Irregular dynamic systems according to R.J. DiPerna and P.L. Lions

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Abstract These are notes of a seminar held at the Institute for Problems in Mechanics, RAS in 2003 and aimed at presentation of [1]. We discuss the notion of a generalized solution to a singular Ordinary Differential Equation introduced by DiPerna and Lions. We stress importance of singular dynamic systems from the “philosophia naturalis” point of view, and extend and simplify the original approach by R.J. DiPerna and P.L. Lions. Further extensions are discussed.

Keywords Singular dynamic systems · singular ODE · irreversible dynamics

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1 Singular dynamic systems and “philosophia naturalis”

In Newtonian mechanics the Universe is governed by

\[ m\ddot{x} = \frac{\partial U(x)}{\partial x} \]  

(1.1)

where \( U \) is the potential energy. Newton himself discovered that the gravitation corresponds to

\[ U(x) = -G \sum_{i \neq j} \frac{m_im_j}{|x_i - x_j|}, \]  

(1.2)

where \( G \) is a constant, \( m_i \) is the mass of the \( i \)th particle, and \( |y| \) stands for the length of 3-dimensional vector \( y \).

If one takes this this ODE point of view seriously, mathematical facts are to be regarded as philosophical principles. E.g. the Laplace determinism, being a belief that the present determines future, is modelled by a uniqueness theorem for the Cauchy problem for (1.1). Similarly, the existence theorem also has a physical or philosophical meaning as a claim that any present state has a future (however unfavorable it can be). So, the existence and uniqueness theorem for ODE might be of more than a purely mathematical interest. Unfortunately, the standard existence and uniqueness theorem for ODE (under the Lipschitz condition for the force function \( \frac{\partial U(x)}{\partial x} \)) is not applicable, say, to potential energy (1.2), for the corresponding forces are not Lipschitz, and not even continuous.

Physical arguments due to Boltzmann and Loschmidt make the issue of existence and uniqueness for (1.1) rather problematic. For starters, the Newton law (1.1) is (or at least seems to be) reversible in time: if \( t \mapsto x(t) \) is a solution, then \( t \mapsto x(-t) \) is a solution as well. However, the Universe has entropy which increases in time according to the second law of thermodynamics. It is clear, that if the phase flow is reversible, the entropy should oscillate, and cannot be monotone increasing.
One can imagine, however, that the reversibility does not hold: the “solution” \( x(-t) \) is not, in fact, a solution. Indeed, the trivial “proof” that it is a solution is a formal application of rules of differentiation. However, the “true” solution \( t \mapsto x(t) \) might obey (1.1) in a generalized sense, and may not be differentiable. Then, it is possible that \( x(-t) \) does not satisfy (1.1) in the generalized sense.

These remarks suggest that the search for a proper notion of a solution of a singular ODE and a further study of its properties rather belongs to natural philosophy than to a routine mathematics and is totally justified.

In fact, this issue is relevant well beyond reconciliation of classical mechanics and thermodynamics. For instance, a natural source of singular dynamic systems is the Control Theory, where the Pontryagin Hamiltonians usually are not everywhere differentiable, and the corresponding vector fields have jumps.

## 2 Cauchy-Lipschitz theorem

This is a well-known theorem on existence and uniqueness of solutions to the Cauchy problem for ODE. It is so well-known that mathematicians usually do not suspect that anybody need something else in this area. Its statement is as follows:

**Theorem 1.** Suppose we are given ODE

\[
\dot{x} = b(x), \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n,
\]

where \( b \) is a Lipschitz continuous vector field. Then, for any \( x_0 \in \mathbb{R}^n \) there is a unique \( C^1 \) solution to (2.1) such that \( x(0) = x_0 \). Moreover, if the phase flow \( X_t : \mathbb{R}^n \to \mathbb{R}^n \) is defined by \( X_t(x_0) = x(t) \) then the phase flow is Lipschitz continuous w.r.t. the variable \( x_0 \in \mathbb{R}^n \), and \( C^1 \) w.r.t. the variable \( t \in \mathbb{R} \).

Lipschitz functions are exactly the functions of the Sobolev class \( W^{1,\infty} \) with uniformly bounded generalized derivative. In other words, if \( b \) belongs to \( W^{1,\infty} \), then the corresponding ODE has good properties. In DiPerna–Lions paper [1] it is shown that rich theory exists for \( b \in W^{1,1} \). Recall, that a function \( u \) which locally belongs to \( L^1 \) is said to belong to \( W^{1,p} \) if \( u \in L^p \), and its first derivative \( \partial u \) in the sense of distributions belongs to \( L^p \). Here, \( p \in [1, \infty] \). We also use the space \( W^{1,1}_s \) of \( L^1 \) functions such that their distributional first partial derivatives are bounded measures. This latter space can be regarded as a “weak version” of \( W^{1,1} \).

In this paper we extend the DiPerna–Lions theory to vector fields \( b \in W^{1,1}_s \). This is important because most commonly used singular vector fields like that with jump singularities along a hypersurface belong to this class, and do not belong to the original DiPerna–Lions class \( W^{1,1} \).

## 3 Extended DiPerna–Lions theory

We build our exposition for simplicity not around differential equations in \( \mathbb{R}^n \), but around differential equations on a torus \( \mathcal{T} = \mathbb{R}^n / \mathbb{Z}^n \). Any other closed manifold is as good as torus for our purposes, but in the torus case we can utilize almost the same classical notations as in the euclidean case. First, state the extended DiPerna–Lions conditions on the vector field \( b \):

\[
\operatorname{div} b \in L^\infty(\mathcal{T}), \quad b \in W^{1,1}_s(\mathcal{T}).
\]
The original DiPerna–Lions conditions were stated in the Euclidean setting and require that

\[
\text{div } b \in L^\infty(\mathbb{R}^n), \quad b \in W^{1,1}_\text{loc}(\mathbb{R}^n), \quad (3.2)
\]

\[
\frac{b(x)}{1 + |x|} \in L^\infty(\mathbb{R}^n) + L^1(\mathbb{R}^n)
\]

which is clearly more involved. We can extend the DiPerna–Lions theory to the Euclidean setup by requiring

\[
\text{div } b \in L^\infty(\mathbb{R}^n), \quad b \in W^{1,1}_\text{*loc}(\mathbb{R}^n), \quad (3.4)
\]

\[
\frac{b(x)}{1 + |x|} \in L^\infty(\mathbb{R}^n) + L^1(\mathbb{R}^n)
\]

instead of (3.2), (3.3).

The only difference is the replacement of the “weak” space \(W^{1,1}_\text{*loc}\) with its “strong” version \(W^{1,1}\).

A typical example of a singular \(b\) which satisfies the DiPerna–Lions conditions (3.2) is the Hamiltonian field with the Hamiltonian function

\[
H(p, q) = \sum_i p_i^2 2m_i + \frac{1}{2} \sum_{i \neq j} e_i e_j |q_i - q_j|^\alpha,
\]

where \(q_i \in \mathbb{R}^3\) and \(\alpha \in (0, 1)\). Note that the Coulomb system (\(\alpha = 1\)) does not satisfy both (3.2) and (3.4). The growth condition (3.3) does not hold for the Hamiltonian field (3.6).

The “less singular” Hamiltonian \(H(p, q) = |p|\), which is typical for the control theory, does not fit the original DiPerna–Lions conditions (3.2), but fits (3.4).

In fact, there is no such a thing as the DiPerna–Lions theorem parallel to that of Cauchy-Lipschitz. What does exist is the DiPerna–Lions theory, which is only partially concerned with ODE.

### 3.1 Transport equation

This is the equation

\[
\frac{\partial u}{\partial t} = \sum_i b_i \frac{\partial u}{\partial x_i}, \quad (3.7)
\]

which is dual to the conservation law equation

\[
\frac{\partial \rho}{\partial t} + \text{div } b \rho = 0 \quad (3.8)
\]

describing evolution of the density of particles moved by the phase flow of (2.1). If the vector field \(b\) is sufficiently regular, say, if the Cauchy-Lipschitz condition holds, one can write down the general solution of (3.7) in terms of the phase flow. Namely,

\[
u(x, t) = u^0(X_t(x)), \quad (3.9)
\]

where the function \(u^0(x) = u(x, 0)\) is arbitrary. In other words, the phase flow of ODE defines and is defined simultaneously by the solution of the Cauchy problem for the transport equation

\[
\frac{\partial u}{\partial t} = b \cdot \nabla u, \quad u(x, 0) = u^0(x), \quad (3.10)
\]
3.2 Renormalizable and approximable solutions

The approach adopted by DiPerna–Lions is to study the Cauchy problem (3.10) for the transport equation without recourse to the corresponding ODE, and then define the phase flow via (3.9).

The first step is to define what is the solution to (3.10). Of course we have a notion of the classical solution: a differentiable function \( u \) which satisfy (3.10). However, one cannot expect to solve (3.10) in the classical sense if the vector field \( b \) is not sufficiently regular. The correct definition of the solution is achieved in two steps. First, we recall the old and well known notion of the weak solution. A function \( u(x, t) \) is a weak solution of (3.10) if for every smooth function \( \phi \) with compact support in \( T \times [0, T) \) (test function) we have

\[
\int_0^T dt \int dx \ u \frac{\partial \phi}{\partial t} = -\int dx \ u^0(x) \phi(x, 0) + \int dx \ u \text{div}(b\phi).
\] (3.11)

In other words, we multiply the formal equality (3.10) by \( \phi \) and formally integrate by part. One can prove the existence of the weak solution of (3.10) under very weak assumptions on the vector field \( b \). E.g., it suffices to assume that \( b \in L^1 \) and \( \text{div} b \in L^1 \).

One cannot, however, guarantee the uniqueness of the weak solution. To restore the uniqueness DiPerna and Lions invented a new notion of renormalizable solution. This requires a new set of test functions. Suppose that \( \beta : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \)-function which is bounded itself and has bounded derivative.

A function function \( u(x, t) \) is a renormalizable solution of (3.10) if for each above \( \beta \) the function \( \beta(u) \) is a weak solution of (3.10). It is clear that if \( u \) is a classical solution, then \( \beta(u) \) also is, but for the weak solutions this transformation may fail to give a solution. In fact, this notion of renormalizability is very close to well known entropy conditions for solutions of nonlinear equations of conservation laws [2], [3].

However, for the extended DiPerna–Lions conditions (3.1) we find it more appropriate to define and work with another type of solutions — approximable solutions. We say that a function \( u \in L^\infty(T \times [0, T]) \) is an approximable solution of (3.10) if \( u \) is a limit \( u_\epsilon(t) \to u(t) \) in \( H^{-s} \) uniformly in \( [0, T] \) of functions \( u_\epsilon \), which are smooth w.r.t. the space variables and satisfy

\[
\frac{\partial u_\epsilon}{\partial t} = \sum_i b_i \frac{\partial u_\epsilon}{\partial x_i} + r_\epsilon, \quad u_\epsilon(x, 0) = u^0_\epsilon(x),
\] (3.12)

where \( r_\epsilon \) are measures w.r.t. \( x \) such that their total variations \( \|r_\epsilon^\prime\| \) are uniformly bounded for each \( t \in [0, T] \) and tend to zero as \( \epsilon \to 0 \). Note that J. Moser [7] stressed that for numerous problems approximate solutions can be more valuable then the exact ones. Our main theorem (extended DiPerna–Lions theorem) is about existence and uniqueness of approximable solution to (3.10).

**Theorem 2.** Suppose that the extended DiPerna–Lions conditions (3.1) hold, and \( u^0 \in L^\infty(T) \). Then there exist a unique approximable solution \( u \) to (3.10). This solution is renormalizable and belongs to

\[
u \in L^\infty(0, T; L^\infty(T)) \cap C([0, T]; L^p(T))
\] (3.13)

for each \( 1 \leq p < \infty \).

In other words, the solution \( u \) with bounded initial condition is bounded and depends on time \( t \) in a continuous way. Denote by \( T_t \) the Cauchy operator

\[
T_t(u^0) = u^t,
\] (3.14)
where \( u'(x) = u(x, t) \). By using Theorem 2 one can restore the phase flow. This requires a general result from functional analysis.

**Theorem 3.** Suppose that

\[
A : L^\infty(\mathcal{T}) \to L^\infty(\mathcal{T})
\]  

(3.15)

is a (automatically continuous) homomorphism of rings with unit \((A(fg) = A(f)A(g), A1 = 1)\). Then, \( A \) is a measurable change of variables: there exists \( \Phi : \mathcal{T} \to \mathcal{T} \) such that \( Au(x) = u(\Phi(x)) \).

One can see easily from the definition of an approximable solution that \( T_t(u^2) = T_t(u)^2 \) (here, \( T_t \) the Cauchy operator and \( u^2 = u \times u \)), and, therefore, one can apply Theorem 3 to \( T_t \). We obtain

\[
T_t(u)(x) = u(X_t(x)),
\]  

(3.16)

where \( X_t \) is a one parameter group of measurable transformations of \( \mathcal{T} \). This is the phase flow we were looking for.

Notice that our construction of the phase flow requires studying the Cauchy problem (3.10) only for bounded initial data. At that point the original approach of DiPerna–Lions is different. They have built the flow \( X_t : \mathbb{R}^n \to \mathbb{R}^n \) as a renormalizable solution to

\[
\frac{\partial X(x, t)}{\partial t} = \sum_i b_i \frac{\partial X(x, t)}{\partial x_i}, \quad X(x, 0) = x,
\]  

(3.17)

where the initial data is surely unbounded. In our approach this primary motivation for introducing and studying renormalizable solutions disappear. Basically by the same reason we do not study the Cauchy problem (3.10) with initial data from \( L^p, p < \infty \).

Another important aspect of our extended DiPerna–Lions theory is the stability theorem for approximable solutions.

**Theorem 4.** Suppose that vector fields \( b_n \in L^1(\mathcal{T}) \) are such that \( b_n, \text{div} (b_n) \) converge in \( L^1 \) to (respectively) \( b, \text{div} b \), where \( b \) satisfy the extended DiPerna–Lions conditions (3.1). Suppose also, that \( u_n \) is a bounded sequence in \( L^\infty(0, T; L^\infty) \) of approximable solutions of (3.10) with \( b \) replaced by \( b_n \), and assume that \( u_n^0 \to u^0 \) in \( L^1 \). Then, \( u_n \) converges as \( n \to \infty \) in \( C([0, T]; L^1) \) to the approximable solution of (3.10) corresponding to the initial condition \( u^0 \).

In terms of the phase flow this means that disturbances of the vector fields which are small in \( L^1 \) and produce small disturbances of the divirgences in \( L^1 \), give a small change of the flow.

### 4 Open questions

We mention only a few arbitrarily chosen issues.

#### 4.1 Formally reversible system with irreversible dynamics

The Coulomb system does not satisfy DiPerna–Lions conditions (3.1), and hypothetically, in general, there is no phase flow and renormalizable solutions in the sense of DiPerna–Lions. One can expect, though, that it is possible to solve the corresponding Cauchy
problem by using the vanishing viscosity method. In other words, we are going to solve the Cauchy problem
\[ \frac{\partial u_\epsilon}{\partial t} = \sum_i b_i \frac{\partial u_\epsilon}{\partial x_i} + \epsilon^2 \Delta u_\epsilon, \quad u_\epsilon(x,0) = u^0(x) \] (4.1)
and then put \( u = \lim_{\epsilon \to 0} u_\epsilon \). The solution of (4.1) should exist only for \( t \geq 0 \) and so should the viscosity solution \( u \). Therefore, the corresponding phase flow is irreversible w.r.t. time.

In the classical language, this probably means that the set of initial conditions for a general Coulomb system, which approach a singular set \( \{x_i = x_j\} \) at finite time, has a positive Liouville measure.

If the above picture is correct, it follows that a formally reversible Newtonian dynamics can be, in fact, irreversible. This is a way to avoid logical contradiction between mechanics and thermodynamics at least at this particular point.

### 4.2 Generalization of the Osgood conditions

The question is: is it possible to find a proper generalization of the Osgood condition in the spirit of the DiPerna–Lions theory. The Osgood condition (which generalizes the Lipschitz one and guarantees the existence and uniqueness of the phase flow) is that
\[ \int_0^1 \frac{dt}{\omega(t)} = \infty, \] (4.2)
where \( \omega \) is a modulus of continuity for the vector field \( b \).

For example, \( \omega(t) = t \log(1/t) \) is a typical modulus satisfying (4.2). One can show that there exists a function \( u \) with this modulus of continuity such that the distributional derivative \( \nabla u \) is not a measure. For instance, the Weierstrass function
\[ u(t) = \sum_{k=1}^{\infty} 2^{-k} \exp(i2^k t) \]
is so. Indeed, the difference \( u(t+h) - u(t) \) is equal to
\[ \sum_{n=1}^M 2^{-n} (\exp(i2^n h) - 1) \exp(i2^n t) + \sum_{n=M+1}^{\infty} 2^{-n} (\exp(i2^n h) - 1) \exp(i2^n t) \] (4.3)
for any \( M \). We choose \( M \) so that \( 2^M h = o(1) \) as \( h \to 0 \). For instance,
\[ M = \log_2(1/h \log(1/h)) \]
is a good choice. Now the first sum in (4.3) can be estimated as \( O(hM) = O(h \log(1/h)) \), because each term \( 2^{-n} (\exp(i2^n h) - 1) \) is \( O(h) \) since \( 2^n h = o(1) \), while the second sum is \( O(2^{-M}) = O(h \log(1/h)) \). This proves that \( h \log(1/h) \) is a modulus of continuity for \( u \).

If the derivative \( f(t) = \sum_{k=1}^{\infty} \exp(i2^k t) \) is a measure, then by the Riesz brothers theorem \[ \text{(if all negative Fourier coefficients of a measure vanish, then it is absolutely continuous w.r.t. the Lebesgue measure)} \] it is an \( L^1 \)-function on the circle \( \mathbb{R}/2\pi\mathbb{Z} \). One can see immediately that
\[ f(2t) = f(t) - e^{2it}, \] (4.4)
and thus,
\[ f(2^m t) = f(t) - \sum_{k=1}^{m} e^{2k it}. \] (4.5)

We have the equality of \(L^1\)-norms
\[ \int_0^{2\pi} |f(2^m t)| dt = \frac{1}{2^m} \int_0^{2\pi} |f(t)| dt = \int_0^{2\pi} |f(t)| dt \]
for any natural \(m\). Therefore, the \(L^1\)-norm of the trigonometric polynomial \(\sum_{k=1}^{m} e^{2k it}\) remains bounded as \(m \to \infty\). This, however, contradicts the now proved Littlewood conjecture \[9], [10] that the \(L^1\)-norm of a polynomial \(\sum_{k=1}^{N} a_k e^{i n_k t}\) such that \(|a_k| \geq 1\) and the integers \(n_k\) are distinct, grows at least like \(C \log N\).

### 4.3 Continuity of the phase flow

The problem is to indicate conditions in the spirit of the DiPerna–Lions theory which guarantee the continuity of the phase flow \(X_t(x)\) w.r.t. \(x\). Another face of the issue is to find \textit{a priori} Sobolev smoothness for the phase flow.

### 4.4 Classical interpretation of measurable phase flow

An example of a related problem is as follows: Does it follow from the existence of measurable state flow for the Hamiltonian \(\text{(3.6)}\) that classical trajectories never hit the singular set \(\{x_i = x_j\}\) for a set of initial points of full measure?

Another problem in this area is the comparison of DiPerna–Lions flows with another kind of flows for discontinuous vector fields, like vibrosolutions, or Filippov’s trajectories.

### 5 Details and proofs

#### 5.1 A priori estimates

We note that the DiPerna–Lions theory is not totally independent of the classical theory around the Cauchy-Lipschitz theorem. All the arguments in the DiPerna–Lions paper go via regularization of the Cauchy problem and then taking a limit as the small regularization parameter \(\epsilon \to 0\). To say something about regularized problem we utilize the classical theory. The following statement about classical solutions of \(\text{(3.10)}\) is trivial.

**Proposition 5.** Let \(u^t(x) = u(x,t)\) be the classical solution of \(\text{(3.10)}\) at time \(t\). Suppose that all data \((b\) and \(u^0)\) is regular. Then
\[ \|u^t\|_{L^\infty} \leq \|u^0\|_{L^\infty}. \] (5.1)

As usual, we will use this a priori estimate to construct solutions to our initial irregular Cauchy problem, so that those solutions satisfy the same estimate. The next \(L^1\) estimate of the classical solutions is almost as easy as the previous \(L^\infty\) one. We present it, in particular, for the sake of explaining the role of the condition \(\text{div } b \in L^\infty\).

**Proposition 6.** Let \(u^t(x) = u(x,t)\) be the classical solution of \(\text{(3.11)}\) at time \(t \in [0,T]\). Suppose that all data \((b\) and \(u^0)\) is regular. Then
\[ \|u^t\|_{L^1} \leq C\|u^0\|_{L^1}, \] (5.2)
where the constant $C$ depends only on $T$ and $M = \sup_{x \in T} |\text{div} b(x)|$.

Proof. If $u$ is a classical solution of (3.10) then $v = |u|$ is a weak solution. By integrating we obtain

$$\int_T v(x, t)dx - \int_T v(x, 0)dx = - \int_0^t \int_T \text{div}(b(x))v(x, s)dx ds$$

and the modulus of the right-hand side is $\leq M \int_0^t \int_T v(x, s)dx ds$. Now, we have for the positive function $f(t) = \int_T v(x, t)dx$ the inequality $f(t) - f(0) \leq M \int_0^t f(s)ds$. It remains to apply the Gronwall lemma to get the desired estimate for $f(t) = \|u_t\|_{L^1}$.

5.2 Regularization

For the sake of regularization we utilize a classical tool: convolution with a $\delta$-shaped sequence of $C^\infty_0$ functions. More precisely, let $\rho \in C^\infty_0(T)$ be a smooth function with a compact support such that $\int_T \rho(x)dx = 1$. We assume that the support lies within a ball on the torus, and the ball lifts homeomorphically to the universal covering $\mathbb{R}^n$. This allows to regard $\rho$ as a function on $\mathbb{R}^n$ and apply to it some simple constructions related to $\mathbb{R}^n$. In particular, for $0 < \epsilon \leq 1$ the function $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$ is well defined, and $\rho_\epsilon \to \delta_0$ in the space of distributions as $\epsilon \to 0$. We define the convolution operator

$$C_\epsilon u(x) = \int_T u(x - y)\rho_\epsilon(y)dy,$$

and will often write $u_\epsilon$ instead of $C_\epsilon u$ for brevity.

The analytic heart of the DiPerna–Lions paper is a statement about commutator of the operators $C_\epsilon$ and our main differential operator (vector field)

$$Bu = \sum_i b_i \frac{\partial u}{\partial x_i} = b \cdot \nabla u.$$ 

Namely, under extended DiPerna–Lions conditions (3.1) the commutator $[B, C_\epsilon]$ is small in a suitable sense as $\epsilon \to 0$. This is not surprising since the operator $C_\epsilon$ becomes arbitrary close to identity as $\epsilon \to 0$, and statements of this kind are well known in the realm of PDE since [8].

**Theorem 7.** Suppose that the extended DiPerna–Lions conditions (3.1) hold. Then the operators $[B, C_\epsilon] : C(T) \to L^1(T)$ are uniformly bounded and tend to zero strongly. In other words, if $u \in C(T)$, then the difference $Bu_\epsilon - (Bu)_\epsilon$ has small $L^1$-norm if $\epsilon$ is small enough.

**Remark.** Under the original DiPerna–Lions conditions (3.2) the corresponding statement is stronger: $[B, C_\epsilon] : L^\infty(T) \to L^1(T)$ is uniformly bounded and tends to zero strongly. In other words, under original DiPerna–Lions conditions $u$ in the proposition may not be continuous.

Proof. As usually, proof is performed in two steps: first, we prove that the operators $[B, C_\epsilon]$ are uniformly bounded for all $\epsilon$ and $B$ subject to uniform bounds (3.1), second, we check that $[B, C_\epsilon]u$ is small if the vector field $b$ and function $u$ are smooth. The second part is, in fact, trivial or, at least, well known, and we skip it.
To prove the first part we start with the explicit formula

$$[B, C_ε]u(x) = \int u(y) \{(b(y) - b(x)) \cdot \nabla ρ_ε(x - y)\} dy - (u \operatorname{div} (b)) \ast ρ_ε. \tag{5.6}$$

We have to bound uniformly the right-hand side in $L^1$, provided that $u$ is continuous, $\|u\|_{L^∞}$, $\|\operatorname{div} (b)\|_{L^∞}$, and $\|b\|_{W^{1,1}}$ are uniformly bounded. This is trivial for the second term $(u \operatorname{div} (b)) \ast ρ_ε$ because of the well known properties of the convolution operator $C_ε : L^∞ \to L^1$.

It remains to estimate

$$\int \left| \int u(y) \{(b(y) - b(x)) \cdot \nabla ρ_ε(x - y)\} dy \right| dx \tag{5.7}$$

which, as one can see after the change of variables $x = y + \epsilon z$, is not greater than

$$C \int_\mathcal{B} \left| \int_\mathcal{T} u(y) \frac{(b(y) - b(y + \epsilon z))}{\epsilon} dy \right| dz, \tag{5.8}$$

where the constant $C = \sup_{z \in \mathcal{B}} |\nabla ρ(z)|$. Here, $\mathcal{B}$ is a small ball, where the support of the mollifier $ρ$ is located. We can regard it as a ball in $\mathbb{R}^n$, and the expression like $y + \epsilon z$ makes sense for $z \in \mathcal{B}$ and $y \in \mathcal{T}$.

Now we note that for $u \in C(\mathcal{T})$

$$\sup_{ε, z \in \mathcal{B}} \left| \int_\mathcal{T} u(y) \frac{(b(y) - b(y + \epsilon z))}{\epsilon} dy \right| dx \leq C \|b\|_{W^{1,1}} \|u\|_{L^∞}, \tag{5.9}$$

where $C$ is an absolute constant, and this gives the desired estimate for the first term in (5.6).

### 5.3 Approximable solutions

**Existence.** Now we apply the above regularization estimates to the Cauchy problem (3.10). We start with an “approximate” Cauchy problem

$$\frac{∂u_δ}{∂t} = b_δ \cdot \nabla u_δ, \quad u_δ(x, 0) = u_0^δ(x), \tag{5.10}$$

where the subscript $δ$ in $b$ and $u^0$ indicates the convolution with the mollifier $ρ_δ$. This problem is regular and can be solved in the classical sense by using the Cauchy-Lipschitz theorem. In particular, the functions $u_δ$ are smooth. Now we consider the “approximate solutions” $u_{δ, ε} = u_δ \ast ρ_ε$ of the Cauchy problem (5.10). Denote by $B_δ$ the operator $b_δ \cdot \nabla$. We have

$$\frac{∂u_{δ, ε}}{∂t} = (b_δ \cdot \nabla u_δ) \ast ρ_ε = b_δ \cdot \nabla u_{δ, ε} - [B_δ, C_ε]u_δ. \tag{5.11}$$

In view of Theorem [7] the remainder $r_{δ, ε} = -[B_δ, C_ε]u_δ$ is uniformly w.r.t. $δ$ small if $ε$ is small. Indeed, $u_δ$ is a continuous function which is a priori bounded by $\|u^0\|_{L^∞}$, while the $W^{1,1}_*$ norm of $b_δ$ is bounded by the $W^{1,1}_*$ norm of $b$. Therefore, we get

$$\frac{∂u_{δ, ε}}{∂t} = b_δ \cdot \nabla u_{δ, ε} + r_{δ, ε}, \quad u_{δ, ε}(x, 0) = u_0^δ(x), \tag{5.12}$$

where the remainder $r_{δ, ε} \in L^∞([0, T]; L^1(\mathcal{T}))$ is uniformly small.
Lipschitz bound. Fix an \( \epsilon > 0 \) and put \( \delta \to 0 \). One can see easily from (6.12) that the functions \( [0, T] \ni t \mapsto u_{\delta, \epsilon}(t) \) are uniformly Lipschitz as functions with values in a “negative” Sobolev space \( H^{-s}(T) \) for some sufficiently large \( s \). Indeed, consider the scalar product \( P \) of the right-hand side of (6.12) with a smooth test function \( \phi(x) \). We have

\[
P = \int (b_{\delta} \cdot \nabla u_{\delta, \epsilon} + r_{\delta, \epsilon}) \phi \, dx = \int (-u_{\delta, \epsilon} \operatorname{div} b_{\delta} + r_{\delta, \epsilon}) \phi - u_{\delta, \epsilon} b_{\delta} \cdot \nabla \phi \, dx
\]

In view of condition (5.11), estimates for \( r_{\delta, \epsilon} \) and \( L^\infty \) a priori estimates for \( u_{\delta, \epsilon} \) one obtains the bound \( |P| \leq C \| \phi \| \), where \( C \) is an absolute constant and \( \| \phi \| = \sup |\phi(x)| + \sup \| \nabla \phi(x) \| \). Thus, the right-hand side of (6.12) is uniformly bounded in \( H^{-s} \) if \( s \) is such that the norm \( \| \phi \| \) is continuous on \( H^s \). By the Sobolev lemma we can take any \( s > \frac{n}{2} + 1 \).

Existence (continued). Now, we can extract a subsequence \( \delta \to 0 \) such that \( u_{\delta, \epsilon}(t) \to u_\epsilon(t) \) in \( H^{-s} \) uniformly on \([0, T]\). Moreover, we can assume that the functions \( r_{\delta, \epsilon} \) converge as distributions to a measure \( r_\epsilon \in L^\infty([0, T]; M(T)) \) which is small with \( \epsilon \). Here, \( M(T) \) stands for the space of measures with the norm given by the total variation:

\[
\| r \|_M = \sup_\phi \int r \phi,
\]

where \( \phi \) runs over continuous functions such that \( |\phi| \leq 1 \). This implies, that

\[
\frac{\partial u_\epsilon}{\partial t} = b \cdot \nabla u_\epsilon + r_\epsilon, \quad u_\epsilon(x, 0) = u_\epsilon^0(x),
\]

where both sides are in \( H^{-s} \). Moreover, all the functions \( u_\epsilon(t) \) are smooth w.r.t. \( x \in T \), so that equation (5.14) is valid in the classical sense. Now we can again apply the same arguments on compactness in \( C([0, T]; H^{-s}) \) and extract a subsequence \( \epsilon \to 0 \) such that \( u_\epsilon(t) \to u(t) \) in \( H^{-s} \) uniformly on \([0, T]\). In other words, we constructed an approximable solution \( u \) of the Cauchy problem (5.10), and the existence part of Theorem 2 is done.

Uniqueness. To prove uniqueness, we have to show that an approximable solution \( u \) with the zero initial condition \( u^0 = 0 \) is zero. Let \( u_\epsilon \) be an approximate classical solutions, satisfying (5.14), where \( u^\prime_\epsilon = 0 \). Put \( v_\epsilon = u^2_\epsilon \). One can see immediately that \( v_\epsilon \) satisfies similar equation

\[
\frac{\partial v_\epsilon}{\partial t} = b \cdot \nabla v_\epsilon + r'_\epsilon, \quad v_\epsilon(x, 0) = 0,
\]

where \( r'_\epsilon = 2u_r r_\epsilon \) is again a measure with a (uniformly w.r.t. \( t \)) small with \( \epsilon \) total variation. This proves, in particular, that \( v = u^2 \) is an approximable solution of (5.10) with initial value \( v^0(x) = u^0(x)^2 \).

Renormalizability. Similar arguments, where we consider \( v_\epsilon = \beta(u_\epsilon) \) instead of \( v_\epsilon = u^\prime_\epsilon \) prove that an approximable solution is renormalizable.

Uniqueness (continued). Now we consider the integral \( I(t) = \int v_\epsilon(x, t) \, dx \), and take the integral of both sides of (6.15). From the bound (5.11) on \( \operatorname{div} b_\epsilon \) we obtain immediately that \( |I(t)| \leq CI(t) \), where \( C \) is an absolute constant, and the integral \( R_\epsilon(t) = \int_T r'_\epsilon(t) \, dx \) is uniformly small. This implies (in view of the Gronwall lemma as applied to the integral inequality \( I(t) \leq C \int_0^t I(s) \, ds + \int_0^t R_\epsilon(s) \, ds \) that \( I(t) \to 0 \) as \( \epsilon \to 0 \) uniformly w.r.t. \( t \). Therefore, \( u_\epsilon \to 0 \) in \( L^2 \), and, therefore, in \( H^{-s} \). Thus, \( u = 0 \) and the uniqueness is done.

Regularity. Now, the only part to be proved of Theorem 2 is the inclusion (5.13). The part \( u \in L^\infty([0, T] \times T) \) is trivial in view of Proposition 5 and it remains to show that
$u \in C([0, T]; L^p(T))$. In other words, we have to prove that if $t_n \to t$ then $u(t_n) \to u(t)$ in $L^p$ for any $p \geq 1$.

What we already know is that $u(t_n) \to u(t)$ as distributions (even in $H^{-s}$). On the other hand, $u(t_n)$ are uniformly bounded (in $L^\infty$). These facts combined imply that $u(t_n) \to u(t)$ weakly in $L^2$. Since $v = u^2$ is also an approximable solution of (3.10) we also get that $u(t_n)^2 \to u(t)^2$ weakly in $L^2$. This allow us to prove easily that $u(t_n) \to u(t)$ (strongly) in $L^2$. Indeed,

$$\int (u(t_n) - u(t))^2 = \int u(t_n)^2 + \int u(t)^2 - 2 \int u(t_n)u(t), \quad (5.16)$$

where the notation $\int f = \int_\mathcal{T} f(x)dx$ is utilized. Since $u(t_n)^2 \to u(t)^2$ weakly, we conclude that $\lim \int u(t_n)^2 = \int u(t)^2$, and

$$\lim \int u(t_n)u(t) = \int u(t)^2$$

in view of the weak convergence $u(t_n) \to u(t)$. Thus, $\lim \int (u(t_n) - u(t))^2 = 0$. Now, since $u$ is a uniformly bounded function, we obtain immediately that $\lim \int |u(t_n) - u(t)|^p = 0$ for any $p \geq 1$, and we are done.

### 5.4 Functional analysis

Here, we prove Theorem 3 about recovering a measurable map from its action on bounded measurable functions. This follows easily from the Dunford–Pettis theorem (cf. [1], [3]). We give the statement of a particular case we need.

**Theorem 8. (Dunford–Pettis theorem)** Suppose that

$$A : L^\infty(\mathcal{T}) \to L^\infty(\mathcal{T}) \quad (5.17)$$

is a bounded operator. Then there exists a unique (modulo null sets of the Lebesgue measure) measure $K(x, dy)$ on $\mathcal{T}$ which is measurable w.r.t. $x \in \mathcal{T}$, and such that for almost every $x \in \mathcal{T}$

$$Af(x) = \int K(x, dy)f(y). \quad (5.18)$$

We have to show that if $A$ is a homomorphism of rings with unit then the measure $K(x, dy)$ is a $\delta$-measure: i.e., is supported by a single point $y = \Phi(x)$.

Indeed, consider the quadratic form $Q(f) = \int (Af^2(x) - Af(x)^2)dx$ which is $\equiv 0$, since $A$ is a homomorphism. In terms of the measure $K(x, dy)$ it is equal to

$$Q(f) = \int_\mathcal{T} \int K(x, dy) [f(y) - Af(x)]^2 dx \equiv 0. \quad (5.19)$$

Therefore,

$$\int K(x, dy) [f(y) - Af(x)]^2 = 0$$

for almost all $x$. For any such an $x$ we get that any function $f$ takes a single value $Af(x) = \int K(x, dy)f(y)$ modulo null sets for the measure $K(x, dy)$. This means that the support of $K(x, dy)$ consists of a single point which is exactly what we have to prove.
5.5 Stability theorem

Theorem 4 can be proved by methods already used in the proof of Theorem 2. We will show that there is a convergent subsequence \( u_n \to u \) in \( C([0, T]; L^1) \) to an approximable solution \( u \). In view of the uniqueness of the approximable solution this will prove the Theorem.

Let \( u_{n, \epsilon} \) be an approximate solution to the Cauchy problem (3.10) with \( b \), resp. \( u^0 \) replaced by \( b_n \) resp. \( u^0_n \). More precisely, we assume that

\[
\frac{\partial u_{n, \epsilon}}{\partial t} = b_n \cdot \nabla u_{n, \epsilon} + r_{n, \epsilon}, \quad u_{n, \epsilon}(x, 0) = u^0_{n, \epsilon}(x),
\] (5.20)

where \( \|r_{n, \epsilon}\|_M \leq \epsilon \), and \( \|u^0_n - u^0_{n, \epsilon}\|_{L^1} \leq \epsilon \). The arguments utilized in the proof of the Lipshitz bound in the previous subsection 5.3 show that the right-hand side of (5.20) is uniformly bounded in \( H^{-s} \) for any \( s > \frac{n}{2} + 1 \). This, in turn, shows that for a subsequence of indices \( n \) we have \( u_{n, \epsilon} \to u_\epsilon \) in \( C([0, T]; H^{-s}) \) and the functions \( u_\epsilon \) satisfy

\[
\frac{\partial u_\epsilon}{\partial t} = b \cdot \nabla u_\epsilon + r_\epsilon, \quad u_\epsilon(x, 0) = u^0_\epsilon(x),
\] (5.21)

where \( \|r_\epsilon\|_M \leq \epsilon \), \( \|u^0 - u^0_\epsilon\|_{L^1} \leq \epsilon \).

Now we can again apply the same arguments on compactness in \( C([0, T]; H^{-s}) \) and extract a subsequence \( \epsilon \to 0 \) such that \( u_\epsilon(t) \to u(t) \) in \( H^{-s} \) uniformly on \([0, T]\).

This function \( u \) is an approximable solution to (3.10), and \( u_\epsilon \) are its approximations.

Moreover, for a subsequence of indices \( n \) we have \( u_n(t) \to u(t) \) in \( H^{-s} \) uniformly on \([0, T]\).

Now, an easy adaptation of the arguments about Regularity from subsection 5.3 shows that \( u_n(t) \to u(t) \) in \( L^p \) for any \( p < \infty \), \( t \in [0, T] \), and \( u \in C([0, T]; L^p) \) for any \( p < \infty \). This implies \( u_n \to u \) in \( C([0, T]; L^p) \) and we are done.

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