Differential operators associated with the equations of motion and nondissipative heat conduction in the Green–Naghdi theory of thermoelasticity

S. A. Lychev and A. V. Manzhirov

1Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, prosp. Vernadskogo 101-1, Moscow, 119526 Russian Federation
E-mail: lychev@ipmnet.ru

Abstract. We consider a wave (hyperbolic) heat conduction theory of the Green–Naghdi type. In the framework of the continual approach, such a theory permits describing low-temperature phenomena of quantum nature (for example, the second sound effect) that are not within reach of the classical parabolic thermoelasticity. At low temperatures, the medium heat conductivity experiences a substantial increase, which should be reckoned with in the design and analysis of cryogenic devices. This change in heat conduction properties results from the wave character of heat propagation, which cannot be taken into account in classical heat conductivity models of the diffusion type but can be described by hyperbolic models of the Green–Naghdi type. That is why considerable attention has been paid to the development of hyperbolic heat conduction models. The present paper deals with analytic solution methods for the corresponding nonself-adjoint initial-boundary value problems.

1. Introduction

Thermal disturbances in some materials (in particular, liquid and crystalline helium) are known to propagate at low temperatures as progressive waves (Peshkov). This phenomenon is usually referred to as “second sound.” The wave nature of heat propagation substantially changes the medium heat conductivity, and this change should be taken into account when designing cryogenic devices. Various theories have been devised to explain this phenomenon (Tisza–Landau, Atkin et al., etc.). One such theory is the Green–Naghdi theory of nondissipative hyperbolic thermoelasticity, in which the time primitive of the temperature is chosen as an independent constitutive variable. This theory results in very special nonself-adjoint boundary value problems, for which classical approaches apparently fail. The aim of this paper is to obtain closed solutions of related linear problems in the form of spectral expansions in complete biorthogonal systems of root functions (eigenfunctions and associated functions) of the corresponding nonself-adjoint operator pencils.

2. Preliminaries

Consider the motion of a body whose elements are identified with points \( \mathbf{X} \) of a reference configuration. The position \( \mathbf{x} \) occupied by a point \( \mathbf{X} \) in the current configuration at time \( t \) is given by a sufficiently smooth one-to-one mapping \( \mathbf{x} = \chi(\mathbf{X}, t) \). In the current configuration, an arbitrary material volume \( \mathcal{B} \) of the body fills some part \( \mathcal{P} \) of the spatial region occupied
by the entire body. This part is bounded by a closed surface $\partial \mathcal{P}$. The deformation gradient tensor $F = (\nabla_R \mathbf{X})^*$ is defined in a standard manner. (Here the asterisk stands for the transpose.) Further, assume that $J = \det F > 0$. By $\nabla_R$ and $\nabla$ we denote the material and spatial gradients, respectively.

In the framework of the classical field theory [5], the field equations for mass conservation, linear and angular momentum, and energy balance can be postulated as

$$\frac{d}{dt} \int_\mathcal{P} \rho \, dV = 0, \quad \frac{d}{dt} \int_\mathcal{P} \rho \mathbf{v} \, dV = \oint_{\partial \mathcal{P}} \mathbf{t} \, dA + \int_\mathcal{P} \rho \mathbf{b} \, dV,$$

$$\frac{d}{dt} \int_\mathcal{P} \rho \mathbf{X} \times \mathbf{v} \, dV = \oint_{\partial \mathcal{P}} \mathbf{X} \times \mathbf{t} \, dA + \int_\mathcal{P} \mathbf{X} \times \rho \mathbf{b} \, dV,$$

$$\frac{d}{dt} \int_\mathcal{P} \rho \left( \frac{\mathbf{v}^2}{2} + \varepsilon \right) \, dV = \oint_{\partial \mathcal{P}} \rho \left( \mathbf{b} \cdot \mathbf{v} + r \right) \, dA + \int_\mathcal{P} \left( \mathbf{t} \cdot \mathbf{v} - q \right) \, dA.$$

Here $\rho$ is the mass density in the current configuration, $\varepsilon$ is the specific internal energy, $q$ is the surface density of heat flux into $\mathcal{P}$, $r$ is the external heat supply rate per unit mass, $\mathbf{t}$ is the surface traction, and $\mathbf{b}$ represents the external body forces per unit mass.

Taking into account the Euler–Cauchy hypothesis [5], we obtain linear relations between $\mathbf{t}$, $q$, and the outward unit normal $\mathbf{n}$ to $\partial \mathcal{P}$; namely, $\mathbf{t} = \mathbf{n} \cdot \mathbf{T}$ and $q = \mathbf{n} \cdot \mathbf{h}$. Let $\eta$ be the entropy density per unit mass. Suppose that $q = \eta \theta$ and $\eta = \mathbf{n} \cdot \mathbf{S}$, where $\mathbf{S}$ is the entropy flux vector. Since the surface $\partial \mathcal{P}$ is arbitrary, it follows that $\mathbf{h} = \mathbf{S} \theta$. The field entropy balance equation reads

$$\frac{d}{dt} \int_\mathcal{P} \rho \eta \, dV = -\oint_{\partial \mathcal{P}} \mathbf{n} \cdot \mathbf{S} \, dA + \int_\mathcal{P} \rho s \, dV, \quad s = s_i + s_e.$$

Here $s$ is the entropy production rate per unit mass. We suppose that it splits additively into the internal entropy production rate $s_i$ and the external entropy production rate $s_e$. We assume that these terms are independent.

Let $\rho_0$ be the mass density in the reference configuration, let $\mathbf{P}$ be the first Piola–Kirchhoff stress, let $\mathbf{h}_R$ be the material heat flux vector, let $\mathbf{S}_R$ be the material entropy flux vector, and let $\psi$ be the specific Helmholtz free energy; i.e.,

$$\rho_0 = J \rho, \quad \mathbf{P} = J \mathbf{F}^{-1} \mathbf{T}, \quad \mathbf{h}_R = J \mathbf{F}^{-1} \mathbf{h}, \quad \mathbf{S}_R = J \mathbf{F}^{-1} \mathbf{S}, \quad \psi = \varepsilon - \theta \eta.$$

An application of the localization theorem yields the local balance equations

$$\dot{\rho}_0 = 0, \quad \nabla_R \cdot \mathbf{P} - \rho_0 \dot{\mathbf{v}} + \rho_0 \mathbf{b} = 0, \quad \mathbf{P}^* \cdot \mathbf{F}^* = \mathbf{F} \cdot \mathbf{P}, \quad (1)$$

$$\mathbf{P} : \dot{\mathbf{F}} - \mathbf{S}_R : \nabla_R \theta - \rho_0 \dot{\psi} - \rho_0 \theta (s_i + s_e) - \rho_0 \eta \dot{\theta} + \rho_0 r = 0, \quad (2)$$

$$\nabla_R \cdot \mathbf{S}_R - \rho_0 (s_i + s_e) + \rho_0 \dot{\eta} = 0. \quad (3)$$

In conventional thermoelasticity, the temperature gradient $\nabla_R \theta$ is chosen as an independent constitutive variable along with the temperature $\theta$ and the deformation gradient $\mathbf{F}$. In dissipationless thermoelasticity, $\nabla_R \theta$ is is taken instead of $\nabla_R \theta$, where $\theta$ is the so-called thermal displacement field defined as the time primitive

$$\theta(X, t) = \int_{t_0}^{t} \theta(X, \tau) \, d\tau + \theta^0, \quad \theta^0 = \text{const}, \quad (4)$$
of the temperature. Thus, the free energy $\psi$ depends on the following independent variables:

$$\psi = \psi \left( X, \theta, \nabla R \theta, F \right).$$

(5)

We point out that the variable (4) is a rather old notion; it introduced as early as in 1921 by van Dantzig as “thermacy.”

The constitutive equation can be obtained by the Noll–Coleman technique. The time derivative of the free energy is given by

$$\dot{\psi} = \frac{\partial \psi}{\partial \nabla R \theta} \cdot \nabla R \dot{\theta} + \dot{\theta} \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial F} \cdot \dot{F} + \frac{\partial \psi}{\partial \theta} \dot{\theta}. \quad (6)$$

By substituting (6) into the energy balance equation (2), we obtain

$$P : \dot{F} - S_R \cdot \nabla R \theta - \rho_0 \left( \frac{\partial \psi}{\partial \nabla R \theta} \cdot \nabla R \dot{\theta} + \dot{\theta} \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial F} \cdot \dot{F} + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \eta \dot{\theta} \right) + \rho_0 \rho_i = \theta \rho_0 s_i + \theta \rho_0 s_e.$$ 

This equation holds for all thermodynamic processes. Since the internal and external entropy production are independent, we obtain the constitutive equations

$$\eta = -\frac{\partial \psi}{\partial \theta}, \quad P = \rho_0 \frac{\partial \psi}{\partial F}, \quad S_R = -\rho_0 \frac{\partial \psi}{\partial \nabla R \theta}, \quad s_i = \frac{\partial \psi}{\partial \theta}, \quad s_e = \theta^{-1} r. \quad (7)$$

The constitutive equation

$$h_R = -\rho_0 \theta \frac{\partial \psi}{\partial \nabla R \theta}$$

for the material heat flux follows from the third equation in (7). If the internal energy does not explicitly depend on the temperature displacement, then the internal entropy production is zero.\(^1\) This case was considered by Green and Naghdi [4].

3. Linear equations

In this section, we consider three states of the body, namely, the initial state, the first deformed state, and the final deformed state. We assume that the difference between the first and final states is infinitesimal. The thermodynamic variables and forces associated with the final deformed state are marked by an asterisk.

The position $\chi^*$ and temperature $\theta^*$ in the final deformed state can be represented in the incremental form

$$\chi^* = \chi + \varsigma \chi', \quad \theta^* = \theta + \varsigma \theta'.$$ 

(8)

Here $\varsigma$ is a real parameter small enough that its square and higher powers can be neglected, $\varsigma \chi'$ and $\varsigma \theta'$ are the position and temperature increments, and $\chi$ and $\theta$ are the position and temperature in the first deformed state.

The incremental form

$$\theta^* = \int_{t_0}^{t} \left[ \theta(\tau) + \varsigma \theta'(\tau) \right] d\tau = \theta + \varsigma \theta', \quad \theta' = \int_{t_0}^{t} \theta'(\tau) d\tau,$$ 

\(^1\) The lack of dissipation makes it possible to give a variational statement, to which the Noether theorem applies. Thus, the governing equations of the Green–Naghdí theory can be obtained from the variational Lagrangian formalism [6, 7].
for the temperature displacement follows from (4) and (8)\textsubscript{2}. The incremental forms for the deformation gradient and the temperature displacement gradient read
\[ F^\ast = (\nabla R \cdot \chi^\ast) = (\nabla R \cdot (\chi + \xi \chi'))^\ast = F + \xi F', \quad F' = (\nabla R \cdot \chi')^\ast, \quad \nabla R \theta^\ast = \nabla R \theta + \xi \nabla R \theta'. \]

By combining this with (5), we obtain the expansion
\[
\psi^\ast = \psi + \zeta \left( \frac{d\psi^\ast}{d\zeta} \right)_{\zeta=0} + \frac{\zeta^2}{2} \left( \frac{d^2\psi^\ast}{d\zeta^2} \right)_{\zeta=0} + o(\zeta^2)
\]
\[
= \psi + \zeta \left( \frac{\partial \psi}{\partial F} : F' + \frac{\partial \psi}{\partial (\nabla R \theta)} \cdot \nabla R \theta' + \frac{\partial \psi}{\partial \theta} \theta' \right) + \frac{\zeta^2}{2} \left[ F' : \frac{\partial^2 \psi}{\partial F^2} : F' + \nabla R \theta' : \frac{\partial^2 \psi}{\partial (\nabla R \theta)^2} : \nabla R \theta' \right]
\]
\[
+ \frac{\partial^2 \psi}{\partial \theta^2} (\theta')^2 + \nabla R \theta' : \frac{\partial^2 \psi}{\partial (\nabla R \theta) \partial F} : F' + \theta' \frac{\partial^2 \psi}{\partial \theta \partial \theta} : F' + \nabla R \theta' : \frac{\partial^2 \psi}{\partial (\nabla R \theta) \partial \theta} \theta'
\]
\[ + o(\zeta^2) \]
of free energy in powers of \( \zeta \) near the first deformed state. Let us introduce some notation:
\[ \mathfrak{A} = \frac{\partial \psi}{\partial F}, \quad \mathfrak{B} = \frac{\partial \psi}{\partial (\nabla R \theta)}, \quad \mathfrak{C} = \frac{\partial \psi}{\partial \theta}, \quad \mathfrak{D} = \frac{\partial^2 \psi}{\partial (\nabla R \theta)^2}, \quad \mathfrak{G} = \frac{\partial^2 \psi}{\partial \theta^2}, \]
\[ \mathfrak{J} = \frac{\partial^2 \psi}{\partial (\nabla R \theta) \partial F}, \quad \mathfrak{K} = \frac{\partial^2 \psi}{\partial (\nabla R \theta) \partial \theta}, \quad \mathfrak{L} = \frac{\partial^2 \psi}{\partial \theta \partial \theta}. \]

It follows from the commutativity of second derivatives that
\[ \mathfrak{E} = \mathfrak{E}^{(3412)}, \quad \mathfrak{D} = \mathfrak{D}^\ast, \quad \mathfrak{J} = \mathfrak{J}^{(231)}. \]

Now we have
\[ \frac{\partial \psi^\ast}{\partial F^\ast} = \mathfrak{A} + \zeta \left( \mathfrak{E} : F' + \mathfrak{J} : \nabla R \theta' + \mathfrak{L} \theta' \right), \]
\[ \frac{\partial \psi^\ast}{\partial (\nabla R \theta)^\ast} = \mathfrak{B} + \zeta \left( \mathfrak{D} \cdot \nabla R \theta' + \mathfrak{J} : F' + \mathfrak{K} \theta' \right), \quad \frac{\partial \psi^\ast}{\partial \theta^\ast} = \mathfrak{C} + \zeta \left( \mathfrak{G} \theta' + \mathfrak{L} : F' + \nabla R \theta' \mathfrak{K} \right). \]

Since the tensor coefficients \( \mathfrak{A}, \mathfrak{B}, \) and \( \mathfrak{C} \) determine the stress and entropy fields in the first deformed state,
\[ \mathfrak{A} = \rho_0 P, \quad \mathfrak{B} = -\rho_0 S_R, \quad \mathfrak{C} = -\eta, \]
we obtain the expansions
\[ P^\ast = \rho_0 \frac{\partial \psi^\ast}{\partial F^\ast} = P + \zeta P', \quad P' = \rho_0 \left( \mathfrak{E} : F' + \mathfrak{J} : \nabla R \theta' + \mathfrak{L} \theta' \right), \]
\[ S_R^\ast = -\rho_0 \frac{\partial \psi^\ast}{\partial (\nabla R \theta)^\ast} = S_R + \zeta S_R', \quad S_R' = -\rho_0 \left( \mathfrak{D} \cdot \nabla R \theta' + \mathfrak{J} : F' + \mathfrak{K} \theta' \right), \]
\[ \eta^\ast = -\frac{\partial \psi^\ast}{\partial \theta^\ast} = \eta + \zeta \eta', \quad \eta' = -\left( \mathfrak{G} \theta' + \mathfrak{L} : F' + \nabla R \theta' \mathfrak{K} \right). \]

The entropy flux vanishes under the conditions \( \nabla R \theta^\ast = 0 \) and \( \theta^\ast = 0 \). It follows that \( \mathfrak{J} = 0 \). By substituting (9) into (1)\textsubscript{2} and (3), we obtain the incremental equations
\[ \nabla R \cdot P' - \rho_0 \dddot{x}' + \rho_0 b' = 0, \quad \nabla R \cdot S_R' + \rho_0 \dddot{\eta}' - \rho_0 s' = 0. \]
Here $b' = b^* - b$ is the body force increment and $s' = s^* - s$ is the entropy production increment induced by the change in the current configuration.

By taking into account relations (9), one can state the equations of motion and heat conduction in the form

$$
\nabla_R \cdot (\rho_0 \varepsilon : (\nabla_R \chi')^* + \rho_0 \mathcal{L} \dot{\theta}') - \rho_0 \dot{\chi}' + \rho_0 b' = 0,
$$

$$
\nabla_R \cdot (\rho_0 \mathbf{D} : \nabla_R \theta') + (\nabla_R \cdot \rho_0 \mathcal{R}) \dot{\theta}' - \rho_0 \left[ \mathcal{F} \dot{\theta}' + \mathcal{J} : (\nabla_R \chi')^* \right] + \rho_0 s' = 0.
$$

These equations, together with the prescribed boundary conditions and the initial conditions

$$
\chi'|_{t=0} = \chi_0, \quad \theta'|_{t=0} = \theta_0, \quad \dot{\chi}'|_{t=0} = \dot{\chi}_0, \quad \dot{\theta}'|_{t=0} = \dot{\theta}_0,
$$

form a well-posed statement of the initial–boundary value problem.

4. Differential operators

Consider the space of square integrable four-component complex-valued vector functions $(\chi, \theta)$ with the inner product

$$
\langle (\chi_1, \theta_1), (\chi_2, \theta_2) \rangle = \int_{\partial \mathcal{D}} (\chi_1 \cdot \chi_2 + \theta_1 \theta_2) \, dV.
$$

The initial–boundary value problem can be stated as the following Cauchy problem with operator coefficients:

$$
\mathcal{L} (\chi, \theta) = f, \quad \mathcal{L} (\chi', \theta') = A_0 (\chi', \theta') + A_1 \frac{\partial}{\partial t} (\chi', \theta') + A_2 \frac{\partial^2}{\partial t^2} (\chi', \theta'),
$$

$$
A_0 = \begin{pmatrix} A \cdot \nabla_R \nabla_R + B : \nabla_R & C \cdot \nabla_R + D \cr E : \nabla_R \nabla_R + F : \nabla_R & \mathbb{H} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\
0 & \mathbb{G} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \rho_0 & 0 \\
0 & \mathbb{K} \end{pmatrix}.
$$

Here we have used the notation

$$
A = \rho_0 \varepsilon^{(2143)}, \quad B = \nabla_R \cdot \rho_0 \varepsilon^{(1243)}, \quad C = \rho_0 \mathcal{L}, \quad D = \nabla_R \cdot \rho_0 \mathcal{L},
$$

$$
E = \rho_0 \mathbb{D}, \quad F = \nabla_R \cdot \rho_0 \mathbb{D}, \quad G = \nabla_R \cdot \rho_0 \mathbb{R}, \quad H = \rho_0 \mathbb{E}, \quad K = \rho_0 \mathcal{L}^*.
$$

The problem in question gives rise to the polynomial operator pencil

$$
\mathcal{L}_\nu = A_0 + \nu A_1 + \nu^2 A_2,
$$

The domain $\mathcal{D}$ of the pencil $\mathcal{L}_\nu$ is determined by the boundary operator $\mathbb{B}$ as follows:

$$
\mathcal{D} = \left\{ (\chi, \theta) \big| \mathbb{B}(\chi, \theta)|_{\partial \mathcal{D}} = 0 \right\}.
$$

The adjoint pencil $\mathcal{L}_\nu^*$ can be obtained directly from the definition:

$$
\forall (\chi_1, \theta_1) \in \mathcal{D}, \quad (\chi_2, \theta_2) \in \mathcal{D}^* \quad \langle \mathcal{L}_\nu (\chi_1, \theta_1), (\chi_2, \theta_2) \rangle = \langle (\chi_1, \theta_1), \mathcal{L}_\nu^* (\chi_2, \theta_2) \rangle.
$$

Indeed, a standard argument based on the Gauss divergence theorem shows that

$$
\langle \mathcal{L}_\nu (\chi, \theta), (\chi^*, \theta^*) \rangle = \langle (\chi, \theta), \mathcal{L}_\nu^* (\chi^*, \theta^*) \rangle = \langle [\chi^*, \theta^*], [\chi^*, \theta^*] \rangle_{\partial \mathcal{D}} = \int_{\partial \mathcal{D}} \rho_0 n_0 \varepsilon \cdot (\nabla_R \chi^*) \cdot \chi^* - \chi^* \cdot \varepsilon^{(4321)} \cdot (\nabla_R \chi^*)^* \cdot \theta^* \cdot \mathbb{D} \cdot \nabla_R \theta^* + \theta^* \varepsilon^* \cdot \chi^* \cdot \mathbb{D}^* \cdot \nabla_R \theta^* + \mathcal{J} : \nabla_R \chi^* \cdot \chi^*, \theta^* \rangle_{\partial \mathcal{D}}
$$

$$
= \int_{\partial \mathcal{D}} \rho_0 n_0 \varepsilon \cdot (\nabla_R \chi^*) \cdot \chi^* - \chi^* \cdot \varepsilon^{(4321)} \cdot (\nabla_R \chi^*)^* \cdot \theta^* \cdot \mathbb{D} \cdot \nabla_R \theta^* + \theta^* \varepsilon^* \cdot \chi^* \cdot \mathbb{D}^* \cdot \nabla_R \theta^* + \mathcal{J} : \nabla_R \chi^* \cdot \chi^*, \theta^* \rangle_{\partial \mathcal{D}}
$$

$$
= \int_{\partial \mathcal{D}} \rho_0 n_0 \varepsilon \cdot (\nabla_R \chi^*) \cdot \chi^* - \chi^* \cdot \varepsilon^{(4321)} \cdot (\nabla_R \chi^*)^* \cdot \theta^* \cdot \mathbb{D} \cdot \nabla_R \theta^* + \theta^* \varepsilon^* \cdot \chi^* \cdot \mathbb{D}^* \cdot \nabla_R \theta^* + \mathcal{J} : \nabla_R \chi^* \cdot \chi^*, \theta^* \rangle_{\partial \mathcal{D}}
$$

(10)
provided that

\[ L^*_w = A_0^* + \tau A_1 + \tau^2 A_2, \]
\[ A_0^* = \left( \begin{array}{cc} L & 0 \\ K : \nabla R & E^* : \nabla R \end{array} \right), \quad A_2^* = - \left( \begin{array}{cc} \rho_0^* \bigl( C^* \cdot \nabla + D \bigr) \end{array} \right), \]
\[ L^* = \rho_0 E^{(3412)}, \quad E^* = \nabla R^* E^{(3421)}, \quad C^* = \rho_0 C^*, \quad E^* = \rho_0 E. \]

We define the domain \( D^* \) of the adjoint pencil \( L^*_w \) as the set of vector functions \( k^* = (\chi^*, \theta^*) \) such that the bilinear form (10) is zero for arbitrary \((\chi, \theta) \in D\). Hence we obtain the adjoint boundary operator \( \mathcal{B}^* \) in closed form. It is well known that the system of root functions of the pencil \( \mathcal{L}_w \) is biorthogonal to that of the adjoint pencil. Note that if \( D = D^* \), then this system is a basis. This provides a theoretical background for the representation of the solution in the form of a spectral expansion.

5. Solution in the general case

The obtained solutions have the form of expansions based on the systems of eigenfunctions \( k_i \) and \( k^*_j \) that can be found as the solutions of the mutually adjoint boundary value problems

\[ \mathcal{L}_w k_i = 0, \quad \mathcal{L}^*_w k_j = 0, \quad k_i = (\chi_i, \theta_i), \quad k^*_j = (\chi^*_j, \theta^*_j); \quad i = 1, \ldots, \infty; \quad j = 1, \ldots, \infty. \tag{11} \]

We assume that the body \( \mathcal{B} \) is compact. Hence the set of eigenfunctions and the corresponding set of eigenvalues are countable. According to [8], the following biorthogonal relations hold for \( k_i \) and \( k^*_j \):

\[ \langle A_1 k_i, k^*_j \rangle + (\nu_i + \nu_j) \langle A_2 k_i, k^*_j \rangle = 0, \quad \langle A_0 k_i, k^*_j \rangle - \nu_i \nu_j \langle A_2 k_i, k^*_j \rangle = 0. \]

These properties permit one to represent the solution in the form [8]

\[ y = \sum_{i=1}^{\infty} \left[ \left( \langle A_1^* k_i^* \rangle + \nu_i \langle A_2^* k_i^* \rangle, y_0 \right) \exp(\nu_i t) - \int_{0}^{t} \left( f(\tau), k_i^* \right) \exp(\nu_i (t - \tau)) \, d\tau \right] k_i Q_i^{-1}, \]

where

\[ y = (\chi, \theta), \quad y_0 = (\chi_0, \theta_0), \quad f = (\rho_0 b, \rho_0 s), \quad Q_i = \langle A_1 k_i, k_i^* \rangle + 2 \nu_i \langle A_2 k_i, k_i^* \rangle. \]

The spectral expansions in complete biorthogonal systems of root functions (eigenfunctions and associated functions) are of the same form and discussed in details in [8].

In the special case \( A_1 = 0 \) and \( \text{Re} \, \nu_i = 0 \), this representation can be written as follows:

\[ y = \sum_{i=1}^{\infty} \left[ \left( \langle A_2^* k_i^* \rangle, y_0 \right) \cos(\lambda_i t) + \frac{1}{\lambda_i} \left( \langle A_2^* k_i^* \rangle, y_0 \right) \sin(\lambda_i t) - \frac{1}{\lambda_i} \int_{0}^{t} \left( f(\tau), k_i^* \right) \sin(\lambda_i (t - \tau)) \, d\tau \right] k_i N_i^{-1}. \]

Here \( N_i = 2 \langle A_2 k_i, k_i^* \rangle \) and \( \lambda_i = \text{Im} \, \nu_i. \)
6. Solution in a special case
To find a simple example of the proposed representation, consider an isotropic homogenous body and assume that the first deformed state is natural. (That is, the initial stress and entropy flux are zero.) In this case, without loss of generality, it can be assumed that $\chi$ represents the displacements and

$$
\rho = \text{const}, \quad T = 0, \quad S = 0, \quad F \approx I, \quad \nabla_R \approx \nabla, \quad \mathbf{e} = \frac{\lambda}{\rho} I \otimes I + \frac{2\mu}{\rho} (I \otimes I),
$$

where $\lambda$ and $\mu$ are the Lamé elastic moduli and $\Lambda, \kappa,$ and $\alpha$ are the thermomechanical moduli. Thus, the mutually adjoint systems (11) are reduced to

$$
\mu \nabla^2 \chi_i + (\lambda + \mu) \nabla \cdot \chi_i - \alpha \nabla \theta_i - \nu_i^2 \rho \chi_i = 0, \quad \nabla^2 \theta_i - \nu_i^2 \beta \theta_i - \nu_i^2 \gamma \nabla \cdot \chi_i = 0,
$$

$$
\mu \nabla^2 \chi_i^* + (\lambda + \mu) \nabla \cdot \chi_i^* + \sigma_i^2 \alpha \nabla \theta_i^* - \sigma_i^2 \rho \chi_i^* = 0, \quad \nabla^2 \theta_i^* - \sigma_i^2 \beta \theta_i^* + \gamma \nabla \cdot \chi_i^* = 0. \quad (12)
$$

Here $\beta = \kappa/\Lambda$ and $\gamma = \alpha/\Lambda$. Now let us apply the Stokes–Helmholtz resolution

$$
\chi_i = \nabla \phi_i + \nabla \times \psi_i, \quad \nabla \cdot \psi_i = 0, \quad \chi_i^* = \nabla \phi_i^* + \nabla \times \psi_i^*, \quad \nabla \cdot \psi_i^* = 0. \quad (13)
$$

where $\phi_i, \phi_i^*$ and $\psi_i, \psi_i^*$ are scalar and vector functions, respectively. By substituting (13) into (12), we obtain

$$
(\lambda + 2\mu) \nabla^2 \phi_i - \alpha \theta_i - \nu_i^2 \rho \phi_i = 0, \quad \nabla^2 \theta_i - \nu_i^2 \beta \theta_i - \nu_i^2 \gamma \nabla \cdot \chi_i = 0, \quad \mu \nabla \psi_i - \nu_i^2 \rho \psi_i = 0,
$$

$$
(\lambda + 2\mu) \nabla^2 \phi_i^* + \sigma_i^2 \alpha \theta_i^* - \sigma_i^2 \rho \phi_i^* = 0, \quad \nabla^2 \theta_i^* - \sigma_i^2 \beta \theta_i^* + \gamma \nabla \cdot \chi_i^* = 0, \quad \mu \nabla \psi_i^* - \sigma_i^2 \rho \psi_i^* = 0.
$$

In this section, we assume that the desired solutions are generated by scalar potentials alone$^2$ ($\psi_i = \psi_i^* = 0$). It follows in a standard way that

$$
\begin{pmatrix}
\phi_i \\
\theta_i
\end{pmatrix} = \begin{pmatrix}
\varrho \\
\varsigma
\end{pmatrix} \Xi_i.
$$

Here $\varrho$ and $\varsigma$ are constants, $\Xi_i$ is a solution of the auxiliary equation

$$
\nabla^2 \Xi_i = \zeta_i \Xi_i
$$

of the Helmholtz type, and $\zeta_i$ satisfies the algebraic equation

$$
\begin{vmatrix}
(\lambda + 2\mu) \zeta_i - \rho \nu_i^2 & -\alpha \\
-\gamma \nu_i^2 & \zeta_i - \beta \nu_i^2
\end{vmatrix} = 0.
$$

By solving the latter for $\nu_i$, we obtain

$$
\nu_{1,2}^2 = \frac{\zeta_i}{2\rho \beta} \xi_{1,2}, \quad \xi_{1,2} = \rho + \alpha \gamma + \beta (\lambda + 2\mu) \pm \omega, \quad \omega = \sqrt{[\rho + \alpha \gamma - \beta (\lambda + 2\mu)]^2 + 4\alpha \beta \gamma (\lambda + 2\mu)}.
$$

$^2$ This assumption, of course, restricts the problem under consideration to potential displacement fields, body forces, and initial data. However, the rotational components of the above-mentioned fields do not interact with the temperature field. Thus, the rotational components can be omitted in the case of purely thermal excitation.
We point out that the coincidence of the roots $\nu_{11}^2 = \nu_{22}^2$ corresponds to the case of multiple eigenvalues, where the corresponding root space has nonempty chains of associated functions. Further, in the case of simple spectrum the eigenfunctions can be obtained as

$$k_{11} = (\chi_{11}, \theta_{11}) = \left( 1 - \frac{\xi_1}{2\rho}, \nabla \Xi_1, \gamma \nu_{11}^2 \Xi_1 \right), \quad k_{12} = (\chi_{12}, \theta_{12}) = \left( 1 - \frac{\xi_2}{2\rho}, \nabla \Xi_2, \gamma \nu_{12}^2 \Xi_2 \right),$$

$$k_{11}^* = (\chi_{11}^*, \theta_{11}^*) = \left( 1 - \frac{\xi_1}{2\rho}, \nabla \Xi_1, -\alpha \Xi_1 \right), \quad k_{12}^* = (\chi_{12}^*, \theta_{12}^*) = \left( 1 - \frac{\xi_2}{2\rho}, \nabla \Xi_2, -\alpha \Xi_2 \right).$$

It is clear that the solution thus obtained depends on the boundary conditions for the auxiliary scalar function $\Xi$. Consider the Neumann type boundary conditions $n \cdot \nabla \Xi \big|_{\partial B} = 0$. This corresponds to the following boundary conditions in terms of the variables $\chi$ and $\theta$ (where $I$ is the unit tensor)

$$n \cdot \chi = 0, \quad n \cdot T \cdot (I - n \otimes n) = 0, \quad n \cdot \nabla \theta = 0.$$

Now let $B$ be a cube with edge length $\pi$, and let $x, y, z$ be Cartesian coordinates. Assume that the body force and the entropy production are zero, and so are the initial displacements, velocities, and temperature rate; i.e.,

$$b = 0, \quad s = 0, \quad \chi \big|_{t=0} = 0, \quad \dot{\chi} \big|_{t=0} = 0, \quad \theta \big|_{t=0} = 0.$$

Let the initial temperature distribution be given by $\theta_0 = H(z) - H(z - h)$, $0 > h > \pi$, where $H(z)$ is the Heaviside step function. The solution of the auxiliary problem is

$$\Xi_k = \cos nx \cos my \cos lz, \quad \zeta_k = -n^2 - m^2 - l^2.$$

Finally, we obtain the desired solution in the form

$$\chi = w e_z, \quad w = \sum_{i=1}^{\infty} \frac{2\alpha \beta}{\pi i^2 \omega} \sin(\pi i h) \sin(\pi i z) \left( \cos(|\nu_{1i}| t) - \cos(|\nu_{2i}| t) \right),$$

$$\theta = \frac{h}{\pi} + \sum_{i=1}^{\infty} \frac{4\alpha \gamma}{\pi i} \sin(\pi i h) \cos(\pi i z) \left[ \frac{\xi_1 \cos(|\nu_{1i}| t)}{(\xi_1 - 2\rho)^2 + 4\alpha \gamma \rho} + \frac{\xi_2 \cos(|\nu_{2i}| t)}{(\xi_2 - 2\rho)^2 + 4\alpha \gamma \rho} \right].$$

The results can be summarized as follows. The differential operator pencil associated with the equations of motion and nondissipative heat conduction is not self-adjoint. The spectrum of the above-mentioned operators in the isotropic homogeneous case is real. The solution of the corresponding initial-boundary value problem can be represented in the form of a spectral expansion.

**Acknowledgments**

The research was financially supported by the Russian Foundation for Basic Research (under grants Nos. 08-01-00553-a, 08-01-91302-IND a, and 09-08-01194-a) and by the Branch of Power Engineering, Mechanical Engineering, Mechanics, and Control Processes of the Russian Academy of Sciences (Program No. 13 OE).

**References**

[1] Pitaevskii L. P. 1968 Phys. Uspekhi 95 139–44.
[2] Landau L. D. 1967 Phys. Uspekhi 11 495–520.
[3] Atkin R. J., Fox N. and Vasey M. W. 1975 J. Elasticity 5 237–48.
[4] Green A. E. and Naghdi P. M. 1993 J. Elasticity 31 189–209.
[5] Truesdell C. and Toupin R. A. 1960 Handbuch der Physik. The Classical Field Theories. Band III/l pp 226–585.
[6] Kalpakides V. K. and Maugin G. A. 2004 Rep. Math. Phys. 53 371–91.
[7] Lychev S. A. and Semenov D. A. 2008 Vestnik SSU 2 183–217 (in Russian).
[8] Lychev S. A. 2008 Mech. Solids 43 769–784.