Catalan Moments

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Abstract

This paper is essentially devoted to the study of some interesting relations among the well known operators \( I(x) \) (the interpolated Invert), \( L(x) \) (the interpolated Binomial) and Revert (that we call \( \eta \)).

We prove that \( I(x) \) and \( L(x) \) are conjugated in the group \( Y(R) \). Here \( R \) is a commutative unitary ring. In the same group we see that \( \eta \) transforms \( I(x) \) in \( L(-x) \) by conjugation. These facts are proved as corollaries of much more general results.

Then we carefully analyze the action of these operators on the set \( R \) of second order linear recurrent sequences. While \( I(x) \) and \( L(x) \) transform \( R \) in itself, \( \eta \) sends \( R \) in the set of moment sequences \( \mu_n(h,k) \) of particular families of orthogonal polynomials, whose weight functions are explicitly computed.

The moments come out to be generalized Motzkin numbers (if \( R = \mathbb{Z} \), the Motzkin numbers are \( \mu_n(-1,1) \)). We give several interesting expressions of \( \mu_n(h,k) \) in closed forms, and one recurrence relation.

There is a fundamental sequence of moments, that generates all the other ones, \( \mu_n(0,k) \). These moments are strongly related with Catalan numbers. This fact allows us to find, in the final part, a new identity on Catalan numbers by using orthogonality relations.

1 A group acting on sequences

Definition 1.1.

\[
S(R) = \{ A = \{a_n\}_{n=0}^{+\infty} : \forall n \ a_n \in R, \ a_0 = 1 \}
\]

where \( R \) is a commutative unitary ring.

Now we embed \( S(R) \) in \( R[[t]] \) in this way

\[
\forall A \in S(R) \quad \lambda(A) = \sum_{n=0}^{+\infty} a_n t^{n+1}.
\]
In $R[[t]]$ is naturally defined the series composition $\circ$

$$
\sum_{n=0}^{+\infty} a_n t^{n+1} \circ \sum_{k=0}^{+\infty} b_k t^{k+1} = \sum_{n=0}^{+\infty} a_n \left( \sum_{k=0}^{+\infty} b_k t^{k+1} \right)^{n+1}.
$$

Then we may induce the operation $\bullet$ in $S(R)$:

**Definition 1.2.**

$$
\forall A, B \in S(R) \quad A \bullet B = \lambda^{-1}(\lambda(A) \circ \lambda(B)).
$$

Of course $(S(R), \bullet)$ is a group.

**Observation 1.3.** If $R = \mathbb{F}_q$, this is the *Nottingham group* over $\mathbb{F}_q$.

From every element $A \in S(R)$ two operators rise: the left multiplication $L_A$ and the right multiplication $R_A$.

**Definition 1.4.**

$$
\forall B \in S(R) \quad L_A(B) = A \bullet B \\
\forall B \in S(R) \quad R_A(B) = B \bullet A.
$$

We also consider the following two special operators: $\eta$, often called *Revert*, and $\varepsilon$ the alternating sign operator.

**Definition 1.5.**

$$
\forall B \in S(R) \quad \eta(B) = B^{-1} \\
\forall B = \{b_n\}_{n=0}^{+\infty} \in S(R) \quad \varepsilon(B) = \{(-1)^n b_n\}_{n=0}^{+\infty}.
$$

Plainly

**Property 1.6.**

(2) $$
\forall A, B \in S(R) \quad \eta(A \bullet B) = \eta(B) \bullet \eta(A).
$$

(3) $$
\forall A, B \in S(R) \quad \varepsilon(A \bullet B) = \varepsilon(A) \bullet \varepsilon(B).
$$

In other words, the inversion $\eta$ is an anti-isomorphism of $S(R)$, and the alternating sign $\varepsilon$ is an isomorphism of $S(R)$.

**Observation 1.7.** The operator $\eta$ is especially important. If $a = \{a_n\}_{n=0}^{+\infty}$, $b = \{b_n\}_{n=0}^{+\infty}$ and $\eta(a) = b$, then we have the relations

$$
\begin{align*}
\{ u = u(t) = \sum_{n=0}^{+\infty} a_n t^{n+1} \\
t = t(u) = \sum_{n=0}^{+\infty} b_n u^{n+1} \text{ inverse series of } u.
\end{align*}
$$
The operators $L_A, R_A, \eta, \varepsilon$, are invertible and can be compounded by applying one after the other (by the usual operation $\circ$). They generate a group, that we call $\Upsilon(R)$. The group $\Upsilon(R)$ acts on $S(R)$.

**Property 1.8.** For every ring $R$ the following are true

(5) \[ \eta = \eta^{-1} \]

(6) \[ \varepsilon = \varepsilon^{-1} \]

(7) \[ \eta \circ \varepsilon = \varepsilon \circ \eta \]

(8) \[ \forall A, B \in S(R) \quad L_A \circ R_B = R_B \circ L_A \]

(9) \[ \forall A \in S(R) \quad L_A \circ \eta = \eta \circ R_{A^{-1}} \]

(10) \[ \forall A \in S(R) \quad \eta \circ L_A = R_{A^{-1}} \circ \eta \]

**Proof.**

(5) and (6) follow from definition.

(7) Let $d = \eta(\varepsilon(a))$. Because $\varepsilon(a) = \{(−1)^n a_n\}_{n=0}^{+\infty}$, (11) becomes

\[
\begin{align*}
\begin{cases}
u = u(t) = \sum_{n=0}^{+\infty} (-1)^n a_n t^{n+1} \\
t = t(u) = \sum_{n=0}^{+\infty} d_n u^{n+1}.
\end{cases}
\end{align*}
\]

But (11(i)) can be rewritten as

\[-u = \sum_{n=0}^{+\infty} a_n (-t)^{n+1}\]

and if $b = \eta(a)$ then

\[t = \sum_{n=0}^{+\infty} (-1)^n b_n u^{n+1}\]
comparing this result with \((11\text{ ii})\) we obtain \(d = \varepsilon(b)\).

\[\forall A, B, C \in S(R)\]
\[(L_A \circ R_B)(C) = L_A(C \bullet B) = A \bullet C \bullet B = R_B(A \bullet C) = R_B(L_A(C)) = (R_B \circ L_A)(C)\].

\[\forall A, B \in S(R) \quad (L_A \circ \eta)(B) = A \bullet B^{-1}\]

and

\[\forall A, B \in S(R) \quad (\eta \circ R_{A^{-1}})(B) = (B \bullet A^{-1})^{-1} = A \bullet B^{-1}.
\]

To prove this we do operator composition with \(\eta\) both in the front and the back of each side of \((9)\). □

Let us pose \(\gamma = \eta \circ \varepsilon\) and \(X(x) = \{x^n\}_{n=0}^{+\infty}\), with \(x \in R\).

Of course \(X(x) \in S(R)\) and both \(L_X(x)\) and \(R_X(x)\) are in \(Y(R)\).

Plainly

\[\lambda(X(x)) = \sum_{n=0}^{+\infty} x^n t^{n+1} = \frac{t}{1-xt}.
\]

We have

**Property 1.9.**

\[L_X(x) \circ \varepsilon = \varepsilon \circ L_{-x}\]

\[R_X(x) \circ \varepsilon = \varepsilon \circ R_{-x}\]

\[\gamma = \gamma^{-1}\]

\[\gamma \circ L_X(x) \circ \gamma^{-1} = R_X(x)\]

\[\gamma \circ R_X(x) \circ \gamma^{-1} = L_X(x).
\]
Proof.
(13) \[ \forall A \in S(R) \]
\[ (\mathcal{L}_{X(x)} \circ \varepsilon)(A) = X(x) \varepsilon(A) = \varepsilon(X(-x)) \varepsilon(A) = \varepsilon(X(-x) \bullet A) = (\varepsilon \circ \mathcal{L}_{X(-x)})(A) \].

(14) Same proof as for (13).

(15) \( \gamma \) is the composition of two commuting involutions.

(16) \[ \forall A \in S(R) \]
\[ (\gamma \circ \mathcal{L}_{X(x)})(A) = (\eta \circ \varepsilon)(X(x) \bullet A) = \eta(X(-x) \bullet \varepsilon(A)) = \gamma(A) \bullet X(x) = (\mathcal{R}_{X(x)} \circ \gamma)(A) \].

(17) Same proof as for (16).

Let us recall the well known operators Invert and Binomial.

Definition 1.10. The operator \( I \) maps the sequence \( A = \{a_n\}_{n=0}^{+\infty} \) in \( B = \{b_n\}_{n=0}^{+\infty} \) where \( B \) has generating function:
\[
\sum_{n=0}^{\infty} b_n t^n = \frac{\sum_{n=0}^{+\infty} a_n t^n}{1 - t \sum_{n=0}^{+\infty} a_n t^n}.
\]

Definition 1.11. The operator \( L \) maps the sequence \( A = \{a_n\}_{n=0}^{+\infty} \) in \( B = \{b_n\}_{n=0}^{+\infty} \) where
\[ b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \]
These operators can be iterated \[ \| \] and interpolated \[ \| \] becoming \( I^{(x)} \), \( L^{(y)} \) in this way:

Definition 1.12. Given \( x \in R \) \( I^{(x)} \) is called Invert interpolated operator. By definition \( I^{(x)}(A) = P = \{p_n(x)\}_{n=0}^{+\infty} \) where \( P \) is the sequence having generating function
\[
P(t) = \sum_{n=0}^{+\infty} p_n(x) t^n = \frac{\sum_{n=0}^{+\infty} a_n t^n}{1 - xt \sum_{n=0}^{+\infty} a_n t^n}.
\]

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Definition 1.13. Given \( y \in R \) \( L^{(y)} \) is called Binomial interpolated operator. By definition

\[
L^{(y)}(A) = \left\{ l_n = \sum_{j=0}^{n} \binom{n}{j} y^{n-j} a_j \right\}_{n=0}^{+\infty}.
\]

The exponential generating function of \( l = \{l_n\}_{n=0}^{+\infty} \) is:

\[
\mathcal{L}(t) = \sum_{n=0}^{+\infty} l_n \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \sum_{j=0}^{n} \frac{(yt)^{n-j}}{(n-j)!} \frac{a_j t^j}{j!} = \exp(ty)A(t)
\]

being

\[
\exp(ty) = \sum_{n=0}^{+\infty} \frac{(ty)^n}{n!} \quad A(t) = \sum_{n=0}^{+\infty} \frac{a_n t^n}{n!}
\]

so that (recalling that \( a_0 = 1 \)) we have the ordinary generating function

\[
L(t) = \frac{1}{t} A \left( \frac{t}{1 - ty} \right)
\]

with \( A(t) = \sum_{n=0}^{+\infty} a_n t^{n+1} \).

The following facts are immediate consequences of (12).

Property 1.14.

\[
\forall x \in R \quad \eta(X(x)) = X^{-1}(x) = \{(-x)^n\}_{n=0}^{+\infty} = X(-x) = \varepsilon(X(x))
\]

\[
\forall x, y \in R \quad X(x) \bullet X(y) = X(x + y)
\]

From their definitions it is not apparent that the operators \( I^{(x)} \) and \( L^{(x)} \) are strongly related. Indeed we are going to prove that they are, respectively, the left and the right multiplication by \( X(x) \) in the group \( S(R) \).

Theorem 1.15.

\[
I^{(x)} = \mathcal{L}X(x)
\]

\[
L^{(x)} = \mathcal{R}X(x)
\]
Proof. Let $B = X(x) \cdot A$, then

$$
\lambda(B) = \lambda(X(x)) \circ \lambda(A) = \sum_{n=0}^{+\infty} x^n \left( \sum_{k=0}^{+\infty} a_k t^{k+1} \right)^n = \sum_{n=0}^{+\infty} \left( \sum_{k=0}^{+\infty} \left( x \sum_{k=0}^{+\infty} a_k t^{k+1} \right)^n \right) = \sum_{k=0}^{+\infty} a_k t^{k+1} \frac{1}{1 - x \sum_{k=0}^{+\infty} a_k t^{k+1}}
$$

so $B = I^{(x)}(A)$ from (18) and (1).

Let $C = A \cdot X(x)$, then

$$
\lambda(C) = \lambda(A) \circ \lambda(X(x)) = \sum_{n=0}^{+\infty} a_n \left( \sum_{k=0}^{+\infty} x^k t^{k+1} \right)^n = \sum_{n=0}^{+\infty} a_n \left( \frac{t}{1 - xt} \right)^{n+1}
$$

so $C = L^{(x)}(A)$ from (21) and (1).

From Theorem 1.15 and the previous properties we obtain

**Theorem 1.16.** Let $Id$ be the identity operator and $x, y \in R$. For the interpolated Invert and Binomial operators the following are true:

$$
I^{(x)} \circ I^{(-x)} = Id \quad L^{(x)} \circ L^{(-x)} = Id
$$

$$
I^{(x)} \circ I^{(y)} = I^{(x+y)} \quad L^{(x)} \circ L^{(y)} = L^{(x+y)}
$$

$$
I^{(x)} \circ \varepsilon = \varepsilon \circ I^{(-x)} \quad L^{(x)} \circ \varepsilon = \varepsilon \circ L^{(-x)}
$$

$$
I^{(x)} \circ L^{(y)} = L^{(y)} \circ I^{(x)}
$$

$$
I^{(x)} \circ \eta = \eta \circ L^{(-x)} \quad \eta \circ I^{(x)} = L^{(-x)} \circ \eta
$$

$$
\gamma \circ I^{(x)} \circ \gamma^{-1} = L^{(x)} \quad \gamma \circ L^{(x)} \circ \gamma^{-1} = I^{(x)}
$$

So we have seen, by the way, that the operators $I^{(x)}$ and $L^{(x)}$ are conjugated in the group $\Upsilon(R)$!
2 Action on linear recurrent sequences of order 2

In this section we analyze the action of $I(x)$ and $L(x)$ on the particular subset of $S(R)$ formed by linear recurrent sequences of order 2 (starting with 1).

**Definition 2.1.**

$$\mathcal{R}(R) = \{W(1, b, h, k) : b, h, k \in R\}$$

where

$$W(1, b, h, k) = \{W_n(1, b, h, k)\}_{n=0}^{+\infty}$$

satisfies the recurrence $\forall n \geq 2$

$$
\begin{cases}
W_0(1, b, h, k) = 1 \\
W_1(1, b, h, k) = b \\
W_n(1, b, h, k) = hW_{n-1}(1, b, h, k) - kW_{n-2}(1, b, h, k) & \forall n \geq 2.
\end{cases}
$$

$I(x)$ and $L(x)$ map $\mathcal{R}(R)$ into itself in the following way

**Theorem 2.2.** $\forall x, y \in R$ we have

$$I(x)(W(1, b, h, k)) = W(1, b + x, h + x, (h - b)x + k)$$

$$L(y)(W(1, b, h, k)) = W(1, b + y, h + 2y, y^2 + hy + k)$$

$$C(x,y)(W(1, b, h, k)) = W(1, b + y + x, h + x + 2y, y^2 + hy + k + (h - b)x + xy)$$

where $C(x,y) = I(x) \circ L(y) = L(y) \circ I(x)$.

**Proof.** The generating function of $W(1, b, h, k)$ is

$$W(t) = \frac{1 + (b - h)t}{1 - ht + kt^2}.$$ 

If we substitute $W(t)$ in (18), and compute $P(W(t))$, we find

$$\frac{1 + (b - h)t}{1 - (h + x)t + (k + (h - b)x)t^2}.$$ 

This proves (27). In the ring $R[z]/(z^2 - hz + k)$ we pose $\alpha_1 = z$ and $\alpha_2 = h - z$ (the roots of $z^2 - hz + k$). Then we have $W_n(1, b, h, k) = p\alpha_1^n + q\alpha_2^n$. 

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Substituting the sequence $\mathcal{W}(1, b, h, k)$ to the sequence $A$ in (19) we obtain

$$l_n = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} a_i = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} (p\alpha_1^i + q\alpha_2^i) =$$

$$= p \sum_{i=0}^{n} \binom{n}{i} y^{n-i} \alpha_1^i + q \sum_{i=0}^{n} \binom{n}{i} y^{n-i} \alpha_2^i = p(y + \alpha_1)^n + q(y + \alpha_2)^n.$$  

Then posing $y + \alpha_1 = R$ and $y + \alpha_2 = S$, observing that $R + S = 2y + \alpha_1 + \alpha_2 = h + 2y$  $RS = y^2 + (\alpha_1 + \alpha_2)y + \alpha_1\alpha_2 = y^2 + hy + k$ where we used $\alpha_1 + \alpha_2 = h$ and $\alpha_1\alpha_2 = k$, we have:

$$l_n = pR^n + qS^n + pR^{n-1}S + qS^{n-1}R - pR^{n-1}S - qS^{n-1}R =$$

$$= R(pR^{n-1} + qS^{n-1}) + S(pR^{n-1} + qS^{n-1}) - RS(pR^{n-2} + qS^{n-2}) =$$

$$= (R+S)(pR^{n-1} + qS^{n-1}) - RS(pR^{n-2} + qS^{n-2}) = (h+2y)l_{n-1} - (y^2 + hy + k)l_{n-2}.$$  

This proves (28).

(29) follows at once from (27) and (28).

**Observation 2.3.** An important subset $\mathcal{F} \subset \mathcal{R}(R)$ consists of sequences

$$F(h, k) = \mathcal{W}(1, h, h, k) = \{1, h, h^2 - k, ..\}$$

(31)

$$\begin{cases} 
W_0(1, h, h, k) = 1 \\
W_1(1, h, h, k) = h \\
W_n(1, h, h, k) = hw_{n-1}(1, h, h, k) - k\mathcal{W}_{n-2}(1, h, h, k) \quad \forall n \geq 2 .
\end{cases}$$

They are a subset of generalized Fibonacci sequences.

From Theorem 2.2 we can define a polynomial sequence $\mathcal{P}(h, k, x)$ as follows

(32)  
$$\mathcal{P}(h, k, x) = \{P_n(h, k, x)\}_{n=0}^{+\infty} = I^{(x)}(F(h, k)) = F(h + x, k)$$

and we can observe that

(33)  
$$I^{(h)}(\mathcal{W}(1, 0, 0, k)) = I^{(h)}(F(0, k)) = F(h, k).$$

These relations, as we will see in the next section, show a connection between $\mathcal{F}$ and orthogonal polynomials. They also help us to prove what Bacher [1] observes about arithmetical properties of $P_n(h, k, x)$.

**Proposition 2.4.**  \(\forall m, n \text{ such that } m|n \text{ then } P_{m-1}(h, k, x)|P_{n-1}(h, k, x).\)
Proof.

(32) gives the recurrence relation

\[
\begin{align*}
P_0(h, k, x) &= 1 \\
P_1(h, k, x) &= h + x \\
P_n(h, k, x) &= (h + x)P_{n-1}(h, k, x) - kP_{n-2}(h, k, x) \quad \forall n \geq 2
\end{align*}
\]

from which

\[
P_n(h, k, x) = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the roots of the characteristic polynomial

\[
t^2 - (h + x)t + k = 0.
\]

Thus

\[
\frac{P_{n-1}(h, k, x)}{P_{m-1}(h, k, x)} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1^m - \alpha_2^m}
\]

and if \( m|n \) then \( P_{m-1}(h, k, x)|P_{n-1}(h, k, x) \).

Finally we can find a couple of relations on sequences \( W(1, b, h, k) \) involving the \( \eta \) operator.

Corollary 2.5. For all sequences \( W(1, b, h, k) \) we have

\[
\begin{align*}
I(x)(\eta(W(1, b, h, k))) &= \eta(W(1, b - x, h - 2x, x^2 - hx + k)) \\
L(x)(\eta(W(1, b, h, k))) &= \eta(W(1, b - x, h - x, (b - h)x + k))
\end{align*}
\]

Proof. The proof is obvious from Theorem (1.16).

3 Moments generating function

From now on we shall pose \( R = \mathbb{C} \).

We know from (32) that \( I(x) \), applied to elements in \( \mathcal{F} \), gives rise to polynomial sequence \( \mathcal{P}(h, k, x) = \{ P_n(h, k, x) \}^{+\infty}_{n=-1} \) (where indexes have been changed in (32) for convenience in calculation), with recurrence relation

\[
\begin{align*}
P_{-1}(h, k, x) &= 0 \\
P_0(h, k, x) &= 1 \\
P_n(h, k, x) &= (x + h)P_{n-1}(h, k, x) - kP_{n-2}(h, k, x) \quad \forall n \geq 1
\end{align*}
\]
From Favard’s theorem ([5], page 21) this recurrence relation, when $k \neq 0$, is also the one for orthogonal polynomials having a proper moment functional. If $h = 0$ we have $P_n(0, k, x) = E_n(x, k)$ the $n$-th Dickson polynomial of the second kind [8]. Moreover for the moments sequence $\mu(h, k)$ related to the sequence $P(h, k, x)$ the following holds:

**Theorem 3.1.** The sequence $\mu(h, k)$ has generating function

\[
(38) \quad \mu(t) = \sum_{n=0}^{+\infty} \mu_n t^n = \frac{1 - ht - \sqrt{(1 - ht)^2 - 4kt^2}}{2kt^2}.
\]

**Proof.** From known results about orthogonal polynomials theory [5], the moments generating function $\mu(t)$ is equal to a continued fraction:

\[
(39) \quad \mu(t) = \sum_{n=0}^{+\infty} \mu_n t^n = \frac{\lambda_0}{1 + \xi_0 t - \frac{\lambda_1 t^2}{1 + \xi_1 t - \frac{\lambda_2 t^2}{1 + \xi_2 t - \frac{\lambda_3 t^2}{\ddots}}}}.
\]

For $P(h, k, x)$ we have $\forall n \xi_n = -h$, $\forall n > 1 \lambda_n = k$, $\lambda_0 = \mu_0 = 1$ and (39) becomes

\[
(40) \quad \mu(t) = \sum_{n=0}^{+\infty} \mu_n t^n = \frac{1}{1 - ht - \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \ddots}}}.
\]

It can be expressed in closed form posing

\[
y = \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \frac{kt^2}{1 - ht - \ddots}}}
\]

and observing that

\[
(41) \quad y = \frac{kt^2}{1 - ht - y}
\]

and

\[
(42) \quad \mu(t) = \frac{1}{1 - ht - y}.
\]

Finding $y$ from (41) we obtain

\[
y_1 = \frac{1 - ht + \sqrt{(1 - ht)^2 - 4kt^2}}{2}
\]
\[ y_2 = \frac{1 - ht - \sqrt{(1 - ht)^2 - 4kt^2}}{2}. \]

We have to choose \( y = y_2 \) because \( y_1 \) replaced in (42) gives rise to discontinuity at \( t = 0 \). With this value for \( y \) and a rationalization we easily find the exact form of \( \mu(t) \) in (38). □

The explicit moments values are given by the

**Corollary 3.2.** The moments \( \mu_n(h, k) \) related to polynomials \( P(h, k, x) \) are equal to

\[
(43) \quad \mu_n(h, k) = \begin{cases} 
-\frac{1}{2k} \sum_{j=0}^{n/2} \binom{n/2}{j} (-2h)^{n-2j}(h^2 - 4k)^j & n \text{ odd} \\
-\frac{1}{2k} \sum_{j=0}^{n/2} \binom{n/2}{j} (-2h)^{n-2j}(h^2 - 4k)^j & n \text{ even}
\end{cases}
\]

where \( n \geq 1 \) and \( \mu_0 = 1 \).

**Proof.** The result follows developing (38):

\[
\sqrt{(1 - ht)^2 - 4kt^2} = (1 - 2ht + (h^2 - 4k)t^2)^{1/2} = \sum_{i=0}^{+\infty} \binom{1/2}{i} (-2ht + (h^2 - 4k)t^2)^i
\]

and so

\[
\mu(t) = \frac{1}{2kt^2} \left( 1 - ht - (1 - ht) - \frac{1}{2}(h^2 - 4k)t^2 - \sum_{i=2}^{+\infty} \binom{1/2}{i} (-2ht + (h^2 - 4k)t^2)^i \right) = \\
= \frac{1}{4k}(h^2 - 4k) - \frac{1}{2k} \sum_{i=2}^{+\infty} \binom{1/2}{i} \sum_{j=0}^{i} \binom{i}{j} (-2h)^{i-j}(h^2 - 4k)^j t^{i-j-2}.
\]

Ordering the summation with respect to the degree \( n \) of \( t^n \), we observe that the coefficient of \( t^n \) for \( n = 0 \) is \(-h^2/2\) and replacing it in \( \mu(t) \) expression we have

\[
\mu(t) = 1 + \sum_{n=1}^{+\infty} \mu_n(h, k) t^n
\]

\[
\mu_n(h, k) = \begin{cases} 
-\frac{1}{2k} \sum_{j=0}^{n+1} \binom{1/2}{n+1-j} (-2h)^{n+2j}(h^2 - 4k)^j & \text{for odd } n \\
-\frac{1}{2k} \sum_{j=0}^{n+2} \binom{1/2}{n+2-j} (-2h)^{n+2j}(h^2 - 4k)^j & \text{for even } n
\end{cases}
\]

**Observation 3.3.** The moments \( \mu_n(h, k) \) are the generalized Motzkin numbers. We will show a combinatorial interpretation of them in Section 6.
4 Weight function

We want to find the weight function $\omega(t)$ of the functional $V$ related to the sequence $P(h, k, x)$ (see [5]). So $V[f]$ will be defined as follows

$$V[f] = \int_C f(t) d\psi(t)$$

where $C$ will be a suitable integration interval, $\psi(t)$ a distribution such that $\psi'(t) = \omega(t)$. By Stieltjes inversion formula we have

$$\psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^t \text{Im}(F(x + iy, h, k)) dx$$

being $z = x + iy \in \mathbb{C}$ and $F(z, h, k) = z^{-1} \mu(z^{-1})$ where $\mu(t)$ is defined by [38]; thus

$$F(z, h, k) = \frac{z - h - \sqrt{(z - h)^2 - 4k}}{2k}.$$

We can immediately find the corresponding primitive $F(z, h, k)$ of $F(z, h, k)$

$$F(z, h, k) = \frac{z - h - \sqrt{(z - h)^2 - 4k}}{2k}.$$

where the arbitrary constant has been made equal to 0, without loss of generality.

Now we can study, depending on $h, k$, the value of

$$\text{Im} \left( \lim_{y \to 0^+} F(x + iy, h, k) \right)$$

considering all the parts which summed together give $F$:

i. $$\lim_{y \to 0^+} \left( \frac{1}{2k} \left( \frac{z^2}{2} - h z \right) \right) = \frac{1}{2k} \left( \frac{x^2}{2} - h x \right)$$

ii. $$\lim_{y \to 0^+} \left( \frac{1}{4k} (z - h) \sqrt{(z - h)^2 - 4k} \right) = \frac{1}{4k} (x - h) \sqrt{(x - h)^2 - 4k}$$

iii. $$\lim_{y \to 0^+} \left( \log \left( \sqrt{(z - h)^2 - 4k} - (z - h) \right) \right) = \log \left( \sqrt{(x - h)^2 - 4k} - (x - h) \right)$$
remembering the condition $k \neq 0$, we note that:

i) is always real;

ii) is real if $(x - h)^2 - 4k \geq 0$ or $k < 0$, otherwise

$$\lim_{y \to 0^+} \left( \frac{1}{4k} (z - h) \sqrt{(z - h)^2 - 4k} \right) = \frac{i}{4k} (x - h) \sqrt{4k - (x - h)^2}$$

when $h - 2\sqrt{k} < x < h + 2\sqrt{k}$;

iii) if $k < 0$ or $k > 0$ and $x \notin (h - 2\sqrt{k}, h + 2\sqrt{k})$, $(x - h)^2 - 4k$ is real, moreover

$$\sqrt{(x - h)^2 - 4k} - (x - h) > 0$$

surely if $k < 0$, while if $k > 0$ and $x \notin (h - 2\sqrt{k}, h + 2\sqrt{k})$ the logarithm is real if $x \in (-\infty, h - 2\sqrt{k})$ and complex if $x \in (h + 2\sqrt{k}, +\infty)$.

In this ultimate case we have

$$\log \left( \sqrt{(x - h)^2 - 4k} - (x - h) \right) = \log \left( \left| \sqrt{(x - h)^2 - 4k} - (x - h) \right| + i\pi \right).$$

Finally if $k > 0$ and $x \in (h - 2\sqrt{k}, h + 2\sqrt{k})$ then

$$\sqrt{(x - h)^2 - 4k} = i\sqrt{4k - (x - h)^2}$$

and

$$\log \left( \sqrt{(x - h)^2 - 4k} - (x - h) \right) = \log \left( -(x - h) + i\sqrt{4k - (x - h)^2} \right) =$$

$$= \log \left( -(x - h) + i\sqrt{4k - (x - h)^2} \right) + i\text{Arg} \left( -(x - h) + i\sqrt{4k - (x - h)^2} \right)$$

with

$$\text{Arg} \left( -(x - h) + i\sqrt{4k - (x - h)^2} \right) = \begin{cases} -\arctan \left( \frac{\sqrt{4k - (x - h)^2}}{(x-h)} \right) & \text{if } h - 2\sqrt{k} < x < h \\ \frac{\pi}{2} & \text{if } x = h \\ \pi - \arctan \left( \frac{\sqrt{4k - (x - h)^2}}{(x-h)} \right) & \text{if } h < x < h + 2\sqrt{k} \end{cases}.$$

So the limit (10) is zero for $k < 0$ and also for $k > 0$ with $x \in (-\infty, h - 2\sqrt{k})$ while when $k > 0$ and $x \in (h - 2\sqrt{k}, +\infty)$ the limit values are

$$\begin{cases} \frac{(x - h)\sqrt{4k - (x - h)^2}}{4k} + \arctan \left( \frac{\sqrt{4k - (x - h)^2}}{(x-h)} \right) & \text{if } h - 2\sqrt{k} < x < h \\ -\frac{\pi}{2} & \text{if } x = h \\ -\frac{(x - h)\sqrt{4k - (x - h)^2}}{4k} - \pi + \arctan \left( \frac{\sqrt{4k - (x - h)^2}}{(x-h)} \right) & \text{if } h < x < h + 2\sqrt{k} \\ -\pi & \text{if } x > h + 2\sqrt{k} \end{cases}.$$
This gives, together with (44)

\[
\omega(t) = \psi'(t) = \begin{cases} 
\frac{\sqrt{4k - (t-h)^2}}{2k\pi} & \text{if } h - 2\sqrt{k} < t < h + 2\sqrt{k} \land t \neq h \\
0 & \text{otherwise}.
\end{cases}
\]

5 Recurrence relation for \(\mu(h, k)\)

We know from definition that

\[
\mu_n = \mathcal{V}[t^n] = \int_C t^n \, d\psi(t)
\]

and relation (48) becomes, using (47)

\[
\mu_n = \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^n \sqrt{4k - (t-h)^2}}{2k\pi} \, dt.
\]

Now we can prove the

**Theorem 5.1.** The sequence \(\mu(h, k)\) is recurrent with

\[
\begin{align*}
\mu_0 &= 1 \\
\mu_1 &= h \\
\mu_n &= \frac{h(2n+1)\mu_{n-1} - (h^2 - 4k)(n-1)\mu_{n-2}}{n+2} & \forall n \geq 2.
\end{align*}
\]

**Proof.** \(\forall n \geq 2\) we have

\[
\mu_n = \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^n - (t-h)\sqrt{4k - (t-h)^2}}{2k\pi} \, dt = \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{t^n - (t-h)\sqrt{4k - (t-h)^2}}{2k\pi} \, dt + h\mu_{n-1}
\]

using integration by parts we obtain

\[
\mu_n = \left[ \frac{-t^{-n-1}(4k - (t-h)^2)^{3/2}}{6k\pi} \right]_{h-2\sqrt{k}}^{h+2\sqrt{k}} - \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{-n-2}{6k\pi} \sqrt{4k - (t-h)^2} \, dt + h\mu_{n-1}
\]

but

\[
\left[ \frac{-t^{-n-1}(4k - (t-h)^2)^{3/2}}{6k\pi} \right]_{h-2\sqrt{k}}^{h+2\sqrt{k}} = 0
\]

so

\[
\mu_n = \frac{(n-1)}{3} \int_{h-2\sqrt{k}}^{h+2\sqrt{k}} \frac{(h^2 - 4k + t^2 - 2ht)\sqrt{4k - (t-h)^2}}{2k\pi} \, dt + h\mu_{n-1} =
\]

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\[
\begin{align*}
= \frac{(h^2 - 4k)(n-1)}{3} \mu_{n-2} - \frac{(n-1)}{3} \mu_n + \frac{2h(n-1)}{3} \mu_{n-1} + h \mu_{n-1} = \\
= \frac{(h^2 - 4k)(n-1)}{3} \mu_{n-2} - \frac{(n-1)}{3} \mu_n + \frac{h(2n+1)}{3} \mu_{n-1}
\end{align*}
\]

we finally find the recurrence

\[
\mu_n = \frac{h(2n+1)\mu_{n-1} - (h^2 - 4k)(n-1)\mu_{n-2}}{n+2}
\]

while \( \mu_0 \) and \( \mu_1 \) can be easily found calculating (49) for \( n = 0, 1 \).

**Corollary 5.2.** We have \( \forall y \in \mathbb{R} \ L^{(y)}(\mu(h,k)) = \mu(h+y,k) \).

**Proof.** In fact if

\[
\mu'_n = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} \mu_i
\]

using (49) we have

\[
\mu'_n = \int_{h+2\sqrt{k}}^{h+2\sqrt{k}} \sum_{i=0}^{n} \binom{n}{i} y^{n-i} i \sqrt{\frac{4k - (t-h)^2}{2k\pi}} dt
\]

Now

\[
\sum_{i=0}^{n} \binom{n}{i} y^{n-i} t^i = (t + y)^n
\]

and substituting \( u = t + y \) and \( h' = h + y \)

\[
\mu'_n = \int_{h'+2\sqrt{k}}^{h'+2\sqrt{k}} u^n \sqrt{\frac{4k - (u-h')^2}{2k\pi}} du
\]

Thus \( \mu'_n \) is defined with an analogous relation like (49) for \( \mu_n \).

**6 Combinatorial interpretation for \( \mu_n(h,k) \)**

We consider a lattice \((n+1) \times (n+1)\) composed by all the points having non negative integer coordinates. Motzkin paths are all the courses starting from \((0,0)\) and reaching \((n,0)\) with the following rules

\[
\begin{align*}
(i,j) & \rightarrow (i+1,j) \quad \text{horizontal shift to east} \\
(i,j) & \rightarrow (i+1,j+1) \quad \text{diagonal shift to north-east} \\
(i,j) & \rightarrow (i+1,j-1) \quad \text{diagonal shift to south-east}
\end{align*}
\]
For example from (0, 0) to (3, 0) we have only the 4 possible paths

\[\begin{align*}
(0, 0) &\rightarrow (1, 0) &\rightarrow (2, 0) &\rightarrow (3, 0) &\rightarrow (1, 1) &\rightarrow (2, 1) &\rightarrow (3, 0) \\
(0, 0) &\rightarrow (1, 1) &\rightarrow (2, 1) &\rightarrow (3, 0) &\rightarrow (0, 0) &\rightarrow (1, 1) &\rightarrow (2, 0) &\rightarrow (3, 0)
\end{align*}\]

If we weight one shift of a path \( P \) posing:

\[
\begin{cases}
  w((i, j) \rightarrow (i + 1, j)) = h \\
  w((i, j) \rightarrow (i + 1, j + 1)) = 1 \\
  w((i, j) \rightarrow (i + 1, j - 1)) = k
\end{cases}
\]

we can describe \( P \) with weights product. The four paths represented above are respectively represented by: \( h^3, hk, hk, kh \). We observe that the sum of all the weights of these paths from \((0,0)\) to \((3,0)\) is \(h^3 + 3hk = \mu_3(h, k)\). This is a consequence of the

**Theorem 6.1** (Viennot’s Theorem [12]). *Under the rules described above, for every Motzkin path \( P \) the following relation holds*

\[
\mu_n(h, k) = \sum_{P:(0,0)\rightarrow(n,0)} w(P).
\]

As a consequence in \( \mu_n(h, k) \) is codified information about all weighted paths from \((0,0)\) to \((n,0)\):

- the sum of coefficients of \( \mu_n(h, k) \) gives the number of all possible Motzkin paths from \((0,0)\) to \((n,0)\);
- the \( h \) exponent in every term gives the number of horizontal shifts to east;
- the \( k \) exponent in every term gives the number of diagonal shifts to north-east (or to south-east);
- the weight \( h \) may be interpreted as the number of colors among which we can select one to draw horizontal shifts to east;
- the weight \( k \) may be interpreted as the number of colors among which we can select one to draw diagonal shifts to north-east or to south-east;

**Example 6.2.**

From \( \mu_4(h, k) = h^4 + 6h^2k + 2k^2 \) we have \( 9 = 1 + 6 + 2 = \mu_4(1, 1) \) distinct paths from \((0,0)\) to \((4,0)\) traced with one color for all shifts:

- 1 path having \( 4 \) horizontal shifts;
• 6 with 2 horizontal shifts and 1 to north-east (and so 1 to south-east);
• 2 with 2 diagonal shifts to north-east (and so 2 to south-east).

Moreover from \( \mu_3(1, 1) = 4 \) we recover the previous one-colored paths and from \( \mu_3(1, 2) = 7 \) we find all the paths painted with one color for horizontal shifts and two possible colors for diagonal shifts:

\[
\begin{align*}
(0, 0) & \rightarrow (1, 0) \rightarrow (2, 0) \rightarrow (3, 0) \rightarrow (0, 0) \rightarrow (1, 0) \rightarrow (2, 1) \rightarrow (3, 0) \\
(0, 0) & \rightarrow (1, 0) \rightarrow (2, 1) \rightarrow (3, 0) \rightarrow (0, 0) \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow (3, 0) \\
(0, 0) & \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow (3, 0) \rightarrow (0, 0) \rightarrow (1, 1) \rightarrow (2, 0) \rightarrow (3, 0) \\
(0, 0) & \rightarrow (1, 1) \rightarrow (2, 0) \rightarrow (3, 0)
\end{align*}
\]

7 The action of \( \eta \)

We begin with an example

**Example 7.1.** If we consider \( \mu(h, k) \) (in this section we take, without loss of generality, \( -h \) instead of \( h \)) we have from (38)

\[
u = 1 + ht - \sqrt{(1 + ht)^2 - 4kt^2}
\]

which solved as an equation in \( t \) gives

\[
t = \frac{u}{ku^2 - hu + 1} = \sum_{n=0}^{\infty} F_n(h, k)u^{n+1}.
\]

We used (30) with \( b = h \) obtaining \( F(h, k) = \{F_n(h, k)\}_{n=0}^{\infty} \), the generalized Fibonacci sequence. So we note that

\[
\left\{ \begin{array}{l}
\eta(F(h, k)) = \mu(h, k) \\
\eta(\mu(h, k)) = F(h, k)
\end{array} \right.
\]

**Observation 7.2.** Recalling the Corollary 2.5 we have an alternative way to find the relation proved in Corollary 5.2. When \( b = h \)

\[
\mathcal{W}(1, h, h) = F(h, k)
\]

and from example (7.1)

\[
F^{(x)}(\mu(h, k)) = \eta(\mathcal{W}(1, h - x, h - 2x, x^2 - hx + k))
\]

\[
L^{(x)}(\mu(h, k)) = \eta(\mathcal{W}(1, h - x, h - x, k)) = \eta(F(h - x, k)) = \mu(h - x, k)
\]

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The terms of $B = \eta(A)$, can be expressed by means of Lagrange inversion formula [7].

\begin{equation}
(51) \quad b_n = \frac{1}{(n+1)!} \frac{d^n}{du^n} \left\{ \left[ \frac{u}{t(u)} \right]^{n+1} \right\} \bigg|_{u=0} .
\end{equation}

Using Lagrange inversion formula we can find an analogous expression of (43) for $\mu_n(h,k)$. In fact

\[ t(u) = \frac{u}{ku^2 - hu + 1} \]

thus

\[ \frac{u}{t(u)} = ku^2 - hu + 1 \]

and

\[ \mu_n(h,k) = b_n = \frac{1}{(n+1)!} \frac{d^n}{du^n} \left\{ (ku^2 - hu + 1)^{n+1} \right\} \bigg|_{u=0} . \]

The trinomial expansion gives

\[ (ku^2 - hu + 1)^{n+1} = \sum_{p+q+r=n+1} \frac{(n+1)!}{p!q!r!} (-hu)^r (ku^2)^q \]

and so

\[ (ku^2 - hu + 1)^{n+1} = \sum_{p+q=0}^{n+1} \frac{(n+1)!}{p!(n+1-p-q)!q!} (-h)^{n+1-p-q} k^q u^{n+1-p+q} . \]

Differentiating $n$ times

\[ \frac{d^n}{du^n} \left\{ (ku^2 - hu + 1)^{n+1} \right\} = \sum_{p+q=0}^{n+1} \frac{(n+1)!}{p!(n+1-p-q)!q!} (-h)^{n+1-p-q} k^q u^{n-p+q} . \]

For $u = 0$ the only non zero term occurs when $q = p - 1$. Consequently $p + q = 2p - 1$ and $0 \leq p + q \leq n + 1$ implies $1 \leq p \leq \left\lfloor \frac{n+2}{2} \right\rfloor$ then

\[ \mu_n(h,k) = \sum_{p=1}^{\left\lfloor \frac{n+2}{2} \right\rfloor} \frac{n!}{p!(n-2p+2)!(p-1)!} (-h)^{n-2p+2} k^{p-1} . \]

Taking $h = -x$ and considering odd and even values for $n$, we have

\[ \left\{ \begin{array}{ll}
\mu_n(-x,k) = \sum_{j=0}^{n-1} A_j^{(n)} x^{2j+1} & n \text{ odd} \\
\mu_n(-x,k) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} A_j^{(n)} x^{2j} & n \text{ even} .
\end{array} \right. \]
where

\[ A_j^{(n)} = \begin{cases} 
\frac{1}{n+1} \left( \frac{n+1}{2} - j, 2j + 1, \frac{n-1}{2} - j \right) k^{\frac{n-1}{2} - j} & \text{n odd} \\
\frac{1}{n+1} \left( \frac{n}{2} + 1 - j, 2j, \frac{n}{2} - j \right) k^{\frac{n}{2} - j} & \text{n even}.
\end{cases} \]

**Observation 7.3.**
If \( x = 0 \) for \( n = 2m \) we have
\[ \mu_{2m}(0, k) = k^m C_m \quad C_m = \frac{1}{2m + 1} \binom{2m + 1}{m} \]
and \( C_m \) is the \( m \)-th Catalan number, while if \( n = 2m + 1 \) we have
\[ \mu_{2m+1}(0, k) = 0. \]

**Observation 7.4 (Orthogonality relations).** We consider the polynomial \( P_n(x) = \mathcal{W}(1, h + x, \tilde{h} + x, k) \). Its explicit expression can be found observing that from (30) we have
\[ \sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{1 - (h + x)t + kt^2} = \sum_{j=0}^{\infty} ((h + x)t - kt^2)^j = \sum_{j=0}^{\infty} j! \binom{j}{l} (h + x)^{j-l} (-k)^l t^{j+1}. \]
Rearranging indexes and posing \( l + j = n \) we obtain
\[ P_n(x) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} (h + x)^{n-2l} (-k)^l. \]
and the generic coefficient \( P_j^{(n)} \) of \( x^j \) follows from the \( j \)-th derivative:
\[ P_j^{(n)} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \binom{n-2l}{j} (h + x)^{n-2l-j} (-k)^l. \]
Now from definition of the functional \( \mathcal{V} \)
\[ \begin{cases} 
\mathcal{V}[1] = 1 \\
\mathcal{V}[P_m(x)P_n(x)] = 0 \quad \text{if} \quad m \neq n
\end{cases} \]
from (37) \( P_0(x) = 1 \) and from (56) choosing \( m = 0 \) we have
\[ \mathcal{V}[P_n(x)] = \delta(n, 0) \]
which in this case becomes the following relation
\[ \sum_{j=0}^{n} P_j^{(n)} \mu_j(h, k) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \binom{n-2l}{j} (h + x)^{n-2l-j} (-k)^l \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \frac{j!(-h)^j-2p+2k^{p-1}}{p!(j-2p+2)!(p-1)!} = \delta(n, 0). \]
And when \( h = 0 \) we have

\[
P_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (-k)^i x^{n-2i} = E_n(x, k) \quad .
\]

where \( E_n(x, k) \) is the \( n \)-th Dickson polynomial of the second kind \[8\].

So if \( n = 2m \), recalling that \( \mu_{2m}(0, k) = k^mC_m \), we obtain a similar orthogonality relation where Catalan numbers are involved

\[
\sum_{i=0}^{m} \binom{2m-i}{i} (-k)^i k^{m-i}C_{m-i} = \delta(m, 0) \quad .
\]

As far as we know this Catalan identity is new. Of course \[5\] is not difficult to prove (try it with Zeilberger’s program \[9\], for example), but it seems interesting also for the context it rises from.

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