Projective view at Optimization Problem for Multiband Filter

Andrei Bogatyrev *

Abstract

The best uniform rational approximation of the sign function on two intervals separated by zero was explicitly solved by E.I. Zolotarëv in 1877. This optimization problem is the initial step in the staircase of the so called approximation problems for multiband filters which are of great importance for electrical engineering. We show that known in the literature optimality criterion for this problem may be contradictory since it does not take into account the projective invariance of the problem. We propose a new consistently projective formulation of this problem and give a constructive optimality criterion for it.

Keywords: Uniform rational approximation, optimization of electrical filters, Ansatz method, equiripple property, alternation, Stiefel class

MSC2010: 41A20, 41A50, 49K35, 94Cxx

Introduction

The design of modern electronic devices involves a lot of deep and sophisticated mathematics. In particular, synthesis of multiband analogue, digital and microwave electrical filters despite their diverse physical implementation is based on the same uniform rational approximation problem which is a generalization of third and fourth Zolotarëv problems. This problem attracted attention of many mathematicians including such influential characters like E.I.Zolotarëv, N.I.Achieser, E.Stiefel, A.A.Gonchar to say nothing of numerous electrical and electronic engineers who are interested mainly in the algorithmic aspects of the problem.

Roughly, the problem consists in the best uniform approximation of the two-valued (indicator) function defined on the prescribed passbands and stopbands – the collection E of disjoint segments on real frequency axis – by a given degree n rational function. It was nearly immediately discovered that even in the simplest two-band classical variant the problem is multi-extremal, that is admits local optima. It is likely that Ed Stiefel [15] with his pupil R.A.-R.Amer [2] were the first who decomposed the set of rational functions into topological classes which secured the uniqueness of the solution. Each of those solutions may be characterized by the alternation principle – or the equiripple property in terms of engineers – which says that the graph of (error function of the) solution looks like the sequence of sufficiently many ripples of constant amplitude, see Fig. 1.

*Supported by RSF grant 16-11-10349P
With this property taken into account, the solution of the problem becomes in a sense very simple: you merely show a function with the requested oscillatory behaviour. Several approaches to the numerical solution of optimization problem for multiband electrical filter are based on the alternation principle.

Say, Remez-type algorithms specially designed for uniform polynomial and rational approximation [12, 13, 14, 16] iteratively build the necessary alternation set for the error function of approximation. For more traditional gradient descent methods the appearance of the alternation is a signal to terminate iterations: equiripple property serves as a certificate for the optimizer.

Another approach for solution is the recently elaborated Ansatz method, the development of the classical approach. An explicit analytical formula for the solution was proposed by E.I.Zolotarev [7] in case of two bands, later extended by E.Stiefel [15] to three bands and independently by this author [10] for arbitrary number of bands. This formula – the elliptic sine of an abelian integral – generalizes the representation for Zolotarëv fractions and contains unknown parameters, both continuous and discrete, which have to be evaluated given the input data of the problem. The number of Ansatz parameters is usually much less than the number of optimization variables, which makes up the heart of this approach. The Ansatz automatically obeys the equiripple criterion, however it becomes a rational function for only specially chosen sets of parameters.

Careful formulation of optimality criterion is very important, however up to now this was not done fairly well for the solution with the defect, when its degree drops. Known criterion proposed in [3] does not endure the projective transformations of the set \( E \) of the filter workbands, the important property of this optimization problem. Also it does not contain any information of the above mentioned topological classes which guarantee the uniqueness of the optimizer. We propose a modification of Malozemov’s criterion for the defect \( d \) optimizer \( R(x) \) in the fixed topological class of degree \( n \) rational functions:

\[
\text{Alt}(R) \geq 2n + 2 - d - \Sigma^0 + \Sigma^1
\]

where \( \text{Alt}(R) \) is the number of cyclic alternation points on the set \( E \) of workbands of the filter, \( \Sigma^0, \Sigma^1 \) are nonnegative integer indexes which depend on the solution \( R \), the topological class and vanish in the case of full degree solution. We emphasize that both sides of the inequality are now even numbers.

1 Optimization problem for multiband filter

Suppose a finite collection \( E \) of disjoint closed segments of real axis \( \mathbb{R} \) is given. The set has a meaning of frequency bands and is decomposed into two subsets: \( E = E^+ \cup E^- \) which are called the passbands \( E^+ \) and the stopbands \( E^- \). Both subsets are non empty. Optimization problem for electrical filter has several equivalent settings [8, 2, 5, 7, 3].

1.1 Three settings

In each of listed below cases we minimize certain quantity among real rational functions \( R(x) \) of bounded degree \( \deg R \leq n \) (being the maximum of the degrees of the numerator and the denominator of a fraction). The goal function may be one of the following.
Figure 1: The optimizer for the 55 workbands set $E$ obtained by the Ansatz method and computed by S.Lyamaev.

1.1.1 Minimal deviation

$$\frac{\max_{x \in E^+} |R(x)|}{\min_{x \in E^-} |R(x)|} \rightarrow \min =: \theta^2 \leq 1.$$ 

1.1.2 Third Zolotarëv problem

Minimize $\theta$ under the condition that there exist real rational function $R(x)$, $\deg R \leq n$, with the restrictions

$$\min_{x \in E^-} |R(x)| \geq \theta^{-1}, \quad \max_{x \in E^+} |R(x)| \leq \theta.$$ 

1.1.3 Fourth Zolotarëv problem

Define the indicator function $S_E(x) = \pm 1$ when $x \in E^\pm$. Find the best uniform rational approximation $R(x)$ of $S_E(x)$ of the given degree:

$$\|R - S_E\|_{C(E)} := \max_{x \in E} |R(x) - S_E(x)| \rightarrow \min =: \mu.$$ 

It is a good exercise to show that all three settings are equivalent and in particular the value of $\theta$ is the same for the first two settings and $1/\mu = (\theta + 1/\theta)/2$ for the third one.

1.2 Historical remarks

Setting 1.1.1 appears in the papers [15] by E.Stiefel (1961) and [2] by R.A.-R.Amer, H.R.Schwarz (1964). Setting 1.1.2 appears after suitable normalization of the rational function in 1.1.1 and essentially coincides with the third Zolotarëv problem [7]. Setting 1.1.3 corresponds to the (extended) fourth Zolotarëv problem [7] and was studied by N.I.Akhiezer [5] (1929). The latter noticed that already in the classical Zolotarëv case with one pass- and one stop- band, the minimizing function is not unique. This phenomenon
was observed in the Stiefel’s paper [15] and fully explained in the dissertation of his pupil R.-A. Amer [2] who decomposed the space of rational functions of bounded deviation (defined in the left hand side of formula in 1.1.1) into classes. Namely, the competing rational functions have no poles at the passbands and no zeros at the stopbands, hence the sign of the polynomial in the numerator of the fraction on each stopband as well as the sign of denominator polynomial on each passband is fixed. Then in the closure of each nonempty class there is a unique minimum. All mentioned authors detected that (local) optimal functions are characterized by alternation (or equiripple in terms of electrical engineers) property. For instance, in the fourth Zolotarev problem the approximation error $\delta(x) := R(x) - S_E(x)$ of degree $n$ minimizer has $2n + 2$ alternation points $a_s \in E$ where $\delta(a_s) = \pm ||\delta||_{C(E)}$ with consecutive change of sign.

### 1.3 Equiripple criterion

V.N. Malozemov in 1979 suggested [3] a criterion for the defect $d$ solution (i.e. of exact degree $n - d$) which should possess at least $2n + 2 - d$ alternation points to be an optimizer. This however may contradict the projective invariance of the problem. The latter property consists in the 1-1 correspondence between the solutions of the problems with the bands $E$ related by a projective (= linear-fractional) transformations. The number of alternation points will depend on the starting point of their counting once $d$ is odd. Also the criterion does not take into consideration any information about Stiefel classes, which means that a minimizer with defect will be the minimizer for all classes whose closure contain it.

A simple counterexample to the criterion adapted for the setting 1.1.3 is as follows (it can be transformed to any other above setting and to higher degree $n$). Degree three Chebyshev polynomial $T_3(x) := 4x^3 - 3x$ approximates the function $\frac{3}{4}S_E(x)$ with the error $\mu := \frac{1}{4}$ on the set $E$ composed of bands $E^+ = -E^- := T_3^{-1}[\frac{1}{2}, 1] = [\cos(7\pi/9), \cos(5\pi/9)] \cup [\cos(\pi/9), 1]$. The error function has 10 explicitly listed alternation points $x_j = -\cos(j\pi/9)$, $j = 0, \ldots, 9$ shown in the Fig. 2. This is no doubt defect $d = 1$ optimizer among degree $n = 4$ rational functions (in the class where the function belongs to). The number of alternation points is slightly excessive so we can get rid of one of them, say $x_0 = -1$, by pressing inside $E$ its very left endpoint. For the modified set $E^*$, say $E^* := E \setminus [-1, -0.99]$, same polynomial $T_3(x)$ will be again a (local) optimizer among degree 4 rational functions by criterion of [3] since it has 9 alternation points. We expect that the function $T_3 \circ l$ will be an optimizer for the new set $E' := l^{-1}E^*$, say for $l(x) := 1/x$ since the problems are obviously related. However, the same criterion says that $T_3 \circ l$ will be not an optimizer in any class of degree 4 functions as it admits only 8 alternation points: only one of the points $l(x_1)$, $l(x_9)$ may be included in the alternation set since the error of approximation is positive for both points.

Examples of this kind may be given for any (not too low) degree $n$ and the total number of bands $m \geq 3$ with the explicit construction of equiripple rational functions from [10].

### 2 Projective setting

Here we discuss the optimization problem setting which embraces all the formulations we met before in Sect 1.1. We no longer treat the infinity point both in the domain of definition and the range of rational function as an exceptional one. Real line extended
\[ T_3 = -\frac{1}{2} \]

\[ T_3 = \frac{1}{2} \]

\[ T_3 = 1 \]

\[ T_3 = -1 \]

Figure 2: Chebyshev polynomial as a rational optimizer with defect

\[ r_{-1} = x_0 \]

\[ r_{x_1} x_{x_2} x_3 x_4 x_5 x_6 x_7 x_8 \]

\[ r_{x_9} = 1 \]

Figure 3: The ordering of four endpoints of \( F \)

by a point at infinity becomes a real projective line \( \mathbb{RP}^1 := \hat{\mathbb{R}} = \mathbb{R} \cup \{ \infty \} \) which is a topological circle. We consider two collections of disjoint closed segments on the extended real line: \( E \) consisting of \( m \geq 2 \) segments and \( F \) of just two segments. The segments of both \( E \) and \( F \) are of two types: \( E := E^+ \sqcup E^- ; F := F^+ \sqcup F^- \). Neither the stopbands \( E^- \) nor the passbands \( E^+ \) are empty sets.

### 2.1 Cross ratio and Large classes

The set of values \( F \) modulo projective (=linear-fractional) transformations depends on a single value – cross ratio of its four endpoints. Suppose the endpoints \( \partial F \) are cyclically ordered as follows \( \partial^0 F^- , \partial^1 F^- , \partial^0 F^+ , \partial^1 F^+ \) – see Fig. 3 then the cross ratio of four endpoints we define as follows

**Definition 1**

\[ \kappa(F) := \frac{\partial^1 F^+ - \partial^1 F^-}{\partial^0 F^+ - \partial^0 F^-} \cdot \frac{\partial^0 F^- - \partial^1 F^-}{\partial^0 F^- - \partial^0 F^+} > 1 . \]

Cross ratio depends on the order of four participating endpoints and may take six values interchanged by the elements of the unharmonic group, however it survives under the action of Klein’s quadratic group which is one of the assertions of

**Lemma 1**

1) Definition of \( \kappa(F) \) is independent of the projective line orientation (i.e. relabeling \( 0 \leftrightarrow 1 \)) and changing the type of components of \( F \) (i.e. interchange of indexes \( \pm \))

2) \( \kappa(F) \) decreases with the growth of its argument: if \( F' \subset F \) then \( \kappa(F') > \kappa(F) \)

Consider the space \( \mathbb{R}_n(x) \subset \mathbb{R}(x) \) of degree not greater than \( n \) rational functions. Unless otherwise explicitly indicated we use the topology induced by the coefficients of rational functions in this space. Considering homogeneous coordinates as suitably ordered coefficients of a rational function we obtain a natural map \( \mathbb{RP}^{2n+1} \to \mathbb{R}_n(x) \) which is
1 − 1 almost everywhere. It transfers the topology from the projective space to the space of rational functions. Along with the topology of coefficients one can consider other topologies like uniform convergence on the whole projective line of arguments (or its parts) with respect to Fubini-Study metrics on the target projective line. All reasonable topologies on the space \( \mathbb{R}_n(x) \) coincide outside the variety \( \mathbb{R}_{n-1}(x) \) of functions with the degree lower than \( n \). In the vicinity of mentioned discriminant variety (described as zero of appropriate resultant) we may observe the differences: if a real zero and pole of a rational function coalesce, the convergence to the limit function of lower degree takes place in the sense of coefficients, but it is not uniform.

**Definition 2** Large class \( \mathcal{R}_n(E, F) \) consists of real rational functions \( R(x) \) of a fixed degree \( \deg R = n \) with the property

\[
R(E^+) \subseteq F^+ \quad \text{and} \quad R(E^-) \subseteq F^-.
\]  

**Definition 3** Closed (in the topology induced by the coefficients) class \( \overline{\mathcal{R}_n(E, F)} \) is the set of bounded degree \( \deg R \leq n \) functions with the property (1).

The large class usually contains several components which is explained in the next section.

### 2.2 Topological indexes and small classes

Separated values at pass and stopbands induce a decomposition of the set of rational functions \( \mathcal{R}_n(E, F) \) into pieces with different topological properties. This phenomenon was first observed by E.Stiefel [15] in 1961 and explained in detail by R.A.-R. Amer in his PhD thesis [2] in 1964 for the problem setting 1.1.1. Indeed, once the value of the goal function from 1.1.1 is finite, a rational function \( R(x) \) has no zeros on \( E^- \) and no poles on \( E^+ \), hence on each component of the bands \( E \) either the the polynomial in the numerator of \( R(x) \) or the denominator of the same fraction has the definite sign. This array of signs is a topological invariant. Here we reintroduce the decomposition into topological classes for the general projectively invariant setting.

Identification of the opposite points of a circle \( S^1 \) gives a double cover \( S^1 \to \mathbb{R}P^1 \). Trying to lift the mapping \( R(x) : \mathbb{R}P^1 \to \mathbb{R}P^1 \) to the double cover of the target space \( \tilde{R}(x) : \mathbb{R}P^1 \to S^1 \), we encounter a topological obstruction to the existence of \( \tilde{R} \), namely the mapping degree or the winding number of \( R(x) \) modulo 2. A simple calculation shows that this value is equal to the algebraic degree \( \deg R \mod 2 \). However, the lift exists on any simply connected piece of \( \mathbb{R}P^1 \). Consider any transition band \( T \), a component of the complement of \( E \) to a projective line. The set \( F \subset \mathbb{R}P^1 \) lifted to the circle \( S^1 \) consists of four segments, two consecutive of which we call \( \tilde{F}_0 \) and another two as \( \tilde{F}_1 \) so that each of sets \( \tilde{F}_x \) is mapped 1-1 to \( F \). The mapping \( R(x) : T \to \mathbb{R}P^1 \) has two lifts \( \tilde{R} \) to the covering circle \( S^1 \) and exactly one of them has value \( \tilde{R}(x) \in \tilde{F}_0 \) on the left endpoint \( x = \partial^0T \) of \( T \). On the opposite side of the interval \( T \) the same function \( \tilde{R}(x) \) takes values in the set \( \tilde{F}_\sigma \) with well defined \( \sigma = \sigma(R, T) \in \{0, 1\} \) which we call the transition index of the band \( T \). Totally, the function \( R(x) \) defines a locally constant map \( \sigma : \mathbb{R}P^1 \setminus E \to \mathbb{Z}_2 \) with the constraint:

\[
\sum_T \sigma(R, T) = \deg R \mod 2.
\]
A practical way of computing the topological indexes is the following. Consider the universal covering \( \mathbb{R} \to \mathbb{RP}^1 = \mathbb{R}/\pi \mathbb{Z} \) of the kind \( \mathbb{R} \ni \phi \to (\sin(\phi) : \cos(\phi)) \in \mathbb{RP}^1 \). Its local inversion is given by \( \phi(P : Q) = \text{Arg}(Q + iP) \ (+\pi \mathbb{Z}) \) for homogeneous coordinates \((P : Q)\) of the projective line. Fix a union of two segments \( \tilde{F}_0 \subset \mathbb{R} \) which are mapped 1-1 to the set \( F \subset \mathbb{RP}^1 \) under the universal covering. Then the full preimage of \( F \) will have an appearance \( \tilde{F}_0 + \pi \mathbb{Z} \). Representing an element \( R(x) \) of \( \mathcal{R}_n(E,F) \) as the irreducible ratio of real polynomials \( P(x) : Q(x) \) we can lift the rational function to the universal cover of the target space:

\[
\tilde{R}(x) = \text{Arg}(Q(x) + iP(x))
\]

which is normalized by the condition \( \tilde{R}(\partial^1 T) \in \tilde{F}_0 \) (here we possibly have to change signs of both polynomials \( P,Q \)). The value of \( \tilde{R} \) on the opposite side \( \partial^1 T \) of the segment lies in \( \tilde{F}_0 + \pi \sigma \) with uniquely defined integer \( \sigma = \sigma(R,T) \). Of course, this index depends on the choice of the auxiliary set \( \tilde{F}_0 \) but this dependency is well controlled. Introduced above binary index we mostly be interested in in what follows is merely a reduction of integer index \( \sigma \mod 2 \).

The above procedure defines transition indexes (integer or binary) for any interval \( G \) of the projective line, with endpoints in \( E \). Simple calculation shows that they are determined by the indexes of transition bands contained in \( G \):

\[
\sigma(R,G) = \sum_{T \subseteq G} \sigma(R,T).
\]

Transition indexes \( \sigma(R,T) \) do not change with the deformation of \( F \) which induces simultaneous deformation of the auxiliary covering set \( \tilde{F}_0 \). Transition indexes are topological invariants of the rational function \( R \) that is remain intact once we continuously deform the function within the class \( \mathcal{R}_n(E,F) \) (in the usual sense, \( R(x,s) \) is a continuous map from \( T \times [0,1] \to \mathbb{RP}^1 \), rational in the first argument). Indexes may change only iff zeros and poles of \( R \) merge on some transition band in the process of deformation which inevitably brings us to the fall of the degree which shows a simple proposition:

**Lemma 2** Let \( R(x) \in \mathcal{R}_n(E,F) \), \( R(0) \neq 0, \infty \) and \( 0 \in T \), then for small real \( \epsilon \neq 0 \) the integer indexes \( \sigma \) are related as follows:

\[
\sigma(R(x)\frac{x + \epsilon}{x}, T) = \sigma(R,T) - \text{sign}(\epsilon R(0)),
\]

(2)

here \( \text{sign}(\cdot) = \pm 1 \).

Proof: direct calculation. \( \blacksquare \)

Figure 4: Solutions with different sets of transition indexes
Definition 4 The elements of $\mathcal{R}_n(E,F)$ with the fixed set of binary transition indexes $\sigma := \{\sigma(T)\}_T$ make up a subset $\mathcal{R}_n(E,F,\sigma)$ we call the small class.

One readily checks the properties of the introduced small classes:

Lemma 3

1. **Double monotonicity:**
   
   $$\mathcal{R}_n(E,F,\sigma) \subseteq \mathcal{R}_n(E',F',\sigma) \quad F \subseteq F' \text{ and } E \supseteq E'$$
   
   provided each segment of $E$ contains a unique segment of $E'$, which is of the same $\pm$-type.

2. **Double projective invariance**
   
   $$\beta \circ \mathcal{R}_n(E,F,\sigma) \circ \alpha^{-1} = \mathcal{R}_n(\alpha E, \beta F, \sigma'), \quad \alpha, \beta \in PGL_2(\mathbb{R}).$$
   
   here $\sigma'(\alpha T) = \sigma(T)$ for all transition bands $T \subset \mathbb{R}P^1 \setminus E$ and specified below consistent choice of the auxiliary set $\hat{F}_0$ for the definition of transition indexes.

3. **Class $\mathcal{R}_n(E,F,\sigma)$ is empty when $n < \sharp\{ T : \sigma(T) = 1 \}$.**

4. Small classes may also be (and usually are) not connected.

Proof. 1) For a given rational function $R$ the growth of the set $F$ and shrinking of the set $E$ does not spoil the restriction (1) and does not change any transition index $\sigma(R,T)$.

2) The first assertion $\beta \circ \mathcal{R}_n(E,F) \circ \alpha^{-1} = \mathcal{R}_n(\alpha E, \beta F)$ follows immediately from the definition of large classes. It remains to check the transition indexes.

Let us clarify the dependence of the binary index $\sigma(T)$ on the choice of the auxiliary set $\hat{F}_0$. The full preimage of $F = F^+ \sqcup F^-$ under the double cover consists of four segments cyclically labeled e.g. as $\hat{F}^+_0, \hat{F}^-_0, \hat{F}^+_1, \hat{F}^-_1$. Other decompositions of the full preimage arise after the relabeling $\hat{F}^+_0 \leftrightarrow \hat{F}^+_1$ or $\hat{F}^-_0 \leftrightarrow \hat{F}^-_1$ or both applied to the initial labeling. For the first and the fourth labeling the binary index is the same while for the second and the third case the index $\sigma(T)$ reverses (compared to the first labeling) exactly when the transition band $T$ is surrounded by stop and pass bands.

The automorphism $\beta$ of the projective line may be lifted in two ways to the automorphism $\hat{\beta}$ of the doubly covering circle. Those two lifts differ by the (pre- or post-) composition with the antipodal map and we fix one of them. Now we choose the agreed labeling for the auxiliary sets $\hat{F}_0$ and $(\beta F)_0 := \hat{\beta} \hat{F}_0$ which brings us to the above relation for the binary transition indexes. Indeed, one of the lifts for the map $\beta \circ R \circ \alpha^{-1}$ is $\hat{\beta} \circ R \circ \alpha^{-1}$ as we see it from the diagram (3):

$$\begin{array}{ccc}
S^1 & \overset{\hat{\beta}}{\rightarrow} & S^1 \\
\downarrow & & \downarrow \\
\alpha T & \overset{\alpha^{-1}}{\rightarrow} & T & \overset{R}{\rightarrow} & \mathbb{R}P^1 & \overset{\beta}{\rightarrow} & \mathbb{R}P^1
\end{array}$$

(3)

We easily check that on the endpoints of the segment $\alpha T$ the lifted map takes the values from $(\beta F)_0$ and $(\beta F)_\sigma$, with $\sigma = \sigma(R,T)$. The order of values depends on the orientation of $\alpha$, but in any case $\sigma(\beta \circ R \circ \alpha^{-1}, \alpha T) = \sigma(R,T)$.

3) Suppose the converse is true and there is a degree $n$ rational function $R$ whose binary transition index $\sigma$ takes the value 1 at more than $n$ transition bands $T$. The auxiliary sets $\hat{F}_0$ and $\hat{F}_1$ lie in the opposite semicircles of $S^1$. The endpoints of this semicircle are
glued to a point of real projective line which becomes a value of $R(x)$ more than $n$ times which is impossible. If each transition band is surrounded by a stop and a passband (as it usually happens in practice), then possibly relabeling the components of the total lift of the set $F$ (see section 2 of this proof) we arrive to a stronger version of the small class vanishing criterion: $n < \max_{j=0,1} \{ T : \sigma(T) = j \}$. This happens e.g. when $2n$ is smaller than the number of components in $E$.

4) Take any function $R(x)$ of the class $R_n(E,F)$, we assume w.l.o.g. that $x = 0$ is contained in the transition band $T$ and $R(0) \neq 0, \infty$. For small real $\epsilon$ two functions $x \pm \epsilon x$ belong to the same large class $R_n(E,F^*)$ with $F^*$ slightly larger than $F$. From Lemma 2 it follows that two functions have different integer transition indexes but the same binary index at $T$, therefore they belong to the different components of the same (binary) small class $R_n(E,F^*,\sigma)$.

Remark 1 In our terms R.-A.R.Amer [2] combines classes $R_n(E,F,\sigma)$ and $\beta R_n(E,F,\sigma)$ for the projective map $\beta$ reversing the orientation of projective line and conserving the components $F^\pm$. This is why he gets twice less number $2^{m-2}$ of classes with $m$ being the total number of bands in $E$.

2.3 Extremal problem for small classes

Problem. Given the set of bands $E$ and the transition index array $\sigma$ consistent with the fixed degree $n$ – find

$$\kappa(n,E,\sigma) := \inf \{ \kappa(F) : \mathcal{R}_n(E,F,\sigma) = \emptyset \}. \tag{4}$$

The idea behind this optimization is the following: we squeeze the set of values $F$, the nonempty functional class $\mathcal{R}_n(E,F,\sigma)$ shrinks and we have to catch the moment – quantitatively described by the cross ratio $\kappa(F)$ – when the class disappears.

Remark 2 In problem formulation 1.1.2 the set $F^+ = [-\theta,\theta]$ and the set $F^- = [1/\theta, -1/\theta]$; $\kappa(F) = (\theta^2 \theta + 1) / \theta^2$. In setting 1.1.3 the sets $F^\pm = [-1 - \mu, 1 + \mu] \cap R_n(E,F,\sigma)$.

The set $F$ with the extremal value of $\kappa > 1$ may correspond to the empty small class $R_n(E,F,\sigma)$. To take into account solutions with defect (= of degree less than $n$), we introduce the notion of the limit class.

Definition 5 The limit class we call the intersection of nested closed sets

$$\mathcal{R}_n(E,F,\sigma) := \bigcap_{F^\prime \supset F} \mathcal{R}_n(E,F^\prime,\sigma). \tag{5}$$

Let us introduce several characteristics of a rational function $R$ lying in the closed class $\mathcal{R}_n(E,F)$. The set of extremal points $Ext(R)$ consists of points $e \in E$ mapped to the boundary of $F$: $R(e) \in \partial F$. Each extremal point $e$ has a parity $\epsilon(e) = 0/1$ depending to which boundary component $\partial^0 F / \partial^1 F$ (= left/right) does the value $R(e)$ belong to. The complement $\mathbb{RP}^1 \setminus Ext(R)$ to the set of extremal points is a disjoint union of open intervals $G$ (gaps) which fall into two types: for even $G$ its endpoints have the same parity $\epsilon$, for odd gap the endpoints are of different parities. Now for each gap $G$ we introduce the differential transition index of $R$ as the sum over all transition bands $T$ contained in the
gap $G$ of the differences of transition indexes of $R$ and the given $\sigma = \{\sigma(T)\}_T$, all taken modulo 2:

$$\delta \sigma(G, R) := \left( \sum_{T \in G} \sigma(R, T) - \sigma(T) \right)_2 = (\sigma(R, G) - \sigma(G))_2 \in \{0, 1\}, \quad (6)$$

here and in what follows $(\cdot)_2$ is the reduction of integers modulo 2 with the values in the set $\{0, 1\}$. We distinguish between global differences which occur on two types of gaps:

$$\Sigma^0(R) := \sum_{\text{even } G} \delta \sigma(R, G); \quad \Sigma^1(R) := \sum_{\text{odd } G} \delta \sigma(R, G). \quad (7)$$

here the first sum is taken over all even gaps $G$ (endpoints of the same parity) and the other sum is taken over odd gaps (endpoints of different parity).

We note two simple facts for the defect of the functions at the boundary of the closed small class.

**Lemma 4** Let $R \in \overline{R_n}(E, F, \sigma)$ and $\Sigma(R) := \sum_T (\sigma(R, T) - \sigma(T))_2$ be the Hamming distance of two binary arrays: actual transition indexes of $R$ and those inherited from the small class. Then

1. $\Sigma(R) \leq d(R) := n - \deg(R)$, \quad (8)

2. $d + \Sigma^0 + \Sigma^1 \in 2\mathbb{Z}$, \quad $\Sigma(R) + d \in 2\mathbb{Z}$. \quad (9)

Proof. The defect $d$ is the number of cancellations of (simple) zeros/poles of a rational function when approaching the boundary of the small class. The binary index $\sigma(R, T)$ inverses exactly when there are odd number of cancellations at the transition band $T$. \blacksquare

### 2.4 Equiripple property

**Definition 6** We say that cyclically ordered extremal points of the function $R(x) \in \overline{R_n}(E, F)$ make up an alternation set iff any two consecutive points have opposite parity (in particular, their total number is even). The maximal number $\text{Alt}(R)$ of alternation points is equal to the number of odd gaps $G \subset \mathbb{R}P^1 \setminus \text{Ext}(R)$.

**Theorem 1** If the value $\kappa(n, E, \sigma) > 1$, then each extremal limit class $\overline{R_n}(E, F, \sigma)$ with $\kappa(F) = \kappa(n, E, \sigma)$ contains a unique function $R(x)$. This function is completely characterized by the property

$$\text{Alt}(R) \geq n + 2 + \deg R - \Sigma^0(R) + \Sigma^1(R). \quad (10)$$

Note that both sides of the inequality are even.

Proof of this theorem is based on two technical propositions.

**Lemma 5** (Achiezer, [6]) Let $P$ and $Q \in \mathbb{C}[x]$ be two mutually prime polynomials of the same degree $n$ and $S$ be an arbitrary polynomial of degree $m$, then the equation

$$S = pQ - qP \quad (11)$$

admits polynomial solutions $p, q$ of degree $\leq \max(m - n, n - 1)$. 

10
Proof. Consider two Lagrange polynomials $p_1$ and $q_1$ which interpolate $S/P$ at the zeros of $Q$ and $S/Q$ at the zeros of $P$ respectively, deg $p_1,q_1 \leq n-1$. In the case of multiple zeros we require not only the equality of values of polynomials but also the equality of several first derivatives. Now we define the polynomial $2r := (S-p_1Q+q_1P)/(PQ)$ which is identically equal to zero when $m < 2n$ or deg $r = m-2n$ when $m \geq 2n$. One easily checks that $p := p_1 + rP; q := q_1 - rQ$ is a solution of (11) and any other solution has the appearance $p + rP; q + rQ$ with arbitrary $r_1 \in \mathbb{C}[x]$. 

Lemma 6 Let $R(x)$ and $R'(x)$ belong to the same closed class $\overline{R_n(E,F)}$ and $e_1$, $e_2$ be two extremal points of $R(x)$. Suppose the values of $R$ and $R'$ do not coincide at each of $e_j$, then the number of points $t \in [e_1,e_2]$ such that $R(t) = R'(t)$ counted with the multiplicity of tangency has the same parity as

$$e(e_1) + e(e_2) + \sum_{e_1 < T < e_2} (\sigma(R,T) + \sigma(R',T)),$$

where summation is taken over all transition bands in the segment $[e_1,e_2]$ of oriented projective line.

Remark 3 Multiplicity of coincidence of functions $R$, $R'$ at a point $t$ is equal to the order of zero of their difference at this point provided $R(t) = R'(t) \neq \infty$. Otherwise we use another projective coordinate for the target projective line and apply the previous definition. In other words, the multiplicity of tangency at a point is the multiplicity of zero of the determinant $PQ' - QP'$ where $P/Q := R$ and $P'/Q' := R'$ are irreducible representations of rational functions as ratios of polynomials.

Remark 4 The coincidence multiplicity is odd exactly when the difference $R - R'$ changes its sign in the small vicinity of the coincidence point.

Remark 5 Here we do not exclude the case $e_1 = e_2$, the segment between those points then mean either one point or the whole projective line.

Proof. We use identifications $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\hat{R} = \mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$ which are implemented by the functions $\exp(i\phi)$ and $\tan(\phi)$ respectively. A lift of the mapping $R(t) : [e_1,e_2] \to \hat{R}$ to the double cover of the target space is implemented now by the formula $\tilde{R}(t) = \phi(t) = \text{Arg}(Q(t) + iP(t)) \mod 2\pi$ where $R(t) = P(t)/Q(t)$ is the irreducible representation of the rational function. The choice of the lift is twofold and can always be made so that $\tilde{R}(e_1) \in \tilde{F}_0$. The values of this lift at the endpoints of the segment $[e_1,e_2]$ are as follows:

$$\tilde{R}(e_1) = \partial^1\tilde{F}_0^*, \mod 2\pi,$$

$$\tilde{R}'(e_1) \in [\partial^0\tilde{F}_0^*, \partial^1\tilde{F}_0^*] \mod 2\pi,$$

where the sign $* = \pm$ is defined by the type of the band $E^\pm \ni e_1$ and $e_1 := e(e_1)$;

$$\tilde{R}(e_2) = \partial^2\tilde{F}_0^{**} + \pi\sigma(R, [e_1,e_2]), \mod 2\pi$$

$$\tilde{R}'(e_2) \in [\partial^1\tilde{F}_0^{**}, \partial^2\tilde{F}_0^{**}] \mod 2\pi,$$

where the sign $** = \pm$ is defined by the type of the band $E^\pm \ni e_2$, $\sigma(R, [e_1,e_2]) = \sum_{e_1 < T < e_2} \sigma(R,T), e_2 := e(e_2)$.
Consider the difference of the lifted values $\hat{R} = \phi, \hat{R}' = \phi'$ at the endpoints:
\[
\phi'(e_1) - \phi(e_1) \in (-1)^{e_1}(0, |F'|] \subset (-1)^{e_1}(0, \pi) \mod 2\pi, \quad (17)
\]
\[
\phi'(e_2) - \phi(e_2) \in (-1)^{e_2}(0, |F'|] + \pi(\sigma(R', [e_1, e_2]) - \sigma(R, [e_1, e_2])) \subset (-1)^{e_2}(0, \pi) + \pi(\sigma(R, [e_1, e_2]) - \sigma(R, [e_1, e_2])) \mod 2\pi, \quad (18)
\]

where $|F'| < \pi$ is the length of the component of $F$ in Fubini-Study metric on the projective line. The coincidence of the values of $R, R'$ at a point is equivalent to $\phi' - \phi \in \pi\mathbb{Z}$ at the point. The multiplicity of the coincidence $\hat{R} = \hat{R}'$ is equal to the multiplicity of zero of the value:
\[
(\phi'(t) - \phi(t)) \mod \pi \sim \sin(\phi'(t) - \phi(t)) = \text{Im} \exp(i\phi'(t) - i\phi(t)) = \text{Im} \frac{Q - iP'}{\sqrt{P'^2 + Q'^2}} \sim (PQ' - QP')(t).
\]

Finally we see it from the formulas (17) and (18) that the number of transitions of the value $\delta \hat{R}$ on the segment $[e_1, e_2]$ from one semicircle $\pm(0, \pi)$ to the other is equal to the value $[12]$ modulo 2. $
$

3 Main theorem proof

0. Existence. We show that the closed sets in the definition [5] of the limit class make up a centered family, that is intersection of finitely many of them is nonempty. Using the compactness of the ambient set of bounded degree rational functions we easily get that all sets of the family have a common point [17].

Indeed, any finite intersection of the small classes closures $\overline{R}(E, F', \sigma)$ contains at least the small class $R(E, F'', \sigma)$ with $F''$ being the finite intersection of the sets $F'$. Since $\text{Int} F' > F, \kappa(F'') < \kappa(F)$ and the latter small class is not empty because $\kappa(F) > 1$ is the extremal value.

1. Sufficiency. Suppose that $R(x)$ from the closed class $\overline{R}(E, F)$ has big enough (specified in the formulation of the theorem [1]) alternation set with respect to the given binary array $\sigma$. We show that the limit class $\overline{R}(E, F, \sigma)$ is extremal and contains the unique point $R$. Let $R'$ be any function in the limit set $\overline{R}(E, F', \sigma)$ with $F' \subseteq F$.

The set $\text{Ext}(R)$ decomposes the projective line into finitely many opened intervals (gaps) $G$. Consider a chain $[e_1, e_2]$ of $k \geq 1$ consecutive gaps such that $R(e) = R'(e)$ for any interior extremal point $e$ of the chain whereas the values of $R, R'$ are different at the endpoints $e_1, e_2$. Let us assess the number of coincidences of $R, R'$ inside the cluster counting their multiplicities. This number is at least $k - 1$, however Lemma [6] can guarantee an extra coincidence point once the value $[12]$ calculated for the cluster has the same parity as $k$. Let the chain contain $m_0$ even and $m_1$ odd gaps $G, m_0 + m_1 = k$, then the sum of parities of the endpoints of cluster is $m_1$ modulo 2. The difference of transition indexes for the functions $R, R'$ over all transition bands in the cluster amounts to $\sum_{G \subseteq [e_1, e_2]} \delta \sigma(G) + \delta \sigma'(G) \mod 2$, where we for brevity designated $\sigma(G) := \sigma(G, R); \quad \sigma'(G) := \sigma(G, R')$. Hence the number of coincidence points in the cluster is at least
\[
k - (k + m_1 + \sum_{G \subseteq [e_0, e_k+1]} \delta \sigma(G) + \delta \sigma'(G)) \geq 0.
\]
\[ m_1 + \sum_{\text{even } G} (\delta\sigma(G))_2 - \sum_{\text{odd } G} (\delta\sigma(G))_2 - \sum_{G} (\delta\sigma'(G))_2. \]

To prove the last inequality we note that its left hand side is not less than \( k - 1 \), whereas the right hand side is not greater than \( k - 1 \) with one exception: \( (\delta\sigma'(G))_2 = 0 \) for all gaps \( G \), \( (\delta\sigma(G))_2 = 0 \) for odd gaps and \( (\delta\sigma(G))_2 = 1 \) for even gaps. In the latter case we have the equality of both sides.

Summing up the latter estimations over all clusters we get a lower bound for the number of the coincidence points of two rational functions \( R \) and \( R' \) on the projective line. Taking into account the inequalities \((10)\) and \((8)\) the lower bound becomes:

\[ \text{Alt}(R) + \Sigma^0(R) - \Sigma^1(R) - \Sigma(R') > \deg(R) + \deg(R'), \tag{19} \]

where from it follows that \( R = R' \) identically. The omitted case when \( R = R' \) in each extremal point of \( R \) immediately brings us to the assessment in the left hand side of \((19)\) for the number of coincidence points since the number of even gaps \( G \) is not less than \( \Sigma^0(R) \).

2. Necessity. Suppose a function \( R \in \mathcal{R}_{n+1}^+(E,F,\sigma) \) possesses not enough number of alternation points. We explicitly indicate a function \( R' \) in the class \( \mathcal{R}_{n}(E,F',\sigma) \) with \( F' \subset F \). Fix the coordinate in the target projective line so that infinity does not lie at the boundary of \( F \) and is not in the set of extremal points of \( R' \) and \( R(\infty) \) is neither zero nor infinity nor a point of \( \partial F \). This gives us a technical convenience of \( R = P/Q \) being the ratio of mutually prime polynomials of the same degree \( n - d \), where \( d \geq 0 \) is the defect of \( R \).

We are looking for a modification of \( R \) among functions of the kind

\[ R'(x) := \frac{M(x)\prod_{s=1}^{\Sigma}(x-x_s)P(x) - \tau p(x)}{M(x)\prod_{s=1}^{\Sigma}(x-x_s)Q(x) - \tau q(x)} \tag{20} \]

where \( x_s \) is any point in the transition band \( T_s \) with the inverted binary index \( \sigma(R,T_s) \neq \sigma(T_s) \) modulo 2 and \( \Sigma = \Sigma(R) \) is the total number of such bands. Polynomial \( M(x) \) of even degree \( d - \Sigma \) is positive on the real axis; \( \tau \) is a small real parameter. Polynomials \( p(x) \) and \( q(x) \) of degrees not greater than \( n \) determine the direction of deformation in the space of rational functions and will be chosen later. The fraction \( R' \) belongs to some class \( \mathcal{R}_{n}(E,F',\sigma) \) once its numerator and denominator have no common factors. The set \( F' \) here will be strictly smaller than \( F \) for small values of \( \tau \) if the difference \( \delta R(x) := R'(x) - R(x) \) has the sign \((-1)^{\epsilon(x)}\) in each extremal point \( x \in \text{Ext}(R) \) – see e.g.\((5)\) for more details. The sign of the difference \( \delta R(x) \), \( x \in E \) for small \( \tau \) is the same as the sign of the polynomial \( \tau\prod_{s=1}^{\Sigma}(x-x_s)(P(x)q(x) - Q(x)p(x)) =: \tau L(x) \). We can explicitly construct the latter polynomial \( L \) provided the number of alternation points of \( R \) is insufficient.

Polynomial \( L \) should have even/odd number of zeros in each even/odd finite gap \( G \). It already has \( \Sigma \) zeros \( x_s \) used to recover the array \( \sigma \) of transition indexes. So we place additional zero to each gap \( G \) where the parity of \( \delta\sigma(R,G) \) is opposite to the parity of the gap itself. We get a polynomial \( L \) of degree \( \text{Alt}(R) - \Sigma^1 + \Sigma^0 + \Sigma \) or one less as we need not place an additional zero to the gap at infinity. Canceling \( L(x) \) by \( \prod_{s=1}^{\Sigma}(x-x_s) \) we obtain a left hand side \( S(x) \) of the functional equation \((11)\) for polynomials \( p,q \). The latter according to Lemma \((4)\) always has a solution for mutually prime \( P,Q \). The solution

\footnote{If \( \text{Ext}(R) \) is empty, the set \( F \) surely may be squeezed}
has degrees of $p, q$ strictly greater than $n$ exactly when $\deg L - \Sigma - n + d > n$. Taking into account the parity (9) of defect we obtain the required in Theorem 1 number of alternation points for $R$.

It remains to check that fraction $R'(x)$ has no cancellations for small $\tau$. We assume w.l.o.g. that only simple zeros of numerator and denominator cancel when $\tau = 0$. Those zeros smoothly depend on time $\tau$ and have finite velocities at the initial moment which may be explicitly calculated. The velocity of each of those zeros in numerator is different from that for denominator when $\tau = 0$, for otherwise the polynomial $L(x)M(x)$ would have multiple zeros. We have enough freedom in the choice of the latter polynomial to avoid this occasion.

4 Conclusion

We elaborated a universal setting for the optimization problem of multiband electrical filter which apparently embraces all previously known settings. Novel viewpoint at this problem is consistent with the important property of the latter, its projective invariance. A notion of cyclic alternation points is introduced to give the optimality criterion for the new setting. This criterion also involves the information about decomposition of the space of rational functions into topological classes which guarantee the uniqueness of the solution. Extremal functions introduced in [10] give us examples with nontrivial additional terms $\Sigma^0, \Sigma^1$ and will be described in a separate publication. Omitted technical (but very important in the engineering practice) details like the weight of the approximation or possible restrictions on participating rational functions can be also incorporated to the elaborated scheme.

References

[1] Gonchar, A.A., The Problems of E.I. Zolotarëv Which Are Connected with Rational Functions, Mat. Sb., 1969, vol. 78 (120), no. 4, pp. 640–654.
[2] R. A.-R. Amer and H. R. Schwarz, Contributions to the approximation problem of electrical filters. Mitt. Inst. Angew. Math. (1964), No. 9, Birkhäuser, Basel. 99pp. See also R.A.-R. Amer’s PHD Thesis, ETH, 1964.
[3] Malozemov, V.N., The Synthesis Problem for a Multiband Electrical Filter, Zh. Vychisln. Mat. i Mat. Fiz., 1979, vol. 19, no. 3, pp. 601–609.
[4] Achiessér, N.I., Elements of the theory of elliptic functions, (Russian). Leningrad 1948. (Translated from the second Russian edition by H. H. McFaden. Translations of Mathematical Monographs, 79. American Mathematical Society, Providence, RI, 1990. viii+237 pp.)
[5] Achiessér, N.I., Sur un problème de E. Zolotarëv, Bull. Acad. Sci. de l’URSS, VII sér., 1929, no. 10, pp. 919–931.
[6] Achiessér, N.I., Theory of Approximation (Monograph) – Dover Publications, 1992.
[7] Zolotarëv, E.I., Application of Elliptic Functions to Questions on Functions Deviating Least and Most from Zero, Zap. Imp. Akad. Nauk St. Petersburg, vol. 30, no. 5 (1877), pp. 1–71.
[8] Cauer, W., *Theorie der linearen Wechselstromschaltungen*, Bd. 1, Leipzig: Becker und Erler, 1941; Bd. 2, Berlin: Akademie, 1960.

[9] Bogatyrev, A.B., *Ekstremal’nye mnogochleny i rimanovy povrchnosti*, Moscow: MC-CME, 2005. Translated under the title *Extremal Polynomials and Riemann Surfaces*, Berlin: Springer, 2012.

[10] Bogatyrev, A.B., Chebyshev Representation of Rational Functions, *Mat. Sb.*, 2010, vol. 201, no. 11, pp. 19–40 [Sb. Math. (Engl. Transl.), 2010, vol. 201, no. 11–12, pp. 1579–1598].

[11] A. B. Bogatyrev, S. A. Goreinov, S. Yu. Lyamaev, Analytical approach to multiband filter synthesis and comparison to other approaches, *Problems Inform. Transmission*, 53:3 (2017), 260–273, arXiv: 1612.01753.

[12] Remez, E.Ya., *Basics of Numerical Methods of Chebyshev Approximation*, Kiev: Naukova Dumka, 1969.

[13] Veidinger, L., On the Numerical Determination of the Best Approximations in the Chebyshev Sense, *Numer. Math.*, 1960, vol. 2, no. 1, pp. 99–105.

[14] Fuchs, W.J.H., On Chebyshev Approximation on Sets with Several Components, *Aspects of Contemporary Complex Analysis*, Brannan, D.A. and Clunie, J.G., Eds., London; New York: Academic Press, 1980, pp. 399–408.

[15] E. Stiefel, Le problème d’approximation dans la théorie des filtres électriques, //Colloque sur l’analyse numérique à Mons, 1961, pp. 81–87.

[16] E. Stiefel, Methods – old and new – for solving the Chebyshev approximation problem // J.SIAM (B) 1, 164–176.

[17] Kelley, John L. General Topology – Springer, Graduate Texts in Mathematics (27), 1975

Institute for Numerical Math., Russian Academy of Sciences, Address: Russia 119991 Moscow ul. Gubkina, 8 Email: ab.bogatyrev@gmail.com, gourmet@inm.ras.ru