Bohm Trajectories as Approximations to Properly Fluctuating Quantum Trajectories

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We explain the approximate nature of particle trajectories in Bohm’s quantum mechanics. They are streamlines of a superfluid in Madelung’s reformulation of the Schrödinger wave function, around which the proper particle trajectories perform their quantum mechanical fluctuations to ensure Heisenberg’s uncertainty relation between position and momentum.

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1. In order to justify modern work on quantum mechanics (QM), one often hears the citation of a remark in a 1964 lecture by Richard Feynman [1]. “I think it is safe to say that no one understands quantum mechanics”. Similarly, Murray Gell-Mann in his lecture at the 1976 Nobel Conference regrets that “Niels Bohr brainwashed the whole generation of theorists into thinking that the job (of finding an adequate presentation of quantum mechanics) was done 50 years ago” [2]. Thus there is no wonder that even now reputable scientists are trying to get our deterministic thinking in line with quantum theory [3].

A theory of this type has been proposed a long time ago. It is based on an observation made as early as 1926, which the proper particle trajectories perform their quantum mechanical fluctuations to ensure Heisenberg’s uncertainty relation between position and momentum.

\[ H = \frac{\hat{p}^2}{2m} + V(x) \]  

is the action and the Hamiltonian of the system. The Lagrangian multiplier \( \lambda \) guarantees that the particle number

\[ N = \int d^3 x \psi^\dagger \psi = \int d^3 x \rho \]

is fixed to render the specific value \( N_0 \).

In the operator language of QM, the second-quantized theory is formulated in terms of field operators \( \hat{\psi}(x,t) \) which are formed from particle annihilation operators as \( \hat{a}_x \ \hat{\psi}(x,t) = e^{iHt/\hbar} \hat{a}_x e^{-iHt/\hbar} \). The \( N \)-body wave functions arise from this by forming matrix elements of the states \( |\psi(t)\rangle \) in a Fock space \( \langle \hat{a}_{x_1}, \ldots, \hat{a}_{x_N} | \)

\[ \Psi_N(x_1, \ldots, x_N; t) = \langle x_1, \ldots, x_N | \psi(t) \rangle \] (5)

Taking the action (2) in the \( N \)-particle Fock space it reads

\[ A_N = \int dt \int dX \ \Psi_N^\dagger(X,t)(i\hbar \partial_t - \hat{H}_N)\Psi_N(X,t) \] (6)

where \( X \) denotes the \( N \)-particle positions \( (x_1, \ldots, x_N) \), and

\[ \hat{H}_N = -\sum_n \left[ \frac{\hbar^2}{2m} \nabla_{x_n}^2 + V_c(x_n) \right] \] (7)

The \( N \)-body wave function [5] satisfies the Schrödinger equation

\[ \hat{H}_N \Psi_N(x_1, \ldots, x_N; t) = i\hbar \partial_t \Psi_N(x_1, \ldots, x_N; t) \] (8)

At this point Madelung [4][5] replaced in 1926 the wave function by a product

\[ \Psi_N \equiv Re^{iS/\hbar} \] (9)

with \( R = \sqrt{\rho} \), and found from the the Schrödinger equation the classical Hamilton-Jacobi equation for \( S \), apart from an extra quantum potential

\[ V_q = -\sum_{k=1}^N \frac{\hbar^2}{2m} \frac{\Delta_k R}{R} \] (10)
The full equation reads
\[ i\hbar \partial_t R - \frac{1}{\hbar} R \partial_t S = \frac{\hbar}{2m} \sum_{k=1}^{N} \left[ R \left( \frac{1}{\hbar} \nabla_k S \right)^2 - 2i \nabla_k R \cdot \frac{1}{\hbar} \nabla_k S - i R \frac{1}{\hbar} \Delta_k S \right] + \frac{1}{\hbar} (V + V_q) R, \tag{11} \]
where \( \Delta_k \equiv \nabla_k^2 \) is the Laplace operator. In this way Madelung interpreted the Schrödinger field as a probability amplitude of a quantum fluid. In light of present-day experiments on low-temperature Bose-Einstein condensates (BEC), we may identify this liquid as a superfluid.

From the particle current density of the Schrödinger field
\[ \mathbf{J}_k \equiv -i \frac{\hbar}{2m} \Psi_N^*(\mathbf{Q},t) \nabla_k \Psi_N(\mathbf{Q},t), \tag{12} \]
with \( \nabla_k = (\partial_1, \ldots, \partial_D) \), and the particle number density
\[ \rho_N \equiv \Psi_N^* \Psi_N, \tag{13} \]
we may identify the superfluid velocity \( \mathbf{V}_k \) by the relation
\[ \rho_N \mathbf{V}_k = \mathbf{J}_k. \tag{14} \]
The famous Bohmian deterministic QM is based on the assumption that the streamlines of superfluid velocity may be interpreted as the possible actual orbits of the single particle under consideration. By integrating the velocities over time one obtains the actual possible trajectories of the particles under consideration. For an \( N \)-body system, the wave function \( \Psi_N \) is called the pilot wave of the particles.

Collecting the imaginary parts in \[11\] yields the continuity equation
\[ \partial_t R^2 = - \sum_{k=1}^{N} \nabla_k (\mathbf{v}_k \cdot R^2), \tag{15} \]
whereas the real parts give
\[ \partial_t S + \frac{1}{2m} \sum_{k=1}^{N} (|\nabla_k S|^2) + V + V_q = 0. \tag{16} \]

In the presence of an electromagnetic vector potential \((A_0, \mathbf{A})\), these equations become
\[ \partial_t R^2 = - \sum_{k=1}^{N} \nabla_k (\mathbf{v}_k \cdot R^2), \tag{17} \]
where \( m \mathbf{v}_k = \mathbf{p}_k = \nabla_k S - (e/c) \mathbf{A} \), and
\[ \partial_t S + eA_0 + \frac{1}{2m} \sum_{k=1}^{N} (|\nabla_k S - \frac{e}{c} \mathbf{A}|^2) + V + V_q = 0. \tag{18} \]

This is the place where we can make the link between QM and Bohm’s theory. We observe that one can replace the gradient kinetic term in the field action \[2\] by setting \[10\]
\[ \hat{\psi}^\dagger \frac{\mathbf{p}^2}{2m} \hat{\psi} \to m \frac{\mathbf{j}^2}{2\rho}, \tag{19} \]
where
\[ \mathbf{j} \equiv \frac{1}{2m} \hat{\psi}^\dagger \nabla \hat{\psi} \tag{20} \]
is the fluctuating current density. Classically, this may be interpreted as describing a cloud of particle probability streaming with a velocity
\[ \mathbf{v} = \frac{\mathbf{j}}{\rho}. \tag{21} \]
This field can be introduced into the quantum mechanical partition function \[1\] as a dummy auxiliary velocity variable by rewriting it as
\[ Z = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}\psi \mathcal{D}\mathbf{v} e^{i[A' + j dt \lambda(t)|N - N_0||]/\hbar}, \tag{22} \]
where
\[ A' = \int dt d^3x \hat{\psi}^\dagger (i\hbar \partial_t - H) \hat{\psi} + \int dt d^3x m \rho \frac{1}{2} \left( \mathbf{v} - \frac{\mathbf{j}}{\rho} \right)^2. \tag{23} \]
If the auxiliary field \( \mathbf{v} \) is fully integrated out of the partition function, we recover the correct Schrödinger quantum mechanics.

We are now prepared to understand in which way the Bohmian QM differs from this correct QM: We simply take the semiclassical approximation \[12\] of the fluctuating velocity field \( \mathbf{v} \), and interprete it as the velocity field of “Bohm trajectories”. By integrating \( \mathbf{v} \) over the time along the streamlines, we calculate \( x(t) = \int_0^t dt \mathbf{v} \) and interprete this as the deterministic position of the quantum particle. The approximate nature of this quantity for describing the motion of particles in the system is obvious.

3. The reader familiar with the standard path integral representation of QM \[11,12\] will recognize that the partition function \[1\] is simply the second-quantized version of the canonical path integral \[14\]:
\[ (x_0,t_0|x_a,t_a) = \int_{x(t_a) = x_a}^{x(t_a) = x_b} D'x \int \frac{Dp}{2\pi\hbar} e^{i[A|x]/\hbar}, \tag{24} \]
with the canonical action
\[ A[p,x] = \int_{t_a}^{t_b} dt \left[ \mathbf{p}(t) \dot{x}(t) - \frac{\mathbf{p}^2(t)}{2m} - V(x(t)) \right]. \tag{25} \]
We note that the first term in this action guarantees the validity of Heisenberg’s uncertainty relation between \( \mathbf{p} \) and \( x \). If we integrate out the fluctuating momentum paths, the amplitude takes the form
\[ (x_0,t_0|x_a,t_a) = \int_{x(t_a) = x_a}^{x(t_a) = x_b} D'x e^{iA_F|x|/\hbar}, \tag{26} \]
with the action

$$A_F[x] = \frac{m}{2} \int_{t_a}^{t_b} dt \left[ x^2(t) - V(x) \right], \quad (27)$$

which was used by Feynman \[11, 12\] to calculate quantum mechanical amplitudes via path integrals by summing over all histories of $x(t)$ in $x$-space.

The QM of Bohm’s is obtained by approximating the path integrals over the fluctuating momenta in two steps. First, one rewrites the initial path integral (24) with the help of a dummy velocity path $v(t)$ as

$$(x_{b,t_b} | x_{a,t_a}) = \int D^{'}x \int Dv \int \frac{Dp}{2\pi\hbar} e^{iA'[p,v,x]/\hbar} (28)$$

in which the action $A[p,x]$ of (24) has been replaced by

$$A'[p,v,x] = \frac{m}{2} \int_{t_a}^{t_b} dt \left[ v(t) - \frac{p(t)}{m} \right]^2 + A[p,x]. \quad (29)$$

The Gaussian path integral over all $v(t)$’s ensures that (28) is the same as the amplitude (24). Second, one approximates the path integral over $v(t)$ in a certain semiclassical way by selecting only the extremum of the first term in (29), i.e., by assuming the velocity $v(t)$ to be equal to $v(t) = p(t)/m$ at each instant of time, rather than performing its proper harmonic quantum fluctuations dancing around $V(t)$ \[13\], to satisfy $v(t) = V(t)$ only on the average. We note that this approximation destroys the validity of Heisenberg’s uncertainty relation. By integrating $V(t)$ over time one obtains functions $X(t)$ which in Bohm’s theory are considered to be the trajectories of the quantum particle guided by the pilot wave. It is therefore evident that Bohmian mechanics is not equivalent to proper QM.

4. It was shown in \[15\] that the drastic variations of the quantum potential (see Fig. 1) in the direction transverse to electron’s motion from the slits to the screen would inevitably induce radiation if the particle does execute Bohmian deterministic classical trajectory, with the emission angle following the direction of the canyon where the particle crosses. This would result in a discrete pattern of such radiation on the screen, which exactly complements the well-known interference pattern of the electron.

With the realization that the Bohmian trajectories are actually semiclassical approximation to the actual fluctuating QM trajectories, we see that this spurious radiation effect indeed should not occur.

5. This interpretation of Bohm’s QM can in principle be tested experimentally \[16\]. For this, one should run a BEC superfluid through a barrier with a double-slit and show that the flow pattern looks like that in Fig. 2 rather than that in Fig. 8.5 on p. 156 of the most complete textbook on Bohm’s theory by Dürr and Teufel \[7\], where the (undisplayed) left-hand part of the figure consists of horizontal straight lines up to the screen \[17, 18\]. The undulations in the flow pattern are caused by the canyons in the quantum potential \[10\], which we have pictured in Fig. 1.

6. As experimentalists are in the process of investigating detailed properties of Bohmian quantum mechanics \[16\], they should be aware that an important aspect of that theory is still absent in Eqs. (11), (16), and (18). That is, the function $S$ is really a multivalued function of configuration space and time \[10\]. Its derivatives $\nabla_k S(Q,t)$ are defined only modulo integer multiples of $2\pi\hbar$ times a delta function in some area $A$ to be denoted by $\delta_k(Q,A;t)$. It is defined by the integral

$$\delta_k(Q,A;t) \equiv \int_{A(t)} d^{3N-3} \bar{Q} \int dA_k \delta(Q - \bar{Q}), \quad (30)$$

where $\int d^{3N-3} \bar{Q}$ runs only over the configuration space of all $\bar{q}_i$ except $\bar{q}_k$, and the vector $\bar{q}_k$ is integrated over the area $A$ \[10\]. Therefore the Bohm equation (11) for the pilot wave is correct only if the gradients of $S$ in that equation are replaced by

$$\nabla_k S(Q,t) \rightarrow \nabla_k S(Q,t) - 2\pi m\hbar \delta_k(Q,A;t), \quad (31)$$
where \( A \) denotes possible surfaces across which the phase jumps by an amount \( 2\pi nm\hbar \), with some integer \( m \). In analogy, a charged particle circulating around an infinitely thin magnetic flux line along the \( z \)-axis has a wave function \( e^{im\phi} \), where \( \phi \) is the azimuthal angle in cylindrical coordinates. The replacement of \( S_\mu \) in Eq. (16) accounts for this effect in general. By analogy with the theory of plasticity, we shall denote the extra term as \( S_\mu^P = 2\pi m\delta(Q, A; t) \) and call it the plastic deformation of the eikonal \( S \).

Similarly we have to replace the time derivative in the first terms of (11), (16), and (18) as
\[
\frac{\partial_t S(Q, t)}{\partial_t S(Q, t) - 2\pi n\hbar(t - t(Q)) = \frac{\partial_t S(Q, t) - S^P_t(Q, t)}{}}
\]

(32)

After these replacements the Bohm equation (16) gives a complete description of the motion of a gas of Bose particles in a zero-temperature condensate if the gas is sufficiently dilute that there are practically no interactions among the particles. In the presence of electromagnetism, the plastic deformations of the eikonal are modified by the usual minimal replacement rules in (17) and (18).

Note that (11) is also the hydrodynamic description of a field \( \Psi(Q, t) \) emerging from a standard Ginzburg-Landau action (19), the only difference is that here the field depends on all 3\( N \) configuration coordinates in \( Q \), rather than only a single coordinate \( x \), as in the original Ginzburg-Landau action, which is a mean-field approximation to a second-quantized many-body action (20).

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