Many electron theory of $1/f$-noise in hopping conductivity

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We show that $1/f$-noise in the variable range hopping regime is related to transitions of many-electrons clusters (fluctuators) between two almost degenerate states. Giant fluctuation times necessary for $1/f$-noise are provided by slow rate of simultaneous tunneling of many localized electrons and by large activation barriers for their consecutive rearrangements. The Hooge constant steeply grows with decreasing temperature because it is easier to find a slow fluctuator at lower temperatures. Our conclusions qualitatively agree with the low temperature observations of $1/f$-noise in p-type silicon and GaAs.

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I. INTRODUCTION

At low temperatures the variable-range hopping conductivity of doped semiconductors with strongly localized electrons obeys the Efros-Shklovskii (ES) law,

$$\sigma_{ES} = \sigma_0 \exp \left[ -\frac{1}{T_{ES}^2} \right], \tag{1}$$

where the temperature $T_{ES}$ is defined by the electron-electron interaction at the localization radius $a$ of electronic states,

$$T_{ES} = C \frac{e^2}{k_B a}. \tag{2}$$

Here $C \approx 2.7$, $e$ is the electron charge, $k_B$ is the Boltzmann constant, and $\kappa$ is the dielectric constant of the semiconductor. The conductivity behavior Eq. (1) is used for example in ion-implanted silicon (Si:P:B) bolometers working as detectors for high resolution astronomical X-ray spectroscopy. An absorption of an X-ray increases the temperature of the semiconductor and this increase is detected by the change in its conductivity. The performance of some bolometers is limited by a $1/f$-noise, which obeys the Hooge’s law,

$$\frac{\delta \sigma^2}{\sigma^2} = \frac{\alpha_H(\omega, T)}{\omega N_D}, \tag{3}$$

where $N_D$ is the total number of donors,

$$\delta \sigma^2 = \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \delta \sigma(t) \delta \sigma(0) \rangle, \quad \delta \sigma(t) \equiv \sigma(t) - \langle \sigma(t) \rangle,$$

and $\langle \cdots \rangle$ denotes ensemble average. The dimensionless Hooge factor $\alpha_H(\omega, T)$ measured for different doping levels grows by six orders of magnitude with the decreasing temperature following approximate power law,

$$\alpha_H \propto T^{-6}. \tag{4}$$

Ref. gives a strong evidence that the noise is caused by the electron localization. This work investigates the bulk Si:P:B semiconductor in the range of dopant concentrations on both sides of the metal-insulator transition. The increase of the noise strength by several orders of magnitude, when crossing the metal-insulator transition from metal-like samples to insulating samples was observed. The noise intensity continues to increase with decreasing electron localization radius $a$, what proves the principal significance of the electron localization in the noise formation. A similar behavior of the normalized $1/f$-noise power has been recently reported in the low density hole system of a GaAs quantum well. Thus, these and other experiments show that $1/f$-noise is one of the manifestations of the complex correlated electronic state (Coulomb glass) formed by localized electrons coupled by the long-range Coulomb interaction.

The mechanism of $1/f$-noise in hopping conductivity has been investigated by several theoretical groups. It was first suggested that the $1/f$-noise in the nearest neighbor-hopping transport is associated with electronic traps, in a way similar to McWorter’s idea of $1/f$-noise in MOSFETs. Each trap consists of an isolated donor within a spherical pore of the large radius $r$. Such rare configurations form fluctuators, which have two possible states (empty or occupied) switching back and forth with the very slow rate defined by the tunneling rate of electron out or into the pore

$$\nu(r) = \nu_0 \exp(-2r/a), \tag{5}$$

where $\nu_0 \sim 10^{12}$ s$^{-1}$ is the hopping rate determined by the electron-phonon interaction. When a fluctuator is occupied the host electron cannot participate in the transport and thus the transitions of fluctuators change the effective number of “charge carriers” leading to a noise in a conductivity. The exponential sensitivity of the tunneling rate to the size $r$ of the trap [Eq. (5)] makes the statistics of trap blinking close to the logarithmically uni-
form one leading to $\delta \sigma^2_{\alpha} \propto 1/\omega$.

The idea of a single-electron fluctuator has been extended to the regime of the variable range hopping in later works.\textsuperscript{16,17} These theories require that each fluctuator characterized by the slow relaxation rate $\nu$ has no neighboring donors, which belong to the energy band around the Fermi level of the width $E_\nu = k_B T \ln(\nu_0/\nu)$ in the sphere of the radius $R_\nu = (a/2) \ln(\nu_0/\nu)$ (here and everywhere below we choose the chemical potential as a reference point of energy). According to Ref. \textsuperscript{17}, the Hooge parameter is defined by the probability of such “pore” in four-dimensional energy-coordinate space, which has the form

$$\alpha_H(\omega, T) \propto \exp\left[ -\left( \frac{T}{T_{ES}} \right)^{3/5} \ln\frac{\nu_0}{\omega} \ln^6\frac{\nu_0}{\nu} \right],$$

where $\nu_{ES} = \nu_0 e^{-(T_{ES}/T)^{1/2}}$ is the rate of typical hops contributing to Eq. (6). At small $\omega$ this Hooge parameter decreases very fast with decreasing $\omega$. As a result $1/f$-noise is limited to a relatively narrow frequency interval $(T_{ES}/T)^{1/2} < \ln(\nu_0/\omega) < (T_{ES}/T)^{3/5}$, where dependence of $\alpha_H(\omega)$ is still weaker than $1/\omega$. Observation of $1/f$-noise in a wider range of frequencies $\omega \ll \nu_{ES}$ still remains a challenge for the theory.

In this work we suggest the model of many-electron fluctuators, which possess small relaxation rates $\nu$ with much larger probability then single-electron traps. A new fluctuator is made of $N$ occupied (neutral) and $N$ empty (positively charged) donor sites quasi-ordered into the quasi-cubic lattice with the period $R$. They include only donors with energies within a band with the width

$$U_R = \frac{e^2}{\kappa R},$$

which are required to have somewhat diminished disorder energies (see below). We assume that the cluster (Fig. 1) has two energy minima with energy difference of order of $k_B T$ realized by two disordered chess-board configurations (Fig. 1). Optimizing the “lattice constant” $R$ and the number of electrons $N$ we show below that a $1/f$-noise associated with such clusters behaves as

$$\alpha_H(\omega, T) \sim \exp\left[ -\left( \frac{T}{T_{ES}} \right)^{3/5} \ln^6\frac{\nu_0}{\nu} \right],$$

At small frequencies the absolute value of the exponent in Eq. (8) is apparently much smaller then that for the single electron trap, Eq. (9). As a result, $1/f$-noise exists in the much broader range of frequencies.

$$(T_{ES}/T)^{1/2} \ll \ln(\nu_0/\omega) \ll (T_{ES}/T)^3.$$
set of filling factors of donors $n_i = 0, 1$ is determined by the energy minimum of the Hamiltonian of the classical impurity band.

$$\hat{H} = \sum_i \left[ \phi_i n_i + \frac{e^2}{2\kappa} \sum_{k \neq i} \frac{(1 - n_i)(1 - n_k)}{|\mathbf{r}_i - \mathbf{r}_k|} \right], \quad (9)$$

where $\phi_i = \frac{q}{\kappa} \sum_j \frac{1}{e + |\mathbf{r}_i - \mathbf{r}_j|}$ is the random potential created on the donor $i$ by all acceptors (enumerated by $j$). In the ground state energies of donor states are

$$\epsilon_i = \phi_i - \frac{e^2}{\kappa} \sum_{k \neq i} \frac{1 - n_k}{|\mathbf{r}_i - \mathbf{r}_k|}. \quad (10)$$

It is known that the density of states $g(\epsilon)$ of such a system has a double peak structure with the soft Coulomb gap

$$g(\epsilon) = \frac{3e^2\kappa^3}{\pi e^6} \quad (11)$$

in the middle. At $n_a \sim n_d/2$ the width of the Coulomb gap in the density of states is comparable with the total width of the peak of the density of donor states. It is the Coulomb gap, which leads to the ES law, Eq. 2. A characteristic length of hops leading to Eq. 11 is $R_{ES} = (a/2)(T_{ES}/T)^{1/2}$ and characteristic width of the band of energies of states contributing to Eq. 11 is $E_{ES} = (T_{ES}T)^{1/2}$ (see Refs. 1–3).

III. LONG-LIVING CHESS-BOARD CLUSTERS

A. Chess-Board cluster and its probability

Chess-board clusters (see Fig. 1) are formed by the rare fluctuations in positions and energies of donors. We require the following conditions to be satisfied for the cluster. First, all occupied and empty donors $i$ belonging to the cluster are placed approximately near the sites $R_i$ of a cubic lattice with the period $R$. This requires

$$|\mathbf{r}_i - \mathbf{R}_i| < \eta_R R, \quad \eta_R \leq 1/2, \quad (12)$$

where $\eta_R \sim 1$ is the parameter, restricting the deviations of electrons and holes belonging to the cluster from the cell centers.

In addition, all donor sites $i$ of the cluster interact with the environment. This interaction includes the disorder energy $\phi_i$ produced by acceptors and the Coulomb interaction with “external” empty donors, which do not belong to the cluster, because they either have energies larger than $U_R$ or spatially are away from the cluster

$$\varphi_i = \phi_i + \frac{e^2}{\kappa} \sum_k \frac{1 - n_k}{\mathbf{r}_{ik}} \quad (13)$$

where $\sum'$ indicates that the sum is taken over the “external” charged donors only. For all donors of the cluster disorder energy $\varphi_i$ is required to be smaller than the characteristic energy $U_R$, Eq. 4, of the Coulomb interaction within the cluster

$$\varphi_i < \eta_E U_R, \quad \eta_E < 1. \quad (14)$$

This second requirement involves the energy constraint parameter, $\eta_E$, which should be sufficiently small to keep the two minimal energy states chess-board like, Fig. 1.

First we evaluate the probability $p_s$ for each chess-board cell to have the donor, located nearby its center with the diminished disorder energy. This probability is given by the product of the density of electronic states within the domain of the allowed energy, \[ \int_0^{\eta_R} (E) dE, \] [see Eq. 14] and the allowed volume $4\pi R^3/3$, Eq. 12. Then using the definition 11 of the density of states within the Coulomb gap we get

$$p_s = 4\pi R^3/3 \quad (15)$$

As discussed previously, $\eta_R < 1$ and $\eta_E < 1/2$ in order to have some ordering of the structure. These two constraints make the probability $p_s$ substantially smaller than 1. It is convenient to express the probability $p_s$ in the exponential form as $p_s = \exp(-\lambda/2)$. The probability to satisfy constraints Eqs. 12, 14 for all $2N$ sites is given by the product of all $2N$ probabilities $p_s$

$$p_{tot} = p_s^{2N} \approx \exp(-\lambda N). \quad (16)$$

Thus the probability to form the chess-board cluster with two energy minima separated by the large potential barrier decreases exponentially with the number of electrons within the cluster. It is important that within the Coulomb gap the parameter $\lambda$ is the number of order of unity.

To characterize $1/f$ conductivity noise induced by clusters one has to define the cluster unit volume probability density, $P(N, \Delta, R)$, of a transition energy $\Delta$, a cell size $R$, and a number of donors $N$. By definition, the product $P(N, \Delta, R) dN dR d\Delta$ characterizes the number of clusters in the unit volume with the cell size in the domain $(R, R + dR)$, energy in the range $(\Delta, \Delta + d\Delta)$ and the number of donors in the range $(N, N + dN)$. The main dependence of the cluster density on the number of donors $N$ is given by the exponent $\eta_E$. Next, we should define the pre-exponential factor. We will see that since the probability rapidly decreases with increasing $N$ the characteristic $N$ are not very large and, therefore calculating the pre-exponential factor we can assume that $N \sim 1$.

We ignore the weak logarithmic dependence of cluster density on the energy $\Delta$ due to the dipole gap. Then the probability density of clusters, $P(N, \Delta, R)$, does not depend on $\Delta$ and can be expressed in the way

$$P(N, \Delta, R) \approx \frac{e^{-N\lambda}}{R^3 U_R} = \frac{\kappa e^{-N\lambda}}{e^2 R^3} \quad (17)$$
FIG. 2: Thermally activated transition of the “chess-board” cluster between two energy minima A and C along the route $A \to B \to C$. The activation energy is proportional to the number of electrons $N^{2/3}$ in the domain wall shown in the panel B by dashed lines.

Indeed, the density distribution of the parameters $N$, $\Delta$, and $R$ is given by the probability of finding a chess-board cluster having $N$ donors, $p_{\text{tot}} \propto e^{-\lambda N}$, divided by the cluster’s volume, $N R^3$, and the typical energy bandwidth of these donors, $U_R \sqrt{N}$. The probability density of clusters is then

$$P(N, \Delta, R) \propto \frac{p_{\text{tot}}}{N R^3 U_R}.$$ 

Below we will keep track of the cluster size, $N$, only in the exponent. In this way we arrive at Eq. (17). This expression can be written staring just from the dimensionality requirement. The function $P(N, \Delta, R)$ cannot depend on other system parameters because of the universality of excitations within the Coulomb gap.

B. Transition rate

Transition between two states, A and C, of a chess-board cluster (Fig. 2) can be made via consecutive transition of single electrons. In the beginning this leads to monotonous growth of energy until proper saddle point is reached. Thus the rate of a thermally activated transition between the two states of fluctuator can be estimated using the minimum activation energy barrier $E_A$ separating two configurations. For the “chess-board” structure in Fig. 2 a minimum potential barrier (saddle point) is defined by the energy of the “domain wall” formation. One has to move all electrons in the one (left most) column of the cluster A (see Fig. 2) to the next positions B to the right side. The activation energy of that move reads $E_A \approx N^{2/3} U_R$, because the number of electrons belonging to a domain wall is $(N^{1/3})^2 = N^{2/3}$. Then this domain wall can move in the system without acquiring or releasing a significant energy. When the domain wall reaches the right side the transition is completed. The rate of this process can be estimated using the standard Arrhenius law with the activation energy $E_A$

$$\nu_A = \nu_0 e^{-N^{2/3} U_R / k_B T}. \quad (18)$$

An alternative process of transition between energy minima A and C, Fig. 2, is simultaneous tunneling of all electrons of the cluster. This transition of all $N$ electrons takes place in the $N$-th order of the perturbation theory in the weak tunneling amplitude of an electron $t \propto e^{R/a}$, so it can be approximately described by the exponential law

$$\nu_{\text{tun}} \approx \nu_0 e^{-2NR/a}. \quad (19)$$

The overall cluster relaxation rate $\nu$ can be approximately expressed as the sum of the thermally activated and tunneling rates [Eqs. (18), (19)]:

$$\nu = \nu_0 \left[ \exp \left( -\frac{N^{2/3} U_R}{k_B T} \right) + \exp \left( -\frac{2NR}{a} \right) \right]. \quad (20)$$

C. Distribution of clusters over their relaxation rates

The distribution function of “chess-board” clusters over their relaxation rates $\nu$ (inverse relaxation times) can be calculated using Eqs. (17), (20). To calculate this distribution function it is convenient to invert the function $\nu(N, R)$ given by Eq. (20) to obtain $N(\nu, R)$. The function $N(\nu, R)$ crosses over from $N_{\text{ac}}(\nu, R) = (k_B T / U_R)^{3/2} \ln^{3/2} (\nu_0 / \nu)$ to $N_{\text{tun}}(\nu, R) = (a / 2R) \ln(\nu_0 / \nu)$ at

$$R_c \approx a \left( \frac{T_{\text{ES}}}{T} \right)^{3/5} \ln^{-1/5} \frac{\nu_0}{\nu},$$

$$N_c \approx \left( \frac{T_{\text{ES}}}{T} \right)^{-3/5} \ln^{6/5} \frac{\nu_0}{\nu}. \quad (21)$$

These values correspond to the case when the thermal activation and tunneling transition rates in Eq. (20) are equal each other. Indeed, at small $R$ tunneling is more frequent than a thermally activated motion. Then according to Eq. (20) at a given $\nu$ we have $N \propto R^{-1}$. Since the cluster probability decreases with $N$, it grows with $R$. This increase takes place until the crossover point $R_c$, where the thermal activation becomes comparable with the tunneling. Further increase of $R$ will reduce the cluster probability, because for thermally activated processes the number of cluster sites grows as $N \sim R^{3/2}$. Therefore, the quantities $R_c$ and $N_c$ are the optimum parameters, which are defined by the crossover between thermal activation and tunneling regimes

$$\nu / \nu_0 = e^{-2N_c R_c / a} = e^{-N^{2/3} / a} R_c k_B T. \quad (22)$$

The cluster probability density $f(\nu, \Delta)$ can be found using the previously defined probability density
\[ P(N, \Delta, R) \], Eq. \[ \text{[17]} \], as
\[ f(\nu, \Delta) = \int_0^\infty dR \sum_{N=1}^\infty P(N, R, \Delta) \times \delta \left[ \nu - \nu_0 \left(e^{-\frac{N^2/3}{U_R/k_B T} + e^{-2NR/a}}\right) \right] \approx \frac{1}{\nu R_{ES}^3 k_BT_{ES}} e^{-\lambda N_c} \]
\[ = \frac{1}{\nu R_{ES}^3 k_BT_{ES}} \exp \left[ -\lambda \left( \frac{T}{T_{ES}} \right)^{3/5} \ln^{6/5} \frac{\nu_0}{\nu} \right]. \quad (23) \]

The above equation is valid only if \( N_c \gg 1 \), which means \( \nu \ll \nu_{ES} \) and \( R < R_{ES} \). At \( \nu \sim \nu_{ES} \) we arrive at \( R = R_{ES} \) and \( N_c = 1 \), and clusters instead of slow modulating of ES hopping conductivity become a part of the conducting network of the ES variable range hopping. As discussed previously, we set \( N_c \sim 1 \) in the pre-exponential factor, which yields \( R_c \approx R_{ES} \) where \( R_{ES} = a(T_{ES}/T)^{1/2} \) is ES hopping length.

One can approximate Eq. \[ \text{[23]} \] by the power law
\[ f(\nu, \Delta) \propto \frac{1}{\nu} \left( \frac{\nu}{\nu_0} \right)^\beta, \quad (24) \]
with
\[ \beta = -\frac{d \ln f}{d \ln \nu} + 1 = \frac{6 \lambda}{5} \left( \frac{T}{T_{ES}} \right)^{3/5} \ln^{1/5} \frac{\nu_0}{\nu}. \quad (25) \]

Obviously the statistics \( 1/\nu \) is applicable if the exponent \( \beta \ll 1 \). According to Eq. \[ \text{[21]} \] this takes place at frequencies \( \nu \gg \nu_{min} \), where
\[ \nu_{min} \approx \nu_0 \exp \left[ -(5/6\lambda)^5 \left( T_{ES}/T \right)^3 \right]. \quad (26) \]

In this region the exponential term in Eq. \[ \text{[23]} \] depends on \( \nu \) only weakly, so the main dependence is defined by the \( 1/\nu \)-statistics. As we mentioned in Introduction, \( \nu_{min} \) cannot be practically distinguished from zero at low temperature.

Note that the realistic variable range hopping measurements are done typically close to metal-insulator transition, where \( n_d a^3 \approx 1 \). In this case, the inequality \( \nu \gg \nu_{min} \) always means that \( R \gg n_d^{-1/3} \) and \( U_R \) does not exceed the width of the Coulomb gap. However, if in the lightly doped limit \( n_d a^3 \ll 1 \) strictly speaking the distribution Eq. \[ \text{[23]} \] is applicable within the Coulomb gap only, when \( R > n_d^{-1/3} \). It can be shown that the case \( a < R < n_d^{-1/3} \), where \( U_R \) exceeds the width of the Coulomb gap, only a minor revision of our theory is needed. Namely, the parameter \( \lambda \) should be increased by a logarithmically large factor.

**IV. CONDUCTIVITY NOISE**

Below we discuss how cluster transitions induce a \( 1/f \)-noise of the electric current. The hopping conductivity Eq. \[ \text{[11]} \] is provided by the critical network of resistances percolation cluster. According to the Shklovskii-De Gennes model of infinite percolation cluster\[ \text{[2]} \], one can view this network as the random lattice formed by one-dimensional links of the approximate length \( R_L \sim aT_{ES}/T \). Each link is a chain of \( \sqrt{T_{ES}/T} \) donor sites separated by the hopping length \( R_{ES} \sim a\sqrt{T_{ES}/T} \).

Following the ideas of Kozub\[ \text{[15]} \], we consider the conduction noise induced by the cluster transitions affecting adjacent links. A link has the critical hop with the largest resistance \( r_h \propto e^{\sqrt{T_{ES}/T}} \) which is comparable with the resistance of the whole chain. The noise is defined by the clusters located in the vicinity of these critical sites. Assume that the critical hop is separated from the nearest neighbor cluster by the distance \( r \) and this cluster makes slow transitions between its two energy minima. The transition of the cluster changes its characteristic dipole moment by the value \( \mu \sim e\sqrt{T_{ES}/T} \) which is comparable with the thermal energy.

\[ R < R_{int} = a(T_{ES}/T)^{3/4}, \quad (27) \]

the resistance of the critical hops changes substantially, leading to the addition or removal of the whole link. The fluctuation of the sample conductivity induced by a change of one link can be expressed as
\[ \delta \sigma \sim \frac{\sigma_{ES}}{N_l} = \frac{\sigma_{ES} R_{ES}^3}{V}, \quad (28) \]

where \( V \) is the system volume and the ratio \( N_l \approx V/R_{int}^3 \) estimates the total number of links.

Only clusters with the energy \( \Delta \) comparable to the thermal energy are able to contribute to the noise, while the contribution of others is exponentially suppressed. Therefore, the relevant clusters density for the noise at frequency \( \omega \) is \( n_{\omega} \approx k_BT f(\omega, k_BT) \), where \( f(\omega, k_BT) \) is given by Eq. \[ \text{[29]} \]. The probability \( p_l \) that the given link has the cluster located nearby, Eq. \[ \text{[27]} \], can be expressed through the cluster density as \( p_l \approx f(\omega, k_BT) k_BT R_{int} \).

Then the noise at the frequency \( \omega \) is induced by \( N_{\omega} \sim p_l N_l \) clusters making random contributions. Using Eq. \[ \text{[28]} \] one can express the noise intensity as
\[ \frac{\delta \sigma^2}{\sigma_{ES}^2} \sim \frac{N_{\omega}}{N_l^2} = \frac{p_l}{N_l}. \quad (29) \]

Using the latter equation one arrives at the Hooge parameter
\[ \alpha_H = n_d a^3 \left( \frac{T_{ES}}{T} \right)^{11/4} \exp \left[ -\frac{\lambda}{\left( \frac{T}{T_{ES}} \right)^{3/5} \ln^{6/5} \frac{\nu_0}{\nu}} \right]. \quad (30) \]

The main exponential term of Eq. \[ \text{[29]} \] apparently agrees with Eq. \[ \text{[5]} \]. According to this equation, at low temperature the prefactor of \( \alpha_H \) is much larger than unity. This happens because standard normalization of \( \alpha_H \) to
the total number of donors (or electrons) is not natural for the variable-range hopping, where only small fraction of all donors participates in transport.

It is the straightforward consequence of the cluster distribution given by Eqs. (23), (30) that for ES hopping, the fluctuators, defined by the single electron pores have much smaller density (see Eq. (6)) and they can be always neglected. Note that this is not the case for Mott variable-range hopping as discussed in Sec. V.

In the derivations of Eqs. (23) and (30) we assumed that \( E_{\text{ES}} = k_B \sqrt{T} T_{\text{ES}} \ll e^2 n_d^{1/3}/\kappa \), \( R_{\text{ES}} = (a/2)(T_{\text{ES}}/T)^{1/2} \gg n_d^{1/3} \), and the variable range hopping takes place. If these conditions are violated and \( k_B T \gg e^2 n_d^{2/3} a/\kappa \) one deals with the nearest neighbor hopping conductivity, where the Coulomb interaction plays a secondary role, see Ref. [2]. In this case there is a range of not very small frequencies

\[
\frac{1}{n_d^{1/3} a} < \frac{\nu}{\omega} < \frac{1}{n_d^{1/3} a} \left( \frac{k_B e^2}{e^2 n_d^{2/3} a} \right)^{1/3},
\]

where \( 1/\nu \)-noise is determined by single-electron pores. At very small frequencies again the chess-board clusters take over.

Returning to the variable range hopping we can generalize our theory to a two-dimensional system. In this case the density of states is proportional to \( |e| \) rather than to \( e^2 \). In addition, the “domain wall” energy is proportional to \( \sqrt{N} \), while the cluster volume is proportional to \( R^2 \). Taking these changes into account and following the previous derivation we arrive at the expression

\[
\alpha_H^{(2d)}(\omega, T) \propto \exp \left[ -\lambda \left( \frac{T}{T_{\text{ES}}} \right)^{2/3} \ln^{4/3} \left( \frac{T}{T_{\text{ES}}} \right) \right].
\]  

(31)

We see that in 2d the \( 1/\nu \)-noise amplitude \( \alpha_H \) is smaller than in 3d. This happens because of smaller activation energy in 2d systems compared to 3d systems (see Sec. III B). Similarly to a 3d case, in 2d a deviation from \( 1/\nu \)-behavior becomes large at very small frequencies. The criterion for \( 1/\nu \)-noise reads

\[
\nu_{\text{ES}} \gg \omega \gg \omega_{\text{min}} \approx \nu_0 \exp \left[ -\frac{3}{4\lambda} \left( \frac{T_{\text{ES}}}{T} \right)^2 \right].
\]  

(32)

Although the low-frequency cutoff \( \omega_{\text{min}} \) exceeds that for a 3d system it is still hardly distinguishable from zero at temperatures \( T \lesssim T_{\text{ES}}/5 \).

Our theory cannot lead to an observable \( 1/\nu \)-noise in 1d case, because in this case the domain wall contains one electron, its energy is small and it is practically impossible to construct a slow enough trap at a finite temperature.

V. MOTT VARIABLE RANGE HOPPING.

In this section we briefly discuss the case of an amorphous semiconductor, where the bare density of states is diminished by a strong compositional disorder of non-Coulomb nature, so that the Coulomb gap is relatively narrow and the variable range hopping conductivity obeys the Mott law \( \ln \sigma_M \sim (T_{\text{M}}/T)^{-(d+1)/2} \).

For this purpose we depart from the original model Eq. (34) and consider the Efros lattice model characterized by Hamiltonian with a strong non-Coulomb disorder

\[
\hat{H} = \sum_i \phi_i n_i + \frac{1}{2} \sum_{ij} \kappa_{ij} n_i n_j.
\]

(33)

Here \( n_i = \pm 1/2 \) stand for a hole and an electron, respectively. The density of bare states \( \phi_i \) is defined as \( \phi_0 = n/2W \), where \( n \) is the concentration of lattice sites \( i \) and \( W \) is the characteristic energy of disorder, i.e., random potentials \( \phi_i \) are distributed uniformly within the domain \((-W, W)\). We assume that the dimensionless ratio \( A = W \kappa/\epsilon^2 n_1^{1/3} \) is much larger than unity. In this case the Coulomb gap is much narrower than the width of the density of states \( W \).

The results of our theory are summarized in Table I. The left half of this table characterizes the variable-range hopping at different temperatures and dimensionalities \( d = 2, \) and \( 3 \) for both the classical impurity band model and the Efros model. At low temperatures the variable range hopping conductance obeys the ES law, Eq. (1), while at higher temperature it obeys the Mott law.

The right half of Table I compares contributions from chess-board clusters, \( \alpha_H \), and one-electron pores, \( \alpha_{\text{pore}} \). As we mentioned above the pores are not important within the range of validity of the ES law. However, they become competitive in the case of the Mott law. Remarkably, in spite of the fact the Coulomb interaction is irrelevant to the Mott variable range conductivity, the very low frequency \( 1/\nu \)-noise is still determined by the chess-board clusters bound by the Coulomb interaction. Naturally single electron pores dominate at higher frequencies close to the frequency of Mott hops. They also become more important with decreasing dimensionality because in a low-dimensional system it is easier to create a pore, but it is more difficult to create a slow chess-board cluster. Therefore, the range applicability of pores is broader in the 2d case.

VI. COMPARISON OF THE THEORY WITH THE EXPERIMENT.

Our analysis predicts the strong temperature increase of the \( 1/\nu \)-noise at low temperatures due to the exponential factor \( e^{-F(\omega, T)} \). This expectation qualitatively agrees with the experiments, where the Hooge parameter is approximately proportional to \( T^{-6} \).

As an example, let us discuss the results of experiments on ion implanted Si:P:B showing the ES temperature dependence of the conductance. In this material the Hooge parameter, \( \alpha_H \), has been measured as a function of temperature in the wide range from 1.4 K to 44 K.
for different levels of doping leading to different temperatures $T_{\text{ES}}$. According to Ref. 4, behavior of the Hooge parameter can be described by the empirical law

$$\alpha_H = 0.034 T_{\text{ES}}^{0.453} (T/0.153)^{-5.2 - 0.9 \log_{10} T_{\text{ES}}}.$$  (34)

Here $T$ and $T_{\text{ES}}$ are measured in Kelvin degrees. In particular, in the material with $T_{\text{ES}} \approx 11\,\text{K}$ the relative resistance fluctuations at $\omega \sim 1\,\text{Hz}$ increased from $10^{-11}$ to $10^{-7}$ as temperature decreased from 0.3 to 0.008 K.

Let us compare this empirical law with Eq. (30) derived in the previous section. Note that while deriving that equation we have neglected the powers of $N_c$ in the pre-exponential factors since $N_c$ cannot be much larger than unity because of the factor $e^{-\lambda N_c}$, see Eq. (23). It is still important to take into account the temperature dependence of the minimum transition rate $\nu_0$ because the cluster transitions are induced by their interaction with phonons. We assume that $\nu_0 \propto T^3$ and take it in the form $\nu_0 = 10^{12} T^3$, where $T$ is measured in Kelvin degrees, to have $\nu(T) \approx T^3/h$ at the energy comparable to the Debye energy. Then for the sample with $T_{\text{ES}} = 11\,\text{K}$ one can obtain a quite reasonable fit of the experimental data setting $\lambda = 2.395$ and $n_0 d^3 = 8.4 \times 10^{-3}$, as shown in Fig. 9.

The estimate for the dimensionless constant $\lambda$ agrees with the expectation $\lambda/2 \sim 1$. The value of $n_0 d^3$ seems to be too small for the material used in the experiments performed in the vicinity of the metal-insulator transition. However, our fit is not sufficiently accurate in interpretation of pre-exponential factors. In particular, the fitted factor $n_0 d^3$ can be significantly underestimated due to neglecting of the $N_c$-dependent pre-exponential factor. What can be even much more important for the deeper understanding of $1/f$ noise in noisy samples is the fact that the density of dopants in them is not uniform, but depends on the distance from the surface.

The $1/\nu$ distribution of clusters over their relaxation rates, $\nu$. The slow fluctuations of cluster states modulate the critical resistors forming the backbone cluster leading to the $1/f$—noise in the electronic conductivity.

Our predictions regarding the magnitude and temperature dependence of the Hooge parameter, $\alpha_H$, are at least in a qualitative agreement with experiments in ion-implanted silicon (Si:P:B). In particular, both specific

![FIG. 3: Comparison of the theory (solid line, Eq. (30) and the experiment (solid line with circles, Eq. (54) for the case $\omega/2\pi = 1\,\text{Hz}$ and $T_{\text{ES}} = 11\,\text{K}$. We have used $\lambda = 2.395$ and $n_0 d^3 = 0.0084$ to make the optimum data fit in the range of temperatures (0.07-0.3 K) studied in the experiment.](image)
features – dramatic decrease of the noise with the temperature increase and its increase with the increase of $T_{ES}$ – are explained. Moreover, the experimental results can be fitted quantitatively using reasonable values of the adjustable parameters.

The low-temperature ($T < 0.2$ K) experimental values of the Hooge parameter exceed unity. This fact also agrees with Eq. (30), since at low temperatures the main temperature dependence is given by the prefactor. Experimental results obtained in Refs. 8, 9 are also in a qualitative agreement with our theory.

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