{i_3} \cdots & \text{if } i_2 = j_1
\end{cases}
$$

and in both cases we have the required result. \qed

Proof of Theorem 3.10. Let us assume that

$$
\psi_{i_1,j_1}^* \psi_{i_2,j_2}^* \cdots \psi_{i_s,j_s}^* = \psi_{i_1,j_1}^* \psi_{i_2,j_2}^* \cdots \psi_{i_t,j_t}^*\psi_{i_t,j_t}^* \cdots \psi_{i_s,j_s}^*
$$

are two reduced words, and let $t_{i_1} t_{i_2} \cdots t_{i_r} \cdot \text{Per}_\sigma$ and $t_{i_1} t_{i_2} \cdots t_{i_r} \cdot \text{Per}_\sigma$ their associated reduced words in terms of the elements $t_i$ of the Coxeter system $(W,S)\subseteq Q$. Then

$$
\text{Per}_{\sigma' \sigma^{-1}} = t_{i_t} \cdots t_{i_2} \cdot t_1 \cdots t_{i_2} \cdot t_{i_1} \cdots t_{i_r}.
$$
and because the group generated by the $t_i$’s is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ast \ldots \ast \mathbb{Z}/2\mathbb{Z}$, we have that $\text{Per}_{\Sigma_{\sigma'}} = \text{id}$, $\tau' = s'$, and $k_{l} = k'_{l}$ for all $1 \leq l \leq s'$.

Let $w = \psi_{t_{1,j_{1}}}^{*} \psi_{t_{2,j_{2}}}^{*} \cdots \psi_{t_{s,j_{s}}}^{*}$ be a non-trivial freely reduced word such that $w = \text{id}$. By Lemma 3.11 we have that $\ell_{i}(w) \geq 2$, but $\ell_{i}(\text{id}) = 0$ which is a contradiction. Therefore there is no non-trivial relation between the $\psi_{i,j}^{*}$, hence, modulo automorphisms, we have

$$\langle \psi_{i,j} \mid 1 \leq i < j \leq m \rangle \cong \bigstar_{1 \leq i < j \leq m} \langle \psi_{i,j}^{*} \rangle \cong \mathbb{Z} \ast \ldots \ast \mathbb{Z}.$$ 

$\square$

In [Wan22, Theorem 1.3] it is shown that there is a fundamental domain $\Pi$ for the action of $\text{Bir}(X)$ on $\overline{\text{Mov}}^{e}(X)$. Whereas the proof is non-constructive, we can now describe the fundamental domain $\Pi$ explicitly in the general case.

**Proposition 3.12.** Let $X_{0} = X$ be general, and let

$$\Pi = \text{Nef}^{e}(X_{0}) \cup \bigcup_{i=1,\ldots,m} \varphi_{0,i}^{*} \text{Nef}^{e}(X_{i}) \subset \overline{\text{Mov}}^{e}(X).$$

Then $\Pi$ is a fundamental domain for the action of $\text{Bir}(X)$ on $\overline{\text{Mov}}^{e}(X_{0})$, that means

$$\overline{\text{Mov}}^{e}(X) = \bigcup_{g \in \text{Bir}(X)} g^{*}\Pi$$

and $\text{int} \, \Pi \cap \text{int} \, g^{*}\Pi = \emptyset$ unless $g^{*} = \text{id}$.

**Proof.** By [Wan22, Proposition 4.7 and Corollary 7.9] we have

$$\overline{\text{Mov}}^{e}(X) = \bigcup_{(X',\alpha)} \alpha^{*} \text{Nef}^{e}(X')$$

where the union runs over all minimal models of $X$ and markings $\alpha : X \dashrightarrow X'$. Recall that $X$ has exactly $m + 1$ minimal models $X_{0} = X, X_{1}, \ldots, X_{m}$ up to isomorphism. Let $E \in \overline{\text{Mov}}^{e}(X)$. Then there exist a minimal model $X_{j}$, a birational map $\alpha_{j} : X \dashrightarrow X_{j}$ which is a sequence of flops and a divisor $E_{j} \in \text{Nef}^{e}(X_{j})$ such that $E = \alpha_{j}^{*}(E_{j})$. By [Wan22, Corollary 7.7] any flop of a $X_{k}$ is among the $\varphi_{k,l}$ constructed in Subsection 2.2. Thus, up to isomorphism, we have

$$\alpha_{j} = \varphi_{i_{s},j} \circ \varphi_{i_{s-1},i_{s}} \circ \cdots \circ \varphi_{i_{0},i_{1}} \circ \varphi_{0,i_{0}}.$$ 

Now, introducing elements of the form $\varphi_{0,i_{k}} \circ \varphi_{i_{k},0}$ as in the proof of Proposition 3.9, we have

$$\alpha_{j} = \varphi_{0,j} \circ \psi_{i_{s},j} \circ \cdots \circ \psi_{i_{1},i_{2}} \circ \psi_{i_{0},i_{1}}.$$

Consequently, we have

$$E = g^{*}(\varphi_{0,j}^{*} E_{j}) \in g^{*}(\varphi_{0,j}^{*} \text{Nef}^{e}(X_{j})) \subset g^{*}\Pi.$$ 

Finally, let us assume that there is a $g \in \text{Bir}(X)$ such that $\text{int} \, \Pi \cap \text{int} \, g^{*}\Pi \neq \emptyset$. By the definition of $\Pi$ this implies that there exists a divisor $E_{k} \in \text{Amp}(X_{k})$ such that (after composing with some $\varphi_{0,k}$ if necessary) $g^{*}E_{k}$ is ample on $X_{k}$ or some other minimal
model. This implies that \( g \) (composed with some \( \varphi_{0,k} \)) is an automorphism, and hence, \( g^* = \text{id} \) by Proposition 3.6.

To describe the movable cone we will use the connection between the generators of \( \text{Bir}(X) \) and the Coxeter system. To see this, let \( D \) be, as in Section 2.1, the fundamental domain of the action of the dual representation of \( W \), and let \( T \) the Tits cone of \( W \). We can identify \( V^* \) with \( N^1(X)_{\mathbb{R}} \) as follows: Let \( \{\alpha_i\}_{i=1}^m \) be the basis of \( V \), and consider \( \{\beta_i\}_{i=1}^m \) as the vectors in \( V \) such that \( Q(\alpha_i, \beta_j) = \delta_{i,j} \), where \( \delta_{i,j} \) is the Kronecker Delta. Then the fundamental domain \( D \) is the cone \( \text{Cone}(\{\beta_i\}_{i=1}^m) \) generated by the \( \beta_i \)'s and the generators of the dual representation of \( W \) correspond to the elements \( t_i \). Finally, identify the fundamental domain \( D \) of \( W \) with \( \text{Nef}(X_0) \) via the map \( \beta_i \mapsto H_{\beta_i}^0 \). Note that as cones we have \( \varphi_{0,i}^* \text{Nef}(X_0) = t_i \text{Per}_{(1,\ldots,i)} \). \( \text{Nef}(X_0) = t_i \text{Nef}(X_0) \).

**Proposition 3.13.** Identify \( D \) with \( \text{Nef}(X) \) as above. Then the Tits cone \( T \) is isomorphic to \( \overline{\text{Mov}}(X) \).

**Proof.** From Proposition 3.12 we have that \( \Pi = \text{Nef}(X_0) \cup \bigcup_{i=1,\ldots,m} \varphi_{0,i}^* \text{Nef}(X_i) \) is a fundamental domain for the action of \( \text{Bir}(X) \) on \( \overline{\text{Mov}}(X) \). Also, notice that if \( w = t_{i_1} t_{i_2} \cdots t_{i_s} \), then \( w.D = w \cdot \text{Per}_r.D \). In particular, \( \Pi = \text{Nef}(X_0) \cup \bigcup_{i=1}^m t_i \text{Nef}(X_0) \), so \( \overline{\text{Mov}}(X) \subseteq T \).

To prove the opposite inclusion, we need to show that given a freely reduced word \( w = t_{i_1} t_{i_2} \cdots t_{i_s} \), there exists a word \( w' = \psi_{i_1,j_1}' \psi_{i_2,j_2}' \cdots \psi_{i_s,j_s}' \) such that the chamber \( w.D \) of the Tits cone is included in the chamber \( w'.\Pi \) of the movable cone. We construct the word \( w' \) by induction on the length of \( w \). If \( w = t_i \), then \( w.D \subseteq \Pi \), so \( w' = \text{id} \). If \( w = t_{i_1} t_{i_2} \cdots t_{i_s} \), with \( s \geq 2 \), we can write

\[
    w = t_{i_1} t_{i_2} t_{i_3} \cdots t_{i_s} = (t_{i_1} t_{i_2} t_{i_1}) t_{i_1} t_{i_3} \cdots t_{i_s}.
\]

Then we set \( w' = \psi_{i_1,i_2}^* w_1' \), where \( w_1' \) is the word associated to

\[
    w_1 = t_{\sigma(i_1)} t_{\sigma(i_3)} \cdots t_{\sigma(i_s)} = t_{i_2} t_{i_3} \cdots t_{i_s}.
\]

with \( \sigma = (i_1, i_2) \). The word \( w_1' \) exists because the length of \( w_1 \) is less than \( s \). Therefore \( T \subseteq \overline{\text{Mov}}(X) \).

To describe the boundary of \( \overline{\text{Mov}}(X) \) we can use the fact that the underlying Coxeter system \( (W, S)_Q \) is Lorentzian.

**Proposition 3.14.** The Coxeter system \( (W, S)_Q \) is Lorentzian.

**Proof.** The statement follows from the fact that for \( n \geq 1 \), the matrix \( Q \) from Equation (3.1) has eigenvalues \( \lambda_1 = 1 + \frac{n}{2} > 0 \) with eigenvectors \( e_1 - e_i \), for \( 2 \leq i \leq m \); and \( \lambda_2 = 1 - \frac{n(m-1)}{2} < 0 \), with eigenvector \( (1, 1, \ldots, 1) \).

Given that \( \overline{\text{Mov}}(X) \) is equal to the Tits cone, and that the Coxeter system is Lorentzian, we can replicate the proof of [Yá22, Theorem 4.10] to compute the boundary of \( \overline{\text{Mov}}(X) \) (see Remark 3.16 for the case of \( n = 1 \)).

**Proposition 3.15** (cf. [Yá22, Theorem 4.10]). The boundary of \( \overline{\text{Mov}}(X) \) is the closure of the union of the \( W \)-orbits of the cones \( \{a_{\lambda}v_{\lambda} + \sum_{i,j \neq k} a_k H_{\beta_i}^0 \mid a_{\lambda} \geq 0, a_k \geq 0\} \), where \( v_{\lambda} \) is:
• If \( n \geq 3 \), an eigenvector associated to the unique eigenvalue \( \lambda > 1 \) of \( t_i t_j \); or
• If \( n = 2 \), \( v_\lambda = 0 \).

Proof. The proof follows exactly as in [Ya22, Theorem 4.10], with \( J = \{ 1, 2, \ldots, m \} \). □

Remark 3.16. Propositions 3.13 and 3.14 still hold when \( n = 1 \), but in this case the boundary of the Tits cone associated to the Coxeter system \((W, S)_Q\) cannot be described in the same way as for \( n \geq 2 \) shown in Proposition 3.15.

Remark 3.17. The boundary of the movable cone \( \overline{\text{Mov}}(X) \) can also be described in terms of Bir\((X)\)-orbits. This will be done in the examples of Section 4.

4. Examples

4.1. General case. In this subsection, we study the movable cone in concrete examples. We consider general cases with varying \( n \) and fixed Picard number \( m = 3 \).

Remark 4.1. In this section, we provide pictures of the movable cone and the nef cone for \( m = 3 \) created with the software Mathematica [Wol]. To do so, we project from \( \mathbb{P}^N\text{dim}(X) = \mathbb{P}^2 \) to the affine space \( V_1 = \{ v \in \mathbb{R}^3 \mid \varphi(v) = v_0 + v_1 + v_2 = 1 \} \) as described in Remark 2.4.

Example 4.2 \((n = 2 \text{ and } m = 3)\). Let \( X := X_0 \subset \mathbb{P}_0 = \mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}_3^2 \) be a general complete intersection of 3 forms of multidegree \((1,1,1)\) and let \( F \) be the \((1,1,1,1)\)-form in \( \mathbb{P} = \mathbb{P}_0^2 \times \mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}_3^2 \) as defined in Construction 2.11. Then \( X \) is Calabi-Yau variety with

\[
\dim X = n(m - 1) - 1 = 3 \quad \text{and} \quad \rho(X) = m = 3.
\]

The Calabi-Yau varieties \( X_i \subset \widehat{\mathbb{P}}_i \) for \( i = 1, 2, 3 \) from Construction 2.11 are the only minimal models of \( X \) up to isomorphism. By Proposition 3.9 we have

\[
\text{Bir}(X) = \text{Aut}(X) \cdot \langle \psi_{i,j} \mid 1 \leq i < j \leq 3 \rangle
\]

and

\[
\langle \psi_{i,j}^* \mid 1 \leq i < j \leq 3 \rangle \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}
\]

by Theorem 3.10. Using Proposition 3.4 we compute that

\[
\psi_{1,2}^* = \begin{pmatrix} -2 & -3 & 0 \\ 3 & 4 & 0 \\ 6 & 12 & 1 \end{pmatrix}, \quad \psi_{1,3}^* = \begin{pmatrix} -2 & 0 & -3 \\ 6 & 1 & 12 \\ 3 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \psi_{2,3}^* = \begin{pmatrix} 1 & 6 & 12 \\ 0 & -2 & -3 \\ 0 & 3 & 4 \end{pmatrix}
\]

with respect to the bases \( B_0 \) given in (2.2).

We have \( \text{Nef}(X) = \text{Cone}(H_0^0, H_1^0, H_2^0) \) and the fundamental domain \( \Pi \) from Proposition 3.12 in \( \mathbb{P}^N\text{dim}(X) \) is hexagonal with vertices

\[
\{ H_3^0, \varphi_{0,1}^*, H_1^1, H_2^1, \varphi_{0,2}^*, H_3^3, H_1^0, \varphi_{0,3}^* H_0^0 \}.
\]

As mentioned in Remark 2.6, all accumulation points of \( \overline{\text{Mov}}(X) \) lie on the quadratic cone

\[
\{ t_0 t_1 + t_0 t_2 + t_1 t_2 = 0 \}
\]
Figure 1. Projective view of the movable cone of $X$ in the case of $m = 3$ and $n = 2$. On the left, the cone obtained as the union of the images of the fundamental domain, in red, under the action of $\text{Bir}(X)$. The nef cone is drawn in blue. On the right, the boundary of the $\text{Mov}(X)$ obtained using Proposition 3.15. In red, the fundamental domain, and in blue the $\text{Bir}(X)$-orbits of the vertices of the fundamental domain.

In $\mathbb{P}N^1(X)_\mathbb{R} \cong \mathbb{P}^2_\mathbb{R}$ given by the inverse of the symmetric matrix

$$Q = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

from Equation (3.1) with $n = 2$.

Recall that any codimension one face of $\text{Nef}(X_i)$ corresponds to a contraction from $X_i$ to some $W_{i,j}$ and induces the flop $\varphi_{i,j} : X_i \dashrightarrow X_j$. By pulling back to $X$ we can identify each codimension one face of the red hexagon $\Pi$ in Figure 1 with exactly one element in

$$\Psi = \{ \psi_{1,2}^*, \psi_{2,1}^* = (\psi_{1,2}^*)^{-1}, \psi_{1,3}^*, \psi_{2,3}^*, \psi_{3,1}^*, \psi_{3,2}^* \}. \tag{4.1}$$

Indeed, for each $\psi \in \Psi$, the cone $\psi \Pi$ is again hexagonal and from the definition of the $\psi_{i,j}^*$ we deduce that $\psi \Pi$ has exactly one codimension one face in common with $\Pi$. More, generally, for any word $w = \psi_{i_1,j_1}^* \psi_{i_2,j_2}^* \cdots \psi_{i_r,j_r}^*$, the cones $w \Pi$ and $w' \Pi$, for $w' = \psi_{i_1,j_1}^* \psi_{i_2,j_2}^* \cdots \psi_{i_{r-1},j_{r-1}}^*$, have one common codimension one face and lie in the segment corresponding to $\psi_{i_1,j_1}^*$.

**Example 4.3** ($n = 3$ and $m = 3$). We consider a general $X := X_0 \subset \mathbb{P}^1_1 \times \mathbb{P}^2_2 \times \mathbb{P}^3_3$ which is a smooth complete intersection Calabi-Yau fivefold of Picard number $m = 3$. 

As before, $X$ has 4 minimal models $X, X_1, X_2, X_3$ up to isomorphism and 

$$\text{Bir}(X) = \langle \psi_{i,j} \mid 1 \leq i < j \leq 3 \rangle \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$$

by Proposition 3.7 and Theorem 3.10. We compute that

$$\psi^*_{1,2} = \begin{pmatrix} -3 & -8 & 0 \\ 8 & 21 & 0 \\ 12 & 36 & 1 \end{pmatrix}, \psi^*_{1,3} = \begin{pmatrix} -3 & 0 & -8 \\ 12 & 1 & 36 \\ 8 & 0 & 21 \end{pmatrix} \text{ and } \psi^*_{2,3} = \begin{pmatrix} 1 & 12 & 36 \\ 0 & -3 & -8 \\ 0 & 8 & 21 \end{pmatrix}$$

with respect to the basis $B_0$ from Equation (2.2).

As before we have \(Nef(X) = \text{Cone}(H^0_0, H^0_1, H^0_3)\) and the fundamental domain $\Pi$ from Proposition 3.12 is hexagonal in $\mathbb{P}^1(X)_{\mathbb{R}}$ with vertices

$$\{H^0_3, \varphi^*_0, H^1_0, \varphi^*_0, H^2_0, H^3_0, \varphi^*_0, H^2_0, \varphi^*_0, H^1_0, \varphi^*_0, H^0_3\}.$$  

Now, in contrast to the previous example $\psi^*_{i,j}$ is diagonalizable for any $i \neq j$ and the boundary of $\overline{\text{Mov}}(X)$ contains now line segments between the eigenvector of $\psi^*_{i,j}$ associated to the unique eigenvalue $\lambda > 1$ and the vertex corresponding to $H^0_k$ where $1 \leq k \leq 3$ and $k \neq i, j$ as pictured in Figure 2. The inverse of the matrix

$$Q = \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1 & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$

from Equation (3.1) with $n = 3$ defines a quadratic cone

$$\{t_0^2 - 6t_0t_1 + t_1^2 - 6t_0t_2 - 6t_1t_2 + t_2^2 = 0\}$$

in $\mathbb{P}^1(X)_{\mathbb{R}} \cong \mathbb{P}^2_{\mathbb{R}}$ which contains all these eigenvectors. As in the previous example, every edge of the red hexagon in Figure 2 corresponds to one of the elements in

$$\Psi = \{\psi^*_{1,2}, \psi^*_{2,1}, \psi^*_{1,3}, \psi^*_{3,1}, \psi^*_{2,3}, \psi^*_{3,2}\}.$$ 

and $\{((\psi^*_{i,j})^k, \Pi)\}$ gives a sequence of adjacent shrinking cone approaching the eigenvector of $\psi^*_{i,j}$ associated to the unique eigenvalue $\lambda > 1$.

### 4.2. Non-general case with non-trivial automorphism group

In this subsection we present some special cases which have a non-trivial automorphism group.

**Example 4.4** ($n = 2$ and $m = 3$). As before, we consider first the complete intersection of 3 forms of multidegree $(1, 1, 1)$ in $\mathbb{P}_0 = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$. Let $R_0$ be the the Cox ring of $\mathbb{P}_0$, and let $x^j = [x^j_0, x^j_1, x^j_2]$ be the variables of the individual $\mathbb{P}^2_j$. Then we define a homomorphism $g$ by

- $x^j_1 \mapsto x^j_2$,
- $x^j_2 \mapsto x^j_1$,
- $x^j_3 \mapsto x^j_3$,

for $j = 0, 1, 2$.  

Let $f'_0, f'_1, f'_2$ be three $(1, 1, 1)$-forms in $\kP_0$ and set

$$f_i := f'_i + g(f'_i).$$

If the $f'_i$ are chosen generically, the variety $X_0$ defined by $f_0, f_1, f_2$ is a smooth Calabi-Yau threefold in $\kP_0$. Moreover, $X_0$ is invariant under the automorphism of $\kP_0$ induced by $g$. Let

$$F = \sum_{i=0}^{2} x^0_i f_i$$

be the $(1, 1, 1, 1)$-form in $\P$. Then we can extend $g$ to the Cox ring of $\P$ by letting the variables $x^0_i$ be mapped identically. Thus, $F$ is a homogeneous polynomial which is invariant under interchanging the $x^1$- and $x^2$-variables. Recall that the $3 \times 3$-matrix $A_{i,j}$ is the matrix of second derivatives of the polynomial $F$ with respect to the homogeneous coordinates $x^i$ and $x^j$. Now, if we proceed with Wang’s construction we see that the matrix $A_{1,2}$ defining the variety $W_{1,2}$ is symmetric and that the two projections $\pi^1_{1,2}$ and $\pi^2_{1,2}$ are no longer small contractions but divisorial.

Furthermore, the matrix $A_{0,1}$ is the transpose of the matrix $A_{2,0}$ after applying the homomorphism $g$. Thus $W_{0,1}$ and $W_{2,0}$ are isomorphic and from the diagram in Proposition 3.4 we conclude that the two minimal models $X_1$ and $X_2$ are isomorphic with the
isomorphism given by

\[ \varphi_{1,2} : X_1 \to X_2, \]
\[ [p^0, p^2, p^3] \mapsto [q^0, q^1, q^3] := [p^0, p^2, p^3]. \]

The birational automorphism \( \psi_{1,2} \) from Proposition 3.4 is now an automorphism given by

\[ \psi_{1,2} : X_0 \to X_0, \]
\[ [p^1, p^2, p^3] \mapsto [p^2, p^1, p^3]; \]

and

\[ \psi_{1,2}^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

with respect to the basis \( B_0 \). The presentation matrices of \( \psi_{2,3}^* \) and \( \psi_{1,3}^* \) are the same as in Example 4.2 but due to the isomorphism between \( X_1 \) and \( X_2 \) we have

\[ \psi_{2,3} = \psi_{1,2} \circ \psi_{1,3} \circ \psi_{1,2}. \]

Indeed, by the symmetry of \( F \) with respect to \( x^1 \) and \( x^2 \) we have an automorphism

\[ \alpha : X_3 \to X_3, \]
\[ [p^0, p^1, p^2] \mapsto [p^0, p^2, p^1] \]

which coincides with \( \varphi_{2,3} \circ \varphi_{1,2} \circ \varphi_{3,1} \). Then, from the definition of the maps \( \psi_{1,j} \) and \( \alpha \) we see that

\[ \psi_{2,3} = \varphi_{3,0} \circ \varphi_{2,3} \circ \varphi_{0,2} = \varphi_{3,0} \circ \varphi_{2,3} \circ (\varphi_{1,2} \circ \varphi_{0,1} \circ \psi_{1,2}) = \varphi_{3,0} \circ (\varphi_{2,3} \circ \varphi_{1,2} \circ \varphi_{3,1}) \circ \varphi_{1,3} \circ \varphi_{0,1} \circ \psi_{1,2} = \varphi_{3,0} \circ \alpha \circ \varphi_{1,3} \circ \varphi_{0,1} \circ \psi_{1,2} = (\varphi_{3,0} \circ \alpha \circ \varphi_{0,3}) \circ \psi_{1,3} \circ \psi_{1,2} = \psi_{1,2} \circ \psi_{1,3} \circ \psi_{1,2}, \]

where the last equality follows from the fact that the matrix \( A_{0,3} = A_{3,0}^T \) is invariant under interchanging \( x^1 \) and \( x^2 \). Assuming that there are no further automorphisms we have

\[ \text{Bir}(X_0) = (\psi_{1,2}, \psi_{1,3}) \cong \mathbb{Z}/2 \mathbb{Z} \ast \mathbb{Z}. \]

The cones \( \text{Nef}(X_0) \) and \( \varphi_{0,3}^\ast(\text{Nef}(X_3)) \) are preserved under \( \psi_{1,2}^\ast \), whereas \( \varphi_{0,2}^\ast(\text{Nef}(X_2)) \) is mapped to the cone \( \varphi_{0,1}^\ast(\text{Nef}(X_1)) \). Thus, if \( \Pi \) is the convex cone generated by the classes \( H_3^0, \varphi_{0,1}^\ast H_0^1, H_2^0 \) and \( \varphi_{0,3}^\ast H_3^0 \), then

\[ \Pi \cup \psi_{1,2}^\ast \Pi = \text{Nef}^\ast(X_0) \cup \bigcup_{i \in \{1,2,3\}} \varphi_{0,i}^\ast \text{Nef}^\ast(X_i) \]

coincides with the hexagonal cone from Example 4.2 and we obtain

\[ \overline{\text{Mov}}^\ast(X_0) = \bigcup_{g \in \text{Bir}(X)} g^\ast \Pi \]
exactly as in the proof of Proposition 3.12. Consequently, the Kawamata-Morrison cone conjecture is also satisfied in this non-general case. Note that Lemma 7.6 up to Corollary 7.9 from [Wan22] still hold with the only exception that \( \varphi_{1,2} : X_1 \to X_2 \) is an isomorphism and the projections \( \pi_{1,2} \) are divisorial.

**Remark 4.5.** Note that \( X_1 \) and \( X_2 \) are isomorphic as minimal models of \( X_0 \) but not as marked minimal models \( \varphi_{0,1} : X_0 \dashrightarrow X_1 \) and \( \varphi_{0,2} : X_0 \dashrightarrow X_2 \). Thus pulling the respective nef cones back to \( X_0 \), we obtain two different chambers of \( \overline{\text{Mov}}(X) \) whose interiors do not intersect.

The question is how the pseudoeffective cone \( \overline{\text{Eff}}(X) \) looks like in this example. A first natural guess is that \( \overline{\text{Eff}}(X) \) is bounded by the quadratic cone

\[
C_Q := \{ t_0t_1 + t_0t_2 + t_1t_2 = 0 \} \subset \mathbb{P}N^1(X)_\mathbb{R}
\]

from the non-symmetric case of Example 4.2. However, using Macaulay2 (see [GS21] and [HSY22]), we compute that

\[
h^0(X, D_1) = 1
\]
where $D_1$ is an effective integral divisor whose numerical class is equivalent to $-2H_1^0 + 2H_2^0 + 6H_3^0$. We note that the class of this divisor lies outside of the quadratic cone and is the intersection of the two tangent lines of the quadratic cone in $\mathbb{P}N^1(X)_{\mathbb{R}}$ to $H_3^0$ and $\varphi_{0,1}^*(H_1^1)$.

Note that any segment connecting $[D_1]$ to a rational point of the quadratic cone is contained in $\text{Eff}(X)$. As the example is symmetric with respect to changing the first and the second coordinate, we obtain an effective integral divisor $D_2$ whose class is $2H_1^0 - 2H_2^0 + 6H_3^0$.

We expect that the pseudoeffective cone consists of the quadratic cone $C_Q$ together with the $\text{Bir}(X)$-orbit of the cones with vertices $[D_1]$ and $[D_2]$, respectively glued tangentially to the cone as displayed in Figure 4.

![Figure 4](image)

**Figure 4.** The expected pseudoeffective cone of $X$: the quadratic cone $C_Q$ together with the $\text{Bir}(X)$-orbit (drawn in red) of the cone with vertex $[D_1]$ glued tangentially to the quadratic cone $C_Q$.

We close this example with a discussion on the numerical dimension of the pseudoeffective divisor $D_1$ (resp. $D_2$).

**Definition 4.6.** ([Nak04, §5]) The numerical dimension $\kappa_\sigma(X, D)$ is the largest integer $k$ such that for some ample divisor $A$, one has

$$\limsup_{m \to \infty} \frac{h^0(X, \lfloor mD \rfloor + A)}{m^k} > 0.$$ 

If no such $k$ exists, we take $\kappa_\sigma(X, D) = -\infty$.

Taking the supremum over all real numbers $k$, Lesieutre introduced the notion $\kappa_\sigma^R$ and gave an example of a divisor $D$ on the boundary of the pseudoeffective cone of a Calabi-Yau threefold $X$ with $\kappa_\sigma(X, D) \neq \kappa_\sigma^R(X, D)$ (see [Les22]).
We show that $D_1$ and $D_2$, along with the tangents to $C_Q$, are on the boundary of the pseudoeffective cone. In particular, this implies that the Bir$(X)$-orbit of these segments are also on the boundary of $\overline{\text{Eff}}(X)$.

**Proposition 4.7.** The boundary of the pseudoeffective cone $\overline{\text{Eff}}(X)$ contains the Bir$(X)$-orbit of the line segment spanned by the divisor classes $[D_1]$ and $[D_2]$, and the segment spanned by the divisor classes $[D_1]$ and $\varphi_{0,1}^*H_0^1$.

**Proof.** Note that any integral boundary divisor of $\overline{\text{Mov}}(X) \cap \partial C_Q$ is not big. Indeed its numerical dimension $\kappa_\sigma$ is 2 since its linear system corresponds to a projection of $X$ to $\mathbb{P}^2$. In particular, the divisor $H_0^0$ is not big. By the properties of $\kappa_\sigma$ we know that

$$\kappa_\sigma(X, \alpha_1D_1 + \alpha_2D_2)$$

is independent of the positive numbers $\alpha_1, \alpha_2$. Hence, the numerical dimension is constant to $2 = \kappa_\sigma(X, H_0^0)$ in this line segment which shows the claim.

Now consider a divisor class $[D]$ on the segment spanned by $[D_1]$ and $\varphi_{0,1}^*H_0^1$, and assume it is not on the boundary of $\overline{\text{Eff}}(X)$. Therefore, for an ample divisor $A$ and $0 < \varepsilon \ll 1$ we have that $D - \varepsilon A$ is big. Consider the line $L$ connecting $D - \varepsilon A$ and $\varphi_{0,1}^*H_0^1$. Because the segment spanned by $[D_1]$ and $\varphi_{0,1}^*H_0^1$ is tangent to $C_Q$, there exists a divisor class $E$ on the line $L$ in the interior of $C_Q$ such that $E$ is big. This implies that the divisor class $\varphi_{0,1}^*H_0^1$ is also big, which is a contradiction. Therefore, the segment spanned by $[D_1]$ and $\varphi_{0,1}^*H_0^1$ is on the boundary of $\overline{\text{Eff}}(X)$. \qed

**Remark 4.8.** By [Nak04, §5, Proposition 2.7] we have

$$0 \leq \kappa_\sigma(X, D_1) = \kappa_\sigma(X, D_2) \leq \kappa_\sigma(D_1 + D_2) = 2.$$

On Example 4.2 there are three types of divisors on the boundary of $\overline{\text{Mov}}(X)$: rational points, eigenvectors of elements of Bir$(X)$, and accumulation points of eigenvectors. The numerical dimension of the first two is well understood. If the divisor class corresponds to a rational point on the boundary, then it can be pulled back birationally to a rational nef divisor on some minimal model $X_i$, and hence it has an integral numerical dimension $\kappa_{\sigma_i}^\mathbb{R} = \kappa_\sigma$.

In the case of an eigenvector of an element $f \in \text{Bir}(X)$, one can perform a similar computation as done in [Les22] and [HS22] by restricting to the two-dimensional cone generated by the eigenvectors associated to the unique eigenvalue $\lambda > 1$ of $f$ and $f^{-1}$.

In this case we also obtain that $\kappa_\sigma^\mathbb{R} = 3/2$.

The third case, corresponding to accumulation points, is not well understood and could be a source of interesting numerical dimensions.

Example 4.4 provides a new interesting divisor class, specifically $D_1$ and its Bir$(X)$-orbit. Computing its numerical dimension requires a different method than the ones described above, and given that $D_1$ is an integral divisor class, it could be another potential source for an interesting behavior of the numerical dimension $\kappa_{\sigma_i}^\mathbb{R}$.

**Question 4.9.** What is $\kappa_{\sigma_i}^\mathbb{R}(X, D_1)$ and does $\kappa_{\sigma_i}^\mathbb{R}(X, D_1) = \kappa_\sigma(X, D_1)$ hold?

**Example 4.10** ($m = 2$, $n = 4$, Aut$(X)$ non-trivial). As the last example we present a Calabi-Yau threefold $X_0 \subset \mathbb{P}^4 \times \mathbb{P}^2$ with $\rho(X_0) = 2$ which has a non-trivial automorphism group Aut$(X_0)$ acting trivially on $N^1(X_0)_\mathbb{R}$. For the sake of simplicity, we denote by $x_i$
the variables of \( \mathbb{P}^4_0 \), by \( y_j \) the ones of \( \mathbb{P}^4_1 \) and by \( z_k \) the ones of \( \mathbb{P}^4_2 \). Let us consider the element \( g \in \text{Aut}(\mathbb{P}^4_0 \times \mathbb{P}^4_1 \times \mathbb{P}^4_2) \) given by

\[
g = \begin{pmatrix}
-1 & -1 & 1 \\
1 & 1 & 1 \\
1 & 1 & -1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 & 1 \\
1 & 1 & 1 \\
1 & 1 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & -1 \\
1 & 1 & -1
\end{pmatrix}.
\]

A basis of all \((1,1,1)\)-forms invariant under \( g \) is given by 63 monomials \( x_iy_jz_k \) with

| \( i \) | \( j \) | \( k \) | number     |
|------|------|------|-----------|
| 0,1  | 0,1  | 0,1,2| \( 2 \cdot 2 \cdot 3 = 12 \) |
| 0,1  | 2,3,4| 3,4  | 12        |
| 2,3,4| 0,1  | 3,4  | 12        |
| 2,3,4| 2,3,4| 0,1,2| 27        |
|      |      |      | 63        |

Taking any combination of these monomials, we obtain an invariant \((1,1,1)\)-form \( F \) and by differentiating with respect to the \( x_i \) we obtain a variety \( X_0 \subset \mathbb{P}^4_0 \times \mathbb{P}^4_1 \times \mathbb{P}^4_2 \) which is invariant under the induced action on \( \mathbb{P}^4_0 \times \mathbb{P}^4_1 \times \mathbb{P}^4_2 \). We verified computationally with \texttt{Macaulay2} (see [HSY22]) that a random combination of the basis elements yields a non-singular complete intersection Calabi-Yau threefold \( X_0 \) with minimal models \( X_1, X_2 \) and \( X_3 \) which are also non-singular.

The space of all possible invariant forms \( F \) is of dimension at most

\[
63 + 3 \cdot \dim(\text{PGL}_5(\mathbb{C})) - 1 = 134
\]

so we do not get a useful dimension bound as in Proposition 3.7 because the (projective) space of all possible \((1,1,1)\)-forms \( F \) is of dimension 124.

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