FINITE GROUP ACTIONS AND G-MONOPOLE CLASSES
ON SMOOTH 4-MANIFOLDS

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ABSTRACT. On a smooth closed oriented 4-manifold $M$ with a smooth action by a compact Lie group $G$, we define a $G$-monopole class as an element of $H^2(M; \mathbb{Z})$ which is the first Chern class of a $G$-equivariant Spin$^c$ structure which has a solution of the Seiberg-Witten equations for any $G$-invariant Riemannian metric on $M$.

We find $\mathbb{Z}_k$-monopole classes on some $\mathbb{Z}_k$-manifolds such as the connected sum of $k$ copies of a 4-manifold with nontrivial mod 2 Seiberg-Witten invariant or Bauer-Furuta invariant, where the $\mathbb{Z}_k$-action is a cyclic permutation of $k$ summands.

As an application, we produce infinitely many exotic non-free actions of $\mathbb{Z}_k \oplus H$ on some connected sums of finite number of $S^2 \times S^2$, $\mathbb{C}P_2$, $\overline{\mathbb{C}P}_2$, and $K3$ surfaces, where $k \geq 2$, and $H$ is any nontrivial finite group acting freely on $S^3$.

1. INTRODUCTION

Let $M$ be a smooth closed oriented manifold of dimension 4. A second cohomology class of $M$ is called a monopole class if it arises as the first Chern class of a Spin$^c$ structure for which the Seiberg-Witten equations

$$\begin{cases}
D_A \Phi = 0 \\
F_A^+ = \Phi \otimes \Phi^* - \frac{1}{2} |\Phi|^2 \text{Id},
\end{cases}$$

admit a solution for every choice of a Riemannian metric. Clearly a basic class, i.e. the first Chern class of a Spin$^c$ structure with a nonzero Seiberg-Witten invariant is a monopole class. However, ordinary Seiberg-Witten invariants which are gotten by the intersection theory on the moduli space of solutions $(A, \Phi)$ of the above equations is trivial in many important cases, for example connected sums of 4-manifolds with $b_2^+ > 0$.

Bauer and Furuta [7, 8] made a breakthrough in detecting a monopole class on certain connected sums of 4-manifolds. Their new brilliant idea is...
to generalize the Pontryagin-Thom construction to a proper map between infinite-dimensional spaces, which is the sum of a linear Fredholm map and a compact map, and take some sort of a stably-framed bordism class of the Seiberg-Witten moduli space as an invariant. However its applications are still limited in that this new invariant which is expressed as a stable cohomotopy class is difficult to compute, and we are seeking after further refined invariants of the Seiberg-Witten moduli space.

In the meantime, sometimes we need a solution of the Seiberg-Witten equations for a specific metric rather than any Riemannian metric. The case we have in mind is the one when a manifold $M$ and its Spin$^c$ structure $\mathfrak{s}$ admit a smooth action by a compact Lie group $G$ and we are concerned with finding a solution of the Seiberg-Witten equations for any $G$-invariant metric.

Thus for a $G$-invariant metric on $M$ and a $G$-invariant perturbation of the Seiberg-Witten equations, we consider the $G$-monopole moduli space $\mathfrak{X}$ consisting of their $G$-invariant solutions modulo gauge equivalence. One can easily see that the ordinary moduli space $\mathfrak{M}$ is acted on by $G$, and $\mathfrak{X}$ turns out to be a subset of its $G$-fixed point set. The intersection theory on $\mathfrak{X}$ will give the $G$-monopole invariant $SW^G_{M,\mathfrak{s}}$, defined first by Y. Ruan [33], which is expected to be different from the ordinary Seiberg-Witten invariant $SW_{M,\mathfrak{s}}$.

In view of this, the following definition is relevant.

**Definition 1.** Suppose that $M$ admits a smooth action by a compact Lie group $G$ preserving the orientation of $M$.

A second cohomology class of $M$ is called a $G$-monopole class if it arises as the first Chern class of a $G$-equivariant Spin$^c$ structure for which the Seiberg-Witten equations admit a $G$-invariant solution for every $G$-invariant Riemannian metric of $M$.

When the $G$-monopole invariant is nonzero, its first Chern class has to be a $G$-monopole class. As explain in [38], the cases we are aiming at are those for finite $G$. If a compact connected Lie group $G$ has positive dimension and is not a torus $T^n$, then $G$ contains a Lie subgroup isomorphic to $S^3$ or $S^3/\mathbb{Z}_2$, and hence $M$ admitting an effective action of such $G$ must have a $G$-invariant metric of positive scalar curvature by the well-known Lawson-Yau theorem [25] so that $M$ with $b^2_2(M) > 1$ has no $G$-monopole class. On the other hand, the Seiberg-Witten invariants of a 4-manifold with an effective $S^1$ action were extensively studied by S. Baldridge [4, 5, 6].

As another invariant for detecting a $G$-monopole class, one can also generalize the Bauer-Furuta invariant $BF_{M,\mathfrak{s}}$ in [7, 8] to the so-called $G$-Bauer-Furuta invariant $BF^G_{M,\mathfrak{s}}$, which is roughly the stable cohomotopy class of the monopole map between $G$-invariant subsets of the associated Hilbert bundles.
over $T^{b_1(M)}$ where $b_1(M)^G$ is the dimension of the space of $G$-invariant harmonic 1-forms on $M$. This should be distinguished from the $G$-equivariant Bauer-Furuta invariant of $[39, 30]$, which is the stable $G$-homotopy class of the monopole map between the original Hilbert bundles over the Picard torus $T^{b_1(M)}$.

Using these invariants, we find $G$-monopole classes in some connected sums which have vanishing Seiberg-Witten invariants:

**Theorem 1.1.** Let $M$ and $N$ be smooth closed oriented 4-manifolds satisfying $b_2^+(M) > 1$ and $b_2^+(N) = 0$, and $\tilde{M}_k$ for any $k \geq 2$ be the connected sum $\tilde{M}_1 \# \cdots \# M \# N$ where there are $k$ summands of $M$.

Suppose that $N$ admits a smooth orientation-preserving $\mathbb{Z}_k$-action with at least one free orbit such that there exist a $\mathbb{Z}_k$-invariant Riemannian metric of positive scalar curvature and a $\mathbb{Z}_k$-equivariant Spin$^c$ structure $s_N$ with $c_1^2(s_N) = -b_2(N)$.

Define a $\mathbb{Z}_k$-action on $\tilde{M}_k$ induced from that of $N$ and the cyclic permutation of the $k$ summands of $M$ glued along a free orbit in $N$, and let $\tilde{s}$ be the Spin$^c$ structure on $\tilde{M}_k$ obtained by gluing $s_N$ and a Spin$^c$ structure $s$ of $M$.

Then for any $\mathbb{Z}_k$-action on $\tilde{s}$ covering the above $\mathbb{Z}_k$-action on $\tilde{M}_k$, $\text{SW}_{\tilde{M}_k, \tilde{s}}$ mod 2 is nontrivial if $\text{SW}_{M, s}$ mod 2 is nontrivial, and also $\text{BF}_{\tilde{M}_k, \tilde{s}}$ is nontrivial, if $\text{BF}_{M, s}$ is nontrivial.

As an application of a $G$-monopole class to differential topology, we can detect exotic smooth group actions on some smooth 4-manifolds, by which we mean topologically equivalent but smoothly inequivalent actions. We say that two smooth group actions $G_1$ and $G_2$ on a smooth manifold $M$ is $C^m$-equivalent for $m = 0, 1, \cdots, \infty$, if there exists a $C^m$-homeomorphism $f : M \rightarrow M$ such that

$$G_1 = f \circ G_2 \circ f^{-1}.$$ 

Such exotic smooth actions of finite groups on smooth 4-manifolds have been found abundantly, for eg, [15, 9, 17, 2, 18, 42, 16]. But all of them are either free or cyclic actions.

In the final section, we use $G$-monopole invariants to find infinitely many non-free non-cyclic exotic group actions. For example, for $k \geq 2$ and any nontrivial finite group $H$ acting freely on $S^3$, there exist infinitely many exotic non-free actions of $\mathbb{Z}_k \oplus H$ on some connected sums of finite numbers of $S^3 \times S^2$, $\mathbb{C}P_2$, $\mathbb{C}P_2$, and K3 surfaces.

The above theorem may be generalized to other types of $N$, and we leave a more complete study to a future project. Applications to Riemannian geometry such as $G$-invariant Einstein metrics and $G$-Yamabe invariant are
rather straightforward from the curvature estimates, and are dealt with in [38].

2. \(G\)-monopole invariant

Let \(M\) be a smooth closed oriented 4-manifold. Suppose that a compact Lie group \(G\) acts on \(M\) smoothly preserving the orientation, and this action lifts to an action on a Spin\(^c\) structure \(s\) of \(M\). Once there is a lifting, any other lifting differs from it by an element of \(\text{Map}(G \times M, S^1)\). We fix a lifting and put a \(G\)-invariant Riemannian metric \(g\) on \(M\).

The corresponding spinor bundles \(W_\pm\) are also \(G\)-equivariant, and we let \(\mathcal{A}(W_\pm)^G\) be the set of its \(G\)-invariant sections. When we put \(G\) as a superscript on a set, we always mean the subset consisting of its \(G\)-invariant elements. Thus \(\mathcal{A}(W_\pm)^G\) is the space of \(G\)-invariant connections on \(\det(W_\pm)\), which is identified as the space of \(G\)-invariant purely-imaginary valued 1-forms \(\Gamma(\Lambda^2(M; i\mathbb{R}))^G\), and \(\mathcal{G}^G = \text{Map}(M, S^1)^G\) is the set of \(G\)-invariant gauge transformations.

The perturbed Seiberg-Witten equations give a smooth map

\[
H : \mathcal{A}(W_+)^G \times \Gamma(W_+)^G \longrightarrow \Gamma(W_-)^G \times \Gamma(\Lambda^2(M; i\mathbb{R}))^G
\]

defined by

\[
H(A, \Phi, \varepsilon) = (D_A\Phi, F_A^+ - \Phi \otimes \Phi^* + \frac{|\Phi|^2}{2}\text{Id} + \varepsilon),
\]

where the domain and the range are endowed with \(L^2_{l+1}\) and \(L^2_l\) Sobolev norms for a positive integer \(l\) respectively, and \(D_A\) is a Spin\(^c\) Dirac operator.

The \(G\)-monopole moduli space \(\mathcal{X}_\varepsilon\) for a perturbation \(\varepsilon\) is then defined as

\[
\mathcal{X}_\varepsilon := H^{-1}_\varepsilon(0)/\mathcal{G}^G
\]

where \(H_\varepsilon\) denotes \(H\) restricted to \(\mathcal{A}(W_+)^G \times \Gamma(W_+)^G \times \{\varepsilon\}\).

Lemma 2.1. The quotient map

\[
p : (\mathcal{A}(W_+)^G \times (\Gamma(W_+)^G - \{0\}))/\mathcal{G}^G \rightarrow (\mathcal{A}(W_+)^G \times (\Gamma(W_+)^G - \{0\}))/\mathcal{G}
\]

is bijective, and hence \(\mathcal{X}_\varepsilon\) is a subset of the ordinary Seiberg-Witten moduli space \(\mathfrak{M}_\varepsilon\).

Proof. Obviously \(p\) is surjective, and to show that \(p\) is injective, suppose that \((A_1, \Phi_1)\) and \((A_2, \Phi_2)\) in \(\mathcal{A}(W_+)^G \times (\Gamma(W_+)^G - \{0\})\) are equivalent under \(\gamma \in \mathcal{G}\). Then

\[
A_1 = A_2 - 2d \ln \gamma, \quad \Phi_1 = \gamma \Phi_2.
\]

By the first equality, \(d \ln \gamma\) is \(G\)-invariant.
Let $S$ be the subset of $M$ where $\gamma$ is $G$-invariant. By the continuity of $\gamma$, $S$ must be a closed subset. Since $S$ contains a nonempty subset

$$\{ x \in M \vert \Phi_1(x) \neq 0 \},$$

$S$ is nonempty. It suffices to show that $S$ is open. Let $x_0 \in S$. Then we have that for any $g \in G$,

$$g^* \ln \gamma(x_0) = \ln \gamma(x_0), \quad \text{and} \quad g^* d \ln \gamma = d \ln \gamma,$$

which implies that $g^* \ln \gamma = \ln \gamma$ on an open neighborhood of $x_0$ on which $g^* \ln \gamma$ and $\ln \gamma$ are well-defined. By the compactness of $G$, there exists an open neighborhood of $x_0$ on which $g^* \ln \gamma$ is well-defined for all $g \in G$, and $\ln \gamma$ is $G$-invariant. This proves the openness of $S$.

The transversality is obtained by a generic $G$-invariant self-dual 2-form:

**Lemma 2.2.** $H$ is a submersion at each $(A, \Phi, \varepsilon) \in H^{-1}(0)$ for nonzero $\Phi$.

**Proof.** Obviously $dH$ restricted to the last factor of the domain is onto the last factor of the range. Using the surjectivity in the ordinary setting, for any element $\psi \in \Gamma(W_-)^G$, there exists an element $(a, \varphi) \in \mathcal{A}(W_+) \times \Gamma(W_+)$ such that $dH(a, \varphi, 0) = \psi$. The average

$$(\tilde{a}, \tilde{\varphi}) = \int_G h^*(a, \varphi) \, d\mu(h)$$

using a unit-volume $G$-invariant metric on $G$ is an element of $\mathcal{A}(W_+)^G \times \Gamma(W_+)^G$. It follows from the smoothness of the $G$-action that every $h^*(a, \varphi)$ and hence $(\tilde{a}, \tilde{\varphi})$ belong to the same Sobolev space as $(a, \varphi)$. Moreover it still satisfies

$$dH(\tilde{a}, \tilde{\varphi}, 0) = \int_G dH(h^*(a, \varphi, 0)) \, d\mu(h) = \int_G h^* dH((a, \varphi, 0)) \, d\mu(h) = \int_G h^* \psi \, d\mu(h) = \psi,$$

where we used the fact that $dH$ is a $G$-equivariant differential operator. This completes the proof.

We will assume that $b_2^+(M)^G$ which is the dimension of the space of $G$-invariant self-dual harmonic 2-forms of $M$ is nonzero. Then $\mathcal{X}_\varepsilon$ consists of irreducible solutions, and is a smooth manifold for generic $\varepsilon$ by the Sard-Smale theorem. From now on, we will always assume that a generic $\varepsilon$ is
chosen so that $X_\varepsilon$ is smooth, and often omit the notation of $\varepsilon$, if no confusion arises.

Its dimension and orientation can be read from the $G$-equivariant elliptic complexes
\[ d^+ + 2d^* : \Gamma(\Lambda^1)^G \to \Gamma(\Lambda^0 \oplus \Lambda^2)^G \]
and
\[ D_A : \Gamma(W_+)^G \to \Gamma(W_-)^G \]
using the index theory in the same way as the ordinary Seiberg-Witten moduli space.

**Theorem 2.3.** When $G$ is finite, $X_\varepsilon$ for any $\varepsilon$ is compact.

**Proof.** Following the proof of the ordinary Seiberg-Witten moduli space, we need the $G$-equivariant version of the gauge fixing lemma.

**Lemma 2.4.** Let $L$ be a $G$-equivariant complex line bundle over $M$ with a hermitian metric, and $A_0$ be a fixed $G$-invariant smooth unitary connection on it.

Then for any $l \geq 0$ there are constants $K, C > 0$ depending on $A_0$ and $l$ such that for any $G$-invariant $L^2_l$ unitary connection $A$ on $L$ there is a $G$-invariant $L^2_{l+1}$ change of gauge $\sigma$ so that $\sigma^*(A) = A_0 + \alpha$ where $\alpha \in L^2_l(T^*M \otimes i\mathbb{R})^G$ satisfies
\[ d^*\alpha = 0, \quad \text{and} \quad ||\alpha||_{L^2_l}^2 \leq C ||F_A^+||_{L^2_{l-1}}^2 + K. \]

**Proof.** We know that a gauge-fixing $\sigma$ with the above estimate always exists, but we need to prove the existence of $G$-invariant $\sigma$. Write $A$ as $A_0 + a$ where $a \in L^2_l(T^*M \otimes i\mathbb{R})^G$. Let $a = a^{\text{harm}} + df + d^*\beta$ be the Hodge decomposition. By the $G$-invariance of $a$, so are $a^{\text{harm}}, df$, and $d^*\beta$. Applying the ordinary gauge fixing lemma to $A_0 + d^*\beta$, we have
\[ ||d^*\beta||^2_{L^2_l} \leq C' ||F_{A_0 + d^*\beta}^+||^2_{L^2_{l+1}} + K' = C' ||F_A^+||^2_{L^2_{l+1}} + K' \]
for some constants $C', K' > 0$. Defining a $G$-invariant $i\mathbb{R}$-valued function $f_{av} = \frac{1}{|G|} \sum_{g \in G} g^* f$, we have
\[ df = \frac{1}{|G|} \sum_{g \in G} g^* df = d(f_{av}) = -d \ln \exp(-f_{av}), \]
and hence $df$ can be gauged away by a $G$-invariant gauge transformation $\exp(-f_{av})$. Write $a^{\text{harm}}$ as $(n|G| + m)a^h$ for $m \in [0, |G|)$ and an integer $n \geq 0$, where $a^h \in H^1(M; \mathbb{Z})^G$ is not a positive multiple of another element.
of $H^1(M; \mathbb{Z})^G$. There exists $\sigma \in G$ such that $a^h = -d \ln \sigma$. In general $\sigma$ is not $G$-invariant, but
\[ |G|a^h = \sum_{g \in G} g^*a^h = -d \ln \prod_{g \in G} g^*\sigma, \]
and hence $n|G|a^h$ can be gauged away by a $G$-invariant gauge transformation $\prod_{g \in G} g^*\sigma^n$. In summary, $A_0 + a$ is equivalent to $A_0 + ma^h + d^*\beta$ after a $G$-invariant gauge transformation, and
\[
||ma^h + d^*\beta||^2_{L^2_t} \leq (||ma^h||^2_{L^2_t} + ||d^*\beta||^2_{L^2_t})^2 \\
\leq |G|^2||a^h||^2_{L^2_t} + 2|G||a^h||d^*\beta||^2_{L^2_t} + ||d^*\beta||^2_{L^2_t} \\
\leq 3|G|^2||a^h||^2_{L^2_t} + 3||d^*\beta||^2_{L^2_t} \\
= K'' + 3C'||F'^+_{A_t}||^2_{L^2_{t-1}} + 3K'
\]
for a constant $K'' > 0$. This completes the proof. \(\Box\)

Now the rest of the compactness proof proceeds in the same way as the ordinary case using the Weitzenböck formula and standard elliptic and Sobolev estimates. For details the readers are referred to [27]. \(\Box\)

**Remark** If $G$ is not finite, $X_\varepsilon$ may not be compact.

For example, consider $M = S^1 \times Y$ with the trivial Spin$^c$ structure and its obvious $S^1$ action, where $Y$ is a closed oriented 3-manifold. For any $n \in \mathbb{Z}$, $nd\theta$ where $\theta$ is the coordinate along $S^1$ is an $S^1$-invariant reducible solution. Although $nd\theta$ is gauge equivalent to 0, but never via an $S^1$-invariant gauge transformation which is an element of the pull-back of $C^\infty(Y, S^1)$. Therefore as $n \to \infty$, $nd\theta$ diverges modulo $G^{S^1}$, which proves that $X_0$ is non-compact. \(\Box\)

In the rest of this paper, we assume that $G$ is finite. Then note that $G$ induces smooth actions on
\[ \mathcal{C} := \mathcal{A}(W_+) \times \Gamma(W_+), \]
\[ \mathcal{B}^* = (\mathcal{A}(W_+) \times (\Gamma(W_+) - \{0\})) / \mathcal{G}, \]
and also the Seiberg-Witten moduli space $\mathfrak{M}$ whenever it is smooth.

Since $X_\varepsilon$ is a subset of $\varepsilon$-perturbed moduli space $\mathfrak{M}_\varepsilon$, (actually a subset of the fixed locus $\mathfrak{M}_\varepsilon^G$ of a $G$-space $\mathfrak{M}_\varepsilon$), we can define the $G$-monopole invariant $SW_{M,\varepsilon}^G$ by integrating the same cohomology classes as in the ordinary Seiberg-Witten invariant $SW_{M,\varepsilon}$. Thus using the $\mathbb{Z}$-algebra isomorphism
\[ \mu_{M,\varepsilon} : \mathbb{Z}[H_0(M; \mathbb{Z})] \otimes \wedge^* H_1(M; \mathbb{Z}) / \text{torsion} \rightarrow H^*(\mathcal{B}^*; \mathbb{Z}), \]
we define it as a function
\[ SW^G_{M,s} : \mathbb{Z}[H_0(M;\mathbb{Z})] \otimes \wedge^* H_1(M;\mathbb{Z})/\text{torsion} \to \mathbb{Z} \]
\[ \alpha \mapsto \langle X, \mu_{M,s}(\alpha) \rangle, \]
which is set to be 0 when the degree of \( \mu_{M,s}(\alpha) \) does not match \( \dim X \). To be specific, for \([c] \in H_1(M,\mathbb{Z})\),
\[ \mu_{M,s}(c) := Hol_c([d\theta]) \]
where \([d\theta] \equiv 1 \in H^1(S^1,\mathbb{Z})\) and \( Hol_c : \mathcal{B}^* \to S^1 \) is given by the holonomy of each connection around \( c \), and \( \mu_{M,s}(U) \) for \( U \equiv 1 \in H_0(M,\mathbb{Z}) \) is given by the first Chern class of the \( S^1 \)-bundle
\[ \mathcal{B}^* = (\mathcal{A}(W_+) \times (\Gamma(W_+) - \{0\})) / \mathcal{G}_o \]
over \( \mathcal{B}^* \) where \( \mathcal{G}_o = \{ g \in \mathcal{G} | g(o) = 1 \} \) is the based gauge group for a fixed base point \( o \in M \).

As in the ordinary case, \( SW^G_{M,s} \) is independent of the choice of a \( G \)-invariant metric and a \( G \)-invariant perturbation \( \varepsilon \), if \( b_2^+(M)^G > 1 \). When \( b_2^+(M)^G = 1 \), one should get an appropriate wall-crossing formula.

If no confusion arises, we will sometimes abuse notation to denote
\[ SW^G_{M,s}(U \otimes \cdots \otimes U) \]
just by \( SW^G_{M,s} \). For an invariant as a \( G \)-manifold, we define the \( G \)-monopole polynomial of \( M \) as
\[ SW^G_M := \sum_s SW^G_{M,s}PD(c_1(s)) \in \mathbb{Z}[H_2(M;\mathbb{Z})^G], \]
where the summation is over the set of all \( G \)-equivariant Spin\(^c\) structures.

When \( \mathcal{M} \) happens to be smooth for a \( G \)-invariant perturbation, the induced \( G \)-action on it is a smooth action, and hence \( \mathcal{M}^G \) is a smooth submanifold. Moreover if the finite group action is free, then \( \pi : M \to M/G \) is a covering, and \( s \) is the pull-back of a Spin\(^c\) structure on \( M/G \), which is determined up to the kernel of \( \pi^* : H^2(M/G,\mathbb{Z}) \to H^2(M,\mathbb{Z}) \), and all the irreducible solutions of the upstairs is precisely the pull-back of the corresponding irreducible solutions of the downstairs:

**Theorem 2.5** ([34] [29]). Let \( M, s, \) and \( G \) be as above. Under the assumption that \( G \) is finite and the action is free, for a \( G \)-invariant generic perturbation
\[ X_{M,s} = \mathcal{M}_{M/G,s'} \quad \text{and} \quad \mathcal{M}^G_{M,s} \simeq \coprod_{c \in \ker \pi^*} \mathcal{M}_{M/G,s'+c}, \]
where the second one is a homeomorphism in general, and \( s' \) is the Spin\(^c\) structure on \( M/G \) induced from \( s \) and its \( G \)-action.
Finally we remark that the \(G\)-monopole invariant may change when a homotopically different lift of the \(G\)-action to the Spin\(^c\) structure is chosen. For more details, the readers are referred to \([12, 31]\).

### 3. \(G\)-Bauer-Furuta invariant

For a more refined invariant to find a \(G\)-monopole class, one can also define \(G\)-Bauer-Furuta invariant in the same way as the ordinary Bauer-Furuta invariant.

Let \(A_0 \in \mathcal{A}(W_+)^G\). Using the free action of the \(G\)-invariant based gauge group \(\mathcal{G}_o^G\) on \((A_0 + i \ker d)^G \subset \mathcal{A}(W_+)^G\), one gets the quotient

\[
(A_0 + i \ker d)^G / \mathcal{G}_o^G
\]

which is diffeomorphic to \(b_1(M)^G\)-dimensional torus \(T^{b_1(M)^G}\), and will be denoted by \(\text{Pic}^G(M)\).

For an integer \(m \geq 4\), define infinite-dimensional Hilbert bundles \(\mathcal{E}\) and \(\mathcal{F}\) over \(\text{Pic}^G(M)\) by

\[
\mathcal{E} = ((A_0 + i \ker d)^G \times (L_{m+1}^2(W_+)^G \oplus L_m^2(\Lambda^1 M)^G \oplus H^0(M)^G)) / \mathcal{G}_o^G
\]

and

\[
\mathcal{F} = ((A_0 + i \ker d)^G \times (L_m^2(W_-)^G \oplus L_m^2(\Lambda^2 M)^G \oplus L_m^2(\Lambda^0 M)^G \oplus H^1(M)^G)) / \mathcal{G}_o^G
\]

where all the forms are purely-imaginary-valued, and \(\mathcal{G}_o^G\) acts only on the connection part and the spinor parts.

The \(G\)-monopole map \(\tilde{H} : \mathcal{E} \to \mathcal{F}\) is an \(S^1\)-equivariant continuous fiber-preserving map defined as

\[
[A, \Phi, a, f] \mapsto [A, D_{A+ia} \Phi, F_{A+ia}^+ - \Phi \otimes \Phi^* + \frac{|\Phi|^2}{2} \text{Id}, d^* a + f, a^{\text{harm}}],
\]

which is fiberwisely the sum of a linear Fredholm map and a nonlinear compact map. Note that

\[
\tilde{H}^{-1}(\text{zero section}) / S^1
\]

is exactly the \(G\)-monopole moduli space. The important property that the inverse image of any bounded set in \(\mathcal{F}\) is bounded follows directly from the corresponding boundedness property of the ordinary monopole map.

Our \(G\)-Bauer-Furuta invariant is then defined as the homotopy class of this \(G\)-monopole map in the set of the \(S^1\)-equivariant continuous fiber-preserving

\footnote{Here we are using the assumption that \(G\) is finite to show that the rank of the sublattice of \(H^1(M; \mathbb{Z})^G\), which is generated by \(\mathcal{G}_o^G\) is \(b_1(M)^G\). For \(\alpha \in H^1(M; \mathbb{Z})^G\), let \(\alpha = d \ln g\) for \(g \in \mathcal{G}_o\). Then

\[
|G| \alpha = \sum_{h \in G} h^* d \ln g = d \ln \prod_{h \in G} h^* g \in d \ln \mathcal{G}_o^G.
\]
maps which differ from the $G$-monopole map by the fiberwise compact perturbations and have bounded inverse image for any bounded set in $\mathcal{F}$.

To express the $G$-monopole map as an $S^1$-equivariant stable cohomotopy class, we take an $S^1$-equivariant trivialization $\mathcal{F} \simeq \text{Pic}^G(M) \times \mathcal{U}$ with the projection map $\varpi : \mathcal{F} \to \mathcal{U}$, and take finite-dimensional approximations of $\varpi \circ \tilde{H} : \mathcal{E} \to \mathcal{U}$.

Define an $S^1$-equivariant stable cohomotopy group
\[
\pi^0_{S^1, \mathcal{U}}(\text{Pic}^G(M); \ker(D)^G \oplus \text{coker}(D)^G \oplus H^2_+(M)^G)
\]
as
\[
\text{colim}_U [S^n \vdash T(\ker(D)^G) \oplus S^n \varpi(\text{coker}(D)^G) \oplus H^2_+(M)^G \oplus S^n]^S_1,
\]
where $U$ runs all finite dimensional real vector subspaces of $\mathcal{U}$ transversal to $\varpi(\text{coker}(D)^G) \oplus H^2_+(M)^G$. Here $T(E) \wedge S^W$ denotes the smash product of the Thom space of a vector bundle $E$ and the one-point compactification of a vector space $W$, and
\[
\ker(D)^G \oplus \text{coker}(D)^G \oplus H^2_+(M)^G \in K^*(\text{Pic}^G(M))
\]
is the difference of the virtual index bundle of the Spin$^c$ Dirac operator $D_A : L^2_{m+1}(W^+)^G \to L^2_m(W^-)^G$ and the trivial bundle of rank $b^+_2(M)^G$, where $A$ runs over $\text{Pic}^G(M)$.

Then our $G$-monopole map gives an element $BF^G_{M,s}$ in the above stable cohomotopy group, which we call the $G$-Bauer-Furuta invariant of $(M,s)$. Most importantly

**Theorem 3.1.** If $BF^G_{M,s}$ is nontrivial, then $c_1(s)$ is a $G$-monopole class.

**Proof.** This is a consequence of facts from functional analysis, and the proof in [23, Proposition 6] for case of the ordinary Bauer-Furuta invariant $BF_{M,s}$ holds true without any change. 

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4. CONNECTED SUMS AND $\mathbb{Z}_k$-MONOPOLE INVARIANT

For $(\tilde{M}_k, \tilde{s})$ described in Theorem 1.1 there is at least one obvious $\mathbb{Z}_k$-action on $\tilde{s}$ coming from the given $\mathbb{Z}_k$-action on $s_N$ and $\mathbb{Z}_k$-equivariant gluing of $k$-copies of $s$. We will call such an action “canonical” and denote its generator by $\tau$. Any other lifting of the $\mathbb{Z}_k$-action on $\tilde{M}_k$ is given by a group generated by $g \circ \tau$ where $g$ is a gauge transformation of $\tilde{s}$, i.e. the multiplication by a smooth $S^1$-valued function on $\tilde{M}_k$.

In general, there are homotopically inequivalent liftings of the $\mathbb{Z}_k$-action on $\tilde{M}_k$, but we have

**Lemma 4.1.** Any lifted action on $\tilde{s}$ can be homotoped to another lifting which is equal to a canonical lifting on the cylindrical gluing regions.
Proof. Let $\sigma$ be a generator of a $\mathbb{Z}_k$ action on $\mathfrak{s}$ covering the $\mathbb{Z}_k$-action on $\bar{\mathcal{M}}_k$. For the trivialization of the Spin$^c$ structure on the $k$ cylindrical regions $\bigcup_{j=1}^k U_j$ where each $U_j$ is diffeomorphic to $S^3 \times [0,1]$ such that the generator $\tau$ of a canonical action acts as the multiplication by 1 there,

$$\sigma|_{U_j} : \mathfrak{s}|_{U_j} \to \mathfrak{s}|_{U_{j+1}}$$

covering the identity map from $U_j$ to $U_{j+1}$ is given by the multiplication by $e^{i\sigma_j} \in C^\infty(S^3 \times [0,1], S^1)$.

Since $\sigma^k = Id$,

$$\sum_{j=1}^k \sigma_j \equiv 0 \mod 2\pi.$$

By the fact that $H^1(S^3 \times [0,1]; \mathbb{Z}) = 0$, any two gauge transformations on $S^3 \times [0,1]$ can be homotoped to each other, and hence one can easily see that there exists a smooth relative homotopy

$$H_j(x, t) : (S^3 \times [0,1]) \times [0,1] \to S^1$$

such that

$$H_j(\cdot, 0) = e^{i\sigma_j(\cdot)},$$

$$H_j(x, \cdot) = e^{i\sigma_j(x)} \text{ for all } x \in S^3 \times \{0,1\},$$

$$H_j(\cdot, 1)|_{S^3 \times [\frac{1}{3}, \frac{2}{3}] \times \{1\}} = 1.$$

But $\prod_{j=1}^k H_j$ may not be 1 to fail to form a group $\mathbb{Z}_k$. To remedy this, modify only $H_k$ by

$$\tilde{H}_k := H_k \cdot \prod_{j=1}^k H_j$$

which is also a smooth relative homotopy satisfying the above three properties and produces

$$\prod_{j=1}^{k-1} H_j \tilde{H}_k = 1.$$

Therefore we have obtained a homotopy of the initial action to the action which is trivial on each $S^3 \times [\frac{1}{3}, \frac{2}{3}]$, while keeping equal to the initial action outside $\bigcup_{j=1}^k U_j$.

By our assumption of $b_2^+(\tilde{\mathcal{M}}_k) > b_2^+(\mathcal{M}) > 1$, this homotopy of the action does not change the $\mathbb{Z}_k$-monopole invariant and also obviously the $G$-Bauer-Furuta invariant. From now on we always assume that the $\mathbb{Z}_k$-action on $\mathfrak{s}$ is equal to a canonical action on the cylindrical gluing regions, and take a trivialization of the Spin$^c$ structure of $k$ cylindrical gluing regions so that a canonical action acts as the multiplication by 1.
Theorem 4.2. Let $(\bar{M}_k, \bar{s})$ be as in Theorem 1.1 and $d \geq 0$ be an integer.

If $H_1(N; \mathbb{R})^{\mathbb{Z}_k} = 0$, then for $a = 1$ or $a_1 \wedge \cdots \wedge a_j$
$$SW_{\bar{M}_k, \bar{s}}^\mathbb{Z}_k(U^d a) \equiv SW_{M, s}(U^d a) \mod 2,$$
where $U$ denotes the positive generator of the zeroth homology of $\bar{M}_k$ or $M$, and each $a_i \in H_1(M; \mathbb{Z})/\text{torsion}$ also denotes any of $k$ corresponding elements in $H_1(M_k; \mathbb{Z})$ by abuse of notation.

If $H_1(N; \mathbb{R})^{\mathbb{Z}_k} \neq 0$, then
$$SW_{\bar{M}_k, \bar{s}}^\mathbb{Z}_k(U^d a \wedge b_1 \wedge \cdots \wedge b_\nu) \equiv SW_{M, s}(U^d a) \mod 2,$$
where $\{b_1, \cdots, b_\nu\} \subset H_1(\bar{M}_k; \mathbb{Z})$ is a basis of $H_1(N; \mathbb{R})^{\mathbb{Z}_k}$ so that $b_1 \wedge \cdots \wedge b_\nu$ is a generator of $\wedge^{\text{top}}(H_1(N; \mathbb{Z})^{\mathbb{Z}_k}/\text{torsion}).$

Proof. First we consider the case when the action on $N$ is not free.

We take a $\mathbb{Z}_k$-invariant metric of positive scalar curvature on $N$. In order to do the connected sum with $k$ copies of $M$, we perform a Gromov-Lawson type surgery \cite{21, 36} around each point of a free orbit of $\mathbb{Z}_k$ keeping the positivity of scalar curvature to get a Riemannian manifold $\hat{N}$ with cylindrical ends with each end isometric to a Riemannian product of a round $S^3$ and $\mathbb{R}$. We suppose that this is done in a symmetric way so that the $\mathbb{Z}_k$-action on $\hat{N}$ is isometric.

On $M$ part, we put any metric and perform a Gromov-Lawson surgery with the same cylindrical end as above. Let’s denote this by $\bar{M}$. Now chop the cylinders at sufficiently large length and then glue $\hat{N}$ and $k$-copies of $\bar{M}$ along the boundary to get a desired $\mathbb{Z}_k$-invariant metric $g_k$ on $\bar{M}_k$.

Sometimes we abuse the notation $\bar{M}_k$ to mean $(\bar{M}_k, g_k)$.

Let’s find out the ordinary moduli space $M_{\bar{M}_k}$ of $(\bar{M}_k, \bar{s})$. Let $M_{\hat{M}}$ and $M_{\hat{N}}$ be the moduli spaces of finite-energy solutions of $(\hat{M}, s)$ and $(\hat{N}, s_{\hat{N}})$ respectively.

By the gluing theory\footnote{For more details, one may consult \cite{32, 33, 41, 37, 24}.} of the moduli space, which is now a standard method in gauge theory, $M_{\bar{M}}$ is diffeomorphic to $M_M$. In $M_{\hat{M}}$, we use a compact-supported self-dual 2-form for a generic perturbation.

Since $\hat{N}$ has a metric of positive scalar curvature and the property that $b_2^+(\hat{N}) = 0$ and $c_1^2(s_{\hat{N}}) = -b_2(\hat{N})$, $\hat{N}$ also has no gluing obstruction even without perturbation so that
$$M_N = M_{\hat{N}} = M_{\hat{N}}^{\text{red}} \simeq T^{b_2(\hat{N})},$$
which can be identified with the set of $L^2$-harmonic 1-forms on $\hat{N}$ modulo gauge, i.e.
$$H^1_{\text{cpt}}(\hat{N}, \mathbb{R}) / H^1_{\text{cpt}}(\hat{N}, \mathbb{Z}).$$
(Here by $T^0$ we mean a point, and $\mathcal{M}^{red} \subset \mathcal{M}$ denotes the moduli space of reducible solutions.)

As is well-known, approximate solutions on $\tilde{M}_k$ are obtained by chopping-off solutions on each $\hat{M}$ and $\hat{N}$ at a large cylindrical length and then grafting them to $\bar{M}_k$ using a $\mathbb{Z}_k$-invariant partition of unity. This gluing process commutes with gauge transformations which are constant on gluing regions. But one can take local slices so that their transition functions are gauge transformations which are constant on gluing regions. Thus we have a well-defined gluing of moduli spaces to get a so-called approximate moduli space $\tilde{\mathcal{M}}_{\bar{M}_k} = \left(\mathcal{M}_M \times \cdots \times \mathcal{M}_M / S^1\right) \times \mathcal{M}_N$

\begin{align*}
\mathcal{M}_{\bar{M}_k} &= \left(\underbrace{(\mathcal{M}_M^0 \times \cdots \times \mathcal{M}_M^0)}_{k}/S^1\right) \times \mathcal{M}_N \\
&= \left(\underbrace{(\mathcal{M}_M \times \cdots \times \mathcal{M}_M)}_{k} \times T^{k-1}\right) \times T^{b_1(N)},
\end{align*}

where $\mathcal{M}_M^0$ is the based moduli space fibering over $\mathcal{M}_M$ with fiber $G_\rho/G = S^1$, and $\tilde{\times}$ means a $T^{k-1}$-bundle over $\underbrace{\mathcal{M}_M \times \cdots \times \mathcal{M}_M}_{k}$.

There is a diffeomorphism

$$\Upsilon : \tilde{\mathcal{M}}_{\bar{M}_k} \to \mathcal{M}_{\bar{M}_k}$$

given by a very small isotopy in $\mathcal{B}^*$, once we choose a normal bundle of $\tilde{\mathcal{M}}_{\bar{M}_k}$ in $\mathcal{B}^*$. They get close exponentially as the length of the cylinders in $\bar{M}_k$ increases.

An important fact for us is that the same $k$ copies of a compactly supported self-dual 2-form can be used for the perturbation on $\hat{M}$ parts, while no perturbation is put on the $\hat{N}$ part. Along with the $\mathbb{Z}_k$-invariance of the Riemannian metric $g_k$, the perturbed Seiberg-Witten equations for $(\tilde{M}_k, g_k)$ are $\mathbb{Z}_k$-equivariant so that the induced smooth $\mathbb{Z}_k$-action on $\mathcal{B}^*$ maps $\mathcal{M}_{\bar{M}_k}$ to itself.

Let’s describe elements of $\tilde{\mathcal{M}}_{\bar{M}_k}$ more explicitly. Let $[\xi] \in \mathcal{M}_M$, where $[ \ ]$ denotes a gauge equivalence class from now on. Let $\tilde{\xi}$ be an approximate solution of $\xi$ cut-off at large cylindrical length, and $\tilde{\xi}(\theta)$ be its gauge-transform under the gauge transformation by $e^{i\theta} \in C^\infty(\tilde{M}, S^1)$. (From now on, the tilde $\tilde{ }$ of a solution will mean its cut-off.) Any element in $\tilde{\mathcal{M}}_{\bar{M}_k}$ can be written as an ordered $(k + 1)$-tuple

$$[(\tilde{\xi}_1(\theta_1), \cdots , \tilde{\xi}_k(\theta_k), \tilde{\eta})]$$
for each $[\xi_i] \in \mathcal{M}_M$ and constants $\theta_i$'s, where the $i$-th term for $i = 1, \cdots, k$ represents the approximate solution grafted on the $i$-th $M$ summand, and the last term is a chop-off of $\eta \in \mathcal{M}_N^{\text{red}}$. We still have to mod out these $(k + 1)$-tuples by $S^1$, and hence there is a bijective correspondence

$$
\mathcal{M}_{\tilde{M}_k} \cong \{(\xi_1, \cdots, \xi_k, \eta) | [\eta] \in \mathcal{M}_N, [\xi_i] \in \mathcal{M}_M, \theta_i \in [0, 2\pi) \ \forall i\},
$$

when the cut-off is done at sufficiently large cylindrical length.

In taking cut-offs of solutions on $\hat{N}$, we will use special a slice, i.e. a gauge representative of $\mathcal{M}_{\hat{N}} = T^{b_1(N)}$. Fix a $\mathbb{Z}_k$-invariant connection $\eta_0$ such that $[\eta_0] \in \mathcal{M}_N$, which exists by taking the $\mathbb{Z}_k$-average of any reducible solution, and take compact-supported closed 1-forms $\beta_1, \cdots, \beta_{b_1(N)}$ which compose a basis of $H^1_{\text{cpt}}(\hat{N}; \mathbb{Z})$ and are zero on the cylindrical gluing regions. Any element $[\eta] \in \mathcal{M}_N$ can be expressed as

$$
\eta = \eta_0 + \sum_i c_i \beta_i
$$

for $c_i \in \mathbb{R}/\mathbb{Z}$, and the gauge equivalence class of its cut-off

$$
\tilde{\eta} := \rho \eta = \rho \eta_0 + \sum_i c_i \beta_i
$$

using a $\mathbb{Z}_k$-invariant cut-off function $\rho$ which is equal to 1 on the support of every $\beta_i$ is well-defined independently of the mod $\mathbb{Z}$ ambiguity of each $c_i$.

We let $\sigma$ be a generator of the $\mathbb{Z}_k$-action. By Lemma 4.1, $\sigma$ is equal to 1 on the cylindrical gluing regions, and hence $\sigma$ can be obviously extended to an action on the Spin$^c$ structure of $\tilde{N} \cup \coprod_{k=1}^{\hat{M}} \tilde{M}$ and also its moduli space of finite-energy monopoles. By the $\mathbb{Z}_k$-invariance of $\rho$

$$
\sigma^* \tilde{\eta} = \sigma^* (\rho \eta) = \rho \sigma^* \eta = \tilde{\sigma}^* \eta,
$$

and thus

$$
(4.2)
\sigma^* (\tilde{\xi}_1(\theta_1), \cdots, \tilde{\xi}_k(0), \tilde{\eta}) = (\tilde{\xi}_k(\sigma_k), \tilde{\xi}_1(\theta_1 + \sigma_1), \cdots, \tilde{\xi}_{k-1}(\theta_{k-1} + \sigma_{k-1}), \tilde{\sigma}^* \eta),
$$

3Here, grafting over each $M$ part is done via the identification of the Spin$^c$ structure of each $M$ part using a canonical action. Thus a canonical action $\tau$ on it is given by the permutation of approximate solutions on $M$ parts, i.e.

$$
\tau^* (\xi_1(\theta_1), \cdots, \xi_k(\theta_k), \tilde{\eta}) = (\xi_k(\theta_k), \xi_1(\theta_1), \cdots, \xi_{k-1}(\theta_{k-1}), \tau^* \tilde{\eta}).
$$
where all $e^{i\sigma_i} \in C^\infty(\hat{M}, S^1)$ are equal to 1 on the cylindrical gluing regions and satisfy

$$\sum_{i=1}^k \sigma_i \equiv 0 \mod 2\pi.$$

**Lemma 4.3.** The $\mathbb{Z}_k$-action on $B^*$ maps $\overline{\mathcal{M}}_{\hat{M}_k}$ to itself.

*Proof.* Since $\sigma^* \beta_i$ also gives an element of $H^1_{cpt}(\hat{N}, \mathbb{Z})$, let’s let $\sigma^* \beta_i$ be cohomologous to $\sum_j d_{ij} \beta_j$ for each $i$. Thus

$$\tilde{\sigma}^* \eta = \rho \sigma^* (\eta_0 + \sum_i c_i \beta_i) = \rho \eta_0 + \sum_i c_i \sigma^* \beta_i$$

is gauge-equivalent to

$$\rho \eta_0 + \sum_{i,j} c_i d_{ij} \beta_j$$

which is the cut-off of $\eta_0 + \sum_{i,j} c_i d_{ij} \beta_j$. Also using the fact that all $e^{i\sigma_j}$ are also 1 on the cylindrical gluing regions so that they can be gauged away by a global gauge transformation on $\hat{M}_k$ without affecting $\tilde{\sigma}^* \eta$, the RHS of (4.2) is gauge-equivalent to

$$(\tilde{\xi}_k(0), \tilde{\xi}_1(\theta_1), \cdots, \tilde{\xi}_{k-1}(\theta_{k-1}), \rho(\eta_0 + \sum_{i,j} c_i d_{ij} \beta_j))$$

which is an approximate solution. □

Moreover we may assume that $\Upsilon$ is $\mathbb{Z}_k$-equivariant by the following lemma.

**Lemma 4.4.** $\Upsilon$ can be made $\mathbb{Z}_k$-equivariant, and the smooth submanifold $\mathcal{M}_{\hat{M}}^{Z_k} = X_{\hat{N}} \simeq T^\nu$. $\mathcal{M}_{\hat{M}}^{Z_k}$ pointwisely fixed under the action is isotopic to $\overline{\mathcal{M}}_{\hat{M}_k}$, the fixed point set in $\overline{\mathcal{M}}_{\hat{M}_k}$.

*Proof.* To get a $\mathbb{Z}_k$-equivariant $\Upsilon$ which is determined by the choice of a smooth normal bundle of $\overline{\mathcal{M}}_{\hat{M}_k}$ in $\mathcal{B}^*$, we only need to choose the normal bundle in a $\mathbb{Z}_k$-equivariant way. This can be achieved by taking the $\mathbb{Z}_k$-average of any smooth Riemannian metric defined in a small neighborhood of $\overline{\mathcal{M}}_{\hat{M}_k}$. The second statement is now obvious. □

As a preparation for finding $\mathbb{Z}_k$-fixed points of $\overline{\mathcal{M}}_{\hat{M}_k}$, we have

**Lemma 4.5.**

$$\overline{\mathcal{M}}_{\hat{N}}^{Z_k} = X_{\hat{N}} \simeq T^\nu.$$
Proof. It’s enough to show $\tilde{X}_N^Z \subset X_N$. Suppose $[\eta] \in X_N^Z$, i.e. $[\sigma^* \eta] = [\eta]$. Then

$$\eta_{av} := \frac{1}{k} \sum_{i=1}^{k} (\sigma^i)^* \eta$$

satisfies that $[\eta_{av}] = [\eta]$, and $\sigma^* \eta_{av} = \eta_{av}$. Since $(\sigma^i)^* \eta$ for all $i$ is a reducible solution, so is their convex combination $\eta_{av}$. This completes the proof.

$\square$

Lemma 4.6. The fixed point set $\tilde{M}_{M_k}^Z$ is diffeomorphic to $k$ copies of $M_M \times T^\nu$

where $T^0$ means a point.

Proof. The condition for a fixed point is that

$$(\tilde{\xi}(0), \tilde{\xi}_1(\theta_1), \cdots, \tilde{\xi}_{k-1}(\theta_{k-1}), \tilde{\xi}_k(\theta), \tilde{\eta}) \equiv (\tilde{\xi}_1(\theta), \cdots, \tilde{\xi}_k(\theta), \tilde{\eta})$$

modulo gauge transformations. By (4.1) this implies $[\xi_1] = [\xi_2] = \cdots = [\xi_k] \in M_M$, and $[\sigma^* \eta] = [\eta] \in M_N$.

and

$$0 \equiv \theta_1 + \theta, \theta_1 \equiv \theta_2 + \theta, \cdots, \theta_{k-1} \equiv 0 + \theta \mod 2\pi$$

for some constant $\theta \in [0, 2\pi)$. Summing up the above $k$ equations gives

$$0 \equiv k\theta \mod 2\pi,$$

and hence

$$\theta = 0, \frac{2\pi}{k}, \cdots, \frac{2(k-1)\pi}{k},$$

which lead to the corresponding $k$ solutions

$$(4.3) \quad [(\tilde{\xi}((k-1)\theta), \tilde{\xi}((k-2)\theta), \cdots, \tilde{\xi}(\theta), \tilde{\xi}(0), \tilde{\eta})],$$

where we let $\xi_i = \xi$ for all $i$ and $[\eta] \in M_N^Z$. Therefore $\tilde{M}_{M_k}^Z$ is diffeomorphic to $k$ copies of $\tilde{M}_M \times M_N^Z \simeq M_M \times T^\nu$. $\square$

Lemma 4.7. $X_{M_k}$ is diffeomorphic to $M_M \times T^\nu$.

Proof. Let

$$\tilde{\xi}_{\theta} = (\tilde{\xi}((k-1)\theta - \sum_{i=1}^{k-1} \sigma_i), \tilde{\xi}((k-2)\theta - \sum_{i=2}^{k-1} \sigma_i), \cdots, \tilde{\xi}(\theta - \sigma_{k-1}), \tilde{\xi}(0), \tilde{\eta})$$
for $\xi \in \mathcal{M}_M$ and $\eta \in \mathcal{X}_M$. Because all $e^{i\sigma_j}$ are also 1 on the cylindrical gluing regions, the gauge-equivalence class $[\tilde{\Xi}_\theta]$ is equal to the above [4, 3], and moreover it has a nice property

$$\sigma^*\tilde{\Xi}_\theta = e^{i\theta} \cdot \tilde{\Xi}_\theta,$$

where $\cdot$ denotes the gauge action.

Recall that the canonical local slice at non-reducible $(A, \Phi)$ is given by

$$S(A, \Phi) = \{(a, \varphi) \in \Gamma(\Lambda^1(\tilde{M}_k; i\mathbb{R}) \oplus W_+) | 2d^*a - iIm(\varphi \Phi) = 0\}.$$

Let $\Xi_\theta \in S_{\tilde{\Xi}_\theta}$ be the unique solution corresponding to $\tilde{\Xi}_\theta$ under $\Upsilon$. Letting $\tilde{\Xi}_\theta = (A_\theta, \Phi_\theta)$,

$$\sigma^*S_{\tilde{\Xi}_\theta} = \{(a, \varphi) \in \Gamma(\Lambda^1(\tilde{M}_k; i\mathbb{R}) \oplus \Gamma(W_+) | 2d^*a - iIm(\varphi e^{i\theta} \Phi_\theta) = 0\}$$

$$= \{(a, \varphi) \in \Gamma(\Lambda^1(\tilde{M}_k; i\mathbb{R}) \oplus \Gamma(W_+) | 2d^*a - iIm(e^{i\theta} \varphi \Phi_\theta) = 0\}$$

$$= \{(a, e^{i\theta} \varphi) \in \Gamma(\Lambda^1(\tilde{M}_k; i\mathbb{R}) \oplus \Gamma(W_+) | 2d^*a - iIm(\varphi \Phi_\theta) = 0\}$$

$$= e^{i\theta} \cdot S_{\tilde{\Xi}_\theta},$$

where the first equality is due to the $\mathbb{Z}_k$-invariance of the metric $g_k$.

By the $\mathbb{Z}_k$-equivariance of $\Upsilon$,

$$[\sigma^*\Xi_\theta] = \sigma^*[\Xi_\theta]$$

$$= \sigma^*\Upsilon([\tilde{\Xi}_\theta])$$

$$= \Upsilon(\sigma^*[\tilde{\Xi}_\theta])$$

$$= \Upsilon([\sigma^*\tilde{\Xi}_\theta])$$

$$= \Upsilon([e^{i\theta} \cdot \tilde{\Xi}_\theta])$$

$$= [e^{i\theta} \cdot \Xi_\theta].$$

Therefore

$$(4.4) \quad \sigma^*\Xi_\theta = e^{i\theta} \cdot \Xi_\theta,$$

because $\sigma^*\Xi_\theta$ and $e^{i\theta} \cdot \Xi_\theta$ are respectively the unique representatives of $\Upsilon([\sigma^*\tilde{\Xi}_\theta]) = \Upsilon([e^{i\theta} \cdot \tilde{\Xi}_\theta])$ in the slices $\sigma^*S_{\tilde{\Xi}_\theta}$ and $e^{i\theta} \cdot S_{\tilde{\Xi}_\theta}$ which are equal.

Thus $\sigma^*\Xi_0 = \Xi_0$ implies that $\mathcal{X}_{M_k}$ contains a copy of $\mathcal{M}_M \times T^\nu$. The reason why that’s all is the consequence of the following sublemma, completing the proof.

**Sublemma 4.8.** Let $\alpha$ be a solution of the Seiberg-Witten equations of $(\tilde{M}_k, \sh)$ satisfying

$$\sigma^*\alpha = e^{i\theta} \cdot \alpha$$

for $\theta = 0, \frac{2\pi}{\kappa}, \ldots, \frac{2(k-1)\pi}{\kappa}$. Then the subset of $\mathbb{Z}_k$-invariant elements in the gauge orbit $G \cdot \alpha$ is

$$G^{\mathbb{Z}_k} \cdot \alpha \quad \text{if } \theta = 0,$$
and empty otherwise.

Proof. Let $g \in G$. Suppose that $\sigma^*(g \cdot \alpha) = g \cdot \alpha$. Then combined with that
\[
\begin{align*}
\sigma^*(g \cdot \alpha) &= \sigma^*(g) \cdot \sigma^*(\alpha) \\
&= \sigma^*(g) \cdot (e^{i\theta} \cdot \alpha) \\
&= (\sigma^*(g)e^{i\theta}) \cdot \alpha,
\end{align*}
\]
it follows that
\[
g \cdot \alpha = (\sigma^*(g)e^{i\theta}) \cdot \alpha,
\]
which implies that
\[
g = \sigma^*(g)e^{i\theta},
\]
where we used the continuity of $g$ and the fact that the spinor part of $\alpha$ is not identically zero on an open subset by the unique continuation property. Therefore
\[
(\sigma^n)^*(g) = (\sigma^*)^n(g) = ge^{-in\theta}
\]
for any $n \in \mathbb{Z}$.

Now if $\theta = 0$, (4.5) with $n = 1$ gives $g \in G_k^G$ as expected. For other values of $\theta$, choose a point $p \in \bar{M}_k$ with a nontrivial stabilizer group $\mathbb{Z}_l$ under the $\mathbb{Z}_k$-action. Then evaluating (4.5) with $n = l$ at the point $p$ gives
\[
g = ge^{-il\theta}
\]
which yields a desired contradiction. $\square$

We claim that the $\mu$ cocycles on $\mathfrak{M}_M \times T^\nu$ and $\mathfrak{X}_{\bar{M}_k}$ coincide,i.e.
\[
\mu_{\mathfrak{M},s}(a_i) = \mu_{\bar{M}_k,s}(a_i), \quad \mu_{\mathfrak{N},s\mathfrak{N}}(b_i) = \mu_{\bar{M}_k,s}(b_i), \quad \mu_{\mathfrak{M},s}(U) = \mu_{\bar{M}_k,s}(U)
\]
for $a_i$ and $b_i$ as in the theorem statement, where the the equality means the identification under the above diffeomorphism. The first equality comes from that the holonomy maps $\text{Hol}_{a_i}$ defined on $\mathfrak{M}_M$ and $\mathfrak{M}_{\bar{M}_k}$ are just the same, when the representative of $a_i$ is chosen away from the gluing regions. Using the isotopy between $\mathfrak{M}_{\bar{M}_k}$ and $\mathfrak{M}_{\bar{M}_k}$, the induced maps $\text{Hol}_{a_i}^*$ to $H^1(\mathfrak{M}_M; \mathbb{Z})$ and $H^1(\mathfrak{M}_{\bar{M}_k}; \mathbb{Z})$ are the same so that
\[
\mu_{\mathfrak{M},s}(a_i) = \text{Hol}_{a_i}^*([d\theta]) = \mu_{\bar{M}_k,s}(a_i)
\]
for each $i$. Likewise for the second equality. For the third equality, note that the $S^1$-fibrations on $\mathfrak{M}_M$ and $\mathfrak{M}_{\bar{M}_k}$ induced by the $\mathcal{G}/\mathcal{G}_o$ action are isomorphic in an obvious way, which in turn gives an isomorphism of the $S^1$-fibrations.

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4This is the only place we use the condition that the action on $N$ is not free.

5In fact, this is also true for any other component of $\mathfrak{M}_{\bar{M}_k}$. 
on $\mathcal{M}_M \times T^\nu$ and $\mathcal{M}_{Mk}^{\mathbb{Z}_k}$ induced by the $G/G_o$ action, where the $T^\nu$ part is fixed under the $G/G_o$ action.

Finally we come to the evaluation of the Seiberg-Witten invariant on $\bar{X}_{Mk}$. If $\nu \neq 0$, then $\mu(b_1) \wedge \cdots \wedge \mu(b_\nu)$ is a generator of $H^\nu(T^\nu; \mathbb{Z})$, and hence it follows from the above identification of $\mu$-cocycles that

$$SW_{Mk,\bar{s}}^{\mathbb{Z}_k}(U^d b_1 \wedge \cdots \wedge b_\nu) \equiv SW_{M,s}(U^d) \mod 2,$$

and

$$SW_{Mk,\bar{s}}^{\mathbb{Z}_k}(U^d a_1 \wedge \cdots \wedge a_j \wedge b_1 \wedge \cdots \wedge b_\nu) \equiv SW_{M,s}(U^d a_1 \wedge \cdots \wedge a_j) \mod 2.$$

The case of $\nu = 0$ is just a special case. (If the diffeomorphism between $\bar{X}_{Mk}$ and $N = \mathbb{Z}_k \bar{M}_k$ is orientation-preserving, then both invariants are exactly the same.)

If the action on $N$ is free, then applying Theorem 2.5 and the gluing theory, we have diffeomorphisms

$$\bar{X}_{Mk,\bar{s}} \simeq \mathcal{M}_{M#N/\mathbb{Z}_k,\bar{s}#s'_N},$$

$$\simeq \mathcal{M}_{M,s} \times \mathcal{M}_{N/\mathbb{Z}_k,s'_N},$$

$$\simeq \mathcal{M}_{M,s} \times T^\nu,$$

where $s'_N$ is the Spin$^c$ structure on $N/\mathbb{Z}_k$ induced from $s_N$ and its $\mathbb{Z}_k$ action induced from that of $\bar{s}$. The rest is the same as the non-free case. □

**Remark** We conjecture that the above diffeomorphism between $\bar{X}_{Mk}$ and $\mathcal{M}_{M} \times T^\nu$ is orientation-preserving, when the homology orientations are appropriately chosen.

One may try to prove $\bar{X}_{Mk} \simeq \mathcal{M}_{M} \times T^\nu$ by gluing $G$-monopole moduli spaces directly. But the above method of proof by gluing ordinary moduli spaces also shows that $\mathcal{M}_{Mk}^{\mathbb{Z}_k}$ is diffeomorphic to $k$ copies of $\mathcal{M}_{M} \times T^\nu$. □

Theorem 4.2 gives a proof of the statement in theorem 1.1 on the $G$-monopole invariant. The statement about the $G$-Bauer-Furuta invariant will be proved in the next section.

5. Connected sums and $\mathbb{Z}_k$-Bauer-Furuta invariant

**Theorem 5.1.** Let $(\bar{M}_k,\bar{s})$ be as in Theorem 1.1. Then

$$BF_{Mk,\bar{s}}^{\mathbb{Z}_k} = BF_{M,s} \wedge BF_{N,s_N}^{\mathbb{Z}_k},$$

and when $b_1(N)^{\mathbb{Z}_k} = 0$,

$$BF_{Mk,\bar{s}}^{\mathbb{Z}_k} = BF_{M,s}.$$
If $BF_M, s$ is nontrivial, so is $BF_{\tilde{M}_k, \tilde{s}}$.

**Proof.** Let $\tilde{M}_k = N \cup \coprod_{i=1}^k (M \cup S^4)$ be the disjoint union of $N$ and $k$-copies of $M \cup S^4$, and endow it with a Spin$^c$ structure $\tilde{s}$ which is $s_N$ on $N$, $s$ on each $M$, and the trivial Spin$^c$ structure $s_0$ on each $S^4$. Then $(\tilde{M}_k, \tilde{s})$ has an obvious $\mathbb{Z}_k$-action induced from the $\mathbb{Z}_k$-action on $\bar{s}$ in a unique way up to homotopy. (Here $\mathbb{Z}_k$ acts on $\coprod_{i=1}^k S^4$ by the obvious cyclic permutation, and on its Spin$^c$ structure as constants coming from the action on $\bar{s}$ over the cylindrical gluing regions.)

Just as the ordinary monopole maps shown in [8], the stable cohomotopy class of the disjoint union of $G$-monopole maps is equal to the smash product $\wedge$ of those, and hence

$$BF_{\tilde{M}_k, \tilde{s}} = BF_{\tilde{M}_k, \tilde{s}} \wedge BF_{\bar{N}, \bar{s}_N}$$

$$= BF_{M \cup S^4, s_0} \wedge BF_{\bar{N}, \bar{s}_N}$$

$$= BF_{M, s} \wedge BF_{S^4, s_0} \wedge BF_{\bar{N}, \bar{s}_N}$$

where we used the fact that $BF_{S^4, s_0}$ is just $[id] \in \pi_0 S_1 (pt) \cong \mathbb{Z}$, which was shown in [8].

A surgery following S. Bauer [8] can turn $\tilde{M}_k$ into the union of $M_k$ and $k$-copies of $S^4 \Pi S^4$. In the notations of [8], for $X = X_1 \cup X_2 \cup X_3 = \tilde{M}_k$, we take

$$X_1 = N = (N - \coprod_{i=1}^k D^4) \cup (\coprod_{i=1}^k D^4),$$

$$X_2 = \coprod_{i=1}^k M = (\coprod_{i=1}^k D^4) \cup (\coprod_{i=1}^k (M - D^4)),$$

$$X_3 = \coprod_{i=1}^k S^4 = (\coprod_{i=1}^k D^4) \cup (\coprod_{i=1}^k D^4),$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

where $X_3$ is needed to make $\tau$ an even permutation so that “the gluing map $V$” of Hilbert bundles along the necks is well-defined continuously. After interchanging the second half parts of $X_i$’s by $\tau$, we get

$$X^\tau = X_1^\tau \cup X_2^\tau \cup X_3^\tau = \tilde{M}_k \cup (\coprod_{i=1}^k S^4) \cup (\coprod_{i=1}^k S^4)$$

as desired.

Most importantly, we perform the above surgery from $\tilde{M}_k$ to $M_k \cup \coprod_{i=1}^{2k} S^4$ in a $\mathbb{Z}_k$-invariant way, and also “the gluing map $V$” from the Hilbert bundles

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6The gluing theorem 2.1 of [8] was stated when each $X_i$ is connected with one gluing neck, but the proof also works well without this assumption. For more details, readers are referred to [8].
\( \mathcal{E}, \mathcal{F} \) on \( \text{Pic} \tilde{\mathcal{M}}_k \) to the Hilbert bundles on \( \text{Pic} \tilde{\mathcal{M}}_k \cup \bigcup_{i=1}^{2k} S^4 \) in a \( \mathbb{Z}_k \)-invariant way. The homotopy of the ordinary monopole map of \( \tilde{\mathcal{M}}_k \) shown in [8] can also be done in a \( \mathbb{Z}_k \)-invariant way. Then those \( G \)-monopole maps of \( \tilde{\mathcal{M}}_k \) and \( \tilde{\mathcal{M}}_k \cup \bigcup_{i=1}^{2k} S^4 \) are conjugate via “the gluing map \( V \)” up to \( \mathbb{Z}_k \)-invariant homotopy. Therefore their stable cohomotopy classes are equal so that

\[
BF_{\tilde{\mathcal{M}}_k, \tilde{s}}^{\mathbb{Z}_k} = BF_{\tilde{\mathcal{M}}_k, \tilde{s}} \wedge BF_{\mathbb{Z}_k}^{\mathbb{Z}_k} \wedge BF_{\mathbb{Z}_k}^{\mathbb{Z}_k} \wedge BF_{S^4}^{\mathbb{Z}_k} \wedge BF_{N, s}^{\mathbb{Z}_k}
\]

where we again used that \( BF_{S^4, s_0} = [id] \).

Therefore we obtained

\[
BF_{\tilde{\mathcal{M}}_k, \tilde{s}}^{\mathbb{Z}_k} = BF_{M, s} \wedge BF_{N, s}^{\mathbb{Z}_k},
\]

and the case of \( b_1(N)^{\mathbb{Z}_k} = 0 \) is derived from the following lemma :

**Lemma 5.2.** If \( b_1(N)^{\mathbb{Z}_k} = 0 \), then \( BF_{N, s}^{\mathbb{Z}_k} \) is the class of the identity map \( [id] \in \pi_{0}^{0}(pt) \cong \mathbb{Z} \).

**Proof.** The method of proof is basically the same as the ordinary Bauer-Furuta invariant in [8].

First, we need to show that the \( \mathbb{Z}_k \)-index of the \( \text{Spin}^c \) Dirac operator is zero. Take a \( \mathbb{Z}_k \)-invariant metric of positive scalar curvature on \( N \). Using the homotopy invariance of \( G \)-index, we compute the index at a \( \mathbb{Z}_k \)-invariant connection \( A_0 \) whose curvature 2-form is harmonic and hence anti-self-dual.

Applying the Weitzenböck formula with the fact that the scalar curvature of \( N \) is positive, and the curvature 2-form is anti-self-dual, we get zero kernel. Now then from the vanishing of the ordinary index given by \( \left( c^2 - \tau(N) \right) / 8 \), the cokernel must be also zero. In particular, we have vanishing of \( \mathbb{Z}_k \)-invariant kernel and cokernel, implying that the \( \mathbb{Z}_k \)-index is zero.

Then along with \( b_1(N)^{\mathbb{Z}_k} = b_1^2(N)^{\mathbb{Z}_k} = 0 \), we conclude that \( BF_{N, s}^{\mathbb{Z}_k} \) belongs to \( \pi_{S^1}^{0}(pt) \) which is isomorphic to \( \pi_{S^1}^{0}(pt) = \mathbb{Z} \) by the isomorphism induced by restriction to the \( S^1 \)-fixed point set on which the \( G \)-monopole map is just the linear isomorphism :

\[
L_{m+1}(\Lambda^1 N)^{\mathbb{Z}_k} \times H^0(N)^{\mathbb{Z}_k} \rightarrow L_{m}(\Lambda^2 N)^{\mathbb{Z}_k} \times L_{m}(\Lambda^0 N)^{\mathbb{Z}_k} \times H^1(N)^{\mathbb{Z}_k}
\]

\[
(a, c) \rightarrow (d^+ a, d^* a + c, a_{\text{harm}}),
\]
because it has no kernel and cokernel. This completes the proof.

Now let’s consider the case of $b_1(N)\mathbb{Z}_k \geq 1$. Again the $\mathbb{Z}_k$-index bundle of the Spin$^c$ Dirac operator over $Pic^{\mathbb{Z}_k}(N) = T^{b_1(N)\mathbb{Z}_k}$ is zero so that $BF^k_{N,sN}$ belongs to $\pi^0_{S^1}(T^{b_1(N)\mathbb{Z}_k})$.

Following [23], we consider the restriction map $\sigma: \pi^0_{S^1}(T^{b_1(N)\mathbb{Z}_k}) \to \pi^0_{S^1}(pt)$ to the fiber over a point in $Pic^{\mathbb{Z}_k}(N)$. By the same method as the above lemma, $\sigma(BF^k_{N,sN})$ is just the identity map. Then the restriction of $BF^k_{M,\tilde{\mathbb{Z}}_k} = BF^k_{M,\tilde{\mathbb{Z}}_k}$ to

$$Pic^{\mathbb{Z}_k}(\coprod_{i=1}^k (M \cup S^4)) \times \{pt\} \subset Pic^{\mathbb{Z}_k}(\tilde{M}_k) = Pic^{\mathbb{Z}_k}(\tilde{M}_k)$$

is given by

$$BF_{M,\tilde{\mathbb{Z}}_k} \wedge \sigma(BF^k_{N,sN}) = BF_{M,s},$$

completing the proof.

6. Examples of $(N,s_N)$ of Theorem [23]

In this section, $G, H$ and $K$ denote compact Lie groups. Let’s recall some elementary facts on equivariant principal bundles.

**Definition 2.** A principal $G$ bundle $\pi: P \to M$ is said to be $K$-equivariant if $K$ acts left on both $P$ and $M$ in such a way that

1. $\pi$ is $K$-equivariant:

$$\pi(k \cdot p) = k \cdot \pi(p)$$

for all $k \in K$ and $p \in P$,

2. the left action of $K$ commutes with the right action of $G$:

$$k \cdot (p \cdot g) = (k \cdot p) \cdot g$$

for all $k \in K, p \in P$, and $g \in G$.

If $H$ is a normal subgroup of $G$, then one can define a principal $G/H$ bundle $P/H$ by taking the fiberwise quotient of $P$ by $H$. Moreover if $P$ is $K$-equivariant under a left $K$ action, then there exists the induced $K$ action on $P/H$ so that $P/H$ is $K$-equivariant.

**Lemma 6.1.** Let $P$ and $\tilde{P}$ be a principal $G$ and $\tilde{G}$ bundle respectively over a smooth manifold $M$ such that $\tilde{P}$ double-covers $P$ fiberwisely. For a normal subgroup $H$ containing $\mathbb{Z}_2$ in both $\tilde{G}$ and $S^1$, let

$$\tilde{P} \otimes_H S^1 := (\tilde{P} \times_M (M \times S^1))/H$$
be the quotient of the fiber product of \( \tilde{P} \) and the trivial \( S^1 \) bundle \( M \times S^1 \) by \( H \), where the right \( H \) action is given by

\[(p, (x, e^{i\theta})) \cdot h = (p \cdot h, (x, h^{-1} e^{i\theta})).\]

Suppose that \( M \) and \( P \) admit a smooth \( S^1 \) action such that \( P \) is \( S^1 \)-equivariant. Then \( \tilde{P} \otimes_H S^1 \) is also \( S^1 \)-equivariant by lifting the action on \( P \). In particular, any smooth \( S^1 \)-action on a smooth spin manifold lifts to its trivial Spin\(^c\) bundle so that the Spin\(^c\) structure is \( S^1 \)-equivariant.

**Proof.** Any \( S^1 \) action on \( P \) can be lifted to \( \tilde{P} \) uniquely at least locally commuting with the right \( \tilde{G} \) action. If the monodromy is trivial for any orbit, then the \( S^1 \) action can be globally well-defined on \( \tilde{P} \), and hence on \( \tilde{P} \otimes_H S^1 \), where the \( S^1 \) action on the latter \( S^1 \) fiber can be any (left) action, e.g. the trivial action, commuting with the right \( S^1 \) action.

If the monodromy is not trivial, it has to be \( \mathbb{Z}_2 \) for any orbit, because the orbit space is connected. In that case, we need the trivial \( S^1 \) bundle \( M \times S^1 \) with an "ill-defined" \( S^1 \) action with monodromy \( \mathbb{Z}_2 \) defined as follows.

First consider the double covering map from \( M \times S^1 \) to itself defined by \((x, z) \mapsto (x, z^2)\). Equip the downstairs \( M \times S^1 \) with the \( S^1 \) action which acts on the base as given and on the fiber \( S^1 \) by the (left) multiplication. Then this downstairs action can be locally lifted to the upstairs commuting with the right \( S^1 \) action. Most importantly, it has \( \mathbb{Z}_2 \) monodromy as desired. Explicitly, \( e^{i\theta} \) for \( \theta \in [0, 2\pi) \) acts on the fiber \( S^1 \) by the (left) multiplication of \( e^{i\theta} \). Combining this with the local action on \( \tilde{P} \), we get a well-defined \( S^1 \) action on \( \tilde{P} \otimes_H S^1 \), because two \( \mathbb{Z}_2 \) monodromies are canceled each other.

Once the \( S^1 \) action on \( \tilde{P} \otimes_H S^1 \) is globally well-defined, it commutes with the right \( \tilde{G} \otimes_H S^1 \) action, because the local \( S^1 \) action on \( \tilde{P} \times S^1 \) commuted with the right \( \tilde{G} \times S^1 \) action.

If \( S^1 \) acts on a smooth manifold, the orthonormal frame bundle is always \( S^1 \)-equivariant under the action. Then by the above result any \( S^1 \) action on a smooth spin manifold lifts to the trivial Spin\(^c\) bundle which is (spin bundle) \( \otimes_{\mathbb{Z}_2} S^1 \).

**Lemma 6.2.** Let \( P \) be a flat principal \( G \) bundle over a smooth manifold \( M \) with a smooth \( S^1 \) action. Suppose that the action can be lifted to the universal cover \( \tilde{M} \) of \( M \). Then it can be also lifted to \( P \) so that \( P \) is \( S^1 \)-equivariant.

**Proof.** For the covering map \( \pi : \tilde{M} \to M \), the pull-back bundle \( \pi^* P \) is the trivial bundle \( M \times G \). By letting \( S^1 \) act on the fiber \( G \) trivially, \( \pi^* P \) can be made \( S^1 \)-equivariant. For the deck transformation group \( \pi_1(M) \), \( P \) is gotten by an element of \( \text{Hom}(\pi_1(M), G) \). Any deck transformation acts on the fiber \( G \) as the left multiplication of a constant in \( G \) so that it commutes with not only the right \( G \) action but also the left \( S^1 \) action which is trivial on the fiber.
Therefore the $S^1$ action on $\pi^* P$ projects down to an $S^1$ action on $P$. To see whether this $S^1$ action commutes with the right $G$ action, it's enough to check for the local $S^1$ action, which can be seen upstairs on $\pi^* P$. \hfill \Box

**Lemma 6.3.** On a smooth closed oriented 4-manifold $N$ with $b^+_2(N) = 0$, any Spin$^c$ structure $s$ satisfies

$$c_1^2(s) \leq -b_2(N),$$

and the choice of a Spin$^c$ structure $s_N$ satisfying $c_1^2(s_N) = -b_2(N)$ is always possible.

**Proof.** If $b_2(N) = 0$, it is obvious. The case of $b_2(N) > 0$ can be seen as follows. Using Donaldson's theorem [13, 14], we diagonalize the intersection form $Q_N$ on $H^2(N; \mathbb{Z})/\text{torsion}$ over $\mathbb{Z}$ with a basis $\{\alpha_1, \cdots, \alpha_{b_2(N)}\}$ satisfying $Q_N(\alpha_i, \alpha_j) = -1$ for all $i$. Then for any Spin$^c$ structure $s$, the rational part of $c_1(s)$ should be of the form

$$\sum_{i=1}^{b_2(N)} a_i \alpha_i$$

where each $a_i \equiv 1 \mod 2$, because

$$Q_N(c_1(s), \alpha) \equiv Q_N(\alpha, \alpha) \mod 2.$$ 

Consequently $|a_i| \geq 1$ for all $i$ which means

$$c_1^2(s) = \sum_{i=1}^{b_2(N)} -a_i^2 \leq -b_2(N),$$

and we can get a Spin$^c$ structure $s_N$ with

$$c_1(s_N) \equiv \sum_i \alpha_i \mod \text{torsion}$$

by tensoring any $s$ with a line bundle $L$ satisfying

$$2c_1(L) + c_1(s) \equiv \sum_i \alpha_i \mod \text{torsion},$$

completing the proof. \hfill \Box

**Theorem 6.4.** Let $X$ be one of

$$S^4, \quad \mathbb{CP}^2, \quad S^1 \times (L_1 \# \cdots \# L_n), \quad \text{and} \quad \hat{S}^1 \times L$$

where each $L_i$ and $L$ are quotients of $S^3$ by free actions of finite groups, and $\hat{S}^1 \times L$ is the manifold obtained from the surgery on $S^1 \times L$ along an $S^1 \times \{pt\}$. 

Then for any integer \( l \geq 0 \) and any smooth closed oriented 4-manifold \( Z \) with \( b_2^+(Z) = 0 \) admitting a metric of positive scalar curvature,

\[
X \# klZ
\]
satisfies the properties of \( N \) in Theorem 1.1, where the Spin\(^c\) structure of \( X \# klZ \) is given by gluing any Spin\(^c\) structure \( s_X \) on \( X \) and any Spin\(^c\) structure \( s_Z \) on \( Z \) satisfying \( c^2_1(s_X) = -b_2(X) \) and \( c^2_1(s_Z) = -b_2(Z) \) respectively.

**Proof.** First, we will define \( \mathbb{Z}_k \) actions preserving a metric of positive scalar curvature. In fact, each \( X \) can be given such \( S^1 \) actions. For \( X = S^4 \), one can take a \( \mathbb{Z}_k \)-action coming from any action of \( S^1 \subset SO(5) \) preserving a round metric, and for \( X = \mathbb{C}P_2 \), we have the following action

\[
\lambda \cdot [z_0, z_1, z_2] = [z_0, \lambda^{m_1} z_1, \lambda^{m_2} z_2]
\]

for \( \lambda \in S^1 \) and \( m_i \in \mathbb{Z} \), which preserves the Fubini-Study metric.

Before considering the next example, recall that every finite group acting freely on \( S^3 \) is in fact conjugate to a subgroup of \( O(4) \), and hence its quotient 3-manifold admits a metric of constant positive curvature. This follows from the spherical space form conjecture finally proven by G. Perelman. (See [28].)

In \( S^1 \times (L_1 \# \cdots \# L_n) \), the action is defined as a rotation along the \( S^1 \)-factor, which obviously preserves a product metric where \( L_1 \# \cdots \# L_n \) is endowed with a metric of positive scalar curvature via the Gromov-Lawson surgery [21].

Finally the above-mentioned \( S^1 \) action on \( S^1 \times L \) can be naturally extended to \( \widetilde{S^1} \times L \), and moreover the Gromov-Lawson surgery [21] on \( S^1 \times \{ pt \} \) produces an \( S^1 \)-invariant metric of positive scalar curvature. Its fixed point set is \( \{ 0 \} \times S^2 \) in the attached \( D^2 \times S^2 \).

Note that all the above \( S^1 \)-actions except on \( S^1 \times (L_1 \# \cdots \# L_n) \) have fixed points under the given \( S^1 \) actions. Now \( X \# klZ \) has an obvious \( \mathbb{Z}_k \)-action induced from that of \( X \) and a \( \mathbb{Z}_k \)-invariant metric which has positive scalar curvature again by the Gromov-Lawson surgery.

It remains to prove that the above \( \mathbb{Z}_k \)-action on \( X \# klZ \) can be lifted to the Spin\(^c\) structure obtained by gluing the above \( s_X \) and \( s_Z \). For this, we will only prove that any such \( s_X \) is \( \mathbb{Z}_k \)-equivariant. Then one can glue \( k \) copies of \( lZ \) in an obvious \( \mathbb{Z}_k \)-equivariant way. Recalling that the \( \mathbb{Z}_k \) action on \( X \) actually comes from an \( S^1 \) action, we will actually show the \( S^1 \)-equivariance of \( s_X \) on \( X \).

On \( S^4 \), the unique Spin\(^c\) structure is trivial. Any smooth \( S^1 \) action on \( S^4 \) which is spin can be lifted its trivial Spin\(^c\) structure by Lemma 6.1. Any smooth \( S^1 \) action on \( \mathbb{C}P_2 \) is uniquely lifted to its orthonormal frame bundle \( F \), and any Spin\(^c\) structure on \( \mathbb{C}P_2 \) satisfying \( c^2_1 = -1 \) is the double...
cover $P_1$ and $P_2$ of $F \oplus P$ and $F \oplus P^*$ respectively, where $P$ is the principal $S^1$ bundle over $\mathbb{C}P_2$ with $c_1(P) = [H]$ and $P^*$ is its dual. Note that there is a base-preserving diffeomorphism between $P$ and $P^*$ whose total space is $S^5$. Obviously the action (6.6) is extended to $S^5 \subset \mathbb{C}^3$ commuting with the principal $S^1$ action of the Hopf fibration. By Lemma 6.1 the $S^1$-action can be lifted to $P_i \otimes_{S^1} S^1$ in an $S^1$-equivariant way, which is isomorphic to $P_i$ for $i = 1, 2$.

In case of $S^1 \times (L_1 \# \cdots \# L_n)$, any Spin$^c$ structure is the pull-back from $L_1 \# \cdots \# L_n$, and satisfies $c_1^2 = 0 = -b_2$. Because the tangent bundle is trivial, a free $S^1$-action is obviously defined on its trivial spin bundle. Then the action can be obviously extended to any Spin$^c$ structure, because it is pulled-back from $L_1 \# \cdots \# L_n$.

**Lemma 6.5.** $\widehat{S^1 \times L}$ is a rational homology 4-sphere, and

$$H^2(\widehat{S^1 \times L}; \mathbb{Z}) = H_1(L; \mathbb{Z}).$$

Its universal cover is $(|\pi_1(L)| - 1)S^2 \times S^2$ where $0(S^2 \times S^2)$ means $S^4$.

**Proof.** Since the Euler characteristic is easily computed as 2 from the surgery description, and $b_1(\widehat{S^1 \times L}) = b_1(L) = 0$, it follows that $\widehat{S^1 \times L}$ is a rational homology 4-sphere.

By the universal coefficient theorem,

$$H^2(\widehat{S^1 \times L}; \mathbb{Z}) = \text{Hom}(H_2(\widehat{S^1 \times L}; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(\widehat{S^1 \times L}; \mathbb{Z}), \mathbb{Z})$$

$$= H_1(\widehat{S^1 \times L}; \mathbb{Z})$$

$$= H_1(L; \mathbb{Z}).$$

The universal cover is equal to the manifold obtained from $S^1 \times S^3$ by performing surgery along $S^1 \times \{\pi_1(L)\}$ points in $S^3$, and hence it must be $(|\pi_1(L)| - 1)S^2 \times S^2$. \qed

By the above lemma, there are $|H_1(L; \mathbb{Z})|$ Spin$^c$ structures on $\widehat{S^1 \times L}$, all of which are torsion to satisfy $c_1^2 = 0 = -b_2(\widehat{S^1 \times L})$. Since any $S^1$ bundle on $\widehat{S^1 \times L}$ is flat, and the $S^1$-action on $\widehat{S^1 \times L}$ can be obviously lifted to its universal cover, the Lemma 6.2 says that any $S^1$ bundle is $S^1$-equivariant under the $S^1$ action.

By the construction, $\widehat{S^1 \times L}$ is spin, and hence the trivial Spin$^c$ bundle is $S^1$-equivariant by Lemma 6.1. Any other Spin$^c$ structure is given by the tensor product over $S^1$ of the trivial Spin$^c$ bundle and an $S^1$ bundle, both of which are $S^1$-equivariant bundles. Therefore any Spin$^c$ bundle of $\widehat{S^1 \times L}$ is $S^1$-equivariant.

This completes all the proof.
**Remark** In case of $\hat{S}^1 \times L$, there may be other $\mathbb{Z}_k$-actions satisfying our condition. For instance, if $L$ is a Lens space, one can easily see that there are $\mathbb{Z}_k$-actions coming from non-free $S^1$-actions of $S^3$ commuting with the action giving the quotient $L$. Moreover, one can also take a combination of this action and the above defined action so that $\hat{S}^1 \times L/\mathbb{Z}_k$ is an orbifold. □

7. Exotic group actions

Following [20], we say that a simply connected 4-manifold **dissolves** if it is diffeomorphic either to

$n\mathbb{C}P^2 \# m\overline{\mathbb{C}P^2}$ or to $\pm (n(S^2 \times S^2) \# mK3)$

for some $n, m \geq 0$ according to its homeomorphism type. We also use the term **mod 2 basic class** to mean the first Chern class of a Spin$^c$ structure with nonzero mod 2 Seiberg-Witten invariant.

**Theorem 7.1.** Let $M$ be a smooth closed 4-manifold and \{\(M_i | i \in I\)\} be a family of smooth 4-manifolds such that every $M_i$ is homeomorphic to $M$ and the numbers of mod 2 basic classes of $M_i$'s are all mutually different, but each $M_i \# l_i(S^2 \times S^2)$ is diffeomorphic to $M \# l_i(S^2 \times S^2)$ for an integer $l_i \geq 1$.

If $l_{max} := \sup_{i \in I} l_i < \infty$, then for any integers $k \geq 2$ and $l \geq l_{max} + 1$,

$$klM \# (l - 1)(S^2 \times S^2)$$

admits an $\mathcal{I}$-family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ where $H$ is any group of order $l$ acting freely on $S^3$.

**Proof.** Think of $klM \# (l - 1)(S^2 \times S^2)$ as

$$klM_i \# (l - 1)(S^2 \times S^2),$$

and our $H$ action is defined as the deck transformation map of the $l$-fold covering map onto

$$\hat{M}_{i,k} := kM_i \# S^1 \times L$$

where $S^1 \times L$ for $L = S^3 / H$ is defined as in Theorem [6.4]. To define a $\mathbb{Z}_k$-action, note that $\hat{M}_{i,k}$ has a $\mathbb{Z}_k$-action coming from the $\mathbb{Z}_k$-action of $S^1 \times L$ defined in Theorem [6.4], which is basically a rotation along the $S^1$-direction. This $\mathbb{Z}_k$ action is obviously lifted to the above $l$-fold cover, and it commutes with the above defined $H$ action. Thus we have an $\mathcal{I}$-family of $\mathbb{Z}_k \oplus H$ actions.
on $k\text{M} \# (l-1)(S^2 \times S^2)$, which are all topologically equivalent by using the homeomorphism between each $M_i$ and $M$.

Recall from Theorem 6.4 that all the $\text{Spin}^c$ structures on $\tilde{\text{S}}^1 \times L$ are $\mathbb{Z}_k$-equivariant and satisfy $c_1^2 = -b_2(\tilde{\text{S}}^1 \times L)$. By Theorem 4.2 and the fact that $b_1(\tilde{\text{S}}^1 \times L) = 0$, for any $\text{Spin}^c$ structure $\mathfrak{s}_i$ on $M_i$,

$$\text{SW}^{\mathbb{Z}_k}_{M_i, \mathfrak{s}_i} \equiv \text{SW}_{M_i, \mathfrak{s}_i} \mod 2,$$

and hence

$$\text{SW}^{\mathbb{Z}_k}_{M_i, \mathfrak{s}_i} \equiv \text{SW}_{M_i} \sum_{[\alpha] \in H^2(\tilde{\text{S}}^1 \times L; \mathbb{Z})} [\alpha] \mod 2.$$

Therefore $\text{SW}^{\mathbb{Z}_k}_{M_i, \mathfrak{s}_i} \mod 2$ for all $i$ have mutually different numbers of monomials, and hence the $\mathfrak{I}$-family of $\mathbb{Z}_k \oplus H$ actions on $k\text{M} \# (l-1)(S^2 \times S^2)$ cannot be smoothly equivalent, completing the proof. □

Corollary 7.2. Let $H$ be a finite group of order $l \geq 2$ acting freely on $S^3$. For any $k \geq 2$, there exists an infinite family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ on $k\text{M} \# (l-1)(S^2 \times S^2)$ for infinitely many $m$, and any $n, m' \geq 1, n' \geq 2$.

Proof. By the result of B. Hanke, D. Kotschick, and J. Wehrheim [22], $m(S^2 \times S^2)$ for infinitely many $m$ has the property of $M$ in the above theorem with each $l_i = 1$ and $|\mathfrak{I}| = \infty$. The different smooth structures of their examples are constructed by fiber-summing a logarithmic transform of $E(2n)$ and a certain symplectic 4-manifold along a symplectically embedded torus, and different numbers of mod 2 basic classes are due to those different logarithmic transformations. Indeed the Seiberg-Witten polynomial of the multiplicity $r$ logarithmic transform of $E(2n)$ is given by

$$([T]^r - [T]^{-r})^{2n-2}([T]^{r-1} + [T]^{r-3} + \cdots + [T]^{1-r})$$

whose number of terms with coefficients mod 2 can be made arbitrarily large by taking $r$ sufficiently large, and the fiber sum with the other symplectic 4-manifold is performed on a fiber in an $N(2)$ disjoint from the Gompf nucleus $N(2n)$ where the log transform is performed so that all these mod 2 basic classes survive the fiber-summing by the gluing formula of C. Taubes [40]. Therefore $(k\text{M} + l-1)(S^2 \times S^2)$ has desired actions by the above theorem.

For the second example, we use a well-known fact that $E(n)$ also has the above properties of $M$ in the above theorem with each $l_i = 1$, where
its exotica $M_i$’s are $E(n)K$ for a knot $K \subset S^3$ by the Fintushel-Stern knot surgery. Recall the theorem by S. Akbulut [1] and D. Auckly [3] which says that for any smooth closed simply-connected $X$ with an embedded torus $T$ such that $T \cdot T = 0$ and $\pi_1(X - T) = 0$, a knot-surgered manifold $X_K$ along $T$ via a knot $K$ satisfies

$$X_K \# (S^2 \times S^2) = X \# (S^2 \times S^2).$$

And from the formula

$$SW_{E(n)K} = \Delta_K([T]^2)([T] - [T]^{-1})^{n-2}$$

where $\Delta_K$ is the symmetrized Alexander polynomial of $K$, one can easily see that the number of mod 2 basic classes of $E(n)K$ can be made arbitrarily large by choosing $K$ appropriately. (For example, take $K$ with

$$\Delta_K(t) = 1 + \sum_{j=1}^{2d} (-1)^j (t^j + t^{-j})$$

for sufficiently large $d$.) Therefore

$$klE(2n)\#(l - 1)(S^2 \times S^2) = klnK3\#(kl(n - 1) + l - 1)S^2 \times S^2$$

has desired actions, where we used the fact that $S\#(S^2 \times S^2)$ dissolves for any smooth closed simply-connected elliptic surface $S$ by the work of R. Mandelbaum [26] and R. Gompf [19].

For the third example, one can take $M$ to be $E(n')\#m'\mathbb{CP}^2$, where its exotica $M_i$’s are $E(n')K\#m'\mathbb{CP}^2$ for a knot $K$, because

$$SW_{E(n')K\#m'\mathbb{CP}^2} = SW_{E(n')\#m'\mathbb{CP}^2}K$$

$$= \Delta_K([T]^2)([T] - [T]^{-1})^{n'-2} \prod_{i=1}^{m'}([E_i] + [E_i]^{-1}),$$

where $E_i$’s denote the exceptional divisors, and we used the fact that $E(n')$ is of simple type. Since $E(n')\#\mathbb{CP}^2$ for any $n'$ is non-spin,

$$kl(E(n')\#m'\mathbb{CP}^2)\#(l - 1)(S^2 \times S^2) = kl(E(n')\#m'\mathbb{CP}^2)\#(l - 1)\mathbb{CP}^2\#\mathbb{CP}^2,$$

and it dissolves into the connected sum of $\mathbb{CP}^2$’s and $\overline{\mathbb{CP}^2}$’s, using the dissolution ([26, 19]) of $E(n')\#\mathbb{CP}^2$ into $2n'\mathbb{CP}^2\#(10n' - 1)\overline{\mathbb{CP}^2}$. □

**Remark** When $L$ is a Lens space, one can take other $\mathbb{Z}_k$-actions on $\hat{S}^1 \times L$ mentioned in the remark of section 4 such that its $\mathbb{Z}_k$-quotient is an orbifold. Then it follows that

$$\hat{M}_{i,k}/\mathbb{Z}_k = M_i\#(\hat{S}^1 \times L/\mathbb{Z}_k),$$
and one can use orbifold Seiberg-Witten invariants introduced by W. Chen to show that Seiberg-Witten polynomials $SW_{\tilde{M}_{i,k}/\mathbb{Z}_k}$ are all distinct and hence $\tilde{M}_{i,k}/\mathbb{Z}_k$ are mutually non-diffeomorphic as orbifolds. This is another way of proving the smooth inequivalence of such $\mathbb{Z}_k \oplus H$ actions on $klM \# (l - 1)(S^2 \times S^2)$, but Seiberg-Witten theory on orbifolds is not fully developed yet.

For other combinations of $K3$ surfaces and $S^2 \times S^2$’s in the above corollary, one can use B. Hanke, D. Kotschick, and J. Wehrheim’s other examples in [22]. One can also construct many other such examples of $M$ with infinitely many exotica which become diffeomorphic after one stabilization by using the knot surgery.

Theorem 7.1 and Corollary 7.2 can be generalized a little further. (See [38].)

Acknowledgement. The author would like to express sincere thanks to Prof. Ki-Heon Yun for helpful discussions and supports.

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